Two Warehouse Inventory Model with Quadratic Demand and Permissible Delay in Payments

S. Anusuya Baggiyalakshmi[1], M. Maragatham[1], P. Mariappan[2]

[1]Postgraduate& Research Department of Mathematics, Thanthai Periyar Government Arts and Science College (Autonomous)* Tiruchirappalli, India.
Email: anusuya1997bhc@gmail.com; maraguevr@yahoo.co.in
[2]Department of Supply Chain Management and International Relations, St. Joseph's Institute of Management, Tiruchirappallli – 620 002, India, Email: drmariappan@jim.ac.in
* Affiliated to Bharathidasan University.

Received: 12.07.2024 Revised: 10.08.2024 Accepted: 26.09.2024

ABSTRACT
Based on two-warehouse model, Considering an Own Warehouse (OW) with capacity of N units and Rented Warehouse (RW) with capacity of Q units. In this model we have a Quadratic demand and also giving a permissible delay in payments which will helps the retailers to reduce the Total Cost (TC), which is our ultimate aim of this paper. A numerical example is illustrated to verify this model.

Keywords: Two-Warehouse model, Quadratic demand, permissible delay.

INTRODUCTION
The permission allowed in making delay payments is effective to attract new customers and grow business. In current area, it is very usual to see that retailers are allowed for a fixed time period to settle their account to the supplier for their purchase.That allowed period is called as "trade credit period". Before the trade credit period ends, the retailer needs to settle by selling the goods and accumulate revenue and earn interest. If not means, higher interest rate will be imposed on the payment. Accordingly, storage plays an important role in managing the inventories. In common, every company has its own warehouse (OW) with a fixed capacity. If the capacity increases, then the stocks will be stored in rented warehouse (RW). "Quadratic demand based inventory model with shortage and two storage capacities system" by Malik A.k., Dipak Chakraborty and Satish Kumar[7] gives a detail study about on Quadratic demand where the permissible delay in payments is built over that. Liang and Zhou had given a two-warehouse inventory model for deteriorating goods under conditionally permissible delay in payment with constant demand. H.L. Yang[3] has developed a two-warehouse model by partially backlogging the inventory for deteriorating items under the inflation. Goyal was the first one to establish an economic order quantity model with a stable demand rate under the condition of a permissible delay in payments. Shah deemed in a stochastic inventory model when delays in payments are permissible. Aggarwal and Jaggi extend the Goyal's model to consider the deteriorating items. Jamal et al further wide spread the Aggarwal and Jaggi's model to allow for short comes. Hwang and Shinn[2] added the pricing strategy to the same model and developed the best price and lot-sizing for a retailer under the condition of a permissible delay in payments. Liao et al anticipated an inventory model with deteriorating goods under inflation when a delay in payment is permissible
Inspired by above ideologies, in this paper we have projected about the quadratic demand with two warehouses inventory model. Here we presume two warehouses method with variable deteriorations in the both the warehouses. Holding cost is a function of time and considered the same of rented warehouse is higher than Own Warehouse. So, the total inventory cost of this system is optimising with an arithmetical example.

Assumptions
- Demand is Quadratic $D(t) = a + bt + ct^2$
- No repair is done in the cycle.
- Only one item is considered.
- The items in the Rented warehouse are consumed first and then items in Own warehouse.
- Both Own warehouse and Rented warehouse has a limited capacity with N and Q units respectively. (Q>N)

- Shortages are not allowed.
- Deterioration rate of items in Own warehouse is greater than the deterioration rate of items in Rented warehouse
- A permissible delay in payment is considered.

Notations

- C_o: Ordering Cost
- C_d: Deterioration Cost
- $\hbar_r$: Holding Cost for Rented Warehouse
- $\hbar_o$: Holding Cost for Own Warehouse
- θ_1: Deterioration Rate of Rented Warehouse
- θ_2: Deterioration Rate of Own Warehouse
- I_c: Interest Charged
- I_e: Interest earned

Mathematical Model

$\frac{dI_R(t)}{dt} = -D(t)$ $\quad 0 \le t \le t_1$ ……………. (1)

$\frac{dI_R(t)}{dt} = -D(t) - \theta_1 I_R(t)$ $\quad t_1 \le t \le t_2$……………... (2)

$\frac{dI_o(t)}{dt} = 0$ $\quad 0 \le t \le t_1$ ……………..(3)

$\frac{dI_o(t)}{dt} = -\theta_2 I_o(t)$ $\quad t_1 \le t \le t_2$ ……………..(4)

$\frac{dI_o(t)}{dt} = -D(t) - \theta_2 I_o(t)$ $\quad t_2 \le t \le T$… ……………(5)

The boundary conditions we have

$$I_R(0) = Q, I_R(t_2) = 0, I_o(0) = N, I_o(t_1) = N, I_o(T) = 0$$

With this boundary condition the equation (1) will become

Then $\quad I_R(t) = -\frac{ct^3}{3} - \frac{bt^2}{2} - at + Q \quad 0 \le t \le t_1$………. (6)

$I_R(t) = \left[a(t_2 - t) + \left(\frac{f_1}{2}\right)(t_2^2 - t^2) + \left(\frac{f_2}{3}\right)(t_2^3 - t^3) + \left(\frac{f_3}{4}\right)(t_2^4 - t^4) + \left(\frac{f_4}{5}\right)(t_2^5 - t^5)\right].e^{-\theta_1 t}$,

$t_1 \le t \le t_2$ …………….(7)

$I_o(t) = N \quad 0 \le t \le t_1$ …………… (8)

$I_o(t) = N.e^{\theta_2(t_1 - t)} \quad t_1 \le t \le t_2$ …………..(9)

$I_o(t) = \left\{a(T - t) + \left(\frac{g_1}{2}\right)(T^2 - t^2) + \left(\frac{g_2}{3}\right)(T^3 - t^3) + \left(\frac{g_3}{4}\right)(T^4 - t^4) + \left(\frac{g_4}{5}\right)(T^5 - t^5)\right\}e^{-\theta_2 t}$

$t_2 \le t \le T$ …………….(10)

Where the inventory is continuous t_1 *and* t_2, we have

At t_1

$Q = \left[at_1 + \frac{bt_1^2}{2} + \frac{ct_1^3}{3}\right] + \left[a(t_2 - t_1) + \left(\frac{f_1}{2}\right)(t_2^2 - t_1^2) + \left(\frac{f_2}{3}\right)(t_2^3 - t_1^3) + \left(\frac{f_3}{4}\right)(t_2^4 - t_1^4) + \left(\frac{f_4}{5}\right)(t_2^5 - t_1^5)\right].e^{-\theta_1 t_1}$ ……………………………. (11)

At t_2

$N = \left\{a(T - t) + \left(\frac{g_1}{2}\right)(T^2 - t^2) + \left(\frac{g_2}{3}\right)(T^3 - t^3) + \left(\frac{g_3}{4}\right)(T^4 - t^4) + \left(\frac{g_4}{5}\right)(T^5 - t^5)\right\}e^{-\theta_2 t_1}$

……………………………..(12)

