Existence Results for Fractional Integro-Differential Equations(FRIDE) with State-Dependent Delays and Infinite Delay in Fréchet Spaces (FS)

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ABSTRACT

In this paper, we investigate the existence of mild solutions for semilinear integro-differential equations with state-dependent delay of order $0<\mathfrak{b}<1$ and infinite delay in Fréchet spaces. We assume that the linear part possesses a resolvent operator. The existence of mild solutions is established using Kuratowski's measure of noncompactness and the generalized Darbo fixed point theorem in Fréchet spaces. Finally, an example is provided for demonstration.

Keywords: Integro-differential equations; semigroup theory; resolvent operator; Kuratowski measure of noncompactness.

INTRODUCTION

In recent years, researchers have shown considerable interest in integrodifferential equations(IDEs) because of their applications in various fields, including physics, chemistry, and engineering. For further information, see the works by A.A. Kilbas et al. [19], Paulo [25], and Grimmer et al.[14]. In this paper, we explore the existence of mild solutions for fractional integrodifferential equations (FIDEs) of the form:

$$ {}^{c}\mathscr{D}_{\mu}^{\mathfrak{b}}\mathscr{Y}(\mu)=\mathscr{A}\mathscr{Y}(\mu)+\int_{0}^{\mu}\mathscr{K}(\mu-\mathfrak{r})\mathscr{Y}(\mathfrak{r})d\mathfrak{r}+\mathfrak{f}(\mu,\mathscr{Y}_{\rho(\mu,\mathscr{Y}_{\mu})}),\mu>0 \quad (1)$$

$$\mathscr{Y}(\mu)=\phi(\mu),\ \ \mu\in(\infty,0] \quad (2)$$

where ${}^{c}\mathscr{D}_{\mu}^{\mathfrak{b}}$ is Caputo fractional derivative (CFD) of order $0<\mathfrak{b}<1$, $\mathscr{A}:D(\mathscr{A})\subset E\to E$ is closed linear operator on a Banach space E, $\mathscr{K}(\mu)$ is closed linear operator with domain $D(\mathscr{A})\subset D(\mathscr{K})$, $\mathfrak{f}:\mathbb{R}^{+}\times\mathscr{D}\to E$ is a given function, $\mathscr{D}=\{\psi:(\infty,0]\to E,\psi$ is continuous$\}$, $\rho:\mathbb{R}^{+}\times\mathscr{D}\to\mathbb{R}^{+}$ are suitable functions that satisfy appropriate conditions which will be described in the sequel. We denote by $\mathscr{Y}_{\mu}$ the element of $\mathscr{D}$ defined by $\mathscr{Y}_{\mu}(\theta)=\mathscr{Y}(\mu+\theta),\ \theta\in(\infty,0]$. Here $\mathscr{Y}_{\mu}$ represents the history up to the present time t of the state $\mathscr{Y}(\cdot)$. We assume that the histories $\mathscr{Y}_{\mu}$ belongs to some abstract phase $\mathscr{D}$, to specified later, and $\phi\in\mathscr{D}$.

The structure of this paper is as follows: Section 2 covers preliminary concepts that will be utilized in subsequent sections. Section 3 defines the mild solution for problem (1)-(2), drawing inspiration from the works [21], and includes the proof of our main results. Lastly, Section presents an example.

PRELIMINARY OUTCOME

1255To begin, we introduce some notations, definitions, and preliminary concepts.

Let E is a Banach space with norm denoted as $\|\cdot\|$.

• $\mathscr{C}$ be the Banach space of all continuous functions u from$\mathfrak{J} := [0,\mathfrak{b}], \mathfrak{b} > 0$, into E with the norm

$$\|u\|_{\mathscr{C}(\mathfrak{J},E)} = \sup_{\mu\in\mathfrak{J}} \|u(\mu)\|$$

• $L(E)$ represents the Banach space of all bounded linear operators from E into E and corresponding norm denoted by

$$\|M\|_{L(E)} := \sup_{\|u\|=1} \|Mu\|$$

• $L^1[\mathfrak{J},E]$ is the Banach space of measurable functions $u : \mathfrak{J} \longrightarrow E$ which are Bochner integrable normed by

$$\|u\|_{L^1} = \int_0^{\mathfrak{b}} \|u(\mu)\| d\mu$$

. • $\mathscr{C}(\mathbb{R}^+,E)$ be the Fréchet space of all continuous functions u from $\mathbb{R}^+$ into E, equipped with the family of seminorms

$$\|u\|_n := \sup_{\mu\in[0,n]} \|u\|; n \in \mathbb{N}$$

and the distance

$$d(u_1,u_2) := \sum_{1\leq n} 2^{-n} \frac{\|u_1-u_2\|_n}{1+\|u_1-u_2\|_n}, \quad u_1,u_2 \in \mathscr{C}(\mathbb{R},E)$$

Definition 1. *[19] The fractional integral operator $\mathfrak{J}^{\mathfrak{b}}$ of order $\mathfrak{b} > 0$ of a continuous function f is defined by*

$$\mathfrak{J}_\mu^{\mathfrak{b}} f(\mu) := \frac{1}{\Gamma(\mathfrak{b})} \int_0^\mu (\mu-\mathfrak{r})^{\mathfrak{b}-1} f(\mathfrak{r}) d\mathfrak{r}$$

we can write $I_\mu^{\mathfrak{b}} f(\mu) = f(\mu) * \psi_{\mathfrak{b}}(\mu)$, where $\psi_{\mathfrak{b}}(\mu) = \dfrac{\mu^{\mathfrak{b}-1}}{\Gamma(\mathfrak{b})}$ for $\mu > 0$ and $\psi_{\mathfrak{b}}(\mu) = 0$ for $\mu \leq 0$ and $\psi_{\mathfrak{b}}(\mu) \longrightarrow \delta(\mu)$ as $\mathfrak{b} \longrightarrow 0$.

Definition 2. *[19] The Riemann-Liouville fractional derivative (RLFD) of order $\mathfrak{b}$ of f is defined by*

$$\mathscr{D}_a^{\mathfrak{b}} f(\mu) = \frac{1}{\Gamma(n-\mathfrak{b})} \frac{d^n}{d\mathfrak{r}^n} \int_a^\mu (\mu-\mathfrak{r})^{n-\mathfrak{b}-1} f(\mathfrak{r}) d\mathfrak{r}$$

where $n = [\mathfrak{b}]+1$ and $[\mathfrak{b}]$ denotes the integer part of $\mathfrak{b}$.

