Fixed Point Theorem in Soft Parametric Metric Space through C-Class Function

Hrishikesh Tripathi

Department of Mathematics, Govt. Women's Polytechnic College Jabalpur, MP,
Email: tripathihk1@gmail.com

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ABSTRACT
Fixed point theory is a very popular tool to solve the real life problems using soft set and fuzzy set theory. The present paper depends on the concept of soft fixed-point theorems through soft metric spaces. A fixed-point theoremon soft parametric metric space through C-class function is established in this research paper. Proved result extend and generalize new rational inequalities.

Keywords: Soft Parametric metric space, fixed point, C-class Function.

1. INTRODUCTION

Soft Metric Space (SMS) is a generalization of classical metric space (MS) which was studied and introduced by Das and Sumanta[1][15] in 2013and established fixed point theorems which was based on parameterization tool . That is parameterization tool used in soft universe. In 2014, Hussain et. al.[11] studied and defined new notion of generalized metric space and called parametric metric space and proved some fixed point results. This takes values according to parameters when studied the distance between two soft points.In 2021, Tuncay and Cetkin[8] introduced the new concept soft parametric metric space and defined some basic features of these space and established fixed point theorems for continuous soft mapping on soft parametric metric space. In 2021, Demir[17] introduce the concept of a soft complex valued b-metric space and defined its properties and established some fixed soft element results.A lot of investigations on different version of soft parametric metric spaces have been done by several researchers. Some more results on soft metric space can be viewed from [18-28] This manuscript is arranged into two sections. In section 1, we recall some basic definitions. In section 2, we prove unique fixed-point theorem These results extend the existing literature.

2. Preliminaries

In this section, we recall some basic definition which will be utilized in our paper. Throughout in this work, $\widetilde{X}$be an initial universe, E be the set of parameters.

Definition 2.1[8]:A mapping $\tilde{d}: SP(\widetilde{X}) \times SP(\widetilde{X}) \times (0,\infty) \to \mathbb{R}(E)^*$ is said to be a soft parametric metric space on the set $\widetilde{X}$ if $\tilde{d}$ satisfies the following conditions for all$\tilde{x}_{E_1}, \tilde{y}_{E_2}, \tilde{z}_{E_3} \in \widetilde{X}$.

(SP1)$\tilde{d}(\tilde{x}_{E_1}, \tilde{y}_{E_2}, t) = 0 \Leftrightarrow \tilde{x}_{E_1} = \tilde{y}_{E_2}$, for all $t > 0$;

(SP2)$\tilde{d}(\tilde{x}_{E_1}, \tilde{y}_{E_2}, t) = \tilde{d}(\tilde{y}_{E_2}, \tilde{x}_{E_1}, t)$, for all $t > 0$;

(SP3)$\tilde{d}(\tilde{x}_{E_1}, \tilde{z}_{E_3}, t) \le \tilde{d}(\tilde{x}_{E_1}, \tilde{y}_{E_2}, t) + \tilde{d}(\tilde{y}_{E_2}, \tilde{z}_{E_3}, t)$, for all $t > 0$;

Then the pair $(\widetilde{X}, \tilde{d}, E)$ is called soft parametric metric space.

Definition 2.2: Let $(\widetilde{X}, \tilde{d}, E)$ be a soft parametric metric space and $\{\widetilde{x_n}\}$be a sequence in $\widetilde{X}$ then

(i) A sequence $\{\widetilde{x_n}\}$in $\widetilde{X}$is said to be convergent, $\{\widetilde{x_n}\}$convergence to $\tilde{x}$If for any $0 < \tilde{c} \in X^E$ then there exist $n_0 \in \mathbb{N}$ such that $\tilde{d}(\widetilde{x_n}, \tilde{x}) < \tilde{c}$, for all $n > n_0$ and we can denote this $\lim_{n\to\infty}\widetilde{x_n} = \tilde{x}$ or $\widetilde{x_n} \to \tilde{x}$ as $n \to \infty$.

(ii) A sequence$\{\widetilde{x_n}\}$in $\widetilde{X}$is said to be Cauchy sequenceIf for any $0 < \tilde{c} \in X^E$, there exists$n_0 \in \mathbb{N}$, such that $\tilde{d}(\widetilde{x_n}, \widetilde{x_{n+m}}) < \tilde{c}$, where $m, n \in \mathbb{N}$ and $n > n_0$.

(iii) If every Cauchy sequence is convergent in$(\widetilde{X}, \tilde{d}, E)$ then $(\widetilde{X}, \tilde{d}, E)$ is said to be a complete soft parametric metric space.

