Volume 21, Number 6 December 1st, 2016
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of

Computational

Analysis and

Applications

EUDOXUS PRESS,LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE

SCOPE OF THE JOURNAL

An international publication of Eudoxus Press, LLC
(fourteen times annually)

Editor in Chief: George Anastassiou
Department of Mathematical Sciences,
University of Memphis, Memphis, TN 38152-3240, U.S.A
ganastss@memphis.edu
http://www.msci.memphis.edu/~ganastss/jocaaa

The main purpose of "J.Computational Analysis and Applications" is to publish high quality research articles from all subareas of Computational Mathematical Analysis and its many potential applications and connections to other areas of Mathematical Sciences. Any paper whose approach and proofs are computational,using methods from Mathematical Analysis in the broadest sense is suitable and welcome for consideration in our journal, except from Applied Numerical Analysis articles. Also plain word articles without formulas and proofs are excluded. The list of possibly connected mathematical areas with this publication includes, but is not restricted to: Applied Analysis, Applied Functional Analysis, Approximation Theory, Asymptotic Analysis, Difference Equations, Differential Equations, Partial Differential Equations, Fourier Analysis, Fractals, Fuzzy Sets, Harmonic Analysis, Inequalities, Integral Equations, Measure Theory, Moment Theory, Neural Networks, Numerical Functional Analysis, Potential Theory, Probability Theory, Real and Complex Analysis, Signal Analysis, Special Functions, Splines, Stochastic Analysis, Stochastic Processes, Summability, Tomography, Wavelets, any combination of the above, e.t.c.

"J.Computational Analysis and Applications" is a peer-reviewed Journal. See the instructions for preparation and submission of articles to JoCAAA. Assistant to the Editor: Dr.Razvan Mezei,Lenoir-Rhyne University,Hickory,NC 28601, USA.

Journal of Computational Analysis and Applications(JoCAAA) is published by **EUDOXUS PRESS,LLC**,1424 Beaver Trail Drive,Cordova,TN38016,USA,anastassioug@yahoo.com http://www.eudoxuspress.com. **Annual Subscription Prices**:For USA and Canada,Institutional:Print $700, Electronic OPEN ACCESS. Individual:Print $350. For any other part of the world add $130 more(handling and postages) to the above prices for Print. No credit card payments.

JoCAAA is reviewed and abstracted by AMS Mathematical Reviews,MATHSCI,and Zentralblaat MATH.

Dartmouth College
8000 Cummings Hall,
Hanover, NH 03755-8000
603-646-3843 (X 3546 Secr.)
e-mail:george.cybenko@dartmouth.edu
Approximation Theory and Neural
Networks

Sever S. Dragomir
School of Computer Science and
Mathematics, Victoria University,
PO Box 14428,
Melbourne City,
MC 8001, AUSTRALIA
Tel. +61 3 9688 4437
Fax +61 3 9688 4050
sever.dragomir@vu.edu.au
Inequalities, Functional Analysis,
Numerical Analysis, Approximations,
Information Theory, Stochastics.

Oktay Duman
TOBB University of Economics and
Technology,
Department of Mathematics, TR-
06530,
Ankara, Turkey,
oduman@etu.edu.tr
Classical Approximation Theory,
Summability Theory, Statistical
Convergence and its Applications

Saber N. Elaydi
Department Of Mathematics
Trinity University
715 Stadium Dr.
San Antonio, TX 78212-7200
210-736-8246
e-mail: selaydi@trinity.edu
Ordinary Differential Equations,
Difference Equations

Christodoulos A. Floudas
Department of Chemical Engineering
Princeton University
Princeton,NJ 08544-5263
609-258-4595(x4619 assistant)
e-mail: floudas@titan.princeton.edu
Optimization Theory&Applications,
Global Optimization

J .A. Goldstein
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152
901-678-3130
jgoldste@memphis.edu
Partial Differential Equations,
Semigroups of Operators

H. H. Gonska
Department of Mathematics
University of Duisburg
Duisburg, D-47048
Germany
011-49-203-379-3542
e-mail: heiner.gonska@uni-due.de
Approximation Theory, Computer
Aided Geometric Design

John R. Graef
Department of Mathematics
University of Tennessee at
Chattanooga
Chattanooga, TN 37304 USA
John-Graef@utc.edu
Ordinary and functional
differential equations, difference
equations, impulsive systems,
differential inclusions, dynamic
equations on time scales, control
theory and their applications

Weimin Han
Department of Mathematics
University of Iowa
Iowa City, IA 52242-1419
319-335-0770
e-mail: whan@math.uiowa.edu
Numerical analysis, Finite element
method, Numerical PDE, Variational
inequalities, Computational
mechanics

Tian-Xiao He
Department of Mathematics and
Computer Science
P.O. Box 2900, Illinois Wesleyan
University
Bloomington, IL 61702-2900, USA
Tel (309)556-3089
Fax (309)556-3864
the@iwu.edu
Approximations, Wavelet,
Integration Theory, Numerical
Analysis, Analytic Combinatorics

Margareta Heilmann
Faculty of Mathematics and Natural
Sciences, University of Wuppertal
Gaußstraße 20
D-42119 Wuppertal, Germany,
heilmann@math.uni-wuppertal.de

National Technical University of Athens
Zografou campus, 157 80
Athens, Greece
tel:: +30(210) 772 1722
Fax +30(210) 772 1775
papanico@math.ntua.gr
Partial Differential Equations, Probability

Choonkil Park
Department of Mathematics
Hanyang University
Seoul 133-791
S. Korea, baak@hanyang.ac.kr
Functional Equations

Svetlozar (Zari) Rachev,
Professor of Finance, College of Business, and Director of Quantitative Finance Program,
Department of Applied Mathematics & Statistics
Stonybrook University
312 Harriman Hall, Stony Brook, NY 11794-3775
tel: +1-631-632-1998,
svetlozar.rachev@stonybrook.edu

Alexander G. Ramm
Mathematics Department
Kansas State University
Manhattan, KS 66506-2602
e-mail: ramm@math.ksu.edu
Inverse and Ill-posed Problems, Scattering Theory, Operator Theory, Theoretical Numerical Analysis, Wave Propagation, Signal Processing and Tomography

Tomasz Rychlik
Polish Academy of Sciences
Instytut Matematyczny PAN
00-956 Warszawa, skr. poczt. 21
ul. Śniadeckich 8
Poland
trychlik@impan.pl
Mathematical Statistics, Probabilistic Inequalities

Boris Shekhtman
Department of Mathematics
University of South Florida
Tampa, FL 33620, USA
Tel 813-974-9710
shekhtma@usf.edu
Approximation Theory, Banach spaces, Classical Analysis

T. E. Simos
Department of Computer Science and Technology
Faculty of Sciences and Technology
University of Peloponnese
GR-221 00 Tripolis, Greece
Postal Address:
26 Menelaou St.
Anfithea - Paleon Faliron
GR-175 64 Athens, Greece
tsimos@mail.ariadne-t.gr
Numerical Analysis

H. M. Srivastava
Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia V8W 3R4
Canada
tel.250-472-5313; office,250-477-6960 home, fax 250-721-8962
harimsri@math.uvic.ca
Real and Complex Analysis, Fractional Calculus and Appl., Integral Equations and Transforms, Higher Transcendental Functions and Appl.,q-Series and q-Polynomials, Analytic Number Th.

I. P. Stavroulakis
Department of Mathematics
University of Ioannina
451-10 Ioannina, Greece
ipstav@cc.uoi.gr
Differential Equations
Phone +3-065-109-8283

Manfred Tasche
Department of Mathematics
University of Rostock
D-18051 Rostock, Germany
manfred.tasche@mathematik.uni-rostock.de
Numerical Fourier Analysis, Fourier Analysis, Harmonic Analysis, Signal Analysis, Spectral Methods, Wavelets, Splines, Approximation Theory

Roberto Triggiani
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152
PDE, Control Theory, Functional

Analysis, rtrggani@memphis.edu

Juan J. Trujillo
University of La Laguna
Departamento de Analisis Matematico
C/Astr.Fco.Sanchez s/n
38271. LaLaguna. Tenerife.
SPAIN
Tel/Fax 34-922-318209
Juan.Trujillo@ull.es
Fractional: Differential Equations-Operators-Fourier Transforms, Special functions, Approximations, and Applications

Ram Verma
International Publications
1200 Dallas Drive #824 Denton,
TX 76205, USA
Verma99@msn.com
Applied Nonlinear Analysis, Numerical Analysis, Variational Inequalities, Optimization Theory, Computational Mathematics, Operator Theory

Xiang Ming Yu
Department of Mathematical Sciences
Southwest Missouri State University
Springfield, MO 65804-0094
417-836-5931
xmy944f@missouristate.edu
Classical Approximation Theory, Wavelets

Lotfi A. Zadeh
Professor in the Graduate School and Director, Computer Initiative, Soft Computing (BISC)
Computer Science Division
University of California at Berkeley
Berkeley, CA 94720
Office: 510-642-4959
Sec: 510-642-8271
Home: 510-526-2569
FAX: 510-642-1712
zadeh@cs.berkeley.edu
Fuzzyness, Artificial Intelligence, Natural language processing, Fuzzy logic

Richard A. Zalik
Department of Mathematics
Auburn University
Auburn University, AL 36849-5310
USA.
Tel 334-844-6557 office
678-642-8703 home
Fax 334-844-6555
zalik@auburn.edu
Approximation Theory, Chebychev Systems, Wavelet Theory

Ahmed I. Zayed
Department of Mathematical Sciences
DePaul University
2320 N. Kenmore Ave.
Chicago, IL 60614-3250
773-325-7808
e-mail: azayed@condor.depaul.edu
Shannon sampling theory, Harmonic analysis and wavelets, Special functions and orthogonal polynomials, Integral transforms

Ding-Xuan Zhou
Department Of Mathematics
City University of Hong Kong
83 Tat Chee Avenue
Kowloon, Hong Kong
852-2788 9708,Fax:852-2788 8561
e-mail: mazhou@cityu.edu.hk
Approximation Theory, Spline functions, Wavelets

Xin-long Zhou
Fachbereich Mathematik, Fachgebiet Informatik
Gerhard-Mercator-Universitat Duisburg
Lotharstr.65, D-47048 Duisburg, Germany
e-mail:Xzhou@informatik.uni-duisburg.de
Fourier Analysis, Computer-Aided Geometric Design, Computational Complexity, Multivariate Approximation Theory, Approximation and Interpolation Theory

Instructions to Contributors
Journal of Computational Analysis and Applications

An international publication of Eudoxus Press, LLC, of TN.

Editor in Chief: George Anastassiou

Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152-3240, U.S.A.

1. Manuscripts files in Latex and PDF and in English, should be submitted via email to the Editor-in-Chief:

Prof.George A. Anastassiou
Department of Mathematical Sciences
The University of Memphis
Memphis,TN 38152, USA.
Tel. 901.678.3144
e-mail: ganastss@memphis.edu

Authors may want to recommend an associate editor the most related to the submission to possibly handle it.

Also authors may want to submit a list of six possible referees, to be used in case we cannot find related referees by ourselves.

2. Manuscripts should be typed using any of TEX,LaTEX,AMS-TEX,or AMS-LaTEX and according to EUDOXUS PRESS, LLC. LATEX STYLE FILE. (Click HERE to save a copy of the style file.)They should be carefully prepared in all respects. Submitted articles should be brightly typed (not dot-matrix), double spaced, in ten point type size and in 8(1/2)x11 inch area per page. Manuscripts should have generous margins on all sides and should not exceed 24 pages.

3. Submission is a representation that the manuscript has not been published previously in this or any other similar form and is not currently under consideration for publication elsewhere. A statement transferring from the authors(or their employers,if they hold the copyright) to Eudoxus Press, LLC, will be required before the manuscript can be accepted for publication.The Editor-in-Chief will supply the necessary forms for this transfer.Such a written transfer of copyright,which previously was assumed to be implicit in the act of submitting a manuscript,is necessary under the U.S.Copyright Law in order for the publisher to carry through the dissemination of research results and reviews as widely and effective as possible.

4. The paper starts with the title of the article, author's name(s) (no titles or degrees), author's affiliation(s) and e-mail addresses. The affiliation should comprise the department, institution (usually university or company), city, state (and/or nation) and mail code.

The following items, 5 and 6, should be on page no. 1 of the paper.

5. An abstract is to be provided, preferably no longer than 150 words.

6. A list of 5 key words is to be provided directly below the abstract. Key words should express the precise content of the manuscript, as they are used for indexing purposes.

The main body of the paper should begin on page no. 1, if possible.

7. All sections should be numbered with Arabic numerals (such as: 1. INTRODUCTION) .
Subsections should be identified with section and subsection numbers (such as 6.1. Second-Value Subheading).
If applicable, an independent single-number system (one for each category) should be used to label all theorems, lemmas, propositions, corollaries, definitions, remarks, examples, etc. The label (such as Lemma 7) should be typed with paragraph indentation, followed by a period and the lemma itself.

8. Mathematical notation must be typeset. Equations should be numbered consecutively with Arabic numerals in parentheses placed flush right, and should be thusly referred to in the text [such as Eqs.(2) and (5)]. The running title must be placed at the top of even numbered pages and the first author's name, et al., must be placed at the top of the odd numbed pages.

9. Illustrations (photographs, drawings, diagrams, and charts) are to be numbered in one consecutive series of Arabic numerals. The captions for illustrations should be typed double space. All illustrations, charts, tables, etc., must be embedded in the body of the manuscript in proper, final, print position. In particular, manuscript, source, and PDF file version must be at camera ready stage for publication or they cannot be considered.

Tables are to be numbered (with Roman numerals) and referred to by number in the text. Center the title above the table, and type explanatory footnotes (indicated by superscript lowercase letters) below the table.

10. List references alphabetically at the end of the paper and number them consecutively. Each must be cited in the text by the appropriate Arabic numeral in square brackets on the baseline.
References should include (in the following order):
initials of first and middle name, last name of author(s)
title of article,

name of publication, volume number, inclusive pages, and year of publication.

Authors should follow these examples:

<u>Journal Article</u>

1. H.H.Gonska,Degree of simultaneous approximation of bivariate functions by Gordon operators, (journal name in italics) *J. Approx. Theory*, 62,170-191(1990).

<u>Book</u>

2. G.G.Lorentz, (title of book in italics) *Bernstein Polynomials* (2nd ed.), Chelsea,New York,1986.

<u>Contribution to a Book</u>

3. M.K.Khan, Approximation properties of beta operators,in(title of book in italics) *Progress in Approximation Theory* (P.Nevai and A.Pinkus,eds.), Academic Press, New York,1991,pp.483-495.

11. All acknowledgements (including those for a grant and financial support) should occur in one paragraph that directly precedes the References section.

12. Footnotes should be avoided. When their use is absolutely necessary, footnotes should be numbered consecutively using Arabic numerals and should be typed at the bottom of the page to which they refer. Place a line above the footnote, so that it is set off from the text. Use the appropriate superscript numeral for citation in the text.

13. After each revision is made please again submit via email Latex and PDF files of the revised manuscript, including the final one.

14. Effective 1 Nov. 2009 for current journal page charges, contact the Editor in Chief. Upon acceptance of the paper an invoice will be sent to the contact author. The fee payment will be due one month from the invoice date. The article will proceed to publication only after the fee is paid. The charges are to be sent, by money order or certified check, in US dollars, payable to Eudoxus Press, LLC, to the address shown on the Eudoxus <u>homepage</u>.

No galleys will be sent and the contact author will receive one (1) electronic copy of the journal issue in which the article appears.

15. This journal will consider for publication only papers that contain proofs for their listed results.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 21, NO.6, 2016, COPYRIGHT 2016 EUDOXUS PRESS, LLC

Some Results of a New Integral Operator

Roberta Bucur[1], Loriana Andrei[2], Daniel Breaz[3]

[1]Department of Mathematics, University of Piteşti,
Târgul din Vale St., No.1, 110040 Piteşti, România,
E-mail:roberta_bucur@yahoo.com,
[2]Department of Mathematics and Computer Science, University of Oradea,
1 Universitatii St., 410087 Oradea, România,
E-mail:lori_andrei@yahoo.com,
[3]Department of Mathematics,"1 Decembrie 1918" University of Alba Iulia,
N. Iorga St., No. 11-13, 510009 Alba Iulia,România,
E-mail:dbreaz@uab.ro.

Abstract

The main objective of the present paper is to obtain sufficient conditions for the univalence, starlikeness and convexity of a new integral operator defined on the space of normalized analytic functions in the open unit disk. Results presented in this paper may motivate further reserch in this fascinating field.

Keywords: analytic, univalent, starlike and convex functions, integral operator.
2010 Mathematics Subject Classifications: 30C45.

1 Introduction

Let $U = \{z : |z| < 1\}$ be the open unit disk and $\mathcal{A}$ the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \text{ for all } z \in U, \tag{1}$$

which are analytic in U. Consider S the class of all functions in $\mathcal{A}$ which are univalent in U.

A domain $D \subset \mathbb{C}$ is starlike with respect to a point $w_0 \in D$ if the line segment joining any point of D to w_0 lies inside D, while a domain is convex if the line segment joining any two points in D lies entirely in D. We say that the function $f \in \mathcal{A}$ is starlike if $f(U)$ is a starlike domain with respect to origin, and convex if $f(U)$ is convex. Analytically, $f \in \mathcal{A}$ is starlike if and only if

$$Re\left[\frac{zf'(z)}{f(z)}\right] > 0, \text{ for all } z \in U,$$

and $f \in \mathcal{A}$ is convex if and only if

$$Re\left[1 + \frac{zf''(z)}{f'(z)}\right] > 0, \text{ for all } z \in U.$$

The classes consisting of starlike and convex functions are denoted by S^* and K, respectively. Further, we denote by $S^*(\delta)$ and $K(\delta)$ the class of starlike functions of order δ and the class of convex functions of order δ $(0 \leq \delta < 1)$, respectively, where

$$Re\left[\frac{zf'(z)}{f(z)}\right] > \delta \text{ and } Re\left[1 + \frac{zf''(z)}{f'(z)}\right] > \delta.$$

Recently, Frasin and Jahangiri [4] defined the family $B(\mu, \lambda)$, $\mu \geq 0$, $0 \leq \lambda < 1$ consisting of functions $f \in \mathcal{A}$ satisfying the condition
$$\left| f^{'}(z) \left[\frac{z}{f(z)} \right]^{\mu} - 1 \right| < 1 - \lambda, \text{ for all } z \in U.$$
We note that $B(1, \lambda) = S^*(\lambda)$, $B(2, \lambda) = B(\lambda)$ (see [3]) and $B(2, 0) = S$.

For the functions $f, g \in \mathcal{A}$ and $\alpha, \zeta \in \mathbb{C}$ we define the integral operator $I_\alpha^\zeta(f, g)$ given by
$$I_\alpha^\zeta(f, g)(z) = \left[\zeta \int_0^z t^{\alpha+\zeta-1} \left(\frac{f^{'}(t)}{g(t)} \right)^{\alpha} dt \right]^{\frac{1}{\zeta}}. \tag{2}$$
Note that the integral operator $I_\alpha^\zeta(f, g)(z)$ generalizes the integral operator $I_\alpha(f, g)(z)$ introduced in [2].

In this paper our purpose is to derive univalence conditions, starlikeness properties and the order of convexity for the integral operator introduced in (2). Recently, many authors studied the problem of integral operators which preserve the class S (see [5], [9]).

In order to prove our results, we have to recall here the following:

Lemma 1.1 *(Mocanu and Şerb [7]) Let $M_0 = 1,5936...$, the positive solution of equation*
$$(2 - M)e^M = 2. \tag{3}$$
If $f \in \mathcal{A}$ and
$$\left| \frac{f^{''}(z)}{f^{'}(z)} \right| \leq M_0, \text{ for all } z \in U,$$
then
$$\left| \frac{zf^{'}(z)}{f(z)} - 1 \right| < 1, \text{ for all } z \in U.$$
The edge M_0 is sharp.

Lemma 1.2 *(Pascu [8]) Let γ be a complex number, $Re\gamma > 0$ and let the function $f \in \mathcal{A}$. If*
$$\frac{1 - |z|^{2Re\gamma}}{Re\gamma} \cdot \left| \frac{zf^{''}(z)}{f^{'}(z)} \right| \leq 1,$$
for all $z \in U$, then for any complex number ζ, $Re\zeta \geq Re\gamma$, the function
$$F_\zeta(z) = \left[\zeta \int_0^z t^{\zeta-1} f^{'}(t) dt \right]^{\frac{1}{\zeta}}$$
is regular and univalent in U.

Lemma 1.3 *(General Schwarz Lemma [6]) Let f be regular function in the disk $U_R = \{z \in \mathbb{C} : |z| < R\}$ with $|f(z)| < M$, M fixed. If f has in $z = 0$ one zero with multiply bigger than m, then*
$$|f(z)| \leq \frac{M}{R^m} |z|^m, \quad z \in U_R.$$
The equality case hold only if $f(z) = e^{i\theta} \cdot \frac{M}{R^m} \cdot z^m$, where θ is constant.

Lemma 1.4 *(Ready and Padmanabhan [10]) Let the functions p, q be analytic in U with*
$$p(0) = q(0) = 0,$$
and let δ be a real number. If the function q maps the unit disk U onto a region which is starlike with respect to the origin, the inequality
$$Re \left[\frac{p^{'}(z)}{q^{'}(z)} \right] > \delta, \text{ for all } z \in U$$
implies that
$$Re \left[\frac{p(z)}{q(z)} \right] > \delta, \text{ for all } z \in U.$$

Lemma 1.5 *(Wilken and Feng [11]) If $0 \le \delta < 1$ and $f \in K(\delta)$, then $f \in S^*(\nu(\delta))$, where*

$$\nu(\delta) = \begin{cases} \frac{1-2\delta}{2^{2(1-\delta)}-2}, & if\ \delta \neq \frac{1}{2}, \\ \frac{1}{2\log 2}, & if\ \delta = \frac{1}{2}. \end{cases} \tag{4}$$

2 Main results

The univalence condition for the operator $I_\alpha^\zeta(f,g)$ defined in (2) is proved in the next theorem, by using Pascu univalence criterion.

Theorem 2.1 *Let α, γ be complex numbers, $Re\gamma > 0, M_0$ the positive solution of the equation (3), $M_0 = 1,5936...$, and $f, g \in \mathcal{A}$. If*

$$\left|\frac{f''(z)}{f'(z)}\right| \le M_0, \quad \left|\frac{g''(z)}{g'(z)}\right| \le M_0\,, \quad z \in U \tag{5}$$

and

$$2M_0 Re\gamma + (2Re\gamma+1)^{\frac{2Re\gamma+1}{2Re\gamma}} \le \frac{Re\gamma \cdot (2Re\gamma+1)^{\frac{2Re\gamma+1}{2Re\gamma}}}{|\alpha|}, \tag{6}$$

then for any complex number ζ, $Re\zeta \ge Re\gamma$, the integral operator

$$I_\alpha^\zeta(f,g)(z) = \left[\zeta \int_0^z t^{\alpha+\zeta-1}\left(\frac{f'(t)}{g(t)}\right)^\alpha dt\right]^{\frac{1}{\zeta}}$$

is in the class S.

Proof. Let the function

$$h(z) = \int_0^z \left[\frac{tf'(t)}{g(t)}\right]^\alpha dt. \tag{7}$$

The function h is regular in U and $h(0) = h'(0) - 1 = 0$.
From (7) we have

$$h'(z) = \left[\frac{zf'(z)}{g(z)}\right]^\alpha$$

and

$$h''(z) = \alpha\left(\frac{zf'(z)}{g(z)}\right)^{\alpha-1} \cdot \left[\frac{f'(z)}{g(z)} + \frac{zf''(z)}{g(z)} - z \cdot \frac{f'(z)}{g(z)} \cdot \frac{g'(z)}{g(z)}\right].$$

We get

$$\frac{zh''(z)}{h'(z)} = \alpha\left[1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)}\right] = \alpha\left[\frac{zf''(z)}{f'(z)} - \left(\frac{zg'(z)}{g(z)} - 1\right)\right]. \tag{8}$$

From (8) we obtain

$$\frac{1-|z|^{2Re\gamma}}{Re\gamma} \cdot \left|\frac{zh''(z)}{h'(z)}\right| \le \frac{1-|z|^{2Re\gamma}}{Re\gamma} \cdot |z| \cdot |\alpha| \cdot \left|\frac{f''(z)}{f'(z)}\right| + \frac{1-|z|^{2Re\gamma}}{Re\gamma} \cdot |\alpha| \cdot \left|\frac{zg'(z)}{g(z)} - 1\right|.$$

From (5) and applying Lemma 1.1 we obtain

$$\left|\frac{zg'(z)}{g(z)} - 1\right| < 1, \text{ for all } z \in U,$$

which implies that

$$\frac{1-|z|^{2Re\gamma}}{Re\gamma}\cdot\left|\frac{zh''(z)}{h'(z)}\right|\leq\frac{1-|z|^{2Re\gamma}}{Re\gamma}\cdot|z|\cdot|\alpha|\cdot M_0+\frac{1-|z|^{2Re\gamma}}{Re\gamma}\cdot|\alpha|.$$

Since

$$\max_{|z|\leq 1}\frac{1-|z|^{2Re\gamma}}{Re\gamma}\cdot|z|=\frac{2}{(2Re\gamma+1)^{\frac{2Re\gamma+1}{2Re\gamma}}},$$

we have

$$\frac{1-|z|^{2Re\gamma}}{Re\gamma}\cdot\left|\frac{zh''(z)}{h'(z)}\right|\leq\frac{2}{(2Re\gamma+1)^{\frac{2Re\gamma+1}{2Re\gamma}}}\cdot|\alpha|\cdot M_0+\frac{|\alpha|}{Re\gamma}. \tag{9}$$

Using (6) in (9) we obtain

$$\frac{1-|z|^{2Re\gamma}}{Re\gamma}\cdot\left|\frac{zh''(z)}{h'(z)}\right|\leq 1,\quad z\in U, \tag{10}$$

and by applying Lemma 1.2, we obtain that the function $I_\alpha^\zeta(f,g)(z)$ is in the class S. ■

If we put $\zeta=1$ in Theorem 2.1, we obtain

Corollary 2.2 *Let α,γ be complex numbers, $0<Re\gamma\leq 1$, M_0 the positive solution of the equation (3), $M_0=1,5936...$, and $f,g\in\mathcal{A}$. If*

$$\left|\frac{f''(z)}{f'(z)}\right|\leq M_0,\quad\left|\frac{g''(z)}{g'(z)}\right|\leq M_0\,,\quad z\in U,$$

and

$$2M_0Re\gamma+(2Re\gamma+1)^{\frac{2Re\gamma+1}{2Re\gamma}}\leq\frac{Re\gamma\cdot(2Re\gamma+1)^{\frac{2Re\gamma+1}{2Re\gamma}}}{|\alpha|},$$

then the integral operator

$$I_\alpha(f,g)(z)=\int_0^z\left[\frac{tf'(t)}{g(t)}\right]^\alpha dt,$$

is in the class S.

Putting $Re\gamma=1$ in Corrolary 2.2, we obtain

Corollary 2.3 *Let α,γ be complex numbers, $0<Re\gamma\leq 1$, M_0 the positive solution of the equation (3), $M_0=1,5936...$, and $f,g\in\mathcal{A}$. If*

$$\left|\frac{f''(z)}{f'(z)}\right|\leq M_0,\quad\left|\frac{g''(z)}{g'(z)}\right|\leq M_0,\quad z\in U,$$

and

$$|\alpha|\leq\frac{3\sqrt{3}}{2M_0+3\sqrt{3}}$$

then the integral operator

$$I_\alpha(f,g)(z)=\int_0^z\left[\frac{tf'(t)}{g(t)}\right]^\alpha dt,$$

is in the class S.

This result was also obtained in [2].

In the following theorem we give sufficient conditions such that the integral operator $I_\alpha^\zeta(f,g)(z)\in S^*$.

Theorem 2.4 *Let α, ζ be complex numbers, $M \geq 1$, $f \in \mathcal{A}$ and $g \in B(\mu, \lambda)$ such that*

$$\left| \frac{zf''(z)}{f'(z)} \right| < 1 \quad and \quad |g(z)| < M, \quad z \in U.$$

If

$$|\alpha| \leq \frac{|\zeta|}{2 + (2 - \lambda) M^{\mu - 1}},$$

then the integral operator $I_\alpha^\zeta(f, g)(z)$ is in the class S^*.

Proof. Let's consider the function φ given by

$$\varphi(z) = I_\alpha^\zeta(f, g)(z), \quad z \in U. \tag{11}$$

Then, by differentiating φ with respect to z, we obtain

$$\frac{z\varphi'(z)}{\varphi(z)} = \frac{z^{\alpha+\zeta} \left[\frac{f'(z)}{g(z)}\right]^\alpha}{\zeta \int_0^z t^{\alpha+\zeta-1} \left(\frac{f'(t)}{g(t)}\right)^\alpha dt}.$$

Letting

$$p(z) = z\varphi'(z) \text{ and } q(z) = \varphi(z),$$

we find that

$$\frac{p'(z)}{q'(z)} = 1 + \frac{\alpha}{\zeta} \left[1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)}\right].$$

Thus,

$$\begin{aligned} \left| \frac{p'(z)}{q'(z)} - 1 \right| &\leq \frac{|\alpha|}{|\zeta|} \left[1 + \left| \frac{zf''(z)}{f'(z)} \right| + \left| \frac{zg'(z)}{g(z)} \right| \right] \\ &\leq \frac{|\alpha|}{|\zeta|} \left[1 + \left| \frac{zf''(z)}{f'(z)} \right| + \left(\left| g'(z) \cdot \left(\frac{z}{g(z)}\right)^\mu - 1 \right| + 1 \right) \left| \frac{g(z)}{z} \right|^{\mu - 1} \right]. \end{aligned} \tag{12}$$

Since $|g(z)| < M$, $z \in U$, by applying the Schwarz Lemma, we have

$$\left| \frac{g(z)}{z} \right| \leq M, \text{ for all } z \in U. \tag{13}$$

By using the hypothesis and (13) we obtain

$$\left| \frac{p'(z)}{q'(z)} - 1 \right| \leq \frac{|\alpha|}{|\zeta|} \left[2 + (2 - \lambda) \cdot M^{\mu - 1}\right] \leq 1,$$

that is

$$Re \left[\frac{p'(z)}{q'(z)}\right] > 0, \quad z \in U.$$

Therefore, applying Lemma 1.4, we find that

$$Re \left[\frac{p(z)}{q(z)}\right] > 0, \quad z \in U.$$

This completes the proof. of the theorem. ∎

Taking $\mu = 1$ in Theorem 2.4, we have

Corollary 2.5 *Let α, ζ be complex numbers, $M \geq 1$, $f \in \mathcal{A}$ and $g \in S^*(\lambda)$ such that*

$$\left|\frac{zf''(z)}{f'(z)}\right| < 1 \quad and \quad |g(z)| < M, \quad z \in U.$$

If

$$|\alpha| \leq \frac{|\zeta|}{4-\lambda},$$

then the integral operator $I_\alpha(f,g)$ is in the class S^*.

Letting $\lambda = 0$ in Corollary 2.5, we obtain

Corollary 2.6 *Let α, ζ be complex numbers with $|\alpha| = \frac{|\zeta|}{4}$ and $M \geq 1$. If $f \in \mathcal{A}$ and $g \in S^*$ satisfies*

$$\left|\frac{zf''(z)}{f'(z)}\right| < 1 \quad and \quad |g(z)| < M, \quad z \in U,$$

then the integral operator $I_\alpha^\zeta(f,g)$ is in the class S^*.

Next, we find sufficient conditions such that $I_\alpha^\zeta(f,g)(z) \in K(\delta)$.

Theorem 2.7 *Let α, ζ be complex numbers, M, $N \geq 1$, $f \in \mathcal{A}$ and $g \in B(\mu,\lambda)$. If*

$$|g(z)| < M \quad and \quad \left|\frac{f''(z)}{f'(z)}\right| < N,$$

for all $z \in U$ then, the integral operator $I_\alpha^\zeta(f,g)$ is in the class $K(\delta)$, where

$$\delta = 1 - \left|\frac{\alpha}{\zeta}\right|[1 + N + (2-\lambda)M^{\mu-1}] \quad and \quad 0 < \left|\frac{\alpha}{\zeta}\right|[1 + N + (2-\lambda)M^{\mu-1}] \leq 1.$$

Proof. Letting the function φ be given by (11), we have

$$\frac{z\varphi''(z)}{\varphi'(z)} = \frac{\alpha}{\zeta}\left[1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)}\right].$$

Therefore, using the hypothesis of the theorem and applying the Schwarz Lemma, we obtain

$$\left|\frac{z\varphi''(z)}{\varphi'(z)}\right| \leq \left|\frac{\alpha}{\zeta}\right|\left[1 + \left|\frac{zf''(z)}{f'(z)}\right| + \left|\frac{zg'(z)}{g(z)}\right|\right] \leq \left|\frac{\alpha}{\zeta}\right|\left[1 + N + \left|\frac{zg'(z)}{g(z)} \cdot \left(\frac{z}{g(z)}\right)^\mu\right| \cdot \left|\left(\frac{g(z)}{z}\right)^{\mu-1}\right|\right]$$
$$\leq \left|\frac{\alpha}{\zeta}\right|\left[1 + N + \left[\left|g'(z)\left(\frac{z}{g(z)}\right)^\mu - 1\right| + 1\right] \cdot M^{\mu-1}\right] \leq \left|\frac{\alpha}{\zeta}\right|[1 + N + (2-\lambda) \cdot M^{\mu-1}] = 1 - \delta.$$

This evidently completes the proof. ■

Letting $\mu = 1$ in Theorem 2.7, we have

Corollary 2.8 *Let α, ζ be complex numbers, M, $N \geq 1$, $f \in \mathcal{A}$ and $g \in S^*(\lambda)$. If*

$$|g(z)| < M \quad and \quad \left|\frac{f''(z)}{f'(z)}\right| < N,$$

for all $z \in U$ then, the integral operator $I_\alpha^\zeta(f,g)$ is in the class $K(\delta)$, where

$$\delta = 1 - \left|\frac{\alpha}{\zeta}\right|(3 + N - \lambda) \quad and \quad 0 < \left|\frac{\alpha}{\zeta}\right|(3 + N - \lambda) \leq 1.$$

Letting $\delta = \lambda = 0$ in Corollary 2.8, we obtain

Corollary 2.9 *Let α, ζ be complex numbers, M, $N \geq 1$, $f \in \mathcal{A}$ and $g \in S^*$. If*

$$|g(z)| < M \quad \text{and} \quad \left|\frac{f''(z)}{f'(z)}\right| < N$$

for all $z \in U$ then, the integral operator $I_{\alpha}^{\zeta}(f, g)$ is convex in U, where

$$|\alpha| = \frac{|\zeta|}{3+N}.$$

Theorem 2.10 *If α, ζ are complex numbers and $f, g \in K(\delta)$ then $I_{\zeta}^{\alpha}(f, g)$ belongs to the class $K(b)$, where $b = 1 - \left|\frac{\alpha}{\zeta}\right|(2 - \delta - \nu(\delta))$, $0 \leq b < 1$ and $\nu(\delta)$ is given by Lemma 1.5.*

Proof. Letting the function φ be given by (11), we have

$$\left|\frac{z\varphi''(z)}{\varphi'(z)}\right| \leq \left|\frac{\alpha}{\zeta}\right| \left[\left|\frac{zf''(z)}{f'(z)}\right| + \left|\frac{zg'(z)}{g(z)} - 1\right|\right].$$

Since $g \in K(\delta)$, by applying Lemma 1.5, we yield that $g \in S^*(\nu(\delta))$. So,

$$\left|\frac{z\varphi''(z)}{\varphi'(z)}\right| \leq \left|\frac{\alpha}{\zeta}\right|(2 - \delta - \nu(\delta)) = 1 - b, \tag{14}$$

which evidently proves Theorem 2.10. ■

Corollary 2.11 *Let α, ζ be complex numbers with $|\alpha/\zeta| \leq 2/3$. If $f, g \in K$ then $I_{\zeta}^{\alpha}(f, g)$ belongs to the class $K(1 - \frac{3}{2}\left|\frac{\alpha}{\zeta}\right|)$.*

References

[1] J. Becker, *Lownersche Differential gleichung und quasi-konform fortsetzbare schlichte funktionen,* J. Reine Angew. Math., 255 (1972), 23-43.

[2] R. Bucur, L. Andrei, D. Breaz, *Geometric Properties of a New Integral Operator,* Abstr. Appl. Anal., Vol. 2015, Article ID 430197 (2015), 6 pg.

[3] B. A. Frasin, M. Darus, *On certain analytic univalent functions,* Int. J. Math. and Math. Sci., 25(5)(2001), 305-310.

[4] B. A. Frasin, J. Jahangiri, *A new and comprehensive class of analytic functions,* Analele Univ. Oradea, Fasc. Math. XV(2008), 59-62.

[5] V. M. Macarie, *Univalence criterion for a certain general integral operator*, Stud. Univ. Babes-Bolyai Math. 57 (2012), No. 2, 221-225.

[6] O. Mayer, *The functions theory of one variable complex,* Bucuresti, 1981.

[7] P. T. Mocanu, I. Şerb, *A sharp simple criterion for a subclass of starlike functions,* Complex variables, 32 (1997), 161-168.

[8] N. N. Pascu, *An improvement of Becker's univalence criterion* Proceedings of the Commemorative Session Simion Stoilow, Brasov, Preprint (1987), 43-48.

[9] V. Pescar, *New univalence criteria for some integral operators,* Stud. Univ. Babes-Bolyai Math. 59 (2014), No. 2, 167-176.

[10] G. L. Ready, K. S. Padmanabhan, *On analytic function with reference to the Bernardi integral operator,* Bull. Austral. Math. Soc. 25(1982), 387-396.

[11] D. R. Wilken, J. Feng, *A remark on convex and starlike functions*, J. London Math. Soc. 21(2) (1980), 287-290.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 21, NO.6, 2016, COPYRIGHT 2016 EUDOXUS PRESS, LLC

A new generalization of the Ostrowski inequality and Ostrowski type inequality for double integrals on time scales

A. Tuna* and **S. Kutukcu**

Department of Mathematics, Faculty of Science and Arts University of Niğde, 51240, Niğde, Turkey

Department of Mathematics, Faculty of Science and Arts Ondokuz Mayıs University, 55139, Samsun, Turkey

Abstract: A new generalization of the Ostrowski inequality and Ostrowski type inequality for double integrals on time scales are established in this paper. Several interesting inequalities representing special cases of our general results are supplied.

Keywords: Ostrowski type inequalities; Double integrals; Time scales.

1 Introduction

In 1938, Ostrowski [21] proved the following interesting integral inequality.

Theorem 1.1 *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable in (a, b) and its derivative $f' : (a, b) \to \mathbb{R}$ is bounded in (a, b). Then for any $x \in [a, b]$, we have*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \left(\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right) (b-a)\|f'\|_\infty.$$

where $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(x)| < \infty$. The inequality is sharp in the sense that the constant $\frac{1}{4}$ cannot be replaced by a smaller one.

Mohammad Masjed-Jamei and Sever S. Dragomir[11] established the generalization of the Ostrowski inequality for functions in L^p-spaces and applied it to find appropriate error bounds for numerical quadrature rules of equal coefficients type using kernel (3.2) on $[a, b]$.

*Corresponding Author
E-mail: atuna@nigde.edu.tr (A. Tuna), skutukcu@yahoo.com (S. Kutukcu)
2010 MR Subject Classification: 26D15, 26E70

The Ostrowski inequality and the Montgomery identity were generalized by Bohner et. al.[7] to an arbitrary time scale, unifying the discrete, the continuous, and the quantum cases:

Theorem 1.2 *$a, b, s, t \in \mathbb{T}$, $a < b$ and $f : [a, b] \to \mathbb{R}$ be differentiable. Then*

$$\left| f(t) - \frac{1}{b-a} \int_a^b f(\sigma(s)) \triangle s \right| \leq \frac{M}{b-a} \left(h_2(t, a) + h_2(t, b) \right), \tag{1.1}$$

where $h_2(.,.)$ is defined by Definition 8 and $M = \sup_{a<t<b} \left| f^{\triangle}(t) \right| < \infty$. This inequality is sharp in the sense that the right-hand side of (1.1) cannot be replaced by a smaller one.

During the past few years, many researchers have given considerable attention to the Ostrowski inequality on time scales. In [16, 17, 18], variants generalizations, extensions of Ostrowski inequality on time scales have established.

In 1988, S. Hilger [10] introduced the time scales theory to unify continuous and discrete analysis. For other results of Ostrowski type inequalities involving functions of two independent variables for multiple points, the Ostrowski type inequalities involving functions of two independent variables for k^2 points, generalized double integral Ostrowski type inequalities, Ostrowski type inequalities for double integrals, Ostrowski type inequality for double integrals on time scales via $\triangle\triangle-$integral, Ostrowski and Grüss type inequalities for triple integrals, weighted Grüss type inequalities for double integrals, Grüss type inequalities, the Ostrowski type inequality for double integrals, generalized n dimensional Ostrowski type and Grüss type integral inequalities, generalized $2D$ Ostrowski-Grüss type integral inequalities on time scales see the papers [8, 9, 12, 14, 15, 19, 20, 22, 23, 24, 25], respectively.

This paper is organized as follows. In Section 2, we briefly present the general definitions and theorems related to the time scales calculus. A new generalization of the Ostrowski inequality and Ostrowski type inequality for double integrals are derived in Section 3. We also apply our results to the continuous and discrete calculus cases.

2 General Definitions

In this section we briefly introduce the time scales theory. For further details and proofs we refer the reader to Hilger's Ph.D. thesis [10], the books [2, 3, 13], and the survey [1].

Definition 2.1 *A time scale is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$.*

Throughout this work we assume $\mathbb{T}$ is a time scale and $\mathbb{T}$ has the topology that is inherited from the standard topology on $\mathbb{R}$. It is also assumed throughout that in $\mathbb{T}$ the interval $[a, b]$ means the set $\{t \in \mathbb{T}\colon a \leq t \leq b\}$ for the points $a < b$ in $\mathbb{T}$. Since a time scale may not be connected, we need the following concept of jump operators.

Definition 2.2 *The forward and bacward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ are defined by*

$$\sigma(t) = \inf \{s \in \mathbb{T}\colon s > t\}, \qquad \rho(t) = \sup \{s \in \mathbb{T}\colon s < t\},$$

respectively.

The jump operators σ and ρ allow the classification of points in $\mathbb{T}$ as follows.

Definition 2.3 *If $\sigma(t) > t$, then we say that t is right-scattered, while if $\rho(t) < t$ then we say that t is left-scattered. Points that are right-scattered and left-scattered at the same time are called isolated. If $\sigma(t) = t$, then t is called right-dense, anf if $\rho(t) = t$ then t is called left-dense. Points that are both right-dense and left-dense are called dense.*

Definition 2.4 *The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$ for $t \in \mathbb{T}$. The set $\mathbb{T}^k$ is defined as follows: if $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} - \{m\}$; otherwise, $\mathbb{T}^k = \mathbb{T}$.*

If $\mathbb{T} = \mathbb{R}$, then $\mu(t) = 0$, and when $\mathbb{T} = \mathbb{Z}$, we have $\mu(t) = 1$.

Definition 2.5 *Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and fix $t \in \mathbb{T}$. Then the (delta) derivative $f^{\Delta}(t) \in \mathbb{R}$ at $t \in \mathbb{T}^k$ is defined to be number (provided it exists) with property that given for any $\epsilon > 0$ there exists a neighborhood U of t such that*

$$\left| f(\sigma(t)) - f(s) - f^{\Delta}(t) \left[\sigma(t) - s \right] \right| \le \epsilon \left| \sigma(t) - s \right|, \qquad \forall s \in U.$$

If $\mathbb{T} = \mathbb{R}$, then $f^{\Delta}(t) = \frac{df(t)}{dt}$, and if $\mathbb{T} = \mathbb{Z}$, then $\Delta f(t) = f(t+1) - f(t)$.

Theorem 2.6 *Assume $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^k$. Then the product $fg : \mathbb{T} \to \mathbb{R}$ is differentiable at t with*

$$(fg)^{\Delta}(t) = f^{\Delta}(t) g(t) + f(\sigma(t)) g^{\Delta}(t).$$

Definition 2.7 *The function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous(denote $f \in C_{rd}(\mathbb{T}, \mathbb{R})$), if it is continuous at all right-dense points $t \in \mathbb{T}$ and its left-sided limits exist at all left-dense points $t \in \mathbb{T}$.*

It follows from[2, Theorem 1.74] that every rd-continuous function has an anti-derivative.

Definition 2.8 *A function $F : \mathbb{T} \to \mathbb{R}$ is called a delta antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided $F^{\Delta}(t) = f(t)$ for any $t \in \mathbb{T}^k$. In this case, we define the $\Delta-$integral of f as*

$$\int_a^b f(s) \Delta s := F(t) - F(a), \quad t \in \mathbb{T}.$$

Theorem 2.9 *Let f, g be rd-continuous, $a, b, c \in \mathbb{T}$ and $\alpha, \beta \in \mathbb{R}$. Then*

(1) $\int_a^b \left[\alpha f(t) + \beta g(t) \right] \Delta t = \alpha \int_a^b f(t) \Delta t + \beta \int_a^b g(t) \Delta t,$

(2) $\int_a^b f(t) \Delta t = -\int_b^a f(t) \Delta t,$

(3) $\int_a^b f(t) \Delta t = \int_a^c f(t) \Delta t + \int_c^b f(t) \Delta t$

(4) $\int_a^b f(t) g^{\Delta}(t) \Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t) g^{\sigma}(t) \Delta t,$

Theorem 2.10 *If f is $\Delta-$integrable on $[a,b]$, then so is $|f|$, and*

$$\left|\int_a^b f(t)\Delta t\right| \leq \int_a^b |f(t)|\,\Delta t.$$

Definition 2.11 *Let $h_k : \mathbb{T}^2 \to \mathbb{R}$, $k \in \mathbb{N}_0$ be defined by $h_0(t,s) = 1$, for all $s,t \in \mathbb{T}$ and then recursively by $h_{k+1}(t,s) = \int_s^t h_k(\tau,s)\,\Delta\tau$, for all $s,t \in \mathbb{T}$.*

The two-variable time scales calculus and multiple integration on time scales were introduced in [4, 5] (see also [6]). Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be two time scales and put

$$\mathbb{T}_1 \times \mathbb{T}_2 = \{(t,s) : t \in \mathbb{T}_1,\ s \in \mathbb{T}_2\},$$

which is acomplete metric space with the metric d defined by

$$d((t,s),(t',s')) = \sqrt{(t-t')^2 + (s-s')^2}, \quad \forall (t,s),(t',s') \in \mathbb{T}_1 \times \mathbb{T}_2.$$

For a given $\delta > 0$, the $\delta-$neighborhood $U_\delta(t_0,s_0)$ of a given point $(t_0,s_0) \in \mathbb{T}_1 \times \mathbb{T}_2$ is the set of all points $(t,s) \in \mathbb{T}_1 \times \mathbb{T}_2$ such that $d((t,s),(t',s')) < \delta$. Let σ_1, ρ_1 and σ_2, ρ_2 be the forward jump and backward jump operators in $\mathbb{T}_1$ and $\mathbb{T}_2$, respectively.

Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be two time scales. For $i = 1,2$, let σ_i, ρ_i and $\triangle_i$ denote the forward ump operator, the backward jump operator, and the delta differentiation operator, respectively, on $\mathbb{T}_i$. Suppose $a < b$ are points in $\mathbb{T}_1$, $c < d$ are points in $\mathbb{T}_2$, $[a,b)$ is the half-closed bounded interval in $\mathbb{T}_1$, $[c,d)$ is the half-closed bounded interval in $\mathbb{T}_2$ Let us introduce a "rectangle" in $\mathbb{T}_1\times\ \mathbb{T}_2$ by

$$R = [a,b) \times [c,d) = \{(t_1,t_2) : t_1 \in [a,b),\ t_2 \in [c,d)\}.$$

3 Main Results

To derive main results in this section, we need the following Lemma.

