Generalized fractional operators and their image formulas

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Abstract

Numerous image formulae for a diversity of polynomials and functions subjected to a variety of fractional integrals and derivatives have been given. In this paper, we aimto construct image formulae for the product of incomplete H-functions and a general class of polynomials under the Katugampola fractional integral and derivative operators . We also provide some particular instances of our main findings in corollaries, among many others.

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1 Introduction and preliminaries

We begin by recalling the well-known Gamma function Γ defined by (see, e.g., [19, Section 1.1])

$$\Gamma(\mu) = \begin{cases} \int_0^\infty e^{-\mathfrak{v}}\mathfrak{v}^{\mu-1}d\mathfrak{v} & (\Re(\mu) > 0) \\ \frac{\Gamma(\mu+k)}{(\mu)_k} & (\mu \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0};\ k \in \mathbb{N}_0), \end{cases} \tag{1.1}$$

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where the Pochhammer symbol $(\mu)_\nu$ $(\mu, \nu \in \mathbb{C})$ is defined, in terms of Gamma function Γ (see, e.g., [19, p. 2 and p. 5]), by

$$\begin{aligned}(\mu)_\nu &= \frac{\Gamma(\mu+\nu)}{\Gamma(\mu)} \quad (\mu+\nu \in \mathbb{C}\setminus\mathbb{Z}_{\leq 0},\ \nu \in \mathbb{C}\setminus\{0\};\ \mu \in \mathbb{C}\setminus\mathbb{Z}_{\leq 0},\ \nu=0)\\ &= \begin{cases} 1 & (\nu=0,\ \mu \in \mathbb{C}\setminus\{0\}),\\ \mu(\mu+1)\cdots(\mu+n-1) & (\nu=n\in\mathbb{N},\ \mu\in\mathbb{C}),\end{cases}\end{aligned} \tag{1.2}$$

it being assumed that $(0)_0 = 1$. Here and throughout, let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{Z}$, and $\mathbb{N}$ denote the sets of complex numbers, real numbers, positive real numbers, integers, and positive integers, respectively. Also let $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$ and $\mathbb{Z}_{\leq \ell}$ be the set of integers which are less than or equal to some integer $\ell \in \mathbb{Z}$. The incomplete Gamma function $\gamma(\mu, u)$ and its complement $\Gamma(\mu, u)$ defined by

$$\gamma(\mu,u) = \int_0^u e^{-\mathfrak{v}}\mathfrak{v}^{\mu-1}d\mathfrak{v} \qquad (u \geq 0;\ \Re(\mu)>0), \tag{1.3}$$

and

$$\Gamma(\mu,u) = \int_u^\infty e^{-\mathfrak{v}}\mathfrak{v}^{\mu-1}d\mathfrak{v} \qquad (u \geq 0;\ \Re(\mu)>0 \text{ when } u=0), \tag{1.4}$$

respectively, satisfy the following relation:

$$\gamma(\mu,u)+\Gamma(\mu,u)=\Gamma(\mu) \qquad (\Re(\mu)>0). \tag{1.5}$$

Srivastava et al. [21] used the incomplete Gamma functions to introduce the following incomplete H-functions (see also [5]):

$$\begin{aligned}\gamma_{\mathsf{p,q}}^{\mathsf{m,n}}(z) &= \gamma_{\mathsf{p,q}}^{\mathsf{m,n}}\left[z \,\middle|\, \begin{matrix} (\mathsf{e}_1,\mathsf{E}_1,\mathsf{y}),(\mathsf{e}_i,\mathsf{E}_i)_{2,\mathsf{p}} \\ (\mathsf{f}_i,\mathsf{F}_i)_{1,\mathsf{q}} \end{matrix} \right]\\ &= \gamma_{\mathsf{p,q}}^{\mathsf{m,n}}\left[z \,\middle|\, \begin{matrix} (\mathsf{e}_1,\mathsf{E}_1,\mathsf{y}),(\mathsf{e}_2,\mathsf{E}_2),\cdots,(\mathsf{e}_\mathsf{p},\mathsf{E}_\mathsf{p}) \\ (\mathsf{f}_1,\mathsf{F}_1),(\mathsf{f}_2,\mathsf{F}_2),\cdots,(\mathsf{f}_\mathsf{q},\mathsf{F}_\mathsf{q}) \end{matrix} \right]\\ &:= \frac{1}{2\pi \mathrm{i}}\int_{\mathfrak{C}} \mathbb{G}(\xi,\mathsf{y})\ z^{-\xi}d\xi,\end{aligned} \tag{1.6}$$

where

$$\mathbb{G}(\xi,\mathsf{y}) = \frac{\gamma(1-\mathsf{e}_1-\mathsf{E}_1\xi,\mathsf{y})\prod\limits_{i=1}^{\mathsf{m}}\Gamma(\mathsf{f}_i+\mathsf{F}_i\xi)\prod\limits_{i=2}^{\mathsf{n}}\Gamma(1-\mathsf{e}_i-\mathsf{E}_i\xi)}{\prod\limits_{i=\mathsf{m}+1}^{\mathsf{q}}\Gamma(1-\mathsf{f}_i-\mathsf{F}_i\xi)\prod\limits_{i=\mathsf{n}+1}^{\mathsf{p}}\Gamma(\mathsf{e}_i+\mathsf{E}_i\xi)}; \tag{1.7}$$

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$$\begin{aligned}\Gamma_{\mathsf{p,q}}^{\mathsf{m,n}}(z) &= \Gamma_{\mathsf{p,q}}^{\mathsf{m,n}}\left[\, z \;\middle|\; \begin{matrix} (\mathsf{e}_1,\mathsf{E}_1,\mathsf{y}),(\mathsf{e}_i,\mathsf{E}_i)_{2,\mathsf{p}} \\ \\ (\mathsf{f}_i,\mathsf{F}_i)_{1,\mathsf{q}} \end{matrix} \right] \\ &= \Gamma_{\mathsf{p,q}}^{\mathsf{m,n}}\left[\, z \;\middle|\; \begin{matrix} (\mathsf{e}_1,\mathsf{E}_1,\mathsf{y}),(\mathsf{e}_2,\mathsf{E}_2),\cdots,(\mathsf{e}_\mathsf{p},\mathsf{E}_\mathsf{p}) \\ \\ (\mathsf{f}_1,\mathsf{F}_1),(\mathsf{f}_2,\mathsf{F}_2),\cdots,(\mathsf{f}_\mathsf{q},\mathsf{F}_\mathsf{q}) \end{matrix} \right] \\ &:= \frac{1}{2\pi \mathrm{i}} \int_{\mathfrak{C}} \mathbb{F}(\xi,\mathsf{y})\; z^{-\xi} d\xi, \end{aligned} \tag{1.8}$$

where

$$\mathbb{F}(\xi,\mathsf{y}) = \frac{\Gamma(1-\mathsf{e}_1-\mathsf{E}_1\xi,\mathsf{y}) \prod_{i=1}^{\mathsf{m}} \Gamma(\mathsf{f}_i+\mathsf{F}_i\xi) \prod_{i=2}^{\mathsf{n}} \Gamma(1-\mathsf{e}_i-\mathsf{E}_i\xi)}{\prod_{i=\mathsf{m}+1}^{\mathsf{q}} \Gamma(1-\mathsf{f}_i-\mathsf{F}_i\xi) \prod_{i=\mathsf{n}+1}^{\mathsf{p}} \Gamma(\mathsf{e}_i+\mathsf{E}_i\xi)}. \tag{1.9}$$

For convergence conditions of these incomplete H-functions as well as the description of the contour $\mathfrak{C}$, one may refer to [21]. They [21] explored a variety of intriguing properties of these incomplete H-functions, such as decomposition and reduction formulas, derivative formulas, various integral transforms, and computational representations, as well as applied some significantly general RiemannLiouville and Weyl type fractional integral operators to each of these incomplete H-functions.

