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A new techniques applied to Volterra–Fredholm integral equations with discontinuous kernel

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Abstract

The purpose of this paper is to establish the general solution of a Volterra–Fredholm integral equation with discontinuous kernel in a Banach space. Banach's fixed point theorem is used to prove the existence and uniqueness of the solution. By using separation of variables method, the problem is reduced to a Volterra integral equations of the second kind with continuous kernel. Normality and continuity of the integral operator are also discussed.

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Key–Words: Banach space, Volterra–Fredholm integral equation, Separation of variables method.

1 Introduction

It is well-known that the integral equations govern many mathematical models of various phenomena in physics, economy, biology, engineering, even in mathematics and other fields of science. The illustrative examples of such models can be found in the literature, (see, e.g., [5, 6, 9, 11, 12, 14, 18, 20]). Many problems of mathematical physics, applied mathematics, and engineering are reduced to Volterra–Fredholm integral equations, see [1, 2].

Analytical solutions of integral equations, either do not exist or it's hard to compute. Eventual an exact solution is computable, the required calculations may be tedious, or the resulting solution may be difficult to interpret. Due to this, it is required to obtain an efficient numerical solution. There are numerous studies in literature concerning the numerical solution of integral equations such as [4, 8, 10, 13, 16, 17, 21].

In this present paper, the existence and uniqueness solution of the Eq. (1) are discussed and proved in the space $L_2(\Omega) \times C[0, T]$, $0 \leq T < 1$. Moreover, the normality and continuity of the integral operator are obtained. A numerical method is used to translate the Volterra–Fredholm integral equation (1) to a Volterra integral equations of the second kind with continuous kernel,

The outline of the paper is as follows: Sect. 1 is the introduction; In Sect. 2, the existence of a unique solution of the Volterra–Fredholm integral equation is discussed and proved using Picard's method and Banach's fixed point method. Sect. 3, include the general solution of the Volterra–Fredholm integral equation by applying the method of separation of variables. A brief conclusion is presented in Sect. 4.

Consider the following linear Volterra–Fredholm integral equation:

$$\begin{gathered}\mu\psi(x,t)-\lambda\int_0^t\int_\Omega\Phi(t,\tau)k(|x-y|)\psi(y,\tau)\mathrm{d}y\mathrm{d}\tau-\lambda\int_0^t F(t,\tau)\psi(x,\tau)\mathrm{d}\tau=g(x,t),\\ (x=\bar{x}(x_1,x_1,\ldots,x_n),\quad y=\bar{y}(y_1,y_1,\ldots,y_n)),\end{gathered}\tag{1}$$

where μ is a constant, defined the kind of integral equation, λ is constant, may be complex and has many physical meaning. The function $\psi(x,t)$ is unknown in the Banach space $L_2(\Omega)\times C[0,T]$, $0\le T<1$, where Ω is the domain of integration with respect to position and the time $t\in[0,T]$ and it called the potential function of the Volterra–Fredholm integral equation. The kernels of time $\Phi(t,\tau)$, $F(t,\tau)$ are continuous in $C[0,T]$ and the known function $g(x,t)$ is continuous in the space $L_2(\Omega)\times C[0,T]$, $0\le t\le T$. In addition the kernel of position $k(|x-y|)$ is discontinuous function.

2 The existence of a unique solution of the Volterra–Fredholm integral equation

In this paper, for discussing the existence and uniqueness of the solution of Eq. (1), we assume the following conditions:

(i) The kernel of position $k(|x-y|)\in L_2([\Omega]\times[\Omega])$, $x,y\in[\Omega]$ satisfies the discontinuity condition:

$$\left\{\int_\Omega\int_\Omega k^2(|x-y|)\mathrm{d}x\mathrm{d}y\right\}^{\frac{1}{2}}=k^*,\quad k^*\ \textit{is constant}.$$

(ii) The kernels of time $\Phi(t,\tau)$, $F(t,\tau)\in C[0,T]$ and satisfies $|\Phi(t,\tau)|\le M_1$, $|F(t,\tau)|\le M_2$, s.t M_1, M_2 are constants, $\forall t,\tau\in[0,T]$.

