Analytical and Numerical Analysis of Dynamical Behavior in a Nonlinear System with Quadratic Term

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ABSTRACT
In this paper, we study a simple nonlinear system where one of the differential equations has quadratic term. The system has no equilibria under certain condition. When it has equilibrium points, the analytical finding shows that they are all unstable. Numerical analysis is used to study the system behavior when the system has no equilibria. We provide some phase portraits, Poincare maps, and bifurcation diagram using local maximum of the system trajectory.

Keywords: Dynamical System, No Equilibria, Bifurcation, Local Maximum

1. INTRODUCTION

The dynamics of nonlinear systems have become a significant topic in fields such as economics [1], [2], [3], [4], [5], physics [6], [7], and even environmental studies [8]. Dynamic systems describe the evolution of variables within a system over time. In other words, they capture the long-term behavior of a system based on governing equations. Dynamic systems are often modeled using linear differential equations [6], but many real-world phenomena can only be accurately represented by nonlinear equations [1], [8], [9], [10]. Linear systems are generally easier to solve analytically, offering straightforward solutions. However, nonlinear systems are considerably more challenging, as analytical methods may not always yield meaningful insights. In such cases, numerical methods become a crucial alternative for analysis. Recently, a wide array of numerical techniques is available to tackle nonlinear differential equations [11], enabling a deeper understanding of complex systems. Moreover, nonlinear systems have the potential to exhibit a rich variety of behaviors, such as bifurcations, limit cycles, and even chaotic dynamics. These phenomena emerge from the inherent complexity of nonlinear interactions, where small changes in initial conditions or system parameters can lead to dramatically different outcomes. Bifurcations, for instance, involve sudden qualitative changes in the system's behavior, while limit cycles represent stable, repeating oscillations. Chaos, on the other hand, is characterized by highly sensitive dependence on initial conditions, resulting in seemingly random yet deterministic behavior. Such diversity makes the analysis of nonlinear systems both challenging and fascinating.

Bifurcation and chaotic behavior can occur in many systems, from simple to highly complex. A classic example is the logistic map [12], which exhibits period-doubling bifurcations as the control value rises, finally leading to chaos. Another well-known system is the Lorenz system [13], which simulates atmospheric convection and exhibits both bifurcations and chaotic attractors. More sophisticated systems, such as the Rossler system [14] Chua's circuit [15] and Rabinovich system [16], exhibit rich dynamics, including bifurcations, limit cycles, and chaos.

This article examines the system NE-14 presented in [10]. We investigate the mechanism of chaos development in a system without equilibrium points. The study begins by increasing the potential parameter space, and then examines stability around equilibrium points in the case of an extended system. Parameter modifications are then used in each computation until the chaotic behavior dynamics of the system are obtained.

2. RESEARCH METHOD

This research is designed to investigate the mechanisms of chaos emergence in the NE-14 dynamic system as described in [10]. The study employs both analytical and numerical approaches. The analytical analysis aims to identify the stability of equilibrium points and understand the transition to chaos. On the other hand, the numerical analysis is conducted to explore the system dynamics at various parameter values and to observe chaotic behavior through simulations.

Experiments are conducted using Python software equipped with the NumPy, SciPy, and Matplotlib libraries. Python was chosen for its efficiency in performing numerical computations and data visualization. For numerical analysis, the Euler scheme is employed as the integration method to solve the differential equations. The experimental setup involves implementing the Euler scheme in Python code to simulate the system at various parameter values a and to observe changes in system behaviour.

The primary data for this research consists of numerical simulation results from the NE-14 system. Additionally, supplementary data is obtained from [10], which provides crucial information about the system and relevant parameters for analysis. This data includes system parameters, previous simulation results, and findings from the analysis described in [10]. This supplementary data will be used to compare the research results with existing findings and to enhance the validity of the results obtained from the simulations. Data is collected through numerical simulations using Python. The data collection technique involves recording simulation results produced by the Euler scheme, including graphs that display system dynamics and chaotic patterns. Mathematical analysis techniques are also employed to identify the stability of equilibrium points and bifurcations. Data from [10] will be used to complement and compare the simulation and mathematical analysis results obtained in this research.