Srivastava [17] introduced the following general class of polynomials:

$$S_{\mathsf{n}}^{\mathsf{m}}[x] = \sum_{k=0}^{[\mathsf{n}/\mathsf{m}]} \frac{(-\mathsf{n})_{\mathsf{m}k}}{k!} A_{\mathsf{n},k}\, x^k \qquad (\mathsf{m}\in\mathbb{N},\ \mathsf{n}\in\mathbb{N}_0), \tag{1.10}$$

where the coefficients $A_{\mathsf{n},k}$ (n, $k \in \mathbb{N}_0$) are arbitrary real or complex constants. By properly specializing $A_{\mathsf{n},k}$, the general class of polynomials may generate many existing polynomials as special instances, including Jacobi and Laguerre polynomials (see, e.g., [15]). In particular, setting $A_{0,0} = 1$ and $x = 0$ reduces $S_{\mathsf{n}}^{\mathsf{m}}[x]$ to unity.

There have been many introductions and investigations of fractional integrals and derivatives. Two of them are recalled here. The left-sided and right-sided Riemann-Liouville fractional integrals $I_{a+}^{\alpha} f$ and $I_{b-}^{\alpha} f$ of order $\alpha \in \mathbb{C}$ are defined as (see, e.g., [10, 12–14])

$$\left(I_{a+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\tau)^{\alpha-1} f(\tau)\, d\tau \quad (x > a,\ \Re(\alpha) > 0), \tag{1.11}$$

and

$$\left(I_{b-}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (\tau-x)^{\alpha-1} f(\tau)\, d\tau \quad (b > x,\ \Re(\alpha) > 0), \tag{1.12}$$

respectively. The Riemann-Liouville fractional derivatives $D^{\alpha}_{a+}f$ and $D^{\alpha}_{b-}f$ of order $\alpha \in \mathbb{C}$ ($\Re(\alpha) \geq 0$) are defined by

$$(D^{\alpha}_{a+}f)(x) = \left(\frac{d}{dx}\right)^{\mathsf{n}} \left(I^{\mathsf{n}-\alpha}_{a+}f\right)(x) \quad (x > a) \tag{1.13}$$

and

$$(D^{\alpha}_{b-}f)(x) = \left(-\frac{d}{dx}\right)^{\mathsf{n}} \left(I^{\mathsf{n}-\alpha}_{b-}f\right)(x) \quad (x < b), \tag{1.14}$$

where $\mathsf{n} = [\Re(\alpha)] + 1$.

For $\rho \in \mathbb{R} \setminus \{0\}$, the left-sided and right-sided Katugampola fractional integrals, respectively, denoted by ${}^{\rho}I^{\lambda}_{a+}$ and ${}^{\rho}I^{\lambda}_{b-}$ of order $\lambda \in \mathbb{C}$ ($\Re(\lambda) > 0$), are defined as (see [7])

$$\left({}^{\rho}I^{\lambda}_{a+}\phi\right)(s) = \frac{\rho^{1-\lambda}}{\Gamma(\lambda)} \int_a^s \frac{\tau^{\rho-1}\phi(\tau)}{(s^{\rho} - \tau^{\rho})^{1-\lambda}} d\tau \quad (s > a), \tag{1.15}$$

and

$$\left({}^{\rho}I^{\lambda}_{b-}\phi\right)(s) = \frac{\rho^{1-\lambda}}{\Gamma(\lambda)} \int_s^b \frac{\tau^{\rho-1}\phi(\tau)}{(\tau^{\rho} - s^{\rho})^{1-\lambda}} d\tau \quad (b > s). \tag{1.16}$$

It is noted that

(i) when $\rho = 1$, (1.15) and (1.16), respectively, reduce to Riemann-Liouville fractional integrals (1.11) and (1.12);

(ii) taking $\rho \to 0^+$, (1.15) and (1.16), respectively, reduce to the famous Hadamard fractional integrals (see [6]; see also [7]):

$$\left(H^{\lambda}_{a+}\phi\right)(s) = \frac{1}{\Gamma(\lambda)} \int_a^s \left(\log\frac{s}{\tau}\right)^{\lambda-1} \frac{\phi(\tau)}{\tau} d\tau \quad (s > a,\, \Re(\lambda) > 0), \tag{1.17}$$

and

$$\left(H^{\lambda}_{b-}\phi\right)(s) = \frac{1}{\Gamma(\lambda)} \int_s^b \left(\log\frac{\tau}{s}\right)^{\lambda-1} \frac{\phi(\tau)}{\tau} d\tau \quad (s < b,\, \Re(\lambda) > 0). \tag{1.18}$$

The matching Katugampola fractional derivatives on the left and right sides, designated respectively by ${}^{\rho}D^{\lambda}_{a+}$ and ${}^{\rho}D^{\lambda}_{b-}$, are defined as (see [8])

$$\begin{aligned}\left({}^{\rho}D^{\lambda}_{a+}\phi\right)(s) &= \left(s^{1-\rho}\frac{d}{ds}\right)^{\mathsf{n}} \left({}^{\rho}I^{\mathsf{n}-\lambda}_{a+}\phi\right)(s) \\ &= \frac{\rho^{\lambda-\mathsf{n}+1}}{\Gamma(\mathsf{n}-\lambda)} \left(s^{1-\rho}\frac{d}{ds}\right)^{\mathsf{n}} \int_a^s \frac{\tau^{\rho-1}\phi(\tau)}{(s^{\rho} - \tau^{\rho})^{\lambda-\mathsf{n}+1}} d\tau, \end{aligned} \tag{1.19}$$

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and

$$\begin{aligned}\left({}^{\rho}D_{b-}^{\lambda}\phi\right)(s) &= \left(-s^{1-\rho}\frac{d}{ds}\right)^{\mathsf{n}}\left({}^{\rho}I_{b-}^{\mathsf{n}-\lambda}\phi\right)(s)\\ &= \frac{\rho^{\lambda-\mathsf{n}+1}}{\Gamma(\mathsf{n}-\lambda)}\left(-s^{1-\rho}\frac{d}{ds}\right)^{\mathsf{n}}\int_s^b \frac{\tau^{\rho-1}\phi(\tau)}{(\tau^{\rho}-s^{\rho})^{\lambda-\mathsf{n}+1}}\,d\tau, \qquad (1.20)\end{aligned}$$

where $\mathsf{n} = [\Re(\lambda)] + 1$.

The identities in Lemmas 1.1 and 1.2 provide the image formulae for the power function t^{α} when the fractional integral and derivative operators of Katugampola are used. In this case, we make major use of the well-known beta function (see, e.g., [19, p. 8]):

$$B(\alpha,\beta) = \begin{cases} \displaystyle\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,\mathrm{d}t & (\Re(\alpha)>0,\ \Re(\beta)>0) \\ \dfrac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} & (\alpha,\beta\in\mathbb{C}\setminus\mathbb{Z}_{\le 0}). \end{cases} \qquad (1.21)$$

Proofs have been omitted.