(iii) The given function $g(x,t)$ with its partial derivatives with respect to the position and time is continuous in the space $L_2(\Omega)\times C[0,T]$, $0\le\tau\le T<1$ and its norm is defined as,

$$\|g(x,t)\|=\max_{0\le t\le T}\int_0^t\left(\int_\Omega g^2(x,\tau)\mathrm{d}x\right)^{\frac{1}{2}}\mathrm{d}\tau=N,\ N\ \textit{is a constant}.$$

Theorem 1. *If the conditions (i)–(iii) are satisfied, then Eq. (1) has a unique solution $\psi(x,t)$ in the Banach space $L_2(\Omega)\times C[0,T]$, $0\leq T<1$, under the condition,*

$$|\lambda| < \frac{|\mu|}{M_1 k^* + M_2 T}.$$

Proof. To prove the existence of a unique solution of Eq. (1) we use the successive approximations method (*Picard's method*), or we can used Banach's fixed point theorem.

2.1 Picard's method

We assume the solution of Eq. (1) takes the form:

$$\psi(x,t) = \lim_{n\to\infty} \psi_n(x,t),$$

where

$$\psi_n(x,t) = \sum_{i=0}^{n} H_i(x,t), \quad t\in[0,T], \qquad n=1,2,\ldots$$

where the functions $G_i(x,t)$, $i=0,1,\ldots,n$ are continuous functions of the form:

$$\left.\begin{aligned} H_n(x,t) &= \psi_n(x,t) - \psi_{n-1}(x,t), \\ H_0(x,t) &= g(x,t) \end{aligned}\right\}. \tag{2}$$

□

Now we should prove the following lemmas:

Lemma 1. *The series $\sum_{i=0}^{n} H_i(x,t)$ is uniformly convergent to a continuous solution function $\psi(x,t)$.*

Proof. We construct the sequences,

$$\mu\psi_n(x,t) = g(x,t) + \lambda\int_0^t\int_\Omega \Phi(t,\tau)k(|x-y|)\psi_{n-1}(y,\tau)\mathrm{d}y\mathrm{d}\tau + \lambda\int_0^t F(t,\tau)\psi_{n-1}(x,\tau)\mathrm{d}\tau,$$
$$\psi_0(x,t) = g(x,t).$$

Then, we get

$$\begin{aligned}\psi_n(x,t)-\psi_{n-1}(x,t) =& \frac{\lambda}{\mu}\int_0^t\int_\Omega \Phi(t,\tau)k(|x-y|)(\psi_{n-1}(y,\tau)-\psi_{n-2}(y,\tau))\mathrm{d}y\mathrm{d}\tau \\ &+ \frac{\lambda}{\mu}\int_0^t F(t,\tau)(\psi_{n-1}(x,\tau)-\psi_{n-2}(x,\tau))\mathrm{d}\tau.\end{aligned}$$

From Eq. (2), then, we have

$$H_n(x,t) = |\gamma| \int_0^t \int_\Omega \Phi(t,\tau) k(|x-y|) H_{n-1}(y,\tau) \mathrm{d}y \mathrm{d}\tau + |\gamma| \int_0^t F(t,\tau) H_{n-1}(x,\tau) \mathrm{d}\tau; \qquad \gamma = \frac{\lambda}{\mu},$$

using the properties of the norm, we obtain

$$\|H_n(x,t)\| \leq |\gamma| \left\| \int_0^t \int_\Omega \Phi(t,\tau) k(|x-y|) H_{n-1}(y,\tau) \mathrm{d}y \mathrm{d}\tau \right\| + |\gamma| \left\| \int_0^t F(t,\tau) H_{n-1}(x,\tau) \mathrm{d}\tau \right\|. \quad (3)$$

For $n = 1$, the formula (3) yields

$$\|H_1(x,t)\| \leq |\gamma| \left\| \int_0^t \int_\Omega \Phi(t,\tau) k(|x-y|) H_0(y,\tau) \mathrm{d}y \mathrm{d}\tau \right\| + |\gamma| \left\| \int_0^t F(t,\tau) H_0(x,\tau) \mathrm{d}\tau \right\|,$$