Definition 3. *[19] For a function f defined on the interval $[a,b]$, the Caputo fractional derivative (CFD) of order $\mathfrak{b}$ of f, is defined by*

$$({}_0^c\mathscr{D}_\mu^{\mathfrak{b}} f)(\mu) = \frac{1}{\Gamma(n-\mathfrak{b})} \int_0^\mu (\mu-\mathfrak{r})^{n-\mathfrak{b}-1} f^{(n)}(\mathfrak{r}) d\mathfrak{r},$$

Where $n = [\mathfrak{b}]+1$
Therefore, for $0 < \mathfrak{b} < 1$, $n = [\mathfrak{b}]+1 = 1$ and for $a = 0$, the Caputo's fractional derivative for $\mu \in [0,b]$ is given by

$$({}_0^c\mathscr{D}_\mu^{\mathfrak{b}} f)(\mu) = \frac{1}{\Gamma(1-\mathfrak{b})} \int_0^\mu (\mu-\mathfrak{r})^{-\mathfrak{b}} f'(\mathfrak{r}) d\mathfrak{r}$$

Definition 4. *The Laplace transform of the function f is the function $\widehat{f}$ of the variable s defined by*

$$\widehat{f}(\mathfrak{r}) = \int_0^{\infty} e^{-\mathfrak{r}\mu} f(\mu) d\mu$$

.

Definition 5. *The Laplace transform of the convolution,*

$$f(\mu) * g(\mu) = \int_o^{\mu} f(\mu - \mathfrak{r}) g(\mathfrak{r}) d\mathfrak{r}$$

of two functions f, g which are equal to zero for $\mu < 0$, is equal to the product of their Laplace transform

$$\widehat{f(\mu) * g(\mu)} = \widehat{f}(\mathfrak{r})\widehat{g}(\mathfrak{r})$$

Definition 6. *The Laplace transform of the fractional integral of ordre $\mathfrak{b}$ with $0 < \mathfrak{b} < 1$ of the function f is given by*

$$\widehat{\mathscr{I}_{\mu}^{\mathfrak{b}} f(\mu)} = \widehat{f(\mu) * \psi_{\mathfrak{b}}(\mu)} = \widehat{f}(\mathfrak{r})\widehat{\psi_{\mathfrak{b}}}(\mathfrak{r}) = \mathfrak{r}^{-\mathfrak{b}} \widehat{f}(\mathfrak{r})$$

where $\widehat{f}$ denotes the Laplace transform of f.

Definition 7. *[25] A one-parameter family of bounded linear operators $(R_{\mathfrak{b}}(\mu))_{\mu \geq 0}$ on E is called an $\mathfrak{b}$-resolvent operator of the problem homogeneous*

$$^{c}\mathscr{D}_{\mu}^{\mathfrak{b}} u(\mu) = \mathscr{A} u(\mu) + \int_0^{\mu} \mathscr{K}(\mu - \mathfrak{r}) u(\mathfrak{r}) d\mathfrak{r}, \mu > 0 \tag{3}$$

$$u(0) = \phi(0), \tag{4}$$

if the following conditions are verified:
• *The function $R_{\mathfrak{b}}(\cdot) : [0, \infty) \longrightarrow L(E)$ is strongly continuous and $R_{\mathfrak{b}}(0)u = u$ for all $u \in E$ and $\mathfrak{b} \in (0,1)$.*
• *For $u \in D(\mathscr{A})$, $R_{\mathfrak{b}}(\cdot)u \in \mathscr{C}([0,\infty), [D(\mathscr{A})]) \cap \mathscr{C}^{\mathfrak{b}}([0,\infty), E)$, and*

$$^{c}\mathscr{D}_{\mu}^{\mathfrak{b}} R_{\mathfrak{b}}(\mu) u = \mathscr{A} R_{\mathfrak{b}}(\mu) u + \int_0^{\mu} \mathscr{K}(\mu - \mathfrak{r}) R_{\mathfrak{b}}(\mathfrak{r}) u d\mathfrak{r}, \quad \mu \geq 0 \tag{5}$$

$$^{c}\mathscr{D}_{\mu}^{\mathfrak{b}} R_{\mathfrak{b}}(\mu) u = R_{\mathfrak{b}}(\mu) \mathscr{A} u + \int_0^{\mu} R_{\mathfrak{b}}(\mu - \mathfrak{r}) \mathscr{K}(\mathfrak{r}) u d\mathfrak{r}, \quad \mu \geq 0 \tag{6}$$

Definition 8. *[9] An $\mathfrak{b}$-ROF $(R_{\mathfrak{b}}(t))_{\mu \geq 0}$ is called analytic, if the function $R_{\mathfrak{b}}(\cdot) : \mathbb{R}^+ \longrightarrow L(E)$ admits analytic extension to a sector $\Sigma(0, \theta_0)$ for some $0 < \theta_0 \leq \frac{\pi}{2}$. An analytic $\mathfrak{b}$-ROF $(R_{\mathfrak{b}})$ is said to be of analyticity type (ω_0, θ_0) if for each $\theta < \theta_0$ and $\omega > \omega_0$ there exists $M_1 = M_1(\omega, \theta)$ such that $||R_{\mathfrak{b}}(z)|| \leq M_1 e^{\omega Rez}$ for $z \in \Sigma(0, \theta)$ where Rez denotes the real part of z and $\Sigma(\omega, \theta) := \{\lambda \in \mathbb{C} : |arg(\lambda - \omega)| < \theta, \quad \omega, \theta \in \mathbb{R}\}$*

Definition 9. *[9] An $\mathfrak{b}$-ROF$(S_{\mathfrak{b}}(\mu))_{\mu \geq 0}$ is called compact for $\mu > 0$ if for every $\mu > 0$, $R_{\mathfrak{b}}(\mu)$ is a compact operator.*

Theorem 1. *[9] Let Agenerate a compact analytic semigroup $T(\mu)_{\mu \geq 0}$ then for any $\mathfrak{b}$ it also generates a compact analytic resolvent family $(R_{\mathfrak{b}}(\mu))_{\mu \geq 0}$.*

Lemma 1. *[9] Assume that $\mathfrak{b}$-ROF$(R_{\mathfrak{b}}(\mu))_{\mu \geq 0}$ is compact for $t > 0$ and analytic of type (ω_0, θ_0). Then the following assertions hold:*

1. $\lim_{h \longrightarrow 0} \|R_{\mathfrak{b}}(\mu + h) - R_{\mathfrak{b}}(\mu)\| = 0$, *for* $\mu > 0$.
2. $\lim_{h \longrightarrow 0^+} \|R_{\mathfrak{b}}(\mu + h) - R_{\mathfrak{b}}(\mu)\| = 0$, *for* $\mu > 0$.

