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Complex Multivariate Taylor's formula

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Abstract

We derive here a Taylor's formula with integral remainder in the several complex variables and we estimate its remainder.

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1 Main Results

We need the following vector Taylor's formula:

Theorem 1 *(Shilov, [3], pp. 93-94) Let $n \in \mathbb{N}$ and $f \in C^n([a,b], X)$, where $[a,b] \subset \mathbb{R}$ and $(X, \|\cdot\|)$ is a Banach space. Then*

$$f(b) = f(a) + \sum_{i=1}^{n-1} \frac{(b-a)^i}{i!} f^{(i)}(a) + \frac{1}{(n-1)!} \int_a^b (b-x)^{n-1} f^{(n)}(x)\, dx. \quad (1)$$

The remainder here is the Riemann X-valued integral (defined similar to numerical one) given by

$$Q_{n-1} = \frac{1}{(n-1)!} \int_a^b (b-x)^{n-1} f^{(n)}(x)\, dx, \quad (2)$$

with the property:

$$\|Q_{n-1}\| \leq \max_{a \leq x \leq b} \left\| f^{(n)}(x) \right\| \frac{(b-a)^n}{n!}. \quad (3)$$

The derivatives above are defined similar to the numerical ones. We make

Remark 2 *Here Q is an open convex subset of $\mathbb{C}^k$, $k \geq 2$; $z := (z_1, ..., z_k)$, $x_0 := (x_{01}, ..., x_{0k}) \in Q$. Let $f : Q \to \mathbb{C}$ be a coordinate-wise holomorphic* *function. Then, by the famous Hartog's fundamental theorem ([1], [2]) f is jointly holomorphic and jointly continuous on Q. Let $n \in \mathbb{N}$. Each n^{th} order complex partial derivative is denoted by $f_\alpha := \frac{\partial^\alpha f}{\partial x^\alpha}$, where $\alpha := (\alpha_1, ..., \alpha_k)$, $\alpha_i \in \mathbb{Z}_+$, $i = 1, ..., k$ and $|\alpha| := \sum_{i=1}^{k} \alpha_i = n$.*

Consider

$$g_z(t) := f(x_0 + t(z - x_0)), \quad 0 \le t \le 1. \tag{4}$$

Clearly it holds that $x_0 + t(z - x_0) \in Q$ and $g_z(t) \in \mathbb{C}$, $\forall\, t \in [0, 1]$.

Then we derive

$$g_z^{(j)}(t) = \left[\left(\sum_{i=1}^{k} (z_i - x_{0i}) \frac{\partial}{\partial x_i} \right)^j f \right] (x_{01} + t(z_1 - x_{01}), ..., x_{0k} + t(z_k - x_{0k})), \tag{5}$$

for all $j = 0, 1, ..., n$.

Notice here that any mixed partials commute. We remind that $(\mathbb{C}, |\cdot|)$ is a Banach space. By Shilov's Theorem 1, about the Taylor's formula for Banach space valued functions, we obtain

Theorem 3 *It holds*

$$f(z_1, ..., z_k) = g_z(1) = \sum_{j=0}^{n-1} \frac{g_z^{(j)}(0)}{j!} + R_n(z, 0), \tag{6}$$

where

$$R_n(z, 0) = \frac{1}{(n-1)!} \int_0^1 (1 - \theta)^{n-1} g_z^{(n)}(\theta)\, d\theta, \tag{7}$$

and notice that $g_z(0) = f(x_0)$.

