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Non-polynomial fractal quintic spline method for nonlinear boundary-value problems

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Abstract

In this study, we have proposed second, fourth and sixth order convergent numerical techniques for approximating linear and non-linear boundary value problems of second order with the help of fractal non-polynomial spline function. We have discussed the convergence analysis and error bound for sixth order method to prove the theoretical aspects of the presented method. Numerical problems are experimented to validate the theoretical results. Comparison with fractal polynomial and few other existing methods leads us to the conclusion that the proposed technique is more efficient.

Keywords: Difference equations, fractal non-polynomial spline, quasilinearisation, convergence analysis, truncation error.

Mathematics Subject Classification: 28A80, 65D07, 34B15

1. Introduction

With the help of fractal non-polynomial spline, we have developed numerical techniques to find the approximate solution of boundary value problems(BVPs) of the type:

$$\begin{cases} w_{tt}(t) + \mathtt{p}(t)w(t) = \mathtt{f}(t), & t \in (0,\ 1), \\ w(0) = \sigma_0, & w(1) = \sigma_1, \end{cases} \tag{1.1}$$

and

$$\begin{cases} w_{tt}(t) + \mathtt{F}(t, w(t)) = 0, & t \in (0,\ 1), \\ w(0) = \sigma_0, & w(1) = \sigma_1, \end{cases} \tag{1.2}$$

where σ_0 and σ_1 are constants. In (1.1), $\mathtt{p}(t)$ and $\mathtt{f}(t)$ are continuous functions in closed interval $I = [0,\ 1]$. For random choices of $\mathtt{p}$ and $\mathtt{f}$, exact solution of these BVPs cannot be find.

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Therefore we approach numerical methods to get approximate solution of (1.1). In (1.2), presume that for $(t,w(t)) \in \mathtt{D} = \{0 \leq t \leq 1,\ -\infty < w(t) < \infty\}$, $\mathtt{F}$ and $\frac{\partial \mathtt{F}}{\partial w}$ are continuous. We know that (1.2) admits unique solution, if $\sup_{(t,w(t))\in D} \frac{\partial \mathtt{F}}{\partial w} < \pi^2$,[22]. Here we assume that $\frac{\partial \mathtt{F}}{\partial w} \leq 0$ on D and $\frac{\partial \mathtt{F}}{\partial w} < 0$ on $D^* = \{0 < t < 1,\ -\infty < w(t) < \infty\}$. The notation w_{tt} symbolizes second derivative of w with respect to t.

Various authors have used different techniques to find numerical solution of linear as well as non-linear BVPs. Authors in [11] used cubic spline functions to find the approximate solution of nonlinear BVPs. Few numerical techniques derived by various authors for solving non-linear BVPs are given in [1, 2, 8, 14, 23, 27, 28, 32] and fractional differential equations are given in [13, 15, 16, 17, 18, 19, 29, 30].

With the help of quasilinearisation technique[6, 21, 26], the non-linear BVP (1.2) is converted into a system of linear BVPs, which in turn are solved by derived numerical scheme using fractal non-polynomial quintic spline function. A parameter λ called scaling factor is used in fractal spline which is suitably restricted to obtain the approximate solution of the linearized BVPs. Fractal interpolation function was introduced by Barnsley[4] using Iterated function system. Although fractals are difficult to constrain but they are best suitable for generation of various irregular shapes found in nature. It provides the possibility of simulating and describing landscapes precisely with the help of mathematical models. To find the numerical solution of (1.2), Balasubramani et. al.[3] have worked upon fractal quintic polynomial spline functions. In this paper we have worked upon finding the approximate solution using fractal non-polynomial spline functions and observed that the proposed scheme provides better results. The description of paper is as follows:

At the beginning ,we have given a brief description of the presented method which uses fractal non-polynomial quintic spline to get a relation between $w(t)$ and $M(t)$ using continuity conditions. In section 3, we have discussed the truncation error. Thereafter, possible classes of method are discussed in section 4. Then we have discussed the convergence analysis of sixth order method in section 5. Error bounds are carried out. Thereafter, we have given a briefing about finite-difference method and Numerov's method, and experimented four numerical problems to testify the efficacy of proposed method in section 6. Concluding remarks are provided in section 7.

2. Fractal Nonpolynomial spline

Let $0 = t_0 < t_1 < t_2 < \ldots < t_n = 1$ be the partition of the interval $I = [0,1]$ given in (1.1) and (1.2). Let $w(t)$ and $W_{\mathtt{j}}$ denote the analytical and approximate solutions respectively. For $t_{\mathtt{j}} = \mathtt{j}h,\ h = 1/n,\ \mathtt{j} = 0,1,\ldots,n$. Let $M_{\mathtt{j}}$ and $S_{\mathtt{j}}$ denote the approximation corresponding to $w_{tt}(t_{\mathtt{j}})$ and $w_{tttt}(t_{\mathtt{j}})$ respectively.

Concept of Iterated functions system (IFS) is used to develop fractal interpolation functions(FIF). Basic details related to fractal interpolation are provided in [5, 9, 10].

Define $\mathtt{H}_{\mathtt{j}} : I \to I_{\mathtt{j}}$, where $I_{\mathtt{j}} = [t_{\mathtt{j}-1}, t_{\mathtt{j}}]$ such that

$$\mathtt{H}_{\mathtt{j}}(t) = ht + t_{\mathtt{j}-1},\ t \in I.$$

Clearly, $\mathtt{H}_{\mathtt{j}}(t_0) = t_{\mathtt{j}-1}$ and $\mathtt{H}_{\mathtt{j}}(t_n) = t_{\mathtt{j}}$,
and define $\mathbb{F}_{\mathtt{j}} : I \times \mathbb{R} \to \mathbb{R}$ such that

$$\mathbb{F}_{\mathtt{j}}(t,w) = \lambda w + r_{\mathtt{j}}(t),\ (t,w)\ \in\ I \times \mathbb{R},$$

where λ is scaling factor such that $|\lambda| < h^4$ and

$$r_{\mathtt{j}}(t) = \mathtt{A}_{\mathtt{j}} cos\xi(t-t_0) + \mathtt{B}_{\mathtt{j}} sin\xi(t-t_0) + \mathtt{C}_{\mathtt{j}}(t-t_0)^3 + \mathtt{D}_{\mathtt{j}}(t-t_0)^2 + \mathtt{E}_{\mathtt{j}}(t-t_0) + \mathtt{F}_{\mathtt{j}}.$$

Constructing the IFS as follows

$$I \times \mathbb{R}; X_{\mathtt{j}}(t,w) = (\mathtt{H}_{\mathtt{j}}(t), (\mathbb{F}_{\mathtt{j}}(t,w))) : \mathtt{j} = 1,2,\ldots,n,$$

which satisfies the following conditions:

$$\begin{cases} \mathbb{F}_{\mathtt{j}}(t_0,W_0) = W_{\mathtt{j}-1},\ \ \mathbb{F}_{\mathtt{j}}(t_n,W_n) = W_{\mathtt{j}}, \\ \mathbb{F}_{\mathtt{j},1}(t_n,W_{n,1}) = \mathbb{F}_{\mathtt{j}+1,1}(t_0,W_{0,1}), \\ \mathbb{F}_{\mathtt{j},2}(t_0,M_0) = M_{\mathtt{j}-1},\ \ \mathbb{F}_{\mathtt{j},2}(t_n,M_n) = M_{\mathtt{j}}, \\ \mathbb{F}_{\mathtt{j},3}(t_n,W_{n,3}) = \mathbb{F}_{\mathtt{j}+1,3}(t_0,W_{0,3}), \\ \mathbb{F}_{\mathtt{j},4}(t_0,S_0) = S_{\mathtt{j}-1},\ \ \mathbb{F}_{\mathtt{j},4}(t_n,S_n) = S_{\mathtt{j}}, \end{cases}$$

where $\mathtt{j} = 1,2,....,n-1$, and $\mathbb{F}_{\mathtt{j},\mathtt{k}}(t,w) = \frac{\lambda w + r_{\mathtt{j}}^{\mathtt{k}}(t)}{h^{\mathtt{k}}}$, $\mathtt{k} = 1,2,3,4$ and

$$W_{0,1} = \frac{r_1^{(1)}(t_0)}{h-\lambda},\ \ W_{n,1} = \frac{r_n^{(1)}(t_n)}{h-\lambda},\ W_{0,3} = \frac{r_1^{(3)}(t_0)}{h^3-\lambda},\ \ W_{n,3} = \frac{r_n^{(3)}(t_n)}{h^3-\lambda}\ .$$

Clearly, IFS is satisfying C^4-differentiability conditions on FIFs[5, 9, 10].

Let $\mathcal{F} = \{\Phi\ \in\ \mathtt{C}^4(I,\mathbb{R}) \mid \Phi(t_0) = W_0,\ \Phi(t_n) = W_n,\ \Phi^{(2)}(t_0) = M_0,$
$\Phi^{(2)}(t_n) = M_n,\ \Phi^{(4)}(t_0) = S_0,\ \Phi^{(4)}(t_n) = S_n\}$.
Then $(\mathcal{F},d)$ is a complete metric space and d is a metric induced on $\mathcal{F}$ by $\mathtt{C}^4-$norm. Let us define the Read-Bajraktarevic operator $\mathbb{T}$ on $(\mathcal{F},d)$ as

$$\begin{aligned}\mathbb{T}(\Phi(\mathtt{H}_{\mathtt{j}}(t))) = \lambda\Phi(t) + \mathtt{A}_{\mathtt{j}} cos\xi(t-t_0) + \mathtt{B}_{\mathtt{j}} sin\xi(t-t_0) + \mathtt{C}_{\mathtt{j}}(t-t_0)^3 + \mathtt{D}_{\mathtt{j}}(t-t_0)^2 \\ + \mathtt{E}_{\mathtt{j}}(t-t_0) + \mathtt{F}_{\mathtt{j}}\ , \qquad t \in [t_0,t_n], \qquad \mathtt{j} = 1,2,....,n.\end{aligned}$$

As operator $\mathbb{T}$ is contraction map, it must have a unique fixed point φ (say) which will satisfy the following conditions:

$$\varphi(\mathtt{H}_{\mathtt{j}}(t)) = \lambda\varphi(t) + \mathtt{A}_{\mathtt{j}} cos\xi(t-t_0) + \mathtt{B}_{\mathtt{j}} sin\xi(t-t_0) + \mathtt{C}_{\mathtt{j}}(t-t_0)^3 + \mathtt{D}_{\mathtt{j}}(t-t_0)^2$$

$$+\mathtt{E}_\mathtt{j}(t-t_0)+\mathtt{F}_\mathtt{j}, \quad t\in[t_0,t_n], \;\; \mathtt{j}=1,2,....,n. \tag{2.1}$$

From [10], it can be seen that

$$\begin{cases}\mathbb{F}_\mathtt{j}(t_0,W_0)=W_{\mathtt{j}-1},\;\; \mathbb{F}_\mathtt{j}(t_n,W_n)=W_\mathtt{j},\;\; \mathbb{F}_{\mathtt{j},2}(t_0,M_0)=M_{\mathtt{j}-1},\\ \mathbb{F}_{\mathtt{j},2}(t_n,M_n)=M_\mathtt{j},\;\; \mathbb{F}_{\mathtt{j},4}(t_0,S_0)=S_{\mathtt{j}-1},\;\; \mathbb{F}_{\mathtt{j},2}(t_n,S_n)=S_\mathtt{j},\end{cases}$$

are equivalent to

$$\begin{cases}\varphi(t_{\mathtt{j}-1})=W_{\mathtt{j}-1},\; \varphi(t_\mathtt{j})=W_\mathtt{j},\; \varphi^{(2)}(t_{\mathtt{j}-1})=M_{\mathtt{j}-1},\\ \varphi^{(2)}(t_\mathtt{j})=M_\mathtt{j},\; \varphi^{(4)}(t_{\mathtt{j}-1})=S_{\mathtt{j}-1},\; \varphi^{(4)}(t_\mathtt{j})=S_\mathtt{j}.\end{cases} \tag{2.2}$$

