Solution of Fuzzy Linear System by Parametric Forms of Triangular Fuzzy Number

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Received: 16.07.2024 Revised: 22.08.2024 Accepted: 25.09.2024

ABSTRACT
In this paper, a fuzzy linear system is investigated by using single and double parametric forms of triangular fuzzy number. Conditions for the existence of strong and weak fuzzy solutions are derived. Further some problems are solved to illustrate the efficacy and reliability of the proposed method.

Keywords: Fuzzy Number, Triangular fuzzy number, α− cut set, Parametric form of fuzzy numbers

1. INTRODUCTION

The system of simultaneous linear equations is crucial in numerous fields such as mathematics, operational research, physics, statistics, and engineering. In numerous applications the involved parameters are necessary to be solved that are present in the System of simultaneous linear equation.The form of a linear equation that is real is MW = d where the real matrix A and d are crisp and the real vector W is unknown. Having clearly numbered variables for the variables involved in the system of equations will make solving it easier and simpler. Observation, experiments and experience is usually used to estimate variables because of the uncertainty or vagueness of the parameters. Fuzzy numbers can be used instead of crisp numbers to overcome uncertainty and vagueness.

Zadeh [30] first presented the notion of fuzzy sets and fuzzy numbers in 1965. Friedman et al. introduced the generalized ntimesFSLE solution in 1998, The components of them are the coefficient matrix and the right-hand column vector, which are defined as crisp and fuzzy respectively. Allahviranloo[[4],[5]], Abbasbandy et al. [2], Chakraverty and Behera [11] are among the authors who have suggested solutions to this type of systems.

Friedman [22] made it possible to turn the system into a 2n×2n crisp by implementing embedding method. When using the method, a strong solution is determined by the fact that the found solution is a vector of fuzzy numbers, while a weak solution is determined by the fact that it is not a vector of fuzzy numbers. A new method has been proposed by Babbar et al. [10] to find a non-negative solution for a fully fuzzy linear system, in which coefficient elements are included as arbitrary triangular fuzzy numbers of the form (e, f,g). In [[9], [14] Behera describe a method for solving real and complex fuzzy system of linear equations. One of the major applications using fuzzy number is treating linear systems whose parameters are all or partially represented by fuzzy numbers [[1],[3], [6],[7][16][21][28][29]].

In this paper, a fuzzy linear system is investigated by using single and double parametric forms of triangular fuzzy number. Conditions for the existence of strong and weak fuzzy solutions are derived. Further some problems are solved to illustrate the efficacy and reliability of the proposed method.

2. Preliminaries

Definition 2.1 [22] **Fuzzy Number**:

A fuzzy number is a map $f: \mathbb{R} \to [0,1]$ which satisfies:

1. $\exists a_0 \in \mathbb{R}$ that is equal to 1, then f is normal. where f is the membership function of the fuzzy set.
2. $f\{\omega a + (1-\omega)b\} \geq \min\{f(a), f(b)\}, \omega \in [0,1]$ and for every, $a, b \in \mathbb{R}$, then f is said to be convex fuzzy set.
3. $f(a_0)$ is piece wise continuous.

Definition 2.2 [10] A Triangular fuzzy number F is a convex, normalised fuzzy set $\widetilde{A}$ of the real line $\mathbb{R}$ such that,

1. $\exists$ exactly one $f \in \mathbb{R}$ such that $f(a_0) = 1$
2. $f(a_0)$ is piecewise continuous.

Let $\tilde{F} = (e, f, g)$, be an arbitrary fuzzy number then the membership function of triangular fuzzy number is defined as:

$$f(a) = \begin{cases} 0, & a \leq e; \\ \dfrac{a-e}{f-e}, & e \leq a \leq f; \\ \dfrac{g-a}{g-f}, & g \leq a \leq f; \\ 0, & a \geq g \end{cases}$$

Definition 2.3 [2] A fuzzy set A that is defined on X and $\varepsilon \in [0,1]$ can be called the ε-cut set. Since it contains all the elements of the fuzzy set A for which the membership function f is greater or equal to ε. It is denoted by $\tilde{A}_\varepsilon$ or $\tilde{A}^\varepsilon$ and is defined as:

$$\tilde{A}_\varepsilon = \{a \in X | f(a) \geq \varepsilon\}.$$

Definition 2.4 [15] An ordered pair of functions can be used to represent $\tilde{F} = (e, f, g)$, which is the triangular fuzzy number$\tilde{F} = (e, f, g)$.

$$[\underline{F}(\varepsilon), \overline{F}(\varepsilon)] = [(f-e)\varepsilon + e, -(g-f)\varepsilon + g], \varepsilon \in [0,1]$$

The acronym ε-cut form refers to the parametric form or single parametric form of fuzzy numbers.
The upper and lower bounds of fuzzy numbers meet the following criteria, as specified.

1. $\underline{F}(\varepsilon)$ is a bounded, left-continuous, increasing function over [0,1].
2. $\overline{F}(\varepsilon)$ is a bounded, right-continuous, decreasing function over [0,1].
3. $\underline{F}(\varepsilon) \leq \overline{F}(\varepsilon), 0 \leq (\varepsilon) \leq 1$.

