Fixed Point Theorems for Single valued mapping and Set Valued Mapping in Fuzzy Metric spaces through α -admissible approach

Sushma Yadav[1*], Kamal Wadhwa[2]

[1,2]Department of Mathematics, Govt. Narmada PG College Narmadapuram, [M.P], India
Email: yadavsushma615@gmail.com[1], wadhwakamal68@gmail.com[2]
*Corresponding Author

Received: 13.07.2024 Revised: 12.08.2024 Accepted: 22.09.2024

ABSTRACT
Fuzzy metric spaces are an extension of classical metric spaces that incorporate the concept of fuzziness into the measurement of distance. It provides a flexible framework for analyzing situations where precision is not feasible or desirable. It offers valuable insights in areas requiring the handling of vague or imprecise information, broadening the scope of traditional mathematical analysis. The present manuscript deals with establishment of fixed point and α -fuzzy fixed point theorems for Single valued mapping and Set Valued Mapping in Fuzzy Metric spaces through α-admissible approach, which is an extension and generalization of the results of many authors with new rational contractive conditions. Established results are generalizations of previous known results in the field of fixed point theory concerning to fuzzy metric spaces.

Keywords: Single valued Mapping, Set valued mapping,α-admissible, fixed point, t-norms.

1. INTRODUCTION

Fixed point theory is essential to the advancement of many disciplines and has applications in computer science, physics, medicine, applied science, and other subjects. Fuzzy machines were developed in the current situation using the fixed point theory fuzzy locking system concept.The fixed point theorem for contractive mapping in complete metric space was proved by Banach [3] in 1922.After Zadeh[8] first proposed the idea of a fuzzy set in 1969, other authors were able to produce fixed point solutions for fuzzy mapping. Using continuous t-norms, Kramosil and Michalek [18] introduced the novel concept of fuzzy metric space in 1975. George and Veeramani[7] have modified these results.Heilpern[10] established a few fixed point findings for contraction mapping in 1981.In 2008, A few fixed point theorems for fuzzy metric spaces were proven by Mihet [26], who also introduced the idea of $\boldsymbol{\psi}$-contractive mapping. The (α-ψ)-contractive and α-admissible mapping was provided by Samet et al. (2012)[23], who also developed certain fixed point theorems.(α-ψ)-contractive for set valued mapping in fuzzy metric spaces was introduced by Hong[11] in 2014. A novel notion, (α-ψ)-fuzzy contraction, was presented and several new fixed point results on entire fuzzy metric space were proven by Saha et al. in 2015[4]. Fixed point and common fixed point results for multi-valued mapping in b-metric space were proved in 2016 by Joseph[14] and in 2017 by Jinakul [13].Several fixed point findings for single valued and set valued mapping were demonstrated by Vishal Gupta et al. in 2018 [9]. Some more results on related to fuzzy sets can be viewed in [23-30]. In order to examine fuzzy fixed point results for single-valued and set-valued mappings, we use two generalized contractions with novel including rational expressions in the context of fuzzy metric spaces in this paper. We have incorporated examples and applications that highlight and validate our acquired outcomes. Our results have generalized numerous fixed point results found in the relevant literature, as fuzzy mapping is a generalization of multi-valued mapping. The established results are specially motivated by [9]. Also concepts are taken from [11-18].

Preliminaries

Definition 2.1[9]A map $*: [0,1] \times [0,1] \to [0,1]$ is called continuous triangular norm (In short t- norms), if it's satisfied following condition for $a, b, c, d \in [0,1]$:

- **(i)** $a * b = b * a$ (Symmetry);
- **(ii)** $a * b \leq c * d$ if $a \leq c$ and $b \leq d$ (Monotonicity);
- **(iii)** $a * (b * c) = (a * b) * c$ (Associativity);

(iv) $1 * a = a$ (boundary condition);

Definition 2.2[15] Let a triplet $(\Omega, \mathcal{M}, *)$is called FMS if Ω is an arbitrary set, $*$ is t-norm and $\mathcal{M}$ is a FS on $\Omega \times \Omega \times [0, \infty)$ such that for all $x, y, z \in \Omega$and $p, q \geq 0$ then

1. $\mathcal{M}(x, y, 0) = 0$;
2. $\mathcal{M}(x, y, t) = 1, \forall t > 0$ *if and only if* $x = y$;
3. $\mathcal{M}(x, y, t) = \mathcal{M}(y, x, t)$;
4. $\mathcal{M}(x, z, p + q) \geq \mathcal{M}(x, y, p) * \mathcal{M}(y, z, q)$;
5. $\mathcal{M}(x, y, .): (0, \infty) \to [0,1]$ is continuous.

Example 1: Let(Ω, d) be a metric space, Define $u * v = \min\{u, v\}$(or $u * v = uv$) for all $u, v \in [0,1]$. then fuzzy metric may define as $\mathcal{M}(u, v, t) = \frac{t}{t+d(x,y)}$ for all $x, y \in \Omega$ and $t > 0$;

Example 2: Let (Ω, d) be a bounded metric space with $d(u, v) < \alpha$, where α is fixed point constant in$(0, \infty)$andW: $R^+ \to (\alpha, \infty)$be an increasing continuous function. define a function $\mathcal{M}: \Omega \times \Omega \times (0, \infty) \to [0,1]$as $\mathcal{M}(x, y, t) = 1 - \frac{d(x,y)}{w(t)}$; $x, y \in \Omega, t > 0$;

Definition 2.4[15]: Let $(\Omega, d,*)$ be fuzzy metric space then a sequence $\{x_n\}$ in a FMS $(\Omega, d,*)$ to a point $x \in \Omega$ if$\lim_{n\to\infty} \mathcal{M}(x_n, x, t) = 1$ for all $t > 0$;

Definition 2.5[9] Let $(\Omega, d,*)$ be fuzzy metric space then a sequence $\{x_n\}$ in a (Ω, d, t) is called Cauchy sequence if and only if for all $\epsilon \in (0,1)$ and $t > 0$ there exists $n_0 \in N$ such that

$$\lim_{n\to\infty} \mathcal{M}(x_n, x_m, t) = 1 - \epsilon, \text{for all } n, m > n_0;$$

Definition 2.6[9] A fuzzy metric space in which every Cauchy sequence is convergent is complete.

Definition 2.7 A single-valued mapping is one in which the range contains a unique value for every point in the domain. As a result, it is many-to-one or one-to-one.

Definition 2.8 A set-valued mapping, also known as a correspondence, is a type of mathematical function that transfers items from one function domain (a set) to sub-domains of another set. Multi-valued mapping is another term for it.

Definition 2.9[9] Let $(\Omega, \mathcal{M},*)$ be fuzzy metric space and let $f: \Omega \to \Omega$ and $\alpha: \Omega \times \Omega \times (0, \infty) \to (0, \infty)$ be two mapping then a mappingf is called α-admissible if:

$$\alpha(u, v, t) \leq 1 \Rightarrow \alpha(fu, fv, t) \leq 1, u, v \in \Omega \text{ and } t > 0.$$

Definition 2.10[9] Let $(\Omega, \mathcal{M},*)$ be fuzzy metric space and let $f: \Omega \to \Omega$ and $\alpha: \Omega \times \Omega \times (0, \infty) \to (0, \infty)$ be two mapping then a mappingf is called α-admissible mapping approach to η if:

$$\alpha(u, v, t) \leq \eta(u, v, t) \Rightarrow \alpha(fu, fv, t) \leq \eta(fu, fv, t), u, v \in \Omega \text{ and } t > 0.$$

Note: (a) if $\eta(u, v, t) = 1$ then this definition reduce to the definition 2.7.

(b) if $\alpha(u, v, t) = 1$ then mapping f is called η-Sub admissible mapping

Definition 2.11[9]Leta family of function $\boldsymbol{\psi}$ such that $\varphi: [0,1] \to [0,1]$ is continuous non-decreasing and $\varphi(\wp) > p$ *for each* $\wp \in [0,1)$.