by applying Cauchy–Schwarz inequality and using the condition (ii) we get

$$\begin{aligned} \|H_1(x,t)\| \leq & |\gamma| M_1 \left\| \left(\int_\Omega |k(|x-y|)|^2 \mathrm{d}y \right)^{\frac{1}{2}} . \max_{0 \leq t \leq T} \left| \int_0^t \left(\int_\Omega |H_0(y,\tau)|^2 \mathrm{d}y \right)^{\frac{1}{2}} \mathrm{d}\tau \right| \right\| \\ & + |\gamma| M_2 \int_0^t \|H_0(x,\tau)\| \,\mathrm{d}\tau, \end{aligned}$$

using the conditions (i) and (iii), we have

$$\|H_1(x,t)\| \leq |\gamma| M_1 k^* N + |\gamma| M_2 N \|t\|, \quad (4)$$

where $\max_{0 \leq t \leq T} |t| = T$, so that formula (4) becomes

$$\|H_1(x,t)\| \leq |\gamma| N (M_1 k^* + M_2 T),$$

by induction, we get

$$\|H_n(x,t)\| \leq \beta^n N; \qquad \beta = |\gamma|(M_1 k^* + M_2 T) < 1; \quad n = 1, 2, \ldots.$$

Since

$$|\lambda| < \frac{|\mu|}{M_1 k^* + M_2 T},$$

this leads us to say that the sequence $\psi_n(x,t)$ has a convergent solution. So that, for $n \to \infty$, we have

$$\psi(x,t) = \sum_{i=0}^{\infty} H_i(x,t). \quad (5)$$

The above formula represents an infinite convergence series. □

Lemma 2. *The function $\psi(x,t)$ of the series* (5) *represents an unique solution of Eq.* (1).

Proof. To show that $\psi(x,t)$ is the only solution of Eq. (1), we assume the existence of another solution $\varphi(x,t)$ of Eq. (1), then we obtain

$$\begin{aligned}\mu[\psi(x,t)-\varphi(x,t)] =&\lambda\int_0^t\int_\Omega \Phi(t,\tau)k(|x-y|)[\psi(y,\tau)-\varphi(y,\tau)]\mathrm{d}y\mathrm{d}\tau\\&+\lambda\int_0^t F(t,\tau)[\psi(x,\tau)-\varphi(x,\tau)]\mathrm{d}\tau,\end{aligned}$$

which leads us to the following

$$\begin{aligned}\|\psi(x,t)-\varphi(x,t)\| =&|\gamma|\left\|\int_0^t\int_\Omega \Phi(t,\tau)k(|x-y|)(\psi(y,\tau)-\varphi(y,\tau))\mathrm{d}y\mathrm{d}\tau\right\|\\&+|\gamma|\left\|\int_0^t F(t,\tau)(\psi(x,\tau)-\varphi(x,\tau))\mathrm{d}\tau\right\|,\end{aligned}$$

by applying the Cauchy–Schwarz inequality and using the conditions (i) and (ii), we get

$$\begin{aligned}\|\psi(x,t)-\varphi(x,t)\| \le& |\gamma|M_1k^*\int_0^t\int_\Omega \|\psi(y,\tau)-\varphi(y,\tau)\|\mathrm{d}y\mathrm{d}\tau\\&+|\gamma|M_2\int_0^t\|\psi(x,\tau)-\varphi(x,\tau)\|\mathrm{d}\tau,\\&\le \beta\|\psi(x,t)-\varphi(x,t)\|,\qquad \beta=|\gamma|M_1k^*+|\gamma|M_2T<1.\end{aligned}\tag{6}$$

The formula (6) can be adapted as,

$$(1-\beta)\|\psi(x,t)-\varphi(x,t)\|\le 0.$$

Since $\beta<1$, so that $\psi(x,t)=\varphi(x,t)$, that is the solution is unique. □

2.2 Banach's fixed point theorem

When the Picard's method fails to prove the existence of a unique solution for the homogeneous integral equations or for the integral equations of the first kind, we must use Banach's fixed point theorem. For this, we write the formula (1) in the integral operator form:

$$\begin{aligned}(\overline{U}\psi)(x,t)&=\frac{1}{\mu}g(x,t)+(U\psi)(x,t),\\(U\psi)(x,t)&=\frac{\lambda}{\mu}\int_0^t\int_\Omega\Phi(t,\tau)k(|x-y|)\psi(y,\tau)\mathrm{d}y\mathrm{d}\tau+\frac{\lambda}{\mu}\int_0^t F(t,\tau)\psi(x,\tau)\mathrm{d}\tau.\end{aligned}\tag{7}$$

To prove the existence of a unique solution of Eq. (1), using Banach's fixed point theorem, we must prove the normality and continuity of the integral operator (7).