Definition 2.5 [18] In accordance with the parametric form defined in the definition 2.4,

$$\tilde{F} = [\underline{F}(\varepsilon), \overline{F}(\varepsilon)].$$

Then double parametric form, represent in a crisp form as:

$$\tilde{F}(\varepsilon, \lambda) = \lambda\left(\overline{F}(\varepsilon) - \underline{F}(\varepsilon)\right) + \overline{F}(\varepsilon),$$

where$\varepsilon, \lambda \in [0,1]$.

Definition 2.6 [22] If all members of the matrix $[\tilde{M}] = (\tilde{m}_{kj})$ are fuzzy numbers, the matrix is known to be a fuzzy matrix. If all members of $[\tilde{M}]$ are non-negative fuzzy numbers, the fuzzy matrix $[\tilde{M}]$ will be non-negative, as indicated by $[\tilde{M}] \geq 0$.

Definition 2.7 [26] The system of linear equations that involves $n \times n$ is written as:

$$\begin{aligned} \tilde{m}_{11}\tilde{\omega}_1 + \tilde{m}_{12}\tilde{\omega}_2 + \cdots + \tilde{m}_{1n}\tilde{\omega}_n &= \tilde{d}_1 \\ \tilde{m}_{21}\tilde{\omega}_1 + \tilde{m}_{22}\tilde{\omega}_2 + \cdots + \tilde{m}_{2n}\tilde{\omega}_n &= \tilde{d}_2 \\ \vdots \quad \vdots \quad \vdots \quad & \quad \vdots \\ \tilde{m}_{n1}\tilde{\omega}_1 + \tilde{m}_{n2}\tilde{\omega}_2 + \cdots + \tilde{m}_{nn}\tilde{\omega}_n &= \tilde{d}_n \end{aligned}$$

The system can be expressed in matrix notation as $[\tilde{M}]\tilde{\{W\}} = \tilde{\{d\}}$, in which the fuzzy $n \times n$ matrix $[\tilde{M}] = \tilde{(m_{kj})}$ is the coefficient matrix, $\tilde{\{d\}} = \tilde{\{d_k\}}, 1 \leq k$ is a column vector of fuzzy numbers, and $\tilde{\{W\}} = \tilde{\{\omega_j\}}$ is the vector of fuzzy unknowns. In this case, $\tilde{M}, \tilde{d}$, and $\tilde{W} \geq 0$. have been assumed.

Definition 2.8 [24] If $\omega = (\underline{\omega_1}, \underline{\omega_2}, \cdots, \underline{\omega_n}, -\overline{\omega_1}, -\overline{\omega_2}, \cdots, -\overline{\omega_n})^T$ is a solution of fuzzy linear system $S\omega = d$ and $\forall\, 1 \leq \imath \leq n$, the inequalities, $\underline{\omega_\imath} \leq \overline{\omega_\imath}$ holds, then the solution $\omega = (\underline{\omega_1}, \underline{\omega_2}, \cdots, \underline{\omega_n}, -\overline{\omega_1}, -\overline{\omega_2}, \cdots, -\overline{\omega_n})^T$ is a strong solution of the fuzzy linear system. In this case, $S = s_{ij}, 1 \leq i, j \leq 2n$, and S is a $2n \times 2n$ crisp matrix.

Definition 2.9 [24] If $\boldsymbol{\omega} = (\underline{\boldsymbol{\omega_1}}, \underline{\boldsymbol{\omega_2}}, \cdots, \underline{\boldsymbol{\omega_n}}, -\overline{\boldsymbol{\omega_1}}, -\overline{\boldsymbol{\omega_2}}, \cdots, -\overline{\boldsymbol{\omega_n}})^{\mathbf{T}}$ is a solution of fuzzy linear system $\mathbf{S\omega} = \mathbf{d}$ and for some $\boldsymbol{\imath} \in [\mathbf{1}, \mathbf{n}]$ the inequalities, $\underline{\mathbf{a_\imath}} \geq \overline{\mathbf{a_\imath}}$ holds, then the solution $\boldsymbol{\omega} = (\underline{\boldsymbol{\omega_1}}, \underline{\boldsymbol{\omega_2}}, \cdots, \underline{\boldsymbol{\omega_n}}, -\overline{\boldsymbol{\omega_1}}, -\overline{\boldsymbol{\omega_2}}, \cdots, -\overline{\boldsymbol{\omega_n}})^{\mathbf{T}}$ is a weak solution of the fuzzy linear system.
In this case, $S = s_{ij}, 1 \leq i, j \leq 2n$, and S is a $2n \times 2n$ crisp matrix.

3. Solutions for fuzzy system of linear equation

In this section, we mainly discuss two approaches for solving fuzzy linear systems: the first method is based on single parametric form of triangular fuzzy numbes, while the second method is based on double parametric form of triangular fuzzy numbers.

