A Study on 'Point-Free Foundation of Geometry'

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ABSTRACT

The name "point-free geometry" denotes a series of researches on foundation of geometry in which the main primitive notion is the one of "solid body" (or, better, "region"), while "points", "lines", "planes" are defined. The main motivation of these researches is ontological in nature since the existence of solid bodies in the space looks more acceptable than the existence of points, lines, planes. In particular, an entity without extension as a point is not conceivable and it is inconceivable that three-dimensional entities are formed by entities without extension. Regardless of the validity of these motivations, the researches on point-free geometry show that the historically dominant choice of assuming points as primitive entities is only one of the possible choices (maybe the best one) and that alternative are possible. In our opinion this fact alone is sufficient to show the importance of point-free geometry.

Keywords: point, point-free geometry, theory, axioms, region, convex,

INTRODUCTION

This paper starts from the observation that, as far as we know, the existing researchers in point-free geometry paid attention only to the logical aspects of the proposed theory (consistency, categoricity, independence and so on). However, in our opinion, it is interesting to search for point-free axiomatizations taking into account also their possible educational potentialities. We propose simple and intuitive primitive notions and axioms whose validity can be experimented through appropriate laboratory activities.

It is thanks to a relationship between epistemological and didactic reflections on mathematics that we arrive at the debate on the first elements, to understand how there is nocoincidence between primitive elements for a novice student and primitive terms in mathematics (D'Amore & Fandiño Pinilla, 2009).

In this paper, we are interested in the possibility of identifying a system of primitive notions and axioms for a point-free geometry expressing in a direct way the childhood spatial experiences. We refer to experiences inside and outside schools. In particular, we refer to activities devoted to develop the geometrical intuition. We share the following claim.

Experimental and intuitional methods are not identical. … Take the equality of vertically opposite angles. If I measure the angles I am proceeding experimentally; if I open out two
sticks crossed in the form of an X, and say that it is obvious to me that the amount of opening is equal on the two sides, then I am using intuition. (Godfrey & Siddons, 1931, p. 21)

The choice of the point-free approach is in accordance with the following critical remark.

The tendency to reproduce the Euclidean approach in basic school continues today; many teachers introduce this discipline starting from concepts such as points, lines and planes which are important for a rational treatment, but distant from the student's experience orfrom definitions that should instead be considered as a point of arrival of a constructive,
personal learning process (Arrigo & Sbaragli, 2004).

We propose a point-free theory that might be treated not only at University, in a course devoted toanalyze the question of an axiomatic foundation of geometry, a course we consider essential for future teachers, but it could be also introduced in high school, as a rational arrangement of space exploration activities and, under teacher mediation, in primary and secondary school students. In the latter case, we are talking about a theory that should not be directly offered but it could be experimentally derived, even partially, by children during classroom activities. In choosing primitive notions we always look at possible building of teaching-learning laboratory activities. For instance, in plane geometry some interesting possibilities are related to folding and to equide composability activities and therefore to artifacts such as scissors and paper sheets.

