A vacation queueing model with first exceptional service, encouraged arrivals and server breakdowns

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ABSTRACT

This paper examines a single server queueing model with multiple vacations, encouraged customer arrivals and operational breakdowns. Service resumes after repairs with handling first customer of every busy period exceptionally. Additionally vacation is taken by server after each busy period. It takes another vacation if he finds no customer in the queue. Customers arrive according to poisson process with different rates during vacation, active service and repair process. Active arrival occurs in accordance with percentage change due to encouraged customers. For both type of customers (first and followed) uninterrupted service times, vacation times and repair times follow exponential distribution. Using probability generating function we found mean queue length and derived existence condition for steady state queue length. Numerical examples are also presented.

Keywords: Encouraged arrival, First exceptional service, Multiple server vacations, Service breakdown & repair.

1. INTRODUCTION

Encouragement of arrivals is one of the essential parts of business promotion to create brand awareness and brand loyalty. Offering coupons, free samples, rebates, free gifts, free trials, sell on customized product etc. are the ongoing trends to influence people for the product and services.

We have analyzed here a queueing model with server vacations and encouraged arrival where first customer of every busy period receives extra ordinary service. Queueing system also experiences breakdowns and repairs. In 1964, Welch introduced the concept of providing exceptional service to the first customer in an M/G/1 queueing process. A vacation queueing system has been first studied in 1970's. Levy & Yechiali (1976) analyzed M/M/s queueing system in which a server goes on vacation of an exponentially distribution duration if it finds no waiting unit in line. Li.et.al.(1996) suggested a special featured queue model where the server has adaptive learning capability and its performance gradually optimizes over a period of time. Yutaka Baba (1999) provided M/G/1 queue model which characterizes exceptional service to first N customers after every busy period.

Gray et.al. (2000) examined queueing models with multiple vacations, server failures and repairs and gave decomposition theorem for the mean queue length distribution. Ghimire et.al. (2011) proposed a M/M/1 vacation and breakdown queueing system with arrival heterogeneous type and service according to the queue length. Ayyappan et.al. (2013) worked on a multiple vacations $M^{[X]}/G/1$ queue model with two types services and restricted admissibility. Jain et.al.(2014) applied reverse balking concept on M/M/1/N queueing models. Khurana & Sharma (2017) used the generating function approach to derive performance measures for a vacation queue providing the first service of a busy period exceptionally. Kumar Som et.al (2018) obtained stationary probabilities of a Markovian queuing model characterized with encouraged arrivals and multi-server feedback. Khurana & Sharma (2019) extended the research for M/M/ 1 vacation queueing model providing exceptional services to first N customers where server is subject to breakdown & repairs. Suvitha et. al. (2021) considered two heterogeneous server vacation queue with balking and obtained steady state results. Medhi (2021) derived steady state solution for a discouraged arrival Markovian queueing system characterized with reneging, balking and customer's

retention. Malik et. al. (2022) used recursive technique to find steady state solution of a multiple vacation queueing system with encouraged arrival.
In our paper we proposed the M/M/1 queue model with encouraged arrivals, multiple vacations, and exceptional service in each busy period, server breakdowns and repairs. This type of model has a wide range of application in sales & marketing, production & manufacturing units, service sectors etc.

2. Model description

We consider M/M/1 single server queueing model with following assumptions:

(a) Server can take multiple vacations when it finds empty system. After ending up the vacation period if finds customer in the queue it activates otherwise goes on other vacation
(b) Each first customer of busy period receives exceptional service.
(c) Customers arrive with three different rates during vacation, active service and breakdown according to poisson process. Percentage change due to encouraged customer is incorporated in the active arrival rate with parameter $'\eta'$.
(d) Services are subject to breakdown and server goes for the repair process. It resumes service after repair where service was interrupted.
(e) Uninterrupted service time, vacation time and repair time are independent of each other and follow exponential distribution.