3.1. Solution method using single parametric form

The representation of the system

$$[\tilde{M}]\{\tilde{W}\} = \{\tilde{d}\},$$

is formatted as:

$$\sum_{j=1}^{n} m_{\kappa j}\omega_j = \widetilde{d_\kappa} \text{ for } \kappa = 1,2,\dots,n. \qquad (3.1)$$

Using the parametric form of fuzzy elements, the real fuzzy unknown, the right-hand real fuzzy number vector, and the members of the fuzzy coefficient matrix can all be written as follows:

$$\widetilde{m_{\kappa j}} = [\underline{m}_{\kappa j}(\varepsilon), \overline{m}_{\kappa j}(\varepsilon)], \widetilde{\omega_j} = [\underline{\omega}_j(\varepsilon), \overline{\omega}_j(\varepsilon)], \widetilde{d_\kappa} = [\underline{d}_\kappa(\varepsilon), \overline{d}_\kappa(\varepsilon)].$$

Substituting the above expression in equation (3.1), we get;

$$\sum_{j=1}^{n} [\underline{m}_{\kappa j}(\varepsilon), \overline{m}_{\kappa j}(\varepsilon)][\underline{\omega}_j(\varepsilon), \overline{\omega}_j(\varepsilon)] = [\underline{d}_\kappa(\varepsilon), \overline{d}_\kappa(\varepsilon)] \text{ for } \kappa = 1,2,\dots,n \qquad (3.2)$$

Applying the standard rule of fuzzy arithmetic, convert equation 3.2 to the following two equations 3.3 and 3.4.

$$\sum_{j=1}^{n} \underline{m}_{\kappa j}(\varepsilon)\underline{\omega}_j(\varepsilon) = \underline{d}_\kappa(\varepsilon) \qquad (3.3)$$

$$\sum_{j=1}^{n} \overline{m}_{\kappa j}(\varepsilon)\overline{\omega}_j(\varepsilon) = \overline{d}_\kappa(\varepsilon) \qquad (3.4)$$

It is possible to express the combined form of equations 3.3 and 3.4 in an explicit manner.
Now solve the below equation;

$$\begin{pmatrix} S & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix} \qquad (3.5)$$

where,

$$S = \begin{pmatrix} \underline{m_{11}}(\varepsilon) & \underline{m_{12}}(\varepsilon) & \dots & \underline{m_{1n}}(\varepsilon) \\ \underline{m_{21}}(\varepsilon) & \underline{m_{22}}(\varepsilon) & \dots & \underline{m_{2n}}(\varepsilon) \\ \vdots & \vdots & \ddots & \vdots \\ \underline{m_{n1}}(\varepsilon) & \underline{m_{n2}}(\varepsilon) & \dots & \underline{m_{nn}}(\varepsilon) \end{pmatrix} \qquad (3.6)$$

$$D = \begin{pmatrix} \overline{m_{11}}(\varepsilon) & \overline{m_{12}}(\varepsilon) & \dots & \overline{m_{1n}}(\varepsilon) \\ \overline{m_{21}}(\varepsilon) & \overline{m_{22}}(\varepsilon) & \dots & \overline{m_{2n}}(\varepsilon) \\ \vdots & \vdots & \ddots & \vdots \\ \overline{m_{n1}}(\varepsilon) & \overline{m_{n2}}(\varepsilon) & \dots & \overline{m_{nn}}(\varepsilon) \end{pmatrix} \qquad (3.7)$$

$$y = \begin{pmatrix} \underline{\omega_1}(\varepsilon) \\ \underline{\omega_2}(\varepsilon) \\ \vdots \\ \underline{\omega_n}(\varepsilon) \end{pmatrix} \qquad (3.8)$$

$$z = \begin{pmatrix} \overline{\omega}_1(\varepsilon) \\ \overline{\omega}_2(\varepsilon) \\ \vdots \\ \overline{\omega}_n(\varepsilon) \end{pmatrix} \qquad (3.9)$$

$$p = \begin{pmatrix} \underline{d_1}(\varepsilon) \\ \underline{d_2}(\varepsilon) \\ \vdots \\ \underline{d_n}(\varepsilon) \end{pmatrix} \qquad (3.10)$$

$$q = \begin{pmatrix} \bar{d}_1(\varepsilon) \\ \bar{d}_2(\varepsilon) \\ \vdots \\ \bar{d}_n(\varepsilon) \end{pmatrix} \qquad (3.11)$$

Where O represent $n \times n$ zero matrix.
One has the choice of either solving equations (3.3) and (3.4) separately or equation (3.5) directly to obtain the lower and upper bounds of the solution vector.

3.2. Solution method using double parametric form

The fuzzy coefficient matrix, fuzzy unknown vector, and right-side fuzzy number vector of the aforementioned system can be expressed in several ways using the specification of the double parametric form (2.5) in the system (3.2):

$$[\underline{m}_{\kappa J}(\varepsilon), \bar{m}_{\kappa J}(\varepsilon)] = \lambda(\underline{m}_{\kappa J}(\varepsilon) - \bar{m}_{\kappa J}(\varepsilon)) + \underline{m}_{\kappa J}(\varepsilon)$$

$$[\underline{\omega}_J(\varepsilon), \bar{\omega}_J(\varepsilon)] = \lambda(\underline{\omega}_J(\varepsilon) - \bar{\omega}_J(\varepsilon)) + \underline{\omega}_J(\varepsilon)$$

$$[\underline{d}_\kappa(\varepsilon), \bar{d}_\kappa(\varepsilon)] = \lambda\left(\underline{d}_\kappa(\varepsilon) - \bar{d}_\kappa(\varepsilon)\right) + \underline{d}_\kappa(\varepsilon)$$

