The Existence and Uniqueness of Fractional Integrodifferential Equations with Interval Impulses and Infinite Delay

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ABSTRACT
The main focus of this research is on the existence and exclusivity of solutions for the fractional integrodifferential equations with interval impulses and infinite delay. Schauder's fixed point theorem is used to acquire the required results. An example of the main result is also specified.

Keywords: Fractional derivative, Integrodifferential equations, Interval impulses, Fixed point theorem, Nonlocal conditions.

1. INTRODUCTION

The derivatives of the non-integer and the generalization of integral are the main focus of the fractional calculus. Fractional derivatives are an excellent tool for describing memory and heredity qualities in a variety of materials and systems. In comparison to standard integer-order models, this is the main advantage of fractional differential equations. Fractional differential equations are used in a wide range of scientific disciplines and engineering, including modelling systems and processes mathematically. Many other subjects, such as chemistry, physics, complex media electrodynamics, aerodynamics, and so on, utilise fractional differential equations. Furthermore, fractional order derivatives are involved.

In the mathematical modeling of numerous fields like physical, biological phenomena, and engineering sciences a major role is accomplished by the integrodifferential equation both in the theoretical and practical aspects where it is impossible to neglect the consequences of the real-world problem. It is extremely essential to make a study on integrodifferential equations as numerous practical systems are integrodifferential equations in nature.

The goal of this study is to show that fractional infinite delay integrodifferential equations with interval impulses of the type have solutions

$$D_p^\alpha u(p) = f(p, u_p) + \int_0^p k(p, s, u_s)ds, p \in (s_i, p_{i+1}]i = 0,1, \dots, M \quad (1.1)$$

$$u(p) = g_i(p, u(p)), p \in (p_i, s_i]i = 1,2, \dots, M \quad (1.2)$$

$$u(0) + h(u) = \phi_1, \phi_1 \in E_v \quad (1.3)$$

where $0 < \alpha < 1$ and the state $u(.)$pertains to the space of Banach U provided with the expected $\|\cdot\|$, D_p^α represents the fractional derivative of Caputo, f is an appropriate function, $0 = p_0 = s_0 < p_1 \le s_1 \le p_2 < \cdots p_M \le s_M \le p_{M+1} = b$ are a predetermined number, $g_i \in C((p_i, s_i] \times U; U), i = 1,2, \dots, M$. Let $u_p(.)$denote $u_p(\eta) = u(p + \eta), \eta \in (-\infty, 0]$. Hence we contemplate that impulses kicks off with a bang at that point p_i on the interval, and their actions continue $[p_i, s_i]$

Impulsive differential equations belongs to the category which most of the principle models falls in, which outlines most of the advancement processes that unexpectedly transform their state at a particular moment. IDE are most suitable method to model these processes. To obtain more information on this theory and more of its application, we have referred to the monographs of [7] and the papers [13,14], where properties of their results are deliberated and more precise bibliographies are given. In recent years, there has been significant progress in fractional differential and partial differential equations, for reference see the monographs [1,3,4,5,6,9,10,16,17].

The impulsive integrodifferential equations behavior in abstract spaces have already been scrutinized by [12,15] by several authors. To support the study of population dynamics, ecology, epidemic and biology various mathematical models have been used and which could be indicated as impulsive delay differential equations. Many researchers have investigated the theory of impulsive delay differential equations [2,8,9] right away. However, there are a variety of situations in which impulsive behaviour begins immediately and continues for a finite period of time. A new class of abstract impulsive differential equation was established by Eduardo Hernandez and Donal O' Regan [11] for which the impulses are not instantaneous.

2. Preliminaries

This part introduces some of the fundamental notations, definitions, lemmas, and theorems that are utilised throughout the article. Consider a continuous function $v:(-\infty,0]\to(0,+\infty]$ which satisfies $l=\int_{-\infty}^{0}v(p)dt<+\infty$. Here the function v induces a Banach space $\left(E_v,\|\cdot\|_{E(v)}\right)$ it is defined in the following way.

$$E_v=\{\psi:(-\infty,0]\to U;\ \text{in any case } C>0,\psi(\eta) \text{ is a bounded and on a measurable function } [-C,0] \text{ and } \int_{-\infty}^{0} v(p)\sup_{p\leq\eta\leq 0}\|\psi(\eta)\|dt<+\infty$$

provided with the expected $\|\psi\|_{E_v}=\int_{-\infty}^{0}v(s)\sup_{s\leq\eta\leq 0}\|\psi(\eta)\|ds$.

