New Fixed Points Outcomes for Fractal Creation by Applying Different Fixed Point Technique

Narayan partap[1], Sarika Jain and Renu Chugh

December 25, 2021

Abstract

Fractal like Julia set is regarded as one of the striking and significant mathematical fractals in the field of science and technology. There are different numerical iterative techniques which generate these fractals and in fact these numerical iterative techniques are the strength of fractal geometry. In recent past, Julia sets have been studied through numerical techniques like Picard, Mann, and Ishikawa etc. which are the examples of one-step, two-step, and three-step iterative techniques respectively. In this article, we have concentrated our research work on the computation as well as the different features of cubic Julia sets for the complex polynomial $P_{m,n}(z) = z^3 + mz + n$. This generation process has been carried-over through a new numerical four-step iterative technique. We have generated new and ever seen cubic Julia sets for the above complex polynomial. The cubic Julia sets generated through above polynomial have important mathematical properties. It is also fascinating that some of the generated cubic Julia sets are analogous to fractal shaped antennas, butterfly and some categories of ants. Some of the generated cubic Julia sets can also be categorized as wall-decorated pictures.

Key words: Cubic Julia sets; Four-Step Iterative Technique; Escape Criterion; Complex Cubic Equation

Mathematics Subject Classification(2010): 37C25; 28A80; 54E35; 54E50

1

1 Introduction

Fractals have many applications in various fields of science and technology. The production of cell phone has become possible only due to fractal shaped antennas. This was not possible before the introduction of this notion. The Chaos theory is what fractals are all about, and that is at the core of most Cosmological Models which describe our entire Universe on both the scales of very small and large. Fractal geometry is very practical for all sorts of things from biology to physics to cosmology to even the stock market. For instance veins and arteries form fractal trees, some mineral deposits are distributed under the fractal laws through the soil, and the stock market, perhaps not quite a natural phenomenon, behaves according to fractal laws.

If we look back into the history of fractal geometry, we find that the interest in fractal geometry with respect to Julia sets began in the 19th century. Named after

[1]Corresponding author: Haryana Education Department, Haryana, India

the esteemed French mathematician Gaston Julia, the Julia set expresses a complex and rich set of dynamics, giving rise to a large range of breathtaking visuals. Gaston was first who invented Julia sets and examined their features [27] $(p-122)$. In 1918, a masterpiece of him on Julia sets was published which was the major event in the history of fractal geometry. In this masterpiece, Gaston Julia first proposed the latest idea on a Julia set and that made him famous in the world of mathematics. Julia sets live in complex plane and this is the place where all chaotic behavior of complex plane occurs [9] $(p-221)$. When all the computational experiments were far ahead of its time, Herald Cramer gave the first approximate graphical representation to Julia sets. Mathematical items like Julia sets were recognized when computer graphics became user friendly [28].

What makes Julia sets interesting to analysis is that, despite being borne out of apparently easy iterative techniques, they can be very tangled and often fractal in nature. Due to fascination of computer experiments and incredulity of their graphical representation, Mandelbrot and Julia asets remain a topic of modern research with much interest being in evoking the tangled structure of Julia sets and in calculating their properties such as 'fractal dimension' [14]. For detailed investigation on Mandelbrot and Julia sets, one can follow the related work and complex dynamics system in references like [4, 5, 6, 8, 9, 12, 16, 18, 25, 26, 39].These sets (Julia sets) have been computed and examined for cubic [2, 3, 7, 9, 12, 13], quadratic [9, 19, 27], and also for higher degree polynomials using Picard iteration,which is an example of one-step numerical procedure.

Julia sets may be generated for any order complex polynomial. In this research Paper, we have emphasized our research work on cubic Julia sets with reference to complex polynomial of order 3. In 1988, Branner and Hubbard, by a series of papers, paid concerned with the way in which the space of polynomial of degree $d > 3$ is decomposed when the polynomial are classified according to their active behavior using iterative procedure. However their results are satisfied only for $d = 3$. They also dispense with the good framework of the locus connectedness for cubic polynomials [2]. In 1992, their second paper of this series was also dedicated to the account of dynamical system complex cubic polynomial. They also raised the query when the Julia set of a cubic polynomial became a Cantor set [3].

In 1998, Liaw found the regularities for the parameter space of the cubic polynomials in the two dimensional projections. The projection of the parameter points that have non-totally disconnected Julia sets can be observe as a combination of Mandelbrot-like sets [20]. In the same year Cheng and Liaw established asymptotic similarity between the parameter spaces (the Mandelbrot set) and the dynamic space (the Julia sets) of cubic mappings. The dynamic spaces and the parametric spaces of cubic polynomials consist many small copies of the Julia sets and Mandelbrot sets respectively of the standard quadratic mapping [7]. In 2001, Liaw studied the orientations, sizes, and positions of above small copies to understand the formation of the cubic polynomials [21].

In 1999, Yan, Liu, and Zhu worked on a general complex cubic iteration and produced results on the range of Mandelbrot and Julia sets generated from above cubic iteration[41]. These results are helpful in plotting the above generated sets. In 2003, Tomova produced results about the limits of cubic Julia and Mandelbrot sets for complex polynomial $z^3 + pz + q$ and also for Julia and Mandelbrot sets of higher orders polynomial of the form $z^n + c$, [40].