Lemma 2.12 [9]Let a family of function $\boldsymbol{\psi}$ and $\varphi(\wp) > p$ *for each* $\wp > 0$ if and only if $\lim_{n\to\infty} \varphi^n(\wp) = 1$, for each $\wp \in [0,1)$, $\varphi^n$is n-th iterate of φ.

Definition 2.13[15]Let $(\Omega, d,*)$ be fuzzy metric space. We define the Housdorfffuzzymetric space H on $\eta(\Omega) \times \eta(\Omega) \times (0, \infty)$ to $[0,1])$id defined as:

$$H(A, B, \sigma) = \min\{\inf_{x\in A} \sup_{y\in B} d(x, y, \sigma), \inf_{y\in B} \sup_{x\in a} d(x, y, \sigma)\},$$

For all $A, B \in \eta(\Omega)$, where$\eta(\Omega)$ is the collection of all non-empty compact subsets of $\Omega$and $d(x, B) = \inf\{d(x, a): a \in B\}$, for all $x \in \Omega$.

A Fuzzy set X is a function with domain X and values in [0, 1] ,F(X) is the collection of all fuzzy sets in X. If A isa fuzzy set and $x \in X$, then the function value A(x) is called grade of membership or membership value of x in X. The α −cut of a fuzzy set A, is denoted by $[A]_\alpha$, and defined as:

$$[A]_\alpha = \{x: A(x) \geq \alpha\}, \text{ where} \alpha \in (0,1]$$

$$[A]_0 = \{x: A(x) > 0\},$$

Let X be any nonempty set and Y be a metric space. A mapping T is called a fuzzy mapping, if T is a mapping from X into F(X).A fuzzy mapping T is a fuzzy subset on $X \times Y$with membership function T(x)(y). The function T(x)(y) is the grade of membership of y in T(x). For convenience, we denote theα −cut of T(x) by $[Tx]_\alpha$ instead of $[T(x)]_\alpha$[1].

Definition 2.14(15] A point $x \in X$ is called α-fuzzy fixed point of a fuzzy mapping $T: X \to F(X)$ if there exist $\alpha \in (0,1]$suchthat$x \in [Tx]_\alpha$.

Lemma 2.13 (9]) In a FMS $(\Omega, \mathcal{M},*)$, $(\Omega, \mathcal{M}, .)$is non-decreasing for all $x, y \in \Omega$.

Lemma 2.14 (15]) Let $(\eta(\Omega), H,*)$be a Hausdorff fuzzy metric space on$\eta(\Omega)$ if for all $A, B \in \eta(\Omega)$, *for all* $u \in A$, *there exist* $v_u \in B$ satisfies $\mathcal{M}(u, B, t) = \mathcal{M}(u, v_u, \alpha)$Then $H(A, B, \alpha) \leq \mathcal{M}(u, v_u, \alpha)$.

Lemma 2.16 (15]) Let $(\Omega,\mathcal{M},*)$be a complete FMS, if there exist $\delta \in (0,1)$ such that $\mathcal{M}(u,v,\alpha\delta) \geq \mathcal{M}(u,v,\alpha)\ ,for\ all\ u,v \in \Omega\ and\ \alpha \in (0,\infty),\ then\ u = v$.

Lemma 2.17 (15]) Let $(\Omega,\mathcal{M},*)$be a complete FMS. Then, for each $x \in \Omega, B \in \eta(\Omega)$ for $\jmath > 0$ there exists $y_0 \in B$such that $\mathcal{M}(x,y_0,\jmath) = \mathcal{M}(x,B,\jmath)$.

Where, $\eta(\Omega)$ is the collection of all non-empty compact subsets of Ω.

Lemma 2.18(15) Let B be a non-empty subset compact subset of a FMS $(\Omega,\mathcal{M},*)$, for $\partial \in \Omega\ and\ \jmath > 0\ then\ ,\mathcal{M}(x,B,\jmath) = Sup\{\mathcal{M}(x,\partial,\jmath) : \partial \in B\}$.

Now we present our main theorems with new rational contractive conditions.

2. MAIN RESULTS

Theorem (3.1)let $U:\Omega \to \Omega$ is α-admissible single valued mapping approach to η in complete FMS $(\Omega,\mathcal{M},\star)$such that:

$$\alpha(u,v,t) \leq \eta(u,v,t) \Rightarrow \mathcal{M}(Uu,Uv,t) \geq \varphi\left\{Min\left\{\begin{matrix}\mathcal{M}(u,v,t),\mathcal{M}(v,Uv,t),\mathcal{M}(u,Uv)*\mathcal{M}(v,Uv), \\ \left[\dfrac{\{1+\mathcal{M}(u,Uu,t)\}.\mathcal{M}(v,Uv,t)}{1+\mathcal{M}(u,v,t)}\right]\end{matrix}\right\}\right\}$$

(a) There exists $u_0 \in \Omega\ ,such\ that\ \alpha(u_0,Uu_0,t) \leq \eta(u_0,Uu_0,t),t \in (0,1)$.

(b) $U\ is\ converges\ to\ u \in \Omega\ and\quad \alpha(u_n,u_{n+1},t) \leq \eta(u_n,u_{n+1},t),\ and\ t > 0, \alpha(u_n,u,t) \leq \eta(u_n,u,t)$.

Then $U\ have\ a\ fuzzy\ fixed\ point$.

Proof: Let x_0 be an arbitrary point of$\Omega\ such\ that\ \alpha(u_0,Uu_0,t) \leq \eta(u_0,Uu_0,t)$. then define a sequence $\{u_n\}$ in Ω such that $u_n = U^n u_0 = Uu_{n-1}, for\ all\ n \in N$. if $u_n = u_{n+1}$ then existence of fixed point is apparent. Now if $u_n \neq u_{n+1}$. Since U is α-admissible single valued mapping approach to η and $\alpha(u_0,Uu_0,t) \leq \eta(u_0,Uu_0,t)$then

$$\alpha(u_1,u_2,t) = \alpha(Uu_0,U^2u_0,t) \leq \eta(Uu_0,U^2u_0,t) = \eta(u_1,u_2,t)$$

By mathematical induction, $\alpha(u_n,u_{n+1},t) \leq \eta(u_n,u_{n+1},t), for\ all\ n \in N \cup \{0\}$.

Now put $u = u_{n-1}\ and\ \ v = u_n\ in\ inequality\ (3.1)$

$$\mathcal{M}(Uu_{n-1},Uu_n,t) \geq \varphi\left\{Min\left\{\begin{matrix}\mathcal{M}(u_{n-1},u_n,t),\mathcal{M}(u_n,Uu_n,t),\mathcal{M}(u_{n-1},Uu_n)*\mathcal{M}(u_n,Uu_n), \\ \left[\dfrac{\{1+\mathcal{M}(u_{n-1},Uu_{n-1},t)\}.\mathcal{M}(u_n,Uu_n,t)}{1+\mathcal{M}(u_{n-1},u_n,t)}\right]\end{matrix}\right\}\right\}$$

$$\mathcal{M}(u_n,u_{n+1},t) \geq \varphi\left\{Min\left\{\begin{matrix}\mathcal{M}(u_{n-1},u_n,t),\mathcal{M}(u_n,u_{n+1},t),\mathcal{M}(u_{n-1},u_{n+1},t)*\mathcal{M}(u_n,u_{n+1},t), \\ \left[\dfrac{\{1+\mathcal{M}(u_{n-1},u_n,t)\}.\mathcal{M}(u_n,u_{n+1},t)}{1+\mathcal{M}(u_{n-1},u_n,t)}\right]\end{matrix}\right\}\right\}$$

$\mathcal{M}(u_n,u_{n+1},t) \geq \varphi\{Min\{\mathcal{M}(u_{n-1},u_n,t),\mathcal{M}(u_n,u_{n+1},t)\}\}$

If $Min\{\mathcal{M}(u_{n-1},u_n,t),\mathcal{M}(u_n,u_{n+1},t)\} = \mathcal{M}(u_n,u_{n+1},t)$ then

$\mathcal{M}(u_n,u_{n+1},t) \geq \varphi\{\mathcal{M}(u_n,u_{n+1},t)\} > M(u_n,u_{n+1},t)$, which is contradiction.

