A Study on Independence Number in Cluster Hypergraphs

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ABSTRACT
LetXbe a non-empty set and let $V_X$be a subset ofP(X)such that $V_X \neq \emptyset$and $X \subset V_X$.Let E be a multi set whose elements belong to P(P(X))such that $E \neq \phi$ and for each element $e \in E$, there exists atleast one element $v \in V_X$ such that $v \in e$ .Then $H = (V_X, E)$ is said to be cluster hypergraph. This article explores the concept of Independence Number and Strongly Independence Number in Cluster Hypergraphs. Also it is demonstrated that $\beta(H) = \beta([H]_{c2})$, $i(H) = i([H]_{c2})$,also some theorems related to the concept Independence Number in Cluster Hypergraphs have been discussed and demonstrated in this article.

Keywords: cluster hypergraph, independence number, strongly independence number, independent domination number.

1. INTRODUCTION

The major research areas in graph theory is the study of domination and related concepts such as independence, irredundance, and covering. De Jaenisch strived to ascertain the minimum number of queens necessaryto cover $n \times n$ chess board. Three kinds of problems that the chess players facedwere brought out by Rouse Ball at this time period. These includethe following.

1. Covering: To find out the minimum number of chess pieces of a specific type thatis required to attack every square of $n \times n$chess board.
2. Independent covering: To find out the least number of mutually non attackingchess pieces of a given type necessary to dominate every square of $n \times n$ chess board.
3. Independence: To determine the minimum number of chess pieces of a specific typeneeded to attack every square of$n \times n$chess board such that none of the pieces attack one other [1].

This paper focuses solely on the concept of Independence Number and Strongly Independence Number. Let $G = (V, E)$ be a graph. A subset S of v is called an independent set of G if no two vertices in S are adjacent in G. The minimum cardinality of an independent set is called the independence number of G and is denoted by i(G).The maximum cardinality of an independent set is called the independence number of G and is denoted by $\beta_0(G)$ [2].

This study begins by presenting a motivational example that illustrates the study of Independence Number in Cluster Hypergraphs. Most of the Indian Universities have started introducing "free elective courses"as a part of the curriculum, where the free electives are offered by the various departments of the university and each student has to undergo a specified number of free electives to become eligible for the award of the degree. The students are given the freedom to choose any of the free electives offered by any departments in the university. Some departments offer the same electives. So by the similar subject, departments forms a cluster. Therefore, this situation can be modeled as a cluster hypergraph. Let V_X indicates the set of all students and Y be the set of all free electives offered by the various departments in a particular semester. For each free electivei $\in$ Y,let $E_i = \{x \epsilon V_X\ ; x \text{ has chosen the electivesi}\}$ Let $E = \{E_i\ ;\ i \epsilon Y\}$.Then $H = (V_X, E)$ be a clusterhypergraph. Any dominating set S in H corresponds to a set of students withthe property that for any studentsu $\in V_X/S$, then there exists atleast one student vϵSsuch that u and v have chosen common free electives. Thus, in any one particular period the students are enrolled in various free elective courses offered by the university, any information can be conveyed to all the students through the members of S. Any strongly independent set I inHcorresponds to a set of students having the property that no two students in I have chosen the sameelectives. Again, if K is the maximum cardinality of an edge $E_i$in H, then the rankof the cluster hypergraph is K and any edgeE_i with $|E_i| = K$, where Killustrates the free electives that are chosen by the maximum number of students and hence, is the most popular course among the students. In this article the new concept Independence Number in Cluster Hypergraphs have been introduced and the same concept has been extended to prove some theorems related to Independence Number and Strongly Independence Number in Cluster Hypergraphs.

2. Preliminaries

Definition2.1. Let X be a non empty set and let V_X be a subset of P(X) such that $V_X \neq \varphi$ and$X \subset V_X$. Now E be a multi-set whose elements belong to P(P(X))such that

(i) $E \neq \varphi$

(ii) for each element $e \in E$, there exists atleast one element $v \in V_X$ suchthat$v \in e$.

Then $H = (V_X, E)$is said to be Cluster Hyper graph where V_X is said to be a vertex set and E is said to be multi-hyper edge set [3].

Definition 2.2. Let$H = (V_X, E)$be a cluster hypergraph. A set$S \subseteq V_X(H)$is said to be a Dominating Set of H if for every vertex $x \in V_X(H) - S$there exists $y \in S$, such that x and y is adjacent in H.Thatis, there exists$C \in E$ such that $x, y \in C$. The minimum cardinality of a dominating set in a cluster hypergraphH is called the Domination Number of H and is denoted by $\gamma(H)$ [4].

Definition 2.3.A pendant vertex is a vertex that is incident with exactly one edge of the cluster hypergraph H.