In 2006, Fu *et.al.* generated higher order Julia set and Mandelbrot sets by applying fast computing algorithm [15]. Mamta *et.al.* studied and computed superior Julia sets for nth degree complex polynomial [32, 33], for cubic [6] and for quadratic complex polynomials [17, 30, 31] using superior iterates (a two-step feedback procedure). Superior Julia sets have also been analysed under the effect of noises [34, 35]. In 2018, Narayan*et.al.* computed and analysed new collection of antifractals for the complex polynomial $\bar{z}^n + c$ in the GK-orbit [23].

In our previous article, we have computed and examined new collection of fractals using faster iteration with s-convexity [24].Recently, Rahman, Nisar and Golamankaneh, also studied the generalized Riemann–Liouville fractional integral for the functions with fractal support. They explored reverse Minkowski's inequalities and certain other related inequalities by employing generalized Riemann–Liouville fractional integral for the functions with fractal support [29].

Fractals have also vital role in various domains ranging from mathematical to engineering applications, nano technology to bio medical domain. However, for a deep understanding of fractals and to review the relevant areas, one must go through some recent implementations of non-linear concepts in various areas of machine learning, microprocessors and computer science related to data analysis [10, 37]. For more applications of numerical computing techniques one may see [11, 38].

In this research paper, we have explored new and ever seen cubic fractal sets for the complex cubic polynomial $z \to z^3 + mz + n$, using new different numerical iterative technique (a four-step feedback system) with improved escape criterion. We have also analysed the characteristics of the above generated cubic Julia sets.

2 Material and Method

In past years, researchers focused mainly on three types of feedback procedures like Picard iterates, a one-step feedback system [27], Mann iterates, a two-step feedback system [17, 36], Ishikawa iterates, a three-step feedback system [6] etc. In 2014, Mujahid *et. al.* introduced a new four-step iterative procedure and proved that it is faster than all of Mann, Picard and Agarwal *et. al.* processes. They supported analytic proof by a numerical example and mentioned that this process is independent of all above processes. They also demonstrated some strong and weak convergence results for two non-expansive functions [1].

In this research paper, we have computed Julia sets for the cubic complex polynomial using this faster four-step feedback procedure with improved escape criterion which provides the different results than the previous results.

Definition 2.1. Let S be a non-empty set and p be a self-map from S to S. For a point s_0 in S, the **Picard orbit** (generally called orbit of p) is the set of all iterates of a point s_0, that is: $P(p, s_0) = \{s_n : s_n = p(s_{n-1}),\ n = 1, 2, \cdots\}$, where $P(p, s_0)$ of p at the initial point s_0 is the sequence $(p^n s_0)$

Definition 2.2. The filled in Julia set of the function $F(z) = z^n + c$ is defined as

$$K(F) = \{z \in C : F^k(z) \textit{ does not tend to } \infty\}$$

where $F^k(z)$ is the k^{th} iterate of function F, *K(F)* denotes the filled in Julia set and C is the complex space. The boundary of *K(F)* is known as Julia set of the function F. Julia set is the set of those points whose orbits are bounded under $F_c(z) = z^n + c$.

Definition 2.3. (New Iterative Procedure-NIP) Let X be a non-empty set such that $T : X \to X$ and $\{x_n\}$ be a sequence of iterates of initial point $x_0 \in X$ such that

$$\{x_{n+1} : x_{n+1} = (1-u_n)Ty_n + u_nTz_n \ ;$$
$$y_n = (1-v_n)Tx_n + v_nTz_n \ ;$$
$$z_n = (1-w_n)x_n + w_nTx_n; \ n = 0,1,2,\cdots,\},$$

where $u_n, v_n, w_n \in [1,0]$ and $\{u_n\},\{v_n\},\{w_n\}$ are sequences of positive numbers. For the sake of simplicity, we take $u_n = u, v_n = v, w_n = w$.

2.1 Escape Criterion for Cubic Complex Polynomial

In computation of fractals like Julia and Mandelbrot sets, there is always need to establish a criterion called 'Escape criterion' which enacts a chief role in the graphical representation of Mandelbrot as well as Julia sets and these criteria are different for different order complex polynomials.
Here, in this part, we demonstrate the new escape criterion using NIP for the complex cubic polynomials:

$$P_{m,n}(z) = z^3 + mz + n$$

where m and n are complex numbers.
The following theorem and results provide the escape criterion using NIP for the above mentioned complex polynomial.

Theorem 2.1. *Consider that* $|z| > |n| > (|m| + 2/|u|)^{1/2}$, $|z| > |n| > (|m| + 2/|v|)^{1/2}$, *and* $|z| > |n| > (|m| + 2/|w|)^{1/2}$, *where* $0 < u < 1$, $0 < v < 1$, $0 < w < 1$, *and c is in the complex plane.*
Define

$$z_1 = (1-u)P_{m,n}(z) + uP_{m,n}(z) \ ,$$
$$z_2 = (1-u)P_{m,n}(z_1) + uP_{m,n}(z_1) \ ,$$
$$\vdots$$
$$z_n = (1-u)P_{m,n}(z_{n-1}) + uP_{m,n}(z_{n-1}) \ , n = 2,3,4,\cdots$$

where $P_{m,n}(z)$ *is the fuction of* z.
Then $|z| \to \infty$ *as* $n \to \infty$

