Some Approximation Results of Kantorovich type operators

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In this manuscript, we investigate a variant of the operators define by Lupaş. We compute the rate of convergence for different class of functions. In section 3, the weighted approximation results are established. At the end, stated the problems for further research.

Keyword: Positive linear operators; Rate Convergence; Weighted approximation

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1 Introduction

In [1], Lupaş proposed to study the following sequence of linear and positive operators

$$P_n^{[0]}(f,x) = 2^{-nx}\sum_{k=0}^{\infty}\frac{(nx)_k}{2^k k!}f\left(\frac{k}{n}\right), \quad x\geq 0, \quad f:[0,\infty)\to\mathbb{R}, \tag{1.1}$$

where $(nx)_0 = 1$ and $(nx)_k = nx(nx+1)(nx+2)\dots(nx+k-1)$, $k\geq 1$.
We can consider that $P_n^{[0]}$, $n\geq 1$, are defined on E where $E = \bigcup\limits_{a>0} E_a$ and E_a is the subspace of all real valued continuous functions f on $[0,\infty)$ such as $\sup\limits_{x\geq 0}(\exp(-ax)|f(x)|) < \infty$. The space E_a is endowed with the norm $\|f\|_a = \sup\limits_{x\geq 0}(\exp(-ax)|f(x)|)$ with respect to which it becomes a Banach space.
In recent year, Patel and Mishra [2] generalized Jain operators type variant of the Lupaş operators defined as

$$P_n^{[\beta]}(f,x) = \sum_{k=0}^{\infty}\frac{(nx+k\beta)_k}{2^k k!}2^{-(nx+k\beta)}f\left(\frac{k}{n}\right), \quad x\geq 0, \quad f:[0,\infty)\to\mathbb{R}, \tag{1.2}$$

where $(nx+k\beta)_0 = 1$, $(nx+k\beta)_1 = nx$ and $(nx+k\beta)_k = nx(nx+k\beta+1)(nx+k\beta+2)\dots(nx+k\beta+k-1)$, $k\geq 2$.
We mention that $\beta = 0$, the operators $P_n^{[0]}$ reduce to Lupaş operators (1.1). In

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[2], the authors have used following Lagrange's formula to define the operators (1.2):

$$\phi(z) = \phi(0) + \sum_{k=1}^{\infty} \frac{1}{k!} \left[\frac{d^{k-1}}{dz^{k-1}} ((f(z)^k)\phi'(z) \right]_{z=0} \left(\frac{z}{f(z)} \right)^k. \quad (1.3)$$

But, if we use following Lagrange's formula then the generalization of the operators (1.1) is written better way:

$$\phi(z) \left[1 - \frac{z}{f(z)} \frac{df(z)}{dz} \right]^{-1} = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\frac{d^k}{dz^k} ((f(z)^k)\phi(z) \right]_{z=0} \left(\frac{z}{f(z)} \right)^k.$$

By choosing $\phi(z) = (1-z)^{-\alpha}$ and $f(z) = (1-z)^{\beta}$, we have

$$\begin{aligned} &(1-z)^{-\alpha} \left[1 - z\beta(1-z)^{-1} \right]^{-1} \\ = &\sum_{k=0}^{\infty} \frac{1}{k!} (\alpha + k\beta)(\alpha + k\beta + 1) \dots (\alpha + k\beta + k - 1) \left(\frac{z}{(1-z)^{-\beta}} \right)^k. \end{aligned}$$

Taking $z = \frac{1}{2}$, we get

$$1 = (1-\beta) \sum_{k=0}^{\infty} \frac{1}{2^k k!} (\alpha + \beta k)_k 2^{-(\alpha+\beta k)}.$$

Now, we may define the operators as

$$P_n^{[\beta]}(f, x) = \sum_{k=0}^{\infty} \mathbf{p}_{\beta}(k, nx) f\left(\frac{k}{n} \right) \quad (1.4)$$

where $\mathbf{p}_{\beta}(k, nx) = (1-\beta) \sum_{k=0}^{\infty} \frac{1}{2^k k!} (nx + \beta k)_k 2^{-(nx+\beta k)}$. where $(nx + \beta k)_0 = 1$ and $(nx + \beta k)_k = (nx + \beta k)(nx + \beta k + 2) \dots (nx + \beta k + k - 1)$, $k \geq 1$ and $0 \leq \dfrac{\beta + 1}{2} < 1$

The parameter β may depend on the natural number n. It is easy to see that for $\beta = 0$, the operators $P_n^{[\beta]}(f, x)$ reduces to Lupaş operator (1.1). We mention that, the operators (1.2) and (1.4) has no much difference as their moments are same. To calculate the moments of (1.4), we follows techniques developed in [2] and along the lines of [2], we have

$$\begin{aligned}
P_n^{[\beta]}(1,x) &= 1\\
P_n^{[\beta]}(t,x) &= \frac{x}{1-\beta}+\frac{2\beta}{n(1-\beta)^2}\\
P_n^{[\beta]}(t^2,x) &= \frac{x^2}{(1-\beta)^2}+\frac{2x(1+2\beta)}{n(1-\beta)^3}+\frac{6\left(\beta+\beta^2\right)}{n^2(1-\beta)^4}\\
P_n^{[\beta]}(t^3,x) &= \frac{x^3}{(1-\beta)^3}+\frac{6x^2(1+\beta)}{n(1-\beta)^4}+\frac{6x\left(1+6\beta+3\beta^2\right)}{n^2(1-\beta)^5}\\
&\quad+\frac{2\left(13\beta+34\beta^2+13\beta^3\right)}{n^3(1-\beta)^6}\\
P_n^{[\beta]}(t^4,x) &= \frac{x^4}{(1-\beta)^4}+\frac{4x^3(3+2\beta)}{n(1-\beta)^5}+\frac{36x^2\left(1+3\beta+\beta^2\right)}{n^2(1-\beta)^6}\\
&\quad+\frac{2x\left(13+146\beta+209\beta^2+52\beta^3\right)}{n^3(1-\beta)^7}+\frac{30\left(5\beta+23\beta^2+23\beta^3+5\beta^4\right)}{n^4(1-\beta)^8}.
\end{aligned}$$