Let us begin by defining the space

$$E_w^{'}=\{\psi:(-\infty,b]\to U;\psi_k\in C(J_k,U),k=0,1,2,3,\dots,M \text{ in addition to it } \psi(p_k^-)\text{and } \psi(p_k^+)\text{with } \psi(p_k)\psi(p_k^-),\psi(p)=g_k(p,u(p)) \text{ for } p\in(p_k,s_k]\ k=1,2,\dots,M,\psi_0=\psi(0)+h(\psi)=\psi\in E_0$$

We consider the space for the impulsive situations QC(U) it is supported via means of all functions $u:[0,b]\to U$ such that u(.) is continuous at $p\neq p_i,u(p_i^-)=u(p_i)$ and $u(p_i^+)$ exists for all $i=1,2,3,\dots,M$, provided uniform norm for [0, b] signified $\|u\|_{QC(U)}$. We begin $u_i\in C([p_i,p_{i+1}],U)$, which is given by

$$u_i(p)=\{u(p),\ \text{for } p\in(p_i,p_{i+1}]\ u(p),\ \text{for } p=p_i$$

Moreover $E\subseteq QC(U)$ the notation is used here E_i for the set $E_i=\{u_i:u\in E\},i\in\{0,1,2,\dots,N\}$. Theorem 2.1 Assume the following D is a closed bounded convex subset of U, and that A is a completely continuous function from D to D. Then there is a point to be made $Z\in D$ as a result $AZ=Z$.

Lemma 2.1 [8] A set $E\cup QC(U)$ is relatively compact in QC(U) if and only if if each set E_i is relatively compact in $C([p_i,p_{i+1}],U)$.

Definition 2.1 A function $U:(-\infty,b]\to U$ is referred to as a problem solution (1.1) - (1.3) if $u(0)+h(u)=\phi_1\in E_v,u(p)=g_i(p,u(p))$ for all $p\in(p_i,s_i],i=1,2,3,\dots,M$, the integral equations following hold because the restriction of u(.)to the interval $J_h(h=0,1,2,3,\dots,M)$ is continuous.

$$u(p)=\phi(0)-h(u)+\frac{1}{\Gamma(q)}\int_0^p(p-s)^{q-1}\left(f(s,u_s)+\int_0^s k(s,\sigma,u_\sigma)d\sigma\right)ds,\ \text{for all } p\in[0,p_1]$$

$$u(p)=g_i\left(s_i,u(s_i)\right)+\frac{1}{\Gamma(q)}\int_{s_i}^p(p-s)^{q-1}\left(f(s,u_s)+\int_{s_i}^s k(s,\sigma,u_\sigma)d\sigma\right)ds$$

for all $p\in[s_i,p_{i+1}]$ and every $i=1,2,3,\dots,M$.

Lemma 2.2 Assume the following $U\in E_w^{'}$ then, for $p\in[0,b],U_p\in E_v$. Moreover

$$l\|u(p)\|\leq\|u_t\|E_v\leq\|\phi\|E_v+lsuu_{s\in[0,p]}\|u(s)\|$$

3. Main Results

For $\phi\in E_v,\phi\hat{}$ is defined as

$$\phi\hat{}(p)=\{\phi(p),p\in(-\infty,0]\ \phi(0),p\in[0,p_1]\ 0,p\in[p_1,b]$$

then $\phi\hat{}\in E_w^{'}$

Let $u(p)=v(p)+\phi\hat{}(p),-\infty<p<b$. v clearly satisfies. $v_0=0$ for $p\in(-\infty,0]$