(i) Notations & steady state probabilities

λ : Arrival rate at the time of active service
λ_0: Arrival rate at the time of vacations
λ_1 : Arrival rate at the time of breakdowns
μ_1: First customer service rate for every busy period
μ_2 : Second or following customer service rate for every busy period.
b: Server breakdown rate
ν : Server vacation rate
r: Server repair rate
ξ: Percentage change in number of customers determined by using historical or observed data.

$$\rho_0 = \frac{\lambda_0}{\lambda_0 + \alpha + \nu}$$

$\lambda_0, \lambda > 0, \quad \lambda_1 \geq 0$
$\mu_1, \mu_2, b, , \nu, r, \xi > 0$

The steady-state probabilities of the suggested model are as follows:

$P_{0,i}$: The probability of the state that the server has gone on vacation and the number of customers in the system is i. $i \geq 0$.
$P_{1,i,1}$: The probability of the state that service is in active mode, the number of customers in the system is i and the first customer of a busy period is being served. $i \geq 1$.
$P_{1,i,2}$: The probability of the state that service is in active mode, the number of customers in the system is i and the customer being served is the second or the following customer of a busy period. $i \geq 1$.
$P_{2,i,1}$: The probability of the state that service is in active mode, the number of customers in the system is i and the first customer of a busy period is being served. $i \geq 1$.
$P_{2,i,2}$: The probability of the state that service is in active mode, the number of customers in the system is i and the customer being served is the second or the following customer of a busy period. $i \geq 1$.

(ii) Normalization condition

Observing the states of the system, the normalization condition is

$$P_{0,0} + \sum_{i=1}^{\infty} (P_{0,i} + P_{1,i,1} + P_{1,i,2} + P_{2,i,1} + P_{2,i,2}) = 1$$

(iii) Generating Function for the model

In order to apply generating function technique we define in this section following partial generating functions:

$$F_0(z) = \sum_{i=0}^{\infty} P_{0,i}\, z^i, F_{1,1}(z) = \sum_{i=1}^{\infty} P_{1,i,1} z^i\, , F_{1,2}(z) = \sum_{i=1}^{\infty} P_{1,i,2} z^i\, , F_{2,1}(z) = \sum_{i=1}^{\infty} P_{2,i,1} z^i,$$

$F_{2,2}(z) = \sum_{i=1}^{\infty} P_{2,i,2} z^i$,

Defining the generating function to find queue length distribution:

$$F(z) = F_0(z) + \sum_{j=1,2} \{F_{1,j}(z) + F_{2,j}(z)\}$$

3. Analysis of Mean Queue Length

Following are the steady state equations of the proposed model for the subsequent analysis:

$\lambda_0 P_{0,0} = \mu_1 P_{1,1,1} + \mu_2 P_{1,1,2}$ (1)

$(\lambda_0 + \nu) P_{0,i} = \lambda_0 P_{0,i-1}, \ i \geq 1$ (2)

$\{\lambda(1+\xi) + \mu_1 + b\} P_{1,1,1} = \nu P_{0,1} + r P_{2,1,1},$ (3)

$\{\lambda(1+\xi) + \mu_2 + b\} P_{1,1,2} = \mu_1 P_{1,2,1} + \mu_2 P_{1,2,2} + r P_{2,1,2},$ (4)

$\{\lambda(1+\xi) + \mu_1 + b\} P_{1,i,1} = \lambda(1+\xi) P_{1,i-1,1} + \nu P_{0,i} + r P_{2,i,1}, \ i \geq 2$ (5)

$\{\lambda(1+\xi) + \mu_2 + b\} P_{1,i,2} = \lambda(1+\xi) P_{1,i-1,2} + \mu_1 P_{1,i+1,1} + \mu_2 P_{1,i+1,2} + r P_{2,i,2}, \ i \geq 2$ (6)

$(\lambda_1 + r) P_{2,1,1} = b P_{1,1,1}$ (7)

$(\lambda_1 + r) P_{2,1,2} = b P_{1,1,2}$ (8)

$(\lambda_1 + r) P_{2,i,1} = \lambda_1 P_{2,i-1,1} + b P_{1,i,1} \quad i \geq 2$ (9)

$(\lambda_1 + r) P_{2,i,2} = \lambda_1 P_{2,i-1,2} + b P_{1,i,2} \quad i \geq 2$ (10)

(i) The Generating Function

Now, utilizing the generating function method, we obtain from equation (2)

$F_0(z) = \sum_{i=0}^{\infty} P_{0,i} z^i = \frac{P_{0,0}}{1-\rho_0 z}$ (11)