Lemma 1.1 *Let $\rho > 0$, $\Re(\alpha) > 0$, and $\Re(\lambda) > 0$. Then*

$$\left({}^{\rho}I_{0+}^{\lambda}t^{\alpha}\right)(s) = \rho^{-\lambda}\frac{\Gamma\left(\frac{\alpha}{\rho}+1\right)}{\Gamma\left(\frac{\alpha}{\rho}+1+\lambda\right)}s^{\alpha+\rho\lambda} \qquad (1.22)$$

and

$$\left({}^{\rho}I_{0-}^{\lambda}t^{\alpha}\right)(s) = (-\rho)^{-\lambda}\frac{\Gamma\left(\frac{\alpha}{\rho}+1\right)}{\Gamma\left(\frac{\alpha}{\rho}+1+\lambda\right)}s^{\alpha+\rho\lambda}. \qquad (1.23)$$

Lemma 1.2 *Let $\rho > 0$, $\Re(\alpha) > 0$, $\Re(\lambda) > 0$, and $n = [\Re(\lambda)] + 1$. Then*

$$\left({}^{\rho}D_{0+}^{\lambda}t^{\alpha}\right)(s) = \rho^{\lambda}\frac{\Gamma\left(\frac{\alpha}{\rho}+1\right)}{\Gamma\left(\frac{\alpha}{\rho}+1-\lambda\right)}s^{\alpha-\rho\lambda} \qquad (1.24)$$

and

$$\left({}^{\rho}D_{0-}^{\lambda}t^{\alpha}\right)(s) = (-\rho)^{\lambda}\frac{\Gamma\left(\frac{\alpha}{\rho}+1\right)}{\Gamma\left(\frac{\alpha}{\rho}+1-\lambda\right)}s^{\alpha-\rho\lambda}. \qquad (1.25)$$

Numerous image formulae for a diversity of polynomials and functions subjected to a variety of fractional integrals and derivatives have been given (see, e.g., [1], [2], [3], [4], [9], [18], [22], [23], [24], [25]). The purpose of this article isto establish image formulae for the product of incomplete H-functions and a general class of polynomials under the Katugampola fractional integral and derivative operators. Among many others, we also present some specific examples of our major results.

2 Katugampola fractional integral operators involving incomplete H-functions and general class of polynomials

In this part, we state the following theorems that establish the image formulae for product of the incomplete H-functions and the general class of polynomials under the left- and right-sided Katugampola fractional integral operators.

Theorem 2.1 *Let* $\Re(\lambda) > 0$, a, $b \in \mathbb{R}$, ρ, α, $\beta \in \mathbb{R}^+$, $\mathsf{y} \geq 0$, *and* $s > 0$. *Then*

$$\left({}^{\rho}I^{\lambda}_{0+} S^{\mathsf{m}}_{\mathsf{n}}[at^{\alpha}]\Gamma^{\mathsf{m,n}}_{\mathsf{p,q}}\left[bt^{\beta}\left|\begin{matrix}(\mathsf{e}_1,\mathsf{E}_1,\mathsf{y}),(\mathsf{e}_j,\mathsf{E}_j)_{2,\mathsf{p}}\\ (\mathsf{f}_j,\mathsf{F}_j)_{1,\mathsf{q}}\end{matrix}\right.\right]\right)(s) = \rho^{-\lambda}s^{\rho\lambda}\sum_{k=0}^{[\frac{\mathsf{n}}{\mathsf{m}}]}\frac{(-\mathsf{n})_{\mathsf{m}k}}{k!}A_{\mathsf{n},k}(as^{\alpha})^k \times \Gamma^{\mathsf{m,n+1}}_{\mathsf{p+1,q+1}}\left[bs^{\beta}\left|\begin{matrix}(\mathsf{e}_1,\mathsf{E}_1,\mathsf{y}),(\mathsf{e}_j,\mathsf{E}_j)_{2,\mathsf{p}}(\frac{-\alpha k}{\rho},\frac{\beta}{\rho})\\ (\mathsf{f}_j,\mathsf{F}_j)_{1,\mathsf{q}},(-\lambda-\frac{\alpha k}{\rho},\frac{\beta}{\rho})\end{matrix}\right.\right] \tag{2.1}$$

and

$$\left({}^{\rho}I^{\lambda}_{0+} S^{\mathsf{m}}_{\mathsf{n}}[at^{\alpha}]\gamma^{\mathsf{m,n}}_{\mathsf{p,q}}\left[bt^{\beta}\left|\begin{matrix}(\mathsf{e}_1,\mathsf{E}_1,\mathsf{y}),(\mathsf{e}_j,\mathsf{E}_j)_{2,\mathsf{q}}\\ (\mathsf{f}_j,\mathsf{F}_j)_{1,w}\end{matrix}\right.\right]\right)(s) = \rho^{-\lambda}s^{\rho\lambda}\sum_{k=0}^{[\frac{\mathsf{n}}{\mathsf{m}}]}\frac{(-\mathsf{n})_{\mathsf{m}k}}{k!}A_{\mathsf{n},k}(as^{\alpha})^k \times \gamma^{\mathsf{m,n+1}}_{\mathsf{p+1,q+1}}\left[bs^{\beta}\left|\begin{matrix}(\mathsf{e}_1,\mathsf{E}_1,\mathsf{y}),(\mathsf{e}_j,\mathsf{E}_j)_{2,\mathsf{p}}(\frac{-\alpha k}{\rho},\frac{\beta}{\rho})\\ (\mathsf{f}_j,\mathsf{F}_j)_{1,\mathsf{q}},(-\lambda-\frac{\alpha k}{\rho},\frac{\beta}{\rho})\end{matrix}\right.\right]. \tag{2.2}$$

Proof. Let Δ be the left-handed member of (2.1). Using (1.15), (1.10) and (1.8), and changing the order of integrals, which may be readily verified under the constraints, we have

$$\Delta = \sum_{k=0}^{[\mathsf{n}/\mathsf{m}]}\frac{(-\mathsf{n})_{\mathsf{m}k}}{k!}A_{\mathsf{n},k}\,a^k\int_{\mathfrak{C}}\mathbb{F}(\xi,\mathsf{y})b^{-\xi}\left({}^{\rho}I^{\lambda}_{0+}\left[t^{\alpha k-\beta\xi}\right]\right)(s)\,d\xi. \tag{2.3}$$

Employing (1.22) to evaluate the right-handed Katugampola fractional integral in (2.3), we obtain

$$\Delta=\rho^{-\lambda}s^{\rho\lambda}\sum_{k=0}^{[\mathsf{n}/\mathsf{m}]}\frac{(-\mathsf{n})_{\mathsf{m}k}}{k!}A_{\mathsf{n},k}(s^{\alpha}a)^{k}\frac{1}{2\pi i}\int_{\mathfrak{L}}\mathbb{F}(\xi,\mathsf{y})(bs^{\beta})^{-\xi}\frac{\Gamma\left[1+\frac{\alpha k}{\rho}-\frac{\beta}{\rho}\xi\right]}{\left[1+\lambda+\frac{\alpha k}{\rho}-\frac{\beta}{\rho}\xi\right]}d\xi,$$

which, upon expressing the integral in terms of (1.8), yields the desired right-handed member of (2.1).

The proof of (2.2) would run in parallel with that of (2.1). We omit the specific. □

Theorem 2.2 *Let $\Re(\lambda)>0$, $a,b\in\mathbb{R}$, ρ, α, $\beta\in\mathbb{R}^{+}$, $\mathsf{y}\geq 0$, and $s<0$. Then*

$$\left({}^{\rho}I_{0-}^{\lambda}S_{\mathsf{n}}^{\mathsf{m}}[at^{\alpha}]\Gamma_{\mathsf{p},\mathsf{q}}^{\mathsf{m},\mathsf{n}}\left[bt^{\beta}\left|\begin{matrix}(\mathsf{e}_1,\mathsf{E}_1,\mathsf{y}),(\mathsf{e}_j,\mathsf{E}_j)_{2,\mathsf{p}}\\ \\ (\mathsf{f}_j,\mathsf{F}_j)_{1,\mathsf{q}}\end{matrix}\right.\right]\right)(s)=(-\rho)^{-\lambda}s^{\rho\lambda}\sum_{k=0}^{[\frac{\mathsf{n}}{\mathsf{m}}]}\frac{(-\mathsf{n})_{\mathsf{m}k}}{k!}A_{\mathsf{n},k}(as^{\alpha})^{k}$$
$$\times\Gamma_{\mathsf{p}+1,\mathsf{q}+1}^{\mathsf{m},\mathsf{n}+1}\left[b\,s^{\beta}\left|\begin{matrix}(\mathsf{e}_1,\mathsf{E}_1,\mathsf{y}),(\mathsf{e}_j,\mathsf{E}_j)_{2,\mathsf{p}}(-\frac{\alpha k}{\rho},\frac{\beta}{\rho})\\ \\ (\mathsf{f}_j,\mathsf{F}_j)_{1,\mathsf{q}},(-\lambda-\frac{\alpha k}{\rho},\frac{\beta}{\rho})\end{matrix}\right.\right] \tag{2.4}$$