$$v(p)=-h(v+\phi\hat{})+\frac{1}{\Gamma(q)}\int_0^p(p-s)^{q-1}\left[f(s,v_s+\phi\hat{}_s)+\int_0^s k(s,\sigma,v_\sigma+\phi\hat{}_\sigma)d\sigma\right]ds$$

for all $p\in[0,p_1]$

$$v(p)=g_i(p,(v+\phi\hat{})(p)) \text{ for all } p\in[p_1,s_1],\ \text{and each } i=1,2,\dots,M$$

$$v(p)=g_i\left(s_i,(v+\phi\hat{})(s_i)\right)+\frac{1}{\Gamma(q)}(p-s)^{q-1}\left[f(s,v_s+\phi\hat{}_s)+\int_{s_i}^s k(s,\sigma,v_\sigma+\phi\hat{}_\sigma)d\sigma\right]ds$$

for all $p\in[s_i,p_{i+1}]$ every $i=1,2,\dots,M$.

iff u meets the criteria

$u(p) = \phi(p), p \in (-\infty, 0]$ $u(p)$
$= \phi(0) - h(u) + \frac{1}{\Gamma(q)}\int_0^p (p-s)^{q-1}\left[f(s,u_s) + \int_0^s k(s,\sigma,u_\sigma)d\sigma\right]ds$, for all $p \in [0,p_1]$ $u(p) = g_i(p,u(p))$, for all $p \in (p_i, s_i]$ each $i = 1,2,3,\dots,M$

And

$$u(p) = g_i(s_i, u(s_i)) + \frac{1}{\Gamma(q)}\int_{s_i}^p (p-s)^{q-1}\left[f(s,u_s) + \int_{s_i}^s k(s,\sigma,u_\sigma)d\sigma\right]ds$$

for all $p \in [s_i, p_{i+1}]$ every $i = 1,2,\dots,M$

We introduce the following conditions to demonstrate our key findings

(H_1) $f_1: [0,b] \times E_v \to U$ is continuous, and two positive constants K_1, K_2 exist such that $\|f_1(p,\phi_1) - f_1(p,\phi_2)\| \le K_1(\|\phi_1 - \phi_2\|E_v, K_2 = \sup_{p\in[0,b]}\|f_1(p,0)\|$.

(H_2) $K: \Delta \times E_v \to U$, where $\Delta = \{(r,s): 0 \le s \le b\}$, with positive constants P_1, P_2 satisfying $\|K(p,s,\phi_1) - K(p,s,\phi_2)\| \le P_1(\|\phi_1 - \phi_2\|E_w^{'}), P_2 = \sup_{p,s}\|K((p,s,0)\|$.

(H_3) The functions $g_i: (p_i, s_i] \times U \to U$, these are positive constants that are continuous. As a result $Lg_i\|g_i(p,u) - g_i(p,v)\| \le Lg_i\|u - v\|$ for all $u, v \in U, p \in (p_i, s_i]$ and each $i = 0,1,2,\dots,M$.

(H_4) $h: E_w^{'} \to U$ is continuous, and some positive constants δ_1, δ_2 exist such that $\|h(u) - h(v)\| \le \delta_1\|u - v\|E_w^{'}$ and $\|h(u)\| \le \delta_1\|u\|E_w^{'} + \delta_2$

(H_5) $\omega = \max_i\left\{Lg_i + \delta_1 + \frac{lb^q}{\Gamma(q+1)}[K_1 + p_1 b]\right\} < 1, i = 1,2,\dots,M$.

Theorem 3.1 If the requirements $(H_1) - (H_5)$ are met, then the problem (1.1) - (1.3) has a unique solution.