The conditions $\mathbb{F}_{\mathtt{j},1}(t_n,W_{n,1})=\mathbb{F}_{\mathtt{j}+1,1}(t_0,W_{0,1})$, and $\mathbb{F}_{\mathtt{j},3}(t_n,W_{n,3})=\mathbb{F}_{\mathtt{j}+1,3}(t_0,W_{0,3})$, can be reevaluated as $\varphi^{(1)}(\mathtt{H}_\mathtt{j}(t_n))=\varphi^{(1)}(\mathtt{H}_{\mathtt{j}+1}(t_0))$ and $\varphi^{(3)}(\mathtt{H}_\mathtt{j}(t_n))=\varphi^{(3)}(\mathtt{H}_{\mathtt{j}+1}(t_0))$ respectively. The coefficients $\mathtt{A}_j$, $\mathtt{B}_j$, $\mathtt{C}_j$, $\mathtt{D}_j$, $\mathtt{E}_j$ and $\mathtt{F}_j$ used in (2.1) are evaluated using (2.2). We get

$$\begin{aligned}
&\mathtt{A}_\mathtt{j}=\tfrac{h^4}{\xi^4}\Big(S_{\mathtt{j}-1}-\tfrac{\lambda}{h^4}S_0\Big),\\
&\mathtt{B}_\mathtt{j}=\tfrac{h^4}{\xi^4\sin\xi}\Big(S_\mathtt{j}-\tfrac{\lambda}{h^4}S_n\Big)-\tfrac{h^4\cos\xi}{\xi^4\sin\xi}\Big(S_{\mathtt{j}-1}-\tfrac{\lambda}{h^4}S_0\Big),\\
&\mathtt{C}_\mathtt{j}=\tfrac{h^2}{6}\Big(M_\mathtt{j}-\tfrac{\lambda}{h^2}M_n\Big)-\tfrac{h^2}{6}\Big(M_{\mathtt{j}-1}-\tfrac{\lambda}{h^2}M_0\Big)+\tfrac{h^4}{6\xi^2}\Big(S_\mathtt{j}-\tfrac{\lambda}{h^4}S_n\Big)-\tfrac{h^4}{6\xi^2}\Big(S_{\mathtt{j}-1}-\tfrac{\lambda}{h^4}S_0\Big),\\
&\mathtt{D}_\mathtt{j}=\tfrac{h^2}{2}\Big(M_{\mathtt{j}-1}-\tfrac{\lambda}{h^2}M_0\Big)+\tfrac{h^4}{2\xi^2}\Big(S_{\mathtt{j}-1}-\tfrac{\lambda}{h^4}S_0\Big),\\
&\mathtt{E}_\mathtt{j}=\Big(W_\mathtt{j}-\lambda W_n\Big)-\Big(W_{\mathtt{j}-1}-\lambda W_0\Big)-\tfrac{h^4}{6\xi^4}(6+\xi^2)\Big(S_\mathtt{j}-\tfrac{\lambda}{h^4}S_n\Big)+\tfrac{h^4}{6\xi^4}(6-2\xi^2)\Big(S_{\mathtt{j}-1}-\tfrac{\lambda}{h^4}S_0\Big)\\
&\qquad-\tfrac{h^2}{6}\Big(M_\mathtt{j}-\tfrac{\lambda}{h^2}M_n\Big)-\tfrac{2h^2}{6}\Big(M_{\mathtt{j}-1}-\tfrac{\lambda}{h^2}M_0\Big),\\
&\mathtt{F}_\mathtt{j}=\Big(W_{\mathtt{j}-1}-\lambda W_0\Big)+\tfrac{h^4}{\xi^4}\Big(S_{\mathtt{j}-1}-\tfrac{\lambda}{h^4}S_0\Big).
\end{aligned}$$

For continuity of $\varphi^{(1)}$, we have used $\varphi^{(1)}(t_\mathtt{j}^-)=\varphi^{(1)}(t_\mathtt{j}^+)$ i.e., $\varphi^{(1)}(\mathtt{H}_\mathtt{j}(t_n))=\varphi^{(1)}(\mathtt{H}_{\mathtt{j}+1}(t_0))$ and eventually get the following condition:

$$\lambda\varphi^{(1)}(t_n)-\mathtt{A}_\mathtt{j}\xi\sin\xi+\mathtt{B}_\mathtt{j}\xi\cos\xi+3\mathtt{C}_\mathtt{j}+2\mathtt{D}_\mathtt{j}+\mathtt{E}_\mathtt{j}=\lambda\varphi^{(1)}(t_0)+\xi\mathtt{B}_{\mathtt{j}+1}+\mathtt{E}_{\mathtt{j}+1}. \tag{2.3}$$

Similarly for continuity of $\varphi^{(3)}$ we have used $\varphi^{(3)}(t_\mathtt{j}^-)=\varphi^{(3)}(t_\mathtt{j}^+)$ i.e., $\varphi^{(3)}(\mathtt{H}_\mathtt{j}(t_n))=\varphi^{(3)}(\mathtt{H}_{\mathtt{j}+1}(t_0))$ and get

$$\lambda\varphi^{(3)}(t_n)+\mathtt{A}_\mathtt{j}\xi^3\sin\xi-\mathtt{B}_\mathtt{j}\xi^3\cos\xi+6\mathtt{C}_\mathtt{j}=\lambda\varphi^{(3)}(t_0)+\xi^3\mathtt{B}_{\mathtt{j}+1}+6\mathtt{C}_{\mathtt{j}+1}. \tag{2.4}$$

After substituting the values of $\mathtt{A}_\mathtt{j}$, $\mathtt{B}_\mathtt{j}$, $\mathtt{C}_\mathtt{j}$, $\mathtt{D}_\mathtt{j}$, $\mathtt{E}_\mathtt{j}$, $\mathtt{B}_{\mathtt{j}+1}$, $\mathtt{C}_{\mathtt{j}+1}$ and $\mathtt{E}_{\mathtt{j}+1}$ in (2.3) and (2.4), we obtain

$$\Big(S_0+S_n\Big)\Big(\frac{\lambda}{2\xi^2}+\frac{\lambda}{\xi^3}\frac{\cos\xi}{\sin\xi}-\frac{\lambda}{\xi^3}\sin\xi\Big)+\Big(S_{\mathtt{j}-1}+S_{\mathtt{j}+1}\Big)\Big(\frac{h^4}{\xi^3\sin\xi}-\frac{h^4}{6\xi^4}(6+\xi^2)\Big)$$

$$
\begin{aligned}
+S_{\mathrm{j}}\Big(\frac{h^4}{6\xi^4}(12-4\xi^2)-\frac{2h^4}{\xi^3}\frac{\cos\xi}{\sin\xi}\Big) &= \lambda\varphi^{(1)}(t_n)-\lambda\varphi^{(1)}(t_0)-(W_{\mathrm{j}+1}-2W_{\mathrm{j}}+W_{\mathrm{j}-1})\\
&\quad -\frac{\lambda}{2}(M_0+M_n)+\frac{h^2}{6}(M_{\mathrm{j}+1}+4M_{\mathrm{j}}+M_{\mathrm{j}-1}), \qquad (2.5)
\end{aligned}
$$

and

$$
\begin{aligned}
(S_0+S_n)\Big(\frac{\lambda}{\xi}\frac{\cos\xi}{\sin\xi}-\frac{\lambda}{\xi\sin\xi}\Big)+\Big(\frac{h^4}{\xi\sin\xi}-\frac{h^4}{\xi^2}\Big)(S_{\mathrm{j}-1}+S_{\mathrm{j}+1})+S_{\mathrm{j}}\Big(\frac{2h^4}{\xi^2}-\frac{2h^4}{\xi}\frac{\cos\xi}{\sin\xi}\Big)\\
=\lambda(\varphi^{(3)}(t_0)-\varphi^{(3)}(t_n))+h^2(M_{\mathrm{j}-1}-2M_{\mathrm{j}}+M_{\mathrm{j}+1}), \quad (2.6)
\end{aligned}
$$

respectively. From equation (2.5), we have

$$
\begin{aligned}
\Big(\alpha_2S_{\mathrm{j}-1}+2\beta_2S_{\mathrm{j}}+\alpha_2S_{\mathrm{j}+1}\Big) &= -\frac{1}{6h^2}(M_{\mathrm{j}+1}+4M_{\mathrm{j}}+M_{\mathrm{j}-1})+\frac{1}{h^4}k_2\Big(S_0+S_n\Big)-\frac{\lambda}{h^4}(\varphi^{(1)}(t_n)\\
&\quad -\varphi^{(1)}(t_0))+\frac{\lambda}{2h^4}(M_0+M_n)+\frac{1}{h^4}(W_{\mathrm{j}+1}-2W_{\mathrm{j}}+W_{\mathrm{j}+1}), \qquad (2.7)
\end{aligned}
$$

and from equation (2.6), we have

$$
\begin{aligned}
(\alpha_1S_{\mathrm{j}-1}+2\beta_1S_{\mathrm{j}}+\alpha_1S_{\mathrm{j}+1}) &= \frac{1}{h^2}(M_{\mathrm{j}+1}-2M_{\mathrm{j}}+M_{\mathrm{j}-1})-\frac{1}{h^4}k_1(S_0+S_n)\\
&\quad -\frac{\lambda}{h^4}(\varphi^{(3)}(t_n)-\varphi^{(3)}(t_0)), \qquad (2.8)
\end{aligned}
$$

where
$\alpha_1=\frac{1}{\xi^2}\Big(\xi cosec(\xi)-1\Big)$, $\quad\beta_1=\frac{1}{\xi^2}\Big(1-\xi cot(\xi)\Big)$,
$\alpha_2=\frac{1}{\xi^2}\Big(\frac{1}{6}-\alpha_1\Big)$, $\quad\beta_2=\frac{1}{\xi^2}\Big(\frac{1}{3}-\beta_1\Big)$,
$k_1=\frac{cot\xi}{\xi}-\frac{cosec\xi}{\xi}$, $\quad k_2=\frac{1}{\xi^2}\Big(\frac{1}{2}+k_1\Big)$.
Solving (2.7) and (2.8), we get

$$
\begin{aligned}
S_{\mathrm{j}} &= \frac{(S_0+S_n)}{2h^4}\frac{(\alpha_1k_2+\alpha_2k_1)}{(\alpha_1\beta_2-\alpha_2\beta_1)}-\frac{\alpha_1\lambda}{2h^4}\frac{(\varphi^{(1)}(t_n)-\varphi^{(1)}(t_0))}{(\alpha_1\beta_2-\alpha_2\beta_1)}+\frac{\alpha_2\lambda}{2h^4}\frac{(\varphi^{(3)}(t_n)-\varphi^{(3)}(t_0))}{(\alpha_1\beta_2-\alpha_2\beta_1)}\\
&\quad +\frac{\alpha_1\lambda}{4h^4}\frac{(M_0+M_n)}{(\alpha_1\beta_2-\alpha_2\beta_1)}+\frac{\alpha_1}{2h^4}\frac{(W_{\mathrm{j}+1}-2W_{\mathrm{j}}+W_{\mathrm{j}-1})}{(\alpha_1\beta_2-\alpha_2\beta_1)}-\frac{\alpha_1}{12h^2}\frac{(M_{\mathrm{j}+1}+4M_{\mathrm{j}}+M_{\mathrm{j}-1})}{(\alpha_1\beta_2-\alpha_2\beta_1)}\\
&\quad -\frac{\alpha_2}{2h^2}\frac{(M_{\mathrm{j}+1}-2M_{\mathrm{j}}+M_{\mathrm{j}-1})}{(\alpha_1\beta_2-\alpha_2\beta_1)}. \qquad (2.9)
\end{aligned}
$$

