Topological Indices of SK-join of Graphs

Roy John[1*], Akhil B.[2], Manju V.N.[3], Basil Roy Idiculla[4]

[1]Department of Mathematics, St.Stephen's College, Pathanapuram-689695, Kerala, India,
Email: roymaruthoor@gmail.com
[2]Department of Mathematics, University of Kerala,Thiruvananthapuram-695581,Kerala, India,
Email: akhilb@keralauniversity.ac.in
[3]Department of Mathematics, University of Kerala,Thiruvananthapuram-695581, Kerala, India,
Email: manjushaijulal@gmail.com
[4]Department of Mathematics, Marthoma College, Kuttapuzha, Tiruvalla -689103, Kerala,India,
Email: baasilroy@gmail.com
*Corresponding Author

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ABSTRACT
Topological indices are useful tool for characterizing chemical constitution. Graph operations is a significant field of graph theory in which new graphs are constructed from the given graphs. In this paper, topological indices of certain graphs obtained through a noval join operation called SK-join are determined.

Keywords: SK-join, first and second Zagreb indices, Harmonic index, Geometric-Arithmetic index.

1. INTRODUCTION

For a graph $G = (V(G), E(G))$ with order $n = |V(G)|$ and size $m = |E(G)|$, the degree of a vertexv_i, in G is the number of edges incident on v_i and is denoted by $d(v_i)$ or simply d_i . The distance between two vertices u and v in G is the length of the shortest path joining them and it is represented by $d(u, v)$ in G. All graphs under our consideration are simple, finite and undirected graphs. We follow [1] for more terminologies and notations.

Chemical graph theory is an interdisciplinary field in which relevant mathematical questions are explored using graph theoretical and computational methods, and the molecular structure of a chemical product is examined as a graph. We now have numerous noval and distinctive concepts and techniques for these kinds of studies because to the field's recent rapid progress.The concept of "topological indices" is among the most significant ones used in chemical graph theory[5].

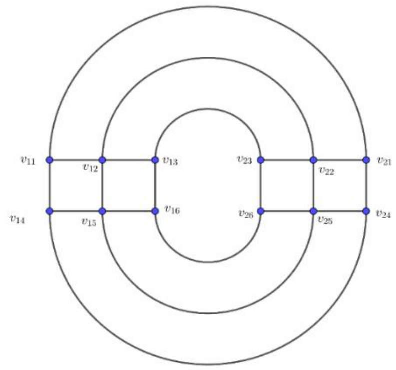
In graph theory, graph operation is a renowned field where new graphs are created by applying rules to existing graphs. Various parameters of a given graph can be made more widely applicable by applying new graph operations to it. In 2020, G. Suresh Singh and Koshy introduced a new join operation called the SK-join of two graphs G_1 and G_2 [4]. In this paper topological indices of SK-join of certain classes of graphs are determined.

2. Basic Concepts

Definition 1. The cartesian product of two graphs G_1 and G_2, denoted by $G_1 \times G_2$ is a graphwith vertex set $V(G_1) \times V(G_2)$ and (u_i , v_j),(u_k , v_l) are adjacent in $G_1 \times G_2$ if and only if $u_i = u_k$ and $v_jv_l \in E(G_2)$ or $u_iu_k \in E(G_1)$ and $v_j = v_l$.

Definition 2.[4]Let G_1 and G_2 be two nonempty, simple, finite and undirected graphs and $G_1 \times G_2$ be their cartesian product. Take two copies of $G_1 \times G_2$ and let it be G' and G''. Let G be a new graph with the vertices v_{ij} where $i = 1, 2$ and j varies from $1, 2, \ldots, |V(G_1 \times G_2)|$. Now new graph, known as SK-join is obtained by joining v_{1j} to v_{2j} for all j. We denote this new graph by $G' \diamondsuit G''$.

Example1. Consider two copies G' and G'' of $P_2 \times P_3$ and join the corresponding vertices as follows:



Note 1. The cartesian product of two graphs G_1 and G_2 is connected if and only if both the graphs are connected

Note 2. Let G' and G'' be the two copies of $G_1 \times G_2$, then the SK-join$G' \diamondsuit G''$, is a connected graph if and only if, G_1 and G_2 is connected.

Definition 3. [2] Let G be a graph, then the first and second Zagreb indices of G are defined as

$$M_1(G) = \sum_{u \in V(G)} d(u) + d(v)$$

$$M_2(G) = \sum_{uv \in E(G)} d(u)d(v) .$$

Definition 4. [3] The Harmonic index of a graph G is defined as

$$H(G) = \sum_{uv \in E(G)} \frac{2}{d(u) + d(v)} .$$

Definition 5. [6] The Geometric-Arithmetic index of a graph G is defined as

$$GA(G) = \sum_{uv \in E(G)} \frac{2\sqrt{d(u)d(v)}}{d(u) + d(v)}$$

3. Topological indices of SK-join of graphs

In this section we compute the first and second Zagreb indices, Harmonic index and Geometric-Arithmetic index of recently constructed new graphs, denoted by $G' \diamondsuit G''$ of some standard graphs.

Theorem 1. For P_2 and P_3, let G' and G'' be two copies of $P_2 \times P_3$, then the first and second Zagreb indices, Harmonicindex and Geometric-Arithmetic index of $G' \diamondsuit G''$ are respectively given by:

1. $M_1(G' \diamondsuit G'') = 2(32n - 28)$
2. $M_2(G' \diamondsuit G'') = 8(16n - 19)$
3. $H(G' \diamondsuit G'') = 2n - \frac{1}{21}$
4. $GA(G' \diamondsuit G'') = 8n + \frac{-84+32\sqrt{3}}{7}$

Proof: We haveG' and G'', the two copies of $P_2 \times P_3$. The edge set of $G' \diamondsuit G''$ can be partitioned as follows:

$$E(G' \diamondsuit G'') = E_1 \cup E_2 \cup E_3,$$

Where,

$$E_1 = \{uv | d_u = 3, d_v = 3\}, |E_1| = 8$$

$$E_2 = \{uv | d_u = 3, d_v = 4\}, |E_2| = 8$$

$$E_3 = \{uv | d_u = 4, d_v = 4\}, |E_3| = 8n - 20.$$

$$M_1(G' \diamondsuit G'') = \sum_{uv \in E(G' \diamondsuit G'')} d(u) + d(v)$$

$$= \sum_{uv \in E_1} d(u) + d(v) + \sum_{uv \in E_2} d(u) + d(v) + \sum_{uv \in E_3} d(u) + d(v)$$

$$= \sum_{uv \in E_1} 3 + 3 + \sum_{uv \in E_2} 3 + 4 + \sum_{uv \in E_3} 4 + 4$$

$$= 6\sum_{uv\in E_1} 1 + 7\sum_{uv\in E_2} 1 + 8\sum_{uv\in E_3} 1$$
$$= 6\times 8 + 7\times 8 + 8\times(8n-20)$$
$$= 2(32n-28).$$

$$M_2(G'\diamond G'') = \sum_{uv\in E(G'\diamond G'')} d(u)d(v)$$
$$= \sum_{uv\in E_1} d(u)d(v) + \sum_{uv\in E_2} d(u)d(v) + \sum_{uv\in E_3} d(u)d(v)$$
$$= \sum_{uv\in E_1} 3\times 3 + \sum_{uv\in E_2} 3\times 4 + \sum_{uv\in E_3} 4\times 4$$
$$= 9\sum_{uv\in E_1} 1 + 12\sum_{uv\in E_2} 1 + 16\sum_{uv\in E_3} 1$$
$$= 9\times 8 + 12\times 8 + 16\times(8n-20)$$
$$= 8(16n-19).$$