and

$$\left({}^{\rho}I_{0-}^{\lambda}S_{\mathsf{n}}^{m}[at^{\alpha}]\gamma_{\mathsf{p},\mathsf{q}}^{\mathsf{m},\mathsf{n}}\left[bt^{\beta}\left|\begin{matrix}(\mathsf{e}_1,\mathsf{E}_1,\mathsf{y}),(\mathsf{e}_j,\mathsf{E}_j)_{2,\mathsf{p}}\\ \\ (\mathsf{f}_j,\mathsf{F}_j)_{1,\mathsf{q}}\end{matrix}\right.\right]\right)(s)=(-\rho)^{-\lambda}s^{\rho\lambda}\sum_{k=0}^{[\frac{\mathsf{n}}{\mathsf{m}}]}\frac{(-\mathsf{n})_{\mathsf{m}k}}{k!}A_{\mathsf{n},k}(as^{\alpha})^{k}$$
$$\times\gamma_{\mathsf{p}+1,\mathsf{q}+1}^{\mathsf{m},\mathsf{n}+1}\left[bs^{\beta}\left|\begin{matrix}(\mathsf{e}_1,\mathsf{E}_1,\mathsf{y}),(\mathsf{e}_j,\mathsf{E}_j)_{2,\mathsf{p}}(-\frac{\alpha k}{\rho},\frac{\beta}{\rho})\\ \\ (\mathsf{f}_j,\mathsf{F}_j)_{1,\mathsf{q}},(-\lambda-\frac{\alpha k}{\rho},\frac{\beta}{\rho})\end{matrix}\right.\right]. \tag{2.5}$$

Proof. The proof would proceed in the same manner as the proof of Theorem 2.1. We omit specifics. □

3 Katugampola fractional derivative operators with incomplete H-functions and general class of polynomials

The following two theorems provide the image formulae for product of the incomplete H-functions and the general class of polynomials under the left- and right-sided Katugampola fractional derivative operators. Since the proofs here would be identical to those used in Theorems 2.1 and 2.2, we omit the required proofs.

Theorem 3.1 *Let* $\Re(\lambda) > 0$, $a, b \in \mathbb{R}$, $\rho, \alpha, \beta \in \mathbb{R}^+$, $\mathsf{y} \geq 0$, *and* $s > 0$. *Then*

$$\left({}^{\rho}\mathcal{D}^{\lambda}_{0+} S^{\mathsf{m}}_{\mathsf{n}}[at^{\alpha}] \Gamma^{\mathsf{m,n}}_{\mathsf{p,q}} \left[bt^{\beta} \middle| \begin{matrix} (\mathsf{e}_1, \mathsf{E}_1, \mathsf{y}), (\mathsf{e}_j, \mathsf{E}_j)_{2,\mathsf{p}} \\ (\mathsf{f}_j, \mathsf{F}_j)_{1,\mathsf{q}} \end{matrix} \right]\right)(s)$$
$$= \rho^{\lambda} s^{\rho\lambda} \sum_{k=0}^{[\mathsf{n/m}]} \frac{(-\mathsf{n})_{\mathsf{m}k}}{k!} A_{\mathsf{n},k} (as^{\alpha})^k \Gamma^{\mathsf{m,n+1}}_{\mathsf{p+1,q+1}} \left[bs^{\beta} \middle| \begin{matrix} (\mathsf{e}_1, \mathsf{E}_1, \mathsf{y}), (\mathsf{e}_j, \mathsf{E}_j)_{2,\mathsf{p}}, (-\frac{\alpha k}{\rho}, \frac{\beta}{\rho}) \\ (\mathsf{f}_j, \mathsf{F}_j)_{1,\mathsf{q}}, (\lambda - \frac{\alpha k}{\rho}, \frac{\beta}{\rho}) \end{matrix} \right] \tag{3.1}$$

and

$$\left({}^{\rho}\mathcal{D}^{\lambda}_{0+} S^{\mathsf{m}}_{\mathsf{n}}[at^{\alpha}] \gamma^{\mathsf{m,n}}_{\mathsf{p,q}} \left[bt^{\beta} \middle| \begin{matrix} (\mathsf{e}_1, \mathsf{E}_1, \mathsf{y}), (\mathsf{e}_j, \mathsf{E}_j)_{2,\mathsf{p}} \\ (\mathsf{f}_j, \mathsf{F}_j)_{1,\mathsf{q}} \end{matrix} \right]\right)(s)$$
$$= \rho^{\lambda} s^{\rho\lambda} \sum_{k=0}^{[\mathsf{n/m}]} \frac{(-\mathsf{n})_{\mathsf{m}k}}{k!} A_{\mathsf{n},k} (as^{\alpha})^k \gamma^{\mathsf{m,n+1}}_{\mathsf{p+1,q+1}} \left[bs^{\beta} \middle| \begin{matrix} (\mathsf{e}_1, \mathsf{E}_1, \mathsf{y}), (\mathsf{e}_j, \mathsf{E}_j)_{2,\mathsf{p}}, (-\frac{\alpha k}{\rho}, \frac{\beta}{\rho}) \\ (\mathsf{f}_j, \mathsf{F}_j)_{1,\mathsf{q}}, (\lambda - \frac{\alpha k}{\rho}, \frac{\beta}{\rho}) \end{matrix} \right]. \tag{3.2}$$

Theorem 3.2 *Let* $\Re(\lambda) > 0$, $a, b \in \mathbb{R}$, $\rho, \alpha, \beta \in \mathbb{R}^+$, $\mathsf{y} \geq 0$, *and* $s < 0$. *Then*

$$\left({}^{\rho}\mathcal{D}^{\lambda}_{0-} S^{\mathsf{m}}_{\mathsf{n}}[at^{\alpha}] \Gamma^{\mathsf{m,n}}_{\mathsf{p,q}} \left[bt^{\beta} \middle| \begin{matrix} (\mathsf{e}_1, \mathsf{E}_1, \mathsf{y}), (\mathsf{e}_j, \mathsf{E}_j)_{2,\mathsf{p}} \\ (\mathsf{f}_j, \mathsf{F}_j)_{1,\mathsf{q}} \end{matrix} \right]\right)(s)$$
$$= (-\rho)^{\lambda} s^{\rho\lambda} \sum_{k=0}^{[\mathsf{n/m}]} \frac{(-\mathsf{n})_{\mathsf{m}k}}{k!} A_{\mathsf{n},k} (as^{\alpha})^k \Gamma^{\mathsf{m,n+1}}_{\mathsf{p+1,q+1}} \left[bs^{\beta} \middle| \begin{matrix} (\mathsf{e}_1, \mathsf{E}_1, \mathsf{y}), (\mathsf{e}_j, \mathsf{E}_j)_{2,\mathsf{p}}, (-\frac{\alpha k}{\rho}, \frac{\beta}{\rho}) \\ (\mathsf{f}_j, \mathsf{F}_j)_{1,\mathsf{q}}, (\lambda - \frac{\alpha k}{\rho}, \frac{\beta}{\rho}) \end{matrix} \right] \tag{3.3}$$

and

$$\left({}^{\rho}\mathcal{D}^{\lambda}_{0-} S^{\mathsf{m}}_{\mathsf{n}}[at^{\alpha}] \gamma^{\mathsf{m,n}}_{\mathsf{p,q}} \left[bt^{\beta} \middle| \begin{matrix} (\mathsf{e}_1, \mathsf{E}_1, \mathsf{y}), (\mathsf{e}_j, \mathsf{E}_j)_{2,\mathsf{p}} \\ (\mathsf{f}_j, \mathsf{F}_j)_{1,\mathsf{q}} \end{matrix} \right]\right)(s)$$
$$= (-\rho)^{\lambda} s^{\rho\lambda} \sum_{k=0}^{[\mathsf{n/m}]} \frac{(-\mathsf{n})_{\mathsf{m}k}}{k!} A_{\mathsf{n},k} (as^{\alpha})^k \gamma^{\mathsf{m,n+1}}_{\mathsf{p+1,q+1}} \left[bs^{\beta} \middle| \begin{matrix} (\mathsf{e}_1, \mathsf{E}_1, \mathsf{y}), (\mathsf{e}_j, \mathsf{E}_j)_{2,\mathsf{p}}, (-\frac{\alpha k}{\rho}, \frac{\beta}{\rho}) \\ (\mathsf{f}_j, \mathsf{F}_j)_{1,\mathsf{q}}, (\lambda - \frac{\alpha k}{\rho}, \frac{\beta}{\rho}) \end{matrix} \right]. \tag{3.4}$$