We make

Remark 4 *Notice that (by (7)) we get*

$$|R_n(z, 0)| \le \left(\max_{0 \le \theta \le 1} \left| g_z^{(n)}(\theta) \right| \right) \frac{1}{n!}. \tag{8}$$

We also have for $j = 0, 1, ..., n$:

$$g_z^{(j)}(0) = \sum_{\substack{\alpha := (\alpha_1, ..., \alpha_k),\ \alpha_j \in \mathbb{Z}^+ \\ i = 1, ..., k;\ |\alpha| := \sum_{i=1}^{k} \alpha_i = j}} \left(\frac{j!}{\prod_{i=1}^{k} \alpha_i!} \right) \left(\prod_{i=1}^{k} (z_i - x_{0i})^{\alpha_i} \right) f_\alpha(x_0). \tag{9}$$

Furthermore it holds

$$g_z^{(n)}(\theta) = \sum_{\substack{\alpha:=(\alpha_1,...,\alpha_k),\ \alpha_j\in\mathbb{Z}^+ \\ i=1,...,k;\ |\alpha|:=\sum_{i=1}^k \alpha_i = n}} \left(\frac{n!}{\prod_{i=1}^k \alpha_i!} \right) \left(\prod_{i=1}^k (z_i - x_{0i})^{\alpha_i} \right) f_\alpha (x_0 + \theta (z - x_0)), \tag{10}$$

$0 \leq \theta \leq 1$.

Another version of (6) is

$$f(z_1, ..., z_k) = g_z(1) = \sum_{j=0}^n \frac{g_z^{(j)}(0)}{j!} + \overline{R_n}(z, 0), \tag{11}$$

where

$$\overline{R_n}(z, 0) = \frac{1}{(n-1)!} \int_0^1 (1-\theta)^{n-1} \left(g_z^{(n)}(\theta) - g_z^{(n)}(0) \right) d\theta. \tag{12}$$

Identities (6) and (11) are the multivariate complex Taylor's formula with integral remainders.

We give

Example 5 *Let $n = k = 2$. Then*

$$g_z(t) = f(x_{01} + t(z_1 - x_{01}), x_{02} + t(z_2 - x_{02})), \quad t \in [0, 1],$$

and

$$g_z'(t) = (z_1 - x_{01}) \frac{\partial f}{\partial x_1}(x_0 + t(z - x_0)) + (x_2 - x_{02}) \frac{\partial f}{\partial x_2}(x_0 + t(z - x_0)). \tag{13}$$

In addition,

$$\begin{aligned} g_z''(t) &= (z_1 - x_{01}) \left(\frac{\partial f}{\partial x_1}(x_0 + t(z - x_0)) \right)' + (x_2 - x_{02}) \left(\frac{\partial f}{\partial x_2}(x_0 + t(z - x_0)) \right)' \\ &= (z_1 - x_{01}) \left\{ (z_1 - x_{01}) \frac{\partial f^2}{\partial x_1^2}(*) + (z_2 - x_{02}) \frac{\partial f^2}{\partial x_2 \partial x_1}(*) \right\} + \\ &\quad (z_2 - x_{02}) \left\{ (z_1 - x_{01}) \frac{\partial f^2}{\partial x_1 \partial x_2}(*) + (z_2 - x_{02}) \frac{\partial f^2}{\partial x_2^2}(*) \right\}. \end{aligned} \tag{14}$$

Hence,

$$\begin{aligned} g_z''(t) &= (z_1 - x_{01})^2 \frac{\partial f^2}{\partial x_1^2}(*) + (z_1 - x_{01})(z_2 - x_{02}) \frac{\partial f^2}{\partial x_2 \partial x_1}(*) + \\ &\quad (z_1 - x_{01})(z_2 - x_{02}) \frac{\partial f^2}{\partial x_1 \partial x_2}(*) + (z_2 - x_{02})^2 \frac{\partial f^2}{\partial x_2^2}(*), \end{aligned} \tag{15}$$

where $* := x_0 + t(z - x_0)$.

Notice that $g_z(t), g_z'(t), g_z''(t) \in \mathbb{C}$.

3

We make

Remark 6 *We define*

$$\|f\|_{p,\overline{zx_0}} := \left(\int_0^1 \left| f\left(x_0 + \theta \left(z - x_0 \right) \right) \right|^p d\theta \right)^{\frac{1}{p}}, \quad p \geq 1, \tag{16}$$

where $\overline{zx_0}$ *denotes the segment* $\overline{zx_0} \subset Q$.