Lemma 2.3[8]: Let $(\widetilde{X}, \tilde{d}, E)$ be a soft parametric metric space and $\{\widetilde{x_n}\}$ be a sequence in $\widetilde{X}$ such that $\tilde{d}(\widetilde{x_n}, \widetilde{x_{n+1}}, t) \le r\tilde{d}(\widetilde{x_{n-1}}, \widetilde{x_n}, t)$, where $r \in [0,1)$ and $n = 1,2, \ldots..$ then $\{\widetilde{x_n}\}$ satisfied Cauchy property in $(\widetilde{X}, \tilde{d}, E)$.

Є-class function has been defined:

Definition 2.6: A Continuous mapping $Ƒ: (0,\infty) \times [0,\infty) \to R$ is said to be Є-class function if it satisfies the following conditions:
$\{C_a\} Ƒ(\eta, Ҡ) \le \eta$ for all $\eta, Ҡ \in [0,\infty)$.
$\{C_b\} Ƒ(\eta, Ҡ) \le \eta \Rightarrow$ either $\eta = 0$ or $Ҡ = 0$.
Example: 2.7: The following functions $Ƒ: (0,\infty) \times [0,\infty) \to R$ are elements of **Є** for all $\eta, Ҡ \in [0,\infty)$;
(I) $Ƒ(\eta, Ҡ) = \eta - Ҡ, Ƒ(\eta, Ҡ) = \eta \Rightarrow Ҡ = 0$;
(II) $Ƒ(\eta, Ҡ) = \beta\eta, 0 < \beta < 1, Ƒ(\eta, Ҡ) = \eta \Rightarrow \eta = 0$;
(III) $Ƒ(\eta, Ҡ) = \eta - \varphi(\eta), Ƒ(\eta, Ҡ) = \eta \Rightarrow \eta = 0$, here $\varphi: [0,\infty) \to [0,\infty)$ is a continuous function such that $\varphi(t) = 0 \Leftrightarrow t = 0$;
(IV) $Ƒ(\eta, Ҡ) = \eta - \frac{t}{k+t}, Ƒ(\eta, Ҡ) = \eta = Ҡ = 0$;
(V) $Ƒ(\eta, Ҡ) = \eta - \frac{2+t}{k+t}, Ƒ(\eta, Ҡ) = \eta \Rightarrow Ҡ = 0$;
(VI) $Ƒ(\eta, Ҡ) = Ƒ(\eta, Ҡ) = \eta \Rightarrow \eta = 0$ or $Ҡ = 0$;
(VII) $Ƒ(\eta, Ҡ) = \eta\beta(\eta), \beta: [0,\infty) \to [0,1)$, and is a continuous function, $Ƒ(\eta, Ҡ) = \eta \Rightarrow \eta = 0$;

Let Ψ denote the set of all continuous and monotone non-decreasing function $\phi: [0,\infty) \to [0,\infty)$ such that $\phi(Ҡ) = 0$ iff $Ҡ = 0$, $\phi(\eta + Ҡ) \le \phi(\eta) + \phi(Ҡ)$ for all $\eta, Ҡ \in [0,\infty)$.
Let Φ_1 denote the all continuous function $\phi: [0,\infty) \to [0,\infty)$ such that $\phi(Ҡ) = 0$ iff $Ҡ = 0$ and Φ_u denote the set of all continuous function $\phi: [0,\infty) \to [0,\infty)$ such that $\phi(0) \ge 0$, note that $\Phi_1 \subset \Phi_u$.

1. Main Result

Theorem 3.1: Let $(\tilde{X}, \tilde{d}, E)$ be a complete soft parametric metric space with E and suppose $P_E : (\tilde{X}, \tilde{d}, E) \to (\tilde{X}, \tilde{d}, E)$ be a soft mapping satisfying:

$$\Gamma\{\tilde{d}(P_E(\tilde{\sigma}), P_E(\tilde{\omega}), t)\} \le \mathbb{H}\left(\Gamma\left(\partial(\tilde{\sigma}, \tilde{\omega}), \rho\left(\partial(\tilde{\sigma}, \tilde{\omega})\right)\right)\right)$$

for all $\tilde{\sigma}, \tilde{\omega} \in X^E$, for $t > 0$ and $\mathbb{H} \in C$, $\Gamma \in \Psi, \rho \in \Phi_u$.

$$\partial(\tilde{\sigma}, \tilde{\omega}) = \alpha\tilde{d}(\tilde{\sigma}, \widetilde{\omega, t}) + \beta\left[\tilde{d}(\tilde{\sigma}, U_E(\tilde{\sigma}), t) + \tilde{d}(\tilde{\omega}, P_E(\tilde{\omega}), t)\right] + \gamma \operatorname{Max}\{\tilde{d}(\tilde{\sigma}, U_E(\tilde{\sigma}), t), \tilde{d}(\tilde{\omega}, P_E(\tilde{\omega}), t)\}$$
$$+ \delta\left[\frac{\tilde{d}(\tilde{\omega}, P_E(\tilde{\omega}), t)((1 + \tilde{d}(\tilde{\sigma}, U_E(\tilde{\sigma}), t)}{1 + \tilde{d}(\tilde{\sigma}, \widetilde{\omega, t})} + \frac{\tilde{d}(\tilde{\sigma}, U_E(\tilde{\sigma}), t)\tilde{d}(U_E(\tilde{\sigma}), \widetilde{\omega, t})}{\tilde{d}(\tilde{\sigma}, U_E(\tilde{\sigma}), t) + \tilde{d}(\tilde{\sigma}, \tilde{\omega}, t)}\right]$$