If we substitute these expressions in system (3.1), we may end up with:

$$\sum_{J=1}^{n} \lambda\{(\underline{m}_{\kappa J}(\varepsilon) - \bar{m}_{\kappa J}(\varepsilon)) + \underline{m}_{\kappa J}(\varepsilon)\}\{\lambda(\underline{\omega}_J(\varepsilon) - \bar{\omega}_J(\varepsilon)) + \underline{\omega}_J(\varepsilon)\} = \lambda(\underline{d}_\kappa(\varepsilon) - \bar{d}_\kappa(\varepsilon)) + \underline{d}_\kappa(\varepsilon). \qquad (3.12)$$

Let us define $\lambda(\underline{\omega}_J(\varepsilon) - \bar{\omega}_J(\varepsilon)) + \underline{\omega}_J(\varepsilon) = \tilde{\omega}_J(\varepsilon, \lambda)$ and then we put this in equation (3.12) to get,

$$\sum_{J=1}^{n} \lambda\{(\underline{m}_{\kappa J}(\varepsilon) - \bar{m}_{\kappa J}(\varepsilon)) + \underline{m}_{\kappa J}(\varepsilon)\}\{\tilde{\omega}_J(\varepsilon, \lambda)\} = \lambda(\underline{d}_\kappa(\varepsilon) - \bar{d}_\kappa(\varepsilon)) + \underline{d}_\kappa(\varepsilon). \qquad (3.13)$$

To determine the value $\tilde{x}_J(\varepsilon, \lambda)$, the equation (3.13) is symbolically solved. Substituting $\lambda = 0$ and 1 after getting the expression $\tilde{x}_J(\varepsilon, \lambda)$ determines the upper and lower bounds of the fuzzy solution vector. The result of this is that $\tilde{x}_J(\varepsilon, 0)$ corresponds to $\underline{x}_J(\varepsilon)$ and $\tilde{x}_J(\varepsilon, 1)$ corresponds to $\bar{x}_J(\varepsilon)$.
The solution procedure does not change the order of the main system. In comparison to other methods, the method is computationally more efficient. The approach is straightforward to use since it converts the fuzzy system into a crisp one utilizing double parametric fuzzy numbers.

4. Theorems and Numerical examples

This section covers fuzzy linear system solutions that are based on the existence of both strong and weak fuzzy solutions. Additionally, we provide some examples to support the proposed methods.

Theorem 4.1 In a single parametric form, the matrix ,

$$P = \begin{pmatrix} S & O \\ O & D \end{pmatrix}$$

is a non singular if and only if S and D non singular, where O is a $n \times n$ zero matrix.

Proof. The matrix,

$$P = \begin{pmatrix} S & O \\ O & D \end{pmatrix}$$

where S is the lower bound of the coefficient matrix and D is the upper bound of the coefficient matrix. Also, the matrix $Q = (S - O)$.

By adding the $(n + \imath)^{th}$ row of P to its $\imath^{th}$ row for $1 \le \imath \le n$, we obtain

$$P = \begin{pmatrix} S & O \\ O & D \end{pmatrix} \qquad (4.1)$$

Next, we substract the $\jmath^{th}$ column of P from its $(n + \jmath)^{th}$ column for $1 \le \imath \le n$ and obtain

$$P_1 = \begin{pmatrix} S + O & O + D \\ O & D \end{pmatrix} \qquad (4.2)$$

Here, in P_2 the absolute value of D and S are same, the only difference is that one is lower bound and one is upper bound. So,

$$P_2 = \begin{pmatrix} S+O & O \\ O & D-O \end{pmatrix} \qquad (4.3)$$

Clearly, $|P| = |P_1| = |P_2| = |S+O||S-O| = |S+O||C|$ and here, $C = D - 0$ or only D which is a upper bound of $n \times n$ matrix.
From above, $|P| \neq 0$ if and only if $|S+O| \neq 0$ and $|C| \neq 0$ or $|S| \neq 0$ and $|D| \neq 0$. Thus the proof is done.

Theorem 4.2 It is necessary for P^{-1} to have the same structure as P if it exists.

$$P^{-1} = \begin{pmatrix} T & O \\ O & E \end{pmatrix} \qquad (4.4)$$

Proof. The entries of P in the $\imath^{th}$ row and $\jmath^{th}$ column are represented by $p_{\imath\jmath}$ and the elements of P^{-1} at the same row and column are denoted by $v_{\imath\jmath}$, then

$$p_{\imath\jmath} = \frac{(-1)^{\imath+\jmath}|P\jmath i|}{|P|} \qquad (4.5)$$

where the matrix obtained by deleting the $\imath^{th}$ column and $\jmath^{th}$ row from P is denoted by $P_{\jmath\imath}$. Let us now consider, for some $1 \leq i, \jmath \leq n$, the entries $v_{\imath,n+\jmath}$ and $v_{n+\imath,\jmath}$ of P^{-1}. $P_{n+\jmath,\imath}$ and $P_{\jmath,n+\imath}$ are the corresponding related matrices.It is simple to demonstrate that by switching rows and columns an even number of times, $P_{n+\jmath,\imath}$ may be created from $P\jmath$, $n+\imath$.