If $Min\{\mathcal{M}(u_{n-1},u_n,t),\mathcal{M}(u_n,u_{n+1},t)\} = \mathcal{M}(u_{n-1},u_n,t)$ then

$\mathcal{M}(u_n,u_{n+1},t) \geq \varphi\{\mathcal{M}(u_{n-1},u_n,t)\} > M(u_{n-1},u_n,t)$

Now for all $p > q,\ p,q \in N$ then

$$\mathcal{M}(u_n,v_m,t) \geq \mathcal{M}(u_n,v_{n+1},t_{n+1}) * \mathcal{M}(u_{n++1},v_{n+2},t_{n+2}) * \dots * \mathcal{M}(u_{m-1},v_m,t_m)$$

$$\mathcal{M}(u_n,v_m,t) \geq \sum_{i=n}^{m-1} \mathcal{M}(u_i,v_{i+1},t_{i+1})$$

$$\mathcal{M}(u_n,v_m,t) \geq \sum_{i=n}^{m-1} \varphi^i\big(\mathcal{M}(u_0,v_1,t_i)\big)$$

$$\sum_{i=n+1}^{m} t_i = t\ and\ t_i > 0,\ \ i = n+1, n+2 \dots \dots m.$$

Then by lemma (2.12), suppose$\varphi^i\big(\mathcal{M}(u_0,v_1,t_i)\big) > 1 - \frac{1}{2^i}, i = large\ enough$.

We know that by Cauchy series $\sum_{i=1}^{\infty}\frac{1}{2^i}$ is convergent therefore $\sum_{i=n}^{m-1} 1 - \frac{1}{2^i}$ is also convergent. i.e $\sum_{i=n}^{m-1} 1 - \frac{1}{2^i} = 1$. hence the sequence $\{u_n\}$ is Cauchy sequence and $(\Omega,\mathcal{M},\star)$ is complete fuzzy metric space.Therefore, there exist $\vartheta \in \Omega$ such that $u_n \to \vartheta$. if assume that U is a continuous function then $U\vartheta = lim_{n\to\infty} Uu_n = lim_{n\to\infty} u_{n+1} = \vartheta$. hence ϑ is a fixed point.

Also $\alpha(u_n,u_{n+1},t) \leq \eta(u_{n-1},u_{n+1},t), for\ all\ n \in N \cup \{0\}$

$\alpha(u_n,\vartheta,t) \leq \eta(u_n,\vartheta,t)$

Now using condition (a) in inequality (3.1) then we get

$$\mathcal{M}(U\vartheta, Uu_n, t) \geq \varphi\left\{Min\left\{\begin{matrix}\mathcal{M}(\vartheta, u_n, t), \mathcal{M}(u_n, Uu_n, t), \mathcal{M}(\vartheta, Uu_n) * \mathcal{M}(u_n, Uu_n), \\ \left[\dfrac{\{1+\mathcal{M}(\vartheta, U\vartheta, t)\}.\mathcal{M}(u_n, Uu_n, t)}{1+\mathcal{M}(\vartheta, u_n, t)}\right]\end{matrix}\right\}\right\}$$

$$\mathcal{M}(U\vartheta, u_{n+1}, t) \geq \varphi\left\{Min\left\{\begin{matrix}\mathcal{M}(\vartheta, u_n, t), \mathcal{M}(u_n, u_{n+1}, t), \mathcal{M}(\vartheta, u_{n+1}) * \mathcal{M}(u_n, u_{n+1}), \\ \left[\dfrac{\{1+\mathcal{M}(\vartheta, U\vartheta, t)\}.\mathcal{M}(u_n, u_{n+1}, t)}{1+\mathcal{M}(\vartheta, u_n, t)}\right]\end{matrix}\right\}\right\}$$

Taking limit as $n \to \infty$ on both sides then we get
$\mathcal{M}(U\vartheta, \vartheta, t) \geq \varphi(\mathcal{M}(\vartheta, U\vartheta, t)) > M(\vartheta, U\vartheta, t)$
$\mathcal{M}(\vartheta, U\vartheta, t) = 1 \Rightarrow U\vartheta = \vartheta.$

Theorem (3.2) let $(\Omega, \mathcal{M}, \star)$be a complete FMS and $U: \Omega \to \mathcal{M}(\Omega)$ beset valued mappingfor all $x, y \in \Omega$, $\alpha \in (0,1]$ and $h \in (0,1)$if satisfying the conditions:

(a) $lim_{t\to\infty} \mathcal{M}(x, y, t) = 1.$

(b) $H([Ux]_{\alpha(x)}, [Uy]_{\alpha(y)}, ht) \geq$

$$Min\left\{\begin{matrix}\mathcal{M}(x, y, t), \mathcal{M}(x, [Ux]_{\alpha(x)}, t), \mathcal{M}(y, [Ux]_{\alpha(x)}, t), \\ \left[\dfrac{\{1+\mathcal{M}(y, [Ux]_{\alpha(x)}, t)\}.\mathcal{M}(y, [Uy]_{\alpha(y)}, t)}{1+\mathcal{M}(y, [Uy]_{\alpha(y)}, t)}\right]\end{matrix}\right\}$$

Then U $have\ a\ fuzzy\ fixed\ point$.

Proof: Let x_0 be an arbitrary point ofΩ and $let\ x_1 \in [Ux_0]_{\alpha(x_0)}$ then by lemma [2.16], we may choose$x_2 \in [Ux_1]_{\alpha(x_1)}$

$$such\ that\ \mathcal{M}(x_1, x_2, t) \geq H\left([Ux_0]_{\alpha(x_0)}, [Ux_1]_{\alpha(x_1)}, \frac{t}{h}\right)$$

.By induction we can write $x_{k+1} \in [Ux_k]_{\alpha(x_k)}$ $, for\ all\ k \in N$, satisfying
$\mathcal{M}(x_k, x_{k+1}, t) \geq H([Ux_{k-1}]_{\alpha(x_{k-1})}, [Ux_k]_{\alpha(x_k)}, t), for\ all\ k \in N,$
Now ,
$\mathcal{M}(x_2, x_3, t) \geq H([Ux_1]_{\alpha(x_1)}, [Ux_2]_{\alpha(x_2)}, t)$
By using inequality 3.2(b) we get

$$\mathcal{M}(x_2, x_3, t) \geq Min\left\{\begin{matrix}\mathcal{M}\left(x_1, x_2, \frac{t}{h}\right), \mathcal{M}\left(x_1, [Ux_1]_{\alpha(x_1)}, \frac{t}{h}\right), \mathcal{M}\left(x_2, [Ux_1]_{\alpha(x_1)}, \frac{t}{h}\right), \\ \left[\dfrac{\left\{1+\mathcal{M}\left(x_2, [Ux_1]_{\alpha(x_1)}, \frac{t}{h}\right)\right\}.\mathcal{M}(x_2, [Ux_1]_{\alpha(x_1)}, \frac{t}{h})}{1+\mathcal{M}(x_2, [Ux_2]_{\alpha(x_2)}, \frac{t}{h})}\right]\end{matrix}\right\}$$

$$\geq Min\left\{\begin{matrix}\mathcal{M}\left(x_1, x_2, \frac{t}{h}\right), \mathcal{M}\left(x_1, x_2, \frac{t}{h}\right), \mathcal{M}\left(x_2, x_3, \frac{t}{h}\right), \\ \left[\dfrac{1+\mathcal{M}\left(x_2, x_3, \frac{t}{h}\right).\mathcal{M}(x_2, x_3, \frac{t}{h})}{1+\mathcal{M}(x_2, x_3, \frac{t}{h})}\right]\end{matrix}\right\}$$

$$\geq Min\left\{\mathcal{M}\left(x_1, x_2, \frac{t}{h}\right), \mathcal{M}\left(x_2, x_3, \frac{t}{h}\right)\right\}$$

