The complement on the existence of fixed points that belong to the zero set of a certain function due to Karapinar et al.

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Abstract

Recently, the idea of φ-fixed point and the elementary results on φ-fixed points were first investigated by Jleli et al. [Jleli M, Samet B, Vetro C (2014) Fixed point theory in partial metric spaces via φ-fixed point's concept in metric spaces. Journal of Inequalities and Applications, 2014(1):1-9.]. Based on this work, Karapinar et al. [Karapinar E, O'Regan D, Samet B (2015) On the existence of fixed points that belong to the zero set of a certain function. Fixed Point Theory and Applications, 2015(1):1-14.] established the new φ-fixed point results, which can be reduced to the famous fixed point result of Boyd and Wong in 1969. However, the main result of Karapinar et al. does not cover the φ-fixed point results of Jleli et al. This paper aims to fulfill this gap by proving φ-fixed point results covering several φ-fixed point results and fixed point results. **Key words**: φ-fixed point; φ-Picard mapping; Control function
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1 Introduction and preliminaries

In 2014, Jleli et al. [1] had initiated the concept of (F, φ)-contraction with the help of some control function, which is one of the interesting generalizations of the classical Banach contraction principle and first introduced the concepts of φ-fixed point and φ-Picard mapping. Moreover, they also proved some φ-fixed point theorems for contractive mappings expanded some fixed point results in metric spaces. Consistent with Jleli et al. [1], we will be needed the following notations, definitions, and results in this research.

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Let X be a nonempty set, $\varphi : X \to [0, \infty)$ be a given function and $T : X \to X$ be a mapping. By F_T and Z_φ the set of all fixed points of T and the set of all zeros of the function φ, respectively, i.e., $F_T := \{x \in X : Tx = x\}$ and $Z_\varphi := \{x \in X : \varphi(x) = 0\}$.

Definition 1.1 ([1])**.** Let X be a nonempty set and $\varphi : X \to [0, \infty)$ be a given function. An element $z \in X$ is said to be a φ-fixed point of the mapping $T : X \to X$ if and only if z is a fixed point of T and $\varphi(z) = 0$ (i.e., $z \in F_T \cap Z_\varphi$).

Definition 1.2 ([1])**.** Let (X, d) be a metric space and $\varphi : X \to [0, \infty)$ be a given function. A mapping $T : X \to X$ is said to be a φ-Picard mapping if and only if, for each $x, z \in X$, the following conditions are satisfied:

(i) $F_T \cap Z_\varphi = \{z\}$ for some $z \in X$;

(ii) $T^n x \to z$ as $n \to \infty$ for each $x \in X$.

To describe the control function, which is an important class of this work, let $\mathcal{F}$ be the family of all functions $F : [0, \infty)^3 \to [0, \infty)$ satisfying the following conditions:

$(F1)$ $\max\{a, b\} \leq F(a, b, c)$ for all $a, b, c \in [0, \infty)$;

$(F2)$ $F(0, 0, 0) = 0$;

$(F3)$ F is continuous.

As examples, the following functions $F_1, F_2, F_3 : [0, \infty)^3 \to [0, \infty)$ belong to $\mathcal{F}$:

(i) $F_1(a, b, c) = a + b + c$ for all $a, b, c \in [0, \infty)$;

(ii) $F_2(a, b, c) = \max\{a, b\} + c$ for all $a, b, c \in [0, \infty)$;

(iii) $F_3(a, b, c) = a + a^2 + b + c$ for all $a, b, c \in [0, \infty)$.