Simulation results will be analyzed using tools from Python libraries, specifically NumPy for numerical computations, SciPy for mathematical analysis, and Matplotlib for data visualization. The analysis will involve examining graphs and numerical data to identify chaotic behavior, periodic patterns, and transitions to chaos. Mathematical analysis techniques will also be applied to determine the stability of equilibrium points and changes in system structure due to variations in the parameter a. Additional data from Sprott's paper will be used as a reference to compare and assess the consistency of the research findings with existing literature.

3. RESULT AND DISCUSSION

Consider a one parameter family of systems of ordinary differential equations in $\mathbb{R}^3$as follows [1]:

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = z$$
$$\frac{dz}{dt} = x^2 - y^2 + 2xz + yz + a$$

where $a \in \mathbb{R}$ is a parameter that influences the system dynamics, while x, y, and z are the coordinate.

It is evident from [1] that system for $a = 1$ exhibits chaotic dynamics. In this paper, we are concentrating on describing the mechanism that produces the chaotic dynamics.

3.1 Equilibrium Point

To determine the equilibrium points of the system, we set the time derivatives equal to zero. This yields the following conditions.

$$y = 0,\ \ z = 0,\ \ x^2 + a = 0\#(1)$$

From (1), it follows that the system has equilibrium points only if $a \leq 0$, where $x = \pm\sqrt{|a|}$. If $a > 0$, then there is no real equilibrium points exist.

3.2 Stability Analysis

Linearization of the system around the equilibrium points is performed by constructing the Jacobian matrix. The Jacobian $J(x, y, z)$ is given by the following partial derivatives of the system with respect to the state of x, y, z.

$$J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2x + 2z & -2y + z & 2x + y \end{bmatrix}\#(2)$$

To investigate the stability, we substitute the equilibrium points $\left(\pm\sqrt{|a|}, 0{,}0\right)$ to (2). When $a = 0$, the equilibrium point is $(0{,}0{,}0)$. Substituting these values into the Jacobian matrix yield

$$J(0{,}0{,}0) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The eigenvalues of the matrix are $\lambda_1 = 0, \lambda_2 = 0$ and $\lambda_3 = 0$, indicating that the equilibrium point is non-hyperbolic. This need more detailed analysis to fully understand the system behavior. When $a > 0$ no real equilibrium point exists. Therefore, when $a \geq 0$ the system behavior must be explored through numerical simulations. Next, when$a < 0$ the equilibrium points occur at $\left(\pm\sqrt{|a|}, 0,0\right)$. Substituting this point into the Jacobian matrix we obtain

$$J\left(\pm\sqrt{|a|}, 0,0\right) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \pm 2\sqrt{|a|} & 0 & \pm 2\sqrt{|a|} \end{bmatrix}$$

The characteristic equation of this matrix is determined by solving the equation $\det(J - \lambda I) = 0$ where I is the identity matrix and λ represents the eigenvalues. Thus, the characteristic equations are

$$\lambda^3 \mp 2\sqrt{|a|}\lambda^2 \mp 2\sqrt{|a|} = 0$$

Based on the Routh-Hurwitz criterion, the polynomial $p(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$ has all roots with negative real parts if and only if $a_1 > 0$, $a_3 > 0$ and $a_1 a_2 > a_3$. However, for the equilibrium point $\left(-\sqrt{|a|}, 0,0\right)$ the condition $a_1 a_2 > a_3$ is not satisfied (note that $a_1 a_2 = 2\sqrt{|a|}(0) = 0 < 2\sqrt{|a|} = a_3$), while for the equilibrium point $\left(\sqrt{|a|}, 0,0\right)$, the conditions $a_1 > 0$ and $a_2 > 0$ are violated. Consequently, the characteristic polynomial for both equilibrium points has at least one eigenvalue with a positive real part, implying that both equilibrium points are unstable.

