Certain problems in ordered partial metric space using mixed g-monotone

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December 23, 2021

Abstract

Motivated from the fixed point hypothesis, we demonstrate the presence and uniqueness for coupled coincidence type point including a contractive condition for a map in partially metric utilizing mixed g-monotone. A model is likewise outfitted to exhibit the legitimacy of the speculations of our outcomes.

Key words: Complete Metric Space; Coupled Fixed Point; mixed g-monotone property

Mathematics Subject Classification(2010): 54H25; 47H10

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1 Introduction

A different idea of generalized metric space perceived as partial metric space offered by Matthews [6]. Many authors had given imperative results on such type of spaces [8, 9, 10].

Bhaskar and Lakshmikantam [2] established coupled fixed point and demonstrated certain coupled fixed point results for maps which gratify the property of mixed monotone. Also, present applications for periodic boundary value problem. Authors have extended numerous outcomes on coupled fixed point hypotheses on metric spaces, e.g., in [1, 2, 3, 4, 5, 7, 11].

We foremost verify the presence of coupled coincidence points. Then, we demonstrate uniqueness of coupled coincidence point results for a map having the property of mixed g-monotone in partial metric spaces. At the end we support the result by giving an example.

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The view of partial metric spaces given by Matthews [6].

Definition 2.1. [6] Presuppose Z be a null set. A partial metric on Z defines as a function $p : Z \times Z \to \mathbb{R}_n$ for every $s, t, z \in Z$:

$$\begin{aligned}
&(1) \quad s = t \iff p(s,s) = p(s,t) = p(t,t)\\
&(2) \quad p(s,s) \leq p(s,t),\\
&(3) \quad p(s,t) = p(t,s)\\
&(4) \quad p(s,t) \leq p(s,t) + p(z,t) - p(z,z)
\end{aligned}$$

A pair (Z, p) known as partial metric space, and p is partial metric on Z where, Z is a null set.

If p is a partial metric on Z, the function $p^r : Z \times Z \to \mathbb{R}_+$ defined as

$$p^r(s,t) = 2\,p(s,t) - p(s,s) - p(t,t)$$

is a metric on Z.

The prevailing definitions given by [2].

Definition 2.2. [2] A point $(s, v) \in Z \times Z$ for a map $T : Z \times Z \to Z$ possesses $T(s, v) = s, T(v, s) = h$ then it is known as coupled fixed point.

Definition 2.3. [4] Assume $(S, \leq)$ is partially ordered set, let two mappings $F : S \times S \to S$ and $g : S \to S$. Then F possesses property of the mixed g-monotone if $F(s, w)$ is g-non-decreasing in its starting element and is g-non-increasing in its next element, for $s, w \in S$

$$\begin{aligned}
s_1, s_2 \in S, gs_1 \leq gs_2 &\implies F(s_1, w) \leq F(s_2, w)\\
w_1, w_2 \in S, gw_1 \leq gw_2 &\implies F(s, w_1) \geq F(s, w_2).
\end{aligned}$$

2 Main Theorem

Theorem 2.1. *Presuppose $(Z, \leq, p)$ be a complete partially ordered set. Presuppose mappings $T : Z \times Z \to Z$ and $g : Z \to Z$ possesses property of mixed g-monotone. Presuppose $T(Z \times Z) \subseteq g(Z)$ and for any $s_0, v_0 \in Z$ with $gs_0 \leq T(s_0, v_0)$ and $gv_0 \geq T(v_0, s_0)$*

$$\begin{aligned}
p(T(s,v), T(u,z)) \leq\ & p(gs, gu) - \psi(p(gv, gz))\\
& + L\ min\{p(gs, T(s,v)), p(gu, T(u,z)),\\
& p(gu, T(s,v)), p(gs, T(u,z))\}.
\end{aligned} \tag{2.1}$$

for every $s, u, v, z \in Z$ with $gs \geq gu$, $gv \leq gz$, and $L \geq 0$. Here ψ : [0,∞)$\to$[0,∞) is a map which is non-decreasing, continuous and non-negative in (0,∞), ψ(0) = 0 and $\lim_{t\to\infty} \psi(t) = \infty$. Presuppose either Z has the subsequent properties or T is continuous.