Definition 1.3 ([1])**.** Let (X, d) be a metric space, $\varphi : X \to [0, \infty)$ be a given function, and $F \in \mathcal{F}$. A mapping $T : X \to X$ is said to be an (F, φ)-contraction mapping if there exists $k \in [0, 1)$ such that

$$F(d(Tx, Ty), \varphi(Tx), \varphi(Ty)) \leq kF(d(x, y), \varphi(x), \varphi(y)) \quad \forall (x, y) \in X^2. \tag{1.1}$$

Theorem 1.4 ([1])**.** *Let (X, d) be a complete metric space, $\varphi : X \to [0, \infty)$ be a lower semi-continuous function, $F \in \mathcal{F}$ and $T : X \to X$ be an (F, φ)-contraction mapping. Then $F_T \subseteq Z_\varphi$ and T is a φ-Picard mapping.*

Remark 1.5. Note that if we set $F(a, b, c) = a + b + c$ for all $a, b, c \in [0, \infty)$ and $\varphi(x) = 0$ for all $x \in X$ in (1.1), then the contractive condition (1.1) reduces to the Banach contractive condition.

In recent years, Jleli et al.'s fixed point theorem has been generalized and extended in several directions. One such generalization was introduced by Karapinar et al. [2] by replacing the constant k of the contractive condition (1.1) with the control function, which was first introduced by Boyd and Wong [3]. They also proved the existence and uniqueness results of a φ-fixed point for new nonlinear mappings. Nevertheless, this result expands all conditions of results of [1], except that the condition $(F2)$ is replaced by

$(F2^\star)$ $F(a,0,0) = a$ for all $a \geq 0$.

Here, we recall the definition of the following class as given by Boyd and Wong [3]. Denote Ψ the set of functions $\psi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

$(\psi 1)$ ψ is upper semi-continuous from the right;

$(\psi 2)$ $\psi(t) < t$ for each $t > 0$.

Combining this definition with Jleli et al.'s theorem, Karapinar et al. [2] proved the following theorem:

Theorem 1.6 ([2])**.** *Let (X,d) be a complete metric space. Suppose that the mapping $T : X \to X$ satisfies the following condition:*

$$F(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) \leq \psi(F(d(x,y),\varphi(x),\varphi(y))) \quad \forall (x,y) \in X^2, \quad (1.2)$$

where $\varphi : X \to [0,\infty)$ is lower semi-continuous, $\psi \in \Psi$, and $F : [0,\infty)^3 \to [0,\infty)$ is a function satisfying the following conditions:

$(F1)$ $\max\{a,b\} \leq F(a,b,c)$ *for all* $a,b,c \in [0,\infty)$*;*

$(F2^\star)$ $F(a,0,0) = a$ *for all* $a \geq 0$*;*

$(F3)$ *F is continuous.*

Then $F_T \subseteq Z_\varphi$ and T is a φ-Picard mapping.

In the case of ψ defined by $\psi(t) = kt$ for some $k \in [0,1)$, Theorem 1.6 seem almost similar to a generalization of Theorem 1.4 except that Theorem 1.6 use the control function F satisfying conditions $(F1)$, $(F2^\star)$, $(F3)$ rather than Theorem 1.4 use the control function F satisfying conditions $(F1)$, $(F2)$, $(F3)$. It is easy to see that the condition $(F2^\star)$ is stronger than the condition $(F2)$ since there are many functions satisfying the condition $(F2)$ but it does not satisfy the condition $(F2^\star)$. For example, functions $F_1, F_2, F_3 : [0,\infty)^3 \to [0,\infty)$ defined by $F_1(a,b,c) = a+a^2+b+c$, $F_2(a,b,c) = \ln(a+1)+(a+b)e^c+\max\{a,b\}$, and $F_3(a,b,c) = \max\{2a,b\}+c$ for all $a,b,c \geq 0$. From the above observation, we can conclude that the main theorem of [2] is not a proper extension of Theorem 1.4.

The main goal of this work is to fulfill the mentioned gap by using the new technique for improving Theorem 1.6 via the original control function, which was introduced by Jleli et al. in [1]. For simplicity, the following diagram shows the relation of Karapinar et al.'s results and our results, which describes the objectives of this research.