Definition 10. *[9] An $\mathfrak{b}$-ROF$(R_{\mathfrak{b}}(t))_{t\geq 0}$ is said to be exponentially bounded if there exist constants $M \geq 1$, $\omega \geq 0$ such that*

$$\|R_{\mathfrak{b}}(\mu)\| \leq Me^{\omega\mu} \ \ for\ \mu \geq 0$$

in this case we write $A \in \mathscr{C}_{\mathfrak{b}}(M,\omega)$.

Theorem 2. *[25] Assume that the following hypotheses hold:*

($\mathbf{H_1}$) *The operator $\mathscr{A} : D(\mathscr{A}) \subseteq E \longrightarrow E$ is is closed linear operator with $[D(\mathscr{A})]$ dense in E, for some $\Phi \in (\frac{\pi}{2},\pi)$ there is positive constants $C_0 = C_0(\Phi)$ such that $\lambda \in \rho(A)$*

$$\Sigma_{0,\phi} = \{\lambda \in \mathbb{C} | \arg(\lambda) < \phi\} \subset \rho(\mathscr{A}),$$

and $\|R(\lambda,\mathscr{A})\| \leq \frac{C_0}{|\lambda|}$ for all $\lambda \in \Sigma_{0,\phi}$

($\mathbf{H_2}$) *For all $0 \leq t$, $\mathscr{K}(\mu) : D(\mathscr{K}(\mu)) \subseteq E \longrightarrow E$ is closed linear operator and $\mathscr{K}(\cdot)u$ is strongly measurable on $(0,\infty)$ For any $u \in D(\mathscr{A})$, there exists $\|(\cdot) \in L^1_{loc}(\mathbb{R}^+)$ such that $\widehat{k}(\lambda)$ exists for $Re(\lambda) > 0$ and $\|\mathscr{K}(\mu)u\| \leq \|(\mu)\|u\|$ for all $\mu > 0$ and $u \in D(\mathscr{A})$.*
Moreover, the operator-valued function $\widehat{\mathscr{K}}$ has an analytical extension to $\Sigma_{0,\phi}$ such that $\|\widehat{\mathscr{K}}(\lambda)u\| \leq \|\widehat{\mathscr{K}}(\lambda)\|\|u\|$ for all $u \in D(\mathscr{A})$, and $\|\widehat{\mathscr{K}}(\lambda)\| = o(\frac{1}{|\lambda|})$, $|\lambda| \longrightarrow \infty$.

($\mathbf{H_3}$) *There exists a subspace $D \subseteq D(\mathscr{A})$ dense in $[D(\mathscr{A})]$ and positive constants C_1, such that $\mathscr{A}(D) \subseteq D(\mathscr{A}), \|\mathscr{A}\widehat{\mathscr{K}}(\lambda)u\| \leq C_1\|u\|$ for all $u \in D(\mathscr{A})$ and $\lambda \in \Sigma_{0,\phi}$,*
Then a resolvent operator exists the problem $(3)-(4)$.

Definition 11. *We define the following sets :*

$$\begin{aligned}
\rho_{\mathfrak{b}}(G_{\mathfrak{b}}) &= \{\lambda \in \mathbb{C} : G_{\mathfrak{b}}(\lambda) := \lambda^{\mathfrak{b}-1}(\lambda^{\mathfrak{b}}\mathscr{I} - \mathscr{A} - \widehat{\mathscr{K}}(\lambda))^{-1} \in L(E)\} \\
\rho_{\mathfrak{b}}(F_{\mathfrak{b}}) &= \{\lambda \in \mathbb{C} : F_{\mathfrak{b}}(\lambda) := (\lambda^{\mathfrak{b}}\mathscr{I} - \mathscr{A} - \widehat{\mathscr{K}}(\lambda))^{-1} \in L(E)\} \\
\Gamma^1_{r,\theta} &= \{\mu e^{i\theta} : \mu \geq r\} \\
\Gamma^2_{r,\theta} &= \{re^{i\zeta} : -\theta \leq \zeta \leq \theta\} \\
\Gamma^3_{r,\theta} &= \{\mu e^{-i\theta} : \mu \geq r\} \\
\Gamma_{r,\theta} &= \cup_{i=1}^{3}\Gamma^i_{r,\theta}
\end{aligned}$$

Definition 12. *We define the opeartor family $(R_{\mathfrak{b}}(\mu))_{\mu\geq 0}$ by*

$$R_{\mathfrak{b}}(\mu) := \frac{1}{2\pi i}\int_{\Gamma_{r,\theta}} e^{\lambda\mu}G_{\mathfrak{b}}(\lambda)d\lambda, \mu \geq 0 \tag{7}$$

and the auxiliary resolvent opearator family $(S_{\mathfrak{b}}(\mu))_{t\geq 0}$ by

$$S_{\mathfrak{b}}(\mu) := \frac{\mu^{1-\mathfrak{b}}}{2\pi i}\int_{\Gamma_{r,\theta}} e^{\lambda\mu}F_{\mathfrak{b}}(\lambda)d\lambda, \mu \geq 0 \tag{8}$$

Theorem 3. *[25]Assume that hypotheses $(H_1)-(H_3)$ holds. The operator function $R_{\mathfrak{b}}(\cdot)$ which defined by (7) is :*

(i) Exponentially bounded in $L(E)$.

(ii) Exponentially bounded in $L[D(\mathscr{A})]$).

(iii) Strongly continuous on $[0,\infty)$ *and uniformly continuous on* $[0,\infty)$.

(iv) Strongly continuous on $[0,\infty)$ *in* $L([D(\mathscr{A})])$.

Lemma 2. *[25]For every* $\lambda \in \mathbb{C}$ *with* $Re(\lambda) > \max\{0,r\}$, $\widehat{R_{\mathfrak{b}}}(\lambda) = G_{\mathfrak{b}}(\lambda)$ *and* $\widehat{\mu^{1-\mathfrak{b}}S_{\mathfrak{b}}(\mu)}(\lambda) = F_{\mathfrak{b}}(\lambda)$.

Theorem 4. *[25]Assume that hypotheses* $(H_1)-(H_3)$ *holds. The function* $R_{\mathfrak{b}}(\cdot)$ *which defined by* (7) *is a* $\mathfrak{b}$*-resolvent operator for the system* (3)- (4).