We obtain from equations (3) and (5)

$\left(\lambda(1+\xi) + \mu_1 + b - \lambda(1+\xi)z\right) F_{1,1}(z) = \nu[F_0(z) - P_{0,0}] + r F_{2,1}(z)$ (12)

Equation (4) and equation (6) give us

$\left(\lambda(1+\xi) + \mu_2 + b - \lambda(1+\xi)z\right) F_{1,2}(z) = \frac{\mu_1}{z} F_{1,1}(z) + \frac{\mu_2}{z} F_{1,2}(z) + r F_{2,2}(z) - \lambda_0 P_{0,0}$ (13)

From equation (7) and (10), we get

$F_{2,1}(z) = \frac{b}{\lambda_1 + r - \lambda_1 z} F_{1,1}(z)$ (14)

From equation (8) and (10), we get

$F_{2,2}(z) = \frac{b}{\lambda_1 + r - \lambda_1 z} F_{1,2}(z)$ (15)

Utilizing equation (14) in (12)

$\left(\lambda(1+\xi) + \mu_1 + b - \lambda(1+\xi)z\right) F_{1,1}(z) = \nu[F_0(z) - P_{0,0}] + \frac{rb}{\lambda_1 + r - \lambda_1 z} F_{1,1}(z)$

$$\Rightarrow \left(\lambda(1+\xi) + \mu_1 + b - \lambda(1+\xi)z - \frac{rb}{\lambda_1 + r - \lambda_1 z}\right) F_{1,1}(z) = \nu\left(\frac{1}{1-\rho_0 z} - 1\right) P_{0,0}$$

$\Rightarrow F_{1,1}(z) = \frac{\nu P_{0,0} \rho_0 z}{A(1-\rho_0 z)}$ (16)

Similarly using (15) in (13)

$$\left(\lambda(1+\xi) + \mu_2 + b - \lambda(1+\xi)z\right) F_{1,2}(z) = \frac{\mu_1}{z} \frac{\nu P_{0,0} \rho_0 z}{A(1-\rho_0 z)} + \frac{\mu_2}{z} F_{1,2}(z) + \frac{rb}{\lambda_1 + r - \lambda_1 z} F_{1,2}(z) - \lambda_0 P_{0,0}$$

$$\Rightarrow \left(\lambda(1+\xi) + \mu_2 + b - \lambda(1+\xi)z - \frac{\mu_2}{z} - \frac{rb}{\lambda_1 + r - \lambda_1 z}\right) F_{1,2}(z) = \frac{\mu_1}{z} \frac{\nu P_{0,0} \rho_0 z}{A(1-\rho_0 z)} - \lambda_0 P_{0,0}$$

$\Rightarrow F_{1,2}(z) = \frac{1}{B}\left[\frac{\mu_1}{z} \frac{\nu P_{0,0} \rho_0 z}{A(1-\rho_0 z)} - \lambda_0 P_{0,0}\right]$ (17)

Where,

$$A = \lambda(1+\xi) + \mu_1 + b - \lambda(1+\xi)z - \frac{rb}{\lambda_1 + r - \lambda_1 z}$$

$$B = \lambda(1+\xi) + \mu_2 + b - \lambda(1+\xi)z - \frac{\mu_2}{z} - \frac{rb}{\lambda_1 + r - \lambda_1 z}$$

Generating function F(z) using (11), (16) & (17)

$$F(z) = \frac{P_{0,0}}{1-\rho_0 z} + \left(1 + \frac{b}{\lambda_1 + r - \lambda_1 z}\right)\left(F_{1,1}(z) + F_{1,2}(z)\right)$$

Where$\sum_{j=1,2} F_{1,j}(z) = \frac{\nu P_{0,0}\rho_0 z}{A(1-\rho_0 z)} + \frac{1}{B}\left[\frac{\mu_1}{z}\frac{\nu P_{0,0}\rho_0 z}{A(1-\rho_0 z)} - \lambda_0 P_{0,0}\right]$ (18)