Proof. Consider

$$\begin{aligned}
|P'_{m,n}(z)| &= |(1-w)z + wP''_{m,n}(z)| \ \ where P''_{m,n}(z) = z^3 + mz + n \\
&= |(1-w)z + w(z^3 + mz + n)| \\
&= |z - wz + wz^3 + mwz + wn)| \\
&\geq |wz^3 + mwz + z - wz| - |wn| \\
&\geq |z|(|wz^2 + mw + 1 - w|) - w|z|, && (\because |z| \geq n) \\
&\geq |z|(|wz^2 + mw| - |1 - w|) - w|z| \\
&= |z|(w|z^2 + m| - 1)
\end{aligned}$$

i.e.

$$|P'_{m,n}(z)| \geq |z|(w|z^2 + m| - 1) \tag{2.1}$$

Also

$$\begin{aligned}
|P_{m,n}(z)| &= |(1-v)P''_{m,n}(z)+vP'_{m,n}(z)| \\
&\geq |(1-v)(z^3+mz+n+v|z|(w|z^2+m|-1|)) \qquad [By\ using\ Eq.(2.1)] \\
&= |(1-v)z^3+m(1-v)z+n(1-v)+vw|z|(|z^2+m|-v|z||) \\
&\geq |(1-v)z^3+m(1-v)z|-(1-v)|z|+vw|z|(|z^2+m|-v|z|) \\
&= |z|[(1-v)|z^2+m|-1]+vw|z||z^2+m| \\
&\geq |z|[(vw-v+1)(|z^2|-|m|-1)]
\end{aligned}$$

Since,

$$z_n = (1-u)P_{m,n}(z_{n-1})+uP'_{m,n}(z_{n-1})$$

We have

$$\begin{aligned}
|z_1| &= |(1-u)P_{m,n}(z)+uP'_{m,n}(z)| \\
&\geq |(1-u)[|z|(vw-v+1)|z^2+m|-1]+u[|z|(w|z^2+m|-1)] \\
&\geq |z||(uv+vw-uvw-u-v+1)|z^2+m|-|(1-u)|z|| \\
&\quad +uw|z||z^2+m|-u|z| \\
&= |z|[(uv+vw+uw-uvw-u-v+1)|z^2+m|-1] \\
&\geq |z|[(uv+vw+uw-uvw-u-v+1)(|z^2|-|m|)-1] \\
&= |z|R[|z^2|-(|m|+1/R)]
\end{aligned}$$

where

$$R = uv+vw+uw-uvw-u-v+1$$

Since, $|z|>|n|>(|m|+2/|u|)^{1/2}$, $|z|>|n|>(|m|+2/|v|)^{1/2}$, and $|z|>|n|>(|m|+2/|w|)^{1/2}$, so that we have $|z|>(|m|+2/R)^{1/2}$, this mean $|z^2|-(|m|+1/R)>(1/R)$ so that $R(|z^2|-(|m|+1/R))>1$. Therefore there exists a $\lambda>0$ such that

$$|z_1| > (1+\lambda)|z|$$

Repeating the same argument, we get

$$|z_n| > (1+\lambda)^n|z|$$

Thus, orbit of z tends to infinity as $n\to\infty$. This completes the proof. □

Corollary 2.1.1. *Consider the complex cubic polynomial $P_{m,n}(z)=z^3+mz+n$ where m, n are complex numbers and assume $|z|>max[\ |n|,(|m|+2/|u|)^{1/2},(|m|+2/|v|)^{1/2},(|m|+2/|w|)^{1/2}]$ then $|z_n|>(1+\lambda)^n|z|$ and $|z_n|\to\infty$ as $n\to\infty$. This provides the 'escape criterion' for the above complex cubic polynomial.*

Corollary 2.1.2. *Assume*
$|z_k| > max[\ |n|, (|m|+2/|u|)^{1/2}, (|m|+2/|v|)^{1/2}, (|m|+2/|w|)^{1/2}]$ *for some* $k \geq 0$ *then* $|z_k| > (1+\lambda)^n |z_{k-1}|$ *and* $|z_n| \to \infty$ *as* $n \to \infty$.
This result provides an algorithm to generate cubic Julia sets for the above mentioned complex polynomial.

3 Result And Discussion

3.1 Fractals as Cubic Julia sets

Cubic Julia sets, applying new different fixed point technique, are generated through complex polynomial $P_{m,n}(z) = z^3 + mz + n$, using cubic escape criterion with the software Mathematica 10.0, See Figures 1-20.