$$H(G'\diamond G'') = \sum_{uv\in E(G'\diamond G'')} \frac{2}{d(u)+d(v)}$$
$$= \sum_{uv\in E_1} \frac{2}{d(u)+d(v)} + \sum_{uv\in E_2} \frac{2}{d(u)+d(v)} + \sum_{uv\in E_3} \frac{2}{d(u)+d(v)}$$
$$= \sum_{uv\in E_1} \frac{2}{3+3} + \sum_{uv\in E_2} \frac{2}{3+4} + \sum_{uv\in E_3} \frac{2}{4+4}$$
$$= \frac{1}{3}\sum_{uv\in E_1} 1 + \frac{2}{7}\sum_{uv\in E_2} 1 + \frac{1}{4}\sum_{uv\in E_3} 1$$
$$= \tfrac{1}{3}\times 8 + \tfrac{2}{7}\times 8 + \tfrac{1}{4}\times(8n-20)$$
$$= 2n - \tfrac{1}{21}.$$

$$GA(G'\diamond G'') = \sum_{uv\in E(G'\diamond G'')} \frac{2\sqrt{d(u)d(v)}}{d(u)+d(v)}$$
$$= \sum_{uv\in E_1} \frac{2\sqrt{d(u)d(v)}}{d(u)+d(v)} + \sum_{uv\in E_2} \frac{2\sqrt{d(u)d(v)}}{d(u)+d(v)} + \sum_{uv\in E_3} \frac{2\sqrt{d(u)d(v)}}{d(u)+d(v)}$$
$$= \sum_{uv\in E_1} \frac{2\sqrt{3\times 3}}{3+3} + \sum_{uv\in E_2} \frac{2\sqrt{3\times 4}}{3+4} + \sum_{uv\in E_3} \frac{2\sqrt{4\times 4}}{4+4}$$
$$= \sum_{uv\in E_1} 1 + \frac{4\sqrt{3}}{7}\sum_{uv\in E_2} 1 + \sum_{uv\in E_3} 1$$
$$= 8 + \frac{4\sqrt{3}}{7}\times 8 + 8n - 20$$
$$= 8n + \frac{-84+32\sqrt{3}}{7}.$$

Hence the proof.

Theorem 2. For the graphs C_3 and C_n, let the two copies of $C_3 \times C_n$ be denoted by G' and G''. Then for the graph $G'\diamond G''$, first and second Zagreb indices, Harmonic index and Geometric-Arithmetic index are respectively given by:

1. $M_1(G'\diamond G'') = 150n$
2. $M_2(G'\diamond G'') = 375n$
3. $H(G'\diamond G'') = 3n$
4. $GA(G'\diamond G'') = 15n.$

Proof. The graph $G'\diamond G''$ is a 5- regular graph with $15n$ edges.

$$M_1(G'\diamond G'') = \sum_{uv\in E(G'\diamond G'')} d(u)+d(v) = \sum_{uv\in E(G'\diamond G'')} 5+5 = 10\sum_{uv\in E(G'\diamond G'')} 1 = 10\times 15n = 150n.$$

$$M_2(G'\diamond G'') = \sum_{uv\in E(G'\diamond G'')} d(u)d(v) = \sum_{uv\in E(G'\diamond G'')} 5\times 5 = 25 \sum_{uv\in E(G'\diamond G'')} 1 = 25\times 15n = 375n.$$

$$H(G'\diamond G'') = \sum_{uv\in E(G'\diamond G'')} \frac{2}{d(u)+d(v)} = \sum_{uv\in E(G'\diamond G'')} \frac{2}{5+5}$$

$$= \frac{1}{5} \sum_{uv\in E(G'\diamond G'')} 1 = \frac{1}{5}\times 15n = 3n.$$

$$GA(G'\diamond G'') = \sum_{uv\in E(G'\diamond G'')} \frac{2\sqrt{d(u)d(v)}}{d(u)+d(v)} = \sum_{uv\in E(G'\diamond G'')} \frac{2\sqrt{5\times 5}}{5+5}$$

$$= \sum_{uv\in E(G'\diamond G'')} 1 = 15n.$$

Thus, the theorem is proved

Theorem 3. Let the two copies of $K_m \times K_n$ be denoted by G' and G''. Then first and second Zagreb indices, Harmonic index and Geometric-Arithmetic index of the graph $G'\diamond G''$ are respectively given by:

1. $M_1(G'\diamond G'') = 2mn(m+n-1)^2$
2. $M_2(G'\diamond G'') = mn(m+n-1)^3$
3. $H(G'\diamond G'') = mn$
4. $GA(G'\diamond G'') = mn(m+n-1)$.

Proof. The graph of $G'\diamond G''$ is $m-1+n-1+1 = m+n-1$ regular with $2\left(m\binom{n}{2}+n\binom{m}{2}\right)+mn = mn(m+n-1)$ edges.

$$M_1(G'\diamond G'') = \sum_{uv\in E(G'\diamond G'')} d(u)+d(v) = \sum_{uv\in E(G'\diamond G'')} (m+n-1)+(m+n-1)$$

$$= 2(m+n-1) \sum_{uv\in E(G'\diamond G'')} 1 = 2(m+n-1)\left(2\left(m\binom{n}{2}+n\binom{n}{2}\right)+mn\right)$$

$$= (m+n-1)^2\left(2\left(m\frac{n(n-1)}{2}+n\frac{m(m-1)}{2}\right)+mn\right)$$

$$= 2(m+n-1)\left(\left(mn(n-1)+mn(m-1)\right)+mn\right)$$

$$= 2(m+n-1)mn(n-1+m-1+1) = 2mn(m+n-1)^2.$$

$$M_2(G'\diamond G'') = \sum_{uv\in E(G'\diamond G'')} d(u)d(v) = \sum_{uv\in E(G'\diamond G'')} (m+n-1)(m+n-1)$$

$$= (m+n-1)^2 \sum_{uv\in E(G'\diamond G'')} 1$$

$$= (m+n-1)^2\left(2\left(m\binom{n}{2}+n\binom{n}{2}\right)+mn\right)$$

$$= (m+n-1)^2\left(2\left(m\frac{n(n-1)}{2}+n\frac{m(m-1)}{2}\right)+mn\right)$$

$$= (m+n-1)^2\left(\left(mn(n-1)+mn(m-1)\right)+mn\right)$$

$$= (m+n-1)^2 mn(n-1+m-1+1) = mn(m+n-1)^3.$$

$$H(G'\diamond \mathrm{G}'') = \sum_{uv\in E(G'\diamond G'')} \frac{2}{d(u)+d(v)}$$

$$= \sum_{uv\in E(G'\diamond G'')} \frac{2}{2(m+n-1)}$$

$$= \frac{1}{m+n-1} \sum_{uv\in E(G'\diamond G'')} 1$$

$$= \frac{1}{m+n-1} mn(m+n-1) = mn.$$

$$GA(G'\diamond G'') = \sum_{uv\in E(G'\diamond G'')} \frac{2\sqrt{d(u)d(v)}}{d(u)+d(v)} = \sum_{uv\in E(G'\diamond G'')} \frac{2\sqrt{(m+n-1)(m+n-1)}}{2(m+n-1)} = \sum_{uv\in E(G'\diamond G'')} 1$$
$$= mn(m+n-1).$$

Hence the proof.