4 Particular cases and remarks

Due to the generality of both incomplete H-functions and the general class polynomials, the main identities established in the preceding sections may result in a variety of simpler formulae as special instances. For example, the case $\mathsf{y} = 0$ of (1.8) reduces to the Fox's H-function (see, e.g., [20, p. 10]; see also [11], [16]):

$$\Gamma^{\mathsf{m},\mathsf{n}}_{\mathsf{p},\mathsf{q}}\left[z \,\middle|\, \begin{matrix} (\mathsf{e}_1,\mathsf{E}_1),(\mathsf{e}_i,\mathsf{E}_i)_{2,\mathsf{p}} \\ \\ (\mathsf{f}_i,\mathsf{F}_i)_{1,\mathsf{q}} \end{matrix} \right] = H^{\mathsf{m},\mathsf{n}}_{\mathsf{p},\mathsf{q}}\left[z \,\middle|\, \begin{matrix} (\mathsf{e}_1,\mathsf{E}_1),(\mathsf{e}_i,\mathsf{E}_i)_{2,\mathsf{p}} \\ \\ (\mathsf{f}_i,\mathsf{F}_i)_{1,\mathsf{q}} \end{matrix} \right]$$
$$= H^{\mathsf{m},\mathsf{n}}_{\mathsf{p},\mathsf{q}}\left[z \,\middle|\, \begin{matrix} (\mathsf{e}_1,\mathsf{E}_1),(\mathsf{e}_2,\mathsf{E}_2),\cdots,(\mathsf{e}_\mathsf{p},\mathsf{E}_\mathsf{p}) \\ \\ (\mathsf{f}_1,\mathsf{F}_1),(\mathsf{f}_2,\mathsf{F}_2),\cdots,(\mathsf{f}_\mathsf{q},\mathsf{F}_\mathsf{q}) \end{matrix} \right]. \tag{4.1}$$

For another example, putting $\mathsf{m} = 1$, $\mathsf{n} = \mathsf{p}$, q being replaced by $\mathsf{q}+1$ and taking appropriate parameters, the functions (1.6) and (1.8) reduce, respectively, to the incomplete Fox-Wright Ψ-functions ${}_\mathsf{p}\Psi^{(\gamma)}_\mathsf{q}$ and ${}_\mathsf{p}\Psi^{(\Gamma)}_\mathsf{q}$ (see [21, Eqs. (6.3) and (6.4)]; see also [2, Eqs. (14) and (15)]):

$$\gamma^{1,\mathsf{p}}_{\mathsf{p},\mathsf{q}+1}\left[-z \,\middle|\, \begin{matrix} (1-\mathsf{e}_1,\mathsf{E}_1,\mathsf{y}),(1-\mathsf{e}_j,\mathsf{E}_j)_{2,\mathsf{p}} \\ \\ (0,1),(1-\mathsf{b}_j,\mathsf{B}_j)_{1,\mathsf{q}} \end{matrix} \right] = {}_\mathsf{p}\Psi^{(\gamma)}_\mathsf{q}\left[\begin{matrix} (\mathsf{e}_1,\mathsf{E}_1,\mathsf{y}),(\mathsf{e}_j,\mathsf{E}_j)_{2,\mathsf{p}}; \\ \\ (\mathsf{b}_j,\mathsf{B}_j)_{1,\mathsf{q}}; \end{matrix} \; z \right] \tag{4.2}$$

and

$$\Gamma^{1,\mathsf{p}}_{\mathsf{p},\mathsf{q}+1}\left[-z \,\middle|\, \begin{matrix} (1-\mathsf{e}_1,\mathsf{E}_1,\mathsf{y}),(1-\mathsf{e}_j,\mathsf{E}_j)_{2,\mathsf{p}} \\ \\ (0,1),(1-\mathsf{b}_j,\mathsf{B}_j)_{1,\mathsf{q}} \end{matrix} \right] = {}_\mathsf{p}\Psi^{(\Gamma)}_\mathsf{q}\left[\begin{matrix} (\mathsf{e}_1,\mathsf{E}_1,\mathsf{y}),(\mathsf{e}_j,\mathsf{E}_j)_{2,\mathsf{p}}; \\ \\ (\mathsf{b}_j,\mathsf{B}_j)_{1,\mathsf{q}}; \end{matrix} \; z \right]. \tag{4.3}$$

The following corollaries cover some of them.

Corollary 4.1 *Let $\Re(\lambda) > 0$, a, $b \in \mathbb{R}$, and ρ, α, $\beta \in \mathbb{R}^+$. Then*

$$\left({}^\rho I^\lambda_{0+} S^\mathsf{m}_\mathsf{n}[at^\alpha] H^{\mathsf{m},\mathsf{n}}_{\mathsf{p},\mathsf{q}}\left[bt^\beta \,\middle|\, \begin{matrix} (\mathsf{e}_j,\mathsf{E}_j)_{1,\mathsf{p}} \\ \\ (\mathsf{f}_j,\mathsf{F}_j)_{1,\mathsf{q}} \end{matrix} \right] \right)(s) = \rho^{-\lambda} s^{\rho\lambda} \sum_{k=0}^{[\mathsf{n}/\mathsf{m}]} \frac{(-\mathsf{n})_{\mathsf{m}k}}{k!} A_{\mathsf{n},k} (as^\alpha)^k$$
$$\times H^{\mathsf{m},\mathsf{n}+1}_{\mathsf{p}+1,\mathsf{q}+1}\left[bs^\beta \,\middle|\, \begin{matrix} (\mathsf{e}_j,\mathsf{E}_j)_{1,\mathsf{p}}(-\frac{\alpha k}{\rho},\frac{\beta}{\rho}) \\ \\ (\mathsf{f}_j,\mathsf{F}_j)_{1,\mathsf{q}},(-\lambda-\frac{\alpha k}{\rho},\frac{\beta}{\rho}) \end{matrix} \right] \tag{4.4}$$

$$(s > 0)$$

and

$$\left({}^\rho I^\lambda_{0-} S^\mathsf{m}_\mathsf{n}[at^\alpha] H^{\mathsf{m},\mathsf{n}}_{\mathsf{p},\mathsf{q}}\left[bt^\beta \,\middle|\, \begin{matrix} (\mathsf{e}_j,\mathsf{E}_j)_{1,\mathsf{p}} \\ \\ (\mathsf{f}_j,\mathsf{F}_j)_{1,\mathsf{q}} \end{matrix} \right] \right)(s) = (-\rho)^{-\lambda} s^{\rho\lambda} \sum_{k=0}^{[\mathsf{n}/\mathsf{m}]} \frac{(-\mathsf{n})_{\mathsf{m}k}}{k!} A_{\mathsf{n},k} (as^\alpha)^k$$
$$\times H^{\mathsf{m},\mathsf{n}+1}_{\mathsf{p}+1,\mathsf{q}+1}\left[bs^\beta \,\middle|\, \begin{matrix} (\mathsf{e}_j,\mathsf{E}_j)_{1,\mathsf{p}}(\frac{-\alpha k}{\rho},\frac{\beta}{\rho}) \\ \\ (\mathsf{f}_j,\mathsf{F}_j)_{1,\mathsf{q}},(-\lambda-\frac{\alpha k}{\rho},\frac{\beta}{\rho}) \end{matrix} \right] \tag{4.5}$$

$$(s<0).$$

Proof. Taking $\mathsf{y}=0$ in (2.1) and (2.4), we get the required results. $\square$

