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Non-polynomial spline solution of one dimensional singularly perturbed parabolic equations

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Abstract

In this paper, two-parameter singularly perturbed parabolic equations are examined by two level method using non-polynomial spline. We have used non-polynomial quadratic spline in space and finite difference discretization in time. Stability analysis is carried out. The approximate solution is shown to converge point-wise to the true solution. Numerical solution of singularly perturbed parabolic equations consisting of linear as well as non-linear has been solved. Three numerical examples are presented to show the efficiency and effectiveness of the developed method.

Key words: Two-parameter singularly perturbed problems; Non-polynomial splines; Stability analysis
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[1]

1 Introduction

We consider the two-parameter singularly perturbed one dimensional parabolic partial differential equation(PDE) of the form:

$$z_\kappa - \epsilon_d z_{\lambda\lambda} + \epsilon_c r(\lambda) z_\lambda + s(\lambda) z \quad = \quad g(\lambda, \kappa), \qquad (\lambda, \kappa) \in Q_T, \tag{1.1}$$

subject to

$$z \quad = \quad 0, \ (\lambda, \kappa) \in \partial S \times I, \tag{1.2}$$

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and

$$z(\lambda,0) \quad = \quad z_0(\lambda),\ \lambda \in S, \tag{1.3}$$

where, $Q_T = S \times I,\ S \equiv r : l < r < m,\ \partial S \equiv \{l\} \cup \{m\},\ I \equiv (0,T)$ and $l(>0), m(>0) \in \mathbb{R},\ z = z(\lambda,\kappa),\ r(\lambda)$ and $s(\lambda)$ are continously differentiable functions and $g(\lambda,\kappa)$ is continuous function defined on Q_T. Also $0 < \epsilon_c \ll 1$ and $0 < \epsilon_d \ll 1$. The above problems occur in various fields of sciences, such as, elasticity, mechanics, chemical reactor theory and convection-diffusion process. There are numerous asymptotic expansion methods available for solution of problems of the above type. But there were difficulties in applying these asymptotic expansions in the inner and outer regions. Many researchers have derived numerical methods for solving singularly perturbed boundary value problems(SPBVPs). Scheme based on parametric spline functions has been developed by Khan et al.[5]. Fractional Kersten-Krasil'shchik coupled KdV mKdV System arising in multi-component plasmas have been numerically solved by Goswami et al.[3]. A uniform convergent numerical method is given by Clavero et al.[2] and Kadalbajoo et al.[4] to solve the one-dimensional time-dependent convection-diffusion problem. Sharma and Kaushik [8] solved a singularly perturbed time delayed convection diffusion problem on a domain which is rectangular. Zahra et al.[10], Aziz and Khan [1] have also used spline methods for solution of SPBVPs. An efficient numerical approach for fractional multidimensional diffusion equations with exponential memory is given by Singh et al.[9]. In recent past, Mohanty et al. [7] have solved singularly perturbed parabolic equations using methods based on spline in tension. In this paper, we develop a new algorithm for solving SPBVPs associated with homogeneous Dirichlet boundary conditions.

This paper is divided into 5 sections as follows: In Section 2, the non-polynomial spline scheme is derived. In Section 3, we discuss application of the method for SPBVPs with scheme of $O(k+h^2)$. Truncation error is also discussed in Section 3. In Section 4, stability analysis is carried out. In Section 5 three problems are solved which confirm theoretical behaviour along with the rate of convergence.

2 Non-polynomial Spline

We divide the *[l,m]* interval uniformaly as

$$l = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_{n-1} < \lambda_n = m,$$

where

$$\lambda_i = l + ih, \quad 0 \le i \le n \quad and \quad h = \frac{(m-l)}{n}.$$

Let

$$R_i(\lambda) = a_i \cos\tau(\lambda - \lambda_{i-1/2}) + b_i \sin\tau(\lambda - \lambda_{i-1/2}) + c_i, \tag{2.1}$$