Our hope is that in an experimental teaching-learning activity we can realize the social con struction of a "germ-theory":
It is still an initial theory built gradually, but with a potential for expansion and a tendency todevelop into a complete theory. In other words, it contains some statements which, although they do not exhaust the axioms traditionally assumed in one of the possible theoretical arrangements allow the production of conjectures or construction methods andtheir proofs (Bartolini Bussi, 1999).
A germ of theory, therefore, although playing from a logical point of view the same role as a theory, does not aim to establish a branch of mathematical knowledge (Ferrari & Gerla, 2015). Wehope to find a formal basis, a theory or a seed of theory, which provides tools that help the teaching and the learning of geometry.
Another reason leading us to consider the point-free approach to geometry in a teaching- learning perspective is the gap between the theoretical approach to geometry proposed at school and the geometric knowledge in the reality.
The importance of the study of the contrast between the learning of mathematics at school and the extra-scholastic or pre-scholastic mathematical knowledge, between capacity to work with mathematics "taught" at school and capacity to use mathematics in a spontaneous way in reality is emphasized in the following statement.
A clear gap between the mathematically rich situations, mainly in the numerical field, that children experience in out-of-school and the classroom practice." (Bonotto, 2001).
We think that this contrast occurs also between knowledge and intuitions acquired in the activities and laboratories of the first years of life, during the pre-school, and the theoretical knowledge of the following years.
We believe that the existing gap between simple and intuitive primitive notions coming from experience and abstract entities proposed at school, such as points and lines, could be sought in the context of point-free geometry.
The researches in point-free geometry originate from the proposals by S. Leśniewski and the Polish logic school (Leśniewski, 1992) and from the analysis by A. N. Whitehead (Whitehead, 1919, 1920).Focusing on what Whitehead said, the notion of "event" and of "inclusion" of events are assumed as primitive. Instead, the points (and other "abstract" entities such as lines and surfaces) are defined by "abstraction processes". Several papers proposing various kinds of point-free axiomatic approaches to geometry have been produced in this last century, as we will emphasize in Section 2.
The novelty of our approach lies in the fact that in choosing primitive notions and axioms we always look at possible building of teaching-learning laboratory activities.
In our approach, inspired by Sniatycki (1968), the notions of convexity and half-planes play a crucial role. Indeed, in Section 3, we introduce an n-dimensional prototype of point-free geometry by using the primitive notion of convexity. The 2-dimensional prototype, briefly denoted by PFP, enable us to define the notion of Re-half-plane, fundamental to define a lot of other concepts, such as Re-lines, Re-points, polygons. Moreover, it gives us the possibility to face with success the important question to put an order on a Re-line.
The theory T, introduced in Section 5, able to capture the prototypical point-free model, has been deduced starting from the following statements.

(a) T must at least make meaningful the definitions
(b) The axioms in T must be satisfied in PFP so that every theorem of T is valid in this model.
(c) T must be categorical.
(d) The axioms in T must refer to properties of regions and ovals as directly as possible (evident and intuitive)
(e) The axioms have to be satisfactory from a didactic and not only from a logical point o some open problems related to this point of view emerged. The paper is only the first step in a research project that will develop in different directions.

Thanks to the omnipresence of analytic geometry, in the present time the role of the points is absolute. Indeed, lines, planes, and all the geometric elements are defined as sets of points satisfying certain algebraic conditions. The role is also absolute in the metrical approaches to geometry and to the ones based on the notions of betweenness or equidistance. Nevertheless, in general, the axiomatization of geometry, in accordance with Euclid and Hilbert, requires also to assume as a primitive the notions of line and plane. Traces of this point of view are also in frequent expressions as "the point P lies on the line r" or "the line r passes through the point P" (instead of "the point P belongs to the line r"), "the line r lies on the plane σ" (instead of "the line r is included in the plane σ"), the two circles meet at a point P" (instead of "there is a point P that belongs to both the two circles").

As we have already said, in point-free geometry one proceeds in a totally different way since the only primitive notion is that of region[1] while points are defined as suitable sets of regions. In otherwords, with a radical reversal of the usual point of view, instead of considering a region as a set ofpoints, one defines a point as a set of regions. Then, according to the language of mathematical logic, a region is a first-order object and a point a second-order object.

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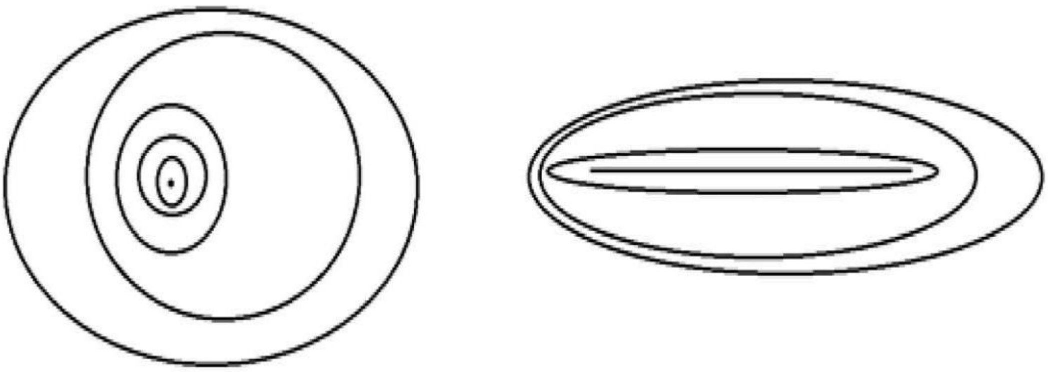
Let us assume the notion of region and the inclusion relation as primitive.

