Journal of

Computational

Analysis and

Applications

The main purpose of "J.Computational Analysis and Applications" is to publish high quality research articles from all subareas of Computational Mathematical Analysis and its many potential applications and connections to other areas of Mathematical Sciences. Any paper whose approach and proofs are computational,using methods from Mathematical Analysis in the broadest sense is suitable and welcome for consideration in our journal, except from Applied Numerical Analysis articles. Also plain word articles without formulas and proofs are excluded. The list of possibly connected mathematical areas with this publication includes, but is not restricted to: Applied Analysis, Applied Functional Analysis, Approximation Theory, Asymptotic Analysis, Difference Equations, Differential Equations, Partial Differential Equations, Fourier Analysis, Fractals, Fuzzy Sets, Harmonic Analysis, Inequalities, Integral Equations, Measure Theory, Moment Theory, Neural Networks, Numerical Functional Analysis, Potential Theory, Probability Theory, Real and Complex Analysis, Signal Analysis, Special Functions, Splines, Stochastic Analysis, Stochastic Processes, Summability, Tomography, Wavelets, any combination of the above, e.t.c.

"J.Computational Analysis and Applications" is a peer-reviewed Journal. See the instructions for preparation and submission of articles to JoCAAA. Assistant to the Editor:
Dr.Razvan Mezei,Lenoir-Rhyne University,Hickory,NC   28601, USA.

Dartmouth College
8000 Cummings Hall,
Hanover, NH 03755-8000
603-646-3843 (X 3546 Secr.)
e-mail:george.cybenko@dartmouth.edu
Approximation Theory and Neural
Networks

Sever S. Dragomir
School of Computer Science and
Mathematics, Victoria University,
PO Box 14428,
Melbourne City,
MC 8001, AUSTRALIA
Tel. +61 3 9688 4437
Fax +61 3 9688 4050
sever.dragomir@vu.edu.au
Inequalities, Functional Analysis,
Numerical Analysis, Approximations,
Information Theory, Stochastics.

Oktay Duman
TOBB University of Economics and
Technology,
Department of Mathematics, TR-
06530,
Ankara, Turkey,
oduman@etu.edu.tr
Classical Approximation Theory,
Summability Theory, Statistical
Convergence and its Applications

Saber N. Elaydi
Department Of Mathematics
Trinity University
715 Stadium Dr.
San Antonio, TX 78212-7200
210-736-8246
e-mail: selaydi@trinity.edu
Ordinary Differential Equations,
Difference Equations

Christodoulos A. Floudas
Department of Chemical Engineering
Princeton University
Princeton,NJ 08544-5263
609-258-4595(x4619 assistant)
e-mail: floudas@titan.princeton.edu
Optimization Theory&Applications,
Global Optimization

J .A. Goldstein
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152
901-678-3130
jgoldste@memphis.edu
Partial Differential Equations,
Semigroups of Operators

H. H. Gonska
Department of Mathematics
University of Duisburg
Duisburg, D-47048
Germany
011-49-203-379-3542
e-mail: heiner.gonska@uni-due.de
Approximation Theory, Computer
Aided Geometric Design

John R. Graef
Department of Mathematics
University of Tennessee at
Chattanooga
Chattanooga, TN 37304 USA
John-Graef@utc.edu
Ordinary and functional
differential equations, difference
equations, impulsive systems,
differential inclusions, dynamic
equations on time scales, control
theory and their applications

Weimin Han
Department of Mathematics
University of Iowa
Iowa City, IA 52242-1419
319-335-0770
e-mail: whan@math.uiowa.edu
Numerical analysis, Finite element
method, Numerical PDE, Variational
inequalities, Computational
mechanics

Tian-Xiao He
Department of Mathematics and
Computer Science
P.O. Box 2900, Illinois Wesleyan
University
Bloomington, IL 61702-2900, USA
Tel (309)556-3089
Fax (309)556-3864
the@iwu.edu
Approximations, Wavelet,
Integration Theory, Numerical
Analysis, Analytic Combinatorics

Margareta Heilmann
Faculty of Mathematics and Natural
Sciences, University of Wuppertal
Gaußstraße 20
D-42119 Wuppertal, Germany,
heilmann@math.uni-wuppertal.de

Approximation Theory (Positive
Linear Operators)

Xing-Biao Hu
Institute of Computational
Mathematics
AMSS, Chinese Academy of Sciences
Beijing, 100190, CHINA
hxb@lsec.cc.ac.cn
Computational Mathematics

Jong Kyu Kim
Department of Mathematics
Kyungnam University
Masan Kyungnam,631-701,Korea
Tel 82-(55)-249-2211
Fax 82-(55)-243-8609
jongkyuk@kyungnam.ac.kr
Nonlinear Functional Analysis,
Variational Inequalities, Nonlinear
Ergodic Theory, ODE, PDE,
Functional Equations.

Robert Kozma
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152, USA
rkozma@memphis.edu
Neural Networks, Reproducing Kernel
Hilbert Spaces,
Neural Percolation Theory

Mustafa Kulenovic
Department of Mathematics
University of Rhode Island
Kingston, RI 02881,USA
kulenm@math.uri.edu
Differential and Difference
Equations

Irena Lasiecka
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152
PDE, Control Theory, Functional
Analysis, lasiecka@memphis.edu

Burkhard Lenze
Fachbereich Informatik
Fachhochschule Dortmund
University of Applied Sciences
Postfach 105018
D-44047 Dortmund, Germany
e-mail: lenze@fh-dortmund.de
Real Networks, Fourier Analysis,
Approximation Theory

Hrushikesh N. Mhaskar
Department Of Mathematics
California State University
Los Angeles, CA 90032
626-914-7002
e-mail: hmhaska@gmail.com
Orthogonal Polynomials,
Approximation Theory, Splines,
Wavelets, Neural Networks

Ram N. Mohapatra
Department of Mathematics
University of Central Florida
Orlando, FL 32816-1364
tel.407-823-5080
ram.mohapatra@ucf.edu
Real and Complex Analysis,
Approximation Th., Fourier
Analysis, Fuzzy Sets and Systems

Gaston M. N'Guerekata
Department of Mathematics
Morgan State University
Baltimore, MD 21251, USA
tel: 1-443-885-4373
Fax 1-443-885-8216
Gaston.N'Guerekata@morgan.edu
nguerekata@aol.com
Nonlinear Evolution Equations,
Abstract Harmonic Analysis,
Fractional Differential Equations,
Almost Periodicity & Almost
Automorphy

M.Zuhair Nashed
Department Of Mathematics
University of Central Florida
PO Box 161364
Orlando, FL 32816-1364
e-mail: znashed@mail.ucf.edu
Inverse and Ill-Posed problems,
Numerical Functional Analysis,
Integral Equations, Optimization,
Signal Analysis

Mubenga N. Nkashama
Department OF Mathematics
University of Alabama at Birmingham
Birmingham, AL 35294-1170
205-934-2154
e-mail: nkashama@math.uab.edu
Ordinary Differential Equations,
Partial Differential Equations

Vassilis Papanicolaou
Department of Mathematics

National Technical University of
Athens
Zografou campus, 157 80
Athens, Greece
tel:: +30(210) 772 1722
Fax   +30(210) 772 1775
papanico@math.ntua.gr
Partial Differential Equations,
Probability

Choonkil Park
Department of Mathematics
Hanyang University
Seoul 133-791
S. Korea, baak@hanyang.ac.kr
Functional Equations

Svetlozar (Zari) Rachev,
Professor of Finance, College of
Business, and Director of
Quantitative Finance Program,
Department of Applied Mathematics &
Statistics
Stonybrook University
312 Harriman Hall, Stony Brook, NY
11794-3775
tel: +1-631-632-1998,
svetlozar.rachev@stonybrook.edu

Alexander G. Ramm
Mathematics Department
Kansas State University
Manhattan, KS 66506-2602
e-mail: ramm@math.ksu.edu
Inverse and Ill-posed Problems,
Scattering Theory, Operator Theory,
Theoretical Numerical Analysis,
Wave Propagation, Signal Processing
and Tomography

Tomasz Rychlik
Polish Academy of Sciences
Instytut Matematyczny PAN
00-956 Warszawa, skr. poczt. 21
ul. Śniadeckich 8
Poland
trychlik@impan.pl
Mathematical Statistics,
Probabilistic Inequalities

Boris Shekhtman
Department of Mathematics
University of South Florida
Tampa, FL 33620, USA
Tel 813-974-9710
shekhtma@usf.edu

Approximation Theory, Banach
spaces, Classical Analysis

T. E. Simos
Department of Computer
Science and Technology
Faculty of Sciences and Technology
University of Peloponnese
GR-221 00 Tripolis, Greece
Postal Address:
26 Menelaou St.
Anfithea - Paleon Faliron
GR-175 64 Athens, Greece
tsimos@mail.ariadne-t.gr
Numerical Analysis

H. M. Srivastava
Department of Mathematics and
Statistics
University of Victoria
Victoria, British Columbia V8W 3R4
Canada
tel.250-472-5313; office,250-477-
6960 home, fax 250-721-8962
harimsri@math.uvic.ca
Real and Complex Analysis,
Fractional Calculus and Appl.,
Integral Equations and Transforms,
Higher Transcendental Functions and
Appl.,q-Series and q-Polynomials,
Analytic Number Th.

I. P. Stavroulakis
Department of Mathematics
University of Ioannina
451-10 Ioannina, Greece
ipstav@cc.uoi.gr
Differential Equations
Phone +3-065-109-8283

Manfred Tasche
Department of Mathematics
University of Rostock
D-18051 Rostock, Germany
manfred.tasche@mathematik.uni-
rostock.de
Numerical Fourier Analysis, Fourier
Analysis, Harmonic Analysis, Signal
Analysis, Spectral Methods,
Wavelets, Splines, Approximation
Theory

Roberto Triggiani
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152
PDE, Control Theory, Functional

Analysis, rtrggani@memphis.edu

Juan J. Trujillo
University of La Laguna
Departamento de Analisis Matematico
C/Astr.Fco.Sanchez s/n
38271. LaLaguna. Tenerife.
SPAIN
Tel/Fax 34-922-318209
Juan.Trujillo@ull.es
Fractional: Differential Equations-
Operators-Fourier Transforms,
Special functions, Approximations,
and Applications

Ram Verma
International Publications
1200 Dallas Drive #824 Denton,
TX 76205, USA
Verma99@msn.com
Applied Nonlinear Analysis,
Numerical Analysis, Variational
Inequalities, Optimization Theory,
Computational Mathematics, Operator
Theory

Xiang Ming Yu
Department of Mathematical Sciences
Southwest Missouri State University
Springfield, MO 65804-0094
417-836-5931
xmy944f@missouristate.edu
Classical Approximation Theory,
Wavelets

Lotfi A. Zadeh
Professor in the Graduate School
and Director, Computer Initiative,
Soft Computing (BISC)
Computer Science Division
University of California at
Berkeley
Berkeley, CA 94720
Office: 510-642-4959
Sec: 510-642-8271
Home: 510-526-2569
FAX: 510-642-1712
zadeh@cs.berkeley.edu
Fuzzyness, Artificial Intelligence,
Natural language processing, Fuzzy
logic

Richard A. Zalik
Department of Mathematics
Auburn University
Auburn University, AL 36849-5310
USA.
Tel 334-844-6557 office
       678-642-8703 home
Fax 334-844-6555
zalik@auburn.edu
Approximation Theory, Chebychev
Systems, Wavelet Theory

Ahmed I. Zayed
Department of Mathematical Sciences
DePaul University
2320 N. Kenmore Ave.
Chicago, IL 60614-3250
773-325-7808
e-mail: azayed@condor.depaul.edu
Shannon sampling theory, Harmonic
analysis and wavelets, Special
functions and orthogonal
polynomials, Integral transforms

Ding-Xuan Zhou
Department Of Mathematics
City University of Hong Kong
83 Tat Chee Avenue
Kowloon, Hong Kong
852-2788 9708,Fax:852-2788 8561
e-mail: mazhou@cityu.edu.hk
Approximation Theory, Spline
functions, Wavelets

Xin-long Zhou
Fachbereich Mathematik, Fachgebiet
Informatik
Gerhard-Mercator-Universitat
Duisburg
Lotharstr.65, D-47048 Duisburg,
Germany
e-mail:Xzhou@informatik.uni-
duisburg.de
Fourier Analysis, Computer-Aided
Geometric Design, Computational
Complexity, Multivariate
Approximation Theory, Approximation
and Interpolation Theory

Instructions to Contributors
Journal of Computational Analysis and Applications

An international publication of Eudoxus Press, LLC, of TN.

Editor in Chief: George Anastassiou

Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152-3240, U.S.A.

1. Manuscripts files in Latex and PDF and in English, should be submitted via email to the Editor-in-Chief:

Prof.George A. Anastassiou
Department of Mathematical Sciences
The University of Memphis
Memphis,TN 38152, USA.
Tel. 901.678.3144
e-mail: ganastss@memphis.edu

Authors may want to recommend an associate editor the most related to the submission to possibly handle it.

Also authors may want to submit a list of six possible referees, to be used in case we cannot find related referees by ourselves.

2. Manuscripts should be typed using any of TEX,LaTEX,AMS-TEX,or AMS-LaTEX and according to EUDOXUS PRESS, LLC. LATEX STYLE FILE. (Click HERE to save a copy of the style file.)They should be carefully prepared in all respects. Submitted articles should be brightly typed (not dot-matrix), double spaced, in ten point type size and in 8(1/2)x11 inch area per page. Manuscripts should have generous margins on all sides and should not exceed 24 pages.

3. Submission is a representation that the manuscript has not been published previously in this or any other similar form and is not currently under consideration for publication elsewhere. A statement transferring from the authors(or their employers,if they hold the copyright) to Eudoxus Press, LLC, will be required before the manuscript can be accepted for publication.The Editor-in-Chief will supply the necessary forms for this transfer.Such a written transfer of copyright,which previously was assumed to be implicit in the act of submitting a manuscript,is necessary under the U.S.Copyright Law in order for the publisher to carry through the dissemination of research results and reviews as widely and effective as possible.

4. The paper starts with the title of the article, author's name(s) (no titles or degrees), author's affiliation(s) and e-mail addresses. The affiliation should comprise the department, institution (usually university or company), city, state (and/or nation) and mail code.

The following items, 5 and 6, should be on page no. 1 of the paper.

5. An abstract is to be provided, preferably no longer than 150 words.

6. A list of 5 key words is to be provided directly below the abstract. Key words should express the precise content of the manuscript, as they are used for indexing purposes.

The main body of the paper should begin on page no. 1, if possible.

7. All sections should be numbered with Arabic numerals (such as: 1. INTRODUCTION) .
Subsections should be identified with section and subsection numbers (such as 6.1. Second-Value Subheading).
If applicable, an independent single-number system (one for each category) should be used to label all theorems, lemmas, propositions, corollaries, definitions, remarks, examples, etc. The label (such as Lemma 7) should be typed with paragraph indentation, followed by a period and the lemma itself.

8. Mathematical notation must be typeset. Equations should be numbered consecutively with Arabic numerals in parentheses placed flush right, and should be thusly referred to in the text [such as Eqs.(2) and (5)]. The running title must be placed at the top of even numbered pages and the first author's name, et al., must be placed at the top of the odd numbed pages.

9. Illustrations (photographs, drawings, diagrams, and charts) are to be numbered in one consecutive series of Arabic numerals. The captions for illustrations should be typed double space. All illustrations, charts, tables, etc., must be embedded in the body of the manuscript in proper, final, print position. In particular, manuscript, source, and PDF file version must be at camera ready stage for publication or they cannot be considered.

Tables are to be numbered (with Roman numerals) and referred to by number in the text. Center the title above the table, and type explanatory footnotes (indicated by superscript lowercase letters) below the table.

10. List references alphabetically at the end of the paper and number them consecutively. Each must be cited in the text by the appropriate Arabic numeral in square brackets on the baseline.
References should include (in the following order):
initials of first and middle name, last name of author(s)
title of article,

name of publication, volume number, inclusive pages, and year of publication.

Authors should follow these examples:

Journal Article

1. H.H.Gonska,Degree of simultaneous approximation of bivariate functions by Gordon operators, (journal name in italics) *J. Approx. Theory*, 62,170-191(1990).

Book

2. G.G.Lorentz, (title of book in italics) *Bernstein Polynomials* (2nd ed.), Chelsea,New York,1986.

Contribution to a Book

3. M.K.Khan, Approximation properties of beta operators,in(title of book in italics) *Progress in Approximation Theory* (P.Nevai and A.Pinkus,eds.), Academic Press, New York,1991,pp.483-495.

11. All acknowledgements (including those for a grant and financial support) should occur in one paragraph that directly precedes the References section.

12. Footnotes should be avoided. When their use is absolutely necessary, footnotes should be numbered consecutively using Arabic numerals and should be typed at the bottom of the page to which they refer. Place a line above the footnote, so that it is set off from the text. Use the appropriate superscript numeral for citation in the text.

13. After each revision is made please again submit via email Latex and PDF files of the revised manuscript, including the final one.

14. Effective 1 Nov. 2009 for current journal page charges, contact the Editor in Chief. Upon acceptance of the paper an invoice will be sent to the contact author. The fee payment will be due one month from the invoice date. The article will proceed to publication only after the fee is paid. The charges are to be sent, by money order or certified check, in US dollars, payable to Eudoxus Press, LLC, to the address shown on the Eudoxus **homepage**.

No galleys will be sent and the contact author will receive one (1) electronic copy of the journal issue in which the article appears.

15. This journal will consider for publication only papers that contain proofs for their listed results.

Mixed problems of fractional coupled systems of Riemann-Liouville differential equations and Hadamard integral conditions

S.K. Ntouyas[1,2] Jessada Tariboon[3] and Phollakrit Thiramanus[3]

[1]Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece
e-mail: sntouyas@uoi.gr
[2]Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

[3]Nonlinear Dynamic Analysis Research Center, Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok, 10800 Thailand
e-mail: jessadat@kmutnb.ac.th, phollakritt@kmutnb.ac.th

Abstract

IIn this paper we study existence and uniqueness of solutions for mixed problems consisting nonlocal Hadamard fractional integrals for coupled systems of Riemann-Liouville fractional differential equations. The existence and uniqueness of solutions is established by using the Banach's contraction principle, while the existence of solutions is derived by applying Leray-Schauder's alternative. Examples illustrating our results are also presented.

Key words and phrases: Riemann-Liouville fractional derivative; Hadamard fractional integral; coupled system; existence; uniqueness; fixed point theorems.
AMS (MOS) Subject Classifications: 34A08; 34A12; 34B15.

1 Introduction

The aim of this paper is to investigate the existence and uniqueness of solutions for nonlocal Hadamard fractional integrals for a coupled system of Riemann-Liouville fractional differential equations of the form:

$$\begin{cases} {}_{RL}D^p x(t) = f(t, x(t), y(t)), & t \in [0, T], \quad 1 < p \le 2, \\ {}_{RL}D^q y(t) = g(t, x(t), y(t)), & t \in [0, T], \quad 1 < q \le 2, \\ x(0) = 0, \quad \sum_{i=1}^{m_1} \mu_{i\,H} I^{\alpha_i} x(\eta_i) = \sum_{j=1}^{n_1} \delta_{j\,H} I^{\beta_j} y(\xi_j) + \lambda_1, \\ y(0) = 0, \quad \sum_{k=1}^{m_2} \tau_{k\,H} I^{\sigma_k} x(\gamma_k) = \sum_{l=1}^{n_2} \omega_{l\,H} I^{\nu_l} y(\theta_l) + \lambda_2, \end{cases} \tag{1}$$

where ${}_{RL}D^q$, ${}_{RL}D^p$ are the standard Riemann-Liouville fractional derivative of orders q, p, two continuous functions $f, g : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$, ${}_H I^{\alpha_i}$, ${}_H I^{\beta_j}$, ${}_H I^{\sigma_k}$ and ${}_H I^{\nu_l}$ are the Hadamard fractional integral of orders $\alpha_i, \beta_j, \sigma_k, \nu_l > 0$, $\lambda_1, \lambda_2 \in \mathbb{R}$ are given constants, $\eta_i, \xi_j, \gamma_k, \theta_l \in (0, T)$, and $\mu_i, \delta_j, \tau_k, \omega_l \in \mathbb{R}$, for $m_1, m_2, n_1, n_2 \in \mathbb{N}$, $i = 1, 2, \ldots, m_1$, $j = 1, 2, \ldots, n_1$, $k = 1, 2, \ldots, m_2$, $l = 1, 2, \ldots, n_2$ are real constants such that

$$\left(\sum_{i=1}^{m_1} \frac{\mu_i \eta_i^{p-1}}{(p-1)^{\alpha_i}} \right) \left(\sum_{l=1}^{n_2} \frac{\omega_l \theta_l^{q-1}}{(q-1)^{\nu_l}} \right) \ne \left(\sum_{j=1}^{n_1} \frac{\delta_j \xi_j^{q-1}}{(q-1)^{\beta_j}} \right) \left(\sum_{k=1}^{m_2} \frac{\tau_k \gamma_k^{p-1}}{(p-1)^{\sigma_k}} \right).$$

Fractional calculus has a long history with more than three hundred years. Up to now, it has been proved that fractional calculus is very useful. Many mathematical models of real problems arising

S. K. NTOUYAS, J. TARIBOON AND P. THIRAMANUS

in various fields of science and engineering were established with the help of fractional calculus, such as viscoelastic systems, dielectric polarization, electrode-electrolyte polarization, and electromagnetic waves. For examples and recent development of the topic, see ([1, 2, 3, 4, 5, 6, 7, 14, 16, 17, 18, 19, 20, 21]). However, it has been observed that most of the work on the topic involves either Riemann-Liouville or Caputo type fractional derivative. Besides these derivatives, Hadamard fractional derivative is another kind of fractional derivatives that was introduced by Hadamard in 1892 [12]. This fractional derivative differs from the other ones in the sense that the kernel of the integral (in the definition of Hadamard derivative) contains logarithmic function of arbitrary exponent. For background material of Hadamard fractional derivative and integral, we refer to the papers [8, 9, 10, 13, 14, 15].

The paper is organized as follows: In Section 2 we will present some useful preliminaries and lemmas. The main results are given in Section 3, where existence and uniqueness results are obtained by using Banach's contraction principle and Leray-Schauder's alternative. Finally the uncoupled integral boundary conditions case is studied in Section 4. Examples illustrating our results are also presented.

2   Preliminaries

In this section, we introduce some notations and definitions of fractional calculus and present preliminary results needed in our proofs later [18, 14].

Definition 2.1 *The Riemann-Liouville fractional derivative of order $q > 0$ of a continuous function $f : (0, \infty) \to \mathbb{R}$ is defined by*

$$_{RL}D^q f(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-q-1} f(s)ds, \quad n-1 < q < n,$$

where $n = [q]+1$, $[q]$ denotes the integer part of a real number q, provided the right-hand side is point-wise defined on $(0, \infty)$, where Γ is the gamma function defined by $\Gamma(q) = \int_0^\infty e^{-s}s^{q-1}ds$.

Definition 2.2 *The Riemann-Liouville fractional integral of order $q > 0$ of a continuous function $f : (0, \infty) \to \mathbb{R}$ is defined by*

$$_{RL}I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)ds,$$

provided the right-hand side is point-wise defined on $(0, \infty)$.

Definition 2.3 *The Hadamard derivative of fractional order q for a function $f : (0, \infty) \to \mathbb{R}$ is defined as*

$$_H D^q f(t) = \frac{1}{\Gamma(n-q)} \left(t\frac{d}{dt}\right)^n \int_0^t \left(\log \frac{t}{s}\right)^{n-q-1} \frac{f(s)}{s}ds, \quad n-1 < q < n, \ n = [q]+1,$$

where $\log(\cdot) = \log_e(\cdot)$.

Definition 2.4 *The Hadamard fractional integral of order $q \in \mathbb{R}^+$ of a function $f(t)$, for all $t > 0$, is defined as*

$$_H I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t \left(\log \frac{t}{s}\right)^{q-1} f(s) \frac{ds}{s},$$

provided the integral exists.

Lemma 2.5 *([14], page 113) Let $q > 0$ and $\beta > 0$. Then the following formulas*

$$_H I^q t^\beta = \beta^{-q} t^\beta \quad and \quad _H D^q t^\beta = \beta^q t^\beta$$

hold.

MIXED PROBLEMS OF FRACTIONAL COUPLED SYSTEMS

Lemma 2.6 *Let $q > 0$ and $x \in C(0,T) \cap L(0,T)$. Then the fractional differential equation $_{RL}D^q x(t) = 0$ has a unique solution $x(t) = c_1 t^{q-1} + c_2 t^{q-2} + \ldots + c_n t^{q-n}$, where $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, and $n - 1 < q < n$.*

Lemma 2.7 *Let $q > 0$. Then for $x \in C(0,T) \cap L(0,T)$ it holds*

$$_{RL}I^q {}_{RL}D^q x(t) = x(t) + c_1 t^{q-1} + c_2 t^{q-2} + \ldots + c_n t^{q-n},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, and $n - 1 < q < n$.

Lemma 2.8 *Given $\phi, \psi \in C([0,T], \mathbb{R})$, the unique solution of the problem*

$$
\begin{cases}
_{RL}D^p x(t) = \phi(t), & t \in [0,T], \quad 1 < p \leq 2, \\
_{RL}D^q y(t) = \psi(t), & t \in [0,T], \quad 1 < q \leq 2, \\
x(0) = 0, \quad \displaystyle\sum_{i=1}^{m_1} \mu_{i\,H} I^{\alpha_i} x(\eta_i) = \sum_{j=1}^{n_1} \delta_{j\,H} I^{\beta_j} y(\xi_j) + \lambda_1, \\
y(0) = 0, \quad \displaystyle\sum_{k=1}^{m_2} \tau_{k\,H} I^{\sigma_k} x(\gamma_k) = \sum_{l=1}^{n_2} \omega_{l\,H} I^{\nu_l} y(\theta_l) + \lambda_2,
\end{cases}
\tag{2}
$$

is

$$
\begin{aligned}
x(t) &= {}_{RL}I^p \phi(t) + \frac{t^{p-1}}{\Omega} \Bigg\{ \sum_{l=1}^{n_2} \frac{\omega_l \theta_l^{q-1}}{(q-1)^{\nu_l}} \left(\sum_{j=1}^{n_1} \delta_{j\,H} I^{\beta_j} {}_{RL}I^q \psi(\xi_j) - \sum_{i=1}^{m_1} \mu_{i\,H} I^{\alpha_i} {}_{RL}I^p \phi(\eta_i) + \lambda_1 \right) \\
&\quad - \sum_{j=1}^{n_1} \frac{\delta_j \xi_j^{q-1}}{(q-1)^{\beta_j}} \left(\sum_{l=1}^{n_2} \omega_{l\,H} I^{\nu_l} {}_{RL}I^q \psi(\theta_l) - \sum_{k=1}^{m_2} \tau_{k\,H} I^{\sigma_k} {}_{RL}I^p \phi(\gamma_k) + \lambda_2 \right) \Bigg\},
\end{aligned}
\tag{3}
$$

and

$$
\begin{aligned}
y(t) &= {}_{RL}I^q \psi(t) + \frac{t^{q-1}}{\Omega} \Bigg\{ \sum_{k=1}^{m_2} \frac{\tau_k \gamma_k^{p-1}}{(p-1)^{\sigma_k}} \left(\sum_{j=1}^{n_1} \delta_{j\,H} I^{\beta_j} {}_{RL}I^q \psi(\xi_j) - \sum_{i=1}^{m_1} \mu_{i\,H} I^{\alpha_i} {}_{RL}I^p \phi(\eta_i) + \lambda_1 \right) \\
&\quad - \sum_{i=1}^{m_1} \frac{\mu_i \eta_i^{p-1}}{(p-1)^{\alpha_i}} \left(\sum_{l=1}^{n_2} \omega_{l\,H} I^{\nu_l} {}_{RL}I^q \psi(\theta_l) - \sum_{k=1}^{m_2} \tau_{k\,H} I^{\sigma_k} {}_{RL}I^p \phi(\gamma_k) + \lambda_2 \right) \Bigg\},
\end{aligned}
\tag{4}
$$

where

$$
\Omega = \sum_{i=1}^{m_1} \frac{\mu_i \eta_i^{p-1}}{(p-1)^{\alpha_i}} \sum_{l=1}^{n_2} \frac{\omega_l \theta_l^{q-1}}{(q-1)^{\nu_l}} - \sum_{j=1}^{n_1} \frac{\delta_j \xi_j^{q-1}}{(q-1)^{\beta_j}} \sum_{k=1}^{m_2} \frac{\tau_k \gamma_k^{p-1}}{(p-1)^{\sigma_k}} \neq 0.
\tag{5}
$$

Proof. Using Lemmas 2.6-2.7, the equations in (2) can be expressed as equivalent integral equations

$$x(t) = {}_{RL}I^p \phi(t) + c_1 t^{p-1} + c_2 t^{p-2}, \tag{6}$$

$$y(t) = {}_{RL}I^q \psi(t) + d_1 t^{q-1} + d_2 t^{q-2}, \tag{7}$$

for $c_1, c_2, d_1, d_2 \in \mathbb{R}$. The conditions $x(0) = 0, y(0) = 0$ imply that $c_2 = 0, d_2 = 0$. Taking the Hadamard fractional integral of order $\alpha_i > 0$, $\sigma_k > 0$ for (6) and $\beta_j > 0$, $\nu_l > 0$ for (7) and using the property of the Hadamard fractional integral given in Lemma 2.5 we get the system

$$
\sum_{i=1}^{m_1} \mu_{i\,H} I^{\alpha_i} {}_{RL}I^p \phi(\eta_i) + c_1 \sum_{i=1}^{m_1} \frac{\mu_i \eta_i^{p-1}}{(p-1)^{\alpha_i}} = \sum_{j=1}^{n_1} \delta_{j\,H} I^{\beta_j} {}_{RL}I^q \psi(\xi_j) + d_1 \sum_{j=1}^{n_1} \frac{\delta_j \xi_j^{q-1}}{(q-1)^{\beta_j}} + \lambda_1,
$$

$$
\sum_{k=1}^{m_2} \tau_{k\,H} I^{\sigma_k} {}_{RL}I^p \phi(\gamma_k) + c_1 \sum_{k=1}^{m_2} \frac{\tau_k \gamma_k^{p-1}}{(p-1)^{\sigma_k}} = \sum_{l=1}^{n_2} \omega_{l\,H} I^{\nu_l} {}_{RL}I^q \psi(\theta_l) + d_1 \sum_{l=1}^{n_2} \frac{\omega_l \theta_l^{q-1}}{(q-1)^{\nu_l}} + \lambda_2,
$$

S. K. NTOUYAS, J. TARIBOON AND P. THIRAMANUS

from which we have

$$
\begin{aligned}
c_1 = \frac{1}{\Omega}\Bigg\{ & \sum_{l=1}^{n_2}\frac{\omega_l\theta_l^{q-1}}{(q-1)^{\nu_l}}\Bigg(\sum_{j=1}^{n_1}\delta_{j\,H}I^{\beta_j}{}_{RL}I^q\psi(\xi_j)-\sum_{i=1}^{m_1}\mu_{i\,H}I^{\alpha_i}{}_{RL}I^p\phi(\eta_i)+\lambda_1\Bigg) \\
& -\sum_{j=1}^{n_1}\frac{\delta_j\xi_j^{q-1}}{(q-1)^{\beta_j}}\Bigg(\sum_{l=1}^{n_2}\omega_{l\,H}I^{\nu_l}{}_{RL}I^q\psi(\theta_l)-\sum_{k=1}^{m_2}\tau_{k\,H}I^{\sigma_k}{}_{RL}I^p\phi(\gamma_k)+\lambda_2\Bigg)\Bigg\}
\end{aligned}
$$

and

$$
\begin{aligned}
d_1 = \frac{1}{\Omega}\Bigg\{ & \sum_{k=1}^{m_2}\frac{\tau_k\gamma_k^{p-1}}{(p-1)^{\sigma_k}}\Bigg(\sum_{j=1}^{n_1}\delta_{j\,H}I^{\beta_j}{}_{RL}I^q\psi(\xi_j)-\sum_{i=1}^{m_1}\mu_{i\,H}I^{\alpha_i}{}_{RL}I^p\phi(\eta_i)+\lambda_1\Bigg) \\
& -\sum_{i=1}^{m_1}\frac{\mu_i\eta_i^{p-1}}{(p-1)^{\alpha_i}}\Bigg(\sum_{l=1}^{n_2}\omega_{l\,H}I^{\nu_l}{}_{RL}I^q\psi(\theta_l)-\sum_{k=1}^{m_2}\tau_{k\,H}I^{\sigma_k}{}_{RL}I^p\phi(\gamma_k)+\lambda_2\Bigg)\Bigg\}.
\end{aligned}
$$

Substituting the values of c_1, c_2, d_1 and d_2 in (6) and (7), we obtain the solutions (3) and (4). $\qquad\square$

3 Main Results

Throughout this paper, for convenience, we use the following expressions

$$
{}_{RL}I^w h(s,x(s),y(s))(v)=\frac{1}{\Gamma(w)}\int_0^v (v-s)^{w-1}h(s,x(s),y(s))ds,
$$

and

$$
{}_H I^u{}_{RL}I^w h(s,x(s),y(s))(v)=\frac{1}{\Gamma(u)\Gamma(w)}\int_0^v\int_0^t\left(\log\frac{v}{t}\right)^{u-1}(t-s)^{w-1}\frac{h(s,x(s),y(s))}{t}dsdt,
$$

where $u\in\{\rho_i,\gamma_j\}$, $v\in\{t,T,\eta_i,\theta_j\}$, $w=\{p,q\}$ and $h=\{f,g\}$, $i=1,2,\dots,n$, $j=1,2,\dots,m$.

Let $\mathcal{C}=C([0,T],\mathbb{R})$ denotes the Banach space of all continuous functions from $[0,T]$ to $\mathbb{R}$. Let us introduce the space $X=\{x(t)|x(t)\in C([0,T])\}$ endowed with the norm $\|x\|=\max\{|x(t)|,t\in[0,T]\}$. Obviously $(X,\|\cdot\|)$ is a Banach space. Also let $Y=\{y(t)|y(t)\in C([0,T])\}$ be endowed with the norm $\|y\|=\max\{|y(t)|,t\in[0,T]\}$. Obviously the product space $(X\times Y,\|(x,y)\|)$ is a Banach space with norm $\|(x,y)\|=\|x\|+\|y\|$.

In view of Lemma 2.8, we define an operator $\mathcal{T}:X\times Y\to X\times Y$ by $\mathcal{T}(x,y)(t)=\begin{pmatrix}\mathcal{T}_1(x,y)(t)\\\mathcal{T}_2(x,y)(t)\end{pmatrix}$, where

$$
\begin{aligned}
\mathcal{T}_1(x,y)(t) = & {}_{RL}I^p f(s,x(s),y(s))(t)+\frac{t^{p-1}}{\Omega}\Bigg\{\sum_{l=1}^{n_2}\frac{\omega_l\theta_l^{q-1}}{(q-1)^{\nu_l}}\Bigg(\sum_{j=1}^{n_1}\delta_{j\,H}I^{\beta_j}{}_{RL}I^q g(s,x(s),y(s))(\xi_j)\\
& -\sum_{i=1}^{m_1}\mu_{i\,H}I^{\alpha_i}{}_{RL}I^p f(s,x(s),y(s))(\eta_i)+\lambda_1\Bigg)-\sum_{j=1}^{n_1}\frac{\delta_j\xi_j^{q-1}}{(q-1)^{\beta_j}}\Bigg(\sum_{l=1}^{n_2}\omega_{l\,H}I^{\nu_l}{}_{RL}I^q g(s,x(s),y(s))(\theta_l)\\
& -\sum_{k=1}^{m_2}\tau_{k\,H}I^{\sigma_k}{}_{RL}I^p f(s,x(s),y(s))(\gamma_k)+\lambda_2\Bigg)\Bigg\},
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{T}_2(x,y)(t) = & {}_{RL}I^q g(s,x(s),y(s))(t)+\frac{t^{q-1}}{\Omega}\Bigg\{\sum_{k=1}^{m_2}\frac{\tau_k\gamma_k^{p-1}}{(p-1)^{\sigma_k}}\Bigg(\sum_{j=1}^{n_1}\delta_{j\,H}I^{\beta_j}{}_{RL}I^q g(s,x(s),y(s))(\xi_j)\\
& -\sum_{i=1}^{m_1}\mu_{i\,H}I^{\alpha_i}{}_{RL}I^p f(s,x(s),y(s))(\eta_i)+\lambda_1\Bigg)-\sum_{i=1}^{m_1}\frac{\mu_i\eta_i^{p-1}}{(p-1)^{\alpha_i}}\Bigg(\sum_{l=1}^{n_2}\omega_{l\,H}I^{\nu_l}{}_{RL}I^q g(s,x(s),y(s))(\theta_l)
\end{aligned}
$$

MIXED PROBLEMS OF FRACTIONAL COUPLED SYSTEMS

$$-\sum_{k=1}^{m_2} \tau_k {}_H I^{\sigma_k} {}_{RL} I^p f(s, x(s), y(s))(\gamma_k) + \lambda_2 \bigg)\bigg\}.$$

Let us introduce the following assumptions which are used hereafter.

(H_1) Assume that $f, g : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions and there exist constants $m_i, n_i, i = 1, 2$ such that for all $t \in [0, T]$ and $u_i, v_i \in \mathbb{R}, i = 1, 2,$

$$|f(t, u_1, u_2) - f(t, v_1, v_2)| \le K_1|u_1 - v_1| + K_2|u_2 - v_2|$$

and

$$|g(t, u_1, u_2) - g(t, v_1, v_2)| \le L_1|u_1 - v_1| + L_2|u_2 - v_2|.$$

(H_2) Assume that there exist real constants $k_i, \ l_i \ge 0 \ (i = 1, 2)$ and $k_0 > 0, l_0 > 0$ such that $\forall x_i \in \mathbb{R}, \ (i = 1, 2)$ we have

$$|f(t, x_1, x_2)| \le k_0 + k_1|x_1| + k_2|x_2|, \quad |g(t, x_1, x_2)| \le l_0 + l_1|x_1| + l_2|x_2|.$$

For the sake of convenience, we set

$$M_1 = \frac{1}{\Gamma(p+1)}\left(T^p + \frac{T^{p-1}}{|\Omega|}\sum_{l=1}^{n_2}\frac{|\omega_l|\theta_l^{q-1}}{(q-1)^{\nu_l}}\sum_{i=1}^{m_1}\frac{|\mu_i|\eta_i^p}{p^{\alpha_i}} + \frac{T^{p-1}}{|\Omega|}\sum_{j=1}^{n_1}\frac{|\delta_j|\xi_j^{q-1}}{(q-1)^{\beta_j}}\sum_{k=1}^{m_2}\frac{|\tau_k|\gamma_k^p}{p^{\sigma_k}}\right), \qquad (8)$$

$$M_2 = \frac{T^{p-1}}{|\Omega|\Gamma(q+1)}\left(\sum_{l=1}^{n_2}\frac{|\omega_l|\theta_l^{q-1}}{(q-1)^{\nu_l}}\sum_{j=1}^{n_1}\frac{|\delta_j|\xi_j^q}{q^{\beta_j}} + \sum_{j=1}^{n_1}\frac{|\delta_j|\xi_j^{q-1}}{(q-1)^{\beta_j}}\sum_{l=1}^{n_2}\frac{|\omega_l|\theta_l^q}{q^{\nu_l}}\right), \qquad (9)$$

$$M_3 = \frac{T^{p-1}}{|\Omega|}\left(|\lambda_1|\sum_{l=1}^{n_2}\frac{|\omega_l|\theta_l^{q-1}}{(q-1)^{\nu_l}} + |\lambda_2|\sum_{j=1}^{n_1}\frac{|\delta_j|\xi_j^{q-1}}{(q-1)^{\beta_j}}\right), \qquad (10)$$

$$M_4 = \frac{1}{\Gamma(q+1)}\left(T^q + \frac{T^{q-1}}{|\Omega|}\sum_{k=1}^{m_2}\frac{|\tau_k|\gamma_k^{p-1}}{(p-1)^{\sigma_k}}\sum_{j=1}^{n_1}\frac{|\delta_j|\xi_j^q}{q^{\beta_j}} + \frac{T^{q-1}}{|\Omega|}\sum_{i=1}^{m_1}\frac{|\mu_i|\eta_i^{p-1}}{(p-1)^{\alpha_i}}\sum_{l=1}^{n_2}\frac{|\omega_l|\theta_l^q}{q^{\nu_l}}\right), \qquad (11)$$

$$M_5 = \frac{T^{q-1}}{|\Omega|\Gamma(p+1)}\left(\sum_{k=1}^{m_2}\frac{|\tau_k|\gamma_k^{p-1}}{(p-1)^{\sigma_k}}\sum_{i=1}^{m_1}\frac{|\mu_i|\eta_i^p}{p^{\alpha_i}} + \sum_{i=1}^{m_1}\frac{|\mu_i|\eta_i^{p-1}}{(p-1)^{\alpha_i}}\sum_{k=1}^{m_2}\frac{|\tau_k|\gamma_k^p}{p^{\sigma_k}}\right), \qquad (12)$$

$$M_6 = \frac{T^{q-1}}{|\Omega|}\left(|\lambda_1|\sum_{k=1}^{m_2}\frac{|\tau_k|\gamma_k^{p-1}}{(p-1)^{\sigma_k}} + |\lambda_2|\sum_{i=1}^{m_1}\frac{|\mu_i|\eta_i^{p-1}}{(p-1)^{\alpha_i}}\right), \qquad (13)$$

and

$$M_0 = \min\{1 - (M_1 + M_5)k_1 - (M_2 + M_4)l_1, \ 1 - (M_1 + M_5)k_2 - (M_2 + M_4)l_2\}, \qquad (14)$$

$k_i, \ l_i \ge 0 \ (i = 1, 2).$

The first result is concerned with the existence and uniqueness of solutions for the problem (1) and is based on Banach's contraction mapping principle.

Theorem 3.1 *Assume that (H_1) holds. In addition, suppose that*

$$(M_1 + M_5)(K_1 + K_2) + (M_2 + M_4)(L_1 + L_2) < 1,$$

where $M_i, i = 1, 2, 4, 5$ are given by (3.1)-(3.2) and (3.4)-(3.5). Then the boundary value problem (1) has a unique solution.

Proof. Define $\sup_{t \in [0,T]} f(t, 0, 0) = N_1 < \infty$ and $\sup_{t \in [0,T]} g(t, 0, 0) = N_2 < \infty$ such that

$$r \ge \max\left\{\frac{M_1 N_1 + M_2 N_2 + M_3}{1 - (M_1 K_1 + M_2 L_1 + M_1 K_2 + M_2 L_2)}, \frac{M_4 N_2 + M_5 N_1 + M_6}{1 - (M_4 L_1 + M_5 K_1 + M_4 L_2 + M_5 K_2)}\right\},$$

where M_3 and M_6 are defined by (3.3) and (3.6), respectively.

We show that $\mathcal{T}B_r \subset B_r$, where $B_r = \{(x, y) \in X \times Y : \|(x, y)\| \le r\}$.

For $(x, y) \in B_r$, we have

$$
\begin{aligned}
|\mathcal{T}_1(x,y)(t)| \quad &= \max_{t \in [0,T]} \Bigg\{ {}_{RL}I^p f(s, x(s), y(s))(t) + \frac{t^{p-1}}{\Omega} \Bigg[\sum_{l=1}^{n_2} \frac{\omega_l \theta_l^{q-1}}{(q-1)^{\nu_l}} \\
&\times \bigg(\sum_{j=1}^{n_1} \delta_{j\,H} I^{\beta_j}{}_{RL}I^q g(s, x(s), y(s))(\xi_j) - \sum_{i=1}^{m_1} \mu_{i\,H} I^{\alpha_i}{}_{RL}I^p f(s, x(s), y(s))(\eta_i) + \lambda_1 \bigg) \\
&- \sum_{j=1}^{n_1} \frac{\delta_j \xi_j^{q-1}}{(q-1)^{\beta_j}} \bigg(\sum_{l=1}^{n_2} \omega_{l\,H} I^{\nu_l}{}_{RL}I^q g(s, x(s), y(s))(\theta_l) \\
&- \sum_{k=1}^{m_2} \tau_{k\,H} I^{\sigma_k}{}_{RL}I^p f(s, x(s), y(s))(\gamma_k) + \lambda_2 \bigg) \Bigg] \Bigg\} \\[2mm]
&\le \ {}_{RL}I^p(|f(s, x(s), y(s)) - f(s, 0, 0)| + |f(s, 0, 0)|)(T) + \frac{T^{p-1}}{|\Omega|} \Bigg[\sum_{l=1}^{n_2} \frac{|\omega_l|\theta_l^{q-1}}{(q-1)^{\nu_l}} \\
&\times \bigg(\sum_{j=1}^{n_1} |\delta_j|_H I^{\beta_j}{}_{RL}I^q(|g(s, x(s), y(s)) - g(s, 0, 0)| + |g(s, 0, 0)|)(\xi_j) \\
&+ \sum_{i=1}^{m_1} |\mu_i|_H I^{\alpha_i}{}_{RL}I^p(|f(s, x(s), y(s)) - f(s, 0, 0)| + |f(s, 0, 0)|)(\eta_i) + |\lambda_1| \bigg) \\
&+ \sum_{j=1}^{n_1} \frac{|\delta_j|\xi_j^{q-1}}{(q-1)^{\beta_j}} \bigg(\sum_{l=1}^{n_2} |\omega_l|_H I^{\nu_l}{}_{RL}I^q(|g(s, x(s), y(s)) - g(s, 0, 0)| + |g(s, 0, 0)|)(\theta_l) \\
&+ \sum_{k=1}^{m_2} |\tau_k|_H I^{\sigma_k}{}_{RL}I^p(|f(s, x(s), y(s)) - f(s, 0, 0)| + |f(s, 0, 0)|)(\gamma_k) + |\lambda_2| \bigg) \Bigg] \\[2mm]
&\le \ {}_{RL}I^p(K_1\|x\| + K_2\|y\| + N_1)(T) + \frac{T^{p-1}}{|\Omega|} \Bigg[\sum_{l=1}^{n_2} \frac{|\omega_l|\theta_l^{q-1}}{(q-1)^{\nu_l}} \\
&\times \bigg(\sum_{j=1}^{n_1} |\delta_j|_H I^{\beta_j}{}_{RL}I^q(L_1\|x\| + L_2\|y\| + N_2)(\xi_j) \\
&+ \sum_{i=1}^{m_1} |\mu_i|_H I^{\alpha_i}{}_{RL}I^p(K_1\|x\| + K_2\|y\| + N_1)(\eta_i) + |\lambda_1| \bigg) \\
&+ \sum_{j=1}^{n_1} \frac{|\delta_j|\xi_j^{q-1}}{(q-1)^{\beta_j}} \bigg(\sum_{l=1}^{n_2} |\omega_l|_H I^{\nu_l}{}_{RL}I^q(L_1\|x\| + L_2\|y\| + N_2)(\theta_l) \\
&+ \sum_{k=1}^{m_2} |\tau_k|_H I^{\sigma_k}{}_{RL}I^p(K_1\|x\| + K_2\|y\| + N_1)(\gamma_k) + |\lambda_2| \bigg) \Bigg] \\[2mm]
&= (K_1\|x\| + K_2\|y\| + N_1) \Bigg\{ {}_{RL}I^p(1)(T) + \frac{T^{p-1}}{|\Omega|} \sum_{l=1}^{n_2} \frac{|\omega_l|\theta_l^{q-1}}{(q-1)^{\nu_l}} \sum_{i=1}^{m_1} |\mu_i|_H I^{\alpha_i}{}_{RL}I^p(1)(\eta_i) \\
&+ \frac{T^{p-1}}{|\Omega|} \sum_{j=1}^{n_1} \frac{|\delta_j|\xi_j^{q-1}}{(q-1)^{\beta_j}} \sum_{k=1}^{m_2} |\tau_k|_H I^{\sigma_k}{}_{RL}I^p(1)(\gamma_k) \Bigg\} + (L_1\|x\| + L_2\|y\| + N_2) \Bigg\{ \frac{T^{p-1}}{|\Omega|} \\
&\times \sum_{l=1}^{n_2} \frac{|\omega_l|\theta_l^{q-1}}{(q-1)^{\nu_l}} \sum_{j=1}^{n_1} |\delta_j|_H I^{\beta_j}{}_{RL}I^q(1)(\xi_j) + \frac{T^{p-1}}{|\Omega|} \sum_{j=1}^{n_1} \frac{|\delta_j|\xi_j^{q-1}}{(q-1)^{\beta_j}} \sum_{l=1}^{n_2} |\omega_l|_H I^{\nu_l}{}_{RL}I^q(1)(\theta_l) \Bigg\}
\end{aligned}
$$

MIXED PROBLEMS OF FRACTIONAL COUPLED SYSTEMS

$$+|\lambda_1|\frac{T^{p-1}}{|\Omega|}\sum_{l=1}^{n_2}\frac{|\omega_l|\theta_l^{q-1}}{(q-1)^{\nu_l}}+|\lambda_2|\frac{T^{p-1}}{|\Omega|}\sum_{j=1}^{n_1}\frac{|\delta_j|\xi_j^{q-1}}{(q-1)^{\beta_j}}$$

$$= (K_1\|x\|+K_2\|y\|+N_1)\left\{\frac{T^p}{\Gamma(p+1)}+\frac{T^{p-1}}{|\Omega|\Gamma(p+1)}\sum_{l=1}^{n_2}\frac{|\omega_l|\theta_l^{q-1}}{(q-1)^{\nu_l}}\sum_{i=1}^{m_1}\frac{|\mu_i|\eta_i^p}{p^{\alpha_i}}\right.$$

$$\left.+\frac{T^{p-1}}{|\Omega|\Gamma(p+1)}\sum_{j=1}^{n_1}\frac{|\delta_j|\xi_j^{q-1}}{(q-1)^{\beta_j}}\sum_{k=1}^{m_2}\frac{|\tau_k|\gamma_k^p}{p^{\sigma_k}}\right\}+(L_1\|x\|+L_2\|y\|+N_2)\left\{\frac{T^{p-1}}{|\Omega|\Gamma(q+1)}\right.$$

$$\left.\times\sum_{l=1}^{n_2}\frac{|\omega_l|\theta_l^{q-1}}{(q-1)^{\nu_l}}\sum_{j=1}^{n_1}\frac{|\delta_j|\xi_j^q}{q^{\beta_j}}+\frac{T^{p-1}}{|\Omega|\Gamma(q+1)}\sum_{j=1}^{n_1}\frac{|\delta_j|\xi_j^{q-1}}{(q-1)^{\beta_j}}\sum_{l=1}^{n_2}\frac{|\omega_l|\theta_l^q}{q^{\nu_l}}\right\}$$

$$+|\lambda_1|\frac{T^{p-1}}{|\Omega|}\sum_{l=1}^{n_2}\frac{|\omega_l|\theta_l^{q-1}}{(q-1)^{\nu_l}}+|\lambda_2|\frac{T^{p-1}}{|\Omega|}\sum_{j=1}^{n_1}\frac{|\delta_j|\xi_j^{q-1}}{(q-1)^{\beta_j}}$$

$$= (K_1\|x\|+K_2\|y\|+N_1)M_1+(L_1\|x\|+L_2\|y\|+N_2)M_2+M_3$$

$$= (M_1K_1+M_2L_1)\|x\|+(M_1K_2+M_2L_2)\|y\|+M_1N_1+M_2N_2+M_3$$

$$\leq (M_1K_1+M_2L_1+M_1K_2+M_2L_2)r+M_1N_1+M_2N_2+M_3\leq r.$$

In the same way, we can obtain that

$$|\mathcal{T}_2(x,y)(t)| \leq (L_1\|x\|+L_2\|y\|+N_2)\left\{\frac{T^q}{\Gamma(q+1)}+\frac{T^{q-1}}{|\Omega|\Gamma(q+1)}\sum_{k=1}^{m_2}\frac{|\tau_k|\gamma_k^{p-1}}{(p-1)^{\sigma_k}}\sum_{j=1}^{n_1}\frac{|\delta_j|\xi_j^q}{q^{\beta_j}}\right.$$

$$\left.+\frac{T^{q-1}}{|\Omega|\Gamma(q+1)}\sum_{i=1}^{m_1}\frac{|\mu_i|\eta_i^{p-1}}{(p-1)^{\alpha_i}}\sum_{l=1}^{n_2}\frac{|\omega_l|\theta_l^q}{q^{\nu_l}}\right\}+(K_1\|x\|+K_2\|y\|+N_1)\left\{\frac{T^{q-1}}{|\Omega|\Gamma(p+1)}\right.$$

$$\left.\times\sum_{k=1}^{m_2}\frac{|\tau_k|\gamma_k^{p-1}}{(p-1)^{\sigma_k}}\sum_{i=1}^{m_1}\frac{|\mu_i|\eta_i^p}{p^{\alpha_i}}+\frac{T^{q-1}}{|\Omega|\Gamma(p+1)}\sum_{i=1}^{m_1}\frac{|\mu_i|\eta_i^{p-1}}{(p-1)^{\alpha_i}}\sum_{k=1}^{m_2}\frac{|\tau_k|\gamma_k^p}{p^{\sigma_k}}\right\}$$

$$+|\lambda_1|\frac{T^{q-1}}{|\Omega|}\sum_{k=1}^{m_2}\frac{|\tau_k|\gamma_k^{p-1}}{(p-1)^{\sigma_k}}+|\lambda_2|\frac{T^{q-1}}{|\Omega|}\sum_{i=1}^{m_1}\frac{|\mu_i|\eta_i^{p-1}}{(p-1)^{\alpha_i}}$$

$$= (L_1\|x\|+L_2\|y\|+N_2)M_4+(K_1\|x\|+K_2\|y\|+N_1)M_5+M_6$$

$$= (M_4L_1+M_5K_1)\|x\|+(M_4L_2+M_5K_2)\|y\|+M_4N_2+M_5N_1+M_6$$

$$\leq (M_4L_1+M_5K_1+M_4L_2+M_5K_2)r+M_4N_2+M_5N_1+M_6\leq r.$$

Consequently, $\|\mathcal{T}(x,y)(t)\|\leq r$.

Now for $(x_2,y_2),(x_1,y_1)\in X\times Y$, and for any $t\in[0,T]$, we get

$$|\mathcal{T}_1(x_2,y_2)(t)-\mathcal{T}_1(x_1,y_1)(t)|$$

$$\leq {}_{RL}I^p|f(s,x_2(s),y_2(s))-f(s,x_1(s),y_1(s))|(T)+\frac{T^{p-1}}{|\Omega|}\left[\sum_{l=1}^{n_2}\frac{|\omega_l|\theta_l^{q-1}}{(q-1)^{\nu_l}}\right.$$

$$\times\left(\sum_{j=1}^{n_1}|\delta_j|_H I^{\beta_j}{}_{RL}I^q(|g(s,x_2(s),y_2(s))-g(s,x_1(s),y_1(s))|)(\xi_j)\right.$$

$$\left.+\sum_{i=1}^{m_1}|\mu_i|_H I^{\alpha_i}{}_{RL}I^p(|f(s,x_2(s),y_2(s))-f(s,x_1(s),y_1(s))|)(\eta_i)\right)$$

$$+\sum_{j=1}^{n_1}\frac{|\delta_j|\xi_j^{q-1}}{(q-1)^{\beta_j}}\left(\sum_{l=1}^{n_2}|\omega_l|_H I^{\nu_l}{}_{RL}I^q(|g(s,x_2(s),y_2(s))-g(s,x_1(s),y_1(s))|)(\theta_l)\right.$$

$$\left.\left.+\sum_{k=1}^{m_2}|\tau_k|_H I^{\sigma_k}{}_{RL}I^p(|f(s,x_2(s),y_2(s))-f(s,x_1(s),y_1(s))|)(\gamma_k)\right)\right]$$

$$
\begin{aligned}
\leq \quad & (K_1\|x_2 - x_1\| + K_2\|y_2 - y_1\|)\left\{ \frac{T^p}{\Gamma(p+1)} + \frac{T^{p-1}}{|\Omega|\Gamma(p+1)} \sum_{l=1}^{n_2} \frac{|\omega_l|\theta_l^{q-1}}{(q-1)^{\nu_l}} \sum_{i=1}^{m_1} \frac{|\mu_i|\eta_i^p}{p^{\alpha_i}} \right. \\
& \left. + \frac{T^{p-1}}{|\Omega|\Gamma(p+1)} \sum_{j=1}^{n_1} \frac{|\delta_j|\xi_j^{q-1}}{(q-1)^{\beta_j}} \sum_{k=1}^{m_2} \frac{|\tau_k|\gamma_k^p}{p^{\sigma_k}} \right\} + (L_1\|x_2 - x_1\| + L_2\|y_2 - y_1\|) \\
& \times \left\{ \frac{T^{p-1}}{|\Omega|\Gamma(q+1)} \sum_{l=1}^{n_2} \frac{|\omega_l|\theta_l^{q-1}}{(q-1)^{\nu_l}} \sum_{j=1}^{n_1} \frac{|\delta_j|\xi_j^q}{q^{\beta_j}} + \frac{T^{p-1}}{|\Omega|\Gamma(q+1)} \sum_{j=1}^{n_1} \frac{|\delta_j|\xi_j^{q-1}}{(q-1)^{\beta_j}} \sum_{l=1}^{n_2} \frac{|\omega_l|\theta_l^q}{q^{\nu_l}} \right\} \\
= \quad & (K_1\|x_2 - x_1\| + K_2\|y_2 - y_1\|)M_1 + (L_1\|x_2 - x_1\| + L_2\|y_2 - y_1\|)M_2 \\
= \quad & (M_1 K_1 + M_2 L_1)\|x_2 - x_1\| + (M_1 K_2 + M_2 L_2)\|y_2 - y_1\|,
\end{aligned}
$$

and consequently we obtain

$$
\|\mathcal{T}_1(x_2, y_2)(t) - \mathcal{T}_1(x_1, y_1)\| \leq (M_1 K_1 + M_2 L_1 + M_1 K_2 + M_2 L_2)[\|x_2 - x_1\| + \|y_2 - y_1\|]. \tag{15}
$$

Similarly,

$$
\|\mathcal{T}_2(x_2, y_2)(t) - \mathcal{T}_2(x_1, y_1)\| \leq (M_4 L_1 + M_5 K_1 + M_4 L_2 + M_5 K_2)[\|x_2 - x_1\| + \|y_2 - y_1\|]. \tag{16}
$$

It follows from (15) and (16) that

$$
\|\mathcal{T}(x_2, y_2)(t) - \mathcal{T}(x_1, y_1)(t)\| \leq [(M_1 + M_5)(K_1 + K_2) + (M_2 + M_4)(L_1 + L_2)](\|x_2 - x_1\| + \|y_2 - y_1\|).
$$

Since $(M_1 + M_5)(K_1 + K_2) + (M_2 + M_4)(L_1 + L_2) < 1$, therefore, $\mathcal{T}$ is a contraction operator. So, By Banach's fixed point theorem, the operator $\mathcal{T}$ has a unique fixed point, which is the unique solution of problem (1). This completes the proof. $\qquad\square$

In the next result, we prove the existence of solutions for the problem (1) by applying Leray-Schauder alternative.

Lemma 3.2 *(Leray-Schauder alternative) ([11], page.4.) Let $F : E \to E$ be a completely continuous operator (i.e., a map that restricted to any bounded set in E is compact). Let*

$$
\mathcal{E}(F) = \{x \in E : x = \lambda F(x) \text{ for some } 0 < \lambda < 1\}.
$$

Then either the set $\mathcal{E}(F)$ is unbounded, or F has at least one fixed point.

Theorem 3.3 *Assume that (H_2) holds. In addition it is assumed that*

$$
(M_1 + M_5)k_1 + (M_2 + M_4)l_1 < 1 \quad and \quad (M_1 + M_5)k_2 + (M_2 + M_4)l_2 < 1,
$$

where M_1, M_2, M_4, M_5 are given by (3.1)-(3.2) and (3.4)-(3.5). Then there exists at least one solution for the boundary value problem (1).

Proof. First we show that the operator $\mathcal{T} : X \times Y \to X \times Y$ is completely continuous. By continuity of functions f and g, the operator $\mathcal{T}$ is continuous.

Let $\Theta \subset X \times Y$ be bounded. Then there exist positive constants P_1 and P_2 such that

$$
|f(t, x(t), y(t))| \leq P_1, \quad |g(t, x(t), y(t))| \leq P_2, \quad \forall (x, y) \in \Theta.
$$

Then for any $(x, y) \in \Theta$, we have

$$
\begin{aligned}
\|\mathcal{T}_1(x, y)\| \leq \quad & {}_{RL}I^p|f(s, x(s), y(s))|(T) + \frac{T^{p-1}}{|\Omega|}\left[\sum_{l=1}^{n_2} \frac{|\omega_l|\theta_l^{q-1}}{(q-1)^{\nu_l}}\left(\sum_{j=1}^{n_1} |\delta_j|\,{}_H I^{\beta_j}{}_{RL}I^q|g(s, x(s), y(s))|(\xi_j) \right.\right. \\
& \left.\left. + \sum_{i=1}^{m_1} |\mu_i|\,{}_H I^{\alpha_i}{}_{RL}I^p|f(s, x(s), y(s))|(\eta_i) + |\lambda_1| \right) \right.
\end{aligned}
$$

MIXED PROBLEMS OF FRACTIONAL COUPLED SYSTEMS

$$
+ \sum_{j=1}^{n_1} \frac{|\delta_j| \xi_j^{q-1}}{(q-1)^{\beta_j}} \left(\sum_{l=1}^{n_2} |\omega_l| {}_H I^{\nu_l}{}_{RL} I^q |g(s, x(s), y(s))|(\theta_l) \right.
$$

$$
\left. + \sum_{k=1}^{m_2} |\tau_k| {}_H I^{\sigma_k}{}_{RL} I^p |f(s, x(s), y(s))|(\gamma_k) + |\lambda_2| \right) \Bigg]
$$

$$
\leq \left(\frac{T^p}{\Gamma(p+1)} + \frac{T^{p-1}}{|\Omega|\Gamma(p+1)} \sum_{l=1}^{n_2} \frac{|\omega_l| \theta_l^{q-1}}{(q-1)^{\nu_l}} \sum_{i=1}^{m_1} \frac{|\mu_i| \eta_i^p}{p^{\alpha_i}} \right.
$$

$$
+ \frac{T^{p-1}}{|\Omega|\Gamma(p+1)} \sum_{j=1}^{n_1} \frac{|\delta_j| \xi_j^{q-1}}{(q-1)^{\beta_j}} \sum_{k=1}^{m_2} \frac{|\tau_k| \gamma_k^p}{p^{\sigma_k}} \right) P_1 + \left(\frac{T^{p-1}}{|\Omega|\Gamma(q+1)} \right.
$$

$$
\times \sum_{l=1}^{n_2} \frac{|\omega_l| \theta_l^{q-1}}{(q-1)^{\nu_l}} \sum_{j=1}^{n_1} \frac{|\delta_j| \xi_j^{q}}{q^{\beta_j}} + \frac{T^{p-1}}{|\Omega|\Gamma(q+1)} \sum_{j=1}^{n_1} \frac{|\delta_j| \xi_j^{q-1}}{(q-1)^{\beta_j}} \sum_{l=1}^{n_2} \frac{|\omega_l| \theta_l^{q}}{q^{\nu_l}} \right) P_2
$$

$$
+ |\lambda_1| \frac{T^{p-1}}{|\Omega|} \sum_{l=1}^{n_2} \frac{|\omega_l| \theta_l^{q-1}}{(q-1)^{\nu_l}} + |\lambda_2| \frac{T^{p-1}}{|\Omega|} \sum_{j=1}^{n_1} \frac{|\delta_j| \xi_j^{q-1}}{(q-1)^{\beta_j}}
$$

$$
= M_1 P_1 + M_2 P_2 + M_3.
$$

Similarly, we get

$$
\|\mathcal{T}_2(x, y)\| \leq \left(\frac{T^q}{\Gamma(q+1)} + \frac{T^{q-1}}{|\Omega|\Gamma(q+1)} \sum_{k=1}^{m_2} \frac{|\tau_k| \gamma_k^{p-1}}{(p-1)^{\sigma_k}} \sum_{j=1}^{n_1} \frac{|\delta_j| \xi_j^{q}}{q^{\beta_j}} \right.
$$

$$
+ \frac{T^{q-1}}{|\Omega|\Gamma(q+1)} \sum_{i=1}^{m_1} \frac{|\mu_i| \eta_i^{p-1}}{(p-1)^{\alpha_i}} \sum_{l=1}^{n_2} \frac{|\omega_l| \theta_l^{q}}{q^{\nu_l}} \right) P_2 + \left(\frac{T^{q-1}}{|\Omega|\Gamma(p+1)} \right.
$$

$$
\times \sum_{k=1}^{m_2} \frac{|\tau_k| \gamma_k^{p-1}}{(p-1)^{\sigma_k}} \sum_{i=1}^{m_1} \frac{|\mu_i| \eta_i^p}{p^{\alpha_i}} + \frac{T^{q-1}}{|\Omega|\Gamma(p+1)} \sum_{i=1}^{m_1} \frac{|\mu_i| \eta_i^{p-1}}{(p-1)^{\alpha_i}} \sum_{k=1}^{m_2} \frac{|\tau_k| \gamma_k^p}{p^{\sigma_k}} \right) P_1
$$

$$
+ |\lambda_1| \frac{T^{q-1}}{|\Omega|} \sum_{k=1}^{m_2} \frac{|\tau_k| \gamma_k^{p-1}}{(p-1)^{\sigma_k}} + |\lambda_2| \frac{T^{q-1}}{|\Omega|} \sum_{i=1}^{m_1} \frac{|\mu_i| \eta_i^{p-1}}{(p-1)^{\alpha_i}}
$$

$$
= M_4 P_2 + M_5 P_1 + M_6.
$$

Thus, it follows from the above inequalities that the operator $\mathcal{T}$ is uniformly bounded.

Next, we show that $\mathcal{T}$ is equicontinuous. Let $t_1, t_2 \in [0, T]$ with $t_1 < t_2$. Then we have

$$
|\mathcal{T}_1(x(t_2), y(t_2)) - \mathcal{T}_1(x(t_1), y(t_1))|
$$

$$
\leq \frac{1}{\Gamma(p)} \int_0^{t_1} [(t_2 - s)^{p-1} - (t_1 - s)^{p-1}] |f(s, x(s), y(s))| ds
$$

$$
+ \frac{1}{\Gamma(p)} \int_{t_1}^{t_2} (t_2 - s)^{p-1} |f(s, x(s), y(s))| ds + \frac{t_2^{p-1} - t_1^{p-1}}{|\Omega|} \left[\sum_{l=1}^{n_2} \frac{|\omega_l| \theta_l^{q-1}}{(q-1)^{\nu_l}} \right.
$$

$$
\times \left(\sum_{j=1}^{n_1} |\delta_j| {}_H I^{\beta_j}{}_{RL} I^q |g(s, x(s), y(s))|(\xi_j) + \sum_{i=1}^{m_1} |\mu_i| {}_H I^{\alpha_i}{}_{RL} I^p |f(s, x(s), y(s))|(\eta_i) + |\lambda_1| \right)
$$

$$
+ \sum_{j=1}^{n_1} \frac{|\delta_j| \xi_j^{q-1}}{(q-1)^{\beta_j}} \left(\sum_{l=1}^{n_2} |\omega_l| {}_H I^{\nu_l}{}_{RL} I^q |g(s, x(s), y(s))|(\theta_l) \right.
$$

$$
\left. + \sum_{k=1}^{m_2} |\tau_k| {}_H I^{\sigma_k}{}_{RL} I^p |f(s, x(s), y(s))|(\gamma_k) + |\lambda_2| \right) \Bigg]
$$

$$
\leq \frac{P_1}{\Gamma(p)} \int_0^{t_1} [(t_2 - s)^{p-1} - (t_1 - s)^{p-1}] ds + \frac{P_1}{\Gamma(p)} \int_{t_1}^{t_2} (t_2 - s)^{p-1} ds
$$

S. K. NTOUYAS, J. TARIBOON AND P. THIRAMANUS

$$
+ \frac{t_2^{p-1} - t_1^{p-1}}{|\Omega|} \left[\sum_{l=1}^{n_2} \frac{|\omega_l| |\theta_l^{q-1}|}{(q-1)^{\nu_l}} \left(P_2 \sum_{j=1}^{n_1} \frac{|\delta_j| \xi_j^q}{q^{\beta_j} \Gamma(q+1)} \right) + P_1 \sum_{i=1}^{m_1} \frac{|\mu_i| \eta_i^p}{p^{\alpha_i} \Gamma(p+1)} + |\lambda_1| \right)
$$

$$
+ \sum_{j=1}^{n_1} \frac{|\delta_j| \xi_j^{q-1}}{(q-1)^{\beta_j}} \left(P_2 \sum_{l=1}^{n_2} \frac{|\omega_l| \theta_l^q}{q^{\nu_l} \Gamma(q+1)} + P_1 \sum_{k=1}^{m_2} \frac{|\tau_k| \gamma_k^p}{p^{\sigma_k} \Gamma(p+1)} + |\lambda_2| \right) \Bigg].
$$

Analogously, we can obtain

$$
|\mathcal{T}_2(x(t_2), y(t_2)) - \mathcal{T}_2(x(t_1), y(t_1))|
$$

$$
\leq \quad \frac{P_2}{\Gamma(q)} \int_0^{t_1} [(t_2 - s)^{q-1} - (t_1 - s)^{q-1}] ds + \frac{P_2}{\Gamma(q)} \int_{t_1}^{t_2} (t_2 - s)^{q-1} ds
$$

$$
+ \frac{t_2^{q-1} - t_1^{q-1}}{|\Omega|} \left[\sum_{k=1}^{m_2} \frac{|\tau_k| \gamma_k^{p-1}}{(p-1)^{\sigma_k}} \left(P_2 \sum_{j=1}^{n_1} \frac{|\delta_j| \xi_j^q}{q^{\beta_j} \Gamma(q+1)} \right) + P_1 \sum_{i=1}^{m_1} \frac{|\mu_i| \eta_i^p}{p^{\alpha_i} \Gamma(p+1)} + |\lambda_1| \right)
$$

$$
+ \sum_{i=1}^{m_1} \frac{|\mu_i| \eta_i^{p-1}}{(p-1)^{\alpha_i}} \left(P_2 \sum_{l=1}^{n_2} \frac{|\omega_l| \theta_l^q}{q^{\nu_l} \Gamma(q+1)} + P_1 \sum_{k=1}^{m_2} \frac{|\tau_k| \gamma_k^p}{p^{\sigma_k} \Gamma(p+1)} + |\lambda_2| \right) \Bigg].
$$

Therefore, the operator $\mathcal{T}(x, y)$ is equicontinuous, and thus the operator $\mathcal{T}(x, y)$ is completely continuous.

Finally, it will be verified that the set $\mathcal{E} = \{(x, y) \in X \times Y | (x, y) = \lambda \mathcal{T}(x, y), 0 \leq \lambda \leq 1\}$ is bounded. Let $(x, y) \in \mathcal{E}$, then $(x, y) = \lambda \mathcal{T}(x, y)$. For any $t \in [0, T]$, we have

$$
x(t) = \lambda \mathcal{T}_1(x, y)(t), \quad y(t) = \lambda \mathcal{T}_2(x, y)(t).
$$

Then

$$
|x(t)| \quad \leq \quad (k_0 + k_1 \|x\| + k_2 \|y\|) \left(\frac{T^p}{\Gamma(p+1)} + \frac{T^{p-1}}{|\Omega| \Gamma(p+1)} \sum_{l=1}^{n_2} \frac{|\omega_l| \theta_l^{q-1}}{(q-1)^{\nu_l}} \sum_{i=1}^{m_1} \frac{|\mu_i| \eta_i^p}{p^{\alpha_i}} \right.
$$

$$
+ \frac{T^{p-1}}{|\Omega| \Gamma(p+1)} \sum_{j=1}^{n_1} \frac{|\delta_j| \xi_j^{q-1}}{(q-1)^{\beta_j}} \sum_{k=1}^{m_2} \frac{|\tau_k| \gamma_k^p}{p^{\sigma_k}} \right) + (l_0 + l_1 \|x\| + l_2 \|y\|)
$$

$$
\times \left(\frac{T^{p-1}}{|\Omega| \Gamma(q+1)} \sum_{l=1}^{n_2} \frac{|\omega_l| \theta_l^{q-1}}{(q-1)^{\nu_l}} \sum_{j=1}^{n_1} \frac{|\delta_j| \xi_j^q}{q^{\beta_j}} + \frac{T^{p-1}}{|\Omega| \Gamma(q+1)} \sum_{j=1}^{n_1} \frac{|\delta_j| \xi_j^{q-1}}{(q-1)^{\beta_j}} \sum_{l=1}^{n_2} \frac{|\omega_l| \theta_l^q}{q^{\nu_l}} \right)
$$

$$
+ |\lambda_1| \frac{T^{p-1}}{|\Omega|} \sum_{l=1}^{n_2} \frac{|\omega_l| \theta_l^{q-1}}{(q-1)^{\nu_l}} + |\lambda_2| \frac{T^{p-1}}{|\Omega|} \sum_{j=1}^{n_1} \frac{|\delta_j| \xi_j^{q-1}}{(q-1)^{\beta_j}}
$$

and

$$
|y(t)| \quad \leq \quad (l_0 + l_1 \|x\| + l_2 \|y\|) \left(\frac{T^q}{\Gamma(q+1)} + \frac{T^{q-1}}{|\Omega| \Gamma(q+1)} \sum_{k=1}^{m_2} \frac{|\tau_k| \gamma_k^{p-1}}{(p-1)^{\sigma_k}} \sum_{j=1}^{n_1} \frac{|\delta_j| \xi_j^q}{q^{\beta_j}} \right.
$$

$$
+ \frac{T^{q-1}}{|\Omega| \Gamma(q+1)} \sum_{i=1}^{m_1} \frac{|\mu_i| \eta_i^{p-1}}{(p-1)^{\alpha_i}} \sum_{l=1}^{n_2} \frac{|\omega_l| \theta_l^q}{q^{\nu_l}} \right) + (k_0 + k_1 \|x\| + k_2 \|y\|)
$$

$$
\times \left(\frac{T^{q-1}}{|\Omega| \Gamma(p+1)} \sum_{k=1}^{m_2} \frac{|\tau_k| \gamma_k^{p-1}}{(p-1)^{\sigma_k}} \sum_{i=1}^{m_1} \frac{|\mu_i| \eta_i^p}{p^{\alpha_i}} + \frac{T^{q-1}}{|\Omega| \Gamma(p+1)} \sum_{i=1}^{m_1} \frac{|\mu_i| \eta_i^{p-1}}{(p-1)^{\alpha_i}} \sum_{k=1}^{m_2} \frac{|\tau_k| \gamma_k^p}{p^{\sigma_k}} \right)
$$

$$
+ |\lambda_1| \frac{T^{q-1}}{|\Omega|} \sum_{k=1}^{m_2} \frac{|\tau_k| \gamma_k^{p-1}}{(p-1)^{\sigma_k}} + |\lambda_2| \frac{T^{q-1}}{|\Omega|} \sum_{i=1}^{m_1} \frac{|\mu_i| \eta_i^{p-1}}{(p-1)^{\alpha_i}}.
$$

Hence we have

$$
\|x\| \leq (k_0 + k_1 \|x\| + k_2 \|y\|) M_1 + (l_0 + l_1 \|x\| + l_2 \|y\|) M_2 + M_3
$$

MIXED PROBLEMS OF FRACTIONAL COUPLED SYSTEMS

and

$$\|y\| \le (l_0 + l_1\|x\| + l_2\|y\|)M_4 + (k_0 + k_1\|x\| + k_2\|y\|)M_5 + M_6,$$

which imply that

$$
\begin{aligned}
\|x\| + \|y\| \le & (M_1 + M_5)k_0 + (M_2 + M_4)l_0 + [(M_1 + M_5)k_1 + (M_2 + M_4)l_1]\|x\| \\
& + [(M_1 + M_5)k_2 + (M_2 + M_4)l_2]\|y\| + M_3 + M_6.
\end{aligned}
$$

Consequently,

$$\|(x,y)\| \le \frac{(M_1 + M_5)k_0 + (M_2 + M_4)l_0 + M_3 + M_6}{M_0},$$

for any $t \in [0, T]$, where M_0 is defined by (14), which proves that $\mathcal{E}$ is bounded. Thus, by Lemma 3.2, the operator $\mathcal{T}$ has at least one fixed point. Hence the boundary value problem (1) has at least one solution. The proof is complete. $\qquad\square$

3.1 Examples

Example 3.4 *Consider the following system of coupled Riemann-Liouville fractional differential equations with Hadamard type fractional integral boundary conditions*

$$
\begin{cases}
{}_{RL}D^{4/3}x(t) = \dfrac{t}{(t+6)^2}\dfrac{|x(t)|}{(1+|x(t)|)} + \dfrac{e^{-t}}{(t^2+3)^3}\dfrac{|y(t)|}{(1+|y(t)|)} + \dfrac{3}{4}, \ t \in [0,2], \\[2mm]
{}_{RL}D^{3/2}y(t) = \dfrac{1}{18}\sin x(t) + \dfrac{1}{2^{2t}+19}\cos y(t) + \dfrac{5}{4}, \ \ t \in [0,2], \\[2mm]
x(0) = 0, \quad 2_H I^{2/3}x(3/5) + \pi_H I^{7/5}x(1) = \sqrt{2}_H I^{3/2}y(1/3) + e^2{}_H I^{5/4}y(\sqrt{3}) + 4, \\[2mm]
y(0) = 0, \quad -3_H I^{9/5}x(2/3) + 4_H I^{7/4}x(9/7) + \dfrac{2}{5}_H I^{1/3}x(\sqrt{2}) \\[2mm]
\qquad\qquad = \dfrac{e}{2}_H I^{11/6}y(8/5) - 2_H I^{12/11}y(1/4) - 10.
\end{cases}
\tag{17}
$$

Here $p = 4/3$, $q = 3/2$, $T = 2$, $\lambda_1 = 4$, $\lambda_2 = -10$, $m_1 = 2$, $n_1 = 2$, $m_2 = 3$, $n_2 = 2$, $\mu_1 = 2$, $\mu_2 = \pi$, $\alpha_1 = 2/3$, $\alpha_2 = 7/5$, $\eta_1 = 3/5$, $\eta_2 = 1$, $\delta_1 = \sqrt{2}$, $\delta_2 = e^2$, $\beta_1 = 3/2$, $\beta_2 = 5/4$, $\xi_1 = 1/3$, $\xi_2 = \sqrt{3}$, $\tau_1 = -3$, $\tau_2 = 4$, $\tau_3 = 2/5$, $\sigma_1 = 9/5$, $\sigma_2 = 7/4$, $\sigma_3 = 1/3$, $\gamma_1 = 2/3$, $\gamma_2 = 9/7$, $\gamma_3 = \sqrt{2}$, $\omega_1 = e/2$, $\omega_2 = -2$, $\nu_1 = 11/6$, $\nu_2 = 12/11$, $\theta_1 = 8/5$, $\theta_2 = 1/4$ and $f(t,x,y) = (t|x|)/(((t+6)^2)(1+|x|)) + (e^{-t}|y|)/(((t^2+3)^3)(1+|y|)) + (3/4)$ and $g(t,x,y) = (\sin x/18) + (\cos y)/(2^{2t}+19) + (5/4)$. Since $|f(t,x_1,y_1) - f(t,x_2,y_2)| \le ((1/18)|x_1 - x_2| + (1/27)|y_1 - y_2|)$ and $|g(t,x_1,y_1) - g(t,x_2,y_2)| \le ((1/18)|x_1 - x_2| + (1/20)|y_1 - y_2|)$. By using the Maple program, we can find

$$\Omega = \sum_{i=1}^{m_1}\frac{\mu_i\eta_i^{p-1}}{(p-1)^{\alpha_i}}\sum_{l=1}^{n_2}\frac{\omega_l\theta_l^{q-1}}{(q-1)^{\nu_l}} - \sum_{j=1}^{n_1}\frac{\delta_j\xi_j^{q-1}}{(q-1)^{\beta_j}}\sum_{k=1}^{m_2}\frac{\tau_k\gamma_k^{p-1}}{(p-1)^{\sigma_k}} \approx -218.9954766 \ne 0.$$

With the given values, it is found that $K_1 = 1/18$, $K_2 = 1/27$, $L_1 = 1/18$, $L_2 = 1/20$, $M_1 \simeq 2.847852451$, $M_2 \simeq 0.5295490231$, $M_4 \simeq 4.723846069$, $M_5 \simeq 1.276954854$, and

$$(M_1 + M_5)(K_1 + K_2) + (M_2 + M_4)(L_1 + L_2) \simeq 0.9364516398 < 1.$$

Thus all the conditions of Theorem 3.1 are satisfied. Therefore, by the conclusion of Theorem 3.1, the problem (17) has a unique solution on $[0, 2]$.

Example 3.5 *Consider the following system of coupled Riemann-Liouville fractional differential equa-*

S. K. NTOUYAS, J. TARIBOON AND P. THIRAMANUS

tions with Hadamard type fractional integral boundary conditions

$$
\begin{cases}
{}_{RL}D^{\pi/2}x(t) = \dfrac{2}{5} + \dfrac{1}{(t+6)^2}\tan^{-1}x(t) + \dfrac{1}{20e}y(t), \ t\in[0,3], \\[2mm]
{}_{RL}D^{7/4}y(t) = \dfrac{\sqrt{\pi}}{2} + \dfrac{1}{42}\sin x(t) + \dfrac{1}{t+20}y(t)\cos x(t), \ \ t\in[0,3], \\[2mm]
x(0)=0, \quad 3\,{}_{H}I^{1/4}x(5/2) + \sqrt{5}\,{}_{H}I^{\sqrt{2}}x(7/8) + \tan(4)\,{}_{H}I^{\sqrt{3}}x(9/4) \\[2mm]
\qquad\qquad = \dfrac{\sqrt{8}\pi}{3}\,{}_{H}I^{5/3}y(5/4) - 2\,{}_{H}I^{6/11}y(\pi/3) + 2, \\[2mm]
y(0)=0, \quad -\dfrac{2}{3}\,{}_{H}I^{2/3}x(\pi/2) + 3\,{}_{H}I^{6/5}x(5/3) + \dfrac{\sqrt{2}}{\pi}\,{}_{H}I^{1/3}x(\sqrt{2}) \\[2mm]
\qquad\qquad + \dfrac{7}{9}\,{}_{H}I^{11/9}x(\sqrt{5}) = e\,{}_{H}I^{7/6}y(\pi/6) - \log(9)\,{}_{H}I^{3/4}y(7/4) - 1.
\end{cases}
\tag{18}
$$

Here $p=\pi/2$, $q=7/4$, $T=3$, $\lambda_1=2$, $\lambda_2=-1$, $m_1=3$, $n_1=2$, $m_2=4$, $n_2=2$, $\mu_1=3$, $\mu_2=\sqrt{5}$, $\mu_3=\tan(4)$, $\alpha_1=1/4$, $\alpha_2=\sqrt{2}$, $\alpha_3=\sqrt{3}$, $\eta_1=5/2$, $\eta_2=7/8$, $\eta_3=9/4$, $\delta_1=\sqrt{8}\pi/3$, $\delta_2=-2$, $\beta_1=5/3$, $\beta_2=6/11$, $\xi_1=5/4$, $\xi_2=\pi/3$, $\tau_1=-2/3$, $\tau_2=3$, $\tau_3=\sqrt{2}/\pi$, $\tau_4=7/9$, $\sigma_1=2/3$, $\sigma_2=6/5$, $\sigma_3=1/3$, $\sigma_4=11/9$, $\gamma_1=\pi/2$, $\gamma_2=5/3$, $\gamma_3=\sqrt{2}$, $\gamma_4=\sqrt{5}$, $\omega_1=e$, $\omega_2=-\log(9)$, $\nu_1=7/6$, $\nu_2=3/4$, $\theta_1=\pi/6$, $\theta_2=7/4$, $f(t,x,y)=(2/5)+(\tan^{-1}x)/((t+6)^2)+(y)/(20e)$ and $g(t,x,y)=(\sqrt{\pi}/2)+(\sin x)/(42)+(y\cos x)/(t+20)$. By using the Maple program, we get

$$
\Omega = \sum_{i=1}^{m_1}\frac{\mu_i\eta_i^{p-1}}{(p-1)_{\alpha_i}}\sum_{l=1}^{n_2}\frac{\omega_l\theta_l^{q-1}}{(q-1)_{\nu_l}} - \sum_{j=1}^{n_1}\frac{\delta_j\xi_j^{q-1}}{(q-1)_{\beta_j}}\sum_{k=1}^{m_2}\frac{\tau_k\gamma_k^{p-1}}{(p-1)_{\sigma_k}} \approx -59.01857601 \neq 0.
$$

Since $|f(t,x,y)| \le k_0 + k_1|x| + k_2|y|$ and $|g(t,x,y)| \le l_0 + l_1|x| + l_2|y|$, where $k_0=2/5$, $k_1=1/36$, $k_2=1/(20e)$, $l_0=\sqrt{\pi}/2$, $l_1=1/42$, $l_2=1/20$, it is found that $M_1\simeq 7.406711671$, $M_2\simeq 1.110132269$, $M_4\simeq 6.802999724$, $M_5\simeq 7.790182643$. Furthermore, we can find that

$$(M_1+M_5)k_1 + (M_2+M_4)l_1 \approx 0.6105438577 < 1,$$

and

$$(M_1+M_5)k_2 + (M_2+M_4)l_2 \approx 0.6751878489 < 1.$$

Thus all the conditions of Theorem 3.3 holds true and consequently the conclusion of Theorem 3.3, the problem (18) has at least one solution on $[0,3]$.