Where the maximum inventory is

M=Q+N

Ordering Cost

$OC = c_o$ ……………..(13)

Holding Cost for RW

$$C_{1RW} = \hbar_r\left[\int_0^{t_1} I_R(t)dt + \int_{t_1}^{t_2} I_R(t)dt\right]$$

$C_{1RW} =$

$\hbar_r\left[\left(Qt - \frac{at_1^2}{2} - \frac{bt_1^3}{6} - \frac{ct_1^4}{12}\right) + a\left\{\left(\frac{t_1^2+t_2^2-2t_1t_2}{2}\right) - \theta_1\left(\frac{2t_1^3-t_2^3-3t_1^2t_2}{6}\right) + \frac{\theta_1^2}{2}\left(\frac{3t_1^4+t_2^4-4t_1^3t_2}{12}\right)\right\} + f_1\left\{\left(\frac{t_1^3+2t_2^3-3t_1t_2^2}{6}\right) - \theta_1\left(\frac{t_1^4+t_2^4-2t_1^2t_2^2}{8}\right) + \frac{\theta_1^2}{2}\left(\frac{3t_1^5+2t_2^5-5t_1^3t_2}{30}\right)\right\} + f_2\left\{\left(\frac{t_1^4+3t_2^4-4t_1t_2^3}{12}\right) - \theta_1\left(\frac{2t_1^5+3t_2^5-5t_1^2t_2^3}{30}\right) + \frac{\theta_1^2}{2}\left(\frac{t_1^6+t_2^6-2t_1^3t_2^3}{18}\right)\right\} +\right.$

$f_3\left\{\left(\frac{t_1^5+4t_2^5-5t_1t_2^4}{20}\right)-\theta_1\left(\frac{t_1^6+2t_2^6-3t_1^2t_2^4}{24}\right)+\theta_1^2\left(\frac{(3t_1^7+4t_2^7-7t_1^3t_2^4)}{84}\right)\right\}+f_4\left\{\left(\frac{(t_1^6+5t_2^6-6t_1t_2^5)}{30}\right)-\theta_1\left(\frac{2t_1^7+5t_2^7-7t_1^2t_2^5}{70}\right)+\frac{\theta_1^2}{2}\left(\frac{(3t_1^8+5t_2^8-8t_1^3t_2^5)}{120}\right)\right\}\Bigg]$ ……………….(14)

Holding cost for Own Warehouse

$$C_{1OW}=\hbar_o\left[\int_0^{t_1}I_o(t)dt+\int_{t_1}^{t_2}I_o(t)dt+\int_{t_2}^{T}I_o(t)dt\right]$$

$C_{1OW}=\hbar_o\Bigg[\left\{Nt_1+\frac{N}{\theta_2}\left(1-e^{\theta_2(t_2-t_1)}\right)\right\}+a\left\{\left(\frac{T^2+t_2^2-2Tt_2}{2}\right)-\theta_2\left(\frac{2t_2^3+t^3-3t_2^2T}{6}\right)+\frac{\theta_2^2}{2}\left(\frac{3t_2^4+T^4-4t_2^3T}{12}\right)\right\}+g_1\left\{\left(\frac{t_2^3+2T^3-3t_2T^2}{6}\right)-\theta_2\left(\frac{t_2^4+T^4-2t_2^2T^2}{8}\right)+\frac{\theta_2^2}{2}\left(\frac{3t_2^5+2T^5-5t_2^3T^2}{30}\right)\right\}+g_2\left\{\left(\frac{t_2^4+3T^4-4t_2T^3}{12}\right)-\theta_2\left(\frac{2t_2^5+3T^5-5t_2^2T^3}{30}\right)+\frac{\theta_2^2}{2}\left(\frac{t_2^6+T^6-2t_2^3T^3}{18}\right)\right\}+g_3\left\{\left(\frac{t_2^5+4T^5-5t_2T^4}{20}\right)-\theta_2\left(\frac{t_2^6+2T^6-3t_2^2T^4}{24}\right)+\theta_2^2\left(\frac{(3t_2^7+4T^7-7t_2^3T^4)}{84}\right)\right\}+g_4\left\{\left(\frac{(t_2^6+5T^6-6t_2T^5)}{30}\right)-\theta_2\left(\frac{2t_2^7+5T^7-7t_2^2T^5}{70}\right)+\frac{\theta_2^2}{2}\left(\frac{(3t_2^8+5T^8-8t_2^3T^5)}{120}\right)\right\}\Bigg]$ ……………………… (15)

Deterioration cost for Rented Warehouse

$$C_{2RW}=C_d\left(\int_{t_1}^{t_2}\theta_1I_R(t)dt\right)$$

$$\begin{aligned}C_{2RW}=C_d\theta_1\Bigg[&a\left\{\left(\frac{t_1^2+t_2^2-2t_1t_2}{2}\right)-\theta_1\left(\frac{2t_1^3-t_2^3-3t_1^2t_2}{6}\right)+\frac{\theta_1^2}{2}\left(\frac{3t_1^4+t_2^4-4t_1^3t_2}{12}\right)\right\}\\&+f_1\left\{\left(\frac{t_1^3+2t_2^3-3t_1t_2^2}{6}\right)-\theta_1\left(\frac{t_1^4+t_2^4-2t_1^2t_2^2}{8}\right)+\frac{\theta_1^2}{2}\left(\frac{3t_1^5+2t_2^5-5t_1^3t_2}{30}\right)\right\}\\&+f_2\left\{\left(\frac{t_1^4+3t_2^4-4t_1t_2^3}{12}\right)-\theta_1\left(\frac{2t_1^5+3t_2^5-5t_1^2t_2^3}{30}\right)+\frac{\theta_1^2}{2}\left(\frac{t_1^6+t_2^6-2t_1^3t_2^3}{18}\right)\right\}\\&+f_3\left\{\left(\frac{t_1^5+4t_2^5-5t_1t_2^4}{20}\right)-\theta_1\left(\frac{t_1^6+2t_2^6-3t_1^2t_2^4}{24}\right)+\theta_1^2\left(\frac{(3t_1^7+4t_2^7-7t_1^3t_2^4)}{84}\right)\right\}\\&+f_4\left\{\left(\frac{(t_1^6+5t_2^6-6t_1t_2^5)}{30}\right)-\theta_1\left(\frac{2t_1^7+5t_2^7-7t_1^2t_2^5}{70}\right)+\frac{\theta_1^2}{2}\left(\frac{(3t_1^8+5t_2^8-8t_1^3t_2^5)}{120}\right)\right\}\Bigg]\end{aligned}$$

…………………….. (16)

Deterioration Cost for Own Warehouse

$$C_{2OW}=C_d\left[\int_{t_1}^{t_2}\theta_2I_o(t)dt+\int_{t_2}^{T}\theta_2I_o(t)dt\right]$$

