Enhancing Data Security with $S-(a,d)$ Vertex Antimagic Labeling

Sravya P.R.

Department of Mathematics, Saveetha Engineering College, Email: sravyarajendran@gmail.com

Received: 10.07.2024 Revised: 17.08.2024 Accepted: 22.09.2024

ABSTRACT
The cloud, a metaphor for the Internet, allows us to store and access data globally in minutes. Uploading data outside the community raises data security concerns. Graph labeling is an encryption technique that ensures authorized parties can access the network data. This research explores a new graph labeling method called $S-(a,d)$ vertex antimagic labeling and $S-(a,d)$ vertex antimagic total labeling, and further investigates these methods for a certain graph.

Keywords: Data security, Graph Labeling, generalized Petersen graph, generalized sunlet graph.

1. INTRODUCTION
In this digital era, smart devices and the Internet have profoundly impacted every aspect of our lives. The usage of the internet generates continuous data, accelerating at a rapid pace. Managing such vast amounts of data within local networks is challenging, prompting both commercial and non-commercial communities to utilize the cloud, a metaphor for the Internet, for global data storage and accessibility within minutes. While the cloud offers reliability, ensuring data security becomes crucial when uploading data beyond community boundaries. On the other, visualizing data as networks allows us to comprehend connections within data, revealing previously unseen insights and unlocking new sources of business intelligence. This research explores a graph labeling based two-scan algorithm for cloud security.

1.1 Graphical Insights: Essential Graph Labeling
In this subsection, we consider fundamental definitions from literature to gain deeper insights about the context. For an extensive survey of graph labeling we refer [5].

Definition 1.1 A graph $G=(V(G),E(G),\psi(G))$ is an ordered triple consisting of non-empty set $V(G)$, a set $E(G)$, disjoint from $V(G)$ and an incidence function $\psi(G)$ that associates with each element of $E(G)$, an unordered pair of $V(G)$. The set $V(G)$ is called the vertex set of G and $E(G)$ is the edge set. Elements of $V(G)$ are called vertices of G and elements of $E(G)$ are called edges of G[4].

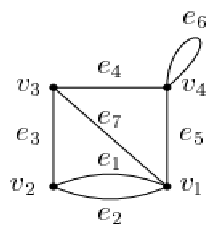

Figure 1: Graph G

Definition 1.2 Let G be a graph with p vertices denoted by$v_1,v_2,\dots,v_p$. The adjacency matrix ($p\times p$ matrix)$A(G)=[a_{ij}]$, in which a_{ij} is the number of edges joining $v_i$and v_j[4].

Definition 1.3 Let G be a graph with p vertices and q edges, represented as $v_1,v_2,\dots,v_p$and $e_1,e_2,\dots,e_q$. The incidence matrix ($p\times q$ matrix) $M(G)=[\ m_{ij}\ ]$, where $m_{ij}$the number of times that v_i and e_j are incident [4].

Definition 1.4 Weight of a vertex v is defined as the sum of the labels of all the edges incident with the vertex v. Denote$W$, the collection of all $w(v)$ where $v\in V(G)$ [6].

Definition 1.5 A graph labeling is an assignment of integers to the vertices, or edges, or both, subject to certain conditions [5].