Lemma 3.1 *Let $a,b,t \in \mathbb{T}_1$ and $c,d,s \in \mathbb{T}_2$ and $f \in CC^1_{rd}([a,b]\times[c,d],\mathbb{R})$. Then we have*

$$\begin{aligned} & w_1 w_2 f(x,y) \\ = \; & \int_a^b\int_c^d K_{w_1}(x,t)\,K_{w_2}(y,s)\,\frac{\partial^2 f(t,s)}{\triangle_2 s\triangle_1 t}\triangle_2 s\triangle_1 t - F_1(x,y) - F_2 + w_2\int_a^b f(\sigma(t),y)\triangle_1 t \\ & + w_1\int_c^d f(x,\sigma(s))\triangle_2 s - \int_a^b [(c-\beta_1) f(\sigma(t),c) - (d-\beta_2) f(\sigma(t),d)]\triangle_1 t \\ & - \int_c^d [(a-\theta_1) f(a,\sigma(s)) - (b-\theta_2) f(b,\sigma(s))]\triangle_2 s - \int_a^b\int_c^d f(\sigma(t),\sigma(s))\triangle_2 s\triangle_1 t \end{aligned} \quad (3.1)$$

4

for all $(x,y)\in[a,b]\times[c,d]$, *where*

$$K_{w_1}(x,t)=\begin{cases} t-\frac{(b-w_1)f(b)-af(a)}{f(b)-f(a)}=t-\theta_1, & a\le t\le x\\ t-\frac{bf(b)-(a+w_1)f(a)}{f(b)-f(a)}=t-\theta_2, & x<t\le b\end{cases} \tag{3.2}$$

$$K_{w_2}(y,s)=\begin{cases} s-\frac{(d-w_2)f(d)-cf(c)}{f(d)-f(c)}=s-\beta_1, & c\le s\le y\\ s-\frac{df(d)-(c+w_2)f(c)}{f(d)-f(c)}=s-\beta_2, & y<s\le d\end{cases} \tag{3.3}$$

in which $w_1,w_2\in\mathbb{R}$, $f(b)\neq f(a)$, $f(d)\neq f(c)$, $\theta_2-\theta_1=w_1$, $\beta_2-\beta_1=w_2$,

$$F_1(x,y)=w_1\left[(d-\beta_2)f(x,d)-(c-\beta_1)f(x,c)\right]+w_2\left[(b-\theta_2)f(b,y)-(a-\theta_1)f(a,y)\right],$$

and

$$F_2=(a-\theta_1)\left[(c-\beta_1)f(a,c)-(d-\beta_2)f(a,d)\right]+(b-\theta_2)\left[(d-\beta_2)f(b,d)-(c-\beta_1)f(b,c)\right].$$

Proof. Integrating by parts and considering (3.2) and (3.3), we get

$$\begin{aligned}&\int_a^b\int_c^d K_{w_1}(x,t)K_{w_2}(y,s)\frac{\partial^2 f(t,s)}{\triangle_2 s\triangle_1 t}\triangle_2 s\triangle_1 t\\ =&\int_a^x\int_c^y (t-\theta_1)(s-\beta_1)\frac{\partial^2 f(t,s)}{\triangle_2 s\triangle_1 t}\triangle_2 s\triangle_1 t+\int_a^x\int_y^d (t-\theta_1)(s-\beta_2)\frac{\partial^2 f(t,s)}{\triangle_2 s\triangle_1 t}\triangle_2 s\triangle_1 t\\ &+\int_x^b\int_c^y (t-\theta_2)(s-\beta_1)\frac{\partial^2 f(t,s)}{\triangle_2 s\triangle_1 t}\triangle_2 s\triangle_1 t+\int_x^b\int_y^d (t-\theta_2)(s-\beta_2)\frac{\partial^2 f(t,s)}{\triangle_2 s\triangle_1 t}\triangle_2 s\triangle_1 t\end{aligned} \tag{3.4}$$

We have

$$\begin{aligned}&\int_a^x\int_c^y (t-\theta_1)(s-\beta_1)\frac{\partial^2 f(t,s)}{\triangle_2 s\triangle_1 t}\triangle_2 s\triangle_1 t\\ =&\int_a^x (t-\theta_1)\left[(y-\beta_1)\frac{\partial f(t,y)}{\triangle_1 t}-(c-\beta_1)\frac{\partial f(t,c)}{\triangle_1 t}-\int_c^y\frac{\partial f(t,\sigma(s))}{\triangle_1 t}\triangle_2 s\right]\triangle_1 t\\ =&(y-\beta_1)\int_a^x (t-\theta_1)\frac{\partial f(t,y)}{\triangle_1 t}\triangle_1 t-(c-\beta_1)\int_a^x (t-\theta_1)\frac{\partial f(t,c)}{\triangle_1 t}\triangle_1 t\\ &-\int_c^y\left(\int_a^x (t-\theta_1)\frac{\partial f(t,\sigma(s))}{\triangle_1 t}\triangle_1 t\right)\triangle_2 s\end{aligned}$$

$$
\begin{aligned}
= \quad & (y-\beta_1)\left[(x-\theta_1) f(x,y) - (a-\theta_1) f(a,y) - \int_a^x f(\sigma(t),y)\triangle_1 t\right] \\
& -(c-\beta_1)\left[(x-\theta_1) f(x,c) - (a-\theta_1) f(a,c) - \int_a^x f(\sigma(t),c)\triangle_1 t\right] \\
& -\int_c^y \left[(x-\theta_1) f(x,\sigma(s)) - (a-\theta_1) f(a,\sigma(s)) - \int_a^x f(\sigma(t),\sigma(s))\triangle_1 t\right]\triangle_2 s \\
= \quad & (x-\theta_1)(y-\beta_1) f(x,y) - (a-\theta_1)(y-\beta_1) f(a,y) - (y-\beta_1)\int_a^x f(\sigma(t),y)\triangle_1 t \\
& -(c-\beta_1)(x-\theta_1) f(x,c) + (a-\theta_1)(c-\beta_1) f(a,c) + (c-\beta_1)\int_a^x f(\sigma(t),c)\triangle_1 t \\
& -(x-\theta_1)\int_c^y f(x,\sigma(s))\triangle_2 s + (a-\theta_1)\int_c^y f(a,\sigma(s))\triangle_2 s \\
& +\int_a^x\int_c^y f(\sigma(t),\sigma(s))\triangle_2 s\triangle_1 t, \qquad (3.5)
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
& \int_a^x\int_y^d (t-\theta_1)(s-\beta_2)\frac{\partial^2 f(t,s)}{\triangle_2 s\triangle_1 t}\triangle_2 s\triangle_1 t \\
= \quad & (d-\beta_2)(x-\theta_1) f(x,d) - (a-\theta_1)(d-\beta_2) f(a,d) - (d-\beta_2)\int_a^x f(\sigma(t),d)\triangle_1 t \\
& -(x-\theta_1)(y-\beta_2) f(x,y) + (a-\theta_1)(y-\beta_2) f(a,y) + (y-\beta_2)\int_a^x f(\sigma(t),y)\triangle_1 t \\
& -(x-\theta_1)\int_y^d f(x,\sigma(s))\triangle_2 s + (a-\theta_1)\int_y^d f(a,\sigma(s))\triangle_2 s \\
& +\int_a^x\int_y^d f(\sigma(t),\sigma(s))\triangle_2 s\triangle_1 t, \qquad (3.6)
\end{aligned}
$$

6

and

$$\begin{aligned}
&\int_x^b \int_c^y (t-\theta_2)(s-\beta_1) \frac{\partial^2 f(t,s)}{\triangle_2 s \triangle_1 t} \triangle_2 s \triangle_1 t \\
&= (b-\theta_2)(y-\beta_1) f(b,y) - (x-\theta_2)(y-\beta_1) f(x,y) - (y-\beta_1) \int_x^b f(\sigma(t),y) \triangle_1 t \\
&\quad - (b-\theta_2)(c-\beta_1) f(b,c) + (c-\beta_1)(x-\theta_2) f(x,c) + (c-\beta_1) \int_x^b f(\sigma(t),c) \triangle_1 t \\
&\quad - (b-\theta_2) \int_c^y f(b,\sigma(s)) \triangle_2 s + (x-\theta_2) \int_c^y f(x,\sigma(s)) \triangle_2 s + \int_x^b \int_c^y f(\sigma(t),\sigma(s)) \triangle_2 s \triangle_1 t, \qquad (3.7)
\end{aligned}$$

and finally

$$\begin{aligned}
&\int_x^b \int_y^d (t-\theta_2)(s-\beta_2) \frac{\partial^2 f(t,s)}{\triangle_2 s \triangle_1 t} \triangle_2 s \triangle_1 t \\
&= (b-\theta_2)(d-\beta_2) f(b,d) - (d-\beta_2)(x-\theta_2) f(x,d) - (d-\beta_2) \int_x^b f(\sigma(t),d) \triangle_1 t \\
&\quad - (b-\theta_2)(y-\beta_2) f(b,y) + (x-\theta_2)(y-\beta_2) f(x,y) + (y-\beta_2) \int_x^b f(\sigma(t),y) \triangle_1 t \\
&\quad - (b-\theta_2) \int_y^d f(b,\sigma(s)) \triangle_2 s + (x-\theta_2) \int_y^d f(x,\sigma(s)) \triangle_2 s \\
&\quad + \int_x^b \int_y^d f(\sigma(t),\sigma(s)) \triangle_2 s \triangle_1 t, \qquad (3.8)
\end{aligned}$$

Substituting (3.5)-(3.8) into (3.4), we obtain the result (3.1). ■

Corollary 3.2 *In the Lemma 3.1, we choose* $w_1 = b-a$, $w_2 = d-c$ *and hence* $\theta_1 = a$, $\theta_2 = b$, $\beta_1 = c$, $\beta_2 = d$. *Then by simple computation, we get*

$$\begin{aligned}
\int_a^b \int_c^d K_{w_1}(x,t) K_{w_2}(y,s) \frac{\partial^2 f(t,s)}{\triangle_2 s \triangle_1 t} \triangle_2 s \triangle_1 t &= (b-a)(d-c) f(x,y) - (d-c) \int_a^b f(\sigma(t),y) \triangle_1 t \\
&\quad - (b-a) \int_c^d f(x,\sigma(s)) \triangle_2 s + \int_a^b \int_c^d f(\sigma(t),\sigma(s)) \triangle_2 s \triangle_1 t
\end{aligned}$$

This is the result given in [22, Lemma 2.3].

7

The following Theorem is a new generalization of the Ostrowski inequality for double integrals on time scales.

Theorem 3.3 *Let the assumptions of Lemma 3.1 hold. Assume that* $\sup_{a<t<b;c<s<d}\left|\frac{\partial^2 f(t,s)}{\triangle_2 s\triangle_1 t}\right| < \infty$. *Then we have the inequality*

$$\begin{aligned}
&\left| w_1 w_2 f(x,y) + F_1(x,y) + F_2 - w_2 \int_a^b f(\sigma(t),y)\triangle_1 t \right. \\
&-w_1 \int_c^d f(x,\sigma(s))\triangle_2 s + \int_a^b \left[(c-\beta_1) f(\sigma(t),c) - (d-\beta_2) f(\sigma(t),d)\right]\triangle_1 t \\
&\left. + \int_c^d \left[(a-\theta_1) f(a,\sigma(s)) - (b-\theta_2) f(b,\sigma(s))\right]\triangle_2 s + \int_a^b \int_c^d f(\sigma(t),\sigma(s))\triangle_2 s\triangle_1 t \right| \\
\leq\ & K\left[\int_a^x |t-\theta_1|\triangle_1 t + \int_x^b |t-\theta_2|\triangle_1 t\right]\left[\int_c^y |s-\beta_1|\triangle_2 s + \int_y^d |s-\beta_2|\triangle_2 s\right] \qquad (3.9)
\end{aligned}$$

where $K = \sup_{a<t<b;c<s<d}\left|\frac{\partial^2 f(t,s)}{\triangle_2 s\triangle_1 t}\right|$,

$$F_1(x,y) = w_1\left[(d-\beta_2) f(x,d) - (c-\beta_1) f(x,c)\right] + w_2\left[(b-\theta_2) f(b,y) - (a-\theta_1) f(a,y)\right],$$

and

$$F_2 = (a-\theta_1)\left[(c-\beta_1) f(a,c) - (d-\beta_2) f(a,d)\right] + (b-\theta_2)\left[(d-\beta_2) f(b,d) - (c-\beta_1) f(b,c)\right].$$

Proof. By applying Lemma 3.1 and using the properties of modulus, we can state that

$$\begin{aligned}
&\left| w_1 w_2 f(x,y) + F_1(x,y) + F_2 - w_2 \int_a^b f(\sigma(t),y)\triangle_1 t \right. \\
&-w_1 \int_c^d f(x,\sigma(s))\triangle_2 s + \int_a^b \left[(c-\beta_1) f(\sigma(t),c) - (d-\beta_2) f(\sigma(t),d)\right]\triangle_1 t \\
&\left. + \int_c^d \left[(a-\theta_1) f(a,\sigma(s)) - (b-\theta_2) f(b,\sigma(s))\right]\triangle_2 s + \int_a^b \int_c^d f(\sigma(t),\sigma(s))\triangle_2 s\triangle_1 t \right| \\
\leq\ & \int_a^b \int_c^d |K_{w_1}(x,t)|\,|K_{w_2}(y,s)| \left|\frac{\partial^2 f(t,s)}{\triangle_2 s\triangle_1 t}\right| \triangle_2 s\triangle_1 t
\end{aligned}$$

where $K_{w_1}(x,t)$ and $K_{w_2}(y,s)$ are given by (3.2) and (3.3). The proof is complete. ■

Theorem 3.4 *Let $a,b,t\in\mathbb{T}_1$ and $c,d,s\in\mathbb{T}_2$ and $f\in CC^1_{rd}\left([a,b]\times[c,d],\mathbb{R}\right)$. Assume that $\sup\limits_{a<t<b;c<s<d}\left|\frac{\partial^2 f(t,s)}{\triangle_2 s\triangle_1 t}\right|<\infty$ and $\sup\limits_{a<t<b;c<s<d}\left|\frac{\partial^2 g(t,s)}{\triangle_2 s\triangle_1 t}\right|<\infty$. Then for all $(x,y)\in[a,b]\times[c,d]$, we have the inequality*

$$
\begin{aligned}
&\Bigg|\int_a^b\int_c^d f\left(\sigma(t),\sigma(s)\right)\triangle_2 s\triangle_1 t\int_a^b\int_c^d g\left(\sigma(t),\sigma(s)\right)\triangle_2 s\triangle_1 t\\
&+\frac{w_1w_2}{2}\left[f(x,y)\int_a^b\int_c^d g\left(\sigma(t),\sigma(s)\right)\triangle_2 s\triangle_1 t+g(x,y)\int_a^b\int_c^d f\left(\sigma(t),\sigma(s)\right)\triangle_2 s\triangle_1 t\right]\\
&+\frac{1}{2}\left[F_1(x,y)\int_a^b\int_c^d g\left(\sigma(t),\sigma(s)\right)\triangle_2 s\triangle_1 t+G_1(x,y)\int_a^b\int_c^d f\left(\sigma(t),\sigma(s)\right)\triangle_2 s\triangle_1 t\right]\\
&+\frac{1}{2}\left[F_2\int_a^b\int_c^d g\left(\sigma(t),\sigma(s)\right)\triangle_2 s\triangle_1 t+G_2\int_a^b\int_c^d f\left(\sigma(t),\sigma(s)\right)\triangle_2 s\triangle_1 t\right]\\
&-\frac{w_2}{2}\left[\int_a^b f\left(\sigma(t),y\right)\triangle_1 t\int_a^b\int_c^d g\left(\sigma(t),\sigma(s)\right)\triangle_2 s\triangle_1 t\right.\\
&\left.+\int_a^b g\left(\sigma(t),y\right)\triangle_1 t\int_a^b\int_c^d f\left(\sigma(t),\sigma(s)\right)\triangle_2 s\triangle_1 t\right]\\
&-\frac{w_1}{2}\left[\int_c^d f\left(x,\sigma(s)\right)\triangle_2 s\int_a^b\int_c^d g\left(\sigma(t),\sigma(s)\right)\triangle_2 s\triangle_1 t\right.\\
&\left.+\int_c^d g\left(x,\sigma(s)\right)\triangle_2 s\int_a^b\int_c^d f\left(\sigma(t),\sigma(s)\right)\triangle_2 s\triangle_1 t\right]\\
&+\frac{1}{2}\left[\int_a^b\left[(c-\beta_1)f\left(\sigma(t),c\right)-(d-\beta_2)f\left(\sigma(t),d\right)\right]\triangle_1 t\int_a^b\int_c^d g\left(\sigma(t),\sigma(s)\right)\triangle_2 s\triangle_1 t\right.\\
&\left.+\int_a^b\left[(c-\beta_1)g\left(\sigma(t),c\right)-(d-\beta_2)g\left(\sigma(t),d\right)\right]\triangle_1 t\int_a^b\int_c^d f\left(\sigma(t),\sigma(s)\right)\triangle_2 s\triangle_1 t\right]\\
&+\frac{1}{2}\left[\int_c^d\left[(a-\theta_1)f\left(a,\sigma(s)\right)-(b-\theta_2)f\left(b,\sigma(s)\right)\right]\triangle_2 s\int_a^b\int_c^d g\left(\sigma(t),\sigma(s)\right)\triangle_2 s\triangle_1 t\right.\\
&\left.+\int_c^d\left[(a-\theta_1)g\left(a,\sigma(s)\right)-(b-\theta_2)g\left(b,\sigma(s)\right)\right]\triangle_2 s\int_a^b\int_c^d f\left(\sigma(t),\sigma(s)\right)\triangle_2 s\triangle_1 t\right]\Bigg|
\end{aligned}
$$

$$\leq \frac{1}{2}\left[K\int_a^b\int_c^d |g(\sigma(t),\sigma(s))|\triangle_2 s\triangle_1 t + L\int_a^b\int_c^d |f(\sigma(t),\sigma(s))|\triangle_2 s\triangle_1 t\right]$$
$$\times\left[\int_a^x |t-\theta_1|\triangle_1 t + \int_x^b |t-\theta_2|\triangle_1 t\right]\left[\int_c^y |s-\beta_1|\triangle_2 s + \int_y^d |s-\beta_2|\triangle_2 s\right] \tag{3.10}$$

where $K = \sup_{a<t<b;c<s<d}\left|\frac{\partial^2 f(t,s)}{\triangle_2 s\triangle_1 t}\right|$, $L = \sup_{a<t<b;c<s<d}\left|\frac{\partial^2 g(t,s)}{\triangle_2 s\triangle_1 t}\right|$.

Proof. From (3.1), we have following identties

$$\begin{aligned}
\int_a^b\int_c^d f(\sigma(t),\sigma(s))\triangle_2 s\triangle_1 t &= -w_1 w_2 f(x,y) - F_1(x,y) - F_2 \\
&\quad + w_2\int_a^b f(\sigma(t),y)\triangle_1 t + w_1\int_c^d f(x,\sigma(s))\triangle_2 s \\
&\quad - \int_a^b \left[(c-\beta_1) f(\sigma(t),c) - (d-\beta_2) f(\sigma(t),d)\right]\triangle_1 t \\
&\quad - \int_c^d \left[(a-\theta_1) f(a,\sigma(s)) - (b-\theta_2) f(b,\sigma(s))\right]\triangle_2 s \\
&\quad + \int_a^b\int_c^d K_{w_1}(x,t) K_{w_2}(y,s)\frac{\partial^2 f(t,s)}{\triangle_2 s\triangle_1 t}\triangle_2 s\triangle_1 t
\end{aligned} \tag{3.11}$$

in which

$$F_1(x,y) = w_1\left[(d-\beta_2) f(x,d) - (c-\beta_1) f(x,c)\right] + w_2\left[(b-\theta_2) f(b,y) - (a-\theta_1) f(a,y)\right],$$

and

$$F_2 = (a-\theta_1)\left[(c-\beta_1) f(a,c) - (d-\beta_2) f(a,d)\right] + (b-\theta_2)\left[(d-\beta_2) f(b,d) - (c-\beta_1) f(b,c)\right]$$

and similarly

$$\begin{aligned}
\int_a^b\int_c^d g(\sigma(t),\sigma(s))\triangle_2 s\triangle_1 t &= -w_1 w_2 g(x,y) - G_1(x,y) - G_2 \\
&\quad + w_2\int_a^b g(\sigma(t),y)\triangle_1 t + w_1\int_c^d g(x,\sigma(s))\triangle_2 s \\
&\quad - \int_a^b \left[(c-\beta_1) g(\sigma(t),c) - (d-\beta_2) g(\sigma(t),d)\right]\triangle_1 t
\end{aligned}$$

10

$$-\int_c^d \left[(a-\theta_1)\, g(a,\sigma(s)) - (b-\theta_2)\, g(b,\sigma(s))\right] \triangle_2 s$$
$$+\int_a^b\int_c^d K_{w_1}(x,t)\, K_{w_2}(y,s)\, \frac{\partial^2 g(t,s)}{\triangle_2 s \triangle_1 t} \triangle_2 s \triangle_1 t \tag{3.12}$$

$$G_1(x,y) = w_1\left[(d-\beta_2)\, g(x,d) - (c-\beta_1)\, g(x,c)\right] + w_2\left[(b-\theta_2)\, g(b,y) - (a-\theta_1)\, g(a,y)\right],$$

and

$$G_2 = (a-\theta_1)\left[(c-\beta_1)\, g(a,c) - (d-\beta_2)\, g(a,d)\right] + (b-\theta_2)\left[(d-\beta_2)\, g(b,d) - (c-\beta_1)\, g(b,c)\right].$$

Now, multiplying both sides (3.11) and (3.12) by $\int_a^b\int_c^d g(\sigma(t),\sigma(s)) \triangle_2 s \triangle_1 t$ and $\int_a^b\int_c^d f(\sigma(t),\sigma(s)) \triangle_2 s \triangle_1 t$, adding the resulting identities and taking absolute values, we get

$$\Bigg| \int_a^b\int_c^d f(\sigma(t),\sigma(s)) \triangle_2 s \triangle_1 t \int_a^b\int_c^d g(\sigma(t),\sigma(s)) \triangle_2 s \triangle_1 t$$
$$+\frac{w_1 w_2}{2}\left[f(x,y)\int_a^b\int_c^d g(\sigma(t),\sigma(s)) \triangle_2 s \triangle_1 t + g(x,y)\int_a^b\int_c^d f(\sigma(t),\sigma(s)) \triangle_2 s \triangle_1 t\right]$$
$$+\frac{1}{2}\left[F_1(x,y)\int_a^b\int_c^d g(\sigma(t),\sigma(s)) \triangle_2 s \triangle_1 t + G_1(x,y)\int_a^b\int_c^d f(\sigma(t),\sigma(s)) \triangle_2 s \triangle_1 t\right]$$
$$+\frac{1}{2}\left[F_2\int_a^b\int_c^d g(\sigma(t),\sigma(s)) \triangle_2 s \triangle_1 t + G_2\int_a^b\int_c^d f(\sigma(t),\sigma(s)) \triangle_2 s \triangle_1 t\right]$$
$$-\frac{w_2}{2}\left[\int_a^b f(\sigma(t),y) \triangle_1 t\int_a^b\int_c^d g(\sigma(t),\sigma(s)) \triangle_2 s \triangle_1 t + \int_a^b g(\sigma(t),y) \triangle_1 t\int_a^b\int_c^d f(\sigma(t),\sigma(s)) \triangle_2 s \triangle_1 t\right]$$

$$-\frac{w_1}{2}\left[\int_c^d f(x,\sigma(s)) \triangle_2 s\int_a^b\int_c^d g(\sigma(t),\sigma(s)) \triangle_2 s \triangle_1 t + \int_c^d g(x,\sigma(s)) \triangle_2 s\int_a^b\int_c^d f(\sigma(t),\sigma(s)) \triangle_2 s \triangle_1 t\right]$$
$$+\frac{1}{2}\Bigg[\int_a^b \left[(c-\beta_1)\, f(\sigma(t),c) - (d-\beta_2)\, f(\sigma(t),d)\right] \triangle_1 t\int_a^b\int_c^d g(\sigma(t),\sigma(s)) \triangle_2 s \triangle_1 t$$
$$+\int_a^b \left[(c-\beta_1)\, g(\sigma(t),c) - (d-\beta_2)\, g(\sigma(t),d)\right] \triangle_1 t\int_a^b\int_c^d f(\sigma(t),\sigma(s)) \triangle_2 s \triangle_1 t\Bigg]$$

11

$$+\frac{1}{2}\left[\int_{c}^{d}\left[(a-\theta_1) f(a,\sigma(s))-(b-\theta_2) f(b,\sigma(s))\right]\triangle_2 s\int_{a}^{b}\int_{c}^{d} g(\sigma(t),\sigma(s))\triangle_2 s\triangle_1 t\right.$$

$$\left.\left.+\int_{c}^{d}\left[(a-\theta_1) g(a,\sigma(s))-(b-\theta_2) g(b,\sigma(s))\right]\triangle_2 s\int_{a}^{b}\int_{c}^{d} f(\sigma(t),\sigma(s))\triangle_2 s\triangle_1 t\right]\right|$$

$$\leq \frac{1}{2}\left[K\int_{a}^{b}\int_{c}^{d}|g(\sigma(t),\sigma(s))|\triangle_2 s\triangle_1 t+L\int_{a}^{b}\int_{c}^{d}|f(\sigma(t),\sigma(s))|\triangle_2 s\triangle_1 t\right]$$

$$\times\int_{a}^{b}\int_{c}^{d}|K_w(x,t)||K_w(y,s)|\triangle_2 s\triangle_1 t$$

Hence, we get the inequality (3.10). The proof is complete. ■

If we apply the Theorem 3.2 to different time scales, we will get some new results.

Corollary 3.5 *If we let* $\mathbb{T}_1=\mathbb{T}_2=\mathbb{R}$ *in Theorem 3.4, then we obtain the inequality*

$$\left|\int_{a}^{b}\int_{c}^{d} f(t,s)\,dsdt\int_{a}^{b}\int_{c}^{d} g(t,s)\,dsdt\right.$$

$$+\frac{w_1w_2}{2}\left[f(x,y)\int_{a}^{b}\int_{c}^{d} g(t,s)\,dsdt+g(x,y)\int_{a}^{b}\int_{c}^{d} f(t,s)\,dsdt\right]$$

$$+\frac{1}{2}\left[F_1(x,y)\int_{a}^{b}\int_{c}^{d} g(t,s)\,dsdt+G_1(x,y)\int_{a}^{b}\int_{c}^{d} f(t,s)\,dsdt\right]$$

$$+\frac{1}{2}\left[F_2\int_{a}^{b}\int_{c}^{d} g(t,s)\,dsdt+G_2\int_{a}^{b}\int_{c}^{d} f(t,s)\,dsdt\right]$$

$$-\frac{w_2}{2}\left[\int_{a}^{b} f(t,y)\triangle_1 t\int_{a}^{b}\int_{c}^{d} g(t,s)\,dsdt+\int_{a}^{b} g(t,y)\triangle_1 t\int_{a}^{b}\int_{c}^{d} f(t,s)\,dsdt\right]$$

$$-\frac{w_1}{2}\left[\int_{c}^{d} f(x,s)\,ds\int_{a}^{b}\int_{c}^{d} g(t,s)\,dsdt+\int_{c}^{d} g(x,s)\,ds\int_{a}^{b}\int_{c}^{d} f(t,s)\,dsdt\right]$$

$$+\frac{1}{2}\left[\int_{a}^{b}\left[(c-\beta_1) f(t,c)-(d-\beta_2) f(t,d)\right]dt\int_{a}^{b}\int_{c}^{d} g(t,s)\,dsdt\right.$$

$$\left.+\int_{a}^{b}\left[(c-\beta_1) g(t,c)-(d-\beta_2) g(t,d)\right]dt\int_{a}^{b}\int_{c}^{d} f(t,s)\,dsdt\right]$$

12

$$
\begin{aligned}
&+\frac{1}{2}\left[\int_c^d \left[(a-\theta_1) f(a,s)-(b-\theta_2) f(b,s)\right] ds \int_a^b \int_c^d g(\sigma(t),s)\, dsdt\right. \\
&\left.\left.+\int_c^d \left[(a-\theta_1) g(a,s)-(b-\theta_2) g(b,d)\right] ds \int_a^b \int_c^d f(t,s)\, dsdt\right]\right| \\
\leq\ & \frac{1}{2}\left[K\int_a^b\int_c^d |g(t,s)|\, dsdt + L\int_a^b\int_c^d |f(t,s)|\, dsdt\right] \\
&\times\left[\int_a^x |t-\theta_1|\, dt+\int_x^b |t-\theta_2|\, dt\right]\left[\int_c^y |s-\beta_1|\, ds+\int_y^d |s-\beta_2|\, ds\right]
\end{aligned}
$$

for all $(x,y)\in[a,b]\times[c,d]$*, where* $K=\sup\limits_{a<t<b;c<s<d}\left|\frac{\partial^2 f(t,s)}{\partial s\partial t}\right|$, $L=\sup\limits_{a<t<b;c<s<d}\left|\frac{\partial^2 g(t,s)}{\partial s\partial t}\right|$. *This inequality is a new Ostrowski type inequality for double integrals in continuous case.*

Corollary 3.6 *If we let* $\mathbb{T}_1=\mathbb{T}_2=\mathbb{Z}$ *in Theorem 3.4, then we obtain the inequality*

$$
\begin{aligned}
&\left|\sum_{t=a}^{b-1}\sum_{s=c}^{d-1} f(t+1,s+1)\sum_{t=a}^{b-1}\sum_{s=c}^{d-1} g(t+1,s+1)\right. \\
&+\frac{w_1w_2}{2}\left[f(x,y)\sum_{t=a}^{b-1}\sum_{s=c}^{d-1} g(t+1,s+1)+g(x,y)\sum_{t=a}^{b-1}\sum_{s=c}^{d-1} f(t+1,s+1)\right] \\
&+\frac{1}{2}\left[F_1(x,y)\sum_{t=a}^{b-1}\sum_{s=c}^{d-1} g(t+1,s+1)+G_1(x,y)\sum_{t=a}^{b-1}\sum_{s=c}^{d-1} f(t+1,s+1)\right] \\
&+\frac{1}{2}\left[F_2\sum_{t=a}^{b-1}\sum_{s=c}^{d-1} g(t+1,s+1)+\sum_{t=a}^{b-1}\sum_{s=c}^{d-1} f(t+1,s+1)\right] \\
&-\frac{w_2}{2}\left[\sum_{t=a}^{b-1} f(t+1,y)\sum_{t=a}^{b-1}\sum_{s=c}^{d-1} g(t+1,s+1)+\sum_{t=a}^{b-1} g(t+1,y)\sum_{t=a}^{b-1}\sum_{s=c}^{d-1} f(t+1,s+1)\right] \\
&-\frac{w_1}{2}\left[\sum_{s=c}^{d-1} f(x,s+1)\sum_{t=a}^{b-1}\sum_{s=c}^{d-1} g(t+1,s+1)+\int_c^d g(x,s+1)\sum_{t=a}^{b-1}\sum_{s=c}^{d-1} f(t+1,s+1)\right] \\
&+\frac{1}{2}\left[\sum_{t=a}^{b-1}\left[(c-\beta_1) f(t+1,c)-(d-\beta_2) f(t+1,d)\right]\sum_{t=a}^{b-1}\sum_{s=c}^{d-1} g(t+1,s+1)\right. \\
&\left.+\sum_{t=a}^{b-1}\left[(c-\beta_1) g(t+1,c)-(d-\beta_2) g(t+1,d)\right]\sum_{t=a}^{b-1}\sum_{s=c}^{d-1} f(t+1,s+1)\right] \\
&+\frac{1}{2}\left[\sum_{s=c}^{d-1}\left[(a-\theta_1) f(a,s+1)-(b-\theta_2) f(b,s+1)\right]\sum_{t=a}^{b-1}\sum_{s=c}^{d-1} g(t+1,s+1)\right.
\end{aligned}
$$

13

$$
\begin{aligned}
& + \sum_{s=c}^{d-1} \left[(a-\theta_1) g(a,s+1) - (b-\theta_2) g(b,s+1) \right] \sum_{t=a}^{b-1} \sum_{s=c}^{d-1} f(t+1,s+1) \Bigg] \Bigg| \\
\leq \quad & \frac{1}{2} \left[K \sum_{t=a}^{b-1} \sum_{s=c}^{d-1} |g(t+1,s+1)| + L \sum_{t=a}^{b-1} \sum_{s=c}^{d-1} |f(t+1,s+1)| \right] \\
& \times \left[\sum_{t=a}^{x-1} |t-\theta_1| + \sum_{t=x}^{b-1} |t-\theta_2| \right] \left[\sum_{s=c}^{y-1} |s-\beta_1| + \sum_{s=y}^{d-1} |s-\beta_2| \right]
\end{aligned}
$$

for all $(x,y) \in [a,b-1] \times [c,d-1]$, where K denotes the maximum value of the absolute value of the difference $\triangle_2 \triangle_1 f$ over $[a,b-1]_{\mathbb{Z}} \times [c,d-1]_{\mathbb{Z}}$ and L denotes the maximum value of the absolute value of the difference $\triangle_2 \triangle_1 g$ over $[a,b-1]_{\mathbb{Z}} \times [c,d-1]_{\mathbb{Z}}$.

This inequality is a new Ostrowski type inequality for double integrals in discrete case.

Note that to compute the integrals of the right hand side of inequalities (3.9) and (3.10), we need the following general identities:

$$
\int_a^b |t-\theta| \triangle_1 t = \begin{cases} [h_2(a,\theta) + h_2(b,\theta)], & a < \theta < b \\ [h_2(b,\theta) - h_2(a,\theta)], & \theta < a < b \\ [h_2(a,\theta) - h_2(b,\theta)], & a < b < \theta, \end{cases}
$$

and

$$
\int_c^d |s-\beta| \triangle_2 s = \begin{cases} [h_2(c,\beta) + h_2(d,\beta)], & c < \beta < d \\ [h_2(d,\beta) - h_2(c,\beta)], & \beta < c < d \\ [h_2(c,\beta) - h_2(d,\beta)], & c < d < \beta. \end{cases}
$$

References

[1] R. Agarwal, M. Bohner and A. Peterson, Inequalities on time scales: a survey, Math. Inequal. Appl., Vol. 4, No. 4, pp. 535–557 (2001).

[2] M. Bohner and A. Peterson, Dynamic Equations on Time Scales, Birkhäuser Boston Inc., Boston, MA, An Introduction with Applications, 2001.

[3] M. Bohner and A. Peterson, Advances in Dynamic Equations on Time Scales, Birkhäuser Boston, Boston, MA, 2003.

[4] M. Bohner and G. S. Guseinov, Partial differentiation on time scales. Dyn. Syst. Appl. 13(3–4), 351–379 (2004).

[5] M. Bohner and G. S. Guseinov, Multiple integration on time scales. Dyn. Syst. Appl. 14(3–4), 579–606 (2005).

[6] M. Bohner and G. S. Guseinov, Double integral calculus of variations on time scales. Comput. Math. Appl., 54, 45–57 (2007).

[7] M. Bohner and T. Matthews, Ostrowski inequalities on time scales, JIPAM. J. Inequal. Pure Appl. Math. 9, Art. 6, 8 pp (2008).

[8] Q. Feng and F. W. Meng, Some Ostrowski type inequalities on time scales involving functions of two independent variables, WSEAS Trans. Math. 11 (4), 303–312 (2012).

[9] Q. Feng and F. W. Meng, Generalized Ostrowski type inequalities for multiple points on time scales involving functions of two independent variables, Journal of Inequalities and Applications, 74 (2012).

[10] S. Hilger, Ein Maβkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten, Ph.D. thesis, Universität Würzburg, Würzburg, Germany, 1988.

[11] M. M. Jamei and S.S. Dragomir, A new generalization of the Ostrowski inequality and applications, Filomat 25:1, 115-123, doi: 10.2298/FIL1101115M (2011).

[12] S. Hussain, M. A. Latif and M. Alomari, Generalized double integral Ostrowski type inequalities on time scales, Applied Mathematics Letters, vol. 24, no. 8, pp. 1461–1467 (2011).

[13] V. Lakshmikantham, S. Sivasundaram and B. Kaymakcalan, Dynamic Systems on Measure Chains, Mathematics and its Applications, 370, Kluwer Academic Publishers Group, Dordrecht, 1996.

[14] W. J. Liu, Q. A. Ngô and W. Chen, Ostrowski type inequalities on time scales for double integrals, Acta Applicandae Mathematicae, vol. 110, no. 1, pp. 477–497 (2010).

[15] W. J. Liu, Q. A. Ngô and W. Chen, On new Ostrowski type inequalities for double integrals on time scales, Dynamic Systems and Applications 19, 189-198 (2010).

[16] W. J. Liu and Q. A. Ngô, A generalization of Ostrowski inequality on time scales for k points, Appl. Math. Comput. 203, pp. 754–760 (2008).

[17] W. J. Liu and Q. A. Ngô, An Ostrowski type inequality on time scales for functions whose second derivatives are bounded, to appear in Inequality Theory and Applications 6, Nova Science Publishers, Inc., New York (2009).

[18] W. J. Liu, Q. A. Ngô and W. B. Chen, A perturbed Ostrowski type inequality on time scales for k points for functions whose second derivatives are bounded, J. Inequal. Appl., Volume 2008, Article ID 597241, 12 pages.

[19] N. A. Mir and R. Ullah, Some inequalities of Ostrowski and Grüss type for triple integrals on time scales, Tamkang J. Math. 42 (4), 15–426 (2011).

[20] N. A.Mir, R. Ullah and Z. Khan, Weighted Grüss type inequalities for double integrals on time scales, Int. J. Pure Appl. Sci. Technol., 8(2), pp. 1-25 (2012).

[21] A.M. Ostrowski, Über die Absolutabweichung einer differentiebaren Funktion von ihrem Integralmitelwert. Comment. Math. Helv. 10, 226–227 (1938).

[22] U. M. Özkan and H. Yildirim, Grüss type inequalities for double integrals on time scales, Computers & Mathematics with Applications, vol. 57, no. 3, pp. 436–444 (2009).

[23] U. M. Özkan and H. Yildirim, Ostrowski type inequality for double integrals on time scales, Acta Applicandae Mathematicae, vol. 110, no. 1, pp. 283–288 (2010).

[24] B. Zheng and Q. Feng, Generalized n Dimensional Ostrowski Type and Grüss Type Inequalities on Time Scales, Journal of Applied Mathematics Volume 2014, Article ID 434958, 11 pages, http://dx.doi.org/10.1155/2014/434958.

[25] B. Zheng, Some new generalized $2D$ Ostrowski-Grüss type inequalities on time scales, Arab J Math Sci 19(2), 159–172 (2013).

Approximate ternary Jordan bi-homomorphisms in Banach Lie triple systems

Madjid Eshaghi Gordji[1], Amin Rahimi[1], Choonkil Park[2] and Dong Yun Shin[3*]

[1]Department of Mathematics, Semnan University, P. O. Box 35195-363, Semnan, Iran

[2]Research Institute for Naturan Sciences, Hanyang University, Seoul 133-791, Korea

[3]Department of Mathematics, University of Seoul, Seoul 130-743, Korea

E-mail: meshaghi@semnan.ac.kr, amin.rahimi2014@hotmail.com, baak@hanyang.ac.kr, dyshin@uos.ac.kr

Abstract. We prove the Hyers-Ulam stability of ternary Jordan bi-homomorphism in Banach Lie triple systems associated to the Cauchy functional equation.

1. Introduction and preliminaries

We say that a functional equation (Q) is stable if any function g satisfying the equation (Q) approximately is near to true solution of (Q).

Ternary algebraic operations were considered in the 19th century by several mathematicians and physicists. Cayley [8] introduced the notion of cubic matrix which in turn was generalized by Kapranov, Gelfand and Zelevinskii [6]. As an application in physics, the quark model inspired a particular brand of ternary algebraic systems. The so-called Nambu mechanics which has been proposed by Nambu [11], is based on such structures. There are also some applications, although still hypothetical, in the fractional quantum Hall effect, the non-standard statistics (the anyons), supersymmetric theories, Yang-Baxter equation, etc, (cf. [15, 27]).

The comments on physical applications of ternary structures can be found in [1, 5, 10, 14, 17, 23, 24, 29].

A C^*-ternary algebra is a complex Banach space, equipped with a ternary product $(x,y,z) \to [x,y,z]$ of A^3 into A, which is $\mathbb{C}$-linear in the outer variables, conjugate $\mathbb{C}$-linear in the middle variable, and associative in the sense that $\Big[x,y,[z,u,v]\Big] = \Big[x,[y,z,u]v\Big] = \Big[[x,y,z],u,v\Big]$, and satisfies

$$\|[x,y,z]\| \le \|x\|.\|y\|.\|z\|, \ \ \|[x,x,x]\| = \|x\|^3$$

A normed (Banach) Lie triple system is a normed (Banach) space $(A, \|\cdot\|)$ with a trilinear mapping $(x,y,z) \mapsto [x,y,z]$ from $A \times A \times A$ to A satisfying the following axioms:

$$\begin{aligned} [x,y,z] &= -[y,x,z], \\ [x,y,z] &= -[y,z,x] - [z,x,y], \\ [u,v,[x,y,z]] &= [[u,v,x],y,z] + [x,[u,v,y],z] + [x,y,[u,v,z]], \\ \|[x,y,z]\| &\le \|x\|\|y\|\|z\| \end{aligned}$$

for all $u,v,x,y,z \in A$ (see [12, 16]).

---

[0]2010 Mathematics Subject Classification. Primary 39B52; 39B82; 46B99; 17A40.

[0]Keywords: Hyers-Ulam stability; bi-additive mapping; Lie triple system; ternary Jordan bi-homomorphism.

[0*]Corresponding author.

Definition 1.1. Let A and B be normed Lie triple systems. A $\mathbb{C}$-bilinear mapping $H : A \times A \to B$ is called a ternary Jordan bi-homomorphism if it satisfies

$$H([x,x,x],[w,w,w]) \quad = \quad [H(x,w), H(x,w), H(x,w)]$$

for all $x, w \in A$.

The stability problem of functional equations originated from a question of Ulam [28] concerning the stability of group homomorphisms. Hyers [13] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [3] for additive mappings and by Th.M. Rassias [21] for linear mappings by considering an unbounded Cauchy difference. J.M. Rassias [20] followed the innovative approach of the Th.M. Rassias theorem in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p\|y\|^p$ for $p, q \in \mathbb{R}$ with $p + q \neq 1$. The stability problems of various functional equations have been extensively investigated by a number of authors (see [2, 7, 9, 10, 18, 19, 22, 23, 24, 25, 26, 30, 31]).

2. Hyers-Ulam stability of ternary Jordan bi-homomorphisms in Banach Lie triple systems

Throughout this section, assume that A is a normed Lie triple system and B is a Banach Lie triple systems.

For a given mapping $f : A \times A \to B$, we define

$$\begin{aligned} D_{\lambda,\mu} f(x,y,z,w) &= f(\lambda x + \lambda y, \mu z + \mu w) + f(\lambda x + \lambda y, \mu z - \mu w) \\ &+ f(\lambda x - \lambda y, \mu z + \mu w) + f(\lambda x - \lambda y, \mu z - \mu w) - 4\lambda\mu f(x,z) \end{aligned}$$

for all $x, y, z, w \in A$ and all $\lambda, \mu \in \mathbb{T}^1 := \{\nu \in \mathbb{C} : \ |\nu| = 1\}$.

From now on, assume that $f(0,z) = f(x,0) = 0$ for all $x, z \in A$.

We need the following lemma to obtain the main results.

Lemma 2.1. ([4]) *Let $f : A \times A \to B$ be a mapping satisfying $D_{\lambda,\mu} f(x,y,z,w) = 0$ for all $x,y,z,w \in A$ and all $\lambda, \mu \in \mathbb{T}^1$. Then the mapping $f : A \times A \to B$ is $\mathbb{C}$-bilinear.*

Lemma 2.2. *Let $f : A \times A \to B$ be a bi-additive mapping. Then the following assertions are equivalent:*

$$f([a,a,a],[w,w,w]) = [f(a,w), f(a,w), f(a,w)] \tag{2.1}$$

for all $a, w \in A$, and

$$\begin{aligned} &f([a,b,c] + [b,c,a] + [c,a,b], [w,w,w]) \\ &= [f(a,w), f(b,w), f(c,w)] + [f(b,w), f(c,w), f(a,w)] + [f(c,w), f(a,w), f(b,w)], \\ &f([a,a,a],[b,c,w] + [c,w,b] + [w,b,c]) \\ &= [f(a,b), f(a,c), f(a,w)] + [f(a,c), f(a,w), f(a,b)] + [f(a,w), f(a,b), f(a,c)], \end{aligned} \tag{2.2}$$

for all $a, b, c, w \in A$.

Proof. Replacing a by $a + b + c$ in (2.1), we get

$$f\Big([(a+b+c),(a+b+c),(a+b+c)],[w,w,w]\Big) = \Big[f(a+b+c,w), f(a+b+c,w), f(a+b+c,w)\Big].$$

The we have

$$\begin{aligned}
&f([(a+b+c),(a+b+c),(a+b+c)],[w,w,w])\\
&=f([a,a,a]+[a,b,a]+[a,c,a]+[b,a,a]+[b,b,a]+[b,c,a]+[c,a,a]+[c,b,a]+[c,c,a]\\
&+[a,a,b]+[a,b,b]+[a,c,b]+[b,a,b]+[b,b,b]+[b,c,b]+[c,a,b]+[c,b,b]+[c,c,b]\\
&+[a,a,c]+[a,b,c]+[a,c,c]+[b,a,c]+[b,b,c]+[b,c,c]+[c,a,c]+[c,b,c]+[c,c,c],[w,w,w])\\
&=[f(a,w),f(a,w),f(a,w)]+[f(a,w),f(b,w),f(a,w)]+[f(a,w),f(c,w),f(a,w)]+[f(b,w),f(a,w),f(a,w)]\\
&+[f(b,w),f(b,w),f(a,w)]+[f(b,w),f(c,w),f(a,w)]+[f(c,w),f(a,w),f(a,w)]+[f(c,w),f(b,w),f(a,w)]\\
&+[f(c,w),f(c,w),f(a,w)]+[f(a,w),f(a,w),f(b,w)]+[f(a,w),f(b,w),f(b,w)]+[f(a,w),f(c,w),f(b,w)]\\
&+[f(b,w),f(a,w),f(b,w)]+[f(b,w),f(b,w),f(b,w)]+[f(b,w),f(c,w),f(b,w)]+[f(c,w),f(a,w),f(b,w)]\\
&+[f(c,w),f(b,w),f(b,w)]+[f(c,w),f(c,w),f(b,w)]+[f(a,w),f(a,w),f(c,w)]+[f(a,w),f(b,w),f(c,w)]\\
&+[f(a,w),f(c,w),f(c,w)]+[f(b,w),f(a,w),f(c,w)]+[f(b,w),f(b,w),f(c,w)]+[f(b,w),f(c,w),f(c,w)]\\
&+[f(c,w),f(a,w),f(c,w)]+[f(c,w),f(b,w),f(c,w)]+[f(c,w),f(c,w),f(c,w)]
\end{aligned}$$

for all $a,b,c,w\in A$.