Definition 2.1. An abstraction process is an order-reversing sequence $(r_n)_{n\in N}$ of regions such that there is no region r such that $r \leq r_n$ for all $n\in N$ (see Figure 1).

We denote by AP the class of abstraction processes.



Abstraction process converging to a point Abstraction process converging to a segment

Figure 1

In some sense, an abstraction process represents the "abstract limit" to which the process converges. In Figure 1 we illustrate an abstraction process that "converges" to a point and an abstraction process that "converges" to a segment. As a matter of fact, Whitehead does not deal with sequences but with classes of regions. We refer to sequences because this most closely resembles the method of constructing real numbers through Cauchy sequences of rational numbers. Both of them rely on identifying an endless approximation process with the object you want to approximate.

Of course, as in the case of Cauchy sequences, two different processes may characterize the same abstract entity. For example, in Figure 2 the sequence of squares and one of the circles "converges" to the same point. Then, Whitehead proposed the following definition of a pre-order and, consequently, of an equivalence relation. This definition is probably inspired by the fact that if $(X_n)_{n\in N}$ and $(Y_n)_{n\in N}$ are two decreasing sequences of subsets of a given set and if for every Y_n there exists X_m such that $X_m \leq Y_n$, then $\cap_{n\in N}X_{n\in N} \subseteq \cap_{n\in N}Y_n$.

Definition 2.2. Given two abstraction processes $(x_n)_{n\in N}$ and $(y_n)_{n\in N}$, we define the relation $\leq^*$ by setting $(x_n)_{n\in N} \leq^* (y_n)_{n\in N}$ if for every y_n there exists x_m such that $x_m \leq y_n$. We define the relation $\equiv^*$ by putting $(x_n)_{n\in N} \equiv^* (y_n)_{n\in N}$ if $(x_n)_{n\in N} \leq^* (y_n)_{n\in N}$ and $(y_n)_{n\in N} \leq^* (x_n)_{n\in N}$. In this case we say that the two sequences are equiconvergent.

Since $\leq^*$ is a pre-order, $\equiv^*$ is an equivalence relation in the class AP of abstraction processes. Also, in the quotient $AP/\equiv^*$ an order relation $\leq^{**}$, is defined putting $[(x_n)_{n\in N}] \leq^{**} [(y_n)_{n\in N}]$ if and onlyif $(x_n)_{n\in N} \leq^* (y_n)_{n\in N}$. In this case, we say that $[(x_n)_{n\in N}]$ is a part of $[(y_n)_{n\in N}]$.

Definition 2.3. The elements of the quotient $AP/\equiv^*$ are named "abstract geometric entities". We call point an abstract geometric entity minimal with respect to $\leq^{**}$.

It should be noted that the request that no region is contained in all the regions of the process stems from the fact that we want to avoid abstract geometric figures containing a region, since wewant the abstract geometric figures to be smaller than the regions. Now, this way of proceeding has some technical difficulties as this example shows. In the Euclidean plane denote by G_{-0}, G_0 and G_{+0} the abstractive sequences of open balls with radius 1/n and centre in (−1/n,0), (0,0) and (1/n,0), respectively (see Figure

3). From an intuitive point of view, G_0 represents the point O ≡ (0,0). Unfortunately, $[G_0]$ is not a point since G_0 coversboth G_{-0} and G_{+0} but these abstractive sets are not equivalent with G_0.

