A Robust Semi-Analytical Approach to Study Time-Fractional Black-Scholes Equation with Non-Local Derivative

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ABSTRACT
The time-fractional Black-Scholes equation has a significant impact on market anomalies and irregularities, which offer long-range dependence and heavy-tailed distributions, which led to a more precise depiction of financial markets, particularly in predicting extreme events and in the valuation of derivatives. The primary goal of this work is to examine Black-Scholes equations of arbitrary order with the assistance of the Caputo fractional derivative. Here, we apply an effective semi-analytical technique called an approximate analytical method. We briefly introduce the Black-Scholes equation, its history, and its applications in the field of economics. The aforementioned equation in financial problems is addressed by employing the analytical method, and this concept is used to assess the value of the option (buy or sell an asset) without a transaction cost. Solutions from the proposed method are obtained in series form, which converges swiftly and also carries out numerical simulations by comparing to different methods. The obtained outcomes are discussed through the 3D plots and graphs with the minimum error that expresses the physical representation of the considered equation. The preferred method to examine fractional Black-Scholes equations is efficient, reliable, and robust.

Keywords: Fractional Black-Scholes equation; Riemann-Liouville fractional integral; Caputo-Fractional derivative; Approximate Analytical method.

1. INTRODUCTION

The theory of fractional calculus (FC) is a firmly established and continuously evolving subject in the area of mathematics. FC is a branch of calculus that is concerned with integrals and derivatives of arbitrary order which came into existence around 300 years ago and it has become an efficient mathematical tool with a wide variety of applications in the field of science and technology. The concept of FC was introduced by German mathematicians Leibniz and L'Hospital in 1695 and the most fascinating work in scientific and engineering applications has been found using FC in the past few years [1–4]. The glorious developments envisioned in FC and their most significant applications lie in biology [5, 6], fluid mechanics [7, 8], biochemistry [9], human diseases [10], physics [11], plasma physics [12], and so on [13–22]. The main advantage of FC is that we can find the arbitrary derivative of a function, which is somewhat restricted to the integer-order in classical calculus. Due to the non-local property of FC, it is easy to analyze memory effects and the hereditary property of the considered problem and it also captures the significant effects and provides more details about the corresponding phenomena. Compared to classical calculus, it has more applications in various fields as it gives solutions in between the intervals. Using this theory, one can examine the behaviour of a wide variety of physical systems in real-world phenomena and it has been used in analyzing and solving problems related to natural phenomena with complex systems, long-range waves, genetic characteristics and so on.

For understanding and modelling the linear and non-linear systems that arise in various scientific and technological sectors, fractional differential equations (FDE) serve a crucial and significant role. Numerous researchers have defined fractional operators such as Riemann-Liouvelli, Caputo, Caputo-Fabrizio, Atangana-Baleanu, Hadamard, Grunwald-Letnikov, Reisz, and Hilfer etc. Each operator has its own limitations for instance, the Riemann-Liouville operator is unable to prove that the derivative of a constant function is zero. When using this operator, we cannot make assumptions about the starting solution in classical form. FDEs may successfully replicate physical phenomena that are dependent on both current and historical events. The solutions obtained from solving FDE using these operators help to describe the nature of complex systems that emerge in our daily lives. Finding the solutions and attaining the exact and approximate solutions of these equations is challenging due to the complexity produced by the arbitrary-order derivatives and integrals. Various numerical and approximating techniques have been suggested in the literature to solve the physical and biological systems of fractional order and determine their outcomes. These systems have been solved by many authors, who have proposed different techniques to evaluate their results. Namely, Veeresha et al. [23] have presented the coupled fractional reduced differential transform method (RDTM) to find the numerical solution for Jaulent-Miodek equations of fractional order.

The work on financial markets has indeed gained more recognition in the field of mathematical research. Nowadays, fractional partial differential equations (FPDE) have been introduced in financial theory. In 1973, Fischer Black and Myron Scholes introduced a renowned theoretical model for option valuation [24]. After introducing the B-S equation, the currently accessible research results primarily focus on two areas: first, they provide option values using more powerful analytical and numerical techniques, and second, they develop new option pricing models. An option is a type of security that permits the buying or selling of an asset within a certain time frame, subject to certain conditions [25,26]. American options and European options are the two main types of options in the financial market. American options can be exercised at any time until the expiration date, while European options can only be used on a specific future date [27]. The primary objective of the B-S equation aims to create a risk-free portfolio, using a combination of bonds (cash), underlying stock, and options. In this paper we study the time-fractional B-S equation is given by [28]

$$\frac{\partial^\alpha v}{\partial t^\alpha}+\frac{1}{2}\sigma^2 s^2\frac{\partial^2 v}{\partial s^2}+rs\frac{\partial v}{\partial s}-rv=0,\ \ (x,t)\epsilon R^+\times(0.T) \qquad 0<\alpha\leq 1. \tag{1}$$

where r denotes the risk free interest rate, ζ is a function of stock price, v is a call price of the option, σ be the volaticity of the stock and time t subjected to the condition

$$v=(x-E,0),\quad x\in R^+,\qquad v(0,t)=0. \tag{2}$$

where E is the expiration price [29]. Many researchers used various methodologies to examine the existence of solutions to the Black-Scholes(B-S) equation. B-S models have been extensively studied in the paper and the references therein [30–32]. The ease and clarity with which the price of the option can be obtained through the B-S equation has led to a significant increase in trading activity. The estimation of stock options is the primary use of the B-S equation. Over the last couple of decades, there has been an increasing interest in this equation due to its ability to provide the values of options effectively. The key elements of the B-S equations are risk-free rate, expiration date, strike price, underlying stock price, and volatility. Fractional B-S equations are solved using various methods namely, Sunil Kumar et al. [33] solved the B-S equation using the homotopy perturbation technique. For the time-fractional B-S equations, Liaqat and Okyere [34] proposed a Laplace residual power series method (LRPSM). A finite-difference technique was used by Song and Wang [35] to resolve the fractional B-S option pricing model. Vijayan et al. [36] solved the fractional B-S equations using the homotopy analysis method with the Shehu transform. Company et al. [37] employed a semi-discretization method to solve the partial differential equations that constitute the B-S option pricing. Morais and Grossinho have examined the existence and localisation results for generalised Block-Scholes models using the upper and lower solutions method [38]. The non-linear B-S equation governing the European option pricing problem was solved by Wang and Lesmana using an upwind finite difference method [39]. Deriving the closed-form solution to the B-S equation relies on the fundamental solution of the heat equation. While the Black-Scholes model performs better at identifying closed-form solutions for European options, it is less successful for non-European options [40].