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be a non-polynomial spline defined on closed interval *[l,m]* reduces to polynomial spline which is quadratic as $\tau \longrightarrow 0$ and $\tau > 0$.
To calculate a_i, b_i and c_i, we define

$$\begin{aligned} R_i(\lambda_i) &= z_i,\ R_i'(\lambda_{i-1/2}) = P_{i-1/2}, \\ R_i''(\lambda_i) &= D_i,\ \ 0 \le i \le n-1. \end{aligned} \tag{2.2}$$

Using above interpolatory conditions we get

$$\begin{aligned} a_i &= -\frac{1}{\tau^2} D_i \sec\left(\frac{\vartheta}{2}\right) - \frac{1}{\tau} P_{i-1/2} \tan(\frac{\vartheta}{2}), \\ b_i &= \frac{1}{\tau} P_{i-1/2}, \\ c_i &= z_{i+1} - \frac{1}{\tau^2} D_i, \end{aligned}$$

where, $\vartheta = \tau h$.
Using continuity conditions, $R_{i-1}^{(m)}(\lambda_{i-1/2}) = R_i^{(m)}(\lambda_{i-1/2}), m = 0, 1$ we get the expression as follows:

$$z_{i-1} - 2z_i + z_{i+1} = h^2(\delta D_{i-1} + \eta D_i + \zeta D_{i+1}), 0 \le i \le n-1 \tag{2.3}$$

where,

$$\begin{aligned} \delta &= \frac{\sec(\frac{\vartheta}{2}) - 1}{\vartheta^2}, \\ \eta &= \frac{4\sec(\frac{\vartheta}{2})(1-\cos^2(\frac{\vartheta}{2})) + 2(1-\sec(\frac{\vartheta}{2}))}{\vartheta^2}, \\ \zeta &= \delta. \end{aligned}$$

When $\tau \to 0$, it means $\vartheta \to 0$, then $(\delta, \eta, \zeta) \to (1/8, 6/8, 1/8)$, and the scheme given by (2.3) reduces into polynomial quadratic spline relation as:

$$z_{i-1} - 2z_i + z_{i+1} = \frac{1}{8} h^2(D_{i-1} + 6D_i + D_{i+1}), 0 \le i \le n-1. \tag{2.4}$$

3 Application of the scheme

We consider a SPBVP of the form

$$z_\kappa - \epsilon_d z_{\lambda\lambda} + \epsilon_c r(\lambda) z_\lambda + s(\lambda) z = g(\lambda, \kappa), \quad (\lambda, \kappa) \in Q_T, \tag{3.1}$$

where, $Q_T = S \times I,\ S \equiv r : l < r < m,\ \partial S \equiv \{l\} \cup \{m\},\ I \equiv (0, T)$ and $l(>0), m(>0) \in \mathbb{R}$. The above equation with

$$z = 0,\ (\lambda, \kappa) \in \partial S \times I,$$

and

$$z(\lambda, 0) = z_0(\lambda),\ \lambda \in S.$$

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Here, we use the following derivative approximations of higher order as:

$$
\begin{aligned}
z'_i &= \frac{z_{i+1} - z_{i-1}}{2h}, \\
z'_{i-1} &= \frac{-3z_{i-1} + 4z_i - z_{i+1}}{2h}, \\
z'_{i+1} &= \frac{z_{i-1} - 4z_i + 3z_{i+1}}{2h}, \\
z^j_{ti} &= \frac{z^{j+1}_i - z^j_i}{k}, \\
z^j_{ti-1} &= \frac{z^{j+1}_{i-1} - z^j_{i-1}}{k}, \\
z^j_{ti+1} &= \frac{z^{j+1}_{i+1} - z^j_{i+1}}{k}.
\end{aligned}
$$

We consider the following ordinary differential equation

$$
\begin{aligned}
\epsilon_d \frac{d^2 z}{d\lambda^2} &= \epsilon_c r(\lambda)\frac{dz}{d\lambda} + s(\lambda)z - g(\lambda) \\
&\equiv G(\lambda, z, z'). \qquad (3.2)
\end{aligned}
$$