In the present paper, we modify the operators defined by (1.4) into integral form in Kantorovich sense, see also G.G. Lorentz [3, Ch.II, p.30]. Actually, we replace $f\left(\frac{k}{n}\right)$ by an integral mean of $f(x)$ over a small interval around the point $\frac{k}{n}$ as follows

$$K_n^{[\beta]}(f,x)=n\sum_{k=0}^{\infty}\mathbf{p}_\beta(k,nx)\int_{k/n}^{(k+1)/n}f(t)dt, \tag{1.5}$$

where $\mathbf{p}_\beta(k,nx)$ was as defined in (1.4) and f belongs to the class of local integrable functions defined on $[0,\infty)$.
The focus of the paper is to investigate these linear and positive operators. Section 2, provided results in connection with the rate of convergence for $K_n^{[\beta]}$ under different assumptions of the function f.

2 Approximation properties

For any integer $s\geq 0$, we denote by e_s the test function, $e_s(x)=x^s$, $x\geq 0$, and we also introduce the s-th order central moment of the operator $K_n^{[\beta]}$, that is

$$\Omega_{n,s}(x)=K_n^{[\beta]}(\psi_{x,s},x),\ \text{where } \psi_{x,s}(t)=(t-x)^s, x\geq 0, t\geq 0.$$

Lemma 1 *The operators $K_n^{[\beta]}$, $n\in\mathbf{N}$ defined by (1.5), verify*

1. $K_n^{[\beta]}(1,x)=1$;
2. $K_n^{[\beta]}(t,x)=\frac{x}{1-\beta}+\frac{(1+\beta)^2}{2n(1-\beta)^2}$;

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3. $K_n^{[\beta]}(t^2,x) = \dfrac{x^2}{(1-\beta)^2} + \dfrac{x\left(3+2\beta+\beta^2\right)}{n(1-\beta)^3} + \dfrac{1+20\beta+12\beta^2+2\beta^3+\beta^4}{3n^2(1-\beta)^4}$;

4. $$K_n^{[\beta]}(t^3,x) = \frac{x^3}{(1-\beta)^3} + \frac{3x^2\left(5+2\beta+\beta^2\right)}{2n(1-\beta)^4} + \frac{x\left(10+32\beta+15\beta^2+2\beta^3+\beta^4\right)}{n^2(1-\beta)^5} + \frac{1+142\beta+219\beta^2+96\beta^3+19\beta^4+2\beta^5+\beta^6}{4n^3(1-\beta)^6}$$

5. $$\begin{aligned} K_n^{[\beta]}(t^4,x) &= \frac{x^4}{(1-\beta)^4} + \frac{2x^3\left(7+2\beta+\beta^2\right)}{n(1-\beta)^5} \\ &+ \frac{2x^2\left(25+44\beta+18\beta^2+2\beta^3+\beta^4\right)}{n^2(1-\beta)^6} \\ &+ \frac{x\left(43+326\beta+329\beta^2+116\beta^3+23\beta^4+2\beta^5+\beta^6\right)}{n^3(1-\beta)^7} \\ &+ \frac{1+1072\beta+3398\beta^2+2824\beta^3+900\beta^4+174\beta^5+28\beta^6+2\beta^7+\beta^8}{5n^4(1-\beta)^8}. \end{aligned}$$

Proof: Observe that $K_n^{[\beta]}(1,x) = P_n^{[\beta]}(1,x) = 1$.
Now,

$$\begin{aligned} K_n^{[\beta]}(t,x) &= n\sum_{k=0}^{\infty} \mathbf{p}_\beta(k,nx) \int_{k/n}^{(k+1)/n} t \ \ dt \\ &= \sum_{k=0}^{\infty} \mathbf{p}_\beta(k,nx) \frac{1+2k}{2n} = \frac{1}{2n} P_n^{[\beta]}(1,x) + P_n^{[\beta]}(t,x) \\ &= \frac{x}{1-\beta} + \frac{(1+\beta)^2}{2n(1-\beta)^2}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} K_n^{[\beta]}(t^2,x) &= n\sum_{k=0}^{\infty} \mathbf{p}_\beta(k,nx) \int_{k/n}^{(k+1)/n} t^2 \ \ dt \\ &= \sum_{k=0}^{\infty} \mathbf{p}_\beta(k,nx) \frac{1+3k+3k^2}{3n^2} \\ &= \frac{1}{3n^2} P_n^{[\beta]}(1,x) + \frac{1}{n} P_n^{[\beta]}(t,x) + P_n^{[\beta]}(t^2,x) \\ &= \frac{x^2}{(1-\beta)^2} + \frac{x\left(3+2\beta+\beta^2\right)}{n(1-\beta)^3} + \frac{1+20\beta+12\beta^2+2\beta^3+\beta^4}{3n^2(1-\beta)^4}. \end{aligned}$$