Using equation (2.9) in equation (2.8), we have

$$
\begin{aligned}
&\alpha_1(W_{\mathrm{j}+2}+W_{\mathrm{j}-2})+2(\beta_1-\alpha_1)(W_{\mathrm{j}+1}+W_{\mathrm{j}-1})+(2\alpha_1-4\beta_1)W_{\mathrm{j}}\\
&\quad =-2(\alpha_1+\beta_1)\lambda(\varphi^{(1)}(t_0)-\varphi^{(1)}(t_n))+2(\alpha_2+\beta_2)\lambda(\varphi^{(3)}(t_0)-\varphi^{(3)}(t_n))\\
&\quad -(\alpha_1+\beta_1)\lambda(M_0+M_n)+h^2(pM_{\mathrm{j}+2}+qM_{\mathrm{j}+1}+rM_{\mathrm{j}}+qM_{\mathrm{j}-1}+pM_{\mathrm{j}-2}), \qquad (2.10)
\end{aligned}
$$

where
$p=\alpha_2+\frac{\alpha_1}{6}$,
$q=2\Big[\frac{1}{6}(2\alpha_1+\beta_1)-(\alpha_2-\beta_2)\Big]$,
$r=2\Big[\frac{1}{6}(\alpha_1+4\beta_1)+(\alpha_2-2\beta_2)\Big]$.
Remark 1: When $\Big(\alpha_1,\beta_1,\alpha_2,\beta_2\Big)=\Big(\frac{1}{6},\frac{2}{6},\frac{-7}{360},\frac{-8}{360}\Big)$ equation (2.10) reduces to (2.5) of Balasubramani et al.[3].
Remark 2: When $\lambda=0$, equation (2.10) reduces to quintic non-polynomial spline method by P. Srivastav et al.[31].

2.1. **Spline Solution for Linear BVPs**

Equation (1.1) is discretized at $t=t_{\mathtt{j}}$, since $M_{\mathtt{j}}+\mathtt{p}_{\mathtt{j}}W_{\mathtt{j}}=\mathtt{f}_{\mathtt{j}}$, where $\mathtt{p}_{\mathtt{j}}=\mathtt{p}(t_{\mathtt{j}})$, $\mathtt{f}_{\mathtt{j}}=\mathtt{f}(t_{\mathtt{j}})$. The boundary equations are discretized as $W_0=\sigma_0,\ \ W_n=\sigma_1$.
Substitute

$$\varphi^{(3)}(t_0)=\frac{-W_0+3W_1-3W_2+W_3}{h^3},\qquad \varphi^{(3)}(t_n)=\frac{W_n-3W_{n-1}+3W_{n-2}-W_{n-3}}{h^3},$$
$$\varphi^{(1)}(t_0)=\frac{W_1-W_0}{h},\qquad \varphi^{(1)}(t_n)=\frac{W_n-W_{n-1}}{h},$$
$$M_{\mathtt{j}}=\mathtt{f}_{\mathtt{j}}-\mathtt{p}_{\mathtt{j}}W_{\mathtt{j}},$$

in (2.10), and after some calculations we get,

$$\begin{cases} -\Big[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\Big]W_1+\Big[\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\Big]W_2-\Big[\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\Big]W_3-\Big[\alpha_1+ph^2\mathtt{p}_{\mathtt{j}-2}\Big]W_{\mathtt{j}-2} \\ -\Big[2(\beta_1-\alpha_1)+qh^2\mathtt{p}_{\mathtt{j}-1}\Big]W_{\mathtt{j}-1}-\Big[(2\alpha_1-4\beta_1)+rh^2\mathtt{p}_{\mathtt{j}}\Big]W_{\mathtt{j}}-\Big[2(\beta_1-\alpha_1) \\ +qh^2\mathtt{p}_{\mathtt{j}+1}\Big]W_{\mathtt{j}+1}-\Big[\alpha_1+ph^2\mathtt{p}_{\mathtt{j}+2}\Big]W_{\mathtt{j}+2}-\Big[\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\Big]W_{n-3}+\Big[\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\Big]W_{n-2} \\ -\Big[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\Big]W_{n-1}=-h^2\Big[p(\mathtt{f}_{\mathtt{j}+2}+\mathtt{f}_{\mathtt{j}-2})+q(\mathtt{f}_{\mathtt{j}+1}+\mathtt{f}_{\mathtt{j}-1})+r\mathtt{f}_{\mathtt{j}}\Big] \\ +\lambda(\alpha_1+\beta_1)[(\mathtt{f}_0+\mathtt{f}_n)-(\mathtt{p}_0\sigma_0+\mathtt{p}_n\sigma_n)]-\Big[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\Big]\sigma_0 \\ -\Big[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\Big]\sigma_1, \qquad \mathtt{j}=2,3,\ldots,(n-2). \end{cases} \tag{2.11}$$

In (2.11) we have $(n-1)$ unknowns $W_1,\ W_2,\ldots W_{n-1}$ and $(n-3)$ equations. Therefore two more equations are required to find unique solution. Hence we derive two boundary equations as follows:

Boundary equations

Let the equation at $\mathtt{j}=1$ and $\mathtt{j}=n-1$ be

$$\begin{cases} \Big[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\Big]W_0-\Big[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\Big]W_1+\Big[\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\Big]W_2 \\ -\Big[\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\Big]W_3-\Big[\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\Big]W_{n-3}+\Big[\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\Big]W_{n-2}-\Big[\frac{2(\alpha_1+\beta_1)\lambda}{h} \\ +\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\Big]W_{n-1}+\Big[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\Big]W_n=\lambda(\alpha_1+\beta_1)[(\mathtt{f}_0+\mathtt{f}_n) \\ -(q_0\sigma_0+q_n\sigma_n)]+\sum_{\mathtt{k}=0}^{\mathtt{k}=5}\big(l_{\mathtt{k}}w(t_{\mathtt{k}})+\mathtt{m}_{\mathtt{k}}h^2w_{tt}(t_{\mathtt{k}})\big), \end{cases} \tag{2.12}$$

and

$$\begin{cases} \left[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]W_0-\left[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W_1+\left[\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W_2 \\ -\left[\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]W_3-\left[\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]W_{n-3}+\left[\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W_{n-2}-\left[\frac{2(\alpha_1+\beta_1)\lambda}{h} \right.\\ \left.+\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W_{n-1}+\left[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]W_n=\lambda(\alpha_1+\beta_1)[(\mathtt{f}_0+\mathtt{f}_n) \\ -(q_0\sigma_0+q_n\sigma_n)]+\sum_{\mathtt{k}=n-5}^{\mathtt{k}=n}\left(l_{\mathtt{k}}w(t_{\mathtt{k}})+\mathtt{m}_{\mathtt{k}}h^2w_{tt}(t_{\mathtt{k}})\right), \end{cases} \tag{2.13}$$

respectively. The system (2.11), (2.12) and (2.13) provides the numerical solution $W_{\mathtt{j}},\ \mathtt{j}=1,2,\ldots,n-1$ for linear BVPs.

2.2. Spline Solution for nonlinear BVPs

2.2.1. Quasilinearisation technique

We use quasilinearisation technique to convert the non-linear BVP given in (1.2) into a system of linear BVPs. Here $w^{(0)}(t)$ denotes the initial approximation and the function $\mathtt{F}(t,w(t))$ is expanded around the $w^{(0)}(t)$ to obtain

$$\mathtt{F}(t,w^{(1)}(t))=\mathtt{F}(t,w^{(0)}(t))+(w^{(1)}-w^{(0)})\big(\frac{\partial \mathtt{F}}{\partial w}\big)_{(t,w^{(0)}(t))}+\cdots.$$

In general,

$$\mathtt{F}(t,w^{(\mathtt{r}+1)}(t))=\mathtt{F}(t,w^{(\mathtt{r})}(t))+(w^{(\mathtt{r}+1)}-w^{(\mathtt{r})})\big(\frac{\partial \mathtt{F}}{\partial w}\big)_{(t,w^{(\mathtt{r})}(t))}+\cdots,$$

where $\mathtt{r}$ is the iteration index such that $\mathtt{r}=0,1,2,...$
The nonlinear BVP (1.2) can be written as

$$\begin{cases} w_{tt}^{(\mathtt{r}+1)}(t)+\mathtt{F}(t,w^{(\mathtt{r}+1)}(t))=0, \quad t\in(0,1), \\ w^{(\mathtt{r}+1)}(0)=\sigma_0, \quad w^{(\mathtt{r}+1)}(1)=\sigma_1. \end{cases} \tag{2.14}$$

By substituting

$$\mathtt{F}(t,w^{(\mathtt{r}+1)}(t))=\mathtt{F}(t,w^{(\mathtt{r})}(t))+(w^{(\mathtt{r}+1)}-w^{(\mathtt{r})})\big(\frac{\partial \mathtt{F}}{\partial w}\big)_{(t,w^{(\mathtt{r})}(t))}$$

in (2.14), we get

$$\begin{cases} w_{tt}^{(\mathtt{r}+1)}(t)+q^{(\mathtt{r})}(t)w^{(\mathtt{r}+1)}(t)=f^{(\mathtt{r})}(t), \quad t\in(0,1), \quad \mathtt{r}=0,1,..., \\ w^{(\mathtt{r}+1)}(0)=\sigma_0, \quad w^{(\mathtt{r}+1)}(1)=\sigma_1, \end{cases} \tag{2.15}$$

where

$$q^{(\mathtt{r})}(t)=\big(\frac{\partial \mathtt{F}}{\partial w}\big)_{(t,w^{(\mathtt{r})}(t))}, \quad f^{(\mathtt{r})}(t)=w^{(\mathtt{r})}(t)\big(\frac{\partial \mathtt{F}}{\partial w}\big)_{(t,w^{(\mathtt{r})}(t))}-\mathtt{F}(t,w^{(\mathtt{r})}(t)).$$

Hence the non-linear BVP (1.2) is converted into a system of linear BVPs. Now we will proceed to solve this system numerically.