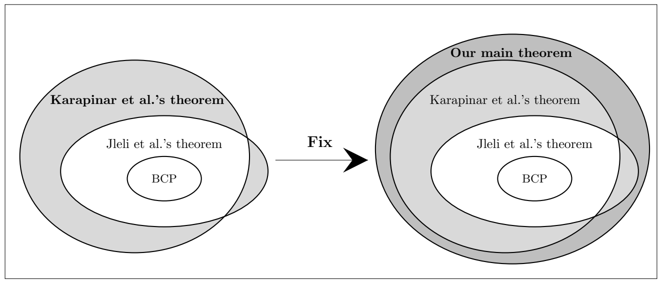

Figure 1: The conceptual research framework

2 Main results

In section, we will prove the generalized φ-fixed point results by using the new technique, which is the improved version of the φ-fixed point theorem of Karapinar et al. [2], but it replaces the condition $(F2^\star)$ by the condition $(F2)$.

Theorem 2.1. *Let (X, d) be a complete metric space and T be a self mapping on X such that*

$$F(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) \leq \psi(F(d(x,y),\varphi(x),\varphi(y))) \quad \forall (x,y) \in X^2, \tag{2.1}$$

where $\varphi : X \to [0,\infty)$ is lower semi-continuous, $F \in \mathcal{F}$ and $\psi \in \Psi$. Then $F_T \subseteq Z_\varphi$ and T is a φ-Picard mapping.

Proof. The first step is to prove that $F_T \subseteq Z_\varphi$. Let $x \in F_T$. Letting $y = x$ in (2.1), we have

$$F(0,\varphi(x),\varphi(x)) \leq \psi(F(0,\varphi(x),\varphi(x))). \tag{2.2}$$

Assume that $\varphi(x) > 0$. It follows from $(F1)$ that $F(0,\varphi(x),\varphi(x)) > 0$. By (2.2) and $(\psi 1)$, we get

$$F(0,\varphi(x),\varphi(x)) \leq \psi(F(0,\varphi(x),\varphi(x))) < F(0,\varphi(x),\varphi(x)),$$

which is a contradiction. Therefore, $\varphi(x) = 0$, which implies that

$$F_T \subseteq Z_\varphi. \tag{2.3}$$

Next, we will show that T is a φ-Picard mapping. Let x_0 be an arbitrary point in X. Define the sequence $\{x_n\} \subseteq X$ by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. If $x_{n^*} = x_{n^*-1}$ for some $n^* \in \mathbb{N}$, then x_{n^*} is a fixed point of T. Hence, for the rest of the proof, we assume that $x_n = x_{n-1}$ for all $n \in \mathbb{N}$, that is,

$$d(x_n, x_{n-1}) > 0 \tag{2.4}$$

for each $n \in \mathbb{N}$. Now, we will claim that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = \lim_{n\to\infty} \varphi(x_n) = 0. \tag{2.5}$$

From $(F1)$ and (2.4), we obtain

$$F(d(x_n, x_{n-1}), \varphi(x_n), \varphi(x_{n-1})) > 0$$

for all $n \in \mathbb{N}$. This allows to use the condition $(\psi 2)$ and so by using the contractive condition (2.1), we obtain

$$\begin{aligned} F(d(x_{n+1}, x_n), \varphi(x_{n+1}), \varphi(x_n)) &\leq \psi(F(d(x_n, x_{n-1}), \varphi(x_n), \varphi(x_{n-1}))) \\ &< F(d(x_n, x_{n-1}), \varphi(x_n), \varphi(x_{n-1})) \end{aligned} \tag{2.6}$$

for all $n \in \mathbb{N}$. This shows that $\{F(d(x_{n+1}, x_n), \varphi(x_{n+1}), \varphi(x_n))\}$ is a decreasing sequence. Furthermore, it is easy to see that it is also bounded below by 0 and hence it converges to some point $r \geq 0$, that is,

$$\lim_{n\to\infty} F(d(x_{n+1}, x_n), \varphi(x_{n+1}), \varphi(x_n)) = r. \tag{2.7}$$

