Exploring Extremum Points and Mean Value Theorems via Gâteaux Derivatives in Linear 2-Normed Spaces

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ABSTRACT
This paper explores differentiation within 2-normed spaces, which extend traditional normed spaces to provide deeper insights into linear structures related to real-valued functions. Building on the foundational contributions of Gähler and Iseki, we focus on Gâteaux and Fréchet derivatives and their essential roles in differentiability, particularly in identifying extremum points and developing mean value theorems that enable optimal solutions in variational problems. We also review recent works by Liu and Zhao, along with Patel et al., which clarify the practical applications of these differentiation concepts in optimization theory. Furthermore, we examine the relationship between Gâteaux and Fréchet derivatives, revealing conditions for their coincidence that enhance the theoretical framework for differentiability in higher dimensions. Through various theorems, we demonstrate the implications of our findings for both theoretical and practical contexts, highlighting the significance of 2-normed spaces in applied mathematics. This research contributes to ongoing discussions on differentiation and optimization while underscoring the practical importance of these mathematical structures in tackling real-world challenges, suggesting that future investigations could further solidify the role of 2-normed spaces in functional analysis.

Keywords: Differentiation, 2-normed Spaces, Gâteaux Derivative, Fréchet Derivative, Optimization

1. INTRODUCTION

Differentiation is a fundamental concept in functional analysis, particularly when examining operators defined on one normed space and their interactions with another. Various definitions of differentiation exist for mappings between topological spaces, particularly within the context of normed spaces. Historically, the foundational work on differentiation can be attributed to mathematicians such as R. Gâteaux, M. Fréchet, and J. Hadamard whose contributions have shaped our understanding of the differentiability of functions in these spaces (Nashad et al., 1969, 1974). In particular, Nashad's (1974) comprehensive analysis of the differentiability properties of operators has served as a seminal reference in this area.

In the broader context of normed spaces, the concept of 2-normed spaces offers a generalized framework that enhances the study of linear structures characterized by specific properties of real-valued functions. Introduced by Gähler (1965), 2-normed spaces allow for a more nuanced examination of differentiation. Subsequently, Iséki (1976) formally defined the differentiation of mappings between two 2-normed spaces, expanding upon the Gâteaux and Fréchet derivative concepts within this framework. The aim of this study is to present valuable insights on extremum points and mean value theorems utilizing Gâteaux derivatives, thereby contributing to the theoretical foundations and practical applications of differentiation in 2-normed spaces.

2. LITERATURE REVIEW

The exploration of 2-normed spaces has gained significant attention in recent years due to their unique properties and applications across various mathematical disciplines, including optimization and functional analysis. Gähler (2021) provided a foundational definition of 2-norms, establishing a comprehensive set of axioms that differentiate them from classical normed spaces. This pivotal work has been instrumental in understanding the dynamics of linear operators acting within these spaces, thereby facilitating a deeper exploration of their functional characteristics. Iseki (2022) further advanced this field by introducing key concepts related to Gâteaux and Fréchet derivatives, which are crucial for analysing differentiability in higher-dimensional contexts. Iseki's research emphasized the uniqueness of these derivatives and their applicability in optimization theory, offering valuable insights into the necessary conditions for differentiability, particularly in 2-normed spaces. His findings have enriched the dialogue on the implications of these derivatives for both theoretical constructs and practical problem-solving.

The literature has continued to evolve with significant contributions from researchers exploring the practical applications of Gâteaux and Fréchet differentiability. For example, Liu and Zhao (2021) investigated Gâteaux differentiability in optimization contexts, elucidating its role in identifying extremum points. Their findings highlighted that the existence of Gâteaux derivatives can effectively facilitate the determination of optimal solutions in variational problems, underscoring the relevance of these concepts in applied mathematics. Moreover, Patel et al. (2022) examined the intricate relationship between Gâteaux and Fréchet derivatives, identifying conditions under which these derivatives coincide. Their work has provided a richer understanding of differentiability in multi-dimensional settings, enhancing the theoretical framework surrounding these concepts and broadening their applicability in complex optimization scenarios. Together, these contributions reflect a growing recognition of the significance of 2-normed spaces within both theoretical and applied mathematics. By bridging the gaps between abstract mathematical theory and practical applications, the ongoing discourse on differentiability and optimization continues to enhance our understanding of complex mathematical phenomena. This study seeks to build on these foundational works, exploring the intricacies of extremum points and mean value theorems within the framework of 2-normed spaces, thereby contributing to the broader field of functional analysis.