2.2.2. **Numerical scheme**

Let $W_{\mathrm{j}}^{(\mathrm{r})}$ is the approximate value of $w^{(\mathrm{r})}(t_{\mathrm{j}})$ and $M_{\mathrm{j}}^{(\mathrm{r})}$ is the approximate value of $w_{tt}^{(\mathrm{r})}(t_{\mathrm{j}})$. Now, at $t = t_{\mathrm{j}}$, the differential equation (2.15) can be discretized as

$$M_{\mathrm{j}}^{(\mathrm{r+1})} + q_{\mathrm{j}}^{(\mathrm{r})} W_{\mathrm{j}}^{(\mathrm{r+1})} = f_{\mathrm{j}}^{(\mathrm{r})},$$

where

$$q_{\mathrm{j}}^{(\mathrm{r})} = \Big(\frac{\partial F}{\partial w}\Big)_{(t_{\mathrm{j}}, w_{\mathrm{j}}^{(\mathrm{r})})}, \quad f_{\mathrm{j}}^{(\mathrm{r})} = w_{\mathrm{j}}^{(\mathrm{r})} \Big(\frac{\partial F}{\partial w}\Big)_{(t_{\mathrm{j}}, w_{\mathrm{j}}^{(\mathrm{r})})} - \mathrm{F}(t_{\mathrm{j}}, w_{\mathrm{j}}^{(\mathrm{r})}).$$

Also, the boundary conditions can be discretised as $W_0^{(\mathrm{r+1})} = \sigma_0, \quad W_n^{(\mathrm{r+1})} = \sigma_1$.

Substitute

$$\varphi^{(3)}(t_0) = \frac{-W_0^{(\mathrm{r+1})} + 3W_1^{(\mathrm{r+1})} - 3W_2^{(\mathrm{r+1})} + W_3^{(\mathrm{r+1})}}{h^3},$$
$$\varphi^{(3)}(t_n) = \frac{W_n^{(\mathrm{r+1})} - 3W_{n-1}^{(\mathrm{r+1})} + 3W_{n-2}^{(\mathrm{r+1})} - W_{n-3}^{(\mathrm{r+1})}}{h^3},$$
$$\varphi^{(1)}(t_0) = \frac{W_1^{(\mathrm{r+1})} - W_0^{(\mathrm{r+1})}}{h}, \qquad \varphi^{(1)}(t_n) = \frac{W_n^{(\mathrm{r+1})} - W_{n-1}^{(\mathrm{r+1})}}{h},$$
$$M_{\mathrm{j}}^{(\mathrm{r+1})} = f_{\mathrm{j}}^{(\mathrm{r})} - q_{\mathrm{j}}^{(\mathrm{r})} W_{\mathrm{j}}^{(\mathrm{r+1})},$$

in equation (2.10) we have

$$\begin{cases} -\Big[\frac{2(\alpha_1+\beta_1)\lambda}{h} + \frac{6(\alpha_2+\beta_2)\lambda}{h^3}\Big] W_1^{(\mathrm{r+1})} + \Big[\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\Big] W_2^{(\mathrm{r+1})} - \Big[\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\Big] W_3^{(\mathrm{r+1})} - \Big[\alpha_1 \\ +ph^2 q_{\mathrm{j}-2}^{(\mathrm{r})}\Big] W_{\mathrm{j}-2}^{(\mathrm{r+1})} - \Big[2(\beta_1-\alpha_1) + qh^2 q_{\mathrm{j}-1}^{(\mathrm{r})}\Big] W_{\mathrm{j}-1}^{(\mathrm{r+1})} - \Big[(2\alpha_1 - 4\beta_1) + rh^2 q_{\mathrm{j}}^{(\mathrm{r})}\Big] W_{\mathrm{j}}^{(\mathrm{r+1})} \\ -\Big[2(\beta_1-\alpha_1) + qh^2 q_{\mathrm{j}+1}^{(\mathrm{r})}\Big] W_{\mathrm{j}+1}^{(\mathrm{r+1})} - \Big[\alpha_1 + ph^2 q_{\mathrm{j}+2}^{(\mathrm{r})}\Big] W_{\mathrm{j}+2}^{(\mathrm{r+1})} - \Big[\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\Big] W_{n-3}^{(\mathrm{r+1})} \\ +\Big[\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\Big] W_{n-2}^{(\mathrm{r+1})} - \Big[\frac{2(\alpha_1+\beta_1)\lambda}{h} + \frac{6(\alpha_2+\beta_2)\lambda}{h^3}\Big] W_{n-1}^{(\mathrm{r+1})} = -h^2\Big[p(f_{\mathrm{j}+2}^{(\mathrm{r})} + f_{\mathrm{j}-2}^{(\mathrm{r})}) \\ +q(f_{\mathrm{j}+1}^{(\mathrm{r})} + f_{\mathrm{j}-1}^{(\mathrm{r})}) + rf_{\mathrm{j}}^{(\mathrm{r})}\Big] + \lambda(\alpha_1+\beta_1)[(f_0^{(\mathrm{r})} + f_n^{(\mathrm{r})}) - (q_0^{(\mathrm{r})}\sigma_0 + q_n^{(\mathrm{r})}\sigma_n)] \\ -\Big[\frac{2(\alpha_1+\beta_1)\lambda}{h} + \frac{2(\alpha_2+\beta_2)\lambda}{h^3}\Big]\sigma_0 - \Big[\frac{2(\alpha_1+\beta_1)\lambda}{h} + \frac{2(\alpha_2+\beta_2)\lambda}{h^3}\Big]\sigma_1, \quad \mathrm{j} = 2,3,\ldots,(n-2). \end{cases} \tag{2.16}$$

In (2.16) we have $(n-1)$ unknowns $W_1^{(\mathrm{r+1})}$, $W_2^{(\mathrm{r+1})}, \ldots W_{n-1}^{(\mathrm{r+1})}$ and $(n-3)$ equations. Therefore two more equations are required to find unique solution. Hence we derive two boundary equations as follows:

Boundary equations

Let the equation at $\mathtt{j}=1$ and $\mathtt{j}=n-1$ be

$$\begin{cases} \left[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]W_0^{(\mathtt{r}+1)}-\left[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W_1^{(\mathtt{r}+1)}+\left[\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W_2^{(\mathtt{r}+1)} \\ -\left[\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]W_3^{(\mathtt{r}+1)}-\left[\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]W_{n-3}^{(\mathtt{r}+1)}+\left[\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W_{n-2}^{(\mathtt{r}+1)}-\left[\frac{2(\alpha_1+\beta_1)\lambda}{h}\right. \\ \left.+\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W_{n-1}^{(\mathtt{r}+1)}+\left[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]W_n^{(\mathtt{r}+1)}=\lambda(\alpha_1+\beta_1)[(f_0^{(\mathtt{r})}+f_n^{(\mathtt{r})}) \\ -(q_0^{(\mathtt{r})}\sigma_0+q_n^{(\mathtt{r})}\sigma_n)]+\sum_{\mathtt{k}=0}^{\mathtt{k}=5}\left(l_{\mathtt{k}}w^{(\mathtt{r}+1)}(t_{\mathtt{k}})+\mathtt{m}_{\mathtt{k}}h^2w_{tt}^{(\mathtt{r}+1)}(t_{\mathtt{k}})\right), \end{cases} \tag{2.17}$$

and

$$\begin{cases} \left[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]W_0^{(\mathtt{r}+1)}-\left[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W_1^{(\mathtt{r}+1)}+\left[\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W_2^{(\mathtt{r}+1)} \\ -\left[\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]W_3^{(\mathtt{r}+1)}-\left[\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]W_{n-3}^{(\mathtt{r}+1)}+\left[\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W_{n-2}^{(\mathtt{r}+1)}-\left[\frac{2(\alpha_1+\beta_1)\lambda}{h}\right. \\ \left.+\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W_{n-1}^{(\mathtt{r}+1)}+\left[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]W_n^{(\mathtt{r}+1)}=\lambda(\alpha_1+\beta_1)[(f_0^{(\mathtt{r})}+f_n^{(\mathtt{r})}) \\ -(q_0^{(\mathtt{r})}\sigma_0+q_n^{(\mathtt{r})}\sigma_n)]+\sum_{\mathtt{k}=n-5}^{\mathtt{k}=n}\left(l_{\mathtt{k}}w^{(\mathtt{r}+1)}(t_{\mathtt{k}})+\mathtt{m}_{\mathtt{k}}h^2w_{tt}^{(\mathtt{r}+1)}(t_{\mathtt{k}})\right), \end{cases} \tag{2.18}$$

respectively. For non-linear BVPs, system (2.16), (2.17) and (2.18) gives the approximate solution $W_{\mathtt{j}}^{(\mathtt{r}+1)}$, $\mathtt{j}=1,2,\ldots,n-1$.

3. Truncation error

From (2.16), we have

$$\begin{cases} T_{\mathtt{j}}^{(\mathtt{r})}(h)=\left[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]W^{(\mathtt{r}+1)}(t_0)-\left[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W^{(\mathtt{r}+1)}(t_1) \\ \qquad +\left[\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W^{(\mathtt{r}+1)}(t_2)-\left[\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]W^{(\mathtt{r}+1)}(t_3)-\Big[\alpha_1 \\ \qquad +ph^2q^{(\mathtt{r})}(t_{\mathtt{j}-2})\Big]W^{(\mathtt{r}+1)}(t_{\mathtt{j}-2})-\Big[2(\beta_1-\alpha_1)+qh^2q^{(\mathtt{r})}(t_{\mathtt{j}-1})\Big]W^{(\mathtt{r}+1)}(t_{\mathtt{j}-1}) \\ \qquad -\Big[(2\alpha_1-4\beta_1)+rh^2q^{(\mathtt{r})}(t_{\mathtt{j}})\Big]W^{(\mathtt{r}+1)}(t_{\mathtt{j}})-\Big[2(\beta_1-\alpha_1) \\ \qquad +qh^2q^{(\mathtt{r})}(t_{\mathtt{j}+1})\Big]W^{(\mathtt{r}+1)}(t_{\mathtt{j}+1})-\Big[\alpha_1+ph^2q^{(\mathtt{r})}(t_{\mathtt{j}+2})\Big]W^{(\mathtt{r}+1)}(t_{\mathtt{j}+2}) \\ \qquad -\left[\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]W^{(\mathtt{r}+1)}(t_{n-3})+\left[\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W^{(\mathtt{r}+1)}(t_{n-2}) \\ \qquad -\left[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W^{(\mathtt{r}+1)}(t_{n-1})+\left[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]W^{(\mathtt{r}+1)}(t_n) \\ \qquad +h^2\Big[\mathtt{p}(f^{(\mathtt{r})}(t_{\mathtt{j}+2})+f^{(\mathtt{r})}(t_{\mathtt{j}-2}))+q(f^{(\mathtt{r})}(t_{\mathtt{j}+1})+f^{(\mathtt{r})}(t_{\mathtt{j}-1}))+rf^{(\mathtt{r})}(t_{\mathtt{j}})\Big] \\ \qquad -\lambda(\alpha_1+\beta_1)[(f^{(\mathtt{r})}(t_0)+f^{(\mathtt{r})}(t_n))-(q^{(\mathtt{r})}(t_0)W^{(\mathtt{r}+1)}(t_0) \\ \qquad +q^{(\mathtt{r})}W^{(\mathtt{r}+1)}(t_n))], \quad \mathtt{j}=2,3,\ldots,(n-2). \end{cases} \tag{3.1}$$