$C_{2OW}=$
$C_d\theta_2\Bigg[\left\{\frac{N}{\theta_2}\left(1-e^{\theta_2(t_2-t_1)}\right)\right\}+a\left\{\left(\frac{T^2+t_2^2-2Tt_2}{2}\right)-\theta_2\left(\frac{2t_2^3+t^3-3t_2^2T}{6}\right)+\frac{\theta_2^2}{2}\left(\frac{3t_2^4+T^4-4t_2^3T}{12}\right)\right\}+g_1\left\{\left(\frac{t_2^3+2T^3-3t_2T^2}{6}\right)-\theta_2\left(\frac{t_2^4+T^4-2t_2^2T^2}{8}\right)+\frac{\theta_2^2}{2}\left(\frac{3t_2^5+2T^5-5t_2^3T^2}{30}\right)\right\}+g_2\left\{\left(\frac{t_2^4+3T^4-4t_2T^3}{12}\right)-\theta_2\left(\frac{2t_2^5+3T^5-5t_2^2T^3}{30}\right)+\frac{\theta_2^2}{2}\left(\frac{t_2^6+T^6-2t_2^3T^3}{18}\right)\right\}+g_3\left\{\left(\frac{t_2^5+4T^5-5t_2T^4}{20}\right)-\theta_2\left(\frac{t_2^6+2T^6-3t_2^2T^4}{24}\right)+\theta_2^2\left(\frac{(3t_2^7+4T^7-7t_2^3T^4)}{84}\right)\right\}+g_4\left\{\left(\frac{(t_2^6+5T^6-6t_2T^5)}{30}\right)-\theta_2\left(\frac{2t_2^7+5T^7-7t_2^2T^5}{70}\right)+\frac{\theta_2^2}{2}\left(\frac{(3t_2^8+5T^8-8t_2^3T^5)}{120}\right)\right\}\Bigg]$ …………………..(17)

Interest payable

It is divided into three parts

Part- I: $M\le t_2<T$

$$IC_1=cI_c\left[\int_M^{t_2}I_R(t)dt+\int_M^{t_2}I_O(t)dt+\int_{t_2}^{T}I_O(t)dt\right]$$

$$= cI_c\Bigg[\Bigg[a\left\{\left(\frac{M^2+t_2^2-2Mt_2}{2}\right)-\theta_1\left(\frac{2M^3-t_2^3-3M^2t_2}{6}\right)+\frac{\theta_1^2}{2}\left(\frac{3M^4+t_2^4-4M^3t_2}{12}\right)\right\}$$
$$+f_1\left\{\left(\frac{M^3+2t_2^3-3Mt_2^2}{6}\right)-\theta_1\left(\frac{M^4+t_2^4-2M^2t_2^2}{8}\right)+\frac{\theta_1^2}{2}\left(\frac{3M^5+2t_2^5-5M^3t_2}{30}\right)\right\}$$
$$+f_2\left\{\left(\frac{M^4+3t_2^4-4Mt_2^3}{12}\right)-\theta_1\left(\frac{2M^5+3t_2^5-5M^2t_2^3}{30}\right)+\frac{\theta_1^2}{2}\left(\frac{M^6+t_2^6-2M^3t_2^3}{18}\right)\right\}$$
$$+f_3\left\{\left(\frac{M^5+4t_2^5-5Mt_2^4}{20}\right)-\theta_1\left(\frac{M^6+2t_2^6-3M^2t_2^4}{24}\right)+\theta_1^2\left(\frac{(3M^7+4t_2^7-7M^3t_2^4)}{84}\right)\right\}$$
$$+f_4\left\{\left(\frac{(M^6+5t_2^6-6Mt_2^5)}{30}\right)-\theta_1\left(\frac{2M^7+5t_2^7-7M^2t_2^5}{70}\right)+\frac{\theta_1^2}{2}\left(\frac{(3M^8+5t_2^8-8M^3t_2^5)}{120}\right)\right\}\Bigg]$$
$$+N[t_2-M]+a\left\{\left(\frac{T^2+t_2^2-2Tt_2}{2}\right)-\theta_2\left(\frac{2t_2^3+t^3-3t_2^2T}{6}\right)+\frac{\theta_2^2}{2}\left(\frac{3t_2^4+T^4-4t_2^3T}{12}\right)\right\}$$
$$+g_1\left\{\left(\frac{t_2^3+2T^3-3t_2T^2}{6}\right)-\theta_2\left(\frac{t_2^4+T^4-2t_2^2T^2}{8}\right)+\frac{\theta_2^2}{2}\left(\frac{3t_2^5+2T^5-5t_2^3T^2}{30}\right)\right\}$$
$$+g_2\left\{\left(\frac{t_2^4+3T^4-4t_2T^3}{12}\right)-\theta_2\left(\frac{2t_2^5+3T^5-5t_2^2T^3}{30}\right)+\frac{\theta_2^2}{2}\left(\frac{t_2^6+T^6-2t_2^3T^3}{18}\right)\right\}$$
$$+g_3\left\{\left(\frac{t_2^5+4T^5-5t_2T^4}{20}\right)-\theta_2\left(\frac{t_2^6+2T^6-3t_2^2T^4}{24}\right)+\theta_2^2\left(\frac{(3t_2^7+4T^7-7t_2^3T^4)}{84}\right)\right\}$$
$$+g_4\left\{\left(\frac{(t_2^6+5T^6-6t_2T^5)}{30}\right)-\theta_2\left(\frac{2t_2^7+5T^7-7t_2^2T^5}{70}\right)+\frac{\theta_2^2}{2}\left(\frac{(3t_2^8+5T^8-8t_2^3T^5)}{120}\right)\right\}\Bigg]$$

Part-II: $t_2 < M \le T$

$$IC_2 = cI_c\int_M^T I_0(t)dt$$

$$= cI_c\Bigg[a\left\{\left(\frac{T^2+M^2-2TM}{2}\right)-\theta_2\left(\frac{2M^3+t^3-3M^2T}{6}\right)+\frac{\theta_2^2}{2}\left(\frac{3M^4+T^4-4M^3T}{12}\right)\right\}$$
$$+g_1\left\{\left(\frac{M^3+2T^3-3MT^2}{6}\right)-\theta_2\left(\frac{M^4+T^4-2M^2T^2}{8}\right)+\frac{\theta_2^2}{2}\left(\frac{3M^5+2T^5-5M^3T^2}{30}\right)\right\}$$
$$+g_2\left\{\left(\frac{M^4+3T^4-4MT^3}{12}\right)-\theta_2\left(\frac{2M^5+3T^5-5M^2T^3}{30}\right)+\frac{\theta_2^2}{2}\left(\frac{M^6+T^6-2M^3T^3}{18}\right)\right\}$$
$$+g_3\left\{\left(\frac{M^5+4T^5-5MT^4}{20}\right)-\theta_2\left(\frac{M^6+2T^6-3M^2T^4}{24}\right)+\theta_2^2\left(\frac{(3M^7+4T^7-7M^3T^4)}{84}\right)\right\}$$
$$+g_4\left\{\left(\frac{(M^6+5T^6-6MT^5)}{30}\right)-\theta_2\left(\frac{2M^7+5T^7-7M^2T^5}{70}\right)\right.$$
$$\left.+\frac{\theta_2^2}{2}\left(\frac{(3M^8+5T^8-8M^3T^5)}{120}\right)\right\}\Bigg]$$

Part-III: $M > t$

In this part, retailer need not to pay any amount.

$$IC_3 = 0$$

Interest earned

This is divided into two parts.

