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Journal of Computational Analysis and Applications

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Second order duality for multiobjective optimization problems

Meraj Ali Khan, Falleh R. Al-Solamy

Abstract

In this paper, we first introduce a new class of generalized convex functions, called second order (F, α, ρ, d)-V-convex functions and then discuss appropriate duality results for second order Mangasarian type, Mond-Weir type and general Mond-Weir type multiobjective duals.

Keywords: Multiobjective optimization; Second order generalized convex functions; Weak efficiency; Duality

Mathematics Subject Classification 2000: 90C29, 90C30, 90C46, 49N15

1. Introduction

Optimization theory is one of the most lively and exciting branch in modern mathematics, in which the importance of convexity is well known. But the notion of convexity does no longer suffice for many mathematical models used in decision sciences, economics, management sciences, stochastics, applied mathematics and engineering. Therefore, various generalizations of convex functions have been provided for the validity of results to larger classes of optimization problems. The generalization of convex functions was originally proposed by Hanson [7], which were named as invex functions by Craven [4], and η-convex functions by Kaul and Kaur [10]. In [9], Jeyakumar and Mond introduced V-invexity and its generalization for vector functions. More specifically, Preda [16] introduced the concept of (F, ρ)-convexity, an extension of F-convexity defined by Hanson and Mond [8] and ρ-convexity given by Vial [17]. Recently, Agarwal et al. [1] introduced a new class of generalized V-type I functions for a multiobjective problem and discussed sufficiency and duality results.

Second order duality was first introduced by Mangasarian [11] for a scalar programming problem. Mond [13] reproved second order duality results of Mangasarian [11] under simpler assumptions, and showed that the second order dual has computational advantages over the first order dual. Zhang and Mond [19] extended the class of (F, ρ)-convex functions to second order (F, ρ)-convex functions and discussed duality results for Mangasarian type, Mond-Weir type and general Mond-Weir type multiobjective duals. Aghezzaf [2] introduced new classes of generalized second order (F, ρ)-convexity for vector-valued functions and established various duality results for mixed type vector dual. In [6], Hachimi and Aghezzaf proposed a new class of generalized second order type I vector-valued functions for multiobjective programming problem and obtained mixed type duality theorems. Ahmad and Husain [3] defined a class of generalized second order (F, α, ρ, d)-convex functions and established duality results for Mond-Weir type multiobjective dual. Gulati and Agarwal [5] established Huard type converse duality theorems for second-order scalar and multiobjective dual problems showing certain inconsistencies in the earlier work of Yang et al. [18] and Mond and Zhang [15].

Being inspired by the excellent work of Mond and Zhang [15], Zhang and Mond [19] and Ahmad and Husain [3], we introduce the concept of second order $(\mathcal{F}, \alpha, \rho, d)$-V-convex function and its generalizations, which includes most of the introduced classes of generalized convex functions. To characterize the introduced definitions, an example of second order (F, α, ρ, d)-V-convex function is given. Weak, strong and strict converse duality theorems are proved for second order Mangasarian type, Mond-Weir type and general Mond-Weir type multiobjective duals. These results extend the results appeared in [3, 15, 16, 19].

2. Notations and preliminaries

The following conventions for vectors in $\mathbb{R}^n$ will be followed: $x \geqq y \Leftrightarrow x_i \geqq y_i,\ i = 1, 2, \ldots, n$; $x \geq y \Leftrightarrow x \geqq y$, and there exists at least one i such that $x_i > y_i$; $x > y \Leftrightarrow x_i > y_i,\ i = 1, 2, \ldots, n$. The index sets are $K = \{1, 2, \ldots, k\}$ and $M = \{1, 2, \ldots, m\}$.

Consider the following nonlinear multiobjective programming problem:

(P) Minimize $f(x) = [f_1(x), f_2(x), \ldots, f_k(x)]$

subject to $x \in S = \{x \in X :\ g(x) \leqq 0\}$,

where $X \subseteq \mathbb{R}^n$ is a nonempty open set and the functions $f = (f_1, f_2, \ldots, f_k) : X \to \mathbb{R}^k$ and $g = (g_1, g_2, \ldots, g_m) : X \to \mathbb{R}^m$ are twice differentiable at $\bar{x} \in X$.