Proof. Define $\zeta: E_w^{'} \to E_w^{'}$ as follows

$$\zeta v(p) = 0, p \in (-\infty, 0]$$

$$\zeta v(p) = -h(v + \phi^\wedge) + \frac{1}{\Gamma(q)}\int_0^p (p-s)^{q-1}\left[f(s, v_s + \phi^\wedge_s) + \int_0^s k(s,\sigma,v_\sigma + \phi^\wedge_\sigma)d\sigma\right]ds,$$

for all $p \in [0, p_1]$

$$\zeta v(p) = g_i(p, (v + \phi^\wedge)(p)), \text{ for all } p \in (p_i, s_i] \text{ and each } i = 1,2,3,\dots,M$$

and

$$\zeta v(p) = g_i(s_i, (v + \phi^\wedge)(s_i)) + \frac{1}{\Gamma(q)}\int_{s_i}^p (p-s)^{q-1}\left[f(s, v_s + \phi^\wedge_s) + \int_{s_i}^s k(s,\sigma,v_\sigma + \phi^\wedge_\sigma)d\sigma\right]ds \text{ for all } p$$

$\in [s_i, p_{i+1}]$ and every $i = 1,2,\dots,M$

$v + \phi^\wedge$ is a solution of the (1.1) - (1.3) system, if v is a fixed point of ζ. We'll prove that ζ fulfils Theorem 2.1's hypothesis.

Define the Banach space $(E_w^{'}, \|\cdot\|E_w^{'})$ defined by $E_w^{'}, E_w^{'} = \{v \in E_w^{'}: v_0 = 0 \in E_v\}$ with norm $\|v\|E_w^{'} = \sup\{\|v(s)\|: s \in [0,b]\}$ set $E_r = \{v \in E_w^{'}: \|v\|E_w^{'} \le r\}$ for some $r > 0$.

In any case $v \in E_r, p \in [0,b]$ as well as lemma 2.2, We've got

$$\|v_t + \phi^\wedge_t\|_{E_v} \le \|\phi\|_{E_v} + l[r + \|\phi(0)\|]$$
$$\|v + \phi^\wedge\|_{E_w^{'}} \le r + \|\phi\|_{E_v} + \|\phi(0)\|]$$

It is clear from the premise that ζ is clearly defined. Moreover, for $v_1, v_2 \in E_w^{'}, i \in \{1,2,\dots,M\}$, and $p \in [s_i, p_{i+1}]$ we get

$$\|\zeta v_1(p) - \zeta v_2(p)\|$$
$$\le \|g_i(s_i, (v_1 + \phi^\wedge)(s_i)) - g_i(s_i, (v_2 + \phi^\wedge)(s_i))\| + \frac{1}{\Gamma(q)}\int_{s_i}^p (p-s)^{q-1}\|[f(s, v_{1s} + \phi^\wedge_s)$$
$$+ \int_{s_i}^s k(s,\sigma,v_{1\sigma} + \phi^\wedge_\sigma)d\sigma] - \left[f(s, v_{2s} + \phi^\wedge_s) + \int_{s_i}^s k(s,\sigma,v_{2\sigma} + \phi^\wedge_\sigma)d\sigma\right]\|ds$$
$$\le \left[Lg_i + \frac{lb^q}{\Gamma(q+1)}[K_1 + P_1 b]\right]\|v_1 - v_2\|E_w^{'}$$

Hence $\|\zeta v_1 - \zeta v_2\|_{c([s_i,p_{i+1}];U)} \le \Omega\|v_1 - v_2\|E_w^{'}, i = 1,2,3,\dots,M.$

Using the same procedure for the interval $[0, p_1]$, we get

$$\|\zeta v_1(p) - \zeta v_2(p)\|$$
$$\le \| - h(v_1 + \phi^\wedge) + h(v_2 + \phi^\wedge)\| + \frac{1}{\Gamma(q)}\int_0^p (p-s)^{q-1}\|[f(s, v_{1s} + \phi^\wedge_s)$$

$$+\int_{s_i}^{s} k(s,\sigma,v_{1\sigma}+\phi\hat{}_\sigma)d\sigma] - \left[f(s,v_{2s}+\phi\hat{}_s)+\int_{s_i}^{s} k(s,\sigma,v_{2\sigma}+\phi\hat{}_\sigma)d\sigma\right]\|ds$$
$$\le \left[\delta_1 + \frac{ll^q}{\Gamma(q+1)}[K_1+P_1b]\right]\|v_1-v_2\|E_w^{'}$$