4 Uncoupled integral boundary conditions case

In this section we consider the following system

$$
\begin{cases}
{}_{RL}D^p x(t) = f(t,x(t),y(t)), & t\in[0,T], \quad 1<p\le 2, \\
{}_{RL}D^q y(t) = g(t,x(t),y(t)), & t\in[0,T], \quad 1<q\le 2, \\
x(0)=0, \quad \displaystyle\sum_{i=1}^{m_1}\mu_i\,{}_{H}I^{\alpha_i}x(\eta_i) = \sum_{j=1}^{n_1}\delta_j\,{}_{H}I^{\beta_j}x(\xi_j) + \lambda_1, \\
y(0)=0, \quad \displaystyle\sum_{k=1}^{m_2}\tau_k\,{}_{H}I^{\sigma_k}y(\gamma_k) = \sum_{l=1}^{n_2}\omega_l\,{}_{H}I^{\nu_l}y(\theta_l) + \lambda_2.
\end{cases}
\tag{19}
$$

Lemma 4.1 *(Auxiliary Lemma) For $h\in C([0,T],\mathbb{R})$, the unique solution of the problem*

$$
\begin{cases}
{}_{RL}D^p x(t) = h(t), & 1<p\le 2, \quad t\in[0,T] \\
x(0)=0, \quad \displaystyle\sum_{i=1}^{m_1}\mu_i\,{}_{H}I^{\alpha_i}x(\eta_i) = \sum_{j=1}^{n_1}\delta_j\,{}_{H}I^{\beta_j}x(\xi_j) + \lambda_1,
\end{cases}
\tag{20}
$$

MIXED PROBLEMS OF FRACTIONAL COUPLED SYSTEMS

is given by

$$x(t) = {}_{RL}I^p h(t) + \frac{t^{p-1}}{\Lambda} \left(\sum_{j=1}^{n_1} \delta_{jH} I^{\beta_j}{}_{RL} I^p h(\xi_j) - \sum_{i=1}^{m_1} \mu_{iH} I^{\alpha_i}{}_{RL} I^p h(\eta_i) + \lambda_1 \right), \tag{21}$$

where

$$\Lambda := \sum_{i=1}^{m_1} \frac{\mu_i \eta_i^{p-1}}{(p-1)^{\alpha_i}} - \sum_{j=1}^{n_1} \frac{\delta_j \xi_j^{p-1}}{(p-1)^{\beta_j}} \neq 0. \tag{22}$$

4.1   Existence results for uncoupled case

In view of Lemma 4.1, we define an operator $\mathfrak{T} : X \times Y \to X \times Y$ by $\mathfrak{T}(u,v)(t) = \begin{pmatrix} \mathfrak{T}_1(u,v)(t) \\ \mathfrak{T}_2(u,v)(t) \end{pmatrix}$ where

$$\begin{aligned} \mathfrak{T}_1(u,v)(t) &= {}_{RL}I^p f(s,u(s),v(s))(t) + \frac{t^{p-1}}{\Lambda} \left(\sum_{j=1}^{n_1} \delta_{jH} I^{\beta_j}{}_{RL} I^p f(s,u(s),v(s))(\xi_j) \right. \\ &\quad \left. - \sum_{i=1}^{m_1} \mu_{iH} I^{\alpha_i}{}_{RL} I^p f(s,u(s),v(s))(\eta_i) + \lambda_1 \right), \end{aligned}$$

and

$$\begin{aligned} \mathfrak{T}_2(u,v)(t) &= {}_{RL}I^q g(s,u(s),v(s))(t) + \frac{t^{q-1}}{\Phi} \left(\sum_{l=1}^{n_2} \omega_{lH} I^{\nu_l}{}_{RL} I^q g(s,u(s),v(s))(\theta_l) \right. \\ &\quad \left. - \sum_{k=1}^{m_2} \tau_{kH} I^{\sigma_k}{}_{RL} I^q g(s,u(s),v(s))(\gamma_k) + \lambda_2 \right), \end{aligned}$$

where

$$\Phi = \sum_{k=1}^{m_2} \frac{\tau_k \gamma_k^{q-1}}{(q-1)^{\sigma_k}} - \sum_{l=1}^{n_2} \frac{\omega_l \theta_l^{q-1}}{(q-1)^{\nu_l}} \neq 0.$$

In the sequel, we set

$$\overline{M}_1 = \frac{1}{\Gamma(p+1)} \left(T^p + \frac{T^{p-1}}{|\Lambda|} \sum_{j=1}^{n_1} \frac{|\delta_j| \xi_j^p}{p^{\beta_j}} + \frac{T^{p-1}}{|\Lambda|} \sum_{i=1}^{m_1} \frac{|\mu_i| \eta_i^p}{p^{\alpha_i}} \right), \tag{23}$$

$$\overline{M}_2 = \frac{T^{p-1}\lambda_1}{|\Lambda|}, \tag{24}$$

$$\overline{M}_3 = \frac{1}{\Gamma(q+1)} \left(T^q + \frac{T^{q-1}}{|\Phi|} \sum_{l=1}^{n_2} \frac{|\omega_l| \theta_l^q}{q^{\nu_l}} + \frac{T^{q-1}}{|\Phi|} \sum_{k=1}^{m_2} \frac{|\tau_k| \gamma_k^q}{q^{\sigma_k}} \right), \tag{25}$$

$$\overline{M}_4 = \frac{T^{q-1}\lambda_2}{|\Phi|}. \tag{26}$$

Now we present the existence and uniqueness result for the problem (19). We do not provide the proof of this result as it is similar to the one for Theorem 3.1.

Theorem 4.2 *Assume that $f, g : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions and there exist constants $\overline{K}_i, \overline{L}_i, i = 1, 2$ such that for all $t \in [0, T]$ and $u_i, v_i \in \mathbb{R}, i = 1, 2,$*

$$|f(t, u_1, u_2) - f(t, v_1, v_2)| \le \overline{K}_1 |u_1 - v_1| + \overline{K}_2 |u_2 - v_2|$$

$$S.\ K.\ NTOUYAS,\ J.\ TARIBOON\ AND\ P.\ THIRAMANUS$$

and

$$|g(t, u_1, u_2) - g(t, v_1, v_2)| \leq \overline{L}_1|u_1 - v_1| + \overline{L}_2|u_2 - v_2|.$$

In addition, assume that

$$\overline{M}_1(\overline{K}_1 + \overline{K}_2) + \overline{M}_3(\overline{L}_1 + \overline{L}_2) < 1,$$

where $\overline{M}_1$ and $\overline{M}_3$ are given by (23) and (25) respectively. Then the boundary value problem (19) has a unique solution.

Example 4.3 *Consider the following system of coupled Riemann-Liouville fractional differential equations with uncoupled Hadamard type fractional integral boundary conditions*

$$
\begin{cases}
{}_{RL}D^{e/2}x(t) = \dfrac{\cos(\pi t)}{(\pi^t + 4)^2}\dfrac{|x(t)|}{|x(t)| + 2} + \dfrac{3e^{t/2}}{(t+5)^3}\dfrac{|y(t)|}{|y(t)| + 3} + \dfrac{2}{e}, \quad t \in [0, 4], \\[2ex]
{}_{RL}D^{\sqrt{3}}y(t) = \dfrac{\sin x(t)}{15(e^t + 3)} + \dfrac{2\sqrt{|y(t)| + 1}}{7\pi(t+3)} + 5, \quad t \in [0, 4], \\[2ex]
x(0) = 0, \quad \sqrt{11}\,{}_H I^{5/2}x(2/3) + \dfrac{\tan^2(5)}{20}\,{}_H I^{10/3}x(\pi) = \dfrac{5}{e}\,{}_H I^{3/7}x(e) \\[2ex]
\qquad\qquad - \dfrac{7}{6}\,{}_H I^{\sqrt{5}}x(\sqrt{2}) + \dfrac{\pi}{2}\,{}_H I^{2/5}x(12/7) + 11, \\[2ex]
y(0) = 0, \quad \dfrac{\log(15)}{9}\,{}_H I^{7/4}y(1/4) + 2\,{}_H I^{5/6}y(\sqrt{7}) \\[2ex]
\qquad\qquad = \dfrac{\pi^2}{15}\,{}_H I^{4/3}y(1/e) + \sqrt{5}\,{}_H I^{9/7}y(7/2) + \sqrt{8}/3.
\end{cases}
\tag{27}
$$

Here $p = e/2$, $q = \sqrt{3}$, $T = 4$, $\lambda_1 = 11$, $\lambda_2 = \sqrt{8}/3$, $m_1 = 2$, $n_1 = 3$, $m_2 = 2$, $n_2 = 2$, $\mu_1 = \sqrt{11}$, $\mu_2 = \tan^2(5)/20$, $\alpha_1 = 5/2$, $\alpha_2 = 10/3$, $\eta_1 = 2/3$, $\eta_2 = \pi$, $\delta_1 = 5/e$, $\delta_2 = -7/6$, $\delta_3 = \pi/2$, $\beta_1 = 3/7$, $\beta_2 = \sqrt{5}$, $\beta_3 = 2/5$, $\xi_1 = e$, $\xi_2 = \sqrt{2}$, $\xi_3 = 12/7$, $\tau_1 = \log(15)/9$, $\tau_2 = 2$, $\sigma_1 = 7/4$, $\sigma_2 = 5/6$, $\gamma_1 = 1/4$, $\gamma_2 = \sqrt{7}$, $\omega_1 = \pi^2/15$, $\omega_2 = \sqrt{5}$, $\nu_1 = 4/3$, $\nu_2 = 9/7$, $\theta_1 = 1/e$, $\theta_2 = 7/2$, $f(t, x, y) = (\cos(\pi t)|x|)/(((\pi^t + 4)^2)(|x| + 2)) + (3e^{t/2}|y|)/(((t+5)^3)(|y| + 3)) + (2/e)$ and $g(t, x, y) = (\sin x(t))/(15(e^t + 3)) + (2\sqrt{|y| + 1})/(7\pi(t+3)) + 5$. Since $|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq ((1/50)|x_1 - x_2| + (e^2/125)|y_1 - y_2|)$ and $|g(t, x_1, y_1) - g(t, x_2, y_2)| \leq ((1/60)|x_1 - x_2| + (1/(21\pi))|y_1 - y_2|)$. By using the Maple program, we can find

$$\Lambda := \sum_{i=1}^{m_1}\frac{\mu_i\eta_i^{p-1}}{(p-1)^{\alpha_i}} - \sum_{j=1}^{n_1}\frac{\delta_j\xi_j^{p-1}}{(p-1)^{\beta_j}} \approx 69.35947949 \neq 0$$

and

$$\Phi = \sum_{k=1}^{m_2}\frac{\tau_k\gamma_k^{q-1}}{(q-1)^{\sigma_k}} - \sum_{l=1}^{n_2}\frac{\omega_l\theta_l^{q-1}}{(q-1)^{\nu_l}} \approx -3.358717154 \neq 0.$$

With the given values, it is found that $\overline{K}_1 = 1/50$, $\overline{K}_2 = e^2/125$, $\overline{L}_1 = 1/60$, $\overline{L}_2 = 1/(21\pi)$, $\overline{M}_1 \simeq 5.673444294$, $\overline{M}_3 \simeq 15.54186374$. In consequence,

$$\overline{M}_1(\overline{K}_1 + \overline{K}_2) + \overline{M}_3(\overline{L}_1 + \overline{L}_2) \approx 0.9434486991 < 1.$$

Thus all the conditions of Theorem 4.2 are satisfied. Therefore, there exists a unique solution for the problem (27) on $[0, 4]$.

The second result dealing with the existence of solutions for the problem (19) is analogous to Theorem 3.3 and is given below.

Theorem 4.4 *Assume that there exist real constants $\bar{k}_i$, $\bar{l}_i \geq 0$ $(i = 1, 2)$ and $\bar{k}_0 > 0, \bar{l}_0 > 0$ such that $\forall x_i \in \mathbb{R}$, $(i = 1, 2)$ we have*

$$|f(t, x_1, x_2)| \leq \bar{k}_0 + \bar{k}_1|x_1| + \bar{k}_2|x_2|,$$

$$|g(t, x_1, x_2)| \leq \bar{l}_0 + \bar{l}_1|x_1| + \bar{l}_2|x_2|.$$

MIXED PROBLEMS OF FRACTIONAL COUPLED SYSTEMS

In addition it is assumed that

$$\bar{k}_1\overline{M}_1 + \bar{l}_1\overline{M}_3 < 1 \ \ and \ \ \bar{k}_2\overline{M}_1 + \bar{l}_2\overline{M}_3 < 1,$$

where $\overline{M}_1$ and $\overline{M}_3$ are given by (23) and (25) respectively. Then the boundary value problem (19) has at least one solution.

Proof. Setting

$$\overline{M}_0 = \min\{1 - \bar{k}_1\overline{M}_1 - \bar{l}_1\overline{M}_3, \ 1 - \bar{k}_2\overline{M}_1 - \bar{l}_2\overline{M}_3\}, \ \ \bar{k}_i, \ \bar{l}_i \geq 0 \ (i = 1, 2),$$

the proof is similar to that of Theorem 3.3. So we omit it. $\square$

References

[1] R.P. Agarwal, Y. Zhou, Y. He, Existence of fractional neutral functional differential equations, *Comput. Math. Appl.* **59** (2010), 1095-1100.

[2] B. Ahmad, J.J. Nieto, Boundary value problems for a class of sequential integrodifferential equations of fractional order, *J. Funct. Spaces Appl.* 2013, Art. ID 149659, 8 pp.

[3] B. Ahmad, J.J. Nieto, Riemann-Liouville fractional integro-differential equations with fractional nonlocal integral boundary conditions, *Bound. Value Probl.* **2011**, 2011:36, 9 pp.

[4] B. Ahmad, S.K. Ntouyas, A. Alsaedi, A study of nonlinear fractional differential equations of arbitrary order with Riemann-Liouville type multistrip boundary conditions, *Math. Probl. Eng.* (2013), Art. ID 320415, 9 pp.

[5] B. Ahmad, S.K. Ntouyas, A. Alsaedi, New existence results for nonlinear fractional differential equations with three-point integral boundary conditions, *Adv. Differ. Equ.* (2011) Art. ID 107384, 11pp.

[6] D. Baleanu, K. Diethelm, E. Scalas, J.J.Trujillo, *Fractional Calculus Models and Numerical Methods. Series on Complexity, Nonlinearity and Chaos*, World Scientific, Boston, 2012.

[7] D. Baleanu, O.G. Mustafa, R.P. Agarwal, On L^p-solutions for a class of sequential fractional differential equations, *Appl. Math. Comput.* **218** (2011), 2074-2081.

[8] P.L. Butzer, A.A. Kilbas, J.J. Trujillo, Compositions of Hadamard-type fractional integration operators and the semigroup property, *J. Math. Anal. Appl.* **269** (2002), 387-400.

[9] P.L. Butzer, A.A. Kilbas, J.J. Trujillo, Fractional calculus in the Mellin setting and Hadamard-type fractional integrals, *J. Math. Anal. Appl.* **269** (2002), 1-27.

[10] P.L. Butzer, A.A. Kilbas, J.J. Trujillo, Mellin transform analysis and integration by parts for Hadamard-type fractional integrals, *J. Math. Anal. Appl.* **270** (2002), 1-15.

[11] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer-Verlag, New York, 2003.

[12] J. Hadamard, Essai sur l'etude des fonctions donnees par leur developpment de Taylor, *J. Mat. Pure Appl. Ser.* **8** (1892) 101-186.

[13] A.A. Kilbas, Hadamard-type fractional calculus, *J. Korean Math. Soc.* **38** (2001), 1191-1204.

[14] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.

[15] A.A. Kilbas, J.J. Trujillo, Hadamard-type integrals as G-transforms, *Integral Transform. Spec. Funct.* **14** (2003), 413-427.

S. K. NTOUYAS, J. TARIBOON AND P. THIRAMANUS

[16] X. Liu, M. Jia, W. Ge, Multiple solutions of a p-Laplacian model involving a fractional derivative, *Adv. Differ. Equ.* 2013, **2013:126**.

[17] D. O'Regan, S. Stanek, Fractional boundary value problems with singularities in space variables, *Nonlinear Dynam.* **71** (2013), 641-652.

[18] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[19] J. Tariboon, T. Sitthiwirattham, S. K. Ntouyas, Existence results for fractional differential inclusions with multi–point and fractional integral boundary conditions, *J. Comput. Anal. Appl.* **17** (2014), 343-360.

[20] P. Thiramanus, J. Tariboon, S. K. Ntouyas, Average value problems for nonlinear second-order impulsive q-difference equations, *J. Comput. Appl. Anal.* **18** (2015), 590-611.

[21] L. Zhang, B. Ahmad, G. Wang, R.P. Agarwal, Nonlinear fractional integro-differential equations on unbounded domains in a Banach space, *J. Comput. Appl. Math.* **249** (2013), 51–56.

Ternary Jordan ring derivations on Banach ternary algebras: A fixed point approach

Madjid Eshaghi Gordji[1], Shayan Bazeghi[1], Choonkil Park[2*] and Sun Young Jang[3*]

[1]Department of Mathematics, Semnan University, P. O. Box 35195-363, Semnan, Iran

[2]Research Institute for Natural Sciences, Hanyang University, Seoul 133-791, Korea

[3]Department of Mathematics, University of Ulsan, Ulsan 680-749, Korea

e-mail: `m.eshaghi@semnan.ac.ir`, `v.keshavarz68@yahoo.com`, `baak@hanyang.ac.kr`, `jsym@ulsan.ac.kr`

Abstract. Let A be a Banach ternary algebra. An additive mapping $D : (A, [\,]) \to (A, [\,])$ is called a ternary Jordan ring derivation if $D([xxx]) = [D(x)xx] + [xD(x)x] + [xxD(x)]$ for all $x \in A$.

In this paper, we prove the Hyers-Ulam stability of ternary Jordan ring derivations on Banach ternary algebras.

1. Introduction

We say that a functional equation (Q) is stable if any function g satisfying the equation (Q) approximately is near to true solution of (Q). Also, we say that a functional equation is superstable if every approximately solution is an exact solution of it.

Recently, Bavand Savadkouhi et al. [4] investigate the stability of ternary Jordan derivations on Banach ternary algebras by direct methods.

Ternary algebraic operations were considered in the 19th century by several mathematicians. Cayley [7] introduced the notion of cubic matrix, which in turn was generalized by Kapranov, Gelfand and Zelevinskii [17].

The comments on physical applications of ternary structures can be found in [3, 12, 13, 14, 22, 23, 26, 28, 31, 32].

Let A be a Banach ternary algebra. An additive mapping $D : (A, [\,]) \to (A, [\,])$ is called a ternary ring derivation if

$$D([xyz]) = [D(x)yz] + [xD(y)z] + [xyD(z)]$$

for all $x, y, z \in A$.

An additive mapping $D : (A, [\,]) \to (A, [\,])$ is called a ternary Jordan ring derivation if

$$D([xxx]) = [D(x)xx] + [xD(x)x] + [xxD(x)]$$

for all $x \in A$.

Theorem 1.1. ([11]) *Suppose that (Ω, d) is a complete generalized metric space and $T : \Omega \to \Omega$ is a strictly contractive mapping with the Lipschitz constant L. Then, for any $x \in \Omega$, either*

$$d(T^n x, T^{n+1} x) = \infty, \quad \forall n \geq 0,$$

or there exists a positive integer n_0 such that

(1) $d(T^n x, T^{n+1} x) < \infty$ for all $n \geq n_0$;

(2) the sequence $\{T^n x\}$ is convergent to a fixed point y^* of T;

(3) y^* is the unique fixed point of T in $\Lambda = \{y \in \Omega : d(T^{n_0} x, y) < \infty\}$;

(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Ty)$ for all $y \in \Lambda$.

The study of stability problems originated from a famous talk given by Ulam [30] in 1940: "Under what condition does there exist a homomorphism near an approximate homomorphism?" In the next year 1941, Hyers [15] answered affirmatively the question of Ulam for additive mappings between Banach spaces. A generalized version of the theorem of Hyers for approximately additive mappings was given by Rassias [24] in 1978. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1, 5, 8, 10, 18, 19, 20, 21, 25, 27, 29, 33, 34]).

In this paper, we prove the Hyers-Ulam stability and superstability of ternary Jordan ring derivations on Banach ternary algebras by the fixed point method.

M. Eshaghi Gordji, Sh. Bazeghi, C. Park, S. Y. Jang

2. Hyers-Ulam stability of ternary Jordan ring derivations

In this section, we prove the Hyers-Ulam stability of ternary Jordan ring derivations on Banach ternary algebras. Throughout this section, assume that A is a Banach ternary algebra.

Lemma 2.1. *Let $f : A \to A$ be an additive mapping. Then the following assertions are equivalent.*

$$f([a,a,a]) = [f(a),a,a] + [a,f(a),a] + [a,a,f(a)] \tag{2.1}$$

for all $a \in A$, and

$$
\begin{aligned}
f([a,b,c] + [b,c,a] + [c,a,b]) =& [f(a),b,c] + [a,f(b),c] + [a,b,f(c)] + [f(b),c,a] + [b,f(c),a] \\
&+ [b,c,f(a)] + [f(c),a,b] + [c,f(a),b] + [c,a,f(b)]
\end{aligned}
\tag{2.2}
$$

for all $a,b,c \in A$.

Proof. Replacing a by $a + b + c$ in (2.1), we have

$$
\begin{aligned}
f([(a+b+c),(a+b+c),(a+b+c)]) =& [f(a+b+c),(a+b+c),(a+b+c)] + [(a+b+c),f(a+b+c),(a+b+c)] \\
&+ [(a+b+c),(a+b+c),f(a+b+c)]
\end{aligned}
$$

and so

$f([(a+b+c),(a+b+c),(a+b+c)])$

$= f([a,a,a] + [a,b,a] + [a,c,a] + [b,a,a] + [b,b,a] + [b,c,a] + [c,a,a] + [c,b,a] + [c,c,a]$

$+ [a,a,b] + [a,b,b] + [a,c,b] + [b,a,b] + [b,b,b] + [b,c,b] + [c,a,b] + [c,b,b] + [c,c,b]$

$+ [a,a,c] + [a,b,c] + [a,c,c] + [b,a,c] + [b,b,c] + [b,c,c] + [c,a,c] + [c,b,c] + [c,c,c])$

$= f([a,a,a]) + f([a,b,a]) + f([a,c,a]) + f([b,a,a]) + f([b,b,a]) + f([b,c,a]) + f([c,a,a]) + f([c,b,a]) + f([c,c,a])$

$+ f([a,a,b]) + f([a,b,b]) + f([a,c,b]) + f([b,a,b]) + f([b,b,b]) + f([b,c,b]) + f([c,a,b]) + f([c,b,b]) + f([c,c,b])$

$+ f([a,a,c]) + f([a,b,c]) + f([a,c,c]) + f([b,a,c]) + f([b,b,c]) + f([b,c,c]) + f([c,a,c]) + f([c,b,c]) + f([c,c,c])$

$= [f(a),a,a] + [a,f(a),a] + [a,a,f(a)] + [f(a),b,a] + [a,f(b),a] + [a,b,f(a)] + [f(a),c,a] + [a,f(c),a] + [a,c,f(a)]$

$+ [f(b),a,a] + [b,f(a),a] + [b,a,f(a)] + [f(b),b,a] + [b,f(b),a] + [b,b,f(a)] + [f(b),c,a] + [b,f(c),a] + [b,c,f(a)]$

$+ [f(c),a,a] + [c,f(a),a] + [c,a,f(a)] + [f(c),b,a] + [c,f(b),a] + [c,b,f(a)] + [f(c),c,a] + [c,f(c),a] + [c,c,f(a)]$

$+ [f(a),a,b] + [a,f(a),b] + [a,a,f(b)] + [f(a),b,b] + [a,f(b),b] + [a,b,f(b)] + [f(a),c,b] + [a,f(c),b] + [a,c,f(b)]$

$+ [f(b),a,b] + [b,f(a),b] + [b,a,f(b)] + [f(b),b,b] + [b,f(b),b] + [b,b,f(b)] + [f(b),c,b] + [b,f(c),b] + [b,c,f(b)]$

$+ [f(c),a,b] + [c,f(a),b] + [c,a,f(b)] + [f(c),b,b] + [c,f(b),b] + [c,b,f(b)] + [f(c),c,b] + [c,f(c),b] + [c,c,f(b)]$

$+ [f(a),a,c] + [a,f(a),c] + [a,a,f(c)] + [f(a),b,c] + [a,f(b),c] + [a,b,f(c)] + [f(a),c,c] + [a,f(c),c] + [a,c,f(c)]$

$+ [f(b),a,c] + [b,f(a),c] + [b,a,f(c)] + [f(b),b,c] + [b,f(b),c] + [b,b,f(c)] + [f(b),c,c] + [b,f(c),c] + [b,c,f(c)]$

$+ [f(c),a,c] + [c,f(a),c] + [c,a,f(c)] + [f(c),b,c] + [c,f(b),c] + [c,b,f(c)] + [f(c),c,c] + [c,f(c),c] + [c,c,f(c)]$

for all $a,b,c \in A$.

On the other hand, we have

$f([(a+b+c),(a+b+c),(a+b+c)])$

$= [f(a),a,a] + [f(a),a,b] + [f(a),a,c] + [f(a),b,a] + [f(a),b,b] + [f(a),b,c] + [f(a),c,a] + [f(a),c,b] + [f(a),c,c]$

$+ [f(b),a,a] + [f(b),a,b] + [f(b),a,c] + [f(b),b,a] + [f(b),b,b] + [f(b),b,c] + [f(b),c,a] + [f(b),c,b] + [f(b),c,c]$

$+ [f(c),a,a] + [f(c),a,b] + [f(c),a,c] + [f(c),b,a] + [f(c),b,b] + [f(c),b,c] + [f(c),c,a] + [f(c),c,b] + [f(c),c,c]$

$+ [a,f(a),a] + [a,f(a),b] + [a,f(a),c] + [b,f(a),a] + [b,f(a),b] + [b,f(a),c] + [c,f(a),a] + [c,f(a),b] + [c,f(a),c]$

$+ [a,f(b),a] + [a,f(b),b] + [a,f(b),c] + [b,f(b),a] + [b,f(b),b] + [b,f(b),c] + [c,f(b),a] + [c,f(b),b] + [c,f(b),c]$

$+ [a,f(c),a] + [a,f(c),b] + [a,f(c),c] + [b,f(c),a] + [b,f(c),b] + [b,f(c),c] + [c,f(c),a] + [c,f(c),b] + [c,f(c),c]$

$+ [a,a,f(a)] + [a,b,f(a)] + [a,c,f(a)] + [b,a,f(a)] + [b,b,f(a)] + [b,c,f(a)] + [c,a,f(a)] + [c,b,f(a)] + [c,c,f(a)]$

$+ [a,a,f(b)] + [a,b,f(b)] + [a,c,f(b)] + [b,a,f(b)] + [b,b,f(b)] + [b,c,f(b)] + [c,a,f(b)] + [c,b,f(b)] + [c,c,f(b)]$

$+ [a,a,f(c)] + [a,b,f(c)] + [a,c,f(c)] + [b,a,f(c)] + [b,b,f(c)] + [b,c,f(c)] + [c,a,f(c)] + [c,b,f(c)] + [c,c,f(c)]$

for all $a,b,c \in A$.

Ternary Jordan ring derivations on Banach ternary algebras

It follows that

$$([f(b), c, a] + [b, f(c), a] + [b, c, f(a)]) + ([f(c), a, b] + [c, f(a), b] + [c, a, f(b)]) + ([f(a), b, c] + [a, f(b), c] + [a, b, f(c)])$$
$$= f([b, c, a]) + f([c, a, b]) + f([a, b, c]) = f([b, c, a] + [c, a, b] + [a, b, c]) = f([a, b, c] + [b, c, a] + [c, a, b])$$

and

$$[f(a), b, c] + [f(b), c, a] + [f(c), a, b] + [c, f(a), b] + [a, f(b), c] + [b, f(c), a] + [b, c, f(a)] + [c, a, f(b)] + [a, b, f(c)]$$
$$= ([f(a), b, c] + [a, f(b), c] + [a, b, f(c)]) + ([f(b), c, a] + [b, f(c), a] + [b, c, f(a)]) + ([f(c), a, b] + [c, f(a), b] + [c, a, f(b)])$$

for all $a, b, c \in A$. Then

$$f([a, b, c] + [b, c, a] + [c, a, b]) = ([f(a), b, c] + [a, f(b), c] + [a, b, f(c)]) + ([f(b), c, a] + [b, f(c), a]$$
$$+ [b, c, f(a)]) + ([f(c), a, b] + [c, f(a), b] + [c, a, f(b)])$$

for all $a, b, c \in A$. Hence (2.2) holds true.

For the converse, replacing b and c by a in (2.2), we have

$$f([a, a, a] + [a, a, a] + [a, a, a]) = [f(a), a, a] + [a, f(a), a] + [a, a, f(a)] + [f(a), a, a] + [a, f(a), a]$$
$$+ [a, a, f(a)] + [f(a), a, a] + [a, f(a), a] + [a, a, f(a)]$$

and so

$$f(3[a, a, a]) = 3([f(a), a, a] + [a, f(a), a] + [a, a, f(a)])$$

for all $a \in A$. Thus

$$f([a, a, a]) = [f(a), a, a] + [a, f(a), a] + [a, a, f(a)]$$

for all $a \in A$. This completes the proof. $\square$

Theorem 2.2. *Let $f : A \to A$ be a mapping for which there exists function $\varphi : A \times A \times A \to [0, \infty)$ such that*

$$\|f(x + y) - f(x) - f(y)\| \leq \varphi(x, y, 0), \tag{2.3}$$

$$\|f([x, y, z] + [y, z, x] + [z, x, y]) - [f(x), y, z] - [x, f(y), z] - [x, y, f(z)] - [f(y), z, x]$$
$$- [y, f(z), x] - [y, z, f(x)] - [f(z), x, y] - [z, f(x), y] - [z, x, f(y)]\| \leq \varphi(x, y, z) \tag{2.4}$$

for all $x, y, z \in A$. If there exists a constant $0 < L < 1$ such that

$$\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \leq \frac{L}{8}\varphi(x, y, z) \tag{2.5}$$

for all $x, y, z \in A$, then there exists a unique ternary Jordan ring derivation $D : A \to A$ such that

$$\|f(x) - D(x)\| \leq \frac{L}{8 - 2L}\varphi(x, x, 0) \tag{2.6}$$

for all $x \in A$.

Proof. It follows from (2.5) that

$$\lim_{n \to \infty} 2^{3n}\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) = 0 \tag{2.7}$$

for all $x, y, z \in A$. By (2.5), $\varphi(0, 0, 0) = 0$. Letting $x = y = 0$ in (2.3), we get $\|f(0)\| \leq \varphi(0, 0, 0) = 0$ and so $f(0) = 0$. Let $\Omega = \{g : A \to X, \ g(0) = 0\}$. We introduce a generalized metric on Ω as follows:

$$d(g, h) = d_\varphi(g, h) = \inf\{C \in (0, \infty) : \ \|g(x) - h(x)\| \leq C\varphi(x, x, 0), \ \forall x \in A\}.$$

It is easy to show that (Ω, d) is a generalized complete metric space [16]. Now, we consider the mapping $T : \Omega \to \Omega$ defined by $Tg(x) = 2g(\frac{x}{2})$ for all $x \in A$ and $g \in \Omega$. Note that, for all $g, h \in \Omega$ and $x \in A$,

$$d(g, h) < C \implies \|g(x) - h(x)\| \leq C\varphi(x, x, 0)$$
$$\implies \|2g(\frac{x}{2}) - 2h(\frac{x}{2})\| \leq 2 \ C \ \varphi(\frac{x}{2}, \frac{x}{2}, 0)$$
$$\implies \|2g(\frac{x}{2}) - 2h(\frac{x}{2})\| \leq \frac{L}{4} \ C \ \varphi(x, x, 0)$$
$$\implies d(Tg, Th) \leq \frac{L}{4} \ C.$$

Hence we obtain that

$$d(Tg, Th) \leq \frac{L}{4} \ d(g, h)$$

M. Eshaghi Gordji, Sh. Bazeghi, C. Park, S. Y. Jang

for all $g, h \in \Omega$, that is, T is a strictly contractive mapping of Ω with the Lipschitz constant L. Putting $y = x$ in (2.3), we have

$$\|f(2x) - 2f(x)\| \leq \varphi(x, x, 0), \tag{2.8}$$

and so

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| \leq \varphi\left(\frac{x}{2}, \frac{x}{2}, 0\right) \leq \frac{L}{8}\varphi(x, x, 0)$$

for all $x \in A$. Let us denote

$$D(x) = \lim_{n \to \infty} 2^n f\left(\frac{x}{2^n}\right) \tag{2.9}$$

for all $x \in A$. By the result in ([2, 6]), D is an additive mapping and so it follows from the definition of D, (2.4) and (2.7) that

$$\|D([x, y, z] + [y, z, x] + [z, x, y]) - [D(x), y, z] - [x, D(y), z] - [x, y, D(z)] - [D(y), z, x] - [y, D(z), x]$$
$$- [y, z, D(x)] - [D(z), x, y] - [z, D(x), y] - [z, x, D(y)]\|$$
$$= \lim_{n \to \infty} 8^n \|f([\frac{x, y, z}{2^{3n}}] + [\frac{y, z, x}{2^{3n}}] + [\frac{z, x, y}{2^{3n}}]) - [f(\frac{x}{2^n}), \frac{y}{2^n}, \frac{z}{2^n}] - [\frac{x}{2^n}, f(\frac{y}{2^n}), \frac{z}{2^n}] - [\frac{x}{2^n}, \frac{y}{2^n}, f(\frac{z}{2^n})]$$
$$- [f(\frac{y}{2^n}), \frac{z}{2^n}, \frac{x}{2^n}] - [\frac{y}{2^n}, f(\frac{z}{2^n}), \frac{x}{2^n}] - [\frac{y}{2^n}, \frac{z}{2^n}, f(\frac{x}{2^n})] - [f(\frac{z}{2^n}), \frac{x}{2^n}, \frac{y}{2^n}] - [\frac{z}{2^n}, f(\frac{x}{2^n}), \frac{y}{2^n}] - [\frac{z}{2^n}, \frac{x}{2^n}, f(\frac{y}{2^n})]\|$$
$$\leq \lim_{n \to \infty} 8^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) = 0$$

for all $x, y, z \in A$ and so $D([x, y, z] + [y, z, x] + [z, x, y]) = [D(x), y, z] + [x, D(y), z] + [x, y, D(z)] + [D(y), z, x] + [y, D(z), x] + [y, z, D(x)] + [D(z), x, y] + [z, D(x), y] + [z, x, D(y)]$, which implies that D is a ternary Jordan ring derivation, by Lemma 2.1. According to Theorem 1.1, since D is the unique fixed point of T in the set $\Lambda = \{g \in \Omega : d(f, g) < \infty\}$, D is the unique mapping such that

$$\|f(x) - D(x)\| \leq C\, \varphi(x, x, 0)$$

for all $x \in A$ and $C > 0$. By Theorem 1.1, we have

$$d(f, D) \leq \frac{1}{1 - \frac{L}{4}} d(f, Tf) \leq \frac{4L}{8(4 - L)}$$

and so

$$\|f(x) - D(x)\| \leq \frac{L}{8 - 2L}\, \varphi(x, x, 0)$$

for all $x \in A$. This completes the proof. □

Corollary 2.3. *Let θ, r be nonnegative real numbers with $r > 1$. Suppose that $f : A \to A$ is a mapping such that*

$$\|f(x + y) - f(x) - f(y)\| \leq \theta(\|x\|^r + \|y\|^r), \tag{2.10}$$

$$\|f([x, y, z] + [y, z, x] + [z, x, y]) - [f(x), y, z] - [x, f(y), z] - [x, y, f(z)] - [f(y), z, x]$$
$$-[y, f(z), x] - [y, z, f(x)] - [f(z), x, y] - [z, f(x), y] - [z, x, f(y)]\| \leq \theta(\|x\|^r + \|y\|^r + \|z\|^r) \tag{2.11}$$

for all $x, y, z \in A$. Then there exists a unique ternary Jordan ring derivation $D : A \to A$ satisfying

$$\|f(x) - D(x)\| \leq \frac{\theta}{2^{r+1} - 1}\|x\|^r$$

for all $x \in A$.

Proof. The proof follows from Theorem 2.2 by taking

$$\varphi(x, y, z) := \theta(\|x\|^r + \|y\|^r + \|z\|^r)$$

for all $x, y, z \in A$. Then we can choose $L = 2^{1-r}$ and so we obtain the desired conclusion. □

Remark 2.4. Let $f : A \to A$ be a mapping with $f(0) = 0$ such that there exists a function $\varphi : A \times A \times A \to [0, \infty)$ satisfying (2.3) and (2.4). Let $0 < L < 1$ be a constant such that

$$\varphi(2x, 2y, 2z) \leq 2L\varphi(x, y, z)$$

for all $x, y, z \in A$. By a similar method as in the proof of Theorem 2.2, one can show that there exists a unique ternary Jordan ring derivation $D : A \to A$ satisfying

$$\|f(x) - D(x)\| \leq \frac{2}{4 - L}\varphi(x, x, 0)$$

for all $x \in A$. For the case

$$\varphi(x, y, z) := \delta + \theta(\|x\|^r + \|y\|^r + \|z\|^r),$$

Ternary Jordan ring derivations on Banach ternary algebras

(where θ, δ are nonnegative real numbers and $0 < r < 1$, there exists a unique ternary Jordan ring derivation $D : A \to X$ satisfying

$$\|f(x) - D(x)\| \leq \frac{4\delta}{8 - 2^r} + \frac{8\theta}{8 - 2^r}\|x\|^r$$

for all $x \in A$.

Now, we formulate a theorem for the superstability of ternary Jordan ring derivations.

Theorem 2.5. *Suppose that there exist a function $\varphi : A \times A \times A \to [0, \infty)$ and a constant $0 < L < 1$ such that*

$$\psi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \leq \frac{L}{8}\varphi(x, y, z)$$

for all $x, y, z \in A$. Moreover, if $f : A \to A$ is an additive mapping such that

$$\|f([x, y, z] + [y, z, x] + [z, x, y]) - [f(x), y, z] - [x, f(y), z] - [x, y, f(z)] - [f(y), z, x]$$
$$-[y, f(z), x] - [y, z, f(x)] - [f(z), x, y] - [z, f(x), y] - [z, x, f(y)]\| \leq \varphi(x, y, z)$$

for all $x, y, z \in A$, then f is a ternary Jordan ring derivation.

Proof. The proof is similar to the proof of Theorem 2.2. We will omit it. $\square$

Corollary 2.6. *Let θ, s be nonnegative real numbers and $s > 3$. If $f : A \to A$ is an additive mapping such that*

$$\|f([x, y, z] + [y, z, x] + [z, x, y]) - [f(x), y, z] - [x, f(y), z] - [x, y, f(z)] - [f(y), z, x]$$
$$-[y, f(z), x] - [y, z, f(x)] - [f(z), x, y] - [z, f(x), y] - [z, x, f(y)]\| \leq \theta(\|x\|^s + \|y\|^s + \|z\|^s)$$

for all $x, y, z \in A$, then f is a ternary Jordan ring derivation.

Remark 2.7. Suppose that there exist a function $\psi : A \times A \times A \to [0, \infty)$ and a constant $0 < L < 1$ such that

$$\varphi(2x, 2y, 2z) \leq 2L\varphi(x, y, z)$$

for all $x, y, z \in A$. Moreover, if $f : A \to A$ is an additive mapping such that

$$\|f([x, y, z] + [y, z, x] + [z, x, y]) - [f(x), y, z] - [x, f(y), z] - [x, y, f(z)] - [f(y), z, x]$$
$$-[y, f(z), x] - [y, z, f(x)] - [f(z), x, y] - [z, f(x), y] - [z, x, f(y)]\| \leq \varphi(x, y, z)$$

for all $x, y, z \in A$, then f is a ternary Jordan ring derivation.

References

[1] M. Adam, On the stability of some quadratic functional equation, J. Nonlinear Sci. Appl. **4** (2011), 50-59.

[2] J. Bae, W. Park, A functional equation having monomials as solutions, Appl. Math. Comput. **216** (2010), 87-94.

[3] F. Bagarello, G. Morchio, Dynamics of mean-field spin models from basic results in abstract differential equations, *J. Stat. Phys.* **66** (1992), 849-866.

[4] M. Bavand Savadkouhi, M. Eshaghi Gordji, J. M. Rassias and N. Ghobadipour, Approximate ternary Jordan derivations on Banach ternary algebras, J. Math. Phys. **50**, Art. ID 042303 (2009).

[5] L. Cădariu, L. Găvruta, P. Găvruta, On the stability of an affine functional equation, J. Nonlinear Sci. Appl. **6** (2013), 60-67.

[6] L. Cădariu, V. Radu, On the stability of the Cauchy functional equation: A fixed point approach, Grazer Math. Ber. **346** (2004), 43-52.

[7] A. Cayley, On the 34concomitants of the ternary cubic. Amer. J. Math. **4** (1881), 1-15.

[8] A. Chahbi, N. Bounader, On the generalized stability of d'Alembert functional equation, J. Nonlinear Sci. Appl. **6** (2013), 198-204.

[9] Y. Cho, C. Park, M. Eshaghi Gordji, Approximate additive and quadratic mappings in 2-Banach spaces and related topics, Int. J. Nonlinear Anal. Appl. **3** (2012), No. 1, 75-81.

[10] Y. Cho, C. Park, M. Eshaghi Gordji, Approximate additive and quadratic mappings in 2-Banach spaces and related topics, Int. J. Nonlinear Anal. Appl. **3** (2012), 75-81.

[11] J. Diaz, B. Margolis, A fixed point theorem of the alternative for contractions on a generalized complete metric space, Bull. Amer. Math. Soc. **74** (1968), 305-309.

[12] M. Eshaghi Gordji, A. Ebadian, N. Ghobadipour, J. M. Rassias, M. B. Savadkouhi, Approximately ternary homomorphisms and derivations on C^*-ternary algebras, Abs. Appl. Anal. **2012**, Art. ID 984160 (2012).

M. Eshaghi Gordji, Sh. Bazeghi, C. Park, S. Y. Jang

[13] P. Gavruta, L. Gavruta, A new method for the generalized Hyers-Ulam-Rassias stability, Int. J. Nonlinear Anal. Appl. **1** (2010), 11-18.

[14] N. Ghobadipour, C. Park, Cubic-quartic functional equations in fuzzy normed spaces, Int. J. Nonlinear Anal. Appl. **1** (2010), 12-21.

[15] D. H. Hyers, On the stability of the linear functional equation, Proc. Natl Acad. Sci. USA **27** (1941), 222-224.

[16] K. Jun, H. Kim, The generalized Hyers-Ulam-Rassias stability of a cubic functional equation. J. Math. Anal. Appl. **274** (2002), 867-878.

[17] M. Kapranov, I. M. Gelfand, A. Zelevinskii, Discriminants, Resultants and Multidimensional Determinants, Birkhäuser, Berlin, 1994.

[18] M. Kim, Y. Kim, G. A. Anastassiou, C. Park, An additive functional inequality in matrix normed modules over a C^*-algebra, J. Comput. Anal. Appl. **17** (2014), 329-335.

[19] M. Kim, S. Lee, G. A. Anastassiou, C. Park, Functional equations in matrix normed modules, J. Comput. Anal. Appl. **17** (2014), 336-342.

[20] C. Park, K. Ghasemi, S. G. Ghaleh, S. Jang, Approximate n-Jordan $*$-homomorphisms in C^*-algebras, J. Comput. Anal. Appl. **15** (2013), 365-368.

[21] C. Park, A. Najati, S. Jang, Fixed points and fuzzy stability of an additive-quadratic functional equation, J. Comput. Anal. Appl. **15** (2013), 452-462.

[22] C. Park, A. Najati, Generalized additive functional inequalities in Banach algebras, Int. J. Nonlinear Anal. Appl. **1** (2010), 54-62.

[23] C. Park, Th. M. Rassias, Isomorphisms in unital C^*-algebras, Int. J. Nonlinear Anal. Appl. **1** (2010), 1-10.

[24] Th. M. Rassias, On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. **72** (1978), 297-300.

[25] S. Schin, D. Ki, J. Chang, M. Kim, Random stability of quadratic functional equations: a fixed point approach, J. Nonlinear Sci. Appl. **4** (2011), 37-49.

[26] S. Shagholi, M. Eshaghi Gordji, M. Bavand Savadkouhi, Stability of ternary quadratic derivation on ternary Banach algebras, J. Comput. Anal. Appl. **13** (2011), 1097-1105.

[27] S. Shagholi, M. Bavand Savadkouhi, M. Eshaghi Gordji, Nearly ternary cubic homomorphism in ternary Fréchet algebras, J. Comput. Anal. Appl. **13** (2011), 1106-1114.

[28] D. Shin, C. Park, Sh. Farhadabadi, On the superstability of ternary Jordan C^*-homomorphisms, J. Comput. Anal. Appl. **16** (2014), 964-973.

[29] D. Shin, C. Park, Sh. Farhadabadi, Stability and superstability of J^*-homomorphisms and J^*-derivations for a generalized Cauchy-Jensen equation, J. Comput. Anal. Appl. **17** (2014), 125-134.

[30] S. M. Ulam, Problems in Modern Mathematics, Chapter VI, Science ed. Wiley, New York, 1940.

[31] L. Vainerman, R. Kerner, On special classes of n-algebras, J. Math. Phys. **37**, Art. ID 2553 (1996).

[32] S. Zolfaghari, Stability of generalized QCA-functional equation in p-Banach spaces, Int. J. Nonlinear Anal. Appl. **1**, (2010) , 84-99.

[33] C. Zaharia, On the probabilistic stability of the monomial functional equation, J. Nonlinear Sci. Appl. **6** (2013), 51-59.

[34] S. Zolfaghari, Approximation of mixed type functional equations in p-Banach spaces, J. Nonlinear Sci. Appl. **3** (2010), 110-122.

Initial value problems for a nonlinear integro-differential equation of mixed type in Banach spaces[*]

Xiong-Jun Zheng[†], Jin-Ming Wang

College of Mathematics and Information Science, Jiangxi Normal University

Nanchang, Jiangxi 330022, People's Republic of China

Abstract

In this paper, we discuss the following initial value problem for first order nonlinear integro-differential equations of mixed type in a Banach space:

$$\begin{cases} u' = f(t, u, Tu, Su) \\ u(t_0) = u_0. \end{cases}$$

In the case of the integral kernel $k(t, s)$ of the operator $(Tu)(t) = \int_{t_0}^{t} k(t, s)u(s)ds$ being unbounded, we obtain the existence of maximal and minimal solutions for the above problem by establishing a new comparison theorem.

Keywords: noncompactness measure, unbounded integral kernel, maximal and minimal solutions, integro-differential equations.

1   Introduction and Preliminaries

Suppose that E is a Banach space. In this paper, We consider the following initial value problem for first order nonlinear integro-differential equations of mixed type in E:

$$\begin{cases} u = f(t, u, Tu, Su) \\ u(t_0) = u_0, \end{cases} \tag{1.1}$$

where $f \in C[J \times E \times E \times E, \ E]$, $J = [t_0, t_0 + a](a > 0)$, $u_0 \in E$, and

$$(Tu)(t) = \int_{t_0}^{t} k(t, s)u(s)ds, \ (Su)(t) = \int_{t_0}^{t_0+a} h(t, s)u(s)ds. \tag{1.2}$$

1

In (1.2), $k(t,s) = \frac{\rho(t,s)}{(t-s)^\alpha}$ $(0 < \alpha < 1)$, $\rho(t,s) \in C[D, R^+]$, and $h(t,s) \in C[D_0, R^+]$, where $R^+ = [0, +\infty)$, $D = \{(t,s) \in R^2 | t_0 \leq s \leq t \leq t_0 + a\}$, $D_0 = \{(t,s) \in R^2 | (t,s) \in J \times J\}$. Here, $k(t,s)$ is unbounded on D, $\rho(t,s)$ is bounded on D, and $h(t,s)$ is bounded on D_0. Set $R_0 = \max\{\rho(t,s) | (t,s) \in D\}$, $h_0 = \max\{h(t,s) | (t,s) \in D_0\}$.

The study of initial value problems for nonlinear integro-differential equations has been of great interest for many researchers for its physical backgrounds and applications in mathematical models. We refer the reader to [1, 5–12] and references therein for some recent results on equation (1.1). However, in many earlier results, the kernel $k(t,s)$ of the operator T is bounded. In this paper, we will make further study on the initial value problem (1.1) in the case of $k(t,s)$ being unbounded. By establishing a comparison theorem, we achieve an existence theorem about minimal and maximal solutions for equation (1.1).

Throughout the rest of this paper, let $(E, \|\cdot\|)$ be a real Banach space and P be a cone in E which defines a partial ordering in E denoted by "$\leq$".

Suppose that E^* is the dual space of E, the dual cone of the cone P is $P^* = \{\varphi \in E^* | \varphi(x) \geq 0, \forall x \in P\}$. A cone $P \subset E$ is said to be normal there exists a constant $\gamma > 0$ such that

$$\theta \leq x \leq y \Longrightarrow \|x\| \leq \gamma\|y\|, \forall x, y \in E.$$

The cone P is normal if and only if any order interval $[x,y] = \{z \in E | x \leq z \leq y\}$ is bounded in norm(see [3]). Set

$$C[J, E] = \left\{u(t) : J \to E \middle| u(t) \text{ is continuous on } J\right\},$$

$$C^1[J, E] = \left\{u(t) : J \to E \middle| u(t) \text{ and } u'(t) \text{ are continuous on } J\right\}.$$

Let $\|u\|_c = \max\limits_{t \in J} \|u(t)\|$ be a norm for $u \in C[J, E]$, then $C[J, E]$ will be a Banach space with norm $\|\cdot\|_c$. It is easy to know $P_c = \{u \in C[J, E] | u(t) \geq \theta, \forall t \in J\}$ is a cone in $C[J, E]$. The cone P_c defines an ordering in $C[J, E]$ which also denoted by "$\leq$" here. Obviously, when the cone P is normal, P_c is a normal cone in $C[J, E]$.

Assume that V is a bounded set in E. The Kuratowski measure of noncompactness $\alpha(V)$ and the Hausdorff measure of noncompactness $\beta(V)$ are defined respectively as follow:

$$\alpha(V) = \inf\{\delta > 0 | V \text{ can be expressed as the union } S = \bigcup_{i=1}^{m} V_i \text{ of a finite number of sets } V_i \text{ with diameter } diam(V_i) \leq \delta\},$$

$$\beta(V) = \inf\left\{\delta > 0 \middle| V \text{ can be covered by a finite number of closed balls } V_i \text{ with diameter } diam(V_i) \leq \delta\right\}.$$

The relationship of the two noncompactness measures is

$$\beta(V) \leq \alpha(V) \leq 2\beta(V). \tag{1.3}$$

2

For the basic properties of cones and noncompactness measures, we refer the reader to [2–4]. For convenience, the Kuratowski measure of noncompactness for bounded sets in E and $C[J, E]$ are all denoted by $\alpha(\cdot)$. In the sequel, we denote $B(t) = \{u(t)|u \in B\}$, $(TB)(t) = \{(Tu)(t)|u \in B\}$, $(SB)(t) = \{(Su)(t)|u \in B\}$ for all $B \subset C[J, E]$ with $t \in J$.

Lemma 1.1. *Let* $m \in C^1[J, R^1]$ *be such that*

$$m'(t) \geq -Mm(t) - N \int_{t_0}^{t} k(t, s)m(s)ds, \ m(t_0) \geq 0, \ t \in J, \tag{1.4}$$

where $M \geq 0$ *and* $N \geq 0$ *are two constants satisfying one of the following conditions:*
   (i)

$$NR_0 e^{Ma} \frac{a^{2-\alpha}}{1 - \alpha} \leq 1; \tag{1.5}$$

   (ii)

$$aM + \frac{NR_0 a^{2-\alpha}}{1 - \alpha} \leq 1. \tag{1.6}$$

Then $m(t) \geq 0$ *for all* $t \in J$.

Proof. **Case 1.** If the condition (i) is established, let $v(t) = m(t)e^{Mt}$. From (1.4), we have

$$v'(t) \geq -N \int_{t_0}^{t} k^*(t, s)v(s)ds, \ \forall t \in J, \ v(t_0) \geq 0, \tag{1.7}$$

where $k^*(t, s) = k(t, s)e^{M(t-s)}$. Now, we prove that

$$v(t) \geq 0, \ \forall t \in J. \tag{1.8}$$

In fact, if there exists $t_0 \leq t_1 \leq t_0 + a$ such that $v(t_1) < 0$ and let $\max\{v(t) : t_0 \leq t \leq t_1\} = b$, then $b \geq 0$. If $b = 0$, then $v(t) \leq 0$ for all $t_0 \leq t \leq t_1$ and so (1.7) implies that

$$v'(t) \geq 0, \ \forall t_0 \leq t \leq t_1.$$

Hence we have $v(t_1) \geq v(t_0) = m(t_0)e^{Mt_0} \geq 0$, which contradicts $v(t_1) < 0$.
   If $b > 0$, then there exists $t_0 \leq t_2 < t_1$ such that $v(t_2) = b > 0$ and so there exists $t_2 < t_3 < t_1$ such that $v(t_3) = 0$. Then, by the mean value theorem, there exists $t_2 < t_4 < t_3$ such that

$$v'(t_4) = \frac{v(t_3) - v(t_2)}{t_3 - t_2} = \frac{-v(t_2)}{t_3 - t_2} = \frac{-b}{t_3 - t_2} < -\frac{b}{a}. \tag{1.9}$$

On the other hand, from (1.7), we have

$$v'(t_4) \geq -N \int_{t_0}^{t_4} k^*(t_4, s)v(s)ds$$

3

$$
\begin{aligned}
&\geq\ -Nb\int_{t_0}^{t_4} k^*(t_4,s)ds \\
&=\ -Nb\int_{t_0}^{t_4} \frac{\rho(t_4,s)}{(t_4-s)^\alpha}e^{M(t_4-s)}ds \\
&\geq\ -NbR_0\int_{t_0}^{t_4} (t_4-s)^{-\alpha}e^{M(t_4-s)}ds \\
&\geq\ -NbR_0 e^{Ma}\int_{t_0}^{t_4} (t_4-s)^{-\alpha}ds \\
&=\ -NbR_0 e^{Ma}\frac{(t_4-t_0)^{1-\alpha}}{1-\alpha} \\
&\geq\ -NbR_0 e^{Ma}\frac{a^{1-\alpha}}{1-\alpha}.
\end{aligned}
$$

Then from (1.9), we have $NR_0 e^{Ma}\frac{a^{2-\alpha}}{1-\alpha} > 1$ which contradicts (1.5). Therefore, (1.8) is true and so $m(t) \geq 0$ for all $t \in J$.

Case 2. If the assumption (ii) holds, but the conclusion does not hold, then there exists $t_1 \in (t_0, t_0+a]$ such that

$$
m(t_1) = \min_{t\in J} m(t) < 0,
$$

and so $m'(t_1) \leq 0$. If $\max\limits_{t_0\leq t\leq t_1} m(t) \leq 0$, from (1.4), we have

$$
0 \geq m'(t_1) \geq -Mm(t_1) - N\int_{t_0}^{t_1} k(t_1,s)m(s)ds \geq -Mm(t_1) > 0,
$$

which is a contradictory statement. Therefore, there exists $t_2 \in [t_0, t_1)$ such that $m(t_2) = \max\limits_{t_0\leq t\leq t_1} m(t) = \mu > 0$. Then, by the mean value theorem, there exists $t_3 \in (t_2, t_1)$ such that

$$
m'(t_3) = \frac{m(t_1)-m(t_2)}{t_1-t_2} < -\frac{\mu}{a}.
$$

It follows from (1.4) that

$$
\begin{aligned}
-\frac{\mu}{a} > m'(t_3)\ &\geq\ -Mm(t_3) - N\int_{t_0}^{t_3}\frac{\rho(t_3,s)}{(t_3-s)^\alpha}m(s)ds \\
&\geq\ -M\mu - NR_0\mu\int_{t_0}^{t_3}\frac{1}{(t_3-s)^\alpha}ds \\
&=\ -M\mu - NR_0\mu\frac{(t_3-t_0)^{1-\alpha}}{1-\alpha} \\
&\geq\ -M\mu - NR_0\mu\frac{a^{1-\alpha}}{1-\alpha},
\end{aligned}
$$

i.e. $aM + NR_0\frac{a^{2-\alpha}}{1-\alpha} > 1$ which contradicts (1.6). The Lemma is proved. $\square$

4

Lemma 1.2. *Let $m \in C[J, R^+]$ be such that*

$$m(t) \leq M_1 \int_{t_0}^{t} m(s)ds + M_2(t - t_0) \int_{t_0}^{t_0+a} m(s)ds, \ t \in J \qquad (1.10)$$

where $M_1 > 0$, $M_2 \geq 0$, are constants for satisfying one of the following conditions: (i)$aM_2(e^{aM_1} - 1) < M_1$, (ii)$a(2M_1 + aM_2) < 2$. Then $m(t) \equiv 0$, $t \in J$.

Proof. **Case 1.** If the condition (i) holds, letting $v(t) = m(t)e^{Mt}$, then $m_1(t_0) = 0$, $m_1'(t) = m(t)$, $t \in J$. If $m_1(t_0 + a) \neq 0$, it follows from (1.10) that

$$m_1'(t) \leq M_1 m_1(t) + aM_2 m_1(t_0 + a), \ t \in J$$

and from $e^{-M_1(t-t_0)} > 0$ we have

$$\left(m_1(t)e^{-M_1(t-t_0)}\right)' \leq aM_2 m_1(t_0 + a)e^{-M_1(t-t_0)}, \ t \in J.$$

Now, we integrate the above inequality between t_0 and t with noticing $m_1(t_0) = 0$, we can obtain

$$
\begin{aligned}
m_1(t)e^{-M_1(t-t_0)} &\leq aM_2 m_1(t_0 + a) \int_{t_0}^{t} e^{-M_1(s-t_0)}ds \\
&\leq \frac{aM_2}{M_1} m_1(t_0 + a)\left(1 - e^{-M_1(t-t_0)}\right), \ t \in J.
\end{aligned}
$$

By choosing $t = t_0 + a$, we can get

$$aM_2\left(e^{aM_1} - 1\right) \geq M_1$$

which contradicts (i). Consequently, $m_1(t_0 + a) = \int_{t_0}^{t_0+a} m(s)ds = 0$ which implies $m(t) \equiv 0$, $t \in J$.

Case 2. If the condition (ii) is established, it follows from (1.10) that

$$m(t) \leq [M_1 + M_2(t - t_0)] \int_{t_0}^{t_0+a} m(s)ds.$$

Integrating the above inequality between t_0 and $t_0 + a$, we get

$$\int_{t_0}^{t_0+a} m(t)dt \leq \left[aM_1 + \frac{a^2 M_2}{2}\right] \int_{t_0}^{t_0+a} m(s)ds.$$

From the above inequality and conditio (ii), it follows that $\int_{t_0}^{t_0+a} m(t)dt = 0$, so $m(t) \equiv 0, t \in J$. This completes the proof. □

Lemma 1.3. *If B is a equicontinuous bounded set in $\subset C[J, E]$, then $\alpha(B) = \max_{t \in J} \alpha(B(t))$.*

5

Lemma 1.4. *If B is a equicontinuous bounded set in $\subset C[J, E]$ with $J = [a, b]$, then $\alpha(\{u(t) | u \in B\})$ is continuous with respect to $t \in J$ and*

$$\alpha\left(\left\{\int_a^b u(t)dt \,\middle|\, u \in B\right\}\right) \leq \int_a^b \alpha\{u(t) | u \in B\}\, dt.$$

Lemma 1.5. *(see [2]) Let E be a separable Banach space, $J = [a, b]$ and $\{u_n\} : J \to E$ be continuous abstract function sequences. If there exists a function $\phi \in L[a, b]$ such that $\|u_n(t)\| \leq \phi(t)$, $t \in J$, $n = 1, 2, 3, \cdots$, then $\beta\left(\{u_n(t) | n = 1, 2, 3, \cdots\}\right)$ is integrable on J and*

$$\beta\left(\left\{\int_a^b u_n(t)dt \,\middle|\, n = 1, 2, 3, \cdots\right\}\right) \leq \int_a^b \beta\left(\{u_n(t) | n = 1, 2, 3, \cdots\}\right) dt.$$

Now, we give our assumptions:

(H_1) There exist $v_0, \omega_0 \in C^1[J, E]$ such that $v_0(t) \leq \omega_0(t)(t \in J)$ and v_0, ω_0 are a lower solution and an upper solution respectively for the initial value problem (1.1), that is

$$v_0' \leq f(t, v_0, Tv_0, Sv_0), \ \forall t \in J; \ v_0(t_0) \leq u_0,$$

$$\omega_0' \geq f(t, \omega_0, T\omega_0, S\omega_0), \ \forall t \in J; \ \omega_0(t_0) \geq u_0.$$

(H_2) For any $t \in J$, any $u, v \in [v_0, \omega_0] = \{u \in C[J, E] | v_0 \leq u \leq \omega_0\}$ and $u \leq v$, we have

$$f(t, v, Tv, Sv) - f(t, u, Tu, Su) \geq -M(v - u) - NT(v - u),$$

where $M > 0$, $N \geq 0$ are constants satisfying the condition (i) or (ii) in Lemma 1.1.

(H_3) For any $t \in J$ and equicontinuous bounded monotone sequences $B = \{u_n\} \subset [v_0, \omega_0]$, we have

$$\alpha(f(t, B(t), (TB)(t), (SB)(t))) \leq c_1 \alpha(B(t)) + c_2 \alpha((TB)(t)) + c_3 \alpha((SB)(t)),$$

where $c_i(i = 1, 2, 3)$ are constants satisfying one of the following two conditions:

(i)$ah_0c_3\left(e^{2a(c_1 + M + \frac{c_2 R_0 a^{1-\alpha}}{1-\alpha} + \frac{2N R_0 a^{1-\alpha}}{1-\alpha})} - 1\right) < c_1 + M + \frac{c_2 R_0 a^{1-\alpha}}{1-\alpha} + \frac{2N R_0 a^{1-\alpha}}{1-\alpha}$;

(ii)$a\left(2c_1 + 2M + \frac{2c_2 R_0 a^{1-\alpha}}{1-\alpha} + \frac{4N R_0 a^{1-\alpha}}{1-\alpha} + ah_0c_3\right) < 1.$

2 Main results

Theorem 2.1. *Let E be a real Banach space, $P \subset E$ be a normal cone and the conditions (H_1), (H_2), (H_3) be satisfied. Then the initial value problem (1.1) has a minimal solution and a maximal solution $\overline{u}$, $u^* \in C^1[J, E]$ in $[v_0, \omega_0]$, and for the initial value v_0 and ω_0, the iterative sequences $\{v_n(t)\}$ and $\{\omega_n(t)\}$ defined by the following formulas converge uniformly to $\overline{u}(t)$, $u^*(t)$ on J according to the norm in E respectively:*

$$v_n(t) = u_0 e^{-M(t-t_0)} + \int_{t_0}^t e^{M(s-t)} \left[f\left(s, v_{n-1}(s), (Tv_{n-1})(s), (Sv_{n-1})(s)\right) \right.$$

6

$$+Mv_{n-1}(s) - NT(v_n - v_{n-1})(s)\big]ds, \ \forall t \in J, \tag{2.1}$$

$$\omega_n(t) = u_0 e^{-M(t-t_0)} + \int_{t_0}^{t} e^{M(s-t)}\big[f\big(s, \omega_{n-1}(s), (T\omega_{n-1})(s), (S\omega_{n-1})(s)\big)$$

$$+M\omega_{n-1}(s) - NT(\omega_n - \omega_{n-1})(s)\big]ds, \ \forall t \in J \ (n = 1,2,3,\cdots). \tag{2.2}$$

Moreover, there holds

$$v_0 \le v_1 \le \cdots \le v_n \le \cdots \le \bar{u} \le u^* \le \cdots \le \omega_1 \le \omega_0. \tag{2.3}$$

Proof. For any $\eta \in [v_0, \omega_0]$, we consider the initial value problem of linear integro-differential equation in Banach space E:

$$u' = g(t) - Mu - NTu, \ u(t_0) = u_0, \tag{2.4}$$

where $g(t) = f\big(t, \eta(t), (T\eta)(t), (S\eta)(t)\big) + M\eta(t) + N(T\eta)(t)$. It is easy to show that u is a solution of the linear initial value problem (2.4) if and only if u is the fixed point in $C[J, E]$ of the following operator

$$(Au)(t) = u_0 e^{-M(t-t_0)} + \int_{t_0}^{t} e^{M(s-t)}\big[g(s) - N(Tu)(s)\big]ds. \tag{2.5}$$

In the following, we will prove there exists n_0 such that A^{n_0} is a contraction operator. For any $u, v \in C[J, E]$, $t \in J$, it follows from (2.5) that

$$
\begin{aligned}
\|(Au)(t) - (Av)(t)\| &\le N \int_{t_0}^{t} \|T(u-v)(s)\| ds \\
&\le N \int_{t_0}^{t} \left[\int_{t_0}^{s} k(s,\tau)\|u(\tau) - v(\tau)\| d\tau \right] ds \\
&= N \int_{t_0}^{t} \left[\int_{t_0}^{s} \frac{\rho(s,\tau)}{(s-\tau)^\alpha}\|u(\tau) - v(\tau)\| d\tau \right] ds \\
&\le NR_0 \|u-v\|_c \int_{t_0}^{t} \int_{t_0}^{s} \frac{1}{(s-\tau)^\alpha} d\tau ds \\
&= \frac{NR_0(t-t_0)^{2-\alpha}}{(1-\alpha)(2-\alpha)} \|u-v\|_c.
\end{aligned}
\tag{2.6}
$$

In the same way, by (2.5) and (2.6), we have

$$
\begin{aligned}
\left\|(A^2 u)(t) - (A^2 v)(t)\right\| &\le N \int_{t_0}^{t} \|T(Au - Av)(s)\| ds \\
&\le N \int_{t_0}^{t} \left[\int_{t_0}^{s} k(s,\tau)\|(Au)(\tau) - (Av)(\tau)\| d\tau \right] ds
\end{aligned}
$$

7

$$
\begin{aligned}
&\leq\ NR_0 \int_{t_0}^{t} \left[\int_{t_0}^{s} \frac{1}{(s-\tau)^{\alpha}} \frac{NR_0(\tau - t_0)^{2-\alpha}}{(1-\alpha)(2-\alpha)} \|u-v\|_c d\tau \right] ds \\
&=\ \frac{(NR_0)^2}{(1-\alpha)(2-\alpha)} \|u-v\|_c \int_{t_0}^{t} \left[\int_{t_0}^{s} \frac{(\tau-t_0)^{2-\alpha}}{(s-\tau)^{\alpha}} d\tau \right] ds \\
&\leq\ \frac{(NR_0)^2 \|u-v\|_c}{(1-\alpha)(2-\alpha)} \int_{t_0}^{t} \int_{t_0}^{s} (s-\tau)^{-\alpha}(s-t_0)^{2-\alpha} d\tau ds \\
&=\ \frac{(NR_0)^2 \|u-v\|_c}{(1-\alpha)(2-\alpha)} \int_{t_0}^{t} \frac{(s-t_0)^{3-2\alpha}}{1-\alpha} ds \\
&=\ \frac{(NR_0)^2}{(1-\alpha)^2 2(2-\alpha)^2} \|u-v\|_c (t-t_0)^{4-2\alpha}.
\end{aligned}
$$

It is easy to prove that by mathematical induction

$$
\left\| (A^n u)(t) - (A^n v)(t) \right\| \leq \frac{(NR_0)^n}{n![(1-\alpha)(2-\alpha)]^n} (t-t_0)^{n(2-\alpha)} \|u-v\|_c,\ t \in J,\ n = 1,2,3,\cdots.
$$

Thus

$$
\|A^n u - A^n v\|_c \leq \frac{(NR_0 a^{2-\alpha})^n}{n![(1-\alpha)(2-\alpha)]^n} \|u-v\|_c,\ n=1,2,3,\cdots.
$$

We can choose $n_0 \in \{1,2,3,\cdots\}$ such that $\frac{(NR_0 a^{2-\alpha})^n}{n![(1-\alpha)(2-\alpha)]^n} < 1$, and so A^{n_0} a contraction operator in $C[J,E]$. Therefore, it follows from the principle of contraction mapping that A^{n_0}, that is, A has a unique fixed point u_η in $C[J,E]$ which implies the linear initial value problem (2.4) has a unique solution u_η in $C[J,E]$. Now, we define a operator

$$
B\eta = u_\eta \tag{2.7}
$$

where u_η is a unique solution for η of the linear initial value problem (2.4), and satisfies

$$
u_\eta' = f(t,\eta(t),(T\eta)(t),(S\eta)(t)) - M(u_\eta(t)-\eta(t)) - NT(u_\eta - \eta)(t), u_\eta(t_0) = u_0.
$$

Then $B : [v_0, \omega_0] \longrightarrow C[J,E]$, and the iterative sequences (2.1)(2.2) can be written

$$
v_n = Bv_{n-1},\ \omega_n = B\omega_{n-1},\ n = 1,2,3,\cdots. \tag{2.8}
$$

Moreover, we claim that the operator B defined by (2.7) satisfies
i)

$$
v_0 \leq Bv_0,\ B\omega_0 \leq \omega_0; \tag{2.9}
$$

ii)

$$
B\eta_1 \leq B\eta_2,\ \forall \eta_1,\ \eta_2 \in [v_0, \omega_0],\ \eta_1 \leq \eta_2. \tag{2.10}
$$

Next, we will prove i) and ii). Firstly, we prove the result i). Set $v_1 = Bv_0$, it follows from the definition of B that

8

$$v_1' = f(t, v_0, Tv_0, Sv_0) - M(v_1 - v_0) - NT(v_1 - v_0), \ v_1(t_0) = u_0. \qquad (2.11)$$

For any $\varphi \in P^*$, let $m(t) = \varphi(v_1(t) - v_0(t))$, it follows from (2.11) and the assumption (H_1) that

$$m'(t) \geq -Mm(t) - N \int_{t_0}^{t} k(t, s)m(s)ds, \ m(t_0) \geq 0.$$

Thus, by lemma 1.1, it follows that $m(t) \geq 0$ for all $t \in J$, which implies $v_1(t) - v_0(t) \geq 0$ for all $t \in J$. It follows theorem 2.4.3 in [3] that $v_0 \leq Bv_0$. Similarly, we can prove that $B\omega_0 \leq \omega_0$. Consequently, the result i) is proved.

Next, we prove ii). Let $u_{\eta_1} = B\eta_1$, $u_{\eta_2} = B\eta_2$, it follows from the hypothesis (H_2) and the definition of B that

$$
\begin{aligned}
u_{\eta_1}' - u_{\eta_2}' &= f(t, \eta_2, T\eta_2, S\eta_2) - M(u_{\eta_2} - \eta_2) - NT(u_{\eta_2} - \eta_2) \\
&\quad -f(t, \eta_1, T\eta_1, S\eta_1) + M(u_{\eta_1} - \eta_1) + NT(u_{\eta_1} - \eta_1) \\
&\geq -M(u_{\eta_2} - u_{\eta_1}) - NT(u_{\eta_2} - u_{\eta_1}) \qquad (2.12)
\end{aligned}
$$

and

$$u_{\eta_2}(t_0) - u_{\eta_1}(t_0) = u_0 - u_0 = \theta. \qquad (2.13)$$

For any $\varphi \in P^*$, let $m(t) = \varphi(u_{\eta_2}(t) - u_{\eta_1}(t))$. From (2.12) and (2.13), it follows that

$$m'(t) \geq -Mm(t) - N \int_{t_0}^{t} k(t, s)m(s)ds, \ m(t_0) = 0$$

Thus, by lemma 1.1, it follows that $m(t) \geq 0$ for all $t \in J$, which implies $u_{\eta_2}(t) - u_{\eta_1}(t) \geq \theta$, $t \in J$, that is, $B\eta_1 \leq B\eta_2$. So the result ii) is proved.

Form (2.8)-(2.10) and observing that $v_0 \leq \omega_0$, it follows that

$$v_0 \leq v_1 \leq \cdots \leq v_n \leq \cdots \leq \omega_n \leq \cdots \leq \omega_1 \leq \omega_0. \qquad (2.14)$$

and B is a mapping with $[v_0, \omega_0]$ into $[v_0, \omega_0]$.

In the following, we prove that $\{v_n(t)\}$ converges uniformly to some element $\overline{u} \in C[J, E]$ in J. By the normality of P, the cone P_c is normal in $C[J, E]$ which implies the order interval $[v_0, \omega_0]$ is a bounded set in $C[J, E]$. Then, it follows from (2.14) that $\{v_n\}$ is a bounded set in $C[J, E]$. On the one hand, for any $\eta \in [v_0, \omega_0]$, by the conditions (H_1) and (H_2), we have

$$
\begin{aligned}
v_0' + Mv_0 + NTv_0 &\leq f(t, v_0, Tv_0, Sv_0) + Mv_0 + NTv_0 \\
&\leq f(t, \eta, T\eta, S\eta) + M\eta + NT\eta \\
&\leq f(t, \omega_0, T\omega_0, S\omega_0) + M\omega_0 + NT\omega_0 \\
&\leq \omega_0' + M\omega_0 + NT\omega_0.
\end{aligned}
$$

9

Then, by the normality of P_c, the set $\big\{f(t,\eta,T\eta,S\eta)+M\eta+NT\eta\big|\eta\in[v_0,\omega_0]\big\}$ is a bounded set in $C[J,E]$. On the other hand, the set $\big\{T\eta\big|\eta\in[v_0,\omega_0]\big\}$ is also a bounded set in $C[J,E]$, because it follows from the boundedness of $[v_0,\omega_0]$ that for any $\eta\in[v_0,\omega_0]$,

$$
\begin{aligned}
\|T\eta(t)\| &\leq \int_{t_0}^{t} k(t,s)\|\eta(s)\|ds\\
&\leq \|\eta\|_c \int_{t_0}^{t} \frac{\rho(t,s)}{(t-s)^\alpha}ds\\
&\leq R_0\|\eta\|_c \int_{t_0}^{t} \frac{1}{(t-s)^\alpha}ds\\
&= R_0\|\eta\|_c \frac{(t-t_0)^{1-\alpha}}{1-\alpha}.
\end{aligned}
$$

Therefore, $\{f(t,\eta,T\eta,S\eta)|\eta\in[v_0,\omega_0]\}$ is a bounded set in $C[J,E]$. Thus, from

$$
v_n'=f(t,v_{n-1},Tv_{n-1},Sv_{n-1})-M(v_n-v_{n-1})-NT(v_n-v_{n-1}),t\in J,n=1,2,3,\cdots,\quad(2.15)
$$

it follows that $\{v_n'|n=1,2,3,\cdots\}$ is a bounded set in $C[J,E]$. Applying the mean value theorem, we see that all the functions $\{v_n(t)|n=1,2,3,\cdots\}$ is equicontinuous on J. From Lemma 1.3, we have

$$
\alpha(\{v_n|n=1,2,3,\cdots\})=\max_{t\in J}\alpha(\{v_n(t)|n=1,2,3,\cdots\}).\quad(2.16)
$$

Now, we prove $\alpha(\{v_n|n=1,2,3,\cdots\})=0$. From (2.4), (2.5), (2.7) and (2.8), it follows that

$$
\begin{aligned}
v_n(t) &= u_0 e^{-M(t-t_0)}+\int_{t_0}^{t} e^{M(s-t)}\big[f\left(s,v_{n-1}(s),(Tv_{n-1}),(Sv_{n-1})(s)\right)\\
&\quad +Mv_{n-1}(s)-NT(v_n-v_{n-1})(s)\big]ds.\quad(2.17)
\end{aligned}
$$

Let $m(t)=\alpha\{v_n(t)|n=1,2,3,\cdots\}$, then $m(t_0)=\alpha(\{u_0\})=0$, $m\in C[J,R^+]$. For every n, by the continuity of $v_n(t)$, $\{v_n(t)|t\in J\}$ is a separable set in E, so $\{v_n(t)|t\in J,n=1,2,3,\cdots\}$ is a separable set in E. Thus, we can assume that E is a separable Banach space without loss of generality (otherwise, the closed subspace in E is spanned by $\{v_n(t)|t\in J,n=1,2,3,\cdots\}$ can be used in place of E). By (2.17), (1.3) and Lemma 1.5 and observing $0<e^{M(s-t)}\leq1$, $(t,s)\in D$, we can obtain

$$
\begin{aligned}
m(t) &\leq \alpha\left(\int_{t_0}^{t} e^{M(s-t)}\big[f(s,B(s),(TB)(s),(SB)(s))+MB(s)-NT(B_1-B)(s)\big]ds\right)\\
&\leq 2\beta\left(\int_{t_0}^{t} e^{M(s-t)}\big[f(s,B(s),(TB)(s),(SB)(s))+MB(s)-NT(B_1-B)(s)\big]ds\right)\\
&\leq 2\int_{t_0}^{t}\beta\big[f(s,B(s),(TB)(s),(SB)(s))+MB(s)-NT(B_1-B)(s)\big]ds
\end{aligned}
$$

10

$$\leq \; 2\int_{t_0}^{t} \Big[\beta\left(f\left(s,B(s),(TB)(s),(SB)(s)\right)\right)$$
$$+ M\beta(B(s)) + N\beta(T(B_1 - B)(s))\Big]ds. \tag{2.18}$$

where $B(s) = \{v_n(s)|n = 0,1,2,\cdots\}$, $B_1(s) = \{v_n(s)|n = 1,2,3,\cdots\}$. By the condition (H_3) and (1.3), we have

$$\beta\left(f(s,B(s),(TB)(s),(SB)(s))\right)$$
$$\leq \; \alpha\left(f(s,B(s),(TB)(s),(SB)(s))\right)$$
$$\leq \; c_1\alpha(B(s)) + c_2\alpha(((TB))(s)) + c_3\alpha((SB)(s)). \tag{2.19}$$

From the uniform boundedness of $B(s)$ and uniform continuity of $h(t,s)$, it easy to prove $(SB)(s)$ is a equicontinuous bounded set, so it follows from Lemma 1.4 that

$$\alpha((SB)(s)) = \alpha\left(\int_{t_0}^{t_0+a} h(s,\tau)B(\tau)d\tau\right) \leq h_0\int_{t_0}^{t_0+a} m(\tau)d\tau. \tag{2.20}$$

Now, we consider dealing with $\alpha\big((TB)(s)\big)$. Firstly,

$$\int_{t_0}^{s} k(s,\tau)d\tau = \int_{t_0}^{s} \frac{\rho(s,\tau)}{(s-\tau)^\alpha}d\tau \leq R_0\int_{t_0}^{s} \frac{1}{(s-t)^\alpha}d\tau \leq \frac{R_0 a^{1-\alpha}}{1-\alpha}.$$

Since $B(s)$ is equicontinuous bounded sequences and $\alpha(B(s)) = m(s)$, there exists a partition $B(s) = \bigcup\limits_{i=1}^{l} B_i$ such that the partition $(TB)(s) = \bigcup\limits_{i=1}^{l} TB_i$ exists, where $TB_i = \left\{\int_{t_0}^{s} k(s,\tau)v_i(\tau)d\tau \big| v_i \in B_i\right\}$, so we have

$$\begin{aligned}
diam(TB_i) &= \sup_{\forall v_i^1, v_i^2 \in B_i} \left\| \int_{t_0}^{s} k(s,\tau)\left[v_i^1(\tau) - v_i^2(\tau)\right]d\tau \right\| \\
&\leq \frac{R_0 a^{1-\alpha}}{1-\alpha} \sup_{\forall v_i^1, v_i^2 \in B_i} \left\| v_i^1(\tau) - v_i^2(\tau) \right\| \\
&= \frac{R_0 a^{1-\alpha}}{1-\alpha} diam(B_i) \\
&< \frac{R_0 a^{1-\alpha}}{1-\alpha}\alpha(B(s)) + \frac{R_0 a^{1-\alpha}}{1-\alpha}\cdot\varepsilon.
\end{aligned}$$

By using the arbitrariness of ε, we have

$$\alpha(TB(s) \leq \frac{R_0 a^{1-\alpha}}{1-\alpha}\alpha(B(s)) = \frac{R_0 a^{1-\alpha}}{1-\alpha}m(s), \tag{2.21}$$

and by (1.3), we have

$$\beta\left(T(B_1 - B)(s)\right) \leq \alpha\left(T(B_1 - B)(s)\right) \leq \frac{2R_0 a^{1-\alpha}}{1-\alpha}m(s). \tag{2.22}$$

11

Thus, it follows from (2.18)-(2.22) that

$$
\begin{aligned}
m(t) &\leq 2\int_{t_0}^{t}\left[c_1 m(s) + \frac{c_2 R_0 a^{1-\alpha}}{1-\alpha}m(s) + c_3 h_0 \int_{t_0}^{t_0+a} m(\tau)d\tau \right.\\
&\left.\quad + Mm(s) + \frac{2NR_0 a^{1-\alpha}}{1-\alpha}m(s)\right]ds\\
&= 2\left(c_1 + M + \frac{c_2 R_0 a^{1-\alpha}}{1-\alpha} + \frac{2NR_0 a^{1-\alpha}}{1-\alpha}\right)\int_{t_0}^{t} m(s)ds\\
&\quad + 2h_0 c_3 (t-t_0)\int_{t_0}^{t_0+a} m(s)ds.
\end{aligned}
$$

Therefore, from Lemma 1.2 and the conditions (i)(ii) of the assumption (H_3), we have $m(t)\equiv 0$, $t\in J$ which implies $\alpha\{v_n|n=1,2,3,\cdots\}=0$ from (2.16), that is, $v_n \subset [v_0,\omega_0]$ is a relatively compact set in $C[J,E]$. Thus there exists a subsequence $\{v_{n_k}\}\subset\{v_n\}$ and some $\overline{u}\in[v_0,\omega_0]$ such that $\{v_{n_k}\}$ converges to $\overline{u}$ in norm $\|\cdot\|_c$. Further, from (2.14) and the normality of P_c, it is easy to prove that $\{v_n\}$ converges to $\overline{u}$ in norm $\|\cdot\|_c$, that is, $\{v_n(t)\}$ converges uniformly to $\overline{u}(t)$ on J according to the norm in E. Similarly, we can prove that $\{\omega_n(t)\}$ converges uniformly to some $u^*\in[v_0,\omega_0]$ on J according to the norm in E. Clearly, the result (2.3) is true.

Finally, we prove that $\overline{u}$ and u^* are a minimal solution and a maximal solution respectively of the initial value problem (1.1). Let

$$u_n(t) = -M(v_n(t) - v_{n-1}(t)) - NT(v_n - v_{n-1})(t),\ t\in J,\ n=1,2,3,\cdots.$$

Since $\{v_n(t)\}$ converges uniformly to $\overline{u}(t)$ on J, it is easy to prove $\|u_n\|_c \to 0(n\to\infty)$. Setting $\varepsilon_n = \|u_n\|_c$, from (2.15), we get

$$v_n' = f(t, v_{n-1}, Tv_{n-1}, Sv_{n-1}) + u_n(t),\ v_n(t_0)=u_0,\ \|u_n(t)\|\leq\varepsilon_n,\ t\in J.$$

Applying Corollary 2.1.1 in [4], we know $\overline{u}$ is a solution of the initial value problem (1.1). Similarly, we can prove that ω^* is also a solution of the initial value problem (1.1). If u is a solution in $[v_0,\omega_0]$ of the initial value problem (1.1), then $Bu=u$, so by $v_0\leq u\leq\omega_0$ and (2.8)-(2.10), it is easy to obtain

$$v_n \leq u \leq \omega_n, n=1,2,3,\cdots.$$

Letting $n\to\infty$ in above formula, we get $\overline{u}\leq u\leq u^*$. Consequently, $\overline{u},\ u^*$ are the minimal solution and maximal solution of the initial value problem (1.1) respectively. This completes the proof. □

References

[1] B. Ahmad, et al., Some existence theorems for fractional integrodifferential equations and inclusions with initial and non-separated boundary conditions, Bound. Value Probl. 2014, 2014:249.

12

[2] K. Deimling, Nonlinear Functional Analysis, Springer-Verlag, Berlin, 1985.

[3] D. Guo, Partial ordering methods in nonlinear problems, Shandong Science and Technology Press, Jinan, China, 1997 (in Chinese).

[4] D. Guo, J. Sun, Ordinary differential equations in abstract spaces, Shandong Science and Technology Press, Jinan, China, 1989 (in Chinese).

[5] A. Jawahdou, Initial value problem of fractional integro-differential equations in Banach space, Fract. Calc. Appl. Anal. 18 (2015), no. 1, 20–37.

[6] H. Lan, Y. Cui, Perturbation technique for a class of nonlinear implicit semilinear impulsive integrodifferential equations of mixed type with noncompactness measure, Adv. Difference Equ. 2015, no. 1, 2015:11.

[7] L. Liu, C. Wu, F. Guo, A unique solution of initial value problems for first order impulsive integro-differential equations of mixed type in Banach spaces, J. Math. Anal. Appl., 275 (2002), 369–385.

[8] J. Machado et al., Controllability results for impulsive mixed-type functional integro-differential evolution equations with nonlocal conditions, Fixed Point Theory Appl. 2013, 2013:66.

[9] N. I. Mahmudov, S. Zorlu, Approximate controllability of fractional integrodifferential equations involving nonlocal initial conditions, Bound. Value Probl. 2013, 2013:118.

[10] Z. Šmarda, Y. Khan, Singular initial value problem for a system of integro-differential equations unsolved with respect to the derivative, Appl. Math. Comput. 222 (2013), 290–296.

[11] Y. Sun, Positive solutions for nonlinear integrodifferential equations of mixed type in Banach spaces, Abstr. Appl. Anal. 2013, Art. ID 787038.

[12] S. Xie, Existence of solutions for nonlinear mixed type integro-differential functional evolution equations with nonlocal conditions, Bound. Value Probl. 2012, 2012:100.

13

Solving the multicriteria transportation equilibrium system problem with nonlinear path costs

Chaofeng Shi,   Yingrui Wang

Abstract. In this paper, we present an self-adaptive algorithm for solving the multicriteria transportation equilibrium system problem with variable demand and nonlinear path costs. The path cost function considered is comprised of three attributes, travel time, toll and travel fares, that are combined into a nonlinear generalized cost. Travel demand is determined endogenously according to a travel disutility function. Travelers choose routes with the minimum overall generalized costs. Numerical experiments are conducted to demonstrate the feasibility of the algorithm to this class of transportation equilibrium system problems.

1   Introduction

Usually, there are more than one kind of goods transported through the traffic network, in reality. As we know, the transportation cost of one kind of goods can be affected by other kinds of goods under the same traffic network. In detail, the flows of different kinds of goods are not independent. Generally, in 2010, He et. al . [1] called this problem as dynamic traffic network equilibrium system. Several authors (see, for instance, [2-5]) study the model with elastic demands and develop some results in this context theoretical features and numerical procedures. For example, in the general economic case, the equilibrium cost will affect to the market demand of goods, so the O-D pair demand of these goods depends on the equilibrium cost and the equilibrium distribution. Therefore, it is reasonable to consider the traffic equilibrium problem with elastic demand when there are many kinds of goods transported through the same traffic network. At the same time, the travel cost function is considered widely and deeply. It is generally accepted that travelers consider a number of criteria (e.g., time, money, distance, safety, route complexity, etc.) when selecting routes. Presumably, these criteria are then combined in some manner to form a generalized cost for each particular route or path under consideration, and a route selected based on minimization of the generalized cost of the trip. Most commonly, it is assumed that travelers select the 'best' route based on either a single criterion, such as travel time, or several criteria using a linear (or additive) path cost function. However, as pointed out by Gabriel and Bernstein [6], there are many situations in which the linear path cost function is inadequate for addressing factors affecting a variety of transportation policies. Such factors include: (i) Nonlinear valuation of travel time–small amounts of time are valued proportionately less than larger amounts of time. (ii) Emissions fees–emissions of hydrocarbons and carbon monoxide are a nonlinear function of travel times. (iii) Path-specific tolls and fares–most existing fare and toll pricing structures are not directly proportional to either travel time or distance. These, and other such factors, are generally difficult to accommodate without explicitly using path flows in the formulation and solution, particularly for traffic equilibrium problems involving multi-dimensional nonlinear path costs. Despite the obvious usefulness of incorporating multiple criteria and relaxing the assumption of linear path costs for an important class of traffic equilibrium problems, there have been relatively few attempts to incorporate multiple criteria within route choice modeling. Under the assumption that the nonlinear path cost function is known a priori, Scott and Bernstein [7] solved a constrained shortest path problem (CSPP) to generate a set of Pareto optimal paths and then identify the best path by evaluating the cost values of the alternative paths. In a later study, Scott and Bernstein [8] embedded the CSPP into the gradient projection method to solve the non-additive

traffic equilibrium problem. Using a new gap function recently proposed by Facchinei and Soares [9], Lo and Chen [10] reformulated the nonadditive traffic equilibrium problem as an equivalent unconstrained optimization and solved a special case involving fixed demand and route-specific costs. Chen et al. [11] provided a projection and contraction algorithm for solving the elastic traffic equilibrium problem with route-specific costs. Recently, some formulations and properties of the non-additive traffic equilibrium models were also explored, such as the nonlinear time/money relation [12], the uniqueness and convexity of the bicriteria traffic equilibrium problem [13]. Furthermore, Altman and Wynter [14] discussed the non-additive cost structures in both transportation and telecommunication networks. Howerver, there are few results to discuss the problem related to the transportation network system for the nonlinear multiciteria transportation cost functions. On the other hand, Verma [15] investigated the approximation solvability of a new system of nonlinear variational inequalities involving strongly monotone mappings. In 2005, Bnouhachem [16] presented a new self-adaptive method for solving general mixed variational inequalities. In 2007, Shi [17] proposed a new self-adaptive iterative method for solving nonlinear variational ineuality system (SNVI) and proved the convergence of the proposed method. The numerical examples were given to illustrate the efficiency of the proposed method. In this paper, we consider the traffic equilibrium problem with variable demand, fixed tolls, and a nonlinear path cost function. We first discuss the multicriteria traffic equilibrium problem and its equivalent nonlinear variational inequality formulation, and present the associated multicriteria shortest path problem and solution algorithm. We then explore a new self-adaptive iterative method (SI) developed by Shi [17] to solve SNVI that characterizes this class of traffic equilibrium system problem. The SI method is simple and can handle a general monotone mapping. Unlike the non-smooth equations/sequential quadratic programming (NE/SQP) method proposed by Gabriel and Bernstein [6] to solve the non-additive traffic equilibrium problem, the SI method does not require the mapping to be differentiable.

