J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 30, NO.1, 2022, COPYRIGHT 2022 EUDOXUS PRESS, LLC

Riesz Basis in de Branges Spaces of Entire Functions

Sa'ud Al-Sa'di[1*] and Hamed Obiedat[2]

[1,2] Department of Mathematics, Faculty of Science, The Hashemite University, P.O Box 330127, Zarqa 13133, Jordan

Abstract

In this paper we consider the problem of Riesz basis in de Branges spaces of entire functions $\mathcal{H}(E)$ with the condition that $\varphi'(x) \geq \alpha > 0$, where φ is the corresponding phase function. We are concerned with the sets of real numbers $\{\lambda_n\}$ such that the normalized reproducing kernels $k(\lambda_n, .)/\|k(\lambda_n, .)\|$ satisfies the restricted isometry property, which in turn constitute a Riesz basis in $\mathcal{H}(E)$. Then we give a criterion on stability of reproducing kernels corresponding to real points which form a Riesz basis in $\mathcal{H}(E)$ with respect to small perturbations, which generalize some well-known Riesz basis perturbation results in the Paley-Wiener space.

2010 Mathematics Subject Classification: 46E22; 41A99; 30B99; 30D10

Key words and phrases: de Branges Spaces; Reproducing kernels; phase function; Restricted isometry property; Riesz basis.

1 Introduction

Compressive sensing provides an alternative method for efficiently acquiring and reconstructing a signal to the Shannon sampling theorem when the signal under acquisition is known to be sparse or compressible. Recently, Candès and Tao [4] introduced very intense activity related to compressed sensing, known as the restricted isometry property, which is also known as the uniform uncertainty principle. The restricted isometry property generalizes the notion of coherence, and allow recovering and extending many known compressive sampling results.

In this paper we work in the context of a reproducing kernel Hilbert spaces. In these spaces the restricted isometry property is a very convenient tool which allows one to reconstruct a signal from its sampling values. It is known that a frame which satisfies a restricted isometry property with isometry constant $\delta < 1$ act as an orthogonal basis. For this reason, one of the main interests of the present paper is to understand what properties of a sequence $\{\lambda_n\}$ of real numbers guarantee that the corresponding normalized reproducing kernels

* Corresponding author: saud@hu.edu.jo

satisfies a restricted isometry property in de Branges spaces $\mathcal{H}(E)$ of entire functions as a special class of reproducing kernel Hilbert spaces. Theory of de Branges spaces is an important branch of modern analysis having numerous interesting applications in mathematical physics, harmonic analysis and even number theory.

The problem of description of Riesz bases of normalized reproducing kernels is one of intriguing open problems in the area, results in this direction would be of interests for specialists in de Branges theory and its applications. In spite of many deep and important results, there is still no explicit description of bases in general de Branges spaces. The present paper studies stability of Riesz bases of reproducing kernels in the class of de Branges spaces with the condition that $\varphi'(x) \geq \alpha > 0$ on $\mathbb{R}$, where φ is an important characteristic of a de Branges space known as a phase function. Specifically, we are concerned with the sets of real numbers $\Lambda = \{\lambda_n\}$ such that the normalized reproducing kernels $k(\lambda_n, .)/\|k(\lambda_n, .)\|$ constitute a Riesz basis. We also prove new results on stability of reproducing kernels corresponding to real points which form a Riesz basis in $\mathcal{H}(E)$ with respect to small perturbations, which generalize some well-known Riesz basis perturbation results in the Paley-Wiener space.

In order to properly state our results, we need to review the main concepts and terminology of the theory of de Branges spaces of entire functions introduced by L. de Branges [13] in connection with inverse spectral problems for differential operators. These spaces generalize the classical Paley-Wiener space which consists of the entire functions of exponential type and square integrable on the real line. More information about these spaces can be found in [8–11].

2 Theory of de Branges spaces

In this section, we present a brief review and some relevant results on de Branges spaces theory. Assume f is an analytic function on the upper half-plane $\mathbb{C}^+ = \{z \in \mathbb{C} : \Im z > 0\}$, then f is said to be of *bounded type* in $\mathbb{C}^+$ if it can be written as a quotient of two bounded analytic functions in $\mathbb{C}^+$. The *mean type* of f in $\mathbb{C}^+$ is defined by

$$\mathrm{mt}_+(f) := \limsup_{y \to +\infty} \frac{\log |f(iy)|}{y}.$$

For an entire function f, we define the function f^* as $f^*(z) := \overline{f(\bar{z})}$. The *Hermite-Biehler* class, denoted by $\mathcal{HB}$, consists of all entire functions $E(z)$ that has no zeros in the upper half-plane and satisfies the condition

$$|E(\bar{z})| < |E(z)|, \quad \text{whenever } \Im z > 0. \tag{1}$$

Given a function $E \in \mathcal{HB}$, the associated de Branges space $\mathcal{H}(E)$ consists of all entire functions $f(z)$ such that

$$||f||_E^2 := \int_{\mathbb{R}} \left| \frac{f(t)}{E(t)} \right|^2 dt < \infty, \tag{2}$$

and $f(z)/E(z)$ and $f^*(z)/E(z)$ are of bounded type and nonpositive mean type in the upper half-plane. This is a Hilbert space with respect to the inner product

$$\langle f, g\rangle_E = \int_{\mathbb{R}} \frac{f(t)\overline{g(t)}}{|E(t)|^2}\, dt.$$