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$$\begin{aligned}
K_n^{[\beta]}(t^3,x) &= n\sum_{k=0}^{\infty}\mathbf{p}_{\beta}(k,nx)\int_{k/n}^{(k+1)/n} t^3 \quad dt \\
&= \sum_{k=0}^{\infty}\mathbf{p}_{\beta}(k,nx)\left(\frac{1+4k+6k^2+4k^3}{4n^3}\right) \\
&= \frac{1}{4n^3}P_n^{[\beta]}(1,x)+\frac{1}{n^2}P_n^{[\beta]}(t,x)+\frac{6}{4n}P_n^{[\beta]}(t^2,x)+P_n^{[\beta]}(t^3,x) \\
&= \frac{x^3}{(1-\beta)^3}+\frac{3x^2\left(5+2\beta+\beta^2\right)}{2n(1-\beta)^4}+\frac{x\left(10+32\beta+15\beta^2+2\beta^3+\beta^4\right)}{n^2(1-\beta)^5} \\
&\quad +\frac{1+142\beta+219\beta^2+96\beta^3+19\beta^4+2\beta^5+\beta^6}{4n^3(1-\beta)^6}
\end{aligned}$$

and

$$\begin{aligned}
K_n^{[\beta]}(t^4,x) &= n\sum_{k=0}^{\infty}\mathbf{p}_{\beta}(k,nx)\int_{k/n}^{(k+1)/n} t^4 \quad dt \\
&= \sum_{k=0}^{\infty}\mathbf{p}_{\beta}(k,nx)\left(\frac{1}{5n^4}+\frac{k}{n^4}+\frac{2k^2}{n^4}+\frac{2k^3}{n^4}+\frac{k^4}{n^4}\right) \\
&= \frac{1}{5n^4}P_n^{[\beta]}(1,x)+\frac{1}{n^3}P_n^{[\beta]}(t,x)+\frac{2}{n^2}P_n^{[\beta]}(t^2,x) \\
&\quad +\frac{2}{n}P_n^{[\beta]}(t^3,x)+P_n^{[\beta]}(t^4,x) \\
&= \frac{x^4}{(1-\beta)^4}+\frac{2x^3\left(7+2\beta+\beta^2\right)}{n(1-\beta)^5}+\frac{2x^2\left(25+44\beta+18\beta^2+2\beta^3+\beta^4\right)}{n^2(1-\beta)^6} \\
&\quad +\frac{x\left(43+326\beta+329\beta^2+116\beta^3+23\beta^4+2\beta^5+\beta^6\right)}{n^3(1-\beta)^7} \\
&\quad +\frac{1+1072\beta+3398\beta^2+2824\beta^3+900\beta^4+174\beta^5+28\beta^6+2\beta^7+\beta^8}{5n^4(1-\beta)^8}.
\end{aligned}$$

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Lemma 1 implies the following identities

$$\Omega_{n,0}(x) = 1, \quad \Omega_{n,1}(x) = \frac{x\beta}{1-\beta} + \frac{(1+\beta)^2}{2n(1-\beta)^2}, \tag{2.1}$$

$$\Omega_{n,2}(x) = \frac{x^2\beta^2}{(1-\beta)^2} + \frac{x\left(2+\beta+2\beta^2+\beta^3\right)}{n(1-\beta)^3} + \frac{1+20\beta+12\beta^2+2\beta^3+\beta^4}{3n^2(1-\beta)^4}, \tag{2.2}$$

$$\begin{aligned}\Omega_{n,3}(x) &= \frac{x^3\beta^3}{(1-\beta)^3} + \frac{3x^2\beta\left(4+\beta+2\beta^2+\beta^3\right)}{2n(1-\beta)^4} \\ &+ \frac{x\left(9+13\beta+23\beta^2+12\beta^3+2\beta^4+\beta^5\right)}{n^2(1-\beta)^5} \\ &+ \frac{1+142\beta+219\beta^2+96\beta^3+19\beta^4+2\beta^5+\beta^6}{4n^3(1-\beta)^6},\end{aligned} \tag{2.3}$$

$$\begin{aligned}\Omega_{n,4}(x) &= \frac{x^4\beta^4}{(1-\beta)^4} + \frac{2x^3\beta^2\left(6+\beta+2\beta^2+\beta^3\right)}{n(1-\beta)^5} \\ &+ \frac{2x^2\left(6+18\beta+25\beta^2+26\beta^3+12\beta^4+2\beta^5+\beta^6\right)}{n^2(1-\beta)^6} \\ &+ \frac{x\left(42+185\beta+252\beta^2+239\beta^3+100\beta^4+19\beta^5+2\beta^6+\beta^7\right)}{n^3(1-\beta)^7} \\ &+ \frac{1+1072\beta+3398\beta^2+2824\beta^3+900\beta^4+174\beta^5+28\beta^6+2\beta^7+\beta^8}{5n^4(1-\beta)^8}.\end{aligned} \tag{2.4}$$

Remark 1 *Since* $\beta \in [0,1)$, $(1-\beta)^2 \leq 1$ *and* $(1-\beta)^{-2} \leq (1-\beta)^{-3} \leq (1-\beta)^{-4}$, *we have*

$$\Omega_{n,1}(x) \leq \frac{2xn\beta + (1+\beta)^2}{2n(1-\beta)^2} \tag{2.5}$$

and

$$\begin{aligned}\Omega_{n,2}(x) &\leq \frac{3n^2x^2\beta^2 + 3nx\left(2+\beta+2\beta^2+\beta^3\right) + 1+20\beta+12\beta^2+2\beta^3+\beta^4}{3n^2(1-\beta)^4} \\ &\leq \frac{3n^2x^2\beta^2 + 6nx\,(1+2\beta) + 1 + 35\beta}{3n^2(1-\beta)^4}.\end{aligned} \tag{2.6}$$

Also, using $\max\{1, x, x^2, x^3, x^4\} \leq (1+x+x^2+x^3+x^4)$, *we have*

$$\begin{aligned}\Omega_{n,4}(x) &\leq \left(\frac{\beta^4}{(1-\beta)^8} + \frac{20}{n(1-\beta)^8} + \frac{180}{n^2(1-\beta)^8} \right. \\ &\left. + \frac{840}{n^3(1-\beta)^8} + \frac{8400}{5n^4(1-\beta)^8}\right)(1+x+x^2+x^3+x^4) \\ &\leq B_\beta(n)(1+x+x^2+x^3+x^4),\end{aligned} \tag{2.7}$$

where

$$B_\beta(n) = \frac{\beta^4}{(1-\beta)^8} + \frac{20}{n(1-\beta)^8} + \frac{180}{n^2(1-\beta)^8} + \frac{840}{n^3(1-\beta)^8} + \frac{8400}{5n^4(1-\beta)^8}. \tag{2.8}$$