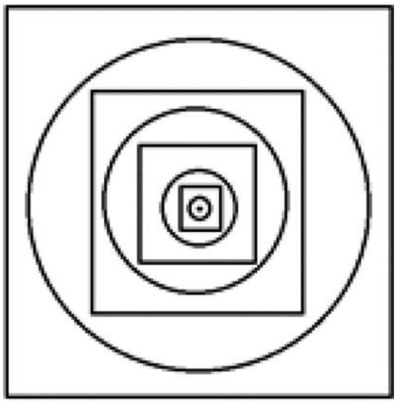


Figure 2. Two equivalent abstraction processes.

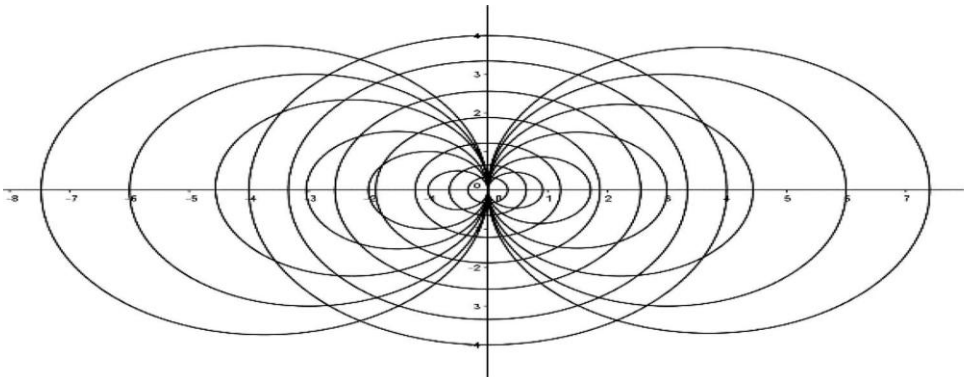


Figure 3. Non-minimal abstractgeometric entities.

On the other hand, neither $[G_{-0}]$ nor $[G_{+0}]$ are points. In fact, denote by OA_nB_n the triangle defined by the points O = (0,0), $A_n = (1/n,1/n)$, $B_n = (1/n, -1/n)$ (see Figure 4), then $<OA_nB_n>_{n\in N}$ is anabstractive sequence which is covered by G_{+0} but is not equivalent to G_{+0}. This means that the geometrical element $[G_{+0}]$ is not a point since it is not minimal in the class of geometrical elements. In a similar way, one proves that $[G_{-0}]$ is not a point. Again, there is no difficulty to prove that $[<OA_nB_n>_{n\in N}]$ is not a point, and one begins to suspect that Whitehead's definition of point is empty.

These difficulties led Whitehead to formulate in Process and Reality (Whitehead, 1929) a different notion of abstraction process. Indeed, he considers as a primitive the topological relation

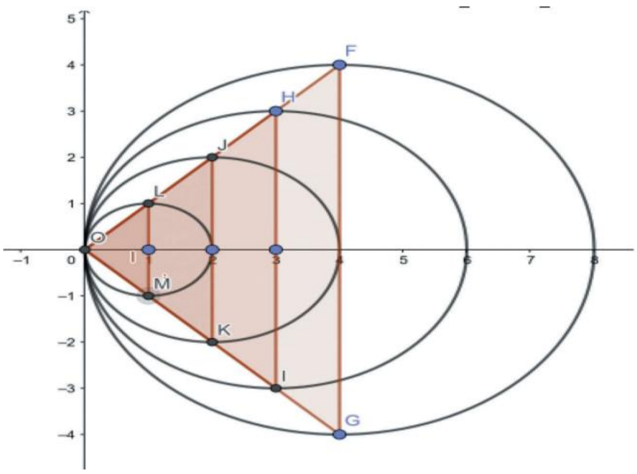


Figure 4 . An abstractive class of triangles covered by an abstractive class of tangent balls

of connection (i.e. to be either overlapping or in contact) as a tool to define the non-tangential inclusion.[2] This will allow a definition of abstraction process in which the inclusion is non-tangential.

It should be kept in mind that Whitehead's research does not want to have a mathematical character but only to be a qualitative analysis of how points, lines and all "non-concrete" entities can be defined through the regions (namely through the "events", i.e. objects in 4-dimension space). However, this analysis has inspired a long series of researches, mathematical in nature, on point-free geometry.