If $Min\left\{\mathcal{M}\left(x_1, x_2, \frac{t}{h}\right), \mathcal{M}\left(x_2, x_3, \frac{t}{h}\right)\right\} = \mathcal{M}\left(x_2, x_3, \frac{t}{h}\right)$ then
We have $\mathcal{M}(x_2, x_3, t) \geq \mathcal{M}\left(x_2, x_3, \frac{t}{h}\right)$
So, by lemma 6 nothing left to prove. Now if we have
$Min\left\{\mathcal{M}\left(x_1, x_2, \frac{t}{h}\right), \mathcal{M}\left(x_2, x_3, \frac{t}{h}\right)\right\} = \mathcal{M}\left(x_1, x_2, \frac{t}{h}\right)$ then
Then again by lemma 2.16, we have $\mathcal{M}(x_2, x_3, t) \geq \mathcal{M}\left(x_1, x_2, \frac{t}{h}\right)$
Again $\mathcal{M}(x_2, x_3, t) \geq H\left([Ux_0]_{\alpha(x_0)}, [Ux_1]_{\alpha(x_1)}, \frac{t}{h}\right)$

$$\geq Min\left\{\begin{matrix}\mathcal{M}\left(x_0, x_1, \frac{t}{h^2}\right), \mathcal{M}\left(x_0, [Ux_0]_{\alpha(x_0)}, \frac{t}{h^2}\right), \mathcal{M}\left(x_1, [Ux_0]_{\alpha(x_0)}, \frac{t}{h^2}\right), \\ \left[\dfrac{\left\{1+\mathcal{M}\left(x_1, [Ux_0]_{\alpha(x_0)}, \frac{t}{h^2}\right)\right\}.\mathcal{M}(x_1, [Ux_0]_{\alpha(x_0)}, \frac{t}{h^2})}{1+\mathcal{M}(x_1, [Ux_1]_{\alpha(x_1)}, \frac{t}{h^2})}\right]\end{matrix}\right\}$$

$$\geq Min\left\{\begin{array}{c}\mathcal{M}\left(x_0,x_1,\frac{t}{h^2}\right),\mathcal{M}\left(x_0,x_1,\frac{t}{h^2}\right),\mathcal{M}\left(x_1,x_2,\frac{t}{h^2}\right),\\ \left[\frac{1+\mathcal{M}\left(x_1,x_2,\frac{t}{h^2}\right).\mathcal{M}(x_1,x_2,\frac{t}{h^2})}{1+\mathcal{M}(x_1,x_2,\frac{t}{h^2})}\right]\end{array}\right\}$$

$$\geq Min\left\{\mathcal{M}\left(x_0,x_1,\frac{t}{h^2}\right),\mathcal{M}\left(x_1,x_2,\frac{t}{h^2}\right)\right\}$$

If $Min\left\{\mathcal{M}\left(x_0,x_1,\frac{t}{h^2}\right),\mathcal{M}\left(x_1,x_2,\frac{t}{h^2}\right)\right\}=\mathcal{M}\left(x_1,x_2,\frac{t}{h^2}\right)$ then

We have $\mathcal{M}(x_2,x_3,t)\geq\mathcal{M}\left(x_1,x_2,\frac{t}{h^2}\right)$

So, by lemma 2.17 nothing left to prove. Now if we have

$Min\left\{\mathcal{M}\left(x_0,x_1,\frac{t}{h^2}\right),\mathcal{M}\left(x_1,x_2,\frac{t}{h^2}\right)\right\}=\mathcal{M}\left(x_0,x_1,\frac{t}{h}\right)$then

Then again by lemma 2.16, we have $\mathcal{M}(x_2,x_3,t)\geq\mathcal{M}\left(x_0,x_1,\frac{t}{h^2}\right)$

In the same manner, $\mathcal{M}(x_n,x_{n+1},t)\geq\mathcal{M}\left(x_0,x_1,\frac{t}{h^n}\right)$

Now let $p=n+m$, $(p>n)$ then we have

$$\mathcal{M}(x_n,x_{n+m},t)\geq\mathcal{M}\left(x_n,x_{n+1},\frac{t}{m}\right)*\ldots..*\mathcal{M}\left(x_{n+m-1},x_{n+m},\frac{t}{m}\right),\ \ m-times$$

By applying above results then we get

$$\mathcal{M}(x_n,x_{n+m},t)\geq\mathcal{M}\left(x_0,x_1,\frac{t}{mh^n}\right)*\ldots..*\mathcal{M}\left(x_0,x_1,\frac{t}{mh^n+m-1}\right),\ \ m-times$$

Letting limit as $n\to\infty$ and using inequality 3.2(a) then we get

$lim_{n\to\infty}\mathcal{M}(x_n,x_{n+m},t)=1,$

Hence $\{x_n\}$ follows Cauchy sequence. So by completeness there exists $\vartheta\in\Omega$ $such\ that\ lim_{n\to\infty}x_n=\vartheta$.

Now to show ϑ is a fuzzy fixed point of U. we have

$\mathcal{M}(\vartheta,[U\vartheta]_{\alpha(\vartheta)},t)\geq\mathcal{M}(\vartheta,x_{n+1},(1-h)t)*\mathcal{M}(x_{n+1},[U\vartheta]_{\alpha(\vartheta)},ht)$

$\mathcal{M}(\vartheta,[U\vartheta]_{\alpha(\vartheta)},t)\geq\mathcal{M}(\vartheta,x_{n+1},(1-h)t)*H([Ux_n]_{\alpha(x_n)},[U\vartheta]_{\alpha(\vartheta)},ht)$

$\mathcal{M}(\vartheta,[U\vartheta]_{\alpha(\vartheta)},t)\geq\mathcal{M}(\vartheta,x_{n+1},(1-h)t)*$

$$\geq Min\left\{\begin{array}{c}\mathcal{M}(x_n,\vartheta,t),\mathcal{M}\left(x_n,[Ux_n]_{\alpha(x_n)},t\right),\mathcal{M}\left(\vartheta,[Ux_n]_{\alpha(x_n)},t\right),\\ \left[\frac{\left\{1+\mathcal{M}\left(x_n,[Ux_n]_{\alpha(x_n)},t\right)\right\}.\mathcal{M}(\vartheta,[Ux_n]_{\alpha(x_n)},t)}{1+\mathcal{M}(\vartheta,[U\vartheta]_{\alpha(\vartheta)},t)}\right]\end{array}\right\}$$

Taking $n\to\infty$ $then\ we\ get$

$$\mathcal{M}(\vartheta,[U\vartheta]_{\alpha(\vartheta)},t)\geq\mathcal{M}(\vartheta,x_{n+1},(1-h)t)*\geq Min\left\{\begin{array}{c}\mathcal{M}(\vartheta,\vartheta,t),\mathcal{M}(\vartheta,[U\vartheta]_{\alpha(\vartheta)},t),\mathcal{M}(\vartheta,[U\vartheta]_{\alpha(\vartheta},t),\\ \left[\frac{\{1+\mathcal{M}(\vartheta,[U\vartheta]_{\alpha(\vartheta)},t)\}.\mathcal{M}(\vartheta,[U\vartheta]_{\alpha(\vartheta},t)}{1+\mathcal{M}(\vartheta,[U\vartheta]_{\alpha(\vartheta)},t)}\right]\end{array}\right\}$$

$\mathcal{M}(\vartheta,[U\vartheta]_{\alpha(\vartheta)},t)\geq Min\{\mathcal{M}(\vartheta,[U\vartheta]_{\alpha(\vartheta)},t),1,1,1\}$

If $Min\{\mathcal{M}(\vartheta,[U\vartheta]_{\alpha(\vartheta)},t),1,1,1\}=1$ then we get

$\mathcal{M}(\vartheta,[U\vartheta]_{\alpha(\vartheta)},t)\geq 1$

Hence we get ϑ is a fuzzy fixed point of U.