Theorem 1 *Let $K_n^{[\beta_n]}$ be defined by (1.5) and $\beta_n \in [0,1)$ with $\beta_n \to 0$. Then for $f \in C[0,\infty)$ one has $\lim_{n\to\infty} K_n^{[\beta_n]}(f,\cdot) = f$ uniformly on any compact $K \subset [0,\infty)$.*

Proof: By making use of Lemma 1, we have

$$\lim_{n\to\infty} K_n^{[\beta_n]}(t^j, x) = x^j, \text{ with } \beta_n \to 0$$

$j = 0, 1, 2$, uniformly on any compact $K \subset [0,\infty)$. Consequently, our assertion follows directly from the well-known theorem of Bohman-Korovkin.
Let $C_B[0,\infty)$ denote the space of real valued continuous and bounded functions f on the interval $[0,\infty)$, endowed with the norm

$$\|f\| = \sup_{0\le x\le\infty} |f(x)|$$

For any $\delta > 0$, Peetre's K-functional is define by

$$K_2(f,\delta) = \inf_{g\in C_B^2[0,\infty)} \{\|f-g\| + \delta\|g''\|\},$$

where $C_B^2[0,\infty) = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$. By DeVore and Lorentz [4, P.177, Theorem 2.4], there exists an absolute constant $C > 0$ such that

$$K_2(f,\delta) \le C\omega_2\left(f, \sqrt{\delta}\right), \tag{2.9}$$

where the second order modulus of smoothness of $g \in C_B[0,\infty)$ is defined as

$$\omega_2(g;\delta) = \sup_{0<h\le\delta} \sup_{x\ge} |g(x) - 2g(x+h) + g(x+2h)|, \qquad \delta > 0,$$

also usual modulus of continuity of $f \in C_B[0,\infty)$ is defined by

$$\omega_1(g;\delta) = \sup_{0<h\le\delta} \sup_{x\ge 0} |g(x+h) - g(x)|, \qquad \delta > 0.$$

Theorem 2 *Let $K_n^{[\beta]}$ be defined by (1.5) and $\beta \in [0,1)$ then for each $x \ge 0$ the following inequality*

$$|K_n^{[\beta]}(f,x) - f(x)| \le \frac{4}{3}\omega_1\left(f; \frac{\sqrt{3n^2x^2\beta^2 + 6nx\,(1+2\beta) + 1 + 35\beta}}{3n(1-\beta)^2}\right)$$

holds.

Proof: Since $K_n^{[\beta]}(1,x) = 1$ and $\mathbf{p}_\beta(k,nx) \ge 0$, we can write

$$|K_n^{[\beta]}(f,x) - f(x)| \le n\sum_{k=0}^{\infty} \mathbf{p}_\beta(k,nx) \int_{k/n}^{(k+1)/n} |f(t) - f(x)|dt. \tag{2.10}$$

On the other hand

$$|f(t) - f(x)| \le \omega_1(f; |t-x|) \le (1 + \delta^{-2}(t-x)^2)\omega_1(f;\delta).$$

For $|t-x|<\delta$ the last increase is clear. For $|t-x|\geq\delta$, we use the following properties

$$\omega_1(f;\lambda\delta)\leq(1+\lambda)\omega_1(f;\delta)\leq(1+\lambda^2)\omega_1(f;\delta),$$

where one can choose $\lambda=\delta^{-1}|t-x|$. This way the relation (2.6) implies

$$\begin{aligned}|K_n^{[\beta]}(f,x)-f(x)| &\leq n\sum_{k=0}^{\infty}\mathbf{p}_\beta(k,nx)\int_{k/n}^{(k+1)/n}(1+\delta^{-2}(x-t)^2)\omega_1(f;\delta)dt\\ &= \left(\Omega_{n,0}(x)+\delta^{-2}\Omega_{n,2}(x)\right)\omega_1(f;\delta)\\ &= \left(1+\delta^{-2}\left\{\frac{3n^2x^2\beta^2+6nx\,(1+2\beta)+1+35\beta}{3n^2(1-\beta)^4}\right\}\right)\omega_1(f;\delta).\end{aligned}$$

Choosing $\delta=\left(\dfrac{3n^2x^2\beta^2+6nx\,(1+2\beta)+1+35\beta}{n^2(1-\beta)^4}\right)^{1/2}$, we obtain the desired result.

Further, we estimate the rate of convergence for smooth functions.

Theorem 3 *Let $K_n^{[\beta]}$ be defined by (1.5) and $\beta\in[0,1)$. Then for $f\in C^1[0,\infty)$ and $a>0$ one has*

$$|K_n^{[\beta]}(f,x)-f(x)|\leq\frac{1}{2n(1-\beta)^2}\left(b_n\|f'\|_{C[0,a]}+c_n\omega_1\left(f';\frac{1}{\sqrt{n}}\right)\right),$$

where $b_n=2an\beta+(1+\beta)^2$ and
$c_n=2\sqrt{n^2a^2\beta^2+2na\,(1+2\beta)+1+35\beta}$
$\left(1+(1-\beta)^{-2}\sqrt{na^2\beta^2+2a\,(1+2\beta)+(1+35\beta)n^{-1}}\right)$.

