Tauberian conditions for Statistical Convergence from Statistical Summability using Nörlund-Euler means in Non-Archimedean Fields

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ABSTRACT
All through this research article, K stands for a non-archimedean field which is both non-trivially valued and complete. Tauberian conditions for statistical convergence that arises from the fact of statistical summability by Nörlund method over K has recently been investigated by us. We present here, the if and only if conditions for statistical convergence due to statistical summability by Nörlund-Euler methods over non-archimedean fields.

Keywords: Non-archimedean fields, Nörlund-Euler means, statistical convergence, statistical summability $(N,p,q)(E,1)$ method, Tauberian conditions.

1. INTRODUCTION

It was Fast [4] who first brought about this concept of statistical convergence in the year 1951, which is very significant in summability theory. Schoenberg established a relation between summability and statistical convergence. Hardy's and Landau's Tauberian theorems in classical nature of statistical convergence were studied by Fridy and Khan. In this paper, investigation for Tauberian conditions for sequences that are statistically $(N,p,q)(E,1)$ summable by Nörlund-Euler means over K is done.
Statistical convergence of a sequence (x_k), is defined as
$\lim_{n\to\infty}\frac{1}{n}|\{k\le n: |x_k-L|\ge\epsilon\}|=0$,
For$x_k\in K$, $k=1,2,\ldots$, for every $\epsilon>0$. Here L is the statistical limit and the outer vertical lines stand for the set's cardinal number (see [12]). This is written as,

$$\text{st}-\lim_{k\to\infty}x_k=L\#(1.1)$$

Consider two sequences $p=(p_n)$ and $q=(q_n)$ in K, with

$$P_n=\sum_{i=0}^{n}p_i,\quad p_i\ne 0\quad\text{and}\quad Q_n=\sum_{i=0}^{n}q_i,\quad q_i\ne 0$$

The convolution of (p_n) and (q_n) is given by

$$R_n=\sum_{k=0}^{n}p_kq_{n-k},\qquad n=0,1,2,\ldots$$

Definition 1.1. The summability of $x=(x_n)$ by the generalized Nörlund method (N,p,q), by (p_n) and (q_n) to s is defined as

$$t_n^{p,q}=\frac{1}{R_n}\sum_{k=0}^{n}p_kq_{n-k}x_k\to s,\quad\text{as}\quad n\to\infty$$

Definition 1.2. A sequence (x_n) is $(E,1)$ summable to s if,

$$E_n^1=\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}x_k\to s,\quad\text{as}\quad n\to\infty$$

Definition 1.3. The Nörlund-Euler method is defined as

$$t_n^{p,q,E} = \frac{1}{R_n}\sum_{k=0}^{n} p_k q_{n-k} E_k^1$$
$$= \frac{1}{R_n}\sum_{k=0}^{n} p_k q_{n-k} \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j} x_j$$

Definition 1.4. A sequence (x_n) is summable by the generalized Nörlund-Euler $(N,p,q)(E,1)$ method to s, if, $t_n^{p,q,E} \to s$ as $n \to \infty$.

We now define $(N,p,q)(E,1)$ statistical summability of a sequence as follows.

Definition 1.5. $x = (x_n)$ is said to be statistically $(N,p,q)(E,1)$ summable to a limit L if,
$$st - \lim_{n\to\infty} t_n^{p,q,E} = L\#(1.2)$$

2. Key Results

Theorem 2.1. Consider sequences (p_m) and(q_m) in$K$ such that $p_0 > 0,\ p_m \neq 0,\ q_0 > 0,\ q_m \neq 0$. Consider $(\lambda_k),\ \lambda_k \in K$ such that $\lim_{k\to\infty}\lambda_k = 0$ and
$$\text{st} - \lim_{m\to\infty}\frac{R_m}{R_{\lambda_m}} < 1 \quad \text{for every} \quad 0 < \lambda_m < 1\#(2.1)$$
If (x_k) is statistically $(N,p,q)(E,1)$ summable to a limit L, then (x_k) is said to be statistically convergent to L if and only if,
$$st - \lim_{m\to\infty}\frac{1}{\left(R_m - R_{\lambda_m}\right)}\sum_{k=\lambda_m+1}^{m} p_k q_{m-k}\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}\left(x_m - x_j\right) = 0\#(2.2)$$
To prove this theorem, we require the lemmas given below.