On the other hand, for the right side of equation, we have

$$\begin{aligned}
&[f(a+b+c,w),f(a+b+c,w),f(a+b+c,w)]\\
&=[f(a,w),f(a,w),f(a,w)]+[f(a,w),f(a,w),f(b,w)]+[f(a,w),f(a,w),f(c,w)]+[f(a,w),f(b,w),f(a,w)]\\
&+[f(a,w),f(b,w),f(b,w)]+[f(a,w),f(b,w),f(c,w)]+[f(a,w),f(c,w),f(a,w)]+[f(a,w),f(c,w),f(b,w)]\\
&+[f(a,w),f(c,w),f(c,w)]+[f(b,w),f(a,w),f(a,w)]+[f(b,w),f(a,w),f(b,w)]+[f(b,w),f(a,w),f(c,w)]\\
&+[f(b,w),f(b,w),f(a,w)]+[f(b,w),f(b,w),f(b,w)]+[f(b,w),f(b,w),f(c,w)]+[f(b,w),f(c,w),f(a,w)]\\
&+[f(b,w),f(c,w),f(b,w)]+[f(b,w),f(c,w),f(c,w)]+[f(c,w),f(a,w),f(a,w)]+[f(c,w),f(a,w),f(b,w)]\\
&+[f(c,w),f(a,w),f(c,w)]+[f(c,w),f(b,w),f(a,w)]+[f(c,w),f(b,w),f(b,w)]+[f(c,w),f(b,w),f(c,w)]\\
&+[f(c,w),f(c,w),f(a,w)]+[f(c,w),f(c,w),f(b,w)]+[f(c,w),f(c,w),f(c,w)]
\end{aligned}$$

for all $a,b,c,w\in A$.

It follows that

$$f([a,b,c]+[b,c,a]+[c,a,b],[w,w,w])=[f(a,w),f(b,w),f(c,w)]+[f(b,w),f(c,w),f(a,w)]+[f(c,w),f(a,w),f(b,w)]$$

for all $a,b,c,w\in A$. Hence (2.2) holds.

Similarly, we can show that

$$f([a,a,a],[b,c,w]+[c,w,b]+[w,b,c])=[f(a,b),f(a,c),f(a,w)]+[f(a,c),f(a,w),f(a,b)]+[f(a,w),f(a,b),f(a,c)]$$

for all $a,b,c,w\in A$.

For the converse, replacing b and c by a in (2.2), we have

$$f([a,a,a]+[a,a,a]+[a,a,a],[w,w,w])=[f(a,w),f(a,w),f(a,w)]+[f(a,w),f(a,w),f(a,w)]+[f(a,w),f(a,w),f(a,w)]$$

and so $f(3[a,a,a],[w,w,w]) = 3([f(a,w),f(a,w),f(a,w)])$. Thus

$$f([a,a,a],[w,w,w]) = [f(a,w),f(a,w),f(a,w)]$$

for all $a, w \in A$. This completes the proof. □

Now we prove the Hyers-Ulam stability of ternaty Jordan bi-homomorphisms in Banach Lie triple systems.

Theorem 2.3. *Let p and θ be positive real numbers with $p < 2$, and let $f : A \times A \to B$ be a mapping such that*

$$\|D_{\lambda,\mu} f(x,y,z,w)\|_B \le \theta(\|x\|_A^p + \|y\|_A^p + \|z\|_A^p + \|w\|_A^p), \tag{2.3}$$

$$\begin{aligned} &\|f\Big(([x,y,z]+[y,z,x]+[z,x,y]),[w,w,w]\Big) - [f(x,w),f(y,w),f(z,w)] - [f(y,w),f(z,w),f(x,w)] \\ &- [f(z,w),f(x,w),f(y,w)]\|_B + \|f\Big([x,x,x],([y,z,w]+[z,w,y]+[w,y,z])\Big) - [f(x,y),f(x,z),f(x,w)] \\ &- [f(x,z),f(x,w),f(x,y)] - [f(x,w),f(x,y),f(x,z)]\|_B \le \theta(\|x\|_A^p + \|y\|_A^p + \|z\|_A^p + \|w\|_A^p) \end{aligned} \tag{2.4}$$

for all $\lambda, \mu \in \mathbb{T}^1$ and all $x, y, z, w \in A$. Then there exists a unique ternary Jordan bi-homomorphism $H : A \times A \to B$ such that

$$\|f(x,y) - H(x,y)\|_B \le \frac{2\theta}{4 - 2^p}(\|x\|_A^p + \|y\|_A^p) \tag{2.5}$$

for all $x, y \in A$.

Proof. By the same reasoning as in the proof of [4, Theorem 2.3], there exists a unique $\mathbb{C}$-bilinear mapping $H : A \times A \to B$ satisfying (2.5). The $\mathbb{C}$-bilinear mapping $H : A \times A \to B$ is given by

$$H(x,y) := \lim_{n\to\infty} \frac{1}{4^n} f(2^n x, 2^n y),$$

for all $x, y \in A$.

It follows from (2.4) that

$$\begin{aligned} &\|H\Big(([x,y,z]+[y,z,x]+[z,x,y]),[w,w,w]\Big) - [H(x,w),H(y,w),H(z,w)] - [H(y,w),H(z,w),H(x,w)] \\ &- [H(z,w),H(x,w),H(y,w)]\|_B + \|H\Big([x,x,x],([y,z,w]+[z,w,y]+[w,y,z])\Big) - [H(x,y),H(x,z),H(x,w)] \\ &- [H(x,z),H(x,w),H(x,y)] - [H(x,w),H(x,y),H(x,z)]\|_B \\ &= \lim_{n\to\infty} \frac{1}{64^n}\Big(\|f\Big(([2^n x,2^n y,2^n z]+[2^n y,2^n z,2^n x]+[2^n z,2^n x,2^n y]),[2^n w,2^n w,2^n w]\Big) \\ &- [f(2^n x,2^n w),f(2^n y,2^n w),f(2^n z,2^n w)] - [f(2^n y,2^n w),f(2^n z,2^n w),f(2^n x,2^n w)] \\ &- [f(2^n z,2^n w),f(2^n x,2^n w),f(2^n y,2^n w)]\|_B + \|f\Big([2^n x,2^n x,2^n x],([2^n y,2^n z,2^n w]+[2^n z,2^n w,2^n y]+[2^n w,2^n y,2^n z]\Big) \\ &- [f(2^n x,2^n y),f(2^n x,2^n z),f(2^n x,2^n w)] - [f(2^n x,2^n z),f(2^n x,2^n w),f(2^n x,2^n y)] \\ &- [f(2^n x,2^n w),f(2^n x,2^n y),f(2^n x,2^n z)]\|_B\Big) \le \lim_{n\to\infty} \frac{2^{np}}{64^n}\theta(\|x\|_A^p + \|y\|_A^p + \|z\|_A^p + \|w\|_A^p) = 0 \end{aligned} \tag{2.6}$$

for all $x, y, z, w \in A$. So

$$H\Big(([x,y,z]+[y,z,x]+[z,x,y]),[w,w,w]\Big) = [H(x,w),H(y,w),H(z,w)]+[H(y,w),H(z,w),H(x,w)]+[H(z,w),H(x,w),H(y,w)]$$

and

$$H\Big([x,x,x],([y,z,w]+[z,w,y]+[w,y,z])\Big) = [H(x,y),H(x,z),H(x,w)]+[H(x,z),H(x,w),H(x,y)]+[H(x,w),H(x,y),H(x,z)]$$

for all $x, y, z, w \in A$. By Lemma 2.2, the bi-additive mapping H is a unique ternary Jordan bi-homomorphism satisfying (2.5). □

Theorem 2.4. *Let p and θ be positive real numbers with $p > 6$, and let $f : A \times A \to B$ be a mapping satisfying* (2.3) *and* (2.4). *Then there exists a unique ternary Jordan bi-homomorphism $H : A \times A \to B$ such that*

$$\|f(x,y) - H(x,y)\|_B \leq \frac{2\theta}{2^p - 4}(\|x\|_A^p + \|y\|_A^p)$$

for all $x, y \in A$.

Proof. The proof is similar to the proof of Theorem 2.3. □

Theorem 2.5. *Let p and θ be positive real numbers with $p < \frac{1}{2}$, and let $f : A \times A \to B$ be a mapping such that*

$$\|D_{\lambda,\mu} f(x,y,z,w)\|_B \leq \theta(\|x\|_A^p \cdot \|y\|_A^p \cdot \|z\|_A^p \cdot \|w\|_A^p),$$

$$\begin{aligned}
&\|f\Big(([x,y,z] + [y,z,x] + [z,x,y]), [w,w,w]\Big) - [f(x,w), f(y,w), f(z,w)] - [f(y,w), f(z,w), f(x,w)] \\
&- [f(z,w), f(x,w), f(y,w)]\|_B \\
&+ \|f\Big([x,x,x], ([y,z,w] + [z,w,y] + [w,y,z])\Big) - [f(x,y), f(x,z), f(x,w)] - [f(x,z), f(x,w), f(x,y)] \\
&- [f(x,w), f(x,y), f(x,z)]\|_B \leq \theta(\|x\|_A^p \cdot \|y\|_A^p \cdot \|z\|_A^p \cdot \|w\|_A^p)
\end{aligned}$$

for all $\lambda, \mu \in \mathbb{T}^1$ and all $x, y, z, w \in A$. Then there exists a unique ternary Jordan bi-homomorphism $H : A \times A \to B$ such that

$$\|f(x,y) - H(x,y)\|_B \leq \frac{\theta}{4 - 2^{4p}}\|x\|_A^{2p}\|y\|_A^{2p} \tag{2.7}$$

for all $x, y \in A$.

Proof. By the same reasoning as in the proof of [4, Theorem 2.6], there exists a unique $\mathbb{C}$-bilinear mapping $H : A \times A \to A$ satisfying (2.7). The $\mathbb{C}$-bilinear mapping $H : A \times A \to A$ is given by

$$H(x,y) := \lim_{n\to\infty} \frac{1}{4^n} f(2^n x, 2^n y),$$

for all $x, y \in A$.

The rest of the proof is similar to the proof of Theorem 2.3. □

References

[1] V. Abramov, R. Kerner, B. Le Roy, Hypersymmetry: A Z_3 graded generalization of supersymmetry, J. Math. Phys. **38** (1997), 1650–1669.

[2] M. Adam, On the stability of some quadratic functional equation, J. Nonlinear Sci. Appl. **4** (2011), 50–59.

[3] T. Aoki, On the stability of the linear transformation in Banach spaces, J. Math. Soc. Japan **2** (1950), 64–66.

[4] J. Bae, W. Park, Approximate bi-homomorphisms and bi-derivations in C^*-ternary algebras, Bull. Korean Math. Soc. **47** (2010), 195–209.

[5] F. Bagarello, G. Morchio, Dynamics of mean-field spin models from basic results in abstract differential equations, J. Stat. Phys. **66** (1992), 849–866.

[6] M. Bavand Savadkouhi, M. Eshaghi Gordji, J. M. Rassias, N. Ghobadipour, Approximate ternary Jordan derivations on Banach ternary algebras, J. Math. Phys. **50**, Art. ID 042303 (2009).

[7] L. Cădariu, L. Găvruta, P. Găvruta, On the stability of an affine functional equation, J. Nonlinear Sci. Appl. **6** (2013), 60–67.

[8] A. Cayley, On the 34concomitants of the ternary cubic. Amer. J. Math. **4** (1881), 1–15.

[9] A. Chahbi, N. Bounader, On the generalized stability of d'Alembert functional equation, J. Nonlinear Sci. Appl. **6** (2013), 198–204.

[10] Y. Cho, C. Park, M. Eshaghi Gordji, Approximate additive and quadratic mappings in 2-Banach spaces and related topics, Int. J. Nonlinear Anal. Appl. **3** (2012), No. 1, 75–81.
[11] Y. L. Daletskii, L. A. Takhtajan, Leibniz and Lie algebra structures for Nambu algebra, Lett. Math. Phys. **39** (1997), 127–141.
[12] T. Hopkins, Nilpotent ideals in Lie and anti-Lie triple systems, J. Algebra **178** (1995), 480–492.
[13] D.H. Hyers, On the stability of the linear functional equation, Proc. Natl. Acad. Sci. U.S.A. **27** (1941), 222–224.
[14] M. Kapranov, I. M. Gelfand and A. Zelevinskii, Discriminants, Resultants and Multidimensional Determinants, Birkhäuser, Berlin, 1994.
[15] R. Kerner, The cubic chessboard: Geometry and physics, Class. Quantum Grav. **14** (1997), A203, 1997.
[16] W.G. Lister, *A structure theory of Lie triple systems*, Trans. Amer. Math. Soc. **72** (1952), 217–242.
[17] J. Nambu, Generalized Hamiltonian dynamics, Physical Review **D (3) 7** (1973), 2405–2412.
[18] C. Park, K. Ghasemi, S. G. Ghaleh, S. Jang, Approximate n-Jordan $*$-homomorphisms in C^*-algebras, J. Comput. Anal. Appl. **15** (2013), 365-368.
[19] C. Park, A. Najati, S. Jang, Fixed points and fuzzy stability of an additive-quadratic functional equation, J. Comput. Anal. Appl. **15** (2013), 452–462.
[20] J. M. Rassias, On approximation of approximately linear mappings by linear mappings, J. Funct. Anal. **46** (1982), 126–130.
[21] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.
[22] S. Schin, D. Ki, J. Chang, M. Kim, Random stability of quadratic functional equations: a fixed point approach, J. Nonlinear Sci. Appl. **4** (2011), 37–49.
[23] S. Shagholi, M. Bavand Savadkouhi, M. Eshaghi Gordji, Nearly ternary cubic homomorphism in ternary Fréchet algebras, J. Comput. Anal. Appl. **13** (2011), 1106–1114.
[24] S. Shagholi, M. Eshaghi Gordji, M. Bavand Savadkouhi, Stability of ternary quadratic derivation on ternary Banach algebras, J. Comput. Anal. Appl. **13** (2011), 1097–1105.
[25] D. Shin, C. Park, Sh. Farhadabadi, On the superstability of ternary Jordan C^*-homomorphisms, J. Comput. Anal. Appl. **16** (2014), 964–973.
[26] D. Shin, C. Park, Sh. Farhadabadi, Stability and superstability of J^*-homomorphisms and J^*-derivations for a generalized Cauchy-Jensen equation, J. Comput. Anal. Appl. **17** (2014), 125–134.
[27] L. A. Takhtajan, On foundation of the generalized Nambu mechanics, Commun. Math. Phys. **160** (1994), 295–315.
[28] S.M. Ulam, Problems in Modern Mathematics, Chapter VI, Science ed., Wiley, New York, 1940.
[29] L. Vainerman, R. Kerner, On special classes of n-algebras, J. Math. Phys. **37** (1996), 2553–2565.
[30] C. Zaharia, On the probabilistic stability of the monomial functional equation, J. Nonlinear Sci. Appl. **6** (2013), 51–59.
[31] S. Zolfaghari, Approximation of mixed type functional equations in p-Banach spaces, J. Nonlinear Sci. Appl. **3** (2010), 110–122.

BOREL DIRECTIONS AND UNIQUENESS OF MEROMORPHIC FUNCTIONS SHARING FIVE VALUES

JIANREN LONG AND CHUNHUI QIU

ABSTRACT. We study a problem uniqueness of meromorphic functions in an angular domain concerning a Borel direction, and obtain some uniqueness results by using Nevanlinn theory of angular domain and angular distributions, that is, if the zeros of $f - a_j$ $(j = 1, 2, \cdots, 5)$ is also zeros of $g - a_j$ in the angular domain, then $f = g$.

1. Introduction and main results

As usual, the abbreviations IM and CM refer to sharing values ignoring multiplicities and counting multiplicities in domain $D \subseteq \mathbb{C}$, respectively, where $\mathbb{C}$ denotes the complex plane. In addition, $\rho(f)$ denotes the order of growth of a meromorphic function f in $\mathbb{C}$. The standard notation and basic results in Nevanlinna theory of meromorphic functions can be found in [7] or [20].

In [12], Nevanlinna proved the remarkable five-value theorem and four-value theorem by using his value distribution theory, here the five-value theorem is stated as follows.

Theorem A. *Let f and g be two non-constant meromorphic functions in $\mathbb{C}$ and let $a_i \in \bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ $(i = 1, 2, 3, 4, 5)$ be five distinct values. If f and g share the values a_i $(i = 1, 2, 3, 4, 5)$ IM in $D(= \mathbb{C})$, then $f = g$.*

After his work, lots of uniqueness results of meromorphic functions in the complex plane have been obtained, which are introduced systematically in [18]. In [24, 25], Zheng first took into the uniqueness question of meromorphic functions related shared values in an angular domain, and obtained some five-value theorem and four-value theorem in some angular domain, while he posed the question: Under what conditions, must two meromorphic functions on $D(\neq \mathbb{C})$ be identical? After his work, a lot of uniqueness results of meromorphic functions in an angular domain concerning this problem were obtained. In [1, 17, 23], Nevanlinna's five value theorem and four value theorem were extended to some angular domain by using sectorial

2010 *Mathematics Subject Classification.* Primary 30D30; Secondary 30D35.

Key words and phrases. Meromorphic functions, Uniqueness, Sharing values, Borel directions, Angular domain.

Nevanlinna characteristic, respectively. It is an interesting topic how to extend some interesting uniqueness results in the whole complex plane to an angular domain, more uniqueness results concerning this problem can be found in [9, 10]. Recently, this problem was studied [11] by using new idea that angular distributions of meromorphic functions is considered. In order to make our statements understand easily, we first recall the following definition and Theorem B.

Theorem B. *Let $B(r)$ be a positive and continuous function in $[0,\infty)$ which satisfies $\limsup_{r\to\infty}\dfrac{\log B(r)}{\log r}=\infty$. Then there exists a continuously differentiable function $\rho(r)$, which satisfies the following conditions.*

(i) *$\rho(r)$ is continuous and nondecreasing for $r\geq r_0$ $(r_0>0)$ and tends to ∞ as $r\to\infty$;*

(ii) *The function $U(r)=r^{\rho(r)}$ $(r\geq r_0)$ satisfies the condition*

$$\lim_{r\to\infty}\frac{\log U(R)}{\log U(r)}=1,\ \ R=r+\frac{r}{\log U(r)};$$

(iii)

$$\limsup_{r\to\infty}\frac{\log B(r)}{\log U(r)}=1.$$

Theorem B is due to K.L.Hiong [8]. A simple proof of the existence of $\rho(r)$ was given by Chuang [2].

Definition 1. We define $\rho(r)$ and $U(r)$ in Theorem B by the proximate order and type function of $B(r)$, respectively. For a meromorphic function $f(z)$ of infinite order, we define its proximate order and type function as the proximate order and type function of $T(r,f)$. Let $\rho(r)$ be a proximate order of meromorphic function f of infinite order in $\mathbb{C}$, and let $M(\rho(r))$ be the set of all meromorphic functions g in $\mathbb{C}$ such that

$$\limsup_{r\to\infty}\frac{\log T(r,g)}{\rho(r)\log r}\leq 1.$$

Let $\alpha<\beta$ such that $\beta-\alpha<2\pi$ and $r>0$, we denote

$$\Omega(\alpha,\beta)=\{z:\alpha\leq\arg z\leq\beta\},$$
$$\Omega(\alpha,\beta;r)=\{z:\alpha\leq\arg z\leq\beta\}\cap\{z:0<|z|\leq r\}.$$

The following definition, originally due to Hiong [8], which also be found in [3] or [4, p. 140].

Definition 2. Suppose that $\rho(r)$ is a proximate order of meromorphic function f of infinite order in $\mathbb{C}$. A ray $\arg z=\theta\in[0,2\pi)$ from the origin is

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 21, NO.6, 2016, COPYRIGHT 2016 EUDOXUS PRESS, LLC

called a Borel direction order $\rho(r)$ of f, if for any $\varepsilon > 0$ and any complex value $a \in \bar{\mathbb{C}}$, possibly with two exceptions, the following equality

$$\limsup_{r\to\infty} \frac{\log n(\Omega(\theta-\varepsilon,\theta+\varepsilon;r),\frac{1}{f-a})}{\rho(r)\log r} = 1 \tag{1.1}$$

holds, where $n(\Omega(\theta-\varepsilon,\theta+\varepsilon;r),\frac{1}{f-a})$ is the number of zeros, counting multiplicities, of $f-a$ in the region $\Omega(\theta-\varepsilon,\theta+\varepsilon;r)$.

It is well known that every meromorphic function of infinite order must have at least one Borel direction of order $\rho(r)$. The proof can be found in [4, pp. 140-145]. In Nevanlinna theory of meromorphic functions, the angular distributions is one of main topics. Borel direction plays a basic role in the theory of angular distributions of meromorphic functions, lots of results can be found in [5, 13, 15, 16, 19, 21, 22]. In [11], the authors investigated the uniqueness of meromorphic functions in an angular domain by using theory of angular distributions, and proved the following version of five value theorem.

Theorem C. *Let $\rho(r)$ be a proximate order of meromorphic function f of infinite order in $\mathbb{C}$ and let $g \in M(\rho(r))$. Suppose that $\arg z = \theta \in [0, 2\pi)$ is a Borel direction of order $\rho(r)$ of f. For any $\varepsilon > 0$, if f and g share five distinct values $a_i \in \bar{\mathbb{C}}$ $(i = 1, 2, 3, 4, 5)$ IM in $\Omega(\theta-\varepsilon,\theta+\varepsilon)$, then $f = g$.*

In order to state the next result, we also need the following notation. Let f be a non-constant meromorphic function in $\mathbb{C}$, and let a be an arbitrary complex number. We use $\bar{E}(a, D, f)$ to denote the zeros set of $f-a$ in $D \subseteq \mathbb{C}$, in which each zero is counted only once. Clearly, we say that f and g share a IM in D, if $\bar{E}(a, D, f) = \bar{E}(a, D, g)$. We use $\bar{E}(a, f)$ to denote the zeros set of $f-a$ in $D = \mathbb{C}$. In [18, Theorem 3.2], C.C.Yang improved Theorem A by proving

Theorem D. *Let f and g be two non-constant meromorphic functions in $\mathbb{C}$ and $a_i \in \bar{\mathbb{C}}$ $(i = 1, 2, 3, 4, 5)$ be five distinct values. If*

$$\bar{E}(a_i, f) \subseteq \bar{E}(a_i, g), \quad i = 1, 2, 3, 4, 5, \tag{1.2}$$

and

$$\liminf_{r\to\infty} \sum_{i=1}^{5} \bar{N}(r, \frac{1}{f-a_i}) / \sum_{i=1}^{5} \bar{N}(r, \frac{1}{g-a_i}) > \frac{1}{2}, \tag{1.3}$$

then $f = g$.

Now, it is natural to ask the following question.

Question 1. *Do f and g coincide if they satisfy the conditions of Theorem D in an angular domain?*

In the present paper, we answer to Question 1 is affirmative for some class of meromorphic functions by using Nevanlinna theory in an angular domain which is recalled in Lemma 2.1 below. The first result is stated as follows.

Theorem 1.1. *Let $\rho(r)$ be a proximate order of meromorphic function f of infinite order in $\mathbb{C}$ and let $g \in M(\rho(r))$. Let $a_i \in \bar{\mathbb{C}}$ $(i = 1, 2, 3, 4, 5)$ be five distinct values. Suppose that $\arg z = \theta \in [0, 2\pi)$ is a Brole direction of order $\rho(r)$ of f. For any given $\varepsilon > 0$, if*

$$\bar{E}(a_i, \Omega(\theta - \varepsilon, \theta + \varepsilon), f) \subseteq \bar{E}(a_i, \Omega(\theta - \varepsilon, \theta + \varepsilon), g), \quad i = 1, 2, 3, 4, 5, \tag{1.4}$$

and

$$\liminf_{r\to\infty} \sum_{i=1}^{5} \bar{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{f - a_i}) / \sum_{i=1}^{5} \bar{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{g - a_i}) > \frac{1}{2}, \tag{1.5}$$

then $f = g$.

Before stating the following result, we need some notation concerning Ahlfors theory in an angular domain $\Omega(\alpha, \beta)$ which can be found [14, pp. 258-259], or for reference [26, pp. 66-76].

$$S_A(r, \Omega(\alpha, \beta), f) = \frac{1}{\pi} \int_0^r \int_\alpha^\beta (\frac{|f'(te^{i\varphi})|}{1 + |f(te^{i\varphi})|^2})^2 tdtd\varphi,$$

$$T(r, \Omega(\alpha, \beta), f) = \int_0^r \frac{S_A(t, \Omega(\alpha, \beta), f)}{t} dt.$$

Especially the corresponding notation in the whole complex plane are denoted by

$$S_A(r, f) = \frac{1}{\pi} \int_0^r \int_0^{2\pi} (\frac{|f'(te^{i\varphi})|}{1 + |f(te^{i\varphi})|^2})^2 tdtd\varphi,$$

$$T(r, f) = \int_0^r \frac{S_A(t, f)}{t} dt.$$

By using the relationship between Ahlfors characteristic function in an angular domain and sectorial Nevanlinna characteristic function which is introduced in Lemma 2.7 of Section 2, we can prove the following result.

Theorem 1.2. *Let f and g be two non-constant meromorphic functions of finite order in $\mathbb{C}$ and $a_i \in \bar{\mathbb{C}}$ $(i = 1, 2, 3, 4, 5)$ be five distinct values. Suppose* *that* $\Omega(\alpha,\beta)$ *is an angular domain such that* f *satisfies*

$$\limsup_{\varepsilon\to 0^+}\limsup_{r\to\infty}\frac{\log T(r,\Omega(\alpha+\varepsilon,\beta-\varepsilon),f)}{\log r}>\omega, \tag{1.6}$$

where $\omega=\frac{\pi}{\beta-\alpha}$. *If*

$$\bar{E}(a_i,\Omega(\alpha,\beta),f)\subseteq\bar{E}(a_i,\Omega(\alpha,\beta),g),\quad i=1,2,3,4,5, \tag{1.7}$$

and

$$\liminf_{r\to\infty}\sum_{i=1}^{5}\bar{C}_{\alpha,\beta}(r,\frac{1}{f-a_i})/\sum_{i=1}^{5}\bar{C}_{\alpha,\beta}(r,\frac{1}{g-a_i})>\frac{1}{2}, \tag{1.8}$$

then $f=g$.

Theorem 1.3. *Let* f *and* g *be two non-constant meromorphic functions of finite order in* $\mathbb{C}$ *and* $a_i\in\bar{\mathbb{C}}$ $(i=1,2,3,4,5)$ *be five distinct values. Suppose that* $\Omega(\alpha,\beta)$ *is an angular domain such that for any* $\varepsilon>0$ *and for some* $a\in\bar{\mathbb{C}}$

$$\limsup_{r\to\infty}\frac{\log n(r,\Omega(\alpha+\varepsilon,\beta-\varepsilon),\frac{1}{f-a})}{\log r}>\omega, \tag{1.9}$$

where $\omega=\frac{\pi}{\beta-\alpha}$. *If* f *and* g *satisfy* (1.7) *and* (1.8), *then* $f=g$.

Remark 1.4. It is well know that every meromorphic function of order $\rho\in(0,\infty)$ must have at least one direction $\arg z=\theta\in[0,2\pi)$ such that for sufficiently small $\varepsilon>0$,

$$\limsup_{r\to\infty}\frac{\log n(r,\Omega(\alpha+\varepsilon,\beta-\varepsilon),\frac{1}{f-a})}{\log r}=\rho$$

holds for all $a\in\bar{\mathbb{C}}$ with at most two exceptional values, which can be found in [20, Chapter 3]. So the angular domain satisfying (1.9) must exist when f is of order $\rho\in(\frac{1}{2},\infty)$.

This paper is organized as follows. In Section 2, we recall the properties of sectorial Nevanlinna characteristic and state some Lemmas which are needed in proving our results. The proof of Theorem 1.1 is given in Section 3. Finally, we prove Theorem 1.2 and 1.3 in Section 4.

2. Auxiliary results

Let f be a meromorphic function in the angular domain $\Omega(\alpha,\beta)=\{z:\alpha\le\arg z\le\beta\}$, where $\alpha<\beta$ and $\beta-\alpha<2\pi$. We recall the following definitions that were found in [6, Chapter 1].

$$\begin{aligned}
A_{\alpha,\beta}(r,f) &= \frac{\omega}{\pi}\int_1^r (\frac{1}{t^\omega}-\frac{t^\omega}{r^{2\omega}})\{\log^+|f(te^{i\alpha})|+\log^+|f(te^{i\beta})|\}\frac{dt}{t},\\
B_{\alpha,\beta}(r,f) &= \frac{2\omega}{\pi r^\omega}\int_\alpha^\beta \log^+|f(te^{i\theta})|\sin\omega(\theta-\alpha)d\theta,\\
C_{\alpha,\beta}(r,f) &= 2\sum_{1<|b_n|<r}(\frac{1}{|b_n|^\omega}-\frac{|b_n|^\omega}{r^{2\omega}})\sin\omega(\theta_n-\alpha),
\end{aligned}$$

where $\omega=\frac{\pi}{\beta-\alpha}$ and $b_n=|b_n|e^{i\theta_n}$ are the poles of f in $\Omega(\alpha,\beta)$ counting multiplicities. The function $C_{\alpha,\beta}(r,f)$ is called the sectorial counting function of the poles of f in $\Omega(\alpha,\beta)$. In the corresponding counting function $\bar{C}_{\alpha,\beta}(r,f)$ these multiplicities are ignored. For $a\in\mathbb{C}$, the definitions of $A_{\alpha,\beta}(r,\frac{1}{f-a})$, $B_{\alpha,\beta}(r,\frac{1}{f-a})$, and $C_{\alpha,\beta}(r,\frac{1}{f-a})$ are immediate. Finally, the sectorial Nevanlinna characteristic function is given by

$$S_{\alpha,\beta}(r,f)=A_{\alpha,\beta}(r,f)+B_{\alpha,\beta}(r,f)+C_{\alpha,\beta}(r,f).$$

We state sectorial analogues of Nevanlinna's first and second main theorems as follows.

Lemma 2.1 ([6])**.** *Let f be a meromorphic function in $\mathbb{C}$ and let $\Omega(\alpha,\beta)$ be an angular domain. Then, for any $a\in\mathbb{C}$,*

$$S_{\alpha,\beta}(r,\frac{1}{f-a})=S_{\alpha,\beta}(r,f)+O(1).$$

Moreover, for any $q\geq 3$ distinct values, $a_j\in\bar{\mathbb{C}}$ $(j=1,2,\cdots,q)$,

$$(q-2)S_{\alpha,\beta}(r,f)\leq\sum_{j=1}^q\bar{C}_{\alpha,\beta}(r,\frac{1}{f-a_j})+R_{\alpha,\beta}(r,f),$$

where

$$\begin{aligned}
R_{\alpha,\beta}(r,f) &= A_{\alpha,\beta}(r,\frac{f'}{f})+B_{\alpha,\beta}(r,\frac{f'}{f})\\
&\quad+\sum_{j=1}^q\{A_{\alpha,\beta}(r,\frac{f'}{f-a_j})+B_{\alpha,\beta}(r,\frac{f'}{f-a_j})\}+O(1). \qquad (2.1)
\end{aligned}$$

Lemma 2.2 ([6])**.** *Let f be a meromorphic function in $\mathbb{C}$ and let $\Omega(\alpha,\beta)$ be an angular domain. Then*

$$\begin{aligned}
A_{\alpha,\beta}(r,\frac{f'}{f}) &\leq K\{(\frac{R}{r})^\omega\int_r^R\frac{\log^+T(t,f)}{t^{\omega+1}}dt+\log^+\frac{r}{R-r}+\log\frac{R}{r}+1\},\\
B_{\alpha,\beta}(r,\frac{f'}{f}) &\leq \frac{4\omega}{r^\omega}m(r,\frac{f'}{f}),
\end{aligned}$$

where $\omega=\frac{\pi}{\beta-\alpha}$, $1<r<R<\infty$, K is a nonzero constant.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 21, NO.6, 2016, COPYRIGHT 2016 EUDOXUS PRESS, LLC

The next result follows from Lemma 2.2 and Lemma on the logarithmic derivative.

Lemma 2.3. *Let f be a meromorphic function in $\mathbb{C}$ and let $\Omega(\alpha,\beta)$ be an angular domain. Then*

$$R_{\alpha,\beta}(r,f)=\begin{cases} O(1), & f \text{ is of finite order;}\\ O(\log U(r)), & f \text{ is of infinite order;}\end{cases}$$

where $R_{\alpha,\beta}(r,f)$ is defined as in (2.1), $U(r)=r^{\rho(r)}$ and $\rho(r)$ is a proximate order of the meromorphic function f of infinite order.

Lemma 2.4 ([3])**.** *Suppose that $\rho(r)$ is a proximate order of meromorphic function f of infinite order in $\mathbb{C}$. Then, a ray $\arg z=\theta\in[0,2\pi)$ from the origin is a Borel direction of order $\rho(r)$ of f if and only if for any $\varepsilon\in(0,\frac{\pi}{2})$, we have*

$$\limsup_{r\to\infty}\frac{\log S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)}{\rho(r)\log r}=1.$$

Lemma 2.5 ([23])**.** *Let f be a meromorphic function in $\mathbb{C}$, $\Omega(\alpha,\beta)$ be an angular domain. If the order of f is finite order and satisfy*

$$\limsup_{\varepsilon\to0^+}\limsup_{r\to\infty}\frac{\log T(r,\Omega(\alpha+\varepsilon,\beta-\varepsilon),f)}{\log r}=\lambda>\omega,$$

where $\omega=\frac{\pi}{\beta-\alpha}$. Then

$$\limsup_{\varepsilon\to0^+}\limsup_{r\to\infty}\frac{\log S_{\alpha+\varepsilon,\beta-\varepsilon}(r,f)}{\log r}=\lambda-\omega.$$

In order to describe the relationship between Ahlfors characteristic function in an angular domain and sectorial Nevanlinna characteristic function, we also need some notation and definition. Since $S_{\alpha,\beta}(r,f)$ is not increasing with respect to r, hence Nevanlinn defined the following function $\dot{S}_{\alpha,\beta}(r,f)$ that is increasing with respect to r,

$$\dot{S}_{\alpha,\beta}(r,f)=\frac{1}{\pi}\int_1^r\int_\alpha^\beta(\frac{1}{t^\omega}-\frac{t^\omega}{r^{2\omega}})(\frac{|f'(te^{i\theta})|}{1+|f(te^{i\theta})|^2})^2\sin\omega(\theta-\alpha)tdtd\theta,$$

where $\omega=\frac{\pi}{\beta-\alpha}$. $\dot{S}_{\alpha,\beta}(r,f)$ and $S_{\alpha,\beta}(r,f)$ have following relationship.

Lemma 2.6. [26, Lemma 2.2.1] *Let f be a meromorphic function in $\Omega(\alpha,\beta)$. Then*

$$\dot{S}_{\alpha,\beta}(r,f)=S_{\alpha,\beta}(r,f)+O(1).$$

In [26], we can also find the relationship between $\dot{S}_{\alpha,\beta}(r,f)$ and $T(r,\Omega(\alpha,\beta),f)$ as follows.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 21, NO.6, 2016, COPYRIGHT 2016 EUDOXUS PRESS, LLC

Lemma 2.7. [26, Theorem 2.4.7] *Let f be a meromorphic function in $\Omega(\alpha,\beta)$. Then*

$$\dot{S}_{\alpha,\beta}(r,f) \leq 2\omega \frac{T(r,\Omega(\alpha,\beta),f)}{r^{\omega}} + \omega^2 \int_1^r \frac{T(t,\Omega(\alpha,\beta),f)}{t^{\omega+1}}dt,$$

where $\omega = \frac{\pi}{\beta-\alpha}$.

Lemma 2.8. *Let f be a meromorphic function in $\mathbb{C}$, and $\Omega(\alpha,\beta)$ be an angular domain. For any $\varepsilon > 0$ and for some $a \in \bar{\mathbb{C}}$, if f satisfies*

$$\limsup_{r\to\infty} \frac{\log n(r,\Omega(\alpha+\varepsilon,\beta-\varepsilon),\frac{1}{f-a})}{\log r} > \omega, \tag{2.2}$$

where $\omega = \frac{\pi}{\beta-\alpha}$, then

$$\limsup_{\varepsilon\to 0^+} \limsup_{r\to\infty} \frac{\log T(r,\Omega(\alpha+\varepsilon,\beta-\varepsilon),f)}{\log r} > \omega.$$

Proof. For any given $\varepsilon > 0$, from (2.2), there exists a sequence $\{r_n\}$, $r_n \to \infty$ as $n \to \infty$, such that

$$\lim_{n\to\infty} \frac{\log n(r_n,\Omega(\alpha+\varepsilon,\beta-\varepsilon),\frac{1}{f-a})}{\log r_n} = \lambda > \omega.$$

Let σ be a real number such that $\omega < \sigma < \lambda$, we have

$$n(r_n,\Omega(\alpha+\varepsilon,\beta-\varepsilon),\frac{1}{f-a}) > r_n^{\sigma} > r_n^{\omega}, \quad n \geq n_0.$$

By this and

$$C_{\alpha+\frac{\varepsilon}{2},\beta-\frac{\varepsilon}{2}}(r,\frac{1}{f-a}) \geq 2\omega\sin(\frac{\omega\varepsilon}{2})\frac{N(r,\Omega(\alpha+\varepsilon,\beta-\varepsilon),\frac{1}{f-a})}{r^{\omega}} + 2\omega^2\sin(\frac{\omega\varepsilon}{2})\int_1^r \frac{N(t,\Omega(\alpha+\varepsilon,\beta-\varepsilon),\frac{1}{f-a})}{t^{\omega+1}},$$

which can be found in [26, Lemma 2.2.2], we have

$$C_{\alpha+\frac{\varepsilon}{2},\beta-\frac{\varepsilon}{2}}(r_n,\frac{1}{f-a}) > r_n^{\sigma-\omega}. \tag{2.3}$$

By using Lemma 2.1 and (2.3), we get

$$S_{\alpha+\frac{\varepsilon}{2},\beta-\frac{\varepsilon}{2}}(r_n,f) > r_n^{\sigma-\omega}.$$

It follows from Lemmas 2.6 and 2.7 that

$$T(r_n,\Omega(\alpha+\frac{\varepsilon}{2},\beta-\frac{\varepsilon}{2}),f) > r_n^{\sigma}.$$

Thus,

$$\limsup_{r\to\infty} \frac{\log T(r,\Omega(\alpha+\frac{\varepsilon}{2},\beta-\frac{\varepsilon}{2}),f)}{\log r} > \sigma > \omega.$$

Noting ε is arbitrary small, hence lemma holds. ☐

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 21, NO.6, 2016, COPYRIGHT 2016 EUDOXUS PRESS, LLC

3. Proof of Theorem 1.1

Suppose that $\rho(r)$ is a proximate order of meromorphic function f of infinite order, $g \in M(\rho(r))$ and that $\arg z = \theta \in [0, 2\pi)$ is a Borel direction of order $\rho(r)$ of f. For any given $\varepsilon > 0$, f and g satisfy (1.4) and (1.5) in the angular domain $\Omega(\theta-\varepsilon, \theta+\varepsilon) = \{z : \theta - \varepsilon \leq \arg z \leq \theta + \varepsilon\}$.

Firstly, we claim that $\arg z = \theta$ is also a Borel direction of order $\rho(r)$ of g. Since $\arg z = \theta$ is a Borel direction of order $\rho(r)$ of f, for above given ε, by using Lemmas 2.1, 2.3 and 2.4, then there exists a value a such that

$$\limsup_{r\to\infty} \frac{\log \bar{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{f-a})}{\rho(r)\log r} \geq 1.$$

Without loss of generality, we may assume that $a = a_1$. Thus,

$$\limsup_{r\to\infty} \frac{\log \bar{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{f-a_1})}{\rho(r)\log r} \geq 1.$$

It follows from (1.4) that

$$\limsup_{r\to\infty} \frac{\log \bar{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{g-a_1})}{\rho(r)\log r} \geq 1.$$

Therefore, we get

$$\limsup_{r\to\infty} \frac{\log S_{\theta-\varepsilon,\theta+\varepsilon}(r, g)}{\rho(r)\log r} \geq 1.$$

Combining this and $g \in M(\rho(r))$, we have

$$\limsup_{r\to\infty} \frac{\log S_{\theta-\varepsilon,\theta+\varepsilon}(r, g)}{\rho(r)\log r} = 1.$$

By using Lemma 2.4, we know that $\arg z = \theta$ is a Borel direction of order $\rho(r)$ of g.

In order to prove that $f = g$, we assume on the contrary to the assertion that $f \neq g$. Now we use the similar method of [23] to complete the proof. To this end, we consider two cases.

Case 1. We may assume that all a_i $(i = 1, 2, 3, 4, 5)$ are finite. By using Lemma 2.1, we can obtain

$$3S_{\theta-\varepsilon,\theta+\varepsilon}(r, f) \leq \sum_{i=1}^{5} \bar{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{f-a_i}) + R_{\theta-\varepsilon,\theta+\varepsilon}(r, f), \tag{3.1}$$

and

$$3S_{\theta-\varepsilon,\theta+\varepsilon}(r, g) \leq \sum_{i=1}^{5} \bar{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{g-a_i}) + R_{\theta-\varepsilon,\theta+\varepsilon}(r, g). \tag{3.2}$$

From (1.4), we have

$$\begin{aligned}\sum_{i=1}^{5} \bar{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{f-a_i}) &\leq C_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{f-g}) \\ &\leq S_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{f-g}) \\ &\leq S_{\theta-\varepsilon,\theta+\varepsilon}(r, f) + S_{\theta-\varepsilon,\theta+\varepsilon}(r, g) + O(1). \qquad (3.3)\end{aligned}$$

Since $\arg z = \theta$ is a Borel direction of order $\rho(r)$ of f, by using Lemma 2.4, then we have

$$\limsup_{r\to\infty} \frac{\log S_{\theta-\varepsilon,\theta+\varepsilon}(r, f)}{\rho(r)\log r} = 1.$$

It follows from this and Lemma 2.3, we have

$$\limsup_{r\to\infty} \frac{R_{\theta-\varepsilon,\theta+\varepsilon}(r, f)}{S_{\theta-\varepsilon,\theta+\varepsilon}(r, f)} = 0. \qquad (3.4)$$

Similarly, we have

$$\limsup_{r\to\infty} \frac{R_{\theta-\varepsilon,\theta+\varepsilon}(r, g)}{S_{\theta-\varepsilon,\theta+\varepsilon}(r, g)} = 0. \qquad (3.5)$$

Combining (3.1)-(3.5), for sufficiently large r, we have

$$\begin{aligned}\sum_{i=1}^{5} \bar{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{f-a_i}) \leq (\frac{1}{3}+o(1)) &\sum_{i=1}^{5} \bar{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{f-a_i}) \\ + (\frac{1}{3}+o(1)) &\sum_{i=1}^{5} \bar{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{g-a_i}).\end{aligned}$$

Therefore,

$$(\frac{2}{3}+o(1)) \sum_{i=1}^{5} \bar{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{f-a_i}) \leq (\frac{1}{3}+o(1)) \sum_{i=1}^{5} \bar{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{g-a_i}).$$

It follows that

$$\liminf_{r\to\infty} \sum_{i=1}^{5} \bar{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{f-a_i}) / \sum_{i=1}^{5} \bar{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{g-a_i}) \leq \frac{1}{2}.$$

This contradicts to (1.5), and hence $f = g$.

Case 2. If one of the values a_i $(i = 1, 2, 3, 4, 5)$ is ∞, without loss of generality, we may assume that $a_5 = \infty$. Take a finite value c such that $c \neq a_i (i = 1, 2, 3, 4)$ and set $F = \frac{1}{f-c}$, $G = \frac{1}{g-c}$, $b_i = \frac{1}{a_i - c}$ $(i = 1, 2, 3, 4)$ and $b_5 = 0$, then F and G satisfy $\bar{E}(b_i, \Omega(\theta-\varepsilon, \theta+\varepsilon), F) \subseteq \bar{E}(b_i, \Omega(\theta-\varepsilon, \theta+\varepsilon), G)$ $(i = 1, 2, 3, 4, 5)$, and

$$\liminf_{r\to\infty} \sum_{i=1}^{5} \bar{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{F-b_i}) / \sum_{i=1}^{5} \bar{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{G-b_i}) > \frac{1}{2}.$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 21, NO.6, 2016, COPYRIGHT 2016 EUDOXUS PRESS, LLC

From Lemma 2.1, we also know that $S_{\theta-\varepsilon,\theta+\varepsilon}(r,F)=S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)+O(1)$ and $S_{\theta-\varepsilon,\theta+\varepsilon}(r,G)=S_{\theta-\varepsilon,\theta+\varepsilon}(r,g)+O(1)$. From the previous proof, we know $F=G$. Therefore $f=g$. The proof is completed.

4. Proofs of Theorems 1.2 and 1.3

Proof of Theorem 1.2. Suppose that f and g be two non-constant meromorphic functions of finite order in $\mathbb{C}$ satisfying (1.6)-(1.8), $\Omega(\alpha,\beta)=\{z:\alpha\le\arg z\le\beta\}$ is an angular domain and $\omega=\frac{\pi}{\beta-\alpha}$. Set

$$\limsup_{\varepsilon\to0^+}\limsup_{r\to\infty}\frac{\log T(r,\Omega(\alpha+\varepsilon,\beta-\varepsilon),f)}{\log r}=\lambda. \tag{4.1}$$

Firstly, we claim that

$$\limsup_{\varepsilon\to0^+}\limsup_{r\to\infty}\frac{\log S_{\alpha+\varepsilon,\beta-\varepsilon}(r,g)}{\log r}\ge\lambda-\omega. \tag{4.2}$$

From (4.1), for any given $\varepsilon_1\in(0,\frac{\lambda-\omega}{2})$, there exists at least some $\varepsilon_2\in(0,\varepsilon_1)$ such that

$$\limsup_{r\to\infty}\frac{\log T(r,\Omega(\alpha+\varepsilon_2,\beta-\varepsilon_2),f)}{\log r}=\lambda'\ge\lambda-\varepsilon_1,$$

where $\lambda'(\le\lambda)$ is a constant. It follows from Lemma 2.5 and (1.6) that

$$\limsup_{r\to\infty}\frac{\log S_{\alpha+\varepsilon_2,\beta-\varepsilon_2}(r,f)}{\log r}=\lambda'-\omega\ge\lambda-\omega-\varepsilon_1.$$

By using Lemmas 2.1 and 2.3, then there exists a value a such that

$$\limsup_{r\to\infty}\frac{\log\bar{C}_{\alpha+\varepsilon_2,\beta-\varepsilon_2}(r,\frac{1}{f-a})}{\log r}\ge\lambda-\omega-\varepsilon_1.$$

Without loss of generality, we may assume that $a=a_1$. Thus,

$$\limsup_{r\to\infty}\frac{\log\bar{C}_{\alpha+\varepsilon_2,\beta-\varepsilon_2}(r,\frac{1}{f-a_1})}{\log r}\ge\lambda-\omega-\varepsilon_1.$$

It follows from (1.7) that

$$\limsup_{r\to\infty}\frac{\log\bar{C}_{\alpha+\varepsilon_2,\beta-\varepsilon_2}(r,\frac{1}{g-a_1})}{\log r}\ge\lambda-\omega-\varepsilon_1.$$

Therefore, we get

$$\limsup_{r\to\infty}\frac{\log S_{\alpha+\varepsilon_2,\beta-\varepsilon_2}(r,g)}{\log r}\ge\lambda-\omega-\varepsilon_1.$$

Noting ε_1 is arbitrary and $\varepsilon_2<\varepsilon_1$, so (4.2) holds.

We assume on the contrary to the assertion that $f\not\equiv g$. We consider two cases.

Case 1. We may assume that all a_i $(i=1,2,3,4,5)$ are finite.