Theorem 4. Let G' and G'' be the two copies of$K_m \times W_n$, where W_n is a wheel of order$n+1$. Then for the graph $G'\diamond G''$, first and second Zagreb indices, Harmonic index and Geometric-Arithmetic index are respectively given by:

1. $M_1(G'\diamond G'') = 2(m+n)m^2 + 2mn(m^2+7m+n+9)$
2. $M_2(G'\diamond G'') = m^2(m+n)^2 + 2mn(m+n)(m+3) + mn(m+2)(m+3)^2$
3. $H(G'\diamond G'') = \dfrac{m^2}{m+n} + \dfrac{4mn}{2m+n+3} + \dfrac{mn(m+2)}{m+3}$
4. $GA(G'\diamond G'') = m^2 + mn(m+2) + \dfrac{4mn\sqrt{(m+n)(m+3)}}{2m+n+3}.$

Proof. The edge set of $G'\diamond G''$can be partitioned as follows:

$$E_1 = \{uv | d_u = m+n, d_v = m+n\}, |E_1| = 2\binom{m}{2} + m = 2\frac{m(m-1)}{2} + m = m^2$$
$$E_2 = \{uv | d_u = m+n, d_v = m+3\}, |E_2| = 2mn$$
$$E_3 = \{uv | d_u = m+3, d_v = m+3\},$$
$$|E_3| = 2\left[n\binom{m}{2} + mn\right] + mn = 2n\frac{m(m-1)}{2} + 3m = mn(n+2).$$

$$M_1(G'\diamond G'') = \sum_{uv\in E(G'\diamond G'')} d(u)+d(v) = \sum_{uv\in E_1} d(u)+d(v) + \sum_{uv\in E_2} d(u)+d(v) + \sum_{uv\in E_3} d(u)+d(v)$$
$$= \sum_{uv\in E_1} (m+n)+(m+n) + \sum_{uv\in E_2} (m+n)+(m+3) + \sum_{uv\in E_3} 4\,(m+3)$$
$$+(m+3) = 2(m+3)\sum_{uv\in E_1} 1 + (2m+n+3)\sum_{uv\in E_2} 1 + 2(m+3)\sum_{uv\in E_3} 1$$
$$= 2(m+n)m^2 + (2m+n+3)2mn + 2(m+3)mn(m+2)$$
$$= 2(m+n)m^2 + 2mn(m^2+7m+n+9).$$

$$M_2(G'\diamond G'') = \sum_{uv\in E(G'\diamond G'')} d(u)d(v) = \sum_{uv\in E_1} d(u)d(v) + \sum_{uv\in E_2} d(u)d(v) + \sum_{uv\in E_3} d(u)d(v)$$
$$= \sum_{uv\in E_1} (m+n)(m+n) + \sum_{uv\in E_2} (m+n)(m+3) + \sum_{uv\in E_3} 4\,(m+3)(m+3)$$
$$= (m+3)^2\sum_{uv\in E_1} 1 + (m+n)(m+3)\sum_{uv\in E_2} 1 + (m+3)^2\sum_{uv\in E_3} 1$$
$$= (m+n)^2m^2 + (m+n)(m+3)2mn + (m+3)^2mn(m+2)$$
$$= m^2(m+n)^2 + 2mn(m+n)(m+3) + mn(m+2)(m+3)^2.$$

$$H(G'\diamond G'') = \sum_{uv\in E(G'\diamond G'')} \frac{2}{d(u)+d(v)}$$
$$= \sum_{uv\in E_1} \frac{2}{m+n+m+n} + \sum_{uv\in E_2} \frac{2}{m+n+m+3} + \sum_{uv\in E_3} \frac{2}{m+3+m+3}$$
$$= \frac{1}{m+n}\sum_{uv\in E_1} 1 + \frac{2}{2m+n+3}\sum_{uv\in E_2} 1 + \frac{1}{m+3}\sum_{uv\in E_3} 1$$
$$= \frac{m^2}{m+n} + \frac{4mn}{2m+n+} + \frac{mn(m+2)}{m+3}.$$

$$GA(G^{'}\diamond G^{''}) = \sum_{uv\in E(G^{'}\diamond G^{''})} \frac{2\sqrt{d(u)d(v)}}{d(u)+d(v)}$$
$$= \sum_{uv\in E_1} \frac{2\sqrt{(m+n)(m+n)}}{m+n+m+n} + \sum_{uv\in E_2} \frac{2\sqrt{(m+n)(m+3)}}{m+n+m+3} + \sum_{uv\in E_3} \frac{2\sqrt{(m+3)(m+3)}}{m+3+m+3}$$
$$= \sum_{uv\in E_1} 1 + \frac{2\sqrt{(m+n)(m+3)}}{2m+n+3} \sum_{uv\in E_2} 1 + \sum_{uv\in E_3} 1$$
$$= m^2 + \frac{4mn\sqrt{(m+n)(m+3)}}{2m+n+3} + mn(n+2)$$
$$= m^2 + mn(m+2) + \frac{4mn\sqrt{(m+n)(m+3)}}{2m+n+3}.$$

Hence the proof.

Theorem 5. Let G' and G'' be the two copies of $P_2 \times W_n$, where W_n is a wheel of order$n+1$, with $n \geq 3$. Then for the graph $G'\diamond G''$, first and second Zagreb indices, Harmonic index and Geometric-Arithmetic index are respectively given by:

1. $M_1(G^{'}\diamond G^{''}) = 4n^2 + 116n + 16$
2. $M_2(G^{'}\diamond G^{''}) = 4(n+2)^2 + 20n(n+12)$
3. $H(G^{'}\diamond G^{''}) = \frac{4}{n+2} + \frac{8n}{n+7} + \frac{8n}{5}$
4. $GA(G^{'}\diamond G^{''}) = 8n + \frac{8n\sqrt{5(n+2)}}{n+7} + 4.$

Proof. The edge set of $G^{'}\diamond G^{''}$ can be partitioned as follows:

$$E_1 = \{uv \mid d_u = n+2, d_v = n+2\}, |E_1| = 4$$
$$E_2 = \{uv \mid d_u = n+2, d_v = 5\}, |E_2| = 4n$$
$$E_3 = \{uv \mid d_u = 5, d_v = 5\}, |E_3| = 8n.$$

$$M_1(G^{'}\diamond G^{''}) = \sum_{uv\in E(G^{'}\diamond G^{''})} d(u)+d(v) = \sum_{uv\in E_1} (n+2)+(n+2) + \sum_{uv\in E_2} (n+2)+5 + \sum_{uv\in E_3} 5+5$$
$$= 2(n+2)\sum_{uv\in E_1} 1 + (n+7)\sum_{uv\in E_2} 1 + 10\sum_{uv\in E_3} 1$$
$$= 2(n+2)\times 4 + (n+7)\times 4n + 10\times 8n$$
$$= 4n\text{^}2 + 116n + 16.$$

$$M_2(G^{'}\diamond G^{''}) = \sum_{uv\in E(G^{'}\diamond G^{''})} d(u)d(v) = \sum_{uv\in E_1} (n+2)(n+2) + \sum_{uv\in E_2} (n+2)5 + \sum_{uv\in E_3} 5\times 5$$
$$= (n+2)^2 \sum_{uv\in E_1} 1 + 5(n+2)\sum_{uv\in E_2} 1 + 25\sum_{uv\in E_3} 1$$
$$= (n+2)^2 \times 4 + 5(n+2)\times 4n + 25\times 8n$$
$$= 4(n+2)^2 + 20n(n+12).$$

$$H(G^{'}\diamond G^{''}) = \sum_{uv\in E(G^{'}\diamond G^{''})} \frac{2}{d(u)+d(v)}$$
$$= \sum_{uv\in E_1} \frac{2}{n+2+n+2} + \sum_{uv\in E_2} \frac{2}{n+2+5} + \sum_{uv\in E_3} \frac{2}{5+5}$$
$$= \frac{1}{n+2}\sum_{uv\in E_1} 1 + \frac{2}{n+7}\sum_{uv\in E_2} 1 + \frac{1}{5}\sum_{uv\in E_3} 1$$
$$= \frac{4}{n+2} + \frac{8n}{n+7} + \frac{8n}{5}.$$

$$GA(G'\diamond G'') = \sum_{uv\in E(G'\diamond G'')} \frac{2\sqrt{d(u)d(v)}}{d(u)+d(v)} = \sum_{uv\in E_1} \frac{2\sqrt{(n+2)(n+2)}}{n+2+n+2} + \sum_{uv\in E_2} \frac{2\sqrt{(n+2)5}}{n+2+5} + \sum_{uv\in E_3} \frac{2\sqrt{5\times 5}}{5+5}$$
$$= \sum_{uv\in E_1} 1 + \frac{2\sqrt{5(n+2)}}{n+7} \sum_{uv\in E_2} 1 + \sum_{uv\in E_3} 1 = 4 + \frac{2\sqrt{5(n+2)}}{n+7} 4n + 8n$$
$$= 8n + \frac{8n\sqrt{5(n+2)}}{n+7} + 4.$$

Hence the theorem.