Proof: We can write

$$f(x)-f(t)=(x-t)f'(x)+(x-t)(f'(\xi)-f'(x)),$$

where $\xi=\xi(t,x)$ is a point of the interval determinate by x and t. If we multiply both members of this inequality by $n\mathbf{p}_\beta(k,nx)\int_{k/n}^{(k+1)/n}dt$ and sum over k , there follows

$$\begin{aligned}|K_n^{[\beta]}(f,x)-f(x)| &\leq |f'(x)|\Omega_{n,1}(x)\\ &\quad+n\sum_{k=0}^{\infty}\mathbf{p}_\beta(k,nx)\int_{k/n}^{(k+1)/n}|x-t|\cdot|f'(\xi)-f'(t)|dt\\ &\leq \left(\frac{2xn\beta+(1+\beta)^2}{2n(1-\beta)^2}\right)\max_{x\in[0,a]}|f'(x)| \qquad (2.11)\\ &\quad+n\sum_{k=0}^{\infty}\mathbf{p}_\beta(k,nx)\int_{k/n}^{(k+1)/n}|x-t|(1+\delta^{-1}|t-x|)\omega_1(f';\delta)dt.\end{aligned}$$

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According to Cauchy's inequality, we have

$$\begin{aligned}
& n\sum_{k=0}^{\infty}\mathbf{p}_{\beta}(k,nx)\int_{k/n}^{(k+1)/n}|x-t|dt \\
& \leq \sqrt{n}\sum_{k=0}^{\infty}\mathbf{p}_{\beta}(k,nx)\left\{\int_{k/n}^{(k+1)/n}(x-t)^2dt\right\}^{1/2} \\
& \leq \sqrt{n}\left\{\left[\sum_{k=0}^{\infty}\mathbf{p}_{\beta}(k,nx)\right]\left[\sum_{k=0}^{\infty}\mathbf{p}_{\beta}(k,nx)\int_{k/n}^{(k+1)/n}(x-t)^2dt\right]\right\}^{1/2}.
\end{aligned}$$

Hence,

$$n\sum_{k=0}^{\infty}\mathbf{p}_{\beta}(k,nx)\int_{k/n}^{(k+1)/n}|x-t|dt \quad \leq \quad \sqrt{\Omega_{n,2}(x)}. \tag{2.12}$$

Using inequalities (2.12) in (2.11), we write

$$\begin{aligned}
|K_n^{[\beta]}(f,x)-f(x)| &\leq \left(\tfrac{2an\beta+(1+\beta)^2}{2n(1-\beta)^2}\right)\|f'\|_{C[0,a]} \\
&+\sqrt{\Omega_{n,2}(x)}\left(1+\delta^{-1}\sqrt{\Omega_{n,2}(x)}\right)\omega_1(f';\delta)dt.
\end{aligned} \tag{2.13}$$

Inserting $\delta=\dfrac{1}{\sqrt{n}}$ and using $\sqrt{\Omega_{n,2}(x)}\leq\dfrac{\sqrt{n^2a^2\beta^2+2na\,(1+2\beta)+1+35\beta}}{n(1-\beta)^2}$, $x\in[0,a]$, the proof of our theorem is complete.

Theorem 4 *Let $f\in C_B[0,\infty)$. Then for all $x\in[0,\infty)$ there exists a constant $A>0$ such that*

$$|K_n^{[\beta]}(f,x)-f(x)|\leq A\omega_2\left(f,\xi_n(x)\right)+\omega_1\left(f,\frac{x}{1-\beta}+\frac{(1+\beta)^2}{2n(1-\beta)^2}\right),$$

where $\xi_n(x)=\frac{3n^2x^2\beta^2+6nx(1+2\beta)+1+35\beta}{3n^2(1-\beta)^4}+\left(\frac{2xn\beta+(1+\beta)^2}{2n(1-\beta)^2}\right)^2.$

Proof: Consider the following operator

$$\hat{K}_n^{[\beta]}(f,x)=K_n^{[\beta]}(f,x)-f\left(\frac{x}{1-\beta}+\frac{(1+\beta)^2}{2n(1-\beta)^2}\right)+f(x). \tag{2.14}$$

By the definition of the operators $\hat{K}_n^{[\beta]}$ and Lemma 1, we have

$$\hat{K}_n^{[\beta]}(t-x,x)=0.$$

Let $g\in C_B^2[0,\infty)$ and $x\in[0,\infty)$. By Taylor's formula of g, we get

$$g(t)-g(x)=(t-x)g'(x)+\int_x^t(t-u)g''(u)du,\quad t\in[0,\infty).$$

One may write

$$\begin{aligned}\hat{K}_n^{[\beta]}(g,x)-g(x) &= g'(x)\hat{K}_n^{[\beta]}(t-x,x)+\hat{K}_n^{[\beta]}\left(\int_x^t (t-u)g''(u)du,x\right)\\ &= \hat{K}_n^{[\beta]}\left(\int_x^t (t-u)g''(u)du,x\right)\\ &= K_n^{[\beta]}\left(\int_x^t (t-u)g''(u)du,x\right)\\ &\quad -\int_x^{\frac{x}{1-\beta}+\frac{(1+\beta)^2}{2n(1-\beta)^2}}\left(\frac{x}{1-\beta}+\frac{(1+\beta)^2}{2n(1-\beta)^2}-u\right)du.\end{aligned}$$

Now, using the following inequalities

$$\left|\int_x^t (t-u)g''(u)du\right| \le (t-x)^2\|g''\| \tag{2.15}$$

and

$$\left|\int_x^{\frac{x}{1-\beta}+\frac{(1+\beta)^2}{2n(1-\beta)^2}}\left(\frac{x}{1-\beta}+\frac{(1+\beta)^2}{2n(1-\beta)^2}-u\right)du\right| \le \left[\frac{x}{1-\beta}+\frac{(1+\beta)^2}{2n(1-\beta)^2}\right]^2\|g''\|,$$

we reach to

$$\begin{aligned}|\hat{K}_n^{[\beta]}(g,x)-g(x)| &\le \left\{K_n^{[\beta]}((t-x)^2,x)+\left[\frac{x}{1-\beta}+\frac{(1+\beta)^2}{2n(1-\beta)^2}\right]^2\right\}\|g''\|\\ &\le \left\{\frac{3n^2x^2\beta^2+6nx\,(1+2\beta)+1+35\beta}{3n^2(1-\beta)^4}\right.\\ &\quad \left.+\left(\frac{2xn\beta+(1+\beta)^2}{2n(1-\beta)^2}\right)^2\right\}\|g''\|. \end{aligned}\tag{2.16}$$