By arguing similar to that proof of Theorem 1.1, we can obtain the following inequalities,

$$3S_{\alpha+\varepsilon,\beta-\varepsilon}(r,f) \leq \sum_{i=1}^{5} \bar{C}_{\alpha+\varepsilon,\beta-\varepsilon}(r,\frac{1}{f-a_i}) + R_{\alpha+\varepsilon,\beta-\varepsilon}(r,f), \tag{4.3}$$

$$3S_{\alpha+\varepsilon,\beta-\varepsilon}(r,g) \leq \sum_{i=1}^{5} \bar{C}_{\alpha+\varepsilon,\beta-\varepsilon}(r,\frac{1}{g-a_i}) + R_{\alpha+\varepsilon,\beta-\varepsilon}(r,g), \tag{4.4}$$

$$\begin{aligned}\sum_{i=1}^{5} \bar{C}_{\alpha+\varepsilon,\beta-\varepsilon}(r,\frac{1}{f-a_i}) &\leq C_{\alpha+\varepsilon,\beta-\varepsilon}(r,\frac{1}{f-g}) \\ &\leq S_{\alpha+\varepsilon,\beta-\varepsilon}(r,f) + S_{\alpha+\varepsilon,\beta-\varepsilon}(r,g) + O(1).\end{aligned} \tag{4.5}$$

By using (1.6), Lemmas 2.3 and 2.5, we get

$$\limsup_{r\to\infty} \frac{R_{\alpha+\varepsilon,\beta-\varepsilon}(r,f)}{S_{\alpha+\varepsilon,\beta-\varepsilon}(r,f)} = 0. \tag{4.6}$$

Similarly, it follows from (4.2) that

$$\limsup_{r\to\infty} \frac{R_{\alpha+\varepsilon,\beta-\varepsilon}(r,g)}{S_{\alpha+\varepsilon,\beta-\varepsilon}(r,g)} = 0. \tag{4.7}$$

Combining (4.3)-(4.7), for sufficiently large r, we have

$$(\frac{2}{3}+o(1))\sum_{i=1}^{5} \bar{C}_{\alpha+\varepsilon,\beta-\varepsilon}(r,\frac{1}{f-a_i}) \leq (\frac{1}{3}+o(1))\sum_{i=1}^{5} \bar{C}_{\alpha+\varepsilon,\beta-\varepsilon}(r,\frac{1}{g-a_i}).$$

It follows that

$$\liminf_{r\to\infty} \sum_{i=1}^{5} \bar{C}_{\alpha+\varepsilon,\beta-\varepsilon}(r,\frac{1}{f-a_i}) / \sum_{i=1}^{5} \bar{C}_{\alpha+\varepsilon,\beta-\varepsilon}(r,\frac{1}{g-a_i}) \leq \frac{1}{2}.$$

Noting $\varepsilon \to 0$, this contradicts to (1.8), and hence $f = g$.

Case 2. If one of the values a_i $(i = 1,2,3,4,5)$ is ∞, without loss of generality, we may assume that $a_5 = \infty$. By using similar way of the proof of Theorem 1.1, we can easily obtain $f = g$. The proof is completed. □

Proof of Theorem 1.3. By Lemma 2.8, (1.9) implies (1.6). So combining Theorem 1.2 we get the conclusion of Theorem 1.3. □

Acknowledgements. This research was partly supported by the United Technology Foundation of Science and Technology Department of Guizhou Province and Guizhou Normal University (Grant No. LKS[2012]12), the National Natural Science Foundation of China (Grant No. 11171277).

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 21, NO.6, 2016, COPYRIGHT 2016 EUDOXUS PRESS, LLC

References

[1] T. B. Cao and H. X. Yi, *On the uniqueness of meromorphic functions that share four values in one angular domain*, J. Math. Anal. Appl. 358 (2009), 81-97.

[2] C. T. Chuang, *Sur les fonctions-types*, Sci. Sinica 10 (1961), 171-181.

[3] C. T. Chuang, *On Borel directions of meromorphic functions of infinite order*(II), Bull. HongKong Math. Soc. 2 (1999), no. 2, 305-323.

[4] C. T. Chuang, *Singular directions of meromorphic functions*, Science Press, Beijing, 1982. (In Chinese)

[5] D. Drasin and A. Weitsman, *On the Julia Directions and Borel Directions of Entire Functions,* Proc. London Math. Soc. 32 (1976), no. 3, 199-212.

[6] A. A. Gol'dberg and I. V. Ostrovshii, *Value Distribution of Meromorphic Functions*, Translations of Mathematical Monographs, vol. 236, American Mathematical Society, Providence, RI, 2008.

[7] W. K. Hayman, *Meromorphic Functions*, Clarendon Press, Oxford, 1964.

[8] K. L. Hiong, *Sur les fonctions entiéres et les fonctions méromorphes d'ordre infini*, J. Math. Pures Appl. 14 (1935), 233-308.

[9] W. C. Lin, S. Mori and H. X. Yi, *Uniqueness theorems of entire functions with shared-set in an angular domain*, Acta Math. Sinica 24 (2008), 1925-1934.

[10] W. C. Lin, S. Mori and K. Tohge, *Uniqueness theorems in an angular domain*, Tohoku Math. J. 58 (2006), 509-527.

[11] J. R. Long and P. C. Wu, *Borel directions and Uniqueness of meromorphic functions*, Chinese Ann. Math. 33A (2012), no. 3, 261-266.

[12] R. Nevanlinna, *Le théoréme de Picard-Borel et la théorie de fonctions méromorphes*, Paris, 1929.

[13] M. Tsuji, *On Borel directions of meromorphic functions of finite order, I,* Tohoku Math. J. 2 (1950), no. 2, 97-112.

[14] M. Tsuji, *Potential theoty in modern function theory*, Maruzen, Tokyo, 1959.

[15] S. J. Wu, *Further results on Borel removable set s of entire functions*, Ann. Acad. Sci. Fenn. Ser. A. I Math. 19 (1994), 67-81.

[16] S. J. Wu, *On the distribution of Borel directions of entire function*, Chinese Ann. Math. 14A (1993), no. 4, 400-406.

[17] J. F. Xu and H. X. Yi, *On uniqueness of meromorphic functions with shared four values in some angular domains*, Bull. Malays Sci. Soc. (2) 31 (2008), no. 1, 57-65.

[18] C. C. Yang and H. X. Yi, *Uniqueness Theory of Meromorphic Functions*, Kluwer Publ., Dordrecht, 2003.

[19] L. Yang and C. C. Yang, *Angular distribution of ff'*, Sci. China Ser. A 37 (1994), no. 3, 284- 294.

[20] L. Yang, *Value Distribution Theory,* Springer-Verlag, Berlin, 1993.

[21] L. Yang, *Borel directions of meromorphic functions in an angular domain*, Sci. China Math. I (1979), 149-164.

[22] L. Yang and G. H. Zhuang, *The distribution of Borel directions of entire functions*, Sci. China Ser. A 3 (1976), 157-168.

[23] Q. C. Zhang, *Meromorphic functions sharing values in an angular domain*, J. Math. Anal. Appl. 349 (2009), 100-112.

[24] J. H. Zheng, *On uniqueness of meromorphic functions shared values in some angular domains*, Canad. Math. Bull. 47 (2004), no. 1, 152-160.

[25] J. H. Zheng, *On uniqueness of meromorphic functions shared values in one angular domain,* Complex Variables 48 (2003), no. 9, 777-785.

[26] J. H. Zheng, *Value Distribution of Meromorphic Functions*, Springer and Tsinghua University Publishing House, 2010.

Jianren Long
School of Mathematics and Computer Science, Guizhou Normal University, 550001, Guiyang, P.R. China.
The current address.
School of Mathematical Sciences, Xiamen University, 361005, Xiamen, P.R. China.
E-mail address: longjianren2004@163.com, jrlong@gznu.edu.cn

Chunhui Qiu
School of Mathematical Sciences, Xiamen University, 361005, Xiamen, P.R. China.
E-mail address: chqiu@xmu.edu.cn

ON AN INTERVAL-REPRESENTABLE GENERALIZED PSEUDO-CONVOLUTION BY MEANS OF THE INTERVAL-VALUED GENERALIZED FUZZY INTEGRAL AND THEIR PROPERTIES

JEONG GON LEE AND LEE-CHAE JANG

Division of Mathematics and Informational Statistics,
and Nanoscale Science and Technology Institute,
Wonkwang University, Iksan 570-749, Republic of Korea
E-mail : jukolee@wku.ac.kr

General Education Institute, Konkuk University, Chungju 138-701, Korea
E-mail : leechae.jang@kku.ac.kr

Abstract. In this paper, we consider the generalized pseudo-convolution in the theory of probabilistic metric space and their properties which was introduced by Pap-Stajner (1999). Wu-Wang-Ma(1993) and Wu-Ma-Song(1995) studied the generalized fuzzy integral and their properties. Recently, Jang(2013) defined the interval-valued generalized fuzzy integral by using an interval-representable pseudo-multiplication. From the generalized fuzzy integral, we define a generalized pseudo-convolution by means of the generalized fuzzy integral and investigate their properties.

In particular, we also define an interval-representable generalized pseudo-convolution of interval-valued functions by means of the interval-valued generalized fuzzy integral and investigate their properties.

1. Introduction

Fang [8-10], Wu-Wang-Ma [35], Wu-Ma-Song [36], Xie-Fang [37] have studied the generalized fuzzy integral(for short, the (G) fuzzy integral) by using a pseudo-multiplication which is a generalization of fuzzy integrals in [5, 25, 26, 29, 31, 33, 39]. Pap-Stajner [28] introduced a notion of the generalized pseudo-convolution of functions based on pseudo-operations and proved their mathematical theories such as optimization, probabilistic metric spaces, and information theory

Many researchers [1,2,7,13-19, 21, 30, 34, 38, 40] have been studying various integrals of measurable multi-valued functions which are used for representing uncertain functions, for examples, the Aumann integral, the fuzzy integral, and the Choquet integral of measurable interval-valued functions in many different mathematical theories and their applications. Recently, Jang [20] defined the interval-valued generalized fuzzy integral (for short, the (IG) fuzzy integral) with respect to a fuzzy measure by using an interval-representable pseudo-multiplication of measurable interval-valued functions and investigated some convergence properties of them.

1991 *Mathematics Subject Classification.* 28E10, 28E20, 03E72, 26E50 11B68.

Key words and phrases. fuzzy measure, generalized fuzzy integral, interval-representable pseudo-multiplication, interval-valued function, generalized pseudo-convolution.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 21, NO.6, 2016, COPYRIGHT 2016 EUDOXUS PRESS, LLC

The purpose of this study is to define the generalized pseudo-convolution of functions by means of the (G) fuzzy integral and to investigate some properties of them. In particular, we also define the interval-valued generalized pseudo-convolution of interval-valued functions by means of the (IG) fuzzy integral and to investigate some properties of them.

The paper is organized in five sections. In section 2, we list definitions and some properties of the generalized fuzzy integral with respect to a fuzzy measure by using generalized pseudo-multiplication and the interval-valued generalized fuzzy integral with respect to a fuzzy measure by using interval-representable generalized pseudo-multiplication. In section 3, we define the generalized pseudo-convolution of integrable nonnegative functions by means of the (G) fuzzy integral and investigate their properties. Furthermore, we give an example of the generalized pseudo-convolution of integrable nonnegative functions. In section 4, we define a interval-representable semigroup and the interval-valued generalized pseudo-convolution of integrable interval-valued functions by means of the (IG) fuzzy integral and investigate their properties. Furthermore, we give an example of the interval-valued generalized pseudo-convolution of integrable interval-valued functions. In section 5, we give a brief summary results and some conclusions.

2. Definitions and Preliminaries

In this section, we introduce some definitions and properties of a fuzzy measure, a pseudo-multiplication, a pseudo-addition, the (G) fuzzy integral with respect to a fuzzy measure by using a pseudo-multiplication of a measurable functions. Let X be a set and $(X, \mathcal{A})$ be a measurable space. Denote by $\mathfrak{F}(X)$ the set of all measurable nonnegative functions on X.

Definition 2.1. ([25, 26]) (1) A fuzzy measure $\mu : \mathcal{A} \longrightarrow [0, \infty]$ is a set function satisfying

$$\begin{array}{ll} \text{(i)} & \mu(\emptyset) = 0 \\ \text{(ii)} & \mu(A) \leq \mu(B) \;\; \text{whenever} \;\; A, B \in \mathcal{A} \;\; \text{and} \;\; A \subset B. \end{array}$$

(2) A fuzzy measure μ is said to be finite if $\mu(X) < \infty$.

Definition 2.2. ([10, 33, 37]) (1) A binary operation $\oplus : [0, \infty]^2 \longrightarrow [0, \infty]$ is called a pseudo-addition if it is non-decreasing in both components, associative, and 0 is its neutral element.

(2) A binary operation $\odot : [0, \infty]^2 \longrightarrow [0, \infty]$ is called a pseudo-multiplication corresponding to $\oplus$ if it satisfies the following axioms:

(i) $a \odot b = b \odot a$,

(ii) $a \odot (x \oplus y) = (a \odot x) \oplus (a \odot y)$,

(iii) $a \leq b \Longrightarrow a \odot x \leq b \odot x$,

(vi) $a \odot x = 0 \Longleftrightarrow a = 0$ or $x = 0$,

(v) there exists a unit element, that is, $\exists e \in (0, \infty]$ such that $e \odot x = x$ for all $x \in [0, \infty]$,

(vi) $a_n \longrightarrow a \in (0, \infty)$ and $x_n \longrightarrow x \in [0, \infty] \Longrightarrow a_n \odot x_n \longrightarrow a \odot x$ and $\lim_{a \to \infty} a \odot x = \infty \odot x$ for all $x \in (0, \infty]$.

Definition 2.3. ([20, 33, 37]) (1) Let $(X, \mathcal{A}, \mu)$ be a fuzzy measure space,$f \in \mathfrak{F}(X)$, and $A \in \mathcal{A}$. The (G) fuzzy integral with respect to a fuzzy measure μ by using a pseudo-multiplication $\odot$ corresponding to the pseudo-addition $\oplus = \max$(maximum) of f on A is defined by

$$(G)\int_A^{\odot} f d\mu = \sup_{\alpha>0} \alpha \odot \mu_{A,f}(\alpha), \tag{1}$$

where $\mu_{A,f}(\alpha) = \mu(\{x \in A | f(x) \geq \alpha\})$ for all $\alpha \in (0, \infty)$.

(2) f is said to be integrable if $(G)\int_A^{\odot} f d\mu$ is finite.

Let $\mathfrak{F}(X)^*$ be the set of all nonnegative integrable functions on X. We consider the intervals, a standard interval-valued pseudo-multiplication, and an extended interval-valued pseudo-multiplication. Let $I(Y)$ be the set of all bounded closed intervals (intervals, for short) in Y as follows:

$$I(Y) = \{\overline{a} = [a_l, a_r] \mid a_l, a_r \in Y \text{ and } a_l \leq a_r\}, \tag{2}$$

where Y is $[0, \infty)$ or $[0, \infty]$. For any $a \in Y$, we define $a = [a, a]$. Obviously, $a \in I(Y)$ (see[4, 7, 16-21, 30, 34, 38-40]). Denote by $I\mathfrak{F}(X)$ the set of all measurable interval-valued functions on X.

Definition 2.4. ([20]) If $\overline{a} = [a_l, a_r], \overline{b} = [b_l, b_r], \overline{a}_n = [a_{ln}, a_{rn}], \overline{a}_\alpha = [a_{l\alpha}, a_{r\alpha}] \in I(Y)$ for all $n \in \mathbb{N}$ and $\alpha \in [0, \infty)$, and $k \in [0, \infty)$, then we define arithmetic, maximum, minimum, order, inclusion, superior, inferior operations as follows:

(1) $\overline{a} + \overline{b} = [a_l + b_l, a_r + b_r]$,
(2) $k\overline{a} = [ka_l, ka_r]$,
(3) $\overline{a}\overline{b} = [a_l b_l, a_r b_r]$,
(4) $\overline{a} \vee \overline{b} = [a_l \vee b_l, a_r \vee b_r]$,
(5) $\overline{a} \wedge \overline{b} = [a_l \wedge b_l, a_r \wedge b_r]$,
(6) $\overline{a} \leq \overline{b}$ if and only if $a_l \leq b_l$ and $a_r \leq b_r$,
(7) $\overline{a} < \overline{b}$ if and only if $a_l \leq b_l$ and $a_l \neq b_l$,
(8) $\overline{a} \subset \overline{b}$ if and only if $b_l \leq a_l$ and $a_r \leq b_r$],
(9) $\sup_n \overline{a}_n = [\sup_n a_{nl}, \sup_n a_{nr}]$,
(10) $\inf_n \overline{a}_n = [\inf_n a_{nl}, \inf_n a_{nr}]$,
(11) $\sup_\alpha \overline{a}_\alpha = [\sup_\alpha a_{\alpha l}, \sup_\alpha a_{\alpha r}]$, and
(12) $\inf_\alpha \overline{a}_\alpha = [\inf_\alpha a_{\alpha l}, \inf_\alpha a_{\alpha r}]$.

Definition 2.5. ([20]) (1) A mapping $\bigodot_I : I([0,\infty])^2 \longrightarrow I([0,\infty])$ is called a standard interval-valued pseudo-multiplication if there exist pseudo-multiplications $\odot_l$ and $\odot_r$ such that $x \odot_l y \leq x \odot_r y$ for all $x, y \in [0, \infty]$, and such that for all $\overline{a} = [a_l, a_r], \overline{b} = [b_l, b_r] \in I([0, \infty])$,

$$\overline{a} \bigodot_I \overline{b} = [a_l \odot_l b_l, a_r \odot_r b_r]. \tag{3}$$

Then $\odot_l$ and $\odot_r$ are called the representants of $\bigodot_I$.

(2) A mapping $\bigodot_{II} : I([0,\infty])^2 \longrightarrow I([0,\infty])$ is called an extended interval-valued pseudo-multiplication if there exists a pseudo-multiplication $\odot$ such that for any $\overline{a} = [a_l, a_r], \overline{b} = [b_l, b_r] \in I([0, \infty])$,

$$\overline{a} \bigodot_{II} \overline{b} = [a_l \odot b_l, \max\{a_l \odot b_r, a_r \odot b_l\}]. \tag{4}$$

Then $\odot$ is called the representant of $\bigodot_{II}$.

We also introduce the (IG) fuzzy integral with respect to a fuzzy measure by using two interval-representable pseudo-multiplications which are used to define the interval-valued generalized pseudo-convolution in the next section 4.

Definition 2.6. ([20]) Let $(X, \mathcal{A}, \mu)$ be a fuzzy measure space. (1) An interval-valued function $\overline{f} : X \to I([0,\infty) \setminus \{\emptyset\}$ is said to be measurable if for any open set $O \subset [0,\infty)$,

$$\overline{f}^{-1}(O) = \{x \in X \mid \ \overline{f} \cap O \neq \emptyset\} \in \mathcal{A}. \tag{5}$$

(2) If $\bigodot : I([0,\infty])^2 \longrightarrow I([0,\infty])$ is an interval-representable pseudo-multiplication and $\overline{f} \in I\mathfrak{F}(X)$ and $A \in \mathcal{A}$, then the (IG) fuzzy integral with respect to μ by using $\bigodot$ of $\overline{f}$ on A is defined by

$$(IG) \int_A^{\bigodot} \overline{f} d\mu = \sup_{\alpha > 0} \alpha \bigodot \mu_{A,\overline{f}}(\alpha), \tag{6}$$

where $\mu_{A,\overline{f}}(\alpha) = [\mu_{A,f_l}(\alpha), \mu_{A,f_r}(\alpha)]$ for all $\alpha \in [0,\infty)$.

(3) $\overline{f}$ is said to be integrable on A if

$$(IG) \int_A^{\bigodot} \overline{f} d\mu \in \mathcal{P}([0,\infty)) \setminus \{\emptyset\}, \tag{7}$$

where $\mathcal{P}(\mathbb{R}^+)$ is the set of all subsets of $[0,\infty)$.

Let $I\mathfrak{F}(X)^*$ be the set of all integrable interval-valued functions. We consider the following theorem which is used to investigate some characterizations of the interval-valued generalized pseudo-convolution by means of the (IG) fuzzy integral.

Theorem 2.1. (1) *Let $\odot_l$ and $\odot_r$ be pseudo-multiplications on $[0,\infty]$ corresponding to a pseudo-addition $\oplus = \max$. If $\bigodot_I$ is a standard interval-valued pseudo-multiplication, $A \in \mathcal{A}$, and $f \in I\mathfrak{F}(X)^*$, then we have*

$$(IG) \int_A^{\bigodot_I} \overline{f} d\mu = \left[(G) \int_A^{\odot_l} f_l d\mu, (G) \int_A^{\odot_r} f_r d\mu \right]. \tag{8}$$

(2) *Let $\odot : [0,\infty]^2 \longrightarrow [0,\infty]$ be a pseudo-multiplication, $\overline{f} = [f_l, f_r] \in I\mathfrak{F}(X)^*$, and $A \in \mathcal{A}$. If $\bigodot_{II}$ is an extended interval-valued pseudo-multiplication, then we have*

$$(IG) \int_A^{\bigodot_{II}} \overline{f} d\mu = \left[(G) \int_A^{\odot} f_l d\mu, (G) \int_A^{\odot} f_r d\mu \right]. \tag{9}$$

3. The generalized pseudo-convolution on $\mathfrak{F}(X)^*$

In this section, we consider a semigroup $([0,\infty), \otimes)$ and define the generalized pseudo-convolution on $\mathfrak{F}(X)^*$.

Definition 3.1. Let $f, h \in \mathfrak{F}(X)^*$ and $t \in [0, \infty)$. The generalized pseudo-convolution of f and h by means of the (G) fuzzy integral is defined by

$$(f * h)(t) = (G) \int_{[0,t]}^{\odot} f(t-u) \otimes h(u) d\mu(u). \tag{10}$$

Then we obtain the following basic properties and examples of the generalized pseudo-convolution of nonnegative measurable functions.

Theorem 3.1. (1) *If $f, h \in \mathfrak{F}(X)^*$ and $t \in [0, \infty)$ and $\otimes$ is a minimum operation(min) and $f(t-u) \leq h(u)$ for all $u \in [0,t]$, then we have*

$$(f * h)(t) = \sup_{\alpha \in [0,t]} \alpha \odot \mu_{[0,t],f}(\alpha). \tag{11}$$

(2) *If $f, h \in \mathfrak{F}(X)^*$ and $t \in [0, \infty)$ and $\otimes$ is a multiplication operation($\cdot$) and $f(x) = c$ for all $x \in [0, \infty)$, then we have*

$$(f * h)(t) = \sup_{\alpha \in [0,t]} \alpha \odot \mu_{[0,t],h}\left(\frac{\alpha}{c}\right). \tag{12}$$

(3) *If $f, h \in \mathfrak{F}(X)^*$ and $t \in [0, \infty)$ and $\{t - x | f(x) > 0\} \cap \{x | h(x) > 0\} = \emptyset$ and $a \otimes 0 = 0$ for all $a \in [0,t]$, then we have*

$$(f * h)(t) = 0. \tag{13}$$

(4) *If $f, h \in \mathfrak{F}(X)^*$ and $t \in [0, \infty)$ and $\odot$ is a minimum operation(min) and $\mu(\{u \in [0,t] | f(t-u) \otimes h(u) > \alpha\}) = g(\alpha) \geq \alpha$ for all $\alpha \in [0, \infty)$, then we have*

$$(f * h)(t) = t. \tag{14}$$

Proof.(1) Suppose that $\otimes$ is a minimum operation(min) and $f(t-u) \leq h(u)$ for all $u \in [0,t]$. Then we have

$$\begin{aligned} \mu_{[0,t],f(t-\cdot)\otimes h(\cdot)}(\alpha) &= \mu(\{u \in [0,t] | \min\{f(t-u), h(u)\} > \alpha\}) \\ &= \mu(\{u \in [0,t] | h(u) > \alpha\}) = \mu_{[0,t],h}(\alpha). \end{aligned} \tag{15}$$

By (15), we have

$$\begin{aligned} (f * h)(t) &= (G) \int_{[0,t]}^{\odot} f(t-u) \otimes h(u) d\mu(u) \\ &= \sup_{\alpha \in [0,t]} \alpha \odot \mu_{[0,t],f(t-\cdot)\otimes h(\cdot)}(\alpha) \\ &= \sup_{\alpha \in [0,t]} \alpha \odot \mu_{[0,t],h}(\alpha). \end{aligned} \tag{16}$$

(2) Suppose that $\otimes$ is a multiplication operation($\cdot$) and $f(x) = c$ for all $x \in [0, \infty)$. Then we have

$$\begin{aligned} \mu_{[0,t],fh}(\alpha) &= \mu(\{u \in [0,t] | f(t-u)h(u) > \alpha\}) \\ &= \mu(\{u \in [0,t] | ch(u) > \alpha\}) \\ &= \mu\left(\left\{u \in [0,t] | h(u) > \frac{\alpha}{c}\right\}\right) = \mu_{[0,t],h}\left(\frac{\alpha}{c}\right). \end{aligned} \tag{17}$$

By (17), we have

$$(f * h)(t) = (G) \int_{[0,t]}^{\odot} f(t-u) \otimes h(u) d\mu(u)$$

$$\begin{aligned} &= (G)\int_{[0,t]}^{\odot} ch(u)d\mu(u) \\ &= \sup_{\alpha\in[0,t]} \alpha\odot\mu_{[0,t],ch}(\alpha) \\ &= \sup_{\alpha\in[0,t]} \alpha\odot\mu_{[0,t],h}\left(\frac{\alpha}{c}\right). \end{aligned} \tag{18}$$

(3) Suppose that $\{t-x|f(x)>0\}\cap\{x|h(x)>0\}=\emptyset$ and $u\otimes 0=0$ for all $u\in[0,t]$. Then we have

$$\begin{aligned} \mu_{[0,t],f(t-\cdot)\otimes h(\cdot)}(\alpha) &= \mu(\{u\in[0,t]|f(t-u)\otimes h(u)>\alpha\}) \\ &= \mu(\emptyset)=0. \end{aligned} \tag{19}$$

By (19) and Definition 2.2 (2)(vi), we have

$$\begin{aligned} (f*h)(t) &= \sup_{\alpha\in[0,t]} \alpha\odot\mu_{[0,t],f(t-\cdot)\otimes h(\cdot)}(\alpha) \\ &= \sup_{\alpha\in[0,t]} \alpha\odot 0=0. \end{aligned} \tag{20}$$

(4) Suppose that $\odot$ is a minimum operation(min) and $\mu(\{u\in[0,t]|f(t-u)\otimes h(u)>\alpha\})=g(\alpha)\geq\alpha$ for all $\alpha\in[0,\infty)$. Then we have

$$\begin{aligned} (f*h)(t) &= \sup_{\alpha\in[0,t]} \alpha\odot\mu_{[0,t],f(t-\cdot)\otimes h(\cdot)}(\alpha) \\ &= \sup_{\alpha\in[0,t]} \min\{\alpha,g(\alpha)\} \\ &= \sup_{\alpha\in[0,t]} \alpha=t. \end{aligned} \tag{21}$$

Theorem 3.2. *Let $([0,\infty),\otimes)$ be a semigroup and e be a unit element with respect to $\otimes$, that is, $e\otimes u=u$ for all $u\in[0,\infty)$. If $f\in\mathfrak{F}(X)^*$, then we have*

$$(e*f)(t)=(G)\int_{[0,t]}^{\odot} fd\mu. \tag{22}$$

Proof. Since $(e\otimes f)(u)=e\otimes f(u)=f(u)$ for all $u\in[0,\infty)$, we have

$$\begin{aligned} (e*f)(t) &= \sup_{\alpha\in[0,t]} \alpha\odot\mu_{[0,t],e\otimes f}(\alpha) \\ &= \sup_{\alpha\in[0,t]} \alpha\odot\mu_{[0,t],f}(\alpha) \\ &= (G)\int_{[0,t]}^{\odot} fd\mu. \end{aligned} \tag{23}$$

Remark 3.3. *A function $f:X\longrightarrow[0,\infty)$ is an idempotent with respect to the generalized pseudo-convolution $*$ induced by semigroup $([0,\infty),\otimes)$ if and only if $f*f=f$. It is easy to see that if e is a unit element as in Theorem 3.3, that is, $f*e=f$ for all $f\in\mathfrak{F}(X)$, then we also have $e*e=e$. Therefore, e is an idempotent with respect to $*$.*

Example 3.1. Let $u\odot v=\min\{u,v\}$ and $u\otimes v=u\cdot v$ for all $u,v\in[0,\infty)$, and $f(x)=1$ and $h(x)=x^2$ for all $x\in[0,\infty)$, and m be the Lebesgue measure on $[0,\infty)$. If $\mu=m^2$, then clearly μ is a fuzzy measure. Thus, we have

$$\mu_{[0,t],f(t-\cdot)\otimes h(\cdot)}(\alpha) = \mu(\{u\in[0,t]|1\otimes u^2>\alpha\})$$

$$= \quad \mu([\sqrt{\alpha}, t]) = (t-\sqrt{\alpha})^2. \tag{24}$$

By (24), we have

$$\begin{aligned}(f*h)(t) &= \sup_{\alpha\in[0,t]} \min\{\alpha, (t-\sqrt{\alpha})^2\} \\ &= \frac{t^2}{4}. \end{aligned}\tag{25}$$

4. The interval-valued generalized pseudo-convolution on $I\mathfrak{F}(X)^*$

In this section, we define a standard interval-valued semigroup $(I([0,\infty), \bigotimes)$ and the interval-representable generalized pseudo-convolution of interval-valued functions by means of the (IG) fuzzy integral on $I\mathfrak{F}(X)^*$.

Definition 4.1. A pair $(I([0,\infty), \bigotimes)$ is called a standard interval-valued semigroup if there exist two semigroups $([0,\infty), \otimes_l)$ and $([0,\infty), \otimes_r)$ such that

$$\overline{u} \bigotimes \overline{v} = [u_l \otimes_l v_l, u_r \otimes_r v_r], \tag{26}$$

for all $\overline{u} = [u_l, u_r], \overline{v} = [v_l, v_r] \in I([0,\infty))$.

Definition 4.2. Let $\overline{f}, \overline{h} \in I\mathfrak{F}(X)^*$ and $t \in [0,\infty)$. The interval-valued generalized pseudo-convolution of $\overline{f}$ and $\overline{h}$ by means of the (IG) fuzzy integral is defined by

$$(\overline{f} * \overline{h})(t) = (IG)\int_{[0,t]}^{\bigodot} \overline{f}(t-u) \otimes \overline{h}(u) d\mu(u) \tag{27}$$

where $\bigodot$ is an interval-representable pseudo-multiplication.

Then we obtain the following basic properties and examples of the interval-valued generalized pseudo-convolution of measurable interval-valued functions.

Theorem 4.1. (1) *Let $\odot_l$ and $\odot_r$ be pseudo-multiplications on $[0,\infty]$ corresponding to a pseudo-addition $\oplus = \max$. If $\bigodot_I$ is a standard interval-valued pseudo-multiplication and $\overline{f} = [f_l, f_r], \overline{h} = [h_l, h_r] \in I\mathfrak{F}^*(X)$, $t \in [0,\infty]$ and*

$$(\overline{f} *_1 \overline{f})(t) = (IG)\int_{A}^{\bigodot_I} \overline{f}(t-u) \bigotimes \overline{h}(u) d\mu(u), \tag{28}$$

then we have

$$(\overline{f} *_1 \overline{f})(t) = [(f_l *_{1l} h_l, f_r *_{1r} h_r], \tag{29}$$

where $(f_l *_{1l} h_l)(t) = (G)\int_{[0,t]}^{\odot_l} f_l(t-u) \otimes_l h_l(u) d\mu(u)$ and $(f_r *_{1r} h_r)(t) = (G)\int_{[0,t]}^{\odot_r} f_r(t-u) \otimes_r h_r(u) d\mu(u)$.

(2) *Let $\odot$ be a pseudo-multiplications on $[0,\infty]$ corresponding to a pseudo-addition $\oplus = \max$. If $\bigodot_{II}$ is an extended interval-valued pseudo-multiplication and $\overline{f} = [f_l, f_r], \overline{h} = [h_l, h_r] \in I\mathfrak{F}^*(X)$, $t \in [0,\infty]$ and*

$$(\overline{f} *_2 \overline{f})(t) = (IG)\int_{A}^{\bigodot_{II}} \overline{f}(t-u) \bigotimes \overline{h}(u) d\mu(u), \tag{30}$$

then we have

$$(\overline{f} *_2 \overline{f})(t) = [(f_l *_{2l} h_l, f_r *_{2r} h_r], \tag{31}$$

where $(f_l *_{2l} h_l)(t) = (G)\int_{[0,t]}^{\odot} f_l(t-u) \otimes_l h_l(u) d\mu(u)$ *and* $(f_r *_{2r} h_r)(t) = (G)\int_{[0,t]}^{\odot} f_r(t-u) \otimes_r h_r(u) d\mu(u)$.

Proof. (1) Since $\overline{f} \bigotimes \overline{h} = [f_l \otimes_l h_l, f_r \otimes_r h_r]$, by Theorem 2.7 (1), we have

$$\begin{aligned}(\overline{f} *_1 \overline{f})(t) &= (IG)\int_A^{\odot_I} \overline{f}(t-u) \bigotimes \overline{h}(u) d\mu(u) \\ &= \left[(G)\int_{[0,\infty]}^{\odot_l} f_l \otimes_l h_l d\mu, (G)\int_{[0,\infty]}^{\odot_r} f_r \otimes_r h_r d\mu\right] \\ &= [(f_l *_{1l} h_l, f_r *_{1r} h_r].\end{aligned}$$

(2) Since $\overline{f} \bigotimes \overline{h} = [f_l \otimes_l h_l, f_r \otimes_r h_r]$, by Theorem 2.7 (2), we have

$$\begin{aligned}(\overline{f} *_2 \overline{f})(t) &= (IG)\int_A^{\odot_{II}} \overline{f}(t-u) \bigotimes \overline{h}(u) d\mu(u) \\ &= \left[(G)\int_{[0,\infty]}^{\odot} f_l \otimes_l h_l d\mu, (G)\int_{[0,\infty]}^{\odot} f_r \otimes_r h_r d\mu\right] \\ &= [(f_l *_{2l} h_l, f_r *_{2r} h_r].\end{aligned}$$

Theorem 4.2. (1) *If* $\overline{f} = [f_l, f_r], \overline{h} = [h_l, h_r] \in I\mathfrak{F}(X)^*$ *and* $t \in [0,\infty)$, *and* $\otimes_l = \otimes_r$ *are minimum operation(min) and* $\overline{f}(t-u) \leq \overline{h}(u)$ *for all* $u \in [0,t]$, *then we have*

$$(\overline{f} *_1 \overline{h})(t) = \left[\sup_{\alpha\in[0,t]} \alpha \odot_l \mu_{[0,t],f_l}(\alpha), \sup_{\alpha\in[0,t]} \alpha \odot_r \mu_{[0,t],f_r}(\alpha)\right] \tag{32}$$

and

$$(\overline{f} *_2 \overline{h})(t) = \left[\sup_{\alpha\in[0,t]} \alpha \odot \mu_{[0,t],f_l}(\alpha), \sup_{\alpha\in[0,t]} \alpha \odot \mu_{[0,t],f_r}(\alpha)\right]. \tag{33}$$

(2) *If* $\overline{f} = [f_l, f_r], \overline{h} = [h_l, h_r] \in I\mathfrak{F}(X)^*$ *and* $t \in [0,\infty)$ *and* $\otimes_l = \otimes_r$ *is multiplication operation(·) and* $\overline{f}(x) = [c,d] \in I([0,\infty))$ *for all* $x \in [0,\infty)$, *then we have*

$$(\overline{f} *_1 \overline{h})(t) = \left[\sup_{\alpha\in[0,t]} \alpha \odot_l \mu_{[0,t],h_l}\left(\frac{\alpha}{c}\right), \sup_{\alpha\in[0,t]} \alpha \odot_r \mu_{[0,t],f_r}\left(\frac{\alpha}{d}\right)\right] \tag{34}$$

and

$$(\overline{f} *_2 \overline{h})(t) = \left[\sup_{\alpha\in[0,t]} \alpha \odot \mu_{[0,t],h_l}\left(\frac{\alpha}{c}\right), \sup_{\alpha\in[0,t]} \alpha \odot \mu_{[0,t],f_r}\left(\frac{\alpha}{d}\right)\right]. \tag{35}$$

(3) *If* $\overline{f} = [f_l, f_r], \overline{h} = [h_l, h_r] \in I\mathfrak{F}(X)^*$ *and* $t \in [0,\infty)$ *and* $\{t-x|\overline{f}(x) > [0,0]\} \cap \{x|\overline{h}(x) > [0,0]\} = \emptyset$ *and* $\overline{a} \bigotimes [0,0] = [0,0]$ *for all* $\overline{a} \in I([0,t])$, *then we have*

$$(\overline{f} *_1 \overline{h})(t) = 0 \tag{36}$$

and

$$(\overline{f} *_2 \overline{h})(t) = 0. \tag{37}$$

(4) *If* $\overline{f} = [f_l, f_r], \overline{h} = [h_l, h_r] \in I\mathfrak{F}(X)^*$ *and* $t \in [0,\infty)$ *and* $\mu(\{u \in [0,t] | f_l(t-u) \otimes_l h_l(u) > \alpha\}) = g_l(\alpha)$ *and* $\mu(\{u \in [0,t] | f_r(t-u) \otimes_r h_r(u) > \alpha\}) = g_r(\alpha)$ *for all* $\alpha \in [0,\infty)$*, then we have*

$$(\overline{f} *_1 \overline{h})(t) = [\sup_{\alpha\in[0,t]} \alpha \odot_l g_l(\alpha), \sup_{\alpha\in[0,t]} \alpha \odot_r g_r(\alpha)] \tag{38}$$

and

$$(\overline{f} *_2 \overline{h})(t) = [\sup_{\alpha\in[0,t]} \alpha \odot g_l(\alpha), \sup_{\alpha\in[0,t]} \alpha \odot g_r(\alpha)]. \tag{39}$$

Proof.(1) Suppose that $\otimes_l = \otimes_r$ are minimum operation(min) and $\overline{f}(t-u) \le \overline{h}(u)$ for all $u \in [0,t]$. Then we have $f_l(t-u) \le h_l(u)$ and $f_r(t-u) \le h_r(u)$ for all $u \in [0,t]$. Thus, by Theorem 4.1(1) and Theorem 3.1 (1), we have

$$\begin{aligned}(\overline{f} *_1 \overline{h})(t) &= [(f_l *_{1l} h_l)(t), (f_r *_{1l} h_r)(t)] \\ &= \left[(G)\int_{[0,t]}^{\odot_l} f_l \otimes_l h_l d\mu, (G)\int_{[0,t]}^{\odot_r} f_r \otimes_r h_r d\mu\right] \\ &= \left[\sup_{\alpha\in[0,t]} \alpha \odot_l \mu_{[0,t],f_l}(\alpha), \sup_{\alpha\in[0,t]} \alpha \odot_r \mu_{[0,t],f_r}(\alpha)\right].\end{aligned} \tag{40}$$

By Theorem 4.1(2) and Theorem 3.1 (1), we have

$$\begin{aligned}(\overline{f} *_2 \overline{h})(t) &= [(f_l *_{2l} h_l)(t), (f_r *_{2l} h_r)(t)] \\ &= \left[(G)\int_{[0,t]}^{\odot} f_l \otimes_l h_l d\mu, (G)\int_{[0,t]}^{\odot} f_r \otimes_r h_r d\mu\right] \\ &= \left[\sup_{\alpha\in[0,t]} \alpha \odot \mu_{[0,t],f_l}(\alpha), \sup_{\alpha\in[0,t]} \alpha \odot \mu_{[0,t],f_r}(\alpha)\right].\end{aligned} \tag{41}$$

(2) Suppose that $\overline{f}(x) = [c,d] \in I([0,\infty))$ for all $x \in [0,\infty)$. By Theorem 3.1 (2) and Theorem 4.1 (1), we have

$$\begin{aligned}(\overline{f} *_1 \overline{h})(t) &= [(f_l *_{1l} h_l)(t), (f_r *_{1l} h_r)(t)] \\ &= \left[(G)\int_{[0,t]}^{\odot_l} f_l(t-u)\cdot h_l(u)d\mu(u), (G)\int_{[0,t]}^{\odot_r} f_r(t-u)\cdot h_r(u)d\mu(u)\right] \\ &= \left[(G)\int_{[0,t]}^{\odot_l} c\cdot h_l(u)d\mu(u), (G)\int_{[0,t]}^{\odot_r} d\cdot h_r(u)d\mu(u)\right] \\ &= \left[\sup_{\alpha\in[0,t]} \alpha \odot_l \mu_{[0,t],h_l}\left(\frac{\alpha}{c}\right), \sup_{\alpha\in[0,t]} \alpha \odot_r \mu_{[0,t],h_r}\left(\frac{\alpha}{d}\right)\right].\end{aligned} \tag{42}$$

By Theorem 3.1 (2) and Theorem 4.1 (2), we have

$$\begin{aligned}(\overline{f} *_2 \overline{h})(t) &= [(f_l *_{2l} h_l)(t), (f_r *_{2l} h_r)(t)] \\ &= \left[(G)\int_{[0,t]}^{\odot} f_l(t-u)\cdot h_l(u)d\mu(u), (G)\int_{[0,t]}^{\odot} f_r(t-u)\cdot h_r(u)d\mu(u)\right] \\ &= \left[(G)\int_{[0,t]}^{\odot} c\cdot h_l(u)d\mu(u), (G)\int_{[0,t]}^{\odot} d\cdot h_r(u)d\mu(u)\right] \\ &= \left[\sup_{\alpha\in[0,t]} \alpha \odot \mu_{[0,t],h_l}\left(\frac{\alpha}{c}\right), \sup_{\alpha\in[0,t]} \alpha \odot \mu_{[0,t],h_r}\left(\frac{\alpha}{d}\right)\right].\end{aligned} \tag{43}$$

(3) Suppose that $\{t-x|\overline{f}(x)>[0,0]\}\cap\{x|\overline{h}(x)>[0,0]\}=\emptyset$ and $\overline{a}\bigotimes[0,0]=[0,0]$ for all $\overline{a}\in I([0,t])$. Then we have that $\{t-x|f_l(x)>0\}\cap\{x|h_l(x)>0\}=\emptyset$ and $a_l\otimes 0=0$ for all $a_l\in[0,t]$, and $\{t-x|f_r(x)>0\}\cap\{x|h_r(x)>0\}=\emptyset$ and $a_r\otimes 0=0$ for all $a_r\in[0,t]$. By Theorem 3.1(3), we have

$$(f_l *_{1l} h_l)(t)=0 \text{ and } (f_r *_{1r} h_r)(t)=0 \tag{44}$$

and

$$(f_l *_{2l} h_l)(t)=0 \text{ and } (f_r *_{2r} h_r)(t)=0. \tag{45}$$

By (44) and Theorem 4.1(1), we have

$$(\overline{f} *_1 \overline{h})(t) = [(f_l *_{1l} h_l)(t),(f_r *_{1r} h_r)(t)]=0. \tag{46}$$

By (45) and Theorem 4.1(2), we have

$$(\overline{f} *_2 \overline{h})(t) = [(f_l *_{2l} h_l)(t),(f_r *_{2r} h_r)(t)]=0. \tag{47}$$

(4) Suppose that f $\overline{f}=[f_l,f_r],\overline{h}=[h_l,h_r]\in I\mathfrak{F}(X)$ and $t\in[0,\infty)$ and $\mu(\{u\in[0,t]|f_l(t-u)\otimes_l h_l(u)>\alpha\})=g_l(\alpha)$ and $\mu(\{u\in[0,t]|f_r(t-u)\otimes_r h_r(u)>\alpha\})=g_r(\alpha)$ for all $\alpha\in[0,\infty)$. By Theorem 3.1 (4), we have

$$(f_l *_{1l} h_l)(t)=\sup_{\alpha\in[0,t]}\alpha\odot_l g_l(\alpha) \text{ and } (f_r *_{1r} h_r)(t)=\sup_{\alpha\in[0,t]}\alpha\odot_r g_r(\alpha), \tag{48}$$

and

$$(f_l *_{2l} h_l)(t)=\sup_{\alpha\in[0,t]}\alpha\odot g_l(\alpha) \text{ and } (f_r *_{2r} h_r)(t)=\sup_{\alpha\in[0,t]}\alpha\odot g_r(\alpha). \tag{49}$$

By (48) and Theorem 4.1(1), we have

$$\begin{aligned}(\overline{f} *_1 \overline{h})(t) &= [(f_l *_{1l} h_l)(t),(f_r *_{1r} h_r)(t)]\\ &= [\sup_{\alpha\in[0,t]}\alpha\odot_l g_l(\alpha),\ \sup_{\alpha\in[0,t]}\alpha\odot_r g_r(\alpha)].\end{aligned} \tag{50}$$

By (49) and Theorem 4.1(2), we have

$$\begin{aligned}(\overline{f} *_2 \overline{h})(t) &= [(f_l *_{2l} h_l)(t),(f_r *_{2r} h_r)(t)]\\ &= [\sup_{\alpha\in[0,t]}\alpha\odot g_l(\alpha),\ \sup_{\alpha\in[0,t]}\alpha\odot g_r(\alpha)].\end{aligned} \tag{51}$$

Theorem 4.3. *Let $(I([0,\infty)),\bigotimes=[\otimes_l,\otimes_r])$ be a standard interval-valued semigroup and e_l be a unit element with respect to $\otimes_l$ and and e_r be a unit element with respect to $\otimes_r$. If $\overline{f}\in I\mathfrak{F}(X)^*$, then we have*

$$(\overline{e} *_1 \overline{f})(t)=(IG)\int_{[0,t]}^{\bigotimes_I}\overline{f}d\mu \tag{52}$$

and

$$(\overline{e} *_2 \overline{f})(t)=(IG)\int_{[0,t]}^{\bigotimes_{II}}\overline{f}d\mu \tag{53}$$

where $\overline{e}=[e_l,e_r]$.

