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On the Barnes-type multiple twisted q-Euler zeta function of the second kind

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Abstract :

In this paper we introduce the Barnes-type multiple twisted q-Euler numbers and polynomials of the second kind, by using fermionic p-adic invariant integral on $\mathbb{Z}_p$. Using these numbers and polynomials, we construct the Barnes-type multiple twisted q-Euler zeta function of the second kind. Finally, we obtain the relations between these numbers and polynomials and Barnes-type multiple twisted q -Euler zeta function.

Key words : p-adic invariant integral on $\mathbb{Z}_p$, Euler numbers and polynomials of the second kind, q-Euler numbers and polynomials of the second kind, Barnes-type multiple twisted q-Euler numbers and polynomials of the second kind, Barnes-type multiple twisted q -Euler zeta function.

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1. Introduction

Recently, Bernoulli numbers, Bernoulli polynomials, q-Bernoulli numbers, q-Bernoulli polynomials, the second kind Bernoulli number, the second kind Bernoulli polynomials, Euler numbers of the second kind , Euler polynomials of the second kind, Genocchi numbers, Genocchi polynomials, tangent numbers, tangent polynomials, and Bell polynomials were studied by many authors(see [1, 2, 3, 4, 9]). Euler numbers and polynomials possess many interesting properties and arising in many areas of mathematics and physics. In [5], by using Euler numbers E_j and polynomials $E_j(x)$ of the second kind, we investigated the alternating sums of powers of consecutive odd integers. Also we carried out computer experiments for doing demonstrate a remarkably regular structure of the complex roots of the second kind Euler polynomials $E_n(x)$(see [6]). The outline of this paper is as follows. We introduce the Barnes-type multiple twisted q-Euler numbers and polynomials of the second kind, by using fermionic p-adic invariant integral on $\mathbb{Z}_p$. In Section 2, we construct the Barnes-type multiple twisted q-Euler zeta function of the second kind. Finally, we obtain the relations between these numbers and polynomials and Barnes-type multiple twisted q -Euler zeta function.

Throughout this paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of rational numbers, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{C}$ denotes the complex number field, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. For

$$g \in UD(\mathbb{Z}_p) = \{g | g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\},$$

the fermionic p-adic invariant integral on $\mathbb{Z}_p$ of the function $g \in UD(\mathbb{Z}_p)$ is defined by

$$I_{-1}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} g(x)(-1)^x, \text{ see [1, 2, 4].} \tag{1.1}$$

From (1.1), we note that

$$\int_{\mathbb{Z}_p} g(x+1)d\mu_{-1}(x) + \int_{\mathbb{Z}_p} g(x)d\mu_{-1}(x) = 2g(0). \tag{1.2}$$

First, we introduced the second kind Euler numbers E_n. The second kind Euler numbers E_n are defined by the generating function:

$$\frac{2e^t}{e^{2t}+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}. \tag{1.3}$$

We introduce the second kind Euler polynomials $E_n(x)$ as follows:

$$\frac{2e^t}{e^{2t}+1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}. \tag{1.4}$$

In [5, 6], we studied the second kind Euler numbers E_n and polynomials $E_n(x)$ and investigate their properties.

2. Barnes-type multiple twisted q-Euler numbers and polynomials of the second kind

In this section, we assume that $w_1, \dots, w_k \in \mathbb{Z}_p$ and $a_1, \dots, a_k \in \mathbb{Z}$. Let $T_p = \cup_{N\geq 1} C_{p^N} = \lim_{N\to\infty} C_{p^N}$, where $C_{p^N} = \{\omega | \omega^{p^N} = 1\}$ is the cyclic group of order p^N. For $\omega \in T_p$, we denote by $\phi_\omega : \mathbb{Z}_p \to \mathbb{C}_p$ the locally constant function $x \longmapsto \omega^x$.

We construct the Barnes-type multiple twisted q-Euler polynomials of the second kind,

$$E_{n,\omega,q}(w_1, \dots, w_k; a_1, \dots, a_k \mid x).$$

For $k \in \mathbb{N}$, we define Barnes-type multiple twisted q-Euler polynomials of the second kind as follows:

$$\begin{aligned} &\underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k-\text{times}} \omega^{x_1+\cdots+x_k} q^{a_1x_1+\cdots+a_kx_k} e^{(x+2w_1x_1+\cdots+2w_kx_k+k)t} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k) \\ &= \frac{2^k e^{kt}}{(\omega q^{a_1} e^{2w_1t}+1)(\omega q^{a_2} e^{2w_2t}+1)\cdots(\omega q^{a_k} e^{2w_kt}+1)} e^{xt} \\ &= \sum_{n=0}^{\infty} E_{n,\omega,q}(w_1, \dots, w_k; a_1, \dots, a_k \mid x) \frac{t^n}{n!}. \end{aligned} \tag{2.1}$$

In the special case, $x = 0$, $E_{n,\omega,q}(w_1, \dots, w_k; a_1, \dots, a_k \mid 0) = E_{n,\omega,q}(w_1, \dots, w_k; a_1, \dots, a_k)$ are called the n-th Barnes-type multiple twisted q-Euler numbers of the second kind. By (2.1) and Taylor expansion of $e^{(x+2w_1x_1+\cdots+2w_kx_k+k)t}$, we have the following theorem.