2 Preliminaries

Without loss of generality, we consider the case that there are only two kinds of goods transported through the network. Suppose that a traffic network consists of a set N of nodes, a set Ω of origin-destination (O/D) pairs, and a set R of routes. Each route $r \in R$ links one given origin-destination pair $\omega \in \Omega$. The set of all $r \in R$ which links the same origin-destination pair $\omega \in \Omega$ is denoted by $R(\omega)$. Assume that n is the number of the route in R and m is the number of origin-destination (O/D) pairs in Ω. Let vector $H^i = (H_1^i, H_2^i, \cdots, H_r^i, \cdots, H_n^i)^T \in R^n$ $i = 1, 2$ denote the flow vector for the two kinds of goods, where $H_r^i, r \in R$, denotes the flow in route $r \in R$. A feasible flow has to satisfy the capacity restriction principle: $\lambda_r^i \leq H_r^i \leq \mu_r^i$, for all $r \in R$, and a traffic conservation law:$\sum_{r \in R(\omega)} H_r^i = \rho_\omega^i(H^1, H^2)$, for all $\omega \in \Omega$, where λ and μ are given in R^n, is the travel demand related to the given pair $\omega \in \Omega$, and $\rho_\omega^i(H^1, H^2) \geq 0$ denotes the travel demand vector, which generally depends on equilibrium cost and, essentially, on the equilibrium distribution H^1 and H^2. Thus the set of all feasible flows is given by

$$K_i(H^1, H^2) := \{H \in R^n \| \lambda^i \leq H \leq \mu^i, \Phi H = \rho^i(H^1, H^2)\}, \tag{2.1}$$

where $\Phi = (\delta_{\omega r})_m \times n$ is defined as

$$\delta_{\omega r} = \begin{cases} 1 & \text{if } r \in R(\omega) \\ 0 & \text{Otherwise} \end{cases}$$

Thus the set of feasible flows is given by $K_1(H^1, H^2) \times K_2(H^1, H^2)$. We call that is a flow of the traffic network system with elastic demands. As pointed out by Gabriel and Bernstein [6], the linear assumption is rather restrictive and cannot adequately model certain important applications.

Let mapping$C^i : K \to R^n$ be the cost function of the ith kinds of goods for $i = 1, 2$. $C_r^i(H^1, H^2)$ gives the marginal cost of transporting one additional unit of the ith kind of goods under the rth route. For the multicriteria traffic equilibrium problem with nonlinear path costs based on travel time, toll and transportation fares, a possible nonlinear path cost function can be the following form:

$$C_r^i(H^1, H^2) = g_r(\sum_{a \in A} \delta_{pa}^{rs} t_a^i(H^1, H^2)) + \sum_{a \in A} \delta_{pa}^{rs} \tau_a + \sum_{a \in A} \delta_{pa}^{rs} f_a^i(H^1, H^2), \tag{2.2}$$

2

Where g_r is a nonlinear function describing the value-of-time for path r, τ_a is the toll on link a, and f_a is the transportation fares function on link a.

Definition 2.1. $(H^1, H^2) \in K_1(H^1, H^2) \times K_2(H^1, H^2)$ is an equilibrium flow if and only if for all $\omega \in \Omega$ and $q, s, p, r \in R(\omega)$ there holds

$$
\begin{aligned}
C_q^1(H^1, H^2) < C_s^1(H^1, H^2) &\Rightarrow H_q^1 = \mu_q^1 \quad or \quad H_s^1 = \lambda_s^1, \\
C_p^2(H^1, H^2) < C_r^2(H^1, H^2) &\Rightarrow H_p^2 = \mu_p^2 \quad or \quad H_r^2 = \lambda_r^2,
\end{aligned}
\tag{2.3}
$$

3 Existence and Uniqueness of the solution for the multicriteria transportation equilibrium system problem

The following result establishes relationship between the system of dynamic traffic equilibrium problem and a system of variational inequalities.

Theorem 3.1. $(H^1, H^2) \in K_1(H^1, H^2) \times K_2(H^1, H^2)$ is an equilibrium flow if and only if ,

$$
\begin{aligned}
< C^1(H^1, H^2), F^1 - H^1 > &\geq 0 \quad \forall F^1 \in K_1(H^1, H^2), \\
< C^2(H^1, H^2), F^2 - H^2 > &\geq 0 \quad \forall F^2 \in K_2(H^1, H^2),
\end{aligned}
\tag{3.1}
$$

Proof. First assume that (3.1) holds and (2.3) does not hold. Then there exist $\omega \in \Omega$ and $q, s \in R(\omega)$ such that

$$
C_q^i(H^1, H^2) < C_s^i(H^1, H^2), \quad H_q^i < \mu_s^i, \quad H_q^i > \lambda_s^i, i = 1, 2.
\tag{3.2}
$$

Let $\delta_i = \min\{\mu_q^i - H_q^i, h_s^i - \lambda_s^i\}, i = 1, 2$.
Then $\delta_i > 0, i = 1, 2$.
We define a vector $F_i \in K_i(H^1, H^2), i = 1, 2$, whose components are

$$
F_q^i(t) = H_q^i + \delta_i, \quad F_s^i(t) = H_s^i - \delta_i, \quad F_r^i = H_r^i,
\tag{3.3}
$$

when $r \neq q, s$.
Thus,

$$
< C^i(H^1, H^2), F^i - H^i > = \sum_{j=1}^{n} C_j^i(H^1, H^2)(F_j^i - H_j^i) = \delta_i(C_q^i(H^1, H^2) - C_s^i(H^1, H^2)) < 0,
\tag{3.4}
$$

and so (3.1) is not satisfied. Therefore, it is proved that (3.1) implies (2.4).
Next, assume that (2.4) holds. That is

$$
C_q^i(H^1, H^2) < C_s^i(H^1, H^2) \Rightarrow H_q^i = \mu_q^i, \quad or \quad H_s^i = \mu_s^i, i = 1, 2.
\tag{3.5}
$$

Let $F^i \in K_i(H^1, H^2)$ for $i = 1, 2$. Then (3.1) holds from Definition 2.1. The proof is completed. Furthermore, we discuss the existence and uniqueness of the solution for the dynamic traffic equilibrium system (3.1). In order to get our main results, the following definitions will be employed.

Definition 3.2. $C^i(x, y)(i = 1, 2)$ is said to be θ-strictly monotone with respect to x on $K_1(H^1, H^2) \times K_2(H^1, H^2)$ if there exists $\theta > 0$ such that

$$
< C^i(x_1, y) - C^i(x_2, y), x_1 - x_2 > \geq \theta\|x_1 - x_2\|_2^2,
\tag{3.6}
$$

$\forall x_1, x_2 \in K_1(H^1, H^2), \forall y \in K_2(H^1, H^2)$.

3

Definition 3.3. $C^i(x, y)(i = 1, 2)$ is said to be L-Lipschitz continuous with respect to x on $K_1(H^1, H^2) \times K_2(H^1, H^2)$ if there exists $\theta > 0$such that

$$\|C^i(x_1, y) - C^i(x_2, y)\|_2 \leq L\|x_1 - x_2\|_2, \tag{3.7}$$

$\forall x_1, x_2 \in K_1(H^1, H^2), \forall y \in K_2(H^1, H^2)$.

Remark 3.4. Based on Definitions 3.2 and 3.3, we can similarly define the θ-strict monotonicity and L-Lipschitz continuity of $C^i(x, y)$ with respect to y on $K_1(H^1, H^2) \times K_2(H^1, H^2)$, for $i = 1, 2$.

Theorem 3.5. $(H^1, H^2) \in K_1(H^1, H^2) \times K_2(H^1, H^2)$ is an equilibrium flow if and only if there exist $\alpha > 0$ and $\beta > 0$ such that

$$\begin{aligned} H^1 &= P_{K_1}(H^2 - \alpha C^1(H^1, H^2)), \\ H^2 &= P_{K_2}(H^2 - \beta C^1(H^1, H^2)), \end{aligned} \tag{3.8}$$

where $P_{k_i} : R^n \to K_i(H^1, H^2)$ is a projection operator for $i = 1, 2$.
Proof. The proof is analogous to that of Theorem 5.2.4 of [18].
Let $\|(x, y)_1\|$ be the norm on space $K_1(H^1, H^2) \times K_2(H^1, H^2)$ defined as follows:

$$\|(x, y)\|_1 = \|x\|_2 + \|y\|_2, \forall x \in K_1(H^1, H^2), y \in K_2(H^1, H^2). \tag{3.9}$$

It is easy to see that $(K_1(H^1, H^2) \times K_2(H^1, H^2), \|.\|_1)$is a Banach space. Similar to Theorem 3.9 in He et. al. [1], one can easily obtain the following theorem, the proof is omitted.

Theorem 3.6. Suppose that $C^1(H^1, H^2)$ is θ_1-strictly monotone and L_{11}-Lipschitz continuous with respect to H^1, and L_{12}-Lipschitz continuous with respect to H^2 on $K_1(H^1, H^2) \times K_2(H^1, H^2)$. Suppose that $C^2(H^1, H^2)$ is L_{21}-Lipschitz continuous with respect to H^1, θ_2-strictly monotone, and L_{22}-Lipschitz continuous with respect to H^2 on $K_1(H^1, H^2) \times K_2(H^1, H^2)$. If there exist $\alpha > 0$ and $\beta > 0$ such that

$$\begin{aligned} \sqrt{1 - 2\gamma\theta_1 + \alpha^2 L_{11}^2} + \beta L_{21} &< 1, \\ \sqrt{1 - 2\eta\theta_2 + \beta^2 L_{22}^2} + \alpha L_{12} &< 1, \end{aligned} \tag{3.10}$$

then problem (3.1) admits unique solution.

Remark 3.7. If $f_j^1(H^1, H^2)$ is $\hat{\theta}_j^1$-strictly monotone with respect to $H^1$and $g_j^1 \circ \sum_{j=1}^n \delta_{pj}^{rs} t_j^1$ is $\overline{\theta}_j^1$-strictly monotone with respect to H^1, then

$$\theta_1 = \sum_{j=1}^n (\overline{\theta}_j^1 + \delta_{pj}^{rs} \hat{\theta}_j^1).$$

In fact,

$$< C_j^1(H^1, H^2) - C_j^2(\hat{H}^1, H^2), H^1 - \hat{H}^1 >$$

$$= < g_r(\sum_{j=1}^n \delta_{pj}^{rs} t_j^1(H^1, H^2)) + \sum_{j=1}^n \delta_{pj}^{rs} \tau_j + \sum_{j=1}^n \delta_{pj}^{rs} f_j^1(H^1, H^2) - g_r(\sum_{j=1}^n \delta_{pj}^{rs} t_j^1(\hat{H}^1, H^2)) \tag{3.11}$$

$$+ \sum_{j=1}^n \delta_{pj}^{rs} \tau_j + \sum_{j=1}^n \delta_{pj}^{rs} f_j^1(\hat{H}^1, H^2), H^1 - \hat{H}^1 >$$

$$\geq \sum_{j=1}^n (\overline{\theta}_j^1 + \delta_{pj}^{rs} \hat{\theta}_j^1)\|H^1 - \hat{H}^1\|_2^2$$

So,

$$< C_j(H^1, H^2) - C_j(\hat{H}^1, H^2), H^1 - \hat{H}^1 > \geq \theta_1 \|H^1 - \hat{H}^1\|.$$

4

4   Algorithms for solving the multicriteria transportation equilibrium system problem

Here, we describe an iterative algorithm with fixed step-sizes, and also describe a self-adaptive algorithm, which uses a self-adaptive stratedgy of step-size choice.

Algorithm 4.1 Iterative Method with fixed step-sizes
Step 1. Given $\epsilon > 0, \alpha, \beta \in [0, 1)$, and $(H_1^0, H_2^0) \in K_1(H^1, H^2) \times K_2(H^1, H^2)$, set $k = 0$.
Step 2. Get the next iterate:

$$H^{1,k+1} = P_{K_1}(H^{2,k} - \alpha C^1(H^{1,k}, H^{2,k}),$$

$$H^{2,k+1} = P_{K_2}(H^{2,k} - \beta C^1(H^{1,k}, H^{2,k}),$$

Step 3. Compute $r_1 = \|H_1^{(k+1)} - H_1^{(k)}\|, r_2 = \|H_2^{(k+1)} - H_2^{(k)}\|$, if $r_1, r_2 < \epsilon$, then stop; Otherwise, $k = k+1$, go to step 2.

Algorithm 4.2 SI method
Step 1. Given $\epsilon > 0, \gamma \in [1, 2), \mu \in (0, 1), \rho > 0, \delta \in (0, 1), \delta_0 \in (0, 1)$, and $\mu_0 \in H$, set $k = 0$.
Step 2. Set $\rho_k = \rho$, if $\|r^1(H^{1K}, \rho)\| < \epsilon$ and $\|r^1(H^{1K}, \rho)\| < \epsilon$, then stop; otherwise, find the smallest nonnegative integer m_k, such that $\rho_k = \rho \mu^{m_k}$ satisfying

$$\|\rho_k(C^1(H^{1k}, H^{2k}) - C^1(w^k, H^{2k})\| \leq \delta \|r(x^k, \rho_k)\|, \tag{4.1}$$

where $w^k = P_K[H^{1k} - \rho_k C^1(H^{1k}, H^{2k})]$.
Step 3. Compute

$$d(H^{1k}, \rho_k) = r(H^{1k}, \rho_k) - \rho_k C^2(H^{1k}, H^{2k}) + \rho_k C^2(P_K[H^{1k} - \rho_k C(H^{1k}, H^{2k})], H^{2k}), \tag{4.2}$$

where $r(H^{1k}, \rho) = H^{1k} - P_K[H^{1k} - \rho C^2(H^{1k}, H^{2k})]$.
Step 4. Get the next iterate:

$$H^{2k} = P_K[H^{1k} - \gamma d(H^{1k}, \rho_k) - \gamma C^2(H^{1k}, H^{2k})]; \tag{4.3}$$
$$H^{1,k+1} = P_K[H^{1k} - \rho C^1(H^{1k}, H^{2k})]$$

Step 5. If $\|\rho_k(C(H^{1k}, H^{2k}) - C(w^k, H^{2k})\| \leq \delta_0 \|r(x^k, \rho_k)\|$, then set $\rho = \rho_k/\mu$, else set $\rho = \rho_k$. Set $k = k + 1$, and go to Step 2.

Remark 4.2. Note that Algorithm 4.2 is obviously a modification of the standard procedure. In Algorithm 4.2, the searching direction is taken as $H^{1k} - \gamma d(H^{1k}, \rho_k) - \gamma C(H^{1k}, H^{2k})$, which is closely related to the projection residue, and differs from the standard procedure. In addition, the self-adaptive strategy of step-size choice is used. The numerical results show that these modifications can introduce computational efficiency substantially.

Theorem 4.3. Suppose that $C^1(H^1, H^2)$ is θ_1 -strictly monotone and L_{11}-Lipschitz continuous with respect to H^1, and L_{12}-Lipschitz continuous with respect to H_2 on $K_1(H^1, H^2) \times K_2(H^1, H^2)$. Suppose that $C^2(H^1, H^2)$ is L_{21}-Lipschitz continuous with respect to H^1, θ_2 -strictly monotone, and L_{22}-Lipschitz continuous with respect to H^2 on $K_1(H^1, H^2) \times K_2(H^1, H^2)$. Let $H^{1*}, H^{2*} \in K$ form a solution set for the SNVI (2.1) and let the sequences $\{H^{1k}\}$ and $\{H^{2k}\}$ be generated by Algorithm 4.2. If $0 < \overline{\theta} < \sqrt{1 - 2\rho\theta_1 + 2\rho^2 L_{12}^2}(1 + \gamma L_{21})/(1 - \gamma L_{22}) + \sqrt{2\rho L_{11}^2 + 2\rho L_{11}} < 1$, then the sequence $\{H^{1k}\}$ converges to H^{1*} and the sequence $\{H^{2k}\}$ converges to H^{2*}, for $0 < \rho < 2r/s^2$.
Proof. Since (H^{1*}, H^{2*}) is a solution of transportation equilibrium system (3.2), it follows from Theorem 3.5 that

$$H^{1*} = P_{K_1}[H^{2*} - \rho C^1(H^{1*}, H^{2*})], \tag{4.4}$$
$$H^{2*} = P_{K_2}[H^{1*} - \gamma C^2(H^{1*}, H^{2*})]$$

5

Applying Algorithm 4.2, we know

$$\|H^{1,k+1} - H^{1*}\| = \|P_{K_1}[H^{2k} - \rho C^1(H^{1k}, H^{2k})] - P_{K_1}[H^{2*} - \rho C^1(H^{1*}, H^{2*})]$$

$$\leq \|H^{2k} - H^{2*} - \rho C^1(H^{1k}, H^{2k}) + \rho C^1(H^{1*}, H^{2*})\|$$

Since T is r-strongly monotone and s-Lipschitz continuous, we know

$$\|H^{2k} - H^{2*} - \rho C^1(H^{1*}, H^{2*})\|^2$$

$$\leq \|H^{2k} - H^{2*}\|^2 - 2\rho < C^1(H^{1k} - H^{2k} - C^1(H^{1*}, H^{2*}, H^{2k} - H^{2*} > +\rho^2 \|C^1(H^{1k}, H^{2k}) - C^1(H^{1*}, H^{2*})\|$$

$$\leq \|H^{2k} - H^{2*}\|^2 - 2\rho\theta_1\|H^{2k} - H^{2*}\|^2 + 2\rho L_{11}\|H^{1k} - H^{1*}\|^2 + \rho^2 \|C^1(H^{1k}, H^{2k}) - C^1(H^{1*}, H^{2*})\|$$

$$\leq \|H^{2k} - H^{2*}\|^2 - 2\rho\theta_1\|H^{2k} - H^{2*}\|^2 + 2\rho L_{11}\|H^{1k} - H^{1*}\|^2 + 2\rho^2 L_{11}^2\|H^{1k} - H^{1*}\|^2 + 2\rho^2 L_{12}^2\|H^{2k} - H^{2*}\|^2$$

$$\leq (1 - 2\rho\theta_1 + 2\rho^2 L_{12}^2)\|H^{2k} - H^{2*}\|^2 + (2\rho^2 L_{11}^2 + 2\rho L_{11})\|H^{1k} - H^{1*}\|^2$$

It follows that

$$\|H^{1,k+1} - H^{1*}\| \leq \sqrt{1 - 2\rho\theta_1 + 2\rho^2 L_{12}^2}\|H^{2k} - H^{2*}\| + \sqrt{2\rho^2 L_{11}^2 + 2\rho L_{11}}\|H^{1k} - H^{1*}\|. \qquad (4.5)$$

Next, we consider

$$\|H^{2k} - H^{2*}\| = \|P_{K_2}[H^{1k} - \gamma d(H^{1k}, \rho_k) - \gamma C^2(H^{1k}, H^{2k})] - P_{K_2}[H^{1*} - \gamma C^2(H^{1*}, H^{2*})]\| \qquad (4.6)$$

$$\leq \|H^{1k} - \gamma d(H^{1k}, \rho_k) - \gamma C^2(H^{1k}, H^{2k}) - H^{1*} + \gamma C^2(H^{1*}, H^{2*})\|$$

$$\leq \|H^{1k} - \gamma d(H^{1k}, \rho_k) - H^{1*}\| + \gamma\|C^2(H^{1k}, H^{2k}) - C^2(H^{1*}, H^{2*})\|$$

where we use the property of the operator P_K. Now, we consider

$$\|H^{1k} - H^{1*} - \gamma d(H^{1k}, \rho_k)\|^2 \qquad (4.7)$$

$$\leq \|H^{1k} - H^{1*}\|^2 - 2\gamma < H^{1k} - H^{1*}, d(H^{1k}, \rho_k) > +\gamma^2\|d(H^{1k}, \rho_k)\|^2$$

$$\leq \|H^{1k} - H^{1*}\|^2,$$

where we use the definition of $d(H^{2k}, \rho_k)$.
It follows that

$$\|H^{2,k} - H^{2*}\| \leq (1 + \gamma L_{21})\|H^{1k} - H^{1*}\| + \gamma L_{22}\|H^{2k} - H^{2*}\|. \qquad (4.8)$$

From (4.5) to (4.8), we know

$$\|H^{1,k+1} - H^{1*}\| \leq (\sqrt{1 - 2\rho\theta_1 + 2\rho^2 L_{12}^2}(1 + \gamma L_{21})/(1 - \gamma L_{22}) + \sqrt{2\rho L_{11}^2 + 2\rho L_{11}})\|H^{1k} - H^{1*}\|. \qquad (4.9)$$

Since $0 < \overline{\theta} < 1$, from (4.9), we know $H^{1k} \to H^{1*}$. Thus from (4.8), we know $H^{2k} \to H^{2*}$.

5   Numerical results

In this section, we presented some numerical results for the proposed method. we consider a simple traffic network consisting of two nodes, only a origin-destination (O/D) pair, and a set R of routes. Each route $r \in R$ links the origin-destination pair in parallel. Assume that n is the number of the route in R. Let $C^1(H_1, H_2) = DH_1(t) + c_1^T H_2(t)$, $C^2(H_1, H_2) = DH_1(t) + c_2^T H_2(t)$, where

$$D = \begin{bmatrix} 4 & -2 & \cdots & \cdots \\ 1 & 4 & \cdots & \cdots \\ \cdots & \cdots & 4 & -2 \\ \cdots & \cdots & 1 & 4 \end{bmatrix},$$

$c_1 = (-1, -1, \cdots, -1)^T$, $c_2 = (1, 1, \cdots, 1)^T$, $H_1(t) = H_1 \in R^n$, $H_1(t) = H_2 \in R^n$. let

$$K_1(H_1, H_2) = \{H_1 | H_1 \in [l, u], H_1^i + H_2^i \leq 2000, i = 1, 2, \cdots, n\},$$

6

$$K_2(H_1, H_2) = \{H_1 | H_1 \in [l, u], H_1^i + H_2^i \leq 2000, i = 1, 2, \cdots, n\}.$$

where $l = (0, 0, \cdots, 0)^T$, $u = (1000, 1000, \cdots, 1000)^T$. The calculations are started with vectors $H_1 = (0, 0, \cdots, 0)^T$, $H_2 = (5, 5, \cdots, 5)^T$ and stopped whenever $r_1, r_2 < 10^{-5}$. Table 1 gives the numerical results of Algorithms 4. 1. Table 2 gives the numerical results of Algorithms 4. 2.

Comparing Table 2 and Table 1, it show that Algorithm 4.2 is very effective for the problem tested. In addition, it seems that the computational time and the iteration numbers are not very sensitive to the problem size.

Table 1: Computation performance with different scales by Algorithm 4.1

n	Iteration	CPU(s)
50	366	29.5469
100	183	28.5469
200	93	27.2813
300	63	30.4375

Table 2: Computation performance with different scales by Algorithm 4.2

n	Iteration	CPU(s)
50	220	17.7282
100	110	16.1281
200	56	15.3687
300	38	18.2625

References

[1] Y. P. He, J. P. Xu, N. J. Huang, and M. Wu, Dynamic traffic network equilibrium system, Fixed Point Theory and Application, vol. 2010, Art. 873025.

[2] S. C. Dafermos, The general multinodal network equilibrium problem with elastic demand, Networks, vol. 12, pp. 57-72, 1982.

[3] M. Fukushima, On the dual approach to the traffic assignment problem, Transportation Res. Part B, vol. 18, pp. 235-245, 1984.

[4] M. Fukushima and T. Itoh, A dual approach to asymmetric traffic equilibrium problems, Math. Japon. vol. 32 , pp. 701-721, 1987.

[5] A. Maugeri, Stability results for variational inequalities and applications to traffic equilibrium problem, Supplemento Rendiconti Circolo Matematico di Palermo vol. 8, pp. 269-280, 1985.

[6] S. Gabriel, D. Bernstein, The traffic equilibrium problem with nonadditive path costs, Transportation Science, Vol 31, pp. 337-348, 1997.

[7] K. Scott, D. Bernstein, Solving a best path problem when the value of time function is nonlinear, in: The 77th Annual Meeting of Transportation Research Board, 1998.

[8] K. Scott, D. Bernstein, Solving a traffic equilibrium problem when paths are not additive, in: The 78th Annual Meeting of Transportation Research Board, 1999.

[9] F. Facchinei, J. Soares, A new merit function for nonlinear complementarity problems and a related algorithm, SIAM Journal of Optimization, Vol. 7, pp. 225-247 , 1997.

[10] H.K. Lo, A. Chen, Traffic equilibrium problem with route-specific costs: formulation and algorithms, Transportation Research Part B, Vol. 34 , pp. 493-513 , 2000.

7

[11] A. Chen, H.K. Lo, H. Yang, A self-adaptive projection and contraction algorithm for the traffic assignment problem with path-specific costs, European Journal of Operational Research, Vol. 135, pp. 27-41 , 2001.

[12] T. Larsson, P.O. Lindberg, J. Lundgren, M. Patriksson, C. Rydergren, On traffic equilibrium models with a nonlinear time/money relation, in: M. Patriksson, M. Labbe (Eds.), Transportation Planning: State of the Art, Kluwer Academic Publishers, 2002.

[13] D.H. Bernstein, L. Wynter, Issues of uniqueness and convexity in non-additive bi-criteria equilibrium models, in: Proceedings of the EURO Working Group on Transportation, Rome, Italy, 2000.

[14] E. Altman, L. Wynter, Equilibrium, games, and pricing in transportation and telecommunication networks, Networks and Spatial Economics , Vol. 4, pp. 7-21, 2004.

[15] R. U. Verma, Projection methods, algorithms, and a new system of nonlinear variational inequalities, Computers and Mathematics with Applications, vol. 41, no. 7-8, pp. 1025-1031, 2001.

[16] A. Bnouhachem, A self-adaptive method for solving general mixed variational inequalities, Journal of Mathematical Analysis and Applications, vol. 309, no. 1, pp. 136-150, 2005.

[17] C. F. Shi, A self-adaptive method for solving a system of nonlinear variational inequalities, Mathematical Problems in Engineering, Art. 23795, 2007.

[18] P. Daniele, Dynamic Networks and Evolutionary Variational Inequalities, New Dimensions in Networks, Edward Elgar, Cheltenham, UK, 2006.

Chaofeng Shi
Department of Financial and Economics
Chongqing Jiaotong University
Chongqing, 400074, P. R. China
and
Department of Economics
University of California
Riverside, CA 92507, USA.
Correspondence should be addressed to Chaofeng Shi, E-mail: shichf@163.com

Yingrui Wang
Department of Financial and Economics
Chongqing Jiaotong University
Chongqing, 400074, P. R. China

8

BARNES' MULTIPLE FROBENIUS-EULER AND HERMITE MIXED-TYPE POLYNOMIALS

DAE SAN KIM, DMITRY V. DOLGY, AND TAEKYUN KIM

ABSTRACT. In this paper, we consider the Barnes' multiple Frobenius-Euler and Hermite mixed-type polynomials. Using the umbral calculus, we derive several explicit formulas and recurrence relations for these polynomials. Also, we establish connections between our polynomials and several known families of polynomials.

1. INTRODUCTION

For $\lambda \neq 1$, $s \in \mathbb{N}$, the Frobenius-Euler polynomials of order s are defined by the generating function

$$(1.1) \qquad \left(\frac{1 - \lambda}{e^t - \lambda} \right)^s e^{xt} = \sum_{n=0}^{\infty} \mathbb{H}_n^{(s)} (x \mid \lambda) \frac{t^n}{n!}, \quad (\text{see } [7, 12, 19]).$$

Let $a_1, a_2, \ldots, a_r, \lambda_1, \lambda_2, \ldots, \lambda_r \in \mathbb{C}$ with $a_1, \ldots, a_r \neq 0$, $\lambda_1, \ldots, \lambda_r \neq 1$. Then the Barnes' multiple Frobenius-Euler polynomials $H_n (x \mid a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r)$ are given by the generating funciton

$$(1.2)$$
$$\prod_{j=1}^{r} \left(\frac{1 - \lambda_j}{e^{a_j t} - \lambda_j} \right) e^{xt} = \sum_{n=0}^{\infty} H_n (x \mid a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r) \frac{t^n}{n!}, \quad (\text{see } [13, 15]).$$

When $x = 0$, $H_n (a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r) = H_n (0 \mid a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r)$ are called the Barnes' multiple Frobenius-Euler numbers (see [13]).

For $a_1 = a_2 = \cdots = a_r = 1$, $\lambda_1 = \lambda_2 = \cdots = \lambda_r = \lambda$, we have $H_n(x \mid \underbrace{1, 1, \ldots, 1}_{r-\text{times}}; \underbrace{\lambda, \lambda, \ldots, \lambda}_{r-\text{times}}) = \mathbb{H}_n^{(r)} (x \mid \lambda)$. When $x = 0$, $\mathbb{H}_n^{(r)} (\lambda) = \mathbb{H}_n^{(r)} (0 \mid \lambda)$ are called the Frobenius-Euler numbers of order r.

The Hermite polynomials $H_n^{(\nu)} (x)$ of variance ν $(0 \neq \nu \in \mathbb{R})$ are given by the generating function

$$(1.3) \qquad e^{-\nu t^2/2} e^{xt} = \sum_{n=0}^{\infty} H_n^{(\nu)} (x) \frac{t^n}{n!}, \quad (\text{see } [24]).$$

When $x = 0$, $H_n^{(\nu)} = H_n^{(\nu)} (0)$ are called the Hermite numbers of variance ν. It is well known that the Bernoulli polynomials of order r $(\in \mathbb{N})$ are defined by the generating function

---

2010 *Mathematics Subject Classification.* 05A19, 05A40, 11B75, 11B83.

Key words and phrases. Barnes' multiple Frobenius-Euler and Hermite mixed-type polynomials, umbral calculus.

1

2   DAE SAN KIM, DMITRY V. DOLGY, AND TAEKYUN KIM

$$(1.4) \qquad \left(\frac{t}{e^t - 1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x) \frac{t^n}{n!}, \quad (\text{see } [1\text{--}24, 26]).$$

When $x = 0$, $B_n^{(r)} = B_n^{(r)}(0)$ are called the Bernoulli numbers of order r. For $n \geq 0$, the Stirling numbers of the first kind are given by

$$(1.5) \qquad (x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l) x^l, \quad (\text{see } [24]).$$

The Stirling numbers of the second kind are defined by the generating function

$$(1.6) \qquad (e^t - 1)^n = n! \sum_{l=n}^{\infty} S_2(l,n) \frac{x^l}{l!}, \quad (\text{see } [24]).$$

Let $\mathbb{C}$ be the complex field and let $\mathcal{F}$ be the set of all formal power series in the variable t:

$$(1.7) \qquad \mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \,\middle|\, a_k \in \mathbb{C} \right\}.$$

Let $\mathbb{P} = \mathbb{C}[x]$ and let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. We use the notation $\langle L \mid p(x) \rangle$ to denote the action of the linear functional L on the polynomial $p(x)$, and we recall that the vector space operations on $\mathbb{P}^*$ are defined by $\langle L + M \mid p(x) \rangle = \langle L \mid p(x) \rangle + \langle M \mid p(x) \rangle$, and $\langle cL \mid p(x) \rangle = c \langle L \mid p(x) \rangle$, where c is a complex constant in $\mathbb{C}$. The linear functional $\langle f(t) \mid \cdot \rangle$ on $\mathbb{P}$ is defined by

$$(1.8) \qquad \langle f(t) \mid x^n \rangle = a_n, \quad (n \geq 0), \quad \text{where } f(t) \in \mathcal{F}, \quad (\text{see } [17, 21, 24]).$$

By (1.8), we easily get

$$(1.9) \qquad \langle t^k \mid x^n \rangle = n! \delta_{n,k}, \quad (n, k \geq 0), \quad (\text{see } [8, 21, 24]),$$

where $\delta_{n,k}$ is the Kronecker's symbol.

Let $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L \mid x^k \rangle}{k!} t^k$. Then, by (1.9), we get $\langle f_L(t) \mid x^n \rangle = \langle L \mid x^n \rangle$. So, the map $L \mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ denotes both the algebra of formal power series in t and the vector space of all linear functionals on $\mathbb{P}$, and so an element $f(t)$ of $\mathcal{F}$ will be thought of as both a formal power series and a linear functional. We call $\mathcal{F}$ the umbral algebra and the umbral calculus is the study of umbral algebra. The order $o(f(t))$ of a power series $f(t) \neq 0$ is the smallest integer k for which the coefficient of t^k does not vanish. If the order of $f(t)$ is 1, then $f(t)$ is called a delta series; if the order $g(t)$ is 0, then $g(t)$ is called an invertible series. Let $f(t), g(t) \in \mathcal{F}$ with $o(f(t)) = 1$ and $o(g(t)) = 0$. Then there exists a unique sequence $s_n(x)$ ($\deg s_n(x) = n$) such that $\left\langle g(t) f(t)^k \mid s_n(x) \right\rangle = n! \delta_{n,k}$ for $n, k \geq 0$. Such a sequence $s_n(x)$ is called the Sheffer sequence for $(g(t), f(t))$ which is denoted by $s_n(x) \sim (g(t), f(t))$ (see [21, 24]). In particular, if $s_n(x) \sim (g(t), t)$, then $s_n(x)$ is called an Appell sequence for $g(t)$. For $f(t), g(t) \in \mathcal{F}$, we have
(1.10)
$$\langle f(t) g(t) \mid p(x) \rangle = \langle f(t) \mid g(t) p(x) \rangle = \langle g(t) \mid f(t) p(x) \rangle = \langle 1 \mid f(t) g(t) p(x) \rangle,$$

$$(1.11) \qquad f(t) = \sum_{k=0}^{\infty} \left\langle f(t) \mid x^k \right\rangle \frac{t^k}{k!}, \quad p(x) = \sum_{k=0}^{\infty} \left\langle t^k \mid p(x) \right\rangle \frac{x^k}{k!}, \quad (\text{see } [24]).$$

Thus, by (1.11), we get
(1.12)
$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k}, \quad e^{yt} p(x) = p(x+y), \quad \text{and} \quad \langle e^{yt} | p(x) \rangle = p(y).$$

The sequence $s_n(x)$ is Sheffer for $(g(t), f(t))$ if and only if

(1.13)
$$\frac{1}{g(\overline{f}(t))} e^{y\overline{f}(t)} = \sum_{k=0}^{\infty} \frac{s_k(x)}{k!} t^k, \quad (y \in \mathbb{C}), \quad (\text{see } [17, 21, 24]),$$

where $\overline{f}(t)$ is the compositional inverse of $f(t)$ with $\overline{f}(f(t)) = f(\overline{f}(t)) = t$. It is well known that the Sheffer identity is given by
(1.14)
$$s_n(x+y) = \sum_{j=0}^{\infty} \binom{n}{j} s_j(x) p_{n-j}(y), \quad \text{where } p_n(x) = g(t) s_n(x), \quad (\text{see } [17, 24]).$$

For $s_n(x) \sim (g(t), f(t))$, we have

(1.15)
$$s_{n+1}(x) = \left(x - \frac{g'(x)}{g(x)} \right) \frac{1}{f'(x)} s_n(x), \quad (n \geq 0),$$

(1.16)
$$s_n(x) = \sum_{j=0}^{n} \frac{1}{j!} \left\langle g(\overline{f}(t))^{-1} \overline{f}(t)^j \middle| x^n \right\rangle x^j,$$

and

(1.17)
$$\langle f(t) | x p(x) \rangle = \langle \partial_t f(t) | p(x) \rangle, \quad f(t) s_n(x) = n s_{n-1}(x), \quad (n \geq 1).$$

Let $s_n(x) \sim (g(t), f(t))$ and $r_n(x) \sim (h(t), l(t))$, $(n \geq 0)$. Then we have

(1.18)
$$s_n(x) = \sum_{m=0}^{n} C_{n,m} r_m(x), \quad (n \geq 0),$$

where

(1.19)
$$C_{n,m} = \frac{1}{m!} \left\langle \frac{h(\overline{f}(t))}{g(\overline{f}(t))} l(\overline{f}(t))^m \middle| x^n \right\rangle, \quad (\text{see } [17, 21, 24]).$$

In this paper, we consider the polynomials $FH_n^{(\nu)}(x \mid a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r)$ whose generating function is given by

(1.20)
$$\prod_{j=1}^{r} \left(\frac{1 - \lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} e^{xt} = \prod_{j=1}^{r} \left(\frac{1 - \lambda_j}{e^{a_j t} - \lambda_j} \right) e^{xt - \nu t^2/2}$$

$$= \sum_{n=0}^{\infty} FH_n^{(\nu)}(x \mid a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r) \frac{t^n}{n!},$$

where $r \in \mathbb{Z}_{>0}$, $a_1, \ldots, a_r, \lambda_1, \ldots, \lambda_r \in \mathbb{C}$ with $a_1, \ldots, a_r \neq 0$, $\lambda_1, \ldots, \lambda_r \neq 1$, and $\nu \in \mathbb{R}$ with $\nu \neq 0$. $FH_n^{(\nu)}(x \mid a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r)$ are called Barnes' multiple Frobenius-Euler and Hermite mixed-type polynomials.

When $x = 0$, $FH_n^{(\nu)}(a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r) = FH_n^{(\nu)}(0 \mid a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r)$ are called the Barnes' multi[ple Frobenius-Euler and Hermite mixed-type numbers. We observe here that $FH_n^{(\nu)}(x \mid a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r)$, $H_n(x \mid a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r)$,

and $H_n^{(\nu)}(x)$ are respectively Appell sequences for $\prod_{j=1}^{r}\left(\frac{e^{a_j t}-\lambda_j}{1-\lambda_j}\right)e^{\nu t^2/2}$, $\prod_{j=1}^{r}\left(\frac{e^{a_j t^n}-\lambda_j}{1-\lambda_j}\right)$, and $e^{\nu t^2/2}$. That is,

$$(1.21) \qquad FH_n^{(\nu)}\left(x \mid a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r\right) \sim \left(\prod_{j=1}^{r}\left(\frac{e^{a_j t}-\lambda_j}{1-\lambda_j}\right)e^{\nu t^2/2}, t\right),$$

$$(1.22) \qquad H_n\left(x \mid a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r\right) \sim \left(\prod_{j=1}^{r}\left(\frac{e^{a_j t}-\lambda_j}{1-\lambda_j}\right), t\right),$$

and

$$(1.23) \qquad H_n^{(\nu)}(x) \sim \left(e^{\nu t^2/2}, t\right).$$

From the Barnes' multiple Frobenius-Euler and Hermite mixed-type polynomials, we investigate some properties of those polynomials. Finally, we give some new and interesting identities which are derived from umbral calculus.

2. Barnes' multiple Frobenius-Euler and Hermite mixed-type polynomials

From (1.21), (1.22) and (1.23), we note that

$$(2.1) \quad tFH_n^{(\nu)}\left(x \mid a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r\right) = \frac{d}{dx}FH_n^{(\nu)}\left(x \mid a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r\right)$$
$$= nFH_{n-1}^{(\nu)}\left(x \mid a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r\right),$$

$$(2.2) \qquad tH_n\left(x \mid a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r\right) = \frac{d}{dx}H_n\left(x \mid a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r\right)$$
$$= nH_{n-1}\left(x \mid a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r\right),$$

and

$$(2.3) \qquad tH_n^{(\nu)}(x) = \frac{d}{dx}H_n^{(\nu)}(x) = nH_{n-1}^{(\nu)}(x).$$

Now, we give explicit expressions related to the Barnes' multiple Frobenius-Euler and Hermite mixed-type polynomials.

From (1.13), we note that

$$(2.4) \qquad FH_n^{(\nu)}\left(x \mid a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r\right) = e^{-\nu t^2/2}\prod_{j=1}^{r}\left(\frac{1-\lambda_j}{e^{a_j t}-\lambda_j}\right)x^n$$

$$= e^{-\nu t^2/2}H_n\left(x \mid a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r\right)$$

$$= \sum_{m=0}^{\infty}\frac{1}{m!}\left(-\frac{\nu}{2}\right)^m t^{2m}H_n\left(x \mid a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r\right)$$

$$= \sum_{m=0}^{\left[\frac{n}{2}\right]}\frac{1}{m!}\left(-\frac{\nu}{2}\right)^m (n)_{2m}H_{n-2m}\left(x \mid a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r\right)$$

$$= \sum_{m=0}^{\left[\frac{n}{2}\right]}\binom{n}{2m}\frac{(2m)!}{m!}\left(-\frac{\nu}{2}\right)^m H_{n-2m}\left(x \mid a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r\right).$$

By (1.9), we get

(2.5)
$$FH_n^{(\nu)}(y \mid a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r)$$

$$= \left\langle \sum_{i=0}^{\infty} FH_i^{(\nu)}(y \mid a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r) \frac{t^i}{i!} \middle| x^n \right\rangle$$

$$= \left\langle \prod_{j=1}^{r} \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} e^{yt} \middle| x^n \right\rangle$$

$$= \left\langle \prod_{j=1}^{r} \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) \middle| e^{-\nu t^2/2} e^{yt} x^n \right\rangle$$

$$= \left\langle \prod_{j=1}^{r} \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) \middle| \sum_{l=0}^{\infty} H_l^{(\nu)}(y) \frac{t^l}{l!} x^n \right\rangle$$

$$= \sum_{l=0}^{n} \binom{n}{l} H_l^{(\nu)}(y) \left\langle \prod_{j=1}^{r} \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) \middle| x^{n-l} \right\rangle$$

$$= \sum_{l=0}^{n} \binom{n}{l} H_l^{(\nu)}(y) \left\langle \sum_{i=0}^{\infty} H_i(a_1, \ldots, a_r; \lambda_1, \ldots \lambda_r) \frac{t^i}{i!} \middle| x^{n-l} \right\rangle$$

$$= \sum_{l=0}^{n} \binom{n}{l} H_{n-l}(a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r) H_l^{(\nu)}(y).$$

Thus, by (2.5), we get
(2.6)
$$FH_n^{(\nu)}(x \mid a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r) = \sum_{l=0}^{n} \binom{n}{l} H_{n-l}(a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r) H_l^{(\nu)}(x).$$

Therefore, by (2.4) and (2.6), we obtain the following theorem.

Theorem 2.1. *For $n \geq 0$, we have*

$$FH_n^{(\nu)}(x \mid a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r)$$

$$= \sum_{m=0}^{\left[\frac{n}{2}\right]} \binom{n}{2m} \frac{(2m)!}{m!} \left(-\frac{\nu}{2}\right)^m H_{n-2m}(x \mid a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r)$$

$$= \sum_{l=0}^{n} \binom{n}{l} H_{n-l}(a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r) H_l^{(\nu)}(x).$$

From (1.9), we have

(2.7)
$$FH_n^{(\nu)}(y \mid a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r)$$

$$= \left\langle \sum_{i=0}^{\infty} FH_i^{(\nu)}(y \mid a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r) \frac{t^i}{i!} \middle| x^n \right\rangle$$

$$= \left\langle \prod_{j=1}^{r} \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} e^{yt} \middle| x^n \right\rangle$$

6   DAE SAN KIM, DMITRY V. DOLGY, AND TAEKYUN KIM

$$= \left\langle e^{-\nu t^2/2} \left| \prod_{j=1}^{r} \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) e^{yt} x^n \right. \right\rangle$$

$$= \left\langle e^{-\nu t^2/2} \left| \sum_{l=0}^{\infty} H_l \left(y \mid a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r \right) \frac{t^l}{l!} x^n \right. \right\rangle$$

$$= \sum_{l=0}^{n} \binom{n}{l} H_l \left(y \mid a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r \right) \left\langle \sum_{i=0}^{\infty} H_i^{(\nu)} \frac{t^i}{i!} \left| x^{n-l} \right. \right\rangle$$

$$= \sum_{l=0}^{n} \binom{n}{l} H_l \left(y \mid a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r \right) H_{n-l}^{(\nu)}.$$

Thus, by (2.7), we get

(2.8)
$$FH_n^{(\nu)} \left(x \mid a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r \right) = \sum_{l=0}^{n} \binom{n}{l} H_l \left(x \mid a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r \right) H_{n-l}^{(\nu)}.$$

Now, we will use the conjugation representation in (1.16). For $FH_n^{(\nu)} \left(x \mid a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r \right) \sim \left(g(t) = \prod_{j=1}^{r} \left(\frac{e^{a_j t} - \lambda_j}{1 - \lambda_j} \right) e^{\nu t^2/2}, f(t) = t \right)$, we observe that

(2.9) $\left\langle g \left(\overline{f}(t) \right)^{-1} \overline{f}(t)^j \left| x^n \right. \right\rangle$

$$= \left\langle \prod_{j=1}^{r} \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} t^j \left| x^n \right. \right\rangle$$

$$= \left\langle \prod_{j=1}^{r} \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} \left| t^j x^n \right. \right\rangle$$

$$= (n)_j \left\langle \prod_{j=1}^{r} \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} \left| x^{n-j} \right. \right\rangle$$

$$= (n)_j \left\langle e^{-\nu t^2/2} \left| H_{n-j} \left(x \mid a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r \right) \right. \right\rangle$$

$$= (n)_j \left\langle \sum_{m=0}^{\infty} \frac{1}{m!} \left(-\frac{\nu}{2} \right)^m t^{2m} \left| H_{n-j} \left(x \mid a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r \right) \right. \right\rangle$$

$$= (n)_j \sum_{m=0}^{\left[\frac{n-j}{2} \right]} \frac{1}{m!} \left(-\frac{\nu}{2} \right)^m (n-j)_{2m} H_{n-j-2m} \left(a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r \right).$$

From (1.16) and (2.9), we can derive the following equation:

(2.10) $FH_n^{(\nu)} \left(x \mid a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r \right)$

$$= \sum_{j=0}^{n} \binom{n}{j} \sum_{m=0}^{\left[\frac{n-j}{2} \right]} \frac{1}{m!} \left(-\frac{\nu}{2} \right)^m (n-j)_{2m} H_{n-j-2m} \left(a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r \right) x^j.$$

Therefore, by (2.8) and (2.10), we obtain the following theorem.

Theorem 2.2. *For $n \geq 0$, we have*

$$FH_n^{(\nu)}(x \mid a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r)$$

$$= \sum_{l=0}^{n} \binom{n}{l} H_{n-l}^{(\nu)} H_l(x \mid a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r)$$

$$= \sum_{j=0}^{n} \binom{n}{j} \sum_{m=0}^{\left[\frac{n-j}{2}\right]} \frac{1}{m!} \left(-\frac{\nu}{2}\right)^m (n-j)_{2m} H_{n-j-2m}(a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r) x^j.$$

Remark. From (1.14), we have

(2.11) $$FH_n^{(\nu)}(x + y \mid a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r)$$

$$= \sum_{j=0}^{n} \binom{n}{j} FH_j^{(\nu)}(x \mid a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r) y^{n-j}.$$

By (1.15) and (1.21), we get

(2.12) $$FH_{n+1}^{(\nu)}(x \mid a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r)$$

$$= \left(x - \frac{g'(t)}{g(t)}\right) FH_n^{(\nu)}(x \mid a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r),$$

where $g(t) = \prod_{j=1}^{r} \left(\frac{e^{a_j t} - \lambda_j}{1 - \lambda_j}\right) e^{\nu t^2/2}$.

Now, we compute that

(2.13) $$\frac{g'(t)}{g(t)} = (\log g(t))'$$

$$= \left(\sum_{j=1}^{r} \log(e^{a_j t} - \lambda_j) - \sum_{j=1}^{r} \log(1 - \lambda_j) + \frac{1}{2}\nu t^2\right)'$$

$$= \sum_{j=1}^{r} \frac{a_j e^{a_j t}}{e^{a_j t} - \lambda_j} + \nu t.$$

So

(2.14) $$\frac{g'(t)}{g(t)} FH_n^{(\nu)}(x \mid a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r)$$

$$= \sum_{j=1}^{r} \frac{a_j e^{a_j t}}{1 - \lambda_j} \cdot \frac{1 - \lambda_j}{e^{a_j t} - \lambda_j} \prod_{i=1}^{r} \left(\frac{1 - \lambda_i}{e^{a_i t} - \lambda_i}\right) e^{-\nu t^2/2} x^n$$

$$+ \nu t FH_n^{(\nu)}(x \mid a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r)$$

$$= \sum_{j=1}^{r} \frac{a_j}{1 - \lambda_j} FH_n^{(\nu)}(x + a_j \mid a_1, \ldots, a_r, a_j; \lambda_1, \ldots, \lambda_r, \lambda_j)$$

$$+ n\nu FH_{n-1}^{(\nu)}(x \mid a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r)$$

$$= \sum_{j=1}^{r} \frac{a_j}{1 - \lambda_j} FH_n^{(\nu)}(x + a_j \mid a_1, \ldots, a_r, a_j; \lambda_1, \ldots, \lambda_r, \lambda_j)$$

$$+ n\nu FH_{n-1}^{(\nu)}(x \mid a_1, \ldots, a_r, a_j; \lambda_1, \ldots, \lambda_r).$$

By (2.12) and (2.14), we get

$$(2.15) \quad FH_{n+1}^{(\nu)}(x \mid a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r)$$

$$= x FH_n^{(\nu)}(x \mid a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r)$$

$$- \sum_{j=1}^{r} \frac{a_j}{1 - \lambda_j} FH_n^{(\nu)}(x + a_j \mid a_1, \ldots, a_r, a_j; \lambda_1, \ldots, \lambda_r, \lambda_j)$$

$$- n\nu FH_{n-1}^{(\nu)}(x \mid a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r).$$

For $n \geq 2$, by (1.9), we get

$$(2.16) \quad FH_n^{(\nu)}(y \mid a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r)$$

$$= \left\langle \sum_{i=0}^{\infty} FH_i^{(\nu)}(y \mid a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r) \frac{t^i}{i!} \middle| x^n \right\rangle$$

$$= \left\langle \prod_{j=1}^{r} \left(\frac{1 - \lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} e^{yt} \middle| x^n \right\rangle$$

$$= \left\langle \partial_t \left(\prod_{j=1}^{r} \left(\frac{1 - \lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} e^{yt} \right) \middle| x^{n-1} \right\rangle$$

$$= \left\langle \left(\partial_t \prod_{j=1}^{r} \left(\frac{1 - \lambda_j}{e^{a_j t} - \lambda_j} \right) \right) e^{-\nu t^2/2} e^{yt} \middle| x^{n-1} \right\rangle$$

$$+ \left\langle \prod_{j=1}^{r} \left(\frac{1 - \lambda_j}{e^{a_j t} - \lambda_j} \right) \left(\partial_t e^{-\nu t^2/2} \right) e^{yt} \middle| x^{n-1} \right\rangle$$

$$+ \left\langle \prod_{j=1}^{r} \left(\frac{1 - \lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} \left(\partial_t e^{yt} \right) \middle| x^{n-1} \right\rangle.$$

The third term is

$$(2.17) \quad y \left\langle \prod_{j=1}^{r} \left(\frac{1 - \lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} e^{yt} \middle| x^{n-1} \right\rangle$$

$$= y FH_{n-1}^{(\nu)}(y \mid a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r).$$

The second term is

$$(2.18) \quad -\nu \left\langle \prod_{j=1}^{r} \left(\frac{1 - \lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} e^{yt} \middle| t x^{n-1} \right\rangle$$

$$= -\nu(n-1) \left\langle \prod_{j=1}^{r} \left(\frac{1 - \lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} e^{yt} \middle| x^{n-2} \right\rangle$$

$$= -\nu(n-1) FH_{n-2}^{(\nu)}(y \mid a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r).$$

We observe that

$$(2.19) \quad \partial_t \left(\prod_{j=1}^{r} \left(\frac{1 - \lambda_j}{e^{a_j t} - \lambda_j} \right) \right)$$

$$= \sum_{i=1}^{r} \left(\frac{1 - \lambda_i}{e^{a_i t} - \lambda_i} \right)' \prod_{j \neq i} \left(\frac{1 - \lambda_j}{e^{a_j t} - \lambda_j} \right)$$

$$= - \prod_{j=1}^{r} \left(\frac{1 - \lambda_j}{e^{a_j t} - \lambda_j} \right) \sum_{i=1}^{r} \frac{a_i e^{a_i t}}{e^{a_i t} - \lambda_i},$$

where
$$(2.20)$$
$$\sum_{i=1}^{r} \frac{a_i e^{a_i t}}{e^{a_i t} - \lambda_i} = \sum_{i=1}^{r} \frac{a_i}{1 - \lambda_i} e^{a_i t} \frac{1 - \lambda_i}{e^{a_i t} - \lambda_i} = \sum_{i=1}^{r} \frac{a_i}{1 - \lambda_i} e^{a_i t} \sum_{m=0}^{\infty} \mathbb{H}_m (\lambda_i) \frac{a_i^m}{m!} t^m.$$

So, by (2.19) and (2.20), we get
$$(2.21)$$
$$\partial_t \left(\prod_{j=1}^{r} \frac{1 - \lambda_j}{e^{a_j t} - \lambda_j} \right) = - \prod_{j=1}^{r} \left(\frac{1 - \lambda_j}{e^{a_j t} - \lambda_j} \right) \sum_{i=1}^{r} \frac{a_i}{1 - \lambda_i} e^{a_i t} \sum_{m=0}^{\infty} \mathbb{H}_m (\lambda_i) \frac{a_i^m}{m!} t^m.$$

Now, the first term is

$$(2.22)$$

$$- \sum_{i=1}^{r} \frac{a_i}{1 - \lambda_i} \left\langle e^{(y + a_i) t} \prod_{j=1}^{r} \left(\frac{1 - \lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2 / 2} \,\middle|\, \sum_{m=0}^{n-1} \mathbb{H}_m (\lambda_i) \frac{a_i^m}{m!} t^m x^{n-1} \right\rangle$$

$$= - \sum_{i=1}^{r} \frac{a_i}{1 - \lambda_i} \sum_{m=0}^{n-1} \binom{n-1}{m} \mathbb{H}_m (\lambda_i) a_i^m$$

$$\times \left\langle e^{(y + a_i) t} \prod_{j=1}^{r} \left(\frac{1 - \lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2 / 2} \,\middle|\, x^{n-1-m} \right\rangle$$

$$= - \sum_{i=1}^{r} \frac{a_i}{1 - \lambda_i} \sum_{m=0}^{n-1} \binom{n-1}{m} \mathbb{H}_m (\lambda_i) a_i^m$$

$$\times \left\langle \sum_{l=0}^{\infty} FH_l^{(\nu)} (y + a_i \mid a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r) \frac{t^l}{l!} \,\middle|\, x^{n-1-m} \right\rangle$$

$$= - \sum_{i=1}^{r} \sum_{m=0}^{n-1} \binom{n-1}{m} \frac{a_i^{m+1}}{1 - \lambda_i} \mathbb{H}_m (\lambda_i) FH_{n-1-m}^{(\nu)} (y + a_i \mid a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r).$$

Therefore, by (2.16), (2.17), (2.18) and (2.22), we obtain the following theorem.

Theorem 2.3. *For $n \geq 2$, we have*

$$FH_n^{(\nu)} (x \mid a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r)$$
$$= x FH_{n-1}^{(\nu)} (x \mid a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r) - \nu (n-1) FH_{n-2}^{(\nu)} (x \mid a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r)$$

10   DAE SAN KIM, DMITRY V. DOLGY, AND TAEKYUN KIM

$$-\sum_{i=1}^{r}\sum_{m=0}^{n-1}\binom{n-1}{m}\frac{a_i^{m+1}}{1-\lambda_i}\mathbb{H}_m\left(\lambda_i\right)FH_{n-1-m}^{(\nu)}\left(x+a_i\mid a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r\right).$$

Remark. We compute the following in two different ways in order to derive an identity:

$$\left\langle\prod_{j=1}^{r}\left(\frac{1-\lambda_j}{e^{a_j t}-\lambda_j}\right)e^{-\nu t^2/2}t^m\middle|x^n\right\rangle,\quad(m,n\geq 0).$$

On one hand, it is

$$(2.23)\qquad\left\langle\prod_{j=1}^{r}\left(\frac{1-\lambda_j}{e^{a_j t}-\lambda_j}\right)e^{-\nu t^2/2}t^m\middle|x^n\right\rangle$$

$$=(n)_m\left\langle\prod_{j=1}^{r}\left(\frac{1-\lambda_j}{e^{a_j t}-\lambda_j}\right)e^{-\nu t^2/2}\middle|x^{n-m}\right\rangle$$

$$=(n)_m FH_{n-m}^{(\nu)}\left(a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r\right).$$

On the other hand, it is

$$(2.24)\qquad\left\langle\partial_t\left(\prod_{j=1}^{r}\left(\frac{1-\lambda_j}{e^{a_j t}-\lambda_j}\right)e^{-\nu t^2/2}t^m\right)\middle|x^{n-1}\right\rangle$$

$$=\left\langle\left(\partial_t\prod_{j=1}^{r}\left(\frac{1-\lambda_j}{e^{a_j t}-\lambda_j}\right)\right)e^{-\nu t^2/2}t^m\middle|x^{n-1}\right\rangle$$

$$+\left\langle\prod_{j=1}^{r}\left(\frac{1-\lambda_j}{e^{a_j t}-\lambda_j}\right)\left(\partial_t e^{-\nu t^2/2}\right)t^m\middle|x^{n-1}\right\rangle$$

$$+\left\langle\prod_{j=1}^{r}\left(\frac{1-\lambda_j}{e^{a_j t}-\lambda_j}\right)e^{-\nu t^2}\left(\partial_t t^m\right)\middle|x^{n-1}\right\rangle.$$

From (2.23) and (2.24), we can derive the following equation: for $n\geq m+2$,

(2.25)

$$FH_{n-m}^{(\nu)}\left(a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r\right)$$

$$=-\nu\left(n-m-1\right)FH_{n-m-2}^{(\nu)}\left(a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r\right)$$

$$-\sum_{i=1}^{r}\sum_{l=0}^{n-m-1}\binom{n-m-1}{l}\frac{a_i^{l+1}}{1-\lambda_i}\mathbb{H}_l\left(\lambda_i\right)FH_{n-1-l-m}^{(\nu)}\left(a_i;a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r\right).$$

For $FH_n^{(\nu)}\left(x\mid a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r\right)\sim\left(\prod_{j=1}^{r}\left(\frac{e^{a_j t}-\lambda_j}{1-\lambda_j}\right)e^{\nu t^2/2},t\right),(x)_n\sim(1,e^t-1),$ we have

$$(2.26)\qquad FH_n^{(\nu)}\left(x\mid a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r\right)=\sum_{m=0}^{n}C_{n,m}\left(x\right)_m,$$

$$(2.27)\qquad C_{n,m}$$

$$= \frac{1}{m!} \left\langle \prod_{j=1}^{r} \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} \left(e^t - 1 \right)^m \middle| x^n \right\rangle$$

$$= \left\langle \prod_{j=1}^{r} \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} \middle| \frac{1}{m!} \left(e^t - 1 \right)^m x^n \right\rangle$$

$$= \left\langle \prod_{j=1}^{r} \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} \middle| \sum_{l=m}^{\infty} S_2\left(l,m\right) \frac{t^l}{l!} x^n \right\rangle$$

$$= \sum_{l=m}^{n} \binom{n}{l} S_2\left(l,m\right) \left\langle \prod_{j=1}^{r} \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} \middle| x^{n-l} \right\rangle$$

$$= \sum_{l=m}^{n} \binom{n}{l} S_2\left(l,m\right) FH_{n-l}^{(\nu)}\left(a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r\right).$$

Therefore, by (2.26) and (2.27), we obtain the following theorem.

Theorem 2.4. *For $n \geq 0$, we have*

$$FH_n^{(\nu)}\left(x \mid a_1,\ldots,a_r;\lambda_1\ldots,\lambda_r\right) = \sum_{m=0}^{n} \sum_{l=m}^{n} \binom{n}{l} S_2\left(l,m\right) FH_{n-l}^{(\nu)}\left(a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r\right)(x)_m.$$

It is easy to show that

$$x^{(n)} = x\left(x+1\right)\cdots\left(x+n-1\right) \sim \left(1, 1-e^{-t}\right).$$

From (1.18) and (1.19), we have

$$(2.28) \qquad FH_n^{(\nu)}\left(x \mid a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r\right) = \sum_{m=0}^{n} C_{n,m} x^{(m)},$$

where

$$(2.29) \quad C_{n,m}$$

$$= \frac{1}{m!} \left\langle \prod_{j=1}^{r} \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} \left(1 - e^{-t} \right)^m \middle| x^n \right\rangle$$

$$= \frac{1}{m!} \left\langle \prod_{j=1}^{r} \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} e^{-mt} \left(e^t - 1 \right)^m \middle| x^n \right\rangle$$

$$= \left\langle \prod_{j=1}^{r} \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} e^{-mt} \middle| \frac{1}{m!} \left(e^t - 1 \right)^m x^n \right\rangle$$

$$= \left\langle \prod_{j=1}^{r} \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} e^{-mt} \middle| \sum_{l=m}^{\infty} S_2\left(l,m\right) \frac{t^l}{l!} x^n \right\rangle$$

$$= \sum_{l=m}^{n} \binom{n}{l} S_2\left(l,m\right) \left\langle \prod_{j=1}^{r} \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} e^{-mt} \middle| x^{n-l} \right\rangle$$

$$= \sum_{l=m}^{n} \binom{n}{l} S_2(l,m) \left\langle \sum_{i=0}^{\infty} FH_i^{(\nu)}(-m \mid a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r) \frac{t^i}{i!} \middle| x^{n-l} \right\rangle$$

$$= \sum_{l=m}^{n} \binom{n}{l} S_2(l,m) \, FH_{n-l}^{(\nu)}(-m \mid a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r).$$

Therefore, by (2.28) and (2.29), we obtain the following theorem.

Theorem 2.5. *For $n \geq 0$, we have*

$$FH_n^{(\nu)}(x \mid a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r)$$

$$= \sum_{m=0}^{n} \sum_{l=m}^{n} \binom{n}{l} S_2(l,m) \, FH_{n-l}^{(\nu)}(-m \mid a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r) \, x^{(m)}.$$

From (1.4), (1.13), (1.18), (1.19) and (1.21), we have

$$(2.30) \qquad FH_n^{(\nu)}(x \mid a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r) = \sum_{m=0}^{n} C_{n,m} B_m^{(s)}(x), \qquad (s \in \mathbb{N}),$$

where

$$(2.31)$$

$$C_{n,m}$$

$$= \frac{1}{m!} \left\langle \prod_{j=1}^{r} \left(\frac{1-\lambda_j}{e^{a_j t}-\lambda_j} \right) e^{-\nu t^2/2} \left(\frac{e^t-1}{t} \right)^s t^m \middle| x^n \right\rangle$$

$$= \binom{n}{m} \left\langle \prod_{j=1}^{r} \left(\frac{1-\lambda_j}{e^{a_j t}-\lambda_j} \right) e^{-\nu t^2/2} \middle| \left(\frac{e^t-1}{t} \right)^s x^{n-m} \right\rangle$$

$$= \binom{n}{m} \left\langle \prod_{j=1}^{r} \left(\frac{1-\lambda_j}{e^{a_j t}-\lambda_j} \right) e^{-\nu t^2/2} \middle| \sum_{l=0}^{\infty} \frac{s!}{(l+s)!} S_2(l+s,s) t^l x^{n-m} \right\rangle$$

$$= \binom{n}{m} \sum_{l=0}^{n-m} \frac{s!}{(l+s)!} S_2(l+s,s)(n-m)_l \left\langle \prod_{j=1}^{r} \left(\frac{1-\lambda_j}{e^{a_j t}-\lambda_j} \right) e^{-\nu t^2/2} \middle| x^{n-m-l} \right\rangle$$

$$= \binom{n}{m} \sum_{l=0}^{n-m} \frac{\binom{n-m}{l}}{\binom{l+s}{s}} S_2(l+s,s) \, FH_{n-m-l}^{(\nu)}(a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r).$$

Therefore, by (2.30) and (2.31), we obtain the following theorem.

Theorem 2.6. *For $n \geq 0$, and $s \in \mathbb{N}$, we have*

$$FH_n^{(\nu)}(x \mid a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r)$$

$$= \sum_{m=0}^{n} \binom{n}{m} \sum_{l=0}^{n-m} \frac{\binom{n-m}{l}}{\binom{l+s}{s}} S_2(l+s,s) \, FH_{n-m-l}^{(\nu)}(a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r) B_m^{(s)}(x).$$

From (1.1), (1.18), (1.19) and (1.21), we have

$$(2.32) \quad FH_n^{(\nu)}(x \mid a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r) = \sum_{m=0}^{n} C_{n,m} \mathbb{H}_m^{(s)}(x \mid \lambda), \quad (s \in \mathbb{N}),$$

where

$$(2.33) \quad C_{n,m}$$

$$= \frac{1}{m!} \left\langle \prod_{j=1}^{r} \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} \left(\frac{e^t - \lambda}{1 - \lambda} \right)^s t^m \middle| x^n \right\rangle$$

$$= \frac{1}{m!(1-\lambda)^s} \left\langle \prod_{j=1}^{r} \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} \left(e^t - \lambda \right)^s \middle| t^m x^n \right\rangle$$

$$= \frac{\binom{n}{m}}{(1-\lambda)^s} \left\langle \prod_{j=1}^{r} \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} \middle| \sum_{j=0}^{s} \binom{s}{j} (-\lambda)^{s-j} e^{jt} x^{n-m} \right\rangle$$

$$= \frac{\binom{n}{m}}{(1-\lambda)^s} \sum_{j=0}^{s} \binom{s}{j} (-\lambda)^{s-j} \left\langle \prod_{j=1}^{r} \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} e^{jt} \middle| x^{n-m} \right\rangle$$

$$= \frac{\binom{n}{m}}{(1-\lambda)^s} \sum_{j=0}^{s} \binom{s}{j} (-\lambda)^{s-j} FH_{n-m}^{(\nu)}(j \mid a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r).$$

Therefore, by (2.32) and (2.33), we obtain the following theorem.

Theorem 2.7. *For $n \geq 0$, we have*

$$FH_n^{(\nu)}(x \mid a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r)$$

$$= \frac{1}{(1-\lambda)^s} \sum_{m=0}^{n} \binom{n}{m} \sum_{j=0}^{s} \binom{s}{j} (-\lambda)^{s-j} FH_{n-m}^{(\nu)}(j \mid a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r) \mathbb{H}_m^{(s)}(x \mid \lambda).$$

REFERENCES

1. T. Agoh and M. Yamanaka, *A study of Frobenius-Euler numbers and polynomials*, Ann. Sci. Math. Québec **34** (2010), no. 1, 1–14. MR 2744192 (2012a:11024)
2. A. Bayad and T. Kim, *Results on values of Barnes polynomials*, Rocky Mountain J. Math. **43** (2013), no. 6, 1857–1869. MR 3178446
3. A. Bayad, T. Kim, W.J. Kim, S. H. Lee, *Arithmetic properties of q-Barnes polynomials*, J. Comput. Anal. Appl. **15** (2013), no. 1, 111–117. MR 3076723
4. M. Can, M. Cenkci, V. Kurt, and Y. Simsek, *Twisted Dedekind type sums associated with Barnes' type multiple Frobenius-Euler l-functions*, Adv. Stud. Contemp. Math. (Kyungshang) **18** (2009), no. 2, 135–160. MR 2508979 (2010a:11072)
5. İ. N. Cangül, V. Kurt, H. Ozden, and Y. Simsek, *higher-order w-q-Genocchi numbers*, Adv. Stud. Contemp. Math. (Kyungshang) **19** (2009), no. 1, 39–57. MR 2542124 (2011b:05010)
6. L. Carlitz, *Some polynomials related to the Bernoulli and Euler polynomials*, Utilitas Math. **19** (1981), 81–127. MR 624049 (82j:10023)

7. _____, *Some remarks on the multiplication theorems for the Bernoulli and Euler polynomials*, Glas. Mat. Ser. III **16(36)** (1981), no. 1, 3–23. MR 634291 (83b:10009)

8. R. Dere and Y. Simsek, *Applications of umbral algebra to some special polynomials*, Adv. Stud. Contemp. Math. (Kyungshang) **22** (2012), no. 3, 433–438. MR 2976601

9. S. Gaboury, R. Tremblay, and B.-J. Fugère, *Some explicit formulas for certain new classes of Bernoulli, Euler and Genocchi polynomials*, Proc. Jangjeon Math. Soc. **17** (2014), no. 1, 115–123. MR 3184467

10. H. W. Gould, *Explicit formulas for Bernoulli numbers*, Amer. Math. Monthly **79** (1972), 44–51. MR 0306102 (46 #5229)

11. D. S. Kim and T. Kim, *Higher-order Frobenius-Euler and poly-Bernoulli mixed-type polynomials*, Adv. Difference Equ. (2013), 2013:251, 13. MR 3108262

12. D. S. Kim, T. Kim, S.-H. Lee, and S.-H. Rim, *A note on the higher-order Frobenius-Euler polynomials and Sheffer sequences*, Adv. Difference Equ. (2013), 2013:41, 12. MR 3032702

13. D. S. Kim, T. Kim, *q-Bernoulli polynomials and q-umbral calculus*, Sci. China Math. no. 9, **57** (2014), no. 9, 1867–1874. MR 3249396

14. D.S. Kim, T. Kim, T. Komatsu, S.-H. Lee *Barnes-type Daehee of the first kind and poly-Cauchy of the first kind mixed-type polynomials*, Adv. Difference Equ. **2014** (2014), 2014:140, 22 pp. MR 3259855

15. T. Kim, *On Euler-Barnes multiple zeta functions*, Russ. J. Math. Phys. **10** (2003), no. 3, 261–267. MR 2012900 (2004j:11106)

16. _____, *An identity of the symmetry for the Frobenius-Euler polynomials associated with the fermionic p-adic invariant q-integrals on $\mathbf{Z}_p$*, Rocky Mountain J. Math. **41** (2011), no. 1, 239–247. MR 2845943 (2012k:11027)

17. _____, *Identities involving Laguerre polynomials derived from umbral calculus*, Russ. J. Math. Phys. **21** (2014), no. 1, 36–45. MR 3182545

18. T. Kim and J. Choi, *A note on the product of Frobenius-Euler polynomials arising from the p-adic integral on $\mathbb{Z}_p$*, Adv. Stud. Contemp. Math. (Kyungshang) **22** (2012), no. 2, 215–223. MR 2961614

19. T. Kim and B. Lee, *Some identities of the Frobenius-Euler polynomials*, Abstr. Appl. Anal. (2009), Art. ID 639439, 7. MR 2487361 (2010a:11231)

20. T. Kim, *A note on poly-Bernoulli and higher-order poly-Bernoulli polynomials*, Russ. J. Math. Phys. **22** (2015), no. 1, 26–33.

21. T. Kim and T. Mansour, *Umbral calculus associated with Frobenius-type Eulerian polynomials*, Russ. J. Math. Phys. **21** (2014), no. 4, 484–493. MR 3284958

22. Q.-M. Luo and F. Qi, *Relationships between generalized Bernoulli numbers and polynomials and generalized Euler numbers and polynomials*, Adv. Stud. Contemp. Math. (Kyungshang) **7** (2003), no. 1, 11–18. MR 1981601

23. H. Ozden, I. N. Cangul, and Y. Simsek, *Remarks on q-Bernoulli numbers associated with Daehee numbers*, Adv. Stud. Contemp. Math. (Kyungshang) **18** (2009), no. 1, 41–48. MR 2479746 (2009k:11037)

24. S. Roman, *The umbral calculus*, Pure and Applied Mathematics, vol. 111, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1984. MR 741185 (87c:05015)

25. K. Shiratani, *On Euler numbers*, Mem. Fac. Sci. Kyushu Univ. Ser. A **27** (1973), 1–5. MR 0314755 (47 #3307)

26. Y. Simsek, O. Yurekli, and V. Kurt, *On interpolation functions of the twisted generalized Frobenius-Euler numbers*, Adv. Stud. Contemp. Math. (Kyung-shang) **15** (2007), no. 2, 187–194. MR 2356176 (2008g:11193)

DEPARTMENT OF MATHEMATICS, SOGANG UNIVERSITY, SEOUL 121-742, REPUBLIC OF KOREA
E-mail address: dskim@sogang.ac.kr

INSTITUTE OF MATHEMATICS AND COMPUTER SCIENCE, FAR EASTERN FEDERAL UNIVERSITY, 690950 VLADIVOSTOK, RUSSIA
E-mail address: *d_dol@mail.ru*

DEPARTMENT OF MATHEMATICS, KWANGWOON UNIVERSITY, SEOUL 139-701, REPUBLIC OF KOREA
E-mail address: tkkim@kw.ac.kr

Robust stability and stabilization of linear uncertain stochastic systems with Markovian switching

Yifan Wu[1]

Department of Basic Courses Jiangsu Food & Pharmaceutical Science College,

HuaiAn, Jiangsu, 223003, China

Abstract.   This paper is concerned with robust stability and stabilization problem for a class of linear uncertain stochastic systems with Markovian switching. The uncertain system under consideration involves parameter uncertainties both in the drift part and in the diffusion part. New criteria for testing the robust stability of such systems are established in terms of bi-linear matrix inequalities (BLMIs), and sufficient conditions are proposed for the design of robust state-feedback controllers. An example illustrates the proposed techniques.

Keywords: Bi-linear matrix inequalities (BLMIs); Robust stabilization; Stochastic system with Markovian switching; Uncertainty

1   Introduction

Stochastic systems with Markovian switching have been used to model many practical systems where they may experience abrupt changes in their structure and parameters. Such systems have played a crucial role in many applications, such as hierarchical control of manufacturing systems([4, 5, 16]), financial engineering ([19]) and wireless communications ([6]).

In the past decades, the stability and control of Markovian jump systems have recently received a lot of attention. For example, [3] and [15] systematically studied stochastic stability properties of jump linear systems. [1] discussed the stability of a semi-linear stochastic differential equation with Markovian switching. [7, 9, 10, 12] discussed the exponential stability of general nonlinear stochastic systems with Markovian switching of the form

$$dx(t) = f(x(t), t, r(t))dt + g(x(t), t, r(t))dB(t). \tag{1.1}$$

---

[1]E-mail address: yifanwu1980@126.com

Over the last decade, stochastic control problems governed by stochastic differential equation with Markovian switching have attracted considerable research interest, and we here mention[2, 11, 20, 23, 24]. It is well known that uncertainty occurs in many dynamic systems and is frequently a cause of instability and performance degradation. In the past few years, considerable attention has been given to the problem of designing robust controllers for linear systems with parameter uncertainty, such as[8, 13, 17, 21, 22]. However, a literature search reveals that the issue of stabilization of uncertain system under consideration involves parameter uncertainties both in the drift part and in the diffusion part has not been fully investigated and remains important and challenging. This situation motivates the present study on the robust stabilization of linear uncertain stochastic systems with Markovian switching. We aim at designing a robust state-feedback controller such that, for all admissible uncertainties , the closed-loop system is exponentially stable in mean square.

The structure of this paper is as follows. In Section 2, we introduce notations, definitions and results required from the literature. In Section 3, we shall discuss the problem of mean square exponential stabilization for a linear jump stochastic system. In Section 4, sufficient conditions are proposed for the design of robust state-feedback controllers. An example is discussed for illustrating our main results in Section 5.

2 Preliminaries

In this paper, we will employ the following notation. Let $|.|$ be the Euclidean norm in $\mathcal{R}^n$. The interval $[0, \infty)$ be denoted by $\mathcal{R}_+$. If A is a vector or matrix, its transpose is denoted by A^T. I_n denotes the $n \times n$ identity matrix. If A is a symmetric matrix $\lambda_{min}(A)$ and $\lambda_{max}(A)$ mean the smallest and largest eigenvalue, respectively. If A and B are symmetric matrices, by $A > B$ and $A \geq B$ we mean that $A - B$ is positive definite and nonnegative definite, respectively. And $\mathcal{C}^{2,1}(\mathcal{R}^n \times \mathcal{R}^+ \times S; \mathcal{R}^+)$ denotes the family of all $\mathcal{R}^+$-valued functions on $\mathcal{R}^n \times \mathcal{R}^+ \times S$ which are continuously twice differentiable in x and once differentiable in t. We write $diag(a_1, ..., a_n)$ for a diagonal matrix whose diagonal entries starting in the upper left corner are $a_1, ..., a_n$.

Let $(\Omega, \mathcal{F}, (\mathcal{F})_t, P)$ be a complete probability space with a filtration $(\mathcal{F})_t$ satisfying the usual conditions. Let $r(t), t \geq 0$, be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, ..., N\}$ with generator $Q = (q_{ij})_{N \times N}$ given by

$$P(r(t + \Delta) = j \mid r(t) = i) = \begin{cases} q_{ij}\Delta + o(\Delta), & if \ i \neq j \\ 1 + q_{ii}\Delta + o(\Delta), & if \ i = j \end{cases}$$

where $\Delta > 0$, and $q_{ij} \geq 0$ denotes the switching rate from i to j if $i \neq j$ while $q_{ii} = -\sum_{i \neq j} q_{ij}$.

2

Definition 1 ([9]) The trivial solution of system (1), or simply system (1) is said to be exponentially stable in mean square if

$$\limsup_{t\to\infty} \frac{1}{t} log(E|x(t;t_0,x_0,r_0)|^2) < 0,$$

for all $(t_0, x_0, r_0) \in \mathcal{R}_+ \times \mathcal{R}^n \times S$.

If $V \in \mathcal{C}^{2,1}(\mathcal{R}^n \times \mathcal{R}^+ \times S; \mathcal{R}^+)$, define operator $\mathcal{L}V(x,t,i)$ associated with system (1) by

$$\mathcal{L}V(x,t,i) = \frac{\partial V(x,t,i)}{\partial t} + \frac{\partial V(x,t,i)}{\partial x} f(x,t,i)$$
$$+ \frac{1}{2} tr[g^T(x,t,i)\frac{\partial^2 V(x,t,i)}{\partial x^2} g(x,t,i)] + \sum_{j=1}^{N} q_{ij} V(x,t,i).$$

We have the following lemma.

Lemma 2.1([9]) Let λ, c_1, c_2 be positive numbers. Assume that there exists a function $V(x,t,i) \in \mathcal{C}^{2,1}(\mathcal{R}^n \times \mathcal{R}^+ \times S; \mathcal{R}^+)$ such that

$$c_1|x(t)|^2 \le V(x,t,i) \le c_2|x(t)|^2$$

and

$$\mathcal{L}V(x,t,i) \le -\lambda|x(t)|^2$$

for all $(x,t,i) \in \mathcal{R}^n \times \mathcal{R}^+ \times S$, then system (1) is exponentially stable in mean square.

In this note, we consider the following linear uncertain stochastic systems with Markovian switching:

$$dx(t) = \tilde{A}(r(t))x(t)dt + \sum_{k=1}^{d} \tilde{B}_k(r(t))x(t)dw_k(t),$$

$$x(t_0) = x_0 \in R^n, t \ge t_0,$$

(2.2)

where $w(t) = (w_1(t), w_2(t), \cdots, w_d(t))^T$ denotes a d-dimensional Brownian motion or Wiener process, $x(t) \in \mathcal{R}^n$ is the system state, we assume that $w(t)$ and $r(t)$ are independent. For any $i \in S, 1 \le k \le d$, $\tilde{A}_i = \tilde{A}(r(t) = i)$ and $\tilde{B}_{ki} = \tilde{B}_k(r(t) = i)$ are not precisely known a priori, but belong to the following admissible uncertainty domains:

$$\mathcal{D}_a = \{A_i + D_{0i}F_{0i}(t)E_{0i} : F_{0i}(t)^T F_{0i}(t) \le I, i \in S\},$$

$$\mathcal{D}_{bk} = \{B_{ki} + D_{ki}F_{ki}(t)E_{ki} : F_{ki}(t)^T F_{ki}(t) \le I, i \in S\},$$

where $A_i, B_{ki}, D_{0i}, E_{0i}, D_{ki}, E_{ki}$ are known constant real matrices with appropriate dimensions, while $F_{0i}(t)$ and $F_{ki}(t)$ denotes the uncertainties in the system matrices, for all $i \in S$.

3

Lemma 2.2 ([14, 18]) Let A, D, E, W and $F(t)$ be real matrices of appropriate dimensions such that $F^T(t)F(t) \leq I$ and $W > 0$, then ,

1) For scalar $\varepsilon > 0$, $DF(t)E + (DF(t)E)^T \leq \varepsilon DD^T + \frac{1}{\varepsilon}E^T E$

2) For any scalar $\varepsilon > 0$ such that $W - \varepsilon DD^T > 0$,

$$(A + DF(t)E)^T W^{-1}(A + DF(t)E) \leq A^T(W - \varepsilon DD^T)^{-1}A + \frac{1}{\varepsilon}E^T E.$$

3   Robust stability analysis

This section, we discuss the robust stability for system (2). For convenience, we will let the initial values x_0 and r_0 be non-random, namely $x_0 \in \mathcal{R}^n$ and $r_0 \in S$, but the theory developed in this paper can be generalized without any difficulty to cope with the case of random initial values, and we write $x(t; t_0, x_0, r_0) = x(t)$ simply.

Theorem 3.1 Suppose that there exist N symmetric positive-definite matrices P_i and positive scalars ε_i, γ_i, and λ_i, such that $\forall\, i \in S$, the following BLMIs hold:

$$
\begin{pmatrix}
\Pi_{11} & * & * & * & * \\
E_{0i}P_i & -\gamma_i I & * & * & * \\
\Pi_{31} & 0 & \Pi_{33} & * & * \\
\Pi_{41} & 0 & 0 & -\varepsilon_i I & * \\
\Pi_{51} & 0 & 0 & 0 & \Pi_{55}
\end{pmatrix} < 0,
\tag{3.3}
$$

where the symbol '*' denotes the transposed element at the symmetric position, and

$$\Pi_{11} = A_i P_i + P_i A_i^T + q_{ii} P_i + \lambda_i P_i + \gamma_i D_{0i} D_{0i}^T,$$

$$\Pi_{31} = [P_i B_{1i}^T, P_i B_{2i}^T, \ldots, P_i B_{di}^T]^T,$$

$$\Pi_{41} = [P_i E_{1i}^T, P_i E_{2i}^T, \ldots, P_i E_{di}^T]^T,$$

$$\Pi_{33} = diag[\varepsilon_i D_{1i} D_{1i}^T - P_i, \ldots, \varepsilon_i D_{di} D_{di}^T - P_i],$$

$$\Pi_{51} = [\underbrace{P_i, P_i, \ldots, P_i}_{N-1}]^T,$$

$$\Pi_{55} = diag[\frac{-1}{q_{i1}}P_1, \ldots, \frac{-1}{q_{i(i-1)}}P_{i-1}, \frac{-1}{q_{i(i+1)}}P_{i+1}, \ldots, \frac{-1}{q_{iN}}P_N],$$

then system (2) is exponentially stable in mean square.

Proof Let $X_i = P_i^{-1}$ and define $V(x, i) = x^T X_i x$ for all $i \in S$. And let $c_1 = min\{\lambda_{min}(X_i) : i \in S\}$, $c_2 = max\{\lambda_{max}(X_i) : i \in S\}$, it is clear that

$$c_1|x(t)|^2 \leq V(x, i) \leq c_2|x(t)|^2.
\tag{3.4}$$

4

On the other hand, a calculation shows that

$$\mathcal{L}V(x,i) = x(t)^T \Big[X_i(A_i + D_{0i}F_{0i}E_{0i}) + (A_i + D_{0i}F_{0i}E_{0i})^T X_i + \sum_{j=1}^{N} q_{ij}X_j$$

$$+ \sum_{k=1}^{d} (B_{ki} + D_{ki}F_{ki}E_{ki})^T X_i (B_{ki} + D_{ki}F_{ki}E_{ki}) \Big] x(t),$$

by Lemma 2.2, for all $i \in S$, if there exist positive scalars ε_i and γ_i such that $\varepsilon_i D_{ki}D_{ki}^T - P_i < 0$, $1 \leq k \leq d$, then we have

$$\mathcal{L}V(x,i) \leq x(t)^T \Big[X_iA_i + A_i^T X_i + \gamma_i X_i D_{0i}D_{0i}^T X_i + \frac{1}{\gamma_i} E_{0i}^T E_{0i}$$

$$+ \sum_{k=1}^{d} B_{ki}^T (P_i - \varepsilon_i D_{ki}D_{ki}^T)^{-1} B_{ki} + \sum_{k=1}^{d} \frac{1}{\varepsilon_i} E_{ki}^T E_{ki} + \sum_{j=1}^{N} q_{ij}X_j \Big] x(t).$$

Thus, there exists a $\lambda > 0$ such that

$$\mathcal{L}V(x,i) \leq -\lambda |x(t)|^2$$

will hold if for any $i \in S$ there exists a $\lambda_i > 0$ such that

$$X_iA_i + A_i^T X_i + \gamma_i X_i D_{0i}D_{0i}^T X_i + \frac{1}{\gamma_i} E_{0i}^T E_{0i}$$

$$+ \sum_{k=1}^{d} B_{ki}^T (P_i - \varepsilon_i D_{ki}D_{ki}^T)^{-1} B_{ki} + \sum_{k=1}^{d} \frac{1}{\varepsilon_i} E_{ki}^T E_{ki} + \sum_{j=1}^{N} q_{ij}X_j + \lambda_i X_i \qquad (3.5)$$

$$< 0.$$

Pre- and post-multiplying (5) by P_i yields

$$A_iP_i + P_iA_i^T + \gamma_i D_{0i}D_{0i}^T + \frac{1}{\gamma_i} P_i E_{0i}^T E_{0i} P_i$$

$$+ \sum_{k=1}^{d} P_i B_{ki}^T (P_i - \varepsilon_i D_{ki}D_{ki}^T)^{-1} B_{ki} P_i$$

$$+ \sum_{k=1}^{d} \frac{1}{\varepsilon_i} P_i E_{ki}^T E_{ki} P_i + \sum_{j \neq i} q_{ij} P_i P_j^{-1} P_i + q_{ii}P_i + \lambda_i P_i$$

$$< 0,$$

which is equivalent to inequality (3) in view of Schur complement equivalence. The assertion of this theorem follows from Lemma 2.1 immediately.

Remark 1 Theorem 3.1 provides the analysis results for the exponential stability of the system (2). It can be seen from (3) that we need to check whether there exist N symmetric positive-definite matrices P_i and positive scalars ε_i, γ_i, and λ_i meeting the N coupled matrix inequalities. It is clear that inequality (3) is BLMIs, and it is LMIs for a prescribed λ_i, then we are able to determine exponential stability of the system (3) readily by checking the solvability of the LMIs.

5

4 Robust stabilization synthesis

This section deals with the robust stabilization problem for linear uncertain stochastic systems with Markovian switching. Let us consider the uncertain stochastic control system of the form

$$dx(t) = \left[\tilde{A}(r(t))x(t) + C(r(t))u(t)\right]dt + \sum_{k=1}^{d}\left[\tilde{B}_k(r(t))x(t) + C_k(r(t))u(t)\right]dw_k(t),$$

$$x(t_0) = x_0 \in R^n, t \geq t_0. \tag{4.6}$$

We aim to design a state-feedback controller $u(t) = K(r(t))x(t)$ such that the resulting closed-loop system

$$dx(t) = \left[\tilde{A}(r(t)) + C(r(t))K(r(t))\right]x(t)dt + \sum_{k=1}^{d}\left[\tilde{B}_k(r(t)) + C_k(r(t))K(r(t))\right]x(t)dw_k(t),$$

$$x(t_0) = x_0 \in \mathcal{R}^n, t \geq t_0. \tag{4.7}$$

is exponentially stable in mean square over all admissible uncertainty domains $\mathcal{D}_a$ and $\mathcal{D}_{bk}$, where $K_i = K(r(t) = i)$ $(i \in S)$ is the controller to be determined.

 The following results solve the robust stabilization problem for system (6).