Proof. Let $u \in D(\mathscr{A})$. From3, for $Re(\lambda) > \max\{0,r\}$,

$$\widehat{R_{\mathfrak{b}}}(\lambda)[\lambda^{1-\mathfrak{b}}(\lambda^{\mathfrak{b}}\mathscr{I} - \mathscr{A} - \widehat{\mathscr{K}}(\lambda))]u = u,$$

which implies

$$\lambda\widehat{R_{\mathfrak{b}}}(\lambda)u - u = \lambda^{1-\mathfrak{b}}\widehat{R_{\mathfrak{b}}}(\lambda)\mathscr{A}u + \lambda^{1-\mathfrak{b}}\widehat{R_{\mathfrak{b}}}(\lambda)\widehat{\mathscr{K}}(\lambda)u,$$

we get

$$\lambda^{\mathfrak{b}}\widehat{R_{\mathfrak{b}}}(\lambda)u - \lambda^{\mathfrak{b}-1}u = \widehat{R_{\mathfrak{b}}}(\lambda)\mathscr{A}u + \widehat{R_{\mathfrak{b}}}(\lambda)\widehat{\mathscr{K}}(\lambda)u,$$

So

$$\widehat{{}^{c}\mathscr{D}^{\mathfrak{b}}_{\mu}R_{\mathfrak{b}}}(\lambda)u = \widehat{R_{\mathfrak{b}}}(\lambda)\mathscr{A}u + (\widehat{R_{\mathfrak{b}} * \mathscr{K}})(\lambda)u,$$

By the uniqueness of the Laplace transform we get

$${}^{c}D^{\mathfrak{b}}_{\mu}R_{\mathfrak{b}}(\mu)u = R_{\mathfrak{b}}(\mu)Au + \int_0^{\mu} R_{\mathfrak{b}}(\mu-\mathfrak{r})\mathscr{K}(\mathfrak{r})ud\mathfrak{r}.$$

Arguing as above but using the equality $[\lambda^{1-\mathfrak{b}}(\lambda^{\mathfrak{b}}\mathscr{I} - \mathscr{A} - \widehat{\mathscr{K}}(\lambda))]\widehat{R_{\mathfrak{b}}}(\lambda)u = u$, we obtain that 5 holds. The proof is now completed.

Theorem 5. *[25]Assume that hypotheses* $(H_1)-(H_3)$ *holds. A continuous function u is a mild solution of*

$${}^{c}\mathscr{D}^{\mathfrak{b}}_{\mu}u(\mu) = \mathscr{A}u(\mu) + \int_0^{\mu} \mathscr{K}(\mu-\mathfrak{r})u(\mathfrak{r})d\mathfrak{r} + f(\mu,u(\mu)) \quad \mu > 0 \tag{9}$$

$$u(0) = \phi(0), \tag{10}$$

where $f : [0,\infty) \longrightarrow E$ *is a continuous function. If*

$$u(\mu) = R_{\mathfrak{b}}(\mu)\phi(0) + \int_0^{\mu} (\mu-\mathfrak{r})^{\mathfrak{b}-1}S_{\mathfrak{b}}(\mu-\mathfrak{r})f(\mathfrak{r},u(\mathfrak{r}))d\mathfrak{r} \quad ,\mu \in (0,\infty)$$

Proof. Let the probem (9)-(10) and $R_{\mathfrak{b}}(\mu), S_{\mathfrak{b}}(\mu)$ the $\mathfrak{b}$-resolvent operator and auxiliary resolvent operator defined by (7), (8) respectively.
Let $u(\mu)$ continous function satisfying (9) - (10). Then applying $\mathscr{I}^{\mathfrak{b}}_{\mu}$ at both sides of equation (9) we have

$$\begin{aligned} u(\mu) &= \phi(0) + \mathscr{I}^{\mathfrak{b}}_{\mu}\mathscr{A}u(\mu) + \mathscr{I}^{\mathfrak{b}}_{\mu}(\mathscr{K}(\mu) * u(\mu)) + \mathscr{I}^{\mathfrak{b}}_{\mu}f(\mu,u(\mu)) \\ &= \phi(0) + \psi_{\mathfrak{b}}(\mu) * \mathscr{A}u(\mu) + \psi_{\mathfrak{b}}(\mu) * (\mathscr{K}(\mu) * u(\mu)) + \psi_{\mathfrak{b}}(\mu) * f(\mu,u(\mu)). \end{aligned}$$

Now assuming that this function is of exponential type and is locally integrable, we apply that Laplace transform os both sides we obtain

$$\widehat{u}(\lambda) = \frac{\phi(0)}{\lambda} + \frac{\mathscr{A}\widehat{u}(\lambda)}{\lambda^{\mathfrak{b}}} + \frac{\widehat{\mathscr{K}}(\lambda)\widehat{u}(\lambda)}{\lambda^{\mathfrak{b}}} + \frac{\widehat{f(u)}(\lambda)}{\lambda^{\mathfrak{b}}},$$

where $\widehat{f(u)}(\lambda)$ is a Laplace transform of $f(t,u(t))$. We infer

$$\begin{aligned}\widehat{u}(\lambda) &= \lambda^{\mathfrak{b}-1}(\lambda^{\mathfrak{b}}\mathscr{I}-\mathscr{A}-\widehat{\mathscr{K}}(\lambda))^{-}1\widehat{f(u)}(\lambda)\\ &= G_{\mathfrak{b}}(\lambda)\phi(0)+F_{\mathfrak{b}}(\lambda)\widehat{f(u)}(\lambda)\\ &= \widehat{R_{\mathfrak{b}}}(\mu)\phi(0)+\widehat{t^{\mathfrak{b}-1}S_{\mathfrak{b}}}(\mu)\widehat{f(u)}(\lambda)\\ &= \widehat{R_{\mathfrak{b}}}(\mu)\phi(0)+\mu^{\mathfrak{b}-1}\widehat{S_{\mathfrak{b}}(\mu)*}f(\mu,u(\mu)).\end{aligned}$$

Finally applying the inverse of the Laplace transform, we find the formula

$$u(\mu)=R_{\mathfrak{b}}(\mu)\phi(0)+\int_0^{\mu}(\mu-\mathfrak{r})^{\mathfrak{b}-1}S_{\mathfrak{b}}(\mu-\mathfrak{r})f(\mathfrak{r},u(\mathfrak{r}))d\mathfrak{r}$$

We recall the following definition of the notion of a sequence of measures of noncompactness:

Definition 13. *The Kuratowski measure of noncompactness $\nu(\cdot)$ define for a bounded subset D of Banach space E is given by*

$$\nu(D):=\inf\{\varepsilon>0 : D=\cup_{i=1}^{k}D_i \quad and \quad diam(D_i)\leq\varepsilon\}$$