Since the numerator of equation (18) vanishes at z=1, the distribution of queue length exists. Now, let us examine the denominator to find the related roots. The denominator of equation (20) is $AB(1-\rho_0 z)$, where

$$A = \frac{\lambda(1+\xi)\lambda_1 z^2 - (2\lambda(1+\xi)\lambda_1 + \lambda(1+\xi)r + \mu_1\lambda_1 + b\lambda_1)z + (\lambda(1+\xi) + \mu_1 + b)(\lambda_1 + r) - rb}{\lambda_1 + r - \lambda_1 z}$$

$$= \frac{Q_1(z)}{\lambda_1 + r - \lambda_1 z}$$

and$B = \frac{(z-1)\{\lambda(1+\xi)\lambda_1 z^2 - (\lambda(1+\xi)\lambda_1 + \lambda(1+\xi)r + \mu_2\lambda_1 + b\lambda_1)z + \mu_2(\lambda_1+r)\}}{z(\lambda_1+r-\lambda_1 z)} = \frac{(z-1)Q_2(z)}{z(\lambda_1+r-\lambda_1 z)}$

Determinant of $Q_1(z)$, $\Delta_1 = (\mu_1\lambda_1 + b\lambda_1 - \lambda(1+\xi)r)^2 + 4\lambda\lambda_1 rb\{\lambda(1+\xi) + \mu_1 + b\} > 0$

Determinant of $Q_2(z)$,

$\Delta_2 = (\lambda(1+\xi)\lambda_1 + \lambda(1+\xi)r + \mu_2\lambda_1 + b\lambda_1)^2 - 4\lambda(1+\xi)\lambda_1\mu_2(\lambda_1 + r)$

$\geq (\lambda(1+\xi)\lambda_1 + \lambda(1+\xi)r - \mu_2\lambda_1)^2 + b^2{\lambda_1}^2$

>0

Both the roots of $Q_2(z) = 0$ must be greater than 1 for the steady state queue length to exist, , which is only feasible if $Q_2(1) > 0$ and $Q_2'(1) < 0$. Applying this we get $\mu_2 r > \lambda(1+\xi)r + b\lambda_1$

Hence with the condition$\frac{\lambda(1+\xi)}{\mu_2} + \frac{b\lambda_1}{\mu_2 r} < 1$ both the roots will be greater than 1.

Using $F(1) = 1$ (Normalization condition) for calculating $P_{0,0}$

$[F_0(z) + F_{1,1}(z) + F_{1,2}(z) + F_{2,1}(z) + F_{2,2}(z)]_{z=1} = 1$

$$\frac{P_{0,0}}{1-\rho_0} + \left(1+\frac{b}{r}\right)\left(F_{1,1}(1) + F_{1,2}(1)\right) = 1$$

$$\frac{P_{0,0}}{1-\rho_0} + \left(1+\frac{b}{r}\right)\left(\frac{\nu P_{0,0}\rho_0}{A(1-\rho_0)} + \frac{1}{B}\left[\frac{\mu_1\nu P_{0,0}\rho_0}{A(1-\rho_0)} - \lambda_0 P_{0,0}\right]\right) = 1$$

After simplification we get

$$P_{0,0} = \frac{\mu_1(1-\rho_0)\{-\lambda(1+\xi)r + \mu_2 r - b\lambda_1\}}{\mu_1(1-\rho_0)\{-\lambda(1+\xi)r + \mu_2 r - b\lambda_1\} + \mu_2(r+b)\nu\rho_0 + \mu_1\lambda_0\rho_0(r+b)}$$

(ii) Mean Queue Length

In this section, F'(1) is used to calculate the mean queue length.

$$L = \left.\frac{dF(z)}{dz}\right|_{z=1} = \frac{\rho_0 P_{0,0}}{(1-\rho_0)^2} + \left(1+\frac{b}{r}\right)\frac{d}{dz}\left[F_{1,1}(z) + F_{1,2}(z)\right]_{z=1} + \frac{b\lambda_1}{r^2}\left[F_{1,1}(1) + F_{1,2}(1)\right]$$