After implementing scheme (2.3) on BVP (3.2), we obtain:

$$
z_{i-1} - 2z_i + z_{i+1} = h^2(\delta G_{i-1} + \eta G_i + \zeta G_{i+1}),\ 1 \le i \le n-1 \qquad (3.3)
$$

where,

$$
\begin{aligned}
G_{i-1} &= G(\lambda_{i-1}, z_{i-1}, z'_{i-1}), \\
G_i &= G(\lambda_i, z_i, z'_i), \\
G_{i+1} &= G(\lambda_{i+1}, z_{i+1}, z'_{i+1}),
\end{aligned}
$$

Using derivative approximations, we obtain

$$
\tilde{A}z_{i-1} + \tilde{B}z_i + \tilde{C}z_{i+1} = -h^2(\delta g_{i-1} + \eta g_i + \zeta g_{i+1}), 1 \le i \le n-1 \qquad (3.4)
$$

where,

$$
\begin{aligned}
\tilde{A} &= \epsilon_d + \frac{3}{2}h\delta\epsilon_c r_{i-1} - \delta h^2 s_{i-1} + \frac{h}{2}\eta\epsilon_c r_i - \frac{h}{2}\zeta\epsilon_c r_{i+1}, \\
\tilde{B} &= -2\epsilon_d - 2h\delta\epsilon_c r_{i-1} - \eta h^2 s_i + 2h\zeta\epsilon_c r_{i+1}, \\
\tilde{C} &= \epsilon_d + \frac{1}{2}h\delta\epsilon_c r_{i-1} - \zeta h^2 s_{i+1} - \frac{h}{2}\eta\epsilon_c r_i - \frac{3h}{2}\zeta\epsilon_c r_{i+1}.
\end{aligned}
$$

For solving parabolic equation (3.1) we obtain the two level spline scheme by replacing z_i by $\frac{1}{2}(z^{j+1}_i + z^j_i)$, z_{i+1} by $\frac{1}{2}(z^{j+1}_{i+1} + z^j_{i+1})$, z_{i-1} by $\frac{1}{2}(z^{j+1}_{i-1} + z^j_{i-1})$, g_i by $(\frac{z^{j+1}_i - z^j_i}{k} + g^j_i)$, g_{i+1} by $(\frac{z^{j+1}_{i+1} - z^j_{i+1}}{k} + g^j_{i+1})$ and g_{i-1} by $(\frac{z^{j+1}_{i-1} - z^j_{i-1}}{k} + g^j_{i-1})$ in (3.4) and hence we obtain as follows:

$$
\begin{aligned}
A_1 z^{j+1}_{i-1} + A_2 z^{j+1}_i + A_3 z^{j+1}_{i+1} &= A_4 z^j_{i-1} + A_5 z^j_i + A_6 z^j_{i+1} - h^2(\delta g^j_{i-1} + \eta g^j_i + \zeta g^j_{i+1}), \\
&\quad 1 \le i \le n-1 \qquad (3.5)
\end{aligned}
$$

where,

$$\begin{aligned}
A_1 &= \frac{-h^2\delta}{k} + \frac{1}{2}(\epsilon_d + \frac{3}{2}h\delta\epsilon_c r_{i-1} - \delta h^2 s_{i-1} + \frac{h}{2}\eta\epsilon_c r_i - \frac{h}{2}\zeta\epsilon_c r_{i+1}),\\
A_2 &= \frac{-h^2\eta}{k} + \frac{1}{2}(-2\epsilon_d - 2h\delta\epsilon_c r_{i-1} - \eta h^2 s_i + 2h\zeta\epsilon_c r_{i+1}),\\
A_3 &= \frac{-h^2\zeta}{k} + \frac{1}{2}(\epsilon_d + \frac{1}{2}h\delta\epsilon_c r_{i-1} - \zeta h^2 s_{i+1} - \frac{h}{2}\eta\epsilon_c r_i - \frac{3h}{2}\zeta\epsilon_c r_{i+1}),\\
A_4 &= \frac{-h^2\delta}{k} + \frac{1}{2}(-\epsilon_d - \frac{3}{2}h\delta\epsilon_c r_{i-1} + \delta h^2 s_{i-1} - \frac{h}{2}\eta\epsilon_c r_i + \frac{h}{2}\zeta\epsilon_c r_{i+1}),\\
A_5 &= \frac{-h^2\eta}{k} + \frac{1}{2}(2\epsilon_d + 2h\delta\epsilon_c r_{i-1} + \eta h^2 s_i - 2h\zeta\epsilon_c r_{i+1}),\\
A_6 &= \frac{-h^2\zeta}{k} + \frac{1}{2}(-\epsilon_d - \frac{1}{2}h\delta\epsilon_c r_{i-1} + \zeta h^2 s_{i+1} + \frac{h}{2}\eta\epsilon_c r_i + \frac{3h}{2}\zeta\epsilon_c r_{i+1}).
\end{aligned}$$