Theorem 6. *[25] Let $\mathscr{D}_{\mathscr{F}}$ be the family of all non-empty and bounded subsets of a Fréchet space $\mathscr{F}$. A family of function $(\Theta_n)_{n\in\mathbb{N}}$ where $\Theta_n:\mathscr{D}_{\mathscr{F}}\longrightarrow[0,+\infty)$ is said to be a family of measure of noncompactness in the real Fréchet space $\mathscr{F}$ if it satisfies the following conditions for $D_1,D_2\in\mathscr{D}_{\mathscr{F}}$:*

(**a**) *D_1 is pre-compact if only if $\Theta_n(D_1)=0$ for all $n\in\mathbb{N}$.*

(**b**) *$\Theta_n(D_1)\leq\Theta_n(D_2)$ where $D_1\subset D_2$ for all $n\in\mathbb{N}$.*

(**c**) *$\Theta_n(D)=\Theta_n(\overline{coD})=\Theta_n(\overline{D})$ for all $n\in IN$, where $\overline{D}$, $\overline{coD}$ are the closure and convex hull of D, respectively.*

(**d**) *If $\{D_i\}_{i=1}^{\infty}$ is a sequence of closed sets of $\mathscr{D}_{\mathscr{F}}$ such that $D_{i+1}\subset D_i, i=1,...,$ and if $\lim_{i\longrightarrow}\Theta_n(D_i)=0$ for each $n\in\mathbb{N}$, then the intersection set $D_\infty=\cap_{i=1}^{\infty}D_i$ is nonempty.*

Lemma 3. *[25] If D is a bounded subset of a Banach space E, then for each $\varepsilon>0$ there is sequence $\{v_k\}_{k\geq1}\subset D$ such that*

$$\nu(D)\leq 2\nu(\{v_k\}_{k\geq1})+\varepsilon$$

Lemma 4. *[25] If $\{v_k\}_{k\geq1}\subset L^1(\mathbb{R}^+,\mathscr{F})$ is uniformly integrable, then $\nu(\{v_k((\cdot)\}_{k\geq1})$ is measurable and*

$$\nu(\{\int_0^{\mu}v_k((\mathfrak{r})d\mathfrak{r}\}_{k\geq1})\leq 2\int_0^{\mu}\nu(\{v_k(\mathfrak{r})\}_{k\geq1})d\mathfrak{r}, \quad \mu\geq0$$

where ν is a Kuratawski measure noncompactness on $\mathscr{F}$

Definition 14. *[5] Let $\mathscr{D}$ be a nonempty subset of Fréchet space $\mathscr{W}$ and let $N:\mathscr{D}\longrightarrow\mathscr{W}$ be a continuous operator which transforms bounded subsets into bounded ones. One says N satisfies the Darbo condition with constants $\{q_n : n\in\mathscr{N}\}$ with respect to a family of measure of noncompactness $(\Theta_n)_{n\in\mathbb{N}}$, if*

$$\Theta_n(N(D))\leq q_n\Theta_n(D)$$

for each bounded set $D\subset\mathscr{D}$ and $n\in\mathbb{N}$. If $q_n<1, n\in\mathbb{N}$ then N is called a contraction with respect to $(\Theta_n)_{n\in\mathbb{N}}$.

The following generalization of the classical Darbo fixed point theorem for Fréchet space plays an important role in proving the main result.

Theorem 7. *[5] Let D be a nonempty, bounded, closed, and convex subset of a Fréchet space $\mathscr{F}$ and let $N : D \longrightarrow D$ be continuous mapping. Suppose that N is a contraction concerning a family of measures of noncompactness $(\Theta_n)_{n\in\mathbb{N}}$. Then N has at least one fixed point in the set D.*

Lemma 5. *Let Set $\mathscr{R}(\rho^-) = \{\rho(\mathfrak{r},\varphi) : (\mathfrak{r},\varphi) \in J\times\mathscr{D}, \rho(\mathfrak{r},\varphi) \leq 0\}$, such that $\rho : \mathscr{J}\times\mathscr{D} \to (\infty,b]$ is continuous. If $\mathscr{Y} : (\infty,b] \longrightarrow E$ is a function such that $\mathscr{Y}_0 = \phi$, then*

$$\|\mathscr{Y}_\mu\|_D \leq (M_b + L^\phi)\|\phi\|_{\mathscr{D}} + K_b \sup\{|\mathscr{Y}(\mathfrak{r})|; \mathfrak{r} \in [0, \max\{0,\mu\}]\}$$

where $M_b = \sup_{\mu\in\mathscr{J}} M(\mu)$, $K_b = \sup_{\mu\in\mathscr{J}} K(\mu)$ and $L^\phi = \sup_{\mu\in\mathscr{R}(\rho^-)} L^\phi(\mu)$.

MAIN RESULT

In this section, we establish and prove the existence of a mild solution for our problem (1)-(2). First, we give its definition.

Definition 15. *A function $\mathscr{Y} \in \mathscr{C}(\mathbb{R},E)$ is called the mild solution for the problem (1)-(2), if $\mathscr{Y}(\mu) = \phi(\mu)$, for all $\mu \in \mathbb{R}^-$ and $\mathscr{Y}(\mu)$ satisfies the following integral equation :*

$$\mathscr{Y}(\mu) = R_{\mathfrak{b}}(\mu)\phi(0) + \int_0^\mu (\mu-\mathfrak{r})^{\mathfrak{b}-1} S_{\mathfrak{b}}(\mu-\mathfrak{r})\mathfrak{f}(\mathfrak{r}, \mathscr{Y}_{\rho(\mathfrak{r},\mathscr{Y}_\mathfrak{r})})d\mathfrak{r}, \quad \mu \geq 0$$

In this work, we will work under the following assumptions:

$(\mathbf{A_1})$ There exist constant $M_0 > 1$ such that $\|R_{\mathfrak{b}}(\mu)\|_{L(E)} \leq M_0$ for every $\mu \in \mathbb{R}^+$.

$(\mathbf{A_2})$(**i**) The function $\mu \longmapsto \mathfrak{f}(\mu,\mathscr{Y})$ is continuous on E for a.e.$t \in \mathbb{R}^+$.

(**ii**) There exists a function $P \in L^1(\mathbb{R}^+,\mathbb{R}^+)$ and a continuous nondecreasing function $\Psi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that

$$\mathfrak{f}(\mu,\mathscr{Y}) \leq P(\mu)\Psi(\|\mathscr{Y}\|) \quad \textit{for a.e} \mu \in \mathbb{R}^+ \quad \textit{and each } \mathscr{Y} \in E$$

(**iii**) For each bounded set $D \subset E$ and for each $\mu \in [0,n]$, $n \in \mathbb{N}$, we have

$$\nu(\mathfrak{f}(\mu,D) \leq \sup_{\mathfrak{r}\leq 0} \mu(D(\mathfrak{r})),$$

where ν is a measure of noncompactness on the Banach space E.