Theorem 6. Let the two copies of $P_3 \times W_n$ be G' and G'', W_n is a wheel of order $n + 1$with $n \geq 3$. Then for the graph $G'\diamond G''$, first and second Zagreb indices, Harmonic index and Geometric-Arithmetic index are respectively given by:

1. $M_1(G'\diamond G'') = \begin{cases} 688, & for\ n = 3 \\ 930, & for\ n = 4 \\ 2(2n^2 + 100n + 17), & for\ n \geq 5 \end{cases}$

2. $M_2(G'\diamond G'') = \begin{cases} 1856, & for\ n = 3 \\ 2617, & for\ n = 4 \\ (n+3)^2 + 8(n+3)(2n+1) + 2(n+2)(11n+2) + 378n, & for\ n \geq 5 \end{cases}$

3. $H(G'\diamond G'') = \begin{cases} 11.98, & for\ n = 3 \\ 11.10, & for\ n = 4 \\ \frac{1}{n+3} + \frac{8}{2n+5} + \frac{4n}{n+9} + \frac{8n}{n+7} + \frac{2}{n+2} + \frac{n}{2} + \frac{8n}{11} + \frac{6n}{5}, & for\ n \geq 5. \end{cases}$

4. $GA(G'\diamond G'') = \begin{cases} 63.93, & for\ n = 3 \\ 82.83, & for\ n = 4 \\ \frac{8\sqrt{(n+3)(n+2)}}{2n+5} + \frac{4n\sqrt{6(n+3)}}{n+9} + \frac{8n\sqrt{5(n+2)}}{n+7} + \frac{8n\sqrt{30}}{11} + 9n + 3, & for\ n \geq 5 \end{cases}$

Proof: For $n = 3$, The edge set can be partitioned as follows:

$$E_1 = \{uv | d_u = 6, d_v = 6\}, |E_1| = 16$$
$$E_2 = \{uv | d_u = 6, d_v = 5\}, |E_2| = 16$$
$$E_3 = \{uv | d_u = 5, d_v = 5\}, |E_3| = 32.$$

$$M_1(G'\diamond G'') = \sum_{uv\in E(G'\diamond G'')} d(u)+d(v) = \sum_{uv\in E_1} 6+6 + \sum_{uv\in E_2} 6+5 + \sum_{uv\in E_3} 5+5$$
$$= 12\sum_{uv\in E_1} 1 + 11\sum_{uv\in E_2} 1 + 10\sum_{uv\in E_3} 1$$
$$= 12\times 16 + 11\times 16 + 10\times 32 = 688.$$

$$M_2(G'\diamond G'') = \sum_{uv\in E(G'\diamond G'')} d(u)d(v) = \sum_{uv\in E_1} 6\times 6 + \sum_{uv\in E_2} 6\times 5 + \sum_{uv\in E_3} 5\times 5$$
$$= 36\sum_{uv\in E_1} 1 + 30\sum_{uv\in E_2} 1 + 25\sum_{uv\in E_3} 1$$
$$= 36\times 16 + 30\times 16 + 25\times 32 = 1856.$$

$$H(G'\diamond G'') = \sum_{uv\in E(G'\diamond G'')} \frac{2}{d(u)+d(v)} = \sum_{uv\in E_1} \frac{2}{6+6} + \sum_{uv\in E_2} \frac{2}{6+5} + \sum_{uv\in E_3} \frac{2}{5+5}$$
$$= \frac{1}{6}\sum_{uv\in E_1} 1 + \frac{2}{11}\sum_{uv\in E_2} 1 + \frac{1}{5}\sum_{uv\in E_3} 1$$
$$= \frac{1}{6}\times 16 + \frac{2}{11}\times 16 + \frac{1}{5}\times 32 = 11.98.$$

$$GA(G'\diamond G'') = \sum_{uv\in E(G'\diamond G'')} \frac{2\sqrt{d(u)d(v)}}{d(u)+d(v)} = \sum_{uv\in E_1} \frac{2\sqrt{6\times 6}}{6+6} + \sum_{uv\in E_2} \frac{2\sqrt{6\times 5}}{6+5} + \sum_{uv\in E_3} \frac{2\sqrt{5\times 5}}{5+5}$$
$$= \sum_{uv\in E_1} 1 + \frac{2\sqrt{30}}{11}\sum_{uv\in E_2} 1 + \sum_{uv\in E_3} 1$$

$$= 16 + \frac{2\sqrt{30}}{11} \times 16 + 32 = 63.93.$$

For $n = 4$, the edge set can be partitioned as follows:

$$E_1 = \{uv|d_u = 7, d_v = 7\}, |E_1| = 1$$
$$E_2 = \{uv|d_u = 7, d_v = 6\}, |E_2| = 12$$
$$E_3 = \{uv|d_u = 6, d_v = 6\}, |E_3| = 14$$
$$E_4 = \{uv|d_u = 6, d_v = 5\}, |E_4| = 32$$
$$E_5 = \{uv|d_u = 5, d_v = 5\}, |E_5| = 24.$$

$$M_1(G' \diamond G'') = \sum_{uv\in E(G'\diamond G'')} d(u) + d(v) = \sum_{uv\in E_1} d(\mathrm{u}) + d(v) + \sum_{uv\in E_2} d(u) + d(v) + \sum_{uv\in E_3} d(u) + d(v) + \sum_{uv\in E_4} d(u) + d(v) + \sum_{uv\in E_5} d(u) + d(v)$$
$$= \sum_{uv\in E_1} 7+7 + \sum_{uv\in E_2} 7+6 + \sum_{uv\in E_3} 6+6 + \sum_{uv\in E_4} 6+5 + \sum_{uv\in E_5} 5+5$$
$$= 14\sum_{uv\in E_1} 1 + 13\sum_{uv\in E_2} 1 + 12\sum_{uv\in E_3} 1 + 11\sum_{uv\in E_4} 1 + 10\sum_{uv\in E_5} 1$$
$$= 14\times 1 + 13\times 12 + 12\times 14 + 11\times 32 + 10\times 24$$
$$= 930.$$

$$M_2(G' \diamond G'') = \sum_{uv\in E(G'\diamond G'')} d(u)d(v)$$
$$= \sum_{uv\in E_1} d(u)d(v) + \sum_{uv\in E_2} d(u)d(v) + \sum_{uv\in E_3} d(u)d(v) + \sum_{uv\in E_4} d(u)d(v) + \sum_{uv\in E_5} d(u)d(v)$$
$$= \sum_{uv\in E_1} 7\times 7 + \sum_{uv\in E_2} 7\times 6 + \sum_{uv\in E_3} 6\times 6 + \sum_{uv\in E_4} 6\times 5 + \sum_{uv\in E_5} 5\times 5$$
$$= 49\sum_{uv\in E_1} 1 + 42\sum_{uv\in E_2} 1 + 36\sum_{uv\in E_3} 1 + 30\sum_{uv\in E_4} 1 + 25\sum_{uv\in E_5} 1$$
$$= 49\times 1 + 42\times 12 + 36\times 14 + 30\times 32 + 25\times 24 = 2617.$$

$$H(G' \diamond G'') = \sum_{uv\in E(G'\diamond G'')} \frac{2}{d(u)+d(v)}$$
$$= \sum_{uv\in E_1} \frac{2}{7+7} + \sum_{uv\in E_2} \frac{2}{7+6} + \sum_{uv\in E_3} \frac{2}{6+6} + \sum_{uv\in E_4} \frac{2}{6+5} + \sum_{uv\in E_5} \frac{2}{5+5}$$
$$= \frac{1}{7}\sum_{uv\in E_1} 1 + \frac{2}{13}\sum_{uv\in E_2} 1 + \frac{1}{6}\sum_{uv\in E_3} 1 + \frac{2}{11}\frac{2}{9}\sum_{uv\in E_4} 1 + \frac{1}{5}\sum_{uv\in E_5} 1$$
$$= \frac{1}{7}\times 1 + \frac{2}{13}\times 12 + \frac{1}{6}\times 14 + \frac{2}{11}\times 32 + \frac{1}{5}\times 24 = 11.10.$$