Error

Here, we expand the scheme(3.5) in terms of $z(\lambda_i, \kappa_j)$ using Taylor's series and get the expression for truncation error as follows:

$$\begin{aligned}
t_i = \Big[&h^2[1-(\delta+\eta+\zeta)]D_\lambda^2 - \frac{1}{2}k[\delta h^2 s_{i-1} + \eta h^2 s_i + \zeta h^2 s_{i+1}]D_t + h^3[\delta-\zeta]D_\lambda^3\\
&+h^4[\frac{1}{12} - \frac{\delta+\zeta}{2}]D_\lambda^4 - \frac{1}{4}k^2[\delta h^2 s_{i-1} + \eta h^2 s_i + \zeta h^2 s_{i+1}]D_t^2 + h^5[\frac{\delta-\zeta}{3!}]D_\lambda^5\\
&+h^6[\frac{1}{360} - \frac{\delta+\zeta}{24}]D_\lambda^6 + \ldots\Big] z_i^j,\ 1 \le i \le n-1. \qquad (3.6)
\end{aligned}$$

For $\delta + \eta + \zeta = 1$ and $\delta = \zeta$, the method of $O(k + h^2)$ is obtained.

4 Stability Analysis

Here, we obtain the expression which gives information regarding stability of the scheme (3.5). We take Z_i^j as actual solution which satisfies the equation

$$\begin{aligned}
A_1 Z_{i-1}^{j+1} + A_2 Z_i^{j+1} + A_3 Z_{i+1}^{j+1} &= A_4 Z_{i-1}^j + A_5 Z_i^j + A_6 Z_{i+1}^j - h^2(\delta g_{i-1}^j + \eta g_i^j\\
&\quad + \zeta g_{i+1}^j),\ 1 \le i \le n-1. \qquad (4.1)
\end{aligned}$$

We assume that an error $e_i^j = Z_i^j - z_i^j$ exist at each point (λ_i, κ_j), then by subtracting (3.5) from (4.1) we get the expression as

$$\begin{aligned}
A_1 e_{i-1}^{j+1} + A_2 e_i^{j+1} + A_3 e_{i+1}^{j+1} &= A_4 e_{i-1}^j + A_5 e_i^j + A_6 e_{i+1}^j,\\
&\quad 1 \le i \le n-1. \qquad (4.2)
\end{aligned}$$

To derive stability analysis for the scheme (3.5), we assume that the solution of the homogeneous part of (4.2) is of the form $e_i^j = \varpi^j e^{i\rho}$, where $\varpi \in \mathbb{C}$, $i = \sqrt{-1}$

and $\rho \in \mathbb{R}$. Finally, we get the amplification factor as

$$\varpi = \frac{A_4 e^{-i\rho} + A_5 + A_6 e^{i\rho}}{A_1 e^{-i\rho} + A_2 + A_3 e^{i\rho}}, \tag{4.3}$$

then,

$$\varpi = \frac{-\frac{h^2}{k\epsilon_d}(\delta+\rho) - \frac{h^2}{\epsilon_d}(\delta q_{i-1} - \eta q_i + \zeta q_{i+1}) + 2B_1 \sin^2(\frac{\rho}{2}) + iB_2 \sin(\frac{\rho}{2})}{-\frac{h^2}{k\epsilon_d}(\delta+\rho) + \frac{h^2}{\epsilon_d}(\delta q_{i-1} - \eta q_i + \zeta q_{i+1}) - 2B_1 \sin^2(\frac{\rho}{2}) + iB_2 \sin(\frac{\rho}{2})},$$

where

$$B_1 = 1 + \frac{h^2}{k\epsilon_d}(\delta+\zeta) + h\frac{\epsilon_c}{\epsilon_d}(\delta p_{i-1} - 2\zeta p_{i+1}) + h^2\frac{1}{\epsilon_d}(\delta q_{i-1} + \zeta q_{i+1}),$$

$$B_2 = \frac{h^2}{k\epsilon_d}(\delta-\rho) + \frac{1}{2\epsilon_d}[h\epsilon_c(\delta p_{i-1} + \eta p_i - \zeta p_{i+1}) + h^2(\zeta q_{i+1} - \delta q_{i-1})].$$

