Proposed Method for Solving Quasi-Linear Fractional Partial Differential Equations Using α-Fractional Derivative

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ABSTRACT
The fractional partial differential equation (FPDE) plays a big role in engineering and applied science. Finding the solutions to the FPDEs is a significant subject and a wide field. The objective of this article is to use the α-fractional derivative definition for converting the FPDE to a partial differential equation (PDE) and then, we use the method of lines for solving a quasi-linear PDE. The characteristic of the α-fractional derivative definition is appropriate, significant, and powerful. Additionally, the properties of the definition ofα-fractional derivative are used for converting the quasi-linear (FPDE) to a PDE. Hence, the PDE is converted to a system of ordinary differential equations(ODEs) by using the method of lines (MOL). Some implementations are solved using the proposed method and then, compared with exact or numerical solutions. The test implementations showed that the proposed method agrees well with their solutions. Hence, the algorithm of this method proved to be efficient and accurate.

Keyword: α-Fractional Derivative, ODEs, PDEs, FPDEs.

1. INTRODUCTION

Fractional calculus is a subject of great interest in the 20^{th} and 21^{st} centuries, especially over the past three decades. It appears extensively in both pure and applied mathematics, as well as in the applied fields of science and engineering. A lot of researchers have examined some definitions of FDEs. For instance, they have presented some modern concepts in the definitions of fractional derivatives (FDs),
as well as development of the methods of solving FDEs [1–5]. However, several authors introduced some new definitions for FDs. For example,a novel definition of the FD is introduced by Khalil et al [6], while a significant definition of the α-fractional integral and the α-FD of real functions is introduced by Mechee et al. [7]. Also, Zheng et al. were proposed a definition of Caputo type for FD then, they studied its properties [8]. Furthermore, we will use some properties of these definitions of FDs in solving the FPDEs.
Accordingly, the literature review on FDs and solving FDEs can introduced as follows: conformable fractional differential transform(CFDT) and its application introduced by Unal and Gan [9]. On the other hand, the second-order conjugate boundary value problems (BVPs) have been reformulated by Anderson and Avery using the new definition of FD [10] while Hammad and Khalil have examined Legendre conformable FDEs and their fundamental properties [11] Similarly, the existence of conformable fractional is demonstrated by Abdel Hakim [12]. Khalil and Abu-Hammad have studied the exact solution of the heat-conformable FDE. Furthermore, Abdel Jawad is established the fundamental concepts in FDs, and then, he developed the definition of the conformable FD[13].
Additionally, Ortega and Rosales were presented the conformable FD features[14], while the way how to solve specific composition DEs of fractional order types which related to optimal control issues is investigated by Qasim and Holel [15].In a similar, Euler and RK methods for solving some classes of FDEs have been generalized by the authors in [16–17].Lastly, some authors analyze some types of the FDEs and then, they studied their solutions [18-23].
In this article, we solved the quasi-linear FPDEs by using some properties of the specific definition of α-fractional FD of real functions [7]. Accordingly, FPDE is transformed into a PDE using α-fractional

transform. This α-transformation proved to be the most effective approach for transforming the FPDE into an PDE.

2. Background of Fractional Derivatives

Some definitions and concepts relevant to this subject have been introduced in this section. The classical and modern definitions of FDs of the function $f(\tau)$ have introduced.

Firstly, Riemann-Liouville Derivative [1] defined the fractional Riemann-Liouville integral operator of left side with order $\alpha > 0$ as follows:

$$ {}^{a}D_{\tau}^{\alpha}\, \emptyset(\gamma) = \frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dt^m}\int_a^{\gamma}\frac{\emptyset^{(m)}(\gamma)}{(\gamma-\mu)^{m-\alpha-1}}\,d\mu, \qquad (1)$$

where the fractional number α satisfy the in equalitym - $1 \le \alpha <$ mwherem∈ N and the function $\emptyset: [a,\infty) \to \mathcal{R}$ is continuous in its domain.