Proof. By Theorem 3.2, we have

$$(e_l *_{1l} f_l)(t)=(G)\int_{[0,t]}^{\odot_l}f_l d\mu \text{ and } (e_r *_{1r} f_r)(t)=(G)\int_{[0,t]}^{\odot_r}f_r d\mu \tag{54}$$

and

$$(e_l *_{2l} f_l)(t) = (G)\int_{[0,t]}^{\odot} f_l d\mu \text{ and } (e_r *_{2r} f_r)(t) = (G)\int_{[0,t]}^{\odot} f_r d\mu. \tag{55}$$

By Theorem 4.1(1) and (54), we have

$$\begin{aligned}
(\overline{e} *_1 \overline{f})(t) &= [e_l *_{1l} f_l, e_r *_{1r} f_r] \\
&= \left[(G)\int_{[0,t]}^{\odot_l} e_l \otimes_l f_l d\mu, (G)\int_{[0,t]}^{\odot_r} e_r \otimes_r f_r d\mu\right] \\
&= \left[(G)\int_{[0,t]}^{\odot_l} f_l d\mu, (G)\int_{[0,t]}^{\odot_r} f_r d\mu\right] \\
&= (IG)\int_{[0,t]}^{\bigotimes_I} \overline{f} d\mu.
\end{aligned} \tag{56}$$

By (55) and Theorem 4.1(2), we have

$$\begin{aligned}
(\overline{e} *_2 \overline{f})(t) &= [e_l *_{2l} f_l, e_r *_{2r} f_r] \\
&= \left[(G)\int_{[0,t]}^{\odot} e_l \otimes_l f_l d\mu, (G)\int_{[0,t]}^{\odot} e_r \otimes_r f_r d\mu\right] \\
&= \left[(G)\int_{[0,t]}^{\odot} f_l d\mu, (G)\int_{[0,t]}^{\odot} f_r d\mu\right] \\
&= (IG)\int_{[0,t]}^{\bigotimes_{II}} \overline{f} d\mu.
\end{aligned} \tag{57}$$

Remark 4.4. *A function $\overline{f} : X \longrightarrow I([0,\infty))$ is an interval-valued idempotent with respect to the standard interval-valued generalized pseudo-convolution $*_i$ (for $i = 1,2$) induced by a standard interval-valued semigroup $(I([0,\infty)), \bigotimes)$ if and only if $\overline{f} *_i \overline{f} = \overline{f}$ for $i = 1,2$. It is easy to see that if $\overline{e} = [e_l, e_r]$ is a unit element as in Theorem 4.2, that is, $\overline{f} *_i E = \overline{f}$ for all $\overline{f} \in I\mathfrak{F}(X)^*$, then we also have $\overline{e} *_i \overline{e} = \overline{e}$ for $i = 1,2$. Therefore, $\overline{e}$ is an interval-valued idempotent with respect to $*_i$ for $i = 1,2$.*

Example 4.1. Suppose that $\odot_l = \odot_r = \odot$ and $u \odot v = \min\{u, v\}$ and $u \otimes_l v = u \otimes_r v = u \cdot v$ for all $u, v \in [0,\infty)$, and $\overline{f}(x) = [1,2]$ and $\overline{h}(x) = [x^2, 2x^2]$ for all $x \in [0,\infty)$, and m be the Lebesgue measure on $[0,\infty)$. If $\mu = m^2$, then clearly μ is a fuzzy measure. Thus, we have

$$\begin{aligned}
\mu_{[0,t], f_l(t-\cdot)\otimes_l h_l(\cdot)}(\alpha) &= \mu(\{u \in [0,t] | 1 \otimes u^2 > \alpha\}) \\
&= \mu([\sqrt{\alpha}, t]) = (t - \sqrt{\alpha})^2
\end{aligned} \tag{58}$$

and

$$\begin{aligned}
\mu_{[0,t], f_r(t-\cdot)\otimes_r h_r(\cdot)}(\alpha) &= \mu(\{u \in [0,t] | 2 \otimes 2u^2 > \alpha\}) \\
&= \mu\left(\left[\frac{\sqrt{\alpha}}{2}, t\right]\right) = \left(t - \frac{\sqrt{\alpha}}{2}\right)^2.
\end{aligned} \tag{59}$$

By (58) and Theorem 4.1(1), we have

$$\begin{aligned}
&(\overline{f} *_1 \overline{h})(t) \\
= &\left[\sup_{\alpha\in[0,t]} \min\left\{\alpha, \mu_{[0,t], f_l(t-\cdot)\otimes_l h_l(\cdot)}(\alpha)\right\}, \sup_{\alpha\in[0,t]} \min\left\{\alpha, \mu_{[0,t], f_r(t-\cdot)\otimes_r h_r(\cdot)}(\alpha)\right\}\right]
\end{aligned}$$

$$\begin{aligned}
&= \left[\sup_{\alpha \in [0,t]} \min \left\{ \alpha, (t-\sqrt{\alpha})^2 \right\}, \sup_{\alpha \in [0,t]} \min \left\{ \alpha, \left(t - \frac{\sqrt{\alpha}}{2} \right)^2 \right\} \right] \\
&= \left[\frac{t^2}{4}, 4t^2 \right].
\end{aligned}$$

5. Conclusions

This study was to define the generalized pseudo-convolution of integrable functions by means of the (G) fuzzy intgeral(see Definition 3.1) and to investigate some properties and an example of the generalized pseudo-convolution on $\mathfrak{F}(X)^*$ in Theorems 3.2, 3.3 and Example 3.1.

By using the concept of an interval-representable pseudo-multiplication(see Definitions 2.5 and 2.6), we can define a standard interval-valued semigroup (see Definition 4.1) and the interval-representable generalized pseudo-convolution on $I\mathfrak{F}(X)^*$(see definition 4.2). From Theorems 4.3, 4.4, and 4.5, we investigate some characterizations of the interval-representable generalized pseudo-convolution of integrable interval-valued functions.

Furthermore, some applications of the interval-representable generalized pseudo-convolution are focused on various transform operations including pseudo-Laplace transform. For this reason, the future work can also be directed to interval-representable generalized pseudo-transform operations by means of the (IG) fuzzy integral.

Acknowledgement: This paper was supported by Wonkwang University in 2014.

References

[1] J.P. Aubin, *Set-valued Analysis*, *Birkhauser Boston,* (1990).

[2] R.J. Aumann, *Integrals of set-valued functions*, *J. Math. Anal. Appl.,* **12** (1965), 1-12.

[3] F. Baccelli, G. Cohen, G.J. Olsder, *Qudrat, synechronization and linearity: an algebra for discrete event systems*, Wiley, NewYork, 1992.

[4] G. Beliakov, H. Bustince, D.P. Goswami, U.K. Mukherjee, *On averaging operators for Atanassov's intuitionistic fuzzy sets*, *Information Sciences*, **181** (2011), 1116-1124.

[5] P. Benvenuti, R. Mesiar, D. Vivona, *Monotone set functions-based integrals Handbook of measure theory,* **Volume II**, 2011, 1116-1124.

[6] P. Benvenuti, R. Mesiar *Pseudo-arithmetical operations as a basis for the general measure and integration theory*, *Information Sciences*, **160** (2004), 1-11.

[7] G. Deschrijver, *Generalized arithemetic operators and their relatioship to t-norms in interval-valued fuzzy set theory*, *Fuzzy Sets and Systems*, **160** (2009), 3080-3102.

[8] J. Fang, *A note on the convergence theorem of generalized fuzzy integrals*, *Fuzzy Sets and Systems,* **127** (2002), 377-381.

[9] J. Fang, *On the convergence theorems of generalized fuzzy integral sequence*, *Fuzzy Sets and Systems,* **124** (2001), 117-123.

[10] J. Fang, *Some properties of sequences of generalized fuzzy integrable functions*, *Fuzzy Sets and Systems,* **158** (2007), 1832-1842.

[11] M. Grabisch, *Fuzzy integral in multicriteria decision making*, *Fuzzy Sets and Systems,* **69**(1995), 279-298.

[12] M. Ha, C. Wu, *Fuzzy measurs and integral theory, Science Press, Beijing,* 1998.

[13] L.C. Jang, B.M. Kil, Y.K. Kim, J.S. Kwon, *Some properties of Choquet integrals of set-valued functions*, *Fuzzy Sets and Systems,* **91** (1997), 61-67.

[14] L.C. Jang, J.S. Kwon, *On the representation of Choquet integrals of set-valued functions and null sets*, *Fuzzy Sets and Systems,* **112** (2000), 233-239.

[15] L.C. Jang, T. Kim, J.D. Jeon,*On the set-valued Choquet integrals and convergence theorems(II), Bull. Korean Math. Soc.* **40(1)** (2003), 139-147.

[16] L.C. Jang, *Interval-valued Choquet integrals and their apllications*, *J. Appl. Math. and Computing,* **16(1-2)** (2004), 429-445.

[17] L.C. Jang, *A note on the monotone interval-valued set function defined by the interval-valued Choquet integral*, *Commun. Korean Math. Soc.,* **22** (2007), 227-234.
[18] L.C. Jang, *On properties of the Choquet integral of interval-valued functions*, *Journal of Applied Mathematics,* **2011** (2011), Article ID 492149, 10pages.
[19] L.C. Jang, *A note on convergence properties of interval-valued capacity functionals and Choquet integrals*, *Information Sciences,* **183** (2012), 151-158.
[20] L.C. Jang, *A note on the interval-valued generalized fuzzy integral by means of an interval-representable pseudo-multiplication and their convergence properties*, *Fuzzy Sets and Systems,* **222** (2013), 45-57.
[21] L.C. Jang, *Some characterizations of the Choquet integral with respect to a monotone interval-valued set function*, *International Journal of Fuzzy Logic and Intelligent Systems,* **13(1)** (2013), 75-81.
[22] E.P. Klement, R. Mesiar, E. Pap, *Triangular norms*, Kluwer Academic Publishers, Dordrecht, 2000.
[23] V.P. Maslov, S.N. Samborskij, *Idempotent Analysis, Advances in Soviet Mathematics 13*, Amer. Math. Soc. Providence, Rhode Island, 1992.
[24] R. Mesiar, J. Rybarisk, *PAN-operations structure*, *Fuzzy Sets and Systems,* **74** (1995), 365-369.
[25] T. Murofushi, M. Sugeno, *A theory of fuzzy measures: representations, the Choquet integral, and null sets*, *J. Math. Anal. Appl.,* **159** (1991), 532-549.
[26] T. Murofushi, M. Sugeno, M. Suzaki, *Autocontinuity, convergence in measure, and convergence indistribution, Fuzzy Sets and Systems,* **92**(1997) 197-203.
[27] E. Pap, *Null-additive set functions*, Kluwer Academic Publishers,Dordrecht,1995.
[28] E. Pap, I. Stajner, *Generalized pseudo-convolution in the theory of probabilistic metric spaces, information, fuzzy numbers, optimatation, system theory* , *Fuzzy Sets and Systems,* **102**(1999),393-415.
[29] D.A. Ralescu, G. Adams, *The fuzzy integral*, *J. Math. Anal. Appl.,* **75(2)** (1980), 562-570.
[30] H. Schjear-Jacobsen, *Representation and calculation of economic uncertains: intervals, fuzzy numbers and probabilities*, *Int. J. of Production Economics,* **78**(2002), 91-98.
[31] N. Shilkret, *Maxitive measures and integration*, *Indag Math,* **33**(1971), 109-116.
[32] M. Sugeno, *Theory of fuzzy integrals and its applications*, *Doctorial Thesis, Tokyo Institute of Techonology, Tokyo*,(1974).
[33] M. Sugeno, T. Murofushi, *Pseudo-additive measures and integrals*, *J. Math. Anal. Appl.*, **122**(1987), 197-222.
[34] K. Wechselberger, *The theory of interval-probability as a unifying concept for uncertainty*, *Int. J. Approximate Reasoning,* **24**(2000), 149-170.
[35] C. Wu, S. Wang, M. Ma, *Generalized fuzzy integrals: Part1. Fundamental concept*, *Fuzzy Sets and Systems,* **57**(1993), 219-226.
[36] C. Wu, M. Ma, S. Song, S. Zhang *Generalized fuzzy integrals: Part3. convergence theorems*, *Fuzzy Sets and Systems*, *Fuzzy Sets and Systems,* **70** (1995), 75-87.
[37] Q. Xie, J. Fang, *Corrections and remarks to the paper in Fuzzy Sets and Systems 124(2001) 117-123*, *Fuzzy Sets and Systems*, *Fuzzy Sets and Systems,* **157**(2006), 699-704.
[38] D. Zhang, C. Guo, D. Lin, *Set-valued Choquet integrals revisited*, *Fuzzy Sets and Systems,* **147**(2004), 475-485.
[39] D. Zhang, C. Guo, *On the convergence of sequences of fuzzy measures and generalized convergences theorems of fuzzy integral*, *Fuzzy Sets and Systems,* **72**(1995), 349-356.
[40] D. Zhang, Z. Wang, *Fuzzy integrals of fuzzy-valued functions*, *Fuzzy Sets and Systems,* **54** (1993), 63-67.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 21, NO.6, 2016, COPYRIGHT 2016 EUDOXUS PRESS, LLC

Fixed point and coupled fixed point theorems for generalized cyclic weak contractions in partially ordered probabilistic metric spaces

Chuanxi Zhu, Wenqing Xu†

Department of Mathematics, Nanchang University, Nanchang, 330031, P. R. China

Abstract. In this paper, we introduce the concept of new generalized cyclic weak contraction mappings and prove a class of fixed point theorems for such mappings in partially ordered probabilistic metric spaces. In addition, we also establish a coupled fixed point for mixed monotone mappings under contractive conditions in partially ordered probabilistic metric spaces. Our results extend and generalize Harjani *et al.* (Nonlinear anal. 71(2009)3403-3410) and Wu (Fixed Point Theory Appl. 2014(2014)49). Also, we introduce an example to support the validity of our results. Finally, an application of our results extends fixed point theorems for generalized weak contraction mappings in ordered metric spaces.

Keywords: Menger probabilistic metric space; partially ordered; cyclic weak contractions; fixed point
MR Subject Classification: 47H10, 34B15, 46S50

1 Introduction and preliminaries

Fixed point theory in metric spaces is an important banch of nonlinear analysis, which is closely related to the existence and uniqueness of solutions of differential and integral equations. The celebrated Banach's contraction mapping principle is one of the cornerstones in development of nonlinear analysis.

In the past years, Kirk and Srinvasan [1] presented fixed point theorems for mappings satisfying cyclical contractive conditions. Ran and Reurings [2] introduced fixed point theorems of Banach contraction operator in partially ordered metric spaces. Agarwal *et al.* [3] proved fixed point results of generalized contractive operators in partially ordered metric spaces; Harjani and Sadarangani [4] presented some fixed point theorems for weakly contractive mappings in complete metric spaces endowed with a partial order. Shatanwi [5] introduced nonlinear weakly C-contractive mappings in ordered metric spaces and proved some fixed point theorems. For more detail on fixed point theory and related results, we refer to [6-12] and the references therein.

In 1942, Menger [13] introduced the concept of probabilistic metric spaces, a number of authors have done considerable works on probabilistic metric spaces [14-19]. Recently, the extension of fixed point theory to generalized structures as partially ordered probabilistic metric spaces has received much attention (see, [20-22]).

†To whom correspondence should be addressed. E-mail:wen_qing_xu@163.com(W. Xu).

†This work has been supported by the National Natural Science Foundation of China (11361042,11071108), the Provincial Natural Science Foundation of Jiangxi, China (20132BAB201001,2010GZS0147).

However, we rarely see any work about fixed point theorems for mappings under weakly contractive conditions in partially ordered probabilistic metric spaces.

The aim of this paper is to determine some fixed point theorems for generalized cyclic weak contractions in the framework of partially ordered probabilistic metric spaces. Also, we introduce an example to support the validity of our results. Our results extend and generalize the main results of [3-8,11-12].

We introduce some useful concepts and lemmas for the development of our results.

Let R denote the set of reals and R^+ the nonnegative reals. A mapping $F : R \to R^+$ is called a distribution function if it is nondecreasing and left continuous with $\inf_{t\in R} F(t) = 0$ and $\sup_{t\in R} F(t) = 1$. We will denote by D the set of all distribution functions and $D^+ = \{F \in D : F(t) = 0, t \leq 0\}$.

Let H denote the specific distribution function defined by

$$H(x) = \begin{cases} 0, & x \leq 0; \\ 1, & x > 0. \end{cases}$$

Definition 1.1 ([14]). The mapping $\Delta : [0,1] \times [0,1] \to [0,1]$ is called a triangular norm (for short, a t-norm) if the following conditions are satisfied:
$(\Delta-1)$ $\Delta(a,1) = a$, for all $a \in [0,1]$;
$(\Delta-2)$ $\Delta(a,b) = \Delta(b,a)$;
$(\Delta-3)$ $\Delta(a,b) \leq \Delta(c,d)$, for $c \geq a, d \geq b$;
$(\Delta-4)$ $\Delta(a,\Delta(b,c)) = \Delta(\Delta(a,b),c)$.

Two typical examples of continuous t-norm are $\Delta_1(a,b) = \max\{a+b-1, 0\}$ and $\Delta_2(a,b) = ab$, for all $a, b \in [0,1]$.

Definition 1.2 ([14]). A triplet (X, F, Δ) is called a Menger probabilistic metric space (for short, Menger PM-space), if X is a nonempty set, Δ is a t-norm and F is a mapping from $X \times X \to D^+$ satisfying the following conditions (for $x, y \in X$, we denote $F(x,y)$ by $F_{x,y}$):
(MS-1) $F_{x,y}(t) = H(t)$, for all $t \in R$, if and only if $x = y$;
(MS-2) $F_{x,y}(t) = F_{y,x}(t)$, for all $x, y \in X$ and $t \in R$;
(MS-3) $F_{x,z}(s+t) \geq \Delta(F_{x,y}(s), F_{y,z}(t))$, for all $x, y, z \in X$ and $s, t \geq 0$.

Definition 1.3 ([15]). (X, F, Δ) is called a non-Archimedean Menger PM-space (shortly, a N.A Menger PM-space), if (X, F, Δ) is a Menger PM-space and Δ satisfies the following condition: for all $x, y, z \in X$ and $t_1, t_2 \geq 0$,

$$F_{x,z}(\max\{t_1, t_2\}) \geq \Delta(F_{x,y}(t_1), F_{y,z}(t_2)). \tag{1.1}$$

Definition 1.4 ([15]). A non-Archimedean Menger PM-space (X, F, Δ) is said to be type $(D)_g$ if there exists a $g \in \Omega$ such that

$$g(\Delta(s,t)) \leq g(s) + g(t),$$

for all $s, t \in [0,1]$, where $\Omega = \{g : g : [0,1] \to [0,\infty)$ is continuous, strictly decreasing, $g(1) = 0\}$.

Example 1.1 . (X, F, Δ) is a N.A Menger PM-space, and $\Delta \geq \Delta_1$, where $\Delta_1(s,t) = \max\{s+t-1, 0\}$, then (X, F, Δ) is of $(D)_g$-type for $g \in \Omega$ defined by $g(t) = 1 - t$.

Remark 1.1 Schweizer and Sklar [14] point out that if (X, F, Δ) is a Menger probabilistic metric space and Δ is continuous, then (X, F, Δ) is a Hausdorff topological space in the (ε, λ)-topology T, i.e., the family of sets $\{U_x(\varepsilon, \lambda) : \varepsilon > 0, \lambda \in (0, 1]\}$ $(x \in X)$ is a basis of neighborhoods of a point x for T, where $U_x(\varepsilon, \lambda) = \{y \in X : F_{x,y}(\varepsilon) > 1 - \lambda\}$.

Lemma 1.1 ([15]). Let $\{x_n\}$ be a sequence in X such that $\lim\limits_{n\to\infty} F_{x_n,x_{n+1}}(t) = 1$ for all $t > 0$. If the sequence $\{x_n\}$ is not a Cauchy sequence in X, then there exist $\varepsilon_0 > 0$, $t_0 > 0$ and two sequences $\{k(i)\}$, $\{m(i)\}$ of positive integers such that
(1) $m(i) > k(i)$, and $m(i) \to \infty$ as $i \to \infty$;
(2) $F_{x_{m(i)},x_{k(i)}}(t_0) < 1 - \varepsilon_0$ and $F_{x_{m(i)-1},x_{k(i)}}(t_0) \geq 1 - \varepsilon_0$, for $i = 1, 2, \cdots$.

Definition 1.5 ([1]). Let X be a non-empty set, m be a positive integer, $A_1, A_2, \ldots, A_m$ be subsets of X, $Y = \cup_{i=1}^m A_i$ and a mapping $f : Y \to Y$. Then Y is said to be a cyclic representation of Y with respect to f, if
(i) A_i, $i = 1, 2, \ldots, m$, are nonempty closed sets;
(ii) $f(A_1) \subseteq A_2$, $\ldots$, $f(A_{m-1}) \subseteq A_m$, $f(A_m) \subseteq A_1$.

Example 1.2 Let $X = R^+$. Let $A_1 = [0, 2]$, $A_2 = [\frac{1}{2}, \frac{3}{2}]$, $A_3 = [\frac{3}{4}, \frac{5}{4}]$, and $Y = \bigcup_{i=1}^3 A_i$. Defined $f : Y \to Y$ by $fx = \frac{1}{2} + \frac{1}{2}x$, for all $x \in Y$.

Clearly $Y = \bigcup_{i=1}^3 A_i$ is a cyclic representation of Y with respect to f.

Definition 1.6 ([9]). The function $h : [0, \infty) \to [0, \infty)$ is called an altering distance function, if the following properties are satisfied: (a) h is continuous and nondecreasing; (b) $h(t) = 0$ if and only if $t = 0$.

In [10], Bhasker and Lakshmikantham introduced the concepts of mixed monotone mappings and coupled fixed point.

Definition 1.7 ([10]). Let $(X, \leq)$ be a partially ordered set and $A : X \times X \to X$. The mapping A is said to have the mixed monotone property if A is monotone nondecreasing in its first argument and is monotone nonincreasing in its second argument, that is, for any $x, y \in X$,

$$\begin{aligned} x_1, x_2 \in X, \quad x_1 \leq x_2 \quad &\Longrightarrow \quad A(x_1, y) \leq A(x_2, y), \\ y_1, y_2 \in X, \quad y_2 \leq y_1 \quad &\Longrightarrow \quad A(x, y_1) \leq A(x, y_2). \end{aligned}$$

Definition 1.8 ([10]). An element $(x, y) \in X^2$ is said to be a coupled fixed point of the mapping $A : X^2 \to X$ if $A(x, y) = x$ and $A(y, x) = y$.

For $\tilde{a} = (x, y), \tilde{b} = (u, v) \in X^2$, we introduce a distribution function $\tilde{F}$ from X^2 into D^+ defined by

$$\tilde{F}_{\tilde{a},\tilde{b}}(t) = \min\{F_{x,u}(t), F_{y,v}(t)\}, \text{ for all } t > 0.$$

In [20], Wu proved the following results:

Lemma 1.2 ([20]). If (X, F, Δ) is a complete Menger PM space, then $(X^2, \tilde{F}, \Delta)$ is also a complete Menger PM space.

In the section 3 of this paper, we establish some coupled point theorems under contractive conditions in partially ordered probabilistic metric spaces. The obtained results extend and generalized the main results of [20-22]. Finally, we also obtain the corresponding fixed point theorems for generalized weak contraction mapping in ordered metric spaces.

2 Fixed point theorems for generalized cyclic weak contractions

We start with the definition of generalized cyclic weak contraction mappings in probabilistic metric spaces.

Definition 2.1 Let $(X, \leq)$ be a partially ordered set and (X, F, Δ) be a N.A Menger PM-space of type $(D)_g$. Let m be a positive integer, $A_1, A_2, \ldots, A_m$ be subsets of X, $Y = \cup_{i=1}^m A_i$. A mapping $T : X \to X$ is said to be a generalized cyclic weak contraction, if Y is a cyclic representation of Y with respect to T, $A_{m+1} = A_1$ and for $k \in \{1, 2, \ldots, m\}$, and for all $x, y \in X$, $x \in A_k$ and $y \in A_{k+1}$ are comparable with

$$h(g(F_{Tx,Ty}(t))) \leq h(M_t(x, y)) - \phi(M_t(x, y)), \quad \text{for all } t > 0, \tag{2.1}$$

where $M_t(x, y) = \max\{g(F_{x,y}(t)), g(F_{x,Tx}(t)), g(F_{y,Ty}(t)), \frac{1}{2}[g(F_{x,Ty}(t)) + g(F_{y,Tx}(t))]\}$, h is a altering distance function, $\phi : [0, \infty) \to [0, \infty)$ is a continuous function such that $\phi(s) = 0$ if and only if $s = 0$.

Theorem 2.1 Let $(X, \leq)$ be a partially ordered set and (X, F, Δ) be a complete N.A Menger PM-space of type $(D)_g$. Let m be a positive integer, $A_1, A_2, \ldots, A_m$ be subsets of X, $Y = \cup_{i=1}^m A_i$, $T : Y \to Y$ be a generalized cyclic weak contraction, and T be nondecreasing. Also assume that either
(a) T is continuous or,
(b) if a nondecreasing sequence $x_n \to x$, then $x_n \leq x$, for all $n \in N$.

If there exists $x_0 \in A_1$ such that $x_0 \leq Tx_0$, then T has a fixed point. Furthermore, the set of fixed points of T is well ordered if and only if T has a unique fixed point.

Proof. Since $T(A_1) \subseteq A_2$, there exists an $x_1 \in A_2$, such that $x_1 = Tx_0$. Since $T(A_2) \subseteq A_3$, there exists an $x_2 \in A_3$, such that $x_2 = Tx_1$. Continuing this process, we can construct a sequence $\{x_n\}$ such that $x_{n+1} = Tx_n$, for all $n \in N$, and there exists $i_n \in \{1, 2, \ldots, m\}$ such that $x_n \in A_{i_n}$ and $x_{n+1} \in A_{i_n+1}$.

Since $x_0 \leq Tx_0 = x_1$ and T is nondecreasing, we have $Tx_0 \leq Tx_1$, that is, $x_1 \leq x_2$. By induction, we get that $x_0 \leq x_1 \leq \cdots \leq x_n \leq \cdots$, for all $n \in N$.

Without loss of generality, assume that $x_{n+1} \neq x_n$, for all $n \in N$ (otherwise, $x_{n+1} = Tx_n = x_n$, then the conclusion holds).

Since $x_n \in A_{i_n}$ and $x_{n+1} \in A_{i_n+1}$ are comparable, for $i_n \in \{1, 2, \ldots, m\}$, by inequality (2.1), we get

$$h[g(F_{x_{n+1},x_n}(t))] \leq h[M_t(x_n, x_{n-1})] - \phi(M_t(x_n, x_{n-1})), \quad \text{for all } t > 0, \tag{2.2}$$

where

$$\begin{aligned}M_t(x_n,x_{n-1}) &= \max\{g(F_{x_n,x_{n-1}}(t)), g(F_{x_n,x_{n-1}}(t)), g(F_{x_n,x_{n+1}}(t)), \frac{1}{2}g(F_{x_{n-1},x_{n+1}}(t))\}\\ &\leq \max\{g(F_{x_n,x_{n-1}}(t)), g(F_{x_n,x_{n+1}}(t)), \frac{1}{2}g(\Delta(F_{x_{n-1},x_n}(t), F_{x_n,x_{n+1}}(t)))\}\\ &\leq \max\{g(F_{x_n,x_{n-1}}(t)), g(F_{x_n,x_{n+1}}(t)), \frac{1}{2}[g((F_{x_{n-1},x_n}(t)) + g(F_{x_n,x_{n+1}}(t))]\}\\ &= \max\{g(F_{x_n,x_{n-1}}(t)), g(F_{x_n,x_{n+1}}(t))\} = M_t(x_n,x_{n-1}).\end{aligned}$$

Suppose that $M_t(x_n,x_{n-1}) = g(F_{x_n,x_{n+1}}(t))$, by (2.2), we have

$$h[g(F_{x_{n+1},x_n}(t))] \leq h[g(F_{x_n,x_{n+1}}(t))] - \phi(g(F_{x_n,x_{n+1}}(t))), \quad \text{for all } t > 0,$$

which implies that $\phi(g(F_{x_n,x_{n+1}}(t))) = 0$. Thus, $g(F_{x_n,x_{n+1}}(t)) = 0$, that is, $F_{x_n,x_{n+1}}(t) = 1$ for all $t > 0$. Then $x_n = x_{n+1}$, which is in contradiction to $x_n \neq x_{n+1}$, for any $n \in N$.

Hence, $M_t(x_n,x_{n-1}) = g(F_{x_n,x_{n-1}}(t))$, it follows from (2.2) that

$$h[g(F_{x_{n+1},x_n}(t))] \leq h[g(F_{x_n,x_{n-1}}(t))] - \phi(g(F_{x_n,x_{n-1}}(t))) \leq h[g(F_{x_n,x_{n-1}}(t))], \quad \forall t > 0, \tag{2.3}$$

Since h is nondecreasing, it follows from (2.3) that $\{g(F_{x_{n+1},x_n}(t))\}$ is a decreasing sequence, for every $t > 0$. Hence, there exists $r_t \geq 0$ such that $\lim_{n\to\infty} g(F_{x_{n+1},x_n}(t)) = r_t$.

By using the continuities of h and ϕ, letting $n \to \infty$ in (2.3), we get $h(r_t) \leq h(r_t) - \phi(r_t)$, which implies that $\phi(r_t) = 0$. Then $r_t = 0$, that is, $\lim_{n\to\infty} g(F_{x_{n+1},x_n}(t)) = 0$ and $\lim_{n\to\infty} F_{x_{n+1},x_n}(t) = 1$, for all $t > 0$.

In the sequel, we will prove that $\{x_n\}$ is Cauchy sequence. To prove this fact, we first prove the following claim.

Claim: for every $t > 0$, $\varepsilon > 0$, there exists $n_0 \in N$, such that $p, q \geq n_0$ with $p - q \equiv 1 \mod m$ then $F_{x_p,x_q}(t) > 1 - \varepsilon$, that is, $g(F_{x_p,x_q}(t)) < g(1-\varepsilon)$.

In fact, suppose to the contrary, there exist $t_0 > 0$ and $\varepsilon_0 > 0$, such that for any $n \in N$, we can find $p(n) > q(n) \geq n$ with $p(n) - q(n) \equiv 1 \mod m$ satisfying $F_{x_{p(n)},x_{q(n)}}(t_0) \leq 1 - \varepsilon_0$, that is, $g(F_{x_{p(n)},x_{q(n)}}(t_0)) \geq g(1-\varepsilon_0)$.

Now, we take $n > 2m$. Then corresponding to $q(n) \geq n$, we can choose $p(n)$ in such a way that it is the smallest integer with $p(n) > q(n)$ satisfying $p(n) - q(n) \equiv 1 \mod m$ and $g(F_{x_{p(n)},x_{q(n)}}(t_0)) \geq g(1-\varepsilon_0)$. Therefore, $g(F_{x_{p(n)-m},x_{q(n)}}(t_0)) < g(1-\varepsilon_0)$. Using the non-Archimedean Menger triangular inequality and Definition 1.5, we have

$$\begin{aligned}g(1-\varepsilon_0) &\leq g(F_{x_{q(n)},x_{p(n)}}(t_0)) \leq g(\Delta(F_{x_{q(n)},x_{q(n)+1}}(t_0), F_{x_{q(n)+1},x_{p(n)}}(t_0)))\\ &\leq g(F_{x_{q(n)},x_{q(n)+1}}(t_0)) + g(F_{x_{q(n)+1},x_{p(n)}}(t_0))\\ &\leq g(F_{x_{q(n)},x_{q(n)+1}}(t_0)) + g(F_{x_{q(n)+1},x_{p(n)+1}}(t_0)) + g(F_{x_{p(n)+1},x_{p(n)}}(t_0))\\ &\leq 2g(F_{x_{q(n)},x_{q(n)+1}}(t_0)) + g(F_{x_{q(n)},x_{p(n)+1}}(t_0)) + g(F_{x_{p(n)+1},x_{p(n)}}(t_0))\\ &\leq 2g(F_{x_{q(n)},x_{q(n)+1}}(t_0)) + g(F_{x_{q(n)},x_{p(n)}}(t_0)) + 2g(F_{x_{p(n)+1},x_{p(n)}}(t_0))\\ &\leq 2g(F_{x_{q(n)},x_{q(n)+1}}(t_0)) + g(F_{x_{q(n)},x_{p(n)-m}}(t_0)) + g(F_{x_{p(n)-m},x_{p(n)}}(t_0)) + 2g(F_{x_{p(n)+1},x_{p(n)}}(t_0))\\ &\leq 2g(F_{x_{q(n)},x_{q(n)+1}}(t_0)) + g(1-\varepsilon_0) + \sum_{i=1}^{m} g(F_{x_{p(n)-i},x_{p(n)-i+1}}(t_0)) + 2g(F_{x_{p(n)+1},x_{p(n)}}(t_0)).\end{aligned} \tag{2.4}$$

Since $\lim_{n\to\infty} g(F_{x_{n+1},x_n}(t)) = 0$ for all $t > 0$, letting $n \to \infty$ in (2.4), we have

$$\begin{aligned} g(1-\varepsilon_0) &= \lim_{n\to\infty} g(F_{x_{q(n)},x_{p(n)}}(t_0)) = \lim_{n\to\infty} g(F_{x_{q(n)+1},x_{p(n)}}(t_0)) \\ &= \lim_{n\to\infty} g(F_{x_{q(n)+1},x_{p(n)+1}}(t_0)) = \lim_{n\to\infty} g(F_{x_{q(n)},x_{p(n)+1}}(t_0)). \end{aligned} \tag{2.5}$$

By $p(n) - q(n) \equiv 1 \mod m$, we know that $x_{p(n)}$ and $x_{q(n)}$ lie in different adjacently labeled sets A_i and A_{i+1}, for $1 \le i \le m$. Using the fact that T is a generalized cyclic weak contraction, we have

$$h[g(F_{x_{q(n)+1},x_{p(n)+1}}(t_0))] = h[g(F_{Tx_{q(n)},Tx_{p(n)}}(t_0))] \le h[M_{t_0}(x_{q(n)},x_{p(n)})] - \phi(M_{t_0}(x_{q(n)},x_{p(n)})), \tag{2.6}$$

where

$$\begin{aligned} M_{t_0}(x_{q(n)},x_{p(n)}) = \max\{&g(F_{x_{q(n)},x_{p(n)}}(t_0)), g(F_{x_{q(n)},x_{q(n)+1}}(t_0)), g(F_{x_{p(n)},x_{p(n)+1}}(t_0)), \\ &\frac{1}{2}[g(F_{x_{q(n)},x_{p(n)+1}}(t_0)) + g(F_{x_{p(n)},x_{q(n)+1}}(t_0))]\}. \end{aligned}$$

By (2.5), we have $\lim_{n\to\infty} M_{t_0}(x_{q(n)},x_{p(n)}) = \max\{g(1-\varepsilon_0), 0, 0, \frac{1}{2}[g(1-\varepsilon_0) + g(1-\varepsilon_0)]\} = g(1-\varepsilon_0)$. According to the continuities of h and ϕ, letting $n \to \infty$ in (2.6), we get

$$h[g(1-\varepsilon_0)] \le h[g(1-\varepsilon_0)] - \phi(g(1-\varepsilon_0)).$$

Thus, $\phi(g(1-\varepsilon)) = 0$, that is $g(1-\varepsilon_0) = 0$. Then $\varepsilon_0 = 0$, which is in contradiction to $\varepsilon_0 > 0$.

Therefore, our claim is proved. In the sequel, we will prove that $\{x_n\}$ is Cauchy sequence.

By the continuity of g and $g(1) = 0$, we have $\lim_{a\to 0^+} g(1-a\epsilon) = 0$, for any given $\varepsilon > 0$. Since g is strictly decreasing, then there exists $a > 0$ such that $g(1-a\varepsilon) \le \frac{g(1-\varepsilon)}{2}$.

For any given $t > 0$, $\varepsilon > 0$, there exists $a > 0$ such that $g(1-a\varepsilon) \le \frac{g(1-\varepsilon)}{2}$. By the claim, we find $n_0 \in N$ such that if $p, q \ge n_0$ with $p - q \equiv 1 \mod m$, then

$$F_{x_p,x_q}(t) > 1 - a\varepsilon, \quad \textit{and} \quad g(F_{x_p,x_q}(t)) < g(1-a\varepsilon) \le \frac{g(1-\varepsilon)}{2}. \tag{2.7}$$

Since $\lim_{n\to\infty} g(F_{x_{n+1},x_n}(t)) = 0$, we also find $n_1 \in N$ such that ree

$$g(F_{x_{n+1},x_n}(t)) \le \frac{g(1-\varepsilon)}{2m}, \tag{2.8}$$

for any $n > n_1$.

Suppose that $r, s \ge \max\{n_0, n_1\}$ and $s > r$. Then there exists $k \in \{1, 2, \ldots, m\}$ such that $s - r \equiv k \mod m$. Therefore, $s - r + j \equiv 1 \mod m$, for $j = m - k + 1$, $j \in \{0, 1, \ldots, m-1\}$. So, we have

$$g(F_{x_r,x_s}(t)) \le g(F_{x_r,x_{s+j}}(t)) + g(F_{x_{s+j},x_{s+j-1}}(t)) + \cdots + g(F_{x_{s+1},x_s}(t)).$$

From (2.7), (2.8) and the last inequality, we get

$$g(F_{x_r,x_s}(t)) < \frac{g(1-\varepsilon)}{2} + j \cdot \frac{g(1-\varepsilon)}{2m} \le \frac{g(1-\varepsilon)}{2} + \frac{g(1-\varepsilon)}{2} = g(1-\varepsilon). \tag{2.9}$$

Since g is strictly decreasing, by (2.9), we obtain $F_{x_r,x_s}(t) > 1 - \varepsilon$. Therefore $\{x_n\}$ is Cauchy sequence.

Since X is a complete PM-space, $Y = \cup_{i=0}^{m} A_i$ is closed, then Y also is a complete space. Thus there exists $x^* \in Y$ such that $x_n \to x^*$. As $Y = \cup_{i=1}^{m} A_i$ is a cyclic representation of Y with respect to T, then the sequence $\{x_n\}$ has infinite terms in each A_i for $i \in \{1, 2, \dots, m\}$.

First, suppose that $x^* \in A_i$, then $Tx^* \in A_{i+1}$, and we take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $x_{n_k} \in A_{i-1}$(the existence of this subsequence is guaranteed by above mentioned comment).

Case (a): If T is continuous. Since $\lim\limits_{n\to\infty} x_n = x^*$, we have $Tx^* = x^*$.

Case (b): If it satisfies a nondecreasing sequence $x_n \to x^*$, such that $x_n \le x^*$, then $x_{n_k} \in A_{i-1}$ and $x^* \in A_i$ are comparable. By (2.1), we have

$$h[g(F_{x_{n_k+1},Tx^*}(t))] = h[g(F_{Tx_{n_k},Tx^*}(t))] \le h[M_t(x_{n_k}, x^*)] - \phi(M_t(x_{n_k}, x^*)), \tag{2.10}$$

where

$$\begin{aligned} M_t(x_{n_k}, x^*) = \max\{&g(F_{x_{n_k},x^*}(t)), g(F_{x_{n_k},x_{n_{k+1}}}(t)), g(F_{x^*,Tx^*}(t)), \\ &\frac{1}{2}[g(F_{x_{n_k},Tx^*}(t)) + g(F_{x_{n_{k+1}},x^*}(t))]\}. \end{aligned}$$

Let G_0 be the set of all the discontinuous points of $F_{x^*,Tx^*}(t)$. Since g, h, and ϕ are continuous, we obtain that G_0 also is the set of all the discontinuous points of $g(F_{x^*,Tx^*}(t))$, $h[g(F_{x^*,Tx^*}(t))]$ and $\phi(g(F_{x^*,Tx^*}(t)))$. Moreover, we know that G_0 is a countable set. Let $G = R^+ \backslash G_0$. When $t \in G \backslash \{0\}$ (t is a continuity point of $F_{x^*,Tx^*}(t)$), we have

$$\lim_{k\to\infty} M_t(x_{n_k}, x^*) = \max\{0, 0, g(F_{x^*,Tx^*}(t)), \frac{1}{2}[g(F_{x^*,Tx^*}(t)) + 0]\} = g(F_{x^*,Tx^*}(t)).$$

Letting $n \to \infty$ in (2.10), we get

$$h[g(F_{x^*,Tx^*}(t))] \le h[g(F_{x^*,Tx^*}(t))] - \phi(g(F_{x^*,Tx^*}(t))).$$

Thus, $\phi(g(F_{x^*,Tx^*}(t))) = 0$, that is, $g(F_{x^*,Tx^*}(t)) = 0$. Then

$$F_{x^*,Tx^*}(t) = H(t), \quad \text{for all } t \in G. \tag{2.11}$$

When $t \in G_0$ with $t > 0$, by the density of real numbers, there exist $t_1, t_2 \in G$ such that $0 < t_1 < t < t_2$. Since the distribution is nondecreasing, we have

$$1 = H(t_1) = F_{x^*,Tx^*}(t_1) \le F_{x^*,Tx^*}(t) \le F_{x^*,Tx^*}(t_2) = 1.$$

This shows that, for all $t \in G_0$ with $t > 0$,

$$F_{x^*,Tx^*}(t) = H(t). \tag{2.12}$$

Combing (2.11) with (2.12), we have $F_{x^*,Tx^*}(t) = H(t)$, for all $t > 0$, that is, $Tx^* = x^*$.

Hence, in all case, we have $Tx^* = x^*$.

Finally, we prove the uniqueness of the fixed point under the additional conditions. In fact, suppose that there exist $x^*, y^* \in Y$ such that $Tx^* = x^*$, $Ty^* = y^*$, then we have $x^*, y^* \in \cap_{i=1}^{m} A_i$.

Since the set of fixed points of T is well ordered, we have $x^* \in A_i$ and $y^* \in A_{i+1}$ are comparable. By $(2,1)$, we have

$$h[g(F_{x^*,y^*}(t))] \leq h[M_t(x^*,y^*)] - \phi(M_t(x^*,y^*)), \quad \text{for all } t > 0,$$

where

$$M_t(x^*,y^*) = \max\{g(F_{x^*,y^*}(t)), g(F_{x^*,x^*}(t)), g(F_{y^*,y^*}(t)), \frac{1}{2}[g(F_{x^*,y^*}(t)) + g(F_{x^*,y^*}(t))]\} = g(F_{x^*,y^*}(t)).$$

Thus, $\phi(g(F_{x^*,y^*}(t))) = 0$, that is, $g(F_{x^*,y^*}(t)) = 0$. Hence, $F_{x^*,y^*}(t) = 1$, for all $t > 0$. Then $x^* = y^*$.

Remark 2.1 Theorem 2.1 generalizes and extends Theorem 2.1 in [6] and Theorem 2.4 in [7].

Corollary 2.1 Let $(X, \leq)$ be a partially ordered set and (X, F, Δ) be a complete N.A Menger PM-space, $T : X \to X$ be a nondecreasing mapping. Suppose that for comparable $x, y \in X$, we have

$$g(F_{Tx,Ty}(t)) \leq \Phi(\max\{g(F_{x,y}(t)), g(F_{x,Tx}(t)), g(F_{y,Ty}(t)), \frac{1}{2}[g(F_{x,Ty}(t)) + g(F_{y,Tx}(t))]\}),$$

for all $t > 0$, where $\Phi : [0,\infty) \to [0,\infty)$ is a continuous function, $\Phi(t) < t$, for $t > 0$ and $\Phi(0) = 0$. Also assume that either
(a) T is continuous or, (b) if a nondecreasing sequence $x_n \to x$, then $x_n \leq x$, for all $n \in N$.

If there exists $x_0 \in X$ such that $x_0 \leq Tx_0$, then T has a fixed point. Furthermore, the set of fixed points of T is well ordered if and only if T has a unique fixed point.
Proof. Taking $h(x) = x$ and $\Phi(t) = t - \phi(t)$ in Theorem 2.1, we can easily obtain the above corollary.

Corollary 2.2 Let $(X, \leq)$ be a partially ordered set and (X, F, Δ) be a complete N.A Menger PM-space, $T : X \to X$ be a nondecreasing mapping. Suppose that for comparable $x, y \in X$, we have

$$F_{Tx,Ty}(t) \geq \psi(\min\{F_{x,y}(t), F_{x,Tx}(t), F_{y,Ty}(t), \frac{1}{2}[F_{x,Ty}(t) + F_{y,Tx}(t)]\}), \quad \text{for all } t > 0,$$

where $\varphi : [0,1] \to [0,1]$ is a continuous function, $t < \psi(t) < 1$ for $t \in [0,1)$, $\psi(t) = 1$ if and only if $t = 1$. Also assume that either
(a) T is continuous or, (b) if a nondecreasing sequence $x_n \to x$, then $x_n \leq x$, for all $n \in N$.

If there exists $x_0 \in X$ such that $x_0 \leq Tx_0$, then T has a fixed point. Furthermore, the set of fixed points of T is well ordered if and only if T has a unique fixed point.
Proof. Taking $h(x) = x$ and $g(t) = 1 - t$, $\psi(t) = t + \phi(1-t)$ in Theorem 2.1, we can easily obtain the above corollary.

Remark 2.2 Corollary 2.2 generalizes and extends Theorem 2.1 in [22].

Now, we give an example to demonstrate Theorem 2.1.

Example 2.1 . Let $X = R^+$, $\Delta_1(a,b) = \max\{a+b-1, 0\}$, F be defined by

$$F_{x,y}(t) = \begin{cases} 0, & t \leq 0, \\ \frac{\min\{x,y\}}{\max\{x,y\}}, & 0 < t \leq 1, \\ 1, & t > 1. \end{cases}$$

for all $x, y \in X$. Then, for every given $x, y \in X$, it is easy to verify that $F_{x,y}$ is a distribution function and (X, F, Δ) is a complete N.A Menger PM-space.

In fact, $(MS-1)$ and $(MS-2)$ are easy to check. To prove inequality (1.1). We consider the case:

Case 1. If $t_1 > 1$ or $t_2 > 1$, then $F_{x,z}(\max\{t_1, t_2\}) \geq \Delta_1(F_{x,y}(t_1), F_{y,z}(t_2))$, for any $x, y, z \in X$.

Case 2. If $0 < t_1, t_2 \leq 1$ and $x \leq y \leq z$, for $x, y, z \in R^+$, then

$$F_{x,z}(\max\{t_1, t_2\}) - \Delta_1(F_{x,y}(t_1), F_{y,z}(t_2)) = \frac{x}{z} + 1 - (\frac{x}{y} + \frac{y}{z}) = \frac{(y-x)(z-y)}{yz} \geq 0.$$

Case 3. If $0 < t_1, t_2 \leq 1$ and $y \leq x \leq z$, for $x, y, z \in R^+$, then

$$F_{x,z}(\max\{t_1, t_2\}) - \Delta_1(F_{x,y}(t_1), F_{y,z}(t_2)) = \frac{x}{z} + 1 - (\frac{y}{x} + \frac{y}{z}) = \frac{(x+z)(x-y)}{xz} \geq 0.$$

Case 4. If $0 < t_1, t_2 \leq 1$ and $x \leq z \leq y$, for $x, y, z \in R^+$, then

$$F_{x,z}(\max\{t_1, t_2\}) - \Delta_1(F_{x,y}(t_1), F_{y,z}(t_2)) = \frac{x}{z} + 1 - (\frac{x}{y} + \frac{z}{y}) = \frac{(x+z)(y-z)}{yz} \geq 0.$$

Hence, in all case, we have $F_{x,z}(\max\{t_1, t_2\}) \geq \Delta_1(F_{x,y}(t_1), F_{y,z}(t_2))$, for all $t_1, t_2 \in R^+$, that is, 1.1 holds.

Suppose that $A_1 = [0, 1]$, $A_2 = [\frac{1}{2}, 1]$, $A_3 = [\frac{3}{4}, 1]$, and $Y = \bigcup_{i=1}^{3} A_i$. Let $f : Y \to Y$ and $fx = \frac{1}{2} + \frac{1}{2}x$, for all $x \in Y$,

Clearly $Y = \bigcup_{i=1}^{3} A_i$ is a cyclic representation of Y with respect to f.

We next prove that it satisfies the conditions of Theorem 2.1, where $h(x) = \frac{1}{2}x$, $\phi(x) = \frac{1}{6}x$, and $g(t) = 1-t$. By the definitions of F, g, h and ϕ, we only need to prove that

$$F_{fx,fy}(t) \geq Q_t(x, y) + \frac{1}{3}(1 - Q_t(x, y)) = \frac{2}{3}Q_t(x, y) + \frac{1}{3}, \tag{2.13}$$

where $Q_t(x, y) = \min\{F_{x,y}(t), F_{x,Tx}(t), F_{y,Ty}(t), \frac{1}{2}[F_{x,Ty}(t) + F_{y,Tx}(t)]\}$.

Since $fx = \frac{1}{2} + \frac{1}{2}x$. If $0 \leq x \leq y$, for $x, y \in [0, 1]$, then we have

$$Q_t(x, y) = \min\{F_{x,y}(t), F_{x,Tx}(t), F_{y,Ty}(t), \frac{1}{2}[F_{x,Ty}(t) + F_{y,Tx}(t)]\} \leq F_{x,y}(t) = \frac{x}{y}.$$

Hence, we consider the following two cases:

Case 1. If $0 < t \leq 1$, we have

$$F_{fx,fy}(t) - \frac{2}{3}Q_t(x, y) - \frac{1}{3} \geq \frac{x+1}{y+1} - \frac{2x}{3y} - \frac{1}{3} = \frac{(2-y)(y-x)}{3(y+1)y} \geq 0,$$

which implies that (2.13) holds.