The first rigorous mathematical paper is due to the famous Polish logician Alfred Tarski who in a few pages proposes a point-free theory based only on the notions of inclusion and sphere (Tarski, 1929). Then, Tarski studies structures as (Re, ≤, S) where

- Re is intended as the class of "regions",
- the relation ≤ as "inclusion"
- the subset S of Re as the class of "spheres".

Obviously, suitable axioms are imposed to this class of structures. For example, one requires that (Re, ≤) is an order relation. The main step of Tarski's paper is the definition of various types of tangency between spheres in order to define in S the relation of "being concentric" we denote by ≡. This relation is an equivalence (Gruszczyński & Pietruszczak, 2008) and this enables us to definea point as an element of the quotient of S modulo ≡. In other words, a point is a complete class of concentric spheres. Successively, Tarski defines in the set of points the ternary relation "X and Y areequidistant from Z". This allows him to "cannibalize" the proposal for a foundation of geometry made by M. Pieri based on this relationship (Pieri, 1908). Indeed, by the following postulate he adds the whole Pieri's system of axioms to the axioms referring directly to spheres and regionsproperties.

Postulate 1. (Cannibalization axiom) The defined notions of point and equidistance of two points from a third point satisfy Pieri's postulates of three-dimensional Euclidean geometry.[3]

This postulate has been named by us "cannibalization axiom" since Tarski constructs his theory starting from Pieri's axioms transforming them in axioms of his own language, and therefore in formulas expressing properties of the structures (Re, ≤, S).

After Tarski's paper, research focused on point-free topology in which a topology could be viewedfrom a lattice theoretical point of view. Indeed, the class of open sets is a complete lattice satisfying an infinitely distributive property. However, it is evident that topological notions alone are not sufficient to establish Euclidean geometry. A point-free metric space theory was proposed by F. Previale (1966a). In this case, the lattice is equipped with a diameter function and this allowselegant definitions of point and of a metric in the set of points. A further step (Gerla, 1990) was done considering both diameter and distance between regions as undefined notions (see also Di Concilio & Gerla, 2006). Every model of the proposed theories is associated with a metric space. This makes possible to obtain an implicit point-free axiomatization of Euclidean geometry by cannibalizing, for example, the metric-based foundation of geometry proposed in (Blumenthal, 1970).

The literature on point-free geometry is rather large and it is not possible to cite all the papers on this subject. A survey is contained in two papers (Gerla, 2019; Gerla & Miranda, 2008). Various routes to define points have been crossed (see for example, Coppola et al., 2010; Gerla, 2001; Gerla& Miranda, 2004; Gerla & Tortora, 1992). This paper is partially inspired by the approach proposed by Sniatycki (1968) and by a successive paper (Gerla & Gruszczyński, 2017) in which both the notions of half-plane and convex region play a crucial role.

Two options are possible to identify an adequate system of axioms for"geometry of regions". The first one consists in referring to an intuitive idea of region and in proposing axioms that reflectthis intuition. It is the way followed by Euclid to construct his theory. The second option is to lean on the existing analytical geometry and therefore on the linear structure of $\mathbb{R}^n$ (where $\mathbb{R}$ is the ordered field of real numbers). Specifically, the tools provided by $\mathbb{R}^n$ will help usto construct a structure whose domain is a particular class of subsets of $\mathbb{R}^n$ which we will call regions. This structure will then be used as a guiding structure for the formulation of an axiomatic theory whose axioms will be chosen among the properties valid in it. Notice that it is no longer shameful to lean on a structure for constructing an alternative to the structure itself. Non-Euclidean geometries developed in a similar way by constructing their own non-Euclidean models within the classical Euclidean plane.