Corollary 4.2 *Let $\Re(\lambda)>0$, a, $b\in\mathbb{R}$, and ρ, α, $\beta\in\mathbb{R}^+$. Then*

$$\left({}^{\rho}\mathcal{D}^{\lambda}_{0+}S^{\mathsf{m}}_{\mathsf{n}}[at^{\alpha}]H^{\mathsf{m,n}}_{\mathsf{p,q}}\left[bt^{\beta}\middle|\begin{array}{c}(\mathsf{e}_j,\mathsf{E}_j)_{1,\mathsf{p}}\\ \\(\mathsf{f}_j,\mathsf{F}_j)_{1,\mathsf{q}}\end{array}\right]\right)(s)=\rho^{\lambda}s^{\rho\lambda}\sum_{k=0}^{[\frac{\mathsf{n}}{\mathsf{m}}]}\frac{(-\mathsf{n})_{\mathsf{m}k}}{k!}A_{\mathsf{n},k}(as^{\alpha})^{k}$$
$$\times H^{\mathsf{m,n+1}}_{\mathsf{p+1,q+1}}\left[bs^{\beta}\middle|\begin{array}{c}(\mathsf{e}_j,\mathsf{E}_j)_{1,\mathsf{p}},(-\frac{\alpha k}{\rho},\frac{\beta}{\rho})\\ \\(\mathsf{f}_j,\mathsf{F}_j)_{1,\mathsf{q}},(\lambda-\frac{\alpha k}{\rho},\frac{\beta}{\rho})\end{array}\right] \tag{4.6}$$

$$(s>0)$$

and

$$\left({}^{\rho}\mathcal{D}^{\lambda}_{0-}S^{\mathsf{m}}_{\mathsf{n}}[at^{\alpha}]H^{\mathsf{m,n}}_{\mathsf{p,q}}\left[bt^{\beta}\middle|\begin{array}{c}(\mathsf{e}_j,\mathsf{E}_j)_{1,\mathsf{p}}\\ \\(\mathsf{f}_j,\mathsf{F}_j)_{1,\mathsf{q}}\end{array}\right]\right)(s)=\rho^{\lambda}s^{\rho\lambda}\sum_{k=0}^{[\mathsf{n}/\mathsf{m}]}\frac{(-\mathsf{n})_{\mathsf{m}k}}{k!}A_{\mathsf{n},k}(as^{\alpha})^{k}$$
$$\times H^{\mathsf{m,n+1}}_{\mathsf{p+1,q+1}}\left[bs^{\beta}\middle|\begin{array}{c}(\mathsf{e}_j,\mathsf{E}_j)_{1,\mathsf{p}},(-\frac{\alpha k}{\rho},\frac{\beta}{\rho})\\ \\(\mathsf{f}_j,\mathsf{F}_j)_{1,\mathsf{q}},(\lambda-\frac{\alpha k}{\rho},\frac{\beta}{\rho})\end{array}\right] \tag{4.7}$$

$$(s<0).$$

Proof. Taking $\mathsf{y}=0$ in (3.1) and (3.3), we get the required results. $\square$

Corollary 4.3 *Let $\Re(\lambda)>0$, a, $b\in\mathbb{R}$, ρ, α, $\beta\in\mathbb{R}^+$, $\mathsf{y}\geq 0$, and $s>0$. Then*

$$\left({}^{\rho}I^{\lambda}_{0+}S^{\mathsf{m}}_{\mathsf{n}}[at^{\alpha}]_{\mathsf{p}}\Psi^{(\Gamma)}_{\mathsf{q}}\left[bt^{\beta}\middle|\begin{array}{r}(\mathsf{e}_1,\mathsf{E}_1,\mathsf{y}),(\mathsf{e}_j,\mathsf{E}_j)_{2,\mathsf{p}}\\ \\(\mathsf{f}_j,\mathsf{F}_j)_{1,\mathsf{q}}\end{array}\right]\right)(s)=\rho^{-\lambda}s^{\rho\lambda}\sum_{k=0}^{[\mathsf{n}/\mathsf{m}]}\frac{(-\mathsf{n})_{\mathsf{m}k}}{k!}A_{\mathsf{n},k}(as^{\alpha})^{k}$$
$$\times\,{}_{\mathsf{p}+1}\Psi^{(\Gamma)}_{\mathsf{q}+1}\left[bs^{\beta}\middle|\begin{array}{r}(\mathsf{e}_1,\mathsf{E}_1,\mathsf{y}),(\mathsf{e}_j,\mathsf{E}_j)_{2,\mathsf{p}}(1+\frac{\alpha k}{\rho},\frac{\beta}{\rho})\\ \\(\mathsf{f}_j,\mathsf{F}_j)_{1,\mathsf{q}},(1+\lambda+\frac{\alpha k}{\rho},\frac{\beta}{\rho})\end{array}\right] \tag{4.8}$$

and

$$\left({}^{\rho}I^{\lambda}_{0+}S^{\mathsf{m}}_{\mathsf{n}}[at^{\alpha}]_{\mathsf{p}}\Psi^{(\gamma)}_{\mathsf{q}}\left[bt^{\beta}\middle|\begin{array}{r}(\mathsf{e}_1,\mathsf{E}_1,\mathsf{y}),(\mathsf{e}_j,\mathsf{E}_j)_{2,\mathsf{p}}\\ \\(\mathsf{f}_j,\mathsf{F}_j)_{1,\mathsf{q}}\end{array}\right]\right)(s)=\rho^{-\lambda}s^{\rho\lambda}\sum_{k=0}^{[\mathsf{n}/\mathsf{m}]}\frac{(-\mathsf{n})_{\mathsf{m}k}}{k!}A_{\mathsf{n},k}(as^{\alpha})^{k}$$
$$\times\,{}_{\mathsf{p}+1}\Psi^{(\gamma)}_{\mathsf{q}+1}\left[bs^{\beta}\middle|\begin{array}{r}(\mathsf{e}_1,\mathsf{E}_1,\mathsf{y}),(\mathsf{e}_j,\mathsf{E}_j)_{2,\mathsf{p}}(1+\frac{\alpha k}{\rho},\frac{\beta}{\rho})\\ \\(\mathsf{f}_j,\mathsf{F}_j)_{1,\mathsf{q}},(1+\lambda+\frac{\alpha k}{\rho},\frac{\beta}{\rho})\end{array}\right]. \tag{4.9}$$