The condition for the scheme to be stable is $|\varpi| \leq 1$. As we know that $0 \leq \sin^2(\frac{\rho}{2}) \leq 1$ and $\epsilon_d \propto h$, then from above relation it is easily verified that $|\varpi| \leq 1$ for every ρ. Hence the developed method is unconditionally stable.

5 Numerical Illustrations

We consider three second order SPBVPs. The maximum absolute errors(MAE) are tabulated in Tables 1-4 depending upon the choice of parameters. The convergence rate is denoted by α_n and is computed by following expression:

$$\alpha_n = \ln_2(Er_{n,k}/Er_{2n,k}),$$

and there is a different way to find rate of convergence denoted by $\tilde{\alpha}_n$ and is computed by using

$$\tilde{\alpha}_n = \ln_2(Er_{n,k}/Er_{2n,k/2}).$$

Example 1:

Consider the following problem from Zahra et al.[10].

$$z_\kappa - \epsilon_d\, z_{\lambda\lambda} + z_\lambda = g(\lambda, \kappa), T = 1,$$

in [0,1] associated with $z(0,\kappa) = 0, z(1,\kappa) = 0$ and $z_0(\lambda) = \exp(-1/\epsilon_d) + (1 - \exp(-1/\epsilon_d))\lambda - \exp(-(1-\lambda)/\epsilon_d)$, where
$g(\lambda,\kappa) = \exp(-\kappa)(-c_1 + c_2(1-\lambda) + \exp(-(1-\lambda)/\epsilon_d))$.

The analytical solution is $z(\lambda,\kappa) = \exp(-\kappa)(c_1 + c_2\lambda - \exp(-(1-\lambda)/\epsilon_d))$, where $c_1 = \exp(-1/\epsilon_d), c_2 = 1 - \exp(-1/\epsilon_d)$. The numerical results for $N = 2^4, 2^5, 2^6, 2^7$ and $\epsilon_d = 1/2^8, 1/2^{10}, 1/2^{12}, 1/2^{24}, 1/2^{26}$ using parameters $(\delta,\eta,\zeta) = \frac{1}{8}(1,6,1)$ compared with Zahra et al.[10] are tabulated in Table 1. And for $N = 2^4, 2^5, 2^6, 2^7, 2^8, 2^9$ and $\epsilon_d = 1, 1/4, 1/16, 1/64$ using parameters $(\delta,\eta,\zeta) = \frac{1}{8}(1,6,1)$ compared with Clavero et al.[2] are tabulated in Table 2.

Example 2:

Consider the following PDE from Zahra et al.[10]

$$z_\kappa - \epsilon_d\ z_{\lambda\lambda} + \epsilon_c\ z_\lambda \quad = \quad g(\lambda,\kappa),\ T = 1,$$

in [0,1] associated with $z(0,\kappa) = 0, z(1,\kappa) = 0$ and $z_0(\lambda) = [\phi_1 \cos(\pi\lambda) + \phi_2 \sin(\pi\lambda) + \psi_1 \exp(\theta_1\lambda) + \psi_2 \exp(-\theta_2(1-\lambda))]$, where
$g(\lambda,\kappa) = \exp(-\kappa)[\{-\phi_1 \cos(\pi\lambda) - \phi_2 \sin(\pi\lambda) - \psi_1 \exp(\theta_1\lambda) - \psi_2 \exp(-\theta_2(1-\lambda))\} + \epsilon_d\{\phi_1\pi^2 \cos(\pi\lambda) + \phi_2\pi^2 \sin(\pi\lambda) - \frac{\psi_1}{\theta_1^2} \exp(\theta_1\lambda) - \frac{\psi_2}{\theta_2^2} \exp(-\theta_2(1-\lambda))\} + \epsilon_c\{-\phi_1\pi \cos(\pi\lambda) + \phi_2\pi \sin(\pi\lambda) + \frac{\psi_1}{\theta_1} \exp(\theta_1\lambda) + \frac{\psi_2}{\theta_2} \exp(-\theta_2(1-\lambda))\}]$. The analytical solution is $z(\lambda,\kappa) = \exp(-\kappa)[\phi_1 \cos(\pi\lambda) + \phi_2 \sin(\pi\lambda) + \psi_1 \exp(\theta_1\lambda) + \psi_2 \exp(-\theta_2(1-\lambda))]$ where,