$$GA(G' \diamond G'') = \sum_{uv\in E(G'\diamond G'')} \frac{2\sqrt{d(u)d(v)}}{d(u)+d(v)} = \sum_{uv\in E_1} \frac{2\sqrt{7\times 7}}{7+7} + \sum_{uv\in E_2} \frac{2\sqrt{7\times 6}}{7+6} + \sum_{uv\in E_3} \frac{2\sqrt{6\times 6}}{6+6} + \sum_{uv\in E_4} \frac{2\sqrt{6\times 5}}{6+5} + \sum_{uv\in E_5} \frac{2\sqrt{5\times 5}}{5+5}$$
$$= \sum_{uv\in E_1} 1 + \frac{2\sqrt{42}}{13}\sum_{uv\in E_2} 1 + \sum_{uv\in E_3} 1 + \frac{2\sqrt{30}}{11}\sum_{uv\in E_4} 1 + \sum_{uv\in E_5} 1$$
$$= 1 + \frac{2\sqrt{42}}{13} + 14 + \frac{2\sqrt{30}}{11}32 + 24 = 82.83.$$

For $n \geq 5$, The edge set of the graph $G' \diamond G''$ can be partitioned as follows:

$$E_1 = \{uv|d_u = n+3, d_v = n+3\}, |E_1| = 1$$
$$E_2 = \{uv|d_u = n+3, d_v = n+2\}, |E_2| = 4$$
$$\mathrm{E}_3 = \{uv|d_u = n+3, d_v = 6\}, |E_3| = 2n$$

$$E_4 = \{uv|d_u = n+2, d_v = 5\}, |E_4| = 4n$$
$$E_5 = \{uv|d_u = n+2, d_v = n+2\}, |E_5| = 2$$
$$E_6 = \{uv|d_u = 6, d_v = 6\}, |E_6| = 3n$$
$$E_7 = \{uv|d_u = 6, d_v = 5\}, |E_7| = 4n$$
$$E_8 = \{uv|d_u = 5, d_v = 5\}, |E_8| = 6n.$$

$$\begin{aligned}
M_1(G'\diamond G'') &= \sum_{uv\in E(G'\diamond G'')} d(u)+d(v)\\
&= \sum_{uv\in E_1} d(u)+d(v) + \sum_{uv\in E_2} d(u)+d(v) + \sum_{uv\in E_3} d(u)+d(v) + \sum_{uv\in E_4} d(u)\\
&\quad +d(v) + \sum_{uv\in E_5} d(u)+d(v) + \sum_{uv\in E_6} d(u)+d(v)\\
&\quad + \sum_{uv\in E_7} d(u)+d(v) + \sum_{uv\in E_8} d(u)+d(v)\\
&= \sum_{uv\in E_1} (n+3)+(n+3) + \sum_{uv\in E_2} (n+3)+(n+2) + \sum_{uv\in E_3} (n+3)+6 + \sum_{uv\in E_4} (n+2)+5\\
&\quad + \sum_{uv\in E_5} (n+2)+(n+2) + \sum_{uv\in E_5} (n+2)+(n+2) + \sum_{uv\in E_6} 6+6\\
&\quad + \sum_{uv\in E_7} 6+5 + \sum_{uv\in E_8} 5+5\\
&= 2(n+3)\sum_{uv\in E_1} 1 + (2n+5)\sum_{uv\in E_2} 1 + (n+9)\sum_{uv\in E_3} 1 + (n+7)\sum_{uv\in E_4} 1\\
&\quad + 2(n+2)\sum_{uv\in E_5} 1 + 12\sum_{uv\in E_6} 1 + 11\sum_{uv\in E_7} 1 + 10\sum_{uv\in E_8} 1\\
&= 2(n+3)\times 1 + (2n+5)\times 4 + (n+9)2n + (n+7)4n + 2(n+2)2 + 12\times 3n + 11\\
&\quad \times 4n + 10\times 6n\\
&= 2(2n^2+100n+17)
\end{aligned}$$

$$\begin{aligned}
M_2(G'\diamond G'') &= \sum_{uv\in E(G'\diamond G'')} d(u)d(v)\\
&= \sum_{uv\in E_1} d(u)d(v) + \sum_{uv\in E_2} d(\mathrm{u})d(v) + \sum_{uv\in E_3} d(u)d(v) + \sum_{uv\in E_4} d(u)d(v)\\
&\quad + \sum_{uv\in E_5} d(u)d(v) + \sum_{uv\in E_6} d(u)d(v) + \sum_{uv\in E_7} d(u)d(v) + \sum_{uv\in E_8} d(u)d(v)\\
&= \sum_{uv\in E_1} (n+3)\times(n+3) + \sum_{uv\in E_2} (n+3)\times(n+2) + \sum_{uv\in E_3} (n+3)6 + \sum_{uv\in E_8} 5\times 5\\
&\quad + \sum_{uv\in E_4} (n+2)\times 5 + \sum_{uv\in E_5} (n+2)\times(n+2) + \sum_{uv\in E_6} 6\times 6 + \sum_{uv\in E_7} 6\times 5\\
&= (n+3)^2\sum_{uv\in E_1} 1 + (n+3)(n+2)\sum_{uv\in E_2} 1 + 6(n+3)\sum_{uv\in E_3} 1 + 5(n+2)\sum_{uv\in E_4} 1\\
&\quad + (n+2)^2\sum_{uv\in E_5} 1 + 36\sum_{uv\in E_6} 1 + 30\sum_{uv\in E_7} 1 + 25\sum_{uv\in E_8} 1\\
&= (n+3)^2\times 1 + (n+3)(n+2)\times 4 + 6(n+3)2n + 5(n+2)4n + (n+2)^2 2 + 36\times 3n\\
&\quad + 30\times 4n + 25\times 6n\\
&= (n+3)^2 + 4(n+3)(4n+2) + 2(n+2)(11n+2) + 378n\\
&= (n+3)^2 + 8(n+3)(2n+1) + 2(n+2)(11n+2) + 378n.
\end{aligned}$$

$$\begin{aligned}
H(G'\diamond G'') &= \sum_{uv\in E(G'\diamond G'')} \frac{2}{d(u)+\mathrm{d}(v)}\\
&= \sum_{uv\in E_1} \frac{2}{n+3+n+3} + \sum_{uv\in E_2} \frac{2}{n+3+n+2} + \sum_{uv\in E_3} \frac{2}{n+3+6} + \sum_{uv\in E_4} \frac{2}{n+2+5}\\
&\quad + \sum_{uv\in E_5} \frac{2}{n+2+n+2} + \sum_{uv\in E_6} \frac{2}{6+6} + \sum_{uv\in E_7} \frac{2}{6+5} + \sum_{uv\in E_8} \frac{2}{5+5}
\end{aligned}$$

$$= \frac{1}{n+3}\sum_{uv\in E_1} 1 + \frac{2}{2n+5}\sum_{uv\in E_2} 1 + \frac{2}{n+9}\sum_{uv\in E_3} 1 + \frac{2}{n+7}\sum_{uv\in E_4} 1 + \frac{1}{n+2}\sum_{uv\in E_5} 1 + \frac{1}{6}\sum_{uv\in E_6} 1 + \frac{2}{11}\sum_{uv\in E_7} 1 + \frac{1}{5}\sum_{uv\in E_8} 1$$

$$= \frac{1}{n+3} + \frac{8}{2n+5} + \frac{4n}{n+9} + \frac{8n}{n+7} + \frac{2}{n+2} + \frac{n}{2} + \frac{8n}{11} + \frac{6n}{5}.$$

$$GA(G' \diamond G'') = \sum_{uv\in E(G'\diamond G'')} \frac{2\sqrt{d(u)d(v)}}{d(u)+d(v)}$$