Proof. Using (4.2) and (4.3) in (2.1) and (2.2) gives the required identities. □

Corollary 4.4 *Let* $\Re(\lambda) > 0$*,* a*,* $b \in \mathbb{R}$*,* ρ*,* α*,* $\beta \in \mathbb{R}^+$*,* $\mathsf{y} \geq 0$*, and* $s < 0$*. Then*

$$\left({}^{\rho}I^{\lambda}_{0-} S^{\mathsf{m}}_{\mathsf{n}}[at^{\alpha}]_{\mathsf{p}} \Psi^{(\Gamma)}_{\mathsf{q}} \left[bt^{\beta} \middle| \begin{matrix} (\mathsf{e}_1, \mathsf{E}_1, \mathsf{y}), (\mathsf{e}_j, \mathsf{E}_j)_{2,\mathsf{p}} \\ (\mathsf{f}_j, \mathsf{F}_j)_{1,\mathsf{q}} \end{matrix}\right]\right)(s)$$
$$= (-\rho)^{-\lambda} s^{\rho\lambda} \sum_{k=0}^{[\mathsf{n}/\mathsf{m}]} \frac{(-\mathsf{n})_{\mathsf{m}k}}{k!} A_{\mathsf{n},k} (as^{\alpha})^k {}_{\mathsf{p}+1}\Psi^{(\Gamma)}_{\mathsf{q}+1} \left[bs^{\beta} \middle| \begin{matrix} (\mathsf{e}_1, \mathsf{E}_1, \mathsf{y}), (\mathsf{e}_j, \mathsf{E}_j)_{2,\mathsf{p}} (1 + \frac{\alpha k}{\rho}, \frac{\beta}{\rho}) \\ (\mathsf{f}_j, \mathsf{F}_j)_{1,\mathsf{q}}, (1 + \lambda + \frac{\alpha k}{\rho}, \frac{\beta}{\rho}) \end{matrix}\right] \tag{4.10}$$

and

$$\left({}^{\rho}I^{\lambda}_{0-} S^{\mathsf{m}}_{\mathsf{n}}[at^{\alpha}]_{\mathsf{p}} \Psi^{(\gamma)}_{\mathsf{q}} \left[bt^{\beta} \middle| \begin{matrix} (\mathsf{e}_1, \mathsf{E}_1, \mathsf{y}), (\mathsf{e}_j, \mathsf{E}_j)_{2,\mathsf{p}} \\ (\mathsf{f}_j, \mathsf{F}_j)_{1,\mathsf{q}} \end{matrix}\right]\right)(s)$$
$$= (-\rho)^{-\lambda} s^{\rho\lambda} \sum_{k=0}^{[\frac{\mathsf{n}}{\mathsf{m}}]} \frac{(-\mathsf{n})_{\mathsf{m}k}}{k!} A_{\mathsf{n},k} (as^{\alpha})^k {}_{\mathsf{p}+1}\Psi^{(\gamma)}_{\mathsf{q}+1} \left[bs^{\beta} \middle| \begin{matrix} (\mathsf{e}_1, \mathsf{E}_1, \mathsf{y}), (\mathsf{e}_j, \mathsf{E}_j)_{2,\mathsf{p}} (1 + \frac{\alpha k}{\rho}, \frac{\beta}{\rho}) \\ (\mathsf{f}_j, \mathsf{F}_j)_{1,\mathsf{q}}, (1 + \lambda + \frac{\alpha k}{\rho}, \frac{\beta}{\rho}) \end{matrix}\right] \tag{4.11}$$

Proof. Employing (4.2) and (4.3) in (2.4) and (2.5) provides the desired identities. □

Corollary 4.5 *Let* $\Re(\lambda) > 0$*,* a*,* $b \in \mathbb{R}$*,* ρ*,* α*,* $\beta \in \mathbb{R}^+$*,* $\mathsf{y} \geq 0$*, and* $s > 0$*. Then*

$$\left({}^{\rho}\mathcal{D}^{\lambda}_{0+} S^{\mathsf{m}}_{\mathsf{n}}[at^{\alpha}]_{\mathsf{p}} \Psi^{(\Gamma)}_{\mathsf{q}} \left[bt^{\beta} \middle| \begin{matrix} (\mathsf{e}_1, \mathsf{E}_1, \mathsf{y}), (\mathsf{e}_j, \mathsf{E}_j)_{2,\mathsf{p}} \\ (\mathsf{f}_j, \mathsf{F}_j)_{1,\mathsf{q}} \end{matrix}\right]\right)(s)$$
$$= \rho^{\lambda} s^{\rho\lambda} \sum_{k=0}^{[\mathsf{n}/\mathsf{m}]} \frac{(-\mathsf{n})_{\mathsf{m}k}}{k!} A_{\mathsf{n},k} (as^{\alpha})^k {}_{\mathsf{p}+1}\Psi^{(\Gamma)}_{\mathsf{q}+1} \left[bs^{\beta} \middle| \begin{matrix} (\mathsf{e}_1, \mathsf{E}_1, \mathsf{y}), (\mathsf{e}_j, \mathsf{E}_j)_{2,\mathsf{p}}, (1 + \frac{\alpha k}{\rho}, \frac{\beta}{\rho}) \\ (\mathsf{f}_j, \mathsf{F}_j)_{1,\mathsf{q}}, (1 - \lambda + \frac{\alpha k}{\rho}, \frac{\beta}{\rho}) \end{matrix}\right] \tag{4.12}$$

and

$$\left({}^{\rho}\mathcal{D}^{\lambda}_{0+} S^{\mathsf{m}}_{\mathsf{n}}[at^{\alpha}]_{\mathsf{p}} \Psi^{(\gamma)}_{\mathsf{q}} \left[bt^{\beta} \middle| \begin{matrix} (\mathsf{e}_1, \mathsf{E}_1, \mathsf{y}), (\mathsf{e}_j, \mathsf{E}_j)_{2,\mathsf{p}} \\ (\mathsf{f}_j, \mathsf{F}_j)_{1,\mathsf{q}} \end{matrix}\right]\right)(s)$$
$$= \rho^{\lambda} s^{\rho\lambda} \sum_{k=0}^{[\frac{\mathsf{n}}{\mathsf{m}}]} \frac{(-\mathsf{n})_{\mathsf{m}k}}{k!} A_{\mathsf{n},k} (as^{\alpha})^k {}_{\mathsf{p}+1}\Psi^{(\Gamma)}_{\mathsf{q}+1} \left[bs^{\beta} \middle| \begin{matrix} (\mathsf{e}_1, \mathsf{E}_1, \mathsf{y}), (\mathsf{e}_j, \mathsf{E}_j)_{2,\mathsf{p}}, (1 + \frac{\alpha k}{\rho}, \frac{\beta}{\rho}) \\ (\mathsf{f}_j, \mathsf{F}_j)_{1,\mathsf{q}}, (1 - \lambda + \frac{\alpha k}{\rho}, \frac{\beta}{\rho}) \end{matrix}\right]. \tag{4.13}$$

Proof. Applying (4.2) and (4.3) to (3.1) and (3.2) offers the desired results. □

Corollary 4.6 *Let $\Re(\lambda) > 0$, $\delta \in \mathbb{C} \setminus \mathbb{Z}_{\leq -1}$, $a, b \in \mathbb{R}$, $\rho, \alpha, \beta \in \mathbb{R}^+$, $\mathsf{y} \geq 0$, and $s > 0$. Then*

$$\left({}^{\rho}I_{0+}^{\lambda} L_{\mathsf{n}}^{(\delta)}(at^{\alpha}) \Gamma_{\mathsf{p},\mathsf{q}}^{1,\mathsf{n}} \left[bt^{\beta} \middle| \begin{matrix} (\mathsf{e}_1, \mathsf{E}_1, \mathsf{y}), (\mathsf{e}_j, \mathsf{E}_j)_{2,\mathsf{p}} \\ (\mathsf{f}_j, \mathsf{F}_j)_{1,\mathsf{q}} \end{matrix} \right] \right)(s) = \rho^{-\lambda} s^{\rho\lambda} \sum_{k=0}^{[\mathsf{n}]} \frac{(-\mathsf{n})_k}{k!\,\mathsf{n}!} \frac{(1+\delta)_{\mathsf{n}}}{(1+\delta)_k} (as^{\alpha})^k \times \Gamma_{\mathsf{p}+1,\mathsf{q}+1}^{1,\mathsf{n}+1} \left[bs^{\beta} \middle| \begin{matrix} (\mathsf{e}_1, \mathsf{E}_1, \mathsf{y}), (\mathsf{e}_j, \mathsf{E}_j)_{2,\mathsf{p}} (-\frac{\alpha k}{\rho}, \frac{\beta}{\rho}) \\ (\mathsf{f}_j, \mathsf{F}_j)_{1,\mathsf{q}}, (-\lambda - \frac{\alpha k}{\rho}, \frac{\beta}{\rho}) \end{matrix} \right] \tag{4.14}$$