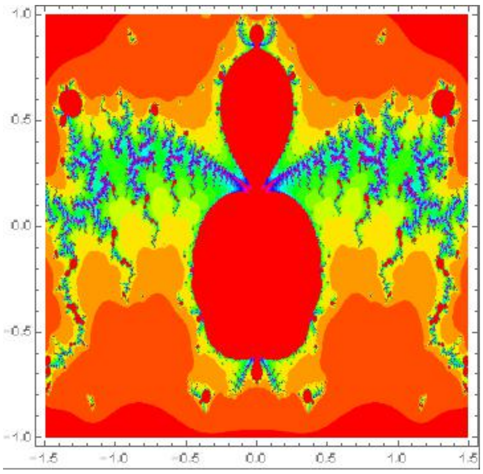

Figure 1: Ants Cubic Julia set for u= v=.48, w=.28, m=-3.2, n=.56I

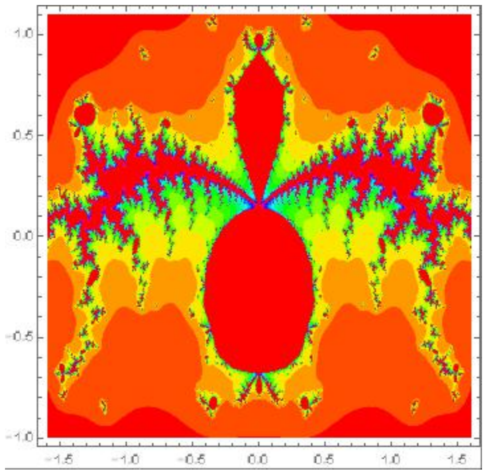

Figure 2: Ants Cubic Julia set for u= v=.477, w=.283, m=-3.13, n=.59I

3.2 Graphical Execution of Cubic Julia Sets Employed NIP

For the graphical execution of cubic Julia sets we iterate complex cubic polynomial $P_{m,n}(z) = z^3 + mz + n$, and define the prisoner set using escape criterion under the above new different iterative procedure. We have also generated some interested and ever seemed cubic Julia sets.

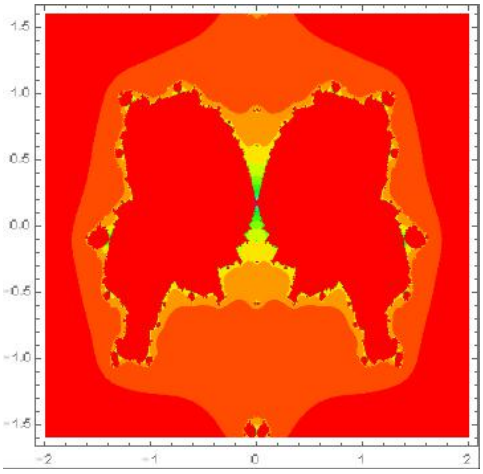

Figure 3: Cubic Julia set for u= v=.42, w=.05, m=-1.6, n=.45I

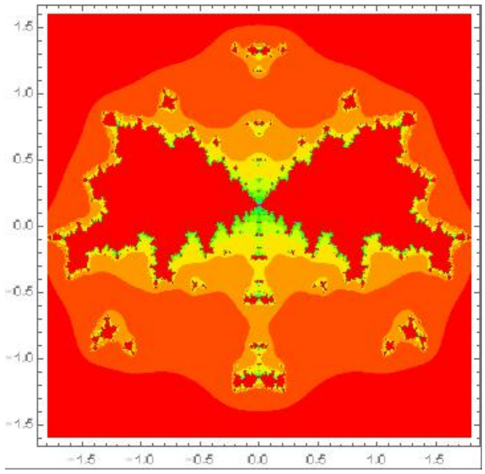

Figure 4: Cubic Julia set for u= v=.45, w=.09, m=-2.3, n=.52I

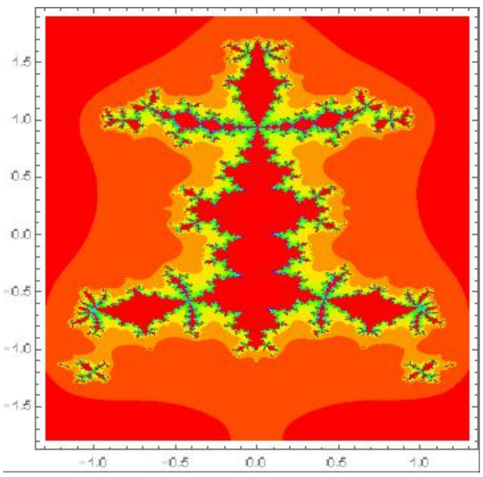

Figure 5: Cubic Julia set for u= v=.44 w=.07 m=.6-.01I n=1.2I

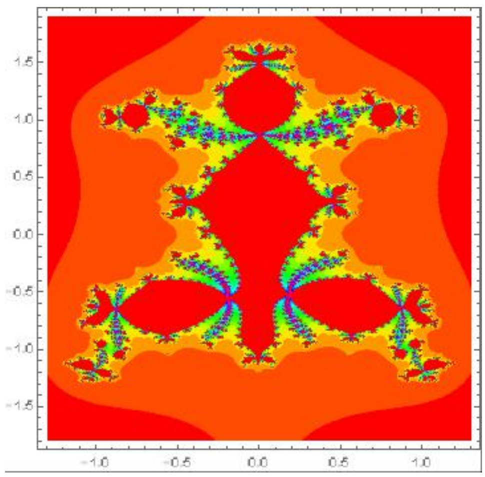

Figure 6: Cubic Julia set for u= v=.44 w=.03 m=.62-.01I n=1I

In this paper we have used two sets of parameters, $u, v, w$and m, n, and by changing parametric cost of these set of pairs, we have also noticed many important observations as follow:

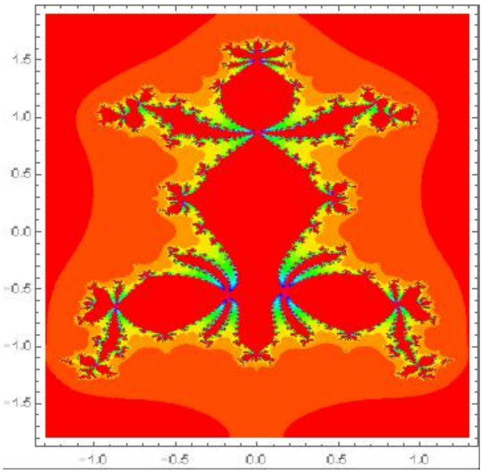

Figure 7: Cubic Julia set for u= v=.44 w=.07 m=.62-.01I n=1I

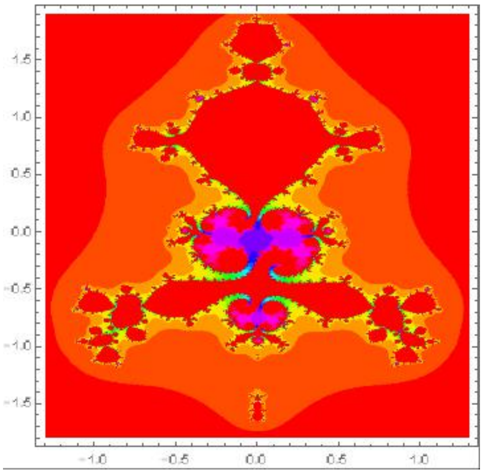

Figure 8: Cubic Julia set for u= v=.57 w=.05 m=.6+.01I n=1.2I

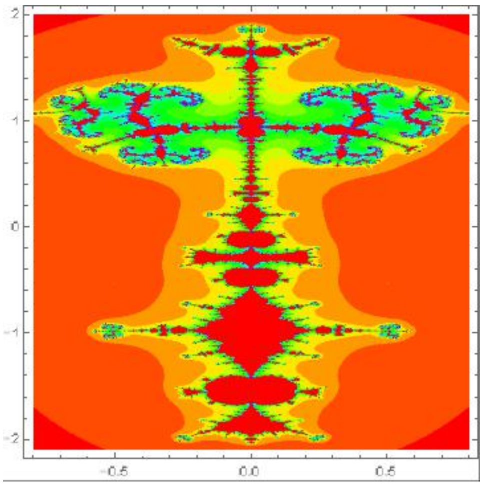

Figure 9: Antenna Cubic Julia set for u= v=.15 w=.83 m=2.5 n=-.80I

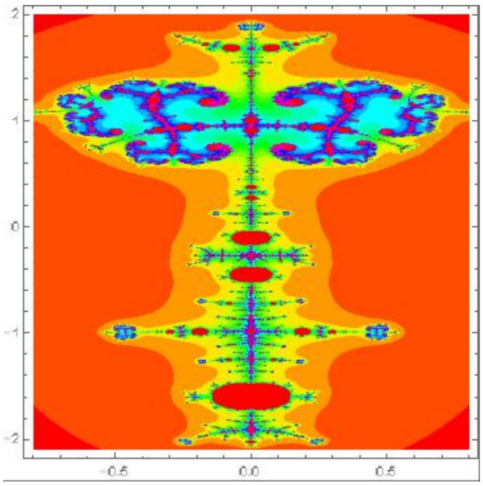

Figure 10: Antenna Cubic Julia set for u= v=.15 w=.83 m=2.62 n=-.80I

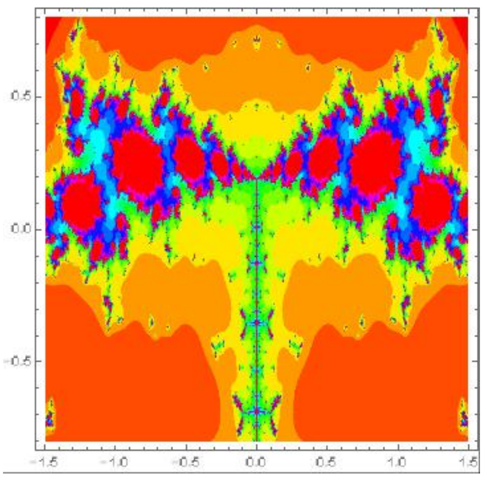

Figure 11: Antenna Cubic Julia set for u= v=.46 w=.366 m=-29.1 n=.75I

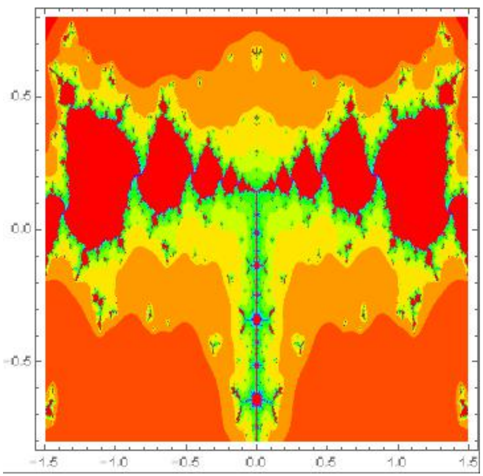

Figure 12: Antenna Cubic Julia set for u= v=.46 w=.366 m=-3.1 n=.59I

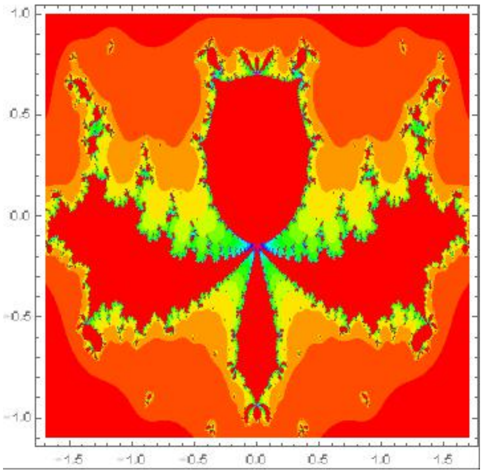

Figure 13: Cubic Julia set for u= v=.48,w=.32 m=-3.13, n=-.59I

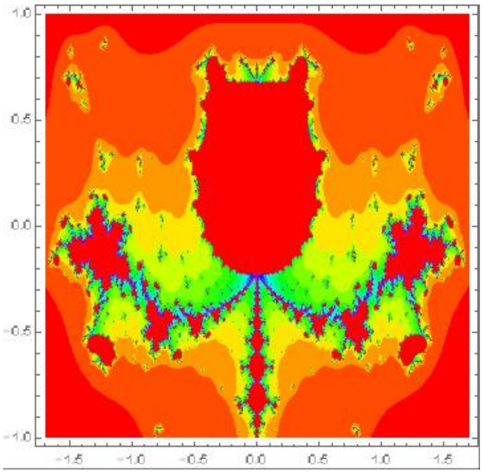