Hence $\|\zeta v_1 - \zeta v_2\|_{C([0,p_1];U)} \le \Omega\|v_1-v_2\|E_w^{'}$

Moreover, for $p \in (p_i, s_i]$ we have

$$\|\zeta v_1(p) - \zeta v_2(p)\| \le Lg_i\|v_1-v_2\|E_w^{'}$$

Hence

$\|\zeta v_1 - \zeta v_2\|_{C((p_i,s_i];U)} \le \Omega\|v_1-v_2\|E_w^{'}, i = 1,2,\dots,M.$

From the above we have that $\|\zeta v_1 - \zeta v_2\| \le \Omega\|v_1-v_2\|E_w^{'}$. As a result, ζ is a contradiction, and the solution (1.1) - (1.3) is unique. This brings the proof to a close.

Theorem 3.2 Assuming that the hypotheses $(H_1)-(H_5)$ are true and that the functions $g_i(.,0)$ are limited, there is a solution for the system (1.1) - (1.3).

Proof. We define ζ as in the theorem 3.1

The proof is then divided into five steps.

Step I: To prove $\zeta E_r \subset E_r$ There is a positive integer r such that E_r is a closed bounded convex set in $E_w^{'}.\zeta E_r \subset E_r$, if this isn't true for every positive number r, there are some $v \in E_r$ and $p \in (-\infty, b]$ such that $\|\zeta(u)(p)\| > r$, where p is determined by r.

$$r < \|\zeta v(p)\| \le \|g_i(p,(v+\phi\hat{})(p))\| \le L_{g_i}\|v+\phi\hat{}\|_{E_w^{'}}^{'} + \|g_i(p,0)\|$$
$$\le L_{g_i}\left(r+\|\phi\|_{E_v}\right) + \|g_i(.,0)\|_{C([p_i,s_i];U)}$$

Taking the lower limit and dividing by r on both sides $r \to +\infty$, we get $1 \le Lg_i$. This is in direct Opposition to (H_5). Therefore $||\zeta v||_{C([p_i,s_i];U)} \le r$ for $i \ge 1$.

Following the steps outlined above for $p \in [s_i, p_{i+1}]$ and $p \in [0,p_1], i \ge 1$ we obtain that $r \le L_{g_i}\big(r + \|\phi\|_{E_v}\big) + \|g_i(.,0)\|_{c([p_i,s_i];U)} + \frac{b^q}{\Gamma(q+1)}\left[K_1\left[(lr+\|\phi\|+E_v)+p_1b\left(lr+\|\phi\|_{E_v}\right)+p_2b\right]+K_2\right]$ Taking the lower limit and dividing by r on both sides $p \to +\infty$, we get $1 \le L_{g_i} + \frac{lb^q}{\Gamma(q+1)}(K_1+P_1b)$ and for $p \in [0,p_1]$, we obtain that $1 \le \delta_1 + \frac{lb^q}{\Gamma(q+1)}(K_1+P_1b)$, as a result of which there is a discrepancy (H_5). As a result, $\zeta E_r \subset E_r$

The decomposition is then introduced $\zeta = \zeta_1 + \zeta_2 = \sum_{i=0}^{M}\zeta_i^1 + \sum_{i=0}^{M}\zeta_i^2$ where $\zeta_i^j : E_r \to E_r, i = 1,2,\dots,M, j = 1,2,3,\dots$ provided

$$\zeta_i^1 v(p) = \{0,\ for\, p \in (-\infty,0]\ -h(v+\phi\hat{}),\ for\, p \in [0,t_1]\ g_i(p,(v+\phi\hat{})(p)),\ for\, p \in [p_i,s_i], i \ge 1\ g_i\big(s_i,(v+\phi\hat{})(s_i)\big),\ for\, p \in [s_i,p_{i+1}], i \ge 1\ 0,\ for\, p \in [p_i,p_{i+1}], i \ge 0$$

$$\zeta_i^2 v(p) = \{0,\ for\, p \in (-\infty,0]\ \frac{1}{\Gamma(q)}\int_{s_i}^{p}(p-s)^{q-1}\left[f(s,v_s+\phi\hat{}_s)+\int_{s_i}^{s}k(s,\sigma,x_\sigma)d\sigma\right]ds,\ for\, p \in (s_i,p_{i+1}], i \ge 0\ 0,\ for\, p \in [s_i,p_{i+1}], i \ge 0$$