Where $\alpha, \beta, \gamma, \delta \in [0, \frac{1}{2})$ and $\alpha + 2\beta + \gamma + \delta < 1$ then P_E has a unique fixed point.

Proof: Choose $\sigma_0 \in X^E$ and define sequences $\{\sigma_n\}$ as follows $P_E(\widetilde{\sigma_n}) = \widetilde{\sigma_{n+1}}$ and $P_E(\widetilde{\sigma_{n+1}}) = \widetilde{\sigma_{n+2}}$, for all $n \in 0,1,2,3 \ldots$ then we have

$$\Gamma\{\tilde{d}(\widetilde{\sigma_{n+1}}, \widetilde{\sigma_{n+2}}, t)\} = \Gamma\{\tilde{d}(P_E(\widetilde{\sigma_n}), P_E(\widetilde{\sigma_{n+1}}), t)\}$$

$$\le \mathbb{H}\left[\begin{array}{l}\Gamma\left\{\begin{array}{c}\alpha\tilde{d}(\widetilde{\sigma_n}, \widetilde{\sigma_{n+1}}, t) + \beta\left[\tilde{d}(\widetilde{\sigma_n}, P_E(\widetilde{\sigma_n}), t) + \tilde{d}(\widetilde{\sigma_{n+1}}, P_E(\widetilde{\sigma_{n+1}}), t)\right] \\ +\gamma\left[\frac{\tilde{d}(\widetilde{\sigma_{n+1}}, P_E(\widetilde{\sigma_{n+1}}), t)(1 + \tilde{d}(\widetilde{\sigma_n}, P_E(\widetilde{\sigma_n}), t)}{1 + \tilde{d}(\widetilde{\sigma_n}, \widetilde{\sigma_{n+1}}, t)} + \frac{\tilde{d}(\widetilde{\sigma_n}, P_E(\widetilde{\sigma_n}), t)\tilde{d}(P_E(\widetilde{\sigma_n}), \widetilde{\sigma_{n+1}}, t)}{\tilde{d}(\widetilde{\sigma_n}, P_E(\widetilde{\sigma_n}, t)) + \tilde{d}(\widetilde{\sigma_n}, \widetilde{\sigma_{n+1}}, t)}\right] \\ +\delta \operatorname{Max}\{\tilde{d}(\widetilde{\sigma_n}, P_E(\widetilde{\sigma_n}), t), \tilde{d}(\widetilde{\sigma_{n+1}}, P_E(\widetilde{\sigma_{n+1}}), t)\}\end{array}\right\}, \\ \rho\left\{\begin{array}{c}\alpha\tilde{d}(\widetilde{\sigma_n}, \widetilde{\sigma_{n+1}}, t) + \beta\left[\tilde{d}(\widetilde{\sigma_n}, P_E(\widetilde{\sigma_n}), t) + \tilde{d}(\widetilde{\sigma_{n+1}}, P_E(\widetilde{\sigma_{n+1}}), t)\right] \\ +\gamma\left[\frac{\tilde{d}(\widetilde{\sigma_{n+1}}, P_E(\widetilde{\sigma_{n+1}}), t)(1 + \tilde{d}(\widetilde{\sigma_n}, P_E(\widetilde{\sigma_n}), t)}{1 + \tilde{d}(\widetilde{\sigma_n}, \widetilde{\sigma_{n+1}}, t)} + \frac{\tilde{d}(\widetilde{\sigma_n}, P_E(\widetilde{\sigma_n}), t)\tilde{d}(P_E(\widetilde{\sigma_n}), \widetilde{\sigma_{n+1}}, t)}{\tilde{d}(\widetilde{\sigma_n}, P_E(\widetilde{\sigma_n}, t)) + \tilde{d}(\widetilde{\sigma_n}, \widetilde{\sigma_{n+1}}, t)}\right] \\ +\delta \operatorname{Max}\{\tilde{d}(\widetilde{\sigma_n}, P_E(\widetilde{\sigma_n}), t), \tilde{d}(\widetilde{\sigma_{n+1}}, P_E(\widetilde{\sigma_{n+1}}), t)\}\end{array}\right\}\end{array}\right]$$