Theorem 4.1 The closed-loop system (7) is exponentially stable in mean square with respect to state-feedback gain $K_i = Y_i P_i^{-1}$, if there exist N symmetric positive-definite matrices P_i, N matrices Y_i and positive scalars ε_i, γ_i, and λ_i, such that $\forall$ $i \in S$, the following BLMIs hold:

$$\begin{pmatrix} \Pi_{11} & * & * & * & * \\ E_{0i}P_i & -\gamma_i I & * & * & * \\ \Pi_{31} & 0 & \Pi_{33} & * & * \\ \Pi_{41} & 0 & 0 & -\varepsilon_i I & * \\ \Pi_{51} & 0 & 0 & 0 & \Pi_{55} \end{pmatrix} < 0, \tag{4.8}$$

where

$$\Pi_{11} = (A_i P_i + C_i Y_i) + (A_i P_i + C_i Y_i)^T + q_{ii} P_i$$

$$+ \lambda_i P_i + \gamma_i D_{0i} D_{0i}^T,$$

$$\Pi_{31} = [(B_{1i}P_i + C_{1i}Y_i)^T, \ldots, (B_{di}P_i + C_{di}Y_i)^T]^T,$$

$$\Pi_{41} = [P_i E_{1i}^T, P_i E_{2i}^T, \ldots, P_i E_{di}^T]^T,$$

$$\Pi_{33} = diag[\varepsilon_i D_{1i} D_{1i}^T - P_i, \ldots, \varepsilon_i D_{di} D_{di}^T - P_i],$$

$$\Pi_{51} = [\underbrace{P_i, P_i, \ldots, P_i}_{N-1}]^T,$$

$$\Pi_{55} = diag[\frac{-1}{q_{i1}}P_1, \ldots, \frac{-1}{q_{i(i-1)}}P_{i-1}, \frac{-1}{q_{i(i+1)}}P_{i+1}, \ldots, \frac{-1}{q_{iN}}P_N].$$

6

Proof The proof is similar to that of Theorem 3.1, so we only give an outlined one. Let $X_i = P_i^{-1}$ and define $V(x, i) = x^T X_i x$. There exists a $\lambda > 0$ such that $\mathcal{L}V(x, i) \leq -\lambda|x(t)|^2$ will hold if for any $i \in S$ there exist positive scalars ε_i, γ_i and λ_i, where $\varepsilon_i D_{ki} D_{ki}^T - P_i < 0$, $1 \leq k \leq d$, such that

$$
\begin{aligned}
&X_i(A_i + C_i K_i) + (A_i + C_i K_i)^T X_i + \gamma_i X_i D_{0i} D_{0i}^T X_i + \tfrac{1}{\gamma_i} E_{0i}^T E_{0i} \\
&+ \sum_{k=1}^{d}(B_{ki} + C_{ki} K_i)^T (P_i - \varepsilon_i D_{ki} D_{ki}^T)^{-1}(B_{ki} + C_{ki} K_i) \\
&+ \sum_{k=1}^{d} \frac{1}{\varepsilon_i} E_{ki}^T E_{ki} + \sum_{j=1}^{N} q_{ij} X_j + \lambda_i X_i < 0.
\end{aligned}
\tag{4.9}
$$

Noting that $Y_i = K_i P_i$, and Pre- and post-multiplying (9) by P_i yields

$$
\begin{aligned}
&(A_i P_i + C_i Y_i) + (A_i P_i + C_i Y_i)^T + \gamma_i D_{0i} D_{0i}^T + \tfrac{1}{\gamma_i} P_i E_{0i}^T E_{0i} P_i \\
&+ \sum_{k=1}^{d}(B_{ki} P_i + C_{ki} Y_i)^T (P_i - \varepsilon_i D_{ki} D_{ki}^T)^{-1}(B_{ki} P_i + C_{ki} Y_i) \\
&+ \sum_{k=1}^{d} \frac{1}{\varepsilon_i} P_i E_{ki}^T E_{ki} P_i + \sum_{j \neq i} q_{ij} P_i P_j^{-1} P_i + q_{ii} P_i + \lambda_i P_i < 0,
\end{aligned}
$$

which is equivalent to (8) in view of Schur complement equivalence. The assertion of this theorem follows from Lemma 2.1 immediately.

Remark 2 It is shown in Theorem 4.1 that the robust exponentially stabilization of system (6)-(7) is guaranteed if the inequalities (8) are valid. And the inequality (8) is linear in Y_i and P_i for a prescribed λ_i, thus the standard LMI techniques can be exploited to check the exponential stability of the closed-loop system (7).

5   Example

Let $w(t)$ be a one-dimensional Brownian motion, let r(t) be a right-continuous Markov chain taking values in $S = \{1, 2\}$ with generator $Q = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$, consider a two-dimensional stochastic systems with Markovian switching of the form

$$
\begin{aligned}
dx(t) &= \Big[\big(A(r(t)) + D_0(r(t))F_0(r(t), t)E_0(r(t))\big)x(t) + C(r(t))u(t) \Big] dt \\
&+ \Big[\big(B(r(t)) + D_1(r(t))F_1(r(t), t)E_1(r(t))\big)x(t) + C_1(r(t))u(t) \Big] dw(t),
\end{aligned}
\tag{5.10}
$$

where

$$
A_1 = \begin{pmatrix} 0.5 & 0.2 \\ 0.3 & 0.8 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 0.1 \\ 0.2 & 2 \end{pmatrix}, B_1 = \begin{pmatrix} -1 & 0.5 \\ 0.5 & -1 \end{pmatrix}, B_2 = \begin{pmatrix} -2 & 0.1 \\ 0.1 & 1 \end{pmatrix},
$$

7

$$D_{01} = diag(-1, -2), D_{02} = diag(0.2, 0.3), D_{11} = diag(-1, -1), D_{12} = diag(5, -0.5),$$

$$E_{01} = diag(0.2, 0.2), E_{02} = diag(-3, -5), E_{11} = diag(-0.9, -0.9), E_{12} = diag(0.5, 1),$$

$$C_1 = \begin{pmatrix} -8 & 0.1 \\ 0.05 & -10 \end{pmatrix}, C_2 = \begin{pmatrix} -20 & 0 \\ 0 & -30 \end{pmatrix}, C_{11} = \begin{pmatrix} -1 & 0.5 \\ 2 & 3 \end{pmatrix}, C_{12} = \begin{pmatrix} -2 & 1 \\ 0.5 & -4 \end{pmatrix},$$

for $i = 1, 2$, $F_{0i}(t)$ and $F_{1i}(t)$ denote the uncertainties of system (10). Let $\lambda_1 = 1, \lambda_2 = 2$, by solving LMIs (8), we find the feasible solution:

$$P_1 = \begin{pmatrix} 98.708 & 4.383 \\ 4.383 & 85.385 \end{pmatrix}, P_2 = \begin{pmatrix} 233.108 & -0.786 \\ -0.786 & 180.327 \end{pmatrix}, Y_1 = \begin{pmatrix} 93.468 & -16.376 \\ -20.698 & 70.862 \end{pmatrix},$$

$$Y_2 = \begin{pmatrix} 171.947 & -64.056 \\ 75.520 & 82.513 \end{pmatrix}, \gamma_1 = 0.082, \quad \gamma_2 = 1.170, \varepsilon_1 = 0.034, \quad \varepsilon_2 = 0.004,$$

therefore, by Theorem 4.1, closed-loop system (10) is exponentially stable in mean square with respect to state-feedback gain $K_i = Y_i P_i^{-1}$.

6   Conclusions

Based on the exponential stability theory, we have investigated the robust stochastic stability of the uncertain stochastic system with Markovian switching, sufficient stability conditions were developed. The robust stability of such systems can be tested based on the feasibility of bi-linear matrix inequalities An example has been presented to illustrate the effectiveness of the main results. It is believed that this approach is one step further toward the descriptions of the uncertain stochastic systems.

References

[1] G.K. Basak, A. Bisi, M.K. Ghosh, Stability of a random diffusion with linear drift. J. Math. Anal. Appl. 202, 604-622 (1996)

[2] Y. Dong, J. Sun, On hybrid control of a class of stochastic non-linear Markovian switching systems. Automatica. 44, 990-995 (2008)

8

[3] X. Feng, K.A. Loparo, Y. Ji, H.J. Chizeck, Stochastic stability properties of jump linear systems. IEEE Trans. Automat. Contr. 37, 38-52 (1992)

[4] M.K. Ghosh, A. Arapostathis, S.I. Marcus, Ergodic control of switching diffusions. SIAM J. Control Optim. 35, 1952-1988 (1997)

[5] M.K. Ghosh, A. Arapostathis, S.I. Marcus, Optimal control of switching diffusions with application to flexible manufacturing systems. SIAM J. Contr. Optim. 35, 1183-204 (1993)

[6] V. Krishnamurthy, X. Wang, G. Yin,Spreading code optimization and adaptation in CDMA via discrete stochastic approximation. IEEE Trans. Inform. Theory. 50, 1927-1949 (2004)

[7] R. Khasminskii, C. Zhu, G. Yin, Stability of regime-switching diffusions. Stoch. Proc. Appl. 117, 1037-1051 (2007)

[8] J. Lian, F. Zhang, P. Shi, Sliding mode control of uncertain stochastic hybrid delay systems with average dwell time. Circuits Syst. Signal Process. 31, 539-553 (2012)

[9] X. Mao, Stability of stochastic differential equations with Markovian switching. Stoch. Proc. Appl. 79, 45-67(1999)

[10] X. Mao, G. Yin, C. Yuan, Stabilization and destabilization of hybrid systems of stochastic differential equations. Automatica. 43, 264-273(2007)

[11] X. Mao, J. Lam, L. Huang, Stabilisation of hybrid stochas- tic differential equations by delay feedback control. Syst. Control Lett. 57, 927-935 (2008)

[12] S. Pang, F. Deng, X. Mao, Almost sure and moment exponential stability of eulercmaruyama discretizations for hybrid stochastic differential equations. J. Comput. Appl. Math. 213, 127-141 (2008)

[13] P.V. Pakshin, Robust stability and stabilization of family of jumping stochastic systems. Nonlinear Analysis. 30, 2855-2866 (1997)

[14] I.R. Petersen, A stabilization algorithm for a class of uncertain linear systems. Syst. Control Lett., 8:351-357,1987.

[15] M.A. Rami, L.E. Ghaoui, LMI optimization for nonstandard Riccati equations arising in stochastic control. IEEE Trans. Automat. Contr. 41, 1666-1671(1996)

[16] S.P. Sethi, Q. Zhang. *Hierarchical decision making in stochastic manufacturing systems*(Birkh*ä*user, Boston, 1994)

9

[17] Z. Wang, H. Qiao, K.J. Burnham, On stabilization of bilinear uncertain time-delay stochastic systems with Markovian jumping parameters. IEEE Trans. Automat. Contr. 47, 640-646 (2002)

[18] S. Xu, T. Chen. Robust H_∞ control for uncertain stochastic systems with state delay, IEEE Trans. Automat. Control. 47, 2089-2094 (2002)

[19] G. Yin, R.H. Liu, Q. Zhang, Recursive algorithms for stock liquidation: A stochastic optimization approach. SIAM J. Contr. Optim. 13, 240-263(2002)

[20] C. Yuan, J. Lygeros, On the exponential stability of switching diffusion processes, IEEE Trans. Automat. Control. 50, 1422-1426 (2005)

[21] C. Yuan, J. Lygeros, Stabilization of a class of stochastic differential equations with Markovian switching. Syst. Control Lett. 54, 819-833 (2005)

[22] C. Yuan, X. Mao, Robust stability and controllability of stochastic differential delay equations with Markovian switching. Automatica. 40, 343-354(2004)

[23] C. Zhu, Optimal control of the risk process in a regime-switching environment. Automatica. 47, 1570-1579 (2011)

[24] F.Zhu, Z.Han, J.Zhang, Stability analysis of stochastic differential equations with Markovian switching. Syst. Control Lett. 61, 1209-1214(2012)

10

On interval valued functions and Mangasarian type duality involving Hukuhara derivative

Izhar Ahmad[1,*] Deepak Singh[2], Bilal Ahmad Dar[3], S. Al-Homidan[4]

[1,4] Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia.
[2]Department of Applied Sciences, NITTTR (under Ministry of HRD, Govt. of India), Bhopal, M.P., India.
[3]Department of Applied Mathematics, Rajiv Gandhi Proudyogiki Vishwavidyalaya (State Technological University of M.P.), Bhopal, M.P., India.

Abstract

In this paper, we introduce twice weakly differentiable and twice H-differentiable interval valued functions. The existence of twice H-differentiable interval-valued function and its relation with twice weakly differentiable functions are presented. Interval valued bonvex and generalized bonvex functions involving twice H-differentiability are proposed. Under the proposed settings, necessary conditions are elicited naturally in order to achieve LU-efficient solution. Mangasarian type dual is discussed for a nondifferentiable multiobjective programming problem and appropriate duality results are also derived. The theoretical developments are illustrated through non-trivial numerical examples.

Keyword: Interval valued functions; twice weak differentiability; twice H- differentiability, LU-efficient solution; generalized bonvexity; duality.

Mathematics Subject Classification: 90C25, 90C29, 90C30.

1 Introduction

The study of uncertain programming is always challenging in its modern face. Several attempts to achieve optimal in the same have been made in several directions. However optimality conditions still needs to be optimized. In this direction interval valued programming is one of the several techniques which has got attention of researchers in the recent past. Existing literature [2, 4, 5, 7, 8, 9, 10, 11, 12, 13, 17, 19, 20, 21, 22] contains many interesting results on the study of interval valued programming involving different types of differentiability concepts and various types of convexity concepts of interval valued functions.

Second order duality gives tighter bounds for the value of the objective function when approximations are used. For more information, authors may see ([11], pp

---

*Corresponding author.

E-mail addresses: drizhar@kfupm.edu.sa (Izhar Ahmad), dk.singh1002@gmail.com (Deepak Singh), sahilbilal99in@gmail.com (Bilal Ahmad Dar), homidan@kfupm.edu.sa (S.Al-Homidan)

1

93). One more advantage is that if a feasible point in the primal problem is given and first order duality does not use, then we can apply second order duality to provide a lower bound of the value of the primal problem.

Note that the study of nondifferentiable interval valued programming problems has not been studied extensively as quoted in Sun and Wang [18] therefore to study the second order duals of the aforesaid problem is an interesting move, we consider the following nondifferentiable vector programming problem with interval valued objective functions and constraint conditions and study its second order dual of Mangasarian type.
(IP)

$$\min f(x) + (x^T B x)^{\frac{1}{2}} = \left(f_1(x) + (x^T B x)^{\frac{1}{2}}, ..., f_k(x) + (x^T B x)^{\frac{1}{2}} \right)$$
$$\text{subject to } g_j(x) \preceq_{LU} [0, 0], j \in \Lambda_m$$

where $f_i = [f_i^L, f_i^U], i \in \Lambda_k$ and $g_j = [g_j^L, g_j^U], j \in \Lambda_m$ are interval valued functions with $f_i^L, f_i^U, g_j^L, g_j^U : R^n \to R, i \in \Lambda_k, j \in \Lambda_m$ be twice differentiable functions.

The remaining paper is designed as: section 2 is devoted to preliminaries. Section 3 represents the differentiation of interval valued functions with the introduction of twice weakly differentiable and twice H-differentiable interval valued functions. Some properties of these functions are also presented. Section 4 highlights the concept of so-called bonvexity and its quasi and pseudo forms of interval valued functions and their properties. In section 5, the necessary conditions for proposed solution concept are elicited naturally by considering above settings. Finally with the proposed settings the section 6 is devoted to study the Mangasarian type dual of primal problem (IP). Lastly we conclude in section 7.

2 Preliminaries

Let $\mathcal{I}_c$ denote the class of all closed and bounded intervals in R. i.e.,

$$\mathcal{I}_c = \{[a, b] : a, b \in R \ and \ a \leq b\}.$$

And $b - a$ is the width of the interval $[a, b] \in \mathcal{I}_c$. Then for $A \in \mathcal{I}_c$ we adopt the notation $A = [a^L, a^U]$, where a^L and a^U are respectively the lower and upper bounds of A. Let $A = [a^L, a^U], B = [b^L, b^U] \in \mathcal{I}_c$ and $\lambda \in R$, we have the following operations.

$(i) \qquad A + B = \{a + b : a \in A \ and \ b \in B\} = [a^L + b^L, a^U + b^U]$
(ii)

$$\lambda A = \lambda[a^L, a^U] = \left\{ \begin{array}{l} [\lambda a^L, \lambda a^U] \ if \ \lambda \geq 0 \\ [\lambda a^U, \lambda a^L] \ if \ \lambda < 0; \end{array} \right.$$

(iii)

$$A \times B = [\min_{ab}, \max_{ab}],$$

where

$$\min_{ab} = \min\{a^L b^L, a^L b^U, a^U b^L, a^U b^U\}$$

2

and

$$\max_{ab} = \max\{a^L b^L, a^L b^U, a^U b^L, a^U b^U\}$$

In view of (i) and (ii) we see that

$$-B = -[b^L, b^U] = [-b^U, -b^L] \quad and \quad A - B = A + (-B) = [a^L - b^L, a^U - b^L].$$

Also the real number $a \in R$ can be regarded as a closed interval $A_a = [a, a]$, then we have for $B \in \mathcal{I}_c$

$$a + B = A_a + B = [a + b^L, a + b^U].$$

Note that the space $\mathcal{I}_c$ is not a linear space with respect to the operations (i) and (ii), since it does not contain inverse elements.

3   Differentiation of interval valued functions

Definition 1. *[20] Let X be open set in R. An interval-valued function $f : X \to \mathcal{I}_c$ is called weakly differentiable at x^* if the real-valued functions f^L and f^U are differentiable at x^* (in the usual sense).*

Given $A, B \in \mathcal{I}_c$, if there exists $C \in \mathcal{I}_c$ such that $A = B + C$, then C is called the Hukuhara difference of A and B. We also write $C = A \ominus_H B$ when the Hukuhara difference C exists, which means that $a^L - b^L \le a^U - b^U$ and $C = [a^L - b^L, a^U - b^U]$.

Proposition 1. *[20] Let $A = [a^L, a^U]$ and $B = [b^L, b^U]$ be two closed intervals in R. If $a^L - b^L \le a^U - b^U$, then the Hukuhara difference C exists and $C = [a^L - b^L, a^U - b^U]$.*

Definition 2. *[20] Let X be an open set in R. An interval-valued function $f : X \to \mathcal{I}_c$ is called H-differentiable at x^* if there exists a closed interval $A(x^*) \in \mathcal{I}_c$ such that*

$$\lim_{h \to 0+} \frac{f(x^* + h) \ominus_H f(x^*)}{h} \quad and \quad \lim_{h \to 0+} \frac{f(x^*) \ominus_H f(x^* + h)}{h}$$

both exist and are equal to $A(x^*)$. In this case, $A(x^*)$ is called the H-derivative of f at x^*.

Proposition 2. *[20] Let f be an interval-valued function defined on $X \subseteq R^n$. If f is H-differentiable at $x^* \in X$, then f is weakly differentiable at x^*.*

Next we introduce twice differentiable interval valued functions and study some properties.

Definition 3. *Let X be an open set in R^n, and let $x^* = (x_1^*, ..., x_n^*) \in X$ be fixed. Then we say that f is twice weakly differentiable interval valued function at x^* if*

3

f^L and f^U are twice differentiable functions at x^* (in usual sense). We denote by $\nabla^2 f$ the second differential of f, then we have

$$
\begin{aligned}
\nabla^2 f(x^*) &= \nabla(\nabla f(x))_{x=x^*} \\
&= \nabla(\nabla[f^L(x), f^U(x)])_{x=x^*} \\
&= \nabla([\nabla f^L(x), \nabla f^U(x)])_{x=x^*} \\
&= [\nabla^2 f^L(x), \nabla^2 f^U(x)]_{x=x^*} \\
&= \left[\left(\frac{\partial^2 f^L}{\partial x_i \partial x_j} \right)_{i,j}(x), \left(\frac{\partial^2 f^U}{\partial x_i \partial x_j} \right)_{i,j}(x) \right]_{x=x^*}.
\end{aligned}
$$

Definition 3 is illustrated by the following example.

Example 1. *Consider the interval valued function*

$$
f(x_1, x_2) = [f^L = 2x_1 + x_2^2, f^U = x_1^2 + x_2^2 + 1]. \tag{1}
$$

Therefore we have

$$
\nabla f(x) = [(2, 2x_2), (2x_1, 2x_2)]^T
$$

and

$$
\nabla^2 f(x) = \left[\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right].
$$

Definition 4. *Let X be an open set in R^n, and let $x^* = (x_1^*, ..., x_n^*) \in X$ be fixed. Then we say that f is twice H-differentiable interval valued function if f' is H-differentiable at x^*, where f' is H-derivative of f. We denote by $\nabla_H^2 f$ the second order H-differential of f, then we have*

$$
\begin{aligned}
\nabla_H^2 f(x^*) &= \nabla_H(\nabla_H f(x))_{x=x^*} \\
&= \nabla_H \left(\frac{\partial f}{\partial x_1}(x), ..., \frac{\partial f}{\partial x_n}(x) \right)^T_{x=x^*} \\
&= \left(\nabla_H \left[\frac{\partial f^L}{\partial x_1}(x), \frac{\partial f^U}{\partial x_1}(x) \right], ..., \nabla_H \left[\frac{\partial f^L}{\partial x_n}(x), \frac{\partial f^U}{\partial x_n}(x) \right] \right)^T_{x=x^*} \\
&= \begin{pmatrix} \left[\frac{\partial^2 f^L}{\partial^2 x_1}(x), \frac{\partial^2 f^U}{\partial^2 x_1}(x) \right] & \cdots & \left[\frac{\partial^2 f^L}{\partial x_1 \partial x_n}(x), \frac{\partial^2 f^U}{\partial x_1 \partial x_n}(x) \right] \\ \vdots & \ddots & \vdots \\ \left[\frac{\partial^2 f^L}{\partial x_n \partial x_1}(x), \frac{\partial^2 f^U}{\partial x_n \partial x_1}(x) \right] & \cdots & \left[\frac{\partial^2 f^L}{\partial^2 x_n}(x), \frac{\partial^2 f^U}{\partial^2 x_n}(x) \right] \end{pmatrix}_{n \times n, x=x^*}.
\end{aligned}
$$

$$ \tag{2} $$

Following example justifies the existence of twice H-differentiable interval valued function.

Example 2. *Consider the interval valued function (1), then by definition we have*

$$
\nabla_H f(x) = ([2, 2x_1], [2x_2, 2x_2])^T
$$

4

which exist for $x_1 \geq 1$. Therefore we have

$$\nabla_H^2 f(x) = \nabla_H([2, 2x_1], [2x_2, 2x_2])^T$$
$$= \left(\begin{array}{cc} [0, 2] & [0, 0] \\ [0, 0] & [2, 2] \end{array} \right).$$

The relation between twice weakly differentiable and twice H-differentiable interval valued functions is furnished as follows.

Proposition 3. *Let f be an interval-valued function defined on $X \subseteq R^n$. If f is twice H-differentiable at $x^* \in X$, then f is twice weakly differentiable at x^*.*

Proof. From (2) we have

$$\nabla_H^2 f(x^*) = \left[\left(\begin{array}{ccc} \frac{\partial^2 f^L}{\partial^2 x_1}(x) & \cdots & \frac{\partial^2 f^L}{\partial x_1 \partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f^L}{\partial x_n \partial x_1}(x) & \cdots & \frac{\partial^2 f^L}{\partial^2 x_n}(x) \end{array} \right)_{n \times n}, \left(\begin{array}{ccc} \frac{\partial^2 f^U}{\partial^2 x_1}(x) & \cdots & \frac{\partial^2 f^U}{\partial x_1 \partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f^U}{\partial x_n \partial x_1}(x) & \cdots & \frac{\partial^2 f^U}{\partial^2 x_n}(x) \end{array} \right)_{n \times n} \right]_{x = x^*}$$
$$= [\nabla^2 f^L(x), \nabla^2 f^U(x)]_{x=x^*}$$
$$= \nabla^2 f(x^*).$$

$\square$

We authenticate Proposition 3 by following example.

Example 3. *From Example 2 we have*

$$\nabla_H^2 f(x) = \left(\begin{array}{cc} [0, 2] & [0, 0] \\ [0, 0] & [2, 2] \end{array} \right)$$
$$= \left[\left(\begin{array}{cc} 0 & 0 \\ 0 & 2 \end{array} \right), \left(\begin{array}{cc} 2 & 0 \\ 0 & 2 \end{array} \right) \right]$$
$$= \left[\nabla^2 f^L(x), \nabla^2 f^U(x) \right]$$
$$= \nabla^2 f(x). \text{ (see Example 1).}$$

The converse of Proposition 3 is not true in general, however we have the following result.

Proposition 4. *Let $f \in T$, be twice weakly differentiable function at x^*, with $(f^L)''(x^*) = a^{L'}(x^*)$ and $(f^U)''(x^*) = a^{U'}(x^*)$.*

  1. if $(f^L)'(x^*+h) - (f^L)'(x^*) \leq (f^U)'(x^*+h) - (f^U)'(x^*)$ and $(f^L)'(x^*) - (f^L)'(x^*-h) \leq (f^U)'(x^*) - (f^U)'(x^*-h)$ for every $h > 0$, then f is twice H-differentiable with second H-derivative $[a^{L'}(x^*), a^{U'}(x^*)]$.

  2. if $a^{L'}(x^*) > a^{U'}(x^*)$, then f is not twice H-differentiable at x^*.

Proof. The proof is similar as that of Proposition 4.3 of [20]. $\square$

The existence of twice weakly differentiable interval valued functions which are not twice H-differentiable is proved by following example.

Example 4. *Consider $f : [0, 2] \to [x^3 + x^2 + 1, x^3 + 2x + 2]$ be an interval valued function defined on $[0, 2]$. Then f is twice weakly differentiable on $(0, 2)$ but f is not twice H-differentiable on $(0, 2)$ as $a^{L'}(x^*) > a^{U'}(x^*)$.*

5

4   Interval valued bonvex functions

Convexity is an important concept in studying the theory and methods of mathematical programming, which has been generalized in several ways. For differentiable functions numerous generalizations of convexity exist in the literature. An important concept so-called second order convexity for twice differentiable real valued functions was introduced in Mond [14], however Bector and Chandra [6] named them as bonvex functions. Now consider f to be real valued twice differentiable function, then for the definitions of (strictly) bonvexity, (strictly) pseudobonvexity and (strictly) quasibonvexity, one is refered to [3].

In this section, we introduce LU-bonvex, LU-pseudobonvex and LU-quasibonvex interval valued functions and their strict conditions. We consider T to be the set of all interval valued functions defined on $X \subseteq R^n$.

Definition 5. *Let $f \in T$ be twice H-differentiable function at $x^* \in X$. If we have for every $x \in X$ and $P = (P_1, ..., P_n)$ with $P_i \in \mathcal{I}_c$ such that $P_i^L \geq 0, i \in \Lambda_k$.*

1.

$$f(x) \ominus_H f(x^*) + \frac{1}{2}P^T \nabla_H^2 f(x^*)P \succeq_{LU} \left\{ \nabla_H f(x^*) + \nabla_H^2 f(x^*)P \right\} (x - x^*)$$

then we say that f is LU-bonvex at x^*. We also say that f is strictly LU-bonvex at $x^*(\neq x)$ if the inequality is strict.

2. *If*

$$f(x) \ominus_H f(x^*) + \frac{1}{2}P^T \nabla_H^2 f(x^*)P \preceq_{LU} [0,0],$$

$$\Rightarrow \left\{ \nabla_H f(x^*) + \nabla_H^2 f(x^*)P \right\} (x - x^*) \preceq_{LU} [0,0]$$

then we say that f is LU-quasibonvex at x^*. We also say that f is strictly LU-quasibonvex $x^*(\neq x)$ if the inequality is strict.

3. *If*

$$\left\{ \nabla_H f(x^*) + \nabla_H^2 f(x^*)P \right\} (x - x^*) \succeq_{LU} [0,0],$$

$$\Rightarrow f(x) \ominus_H f(x^*) + \frac{1}{2}P^T \nabla_H^2 f(x^*)P \succeq_{LU} [0,0]$$

then we say that f is LU-pseudobonvex at x^*. We also say that f is strictly LU-pseudobonvex at $x^*(\neq x)$ if the inequality is strict.

Now we present some non-trivial examples which authenticates that the class of interval valued functions introduced in this section is non-empty.

Example 5. *Consider an interval valued function $f(x) = [x^2 + 3x + 2, x^2 + 3x + 5], x \geq 0$. Then we have*

$$\nabla_H f(x) = ([2x + 3, 2x + 3])$$

6

$$= ([3,3])_{x=0}$$

and

$$\nabla_H^2 f(x) = ([2,2])$$

we have

$$[x^2 + 3x + 2, x^2 + 3x + 5] \ominus_H [2,5] + \frac{1}{2}([0,1])^T [2,2][0,1] = [x^2 + 3x + 2, x^2 + 3x + 2]$$

$$\succeq_{LU} ([3,3] + [2,2][0,1])(x)$$

$$= [3x, 5x]$$

therefore f is LU-bonvex at $x = 0$.

Next consider another interval valued functions defined as

$$f(x_1, x_2) = [x_1^2 + x_2^2 + 3, x_1^2 + x_2^2 + 5], x \geq 0.$$

Then we have

$$\nabla_H f(x_1, x_2) = ([2x_1, 2x_1], [2x_2, 2x_2])^T$$

$$= ([4,4], [4,4]^T)^T_{(x_1, x_2)=(2,2)}$$

and

$$\nabla_H^2 f(x) = \begin{pmatrix} [2,2] & [0,0] \\ [0,0] & [2,2] \end{pmatrix}$$

Now let

$$[x_1^2 + x_2^2 + 3, x_1^2 + x_2^2 + 5] \ominus_H [11,13] + \frac{1}{2} \begin{pmatrix} [1,1] \\ [1,1] \end{pmatrix}^T \begin{pmatrix} [2,2] & [0,0] \\ [0,0] & [2,2] \end{pmatrix} \begin{pmatrix} [1,1] \\ [1,1] \end{pmatrix}$$

$$\preceq_{LU} [0,0]$$

then

$$\left(\begin{pmatrix} [4,4] \\ [4,4] \end{pmatrix} + \begin{pmatrix} [2,2] & [0,0] \\ [0,0] & [2,2] \end{pmatrix} \begin{pmatrix} [1,1] \\ [1,1] \end{pmatrix} \right) \begin{pmatrix} x_1 - 2 \\ x_2 - 2 \end{pmatrix} \preceq_{LU} [0,0].$$

this shows that f is LU-quasibonvex at $(2,2)$.
However if

$$\left(\begin{pmatrix} [4,4] \\ [4,4] \end{pmatrix} + \begin{pmatrix} [2,2] & [0,0] \\ [0,0] & [2,2] \end{pmatrix} \begin{pmatrix} [1,1] \\ [1,1] \end{pmatrix} \right) \begin{pmatrix} x_1 - 2 \\ x_2 - 2 \end{pmatrix} \succeq_{LU} [0,0].$$

then

$$[x_1^2 + x_2^2 + 3, x_1^2 + x_2^2 + 5] \ominus_H [11,13] + \frac{1}{2} \begin{pmatrix} [1,1] \\ [1,1] \end{pmatrix}^T \begin{pmatrix} [2,2] & [0,0] \\ [0,0] & [2,2] \end{pmatrix} \begin{pmatrix} [1,1] \\ [1,1] \end{pmatrix}$$

$$\succeq_{LU} [0,0]$$

this shows that f is LU-pseudobonvex at $(2,2)$.

7

Proposition 5. *Let $f \in T$ be twice H-differentiable function at x^* and $P = (P_1, ..., P_n)$ with $P_i \in \mathcal{I}_c$ such that $P_i^L \geq 0, i \in \Lambda_n$.*

1. *if f is LU-bonvex at x^* then f^L and f^U are bonvex functions at x^*.*

2. *if f is LU-quasibonvex at x^* then f^L and f^U are quasibonvex functions at x^*.*

3. *if f is LU-pseudobonvex at x^* then f^L and f^U are pseudobonvex functions at x^*.*

Proof. (i) Let f is LU-bonvex at x^*, then by definition we have

$$f(x) \ominus_H f(x^*) + \frac{1}{2}P^T \nabla_H^2 f(x^*)P \succeq_{LU} \left\{ \nabla_H f(x^*) + \nabla_H^2 f(x^*)P \right\} (x - x^*)$$

Since f is twice H-differentiable at x^*, then by Proposition 3 and Definition 3 f^L and f^U are twice differentiable at x^*. Also since $P_i^L \geq 0$, therefore we have

$$f^L(x) - f^L(x^*) + \frac{1}{2}P^{L^T} \nabla^2 f^L(x^*)P^L \geq \left\{ \nabla f^L(x^*) + \nabla^2 f^L(x^*)P^L \right\} (x - x^*),$$

and

$$f^U(x) - f^U(x^*) + \frac{1}{2}P^{U^T} \nabla^2 f^U(x^*)P^U \geq \left\{ \nabla f^U(x^*) + \nabla^2 f^U(x^*)P^U \right\} (x - x^*).$$

Therefore f^L and f^U are bonvex functions at x^*.
(ii) and (iii) follows by similar treatment. $\quad\square$

Note that the converse of Proposition 5 follows in the light of Proposition 4.

Proposition 6. *Let $f \in T$ be twice H-differentiable function at x^* and $P = (P_1, ..., P_n)$ with $P_i \in \mathcal{I}_c$ such that $P_i^L \geq 0, i \in \Lambda_n$.*

1. *if f is strictly LU-bonvex at x^* then either f^L or f^U or both are strictly bonvex functions at x^*.*

2. *if f is strictly LU-quasibonvex at x^* then either f^L or f^U or both are strictly quasibonvex functions at x^*.*

3. *if f is strictly LU-pseudobonvex at x^* then either f^L or f^U or both are strictly pseudobonvex functions at x^*.*

Proof. Proof is same as that of Proposition 5. $\quad\square$

Remark 1. *If we assume that $f^L = f^U$, then bonvexity comes as a sub-case of LU-bonvexity, and similarly for quasi and pseudobonvexity.*

8

5 Solution concept and necessary conditions

In this section we shall propose solution concept and derive some necessary conditions for problem (IP). We define by S_{IP} the set of feasible solutions of (IP).

Definition 6. *Let* $x^* \in S_{IP}$. *We say that* x^* *is an efficient solution of* (IP) *if there exist no* $\hat{x} \in S_{IP}$, *such that*

$$f_i(\hat{x}) \preceq_{LU} f_i(x^*), i \in \Lambda_k \text{ and } f_h(\hat{x}) \prec_{LU} f_h(x^*), \text{ for at least one index } h.$$

An efficient solution x^* *is said to be properly efficient solution of* (IP) *if there exist scalar* $M > 0$, *such that for all* $i \in \Lambda_k, f_i(x) \prec_{LU} f_i(x^*)$ *and* $x \in S_{IP}$ *imply that*

$$f_i(x^*) \ominus_H f_i(x) \preceq_{LU} M\{f_h(x) \ominus_H f_h(x^*)\}$$

for atleast one index $h \in \Lambda_k - i$ *such that* $f_h(x^*) \prec_{LU} f_h(x)$.

Theorem 1. *(Mond et al. [16]) Let* x^* *be a properly efficient solution of* (P) *(see, [3]) at which constraint qualification [15] is satisfied. Then there exist* $\lambda^* \in R^k, u^* \in R^m$ *and* $v_i^* \in R^n, i \in \Lambda_K$ *such that*

$$\sum_{i=1}^{k} \lambda_i^*(f_i(x^*) + B_i v_i^*) + \nabla u^{*T} g(x^*) = 0,$$

$$u^{*T} g(x^*) = 0,$$
$$(x^{*T} B_i x^*)^{\frac{1}{2}} = x^{*T} B_i v_i^*, i \in \Lambda_k,$$
$$v_i^{*T} B_i v_i^* \leq 1, i \in \Lambda_k,$$
$$\lambda^* > 0, \sum_{i=1}^{k} \lambda_i^* = 1, u^* \geq 0.$$

Now we present the necessary conditions for problem(IP). Consider the following **constraint qualification** $CQ1$

$$d^T \nabla_H g_j(x^*) \succeq_{LU} [0,0], j \in J_0(x^*)$$

$$d^T \nabla_H f_i(x^*) + d^T B_i x^* / (x^{*T} B_i x^*)^{\frac{1}{2}} \preceq_{LU} [0,0], \quad if \quad x^{*T} B_i x^* > 0$$
$$d^T \nabla_H f_i(x^*) + (d^T B_i d)^{\frac{1}{2}} \preceq_{LU} [0,0], \quad if \quad x^{*T} B_i x^* = 0$$

Theorem 2. *Let* x^* *be a properly efficient solution of* (IP) *at which a constraint qualification* $CQ1$ *is satisfied. Then there exist* $\lambda^* \in R^k, u^* \in R^m$ *and* $v_i^* \in R^n, i \in \Lambda_K$ *such that*

$$\sum_{i=1}^{k} \lambda_i^*(\nabla_H f_i(x^*) + B_i v_i^*) + \nabla_H u^{*T} g(x^*) = [0,0],$$

$$u^{*T} g(x^*) = [0,0],$$
$$(x^{*T} B_i x^*)^{\frac{1}{2}} = x^{*T} B_i v_i^*, i \in \Lambda_k,$$
$$v_i^{*t} B_i v_i^* \leq 1, i \in \Lambda_k,$$
$$\lambda^* > 0, \sum_{i=1}^{k} \lambda_i^* = 1, u^* \geq 0.$$

<center>9</center>

Proof. Since x^* is properly efficient solution of (IP) at which a constraint qualification $CQ1$ is satisfied. Then using the property of intervals and twice H-derivative, for $0 < \xi_i^L, \xi_i^U \in R, i \in \Lambda_k$ with $\xi_i^L + \xi_i^U = 1, i \in \Lambda_k$, we have
$CQ2$

$$d^T \nabla g_j^L(x^*) > 0, j \in J_0(x^*)$$

$$d^T \nabla g_j^U(x^*) > 0, j \in J_0(x^*)$$

$$d^T(\xi_i^L \nabla f_i^L(x^*) + \xi_i^U \nabla f_i^U(x^*)) + d^T B_i x^* / (x^{*T} B_i x^*)^{\frac{1}{2}} < 0, \quad if \quad x^{*T} B_i x^* > 0$$

$$d^T(\xi_i^L \nabla f_i^L(x^*) + \xi_i^L \nabla f_i^U(x^*)) + (d^T B_i d)^{\frac{1}{2}} < 0, \quad if \quad x^{*T} B_i x^* = 0$$

Further using the property of intervals and twice H-derivative, for $0 < \xi_i^L, \xi_i^U \in R, i \in \Lambda_k$ with $\xi_i^L + \xi_i^U = 1, i \in \Lambda_k$ we have new conditions as

$$\sum_{i=1}^{k} \lambda_i^*((\xi_i^L \nabla f_i^L(x^*) + \xi_i^L \nabla f_i^U(x^*)) + B_i v_i^*) + \nabla u^{*T}(g^L(x^*) + g^U(x^*)) = 0,$$

$$u^{*T} g^L(x^*) = 0,$$

$$u^{*T} g^U(x^*) = 0,$$

$$(x^{*T} B_i x^*)^{\frac{1}{2}} = x^{*T} B_i v_i^*, i \in \Lambda_k,$$

$$v_i^{*T} B_i v_i^* \leq 1, i \in \Lambda_k,$$

$$\lambda^* > 0, \sum_{i=1}^{k} \lambda_i^* = 1, u^* \geq 0.$$

Now using constraint qualification $CQ2$ the above conditions are justified by Theorem 1 for the problem (say $(IP1)$) heaving objective function $(\xi_1^L f_1^L(x) + \xi_1^L f_1^U(x), ..., \xi_k^L f_k^L(x) + \xi_k^L f_k^U(x))$ and constraint functions $g_j^L(x), g_j^U(x) \leq 0, j \in \Lambda_m$. Now it is easy to see that the optimal solutions of (IP) and $(IP1)$ are same. This completes the proof. $\qquad\square$

6   Mangasarian type duality

In this section, we propose the following Mangasarian type dual of primal problem (IP).
(MSD) V-maximize

$$\left(f_1(y) + u^T g(y) + y^T B_1 v_1 \ominus_H \frac{1}{2} P^T \nabla_H^2 \{f_1(y) + u^T g(y)\} P, ..., \right.$$

$$\left. f_k(y) + u^T g(y) + y^T B_k v_k \ominus_H \frac{1}{2} P^T \nabla_H^2 \{f_k(y) + u^T g(y)\} P \right)$$

subject to

$$\sum_{i=1}^{k} \lambda_i(\nabla_H f_i(y) + \nabla_H^2 f_i(y)P + B_i v_i) + \nabla_H u^T g(y) + \nabla_H^2 u^T g(y)P = [0, 0] \qquad (3)$$

$$v_i^T B_i v_i \leq 1, i \in \Lambda_k \qquad (4)$$

$$\lambda > 0, \sum_{i=1}^{r} \lambda_i = 1 \qquad (5)$$

$u = (u_1, ..., u_m)^T \geq 0, g = (g_1, ..., g_m)$ such that $g_j = [g_j^L, g_j^U], j = 1, ..., m, P = (P_1, ..., P_n)$ with $P_i \in \mathcal{I}_c$ such that $P_i^L \geq 0, i \in \Lambda_k.$ and $y, v_i \in R^n.$

We define by S_{MSD} the set of all feasible solutions of (MSD), therefore if $z \in S_{MSD}$ then $z = (y, u, v, \lambda, P)$, such that $v \in R^k$ with $v_i \in R^n$, and $P_i \in \mathcal{I}_c$ such that $P_i^L \geq 0, i \in \Lambda_k$. We shall use the following generalized Schwartz inequality:

$$x^T A z \leq (x^T A x)^{1/2} (z^T A z)^{1/2},$$

where $x, z \in R^n$ and A is positive semidefinite symmetric matrix of order n.

Theorem 3. *(weak duality) Let* $x \in S_{IP}$ *and* $z \in S_{MSD}$. *Assume that* $f_i(.) + (.)^T B_i v_i, i \in \Lambda_k$ *and* $g_j(.), j \in \Lambda_m$ *are LU-bonvex at* y, *then the following can not hold.*

$$f_i(x) + (x^T B_i x)^{\frac{1}{2}} \preceq_{LU} f_i(y) + u^T g(y) + y^T B_i v_i \ominus_H \frac{1}{2} P^T \{\nabla_H^2 f_i(y) + u^T g(y)\} P, i \in \Lambda_k. \qquad (6)$$

and

$$f_h(x) + (x^T B_h x)^{\frac{1}{2}} \prec_{LU} f_h(y) + u^T g(y) + y^T B_h v_h \ominus_H \frac{1}{2} P^T \{\nabla_H^2 f_h(y) + u^T g(y)\} P, \qquad (7)$$

for at least one index h.

Proof. From (3) we have

$$\sum_{i=1}^{k} \lambda_i \left(\nabla f_i^L(y) + \nabla^2 f_i^L(y) P^L + B_i v_i\right) + \nabla u^T g^L(y) + \nabla^2 u^T g^L(y) P^L = 0.$$

$$\sum_{i=1}^{k} \lambda_i \left(\nabla f_i^U(y) + \nabla^2 f_i^U(y) P^U + B_i v_i\right) + \nabla u^T g^U(y) + \nabla^2 u^T g^U(y) P^U = 0.$$

Adding we get,

$$\sum_{i=1}^{k} \lambda_i \left(\nabla f_i^L(y) + \nabla f_i^U(y) + \nabla^2 f_i^L(y) P^L + \nabla^2 f_i^U(y) P^U + 2 B_i v_i\right) + \nabla u^T g^L(y)$$

$$+ \nabla u^T g^U(y) + \nabla^2 u^T g^L(y) P^L + \nabla^2 u^T g^U(y) P^U = 0. \qquad (8)$$

If possible let (6) and (7) holds then by definition we have

$$\begin{cases} f_i^L(x) + (x^T B_i x)^{\frac{1}{2}} \leq f_i^L(y) + u^T g^L(y) + y^T B_i v_i - \frac{1}{2} P^{L^T} \nabla^2 \{f_i^L(y) + u^T g^L(y)\} P^L. \\ f_i^U(x) + (x^T B_i x)^{\frac{1}{2}} \leq f_i^U(y) + u^T g^U(y) + y^T B_i v_i - \frac{1}{2} P^{U^T} \nabla^2 \{f_i^U(y) + u^T g^U(y)\} P^U. \end{cases}$$

for $i \in \Lambda_k$, and

$$\begin{cases} f_h^L(x) + (x^T B_h x)^{\frac{1}{2}} < f_h^L(y) + u^T g^L(y) + y^T B_h v_h - \frac{1}{2} P^{L^T} \nabla^2 \{f_h^L(y) + u^T g^L(y)\} P^L. \\ f_h^U(x) + (x^T B_h x)^{\frac{1}{2}} \leq f_h^U(y) + u^T g^U(y) + y^T B_h v_h - \frac{1}{2} P^{U^T} \nabla^2 \{f_h^U(y) + u^T g^U(y)\} P^U. \end{cases}$$

11

or

$$
\begin{cases}
f_h^L(x) + (x^T B_h x)^{\frac{1}{2}} \le f_h^L(y) + u^T g^L(y) + y^T B_h v_h - \frac{1}{2} P^{L^T} \nabla^2 \{f_h^L(y) + u^T g^L(y)\} P^L. \\
f_h^U(x) + (x^T B_h x)^{\frac{1}{2}} < f_h^U(y) + u^T g^U(y) + y^T B_h v_h - \frac{1}{2} P^{U^T} \nabla^2 \{f_h^U(y) + u^T g^U(y)\} P^U.
\end{cases}
$$

or

$$
\begin{cases}
f_h^L(x) + (x^T B_h x)^{\frac{1}{2}} < f_h^L(y) + u^T g^L(y) + y^T B_h v_h - \frac{1}{2} P^{L^T} \nabla^2 \{f_h^L(y) + u^T g^L(y)\} P^L. \\
f_h^U(x) + (x^T B_h x)^{\frac{1}{2}} < f_h^U(y) + u^T g^U(y) + y^T B_h v_h - \frac{1}{2} P^{U^T} \nabla^2 \{f_h^U(y) + u^T g^U(y)\} P^U.
\end{cases}
$$

for atleast one index h.

This yields for $\lambda = (\lambda_1, ..., \lambda_r); \lambda_i > 0$

$$
\sum_{i=1}^{k} \lambda_i \left\{ \left(f_i^L(x) + (x^T B_i x)^{\frac{1}{2}} \right) + \left(f_i^U(x) + (x^T B_i x)^{\frac{1}{2}} \right) \right\} <
$$

$$
\sum_{i=1}^{k} \lambda_i \left\{ f_i^L(y) + y^T B_i v_i - \frac{1}{2} P^{L^T} \nabla^2 f_i^L(y) P^L \right\} + u^T g^L(y) - \frac{1}{2} P^{L^T} \nabla^2 u^T g^L(y) P^L +
$$

$$
\sum_{i=1}^{k} \lambda_i \left\{ f_i^U(y) + y^T B_i v_i - \frac{1}{2} P^{U^T} \nabla^2 f_i^U(y) P^U \right\} + u^T g^U(y) - \frac{1}{2} P^{U^T} \nabla^2 u^T g^U(y) P^U.
$$

$$(9)$$

From the hypothesis that $f_i(.) + (.)^T B_i x, i \in \Lambda_k$ and $g_j, j \in \Lambda_m$ are LU-bonvex at y, we have

$$
f_i(x) + x^T B_i v_i \ominus_H (f_i(y) + y^T B_i v_i) + \frac{1}{2} P^T \nabla_H^2 f_i(y) P \succ_{LU}
$$

$$
\left(\nabla_H f_i(y) + \nabla_H^2 f_i(y) P + B_i v_i \right) (x - y), i \in \Lambda_k \qquad (10)
$$

and

$$
g_j(x) \ominus_H g_j(y) + \frac{1}{2} P^T \nabla_H^2 g_j(y) P \succ_{LU} \left(\nabla_H g_j(y) + \nabla_H^2 g_j(y) P \right) (x - y), j \in \Lambda_m. \quad (11)
$$

After multiplying (10) by $\lambda_i, i \in \Lambda_k$ and (11) by $u_j, j \in \Lambda_m$ and adding, yields

$$
\sum_{i=1}^{k} \lambda_i \left\{ f_i^L(x) + x^T B_i v_i - f_i^L(y) - y^T B_i v_i + \frac{1}{2} P^{L^T} \nabla^2 f_i^L(y) P^L \right\} + u^T g^L(x) - u^T g^L(y) +
$$

$$
\frac{1}{2} P^{L^T} \nabla^2 u^T g^L(y) P^L + \sum_{i=1}^{k} \lambda_i \left\{ f_i^U(x) + x^T B_i v_i - f_i^U(y) - y^T B_i v_i + \frac{1}{2} P^{U^T} \nabla^2 f_i^U(y) P^U \right\} +
$$

$$
u^T g^U(x) - u^T g^U(y) + \frac{1}{2} P^{U^T} \nabla^2 u^T g^U(y) P^U \ge
$$

$$
\left\{ \sum_{i=1}^{k} \lambda_i (\nabla f_i^L(y) + \nabla^2 f_i^L(y) P^L + B_i v_i) + \nabla u^T g^L(y) + \nabla^2 u^T g^L(y) P^L \right\} (x - y) +
$$

$$
\left\{ \sum_{i=1}^{k} \lambda_i (\nabla f_i^U(y) + \nabla^2 f_i^U(y) P^U + B_i v_i) + \nabla u^T g^U(y) + \nabla^2 u^T g^U(y) P^U \right\} (x - y).
$$

12

Now by (4), (9), Schewartz inequality and $u^T g(x) \preceq_{LU} [0,0]$, we get

$$\sum_{i=1}^{k} \lambda_i \Big\{ \nabla f_i^L(y) + \nabla^2 f_i^L(y) P^L + \nabla f_i^U(y) + \nabla^2 f_i^U(y) P^U + 2Bv_i \Big\} + \nabla u^T g^L(y) +$$

$$\nabla u^T g^U(y) + \nabla^2 u^T g^L(y) P^L + \nabla^2 u^T g^U(y) P^U < 0.$$

which is a contradiction to (8). This completes the proof. $\qquad \square$

Theorem 4. *(Strong duality theorem) Assume that x^* is properly efficient solution of problem (IP) at which constraint qualification $CQ1$ is satisfied. Then there exist $\lambda^* \in R^k, u^* \in R^m$ and $v_i^* \in R^n, i \in \Lambda_k$, such that $(x^*, u^*, v_i^*, \lambda^*, P^{*T} = ([0,0], ..., [0,0]))$ is feasible for (MSD) and the corresponding objective values of (IP) and (MSD) are equal. Moreover assume that the weak duality between (IP) and (MSD) in Theorem are satisfied, then $(x^*, u^*, v_i^*, \lambda^*, P^{*T} = ([0,0], ..., [0,0]))$ is an efficient solution of (MSD).*

Proof. Since x^* is efficient solution of problem (IP) at which constraint qualification $CQ1$ is satisfied. Then by Theorem 2 there exist $\lambda^* \in R^k, u^* \in R^m$ and $v_i^* \in R^n, i \in \Lambda_k$, such that

$$\sum_{i=1}^{k} \lambda_i^* (\nabla_H f_i(x^*) + B_i v_i^*) + \nabla_H u^{*T} g(x^*) = [0,0],$$

$$u^{*T} g(x^*) = [0,0],$$

$$(x^{*T} B_i x^*)^{\frac{1}{2}} = x^{*T} B_i v_i^*, i \in \Lambda_k,$$

$$v_i^{*T} B_i v_i^* \leq 1, i \in \Lambda_k,$$

$$\lambda^* > 0, \sum_{i=1}^{k} \lambda_i^* = 1, u^* \geq 0.$$

Which yields that $(x^*, u^*, v_i^*, \lambda^*, P^{*T} = ([0,0], ..., [0,0])) \in S_{MSD}$ and the corresponding objective values of (IP) and (MSD) are equal.

Now let $(x^*, u^*, v_i^*, \lambda^*, P^{*T} = ([0,0], ..., [0,0]))$ is not efficient solution of dual problem (MSD), then by Definition there exist $(y^*, u^*, v_i^*, \lambda^*, P^*) \in S_{MSD}$, such that

$$f_i(x^*) + x^{*T} B_i v_i^* + u^{*T} g(x^*) \preceq_{LU} f_i(y^*) + u^{*T} g(y^*) + y^{*T} B_i v_i^*$$

$$\ominus_H \frac{1}{2} P^{*T} \nabla_H^2 \{f_i(y^*) + u^{*T} g(y^*)\} P^*, i \in \Lambda_k$$

and

$$f_i(x^*) + x^{*T} B_i v_i^* + u^{*T} g(x^*) \prec_{LU} f_i(y^*) + u^{*T} g(y^*) + y^{*T} B_i v_i^*$$

$$\ominus_H \frac{1}{2} P^{*T} \nabla_H^2 \{f_i(y^*) + u^{*T} g(y^*)\} P^*,$$

for atleast one index h.

Now using $(x^{*T} B_i x^*)^{\frac{1}{2}} = x^{*T} B_i v_i^*, i \in \Lambda_k$ and $u^{*T} g(y^*) = [0,0]$, we get a contradiction to weak duality theorem. Therefore $(x^*, u^*, v_i^*, \lambda^*, P^{*T} = ([0,0], ..., [0,0]))$ is an efficient solution of dual problem (MSD). $\qquad \square$

13

Theorem 5. *(Strict converse duality) Let* $x^* \in S_{IP}$ *and* $z^* \in S_{MSP}$ *such that*

$$\sum_{i=1}^{k} \lambda_i^* \{f_i(x^*) + x^{*T} B_i v_i^*\} \preceq_{LU} \sum_{i=1}^{k} \lambda_i^* \left\{ f_i(y^*) + y^{*T} B_i v_i^* \ominus_H \frac{1}{2} P^{*T} \nabla_H^2 f_i(y^*) P^* \right\}$$

$$+ u^{*T} g(y^*) \ominus_H \frac{1}{2} P^{*T} \nabla_H^2 u^{*T} g(y^*) P^*. \tag{12}$$

Assume that $f_i(.) + (.)^T B_i v_i^*, i \in \Lambda_k$ *are strictly LU-bonvex at* y^* *and* $g_j(.), j \in \Lambda_m$ *is LU-bonvex at* y^* *then* $x^* = y^*$.

Proof. If possible let $x^* \neq y^*$. Now since $f_i(.) + (.)^T B_i v_i^*, i \in \Lambda_k$ are strictly *LU*-bonvex at y^*, we have

$$f_i(x^*) + x^{*T} B_i v_i^* \ominus_H (f_i(y^*) + y^{*T} B_i v_i^*) + \frac{1}{2} P^{*T} \nabla_H^2 f_i(y^*) P^* \succ_{LU}$$

$$\left(\nabla_H f_i(y^*) + \nabla_H^2 f_i(y^*) P^* + B_i v_i^* \right) (x^* - y^*), i \in \Lambda_k. \tag{13}$$

and

$$g_j(x^*) \ominus_H g_j(y^*) + \frac{1}{2} P^{*T} \nabla_H^2 g_j(y^*) P^* \succeq_{LU} (\nabla_H g_j(y^*) + \nabla_H^2 g_j(y^*) P^*)(x^* - y^*), j \in \Lambda_m. \tag{14}$$

Now multiplying (13) by $\lambda_i^*, i \in \Lambda_k$ and (14) by $u_j^*, j \in \Lambda_m$ and then summing up we get

$$\sum_{i=1}^{k} \lambda_i^* \left\{ f_i(x^*) + x^{*T} B_i v_i^* \right\} + u^{*T} g(x^*) \ominus_H \sum_{i=1}^{k} \lambda_i^* \left\{ f_i(y^*) + y^{*T} B_i v_i \ominus_H \right.$$

$$\left. \frac{1}{2} P^{*T} \nabla_H^2 f_i(y^*) P^* \right\} \ominus_H u^{*T} g(y^*) + \frac{1}{2} P^{*T} \nabla_H^2 u^{*T} g_j(y^*) P^* \succ_{LU}$$

$$\left\{ \sum_{i=1}^{k} \lambda_j^* (\nabla_H f_i(y^*) + \nabla_H^2 f_i(y^*) P^* + B_i v_i^*) + \nabla_H u^{*T} g(y^*) + \nabla_H^2 u^{*T} g(y^*) P^* \right\} (x^* - y^*).$$

The above inequality on using (3) and $u^{*T} g(x^*) \preceq_{LU} [0,0]$ gives

$$\sum_{i=1}^{k} \lambda_i^* \{f_i(x^*) + x^{*T} B_i v_i^*\} \succ_{LU} \sum_{i=1}^{k} \lambda_i^* \left\{ f_i(y^*) + y^{*T} B_i v_i^* \ominus_H \right.$$

$$\left. \frac{1}{2} P^{*T} \nabla_H^2 f_i(y^*) P^* \right\} + u^{*T} g(y^*) \ominus_H \frac{1}{2} P^{*T} \nabla_H^2 u^{*T} g(y^*) P^*.$$

which is a contradiction to (12). Hence $x^* = y^*$ □

7 Conclusions

This paper represents the study of nondifferentiable vector problem in which objective functions and constraints are interval valued. Firstly the twice H- differentiable interval valued functions are introduced, secondly the concepts of LU-bonvexity, LU-quasibonvexity and LU-pseudobonvexity are introduced, thirdly the necessary conditions for proposed solution concept are obtained. And lastly the Mangasarian

14

type dual is proposed and the corresponding duality results are obtained. Although the interval valued equality constraints are not considered in this paper, the similar methodology proposed in this paper can also be used to handle the interval valued equality constraints. However it will be interesting to study the Mond-Weir type duality results [1] for the problem (IP). Future research is oriented to consider the uncertain environment in order to study the optimality conditions involving Fuzzy parameters.

Acknowledgements

References

[1] I. Ahmad, Z. Husain, Second order (F, α, ρ, d)-convexity and duality in multi-objective programming, Inform Sci 176 (2006) 3094-3103.

[2] I. Ahmad, A. Jayswal, J. Banerjee: On interval-valued optimization problems with generalized invex functions, J Inequal Appl (2013) 1-14.

[3] I. Ahmad, S. Sharma, Second order duality for nondifferentiable multiobjective programming problems, Num Func Anal Optim 28 (9-10) (2007) 975-988.

[4] I. Ahmad, D. Singh, B. A. Dar, Optimality conditions for invex interval-valued nonlinear programming problems involving generalized H-derivative, Filomat (2015) (Accepted).

[5] A. Jayswal, I. Stancu-Minasian, J. Banerjee, A.M. Stancu, Sufficieny and duality for optimization problems involving interval-valued invex functions in parametric form, Oper Res Int J 15(2015) 137-161.

[6] C. R. Bector, S. Chandra, Generalized-bonvexity and higher order duality for fractional programming, Opsearch 24 (1987)143-154.

[7] A. Bhurjee, G. Panda, Efficient solution of interval optimization problem, Math Meth Oper Res 76 (2012) 273-288.

[8] A. Bhurjee, G. Panda, Multiobjective optimization problem with bounded parameters, Rairo-Oper. Res. 48(2014), 545-558.

[9] Y. Chalco-Cano, H. Roman-Flores, MD. Jimenez-Gamero, Generalized derivative and π-derivative for set valued functions, Inform Sci 181 (2011) 2177-2188.

[10] Y. Chalco-Cano, W.A. Lodwick, A. Rufian-Lizana, Optimality conditions of type KKT for optimization problem with interval-valued objective function via generalized derivative, Fuzzy Optim Dec Making 12 (2013) 305-322.

15

[11] Y. Chalco-Cano, A. Rufian-Lizana, H. Roman-Flores, M.D. Jimenez-Gamero, Calculus for interval-valued functions using generalized Hukuhara derivative and applications, Fuzzy Sets and Systems 219 (2013) 49-67.

[12] H. Ishibuchi, H. Tanaka, Multiobjective programming in optimization of interval valued objective functions, Eur J Oper Res 48 (1990) 219-225.

[13] L. Li, S. Liu, J. Zhang, Univex interval-valued mapping with differentiability and its application in nonlinear programming, J Appl Math Art. ID 383692, (2013). http://dx.doi.org/10.1155/2013/383692.

[14] B. Mond, Second order duality for nonlinear programs, Opsearch 11 (1974) 90-99.

[15] B. Mond, A class of nondifferentiable mathematical programming problems. J. Math Anal Appl 46 (1974) 169-174.

[16] B. Mond, I. Husain, M.V. Durgaprasad, Duality for a class of nondifferentiable multiple objective programming problems, J Inform Optim Sci 9 (1988) 331-341.

[17] L. Stefanini, B. Bede, Generalized Hukuhara differentiability of interval-valued functions and interval differential equations, Nonlinear Anal 71 (2009) 1311-1328.

[18] Y. Sun, L. Wang, Optimality conditions and duality in nondifferentiable interval valued progra-mming, J Indust Manag Optim 9 no. 1 (2013) 131-142.

[19] D. Singh, B. A. Dar, A. Goyal, KKT optimality conditions for interval valued optimization problems, J Nonl Anal Optim 5 no. 2 (2014) 91-103.

[20] H. C. Wu, The karush Kuhn tuker optimality conditions in an optimization problem with interval valued objective functions, Eur J Oper Res 176 (2007) 46-59.

[21] H. C. Wu, The karush Kuhn tuker optimality conditions in multiobjective programming problems with interval valued objective functions, Eur J Oper Res 196 (2009) 49-60.

[22] J Zhang, S. Liu, L. Li, Q. Feng, The KKT optimality conditions in a class of generalized convex optimization problems with an interval-valued objective function, Optim Lett 8 (2014) 607-631.

16

ADDITIVE-QUADRATIC ρ-FUNCTIONAL INEQUALITIES IN β-HOMOGENEOUS NORMED SPACES

SUNGSIK YUN, GEORGE A. ANASTASSIOU AND CHOONKIL PARK*

ABSTRACT. In this paper, we solve the following additive-quadratic ρ-functional inequalities

$$\|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)\| \tag{0.1}$$
$$\leq \left\| \rho \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - \frac{3}{2}f(x) + \frac{1}{2}f(-x) - \frac{1}{2}f(y) - \frac{1}{2}f(-y) \right) \right\|,$$

where ρ is a fixed complex number with $|\rho| < 1$, and

$$\left\| 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - \frac{3}{2}f(x) + \frac{1}{2}f(-x) - \frac{1}{2}f(y) - \frac{1}{2}f(-y) \right\|$$
$$\leq \|\rho(f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y))\|, \tag{0.2}$$

where ρ is a fixed complex number with $|\rho| < \frac{1}{2}$, and prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequalities (0.1) and (0.2) in β-homogeneous complex Banach spaces and prove the Hyers-Ulam stability of additive-quadratic ρ-functional equations associated with the additive-quadratic ρ-functional inequalities (0.1) and (0.2) in β-homogeneous complex Banach spaces.

1. INTRODUCTION AND PRELIMINARIES

The stability problem of functional equations originated from a question of Ulam [24] concerning the stability of group homomorphisms. The functional equation $f(x+y) = f(x) + f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [11] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [15] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [8] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. The functional equation $f\left(\frac{x+y}{2}\right) = \frac{1}{2}f(x) + \frac{1}{2}f(y)$ is called the *Jensen equation*.

The functional equation $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The stability of quadratic functional equation was proved by Skof [23] for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [6] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. The functional equation $2f\left(\frac{x+y}{2}\right) + 2\left(\frac{x-y}{2}\right) = f(x) + f(y)$ is called a *Jensen type quadratic equation*. The stability problems of various functional equations have been extensively investigated by a number of authors (see [1, 4, 5, 13, 14, 18, 19, 20, 21, 22]).

---

*Corresponding author: Choonkil Park (email: baak@hanyang.ac.kr).

S. YUN, G. A. ANASTASSIOU, C. PARK

In [9], Gilányi showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(xy^{-1})\| \leq \|f(xy)\| \tag{1.1}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(xy) + f(xy^{-1}).$$

See also [16]. Gilányi [10] and Fechner [7] proved the Hyers-Ulam stability of the functional inequality (1.1). Park, Cho and Han [12] proved the Hyers-Ulam stability of additive functional inequalities.

Definition 1.1. Let X be a linear space. A nonnegative valued function $\|\cdot\|$ is an F-norm if it satisfies the following conditions:

(FN$_1$) $\|x\| = 0$ if and only if $x = 0$;
(FN$_2$) $\|\lambda x\| = \|x\|$ for all $x \in X$ and all λ with $|\lambda| = 1$;
(FN$_3$) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$;
(FN$_4$) $\|\lambda_n x\| \to 0$ provided $\lambda_n \to 0$;
(FN$_5$) $\|\lambda x_n\| \to 0$ provided $x_n \to 0$.
Then $(X, \|\cdot\|)$ is called an F^*-space. An F-space is a complete F^*-space.

An F-norm is called β-homogeneous ($\beta > 0$) if $\|tx\| = |t|^\beta \|x\|$ for all $x \in X$ and all $t \in \mathbb{C}$ (see [17]).

In Section 2, we solve the additive-quadratic ρ-functional inequality (0.1) and prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (0.1) in β-homogeneous complex Banach spaces. We moreover prove the Hyers-Ulam stability of an additive-quadratic ρ-functional equation associated with the additive-quadratic ρ-functional inequality (0.1) in β-homogeneous complex Banach spaces.

In Section 3, we solve the additive-quadratic ρ-functional inequality (0.2) and prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (0.2) in β-homogeneous complex Banach spaces. We moreover prove the Hyers-Ulam stability of an additive-quadratic ρ-functional equation associated with the additive-quadratic ρ-functional inequality (0.2) in β-homogeneous complex Banach spaces.

Throughout this paper, let β_1, β_2 be positive real numbers with $\beta_1 \leq 1$ and $\beta_2 \leq 1$. Assume that X is a β_1-homogeneous real or complex normed space with norm $\|\cdot\|$ and that Y is a β_2-homogeneous complex Banach space with norm $\|\cdot\|$.

2. Additive-quadratic ρ-functional inequality (0.1)

Throughout this section, assume that ρ is a fixed complex number with $|\rho| < 1$.

In this section, we investigate the additive-quadratic ρ-functional inequality (0.1) in β-homogeneous complex Banach spaces.

Lemma 2.1. *An even mapping $f : X \to Y$ satisfies*

$$\|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)\| \tag{2.1}$$
$$\leq \left\| \rho \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - \frac{3}{2}f(x) + \frac{1}{2}f(-x) - \frac{1}{2}f(y) - \frac{1}{2}f(-y) \right) \right\|$$

for all $x, y \in X$ if and only if $f : X \to Y$ is quadratic.

Proof. Assume that $f : X \to Y$ satisfies (2.1).

Letting $x = y = 0$ in (2.1), we get $\|2f(0)\| \leq |\rho|^{\beta_2}\|2f(0)\|$. So $f(0) = 0$.

Letting $y = x$ in (2.1), we get $\|f(2x) - 4f(x)\| \leq 0$ and so $f(2x) = 4f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{4}f(x) \tag{2.2}$$

ADDITIVE-QUADRATIC ρ-FUNCTIONAL INEQUALITIES

for all $x \in X$.

It follows from (2.1) and (2.2) that

$$\|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)\|$$
$$\leq \left\| \rho \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - \frac{3}{2}f(x) + \frac{1}{2}f(-x) - \frac{1}{2}f(y) - \frac{1}{2}f(-y) \right) \right\|$$
$$= \frac{|\rho|^{\beta_2}}{2^{\beta_2}} \|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)\|$$

and so

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in X$.

The converse is obviously true. $\qquad\square$

Corollary 2.2. *An even mapping $f : X \to Y$ satisfies*

$$f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y) \qquad (2.3)$$
$$= \rho \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - \frac{3}{2}f(x) + \frac{1}{2}f(-x) - \frac{1}{2}f(y) - \frac{1}{2}f(-y) \right)$$

for all $x, y \in X$ if and only if $f : X \to Y$ is quadratic.

The functional equation (2.3) is called an *additive-quadratic ρ-functional equation*.

We prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (2.1) in β-homogeneous complex Banach spaces for an even mapping case.

Theorem 2.3. *Let $r > \frac{2\beta_2}{\beta_1}$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an even mapping such that*

$$\|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)\| \qquad (2.4)$$
$$\leq \left\| \rho \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - \frac{3}{2}f(x) + \frac{1}{2}f(-x) - \frac{1}{2}f(y) - \frac{1}{2}f(-y) \right) \right\| + \theta(\|x\|^r + \|y\|^r)$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{2\theta}{2^{\beta_1 r} - 4^{\beta_2}} \|x\|^r \qquad (2.5)$$

for all $x \in X$.

Proof. Letting $x = y = 0$ in (2.4), we get $\|2f(0)\| \leq |\rho|^{\beta_2} \|2f(0)\|$. So $f(0) = 0$.

Letting $y = x$ in (2.4), we get

$$\|f(2x) - 4f(x)\| \leq 2\theta \|x\|^r \qquad (2.6)$$

for all $x \in X$. So $\|f(x) - 4f\left(\frac{x}{2}\right)\| \leq \frac{2}{2^{\beta_1 r}} \theta \|x\|^r$ for all $x \in X$. Hence

$$\left\| 4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right) \right\| \leq \sum_{j=l}^{m-1} \left\| 4^j f\left(\frac{x}{2^j}\right) - 4^{j+1} f\left(\frac{x}{2^{j+1}}\right) \right\| \leq \frac{2}{2^{\beta_1 r}} \sum_{j=l}^{m-1} \frac{4^{\beta_2 j}}{2^{\beta_1 r j}} \theta \|x\|^r \qquad (2.7)$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.7) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n \to \infty} 4^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.7), we get (2.5).

Since $f : X \to Y$ is even, the mapping $Q : X \to Y$ is even.

S. YUN, G. A. ANASTASSIOU, C. PARK

It follows from (2.4) that

$$\|Q(x+y) + Q(x-y) - 2Q(x) - Q(y) - Q(-y)\|$$
$$= \lim_{n\to\infty} 4^{\beta_2 n} \left\| f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) - f\left(\frac{-y}{2^n}\right) \right\|$$
$$\le \lim_{n\to\infty} 4^{\beta_2 n} |\rho|^{\beta_2} \left(\left\| 2f\left(\frac{x+y}{2^{n+1}}\right) + 2f\left(\frac{x-y}{2^{n+1}}\right) - \frac{3}{2}f\left(\frac{x}{2^n}\right) + \frac{1}{2}f\left(\frac{-x}{2^n}\right) \right. \right.$$
$$\left. \left. - \frac{1}{2}f\left(\frac{y}{2^n}\right) - \frac{1}{2}f\left(\frac{-y}{2^n}\right) \right\| \right) + \lim_{n\to\infty} \frac{4^{\beta_2 n}\theta}{2^{\beta_1 nr}}(\|x\|^r + \|y\|^r)$$
$$= |\rho|^{\beta_2} \left\| 2Q\left(\frac{x+y}{2}\right) + 2Q\left(\frac{x-y}{2}\right) - \frac{3}{2}Q(x) + \frac{1}{2}Q(-x) - \frac{1}{2}Q(y) - \frac{1}{2}Q(-y) \right\|$$

for all $x, y \in X$. So

$$\|Q(x+y) + Q(x-y) - 2Q(x) - Q(y) - Q(-y)\|$$
$$\le \left\| \rho\left(2Q\left(\frac{x+y}{2}\right) + 2Q\left(\frac{x-y}{2}\right) - \frac{3}{2}Q(x) + \frac{1}{2}Q(-x) - \frac{1}{2}Q(y) - \frac{1}{2}Q(-y) \right) \right\|$$

for all $x, y \in X$. By Lemma 2.1, the mapping $Q : X \to Y$ is quadratic.

Now, let $T : X \to Y$ be another quadratic mapping satisfying (2.5). Then we have

$$\|Q(x) - T(x)\| = 4^{\beta_2 n} \left\| Q\left(\frac{x}{2^n}\right) - T\left(\frac{x}{2^n}\right) \right\|$$
$$\le 4^{\beta_2 n} \left(\left\| Q\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right) \right\| + \left\| T\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right) \right\| \right)$$
$$\le \frac{4 \cdot 4^{\beta_2 n}}{(2^{\beta_1 r} - 4^{\beta_2})2^{\beta_1 nr}} \theta\|x\|^r,$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $Q(x) = T(x)$ for all $x \in X$. This proves the uniqueness of Q. Thus the mapping $Q : X \to Y$ is a unique quadratic mapping satisfying (2.5). □

Theorem 2.4. *Let $r < \frac{2\beta_2}{\beta_1}$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an even mapping satisfying* (2.4). *Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \le \frac{2\theta}{4^{\beta_2} - 2^{\beta_1 r}}\|x\|^r \tag{2.8}$$

for all $x \in X$.

Proof. It follows from (2.6) that $\left\| f(x) - \frac{1}{4}f(2x) \right\| \le \frac{2\theta}{4^{\beta_2}}\|x\|^r$ for all $x \in X$. Hence

$$\left\| \frac{1}{4^l}f(2^l x) - \frac{1}{4^m}f(2^m x) \right\| \le \sum_{j=l}^{m-1} \left\| \frac{1}{4^j}f(2^j x) - \frac{1}{4^{j+1}}f(2^{j+1}x) \right\| \le \sum_{j=l}^{m-1} \frac{2^{\beta_1 rj}}{4^{\beta_2 j}} \frac{2\theta}{4^{\beta_2}}\|x\|^r \tag{2.9}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.9) that the sequence $\{\frac{1}{4^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n}f(2^n x)\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} \frac{1}{4^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.9), we get (2.8).

The rest of the proof is similar to the proof of Theorem 2.3. □

Lemma 2.5. *An odd mapping $f : X \to Y$ satisfies (2.1) if and only if $f : X \to Y$ is additive.*

Proof. Since $f : X \to Y$ is an odd mapping, $f(0) = 0$.

Assume that $f : X \to Y$ satisfies (2.1).

Letting $y = x$ in (2.1), we get $\|f(2x) - 2f(x)\| \le 0$ and so $f(2x) = 2f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{2}f(x) \tag{2.10}$$

for all $x \in X$.

It follows from (2.1) and (2.10) that

$$\|f(x + y) + f(x - y) - 2f(x) - f(y) - f(-y)\|$$
$$\le \left\|\rho\left(2f\left(\frac{x + y}{2}\right) + 2f\left(\frac{x - y}{2}\right) - \frac{3}{2}f(x) + \frac{1}{2}f(-x) - \frac{1}{2}f(y) - \frac{1}{2}f(-y)\right)\right\|$$
$$= |\rho|^{\beta_2}\|f(x + y) + f(x - y) - 2f(x) - f(y) - f(-y)\|$$

and so

$$f(x + y) + f(x - y) = 2f(x) \tag{2.11}$$

for all $x, y \in X$. Letting $z = x + y$ and $w = z - y$ in (2.11), we get

$$f(z) + f(w) = 2f\left(\frac{z + w}{2}\right) = f(z + w)$$

for all $z, w \in X$.

The converse is obviously true. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

Corollary 2.6. *An odd mapping $f : X \to Y$ satisfies (2.3) if and only if $f : X \to Y$ is additive.*

We prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (2.1) in β-homogeneous complex Banach spaces for an odd mapping case.

Theorem 2.7. *Let $r > \frac{\beta_2}{\beta_1}$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an odd mapping satisfying (2.4). Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \le \frac{2\theta}{2^{\beta_1 r} - 2^{\beta_2}}\|x\|^r \tag{2.12}$$

for all $x \in X$.

Proof. Letting $x = y = 0$ in (2.4), we get $\|2f(0)\| \le |\rho|^{\beta_2}\|2f(0)\|$. So $f(0) = 0$.

Letting $y = x$ in (2.4), we get

$$\|f(2x) - 2f(x)\| \le 2\theta\|x\|^r \tag{2.13}$$

for all $x \in X$. So $\|f(x) - 2f\left(\frac{x}{2}\right)\| \le \frac{2}{2^{\beta_1 r}}\theta\|x\|^r$ for all $x \in X$. Hence

$$\left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| \le \sum_{j=l}^{m-1}\left\|2^j f\left(\frac{x}{2^j}\right) - 2^{j+1}f\left(\frac{x}{2^{j+1}}\right)\right\| \le \frac{2}{2^{\beta_1 r}}\sum_{j=l}^{m-1}\frac{2^{\beta_2 j}}{2^{\beta_1 r j}}\theta\|x\|^r \tag{2.14}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.14) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n \to \infty} 2^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.14), we get (2.12).

Since $f : X \to Y$ is odd, the mapping $A : X \to Y$ is odd.

S. YUN, G. A. ANASTASSIOU, C. PARK

It follows from (2.4) that

$$\|A(x+y) + A(x-y) - 2A(x) - A(y) - A(-y)\|$$

$$= \lim_{n\to\infty} 2^{\beta_2 n} \left\| f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) - f\left(\frac{-y}{2^n}\right) \right\|$$

$$\leq \lim_{n\to\infty} 2^{\beta_2 n} |\rho|^{\beta_2} \left(\left\| 2f\left(\frac{x+y}{2^{n+1}}\right) + 2f\left(\frac{x-y}{2^{n+1}}\right) - \frac{3}{2}f\left(\frac{x}{2^n}\right) + \frac{1}{2}f\left(\frac{-x}{2^n}\right) \right. \right.$$

$$\left. \left. - \frac{1}{2}f\left(\frac{y}{2^n}\right) - \frac{1}{2}f\left(\frac{-y}{2^n}\right) \right\| \right) + \lim_{n\to\infty} \frac{2^{\beta_2 n}\theta}{2^{\beta_1 nr}}(\|x\|^r + \|y\|^r)$$

$$= |\rho|^{\beta_2} \left\| 2A\left(\frac{x+y}{2}\right) + 2A\left(\frac{x-y}{2}\right) - \frac{3}{2}A(x) + \frac{1}{2}A(-x) - \frac{1}{2}A(y) - \frac{1}{2}A(-y) \right\|$$

for all $x, y \in X$. So

$$\|A(x+y) + A(x-y) - 2A(x) - A(y) - A(-y)\|$$

$$\leq \left\| \rho \left(2A\left(\frac{x+y}{2}\right) + 2A\left(\frac{x-y}{2}\right) - \frac{3}{2}A(x) + \frac{1}{2}A(-x) - \frac{1}{2}A(y) - \frac{1}{2}A(-y) \right) \right\|$$

for all $x, y \in X$. By Lemma 2.5, the mapping $A : X \to Y$ is additive.

Now, let $T : X \to Y$ be another additive mapping satisfying (2.12). Then we have

$$\|A(x) - T(x)\| = 2^{\beta_2 n} \left\| A\left(\frac{x}{2^n}\right) - T\left(\frac{x}{2^n}\right) \right\|$$

$$\leq 2^{\beta_2 n} \left(\left\| A\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right) \right\| + \left\| T\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right) \right\| \right)$$

$$\leq \frac{4 \cdot 2^{\beta_2 n}}{(2^{\beta_1 r} - 2^{\beta_2})2^{\beta_1 nr}} \theta \|x\|^r,$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$. This proves the uniqueness of A. Thus the mapping $A : X \to Y$ is a unique additive mapping satisfying (2.12). $\square$

Theorem 2.8. *Let $r < \frac{\beta_2}{\beta_1}$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an odd mapping satisfying (2.4). Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \leq \frac{2\theta}{2^{\beta_2} - 2^{\beta_1 r}}\|x\|^r \tag{2.15}$$

for all $x \in X$.

Proof. It follows from (2.13) that $\left\| f(x) - \frac{1}{2}f(2x) \right\| \leq \frac{2\theta}{2^{\beta_2}}\|x\|^r$ for all $x \in X$. Hence

$$\left\| \frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x) \right\| \leq \sum_{j=l}^{m-1} \left\| \frac{1}{2^j}f(2^j x) - \frac{1}{2^{j+1}}f(2^{j+1} x) \right\| \leq \sum_{j=l}^{m-1} \frac{2^{\beta_1 rj}}{2^{\beta_2 j}} \frac{2\theta}{2^{\beta_2}}\|x\|^r \tag{2.16}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.16) that the sequence $\{\frac{1}{2^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n}f(2^n x)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} \frac{1}{2^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.16), we get (2.15).

The rest of the proof is similar to the proof of Theorem 2.7. $\square$

ADDITIVE-QUADRATIC ρ-FUNCTIONAL INEQUALITIES

By the triangle inequality, we have

$$\|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)\|$$
$$- \left\| \rho \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - \frac{3}{2}f(x) + \frac{1}{2}f(-x) - \frac{1}{2}f(y) - \frac{1}{2}f(-y) \right) \right\|$$
$$\leq \| f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)$$
$$- \rho \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - \frac{3}{2}f(x) + \frac{1}{2}f(-x) - \frac{1}{2}f(y) - \frac{1}{2}f(-y) \right) \|.$$

As corollaries of Theorems 2.3, 2.4, 2.7 and 2.8, we obtain the Hyers-Ulam stability results for the additive-quadratic ρ-functional equation (2.3) in β-homogeneous complex Banach spaces.

Corollary 2.9. *Let* $r > \frac{2\beta_2}{\beta_1}$ *and* θ *be nonnegative real numbers, and let* $f : X \to Y$ *be an even mapping such that*

$$\|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y) \tag{2.17}$$
$$- \rho \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - \frac{3}{2}f(x) + \frac{1}{2}f(-x) - \frac{1}{2}f(y) - \frac{1}{2}f(-y) \right) \| \leq \theta(\|x\|^r + \|y\|^r)$$

for all $x, y \in X$. *Then there exists a unique quadratic mapping* $Q : X \to Y$ *satisfying* (2.5).

Corollary 2.10. *Let* $r < \frac{2\beta_2}{\beta_1}$ *and* θ *be nonnegative real numbers, and let* $f : X \to Y$ *be an even mapping satisfying* (2.17). *Then there exists a unique quadratic mapping* $Q : X \to Y$ *satisfying* (2.8).

Corollary 2.11. *Let* $r > \frac{\beta_2}{\beta_1}$ *and* θ *be nonnegative real numbers, and let* $f : X \to Y$ *be an odd mapping satisfying* (2.17). *Then there exists a unique additive mapping* $A : X \to Y$ *satisfying* (2.12).

Corollary 2.12. *Let* $r < \frac{\beta_2}{\beta_1}$ *and* θ *be nonnegative real numbers, and let* $f : X \to Y$ *be an odd mapping satisfying* (2.17). *Then there exists a unique additive mapping* $A : X \to Y$ *satisfying* (2.15).

Remark 2.13. If ρ is a real number such that $-1 < \rho < 1$ and Y is a β_2-homogeneous real Banach space, then all the assertions in this section remain valid.

3. ADDITIVE-QUADRATIC ρ-FUNCTIONAL INEQUALITY (0.2)

Throughout this section, assume that ρ is a fixed complex number with $|\rho| < \frac{1}{2}$.

In this section, we investigate the additive-quadratic ρ-functional inequality (0.2) in β-homogeneous complex Banach spaces.

Lemma 3.1. *An even mapping* $f : X \to Y$ *satisfies*

$$\left\| 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - \frac{3}{2}f(x) + \frac{1}{2}f(-x) - \frac{1}{2}f(y) - \frac{1}{2}f(-y) \right\|$$
$$\leq \| \rho(f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)) \| \tag{3.1}$$

for all $x, y \in X$ *if and onlt if* $f : X \to Y$ *is quadratic.*

Proof. Assume that $f : X \to Y$ satisfies (3.1).

Letting $x = y = 0$ in (3.1), we get $\|2f(0)\| \leq |\rho|^{\beta_2}\|2f(0)\|$. So $f(0) = 0$.

Letting $y = 0$ in (3.1), we get

$$\left\| 4f\left(\frac{x}{2}\right) - f(x) \right\| \leq 0 \tag{3.2}$$

S. YUN, G. A. ANASTASSIOU, C. PARK

for all $x \in X$. So $f\left(\frac{x}{2}\right) = \frac{1}{4}f(x)$ for all $x \in X$.

It follows from (3.1) and (3.2) that

$$\frac{1}{2^{\beta_2}}\|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)\|$$

$$= \left\|2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - \frac{3}{2}f(x) + \frac{1}{2}f(-x) - \frac{1}{2}f(y) - \frac{1}{2}f(-y)\right\|$$

$$\leq |\rho|^{\beta_2}\|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)\|$$

and so

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in X$.

The converse is obviously true. $\qquad\square$

Corollary 3.2. *An even mapping $f : X \to Y$ satisfies*

$$2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - \frac{3}{2}f(x) + \frac{1}{2}f(-x) - \frac{1}{2}f(y) - \frac{1}{2}f(-y)$$

$$= \rho\left(f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)\right) \tag{3.3}$$

for all $x, y \in X$ if and only if $f : X \to Y$ is quadratic.

The functional equation (3.3) is called an *additive-quadratic ρ-functional equation*.

We prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (3.1) in β-homogeneous complex Banach spaces for an even mapping case.

Theorem 3.3. *Let $r > \frac{2\beta_2}{\beta_1}$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an even mapping such that*

$$\left\|2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - \frac{3}{2}f(x) + \frac{1}{2}f(-x) - \frac{1}{2}f(y) - \frac{1}{2}f(-y)\right\| \tag{3.4}$$

$$\leq \|\rho(f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y))\| + \theta(\|x\|^r + \|y\|^r)$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{2^{\beta_1 r}\theta}{2^{\beta_1 r} - 4^{\beta_2}}\|x\|^r \tag{3.5}$$

for all $x \in X$.

Proof. Letting $x = y = 0$ in (3.4), we get $\|2f(0)\| \leq |\rho|^{\beta_2}\|2f(0)\|$. So $f(0) = 0$.

Letting $y = 0$ in (3.4), we get

$$\left\|4f\left(\frac{x}{2}\right) - f(x)\right\| \leq \theta\|x\|^r \tag{3.6}$$

for all $x \in X$. So

$$\left\|4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\| \leq \sum_{j=l}^{m-1}\left\|4^j f\left(\frac{x}{2^j}\right) - 4^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \leq \sum_{j=l}^{m-1}\frac{4^{\beta_2 j}}{2^{\beta_1 r j}}\theta\|x\|^r \tag{3.7}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.7) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n \to \infty} 4^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.7), we get (3.5).

Since $f : X \to Y$ is even, the mapping $Q : X \to Y$ is even.
It follows from (3.4) that

$$
\left\| 2Q\left(\frac{x+y}{2}\right) + 2Q\left(\frac{x-y}{2}\right) - \frac{3}{2}Q(x) + \frac{1}{2}Q(-x) - \frac{1}{2}Q(y) - \frac{1}{2}Q(-y) \right\|
$$

$$
= \lim_{n\to\infty} 4^{\beta_2 n} \left(\left\| 2f\left(\frac{x+y}{2^{n+1}}\right) + 2f\left(\frac{x-y}{2^{n+1}}\right) - \frac{3}{2}f\left(\frac{x}{2^n}\right) + \frac{1}{2}f\left(\frac{-x}{2^n}\right) - \frac{1}{2}f\left(\frac{y}{2^n}\right) - \frac{1}{2}f\left(\frac{-y}{2^n}\right) \right\| \right)
$$

$$
\leq \lim_{n\to\infty} 4^{\beta_2 n} \left\| \rho\left(f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) - f\left(\frac{-y}{2^n}\right) \right) \right\|
$$

$$
+ \lim_{n\to\infty} \frac{4^{\beta_2 n}\theta}{2^{\beta_1 n r}} (\|x\|^r + \|y\|^r)
$$

$$
= \|\rho(Q(x+y) + Q(x-y) - 2Q(x) - Q(y) - Q(-y))\|
$$

for all $x, y \in X$. So

$$
\left\| 2Q\left(\frac{x+y}{2}\right) + 2Q\left(\frac{x-y}{2}\right) - \frac{3}{2}Q(x) + \frac{1}{2}Q(-x) - \frac{1}{2}Q(y) - \frac{1}{2}Q(-y) \right\|
$$

$$
\leq \|\rho(Q(x+y) + Q(x-y) - 2Q(x) - Q(y) - Q(-y))\|
$$

for all $x, y \in X$. By Lemma 3.1, the mapping $Q : X \to Y$ is quadratic.

Now, let $T : X \to Y$ be another quadratic mapping satisfying (3.5). Then we have

$$
\begin{aligned}
\|Q(x) - T(x)\| &= 4^{\beta_2 n} \left\| Q\left(\frac{x}{2^n}\right) - T\left(\frac{x}{2^n}\right) \right\| \\
&\leq 4^{\beta_2 n} \left(\left\| Q\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right) \right\| + \left\| T\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right) \right\| \right) \\
&\leq \frac{2 \cdot 4^{\beta_2 n} \cdot 2^{\beta_1 r}}{(2^{\beta_1 r} - 4^{\beta_2})2^{\beta_1 n r}} \theta \|x\|^r,
\end{aligned}
$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $Q(x) = T(x)$ for all $x \in X$. This proves the uniqueness of Q. Thus the mapping $Q : X \to Y$ is a unique quadratic mapping satisfying (3.5). $\qquad\square$

Theorem 3.4. *Let $r < \frac{2\beta_2}{\beta_1}$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an even mapping satisfying* (3.4). *Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$
\|f(x) - Q(x)\| \leq \frac{2^{\beta_1 r}\theta}{4^{\beta_2} - 2^{\beta_1 r}} \|x\|^r \tag{3.8}
$$

for all $x \in X$.