(a) For the normality, we use Eq. (7) to get

$$\|(U\psi)(x,t)\|=\left|\frac{\lambda}{\mu}\right|\left\|\int_0^t\int_\Omega\Phi(t,\tau)k(|x-y|)\psi(y,\tau)\mathrm{d}y\mathrm{d}\tau\right\|+\left|\frac{\lambda}{\mu}\right|\left\|\int_0^t F(t,\tau)\psi(x,\tau)\mathrm{d}\tau\right\|;\quad \mu\neq 0.$$

Using the condition (ii), then applying Cauchy–Schwarz inequality, we get

$$
\begin{aligned}
\|(U\psi)(x,t)\| \le \left|\frac{\lambda}{\mu}\right| M_1 \left\| \left(\int_\Omega |k(|x-y|)|^2 \mathrm{d}y\right)^{\frac{1}{2}} \cdot \max_{0\le t\le T} \left| \int_0^t \left(\int_\Omega |H_0(y,\tau)|^2 \mathrm{d}y\right)^{\frac{1}{2}} \mathrm{d}\tau \right| \right\| \\
+ \left|\frac{\lambda}{\mu}\right| M_2 \left\| \int_0^t \|H_0(x,\tau)\| \mathrm{d}\tau \right\|,
\end{aligned}
$$

using the condition (i), we obtain

$$
\|(U\psi)(x,t)\| \le \left|\frac{\lambda}{\mu}\right| (M_1 k^* + M_2 T)\|\psi(x,t)\|,
$$

since

$$
\|(U\psi)(x,t)\| \le \beta\|\psi(x,t)\|; \qquad \beta = \left|\frac{\lambda}{\mu}\right| (M_1 k^* + M_2 T) < 1,
$$

where

$$
|\lambda| < \frac{|\mu|}{M_1 k^* + M_2 T}.
$$

Therefore, the integral operator U has a normality, which leads immediately after using the condition (iii) to the normality of the operator $\overline{U}$.

(b) For the continuity, we suppose the two potential functions $\psi_1(x,t)$ and $\psi_2(x,t)$ in the space $L_2(\Omega) \times C[0,T]$ are satisfied Eq. (7), then

$$
\begin{aligned}
(\overline{U}\psi_1)(x,t) &= \frac{1}{\mu} g(x,t) + \frac{\lambda}{\mu} \int_0^t \int_\Omega \Phi(t,\tau) k(|x-y|)\psi_1(y,\tau)\mathrm{d}y\mathrm{d}\tau + \frac{\lambda}{\mu}\int_0^t F(t,\tau)\psi_1(x,\tau)\mathrm{d}\tau, \\
(\overline{U}\psi_2)(x,t) &= \frac{1}{\mu} g(x,t) + \frac{\lambda}{\mu} \int_0^t \int_\Omega \Phi(t,\tau) k(|x-y|)\psi_2(y,\tau)\mathrm{d}y\mathrm{d}\tau + \frac{\lambda}{\mu}\int_0^t F(t,\tau)\psi_2(x,\tau)\mathrm{d}\tau,
\end{aligned} \tag{8}
$$

Using equations (8), we get

$$
\begin{aligned}
\overline{U}[\psi_1(x,t) - \psi_2(x,t)] =& \frac{\lambda}{\mu} \int_0^t \int_\Omega \Phi(t,\tau) k(|x-y|)[\psi_1(y,\tau) - \psi_2(y,\tau)]\mathrm{d}y\mathrm{d}\tau \\
&+ \frac{\lambda}{\mu}\int_0^t F(t,\tau)[\psi_1(x,\tau) - \psi_2(x,\tau)]\mathrm{d}\tau.
\end{aligned}
$$

using the condition (ii) and applying the Cauchy–Schwarz inequality we get ,

$$
\begin{aligned}
\|\overline{U}[\psi_1(x,t) - \psi_2(x,t)]\| \le \left|\frac{\lambda}{\mu}\right| M_1 \left\| \left(\int_\Omega |k(|x-y|)|^2 \mathrm{d}y\right)^{\frac{1}{2}} \right. \\
\left. \cdot \max_{0\le t\le T} \left| \int_0^t \left(\int_\Omega |\psi_1(y,\tau) - \psi_2(y,\tau)|^2 \mathrm{d}y\right)^{\frac{1}{2}} \mathrm{d}\tau \right| \right\| + \left|\frac{\lambda}{\mu}\right| M_2 \left\| \int_0^t |\psi_1(x,\tau) - \psi_2(x,\tau)| \mathrm{d}\tau \right\|.
\end{aligned}
$$