Definition 1. A point $\bar{x} \in S$ is said to be a weakly efficient solution of (P), if there exists no other $x \in S$ such that

$$f(x) < f(\bar{x}).$$

The following definitions are due to Mond and Zhang [15]:

Definition 2. Function $f : X \to \mathbb{R}^k$ is said to be second order V-invex at $\bar{x} \in X$, if there exist functions $\eta : X \times X \to \mathbb{R}^n$ and $\alpha_i : X \times X \to \mathbb{R}_+ \setminus \{0\},\ i \in K$ such that

$$f_i(x) - f_i(\bar{x}) + \frac{1}{2}p^T \nabla^2 f_i(\bar{x})p \geqq \alpha_i(x, \bar{x})[\nabla f_i(\bar{x}) + \nabla^2 f_i(\bar{x})p]\eta(x, \bar{x})$$

holds for all $p \in \mathbb{R}^n$ and for all $x \in X$.

Definition 3. Function $f : X \to \mathbb{R}^k$ is said to be second order V-quasiinvex at $\bar{x} \in X$, if there exist functions $\eta : X \times X \to \mathbb{R}^n$ and $\gamma_i : X \times X \to \mathbb{R}_+ \setminus \{0\},\ i \in K$ such that

$$\sum_{i=1}^{k} \gamma_i(x,\bar{x}) f_i(x) \leqq \sum_{i=1}^{k} \gamma_i(x,\bar{x})[f_i(\bar{x}) - \tfrac{1}{2} p^T \nabla^2 f_i(\bar{x}) p]$$
$$\Rightarrow \quad \sum_{i=1}^{k} [\nabla f_i(\bar{x}) + \nabla^2 f_i(\bar{x}) p] \eta(x,\bar{x}) \leqq 0$$

holds for all $p \in \mathbb{R}^n$ and for all $x \in X$.

Definition 4. Function $f : X \to \mathbb{R}^k$ is said to be second order V-pseudoinvex at $\bar{x} \in X$, if there exist functions $\eta : X \times X \to \mathbb{R}^n$ and $\beta_i : X \times X \to \mathbb{R}_+ \setminus \{0\},\ i \in K$ such that

$$\sum_{i=1}^{k} [\nabla f_i(\bar{x}) + \nabla^2 f_i(\bar{x}) p] \eta(x,\bar{x}) \geqq 0 \Rightarrow \sum_{i=1}^{k} \beta_i(x,\bar{x}) f_i(x) \geqq \sum_{i=1}^{k} \beta_i(x,\bar{x})[f_i(\bar{x}) - \tfrac{1}{2} p^T \nabla^2 f_i(\bar{x}) p]$$

holds for all $p \in \mathbb{R}^n$ and for all $x \in X$.

Definition 5. A functional $F : X \times X \times \mathbb{R}^n \to \mathbb{R}$ is said to be sublinear in its third argument, if for any $x, \bar{x} \in X$,

(i) $F(x,\bar{x};a_1 + a_2) \leqq F(x,\bar{x};a_1) + F(x,\bar{x};a_2)\ \ \forall\ \ a_1, a_2 \in \mathbb{R}^n$,

(ii) $F(x,\bar{x};\alpha a) = \alpha F(x,\bar{x};a)\ \ \forall\ \alpha \in R, \alpha \geqq 0$ and $\forall\ a \in \mathbb{R}^n$.

The following definitions of second order (F,ρ)-convexity and its generalization were introduced by Zhang and Mond [19]. Let F be a functional sublinear in its third argument, $\phi : X \to \mathbb{R}$ be twice differentiable at $\bar{x} \in X$, $d : X \times X \to \mathbb{R}$ be a metric and $\rho \in \mathbb{R}$.

Definition 6. Function $\phi : X \to \mathbb{R}$ is said to be second order (F,ρ)-convex at $\bar{x} \in X$, if

$$\phi(x) - \phi(\bar{x}) + \frac{1}{2} p^T \nabla^2 \phi(\bar{x}) p \geqq F(x,\bar{x};\nabla\phi(\bar{x}) + \nabla^2\phi(\bar{x}) p) + \rho d(x,\bar{x})$$

holds for all $p \in \mathbb{R}^n$ and for all $x \in X$.

Definition 7. Function $\phi : X \to \mathbb{R}$ is said to be second order (F,ρ)-quasiconvex at $\bar{x} \in X$, if

$$\phi(x) \leqq \phi(\bar{x}) - \frac{1}{2} p^T \nabla^2 \phi(\bar{x}) p \Rightarrow F(x,\bar{x};\nabla\phi(\bar{x}) + \nabla^2\phi(\bar{x}) p) \leqq -\rho d(x,\bar{x})$$

holds for all $p \in \mathbb{R}^n$ and for all $x \in X$.