Theorem 1. For positive integers n and k, we have

$$\begin{aligned} &E_{n,\omega,q}(w_1, \dots, w_k; a_1, \dots, a_k \mid x) \\ &= \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k-\text{times}} \omega^{x_1+\cdots+x_k} q^{a_1x_1+\cdots+a_kx_k} (x+2w_1x_1+\cdots+2w_kx_k+k)^n d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k). \end{aligned}$$

In the case when $x = 0$ in (2.1), we have the following corollary.

Corollary 2. For positive integers n, we have

$$\begin{aligned} &E_{n,\omega,q}(w_1, \dots, w_k; a_1, \dots, a_k) \\ &= \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k-\text{times}} \omega^{\sum_{i=1}^k x_i} q^{\sum_{i=1}^k a_ix_i} (2w_1x_1+\cdots+2w_kx_k+k)^n d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k). \end{aligned} \tag{2.2}$$

By Theorem 1 and (2.2), we obtain

$$E_{n,\omega,q}(w_1,\ldots,w_k;a_1,\ldots,a_k \mid x)=\sum_{l=0}^{n}\binom{n}{l}E_{l,\omega,q}(w_1,\ldots,w_k;a_1,\ldots,a_k)x^{n-l}, \tag{2.3}$$

where $\binom{n}{k}$ is a binomial coefficient.

We define distribution relation of Barnes-type multiple twisted q-Euler polynomials of the second kind as follows: For $m\in\mathbb{N}$ with $m\equiv 1(\mod 2)$, we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty}E_{n,\omega,q}(w_1,\ldots,w_k;a_1,\ldots,a_k \mid x)\frac{t^n}{n!}\\&=\frac{2^k e^{kmt}}{(\omega^m q^{a_1m}e^{2w_1mt}+1)(\omega^m q^{a_2m}e^{2w_2mt}+1)\cdots(\omega^m q^{a_km}e^{2w_kmt}+1)}\\&\quad\times\sum_{l_1,\ldots,l_k=0}^{m-1}(-1)^{l_1+\cdots+l_k}\omega^{\sum_{i=1}^k l_i}q^{\sum_{i=1}^k a_il_i}e^{\left(\frac{x+2w_1l_1+\cdots+2w_kl_k+k-mk}{m}\right)(mt)}.\end{aligned}$$

From the above equation, we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty}E_{n,\omega,q}(w_1,\ldots,w_k;a_1,\ldots,a_k \mid x)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}m^n\sum_{l_1,\ldots,l_k=0}^{m-1}(-1)^{l_1+\cdots+l_k}\omega^{\sum_{i=1}^k l_i}q^{\sum_{i=1}^k a_il_i}\\&\quad\times E_{n,\omega^m,q^m}\left(w_1,\ldots,w_k;a_1,\ldots,a_k \mid \frac{x+2w_1l_1+\cdots+2w_kl_k+k-mk}{m}\right)\frac{t^n}{n!}.\end{aligned}$$

By comparing coefficients of $\frac{t^n}{n!}$ in the above equation, we arrive at the following theorem.

Theorem 3 (Distribution relation). For $m\in\mathbb{N}$ with $m\equiv 1(\mod 2)$, we have

$$\begin{aligned}&E_{n,\omega,q}(w_1,\ldots,w_k;a_1,\ldots,a_k \mid x)\\&=m^n\sum_{l_1,\ldots,l_k=0}^{m-1}(-1)^{l_1+\cdots+l_k}\omega^{\sum_{i=1}^k l_i}q^{\sum_{i=1}^k a_il_i}\\&\quad\times E_{n,\omega^m,q^m}\left(w_1,\ldots,w_k;a_1,\ldots,a_k \mid \frac{x+2w_1l_1+\cdots+2w_kl_k+k-mk}{m}\right).\end{aligned}$$

From (2.1), we derive

$$\begin{aligned}&\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{k-\text{times}}\omega^{x_1+\cdots+x_k}q^{a_1x_1+\cdots+a_kx_k}e^{(x+2w_1x_1+\cdots+2w_kx_k+k)t}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k)\\&=2^k\sum_{m_1,\ldots m_k=0}^{\infty}(-1)^{m_1+\cdots+m_k}\omega^{\sum_{i=1}^k m_i}q^{\sum_{i=1}^k a_im_i}e^{(x+2w_1m_1+\cdots+2w_km_k+k)t}.\end{aligned} \tag{2.4}$$

From (2.2) and (2.4), we note that

$$\begin{aligned}&E_{n,\omega,q}(w_1,\ldots,w_k;a_1,\ldots,a_k \mid x)\\&=2^k\sum_{m_1,\ldots m_r=0}^{\infty}(-1)^{m_1+\cdots+m_r}q^{\sum_{i=1}^k a_im_i}(x+2w_1m_1+\cdots+2w_km_k+k)^n.\end{aligned} \tag{2.5}$$

By using binomial expansion and (2.1), we have the following addition theorem.