Here, we implement an efficient technique called an approximate analytical method (AAM) to solve fractional B-S equation and this method provides an approximate solution. AAM is a semi-analytical method that can be used to resolve highly non-linear problems since it provides a series solution, which enables us to analyze the answer more thoroughly. KDV equations, fluid flow models, and solute problems have all been resolved using AAM [41–44]. This paper has been presented in the following manner: Section 2 offers the definitions and properties of the Laplace transform and fractional calculus. In Section 3, we proposed the AAM algorithm to examine the solutions of Black-Scholes equations of fractional order in terms of the Caputo operator. In Section 4, using the projected method the solutions and their graphical representation of three different examples of fractional B-S equations have been presented. The obtained results and graphs have been discussed for considered problems in Section 5. Finally, the conclusions of our work are reported in Section 6.

2. Preliminaries

Fractional derivatives and integrals have many different definitions and characteristics. The definitions and preliminary statements of the FC that are utilised in the present study and found in [1–3].

Definition 1. Let $\alpha \in \mathbb{R}$ and $\alpha \geq 0$. The fractional integral operator of Riemann-Liouville (RL) sense is denoted by J α t for the function $v(\zeta,t) \in C_\alpha$, $(\alpha \geq -1)$ is defined as

Definition 2. The fractional partial derivative of $v(\zeta,t) \in C_{-1}^{n}$ in the Caputo sense is defined as

$$J^{\alpha}v(\zeta,t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\vartheta)^{\alpha-1}v(\zeta,t)d\vartheta,$$

$$J^{0}v(\zeta,t) = v(\zeta,t). \quad (3)$$

$$D_t^{\alpha}v(\zeta,t) = \begin{cases} \dfrac{d^m v(\zeta,t)}{dt^m}, & \alpha = m \in \mathbb{N}, \\ \dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_0^t (t-\vartheta)^{n-\alpha-1}f^{(m)}(\vartheta)d\vartheta, & \alpha \in (m-1,m), m \in \mathbb{N}. \end{cases} \quad (4)$$

Definition 3. The Laplace transform (LT) of the Caputo fractional derivative $D_t^{\alpha}v(\zeta,t)$ is denoted as

$$L[D_t^{\alpha}v(\zeta,t)] = s^{\alpha}V(\zeta,s) - \sum_{r=0}^{m-1} s^{\alpha-r-1} f^{(r)}(0^+), (n-1 < \alpha \leq n), \quad (5)$$

where $v(\zeta,s)$ denote the LT of the function $v(\zeta,s)$).

Theorem 2.1. Let $b > -1$ and $\alpha_1,\alpha_2 \in \mathbb{R}, \alpha_1,\alpha_2 \geq 0$. Then, the RL fractional partial integral operator ${}_0J_t^{\alpha}$ satisfies the following properties for the function $v(\zeta,t) \in C_{\mu,}\mu > -1$,

$${}_0J_t^{\alpha_1}\ {}_0J_t^{\alpha_2}v(\zeta,t) = {}_0J_t^{\alpha_1+\alpha_2}u(x,y,t),$$

$${}_0J_t^{\alpha_1}\ {}_0J_t^{\alpha_2}v(\zeta,t) = {}_0J_t^{\alpha_2}\ {}_0J_t^{\alpha_1}u(x,y,t), \quad (6)$$

$${}_0J_t^{\alpha}t^{b} = \frac{\Gamma(b+1)}{\Gamma(b+\alpha+1)}t^{\alpha+b}.$$

Theorem 2.2. Let $\alpha, t \in \mathbb{R}, t \geq 0, m-1 < \partial < m \in N$. Then, $D_t^{\alpha}\ {}_0J_t^{\alpha}v(\zeta,t) = v(\zeta,t)$,

$$D_t^{\alpha}\ {}_0J_t^{\alpha}v(\zeta,t) = v(\zeta,t) - \sum_{r=0}^{m-1}\frac{t^k}{k!}\frac{\partial^k v(\zeta,0^+)}{\partial t^k}, \quad (7)$$

3. Methodology of Approximate Analytical Method (AAM).

Aiming to demonstrate the reliability of this method, we will examine the nonlinear fractional partial differential equation with the following beginning conditions.

$$D_t^{\alpha}v\left(\bar{x},\bar{y},t\right) = f\left(\bar{x},\bar{y},t\right) + L\bar{v} + N\bar{v},\ m-1<\partial<m\in N\quad \frac{\partial^i v\left(\bar{x},\bar{y},0\right)}{\partial t^i} = f_i\left(\bar{x},\bar{y}\right). \quad (8)$$

$$i = 0,1,2,3,\ldots m-1$$

where $v(\zeta,t)$ is the source term and α is the order of the Caputo fractional partial derivative, which is an analytical function L is linear and Nis non-linear operators, and $\bar{\zeta} = (\zeta_1,\zeta_2 \ldots \zeta_n)$.To get the analytical solution of the problem that has been considered, a novel technique called the approximate analytical method can be used. This technique provides computational accuracy to offer appropriate piecewise analytical solutions, making it a useful tool for solving non-linear fractional differential equations. It's crucial to examine the results in order to illustrate AAM.

Lemma 1. For $v(\bar{\zeta},t) = \sum_{k=0}^{\infty} r^k v(\bar{\zeta},t)$,linear operator $L(v)$ satisfies the given below property;

$$L(v(\bar{\zeta},t)) = L\left(\sum_{k=0}^{\infty} r^k v(\bar{\zeta},t)\right) = \sum_{k=0}^{\infty} r^k Lv(\bar{\zeta},t). \quad (9)$$

Theorem 3.1. Let $v(\bar{\zeta},t) = \sum_{k=0}^{\infty} v(\bar{\zeta},t)$,and $v_\lambda(\bar{\zeta},t) = \sum_{k=0}^{\infty} \lambda^k v_\lambda(\bar{\zeta},t)$.where λ is the non-zero parameter such that $0 \leq \lambda \leq 1$, subsequently, non-linear operator $N(v_\lambda)$ satisfies the below conditions [38]:

$$N(u_\lambda) = N\sum_{k=0}^{\infty}\left(\lambda^k u_k\right) = \sum_{n=0}^{\infty}\left(\frac{1}{n!}\frac{\delta}{\delta\lambda^n}\left(N\left(\sum_{k=0}^{\infty}\lambda^k u_k\right)\right)_{\lambda=0}\right)\lambda^n. \quad (10)$$

Definition 3. The polynomials explained as follows $P_n(v_0, v_1 \ldots v_n)$ is defined as follows:

$P_n(u_0, u_1, u_2, \dots u_n) = \frac{1}{n!}\frac{\delta^n}{\delta\lambda^n}\left(N\left(\sum_{k=0}^{n}\lambda^k v_k\right)\right)_{\lambda=0}$ (11)

Remark 3.1 Let $P_n = P_n(v_0, v_1, v_2, \dots v_n)$ be as in Definition 3. The nonlinear term $N(v_\lambda)$ can be defined in terms of P_n by using Theorem 3.2. as follows:

$$N(v_\lambda) = \sum_{n=0}^{\infty}\lambda^n P_n. \tag{12}$$

Now, we interpret the statements of existence, convergence and the maximum absolute error theorem for the considered equation Eq. (8) using AAM.