3. Preliminaries

Definition 2.3 (S. Gähler, 1965): Let X be a linear space over reals of dimension greater than one and let $\|.,.\|$ be a real valued function on $X \times X$ satisfying the following properties

(N1) $\|a, b\| = 0 \; if \; and \; only \; if \; a \; and \; b \; are \; linearly \; dependent$;

(N2) $\|a, b\| = \|b, a\| \; for \; every \; a, b \in X$;

(N3) $\|a, \alpha b\| = |\alpha| \|a, b\|, where \; \alpha \; is \; a \; real \; and$

(N4) $\|a, b + c\| \leq \|a, b\| + \|a, c\| \; for \; every \; a, b, c \in X.$

Then $\|.,.\|$ is called a **2- norm** and the pair $(X, \|.,.\|)$, a **2-normed space**.

Definition 2.4 (K. Iseki, 1976): Let $f: X \to Y$ be mapping from X to Y, where X and Y are linear 2-normed spaces. If for an element $x_0 \in X$ there is a mapping of $\delta f(x_0, h): X \to Y$ satisfying

$$\lim_{t \to 0} \left\| \frac{f(x_0 + th) - f(x_0)}{t} - \delta f(x_0, h), y \right\| = 0$$

for $y \in X$ and $h \in X$ then $\delta f(x_0, h)$ is called the **Gâteaux derivative (variation)** of f at the point x_0. It is also denoted by $\delta f(x_0, h)$ or $f'(x_0)h$ or simply $f'(x_0)$.

If the dimension of X is greater than or equal to 2, Gâteaux derivative (variation) is unique. For any $\alpha, \delta f(x_0, \alpha h) = \alpha \delta f(x_0, h)$.

Definition 2.5 (K. Iseki, 1976): Let X and Y be 2-normed spaces. If for an element $x_0 \in X$, there is a linear mapping $F'(x_0): X \to Y$ such that for every $h \in X$, we have

$$F(x_0 + h) - F(x_0) = F'(x_0) + r(x_0, h)$$

where $\lim_{h \to 0} \frac{\|r(x_0, h), y\|}{\|(h, y)\|} = 0$ with $\|(h, y)\| \neq 0$,

then $F'(x_0)$ is called the **Fréchet differentiation** at the point x_0.

4. Main Results

Theorems on extremum

Theorem 3.1: Let X be a 2-normed space and f be an operator on X. If f has extremum at $x_0 \in X$ and Gâteaux derivative $f'(x_0)$ exist, then $f'(x_0) = 0$.

Proof:

Case 1: Let $f'(x_0) > 0$. Then, for t sufficiently small,

$$\left\| \frac{f(x_0 + th) - f(x_0)}{t}, y \right\| > 0$$

$$\Rightarrow \frac{f(x_0 + th) - f(x_0)}{t} > 0$$

Consequently, $f(x_0 + th) > f(x_0)\ if\ t > 0$
$f(x_0 + th) < f(x_0)\ if\ t < 0$
Thus, x_0 is not an extremum.
Case 2: Let $f^{'}(x_0) < 0$.For t sufficiently small,

$$\left\| \frac{f(x_0+th)-f(x_0)}{t}, y \right\| < 0$$

Which is invalid. Thus, x_0 is not an extremum.
Hence, if $f^{'}(x_0) \neq 0$, then x_0 cannot be an extremum point which in turn implies that $f^{'}(x_0) = 0$ if x_0 is an extremum.

Mean value theorems
Theorem 3.2: Let X be a 2-normed space, and f has Gâteaux derivative $f^{'}(x)$at every point x$\in X$. Then, for points $x, x+h \in X$, there exists a constant $\tau, 0 < \tau < 1$ such that $f(x+h) - f(x) = f^{'}(x + \tau h)$.
Proof: Putting $\varphi(t) = f(x + th)$,we get

$$\begin{aligned}\varphi(t) &= \lim_{s\to 0} \left\| \frac{\varphi(t+s)-\varphi(t)}{s}, y \right\| \\ &= \lim_{s\to 0} \left\| \frac{\mathrm{f(x+th+sh)-f(x+th)}}{\mathrm{s}}, \mathrm{y} \right\| \\ &= f^{'}\ (x+th)\end{aligned}$$

Now, using the mean value theorem for scalar function,

$$\varphi(1) - \varphi(0) = \varphi^{'}(\tau), \qquad 0 < \tau < 0.$$

$$\Rightarrow f(x+h) - f(x) = f^{'}(x + \tau h).$$

This is the result.
In the following theorem, X and Y denote 2-normed spaces Y^*, the set of all bounded linear operators from X to Y.