By using the condition (i), the last inequality becomes,

$$\|\overline{U}[\psi_1(x,t)-\psi_2(x,t)]\| \leq \left|\frac{\lambda}{\mu}\right| (M_1k^* + M_2T)\|\psi_1(x,t)-\psi_2(x,t)\|,$$

hence, we have

$$\|\overline{U}[\psi_1(x,t)-\psi_2(x,t)]\| \leq \beta\|\psi_1(x,t)-\psi_2(x,t)\|; \qquad \beta = \left|\frac{\lambda}{\mu}\right| (M_1k^* + M_2T) < 1, \tag{9}$$

with

$$|\lambda| < \frac{|\mu|}{(M_1k^* + M_2T)}.$$

Inequality (9) leads us to the continuity of the integral operator $\overline{U}$. So that, $\overline{U}$ is a contraction operator. Therefore by Banach's fixed point theorem, there is an unique fixed point $\psi(x,t)$, which is the solution of the linear mixed integral equation (1).

3 Separation of variables method

To obtain the general solution of Eq. (1), we do the following:
For $t = 0$, the formula (1) becomes

$$\mu\psi(x,0) = g(x,0). \tag{10}$$

Then, seek the solution of equation (1) in the form:

$$\psi(x,t) = \sum_{n=1}^{\infty} c_n(t)\psi_n(x),$$

in this aim, we write

$$\psi(x,t) = \psi_0(x,t) + \psi_1(x,t), \tag{11}$$

where $\psi_0(x,t)$, $\psi_1(x,t)$ are called, respectively, the complementary and particularly solution of (1). Using Eq. (11) in Eq. (1), we get

$$\mu\psi_k(x,t) - \lambda\int_0^t\int_\Omega \Phi(t,\tau)k(|x-y|)\psi_k(y,\tau)\mathrm{d}y\mathrm{d}\tau - \lambda\int_0^t F(t,\tau)\psi_k(x,\tau)\mathrm{d}\tau = \delta_k g(x,t);\ k=0,1, \tag{12}$$

also, for Eq. (10), we have

$$\mu\psi_k(x,0) = \delta_k g(x,0), \tag{13}$$

where,

$$\delta_k = \begin{cases} 0; & k=0 \\ 1; & k=1 \end{cases}.$$

From the two Eqs. (12), (13), we get

$$\begin{aligned}\mu[\psi_k(x,t)-\psi_k(x,0)] - \lambda\int_0^t\int_\Omega \Phi(t,\tau)k(|x-y|)\psi_k(y,\tau)\mathrm{d}y\mathrm{d}\tau \\ - \lambda\int_0^t F(t,\tau)\psi_k(x,\tau)\mathrm{d}\tau = \delta_k[g(x,t)-g(x,0)].\end{aligned} \tag{14}$$

Now, we can represent the solution of (11) in the series form

$$\psi_k(x,t) = \sum_{n=1}^{\infty}\left(c_{2n}^{(k)}(t)\psi_{2n}(x) + c_{2n-1}^{(k)}(t)\psi_{2n-1}(x)\right), \tag{15}$$

where $\psi_{2n}(x),\ \psi_{2n-1}(x)$ are the even and odd functions respectively.
Using Eq. (15) in Eq. (14), we obtain