Secondly, Khalil et. al [5] introduced the comfortable FD of the function $\emptyset(\gamma) : [a,\infty) \to \mathcal{R}$. as follows:

$$T_\alpha\big(\emptyset(\gamma)\big) = \emptyset^{(\alpha)}(\gamma) = \lim_{\epsilon\to 0}\frac{\emptyset(\gamma+\epsilon\gamma^{\,1-\alpha}) - \emptyset(\gamma)}{2\epsilon}. \qquad (2)$$

for $\alpha \in (0,1]$.

Lastly, Mechee et al. [6] has introduced α-FD of the function $f(\tau)$is as follows:

$$T_\alpha\big(\emptyset(\gamma)\big) = \emptyset^{(\alpha)}(\gamma) = \lim_{\epsilon\to 0}\frac{\emptyset(\gamma+\epsilon\gamma^{\,1-\alpha}) - \emptyset(\gamma-\epsilon\gamma^{\,1-\alpha})}{2\epsilon}. \qquad (3)$$

3. Fractional Diffusion Equation

In general, the quasi-linear FPDEs of order n have the following form

$$\frac{\partial^\alpha u(\tau,\xi)}{\partial\tau^\alpha} = \emptyset\left(u(\tau,\xi), \frac{\partial u(\tau,\xi)}{\partial\xi}, \frac{\partial^2 u(\tau,\xi)}{\partial\xi^2}\right), \qquad 0<\xi<l, \tau>0, \qquad (4)$$

with the initial-condition (IC)

$u(0,\xi) = f(\xi)$, for $0 < \xi < l$, (5)

and the Dirichlet-boundary conditions(DBCs)

$$u(\tau,0) = u(\tau,l) = 0 \text{ for } \tau > 0. \qquad (6)$$

In special case consider a fractional diffusion equation in one dimension:

$$\frac{\partial^\alpha u(\tau,\xi)}{\partial\tau^\alpha} = \beta^2\frac{\partial^2 u(\tau,\xi)}{\partial\xi^2}, \qquad 0<\xi<l, \qquad \tau>0, \qquad (7)$$

with the IC and the DBCs in equations (5)-(6)and

Proposition 1: Let $u(\tau,x)$ be differential function in the domain $\mathfrak{H} \sqsubset \mathbb{R}\times\mathbb{R}$

Then, $\frac{\partial^\alpha u(\tau,x)}{\partial\tau^\alpha} = \tau^{1-\alpha}\, u_\tau(\tau,x)and\frac{\partial^\alpha u(\tau,x)}{\partial x^\alpha} = x^{1-\alpha}\, u_x(\tau,x)$.

Proof

From the definitions in the Equation (2) and (3), we get the following

$$T_\alpha\big(\emptyset(\tau)\big) = \emptyset^{(\alpha)}(\tau) = \tau^{\,1-\alpha}\emptyset'(\tau). \qquad (8)$$

However, we can convert the fractional partial derivative to a partial derivative as follows:

$$\frac{\partial^\alpha u(\tau,x)}{\partial\tau^\alpha} = \lim_{\epsilon\to 0}\frac{u(\tau+\epsilon\tau^{\,1-\alpha},x) - u(\tau,x)}{\epsilon},$$

$$= \lim_{\epsilon\to 0}\frac{u(\tau,x) + \epsilon\tau^{\,1-\alpha}u_\tau(\tau,x) + \frac{(\epsilon\tau^{\,1-\alpha})^2}{2!}u_{\tau\tau}(\tau,x) + \cdots - u(\tau,x)}{\epsilon}.$$

Then,$\frac{\partial^\alpha u(\tau,x)}{\partial\tau^\alpha} = \tau^{1-\alpha}\, u_\tau(\tau,x)$.