$$v_{\imath,n+\jmath} = (-1)^{\imath+n+\jmath}\frac{P_{n+\jmath,\imath}}{|P|} \qquad (4.6)$$

$$v_{n+\imath,\jmath} = (-1)^{\imath+n+\jmath}\frac{P_{\jmath,n+\imath}}{|P|} \qquad (4.7)$$

Similar to this, for all $\imath$ and $\jmath$, $v_{\imath\jmath} = v_{n+\imath,n+\jmath}$; so, P^{-1} must have the structure determined by equation (4.4), which completes the proof.
The computation of T and E is done as follows:

$$\begin{pmatrix} S & O \\ O & D \end{pmatrix}\begin{pmatrix} T & O \\ O & E \end{pmatrix} = \begin{pmatrix} I & O \\ O & I \end{pmatrix}$$

thus we get

$$ST + O = 1 \qquad (4.8)$$

$$O + DE = 1 \qquad (4.9)$$

From equation (4.8) and (4.9), we get the structure of $T = S^{-1}$ and $E = D^{-1}$. Now, assuming that S and D are non-singular, we obtain $X = P^{-1}Y$. Thus P^{-1} has the same structure as P have. Thus the proof is completed

Theorem 4.3 Assume $P = \begin{pmatrix} S & O \\ O & D \end{pmatrix}$ is a matrix that is non-singular matrix. If and only if $(S+O)^{-1}(\underline{d} - \overline{d}) \leq 0$, then the system $MW = d$ has a strong solution.
Proof. As we define,

$$\underline{\omega} = (\underline{\omega_1}, \underline{\omega_2}, \ldots, \underline{\omega_n})$$

$$\overline{\omega} = (\overline{\omega_1}, \overline{\omega_2}, \ldots, \overline{\omega_n})$$

Next, we get the following from the system $MW = d$:

$$\begin{pmatrix} S & O \\ O & D \end{pmatrix}\begin{pmatrix} \underline{\omega} \\ -\overline{\omega} \end{pmatrix} = \begin{pmatrix} \underline{d} \\ -\overline{d} \end{pmatrix}$$

Hence,

$$S\underline{\omega} - O\overline{\omega} = \underline{d} \qquad (4.10)$$

$$-O\underline{\omega} + D\overline{\omega} = \overline{d} \qquad (4.11)$$

From equation (4.10) and (4.11)

$$\underline{\omega} + O\overline{\omega} - O\underline{\omega} - D\overline{\omega} = \underline{d} - \overline{d}$$

$$(S+O)\underline{\omega}-(O+D)\overline{\omega}=\underline{d}-\overline{d}$$

As we know that the absolute value of S and D are equal. So, we can write the absolute value of D = S and vice-versa.
Therefore,

$$(S+O)\underline{\omega}-(O+D)\overline{\omega}=\underline{d}-\overline{d}.$$

Also,

$$(S+O)(\underline{\omega}-\overline{\omega})=\underline{d}-\overline{d}$$

(S + O) is a non-singular, as demonstrated by theorem (5.1). As a result,

$$(\underline{\omega}-\overline{\omega})=(S+O)^{-1}(\underline{d}-\overline{d}) \qquad (4.12)$$

By definition, (2.9), if there is a strong solution to the system, then $(\underline{\omega}-\overline{\omega}) \leq 0$. This means that the inequality $(S+O)^{-1}(\underline{d}-\overline{d}) \leq 0$ holds. Accordingly, we have $(\underline{\omega}-\overline{\omega}) \leq 0$ by equation (4.12). Which completes the proof.
By combining Theorems 4.1 and 4.3, we get following result.

Corollary 4.4 The system has a unique strong solution, if the following conditions hold:
1. The matrices $Q=(S-O)$ and $(S+O)$ both are non-singular.
2. $(S+O)^{-1}(\underline{d}-\overline{d}) \leq 0$.

Theorem 4.5 If and only if P^{-1} is non-negative, that is, $(P^{-1}) \geq 0$, and $1 \leq \imath, \jmath \leq 2n$, then the unique solution W of $W = P^{-1}D$ is a fuzzy vector for any D.
Proof. Take $P^{-1} = v_{\imath\jmath}$, where $1 \leq \imath, \jmath \leq 2n$,
then

$$\underline{\omega_\imath}=\sum_{\jmath=1}^{n} \underline{v_{\imath\jmath}} d_\jmath-\sum_{\jmath=1}^{n} v_{\imath,n+\jmath}\overline{d_\jmath}, 1 \leq \imath \leq n \qquad (4.13)$$

$$-\overline{\omega_\imath}=\sum_{\jmath=1}^{n} v_{n+\imath,\jmath}\underline{d_\jmath}-\sum_{\jmath=1}^{n} v_{n+\imath,n+\jmath}\overline{d_\jmath} \qquad (4.14)$$