Existence Theorem

Theorem 3.2 By defining the functions $f(\bar{\zeta}, t), f_i(\bar{\zeta})$ as in Eq.(8) and for $m-1 < \alpha < m \in \mathbb{N}$ Eq.(8) gives at least one solution, which is provided by[38]

$$v(\bar{\zeta}, t) = f_t^{(-\alpha)}(\bar{\zeta}, t) + \sum_{i=0}^{m-1}\frac{t^i}{i!}f_i(\bar{\zeta}) + \sum_{i=0}^{m-1}[L_t^{-\alpha}v_{(k-1)} + p_{(k-1)t}^{(-\alpha)}], \tag{13}$$

where $p_{(k-1)t}^{(-\alpha)}$ and $L_t^{-\alpha}v_{(k-1)}$ are the Riemann-Liouville partial fractional integral of order α for P_{k-1} and $L(v_{k-1})$ with regard to t respectively.

Proof: Let us consider the solution $v(\bar{\zeta}, t)$ of Eq (8) in analytical form

$$v(\bar{\zeta}, t) = \sum_{k=0}^{\infty}v_k(\bar{\zeta}, t). \tag{14}$$

Let us take the given below expression to solve IVP Eq. (8)

$$D_t^\alpha v_\lambda(\bar{\zeta}, t) = \lambda[f(\bar{\zeta}, t) + L(v_\lambda) + N(v_\lambda)], \qquad 0 \le \lambda \le 1, \tag{15}$$

with the starting solution

$$\frac{\partial^i v(\bar{\zeta}, t)}{\partial t^i} = f_i(\bar{\zeta}), \quad i = 0,1,2,3, \dots . m-1 \;\; i = 0,1,2,3 \dots . , m-1. \tag{16}$$

Let us assume that Eq.(15) has the solution in the form

$$v_\lambda(\bar{\zeta}, t) = \sum_{k=0}^{\infty}\lambda^k v_k(\bar{\zeta}, t). \tag{17}$$

Using Theorem 3.1 and taking Eq. (8) with Riemann-Liouville partial integral, we get

$$v_\lambda(\bar{\zeta}, t) = \sum_{i=0}^{m-1}\frac{t^i}{i!}\frac{\partial^i v_\lambda(\bar{\zeta}, 0)}{\partial t^i} + \lambda\, {}_0J_t^\alpha[f((\bar{\zeta}, t) + L(v_\lambda) + N(v_\lambda)]. \tag{18}$$

Using Eq (16)we can express Eq. (18), as follows:

$$v_\lambda(\bar{\zeta}, t) = \sum_{i=0}^{m-1}\frac{t^i}{i!}g_i(\bar{\zeta}) + \lambda\left[f_t^{(-\alpha)}(\bar{\zeta}, t) + J_t^\alpha[L(v_\lambda)] + J_t^\alpha[N(v_\lambda)]\right]. \tag{19}$$

Substitute Eq. (17) into Eq. (19), which gives

$$\sum_{k=0}^{\infty}\lambda^k v_k(\bar{\zeta}, t) = \sum_{i=0}^{m-1}\frac{t^i}{i!}g_i(\bar{\zeta}) + \lambda f_t^{(-\alpha)}(\bar{\zeta}, t) + J_t^\alpha\lambda\sum_{k=0}^{\infty}\left[L\left(\lambda^k v_k\right)\right] + J_t^\alpha\lambda\sum_{n=0}^{\infty}\left[\frac{1}{n!}\frac{\partial^n}{\partial\lambda^n}\left(N\left(\sum_{k=0}^{\infty}\lambda^k v_k\right)\right)_{\lambda=0}\right]\lambda^n. \tag{20}$$

With the help of Definition 4 and Eq (20), we arrive at

$$\sum_{k=0}^{\infty}\lambda^k v_k(\bar{\zeta}, t) = \sum_{i=0}^{m-1}\frac{t^i}{i!}g_i(\bar{\zeta}, t) + \lambda f_t^{(-\alpha)}(\bar{\zeta}, t) + J_t^\alpha\lambda\sum_{k=0}^{\infty}\left[L\left(\lambda^k v_k\right)\right] + J_t^\alpha\lambda\sum_{n=0}^{\infty}p_n\lambda^n. \tag{21}$$

We can obtain the following terms by equating the coefficients of identical powers of λ in Eq. (21) we get below terms

$$v_o(\bar{\zeta}, t) = \sum_{i=0}^{m-1}\frac{t^i}{i!}g_i(\bar{\zeta}),$$
$$v_1(\bar{\zeta}, t) = f_t^{(-\alpha)}(\bar{\zeta}, t) + L_t^{(-\alpha)}v_0 + P_{ot}^{(-\alpha)},$$
$$v_2(\bar{\zeta}, t) = L_t^{(-\alpha)}v_{k-1} + P_{(k-1)t}^{(-\alpha)},$$
$$k = 2,3,\dots \tag{22}$$

Substituting Eq.(22) in Eq(17), which gives the solution of Eq(8) . By the help of Eq. (14) and Eq (17) gives

$$v(\bar{\zeta}, t) = \lim_{\lambda\to 1} v_\lambda(\bar{\zeta}, t) = v_0(\bar{\zeta}, t) + v_1(\bar{\zeta}, t) + \sum_{k=2}^{\infty}v_k(\bar{\zeta}, t). \tag{23}$$

We can see that, $\frac{\partial^i v(\bar{\zeta}, 0)}{\partial t^i} = \lim_{\lambda\to 1}\frac{\partial^i v_\lambda(\bar{\zeta}, t)}{\partial t^i} \Longrightarrow g_i(\bar{\zeta}) = f_i(\bar{\zeta})$. Replacing Eq (22) in Eq (23).

Ends the proof.

4. Solution of equation using AAM.

We consider the fractional Black-Scholes equation to evaluate the clarity precision of the proposed method. Here, we solve the fractional Black-Scholes equation in terms of Caputo fractional derivative.

Example 1.