From (2.6), (2.7) and the squeeze theorem, we get

$$\lim_{n\to\infty} \psi(F(d(x_n, x_{n-1}), \varphi(x_n), \varphi(x_{n-1}))) = r. \tag{2.8}$$

Assume that $r > 0$. So we have

$$\begin{aligned} r &\overset{(2.8)}{=} \limsup_{n\to\infty} \psi(F(d(x_n, x_{n-1}), \varphi(x_n), \varphi(x_{n-1}))) \\ &\overset{(\psi 1)}{\leq} \psi(r) \\ &\overset{(\psi 2)}{<} r \end{aligned}$$

which provides a contradiction. Therefore, $r = 0$, that is,

$$\lim_{n\to\infty} F(d(x_{n+1}, x_n), \varphi(x_{n+1}), \varphi(x_n)) = 0,$$

and thus, by $(F1)$, we get

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = \lim_{n\to\infty} \varphi(x_n) = 0,$$

that is, Equation (2.5) holds.

Now, we shall prove that $\{x_n\}$ is a Cauchy sequence. Assume on the contrary that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\epsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) \geq k$ and

$$d\left(x_{m(k)}, x_{n(k)}\right) \geq \epsilon \tag{2.9}$$

for all $k \in \mathbb{N}$. Corresponding to $m(k)$, we may choose $n(k)$ such that it is the smallest integer satisfying (2.9). Then we have

$$d\left(x_{m(k)}, x_{n(k)-1}\right) < \epsilon.$$

By the triangular inequality, we have

$$\begin{aligned} \epsilon &\leq d\left(x_{m(k)}, x_{n(k)}\right) \\ &\leq d\left(x_{m(k)}, x_{n(k)-1}\right) + d\left(x_{n(k)-1}, x_{n(k)}\right) \\ &< \epsilon + d\left(x_{n(k)-1}, x_{n(k)}\right). \end{aligned}$$

Letting $k \to \infty$ in the above inequality and using (2.5), we have

$$\lim_{k\to\infty} d\left(x_{m(k)}, x_{n(k)}\right) = \epsilon. \tag{2.10}$$

By a similar way, we can show that

$$\lim_{k\to\infty} d\left(x_{m(k)+1}, x_{n(k)+1}\right) = \epsilon. \tag{2.11}$$

Using $(F3)$, (2.5), (2.10) and (2.11), it follows that

$$\lim_{k\to\infty} F\left(d\left(x_{m(k)}, x_{n(k)}\right), \varphi\left(x_{m(k)}\right), \varphi\left(x_{n(k)}\right)\right) = F(\epsilon, 0, 0) \tag{2.12}$$

and

$$\lim_{k\to\infty} F\left(d\left(x_{m(k)+1}, x_{n(k)+1}\right), \varphi\left(x_{m(k)+1}\right), \varphi\left(x_{n(k)+1}\right)\right) = F(\epsilon, 0, 0). \tag{2.13}$$

Now, we choose $x = x_{m(k)}$ and $y = x_{n(k)}$ in (2.1), we infer

$$F\left(d\left(x_{m(k)+1}, x_{n(k)+1}\right), \varphi\left(x_{m(k)+1}\right), \varphi\left(x_{n(k)+1}\right)\right) \leq \psi\left(F\left(d(x_{m(k)}, x_{n(k)}), \varphi(x_{m(k)}), \varphi(x_{n(k)})\right)\right).$$

Taking the limit superior as $k \to \infty$ on both sides of the above inequality and using (2.13), we deduce

$$F(\epsilon, 0, 0) \leq \limsup_{k\to\infty} \psi\left(F\left(d(x_{m(k)}, x_{n(k)}), \varphi(x_{m(k)}), \varphi(x_{n(k)})\right)\right). \tag{2.14}$$