The Hilbert space $\mathcal{H}(E)$ has the special property that, for every nonreal number w, the linear functional defined on the space by $f \mapsto f(w)$ is continuous. Therefore, for every nonreal $w \in \mathbb{C}$ there exists a function $k(w, z)$ in $\mathcal{H}(E)$ such that

$$f(w) = \langle f(t), k(w, t)\rangle_E, \tag{3}$$

for every $f \in \mathcal{H}(E)$. Property (3) is known as the *reproducing kernel property*. The function $k(w, z)$ is called the *reproducing kernel* of $\mathcal{H}(E)$, which is given by (see [13, Theorem 19])

$$k(w, z) = \frac{\bar{E}(w)E(z) - E(\bar{w})E^*(z)}{2\pi i(\bar{w} - z)}. \tag{4}$$

An important feature of the de Branges space $\mathcal{H}(E)$ is the phase function corresponding to the generating function E, that is, for any entire function $E \in \mathcal{HB}$, there exists a continuous and strictly increasing function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $E(x)e^{i\varphi(x)} \in \mathbb{R}$ for all $x \in \mathbb{R}$, essentially, $\varphi = -\arg(E)$ on $\mathbb{R}$, and $E(x)$ can be written as

$$E(x) = |E(x)|e^{-i\varphi(x)}, \;\; x \in \mathbb{R}. \tag{5}$$

If a function φ has these properties then it is referred to as a *phase function* of E. It follows that a phase function of E is defined uniquely up to an additive constant, a multiple of 2π. If $\varphi(x)$ is any such function, and $E(x) \neq 0$, then using (4) and (5), an easy computation gives

$$\|k(x, .)\|^2 = k(x, x) = \frac{1}{\pi}\varphi'(x)|E(x)|^2. \tag{6}$$

The leading example of de Branges spaces is the Paley-Wiener space

$$\mathcal{H}(e^{-i\pi z}) = \mathcal{P}\mathrm{W}_\pi,$$

consists of square-integrable functions on the real line whose Fourier transforms are supported on $[-\pi, \pi]$. The reproducing kernel for $\mathcal{P}\mathrm{W}_\pi$ is $k(w, z) = \frac{\sin \pi(z-\bar{w})}{\pi(z-\bar{w})}$, $w, z \in \mathbb{C}$, $z \neq \bar{w}$, and the corresponding phase function $\varphi(x) = \pi x$.

A key feature of a de Branges space is that it always has a basis consisting of reproducing kernels corresponding to real points, [2].

Theorem 2.1. *Let $\mathcal{H}(E)$ be a de Branges space and $\varphi(x)$ be a phase function associated with E. If $\alpha \in \mathbb{R}$, and $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}}$ is a sequence of real numbers, such that $\varphi(\lambda_n) = \alpha + \pi n$, $n \in \mathbb{Z}$, then The functions $\{k(\lambda_n, z)\}_{n\in\mathbb{Z}}$ form an orthogonal set in $\mathcal{H}(E)$.*

If $e^{i\alpha}E(z)-e^{-i\alpha}E^*(z) \notin \mathcal{H}(E)$, then $\left\{\frac{k(\lambda_n,z)}{\|k(\lambda_n,.)\|}\right\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $\mathcal{H}(E)$. Moreover, for every $f(z) \in \mathcal{H}(E)$,

$$f(z) = \sum_{n\in\mathbb{Z}} f(\lambda_n) \frac{k(\lambda_n, z)}{\|k(\lambda_n, .)\|^2}, \tag{7}$$

and

$$\|f\|^2 = \sum_{n\in\mathbb{Z}} \left|\frac{f(\lambda_n)}{E(\lambda_n)}\right|^2 \frac{\pi}{\varphi'(\lambda_n)}. \tag{8}$$

A central tool in our proofs is the following Bernstein inequality in de Branges spaces introduced by A. Baranov, whose proof can be found in [2]:

Lemma 2.2. *Let $E \in \mathcal{HB}$ be such that $E'/E \in \mathbb{H}^\infty(\mathbb{C}^+)$, then*

$$\|f'/E\|_2 \leq C_{_{Ber}} \|f\|_E$$

for all $f \in \mathcal{H}(E)$, where $C_{_{Ber}} = (4+\sqrt{6})\|E'/E\|_\infty$.

3 Basis Theory

In this section we recall some basic concept of frames and Riesz bases for Hilbert spaces (see for example, Daubechies [7]; Duffin and Schaeffer [14]).

A family of elements $\{f_n\}_{n=1}^\infty$ in a separable Hilbert space $\mathcal{H}$ forms a frame if there exist $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{n=1}^\infty |\langle f, f_n\rangle|^2 \leq B\|f\|^2, \quad \text{for all } \ f \in \mathcal{H}. \tag{9}$$

The constants A, B in (9) are called the *frame bounds* for $\{f_n\}_{n=1}^\infty$. If the two frame bounds are equal we call a frame $\{f_n\}_{n=1}^\infty$ a *tight frame*. For each $f \in \mathcal{H}$ we have the *frame expansions*

$$f = \sum_{n=1}^\infty \langle f, f_n\rangle \tilde{f}_n = \sum_{n=1}^\infty \langle f, \tilde{f}_n\rangle f_n, \tag{10}$$

with unconditional convergence of these series, where $\{\tilde{f}_n\}$ is the dual frame of $\{f_n\}$. If, in addition to (9), $\{f_n\}_{n=1}^\infty$ is a linearly independent set, we call it a *Riesz basis* for $\mathcal{H}$. An equivalent characterization for a sequence $\{f_n\}_{n=1}^\infty$ to be a Riesz basis is that $\{f_n\}_{n=1}^\infty$ be a complete sequence in $\mathcal{H}$ and there exist positive constants A and B such that

$$A\sum_n |c_n|^2 \leq \left\|\sum_n c_n f_n\right\|_{\mathcal{H}}^2 \leq B\sum_n |c_n|^2, \tag{11}$$

for all finite sequences of scalars $\{c_n\}$, see [20].