$$\begin{aligned}&\mu\sum_{n=1}^{\infty}\left(c_{2n}^{(k)}(t)-c_{2n}^{(k)}(0)\right)\psi_{2n}(x) + \mu\sum_{n=1}^{\infty}\left(c_{2n-1}^{(k)}(t)-c_{2n-1}^{(k)}(0)\right)\psi_{2n-1}(x)\\ &-\lambda\int_0^t\int_\Omega \Phi(t,\tau)k(|x-y|)\sum_{n=1}^{\infty}\left(c_{2n}^{(k)}(\tau)\psi_{2n}(y)+c_{2n-1}^{(k)}(\tau)\psi_{2n-1}(y)\right)\mathrm{d}y\mathrm{d}\tau\\ &-\lambda\int_0^t F(t,\tau)\sum_{n=1}^{\infty}\left(c_{2n}^{(k)}(\tau)\psi_{2n}(x)+c_{2n-1}^{(k)}(\tau)\psi_{2n-1}(x)\right)\mathrm{d}\tau = \delta_k[g(x,t)-g(x,0)].\end{aligned} \tag{16}$$

Taking $k=0$, in Eq. (14), yields

$$\begin{aligned}&\mu\sum_{n=1}^{\infty}\left(c_{2n}^{(0)}(t)-c_{2n}^{(0)}(0)\right)\psi_{2n}(x) + \mu\sum_{n=1}^{\infty}\left(c_{2n-1}^{(0)}(t)-c_{2n-1}^{(0)}(0)\right)\psi_{2n-1}(x)\\ &-\lambda\int_0^t\int_\Omega \Phi(t,\tau)k(|x-y|)\sum_{n=1}^{\infty}\left(c_{2n}^{(0)}(\tau)\psi_{2n}(y)+c_{2n-1}^{(0)}(\tau)\psi_{2n-1}(y)\right)\mathrm{d}y\mathrm{d}\tau\\ &-\lambda\int_0^t F(t,\tau)\sum_{n=1}^{\infty}\left(c_{2n}^{(0)}(\tau)\psi_{2n}(x)+c_{2n-1}^{(0)}(\tau)\psi_{2n-1}(x)\right)\mathrm{d}y\mathrm{d}\tau = 0.\end{aligned} \tag{17}$$

Theorem 2. *(see [3, 19]). For a symmetric and positive kernel of Fredholm integral term of Eq.* (1)*, the integral operator,*

$$(K\psi_n)(x) = \int_\Omega k(|x-y|)\psi_n(y)\mathrm{d}y,$$

through the time interval $0 \le t \le T < 1$ is compact and self–adjoint operator. So, we may write $(K\psi_n)(x) = \alpha_n\psi_n(x)$, where α_n and $\psi_n(x)$ are the eigenvalues and the eigenfunctions of the integral operator, respectively.

In view of theorem 2, and Eq. (17), we arrive to the following

$$\begin{aligned}&\mu\sum_{n=1}^{\infty}\left(c_{2n}^{(0)}(t)-c_{2n}^{(0)}(0)\right)\psi_{2n}(x)+\mu\sum_{n=1}^{\infty}\left(c_{2n-1}^{(0)}(t)-c_{2n-1}^{(0)}(0)\right)\psi_{2n-1}(x)\\&-\lambda\int_0^t\Phi(t,\tau)\sum_{n=1}^{\infty}\left(\alpha_{2n}c_{2n}^{(0)}(\tau)\psi_{2n}(x)+\alpha_{2n-1}c_{2n-1}^{(0)}(\tau)\psi_{2n-1}(x)\right)\mathrm{d}\tau\\&-\lambda\int_0^t F(t,\tau)\sum_{n=1}^{\infty}\left(c_{2n}^{(0)}(\tau)\psi_{2n}(x)+c_{2n-1}^{(0)}(\tau)\psi_{2n-1}(x)\right)\mathrm{d}y\mathrm{d}\tau=0.\end{aligned}$$

Separating the odd and even terms, we obtain

$$c_{2n}^{(0)}(t)-\gamma\int_0^t(\alpha_{2n}\Phi(t,\tau)+F(t,\tau))c_{2n}^{(0)}(\tau)\mathrm{d}\tau=c_{2n}^{(0)}(0);\quad\gamma=\frac{\lambda}{\mu},\tag{18}$$

and,

$$c_{2n-1}^{(0)}(t)-\gamma\int_0^t(\alpha_{2n-1}\Phi(t,\tau)+F(t,\tau))c_{2n-1}^{(0)}(\tau)\mathrm{d}\tau=c_{2n-1}^{(0)}(0),\tag{19}$$

the two Eqs. (18) and (19) give the same results for even and odd functions, so it is suffice to study the following equation,

$$c_{n}^{(0)}(t)-\gamma\int_0^t(\alpha_{n}\Phi(t,\tau)+F(t,\tau))c_{n}^{(0)}(\tau)\mathrm{d}\tau=c_{n}^{(0)}(0);\quad\gamma=\frac{\lambda}{\mu},\tag{20}$$

where $c_n^{(0)}(0)$ is constant will be determined.