Figure 14: Cubic Julia set for u= v=.48, w=.32 m=-3.03, n=-.89I

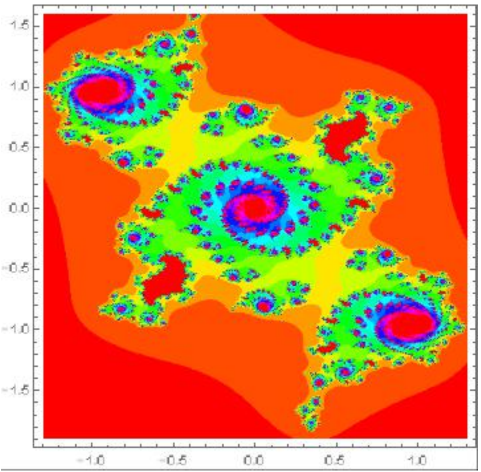

Figure 15: Cubic Julia set for u= v=.5, w=.03, m=1.2I n=0

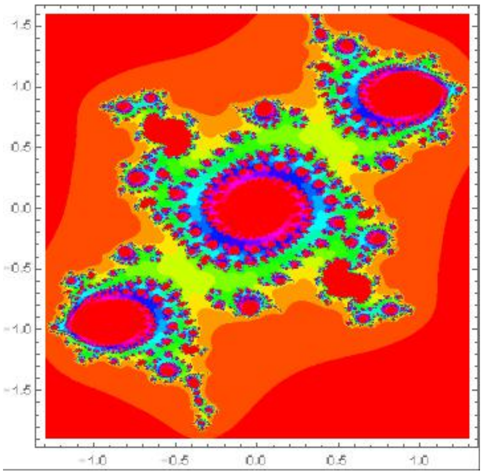

Figure 16: Cubic Julia set for u= v=.5, w=.05,m=-1.17I n=0

- It has been observed that the Figures 1 - 14 and Figures 17 - 20 show the perfect mathematical reflexive symmetry about imaginary axes only whereas the Figures 15 - 16 show the perfect mathematical reflexive symmetry about both the axes (real as well as imaginary axes).
- It is noticed that the prisoner sets of the cubic Julia sets in the Figures 1 - 5, Figure 17 and Figures 11 - 16 are mathematically disconnected whereas the prisoner sets of the remaining cubic Julia sets are mathematically connected.