Proof. It follows from (3.6) that $\left\| f(x) - \frac{1}{4}f(2x) \right\| \leq \frac{2^{\beta_1 r}\theta}{4^{\beta_2}}\|x\|^r$ for all $x \in X$. Hence

$$
\left\| \frac{1}{4^l}f(2^l x) - \frac{1}{4^m}f(2^m x) \right\| \leq \sum_{j=l}^{m-1} \left\| \frac{1}{4^j}f(2^j x) - \frac{1}{4^{j+1}}f(2^{j+1} x) \right\| \leq \frac{2^{\beta_1 r}\theta}{4^{\beta_2}} \sum_{j=l}^{m-1} \frac{2^{\beta_1 r j}}{4^{\beta_2 j}} \|x\|^r \tag{3.9}
$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.9) that the sequence $\{\frac{1}{4^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n}f(2^n x)\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$
Q(x) := \lim_{n\to\infty} \frac{1}{4^n}f(2^n x)
$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.9), we get (3.8).

The rest of the proof is similar to the proof of Theorem 3.3. $\qquad\square$

S. YUN, G. A. ANASTASSIOU, C. PARK

Lemma 3.5. *An odd mapping $f : X \to Y$ satisfies (3.1) if and only if $f : X \to Y$ is additive.*

Proof. Assume that $f : X \to Y$ satisfies (3.1).

Letting $x = y = 0$ in (3.1), we get $\|2f(0)\| \leq |\rho|^{\beta_2} \|2f(0)\|$. So $f(0) = 0$.

Letting $y = 0$ in (3.1), we get

$$\left\| 4f\left(\frac{x}{2}\right) - 2f(x) \right\| \leq 0 \tag{3.10}$$

for all $x \in X$. So $f\left(\frac{x}{2}\right) = \frac{1}{2}f(x)$ for all $x \in X$.

It follows from (3.1) and (3.10) that

$$\frac{1}{2^{\beta_2}} \|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)\|$$

$$= \left\| 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - \frac{3}{2}f(x) + \frac{1}{2}f(-x) - \frac{1}{2}f(y) - \frac{1}{2}f(-y) \right\|$$

$$\leq |\rho|^{\beta_2} \|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)\|$$

and so

$$f(x+y) + f(x-y) = 2f(x)$$

for all $x, y \in X$.

The converse is obviously true. $\qquad\square$

Corollary 3.6. *An odd mapping $f : X \to Y$ satisfies (3.3) if and only if $f : X \to Y$ is additive.*

We prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (3.1) in β-homogeneous complex Banach spaces for an odd mapping case.

Theorem 3.7. *Let $r > \frac{\beta_2}{\beta_1}$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an odd mapping satisfying (3.4). Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \leq \frac{2^{\beta_1 r} \theta}{(2^{\beta_1 r} - 2^{\beta_2}) 2^{\beta_2}} \|x\|^r \tag{3.11}$$

for all $x \in X$.

Proof. Letting $x = y = 0$ in (3.4), we get $\|2f(0)\| \leq |\rho|^{\beta_2} \|2f(0)\|$. So $f(0) = 0$.

Letting $y = 0$ in (3.4), we get

$$\left\| 4f\left(\frac{x}{2}\right) - 2f(x) \right\| \leq \theta \|x\|^r \tag{3.12}$$

for all $x \in X$. So

$$\left\| 2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right) \right\| \leq \sum_{j=l}^{m-1} \left\| 2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right) \right\| \leq \sum_{j=l}^{m-1} \frac{2^{\beta_2 j}}{2^{\beta_1 r j}} \frac{\theta}{2^{\beta_2}} \|x\|^r \tag{3.13}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.13) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n \to \infty} 2^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.13), we get (3.11).

Since $f : X \to Y$ is odd, the mapping $A : X \to Y$ is odd.

ADDITIVE-QUADRATIC ρ-FUNCTIONAL INEQUALITIES

It follows from (3.4) that

$$\left\| 2A\left(\frac{x+y}{2}\right) + 2A\left(\frac{x-y}{2}\right) - \frac{3}{2}A(x) + \frac{1}{2}A(-x) - \frac{1}{2}A(y) - \frac{1}{2}A(-y) \right\|$$

$$= \lim_{n\to\infty} 2^{\beta_2 n} \left(\left\| 2f\left(\frac{x+y}{2^{n+1}}\right) + 2f\left(\frac{x-y}{2^{n+1}}\right) - \frac{3}{2}f\left(\frac{x}{2^n}\right) + \frac{1}{2}f\left(\frac{-x}{2^n}\right) - \frac{1}{2}f\left(\frac{y}{2^n}\right) - \frac{1}{2}f\left(\frac{-y}{2^n}\right) \right\| \right)$$

$$\leq \lim_{n\to\infty} 2^{\beta_2 n} \left\| \rho\left(f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) - f\left(\frac{-y}{2^n}\right) \right) \right\|$$

$$+ \lim_{n\to\infty} \frac{2^{\beta_2 n}\theta}{2^{\beta_1 nr}}(\|x\|^r + \|y\|^r)$$

$$= \|\rho(A(x+y) + A(x-y) - 2A(x) - A(y) - A(-y))\|$$

for all $x, y \in X$. So

$$\left\| 2A\left(\frac{x+y}{2}\right) + 2A\left(\frac{x-y}{2}\right) - \frac{3}{2}A(x) + \frac{1}{2}A(-x) - \frac{1}{2}A(y) - \frac{1}{2}A(-y) \right\|$$

$$\leq \|\rho(A(x+y) + A(x-y) - 2A(x) - A(y) - A(-y))\|$$

for all $x, y \in X$. By Lemma 3.5, the mapping $A : X \to Y$ is additive.

Now, let $T : X \to Y$ be another additive mapping satisfying (3.11). Then we have

$$\|A(x) - T(x)\| = 2^{\beta_2 n} \left\| A\left(\frac{x}{2^n}\right) - T\left(\frac{x}{2^n}\right) \right\|$$

$$\leq 2^{\beta_2 n} \left(\left\| A\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right) \right\| + \left\| T\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right) \right\| \right)$$

$$\leq \frac{2 \cdot 2^{\beta_2 n} \cdot 2^{\beta_1 r}}{(2^{\beta_1 r} - 2^{\beta_2})2^{\beta_1 nr}} \frac{\theta}{2^{\beta_2}} \|x\|^r,$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$. This proves the uniqueness of A. Thus the mapping $A : X \to Y$ is a unique additive mapping satisfying (3.11). $\square$

Theorem 3.8. *Let $r < \frac{\beta_2}{\beta_1}$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an odd mapping satisfying (3.4). Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \leq \frac{2^{\beta_1 r}\theta}{(2^{\beta_2} - 2^{\beta_1 r})2^{\beta_2}}\|x\|^r \tag{3.14}$$

for all $x \in X$.

Proof. It follows from (3.12) that $\left\| f(x) - \frac{1}{2}f(2x) \right\| \leq \frac{2^{\beta_1 r}\theta}{4^{\beta_2}}\|x\|^r$ for all $x \in X$. Hence

$$\left\| \frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x) \right\| \leq \sum_{j=l}^{m-1} \left\| \frac{1}{2^j}f(2^j x) - \frac{1}{2^{j+1}}f(2^{j+1}x) \right\| \leq \frac{2^{\beta_1 r}\theta}{4^{\beta_2}} \sum_{j=l}^{m-1} \frac{2^{\beta_1 rj}}{2^{\beta_2 j}}\|x\|^r \tag{3.15}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.15) that the sequence $\{\frac{1}{2^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n}f(2^n x)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} \frac{1}{2^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.15), we get (3.14).

The rest of the proof is similar to the proof of Theorem 3.7. $\square$

S. YUN, G. A. ANASTASSIOU, C. PARK

By the triangle inequality, we have

$$\left\| 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - \frac{3}{2}f(x) + \frac{1}{2}f(-x) - \frac{1}{2}f(y) - \frac{1}{2}f(-y) \right\|$$
$$- \left\| \rho\left(f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)\right) \right\|$$
$$\leq \left\| 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - \frac{3}{2}f(x) + \frac{1}{2}f(-x) - \frac{1}{2}f(y) - \frac{1}{2}f(-y) \right.$$
$$\left. - \rho\left(f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)\right) \right\|.$$

As corollaries of Theorems 3.3, 3.4, 3.7 and 3.8, we obtain the Hyers-Ulam stability results for the additive-quadratic ρ-functional equation (3.3) in β-homogeneous complex Banach spaces.

Corollary 3.9. *Let* $r > \frac{2\beta_2}{\beta_1}$ *and* θ *be nonnegative real numbers, and let* $f : X \to Y$ *be an even mapping such that*

$$\left\| 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - \frac{3}{2}f(x) + \frac{1}{2}f(-x) - \frac{1}{2}f(y) - \frac{1}{2}f(-y) \right. \tag{3.16}$$
$$\left. - \rho\left(f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)\right) \right\| \leq \theta(\|x\|^r + \|y\|^r)$$

for all $x, y \in X$. *Then there exists a unique quadratic mapping* $Q : X \to Y$ *satisfying* (3.5).

Corollary 3.10. *Let* $r < \frac{2\beta_2}{\beta_1}$ *and* θ *be nonnegative real numbers, and let* $f : X \to Y$ *be an even mapping satisfying* (3.16). *Then there exists a unique quadratic mapping* $Q : X \to Y$ *satisfying* (3.8).

Corollary 3.11. *Let* $r > \frac{\beta_2}{\beta_1}$ *and* θ *be nonnegative real numbers, and let* $f : X \to Y$ *be an odd mapping satisfying* (3.16). *Then there exists a unique additive mapping* $A : X \to Y$ *satisfying* (3.11).

Corollary 3.12. *Let* $r < \frac{\beta_2}{\beta_1}$ *and* θ *be nonnegative real numbers, and let* $f : X \to Y$ *be an odd mapping satisfying* (3.16). *Then there exists a unique additive mapping* $A : X \to Y$ *satisfying* (3.14).

Remark 3.13. If ρ is a real number such that $-\frac{1}{2} < \rho < \frac{1}{2}$ and Y is a β_2-homogeneous real Banach space, then all the assertions in this section remain valid.

REFERENCES

[1] M. Adam, *On the stability of some quadratic functional equation*, J. Nonlinear Sci. Appl. **4** (2011), 50–59.
[2] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64–66.
[3] J. Bae and W. Park, *Approximate bi-homomorphisms and bi-derivations in C^*-ternary algebras,* Bull. Korean Math. Soc. **47** (2010), 195–209.
[4] L. Cădariu, L. Găvruta and P. Găvruta, *On the stability of an affine functional equation,* J. Nonlinear Sci. Appl. **6** (2013), 60–67.
[5] A. Chahbi and N. Bounader, *On the generalized stability of d'Alembert functional equation*, J. Nonlinear Sci. Appl. **6** (2013), 198–204.
[6] P. W. Cholewa, *Remarks on the stability of functional equations*, Aequationes Math. **27** (1984), 76–86.
[7] W. Fechner, *Stability of a functional inequalities associated with the Jordan-von Neumann functional equation*, Aequationes Math. **71** (2006), 149–161.
[8] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.
[9] A. Gilányi, *Eine zur Parallelogrammgleichung äquivalente Ungleichung*, Aequationes Math. **62** (2001), 303–309.
[10] A. Gilányi, *On a problem by K. Nikodem*, Math. Inequal. Appl. **5** (2002), 707–710.
[11] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. U.S.A. **27** (1941), 222–224.
[12] C. Park, Y. Cho and M. Han, *Functional inequalities associated with Jordan-von Neumann-type additive functional equations*, J. Inequal. Appl. **2007** (2007), Article ID 41820, 13 pages.

ADDITIVE-QUADRATIC ρ-FUNCTIONAL INEQUALITIES

[13] C. Park, K. Ghasemi, S. G. Ghaleh and S. Jang, *Approximate n-Jordan ∗-homomorphisms in C^*-algebras,* J. Comput. Anal. Appl. **15** (2013), 365-368.
[14] C. Park, A. Najati and S. Jang, *Fixed points and fuzzy stability of an additive-quadratic functional equation,* J. Comput. Anal. Appl. **15** (2013), 452–462.
[15] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.
[16] J. Rätz, *On inequalities associated with the Jordan-von Neumann functional equation*, Aequationes Math. **66** (2003), 191–200.
[17] S. Rolewicz, *Metric Linear Spaces*, PWN-Polish Scientific Publishers, Warsaw, 1972.
[18] S. Schin, D. Ki, J. Chang and M. Kim, *Random stability of quadratic functional equations: a fixed point approach*, J. Nonlinear Sci. Appl. **4** (2011), 37–49.
[19] S. Shagholi, M. Bavand Savadkouhi and M. Eshaghi Gordji, *Nearly ternary cubic homomorphism in ternary Fréchet algebras,* J. Comput. Anal. Appl. **13** (2011), 1106–1114.
[20] S. Shagholi, M. Eshaghi Gordji and M. Bavand Savadkouhi, *Stability of ternary quadratic derivation on ternary Banach algebras,* J. Comput. Anal. Appl. **13** (2011), 1097–1105.
[21] D. Shin, C. Park and Sh. Farhadabadi, *On the superstability of ternary Jordan C^*-homomorphisms,* J. Comput. Anal. Appl. **16** (2014), 964–973.
[22] D. Shin, C. Park and Sh. Farhadabadi, *Stability and superstability of J^*-homomorphisms and J^*-derivations for a generalized Cauchy-Jensen equation,* J. Comput. Anal. Appl. **17** (2014), 125–134.
[23] F. Skof, *Propriet locali e approssimazione di operatori*, Rend. Sem. Mat. Fis. Milano **53** (1983), 113–129.
[24] S. M. Ulam, *A Collection of the Mathematical Problems*, Interscience Publ. New York, 1960.
[25] L. G. Wang and B. Liu, *The Hyers-Ulam stability of a functional equation deriving from quadratic and cubic functions in quasi-β-normed spaces*, Acta Math. Sin., Engl. Ser. **26** (2010), 2335–2348.
[26] Z. H. Wang and W. X. Zhang, *Fuzzy stability of quadratic-cubic functional equations*, Acta Math. Sin., Engl. Ser. **27** (2011), 2191–2204.
[27] T. Z. Xu, J. M. Rassias and W. X. Xu, *Generalized Hyers-Ulam stability of a general mixed additive-cubic functional equation in quasi-Banach spaces*, Acta Math. Sin., Engl. Ser. **28** (2012), 529–560.

Sungsik Yun

Department of Financial Mathematics, Hanshin University, Gyeonggi-do 447-791, Korea
E-mail address: ssyun@hs.ac.kr

George A. Anastassiou

Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152, USA
E-mail address: ganastss@memphis.edu

Choonkil Park

Research Institute for Natural Sciences, Hanyang University, Seoul 133-791, Korea
E-mail address: baak@hanyang.ac.kr

A note on stochastic functional differential equations driven by G-Brownian motion with discontinuous drift coefficients

Faiz Faizullah[*], **Aamir Mukhtar**[1], **M. A. Rana**[1]

[*]Department of BS and H, College of E and ME, National University
of Sciences and Technology (NUST) Pakistan.
[1]Department of Basic Sciences, Riphah International University, Islamabad, Pakistan.

March 27, 2015

Abstract

In the fields of sciences and engineering, the role of discontinuous functions is of immense importance. Heaviside function, for instance, describes the switching process of voltage in an electrical circuit through mathematical process. The current paper aims at exploring the existence theory for stochastic functional differential equations driven by G-Brownian motion (G-SFDEs) whose drift coefficients may not be continuous. It is ascertain that G-SFDEs with discontinuous drift coefficients have more than one bounded and continuous solutions.

Key words: Stochastic functional differential equations, discontinuous drift coefficints, G-Brownian motion, existence.

1 Introduction

For the purpose of analysis and formulation of systems pertaining to engineering, economics and social sciences, stochastic dynamical systems play an important role. Through these equations, while considering the present status, one reconstructs the history and predicts the future of the dynamical systems. On the other hand, in several applications, analysis of the modeling system predicts that the change rate of the system's existing status depends not only on the state that is prevalent but also on the precedent record of the system. This leads to stochastic functional differential equations. The stochastic functional differential equations driven by G-Brownian motion (G-SFDEs) with Lipschitz continuous coefficients was initiated by Ren et.al. [12]. Afterwards, Faizullah used the Caratheodory approximation scheme for developing the existence and uniqueness of solution for G-SFDEs with continuous coefficients [3]. On the other hand, in this case, we study

---

[*]Corresponding author, E-mail: faiz_math@yahoo.com

1

the existence theory for G-SFDEs with discontinuous drift coefficients, such as in the following G-SFDE

$$dX(t) = H(X_t)dt + d\langle B \rangle(t) + dB(t),$$

where $H : \mathbb{R} \to \mathbb{R}$ is the Heaviside function defined by

$$H(x) = \begin{cases} 0, & \text{if } x < 0 \ ; \\ 1, & \text{if } x \geq 0. \end{cases}$$

The above mentioned equations arise, when we take into account the effects of background noise switching systems with delays [5]. For more details on SDEs with discontinuous drift coefficients see [4, 7]. The following stochastic functional differential equation driven by G-Brownian motion (G-SFDE) with finite delay is considered

$$dX(t) = \alpha(t, X_t)dt + \beta(t, X_t)d\langle B, B \rangle(t) + \sigma(t, X_t)dB(t), \ 0 \leq t \leq T, \tag{1.1}$$

where $X(t)$ is the value of stochastic process at time t and $X_t = \{X(t + \theta) : -\tau \leq \theta \leq 0\}$ is a $BC([-\tau, 0]; \mathbb{R})$-valued stochastic process, which represents the family of bounded continuous $\mathbb{R}$-valued functions φ defined on $[-\tau, 0]$ having norm $\|\varphi\| = \sup_{-\tau \leq \theta \leq 0} |\varphi(\theta)|$. Let $\alpha : [0, T] \times BC([-\tau, 0]; \mathbb{R}) \to \mathbb{R}$, $\beta : [0, T] \times BC([-\tau, 0]; \mathbb{R}) \to \mathbb{R}$ and $\sigma : [0, T] \times BC([-\tau, 0]; \mathbb{R}) \to \mathbb{R}$ are Borel measurable. The condition $\xi(0) \in \mathbb{R}$ is given , $\{\langle B, B \rangle(t), t \geq 0\}$ is the quadratic variation process of G-Brownian motion $\{B(t), t \geq 0\}$ and $\alpha, \beta, \sigma \in M_G^2([-\tau, T]; \mathbb{R})$. Let $\mathbb{L}^2$ denote the space of all $\mathcal{F}_t$-adapted process $X(t), 0 \leq t \leq T$, such that $\| X \|_{\mathbb{L}^2} = \sup_{-\tau \leq t \leq T} |X(t)| < \infty$. We define the initial condition of equation (1.1) as follows;

$$X_{t_0} = \xi = \{\xi(\theta) : -\tau < \theta \leq 0\} \ is \ \mathcal{F}_0 - measurable, \ BC([-\tau, 0]; \mathbb{R}) - \ valued$$
$$random \ variable \ such \ that \ \xi \in M_G^2([-\tau, 0]; \mathbb{R}) \,. \tag{1.2}$$

G-SFDEs (1.1) with initial condition (1.2) can be written in the following integral form;

$$X(t) = \xi(0) + \int_0^t \alpha(s, X_s)ds + \int_0^t \beta(s, X_s)d\langle B, B \rangle(s) + \int_0^t \sigma(s, X_s)dB(s).$$

Consider the following linear growth and Lipschitz conditions respectively.

(i) For any $t \in [0, T]$, $|\alpha(t, x)|^2 + |\beta(t, x)|^2 + |\sigma(t, x)|^2 \leq K(1 + |x|^2), K > 0$.

(ii) For all $x, y \in (BC[-\tau, 0]; \mathbb{R})$ and $t \in [0, T]$, $|\alpha(t, x) - \alpha(t, y)|^2 + |\beta(t, x) - |\beta(t, y)|^2 + |\sigma(t, x) - \sigma(t, y)|^2 \leq K(x - y)^2, K > 0$.

The above G-SFDE has a unique solution $X(t) \in M_G^2([-\tau, T]; \mathbb{R})$ if all the coefficients α, β and σ satisfy the Linear growth and Lipschitz conditions [3, 12]. However, we suppose that the drift coefficient α does not need to be continuous. The solution of equation 1.1 with initial condition 1.2 is an $\mathbb{R}$ valued stochastic processes $X(t), t \in [-\tau, T]$ if

2

(i) $X(t)$ is path-wise continuous and $\mathcal{F}_t$-adapted for all $t \in [0, T]$;

(ii) $\alpha(t, X_t) \in \mathcal{L}^1([o, T]; \mathbb{R})$ and $\beta(t, X_t), \sigma(t, X_t) \in \mathcal{L}^2([o, T]; \mathbb{R})$;

(iii) $X_0 = \xi$ and for each $t \in [0, T]$, $dX(t) = \alpha(t, X_t)dt + \beta(t, X_t)d\langle B, B\rangle(t) + \sigma(t, X_t)dB(t)$ q.s.

In the subsequent section, some preliminaries are given whereas in section 3, the comparison theorem is developed. The last section, shows that under some suitable conditions, the G-SFDE (1.1), provides more than one solutions.

2 Basic concepts and notions

In this section, we give some notions and basic definitions of the sublinear expectation [1, 2, 10, 11, 13]. Let Ω be a (non-empty) basic space and $\mathcal{H}$ be a linear space of real valued functions defined on Ω such that any arbitrary constant $c \in \mathcal{H}$ and if $X \in \mathcal{H}$ then $|X| \in \mathcal{H}$. We consider that $\mathcal{H}$ is the space of random variables.

Definition 2.1. A functional $\mathbb{E} : \mathcal{H} \to \mathbb{R}$ is called sub-linear expectation, if $\forall\ X, Y \in \mathcal{H}$, $c \in \mathbb{R}$ and $\lambda \geq 0$ it satisfies the following properties

(1) (Monotonicity): If $X \geq Y$ then $\mathbb{E}[X] \geq \mathbb{E}[Y]$.

(2) (Constant preserving): $\mathbb{E}[c] = c$.

(3) (Sub-additivity): $\mathbb{E}[X + Y] \leq \mathbb{E}[X] + \mathbb{E}[Y]$.

(4) (Positive homogeneity): $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X]$.

The triple $(\Omega, \mathcal{H}, \mathbb{E})$ is called a sublinear expectation space. Consider the space of random variables $\mathcal{H}$ such that if $X_1, X_2, ..., X_n \in \mathcal{H}$ then $\varphi(X_1, X_2, ..., X_n) \in \mathcal{H}$ for each $\varphi \in \mathbb{C}_{l.Lip}(\mathbb{R}^n)$, where $\mathbb{C}_{l.Lip}(\mathbb{R}^n)$ is the space of linear functions φ defined as the following

$$\mathbb{C}_{l.Lip}(\mathbb{R}^n) = \{\varphi : \mathbb{R}^n \to \mathbb{R} \mid \exists C \in [0, \infty)\ : \forall\ x, y \in \mathbb{R}^n,$$
$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^C + |y|^C)|x - y|\}.$$

G-expectation and G-Brownian Motion. Let $\Omega = C_0([0, \infty))$, that is, the space of all $\mathbb{R}$-valued continuous paths $(w_t)_{t \in [0, \infty)}$ with $w_0 = 0$ equipped with the distance

$$\rho(w^1, w^2) = \sum_{k=1}^{\infty} \frac{1}{2^k} (\max_{t \in [0, k]} |w_t^1 - w_t^2| \wedge 1),$$

and consider the canonical process $B_t(w) = w_t$ for $t \in [0, \infty)$, $w \in \Omega$ then for each fixed $T \in [0, \infty)$ we have

$$L_{ip}(\Omega_T) = \{\varphi(B_{t_1}, B_{t_2}, ..., B_{t_n}) : t_1, ..., t_n \in [0, T], \varphi \in \mathbb{C}_{l.Lip}(\mathbb{R}^n), n \in \mathbb{N}\},$$

3

where $L_{ip}(\Omega_t) \subseteq L_{ip}(\Omega_T)$ for $t \leq T$ and $L_{ip}(\Omega) = \cup_{m=1}^{\infty} L_{ip}(\Omega_m)$.

Consider a sequence $\{\xi_i\}_{i=1}^{\infty}$ of random variables on a sublinear expectation space $(\hat{\Omega}, \hat{\mathcal{H}}_p, \hat{\mathbb{E}})$ such that ξ_{i+1} is independent of $(\xi_1, \xi_2, ..., \xi_i)$ for each $i = 1, 2, ...$ and ξ_i is G-normally distributed for each $i \in \{1, 2, ...\}$. Then a sublinear expectation $\mathbb{E}[.]$ defined on $L_{ip}(\Omega)$ is introduced as follows.

For $0 = t_0 < t_1 < ... < t_n < \infty$ $(t_0, t_1, ..., t_n \in [t, \infty))$ [13], $\varphi \in \mathbb{C}_{l.Lip}(\mathbb{R}^n)$ and each

$$X = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}}) \in L_{ip}(\Omega),$$

$$\mathbb{E}[\varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}})]$$
$$= \hat{\mathbb{E}}[\varphi(\sqrt{t_1 - t_0}\xi_1, ..., \sqrt{t_n - t_{n-1}}\xi_n)].$$

The conditional sublinear expectation of $X \in L_{ip}(\Omega_t)$ is defined by

$$\mathbb{E}[X|\Omega_t] = \mathbb{E}[\varphi(B_{t_1}, B_{t_2} - B_{t_1}, ..., B_{t_m} - B_{t_{m-1}})|\Omega_t]$$
$$= \psi(B_{t_1}, B_{t_2} - B_{t_1}, ..., B_{t_j} - B_{t_{j-1}}),$$

where
$$\psi(x_1, ..., x_j) = \hat{\mathbb{E}}[\varphi(x_1, ..., x_j, \sqrt{t_{j+1} - t_j}\xi_{j+1}, ..., \sqrt{t_n - t_{n-1}}\xi_n)].$$

Definition 2.2. The sublinear expectation $\mathbb{E} : L_{ip}(\Omega) \to \mathbb{R}$ defined above is called a G-expectation and the corresponding canonical process $\{B_t, t \geq 0\}$ is called a G-Brownian motion.

The completion of $L_{ip}(\Omega)$ under the norm $\|X\|_p = (\mathbb{E}[|X|^p])^{1/p}$ [11, 13] for $p \geq 1$ is denoted by $L_G^p(\Omega)$ and $L_G^p(\Omega_t) \subseteq L_G^p(\Omega_T) \subseteq L_G^p(\Omega)$ for $0 \leq t \leq T < \infty$. The filtration generated by the canonical process $(B_t)_{t \geq 0}$ is denoted by $\mathcal{F}_t = \sigma\{B_s, 0 \leq s \leq t\}$, $\mathcal{F} = \{\mathcal{F}_t\}_{t \geq 0}$.

Itô's Integral of G-Brownian motion. For any $T \in [0, \infty)$, a finite ordered subset $\pi_T = \{t_0, t_1, ..., t_N\}$ such that $0 = t_0 < t_1 < ... < t_N = T$ is a partition of $[0, T]$ and

$$\mu(\pi_T) = \max\{|t_{i+1} - t_i| : i = 0, 1, ..., N - 1\}.$$

A sequence of partitions of $[0, T]$ is denoted by $\pi_T^N = \{t_0^N, t_1^N, ..., t_N^N\}$ such that $\lim_{N \to \infty} \mu(\pi_T^N) = 0$.

Consider the following simple process:
Let $p \geq 1$ be fixed. For a given partition $\pi_T = \{t_0, t_1, ..., t_N\}$ of $[0, T]$,

$$\eta_t(w) = \sum_{i=0}^{N-1} \xi_i(w) I_{[t_i, t_{i+1}]}(t), \tag{2.1}$$

where $\xi_i \in L_G^p(\Omega_{t_i})$, $i = 0, 1, ..., N - 1$. The collection containing the above type of processes, that is, containing $\eta_t(w)$ is denoted by $M_G^{p,0}(0, T)$. The completion of $M_G^{p,0}(0, T)$ under the norm $\|\eta\| = \{\int_0^T \mathbb{E}[|\eta_u|^p]du\}^{1/p}$ is denoted by $M_G^p(0, T)$ and for $1 \leq p \leq q$, $M_G^p(0, T) \supset M_G^q(0, T)$.

4

Definition 2.3. For each $\eta_t \in M_G^{2,0}(0,T)$, the Itô's integral of G-Brownian motion is defined as

$$I(\eta) = \int_0^T \eta_u dB_u = \sum_{i=0}^{N-1} \xi_i (B_{t_{i+1}} - B_{t_i}).$$

Definition 2.4. An increasing continuous process $\{\langle B \rangle_t : t \geq 0\}$ with $\langle B \rangle_0 = 0$, defined by

$$\langle B \rangle_t = B_t^2 - 2 \int_0^t B_u dB_u,$$

is called the quadratic variation process of G-Brownian motion.

3 Comparison theorem for G-SFDEs

The purpose of this section is to establish comparison result for problem (1.1) with initial data (1.2). Consider the following two stochastic functional differential equations

$$X(t) = \xi^1(0) + \int_{t_0}^t \alpha_1(s, X_s)ds + \int_{t_0}^t \beta(s, X_s)d\langle B, B \rangle(s) + \int_{t_0}^t \sigma(s, X_s)dB(s), \quad t \in [0, T], \quad (3.1)$$

$$X(t) = \xi^2(0) + \int_{t_0}^t \alpha_2(s, X_s)ds + \int_{t_0}^t \beta(s, X_s)d\langle B, B \rangle(s) + \int_{t_0}^t \sigma(s, X_s)dB(s), \quad t \in [0, T]. \quad (3.2)$$

Theorem 3.1. *Assume that:*

(**i**) X^1 *and* X^2 *are unique strong solutions of problems* (3.1) *and* (3.2) *respectively.*

(**ii**) $\alpha_1(s, X_s) \leq \alpha_2(s, X_s)$ *componentwise for all* $t \in [t_0, T]$, $x \in BC([-\tau, 0]; \mathbb{R}^d)$ *and* $\xi^1 \leq \xi^2$.

(**iii**) α_1 *or* α_2 *is increasing such that* $f(t, x) \leq f(t, y)$ *when* $x \leq y$ *for all* $x, y \in C([-\tau, 0]; \mathbb{R})$.

Then for all $t > 0$ *we have* $X^1 \leq X^2$ *q.s.*

Proof. First, we define an operator $q(.,.) : C([-\tau, 0]; \mathbb{R}) \times C([-\tau, 0]; \mathbb{R}) \to C([-\tau, 0]; \mathbb{R})$ such that

$$q(x, y) = \max[x, y].$$

Obviously, $y \to q(x, y)$ satisfies the linear growth and Lipschitz conditions. Now we suppose that α_2 is increasing and consider the following equation

$$Y(t) = \xi^2(0) + \int_{t_0}^t \alpha_2(s, q(X_s^1, Y_s))ds + \int_{t_0}^t \beta(s, q(X_s^1, Y_s))d\langle B, B \rangle(s)$$
$$+ \int_{t_0}^t \sigma(s, q(X_s^1, Y_s))dB(s), \quad t_0 \leq t \leq T. \quad (3.3)$$

5

Thus it is easy to see that the coefficients satisfy the linear growth and Lipschitz conditions, so 3.3 has a unique solution $Y(t)$. We shall prove that $Y(t) \geq X_s^1$ q.s. We define the following two stopping times. For more details on stopping times we refere the reader to [7, 8].

$$\tau_1 = \inf\{t \in [t_0, T] : X_s^1 - Y(t) > 0\} \ where \ \tau_1 < T,$$
$$\tau_2 = \inf\{t \in [\tau_1, T] : X_s^1 - Y(t) < 0\}.$$

Contrary suppose that there exist an interval $(\tau_1, \tau_2) \subset [t_0, T]$ such that $Y(\tau_1) = X^1(\tau_1) = \xi^*(0)$ and $Y(t) \leq X^1(t)$ for all $t \in (\tau_1, \tau_2)$. Then,

$$Y(t) - X^1(t) = \xi^*(0) + \int_{\tau_1}^t \alpha_2(s, q(X_s^{\;1}, Y_s))ds + \int_{\tau_1}^t \beta(s, q(X_s^{\;1}, Y_s))d\langle B, B\rangle(s)$$

$$+ \int_{\tau_1}^t \sigma(s, q(X_s^{\;1}, Y_s))dB(s) - \xi^*(0) - \int_{\tau_1}^t \alpha_1(s, X_s^{\;1})ds$$

$$- \int_{\tau_1}^t \beta(s, X_s^{\;1})d\langle B, B\rangle(s) - \int_{\tau_1}^t \sigma(s, X_s^{\;1})dB(s), \ \ t \in (\tau_1, \tau_2).$$

$$Y(t) - X^1(t) = \int_{\tau_1}^t [\alpha_2(s, q(X_s^{\;1}, Y_s)) - \alpha_1(s, X_s^{\;1})]ds$$

$$+ \int_{\tau_1}^t [\beta(s, q(X_s^{\;1}, Y_s)) - \beta(s, X_s^{\;1})]d\langle B, B\rangle(s)$$

$$+ \int_{\tau_1}^t [\sigma(s, q(X_s^{\;1}, Y_s)) - \sigma(s, X_s^{\;1})]dB(s), \ \ t \in (\tau_1, \tau_2).$$

But our supposition $Y(t) \leq X^1(t)$ yields $q(X^1, Y) = \max[X^1, Y] = X^1$. So, we have

$$Y(t) - X^1(t) = \int_{\tau_1}^t [\alpha_2(s, X_s^{\;1}) - \alpha_1(s, X_s^{\;1})]ds$$

$$+ \int_{\tau_1}^t [\beta(s, X_s^{\;1}) - \beta(s, X_s^{\;1})]d\langle B, B\rangle(s)$$

$$+ \int_{\tau_1}^t [\sigma(s, X_s^{\;1}) - \sigma(s, X_s^{\;1})]dB(s)$$

$$Y(t) - X^1(t) = \int_{\tau_1}^t [\alpha_2(s, X_s^{\;1}) - \alpha_1(s, X_s^{\;1})]ds \geq 0,$$

because $\alpha_2(t, x) \geq \alpha_1(t, x)$. Which gives contradiction. So, our supposition $Y(t) \leq X^1(t)$ for all $t \in (\tau_1, \tau_2)$ is wrong. Thus $Y(t) \geq X^1(t)$ q.s. and so $p(X^1, Y) = Y$. It means that $Y = X^2 \geq X^1$ because G-SFDE (3.3) has a unique solution X^2. □

4 G-SFDEs with discontinuous drift coefficients

We now suppose that α is left continuous, increasing and $\alpha(t, x) \geq 0$ for all $(t, x) \in [0, T] \times BC([-\tau, 0]; \mathbb{R})$ but not continuous. Consider the following sequence of problems.

$$X^n(t) = \xi(0) + \int_0^t \alpha(s, X_s^{n-1})ds + \int_0^t \beta(s, X_s^n)d\langle B, B\rangle(s) + \int_0^t \sigma(s, X_s^n)dB(s), \ t \in [0, T], \quad (4.1)$$

6

where $X^0 = L_t$, L_t is the unique solution of the following problem

$$L_t = \xi + \int_0^t \beta(s, L_s) d\langle B, B\rangle(s) + \int_0^t \sigma(s, L_s) dB(s), t \in [0, T]. \tag{4.2}$$

Thus using the comparison theorem and the fact that $\alpha(t, x) \geq 0$, we have $X^1 \geq L_t$. So, we can see that X^n is an increasing sequence. Now we shall prove that X^n is bounded in $\mathbb{L}^2$ norm.

Lemma 4.1. *Suppose* $X^n(t)$ *be a solution of problem* (4.1) *then there exists a positive constant* C *independent of* n *such that,*

$$E\left(\sup_{-\tau \leq s \leq T} |X^n(s)|^2\right) \leq C.$$

Proof. For any $n \geq 1$ we define the following stopping time in a similar way as given in [9]

$$\tau_m = T \wedge \inf\{t \in [t_0, T] : \|X_t^n\| \geq m\}.$$

We have $\tau_m \uparrow T$ and define $X^{n,m}(t) = X^n(t \wedge \tau_m)$ for $t \in (-\tau, T)$. Then for $t \in [0, T]$,

$$X^{n,m}(t) = \xi(0) + \int_0^t \alpha(t, X_t^{n-1,m}) I_{[o,\tau_m]} dt + \int_0^t \beta(t, X_t^{n,m}) I_{[o,\tau_m]} d\langle B, B\rangle_t + \int_0^t \sigma(t, X_t^{n,m}) I_{[o,\tau_m]} dB_t.$$

$$|X^{n,m}(t)|^2 = |\xi(0) + \int_0^t \alpha(t, X_t^{n-1,m}) I_{[0,\tau_m]} dt + \int_0^t \beta(t, X_t^{n,m}) I_{[0,\tau_m]} d\langle B, B\rangle_t$$

$$+ \int_0^t \sigma(t, X_t^{n,m}) I_{[0,\tau_m]} dB_t|^2$$

$$\leq 4|\xi(0)|^2 + 4|\int_0^t \alpha(t, X_t^{n-1,m}) I_{[0,\tau_m]} dt|^2 + 4|\int_0^t \beta(t, X_t^{n,m}) I_{[0,\tau_m]} d\langle B, B\rangle_t|^2$$

$$+ 4|\int_0^t \sigma(t, X_t^{n,m}) I_{[0,\tau_m]} dB_t|^2$$

Taking G-expectation, using properties of G-integral, G-quadratic variation process [10, 11] and linear growth condition we get

$$E[|X^{n,m}(t)|^2] \leq 4E|\xi(0)|^2 + 4C_1 \int_0^t [1 + E|X_t^{n-1,m}|^2] dt + 4C_2 \int_0^t [1 + E|X_t^{n,m}|^2] dt|$$

$$+ 4C_3 \int_0^t [1 + E|X_t^{n,m}|^2] dt$$

$$\leq 4E|\xi(0)|^2 + 4C_1 \int_0^t dt + 4C_1 \int_0^t E|X_t^{n-1,m}|^2 dt + 4C_2 \int_0^t dt + 4C_2 \int_0^t E|X_t^{n,m}|^2 dt$$

$$+ 4C_3 \int_0^t dt + 4C_3 \int_0^t E|X_t^{n,m}|^2 dt$$

$$= 4E|\xi(0)|^2 + 4C_1 T + 4C_1 \int_0^t E|X_t^{n-1,m}|^2 dt + 4C_2 T + 4C_2 \int_0^t E|X_t^{n,m}|^2 dt$$

$$+ 4C_3 T + 4C_3 \int_0^t E|X_t^{n,m}|^2 dt.$$

7

Then for any $k \in \mathbb{N}$ we have,

$$\max_{1 \leq n \leq k} E[|X^{n,m}(t)|^2] \leq C_4 + 4C_1 \int_0^t \max_{1 \leq n \leq k} E|X_t^{n-1,m}|^2 dt + 4C_2 \int_0^t \max_{1 \leq n \leq k} E|X_t^{n,m}|^2 dt + 4C_3 \int_0^t \max_{1 \leq n \leq k} E|X_t^{n,m}|^2 dt,$$

where $C_4 = 4[E|\xi|^2 + C_1 T + C_2 T + C_3 T]$ and thus using Doob's martingale inequality for any $n, m \in \mathbb{N}$ we have,

$$E[\sup_{0 \leq s \leq t} |X^{n,m}(s)|^2] \leq C_4 + C_5 \int_0^t E|X_s^{n,m}|^2 dt, \tag{4.3}$$

where $C_5 = 4(C_1 + C_2 + C_3)$. One can observe the fact [9],

$$\sup_{-\tau \leq s \leq t} |X^{n,m}(s)|^2 \leq \|\xi\| + \sup_{0 \leq s \leq t} |X^{n,m}(s)|^2,$$

and thus 4.3 yields

$$E[\sup_{-\tau \leq s \leq t} |X^{n,m}(s)|^2] \leq E[\|\xi\|] + C_4 + C_5 \int_0^t E|X_s^{n,m}|^2 dt$$

$$\leq C_6 + C_5 \int_0^t E[\sup_{-\tau \leq r \leq s} |X^{n,m}(r)|^2] dt,$$

where $C_6 = E[\|\xi\|] + C_4$. So, using the Gronwall inequality and taking $m \to \infty$ we have,

$$E[\sup_{-\tau \leq s \leq t} |X^n(s)|^2] \leq C_6 e^{C_4 t}.$$

Letting $t = T$ we get the desired result,

$$E[\sup_{-\tau \leq s \leq T} |X^n(s)|^2] \leq C^*, \quad C^* = C_4 e^{CT}.$$

$\square$

Theorem 4.2. *Suppose that:*

  (i) *The coefficient α be left continuous and increasing in the second variable x.*

  (ii) *For all $(t, x) \in [0, T] \times BC([-\tau, 0]; \mathbb{R})$, $\alpha(t, x) \geq 0$.*

Then the G-SFDE (1.1) has more than one solution $X(t) \in M_G^2([-\tau, T]; \mathbb{R})$.

Proof. By theorem 3.1 we know that $\{X^n\}$ is increasing and by Lemma 4.1 it is bounded in $\mathbb{L}^2$. Then by Dominated Convergence theorem we can deduce that X^n converges in $\mathbb{L}^2$. Denoting the limit of X^n by X and thus for almost all w, we get

$$\alpha(t, X^n(t)) \to \alpha(t, X(t)) \ as \ n \to \infty,$$

and

$$|\alpha(t, X^n(t))| \leq K(1 + \sup_n |X^n(t)|) \in L^1([t_0, T]).$$

8

Thus, for almost all w and uniformly in t

$$\int_0^t \alpha(s, X^n(s))ds \to \int_0^t \alpha(s, X(s))ds, \ n \to \infty.$$

By the properties of β, σ and by the continuity properties of G-integral and its quadratic variation process we have,

$$\sup_{0 \leq t \leq T} \left| \int_0^t \beta(s, X^n(s))d\langle B, B\rangle_s - \int_0^t \beta(s, X(s))d\langle B, B\rangle_s \right| \to 0 \ (q.s), \ n \to \infty.$$

$$\sup_{0 \leq t \leq T} \left| \int_0^t \sigma(s, X^n(s))dB(s) - \int_0^t \sigma(s, X(s))dB(s) \right| \to 0 \ (q.s), \ n \to \infty.$$

It is easy to conclude that X^n converges uniformly to X in t, hence X is continuous. Taking limit in equation (4.1), we get that X is the desired solution for stochastic functional differential equation (1.1) with initial condition (1.2). $\square$

5 Acknowledgments

References

[1] Denis L, Hu M, Peng S. Function spaces and capacity related to a sublinear expectation: Application to G-Brownian motion paths. Potential Anal., 2010; 34: 139-161.

[2] Faizullah F. A note on the Caratheodory approximation scheme for stochastic differential equations under G-Brownian motion. Zeitschrift fr Naturforschung A. 2012; 67a: 699-704.

[3] Faizullah F. Existence of solutions for G-SFDEs with Caratheodory Approximation Scheme. Abstract and Applied Analysis. http://dx.doi.org/10.1155/2014/809431, 2014; volume 2014: pages 8.

[4] Faizullah F, Piao D. Existence of solutions for G-SDEs with upper and lower solu- tions in the reverse order. International Journal of the Physical Sciences. 2012; 16(12): 1820-1829.

[5] Halidias N, Ren Y. An existence theorem for stochastic functional differential equations with delays under weak conditions. Statistics and Probability Letters. 2008; 78: 2864-2867.

[6] Halidias N, Kloeden P. A note on strong solutions for stochastic differential equations with discontinuous drift coefficient. J. Appl. Math. Stoch. Anal. doi:10.1155/JAMSA/2006/73257. Article ID 73257., 2006; 78: 1-6.

9

[7] Hu M, Peng S. Extended conditional G-expectations and related stopping times. arXiv:1309.3829v1[math.PR] 16 Sep 2013.

[8] Li X, Peng S. Stopping times and related Ito's calculus with G-Brownian motion. Stochastic Processes and thier Applications. 2011; 121: 1492-1508.

[9] Mao X. *Stochastic differential equations and their applications*. Horwood Publishing Chichester 1997.

[10] Peng S. G-expectation, G-Brownian motion and related stochastic calculus of Ito's type. The abel symposium 2005, Abel symposia 2, edit. benth et. al., Springer-vertag. 2006; 541-567.

[11] Peng S. Multi-dimentional G-Brownian motion and related stochastic calculus under G-expectation. Stochastic Processes and thier Applications. 2008; 12: 2223-2253.

[12] Ren Y, Bi Q, Sakthivel R. Stochastic functional differential equations with infinite delay driven by G-Brownian motion. Mathematical Methods in the Applied Sciences, 2013; 36(13): 1746-1759.

[13] Song Y. Properties of hitting times for G-martingale and their applications. Stochastic Processes and thier Applications. 2011; 8(121): 1770-1784.

10

SUBCLASSES OF JANOWSKI-TYPE FUNCTIONS DEFINED BY CHO-KWON-SRIVASTAVA OPERATOR

SAIMA MUSTAFA, TEODOR BULBOACĂ, AND BADR S. ALKAHTANI

ABSTRACT. We introduce a new subclass of analytic functions in the unit disk U defined by using Cho-Kwon Srivastava integral operator. Inclusion results radius problem and integral preserving properties are investigated.

1. INTRODUCTION

Let $\mathcal{A}_p$ be the class of analytic functions in U of the form

$$(1.1) \qquad f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}, \ z \in \mathrm{U}, \quad (p \in \mathbb{N}),$$

where $\mathbb{N} := \{1, 2, \dots\}$. For $p = 1$ we denotes $\mathcal{A} := \mathcal{A}_1$. Note that the class $\mathcal{A}_p$ is closed under *the convolution (or Hadamard) product*, that is

$$f(z) * g(z) := z^p + \sum_{n=1}^{\infty} a_{p+n} b_{p+n} z^{p+n}, \ z \in \mathrm{U}, \quad (p \in \mathbb{N}),$$

where f is given by (1.1) and $g(z) = z^p + \sum_{n=1}^{\infty} b_{p+n} z^{p+n}, \ z \in \mathrm{U}$.

The operator $L^p(d, e) : \mathcal{A}_p \to \mathcal{A}_p$ is defined by using the Hadamard (convolution) product, that is

$$(1.2) \qquad L^p(d, e) f(z) := f(z) * \varphi_p(d, e; z),$$

where

$$\varphi_p(d, e; z) := z^p + \sum_{n=1}^{\infty} \frac{(d)_n}{(e)_n} z^{p+n}, \quad \left(d \in \mathbb{C}, \ e \in \mathbb{C} \setminus \mathbb{Z}_0^-\right),$$

and $(d)_n = d(d+1)\dots(d+n-1)$, with $(d)_0 = 1$, represents the well-known *Pochhammer symbol*.

From (1.2) it follows immediately that

$$z\left(L^p(d, e) f(z)\right)' = d L^p(d+1, e) f(z) - (d-p) L^p(d, e) f(z).$$

The operator $L^p(d, e)$ was introduced by Saitoh [16] and this is an extension of the operator $L(d, e)$ which was defined by Carlson and Shaffer [2].

1

2   S. MUSTAFA, T. BULBOACĂ, AND B. S. ALKAHTANI

Analogous to the $L^p(d,e)$ operator, Cho et. al. [4] introduced the operator $I_\mu^p(d,e) : \mathcal{A}_p \to \mathcal{A}_p$ defined by

$$(1.3) \qquad I_\mu^p(d,e)f(z) := f(z) * \varphi_p^\dagger(d,e;z),$$

where

$$\varphi_p^\dagger(d,e;z) := z^p + \sum_{n=1}^\infty \frac{(\mu+p)_n(e)_n}{n!(d)_n} z^{p+n}, \quad \left(d,e \in \mathbb{C} \setminus \mathbb{Z}_0^-, \ \mu > -p\right).$$

We notice that

$$\varphi_p^\dagger(d,e;z) * \varphi_p(d,e;z) = \frac{z^p}{(1-z)^{\mu+p}}, \ z \in \mathrm{U}.$$

From (1.3), the following identities can be easily obtained [4]:

$$(1.4) \quad z\left(I_\mu^p(d+1,e)f(z)\right)' = dI_\mu^p(d,e)f(z) - (d-p)\,I_\mu^p(d+1,e)f(z),$$

$$(1.5) \quad z\left(I_\mu^p(d,e)f(z)\right)' = (\mu+p)I_{\mu+1}^p(d,e)f(z) - \mu I_\mu^p(d,e)f(z).$$

We may easily remark the following relations

$$I_1^p(p+1,1)f(z) = f(z), \quad I_1^p(p,1)f(z) = \frac{zf'(z)}{p},$$

and remark that the operator $I_\mu^1(a+2,1)$, with $\mu > -1$ and $a > -2$, was studied in [5].

If f and g are two analytic functions in U, we say that *f is subordinate to g*, written symbolically as $f(z) \prec g(z)$, if there exists a *Schwarz function w*, which (by definition) is analytic in U, with $w(0) = 0$, and $|w(z)| < 1$, $z \in \mathrm{U}$, such that $f(z) = g(w(z))$, for all $z \in \mathrm{U}$. Furthermore, if the function g is univalent in U, then we have the following equivalence, (cf., e.g., [10], see also [11, p. 4]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathrm{U}) \subset g(\mathrm{U}).$$

Definition 1.1. *1. Like in [3], for arbitrary fixed numbers A, B and β, with $-1 \le B < A \le 1$ and $0 \le \beta < 1$, let $P[A,B,\beta]$ denote the family of functions p that are analytic in U, with $p(0) = 1$, and such that*

$$p(z) \prec \frac{1 + [(1-\beta)A + \beta B]\,z}{1 + Bz}.$$

We will use the notations $P[A,B] := P[A,B,0]$ and $P(0) := P[1,-1,0]$.

2. Let $P_l[A,B,\beta]$ denote the class of functions p that are analytic in U, with $p(0) = 1$, that are represented by

$$(1.6) \qquad p(z) = \left(\frac{l}{4} + \frac{1}{2}\right)K_1(z) - \left(\frac{l}{4} - \frac{1}{2}\right)K_2(z),$$

where $K_1, K_2 \in P[A,B,\beta]$, $-1 \le B < A \le 1$, $0 \le \beta < 1$, and $l \ge 2$.

Remarks 1.1. (*i*) Remark that the class $\mathcal{P}_l(\beta) := P_l[1,-1,\beta]$ was defined and studied in [12], while for $l = 2$ and $\beta = 0$ the above class was introduced by Janowski [8]. Moreover, the class $P_l := P_l[1,-1,0]$ is the well-known class of Pinchuk [15].

Also, we see that $P_l[A,B,\beta] \subset \mathcal{P}_l(\widetilde{\beta})$, where $\widetilde{\beta} = \dfrac{1 - A_1}{1 - B}$ and $A_1 = (1-\beta)A + \beta B$.

(*ii*) Notice that, if g is analytic in U with $g(0) = 1$, then there exit functions g_1 and g_2 analytic in U with $g_1(z) = g_2(z) = 1$, such that the function g could be written in the form (1.6). For example, taking

$$g_1(z) = \frac{g(z) - 1}{k} + \frac{g(z) + 1}{2} \quad \text{and} \quad g_1(z) = \frac{g(z) + 1}{2} - \frac{g(z) - 1}{k},$$

then g_1 and g_2 are analytic in U, and $g_1(z) = g_2(z) = 1$.

We will assume throughout our discussion, unless otherwise stated, that $\lambda > 0$, $d, e \in \mathbb{R} \setminus \mathbb{Z}_0^-$, $\mu > -p$, $-1 \le B < A \le 1$, $\vartheta \ge 0$, and $p \in \mathbb{N}$. Moreover, all the powers are the principal ones.

Using the Cho-Kwon-Srivastava integral operator $I_\mu^p(d,e)$ defined by (1.4), we will define the following subclasses of $\mathcal{A}_p$.

Definition 1.2. *Let* $d, e \in \mathbb{R} \setminus \mathbb{Z}_0^-$, $\lambda > 0$, $\mu > -p$, $0 \le \beta < 1$, *and* $\vartheta \ge 0$. *For the function* $f \in \mathcal{A}_p$, $p \in \mathbb{N}$, *we say that* $f \in \mathcal{N}_{l,p}^{\lambda,\vartheta}(d,e;\mu;\beta,A,B)$, *with* $l \ge 2$, *if and only if*

$$(1+\vartheta)\left(\frac{z^p}{I_\mu^p(d,e)f(z)}\right)^\lambda - \vartheta\frac{I_{\mu+1}^p(d,e)f(z)}{I_\mu^p(d,e)f(z)}\left(\frac{z^p}{I_\mu^p(d,e)f(z)}\right)^\lambda \in P_l[A,B,\beta].$$

We need to notice that, since the left-hand side function from the above definition need to be analytic in U, we implicitly assumed that $I_\mu^p(d,e)f(z) \ne 0$ for all $z \in \dot{\text{U}}$.

Remarks 1.2. We remark the following special cases of the above classes:
(*i*) for $\beta = 0$ and $l = 2$ we obtain the subclass of non-Bazilević functions defined by [18];
(*ii*) for $\mu = 0$, $l = 2$, $\vartheta = B = -1$, $A = 1$ and $\lambda > 0$, the above class reduces to the class $Q(\lambda)$ of p–valent non-Bazilević functions (see [14]).

2. Preliminaries

The following definitions and lemmas will be required in our present investigation.

Lemma 2.1. [7] *Let* h *be a convex function in* U *with* $h(0) = 1$. *Suppose also that the function* p *given by*

$$p(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \dots, \ z \in \text{U},$$

is analytic in U. *Then*

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z) \quad (\text{Re}\,\gamma \ge 0,\ \gamma \ne 0),$$

4   S. MUSTAFA, T. BULBOACĂ, AND B. S. ALKAHTANI

implies

$$(2.1) \qquad p(z) \prec q(z) = \frac{\gamma}{n} z^{-\frac{\gamma}{n}} \int_0^z t^{\frac{\gamma}{n}-1} h(t)\,\mathrm{d}\,t \prec h(z),$$

and q is the best dominant of (2.1).

For real or complex numbers a, b and c, the *Gauss hypergeometric function* is defined by

$$_2F_1(a, b, c; z) = 1 + \frac{a \cdot b}{c} \frac{z}{1!} + \frac{a(a+1) \cdot b(b+1)}{c(c+1)} \frac{z^2}{2!} + \dots$$

$$(2.2) \qquad = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \quad a, b \in \mathbb{C}, \ c \in \mathbb{C} \setminus \{0, -1, -2, \dots\},$$

where $(d)_k$ is the previously recalled Pochhammer symbol. The series (2.2) converges absolutely for $z \in U$, hence it represents an analytic function in U (see [19, Chapter 14]).

Each of the following identities are fairly well-known:

Lemma 2.2. [19, Chapter 14] *For all real or complex numbers a, b and c, with* $c \neq 0, -1, -2, \dots, $ *, the next equalities hold:*

$$(2.3) \qquad \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,\mathrm{d}\,t = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} {}_2F_1(a, b, c; z)$$

$$where \quad \operatorname{Re} c > \operatorname{Re} b > 0,$$

$$(2.4) \qquad {}_2F_1(a, b, c; z) = (1-z)^{-a} {}_2F_1\left(a, c-b, c; \frac{z}{z-1}\right),$$

and

$$(2.5) \qquad {}_2F_1(a, b, c; z) = {}_2F_1(b, a, c; z).$$

Lemma 2.3. [17] *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$ *be analytic in* U *and* $g(z) = \sum_{k=0}^{\infty} b_k z^k$ *be analytic and convex in* U. *If* $f(z) \prec g(z)$, *then*

$$|a_k| \le |b_1|, \ k \in \mathbb{N}.$$

3. MAIN RESULTS FOR THE CLASS $\mathcal{N}_{l,p}^{\lambda,\vartheta}(d, e; \mu; \beta, A, B)$

In this section, some properties of the class $\mathcal{N}_{l,p}^{\lambda,\vartheta}(d, e; \mu; \beta, A, B)$ such as inclusion results, integral preserving property, radius problem, coefficient bound will be discussed.

Theorem 3.1. *1. If* $f \in \mathcal{N}_{l,p}^{\lambda,\vartheta}(d, e; \mu; \beta, A, B)$, *then*

$$\left(\frac{z^p}{I_\mu^p(d, e)f(z)}\right)^\lambda \in P_l[A, B, \beta].$$

2. Moreover, if $f \in \mathcal{N}_{l,p}^{\lambda,\vartheta}(d,e;\mu;\beta,\gamma)$ with $\vartheta \neq 0$, then

$$\left(\frac{z^p}{I_\mu^p(d,e)f(z)}\right)^\lambda \in \mathcal{P}_l(\beta_1),$$

where

$$\beta_1 := \beta + (1-\beta)\vartheta_1$$

and

$$\vartheta_1 := \vartheta_1(p,\lambda,\vartheta,\mu;A,B) =$$

$$\begin{cases} \frac{A}{B} + \left(1-\frac{A}{B}\right)(1-B)^{-1}\,_2F_1\left(1,1,\frac{\lambda(\mu+p)}{\vartheta}+1,\frac{B}{B-1}\right), & B \neq 0, \\ 1 - \frac{\lambda(\mu+p)}{\lambda(\mu+p)+\vartheta}A, & B = 0. \end{cases}$$

(All the powers are the principal ones).

Proof. Since the implication is obvious for $\vartheta = 0$, suppose that $\vartheta > 0$. Letting

(3.1) $$K(z) = \left(\frac{z^p}{I_\mu^p(d,e)f(z)}\right)^\lambda.$$

It follows that K is analytic in U, with $K(0) = 1$, and according to the part (ii) of Remarks 1.1 the function K could be written in the form

(3.2) $$K(z) = \left(\frac{k}{4}+\frac{1}{2}\right)K_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)K_2(z),$$

where K_1 and K_2 are analytic in U, with $K_1(z) = K_2(z) = 1$.

From the part 2. of Definition 1.1 we have that $K \in P_l[A,B,\beta]$, if and only if the function K has the representation given by the above relation, where $K_1, K_2 \in P[A,B,\beta]$. Consequently, supposing that K is of the form (3.2), we will prove that $K_1, K_2 \in P[A,B,\beta]$.

Differentiating the relation (3.1) and using the identity (1.5), we have

$$\frac{zK'(z)}{\lambda(\mu+p)} = K(z) - \frac{I_{\mu+1}^p(d,e)f(z)}{I_\mu^p(d,e)f(z)}\left(\frac{z^p}{I_\mu^p(d,e)f(z)}\right)^\lambda,$$

and from this relation we deduce that

$$(1+\vartheta)\left(\frac{z^p}{I_\mu^p(d,e)f(z)}\right)^\lambda - \vartheta\frac{I_{\mu+1}^p(d,e)f(z)}{I_\mu^p(d,e)f(z)}\left(\frac{z^p}{I_\mu^p(d,e)f(z)}\right)^\lambda =$$

$$K(z) + \frac{\vartheta}{\lambda(\mu+p)}zK'(z).$$

Since $f \in \mathcal{N}_{l,p}^{\lambda,\vartheta}(d,e;\mu;\beta,A,B)$, from the above relation it follows that

$$K(z) + \frac{\vartheta}{\lambda(\mu+p)}zK'(z) \in P_l[A,B,\beta],$$

and according to the second part of the Definition 1.1, this is equivalent to

$$K_i(z) + \frac{\vartheta}{\lambda(\mu + p)}\, zK_i'(z) \in P[A, B, \beta], \ (i = 1, 2),$$

that is

$$\frac{1}{1 - \beta}\left[K_i(z) + \frac{\vartheta}{\lambda(\mu + p)} zK_i'(z) - \beta\right] \in P[A, B], \ (i = 1, 2).$$

Writing

$$(3.3) \qquad\qquad K_i(z) = (1 - \beta)p_i(z) + \beta, \ (i = 1, 2),$$

from the previous relation we have

$$p_i(z) + \frac{\vartheta}{\lambda(\mu + p)}\, zp_i'(z) \in P[A, B], \ (i = 1, 2).$$

By using Lemma 2.1 for $\gamma = \dfrac{\lambda(\mu + p)}{\vartheta}$ and $n = 1$, from the above relation we deduce that

$$p_i(z) \prec q(z) \prec \frac{1 + Az}{1 + Bz}, \ (i = 1, 2),$$

where

$$q(z) = \frac{\lambda(\mu + p)}{\vartheta} z^{-\frac{\lambda(\mu+p)}{\vartheta}} \int_0^z t^{\frac{\lambda(\mu+p)}{\vartheta} - 1} \frac{1 + At}{1 + Bt}\, \mathrm{d}\, t$$

is the best dominant for p_i, $i = 1, 2$.

Since $p_i(z) \prec \dfrac{1 + Az}{1 + Bz}$, $i = 1, 2$, from (3.3) it follows that $K_i \in P[A, B, \beta]$, $i = 1, 2$, and according to (3.1) we conclude that $K \in P_l[A, B, \beta]$, which proves the first part of the theorem.

For the second part of our result, we distinguish the following two cases:

(i) For $B = 0$, a simple computation shows that

$$p_i(z) \prec q(z) = 1 + \frac{\lambda(\mu + p)}{\lambda(\mu + p) + \vartheta}\, Az, \ (i = 1, 2).$$

(ii) For $B \neq 0$, making the change of variables $s = zt$, followed by the use of the identities (2.3), (2.4) and (2.5) of Lemma 2.2, we obtain

$$p_i(z) \prec q(z) = \frac{\lambda(\mu + p)}{\vartheta} \int_0^1 s^{\frac{\lambda(\mu+p)}{\vartheta} - 1} \frac{1 + Asz}{1 + Bsz}\, \mathrm{d}\, s =$$

$$\frac{A}{B} + \left(1 - \frac{A}{B}\right)(1 + Bz)^{-1}\, {}_2F_1\left(1, 1, \frac{\lambda(\mu + p)}{\vartheta} + 1, \frac{Bz}{Bz + 1}\right), \ (i = 1, 2).$$

Now, it is sufficient to show that

$$(3.4) \qquad\qquad \inf\{\operatorname{Re} q(z) : z \in \mathrm{U}\} = q(-1).$$

We may easily show that

$$\operatorname{Re} \frac{1 + Az}{1 + Bz} \geq \frac{1 - Ar}{1 - Br}, \quad \text{for} \quad |z| \leq r < 1.$$

Denoting $G(t,z) = \dfrac{1 + Atz}{1 + Btz}$ and $\mathrm{d}\,\mu(t) = \dfrac{\lambda(\mu + p)}{\vartheta}\, t^{\frac{\lambda(\mu+p)}{\vartheta} - 1}\, \mathrm{d}\,t$, which is a positive measure on $[0, 1]$, we have

$$q(z) = \int_0^1 G(t, z)\,\mathrm{d}\,\mu(t),$$

hence it follows

$$\operatorname{Re} q(z) \geq \int_0^1 \frac{1 - Atr}{1 - Btr}\,\mathrm{d}\,\mu(t) = q(-r), \quad \text{for} \quad |z| \leq r < 1.$$

By letting $r \to 1^-$ we obtain (3.4), and from (3.3) and (3.1) we conclude that $K \in \mathcal{P}_l(\beta_1)$, which completes our proof. $\qquad\square$

Theorem 3.2. *If $0 \leq \vartheta_1 < \vartheta_2$, then*

$$\mathcal{N}_{l,p}^{\lambda,\vartheta_2}(d, e; \mu; \beta, A, B) \subset \mathcal{N}_{l,p}^{\lambda,\vartheta_1}(d, e; \mu; \beta, A, B)$$

Proof. The first part of Theorem 3.1 shows that the above inclusion holds whenever $\vartheta_1 = 0$.

If $0 < \vartheta_1 < \vartheta_2$, for an arbitrary $f \in \mathcal{N}_{l,p}^{\lambda,\vartheta_2}(d, e; \mu; \beta, A, B)$ let denote

$$U_1(z) = (1 + \vartheta_1)\left(\frac{z^p}{I_\mu^p(d, e)f(z)}\right)^\lambda - \vartheta_1 \frac{I_{\mu+1}^p(d, e)f(z)}{I_\mu^p(d, e)f(z)}\left(\frac{z^p}{I_\mu^p(d, e)f(z)}\right)^\lambda$$

and

$$U_0(z) = \left(\frac{z^p}{I_\mu^p(d, e)f(z)}\right)^\lambda.$$

A simple computation shows that

$$(1 + \vartheta_1)\left(\frac{z^p}{I_\mu^p(d, e)f(z)}\right)^\lambda - \vartheta_1 \frac{I_{\mu+1}^p(d, e)f(z)}{I_\mu^p(d, e)f(z)}\left(\frac{z^p}{I_\mu^p(d, e)f(z)}\right)^\lambda =$$

$$\left(1 - \frac{\vartheta_1}{\vartheta_2}\right)U_0(z) + \frac{\vartheta_1}{\vartheta_2}U_2(z).$$

Since $P_l[A, B, \beta]$ is a convex set, from the first part of Theorem 3.1, according to the above notations it follows that

$$\left(1 - \frac{\vartheta_1}{\vartheta_2}\right)U_0(z) + \frac{\vartheta_1}{\vartheta_2}U_2(z) \in P_l[A, B, \beta],$$

that is $f \in \mathcal{N}_{l,p}^{\lambda,\vartheta_1}(d, e; \mu; \beta, A, B)$. $\qquad\square$

Theorem 3.3. *If $f \in \mathcal{N}_{l,p}^{\lambda,0}(d, e; \mu; \beta, 1, -1)$, then $f(\rho z) \in \mathcal{N}_{l,p}^{\lambda,\vartheta}(d, e; \mu; \beta, 1, -1)$, where ρ is given by*

$$(3.5) \qquad \rho = \frac{-\left(\beta + \frac{\vartheta}{\lambda(\mu+p)}\right) + \sqrt{\left(\beta + \frac{\vartheta}{\lambda(\mu+p)}\right)^2 + 1 - \beta^2}}{1 + \beta}.$$

8   S. MUSTAFA, T. BULBOACĂ, AND B. S. ALKAHTANI

Proof. For an arbitrary $f \in \mathcal{N}_{l,p}^{\lambda,0}(d,e;\mu;\beta,1,-1)$, let denote

$$\left(\frac{z^p}{I_\mu^p(d,e)f(z)}\right)^\lambda = K(z) = \left(\frac{l}{4}+\frac{1}{2}\right)K_1(z) - \left(\frac{l}{4}-\frac{1}{2}\right)K_2(z),$$

where $K_1, K_2 \in P[1,-1,\beta]$, which in equivalent to $K_1(0) = K_2(0) = 1$ and $\operatorname{Re} K_1(z) > \beta$, $\operatorname{Re} K_2(z) > \beta$, $z \in U$.

With this notation, like in the proof of Theorem 3.1 we obtain

$$(1+\vartheta)\left(\frac{z^p}{I_\mu^p(d,e)f(z)}\right)^\lambda - \vartheta\frac{I_{\mu+1}^p(d,e)f(z)}{I_\mu^p(d,e)f(z)}\left(\frac{z^p}{I_\mu^p(d,e)f(z)}\right)^\lambda =$$

$$K(z) + \frac{\vartheta}{\lambda(\mu+p)}zK'(z) =$$

$$\left(\frac{l}{4}+\frac{1}{2}\right)\left[K_1(z) + \frac{\vartheta}{\lambda(\mu+p)}zK_1'(z)\right] - \left(\frac{l}{4}-\frac{1}{2}\right)\left[K_2(z) + \frac{\vartheta}{\lambda(\mu+p)}zK_2'(z)\right].$$

In order to have $f(\rho z) \in \mathcal{N}_{l,p}^{\lambda,\vartheta}(d,e;\mu;\beta,1,-1)$, according to the above formula, we need to find the (bigger) value of ρ, such that

$$\operatorname{Re}\left[K_i(z) + \frac{\vartheta}{\lambda(\mu+p)}zK_i'(z)\right] > \beta, \ |z| < \rho, \ (i = 1, 2).$$

From the well-known estimates for the class $P(0)$ (see, eq., [6]) we have

$$|K_i'(z)| \le \frac{2\operatorname{Re} K_i(z)}{1-r^2}, \ |z| \le r < 1, \ (i = 1, 2),$$

$$\operatorname{Re} K_i(z) \ge \frac{1-r}{1+r}, \ |z| \le r < 1, \ (i = 1, 2),$$

thus, we deduce that

$$\operatorname{Re}\left[K_i(z) + \frac{\vartheta}{\lambda(\mu+p)}zK_i'(z)\right] \ge \operatorname{Re} K_i(z) - \frac{\vartheta}{\lambda(\mu+p)}|zK_i'(z)| \ge$$

$$(3.6) \quad \operatorname{Re} K_i(z)\left[1 - \frac{\vartheta}{\lambda(\mu+p)}\frac{2r}{1-r^2}\right], \ |z| \le r < 1, \ (i = 1, 2).$$

A simple computation shows that

$$(3.7) \qquad\qquad 1 - \frac{\vartheta}{\lambda(\mu+p)}\frac{2r}{1-r^2} \ge 0,$$

for $0 \le r \le r_0$, where

$$r_0 := \frac{-\vartheta + \sqrt{\vartheta^2 + \lambda^2(\mu+p)^2}}{\lambda(\mu+p)} \in (0, 1).$$

Now, from the inequality (3.6) we have

$$\operatorname{Re}\left[K_i(z) + \frac{\vartheta}{\lambda(\mu+p)}zK_i'(z)\right] \ge \frac{1-r}{1+r}\left[1 - \frac{\vartheta}{\lambda(\mu+p)}\frac{2r}{1-r^2}\right], \ |z| \le r_0 < 1,$$

for $i = 1, 2$. It is easy to check that

$$\frac{1-r}{1+r}\left[1 - \frac{\vartheta}{\lambda(\mu+p)}\frac{2r}{1-r^2}\right] > \beta$$

for $0 \leq r < \rho$, where ρ is given by (3.5). Moreover, since the above inequality is equivalent to

$$1 - \frac{\vartheta}{\lambda(\mu+p)}\frac{2r}{1-r^2} > \frac{1+r}{1-r}\beta$$

for $r \in [0, 1)$, if follows that (3.7) holds for $r \in [0, \rho)$, and our theorem is completely proved. $\square$

Next we will consider some properties of generalized p–valent Bernardi integral operator. Thus, for $f \in \mathcal{A}_p$, let $F_{\eta,p} : \mathcal{A}_p \to \mathcal{A}_p$ be defined by

$$(3.8) \qquad F_{\eta,p}f(z) = \frac{\eta+p}{z^\eta}\int_0^z f(t)t^{\eta-1}\,\mathrm{d}\,t, \quad (\eta > -p).$$

We will give a short proof that this operator is well-defined, as follows. If the function $f \in \mathcal{A}_p$ is of the form (1.1), then the definition relation (3.8) could be written as

$$F_{\eta,p}f(z) = \frac{\eta+p}{z^\eta}\int_0^z f(t)t^{\eta-1}\,\mathrm{d}\,t = (\eta+p)I_{\eta,p}f(z),$$

where

$$I_{\eta,p}f(z) = \frac{1}{z^\eta}\int_0^z f(t)t^{\eta-1}\,\mathrm{d}\,t.$$

We see that integral operator $I_{\eta,p}$ defined above is similar to that of Lemma 1.2c. of [11]. According to this lemma, it follows that $I_{\eta,p}$ is an analytic integral operator for any function f of the form (1.1) whenever $\mathrm{Re}\,\eta > -p$, and $F_{\eta,p}f \in \mathcal{A}_p$ has the form

$$F_{\eta,p}f(z) = z^p + (\eta+p)\sum_{n=1}^\infty \frac{a_{p+n}}{p+n+\eta}z^{p+n}, \; z \in \mathrm{U}.$$

The operator defined in (3.8) is called *the generalized p–valent Bernardi integral operator*, and for special case $p = 1$ we get *the generalized Bernardi integral operator*. Thus, for $p = 1$ and $\eta \in \mathbb{N}$, the operator $F_\eta := F_{\eta,1}$ was introduced by Bernardi [1], and in particular, if $\eta = 1$ it reduces to the operator F_1 that was earlier introduced by Livingston [9].

Theorem 3.4. *Let $f \in \mathcal{A}_p$ and $F = F_{\eta,p}f$, where $F_{\eta,p}$ is given by* (3.8). *If*
(3.9)

$$(1+\vartheta)\left(\frac{z^p}{I_\mu^p(d,e)F(z)}\right)^\lambda - \vartheta\frac{I_\mu^p(d,e)f(z)}{I_\mu^p(d,e)F(z)}\left(\frac{z^p}{I_\mu^p(d,e)F(z)}\right)^\lambda \in P_l[A, B, \beta],$$

10   S. MUSTAFA, T. BULBOACĂ, AND B. S. ALKAHTANI

where $d, e \in \mathbb{R} \setminus \mathbb{Z}_0^-$, $\lambda > 0$, $\mu > -p$, $0 \le \beta < 1$, $\vartheta \ge 0$ and $l \ge 2$, then

$$\left(\frac{z^p}{I_\mu^p(d,e)F(z)} \right)^\lambda \in P_l[A, B, \beta].$$

(All the powers are the principal ones).

Proof. Like we mentioned after the Definition 1.2, since the left-hand side function from the relation (3.9) need to be analytic in U, we implicitly assumed that $I_\mu^p(d,e)F(z) \ne 0$ for all $z \in \dot{U}$.

The implication is obvious for $\vartheta = 0$, hence suppose that $\vartheta > 0$. Letting

(3.10)
$$\left(\frac{z^p}{I_\mu^p(d,e)F(z)} \right)^\lambda = K(z),$$

from the assumption (3.9) it follows that K is analytic in U, with $K(0) = 1$.

It is easy to check that, if $f, g \in \mathcal{A}_p$, then

(3.11)
$$\frac{z}{p}(f(z) * g(z))' = \left(\frac{z}{p} f'(z) \right)' * g(z).$$

Moreover, since $F = F_{\eta,p} f$, where $F_{\eta,p}$ is given by (3.8), a simple differentiation shows that

(3.12) $z \left(I_\mu^p(d,e)F(z) \right)' = (\eta + p) I_\mu^p(d,e)f(z) - \eta I_\mu^p(d,e)F(z).$

Taking the logarithmical differentiation of (3.10), we have

$$\lambda \left[p - \frac{z \left(I_\mu^p(d,e)F(z) \right)'}{I_\mu^p(d,e)F(z)} \right] = \frac{zK'(z)}{K(z)},$$

and using the relations (3.11) and (3.12), it follows that

$$\frac{I_\mu^p(d,e)f(z)}{I_\mu^p(d,e)F(z)} = 1 - \frac{1}{\lambda(\eta + p)} \frac{zK'(z)}{K(z)},$$

and thus

$$(1 + \vartheta) \left(\frac{z^p}{I_\mu^p(d,e)F(z)} \right)^\lambda - \vartheta \frac{I_\mu^p(d,e)f(z)}{I_\mu^p(d,e)F(z)} \left(\frac{z^p}{I_\mu^p(d,e)F(z)} \right)^\lambda =$$
$$K(z) + \frac{\vartheta}{\lambda(p + \eta)} zK'(z).$$

From the assumption (3.9), the above relation gives that

$$K(z) + \frac{\vartheta}{\lambda(p + \eta)} zK'(z) \in P_l[A, B, \beta],$$

and using a similar proof with those of the first part of Theorem 3.1 we obtain that $K \in P_l[A, B, \beta]$, which proves our result. $\square$

Theorem 3.5. *If*

$$(3.13) \qquad f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \in \mathcal{N}_{l,p}^{\lambda,\vartheta}(d, e; \mu; \beta, A, B),$$

where $d, e \in \mathbb{R} \setminus \mathbb{Z}_0^-$, $\lambda > 0$, $\mu > -p$, $0 \le \beta < 1$, $\vartheta \ge 0$ and $l \ge 2$, then

$$(3.14) \qquad |a_{p+1}| \le \left| \frac{d}{e} \right| \frac{(1-\beta)(A-B)}{|\vartheta + \lambda(\mu+p)|}.$$

The inequality (3.14) is sharp.

Proof. If we let
(3.15)

$$(1+\vartheta)\left(\frac{z^p}{I_\mu^p(d,e)f(z)}\right)^\lambda - \vartheta \frac{I_{\mu+1}^p(d,e)f(z)}{I_\mu^p(d,e)f(z)}\left(\frac{z^p}{I_\mu^p(d,e)f(z)}\right)^\lambda = p(z),$$

using the fact that

$$I_\mu^p(d,e)f(z) = z^p + \sum_{n=1}^{\infty} \frac{(\mu+p)_n (e)_n}{n!(d)_n} a_{p+n} z^{p+n},$$

we have

$$(1+\vartheta)\left(\frac{z^p}{I_\mu^p(d,e)f(z)}\right)^\lambda - \vartheta \frac{I_{\mu+1}^p(d,e)f(z)}{I_\mu^p(d,e)f(z)}\left(\frac{z^p}{I_\mu^p(d,e)f(z)}\right)^\lambda =$$

$$1 - \left(1 + \frac{\vartheta}{\lambda(\mu+p)}\right)\frac{(\mu+p)_1 (e)_1}{1!(d)_1}\lambda\, a_{p+1} z + \cdots =$$

$$(3.16) \qquad 1 - \frac{e}{d}[\vartheta + \lambda(\mu+p)]\, a_{p+1} z + \ldots, \quad z \in \mathrm{U}.$$

Since $f \in \mathcal{N}_{l,p}^{\lambda,\vartheta}(d, e; \mu; \beta, A, B)$, it follows that the function p defined by (3.15) is of the form

$$p(z) = \left(\frac{l}{4} + \frac{1}{2}\right)p_1(z) - \left(\frac{l}{4} - \frac{1}{2}\right)p_2(z),$$

where $p_1, p_2 \in P[A, B, \beta]$. It follows that

$$p_i(z) \prec \frac{1 + [(1-\beta)A + \beta B]z}{1 + Bz} \quad (i = 1, 2),$$

and from the above relation we deduce that

$$(3.17) \qquad p(z) \prec \frac{1 + [(1-\beta)A + \beta B]z}{1 + Bz}.$$

According to (3.16), from the subordination (3.17) we obtain

$$1 - \frac{e}{d}\frac{\vartheta + \lambda(\mu+p)}{1-\beta} a_{p+1} z + \cdots = \frac{p(z) - \beta}{1-\beta} \prec \frac{1+Az}{1+Bz},$$

and from Lemma 2.3 we conclude that

$$\left| -\frac{e}{d}\frac{\vartheta + \lambda(\mu+p)}{1-\beta} \right| |a_{p+1}| \le |A-B|,$$

12   S. MUSTAFA, T. BULBOACĂ, AND B. S. ALKAHTANI

which proves our result.

To prove that the inequality (3.14) is sharp we need to show that there exists a function $f \in \mathcal{N}_{l,p}^{\lambda,\vartheta}(d, e; \mu; \beta, A, B)$ of the form (3.13), such that for this function we have equality in (3.14).

Thus, we will prove that there exists $f \in \mathcal{N}_{l,p}^{\lambda,\vartheta}(d, e; \mu; \beta, A, B)$, such that the identity (3.15) holds for the special case

$$p(z) = \frac{1 + [(1 - \beta)A + \beta B]\, z}{1 + Bz}.$$

Setting

$$(3.18) \qquad K(z) = \left(\frac{z^p}{I_\mu^p(d, e) f(z)} \right)^\lambda,$$

like in the proof of Theorem 3.1 we deduce that the relation (3.15) is equivalent to

$$(3.19) \qquad K(z) + \frac{1}{\gamma} zK'(z) = p(z), \quad \text{where} \quad \gamma := \frac{\lambda(\mu + p)}{\vartheta}.$$

(i) If $\vartheta = 0$, the above differential equation has the solution $K = p$.

(ii) If $\vartheta > 0$, then $\gamma > 0$ whenever $\lambda > 0$ and $\mu > -p$. Since the function p is convex in the unit disk U, according to Lemma 2.1 it follows that this differential equation has the solution

$$\widetilde{K}(z) = \frac{\gamma}{z^\gamma} \int_0^z t^{\gamma-1} p(t)\, \mathrm{d}\, t \prec p(z).$$

It is easy to check that $p(z) \neq 0$ for all $z \in$ U, and from the above subordination we get that $\widetilde{K}(z) \neq 0$, $z \in$ U.

Now, if we define the function K_0 by

$$K_0(z) = \begin{cases} p(z), & \text{if} \quad \vartheta = 0, \\ \widetilde{K}(z), & \text{if} \quad \vartheta > 0, \end{cases}$$

then K_0 is the analytic solution of the differential equation (3.19), and moreover $K_0(z) \neq 0$, $z \in$ U.

Thus, for $K = K_0$ the relation (3.18) is equivalent to

$$I_\mu^p(d, e) f(z) = z^p K_0^{-1/\lambda}(z),$$

and this equation has the solution

$$(3.20) \qquad f_0(z) := \psi_p(d, e; z) * \left(z^p K_0^{-1/\lambda}(z) \right),$$

where

$$\psi_p(d, e; z) := z^p + \sum_{n=1}^\infty \frac{n!(d)_n}{(\mu + p)_n (e)_n} z^{p+n}. \quad \left(d, e \in \mathbb{C} \setminus \mathbb{Z}_0^-,\ \mu > -p \right),$$

Consequently, for the function f_0 defined by (3.20) we get equality in (3.14), hence the sharpness of our result is proved. $\square$

As a special case, for $l = 2$ and $\beta = 0$ we obtain the corresponding result for the class $\mathcal{N}_{2,p}^{\lambda,\vartheta}(d, e; \mu; 0, A, B)$ (see [18] for $n = 1$).

References

[1] S. D. Bernardi, *Convex and starlike univalent functions*, Trans. Amer. Math. Soc., 135(1969), 429–446.

[2] B. C. Carlson and D. B. Shaffer, *Starlike and prestarlike hypergeometric functions*, SIAM J. Math. Anal., 159(1984), 737–745.

[3] M. Çağlar, Y. Polatoğlu, E. Yavuz, *λ–Fractional properties of generalized Janowski functions in the unit disc*, Mat. Vesnik, 50(2008), 165–171.

[4] N. E. Cho, O. S. Kwon and H. M. Srivastava, *Inclusion relationships and argument properties for certain subclasses of multivalent functions associated with a family of linear operators*, J. Math. Anal. Appl., 292(2004), 470–483.

[5] J. H. Choi, M. Saigo and H. M. Srivastava, *Some inclusion properties of certain family of integral operators*, J. Math. Anal. Appl., 276(2002) 432–444.

[6] A. W. Goodman, Univalent Functions, Vol. I, II, Polygonal Publishing House, Washington, New Jersey, 1983.

[7] D. J. Hallenbeck and St. Ruscheweyh, *Subordination by convex functions*, Proc. Amer. Math. Soc., 52(1975), 191–195.

[8] W. Janowski, *Some extremal problems for certain families of analytic functions*, Ann. Polon. Math., 28(1973), 297–327.

[9] A. E. Livingston, *On the radius of univalence of certain analytic functions*, Proc. Amer. Math. Soc., 17(1966), 352–357.

[10] S. S. Miller and P. T. Mocanu, *Second order differential inequalities in the complex plane*, J. Math. Anal. Appl., 65(1978), 289–305.

[11] S. S. Miller and P. T. Mocanu, Differential Subordinations. Theory and Applications, Marcel Dekker Inc., New York Basel, 2000.

[12] K. I. Noor and M. A. Noor, *On certain classes of analytic functions defined by Noor integral operator*, J. Math. Anal. Appl., 281(2003), 244-252.

[13] K. I. Noor, *Applications of certain operators to the classes related with generalized Janowski functions*, Integral Transforms Spec. Funct., 2(2010), 557–567.

[14] K. I. Noor, M. Ali, M. Arif, *On a class of p–valent non-Bazilević functions*, Gen. Math., 18(2)(2010), 31–46

[15] B. Pinchuk, *Functions with bounded boundary rotation*, Israel J. Math., 10(1971), 7–16.

[16] H. Saitoh, *A linear operator and its applications of first order differential subordinations*, Math. Japon., 44(1996), 31–38.

[17] W. Rogosinski, *On the coefficients of subordinate functions*, Proc. London Math. Soc. (Ser. 2), 48(1943), 48–82.

[18] Z.-G. Wang, H.-T. Wang and Y. Sun, *A class of multivalent non-Bazilević functions involving the Cho-Kwon-Srivastava operator*, Tamsui Oxf. J. Inf. Math. Sci., 26(1)(2010), 1–19.

[19] E. T. Whittaker and G. N. Watson, *A Course on Modern Analysis: An Introduction to the General Theory of Infinite Processes and of Analytic Functions;*

14   S. MUSTAFA, T. BULBOACĂ, AND B. S. ALKAHTANI

With an Account of the Principal Transcendental Functions, Fourth Edition, Cambridige Univ. Press, Cambridge, 1927.

Department of Mathematics, Pir Mehr Ali Shah Arid Agriculture University, Rawalpindi, Pakistan
  E-mail address: saimamustafa28@gmail.com

Faculty of Mathematics and Computer Science, Babeş-Bolyai University, 400084 Cluj-Napoca, Romania
  E-mail address: bulboaca@math.ubbcluj.ro

Mathematics Department, College of Science, King Saud University, P.O.Box 1142, Riyadh 11989, Saudi Arabia
  E-mail address: alhaghog@gmail.com

ON A PRODUCT-TYPE OPERATOR FROM MIXED-NORM SPACES TO BLOCH-ORLICZ SPACES

HAIYING LI AND ZHITAO GUO

ABSTRACT. The boundedness and compactness of a product-type operator DM_uC_ψ from mixed-norm spaces to Bloch-Orlicz spaces are characterized in this paper.

1. INTRODUCTION

Let $\mathbb{D}$ denote the unit disk in the complex plane $\mathbb{C}$, $\mathcal{H}(\mathbb{D})$ the class of all analytic functions on $\mathbb{D}$ and $\mathbb{N}$ the set of nonnegative integers.

A positive continuous function ϕ on $[0,1)$ is called normal if there exist two positive numbers s and t with $0 < s < t$, and $\delta \in [0,1)$ such that (see [19])

$$\frac{\phi(r)}{(1-r)^s} \text{ is decreasing on } [\delta,1), \ \lim_{r \to 1} \frac{\phi(r)}{(1-r)^s} = 0;$$

$$\frac{\phi(r)}{(1-r)^t} \text{ is increasing on } [\delta,1), \ \lim_{r \to 1} \frac{\phi(r)}{(1-r)^t} = \infty.$$

For $p,q \in (0,\infty)$ and ϕ normal, the mixed-norm space $H(p,q,\phi)(\mathbb{D}) = H(p,q,\phi)$ is the space of all functions $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|_{H(p,q,\phi)} = \left(\int_0^1 M_q^p(f,r) \frac{\phi^p(r)}{1-r} dr \right)^{\frac{1}{p}} < \infty,$$

where

$$M_q(f,r) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^q d\theta \right)^{\frac{1}{q}}.$$

For $1 \le p,q < \infty$, $H(p,q,\phi)$, equipped with the norm $\|f\|_{H(p,q,\phi)}$, is a Banach space, while for the other vales of p and q, $\|\cdot\|_{H(p,q,\phi)}$ is a quasinorm on $H(p,q,\phi)$, $H(p,q,\phi)$ is a Fréchet space but not a Banach space. Note that if $\phi(r) = (1-r)^{\frac{\alpha+1}{p}}$, then $H(p,q,\phi)$ is equivalent to the weighted Bergman space $A_\alpha^p(\mathbb{D}) = A_\alpha^p$ defined for $0 < p < \infty$ and $\alpha > -1$, as the spaces of all $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|_{A_\alpha^p}^p = (\alpha+1) \int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha dm(z) < \infty,$$

where $dm(z) = \frac{1}{\pi} r dr d\theta$ is the normalized Lebesgue area measure on $\mathbb{D}$ ([8, 12, 18, 25, 27, 33, 35, 48, 51]). For more details on the mixed-norm space on various domains and operators on them, see, e.g., [1, 7, 10, 20, 22, 23, 24, 28, 29, 34, 36, 37, 38, 41, 42, 43, 44, 46, 47, 54].