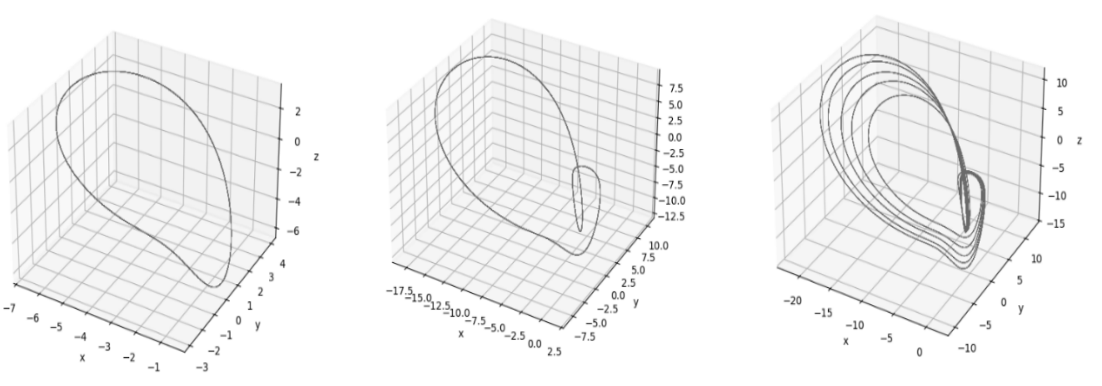
3.3 Numerical Simulation

Based on the analytical results near the equilibrium point, only limited information can be obtained. Furthermore, under conditions where $a \geq 0$, the dynamic phenomena cannot be captured analytically. Therefore, numerical analysis is necessary to understand the behavior of the system and to validate the analytical results. In performing numerical computations, the continuous-time differential equations are discretized using the Euler method [11]. The NE-14 system is discretized into

$$\begin{aligned} x_{n+1} &= x_n + \square\, y_n \\ y_{n+1} &= y_n + \square\, z_n \\ z_{n+1} &= z_n + \square(x_n^2 - y_n^2 + 2x_n z_n + y_n z_n + a) \end{aligned}$$

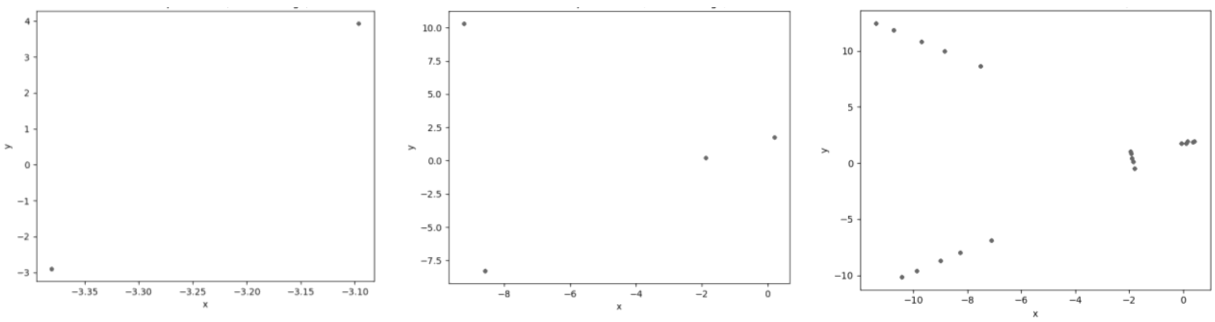
where $\square > 0$ represents the time step increment and x_n, y_n, z_n are the state variables at the n-th time step. By iterating these equations over a specified range of time, we obtain a numerical solution that approximates the systems trajectory and allows us to observe the systems behavior in response to parameter variations.

To investigate the influence of different parameters on the system's trajectory, we vary the parameter a and observe the resulting changes in the system's behavior. For each simulation, we initialize the state variables$(1,0,-4)$, with a fixed time step $\square$. The numerical integration is carried out over a sufficiently large time span to allow transient behavior to settle, revealing the long-term dynamics of the system. Through this process, we aim to identify the occurrence of periodic solutions, bifurcations, and chaotic regimes. By analyzing the resulting time series and phase portraits, we can detect the presence of periodic attractors, as well as any transitions to chaos, which may occur through bifurcation mechanisms such as period-doubling.