Because of P^{-1}'s unique structure, we substitute equation (4.14) with

$$\overline{\omega_\jmath}=-\sum_{\jmath=1}^{n} v_{\imath,n+\jmath}\underline{d_\jmath}+\sum_{\jmath=1}^{n} v_{\imath,\jmath}\overline{d_\jmath} \qquad (4.15)$$

then deduct equation 4.13 from (4.15). We receive

$$\overline{\omega_\imath}-\underline{\omega_\imath}=\sum_{\jmath=1}^{n} v_{\imath\jmath}(\overline{d_\jmath}-\underline{d_\jmath})+\sum_{\jmath=1}^{n} v_{\imath,n+\jmath}(\overline{d_\jmath}-\underline{d_\jmath}) \qquad (4.16)$$

Accordingly, if D is an arbitrary input vector that represents a fuzzy vector, such that $\overline{d_\jmath}-\underline{d_\jmath} \geq 0$, then $v_{\imath\jmath} \geq 0$ for all $\imath$ and $\jmath$ is a necessary and sufficient condition for $\overline{\omega_\imath}-\underline{\omega_\imath} \geq 0, 1 \leq \imath \leq n$.
As a consequence, for each $\jmath$, $\overline{d_\jmath}$ is monotonically decreasing and $\underline{d_\jmath}$ is monotonically increasing. Equations (4.13) and (4.15) require that $\underline{\omega_\imath}$ and $\overline{\omega_\imath}$ be monotonically increasing and decreasing, respectively, in addition to the prior criterion.
The following example illustrates that the fuzzy matrix has a strong solution.

Example 4.6 Consider a 2 × 2 fuzzy linear system;

$$l_1 - l_2 = (q, 2-q)$$
$$l_1 + 3l_2 = (4+q, 7-2q)$$

where $q \in [0,1]$.
The 4 × 4 extended matrix is

$$\underline{l_1} + 0\underline{l_2} = \underline{q}$$
$$\underline{l_1} + 3\underline{l_2} = \underline{4+q}$$
$$(-\overline{l_1}) + \overline{0l_2} = -\overline{2-q}$$
$$(-\overline{l_1}) + (-\overline{3l_2}) = -\overline{7-2q}$$

Using the single parametric form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} \underline{l_1} \\ \underline{l_2} \\ -\overline{l_1} \\ -\overline{l_2} \end{pmatrix} = \begin{pmatrix} q \\ 4+q \\ 2-q \\ 7-2q \end{pmatrix}$$

On solving the above matrix, we get:

$$\underline{l_1}(\varepsilon) = q$$

$$\underline{l_2}(\varepsilon) = \frac{4}{3}$$
$$\overline{l_1}(\varepsilon) = 2 - q$$
$$\overline{l_2}(\varepsilon) = \frac{5 - q}{3}$$

Now, using double parametric form to find the solution we get,

$$\{\delta(1-1)+1\}\{\delta(0-0)+0\}\{\delta(\overline{l_1}-\underline{l_1})+\underline{l_1}\}\{\delta(\overline{l_2}-\underline{l_2})+\underline{l_2}\} = \{\delta((2-q)-(q))+q\} \quad (4.17)$$

$$\{\delta(1-1)+1\}\{\delta(3-3)+3\}\{\delta(\overline{l_1}-\underline{l_1})+\underline{l_1}\}\{\delta(\overline{l_2}-\underline{l_2})+\underline{l_2}\} = \{\delta((7-2q)-(4+q))+(4+q)\}\} \quad (4.18)$$

Let us consider $\delta(\overline{l_j}(\varepsilon)-\underline{l_j}(\varepsilon))+\underline{l_j}(\varepsilon) = \widetilde{x_j}(\varepsilon,\delta)$ for $j = 1,2$. So, putting this value in the above equation, it can be written as:

$$\{\delta(1-1)+1\}\widetilde{l_1}(\varepsilon,\delta)\{\delta(0-0)+0\}\widetilde{l_2}(\varepsilon,\delta) = \{\delta((2-q)-(q))+q\} \quad (4.19)$$
$$\{\delta(1-1)+1\}\widetilde{l_1}(\varepsilon,\delta)\{\delta(3-3)+3\}\widetilde{l_2}(\varepsilon,\delta) = \{\delta((7-2q)-(4+q))+(4+q)\}\} \quad (4.20)$$

Now, put δ = 0 and 1 in $\widetilde{l_j}(\varepsilon,\delta)$ in both the equation to get lower and upper bounds of the fuzzy solution respectively. Therefore, put δ = 0 in equation (4.21) and (4.22) we get,

$$\widetilde{l_1}(\varepsilon,0) = \underline{l_1}(\varepsilon) = q$$
$$\widetilde{l_2}(\varepsilon,0) = \underline{l_2}(\varepsilon) = \frac{4}{3}$$

Now putting δ = 1 in equation (4.21) and (4.22) we get,

$$\widetilde{l_1}(\varepsilon,1) = \overline{l_1}(\varepsilon) = 2 - q$$
$$\widetilde{l_2}(\varepsilon,1) = \overline{l_2}(\varepsilon) = \frac{5-q}{3}$$

for $0 \le q \le 1$, we check whether $l_1 = (\underline{l_1},\overline{l_1})$ and $l_2 = (\underline{l_2},\overline{l_2})$ are strong fuzzy solutions or weak fuzzy solutions. Now for finding the strong and weak fuzzy solution we can replace (ε) by (q), so we write the equation as:

$$\underline{l_1}(q) = q, \qquad \overline{l_1}(q) = 2 - q$$
$$\underline{l_2}(q) = \frac{4}{3}, \qquad \overline{l_2}(q) = \frac{5-q}{3}$$