1

For every $0 < \alpha < \infty$, the α-Bloch space, denoted by $\mathcal{B}^\alpha$, consists of all functions $f \in \mathcal{H}(\mathbb{D})$ such that

$$\sup_{z \in \mathbb{D}}(1 - |z|^2)^\alpha |f'(z)| < \infty.$$

$\mathcal{B}^\alpha$ is a Banach space under the norm

$$\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \sup_{z \in \mathbb{D}}(1 - |z|^2)^\alpha |f'(z)|.$$

For $\alpha = 1$ is obtained the Bloch space. α-Bloch space is introduced and studied by numerous authors. Recently, many authors studied different classes of Bloch-type spaces, where the typical weight function, $\omega(z) = 1 - |z|^2 (z \in \mathbb{D})$ is replaced by a bounded continuous positive function μ defined on $\mathbb{D}$. More precisely, a function $f \in \mathcal{H}(\mathbb{D})$ is called a μ-Bloch function, denoted by $f \in \mathcal{B}^\mu$, if

$$\|f\|_\mu = \sup_{z \in \mathbb{D}} \mu(z)|f'(z)| < \infty.$$

Clearly, if $\mu(z) = \omega(z)^\alpha$ with $\alpha > 0$, $\mathcal{B}^\mu$ is just the α-Bloch space $\mathcal{B}^\alpha$. It is readily seen that $\mathcal{B}^\mu$ is a Banach space with the norm

$$\|f\|_{\mathcal{B}^\mu} = |f(0)| + \|f\|_\mu.$$

For some information on the Bloch, α-Bloch and Bloch-type spaces, as well as some operators on them see, e.g., [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 21, 23, 25, 26, 27, 29, 30, 31, 32, 34, 37, 38, 39, 40, 41, 43, 44, 45, 46, 47, 50, 51, 52, 53, 55].

Recently, Fernández in [17] used Young's functions to define the Bloch-Orlicz space. More precisely, let $\varphi : [0, \infty) \to [0, \infty)$ be a strictly increasing convex function such that $\varphi(0) = 0$ and $\lim_{t \to \infty} \varphi(t) = \infty$. The Bloch-Orlicz space associated with the function φ, denoted by $\mathcal{B}^\varphi$, is the class of all analytic functions f in $\mathbb{D}$ such that

$$\sup_{z \in \mathbb{D}}(1 - |z|^2)\varphi(\lambda|f'(z)|) < \infty$$

for some $\lambda > 0$ depending on f. Also, since φ is convex, it is not hard to see that the Minkowski's functional

$$\|f\|_\varphi = \inf\left\{ k > 0 : S_\varphi\left(\frac{f'}{k}\right) \leq 1 \right\}$$

define a seminorm for $\mathcal{B}^\varphi$, which, in this case, is known as Luxemburg's seminorm, where

$$S_\varphi(f) = \sup_{z \in \mathbb{D}}(1 - |z|^2)\varphi(|f(z)|)$$

We know that $\mathcal{B}^\varphi$ is a Banach space with the norm $\|f\|_{\mathcal{B}^\varphi} = |f(0)| + \|f\|_\varphi$. We also have that the Bloch- Orlicz space is isometrically equal to μ-Bloch space, where

$$\mu(z) = \frac{1}{\varphi^{-1}(\frac{1}{1-|z|^2})}, \ z \in \mathbb{D}.$$

Thus for any $f \in \mathcal{B}^\varphi$, we have

$$\|f\|_{\mathcal{B}^\varphi} = |f(0)| + \sup_{z \in \mathbb{D}} \mu(z)|f'(z)|.$$

It is well known that the differentiation operator D is defined by

$$(Df)(z) = f'(z), \ f \in \mathcal{H}(\mathbb{D}).$$

Let $u \in \mathcal{H}(\mathbb{D})$, then the multiplication operator M_u is defined by

$$(M_u f)(z) = u(z)f(z), \ f \in \mathcal{H}(\mathbb{D}).$$

Let ψ be an analytic self-map of $\mathbb{D}$. The composition operator C_ψ is defined by

$$(C_\psi f)(z) = f(\psi(z)), \ f \in \mathcal{H}(\mathbb{D}).$$

Investigation of products of these and integral-type operators attracted a lot of attention recently (see, e.g., [2]-[49], [51]-[55]). For example, in [3] and [17], the authors investigated bounded superposition operators between Bloch-Orlicz and α-Bloch spaces and composition operators on Bloch-Orlicz type spaces. In [37] and [38], S. Stević investigated extended Cesàro operators between mixed-norm spaces and Bloch-type spaces and an integral-type operator from logarithmic Bloch-type spaces to mixed-norm spaces on the unit ball. In [36] and [41], S. Stević investigated an integral-type operator from logarithmic Bloch-type and mixed-norm spaces to Bloch-type spaces and weighted differentiation composition operators from mixed-norm spaces to weighted-type spaces. In [42] and [46], S. Stević investigated an integral-type operator from Zygmund-type spaces to mixed-norm spaces on the unit ball and weighted differentiation composition operators from the mixed-norm space to the nth weighted-type space on the unit disk. S. Stević in [34] gave the properties of products of integral-type operators and composition operators from the mixed norm space to Bloch-type spaces. In [47], S. Stević investigated weighted radial operator from the mixed-norm space to the nth weighted-type space on the unit ball. In [54], X. Zhu studied extended Cesàro operators from mixed-norm spaces to Zygmund type spaces.

Motivated, among others, by these papers, we will study here the boundedness and compactness of the following operator, which is also a product-type one,

$$(DM_u C_\psi f)(z) = u'(z)f(\psi(z)) + u(z)\psi'(z)f'(\psi(z)), \ f \in \mathcal{H}(\mathbb{D}),$$

from $H(p,q,\phi)$ to $\mathcal{B}^\varphi$.

In what follows,

$$\mu(z) = \frac{1}{\varphi^{-1}\left(\frac{1}{1-|z|^2}\right)},$$

and we use the letter C to denote a positive constant whose value may change at each occurrence.

2. THE BOUNDEDNESS AND COMPACTNESS OF $DM_u C_\phi : H(p,q,\phi) \to \mathcal{B}^\varphi$

In this section, we will give our main results and proofs. In order to prove our main results, we need some auxiliary results. Our first lemma characterizes compactness in terms of sequential convergence. Since the proof is standard, it is omitted here (see, Proposition 3.11 in [4]).

Lemma 1. *Suppose $u \in \mathcal{H}(\mathbb{D})$, ψ is an analytic self-map of $\mathbb{D}$, $0 < p,q < \infty$ and ϕ is normal. Then the operator $DM_u C_\psi : H(p,q,\phi) \to \mathcal{B}^\varphi$ is compact if and only if it is bounded and for each sequence $\{f_n\}_{n\in\mathbb{N}}$ which is bounded in $H(p,q,\phi)$ and converges to zero uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$, we have $\|DM_u C_\psi f_n\|_{\mathcal{B}^\varphi} \to 0$ as $n \to \infty$.*

The following lemma can be found in [36].

Lemma 2. *Assume $0 < p,q < \infty$, ψ is normal and $f \in H(p,q,\phi)$. Then for every $n \in \mathbb{N}$, there is a positive constant C independent of f such that*

$$|f^{(n)}(z)| \le \frac{C\|f\|_{H(p,q,\phi)}}{\phi(|z|)(1-|z|^2)^{\frac{1}{q}+n}}.$$

4   HAIYING LI AND ZHITAO GUO

Theorem 3. *Let* $u \in \mathcal{H}(\mathbb{D})$, ψ *be an analytic self-map of* $\mathbb{D}$, $0 < p, q < \infty$ *and* ϕ *be normal. Then* $DM_uC_\psi : H(p,q,\phi) \to \mathcal{B}^\varphi$ *is bounded if and only if*

$$k_1 = \sup_{z \in \mathbb{D}} \frac{\mu(z)|u''(z)|}{\phi(|\psi(z)|)(1 - |\psi(z)|^2)^{\frac{1}{q}}} < \infty, \tag{1}$$

$$k_2 = \sup_{z \in \mathbb{D}} \frac{\mu(z)|2u'(z)\psi'(z) + u(z)\psi''(z)|}{\phi(|\psi(z)|)(1 - |\psi(z)|^2)^{\frac{1}{q}+1}} < \infty, \tag{2}$$

$$k_3 = \sup_{z \in \mathbb{D}} \frac{\mu(z)|u(z)||\psi'(z)|^2}{\phi(|\psi(z)|)(1 - |\psi(z)|^2)^{\frac{1}{q}+2}} < \infty. \tag{3}$$

Proof. Assume that (1), (2) and (3) hold. By Lemma 2, then we get

$$|f(\psi(z))| \leq \frac{C_1\|f\|_{H(p,q,\phi)}}{\phi(|\psi(z)|)(1 - |\psi(z)|^2)^{\frac{1}{q}}},$$

$$|f'(\psi(z))| \leq \frac{C_2\|f\|_{H(p,q,\phi)}}{\phi(|\psi(z)|)(1 - |\psi(z)|^2)^{\frac{1}{q}+1}},$$

$$|f''(\psi(z))| \leq \frac{C_3\|f\|_{H(p,q,\phi)}}{\phi(|\psi(z)|)(1 - |\psi(z)|^2)^{\frac{1}{q}+2}}.$$

Then for each $f \in H(p,q,\phi) \setminus \{0\}$, we have:

$$S_\varphi\left(\frac{(DM_uC_\psi f)'(z)}{C\|f\|_{H(p,q,\phi)}}\right)$$

$$\leq \sup_{z \in \mathbb{D}}(1 - |z|^2)\varphi\Bigg[\left(\frac{k_1\phi(|\psi(z)|)(1 - |\psi(z)|^2)^{\frac{1}{q}}|f(\psi(z))|}{C\mu(z)\|f\|_{H(p,q,\phi)}}\right)$$

$$+ \left(\frac{k_2\phi(|\psi(z)|)(1 - |\psi(z)|^2)^{\frac{1}{q}+1}|f'(\psi(z))|}{C\mu(z)\|f\|_{H(p,q,\phi)}}\right)$$

$$+ \left(\frac{k_3\phi(|\psi(z)|)(1 - |\psi(z)|^2)^{\frac{1}{q}+2}|f''(\psi(z))|}{C\mu(z)\|f\|_{H(p,q,\phi)}}\right)\Bigg]$$

$$\leq \sup_{z \in \mathbb{D}}(1 - |z|^2)\varphi\left[\frac{k_1C_1 + k_2C_2 + k_3C_3}{C\mu(z)}\right]$$

$$\leq \sup_{z \in \mathbb{D}}(1 - |z|^2)\varphi\left(\varphi^{-1}\left(\frac{1}{1 - |z|^2}\right)\right) = 1$$

where C is a constant such that $C \geq k_1C_1 + k_2C_2 + k_3C_3$. Now, we can conclude that there exists a constant C such that $\|DM_uC_\psi f\|_{\mathcal{B}^\varphi} \leq C\|f\|_{H(p,q,\phi)}$ for all $f \in H(p,q,\phi)$, so the product-type operator $DM_uC_\psi : H(p,q,\phi) \to \mathcal{B}^\varphi$ is bounded.

Conversely, suppose that $DM_uC_\psi : H(p,q,\phi) \to \mathcal{B}^\varphi$ is bounded, i.e., there exists $C > 0$ such that $\|DM_uC_\psi f\|_{B^\varphi} \leq C\|f\|_{H(p,q,\phi)}$ for all $f \in H(p,q,\phi)$. Taking the function $f(z) = 1 \in H(p,q,\phi)$, and $\|f\|_{H(p,q,\phi)} \leq C$, then

$$S_\varphi\left(\frac{(DM_uC_\psi f)'(z)}{C}\right) = S_\varphi\left(\frac{u''(z)}{C}\right) = \sup_{z \in \mathbb{D}}(1 - |z|^2)\varphi\left(\frac{|u''(z)|}{C}\right) \leq 1.$$

It follows that

$$\sup_{z \in \mathbb{D}} \mu(z)|u''(z)| < \infty. \tag{4}$$

Taking the function $f(z) = z \in H(p,q,\phi)$, and $\|f\|_{H(p,q,\phi)} \leq C$, then

$$
S_\varphi\left(\frac{(DM_uC_\psi f)'(z)}{C}\right)
$$
$$
= \sup_{z\in\mathbb{D}}(1-|z|^2)\varphi\left(\frac{|u''(z)\psi(z) + (2u'(z)\psi'(z) + u(z)\psi''(z))|}{C}\right) \leq 1.
$$

Hence

$$
\sup_{z\in\mathbb{D}}\mu(z)|u''(z)\psi(z) + 2u'(z)\psi'(z) + u(z)\psi''(z)| < \infty.
$$

By (4) and the boundedness of $\psi(z)$, we can see that

$$
\sup_{z\in\mathbb{D}}\mu(z)|2u'(z)\psi'(z) + u(z)\psi''(z)| < \infty. \tag{5}
$$

Taking the function $f(z) = \frac{z^2}{2} \in H(p,q,\phi)$, similarly, we can get

$$
\sup_{z\in\mathbb{D}}\mu(z)|u(z)||\psi'(z)|^2 < \infty. \tag{6}
$$

For a fixed $\omega \in \mathbb{D}$, set

$$
f_{\psi(\omega)}(z) = \frac{A(1-|\psi(\omega)|^2)^{t+1}}{\phi(|\psi(\omega)|)(1-\overline{\psi(\omega)}z)^{\frac{1}{q}+t+1}} + \frac{B(1-|\psi(\omega)|^2)^{t+2}}{\phi(|\psi(\omega)|)(1-\overline{\psi(\omega)}z)^{\frac{1}{q}+t+2}}
$$
$$
+ \frac{(1-|\psi(\omega)|^2)^{t+3}}{\phi(|\psi(\omega)|)(1-\overline{\psi(\omega)}z)^{\frac{1}{q}+t+3}}, \tag{7}
$$

where the constant t is from the definition of the normality of the function ϕ.

Then $\sup_{\omega\in\mathbb{D}}\|f_{\psi(\omega)}\|_{H(p,q,\phi)} < \infty$, and we have

$$
f_{\psi(\omega)}(\psi(\omega)) = \frac{A+B+1}{\phi(|\psi(\omega)|)(1-|\psi(\omega)|^2)^{\frac{1}{q}}},
$$
$$
f'_{\psi(\omega)}(\psi(\omega)) = \frac{(AM_1 + BM_2 + M_3)\overline{\psi(\omega)}}{\phi(|\psi(\omega)|)(1-|\psi(\omega)|^2)^{\frac{1}{q}+1}},
$$
$$
f''_{\psi(\omega)}(\psi(\omega)) = \frac{(AM_1M_2 + BM_2M_3 + M_3M_4)\overline{\psi(\omega)}^2}{\phi(|\psi(\omega)|)(1-|\psi(\omega)|^2)^{\frac{1}{q}+2}}.
$$

where $M_i = \frac{1}{q} + t + i, i = 1,2,3,4$.

To prove (1), we choose the corresponding function in (7) with

$$
A = \frac{M_3}{M_1}, \ B = -\frac{2M_3}{M_2},
$$

and denote it by $f_{\psi(\omega)}$, then we have

$$
f_{\psi(\omega)}(\psi(\omega)) = \frac{P}{\phi(|\psi(\omega)|)(1-|\psi(\omega)|^2)^{\frac{1}{q}}}, \ f'_{\psi(\omega)}(\psi(\omega)) = f''_{\psi(\omega)}(\psi(\omega)) = 0, \tag{8}
$$

where $P = \frac{M_3}{M_1} - \frac{2M_3}{M_2} + 1$.

By the boundedness of $DM_uC_\psi : H(p,q,\phi) \to \mathcal{B}^\varphi$, we have $\|DM_uC_\psi f_{\psi(\omega)}\|_{\mathcal{B}^\varphi} \leq C$, then

$$
1 \geq S_\varphi\left(\frac{(DM_uC_\psi f_{\psi(\omega)})'(z)}{C}\right)
$$
$$
\geq \sup_{w\in\mathbb{D}}(1-|\omega|^2)\varphi\left(\frac{P|u''(\omega)|}{C\phi(|\psi(\omega)|)(1-|\psi(\omega)|^2)^{\frac{1}{q}}}\right),
$$

6   HAIYING LI AND ZHITAO GUO

from which we can get (1). To prove (2), we choose the corresponding function in (7) with

$$A = \frac{-2M_2 - M_1M_2 + M_3M_4}{2M_2}, \ B = \frac{M_1M_2 - M_3M_4}{2M_2},$$

and denote it by $g_{\psi(\omega)}$, then we have

$$g'_{\psi(\omega)}(\psi(\omega)) = \frac{E\overline{\psi(\omega)}}{\phi(|\psi(\omega)|)(1 - |\psi(\omega)|^2)^{\frac{1}{q}+1}}, \ g_{\psi(\omega)}(\psi(\omega)) = g''_{\psi(\omega)}(\psi(\omega)) = 0, \quad (9)$$

where

$$E = \frac{-2M_1M_2 - M_1^2M_2 + M_1M_3M_4}{2M_2} + \frac{M_1M_2 - M_3M_4}{2} + M_3.$$

By the boundedness of $DM_uC_\psi : H(p,q,\phi) \to \mathcal{B}^\varphi$, we have $\|DM_uC_\psi g_{\psi(\omega)}\|_{\mathcal{B}^\varphi} \leq C$, then

$$
\begin{aligned}
1 &\geq S_\varphi\left(\frac{(DM_uC_\psi g_{\psi(\omega)})'(z)}{C}\right) \\
&\geq \sup_{\frac{1}{2} < |\psi(\omega)| < 1} (1 - |\omega|^2)\varphi\left(\frac{|(DM_uC_\psi g_{\psi(\omega)})'(\omega)|}{C}\right) \\
&= \sup_{\frac{1}{2} < |\psi(\omega)| < 1} (1 - |\omega|^2)\varphi\left(\frac{E|\psi(\omega)||2u'(\omega)\psi'(\omega) + u(\omega)\psi''(\omega)|}{C\phi(|\psi(\omega)|)(1 - |\psi(\omega)|^2)^{\frac{1}{q}+1}}\right).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\sup_{\frac{1}{2} < |\psi(\omega)| < 1} \frac{\mu(\omega)|2u'(\omega)\psi'(\omega) + u(\omega)\psi''(\omega)|}{\phi(|\psi(\omega)|)(1 - |\psi(\omega)|^2)^{\frac{1}{q}+1}} \\
&\leq 2 \sup_{\frac{1}{2} < |\psi(\omega)| < 1} \frac{\mu(\omega)|\psi(\omega)||2u'(\omega)\psi'(\omega) + u(\omega)\psi''(\omega)|}{\phi(|\psi(\omega)|)(1 - |\psi(\omega)|^2)^{\frac{1}{q}+1}} < \infty. \quad (10)
\end{aligned}
$$

Since ϕ is normal, and using (5), we have

$$
\begin{aligned}
&\sup_{|\psi(\omega)| \leq \frac{1}{2}} \frac{\mu(\omega)|2u'(\omega)\psi'(\omega) + u(\omega)\psi''(\omega)|}{\phi(|\psi(\omega)|)(1 - |\psi(\omega)|^2)^{\frac{1}{q}+1}} \\
&\leq C \sup_{|\psi(\omega)| \leq \frac{1}{2}} \mu(\omega)|2u'(\omega)\psi'(\omega) + u(\omega)\psi''(\omega)| < \infty. \quad (11)
\end{aligned}
$$

From (10) and (11), we can get (2). To prove (3), we choose the corresponding function in (7) with

$$A = 1, \ B = -2,$$

and denote it by $h_{\psi(\omega)}$, then we have

$$h_{\psi(\omega)}(\psi(\omega)) = h'_{\psi(\omega)}(\psi(\omega)) = 0, \ h''_{\psi(\omega)}(\psi(\omega)) = \frac{F\overline{\psi(\omega)}^2}{\phi(|\psi(\omega)|)(1 - |\psi(\omega)|^2)^{\frac{1}{q}+2}}, \quad (12)$$

where $F = M_1M_2 - 2M_2M_3 + M_3M_4$.

By the boundedness of $DM_uC_\psi : H(p,q,\phi) \to \mathcal{B}^\varphi$, we have $\|DM_uC_\psi h_{\psi(\omega)}\|_{\mathcal{B}^\varphi} \leq C$, then

$$
\begin{aligned}
1 &\geq S_\varphi\left(\frac{(DM_uC_\psi h_{\psi(\omega)})'(z)}{C}\right) \\
&\geq \sup_{\frac{1}{2}<|\psi(\omega)|<1} (1-|\omega|^2)\varphi\left(\frac{|(DM_uC_\psi h_{\psi(\omega)})'(\omega)|}{C}\right) \\
&= \sup_{\frac{1}{2}<|\psi(\omega)|<1} (1-|\omega|^2)\varphi\left(\frac{F|\psi(\omega)|^2|u(\omega)||\psi'(\omega)|^2}{C\phi(|\psi(\omega)|)(1-|\psi(\omega)|^2)^{\frac{1}{q}+2}}\right).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\sup_{\frac{1}{2}<|\psi(\omega)|<1} &\frac{\mu(\omega)|u(\omega)||\psi'(\omega)|^2}{\phi(|\psi(\omega)|)(1-|\psi(\omega)|^2)^{\frac{1}{q}+2}} \\
&\leq 4\sup_{\frac{1}{2}<|\psi(\omega)|<1} \frac{\mu(\omega)|\psi(\omega)|^2|u(\omega)||\psi'(\omega)|^2}{\phi(|\psi(\omega)|)(1-|\psi(\omega)|^2)^{\frac{1}{q}+2}} < \infty.
\end{aligned} \tag{13}
$$

Since ϕ is normal, and using (6), we have

$$
\sup_{|\psi(\omega)|\leq\frac{1}{2}} \frac{\mu(\omega)|u(\omega)||\psi'(\omega)|^2}{\phi(|\psi(\omega)|)(1-|\psi(\omega)|^2)^{\frac{1}{q}+2}} \leq C\sup_{|\psi(\omega)|\leq\frac{1}{2}} \mu(\omega)|u(\omega)||\psi'(\omega)|^2 < \infty. \tag{14}
$$

From (13) and (14), we can get (3), finishing the proof of the theorem. $\qquad\square$

Theorem 4. *Let $u \in \mathcal{H}(\mathbb{D})$, ψ be an analytic self-map of $\mathbb{D}$, $0 < p,q < \infty$ and ϕ be normal. Then $DM_uC_\psi : H(p,q,\phi) \to \mathcal{B}^\varphi$ is compact if and only if $DM_uC_\psi : H(p,q,\phi) \to \mathcal{B}^\varphi$ is bounded and*

$$
\lim_{|\psi(z)|\to 1} \frac{\mu(z)|u''(z)|}{\phi(|\psi(z)|)(1-|\psi(z)|^2)^{\frac{1}{q}}} = 0, \tag{15}
$$

$$
\lim_{|\psi(z)|\to 1} \frac{\mu(z)|2u'(z)\psi'(z)+u(z)\psi''(z)|}{\phi(|\psi(z)|)(1-|\psi(z)|^2)^{\frac{1}{q}+1}} = 0, \tag{16}
$$

$$
\lim_{|\psi(z)|\to 1} \frac{\mu(z)|u(z)||\psi'(z)|^2}{\phi(|\psi(z)|)(1-|\psi(z)|^2)^{\frac{1}{q}+2}} = 0. \tag{17}
$$

Proof. Suppose that $DM_uC_\psi : H(p,q,\phi) \to \mathcal{B}^\varphi$ is compact. It is clear that $DM_uC_\psi : H(p,q,\phi) \to \mathcal{B}^\varphi$ is bounded. Let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\psi(z_n)| \to 1$ as $n \to \infty$. Set

$$
f_n(z) = f_{\psi(z_n)}(z),\ g_n(z) = g_{\psi(z_n)}(z),\ h_n(z) = h_{\psi(z_n)}(z).
$$

Then by the proof of Theorem 3,

$$
\sup_{n\in\mathbb{N}}\|f_n\|_{H(p,q,\phi)} < \infty, \sup_{n\in\mathbb{N}}\|g_n\|_{H(p,q,\phi)} < \infty, \sup_{n\in\mathbb{N}}\|h_n\|_{H(p,q,\phi)} < \infty.
$$

Moreover, we can see that f_n, g_n, h_n converges to 0 uniformly on compact subsets of $\mathbb{D}$. Since $DM_uC_\psi : H(p,q,\phi) \to \mathcal{B}^\varphi$ is compact, by Lemma 1, we get

$$
\lim_{n\to\infty}\|DM_uC_\psi f_n\|_{\mathcal{B}^\varphi} = \lim_{n\to\infty}\|DM_uC_\psi g_n\|_{\mathcal{B}^\varphi} = \lim_{n\to\infty}\|DM_uC_\psi h_n\|_{\mathcal{B}^\varphi} = 0.
$$

By (8) we have

$$
f_n(\psi(z_n)) = \frac{P}{\phi(|\psi(z_n)|)(1-|\psi(z_n)|^2)^{\frac{1}{q}}},\ f_n'(\psi(z_n)) = f_n''(\psi(z_n)) = 0,
$$

8   HAIYING LI AND ZHITAO GUO

Then

$$
\begin{aligned}
1 &\geq S_\varphi\left(\frac{(DM_uC_\psi f_n)'(z_n)}{\|DM_uC_\psi f_n\|_{\mathcal{B}^\varphi}}\right) \\
&\geq (1-|z_n|^2)\varphi\left(\frac{P|u''(z_n)|}{\phi(|\psi(z_n)|)(1-|\psi(z_n)|^2)^{\frac{1}{q}}\|DM_uC_\psi f_n\|_{\mathcal{B}^\varphi}}\right).
\end{aligned}
$$

It follows that

$$
\frac{\mu(z_n)|u''(z_n)|}{\phi(|\psi(z_n)|)(1-|\psi(z_n)|^2)^{\frac{1}{q}}} \leq C\|DM_uC_\psi f_n\|_{\mathcal{B}^\varphi}.
$$

Therefore

$$
\begin{aligned}
\lim_{|\psi(z_n)|\to 1} &\frac{\mu(z_n)|u''(z_n)|}{\phi(|\psi(z_n)|)(1-|\psi(z_n)|^2)^{\frac{1}{q}}} \\
&= \lim_{n\to\infty} \frac{\mu(z_n)|u''(z_n)|}{\phi(|\psi(z_n)|)(1-|\psi(z_n)|^2)^{\frac{1}{q}}} = 0. \tag{18}
\end{aligned}
$$

So (15) follows. By (9) we have

$$
g_n'(\psi(z_n)) = \frac{E\cdot\overline{\psi(z_n)}}{\phi(|\psi(z_n)|)(1-|\psi(z_n)|^2)^{\frac{1}{q}+1}},\ g_n(\psi(z_n)) = g_n''(\psi(z_n)) = 0,
$$

Then

$$
\begin{aligned}
1 &\geq S_\varphi\left(\frac{(DM_uC_\psi g_n)'(z_n)}{\|DM_uC_\psi g_n\|_{\mathcal{B}^\varphi}}\right) \\
&\geq (1-|z_n|^2)\varphi\left(\frac{E|\psi(z_n)||2u'(z_n)\psi'(z_n)+u(z_n)\psi''(z_n)|}{\phi(|\psi(z_n)|)(1-|\psi(z_n)|^2)^{\frac{1}{q}+1}\|DM_uC_\psi g_n\|_{\mathcal{B}^\varphi}}\right).
\end{aligned}
$$

It follows that

$$
\frac{\mu(z_n)|\psi(z_n)||2u'(z_n)\psi'(z_n)+u(z_n)\psi''(z_n)|}{\phi(|\psi(z_n)|)(1-|\psi(z_n)|^2)^{\frac{1}{q}+1}} \leq C\|DM_uC_\psi g_n\|_{\mathcal{B}^\varphi}.
$$

Therefore

$$
\begin{aligned}
\lim_{|\psi(z_n)|\to 1} &\frac{\mu(z_n)|2u'(z_n)\psi'(z_n)+u(z_n)\psi''(z_n)|}{\phi(|\psi(z_n)|)(1-|\psi(z_n)|^2)^{\frac{1}{q}+1}} \\
&= \lim_{n\to\infty} \frac{\mu(z_n)|\psi(z_n)||2u'(z_n)\psi'(z_n)+u(z_n)\psi''(z_n)|}{\phi(|\psi(z_n)|)(1-|\psi(z_n)|^2)^{\frac{1}{q}+1}} = 0. \tag{19}
\end{aligned}
$$

So (16) follows. By (12), we have

$$
h_n(\psi(z_n)) = h_n'(\psi(z_n)) = 0,\ h_n''(\psi(z_n)) = \frac{F\cdot\overline{\psi(z_n)}^2}{\phi(|\psi(z_n)|)(1-|\psi(z_n)|^2)^{\frac{1}{q}+2}}.
$$

Then

$$
\begin{aligned}
1 &\geq S_\varphi\left(\frac{(DM_uC_\psi h_n)'(z_n)}{\|DM_uC_\psi h_n\|_{\mathcal{B}^\varphi}}\right) \\
&\geq (1-|z_n|^2)\varphi\left(\frac{F|\psi(z_n)|^2|u(z_n)||\psi'(z_n)|^2}{\phi(|\psi(z_n)|)(1-|\psi(z_n)|^2)^{\frac{1}{q}+2}\|DM_uC_\psi h_n\|_{\mathcal{B}^\varphi}}\right).
\end{aligned}
$$

It follows that

$$\frac{\mu(z_n)|\psi(z_n)|^2|u(z_n)||\psi'(z_n)|^2}{\phi(|\psi(z_n)|)(1-|\psi(z_n)|^2)^{\frac{1}{q}+2}} \leq C\|DM_uC_\psi h_n\|_{\mathcal{B}^\varphi}.$$

Therefore

$$\lim_{|\psi(z_n)|\to 1} \frac{\mu(z_n)|u(z_n)||\psi'(z_n)|^2}{\phi(|\psi(z_n)|)(1-|\psi(z_n)|^2)^{\frac{1}{q}+2}} = \lim_{n\to\infty} \frac{\mu(z_n)|\psi(z_n)|^2|u(z_n)||\psi'(z_n)|^2}{\phi(|\psi(z_n)|)(1-|\psi(z_n)|^2)^{\frac{1}{q}+2}} = 0.$$

So (17) follows.

Conversely, suppose $DM_uC_\psi : H(p,q,\phi) \to \mathcal{B}^\varphi$ is bounded and (15), (16), (17) hold. Then (4), (5), (6) hold by Theorem 3 and for every $\epsilon > 0$, there is a $\delta \in (0,1)$ such that

$$\frac{\mu(z)|u''(z)|}{\phi(|\psi(z)|)(1-|\psi(z)|^2)^{\frac{1}{q}}} < \epsilon, \tag{20}$$

$$\frac{\mu(z)|2u'(z)\psi'(z) + u(z)\psi''(z)|}{\phi(|\psi(z)|)(1-|\psi(z)|^2)^{\frac{1}{q}+1}} < \epsilon, \tag{21}$$

$$\frac{\mu(z)|u(z)||\psi'(z)|^2}{\phi(|\psi(z)|)(1-|\psi(z)|^2)^{\frac{1}{q}+2}} < \epsilon. \tag{22}$$

whenever $\delta < |\psi(z)| < 1$.

Assume that $\{t_n\}_{n\in\mathbb{N}}$ is a sequence in $H(p,q,\phi)$ such that $\sup_{n\in\mathbb{N}}\|t_n\|_{H(p,q,\phi)} \leq L$, and $\{t_n\}$ converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $n\to\infty$. Let $K = \{z \in \mathbb{D} : |\psi(z)| \leq \delta\}$. Then by Lemma 2, (4), (5), (6), (21), (22) and (23), we have

$$\sup_{z\in\mathbb{D}} \mu(z)|(DM_uC_\psi t_n)'(z)|$$

$$\leq \sup_{z\in\mathbb{D}} \mu(z)|u''(z)||t_n(\psi(z))| + \sup_{z\in\mathbb{D}} \mu(z)|2u'(z)\psi'(z) + u(z)\psi''(z)||t_n'(\psi(z))|$$

$$+ \sup_{z\in\mathbb{D}} \mu(z)|u(z)||\psi'(z)|^2|t_n''(\psi(z))|$$

$$\leq \sup_{z\in K} \mu(z)|u''(z)||t_n(\psi(z))| + C_1 \sup_{z\in\mathbb{D}\backslash K} \frac{\mu(z)|u''(z)|\|t_n\|_{H(p,q,\phi)}}{\phi(|\psi(z)|)(1-|\psi(z)|^2)^{\frac{1}{q}}}$$

$$+ \sup_{z\in K} \mu(z)|2u'(z)\psi'(z) + u(z)\psi''(z)||t_n'(\psi(z))|$$

$$+ C_2 \sup_{z\in\mathbb{D}\backslash K} \frac{\mu(z)|2u'(z)\psi'(z) + u(z)\psi''(z)|\|t_n\|_{H(p,q,\phi)}}{\phi(|\psi(z)|)(1-|\psi(z)|^2)^{\frac{1}{q}+1}}$$

$$+ \sup_{z\in K} \mu(z)|u(z)||\psi'(z)|^2|t_n''(\psi(z))| + C_3 \sup_{z\in\mathbb{D}\backslash K} \frac{\mu(z)|u(z)||\psi'(z)|^2\|t_n\|_{H(p,q,\phi)}}{\phi(|\psi(z)|)(1-|\psi(z)|^2)^{\frac{1}{q}+2}}$$

$$\leq C\left(\sup_{|\omega|\leq\delta} |t_n(\omega)| + \sup_{|\omega|\leq\delta} |t_n'(\omega)| + \sup_{|\omega|\leq\delta} |t_n''(\omega)|\right) + 3L\epsilon.$$

So we obtain

$$\|DM_uC_\psi t_n\|_{B^\varphi} = |u'(0)t_n(\psi(0)) + u(0)\psi'(0)t_n'(\psi(0))| + \sup_{z\in\mathbb{D}} \mu(z)|(DM_uC_\psi t_n)'(z)|$$

$$\leq |u'(0)||t_n(\psi(0))| + |u(0)||\psi'(0)||t_n'(\psi(0))|$$

$$+ C\left(\sup_{|\omega|\leq\delta} |t_n(\omega)| + \sup_{|\omega|\leq\delta} |t_n'(\omega)| + \sup_{|\omega|\leq\delta} |t_n''(\omega)|\right) + 3L\epsilon. \tag{23}$$

Since t_n converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$, Cauchy's estimation gives that t_n', t_n'' also do as $n \to \infty$. In particular, since $\{\omega : |\omega| \leq \delta\}$ and $\{\psi(0)\}$ are compact it follows that

$$\lim_{n\to\infty} |u'(0)||t_n(\psi(0))| + |u(0)||\psi'(0)||t_n'(\psi(0))| = 0,$$

$$\lim_{n\to\infty} \sup_{|\omega|\leq\delta} |t_n(\omega)| = \lim_{n\to\infty} \sup_{|\omega|\leq\delta} |t_n'(\omega)| = \lim_{n\to\infty} \sup_{|\omega|\leq\delta} |t_n''(\omega)| = 0.$$

Hence, letting $n \to \infty$ in (24), we get

$$\lim_{n\to\infty} \|DM_uC_\psi t_n\|_{B^\varphi} = 0.$$

Employing Lemma 1 the implication follows. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

REFERENCES

[1] K. Avetisyan and S. Stević. Holomorphic functions on the mixed norm spaces on the polydisc II, J. Comput. Anal. Appl. 11 (2) (2009), 239-251.

[2] K. Avetisyan and S. Stević. Extended Cesàro operators between different Hardy spaces, Appl. Math. Comput. 207 (2009), 346-350.

[3] R. E. Castillo, J. C. Ramos Fernández and M.Salazar. Bounded superposition operators between Bloch-Orlicz and α-Bloch spaces. Appl. Math. Comput. 218 (7) (2011), 3441-3450.

[4] C. C. Cowen and B. D. MacCluer. Composition operators on spaces of analytic functions, Studies in Advanced Mathematics, CRC, Boca Raton, 1995.

[5] S. G. Krantz and S. Stević. On the iterated logarithmic Bloch space on the unit ball. Nonlinear Anal. 71 (2009), 1772-1795.

[6] S. Li. On an integral-type operator from the Bloch space into the $Q_K(p, q)$ space. Filomat, 26 (2) (2012) 331-339.

[7] S. Li and S. Stević. Integral type operators from mixed-norm spaces to α-Bloch spaces, Integral Transforms Spec. Funct. 18 (7) (2007), 485-493.

[8] S. Li and S. Stević. Weighted composition operators from Bergman-type spaces into Bloch spaces, Proc. Indian Acad. Sci. Math. Sci. 117 (3) (2007), 371-385.

[9] S. Li and S. Stević. Weighted composition operators from H^∞ to the Bloch space on the polydisc, Abstr. Appl. Anal. Vol. 2007, Article ID 48478, (2007), 12 pp.

[10] S. Li and S. Stević. Composition followed by differentiation from mixed-norm spaces to α-Bloch spaces, Sb. Math. 199 (12) (2008), 1847-1857.

[11] S. Li and S. Stević. Generalized composition operators on Zygmund spaces and Bloch type spaces, J. Math. Anal. Appl. 338 (2) (2008), 1282-1295.

[12] S. Li and S. Stević. Riemann-Stieltjes operators between different weighted Bergman spaces, Bull. Belg. Math. Soc. Simon Stevin 15 (4) (2008), 677-686.

[13] S. Li and S. Stević. Weighted composition operators from Zygmund spaces into Bloch spaces. Appl. Math. Comput. 206 (2) (2008), 825-831.

[14] S. Li and S. Stević. Integral-type operators from Bloch-type spaces to Zygmund-type spaces. Appl. Math. Comput. 215 (2) (2009), 464-473.

[15] S. Li and S. Stević. Composition followed by differentiation between H^∞ and α-Bloch spaces, Houston J. Math. 35 (1) (2009), 327-340.

[16] K. Madigan and A. Matheson. Compact composition operators on the Bloch space, Trans. Amer. Math. Soc. 347(7) (1995), 2679-2687.

[17] J. C. Ramos Fernández. Composition operators on Bloch-Orlicz type spaces, Applied Mathematics and Computation, 217 (7) (2010), 3392-3402.

[18] A. K. Sharma. Products of composition multiplication and differentiation between Bergman and Bloch type spaces. Turk. J. Math. 35(2011), 275-291.

[19] A. Shields and D. Williams. Bounded projections, duality, and multipliers in spaces of analytic functions, Trans. Amer. Math. Soc., 162 (1971), 287-302.

[20] S. Stević. Boundedness and compactness of an integral operator on mixed norm spaces on the polydisc, Siberian Math. J. 48 (3) (2007), 559-569.

[21] S. Stević. Essential norms of weighted composition operators from the α-Bloch space to a weighted-type space on the unit ball, Abstr. Appl. Anal. Vol. 2008, Article ID 279691, (2008), 11 pp.

[22] S. Stević. Generalized composition operators between mixed norm space and some weighted spaces, Numer. Funct. Anal. Optimization 29 (7-8) (2008), 959-978.

[23] S. Stević. Generalized composition operators from logarithmic Bloch spaces to mixed-norm spaces, Util. Math. 77 (2008), 167-172.

[24] S. Stević. Holomorphic functions on the mixed norm spaces on the polydisc, J. Korean Math. Soc. 45 (1) (2008), 63-78.

[25] S. Stević. Norms of some operators from Bergman spaces to weighted and Bloch-type space, Util. Math. 76 (2008), 59-64.

[26] S. Stević, On a new operator from H^∞ to the Bloch-type space on the unit ball, Util. Math. 77 (2008), 257-263.

[27] S. Stević. On a new integral-type operator from the weighted Bergman space to the Bloch-type space on the unit ball, Discrete Dyn. Nat. Soc. Vol. 2008, Art. ID 154263, (2008), 14 pp.

[28] S. Stević. Boundedness and compactness of an integral operator between H^∞ and a mixed norm space on the polydisk, Siberian Math. J. 50 (3) (2009), 495-497.

[29] S. Stević. Integral-type operators from a mixed norm space to a Bloch-type space on the unit ball, Siberian Math. J. 50 (6) (2009), 1098-1105.

[30] S. Stević. Norm and essential norm of composition followed by differentiation from α-Bloch spaces to H_μ^∞, Appl. Math. Comput. 207 (2009), 225-229.

[31] S. Stević. On an integral operator from the Zygmund space to the Bloch-type space on the unit ball, Glasg. J. Math. 51 (2009), 275-287.

[32] S. Stević. On a new integral-type operator from the Bloch space to Bloch-type spaces on the unit ball, J. Math. Anal. Appl. 354 (2009), 426-434.

[33] S. Stević. Products of composition and differentiation operators on the weighted Bergman space, Bull. Belg. Math. Soc. Simon Stevin 16 (2009), 623-635.

[34] S. Stević. Products of integral-type operators and composition operators from the mixed norm space to Bloch-type spaces, Siberian Math. J. 50 (4) (2009), 726-736.

[35] S. Stević. Weighted composition operators from weighted Bergman spaces to weighted-type spaces on the unit ball, Appl. Math. Comput. 212 (2009), 499-504.

[36] S. Stević. Weighted differentiation composition operators from mixed-norm spaces to weighted-type spaces. Appl. Math. Comput. 211 (1) (2009), 222-233.

[37] S. Stević. Extended Cesàro operators between mixed-norm spaces and Bloch-type spaces in the unit ball. Houston J. Math. 36 (3) (2010), 843-858.

[38] S. Stević. On an integral-type operator from logarithmic Bloch-type spaces to mixed-norm spaces on the unit ball. Appl. Math. Comput. 215 (11) (2010), 3817-3823.

[39] S. Stević. Characterizations of composition followed by differentiation between Bloch-type spaces. Appl. Math. Comput. 218 (8) (2011), 4312-4316.

[40] S. Stević. Boundedness and compactness of an integral-type operator from Bloch-type spaces with normal weights to $F(p, q, s)$ space. Appl. Math. Comput. 218 (9) (2012), 5414-5421.

[41] S. Stević. On an integral-type operator from logarithmic Bloch-type and mixed-norm spaces to Bloch-type spaces. Nonlinear Anal. 71 (12) (2009), 6323-6342.

[42] S. Stević. On an integral-type operator from Zygmund-type spaces to mixed-norm spaces on the unit ball. Abstr. Appl. Anal. 2010, Art. ID 198608, 7 pp.

[43] S. Stević. On operator P_φ^g from the logarithmic Bloch-type space to the mixed-norm space on unit ball, Appl. Math. Comput. 215 (2010), 4248-4255.

[44] S. Stević. Norm estimates of weighted composition operators between Bloch-type spaces. Ars Combin. 93 (2009), 161-164.

12   HAIYING LI AND ZHITAO GUO

[45] S. Stević. Weighted differentiation composition operators from H^∞ and Bloch spaces to nth weigthed-type spaces on the unit disk, Appl. Math. Comput. 216 (2010), 3634-3641.

[46] S. Stević. Weighted differentiation composition operators from the mixed-norm space to the nth weighted-type space on the unit disk. Abstr. Appl. Anal. 2010, Art. ID 246287, 15 pp.

[47] S. Stević. Weighted radial operator from the mixed-norm space to the nth weighted-type space on the unit ball. Appl. Math. Comput. 218 (18) (2012), 9241-9247.

[48] S. Stević and A. K. Sharma. Integral-type operators between weighted Bergman spaces on the unit disk. J. Comput. Anal. Appl. 14 (7) (2012), 1339-1344.

[49] S. Stević and S. Ueki. Integral-type operators acting between weighted-type spaces on the unit ball, Appl. Math. Comput. 215 (2009), 2464-2471.

[50] K. Zhu. Bloch type spaces of analytic functions, Rocky Mountain J. Math. 23 (3) (1993), 1143-1177.

[51] X. Zhu. Generalized weighted composition operators from Bloch-type spaces to weighted Bergman spaces, Indian J. Math. 49 (2) (2007), 139-149.

[52] X. Zhu. Multiplication followed by differentiation on Bloch-type spaces, Bull. Allahbad. Math. Soc. 23 (1) (2008), 25-39.

[53] X. Zhu. Volterra type operators from logarithmic Bloch spaces to Zygmund type space, Inter. J. Modern Math. 3 (3) (2008), 327-336.

[54] X. Zhu. Extended Cesàro operators from mixed norm spaces to Zygmund type spaces. Tamsui Oxf. J. Math. Sci. 26 (4) (2010), 411-422.

[55] X. Zhu, Generalized weighted composition operators from Bers-type spaces into Bloch-type spaces Math. Inequal. Appl. 17 (1) (2014), 187-195.

HAIYING LI, HENAN ENGINEERING LABORATORY FOR BIG DATA STATISTICAL ANALYSIS AND OPTIMAL CONTROL, SCHOOL OF MATHEMATICS AND INFORMATION SCIENCE, HENAN NORMAL UNIVERSITY, 453007 XINXIANG, P.R.CHINA
E-mail address: haiyingli2012@yahoo.com

ZHITAO GUO, HENAN ENGINEERING LABORATORY FOR BIG DATA STATISTICAL ANALYSIS AND OPTIMAL CONTROL, SCHOOL OF MATHEMATICS AND INFORMATION SCIENCE, HENAN NORMAL UNIVERSITY, 453007 XINXIANG, P.R.CHINA

A SHORT NOTE ON INTEGRAL INEQUALITY OF TYPE HERMITE-HADAMARD THROUGH CONVEXITY

MUHAMMAD IQBAL, SHAHID QAISAR, AND MUHAMMAD MUDDASSAR*

ABSTRACT. In this short note, a Riemann-Liouville fractional integral identity including first order derivative of a given function is established. With the help of this fractional-type integral identity, some new Hermite-Hadamard-type inequality involving Riemann-Liouville fractional integrals for $(m, h_1, h_2)-$convex function are considered. Our method considered here may be a stimulant for further investigations concerning Hermite-Hadamard-type inequalities involving fractional integrals.

1. Introduction and Defintions

Many inequalities have been established for convex functions but the most famous is the Hermite-Hadamarad inequality, due to its rich geometrical significance and applications, which is stated as in [12]

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function defined on the closed interval I of real numbers and $a, b \in I$ with $a < b$

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t) dt \leq \frac{f(a) + f(b)}{2}$$

Both the inequalities hold in reversed direction if f is concave. We recall some preliminary concepts about convex functions:

Definition 1. [7]. *A function $f : [0, \infty) \to \mathbb{R}$ is said to be s-convex function or f belongs to the class K_s^i if for all $x, y \in [0, \infty)$ and $\mu, \nu \in [0, 1]$, the following inequality holds*

$$f(\mu x + \nu y) \leq \mu^s f(x) + \nu^s f(y)$$

for some fixed $\alpha \in (0, 1]$.

Note that, if $\mu^s + \nu^s = 1$, the above class of convex functions is called s-convex functions in first sense and represented by K_s^1 and if $\mu + \nu = 1$ the above class is called s-convex in second sense and represented by K_s^2.

Definition 2. [11]. *A function $f : [0, b] \to \mathbb{R}$ is said to be (α, m)-convex, where $(\alpha, m) \in [0, 1]^2$, if for every $x, y \in [0, b]$ and for $\lambda \in [0, 1]$, the following inequality holds*

$$f(\lambda y + m(1 - \lambda) x) \leq \lambda^\alpha f(y) + m(1 - \lambda^\alpha) f(x)$$

where $(\alpha, m) \in [0, 1]^2$ and for some fixed $m \in (0, 1]$.

1

Theorem 1. [3]. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° (interior of I) where $a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then we have*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \le \frac{b-a}{8} \left[|f'(a)| + |f'(b)| \right].$$

Theorem 2. [8]. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I^o (interior of I) where $a, b \in I$ with $a < b$. If the mapping $|f'|^q$ is convex on $[a, b]$, for some $q \ge 1$, then the following inequality holds:*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \le \frac{b-a}{4} \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{\frac{1}{q}}$$

Theorem 3. [13]. *Let $f : I \subset [0, \infty) \to \mathbb{R}$ be differentiable mapping on I^o $a, b \in I^o$ with $a < b$, and If $|f'|^q$ is quasi-convex on $[a, b]$, $p > 1$. Then the following inequality holds:*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \le \frac{(b-a)}{16} \left(\frac{4}{1+p} \right)^{\frac{1}{p}}$$

$$\left\{ \left[|f'(a)|^{\frac{p}{p-1}} + 3\,|f'(b)|^{\frac{p}{p-1}} \right]^{1-\frac{1}{p}} \right.$$

$$\left. + \left[3\,|f'(a)|^{\frac{p}{p-1}} + |f'(b)|^{\frac{p}{p-1}} \right]^{1-\frac{p}{p-1}} \right\}$$

Theorem 4. [9]. *Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a differentiable function on I^0 where $a, b \in I^o$ with $a < b$ such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If the mapping $|f''|^q$ is s-convex on $[a, b]$ for some fixed $s \in (0, 1]$ and $q \ge 1$, then the following inequality holds:*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \le \frac{b-a}{2^{\frac{1}{q}}} \left[\frac{s + \left(\frac{1}{2}\right)^s}{(s+1)(s+2)} \right]^{\frac{1}{q}} \left[|f'(a)|^q + |f'(b)|^q \right]^{\frac{1}{q}}$$

Theorem 5. [4]. *Suppose that $f : [0, \infty) \to [0, \infty)$ is a convex function in the second sensewhere $\alpha \in (0, 1)$ and let $a, b \in [0, \infty)$, $a < b$. if $f \in L_1([a, b])$, then the following inequality holds*

$$2^{\alpha-1} f\left(\frac{a+b}{2} \right) \le \frac{1}{b-a} \int_a^b f(x)dx \le \frac{f(a) + f(b)}{\alpha + 1}.$$

Fraction calculus [2, 6, 1, 5] was introduced at the end of the nineteenth century by Riemann and Liouville the subject of which has become a rapidly growing area and has found applications in diverse fields ranging from physical sciences and engineering to biological sciences and economics. We recall some definitions and preliminary facts of fractional calculus theory which will be used in this paper.

Definition 3. [6]. *Let $f \in L_1[a, b]$. The Riemann-Liouville fractional integrals $J_{\alpha+}^\alpha f$ and $J_{b-}^\alpha f$ of order $\alpha > 0$ with $\alpha \ge 0$ are defined by*

$$J_{\alpha+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt, \quad (a < x),$$

and

$$J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\, dt, \quad (b > x),$$

respectively.

Here $\Gamma(\alpha) = \int_0^\infty e^{-u} u^{\alpha-1} du.$ *Here is* $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x).$

In case of $\alpha = 1$, the fractional integral reduces to the classical integral. The aim of this paper is to establish Hermite-Hadamard type inequalities based on $(m, h_1, h_2) -$ convexity. Using these results we obtained new inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

2. Main Results

Before proceeding to our main results, we present some necessary definition and lemma which are used further in this paper.

Definition 4. *Let* $f : I \subseteq R_0 \to R$, $\quad h_1, h_2 : [0,1] \to R_0$, *and* $m \in (0,1]$, *then* f *is said to be* $(m, h_1, h_2) -$ *convex. if* f *is non–negative and the following inequality*

$$f(\lambda x + m(1-\lambda) y) \le h_1(\lambda) f(x) + m h_2(1-\lambda) f(y),$$

holds for all $x, y \quad \in I$ *and* $\lambda \in [0,1]$.

If the above inequality is reversed then f is said to be $(m, h_1, h_2) -$ concave.

$$M_\alpha(a,b) = \frac{1}{4} \left[f(a) + \frac{1}{2} \left(f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right) + f(b) \right]$$
$$- \frac{\Gamma(\alpha+1) 4^{\alpha-1}}{(b-a)^\alpha} \left[J_{a+}^\alpha f\left(\frac{3a+b}{4}\right) + J_{\frac{3a+b}{4}}^\alpha f\left(\frac{a+3b}{4}\right) + J_{\frac{a+3b}{4}}^\alpha f(b) \right].$$

Specially, when $\alpha = 1$, we have

$$M_1(a,b) = \frac{1}{4} \left[f(a) + \frac{1}{2} \left(f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(t)\, dt.$$

Lemma 1. *Suppose* $f : [a,b] \to R$ *is a differentiable mapping on* (a,b). *If* $f' \in L_1([a,b])$, *then we have the following identity*

$$M_\alpha(a,b) = \frac{b-a}{16} \times \left\{ \begin{array}{l} \int_0^1 (0-\lambda^\alpha) f'\left(\lambda a + (1-\lambda)\frac{3a+b}{4}\right) d\lambda \\ + \int_0^1 \left(\frac{1}{2} - \lambda^\alpha\right) f'\left(\lambda \frac{3a+b}{4} + (1-\lambda)\frac{a+3b}{4}\right) d\lambda \\ + \int_0^1 (1-\lambda^\alpha) f'\left(\lambda \frac{a+3b}{4} + (1-\lambda) b\right) d\lambda \end{array} \right\}$$

Proof. By integrating, and by making use of the substitution $u = \lambda a + (1-\lambda)\frac{3a+b}{4}$ we have

$$\frac{b-a}{16} \left\{ \int_0^1 (-\lambda^\alpha) f'\left(\lambda a + (1-\lambda)\frac{3a+b}{4}\right) d\lambda \right\}$$
$$= \frac{1}{4} \left[f(a) - \alpha \int_0^1 (-\lambda^\alpha) f\left(\lambda a + (1-\lambda)\frac{3a+b}{4}\right) \lambda^{\alpha-1} d\lambda \right]$$
$$= \frac{1}{4} f(a) - \frac{\alpha 4^{\alpha-1}}{(b-a)^\alpha} \int_a^{\frac{3a+b}{4}} \left(\frac{3a+b}{4} - u\right)^{\alpha-1} f(u)\, du$$
$$= \frac{1}{4} f(a) - \frac{\Gamma(\alpha+1) 4^{\alpha-1}}{(b-a)^\alpha} J_{a+}^\alpha f\left(\frac{3a+b}{4}\right)$$

$$\frac{b-a}{16} \left\{ \int_0^1 \left(\frac{1}{2} - \lambda^\alpha\right) f'\left(\lambda \frac{3a+b}{4} + (1-\lambda)\frac{a+3b}{4}\right) d\lambda \right\}$$
$$= \frac{1}{8} \left[\left(f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) \right) \right]$$
$$= \frac{b-a}{16} \left\{ \int_0^1 \left(\frac{1}{2} - \lambda^\alpha\right) f'\left(\lambda \frac{3a+b}{4} + (1-\lambda)\frac{a+3b}{4}\right) d\lambda \right\}$$
$$= \frac{1}{8} \left[\left(f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) \right) \right] - \frac{\Gamma(\alpha+1) 4^{\alpha-1}}{(b-a)^\alpha} J_{\alpha+}^\alpha f\left(\frac{a+3b}{4}\right)$$

$$\int_0^1 (1 - \lambda^\alpha) f' \left(\lambda \tfrac{a+3b}{4} + (1 - \lambda) b\right) d\lambda$$
$$= \tfrac{1}{4} f(b) - \tfrac{\Gamma(\alpha+1)4^{\alpha-1}}{(b-a)^\alpha} J^\alpha_{\frac{a+3b}{4}+} f(b)$$

This proves as required.   □

Theorem 6. *Suppose $f : [a, b] \to R$ is a differentiable mapping on (a, b) with $a < b$. such that $f' \in L_1([a, b])$ for $0 < a < b$. If $|f'|^q$ is $(m, h_1, h_2)-$convex on $[a, b]$ for some fixed $q > 1$ and $h_1, h_2 \in L_1([a, b])$, then we have the following inequality*

$$M_\alpha(a, b) \leq \tfrac{b-a}{16} \times$$
$$\left[\begin{array}{l} \left(\tfrac{q-1}{\alpha q + q - 1}\right)^{1-1/q} \times \left(|f'(a)|^q \|h_1\|_1 + m \left|f'\left(\tfrac{3a+b}{4m}\right)\right|^q \|h_2\|_1\right)^{1/q} \\ + \tfrac{1}{2} \left(\tfrac{\alpha 2^{(\alpha-1)/\alpha} - \alpha + 1}{\alpha + 1}\right)^{1-1/q} \times \left(\left|f'\left(\tfrac{3a+b}{4}\right)\right|^q \|h_1\|_1 + m \left|f'\left(\tfrac{a+3b}{4m}\right)\right|^q \|h_2\|_1\right)^{1/q} \\ + \left(\tfrac{1}{\alpha} B\left(\tfrac{2q-1}{q-1}, \tfrac{1}{\alpha}\right)\right)^{1-1/q} \times \left(\left|f'\left(\tfrac{a+3b}{4}\right)\right|^q \|h_1\|_1 + m \left|f'\left(\tfrac{b}{m}\right)\right|^q \|h_2\|_1\right)^{1/q} \end{array} \right]$$

Suppose $\|h_1\|_p = \left(\int_0^1 h_1^p(\lambda) d\lambda\right)^{1/p}$ for $p \geq 1$ with $B(x, y)$ is the classical Beta function which may be defined $B(x, y) = \int_0^1 \lambda^{x-1}(1 - \lambda)^{y-1}$, $x > 0, y > 0$. for $0 < a < b$.

Proof. Holder integral inequality and Lemma 1 together implies with (m, h_1, h_2)-convexity of $|f'|^q$

$$M_\alpha(a, b) \leq \tfrac{b-a}{16} \times$$
$$\left[\begin{array}{l} \left(\int_0^1 \lambda^{\alpha q/q - 1} d\lambda\right)^{1-1/q} \times \left(|f'(a)|^q \|h_1\|_1 + m \left|f'\left(\tfrac{3a+b}{4m}\right)\right|^q \|h_2\|_1\right)^{1/q} \\ + \left(\tfrac{1}{2}\right)^{1/q} \left(\int_0^1 \left|\tfrac{1}{2} - \lambda^\alpha\right| d\lambda\right)^{1-1/q} \times \left(\left|f'\left(\tfrac{3a+b}{4}\right)\right|^q \|h_1\|_1 + m \left|f'\left(\tfrac{a+3b}{4m}\right)\right|^q \|h_2\|_1\right)^{1/q} \\ + \left(\int_0^1 (1 - \lambda^\alpha)^{q/q - 1} d\lambda\right)^{1-1/q} \times \left(\left|f'\left(\tfrac{a+3b}{4}\right)\right|^q \|h_1\|_1 + m \left|f'\left(\tfrac{b}{m}\right)\right|^q \|h_2\|_1\right)^{1/q} \end{array} \right]$$

Therefore,

$$\int_0^1 \lambda^{\alpha q/q - 1} d\lambda = \tfrac{q-1}{\alpha q + q - 1}, \quad \int_0^1 (1 - \lambda^\alpha)^{q/q - 1} d\lambda = \tfrac{1}{\alpha} B\left(\tfrac{2q-1}{q-1}, \tfrac{1}{\alpha}\right),$$
$$\int_0^1 \left|\tfrac{1}{2} - \lambda^\alpha\right| d\lambda = \tfrac{\alpha 2^{(\alpha-1)/\alpha} - \alpha + 1}{\alpha + 1}$$

This completes the proof .   □

Corollary 1. *In Theorem 6, if we choose $h_1(\lambda) = h(\lambda)$, $h_2(\lambda) = h(1 - \lambda)$, then we have,*

$$M_\alpha(a, b) \leq \tfrac{(b-a)\|h_1\|_1^{1/q}}{16} \times$$
$$\left[\begin{array}{l} \left(\tfrac{q-1}{\alpha q + q - 1}\right)^{1-1/q} \times \left(|f'(a)|^q + m \left|f'\left(\tfrac{3a+b}{4m}\right)\right|^q\right)^{1/q} \\ + \tfrac{1}{2} \left(\tfrac{\alpha 2^{(\alpha-1)/\alpha} - \alpha + 1}{\alpha + 1}\right)^{1-1/q} \times \left(\left|f'\left(\tfrac{3a+b}{4}\right)\right|^q + m \left|f'\left(\tfrac{a+3b}{4m}\right)\right|^q\right)^{1/q} \\ + \left(\tfrac{1}{\alpha} B\left(\tfrac{2q-1}{q-1}, \tfrac{1}{\alpha}\right)\right)^{1-1/q} \times \left(\left|f'\left(\tfrac{a+3b}{4}\right)\right|^q + m \left|f'\left(\tfrac{b}{m}\right)\right|^q\right)^{1/q} \end{array} \right]$$

Furthermore if we choose $m = 1$, we have

$$M_\alpha(a, b) \leq \tfrac{(b-a)\|h_1\|_1^{1/q}}{16} \times$$
$$\left[\begin{array}{l} \left(\tfrac{q-1}{\alpha q + q - 1}\right)^{1-1/q} \times \left(|f'(a)|^q + \left|f'\left(\tfrac{3a+b}{4}\right)\right|^q\right)^{1/q} \\ + \tfrac{1}{2} \left(\tfrac{\alpha 2^{(\alpha-1)/\alpha} - \alpha + 1}{\alpha + 1}\right)^{1-1/q} \times \left(\left|f'\left(\tfrac{3a+b}{4}\right)\right|^q + \left|f'\left(\tfrac{a+3b}{4}\right)\right|^q\right)^{1/q} \\ + \left(\tfrac{1}{\alpha} B\left(\tfrac{2q-1}{q-1}, \tfrac{1}{\alpha}\right)\right)^{1-1/q} \times \left(\left|f'\left(\tfrac{a+3b}{4}\right)\right|^q + |f'(b)|^q\right)^{1/q} \end{array} \right]$$

Corollary 2. *Under the conditions of Corollary 1, if we choose* $h_1(\lambda) = h(\lambda) = \lambda^s, m = 1$, *we have the*

$$M_\alpha(a,b) \leq \frac{(b-a)}{16}\left(\frac{1}{s+1}\right)^{1/q} \times$$

$$\left[\begin{array}{l} \left(\frac{q-1}{\alpha q+q-1}\right)^{1-1/q} \times \left(|f'(a)|^q + \left|f'\left(\frac{3a+b}{4}\right)\right|^q\right)^{1/q} \\ +\frac{1}{2}\left(\frac{\alpha 2^{(\alpha-1)/\alpha}-\alpha+1}{\alpha+1}\right)^{1-1/q} \times \left(\left|f'\left(\frac{3a+b}{4}\right)\right|^q + \left|f'\left(\frac{a+3b}{4}\right)\right|^q\right)^{1/q} \\ +\left(\frac{1}{\alpha}B\left(\frac{2q-1}{q-1},\frac{1}{\alpha}\right)\right)^{1-1/q} \times \left(\left|f'\left(\frac{a+3b}{4}\right)\right|^q + |f'(b)|^q\right)^{1/q} \end{array}\right]$$

Specially, if we choose $\alpha = s = m = 1$, we have the

$$M_\alpha(a,b) \leq \frac{(b-a)}{16}\left(\frac{1}{2}\right)^{1/q} \times \left[\begin{array}{l} \left(\frac{q-1}{2q-1}\right)^{1-1/q} \times \left(|f'(a)|^q + \left|f'\left(\frac{3a+b}{4}\right)\right|^q\right)^{1/q} \\ +\left(\frac{1}{2}\right)^{2-1/q} \times \left(\left|f'\left(\frac{3a+b}{4}\right)\right|^q + \left|f'\left(\frac{a+3b}{4}\right)\right|^q\right)^{1/q} \\ +\left(\frac{q-1}{2q-1}\right)^{1-1/q} \times \left(\left|f'\left(\frac{a+3b}{4}\right)\right|^q + |f'(b)|^q\right)^{1/q} \end{array}\right]$$

Corollary 3. *Under the conditions of Theorem 6, if we choose* $h_1(\lambda) = \lambda^{\alpha_1}$, $h_2(\lambda) = 1 - \lambda^{\alpha_1}$, *we have the*

$$M_\alpha(a,b) \leq \frac{(b-a)}{16}\left(\frac{1}{\alpha_1+1}\right)^{1/q} \times$$

$$\left[\begin{array}{l} \left(\frac{q-1}{\alpha q+q-1}\right)^{1-1/q} \times \left(|f'(a)|^q + m\alpha_1\left|f'\left(\frac{3a+b}{4m}\right)\right|^q\right)^{1/q} \\ +\frac{1}{2}\left(\frac{\alpha 2^{(\alpha-1)/\alpha}-\alpha+1}{\alpha+1}\right)^{1-1/q} \times \left(\left|f'\left(\frac{3a+b}{4}\right)\right|^q + m\left|f'\left(\frac{a+3b}{4m}\right)\right|^q\right)^{1/q} \\ +\left(\frac{1}{\alpha}B\left(\frac{2q-1}{q-1},\frac{1}{\alpha}\right)\right)^{1-1/q} \times \left(\left|f'\left(\frac{a+3b}{4}\right)\right|^q + m\alpha_1\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{1/q} \end{array}\right]$$

Specially, if we choose $m = 1$, *we have the,*

$$M_\alpha(a,b) \leq \frac{(b-a)}{16}\left(\frac{1}{\alpha_1+1}\right)^{1/q} \times$$

$$\left[\begin{array}{l} \left(\frac{q-1}{\alpha q+q-1}\right)^{1-1/q} \times \left(|f'(a)|^q + \alpha_1\left|f'\left(\frac{3a+b}{4}\right)\right|^q\right)^{1/q} \\ +\frac{1}{2}\left(\frac{\alpha 2^{(\alpha-1)/\alpha}-\alpha+1}{\alpha+1}\right)^{1-1/q} \times \left(\left|f'\left(\frac{3a+b}{4}\right)\right|^q + \left|f'\left(\frac{a+3b}{4}\right)\right|^q\right)^{1/q} \\ +\left(\frac{1}{\alpha}B\left(\frac{2q-1}{q-1},\frac{1}{\alpha}\right)\right)^{1-1/q} \times \left(\left|f'\left(\frac{a+3b}{4}\right)\right|^q + \alpha_1|f'(b)|^q\right)^{1/q} \end{array}\right]$$

Theorem 7. *Suppose* $f:[a,b] \to R$ *is a differentiable mapping on* (a,b) *with* $a < b$. *such that* $f' \in L_1([a,b])$ *for* $0 < a < b$. *If* $|f'|^q$ *is and* $(m,h_1,h_2)-$ *convex on* $[a,b]$ *for* $q \geq 1$, *and* $h_1, h_2 \in L_1([a,b])$, *then we have the following inequality*

$$M_\alpha(a,b) \leq \frac{(b-a)}{16}\left(\frac{1}{\alpha+1}\right)^{1-1/q} \times$$

$$\left\{\begin{array}{l} \left[\frac{1}{2}|f'(a)|^q\left(\frac{1}{2\alpha+1}+\|h_1\|_2^2\right) + \frac{m}{2}\left|f'\left(\frac{3a+b}{4m}\right)\right|^q\left(\frac{1}{2\alpha+1}+\|h_2\|_2^2\right)\right]^{1/q} \\ +\frac{1}{2}\left(\frac{\alpha 2^{(\alpha-1)/\alpha}-\alpha+1}{\alpha+1}\right)^{1-1/q} \times \left(\left|f'\left(\frac{3a+b}{4}\right)\right|^q\|h_1\|_1 + m\left|f'\left(\frac{a+3b}{4m}\right)\right|^q\|h_2\|_1\right)^{1/q} \\ +\alpha^{1-1/q}\left[\begin{array}{l}\frac{1}{2}\left|f'\left(\frac{a+3b}{4}\right)\right|^q\left(\frac{2\alpha^2}{(2\alpha+1)(\alpha+1)}+\|h_1\|_2^2\right) \\ +\frac{m}{2}\left|f'\left(\frac{b}{m}\right)\right|^q\left(\frac{2\alpha^2}{(2\alpha+1)(\alpha+1)}+\|h_1\|_2^2\right)\end{array}\right]^{1/q} \end{array}\right\}$$

Where $\frac{1}{p} + \frac{1}{q} = 1$.

6   M. IQBAL, S. QAISAR, AND M. MUDDASSAR

Proof. Using Holder's inequality and by Lemma 1, and $(m, h_1, h_2) -$ convexity of $|f'|^q$, we get

$$M_\alpha(a,b) \le \frac{(b-a)}{16} \times$$
$$\left\{ \begin{array}{l} \left(\int_0^1 \lambda^\alpha d\lambda\right)^{1-1/q} \left[f'(a)|^q \int_0^1 \lambda^\alpha h_1(\lambda)\, d\lambda + \frac{m}{2}\left|f'\left(\frac{3a+b}{4m}\right)\right|^q \int_0^1 \lambda^\alpha h_2(\lambda)\, d\lambda\right]^{1/q} \\ +\left(\frac{1}{2}\right)^{1/q}\left(\int_0^1 \left|\frac{1}{2}-\lambda^\alpha\right| d\lambda\right)^{1-1/q} \times \left(\left|f'\left(\frac{3a+b}{4}\right)\right|^q \int_0^1 h_1(\lambda)\, d\lambda + m\left|f'\left(\frac{a+3b}{4m}\right)\right|^q \int_0^1 h_2(\lambda)\, d\lambda\right)^{1/q} \\ +\left(\int_0^1(1-\lambda^\alpha)\, d\lambda\right)^{1-1/q}\left[\begin{array}{l}\left|f'\left(\frac{a+3b}{4}\right)\right|^q \int_0^1(1-\lambda^\alpha)h_1(\lambda)\, d\lambda \\ +m\left|f'\left(\frac{b}{m}\right)\right|^q \int_0^1(1-\lambda^\alpha)h_2(\lambda)\, d\lambda\end{array}\right]^{1/q} \end{array}\right\}$$

$$M_\alpha(a,b) \le \frac{(b-a)}{16} \times$$
$$\left\{ \begin{array}{l} \left(\int_0^1 \lambda^\alpha d\lambda\right)^{1-1/q} \left[|f'(a)|^q \int_0^1 \frac{\lambda^{2\alpha}+h_1^2(\lambda)}{2}d\lambda + m\left|f'\left(\frac{3a+b}{4m}\right)\right|^q \int_0^1 \frac{\lambda^{2\alpha}+h_2^2(\lambda)}{2}d\lambda\right]^{1/q} \\ +\left(\frac{1}{2}\right)^{1/q}\left(\int_0^1 \left|\frac{1}{2}-\lambda^\alpha\right| d\lambda\right)^{1-1/q} \times \left(\left|f'\left(\frac{3a+b}{4}\right)\right|^q \int_0^1 h_1(\lambda)\, d\lambda + m\left|f'\left(\frac{a+3b}{4m}\right)\right|^q \int_0^1 h_2(\lambda)\, d\lambda\right)^{1/q} \\ +\left(\int_0^1(1-\lambda^\alpha)\, d\lambda\right)^{1-1/q}\left[\begin{array}{l}\left|f'\left(\frac{a+3b}{4}\right)\right|^q \int_0^1 \frac{(1-\lambda^\alpha)^2+h_1^2(\lambda)}{2}d\lambda \\ +m\left|f'\left(\frac{b}{m}\right)\right|^q \int_0^1 \frac{(1-\lambda^\alpha)^2+h_2^2(\lambda)}{2}d\lambda\end{array}\right]^{1/q} \end{array}\right\}$$

$$= \frac{(b-a)}{16}\left(\frac{1}{\alpha+1}\right)^{1-1/q} \times$$
$$\left\{ \begin{array}{l} \left[\frac{1}{2}|f'(a)|^q\left(\frac{1}{2\alpha+1}+\|h_1\|_2^2\right) + \frac{m}{2}\left|f'\left(\frac{3a+b}{4m}\right)\right|^q\left(\frac{1}{2\alpha+1}+\|h_2\|_2^2\right)\right]^{1/q} \\ +\frac{1}{2}\left(\frac{\alpha 2^{(\alpha-1)/\alpha}-\alpha+1}{\alpha+1}\right)^{1-1/q} \times \left(\left|f'\left(\frac{3a+b}{4}\right)\right|^q\|h_1\|_1 + m\left|f'\left(\frac{a+3b}{4m}\right)\right|^q\|h_2\|_1\right)^{1/q} \\ +\alpha^{1-1/q}\left[\begin{array}{l}\frac{1}{2}\left|f'\left(\frac{a+3b}{4}\right)\right|^q\left(\frac{2\alpha^2}{(2\alpha+1)(\alpha+1)}+\|h_1\|_2^2\right) \\ +\frac{m}{2}\left|f'\left(\frac{b}{m}\right)\right|^q\left(\frac{2\alpha^2}{(2\alpha+1)(\alpha+1)}+\|h_1\|_2^2\right)\end{array}\right]^{1/q} \end{array}\right\}$$

This completes the proof .   □

Corollary 4. *In Theorem 7, if we choose* $h_1(\lambda) = \lambda^{\alpha_1}$, $h_2(\lambda) = 1-\lambda^{\alpha_1}$, *we have*

$$M_\alpha(a,b) \le \frac{(b-a)}{16}\left(\frac{1}{\alpha+1}\right)^{1-1/q} \times$$
$$\left\{ \begin{array}{l} \left[\frac{1}{2}|f'(a)|^q\left(\frac{1}{2\alpha+1}+\frac{1}{2\alpha_1+1}\right) + \frac{m}{2}\left|f'\left(\frac{3a+b}{4m}\right)\right|^q\left(\frac{1}{2\alpha+1}+\frac{2\alpha_1^2}{(2\alpha_1+1)(\alpha+1)}\right)\right]^{1/q} \\ +\frac{1}{2}\left(\alpha 2^{(\alpha-1)/\alpha}-\alpha+1\right)^{1-1/q} \times \left(\left|f'\left(\frac{3a+b}{4}\right)\right|^q\frac{1}{\alpha_1+1} + m\left|f'\left(\frac{a+3b}{4m}\right)\right|^q\frac{\alpha_1}{\alpha_1+1}\right)^{1/q} \\ +\alpha^{1-1/q}\left[\begin{array}{l}\frac{1}{2}\left|f'\left(\frac{a+3b}{4}\right)\right|^q\left(\frac{2\alpha^2}{(2\alpha+1)(\alpha+1)}+\frac{1}{2\alpha_1+1}\right) \\ +\frac{m}{2}\left|f'\left(\frac{b}{m}\right)\right|^q\left(\frac{2\alpha^2}{(2\alpha+1)(\alpha+1)}+\frac{2\alpha_1^2}{(2\alpha_1+1)(\alpha_1+1)}\right)\end{array}\right]^{1/q} \end{array}\right\}$$

If we choose $h_1(\lambda) = h(\lambda)$, $h_2(\lambda) = h(1-\lambda)$, $m = 1$ *we have*

$$M_\alpha(a,b) \le \frac{(b-a)}{16} \times$$
$$\left\{ \begin{array}{l} \left(\frac{1}{\alpha+1}\right)^{1-1/q}\left[\frac{1}{2}\left(\frac{1}{2\alpha+1}+\|h_1\|_2^2\right)+\left(|f'(a)|^q+\left|f'\left(\frac{3a+b}{4}\right)\right|^q\right)\right]^{1/q} \\ +\frac{1}{2}\left(\alpha 2^{(\alpha-1)/\alpha}-\alpha+1\right)^{1-1/q}\times\left(\|h_1\|_1\left(\left|f'\left(\frac{3a+b}{4}\right)\right|^q+\left|f'\left(\frac{a+3b}{4}\right)\right|^q\right)\right)^{1/q} \\ +\alpha^{1-1/q}\left[\begin{array}{l}\frac{1}{2}\left(\frac{2\alpha^2}{(2\alpha+1)(\alpha+1)}+\|h_1\|_2^2\right) \\ \times\left(\left|f'\left(\frac{a+3b}{4}\right)\right|^q+|f'(b)|^q\right)\end{array}\right]^{1/q} \end{array}\right\}$$

If we choose $h_1(\lambda) = h(\lambda) = \lambda^s$, $h_2(\lambda) = h(1-\lambda)$, $m = 1$ we have the,

$$M_\alpha(a,b) \leq \frac{(b-a)}{16} \times$$
$$\left\{ \begin{array}{l} \left(\frac{1}{\alpha+1}\right)^{1-1/q} \left[\frac{1}{2}\left(\frac{1}{2\alpha+1} + \frac{1}{2s+1}\right) + \left(|f'(a)|^q + \left|f'\left(\frac{3a+b}{4}\right)\right|^q\right)\right]^{1/q} \\ + \frac{1}{2}\left(\alpha 2^{(\alpha-1)/\alpha} - \alpha + 1\right)^{1-1/q} \times \left(\frac{1}{s+1}\left(\left|f'\left(\frac{3a+b}{4}\right)\right|^q + \left|f'\left(\frac{a+3b}{4}\right)\right|^q\right)\right)^{1/q} \\ + \alpha^{1-1/q}\left[\begin{array}{l}\frac{1}{2}\left(\frac{2\alpha^2}{(2\alpha+1)(\alpha+1)} + \frac{1}{2s+1}\right) \\ \times \left(\left|f'\left(\frac{a+3b}{4}\right)\right|^q + |f'(b)|^q\right)\end{array}\right]^{1/q} \end{array} \right\}$$

In Theorem 7, if we choose $h_1(\lambda) = h(\lambda) = \lambda^s$, $h_2(\lambda) = h(1-\lambda)$, $\alpha = s = m = 1$, we have

$$M_\alpha(a,b) \leq \frac{(b-a)}{16} \times \left\{ \begin{array}{l} \left(\frac{1}{2}\right)^{1-1/q}\left[\frac{1}{3}\left(|f'(a)|^q + \left|f'\left(\frac{3a+b}{4}\right)\right|^q\right)\right]^{1/q} \\ + \frac{1}{2}\left(\frac{1}{2}\left(\left|f'\left(\frac{3a+b}{4}\right)\right|^q + \left|f'\left(\frac{a+3b}{4}\right)\right|^q\right)\right)^{1/q} \\ + \left[\frac{1}{3}\left(\left|f'\left(\frac{a+3b}{4}\right)\right|^q + |f'(b)|^q\right)\right]^{1/q} \end{array} \right\}.$$

3. Acknowledgments

References

[1] Z. Dahmani, New inequalities in fractional integrals, International Journal of Nonlinear Science, 9(4) (2010), 493-497..

[2] K. Diethelm, The Analysis of Fractional Differential Equations, Lecture Notes in Mathematics, Springer, Berlin, 2010.

[3] S. S. Dragomir, R. P. Agarwal. Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula, Appl. Math. Lett., 11(5) (1998) 91-95.

[4] S. S. Dragomir, S. Fitzpatrick. The Hadamard's inequality for s-convex functions in the second sense, Demonstratio Math. 32 (4) (1999), 687-696.

[5] E.K. Godunova and V. I. Levin, Neravenstva dlja funkeii širokogo klassa, zaščego vypuklye, monotonnye i nekotorye drugie vidy funkii, Vyscislitel. Mat. i. Fiz. Mezvuzov. Sb. Nauc.Trudov, MGPI, Moskva, 1985, 138-142.

[6] R. Gorenflo, F. Mainardi, Fractional calculus: integral and differential equations of fractional order, Springer Verlag, Wien (1997), 223-276.

[7] H. Hudzik, L. Maligrada. Some remarks on s-convex functions, Aequationes Math. 48(1994) 100-111.

[8] U.S. Kirmaci. Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, Appl. Math. Comp. 147(1)(2004), 137-146.

[9] U. S. Kirmaci, & Özdemir, M. E. 2004a. On some inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, Applied Mathematics and Computation 153: 361-368.

[10] G. Maksa, Z. S. Pales, The equality case in some recent convexity inequalities, Opuscula Math., 31(2011), no.2, 269-277.

[11] V. G. Mihesan, A generalization of the convexity, Seminar on Functional Equations, Approx.and Convex,Cluj-Napoca (Romania) (1993).

[12] C. Niculescu, L. E. Persson, Convex functions and their application, Springer, Berlin Heidelberg New York, (2004).

8   M. IQBAL, S. QAISAR, AND M. MUDDASSAR

[13] C. E. M. Pearce, & J. Pecric 2000a. Inequalities for differentiable mappings with application to special means and quadrature formula, Applied Mathematics Letters 13: 51-55.

[14] S. Qaisar, C. He. S. Hussain. On New Inequalities of Hermite-Hadamard type for generalized Convex Functions. Italian journal of pure and applied mathematics. In Press.

[15] S. Qaisar, S. Hussain, Some results on Hermite-Hadamard type inequality through convexity.Turkish J.Anal.Num.Theo Vol. 2(2), 53-59 (2014).

[16] S. Qaisar, C. He. S. Hussain, New integral inequalities through invexity with applications. International Journal of Analysis and Applications.(In press).

[17] S. Qaisar, He. C, S. Hussain, On New Inequalities Of Hermite-Hadamard Type For Functions Whose Third Derivative Absolute Values Are Quasi-Convex With Applications,JournalofEgyptianMathematicalSociety.doi.org/10.1016/j.joems.2013.05.01

[18] M.Z. Sarikaya, E. Set, H. Yaldiz, and N. Basak, Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities, Math. Comput. Model, 57(2013), 2403-2407.

[19] S. Varošanec, On h-convexity, J. Math. Anal. Appl., 326 (2007), 303-311.

E-mail address: `miqbal.bki@gmail.com`

GOVERNMENT COLLEGE OF TECHNOLOGY, SAHIWAL, PAKISTAN

E-mail address: `shahidqaisar90@yahoo.com`

COMSATS INSTITUTE OF INFORMATION TECHNOLOGY, SAHIWAL, PAKISTAN

E-mail address: `malik.muddassar@gmail.com`

UNIVERSITY OF ENGINEERING AND TECHNOLOGY, TAXILA, PAKISTAN

DEGENERATE POLY-BERNOULLI POLYNOMIALS OF THE SECOND KIND

DMITRY V. DOLGY, DAE SAN KIM, TAEKYUN KIM, AND TOUFIK MANSOUR

ABSTRACT. In this paper, we introduce the degenerate poly-Bernoulli polynomials of the second kind, which reduce in the limit to the poly-Bernoulli polynomials of the second kind. We present several explicit formulas and recurrence relations for these polynomials. Also, we establish a connection between our polynomials and several known families of polynomials.

1. INTRODUCTION

The *Korobov polynomials of the first kind* $K_n(\lambda, x)$ with $\lambda \neq 0$ introduced by Korobov (actually he defined the polynomials $\frac{1}{n!}K_n(\lambda, x)$) (see [13, 14, 18]) are given by

$$(1.1) \qquad \frac{\lambda t}{(1 + \lambda t)^\lambda - 1}(1 + t)^x = \sum_{n \geq 0} K_n(\lambda, x)\frac{t^n}{n!}.$$

When $x = 0$, we define $K_n(\lambda) = K_n(\lambda, 0)$. These are what would have been called the degenerate Bernoulli polynomials of the second kind, since $\lim_{\lambda \to 0} K_n(\lambda, x) = b_n(x)$, where $b_n(x)$ is the *nth Bernoulli polynomial of the second kind* (see [15]) given by

$$\frac{t}{\log(1 + t)}(1 + t)^x = \sum_{n \geq 0} b_n(x)\frac{t^n}{n!}.$$

On the other hand, the *poly-Bernoulli polynomials of the second kind* $Pb_n^{(k)}(x)$ (of index k) are introduced in [12] (see also [5, 7, 10]) and given by

$$(1.2) \qquad \frac{Li_k(1 - e^{-t})}{\log(1 + t)}(1 + t)^x = \sum_{n \geq 0} Pb_n^{(k)}(x)\frac{t^n}{n!},$$

where $Li_k(x)$ $(k \in \mathbb{Z})$ is the classical *polylogarithm function* given by $Li_k(x) = \sum_{n \geq 1} \frac{x^n}{n^k}$.

In this paper, we introduce the *degenerate poly-Bernoulli polynomials of the second kind* $Pb_n^{(k)}(\lambda, x)$ with $\lambda \neq 0$ (of index k) (see [3, 6, 8]) which are given by

$$(1.3) \qquad \frac{\lambda Li_k(1 - e^{-t})}{(1 + t)^\lambda - 1}(1 + t)^x = \sum_{n \geq 0} Pb_n^{(k)}(\lambda, x)\frac{t^n}{n!}.$$

When $x = 0$, $Pb_n^{(k)}(\lambda, 0)$ are called the *degenerate poly-Bernoulli numbers of the second kind*. Clearly, $Pb_n^{(1)}(\lambda, x) = K_n(\lambda, x)$ and $\lim_{\lambda \to 0} Pb_n^{(k)}(\lambda, x) = Pb_n^{(k)}(x)$.

1

Recall here that the λ-*Daehee polynomials of the first kind* $D_{n,\lambda}(x)$ (see [9]) are given by

$$(1.4) \qquad \frac{\lambda \log(1+t)}{(1+t)^\lambda - 1}(1+t)^x = \sum_{n \geq 0} D_{n,\lambda}(x)\frac{t^n}{n!}.$$

When $x = 0$, $D_{n,\lambda} = D_{n,\lambda}(0)$ are called the λ-*Daehee numbers of the first kind*. Note that, as $\frac{\lambda \log(1+t)}{(1+t)^\lambda - 1}\frac{Li_k(1-e^{-t})}{\log(1+t)}(1+t)^x = \sum_{n \geq 0} Pb_n^{(k)}(\lambda, x)\frac{t^n}{n!}$, the degenerate poly-Bernoulli polynomials of the second kind are mixed-type of the λ-Daehee polynomials of the first kind and the poly-Bernoulli polynomials of the second kind.

The goal of this paper is to use umbral calculus to obtain several new and interesting identities of degenerate poly-Bernoulli polynomials of the second kind. To do that we refer the reader to umbral algebra and umbral calculus as given in [16, 17]. More precisely, we give some properties, explicit formulas, recurrence relations and identities about the degenerate poly-Bernoulli polynomials of the second kind. Also, we establish a connection between our polynomials and several known families of polynomials.

2. Explicit formulas

In this section we present several explicit formulas for the degenerate poly-Bernoulli polynomials of the second kind, namely $Pb_n^{(k)}(\lambda, x)$. It is immediate from (1.3) that the degenerate poly-Bernoulli polynomials of the second kind are given by the Sheffer sequence for the pair

$$(2.1) \qquad Pb_n^{(k)}(\lambda, x) \sim (g_k(t), f(t)) \equiv \left(\frac{e^{\lambda t} - 1}{\lambda Li_k(1 - e^{1-e^t})}, e^t - 1\right).$$

To do so, we recall that Stirling numbers $S_1(n, k)$ of the first kind can be defined by means of exponential generating functions as

$$(2.2) \qquad \sum_{\ell \geq j} S_1(\ell, j)\frac{t^\ell}{\ell} = \frac{1}{j!}\log^j(1+t)$$

and can be defined by means of ordinary generating functions as

$$(2.3) \qquad (x)_n = \sum_{m=0}^{n} S_1(n, m)x^m \sim (1, e^t - 1),$$

where $(x)_n = x(x-1)(x-2)\cdots(x-n+1)$ with $(x)_0 = 1$.

Theorem 2.1. *For all $n \geq 0$,*

$$Pb_n^{(k)}(\lambda, x) = \sum_{j=0}^{n}\left(\sum_{\ell=j}^{n}\binom{n}{\ell}S_1(\ell, j)Pb_{n-\ell}^{(k)}(\lambda, 0)\right)x^j$$

$$= \sum_{j=0}^{n}\left(\sum_{\ell=j}^{n}\sum_{m=0}^{n-\ell}\binom{n}{\ell}\binom{n-\ell}{m}S_1(\ell, j)Pb_m^{(k)}D_{n-\ell-m,\lambda}\right)x^j.$$

Proof. By applying the fact that

$$(2.4) \qquad s_n(x) = \sum_{j=0}^{n} \frac{1}{j!} \langle g(\bar{f}(t))^{-1} \bar{f}(t)^j \mid x^n \rangle x^j,$$

for any $s_n(x) \sim (g(t), f(t))$ (see [16, 17]) in the case of degenerate poly-Bernoulli polynomials of the second kind (see (2.1)), we have

$$\frac{1}{j!} \langle g_k(\bar{f}(t))^{-1} \bar{f}(t)^j \mid x^n \rangle$$

$$(2.5) \quad = \frac{1}{j!} \left\langle \frac{\lambda Li_k(1 - e^{-t})}{(1+t)^\lambda - 1} (\log(1+t))^j \mid x^n \right\rangle = \left\langle \frac{\lambda Li_k(1 - e^{-t})}{(1+t)^\lambda - 1} \mid \frac{\log^j(1+t)}{j!} x^n \right\rangle,$$

which, by (2.3), we have

$$\frac{1}{j!} \langle g_k(\bar{f}(t))^{-1} \bar{f}(t)^j \mid x^n \rangle$$

$$= \left\langle \frac{\lambda Li_k(1 - e^{-t})}{(1+t)^\lambda - 1} \mid \sum_{\ell \geq j} S_1(\ell, j) \frac{t^\ell}{\ell!} x^n \right\rangle = \sum_{\ell=j}^{n} \binom{n}{\ell} S_1(\ell, j) \left\langle \frac{\lambda Li_k(1 - e^{-t})}{(1+t)^\lambda - 1} \mid x^{n-\ell} \right\rangle$$

$$= \sum_{\ell=j}^{n} \binom{n}{\ell} S_1(\ell, j) Pb_{n-\ell}^{(k)}(\lambda, 0),$$

which completes the proof of the first equality.