Contemplate the following time-fractional Black-Scholes equation as [36]

$${}_{0}^{c}D_{t}^{\alpha}v(\zeta,t)-\frac{\partial^{2}v}{\partial\zeta^{2}}-(k-1)\frac{\partial v}{\partial\zeta}+kv=0,\quad 0<\alpha\leq 1, \tag{24}$$

subject to the starting solution

$$v(\zeta,o)=e^{\zeta}-1. \tag{25}$$

This equation contains only two dimensionless parameters such as $k=\frac{2r}{\sigma^{2}}$and the dimensionless time to expiry $\frac{1}{2}\sigma^{2}T$. These parameters represent the balance between the rate of interests and the variability of stock returns. Although there are four dimensional parameters E, T, σ^{2} and r in the original statements of the problem, the equation Eq. (24) can be written as

$$\frac{\partial^{\alpha}v}{\partial t^{\alpha}}=\frac{\partial^{2}v}{\partial\zeta^{2}}+(k-1)\frac{\partial v}{\partial\zeta}-kv, \tag{26}$$

with the assistance of the AAM solutions method. Assume Eq. (24) has the following solution form:

$$v(\zeta,t)=\sum_{k=0}^{\infty}v_{k}(\zeta,t). \tag{27}$$

Consider Eq. (26) to get an approximate solution.

$${}_{0}^{c}D_{t}^{\alpha}v(\zeta,t)=\lambda\left[\frac{\partial^{2}v}{\partial\zeta^{2}}+(k-1)\frac{\partial v}{\partial\zeta}-kv\right], \tag{28}$$

using the assumed starting condition

$$u_{\lambda}(\zeta,0)=g(\zeta). \tag{29}$$

Let us suppose that, Eq. (28) has the solution in the following series form

$$v_{\lambda}(\zeta,t)=\sum_{k=0}^{\infty}\lambda^{k}v(\zeta,t). \tag{30}$$

Implementing the RL fractional partial integral on both sides of the Eq. (28) and also using Eq. (29) and Theorem 2.2 to get the following equation

$$u_{\lambda}(\zeta,t)=g(x)+\lambda\,{}_{0}J_{t}^{\alpha}\left[\frac{\partial^{2}v}{\partial\zeta^{2}}+(k-1)\frac{\partial v}{\partial\zeta}-kv\right]. \tag{31}$$

Substituting Eq. (30) in Eq. (31), we get

$$\sum_{k=0}^{\infty}\lambda^{k}v_{\lambda}(\zeta,t)=g(\zeta)+\lambda\,{}_{0}J_{t}^{\alpha}\left[\sum_{k=0}^{\infty}\lambda^{k}\frac{\partial^{2}v_{k}}{\partial v^{2}}+(k-1)\sum_{k=0}^{\infty}\lambda^{k}\frac{\partial v_{x}}{\partial v}-k\sum_{k=0}^{\infty}\lambda^{k}v_{k}\right]. \tag{32}$$

After equating the coefficients of identical powersλ in Eq. (32), we can obtain the coefficients:

$$v_{0}(\zeta,t)=g(\zeta),$$

$$v_{1}(\zeta,t)=0\,J_{t}^{\alpha}\left[\frac{\partial^{2}v_{0}}{\partial v^{2}}+(k-1)\frac{\partial v}{\partial v}-kv_{0}\right],$$

$$v_{2}(\zeta,t)={}_{0}J_{t}^{\alpha}\left[\frac{\partial^{2}v_{1}}{\partial\zeta^{2}}+(k-1)\frac{\partial v_{1}}{\partial\zeta}-kv_{1}\right], \tag{33}\quad k=3,4,\ldots$$

From Eq. (27) and Eq. (30). We can get the solution to the considered problem as

$$v(\zeta,t)=\lim_{\lambda\to 1}v_{\lambda}(\zeta,t)=\sum_{k=0}^{\infty}v_{k}(\zeta,t). \tag{34}$$

From Eq. (34) we can see that $v(x,0)=v_{\lambda}(x,0)$ which implies $g(x)=v(\zeta,0)$.

Using the components which we found in Eq. (33) and with the assistance of Definition 3, and Eq. (34). We have found some terms for the solution of the considered Black-Scholes equation (Eq. 24). The computation of the solution and its graphical representations are performed using the Mathematica program.

$$v_{0}(\zeta,t)=e^{\zeta}-1,$$

$$v_{1}(\zeta,t)=k\frac{t^{\alpha}}{\Gamma\alpha+1},$$

$$v_{2}(\zeta,t)=-k^{2}\frac{t^{2\alpha}}{\Gamma 2\alpha+1}, \tag{35}$$

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Thus, the approximate results of Eq. (24) is given by

$$v(\zeta,t)=e^{\zeta}-1+k\frac{t^{\alpha}}{\Gamma\alpha+1}-k^{2}\frac{t^{2\alpha}}{\Gamma 2\alpha+1}+\cdots \tag{36}$$

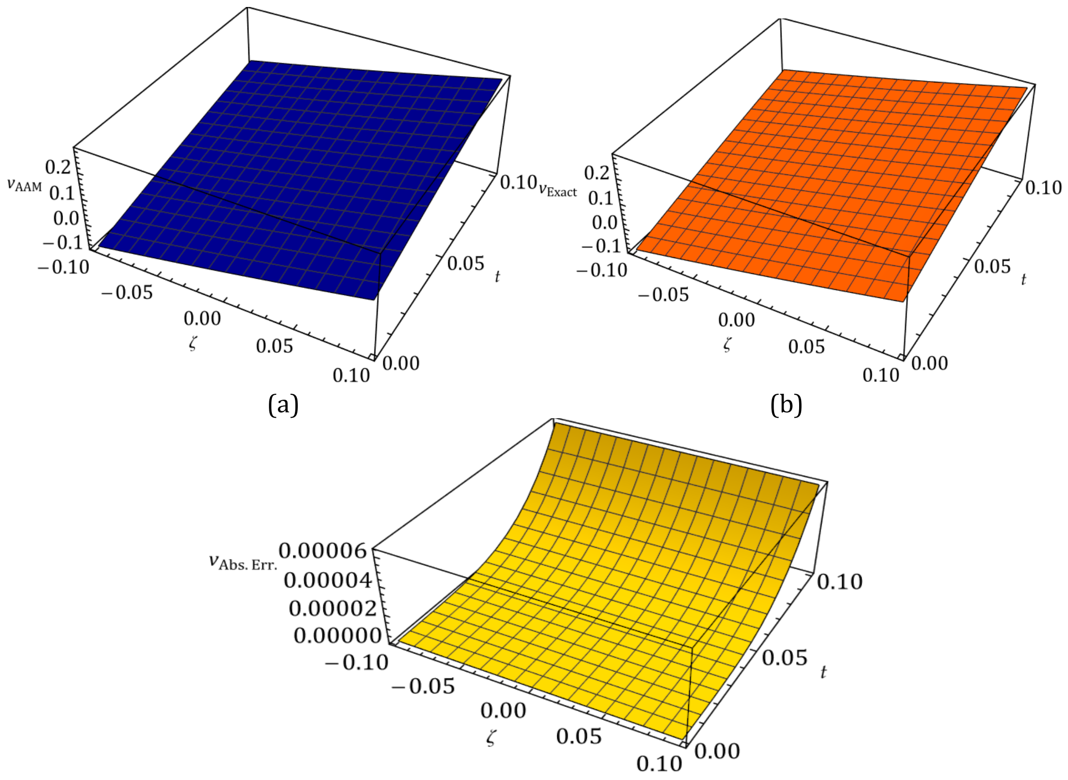


Figure 1. 3D plots of Example 1, when (a) AAM solution, (b) Exact solution, (c) Absolute error= $|u_{exact} - u_{App.}|$, at $\alpha = 1$.