and

$$\left({}^{\rho}I_{0+}^{\lambda} L_{\mathsf{n}}^{(\delta)}(at^{\alpha}) \gamma_{\mathsf{p},\mathsf{q}}^{1,\mathsf{n}} \left[bt^{\beta} \middle| \begin{matrix} (\mathsf{e}_1, \mathsf{E}_1, \mathsf{y}), (\mathsf{e}_j, \mathsf{E}_j)_{2,\mathsf{q}} \\ (\mathsf{f}_j, \mathsf{F}_j)_{1,w} \end{matrix} \right] \right)(s) = \rho^{-\lambda} s^{\rho\lambda} \sum_{k=0}^{[\mathsf{n}]} \frac{(-\mathsf{n})_k}{k!\,\mathsf{n}!} \frac{(1+\delta)_{\mathsf{n}}}{(1+\delta)_k} (as^{\alpha})^k \times \gamma_{\mathsf{p}+1,\mathsf{q}+1}^{1,\mathsf{n}+1} \left[bs^{\beta} \middle| \begin{matrix} (\mathsf{e}_1, \mathsf{E}_1, \mathsf{y}), (\mathsf{e}_j, \mathsf{E}_j)_{2,\mathsf{p}} (-\frac{\alpha k}{\rho}, \frac{\beta}{\rho}) \\ (\mathsf{f}_j, \mathsf{F}_j)_{1,\mathsf{q}}, (-\lambda - \frac{\alpha k}{\rho}, \frac{\beta}{\rho}) \end{matrix} \right], \tag{4.15}$$

where $L_{\mathsf{n}}^{(\delta)}(x)$ are Laguerre polynomials.

Proof. Setting $\mathsf{m} = 1$ and choosing $A_{\mathsf{n},k} = (1+\delta)_{\mathsf{n}} / \{(1+\delta)_k\, \mathsf{n}!\}$ in the results in Theorem 2.1, with the aid of Laguerre polynomials $L_{\mathsf{n}}^{(\delta)}(x)$ (see, e.g., [15, p. 201, Eq. (3)]), we obtain the desired identities here. □

Likewise, as with Corollary 4.6, substituting $\mathsf{m} = 1$ and selecting $A_{\mathsf{n},k} = (1+\delta)_{\mathsf{n}} / \{(1+\delta)_k\, \mathsf{n}!\}$ in the identities in Theorems 2.2–3.2 and Corollaries 4.1–4.5 results in the corresponding formulae involving the Laguerre polynomials.

References

[1] P. Agarwal and J. Choi, Fractional calculus operators and their image formulas, *J. Korean Math. Soc.* **53**(5) (2016), 1183-1210 `http://dx.doi.org/10.4134/JKMS.j150458`

[2] M. K. Bansal and J. Choi, A note on pathway fractional integral formulas associated with the incomplete H-functions, *Int. J. Appl. Comput. Math.* **5**(5) (2019), Article ID 133. `https://doi.org/10.1007/s40819-019-0718-8`

[3] M. K. Bansal, D. Kumar and R. Jain, A study of Marichev-Saigo-Maeda fractional integral operators associated with S-generalized Gauss hypergeometric function, *KYUNGPOOK Math. J.* **59**(3) (2019), 433–443. `https://doi.org/10.5666/KMJ.2019.59.3.433`

[4] M. K. Bansal, D. Kumar, K. S. Nisar and J. Singh, Certain fractional calculus and integral transform results of incomplete ℵ-functions with applications, *Math. Meth. Appl. Sci.* **43**(8)(2020), 5602–5614. https://doi.org/10.1002/mma.6299

[5] M. K. Bansal, D. Kumar, J. Singh and K. S. Nisar, On the solutions of a class of integral equations pertaining to incomplete H-function and incomplete $\overline{H}$-function, *Mathematics* **8**(5) (2020), Article ID 819. https://doi.org/10.3390/math8050819

[6] J. Hadamard, Essai sur ľetude des fonctions donnees par leur developpment de Taylor, *J. Pure Appl. Math.* **4**(8) (1892), 101–186.

[7] U. N. Katugampola, New approach to a generalized fractional integral, *Appl. Math. Comput.* **218**(3) (2011), 860–865. https://doi.org/10.1016/j.amc.2011.03.062

[8] U. N. Katugampola, New approach to generalized fractional derivatives, *Bull. Math. Anal. Appl.* **6**(4) (2014), 1–15.

[9] O. Khan, N. Khan, K. S. Nisar, M. Saif and D. Baleanu, Fractional calculus of a product of multivariable Srivastava polynomial and multi-index Bessel function in the kernel F_3, *AIMS Math.* **5**(2) (2020), 1462–1475. DOI:10.3934/math.2020100

[10] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematical Studies, Vol. **204**, Elsevier (North-Holland) Science Publishers, Amsterdam, London and New York, 2006.

[11] D. Kumar and J. Singh, Application of generalized M-series and $\overline{H}$-function in electric circuit theory, *MESA* **7**(3) (2016), 503–512.

[12] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons, INC, New York, 1993.

[13] K. Oldham and J. Spanier, *Fractional Calculus: Theory and Applications of Differentiation and Integration of Arbitrary Order*, Academic Press, New York, 1974.

[14] I. Podlubny, *Fractional Differential Equations*, Academic Press, California, USA, 1999.

[15] E. D. Rainville, *Special Functions*, Macmillan Company, New York, 1960; Reprinted by Chelsea Publishing Company, Bronx, New York, 1971.

[16] J. Singh and D. Kumar, On the distribution of mixed sum of independent random variables one of them associated with Srivastava's polynomials and $\overline{H}$-function, *J. Appl. Math. Stat. Inform.* **10**(1) (2014), 53–62. https://doi.org/10.2478/jamsi-2014-0005

[17] H. M. Srivastava, A contour integral involving Fox's H-function, *Indian J. Math.* **14** (1972), l–6.

[18] H. M. Srivastava, M. K. Bansal and P. Harjule, A study of fractional integral operators involving a certain generalized multi-index Mittag-Leffler function, *Math. Meth. Appl. Sci.* **41**(16) (2018), 6108–6121. https://doi.org/10.1002/mma.5122

[19] H. M. Srivastava and J. Choi, *Zeta and q-Zeta Functions and Associated Series and Integrals*, Elsevier Science Publishers, Amsterdam, London and New York, 2012.

[20] H. M. Srivastava, K. C. Gupta and S. P. Goyal, *The H-Functions of One and Two Variables with Applications*, South Asian Publishers, New Delhi and Madras, 1982.

[21] H. M. Srivastava, R. K. Saxena and R. K. Parmar, Some families of the incomplete H-functions and the incomplete $\bar{H}$-functions and associated integral transforms and operators of fractional calculus with applications, *Russian J. Math. Phys.* **25**(1) (2018), 116–138. DOI10.1134/S1061920818010119

[22] H. M. Srivastava and Ž. Tomovski, Fractional calculus with an integral operator containing a generalized Mittag-Leffler function in the kernel, *Appl. Math. Comput.* **211**(1) (2009), 198–210. doi:10.1016/j.amc.2009.01.055

[23] K. Jangid, S.D. Purohit, K.S. Nisar and S. Araci, Generating functions involving the incomplete H-functions, *Analysis* **41(4)** (2021), 239–244. https://doi.org/10.1515/anly-2021-0038

[24] A. Bhargava, R.K. Jain and J. Singh, Certain New Results Involving Multivariable Aleph(ℵ) -Function, Srivastava Polynomials, Hypergeometric Functions and $\bar{H}$Function, *Int. J. Appl. Comput. Math* **7** (2021), 196. doi.org/10.1007/s40819-021-01071-w

[25] D.L. Suthar, A.M. Khan, A. Alaria, S.D. Purohit and J. Singh, Extended Bessel-Maitland function and its properties pertaining to integral transforms and fractional calculus, *AIMS Mathematics*, **5(2)** (2020), 1400–1410. doi:10.3934/math.2020096