Now let us calculate $a_j = \frac{1}{j!} \langle g_k(\bar{f}(t))^{-1} \bar{f}(t)^j \mid x^n \rangle$ in another way. By (2.5), we have

$$a_j = \sum_{\ell=j}^{n} \binom{n}{\ell} S_1(\ell, j) \left\langle \frac{\lambda \log(1+t)}{(1+t)^\lambda - 1} \mid \frac{Li_k(1 - e^{-t})}{\log(1+t)} x^{n-\ell} \right\rangle,$$

which, by (1.3) and (1.4), implies

$$a_j = \sum_{\ell=j}^{n} \sum_{m=0}^{n-\ell} \binom{n}{\ell} \binom{n-\ell}{m} S_1(\ell, j) Pb_m^{(k)} \left\langle \frac{\lambda \log(1+t)}{(1+t)^\lambda - 1} \mid x^{n-\ell-m} \right\rangle$$

$$= \sum_{\ell=j}^{n} \sum_{m=0}^{n-\ell} \binom{n}{\ell} \binom{n-\ell}{m} S_1(\ell, j) Pb_m^{(k)} D_{n-\ell-m,\lambda}.$$

Thus,

$$Pb_n^{(k)}(\lambda, x) = \sum_{j=0}^{n} \left(\sum_{\ell=j}^{n} \sum_{m=0}^{n-\ell} \binom{n}{\ell} \binom{n-\ell}{m} S_1(\ell, j) Pb_m^{(k)} D_{n-\ell-m,\lambda} \right) x^j,$$

as required. □

Theorem 2.2. *For all $n \geq 0$,*

$$Pb_n^{(k)}(\lambda, x) = \sum_{m=0}^{n} \binom{n}{m} D_{n-m,\lambda} Pb_m^{(k)}(x) = \sum_{m=0}^{n} \binom{n}{m} Pb_{n-m}^{(k)} D_{m,\lambda}(x).$$

Proof. By (1.3), we have

$$Pb_n^{(k)}(\lambda, y) = \left\langle \frac{\lambda \log(1+t)}{(1+t)^\lambda - 1} \;\middle|\; \frac{Li_k(1 - e^{-t})}{\log(1+t)} (1+t)^y x^n \right\rangle,$$

which, by (1.2), we obtain

$$Pb_n^{(k)}(\lambda, y) = \sum_{m=0}^{n} \binom{n}{m} Pb_m^{(k)}(y) \left\langle \frac{\lambda \log(1+t)}{(1+t)^\lambda - 1} \;\middle|\; x^{n-m} \right\rangle.$$

Therefore, by (1.4), we obtain the first equality. To obtain the second equality, we reverse the order, namely we use at first (1.4) and then (1.2), to obtain

$$Pb_n^{(k)}(\lambda, y) = \sum_{m=0}^{n} \binom{n}{m} D_{m,\lambda}(y) \left\langle \frac{Li_k(1 - e^{-t})}{\log(1+t)} \;\middle|\; x^{n-m} \right\rangle = \sum_{m=0}^{n} \binom{n}{m} D_{m,\lambda}(y) Pb_{n-m}^{(k)},$$

which completes the proof. $\qquad\qquad\square$

Note that it was shown in [9] that $D_{n,\lambda}(x)$ is given by $\sum_{j=0}^{n} S_1(n,j) \lambda^j B_j(x/\lambda)$, where $B_m(x)$ is the mth Bernoulli polynomial. Thus, for $x = 0$, we have

$$D_{n,\lambda} = \sum_{j=0}^{n} S_1(n,j) \lambda^j B_j,$$

where B_m is the mth Bernoulli number. Hence, we obtain

$$Pb_n^{(k)}(\lambda, x) = \sum_{m=0}^{n} \left(\sum_{\ell=0}^{n-m} \binom{n}{m} S_1(n-m, \ell) \lambda^\ell B_\ell \right) Pb_m^{(k)}(x)$$

$$= \sum_{m=0}^{n} \left(\sum_{\ell=m}^{n} \binom{n}{\ell} S_1(\ell, m) \lambda^m Pb_{n-\ell}^{(k)} \right) B_m(x/\lambda).$$

Note that Stirling number $S_2(n,k)$ of the second kind can be defined by the exponential generating functions as

(2.6)
$$\sum_{n \geq k} S_2(n,k) \frac{x^n}{n!} = \frac{(e^t - 1)^k}{k!}.$$

Theorem 2.3. *For all $n \geq 1$,*

$$Pb_n^{(k)}(\lambda, x) = \sum_{r=0}^{n} \left(\sum_{\ell=0}^{n-r} \sum_{m=0}^{n-\ell-r} \binom{n-1}{\ell} \binom{n-\ell}{r} B_\ell^{(n)} Pb_m^{(k)} S_2(n - \ell - r, m) \lambda^r \right) B_r(x/\lambda).$$

Proof. By 2.1, $x^n \sim (1, t)$, and the transfer formula (see [16, 17]), we obtain, for $n \geq 1$,

$$\frac{e^{\lambda t} - 1}{\lambda Li_k(1 - e^{1 - e^t})} Pb_n^{(k)}(\lambda, x)$$

$$= x \frac{t^n}{(e^t - 1)^n} x^{-1} x^n = x \sum_{\ell \geq 0} B_\ell^{(n)} \frac{t^\ell}{\ell!} x^{n-1} = \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} B_\ell^{(n)} x^{n-\ell}.$$

Thus,

$$Pb_n^{(k)}(\lambda, x) = \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} B_\ell^{(n)} \frac{\lambda t}{e^{\lambda t} - 1} \frac{\lambda Li_k(1 - e^{-s})}{\log(1 + s)} \Big|_{s=e^t-1} x^{n-\ell},$$

which, by (1.2) and (2.6), implies

$$Pb_n^{(k)}(\lambda, x) = \sum_{\ell=0}^{n-1} \left(\binom{n-1}{\ell} B_\ell^{(n)} \frac{\lambda t}{e^{\lambda t} - 1} \sum_{m \geq 0} Pb_m^{(k)} \frac{(e^t - 1)^m}{m!} x^{n-\ell} \right)$$

$$= \sum_{\ell=0}^{n-1} \sum_{m=0}^{n-\ell} \sum_{r=m}^{n-\ell} \binom{n-1}{\ell} B_\ell^{(n)} Pb_m^{(k)} S_2(r, m) \left(\frac{\lambda t}{e^{\lambda t} - 1} \frac{t^r}{r!} x^{n-\ell} \right)$$

$$= \sum_{\ell=0}^{n-1} \sum_{m=0}^{n-\ell} \sum_{r=m}^{n-\ell} \binom{n-1}{\ell} \binom{n-\ell}{r} B_\ell^{(n)} Pb_m^{(k)} S_2(r, m) \left(\frac{\lambda t}{e^{\lambda t} - 1} x^{n-\ell-r} \right)$$

$$= \sum_{\ell=0}^{n-1} \sum_{m=0}^{n-\ell} \sum_{r=m}^{n-\ell} \binom{n-1}{\ell} \binom{n-\ell}{r} B_\ell^{(n)} Pb_m^{(k)} S_2(r, m) \lambda^{n-\ell-r} B_{n-\ell-r}(x/\lambda).$$

Here we used the following fact: $\frac{1}{g(\lambda t)} x^n = \lambda^n s_n(x/\lambda)$ for any $s_n(x) \sim (g(t), t)$ and $\lambda \neq 0$. Indeed, $\langle t^k \mid 1/g(\lambda t) x^n \rangle = \lambda^{-k} \langle (\lambda t)^k / g(\lambda t) | x^n \rangle = \lambda^{-k} \langle t^k / g(t) | \lambda^n x^n \rangle = \lambda^{n-k} \langle t^k \mid 1/g(t) x^n \rangle = \lambda^n \langle (t/\lambda)^k \mid s_n(x) \rangle = \langle t^k \mid \lambda^n s_n(x/\lambda) \rangle$.

By exchanging the indices of the summations, we obtain that

$$Pb_n^{(k)}(\lambda, x) = \sum_{\ell=0}^{n-1} \sum_{m=0}^{n-\ell} \sum_{r=0}^{n-\ell-m} \binom{n-1}{\ell} \binom{n-\ell}{r} B_\ell^{(n)} Pb_m^{(k)} S_2(n - \ell - r, m) \lambda^r B_r(x/\lambda)$$

$$= \sum_{r=0}^{n} \left(\sum_{\ell=0}^{n-r} \sum_{m=0}^{n-\ell-r} \binom{n-1}{\ell} \binom{n-\ell}{r} B_\ell^{(n)} Pb_m^{(k)} S_2(n - \ell - r, m) \lambda^r \right) B_r(x/\lambda),$$

as claimed. $\qquad\qquad\square$

Theorem 2.4. *For all $n \geq 0$,*

$$Pb_n^{(k)}(\lambda, x) = \sum_{r=0}^{n} \left(\sum_{\ell=r}^{n} \sum_{m=0}^{\ell-r} \binom{\ell}{r} S_1(n, \ell) S_2(\ell - r, m) \lambda^r Pb_m^{(k)} \right) B_r(x/\lambda).$$

Proof. By (2.1) we have that $\frac{e^{\lambda t} - 1}{\lambda Li_k(1 - e^{1-e^t})} Pb_n^{(k)}(\lambda, x) \sim (1, e^t - 1)$. Thus, by (2.3), we obtain

$$(2.7) \qquad Pb_n^{(k)}(\lambda, x) = \frac{\lambda Li_k(1 - e^{1-e^t})}{e^{\lambda t} - 1} (x)_n = \sum_{\ell=0}^{n} S_1(n, \ell) \frac{\lambda Li_k(1 - e^{1-e^t})}{e^{\lambda t} - 1} x^\ell.$$

By replacing the function $\frac{\lambda Li_k(1-e^{1-e^t})}{e^{\lambda t}-1}$ by

$$\frac{\lambda t}{e^{\lambda t} - 1} \frac{\lambda Li_k(1 - e^{-s})}{\log(1 + s)} \Big|_{s=e^t-1},$$

and by using very similar arguments as in the proof of Theorem 2.3, one can complete the proof. $\qquad\square$

Note that $Li_2(1 - e^{-t}) = \int_0^t \frac{y}{e^y - 1} dy = \sum_{j \geq 0} B_j \frac{1}{j!} \int_0^t y^j dy = \sum_{j \geq 0} \frac{B_j t^{j+1}}{j!(j+1)}$. For general $k \geq 2$, the function $Li_k(1 - e^{-t})$ has integral representation as

$$Li_k(1 - e^{-t}) = \int_0^t \underbrace{\frac{1}{e^y - 1} \int_0^y \frac{1}{e^y - 1} \int_0^y \cdots \frac{1}{e^y - 1} \int_0^y}_{(k-2) \text{ times}} \frac{y}{e^y - 1} dy \cdots dydydy,$$

which, by induction on k, implies

$$(2.8) \qquad Li_k(1 - e^{-t}) = \sum_{j_1 \geq 0} \cdots \sum_{j_{k-1} \geq 0} t^{j_1 + \cdots + j_{k-1} + 1} \prod_{i=1}^{k-1} \frac{B_{j_i}}{j_i!(j_1 + \cdots + j_i + 1)}.$$

Theorem 2.5. *For all $n \geq 0$ and $k \geq 2$,*

$$Pb_n^{(k)}(\lambda, x) = \sum_{\ell=0}^{n} (n)_\ell K_{n-\ell}(\lambda, x) \left(\sum_{j_1 + \ldots + j_{k-1} = \ell} \prod_{i=1}^{k-1} \frac{B_{j_i}}{j_i!(j_1 + \cdots + j_i + 1)} \right).$$

Proof. By (2.1), we have

$$Pb_n^{(k)}(\lambda, y) = \left\langle \frac{\lambda Li_k(1 - e^{-t})}{(1+t)^\lambda - 1}(1+t)^y \mid x^n \right\rangle$$

$$= \left\langle \frac{Li_k(1 - e^{-t})}{t} \mid \frac{\lambda t}{(1+t)^\lambda - 1}(1+t)^y x^n \right\rangle,$$

which, by (1.1), implies

$$Pb_n^{(k)}(\lambda, y) = \left\langle \frac{Li_k(1 - e^{-t})}{t} \mid \sum_{\ell \geq 0} K_\ell(\lambda, y) \frac{t^\ell}{\ell!} x^n \right\rangle$$

$$= \sum_{\ell=0}^{n} \binom{n}{\ell} K_\ell(\lambda, y) \left\langle \frac{Li_k(1 - e^{-t})}{t} \mid x^{n-\ell} \right\rangle.$$

Thus, by (2.8), we complete the proof. $\square$

3. Recurrences

Note that, by (1.3) and the fact that $(x)_n \sim (1, e^t - 1)$, we obtain the following identity. $Pb_n^{(k)}(\lambda, x + y) = \sum_{j=0}^{n} \binom{n}{j} Pb_j^{(k)}(\lambda, x)(y)_{n-j}$. Moreover, in the next results, we present several recurrences for the degenerate poly-Bernoulli polynomials, namely $Pb_n^{(k)}(x)$.

Theorem 3.1. *For all $n \geq 1$, $Pb_n^{(k)}(\lambda, x + 1) = Pb_n^{(k)}(\lambda, x) + nPb_{n-1}^{(k)}(\lambda, x)$,*

Proof. It is well-known that $f(t)s_n(x) = ns_{n-1}(x)$ for all $s_n(x) \sim (g(t), f(t))$ (see [16, 17]). Thus, by (2.1), we have $(e^t - 1)Pb_n^{(k)}(\lambda, x) = nPb_{n-1}^{(k)}(\lambda, x)$, which gives $Pb_n^{(k)}(\lambda, x + 1) - Pb_n^{(k)}(\lambda, x) = nPb_{n-1}^{(k)}(\lambda, x)$, as required. $\square$

Theorem 3.2. *For all $n \geq 1$,*

$$Pb_{n+1}^{(k)}(\lambda, x) = xPb_n^{(k)}(\lambda, x-1)$$

$$- \sum_{m=0}^{n} \sum_{\ell=0}^{m+1} \sum_{j=\ell}^{m+1} \frac{S_1(n,m)S_2(j,\ell)}{m+1} \binom{m+1}{j} Pb_\ell^{(k)}(\lambda, 0)\lambda^{m+1-j} B_{m+1-j}\left(\frac{x+\lambda-1}{\lambda}\right)$$

$$+ \sum_{m=0}^{n} \sum_{\ell=0}^{m+1} \sum_{j=\ell}^{m+1} \frac{S_1(n,m)S_2(j,\ell)}{m+1} \binom{m+1}{j} PB_\ell^{(k-1)}\lambda^{m+1-j} B_{m+1-j}\left(\frac{x}{\lambda}\right).$$

Proof. It is well-known that that $s_{n+1}(x) = (x - g'(t)/g(t))\frac{1}{f'(t)}s_n(x)$ for all $s_n(x) \sim (g(t), f(t))$ (see [16, 17]). Thus, by 2.1, we have

$$(x - g_k'(t)/g_k(t))\frac{1}{f'(t)} = xe^{-t} - e^{-t}g_k'(t)/g_k(t),$$

which gives

$$(3.1) \qquad Pb_{n+1}^{(k)}(\lambda, x) = xPb_n^{(k)}(\lambda, x-1) - e^{-t}g_k'(t)/g_k(t)Pb_n^{(k)}(\lambda, x),$$

where

$$e^{-t}\frac{g_k'(t)}{g_k(t)} = e^{-t}\left(\frac{\lambda e^{\lambda t}}{e^{\lambda t}-1} - \frac{1}{Li_k(1-e^{1-e^t})}\frac{Li_{k-1}(1-e^{1-e^t})}{1-e^{1-e^t}}e^t e^{1-e^t}\right)$$

$$= \frac{1}{t}\left(\frac{\lambda t e^{(\lambda-1)t}}{e^{\lambda t}-1}\frac{\lambda Li_k(1-e^{1-e^t})}{e^{\lambda t}-1} - \frac{\lambda t}{e^{\lambda t}-1}\frac{Li_{k-1}(1-e^{1-e^t})}{e^{e^t-1}-1}\right)\frac{e^{\lambda t}-1}{\lambda Li_k(1-e^{1-e^t})}.$$

Note that the order $te^{-t}\frac{g_k'(t)}{g_k(t)}$ is at least one, and by (2.7) we have $\frac{e^{\lambda t}-1}{\lambda Li_k(1-e^{1-e^t})}Pb_n^{(k)}(\lambda, x) = \sum_{m=0}^{n} S_1(n,m)x^m$. Thus, by (1.3), we have

$$e^{-t}\frac{g_k'(t)}{g_k(t)}Pb_n^{(k)}(\lambda, x)$$

$$= \sum_{m=0}^{n} \frac{S_1(n,m)}{m+1}\left(\frac{\lambda t e^{(\lambda-1)t}}{e^{\lambda t}-1}\frac{\lambda Li_k(1-e^{1-e^t})}{e^{\lambda t}-1} - \frac{\lambda t}{e^{\lambda t}-1}\frac{Li_{k-1}(1-e^{1-e^t})}{e^{e^t-1}-1}\right)x^{m+1}.$$

Therefore, by (1.2) and (1.3), we have

$$e^{-t}\frac{g_k'(t)}{g_k(t)}Pb_n^{(k)}(\lambda, x)$$

$$= \sum_{m=0}^{n} \frac{S_1(n,m)}{m+1}\left(\frac{\lambda t e^{(\lambda-1)t}}{e^{\lambda t}-1}\frac{\lambda Li_k(1-e^{-s})}{(1+s)^\lambda-1} - \frac{\lambda t}{e^{\lambda t}-1}\frac{Li_{k-1}(1-e^{-s})}{e^s-1}\bigg|_{s=e^t-1}\right)x^{m+1}$$

$$= \sum_{m=0}^{n} \frac{S_1(n,m)}{m+1}\frac{\lambda t}{e^{\lambda t}-1}\left(e^{(\lambda-1)t}\sum_{\ell\geq 0} Pb_\ell^{(k)}(\lambda, 0)\frac{(e^t-1)^\ell}{\ell!} - \sum_{\ell\geq 0} PB_\ell^{(k-1)}\frac{(e^t-1)^\ell}{\ell!}\right)x^{m+1},$$

where $PB_\ell^{(k)}$ are the poly-Bernoulli numbers(of index k). So with help of (2.6), we obtain

$$e^{-t}\frac{g'(t)}{g(t)}Pb_n^{(k)}(\lambda, x)$$

$$= \sum_{m=0}^{n}\sum_{\ell=0}^{m+1}\sum_{j=\ell}^{m+1}\frac{S_1(n,m)S_2(j,\ell)}{m+1}\binom{m+1}{j}\left(Pb_\ell^{(k)}(\lambda,0)\frac{\lambda t e^{(\lambda-1)t}}{e^{\lambda t}-1} - PB_\ell^{(k-1)}\frac{\lambda t}{e^{\lambda t}-1}\right)x^{m+1-j}$$

$$= \sum_{m=0}^{n}\sum_{\ell=0}^{m+1}\sum_{j=\ell}^{m+1}\frac{S_1(n,m)S_2(j,\ell)}{m+1}\binom{m+1}{j}Pb_\ell^{(k)}(\lambda,0)\lambda^{m+1-j}B_{m+1-j}(\frac{x+\lambda-1}{\lambda})$$

$$- \sum_{m=0}^{n}\sum_{\ell=0}^{m+1}\sum_{j=\ell}^{m+1}\frac{S_1(n,m)S_2(j,\ell)}{m+1}\binom{m+1}{j}PB_\ell^{(k-1)}\lambda^{m+1-j}B_{m+1-j}(\frac{x}{\lambda}).$$

Therefore, by changing the summation on j, then substituting into (3.1), we complete the proof. $\square$

In next theorem, we find expression for $\frac{d}{dx}Pb_n^{(k)}(\lambda, x)$.

Theorem 3.3. *For all $n \geq 0$,*

$$\frac{d}{dx}Pb_n^{(k)}(\lambda, x) = n!\sum_{\ell=0}^{n-1}\frac{(-1)^{n-\ell-1}}{\ell!(n-\ell)}Pb_n^{(\ell)}(\lambda, x).$$

Proof. It is well-known that $\frac{d}{dx}s_n(x) = \sum_{\ell=0}^{n-1}\binom{n}{\ell}\langle\bar{f}(t) \mid x^{n-\ell}\rangle s_\ell(x)$, for all $s_n(x) \sim (g(t), f(t))$. Thus, in the case of degenerate poly-Bernoulli polynomials of the second kind (see (2.1)), we have

$$\langle\bar{f}(t) \mid x^{n-\ell}\rangle = \langle\log(1+t) \mid x^{n-\ell}\rangle = (-1)^{n-\ell-1}(n-\ell-1)!.$$

Thus

$$\frac{d}{dx}Pb_n^{(k)}(\lambda, x) = \sum_{\ell=0}^{n-1}\binom{n}{\ell}(-1)^{n-\ell-1}(n-\ell-1)!Pb_n^{(\ell)}(\lambda, x),$$

which completes the proof. $\square$

Theorem 3.4. *For all $n \geq 1$,*

$$Pb_n^{(k)}(\lambda, x) - xPb_{n-1}^{(k)}(\lambda, x-1)$$

$$= \frac{1}{n}\sum_{m=0}^{n}\binom{n}{m}\left(Pb_m^{(k-1)}(\lambda, x)B_{n-m} - Pb_m^{(k)}(\lambda, x+\lambda-1)K_{n-m}(\lambda)\right).$$

Proof. By (1.3), we have, for $n \geq 1$,

$$Pb_n^{(k)}(\lambda, y) = \left\langle\frac{\lambda Li_k(1-e^{-t})}{(1+t)^\lambda - 1}(1+t)^y \mid x^n\right\rangle$$

$$(3.2) \qquad\qquad = \left\langle\frac{\lambda Li_k(1-e^{-t})}{(1+t)^\lambda - 1}\frac{d}{dt}(1+t)^y \mid x^{n-1}\right\rangle$$

$$(3.3) \qquad\qquad + \left\langle\frac{d}{dt}\frac{\lambda Li_k(1-e^{-t})}{(1+t)^\lambda - 1}(1+t)^y \mid x^{n-1}\right\rangle.$$

The term in (3.2) is given by

$$(3.4) \qquad y \left\langle \frac{\lambda Li_k(1-e^{-t})}{(1+t)^{\lambda}-1}(1+t)^{y-1)} \mid x^{n-1} \right\rangle = yPb_{n-1}^{(k)}(\lambda, y-1).$$

For the term in (3.3), we observe that $\frac{d}{dt}\frac{\lambda Li_k(1-e^{-t})}{(1+t)^{\lambda}-1} = \frac{1}{t}(A-B)$, where

$$A = \frac{t}{e^t-1}\frac{\lambda Li_{k-1}(1-e^{-t})}{(1+t)^{\lambda}-1}, \quad B = \frac{\lambda t}{(1+t)^{\lambda}-1}\frac{\lambda Li_k(1-e^{-t})}{(1+t)^{\lambda}-1}(1+t)^{\lambda-1}.$$

Note that the expression $A-B$ has order at least 1. Now, we ready to compute the term in (3.3). By (1.3), we have

$$\left\langle \frac{d}{dt}\frac{\lambda Li_k(1-e^{-t})}{(1+t)^{\lambda}-1}(1+t)^y \mid x^{n-1} \right\rangle = \left\langle \frac{1}{t}(A-B)(1+t)^y \mid x^{n-1} \right\rangle$$

$$= \frac{1}{n}\left\langle A(1+t)^y \mid x^n \right\rangle - \frac{1}{n}\left\langle B(1+t)^y \mid x^n \right\rangle$$

$$= \frac{1}{n}\left\langle \frac{t}{e^t-1} \mid \sum_{m\geq 0} Pb_m^{(k-1)}(\lambda, y)\frac{t^m}{m!}x^n \right\rangle$$

$$- \frac{1}{n}\left\langle \frac{\lambda t}{(1+t)^{\lambda}-1} \mid \sum_{m\geq 0} Pb_m^{(k)}(\lambda, y+\lambda-1)\frac{t^m}{m!}x^n \right\rangle$$

$$= \frac{1}{n}\sum_{m=0}^{n}\binom{n}{m}Pb_m^{(k-1)}(\lambda, y)\left\langle \frac{t}{e^t-1} \mid x^{n-m} \right\rangle$$

$$- \frac{1}{n}\sum_{m=0}^{n}\binom{n}{m}Pb_m^{(k)}(\lambda, y+\lambda-1)\left\langle \frac{\lambda t}{(1+t)^{\lambda}-1} \mid x^{n-m} \right\rangle$$

$$(3.5) \qquad = \frac{1}{n}\sum_{m=0}^{n}\binom{n}{m}\left(Pb_m^{(k-1)}(\lambda, y)B_{n-m} - Pb_m^{(k)}(\lambda, y+\lambda-1)K_{n-m}(\lambda)\right).$$

Thus, if we replace (3.2) by (3.4) and (3.3) by (3.5), we obtain

$$Pb_n^{(k)}(\lambda, x) - xPb_{n-1}^{(k)}(\lambda, x-1)$$

$$= \frac{1}{n}\sum_{m=0}^{n}\binom{n}{m}\left(Pb_m^{(k-1)}(\lambda, x)B_{n-m} - Pb_m^{(k)}(\lambda, x+\lambda-1)K_{n-m}(\lambda)\right),$$

as claimed. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4. CONNECTIONS WITH FAMILIES OF POLYNOMIALS

In this section, we present a few examples on the connections with families of polynomials. To do that we use the following fact from [16, 17]: For $s_n(x) \sim (g(t), f(t))$ and $r_n(x) \sim (h(t), \ell(t))$, let $s_n(x) = \sum_{k=0}^{n}c_{n,k}r_k(x)$. Then we have

$$(4.1) \qquad\qquad\qquad c_{n,k} = \frac{1}{k!}\left\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))}(\ell(\bar{f}(t)))^k \mid x^n \right\rangle.$$

We start with the connection to Korobov polynomials $K_n(\lambda, x)$ of the first kind.

Theorem 4.1. *For all* $n \geq 0$,

$$Pb_n^{(k)}(\lambda, x) = \sum_{m=0}^{n} \left(\binom{n}{m} \sum_{\ell=0}^{n-m} \frac{1}{n-m-\ell+1} \binom{n-m}{\ell} PB_\ell^{(k)} \right) K_m(\lambda, x)$$

and

$$Pb_n^{(k)}(\lambda, x) = \frac{1}{n+1} \sum_{m=0}^{n} \left(\sum_{\ell=0}^{n-m+1} (-1)^{n-m+1-\ell} \binom{n+1}{m} \frac{\ell!}{\ell^k} S_2(n-m+1, \ell) \right) K_m(\lambda, x).$$

Proof. By (1.1), we have that $K_n(\lambda, x) \sim \left(\frac{e^{\lambda t}-1}{\lambda(e^t-1)}, e^t - 1 \right)$. Let

$$Pb_n^{(k)}(\lambda, x) = \sum_{m=0}^{n} c_{n,m} K_m(\lambda, x).$$

Thus, by (2.1) and (4.1), we obtain

$$c_{n,m} = \frac{1}{m!} \left\langle \frac{(1+t)^\lambda - 1}{\lambda t} \frac{\lambda Li_k(1 - e^{-t})}{(1+t)^\lambda - 1} t^m \Big| x^n \right\rangle = \frac{1}{m!} \left\langle \frac{Li_k(1 - e^{-t})}{t} \Big| t^m x^n \right\rangle$$

$$= \binom{n}{m} \left\langle \frac{e^t - 1}{t} \Big| \frac{Li_k(1 - e^{-t})}{e^t - 1} x^{n-m} \right\rangle = \binom{n}{m} \left\langle \frac{e^t - 1}{t} \Big| \sum_{\ell \geq 0} PB_\ell^{(k)} \frac{t^\ell}{\ell!} x^{n-m} \right\rangle$$

$$= \binom{n}{m} \sum_{\ell=0}^{n-m} \binom{n-m}{\ell} PB_\ell^{(k)} \left\langle \frac{e^t - 1}{t} \Big| x^{n-m-\ell} \right\rangle$$

$$= \binom{n}{m} \sum_{\ell=0}^{n-m} \binom{n-m}{\ell} PB_\ell^{(k)} \int_0^1 u^{n-m-\ell} du$$

$$= \binom{n}{m} \sum_{\ell=0}^{n-m} \frac{1}{n-m-\ell+1} \binom{n-m}{\ell} PB_\ell^{(k)},$$

which completes the proof of the first identity. Note that we can compute $c_{n,m}$ in another way, as follows. By using

$$c_{n,m} = \binom{n}{m} \left\langle \frac{Li_k(1 - e^{-t})}{t} \Big| x^{n-m} \right\rangle,$$

and $Li_k(1 - e^{-t}) = \sum_{\ell \geq 1} \frac{(1 - e^{-t})^\ell}{\ell^k}$ together with (2.6), we obtain that

$$c_{n,m} = \frac{1}{n+1} \sum_{\ell=0}^{n-m+1} (-1)^{n-m+1-\ell} \binom{n+1}{m} \frac{\ell!}{\ell^k} S_2(n-m+1, \ell),$$

which leads to the second identity. $\qquad \square$

Theorem 4.2. *For all* $n \geq 0$,

$$Pb_n^{(k)}(\lambda, x) = \sum_{m=0}^{n} \left(\binom{n}{m} \sum_{\ell=0}^{n-m} \binom{n-m}{\ell} K_\ell(\lambda) D_{n-m-\ell} \right) Pb_m^{(k)}(x),$$

where D_n *is the nth Daehee number defined by* $\frac{\log(1+t)}{t} = \sum_{n \geq 0} D_n \frac{t^n}{n!}$.

Proof. By (1.2), we have that $Pb_n^{(k)}(x) \sim \left(\frac{t}{Li_k(1-e^{1-e^t})}, e^t - 1 \right)$. Let

$$Pb_n^{(k)}(\lambda, x) = \sum_{m=0}^{n} c_{n,m} Pb_m^{(k)}(x).$$

Thus, by (2.1) and (4.1), we obtain

$$
\begin{aligned}
c_{n,m} &= \frac{1}{m!} \left\langle \frac{\log(1+t)}{Li_k(1-e^{-t})} \frac{\lambda Li_k(1-e^{-t})}{(1+t)^\lambda - 1} t^m \middle| x^n \right\rangle = \binom{n}{m} \left\langle \frac{\log(1+t)}{t} \middle| \frac{\lambda t}{(1+t)^\lambda - 1} x^{n-m} \right\rangle \\
&= \binom{n}{m} \left\langle \frac{\log(1+t)}{t} \middle| \sum_{\ell \geq 0} K_\ell(\lambda) \frac{t^\ell}{\ell!} x^{n-m} \right\rangle \\
&= \binom{n}{m} \sum_{\ell=0}^{n-m} \binom{n-m}{\ell} K_\ell(\lambda) \left\langle \frac{\log(1+t)}{t} \middle| x^{n-m-\ell} \right\rangle \\
&= \binom{n}{m} \sum_{\ell=0}^{n-m} \binom{n-m}{\ell} K_\ell(\lambda) D_{n-m-\ell}
\end{aligned}
$$

which completes the proof. □

We start with the connection to *Bernoulli polynomials* $B_n^{(s)}(x)$ *of order s.* Recall that the *Bernoulli polynomials* $B_n^{(s)}(x)$ *of order s* are defined by the generating function

$$\left(\frac{t}{e^t - 1} \right)^s e^{xt} = \sum_{n \geq 0} B_n^{(s)}(x) \frac{t^n}{n!},$$

or equivalently,

(4.2) $$B_n^{(s)}(x) \sim \left(\left(\frac{e^t - 1}{t} \right)^s, t \right)$$

(see [2,4]). In the next result, we express our polynomials $Pb_n^{(k)}(x)$ in terms of *Bernoulli polynomials of order s.* To do that, we recall that Bernoulli numbers of the second kind $b_n^{(s)}$ of order s are defined as

(4.3) $$\frac{t^s}{\log^s(1+t)} = \sum_{n \geq 0} b_n^{(s)} \frac{t^n}{n!}.$$

Theorem 4.3. *For all* $n \geq 0$,

$$Pb_n^{(k)}(\lambda, x) = \sum_{m=0}^{n} \left(\sum_{\ell=m}^{n} \sum_{j=0}^{n-\ell} \binom{n}{\ell} \binom{n-\ell}{j} S_1(\ell, m) Pb_j^{(k)}(\lambda, 0) b_{n-\ell-j}^{(s)} \right) B_m^{(s)}(x).$$

Proof. Let $Pb_n^{(k)}(\lambda, x) = \sum_{m=0}^{n} c_{n,m} B_m^{(s)}(x)$. By (2.1), (4.1) and (4.2), we have

$$c_{n,m} = \frac{1}{m!} \left\langle \left(\frac{t}{\log(1+t)} \right)^s \frac{\lambda Li_k(1-e^{-t})}{(1+t)^\lambda - 1} (\log(1+t))^m \middle| x^n \right\rangle,$$

which, by (2.3) and (1.3), implies

$$
\begin{aligned}
c_{n,m} &= \left\langle \left(\frac{t}{\log(1+t)}\right)^s \frac{\lambda Li_k(1-e^{-t})}{(1+t)^\lambda - 1} \mid \sum_{\ell \geq m} S_1(\ell, m) \frac{t^\ell}{\ell!} x^n \right\rangle \\
&= \sum_{\ell=m}^n \binom{n}{\ell} S_1(\ell, m) \left\langle \left(\frac{t}{\log(1+t)}\right)^s \mid \frac{\lambda Li_k(1-e^{-t})}{(1+t)^\lambda - 1} x^{n-\ell} \right\rangle \\
&= \sum_{\ell=m}^n \binom{n}{\ell} S_1(\ell, m) \left\langle \left(\frac{t}{\log(1+t)}\right)^s \mid \sum_{j \geq 0} Pb_j^{(k)}(\lambda, 0) \frac{t^j}{j!} x^{n-\ell} \right\rangle \\
&= \sum_{\ell=m}^n \sum_{j=0}^{n-\ell} \binom{n}{\ell} \binom{n-\ell}{j} S_1(\ell, m) Pb_j^{(k)}(\lambda, 0) \left\langle \left(\frac{t}{\log(1+t)}\right)^s \mid x^{n-\ell-j} \right\rangle.
\end{aligned}
$$

Thus, by (4.3), we obtain

$$
c_{n,m} = \sum_{\ell=m}^n \sum_{j=0}^{n-\ell} \binom{n}{\ell} \binom{n-\ell}{j} S_1(\ell, m) Pb_j^{(k)}(\lambda, 0) b_{n-\ell-j}^{(s)},
$$

which completes the proof. □

Similar techniques as in the proof of the previous theorem, we can express our polynomials $Pb_n^{(k)}(\lambda, x)$ in terms of other families. For instance, we can express our polynomials $Pb_n^{(k)}(\lambda, x)$ in terms of Frobenius-Euler polynomials (we leave the proof to the interested reader). Note that the *Frobenius-Euler polynomials* $H_n^{(s)}(x|\mu)$ of order s are defined by the generating function $\left(\frac{1-\mu}{e^t-\mu}\right)^s e^{xt} = \sum_{n \geq 0} H_n^{(s)}(x|\mu) \frac{t^n}{n!}, (\mu \neq 1)$, or equivalently, $H_n^{(s)}(x|\mu) \sim \left(\left(\frac{e^t-\mu}{1-\mu}\right)^s, t\right)$ (see [1, 2, 4, 11]).

Theorem 4.4. *For all $n \geq 0$,*

$$
Pb_n^{(k)}(\lambda, x) = \sum_{m=0}^n \left(\sum_{\ell=m}^n \sum_{r=0}^{n-\ell} \binom{n}{\ell} \binom{n-\ell}{r} \frac{s! S_1(\ell, m) Pb_r^{(k)}(\lambda, 0)}{(1-\mu)^{n-\ell-r}(s+\ell+r-n)!} \right) H_m^{(s)}(x|\mu).
$$

REFERENCES

[1] Araci, S. and Acikgoz, M., A note on the Frobenius-Euler numbers and polynomials associated with Bernstein polynomials, *Adv. Stud. Contemp. Math.* **22(3)** (2012), 399–406.

[2] Bayad A. and Kim, T., Identities involving values of Bernstein, q-Bernoulli, and q-Euler polynomials, *Russ. J. Math. Phys.* **18(2)** (2011), 133–143.

[3] Carlitz, L., Degenerate stirling, Bernoulli and Eulerian numbers, *Utilitas Math.* **15** (1979) 51–88.

[4] Ding, D. and Yang, J., Some identities related to the Apostol-Euler and Apostol-Bernoulli polynomials, *Adv. Stud. Contemp. Math.* **20(1)** (2010), 7–21.

[5] Jolany, H. and Faramarzi, H., Generalization on poly-Eulerian numbers and polynomials, *Scientia Magna* **6** (2010), 9–18.

[6] Kim, D. S. and Kim, T., A note on degenerate poly-Bernoulli numbers and polynomials, arXiv:1503.08418 (2015).

[7] Kim, D. S. and Kim, T., A note on poly-Bernoulli and higher-order poly-Bernoulli polynomilas, *Russ. J. Math. Phys.* **22** (2015), 26–33.

[8] Kim, D. S., Kim, T. and Dolgy, D. V., A note on degenerate Bernoulli numbers and polynomials associated with p-adic invariant integral on $\mathbb{Z}_p$, *Appl. Math. Comput.* **295** (2015), 198–204.

[9] Kim, D. S., Kim, T., Lee, S. H. and Seo, J. J., A Note on the λ-Daehee Polynomials, *Int. Journal of Math. Analysis* **7** (2013), 3069–3080.

[10] Kim, D. S., Kim, T., Mansour, T. and Dolgy, D. V., On poly-Bernoulli polynomials of the second kind with umbral calculus viewpoint, *Adv. Difference Equ.* **2015** (2015), 2015:27.

[11] Kim, T., Indentities involving Frobenius-Euler polynomials arising from non-linear differential equations, *J. Number Theory* **132(12)** (2012), 2854–2865.

[12] Kim, T., Kwon, H. I., Lee, S. H. and Seo, J. J., A note on poly-Bernoulli numbers and polynomials of the second kind, *Adv. Difference Equ.* **2014** (2014), 2014:219.

[13] Korobov, N. M., *On some properties of special polynomials*, Proceedings of the IV International Conference "Modern Problems of Number Theory and its Applications" (Russian) (Tula, 2001), vol. 1, 2001, pp. 40–49.

[14] Korobov, N. M., *Special polynomials and their applications*, in: Diophantine Approximations. Mathematical Notes (Russian), vol. 2, Izd. Moskov. Univ., Moscow, 1996, pp. 77–89.

[15] Prabhakar, T. R. and Gupta, S., Bernoulli polynomials of the second kind and general order, *Indian J. Pure Appl. Math.* **11** (1980), 1361–1368.

[16] Roman, S., More on the umbral calculus, with emphasis on the q-umbral calculus, *J. Math. Anal. Appl.* **107** (1985), 222–254.

[17] Roman, S., The umbral calculus, Dover Publ. Inc. New York, 2005.

[18] Ustinov, A. V., Korobov polynomials and umbral analysis, *(Russian) Chebyshevskii Sb.* **4** (2003), 137–152.

INSTITUTE OF NATURAL SCIENCES, FAR EASTERN FEDERAL UNIVERSITY, VLADIVOSTOK, 690950, RUSSIA

E-mail address: *d_dol@mail.ru*

DEPARTMENT OF MATHEMATICS, SOGANG UNIVERSITY, SEOUL 121-742, S. KOREA

E-mail address: dskim@sogang.ac.kr

DEPARTMENT OF MATHEMATICS, KWANGWOON UNIVERSITY, SEOUL 139-701, S. KOREA

E-mail address: tkkim@kw.ac.kr

UNIVERSITY OF HAIFA, DEPARTMENT OF MATHEMATICS, 3498838 HAIFA, ISRAEL

E-mail address: tmansour@univ.haifa.ac.il

Some results for meromorphic functions of several variables

Yue Wang*

School of Information, Renmin University of China, Beijing, 100872, China

Abstract: Using the Nevanlinna theory of the value distribution of meromorphic functions, we investigate the value distribution of complex partial q-difference polynomials of meromorphic functions of zero order, and also investigate the existence of meromorphic solutions of some types of systems of complex partial q-difference equations in $\mathbb{C}^n$. Some existing results are improved and generalized, and some new results are obtained. Examples show that our results are precise.

Keywords: value distribution; meromorphic solution; complex partial q-difference polynomials; complex partial q-difference equations

§1   Introduction

In this paper, we assume that the reader is familiar with the standard notation and basic results of the Nevanlinna theory of meromorphic functions, see, for example [1].

The reference related to notations of this section are referred to Tu[2].

Let M be a connected complex manifold of dimension n and let

$$A(M) = \sum_{n=0}^{2m} A^n(M)$$

be the graded ring of complex valued differential forms on M. Each set $A^n(M)$ can be split into a direct sum

$$A^n(M) = \sum_{p+q=n} A^{p,q}(M),$$

where $A^{p,q}(M)$ is the forms of type p, q. The differential operators d and d^c on $A(M)$ are defined as

$$d := \partial + \overline{\partial} \quad and \quad d^c := \frac{1}{4\pi i}(\partial - \overline{\partial}).$$

where

$$\partial : A^{p,q}(M) \longrightarrow A^{p+1,q}(M),$$

---

*Corresponding author

E-mail addresses: wy2006518@163.com(Y. Wang)

2

$$\overline{\partial}: A^{p,q}(M) \longrightarrow A^{p,q+1}(M).$$

Let $z = (z_1, ..., z_n) \in \mathbb{C}^n$, and let $r \in \mathbb{R}^+$. We define

$$\omega_n(z) := dd^c \log|z|^2 \quad and \quad \sigma_n(z) := d^c \log|z|^2 \wedge \omega_n^{n-1}(z),$$

where $z \in \mathbb{C}^n \backslash \{0\}$ and $|z|^2 := |z_1|^2 + \cdots + |z_n|^2$.

Let $\mathbb{C}^n < r >= \{z \in \mathbb{C}^n : |z| = r\}$, $\mathbb{C}^n(r) = \{z \in \mathbb{C}^n : |z| < r\}$, $\mathbb{C}^n[r] = \{z \in \mathbb{C}^n : |z| \leq r\}$. Then $\sigma_n(z)$ defines a positive measure on $\mathbb{C}^n < r >$ with total measure one. In addition, by defining

$$\nu_n(z) := dd^c|z|^2 \quad and \quad \rho_n(z) := \nu_n^n(z),$$

for all $z \in \mathbb{C}^n$, it follows that $\rho_n(z)$ is the Lebesgue measure on $\mathbb{C}^n$ normalized such that $\mathbb{C}^n(r)$ has measure r^{2n}.

Let w be a meromorphic function on $\mathbb{C}^n$ in the sense that w can be written as a quotient of two relatively prime holomorphic functions. We will write $w = (w_0, w_1)$ where $w_0 \not\equiv 0$, thus w can be regarded as a meromorphic map $w : \mathbb{C}^n \longrightarrow \mathbb{P}^1$ such that $w^{-1}(\infty) \neq \mathbb{C}^n$.

Let $\mathbb{P}^1$ be the Riemann sphere. For $a, b \in \mathbb{P}^1$, the chordal distance from a to b is denoted by $\| a, b \|$, $\| a, \infty \| = \frac{1}{\sqrt{1+|a|^2}}$, $\| a, b \| = \frac{|a-b|}{\sqrt{1+|a|^2} \cdot \sqrt{1+|b|^2}}$, $a, b \in \mathbb{C}$, where $\| a, a \| = 0$ and $0 \leq \| a, b \| = \| b, a \| \leq 1$. If $a \in \mathbb{P}^1$ and $w^{-1}(a) \neq \mathbb{C}^n$, then we define the proximity function as

$$m(r, w, a) = \int_{|z|=r} \log \frac{1}{\| a, w(z) \|} \sigma_n \geq 0, \quad r > 0.$$

Let ν be a divisor on $\mathbb{C}^n$. We identify ν with its multiplicity function, define

$$\nu(r) = \{z \in \mathbb{C}^n : |z| < r\} \bigcap supp\nu, \quad r > 0.$$

The pre-counting function of ν is defined by

$$n(r, \nu) = \sum_{z \in \nu(r)} \nu(z), \quad if \ n = 1, \quad n(r, \nu) = r^{2-2n} \int_{\nu(r)} \nu \nu_n^{n-1}, \quad if \ n > 1.$$

The counting function of ν is defined by

$$N(r, \nu) = \int_s^r n(t, \nu) \frac{dt}{t}, \quad r > s.$$

Let w be a meromorphic function on $\mathbb{C}^n$. If $a \in \mathbb{P}^1$ and $w^{-1}(a) \neq \mathbb{C}^n$, the a-divisor $\nu(w, a) \geq 0$ is defined, and its pre-counting function and counting function will be denoted by $n(r, w, a)$ and $N(r, w, a)$, respectively.

For a divisor ν on $\mathbb{C}^n$, let

$$\overline{n}(r, \nu) = \sum_{z \in \nu(r)} 1, \quad if \ n = 1, \quad \overline{n}(r, \nu) = r^{2-2n} \int_{\nu(r)} \nu_n^{n-1}, \quad if \ n > 1.$$

$$\overline{N}(r, \nu) = \int_s^r \overline{n}(t, \nu) \frac{dt}{t}, \quad r > s. \quad \overline{N}(r, w, a) = \overline{N}(r, \nu(w, a)).$$

For $0 < s < r$, the characteristic of w is defined by

$$T(r, w) = \int_s^r \frac{1}{t^{2n-1}} \int_{\mathbb{C}^n[t]} w^*(\omega) \wedge \nu_n^{n-1} dt = \int_s^r \frac{1}{t} \int_{\mathbb{C}^n[t]} w^*(\omega) \wedge \omega_n^{n-1} dt.$$

where the pullback $w^*(\omega)$ satisfies $w^*(\omega) = dd^c \log(|w_0|^2 + |w_1|^2)$.

The First Main Theorem states

$$T(r, w) = N(r, w, a) + m(r, w, a) - m(s, w, a).$$

In 2012, Korhonen R has investigated the difference analogues of the lemma on the Logarithmic Derivate and of the Second Main Theorem of Nevanlinna theory for meromorphic functions of several variables, see [3]. Particularly, in 2013, Cao T B, see [4], using different method obtains difference analogues of the second main theorem for meromorphic functions in several complex variables from which difference analogues of Picard-type theorems are also obtained. His results are improvements or extensions of some results of Korhonen R.

Similarly, in 2014, Wen Z T has investigated the q-difference theory for meromorphic functions of several variables, see [5]. Some results that we will use in this paper are as follows.

Theorem A [5] *Let w be a meromorphic function in $\mathbb{C}^n$ of zero order such that $w(0) \neq 0, \infty$, and let $q \in \mathbb{C}^n \backslash \{0\}$. Then,*

$$m(r, \frac{w(qz)}{w(z)}) = o(T(r,w)),$$

on a set of logarithmic density 1.

Theorem B [5] *Let w be a meromorphic function in $\mathbb{C}^n$ of zero order such that $w(0) \neq 0, \infty$, and let $q \in \mathbb{C}^n \backslash \{0\}$. Then,*

$$T(r, w(qz)) = T(r, w(z)) + o(T(r,w)),$$

on a set of logarithmic density 1.

Remark: From the proof of Theorem B in [5], we have

$$N(r, w(qz)) = N(r, w(z)) + o(N(r,w)).$$

The remainder of the paper is organized as follows. In §2, we discuss Theorem A's applications to complex partial q-difference equations. We present q-shift analogues of the Clunie lemmas which can be used to study value distribution of zero-order meromorphic solutions of large classes of complex partial q-difference equations. In §3, we study the existence of meromorphic solutions of complex partial q-difference equation of several variables, and obtain four theorems, and then we give some examples, which show that the results obtained in §3 are, in a sense, the best possible. And finally, we prove these four theorems by a series of lemmas.

§2   Value distribution of complex partial q-difference polynomials

Recently, Laine I, Halburd R G, Korhonen R J, Barnett D, Morgan W, investigate complex q-difference theory, and have obtained some results,see [6,7,8,9]. Especially, in 2007, Barnett D C, Halburd R G have obtained a theorem which is analogous to the Clunie Lemma as follows

Theorem C [7] *Let $w(z)$ be a non-constant zero-order meromorphic solution of*

$$w^{n_1}(z)P_1(z,w) = Q_1(z,w),$$

where $P_1(z,w)$ and $Q_1(z,w)$ are complex q-difference polynomials in $w(z)$ of the form

$$P_1(z,w) = \sum_{\lambda_1 \in I_1'} a_{\lambda_1}(z)w(z)^{l_0^1}(w(q_1 z))^{l_1^1} \cdots (w(q_\nu z))^{l_\nu^1},$$

$$Q_1(z,w) = \sum_{\gamma_1 \in J_1'} b_{\gamma_1}(z)w(z)^{l_0^2}(w(q_1 z))^{l_1^2} \cdots (w(q_\mu z))^{l_\mu^2}.$$

4

If the degree of $Q_1(z, w)$ as a polynomial in $w(z)$ and its q-shifts is at most n_1, then

$$m(r, P_1(z, w)) = S(r, w) = o\{T(r, w)\},$$

for all r on a set of logarithmic density 1.

We will investigate the problem of value distribution of complex partial q-difference polynomials (2.1), (2.2) and (2.3), where $z = (z_1, ..., z_n) \in \mathbb{C}^n$.

$$P(z, w) = \sum_{\lambda \in I_1} a_\lambda(z) w(z)^{l_{\lambda_0}} (w(q_{\lambda_1} z))^{l_{\lambda_1}} \cdots (w(q_{\lambda_{\sigma_\lambda}} z))^{l_{\lambda_{\sigma_\lambda}}}. \qquad (2.1)$$

$$Q(z, w) = \sum_{\mu \in J_1} b_\mu(z) w(z)^{m_{\mu_0}} (w(q_{\mu_1} z))^{m_{\mu_1}} \cdots (w(q_{\mu_{\tau_\mu}} z))^{m_{\mu_{\tau_\mu}}}. \qquad (2.2)$$

$$U(z, w) = \sum_{\nu \in K_1} c_\nu(z) w(z)^{n_{\nu_0}} (w(q_{\nu_1} z))^{n_{\nu_1}} \cdots (w(q_{\nu_{\upsilon_\nu}} z))^{n_{\nu_{\upsilon_\nu}}}. \qquad (2.3)$$

where coefficients $\{a_\lambda(z)\}, \{b_\mu(z)\}, \{c_\nu(z)\}$ are small functions of $w(z)$. I_1, J_1, K_1 are three finite sets of multi-indices, $q_j \in \mathbb{C}^n \backslash \{0\}, (j \in \{\lambda_1, \ldots, \lambda_{\sigma_\lambda}, \mu_1, \ldots, \mu_{\tau_\mu}, \nu_1, \ldots, \nu_{\upsilon_\nu}.\})$.

We will prove

Theorem 2.1. *Let w be a meromorphic function in $\mathbb{C}^n$, and be a non-constant meromorphic solution of zero order of a complex partial q-difference equation of the form*

$$U(z, w)P(z, w) = Q(z, w),$$

where complex partial q-difference polynomials $P(z, w), Q(z, w)$ and $U(z, w)$ are respectively as the form of $(2.1), (2.2), (2.3)$, the total degree $\deg U(z, f) = n_1$ in $w(z)$ and its shifts, and $\deg Q(z, f) \leq n_1$. Moreover, we assume that $U(z, w)$ contains just one term of maximal total degree in $w(z)$ and its shifts. Then, we have

$$m(r, P(z, w)) = S(r, w) = o\{T(r, w)\},$$

for all r on a set of logarithmic density 1.

Corollary 2.1. *Let w be a meromorphic function in $\mathbb{C}^n$, and be a non-constant transcendental meromorphic solution of zero order of a complex partial q-difference equation of the form*

$$H(z, w)P(z, w) = Q(z, w),$$

where $H(z, w)$ is a complex partial q-difference product of total degree n_1 in $w(z)$ and its shifts, and where $P(z, w), Q(z, w)$ are complex partial q-difference polynomials such that the total degree of $Q(z, w)$ is at most n_1. Then, we obtain

$$m(r, P(z, w)) = S(r, w) = o\{T(r, w)\},$$

for all r on a set of logarithmic density 1.

Proof of Theorem 2.1 As the proof of Theorem 1 in [10], we rearrange the expression for the complex partial q-difference polynomial $U(z, w)$ by collecting together all terms having the same total degree and then writing $U(z, w)$ as follows

$$U(z, w) = \sum_{j=0}^{n_1} d_j(z) w^j(z),$$

where $d_j(z) = \sum_{\nu = j} c_\nu(z) (\frac{w(q_{\nu_1} z)}{w(z)})^{n_{\nu_1}} \cdots (\frac{w(q_{\nu_{\upsilon_\nu}} z)}{w(z)})^{n_{\nu_{\upsilon_\nu}}}, \ j = 0, 1, \cdots, n_1.$ Since $\deg U(z, w) = n_1$ in $w(z)$ and its shifts, and $U(z, w)$ contains just one term of maximal total degree n_1 in $w(z)$

and its shifts, therefore, $d_{n_1}(z)$ contains just one product of the described form.

By Theorem A, for all r on a set of logarithmic density 1, we have
$$m(r, d_j(z)) = S(r, w) = o\{T(r, w)\}, j = 0, 1, \cdots, n_1.$$

It follows from the assumption that $d_{n_1}(z)$ has just one term of maximal total degree in $U(z, w)$, thus, for all r on a set of logarithmic density 1, we get
$$m(r, \frac{1}{d_{n_1}(z)}) = S(r, w) = o\{T(r, w)\}.$$

Let
$$A(z) = \max_{1 \leq j \leq n_1} \{1, 2 \mid \frac{d_{n_1-j}}{d_{n_1}} \mid^{\frac{1}{j}}\}.$$

Then
$$m(r, A(z)) \leq \sum_{j=0}^{n_1} m(r, d_{n_1-j}) + m(r, \frac{1}{d_{n_1}}) + O(1) = S(r, w) = o\{T(r, w)\}.$$

Let
$$E_1 = \{z \in \mathbb{C}^n < r >: \mid w(z) \mid \leq A(z)\}, \ \ E_2 = \mathbb{C}^n < r > \backslash E_1.$$

Thus
$$m(r, P(z, w)) = \int_{E_1} \log^+ |P(z, w)|\sigma_n(z) + \int_{E_2} \log^+ |P(z, w)|\sigma_n(z). \tag{2.4}$$

Next we estimate respectively $\int_{E_1} \log^+ |P(z, w)|\sigma_n(z)$ and $\int_{E_2} \log^+ |P(z, w)|\sigma_n(z)$ in (2.4).

When $z \in E_1$, we have
$$
\begin{aligned}
\mid P(z, w) \mid &= \mid \sum_{\lambda \in I_1} a_\lambda(z) w(z)^{l_{\lambda_0}} (w(q_{\lambda_1}z))^{l_{\lambda_1}} \cdots (w(q_{\lambda_{\sigma_\lambda}}z))^{l_{\lambda_{\sigma_\lambda}}} \mid \\
&\leq \sum_{\lambda \in I_1} \mid a_\lambda(z) \mid\mid w(z) \mid^{l_{\lambda_0}} \mid w(q_{\lambda_1}z) \mid^{l_{\lambda_1}} \cdots \mid w(q_{\lambda_{\sigma_\lambda}}z) \mid^{l_{\lambda_{\sigma_\lambda}}} \\
&= \sum_{\lambda \in I_1} \mid a_\lambda(z) \mid\mid w(z) \mid^{l_\lambda} \mid \frac{w(q_{\lambda_1}z)}{w(z)} \mid^{l_{\lambda_1}} \cdots \mid \frac{w(q_{\lambda_{\sigma_\lambda}}z)}{w(z)} \mid^{l_{\lambda_{\sigma_\lambda}}} \\
&\leq \sum_{\lambda \in I_1} \mid a_\lambda(z) \mid\mid A(z) \mid^{l_\lambda} \mid \frac{w(q_{\lambda_1}z)}{w(z)} \mid^{l_{\lambda_1}} \cdots \mid \frac{w(q_{\lambda_{\sigma_\lambda}}z)}{w(z)} \mid^{l_{\lambda_{\sigma_\lambda}}},
\end{aligned}
$$
where $l_\lambda = l_{\lambda_0} + l_{\lambda_1} + \cdots + l_{\lambda_{\sigma_\lambda}}$. By Theorem A, for all r on a set of logarithmic density 1, we have
$$\int_{E_1} \log^+ |P(z, w)|\sigma_n(z) = S(r, w) = o\{T(r, w)\}. \tag{2.5}$$

When $z \in E_2$, we obtain
$$\mid w(z) \mid > A(z) \geq 2 \mid \frac{d_{n_1-j}}{d_{n_1}} \mid^{\frac{1}{j}}, (j = 1, 2, \ldots, n_1),$$
i.e.
$$\frac{\mid w(z) \mid^j}{2^j} \geq \mid \frac{d_{n_1-j}}{d_{n_1}} \mid, (j = 1, 2, \ldots, n_1).$$

6

It follows from $U(z,w) = \sum_{j=0}^{n_1} d_j(z) w^j$ that

$$
\begin{aligned}
\mid U(z,w) \mid \;\geq\; & \mid d_{n_1} \mid\mid w \mid^{n_1} -(\mid d_{n_1-1} \mid\mid w \mid^{n_1-1} + \mid d_{n_1-2} \mid\mid w \mid^{n_1-2} + \ldots \\
& + \mid d_1 \mid\mid w \mid + \mid d_0 \mid) \\
=\; & \mid d_{n_1} \mid\mid w \mid^{n_1} - \mid d_{n_1} \mid\mid w \mid^{n_1} \left(\frac{\mid d_{n_1-1} \mid}{\mid d_{n_1} \mid\mid w \mid} + \frac{\mid d_{n_1-2} \mid}{\mid d_{n_1} \mid\mid w \mid^2} \right. \\
& \left. + \cdots + \frac{\mid d_1 \mid}{\mid d_{n_1} \mid\mid w \mid^{n_1-1}} + \frac{\mid d_0 \mid}{\mid d_{n_1} \mid\mid w \mid^{n_1}} \right) \\
=\; & \mid d_{n_1} \mid\mid w \mid^{n_1} - \mid d_{n_1} \mid\mid w \mid^{n_1} \left(\sum_{j=1}^{n_1} \frac{\mid d_{n_1-j} \mid}{\mid d_{n_1} \mid\mid w \mid^{j}} \right) \\
\geq\; & \mid d_{n_1} \mid\mid w \mid^{n_1} \left(1 - \sum_{j=1}^{n_1} \frac{1}{2^j} \right) \\
=\; & \frac{\mid d_{n_1} \mid\mid w \mid^{n_1}}{2^{n_1}}.
\end{aligned}
$$

Since $z \in E_2$, then

$$\mid w(z) \mid > A(z) \geq 1,$$

that is

$$\frac{1}{\mid w(z) \mid} < 1.$$

Using $U(z,w)P(z,w) = Q(z,w)$ and the total degree of $Q(z,w)$ is at most n_1, we obtain

$$
\begin{aligned}
\mid P(z,w) \mid \;=\; & \mid \frac{Q(z,w)}{U(z,w)} \mid \\
\leq\; & \frac{2^{n_1}}{\mid d_{n_1} \mid\mid w \mid^{n_1}} \sum_{\mu \in J_1} \mid b_\mu(z) \mid\mid w(z) \mid^{m_{\mu_0}} \mid w(q_{\mu_1} z) \mid^{m_{\mu_1}} \\
& \cdots \mid w(q_{\mu_{\tau_\mu}} z) \mid^{m_{\mu_{\tau_\mu}}} \\
\leq\; & \frac{2^{n_1}}{\mid d_{n_1} \mid} \sum_{\mu \in J_1} \mid b_\mu(z) \mid\mid \frac{w(q_{\mu_1} z)}{w(z)} \mid^{m_{\mu_1}} \cdots \mid \frac{w(q_{\mu_{\tau_\mu}} z)}{w(z)} \mid^{m_{\mu_{\tau_\mu}}}.
\end{aligned}
$$

From Theorem A, for all r on a set of logarithmic density 1, we have

$$\int_{E_2} \log^+ |P(z,w)|\sigma_n(z) = S(r,w) = o\{T(r,w)\}. \tag{2.6}$$

Combining (2.4),(2.5),(2.6), yields

$$
\begin{aligned}
m(r, P(z,w)) \;=\; & \int_{E_1} \log^+ |P(z,w)|\sigma_n(z) + \int_{E_2} \log^+ |P(z,w)|\sigma_n(z) \\
=\; & S(r,w) = o\{T(r,w)\}.
\end{aligned}
$$

This completes the proof of Theorem 2.1.

§3   Applications to complex partial q-difference equations

Recently, many authors, such as Chiang Y M, Halburd R G, Korhonen R J, Chen Zongxuan, Gao Lingyun have studied solutions of some types of complex difference equation, and systems of complex difference equations, and also obtained many important results, see[11, 12, 13, 14, 15, 16, 17].

Let w be a non-constant meromorphic function of zero order, if meromorphic function g satisfies $T(r,g) = o\{T(r,w)\}$, for all r outside of a set of upper logarithmic density 0, i.e. outside of a set E such that $\limsup\limits_{r\to\infty} \dfrac{\int_{E\cap[1,r]} \frac{dt}{t}}{\log r} = 0$. The complement of E has lower logarithmic density 1, then g is called small function of w.

Let $q_j \in \mathbb{C}^n\backslash\{0\}$, $j = 1,\dots,n_2$, $w_i : \mathbb{C}^n \to \mathbb{P}^1, i = 1,2$. $z = (z_1,...,z_n) \in \mathbb{C}^n$, $I, J, \overline{I}, \overline{J}$ are four finite sets of multi-indices, complex partial q-difference polynomials $\Omega_1(z,w_1,w_2)$, $\Omega_2(z,w_1,w_2), \Omega_3(z,w_1,w_2), \Omega_4(z,w_1,w_2)$ can be expressed as

$$\Omega_1(z,w_1,w_2) = \sum_{(i)\in I} a_{(i)}(z) \prod_{k=1}^{2} w_k^{i_{k0}} \left(w_k(q_1 z)\right)^{i_{k1}} \cdots \left(w_k(q_{n_2} z)\right)^{i_{kn_2}},$$

$$\Omega_2(z,w_1,w_2) = \sum_{(j)\in J} b_{(j)}(z) \prod_{k=1}^{2} w_k^{j_{k0}} \left(w_k(q_1 z)\right)^{j_{k1}} \cdots \left(w_k(q_{n_2} z)\right)^{j_{kn_2}},$$

$$\Omega_3(z,w_1,w_2) = \sum_{(\bar{i})\in \overline{I}} c_{(\bar{i})}(z) \prod_{k=1}^{2} w_k^{\bar{i}_{k0}} \left(w_k(q_1 z)\right)^{\bar{i}_{k1}} \cdots \left(w_k(q_{n_2} z)\right)^{\bar{i}_{kn_2}},$$

$$\Omega_4(z,w_1,w_2) = \sum_{(\bar{j})\in \overline{J}} d_{(\bar{j})}(z) \prod_{k=1}^{2} w_k^{\bar{j}_{k0}} \left(w_k(q_1 z)\right)^{\bar{j}_{k1}} \cdots \left(w_k(q_{n_2} z)\right)^{\bar{j}_{kn_2}},$$

where coefficients $\{a_{(i)}(z)\}, \{b_{(j)}(z)\}, \{c_{(\bar{i})}(z)\}, \{d_{(\bar{j})}(z)\}$ are small functions of w_1, w_2.

Let $\Phi_1 = \dfrac{\Omega_1(z,w_1,w_2)}{\Omega_2(z,w_1,w_2)}$, $\Phi_2 = \dfrac{\Omega_3(z,w_1,w_2)}{\Omega_4(z,w_1,w_2)}$, for Φ_1, we denote $\lambda_{11} = \max\limits_{(i)}\{\sum\limits_{l=0}^{n_2} i_{1l}\}$, $\lambda_{12} = \max\limits_{(i)}\{\sum\limits_{l=0}^{n_2} i_{2l}\}$, $\lambda_{21} = \max\limits_{(j)}\{\sum\limits_{l=0}^{n_2} j_{1l}\}$, $\lambda_{22} = \max\limits_{(j)}\{\sum\limits_{l=0}^{n_2} j_{2l}\}$, $\lambda_1 = \max\{\lambda_{11}, \lambda_{21}\}$, $\lambda_2 = \max\{\lambda_{12}, \lambda_{22}\}$. For Φ_2, we denote similarly $\overline{\lambda_1}, \overline{\lambda_2}$.

We will investigate the existence of meromorphic solutions of complex partial q-difference equation of several variables (3.1) and systems of complex partial q-difference equations of several variables (3.2) and (3.3), where $z = (z_1,...,z_n) \in \mathbb{C}^n$.

$$\sum_{j=1}^{n_2} w(q_j z) = R\left(z, w(z)\right) = \frac{a_0(z) + a_1(z)w(z) + \cdots + a_p(z)w^p(z)}{b_0(z) + b_1(z)w(z) + \cdots + b_q(z)w^q(z)}, \tag{3.1}$$

where $q_1,\dots,q_{n_2} \in \mathbb{C}^n\backslash\{0\}$, $R(z,w(z))$ is irreducible rational function in $w(z)$, $a_0(z),\dots,a_p(z)$, $b_0(z),\dots,b_q(z)$ are rational functions.

$$\begin{cases} \Omega_1(z,w_1,w_2) = R_1(z,w_1) = \dfrac{a_0(z) + a_1(z)w_1(z) + \cdots + a_{p_1}(z)w_1^{p_1}(z)}{b_0(z) + b_1(z)w_1(z) + \cdots + b_{q_1}(z)w_1^{q_1}(z)}, \\[2mm] \Omega_2(z,w_1,w_2) = R_2(z,w_2) = \dfrac{c_0(z) + c_1(z)w_2(z) + \cdots + c_{p_2}(z)w_2^{p_2}(z)}{d_0(z) + d_1(z)w_2(z) + \cdots + d_{q_2}(z)w_2^{q_2}(z)}, \end{cases} \tag{3.2}$$

where coefficients $\{a_i(z)\}, \{b_j(z)\}$ are small functions of w_1, $\{c_l(z)\}, \{d_m(z)\}$ are small functions of w_2. $a_{p_1} b_{q_1} \neq 0, c_{p_2} d_{q_2} \neq 0$. The definition of $\Omega_1(z,w_1,w_2)$ and $\Omega_2(z,w_1,w_2)$ is as before.

$$\begin{cases} \Phi_1 = R_1(z,w_1,w_2), \\ \Phi_2 = R_2(z,w_1,w_2). \end{cases} \tag{3.3}$$

8

where $R_j (j = 1, 2)$ are irreducible rational functions with the meromorphic coefficients.

Definition 3.1. Let w_1 and w_2 be meromorphic functions in $\mathbb{C}^n$.

$$S(r) = \sum T(r, a_{(i)}) + \sum T(r, b_{(j)}) + \sum T(r, c'_{(\bar{i})}) + \sum T(r, d_{(\bar{j})}) + \sum T(r, d'_{(\bar{j})}),$$

where $\sum T(r, d'_{(\bar{j})})$ means the sum of characteristic functions of all coefficients in $R_j (j = 1, 2)$. (Let be $w_1(z)$) ($w_1(z), w_2(z)$) be a set of meromorphic solutions of (3.2) or (3.3). If one (Let be $w_1(z)$) of meromorphic solutions $(w_1(z), w_2(z))$ of (3.2) or (3.3) satisfies $S(r) = o\{T(r, w_1)\}$, outside a possible exceptional set with finite logarithmic measure, then we say $w_1(z)$ is admissible.

We will prove

Theorem 3.1. *Let w be a meromorphic function in $\mathbb{C}^n$. If the q-difference equation (3.1) admits a transcendental meromorphic solution of zero order, then*

$$\max\{p, q\} \le n_2.$$

Remark 3.1. If we replace the left side of (3.1) by $\prod_{j=1}^{n_2} w(q_j z)$, then the same assertion that $\max\{p, q\} \le n_2$ holds.

Theorem 3.2. *Let w_1 and w_2 be meromorphic functions in $\mathbb{C}^n$, and $(w_1(z), w_2(z))$ be a set of zero order meromorphic solution of (3.2). If*

$$\max\{p_1, q_1\} > \lambda_{11}, \max\{p_2, q_2\} > \lambda_{22},$$

and both w_1 and w_2 are admissible, then

$$[\max\{p_1, q_1\} - \lambda_{11}][\max\{p_2, q_2\} - \lambda_{22}] \le \lambda_{12}\lambda_{21}.$$

Example 3.1. $(w_1, w_2) = (z_1 z_2, \dfrac{1}{z_1 z_2})$ is a set of zero order admissible meromorphic solution of the following system of complex partial q-difference equations

$$\begin{cases} w_2^2(-2z_1, -2z_2) = \dfrac{1}{16 w_1^2}, \\ w_1^2(\dfrac{1}{3} z_1, \dfrac{1}{3} z_2) = \dfrac{1}{81 w_2^2}. \end{cases}$$

Easily, we obtain

$$\lambda_{11} = 0, \lambda_{22} = 0, \lambda_{12} = 2, \lambda_{21} = 2, \max\{p_1, q_1\} = 2, \max\{p_2, q_2\} = 2.$$

Thus

$$[\max\{p_1, q_1\} - \lambda_{11}][\max\{p_2, q_2\} - \lambda_{22}] = 4 = \lambda_{12}\lambda_{21}.$$

This example shows the upper bound in Theorem 3.2 can be reached.

Example 3.2. For a system of complex partial q-difference equations

$$\begin{cases} w_1^2(-2z_1, -2z_2) w_2(-\dfrac{1}{2} z_1, -\dfrac{1}{2} z_2) = \dfrac{w_1^4 - (\frac{7}{2} z_1 z_2 - \frac{49}{16}) w_1^2 - \frac{45}{8} z_1 z_2 - \frac{65}{16}}{w_1^2 - 2w_1 - z_1^2 z_2^2 + 2}, \\ w_1(-2z_1, -2z_2) w_2^2(-\dfrac{1}{2} z_1, -\dfrac{1}{2} z_2) = \dfrac{\frac{1}{256} w_2^2 + \frac{1}{64 z_1 z_2} w_2^3}{\frac{3}{z_1 z_2} w_2 - 3 z_1 z_2 + 1}. \end{cases}$$

$(w_1, w_2) = (z_1 z_2 + 1, z_1^2 z_2^2)$ is a set of non-admissible solutions.

Clearly, we know

$$\lambda_{11} = 2, \lambda_{22} = 2, \lambda_{12} = 1, \lambda_{21} = 1, \max\{p_1, q_1\} = 4, \max\{p_2, q_2\} = 3.$$

Thus

$$[\max\{p_1, q_1\} - \lambda_{11}] [\max\{p_2, q_2\} - \lambda_{22}] = 2 > 1 = \lambda_{12}\lambda_{21}.$$

This example shows that we can not omit 'admissible' in Theorem 3.2.

Theorem 3.3. *Let w_1 and w_2 be meromorphic functions in $\mathbb{C}^n$. Let (w_1, w_2) be a set of zero order meromorphic solution of (3.2). If one of the following conditions is satisfied*

$$(i)\max\{p_1, q_1\} > \lambda_{11}, \ (ii)\max\{p_2, q_2\} > \lambda_{22},$$

then both w_1 and w_2 are admissible or none of w_1 and w_2 is admissible.

Theorem 3.4. *Let w_1 and w_2 be meromorphic functions in $\mathbb{C}^n$. Let (w_1, w_2) be a set of zero order meromorphic solution of (3.3). If one of the following conditions is satisfied*

$$(i) \ p_1 > \lambda_1, \ q_2 > \overline{\lambda_2}, \ (ii) \ p_2 > \lambda_2, \ q_1 > \overline{\lambda_1},$$

then both w_1 and w_2 are admissible or none of w_1 and w_2 is admissible, where p_1 and p_2 are the highest degree of w_1 and w_2 in $R_1(z, w_1, w_2)$, we denote similarly q_1, q_2 in $R_2(z, w_1, w_2)$.

Example 3.3. For a system of complex partial q-difference equations

$$\begin{cases} \dfrac{w_1^2(-\frac{1}{2}z_1, -\frac{1}{2}z_2)w_2(\frac{1}{2}z_1, \frac{1}{2}z_2)}{w_1(3z_1, 3z_2) + w_2(-\sqrt{3}z_1, -\sqrt{3}z_2)} = \\[2ex] \dfrac{5w_1^3w_2^2 + 3w_1^2w_2^2 - w_1^2w_2 + \frac{2}{z_1z_2}w_1w_2^2 + 4w_1w_2 - w_2 + 1}{5 - \frac{21}{z_1z_2}w_1^3w_2^3 + 37w_1w_2^2 - 23z_1z_2w_1^2w_2^2 - 4w_1^2w_2}, \\[3ex] \dfrac{(1 - z_1)(1 - z_2)w_1^3(\frac{1}{\sqrt{2}}z_1, \frac{1}{\sqrt{2}}z_2)}{w_2(\sqrt{3}z_1, \sqrt{3}z_2)} = \\[2ex] \dfrac{(\frac{8}{z_2} + \frac{8}{z_1})w_1^4w_2 + 8w_1^3 - 8w_1^2}{3w_1^4w_2^3 - 7w_1^4w_2^2 + 2w_1^2w_2^2 + 5w_1^2w_2 + 4w_2 + 2}, \end{cases}$$

admits a non-admissible meromorphic solution $(w_1, w_2) = (-\dfrac{1}{z_1z_2}, z_1^2z_2^2)$. Clearly, we obtain

$$\lambda_1 = 2, \lambda_2 = 1, p_1 = 3, p_2 = 3. \ \ \overline{\lambda_1} = 3, \overline{\lambda_2} = 1, q_1 = 4, q_2 = 3.$$

In this case

$$p_1 > \lambda_1, \ q_2 > \overline{\lambda_2}, \ p_2 > \lambda_2, \ q_1 > \overline{\lambda_1}.$$

This example shows that Theorem 3.4 holds.

To prove theorems, we need some lemmas as follows.

Lemma 3.1. [2] *Let $R(z, w(z)) = \dfrac{a_0(z) + a_1(z)w(z) + \cdots + a_{p'}(z)w^{p'}(z)}{b_0(z) + b_1(z)w(z) + \cdots + b_{q'}(z)w^{q'}(z)}$ be an irreducible rational function in $w(z)$ with the meromorphic coefficients $\{a_i(z)\}$ and $\{b_j(z)\}$. If $w(z)$ is a meromorphic function in $\mathbb{C}^n$, then*

$$T(r, R(z, w)) = \max\{p', q'\}T(r, w) + O\{\sum T(r, a_i) + \sum T(r, b_j)\}.$$

Lemma 3.2. *Let w_1 and w_2 be non-constant meromorphic functions of zero order in $\mathbb{C}^n$,*

10

$q_i \in \mathbb{C}^n \backslash \{0\}, i = 1, ..., n_2$. *If*

$$\Omega_1(z, w_1, w_2) = \sum_{(i) \in I} a_{(i)}(z) \prod_{k=1}^{2} w_k^{i_{k0}} \left(w_k(q_1 z) \right)^{i_{k1}} \cdots \left(w_k(q_{n_2} z) \right)^{i_{kn_2}},$$

$\{a_{(i)}(z)\}$ *is a small function of* w_1 *and* w_2. $\lambda_{1k} = \max\{\sum_{l=0}^{n_2} i_{kl}\}(k = 1, 2)$, *then*

$$T(r, \Omega_1(z, w_1, w_2)) \leq \lambda_{11} T(r, w_1) + \lambda_{12} T(r, w_2) + S(r, w_1) + S(r, w_2) + S(r).$$

Proof   It is easy to prove by Theorem B.

As the proof of Theorem 2.1 in [18], we have

Lemma 3.3. *Let* w_1 *and* w_2 *be nonconstant meromorphic functions in* $\mathbb{C}^n$. *If*

$$\lim_{r \to \infty} \sup_{r \notin I_1} \frac{S(r)}{T(r, w_1)} = 0, T(r, w_2) = O\{S(r)\}(r \notin I_2),$$

then

$$\lim_{r \to \infty} \sup_{r \notin I_1 \cup I_2} \frac{T(r, w_2)}{T(r, w_1)} = 0,$$

where I_1, I_2 *are both exceptional sets with upper logarithmic density 0.*

Lemma 3.4. *Let* w_1 *and* w_2 *be non-constant meromorphic functions of zero order in* $\mathbb{C}^n$, $q_i \in \mathbb{C}^n \backslash \{0\}, i = 1, ..., n_2$. *Let* $\Phi_1 = \dfrac{\Omega_1(z, w_1, w_2)}{\Omega_2(z, w_1, w_2)}$, $\{a_{(i)}(z)\}, \{b_{(j)}(z)\}$ *are both small functions of* w_1 *and* w_2. *If*

$$\lambda_1 = \max\{\lambda_{11}, \lambda_{21}\}, \lambda_2 = \max\{\lambda_{12}, \lambda_{22}\},$$

then

$$T(r, \Phi_1) \leq \lambda_1 T(r, w_1) + \lambda_2 T(r, w_2) + S(r, w_1) + S(r, w_2).$$

Proof   Let $\mathbb{C}^n < r >= \{z = (z_1, ..., z_n) \in \mathbb{C}^n : |z_1|^2 + \cdots + |z_n|^2 = r^2\}$.

Firstly, we estimate $m(r, \Phi_1)$. Set

$$u(z) = \max\{|\Omega_1(z, w_1, w_2)|, |\Omega_2(z, w_1, w_2)|\},$$

we have

$$\log^+ |\Phi_1| = \log u(z) - \log |\Omega_2(z, w_1, w_2)|,$$

thus

$$\int_{\mathbb{C}^n <r>} \log^+ |\Phi_1| \sigma_n = \int_{\mathbb{C}^n <r>} \log u(z) \sigma_n - \int_{\mathbb{C}^n <r>} \log |\Omega_2(z, w_1, w_2)| \sigma_n.$$

As the proof of Lemma 3.3 in [18], and using Theorem A and Theorem B, we have

$$T(r, \Phi_1) \leq \lambda_1 T(r, w_1) + \lambda_2 T(r, w_2) + S(r, w_1) + S(r, w_2).$$

This completes the proof of Lemma 3.4.

Proof of Theorem 3.1   Let w be a meromorphic function in $\mathbb{C}^n$, and $w(z)$ be a transcendental meromorphic solution of zero order of (3.1). It follows from Lemma 3.1 and Theorem B

that

$$
\begin{aligned}
\max\{p, q\} T(r, w(z)) &= T(r, R(z, w)) + S(r, w) \\
&= T\left(r, \sum_{j=1}^{n_2} w(q_j z)\right) + S(r, w) \\
&\leq n_2 T(r, w) + S(r, w).
\end{aligned}
$$

Thus, we have

$$
\max\{p, q\} \leq n_2.
$$

This completes the proof of Theorem 3.1.

Proof of Theorem 3.2   Let w_1 and w_2 be meromorphic functions in $\mathbb{C}^n$. Let (w_1, w_2) be a set of admissible meromorphic function of (3.2). From the first and the second equation of (3.2), and also using Lemma 3.1 and Lemma 3.2, we obtain

$$
\max\{p_1, q_1\} T(r, w_1) \leq \lambda_{11} T(r, w_1) + \lambda_{12} T(r, w_2) + S(r, w_1) + S(r, w_2) + S(r). \tag{3.4}
$$

$$
\max\{p_2, q_2\} T(r, w_2) \leq \lambda_{21} T(r, w_1) + \lambda_{22} T(r, w_2) + S(r, w_1) + S(r, w_2) + S(r). \tag{3.5}
$$

By (3.4) and (3.5), we have

$$
\left[\max\{p_1, q_1\} - \lambda_{11} + o(1)\right] T(r, w_1) \leq \left(\lambda_{12} + o(1)\right) T(r, w_2). \tag{3.6}
$$

$$
\left[\max\{p_2, q_2\} - \lambda_{22} + o(1)\right] T(r, w_2) \leq \left(\lambda_{21} + o(1)\right) T(r, w_1). \tag{3.7}
$$

Combining (3.6) and (3.7), we obtain

$$
\left[\max\{p_1, q_1\} - \lambda_{11}\right] \left[\max\{p_2, q_2\} - \lambda_{22}\right] \leq \lambda_{12} \lambda_{21}.
$$

This completes the proof of Theorem 3.2.

Proof of Theorem 3.3   Let w_1 and w_2 be nonconstant meromorphic functions of zero order in $\mathbb{C}^n$. It follows from Lemma 3.1 and Lemma 3.2 that

$$
\max\{p_1, q_1\} T(r, w_1) \leq \lambda_{11} T(r, w_1) + \lambda_{12} T(r, w_2) + S(r, w_1) + S(r, w_2) + S(r). \tag{3.8}
$$

$$
\max\{p_2, q_2\} T(r, w_2) \leq \lambda_{21} T(r, w_1) + \lambda_{22} T(r, w_2) + S(r, w_1) + S(r, w_2) + S(r). \tag{3.9}
$$

If w_1 is admissible and w_2 is non-admissible, then the inequality (3.8) becomes

$$
\max\{p_1, q_1\} \leq \lambda_{11} + (\lambda_{12} + o(1)) \frac{T(r, w_2)}{T(r, w_1)} + \frac{S(r)}{T(r, w_1)},
$$

using Lemma 3.3, we get

$$
\max\{p_1, q_1\} \leq \lambda_{11},
$$

outside of a set with upper logarithmic density 0. It is in contradiction with the condition (i).

If w_2 is admissible and w_1 is non-admissible, then the inequality (3.9) becomes

$$
\max\{p_2, q_2\} \leq \lambda_{22} + (\lambda_{21} + o(1)) \frac{T(r, w_1)}{T(r, w_2)} + \frac{S(r)}{T(r, w_2)},
$$

using Lemma 3.3, we get

$$
\max\{p_2, q_2\} \leq \lambda_{22},
$$

outside of a set with upper logarithmic density 0. It is in contradiction with the condition (ii).

Thus both w_1 and w_2 are admissible or none of w_1 and w_2 is admissible.

This proves Theorem 3.3.

Proof of Theorem 3.4   Let w_1 and w_2 be meromorphic functions in $\mathbb{C}^n$. Let (w_1, w_2) be

12

a zero order meromorphic solution of (3.3). Using Lemma 3.1 and Lemma 3.4, we obtain

$$p_1 T(r, w_1) + O\{T(r, w_2)\} \le \lambda_1 T(r, w_1) + \lambda_2 T(r, w_2) + S(r, w_1) + S(r, w_2) + S(r). \quad (3.10)$$

$$p_2 T(r, w_2) + O\{T(r, w_1)\} \le \lambda_1 T(r, w_1) + \lambda_2 T(r, w_2) + S(r, w_1) + S(r, w_2) + S(r). \quad (3.11)$$

$$q_1 T(r, w_1) + O\{T(r, w_2)\} \le \overline{\lambda_1} T(r, w_1) + \overline{\lambda_2} T(r, w_2) + S(r, w_1) + S(r, w_2) + S(r). \quad (3.12)$$

$$q_2 T(r, w_2) + O\{T(r, w_1)\} \le \overline{\lambda_1} T(r, w_1) + \overline{\lambda_2} T(r, w_2) + S(r, w_1) + S(r, w_2) + S(r). \quad (3.13)$$

If w_1 is admissible and w_2 is non-admissible, then the inequality (3.10) becomes

$$p_1 + \frac{O\{T(r, w_2)\}}{T(r, w_1)} \le \lambda_1 + (\lambda_2 + o(1)) \frac{T(r, w_2)}{T(r, w_1)} + \frac{S(r)}{T(r, w_1)}.$$

Using Lemma 3.3, we get

$$p_1 \le \lambda_1,$$

outside of a set with upper logarithmic density 0. It is in contradiction with the first inequality of (i).

If w_2 is admissible and w_1 is non-admissible, then the inequality (3.13) becomes

$$q_2 + \frac{O\{T(r, w_1)\}}{T(r, w_2)} \le \overline{\lambda_2} + \left(\overline{\lambda_1} + o(1)\right) \frac{T(r, w_1)}{T(r, w_2)} + \frac{S(r)}{T(r, w_2)}.$$

Using Lemma 3.3, we get

$$q_2 \le \overline{\lambda_2},$$

outside of a set with upper logarithmic density 0. It is in contradiction with the second inequality of (i).

Similarly, we can prove for conditions (ii).

Thus both w_1 and w_2 are admissible or none of w_1 and w_2 is admissible.

This completes the proof of Theorem 3.4.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors drafted the manuscript, read and approved the final manuscript.

Acknowledgements

References

[1] Yi, HX, Yang, CC: Theory of the Uniqueness of Meromorphic Function, Science Press, Beijing(1995)(in Chinese)

[2] Tu, ZH: Some Malmquist type theorems of partial differential equations on C^n. J. Math. Anal. Appl. **179**, 41-60(1993)

[3] Korhonen, R: A difference Picard theorem for meromorphic functions of several variables. Computational Methods and Function Theory, **12**, 343-361(2012)

[4] Cao, TB: Difference analogues of the second main theorem for meromorphic functions in several complex variables. Math. Nachr. **287**, 530-545(2014)

[5] Wen, ZT: The q-difference theorems for meromorphic functions of several variables. Abstr. Appl. Anal. Article ID: 736021(2014)

[6] Zhang, JL, Korhonen, R: On the Nevanlinna characteristic of $f(qz)$ and its applications. J. Math. Anal. Appl. **369**, 537-544(2010)

[7] Barnett, DC, Halburd, RG, Korhone, RJ, Morgan, W: Nevanlinna theory for the q-difference operator and meromorphic solutions of q-difference equations. Royal Society of Edinburgh, **137**A, 457-474(2007)

[8] Laine, I, Yang, CC: Clunie theorems for difference and q-difference polynomials. J.London Math. Soc. **76**, 556-566(2007)

[9] Morgan, W: Nevanlinna theory for the q-difference equations. Proc. Roy. SocEdinburgh. **137**, 457-474(2007)

[10] Laine, I, Yang, CC: Value distribution of difference polynomials. Proc. Japan Acad. Ser. A Math. Sci. **83**, 148-151(2007)

[11] Chiang, YM, Feng, SJ: On the Nevanlinna characteristic of $f(z + \eta)$ and difference equations in the complex plane. The Ramanujan J. **16**, 105-129(2008)

[12] Halburd, RG, Korhonen, RJ: Difference analogue of the Lemma on the logarithmic derivative with applications to difference equations. J. Math. Anal. Appl. **314**, 477-487(2006)

[13] Chen, ZX: On difference equations relating to Gamma function. Acta Mathematica Scientia,B, **31**, 1281-1294(2011)

[14] Gao, LY: Solutions of complex higher-order difference equations. Acta Mathematica Sinica, **56**, 451-458(2013)

[15] Gao, LY: On meromorphic solutions of a type of difference equations. Chinese Contemporary Math, **35**, 163-170(2014)

[16] Gao, LY: The growth order of solutions of systems of complex difference equations. Acta Mathematica Scientia, **33**B, 814-820(2013)

[17] Gao, LY: Systems of complex difference equations of Malmquist type. Acta Mathematica Sinica, **55**, 293-300(2012)

[18] Gao, LY: On admissible solution of two types of systems of complex differential equations. Acta Math.Sinica, **43**, 149-156(2000)

Nonstationary refinable functions based on generalized Bernstern polynomials

Ting Cheng and Xiaoyuan Yang

Department of Mathematics, Beihang University,
LMIB of the Ministry of Education, Beijing 100191, China
E-mail:xiaochengting0425@126.com
Corresponding author:xiaoyuanyang@vip.163.com

Abstract In this paper, we introduce a new family of nonstationary refinable functions from Generalized Bernstein Polynomials, which include a class of nonstationary refinable functions generated from the family of masks for the pseudo splines of type II (see [17]). Furthermore, a proof of the convergence of nonstationary cascade algorithms associated with the new masks is completed. We then construct symmetric compacted supported nonstationary C^∞ tight wavelet frames in $L_2(\mathbb{R})$ with the spectral frame approximation order.

1 Introduction

In frame systems, because tight wavelet frames (in the stationary case) can not satisfy compactly supported C^∞ properties, the nonstationary case was considered to obtain C^∞ tight wavelet frames with compacted support. Recently, the development of nonstationary tight wavelet frames has attracted a considerable amount of attention.