We also define

$$\|f\|_{\infty,\overline{zx_0}} := \max_{\theta \in [0,1]} \left| f\left(x_0 + \theta \left(z - x_0 \right) \right) \right|. \tag{17}$$

By (10) we obtain

$$\left| g_z^{(n)}\left(\theta \right) \right| \leq \sum_{\substack{\alpha := (\alpha_1, \ldots, \alpha_k),\ \alpha_j \in \mathbb{Z}^+ \\ i = 1, \ldots, k;\ |\alpha| := \sum_{i=1}^{k} \alpha_i = n}} \left(\frac{n!}{\prod_{i=1}^{k} \alpha_i!} \right) \left(\prod_{i=1}^{k} \left| z_i - x_{0i} \right|^{\alpha_i} \right) \left| f_\alpha\left(x_0 + \theta \left(z - x_0 \right) \right) \right|, \tag{18}$$

$\forall\ \theta \in [0,1]$.

Therefore, by norm properties for $1 \leq p \leq \infty$, *it holds*

$$\left\| g_z^{(n)} \right\|_{p,\overline{zx_0}} \leq \sum_{\substack{\alpha := (\alpha_1, \ldots, \alpha_k),\ \alpha_j \in \mathbb{Z}^+ \\ i = 1, \ldots, k;\ |\alpha| := \sum_{i=1}^{k} \alpha_i = n}} \left(\frac{n!}{\prod_{i=1}^{k} \alpha_i!} \right) \left(\prod_{i=1}^{k} \left| z_i - x_{0i} \right|^{\alpha_i} \right) \|f_\alpha\|_{p,\overline{zx_0}} \tag{19}$$

$$\leq \left(\sum_{i=1}^{k} \left| z_i - x_{0i} \right| \right)^n \|f_\alpha^*\|_{p,\overline{zx_0}},$$

where

$$\|f_\alpha^*\|_{p,\overline{zx_0}} := \max_{|\alpha| = n} \|f_\alpha\|_{p,\overline{zx_0}}, \tag{20}$$

for all $1 \leq p \leq \infty$.

That is

$$\left\| g_z^{(n)} \right\|_{p,\overline{zx_0}} \leq \left(\|z - x_0\|_{l_1} \right)^n \|f_\alpha^*\|_{p,\overline{zx_0}}, \tag{21}$$

for all $1 \leq p \leq \infty$.

Therefore by (8) we obtain

$$\left| R_n\left(z, 0 \right) \right| \leq \frac{\left(\|z - x_0\|_{l_1} \right)^n \|f_\alpha^*\|_{\infty,\overline{zx_0}}}{n!}. \tag{22}$$

Next, we put things together and we further estimate $R_n\left(z, 0 \right)$.

Theorem 7 *Here* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. *It holds*

$$|R_n(z,0)| \leq \min \left\{ \begin{array}{c} \frac{\left\| g_z^{(n)} \right\|_{\infty, \overline{zx_0}}}{n!}, \\ \\ \frac{\left\| g_z^{(n)} \right\|_{1, \overline{zx_0}}}{(n-1)!}, \\ \\ \frac{\left\| g_z^{(n)} \right\|_{p, \overline{zx_0}}}{(n-1)!(q(n-1)+1)^{\frac{1}{q}}} \end{array} \right\} \leq \tag{23}$$

$$\left(\|z - x_0\|_{l_1} \right)^n \min \left\{ \begin{array}{c} \frac{\|f_\alpha^*\|_{\infty, \overline{zx_0}}}{n!}, \\ \\ \frac{\|f_\alpha^*\|_{1, \overline{zx_0}}}{(n-1)!}, \\ \\ \frac{\|f_\alpha^*\|_{p, \overline{zx_0}}}{(n-1)!(q(n-1)+1)^{\frac{1}{q}}}. \end{array} \right. \tag{24}$$

Proof. Based on (7), Hölder's inequality and (21). ■

References

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