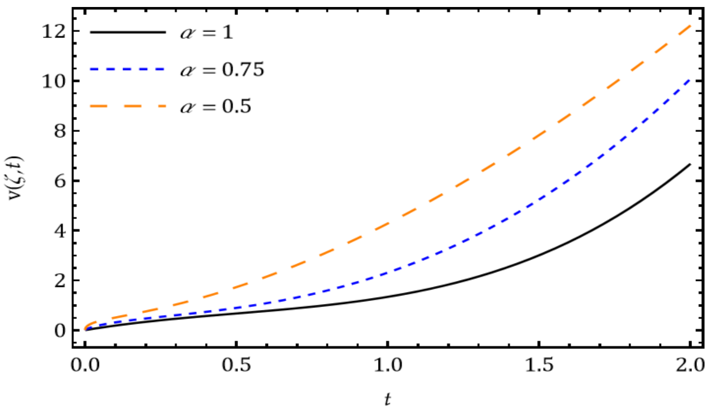


Figure 2. Nature of achieved outcomes for considered problem when $\zeta = 0.01$, for distinct α values.

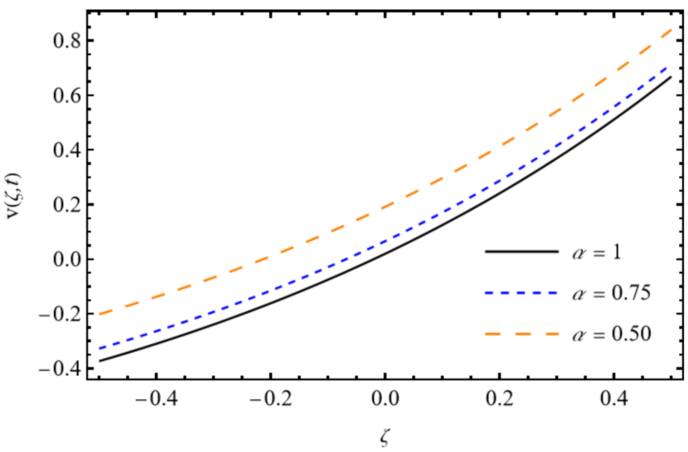


Figure 3. Nature of the solution with respect to ζ at $t = 0.01$.

Table 1: |v exact-v AAM.| of Example 1 at ζ = 1.0 and k = 2.0 when α = 1.0 determined by AAM at feasible locations in the range t ∈ [0, 0.1].

t	Exact solution (v exact)	AAM Solution (v AAM)	LRPSM Solution [34]	Absolute error |v exact-v AAM|
0.01	1.73808	1.73808	1.73808	6.64009×10^{-9}
0.02	1.75749	1.75749	1.75749	1.05819×10^{-7}
0.03	1.77652	1.77652	1.77652	5.33584×10^{-7}
0.04	1.79517	1.79517	1.79517	1.67972×10^{-6}
0.05	1.81344	1.81345	1.81344	4.0847×10^{-6}
0.06	1.83136	1.83137	1.83136	8.43672×10^{-6}
0.07	1.84892	1.84894	1.84892	1.55687×10^{-5}
0.08	1.86614	1.86616	1.86614	2.64556×10^{-5}
0.09	1.88301	1.88305	1.88301	4.22114×10^{-5}
0.1	1.89955	1.89962	1.89955	6.40864×10^{-5}

Example 2.

Next, we consider the following fractional Black-Scholes equation as [36]

$${}_0^cD_t^\alpha v(\zeta,t) + \zeta^2 \frac{\partial^2 v}{\partial \zeta^2} + 0.5\zeta \frac{\partial v}{\partial \zeta} - v = 0, \qquad 0 < \alpha \le 1, \tag{37}$$

with the initial condition

$$v(\zeta,0) = \zeta^3. \tag{38}$$

The above Eq. (37) can be re-written as

$$\frac{\partial^\alpha u}{\partial t^2} + \zeta^2 \frac{\partial^2 v}{\partial \zeta^2} + 0.5\zeta \frac{\partial v}{\partial \zeta} - v, \tag{39}$$

with the assistance of the AAM solutions method. Assume Eq. (37) has the following solution form:

$$v(\zeta,t) = \sum_{k=0}^{\infty} v_k\,(\zeta,t). \tag{40}$$

Consider Eq. (39) to get an approximate solution.

$${}_0^cD_t^\alpha v(\zeta,t) = \lambda\left[-\zeta^2 \frac{\partial^2 v}{\partial \zeta^2} - 0.5\,\zeta \frac{\partial v}{\partial \zeta} + v\right], \tag{41}$$

with the assumed starting solution

$$v(\zeta,t) = g(\zeta). \tag{42}$$

Assume that Eq. (41) has the following series of solutions:

$$v_\lambda(\zeta,t) = \sum_{k=0}^{\infty} \lambda^k v_k\,(\zeta,t). \tag{43}$$

Implementing the RL fractional partial integral on each sides of the Eq. (41) and also using Eq. (42) and Theorem 2.2 to get following equation

$$v_\lambda(\zeta,t) = g(\zeta) + \lambda\,{}_0J_t^\alpha\left[-\zeta^2 \frac{\partial^2 v}{\partial \zeta^2} - 0.5\,\zeta \frac{\partial v}{\partial \zeta} + v\right]. \tag{44}$$

By substituting Eq. (43) in Eq. (44), we get

$$\sum_{k=0}^{\infty} \lambda^k\, v_\lambda(\zeta,t) = g(\zeta) + \lambda\,{}_0J_t^\alpha\left[-\zeta^2 \sum_{k=0}^{\infty} \lambda^k \frac{\partial^2 v_k}{\partial \zeta^2} - 0.5\,\zeta \sum_{k=0}^{\infty} \lambda^k \frac{\partial v_k}{\partial \zeta} - \sum_{k=0}^{\infty} \lambda^k\, u_k\right]. \tag{45}$$

After equating the coefficients of identical powers of λ in Eq. (45), we can obtain the following coefficients:

$$v(\zeta,t) = g(\zeta),$$

$$v_1(\zeta,t) = {}_0J_t^\alpha\left[-\zeta^2 \frac{\partial^2 v_0}{\partial \zeta^2} + 0.5\zeta \frac{\partial v_0}{\partial \zeta} - v_0\right],$$

$$v_2(v,t) = {}_0J_t^\alpha\left[-\zeta^2 \frac{\partial^2 v_1}{\partial \zeta^2} + 0.5\zeta \frac{\partial v_1}{\partial \zeta} - v_1\right],$$

$$k = 3,4,\ldots \tag{46}$$

From the Eq. (40) and (43). We can get the solution to the considered problem as

$$v(\zeta,t) = lim_{\lambda\to 1}\, v_\lambda(\zeta,t) = \sum_{k=0}^{\infty} v_k\,(\zeta,t). \tag{47}$$

From Eq. (47), we can see that $v(\zeta,0) = v_\lambda(\zeta,0)$ which implies $g(\zeta) = v(\zeta,0)$.