If the Reisz basis is an orthogonal basis, then $A = B = 1$. Hence, a Riesz basis is automatically a frame, moreover, inequality in (9) holds with the same constants A and B as the inequality in (11). A Riesz basis $\{f_n\}_{n=1}^{\infty}$ is equivalent to an orthonormal basis $\{e_n\}_{n=1}^{\infty}$ for $\mathcal{H}$, namely, if there is a bounded invertible operator $U : \mathcal{H} \to \mathcal{H}$ such that $Uf_n = e_n$. Consequently, any Riesz basis of $\mathcal{H}$ is an unconditional basis of $\mathcal{H}$ but not conversely in general. Because of this parallelism, the Riesz bases is the appropriate framework from which to obtain nonorthogonal sampling formulas. It follows that every $f \in \mathcal{H}$ has a unique expression

$$f = \sum_n \langle f, \tilde{f}_n \rangle f_n$$

where $\tilde{f}_n = U^* U f_n$ are the elements of the dual basis of $\{f_n\}$.

If $\mathcal{H}$ is a reproducing kernel Hilbert space, a sequence $\Lambda = \{\lambda_n\}$ is *interpolating* for $\mathcal{H}$ if there exists an $f \in \mathcal{H}$ satisfying $f(\lambda_n) = a_n$ for any choice of interpolation data $\{a_n / \|k(\lambda_n, .)\|\} \in \ell^2(\mathbb{C})$. It is *complete interpolating* if in addition f is unique. From an equivalent point of view, it is well known that a sequence Λ is an *interpolating sequence* in $\mathcal{H}$ if and only if $\{k(\lambda_n, .)/\|k(\lambda_n, .)\|\}$ is a Riesz sequence, and Λ is a *complete interpolating sequence* if and only if $\{k(\lambda_n, .)/\|k(\lambda_n, .)\|\}$ is a Riesz basis in $\mathcal{H}$, see [17] for more details and discussions.

Definition 3.1. *A sequence $\{f_n\}_{n=1}^{\infty}$ is said to have the* restricted isometry property *if there exists $\delta \in (0, 1)$ such that*

$$(1-\delta)\sum_{n=1}^{\infty}|c_n|^2 \leq \left\|\sum_{n=1}^{\infty} c_n f_n\right\|^2 \leq (1+\delta)\sum_{n=1}^{\infty}|c_n|^2, \tag{12}$$

for any sequence of scalars $\{c_n\}$, where δ is known as the isometry constant.

Although the restricted isometry property is difficult to verify, small restricted isometry constants are desired; the closed δ to zero, the closer to orthogonal basis. On the other hand, this definition in particular means that $\{f_n\}$ is a Riesz basis for its linear span. Conversely, if $\{f_n\}$ is a Riesz basis satisfying (11) then the scaled sequence $\{\sqrt{\frac{2}{B+A}} f_n\}$ satisfies (12) with $\delta = \frac{B-A}{B+A}$. In this work, we approach the problem of stability of Riesz basis of a Hilbert space $\mathcal{H}$. Specifically, given a family $\{g_n\}_{n=1}^{\infty} \subseteq \mathcal{H}$ which is close, in some sense, to the Riesz basis (or a frame) $\{f_n\}_{n=1}^{\infty} \subseteq \mathcal{H}$, we find conditions to ensure that $\{g_n\}_{n=1}^{\infty}$ is also a Riesz basis (or a frame). This problem is important in practice, and has been studied widely by many authors in the context of bases of exponentials in L^2 on some interval. The first result due to Paley and N. Wiener [18] states that if $\{\lambda_n\}_{n\in\mathbb{Z}} \subseteq \mathbb{R}$ and $\sup_{n\in\mathbb{Z}} |\lambda_n - n| \leq \delta < \frac{1}{\pi^2}$, then the set $\{e^{i\lambda_n x}\}_{n\in\mathbb{Z}}$ is a Riesz basis for the Paley-Wiener space $\mathcal{P}\mathrm{W}_\pi$ (in this cae $f_n = e^{inx}$ and $g_n = e^{i\lambda_n x}$). In [19] M. Kadec proved that the result is true for $\delta < \frac{1}{4}$, whereas the conclusion may fail if $\sup_{n\in\mathbb{Z}} |\lambda_n - n| = \frac{1}{4}$ (see [5]). Recently, some results obtained in [3] on the stability of bases and frames of reproducing kernels based on the estimates of derivatives in terms of Carleson measure in model spaces

$K^2_\Theta = \mathbb{H}^2 \ominus \Theta\mathbb{H}^2$ of the Hardy class $\mathbb{H}^2$ in the upper half plane $\mathbb{C}^+$, where Θ is an inner function in $\mathbb{C}^+$.