Case 2. If $t > 1$, by the definition of F, we have

$$F_{fx,fy}(t) - \frac{2}{3}Q_t(x, y) - \frac{1}{3} = 0,$$

which implies that (2.13) holds.

Hence, in all case, we obtain that (2.13) holds.

Thus, all hypotheses of Theorem 2.1 are satisfied, and we deduce that f has a unique fixed point in Y. Here, $x = 1$ is the unique fixed point of f.

3 Coupled fixed point theorems in partially ordered probabilistic metric spaces

In the section, we will apply the Corollary 2.2 in the Section 2 to prove the coupled fixed point theorems under contractive conditions in partially ordered probabilistic metric spaces.

Lemma 3.1 If (X, F, Δ) is a N.A Menger PM space, then $(X^2, \tilde{F}, \Delta)$ is also a N.A Menger PM space.

Proof. It is sufficient to prove that, for $\tilde{a} = (x, y), \tilde{b} = (u, v), \tilde{c} = (p, q) \in X^2$,

$$\tilde{F}_{\tilde{a},\tilde{c}}(\max\{t_1, t_2\}) \geq \Delta(\tilde{F}_{\tilde{a},\tilde{b}}(t_1), \tilde{F}_{\tilde{b},\tilde{c}}(t_2)),$$

for all $t_1, t_2 \geq 0$. In fact, for all $\tilde{a} = (x, y), \tilde{b} = (u, v), \tilde{c} = (p, q) \in X^2$ and $t_1, t_2 \geq 0$ we have

$$\begin{aligned}\tilde{F}_{\tilde{a},\tilde{c}}(\max\{t_1, t_2\}) &= \min\{F_{x,p}(\max\{t_1, t_2\}), F_{y,q}(\max\{t_1, t_2\})\} \\ &\geq \min\{\Delta(F_{x,u}(t_1), F_{u,p}(t_2)), \Delta(F_{y,v}(t_1), F_{v,q}(t_2))\} \\ &\geq \Delta(\min\{F_{x,u}(t_1), F_{y,v}(t_1)\}, \min\{F_{u,p}(t_2), F_{v,q}(t_2)\}) \\ &= \Delta(\tilde{F}_{\tilde{a},\tilde{b}}(t_1), \tilde{F}_{\tilde{b},\tilde{c}}(t_2)).\end{aligned}$$

The proof is complete.

Theorem 3.1 Let $(X, \leq)$ be a partially ordered set and (X, F, Δ) be a complete N.A Menger PM-space, $A : X \times X \to X$ be a mapping satisfying the mixed monotone property on X. Suppose that for all $x, y, u, v \in X$, $x \leq u$ and $v \leq y$, we have

$$\begin{aligned}F_{A(x,y),A(u,v)}(t) \geq \psi(\min\{&F_{x,u}(t), F_{y,v}(t), F_{x,A(x,y)}(t), F_{u,A(u,v)}(t), F_{y,A(y,x)}(t), F_{v,A(v,u)}(t), \\ &\frac{1}{2}[\min\{F_{x,A(u,v)}(t), F_{y,A(v,u)}(t)\} + \min\{F_{u,A(x,y)}(t), F_{v,A(y,x)}(t)\}]\}),\end{aligned}$$

for all $t > 0$, where $\psi : [0, 1] \to [0, 1]$ is a continuous function, $t < \psi(t) < 1$ for $t \in [0, 1)$, $\psi(t) = 1$ if and only if $t = 1$. Also assume that either

(a) A is continuous or,

(b) if a nondecreasing sequence $x_n \to x$, then $x_n \leq x$, for all $n \in N$;

If a nonincreasing sequence $x_n \to x$, then $y \leq y_n$, for all $n \in N$.

If there exist $x_0, y_0 \in X$ such that $x_0 \leq A(x_0, y_0)$ and $A(y_0, x_0) \leq y_0$, then A has a coupled fixed point, that is, there exist $p, q \in X$ such that $A(p, q) = p$ and $A(q, p) = q$.

Proof. Let $\tilde{X} = X \times X$, for $\tilde{a} = (x, y), \tilde{b} = (u, v) \in \tilde{X}$, we introduce the order $\preceq$ as

$$\tilde{a} \preceq \tilde{b} \text{ if and only if } x \leq u, \ v \leq y.$$

It follows from Lemma 1.2 and Lemma 3.1 that $(X, \preceq, \tilde{F}, \Delta)$ is also a complete partially ordered N.A Menger PM-space, where

$$\tilde{F}_{\tilde{a},\tilde{b}}(t) = \min\{F_{x,u}(t), F_{y,v}(t)\}.$$

The self-mapping $T : \tilde{X} \to \tilde{X}$ is given by

$$T\tilde{a} = (A(x, y), A(y, x)) \text{ for all } \tilde{a} = (x, y) \in \tilde{X}.$$

Then a coupled point of A is a fixed point of T and vice versa.

If $\tilde{a} \preceq \tilde{b}$, then $x \leq u$ and $v \leq y$. Noting the mixed monotone property of A, we see that $A(x, y) \leq A(u, v)$ and $A(v, u) \leq A(y, x)$, then $T\tilde{a} \preceq T\tilde{b}$. Thus T is a nondecreasing mapping with respect to the order $\preceq$ on $\tilde{X}$.

On the other hand, for all $t > 0$ and $\tilde{a} = (x, y), \tilde{b} = (u, v) \in \tilde{X}$ with $\tilde{a} \preceq \tilde{b}$, we have

$$\begin{aligned}
F_{A(x,y),A(u,v)}(t) &\geq \psi(\min\{F_{x,u}(t), F_{y,v}(t), F_{x,A(x,y)}(t), F_{u,A(u,v)}(t), F_{y,A(y,x)}(t), F_{v,A(v,u)}(t)\\
&\quad \frac{1}{2}[\min\{F_{x,A(u,v)}(t), F_{y,A(v,u)}(t)\} + \min\{F_{u,A(x,y)}(t), F_{v,A(y,x)}(t)\}\})\\
&= \psi(\min\{\min\{F_{x,u}(t), F_{y,v}(t)\}, \min\{F_{x,A(x,y)}(t), F_{y,A(y,x)}(t)\}, \min\{F_{u,A(u,v)}(t), F_{v,A(v,u)}(t)\},\\
&\quad \frac{1}{2}[\min\{F_{x,A(u,v)}(t), F_{y,A(v,u)}(t)\} + \min\{F_{u,A(x,y)}(t), F_{v,A(y,x)}(t)\}\})\\
&= \psi(\min\{\tilde{F}_{\tilde{a},\tilde{b}}(t), \tilde{F}_{\tilde{a},T\tilde{a}}(t), F_{\tilde{b},T\tilde{b}}(t), \frac{1}{2}[\tilde{F}_{\tilde{a},T\tilde{b}}(t) + \tilde{F}_{T\tilde{a},\tilde{b}}(t)]\})
\end{aligned}$$

Similarly, $F_{A(y,x),A(v,u)}(t) \geq \psi(\min\{\tilde{F}_{\tilde{a},\tilde{b}}(t), \tilde{F}_{\tilde{a},T\tilde{a}}(t), F_{\tilde{b},T\tilde{b}}(t), \frac{1}{2}[\tilde{F}_{\tilde{a},T\tilde{b}}(t) + \tilde{F}_{T\tilde{a},\tilde{b}}(t)]\})$. Thus,

$$F_{T\tilde{a},T\tilde{b}}(t) \geq \psi(\min\{\tilde{F}_{\tilde{a},\tilde{b}}(t), \tilde{F}_{\tilde{a},T\tilde{a}}(t), F_{\tilde{b},T\tilde{b}}(t), \frac{1}{2}[\tilde{F}_{\tilde{a},T\tilde{b}}(t) + \tilde{F}_{T\tilde{a},\tilde{b}}(t)]\}).$$

Also, there exists an $\tilde{x}_0 = (x_0, y_0) \in \tilde{X}$ such that $\tilde{x}_0 \preceq T\tilde{x}_0 = (A(x_0, y_0), A(y_0, x_0))$.

If a nondecreasing monotone sequence $\{\tilde{x}_n\} = \{(x_n, y_n)\}$ in $\tilde{X}$ tends to $\tilde{x} = (x, y)$, then $\tilde{x}_n = (x_n, y_n) \preceq (x_{n+1}, y_{n+1}) = \tilde{x}_{n+1}$, that is, $x_n \leq x_{n+1}$ and $y_{n+1} \leq y_n$. Thus $\{x_n\}$ is nondecreasing sequence tending to x and $\{y_n\}$ a nonincreasing sequence tending to y. Thus $x_n \leq x$ and $y \leq y_n$ for all $n \in N$. This implies $\tilde{x}_n \preceq x$. Obviously, the continuity of A implies the continuity of T.

Therefore, all hypotheses of Corollary 2.2 are satisfied. Following Corollary 2.2, we deduce that A has a coupled point, that is, there exist $p, q \in \tilde{X}$ such that $A(p, q) = p$ and $A(q, p) = q$.

Remark 3.1 Theorem 3.1 generalizes and extends Theorem 71 in [21] and Corollary 2.1 in [22].

4 An application

In this section, using the Theorem 2.1, we establish some fixed results for generalized weak contractions in partially ordered metric spaces.

Theorem 4.1 Let $(X, d, \leq)$ be an ordered complete metric space, $T : X \to X$ be a nondecreasing mapping. Suppose that for comparable $x, y \in X$, we have

$$d(Tx, Ty) \leq M(x, y) - \varphi(M(x, y)), \quad \forall t > 0, \tag{4.1}$$

where $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\}$, $\varphi : [0, \infty) \to [0, \infty)$ is a continuous function, $\frac{\varphi(s)}{t} \geq \varphi(\frac{s}{t})$, for all $t > 0$, and $\varphi(s) = 0$ if and only if $s = 0$. Also assume that either
(a) T is continuous or, (b) if a nondecreasing sequence $x_n \to x$, then $x_n \leq x$, for all $n \in N$.

If there exists $x_0 \in X$ such that $x_0 \leq Tx_0$, then T has a fixed point. Furthermore, the set of fixed points of T is well ordered if and only if T has a unique fixed point.

Proof. Let (X, F, Δ_2) be the induced N.A Menger PM-space, where F is defined by $F_{x,y}(t) = e^{-\frac{d(x,y)}{t}}$, for $t > 0$, $x, y \in X$. We can easily prove that a sequence $\{x_n\}$ in X converges in the metric d to a point $x^* \in X$ if and if only $\{x_n\}$ in (X, F, Δ_2) τ-converges to x^*. Let $g \in \Omega$, where $g(t) = 1 - t$. Since (X, d) is a complete metric space, then (X, F, Δ_2) is a τ-complete N.A Menger PM-space of type $(D)_g$.

For $x, y \in X$, x and y are comparable, by (4.1), for $t > 0$, we have

$$\begin{aligned} 1 - e^{-\frac{d(Tx,Ty)}{t}} &\leq 1 - e^{-\frac{M(x,y)}{t} + \frac{\varphi(M(x,y))}{t}} \\ &\leq 1 - e^{-\frac{M(x,y)}{t} + \varphi(\frac{M(x,y)}{t})} \\ &= 1 - e^{-\frac{M(x,y)}{t}} - e^{-\frac{M(x,y)}{t}}[e^{\varphi(\frac{M(x,y)}{t})} - 1]. \end{aligned} \tag{4.2}$$

Let $\phi : [0, 1) \to [0, +\infty)$, where $\phi(u) = [1 - u][e^{\varphi(\ln \frac{1}{1-u})} - 1]$, for $u \in [0, 1]$. Since φ is continuous and $\varphi^{-1}(0) = 0$, then ϕ also is continuous and $\phi^{-1}(0) = 0$.

Since $\phi(1 - e^{-\frac{M(x,y)}{t}}) = e^{-\frac{M(x,y)}{t}}[e^{\varphi(\frac{M(x,y)}{t})} - 1]$, $g(s) = 1 - s$, and $F_{x,y}(t) = e^{-\frac{d(x,y)}{t}}$, by (4.2), we get

$$g(F_{Tx,Ty}(t)) \leq M_t(x, y) - \phi(M_t(x, y)),$$

for $t > 0$, where $M_t(x, y) = \max\{g(F_{x,y}(t)), g(F_{x,Tx}(t)), g(F_{y,Ty}(t)), \frac{1}{2}[g(F_{x,Ty}(t)) + g(F_{y,Tx}(t))]\}$.

Thus, all hypotheses of Theorem 2.1 are satisfied, when $h(s) = s$ and $m = 1$. Then the conclusion holds.

References

[1] W.A. Kirk, P.S. Srinivasan and P. Veeramani, Fixed points for mappings satisfying cyclical contractive conditions, *Fixed Point Theory*, 4, 79-89 (2003).

[2] A.C.M. Ran and M.C.B. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, *Proc. Amer. Math. Soc.* 132, 1435-1443 (2004).

[3] R.P. Agarwal, M.A. El-Gebeily and D. O'Regan, Generalized contractions in partially ordered metric spaces, *Appl. Anal.* 87, 1-8 (2008).

[4] J. Harjani and K. Sadarangani, Fixed point theorems for weakly contractive mappings in partially ordered sets, *Nonlinear Anal.* 71, 3403-3410 (2009).

[5] J. Esmaily, S.M. Vaezpour, B.E. Rhoades, Coincidence point theorem for generalized weakly contractions in ordered metric spaces, *Appl. Math. Comput.* 219, 5684-5692 (2013).

[6] J. Harjani, B. Lopez and K. Sadarangani, Fixed point theorems for weakly C-contractive mappings in ordered metric spaces, Comput. *Math. Appl.* 61, 790-796 (2011).

[7] H.K. Nashine and B. Samet, Fixed point results for mappings satisfying (ψ, φ)-weakly contractive condition in partially ordered metric spaces, *Nonlinear Anal.* 74, 2201-2209 (2011).

[8] J. Jachymski, Equivalent conditions for generalized contractions on (ordered) metric spaces. *Nonlinear Anal.* 74, 768-774 (2011).

[9] M.S. Khan, M. Swaleh and S. Sessa, Fixed point theorems by altering distances between the points, *Bull. Austral. Math. Soc.* 30(1), 1-9 (1984).

[10] T.G Bhaskar and V. Lakshmikantham, Fixed point theory in partially ordered metric spaces and applications, *Nonlinear Anal.* 65, 1379-1393 (2006).

[11] M. Abbas, T. Nazir and S. Radenović, Common fixed points of four maps in partially ordered metric spaces, *Appl. Math. Lett.* 24, 1520-1526 (2011).

[12] W. Shatanawi and M. Postolache, Common fixed point results for mappings under nonlinear contraction of cyclic form in ordered metric spaces, *Fixed Point Theory Appl.* 2013, 60 (2013).

[13] K. Menger, Statistical metrics, *Proc. Nat. Acad. Sci. USA.* 28, 535-537 (1942).

[14] B. Schweizer and A. Sklar, Probabilistic Metric Spaces, Elsevier/North-Holland, New York, 1983.

[15] S.S. Chang, Y.J. Cho and S.M. Kang, Probabilistic Metric Spaces and Nonlinear Operator Theory. Chengdu: Si chuan University Press. 1994.

[16] J. Jachymski, On probilistic $\varphi-$contraction on Menger spaces. *Nonlinear Anal.* 73, 2199-2203 (2010).

[17] C.X. Zhu and J.D. Yin, Calculations of a random fixed point index of a random sem-cloosed 1-set-contractive operator. *Math. Comput. Model.* 51, 1135-1139 (2010).

[18] C.X. Zhu, Research on some problems for nonlinear operators. *Nonlinear Anal.* 71, 4568-4571 (2009).

[19] C.X. Zhu, Several nonlinear operator problems in the Menger PN space. *Nonlinear Anal.* 65, 1281-1284 (2006).

[20] X.Q. Hu and X.Y. Ma, Coupled coincidence point theorems under contractive conditions in partially ordered probabilistic metric spaces, *Nonlinear Anal.* 74, 6451-6458 (2011).

[21] Lj. B. Ćirić, R.P. Agarwal and B. Samet, Mixed monotone generalized contractions in partially ordered probabilistic metric spaces, *Fixed Point Theory Appl.* 2011, 56 (2011).

[22] J. Wu, Some fixed-point theorems for mixed monotone operators in partially ordered probabilistic metric spaces, *Fixed Point Theory Appl.* 2014, 49 (2014).

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 21, NO.6, 2016, COPYRIGHT 2016 EUDOXUS PRESS, LLC

Weak Galerkin finite element method for time dependent reaction-diffusion equation

Fuzheng Gao[1]

School of Materials Science and Engineering, Shandong University
School of Mathematics, Shandong University, Jinan, Shandong 250100, China

Guoqun Zhao[2]

School of Materials Science and Engineering, Shandong University, Jinan, Shandong 250061, China

Abstract

We propose a weak Galerkin finite element procedure for time dependent reaction-diffusion equation by using weakly defined gradient operators over discontinuous functions with heterogeneous properties, in which the classical gradient operator is replaced by the discrete weak gradient. Numerical analysis and numerical experiments illustrate and confirm that our new method has effective numerical performances.
Mathematics subject classifications: 65M15, 65M60.

Keywords: Galerkin finite element methods, parabolic equation, weak gradient, error estimate, numerical experiment.

1. Introduction.

Time dependent reaction-diffusion equations are a large important class of equations. In this paper, we consider the following time dependent reaction-diffusion equation:

$$u_t + Au = f(x,t), \quad x \in \Omega, \quad 0 < t \leq T, \tag{1a}$$

$$u = u^0(x), \qquad x \in \Omega, \quad t = 0, \tag{1b}$$

[1]Corresponding author, e-mail: fzgao@sdu.edu.cn (Fuzheng Gao).
[2]e-mail:zhaogq@sdu.edu.cn.

Preprint submitted to JOCAA *April 28, 2015*

with homogenous Dirichlet boundary condition, where Ω is a bounded region in R^2, with a Lipschitz continuous boundary; $u_t = \dfrac{\partial u}{\partial t}$; and A is a second order elliptic differential operator:

$$Au \equiv -\nabla \cdot (a\nabla u) + cu,$$

where a and c are sufficiently smooth functions of x and satisfy $0 < a_* \leq a(x) \leq a^*$ and $c(x) \geq 0$ for fixed a_*, a^*. We define the following bilinear form

$$a(u,v) := \int_\Omega (a\nabla u \cdot \nabla v + cuv)\mathrm{d}x. \tag{2}$$

It is obvious that there is a constant $\alpha_0 > 0$ such that

$$a(u,u) \geq \alpha_0 \|u\|_1^2, \; \forall u \in H_0^1(\Omega). \tag{3}$$

The variational weak form to (1) is: find $u = u(x,t) \in L^2(0,T; H_0^1(\Omega))$, such that

$$(u_t, v) + a(u,v) = (f,v), \quad \forall v \in H_0^1(\Omega), \quad t > 0, \tag{4a}$$
$$u(x,0) = u^0(x), \qquad x \in \Omega, \tag{4b}$$

where $(\cdot,\cdot)$ denotes the inner product of $L^2(\Omega)$.

Many numerical methods for solving such problems have been developed, please see [3, 6, 7, 11, 12, 13, 16] and references in. In [5], a weak Galerkin finite element method (WG-FEM) was introduced and analyzed for parabolic equation based on a discrete weak gradient arising from local Raviart-Thomas (RT) elements [10]. Due to the use of RT elements, the WG finite element formulation of [5] was limited to finite element partitions of triangles for two dimensional problem. To overcome this, we presented a WG-FEM in [4] with a stabilization term for a diffusion equation without reaction term and derived optimal convergence rate in L^2 norm based on a dual argument technique for the solution of the WG-FEM. The WG-FEM was first introduced in [14] for solving second order elliptic problems. Later, the WG-FEMs were studied from implementation point of view in [8] and applied to solve the Helmholtz problem with high wave numbers in [9].

The purpose of this paper is to present a weak Galerkin (WG) finite element procedures using more flexible elements in arbitrary unstructured meshes for time dependent reaction-diffusion problem, and derive optimal convergence rate in the H^1 norm.

The outline of this article is as follows. In Section 3, we define the weak gradient and present semi-discrete and fully-discrete WG-FEMs for problem (1). In Section 4, we establish the optimal order error estimates in H^1-norm to the WG-FEMs for the parabolic problem. Finally in Section 5 we give some numerical examples to verify the theory.

Throughout this paper, the notations of standard Sobolev spaces $L^2(\Omega)$, $H^k(\Omega)$ and associated norms $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$, $\|\cdot\|_k = \|\cdot\|_{H^k(\Omega)}$ are adopted.

2. A weak gradient operator and its discrete approximation

Let T be any polygonal domain with interior T^0 and boundary ∂T. A *weak function* on the region T refers to a function $v = \{v_0, v_b\}$ such that $v_0 \in L^2(T)$ and $v_b \in H^{\frac{1}{2}}(\partial T)$. v_0 represents the value of v on T^0 and v_b represents that of v on ∂T. Note that v_b may not necessarily be related to the trace of v_0 on ∂T. Denote by $W(T)$ the space of weak function associated with T; i.e.,

$$W(T) = \left\{ v = \{v_0, v_b\} : v_0 \in L^2(T), v_b \in H^{\frac{1}{2}}(\partial T) \right\}. \tag{5}$$

Definition 2.1. *[14] The dual of $L^2(T)$ can be identified with itself by using the standard L^2 inner product as action of linear functional. With a similar interpretation, for any $v \in W(T)$, the* weak gradient *of v is defined as a linear functional $\nabla_w v$ in the dual space of $H(div, T)$ whose action on each $q \in H(div, T)$ is given by*

$$(\nabla_w v, q)_T := -\int_T v_0 \nabla \cdot q dT + \int_{\partial T} v_b q \cdot \mathbf{n} d\mathbf{s}, \tag{6}$$

where $\mathbf{n}$ is the outer normal direction to ∂T.

Next, we introduce a discrete weak gradient operator by defining ∇_w in a polynomial subspace of $H(\mathrm{div}, T)$. To this end, for any non-negative integer $r \geq 0$, denote by $P_r(T)$ the set of polynomials on T with degree no more than r. Let $V(K, r) \subset [P_r(T)]^2$ be a subspace of the space of vector-valued polynomials of degree r. A discrete weak gradient operator, denoted by $\nabla_{w,r}$, is defined so that $\nabla_{w,r} v \in V(T, r)$ is the unique solution of the following equation

$$(\nabla_{w,r} v, q)_T := -\int_T v_0 \nabla \cdot q \mathrm{d}T + \int_{\partial T} v_b q \cdot \mathbf{n} \mathrm{d}s, \quad \forall q \in V(T, r). \tag{7}$$

It is easy to know that $\nabla_{w,r}$ is a Galerkin-type approximation of the weak gradient operator ∇_w by using the polynomial space $V(T, r)$.

3. Weak Galerkin finite element methods

Let $\mathcal{T}_h$ be a regular finite element grid on Ω with mesh size h. Assume that the partition $\mathcal{T}_h$ is shape regular so that the routine inverse inequality in the finite element analysis holds true (see [2]). In the general spirit of the Galerkin procedure, we shall design a weak Galerkin finite element method for (4) by the following two basic principles: (1) replace $H^1(\Omega)$ by a space of discrete weak functions defined on the finite element partition $\mathcal{T}_h$ and the boundary of triangular elements; and (2) replace the classical gradient operator by a discrete weak gradient operator ∇_w for weak functions on each triangle T.

For each $T \in \mathcal{T}_h$, denote by $P_j(T^0)$ the set of polynomials on T^0, which is the interior of triangle T, with degree no more than j, and $P_l(\partial T)$ the set of polynomials on ∂T with degree no more than l (i.e., polynomials of degree l on each line segment of ∂T). A discrete weak function $v = \{v_0, v_b\}$ on T refers to a weak function $v = \{v_0, v_b\}$ such that $v_0 \in P_j(T^0)$ and $v_b \in P_l(\partial T)$ with $j \geq 0$ and $l \geq 0$. Denote this space by $W(T, j, l)$, i.e.,

$$W(T, j, l) := \{v = \{v_0, v_b\} : v_0 \in P_j(T^0), v_b \in P_l(\partial T)\}.$$

The corresponding FE space would be defined by matching $W(T, j, l)$ over all the triangles $T \in \mathcal{T}_h$ as

$$V_h := \{v = \{v_0, v_b\} : \{v_0, v_b\}|_T \in W(T, j, l), \forall T \in \mathcal{T}_h\}. \tag{8}$$

Denote by V_h^0 the subspace of V_h with zero boundary values on $\partial\Omega$; i.e.,

$$V_h^0 := \{v = \{v_0, v_b\} \in V_h, v_b \mid_{\partial T \bigcap \partial\Omega} = 0, \forall T \in \mathcal{T}_h\}. \tag{9}$$

According to (7), for each $v = \{v_0, v_b\} \in V_h^0$, the discrete weak gradient $\nabla_{w,r} v$ of v on each element T is given by the following equation:

$$\int_T \nabla_{w,r} v \cdot q \mathrm{d}x = -\int_T v_0 \nabla \cdot q \mathrm{d}x + \int_{\partial T} v_b q \cdot \mathbf{n} \mathrm{d}s, \quad \forall q \in V(T, r). \tag{10}$$

For simplicity of notation, we shall drop the subscript r in the discrete weak gradient operator $\nabla_{w,r}$ from now on. Now, we define the semi-discrete weak Galerkin finite element scheme for (1) as: find $u_h = \{u_0, u_b\}(\cdot, t) \in V_h^0\, (0 \leq t \leq T)$ such that

$$(u_{h,t}, v) + a_w(u_h, v) = (f, v_0), \forall\, v = \{v_0, v_b\} \in V_h^0, t > 0, \tag{11a}$$

$$u_h(x, 0) = Q_h u^0(x), \qquad x \in \Omega, \tag{11b}$$

where the bilinear form $a_w(\cdot,\cdot)$ is defined as

$$a_w(v,w) = \sum_{T\in\mathcal{T}_h} \int_T (a\nabla_w v\cdot\nabla_w w + cv_0w_0)\mathrm{d}x + \sum_{T\in\mathcal{T}_h} h_T^{-1} < v_0 - v_b, w_0 - w_b >_{\partial T}, \tag{12}$$

and $Q_hu = \{Q_0u, Q_bu\}$ is the L^2 projection onto $P_j(T^0)\times P_l(\partial T)$. In other words, on each element T, the function Q_0u is defined as the L^2 projection of u on $P_j(T)$ and on ∂T, Q_bu is the L^2 projection in $P_l(\partial T)$. Hereafter, we choose $l = j$.

Let $\{\varphi_i(x) : i = 1, 2, \cdots, N\}$, where $N = \dim(V_h^0)$, be the bases of V_h^0. For example, when $j = 0$ in $P_j(T)$, φ_i is a function which takes value one in the interior of triangle T of $\mathcal{T}_h$ and zero everywhere else; and φ_i is a function that takes value one on the edge $e\in\partial T$ and zero everywhere else. Then (11) can be expressed as: find a solution of the form

$$u_h = \{u_0, u_b\} = \sum_{i=1}^{N} \mu_j(t)\varphi_i(x),$$

such that its coefficients $\mu_1(t), \mu_2(t), \cdots, \mu_N$ satisfy

$$\sum_{i=1}^{N}[\frac{\mathrm{d}\mu_i(t)}{\mathrm{d}t}(\varphi_i,\varphi_j) + \mu_i a_w(\varphi_i,\varphi_j)] = (f,\varphi_j), \quad t > 0. \tag{13}$$

By Introducing the following matrix and vector notations:

$$\mathbf{M} = [m_{ij}] = [(\varphi_i,\varphi_j)], \quad \mathbf{K} = [k_{ij}] = [a_w(\varphi_i,\varphi_j)],$$

$$\mu = [\mu_1, \mu_2, \cdots, \mu_N]^T, \mathbf{F} = [(f,\varphi_1), (f,\varphi_2), \cdots, (f,\varphi_N)]^T,$$

then (13) can be rewritten as

$$\mathbf{M}\frac{\mathrm{d}\mu}{\mathrm{d}t} + \mathbf{K}\mu = \mathbf{F}. \tag{14}$$

M and $\mathbf{K}$ are positive definite matrix. The ordinary differential equation (ODE) theory tells us that the semi-discrete WG scheme has a unique solution for any $f\in L^2(\Omega)$.

Define a norm $|||\cdot|||_{w,1}$ as

$$|||v|||_{w,1} := \sqrt{\sum_{T\in\mathcal{T}_h} (\|\nabla_w v\|_{0,T}^2 + \|v\|_{0,T}^2 + h_T^{-1}\|v_0 - v_b\|_{0,\partial T}^2)}, \tag{15}$$

which is a H^1-equivalent norm for conventional finite element functions, since the presence of the $L^2(T)$ term renders the norm to be an equivalent H^1 norm for any H^1 function, regardless the value of their zeroth order traces on ∂T; where $\|v\|_{0,T}^2 = \int_T v^2 \mathrm{d}x$ and $\|v_0 - v_b\|_{0,\partial T}^2 = \int_{\partial T}(v_0 - v_b)^2 \mathrm{d}s$. Moreover, the following *Poincaré* inequality holds true for functions in V_h^0.

Lemma 3.1. *Assume that the finite element partition $\mathcal{T}_h$ is shape regular. Then there exists a constant C independent of the mesh size h such that*

$$\|v\| \le |||v|||_{w,1},\ \forall\, v = \{v_0, v_b\} \in V_h^0. \tag{16}$$

Let us now return to our semi-discrete problem in the formulation (11). A basic stability inequality for problem (1) with $f = 0$, for simplicity, is as follows:

Theorem 3.1. *For the numerical solution to scheme (11) with initial setting (11b), there is a L^2-stability as follows*

$$\frac{d}{dt}\int_\Omega u_h^2(x,t)\,dx \le 0. \tag{17}$$

Proof. Taking $v = u_h$ in (11a), with $f = 0$, we get

$$(u_{h,t}(t), u_h(t)) + a_w(u_h(t), u_h(t)) = 0.$$

From the definition of bilinear form $a_w(\cdot,\cdot)$ in (12), we know that

$$a_w(u_h(t), u_h(t)) \ge 0.$$

Based on this fact,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_h^2(t)\mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(u_h(t), u_h(t)) = (u_{h,t}(t), u_h(t)) \le 0.$$

This completes the proof. $\square$

Let τ denote the time step size, and $t_n = n\tau$ $(n = 0, 1, \cdots)$, $u_h^n := u_h(t_n) = \{u_0^n, u_b^n\}$. At time $t = t_n$, using backward difference quotient

$$\bar{\partial}_t u_h^n = (u_h^n - u_h^{n-1})/\tau$$

6

to approximate the differential quotient $u_{h,t}$ in the semi-discrete scheme (11), we get the fully-discrete WG-FE scheme: find $u_h^n = \{u_0^n, u_b^n\} \in V_h^0$ for $n = 1, 2, \cdots$, such that

$$\sum_{T\in\mathcal{T}_h} (\bar{\partial}_t u_h^n, v_h)_T + a_w(u_h^n, v_h) = \sum_{T\in\mathcal{T}_h} (f^n, v_0)_T, \forall v_h = \{v_0, v_b\} \in V_h^0, \tag{18a}$$

$$u_h^0 = Q_h u^0(x). \tag{18b}$$

From (12), for v, $w \in V_h^0$, we get

$$a_w(v, v) \geq \alpha_0 |\|v\||^2_{w,1}, \forall v \in V_h^0,$$

and

$$a_w(v, w) \leq C^* |\|v\||_{w,1} |\|w\||_{w,1},$$

which guarantees the existence and uniqueness of the solution $u_h^n = \{u_0^n, u_b^n\}$ to (18) for a given $u_h^{n-1} = \{u_0^{n-1}, u_b^{n-1}\}$.

4. Error estimate

In this section we will present a priori error estimates in H^1-norm for the semi-discrete scheme (11) and fully-discrete scheme (18) for smooth solutions of (1).

For simplicity, we assume that diffusion coefficient a is piecewise constant with respect to the finite element partition $\mathcal{T}_h$. The corresponding results can be extended to the case of variable coefficients provided that the coefficient function a is sufficiently smooth.

Below we denote C (maybe with indicates) as a positive constant depending solely on the exact solution, which may have different values in each occurrence.

4.1. Preliminaries

4.1.1. Sobolev space definitions and notations

Let Ω be any domain in R^2. In this paper, we adopt the standard definition for the Sobolev space $W^{s,r}(\Omega)$, which consists of functions with (distributional) derivatives of order less than or equal to s in $L^r(\Omega)$ for $1 \leq r \leq +\infty$ and integer s. And their associated inner products $(\cdot, \cdot)_{s,r,\Omega}$, norms $\|\cdot\|_{s,r,\Omega}$, and seminorms $|\cdot|_{s,r,\Omega}$. Further, $\|\cdot\|_{\infty,\Omega}$ represents the norm on $L^\infty(\Omega)$, and $\|\cdot\|_{L^\infty([0,T];W^{s,r}(\Omega))}$ the norm on $L^\infty([0,T]; W^{s,r}(\Omega))$. See Adams [1] for more details.

4.1.2. Properties of finite element space

In our analysis, we shall use two kinds of polynomial finite element spaces associated with each element $T \in \mathcal{T}_h$. one is a scalar polynomial space $P_k(T)$, in which the degree of polynomial is no more than k on T^0 and ∂T, and the other is the vector value polynomial space $[P_{k-1}(T)]^2$ which is used to define the discrete weak gradient ∇_w in (10). For convenience, we denote $[P_{k-1}(T)]^d$ by $G_{k-1}(T)$, which is called a local discrete gradient space.

In addition, we define the local L^2-projection of the vector value function $\mathbf{w}(x)$ in this paper by $\mathcal{Q}_h\mathbf{w}(x)$. It is defined in each element $T \in \mathcal{T}_h$ as the unique vector value function in $G_{k-1}(T)$ such that

$$\int_T \mathcal{Q}_h\mathbf{w}(x) \cdot q(x)\mathrm{d}x = \int_T \mathbf{w}(x) \cdot q(x)\mathrm{d}x, \quad \forall q(x) \in G_{k-1}(T). \tag{19}$$

The following three lemmas are listed without any proof. Their proofs can be found in [14].

Lemma 4.1. *Let Q_h be the L^2 projection operator. Then, on each element $T \in \mathcal{T}_h$, we have the following relation*

$$\nabla_w(Q_h\phi) = \mathcal{Q}_h(\nabla\phi), \quad \forall\phi \in H^1(\Omega). \tag{20}$$

Lemma 4.2. *Let T be an element with $e \in \partial T$ is a portion of its boundary. For any function $\phi \in H^1(T)$, the following trace inequality is valid for general meshes (see [14] for details):*

$$\|\phi\|_e^2 \le C(h_T^{-1}\|\phi\|_T^2 + h_T\|\nabla\phi\|_T^2). \tag{21}$$

Lemma 4.3. *Let $\mathcal{T}_h$ be a finite element partition of domain Ω satisfying corresponding shape regularity assumptions as specified in [15]. Then, for any $\phi \in H^{k+1}(\Omega)$, we have*

$$\sum_{T\in\mathcal{T}_h} \|\phi - Q_0\phi\|_T^2 + \sum_{T\in\mathcal{T}_h} h_T^2\|\nabla(\phi - Q_0\phi)\|_T^2 \le Ch^{2(k+1)}\|\phi\|_{k+1}^2. \tag{22}$$

$$\sum_{T\in\mathcal{T}_h} \|a(\nabla\phi - \mathcal{Q}_h(\nabla\phi))\|_T^2 \le Ch^{2k}\|\phi\|_{k+1}^2. \tag{23}$$

Lemma 4.4. *Assume that $\mathcal{T}_h$ is shape regular. We have the following relation*

$$|\sum_{T\in\mathcal{T}_h} h_T^{-1} < Q_0w - Q_bw, v_0 - v_b >_{\partial T}| \le Ch^k\|w\|_{k+1}|\|v\||_{w,1}, \tag{24}$$

8

and

$$|\sum_{T\in\mathcal{T}_h} < a(\nabla w - \mathcal{Q}_h \nabla w)\cdot \mathbf{n}, v_0 - v_b >_{\partial T}| \le Ch^k \|w\|_{k+1} |\!|\!| v |\!|\!|_{w,1} \qquad (25)$$

for $\forall w \in H^{k+1}(\Omega)$ *and* $v = \{v_0, v_b\} \in V_h^0$.

4.2. Error estimate for semi-discrete WG scheme

In this section, we analyze semi-discrete WG scheme (11) first.

Theorem 4.1. *Let* $u(x,t)$ *and* $u_h(x,t)$ *be the solutions to the problem (1) and the semi-discrete WG scheme (11), respectively. Assume that the exact solution has a regularity such that* $u,\ u_t \in H^{k+1}(\Omega)$. *Then, there exists a constant* C *such that*

$$|\!|\!| u - u_h |\!|\!|_{w,1}^2 \le C[|\!|\!| u^0 - u_h^0 |\!|\!|_{w,1}^2 + h^{2k} \int_0^T (h^2 \|u_t\|_{k+1}^2 + \|u\|_{k+1}^2)dt]. \qquad (26)$$

Proof Let

$$\rho = u - Q_h u, e = Q_h u - u_h. \qquad (27)$$

where Q_h is the local L^2-projection operator and $e = \{e_0, e_b\} = \{Q_0 u - u_0, Q_b u - u_b\}$. Then we have

$$u - u_h = \rho + e. \qquad (28)$$

To estimate ρ, we apply Lemma 4.3 and 4.4. We start by estimating e. Since u and u_h satisfy (4) and (11) respectively, we have

$$(u_t - u_{h,t}, v) + a(u,v) - a_w(u_h, v) = 0, \quad \forall v \in V_h^0.$$

Further,

$$(u_t - Q_h u_t + Q_h u_t - u_{h,t}, v) + a(u,v) - a_w(u_h, v) = 0, \quad \forall v \in V_h^0,$$

i.e.,

$$(e_t, v) + a(u,v) - a_w(u_h, v) = -(\rho_t, v), \quad \forall v \in V_h^0. \qquad (29)$$

In the following, we analyze the term $a(u,v) - a_w(u_h,v)$. Recalling the definitions of $a(u,v)$ and $a_w(u_h,v)$ and noting that $\sum_{T\in\mathcal{T}_h} < a\nabla u \cdot \mathbf{n}, v_b >_{\partial T} = 0$, we derive

$$\begin{aligned}
&a(u,v)-a_w(u_h,v)\\
&=\sum_{T\in\mathcal{T}_h}[(a\nabla u,\nabla v)_T+(cu,v)_T]-\sum_{T\in\mathcal{T}_h}<a\nabla u\cdot\mathbf{n},v>_{\partial T}\\
&\quad-\sum_{T\in\mathcal{T}_h}[(a\nabla_w u_h,\nabla_w v)_T+(cu_h,v)_T]-\sum_{T\in\mathcal{T}_h}h_T^{-1}<u_0-u_b,v_0-v_b>_{\partial T}\\
&=\sum_{T\in\mathcal{T}_h}[(a\nabla u,\nabla v)_T+(cu,v)_T]-\sum_{T\in\mathcal{T}_h}<a\nabla u\cdot\mathbf{n},v_0-v_b>_{\partial T}\\
&\quad-\sum_{T\in\mathcal{T}_h}[(a\nabla_w u_h,\nabla_w v)_T+(cu_h,v)_T]-\sum_{T\in\mathcal{T}_h}h_T^{-1}<u_0-u_b,v_0-v_b>_{\partial T}
\end{aligned}.$$

Further, we have

$$\begin{aligned}
&a(u,v)-a_w(u_h,v)\\
&=\sum_{T\in\mathcal{T}_h}[(a\nabla u,\nabla v)_T-(a\nabla_w u_h,\nabla_w v)_T]\\
&\quad+\sum_{T\in\mathcal{T}_h}[(cu,v)_T-(cu_h,v)_T]-\sum_{T\in\mathcal{T}_h}h_T^{-1}<u_0-u_b,v_0-v_b>_{\partial T}\\
&\qquad-\sum_{T\in\mathcal{T}_h}<a\nabla u\cdot\mathbf{n},v_0-v_b>_{\partial T}\\
&=\sum_{T\in\mathcal{T}_h}[(a\nabla u,\nabla v)_T-(a\nabla_w Q_h u,\nabla_w v)_T\\
&\qquad\qquad+(a\nabla_w Q_h u,\nabla_w v)_T-(a\nabla_w u_h,\nabla_w v)_T]\\
&\quad+\sum_{T\in\mathcal{T}_h}[(cu,v)_T-(cu_h,v)_T]-\sum_{T\in\mathcal{T}_h}h_T^{-1}<u_0-u_b,v_0-v_b>_{\partial T}\\
&\qquad-\sum_{T\in\mathcal{T}_h}<a\nabla u\cdot\mathbf{n},v_0-v_b>_{\partial T}.
\end{aligned}\tag{30}$$

From the definitions of the weak discrete gradient ∇_w and the projection $\mathcal{Q}_h$, as well as the expressions in (10) and (20), we get

$$\begin{aligned}
&(a\nabla_w Q_h u,\nabla_w v)_T=(a\mathcal{Q}_h(\nabla u),\nabla_w v)_T=(\nabla_w v,a\mathcal{Q}_h(\nabla u))_T\\
&=-(v_0,\nabla\cdot(a\mathcal{Q}_h(\nabla u)))_T+<v_b,a\mathcal{Q}_h(\nabla u)\cdot\mathbf{n})>_{\partial T}\\
&=(\nabla v_0,a\mathcal{Q}_h(\nabla u))_T-<v_0-v_b,a\mathcal{Q}_h(\nabla u)\cdot\mathbf{n}>_{\partial T}\\
&=(a\nabla u,\nabla v_0)_T-<v_0-v_b,a\mathcal{Q}_h(\nabla u)\cdot\mathbf{n}>_{\partial T}.
\end{aligned}\tag{31}$$

Substituting (31) into (30) arrives at

$$\begin{aligned}
a(u,v)-a_w(u_h,v)=&\sum_{T\in\mathcal{T}_h}[(a\nabla_w e,\nabla_w v)_T+(c\rho,v)_T+(ce,v)_T]\\
&-\sum_{T\in\mathcal{T}_h}h_T^{-1}<u_0-u_b,v_0-v_b>_{\partial T}\\
&+\sum_{T\in\mathcal{T}_h}<a(\mathcal{Q}_h(\nabla u)-\nabla u)\cdot\mathbf{n},v_0-v_b>_{\partial T}.
\end{aligned}\tag{32}$$

Combining (29) with (32) gives

$$\begin{aligned}(e_t, v) &+ \sum_{T\in\mathcal{T}_h} [(a\nabla_w e, \nabla_w v)_T + (c\rho, v)_T + (ce, v)_T] \\ &- \sum_{T\in\mathcal{T}_h} h_T^{-1} < u_0 - u_b, v_0 - v_b >_{\partial T} \\ &+ \sum_{T\in\mathcal{T}_h} < a(\mathcal{Q}_h(\nabla u) - \nabla u)\cdot \mathbf{n}, v_0 - v_b >_{\partial T} = -(\rho_t, v).\end{aligned} \tag{33}$$

Adding the term $\sum_{T\in\mathcal{T}_h} h_T^{-1} < Q_0 u - Q_b u, v_0 - v_b >_{\partial T}$ to both sides of (33), we have

$$\begin{aligned}&(e_t, v) + a_w(e, v) \\ &= -(\rho_t, v) - \sum_{T\in\mathcal{T}_h} (c\rho, v)_T + \sum_{T\in\mathcal{T}_h} h_T^{-1} < Q_0 u - Q_b u, v_0 - v_b >_{\partial T} \\ &\quad + \sum_{T\in\mathcal{T}_h} < a(\nabla u - \mathcal{Q}_h(\nabla u))\cdot \mathbf{n}, v_0 - v_b >_{\partial T} .\end{aligned} \tag{34}$$

Choosing the test function $v = e_t$ in (34), we have

$$\begin{aligned}&\|e_t\|_0^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} a_w(e, e) \\ &= -(\rho_t, e_t) - \sum_{T\in\mathcal{T}_h} (c\rho, e_t)_T + \sum_{T\in\mathcal{T}_h} h_T^{-1} < Q_0 u - Q_b u, e_{0,t} - e_{b,t} >_{\partial T} \\ &\quad + \sum_{T\in\mathcal{T}_h} < a(\nabla u - \mathcal{Q}_h(\nabla u))\cdot \mathbf{n}, e_{0,t} - e_{b,t} >_{\partial T} \\ &= -(\rho_t, e_t) - (c\rho, e_t) + \sum_{T\in\mathcal{T}_h} h_T^{-1} < Q_0 u - Q_b u, e_{0,t} - e_{b,t} >_{\partial T} \\ &\quad + \sum_{T\in\mathcal{T}_h} < a(\nabla u - \mathcal{Q}_h(\nabla u))\cdot \mathbf{n}, e_{0,t} - e_{b,t} >_{\partial T} \\ &\equiv R_1 + R_2 + R_3 + R_4.\end{aligned} \tag{35}$$

We estimate each term of R_1, R_2, R_3 and R_4, separately.

For R_3 and R_4, we use Lemma 4.4, yielding:

$$|R_3| \le Ch^k \|u\|_{k+1} |||e_t|||_{w,1}, \quad |R_4| \le Ch^k \|u\|_{k+1} |||e_t|||_{w,1}. \tag{36}$$

The other two terms R_1 and R_2 can be bound by applying the *Hölder* inequality and Lemma 3.1, i.e.,

$$|R_1| = |-(\rho_t, e_t)| \le C\|\rho_t\|\|e_t\| \le C\|\rho_t\|_0^2 + \frac{1}{2}\|e_t\|_0^2. \tag{37}$$

$$|R_2| = |-(c\rho, e_t)| \le C\|\rho\|\|e_t\| \le C\|\rho\|_0^2 + \frac{1}{2}\|e_t\|_0^2. \tag{38}$$

Substituting (37), (38) and (36) into (35) leads to:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}a_w(e,e) \le C(\|\rho\|^2 + \|\rho_t\|^2 + h^k\|u\|_{k+1}|||e_t|||_{w,1}) \tag{39}$$

Integrating (39) with respect to t from 0 to T, we have

$$\begin{aligned}&a_w(e(T),e(T)) - a_w(e(0),e(0))\\ &\le C[\int_0^T \|\rho\|^2\mathrm{d}t + \int_0^T \|\rho_t\|^2\mathrm{d}t + h^k\int_0^T \|u\|_{k+1}|||e_t|||_{w,1}\mathrm{d}t)]\\ &\le C[\int_0^T \|\rho\|^2\mathrm{d}t + \int_0^T \|\rho_t\|^2\mathrm{d}t + h^{2k}\int_0^T \|u\|_{k+1}^2\mathrm{d}t + \int_0^T |||e_t|||_{w,1}^2\mathrm{d}t].\end{aligned} \tag{40}$$

By virtue of Lemma 4.3,

$$\|\rho_t\|_0 = \|u_t - Q_h u_t\|_0 \le Ch^{k+1}\|u_t\|_{k+1}. \tag{41}$$

A combination of (22) and (40)-(41) with Gronwall lemma leads to (26). □

4.3. Error estimate for fully discrete WG scheme

Theorem 4.2. *Let u and $\{u_h^n\}$ be the solutions to the parabolic equation (1) and the fully discrete WG scheme (18), respectively. Then*

$$\begin{aligned}&\|u(t_n) - u_h^n\|_{w,1}^2\\ &\le C\{\|u^0 - u_h^0\|_{w,1}^2 + h^{2k}[(\|u^0\|_{k+1}^2 + \int_0^{t_n}\|u_t\|_{k+1}^2 dt) + \tau\sum_{i=1}^n \|u^i\|_{k+1}^2]\\ &\qquad +\tau^2\int_0^{t_n}\|u_{tt}\|_0^2 dt\}.\end{aligned} \tag{42}$$

Proof Set

$$\rho^n = u(t_n) - Q_h u(t_n), \quad e^n = Q_h u(t_n) - u_h^n,$$

then

$$u(t_n) - u_h^n = \rho^n + e^n \tag{43}$$

It follows from Lemma 4.3 that

$$|||\rho^n|||_{w,1} \le C\|\rho^n\|_1 \le Ch^k\|u(t_n)\|_{k+1} \le Ch^k[\|u^0\|_{k+1} + \int_0^{t_n}\|u_\tau\|_{k+1}\mathrm{d}\tau] \tag{44}$$

In (4a), we set $t = t_n$ we have

$$(u_t^n, v) + a(u^n, v) = (f^n, v), \tag{45}$$

12

where u_t^n denotes the value of derivative $\dfrac{\partial u(x,t)}{\partial t}$ at $t = t_n$, and similar definitions to u^n and f^n. Subtracting (18a) from (45), then we have

$$(u_t^n - \bar{\partial}_t u_h^n, v) + a(u^n, v) - a_w(u_h^n, v) = 0, \tag{46}$$

further

$$(\bar{\partial}_t e^n, v) + a(u^n, v) - a_w(u_h^n, v) = (\bar{\partial}_t Q_h u(t^n) - u_t^n, v). \tag{47}$$

For the term $a(u^n, v) - a_w(u_h^n, v)$, taking the same measures used in the analysis course of semi-discrete case, we have

$$\begin{aligned}&(\bar{\partial}_t e^n, v) + a_w(e^n, v)\\ &\quad = (\bar{\partial}_t Q_h u(t^n) - u_t^n, v) - \sum_{T\in\mathcal{T}_h} (c\rho^n, v)_T\\ &\qquad + \sum_{T\in\mathcal{T}_h} h_T^{-1} < Q_0 u^n - Q_b u^n, v_0 - v_b >_{\partial T}\\ &\qquad + \sum_{T\in\mathcal{T}_h} < a(\nabla u^n - \mathcal{Q}_h(\nabla u^n)) \cdot \mathbf{n}, v_0 - v_b >_{\partial T} .\end{aligned} \tag{48}$$

Let LL_1, LL_2 be the two terms of the left hand side (LHS) of the equation (48) and RR_1, RR_2, RR_3, RR_4 be the four terms of the right hand side (RHS) of (48), respectively. Nest, we choose the test function $v = \bar{\partial}_t e^n$ in (48), and estimate these six terms consecutively.