1. *If a decreasing sequence $\{v_m\} \to Z$, therefore $gv_m \geq v$ for every m.*
2. *If an increasing sequence $\{s_m\} \to Z$, therefore $gs_m \leq s$ for every m.*

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Then T and g possesses coupled coincidence point.

Proof. Presuppose $s_0, v_0 \in Z$ such that $gs_0 \leq T(s_0, v_0)$ and $gv_0 \geq T(v_0, s_0)$. Since $T(Z \times Z) \subseteq g(Z)$, select $s_1, v_1 \in Z$ thus $gs_1 = T(s_0, v_0)$ and $gv_1 = T(v_0, z_0)$.

Again, take $s_2, v_2 \in Z$ such that $gs_2 = T(s_1, v_1)$ and $gv_2 = T(v_1, s_1)$. As T possesses the property of mixed g- monotone, we have $gs_0 \leq gs_1 \leq gs_2$ and $gv_2 \leq gv_1 \leq gv_0$. Persistent the same procedure, we can create $\{z_m\}$ and $\{v_m\}$ in Z such that

$$gs_m = T(s_{m-1}, v_{m-1}) \leq gs_{m+1} = T(s_m, v_m)$$

and

$$gv_{m+1} = T(v_m, s_m) \leq gv_m = T(v_{m-1}, s_{m-1}).$$

If, for some integer m, we have $(gs_{m+1}, gv_{m+1}) = (gs_m, gv_m)$, then $T(s_m, v_m) = gs_m$ and $T(v_m, s_m) = gv_m$,thus T and g has a coincidence point (s_m, v_m).
We presume that $(gs_{m+1}, gv_{m+1}) \neq (gs_m, gv_m)$ for all $m \in \mathbb{N}$, that is, we assume that either $gs_{m+1} \neq gs_m$ or $gv_{m+1} \neq gv_m$. we have,

$$\begin{aligned} p(gs_{m+1}, gs_m) &= p(T(s_m, v_m), T(s_{m-1}, v_{m-1})) \\ &\leq p(gs_m, gs_{m-1}) - \psi(p(gv_m, gv_{m-1})) \\ &\quad + L\ min\{p(gs_m, T(s_m, v_m)), p(gs_{m-1}, T(s_{m-1}, v_{m-1})), \\ &\qquad\qquad p(gs_m, T(gs_{m-1}, v_{m-1})), p(gs_{m-1}, T(s_m, v_m))\} \\ &= p(gs_m, gs_{m-1}) - \psi(p(gv_m, gv_{m-1})) \end{aligned} \tag{2.2}$$

similarly,

$$p(gv_{m+1}, gv_m) \leq p(gv_m, gv_{m-1}) - \psi(p(gs_m, gs_{m-1})) \tag{2.3}$$

Let $\delta_m = p(gs_{m+1}, gs_m) + p(gv_{m+1}, gv_m)$.
Add (2) and (3), we have

$$\delta_m \leq \delta_{m-1} - \psi(\delta_{m-1}) \tag{2.4}$$

If $\exists m_1 \in N^*$ s.t. $p(gs_{m_1}, gs_{m_1-1}) = 0, p(gv_{m_1}, gv_{m_1-1}) = 0$, then $gs_{m_1-1} = gs_{m_1} = T(gs_{m_1-1}, gv_{m_1-1}); gv_{m_1-1} = gv_{m_1} = T(gv_{m_1}, gs_{m_1-1})$ and T has coupled coincidence point and the evidence is done. In other case $p(gs_{m+1}, gs_m) \neq 0$; $p(gv_{m+1}, gv_m) \neq 0$ for every $m \in \mathbb{N}$. At that point utilizing presumption on ψ , we get

$$\delta_m \leq \delta_{m-1} - \psi(\delta_{m-1}) \leq \delta_{m-1} \tag{2.5}$$

δ_m is a positive sequence and possesses a limit δ^*. Take limit $m \to \infty$, we have

$$\delta^* \leq \delta^* - \psi(\delta^*)$$

Thus $\psi(\delta^*) = 0$, utilizing supposition on ψ , we accomplished $\delta^* = 0$, ie. $\lim_{m\to\infty}(\delta_m) = 0$

$$\lim_{m\to\infty} p(s_{m+1}, s_m) + p(v_{m+1}, v_m) = 0$$
$$\implies \lim_{m\to\infty} p(s_{m+1}, s_m) = \lim_{m\to\infty} p(v_{m+1}, v_m) = 0 \tag{2.6}$$