In the same way, we can prove the second relation

$$\frac{\partial^\alpha w(\tau,x)}{\partial x^\alpha} = x^{1-\alpha}\, w_x(\tau,x). \qquad (9)$$

From proposition 1, the FPDE in Equation (4) and (5) are converted to the FPDEs $w_\tau(\tau,x) = \tau^{\,\alpha-1}\,\emptyset\left(\frac{\partial^2 w(\tau,x)}{\partial x^2}, \frac{\partial w(\tau,x)}{\partial x}, w(\tau,x)\right), \quad 0<x<l, \tau>0,$ (10)and$w_\tau(\tau,x) = \beta^2\tau^{\,\alpha-1}\frac{\partial^2 w(t,x)}{\partial x^2}$, respectively. The last PDE is heat equation with variable coefficient.

4. Proposed Method

For solving the quasi-linear FPDE in Equation (5) with the IC and DBCs in Equations (5) and (6). The proposed method developed by combining the method of lines (MOL) with the finite difference method and the numerical method of type RK. The following steps of the algorithm should be do

4.1 Proposed Algorithm

1. The FPDE in Equation (4) is converted to PDE in Equation (10) with its ICs and BCs in Equations (5) and (6) respectively.
2. Divide the domain of the problem in the variable x in [0, l] by n subinterval with the norm of partition $h = \frac{l}{n}$and in the variable τ in [0, T] by m subinterval with the norm of partition$k = \frac{T}{m}$.
3. Do steps 4-6 while $1 \le j \le m$.
4. Put instead the pointw (τ, x) by$w_{ij} = (\tau_i, x_j)$ of the PDE in Equation (10)for i=0,1, 2, …, m and j=0,1, 2,…,n.
5. Fix $\tau=\tau_j$ at the left side of Equation (10)and put the formulas of finite difference in the derivatives in the right side which converting the PDE in Equation (10) to following system of ODEs:
   $w_i'(\tau_j) = {\tau_j}^{\alpha-1}\, \emptyset(w_{i-3},\ w_{i-2}, w_{i-1}, w_i, w_{i+1}, w_{i+2}, w_{i+3})$, (11)with the I.C.s, in$w_{0j} = w(\tau_0, x_j) = f(x_j)$, and the Dirichlet-boundary conditions (DBCs)$w_{j0} = w(\tau_j, 0) = w(\tau_j, 0) = 0$, (12)
   for $1 \le j \le m$ and i=1, 2,…,n.
6. Solve the system of ODEs of first-order in Equation (11) and B.Cs in Equation (12) using the RK-type method.

In general, this algorithm can generalize for solving a FPDE of nth-order with I.Cs. and B.Cs.

5. Implementation

In this section, we implemented some examples to use the proposed method.

Example 1: Consider the following FPDE of α-order

$$\frac{\partial^{\frac{1}{2}} u(\tau, x)}{\partial \tau^{\frac{1}{2}}} = \beta^2 \sqrt{\tau} \frac{\partial u(\tau, x)}{\partial x}, \qquad 0 < x < l,\ \ \tau > 0, \qquad (13)$$

with the ICs$u(0, x) = e^{-x}$, for $0 < x < l$,
and DBCs $e^{-1}\ u(\tau, 0) = u(\tau, l) = e^{-1}e^{-\tau}$for $0 < \tau < 1,\ \beta = 1$.
From proposition 1, the FPDE in Equation (12) converted to the following FPDE

$$\frac{\partial u(\tau, x)}{\partial \tau} = \frac{\partial u(\tau, x)}{\partial x}, 0 < x < l, \qquad \tau > 0, \qquad (14)$$

with the same ICs and DBCs. which is PDE of first order with constant coefficients.
By solving Equation (14) with its ICs and DBCs analytically using the separation method, we obtain the exact solution $u(\tau, x) = e^{-\tau}e^{-x}$which satisfy the FPDE in Equation (13). However, this problem give a prove for efficacy of the proposed method.