If $Min\{\mathcal{M}(\vartheta,[U\vartheta]_{\alpha(\vartheta)},t),1,1,1\}=\mathcal{M}(x_n,[U\vartheta]_{\alpha(\vartheta)},t)$ then we get

$\mathcal{M}(\vartheta,[U\vartheta]_{\alpha(\vartheta)},t)\geq\mathcal{M}(\vartheta,[U\vartheta]_{\alpha(\vartheta)},t)$

Which is a contradiction then we have $\mathcal{M}(\vartheta,[U\vartheta]_{\alpha(\vartheta)},t)=1$, thus $\vartheta=[U\vartheta]_{\alpha(\vartheta)}$

Hence ϑ is a fuzzyfixed point of U.

Remark(*)let uswe define new class of ψ $as\ follows$.

Let ψ be the class of all mapping $\Sigma=\{\psi:[0,1]\to[0,1]\}$ be a collection of all continuous function such that

(a) $\psi(a)>a,\ \forall(0<a<1);$

(b) $\psi(1)=1,\psi(0)=0$;

Theorem (3.3) let $(\Omega,\mathcal{M},\star)$be a complete FMS and $U:\Omega\to\mathcal{M}(\Omega)$ be a set valued mapping for all $x,y\in\Omega$, $\alpha\in(0,1]$ and $h\in(0,1)$if satisfying the conditions:

(a) $lim_{t\to\infty}\mathcal{M}(x,y,t)=1.$

(b) $H\left([Ux]_{\alpha(x)}, [Uy, \square t]_{\alpha(y)}\right) \geq$

$$Min\left\{\begin{array}{c}\psi\left(\mathcal{M}(x,y,t)\right), \psi\left(\mathcal{M}\left(x, [Ux]_{\alpha(x)}, t\right)\right), \psi\left(\mathcal{M}\left(y, [Ux]_{\alpha(x)}, t\right)\right), \\ \left[\dfrac{\left\{1+\psi\left(\mathcal{M}\left(y, [Ux]_{\alpha(x)}, t\right)\right)\right\}.\psi\left(\mathcal{M}\left(y, [Uy]_{\alpha(y)}, t\right)\right)}{1+\psi\mathcal{M}\left(y, [Uy]_{\alpha(y)}, t\right)}\right]\end{array}\right\}$$

Then U □$ave\ a\ \ fuzzy\ fixed\ point$.

Proof: Let x_0 be an arbitrary point ofΩ and $let\ x_1 \in [Ux_0]_{\alpha(x_0)}$ then by lemma [2.14], we may choose$x_2 \in [Ux_1]_{\alpha(x_1)}$

$$suc\square\ t\square at\ \ \mathcal{M}(x_1, x_2, t) \geq H\left([Ux_0]_{\alpha(x_0)}, [Ux_1]_{\alpha(x_1)}, \frac{t}{\square}\right)$$

By induction we can write $x_{k+1} \in [Ux_k]_{\alpha(x_k)}$ $, for\ all\ k \in N$, satisfying

$\mathcal{M}(x_k, x_{k+1}, t) \geq H\left([Ux_{k-1}]_{\alpha(x_{k-1})}, [Ux_k]_{\alpha(x_k)}, t\right), for\ all\ k \in N,$

Now ,

$\mathcal{M}(x_2, x_3, t) \geq H\left([Ux_1]_{\alpha(x_1)}, [Ux_2]_{\alpha(x_2)}, t\right)$

By using inequality 3.3(b) we get

$$\mathcal{M}(x_2, x_3, t) \geq\ Min\left\{\begin{array}{c}\psi\left(\mathcal{M}\left(x_1, x_2, \frac{t}{\square}\right)\right), \psi\left(\mathcal{M}\left(x_1, [Ux_1]_{\alpha(x_1)}, \frac{t}{\square}\right)\right), \psi\left(\mathcal{M}\left(x_2, [Ux_1]_{\alpha(x_1)}, \frac{t}{\square}\right)\right), \\ \left[\dfrac{\left\{1+\psi\left(\mathcal{M}\left(x_2, [Ux_1]_{\alpha(x_1)}, \frac{t}{\square}\right)\right)\right\}.\psi\left(\mathcal{M}(x_2, [Ux_1]_{\alpha(x_1)}, \frac{t}{\square})\right)}{1+\psi\left(\mathcal{M}(x_2, [Ux_2]_{\alpha(x_2)}, \frac{t}{\square})\right)}\right]\end{array}\right\}$$

$$\geq Min\left\{\begin{array}{c}\psi\left(\mathcal{M}\left(x_1, x_2, \frac{t}{\square}\right)\right), \psi\left(\mathcal{M}\left(x_1, x_2, \frac{t}{\square}\right)\right), \psi\left(\mathcal{M}\left(x_2, x_3, \frac{t}{\square}\right)\right), \\ \left[\dfrac{1+\psi\left(\mathcal{M}\left(x_2, x_3, \frac{t}{\square}\right)\right).\psi\left(\mathcal{M}(x_2, x_3, \frac{t}{\square})\right)}{1+\psi\left(\mathcal{M}(x_2, x_3, \frac{t}{\square})\right)}\right]\end{array}\right\}$$

$$\geq\ Min\left\{\psi\left(\mathcal{M}\left(x_1, x_2, \frac{t}{\square}\right)\right), \psi\left(\mathcal{M}\left(x_2, x_3, \frac{t}{\square}\right)\right)\right\}$$

If $Min\left\{\psi\left(\mathcal{M}\left(x_1, x_2, \frac{t}{\square}\right)\right), \psi\left(\mathcal{M}\left(x_2, x_3, \frac{t}{\square}\right)\right)\right\} = \psi\left(\mathcal{M}\left(x_2, x_3, \frac{t}{\square}\right)\right)$

By using remark (*) property (a) $i.e.\ \psi(a) > a,\ \ \forall(0 < a < 1)$;

Then we have $\mathcal{M}(x_2, x_3, t) \geq \mathcal{M}\left(x_2, x_3, \frac{t}{\square}\right)$

So, by lemma 2.17 nothing left to prove. Now if we have

$Min\left\{\psi\left(\mathcal{M}\left(x_1, x_2, \frac{t}{\square}\right)\right), \psi\left(\mathcal{M}\left(x_2, x_3, \frac{t}{\square}\right)\right)\right\} = \psi\left(\mathcal{M}\left(x_1, x_2, \frac{t}{\square}\right)\right)$then

Then again by lemma 2.16, we have$\mathcal{M}(x_2, x_3, t) \geq \mathcal{M}\left(x_1, x_2, \frac{t}{\square}\right)$.