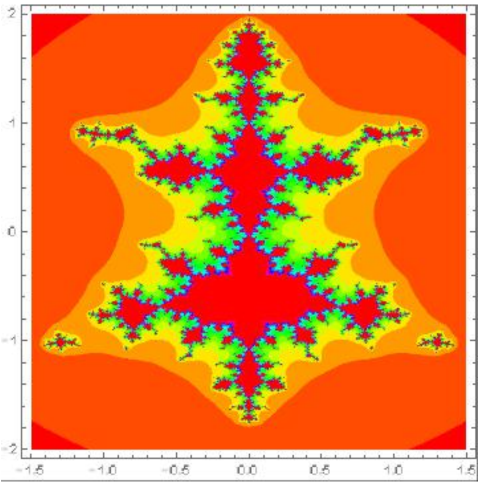

Figure 17: Cubic Julia set for u= v=.03 w=.95 m=1.0 n=1.001I

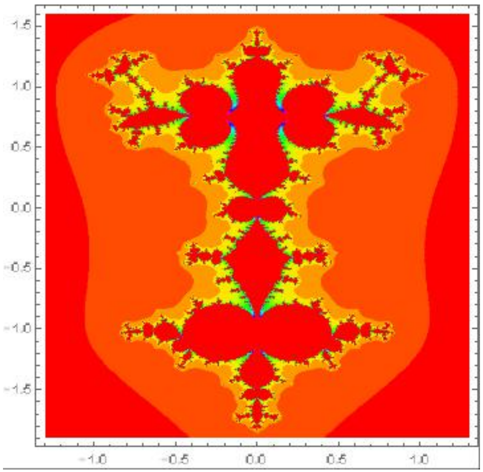

Figure 18: Cubic Julia set for u= v=.5 w=.01 m=1.3 n=-.80I

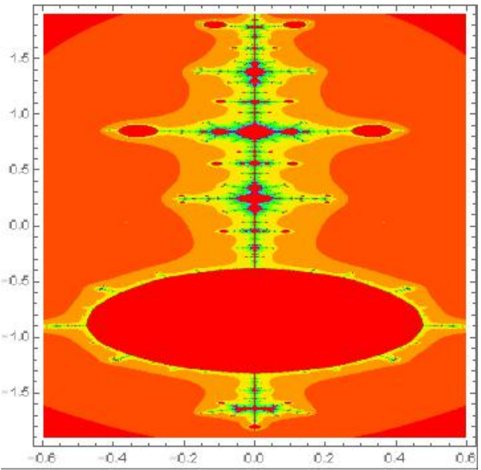

Figure 19: Cubic Julia set for u= v=.19 w=.376 m=2.2 n=.40I

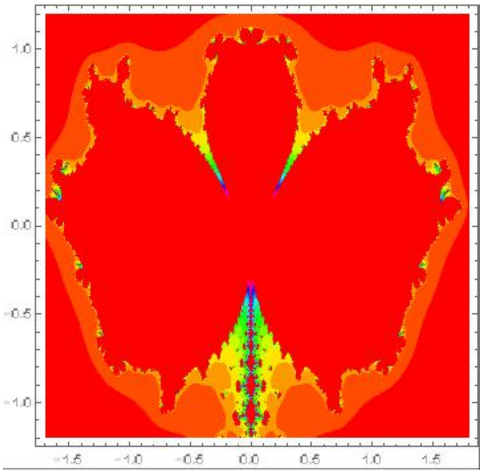

Figure 20: Butterfly Cubic Julia set for u= v=.5 w=.3 m=-2.03 n=-.6I

- It is also noticed that the cubic Julia sets in Figure 1 and Figure 2 capture the shape of ants, Figures 4 - 8, look like as meditating posture, the cubic Julia in Figures 9-12 and Figure 19 take the shape of antennas, the cubic Julia set in Figure 20 take the shape of a butterfly and the Julia sets in Figure 3-4 and Figures 13-14 are wall-decorated pictures.

Conclusion

In this research paper, a new different four-step iterative procedure has been applied to the complex cubic polynomial $P_{m,n}(z) = z^3 + mz + n$, and significant as well as exciting results have been obtained.We have obtained new and ever seen cubic Julia sets as output for the above complex cubic polynomial. Some of the generated cubic Julia sets have perfect mathematical reflexive symmetry about the axes. From above generated

10

cubic Julia sets, some Julia sets are connected and some are disconnected which is also an important mathematical property of fractals.It is also fascinating to see that some of the generated cubic Julia sets take the shape of antennas which is an important application of fractals, some cubic Julia sets capture the shape of ants and the butterfly, some cubic Julia sets look like as meditating posture, and some other look like as wall-decorated pictures.

References

[1] Abbas M and Nazir T (2014) A new iteration process applied to constained minimization and feasibility problems, Matemathykm Bechnk , 66, (2), 223-234.

[2] Branner B and Hubbard JH (1988) The Iteration of Cubic Polynomials Part I: The global topology of parameter space, Acta Math, 160, 143-206.

[3] Branner B and Hubbard JH (1992) The Iteration of Cubic Polynomials Part II: Patterns and para-patterns, Acta Math, 169(3-4), 229-325.

[4] Beardon AF (1991)Iteration of Rational Functions, Springer-Verlag, New York.

[5] Bunde A and Havlin S (1996) Fractals and Disordered Systems, Springer-Verlag, Berlin, Heidelberg.

[6] Chauhan YS, Rana R and Negi A (2010) New Julia sets of Ishikawa Iterates, International Journal of Computer Applications, 7(13), 34-42.

[7] Cheng C and Liaw S (1998) Similarity between the dynamic and parameter spaces in cubic mappings, Fractals, 6(3), 293-299.

[8] Conway JB (1978) Functions of One Complex Variable, Second Edition, Springer-Verlag, New York.

[9] Devaney RL (1992) A First Course in Chaotic Dynamical Systems: Theory and Experiment, Addison Wesley.

[10] Dubey VP, Singh J, Alshehri AM, Dubey S and Kumar D (2021) A comparative analysis of two computational schemes for solving local fractional Laplace equations, Mathematical Methods in the Applied Sciences, 44(17), 13540-13559.

[11] Dubey VP, Dubey S, Kumar D and Singh S (2021) A computational study of fractional model of atmospheric dynamics of carbon dioxide gas, Chaos, Solitons and Fractals, 142, 110375.

[12] Douady A and Hubbard JH (1984) Etude dynamique des pïolynomes complexes, Publications Mathematiques d'Orsay , Universitïe de Paris-Sud.