$$= \sum_{uv\in E_1} \frac{2\sqrt{(n+3)\times(n+3)}}{n+3+n+3} + \sum_{uv\in E_2} \frac{2\sqrt{(n+3)\times(n+2)}}{n+3+n+2} + \sum_{uv\in E_3} \frac{2\sqrt{(n+3)6}}{n+3+6} + \sum_{uv\in E_4} \frac{2\sqrt{(n+2)5}}{n+2+5} + \sum_{uv\in E_5} \frac{2\sqrt{(n+2)\times(n+2)}}{n+2+n+2} + \sum_{uv\in E_6} \frac{2\sqrt{6\times 6}}{6+6} + \sum_{uv\in E_7} \frac{2\sqrt{6\times 5}}{6+5} + \sum_{uv\in E_8} \frac{2\sqrt{5\times 5}}{5+5}$$

$$= \sum_{uv\in E_1} 1 + \frac{2\sqrt{(n+3)(n+2)}}{2n+5}\sum_{uv\in E_2} 1 + \frac{2\sqrt{6(n+3)}}{n+9}\sum_{uv\in E_3} 1 + \frac{2\sqrt{5(n+2)}}{n+7}\sum_{uv\in E_4} 1 + \sum_{uv\in E_5} 1 + \sum_{uv\in E_6} 1 + \frac{2\sqrt{30}}{11}\sum_{uv\in E_7} 1 + \sum_{uv\in E_8} 1$$

$$= 1 + \frac{2\sqrt{(n+3)(n+2)}}{2n+5}\times 4 + \frac{2\sqrt{6(n+3)}}{n+9}\times 2n + \frac{2\sqrt{5(n+2)}}{n+7}\times 4n + 2 + 3n + \frac{2\sqrt{30}}{11}\times 4n + 6n$$

$$= \frac{8\sqrt{(n+3)(n+2)}}{2n+5} + \frac{4n\sqrt{6(n+3)}}{n+9} + \frac{8n\sqrt{5(n+2)}}{n+7} + \frac{8n\sqrt{30}}{11} + 9n + 3$$

This completes the proof.

Theorem 7. Let the two copies of $W_3 \times W_n$ be G' and G'', where W_n is a wheel of order $n+1$. Then for the graph $G' \diamond G''$, the first and second Zagreb indices, Harmonic index and Geometric-Arithmetic index are respectively given by:

1. $M_1(G' \diamond G'') = \begin{cases} 1568, & for\ n = 3 \\ 8(n^2 + 57n + 16), & for\ n \geq 4 \end{cases}$
2. $M_2(G' \diamond G'') = \begin{cases} 5488, & for\ n = 3 \\ 16(n+4)^2 + 56n(n+4) + 1176n & for\ n \geq 4 \end{cases}$
3. $H(G' \diamond G'') = \begin{cases} 16, & for\ n = 4 \\ \frac{16}{n+4} + \frac{16n}{n+11} + \frac{24n}{7}, & for\ n \geq 4 \end{cases}$
4. $GA(G' \diamond G'') = \begin{cases} 112, & for\ n = 3 \\ 24n + \frac{16n\sqrt{7(n+4)}}{n+11} + 16, & for\ n \geq 4 \end{cases}$

Proof: For $n = 3$, The graph is 7 regular with 112 edges.

$$M_1(G' \diamond G'') = \sum_{uv\in E(G'\diamond G'')} d(u) + d(v)$$

$$= \sum_{uv\in E(G'\diamond G'')} 7 + 7 = 14 \sum_{uv\in E(G'\diamond G'')} 1$$

$$= 14 \times 112 = 1568.$$

$$M_2(G' \diamond G'') = \sum_{uv\in E(G'\diamond G'')} d(u)d(v)$$

$$= \sum_{uv\in E(G'\diamond G'')} 7\times 7 = 49 \sum_{uv\in E(G'\diamond G'')} 1$$
$$= 49\times 112 = 5488.$$

$$H(G'\diamond G'') = \sum_{uv\in E(G'\diamond G'')} \frac{2}{d(u)+d(v)} = \sum_{uv\in E(G'\diamond G'')} \frac{2}{7+7}$$
$$= \frac{1}{7}\sum_{uv\in E(G'\diamond G'')} 1 = \frac{1}{7}\times 112 = 16.$$

$$GA(G'\diamond G'') = \sum_{uv\in E(G'\diamond G'')} \frac{2\sqrt{d(u)d(v)}}{d(u)+d(v)} = \sum_{uv\in E(G'\diamond G'')} \frac{2\sqrt{7\times 7}}{7+7}$$
$$= \sum_{uv\in E(G'\diamond G'')} 1 = 112.$$

For $n \geq 4$, The edge set of the graph of $G'\diamond G''$ can be partitioned as follows:

$$E_1 = \{uv|d_u = n+4, d_v = n+4\}, |E_1| = 16$$
$$E_2 = \{uv|d_u = n+4, d_v = 7\}, |E_2| = 8n$$
$$E_3 = \{uv|d_u = 7, d_v = 7\}, |E_3| = 24n.$$

$$M_1(G'\diamond G'') = \sum_{uv\in E(G'\diamond G'')} d(u)+d(v) = \sum_{uv\in E_1}(n+4)+(n+4) + \sum_{uv\in E_2}(n+4)+7 + \sum_{uv\in E_3} 7+7$$
$$= 2(n+4)\sum_{uv\in E_1} 1 + (n+11)\sum_{uv\in E_2} 1 + 14\sum_{uv\in E_3} 1$$
$$= 2(n+4)\times 16 + (n+11)\times 8n + 14\times 24n$$
$$= 8(n^2+57n+16).$$

$$M_2(G'\diamond G'') = \sum_{uv\in E(G'\diamond G'')} d(u)d(v)$$
$$= \sum_{uv\in E_1}(n+4)(n+4) + \sum_{uv\in E_2}(n+4)7 + \sum_{uv\in E_3} 7\times 7$$
$$= (n+4)^2\sum_{uv\in E_1} 1 + 7(n+4)\sum_{uv\in E_2} 1 + 49\sum_{uv\in E_3} 1$$
$$= (n+4)^2\times 16 + 7(n+4)8n + 49\times 24n$$
$$= 16(n+4)^2 + 56n(n+4) + 1176n.$$

$$H(G'\diamond G'') = \sum_{uv\in E(G'\diamond G'')} \frac{2}{d(u)+d(v)} = \sum_{uv\in E_1}\frac{2}{n+4+n+4} + \sum_{uv\in E_2}\frac{2}{n+4+7} + \sum_{uv\in E_3}\frac{2}{7+7}$$
$$= \frac{1}{n+4}\sum_{uv\in E_1} 1 + \frac{2}{n+11}\sum_{uv\in E_2} 1 + \frac{1}{7}\sum_{uv\in E_3} 1 = \frac{1}{n+4}\times 16 + \frac{2}{n+11}\times 8n + \frac{1}{7}\times 24n$$
$$= \frac{16}{n+4} + \frac{16n}{n+11} + \frac{24n}{7}.$$

$$GA(G'\diamond G'') = \sum_{uv\in E(G'\diamond G'')} \frac{2\sqrt{d(u)d(v)}}{d(u)+d(v)}$$
$$= \sum_{uv\in E_1}\frac{2\sqrt{(n+4)(n+4)}}{n+4+n+4} + \sum_{uv\in E_2}\frac{2\sqrt{(n+4)7}}{n+4+7} + \sum_{uv\in E_3}\frac{2\sqrt{7\times 7}}{7+7}$$
$$= \sum_{uv\in E_1} 1 + \frac{2\sqrt{7(n+4)}}{n+11}\sum_{uv\in E_2} 1 + \sum_{uv\in E_3} 1$$
$$= 16 + \frac{2\sqrt{7(n+4)}}{n+11}\times 8n + 24n$$
$$= 24n + \frac{16n\sqrt{7(n+4)}}{n+11} + 16.$$

Thus, we proved the theorem.