For $q = 0$,

$$\underline{l_1}(0) = 0, \overline{l_1}(0) = 2$$
$$\underline{l_2}(0) = 1.33, \overline{l_2}(0) = 1.66$$

For $q = 1$,

$$\underline{l_1}(1) = 1, \overline{l_1}(1) = 1$$
$$\underline{l_1}(1) = 1.33, \overline{l_2}(1) = 1.33$$

For $q = 0.2$,

$$\underline{l_1}(0.2) = 0.2, \overline{l_1}(0.2) = 1.8$$
$$\underline{l_2}(0.2) = 1.33, \overline{l_2}(0.2) = 1.594$$

Here, $\underline{l_1} \le \overline{l_1}$; $\underline{l_2} \le \overline{l_2}$; l_1 and l_2 are monotonic non-increasing function. Therefore, the fuzzy solutions $l_1 = (\underline{l_1},\overline{l_2})$ and $l_2 = (\underline{l_2},\overline{l_2})$ are strong strong fuzzy solutions.

Fuzzy matrix may have both strong and weak solutions as illustrated by the following example.

Example 4.7 A 3 × 3 fuzzy linear system is taken into consideration;

$$l_1 + l_2 - l_3 = (q, 2-q)$$
$$l_1 - 2l_2 + l_3 = (2+q, 3)$$
$$2l_1 + l_2 + 3l_3(-2, -1-q)$$

Using single parametric form,

$$\underline{l_1} + \underline{l_2} + 0\underline{l_3} = q \quad (4.21)$$
$$\underline{l_1} + 0\underline{l_2} + 0\underline{l_3} = 2 + q \quad (4.22)$$

$$2\underline{l_1} + \underline{l_2} + 3\underline{l_3} = -2 \qquad (4.23)$$
$$(-\overline{l_1}) + (-\overline{l_2}) + 0\overline{l_3} = 2 - q \qquad (4.24)$$
$$(-\overline{l_1}) + 0\overline{l_2} + (-\overline{l_3}) = 3 \qquad (4.25)$$
$$(-2\overline{l_1}) + (-\overline{l_2}) + (-3\overline{l_3}) = -1 - q \qquad (4.26)$$

The extended matrix is

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & 1 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} \underline{l_1} \\ \underline{l_2} \\ \underline{l_3} \\ -\overline{l_1} \\ -\overline{l_2} \\ -\overline{l_3} \end{pmatrix} = \begin{pmatrix} q \\ 2+q \\ -2 \\ 2-q \\ 3 \\ -1-q \end{pmatrix}$$

$$\underline{l_1} + \underline{l_2} = q \qquad (4.27)$$
$$\underline{l_1} + \underline{l_3} = 2 - q \qquad (4.28)$$
$$\underline{2l_1} + \underline{l_2} + \underline{3l_3} = -2 \qquad (4.29)$$
$$\overline{l_1} + \overline{l_2} = 2 - q \qquad (4.30)$$
$$\overline{l_1} + \overline{l_3} = 3 \qquad (4.31)$$
$$\overline{2l_1} + \overline{l_2} + \overline{3l_3} = -1 - q \qquad (4.32)$$

Thus, from equation (4.29), (4.30) and (4.31); we get

$$\underline{l_1} = 4 + 2q, \underline{l_2} = -4 - q \text{ and } \underline{l_3} = -2 - q,$$

and from equation (4.32), (4.33) and (4.34)

$$\overline{l_1} = 6, \overline{l_2} = -4 - q \quad \text{and } \overline{l_3} = -3$$

Now, using double parametric form

$$\{\delta(1-1)+1\}\{\delta(1-1)+1\}\{\delta(0-0)+0\}\left\{\delta\left(\overline{l_1}-\underline{l_1}\right)+\underline{l_1}\right\}\left\{\delta\left(\overline{l_2}-\underline{l_2}\right)+\underline{l_2}\right\}\left\{\delta\left(\overline{l_3}-\underline{l_3}\right)+\underline{l_3}\right\} = \{\delta(2-q)-(q)+(q)\} \qquad (4.33)$$
$$\{\delta(1-1)+1\}\{\delta(0-0)+0\}\{\delta(1-1)+1\}\left\{\delta\left(\overline{l_1}-\underline{l_1}\right)+\underline{l_1}\right\}\left\{\delta\left(\overline{l_2}-\underline{l_2}\right)+\underline{l_2}\right\}\{\delta(\overline{l_3}-\underline{l_3})+\underline{l_3}\} = \{\delta(3)-(2+q)+(2+q)\}) \qquad (4.34)$$
$$\{\delta(2-2)+2\}\{\delta(1-1)+1\}\{\delta(3-3)+3\}\left\{\delta\left(\overline{l_1}-\underline{l_1}\right)+\underline{l_1}\right\}\left\{\delta\left(\overline{l_2}-\underline{l_2}\right)+\underline{l_2}\right\}\left\{\delta\left(\overline{l_3}-\underline{l_3}\right)+\underline{l_3}\right\} = \{\delta(-1-q)-(-2)+(-2)\}(4.35)$$