[13] Epstein A and Yampolsky M (1999) Geography of the cubic connectedness locus: intertwining surgery, Ann. Sci. École Norm Sup.4, 32, 151-185.

[14] Fraser J (2009) An Introduction to Julia sets, http://www.mcs.st and.ac.uk/ jmf32/julia.pdf.

[15] Fu C, Jiang H, Yi X and Zhang Z (2006) An accelerated algorithm of constructing general high-order Mandelbrot and Julia sets, Advances in Natural Computation, Proc. Part II ICNC (Sep 24-28, Xi'an, China), Springer, Berlin Heidelberg, 4222, 793-796.

[16] Holmgren RA (1994) A First Course in Discrete Dynamical Systems, Springer-Verlag, New York, Inc.

[17] Kumar M and Rani M (2005) A new approach to superior Julia sets,J. Nature. Phys. Sci., 19(2), 148-155.

[18] Kigami J (2001) Analysis on Fractals, Cambridge Univ. Press, Cambridge.

[19] Lei T (1990) Similarity between the Mandelbrot sets and Julia sets, Commun. Math. Phys., 134(3), 587-617.

[20] Liaw S (1998) The parameter spaces of the cubic polynomials, Fractals, 6(2), 181-189.

[21] Liaw S (2001) Structure of the cubic mappings, Fractals,9(2), 231-235.

[22] Milnor J (2000) Local connectivity of Julia sets: expository lectures, The Mandelbrot Set, Theme and Variations (Edited by Lei Ten), London Math. Soc. Lecture Note Ser., 274, Cambridge University Press, 67-116.

[23] Partap N, Jain S and Chugh R (2018) Computation of Antifractals-Tricorns and Multicorns with their complex nature, Pertanika Journal of science and technology, 26(2), 863 - 872.

[24] Partap N, Jain S and Chugh R (2019) Supremacy of s-convexity scheme and generation of advanced fractal using faster iteration, *Nonlinear Studies*, 26(2), 289 - 301.

[25] Partap N, Jain S and Chugh R (2020) Computation of Newborn J-Sets via Fixed Point Iteration and Their Complex Nature, in: IEEE Proc. ICRITO 2020, Noida, India, 1089 - 1093.

[26] Partap N, Jain S and Chugh R (2020) Reckoning of Newborn M-Sets with Their Complex Nature via Fixed Point Iteration, in: IEEE Proc. ICRITO 2020, Noida, India, 1123 - 1126.

[27] Peitgen HO, Jurgens H and Saupe D (1992) Chaos and Fractals, Springer-Verlag, New York, Inc.

[28] Peitgen HO, Jurgens H and Saupe D (2004) Chaos and Fractals: New frontiers of science (2nd ed.), Springer-Verlag, New York, Inc.

[29] Rahman G, Nisar KS and Golamankaneh AK (2021) The Nonlocal Fractal Integral Reverse Minkowski's and Other Related Inequalities on Fractal Sets, Mathematical Problems in Engineering, 21, 10 pages. Article ID 4764891.

[30] Rani M and Nagi A (2008) New Julia sets for complex Carotid-Kundalini function, Chaos,Solitons and Fractals, 36(2), 226-236.

[31] Rani M and Agarwal R (2009) Generation of fractals from complex logistic map, *Chaos,Solitons and Fractals* **41**, 1, 447-452, 2009.

[32] Rani M (2010) Superior antifractals, in:IEEE Proc. ICCAE 2010,1(1), 798-802.

[33] Rani M (2010a) Superior tricorns and multicorns,in: WSEAS Proc. 9th Int. conf. on Applications of Computer Engineering (ACE '10): Recent Advances and Applications of Computer Engineering, 58-61.

[34] Rani M and Agarwal R (2010) Effect of noise on Julia sets generated by logistic map, in: Proc. 2nd IEEE International Conference on Computer and Automation Engineering (ICCAE 2010), 2, 55-59.

[35] Rani M and Agarwal R (2010) Effect of stochastic noise on superior Julia sets, J. Math. Imaging Vis., 36, 63-68.

[36] Rani M (2011) Cubic superior Julia sets, in: Proc. European Computing Conference,80-84.

[37] Rashid *et.al.*(2020) Multidimensional fixed points in generalized distance spaces,Advances in Difference Equations, 20, 20 pages. Article Number 571.

[38] Singh J,Kumar D,Purohit SD, Mishra AM and Bohra M (2020) An efficient numerical approach for fractional multidimensional diffusion equations with exponential memory,Numerical Methods for Partial Differential Equations, 37(2), 1631-1651.

[39] Steinmnetz N (1993)Rational Iteration: Complex Analytic Dynamical Systems, Walter de Cruyter, Berlin, New York.

[40] Tomova AV(2003) The Mandelbrot set for Julia sets of arbitrary order; A remark on the shape of cubic Mandelbrot and Julia sets, Mandelbrot and Julia sets for the polynomials of arbitrary order, Proc ICT and P, Varna, Bulgaria, FOI-COMMERCE, Sofia, 78-88.

[41] Yan D, Liu X and Zhu W (1999) A Study of Mandelbrot and Julia Sets generated from a general complex cubic Iteration, Fractals, 7(4), 433-437.