Using the condition $(\psi 1)$ and (2.12), we obtain

$$\limsup_{k\to\infty} \psi\left(F\left(d(x_{m(k)}, x_{n(k)}), \varphi(x_{m(k)}), \varphi(x_{n(k)})\right)\right) \leq \psi(F(\epsilon, 0, 0)) < F(\epsilon, 0, 0). \tag{2.15}$$

From (2.14) and (2.15) together with $(\psi 2)$, we obtain

$$F(\epsilon, 0, 0) \leq \psi(F(\epsilon, 0, 0)) < F(\epsilon, 0, 0),$$

which is a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence. By the completeness of X, there exists a point $z \in X$ such that

$$\lim_{n\to\infty} d(x_n, z) = 0. \tag{2.16}$$

Using (2.5), (2.16) and the lower semi-continuity of φ, we get

$$0 \leq \varphi(z) \leq \liminf_{n\to\infty} \varphi(x_n) = 0,$$

which implies that

$$\varphi(z) = 0. \tag{2.17}$$

Next, we will prove that z is a fixed point of T. From $(F2)$, $(F3)$, (2.5) and (2.16), we get

$$\lim_{n\to\infty} F(d(x_n, z), \varphi(x_n), 0) = F(0,0,0) = 0.$$

Note that from $(\psi 2)$, it follows that $\lim_{t\to 0^+} \psi(t) = 0$. Then

$$\lim_{n\to\infty} \psi(F(d(x_n, z), \varphi(x_n), 0)) = \lim_{t\to 0^+} \psi(t) = 0. \tag{2.18}$$

Hence, from $(F1)$, (2.1), (2.17) and (2.18), we conclude that

$$d(x_{n+1}, Tz) \leq \psi(F(d(x_n, z), \varphi(x_n), 0)) \to 0 \text{ as } n \to \infty.$$

Therefore, by the uniqueness of the limit, we obtain $z = Tz$, i.e., z is a fixed point of T.

Finally, we will show that T has a unique fixed point. Suppose that u and v are fixed points of T such that $u \neq v$. Then $d(u, v) > 0$. Therefore,

$$\begin{aligned} F(d(u,v),0,0) &= F(d(Tu,Tv),0,0) \\ &\overset{(2.1)}{\leq} \psi(F(d(u,v),0,0)) \\ &\overset{(\psi 2)}{<} F(d(u,v),0,0), \end{aligned}$$

which is a contradiction. Thus, the fixed point of T is unique. This completes the proof. □

The following example shows that Theorem 2.1 is more applicable than many other results in the literature.

Example 2.2. Let $X = [0,\infty)$ and $d : X \times X \to \mathbb{R}$ be defined by $d(x,y) = |x-y|$ for all $x, y \in X$. Then (X,d) is a complete metric space. Assume that $T : X \to X$ and $\psi : [0,\infty) \to [0,\infty)$ are defined by

$$Tx = \begin{cases} \frac{x^2}{2}, & 0 \leq x < \frac{1}{2}, \\ \frac{1}{8x}, & x \geq \frac{1}{2}, \end{cases} \quad \text{and} \quad \psi(t) = \begin{cases} \frac{t}{2}, & 0 \leq t < 1, \\ \frac{1}{2}\sin\left(\frac{1}{2t-1}\right) + \frac{1}{2}, & t \geq 1. \end{cases}$$

Clearly, by the graph in Figure 2, we have $\psi \in \Psi$.

Now, we will show that the fixed point result of Boyd and Wong [3] can not be applied in this example. For any $x \in \left(0, \frac{1}{2}\right)$ and $y = \frac{1}{2}$, we obtain

$$d(Tx, Ty) = \left|\frac{x^2}{2} - \frac{1}{4}\right| = \frac{1}{4} - \frac{x^2}{2} > \frac{1}{4} - \frac{x}{2} = \left|\frac{x}{2} - \frac{1}{4}\right| = \psi\left(\left|x - \frac{1}{2}\right|\right) = \psi(d(x,y)).$$

Hence, T does not satisfy the Boyd and Wong's contractive condition. Also, the Banach contraction principle is not applicable, since T is not continuous at $\frac{1}{2}$.