In the present paper we are particularly interested in the reproducing kernel Hilbert space $\mathcal{H}(E)$, we shall take for the f_n's the normalized reproducing kernel functions $\frac{k(\lambda_n,.)}{\|k(\lambda_n,.)\|}$, where $\Lambda = \{\lambda_n\}$ is a sequence of real numbers. To be exact, we are interested in stability of the basis $\frac{k(\lambda_n,.)}{\|k(\lambda_n,.)\|}$: given a Riesz basis $\frac{k(\lambda_n,.)}{\|k(\lambda_n,.)\|}$ for $\mathcal{H}(E)$ and a set of points μ_n which, in some sense, close to λ_n, whether the system $\frac{k(\mu_n,.)}{\|k(\mu_n,.)\|}$ is also a Riesz basis for $\mathcal{H}(E)$, which, as a result, leads to a Riesz basis expansion.

We will need below the following lemma which will play the key role in our proofs, see Corollary 15.1.5 in [6].

Lemma 3.1. *Let $\{f_n\}_{n=1}^\infty$ be a frame for a Hilbert space $\mathcal{H}$ with bounds A, B, and let $\{g_n\}_{n=1}^\infty$ be a sequence in $\mathcal{H}$. If there exists a constant $R < A$ such that*

$$\sum_{n=1}^{\infty} |\langle f, f_n - g_n\rangle_{\mathcal{H}}|^2 \leqslant R\,\|f\|^2_{\mathcal{H}}, \quad \forall f \in \mathcal{H},$$

then $\{g_n\}_{n=1}^\infty$ is a frame for $\mathcal{H}$ with bounds

$$A(1-\sqrt{R/A}\,)^2, B(1+\sqrt{R/B}\,)^2.$$

If $\{f_n\}_{n=1}^\infty$ is a Riesz basis, then $\{g_n\}_{n=1}^\infty$ is a Riesz basis.

4 Riesz Basis in de Branges Spaces

Given a de Branges space $\mathcal{H}(E)$ with reproducing kernel $k(w, z)$, we can assume, without loss of generality, that E has no real zeros (see [16]), hence $k(x, x) > 0$ for all $x \in \mathbb{R}$ by (6). Let $\Lambda = \{\lambda_n\}_{n=1}^\infty$ be a sequence of real numbers, from now on, we set

$$f_n(z) := \frac{k(\lambda_n, z)}{\|k(\lambda_n, .)\|}, n \in \mathbb{N}, z \in \mathbb{C}. \tag{13}$$

Definition 4.1. *Let $\Lambda = \{\lambda_n\}_{n=1}^\infty$ be a sequence of distinct points. We say that Λ is* sequentially separated *if $|\lambda_{n+1} - \lambda_n| \geq \sigma_n$, for all $n \geq 1$, and $\sigma_n \leq \sigma_{n+1}$ for all $n \geq 1$.*

Next we derive an estimate of the isometry constant δ. This estimate leads to a sufficient condition for a sequence $\{f_n\}$ to have the Restricted Isometry Property.

Lemma 4.1. *Given a de Branges space $\mathcal{H}(E)$, and $\varphi(x)$ a phase function associated with E such that $\varphi'(x) \geq \alpha > 0$ on $\mathbb{R}$. Let $\{\lambda_n\}_{n=1}^\infty$ be a sequentially* *separated sequence of real numbers such that* $\sigma_n \geq 1$*. If* $\sum_{n=1}^{\infty} \frac{1}{\sigma_n^2} < \frac{3\alpha^2}{\pi^2}$*, then*

$$\delta := \left(\sum_{\substack{m,n=1 \\ m \neq n}}^{\infty} |\langle f_n, f_m \rangle|^2 \right)^{\frac{1}{2}} < 1 \tag{14}$$

Proof. For any real number x, $E(x) = e^{-i\varphi(x)}|E(x)|$, which implies that $\frac{E(x)}{\overline{E}(x)} = e^{-2i\varphi(x)}$. Let $a, b \in \mathbb{R}$, then using (4) and the fact that $k(a,b) = \langle k(a,.), k(b,.) \rangle$ we get,

$$\begin{aligned} \frac{k(a,b)}{\overline{E}(a)} &= \frac{1}{\overline{E}(a)} \frac{\overline{E}(a)E(b) - E(a)\overline{E}(b)}{2\pi i(a-b)} \\ &= \frac{E(b) - e^{-2i\varphi(a)}\overline{E}(b)}{2\pi i(a-b)}. \end{aligned}$$

Simple calculations then shows that

$$\begin{aligned} \langle \frac{k(a,.)}{\overline{E}(a)}, \frac{k(b,.)}{\overline{E}(b)} \rangle &= \frac{1}{E(b)} \frac{k(a,b)}{\overline{E}(a)} \\ &= \frac{1 - e^{2i(\varphi(b)-\varphi(a))}}{2\pi i(a-b)} \end{aligned}$$

and,

$$\frac{k^2(a,b)}{|E(a)|^2|E(b)|^2} = \frac{\sin^2(\varphi(a) - \varphi(b))}{\pi^2(a-b)^2}.$$

Consequently, since $k(x,x) = \frac{1}{\pi}\varphi'(x)|E(x)|^2$ for all $x \in \mathbb{R}$, we have

$$\begin{aligned} \frac{k^2(a,b)}{k(a,a)k(b,b)} &= \pi^2 \frac{k^2(a,b)}{\varphi'(a)\varphi'(b)|E(a)|^2|E(b)|^2} \\ &= \frac{1}{\varphi'(a)\varphi'(b)} \frac{\sin^2(\varphi(a) - \varphi(b))}{(a-b)^2} \end{aligned}$$