Theorem 8. Let the two copies of $W_4 \times W_n$ be G' and G'', where W_n is a wheel of order $n+1$ with $n \geq 4$. Then for the graph $G' \diamondsuit G''$, the first and second Zagreb indices, Harmonic index and Geometric-Arithmetic index are respectively given by:

1. $M_1(G' \diamondsuit G'') = \begin{cases} 2754, & for \quad n = 4 \\ 8n^2 + 590n + 334, & for \quad n \geq 5 \end{cases}$

2. $M_2(G' \diamondsuit G'') = \begin{cases} 10273, & for \quad n = 4 \\ (n+5)\text{^}2 + 8(n+5)(n+16) + 4(n+4)(17n+12) + 1620n, & for \quad n \geq 5 \end{cases}$

3. $H(G' \diamondsuit G'') = \begin{cases} 24.96, & for\, n = 4 \\ \frac{1}{n+5} + \frac{16}{2n+1} + \frac{24}{n+13} + \frac{16n}{n+11} + \frac{12}{n+4} + \frac{16n}{15} + \frac{3n}{8} + \frac{20n}{7}, & for\, n \geq 5 \end{cases}$

4. $GA(G' \diamondsuit G'') = \begin{cases} 184.83, & for\, n = 4 \\ \frac{16\sqrt{(n+5)(n+4)}}{2n+9} + \frac{48\sqrt{2(n+5)}}{n+13} + \frac{16n\sqrt{7(n+4)}}{n+11} + \frac{32n\sqrt{14}}{15} + 23n + 13, & forn \geq 5 \end{cases}$

Proof. For $n = 4$, The edge set the graph $G' \diamondsuit G''$ can be partitioned as follows:

$$E_1 = \{uv | d_u = 9, d_v = 9\}, |E_1| = 1$$
$$E_2 = \{uv | d_u = 9, d_v = 8\}, |E_2| = 16$$
$$E_3 = \{uv | d_u = 8, d_v = 8\}, |E_3| = 24$$
$$E_4 = \{uv | d_u = 8, d_v = 7\}, |E_4| = 64$$
$$E_5 = \{uv | d_u = 7, d_v = 7\}, |E_5| = 80.$$

$$\begin{aligned} M_1(G' \diamondsuit G'') &= \sum_{uv \in E(G' \diamondsuit G'')} d(u) + d(v) \\ &= \sum_{uv \in E_1} d(u) + d(v) + \sum_{uv \in E_2} d(u) + d(v) + \sum_{uv \in E_3} d(u) + d(v) + \sum_{uv \in E_4} d(u) + d(v) \\ &\quad + \sum_{uv \in E_5} d(u) + d(v) \\ &= \sum_{uv \in E_1} 9 + 9 + \sum_{uv \in E_2} 9 + 8 + \sum_{uv \in E_3} 8 + 8 + \sum_{uv \in E_4} 8 + 7 + \sum_{uv \in E_5} 7 + 7 \\ &= 18 \sum_{uv \in E_1} 1 + 17 \sum_{uv \in E_2} 1 + 16 \sum_{uv \in E_3} 1 + 15 \sum_{uv \in E_4} 1 + 14 \sum_{uv \in E_5} 1 \\ &= 18 \times 1 + 17 \times 16 + 16 \times 24 + 15 \times 64 + 14 \times 80 \\ &= 2754. \end{aligned}$$

$$\begin{aligned} M_2(G' \diamondsuit G'') &= \sum_{uv \in E(G' \diamondsuit G'')} d(u)d(v) \\ &= \sum_{uv \in E_1} d(u)d(v) + \sum_{uv \in E_2} d(u)d(v) + \sum_{uv \in E_3} d(u)d(v) + \sum_{uv \in E_4} d(u)d(v) \\ &\quad + \sum_{uv \in E_5} d(u)d(v) \\ &= \sum_{uv \in E_1} 9 \times 9 + \sum_{uv \in E_2} 9 \times 8 + \sum_{uv \in E_3} 8 \times 8 + \sum_{uv \in E_4} 8 \times 7 + \sum_{uv \in E_5} 7 \times 7 \\ &= 81 \sum_{uv \in E_1} 1 + 72 \sum_{uv \in E_2} 1 + 64 \sum_{uv \in E_3} 1 + 56 \sum_{uv \in E_4} 1 + 49 \sum_{uv \in E_5} 1 \\ &= 81 \times 1 + 72 \times 16 + 64 \times 24 + 56 \times 64 + 49 \times 80 \\ &= 10273. \end{aligned}$$

$$\begin{aligned} H(G' \diamondsuit G'') &= \sum_{uv \in E(G' \diamondsuit G'')} \frac{2}{d(u) + d(v)} \\ &= \sum_{uv \in E_1} \frac{2}{9+9} + \sum_{uv \in E_2} \frac{2}{9+8} + \sum_{uv \in E_3} \frac{2}{8+8} + \sum_{uv \in E_4} \frac{2}{8+7} + \sum_{uv \in E_5} \frac{2}{7+7} \end{aligned}$$

$$= \frac{1}{9}\sum_{uv\in E_1} 1 + \frac{2}{17}\sum_{uv\in E_2} 1 + \frac{1}{8}\sum_{uv\in E_3} 1 + \frac{2}{15}\frac{2}{9}\sum_{uv\in E_4} 1 + \frac{1}{7}\sum_{uv\in E_5} 1$$
$$= \frac{1}{9}\times 1 + \frac{2}{17}\times 16 + \frac{1}{8}\times 24 + \frac{2}{15}\times 64 + \frac{1}{7}\times 80$$
$$= 24.96.$$

$$GA(G'\diamond G'') = \sum_{uv\in E(G'\diamond G'')} \frac{2\sqrt{d(u)d(v)}}{d(u)+d(v)}$$
$$= \sum_{uv\in E_1}\frac{2\sqrt{9\times 9}}{9+9} + \sum_{uv\in E_2}\frac{2\sqrt{9\times 8}}{9+8} + \sum_{uv\in E_3}\frac{2\sqrt{8\times 8}}{8+8} + \sum_{uv\in E_4}\frac{2\sqrt{8\times 7}}{8+7} + \sum_{uv\in E_5}\frac{2\sqrt{7\times 7}}{7+7}$$
$$= \sum_{uv\in E_1} 1 + \frac{12\sqrt{2}}{17}\sum_{uv\in E_2} 1 + \sum_{uv\in E_3} 1 + \frac{4\sqrt{14}}{15}\sum_{uv\in E_4} 1 + \sum_{uv\in E_5} 1$$
$$= 1 + \frac{12\sqrt{2}}{17}\times 16 + 24 + \frac{4\sqrt{14}}{15}64 + 80$$
$$= 184.83$$

For $n \geq 5$,
The edge set can be partitioned as follows,

$$E_1 = \{uv|d_u = n+5, d_v = n+5\}, |E_1| = 1$$
$$E_2 = \{uv|d_u = n+5, d_v = n+4\}, |E_2| = 8$$
$$E_3 = \{uv|d_u = n+5, d_v = 8\}, |E_3| = 12$$
$$E_4 = \{uv|d_u = n+4, d_v = 7\}, |E_4| = 8n$$
$$E_5 = \{uv|d_u = n+4, d_v = n+4\}, |E_5| = 12$$
$$E_6 = \{uv|d_u = 8, d_v = 7\}, |E_6| = 8n$$
$$E_7 = \{uv|d_u = 8, d_v = 8\}, |E_7| = 3n$$
$$E_8 = \{uv|d_u = 7, d_v = 7\}, |E_8| = 20n.$$