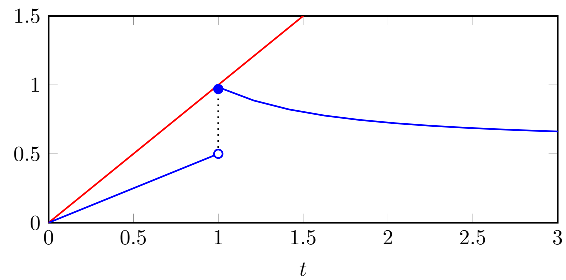

Figure 2: The graph of ψ in blue line

Next, we will show that Theorem 2.1 can be applied in this example. Let $\varphi : X \to [0,\infty)$ and $F : [0,\infty)^3 \to [0,\infty)$ be defined by

$$\varphi(x) = x, \quad x \in X \qquad \text{and} \quad F(a,b,c) = a + a^2 + b + c, \quad a,b,c \geq 0.$$

It is easy to see that $F \in \mathcal{F}$ and φ is lower semi-continuous. Now, we claim that the mapping T satisfies the contractive condition (2.1). Suppose that $x, y \in X$. We have to consider the following cases:

Case 1. If $(x,y) \in \left[0, \frac{1}{2}\right)^2$, then we get

$$\begin{aligned} F(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) &= d(Tx,Ty) + (d(Tx,Ty))^2 + \varphi(Tx) + \varphi(Ty) \\ &= |Tx - Ty| + |Tx - Ty|^2 + Tx + Ty \\ &= \frac{|x^2 - y^2|}{2} + \frac{|x^2 - y^2|^2}{4} + \frac{x^2}{2} + \frac{y^2}{2} \\ &= \frac{|(x+y)(x-y)|}{2} + \frac{|(x+y)(x-y)|^2}{4} + \frac{x^2}{2} + \frac{y^2}{2} \\ &\leq \frac{|x-y|}{2} + \frac{|x-y|^2}{2} + \frac{x}{2} + \frac{y}{2} \qquad (2.19) \\ &\leq \psi(|x-y| + |x-y|^2 + x + y) \\ &= \psi(d(x,y) + (d(x,y))^2 + \varphi(x) + \varphi(y)) \\ &= \psi(F(d(x,y),\varphi(x),\varphi(y))). \end{aligned}$$

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Case 2. If $(x,y) \in \left[\frac{1}{2},\infty\right)^2$, then we get

$$\begin{aligned}
F(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) &= d(Tx,Ty) + (d(Tx,Ty))^2 + \varphi(Tx) + \varphi(Ty) \\
&= |Tx-Ty| + |Tx-Ty|^2 + Tx + Ty \\
&= \left|\frac{1}{8x} - \frac{1}{8y}\right| + \left|\frac{1}{8x} - \frac{1}{8y}\right|^2 + \frac{1}{8x} + \frac{1}{8y} \\
&< \frac{1}{2}\sin\left(\frac{1}{2(|x-y|+|x-y|^2+x+y)+1}\right) + \frac{9}{16} \qquad (2.20) \\
&= \psi(|x-y|+|x-y|^2+x+y) \\
&= \psi(d(x,y) + (d(x,y))^2 + \varphi(x) + \varphi(y)) \\
&= \psi(F(d(x,y),\varphi(x),\varphi(y))).
\end{aligned}$$