Substituting $f^{(\mathtt{r})}(t_{\mathtt{j}})=w_{tt}^{(\mathtt{r}+1)}(t_{\mathtt{j}})+q^{(\mathtt{r})}(t_{\mathtt{j}})w^{(\mathtt{r}+1)}(t_{\mathtt{j}})$ in (3.1), we get

$$\begin{cases} T_{\mathrm{j}}^{(\mathrm{r})}(h) = -2(\alpha_2+\beta_2)\lambda\left[\frac{-W^{(\mathrm{r}+1)}(t_0)+3W^{(\mathrm{r}+1)}(t_1)-3W^{(\mathrm{r}+1)}(t_2)+W^{(\mathrm{r}+1)}(t_3)}{h^3}\right] \\ \qquad +2(\alpha_2+\beta_2)\lambda\left[\frac{W^{(\mathrm{r}+1)}(t_n)-W^{(\mathrm{r}+1)}(t_{n-1})+3W^{(\mathrm{r}+1)}(t_{n-2})-W^{(\mathrm{r}+1)}(t_{n-3})}{h^3}\right] \\ \qquad -2(\alpha_1+\beta_1)\lambda\left[\frac{W_1^{(\mathrm{r}+1)}-W_0^{(\mathrm{r}+1)}}{h}\right]+2(\alpha_1+\beta_1)\lambda\left[\frac{W_n^{(\mathrm{r}+1)}-W_{n-1}^{(\mathrm{r}+1)}}{h}\right] \\ \qquad -(\alpha_1+\beta_1)\lambda W_{tt}^{(\mathrm{r}+1)}(t_0)-(\alpha_1+\beta_1)\lambda W_{tt}^{(\mathrm{r}+1)}(t_n) \\ \qquad -\alpha_1(w^{(\mathrm{r}+1)}(t_{\mathrm{j}+2})+w^{(\mathrm{r}+1)}(t_{\mathrm{j}-2}))-2(\beta_1-\alpha_1)(w^{(\mathrm{r}+1)}(t_{\mathrm{j}+1})+w^{(\mathrm{r}+1)}(t_{\mathrm{j}-1})) \\ \qquad -(2\alpha_1-4\beta_1)w^{(\mathrm{r}+1)}(t_{\mathrm{j}})+ph^2w_{tt}^{(\mathrm{r}+1)}(t_{\mathrm{j}+2})+qh^2w_{tt}^{(\mathrm{r}+1)}(t_{\mathrm{j}+1})+rh^2w_{tt}^{(\mathrm{r}+1)}(t_{\mathrm{j}}) \\ \qquad +qh^2w_{tt}^{(\mathrm{r}+1)}(t_{\mathrm{j}+1})+ph^2w_{tt}^{(\mathrm{r}+1)}(t_{\mathrm{j}+2}). \end{cases} \tag{3.2}$$

After further simplification we obtain,

$$\begin{cases} T_{\mathrm{j}}^{(\mathrm{r})}(h) = -2(\alpha_2+\beta_2)\lambda\left[W_{ttt}^{(\mathrm{r}+1)}(t_0)+O(h)\right]+2(\alpha_2+\beta_2)\lambda\left[W_{ttt}^{(\mathrm{r}+1)}(t_n)+O(h)\right] \\ \qquad -2(\alpha_1+\beta_1)\lambda\left[W_t^{(\mathrm{r}+1)}(t_0)+O(h)\right]+2(\alpha_1+\beta_1)\lambda\left[W_t^{(\mathrm{r}+1)}(t_n)+O(h)\right] \\ \qquad -(\alpha_1+\beta_1)\lambda W_{tt}^{(\mathrm{r}+1)}(t_0)-(\alpha_1+\beta_1)\lambda W_{tt}^{(\mathrm{r}+1)}(t_n) \\ \qquad +\left[\frac{1}{6}(7\alpha_1+\beta_1)-(4p+q)\right]h^4w_{tttt}^{(\mathrm{r}+1)}(t_{\mathrm{j}})+\left[\frac{1}{180}(31\alpha_1+\beta_1)\right. \\ \qquad \left.-\frac{1}{12}(16p+q)\right]h^6w_{tttttt}^{(\mathrm{r}+1)}(t_{\mathrm{j}})+\left[\frac{1}{131040}(1611\alpha_1+31\beta_1)\right. \\ \qquad \left.-\frac{1}{360}(4p+q)\right]h^8w_{tttttttt}^{(\mathrm{r}+1)}(t_{\mathrm{j}})+O(h^9). \end{cases} \tag{3.3}$$

We write

$$T_{\mathrm{j}}^{(\mathrm{r})}(h) = T_{\lambda}^{(\mathrm{r})}(h)+T_{*}^{(\mathrm{r})}(h),$$

where

$$\begin{aligned} T_{\lambda}^{(\mathrm{r})}(h) = {} & -2(\alpha_2+\beta_2)\lambda\left[W_{ttt}^{(\mathrm{r}+1)}(t_0)+O(h)\right]+2(\alpha_2+\beta_2)\lambda\left[W_{ttt}^{(\mathrm{r}+1)}(t_n)+O(h)\right] \\ & -2(\alpha_1+\beta_1)\lambda\left[W_t^{(\mathrm{r}+1)}(t_0)+O(h)\right]+2(\alpha_1+\beta_1)\lambda\left[W_t^{(\mathrm{r}+1)}(t_n)+O(h)\right] \\ & -(\alpha_1+\beta_1)\lambda W_{tt}^{(\mathrm{r}+1)}(t_0)-(\alpha_1+\beta_1)\lambda W_{tt}^{(\mathrm{r}+1)}(t_n), \end{aligned}$$

and

$$\begin{aligned} T_{*}^{(\mathrm{r})}(h) = {} & \left[\frac{1}{6}(7\alpha_1+\beta_1)-(4p+q)\right]h^4w_{tttt}^{(\mathrm{r}+1)}(t_{\mathrm{j}})+\left[\frac{1}{180}(31\alpha_1+\beta_1)-\frac{1}{12}(16p+q)\right]h^6w_{tttttt}^{(\mathrm{r}+1)}(t_{\mathrm{j}}) \\ & +\left[\frac{1}{131040}(1611\alpha_1+31\beta_1)-\frac{1}{360}(4p+q)\right]h^8w_{tttttttt}^{(\mathrm{r}+1)}(t_{\mathrm{j}})+O(h^9). \end{aligned}$$

4. Class of methods

4.1. Second order method

Choose λ such that $|\lambda| < h^4$. For getting method of second order, unknown coefficients must satisfy conditions:

$(\alpha_1+\beta_1) = \frac{1}{2}$.

$\left[\frac{1}{6}(7\alpha_1+\beta_1)-(4p+q)\right] \neq 0.$

One such set of values are:

$(\alpha_1, \beta_1) = \left(\frac{1}{4}, \frac{1}{4}\right)$ and
$p = 1/4$, $q = 0$, $r = 1/2$.
Also

$$\text{at } \mathtt{j} = 1, \qquad \begin{aligned} &\left(l_0, l_1, l_2, l_3, l_4, l_5\right) = \left(0, -1, 2, -1, 0, 0\right), \\ &\left(\mathtt{m}_0, \mathtt{m}_1, \mathtt{m}_2, \mathtt{m}_3, \mathtt{m}_4, \mathtt{m}_5\right) = \left(0, \tfrac{1}{6}, \tfrac{4}{6}, \tfrac{1}{6}, 0, 0\right), \end{aligned}$$

and

$$\text{at } \mathtt{j} = n-1, \quad \begin{aligned} &\left(l_n, l_{n-1}, l_{n-2}, l_{n-3}, l_{n-4}, l_{n-5}\right) = \left(0, -1, 2, -1, 0, 0\right), \\ &\left(\mathtt{m}_n, \mathtt{m}_{n-1}, \mathtt{m}_{n-2}, \mathtt{m}_{n-3}, \mathtt{m}_{n-4}, \mathtt{m}_{n-5}\right) = \left(0, \tfrac{1}{6}, \tfrac{4}{6}, \tfrac{1}{6}, 0, 0\right). \end{aligned}$$

Since $|\lambda| < h^4$, we have $T_\lambda^{(\mathtt{r})}(h) = O(h^4)$ and $T_*^{(\mathtt{r})}(h) = \frac{-2}{3} h^4 w_{tttt}^{(\mathtt{r}+1)}(t_\mathtt{j}) + O(h^5)$.
Therefore

$$T_\mathtt{j}^{(\mathtt{r})}(h) = O(h^4). \tag{4.1}$$

4.2. Fourth order method

Choose λ such that $|\lambda| < h^6$. For getting method of order four, values of unknown coefficients must satisfy conditions:
$(\alpha_1 + \beta_1) = \frac{1}{2}$,
$\left[\frac{1}{6}\left(7\alpha_1 + \beta_1\right) - (4p + q)\right] = 0$,
$\left[\frac{1}{180}\left(31\alpha_1 + \beta_1\right) - \frac{1}{12}(16p + q)\right] \neq 0$.
One such set of values are $(\alpha_1, \beta_1) = \left(\frac{1}{6}, \frac{1}{3}\right)$ and
$p = \frac{1}{120}, q = \frac{26}{120}, r = \frac{66}{120}$.
Also

$$\text{at } \mathtt{j} = 1, \qquad \begin{aligned} &\left(l_0, l_1, l_2, l_3, l_4, l_5\right) = \left(0, -1, 2, -1, 0, 0\right), \\ &\left(\mathtt{m}_0, \mathtt{m}_1, \mathtt{m}_2, \mathtt{m}_3, \mathtt{m}_4, \mathtt{m}_5\right) = \left(0, \tfrac{1}{12}, \tfrac{10}{12}, \tfrac{1}{12}, 0, 0\right), \end{aligned}$$

and

$$\text{at } \mathtt{j} = n-1, \quad \begin{aligned} &\left(l_n, l_{n-1}, l_{n-2}, l_{n-3}, l_{n-4}, l_{n-5}\right) = \left(0, -1, 2, -1, 0, 0\right), \\ &\left(\mathtt{m}_n, \mathtt{m}_{n-1}, \mathtt{m}_{n-2}, \mathtt{m}_{n-3}, \mathtt{m}_{n-4}, \mathtt{m}_{n-5}\right) = \left(0, \tfrac{1}{12}, \tfrac{10}{12}, \tfrac{1}{12}, 0, 0\right). \end{aligned}$$

Since $|\lambda| < h^6$, we have $T_\lambda^{(\mathtt{r})}(h) = O(h^6)$ and $T_*^{(\mathtt{r})}(h) = \frac{7}{5000} h^6 w_{tttt}^{(\mathtt{r}+1)}(t_\mathtt{j}) + O(h^7)$.
Therefore

$$T_\mathtt{j}^{(\mathtt{r})}(h) = O(h^6). \tag{4.2}$$

4.3. Sixth order method

Choose λ such that $|\lambda| < h^8$. For getting method of order six, values of unknown coefficients must satisfy conditions:

$(\alpha_1+\beta_1)=\frac{1}{2}$,
$\frac{1}{6}(7\alpha_1+\beta_1)-(4p+q)=0$,
$\frac{1}{180}(31\alpha_1+\beta_1)-\frac{1}{12}(16p+q)=0$,
$\left[\frac{1}{131040}(1611\alpha_1+31\beta_1)-\frac{1}{360}(4p+q)\right]\neq 0$.
The only set of such values are $(\alpha_1,\beta_1)=\left(\frac{1}{12},\frac{5}{12}\right)$ and $p=\frac{1}{360}, q=\frac{56}{360}, r=\frac{246}{360}$.
Also

at $\mathtt{j}=1$, $\quad \big(l_0,l_1,l_2,l_3,l_4,l_5\big)=\big(-4,7,-2,-1,0,0\big)$,
$\big(\mathtt{m}_0,\mathtt{m}_1,\mathtt{m}_2,\mathtt{m}_3,\mathtt{m}_4,\mathtt{m}_5\big)=\big(\frac{71}{240},\frac{43}{12},\frac{7}{8},\frac{1}{3},\frac{-5}{48},\frac{1}{60}\big)$,

and

at $\mathtt{j}=n-1$, $\quad \big(l_n,l_{n-1},l_{n-2},l_{n-3},l_{n-4},l_{n-5}\big)=\big(-4,7,-2,-1,0,0\big)$,
$\big(\mathtt{m}_n,\mathtt{m}_{n-1},\mathtt{m}_{n-2},\mathtt{m}_{n-3},\mathtt{m}_{n-4},\mathtt{m}_{n-5}\big)=\big(\frac{71}{240},\frac{43}{12},\frac{7}{8},\frac{1}{3},\frac{-5}{48},\frac{1}{60}\big)$.