In 2008, Han and Shen [17] obtained symmetric compactly supported C^∞ tight wavelet frames in $L_2(\mathbb{R})$ with the spectral frame approximation order based on pseudo-splines of type II. In 2009, compactly supported nonstationary C^∞ tight wavelet frames in $L_2(\mathbb{R}^s)$ with the spectral frame approximation order from pseudo box splines were constructed in [22]. Li and

1

NONSTATIONARY REFINABLE FUNCTIONS BASED ON GENERALIZED

Shen [22] generalized univariate pseudo-splines to the multivariate setting and got a new class of refinable functions named pseudo box splines. Next, in [18] and [23], the analysis of characterization of nonstationary tight wavelet frames in Sobolev spaces was given. Han and Shen [18] characterized Sobolev spaces of an arbitrary order of smoothness using nonstationary tight wavelet frames for $L_2(\mathbb{R}^s)$. Also, approximation order of nonstationary tight wavelet frames in Sobolev spaces was obtained in [23]. Recently, the nonstationary subdivision scheme, which nonstationary cascade algorithms is closely related to, was studied in [2–10, 12, 15, 21, 24, 26]. In particular, in [14] and [20], the properties of nonstationary subdivision scheme were performed. Daniel et al.[14] and Jeonga et al.[20] showed C^2 approximating and Hölder regularities of nonstationary subdivision scheme, respectively.

This paper is concerned with the study of symmetric C^∞ nonstationary tight wavelet frames in $L_2(\mathbb{R})$ with compacted support and the spectral frame approximation order, which generalize nonstationary tight wavelet frames from pseudo-splines of type II in [17]. We discover a new extensive function based on Generalized Bernstein polynomials [1]. Furthermore, existence of L_2-solutions of nonstationary refinable functions from the new extensive function is implemented. At last, we prove the convergence of nonstationary cascade algorithms of the new family of nonstationary refinable functions.

The remainder of this paper is organized as follows: Section 2 collects some notations. Section 3 elaborates on existence of L_2-solutions of nonstationary refinable functions. Section 4 implements convergence of nonstationary cascade algorithms. Section 5 constructs symmetric C^∞ nonstationary tight wavelet frames in $L_2(\mathbb{R})$ with compacted support and the spectral frame approximation order. Section 6 gives the conclusion.

2 Preliminaries

For the convenience of the readers, we review some definitions about nonstationary refinable functions in this section.

Generalized Bernstein polynomials [1] are defined as

$$S_k^{(n)}(t) = \binom{n}{k} \frac{t(t+\alpha)\cdots(t+[k-1]\alpha)(1-t)(1-t+\alpha)\cdots(1-t+[n-k-1]\alpha)}{(1+\alpha)(1+2\alpha)\cdots(1+[n-1]\alpha)}, \quad (2.1)$$

where $\alpha \geq 0$. We apply (2.1) to marks of new refinable functions by substituting $t = \sin^2(\frac{\omega}{2})$, $n = m + l$ in (2.1) and the summation of $l + 1$ terms of them as follows:

$$\tau_0^{m,l,\alpha}(\omega) := \sum_{k=0}^{l} \binom{m+l}{k} \left(\prod_{i=0}^{k-1}(\sin^2(\frac{\omega}{2})+i\alpha) \prod_{i=0}^{m+l-k-1}(\cos^2(\frac{\omega}{2})+i\alpha) \right) / \prod_{i=1}^{m+l-1}(1+i\alpha). \quad (2.2)$$

Let $\tau_{0,j}^{m,l,\alpha}(\omega) = \tau_0^{m_j,l_j,\alpha_j}(\omega)$ $(j \in \mathbb{N})$ be defined in (2.2), we obtain

$$\tau_{0,j}^{m,l,\alpha}(\omega) := \sum_{k=0}^{l_j} \binom{m_j+l_j}{k} \left(\prod_{i=0}^{k-1}(\sin^2(\frac{\omega}{2})+i\alpha_j) \prod_{i=0}^{m_j+l_j-k-1}(\cos^2(\frac{\omega}{2})+i\alpha_j) \right) / \prod_{i=1}^{m_j+l_j-1}(1+i\alpha_j), (2.3)$$

2

where two positive integers l_j, m_j and α_j $(j \in \mathbb{N})$ satisfy $l_j < m_j - 5$, $\sum_{j=1}^{\infty} 2^{-j} m_j < \infty$, $\lim_{j \to \infty} m_j = \infty$ and $0 \le \alpha_j < \dfrac{1}{3(m_j + l_j) - 7}$.

A class of 2π-periodic trigonometric polynomials $\widehat{a_j}, j \in \mathbb{N}$, and their associated nonstationary refinable functions $\hat{\phi}_{j-1}, j \in \mathbb{N}$, defined by

$$\hat{\phi}_{j-1}(\omega) := \widehat{a_j}(\omega/2)\hat{\phi}_j(\omega/2) = \prod_{n=1}^{\infty} \hat{a}_{n+j-1}(2^{-n}\omega) \quad \omega \in \mathbb{R}, j \in \mathbb{N}, \qquad (2.4)$$

where the 2π-periodic trigonometric polynomials $\widehat{a_j}, j \in \mathbb{N}$, are called refinement masks. Here the Fourier transform $\hat{f}$ of a function $f \in L_1(\mathbb{R})$ is defined to be $\hat{f}(\omega) := \int_{\mathbb{R}} f(t)e^{-it\omega} dt$ and can be naturally extended to square integrable functions.

The notation $\mathbb{T}$ was introduced in [25], which is defined by

$$\mathbb{T} := \mathbb{R}/[2\pi\mathbb{Z}].$$

Denote $\deg(\hat{a})$ the smallest nonnegative integer such that its Fourier coefficients of $\hat{a}$ vanish outside $[-\deg(\hat{a}), \deg(\hat{a})]$. $\deg(\hat{a})$ here is the minimal integer k such that $[-k, k]$ contains the support of the Fourier coefficients of both $\hat{a}$ and $\hat{a}(-\cdot)$, which was introduced in [17].

In the following, we will adopt some of the notations from [19]. The transition operator $T_{\hat{a}}$ for 2π-periodic functions $\hat{a}$ and f can be defined as

$$[T_{\hat{a}}f](w) := |\hat{a}(\omega/2)|^2 f(\omega/2) + |\hat{a}(\omega/2 + \pi)|^2 f(\omega/2 + \pi), \quad \omega \in \mathbb{R}.$$

For $\tau \in \mathbb{R}$, a quantity is defined by

$$\rho_\tau(\hat{a}, \infty) := \limsup_{n \to \infty} \left\| T_{\hat{a}}^n \left(\left| \sin\left(\frac{\omega}{2}\right) \right|^\tau \right) \right\|_{L_\infty(\mathbb{T})}^{1/n}.$$

The notation $\rho(\hat{a})$ is defined by

$$\rho(\hat{a}) := \inf\{\rho_\tau(\hat{a}, \infty) : |\hat{a}(\omega + \pi)|^2 |\sin(\omega/2)|^\tau \in L_\infty(\mathbb{T}) \text{ and } \tau \ge 0\}.$$

A function $f \in W_2^\nu(\mathbb{R})$ if it satisfys

$$\|f\|_{W_2^\nu(\mathbb{R})}^2 := \int_{\mathbb{R}} (1 + |\omega|^{2\nu}) |\hat{f}(\omega)|^2 d\omega < \infty.$$

As [17], let $\{\widehat{a_j}\}_{j=1}^{\infty}$ be a sequence of 2π-periodic measurable functions. Define $\{f_n\}_{j=1}^{\infty}$ by

$$\widehat{f_n}(\omega) := \chi_{[-\pi,\pi]}(2^{-n}\omega) \prod_{j=1}^{n} \widehat{a_j}(2^{-n}\omega), \omega \in \mathbb{R}, n \in \mathbb{N}, \qquad (2.5)$$

where $\chi_{[-\pi,\pi]}$ denotes the characteristic function of the interval $[-\pi, \pi]$. This can be understood as a representation of the nonstationary cascade algorithm associated with the masks $\{\widehat{a_j}\}_{j=1}^{\infty}$ in the frequency domain. For a sequence of masks $\{\widehat{a_j}\}_{j=1}^{\infty}$ and an initial function $f \in W_2^\nu(\mathbb{R})$, we say that the (nonstationary) cascade algorithm associated with masks $\{\widehat{a_j}\}_{j=1}^{\infty}$ and an initial

3

NONSTATIONARY REFINABLE FUNCTIONS BASED ON GENERALIZED

function f converges in the Sobolev space $W_2^\nu(\mathbb{R})$, if $f_n \in W_2^\nu(\mathbb{R})$ for all $n \in \mathbb{N}$ and the sequence $\{f_n\}_{j=1}^\infty$ is convergent in $W_2^\nu(\mathbb{R})$.

Denote $\rho(\hat{a})$ the spectral radius of the square matrix $(c_{2j-k})_{-K \leq j,k \leq K}$ and define $\nu_2(\hat{a}) := -1/2 - log_2\sqrt{\rho(\hat{a})}$. It is known ([[16], Theorem 4.3 and Proposition 7.2] and [[19], Theorem 2.1]) that the stationary cascade algorithm associated with a 2π-periodic trigonometric polynomial mask $\hat{a}$ converges in $f \in W_2^\nu(\mathbb{R})$ if and only if $\nu_2(\hat{a}) > \nu$.

For a sequence $\{\phi_n\}_{n=0}^\infty$ of functions in $L_2(\mathbb{R})$, we define the linear operators $P_n(f), n \in \mathbb{N}_0$, by

$$P_n(f) := \sum_{k \in \mathbb{Z}} \langle f, \phi_{n;n,k} \rangle \phi_{n;n,k}, \quad f \in L_2(\mathbb{R}) \quad with \quad \phi_{n;n,k} := 2^{n/2} \phi_n(2^n \cdot -k). \tag{2.6}$$

Wavelet functions $\psi_{j-1}^\ell, j \in \mathbb{N}$ and $\ell \in \{1, \cdots, \mathcal{J}_j\}$, are obtained from ϕ_j by

$$\widehat{\psi_{j-1}^\ell}(\omega) := \widehat{b_j^\ell}(\omega/2)\widehat{\phi_j}(\omega/2), \quad \ell \in \{1, \cdots, \mathcal{J}_j\}, \tag{2.7}$$

where $\mathcal{J}_j$ are positive integers and each $b_j^\ell, \ell = 1, \cdots, \mathcal{J}_j$, is called a (high-pass) wavelet mask. Denote $\mathbb{N}_0 := \mathbb{N} \bigcup \{0\}$. We say that $\{\phi_0\} \bigcup \{\psi_j^\ell : j \in \mathbb{N}_0, \ell = 1, \cdots, \mathcal{J}_{j+1}\}$ generates a nonstationary tight wavelet frame in $L_2(\mathbb{R})$ if

$$\{\phi_0(\cdot - k) : k \in \mathbb{Z}\} \bigcup \{\psi_{j;j,k}^\ell := 2^{j/2}\psi_j^\ell(2^j \cdot -k) : j \in \mathbb{N}_0, k \in \mathbb{Z}, \ell = 1, \cdots, \mathcal{J}_{j+1}\} \tag{2.8}$$

is a tight frame of $L_2(\mathbb{R})$.

Finally, we note that the 2π-periodic trigonometric polynomial wavelet masks $\widehat{b_j^\ell}, j \in \mathbb{N}$ and $\ell \in \{1, \cdots, \gamma\}$, can be constructed from the masks $\widehat{a_j}$ by many ways provided that the refinement masks $\widehat{a_j}, j \in \mathbb{N}$, satisfy $|\widehat{a_j}(\omega)|^2 + |\widehat{a_j}(\omega + \pi)|^2 \leq 1, a.e.\omega \in \mathbb{R}$. Define

$$\begin{aligned} \widehat{b_j^1}(\omega) &:= e^{-i\omega}\overline{\widehat{a_j}(\omega + \pi)}, \\ \widehat{b_j^2}(\omega) &:= 2^{-1}[A_j(\omega) + e^{-i\omega}\overline{A_j(\omega)}], \\ \widehat{b_j^3}(\omega) &:= 2^{-1}[A_j(\omega) + e^{-i\omega}\overline{A_j(\omega)}], \end{aligned} \tag{2.9}$$

where A_j is a π-periodic trigonometric polynomial with real coefficients such that

$$|A_j(\omega)|^2 = 1 - |\widehat{a_j}(\omega)|^2 - |\widehat{a_j}(\omega + \pi)|^2.$$

Then, $\widehat{a_j}, \widehat{b_j^1}, \widehat{b_j^2}$ and $\widehat{b_j^3}, j \in \mathbb{N}$, satisfy

$$|\widehat{a_j}(\omega)|^2 + \sum_{\ell=1}^{\mathcal{J}_j}|\widehat{b_j^\ell}(\omega)|^2 = 1 \quad and \quad \widehat{a_j}(\omega)\overline{\widehat{a_j}(\omega + \pi)} + \sum_{\ell=1}^{\mathcal{J}_j}\widehat{b_j^\ell}(\omega)\overline{\widehat{b_j^\ell}(\omega + \pi)} = 0, \tag{2.10}$$

with $\mathcal{J} = 3$. Thus, the wavelet system in (2.8) is a compactly supported tight wavelet frame in $L_2(\mathbb{R})$ (see, [17], Theorem1.1). Furthermore, the corresponding wavelets defined by (2.7) using masks in (2.9) are symmetric or antisymmetric whenever ϕ_j is symmetric.

4

3 Existence of L_2-solutions of nonstationary refinable functions

In this section, demonstration of the existence of L_2-solutions of nonstationary refinable functions is given. For notational simplicity, we will introduce the following two definitions:

$$B_{k,j}(\omega) := \left(\prod_{i=0}^{k-1} (\sin^2(\frac{\omega}{2}) + i\alpha_j) \prod_{i=1}^{m_j+l_j-k-1} (\cos^2(\frac{\omega}{2}) + i\alpha_j) \right) \Big/ \prod_{i=1}^{m_j+l_j-1} (1+i\alpha_j), j \in \mathbb{N}.$$

$$T_{0,j}(\omega) := \sum_{k=0}^{l} \binom{m_j+l_j}{k} \left(\prod_{i=0}^{k-1} (\sin^2(\frac{\omega}{2}) + i\alpha_j) \prod_{i=1}^{m_j+l_j-k-1} (\cos^2(\frac{\omega}{2}) + i\alpha_j) \right) \Big/ \prod_{i=1}^{m_j+l_j-1} (1+i\alpha_j), j \in \mathbb{N}.$$

Two lemmas about the relations of the quantities $\rho_\tau(\tau_{0,j}^{m,l,\alpha}(\omega), \infty), j \in \mathbb{N}$ associated with masks (2.3) will be provided in the following.

Lemma 3.1 *([19], Theorem 4.1) Let $\hat{a}$ be a 2π-periodic measurable function such that $|\hat{a}|^2 \in C^\beta(\mathbb{T})$ with $|\hat{a}|^2(0) \neq 0$ and $\beta > 0$. If $|\hat{a}(\omega)|^2 = |1 + e^{-i\omega}|^{2\tau}|\hat{A}(\omega)|^2$ a.e. $\omega \in \mathbb{R}$ for some $\tau \geq 0$ such that $\hat{A}(\omega) \in L_\infty(\mathbb{T})$, then*

$$\rho_{2\tau}(\hat{a}, \infty) = \inf_{n \in \mathbb{N}} \|T_{\hat{a}}^n 1\|_{L_\infty(\mathbb{T})}^{\frac{1}{n}} = \lim_{n \to \infty} \|T_{\hat{a}}^n 1\|_{L_\infty(\mathbb{T})}^{\frac{1}{n}} = \rho_0(\hat{A}, \infty).$$

Lemma 3.2 *([19], Theorem 4.3) Let $\hat{a}$ and $\hat{c}$ be 2π-periodic measurable functions such that*

$$|\hat{a}(\omega)| \leq |\hat{c}(\omega)|$$

for almost every $\omega \in \mathbb{R}$. Then

$$\rho_\tau(\hat{a}, \infty) \leq \rho_\tau(\hat{c}, \infty), \quad \tau \in \mathbb{R}.$$

The following two lemmas are necessary for proving existence of L_2-solutions of nonstationary refinable functions.

Lemma 3.3 *([17], Lemma 2.1) Let $\widehat{a_j}, j \in \mathbb{N}$ be a 2π-periodic trigonometric polynomials such that $\sup_{j \in \mathbb{N}} \|\widehat{a_j}\|_{L_\infty(\mathbb{R})} < \infty$. If $\sum_{j=1}^\infty 2^{-j} deg(\widehat{a_j}) < \infty$ holds and $\sum_{j=1}^\infty |\widehat{a_j}(0) - 1| < \infty$, then the infinite product (2.4) converges uniformly on every compact set of $\mathbb{R}$ and all $\phi_j, j \in \mathbb{N}_0$, in (2.4) are well-defined compactly supported tempered distributions.*

Lemma 3.4 *([17], Lemma 2.2) Let $\widehat{a_j}, j \in \mathbb{N}$, be 2π-periodic measurable functions satisfying $|\widehat{a_j}(\omega)|^2 + |\widehat{a_j}(\omega + \pi)|^2 \leq 1, a.e.\omega \in \mathbb{R}$ for each $j \in \mathbb{N}$. Assume that, for every $j \in \mathbb{N}_0$, $\widehat{\phi_j}(\omega) := \lim_{N \to \infty} \prod_{n=1}^N \widehat{a_{n+j}(2^{-n}\omega)}$ is well defined for almost every $\omega \in \mathbb{R}$; that is, the infinite product in (2.4) exists for almost every point in $\mathbb{R}$. Then $[\widehat{\phi_j}, \widehat{\phi_j}](\omega) := \sum_{k \in \mathbb{Z}} |\widehat{\phi_j}(\omega + 2\pi k)|^2 \leq 1, a.e.\omega \in \mathbb{R} \forall j \in \mathbb{N}_0$ holds and consequently, $\phi_j \in L_2(\mathbb{R})$ with $\|\phi_j\|_{L_2(\mathbb{R})} \leq 1$ for every $j \in \mathbb{N}_0$.*

A useful condition of establishing existence of L_2-solutions of nonstationary refinable functions is described in the following lemma.

5

NONSTATIONARY REFINABLE FUNCTIONS BASED ON GENERALIZED

Lemma 3.5 *For two positive integers $l_j, m_j, l_j < m_j - 5, j \in \mathbb{N}$, if*

$$0 \leq \alpha_j < \frac{1}{3(m_j + l_j) - 7} \quad (j \in \mathbb{N}), \tag{3.1}$$

then

$$\max_{\omega \in \mathbb{T}} B_{k,j}(\omega) \leq (\frac{1}{2})^{m_j + l_j - 1}, \quad k = 1, 2, \ldots, l_j, \tag{3.2}$$

Proof. For $j \in \mathbb{N}, k = 1, 2, \ldots, l_j$, it is obvious that

$$B_{k,j}(\omega) = \frac{\cos^2(\frac{\omega}{2}) + (m_j + l_j - 1 - j)\alpha_j}{\sin^2(\frac{\omega}{2}) + j\alpha_j} B_{k+1,j}(\omega).$$

We claim that

$$\frac{B_{k,j}(\omega)}{B_{k+1,j}(\omega)} = \frac{\cos^2(\frac{\omega}{2}) + (m_j + l_j - 1 - j)\alpha_j}{\sin^2(\frac{\omega}{2}) + j\alpha_j} > 1. \tag{3.3}$$

Since $l_j < m_j - 5$, for $k = 1, 2, \ldots, l_j$, it holds that

$$k < m_j + l_j - 1 - k. \tag{3.4}$$

There are two cases to consider:

Case I: Suppose that $\cos(\omega) \geq 0$. By (3.1) and (3.4) , it is easy to see that

$$\alpha_j > 0 > \frac{-\cos(\omega)}{m_j + l_j - 1 - 2k}.$$

Then

$$\cos^2(\frac{\omega}{2}) + (m_j + l_j - 1 - k)\alpha_j > \sin^2(\frac{\omega}{2}) + k\alpha_j. \tag{3.5}$$

This implies Condition (3.3).

Case II: Suppose that $\cos(\omega) < 0$. Note that since

$$\frac{(2^{2l_j} - 2^{-1})(m_j - 1 - l_j) - l_j}{l_j(m_j - 1 - l_j)} > \frac{0.5(m_j - 1 - l_j) - l_j}{l_j(m_j - 1 - l_j)} > 0,$$

we can obtain that

$$\frac{2^{m_j + l_j - 1} - 2^{-1}}{l_j} - \frac{1}{m_j - l_j - 1} = \frac{(2^{m_j + l_j - 1} - 2^{-1})(m_j - 1 - l_j) - l_j}{l_j(m_j - 1 - l_j)}$$

$$> \frac{(2^{2l_j} - 2^{-1})(m_j - 1 - l_j) - l_j}{l_j(m_j - 1 - l_j)} > 0.$$

By (3.1), then

$$\alpha_j \geq \frac{2^{m_j + l_j - 1} - 2^{-1}}{l_j} > \frac{1}{m_j - l_j - 1} = \frac{1}{m_j + l_j - 1 - 2l_j} \geq \frac{|\cos(\omega)|}{m_j + l_j - 1 - 2k} = \frac{-\cos(\omega)}{m_j + l_j - 1 - 2k},$$

for $k = 1, 2, \ldots, l_j$. Then (3.5) holds. This concludes the claim (3.3).

By using (3.1), one gets

$$(4(m_j + l_j - 2) - (m_j + l_j - 1))\alpha_j < \frac{4(m_j + l_j - 2) - (m_j + l_j - 1)}{3(m_j + l_j) - 7} = 1.$$

Then

$$\frac{(m_j + l_j - 2)\alpha_j}{(1 + \alpha_j)(1 + (m_j + l_j - 1)\alpha_j)} < \frac{(m_j + l_j - 2)\alpha_j}{1 + (m_j + l_j - 1)\alpha_j} < \frac{1}{4}. \tag{3.6}$$

6

TING CHENG AND XIAOYUAN YANG

Since $l_j < m_j - 5$, we have

$$(3(m_j + l_j) - 7) - (m_j + l_j - 4) = 2m_j + 2l_j - 3 > 0.$$

Then

$$\frac{1}{3(m_j + l_j) - 7} < \frac{1}{m_j + l_j - 4}.$$

Thus

$$(2(m_j + l_j - 3) - (m_j + l_j - 2))\alpha_j < \frac{2(m_j + l_j - 3) - (m_j + l_j - 2)}{m_j + l_j - 4} = 1.$$

Similarly, one has

$$\frac{(m_j + l_j - 3)\alpha_j}{1 + (m_j + l_j - 2))\alpha_j} < \frac{1}{2}. \tag{3.7}$$

For any x, Notice that

$$(\frac{x}{1 + (1 + x)})' > 0 \tag{3.8}$$

and $B_{1,j}(\omega)$ which is a continuous function on $[-\pi, \pi]$ and is differentiable on $(-\pi, \pi)$, has the maximum value at $\omega = \pi$. The reason as follow:

The equation $[B_{1,j}(\omega)]' = 0$ has three zeros, at $\omega = 0, \pm\pi$. Since $[B_{1,j}(\omega)]'' > 0$, $B_{1,j}(0)$ is the minimum of $B_{1,j}(\omega)$ on $[-\pi, \pi]$. Thus $B_{1,j}(\pm\pi)$ is the maximum of $B_{1,j}(\omega)$ on $[-\pi, \pi]$. Therefore, applying (3.3), (3.6), (3.7), (3.8) and

$$
\begin{aligned}
B_{1,j}(\omega) &= \left(\sin^2(\frac{\omega}{2}) \prod_{i=1}^{m_j+l_j-2} (\cos^2(\frac{\omega}{2}) + i\alpha_j) \right) \Big/ \prod_{i=1}^{m_j+l_j-1} (1 + i\alpha_j) \\
&\leq \prod_{i=1}^{m_j+l_j-2} i\alpha_j \Big/ \prod_{i=1}^{m_j+l_j-1} (1 + i\alpha_j) \\
&\leq \prod_{i=1}^{m_j+l_j-3} \frac{i\alpha_j}{1 + (i+1)\alpha_j} \cdot \frac{(m_j + l_j - 2)\alpha_j}{(1 + \alpha_j)(1 + (m_j + l_j - 1)\alpha_j)} \\
&\leq \left(\frac{(m_j + l_j - 3)\alpha_j}{1 + (m_j + l_j - 2)\alpha_j} \right)^{m_j+l_j-3} \cdot \frac{1}{4} \\
&= (\frac{1}{2})^{m_j+l_j-1},
\end{aligned}
$$

we get the inequality (3.2). ¶

Theorem 3.1 *Let $\tau_{0,j}^{m,l,\alpha}(\omega)$ be the mark (2.3), which are defined in (2.4), then the infinite product in (2.4) converges uniformly on every compact set of $\mathbb{R}$.*

Proof. Since $\tau_{0,j}^{m,l,\alpha}(\omega) = \tau_{0,j}^{m,l,\alpha}(\omega + 2\pi), j \in \mathbb{N}$, we obtain $\tau_{0,j}^{m,l,\alpha}(\omega)$ are 2π-periodic trigono-

7

NONSTATIONARY REFINABLE FUNCTIONS BASED ON GENERALIZED

metric polynomials. Applying Lemma 3.5, one has

$$
\begin{aligned}
|\tau_{0,j}^{m,l,\alpha}(\omega)| &= |\sum_{k=0}^{l_j} \binom{m_j+l_j}{k} \left(\prod_{i=0}^{k-1}(\sin^2(\frac{\omega}{2})+i\alpha_j) \prod_{i=0}^{m_j+l_j-k-1}(\cos^2(\frac{\omega}{2})+i\alpha_j) \right) / \prod_{i=1}^{m_j+l_j-1}(1+i\alpha_j)| \\
&\le \left|1+\left(\max_{\omega\in\mathbb{T}} B_{k,j}(\omega)\right)\sum_{k=1}^{l_j}\binom{m_j+l_j}{k}\right| \\
&= 1+(\frac{1}{2})^{m_j+l_j-1}|\sum_{k=1}^{l_j}\binom{m_j+l_j}{k}| \\
&\le 1+(\frac{1}{2})^{m_j+l_j-1}\cdot 2^{m_j+l_j} = 3.
\end{aligned}
$$

Thus, there exists $M=3$, for any $j\in\mathbb{N}$, we derive that $\tau_{0,j}^{m,l,\alpha}(\omega)|\le 3$ holds.
For $\tau_{0,j}^{m,l,\alpha}(0)=1$, we obtain

$$\sum_{j=1}^{\infty}|\tau_{0,j}^{m,l,\alpha}(0)-1|=0<\infty.$$

By using $l_j<m_j-5, \sum_{j=1}^{\infty}2^{-j}m^j<\infty$, ones get

$$
\begin{aligned}
\sum_{j=1}^{\infty}2^{-j}deg(\tau_{0,j}^{m,l,\alpha}(\omega)) &= \sum_{j=1}^{\infty}2^{-j}(2(m_j+l_j)+1) \\
&< \sum_{j=1}^{\infty}2^{-j}(4m_j-9)<\infty.
\end{aligned}
$$

Therefore, by Lemma 3.3, the infinite product in (2.4) converges uniformly on every compact set of $\mathbb{R}$. ¶

Theorem 3.2 *Let $\tau_{0,j}^{m,l,\alpha}(\omega)$ be the mark (2.3), which are defined in (2.4), then the corresponding nonstationary refinable functions $\phi_j \in L_2(\mathbb{R}), j\in\mathbb{N}_0$.*

Proof. By Theorem 3.1, we obtain that

$$\widehat{\phi_j}(\omega)=\lim_{N\to\infty}\prod_{n=1}^{N}\tau_{0,n+j-1}^{m,l,\alpha}(\omega)(2^{-n}\omega)$$

is well defined for almost every $\omega\in\mathbb{R}$. In the following, we claim that

$$|\tau_{0,j}^{m,l,\alpha}(\omega)|^2+|\tau_{0,j}^{m,l,\alpha}(\omega+\pi)|^2\le 1, a.e.\omega\in\mathbb{R}. \tag{3.9}$$

There are two cases to consider:

Case I: Suppose that $\omega=0$. One has

$$|\tau_{0,j}^{m,l,\alpha}(0)|^2+|\tau_{0,j}^{m,l,\alpha}(\pi)|^2=0+1=1, a.e.\omega\in\mathbb{R}.$$

Case II: Suppose that $\omega\ne 0$. Set $E_0=\{0\}$, for $\omega\in\mathbb{R}\setminus E_0$, let $t=2\omega$, ones get

$$
\begin{aligned}
|\tau_{0,j}^{m,l,\alpha}(\omega)|^2 &= 2^{-4}(1+e^{-it})^4|T_{0,j}(t)|^2 \\
&= (1+e^{-it})^4|2^{-2}T_{0,j}(t)|^2.
\end{aligned}
$$

8

TING CHENG AND XIAOYUAN YANG

Applying

$$
\begin{aligned}
B_{0,j}(t) &= \prod_{i=0}^{m+l-1} (\cos^2(\frac{t}{2}) + i\alpha_j) / \prod_{i=1}^{m+l-1} (1 + i\alpha_j) \\
&\leq \prod_{i=1}^{m+l-1} (1 + i\alpha_j) / \prod_{i=1}^{m+l-1} (1 + i\alpha_j) = 1
\end{aligned}
$$

and Lemma 3.5, we obtain

$$
\begin{aligned}
\max_{t \in \mathbb{T}_{\nu,\beth}} 2|2^{-2} T_0^{m,l,\alpha}(t)|^2 &= \max_{t \in \mathbb{T}} 2^{-3} \left| B_{0,j}(t) + \sum_{j=0}^{l} \binom{m+l}{j} B_{k,j}(t) \right|^2 \\
&< \max_{t \in \mathbb{T}} 2^{-3} \left| 1 + \left(\max_{t \in [-\pi,\pi]} B_{k,j}(t) \right) \sum_{k=1}^{l} \binom{m+l}{k} \right|^2 \\
&< \max_{t \in \mathbb{T}} 2^{-3} \left| (1 + (\frac{1}{2})^{m+l-1} (2)^{m+l-1}) \right|^2 = \frac{1}{2}. \qquad (3.10)
\end{aligned}
$$

Bringing Lemma 3.1 and Lemma 3.2 together yields

$$
\rho(\tau_{0,j}^{m,l,\alpha}(t)) \leqslant \rho_4(\tau_{0,j}^{m,l,\alpha}(t), \infty) = \rho_0(2^{-2} T(t), \infty) < 1.
$$

For

$$
\begin{aligned}
\rho_0(2^{-2} T_{0,j}(t), \infty) &= \limsup_{n \to \infty} \| T_{\tau_{0,j}^{m,l,\alpha}(t)}^n \|_{L_\infty(\mathbb{T})}^{1/n} \\
&> T_{\tau_{0,j}^{m,l,\alpha}(t)} \\
&= |\tau_{0,j}^{m,l,\alpha}(\omega)|^2 + |\tau_{0,j}^{m,l,\alpha}(\omega + \pi)|^2.
\end{aligned}
$$

Thus,

$$
|\tau_{0,j}^{m,l,\alpha}(\omega)|^2 + |\tau_{0,j}^{m,l,\alpha}(\omega + \pi)|^2 < 1.
$$

This concludes the claim (3.9).

By Lemma 3.4, the corresponding nonstationary refinable functions $\phi_j \in L_2(\mathbb{R}), j \in \mathbb{N}_0$. ¶

4 Convergence of nonstationary cascade algorithms

In this section, demonstration of the convergence of nonstationary cascade algorithms in the Sobolev space $W_2^\nu(\mathbb{R})$ is given. We will show a lemma about a sufficient condition on the convergence of a nonstationary cascade algorithm which is necessary for the following theorem.

Lemma 4.1 ([17], Proposition 2.6) Let $\widehat{a}_j$ and $\widehat{b}_j (j \in \mathbb{N})$ be 2π-periodic measurable functions such that for all $j \in \mathbb{N}$,

$$
|\widehat{a}_j(\omega)| \leq |\widehat{b}_j(\omega)|, \quad a.e. \ \omega \in \mathbb{R}. \qquad (4.1)
$$

Let $\eta \in W_2^\nu(\mathbb{R})$ such that $\lim_{j \to \infty} \hat{\eta}(2^{-j}\omega) = 1$ for almost every $\omega \in \mathbb{R}$. Define

$$
\widehat{f_n}(\omega) := \hat{\eta}(2^{-n}\omega) \prod_{j=1}^{n} \widehat{a}_j(2^{-n}\omega) \quad and \quad \widehat{g_n}(\omega) := \hat{\eta}(2^{-n}\omega) \prod_{j=1}^{n} \widehat{b}_j(2^{-n}\omega), \omega \in \mathbb{R}.
$$

9

NONSTATIONARY REFINABLE FUNCTIONS BASED ON GENERALIZED

Assume that $\widehat{f_\infty}(\omega) := \lim_{n\to\infty} \prod_{j=1}^{n} \widehat{a_j}(2^{-n}\omega)$ and $\widehat{g_\infty}(\omega) := \lim_{n\to\infty} \prod_{j=1}^{n} \widehat{b_j}(2^{-n}\omega)$ are well defined for almost every $\omega \in \mathbb{R}$. Then, $\lim_{n\to\infty} \|g_n - g_\infty\|_{W_2^\nu(\mathbb{R})} = 0$ implies $\lim_{n\to\infty} \|f_n - f_\infty\|_{W_2^\nu(\mathbb{R})} = 0$. In particular, suppose that there are a positive integer J and a 2π-periodic measurable function $\hat{b}$ such that

$$|\widehat{a_j}(\omega)| \leq |\hat{b}(\omega)|, a.e. \omega \in \mathbb{R}, \ \forall j > J \ and \ \widehat{a_j} \in L_\infty(\mathbb{R}), 1 \leq j \leq J. \tag{4.2}$$

For $n \in \mathbb{N}$, define $\widehat{h_n}(\omega) := \hat{\eta}(2^{-n}\omega) \prod_{j=1}^{n} \widehat{b_j}(2^{-n}\omega)$. If $\{h_n\}_{n=1}^\infty$ converges in $W_2^\nu(\mathbb{R})$, then f_n converges to f_n in $W_2^\nu(\mathbb{R})$, i.e., $\lim_{n\to\infty} \|f_n - f_\infty\|_{W_2^\nu(\mathbb{R})} = 0$.

Theorem 4.1 *Let $\tau_{0,j}^{m,l,\alpha}(\omega)$ be the mark (2.3), which are defined in (2.4), then, for every $n \in \mathbb{N}_0$, the nonstationary cascade algorithm (2.5) associated with $\{\tau_{0,j+n}^{m_j,l_j,\alpha}(\omega)\}_{j=1}^\infty$ converges in $W_2^\nu(\mathbb{R})$, for any $\nu \geq 0$. Consequently, the nonstationary refinable functions $\phi_j^{m_j,l_j,\alpha}$, $j \in \mathbb{N}_0$, in (2.4) must be well-defined compactly supported $C^\infty(\mathbb{R})$ functions.*

Proof. Since $deg(\tau_{0,j}^{m,l,\alpha}(\omega)(\omega)) \leq 2(m_j + l_j) + 1 < 4m_j - 9$, and $\tau_{0,j}^{m,l,\alpha}(\omega)) = 1$, applying $\sum_{j=1}^\infty 2^{-j} m_j < \infty$, we get

$$\sum_{j=1}^\infty 2^{-j} deg(\tau_{0,j}^{m,l,\alpha}(\omega)) \leq \sum_{j=1}^\infty 2^{-j}(4m_j - 9) \leq 4\sum_{j=1}^\infty 2^{-j} m_j < \infty.$$

Moreover, by (3.9), ones obtain $|\tau_{0,j}^{m,l,\alpha}(\omega)| \leq 1$. By using Lemma (3.3), we can derive that $\phi_{0,j}^{m_j,l_j,\alpha}$, $j \in \mathbb{N}_0$ are well defined compactly supported functions.

Because $\tau_{0,j}^{m,l,\alpha}(\omega)$ in the case $\alpha = 0$ have $\nu_2(\tau_{0,j}^{m,l,\alpha}(\omega)) \to \infty$ ([11, 13]). The same proof is carried out for any α. So, there exists a positive integer J such that $\nu_2(\tau_{0,j}^{m,l,\alpha}(\omega)) \geq \nu + 2$. By $\lim_{j\to\infty} m_j = \infty$, there exists a positive integer N such that $m_j > J$ and it is obtained that

$$\nu_2(\tau_{0,j}^{m,l,\alpha}(\omega)) \geq \nu + 2. \tag{4.3}$$

Let $\hat{b}$ be the unique 2π-periodic trigonometric polynomial such that $\tau_{0,j}^{m,l,\alpha}(\omega) = 2^{-1}(1 + e^{-i\omega})\hat{b}(\omega)$. Bringing the definition of $\nu_2(\hat{b})$ and (4.3) together yields $\nu_2(\hat{b}) = \nu_2(\tau_{0,j}^{m,l,\alpha}(\omega)) - 1 \geq \nu + 1 > \nu$, thus, the stationary cascade algorithm associated with the mask $\hat{b}$ converges in $W_2^\nu(\mathbb{R})$ (see [([16]), Theorem 4.3]). Since $|\tau_{0,j}^{m,l,\alpha}(\omega)| \leq |\hat{b}(\omega)|$, applying Lemma (4.1), we derive that the nonstationary cascade algorithm associated with masks $\tau_{0,j}^{m,l,\alpha}(\omega)$ converges in $W_2^\nu(\mathbb{R})$. The same proof is carried out for every $\phi_n^{m_j,l_j,\alpha}$ and for the nonstationary cascade algorithm associated with masks $\{\tau_{0,j}^{m,l,\alpha}(\omega)\}_{j=1}^\infty$. ¶

Remark 4.1 *Notice that when $\alpha_j = 0$ ($j \in \mathbb{N}$), the Theorem 4.1 in this paper is the same as corresponding Theorem 2.8 given in [17].*

5 Construction of nonstationary tight wavelet frames

In this section, we shall construct the symmetric C^∞ tight wavelet frames in $L_2(\mathbb{R})$ with compact support and the spectral frame approximation order based on the masks (2.3). The

10

TING CHENG AND XIAOYUAN YANG

following two lemmas analyze the approximation properties of the operators P_n and relations of $\tau_{0,j}^{m,l,\alpha}(\omega)$, respectively, which are useful for construction of C^∞ tight wavelet frames.

Lemma 5.1 *([17], Theorem 3.2) Let $\widehat{a_j}, j \in \mathbb{N}$ be 2π-periodic measurable functions such that $|\widehat{a_j}(\omega)|^2 + |\widehat{a_j}(\omega + \pi)|^2 \leq 1, a.e.\omega \in \mathbb{R}$ holds for all $j \in \mathbb{N}$ and for every $n \in \mathbb{N}_0$, the function $\widehat{\phi_n}(\omega) := \lim_{J \to \infty} \prod_{j=1}^J \widehat{a_{j+n}}(2^{-j}\omega)$ is well defined for almost every $\omega \in \mathbb{R}$. Let $\nu \geq 0$. If, for $n \in \mathbb{N}$,*

$$\left| 1 - |\widehat{\phi_n}(\omega)|^2 \right|^2 \leq C_{\phi_n}|\omega|^{2\nu}, \quad a.e. \ \omega \in [-\pi, \pi],$$
$$\sum_{k \in \mathbb{Z}\backslash\{0\}} |\widehat{\phi_n}(\omega)|^2 |\widehat{\phi_n}(\omega + 2\pi k)|^2 \leq C_{\widehat{\phi_n}}|\omega|^{2\nu}, \quad a.e. \ \omega \in [-\pi, \pi], \tag{5.1}$$

where C_{ϕ_n} is a constant depending only on ϕ_n, then for the linear operators P_n in (2.6),

$$\|f - P_n(f)\|_{L_2(\mathbb{R})} \leq \max(2, \sqrt{C_{\phi_n}})2^{-\nu n}|f|_{W_2^\nu(\mathbb{R})} \quad \forall f \in W_2^\nu(\mathbb{R}) \quad and \ n \in \mathbb{N}. \tag{5.2}$$

In particular, (5.1) is satisfied if

$$1 - |\widehat{\phi_n}(\omega)|^2 \leq C_{\phi_n}|\omega|^{2\nu}. \tag{5.3}$$

Lemma 5.2 *Let $\tau_{0,j}^{m,l,\alpha}(\omega)$ be the mark (2.3), which are defined in (2.4). For any $0 < \rho \leq 1$ and $\nu \geq 0$, there exist a positive integer N and a positive constant C, (both of them depend only on ρ and ν), such that for all $N \leq \rho m < l \leq m$,*

$$0 \leq 1 - |\tau_{0,j}^{m,l,\alpha}(\omega)|^2 \leq C|\omega|^{2\nu} \quad \forall \omega \in [-\pi, \pi]. \tag{5.4}$$

Proof. Suppose that $\alpha = 0$, the case holds (see [17], Lemma 3.3). Assume that (5.4) holds for $\alpha = k - 1$. Then suppose that $\alpha = k$, since $\tau_{0,j}^{m,l,\alpha}(\omega)$ increases as α increases, we have

$$0 \leq 1 - |\tau_{0,j}^{m,l,\alpha}(\omega)|^2 \leq 1 - |\tau_{0,j}^{m,l,\alpha}(\omega)|^2 \leq C|\omega|^{2\nu}.$$

This completes the claim (5.4). ¶

Theorem 5.1 *Let $\tau_{0,j}^{m,l,\alpha}(\omega)$ be the mark (2.3). For $j \in \mathbb{N}$, define ϕ_{j-1} as in (2.4) and ψ_{j-1}^1, ψ_{j-1}^2, ψ_{j-1}^3 as in (2.7) with the wavelet masks $\widehat{b_j^1}$, $\widehat{b_j^2}$ and $\widehat{b_j^3}$ being derived from $\widehat{a_j}$ in (2.8). Then the following hold:*

(1) *Each nonstationary refinable function $\phi_j, j \in \mathbb{N}_0$, is a compactly supported C^∞ real-valued function that is symmetric about the origin: $\phi_j(-\cdot) = \phi_j$.*

(2) *Each wavelet function $\psi_j^\ell, \ell = 1, 2, 3$ and $j \in \mathbb{N}_0$, is a compactly supported C^∞ real-valued function with l_{j+1} vanishing moments and satisfies $\psi_j^\ell(1 - \cdot) = \psi_j^\ell$ for $\ell = 1, 2$ and $\psi_j^3(1 - \cdot) = -\psi_j^3$.*

(3) *The system $\{\phi_0(\cdot - k) : k \in \mathbb{Z}\} \bigcup \{\psi_{j;j,k}^\ell := 2^{j/2}\psi_j^\ell(2^j \cdot -k) : j \in \mathbb{N}_0, k \in \mathbb{Z}, \ell = 1, 2, 3\}$ is a compactly supported symmetric C^∞ tight wavelet frame in $L_2(\mathbb{R})$.*

11

NONSTATIONARY REFINABLE FUNCTIONS BASED ON GENERALIZED

(4) *If in addition* $\liminf_{j\to\infty} l_j/m_j > 0$, *then the tight wavelet frame in item (3) has the spectral frame approximation order.*

Proof. For item (1), by using Theorem 4.1, it is derived that all $\phi_j, j \in \mathbb{N}_0$ are compactly supported functions in $C^\infty(\mathbb{R})$. Combining all the masks $\widehat{a_j}$, which are 2π-periodic trigonometric polynomials with real coefficients and are symmetric about the origin: $\overline{\widehat{a_j}} = \widehat{a_j}$, and the definition of $\widehat{\phi_j}$ in (2.4) yields,

$$\overline{\widehat{\phi_j}(\omega)} = \widehat{\phi_j}(\omega),$$

which ϕ_j are real-valued. By the definition of $\tau_{0,j}^{m,l,\alpha}(\omega)$ ($j \in \mathbb{N}$) in (2.3), we get

$$1 - |\tau_{0,j}^{m,l,\alpha}(\omega)|^2 = O(|\omega|^{2l}), \omega \to 0. \tag{5.5}$$

$\widehat{b_j^\ell} = O(|\omega|^{l_j}), \omega \to 0$ follows from the fact that (2.10) and (5.5). Thus, ψ_j^ℓ has l_{j+1} vanishing moments.

For item (3), notice that the the definition of $\widehat{b_j^\ell}$ in (2.9), we can straightforward obtain that (2.10). Therefore, the wavelet system in (2.8) is a compactly supported tight wavelet frame in $L_2(\mathbb{R})$ (see [17], Theorem1.1). For item (4), let ν be an arbitrary positive integer and denote $\widehat{d_j} := |\tau_{0,j}^{m,l,\alpha}(\omega)|^2$. For $\liminf_{j\to\infty} l_j/m_j > 0$, there exist a positive integer N and $0 < \rho < \liminf_{j\to\infty} l_j/m_j$ such that $2\nu < N < \rho m_j < l_j \leq m_j$ for all $j \geq N$. By using Lemma 5.2, it is to see that (5.4) holds. That is, there exists a positive constant C, independent of j, such that

$$0 \leq 1 - \widehat{d_j}(\omega) \leq C|\omega|^{2\nu}, \quad \omega \in [-\pi, \pi] \ \ and \ \ j \geq N.$$

For $n \geq N$ and $\ell \in \mathbb{N}$, since $\widehat{d_{\ell+n}}(0) = 1$, ones get

$$|\widehat{d_{\ell+n}}(0) - \widehat{d_{\ell+n}}(2^{-\ell}\omega)| = |1 - \widehat{d_{\ell+n}}(2^{-\ell}\omega)| \leq C2^{-2\nu\ell}|\omega|^{2\nu}, \quad \forall \omega \in [-\pi, \pi].$$

Since $\widehat{d_{\ell+n}}(0) = 1, 0 \leq \widehat{d_{\ell+n}}(\omega) \leq 1$ and (3.9), we derive that

$$0 \leq 1 - |\widehat{\phi_n}(\omega)|^2 \leq \sum_{\ell=1}^{\infty} |\widehat{d_{\ell+n}}(0) - \widehat{d_{\ell+n}}(2^{-\ell}\omega)| \quad \omega \in \mathbb{R}. \tag{5.6}$$

Applying (5.6), it is obtained that

$$1 - |\widehat{\phi_n}(\omega)|^2 \leq C|\omega|^{2\nu} \sum_{\ell=1}^{\infty} 2^{-2\nu\ell}, \quad \omega \in [-\pi, \pi].$$

Thus, (5.3) holds with

$$C_{\phi_n} := C \sum_{\ell=1}^{\infty} 2^{-2\nu\ell} = \frac{C}{1 - 2^{-2\nu}} < \infty.$$

Combining $Q_n = P_n$ and Theorem 5.1, one has

$$\|f - Q_n(f)\|_{L_2(\mathbb{R})} \leq C_1 2^{-\nu n} |f|_{W_2^\nu(\mathbb{R})}, \quad \forall f \in W_2^\nu(\mathbb{R}) \ \ and \ \ n \in \mathbb{N},$$

where $C_1 := \max(2, \sqrt{\frac{C}{1-2^{-2\nu}}})$ is independent of f and n. Since ν is arbitrary, the tight wavelet frame has the desired spectral frame approximation order. ¶

12

Remark 5.1 *Under the condition $\alpha_j = 0$ $(j \in \mathbb{N})$ of Theorem 5.1, one can derive that it is consistent with the claim of Theorem 1.2 given in [17].*

Acknowledgments

References

[1] R.P. Boyer and L.C. Thiel, Generalized Bernstein Polynomials and Symmetric Functions, *Adv. Appl. Math.*, 28,17-39(2001).

[2] C. Costanza, Stationary and nonstationary affine combination of subdivision masks, *Math. Comput. Simulation*, 81,623-635(2010) .

[3] A. Cohen and N. Dyn, Nonstationary Subdivision Schemes and Multiresolution Analysis, *SIAM J. Math. Anal.*, 27,1745-1769(1996).

[4] C. Conti, L. Gori and F. Pitolli, Totally positive functions through nonstationary subdivision schemes, *J. Comput. Appl. Math.*, 200,255-265 (2007) .

[5] W. S. Chen, On the convergence of multidimensional nonstationary subdivision schemes, *Appl. Anal.*, 72,179-189(1999).

[6] W. S. Chen, C. Xu and W. Lin, Spectral approximation orders of multidimensional nontationary biorthogonal semi-multiresolution analysis in sobolev space, *J. Comput. Math.*, 24,81-90(2006).

[7] M. Cecchi and M. Fornasier, Fast homogenization algorithm based on asymptotic theory and multiscale schemes, *Numerical algorithms*, 40,171-186(2005).

[8] N. Dyn, D. Levin and J. Yoon, Analysis of Univariate Nonstationary Subdivision Schemes with Application to Gaussian-Based Interpolatory Schemes, *SIAM J. Math. Anal.*, 39,470-488(2007).

[9] N. Dyn, Exponentials Reproducing Subdivision Schemes, *Found Comput. Math.*, 3,187 (2004).

[10] N. Dyn and D. Levin, Subdivision schemes in geometric modelling, *Acta Numerica*, 11,73-144(2002).

[11] I. Daubechies, *Ten Lectures on Wavelets*, CBMS-NSF Regional Conf. Ser. Appl. Math. 61, SIAM, Philadelphia,1992.

[12] G. Derfel, N. Dynnd and D. Levin, Generalized Refinement Equations and Subdivision Processes, *J. Approx. Theory*, 80,272-297(1995).

[13] I. Daubechies, Orthonormal bases of compactly supported wavelets, *Comm. Pure Appl. Math.*, 41,909-996 (1988).

[14] S. Daniel and P. Shunmugaraj, An approximating C^2 non-stationary subdivision scheme, *Computer Aided Geometric Design*, 26,810-821(2009).

[15] K. Gitta and S. Tomas, Adaptive directional subdivision schemes and shearlet multiresolution analysis, *SIAM J. Math. Anal.*, 41,1436-1471(2009).

[16] B. Han, Vector cascade algorithms and refinable function vectors in Sobolev spaces, *J. Approx. Theory*, 124,44-88(2003).

13

[17] B. Han and Z. Shen, Compactly supported symmetric C^∞ wavelets with spectral approximation order, *SIAM J. Math. Anal.*, 40,905-938(2008).

[18] B. Han and Z.W. Shen, Characterization of Sobolev spaces of arbitrary smoothness using nonstationary tight wavelet frames, *Israe J. Math.*, 172,371-398(2009).

[19] B. Han, Refinable functions and cascade algorithms in weighted spaces with Hölder continuous masks, *SIAM J. Math. Anal.*, 40,70-102(2008).

[20] B. Jeonga, Y. J. Leea and J. Yoon, A family of non-stationary subdivision schemes reproducing exponential polynomials, *J. Math. Anal. Appl.*, 402,207-219(2013).

[21] L. Kobbelt, Using the Discrete Fourier Transform to Analyze the Convergence of Subdivision Schemes, *Appl. Comput. Harmon. A.*, 1,68-91(1998).

[22] S. Li and Y. Shen, Pseudo box splines, *J. Appl. Comput. Harmon. Anal.*, 26,344-356(2009).

[23] L.R. Liang, Approximation order of nonstationary tight wavelet frames in Sobolev spaces, *Int. J. Pure Appl. Math.*, 72,481-489(2011).

[24] G. Laura and P. Francesca, Multiresolution analyses originated from nonstationary subdivision schemes, *J.Comput. Appl. MAth.*, 221,406-415(2008).

[25] Y. Shen and S. Li, Cascade algorithm associated with Hlölder continuous masks. *Appl. Math. Lett.*, 22,1213-1216(2009).

[26] L. Yue, S. Yang and Z. Xia, Multiscale decomposition of divided difference operators and nonstationary subdivision schemes of B-spline series, *Numer. Math. J. Chinese Univ.*, 26,1-11(2004).

The Order and Type of Meromorphic Functions and Entire Functions of Finite Iterated Order [‡]

Jin Tu[1,*], Yun Zeng[1], Hong-Yan Xu[2]

1.College of Mathematics and Information Science, Jiangxi Normal University, Nanchang 330022, China

2. Department of Informatics and Engineering, Jingdezhen Ceramic Institute, Jingdezhen, Jiangxi, 333403, China

Abstract

In this paper, the authors investigate the p-iterated order and p-iterated type of $f_1 + f_2$, $f_1 f_2$, where $f_1(z)$, $f_2(z)$ are meromorphic functions or entire functions with the same p-iterated order and different p-iterated type, and we obtain some results which improve and extend some previous results.

Key words: meromorphic function; entire function; iterated order; iterated type

AMS Subject Classification(2010): 30D35, 30D15

1. Introduction and Notations

In this paper, we assume that readers are familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic functions in the complex plane (e.g. [4, 6-8, 10, 14, 15]). Throughout this paper, by a meromorphic function $f(z)$, we mean a meromorphic function in the complex plane. We use $T(r, f)$ and $M(r, f)$ to denote the characteristic function of a meromorphic function and the maximum modulus of an entire function. In the following, we will recall some notations about meromorphic functions and entire functions.

Definition 1.1. (*see* [4, 8, 10]) The order of a meromorphic function $f(z)$ is defined by

$$\sigma(f) = \varlimsup_{r \to \infty} \frac{\log T(r, f)}{\log r}. \tag{1.1}$$

Especially, if $0 < \sigma(f) < \infty$, then the type of $f(z)$ is defined by

$$\psi(f) = \varlimsup_{r \to \infty} \frac{T(r, f)}{r^{\sigma(f)}}. \tag{1.2}$$

Definition 1.2. (*see* [4, 6 − 8, 10]) The order of an entire function $f(z)$ is defined by

$$\sigma(f) = \varlimsup_{r \to \infty} \frac{\log T(r,f)}{\log r} = \varlimsup_{r \to \infty} \frac{\log \log M(r,f)}{\log r}. \tag{1.3}$$

Especially, if $0 < \sigma(f) < \infty$, then the type of $f(z)$ is defined by

$$\tau(f) = \varlimsup_{r \to \infty} \frac{\log M(r,f)}{r^{\sigma(f)}}. \tag{1.4}$$

The order and type are two important indicators in revealing the growth of meromorphic functions in the complex plane or analytic functions in the unit disc. Many authors have investigated the growth ofmeromorphic functions or analytic functions in the unit disc(e.g. [3,4,7-10,12-15]) and obtain a lot of classical results in the following.

Theorem A. (*see* $[4, 10, 14, 15]$) Let $f_1(z)$ and $f_2(z)$ be meromorphic functions of finite order satisfying $\sigma(f_1) = \sigma_1$ and $\sigma(f_2) = \sigma_2$. Then

$$\sigma(f_1 + f_2) \leq \max\{\sigma_1, \ \sigma_2\}, \ \sigma(f_1 f_2) \leq \max\{\sigma_1, \ \sigma_2\}.$$

Furthermore, if $\sigma_1 \neq \sigma_2$, then $\sigma(f_1 + f_2) = \sigma(f_1 f_2) = \max\{\sigma_1, \sigma_2\}$.

Theorem B. (*see* $[15]$) Let $f_1(z)$ and $f_2(z)$ be meromorphic functions of finite order. Then $\mu(f_1 + f_2) \leq \max\{\sigma(f_1), \ \mu(f_2)\}, \ \mu(f_1 f_2) \leq \max\{\sigma(f_1), \ \mu(f_2)\}$.

Theorem C. (*see* $[11]$) Let $f_1(z)$ and $f_2(z)$ be meromorphic functions of finite order satisfying $\sigma(f_1) < \mu(f_2)$, then $\mu(f_1 + f_2) = \mu(f_1 f_2) = \mu(f_2)$.

Theorem D. (*see* $[7]$) Let $f_1(z)$ and $f_2(z)$ be entire functions of finite order satisfying $\sigma(f_1) = \sigma(f_2) = \sigma$. Then the following two statements hold:
(*i*) If $\tau(f_1) = 0$ and $0 < \tau(f_2) < \infty$, then $\sigma(f_1 f_2) = \sigma$, $\tau(f_1 f_2) = \tau(f_2)$.
(*ii*) If $\tau(f_1) < \infty$ and $\tau(f_2) = \infty$, then $\sigma(f_1 f_2) = \sigma$, $\tau(f_1 f_2) = \infty$.

Theorem E. (*see* $[4, 14, 15]$) Let $f(z)$ be a meromorphic function of finite order, then $\sigma(f') = \sigma(f)$.

From Theorems $A - E$, a natural question is : can we get the similar results for entire functions and meromorphic functions of infinite order (i.e. finite iterated order)? In the following, we recall some notations and definitions of finite iterated order. For $r \in (0, +\infty)$, we define $\exp_1 r = e^r$ and $\exp_{i+1} = \exp(\exp_i r), i \in \mathbb{N}$; for sufficiently large $r \in (0, +\infty)$, we define $\log_1 r = \log r$ and $\log_{i+1} r = \log(\log_i r), r \in \mathbb{N}$; we also denote $\exp_0 r = r = \log_0 r$ and $\exp_{-1} r = \log_1 r$. Throughout this paper, we use p to denote a positive integer. We denote the linear measure of a set $E \subset (0, +\infty)$ by $mE = \int_E dt$ and the logarithmic measure of $E \subset (0, +\infty)$ by $m_l E = \int_E \frac{dt}{t}$.

Definition 1.3. (*see* $[1, 5, 11]$) The p-iterated order and p-iterated lower-order of a meromorphic function $f(z)$ are respectively defined by

$$\sigma_p(f) = \varlimsup_{r \to \infty} \frac{\log_p T(r,f)}{\log r}; \ \mu_p(f) = \varliminf_{r \to \infty} \frac{\log_p T(r,f)}{\log r}. \tag{1.5}$$

Definition 1.4. (*see* $[1, 5, 11]$) The p-iterated order and p-iterated lower-order of an entire

function $f(z)$ are respectively defined by

$$\sigma_p(f) = \varlimsup_{r\to\infty} \frac{\log_{p+1} M(r,f)}{\log r} = \varlimsup_{r\to\infty} \frac{\log_p T(r,f)}{\log r}; \tag{1.6}$$

$$\mu_p(f) = \varliminf_{r\to\infty} \frac{\log_{p+1} M(r,f)}{\log r} = \varliminf_{r\to\infty} \frac{\log_p T(r,f)}{\log r}. \tag{1.7}$$

Definition 1.5. Let $f(z)$ be a meromorphic function satisfying $0 < \sigma_p(f) = \sigma < \infty$ or $0 < \mu_p(f) = \mu < \infty$. Then the p-iterated type of order and the p-iterated lower-type of lower-order of $f(z)$ are respectively defined by

$$\psi_p(f) = \varlimsup_{r\to\infty} \frac{\log_{p-1} T(r,f)}{r^\sigma}; \ \underline{\psi}_p(f) = \varliminf_{r\to\infty} \frac{\log_{p-1} T(r,f)}{r^\mu}. \tag{1.8}$$

Definition 1.6. Let $f(z)$ be an entire function satisfying $0 < \sigma_p(f) = \sigma < \infty$ or $0 < \mu_p(f) = \mu < \infty$. Then the p-iterated type of order and the p-iterated lower-type of lower-order of $f(z)$ are respectively defined by

$$\tau_p(f) = \varlimsup_{r\to\infty} \frac{\log_p M(r,f)}{r^\sigma}; \ \underline{\tau}_p(f) = \varliminf_{r\to\infty} \frac{\log_p M(r,f)}{r^\mu}. \tag{1.9}$$

From the above definitions, we can easily obtain the following propositions:

(i) If $f_1(z)$ and $f_2(z)$ are meromorphic functions with $\sigma_p(f_1) = \sigma_1 < \infty$ and $\sigma_p(f_2) = \sigma_2 < \infty$, then $\sigma_p(f_1 + f_2) \le \max\{\sigma_1, \sigma_2\}, \sigma_p(f_1 f_2) \le \max\{\sigma_1, \sigma_2\}$. Furthermore, if $\sigma_1 \neq \sigma_2$, then $\sigma_p(f_1 + f_2) = \sigma_p(f_1 f_2) = \max\{\sigma_1, \sigma_2\}$.

(ii) If $f_1(z)$ and $f_2(z)$ are meromorphic functions with $\sigma_p(f_1) < \infty$ and $\mu_p(f_2) < \infty$, then $\max\{\mu_p(f_1 + f_2), \mu_p(f_1 f_2)\} \le \max\{\sigma_p(f_1), \mu_p(f_2)\}$ or $\max\{\mu_p(f_1 + f_2), \mu_p(f_1 f_2)\} \le \max\{\mu_p(f_1), \sigma_p(f_2)\}$.

(iii) If $f_1(z)$ and $f_2(z)$ are meromorphic functions satisfying $\sigma_p(f_1) < \mu_p(f_2) \le \infty$, then $\mu_p(f_1 + f_2) = \mu_p(f_1 f_2) = \mu_p(f_2)$.

(iv) If $f(z)$ is an entire function with $0 < \sigma_p(f) < \infty$, then $\psi_p(f) = \tau_p(f), \underline{\psi}_p(f) = \underline{\tau}_p(f)$ for $p \ge 2$ and $\psi(f) \le \tau(f), \underline{\psi}(f) \le \underline{\tau}(f)$ for $p = 1$.

2.   Main Results

Here our first question is : can we get the similar results as Theorem D for meromorphic function or entire function of finite iterated order? Since it is easy to see $\sigma_p(f') = \sigma_p(f)(p \ge 1)$ for meromorphic function $f(z)$ of finite iterated order; our second question is : can we get $\psi_p(f') = \psi_p(f)$ or $\tau_p(f') = \tau_p(f)$ for meromorphic function or entire function of finite iterated order? In fact, we obtain the following results.

Theorem 2.1. Let $f_1(z)$ and $f_2(z)$ be meromorphic functions satisfying $0 < \sigma_p(f_1) = \sigma_p(f_2) = \sigma < \infty$, $0 \le \psi_p(f_1) < \psi_p(f_2) \le \infty$. Set $\alpha = \psi(f_2) - \psi(f_1)$, $\beta = \psi(f_1) + \psi(f_2)$, then
(i) $\sigma_p(f_1 + f_2) = \sigma_p(f_1 f_2) = \sigma(p \ge 1)$; ($ii$) If $p > 1$, we have $\psi_p(f_1 + f_2) = \psi_p(f_1 f_2) = \psi_p(f_2)$;
(iii) If $p = 1$, we have $\alpha \le \psi(f_1 + f_2) \le \beta$, $\alpha \le \psi(f_1 f_2) \le \beta$.

Theorem 2.2. Let $f_1(z)$ and $f_2(z)$ be entire functions satisfying $0 < \sigma_p(f_1) = \sigma_p(f_2) = \sigma < \infty$, $0 \leq \tau_p(f_1) < \tau_p(f_2) \leq \infty$. Then the following statements hold:
(i) If $p \geq 1$, then $\sigma_p(f_1 + f_2) = \sigma$, $\tau_p(f_1 + f_2) = \tau_p(f_2)$;
(ii) If $p > 1$, then $\sigma_p(f_1 f_2) = \sigma$, $\tau_p(f_1 f_2) = \tau(f_2)$.

Remark 2.1. When $p = 1$, (ii) of Theorem 2.2 does not hold. For example, set $f_1 = e^{-z}, f_2 = e^{2z}$ satisfy $\tau(f_1) = 1 < \tau(f_2) = 2$, but $\tau(f_1 f_2) = 1 < \tau(f_2) = 2$.

Theorem 2.3. Let $f_1(z)$ and $f_2(z)$ be meromorphic functions satisfying $\sigma_p(f_1) = \mu_p(f_2) = \mu$ $(0 < \mu < \infty)$, $0 \leq \psi_p(f_1) < \underline{\psi}_p(f_2) \leq \infty$. Then the following statements hold:
(i) $\mu_p(f_1 + f_2) = \mu_p(f_1 f_2) = \mu (p \geq 1)$; (ii) If $p > 1$, then $\underline{\psi}_p(f_1 + f_2) = \underline{\psi}_p(f_1 f_2) = \underline{\psi}_p(f_2)$;
(iii) If $p = 1$, then $\underline{\psi}(f_2) - \psi(f_1) \leq \max\{\underline{\psi}(f_1 + f_2), \underline{\psi}(f_1 f_2)\} \leq \psi(f_1) + \underline{\psi}(f_2)$.

Theorem 2.4. Let $f_1(z)$ and $f_2(z)$ be entire functions satisfying $\sigma_p(f_1) = \mu_p(f_2) = \mu$ $(0 < \mu < \infty)$, $0 \leq \tau_p(f_1) < \underline{\tau}_p(f_2) \leq \infty$. Then the following statements hold:
(i) $\mu_p(f_1 + f_2) = \mu_p(f_1 f_2) = \mu (p \geq 1)$; (ii) If $p \geq 1$, then $\underline{\tau}_p(f_1 + f_2) = \underline{\tau}_p(f_2)$;
(iii) If $p > 1$, then $\underline{\tau}_p(f_1 f_2) = \underline{\tau}_p(f_2)$; if $p = 1$, then $\underline{\tau}(f_2) - \tau(f_1) \leq \underline{\tau}(f_1 f_2) \leq \tau(f_1) + \underline{\tau}(f_2)$.

Theorem 2.5. Let $p > 1$, $f(z)$ be a meromorphic function satisfying $0 < \sigma_p(f) < \infty$. Then $\psi_p(f') = \psi_p(f)$.

Theorem 2.6. Let $p \geq 1$, $f(z)$ be an entire function satisfying $0 < \sigma_p(f) < \infty$. Then $\tau_p(f') = \tau_p(f)$.

3.   Preliminary Lemmas

Lemma 3.1. $(see\ [11])$ Let $f(z)$ be an entire function of p-iterated order satisfying $0 < \sigma_p(f) = \sigma < \infty$, $0 < \tau_p(f) = \tau \leq \infty$. Then for any given $\beta < \tau$, there exists a set $E \subset [1, +\infty)$ having infinite logarithmic measure such that for all $r \in E$, we have

$$\log_p M(r, f) > \beta r^\sigma.$$

Lemma 3.2. $(see\ [7])$ Let $f(z)$ be an analytic function in the circle $|z| \leq R$ and has no zeros in this circle, and if $f(0) = 1$, then its modulus in the circle $|z| \leq r < R$ satisfies the inequality

$$\ln |f(z)| \geq -\frac{2r}{R-r} \ln M(R, f).$$

Lemma 3.3. $(see\ [2])$ Given any number $H > 0$ and complex numbers $a_1, a_2, \cdots, a_n$, there is a system of circles in the complex plane, with the sum of the radii equal to $2H$, such that for each point z lying outside these circles one has the inequality

$$\prod_{k=1}^{n} |z - a_k| \geq \left(\frac{H}{e}\right)^n.$$

Lemma 3.4. Let $f(z)$ be an analytic function in the circle $|z| \leq \beta R$ $(\beta > 1)$ with $f(0) = 1$, and let ε be an arbitrary positive number not exceeding 2. Then inside the circle $|z| \leq R$, but outside of a family of excluded circles the sum of whose radii is not greater than $2\varepsilon e \beta R$, we have

$$\ln |f(z)| > -\left(\frac{2}{\beta - 1} + \frac{\ln 2 - \ln \varepsilon}{\ln \beta} \right) \ln M(\beta^2 R, f).$$

Proof. By the similar proof in [7, p.21], constructing the function

$$h(z) = \frac{(-\beta R)^n}{a_1 a_2 \cdots a_n} \prod_{k=1}^{n} \frac{\beta R(z - a_k)}{(\beta R)^2 - \overline{a}_k z},$$

where $a_1, a_2, \cdots, a_n$ are the zeros of $f(z)$ in the circle $|z| < \beta R$, then we have $h(0) = 1$ and $|h(\beta R e^{i\theta})| = \frac{(\beta R)^n}{|a_1 a_2 \cdots a_n|} > 1$, then function $g(z) = \frac{f(z)}{h(z)}$ has no zeros in the circle $|z| < \beta R$. Therefore, by Lemma 3.2, for any z satisfying $|z| \leq R < \beta R$, we have

$$\begin{aligned}
\ln |g(z)| &\geq -\frac{2R}{\beta R - R} \ln M(\beta R, g) \\
&= -\frac{2}{\beta - 1} (\ln M(\beta R, f) - \ln |h(\beta R e^{i\theta})|) \\
&\geq -\frac{2}{\beta - 1} \ln M(\beta R, f).
\end{aligned} \tag{3.1}$$

For $|z| \leq \beta R$, we get $\prod_{k=1}^{n} |(\beta R)^2 - \overline{a}_k z| \leq [2(\beta R)^2]^n = 2^n (\beta R)^{2n}$. By Lemma 3.3, we get outside of a family of excluded circles the sum of whose radii are not greater than $2\varepsilon e \beta R$, we have

$$\prod_{k=1}^{n} |\beta R(z - a_k)| > (\beta R)^n (\beta \varepsilon R)^n = \varepsilon^n (\beta R)^{2n},$$

where $n = n(\beta R)$ denotes the number of zeros of $f(z)$ in $|z| < \beta R$. So we have

$$|h(z)| \geq \frac{(\beta R)^n}{|a_1 a_2 \cdots a_n|} \frac{\varepsilon^n (\beta R)^{2n}}{2^n (\beta R)^{2n}} \geq \left(\frac{\varepsilon}{2} \right)^n. \tag{3.2}$$

Since $0 < \varepsilon < 2$, by (3.2), we have

$$\ln |\psi(z)| \geq -n \ln \frac{2}{\varepsilon}. \tag{3.3}$$

On the other hand, by Jensen's formula, we have

$$M(\beta^2 R, f) \geq \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} \log |f(\beta^2 R e^{i\theta})| d\theta \right\} = \prod_{k=1}^{n(\beta^2 R)} \frac{\beta^2 R}{|a_k|} \geq \prod_{k=1}^{n(\beta R)} \frac{\beta^2 R}{|a_k|} \geq \beta^n.$$

Therefore,

$$n \leq \frac{\ln M(\beta^2 R, f)}{\ln \beta}. \tag{3.4}$$

6   J. Tu, Y. Zeng, H. Y. Xu

By (3.1), (3.3) and (3.4), we have

$$\ln |f(z)| \geq -\frac{2}{\beta - 1} \ln M(\beta R, f) - \frac{\ln 2 - \ln \varepsilon}{\ln \beta} \ln M(\beta^2 R, f)$$

$$\geq -\left(\frac{2}{\beta - 1} + \frac{\ln 2 - \ln \varepsilon}{\ln \beta} \right) \ln M(\beta^2 R, f). \qquad (3.5)$$

Lemma 3.5. Let $p > 1$, $f(z)$ be an entire function satisfying $0 < \sigma_p(f) = \sigma < \infty$ and $\tau_p(f) < \infty$. Then for any given $\varepsilon > 0$, there exists a set $E \subset (0, +\infty)$ having finite logarithmic measure, such that for all $|z| = r \notin E$, we have

$$\exp\{-\exp_{p-1}\{(\tau_p(f) + \varepsilon)r^\sigma\}\} < |f(z)| < \exp_p\{(\tau_p(f) + \varepsilon)r^\sigma\}.$$

Proof. By (1.9), for any given $\varepsilon > 0$ and for sufficiently large r, we have

$$|f(z)| < \exp_p\{(\tau_p(f) + \varepsilon)r^\sigma\}, \quad M(\beta^2 r, f) < \exp_p\{(\tau_p(f) + \frac{\varepsilon}{2})\beta^{2\sigma}r^\sigma\} \ (\beta > 1). \qquad (3.6)$$

For any given $\varepsilon (0 < \varepsilon < 2)$ and any $\beta > 1$, we choose ε, β satisfying $(\tau_p(f) + \frac{\varepsilon}{2})\beta^{2\sigma} < \tau_p(f) + \varepsilon$, by Lemma 3.4 and (3.5), there exists a set $E \subset (0, +\infty)$ having finite logarithmic measure, such that for all $|z| = r \notin E$, we have

$$|f(z)| > \exp\{-\exp_{p-1}\{(\tau_p(f) + \varepsilon)r^\sigma\}\} \ \ (p > 1). \qquad (3.7)$$

By (3.6), (3.7), we obtain that Lemma 3.5 holds.

Lemma 3.6. (see [4, 14]) Let $f(z)$ be a meromorphic function satisfying $f(0) \neq \infty$. Then for any $\tau > 1$ and $r > 0$, we have

$$T(r, f) < C_\tau T(\tau r, f') + \log^+(\tau r) + 4 + \log^+ |f(0)|,$$

where $C_\tau > 0$ is a constant related to τ.

Lemma 3.7. (see [6]) Let $g : (0, \infty) \to R$ and $h : (0, \infty) \to R$ be monotone non-decreasing functions such that $g(r) \leq h(r)$ outside of an exceptional set E of finite linear measure. Then for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \leq h(\alpha r)$ for all $r > r_0$.

4. Proofs of Theorems 2.1 - 2.6

Proof of Theorem 2.1. (i) Without loss of generality, set $0 \leq \psi_p(f_1) < \psi_p(f_2) < \infty$, by (1.8), for any $\varepsilon > 0$ and for sufficiently large r, we have

$$T(r, f_1 + f_2) \leq T(r, f_1) + T(r, f_2) + \ln 2$$

$$\leq \exp_{p-1}\{(\psi_p(f_1) + \varepsilon)r^\sigma\} + \exp_{p-1}\{(\psi_p(f_2) + \varepsilon)r^\sigma\}$$

$$\leq 2\exp_{p-1}\{(\psi_p(f_2) + \varepsilon)r^\sigma\} \qquad (4.1)$$

The Order and Type of Meromorphic Functions and Entire Functions of Finite Iterated Order   7

By (4.1), we get $\sigma_p(f_1 + f_2) \le \sigma$. On the other hand, by (1.8), for any $\varepsilon(0 < 2\varepsilon < \psi_p(f_2) - \psi_p(f_1))$, there exists a sequence $\{r_n\}_{n=1}^{\infty}$ tending to ∞ such that

$$T(r_n, f_1 + f_2) \ge T(r_n, f_2) - T(r_n, f_1) - \ln 2$$
$$\ge \exp_{p-1}\{(\psi_p(f_2) - \varepsilon)r_n^{\sigma}\} - \exp_{p-1}\{(\psi_p(f_1) + \varepsilon)r_n^{\sigma}\} \qquad (4.2)$$

holds for sufficiently large r_n. By (4.2), we get $\sigma_p(f_1 + f_2) \ge \sigma$; therefore $\sigma_p(f_1 + f_2) = \sigma$ $(p \ge 1)$.
(ii)-(iii) By (4.1) and (4.2), we have $\psi_p(f_1 + f_2) = \psi_p(f_2)$ for $p > 1$ and $\alpha \le \psi(f_1 + f_2) \le \beta$ for $p = 1$. Since

$$T(r, f_1 f_2) \le T(r, f_1) + T(r, f_2), \ T(r, f_1 f_2) \ge T(r, f_2) - T(r, f_1) - o(1)$$

and by the similar proof in (4.1) and (4.2), we can easily obtain that the conclusions in cases (ii)-(iii) holds.
By the above proof, we can easily obtain that Theorem 2.1 also holds for $0 \le \psi_p(f_1) < \psi_p(f_2) = \infty$.

Proof of Theorem 2.2. (i) Set $0 \le \tau_p(f_1) < \tau_p(f_2) < \infty$. By (1.9), for any $\varepsilon > 0$ and for sufficiently large r, we have

$$M(r, f_1 + f_2) \le M(r, f_1) + M(r, f_2)$$
$$\le \exp_p\{(\tau_p(f_1) + \varepsilon)r^{\sigma}\} + \exp_p\{(\tau_p(f_2) + \varepsilon)r^{\sigma}\}$$
$$\le 2\exp_p\{(\tau_p(f_2) + \varepsilon)r^{\sigma}\}, \qquad (4.3)$$

by (4.3), we get $\sigma_p(f_1 + f_2) \le \sigma$, $\tau_p(f_1 + f_2) \le \tau_p(f_2)$. On the other hand, by (1.9), for any ε $(0 < 2\varepsilon < \tau_p(f_2) - \tau_p(f_1)$ there exists a sequence $\{r_n\}_{n=1}^{\infty}$ tending to ∞ such that

$$M(r_n, f_1) < \exp_p\{(\tau_p(f_1) + \varepsilon)r_n^{\sigma}\}, \ M(r_n, f_2) > \exp_p\{(\tau_p(f_2) - \varepsilon)r_n^{\sigma}\} \qquad (4.4)$$

holds for sufficiently large r_n. In each circle $|z| = r_n (n = 1, 2, \cdots)$, we choose a sequence $\{z_n\}_{n=1}^{\infty}$ satisfying $|f_2(z_n)| = M(r_n, f_2)$ such that

$$M(r_n, f_1 + f_2) \ge |f_1(z_n) + f_2(z_n)| \ge |f_2(z_n)| - |f_1(z_n)|$$
$$\ge M(r_n, f_2) - M(r_n, f_1)$$
$$\ge \exp_p\{(\tau_p(f_2) - \varepsilon)r_n^{\sigma}\} - \exp_p\{(\tau_p(f_1) + \varepsilon)r_n^{\sigma}\}$$
$$\ge \frac{1}{2}\exp_p\{(\tau_p(f_2) - \varepsilon)r_n^{\sigma}\} \ (r_n \to \infty), \qquad (4.5)$$

by (4.5), we get $\sigma_p(f_1 + f_2) \ge \sigma$, $\tau_p(f_1 + f_2) \ge \tau_p(f_2)$; therefore $\sigma_p(f_1 + f_2) = \sigma$, $\tau_p(f_1 + f_2) = \tau_p(f_2)$.
(ii) By (1.9), for any $\varepsilon > 0$ and for sufficiently large r, we have

$$M(r, f_1 f_2) \le M(r, f_1)M(r, f_2)$$
$$\le \exp_p\{(\tau_p(f_1) + \varepsilon)r^{\sigma}\}\exp_p\{(\tau_p(f_2) + \varepsilon)r^{\sigma}\}$$
$$\le \exp_p\{(\tau_p(f_2) + 2\varepsilon)r^{\sigma}\} \ (p > 1), \qquad (4.6)$$

Jin Tu et al 994-1003

by (4.6), we get $\sigma_p(f_1 f_2) \leq \sigma$, $\tau_p(f_1 f_2) \leq \tau_p(f_2)(p > 1)$. On the other hand, by Lemma 3.1, for any $\varepsilon > 0$ there exists a set E_1 having infinite logarithmic measure such that for all $r \in E_1$, we have

$$M(r, f_2) > \exp_p\{(\tau_p(f_2) - \varepsilon)r^\sigma\}. \tag{4.7}$$

By Lemma 3.5, for any $\varepsilon > 0$, there exists a set E_2 having finite logarithmic measure such that $|z| = r \notin E_2$, we have

$$|f_1(z)| > \exp\{-\exp_{p-1}\{(\tau_p(f_1) + \varepsilon)r^\sigma\}\} \ (p > 1). \tag{4.8}$$

Therefore, by (4.7) and (4.8), for any $\varepsilon > 0$ and for all $|z| = r \in E_1 \backslash E_2$, we have

$$\begin{aligned}
M(r, f_1 f_2) &\geq M(r, f_2) \exp\{-\exp_{p-1}\{(\tau_p(f_1) + \varepsilon)r^\sigma\}\} \\
&\geq \exp_p\{(\tau_p(f_2) - \varepsilon)r^\sigma\} \exp\{-\exp_{p-1}\{(\tau_p(f_1) + \varepsilon)r^\sigma\}\}. \tag{4.9}
\end{aligned}$$

Since $\tau_p(f_1) < \tau_p(f_2)$, we choose ε satisfying $\tau_p(f_2) - \varepsilon > \tau_p(f_1) + \varepsilon$. By (4.9), for any $\varepsilon(0 < 2\varepsilon < \tau_p(f_2) - \tau_p(f_1))$ and for all $r \in E_1 \backslash E_2$, we have

$$M(r, f_1 f_2) > \exp_p\{(\tau_p(f_2) - 2\varepsilon)r^\sigma\} \ \ (p > 1). \tag{4.10}$$

By (4.10), we have $\sigma_p(f_1 f_2) \geq \sigma, \tau_p(f_1 f_2) \geq \tau_p(f_2)$ for $p > 1$; Therefore $\sigma_p(f_1 f_2) = \sigma$, $\tau_p(f_1 f_2) = \tau_p(f_2)$ for $p > 1$.

By the similar proof in cases (i) and (ii), we can easily obtain that Theorem 2.2 holds for $0 \leq \tau_p(f_1) < \tau_p(f_2) = \infty$.

Proof of Theorem 2.3. Set $0 \leq \psi_p(f_1) < \underline{\psi}_p(f_2) < \infty$. By (1.8), for any $\varepsilon > 0$ and for sufficiently large r, we have

$$T(r, f_1) < \exp_{p-1}\{(\psi_p(f_1) + \varepsilon)r^\mu\}, \ T(r, f_2) > \exp_{p-1}\{(\underline{\psi}_p(f_2) - \varepsilon)r^\mu\}. \tag{4.11}$$

By (4.11), for any ε $(0 < 2\varepsilon < \underline{\psi}_p(f_2) - \psi_p(f_1))$ and for sufficiently large r, we have

$$\begin{aligned}
T(r, f_1 + f_2) &\geq T(r, f_2) - T(r, f_1) - \ln 2 \\
&\geq \exp_{p-1}\{(\underline{\psi}_p(f_2) - \varepsilon)r^\mu\} - \exp_{p-1}\{(\psi_p(f_1) + \varepsilon)r^\mu\}. \tag{4.12}
\end{aligned}$$

By (4.12), we have

$$\mu_p(f_1 + f_2) \geq \mu(p \geq 1), \ \underline{\psi}_p(f_1 + f_2) \geq \underline{\psi}_p(f_2)(p > 1), \ \underline{\psi}(f_1 + f_2) \geq \underline{\psi}(f_2) - \underline{\psi}(f_1). \tag{4.13}$$

On the other hand, by (1.8), for any $\varepsilon > 0$ there exists a sequence $\{r_n\}_{n=1}^\infty$ tending to ∞ such that

$$T(r_n, f_2) < \exp_{p-1}\{(\underline{\psi}_p(f_2) + \varepsilon)r_n^\mu\} \tag{4.14}$$

for sufficiently large r_n. By (4.11) and (4.14), for any $\varepsilon > 0$ and for sufficiently large r_n, we have

$$\begin{aligned}
T(r, f_1 + f_2) &\leq T(r_n, f_1) + T(r_n, f_2) + \ln 2 \\
&\leq \exp_{p-1}\{(\psi_p(f_1) + \varepsilon)r_n^\mu\} + \exp_{p-1}\{(\underline{\psi}_p(f_2) + \varepsilon)r_n^\mu\}. \tag{4.15}
\end{aligned}$$

By (4.15), we have

$$\mu_p(f_1 + f_2) \le \mu(p \ge 1), \ \underline{\psi}_p(f_1 + f_2) \le \underline{\psi}_p(f_2)(p > 1), \ \underline{\psi}(f_1 + f_2) \le \underline{\psi}(f_2) + \underline{\psi}(f_1). \qquad (4.16)$$

Therefore, by (4.13) and (4.16), the conclusions of Theorem 2.3 hold for $f_1 + f_2$. Since $T(r, f_1 f_2) \le T(r, f_1) + T(r, f_2), T(r, f_1 f_2) \ge T(r, f_2) - T(r, f_1) - o(1)$, we can easily obtain that the conclusions of Theorem 2.3 hold for $f_1 f_2$.

Theorem 2.3 also holds for $0 \le \psi_p(f_1) < \underline{\psi}_p(f_2) = \infty$ by the above proof.

Proof of Theorem 2.4. We can obtain the conclusions of Theorem 2.4 by the similar proof in Theorem 2.2 and Theorem 2.3.

Proof of Theorem 2.5. By the Lemma of logarithmic derivative, we have that

$$T(r, f') \le 3T(r, f) + O\{\log r\}$$

holds outside of an exceptional set E of finite linear measure. By Lemma 3.7, there exists $\alpha > 1$ such that $T(r, f') \le 3T(\alpha r, f) + O\{\log(\alpha r)\}$ holds for sufficiently large r, so we have $\tau_p(f') \le \tau_p(f)$ $(p > 1)$. On the other hand, by Lemma 3.6, for any $\tau > 1$, there exists a constant C_τ such that

$$T(r, f) < C_\tau T(\tau r, f') + \log^+(\tau r) + 4 + \log^+ |f(0)|.$$

Set $\tau \to 1^+$, by the above inequality, we have $\tau_p(f') \ge \tau_p(f)(p > 1)$. Therefore $\tau_p(f') = \tau_p(f)$ for $p > 1$.

Proof of Theorem 2.6. For an entire function $f(z)$, we have

$$f(z) = f(0) + \int_0^z f'(\zeta)d\zeta, \qquad (4.17)$$

where the integral route is a line from 0 to z. By (4.17), we have

$$M(r, f) \le |f(0)| + |\int_0^z f'(\zeta)d\zeta| \le |f(0)| + rM(r, f'), \qquad (4.18)$$

By (4.18), we have

$$M(r, f') \ge \frac{M(r, f) - |f(0)|}{r}. \qquad (4.19)$$

By (1.9) and (4.19), we have $\tau_p(f') \ge \tau_p(f)$. On the other hand, by Cauchy's integral formula, we have

$$f'(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^2} d\zeta, \qquad (4.20)$$

where integral curve is the circle $|\zeta| = R$. By (4.20), for any $|z| = r < R$,

$$M(r, f') \le \frac{1}{2\pi} \int_C \left\{ \max \left| \frac{f(\zeta)}{(\zeta - z)^2} \right| \right\} |d\zeta| \le \frac{1}{2\pi} \frac{M(R, f)}{(R - r)^2} \cdot 2\pi R. \qquad (4.21)$$

Set $R = \beta r$ $(\beta > 1)$, then by (4.21), we have

$$M(r, f') \le \frac{\beta}{(\beta - 1)^2 r} M(\beta r, f) \le \frac{\beta}{(\beta - 1)^2 r} \exp_p\{(\tau_p(f) + \varepsilon)(\beta r)^{\sigma_p(f)}\}, \quad r \to \infty. \qquad (4.22)$$

Since $\sigma_p(f') = \sigma_p(f)$, set $\beta \to 1$, we have $\tau_p(f') \le \tau_p(f)$; therefore $\tau_p(f') = \tau_p(f)$.

References

[1] L.G. Bernal, On Growth k-order of Solutions of a Complex Homongeneous Linear Differential Equantials, Proc. Amer. Math. Soc. 101 (1987) 317-322.

[2] H. Cartan, Sur les systemes de fonctions holomorphes a varietes lineaires lacunaires et leurs applications, Ann. Sci.Ecole Norm. Sup. (3) 45 (1928), 255-346.

[3] A. Edrei and W. H. J. Fuchs, On the growth of meromorphic functions with several deficient values, Trans. Amer. Math. Soc. 93 (2) (1959), 292-328.

[4] W. Hayman, Meromorphic Functions, Clarendon Press, Oxford, 1964.

[5] L. Kinnunen, Linear differential equations with solutions of finite iterated order, Southeast Asian Bull. Math. (4) 22 (1988), 385-405.

[6] I. Laine, Nevanlinna Theory and Complex Differential Equations, Walter de Gruyter, Berlin, 1993.

[7] B. Ja. Levin, Distribution of Zeros of Entire Functions, revised edition, Transl. Math. Monographs Vol. 5, Amer. Math. Soc. Providence, 1980.

[8] R. Nevanlinna, Analytic Function, Springer-Verlag, New-York, 1970.

[9] W. J. Pugh and S. M. Shah, On the growth of entire functions of bounded index, Pacific J. Math. 33 (1) (1970), 191-201.

[10] M. Tsuji, Potential Theory in Modern Function Theory, Chelsea, New York, 1975, reprint of the 1959 edition.

[11] J. Tu, H. Y. Xu, C. Y. Zhang, On the zeros of solutions of any order of derivative of second order linear differential equations taking small functions, E. J. Qualitative Theory of Diff. Equ., No. 23 (2011), 1-17.

[12] J. Tu, H. X. Huang, H. Y. Xu, C. F. Chen, The order and type of of meromorphic functions and analytic functions in the unit disc. J. Jiangxi Norm. Univ., Nat. Sci. 37 (5), (2013), 52-55.

[13] J. Tu, J. S. Wei, H. Y. Xu, The order and type of meromorphic functions and analytic functions of [p,q]-$\varphi(r)$ order in the unit disc, J. Jiangxi Norm. Univ., Nat. Sci. 39 (2), (2015), 207-211.

[14] L. Yang, Vaule Distribution Theory and Its New Research, Science Press, Beijing, 1982(in Chinese).

[15] H. X. Yi, C. C.Yang, Uniqueness Theory of Meromorphic Functions, Science Press/Kluwer Academic, Beijing/New York (1995/2003).

TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 21, NO. 5, 2016

TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 21, NO. 5, 2016

(continued)