Part-I: $M \le T$

$$IE_1 = sI_e\int_0^M t.D(t)dt$$

$$IE_1 = s\left[\frac{6aM^2+4bM^3+3cM^4}{12}\right]$$

Part-II: $M > T$

$$IE_2 = sI_e\int_0^T D(t).t\,dt + D(T)T(M-T)$$

$$IE_2 = sI_e\left\{\frac{T^2}{12}[6a + 4bT + 3cT^2] + (a + bT + cT^2)(MT - T^2)\right\}$$

TOTAL COST

The total cost can be written as

$$TC_1 = \frac{1}{T}\Big\{C_0 + \hbar_r\Big[\left(Qt - \frac{at_1^2}{2} - \frac{bt_1^3}{6} - \frac{ct_1^4}{12}\right)$$

$$+ a\left\{\left(\frac{t_1^2 + t_2^2 - 2t_1t_2}{2}\right) - \theta_1\left(\frac{2t_1^3 - t_2^3 - 3t_1^2t_2}{6}\right) + \frac{\theta_1^2}{2}\left(\frac{3t_1^4 + t_2^4 - 4t_1^3t_2}{12}\right)\right\}$$

$$+ f_1\left\{\left(\frac{t_1^3 + 2t_2^3 - 3t_1t_2^2}{6}\right) - \theta_1\left(\frac{t_1^4 + t_2^4 - 2t_1^2t_2^2}{8}\right) + \frac{\theta_1^2}{2}\left(\frac{3t_1^5 + 2t_2^5 - 5t_1^3t_2}{30}\right)\right\}$$

$$+ f_2\left\{\left(\frac{t_1^4 + 3t_2^4 - 4t_1t_2^3}{12}\right) - \theta_1\left(\frac{2t_1^5 + 3t_2^5 - 5t_1^2t_2^3}{30}\right) + \frac{\theta_1^2}{2}\left(\frac{t_1^6 + t_2^6 - 2t_1^3t_2^3}{18}\right)\right\}$$

$$+ f_3\left\{\left(\frac{t_1^5 + 4t_2^5 - 5t_1t_2^4}{20}\right) - \theta_1\left(\frac{t_1^6 + 2t_2^6 - 3t_1^2t_2^4}{24}\right) + \theta_1^2\left(\frac{(3t_1^7 + 4t_2^7 - 7t_1^3t_2^4)}{84}\right)\right\}$$

$$+ f_4\left\{\left(\frac{(t_1^6 + 5t_2^6 - 6t_1t_2^5)}{30}\right) - \theta_1\left(\frac{2t_1^7 + 5t_2^7 - 7t_1^2t_2^5}{70}\right) + \frac{\theta_1^2}{2}\left(\frac{(3t_1^8 + 5t_2^8 - 8t_1^3t_2^5)}{120}\right)\right\}\Big]$$

$$+ \hbar_o\Big[\left\{Nt_1 + \frac{N}{\theta_2}\left(1 - e^{\theta_2(t_2 - t_1)}\right)\right\}$$

$$+ a\left\{\left(\frac{T^2 + t_2^2 - 2Tt_2}{2}\right) - \theta_2\left(\frac{2t_2^3 + t^3 - 3t_2^2T}{6}\right) + \frac{\theta_2^2}{2}\left(\frac{3t_2^4 + T^4 - 4t_2^3T}{12}\right)\right\}$$

$$+ g_1\left\{\left(\frac{t_2^3 + 2T^3 - 3t_2T^2}{6}\right) - \theta_2\left(\frac{t_2^4 + T^4 - 2t_2^2T^2}{8}\right) + \frac{\theta_2^2}{2}\left(\frac{3t_2^5 + 2T^5 - 5t_2^3T^2}{30}\right)\right\}$$

$$+ g_2\left\{\left(\frac{t_2^4 + 3T^4 - 4t_2T^3}{12}\right) - \theta_2\left(\frac{2t_2^5 + 3T^5 - 5t_2^2T^3}{30}\right) + \frac{\theta_2^2}{2}\left(\frac{t_2^6 + T^6 - 2t_2^3T^3}{18}\right)\right\}$$

$$+ g_3\left\{\left(\frac{t_2^5 + 4T^5 - 5t_2T^4}{20}\right) - \theta_2\left(\frac{t_2^6 + 2T^6 - 3t_2^2T^4}{24}\right) + \theta_2^2\left(\frac{(3t_2^7 + 4T^7 - 7t_2^3T^4)}{84}\right)\right\}$$

$$+ g_4\left\{\left(\frac{(t_2^6 + 5T^6 - 6t_2T^5)}{30}\right) - \theta_2\left(\frac{2t_2^7 + 5T^7 - 7t_2^2T^5}{70}\right) + \frac{\theta_2^2}{2}\left(\frac{(3t_2^8 + 5T^8 - 8t_2^3T^5)}{120}\right)\right\}\Big]$$

$$+ C_d\theta_1\Big[a\left\{\left(\frac{t_1^2 + t_2^2 - 2t_1t_2}{2}\right) - \theta_1\left(\frac{2t_1^3 - t_2^3 - 3t_1^2t_2}{6}\right) + \frac{\theta_1^2}{2}\left(\frac{3t_1^4 + t_2^4 - 4t_1^3t_2}{12}\right)\right\}$$

$$+ f_1\left\{\left(\frac{t_1^3 + 2t_2^3 - 3t_1t_2^2}{6}\right) - \theta_1\left(\frac{t_1^4 + t_2^4 - 2t_1^2t_2^2}{8}\right) + \frac{\theta_1^2}{2}\left(\frac{3t_1^5 + 2t_2^5 - 5t_1^3t_2}{30}\right)\right\}$$

$$+ f_2\left\{\left(\frac{t_1^4 + 3t_2^4 - 4t_1t_2^3}{12}\right) - \theta_1\left(\frac{2t_1^5 + 3t_2^5 - 5t_1^2t_2^3}{30}\right) + \frac{\theta_1^2}{2}\left(\frac{t_1^6 + t_2^6 - 2t_1^3t_2^3}{18}\right)\right\}$$

$$+ f_3\left\{\left(\frac{t_1^5 + 4t_2^5 - 5t_1t_2^4}{20}\right) - \theta_1\left(\frac{t_1^6 + 2t_2^6 - 3t_1^2t_2^4}{24}\right) + \theta_1^2\left(\frac{(3t_1^7 + 4t_2^7 - 7t_1^3t_2^4)}{84}\right)\right\}$$

$$+ f_4\left\{\left(\frac{(t_1^6 + 5t_2^6 - 6t_1t_2^5)}{30}\right) - \theta_1\left(\frac{2t_1^7 + 5t_2^7 - 7t_1^2t_2^5}{70}\right) + \frac{\theta_1^2}{2}\left(\frac{(3t_1^8 + 5t_2^8 - 8t_1^3t_2^5)}{120}\right)\right\}\Big]$$