$$\begin{aligned}
\phi_1 &= \frac{\epsilon_d\pi^2 + 1}{\epsilon_c^2\pi^2 + (\epsilon_d\pi^2 + 1)^2}\ ,\\
\phi_2 &= \frac{\epsilon_c\pi}{\epsilon_c^2\pi^2 + (\epsilon_d\pi^2 + 1)^2}\ ,\\
\psi_1 &= -\phi_1 \frac{1 + \exp(-\theta_2)}{1 - \exp(\theta_1 - \theta_2)}\ ,\\
\psi_2 &= \phi_1 \frac{1 + \exp(\theta_1)}{1 - \exp(\theta_1 - \theta_2)}\ ,\\
\theta_1 &= \frac{\epsilon_c - \sqrt{\epsilon_c^2 + 4\epsilon_d}}{2\epsilon_d}\ ,\\
\theta_2 &= \frac{\epsilon_c + \sqrt{\epsilon_c^2 + 4\epsilon_d}}{2\epsilon_d}\ .
\end{aligned}$$

The numerical results for $N = 2^4, 2^5, 2^6, 2^7, 2^8,\ \epsilon_d = 1, 1/4, 1/16$ and $\epsilon_c = 10^{-3}, 10^{-4}, 10^{-5}$ using parameters $(\delta,\eta,\zeta) = \frac{1}{8}(1,6,1)$ are tabulated in Table 3.

Example 3:

Consider the following PDE from Mohanty et al.[6]

$$\epsilon_d\ z_{\lambda\lambda} - z_\kappa + \frac{1}{\lambda} z_\lambda \quad = \quad g(\lambda,\kappa),\ 0 \le \lambda \le 1,\ \kappa > 0$$

The analytical solution is $z(\lambda,\kappa) = \exp(-\kappa)\sinh\lambda$. The right-hand-side functions, initial and boundary conditions may be obtained using the actual solution given above as a test procedure. The numerical results for $N = 2^4, 2^5, 2^6, 2^7, 2^8$ and $\epsilon_d = 1/2, 1/8, 1/16, 1/32, 1/64, 1/128$ using parameters $(\delta,\eta,\zeta) = \frac{1}{12}(1,10,1)$ compared with Mohanty et al.[6] are tabulated in Table 4.

Table 1: MAE of example 1 for $(\delta, \eta, \zeta) = \frac{1}{8}(1, 6, 1)$

Method	$N \backslash \epsilon_d$	$1/2^8$	$1/2^{10}$	$1/2^{12}$	$1/2^{24}$	$1/2^{26}$
Presented method	2^4	1.2136×10^{-02}	1.3494×10^{-02}	1.3835×10^{-02}	1.3949×10^{-02}	1.3949×10^{-02}
α_n		1.1344	0.9591	0.9231	0.91170	0.91170
Zahra et al.[10]		3.5638×10^{-02}	5.1972×10^{-02}	6.7088×10^{-02}	7.3818×10^{-02}	7.1839×10^{-02}
Presented method	2^5	5.5284×10^{-03}	6.9407×10^{-03}	7.2961×10^{-03}	7.4147×10^{-03}	7.4147×10^{-03}
α_n		1.4946	1.0549	0.9785	0.9554	0.9554
Zahra et al.[10]		1.300×10^{-02}	1.4234×10^{-02}	2.3185×10^{-02}	3.4126×10^{-02}	3.4128×10^{-02}
Presented method	2^6	1.9619×10^{-03}	3.3406×10^{-03}	3.7029×10^{-03}	3.8238×10^{-03}	3.8238×10^{-03}
α_n		2.1590	1.1997	1.0249	0.9776	0.9776
Zahra et al.[10]		9.3378×10^{-03}	8.3305×10^{-03}	8.2129×10^{-03}	1.5756×10^{-02}	1.5761×10^{-02}
Presented method	2^7	4.4173×10^{-04}	1.4543×10^{-03}	1.8198×10^{-03}	1.9419×10^{-03}	1.9419×10^{-03}
Zahra et al.[10]		8.4218×10^{-03}	7.9579×10^{-03}	7.6243×10^{-03}	9.1052×10^{-03}	9.1078×10^{-03}