In particular, for f_n defined in (13) we have

$$\begin{aligned} |\langle f_n, f_m \rangle|^2 &= \left| \langle \frac{k(\lambda_n,.)}{\|k(\lambda_n,.)\|}, \frac{k(\lambda_m,.)}{\|k(\lambda_m,.)\|} \rangle \right|^2 \\ &= \frac{1}{\varphi'(\lambda_m)\varphi'(\lambda_n)} \frac{\sin^2(\varphi(\lambda_m) - \varphi(\lambda_n))}{(\lambda_m - \lambda_n)^2} \\ &\leq \frac{1}{\alpha^2} \frac{1}{(\lambda_m - \lambda_n)^2} \end{aligned}$$

because $\varphi'(x) \geq \alpha$ on $\mathbb{R}$ by the hypothesis. Since $\{\lambda_n\}$ is sequentially separated and $\sigma_n \geq 1$ then for $m > n$, $m = n + k$, for some $k \geq 1$, and

$$(\lambda_m - \lambda_n) \geq (m-n)\sigma_n = k\sigma_n$$

Therefore, for any $n \geq 1$,

$$\begin{aligned}\sum_{m=n+1}^{\infty} |\langle f_n, f_m \rangle|^2 &\leq \frac{1}{\alpha^2} \sum_{m=n+1}^{\infty} \frac{1}{(\lambda_m - \lambda_n)^2} \\ &\leq \frac{1}{\alpha^2} \sum_{m=n+1}^{\infty} \frac{1}{(m-n)^2 \sigma_n^2} \\ &\leq \frac{1}{\alpha^2 \sigma_n^2} \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \frac{1}{\alpha^2 \sigma_n^2}.\end{aligned}$$

Consequently,

$$\sum_{\substack{m,n=1 \\ m \neq n}}^{\infty} |\langle f_n, f_m \rangle|^2 = 2 \sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} |\langle f_n, f_m \rangle|^2 \leq \frac{\pi^2}{3\alpha^2} \sum_{n=1}^{\infty} \frac{1}{\sigma_n^2}.$$

From this the conclusion follows with $\delta < 1$. □

Next we apply the estimate obtained in Lemma 4.1 to give conditions for the sequence $\{f_n\}$ to have the Restricted Isometry Property.

Theorem 4.2. *Given a de Branges space $\mathcal{H}(E)$, and $\varphi(x)$ a phase function associated with E such that $\varphi'(x) \geq \alpha > 0$ on $\mathbb{R}$. Let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequentially separated sequence of real numbers such that $\sigma_n \geq 1$, $\forall n \geq 1$. If $\sum_{n=1}^{\infty} \frac{1}{\sigma_n^2} < \frac{3\alpha^2}{\pi^2}$, then the sequence $\{f_n\}_{n=1}^{\infty}$ satisfies the Restricted Isometry Property.*

Proof. From the definition of f_n, $\|f_n\| = 1$, for $n \geq 1$, then for any finite sequence of complex numbers $\{c_n\}_{n \geq 1}$ we have

$$\begin{aligned}
\Big\|\sum_{n=1}^{\infty} c_n f_n\Big\|^2 &= \sum_{m,n=1}^{\infty} c_n \bar{c}_m \langle f_n, f_m\rangle \\
&= \sum_{n=1}^{\infty} |c_n|^2 \big\|f_n\big\|^2 + \sum_{\substack{m,n=1\\ m\neq n}}^{\infty} c_n \bar{c}_m \langle f_n, f_m\rangle \\
&\leq \sum_{n=1}^{\infty} |c_n|^2 + \sum_{\substack{m,n=1\\ m\neq n}}^{\infty} |c_n \bar{c}_m \langle f_n, f_m\rangle| \\
&\leq \sum_{n=1}^{\infty} |c_n|^2 + \Bigg(\sum_{\substack{m,n=1\\ m\neq n}}^{\infty} |c_n|^2 |c_m|^2\Bigg)^{\frac{1}{2}} \Bigg(\sum_{\substack{m,n=1\\ m\neq n}}^{\infty} |\langle f_n, f_m\rangle|^2\Bigg)^{\frac{1}{2}} \\
&\leq \sum_{n=1}^{\infty} |c_n|^2 + \Bigg(\sum_{n=1}^{\infty} |c_n|^2\Bigg)^{\frac{1}{2}} \Bigg(\sum_{m=1}^{\infty} |c_m|^2\Bigg)^{\frac{1}{2}} \Bigg(\sum_{\substack{m,n=1\\ m\neq n}}^{\infty} |\langle f_n, f_m\rangle|^2\Bigg)^{\frac{1}{2}} \\
&= \sum_{n=1}^{\infty} |c_n|^2 + \sum_{n=1}^{\infty} |c_n|^2 \Bigg(\sum_{\substack{m,n=1\\ m\neq n}}^{\infty} |\langle f_n, f_m\rangle|^2\Bigg)^{\frac{1}{2}} \\
&= \Bigg(1 + \Bigg(\sum_{\substack{m,n=1\\ m\neq n}}^{\infty} |\langle f_n, f_m\rangle|^2\Bigg)^{\frac{1}{2}}\Bigg) \sum_{n=1}^{\infty} |c_n|^2 \\
&= (1+\delta) \sum_{n=1}^{\infty} |c_n|^2
\end{aligned}$$

where $\Bigg(\sum_{\substack{m,n=1\\ m\neq n}}^{\infty} |\langle f_n, f_m\rangle|^2\Bigg)^{\frac{1}{2}} = \delta$, by Lemma 4.1.