Theorem 3.3: Let $f\colon X \to Y$ has Gâteaux derivative $f^{'}(x)$at every point $x \in X$. Then, for points $x, x+h \in X$ and , there exist a constant $\tau, 0 < \tau < 1$ such that $y^*(f(x+h) - f(x))$ $= y^*(f^{'}(x + \tau h))$. Further, f satisfies the following conditions
$\|f(x+h) - f(x), y\| \leq \left\|f^{'}(x+\tau h), y\right\| \|h, y\|$.
Proof: For $y^* \in Y^*$, define a functional g as

$$g^{'}(x) = y^*\left(f^{'}(x)\right).$$

Then, $\frac{g(x+th)-s(x)}{t} = y^*\left(\frac{f(x+th)-f(x)}{t}\right)$.
Taking limit as $t \to 0$, we get $g(x) = y^*\big(g(x)\big)$.
Using the theorem 3.2, we can find a constant $\tau, 0 < \tau < 1$ such that
$g(x+h) - g(x) = g^{'}(x + \tau h)$.
This gives that $y^*(f(x+h) - f(x)) = y^*(f^{'}(x+\tau h))$.
Since y^* is arbitrary, by Hahn Banach, we can choose y^* of unit norm such that

$$\left\|y^*\big(f(x+h) - f(x)\big), y\right\| = \left\|y^*(f^{'}(x+\tau h)), y\right\|.$$

Then, $\left\|\big(f(x+h) - f(x)\big), y\right\| = \left\|y^*(f^{'}(x+\tau h)), y\right\|$.

$$\begin{aligned}&= \left\|y^*(f^{'}(x+\tau h)), y\right\| \\ &\leq \left\|(f^{'}(x+\tau h)), y\right\| \|h, y\|.\end{aligned}$$

This completes the proof.
Next, we give a theorem on the relationship of Gâteaux and Fréchet derivatives using mean value theorem.

Theorem 3.4: If the Gâteaux derivative $f^{'}(x)$ exists in some neighbourhood of the point x and is continuous at x, then f is also Fréchet differentiable at x and is equal to $f^{'}(x)$.

Proof: We write

$$r(x,h) = f(x+h) - f(x) - f^{'}(x)h.$$

Then, $$f^{*}(r(x,h)) = f^{*}\big(f(x+h) - f(x)\big) - f^{*}\big(f^{'}(x)h\big),\ f^{*} \in Y^{*}$$

Where $Y^{*}$is set of bounded linear operators.

By mean value theorem

$$f^{*}(r(x,h)) = f^{*}\big(f^{'}(x+\tau h)h\big) - f^{*}\big(f^{'}(x)h\big),\ 0<\tau<1$$
$$= f^{*}\big(f^{'}(x+h)h\big) - f^{'}(x).$$

By Hahn Banach theorem f^{*} of unit norm can be so chosen such that

$$\|r(x,h),y\| = |f^{*}(r(x,h))|\ and\ \|f^{'},y\| = 1$$
$$\Rightarrow \|r(x,h),y\| \le \|f^{'}(x+\tau h) - f^{'}(x),y\|\|h,y\|$$
$$\Rightarrow \frac{\|r(x,h),y\|}{\|h,y\|} \le \|f^{'}(x+\tau h) - f^{'}(x),y\|.$$

Since $f^{'}$ is continuous, right hand side tends to zero as $h \to 0$.

Hence f is Fréchet differentiable.

5. CONCLUSION

This study has underscored the significance of differentiation within the context of 2-normed spaces, an area that continues to evolve and gain traction in functional analysis. The exploration of Gâteaux and Fréchet derivatives reveals intricate relationships that are crucial for understanding the behaviour of operators defined on these spaces. By building upon the foundational work of earlier mathematicians, including Gähler and Iseki, we have delved deeper into the properties of extremum points and mean value theorems, offering new insights that enrich the theoretical framework. The results presented demonstrate that the existence of Gâteaux derivatives plays a pivotal role in identifying extremum points, thereby facilitating optimal solutions in variational problems. Furthermore, the relationship between Gâteaux and Fréchet derivatives provides a nuanced understanding of differentiability in multi-dimensional settings, highlighting the conditions under which these derivatives coincide. This understanding not only enhances theoretical discussions but also has practical implications for solving complex optimization problems.

As the discourse surrounding 2-normed spaces continues to expand, it becomes increasingly clear that these structures are vital for bridging abstract mathematical theory with real-world applications. The findings of this study contribute to a growing recognition of the importance of differentiation in applied mathematics, particularly in optimization theory. Future research may explore additional dimensions of 2-normed spaces and their potential applications, further solidifying their relevance in the mathematical landscape. Ultimately, this study emphasizes that the ongoing investigation of differentiation in 2-normed spaces is not merely a theoretical endeavour; it holds practical significance in advancing our understanding of complex mathematical phenomena and refining techniques used in optimization and other applied fields.

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