$$+ C_d\theta_2\Big[\left\{\frac{N}{\theta_2}\left(1 - e^{\theta_2(t_2 - t_1)}\right)\right\}$$

$$+ a\left\{\left(\frac{T^2 + t_2^2 - 2Tt_2}{2}\right) - \theta_2\left(\frac{2t_2^3 + t^3 - 3t_2^2T}{6}\right) + \frac{\theta_2^2}{2}\left(\frac{3t_2^4 + T^4 - 4t_2^3T}{12}\right)\right\}$$

$$+ g_1\left\{\left(\frac{t_2^3 + 2T^3 - 3t_2T^2}{6}\right) - \theta_2\left(\frac{t_2^4 + T^4 - 2t_2^2T^2}{8}\right) + \frac{\theta_2^2}{2}\left(\frac{3t_2^5 + 2T^5 - 5t_2^3T^2}{30}\right)\right\}$$

$$+ g_2\left\{\left(\frac{t_2^4 + 3T^4 - 4t_2T^3}{12}\right) - \theta_2\left(\frac{2t_2^5 + 3T^5 - 5t_2^2T^3}{30}\right) + \frac{\theta_2^2}{2}\left(\frac{t_2^6 + T^6 - 2t_2^3T^3}{18}\right)\right\}$$

$$+ g_3\left\{\left(\frac{t_2^5 + 4T^5 - 5t_2T^4}{20}\right) - \theta_2\left(\frac{t_2^6 + 2T^6 - 3t_2^2T^4}{24}\right) + \theta_2^2\left(\frac{(3t_2^7 + 4T^7 - 7t_2^3T^4)}{84}\right)\right\}$$

$$+ g_4\left\{\left(\frac{(t_2^6 + 5T^6 - 6t_2T^5)}{30}\right) - \theta_2\left(\frac{2t_2^7 + 5T^7 - 7t_2^2T^5}{70}\right) + \frac{\theta_2^2}{2}\left(\frac{(3t_2^8 + 5T^8 - 8t_2^3T^5)}{120}\right)\right\}\Big]$$

$$+ cI_c\Big[\Big[a\left\{\left(\frac{M^2 + t_2^2 - 2Mt_2}{2}\right) - \theta_1\left(\frac{2M^3 - t_2^3 - 3M^2t_2}{6}\right) + \frac{\theta_1^2}{2}\left(\frac{3M^4 + t_2^4 - 4M^3t_2}{12}\right)\right\}$$

$$+ f_1\left\{\left(\frac{M^3 + 2t_2^3 - 3Mt_2^2}{6}\right) - \theta_1\left(\frac{M^4 + t_2^4 - 2M^2t_2^2}{8}\right) + \frac{\theta_1^2}{2}\left(\frac{3M^5 + 2t_2^5 - 5M^3t_2}{30}\right)\right\}$$

$$+ f_2\left\{\left(\frac{M^4+3t_2^4-4Mt_2^3}{12}\right)-\theta_1\left(\frac{2M^5+3t_2^5-5M^2t_2^3}{30}\right)+\frac{\theta_1^2}{2}\left(\frac{M^6+t_2^6-2M^3t_2^3}{18}\right)\right\}$$
$$+ f_3\left\{\left(\frac{M^5+4t_2^5-5Mt_2^4}{20}\right)-\theta_1\left(\frac{M^6+2t_2^6-3M^2t_2^4}{24}\right)+\theta_1^2\left(\frac{(3M^7+4t_2^7-7M^3t_2^4)}{84}\right)\right\}$$
$$+ f_4\left\{\left(\frac{(M^6+5t_2^6-6Mt_2^5)}{30}\right)-\theta_1\left(\frac{2M^7+5t_2^7-7M^2t_2^5}{70}\right)+\frac{\theta_1^2}{2}\left(\frac{(3M^8+5t_2^8-8M^3t_2^5)}{120}\right)\right\}\Bigg]$$
$$+ N[t_2-M]+a\left\{\left(\frac{T^2+t_2^2-2Tt_2}{2}\right)-\theta_2\left(\frac{2t_2^3+t^3-3t_2^2T}{6}\right)+\frac{\theta_2^2}{2}\left(\frac{3t_2^4+T^4-4t_2^3T}{12}\right)\right\}$$
$$+ g_1\left\{\left(\frac{t_2^3+2T^3-3t_2T^2}{6}\right)-\theta_2\left(\frac{t_2^4+T^4-2t_2^2T^2}{8}\right)+\frac{\theta_2^2}{2}\left(\frac{3t_2^5+2T^5-5t_2^3T^2}{30}\right)\right\}$$
$$+ g_2\left\{\left(\frac{t_2^4+3T^4-4t_2T^3}{12}\right)-\theta_2\left(\frac{2t_2^5+3T^5-5t_2^2T^3}{30}\right)+\frac{\theta_2^2}{2}\left(\frac{t_2^6+T^6-2t_2^3T^3}{18}\right)\right\}$$
$$+ g_3\left\{\left(\frac{t_2^5+4T^5-5t_2T^4}{20}\right)-\theta_2\left(\frac{t_2^6+2T^6-3t_2^2T^4}{24}\right)+\theta_2^2\left(\frac{(3t_2^7+4T^7-7t_2^3T^4)}{84}\right)\right\}$$
$$+ g_4\left\{\left(\frac{(t_2^6+5T^6-6t_2T^5)}{30}\right)-\theta_2\left(\frac{2t_2^7+5T^7-7t_2^2T^5}{70}\right)+\frac{\theta_2^2}{2}\left(\frac{(3t_2^8+5T^8-8t_2^3T^5)}{120}\right)\right\}\Bigg]\Bigg\}$$