Given an integer $l > 0$, the cluster l −section of $H = (V_X, E)$ is defined as the cluster hypergraph $[H]_{c_l}$ whose edges are the sets $F \subset V_X$satisfies either $|F| = l$ and $F \subseteq C$ for some $C \in E$.

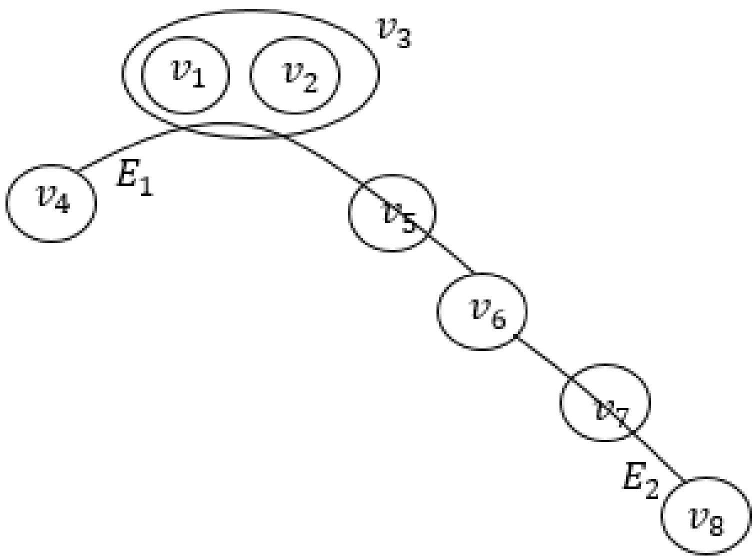
For $l = 2$, the cluster $2 -$ section of $H = (V_X, E)$ is the simple graph $[H]_{c_2} = (V_X, E')$, where $E' = \{\{u, v\} \subseteq V_X /\ u \neq v$ there exist $C \in E$ such that $\{u, v\} \subseteq C\}$. That is, two vertices are adjacent in$[H]_{c_2}$if they belong to the same multi hyperedge in H.

3. Main Results

Definition 3.1. Let $H = (V_X, E)$ be a cluster hypergraph. A subset $S \subseteq V_X(H)$ is said to be independent if it does not contain an edge Ein Hwith $|E| > 1$. The independencenumber or independent number$\alpha(H)$ of a cluster hypergraphHis definedas the maximum cardinality of a maximal independent set in H.

The set $S \subseteq V_X(H)$is called a strongly independent set if no two vertices in S are adjacent. The maximum cardinality of a maximal strongly independent set is denoted by $\beta(H)$and is called the strongly independence number or strongly independent number.

Example 3.2. Consider the cluster hypergraph $H = (V_X, E)$ which is shown in figure 3.1.



Figures 3.1 Cluster Hypergraph H

Here $S = \{v_1, v_2, v_4, v_8\}$be a maximal strongly independent set of maximum cardinality and so$\beta(H) = 4$. It is noted that $S_1 = \{v_1, v_4, v_8\}$ is an independent set because the verticesv_1,v_4 and v_8 are not contained in any edges of H. Also, notice that $S_2 = \{v_1, v_4,$
$v_6\}$is not an independent set, because the edge E_1contains both v_4 and v_6.

Observation 3.3. An independent (strongly independent) set Sis maximal in H if and only if for all $x \in V_X(H)/S$, $S \cup \{x\}$ is not an independent (strongly independent) set.
Observation 3.4. For any cluster hypergraph H, $\beta(H) \leq \alpha(H)$.
Open Problem3.5. In general, illustrate the class of cluster hypergraphs that satisfies the condition$\beta(H) \leq \alpha(H)$.

Theorem 3.6. Consider the cluster hypergraph $H = (V_X, E)$. If H contains no edge of cardinality two, then $\alpha(H) > \beta(H)$.
Proof. Consider the cluster hypergraph $H = (V_X, E)$. Let $S \subseteq V_X(H)$ be a maximal strongly independent set ofH. Then by definition, no two vertices in S areadjacent. Also, by hypothesis,$|C_i| \neq 2$ for all $C_i \in E$. But $|C_i| > 1$, implies that $|C_i| \geq 3$ forall $C_i \in E$. Thus, for any vertexx $\in C_i/S$, $S \cup \{x\}$ is an independent set in H. Hence,$\alpha(H) > \beta(H)$.

Corollary 3.7. Consider the cluster hypergraph $H = (V_X, E)$with $\alpha(H) = \beta(H)$,then H contains atleast one edge $C_i \in E$such that $|C_i| = 2$.
Proof. Consider the cluster hypergraph$H = (V_X, E)$with $\alpha(H) = \beta(H)$. To prove H contains at least one edge $C_i \in E$such that$|C_i| = 2$.Suppose on contrary,H contains no edge of cardinality two. Then by theorem 3.6, $\alpha(H) > \beta(H)$. By observation 3.4, it follows that,$\alpha(H) \neq \beta(H)$which contradicts the previous assertion. Hence,H contains atleast one edge $C_i \in E$with $C_i = 2$.
Observation 3.8. If there exists any two vertices x and y that are adjacent to all the maximal independent sets S, then $\propto (H) > \beta(H)$.