Again $\mathcal{M}(x_2, x_3, t) \geq H\left([Ux_0]_{\alpha(x_0)}, [Ux_1]_{\alpha(x_1)}, \frac{t}{\square}\right)$

$$\geq\ Min\left\{\begin{array}{c}\psi\left(\mathcal{M}\left(x_0, x_1, \frac{t}{\square^2}\right)\right), \psi\left(\mathcal{M}\left(x_0, [Ux_0]_{\alpha(x_0)}, \frac{t}{\square^2}\right)\right), \psi\left(\mathcal{M}\left(x_1, [Ux_0]_{\alpha(x_0)}, \frac{t}{\square^2}\right)\right), \\ \left[\dfrac{\left\{1+\psi\left(\mathcal{M}\left(x_1, [Ux_0]_{\alpha(x_0)}, \frac{t}{\square^2}\right)\right)\right\}.\psi\left(\mathcal{M}(x_1, [Ux_0]_{\alpha(x_0)}, \frac{t}{\square^2})\right)}{1+\psi\left(\mathcal{M}(x_1, [Ux_1]_{\alpha(x_1)}, \frac{t}{\square^2})\right)}\right]\end{array}\right\}$$

$$\geq Min\left\{\begin{array}{c}\psi\left(\mathcal{M}\left(x_0, x_1, \frac{t}{\square^2}\right)\right), \psi\left(\mathcal{M}\left(x_0, x_1, \frac{t}{\square^2}\right)\right), \psi\left(\mathcal{M}\left(x_1, x_2, \frac{t}{\square^2}\right)\right), \\ \left[\dfrac{1+\psi\left(\mathcal{M}\left(x_1, x_2, \frac{t}{\square^2}\right)\right).\psi\left(\mathcal{M}(x_1, x_2, \frac{t}{\square^2})\right)}{1+\psi\left(\mathcal{M}(x_1, x_2, \frac{t}{\square^2})\right)}\right]\end{array}\right\}$$

$$\geq\ Min\left\{\psi\left(\mathcal{M}\left(x_0, x_1, \frac{t}{\square^2}\right)\right), \psi\left(\mathcal{M}\left(x_1, x_2, \frac{t}{\square^2}\right)\right)\right\}$$

If $Min\left\{\psi\left(\mathcal{M}\left(x_0, x_1, \frac{t}{\square^2}\right)\right), \psi\left(\mathcal{M}\left(x_1, x_2, \frac{t}{\square^2}\right)\right)\right\} = \psi\left(\mathcal{M}\left(x_1, x_2, \frac{t}{\square^2}\right)\right)$

By using remark (*) property (a) $i.e.\ \psi(a) > a,\ \ \forall(0 < a < 1)$;

Then we have $\mathcal{M}(x_2,x_3,t) \geq \mathcal{M}\left(x_1,x_2,\frac{t}{h^2}\right)$

So, by lemma 2.17 nothing left to prove. Now if we have

$Min\left\{\psi\left(\mathcal{M}\left(x_0,x_1,\frac{t}{h^2}\right)\right),\psi\left(\mathcal{M}\left(x_1,x_2,\frac{t}{h^2}\right)\right)\right\} = \psi\left(\mathcal{M}\left(x_0,x_1,\frac{t}{h}\right)\right)$

By using remark (*) property (a) $i.e.\ \psi(a) > a,\ \ \forall(0 < a < 1)$;

Then, we have $\mathcal{M}(x_2,x_3,t) \geq \mathcal{M}\left(x_0,x_1,\frac{t}{h^2}\right)$

In the same manner, $\mathcal{M}(x_n,x_{n+1},t) \geq \mathcal{M}\left(x_0,x_1,\frac{t}{h^n}\right)$

Now let $p = n + m,\ (p > n)$ then we have

$$\mathcal{M}(x_n,x_{n+m},t) \geq \mathcal{M}\left(x_n,x_{n+1},\frac{t}{m}\right) * \ldots\ldots * \mathcal{M}\left(x_{n+m-1},x_{n+m},\frac{t}{m}\right),\ \ m-times$$

By applying above results then we get

$$\mathcal{M}(x_n,x_{n+m},t) \geq \mathcal{M}\left(x_0,x_1,\frac{t}{mh^n}\right) * \ldots\ldots * \mathcal{M}\left(x_0,x_1,\frac{t}{mh^n+m-1}\right),\ \ m-times$$

Letting limit as $n \to \infty$ and using inequality 3.2(a) then we get

$lim_{n\to\infty}\mathcal{M}(x_n,x_{n+m},t) = 1$,

Hence $\{x_n\}$ follows Cauchy sequence. So by completeness there exists $\vartheta \in \Omega\ \ such\ that\ \ lim_{n\to\infty} x_n = \vartheta$.

Now to show ϑ is a fuzzy fixed point of U. we have

$$\mathcal{M}\big(\vartheta,[U\vartheta]_{\alpha(\vartheta)},t\big) \geq \mathcal{M}(\vartheta,x_{n+1},(1-h)t) * \mathcal{M}(x_{n+1},[U\vartheta]_{\alpha(\vartheta)},ht)$$

$$\mathcal{M}\big(\vartheta,[U\vartheta]_{\alpha(\vartheta)},t\big) \geq \mathcal{M}(\vartheta,x_{n+1},(1-h)t) * H([Ux_n]_{\alpha(x_n)},[U\vartheta]_{\alpha(\vartheta)},ht)$$

$$\mathcal{M}\big(\vartheta,[U\vartheta]_{\alpha(\vartheta)},t\big) \geq \mathcal{M}(\vartheta,x_{n+1},(1-h)t) *$$

$$\geq Min\left\{\begin{array}{c}\psi\big(\mathcal{M}(x_n,\vartheta,t)\big),\psi\left(\mathcal{M}\left(x_n,[Ux_n]_{\alpha(x_n)},t\right)\right),\psi\left(\mathcal{M}\left(\vartheta,[Ux_n]_{\alpha(x_n)},t\right)\right),\\ \left[\dfrac{\left\{1+\psi\left(\mathcal{M}\left(x_n,[Ux_n]_{\alpha(x_n)},t\right)\right)\right\}.\psi\left(\mathcal{M}(\vartheta,[Ux_n]_{\alpha(x_n)},t)\right)}{1+\psi\big(\mathcal{M}(\vartheta,[U\vartheta]_{\alpha(\vartheta)},t)\big)}\right]\end{array}\right\}$$

Taking $n \to \infty\ \ then\ we\ get$

$$\mathcal{M}\big(\vartheta,[U\vartheta]_{\alpha(\vartheta)},t\big) \geq \mathcal{M}(\vartheta,x_{n+1},(1-h)t) *$$

$$\geq Min\left\{\begin{array}{c}\psi\big(\mathcal{M}(\vartheta,\vartheta,t)\big),\psi\left(\mathcal{M}\big(\vartheta,[U\vartheta]_{\alpha(\vartheta)},t\big)\right),\psi\left(\mathcal{M}\big(\vartheta,[U\vartheta]_{\alpha(\vartheta},t\big)\right),\\ \left[\dfrac{\left\{1+\psi\left(\mathcal{M}\big(\vartheta,[U\vartheta]_{\alpha(\vartheta)},t\big)\right)\right\}.\psi\big(\mathcal{M}(\vartheta,[U\vartheta]_{\alpha(\vartheta},t)\big)}{1+\psi\big(\mathcal{M}(\vartheta,[U\vartheta]_{\alpha(\vartheta)},t)\big)}\right]\end{array}\right\}$$

$$\mathcal{M}\big(\vartheta,[U\vartheta]_{\alpha(\vartheta)},t\big) \geq Min\left\{\psi\left(\mathcal{M}\big(\vartheta,[U\vartheta]_{\alpha(\vartheta)},t\big)\right),1,1,1\right\}$$

By using remark (*) property (a) $i.e.\ \psi(a) > a,\ \ \forall(0 < a < 1)$;

$$\mathcal{M}\big(\vartheta,[U\vartheta]_{\alpha(\vartheta)},t\big) \geq Min\left\{\left(\mathcal{M}\big(\vartheta,[U\vartheta]_{\alpha(\vartheta)},t\big)\right),1,1,1\right\}$$

If $Min\{\mathcal{M}\big(\vartheta,[U\vartheta]_{\alpha(\vartheta)},t\big),1,1,1\} = 1$ then we get

$\mathcal{M}\big(\vartheta,[U\vartheta]_{\alpha(\vartheta)},t\big) \geq 1$

Hence we get ϑ is a fuzzy fixed point of U.