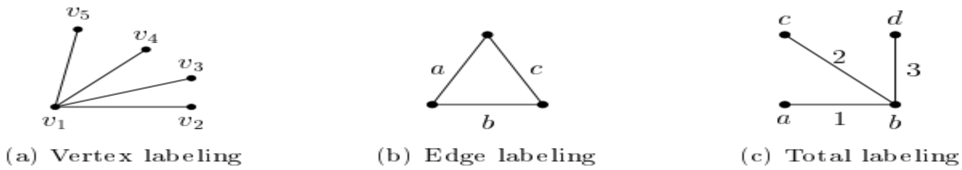

Figure 2: Graph labeling

1.2 Two scan Algorithm and its Relationship with Graph Labeling

Two scan algorithm is an effective algorithm for data security. This algorithm is based on complete labeling process which has two steps. The first scan labels are assigned to objects (objects may be the links, entities or both), and the label equivalences are stored in one dimensional array. After the first scan, the label equivalences are resolved by the use of some searching algorithms (Instead of searching algorithm we use hash function for the second scan)[6]. i.e. We compare the first scan with the edge labeling of the graph G and for the second scan is compared with the induced map g_f. Since cloud is global storage place, we share the network with more than one community, therefore we introduce $S-(a,d)$-vertex antimagic labeling and $S-(a,d)$ vertex antimagic total labeling.

2. $S-(a,d)$ – Vertex Antimagic Labeling

In this section we introduce a graph labeling technique called the $S-(a,d)$ vertex antimagic labeling and $S-(a,d)$ vertex antimagic total labeling.

Definition 2.1 A connected graph $G=(V,E)$ is said to be S - ($,d$) - vertex antimagic, if there exist positive integers a_1,a_2,d_1,d_2 and a bijection $f: E(G) \to \{1,2,3 \ldots, |E(G)|\}$ such that the induced mapping $g_f: V \to W$ is also a bijection where $V = V^{'} \cup V^{''}$ and $W = \{w(v): v \in V(G)\} = \{a_1, a_1+d_1, a_1+2d_1, .., a_1 + (|V^{'}(G)|-1)d_1\} \cup \{a_2, a_2+d_2, a_2+2d_2, .., a_2 + (|V^{''}(G)|-1)d_2\} \cdot$ [7]

Definition 2.2 A connected graph $G=(V,E)$ is said to be S - (a,d) - vertex antimagic total, if there exist positive integers a_1,a_2,d_1,d_2 and a bijection$f: V \cup E \to \{1,2,3 \ldots, |V|+|E|\}$ such that the induced mapping $g_f: V \to W$ is also a bijection where $V = V^{'} \cup V^{''}$ and $W = \{f(v) + \sum f(uv): v \in V, uv \in E\} = \{a_1, a_1+d_1, a_1+2d_1, .., a_1 + (|V^{'}|-1)d_1\} \cup \{a_2, a_2+d_2, a_2+2d_2, .., a_2 + (|V^{''}|-1)d_2\}$ and $\sum f(uv)$ is the sum of the labels of the edges incident with the vertex v. [7]

3. Graph Structures and Their Properties

In this section we recall a simple graph structures called generalized Petersen graph and generalized sunlet graph. Generalized Petersen graph was introduced by Coxeter and named by Watkins.

Definition 3.1 For integers n and k with $2 \le k \le \left\lfloor \frac{n}{2} \right\rfloor$, the generalized Petersen graph $P(n,k)$ is defined to have the vertex set $V(P(n,k)) = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ and edge set $E(P(n,k))$ is of the form $\{(u_i u_{i+1}), (u_i v_i), (v_i v_{i+k})\}(1 \le i \le n)$, where subscripts are taken modulo n. $(u_i u_{i+1})$are the outer edges, $(u_i v_i)$ are spokes and $(v_i v_{i+1})$ are the inner edges [2].

Properties

1. Generalized Petersen graph $P(n,k)$ has order $2n$ and size $3n$.
2. $3-$ Regular graph

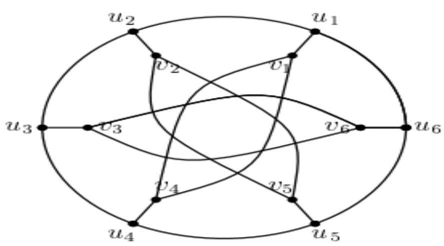

Figure 3: Generalized Petersen Graph $P(6,3)$

Definition 3.2 The generalized sunlet graph (or) n-sunlet graph S_n, is a graph with vertex set $V = \{u_1, u_2, \dots, u_n, v_1, v_2, v_3, \dots v_n\}$ and edge set E consist of all edges of the form $(v_i v_{i+1})$ and $(u_i v_i)$ where $1 \le i \le n$ subscripts of i are to be done in modulo n [7].