By means of the definitions of the operators $\hat{K}_n^{[\beta]}$ and $K_n^{[\beta]}$, we have

$$\begin{aligned}|K_n^{[\beta]}(f,x)-f(x)| &\le |\hat{K}_n^{[\beta]}(f-g,x)|+|(f-g)(x)|+\left|\hat{K}_n^{[\beta]}(g,x)-g(x)\right|\\ &\quad +\left|f\left(\frac{x}{1-\beta}+\frac{(1+\beta)^2}{2n(1-\beta)^2}\right)-f(x)\right|\end{aligned}$$

and

$$\hat{K}_n^{[\beta]}(f,x)\le |K_n^{[\beta]}(f,x)|+2\|f\|\le \|f\|K_n^{[\beta]}(1,x)+2\|f\|=3\|f\|.$$

Thus, we may conclude that

$$\begin{aligned}|K_n^{[\beta]}(f,x)-f(x)| &\le 4\|f-g\|+|\hat{K}_n^{[\beta]}(g,x)-g(x)|\\ &\quad +\left|f\left(\frac{x}{1-\beta}+\frac{(1+\beta)^2}{2n(1-\beta)^2}\right)-f(x)\right|.\end{aligned}$$

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In the light of inequality (2.16), one gets

$$\begin{aligned}
|K_n^{[\beta]}(f,x)-f(x)| &\leq 4\|f-g\| \\
&\quad+\left\{\frac{3n^2x^2\beta^2+6nx\,(1+2\beta)+1+35\beta}{3n^2(1-\beta)^4}\right. \\
&\quad\left.+\left(\frac{2xn\beta+(1+\beta)^2}{2n(1-\beta)^2}\right)^2\right\}\|g''\| \\
&\quad+\omega_1\left(f,\frac{x}{1-\beta}+\frac{(1+\beta)^2}{2n(1-\beta)^2}\right).
\end{aligned}$$

Therefore taking the infimum over all $g\in C_B[0,\infty)$ on the right-hand side of the last inequality and considering (2.9), we find that

$$\begin{aligned}
|K_n^{[\beta]}(f,x)-f(x)| &\leq 4K_2\left(f,\xi_n(x)\right)+\omega_1\left(f,\frac{x}{1-\beta}+\frac{(1+\beta)^2}{2n(1-\beta)^2}\right) \\
&\leq 4C\omega_2\left(f,\xi_n(x)\right)+\omega_1\left(f,\frac{x}{1-\beta}+\frac{(1+\beta)^2}{2n(1-\beta)^2}\right) \\
&\leq A\omega_2\left(f,\xi_n(x)\right)+\omega_1\left(f,\frac{x}{1-\beta}+\frac{(1+\beta)^2}{2n(1-\beta)^2}\right),
\end{aligned}$$

which completes the proof.

Theorem 5 *Let $0<\gamma\leq 1$, $\beta\in[0,1)$ and $f\in C_B[0,\infty)$. Then if $f\in Lip_M(\gamma)$, that is, the inequality $|f(t)-f(x)|\leq M|t-x|^\gamma, x,t\in[0,\infty)$ holds, then for each $x\in[0,\infty)$, we have*

$$|K_n^{[\beta]}(f,x)-f(x)|\leq d_n^{\frac{\gamma}{2}}(x),$$

where $d_n(x)=\dfrac{3n^2x^2\beta^2+6nx\,(1+2\beta)+1+35\beta}{3n^2(1-\beta)^4}$ *and $M>0$ is a constant.*

Proof: Let $f\in C_B[0,\infty)\cap Lip_M(\gamma)$. By the linearity and monotonicity of the operators $K_n^{[\beta]}$, we get

$$\begin{aligned}
|K_n^{[\beta]}(f,x)-f(x)| &\leq K_n^{[\beta]}(|f(t)-f(x)|,x) \\
&\leq MK_n^{[\beta]}(|t-x|^\gamma,x) \\
&= Mn\sum_{k=0}^{\infty}\mathbf{p}_\beta(k,nx)\int_{k/n}^{(k+1)/n}|t-x|^\gamma dt.
\end{aligned}$$

Now, applying the Hölder inequality two times successively with $p = \frac{2}{\gamma}$, $q = \frac{2}{2-\gamma}$, we obtain

$$\begin{aligned} |K_n^{[\beta]}(f,x) - f(x)| &\leq M \sum_{k=0}^{\infty} \mathbf{p}_\beta(k,nx) \left(n \int_{k/n}^{(k+1)/n} |t-x|^\gamma dt \right)^{\frac{\gamma}{2}} \\ &\leq M(\Omega_{n,2}(x))^{\frac{\gamma}{2}} \\ &\leq M \left(\frac{3n^2x^2\beta^2 + 6nx\,(1+2\beta) + 1 + 35\beta}{3n^2(1-\beta)^4} \right)^{\frac{\gamma}{2}}. \end{aligned}$$

This completes the proof.