$(\mathbf{A_3})$ For each $n \in \mathbb{N}$, there exists $q_n > 0$ such that

$$M_0[\|\phi(0)\| + \Psi([M_b + L^\phi]\|\phi\| + K_b[\|\phi(0)\| + q_n])\overline{f}_n] \leq q_n,$$

where for $n \in \mathbb{N}$

$$\overline{f}_n := \sup_{\mu\in[0,n]} P(\mu)$$

$(\mathbf{H}_\varphi)$ The function $\mu \to \varphi_\mu$ is continuous from $\mathscr{R}(\rho^-)$ into $\mathscr{D}$ and there exists a continuous and bounded function $L^\phi : \mathscr{R}(\rho^-) \to (0,\infty)$ such that

$$\|\phi_\mu\|_{\mathscr{D}} \leq L^\phi(\mu)\|\phi\|_{\mathscr{D}}, \qquad \text{for every} \quad \mu \in \mathscr{R}(\rho^-)$$

Theorem 8. *Assume that $(H_1)-(H_3)$, (H_φ), $(A_1)-(A_4)$ hold and for each $n \in \mathbb{N}$,*

$$M_0\|\phi(0)\|\overline{f}_n < \frac{1}{4}$$

Then the problem (1)-(2) has at least one mild solution.

Proof. Let the Frechet space $\mathscr{C}(\mathbb{R},E)$, equipped with the family of seminorms

$$\|u\|_n := \sup_{\mu\in[o,n]} \|u\|; n\in\mathbb{N}$$

Let the family of measure of noncompactness given by

$$\mu_n(D) := \omega_0^n(D) + \sup_{\mu\in[-n,n]} \mu(D(\mu))$$

where

$$\begin{aligned}
\omega^n(D,\varepsilon) &:= \sup\{\omega^n(\mathscr{Y},\varepsilon) : z\in D\}\\
\omega^n(y,\varepsilon) &:= \sup\{\|\mathscr{Y}(\mu)-\mathscr{Y}(\mathfrak{r})\| : \mu,\mathfrak{r}\in[0,n], |\mu-\mathfrak{r}|\le\varepsilon\}\\
\omega_0^n(D) &:= \lim_{\varepsilon\to 0}\omega^n(D,\varepsilon)\\
D(\mu) &:= \{\mathscr{Y}(\mu)\in E, y\in D\}, \quad \mu\in[-n,n]
\end{aligned}$$

We consider the operator N on $C(\mathbb{R},E)$ define by

$$N\mathscr{Y}(\mu) = \begin{cases}\phi(\mu), & \mu\le 0\\ R_{\mathfrak{b}}(\mu)\phi(0)+\int_0^\mu(\mu-\mathfrak{r})^{\mathfrak{b}-1}S_{\mathfrak{b}}(\mu-\mathfrak{r})\mathfrak{f}(\mathfrak{r},\mathscr{Y}_{\rho(\mathfrak{r},\mathscr{Y}_{\mathfrak{r}})})d\mathfrak{r}, & \mu>0\end{cases}$$

Let $x(\cdot):\mathbb{R}\longrightarrow E$, be the function defined by

$$x(\mu) = \begin{cases}\phi(\mu), & \mu\le 0\\ 0, & \mu>0\end{cases}$$

$$\bar{z}(\mu) = \begin{cases}0 & \mu\le 0\\ R_{\mathfrak{b}}(\mu)\phi(0)+\int_0^\mu(\mu-\mathfrak{r})^{\mathfrak{b}-1}S_{\mathfrak{b}}(\mu-\mathfrak{r})\mathfrak{f}(\mathfrak{r},\mathscr{Y}_{\rho(\mathfrak{r},\mathscr{Y}_{\mathfrak{r}})})d\mathfrak{r}, & \mu>0\end{cases}$$

On decompose $\mathscr{Y}(mu)$ at $\mathscr{Y}(\mu) = x(\mu)+\bar{z}(\mu)$, $\mu\in\mathbb{R}^+$, which implies $\mathscr{Y}_\mu = \bar{z}_\mu + x_\mu$ for every $0\lneq\mu$ and the function $\bar{z}(\cdot)$ satisfies

$$\bar{z}(\mu) = \begin{cases}0 & \mu\le 0\\ z(\mu) & \mu>0\end{cases}$$

where

$$z(\mu) = R_{\mathfrak{b}}(\mu)\phi(0)+\int_0^\mu(\mu-\mathfrak{r})^{\mathfrak{b}-1}S_{\mathfrak{b}}(\mu-\mathfrak{r})\mathfrak{f}(\mathfrak{r},x_{\rho(\mathfrak{r},x_{\mathfrak{r}}+\bar{z}_{\mathfrak{r}})}+\bar{z}_{\rho(\mathfrak{r},x_{\mathfrak{r}}+\bar{z}_{\mathfrak{r}}))}d\mathfrak{r}, \tag{11}$$

Transform the problem (11) into a fixed point problem. Consider the operators G

$$Gz(\mu) = R_{\mathfrak{b}}(\mu)\phi(0)+\int_0^\mu(\mu-\mathfrak{r})^{\mathfrak{b}-1}S_{\mathfrak{b}}(\mu-\mathfrak{r})\mathfrak{f}(\mathfrak{r},x_{\rho(\mathfrak{r},x_{\mathfrak{r}}+\bar{z}_{\mathfrak{r}})}+\bar{z}_{\rho(\mathfrak{r},x_{\mathfrak{r}}+\bar{z}_{\mathfrak{r}})}))d\mathfrak{r}$$

We define the ball

$$B_{q_n} := B(0,q_n) = \{z\in C(\mathbb{R},E) : \|z\|_n\le q_n\}$$

$$\begin{aligned}
\|z\|_n &= \|z_0\|_+\sup\{\|z(\mu)\| : -n\lneq\mu\lneq n\}\\
&= \sup\{\|z(t)\| : 0\lneq\mu\lneq n\}
\end{aligned}$$