Now to give a rigorous definition of region in $\mathbb{R}^2$ it is necessary to first establish what topological properties are satisfied by the regions. For instance, is a triangle, intended as a set of points, is an open set? To address this question, assume that T is a set of point in $\mathbb{R}^n$ candidate to be considered a triangle, then its interior i(T) and its closure c(T) have the same sides as T. It follows that, by the congruence principle, there is an isometry between i(T) and c(T). But this is an absurdity since an isometry is a homeomorphism. This forces us to assume that either every triangle is open or every triangle is closed. More in general that either all the regions are open sets or all the regions are closed sets. Unfortunately, this creates difficulties in defining the notion of equidecomposability. Indeed, let ABC be a triangle where AC is a base (see Figure 5). In order to prove the well-known formula for the area, one proceeds as follows:

"Cut into pieces" the triangle: first making a horizontal cut DF at half the height and then a vertical cut BE in the obtained triangle DBF. In this way, we obtain two triangles DBE and EBFand the trapezoid ADFC.

"Rotate" the triangle DBE around the point D and the triangle EBF around point F so that we obtain the rectangle AGHC that has the same base as the triangle ABC and half the height ofthe triangle ABC.

The famous formula for the area of a triangle follows. Now, if we assume that all the regions areclosed set of points, then the regions DBE, EBF and ADFC are not a partition since they have common sides.[4] If we assume that these regions are open, then they do not define a partition since their union is strictly contained in ABC.

Of course, there are various tricks to solve this kind of problems. For example, it is possible to change the notion of partition by calling partition of a shape F a class P of shapes whose union is Fand such that the interiors of two different elements in P are disjoints. However, such a solution and other "ad hoc" solutions involve topological notions which are out of proportion with the immediacy of "cutting shapes into several parts".

Notice that if we interpret an isometry as the result of a movement, it is easy to find further paradoxes. For example, in the rotation around the points D and F the point E splits into two points G and H. Since rectangles are closed sets, no hole is admitted in GH, and this means thatthe point E remains in its position. Therefore, a point becomes three points, in a sense. It should be noted that this kind of paradoxes, which can be easily extended in three-dimensional spaces, has always interested the philosophers. Aristotle posed an analogous problem (Metaphysics 3.5, 1002a28—b11):

For as soon as bodies come into contact or are divided, the boundaries simultaneously become one if they touch and two if they are divided. Hence, when the bodies have been puttogether, one boundary does not exist, but has ceased to exist, and when they have been divided, the boundaries exist which they did not exist before …

It is the case to recall the famous citation related with the contact of two surfaces.

What does it separate air from water? Is it air or water? (Leonardo da Vinci: Codex Atlanticus, UTET 1966: 546)

Now we go forward in searching for a class CR of subsets of $\mathbb{R}^n$ we can consider acceptable candidates to represent the notion of region.

Firstly, in accordance with the previous observation, it is possible to assume, for example, that all the subsets in CR are closed. Obviously, in the case of the space $\mathbb{R}^3$ this class must contain the cubes, the spheres and all the three-dimensional solids usually considered in geometry. Also, since we imagine a region as a part of the space that can be occupied by a solid body, we have to requirethat points, lines and surfaces are not in CR. Moreover, we have to exclude "mixed" figures such as a sphere with a one-dimensional "tail" as the set in Figure 6. Indeed, it is difficult to imagine a partof a solid body able to fill the segment. Obviously, it should be useful that CR is the domain of a suitable algebraic structure. Similar considerations hold true for the regions in the plane $\mathbb{R}^2$. A possible definition of region in accordance with these conditions is furnished by the notion of closed regular subset (Tarski's proposal).

Definition 3.1. We call regulator the operator r: $P(\mathbb{R}^n) \to P(\mathbb{R}^n)$ defined by setting r(X)= c(i(X)) where iand c are the interior and the closure operators in the topological space $\mathbb{R}^n$, respectively. We call closed regular (in brief, regular) a fixed point of r, and we denote by CR_n the class of the closed regular subsets of $\mathbb{R}^n$.

Then, a subset F of $\mathbb{R}^n$ is closed regular if it coincides with the closure of its interior. For everyfigure F we have that r(r(F)) = r(F) and therefore r(F) is regular (see Figure 7).

Notice that the notion of regulator operator can be potentially applicable to the study ofconfining regions extending the previous version of linear spaces (see for example, Shang, 2017).

Moreover, it should also be noted that the choice of representing the notion of region by theclosed regular subsets is not shared by some authors (see for example, Lando & Scott., 2019).

The first important property of CR_n is given by the following theorem.