Properties

1. The size and order of S_n is$2n$.
2. Non-Hamiltonian, not a regular graph.
3. The vertex and edge connectivity is $\kappa(G) = \kappa^{'}(G) = 1$.

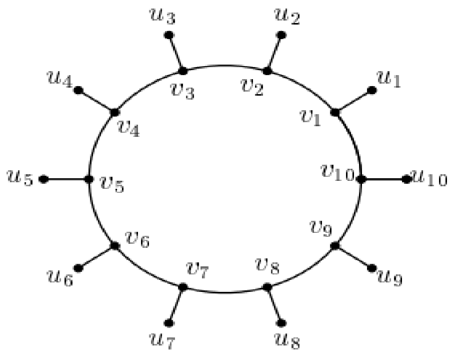

Figure 4: Sunlet graph S_{10}

4. METHODOLOGY

The following steps are involved to solve this graph labeling problem.

1. Label allocation for vertices: Allocate labels to each vertex as v_i or u_i in graph G.
2. Edge labeling with hash function: Label edges with defined hash function f.
3. Verification of bijective hash function: Check whether the given hash function f is bijective. We can use R-program or Matlab to check whether the given function is one-to-one and onto or in some cases which are visible easily.
4. Computation of vertex weights: Calculate weight of the vertex v as sum of the edges adjacent to each vertex.
5. If the vertex weights forms two arithmetic progression then it is $S-(a,d)$-vertex antimagic labeling, otherwise redefine the hash function.

5. RESULTS AND DISCUSSIONS

Theorem 5.1. Generalized Petersen graphs $P(n,k)$ are $S-(a,d)$ - vertex antimagic, for $n \ge 8$and$2 \le k \le \left\lfloor \frac{n}{2} \right\rfloor$.

Proof.

Let $V = u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n$ be the vertices of generalized Petersen graph where u_i, v_i represent the vertices of outer cycle and inner cycle respectively.Define the edge labeling $f: E(G) \to \{1,2,\dots,|E(G)|\}$. The size and order of the generalized Petersen graph is $3n$ and $2n$ respectively.

The following claims shows that generalized Petersen graphs are $S-(a,d)$ vertex antimagic:

1. $f$is a bijective map.
2. The mapping $g_f: V \to W$ is one to one and onto, where W is obtained using f.

Claim 1: Consider the map f from the edge set to its size. The edge labels are defined as follows:

$$f(u_i u_{i+1}) = i \#(1)$$
$$f(v_i v_{i+k}) = 2n + i \#(2)$$
$$f(u_i v_i) = 2n - (i-1) \#(3)$$

Where$1 \le i \le n, 2 \le k \le \left\lfloor \frac{n}{2} \right\rfloor$ and the arithmetic subscripts of i and k are taken modulo n.

Claim 2:$g_f: V \to W$ is a bijection map, and $S-(a,d)$ - vertex antimagic.

Let $W = A \cup B$, the collection of all vertex weights of $P(n,k)$. Consider $A = \bigcup_{i=1}^{n} w(u_i)$and $B = \bigcup_{i=1}^{n} w(v_i)$. The vertex weight for each vertex of $P(n,k)$ is calculated using the sum of the weights (edge labels) adjacent to each vertex.