$$\le \mathbb{H}\left[\begin{array}{l}\Gamma\left\{\begin{array}{c}\alpha\tilde{d}(\widetilde{\sigma_n}, \widetilde{\sigma_{n+1}}, t) + \beta\left[\tilde{d}(\widetilde{\sigma_n}, \widetilde{\sigma_{n+1}}, t) + \tilde{d}(\widetilde{\sigma_{n+1}}, \widetilde{\sigma_{n+2}}, t)\right] \\ +\gamma\left[\frac{\tilde{d}(\widetilde{\sigma_{n+1}}, \widetilde{\sigma_{n+2}}, t)(1 + \tilde{d}(\widetilde{\sigma_n}, \widetilde{\sigma_{n+1}}, t))}{1 + \tilde{d}(\widetilde{\sigma_n}, \widetilde{\sigma_{n+1}}, t)} + \frac{\tilde{d}(\widetilde{\sigma_n}, \widetilde{\sigma_{n+1}}, t)\tilde{d}(\widetilde{\sigma_{n+1}}, \widetilde{\sigma_{n+1}}, t)}{\tilde{d}(\widetilde{\sigma_n}, \widetilde{\sigma_{n+1}}, t) + \tilde{d}(\widetilde{\sigma_n}, \widetilde{\sigma_{n+1}}, t)}\right] \\ +\delta \operatorname{Max}\{\tilde{d}(\widetilde{\sigma_n}, \widetilde{\sigma_{n+1}}, t), \tilde{d}(\widetilde{\sigma_{n+1}}, \widetilde{\sigma_{n+2}}, t)\}\end{array}\right\}, \\ \rho\left\{\begin{array}{c}\alpha\tilde{d}(\widetilde{\sigma_n}, \widetilde{\sigma_{n+1}}, t) + \beta\left[\tilde{d}(\widetilde{\sigma_n}, \widetilde{\sigma_{n+1}}, t) + \tilde{d}(\widetilde{\sigma_{n+1}}, \widetilde{\sigma_{n+2}}, t)\right] \\ +\gamma\left[\frac{\tilde{d}(\widetilde{\sigma_{n+1}}, \widetilde{\sigma_{n+2}}, t)(1 + \tilde{d}(\widetilde{\sigma_n}, \widetilde{\sigma_{n+1}}, t))}{1 + \tilde{d}(\widetilde{\sigma_n}, \widetilde{\sigma_{n+1}}, t)} + \frac{\tilde{d}(\widetilde{\sigma_n}, \widetilde{\sigma_{n+1}}, t)\tilde{d}(\widetilde{\sigma_{n+1}}, \widetilde{\sigma_{n+1}}, t)}{\tilde{d}(\widetilde{\sigma_n}, \widetilde{\sigma_{n+1}}, t) + \tilde{d}(\widetilde{\sigma_n}, \widetilde{\sigma_{n+1}}, t)}\right] \\ +\delta \operatorname{Max}\{\tilde{d}(\widetilde{\sigma_n}, \widetilde{\sigma_{n+1}}, t), \tilde{d}(\widetilde{\sigma_{n+1}}, \widetilde{\sigma_{n+2}}, t)\}\end{array}\right\}\end{array}\right]$$

$$\le \mathbb{H}\left[\begin{array}{l}\Gamma\left\{\begin{array}{c}\alpha\tilde{d}(\widetilde{\sigma_n}, \widetilde{\sigma_{n+1}}, t) + \beta\left[\tilde{d}(\widetilde{\sigma_n}, \widetilde{\sigma_{n+1}}, t) + \tilde{d}(\widetilde{\sigma_{n+1}}, \widetilde{\sigma_{n+2}}, t)\right] + \gamma\tilde{d}(\widetilde{\sigma_{n+1}}, \widetilde{\sigma_{n+2}}, t) \\ +\delta \operatorname{Max}\{\tilde{d}(\widetilde{\sigma_n}, \widetilde{\sigma_{n+1}}, t), \tilde{d}(\widetilde{\sigma_{n+1}}, \widetilde{\sigma_{n+2}}, t)\}\end{array}\right\}, \\ \rho\left\{\begin{array}{c}\alpha\tilde{d}(\widetilde{\sigma_n}, \widetilde{\sigma_{n+1}}, t) + \beta\left[\tilde{d}(\widetilde{\sigma_n}, \widetilde{\sigma_{n+1}}, t) + \tilde{d}(\widetilde{\sigma_{n+1}}, \widetilde{\sigma_{n+2}}, t)\right] + \gamma\tilde{d}(\widetilde{\sigma_{n+1}}, \widetilde{\sigma_{n+2}}, t) \\ +\delta \operatorname{Max}\{\tilde{d}(\widetilde{\sigma_n}, \widetilde{\sigma_{n+1}}, t), \tilde{d}(\widetilde{\sigma_{n+1}}, \widetilde{\sigma_{n+2}}, t)\}\end{array}\right\}\end{array}\right]$$