For the two terms LL_1, LL_2 of the LHS of the error equation, we have

$$|LL_1| = |(\bar{\partial}_t e^n, \bar{\partial}_t e^n)| = \|\bar{\partial}_t e^n\|_0^2. \tag{49}$$

Note that

$$a_w(e^n, e^n) \geq \alpha_0 |||e^n|||_{w,1}^2,$$

and

$$a_w(e^n, e^{n-1}) \leq C^* |||e^n|||_{w,1} |||e^{n-1}|||_{w,1}.$$

using the weighted *Hölder* inequality and choosing a suitable weight ϵ , such that $\epsilon < \alpha_0$ and

$$a_w(e^n, e^{n-1}) \leq \epsilon |||e^n|||_{w,1}^2 + C |||e^{n-1}|||_{w,1}^2.$$

$$|LL_2| \geq \frac{1}{\tau}[(\alpha_0 - \epsilon)|||e^n|||_{w,1}^2 - C|||e^{n-1}|||_{w,1}^2]. \tag{50}$$

13

The terms RR_1 through RR_4 in the RHS of (48) are estimated as follows:

$$|RR_1| \leq C\|\bar{\partial}_t Q_h u(t^n) - u_t^n\|^2 + \frac{1}{2}\|\bar{\partial}_t e^n\|_0^2, |RR_2| \leq C\|\rho^n\|^2 + \frac{1}{2}\|\bar{\partial}_t e^n\|_0^2. \quad (51)$$

$$\begin{aligned} |RR_3| &\leq Ch^k\|u^n\|_{k+1}|||\bar{\partial}_t e^n|||_{w,1} \\ &\leq Ch^{2k}\|u^n\|_{k+1}^2 + \frac{C_1}{\tau}(|||e^n|||_{w,1}^2 + |||e^{n-1}|||_{w,1}^2), \\ |RR_4| &\leq Ch^k\|u^n\|_{k+1}|||\bar{\partial}_t e^n|||_{w,1} \\ &\leq Ch^{2k}\|u^n\|_{k+1}^2 + \frac{C_1}{\tau}(|||e^n|||_{w,1}^2 + |||e^{n-1}|||_{w,1}^2), \end{aligned} \quad (52)$$

where C_1 has to be less than $\frac{1}{2}(\alpha_0 - \epsilon)$. A combination of (48)-(52) leads to

$$\begin{aligned} &|||e^n|||_{w,1}^2 \\ &\leq \beta|||e^{n-1}|||_{w,1}^2 + C\tau(\|\bar{\partial}_t Q_h u(t^n) - u_t^n\|_0^2 + \|\rho^n\|_0^2 + h^{2k}\|u^n\|_{k+1}^2) \\ &\leq \beta|||e^0|||_{w,1}^2 + C\tau\sum_{i=1}^{n}(\|\bar{\partial}_t Q_h u(t^i) - u_t^i\|_0^2 + \|\rho^i\|_0^2 + h^{2k}\|u^i\|_{k+1}^2), \end{aligned} \quad (53)$$

where $\beta = \dfrac{2C_1 + C}{\alpha_0 - \epsilon - 2C_1}$.

Introducing $z^i = \bar{\partial}_t Q_h u(t^i) - u_t^i$, and writing $z^i = z_1^i + z_2^i$, where

$$z_1^i = \bar{\partial}_t Q_h u(t^i) - \bar{\partial}_t u(t^i) = \frac{1}{\tau}\int_{t_{i-1}}^{t_i}(Q_h - I)u_t \mathrm{d}t,$$

and

$$z_2^i = \bar{\partial}_t u(t^i) - u_t(t^i) = -\frac{1}{\tau}\int_{t_{i-1}}^{t_i}(t - t_{i-1})u_{tt}\mathrm{d}t.$$

From Lemma 4.3,

$$\begin{aligned} \sum_{i=1}^{n}\|z_1^i\|_0^2 &\leq C\tau^{-2}\sum_{i=1}^{n}(\textstyle\int_{t_{i-1}}^{t_i} Ch^{k+1}\|u_t\|_{k+1}\mathrm{d}t)^2 \\ &\leq C\tau^{-1}h^{k+1}\textstyle\int_0^{t_n}\|u_t\|_{k+1}^2\mathrm{d}t. \end{aligned} \quad (54)$$

Similarly

$$\sum_{i=1}^{n}\|z_2^i\|_0^2 \leq \sum_{i=1}^{n}(\int_{t_{i-1}}^{t_i}\|u_{tt}\|_0\mathrm{d}t)^2 = \tau\int_0^{t_n}\|u_{tt}\|_0^2\mathrm{d}t \quad (55)$$

Again by Lemma 4.3,

$$\begin{aligned} |||e^0|||_{w,1}^2 &= |||Q_h u^0 - u_h^0|||_{w,1}^2 = |||Q_h u^0 - u^0 + u^0 - u_h^0|||_{w,1}^2 \\ &\leq Ch^{2k}\|u^0\|_{k+1}^2 + |||u^0 - u_h^0|||_{w,1}^2. \end{aligned} \quad (56)$$

A combination of (53), (44) and (54)-(56) leads to (42). □

14

5. Numerical Experiment

In this section, we give three numerical examples using scheme (18) and consider the following parabolic problem [11]

$$u_t - \mathrm{div}(\mathbf{D}\nabla u) = f, \quad \text{in} \quad \Omega \times J, \tag{57}$$

with proper Dirichlet boundary and initial conditions. For simplicity, we let $D = 1,\, 10$; $\Omega = (0,1)\times(0,1)$ be unit square; and the time interval $J = (0,T)$ be $(0,1)$, in all three numerical examples. One can determine the initial and boundary conditions and source term $f(x,t)$ according to the corresponding analytical solution of each example.

We construct triangular mesh as follows. Firstly, we partition the square domain $\Omega = (0,1) \times (0,1)$ into $N \times N$ sub-squares uniformly to obtain the square mesh. Secondly, we divide each square element into two triangles by the diagonal line with a negative slope so that we complete the constructing of triangular mesh.

In the first example, the analytical solution is

$$u = \sin(\pi x)\sin(\pi y)\exp(-t). \tag{58}$$

For a set of simulations, different mesh sizes $h = 1/N(N = 4, 8, 16, 32, 64)$ and different diffusion coefficients $D = 1$ and $D = 10$ are taken, and their corresponding discrete norms errors and convergence rates (CR) are listed in Table 1 for $D = 1$ and $D = 10$. Here $|||\cdot|||_{w,1}$ is defined as discrete version of the definition of (15) without the term $\|v\|^2_{0,T}$.

Table 1: Numerical results of the fist example for $D = 1$ and $D = 10$.

	$D=1$		$D=10$	
h	$\vert\vert\vert u-u_h\vert\vert\vert_{w,1}$	CR	$\vert\vert\vert u-u_h\vert\vert\vert_{w,1}$	CR
2.5000e-01	1.6044e-01		1.2252e+00	
1.2500e-01	8.0594e-02	0.99	6.0921e-01	1.01
6.2500e-02	4.0329e-02	1.00	3.0412e-01	1.00
3.1250e-02	2.0165e-02	1.00	1.5200e-01	1.00
1.5625e-02	1.0082e-02	1.00	7.5990e-02	1.00

In the second example, the analytical solution is

$$u = x(1-x)y(1-y)\exp(x-y-t). \tag{59}$$

15

Numerical error results and CRs are listed in Table 2 for $D = 1$ and $D = 10$ based on the same triangular mesh as those of the first example.

Table 2: Numerical results of the second example for $D = 1$ and $D = 10$.

	$D = 1$		$D = 10$	
h	$\|\|\|u - u_h\|\|\|_{w,1}$	CR	$\|\|\|u - u_h\|\|\|_{w,1}$	CR
2.5000e-01	1.5858e-02		1.2052e-01	
1.2500e-01	7.9786e-03	0.99	5.9967e-02	1.01
6.2500e-02	3.9940e-03	1.00	2.9945e-02	1.00
3.1250e-02	1.9973e-03	1.00	1.4967e-02	1.00
1.5625e-02	9.9864e-04	1.00	7.4829e-03	1.00

In the third example, the analytical solution is

$$u = x(1 - x)y(1 - y)\exp(x + y + t). \tag{60}$$

Numerical error results and CRs are listed in Table 3 for $D = 1$ and $D = 10$ based on the same triangular mesh as those of the first example.

Table 3: Numerical results of the third example for $D = 1$ and $D = 10$.

	$D = 1$		$D = 10$	
h	$\|\|\|u - u_h\|\|\|_{w,1}$	CR	$\|\|\|u - u_h\|\|\|_{w,1}$	CR
2.5000e-01	3.1035e-01		2.3672e+00	
1.2500e-01	1.5916e-01	0.96	1.1977e+00	0.98
6.2500e-02	8.0078e-02	0.99	6.0062e-01	1.00
3.1250e-02	4.0099e-02	1.00	3.0052e-01	1.00
1.5625e-02	2.0056e-02	1.00	1.5029e-01	1.00

All three numerical examples show good agreement with the theoretical results in Section 4, which show that the WG-FEM (18) is stable and first order convergent in H^1 norm.

Acknowldgements The first author's research is partially supported by the Natural Science Foundation of Shandong Province of China grant ZR2013AM023, China Postdoctoral Science Foundation no. 2014M560547, and the fundamental research funds of Shandong university grant 2015JC019.

References

[1] R. A. Adams. *Sobolev spaces*. New York: Academic Press, 1975.

[2] P. G. Ciarlet, *The finite element method for elliptic problems*. Amsterdam: North-Holland, 1978.

[3] M. A. T. Elshebli, *Discrete maximum principle for the finite element solution of linear non-stationary diffusion-reaction problems*, Appl Math Model, 32, 2008: 1530-1541.

[4] F. Gao and L. Mu, L^2 *error estimate for weak Galerkin finite element methods for parabolic problems*, J. Comp. Math. 32(2), 2014: 195-204.

[5] Q. H. Li, J. Wang, *Weak Galerkin finite element methods for parabolic equations*, Numer Methods Partial Differential Equations 29(6), 2013: 2004-2024.

[6] M. Majidi, G. Starke,*Least-square Galerkin methods for parabolic problems: Semidiscretization in time*, SIAM J. Numer. Anal. 39, 2001: 1302-1323.

[7] A. K. Pani and G. Fairweather, H_1-*Galerkin mixed finite element methods for parabolic partial integro-differential equations*, IMA J. Numer. Anal. 22, 2002: 231-252.

[8] L. Mu, J. Wang, Y. Wang and X. Ye, *A computational study of the weak Galerkin method for the second order elliptic equations*, Numer Algor 63, 2013: 753-777.

[9] L. Mu, J. Wang, X. Ye and S. Zhao, *A numerical study on the weak Galerkin method for the Helmholtz equation*, Commun. Comput. Phys. 15(5), 2014: 1461-1479.

[10] P. Raviart, J. Thomas, *A mixed finite element method for second order elliptic problems*, Mathematical Aspects of the Finite Element Method, I.Galligani, E.Magenes, eds., Lectures Notes in Math. 606, Springer-Verlag, New York, 1977.

[11] H. X. Rui, S. D. Kim, S. Kim, *Split least-squares finite element methods for linear and nonlinear parabolic problems*, J. Comput. Appl. Math., 223, 2009: 938-952.

[12] M. Tabata, *Uniform convergence of the upwind finite element approximation for semilinear parabolic problems*, J.Math.Kypto Univ. (JMKYAZ), 18(2), 1978: 327-351.

[13] T. Vejchodský, S. Korotov, A. Hannukainen, *Discrete maximum principle for parabolic problems solved by prismatic finite elements*, Math Comput Simul, 80, 2010: 1758-1770.

[14] J. Wang and X. Ye, *A weak Galerkin method for second order elliptic problems*, J. Comput. Appl. Math., 241, 2013: 103-115.

[15] J. Wang and X. Ye, *A Weak Galerkin mixed finite element method for second-order elliptic problems*, Math. Comp. 83, 2014: 2101C2126.

[16] W. Wu, *Error estimates of generalized difference methods for nonlinear parabolic equations*, Math. Numer. Sinica, 2, 1987: 119-132.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 21, NO.6, 2016, COPYRIGHT 2016 EUDOXUS PRESS, LLC

Some fixed point theorems for generalized expansive mappings in cone metric spaces over Banach algebras

Binghua Jiang[1], Shaoyuan Xu[2]*, Huaping Huang[1], Zelin Cai[1]

1. School of Mathematics and Statistics, Hubei Normal University, Huangshi, 435002, China
2. School of Mathematics and Statistics, Hanshan Normal University, Chaozhou 521041, China

Abstract: In this paper, we prove some fixed point theorems for expansive mappings in cone metric spaces over Banach algebras without the assumption of normality of cones. Moreover, we give some examples to support our results. Our results improve and generalize the recent results of Aage and Salunke(2011).

MSC: 54H25, 47H10, 54E50

Keywords: Generalized expansive mapping, Cone metric space over Banach algebra, Spectral radius

1 Introduction and Preliminaries

In 2007 Huang and Zhang[1] introduced cone metric space and proved some fixed point theorems of contractive mappings in such spaces. Since then, some authors proved lots of fixed point theorems for contractive or expansive mappings in cone metric spaces that expanded certain fixed point results in metric spaces (see [2-14]). However, recently, it is not an attractive topic since some authors have appealed to the equivalence of some metric and cone metric fixed point results (see [21-24]). Recently [13] introduced the concept of cone metric space with Banach algebra and obtained some fixed point theorems in such spaces. Moreover, the authors of [13] gave an example to illustrate that the non-equivalence of fixed point theorems between cone metric spaces over Banach algebras and metric spaces

*Corresponding author: E-mail: xushaoyuan@126.com

(in usual sense). As a result, it is necessary to further investigate fixed point theorems in cone metric spaces over Banach algebras. In this paper, we generalize the famous Banach expansive mapping theorems as follows:

Let (X, d) be a complete cone metric space over Banach algebra $\mathcal{A}$ and K be a cone in $\mathcal{A}$. Suppose the mapping $T : X \to X$ is onto and satisfies the generalized expansive condition:

$$d(Tx, Ty) \succeq kd(x, y),$$

for all $x, y \in X$, where $k, k^{-1} \in K$ are generalized constants with $\rho(k^{-1}) < 1$. Then T has an unique fixed point in X.

Further, we give some other fixed point theorems for expansive mappings with generalized constants in cone metric spaces over Banach algebras. In addition, all cones are not necessarily normal ones. In these cases, our main results are not equivalent to those in metric spaces (see [7]).

For the sake of completeness, we introduce some basic concepts as follows:

Let $\mathcal{A}$ be a Banach algebras with a unit e, and θ the zero element of $\mathcal{A}$. A nonempty closed convex subset K of $\mathcal{A}$ is called a cone if and only if

(i) $\{\theta, e\} \subset K$;

(ii) $K^2 = KK \subset K, K \bigcap (-K) = \{\theta\}$;

(iii) $\lambda K + \mu K \subset K$ for all $\lambda, \mu \geq 0$.

On this basis, we define a partial ordering $\preceq$ with respect to K by $x \preceq y$ if and only if $y - x \in K$, we shall write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$, while $x \ll y$ will indicate that $y - x \in \text{int}K$, where $\text{int}K$ stands for the interior of K. If $\text{int}K \neq \emptyset$, then K is called a solid cone. Write $\|\cdot\|$ as the norm on $\mathcal{A}$. A cone K is called normal if there is a number $M > 0$ such that for all $x, y \in \mathcal{A}$,

$$\theta \preceq x \preceq y \Rightarrow \|x\| \leq M\|y\|.$$

The least positive number satisfying above is called the normal constant of K. An element $x \in \mathcal{A}$ is said to be invertible if there is an element $y \in \mathcal{A}$ such that $yx = xy = e$. The inverse of x is denoted by x^{-1}. For more details, we refer to [10, 13].

In the following we always suppose that $\mathcal{A}$ is a real Banach algebra with a unit e, K is a solid cone in $\mathcal{A}$ and $\preceq$ is a partial ordering with respect to K.

Definition 1.1([13]) Let X be a nonempty set and $\mathcal{A}$ a Banach algebra. Suppose that the mapping $d : X \times X \to \mathcal{A}$ satisfies:

(i) $\theta \prec d(x,y)$ for all $x,y \in X$ with $x \neq y$ and $d(x,y) = \theta$ if and only if $x = y$;

(ii) $d(x,y) = d(y,x)$ for all $x,y \in X$;

(iii) $d(x,y) \preceq d(x,z) + d(z,y)$ for all $x,y,z \in X$.

Then d is called a cone metric on X, and (X,d) is called a cone metric space over Banach algebra $\mathcal{A}$.

Definition 1.2([2]) Let (X,d) be a cone metric space, $x \in X$ and $\{x_n\}$ a sequence in X. Then

(i) $\{x_n\}$ converges to x whenever for every $c \in \mathcal{A}$ with $\theta \ll c$ there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$, we denote this by $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$ (as $n \to \infty$).

(ii) $\{x_n\}$ is a Cauchy sequence whenever for every $c \in \mathcal{A}$ with $\theta \ll c$ there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.

(iii) (X,d) is a complete cone metric space if every Cauchy sequence is convergent.

Lemma 1.3 ([7]) Let $u, v, w \in \mathcal{A}$. If $u \preceq v$ and $v \ll w$, then $u \ll w$.

Lemma 1.4 ([7]) Let $\mathcal{A}$ be a Banach algebra and $\{a_n\}$ a sequence in $\mathcal{A}$. If $a_n \to \theta$ $(n \to \infty)$, then for any $c \gg \theta$, there exists N such that for all $n > N$, one has $a_n \ll c$.

Lemma 1.5 ([10]) Let $\mathcal{A}$ be a Banach algebra with a unit e, $x \in \mathcal{A}$, then $\lim\limits_{n\to\infty} \|x^n\|^{\frac{1}{n}}$ exists and the spectral radius $\rho(x)$ satisfies

$$\rho(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \inf \|x^n\|^{\frac{1}{n}}.$$

If $\rho(x) < |\lambda|$, then $\lambda e - x$ invertible in $\mathcal{A}$, moreover,

$$(\lambda e - x)^{-1} = \sum_{i=0}^{\infty} \frac{x^i}{\lambda^{i+1}} \quad ,$$

where λ is a complex constant.

Lemma 1.6([10]) Let $\mathcal{A}$ be a Banach algebra with a unit $e, a, b \in \mathcal{A}$. If a commutes with b, then

$$\rho(a+b) \leq \rho(a) + \rho(b), \quad \rho(ab) \leq \rho(a)\rho(b).$$

Lemma 1.7([20])) Let K be a cone in a Banach algebra $\mathcal{A}$ and $k \in K$ be a given vector. Let $\{u_n\}$ be a sequence in K. If for each $c_1 \gg \theta$, there exists N_1 such that $u_n \ll c_1$ for all $n > N_1$, then for each $c_2 \gg \theta$, there exists N_2 such that $ku_n \ll c_2$ for all $n > N_2$.

Lemma 1.8([20]) If $\mathcal{A}$ is a Banach algebra with a solid cone K and $\|x_n\| \to 0 (n \to \infty)$, then for any $\theta \ll c$, there exists N such that for all $n > N$, we have $x_n \ll c$.

Remark 1.9 Let $\mathcal{A}$ be a Banach algebra and $k \in \mathcal{A}$. If $\rho(k) < 1$, then $\lim\limits_{n\to\infty} \|k^n\| = 0$.

2 Main results

In this section, we shall prove some fixed point theorems for expansive mappings in the setting of non-normal cone metric spaces over Banach algebras. Furthermore, we display two examples to support our main conclusions.

Theorem 2.1 Let (X, d) be a complete cone metric space over Banach algebra $\mathcal{A}$ and K be a solid cone in $\mathcal{A}$. Suppose that the mapping $T : X \to X$ is onto and satisfies the expansive expansive condition:

$$d(Tx, Ty) \succeq kd(x, y) + ld(Tx, y), \tag{2.1}$$

for all $x, y \in X$, where $k, l, k^{-1} \in K$ are two generalized constants. If $e - l \in K$ and $\rho(k^{-1}) < 1$, then T has a fixed point in X.

Proof Since T is an onto mapping, for each $x_0 \in X$, there exists $x_1 \in X$ such that $Tx_1 = x_0$. Continuing this process, we can define $\{x_n\}$ by $x_n = Tx_{n+1}(n = 0, 1, 2, \ldots)$. Without loss of generality, we assume $x_{n-1} \neq x_n$ for all $n \geq 1$. According to (2.1), we have

$$\begin{aligned} d(x_n, x_{n-1}) &= d(Tx_{n+1}, Tx_n) \\ &\succeq kd(x_{n+1}, x_n) + ld(Tx_{n+1}, x_n) \\ &= kd(x_{n+1}, x_n) + ld(x_n, x_n) \\ &= kd(x_{n+1}, x_n), \end{aligned}$$

then

$$d(x_{n+1}, x_n) \preceq k^{-1}d(x_n, x_{n-1}).$$

Letting $k^{-1} = h$ we get

$$d(x_{n+1}, x_n) \preceq hd(x_n, x_{n-1}) \preceq \cdots \preceq h^n d(x_1, x_0).$$

So by the triangle inequality and $\rho(h) < 1$, for all $m > n$, we see

$$\begin{aligned} d(x_m, x_n) &\preceq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+1}, x_n) \\ &\preceq (h^{m-1} + h^{m-2} + \cdots + h^n) d(x_1, x_0) \\ &= (e + h + \cdots + h^{m-n-1}) h^n d(x_1, x_0) \\ &\preceq \Big(\sum_{i=0}^{\infty} h^i \Big) h^n d(x_1, x_0) \\ &= (e-h)^{-1} h^n d(x_1, x_0). \end{aligned}$$

By Lemma 1.8 and the fact that $\|(e-h)^{-1}h^n d(x_1, x_0)\| \to 0 (n \to \infty)$ (Because of Remark 1.9, $\|h^n\| \to 0$ $(n \to \infty)$), it follows that for any $c \in \mathcal{A}$ with $\theta \ll c$, there exists N such that for all $m > n > N$, we have

$$d(x_m, x_n) \preceq (e-h)^{-1} h^n d(x_1, x_0) \ll c,$$

which implies that $\{x_n\}$ is a Cauchy sequence.

By the completeness of X, there exists $x^* \in X$ such that $x_n \to x^*(n \to \infty)$. Consequently, we can find an $x^{**} \in X$ such that $Tx^{**} = x^*$. Now we show that $x^{**} = x^*$. In fact,

$$\begin{aligned} d(x^*, x_n) &= d(Tx^{**}, Tx_{n+1}) \\ &\succeq kd(x^{**}, x_{n+1}) + ld(Tx^{**}, x_{n+1}) \\ &= kd(x^{**}, x_{n+1}) + ld(x^*, x_{n+1}). \end{aligned}$$

Since

$$d(x^*, x_n) \preceq d(x^*, x_{n+1}) + d(x_{n+1}, x_n),$$

it follows that

$$kd(x^{**}, x_{n+1}) \preceq (e-l) d(x^*, x_{n+1}) + d(x_{n+1}, x_n).$$

Now, we have

$$d(x^{**}, x_{n+1}) \preceq k^{-1}((e-l) d(x^*, x_{n+1}) + d(x_{n+1}, x_n)).$$

Note that $x_n \to x^*(n \to \infty)$, by Lemma 1.7, it follows that for any $c \in \mathcal{A}$ with $\theta \ll c$, there exists N such that for any $n > N$, we have

$$k^{-1}((e-l) d(x^*, x_{n+1}) + d(x_{n+1}, x_n)) \ll c.$$

Thus

$$d(x^{**}, x_{n+1}) \ll c.$$

Since the limit of a convergent sequence in cone metric space over Banach algebra is unique, we get $x^{**} = x^*$, i.e., x^* is a fixed point of T.

Theorem 2.2 Let (X, d) be a complete cone metric space over Banach algebra $\mathcal{A}$ and K be a solid cone in $\mathcal{A}$. Suppose that the mapping $T : X \to X$ is onto and satisfies the generalized expansive condition:

$$d(Tx, Ty) \succeq kd(x, y) + ld(x, Tx) + pd(y, Ty), \tag{2.2}$$

for all $x, y \in X$, where $k, l, p, e - p \in K$ are generalized constants with $(k + l)^{-1} \in K$ and $\rho\big[(k + l)^{-1}(e - p)\big] < 1$. Then T has a fixed point in X.

Proof Since T is an onto mapping, for each $x_0 \in X$, there exists $x_1 \in X$ such that $Tx_1 = x_0$. Continuing this process, we can define $\{x_n\}$ by $x_n = Tx_{n+1} (n = 0, 1, 2, \ldots)$. Without loss of generality, we suppose $x_{n-1} \neq x_n$ for all $n \geq 1$. According to (2.2), we have

$$\begin{aligned} d(x_n, x_{n-1}) &= d(Tx_{n+1}, Tx_n) \\ &\succeq kd(x_{n+1}, x_n) + ld(x_{n+1}, Tx_{n+1}) + pd(x_n, Tx_n) \\ &= kd(x_{n+1}, x_n) + ld(x_{n+1}, x_n) + pd(x_n, x_{n-1}), \end{aligned}$$

which implies that

$$(k + l)d(x_n, x_{n+1}) \preceq (e - p)d(x_n, x_{n-1}).$$

Put $k + l = r$, then

$$rd(x_n, x_{n+1}) \preceq (e - p)d(x_n, x_{n-1}). \tag{2.3}$$

Since r is invertible, to multiply r^{-1} in both sides of (2.3), we have

$$d(x_n, x_{n+1}) \preceq hd(x_n, x_{n-1}),$$

where $h = (k+l)^{-1}(e-p)$. Note that $\rho(h) < 1$ and for all $m > n$,

$$\begin{aligned} d(x_m, x_n) &\preceq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+1}, x_n) \\ &\preceq (h^{m-1} + h^{m-2} + \cdots + h^n) d(x_1, x_0) \\ &= (e + h + \cdots + h^{m-n-1}) h^n d(x_1, x_0) \\ &\preceq \Big(\sum_{i=0}^{\infty} h^i\Big) h^n d(x_1, x_0) \\ &= (e-h)^{-1} h^n d(x_1, x_0). \end{aligned}$$

As is shown in the proof of Theorem 2.1, it follows that $\{x_n\}$ is a Cauchy sequence. Then by the completeness of X, there exists $x^* \in X$ such that $x_n \to x^* (n \to \infty)$. Consequently, we can find a $x^{**} \in X$ such that $Tx^{**} = x^*$. Now we show that $x^{**} = x^*$. Indeed, Since

$$\begin{aligned} d(x^*, x_n) &= d(Tx^{**}, Tx_{n+1}) \\ &\succeq kd(x^{**}, x_{n+1}) + ld(x^{**}, Tx^{**}) + pd(x_{n+1}, Tx_{n+1}) \\ &= kd(x^{**}, x_{n+1}) + ld(x^{**}, x^*) + pd(x_{n+1}, x_n). \end{aligned}$$

Then

$$d(x^*, x_n) \succeq kd(x^{**}, x_{n+1}) + ld(x^{**}, x_{n+1}) - ld(x^*, x_{n+1}) + pd(x_{n+1}, x_n).$$

Note that

$$d(x^*, x_n) \preceq d(x^*, x_{n+1}) + d(x_{n+1}, x_n),$$

thus

$$d(x^*, x_{n+1}) + d(x_{n+1}, x_n) \succeq (k+l)d(x^{**}, x_{n+1}) - ld(x^*, x_{n+1}) + pd(x_{n+1}, x_n),$$

which implies that

$$(k+l)d(x^{**}, x_{n+1}) \preceq (e+l)d(x^*, x_{n+1}) + (e-p)d(x_{n+1}, x_n).$$

Since $k + l = r$ is invertible, we have

$$d(x^{**}, x_{n+1}) \preceq r^{-1}\big((e+l)d(x^*, x_{n+1}) + (e-p)d(x_{n+1}, x_n)\big).$$

Owing to $x_n \to x^* (n \to \infty)$, it follows by Lemma 1.7 that for any $c \in \mathcal{A}$ with $\theta \ll c$ there exists N such that for any $n > N$,

$$r^{-1}\big((e+l)d(x^*, x_{n+1}) + (e-p)d(x_{n+1}, x_n)\big) \ll c,$$

hence

$$d(x^{**}, x_{n+1}) \ll c.$$

Since the limit of a convergent sequence in cone metric space over Banach algebra is unique, we have $x^{**} = x^*$, i.e., x^* is a fixed point of T.

Corollary 2.3 Let (X, d) be a complete cone metric space over Banach algebra $\mathcal{A}$ and K be a cone in $\mathcal{A}$. Suppose the mapping $T : X \to X$ is onto and satisfies the generalized expansive condition:

$$d(Tx, Ty) \succeq kd(x, y), \tag{2.4}$$

for all $x, y \in X$, where $k, k^{-1} \in K$ are generalized constants with $\rho(k^{-1}) < 1$. Then T has an unique fixed point in X.

Proof By using Theorem 2.1 and Theorem 2.2, letting $l = p = \theta$, we need to only prove the fixed point is unique. Indeed, if y^* is another fixed point of T, then

$$d(x^*, y^*) = d(Tx^*, Ty^*) \succeq kd(x^*, y^*),$$

that is,

$$d(x^*, y^*) \preceq k^{-1}d(x^*, y^*) = hd(x^*, y^*).$$

Thus

$$d(x^*, y^*) \preceq hd(x^*, y^*) \preceq h^2d(x^*, y^*) \preceq \cdots \preceq h^nd(x^*, y^*).$$

In view of $\|h^nd(x^*, y^*)\| \to 0 (n \to \infty)$, it establishes that for any $c \in \mathcal{A}$ with $\theta \ll c$, there exists N_2 such that for all $n > N_2$, we have

$$d(x^*, y^*) \preceq h^nd(x^*, y^*) \ll c,$$

so $d(x^*, y^*) = \theta$, which implies that $x^* = y^*$. Hence, the fixed point is unique.

Remark 2.4 Note that Corollary 2.3 only assumes that $\rho(k^{-1}) < 1$, which implies $\rho(k) > 1$, neither $k \succ e$ nor $\|k\| > 1$. This is a vital improvement.

Remark 2.5 Since we get the fixed point theorems in the setting of non-normal cone metric spaces over Banach algebras, our results are never equivalent to the fixed point versions in metric spaces (see [7, 13]). The following examples illustrate our conclusions.

Example 2.6 Let $\mathcal{A} = C^1_{\mathbb{R}}[0, \frac{1}{4}]$ and define a norm on $\mathcal{A}$ by $\|x\| = \|x\|_\infty + \|x'\|_\infty$ for $x \in \mathcal{A}$. Define multiplication in $\mathcal{A}$ as just pointwise multiplication. Then $\mathcal{A}$ is a Banach algebra with a unit $e = 1$. The set $K = \{x \in \mathcal{A} : x \geq 0\}$ is a non-normal cone in $\mathcal{A}$ (see [7]). Let $X = \mathbb{R}$. Define $d : X \times X \to \mathcal{A}$ by $d(x, y)(t) = |x - y|e^t$, for all $t \in [0, \frac{1}{4}]$. Further, let $T : X \to X$ be a mapping defined by $Tx = 2x$ and let $k \in K$ define by $k(t) = \frac{4}{2t+3}$. By careful calculations one sees that all the conditions of Corollary 2.3 are fulfilled. The point $x = 0$ is the unique fixed point of the mapping T.

Example 2.7 Let $\mathcal{A} = \{a = (a_{ij})_{3\times 3} \mid a_{ij} \in \mathbb{R}, 1 \leq i, j \leq 3\}$ and $\|a\| = \frac{1}{3} \sum\limits_{1\leq i,j\leq 3} \mid a_{ij} \mid$. Then the set $K = \{a \in \mathcal{A} \mid a_{ij} \geq 0, 1 \leq i, j \leq 3\}$ is a normal cone in $\mathcal{A}$. Let $X = \{1, 2, 3\}$. Define $d : X \times X \to \mathcal{A}$ by $d(1, 1) = d(2, 2) = d(3, 3) = \theta$ and

$$d(1,2) = d(2,1) = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 4 & 5 & 6 \end{pmatrix},$$

$$d(1,3) = d(3,1) = \begin{pmatrix} 2 & 4 & 6 \\ 0 & 0 & 0 \\ 3 & 4 & 5 \end{pmatrix},$$

$$d(2,3) = d(3,2) = \begin{pmatrix} 2 & 4 & 6 \\ 0 & 0 & 0 \\ 3 & 4 & 5 \end{pmatrix}.$$

We find that (X, d) is a solid cone metric space over Banach algebra $\mathcal{A}$. Let $T : X \to X$ be a mapping defined by $T1 = 2, T2 = 1, T3 = 3$, and let $k, l, p \in K$ be defined by

$$k = \begin{pmatrix} \frac{4}{5} & 0 & 0 \\ 0 & \frac{4}{5} & 0 \\ 0 & 0 & \frac{4}{5} \end{pmatrix},$$

$$p = l = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{10} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $d(Tx, Ty) \succeq kd(x, y) + ld(x, Tx) + pd(y, Ty)$, where $k, l, p, e - p \in K$ are generalized constants. It is easy to prove that $\|e - k - l\| < 1$ and $\|(k + l)^{-1}(e - p)\| < 1$, which imply $\rho(e - k - l) < 1$ and $\rho\big[(k + l)^{-1}(e - p)\big] < 1$. Clearly, all conditions of Theorem 2.2 are fulfilled. Hence T has a fixed point $x = 3$ in X.

Remark 2.8 It needs to emphasis that according to the expansive condition of [11, Theorem 2.1], we are easy to see that the mapping discussed is an injection, and the authors attempt to use [11, Example 2.7] to support this theorem. But unfortunately, this is impossible, since the mapping appearing in this example is not an injection at all. Therefore, it is unreasonable. Basing on the facts above, we may verify that Example 2.7 in this paper is reasonable. It is also interesting, since here we use matrixes as generalized constants.

Competing interests

The authors declare that there have no competing interests.

Acknowledgements

The research is partially supported by the Foundation of Education Ministry, Hubei Province, China (no. Q20122203). The research is also partially supported by the foundation of the research item of Strong Department of Engineering Innovation, Research on Fixed Point Theory with Banach Algebras in Abstract Spaces and Applications, which is sponsored by the Strong School of Engineering Innovation of Hanshan Normal University, China (2013).

References

[1] L.-G. Huang, X. Zhang, "Cone metric space and fixed point theorems of contractive mappings," *Journal of Mathematical Analysis and Applications*, Vol. 332, pp. 1468-1476, 2007.

[2] Sh. Rezapour, R. Hamlbarani, "Some notes on the paper 'Cone metric spaces and fixed point theorems of contractive mappings'," *Journal of Mathematical Analysis and Applications*, Vol. 345, pp 719-724, 2008.

[3] Y. Han, S.-Y. Xu, "New common fixed point results for four maps on cone metric spaces," *Applied Mathematics*, Vol. 2, pp. 1114-1118, 2011.

[4] C.D. Bari, P. Vetro, "φ-pairs and common fixed points in cone metric spaces," *Rendiconti del Circolo Matematico di Palermo*, Vol. 57, pp. 279-285, 2008.

[5] X. Zhang, "Common fixed point theorems of Lipschitz type mappings in cone metric spaces," *Acta Mathematica Sinica(in Chinese)*, Vol. 53, no. 1, pp. 1139-1148, 2010.

[6] M. Abbas, G. Jungck, "Common fixed point results for noncommuting mappings without continuity in cone metic spaces," *Journal of Mathematical Analysis and Applications*, Vol. 341, pp. 416-420, 2008.

[7] S. Janković, Z. Kadelburg, S. Radenović, "On cone metric spaces: A survey," *Nonlinear Analysis*, Vol. 74, pp. 2591-2601, 2011.

[8] S. Radenović, "Common fixed points under contractive conditions in cone metric spaces," *Computers and Mathematics with Applications*, Vol. 58, pp. 1273-1278, 2009.

[9] I. Altun, B. Damjanović, D. Djorić, "Fixed point and common fixed point theorems on ordered cone metric spaces," *Applied Mathematics Letter*, Vol. 23, pp. 310-316, 2010.

[10] W. Rudin, *Functional Analysis*, McGraw-Hill, New York, NY, USA, 2nd edition, 1991.

[11] C. T. Aage, J. N. Salunke, "Some fixed point theorems for expansion onto mappings on cone metric spaces," *Acta Mathematica Sinica(English series)*, Vol. 27, no. 6, pp. 1101-1106, 2011.

[12] Y. Han, S.Y. Xu, "Some new theorems of expanding mappings without continuity in cone metric spaces," *Fixed point Theory and Applications*, Vol. 3, pp. 1-9, 2013.

[13] H. Liu, S. Xu, "Cone metric space with Banach algebras and fixed point theorems of generalized Lipschitz mapping," *Fixed point Theory and Applications*, Vol. 320, pp. 1-10, 2013.

[14] W. Shatanawi, F. Awawdeh, "Some fixed point and coincidence point theorems for expansive maps in cone metric space," *Fixed point Theory and Applications*, Vol. 19, pp. 1-10, 2012.

[15] S. Chouhan, N. Malviya, "A fixed point theorem for expansive type mappings in cone metric spaces," *International Mathematical Forum*, Vol. 6, no. 18, pp. 891-897, 2011.

[16] W. Shatanawi, F. Awawdeh, "Some fixed and coincidence point theorems for expansive maps in cone metric spaces," *Fixed point Theory and Applications*, Vol. 19, pp. 1-10, 2012.

[17] Z. Kadelburg, P. P. Murthy, S. Radenović, "Common fixed points for expansive mappings in cone metric spaces," *International Journal of Mathemathical Analysis*, Vol. 5, no. 27, 1309-1319, 2011.

[18] B.-H. Jiang, S.-Y. Xu, L. Shi, "Coupled coincidence points for mixed monotone random operators in partially ordered metric spaces," *Abstract and Applied Analysis*, Vol. 2014, Article ID 484857, 9 pages, http://dx. doi. org/10. 1155/2014/484857.

[19] A. G. B Ahmad, Z. M. Fadail, M. Abbas, Z. Kadelburg, Stojan Radenović, "Some fixed and periodic points in abstract metric spaces," *Abstract and Applied Analysis*, Vol. 2012, Article ID 908423, 15 pages, doi: 10.1155/2012/908423 62, 1677-1684.

[20] S. -Y. Xu, S. Radenović, "Fixed point theorems of generalized Lipschitz mappings on cone metric spaces over Banach algebras without assumption of normality," *Fixed point Theory and Applications*, Vol. 102, pp. 1-12, 2014.

[21] H. Cakalli, A. Sonmez, C. Genc, "On an equivalence of topological vector space valued cone metric spaces and metric spaces," *Applied Mathematics Letter*, Vol. 25, pp. 429-433, 2012.

[22] W.-S. Du, "A note on cone metric fixed point theory and its equivalence," *Nonlinear Analysis*, Vol. 72, no. 5, pp. 2259-2261, 2010.

[23] Z. Kadelburg, S. Radenović, V. Rakocević, "A note on the equivalence of some metric and cone metric fixed point results," *Applied Mathematics Letter*, Vol. 24, pp. 370-374, 2011.

[24] Y.-Q Feng, W. Mao, "The equivalence of cone metric spaces and metric spaces," *Fixed Point Theory*, Vol. 11, no. 2, pp. 259-263, 2010.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 21, NO.6, 2016, COPYRIGHT 2016 EUDOXUS PRESS, LLC

ADDITIVE ρ-FUNCTIONAL INEQUALITIES IN FUZZY NORMED SPACES

JI-HYE KIM, GEORGE A. ANASTASSIOU AND CHOONKIL PARK*

Abstract. In this paper, we solve the following additive ρ-functional inequalities

$$N(f(x+y)-f(x)-f(y),t) \quad \leq \quad N\left(\rho\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right),t\right) \qquad (0.1)$$

and

$$N\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y),t\right) \quad \leq \quad N(\rho\left(f(x+y)-f(x)-f(y)\right),t) \qquad (0.2)$$

in fuzzy normed spaces, where ρ is a fixed real number with $|\rho|<1$.

Using the fixed point method, we prove the Hyers-Ulam stability of the additive ρ-functional inequalities (0.1) and (0.2) in fuzzy Banach spaces.

1. Introduction and preliminaries

Katsaras [19] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view [11, 23, 48]. In particular, Bag and Samanta [2], following Cheng and Mordeson [8], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [22]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [3].

We use the definition of fuzzy normed spaces given in [2, 27, 28] to investigate the Hyers-Ulam stability of additive ρ-functional inequalities in fuzzy Banach spaces.

Definition 1.1. [2, 27, 28, 29] Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0,1]$ is called a *fuzzy norm* on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

(N_1) $N(x,t)=0$ for $t \leq 0$;
(N_2) $x=0$ if and only if $N(x,t)=1$ for all $t>0$;
(N_3) $N(cx,t)=N(x,\frac{t}{|c|})$ if $c \neq 0$;
(N_4) $N(x+y,s+t) \geq \min\{N(x,s),N(y,t)\}$;
(N_5) $N(x,\cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t)=1$.
(N_6) for $x \neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

The pair (X,N) is called a *fuzzy normed vector space*.

The properties of fuzzy normed vector spaces and examples of fuzzy norms are given in [26, 27].

Definition 1.2. [2, 27, 28, 29] Let (X,N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to *be convergent* or *converge* if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n-x,t)=1$

2010 *Mathematics Subject Classification.* Primary 46S40, 39B52, 47H10, 39B62, 26E50, 47S40.

Key words and phrases. fuzzy Banach space; additive ρ-functional inequality; fixed point method; Hyers-Ulam stability.

*Corresponding author.

for all $t > 0$. In this case, x is called the *limit* of the sequence $\{x_n\}$ and we denote it by N-$\lim_{n\to\infty} x_n = x$.

Definition 1.3. [2, 27, 28, 29] Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called *Cauchy* if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well-known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space*.

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to x_0 in X, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be *continuous* on X (see [3]).

The stability problem of functional equations originated from a question of Ulam [47] concerning the stability of group homomorphisms.

The functional equation $f(x+y) = f(x)+f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [15] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [39] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [12] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach.

The functional equation $f\left(\frac{x+y}{2}\right) = \frac{1}{2}f(x)+\frac{1}{2}f(y)$ is called the *Jensen equation*. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [7, 16, 18, 20, 21, 24, 35, 36, 37, 41, 42, 43, 44, 45, 46]).

Gilányi [13] showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(x - y)\| \leq \|f(x + y)\| \tag{1.1}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(x + y) + f(x - y).$$

See also [40]. Fechner [10] and Gilányi [14] proved the Hyers-Ulam stability of the functional inequality (1.1). Park, Cho and Han [34] investigated the Cauchy additive functional inequality

$$\|f(x) + f(y) + f(z)\| \leq \|f(x + y + z)\| \tag{1.2}$$

and the Cauchy-Jensen additive functional inequality

$$\|f(x) + f(y) + 2f(z)\| \leq \left\|2f\left(\frac{x+y}{2} + z\right)\right\| \tag{1.3}$$

and proved the Hyers-Ulam stability of the functional inequalities (1.2) and (1.3) in Banach spaces.

Park [32, 33] defined additive ρ-functional inequalities and proved the Hyers-Ulam stability of the additive ρ-functional inequalities in Banach spaces and non-Archimedean Banach spaces.

We recall a fundamental result in fixed point theory.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x,z) \le d(x,y) + d(y,z)$ for all $x, y, z \in X$.

Theorem 1.4. [4, 9] Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that
(1) $d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \ge n_0$;
(2) the sequence $\{J^n x\}$ converges to a fixed point y^* of J;
(3) y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;
(4) $d(y, y^*) \le \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

In 1996, G. Isac and Th.M. Rassias [17] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [5, 6, 26, 30, 31, 37, 38]).

In Section 2, we solve the additive ρ-functional inequality (0.1) and prove the Hyers-Ulam stability of the additive ρ-functional inequality (0.1) in fuzzy Banach spaces by using the fixed point method.

In Section 3, we solve the additive ρ-functional inequality (0.2) and prove the Hyers-Ulam stability of the additive ρ-functional inequality (0.2) in fuzzy Banach spaces by using the fixed point method.

2. Additive ρ-functional inequality (0.1)

In this section, we prove the Hyers-Ulam stability of the additive ρ-functional inequality (0.1) in fuzzy Banach spaces. Let ρ be a real number with $|\rho| < 1$. We need the following lemma to prove the main results.

Lemma 2.1. *Let (Y, N) be a fuzzy normed vector spaces. Let $f : X \to Y$ be a mapping such that*

$$N(f(x+y) - f(x) - f(y), t) \ge N\left(\rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right), t\right) \tag{2.1}$$

for all $x, y \in X$ and all $t > 0$. Then f is Cauchy additive, i.e., $f(x+y) = f(x) + f(y)$ for all $x, y \in X$.

Proof. Assume that $f : X \to Y$ satisfies (2.1).

Letting $x = y = 0$ in (2.1), we get $N(f(0), t) = N(0, t) = 1$. So $f(0) = 0$.

Letting $y = x$ in (2.1), we get $N(f(2x) - 2f(x), t) \ge N(0, t) = 1$ and so $f(2x) = 2f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{2} f(x) \tag{2.2}$$

for all $x \in X$.