Lemma 2.1. Consider sequences (p_m) and (q_m) in K such that $p_0 > 0,\ p_m \neq 0,\ \ q_0 > 0,\ \ q_m \neq 0$; and
$$\text{st} - \lim_{m\to\infty}\frac{R_m}{R_{\lambda_m}} < 1 \quad \text{for every} \quad 0 < \lambda_m < 1$$
Let (x_k) be $(N,p,q)(E,1)$ statistically summable to L. Then, for each $0 < \lambda_m < 1$,
$$st - \lim_{m\to\infty} t_{\lambda_m}^{p,q,E} = L\#(2.3)$$
where (R_m) and $\left(t_{\lambda_m}\right)$ are non-decreasing sequences of positive numbers.

Proof. Given that the sequence (x_k) is statistically summable $(N,p,q)(E,1)$ to L.

i.e., $st - \lim\limits_{m\to\infty} t_m^{p,q,E} = L$

i.e., $\lim\limits_{M\to\infty}\frac{1}{M}\left|\left\{m \le M: \left|t_m^{p,q,E} - L\right| \ge \epsilon\right\}\right| = 0$

i.e., $\lim\limits_{M\to\infty}\frac{1}{M}\left|\left\{m \le M: \left|\frac{1}{R_m}\sum_{k=0}^{m} p_k q_{m-k}\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}x_j - L\right| \ge \epsilon\right\}\right| = 0(2.4)$

To prove, st $-\lim_{m\to\infty} t_{\lambda_m}^{p,q,E} = L$,

That is, to prove
$\lim\limits_{M\to\infty}\frac{1}{M}\left|\left\{\lambda_m \le M: \left|\frac{1}{R_{\lambda_m}}\sum_{k=0}^{\lambda_m} p_k q_{\lambda_m-k}\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}x_j - L\right| \ge \epsilon\right\}\right| = 0,$
consider
$$\frac{1}{M}\left|\left\{\lambda_m \le M: \left|\frac{1}{R_{\lambda_m}}\sum_{k=0}^{\lambda_m} p_k q_{\lambda_m-k}\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}x_j - L\right| \ge \epsilon\right\}\right|$$
$$= \frac{1}{M}\left|\left\{\lambda_m \le M: \left|\left(\frac{R_m}{R_{\lambda_m}}\right)\frac{1}{R_m}\sum_{k=0}^{\lambda_m} p_k q_{\lambda_m-k}\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}x_j - L\right| \ge \epsilon\right\}\right|$$
$\le \frac{1}{M}\left|\left\{m \le M: \left|\frac{1}{R_m}\sum_{k=0}^{m} p_k q_{m-k}\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}x_j - L\right| \ge \epsilon\right\}\right|$ using (2.1)

$\to 0$ as $M \to \infty$ (using (2.4))

Therefore,
$$\lim_{M\to\infty}\frac{1}{M}\left|\left\{\lambda_m \le M: \left|\frac{1}{R_{\lambda_m}}\sum_{k=0}^{\lambda_m} p_k q_{\lambda_m-k}\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}x_j - L\right| \ge \epsilon\right\}\right| = 0$$

That is, $st - \lim_{m\to\infty} t_{\lambda_m}^{p,q,E} = L$, which proves the lemma.

Lemma 2.2. For $0 < \lambda_m < 1$,

$$\frac{1}{(R_m - R_{\lambda_m})} \sum_{k=\lambda_m+1}^{m} p_k q_{m-k} \frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j} x_j = t_m^{p,q,E} + \frac{R_{\lambda_m}}{R_m - R_{\lambda_m}} \left(t_m^{p,q,E} - t_{\lambda_m}^{p,q,E} \right)$$

provided $R_m > R_{\lambda_m}$.

Proof. Consider the right hand side.