Let us consider $\delta(\overline{l_j}(\varepsilon) - \underline{l_j}(\varepsilon)) + \underline{l_j}(\varepsilon) = \widetilde{x_j}(\varepsilon,\delta)$ for $j = 1,2,3$. So, putting this value in the above equation, it can be written as:

$$\{\delta(1-1)+1\}\widetilde{l_1}(\varepsilon,\delta)\{\delta(1-1)+1\}\widetilde{l_2}(\varepsilon,\delta)\{\delta(0-0)+0\}\widetilde{l_3}(\varepsilon,\delta) = \{\delta(2-q)-(q)+(q)\} \qquad (4.36)$$
$$\{\delta(1-1)+1\}\widetilde{l_1}(\varepsilon,\delta)\{\delta(0-0)+0\}\widetilde{l_2}(\varepsilon,\delta)\{\delta(1-1)+1\}\widetilde{l_3}(\varepsilon,\delta) = \{\delta(3)-(2+q)+(2+q)\}(4.37)$$
$$\{\delta(2-2)+2\}\widetilde{l_1}(\varepsilon,\delta)\{\delta(1-1)+1\}\widetilde{l_2}(\varepsilon,\delta)\{\delta(3-3)+3\}\widetilde{l_3}(\varepsilon,\delta) = \{\delta(-1-q)-(-2)+(-2)\}(4.38)$$

Now, put δ = 0 and 1 in $\widetilde{l_j}(\varepsilon,\delta)$ in the above equations to get lower and upper bounds of the fuzzy solution respectively. Therefore, put δ = 0 in equation (4.38), (4.39) and (4.40) we get,

$$\underline{l_1}(\varepsilon) + \underline{l_2}(\varepsilon) = q$$
$$\underline{l_1}(\varepsilon) + \underline{l_3}(\varepsilon) = 2 + q$$
$$\underline{2l_1}(\varepsilon) + \underline{l_2}(\varepsilon) + \underline{3l_3} = -2$$

Now put again δ = 1 in the above equation we get,

$$\overline{l_1}(\varepsilon) + \overline{l_2}(\varepsilon) = 2 - q$$
$$\overline{l_1}(\varepsilon) + \overline{l_3}(\varepsilon) = 3$$
$$\overline{2l_1}(\varepsilon) + \overline{l_2}(\varepsilon) + \overline{3l_3} = -1 - q$$

for $0 \le q \le 1$, we check whether $l_1 = (\underline{l_1}, \overline{l_1})$, $l_2 = (\underline{l_2}, \overline{l_2})$ and $l_3 = (\underline{l_3}, \overline{l_3})$ are strong or weak fuzzy solutions.

Now for finding the strong and weak fuzzy solutions, we can replace (ε) by (q), so we write the equation as:

$$\underline{l_1}(q) = 4 + 2q, \overline{l_1}(q) = 6$$
$$\underline{l_2}(q) = -4 - q, \quad \overline{l_2}(q) = -4 - q$$
$$\underline{l_3}(q) = -2 - q, \overline{l_3}(q) = -3$$

Now, we check for $q = 0{,}0.5$ and 1.
For $q = 0$

$$\underline{l_1}(0) = 4, \overline{l_1}(0) = 6$$
$$\underline{l_2}(0) = -4, \overline{l_2}(0) = -4$$
$$\underline{l_3}(0) = -2, \overline{l_3}(0) = -3$$

For $q = 0.5$

$$\underline{l_1}(0.5) = 4, \overline{l_1}(0.5) = 6$$
$$\underline{l_2}(0.5) = -4.5, \overline{l_2}(0.5) = -4.5$$
$$\underline{l_3}(0.5) = -2.5, \overline{l_3}(0.5) = -3$$

For $q = 1$

$$\underline{l_1}(1) = 5, \overline{l_1}(1) = 6$$
$$\underline{l_2}(1) = -5, \overline{l_2}(1) = -5$$
$$\underline{l_3}(1) = -3\overline{l_3}(1) = -3$$

Here, $\underline{l_1} \leq \overline{l_1}$; l_2 and l_3 are not fuzzy numbers, in this case they are weak fuzzy solution. Thus only l_1 = $(\underline{l_1}, \overline{l_1})$ is a strong fuzzy solution and l_2 = $(\underline{l_2}, \overline{l_2})$ and l_3 = $(\underline{l_3}\overline{l_3})$ are weak fuzzy solutions.

5. CONCLUSION

The non-negative solution of a fuzzy linear equation system was examined in this study. This work investigated the strong and weak fuzzy solutions of the fuzzy system of linear equations using the single and double parametric forms of fuzzy numbers. In contrast to the single parametric form of a fuzzy number, the double parametric form is easier to manage and more straightforward, making it computationally more efficient. In contrast to other methods, this technique maintains the same order.

ACKNOWLEDGMENT

The authors would like to acknowledge Integral University, Lucknow, India, for providing the manuscript number IU/R&D/2024-MCN0002348 to the present research work.

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