Where

$$F_{1,1}(1) = \frac{\nu P_{0,0}\rho_0}{\mu_1(1-\rho_0)}$$

$$F_{1,2}(1) = \frac{\nu P_{0,0}\rho_0}{1-\rho_0}\left(\frac{r}{-\lambda(1+\xi)r + \mu_2 r - b\lambda_1}\right)\left(\frac{r\lambda(1+\xi) + b\lambda_1}{r\mu_1} + \frac{\rho_0}{1-\rho_0}\right)$$

$$\left.\frac{d}{dz}F_{1,1}(Z)\right|_{z=1} = \frac{\nu P_{0,0}\rho_0}{\mu_1(1-\rho_0)}\left(\frac{r\lambda(1+\xi) + b\lambda_1}{r\mu_1} + \frac{1}{1-\rho_0}\right)$$

$$\left.\frac{d}{dz}F_{1,2}(Z)\right|_{z=1} = \frac{\nu P_{0,0}\rho_0}{1-\rho_0}\left(\frac{r}{-\lambda(1+\xi)r + \mu_2 r - b\lambda_1}\right)\left(\frac{\mu_2 r^2 + b{\lambda_1}^2}{r(-\lambda(1+\xi)r + \mu_2 r - b\lambda_1)}\right)\left(\left(\frac{r\lambda(1+\xi) + b\lambda_1}{r\mu_1} + \frac{\rho_0}{1-\rho_0}\right) + \left(\frac{\rho_0}{1-\rho_0}\right) +^2 \left(\frac{r\lambda(1+\xi) + b\lambda_1}{r\mu_1}\right)\left(\frac{\rho_0}{1-\rho_0}\right) + \frac{b{\lambda_1}^2}{\mu_1 r^2} + \left(\frac{r\lambda(1+\xi) + b\lambda_1}{r\mu_1}\right)^2\right)$$

Mean number of customer in the queue

$$L_q == \sum_{i=0}^{\infty} iP_{0,i} + \sum_{i=1}^{\infty}(i-1)P_{1,i,1} + \sum_{i=1}^{\infty}(i-1)P_{1,i,2} + \sum_{i=1}^{\infty}(i-1)P_{2,i,1} + \sum_{i=1}^{\infty}(i-1)P_{2,i,2} +$$

4. Numerical Illustrations

In this section, we used MATLAB to graphically illustrate how changing other factors affected the encouragement factor's impact on mean queue length. It has been observed that MQL increases in every case with the increase of η, percentage change in number of customers determined by using historical or observed data. Additionally MQL increases with the rate of server breakdown and arrivals during active service. Whereas with the increase in service rate of both the types, server vacation rate & repair rate MQL decreases.

(i) **ξ vs MQL with varying** μ_1

When $\lambda=9$, $\lambda_0=5$,$\lambda_1=1.5$, $\mu_2=11$, $\nu=6$, $r=18$, $b=1$

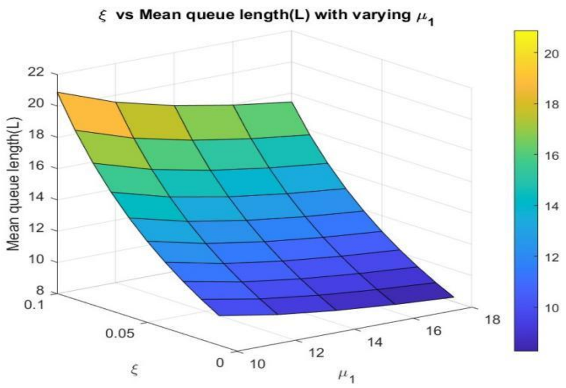

Figure 1: Graphical representation of ξ vs MQL with varying μ_1

(ii) **ξ vs MQL with varying λ**

When $\lambda_0=5$,$\lambda_1=1.5$, , $\mu_1=12$, $\mu_2=11$, $\nu=6$, $r=18$, $b=1$

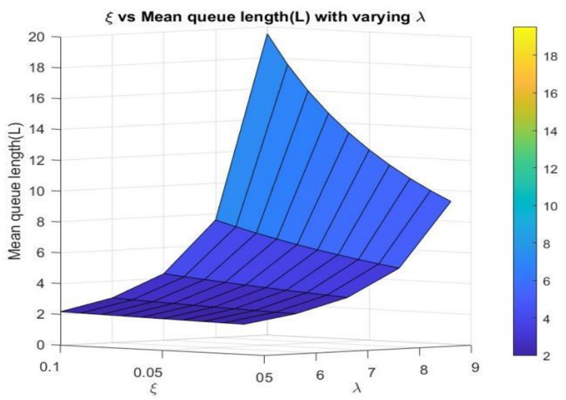

Figure 2: Graphical representation ξ vs MQL with varying λ

(iii) **ξ vs MQL with varying b**

When $\lambda=9$,$\lambda_0=5$,$\lambda_1=1.5$, $\mu_1=12$, $\mu_2=11$, $\nu=6$, $r=18$