Also, taking $k=1$ in formula (16), we obtain

$$\begin{aligned}&\mu\sum_{n=1}^{\infty}\left(c_{2n}^{(1)}(t)-c_{2n}^{(1)}(0)\right)\psi_{2n}(x)+\mu\sum_{n=1}^{\infty}\left(c_{2n-1}^{(1)}(t)-c_{2n-1}^{(1)}(0)\right)\psi_{2n-1}(x)\\&-\lambda\int_0^t\int_\Omega\Phi(t,\tau)k(|x-y|)\sum_{n=1}^{\infty}\left(c_{2n}^{(1)}(\tau)\psi_{2n}(y)+c_{2n-1}^{(1)}(\tau)\psi_{2n-1}(y)\right)\mathrm{d}y\mathrm{d}\tau\\&-\lambda\int_0^t F(t,\tau)\sum_{n=1}^{\infty}\left(c_{2n}^{(1)}(\tau)\psi_{2n}(x)+c_{2n-1}^{(1)}(\tau)\psi_{2n-1}(x)\right)\mathrm{d}y\mathrm{d}\tau=[g(x,t)-g(x,0)].\end{aligned}\tag{21}$$

Using theorem 2 in Eq. (21), to have

$$\begin{aligned}&\mu\sum_{n=1}^{\infty}\left(c_{2n}^{(1)}(t)-c_{2n}^{(1)}(0)\right)\psi_{2n}(x)+\mu\sum_{n=1}^{\infty}\left(c_{2n-1}^{(1)}(t)-c_{2n-1}^{(1)}(0)\right)\psi_{2n-1}(x)\\&-\lambda\int_0^t\Phi(t,\tau)\sum_{n=1}^{\infty}\left(\alpha_{2n}c_{2n}^{(1)}(\tau)\psi_{2n}(x)+\alpha_{2n-1}c_{2n-1}^{(1)}(\tau)\psi_{2n-1}(x)\right)\mathrm{d}\tau\\&-\lambda\int_0^t F(t,\tau)\sum_{n=1}^{\infty}\left(c_{2n}^{(1)}(\tau)\psi_{2n}(x)+c_{2n-1}^{(1)}(\tau)\psi_{2n-1}(x)\right)\mathrm{d}y\mathrm{d}\tau\\&=\sum_{n=1}^{\infty}a_{2n}\psi_{2n}(x)[g(x,t)-g(x,0)],\end{aligned}\tag{22}$$

where,

$$1 = \sum_{n=1}^{\infty} a_{2n}\psi_{2n}(x),$$

the formula (22) can be separated to the following equations

$$\begin{gathered} c_{2n}^{(1)}(t) - \gamma \int_0^t (\alpha_{2n}\Phi(t,\tau) + F(t,\tau))c_{2n}^{(1)}(\tau)\mathrm{d}\tau = \frac{1}{\mu}a_{2n}[g(x,t) - g(x,0)] + c_{2n}^{(1)}(0);\ \gamma = \frac{\lambda}{\mu}, \\ c_{2n-1}^{(1)}(t) - \gamma \int_0^t (\alpha_{2n-1}\Phi(t,\tau) + F(t,\tau))c_{2n-1}^{(1)}(\tau)\mathrm{d}\tau = c_{2n-1}^{(1)}(0). \end{gathered} \tag{23}$$

Eqs. (20) and (23) represent Volterra integral equations of the second kind that have the same continuous kernel $\Phi(t,\tau) \in C([0,T] \times [0,T])$, and each of them has a unique solution in the class $C[0,T]$ the books edited by Linz [15] and Burton [7] contain many different methods to solve the integral equations (20) and (23).