Using the components which we found in Eq. (46) and with assistance of Definition 4, and Eq. (47). The computation of the solution and its graphical representations are performed using the Mathematica program.

$$v_0(\zeta,t) = \zeta^3,$$

$$v_1(\zeta,t) = 6.5\,\zeta \frac{t^\alpha}{\Gamma\alpha+1},$$

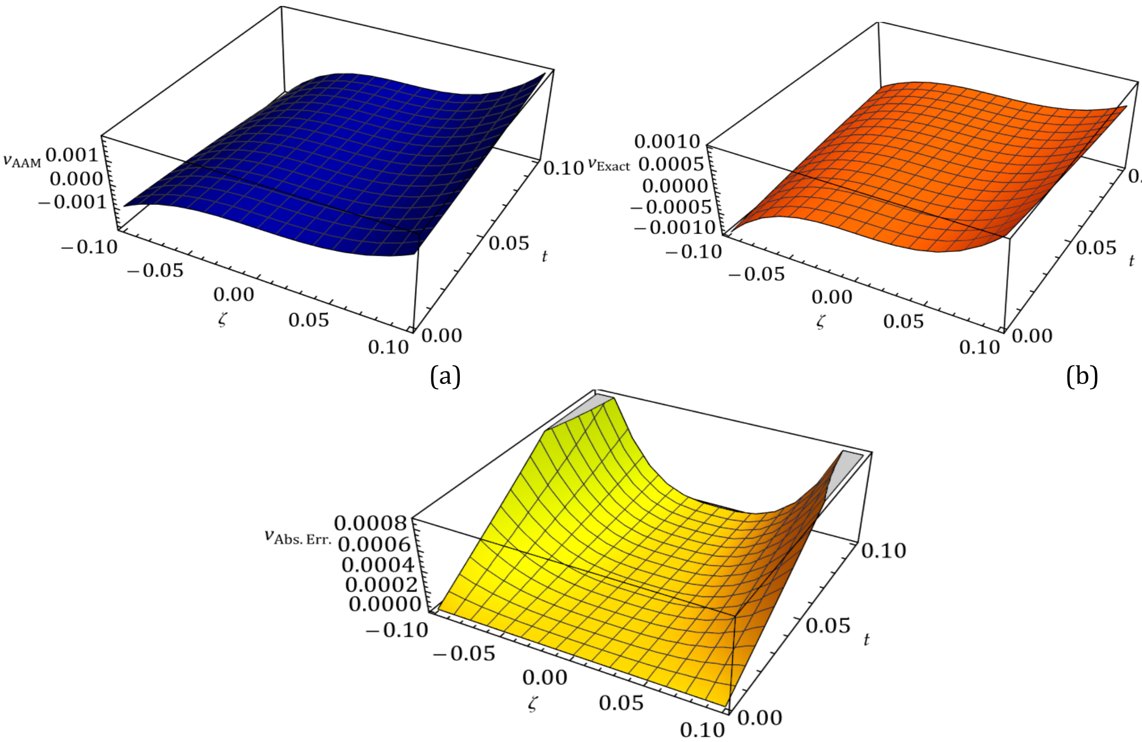
$$v_2(\zeta,t) = 42.25\,\zeta^3 \frac{t^{2\alpha}}{\Gamma 2\alpha+1}, \tag{48}$$

.

.

Thus, the approximate solution of Eq. (37) is given by

$$v(\zeta,t) = \zeta^3 + 6.5\,\zeta\,\frac{t^{\alpha}}{\Gamma\alpha+1} + 42.25\,\zeta^3\,\frac{t^{2\alpha}}{\Gamma 2\alpha+1} + \cdots \qquad (49)$$



(c)

Figure 4. 3D plots of Example 2, when (a) AAM solution, (b) Exact solution, (c) Absolute error= $|u_{exact} - u_{App.}|$, at $\alpha = 1$.

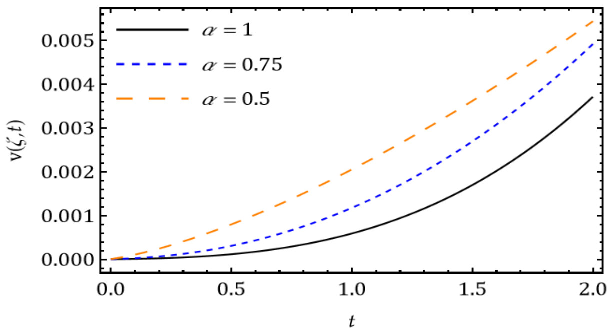


Figure 5. Nature of achieved outcomes for considered problem when $\zeta = 0.02$, for distinct α values**.**

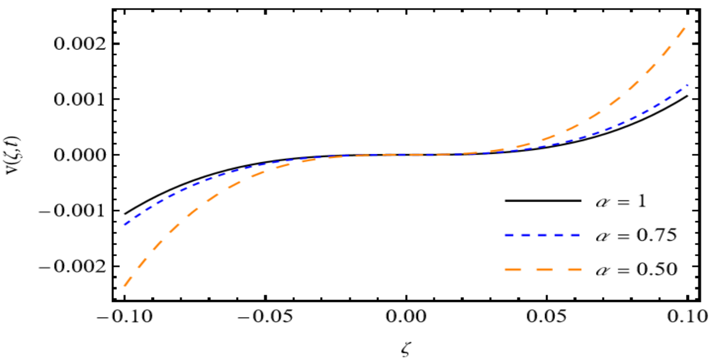


Figure 6. Nature of the solution with respect to ζ at $t = 0.01$.

Table 2: Numerical simulation for Example 2 at ζ = 0.003 when α = 1.0 for different values of t.

t	Exact solution (v exact)	AAM Solution (v AAM)	Absolute error |v exact-v AAM|
0.01	2.53008×10^{-8}	2.53008×10^{-8}	3.51247×10^{-9}
0.02	2.37086×10^{-8}	3.07484×10^{-8}	7.03978×10^{-7}
0.03	2.22165×10^{-8}	3.28133×10^{-8}	1.05968×10^{-7}
0.04	2.08184×10^{-8}	3.50178×10^{-8}	1.41984×10^{-6}
0.05	1.95082×10^{-8}	3.7368×10^{-8}	1.78597×10^{-6}
0.06	1.82805×10^{-8}	3.98763×10^{-8}	2.15958×10^{-6}
0.07	1.71301×10^{-8}	4.25519×10^{-8}	2.54218×10^{-5}
0.08	1.60521×10^{-8}	4.54054×10^{-8}	2.93533×10^{-5}
0.09	1.50419×10^{-8}	4.84477×10^{-8}	3.34058×10^{-5}
0.1	1.40952×10^{-8}	5.16904×10^{-8}	3.75951×10^{-5}

Example 3.