$$\leq \Gamma\left\{\begin{array}{c}\alpha\tilde{d}(\widetilde{\sigma_n},\widetilde{\sigma_{n+1}},t)+\ \beta\left[\tilde{d}(\widetilde{\sigma_n},\widetilde{\sigma_{n+1}},t)+\tilde{d}(\widetilde{\sigma_{n+1}},\widetilde{\sigma_{n+2}},t)\right]+\gamma\tilde{d}(\widetilde{\sigma_{n+1}},\widetilde{\sigma_{n+2}},t)\\ +\delta\ \mathrm{Max}\left\{\tilde{d}(\widetilde{\sigma_n},\widetilde{\sigma_{n+1}},t),\tilde{d}(\widetilde{\sigma_{n+1}},\widetilde{\sigma_{n+2}},t)\right\}\end{array}\right\}$$

There are two possibilities:

Case (1): if $\mathrm{Max}\{\tilde{d}(\widetilde{\sigma_n},\widetilde{\sigma_{n+1}},t),\tilde{d}(\widetilde{\sigma_{n+1}},\widetilde{\sigma_{n+2}},t)\}=\tilde{d}(\widetilde{\sigma_n},\widetilde{\sigma_{n+1}},t)$ then we get,

$$\tilde{d}(\widetilde{\sigma_{n+1}},\widetilde{\sigma_{n+2}},t)\leq\alpha\tilde{d}(\widetilde{\sigma_n},\widetilde{\sigma_{n+1}},t)+\ \beta[\tilde{d}(\widetilde{\sigma_n},\widetilde{\sigma_{n+1}},t)+\tilde{d}(\widetilde{\sigma_{n+1}},\widetilde{\sigma_{n+2}},t)]+\gamma\tilde{d}(\widetilde{\sigma_{n+1}},\widetilde{\sigma_{n+2}},t)+\delta\tilde{d}(\widetilde{\sigma_n},\widetilde{\sigma_{n+1}},t)$$

$$\tilde{d}(\widetilde{\sigma_{n+1}},\widetilde{\sigma_{n+2}},t)\leq(\alpha+\beta+\delta)\tilde{d}(\widetilde{\sigma_n},\widetilde{\sigma_{n+1}},t)+(\beta+\gamma)\tilde{d}(\widetilde{\sigma_{n+1}},\widetilde{\sigma_{n+2}},t)$$

$$[1-(\beta+\gamma)]\tilde{d}(\widetilde{\sigma_{n+1}},\widetilde{\sigma_{n+2}},t)\leq(\alpha+\beta+\delta)\tilde{d}(\widetilde{\sigma_n},\widetilde{\sigma_{n+1}},t)$$

$$\tilde{d}(\widetilde{\sigma_{n+1}},\widetilde{\sigma_{n+2}},t)\leq\frac{(\alpha+\beta+\delta)}{[1-(\beta+\gamma)]}\tilde{d}(\widetilde{\sigma_n},\widetilde{\sigma_{n+1}},t)\text{Where}\Omega=\frac{(\alpha+\beta+\delta)}{[1-(\beta+\gamma)]}$$

$$\tilde{d}(\widetilde{\sigma_{n+1}},\widetilde{\sigma_{n+2}},t)\leq\Omega\tilde{d}(\widetilde{\sigma_n},\widetilde{\sigma_{n+1}},t)$$

By mathematical induction, we get

$$\tilde{d}(\widetilde{\sigma_{n+1}},\widetilde{\sigma_{n+2}},t)\leq\Omega^{n+1}\tilde{d}(\widetilde{\sigma_0},\widetilde{\sigma_1},t)$$

For all u,v $with\ v<u\ then\ we\ have$

$$\tilde{d}(\widetilde{\sigma_v},\widetilde{\sigma_u},t)\leq[\tilde{d}(\widetilde{\sigma_v},\widetilde{\sigma_{v+1}},t)+\tilde{d}(\widetilde{\sigma_{v+1}},\widetilde{\sigma_u},t)]$$

$$\tilde{d}(\widetilde{\sigma_v},\widetilde{\sigma_u},t)\leq\tilde{d}(\widetilde{\sigma_v},\widetilde{\sigma_{v+1}},t)+[\tilde{d}(\widetilde{\sigma_{v+1}},\widetilde{\sigma_{v+2}},t)+\tilde{d}(\widetilde{\sigma_{v+2}},\widetilde{\sigma_u},t)]$$