Example 2: Given the following FPDE of α-order

$$\frac{\partial^{\frac{1}{2}} u(\tau, x)}{\partial \tau^{\frac{1}{2}}} = \beta^2 \sqrt{\tau} \frac{\partial^2 u(\tau, x)}{\partial x^2}, \qquad 0 < x < l, \qquad \tau > 0. \qquad (15)$$

with the IC $u(x, 0) = \sin(\pi x)$, for $0 < x < l$, and the DBCs$u(\tau, 0) = u(\tau, l) = 0$, for τ>0, $\beta = 1$.
From proposition 1, the FPDE in Equation (7) converted to the FPDE

$$\frac{\partial u(\tau, x)}{\partial \tau} = \frac{\partial^2 u(\tau, x)}{\partial x^2}, \qquad 0 < x < l, \qquad \tau > 0. \qquad (16)$$

which is heat equation with constant coefficient. The numerical solutions are evaluated using the proposed algorithmin subsection 4.1, by compound the MOL method with the numerical RK and Euler methods.

Example 3: Given the following FPDE of α-order

$$\frac{\partial^{\frac{1}{4}} u(\tau, x)}{\partial \tau^{\frac{1}{4}}} = \sqrt[4]{\tau^3}\left(-(1 + 2x)e^{-\tau} - u(\tau, x) + \frac{\partial u(\tau, x)}{\partial x} + \frac{\partial^2 u(\tau, x)}{\partial x^2}\right),$$
$$0 < x < l, \qquad \tau > 0, \qquad (17)$$

with the IC u(x,0) =x(x-1), for$0 < x < l$, andDBCs$u(\tau, 0) = u(\tau, l) = 0$ for τ>0.
From proposition 1, the FPDE in Equation (7) converted to the FPDE

$$\frac{\partial \mathrm{u}(\mathrm{t},\mathrm{x})}{\partial \tau} = -(1+2\mathrm{x})\mathrm{e}^{-\tau} - \mathrm{u}(\tau,\mathrm{x}) + \frac{\partial \mathrm{u}(\tau,\mathrm{x})}{\partial \mathrm{x}} + \frac{\partial^2 \mathrm{u}(\tau,\mathrm{x})}{\partial \mathrm{x}^2}, \quad 0 < \mathrm{x} < \mathrm{l}, \qquad \tau > 0, \qquad (18)$$

which is nonhomogeneous PDE with constant coefficient. The numerical solutions of the problem in Equation (18) are evaluated using the proposed algorithm in subsection 4.1, by compound the MOL method with the numerical RK and Euler methods.