$$M_1(G'\diamond G'') = \sum_{uv\in E(G'\diamond G'')} d(u)+d(v)$$
$$= \sum_{uv\in E_1} d(u)+d(v) + \sum_{uv\in E_2} d(u)+d(v) + \sum_{uv\in E_3} d(u)+d(v)$$
$$+ \sum_{uv\in E_4} d(u)+d(v) + \sum_{uv\in E_5} d(u)+d(v) + \sum_{uv\in E_6} d(u)+d(v)$$
$$+ \sum_{uv\in E_7} d(u)+d(v) + \sum_{uv\in E_8} d(u)+d(v)$$
$$= \sum_{uv\in E_1}(n+5)+(n+5) + \sum_{uv\in E_2}(n+5)+(n+4) + \sum_{uv\in E_3}(n+5)+8$$
$$+ \sum_{uv\in E_4}(n+4)+7 + \sum_{uv\in E_5}(n+4)+(n+4) + \sum_{uv\in E_6} 8+7$$
$$+ \sum_{uv\in E_7} 8+8 + \sum_{uv\in E_8} 7+7$$
$$= 2(n+5)\sum_{uv\in E_1} 1 + (2n+9)\sum_{uv\in E_2} 1 + (n+13)\sum_{uv\in E_3} 1 + (n+11)\sum_{uv\in E_4} 1$$
$$+ 2(n+4)\sum_{uv\in E_5} 1 + 15\sum_{uv\in E_6} 1 + 16\sum_{uv\in E_7} 1 + 14\sum_{uv\in E_8} 1$$
$$= (n+5)^2\times 1 + (n+5)(n+4)\times 8 + 8(n+5)\times 12 + 7(n+4)\times 8n$$
$$+(n+4)^2\times 12 + 56\times 8n + 64\times 3n + 49\times 20n$$
$$= (n+5)^2 + 8(n+5)(n+16) + 4(n+4)(17n+12) + 1620n = 8n^2 + 590n + 334.$$

$$
\begin{aligned}
M_2(G'\diamond G'') &= \sum_{uv\in E(G'\diamond G'')} d(u)d(v)\\
&= \sum_{uv\in E_1} d(u)d(v) + \sum_{uv\in E_2} d(u)d(v) + \sum_{uv\in E_3} d(u)d(v) + \sum_{uv\in E_4} d(u)d(v) + \sum_{uv\in E_5} d(u)d(v)\\
&+ \sum_{uv\in E_6} d(u)d(v) + \sum_{uv\in E_7} d(u)d(v) + \sum_{uv\in E_8} d(u)d(v)\\
&= \sum_{uv\in E_1} (n+5)\times(n+5) + \sum_{uv\in E_2} (n+5)\times(n+4) + \sum_{uv\in E_3} (n+5)8\\
&\quad + \sum_{uv\in E_4} (n+4)7 + \sum_{uv\in E_5} (n+4)\times(n+4) + \sum_{uv\in E_6} 8\times 7\\
&\quad + \sum_{uv\in E_7} 8\times 8 + \sum_{uv\in E_8} 7\times 7\\
&= (n+5)^2 \sum_{uv\in E_1} 1 + (n+5)(n+4) \sum_{uv\in E_2} 1 + 8(n+5) \sum_{uv\in E_3} 1 + 7(n+4) \sum_{uv\in E_4} 1\\
&\quad + (n+4)^2 \sum_{uv\in E_5} 1 + 56 \sum_{uv\in E_6} 1 + 64 \sum_{uv\in E_7} 1 + 49 \sum_{uv\in E_8} 1\\
&= (n+5)^2\times 1 + (n+5)(n+4)\times 8 + 8(n+5)\times 12 + 7(n+4)\times 8n + (n+4)^2\times 12\\
&\quad + 56\times 8n + 64\times 3n + 49\times 20n\\
&= (n+5)^2 + 8(n+5)(n+16) + 4(n+4)(17n+12) + 1620n\\
&= (n+5)^2 + 8(n+5)(n+16) + 4(n+4)(17n+12) + 1620n.
\end{aligned}
$$

$$
\begin{aligned}
H(G'\diamond G'') &= \sum_{uv\in E(G'\diamond G'')} \frac{2}{d(u)+d(v)}\\
&= \sum_{uv\in E_1} \frac{2}{n+5+n+5} + \sum_{uv\in E_2} \frac{2}{n+5+n+4} + \sum_{uv\in E_3} \frac{2}{n+5+8} + \sum_{uv\in E_4} \frac{2}{n+4+7}\\
&\quad + \sum_{uv\in E_5} \frac{2}{n+4+n+4} + \sum_{uv\in E_6} \frac{2}{8+7} + \sum_{uv\in E_7} \frac{2}{8+8} + \sum_{uv\in E_8} \frac{2}{7+7}\\
&= \frac{1}{n+5}\sum_{uv\in E_1} 1 + \frac{2}{2n+9}\sum_{uv\in E_2} 1 + \frac{2}{n+13}\sum_{uv\in E_3} 1 + \frac{2}{n+11}\sum_{uv\in E_4} 1 + \frac{1}{n+4}\sum_{uv\in E_5} 1\\
&\quad + \frac{2}{15}\sum_{uv\in E_6} 1 + \frac{1}{8}\sum_{uv\in E_7} 1 + \frac{1}{7}\sum_{uv\in E_8} 1\\
&= \frac{1}{n+5} + \frac{2}{2n+9}\times 8 + \frac{2}{n+13}\times 12 + \frac{2}{n+11}\times 8n + \frac{1}{n+4}\times 12 + \frac{2}{15}\times 8n + \frac{1}{8}\times 3n\\
&\quad + \frac{1}{7}\times 20n\\
&= \frac{1}{n+5} + \frac{16}{2n+1} + \frac{24}{n+13} + \frac{16n}{n+11} + \frac{12}{n+4} + \frac{16n}{15} + \frac{3n}{8} + \frac{20n}{7}.
\end{aligned}
$$

$$
\begin{aligned}
GA(G'\diamond G'') &= \sum_{uv\in E(G'\diamond G'')} \frac{2\sqrt{d(u)d(v)}}{d(u)+d(v)}\\
&= \sum_{uv\in E_1} \frac{2\sqrt{(n+5)\times(n+5)}}{n+5+n+5} + \sum_{uv\in E_2} \frac{2\sqrt{(n+5)\times(n+4)}}{n+5+n+4} + \sum_{uv\in E_3} \frac{2\sqrt{(n+5)8}}{n+5+8}\\
&+ \sum_{uv\in E_4} \frac{2\sqrt{(n+4)7}}{n+4+7} + \sum_{uv\in E_5} \frac{2\sqrt{(n+4)\times(n+4)}}{n+4+n+4} + \sum_{uv\in E_6} \frac{2\sqrt{8\times 7}}{8+7} + \sum_{uv\in E_7} \frac{2\sqrt{8\times 8}}{8+8}\\
&+ \sum_{uv\in E_8} \frac{2\sqrt{7\times 7}}{7+7}\\
&= \sum_{uv\in E_1} 1 + \frac{2\sqrt{(n+5)(n+4)}}{2n+9}\sum_{uv\in E_2} 1 + \frac{4\sqrt{2(n+5)}}{n+13}\sum_{uv\in E_3} 1 + \frac{2\sqrt{7(n+4)}}{n+11}\sum_{uv\in E_4} 1
\end{aligned}
$$

$$+\sum_{uv\in E_5}1+\sum_{uv\in E_6}\frac{4\sqrt{14}}{15}+\sum_{uv\in E_7}1+\sum_{uv\in E_8}1$$

$$=1+\frac{2\sqrt{(n+5)(n+4)}}{2n+9}\sum_{uv\in E_2}1+\frac{4\sqrt{2(n+5)}}{n+13}\sum_{uv\in E_3}1$$

$$+\frac{2\sqrt{7(n+4)}}{n+11}\sum_{uv\in E_4}1+\sum_{uv\in E_5}1+\frac{4\sqrt{14}}{15}\sum_{uv\in E_6}1+\sum_{uv\in E_7}1+\sum_{uv\in E_8}1$$

$$=\frac{16\sqrt{(n+5)(n+4)}}{2n+9}+\frac{48\sqrt{2(n+5)}}{n+13}+\frac{16n\sqrt{7(n+4)}}{n+11}+\frac{32n\sqrt{14}}{15}+23n+13.$$

Hence the theorem.

4. CONCLUSION

In this work, we determined several topological indices of SK-join for specific graph classes. Because the SK-join is a noval graph structure, it is extremely important in the field of topological indices.

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Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this article.

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