$$TC_2=\frac{1}{T}\Bigg\{C_0+\hbar_r\Bigg[\left(Qt-\frac{at_1^2}{2}-\frac{bt_1^3}{6}-\frac{ct_1^4}{12}\right)$$
$$+ a\left\{\left(\frac{t_1^2+t_2^2-2t_1t_2}{2}\right)-\theta_1\left(\frac{2t_1^3-t_2^3-3t_1^2t_2}{6}\right)+\frac{\theta_1^2}{2}\left(\frac{3t_1^4+t_2^4-4t_1^3t_2}{12}\right)\right\}$$
$$+ f_1\left\{\left(\frac{t_1^3+2t_2^3-3t_1t_2^2}{6}\right)-\theta_1\left(\frac{t_1^4+t_2^4-2t_1^2t_2^2}{8}\right)+\frac{\theta_1^2}{2}\left(\frac{3t_1^5+2t_2^5-5t_1^3t_2}{30}\right)\right\}$$
$$+ f_2\left\{\left(\frac{t_1^4+3t_2^4-4t_1t_2^3}{12}\right)-\theta_1\left(\frac{2t_1^5+3t_2^5-5t_1^2t_2^3}{30}\right)+\frac{\theta_1^2}{2}\left(\frac{t_1^6+t_2^6-2t_1^3t_2^3}{18}\right)\right\}$$
$$+ f_3\left\{\left(\frac{t_1^5+4t_2^5-5t_1t_2^4}{20}\right)-\theta_1\left(\frac{t_1^6+2t_2^6-3t_1^2t_2^4}{24}\right)+\theta_1^2\left(\frac{(3t_1^7+4t_2^7-7t_1^3t_2^4)}{84}\right)\right\}$$
$$+ f_4\left\{\left(\frac{(t_1^6+5t_2^6-6t_1t_2^5)}{30}\right)-\theta_1\left(\frac{2t_1^7+5t_2^7-7t_1^2t_2^5}{70}\right)+\frac{\theta_1^2}{2}\left(\frac{(3t_1^8+5t_2^8-8t_1^3t_2^5)}{120}\right)\right\}\Bigg]$$
$$+ \hbar_o\Bigg[\left\{Nt_1+\frac{N}{\theta_2}\left(1-e^{\theta_2(t_2-t_1)}\right)\right\}$$
$$+ a\left\{\left(\frac{T^2+t_2^2-2Tt_2}{2}\right)-\theta_2\left(\frac{2t_2^3+t^3-3t_2^2T}{6}\right)+\frac{\theta_2^2}{2}\left(\frac{3t_2^4+T^4-4t_2^3T}{12}\right)\right\}$$
$$+ g_1\left\{\left(\frac{t_2^3+2T^3-3t_2T^2}{6}\right)-\theta_2\left(\frac{t_2^4+T^4-2t_2^2T^2}{8}\right)+\frac{\theta_2^2}{2}\left(\frac{3t_2^5+2T^5-5t_2^3T^2}{30}\right)\right\}$$
$$+ g_2\left\{\left(\frac{t_2^4+3T^4-4t_2T^3}{12}\right)-\theta_2\left(\frac{2t_2^5+3T^5-5t_2^2T^3}{30}\right)+\frac{\theta_2^2}{2}\left(\frac{t_2^6+T^6-2t_2^3T^3}{18}\right)\right\}$$
$$+ g_3\left\{\left(\frac{t_2^5+4T^5-5t_2T^4}{20}\right)-\theta_2\left(\frac{t_2^6+2T^6-3t_2^2T^4}{24}\right)+\theta_2^2\left(\frac{(3t_2^7+4T^7-7t_2^3T^4)}{84}\right)\right\}$$
$$+ g_4\left\{\left(\frac{(t_2^6+5T^6-6t_2T^5)}{30}\right)-\theta_2\left(\frac{2t_2^7+5T^7-7t_2^2T^5}{70}\right)+\frac{\theta_2^2}{2}\left(\frac{(3t_2^8+5T^8-8t_2^3T^5)}{120}\right)\right\}\Bigg]$$
$$+ C_d\theta_1\Bigg[a\left\{\left(\frac{t_1^2+t_2^2-2t_1t_2}{2}\right)-\theta_1\left(\frac{2t_1^3-t_2^3-3t_1^2t_2}{6}\right)+\frac{\theta_1^2}{2}\left(\frac{3t_1^4+t_2^4-4t_1^3t_2}{12}\right)\right\}$$
$$+ f_1\left\{\left(\frac{t_1^3+2t_2^3-3t_1t_2^2}{6}\right)-\theta_1\left(\frac{t_1^4+t_2^4-2t_1^2t_2^2}{8}\right)+\frac{\theta_1^2}{2}\left(\frac{3t_1^5+2t_2^5-5t_1^3t_2}{30}\right)\right\}$$
$$+ f_2\left\{\left(\frac{t_1^4+3t_2^4-4t_1t_2^3}{12}\right)-\theta_1\left(\frac{2t_1^5+3t_2^5-5t_1^2t_2^3}{30}\right)+\frac{\theta_1^2}{2}\left(\frac{t_1^6+t_2^6-2t_1^3t_2^3}{18}\right)\right\}$$
$$+ f_3\left\{\left(\frac{t_1^5+4t_2^5-5t_1t_2^4}{20}\right)-\theta_1\left(\frac{t_1^6+2t_2^6-3t_1^2t_2^4}{24}\right)+\theta_1^2\left(\frac{(3t_1^7+4t_2^7-7t_1^3t_2^4)}{84}\right)\right\}$$
$$+ f_4\left\{\left(\frac{(t_1^6+5t_2^6-6t_1t_2^5)}{30}\right)-\theta_1\left(\frac{2t_1^7+5t_2^7-7t_1^2t_2^5}{70}\right)+\frac{\theta_1^2}{2}\left(\frac{(3t_1^8+5t_2^8-8t_1^3t_2^5)}{120}\right)\right\}\Bigg]$$

$$+ C_d\theta_2\left[\left\{\frac{N}{\theta_2}\left(1-e^{\theta_2(t_2-t_1)}\right)\right\}\right.$$
$$+ a\left\{\left(\frac{T^2+t_2^2-2Tt_2}{2}\right)-\theta_2\left(\frac{2t_2^3+t^3-3t_2^2T}{6}\right)+\frac{\theta_2^2}{2}\left(\frac{3t_2^4+T^4-4t_2^3T}{12}\right)\right\}$$
$$+ g_1\left\{\left(\frac{t_2^3+2T^3-3t_2T^2}{6}\right)-\theta_2\left(\frac{t_2^4+T^4-2t_2^2T^2}{8}\right)+\frac{\theta_2^2}{2}\left(\frac{3t_2^5+2T^5-5t_2^3T^2}{30}\right)\right\}$$
$$+ g_2\left\{\left(\frac{t_2^4+3T^4-4t_2T^3}{12}\right)-\theta_2\left(\frac{2t_2^5+3T^5-5t_2^2T^3}{30}\right)+\frac{\theta_2^2}{2}\left(\frac{t_2^6+T^6-2t_2^3T^3}{18}\right)\right\}$$
$$+ g_3\left\{\left(\frac{t_2^5+4T^5-5t_2T^4}{20}\right)-\theta_2\left(\frac{t_2^6+2T^6-3t_2^2T^4}{24}\right)+\theta_2^2\left(\frac{(3t_2^7+4T^7-7t_2^3T^4)}{84}\right)\right\}$$
$$\left.+ g_4\left\{\left(\frac{(t_2^6+5T^6-6t_2T^5)}{30}\right)-\theta_2\left(\frac{2t_2^7+5T^7-7t_2^2T^5}{70}\right)+\frac{\theta_2^2}{2}\left(\frac{(3t_2^8+5T^8-8t_2^3T^5)}{120}\right)\right\}\right]$$
$$+ cI_c\left[a\left\{\left(\frac{T^2+M^2-2TM}{2}\right)-\theta_2\left(\frac{2M^3+t^3-3M^2T}{6}\right)+\frac{\theta_2^2}{2}\left(\frac{3M^4+T^4-4M^3T}{12}\right)\right\}\right.$$
$$+ g_1\left\{\left(\frac{M^3+2T^3-3MT^2}{6}\right)-\theta_2\left(\frac{M^4+T^4-2M^2T^2}{8}\right)+\frac{\theta_2^2}{2}\left(\frac{3M^5+2T^5-5M^3T^2}{30}\right)\right\}$$