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$$\tilde{d}(\widetilde{\sigma_v},\widetilde{\sigma_u},t)\leq\Omega^n\tilde{d}(\widetilde{\sigma_0},\widetilde{\sigma_1},t)+\Omega^{n+1}\tilde{d}(\widetilde{\sigma_0},\widetilde{\sigma_1},t)+\Omega^{n+2}\tilde{d}(\widetilde{\sigma_0},\widetilde{\sigma_1},t)+\cdots+\Omega^{v-1}\tilde{d}(\widetilde{\sigma_0},\widetilde{\sigma_1},t)]$$

$$\tilde{d}(\widetilde{\sigma_v},\widetilde{\sigma_u},t)\leq\Omega^n[1+\Omega+\Omega^2+\cdots..+\Omega^{u-v-1}]\tilde{d}(\widetilde{\sigma_0},\widetilde{\sigma_1},t)$$

$$\tilde{d}(\widetilde{\sigma_v},\widetilde{\sigma_u},t)\leq\frac{\Omega^n(1-\Omega^{u-v})}{(1-\Omega)}\tilde{d}(\widetilde{\sigma_0},\widetilde{\sigma_1},t)\to 0\ as\ u,v\to\infty\ (since\ \Omega<1)$$

Since $\Omega\in[0,1)$, Hence the sequence $\{\widetilde{\sigma_n}\}$ is a Cauchy sequence in $(\tilde{X},\tilde{d},E)$. By completeness of $(\tilde{X},\tilde{d},E)\Rightarrow\{\widetilde{\sigma_n}\}$ is convergent. call $\tilde{\sigma}\in X^E$ such that $lim_{n\to\infty}\widetilde{\sigma_n}=\tilde{\sigma}$ and P_E is continuous then

$$P_E(\tilde{\sigma})=P_E\left(\lim_{n\to\infty}\widetilde{\sigma_n}\right)=\lim_{n\to\infty}P_E(\widetilde{\sigma_n})=\lim_{n\to\infty}(\widetilde{\sigma_{n+1}})=\tilde{\sigma}$$

Hence P_E has a fixed point in $\tilde{X}$.

Case (2): if $Max\{\tilde{d}(\widetilde{\sigma_n},\widetilde{\sigma_{n+1}},t),\tilde{d}(\widetilde{\sigma_{n+1}},\widetilde{\sigma_{n+2}},t)\}=\tilde{d}(\widetilde{\sigma_{n+1}},\widetilde{\sigma_{n+2}},t)$ then we get,

$$\tilde{d}(\widetilde{\sigma_{n+1}},\widetilde{\sigma_{n+2}},t)\leq\alpha\tilde{d}(\widetilde{\sigma_n},\widetilde{\sigma_{n+1}},t)+\ \beta\left[\tilde{d}(\widetilde{\sigma_n},\widetilde{\sigma_{n+1}},t)+\tilde{d}(\widetilde{\sigma_{n+1}},\widetilde{\sigma_{n+2}},t)\right]+\gamma\tilde{d}(\widetilde{\sigma_{n+1}},\widetilde{\sigma_{n+2}},t)+\delta\tilde{d}(\widetilde{\sigma_{n+1}},\widetilde{\sigma_{n+2}},t)$$

$$\tilde{d}(\widetilde{\sigma_{n+1}},\widetilde{\sigma_{n+2}},t)\leq(\alpha+\beta)\tilde{d}(\widetilde{\sigma_n},\widetilde{\sigma_{n+1}},t)+(\beta+\gamma+\delta)\tilde{d}(\widetilde{\sigma_{n+1}},\widetilde{\sigma_{n+2}},t)$$

$$[1-(\beta+\gamma+\delta)]\tilde{d}(\widetilde{\sigma_{n+1}},\widetilde{\sigma_{n+2}},t)\leq(\alpha+\beta)\tilde{d}(\widetilde{\sigma_n},\widetilde{\sigma_{n+1}},t)$$

$$\tilde{d}(\widetilde{\sigma_{n+1}},\widetilde{\sigma_{n+2}},t)\leq\frac{(\alpha+\beta)}{[1-(\beta+\gamma+\delta)]}\tilde{d}(\widetilde{\sigma_n},\widetilde{\sigma_{n+1}},t)\quad\text{Where }\zeta=\frac{(\alpha+\beta)}{[1-(\beta+\gamma+\delta)]}$$

$$\tilde{d}(\widetilde{\sigma_{n+1}},\widetilde{\sigma_{n+2}},t)\leq\zeta\tilde{d}(\widetilde{\sigma_n},\widetilde{\sigma_{n+1}},t)$$

By mathematical induction, we get

$$\tilde{d}(\widetilde{\sigma_{n+1}},\widetilde{\sigma_{n+2}},t)\leq\zeta^{n+1}\tilde{d}(\widetilde{\sigma_0},\widetilde{\sigma_1},t)$$