Similarly, we prove the first part of the inequality. We use the claim in equation (14) above, we have

$$\begin{aligned}
\Big\|\sum_{n=1}^{\infty} c_n f_n\Big\|^2 &\geq \Bigg(1 - \Bigg(\sum_{\substack{m,n=1\\ m\neq n}}^{\infty} |\langle f_n, f_m\rangle|^2\Bigg)^{\frac{1}{2}}\Bigg) \sum_{n=1}^{\infty} |c_n|^2 \\
&= (1-\delta) \sum_{n=1}^{\infty} |c_n|^2.
\end{aligned}$$

Therefore, the sequence $\{f_n\}$ satisfies the Restricted Isometry Property for some $\delta \in (0,1)$, completing the proof. □

If $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ is a given sequence, then for $\epsilon > 0$, we define a perturbation sequence

$$\mathcal{M}_\epsilon := \left\{ \mu_n \in \mathbb{R} : \mu_n = \lambda_n + \epsilon_n,\ 0 < \epsilon_n \le \epsilon\, \frac{k(\lambda_n, \lambda_n)}{\tau_n},\ n \ge 1 \right\}, \tag{15}$$

where $\tau_n = \max_{t \in [\lambda_n, \lambda_{n+1}]} k(t,t)$. In what follows, the constant A_f is the lower frame bound of the sequence $\{f_n\}$ in (9) and (11), and C_{Ber} is the Berntein constant from Lemma 2.2.

Theorem 4.3. *Given a de Branges space $\mathcal{H}(E)$, such that $E'/E \in \mathbb{H}^\infty(\mathbb{C}^+)$, and $\varphi(x)$ a phase function associated with E such that $\varphi'(x) \ge \alpha > 0$ on $\mathbb{R}$. If $\{f_n\}$ is a Riesz basis in $\mathcal{H}(E)$, then the sequence $\{\frac{k(\mu_n, z)}{\|k(\lambda_n, .)\|} : \mu_n \in \mathcal{M}_\epsilon\}$ is also a Riesz basis in $\mathcal{H}(E)$ whenever $\epsilon < \frac{\alpha A_f}{\pi C^2_{Ber}}$.*

Proof. Since the function $k(t,t)$ is continuous for all $t \in \mathbb{R}$, the Mean Value Theorem implies that there exists $t_n \in (\lambda_n, \mu_n)$ such that

$$\int_{\lambda_n}^{\mu_n} \frac{k(t,t)}{k(\lambda_n, \lambda_n)}\, dt = \epsilon_n \frac{k(t_n, t_n)}{k(\lambda_n, \lambda_n)}, \text{ for all } n \ge 1.$$

Moreover, since $\mu_n \in \mathcal{M}_\epsilon$, then

$$\epsilon_n \frac{k(t_n, t_n)}{k(\lambda_n, \lambda_n)} \le \epsilon\, \frac{k(\lambda_n, \lambda_n)}{\tau_n} \frac{k(t_n, t_n)}{k(\lambda_n, \lambda_n)} \le \epsilon, \text{ for all } n \ge 1.$$

Let $f \in \mathcal{H}(E)$, and $h_n(z) := \frac{k(\mu_n, z)}{\|k(\lambda_n, .)\|}$, for $\mu_n \in \mathcal{M}_\epsilon$. Then

$$\begin{aligned}
|\langle f, f_n - h_n \rangle|^2 &= \frac{1}{k(\lambda_n, \lambda_n)} |f(\lambda_n) - f(\mu_n)|^2 \\
&= \frac{1}{k(\lambda_n, \lambda_n)} \left| \int_{\lambda_n}^{\mu_n} (f(t))'\, dt \right|^2 \\
&\le \frac{1}{k(\lambda_n, \lambda_n)} \int_{\lambda_n}^{\mu_n} \left| \frac{f'(t)}{E(t)} \right|^2 dt \int_{\lambda_n}^{\mu_n} |E(t)|^2\, dt \\
&= \int_{\lambda_n}^{\mu_n} \left| \frac{f'(t)}{E(t)} \right|^2 dt \int_{\lambda_n}^{\mu_n} \pi\, \frac{k(t,t)}{k(\lambda_n, \lambda_n)} \frac{1}{\varphi'(t)}\, dt \\
&\le \frac{\pi}{\alpha} \int_{\lambda_n}^{\mu_n} \left| \frac{f'(t)}{E(t)} \right|^2 dt \int_{\lambda_n}^{\mu_n} \frac{k(t,t)}{k(\lambda_n, \lambda_n)}\, dt \\
&\le \frac{\pi\, \epsilon}{\alpha} \int_{\lambda_n}^{\mu_n} \left| \frac{f'(t)}{E(t)} \right|^2 dt.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} |\langle f, f_n - h_n \rangle|^2 &\le \frac{\pi\, \epsilon}{\alpha} \int_{\mathbb{R}} \left| \frac{f'(t)}{E(t)} \right|^2 dt \\
&= \frac{\pi\, \epsilon}{\alpha} \|f'/E\|^2 \\
&\le \frac{\pi\, \epsilon}{\alpha} C^2_{\text{Ber}}\, \|f\|^2,
\end{aligned}$$

where the last inequality follows from Lemma 2.2. Consequently, $\{h_n\}$ is a Riesz basis by Lemma 3.1 with $R = \frac{\pi\epsilon}{\alpha} C^2_{\text{Ber}} < A_f$ by the hypothesis.