Step 1: $G(B_{q_n}) \subset B_{q_n}$:
Let $z \in B_{q_n}$ and $\mu \in [0,n]$

$$\begin{aligned}\|Gz(\mu))\| &\leq \|R_{\mathfrak{b}}(\mu)\phi(0)\| + \int_0^t \|(\mu-\mathfrak{r})^{\mathfrak{b}-1}S_{\mathfrak{b}}(\mu-\mathfrak{r})\| |\mathfrak{f}(\mathfrak{r}, x_{\rho(\mathfrak{r},x_{\mathfrak{r}}+\bar{z}_{\mathfrak{r}})} + \bar{z}_{\rho(\mathfrak{r},x_{\mathfrak{r}}+\bar{z}_{\mathfrak{r}})}))|d\mathfrak{r} \\ &\leq M_0\|\phi(0)\| + M_0\int_0^{\mu} |\mathfrak{f}(\mathfrak{r}, x_{\rho(\mathfrak{r},x_{\mathfrak{r}}+\bar{z}_{\mathfrak{r}})} + \bar{z}_{\rho(\mathfrak{r},x_{\mathfrak{r}}+\bar{z}_{\mathfrak{r}})}))|d\mathfrak{r} \\ &\leq M_0\|\phi(0)\| + M_0\int_0^{\mu} p(\mathfrak{r})\Psi(\|x_{\rho(\mathfrak{r},x_{\mathfrak{r}}+\bar{z}_{\mathfrak{r}})} + \bar{z}_{\rho(\mathfrak{r},x_{\mathfrak{r}}+\bar{z}_{\mathfrak{r}})}\|)d\mathfrak{r}\end{aligned}$$

Since

$$\begin{aligned}\|x_{\rho(\mathfrak{r},x_{\mu}+\bar{z}_{\mu})} + \bar{z}_{\rho(\mu,x_{\mu}+\bar{z}_{\mu})}\| &\leq \|x_{\rho(\mu,x_{\mu}+\bar{z}_{\mu})}\| + \|\bar{z}_{\rho(\mu,x_{\mu}+\bar{z}_{\mu})}\| \\ &\leq (M_b + L^{\phi})\|\phi\| + K_b\|\phi(0)\| + K_b q_n\end{aligned}$$

We have

$$\begin{aligned}\|G(B_{q_n})(\mu)\| &\leq M_0\|\phi(0)\| + M_0\int_0^{\mu} P(\mathfrak{r})\Psi((M_b+L^{\phi})\|\phi\| + K_b\|\phi(0)\| + K_b q_n)d\mathfrak{r} \\ &\leq M_0\|\phi(0)\| + M_0\Psi((M_b+L^{\phi})\|\phi\| + K_b\|\phi(0)\| + K_b q_n)\int_0^n P(\mathfrak{r})d\mathfrak{r} \\ &\leq M_0\|\phi(0)\| + M_0\Psi((M_b+L^{\phi})\|\phi\| + K_b\|\phi(0)\| + K_b q_n)\overline{f}_n \\ &\leq M_0[\|\phi(0)\| + \Psi([M_b+L^{\phi}]\|\phi\| + K_b[\|\phi(0)\| + q_n])\overline{f}_n] \\ &\leq q_n\end{aligned}$$

This proves that G transforms the ball B_{q_n} into B_{q_n}. We complete the proof in the following steps.

Step 1: $G(B_{q_n})$ is bounded .

Since $G(B_{q_n}) \subset B_{q_n}$ qnd B_{q_n} is bounded, then $G(B_{q_n}) \subset B_{q_n}$ is bounded.

Step 2: G is continuous.

Let $\{z^j\}_{j\in IN}$ be a sequence such that $z^j \longrightarrow z$ in B_{q_n}. Then for each $t \in [-n,n]$, we have

$$\begin{aligned}\|G(z^j)(\mu) - G(z)(\mu)\| &\leq \int_0^{\mu} \|(\mu-\mathfrak{r})^{\mathfrak{b}-1}S_{\mathfrak{b}}(\mu-\mathfrak{r}) \\ &\quad \|[|\mathfrak{f}(\mathfrak{r}, x_{\rho(\mathfrak{r},x_{\mathfrak{r}}+\bar{z}^j_{\mathfrak{r}})} + \bar{z}^j_{\rho(\mathfrak{r},x_{\mathfrak{r}}+\bar{z}^j_{\mathfrak{r}})}) - \mathfrak{f}(\mathfrak{r}, x_{\rho(\mathfrak{r},x_{\mathfrak{r}}+\bar{z}_{\mathfrak{r}})} + \bar{z}_{\rho(\mathfrak{r},x_{\mathfrak{r}}+\bar{z}_{\mathfrak{r}})}))|]d\mathfrak{r} \\ &\leq M_0\int_0^{\mu} |\mathfrak{f}(\mathfrak{r}, x_{\rho(\mathfrak{r},x_{\mathfrak{r}}+\bar{z}^j_{\mathfrak{r}})} + \bar{z}^j_{\rho(\mathfrak{r},x_{\mathfrak{r}}+\bar{z}^j_{\mathfrak{r}})}) - \mathfrak{f}(\mathfrak{r}, x_{\rho(\mathfrak{r},x_{\mathfrak{r}}+\bar{z}_{\mathfrak{r}})} + \bar{z}_{\rho(\mathfrak{r},x_{\mathfrak{r}}+\bar{z}_{\mathfrak{r}})}))|]d\mathfrak{r}\end{aligned}$$

Since $z^j \longrightarrow z$ as $j \longrightarrow \infty$, the Lebesgue dominated convergence theorem implies that

$$\|G(z^j)(\mu) - G(z)(\mu)\| \longrightarrow 0 \quad as \quad j \longrightarrow +\infty$$

Thus G is continuous.

Step 3: For each equicontinuous subset K of B_{q_n}, $\nu_n(G(k)) \leq K_n\nu_n(K)$.