Theorem 3.2. Let $B_n = (CR_n, \subseteq)$ be the ordered class of regular closed subsets of a topological space $\mathbb{R}^n$. Then B_n is an atomless complete Boolean algebra in which $\emptyset$ and $\mathbb{R}^n$ are the minimum and the maximum, respectively the Boolean operations ·, +, ~ are defined by setting, X · Y ¼ rðX ∖ Yþ; X þ Y ¼ rðX [Yþ; ðXþ ¼ rð−Xþ

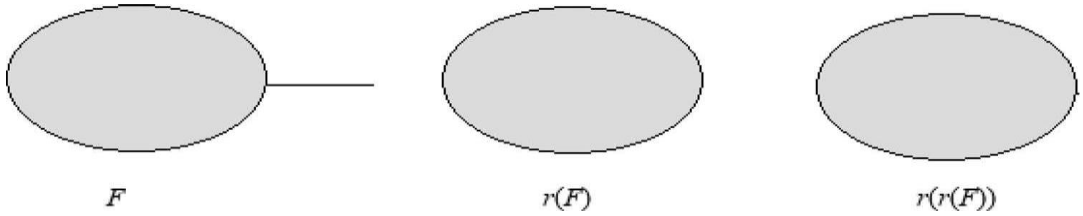
In other words, the Boolean operations are defined by regularizing the set-theoretical operations.[6] In point-free geometry, regular closed sets are frequently assumed as a reference model for the idea of a region. Indeed, it is immediate that all the desired properties are satisfied by the Booleanalgebra (CR_n, ⊆). Moreover, the decomposition of the triangle given in Section 3 is actually a "partition" when we define this notion by the Boolean operations in Theorem 3.2 instead of the set-theoretic operations. Indeed, the product of two different pieces is the empty set since it is the closure of the interior of a segment. Moreover, the sum of all pieces coincides with the whole triangle. This is in complete accordance with the intuition for which by cutting a figure by a pair ofscissors, we obtain a partition of this figure.

Denote by L a language containing symbols ≤, +, ·, ~, 0, U (the usual ones for Boolean algebra theory), a monadic predicate symbol CO (to represent the convexity property) and the relation symbol ≡ (to represent the congruence). The intended interpretation is that objects of which L speaks are regions, ≤ isthe inclusion, +, ·, ~ correspond to the union, intersection, complement, 0 and U are interpreted by the empty region and the universe, respectively and, finally, CO as the convexity property.

Definition 3.3. Given n∈N, we call n-dimensional prototype of point-free geometry the interpretation of L defined by the structure (B_n, CO_n, ≡) where:

B_n denotes the Boolean algebra of regular closed subsets of $\mathbb{R}^n$,

- CO_n is the set of convex elements of B_n,
- ≡ is the usual congruence relation.



We denote by PFP the 2-dimensional prototype.

When we refer to this interpretation we use terms as "Re-union", "Re-intersection", "Re-complement" for the interpretations of +, ·, ~, respectively. Prefix Re reminds us that the elements of the structure are intended to be regular subsets and that the operations are not necessarily set-theoretical in nature.

By these structures, we obtain an elegant solution to the difficulties discussed in Section 2.

Definition 3.4. We say that two regions x and y are Re-disjoint if $x{\cdot}y = 0$. Given a region x, a set {x_1, ..., x_n} of non-empty regions is a Re-partition of x if the elements of this set are mutually Re- disjoint and x is their Re-union, that is $x = x_1+ \dots +x_n$. A Re-partition is said to be convex if every element in the partition is convex.

Hence, two regions x and y are Re-disjoint if the only regular closed set contained in both them isthe empty region. In (B_n, CO_n, ≡) this means that two regions sharing only a common boundary (therefore not disjoint) are Re-disjoint.

Definition 3.5. Two regions x and x are called Re-equidecomposable if there exists a Re-partition
{x_1, ..., x_n} of x and a Re-partition {x_1, ...,x_n} of x with $x_i \equiv x_i$ for i = 1, ...,n.

The triangle and the rectangle mentioned in this Section are Re-equide composable according to this definition.

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