In this example, we consider the [36] time-fractional Black-Scholes equation,

$$ {}_0^cD_t^\alpha v(\zeta,t) + 0.08\,(2+Sin\,\zeta)^2\zeta^2\frac{\partial^2 v}{\partial\zeta^2} + 0.06\zeta\frac{\partial v}{\partial\zeta} - 0.06v = 0 \quad 0<\alpha\le 1, \tag{50}$$

with the starting solution

$$ v(\zeta,0) = \zeta - 25e^{-0.06}. \tag{51}$$

The above Eq. (50) can be re-written as

$$ \frac{\partial^\alpha v}{\partial t^2} = 0.08\,(2+Sin\,\zeta)^2\zeta^2\frac{\partial^2 v}{\partial\zeta^2} + 0.06\zeta\frac{\partial v}{\partial\zeta} - 0.06v, \tag{52}$$

with the help of the AAM procedure, let us assume the solution of the Eq. (50) in a below manner

$$ v(\zeta,t) = \sum_{k=0}^{\infty} v_k\,(\zeta,t). \tag{53}$$

Let us consider the following to obtain approximate solution of Eq. (52)

$$ {}_0^cD_t^\alpha v(\zeta,t) = \lambda\left[0.08\,(2+Sin\,\zeta)^2\zeta^2\frac{\partial^2 v}{\partial\zeta^2} + 0.06\zeta\frac{\partial v}{\partial\zeta} - 0.06v\right], \tag{54}$$

with the assumed starting solution

$$ u_\lambda(\zeta,0) = g(\zeta). \tag{55}$$

Assume that Eq. (54) has the below series of results

$$ u_\lambda(\zeta,t) = \sum_{k=0}^{\infty}\lambda^k v_k\,(\zeta,t). \tag{56}$$

Implementing the RL fractional partial integral on each sides of the Eq. (54) and also using Eq. (55) and Theorem 2.2 to get following equation

$$ v_\lambda(\zeta,t) = g(\zeta) + \lambda\,{}_0J_t^\alpha\left[0.08\,(2+Sin\,\zeta)^2\zeta^2\frac{\partial^2 v}{\partial\zeta^2} + 0.06\zeta\frac{\partial v}{\partial\zeta} - 0.06v\right]. \tag{57}$$

Substituting Eq. (56) in Eq. (57), we get

$$ \sum_{k=0}^{\infty}\lambda^k\,v_\lambda(\zeta,t) = g(\zeta) + \lambda\,{}_0J_t^\alpha\left[0.08\,(2+Sin\,\zeta)^2\zeta^2\sum_{k=0}^{\infty}\lambda^k\frac{\partial^2 v}{\partial\zeta^2} + 0.06\,\zeta\sum_{k=0}^{\infty}\lambda^k\frac{\partial v}{\partial\zeta} - 0.06\,v\right]. \tag{58}$$

After equating the coefficients of identical powers in Eq. (53), we can obtain the coefficients:

$$ \begin{aligned} v_0(\zeta,t) &= g(\zeta),\\ v_1(\zeta,t) &= {}_0J_t^\alpha\left[0.08\,(2+Sin\,\zeta)^2\zeta^2\frac{\partial^2 v_0}{\partial\zeta^2} + 0.06\,\zeta\frac{\partial v_0}{\partial\zeta} - 0.06v_0\right],\\ v_2(\zeta,t) &= {}_0J_t^\alpha\left[0.08\,(2+Sin\,\zeta)^2\zeta^2\frac{\partial^2 v_1}{\partial\zeta^2} + 0.06\,\zeta\frac{\partial v_1}{\partial\zeta} - 0.06v_1\right],\\ k &= 3,4,\ldots \end{aligned} \tag{59}$$

From the Eq. (53) and (56),we can get the solution to the considered problem as

$$ v(\zeta,t) = lim_{\lambda\to 1}\,v_\lambda(\zeta,t) = \sum_{k=0}^{\infty} v_k\,(\zeta,t). \tag{60}$$

From Eq. (60), we can see that $v(\zeta,0) = v_\lambda(\zeta,0)$ which implies $g(\zeta) = v(\zeta,0)$. Using the components which we found in Eq. (59) and with the help of Definition 4 and Eq. (60). The computation of the solution and its graphical representations are performed using the Mathematica program.

$$ \begin{aligned} v_0(\zeta,t) &= \zeta - 25e^{-0.06},\\ v_1(\zeta,t) &= 1.41265\,\frac{t^\alpha}{\Gamma\alpha+1},\\ v_2(\zeta,t) &= -0.084759\,\frac{t^{2\alpha}}{\Gamma 2\alpha+1}, \end{aligned} \tag{61}$$

.

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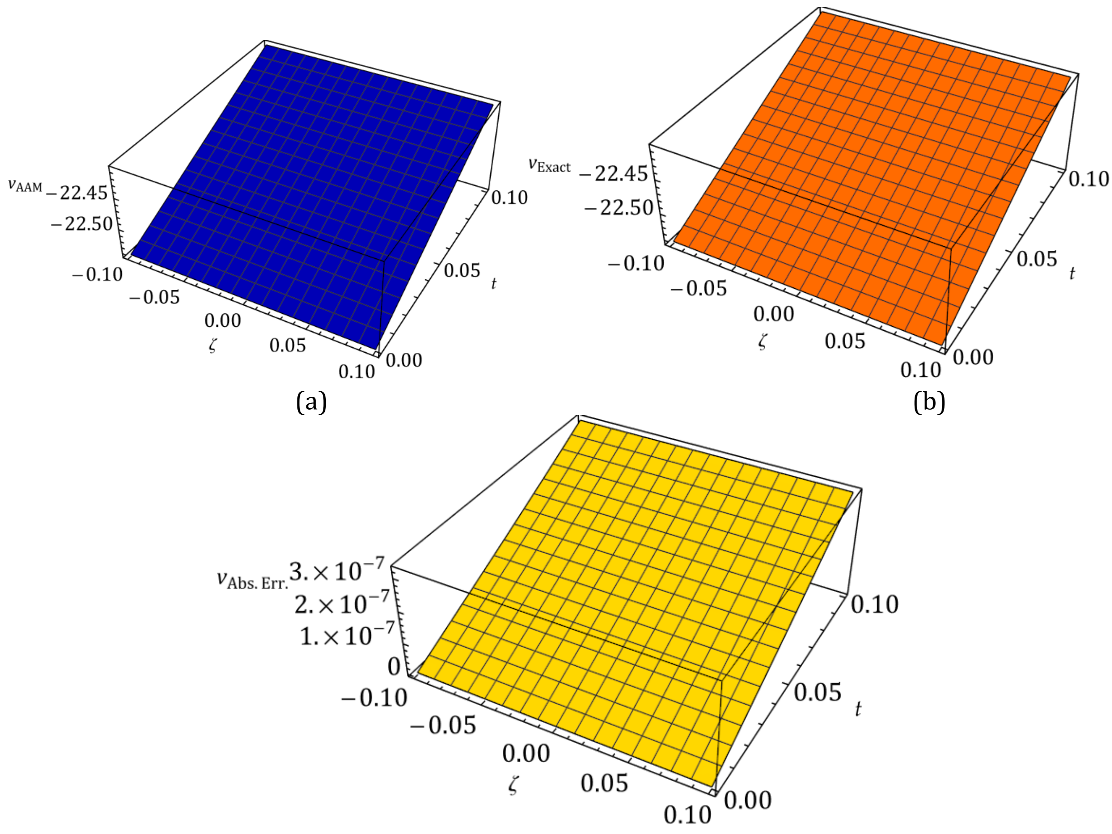
Thus, the approximate solution of Eq. (50) is given by