We will show that $\{gs_m\}$, $\{gv_m\}$ are Cauchy groupings in Z. Assume that in any event one $\{gs_m\}$ or $\{gv_m\}$ be not a Cauchy sequence. At that point there exists $\epsilon > 0$ and two subsequence $m_k > n_k \geq k$ such that

$$r_k = p(gs_{m_k}, gs_{n_k}) + p(gv_{m_k}, gv_{n_k}) \geq \epsilon, \tag{2.7}$$

$\forall\ k = 1, 2, 3, \ldots$. Further, relating to n_k, select m_k such that it is smallest integer $m_k > n_k \geq k$ gratify (2.7), we have

$$p(gs_{m_k}, gs_{n_k}) + p(gv_{m_k}, gv_{n_k}) < \epsilon. \tag{2.8}$$

Using triangle inequality and (2.7) and (2.8), we get

$$\begin{aligned}\epsilon \leq r_k &= p(gs_{m_k}, gs_{n_k}) + p(gv_{m_k}, gv_{n_k}) \\ &\leq p(gs_{m_k}, gs_{n-1_k}) + p(gs_{n-1_k}, gs_{n_k}) + p(gv_{m_k}, gv_{n-1_k}), p(gv_{n-1_k}, gv_{n_k}) \\ &< \epsilon + \delta_{m_{k-1}}\end{aligned}$$

Let $k \to \infty$ and taking equation (2.6), we have $\lim_{n,m\to\infty} r_k = \epsilon > 0$.
Now, we get

$$\begin{aligned}p(gs_{m_{k+1}}, gs_{n_{k+1}}) &= p(T(gs_{m_k}, gv_{m_k}), T(gs_{n_k}, gv_{n_k})) \\ &\leq p(gs_{m_k}, gs_{n_k}) - \psi(p(gv_{m_k}, gv_{n_k}) + L\ min\{p(gs_{m_k}, T(gs_{m_k}, gv_{m_k})), p(gs_{n_k}, T(gs_{n_k}, gv_{n_k})), \\ &\qquad p(gs_{n_k}, T(gs_{n_k}, gv_{n_k})), p(gs_{m_k}, T(gs_{m_k}, gv_{m_k}))\} \\ &\leq p(gs_{m_k}, gs_{n_k}) - \psi(p(gv_{m_k}, gv_{n_k}).\end{aligned} \tag{2.9}$$

Similarly,

$$p(gv_{m_{k+1}}, gv_{n_{k+1}}) \leq p(gv_{m_k}, gv_{n_k}) - \psi(p(gs_{m_k}, gs_{n_k}). \tag{2.10}$$

Using (2.9) and (2.10), we get

$$r_{k+1} \leq r_k - \psi(r_k) \tag{2.11}$$

$\forall\ k \in 1, 2, 3, \ldots$ take $k \to \infty$ in equation (11).

$$\epsilon = \lim_{k\to\infty} r_{k+1} \leq \lim_{k\to\infty} [r_k - \psi(r_k)] < \epsilon. \tag{2.12}$$

a contraction. Thus $\{gs_m\}$ and $\{gv_m\}$ are Cauchy sequence.
Using lemma, $\{gs_m\}$ and $\{gv_m\}$ are Cauchy sequence in (Z, p^t). As, (Z, p) is complete, thus (Z, p^t) is complete, so $\exists\ s, v \in Z$
$\lim_{m\to\infty} p^t(gs_m, s) = \lim_{m\to\infty} p^t(v_m, v) = 0$
By lemma, we get

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$$p(s,s) = \lim_{m\to\infty} p(gs_m, s) = \lim_{m\to\infty} p(gs_m, gs_m)$$
$$p(v,v) = \lim_{m\to\infty} p(gv_m, v) = \lim_{m\to\infty} p(gv_m, gv_m)$$

By condition and equation we get $\lim_{m\to\infty} p(gs_m, gs_m) = 0$.
Thus follows as $p(u,u) = \lim_{m\to\infty} p(gs_m, u) = \lim_{m\to\infty} p(gs_m, gs_m) = 0$, similarly $p(v,v) = \lim_{m\to\infty} p(gv_m, v) = \lim_{m\to\infty} p(gv_m, gv_m) = 0$
We now prove that$T(s,v) = s, T(v,s) = v$.