(a) (b) (c)

Figure 1. Trajectories in the phase space using the initial condition $(1,0,-4)$with parameter (a) a=-1 (b) a=0.8 and (c) a=1.



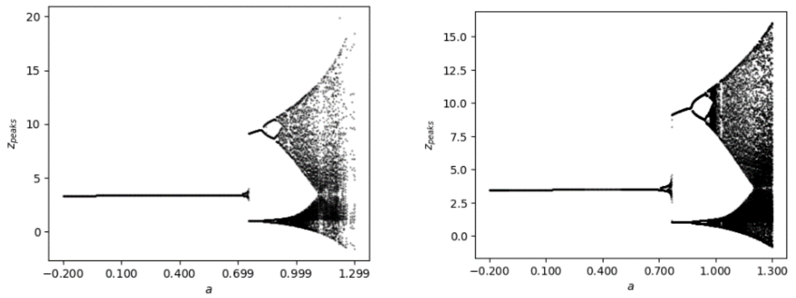
(a) (b) (c)

Figure 2. Poincare map of the trajectory with initial condition $(1,0,-4)$with parameter (a) a=-1 (b) a=0.8 and (c) a=1.

In Figure 1, the trajectories were obtained using the initial condition $(1,0,-4)$ for different values of a. The trajectories were plotted after eliminating the transient time from $t = 0$ to $t = 500$, and the remaining data was visualized in the phase space. Based on the initial phase portraits for the three cases, there is an indication of the emergence of periodic solutions. To further support this observation, a Poincaré map was constructed on the xy-plane from the trajectories in Figure 1, leading to the results shown in Figure 2.

Based on the Poincare map shown in Figure 2, we observe that periodic solutions do indeed occur. Specifically, for positive values of a, exemplified by $a = 1$, there is an increase in the number of periods, indicated by a higher number of generated points. This suggests the possibility of period-doubling bifurcations as the parameter a is varied. To gain a more universal understanding of how variations in a affect the system, we need to illustrate how the number of points changes with different values of a. Therefore, using the same approach as in Figure 2, we plot the x-coordinates from the Poincare map for each value of a ranging from -1 to 1. This will yield a diagram showing the number of points as a function of a.

Next, we present the bifurcation diagram of NE-14 system, where a acts as the bifurcation parameter. Since the NE-14 system has no stable equilibrium, then a bifurcation diagram with the equilibrium x^* as the vertical axis is not appropriate. Therefore, it'd be better to present the bifurcation diagram with z_{peaks} as the vertical axis, where z_{peaks} represents the local maximum of the trajectory at given value of a. The result is presented in Figure 3. We made the simulation by taking the local maximum value of x_t for $t = 2 \times 10^2 + 1, \dots, 10^3$, that is we skip the first two thousand iterations, and consider the eight-hundred thousand iterations. The interval of parameter a is $[-0.2, 1.3]$ where it is divided into 2000 partitions. We use three different values of parameter ⍰. From the figures, we can derive some notes. Higher value of a makes the system has more local maximum, this means that the system behavior becomes more complex.



(a) (b)

Figure 3. Bifurcation diagram of parameter a plotted versus x_{peaks} . Panels (a)-(b) are for ⍰ $= 0.001$ and ⍰ $= 0.01$ respectively.

4. CONCLUSION

This paper analyzes one of interesting dynamical system listed in [1] called as NE-14 system. We have studied the analytical behavior of the system, where all obtained equilibriums are unstable as proved by all the eigenvalues that are not negative. To inspect the behavior of the system when no equilibrium exists, we use numerical analysis by providing the behavior of the phase portrait and Poincare map. We also use the discrete system version for studying the bifurcation diagram of the main parameter, where in this paper we consider the local maximum of the trajectory instead of the equilibrium values. We found that higher value of parameter makes the system becoming more complex. Our study can lead to some interesting research, for example, what happen if the system is studied under non-standard difference equation.

5. ACKNOWLEDGEMENT

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