If $Min\{\mathcal{M}\big(\vartheta,[U\vartheta]_{\alpha(\vartheta)},t\big),1,1,1\} = \mathcal{M}\big(x_n,[U\vartheta]_{\alpha(\vartheta)},t\big)$ then we get

$\mathcal{M}\big(\vartheta,[U\vartheta]_{\alpha(\vartheta)},t\big) \geq \mathcal{M}\big(\vartheta,[U\vartheta]_{\alpha(\vartheta)},t\big)$

Which is a contradiction then we have $\mathcal{M}\big(\vartheta,[U\vartheta]_{\alpha(\vartheta)},t\big) = 1$, thus $\vartheta = [U\vartheta]_{\alpha(\vartheta)}$

Hence ϑ is a fuzzy fixed point of U.

Example (3.4): Let $\Omega = [1,3]$and $U(x,y) < t\ \forall\, x,y \in \Omega$,t $=fix(0,\infty)$and $w: R^+ \to (t,\infty)$ is non-decreasing function defined by $w(t) = t + 2$. Defined a map $\mathcal{M}: \Omega^2 \times (0,\infty) \to [0,1]\ as$

$$\mathcal{M}(x,y,t) = 1 - \frac{d(x,y)}{w(t)};\ x,y \in \Omega, t > 0.$$

Then $(\Omega,\mathcal{M},\star)$be a complete FMS under t-norm *.

Define a fuzzy mapping $U: \Omega \to F(\Omega)\ by\ U(x)(t) = \begin{cases}\frac{1}{2}\ ,1 < t \leq \frac{3}{2}\\ \frac{1}{3}\ ,\frac{3}{2} < t \leq 2\\ 0\ \ ,2 < t \leq 1\end{cases}$

For all $x \in X,\ there\ exists\ \alpha(x) = \frac{1}{2} = \alpha(y), such\ that\ [Ux]_{\frac{1}{2}} = \left[1,\frac{3}{2}\right]$

$$and\ [Uy]_{\frac{1}{2}} = \left[1, \frac{3}{2}\right] t\square en \lim_{t\to\infty} \mathcal{M}(x,y,t) = 1 - \frac{d(x,y)}{w(t)} = 1.$$

and also, we find

$$H\big([Ux]_{\alpha(x)}, [Uy, \square t]_{\alpha(y)}\big) \geq Min \left\{ \begin{matrix} \mathcal{M}(x,y,t), \mathcal{M}\big(x, [Ux]_{\alpha(x)}, t\big), \mathcal{M}\big(y, [Ux]_{\alpha(x)}, t\big), \\ \left[\frac{\{1+\mathcal{M}(y, [\mathrm{Ux}]_{\alpha(x)}, t)\}.\mathcal{M}(y, [Uy]_{\alpha(y)}, t)}{1+\mathcal{M}(y, [Uy]_{\alpha(y)}, t)}\right] \end{matrix} \right\}$$

$$H\left([Ux]_{\frac{1}{2}}, [Uy, \square t]_{\frac{1}{2}}\right) \geq Min \left\{ \begin{matrix} \mathcal{M}(x,y,t), \mathcal{M}\left(x, [Ux]_{\frac{1}{2}}, t\right), \mathcal{M}\left(y, [Ux]_{\frac{1}{2}}, t\right), \\ \left[\frac{\left\{1+\mathcal{M}\left(y, [Ux]_{\frac{1}{2}}, t\right)\right\}.\mathcal{M}(y, [Uy]_{\frac{1}{2}}, t)}{1+\mathcal{M}(y, [Uy]_{\frac{1}{2}}, t)}\right] \end{matrix} \right\} = 0$$

Therefore, $0 \in \Omega$ is the fuzzy fixed point of U.

Corollary (3.5)let $(\Omega, \mathcal{M}, \star)$be a complete FMS and $U: \Omega \to \mathcal{M}(\Omega)$ be a set valued mapping for all $x, y \in \Omega$, $\alpha \in (0,1]$ and $\square \in (0,1)$if satisfying the conditions:

(a) $lim_{t\to} \mathcal{M}(x,y,t) = 1$.
(b) $H\big([Ux]_{\alpha(x)}, [Uy, \square t]_{\alpha(y)}\big) \geq \mathcal{M}(x,y,t)$

Then U $\square ave\ a\ \ fuzzy\ fixed\ point$.

Corollary (3.6)let $(\Omega, \mathcal{M}, \star)$be a complete FMS and $U: \Omega \to \mathcal{M}(\Omega)$ be a set valued mapping for all $x, y \in \Omega$, $\alpha \in (0,1]$ and $\square \in (0,1)$ $and\ \psi \in \boldsymbol{\Sigma}$ if satisfying the conditions:

(a) $lim_{t\to} \mathcal{M}(x,y,t) = 1$.
(b) $H\big([Ux]_{\alpha(x)}, [Uy, \square t]_{\alpha(y)}\big) \geq \psi\big(\mathcal{M}(x,y,t)\big)$

Then U $\square ave\ a\ \ fuzzy\ fixed\ point$.

Applications: To solve an integral if we define a non-decreasing and continuous function $\Upsilon(\alpha): [0,\infty) \to [0,\infty)as\Upsilon(\alpha) = \int_0^\alpha \beta(\alpha)d\alpha$, $\forall\alpha > 0$, for each $\beta\ (\delta) > 0, \delta > 0$ and $\beta(\alpha) = 0\ \ if\ and\ only\ if\ \alpha = 0$. then

Theorem (3.7) let $(\Omega, \mathcal{M}, \star)$be a complete FMS and $U: \Omega \to \mathcal{M}(\Omega)$ be set valued mapping for all $x, y \in \Omega$, $\alpha \in (0,1]$, $\beta(\alpha) \in [0,\infty)$ and $\square \in (0,1)$if satisfying the conditions:

(a) $lim_{t\to\infty} \mathcal{M}(x,y,t) = 1$.
(b) $\int_0^{H([Ux]_{\alpha(x)}, [Uy]_{\alpha(y)}, \square t)} \beta(\alpha)d\alpha \geq$

$$\int_0^{Min\left\{ \begin{matrix} \mathcal{M}(x,y,t), \mathcal{M}(x, [Ux]_{\alpha(x)}, t), \mathcal{M}(y, [Ux]_{\alpha(x)}, t), \\ \left[\frac{\{1+\mathcal{M}(y, [Ux]_{\alpha(x)}, t)\}.\mathcal{M}(y, [Uy]_{\alpha(y)}, t)}{1+\mathcal{M}(y, [Uy]_{\alpha(y)}, t)}\right] \end{matrix} \right\}} \beta(\alpha)d\alpha$$

Such that $[Ux]_{\alpha(x)}\ and\ [Uy]_{\alpha(y)}$ are compact subset of Ω then U has FP.

Proof:if we take $\beta(\alpha) = 1$ then we can easily proof by using theorem (3.2).

Theorem (3.8) let $(\Omega, \mathcal{M}, \star)$be a complete FMS and $U: \Omega \to \mathcal{M}(\Omega)$ be a set valued mapping for all $x, y \in \Omega$, $\alpha \in (0,1], \beta(\alpha) \in [0,\infty)$ and $\square \in (0,1)$if satisfying the conditions:

(a) $lim_{t\to\infty} \mathcal{M}(x,y,t) = 1$.
(b) $\int_0^{H([Ux]_{\alpha(x)}, [Uy]_{\alpha(y)}, \mathrm{ht})} \beta(\alpha)d\alpha \geq$

$$\varphi \left\{ \int_0^{\mathrm{Min}\left\{ \begin{matrix} \mathcal{M}(x,y,t), \mathcal{M}(x, [Ux]_{\alpha(x)}, t), \mathcal{M}(y, [Ux]_{\alpha(x)}, t), \\ \left[\frac{\{1+\mathcal{M}(y, [Ux]_{\alpha(x)}, t)\}.\mathcal{M}(y, [Uy]_{\alpha(y)}, t)}{1+\mathcal{M}(y, [Uy]_{\alpha(y)}, t)}\right] \end{matrix} \right\}} \beta(\alpha)d\alpha \right\}$$

$\varphi \in \boldsymbol{\Sigma}$, such that $[\mathrm{Ux}]_{\alpha(x)}$ and $[\mathrm{Uy}]_{\alpha(y)}$ are compact subset of Ω then U has FP.