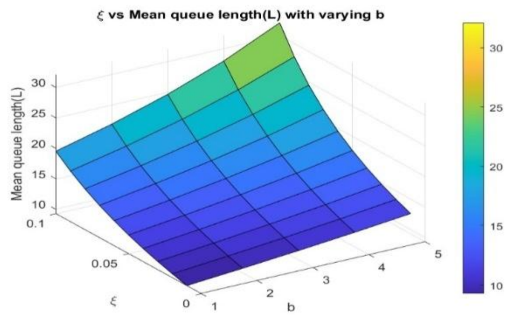

Figure 3: Graphical representationξ vs MQL with varying b

(iv) **ξ vs MQL with varying ν**
When λ=9,λ_0=5,λ_1=1.5, , μ_1 =12, μ_2 =11, r=18, b=1

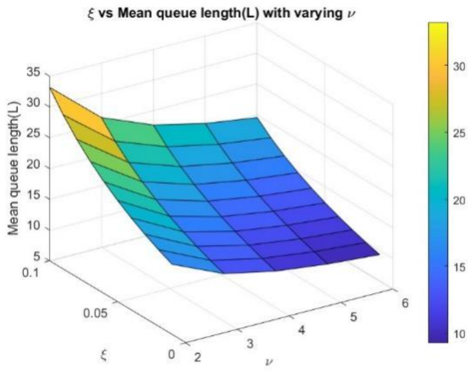

Figure 4: Graphical representationξ vs MQL with varyingν

(v) **ξ vs MQL with varying r**
When λ=9,λ_0=5,λ_1=1.5, μ_1 =12, μ_2 =11,ν=6, b=1

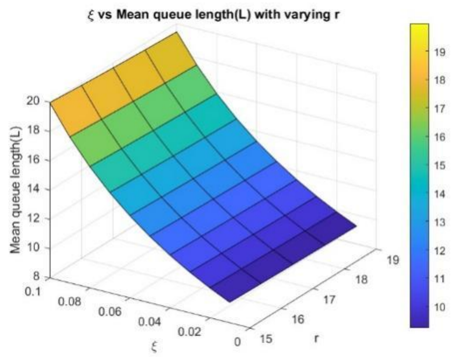

Figure 5: Graphical representation ξ vs MQL with varying r

(vi) **ξ vs MQL with varying** μ_2
Taking parameters λ=9,λ_0=5,λ_1=1.5, μ_1 =12, ν=6, r=18, b=1

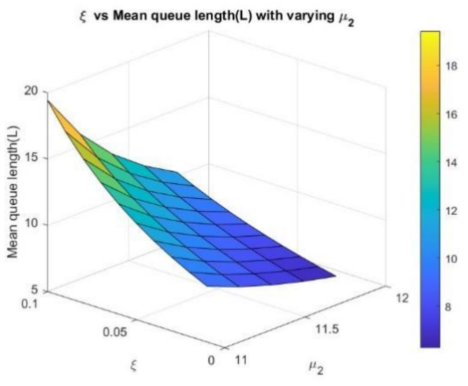

Figure 6: Graphical representation ξ vs MQL with varyingμ_2

5. CONCLUSION

This paper investigates a multiple vacation queuing system featuring encouraged arrival, first exceptional service, breakdowns, and repairs. By employing generating function technique, the study determines steady-state probabilities and mean queue length. Additionally, graph-based numerical analysis is conducted using MATLAB. Numerical examples demonstrate that mean queue length (MQL) consistently rises with increasing encouragement. As anticipated, MQL also increases with higher arrival rates during active service and breakdowns. Conversely, in scenarios with elevated vacation rates, repair rates, and service rates during exceptional and normal services, MQL decreases.

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