It follows from (2.1) and (2.2) that

$$\begin{aligned} N(f(x+y) - f(x) - f(y), t) &\ge N\left(\rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right), t\right) \\ &= N\left(\rho(f(x+y) - f(x) - f(y)), t\right) \\ &= N\left(f(x+y) - f(x) - f(y), \frac{t}{|\rho|}\right) \end{aligned}$$

for all $t > 0$. By (N_5) and (N_6), $N(f(x+y) - f(x) - f(y), t) = 1$ for all $t > 0$. It follows from (N_2) that

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$. $\square$

Theorem 2.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \le \frac{L}{2} \varphi(2x, 2y)$$

for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying

$$N(f(x+y) - f(x) - f(y), t) \ge \min\left\{ N\left(\rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right), t\right), \frac{t}{t + \varphi(x,y)}\right\} \tag{2.3}$$

for all $x, y \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N(f(x) - A(x), t) \ge \frac{(2-2L)t}{(2-2L)t + L\varphi(x,x)} \tag{2.4}$$

for all $x \in X$ and all $t > 0$.

Proof. Letting $y = x$ in (2.3), we get

$$N(f(2x) - 2f(x), t) \ge \frac{t}{t + \varphi(x,x)} \tag{2.5}$$

for all $x \in X$.

Consider the set

$$S := \{g : X \to Y\}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\left\{\mu \in \mathbb{R}_+ : N(g(x) - h(x), \mu t) \ge \frac{t}{t + \varphi(x,x)}, \ \forall x \in X, \forall t > 0\right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [25, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \ge \frac{t}{t + \varphi(x,x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\varepsilon t) &= N\left(2g\left(\frac{x}{2}\right) - 2h\left(\frac{x}{2}\right), L\varepsilon t\right) \\ &= N\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{L}{2}\varepsilon t\right) \\ &\ge \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \varphi\left(\frac{x}{2}, \frac{x}{2}\right)} \ge \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \frac{L}{2}\varphi(x,x)} \\ &= \frac{t}{t + \varphi(x,x)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \le L\varepsilon$. This means that

$$d(Jg, Jh) \le L d(g,h)$$

for all $g, h \in S$.

It follows from (2.5) that

$$N\left(f(x) - 2f\left(\frac{x}{2}\right), \frac{L}{2}t\right) \ge \frac{t}{t + \varphi(x,x)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \le \frac{L}{2}$.

By Theorem 1.4, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, i.e.,

$$A\left(\frac{x}{2}\right) = \frac{1}{2}A(x) \tag{2.6}$$

for all $x \in X$. Since $f : X \to Y$ is odd, $A : X \to Y$ is an odd mapping. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This implies that A is a unique mapping satisfying (2.6) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N(f(x) - A(x), \mu t) \ge \frac{t}{t + \varphi(x,x)}$$

for all $x \in X$;

(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) = A(x)$$

for all $x \in X$;

(3) $d(f, A) \le \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, A) \le \frac{L}{2 - 2L}.$$

This implies that the inequality (2.4) holds.

By (2.3),

$$\begin{aligned} & N\left(2^n\left(f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right), 2^n t\right) \\ & \ge \min\left\{N\left(\rho\left(2^{n+1} f\left(\frac{x+y}{2^{n+1}}\right) - 2^n f\left(\frac{x}{2^n}\right) - 2^n f\left(\frac{y}{2^n}\right)\right), 2^n t\right), \frac{t}{t + \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)}\right\} \end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. So

$$\begin{aligned} & N\left(2^n\left(f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right), t\right) \\ & \ge \min\left\{N\left(\rho\left(2^{n+1} f\left(\frac{x+y}{2^{n+1}}\right) - 2^n f\left(\frac{x}{2^n}\right) - 2^n f\left(\frac{y}{2^n}\right)\right), t\right), \frac{\frac{t}{2^n}}{\frac{t}{2^n} + \frac{L^n}{2^n}\varphi(x,y)}\right\} \end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{\frac{t}{2^n}}{\frac{t}{2^n} + \frac{L^n}{2^n}\varphi(x,y)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$N(A(x+y) - A(x) - A(y), t) \ge N\left(\rho\left(2A\left(\frac{x+y}{2}\right) - A(x) - A(y)\right), t\right)$$

for all $x, y \in X$ and all $t > 0$. By Lemma 2.1, the mapping $A : X \to Y$ is Cauchy additive, as desired. $\square$

Corollary 2.3. *Let $\theta \geq 0$ and let p be a real number with $p > 1$. Let X be a normed vector space with the norm $\|\cdot\|$. Let $f : X \to Y$ be an odd mapping satisfying*

$$\begin{aligned} &N\left(f(x+y) - f(x) - f(y), t\right) \\ &\quad \geq \min\left\{N\left(\rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right), t\right), \frac{t}{t + \theta(\|x\|^p + \|y\|^p)}\right\} \end{aligned}$$

for all $x, y \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n\to\infty} 2^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\left(f(x) - A(x), t\right) \geq \frac{(2^p - 2)t}{(2^p - 2)t + 2\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.2 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{1-p}$, and we get the desired result. $\square$

Theorem 2.4. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq 2L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying (2.3). Then $A(x) := N\text{-}\lim_{n\to\infty} \frac{1}{2^n} f\left(2^n x\right)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\left(f(x) - A(x), t\right) \geq \frac{(2 - 2L)t}{(2 - 2L)t + \varphi(x, x)} \tag{2.7}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2.

It follows from (2.5) that

$$N\left(f(x) - \frac{1}{2}f(2x), \frac{1}{2}t\right) \geq \frac{t}{t + \varphi(x, x)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq \frac{1}{2}$. Hence

$$d(f, A) \leq \frac{1}{2 - 2L},$$

which implies that the inequality (2.7) holds.

The rest of the proof is similar to the proof of Theorem 2.2. $\square$

Corollary 2.5. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 1$. Let X be a normed vector space with the norm $\|\cdot\|$. Let $f : X \to Y$ be an odd mapping satisfying*

$$\begin{aligned} &N\left(f(x+y) - f(x) - f(y), t\right) \\ &\quad \geq \min\left\{N\left(\rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right), t\right), \frac{t}{t + \theta(\|x\|^p + \|y\|^p)}\right\} \end{aligned}$$

for all $x, y \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\left(f(x) - A(x), t\right) \geq \frac{(2 - 2^p)t}{(2 - 2^p)t + 2\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.4 by taking $\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{p-1}$, and we get the desired result. $\square$

3. Additive ρ-functional inequality (0.2)

In this section, we prove the Hyers-Ulam stability of the additive ρ-functional inequality (0.2) in fuzzy Banach spaces. Let ρ be a fuzzy number with $|\rho| < 1$.

Lemma 3.1. *Let (Y, N) be a fuzzy normed vector spaces. A mapping $f : X \to Y$ satisfies $f(0) = 0$ and*

$$N\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y), t\right) \geq N\left(\rho\left(f(x+y) - f(x) - f(y)\right), t\right) \tag{3.1}$$

for all $x, y \in X$ and all $t > 0$. Then f is Cauchy additive, i.e.,$f(x+y) = f(x) + f(y)$ for all $x, y \in X$.

Proof. Assume that $f : X \to Y$ satisfies (3.1).

Letting $y = 0$ in (3.1), we get $N\left(2f\left(\frac{x}{2}\right) - f(x), t\right) \geq N(0, t) = 1$ and so

$$f\left(\frac{x}{2}\right) = \frac{1}{2}f(x) \tag{3.2}$$

for all $x \in X$.

It follows from (3.1) and (3.2) that

$$\begin{aligned} N(f(x+y) - f(x) - f(y), t) &= N\left(2f\left(\frac{x+y}{2} - f(x) - f(y)\right), t\right) \\ &\geq N\left(\rho(f(x+y) - f(x) - f(y)), t\right) \\ &= N\left(f(x+y) - f(x) - f(y), \frac{t}{|\rho|}\right) \end{aligned}$$

for all $t > 0$. By (N_5) and (N_6), $N(f(x+y) - f(x) - f(y), t) = 1$ for all $t > 0$. It follows from (N_2) that

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$. $\square$

Theorem 3.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \leq \frac{L}{2}\varphi(2x, 2y)$$

for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying

$$\begin{aligned} &N\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y), t\right) \\ &\qquad \geq \min\left\{N\left(\rho\left(f(x+y) - f(x) - f(y)\right), t\right), \frac{t}{t + \varphi(x,y)}\right\} \end{aligned} \tag{3.3}$$

for all $x, y \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\left(f(x) - A(x), t\right) \geq \frac{(1-L)t}{(1-L)t + \varphi(x,0)} \tag{3.4}$$

for all $x \in X$ and all $t > 0$.

Proof. Since f is odd, $f(0) = 0$.
Letting $y = 0$ in (3.3), we get

$$N\left(f(x) - 2f\left(\frac{x}{2}\right), t\right) = N\left(2f\left(\frac{x}{2}\right) - f(x), t\right) \geq \frac{t}{t + \varphi(x, 0)} \tag{3.5}$$

for all $x \in X$.

Consider the set

$$S := \{g : X \to Y\}$$

and introduce the generalized metric on S:

$$d(g, h) = \inf\left\{\mu \in \mathbb{R}_+ : N(g(x) - h(x), \mu t) \geq \frac{t}{t + \varphi(x, 0)}, \ \forall x \in X, \forall t > 0\right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [25, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \geq \frac{t}{t + \varphi(x, 0)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\varepsilon t) &= N\left(2g\left(\frac{x}{2}\right) - 2h\left(\frac{x}{2}\right), L\varepsilon t\right) \\ &= N\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{L}{2}\varepsilon t\right) \\ &\geq \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \varphi\left(\frac{x}{2}, 0\right)} \geq \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \frac{L}{2}\varphi(x, 0)} \\ &= \frac{t}{t + \varphi(x, 0)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in S$.

It follows from (3.5) that

$$N\left(f(x) - 2f\left(\frac{x}{2}\right), t\right) \geq \frac{t}{t + \varphi(x, 0)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq 1$.

By Theorem 1.4, there exists a mapping $A : X \to Y$ satisfying the following:
(1) A is a fixed point of J, i.e.,

$$A\left(\frac{x}{2}\right) = \frac{1}{2}A(x) \tag{3.6}$$

for all $x \in X$. Since $f : X \to Y$ is odd, $A : X \to Y$ is an odd mapping. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that A is a unique mapping satisfying (3.6) such that there exists a $\mu \in (0,\infty)$ satisfying

$$N(f(x)-A(x),\mu t) \geq \frac{t}{t+\varphi(x,0)}$$

for all $x \in X$;

(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) = A(x)$$

for all $x \in X$;

(3) $d(f,A) \leq \frac{1}{1-L} d(f,Jf)$, which implies the inequality

$$d(f,A) \leq \frac{1}{1-L}.$$

This implies that the inequality (3.4) holds.

By (3.3),

$$\begin{aligned}
&N\left(2^{n+1} f\left(\frac{x+y}{2^{n+1}}\right) - 2^n f\left(\frac{x}{2^n}\right) - 2^n f\left(\frac{y}{2^n}\right), 2^n t\right)\\
&\geq \min\left\{N\left(\rho\left(2^n\left(f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right), 2^n t\right), \frac{t}{t+\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)}\right\}
\end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. So

$$\begin{aligned}
&N\left(2^{n+1} f\left(\frac{x+y}{2^{n+1}}\right) - 2^n f\left(\frac{x}{2^n}\right) - 2^n f\left(\frac{y}{2^n}\right), t\right)\\
&\geq \min\left\{N\left(\rho\left(2^n\left(f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right), t\right), \frac{\frac{t}{2^n}}{\frac{t}{2^n} + \frac{L^n}{2^n}\varphi(x,y)}\right\}
\end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{\frac{t}{2^n}}{\frac{t}{2^n}+\frac{L^n}{2^n}\varphi(x,y)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$N\left(2A\left(\frac{x+y}{2}\right) - A(x) - A(y), t\right) \geq N\left(\rho\left(A(x+y) - A(x) - A(y)\right), t\right)$$

for all $x, y \in X$ and all $t > 0$. By Lemma 3.1, the mapping $A : X \to Y$ is Cauchy additive, as desired. □

Corollary 3.3. *Let $\theta \geq 0$ and let p be a real number with $p > 1$. Let X be a normed vector space with the norm $\|\cdot\|$. Let $f : X \to Y$ be an odd mapping satisfying*

$$\begin{aligned}
&N\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y), t\right)\\
&\qquad \geq \min\left\{N\left(\rho\left(f(x+y) - f(x) - f(y)\right), t\right), \frac{t}{t+\theta(\|x\|^p+\|y\|^p)}\right\}
\end{aligned}$$

for all $x, y \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n\to\infty} 2^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\left(f(x) - A(x), t\right) \geq \frac{(2^p-2)t}{(2^p-2)t + 2^p\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 3.2 by taking $\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{1-p}$, and we get the desired result. $\square$

Theorem 3.4. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \leq 2L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying (3.3). Then $A(x) := N$-$\lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N(f(x) - A(x), t) \geq \frac{(1-L)t}{(1-L)t + L\varphi(x,0)} \tag{3.7}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S,d) be the generalized metric space defined in the proof of Theorem 3.2.

It follows from (3.5) that

$$N\left(f(x) - \frac{1}{2}f(2x), Lt\right) \geq \frac{t}{t + \varphi(x,0)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq L$. Hence

$$d(f,A) \leq \frac{L}{1-L},$$

which implies that the inequality (3.7) holds.

The rest of the proof is similar to the proof of Theorem 3.2. $\square$

Corollary 3.5. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 1$. Let X be a normed vector space with the norm $\|\cdot\|$. Let $f : X \to Y$ be an odd mapping satisfying*

$$N\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y), t\right)$$
$$\geq \min\left\{N\left(\rho\left(f(x+y) - f(x) - f(y)\right), t\right), \frac{t}{t + \theta(\|x\|^p + \|y\|^p)}\right\}$$

for all $x, y \in X$ and all $t > 0$. Then $A(x) := N$-$\lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N(f(x) - A(x), t) \geq \frac{(2-2^p)t}{(2-2^p)t + 2^p\theta\|x\|^p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.4 by taking $\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{p-1}$, and we get the desired result. $\square$

References

[1] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64–66.

[2] T. Bag and S.K. Samanta, *Finite dimensional fuzzy normed linear spaces*, J. Fuzzy Math. **11** (2003), 687–705.

[3] T. Bag and S.K. Samanta, *Fuzzy bounded linear operators*, Fuzzy Sets and Systems **151** (2005), 513–547.

[4] L. Cădariu and V. Radu, *Fixed points and the stability of Jensen's functional equation*, J. Inequal. Pure Appl. Math. **4**, no. 1, Art. ID 4 (2003).

[5] L. Cădariu and V. Radu, *On the stability of the Cauchy functional equation: a fixed point approach*, Grazer Math. Ber. **346** (2004), 43–52.

[6] L. Cădariu and V. Radu, *Fixed point methods for the generalized stability of functional equations in a single variable*, Fixed Point Theory and Applications **2008**, Art. ID 749392 (2008).

[7] I. Chang and Y. Lee, *Additive and quadratic type functional equation and its fuzzy stability*, Results Math. **63** (2013), 717–730.

[8] S.C. Cheng and J.M. Mordeson, *Fuzzy linear operators and fuzzy normed linear spaces*, Bull. Calcutta Math. Soc. **86** (1994), 429–436.

[9] J. Diaz and B. Margolis, *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc. **74** (1968), 305–309.

[10] W. Fechner, *Stability of a functional inequalities associated with the Jordan-von Neumann functional equation*, Aequationes Math. **71** (2006), 149–161.

[11] C. Felbin, *Finite dimensional fuzzy normed linear spaces*, Fuzzy Sets and Systems **48** (1992), 239–248.

[12] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.

[13] A. Gilányi, *Eine zur Parallelogrammgleichung äquivalente Ungleichung*, Aequationes Math. **62** (2001), 303–309.

[14] A. Gilányi, *On a problem by K. Nikodem*, Math. Inequal. Appl. **5** (2002), 707–710.

[15] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A. **27** (1941), 222–224.

[16] D.H. Hyers, G. Isac and Th.M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Basel, 1998.

[17] G. Isac and Th.M. Rassias, *Stability of ψ-additive mappings: Appications to nonlinear analysis*, Internat. J. Math. Math. Sci. **19** (1996), 219–228.

[18] S. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press lnc., Palm Harbor, Florida, 2001.

[19] A.K. Katsaras, *Fuzzy topological vector spaces II*, Fuzzy Sets and Systems **12** (1984), 143–154.

[20] H. Kim, M. Eshaghi Gordji, A. Javadian and I. Chang, *Homomorphisms and derivations on unital C^*-algebras related to Cauchy-Jensen functional inequality,* J. Math. Inequal. **6** (2012), 557–565.

[21] H. Kim, J. Lee and E. Son, *Approximate functional inequalities by additive mappings,* J. Math. Inequal. **6** (2012), 461–471.

[22] I. Kramosil and J. Michalek, *Fuzzy metric and statistical metric spaces*, Kybernetica **11** (1975), 326–334.

[23] S.V. Krishna and K.K.M. Sarma, *Separation of fuzzy normed linear spaces*, Fuzzy Sets and Systems **63** (1994), 207–217.

[24] J. Lee, C. Park and D. Shin, *An AQCQ-functional equation in matrix normed spaces*, Results Math. **27** (2013), 305–318.

[25] D. Miheţ and V. Radu, *On the stability of the additive Cauchy functional equation in random normed spaces*, J. Math. Anal. Appl. **343** (2008), 567–572.

[26] M. Mirzavaziri and M.S. Moslehian, *A fixed point approach to stability of a quadratic equation*, Bull. Braz. Math. Soc. **37** (2006), 361–376.

[27] A.K. Mirmostafaee, M. Mirzavaziri and M.S. Moslehian, *Fuzzy stability of the Jensen functional equation*, Fuzzy Sets and Systems **159** (2008), 730–738.

[28] A.K. Mirmostafaee and M.S. Moslehian, *Fuzzy versions of Hyers-Ulam-Rassias theorem*, Fuzzy Sets and Systems **159** (2008), 720–729.

[29] A.K. Mirmostafaee and M.S. Moslehian, *Fuzzy approximately cubic mappings*, Inform. Sci. **178** (2008), 3791–3798.

[30] C. Park, *Fixed points and Hyers-Ulam-Rassias stability of Cauchy-Jensen functional equations in Banach algebras*, Fixed Point Theory and Applications **2007**, Art. ID 50175 (2007).

[31] C. Park, *Generalized Hyers-Ulam-Rassias stability of quadratic functional equations: a fixed point approach*, Fixed Point Theory and Applications **2008**, Art. ID 493751 (2008).

[32] C. Park, *Additive ρ-functional inequalities and equations,* J. Math. Inequal. **9** (2015), 17–26.

[33] C. Park, *Additive ρ-functional inequalities in non-Archimedean normed spaces,* J. Math. Inequal. **9** (2015), 397–407.

[34] C. Park, Y. Cho and M. Han, *Stability of functional inequalities associated with Jordan-von Neumann type additive functional equations*, J. Inequal. Appl. **2007**, Art. ID 41820 (2007).

[35] C. Park, K. Ghasemi, S. G. Ghaleh and S. Jang, *Approximate n-Jordan $*$-homomorphisms in C^*-algebras,* J. Comput. Anal. Appl. **15** (2013), 365-368.

[36] C. Park, A. Najati and S. Jang, *Fixed points and fuzzy stability of an additive-quadratic functional equation,* J. Comput. Anal. Appl. **15** (2013), 452–462.

[37] C. Park and Th.M. Rassias, *Fixed points and generalized Hyers-Ulam stability of quadratic functional equations,* J. Math. Inequal. **1** (2007), 515–528.

[38] V. Radu, *The fixed point alternative and the stability of functional equations*, Fixed Point Theory **4** (2003), 91–96.

[39] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.

[40] J. Rätz, *On inequalities associated with the Jordan-von Neumann functional equation*, Aequationes Math. **66** (2003), 191–200.

[41] L. Reich, J. Smítal and M. Štefánková, *Singular solutions of the generalized Dhombres functional equation*, Results Math. **65** (2014), 251–261.

[42] S. Schin, D. Ki, J. Chang and M. Kim, *Random stability of quadratic functional equations: a fixed point approach*, J. Nonlinear Sci. Appl. **4** (2011), 37–49.

[43] S. Shagholi, M. Bavand Savadkouhi and M. Eshaghi Gordji, *Nearly ternary cubic homomorphism in ternary Fréchet algebras,* J. Comput. Anal. Appl. **13** (2011), 1106–1114.

[44] S. Shagholi, M. Eshaghi Gordji and M. Bavand Savadkouhi, *Stability of ternary quadratic derivation on ternary Banach algebras,* J. Comput. Anal. Appl. **13** (2011), 1097–1105.

[45] D. Shin, C. Park and Sh. Farhadabadi, *On the superstability of ternary Jordan C^*-homomorphisms,* J. Comput. Anal. Appl. **16** (2014), 964–973.

[46] D. Shin, C. Park and Sh. Farhadabadi, *Stability and superstability of J^*-homomorphisms and J^*-derivations for a generalized Cauchy-Jensen equation,* J. Comput. Anal. Appl. **17** (2014), 125–134.

[47] S. M. Ulam, *A Collection of the Mathematical Problems*, Interscience Publ. New York, 1960.

[48] J.Z. Xiao and X.H. Zhu, *Fuzzy normed spaces of operators and its completeness*, Fuzzy Sets and Systems **133** (2003), 389–399.

Ji-hye Kim
Department of Mathematics, Hanyang University, Seoul 133-791, Korea
E-mail address: saharin@hanyang.ac.kr

George A. Anastassiou
Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152, USA
E-mail address: ganastss@memphis.edu

Choonkil Park
Research Institute for Natural Sciences, Hanyang University, Seoul 133-791, Korea
E-mail address: baak@hanyang.ac.kr

A NOTE ON BARNES-TYPE BOOLE POLYNOMIALS WITH λ-PARAMETER

TAEKYUN KIM, DMITRY V. DOLGY, AND DAE SAN KIM

ABSTRACT. In this paper, we consider Barnes-type Boole polynomials and give some formulae related to these polynomials.

1. INTRODUCTION

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. The p-adic norm is normalized as $|p|_p = \frac{1}{p}$. Let $C(\mathbb{Z}_p)$ be the space of continuous functions on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, the fermionic p-adic integral on $\mathbb{Z}_p$ is defined by Kim as

(1.1)
$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x, \quad \text{(see [1–19, 21, 22])}.$$

From (1.1), we have

$$I_{-1}(f_n) + (-1)^{n-1} I_{-1}(f) = 2\sum_{l=0}^{n-1} (-1)^{n-1-l} f(l), \quad \text{(see [14])}. \tag{1.2}$$

As is well known, the Boole polynomials are given by the generating function

$$\frac{1}{(1+t)^{\lambda}+1}(1+t)^x = \sum_{n=0}^{\infty} Bl_n(x\mid\lambda)\frac{t^n}{n!}, \quad \text{(see [10])}. \tag{1.3}$$

When $x = 0$, $Bl_n(\lambda) = Bl_n(0 \mid \lambda)$ are called the Boole numbers.

For $a_1, a_2, \ldots, a_r \in \mathbb{C}_p$, the Barnes-type Euler polynomials are given by the generating function

$$\frac{2^r}{(e^{a_1 t}+1)(e^{a_2 t}+1)\cdots(e^{a_r t}+1)}e^{xt} = \sum_{n=0}^{\infty} E_n(x\mid a_1,\ldots,a_r)\frac{t^n}{n!}. \tag{1.4}$$

When $x = 0$, $E_n(a_1,\ldots,a_r) = E_n(0 \mid a_1,\ldots,a_r)$ are called the Barnes-type Euler numbers (see [12, 20]).

From (1.1), we can derive the following equation:

$$\int_{\mathbb{Z}_p} f(x)\, d\mu_{-1}(x) \tag{1.5}$$

2010 *Mathematics Subject Classification.* 11B75, 11B83, 11S80.

Key words and phrases. Barnes-type Boole polynomial, Barnes-type Euler polynomial, fermionic p-adic integral.

1

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 21, NO.6, 2016, COPYRIGHT 2016 EUDOXUS PRESS, LLC

$$
\begin{aligned}
&= \lim_{N\to\infty} \sum_{x=0}^{dp^N-1} f(x)(-1)^x \\
&= \lim_{N\to\infty} \sum_{a=0}^{d-1} \sum_{x=0}^{p^N-1} f(a+dx)(-1)^{a+x} \\
&= \sum_{a=0}^{d-1} (-1)^a \int_{\mathbb{Z}_p} f(a+dx)\, \mu_{-1}(x),
\end{aligned}
$$

where $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$.

In [10], Kim-Kim derived the Witt-type formula for Boole polynomials which are given by

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathbb{Z}_p} (1+t)^{x+\lambda y}\, d\mu_0(y) \\
&= \frac{1}{(1+t)^{\lambda}+1}(1+t)^x \\
&= \sum_{n=0}^{\infty} Bl_n(x \mid \lambda)\frac{t^n}{n!}.
\end{aligned} \tag{1.6}
$$

In this paper, we consider Barnes-type Boole polynomials and give some formulae related to these polynomials.

2. Barnes-type Boole polynomials with λ-parameter

Let $a_1, a_2, \ldots, a_r \in \mathbb{C}_p$. Then, we consider the Barnes-type Boole polynomials which are given by the multivariate fermionic p-adic integral on $\mathbb{Z}_p$ as follows:

$$
\begin{aligned}
&\frac{1}{2^r}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} (1+t)^{\lambda a_1 y_1+\lambda a_2 y_2+\cdots+\lambda a_r y_r+x}\, d\mu_{-1}(y_1)\cdots d\mu_{-1}(y_r) \\
&= \prod_{l=1}^{r}\left(\frac{1}{1+(1+t)^{\lambda a_l}}\right)(1+t)^x \\
&= \sum_{n=0}^{\infty} Bl_{n,\lambda}(x \mid a_1,\ldots,a_r)\frac{t^n}{n!}.
\end{aligned} \tag{2.1}
$$

Note that $Bl_{n,\lambda}^{(1)}(x \mid 1) = Bl_n(x \mid \lambda)$, $(n \geq 0)$. When $x=0$, $Bl_{n,\lambda}(a_1,\ldots,a_r) = Bl_{n,\lambda}(0 \mid a_1,\ldots,a_r)$ are called the Barnes-type Boole numbers.

From (2.1), we have

$$
\begin{aligned}
&\frac{1}{2^r}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} (\lambda a_1 y_1+\cdots+\lambda a_r y_r+x)_n\, d\mu_{-1}(y_1)\cdots d\mu_{-1}(y_r) \\
&= Bl_{n,\lambda}(x \mid a_1,\ldots,a_r), \quad (n \geq 0),
\end{aligned} \tag{2.2}
$$

where $(x)_n = x(x-1)\cdots(x-n+1)$.

We observe that

(2.3)

$$
\frac{1}{2^r}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} (\lambda a_1 y_1+\cdots+\lambda a_r y_r+x)_n\, d\mu_{-1}(y_1)\cdots d\mu_{-1}(y_r)
$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 21, NO.6, 2016, COPYRIGHT 2016 EUDOXUS PRESS, LLC

$$= \frac{1}{2^r}\sum_{l=0}^{n} S_1(n,l)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(\lambda a_1 y_1+\cdots+\lambda a_r y_r+x)^l\, d\mu_{-1}(y_1)\cdots d\mu_{-1}(y_r)$$
$$= \frac{1}{2^r}\sum_{l=0}^{n} S_1(n,l)\lambda^l\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left(a_1 y_1+\cdots+a_r y_r+\frac{x}{\lambda}\right)^l d\mu_{-1}(y_1)\cdots d\mu_{-1}(y_r),$$

where $S_1(n,l)$ is the Stirling number of the first kind.

From (1.2), we have

$$\begin{aligned}\text{(2.4)}\qquad &\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} e^{(a_1x_1+\cdots+a_rx_r+x)t} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\\ &=\left(\frac{2^r}{(e^{a_1t}+1)\cdots(e^{a_rt}+1)}\right)e^{xt}\\ &=\sum_{n=0}^{\infty} E_n(x\mid a_1,\ldots,a_r)\frac{t^n}{n!}.\end{aligned}$$

Thus, by (2.4), we get

$$\begin{aligned}\text{(2.5)}\qquad &\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(a_1x_1+\cdots+a_rx_r+x)^n\, d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\\ &= E_n(x\mid a_1,\ldots,a_r),\quad (n\geq 0).\end{aligned}$$

From (2.2) and (2.5), we obtain the following theorem.

Theorem 1. *For $n \geq 0$, we have*

$$\frac{1}{2^r}\sum_{l=0}^{n} S_1(n,l)\lambda^l E_l\left(\frac{x}{\lambda}\Big| a_1,\ldots,a_r\right) = Bl_{n,\lambda}(x\mid a_1,\ldots,a_r).$$

By (2.1), we get

$$\begin{aligned}\text{(2.6)}\qquad &\frac{1}{2^r}\prod_{l=1}^{r}\left(\frac{2}{e^{a_lt}+1}\right)e^{\frac{x}{\lambda}t}\\ &=\sum_{n=0}^{\infty} Bl_{n,\lambda}(x\mid a_1,\ldots,a_r)\frac{\left(e^{\frac{1}{\lambda}t}-1\right)^n}{n!}\\ &=\sum_{m=0}^{\infty}\left(\lambda^{-m}\sum_{n=0}^{m} Bl_{n,\lambda}(x\mid a_1,\ldots,a_r)S_2(m,n)\right)\frac{t^m}{m!},\end{aligned}$$

where $S_2(m,n)$ is the Stirling number of the second kind.

By (1.4), we get

$$\text{(2.7)}\qquad \prod_{l=1}^{r}\left(\frac{2}{e^{a_lt}+1}\right)e^{\frac{x}{\lambda}t} = \sum_{m=0}^{\infty} E_m\left(\frac{x}{\lambda}\Big| a_1,\ldots,a_r\right)\frac{t^m}{m!}.$$

Therefore, by (2.6) and (2.7), we obtain the following theorem.

Theorem 2. *For $m \geq 0$, we have*

$$\begin{aligned}&\lambda^m E_m\left(\frac{x}{\lambda}\Big| a_1,\ldots,a_r\right)\\ &= 2^r\sum_{n=0}^{m} Bl_{n,\lambda}(x\mid a_1,\ldots,a_r)S_2(m,n).\end{aligned}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 21, NO.6, 2016, COPYRIGHT 2016 EUDOXUS PRESS, LLC

From (1.5), we have

(2.8)

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+t)^{\lambda a_1 y_1+\cdots+\lambda a_r y_r+x}\,d\mu_{-1}(y_1)\cdots d\mu_{-1}(y_r)\\
&=\sum_{k_1,\ldots,k_r=0}^{d-1}(-1)^{k_1+\cdots+k_r}\\
&\quad\times\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+t)^{\lambda a_1(k_1+dy_1)+\cdots+\lambda a_r(k_r+dy_r)+x}\,d\mu_{-1}(y_1)\cdots d\mu_{-1}(y_r)\\
&=\sum_{k_1,\ldots,k_r=0}^{d-1}(-1)^{k_1+\cdots+k_r}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+t)^{\lambda d\left(\frac{a_1k_1+\cdots+a_rk_r+\frac{x}{\lambda}}{d}+a_1y_1+\cdots+a_ry_r\right)}\\
&\quad\times d\mu_{-1}(y_1)\cdots d\mu_{-1}(y_r)\\
&=2^r\sum_{k_1,\ldots,k_r=0}^{d-1}(-1)^{k_1+\cdots+k_r}\sum_{n=0}^{\infty}Bl_{n,\lambda d}(\lambda a_1k_1+\cdots+\lambda a_rk_r+x\mid a_1,\ldots,a_r)\frac{t^n}{n!}\\
&=2^r\sum_{n=0}^{\infty}\left(\sum_{k_1,\ldots,k_r=0}^{d-1}(-1)^{k_1+\cdots+k_r}Bl_{n,\lambda d}(\lambda a_1k_1+\cdots+\lambda a_rk_r+x\mid a_1,\ldots,a_r)\right)\frac{t^n}{n!},
\end{aligned}$$

where $d\in\mathbb{N}$ with $d\equiv 1 \pmod 2$.

From (2.8), we have

(2.9)

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(\sum_{k_1,\ldots,k_r=0}^{d-1}(-1)^{k_1+\cdots+k_r}Bl_{n,d\lambda}(\lambda a_1k_1+\cdots+\lambda a_rk_r+x\mid a_1,\ldots,a_r)\right)\frac{t^n}{n!}\\
&=\frac{1}{2^r}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+t)^{\lambda a_1y_1+\cdots+\lambda a_ry_r+x}\,d\mu_{-1}(y_1)\cdots d\mu_{-1}(y_r)\\
&=\sum_{n=0}^{\infty}Bl_{n,\lambda}(x\mid a_1,\ldots,a_r)\frac{t^n}{n!}.
\end{aligned}$$

By comparing the coefficients on the both sides of (2.9), we obtain the following equation:

Theorem 3. *For $d\in\mathbb{N}$ with $d\equiv 1 \pmod 2$, $n\geq 0$, we have*

$$\begin{aligned}
&Bl_{n,\lambda}(x\mid a_1,\ldots,a_r)\\
&=\sum_{k_1,\ldots,k_r=0}^{d-1}(-1)^{k_1+\cdots+k_r}Bl_{n,\lambda d}(\lambda a_1k_1+\cdots+\lambda a_rk_r+x\mid a_1,\ldots,a_r).
\end{aligned}$$

Let $d\in\mathbb{N}$ with $d\equiv 1 \pmod 2$. From (1.2), we have

$$\int_{\mathbb{Z}_p}e^{a_1(y_1+d)t}d\mu_{-1}(y_1)+\int_{\mathbb{Z}_p}e^{a_1y_1t}d\mu_{-1}(y)=2\sum_{l=0}^{d-1}(-1)^l e^{a_1lt}. \tag{2.10}$$

Thus, by (2.10), we get

$$\int_{\mathbb{Z}_p} e^{a_1 y_1 t} d\mu_{-1}(y) = \frac{2}{e^{a_1 dt}+1} \sum_{l=0}^{d-1} (-1)^l e^{a_1 l t}. \tag{2.11}$$

From (2.11), we can derive

$$\begin{aligned} &\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(a_1 y_1 + a_2 y_2 + \cdots + a_r y_r + x)t} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\ &= \sum_{l_1,\ldots,l_r=0}^{d-1} (-1)^{l_1+\cdots+l_r} \prod_{l=1}^{r} \left(\frac{2}{e^{a_l dt}+1} \right) e^{(a_1 l_1 + \cdots + a_r l_r + x)t} \\ &= \sum_{n=0}^{\infty} d^n \sum_{l_1,\ldots,l_r=0}^{d-1} (-1)^{l_1+\cdots+l_r} E_n \left(\left. \frac{a_1 l_1 + \cdots + a_r l_r + x}{d} \right| a_1, \ldots, a_r \right) \frac{t^n}{n!}. \end{aligned} \tag{2.12}$$

From (2.12) and (2.4), we get

$$\begin{aligned} &E_n(x \mid a_1, \ldots, a_r) \\ &= d^n \sum_{l_1,\ldots,l_r=0}^{d-1} (-1)^{l_1+\cdots+l_r} E_n \left(\left. \frac{a_1 l_1 + \cdots + a_r l_r + x}{d} \right| a_1, \ldots, a_r \right), \end{aligned}$$

where $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$.

On the other hand,

$$\begin{aligned} &\int_{\mathbb{Z}_p} (1+t)^{\lambda a_1 (d+y_1)} d\mu_{-1}(y) + \int_{\mathbb{Z}_p} (1+t)^{\lambda a_1 y_1} d\mu_{-1}(y) \\ &= 2 \sum_{l_1=0}^{d-1} (-1)^{l_1} (1+t)^{\lambda a_1 l_1}, \end{aligned} \tag{2.13}$$

where $d \in \mathbb{N}$ such that $d \equiv 1 \pmod 2$.

By (2.13), we get

$$\begin{aligned} &\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+t)^{\lambda a_1 y_1 + \cdots + \lambda a_r y_r + x} d\mu_{-1}(y_1) \cdots d\mu_{-1}(y_r) \\ &= \prod_{l=1}^{r} \frac{2}{1+(1+t)^{\lambda a_l d}} \sum_{l_1,\ldots,l_r=0}^{d-1} (-1)^{l_1+\cdots+l_r} (1+t)^{\lambda a_1 l_1 + \cdots + \lambda a_r l_r + x} \\ &= 2^r \sum_{m=0}^{\infty} Bl_{m,\lambda d}(a_1, \ldots, a_r) \frac{t^m}{m!} \\ &\quad \times \sum_{k=0}^{\infty} \left(\sum_{l_1,\ldots,l_r=0}^{d-1} (-1)^{l_1+\cdots+l_r} (\lambda a_1 l_1 + \cdots + \lambda a_r l_r + x)_k \right) \frac{t^k}{k!} \\ &= 2^r \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} \sum_{l_1,\ldots,l_r=0}^{d-1} (-1)^{l_1+\cdots+l_r} (\lambda a_1 l_1 + \cdots + \lambda a_r l_r + x)_k Bl_{n-k,\lambda d}(a_1, \ldots, a_r) \right) \frac{t^n}{n!}. \end{aligned} \tag{2.14}$$

From (2.9) and (2.14), we note that

$$\begin{aligned} &Bl_n\left(x \mid a_1, \ldots, a_r\right) \\ &= \sum_{k=0}^{n} \binom{n}{k} \sum_{l_1, \ldots, l_r=0}^{d-1} (-1)^{l_1+\cdots+l_r} \left(x + \lambda a_1 l_1 + \cdots + \lambda a_r l_r\right)_k \\ &\times Bl_{n-k, \lambda d}\left(a_1, \ldots, a_r\right), \end{aligned} \tag{2.15}$$

where $n \geq 0$ and $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$.

Therefore, by (2.15), we obtain the following theorem.

Theorem 4. *For $n \geq 0$ and $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$, we have*

$$\begin{aligned} &Bl_n\left(x \mid a_1, \ldots, a_r\right) \\ &= \sum_{k=0}^{n} \binom{n}{k} \sum_{l_1, \ldots, l_r=0}^{d-1} (-1)^{l_1+\cdots+l_r} \left(x + \lambda a_1 l_1 + \cdots + \lambda a_r l_r\right)_k \\ &\times Bl_{n-k, \lambda d}\left(a_1, \ldots, a_r\right). \end{aligned}$$

From (2.14), we have

(2.16)

$$\begin{aligned} &\frac{1}{2^r} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+t)^{\lambda a_1 y_1 + \cdots + \lambda a_r y_r} \, d\mu_{-1}\left(y_1\right) \cdots d\mu_{-1}\left(y_r\right) \\ &= \sum_{l_1, \ldots, l_r=0}^{d-1} (-1)^{l_1+\cdots+l_r} \left(\prod_{l=1}^{r} \frac{1}{1 + (1+t)^{\lambda a_l d}} \right) (1+t)^{\lambda a_1 l_1 + \cdots + \lambda a_r l_r} \\ &= \sum_{n=0}^{\infty} \left(\sum_{l_1, \ldots, l_r=0}^{d-1} (-1)^{l_1+\cdots+l_r} Bl_{n, \lambda d}\left(\lambda a_1 l_1 + \cdots + \lambda a_r l_r \mid a_1, \ldots, a_r\right) \right) \frac{t^n}{n!}, \end{aligned}$$

and

$$\begin{aligned} &\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+t)^{\lambda a_1 y_1 + \cdots + \lambda a_r y_r} \, d\mu_{-1}\left(y_1\right) \cdots d\mu_{-1}\left(y_r\right) \\ &= 2^r \sum_{n=0}^{\infty} Bl_{n, \lambda}\left(a_1, \ldots, a_r\right) \frac{t^n}{n!}. \end{aligned} \tag{2.17}$$

Therefore, by (2.16) and (2.17), we obtain the following theorem.

Theorem 5. *For $n \geq 0$, $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$, we have*

$$Bl_{n, \lambda}\left(a_1, \ldots, a_r\right) = \sum_{l_1, \ldots, l_r=0}^{d-1} (-1)^{l_1+\cdots+l_r} Bl_{n, \lambda d}\left(\lambda a_1 l_1 + \cdots + \lambda a_r l_r \mid a_1, \ldots, a_r\right).$$

By replacing t by $e^{\frac{1}{\lambda} t} - 1$ in (2.14), we get

$$\begin{aligned} &\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{\left(a_1 y_1 + \cdots + a_r y_r + \frac{x}{\lambda}\right) t} d\mu_{-1}\left(y_1\right) \cdots d\mu_{-1}\left(y_r\right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} \sum_{l_1, \ldots, l_r=0}^{d-1} (-1)^{l_1+\cdots+l_r} \left(\lambda a_1 l_1 + \cdots + \lambda a_r l_r + x\right)_k \right. \end{aligned} \tag{2.18}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 21, NO.6, 2016, COPYRIGHT 2016 EUDOXUS PRESS, LLC

$$\times Bl_{n-k,\lambda d}(a_1,\ldots,a_r))\frac{1}{n!}\left(e^{\frac{1}{\lambda}t}-1\right)^n$$

$$=\sum_{m=0}^{\infty}\lambda^{-m}\left(\sum_{n=0}^{m}\sum_{k=0}^{n}\sum_{l_1,\ldots,l_r=0}^{d-1}\binom{n}{k}S_2(m,n)(-1)^{l_1+\cdots+l_r}\right.$$

$$\left.\times(\lambda a_1l_1+\cdots+\lambda a_rl_r+x)_k Bl_{n-k,\lambda d}(a_1,\ldots,a_r)\right)\frac{t^m}{m!}$$

Thus, by (2.18), we get

$$\lambda^m E_m\left(\frac{x}{\lambda}\middle| a_1,\ldots,a_r\right) \tag{2.19}$$

$$=\sum_{n=0}^{m}\sum_{k=0}^{n}\sum_{l_1,\ldots,l_r=0}^{d-1}\binom{n}{k}S_2(m,n)(-1)^{l_1+\cdots+l_r}$$

$$\times(\lambda a_1l_1+\cdots+\lambda a_rl_r+x)_k Bl_{n-k,\lambda d}(a_1,\ldots,a_r),$$

where $m\geq 0$ and $d\in\mathbb{N}$ with $d\equiv 1 \pmod 2$.

Therefore, by (2.19), we obtain the following theorem.

Theorem 6. *For $m\geq 0$, $d\in\mathbb{N}$ with $d\equiv 1 \pmod 2$, we have*

$$\lambda^m E_m\left(\frac{x}{\lambda}\middle| a_1,\ldots,a_r\right)$$

$$=\sum_{n=0}^{m}\sum_{k=0}^{n}\sum_{l_1,\ldots,l_r=0}^{d-1}\binom{n}{k}S_2(m,n)(-1)^{l_1+\cdots+l_r}$$

$$\times(\lambda a_1l_1+\cdots+\lambda a_rl_r+x)_k Bl_{n-k,\lambda d}(a_1,\ldots,a_r).$$

ACKNOWLEDGEMENTS. This paper is supported by grant NO 14-11-00022 of Russian Scientific Fund

References

1. S. Araci and M. Acikgoz, *A note on the Frobenius-Euler numbers and polynomials associated with Bernstein polynomials*, Adv. Stud. Contemp. Math. (Kyungshang) **22** (2012), no. 3, 399–406.
2. ______, *A note on the values of weighted q-Bernstein polynomials and weighted q-Genocchi numbers*, Adv. Difference Equ. (2015), 2015:30, 9 pp.
3. A. Bayad and J. Chikhi, *Apostol-Euler polynomials and asymtotics for negative binomial reciprocals*, Adv. Stud. Contemp. Math. (Kyungshang) **19** (2014), no. 1, 39–57.
4. A. Bayad and T. Kim, *Results on values of Barnes polynomials*, Rocky Mountain J. Math. **24** (2013), no. 1, 33–37.
5. I. N. Cangul, V. Kurt, H. Ozden, and Y. Simsek, *On the higher-order w-q-Genocchi numbers*, Adv. Stud. Contemp. Math. (Kyungshang) **19** (2009), no. 1, 39–57.
6. D. Ding and J. Yang, *Some identities related to the Apostol-Euler and Apostol-Bernoulli polynomials*, Adv. Stud. Contemp. Math. (Kyungshang) **20** (2010), no. 1, 7–21.
7. S. Gaboury, R. Tremblay, and B.-J. Fugére, *Some explicit formulas for certain new classes of Bernoulli, Euler and Genocchi polynomials*, Adv. Difference Equ. (2013), 2013:246, 10 pp.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 21, NO.6, 2016, COPYRIGHT 2016 EUDOXUS PRESS, LLC

8. Y. He, *Symmetric identities for Carlitz's q-Bernoulli numbers and polynomials*, Adv. Difference Equ. (2013), 2013:246, 10 pp.
9. J.-H. Jeong, J.-H. Jin, J.-W. Park, and S.-H. Rim, *On the twisted weak q-Euler numbers and polynomials with weight 0*, Proc. Jangjeon Math. Soc. **16** (2013), no. 2, 157–163.
10. D. S. Kim and T. Kim, *A note on Boole polynomials*, Integral Transform Spec. Funct. **25** (2014), no. 8, 627–633.
11. D. S. Kim, T. Kim, and J. J. Seo, *A note on q-analogue of Boole polynomials*, Appl. Math. Inf. Sci. **9** (2015), no. 6, 1–6.
12. T. Kim, *On Euler-Barnes multiple zeta functions*, Russ. J. Math. Phys **10** (2003), no. 3, 261–267.
13. ———, *A note on p-adic invariant integral in the rings of p-adic integers*, Adv. Stud. Contemp. Math. (Kyungshang) **13** (2006), no. 1, 95–99.
14. ———, *Symmetric p-adic invariant integral on $\mathbb{Z}_p$ for Bernoulli and Euler polynomials*, J. Difference Equ. Appl. **14** (2008), no. 12, 1267–1277.
15. ———, *Some identities on the q-Euler polynomials of higher order and q-Stirling numbers by fermionic p-adic integral on $\mathbb{Z}_p$*, Russ. J. Math. Phys **16** (2009), no. 4, 484–491.
16. ———, *Barnes-type multiple q-zeta functions and q-euler polynomials*, J. Phys. A. **43** (2010), no. 25, 255201, 11 pp.
17. ———, *Barnes' type multiple degenerate Bernoulli and Euler polynomials*, Appl. Math. Comput. **258** (2015), 556–564.
18. T. Kim, D. V. Dolgy, Y. S. Jang, and J. J. Seo, *A note on symmetric identities for the generalized q-Euler polynomials of the second kind*, Proc. Jangjeon Math. Soc. **17** (2014), no. 3, 375–381.
19. Q.-M. Luo, *q-analogue of some results for the Apostol-Euler polynomials*, Adv. Stud. Contemp. Math. (Kyungshang) **20** (2010), no. 1, 2010.
20. F. Qi, M.-M. Zheng, *Explicit expressions for a family of the Bell polynomials and applications* , Appl. Math. Comput. 258 (2015), 597–607.
21. E. Sen, *Theorems on Apostol-Euler polynomials of higher order arising from Euler basis*, Adv. Stud. Contemp. Math. (Kyungshang) **23** (2013), no. 2, 337–345.
22. Y. Simsek, *Interpolation functions of the Eulerian type polynomials and numbers*, Adv. Stud. Contemp. Math. (Kyungshang) **23** (2013), no. 2, 337–345.
23. Z. Zhang and H. Yang, *Some closed formulas for generalized Bernoulli-Euler numbers and polynomials*, Proc. Jangjeon Math. Soc. **11** (2008), no. 2, 191–198.

Department of Mathematics, Tianjin Polytechnic University, Tianjin, China

Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea
E-mail address: tkkim@kw.ac.kr, taekyun64@hotmail.com

School of Natural Sciences, Far Eastern Federal University, Vladivostok, Russia
E-mail address: d_dol@mail.ru

Department of Mathematics, Sogang University, Seoul 121-742, Republic of Korea
E-mail address: dskim@sogang.ac.kr

TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 21, NO. 6, 2016