(c)

Figure 7. 3D plots of Example 3, when (a) AAM solution, (b) Exact solution, (c) Absolute error= $|u_{exact} - u_{App.}|$, at $\alpha = 1$.

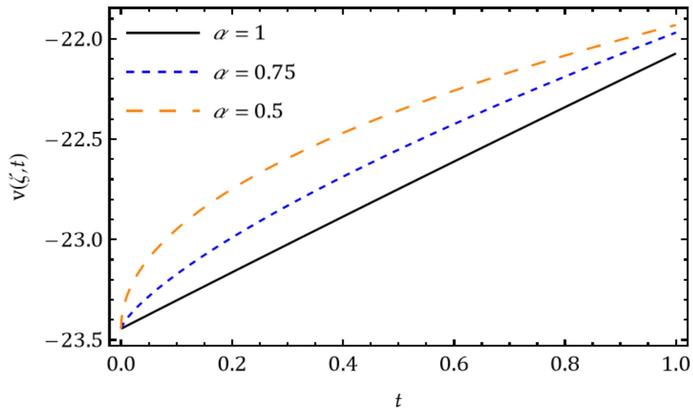


Figure 8. Nature of achieved outcomes for problem when $\zeta = 0.02$, for distinct α values.

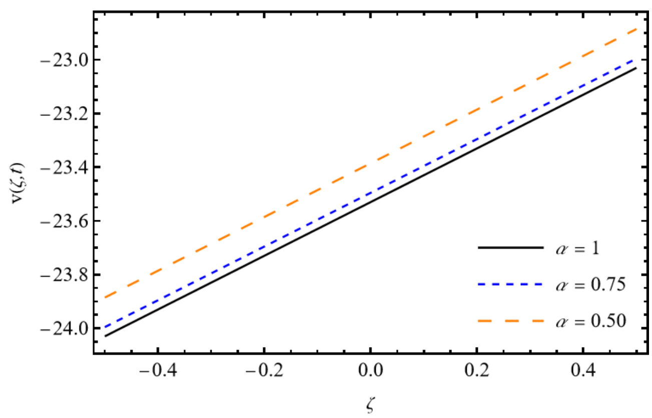


Figure 9. Nature of the solution with respect to ζat $t = 0.01$.

Table 3: Numerical simulation for Example 3 at ζ = 1, and α = 1 for different t.

t	Exact solution (v exact)	AAM Solution (v AAM)	Absolute error |v exact-v AAM|
0.01	-2.253×10^{1}	-2.253×10^{1}	3.67845×10^{-7}
0.02	-2.25159×10^{1}	-2.25159×10^{1}	3.67845×10^{-7}
0.03	-2.25018×10^{1}	-2.25018×10^{1}	3.67845×10^{-7}
0.04	-2.24877×10^{1}	-2.24877×10^{1}	3.67845×10^{-7}
0.05	-2.24736×10^{1}	-2.24736×10^{1}	1.78597×10^{-7}
0.06	-2.24595×10^{1}	-2.24595×10^{1}	3.67845×10^{-7}
0.07	-2.24454×10^{1}	-2.24454×10^{1}	3.67845×10^{-7}
0.08	-2.248314×10^{1}	-2.248314×10^{1}	3.67845×10^{-7}
0.09	-2.24173×10^{1}	-2.24173×10^{1}	3.67845×10^{-7}
0.1	-2.24033×10^{1}	-2.24033×10^{1}	3.67845×10^{-7}

5. Numerical Results and Discussion

In this article, we employed a newlyestablished analytical approximation technique to analyse the iterative process and properties of the Caputo and Reimann-Liouville integral operators. We have to solve some linear time fractional Black-Scholes equations. The solutions are shown via graphs to determine the nature of the considered fractional differential equation. Example 1: Figure 1 displays the solution to the equation in 3D plots at the value of, while Figure 2 illustrates the characteristics and behaviour of the solutions for changing alpha values. Example 2, Figure 3, displays the solution to the equation in 3D plots at the importance of, while Figure 4 illustrates the characteristics and behaviour of the solutions for changing alpha values. Example 3, Figure 5, displays the solution to the equation in 3D plots at the importance of, while Figure 6 illustrates the characteristics and behaviour of the solutions for changing alpha values. Overall, we note that the method is simple for nonlinear and linear fractional differential equations and requires less processing steps. The approximate analytical method (AAM) is particularly efficient in quickly getting the analytical solutions for the Black-Scholes equations without making any assumptions.

6. CONCLUSION

In this paper, an approximate analytical technique is based on iterative processes, Caputo and Reimann-Liouville integral operator properties. The AAM for solving fractional Black-Scholes equations was introduced. The fractional partial differential equation that represents the Black-Sholes equation is being studied using the AAM method. We plotted 3D plots and alpha curves, which shows how accurate the results were. The results are compared with the exact solution to that problem, and we can see that they form the best match. The expected analytical solution for the fractional Black-Scholes equation is studied using AAM. This effective method requires less computational work to solve fractional differential equations.

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Author contributions

Kariyanna Naveen: Methodology, Software, Investigation, Writing-original draft, Writing- review & editing. Doddabhadrappla Gowda Prakasha: Conceptualization, Methodology, Software, Visualization, Investigation, Supervision, Writing original draft. MohdAsif Shah: Methodology, Software, Supervision. Kedaga Channegowda Nandeesh: Formal analysis, Methodology, Writing-review &editing, Resources.

Data availability

No data was used for the research described in the article.

Competing interests

The authors declare that they have no competing interests.

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