Theorem 3.9. Suppose H be a cluster hypergraph, then $\beta(H) = \beta([H]_{c_2})$.
Proof. SupposeHbe a cluster hypergraph. Since any two vertices are adjacent in H if and only if they are adjacent in $\left([H]_{c_2}\right)$. Then $\beta(H) = \beta([H]_{c_2})$.

Theorem 3.10. Consider the cluster hypergraph $H = (V_X, E)$. Then $\alpha(H) = \beta(H)$ if and only if $\alpha(H) = \alpha([H]_{c_2})$.
Proof. Consider the cluster hypergraph$H = (V_X, E)$. Assume that$\alpha(H) = \beta(H)$.It is enough to prove that$\beta(H) = \alpha([H]_{c_2})$.Suppose,S be a maximal independent set inH such that$|S| = \alpha(H) = \beta(H)$.Clearly, by definition no two vertices ofSare adjacent in$\left([H]_{c_2}\right)$.So thatS is a strongly independent set of$([H]_{c_2})$and hence$\alpha\left([H]_{c_2}\right) \geq \beta(H)$. Further, if$S_1$ is a maximal strongly independent set of$[H]_{c_2}$,then by definition no two vertices inS_1 are adjacent inH. It follows that $S_1$is a strongly independent set inHand hence,$\alpha\left([H]_{c_2}\right) \leq \beta(H)$.Thus,$\beta(H) = \alpha([H]_{c_2})$.
Conversely, assume that$\alpha(H) = \alpha\left([H]_{c_2}\right)$.As for any graphG, $\beta(G) = \alpha(G)$.This implies that$\beta([H]_{c_2}) = \alpha([H]_{c_2})$.Hence, by theorem 3.9, it is concluded that $\alpha(H) = \beta(H)$.

Theorem 3.11. For any cluster hypergraph $H = (V_X, E)$, then$\gamma(H) \leq \beta(H)$.
Proof. Consider the cluster hypergraph $H = (V_X, E)$and supposeSbe a strongly independent set inH with $\beta(H) = |S|$. Then, for every vertex $x \in V_X(H)/S$, the set $S \cup \{x\}$ is not a strongly independent set and so x is adjacent to minimum vertices of S. Thus,Sis a dominating set inH. Hence,$\gamma(H) \leq |S| = \beta(H)$.Thus,$\gamma(H) \leq \beta(H)$.

Theorem 3.12. A strongly independent set S of a cluster hypergraph H is a maximal strongly independent set if and only if it is strongly independent and dominating.
Proof. Let S be a maximal strongly independent set of H and let $x \in V_X(H)/S$. Then $S \cup \{x\}$ is not a strongly independent set so there exists a vertex y in S such thatx and y are adjacent in H. Hence,S is a dominating set of the cluster hypergraph H.
Conversely, assume that the set $S \subseteq V_X(H)$ be both strongly independent and dominating. Then for any vertex $x \in V_X(H)/S$, x is adjacent to atleast one vertex inS. This shows that $S \cup \{x\}$ is not a strongly independent set. Hence, S is a maximal strongly independent set inH.

Theorem 3.13. Every maximal strongly independent set of a cluster hypergraph H is a minimal dominating set of H.
Proof. SupposeH be a cluster hypergraph and let S be a maximal strongly independent set in H. Therefore, by theorem 3.12, SdominatesH. Now it is enough to prove that Sis a minimal dominating set of H. Suppose on contrary, for every $x \in S$, S/x is a dominating set in H. This shows that x is adjacent to any vertex in

S/x. It follows that S is not a strongly independent set inH, which is a contradiction. Hence, S is a minimal dominating set ofH.

Corollary 3.14. If S is a strongly independent dominating set of a cluster hypergraph, then S is both a minimal dominating set and maximal strongly independent set. Conversely, if S is a maximal strongly independent set in H, then S is a strongly independent set in H.

Definition 3.15. Let H be a cluster hypergraph. The minimum cardinality of a maximal strongly independent set in H is called the independent domination number and is denoted by i(H).

Observation 3.16. For any cluster hypergraph H, then $i(H) \leq \beta(H)$.

Observation 3.17. For any cluster hypergraph H, $i(H) = i([H]_{c_2})$.

4. CONCLUSION

In this article, the concept Independence Number in Cluster Hypergraph has been introduced and the same concept is extended to prove some theorems related to the Independence Number in Cluster Hypergraphs.

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