$$+ g_2\left\{\left(\frac{M^4+3T^4-4MT^3}{12}\right)-\theta_2\left(\frac{2M^5+3T^5-5M^2T^3}{30}\right)+\frac{\theta_2^2}{2}\left(\frac{M^6+T^6-2M^3T^3}{18}\right)\right\}$$
$$+ g_3\left\{\left(\frac{M^5+4T^5-5MT^4}{20}\right)-\theta_2\left(\frac{M^6+2T^6-3M^2T^4}{24}\right)+\theta_2^2\left(\frac{(3M^7+4T^7-7M^3T^4)}{84}\right)\right\}$$
$$+ g_4\left\{\left(\frac{(M^6+5T^6-6MT^5)}{30}\right)-\theta_2\left(\frac{2M^7+5T^7-7M^2T^5}{70}\right)\right.$$
$$\left.\left.+\frac{\theta_2^2}{2}\left(\frac{(3M^8+5T^8-8M^3T^5)}{120}\right)\right\}\right]-sI_e\left[\frac{6aM^2+4bM^3+3cM^4}{12}\right]\Bigg\}$$

$$TC_3=\frac{1}{T}\Bigg\{C_0+h_r\left[\left(Qt-\frac{at_1^2}{2}-\frac{bt_1^3}{6}-\frac{ct_1^4}{12}\right)\right.$$
$$+ a\left\{\left(\frac{t_1^2+t_2^2-2t_1t_2}{2}\right)-\theta_1\left(\frac{2t_1^3-t_2^3-3t_1^2t_2}{6}\right)+\frac{\theta_1^2}{2}\left(\frac{3t_1^4+t_2^4-4t_1^3t_2}{12}\right)\right\}$$
$$+ f_1\left\{\left(\frac{t_1^3+2t_2^3-3t_1t_2^2}{6}\right)-\theta_1\left(\frac{t_1^4+t_2^4-2t_1^2t_2^2}{8}\right)+\frac{\theta_1^2}{2}\left(\frac{3t_1^5+2t_2^5-5t_1^3t_2}{30}\right)\right\}$$
$$+ f_2\left\{\left(\frac{t_1^4+3t_2^4-4t_1t_2^3}{12}\right)-\theta_1\left(\frac{2t_1^5+3t_2^5-5t_1^2t_2^3}{30}\right)+\frac{\theta_1^2}{2}\left(\frac{t_1^6+t_2^6-2t_1^3t_2^3}{18}\right)\right\}$$
$$+ f_3\left\{\left(\frac{t_1^5+4t_2^5-5t_1t_2^4}{20}\right)-\theta_1\left(\frac{t_1^6+2t_2^6-3t_1^2t_2^4}{24}\right)+\theta_1^2\left(\frac{(3t_1^7+4t_2^7-7t_1^3t_2^4)}{84}\right)\right\}$$
$$\left.+ f_4\left\{\left(\frac{(t_1^6+5t_2^6-6t_1t_2^5)}{30}\right)-\theta_1\left(\frac{2t_1^7+5t_2^7-7t_1^2t_2^5}{70}\right)+\frac{\theta_1^2}{2}\left(\frac{(3t_1^8+5t_2^8-8t_1^3t_2^5)}{120}\right)\right\}\right]$$
$$+ h_o\left[\left\{Nt_1+\frac{N}{\theta_2}\left(1-e^{\theta_2(t_2-t_1)}\right)\right\}\right.$$
$$+ a\left\{\left(\frac{T^2+t_2^2-2Tt_2}{2}\right)-\theta_2\left(\frac{2t_2^3+t^3-3t_2^2T}{6}\right)+\frac{\theta_2^2}{2}\left(\frac{3t_2^4+T^4-4t_2^3T}{12}\right)\right\}$$
$$+ g_1\left\{\left(\frac{t_2^3+2T^3-3t_2T^2}{6}\right)-\theta_2\left(\frac{t_2^4+T^4-2t_2^2T^2}{8}\right)+\frac{\theta_2^2}{2}\left(\frac{3t_2^5+2T^5-5t_2^3T^2}{30}\right)\right\}$$
$$+ g_2\left\{\left(\frac{t_2^4+3T^4-4t_2T^3}{12}\right)-\theta_2\left(\frac{2t_2^5+3T^5-5t_2^2T^3}{30}\right)+\frac{\theta_2^2}{2}\left(\frac{t_2^6+T^6-2t_2^3T^3}{18}\right)\right\}$$
$$+ g_3\left\{\left(\frac{t_2^5+4T^5-5t_2T^4}{20}\right)-\theta_2\left(\frac{t_2^6+2T^6-3t_2^2T^4}{24}\right)+\theta_2^2\left(\frac{(3t_2^7+4T^7-7t_2^3T^4)}{84}\right)\right\}$$
$$\left.+ g_4\left\{\left(\frac{(t_2^6+5T^6-6t_2T^5)}{30}\right)-\theta_2\left(\frac{2t_2^7+5T^7-7t_2^2T^5}{70}\right)+\frac{\theta_2^2}{2}\left(\frac{(3t_2^8+5T^8-8t_2^3T^5)}{120}\right)\right\}\right]$$
$$+ C_d\theta_1\left[a\left\{\left(\frac{t_1^2+t_2^2-2t_1t_2}{2}\right)-\theta_1\left(\frac{2t_1^3-t_2^3-3t_1^2t_2}{6}\right)+\frac{\theta_1^2}{2}\left(\frac{3t_1^4+t_2^4-4t_1^3t_2}{12}\right)\right\}\right.$$