The vertex weight for each vertex u_i is,

$$w(u_i) = f(u_i u_{i-1}) + f(u_i v_i) + f(u_i u_{i+1}) \#(4)$$

Similarly, for any arbitrary vertexv_i, the vertex weight $w(v_i)$ is,

$$w(v_i) = f(v_i v_{i-k}) + f(u_i v_i) + f(v_i v_{i+k}) \#(5)$$

Using (1) and (3) in equation (4), we obtain the following weights:

Case: $i = 1$,

$$\begin{aligned} w(u_1) &= f(u_1 u_{1-1}) + f(u_1 v_1) + f(u_1 u_{1+1}) \\ &= f(u_1 u_n) + f(u_1 v_1) + f(u_1 u_2) \\ &= n + 2n - (i-1) + i \\ &= n + 2n + 1 - 1 + 1 \\ &= 3n + 1 \end{aligned}$$

Case: $2 \le i \le n$.

$$\begin{aligned} w(u_i) &= f(u_i u_{i-1}) + f(u_i v_i) + f(u_i u_{i+1}) \\ &= i - 1 + 2n + 1 - i + i \\ &= 2n + i \end{aligned}$$

Similarly, using (2) and (3) in equation (5), we obtain the following weights

Case: $1 \le i \le k - 1$,

$$\begin{aligned} w(v_i) &= f(v_i v_{i-k}) + f(u_i v_i) + f(v_i v_{i+k}) \\ &= 3n + i - k + 2n + 1 - i + 2n + i \\ &= 7n + 1 - k + i \end{aligned}$$

Case: $i = k$,

$$\begin{aligned} w(v_k) &= f(v_k v_{k-k}) + f(u_k v_k) + f(v_k v_{k+1}) \\ &= f(v_k v_0) + f(u_k v_k) + f(v_k v_{k+1}) \\ &= f(v_k v_n) + f(u_k v_k) + f(v_k v_{k+1}) \\ &= 3n + 2n + 1 - i + 2n + i \\ &= 7n + 1 \end{aligned}$$

Case: $k + 1 \le i \le n$.

$$\begin{aligned} w(v_i) &= f(v_i v_{i-k}) + f(u_i v_i) + f(v_i v_{i+k}) \\ &= 2n + i - k + 2n + 1 - i + 2n + i \\ &= 6n + i - k + 1 \end{aligned}$$

Note that the indices of vertices are taken modulo $n$and k.

Thus $A = \{2n+2, 2n+3, \ldots, 3n, 3n+1\}$ and$B = \begin{cases} 7n + 1 - k + i, & 1 \le i \le k-1 \\ 7n + 1, & i = k \\ 6n + i - k + 1, & k + 1 \le i \le n \end{cases}$

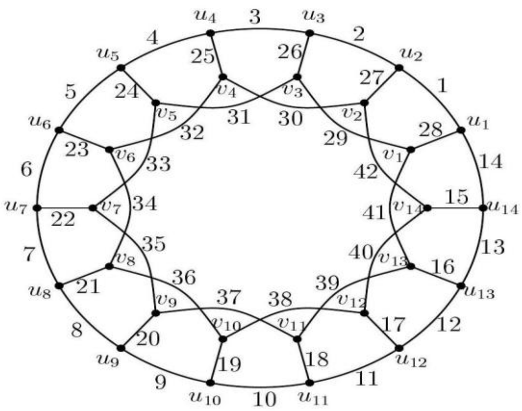

Figure 5: Edge labeling of generalized Petersen graph $P(14,2)$

From above we have, A and$B$ are the two arithmetic progressions with $a_1 = 2n + 2$ and $a_2 = 6n + 2$, and $d_1 = d_2 = 1$.

From claim (1) and claim (2) generalized Petersen graph $P(n,k)$ admit $S-(a,d)$ - vertex antimagic labeling.

Theorem 5.2. Let $P(n,2),(n \geq 8)$, a class of generalized Petersen graph $P(n,k)$ with size and order of $3n$ and $2n$ is $S-(a,d)$ - vertex antimagic total.

Proof.