Step II: The $map\zeta_1 = \sum_{i=0}^{M}\zeta_i^1$ is a contradiction on E_r. Take $v_1, v_2 \in E_r$ arbitrarily. then, for each $p \in (-\infty, b)$ and from (H_3) to (H_5), we have

$$\|\theta_i^1 v_1(p) - \zeta_i^2 v_2(p)\| \le \delta_1\|v_1-v_2\|_{E_w^{'}} + Lg_i\|v_1-v_2\|_{E_w^{'}}$$

which implies that

$$\|\textstyle\sum_{i=0}^{M}\zeta_i^1 v_1 - \sum_{i=0}^{M}\zeta_i^1 v_2\| \le \Omega\|v_1-v_2\|_{E_w^{'}}$$

This proves that ζ_1 is a contradiction on E_r. Next, we use the notation $\zeta_i^2 E_r(p) = \{\zeta_i^2 v(p) : E_r\}$

Step III: For $i = 0,1,2,3\dots,M$ and $s_i < s < p < p_{i+1}$, the set $X_{\sigma\in[s,p]}\zeta_i^2 B_r(\sigma)$ is relatively compact in $E_w^{'}$. Let $s_i < \mu_1 < s$. For $\epsilon > 0$ we make a decision $0 < \lambda_1 < \frac{s-\mu_1}{2}$ such that $\frac{\lambda_1^q}{\Gamma(q+1)}\left[K_1\left(lr+\|\phi\|_{E_v}\right)+K_2\right] \le \epsilon$ for all $E \subset [0,b]$ with $Diam(E) \le \lambda_1$ Then, for $\sigma \in [s,p]$ and $v \in E_r$ we obtain

$$\zeta_i^2 v(\sigma) = \frac{1}{\Gamma(q)}\int_{s_i}^{\sigma-\lambda_1}(\sigma-\lambda_1-s)^{q-1}\left[f(s,v_s+\phi\hat{}_s)+\int_{s_i}^{s}k(s,\sigma,x_\sigma)d\sigma\right]ds + \frac{1}{\Gamma(q)}\int_{\sigma-\lambda_1}^{\sigma}(\sigma-s)^{q-1}\left[f(s,v_s+\phi\hat{}_s)+\int_{s_i}^{s}k(s,\sigma,x_\sigma)d\sigma\right]ds \in E_{r_1} + B_{r_1,\epsilon}$$

where $r_1 = \frac{b^q}{\Gamma(q+1)}\left[K_1\left(lr+\|\phi\|_{E_v}\right)+K_1b\left(P_1\left(lr+\|\phi\|_{E_v}\right)+P_2\right)+K_2\right], r_1,\epsilon = \frac{\lambda_1^q}{\Gamma(q+1)}[K_1(lr + \|\phi\|_{E_v}) + K_1\lambda_1\left(P_1\left(lr+\|\phi\|_{E_v}\right)+P_2\right)+K_2]$, it suggests that $X_{\eta\in[s,p]}\zeta_i^2 E_r(\eta) \subset E_{r1} + E_{r1,\epsilon}.E_{r1}$ is relatively compact

and $Diam(E_\epsilon) \to 0$ as $\epsilon \to 0$, as a result $X_{\eta \in [s,p]} \zeta_i^2 E_r(\eta)$ is relatively compact in $E_w^{'}$.
Step IV: The set of functions $\{\zeta_i^2 E\tilde{}_r\}_i, i = 0,1,2,3,\ldots,M$, is a subset of that is equicontinuous $C([p_i;p_{i+1}];U)$