From lemmas 3 and 3, for any equicontinuous set $K \subset B_{q_n}$ and $\varepsilon > 0$, there exists a sequence $\{z^k\}_{k\in\mathbb{N}} \subset K$, such that for all $\mu \in [-n,n]$, we have

$$\begin{aligned}
\nu(G(k)) &= \nu(\{R_{\mathfrak{b}}(\mu)\phi(0) + \int_0^{\mu} (\mu-\mathfrak{r})^{\mathfrak{b}-1} S_{\mathfrak{b}}(\mu-\mathfrak{r})\mathfrak{f}(\mathfrak{r}, x_{\rho(\mathfrak{r},x_{\mathfrak{r}}+\bar{z}_{\mathfrak{r}})} + \bar{z}_{\rho(\mathfrak{r},x_{\mathfrak{r}}+\bar{z}_{\mathfrak{r}})})d\mathfrak{r}) : z \in K\}) \\
&\leq \nu(\{R_{\mathfrak{b}}(\nu)\phi(0)\}) + \nu(\{\int_0^{\mu} (\mu-\mathfrak{r})^{\mathfrak{b}-1} S_{\mathfrak{b}}(\mu-\mathfrak{r})\mathfrak{f}(\mathfrak{r}, x_{\rho(\mathfrak{r},x_{\mathfrak{r}}+\bar{z}_{\mathfrak{r}})} + \bar{z}_{\rho(\mathfrak{r},x_{\mathfrak{r}}+\bar{z}_{\mathfrak{r}})})d\mathfrak{r} : z \in K\} \\
&\leq M_0\|\phi(0)\| + 4\int_0^{\mu} \nu(\|(\mu-\mathfrak{r})^{\mathfrak{b}-1} S_{\mathfrak{b}}(\mu-\mathfrak{r})\|\{\mathfrak{f}(\mathfrak{r}, x_{\rho(\mathfrak{r},x_{\mathfrak{r}}+\bar{z}^k_{\mathfrak{r}})} + \bar{z}^k_{\rho(\mathfrak{r},x_{\mathfrak{r}}+\bar{z}^k_{\mathfrak{r}})})\})d\mathfrak{r} + \varepsilon \\
&\leq M_0\|\phi(0)\| + 4M_0\int_0^{\mu} P(\mathfrak{r})\nu(\{x_{\rho(\mathfrak{r},x_{\mathfrak{r}}+\bar{z}^k_{\mathfrak{r}})} + \bar{z}^k_{\rho(\mathfrak{r},x_{\mathfrak{r}}+\bar{z}^k_{\mathfrak{r}})})\}_{k\in\mathbb{N}})d\mathfrak{r} + \varepsilon \\
&\leq M_0\|\phi(0)\| + 4M_0\int_0^{\mu} P(\mathfrak{r})\nu(\{x_{\rho(\mathfrak{r},x_{\mathfrak{r}}+\bar{z}^k_{\mathfrak{r}})}\}_{k\in\mathbb{N}})d\mathfrak{r} + 4M_0\int_0^{\mu} P(\mathfrak{r})\nu(\{\bar{z}^k_{\rho(\mathfrak{r},x_{\mathfrak{r}}+\bar{z}^k_{\mathfrak{r}})}\}_{k\in\mathbb{N}})d\mathfrak{r} + \varepsilon \\
&\leq M_0\|\phi(0)\| + 4M_0\overline{f}_n \sup_{|\mathfrak{r}|\leq n} \mu(\{\phi(\mu)\}) + 4M_0\overline{f}_n \nu_n(K) + \varepsilon
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, then

$$\nu(G(k)) \leq 4M_0\overline{f}_n\nu_n(K)$$

Thus

$$\nu_n(G(k)) \leq 4M_0\overline{f}_n\nu_n(K)$$

From Setp 1 to 3, together with Theorem 8, we can conclude that G has at least one fixed point in B_{q_n}, which is a mild solution of problem (1)-(2).

APPLICATION

In this section, an example is provided to illustrate the obtained theory. We consider the following fractional integrodifferential equation with a state-dependent form:

$$\begin{aligned}
\mathscr{D}^{\mathfrak{b}}_{\mu} z(\mu,x) &= \frac{\partial^2 z(\mu,x)}{\partial x^2} + \int_0^t \mu\mathscr{K}(\mu-\mathfrak{r})[\frac{\partial^2 z(\mathfrak{r},x)}{\partial x^2}]ds + g(\mu)|z(1-\rho(z(\mu,x),x)|, \\
&(\mu,x) \in (0,\infty] \times [0,\pi], 0 < \mathfrak{b} < 1, \qquad (12)
\end{aligned}$$

$$z(\mu,0) = z(\mu,\pi) = 0,\ \mu \in\in (0,\infty], \qquad (13)$$

$$z(\mu,x) = \phi(\mu,x),\ \mu \in (\infty,0] \qquad (14)$$

Where$\mathscr{K} : \mathbb{R}^+ \to \mathbb{R}$ is bounded uniformly continuous, continuously differentiable and $\mathscr{K}'$ is bounded uniformly continuous, $\rho \in \mathscr{C}(\mathbb{R},[0,r])$, $g(\cdot)$ is a continuous function from $\mathbb{R}^+$ to $\mathbb{R}$ and $\phi \in \mathscr{D} = \{\psi : (\infty,0] \times [0,\pi] \longrightarrow \mathbb{R}; \psi$.

Set $E = L^2([0,\pi],\mathbb{R})$ and let $D(\mathscr{A}) \subset E \to E$ be the operator $\mathscr{A}z = z''$ with the domain with domain

$$D(\mathscr{A}) = \{z \in E, z, z'\ are\ absolutely\ continuous,\ z'' \in E, z(0) = z(\pi) = 0\}.$$

Then

$$\mathscr{A}z = \sum_{n=1}^{\infty} n^2 < z, e_n > e_n,\ z \in D(\mathscr{A})$$

where$<,>$ is inner product in L^2 and $e_n(z) = \sqrt{\frac{2}{\pi}} sin(nz), z \in [0,\pi], n = 1,2,...$ is the orthonormal set of eigenvectors of A. It is well known that $\mathscr{A}$ is the infinitesimal generator of analytic semigroup $T(\mu)_{\mu\neq 0}$ in E which is given by

$$T(\mu)z = \sum_{n=1}^{\infty} e^{-n^2\mu} < z, e_n > e_n,\ z \in D(\mathscr{A}).$$

From the theorem 1, the operator $\mathscr{A}$ also generates an $\mathfrak{b}$-resolvent family which is compact for $\mu > 0$, and there exists a constant $M \geq 1$ such that $\|S_{\mathfrak{b}}(\mu)\| \leq M$

Let $\mathscr{Y}(\mu) : D(\mathscr{A}) \to E$ be the operator defined by $\mathscr{Y}(\mu)l = \mathscr{K}(\mu)\mathscr{A}l$ for $\mu \neq 0$. Set

$$\mathscr{Y}(\mu)x = z(\mu,x) \quad (\mu,x) \in (0,\infty) \times [0,\pi],$$

$$\mathfrak{f}(\mu,\psi)(x) = g(\mu)|\psi(x)|, \quad x \in [0,\pi], \psi \in E,$$

$$\mathscr{Y}(\mu)x = \phi(\mu,x), \ \mu \in (\infty,0].$$

So, we can check that the assumptions of theorem 8hold. Consequently, the (12)-(14) has at least one mild solution on $\mathbb{R}$.

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