Since $|\lambda|<h^8$, we have $T_{\lambda}^{(\mathtt{r})}(h)=O(h^8)$ and $T_{*}^{(\mathtt{r})}(h)=\frac{7}{5000}h^8 w_{tttttttt}^{(\mathtt{r}+1)}(t_{\mathtt{j}})+O(h^9)$.
Therefore

$$T_{\mathtt{j}}^{(\mathtt{r})}(h)=O(h^8). \tag{4.3}$$

Remark 3: Since $\alpha_2=\frac{1}{\xi^2}\left(\frac{1}{6}-\alpha_1\right)$ and $\beta_2=\frac{1}{\xi^2}\left(\frac{1}{3}-\beta_1\right)$,
i.e.$(\alpha_2+\beta_2)=\frac{1}{\xi^2}\left(\frac{1}{2}-(\alpha_1+\beta_1)\right)$,
therefore $(\alpha_1+\beta_1)=\frac{1}{2}$ implies $(\alpha_2+\beta_2)=0$.

5. Convergence analysis

The system given in (2.16), (2.17) and (2.18) can be written as

$$M^{(\mathtt{r})}W^{(\mathtt{r}+1)}=d^{(\mathtt{r})}, \tag{5.1}$$

where

$$M^{(\mathtt{r})}=\begin{bmatrix}
M_{1,1}^{(\mathtt{r})} & M_{1,2}^{(\mathtt{r})} & M_{1,3}^{(\mathtt{r})} & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & M_{1,n-3}^{(\mathtt{r})} & M_{1,n-2}^{(\mathtt{r})} & M_{1,n-1}^{(\mathtt{r})} \\
M_{2,1}^{(\mathtt{r})} & M_{2,2}^{(\mathtt{r})} & M_{2,3}^{(\mathtt{r})} & M_{2,4}^{(\mathtt{r})} & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & M_{2,n-3}^{(\mathtt{r})} & M_{2,n-2}^{(\mathtt{r})} & M_{2,n-1}^{(\mathtt{r})} \\
M_{3,1}^{(\mathtt{r})} & M_{3,2}^{(\mathtt{r})} & M_{3,3}^{(\mathtt{r})} & M_{3,4}^{(\mathtt{r})} & M_{3,5}^{(\mathtt{r})} & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & M_{3,n-3}^{(\mathtt{r})} & M_{3,n-2}^{(\mathtt{r})} & M_{3,n-1}^{(\mathtt{r})} \\
M_{4,1}^{(\mathtt{r})} & M_{4,2}^{(\mathtt{r})} & M_{4,3}^{(\mathtt{r})} & M_{4,4}^{(\mathtt{r})} & M_{4,5}^{(\mathtt{r})} & M_{4,6}^{(\mathtt{r})} & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & M_{4,n-3}^{(\mathtt{r})} & M_{4,n-2}^{(\mathtt{r})} & M_{4,n-1}^{(\mathtt{r})} \\
M_{5,1}^{(\mathtt{r})} & M_{5,2}^{(\mathtt{r})} & M_{5,3}^{(\mathtt{r})} & M_{5,4}^{(\mathtt{r})} & M_{5,5}^{(\mathtt{r})} & M_{5,6}^{(\mathtt{r})} & M_{5,7}^{(\mathtt{r})} & 0 & \dots & 0 & 0 & 0 & 0 & 0 & M_{5,n-3}^{(\mathtt{r})} & M_{5,n-2}^{(\mathtt{r})} & M_{5,n-1}^{(\mathtt{r})} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
M_{n-5,1}^{(\mathtt{r})} & M_{n-5,2}^{(\mathtt{r})} & M_{n-5,3}^{(\mathtt{r})} & 0 & 0 & 0 & 0 & 0 & \dots & 0 & M_{n-5,n-7}^{(\mathtt{r})} & M_{n-5,n-6}^{(\mathtt{r})} & M_{n-5,n-5}^{(\mathtt{r})} & M_{n-5,n-4}^{(\mathtt{r})} & M_{n-5,n-3}^{(\mathtt{r})} & M_{n-5,n-2}^{(\mathtt{r})} & M_{n-5,n-1}^{(\mathtt{r})} \\
M_{n-4,1}^{(\mathtt{r})} & M_{n-4,2}^{(\mathtt{r})} & M_{n-4,3}^{(\mathtt{r})} & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & M_{n-4,n-6}^{(\mathtt{r})} & M_{n-4,n-5}^{(\mathtt{r})} & M_{n-4,n-4}^{(\mathtt{r})} & M_{n-4,n-3}^{(\mathtt{r})} & M_{n-4,n-2}^{(\mathtt{r})} & M_{n-4,n-1}^{(\mathtt{r})} \\
M_{n-3,1}^{(\mathtt{r})} & M_{n-3,2}^{(\mathtt{r})} & M_{n-3,3}^{(\mathtt{r})} & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & M_{n-3,n-5}^{(\mathtt{r})} & M_{n-3,n-4}^{(\mathtt{r})} & M_{n-3,n-3}^{(\mathtt{r})} & M_{n-3,n-2}^{(\mathtt{r})} & M_{n-3,n-1}^{(\mathtt{r})} \\
M_{n-2,1}^{(\mathtt{r})} & M_{n-2,2}^{(\mathtt{r})} & M_{n-2,3}^{(\mathtt{r})} & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & M_{n-2,n-4}^{(\mathtt{r})} & M_{n-2,n-3}^{(\mathtt{r})} & M_{n-2,n-2}^{(\mathtt{r})} & M_{n-2,n-1}^{(\mathtt{r})} \\
M_{n-1,1}^{(\mathtt{r})} & M_{n-1,2}^{(\mathtt{r})} & M_{n-1,3}^{(\mathtt{r})} & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & M_{n-1,n-3}^{(\mathtt{r})} & M_{n-1,n-2}^{(\mathtt{r})} & M_{n-1,n-1}^{(\mathtt{r})}
\end{bmatrix},$$

where $W^{(\mathtt{r}+1)} = (W_1^{(\mathtt{r}+1)}, W_2^{(\mathtt{r}+1)}, \ldots, W_{n-1}^{(\mathtt{r}+1)})^T$, $M^{(\mathtt{r})}$ is coefficient matrix of $W^{(\mathtt{r}+1)}$and $d^{(\mathtt{r})} = (d_1^{(\mathtt{r})}, d_2^{(\mathtt{r})}, \ldots, d_{n-1}^{(\mathtt{r})})^T$.
Let $N^{(\mathtt{r})}(\mathtt{r})$ be the matrix when $\lambda = 0$. Note that,

$$\|M^{(\mathtt{r})} - N^{(\mathtt{r})}\|_\infty = \max_i \sum_{i=1}^{n-1} \|M_{i,\mathtt{j}}^{(\mathtt{r})} - N_{i,\mathtt{j}}^{(\mathtt{r})}\|.$$

Thus we get

$$\|M^{(\mathtt{r})} - N^{(\mathtt{r})}\|_\infty = 2\left|\frac{2(\alpha_1+\beta_1)\lambda}{h} + \frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right| + 2\left|\frac{-6(\alpha_2+\beta_2)\lambda}{h^3}\right| + 2\left|\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right|.$$

Theorem 5.1. *[7] : Let Q_1andQ_2 be any two matrices having matrix norm as $\|.\|$. If the eigen values of Q_1 are given as $\theta_1, \theta_2, \ldots, \theta_n$ and eigenvalues of Q_2 be given as $\mu_1, \mu_2, \ldots, \mu_{\mathtt{n}}$. Then*

$$\max_{\mathtt{j}} |\theta_{\mathtt{j}} - \mu_{\mathtt{j}}| \leq 2^{\frac{2N-1}{N}} N^{\frac{1}{N}} (2P)^{\frac{N-1}{N}} \|Q_1 - Q_2\|^{\frac{1}{N}}, \tag{5.2}$$

where $P = max(\|Q_1\|,\ \|Q_2\|)$.

In our case, we take the matrices $M^{(\mathtt{r})} = Q_1$, $N^{(\mathtt{r})} = Q_2$, $N = n-1$. Using $\|.\|_\infty$ in theorem 5.1, we get

$$\max_{\mathtt{j}} |\theta_{\mathtt{j}} - \mu_{\mathtt{j}}| \leq 2^{\left(\frac{2n-3}{n-1}\right)} (n-1)^{\left(\frac{1}{n-1}\right)} (2P)^{\left(\frac{n-2}{n-1}\right)} \|M^{(\mathtt{r})} - N^{(\mathtt{r})}\|_\infty^{\left(\frac{1}{n-1}\right)}, \tag{5.3}$$

where $P = max(\|M^{(\mathtt{r})}\|_\infty,\ \|N^{(\mathtt{r})}\|_\infty)$ and $M^{(\mathtt{r})}$ and $N^{(\mathtt{r})}$ have eigenvalues $\theta_{\mathtt{j}}$ *and* $\mu_{\mathtt{j}}, \mathtt{j} = 1, 2, \ldots, n-1$ respectively.

For sufficiently small values of h, $N^{(\mathtt{r})}(\mathtt{r})$ becomes irreducible, $N_{i,i}^{(\mathtt{r})} > 0$, $N_{i,\mathtt{j}}^{(\mathtt{r})} \leq 0$, $i \neq \mathtt{j}$ and the row sums give
$R_1^{(\mathtt{r})} = 4 - \frac{43}{12}h^2 q_1^{(\mathtt{r})} - \frac{7}{8}h^2 q_2^{(\mathtt{r})} - \frac{1}{3}h^2 q_3^{(\mathtt{r})} \quad > 0,$

$R_2^{(\mathtt{r})} = \frac{1}{12} - \frac{56}{360}h^2 q_1^{(\mathtt{r})} - \frac{246}{360}h^2 q_2^{(\mathtt{r})} - \frac{56}{360}h^2 q_3^{(\mathtt{r})} - \frac{1}{360}h^2 q_4^{(\mathtt{r})} \quad > 0,$

$R_{\mathtt{j}}^{(\mathtt{r})} = -\frac{1}{360}h^2 q_{i-2}^{(\mathtt{r})} - \frac{56}{360}h^2 q_{i-1}^{(\mathtt{r})} - \frac{246}{360}h^2 q_i^{(\mathtt{r})} - \frac{56}{360}h^2 q_{i+1}^{(\mathtt{r})} - \frac{1}{360}h^2 q_{i+2}^{(\mathtt{r})} \quad > 0,$
where $\mathtt{j} = 3, 4, \ldots n-3$,

$R_{n-2}^{(\mathtt{r})} = \frac{1}{12} - \frac{56}{360}h^2 q_{n-1}^{(\mathtt{r})} - \frac{246}{360}h^2 q_{n-2}^{(\mathtt{r})} - \frac{56}{360}h^2 q_{n-3}^{(\mathtt{r})} - \frac{1}{360}h^2 q_{n-4}^{(\mathtt{r})} \quad > 0,$