Theorem 4(Addition theorem). Barnes-type multiple twisted q-Euler polynomials of the second kind $E_{n,\omega,q}(w_1,\dots,w_k;a_1,\dots,a_k \mid x)$ satisfies the following relation:

$$E_{n,\omega,q}(w_1,\dots,w_k;a_1,\dots,a_k \mid x+y) = \sum_{l=0}^{n} \binom{n}{l} E_{l,\omega,q}(w_1,\dots,w_k;a_1,\dots,a_k \mid x) y^{n-l}.$$

3. Barnes-type multiple twisted q-Euler zeta function of the second kind

In this section, we assume that $q \in \mathbb{C}$ with $|q| < 1$ and the parameters $w_1,\dots,w_k$ are positive. Let ω be the p^N-th root of unity. By applying derivative operator, $\frac{d^l}{dt^l}|_{t=0}$ to the generating function of Barnes-type multiple twisted q-Euler polynomials of the second kind, $E_{n,\omega,q}(w_1,\dots,w_k;a_1,\dots,a_k \mid x)$, we define Barnes-type multiple twisted q-Euler zeta function of the second kind. This function interpolates the Barnes-type multiple twisted q-Euler polynomials of the second kind at negative integers.

By (2.1), we obtain

$$\begin{aligned} F_{\omega,q}(w_1,\dots,w_k;a_1,\dots,a_k \mid x,t) &= \frac{2^k e^{kt}}{(\omega q^{a_1} e^{2w_1 t}+1)\cdots(\omega q^{a_k} e^{2w_k t}+1)} e^{xt} \\ &= \sum_{n=0}^{\infty} E_{n,\omega,q}(w_1,\dots,w_k;a_1,\dots,a_k \mid x)\frac{t^n}{n!}. \end{aligned} \tag{3.1}$$

Hence, by (3.1), we obtain

$$\begin{aligned} &\sum_{n=0}^{\infty} E_{n,\omega,q}(w_1,\dots,w_k;a_1,\dots,a_k \mid x)\frac{t^n}{n!} \\ &= 2^k \sum_{m_1,\dots m_r=0}^{\infty} (-1)^{m_1+\cdots+m_k} \omega^{\sum_{i=1}^k m_i} q^{\sum_{i=1}^k a_i m_i} e^{(x+2w_1m_1+\cdots+2w_km_k+k)t}. \end{aligned}$$

By applying derivative operator, $\frac{d^l}{dt^l}|_{t=0}$ to the above equation, we have

$$\begin{aligned} &E_{n,\omega,q}(w_1,\dots,w_k;a_1,\dots,a_k \mid x) \\ &= 2^k \sum_{m_1,\dots m_k=0}^{\infty} (-1)^{m_1+\cdots+m_k} \omega^{\sum_{i=1}^k m_i} q^{\sum_{i=1}^k a_i m_i} (x+2w_1m_1+\cdots+2w_km_k+k)^n. \end{aligned} \tag{3.2}$$

By (3.2), we define the Barnes-type multiple twisted q-Euler zeta function of the second kind $\zeta_{\omega,q}(w_1,\dots,w_k;a_1,\dots,a_k \mid s,x)$ as follows:

Definition 5. For $s,x \in \mathbb{C}$ with $Re(x) > 0$, $a_1,\dots,a_k \in \mathbb{C}$, we define

$$\zeta_{\omega,q}(w_1,\dots,w_k;a_1,\dots,a_k \mid s,x) = 2^k \sum_{m_1,\dots,m_k=0}^{\infty} \frac{(-1)^{m_1+\cdots+m_k} \omega^{\sum_{i=1}^k m_i} q^{\sum_{i=1}^k a_i m_i}}{(x+2w_1m_1+\cdots+2w_km_k+k)^s}, \tag{3.3}$$

For $s = -l$ in (3.3) and using (3.2), we arrive at the following theorem.

Theorem 6. For positive integer l, we have

$$\zeta_{\omega,q}(w_1,\dots,w_k;a_1,\dots,a_k \mid -l,x) = E_l(w_1,\dots,w_k;a_1,\dots,a_k \mid x).$$

By (2.6), we define multiple twisted q-Euler zeta function of the second kind $\zeta_{\omega,q}(w_1,\dots,w_k;a_1,\dots,a_k \mid s)$ as follows:

Definition 7. For $s \in \mathbb{C}$, we define

$$\zeta_{\omega,q}(w_1,\ldots,w_k;a_1,\ldots,a_k \mid s) = 2^k \sum_{m_1,\ldots,m_k=0}^{\infty} \frac{(-1)^{m_1+\cdots+m_k}\omega^{\sum_{i=1}^k m_i} q^{\sum_{i=1}^k a_i m_i}}{(2w_1m_1+\cdots+2w_km_k+k)^s}, \tag{3.4}$$

For $s = -l$ in (3.4) and using (2.5), we arrive at the following theorem.

Theorem 8. For positive integer l, we have

$$\zeta_{\omega,q}(w_1,\ldots,w_k;a_1,\ldots,a_k \mid -l) = E_l(w_1,\ldots,w_k;a_1,\ldots,a_k).$$

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