Let u_i and $v_i(1 \leq i \leq n)$ be the vertices join outer and inner cycle of $P(n,2)$ respectively. Define the map $f: V \cup E \rightarrow 1,2,3, \ldots |V|+|E|$ where $|V|=2n$and$|E|=3n$.
Label the vertices of $P(n,2)$ as follows:

$$f(u_i)=\begin{cases} 4n, & i=1 \\ 3n+(i-1) & i=1,2,\cdots n \end{cases} \quad (6)$$

$$f(v_i)=\begin{cases} 5n-1, & i=1 \\ 5n, & i=2 \\ 4n+(i-2), & i=3,4,5\ldots n \end{cases} \quad (7)$$

Similarly label the edges as follows:

$$f(u_i u_{i+1})=i \quad (8)$$
$$f(v_i v_{i+k})=2n+i \quad (9)$$
$$f(u_i v_i)=2n-(i-1) \quad (10)$$

Where the arithmetic indices of i are taken modulo n.

Equations (6), (7), (8), (9) and (10) clearly shows that the function f use the integers $1,2,3,\ldots,5n$ exactly once. Hence the function f is one-to-one and onto.

Claim:$g_f: V \rightarrow W$ is $S-(a,d)$-vertex antimagic total.

Let W the union of vertex weights of the given graph. We compute weight of each vertex as follows:

$$w(u_i)=f(u_i)+f(u_i u_{i-1})+f(u_i v_i)+f(u_i u_{i+1}) \quad (11)$$
$$w(v_i)=f(v_i)+f(v_i v_{i-2})+f(u_i v_i)+f(v_i v_{i+2}) \quad (12)$$

By applying the equations (6) to (10) in (11) we obtain
Case1: $i=1$,

$$\begin{aligned} w(u_i) &= f(u_i)+f(u_i u_{i-1})+f(u_i v_i)+f(u_i u_{i+1}) \\ w(u_1) &= f(u_1)+f(u_1 u_{1-1})+f(u_1 v_1)+f(u_1 u_{1+1}) \\ &= f(u_1)+f(u_n u_1)+f(u_1 v_1)+f(u_1 u_2) \\ &= 4n+n+2n-(1-1)+1 \\ &= 7n+1 \end{aligned}$$

Case 2: $2 \leq i \leq n$,

$$\begin{aligned} w(u_i) &= f(u_i)+f(u_i u_{i-1})+f(u_i v_i)+f(u_i u_{i+1}) \\ &= f(u_i)+f(u_{i-1} u_i)+f(u_i v_i)+f(u_i u_{i+1}) \\ &= (3n+i-1)+i-1+(2n-(i-1))+i \\ &= 5n+2i-1 \end{aligned}$$

Similarly, apply the equations (6) to (10) in (12) we get,
For $i=1$,

$$\begin{aligned} w(v_i) &= f(v_i)+f(v_i v_{i-2})+f(u_i v_i)+f(v_i v_{i+2}) \\ w(v_1) &= f(v_1)+f(v_1 v_{1-2})+f(u_1 v_1)+f(v_1 v_{1+2}) \\ &= (5n-1)+(2n+n-1)+(2n)+(2n+1) \\ &= 12n-1 \end{aligned}$$

When$i=2$,

$$\begin{aligned} w(v_2) &= f(v_2)+f(v_2 v_{2-2})+f(u_2 v_2)+f(v_2 v_{2+2}) \\ &= f(v_2)+f(v_n v_2)+f(u_2 v_2)+f(v_2 v_4) \\ &= (5n)+(3n)+(2n-1)+(2n+2) \\ &= 12n+1 \end{aligned}$$

When$3 \leq i \leq n$.

$$\begin{aligned} w(v_i) &= f(v_i)+f(v_i v_{i-2})+f(u_i v_i)+f(v_i v_{i+2}) \\ w(v_i) &= f(v_1)+f(v_{i-2} v_i)+f(u_i v_i)+f(v_i v_{i+2}) \\ &= (4n+i-2)+(2n+i-2)+(2n-(i-1))+(2n+i) \\ &= 10n+2i-3 \end{aligned}$$

where$a_1 = 5n + 3, a_2 = 10n + 3$ and $d_1 = d_2 = 2$ which shows the vertex weights are unique and follows two arithmetic progression.