$R_{n-1}^{(\mathtt{r})} = 4 - \frac{43}{12}h^2 q_{n-1}^{(\mathtt{r})} - \frac{7}{8}h^2 q_{n-2}^{(\mathtt{r})} - \frac{1}{3}h^2 q_{n-3}^{(\mathtt{r})} \quad > 0.$

Here $N^{(\mathtt{r})}$ is a monotone matrix [20]. Therefore for adequately small values of h, $(N^{(\mathtt{r})})^{-1}$

exist and we get non-zero eigenvalues $\mu_{\mathtt{j}},\ \mathtt{j}=1,2,\dots n-1$. Thus for these values of h (corresponding to which $N^{(\mathtt{r})}$ is a monotone matrix), λ lies in the region $(-h^8,\ h^8)$. We select λ in such a manner that it must satisfy the following two conditions :
(i) $M^{(\mathtt{r})}$ is invertible matrix, since $\|M^{(\mathtt{r})}-N^{(\mathtt{r})}\|_\infty = 2\left|\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right|+2\left|\frac{-6(\alpha_2+\beta_2)\lambda}{h^3}\right|+2\left|\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right|$, and from (5.3) we find that eigenvalues of $M^{(\mathtt{r})}$ are non-zero, whenever λ is sufficiently small.
(ii) Since $N_{\mathtt{j}}^{(\mathtt{r})}>0,\ \mathtt{j}=1,2,\dots,n-1,$, the row sum corresponding to $M^{(\mathtt{r})}$ is

$$S_{\mathtt{j}}^{(\mathtt{r})}=R_{\mathtt{j}}-\frac{4(\alpha_1+\beta_1)\lambda}{h}-\frac{4(\alpha_2+\beta_2)\lambda}{h^3},\quad \mathtt{j}=1,2,\dots,n-1, \tag{5.4}$$

when λ is sufficiently small.
When $N^{(\mathtt{r})}$ is monotone (i.e. when h is adequately small) and $M^{(\mathtt{r})}$ invertible and row sum of $M^{(\mathtt{r})}$ is positive (i.e. for sufficiently small $\lambda\in(-h^8,h^8)$).We derive the error bound as follows:

5.1. Error Bound for Sixth order method

The system (2.16), (2.17), and (2.18) with analytic solutions can be written as

$$M^{(\mathtt{r})}\bar{w}^{(\mathtt{r}+1)}=d^{(\mathtt{r})}+\mathtt{T}^{(\mathtt{r})}(h), \tag{5.5}$$

where

$$\bar{w}^{(\mathtt{r}+1)}=(\bar{w}^{(\mathtt{r}+1)}(t_1),\bar{w}^{(\mathtt{r}+1)}(t_2),\dots,\bar{w}^{(\mathtt{r}+1)}(t_{n-1}))^T,$$

and

$$\mathtt{T}^{(\mathtt{r})}(h)=(\mathtt{T}_1^{(\mathtt{r})}(h),\mathtt{T}_2^{(\mathtt{r})}(h),\dots,\mathtt{T}_{n-1}^{(\mathtt{r})}(h))^T.$$

Since from (5.1) we have

$$M^{(\mathtt{r})}W^{(\mathtt{r}+1)}=d^{(\mathtt{r})}. \tag{5.6}$$

Using (5.5) and (5.6) we get

$$M^{(\mathtt{r})}(\bar{w}^{(\mathtt{r}+1)}-W^{(\mathtt{r}+1)})=\mathtt{T}^{(\mathtt{r})}(h),$$

that is,

$$M^{(\mathtt{r})}\mathtt{E}^{(\mathtt{r}+1)}=\mathtt{T}^{(\mathtt{r})}(h), \tag{5.7}$$

where $\mathtt{E}^{(\mathtt{r}+1)} = (\mathtt{E}_1^{(\mathtt{r}+1)}, \mathtt{E}_2^{(\mathtt{r}+1)}, \ldots, \mathtt{E}_{n-1}^{(\mathtt{r}+1)})^T$, $\mathtt{E}_{\mathtt{j}}^{(\mathtt{r}+1)} = w^{(\mathtt{r}+1)}(t_{\mathtt{j}}) - W_{\mathtt{j}}^{(\mathtt{r}+1)}$.
Consequently, using (5.7) we obtain

$$\mathtt{E}^{(\mathtt{r}+1)} = (M^{(\mathtt{r})})^{-1}\mathtt{T}^{(\mathtt{r})}(h). \tag{5.8}$$

Using the definition of product of inverse of matrix with the matrix itself, we get

$$\sum_{\mathtt{j}=1}^{n-1} M_{i,\mathtt{j}}^{(\mathtt{r})^{-1}} \; S_{\mathtt{j}}^{(\mathtt{r})} = 1, \;\; i = 1, 2, \ldots, n-1.$$

Hence by (5.4) we get

$$\sum_{\mathtt{j}=1}^{n-1} M_{i,\mathtt{j}}^{(\mathtt{r})^{-1}} \leq \frac{1}{\underset{1\leq \mathtt{j}\leq n-1}{S_{\mathtt{j}}^{(\mathtt{r})}}} = \frac{1}{C_i^{(\mathtt{r})} h^2}, \tag{5.9}$$

such that $C^{(\mathtt{r})}$ is constant. Using (5.8) and (5.9) we get

$$\mathtt{E}_i^{(\mathtt{r}+1)} = \sum_{\mathtt{j}=1}^{n-1} M_{i,\mathtt{j}}^{(\mathtt{r})^{-1}} \mathtt{T}_{\mathtt{j}}^{(\mathtt{r})}(h), \quad i = 1, 2, \ldots, n-1. \tag{5.10}$$

Substituting (4.3) and (5.9) in (5.10), we get

$$|\mathtt{E}_i^{(\mathtt{r}+1)}| \leq \frac{qh^8}{C_i^{(\mathtt{r})} h^2},$$

where q is a constant.
Hence we obtain

$$\|E\|_\infty = O(h^6),$$

which proves that the proposed scheme is sixth-order convergent. Similar procedure can be used to derive the convergence of second as well as fourth order methods.

6. Numerical experiments

We take adequate number of iterations till the maximum error between the two succeeding iterations satisfy the following tolerance bound:

$$\max_{\mathtt{j}} |W_{\mathtt{j}}^{(\mathtt{r}+1)} - W_{\mathtt{j}}^{(\mathtt{r})}| < TOL,$$

where TOL is convergence tolerance. When the condition is met, we believe $W^{(\mathtt{r}+1)}$ is the approximate value W of the given problem. Here we have considered $TOL = 10^{-15}$.
For each n, $\mathtt{E}_N$ denotes the maximum point-wise error which is determined by

$$\max_{\mathtt{j}} |w(t_{\mathtt{j}}) - W_{\mathtt{j}}|,$$

where $w(t_{\mathtt{j}})$ and $W_{\mathtt{j}}$ are the analytic and approximate solutions respectively at $t = t_{\mathtt{j}}$. Order of convergence of the proposed method is determined as

$$\mathrm{p}^n = log_2\left(\frac{\mathrm{E}^n}{\mathrm{E}^{2n}}\right).$$

6.1. Numerical Schemes for comparison

As we compare the presented method with Numerov's method and second order finite difference method, here we give a brief particulars about these two methods.

6.1.1. Finite-difference method

Consider BVP given in (1.1) and (1.2), let $W^{(\mathtt{r}+1)}$ be the approximate value of $w^{(\mathtt{r}+1)}(t)$. Putting

$$W_{tt}^{(\mathtt{r}+1)(t)} \approx \frac{1}{h^2}\left[W_{\mathtt{j}-1}^{(\mathtt{r}+1)} - 2W_{\mathtt{j}}^{(\mathtt{r}+1)} + W_{\mathtt{j}+1}^{(\mathtt{r}+1)}\right], \tag{6.1}$$

in (1.2) and after simplifying, we get

$$W_{\mathtt{j}-1}^{(\mathtt{r}+1)} + \left[-2 + h^2 q_{\mathtt{j}}^{(\mathtt{r}+1)}\right] W_{\mathtt{j}}^{(\mathtt{r}+1)} + W_{\mathtt{j}+1}^{(\mathtt{r}+1)} = h^2 \mathtt{f}_{\mathtt{j}}^{(\mathtt{r})}, \tag{6.2}$$

for $\mathtt{j} = 1,2,\ldots n$. Here $W_0 = \sigma_0$ and $W_1 = \sigma_1$.

6.1.2. Numerov's method

For BVP given in (1.1) and (1.2), Numerov's method can be written as

$$W_{\mathtt{j}-1} - 2W_{\mathtt{j}} + W_{\mathtt{j}+1} = \frac{h^2}{12}\left[\mathtt{f}_{\mathtt{j}-1} + 10\mathtt{f}_{\mathtt{j}} + \mathtt{f}_{\mathtt{j}+1}\right], \tag{6.3}$$

where $\mathtt{f}_{\mathtt{j}} = \mathtt{f}(t_{\mathtt{j}}, W_{\mathtt{j}}),\ \mathtt{j} = 0,1\ldots n,\ W_0 = \sigma_0$ and $W_1 = \sigma_1$. To get more details about this method, one can refer [12].

Problem 1: Consider the following linear BVP[25, 31]

$$\begin{cases} w_{tt}(t) + w(t) = -1, & 0 < t < 1, \\ w(0) = 0, \quad w(1) = 0, \end{cases} \tag{6.4}$$

with exact solution $w(t) = cos(t) + \frac{1-cos(1)}{sin(1)} sin(t) - 1$. Approximate results are shown in Table 1 along with results given by Srivastava et al.[31] and Ramadan et al.[25]. λ varies according to the order of method.

Problem 2: Consider the following nonlinear BVP[3]

$$\begin{cases} w_{tt}(t) + exp(-2w(t)) = 0, & 0 < t < 1, \\ w(0) = 0, \qquad w(1) = log(2), \end{cases} \tag{6.5}$$

Table 1: **M.A.E. for problem 1.**

h	**1/8**	**1/16**	**1/32**	**1/64**
Second Order Method $p = 0.04063483994113$, $q = 0.25412730690212$, $r = 0.41047570631347$	1.5516×10^{-03}	2.0410×10^{-04}	3.0770×10^{-05}	5.2534×10^{-06}
p^N	2.9263	2.7296	2.5502	
$(p,q,r) = \left(\frac{1}{4}, 0, \frac{1}{2}\right)$	3.4324×10^{-03}	6.0707×10^{-04}	1.2491×10^{-04}	2.8070×10^{-05}
p^N	2.4992	2.2809	2.1538	
Fourth Order Method $(p,q,r) = \left(\frac{1}{120}, \frac{26}{120}, \frac{66}{120}\right)$	1.9214×10^{-05}	5.8656×10^{-07}	1.7739×10^{-08}	5.2095×10^{-10}
p^N	5.0337	5.0472	5.0896	
$(p,q,r) = \left(\frac{1}{720}, \frac{11}{45}, \frac{183}{360}\right)$	1.9558×10^{-05}	6.0424×10^{-07}	1.8788×10^{-08}	5.8564×10^{-10}
p^N	5.0164	5.0072	5.0036	
Sixth Order Method $(p,q,r) = \left(\frac{1}{360}, \frac{56}{360}, \frac{246}{360}\right)$	2.6594×10^{-07}	2.2124×10^{-09}	1.6972×10^{-11}	1.2678×10^{-13}
p^N	6.9093	7.0262	7.0646	
Srivastava et al.[31]	7.1329×10^{-08}	5.2213×10^{-09}	3.6359×10^{-10}	3.1275×10^{-11}
p^N	3.7720	3.8440	3.5392	
Ramadan et al.[25]	1.7538×10^{-04}	2.1600×10^{-05}	2.6770×10^{-06}	3.3310×10^{-07}
p^N	3.0213	3.0123	3.0065	

with exact solution $w(t) = log(1+t)$. Approximate results are shown in Table 2 along with results given by Balasubramani et al.[3], finite difference method and Mohanty et al.[24].