It is clear that $\{\zeta_i^2 E\tilde{}_r\}_i$ is Right Equicontinuous on $[p_i, s_i)$ and Left Equicontinuous on $(p_i, s_i]$. Let $p \in (s_i; p_{i+1})$, the set $\zeta_i^2 E_r(p)$ is relatively compact in $E_w^{'}$. For $v \in E_r$ and $0 < k < \lambda < p_{i+1} - p$ we get

$$\|\zeta\tilde{}_i^2 v(p+k) - \zeta\tilde{}_i^2 v(p)\| = \|\zeta_i^2 v(p+k) - \zeta_i^2 v(p)\|$$
$$\leq \frac{k^q}{\Gamma(q+1)}\left[K_1\left(lr + \|\phi\|_{E_v}\right) + K_1 b\left(P_1\left(lr + \|\phi\|_{E_v}\right) + P_2\right) + K_2\right]$$
$$+\frac{1}{\Gamma(q)}\int_{s_i}^{p} \|(p+k-s)^{q-1} - (p-s)^{q-1}\|$$
$$\times\left[K_1\left(lr + \|\phi\|_{E_v}\right) + K_1 b\left(P_1\left(lr + \|\phi\|_{E_v}\right) + P_2\right) + K_2\right]ds$$

The Right-Hand side is self-contained $v \in E_r$ as it approaches zero $k \to 0$. This demonstrates that $\{\zeta_i^2 E\tilde{}_r\}_i$ is Right Equicontinuous at p.

We proceed in the same way as we did for $p = s_i$ and $h > 0$, and we get $s_i + k < p_{i+1}$.

$$\|\zeta\tilde{}^2 v(s_{i+k}) - \zeta\tilde{}^2 v(s_i)\| = \|\frac{1}{\Gamma(q)}\int_{s_i}^{s_i+k} (p+k-s)^{q-1}\left[f(s, v_s + \phi\hat{}_s) + \int_{s_i}^{s} k(s,\sigma,u_\sigma)d\sigma\right]ds\|$$
$$\leq \frac{k^q}{\Gamma(q+1)}\left[K_1\left(lr + \|\phi\|_{E_v}\right) + K_1 b\left(P_1\left(lr + \|\phi\|_{E_v}\right) + P_2\right) + K_2\right]$$

this means that $\{\zeta_i^2 E\tilde{}_r\}_i$ is Right Equicontinuous at s_i. Now for $p \in (s_i, p_{i+1}]$. Let $\mu_1 \in (s_i, p]$. Since $X_{s \in [\mu_1, p]}\zeta_i^2 E_r(s)$ is relatively compact in $E_w^{'}$, we choose $0 < \lambda_1 < \frac{p-\mu_1}{2}$ then for $0 < k \leq \lambda_1$ and $v \in B_r$ we obtain, $\|\zeta\tilde{}_i^2 v(p-k) - \zeta\tilde{}_i^2 v(p)\| = \|\zeta_i^2 v(p-k) - \zeta_i^2 v(p)\|$

$$\leq \frac{1}{\Gamma(q)}\int_{p-k}^{p} (p-s)^{q-1}\|f(s, v_s + \phi\hat{}_s) + \int_{s_i}^{s} k(s,\sigma,u_\sigma)d\sigma\|ds$$
$$+\frac{1}{\Gamma(q)}\int_{s_i}^{p-h} \|(p-s)^{q-1} - (p-k-s)^{q-1}\|$$
$$\times\|f(s, v_s + \phi\hat{}_s) + \int_{s_i}^{s} k(s,\sigma,u_\sigma)d\sigma\|ds$$
$$\leq \frac{k^q}{\Gamma(q+1)}\left[K_1\left(lr + \|\phi\|_{B_v}\right) + K_1 b\left(p_1\left(lr + \|\phi\|_{E_v}\right) + p_2\right) + K_2\right]$$
$$+\frac{1}{\Gamma(q)}\int_{s_i}^{p} \|(p-s)^{q-1} - (p-k-s)^{q-1}\| \times [K_1\left(lr + \|\phi\|_{E_v}\right)$$
$$+K_1 b\left(p_1\left(lr + \|\phi\|_{E_v}\right) + p_2\right) + K_2]ds$$

which shows that $\{\zeta_i^2 E\tilde{}_r\}_i$ is left in an equicontinuous state at $p \in (s_i, p_{i+1}]$.