For all u,v $with\ v<u\ then\ we\ have$

$$\tilde{d}(\widetilde{\sigma_{n+1}},\widetilde{\sigma_{n+2}},t)\leq[\tilde{d}(\widetilde{\sigma_v},\widetilde{\sigma_{v+1}},t)+\tilde{d}(\widetilde{\sigma_{v+1}},\widetilde{\sigma_u},t)]$$

$$\tilde{d}(\widetilde{\sigma_{n+1}},\widetilde{\sigma_{n+2}},t)\leq\tilde{d}(\widetilde{\sigma_v},\widetilde{\sigma_{v+1}},t)+[\tilde{d}(\widetilde{\sigma_{v+1}},\widetilde{\sigma_{v+2}},t)+\tilde{d}(\widetilde{\sigma_{v+2}},\widetilde{\sigma_u},t)]$$

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$$\tilde{d}(\widetilde{\sigma_{n+1}},\widetilde{\sigma_{n+2}},t)\leq\zeta^n\tilde{d}(\widetilde{\sigma_0},\widetilde{\sigma_1},t)+\zeta^{n+1}\tilde{d}(\widetilde{\sigma_0},\widetilde{\sigma_1},t)+\zeta^{n+2}\tilde{d}(\widetilde{\sigma_0},\widetilde{\sigma_1},t)+\cdots+\zeta^{v-1}\tilde{d}(\widetilde{\sigma_0},\widetilde{\sigma_1},t)]$$

$$\tilde{d}(\widetilde{\sigma_{n+1}},\widetilde{\sigma_{n+2}},t)\leq\zeta^n[1+\zeta+\zeta^2+\cdots..+\zeta^{u-v-1}]\tilde{d}(\widetilde{\sigma_0},\widetilde{\sigma_1},\tilde{t})$$

$$\tilde{d}(\widetilde{\sigma_{n+1}},\widetilde{\sigma_{n+2}},t)\leq\frac{\zeta^n(1-\zeta^{u-v})}{(1-\zeta)}\tilde{d}(\widetilde{\sigma_0},\widetilde{\sigma_1},t)\to 0\ as\ u,v\to\infty\ (since\ \zeta<1)$$

Since $\zeta\in[0,1)$, Hence the sequence $\{\widetilde{\sigma_n}\}$ is a Cauchy sequence in $(\tilde{X},\tilde{d},E)$. By completeness of $(\tilde{X},\tilde{d},E)\Rightarrow\{\widetilde{\sigma_n}\}$ is convergent. call $\tilde{\hbar}\in X^E$ such that $lim_{n\to\infty}\widetilde{\hbar_n}=\tilde{\hbar}$ and P_E is continuous then

$$P_E(\tilde{\hbar})=P_E\left(\lim_{n\to\infty}\widetilde{\hbar_n}\right)=\lim_{n\to\infty}P_E(\widetilde{\hbar_n})=\lim_{n\to\infty}(\widetilde{\hbar_{n+1}})=\tilde{\hbar}$$

Hence P_E has a fixed point in $\tilde{X}$.

For uniqueness, we assume that $\widetilde{\sigma^*}$is another common fixed soft element of P_E $then$

$$\Gamma\{\tilde{d}(\tilde{\sigma},\widetilde{\sigma^*},t)\}=\Gamma\{\tilde{d}(P_E(\tilde{\sigma}),P_E(\widetilde{\sigma^*}),t)\}$$

$$\leq \mathbb{H}\begin{bmatrix} \Gamma\left\{\begin{matrix} \alpha\tilde{d}(\tilde{\sigma},\widetilde{\sigma^*},t)+\beta\left[\tilde{d}(\tilde{\sigma},P_E(\tilde{\sigma}),t)+\tilde{d}(\widetilde{\sigma^*},P_E(\widetilde{\sigma^*}),t)\right] \\ +\gamma\left[\frac{\tilde{d}(\widetilde{\sigma^*},P_E(\widetilde{\sigma^*}),t)(1+\tilde{d}(\tilde{\sigma},P_E(\tilde{\sigma}),t)}{1+\tilde{d}(\tilde{\sigma},\widetilde{\sigma^*},t)}+\frac{\tilde{d}(\tilde{\sigma},P_E(\tilde{\sigma}),t)\tilde{d}(P_E(\tilde{\sigma}),\widetilde{\sigma^*},t)}{\tilde{d}(\tilde{\sigma},P_E(\tilde{\sigma}),t)+\tilde{d}(\tilde{\sigma},\widetilde{\sigma^*},t)}\right] \\ +\delta\, Max\left\{\tilde{d}(\tilde{\sigma},P_E(\tilde{\sigma}),t),\tilde{d}(\widetilde{\sigma^*},P_E(\widetilde{\sigma^*}),t)\right\} \end{matrix}\right\}, \\ \rho\left\{\begin{matrix} \alpha\tilde{d}(\tilde{\sigma},\widetilde{\sigma^*},t)+\beta\left[\tilde{d}(\tilde{\sigma},P_E(\tilde{\sigma}),t)+\tilde{d}(\widetilde{\sigma^*},P_E(\widetilde{\sigma^*}),t)\right] \\ +\gamma\left[\frac{\tilde{d}(\widetilde{\sigma^*},P_E(\widetilde{\sigma^*}),t)(1+\tilde{d}(\tilde{\sigma},P_E(\tilde{\sigma}),t)}{1+\tilde{d}(\tilde{\sigma},\widetilde{\sigma^*},t)}+\frac{\tilde{d}(\tilde{\sigma},P_E(\tilde{\sigma}),t)\tilde{d}(P_E(\tilde{\sigma}),\widetilde{\sigma^*},t)}{\tilde{d}(\tilde{\sigma},P_E(\tilde{\sigma}),t)+\tilde{d}(\tilde{\sigma},\widetilde{\sigma^*},t)}\right] \\ +\delta\, Max\left\{\tilde{d}(\tilde{\sigma},P_E(\tilde{\sigma}),t),\tilde{d}(\widetilde{\sigma^*},P_E(\widetilde{\sigma^*}),t)\right\} \end{matrix}\right\} \end{bmatrix}$$