Case1: As Z is a complete, $\exists\, s, v \in Z$
$\lim_{m\to\infty} s_m = s, \lim_{m\to\infty} v = v$ we prove that (s,v) is coupled coincidence point of T and g.

$$\begin{aligned} s &= \lim_{m\to\infty} gs_{m+1} = \lim_{n\to\infty} T(s_m, v_m) = T(\lim_{m\to\infty} s_m, \lim_{m\to\infty} v_m) \\ v &= \lim_{m\to\infty} gv_{m+1} = \lim_{m\to\infty} T(v_m, s_m) = T(\lim_{m\to\infty} v_m, \lim_{m\to\infty} s_m) \end{aligned} \tag{2.13}$$

As g is continuous, we attain

$$\lim_{m\to\infty} g(gs_m) = gs, \ \lim_{m\to\infty} g(gv_m) = gv. \tag{2.14}$$

Commutativity of T and g gives

$$\begin{aligned} g(gs_{m+1}) &= g(T(s_m, v_m)) = T(gs_m, gv_m) \\ g(gv_{m+1}) &= g(T(v_m, s_m)) = T(gv_m, gs_m). \end{aligned} \tag{2.15}$$

By continuity of T, $\{g(gs_{m+1})\}$ is converges to $T(s,v)$ and $\{g(gv_{m+1})\}$ converges to $T(v,z)$. From uniqueness of the limit and (2.14), we accomplish $T(s,v) = gs$ and $T(v,s) = gv$, consequently, T and g possesses a coupled incident point.
Case2: Presuppose that the condition (a) and (b) of the result holds.
The sequence $\{gs_m\} \to s$, $\{gv_m\} \to v$

$$\begin{aligned} p(T(s,v), gs) &\le p(T(s,v), gs_{m+1}) + p(gs_{m+1}, gs) \\ &= p(T(s,v), T(s_m, v_m)) + p(gs_{m+1}, gs) \\ &\le p(gs, gs_m) - \psi(p(gv, gv_m)) \\ &+ L\ min\{p(gs, T(s,v)), p(gs_m, T(s_m, v_m)), p(gs_m, T(s,v)), p(gs, T(s_m, v_m))\} + p(gs_{m+1}, gs) \end{aligned}$$

Letting $m\to \infty$, we have $p(T(s,v), s) \le 0$
Thus $T(s,v) = s$, correspondingly, in similar way we can prove that $T(v,s) = v$. □

Theorem 2.2. *Presuppose the assumptions of Theorem 3.1 hold. Presuppose there exists $z \in Z$ which is comparable to s and v for every $s, v \in Z$. Thus T and g possesses only one coupled coincidence point.*

Proof. Succeeding the proof of Theorem 3.1, the arrangement of coupled coincidence points of T and g is non-empty. We will prove that coupled coincidence points are (s, v) and $(\acute{s}, \acute{v})$, then

$$g(s) = T(s, v),\ g(v) = T(v, s)$$
$$\text{and } g(\acute{s}) = T(\acute{s}, \acute{v}),\ g(\acute{v}) = T(\acute{v}, \acute{s}),$$

then

$$gs = g\acute{s} \text{ and } gv = g\acute{v}. \tag{2.16}$$

Select $(d, z) \in Z \times Z$ comparable with both.

Let $d_0 = d, z_0 = z$ and choose $d_1, z_1 \in Z$ so that $gd_1 = T(d_0, z_0)$ and $gz_1 = T(z_0, d_0)$.
Then, similarly to the evidence of Theorem 3.1, we can inductively define sequences $\{gd_m\}$ and $\{gz_m\}$ as follows

$$gd_{m+1} = T(d_m, z_m) \ \ and \ \ gz_{m+1} = T(z_m, d_m).$$

Since $(gs, gv) = (T(s, v), T(v, s))$ and $(T(d, z), T(z, d)) = (gd_1, gz_1)$ are comparable, then $gs \leq gd_1$ and $gv \geq gz_1$. It is easy to prove using the mathematical induction,

$$gs \leq gd_m \ \ gv \geq gz_m \ \ \forall\, m \in \mathbb{N}.$$

Now, from the contractive condition (1)

$$\begin{aligned} p(gs, gs_{m+1}) &= p(T(s, v), T(s_m, v_m)) \\ &\leq p(gs, gs_m) - \psi(p(gv, gv_m)) \\ &\quad + L\ min\{p(gs, T(s, v)), p(gs_m, T(s_m, v_m)), p(gs_m, T(s, v)), p(gs, T(s_m, v_m))\} \\ &\leq p(gs, gs_m) - \psi(p(gv, gv_m)) \end{aligned} \tag{2.17}$$