$$\begin{aligned} t_m^{p,q,E} &+ \frac{R_{\lambda_m}}{R_m - R_{\lambda_m}} \left(t_m^{p,q,E} - t_{\lambda_m}^{p,q,E} \right) \\ &= \frac{R_m t_m^{p,q,E} - R_{\lambda_m} t_m^{p,q,E} + R_{\lambda_m} t_m^{p,q,E} - R_{\lambda_m} t_{\lambda_m}^{p,q,E}}{R_m - R_{\lambda_m}} \\ &= \frac{1}{R_m - R_{\lambda_m}} \left[R_m \left(\frac{1}{R_m} \sum_{k=0}^{m} p_k q_{m-k} \frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j} x_j \right) \right. \\ &\quad \left. - R_{\lambda_m} \left(\frac{1}{R_{\lambda_m}} \sum_{k=0}^{\lambda_m} p_k q_{\lambda_m-k} \frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j} x_j \right) \right] \\ &= \frac{1}{R_m - R_{\lambda_m}} \left[\sum_{k=0}^{\lambda_m} p_k q_{\lambda_m-k} \frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j} x_j \right. \\ &\quad \left. + \sum_{k=\lambda_m+1}^{m} p_k q_{m-k} \frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j} x_j - \sum_{k=0}^{\lambda_m} p_k q_{\lambda_m-k} \frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j} x_j \right] \\ &= \frac{1}{(R_m - R_{\lambda_m})} \sum_{k=\lambda_m+1}^{m} p_k q_{m-k} \frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j} x_j \end{aligned}$$

Thus,

$$\frac{1}{(R_m - R_{\lambda_m})} \sum_{k=\lambda_m+1}^{m} p_k q_{m-k} \frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j} x_j = t_m^{p,q,E} + \frac{R_{\lambda_m}}{(R_m - R_{\lambda_m})} \left(t_m^{p,q,E} - t_{\lambda_m}^{p,q,E} \right)$$

proving the lemma. Rearrangement of terms and addition of x_m results in,

$$\begin{aligned} x_m - t_m^{p,q,E} &= \frac{R_{\lambda_m}}{(R_m - R_{\lambda_m})} \left(t_m^{p,q,E} - t_{\lambda_m}^{p,q,E} \right) \\ &\quad - \frac{1}{(R_m - R_{\lambda_m})} \sum_{k=\lambda_m+1}^{m} p_k q_{m-k} \frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j} x_j + x_m \\ &= \frac{R_{\lambda_m}}{(R_m - R_{\lambda_m})} \left(t_m^{p,q,E} - t_{\lambda_m}^{p,q,E} \right) + \frac{1}{(R_m - R_{\lambda_m})} \sum_{k=\lambda_m+1}^{m} p_k q_{m-k} \frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j} (x_m - x_j) \end{aligned} \tag{2.5}$$

Proof of the theorem.

Necessity: Under the condition that (x_m) is statistically $(N,p,q)(E,1)$ summable to a limit L, we first assume$st - \lim_{m\to\infty} x_m = L$to prove that, for each $0 < \lambda_m < 1$,

$$st - \lim_{m\to\infty} \frac{1}{(R_m - R_{\lambda_m})} \sum_{k=\lambda_m+1}^{m} p_k q_{m-k} \frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j} (x_m - x_j) = 0$$

i.e., we have$st - \lim_{m\to\infty} t_m^{p,q,E} = L$and $st - \lim_{m\to\infty} x_m = L$, which implies that

$$st - \lim_{m\to\infty} \left(x_m - t_m^{p,q,E} \right) = 0$$

i.e.,

$$\lim_{M\to\infty}\frac{1}{M}\left|\left\{m\le M:\left|x_m-t_m^{p,q,E}\right|\ge\epsilon\right\}\right|=0$$

That is,

$$0=\lim_{M\to\infty}\frac{1}{M}\left|\left\{m\le M:\left|\frac{R_{\lambda_m}}{(R_m-R_{\lambda_m})}\left(t_m^{p,q,E}-t_{\lambda_m}^{p,q,E}\right)+\frac{1}{(R_m-R_{\lambda_m})}\sum_{k=\lambda_m+1}^{m}p_kq_{m-k}\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}(x_m-x_j)\right|\ge\epsilon\right\}\right|$$

(using (2.5))

$$=\lim_{M\to\infty}\max\left\{\frac{1}{M}\left|\left\{m\le M:\left|\frac{R_{\lambda_m}}{(R_m-R_{\lambda_m})}\left(t_m^{p,q,E}-t_{\lambda_m}^{p,q,E}\right)\right|\ge\epsilon\right\}\right|,\ \frac{1}{M}\left|\left\{m\le M:\left|\frac{1}{(R_m-R_{\lambda_m})}\sum_{k=\lambda_m+1}^{m}p_kq_{m-k}\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}(x_m-x_j)\right|\ge\epsilon\right\}\right|\right\}$$

$$=\lim_{M\to\infty}\frac{1}{M}\left|\left\{m\le M:\left|\frac{1}{(R_m-R_{\lambda_m})}\sum_{k=\lambda_m+1}^{m}p_kq_{m-k}\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}(x_m-x_j)\right|\ge\epsilon\right\}\right|$$

since, by (1.2) and (2.3) we have,

$\frac{1}{M}\left|\left\{m\le M:\left|\frac{R_{\lambda_m}}{(R_m-R_{\lambda_m})}\left(t_m^{p,q,E}-t_{\lambda_m}^{p,q,E}\right)\right|\ge\epsilon\right\}\right|\to 0$ as $M\to\infty$.