Theorem 5.3. $S_n, (n \geq 3)$is$S - (a, d)$ - vertex antimagic.
Proof.
Consider the generalized sunlet graph $S_n, n \geq 3$ and assign the labels as u_i and $v_i (1 \leq i \leq n)$ for cycle and pendent vertices respectively. Define the edge labeling $f: E \rightarrow \{1,2,3, \dots |E|\}$ as follows:

$$f(v_i v_{i+1}) = i \quad (13)$$
$$f(u_i v_i) = \begin{cases} n + i, & i = 1 \\ 2n + 2 - i, & 2 \leq i \leq n \end{cases} \quad (14)$$

The vertex weights are calculated as follows:

$$w(u_i) = f(u_i v_i)\#(15)$$
$$w(v_i) = f(v_i v_{i+1}) + f(v_i v_{i-1}) + f(u_i v_i)\#(16)$$

Which results $S - (a, d)$ - vertex antimagic labeling.

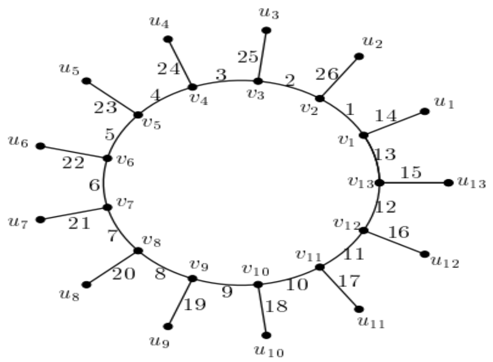

Figure 6: Illustration of $S - (a, d)$- vertex antimagic labeling of S_{13}

The above figure illustrates the $S - (a, d)$- vertex antimagic labeling of S_{13} with $a_1 = n + 1$ and $d_1 = 1$ and $a_2 = 2n + 2$ and $d_2 = 1$.

6. Recapitulation and Future Directions

In this research, we introduce$S - (a, d)$- vertex antimagic, $S - (a, d)$- vertex antimagic total labeling and its applications towards data security. Further we investigate the same, for generalized Petersen graph and generalized sunlet graphs. In future, we intend to study $S - (a, d)$- vertex antimagic and $S - (a, d)$- vertex antimagic total labeling of Petersen derived graphs.

REFERENCES

[1] Alexander Rosa, On Certain Valuations of the Vertices of a Graph, Theory of Graphs, International Symposium, Rome, July1966, Gordon and Breach, N.Y. and Dunod Paris, 349-355,1967.

[2] D.A. Holton and J. Sheehan, The Petersen Graph, Australian Mathematical Society Lecture Series 7, Cambridge University Press, 1993.

[3] N. Hartsfield and G. Ringel, Pearls in Graph Theory: A comprehensive Introduction, Academic Press, Boston, 108-110, 1990.

[4] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications, the Macmillan Press Ltd. New York, 1982.

[5] Joseph A. Gallian, A Dynamic Survey of Graph Labeling, The Electronic Journal of Combinatorics, 21, 2023.

[6] Lifeng He, Yuyan Chao, and Kenji Suzuki A Run-Based Two-Scan Labeling Algorithm.

[7] Sravya PR, Variants of Magic Labeling For Certain Graphs, Thesis (Ph.D.), Dec 2022.

[8] NaliniPriya G. and M. Asswini, A Survey on Vulnerable Attacks in Online Social Networks, *International Conference on Innovation Information in Computing Technologies*, Chennai, India, 2015.

[9] Sreevidya M.V., Ramyadevi R., Chamundeeswari V.V., Node to Node Communication security in IoT networks with hybrid cryptography and steganography system, Artificial Intelligence, Blockchain, Computing and Security - Proceedings of the International Conference on Artificial Intelligence, Blockchain, Computing and Security, ICABCS 2023*, 2024, 2, pp. 534–537.*