Similarly

$$p(gv, gv_{m+1}) = p(gv, gv_m) - \psi(p(gs, gs_m)) \tag{2.18}$$

Adding (2.17) and (2.18), we get

$$p(gs, gs_{m+1}) + p(gv, gv_{m+1}) \leq p(gs, gs_m) + p(gv, gv_m) - [\psi(p(gv, gv_m)) + \psi(p(gs, gs_m))] \tag{2.19}$$

This implies

$$p(gs, gs_{m+1}) + p(gv, gv_{m+1}) \leq p(gs, gs_m) + p(gv, gv_m) \tag{2.20}$$

Thus, the sequence is non-increasing. hence, there exist $\alpha \geq 0$.

$$\lim_{m\to\infty} p(gs, gs_m) + p(gv, gv_m) = \alpha \tag{2.21}$$

We shall prove that $\alpha = 0$. Presuppose in contrary, $\alpha > 0$. Take $m \to \infty$ in equation (2.21), we have

$$\alpha \leq \alpha - \psi(\alpha) < \alpha \tag{2.22}$$

a contradiction. Therefore, $\alpha = 0$, that is

$$\lim_{m\to\infty} p(gs, gs_m) + p(gv, gv_m) = 0.$$

It implies

$$\lim_{m\to\infty} p(gs, gs_m) = \lim_{m\to\infty} p(gv, gv_m) = 0.$$

Similarly, we can prove

$$\lim_{m\to\infty} p(g\acute{s}, gs_m) = \lim_{m\to\infty} p(g\acute{v}, gv_m) = 0.$$

From last equalities, we have $gs = g\acute{s}$ and $gv = g\acute{v}$. □

Example 2.3. Presume $Z = [0, 1]$ with usual partial metric p defined as $p : Z \times Z \to$ [0,1] with $p(s, v) = max(s, v)$. The (Z, p) is complete partial metric space for any $s, v \in Z$.

$$p(s, v) = |s - v|$$

Thus (Z, p^t) is complete Euclidean metric space.
Presume the mapping $T : Z \times Z \to Z$ given as $T(s, v) = \frac{2s-v}{4}; s \geq v$

Take $\psi : [0, \infty) \to [0, \infty)$ such that $\psi(t) = \frac{t}{4}$

As, T has the property of mixed g-monotone property and is continuous.
Now, we discuss the following possibilities for (s, v) and (u, z) with $gs \leq gu,\ gv \geq gz$

Case 1- If $(s, v) = (u, z) = (0, 0)$
Then clearly $p(T(s, v), T(u, z)) = 0$
Thus (1) holds.
Case 2- If $(s, v) = (u, z) = (1, 0)$
Then LHS of (1)
$= p(T(s, v), T(u, z)) = p(T(1, 0), T(1, 0)) = p(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}$,
which is less than RHS of (1)
Thus (1) holds.
Case 3- If $(s, v) = (u, z) = (1, 1)$
Thus (1) holds.
Case 4- If $(s, v) = (1, 0); (u, z) = (0, 0)$
Thus (1) holds.
Case5- If $(s, v) = (1, 0); (u, z) = (1, 1)$
Thus (1) holds.

Therefore, all the properties of Theorem 3.1 are gratified.
Also, g and T possesses unique coupled coincidence point as $(0, 0)$.

Conclusions

As introduced toward the start of this work, Bhaskar and Laksmikantham, stretch out this hypothesis to partially ordered metric spaces and present the idea of coupled fixed point for mixed-monotone map.

Acquiring results as concerns the presence and the uniqueness of certain coupled coincidence point hypotheses for a map possesses the property of mixed g-monotone in partial metric spaces.

Acknowledgement

The authors express their sincere thanks to the reviewers and editors for their valuable comments and suggestions.

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