$$\leq \mathbb{H}\begin{bmatrix} \Gamma\left\{\begin{matrix} \alpha\tilde{d}(\tilde{\sigma},\widetilde{\sigma^*},t)+\beta\left[\tilde{d}(\tilde{\sigma},\widetilde{\sigma},t)+\tilde{d}(\widetilde{\sigma^*},\widetilde{\sigma^*},t)\right] \\ +\gamma\left[\frac{\tilde{d}(\widetilde{\sigma^*},\widetilde{\sigma^*},t)(1+d_B(\tilde{\sigma},\tilde{\sigma},t))}{1+\tilde{d}(\tilde{\sigma},\widetilde{\sigma^*},t)}+\frac{\tilde{d}(\tilde{\sigma},\widetilde{\sigma},t)\tilde{d}(\tilde{\sigma},\widetilde{\sigma^*},t)}{\tilde{d}(\tilde{\sigma},\widetilde{\sigma},t)+\tilde{d}(\tilde{\sigma},\widetilde{\sigma^*},t)}\right] \\ +\delta\, Max\left\{\tilde{d}(\tilde{\sigma},\tilde{\sigma},t),\tilde{d}(\widetilde{\sigma^*},\widetilde{\sigma^*},t)\right\} \end{matrix}\right\}, \\ \rho\left\{\begin{matrix} \alpha\tilde{d}(\tilde{\sigma},\widetilde{\sigma^*},t)+\beta\left[\tilde{d}(\tilde{\sigma},\widetilde{\sigma},t)+\tilde{d}(\widetilde{\sigma^*},\widetilde{\sigma^*},t)\right] \\ +\gamma\left[\frac{\tilde{d}(\widetilde{\sigma^*},\widetilde{\sigma^*},t)(1+d_B(\tilde{\sigma},\tilde{\sigma},t))}{1+\tilde{d}(\tilde{\sigma},\widetilde{\sigma^*},t)}+\frac{\tilde{d}(\tilde{\sigma},\widetilde{\sigma},t)\tilde{d}(\tilde{\sigma},\widetilde{\sigma^*},t)}{\tilde{d}(\tilde{\sigma},\widetilde{\sigma},t)+\tilde{d}(\tilde{\sigma},\widetilde{\sigma^*},t)}\right] \\ +\delta\, Max\left\{\tilde{d}(\tilde{\sigma},\tilde{\sigma},t),\tilde{d}(\widetilde{\sigma^*},\widetilde{\sigma^*},t)\right\} \end{matrix}\right\} \end{bmatrix}$$

$$\leq \mathbb{H}\begin{bmatrix}\Gamma\{\alpha\tilde{d}(\tilde{\sigma},\widetilde{\sigma^*},t)\}, \\ \rho\{\alpha\tilde{d}(\tilde{\sigma},\widetilde{\sigma^*},t)\}\end{bmatrix}$$

$\leq \Gamma\{\alpha\tilde{d}(\tilde{\sigma},\widetilde{\sigma^*},t)\}.\tilde{d}(\tilde{\sigma},\widetilde{\sigma^*}) \leq \alpha\tilde{d}(\tilde{\sigma},\widetilde{\sigma^*},t) \Rightarrow \tilde{d}(\tilde{\sigma},\widetilde{\sigma^*}) = 0 \Rightarrow \tilde{\sigma} = \widetilde{\sigma^*}$

So $\tilde{\sigma}$ is unique common fixed soft element of P_E.

Future Scope of the work

This work further can be generalized for n- dimensional space, soft fuzzy metric spaces, also taken mappings can also be proved for integral type mappings. Established results can be utilized for solving initial value and boundary value problems.

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