□

Theorem 4.4. *Let $\mathcal{H}(E)$ be a de Branges space, with reproducing kernel function $k(w,z)$. Let $\{\lambda_n\},\{\mu_n\}$ be two sequences of real numbers, and $\{h_n(z) := \frac{k(\mu_n,z)}{\|k(\lambda_n,.)\|}\}$ be a Riesz basis in $\mathcal{H}(E)$ with frame bounds A_h and B_h. If there exits positive constants C_1, C_2 such that*

$$C_1 k(\lambda_n,\lambda_n) \leq k(\mu_n,\mu_n) \leq C_2\, k(\lambda_n,\lambda_n), \tag{16}$$

for all $n \geq 1$, then the sequence $\{\frac{k(\mu_n,z)}{\|k(\mu_n,.)\|}\}$ is also a Riesz basis in $\mathcal{H}(E)$, whenever $CB_h < A_h$, where $C = (1 + \frac{1}{C_1} - \frac{2}{\sqrt{C_2}})$.

Proof. Since the sequence $\{h_n\}$ is a Riesz basis, then for all $f \in \mathcal{H}(E)$,

$$A_h\|f\|^2 \leq \sum_{n=1}^{\infty} |\langle f, h_n\rangle|^2 \leq B_h\|f\|^2.$$

Let $f \in \mathcal{H}(E)$, and $g_n(z) := \frac{k(\mu_n,z)}{\|k(\mu_n,.)\|}$. Then

$$\begin{aligned}
|\langle f, h_n - g_n\rangle|^2 &= \left|\frac{f(\mu_n)}{\sqrt{k(\lambda_n,\lambda_n)}} - \frac{f(\mu_n)}{\sqrt{k(\mu_n,\mu_n)}}\right|^2 \\
&= |f(\mu_n)|^2 \left|\frac{1}{\sqrt{k(\lambda_n,\lambda_n)}} - \frac{1}{\sqrt{k(\mu_n,\mu_n)}}\right|^2 \\
&= |f(\mu_n)|^2 \left|\frac{1}{k(\lambda_n,\lambda_n)} + \frac{1}{k(\mu_n,\mu_n)} - \frac{2}{\sqrt{k(\lambda_n,\lambda_n)k(\mu_n,\mu_n)}}\right| \\
&\leq R\frac{|f(\mu_n)|^2}{k(\lambda_n,\lambda_n)}
\end{aligned}$$

where $R = 1 + \frac{1}{C_1} - \frac{2}{\sqrt{C_2}}$. Thus, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} |\langle f, h_n - g_n\rangle|^2 &\leq R\sum_{n=1}^{\infty} \frac{|f(\mu_n)|^2}{k(\lambda_n,\lambda_n)} \\
&= R\sum_{n=1}^{\infty} |\langle f, h_n\rangle|^2 \\
&\leq RB_h\|f\|^2.
\end{aligned}$$

Consequently, $\{g_n\}$ is a Riesz basis by Lemma 3.1 as $RB_h < A_h$. □

Now we state the main result on stability of Riesz basis in de Branges spaces, the proof is an immediate consequence of Theorem 4.3 and Theorem 4.4.

Theorem 4.5. *Given a de Branges space $\mathcal{H}(E)$, such that $E'/E \in \mathbb{H}^\infty(\mathbb{C}^+)$, and $\varphi(x)$ a phase function associated with E such that $\varphi'(x) \geq \alpha > 0$ on $\mathbb{R}$. Let $\{f_n\}$ be a Riesz basis in $\mathcal{H}(E)$ with bounds A_f, B_f. Let $\mathcal{M}_\epsilon$ be the sequence defined in (15), and assume that there exits positive constants C_1, C_2 such that*

$$C_1 k(\lambda_n, \lambda_n) \leq k(\mu_n, \mu_n) \leq C_2\, k(\lambda_n, \lambda_n), \text{ for all } n \geq 1. \tag{17}$$

Then the sequence $\{\frac{k(\mu_n,z)}{\|k(\mu_n,.)\|} : \mu_n \in \mathcal{M}_\epsilon\}$ is also a Riesz basis in $\mathcal{H}(E)$ whenever

$$\epsilon < \frac{\alpha A_f}{\pi C^2_{_{Ber}}} \quad \text{and} \quad C\, B_f (1 + \sqrt{R/B_f}\ )^2 < A_f (1 - \sqrt{R/A_f}\ )^2$$

where $R = \frac{\pi\,\epsilon}{\alpha} C^2_{_{Ber}}$ and $C = (1 + \frac{1}{C_1} - \frac{2}{\sqrt{C_2}})$.

Remark 4.1. *de Branges spaces $\mathcal{H}(E)$ that satisfy the conditions of the previous theorems in general do not have simple analytic characterizations. We would like to emphasize that the best way to construct the corresponding generating functions $E \in \mathcal{HB}$ is via their Weierstrass factorization formula. A special class of Hermite-Biehler functions is the Pólya class where any function can be characterized by its Hadamard factorization formula. For the sake of completeness, we include some examples of such functions, see [1] and [13]:*

(1) Let E have the form

$$E(z) = \gamma\, e^{bz} e^{-iaz} \prod_{n\in\mathbb{Z}} \left(1 - \frac{z}{z_n}\right) e^{zRe(\frac{1}{z_n})}, \tag{18}$$

and let the zeros z_n satisfy the following conditions:

(a). $z_n = \beta n + w_n$, for all $n \in \mathbb{Z}$, where $\beta > 0$, and the sequence $\{w_n\}_{n\in\mathbb{Z}}$ is bounded,

(b). $Im(w_n) \geq \alpha > 0$.