3 Weighted approximation properties

Now, we introduce convergence properties of the operators $K_n^{[\beta]}$ via the weighted Korovkin type theorem given by Gadzhiev in [5, 6]. For this purpose, we recall some definitions and notations.
Let $\rho(x) = 1+x^2$ and $B_\rho[0,\infty)$ be the space of all functions having the property

$$|f(x)| \leq M_f \rho(x),$$

where $x \in [0,\infty)$ and M_f is a positive constant depending only on f. The set $B_\rho[0,\infty)$ is equipped with the norm

$$\|f\|_\rho = \sup_{x\in[0,\infty)} \frac{|f(x)|}{1+x^2}.$$

$C_\rho[0,\infty)$ denotes the space of all continuous functions belonging to $B_\rho[0,\infty)$. By $C_\rho^0[0,\infty)$, we denote the subspace of all functions $f \in C_\rho[0,\infty)$ for which

$$\lim_{x\to\infty} \frac{|f(x)|}{\rho(x)} < \infty.$$

Theorem 6 ([5, 6]) *Let $\{An\}$ be a sequence of positive linear operators acting from $C_\rho[0,\infty)$ to $B_\rho[0,\infty)$ and satisfying the conditions*

$$lim_{n\to\infty}\|A_n(t^v;x) - x^v\|_\rho = 0, v = 0,1,2.$$

Then for any function $f \in C_\rho^0[0,\infty)$,

$$\lim_{n\to\infty} \|A_n(f;\cdot) - f(\cdot)\|_\rho = 0.$$

Note that, a sequence of linear positive operators A_n acts from $C_\rho[0,\infty)$ to $B_\rho[0,\infty)$ if and only if

$$\|A_n(\rho;x)\| \leq M_\rho,$$

where M_ρ is positive constant. This fact also given in [5, 6].

Theorem 7 *Let $\{K_n^{[\beta_n]}\}$ be the sequence of linear positive operators defined by (1.5) and $\beta_n \in [0,1)$ with $\beta_n \to 0$ as $n \to \infty$. Then for each $f \in C_\rho^0[0,\infty)$, we have*

$$\lim_{n\to\infty} \|K_n^{[\beta]}(f;x) - f(x)\|_\rho = 0.$$

Proof: Using Lemma 1, we may write

$$\sup_{x\in[0,\infty)} \frac{|K_n^{[\beta_n]}(\rho,x)|}{1+x^2} \leq \frac{1}{(1-\beta_n)^2} + \frac{\left(3+2\beta_n+\beta_n^2\right)}{n(1-\beta_n)^3} + \frac{1+20\beta_n+12\beta_n^2+2\beta_n^3+\beta_n^4}{3n^2(1-\beta_n)^4} + 1.$$

Since $\lim_{n\to\infty} \beta_n = 0$, , there exists a positive constant M^* such that

$$\frac{1}{(1-\beta_n)^2} + \frac{\left(3+2\beta_n+\beta_n^2\right)}{n(1-\beta_n)^3} + \frac{1+20\beta_n+12\beta_n^2+2\beta_n^3+\beta_n^4}{3n^2(1-\beta_n)^4} \leq M^*$$

for each n. Hence, we get

$$\|K_n^{[\beta_n]}(\rho,x)\|_\rho \leq 1 + M^*,$$

which shows that $\{K_n^{[\beta_n]}\}$ is a sequence of positive linear operators acting from $C_\rho[0,\infty)$ to $B_\rho[0,\infty)$.
In order to complete the proof, it is enough to prove that the conditions of Theorem 6

$$\lim_{n\to\infty} \|K_n^{[\beta_n]}(t^v;x) - x^v\|_\rho = 0, v = 0,1,2$$

are satisfied. It is clear that

$$\lim_{n\to\infty} \|K_n^{[\beta_n]}(1;x) - 1\|_\rho = 0$$

By Lemma 1, we have

$$\begin{aligned} \|K_n^{[\beta_n]}(t;x) - x\|_\rho &= \sup_{x\in\infty} \left| \left(\frac{1}{1-\beta_n} - 1\right) \frac{x}{1+x^2} + \frac{(1+\beta_n)^2}{2n(1-\beta_n)^2} \frac{1}{1+x^2} \right| \\ &\leq \left| \frac{\beta_n}{1-\beta_n} + \frac{(1+\beta_n)^2}{2n(1-\beta_n)^2} \right|. \end{aligned}$$

Thus taking into consideration the conditions $\beta_n \to 0$ as $n \to \infty$, we can conclude that

$$\lim_{n\to\infty} \|K_n^{[\beta_n]}(t;x) - x\|_\rho = 0$$

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Similarly, one gets

$$\begin{aligned}
& \|K_n^{[\beta_n]}(t^2;x)-x^2\|_\rho \\
= & \sup_{x\in\infty}\left|\left(\frac{1}{(1-\beta_n)^2}-1\right)\frac{x^2}{1+x^2}+\frac{\left(3+2\beta_n+\beta_n^2\right)}{n(1-\beta_n)^3}\frac{x}{1+x^2}\right. \\
& \left.+\frac{1+20\beta_n+12\beta_n^2+2\beta_n^3+\beta_n^4}{3n^2(1-\beta_n)^4}\frac{1}{1+x^2}\right| \\
\le & \sup_{x\in\infty}\left|\left(\frac{1}{(1-\beta_n)^2}-1\right)+\frac{\left(3+2\beta_n+\beta_n^2\right)}{n(1-\beta_n)^3}+\frac{1+20\beta_n+12\beta_n^2+2\beta_n^3+\beta_n^4}{3n^2(1-\beta_n)^4}\right| \\
\le & \sup_{x\in\infty}\left|\frac{2\beta_n-\beta_n^2}{(1-\beta_n)^2}+\frac{\left(3+2\beta_n+\beta_n^2\right)}{n(1-\beta_n)^3}+\frac{1+20\beta_n+12\beta_n^2+2\beta_n^3+\beta_n^4}{3n^2(1-\beta_n)^4}\right|
\end{aligned}$$

which leads to

$$\lim_{n\to\infty}\|K_n^{[\beta_n]}(t^2;x)-x^2\|_\rho=0 \text{ with } \beta_n\to 0.$$

Thus the proof is completed.
Now, we compute the order of approximation of the operators $K_n^{[\beta]}$ in terms of the weighted modulus of continuity $\Omega_2(f,\delta)$ (see[7]) defined by

$$\Omega_2(f,\delta)=\sup_{x\ge 0,0<h\le\delta}\frac{|f(x+h)-f(x)|}{1+(x+h)^2},\qquad f\in C_\rho^0[0,\infty)$$

and has the following properties:

(a) $\Omega_2(f,\delta)$ is monotone increasing function of δ,

(b) $\lim\limits_{\delta\to 0+}\Omega_2(f,\delta)=0$,

(c) for each $\lambda\in[0,\infty)$, $\Omega_2(f,\lambda\delta)\le(1+\lambda)\Omega_2(f,\delta)$.