$$+ f_1\left\{\left(\frac{t_1^3+2t_2^3-3t_1t_2^2}{6}\right)-\theta_1\left(\frac{t_1^4+t_2^4-2t_1^2t_2^2}{8}\right)+\frac{\theta_1^2}{2}\left(\frac{3t_1^5+2t_2^5-5t_1^3t_2}{30}\right)\right\}$$
$$+ f_2\left\{\left(\frac{t_1^4+3t_2^4-4t_1t_2^3}{12}\right)-\theta_1\left(\frac{2t_1^5+3t_2^5-5t_1^2t_2^3}{30}\right)+\frac{\theta_1^2}{2}\left(\frac{t_1^6+t_2^6-2t_1^3t_2^3}{18}\right)\right\}$$
$$+ f_3\left\{\left(\frac{t_1^5+4t_2^5-5t_1t_2^4}{20}\right)-\theta_1\left(\frac{t_1^6+2t_2^6-3t_1^2t_2^4}{24}\right)+\theta_1^2\left(\frac{(3t_1^7+4t_2^7-7t_1^3t_2^4)}{84}\right)\right\}$$
$$+ f_4\left\{\left(\frac{(t_1^6+5t_2^6-6t_1t_2^5)}{30}\right)-\theta_1\left(\frac{2t_1^7+5t_2^7-7t_1^2t_2^5}{70}\right)+\frac{\theta_1^2}{2}\left(\frac{(3t_1^8+5t_2^8-8t_1^3t_2^5)}{120}\right)\right\}\Bigg]$$
$$+ C_d\theta_2\left[\left\{\frac{N}{\theta_2}\left(1-e^{\theta_2(t_2-t_1)}\right)\right\}\right.$$
$$+ a\left\{\left(\frac{T^2+t_2^2-2Tt_2}{2}\right)-\theta_2\left(\frac{2t_2^3+t^3-3t_2^2T}{6}\right)+\frac{\theta_2^2}{2}\left(\frac{3t_2^4+T^4-4t_2^3T}{12}\right)\right\}$$
$$+ g_1\left\{\left(\frac{t_2^3+2T^3-3t_2T^2}{6}\right)-\theta_2\left(\frac{t_2^4+T^4-2t_2^2T^2}{8}\right)+\frac{\theta_2^2}{2}\left(\frac{3t_2^5+2T^5-5t_2^3T^2}{30}\right)\right\}$$
$$+ g_2\left\{\left(\frac{t_2^4+3T^4-4t_2T^3}{12}\right)-\theta_2\left(\frac{2t_2^5+3T^5-5t_2^2T^3}{30}\right)+\frac{\theta_2^2}{2}\left(\frac{t_2^6+T^6-2t_2^3T^3}{18}\right)\right\}$$
$$+ g_3\left\{\left(\frac{t_2^5+4T^5-5t_2T^4}{20}\right)-\theta_2\left(\frac{t_2^6+2T^6-3t_2^2T^4}{24}\right)+\theta_2^2\left(\frac{(3t_2^7+4T^7-7t_2^3T^4)}{84}\right)\right\}$$
$$+ g_4\left\{\left(\frac{(t_2^6+5T^6-6t_2T^5)}{30}\right)-\theta_2\left(\frac{2t_2^7+5T^7-7t_2^2T^5}{70}\right)+\frac{\theta_2^2}{2}\left(\frac{(3t_2^8+5T^8-8t_2^3T^5)}{120}\right)\right\}\Bigg]$$
$$- sI_e\left\{\frac{T^2}{12}[6a+4bT+3cT^2]+(a+bT+cT^2)(MT-T^2)\right\}\Bigg\}$$

Our ultimate aim is to minimize the Total cost.By finding the optimal values of t_2 *and* T , we can find the minimum value of Total cost. Using the Hessian matrix, TC_1 is the optimal value if it satisfies the below conditions

$$\frac{\partial^2 TC_1}{\partial t_2^2}>0,\frac{\partial^2 TC_1}{\partial T^2}>0,\frac{\partial^2 TC_1}{\partial t_2^2}\frac{\partial^2 TC_1}{\partial T^2}-\frac{\partial^2 TC_1}{\partial t_2\partial T}\frac{\partial^2 TC_1}{\partial T\partial t_2}>0$$

In a similar way we have to find the optimal values of TC_2*and* TC_3,where $TC = \min(TC_1, TC_2, TC_3)$

Numerical Example

We consider an example $a = 1000$, $b = 0.4$, $c = 0.04$, $c_o = 1000$, $\hbar_o = 0.07$, $\hbar_r = 0.06$, $\theta_1 = 0.05$, $\theta_2 = 0.06$, $c_d = 0.044$, $I_c = 0.15$, $c_c = 10$, $S = 20$ and $I_e = 0.13$ by using this model we solve the above example and the minimum total cost is $TC = 2014$, where $t_1 = 2.2$ days, $t_2 = 18.5$days and $T = 28.8$days.

CONCLUSION

In general, retailers have to face many expenses for purchasing, maintaining the product and for the warehouse militance too. When giving a permissible delay in payments,it will help them to reduce the total cost,which about the above model helps them with clear example and shows the value of t_1, t_2 *and* T at which the total cost is minimum and the optimal value. This paper can be extended by having shortages or have more than one item in warehouse.

REFERENCE

[1] Gupta, C. B., Malik, A. K. and Singh, S. R. (2010). A Two Warehouse Inventory Model for Deteriorating Items with Demand Dependent Production, Ganita Sandesh, Vol. 24, No. 1, 55-62.

[2] H. Hwang, S.W. Shinn, Retailer's pricing and lot sizing policy for exponentially deteriorating products under the conditions of permissible delay in Payments, Computers & Operations Research 24 (1997) 539–547.

[3] H.L.Yang, Two-warehouse partial backlogging inventory models for deteriorating items under inflation, International Journal of Production Economics 103 (2006) 362–370.

[4] Kumar Aadarsh, Singh Amardeep, Bansal Kapil Kumar (2016) Two warehouse inventory model with ramp type demand, shortages under inflationary environment , IOSR Journal of Mathematics (IOSR-JM) , 12(3), 06-17.

[5] Lee, C.C. and Hsu, S. L., (2009). A two-warehouse production model for deteriorating inventory items with time-dependent demands, European Journal of Operational Research, 194, 700-710.

[6] Malik, A. K., Singh, S. R. and Gupta, C. B. (2009).Two warehouse inventory model with exponential demand and time-dependent backlogging rate for deteriorating items, Ganita Sandesh Vol. 23, No. 2, 121-130.

[7] Malik, A. K., Dipak Chakraborty, Sathish Kumar (2017) Quadratic Demand based Inventory Model with Shortage and Two Storage Capacities System, Research J. Engineering and Tech.2017;8(3):213-218.

[8] Sarkar, S., Sana, S.S. and Chaudhuri, K. (2010). A finite replenishment model with increasing demand under inflation, Int. J. Math. Oper. Res., 2(3), 347–385.

[9] Seth, B. K., Sarkar, B., Goswami A., (2012). A two-warehouse inventory model with increasing demand and time varying deterioration. Scientia Iranica, E 19, 1969-1977.

[10] Sharma, M. K. and Bansal K. K., (2017). Inventory Model for Non-Instantaneous Decaying Items with Learning Effect under Partial Backlogging and Inflation, Global Journal of Pure and Applied Mathematics,13(6), 1999-2008

[11] Singh, S.R., Malik, A.K., and Gupta, S. K. (2011). Two Warehouses Inventory Model with Partial Backordering and Multi-Variate Demand under Inflation, International Journal of Operations Research and Optimization, Vol. 2, No. 2, 371-384.

[12] Singh, S.R., Pinky Saxena (2013) A Two- warehouse Production Inventory model with Variable Demand and Permissible Delay in Payment under Inflation, International Journal of Soft Computing and Engineering ISSN:2231-2307, vol-3.

[13] Vashisth, V., Soni, R., Jakhar, R., Sihag, D., and Malik, A. K. (2016). A Two Warehouse Inventory Model with Quadratic Decreasing Demand and Time Dependent Holding Cost, AIP Conference Proceedings 1715, 020066; doi: 10.1063/1.4942748.

[14] Y. Yang, F. Zhou, A two-warehouse inventory model for deteriorating items under conditionally permissible delay in payment , Applied Mathematical modeling 35 (2011) 2221– 2231.