The values of $c_n^{(0)}(0)$, $c_{2n}^{(1)}(0)$ and $c_{2n-1}^{(1)}(0)$ can be obtained, we return to the equation (10), and we seek the solution of this equation in the form,

$$\psi(x,0) = \sum_{n=1}^{\infty} c_n(0)\psi_n(x).$$

Hence, in this respect, we write

$$\psi(x,t) = \psi_0(x,t) + \psi_1(x,t), \tag{24}$$

where $\psi_0(x,t)$ is a complementary solution while $\psi_1(x,t)$ is a particularly solution. So, from Eq. (10) we write

$$\mu\psi_k(x,0) = \delta_k g(x,0), \tag{25}$$

and expand the solution of equation (24) in the form

$$\psi_k(x,0) = \sum_{n=1}^{\infty} \left(c_{2n}^{(k)}(0)\psi_{2n}(x) + c_{2n-1}^{(k)}(0)\psi_{2n-1}(x) \right), \tag{26}$$

using Eq. (26) in Eq. (25), we obtain

$$\mu \sum_{n=1}^{\infty} \left(c_{2n}^{(k)}(0)\psi_{2n}(x) + c_{2n-1}^{(k)}(0)\psi_{2n-1}(x) \right) = \delta_k g(x,0). \tag{27}$$

If we take $k = 0$ in Eq. (27), we obtain

$$\mu \sum_{n=1}^{\infty} \left(c_{2n}^{(0)}(0)\psi_{2n}(x) + c_{2n-1}^{(0)}(0)\psi_{2n-1}(x) \right) = 0,$$

equating the odd and even terms in both sides, we get

$$c_{2n}^{(0)}(0) = 0, \qquad c_{2n-1}^{(0)}(0) = 0,$$

then , we have

$$c_n^{(0)}(0) = 0.$$

Taking $k = 1$ in Eq. (27), we have

$$\mu \sum_{n=1}^{\infty} \left(c_{2n}^{(1)}(0)\psi_{2n}(x) + c_{2n-1}^{(1)}(0)\psi_{2n-1}(x) \right) = g(x, 0).$$

Equating both sides of the last equation, we get

$$c_{2n}^{(1)}(0) = 0, \qquad c_{2n-1}^{(1)}(0) = 0,$$

so, the last two formulas give us

$$c_n^{(1)}(0) = 0.$$

In view of Eqs. (20) and (23), the general solution of (1) can be adapted in the form

$$\psi_N(x, t) = \sum_{n=1}^{N} \left(c_n^{(0)}(t) + c_n^{(1)}(t) \right) \psi_n(x), \tag{28}$$

where $c_n^{(0)}(t)$ and $c_n^{(1)}(t)$ must satisfy the inequality

$$\sum_{n=1}^{N} \left| c_n^{(0)}(t) + c_n^{(1)}(t) \right| < \epsilon, \quad (N \to \infty,\ \epsilon \ll 1,\ 0 \le t \le T < 1). \tag{29}$$

Theorem 3. *If, for $t \in [0, T]$, the inequality (29) holds, the series (28) is uniformly convergent in the space $\ell_2(\Omega) \times C[0, T]$, $N \to \infty$. Hence the solution of the Volterra–Fredholm integral equation (1) can be obtained in a series form of (28).*

Theorem 4. *For the given functions $g(x, t) \in L_2(\Omega) \times C[0, T]$, $\Phi(t, \tau) \in C([0, T] \times [0, T])$, $k(x, y) \in C([\Omega] \times [\Omega])$, and under the condition (29), we have*

$$\|\psi(x, t) - \psi_N(x, t)\| \to 0 \quad as \quad N \to \infty,$$

where $\psi(x, t)$ represents the unique solution of Eq. (1) and the error takes the form:

$$E_N = \|\psi(x, t) - \psi_N(x, t)\|,$$

where

$$E_N \to 0 \quad as \quad N \to 1.$$

4 Conclusion and remarks

From the above results and discussion, the following may be concluded:

1. Equation (1) has a unique solution $\psi(x,t)$ in the space $L_2(\Omega) \times C[0,T]$, under some conditions.

2. The Volterra–Fredholm integral equation of the second kind, in time and position, after using separation of variables method leads to a Volterra integral equations of the second kind with continuous kernel.

3. Solutions of the Volterra integral equations can be obtained by numerical methods.

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