Proof:if we take $\beta(\alpha) = 1$ and also use remark (*) property (a) i.e. $\psi(a) > a$, $\forall(0 < a < 1)$ Then we can easily proof by using theorem (3.3).

4. CONCLUSION

Numerous authors have presented different fixed-point results for self-mapping in a metric space. In this work, we demonstrate the existence and uniqueness of fuzzy fixed point outcomes for single-valued mappings and set-valued using fuzzy metric space. In order to examine fuzzy fixed point results for single-valued and set-valued mappings, we use two generalized contractions with novel including rational expressions in the context of fuzzy metric spaces in this paper. We have incorporated examples and applications that highlight and validate our acquired outcomes. Our results have generalized numerous fixed point results found in the relevant literature, as fuzzy mapping is a generalization of multi-valued mapping.

REFERENCES

[1] Arora, R., & Kumar, M. (2016). Unique fixed point theorems for α–ψ-contractive type mappings in fuzzy metric space. Cogent Mathematics, 3(1), 1183286.
[2] Azam, A. (2011). Fuzzy fixed points of fuzzy mappings via a rational inequality. Hacettepe Journal of Mathematics and Statistics, 40(3), 421-431.
[3] Banach, S. (1922). Sur les operations dans les ensembles abstracts et leur application aux equations integrals. Fundamentamathematicae, 3(1), 133-181.
[4] Das, N. R., & Saha, M. L. (2015). On fixed points in complete fuzzy normed linear spaces. Annals of Fuzzy Mathematics and Informatics, 10(4), 515-524.
[5] Dubey, H., Choudhary, K. W. S., & Bhardwaj, R.(2024). Common Fixed Point Theorems in Fuzzy Metric Space. African journal of Biological sciences,Volume 6, Issue 10, 6173-6181.
[6] Fadhel, F. S., Eidi, J. H., & Wali, H. M. (2021). Contraction mapping theorem in partial fuzzy metric spaces. Journal of Applied Science and Engineering, 25(2), 353-360.
[7] George, A., & Veeramani, P. (1994). On some results in fuzzy metric spaces. Fuzzy sets and systems, 64(3), 395-399.
[8] Goguen, J. A. (1973). LA Zadeh. Fuzzy sets. Information and control, vol. 8 (1965), pp. 338–353.
[9] Gupta, V., Saini, R. K., & Verma, M. (2018). Fixed point theorems for single valued α-ψ-mappings in fuzzy metric spaces. Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics, 68(1), 392-400.
[10] Heilpern, S. (1981). Fuzzy mappings and fixed point theorem. Journal of Mathematical Analysis and Applications, 83(2), 566-569.
[11] Hong, S., & Peng, Y. (2013). Fixed points of fuzzy contractive set-valued mappings and fuzzy metric completeness. Fixed Point Theory and Applications, 2013, 1-13.
[12] Jain, P., Jat, V., & Singh, U. (2022). Transformations And Solutions of Integral Equation Involving Bessel Maitland Function. Journal of Algebraic Statistics, 13(2), 1541-1548.
[13] Jinakul, C., Wiwatwanich, A., &Kaewkhao, A. (2017). Common fixed point theorem for multi-valued mappings on b-metric spaces. International Journal of Pure and Applied Mathematics, 113(1), 167-179.
[14] Joseph, J. M., & Ramganesh, E. (2013). Fixed point theorem on multi-valued mappings. International Journal of Analysis and Applications, 1(2), 123-127.
[15] Maria Joseph, J., Dayana Roselin, D., & Marudai, M. (2016). Fixed point theorems on multi valued mappings in b-metric spaces. SpringerPlus, 5, 1-8.
[16] Kanwal, S., Ali, A., Al Mazrooei, A., & Garcia, G. S. (2023). Existence of fuzzy fixed points of set-valued fuzzy mappings in metric and fuzzy metric spaces. Aims Math, 8(5), 10095-10112.
[17] Kanwal, S., Maham, S., Shagari, M. S., Mohamed, O. K. S., Mustafa, A. O., & Bakery, A. A. (2023). Common coincidence points for Nadler's type hybrid fuzzy contractions. Journal of Inequalities and Applications, 2023(1), 100.
[18] Kanwal, S., Shagari, M. S., Aydi, H., Mukheimer, A., & Abdeljawad, T. (2022). Common fixed-point results of fuzzy mappings and applications on stochastic Volterra integral equations. Journal of Inequalities and Applications, 2022(1), 110.
[19] Kramosil, I., & Michálek, J. (1975). Fuzzy metrics and statistical metric spaces. Kybernetika, 11(5), 336-344.
[20] Murthy, P. P., Kumar, S., & Tas, K. (2010). Common fixed points of self maps satisfying an integral type of contractive condition in fuzzy metric spaces. Mathematical Communications, 15(2), 521-537.
[21] Namdev R., Tiwari R., Singh U., & Bhardwaj, R. (2024). Common Fixed-Point Theorems for Rational Expression in Fuzzy Metric Space using E.A like Property. African Journal of Biological Sciences, 6(10).
[22] Nashine, H. K., Vetro, C., Kumam, W., & Kumam, P. (2014). Fixed point theorems for fuzzy mappings and applications to ordinary fuzzy differential equations. Advances in Difference Equations, 2014, 1-

14.
[23] Rezapour, S., & Samet, m. e. (2014). some fixed-point results for α-ψ-contractive type mappings on intuitionistic fuzzy metric SPACES. Journal of Advanced Mathematical Studies, 7(1).
[24] Singh, U., Singh, N., Singh, R., & Bhardwaj, R. (2022). Common Invariant Point Theorem for Multi-valued Generalized Fuzzy Mapping in b-Metric Space. In Recent Trends in Design, Materials and Manufacturing: Selected Proceedings of ICRADMM 2020 (pp. 23-35). Singapore: Springer Nature Singapore.
[25] Sonam, Bhardwaj, R., & Narayan, S. (2023). Fixed point results in soft fuzzy metric spaces. Mathematics, 11(14), 3189.
[26] Sonam, Vandana Rathore, Amita Pal, Ramakant Bhardwaj, Satyendra Narayan (2023), 'Fixed-Point Results for Mappings Satisfying Implicit Relation in Orthogonal Fuzzy Metric Spaces", Advances in Fuzzy Systems Volume 2023, Article ID 5037401, 8 pages https://doi.org/10.1155/2023/5037401
[27] Sharad Gupta, Ramakant Bhardwaj, Wadkar Balaji Raghunath Rao, Rakesh Mohan Sharraf,(2020) " fixed point theorems in fuzzy metric spaces" Materials Today Proceedings 29 P2,611-616.
[28] Sewani, Godavari A.D. Singh, Ruchi Singh, Ramakant Bhardwaj (2022) "Generalized Intuitionistic Fuzzy b-Metric Space" ECS Transactions, 107 (1) 12415-12434, Doi 10.1149/10701.12415ecst
[29] Turkoglu, D., &Sangurlu, M. (2014). Fixed point theorems for fuzzy ψ–contractive mappings in fuzzy metric spaces. Journal of Intelligent & Fuzzy Systems, 26(1), 137-142.
[30] Wadkar, Balaji Raghunath, Lakshmi Narayan Mishra, Ramakant Bhardwaj and Basant Singh(2017), Fixed Point Theorem In Fuzzy 3-Metric Space Advances in Dynamical Systems and Applications. ISSN 0973-5321, Volume 12, Number 1, pp. 123-134