Table 2: MAE of example 1 for $(\delta, \eta, \zeta) = \frac{1}{8}(1, 6, 1)$

Method	$N \setminus \epsilon_d$	1	1/4	1/16	1/64
Presented Method	2^4	7.8074×10^{-04}	1.2280×10^{-03}	1.0487×10^{-02}	1.0921×10^{-01}
Clavero et al.[2]		1.3076×10^{-03}	1.7398×10^{-03}	4.0133×10^{-02}	5.9664×10^{-02}
	$\tilde{\alpha}_n$	1.7534	1.9008	2.5426	1.3506
Presented Method	2^5	2.3156×10^{-04}	3.2887×10^{-04}	1.8000×10^{-03}	4.2823×10^{-02}
Clavero et al.[2]		7.9078×10^{-04}	9.6845×10^{-03}	2.5552×10^{-03}	3.7372×10^{-02}
	$\tilde{\alpha}_n$	1.8952	1.8851	2.4931	2.0663
Presented Method	2^6	6.2255×10^{-05}	8.9033×10^{-05}	3.1971×10^{-04}	1.0225×10^{-02}
Clavero et al.[2]		3.6986×10^{-04}	5.1056×10^{-03}	1.5865×10^{-02}	2.1792×10^{-02}
	$\tilde{\alpha}_n$	1.9602	1.9124	2.4935	2.6020
Presented Method	2^7	1.5998×10^{-05}	2.3652×10^{-05}	5.6774×10^{-05}	1.6841×10^{-03}
Clavero et al.[2]		1.8894×10^{-04}	2.6223×10^{-03}	9.5603×10^{-03}	1.2381×10^{-03}
	$\tilde{\alpha}_n$	1.9638	1.9347	2.4473	2.5951
Presented Method	2^8	4.1011×10^{-06}	6.1867×10^{-06}	1.0409×10^{-05}	2.7872×10^{-04}
Clavero et al.[2]		9.5517×10^{-05}	1.3289×10^{-03}	5.5999×10^{-03}	6.9704×10^{-03}
	$\tilde{\alpha}_n$	1.9676	1.9514	2.3371	2.6674
Presented Method	2^9	1.0486×10^{-06}	1.5996×10^{-06}	2.0601×10^{-06}	4.3874×10^{-05}
Clavero et al.[2]		4.8028×10^{-05}	6.6891×10^{-04}	3.2019×10^{-03}	3.9052×10^{-03}