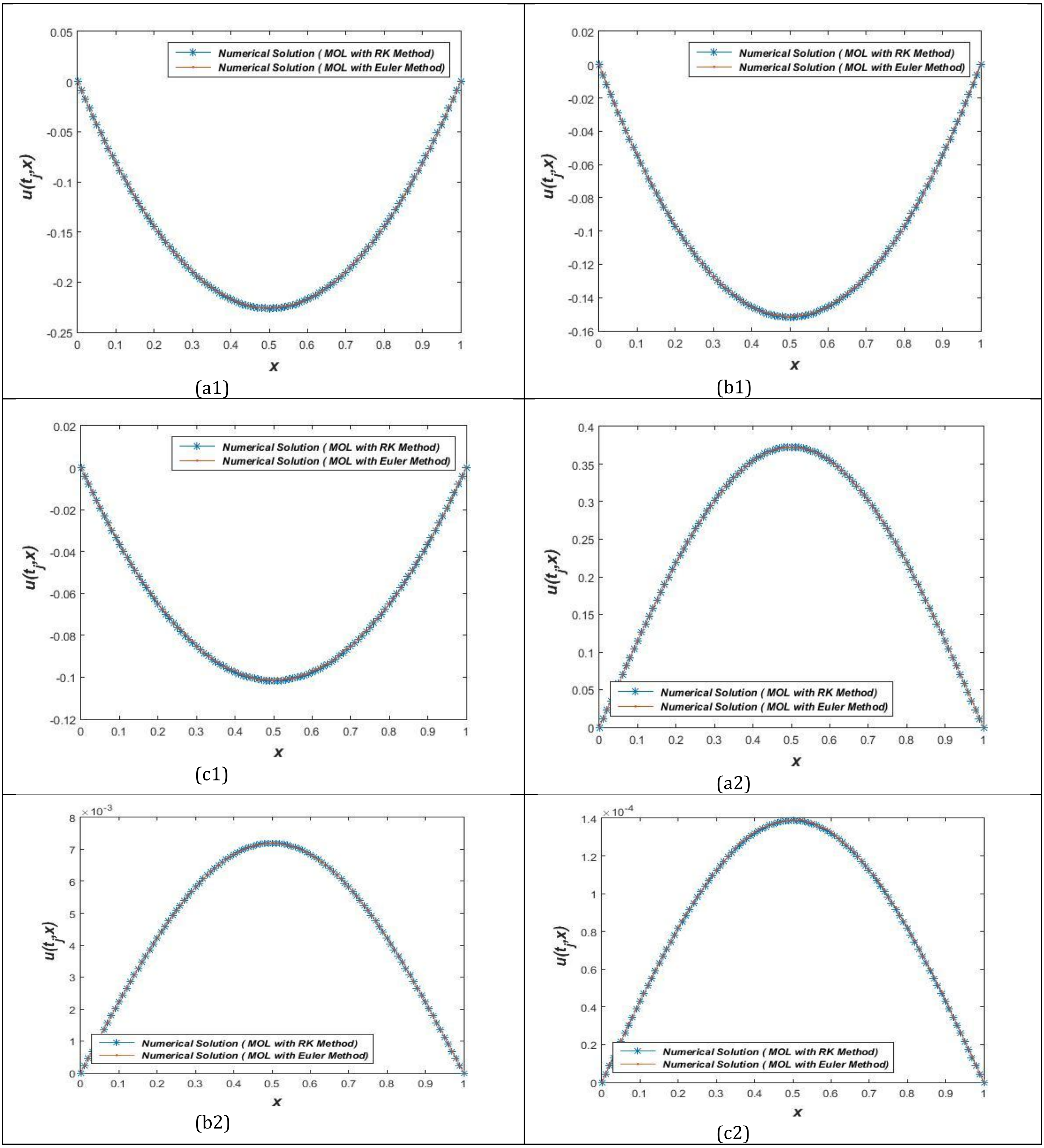

Figure 1: A Numerical Comparison of the Solutions of Example 1 ((a1) $\tau = 0.1$, ((b1) $\tau = 0.5$ and ((c1) $\tau = 0.9$) and Example 2 ((a2) $\tau = 0.1$, ((b2) $\tau = 0.5$ and ((c2) $\tau = 0.9$) Using MOL with RK and Euler Methods.

6. DISCUSSION AND CONCLUSION

The proposed method converts FPDE to PDE, and then it converts it to a system of ODEs that can be solved analytically or numerically. The numerical approach is used by combining MOL with classical numerical RK methods for solving the systems of ODEs. This approach is used for solving some test problems, showing that it agrees well with the exact solutions. These implementations show the accuracy and efficiency of this approach. Accordingly, for the first-case in in Examples 1, the approximated solution is calculated by using the proposed method which is identical with its exact solution, while for the second case, the numerical solutions of Examples 2 and 3 are computed using by compound the MOL method with the numerical RK and Euler methods and then, two numerical solutions of these examples are compared and plotted in Figure 1 for different three cases for each example.

The numerical comparisons for Examples 2 and 3 in Figure 1 and the analytical solution of the FDE in Example 1, we can conclude the powerful and efficiency of the proposed α-FD method.

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