The proof that $\{\zeta_i^2 E\tilde{}_r\}_i$ is equicontinuous is now complete.

Step V: For $i \neq j$, the set $\{\zeta_i^2 E\tilde{}_r\}_j$ is a subset of that is equicontinuous $C\left(\left[p_j, p_{j+1}\right]; U\right)$. The preceding stages, as well as Lemma 2.1, result in, the map ζ_1 is a contraction and the map ζ_2 is completely continuous. Thus, $\zeta = \zeta_1 + \zeta_2$ is an operator for condensing. Based on [13, Theorem 4.3.2], we believe there is a solution of (1.1) - (1.3).

4. Example

Take into consideration the following fractional integrodifferential equation of the form with interval impulsive condition

$$D_p^q u(p) = \frac{1}{(p+2)^2}\frac{|u|}{1+|u|} + \frac{1}{4}\int_0^p e^{\frac{u_s}{3}}ds, p \in \cup_{i=1}^{V}[s_i, p_{i+1}], i = 0,1,2,3,\ldots,M \quad (4.1)$$
$$u(p) = G_i\left(p, u(p)\right), p \in (p_i, s_i], i = 1,2,3,\ldots,M \quad (4.2)$$
$$u(0) + \frac{u}{4+u} = u_0 \quad (4.3)$$

where $0 < q < 1$. Take $[0,b] = [0,1], 0 = p_0 = s_0 < p_1 \leq s_1 \leq \cdots \leq p_N \leq s_M \leq p_{M+1} = 1$ are fixed real numbers, $G_i \in C((p_i, s_i] \times R_1; R_1)$ for all $i = 1,2,\ldots,M$.

Let $u = R_1, K(u_s) = \int_0^p k(p,s,u_s)ds = \int_0^p e^{\frac{u_s}{3}}ds,$ $f(p,u) = \frac{1}{(p+2)^2}\frac{|u|}{1+|u|}$

$$K(p,s,u_s) = \int_0^p e^{\frac{u_s}{3}}ds$$
$$\|f(p,u) - f(p,v)\|$$

$$= \left| \frac{1}{(p+2)^2} \frac{|u|}{1+|u|} - \frac{1}{(p+2)^2} \frac{|v|}{1+|v|} \right|$$
$$= \left| \frac{1}{(p+2)^2} \left(\frac{|u|}{1+|u|} - \frac{|v|}{1+|v|} \right) \right|$$
$$\leq \frac{1}{(p+2)^2} |u-v|$$
$$\leq \frac{1}{4} |u-v|$$
$$\| k(p,s,u_s) - k(p,s,v_s) \|$$
$$= \left| \int_0^p e^{\frac{u_s}{3}} ds - \int_0^p e^{\frac{v_s}{3}} ds \right|$$
$$\leq \frac{1}{3} |u-v|$$

If $u, v \in U$ then we have $\|K(u_s) - K(v_s)\| \leq \frac{1}{3}\|u-v\|$ and $\|f(p,u,K(u_s)) - f(p,v,K(v_s))\| \leq \frac{1}{4}[\|u-v\| + \|K(u_s) - K(v_s)\|]$, here $\delta_1 = \frac{1}{4}, K_1 = \frac{1}{3}, P_1 = \frac{1}{2}$.

Let G_i be Lipschitz functions with Lipschitz constants L_{G_i} satisfied the condition $3L_{g_i} \leq 1$. Let $v(p) = e^p$, therefore $l = \int_{-\infty}^{0} v(p)dt = \int_{-\infty}^{0} e^p dt = [e^p]_{-\infty}^{0} = e^0 - e^{-\infty} = 1 - 0 = 1 < +\infty$. If $q = 1$ then $\Omega = \max_i \left\{ L_{g_i} + \delta_1 + \frac{lb^q}{\Gamma(q+1)}[K_1 + P_1 b] \right\} < 1$. Consequently, all of the Theorem 3.1 and Theorem 3.2 hypotheses. As a result, there is a solution to the problem (4.1) - (4.3).

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