Thus,

$$st-\lim_{m\to\infty}\frac{1}{(R_m-R_{\lambda_m})}\sum_{k=\lambda_m+1}^{m}p_kq_{m-k}\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}(x_m-x_j)=0$$

Sufficiency: Here, we assume that

$$st-\lim_{m\to\infty}\frac{1}{(R_m-R_{\lambda_m})}\sum_{k=\lambda_m+1}^{m}p_kq_{m-k}\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}(x_m-x_j)=0$$

and prove that

$$st-\lim_{m\to\infty}x_m=L.$$

For this, we need to prove that

$$st-\lim_{m\to\infty}\left(x_m-t_m^{p,q,E}\right)=0.$$

i.e., to prove

$$\lim_{M\to\infty}\frac{1}{M}\left|\left\{m\le M:\left|x_m-t_m^{p,q,E}\right|\ge\epsilon\right\}\right|=0.$$

Using equation (2.5),

$$\frac{1}{M}\left|\left\{m \leq M: \left|x_m - t_m^{p,q,E}\right| \geq \epsilon\right\}\right|$$
$$= \frac{1}{M}\left| \left\{m \leq M: \left| \frac{R_{\lambda_m}}{\left(R_m - R_{\lambda_m}\right)}\left(t_m^{p,q,E} - t_{\lambda_m}^{p,q,E}\right) + \frac{1}{\left(R_m - R_{\lambda_m}\right)}\sum_{k=\lambda_m+1}^{m} p_k q_{m-k} \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}\left(x_m - x_j\right)\right| \geq \epsilon\right\} \right|$$
$$\leq \max\left\{\frac{1}{M}\left|\left\{m \leq M: \left|\frac{R_{\lambda_m}}{\left(R_m - R_{\lambda_m}\right)}\left(t_m^{p,q,E} - t_{\lambda_m}^{p,q,E}\right)\right| \geq \epsilon\right\}\right|, \frac{1}{M}\left| \left\{m \leq M: \left| \frac{1}{\left(R_m - R_{\lambda_m}\right)}\sum_{k=\lambda_m+1}^{m} p_k q_{m-k} \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}\left(x_m - x_j\right)\right| \geq \epsilon\right\} \right|\right\}$$

By our assumption,

$$\frac{1}{M}\left|\left\{m \leq M: \left|\frac{1}{\left(R_m - R_{\lambda_m}\right)}\sum_{k=\lambda_m+1}^{m} p_k q_{m-k} \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}\left(x_m - x_j\right)\right| \geq \epsilon\right\}\right| \to 0 \text{ as } M \to \infty$$

Therefore,

$$\frac{1}{M}\left| \left\{m \leq M: \left|x_m - t_m^{p,q,E}\right| \geq \epsilon\right\} \right|$$
$$\leq \max\left\{\frac{1}{M}\left|\left\{m \leq M: \left|\frac{R_{\lambda_m}}{\left(R_m - R_{\lambda_m}\right)}\left(t_m^{p,q,E} - t_{\lambda_m}^{p,q,E}\right)\right| \geq \epsilon\right\}\right|, 0\right\}$$
$$\leq \frac{1}{M}\left|\left\{m \leq M: \left|\frac{R_{\lambda_m}}{\left(R_m - R_{\lambda_m}\right)}\left(t_m^{p,q,E} - t_{\lambda_m}^{p,q,E}\right)\right| \geq \epsilon\right\}\right|$$
$$\to 0 \text{ as } M \to \infty \text{ by (1.2) and (2.3),}$$

by which we have,

$$\lim_{M\to\infty}\frac{1}{M}\left|\left\{m \leq M: \left|x_m - t_m^{p,q,E}\right| \geq \epsilon\right\}\right| = 0$$

That is,

$$st - \lim_{m\to\infty}\left(x_m - t_m^{p,q,E}\right) = 0$$

Thus the theorem is proved.

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