Then $\frac{E'}{E} \in \mathbb{H}^\infty(\mathbb{C}^+)$. If, in addition, $w_n = u_n + iv_n$ where $u_n \in [\alpha_1, \alpha_2]$ and $v_n \in [a_1, a_2]$, $a_1 > 0$ for all $n \in \mathbb{Z}$, then $E'/E \in \mathbb{H}^\infty(\mathbb{C}^+)$. and $\varphi'(x)$ is bounded away from zero.

(2) Let

$$E(z) = \gamma e^{-iaz} S(z) \prod_{n=1}^{\infty} \left(1 - \frac{z}{\bar{z}_n}\right) e^{h_n z},$$

for all $z \in \mathbb{C}$, where the sequence $\{z_n\}_{n=1}^\infty \subset \mathbb{C}^+$ has no condensation points in $\mathbb{C}$ and satisfies the Blaschke condition

$$\sum_{n=1}^{\infty} y_n / \left(x_n^2 + y_n^2\right) < +\infty,$$

which guarantee the convergence of the previous product, and

$$h_n = x_n / \left(x_n^2 + y_n^2\right),\ n \in \mathbb{N},$$

$a > 0$, S is an entire function taking the real values on the real line and having only real zeros, and γ is a complex number with modulus 1. If the sequence $\{z_n\}_{n=1}^{\infty}$ is contained in the set $\Gamma_\tau = \{z \in \mathbb{C}^+ : \tau < \arg z < \pi - \tau\}$, $\tau > 0$, then $\frac{E'}{E} \in \mathbb{H}^\infty(\mathbb{C}^+)$ and $\varphi'(x)$ is bounded away from zero.

Furthermore, a wide class of de Branges spaces for which the previous theorems may be applied is the homogeneous *de Branges spaces. Such spaces are related to the classical Bessel functions and more general confluent hypergeometric functions, and were characterized by L. de Branges [12, 13]. We present a brief review of the construction of these spaces. Let $\nu > -1$. A space $\mathcal{H}(E)$ is said to be* homogeneous *of order ν if, for all $0 < a < 1$ and all $F \in \mathcal{H}(E)$, the function $z \mapsto a^{\nu+1}F(az)$ belongs to $\mathcal{H}(E)$ and has the same norm as F. For $\nu > -1$ consider the real entire functions $A_\nu(z) : \mathbb{C} \to \mathbb{C}$ and $B_\nu(z) : \mathbb{C} \to \mathbb{C}$ given by*

$$A_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}z\right)^{2n}}{n!(\nu+1)(\nu+2)\ldots(\nu+n)} = \Gamma(\nu+1)\left(\frac{1}{2}z\right)^{-\nu} J_\nu(z)$$

and

$$B_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}z\right)^{2n+1}}{n!(\nu+1)(\nu+2)\ldots(\nu+n+1)} = \Gamma(\nu+1)\left(\frac{1}{2}z\right)^{-\nu+1} J_\nu(z)$$

where

$$J_\nu(z) = \sum_{n\geq 0} \frac{(-1)^n \left(\frac{1}{2}z\right)^{2n+\nu}}{n!\Gamma(\nu+n+1)}$$

is the classical Bessel function of the first kind. These special functions have only real, simple zeros and have no common zeros. Furthermore, they satisfy the following differential equations

$$A'_\nu(z) = -B_\nu(z) \quad \text{and} \quad B'_\nu(z) = A_\nu(z) - (2\nu+1)B_\nu(z)/z. \tag{19}$$

If we define

$$E_\nu(z) := A_\nu(z) - iB_\nu(z),$$

then the function $E_\nu(z)$ is a Hermite-Biehler function with no real zeros, of bounded type in the upper-half, and is of exponential type 1 in $\mathbb{C}$. Also we have that

$$c_\nu |x|^{2\nu+1} \leq |E_\nu(x)|^{-2} \leq C_\nu |x|^{2\nu+1},$$

for all real $|x| \geq 1$ and for some $c_\nu, C_\nu > 0$, see [15]. Moreover, it is known that $A_\nu, B_\nu \notin \mathcal{H}(E_\nu)$. Note that if $\nu = -1/2$ we have $A_{-1/2}(z) = \cos z$ and

$B_{-1/2}(z) = \sin z$, hence, $E_{-1/2}(z) = e^{-iz}$ and the space $\mathcal{H}(E_{-1/2})$ coincides with the Paley-Wiener space $\mathcal{P}W_1$. By (19) we have

$$i\frac{E_\nu'(z)}{E_\nu(z)} = 1 - (2\nu+1)\frac{B_\nu(z)}{zE_\nu(z)},$$

for all $z \in \mathbb{C}^+$. Hence $E_\nu'(z)/E_\nu(z) \in H^\infty(\mathbb{C}^+)$. This also implies that the phase function $\varphi_\nu(z)$ associated with $E_\nu(z)$ satisfies

$$\varphi_\nu'(x) = 1 - \frac{(2\nu+1)A_\nu(x)B_\nu(x)}{x\,|E_\nu(x)|^2}.$$

Hence, $\varphi_\nu'(x) \simeq 1$ for all real x.

Acknowledgements

The authors would like to thank Prof. Eric S. Weber, Dept. of Mathematics, Iowa State University, for introducing the topic and for his suggestions on preparing the manuscript.

Conflict of interest

The authors declare that they have no conflict of interest.

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