Theorem 8 *Let $\{K_n^{[\beta]}\}$ be the sequence of linear positive operators defined by (1.5). Then for each $f\in C_\rho^0[0,\infty)$, we have*

$$\sup_{0\le x<\infty}\frac{|K_n^{[\beta]}(f;x)-f(x)|}{(1+x^2)^3}\le C\Omega_2\left(f,(B_\beta(n))^{1/4}\right),$$

where C is positive constant and $B_\beta(n)$ is defined in (2.8).

Proof: For $x\ge 0$ and $t\ge 0$, by the definition of $\Omega_2(f,\delta)$ and the property (c), we may write

$$\begin{aligned}
|f(t)-f(x)| &\le \left(1+\left(x+|t-x|^2\right)\right)\left(1+\frac{|t-x|}{\delta_n}\right)\Omega_2(f,\delta_n) \\
&\le 2(1+x^2)(1+(t-x)^2)\left(1+\frac{|t-x|}{\delta_n}\right)\Omega_2(f,\delta_2).
\end{aligned}$$

By using the monotonicity of $K_n^{[\beta]}$ and the following inequality (see [8])

$$\left(1+(t-x)^2\right)\left(1+\frac{|t-x|}{\delta_n}\right) \leq 2\left(1+\delta_n^2\right)\left(1+\frac{(t-x)^4}{\delta_n^4}\right),$$

one gets

$$\begin{aligned}|K_n^{[\beta]}(f,x)-f(x)| &\leq 2(1+x^2)K_n^{[\beta]}\left(\left(1+(t-x)^2\right)\left(1+\frac{|t-x|}{\delta_n}\right),x\right)\Omega_2(f,\delta_n)\\ &\leq 4(1+x^2)(1+\delta_n^2)K_n^{[\beta]}\left(\left(1+\frac{(t-x)^4}{\delta_n^4}\right),x\right)\Omega_2(f,\delta_n)\\ &\leq 4(1+x^2)(1+\delta_n^2)\left(1+\frac{1}{\delta_n^4}K_n^{[\beta]}((t-x)^4,x)\right)\Omega_2(f,\delta_n)\\ &\leq C_1(1+x^2)\left(1+\frac{1}{\delta_n^4}K_n^{[\beta]}((t-x)^4,x)\right)\Omega_2(f,\delta_n),\end{aligned}$$

with the help of the inequality (2.7) this inequality leads to

$$|K_n^{[\beta]}(f,x)-f(x)| \leq C_1(1+x^2)\left(1+\frac{B_\beta(n)}{\delta_n^4}(1+x+x^2+x^3+x^4)\right)\Omega_2(f,\delta_n),$$

which gives the required result.

Remark 2 *In [9], the authors has consider the generalization of the operators (1.1) as*

$$P_n^{[0]}(f,a_n,b_n,x) = 2^{-a_nx}\sum_{k=0}^{\infty}\frac{(a_nx)_k}{2^kk!}f\left(\frac{k}{b_n}\right), \quad x\geq 0, \quad f:[0,\infty),\to\mathbb{R}, \quad (3.1)$$

where $\{a_n\}$, $\{b_n\}$ are increasing and unbounded sequences of positive numbers such that

$$\lim_{n\to\infty}\frac{1}{b_n}=0, \quad \frac{a_n}{b_n}=1+O\left(\frac{1}{b_n}\right).$$

They studied the convergence properties of these operators in weighted spaces of continuous functions on positive semi-axis. Also, A. Erençin and F Taşdelen [10] consider the generalization of the Kantorovich type operators $P_n^{[0]}(f,a_n,b_n,x)$ given by (3.1) as follows:

$$K_n^{[0]}(f,a_n,b_n,x) = b_n2^{-a_nx}\sum_{k=0}^{\infty}\frac{(a_nx)_k}{2^kk!}\int_{k/b_n}^{(k+1)/b_n}f(t)dt, \qquad (3.2)$$

where f is an integrable function on $[0,\infty)$ and bounded on every compact subinterval of $[0,\infty)$.
Motivated by the operators (3.1) *and* (3.2), *we generalize the operators $P_n^{[\beta]}$ and $K_n^{[\beta]}$ in following way*

$$P_n^{[\beta]}(f,a_n,b_n,x) = \sum_{k=0}^{\infty}\frac{2^{-(a_nx+k\beta)}(a_nx+k\beta)_k}{2^kk!}f\left(\frac{k}{b_n}\right), \quad x\geq 0, \quad f:[0,\infty)\to\mathbb{R}, \qquad (3.3)$$

and

$$K_n^{[\beta]}(f, a_n, b_n, x) = b_n \sum_{k=0}^{\infty} \frac{2^{-(a_n x + k\beta)}(a_n x)_k}{2^k k!} \int_{k/b_n}^{(k+1)/b_n} f(t)dt \tag{3.4}$$

and extend the studies of the present article in a similar direction for the operators (3.3) *and* (3.4)*. The analysis is different so we may discuss that elsewhere.*

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