Table 3: MAE of example 2 for $(\delta, \eta, \zeta) = \frac{1}{8}(1, 6, 1)$

ϵ_d	1			1/4			1/16		
$\epsilon_c \setminus N$	10^{-3}	10^{-4}	10^{-5}	10^{-3}	10^{-4}	10^{-5}	10^{-3}	10^{-4}	10^{-5}
2^4	5.7028×10^{-06}	5.6372×10^{-06}	5.6328×10^{-06}	7.6099×10^{-03}	7.5937×10^{-03}	7.5920×10^{-03}	3.0375×10^{-01}	3.0260×10^{-01}	3.0248×10^{-01}
α_n	1.4451	1.4479	1.4482	1.8084	1.8084	1.8084	1.8258	1.8258	1.8258
2^5	2.0944×10^{-06}	2.0671×10^{-06}	2.0643×10^{-06}	2.1726×10^{-03}	2.1681×10^{-03}	2.1676×10^{-03}	8.5684×10^{-02}	8.5356×10^{-02}	8.5324×10^{-02}
α_n	1.8444	1.8459	1.8461	1.8568	1.8568	1.8568	1.9140	1.9140	1.9140
2^6	5.8324×10^{-07}	5.7502×10^{-07}	5.7420×10^{-07}	5.9986×10^{-04}	5.9857×10^{-04}	5.9845×10^{-04}	2.2736×10^{-02}	2.2650×10^{-02}	2.2640×10^{-02}
α_n	1.9487	1.9496	1.9497	1.9306	1.9306	1.9306	1.9572	1.9572	1.9572
2^7	1.5108×10^{-07}	1.4886×10^{-07}	1.4864×10^{-07}	1.5736×10^{-04}	1.5702×10^{-04}	1.5698×10^{-04}	5.8551×10^{-03}	5.8328×10^{-03}	5.8800×10^{-03}
α_n	1.9810	1.9815	1.9816	1.9658	1.9658	1.9658	1.9786	1.9786	1.9786
2^8	3.8272×10^{-08}	3.7696×10^{-08}	3.7639×10^{-08}	4.0284×10^{-05}	4.0197×10^{-05}	4.0188×10^{-05}	1.4856×10^{-03}	1.4799×10^{-03}	1.4799×10^{-03}

Table 4: MAE of example 3 for $(\delta, \eta, \zeta) = \frac{1}{12}(1, 10, 1)$

Method	$N \setminus \epsilon_d$	1/2	1/8	1/16	1/32	1/64	1/128
Presented method	2^4	7.2294×10^{-04}	8.0022×10^{-04}	8.2241×10^{-04}	8.3576×10^{-04}	8.4320×10^{-04}	8.4714×10^{-04}
Mohanty et al.[6]		0.2924×10^{-03}	0.4454×10^{-03}	0.4777×10^{-03}	0.5054×10^{-03}	0.5344×10^{-03}	0.5615×10^{-03}
Presented method	2^5	1.2613×10^{-05}	1.3899×10^{-05}	1.4267×10^{-04}	1.4488×10^{-04}	1.4610×10^{-04}	1.4675×10^{-04}
Mohanty et al.[6]		0.7286×10^{-04}	0.1129×10^{-03}	0.1239×10^{-03}	0.1410×10^{-03}	0.1869×10^{-03}	0.3134×10^{-03}
Presented method	2^6	2.2181×10^{-05}	2.4391×10^{-05}	2.5022×10^{-05}	2.5721×10^{-05}	2.5610×10^{-05}	2.5721×10^{-05}
Mohanty et al.[6]		0.1814×10^{-04}	0.2835×10^{-04}	0.3166×10^{-04}	0.3984×10^{-04}	0.9429×10^{-04}	0.1684×10^{-03}
Presented method	2^7	3.9130×10^{-06}	4.2982×10^{-06}	4.4079×10^{-06}	4.5295×10^{-06}	4.5102×10^{-06}	4.5295×10^{-06}
Mohanty et al.[6]		0.4524×10^{-05}	0.7091×10^{-05}	0.7987×10^{-05}	0.1088×10^{-04}	0.4743×10^{-04}	0.9120×10^{-04}
Presented method	2^8	6.9111×10^{-07}	7.5878×10^{-07}	7.7796×10^{-07}	7.9929×10^{-07}	7.7590×10^{-07}	7.9929×10^{-07}
Mohanty et al.[6]		0.1129×10^{-05}	0.1771×10^{-05}	0.2002×10^{-05}	0.2844×10^{-05}	0.1821×10^{-05}	0.5283×10^{-04}

Conclusion

We have presented two level scheme using non-polynomial spline for solving singularly perturbed parabolic equations based on one dimension. In examples 1, 2 and 3, we have computed maximum absolute errors for different values of N and ϵ_d for the sake of comparison with references [2,6,10] and results are tabulated in Tables 1-4. From tables it is shown that our method is much better in accuracy than the methods given by Clavero et al.[2], Mohanty et al.[6] and Zahra et al.[10]. It has already been proved that the presented algorithm gives higher numerical rate of convergence. It has also shown that the scheme is unconditionally stable.

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