Table 2: **M.A.E for problem 2.**

h	**1/8**	**1/16**	**1/32**	**1/64**
Second Order Method $(p,q,r) = \left(\frac{1}{4}, 0, \frac{1}{2}\right)$	1.9977×10^{-03}	4.5767×10^{-04}	1.1324×10^{-04}	2.8198×10^{-05}
p^N	2.1259	2.0148	2.0058	
$(p,q,r) = \left(\frac{1}{4}, \frac{1}{4}, 0\right)$	2.7119×10^{-03}	6.2781×10^{-04}	1.5566×10^{-04}	3.8770×10^{-05}
p^N	2.1109	2.0119	2.0053	
Fourth Order Method				
$(p,q,r) = \left(\frac{1}{720}, \frac{11}{45}, \frac{183}{360}\right)$	2.6377×10^{-05}	9.0287×10^{-07}	3.0209×10^{-08}	1.0626×10^{-09}
p^N	4.8686	4.9014	4.8292	
Balasubramani et al.[3] $(p,q,r) = \left(\frac{1}{120}, \frac{26}{120}, \frac{66}{120}\right)$	3.7039×10^{-06}	1.3093×10^{-07}	4.6024×10^{-09}	1.6823×10^{-10}
p^N	4.8222	4.8303	4.7739	
Sixth Order Method $(p,q,r) = \left(\frac{1}{360}, \frac{56}{360}, \frac{246}{360}\right)$	2.4456×10^{-07}	2.1358×10^{-09}	1.6419×10^{-11}	1.1984×10^{-13}
p^N	6.8392	7.0233	7.0980	
Finite difference method	2.2281×10^{-04}	5.6130×10^{-05}	1.4060×10^{-05}	3.5166×10^{-06}
p^N	1.9890	1.9972	1.9993	
Mohanty et al.[24]	1.6424×10^{-05}	1.0481×10^{-06}	6.5976×10^{-08}	3.8966×10^{-09}
p^N	3.9699	3.9896	4.0816	

Problem 3: Consider the following nonlinear BVP[3]

$$\begin{cases} w_{tt}(t) - \frac{(2-t)exp(2w(t))+(1/(t+1))}{3} = 0, & 0 < t < 1, \\ w(0) = 0, \qquad w(1) = log(1/2), \end{cases} \tag{6.6}$$

with exact solution $w(t) = log(1/1+t)$. Approximate results are shown in Table 3 along with results given by Balasubramani et al.[3], finite difference method and Numerov's method.

Table 3: **M.A.E for problem 3.**

h	**1/8**	**1/16**	**1/32**	**1/64**
Second Order Method $(p,q,r)=\left(\frac{1}{4},0,\frac{1}{2}\right)$	1.3688×10^{-03}	4.1286×10^{-04}	1.1600×10^{-04}	3.0846×10^{-05}
p^N	1.7292	1.8314	1.9110	
$(p,q,r)=\left(\frac{1}{4},\frac{1}{4},0\right)$	2.3839×10^{-03}	6.2248×10^{-04}	1.6526×10^{-04}	4.2528×10^{-05}
p^N	1.9372	1.9132	1.9582	
Fourth Order Method $(p,q,r)=\left(\frac{1}{720},\frac{11}{45},\frac{183}{360}\right)$	2.7594×10^{-05}	9.4434×10^{-07}	3.1573×10^{-08}	1.1062×10^{-09}
p^N	4.8689	4.9025	4.8349	
Balasubramani et al.[3] $(p,q,r)=\left(\frac{1}{120},\frac{26}{120},\frac{66}{120}\right)$	3.8662×10^{-06}	1.3680×10^{-07}	4.8082×10^{-09}	1.7524×10^{-10}
p^N	4.8207	4.8304	4.7781	
Sixth Order Method $(p,q,r)=\left(\frac{1}{360},\frac{56}{360},\frac{246}{360}\right)$	1.3851×10^{-07}	1.2157×10^{-09}	6.9262×10^{-12}	1.2062×10^{-13}
p^N	6.8320	7.4555	5.8434	
Finite difference method	2.3261×10^{-04}	5.8573×10^{-05}	1.4670×10^{-05}	3.6702×10^{-06}
p^N	1.9890	1.9974	1.9989	
Numerov′s Method	2.1034×10^{-06}	1.3382×10^{-07}	8.4017×10^{-09}	5.2577×10^{-10}
p^N	3.9743	3.9935	3.9982	

Problem 4: Consider the following nonlinear BVP[3]

$$\begin{cases} w_{tt}(t) - \frac{25t^8 exp(w(t)) - 20t^3}{4+t^5} = 0, \quad 0 < t < 1, \\ w(0) = -log(4), \qquad w(1) = -log(5), \end{cases} \tag{6.7}$$

with exact solution $w(t) = -log(4+t^5)$. Approximate results are shown in Table 4 along with results given by Balasubramani et al.[3], finite difference method and Numerov's method.

Table 4: **M.A.E for problem 4.**

h	**1/8**	**1/16**	**1/32**	**1/64**
Second Order Method $(p,q,r) = \left(\frac{1}{4}, 0, \frac{1}{2}\right)$	5.5212×10^{-03}	1.2773×10^{-03}	3.3060×10^{-04}	8.4161×10^{-05}
p^N	2.1118	1.9500	1.9738	
$(p,q,r) = \left(\frac{1}{4}, \frac{1}{4}, 0\right)$	9.5912×10^{-03}	2.0448×10^{-03}	5.2840×10^{-04}	1.3576×10^{-04}
p^N	2.2296	1.9523	1.9605	
Fourth Order Method $(p,q,r) = \left(\frac{1}{720}, \frac{11}{45}, \frac{183}{360}\right)$	6.2487×10^{-05}	1.0123×10^{-06}	3.8928×10^{-08}	2.7550×10^{-09}
p^N	5.9477	4.7007	3.8206	
Balasubramani el al.[3] $(p,q,r) = \left(\frac{1}{120}, \frac{26}{120}, \frac{66}{120}\right)$	3.9439×10^{-06}	3.3929×10^{-07}	1.4653×10^{-08}	6.6424×10^{-10}
p^N	3.5391	4.5332	4.4633	
Sixth Order Method $(p,q,r) = \left(\frac{1}{360}, \frac{56}{360}, \frac{246}{360}\right)$	5.1118×10^{-06}	1.2322×10^{-08}	2.1551×10^{-10}	3.1186×10^{-12}
p^N	8.6963	5.8374	6.1107	
Finite difference Method	1.1795×10^{-03}	2.9324×10^{-04}	7.3024×10^{-05}	1.8265×10^{-05}
p^N	2.0080	2.0056	1.9992	
Numerov's Method	3.0070×10^{-05}	1.8480×10^{-06}	1.1585×10^{-07}	7.2337×10^{-09}
p^N	4.0242	3.9956	4.0014	

7. Conclusion

This study deals with developing second, fourth and sixth order convergent numerical schemes by using fractal non-polynomial spline function. With the help of quasilinearisation technique, the non-linear BVPs is converted into a system of linear BVPs, which in turn are solved by using the proposed schemes. These schemes are used to find approximate solution

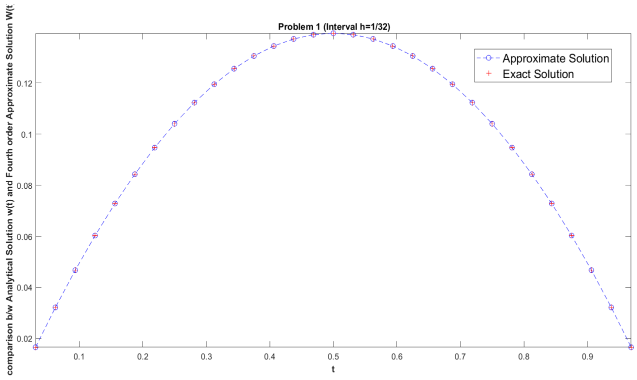

Figure 1: Relationship between analytical and approximate solution for problem 1.

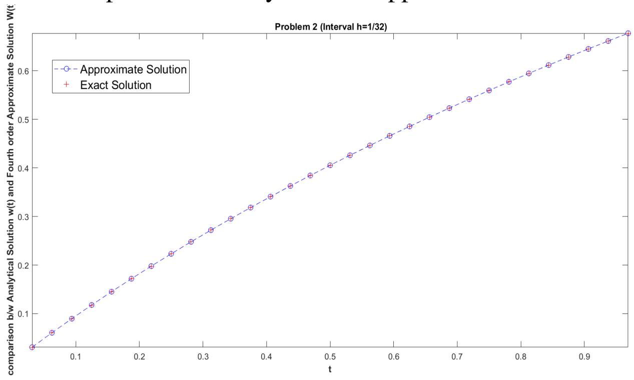

Figure 2: Relationship between analytical and approximate solution for problem 2.

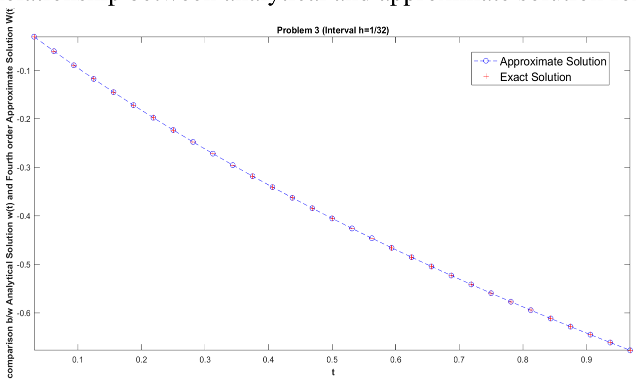

Figure 3: Relationship between analytical and approximate solution for problem 3.

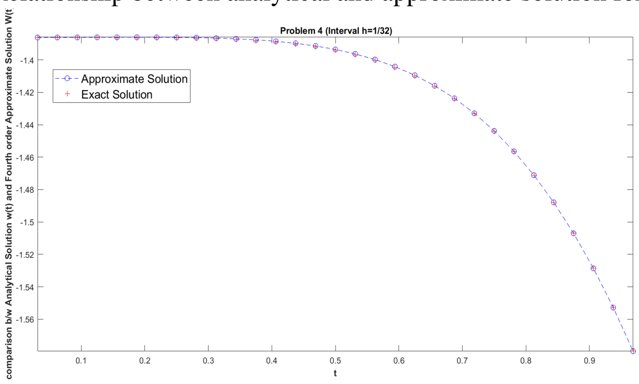

Figure 4: Relationship between analytical and approximate solution for problem 4.

of second order linear as well as nonlinear BVPs. Comparison with polynomial fractal quintic spline and few other methods leads us to the conclusion that the presented methods are more efficient.

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