Case 3. Let $(x,y) \in \left[0,\frac{1}{2}\right) \times \left[\frac{1}{2},\infty\right) \cup \left[\frac{1}{2},\infty\right) \times \left[0,\frac{1}{2}\right)$. Without loss of generality, we may assume that $x \in \left[0,\frac{1}{2}\right)$ and $y \in \left[\frac{1}{2},\infty\right)$ and so $|x-y|+|x-y|^2+x+y > 1$, then

$$\begin{aligned}
F(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) &= d(Tx,Ty) + (d(Tx,Ty))^2 + \varphi(Tx) + \varphi(Ty) \\
&= |Tx-Ty| + |Tx-Ty|^2 + Tx + Ty \\
&= \left|\frac{x^2}{2} - \frac{1}{8y}\right| + \left|\frac{x^2}{2} - \frac{1}{8y}\right|^2 + \frac{x^2}{2} + \frac{1}{8y} \\
&\leq \frac{1}{2}\sin\left(\frac{1}{2(|x-y|+|x-y|^2+x+y)+1}\right) + \frac{9}{16} \qquad (2.21) \\
&= \psi(|x-y|+|x-y|^2+x+y) \\
&= \psi(d(x,y) + (d(x,y))^2 + \varphi(x) + \varphi(y)) \\
&= \psi(F(d(x,y),\varphi(x),\varphi(y))).
\end{aligned}$$

The validity of the conditions (2.19), (2.20) and (2.21) can be checked by plotting 3D surface in MATLAB, shown as Figure 3. Without loss of generality and for the sake of simplicity, we restrict the domain in Figure 3 to $[0,3]$. Therefore, all the required hypotheses of Theorem 2.1 are fulfilled, and so T has a unique φ-fixed point. In this case, the point 0 is a unique φ-fixed point of T.

Remark 2.3. If we take $\varphi(x) = 0$ for all $x \in X$ in Theorem 2.1, then we get the real proper generalization of the Boyd and Wong fixed point theorem. However, if we take the same function φ in Theorem 1.6 and use $(F2^\star)$, we can see that the obtained result is equivalent to the Boyd and Wong fixed point theorem. This yields the advantage of our main result with the several results in the literature as shown in Figure 4.

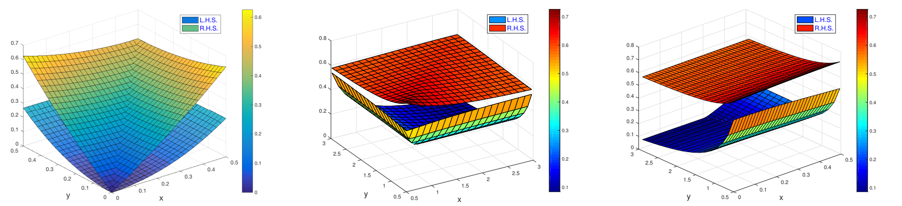

Figure 3: The value of the comparison of the L.H.S. and the R.H.S. of (2.19) and (2.21)

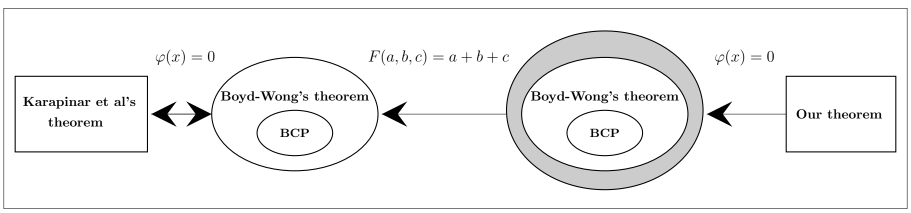

Figure 4: The difference of consequence between our theorem and Karapinar et al.'s theorem

3 Conclusions

Inspired by the problem of the relaxing of the hypothesis of the control function F in Theorem 1.6, we proposed a new technique for solving this problem. By the help of this suggested technique, our main theorem has the new proof, which seems to be simpler than the proof in [2]. The obtained result of this paper is a real proper generalization of the result in [1], and it also covers several famous fixed point results and φ-fixed point results in the literature. For the part of an application, we can use the main result in this work for applying in the homotopy result, and the fixed point results in partial metric spaces like the application in [2] since the class $\mathcal{F}$ is weaker than the class defined in [2].

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