Definition 8. Function $\phi : X \to \mathbb{R}$ is said to be second order (F, ρ)-pseudoconvex at $\bar{x} \in X$, if

$$F(x, \bar{x}; \nabla\phi(\bar{x}) + \nabla^2\phi(\bar{x})p) \geqq -\rho d(x, \bar{x}) \Rightarrow \phi(x) \geqq \phi(\bar{x}) - \frac{1}{2}p^T\nabla^2\phi(\bar{x})p$$

holds for all $p \in \mathbb{R}^n$ and for all $x \in X$.

Finally, in view of Definitions 2-8 and the concept of generalized second order (F, α, ρ, d)-convex functions [3], we propose our definitions of second order (F, α, ρ, d)-V-convex function and its generalizations as follows:

Definition 9. Function $f : X \to \mathbb{R}^k$ is said to be (strictly) second order (F, α, ρ, d)-V-convex at $\bar{x} \in X$, if there exist functions $\alpha_i : X \times X \to \mathbb{R}_+ \setminus \{0\}, i \in K,\ d : X \times X \to \mathbb{R}$ and $\rho = (\rho_1, \rho_2, \ldots, \rho_k) \in \mathbb{R}^k$ such that

$$f_i(x) - f_i(\bar{x}) + \frac{1}{2}p^T\nabla^2 f_i(\bar{x})p\ (>) \geqq F(x, \bar{x}; \alpha_i(x, \bar{x})(\nabla f_i(\bar{x}) + \nabla^2 f_i(\bar{x})p)) + \rho_i d^2(x, \bar{x})$$

holds for all $p \in \mathbb{R}^n$ and for all $x \in X$.

Remark 1.

- (i) For $k = 1$ and $\alpha_i(x, \bar{x}) = 1$, the above definition becomes that of (strictly) second order (F, ρ)-convex function introduced by Zhang and Mond [19].
- (ii) If $\rho_i = 0,\ i \in K$ and $F(x, \bar{x}; a) = a^T\eta(x, \bar{x})$ for a certain mapping $\eta : X \times X \to \mathbb{R}^n$, the inequality reduces to that of (strictly) second order V-invex function introduced by Mond and Zhang [15].
- (iii) If $\alpha_i(x, \bar{x}) = \alpha(x, \bar{x}),\ i \in K$, then we get the definition of (strictly) second order (F, α, ρ, d)-convex function given by Ahmad and Husain [3].

Following example includes earlierly studied classes as special cases of second order (F, α, ρ, d)-V-convex function.

Example 1. Consider the function $f = (f_1, f_2, f_3) : X \to \mathbb{R}^3$, where $X = \mathbb{R}$ such that

$$f_1(x) = (x + 2)^2, \quad f_2(x) = 2 - x^2, \quad f_3(x) = -x^2 - 2x.$$

The feasible region is $S = \{x \in X : x \geqq 2\}$.

Let $F(x, \bar{x}; a) = \frac{a}{12}(x^2 + \bar{x}^2 - 4);\ \alpha_1(x, \bar{x}) = 2;\ \alpha_2(x, \bar{x}) = 4;\ \alpha_3(x, \bar{x}) = 12;$
$\rho_1 = -1;\ \rho_2 = 1;\ \rho_3 = -1;\ d(x, \bar{x}) = |x - \bar{x} + 2|;\ p = 2;\ \bar{x} = 2.$

It can be seen that $f = (f_1, f_2, f_3)$ is second order (F, α, ρ, d)-V-convex for all $x \in X$,

$$f_1(x) - f_1(\bar{x}) + \tfrac{1}{2}p^T\nabla^2 f_1(\bar{x})p = x^2 + 4x - 8$$

$$\geqq F(x, \bar{x}; \alpha_1(x, \bar{x})(\nabla f_1(\bar{x}) + \nabla^2 f_1(\bar{x})p)) + \rho_1 d^2(x, \bar{x}) = x^2, \qquad (I)$$

$$f_2(x) - f_2(\bar{x}) + \tfrac{1}{2}p^T\nabla^2 f_2(\bar{x})p = -\ x^2$$

$$\geqq F(x, \bar{x}; \alpha_2(x, \bar{x})(\nabla f_2(\bar{x}) + \nabla^2 f_2(\bar{x})p)) + \rho_2 d^2(x, \bar{x}) = -\frac{5}{3}x^2, \qquad (II)$$

$$f_3(x) - f_3(\bar{x}) + \tfrac{1}{2}p^T\nabla^2 f_3(\bar{x})p = -x^2 - 2x + 4$$

$$\geqq F(x, \bar{x}; \alpha_3(x, \bar{x})(\nabla f_3(\bar{x}) + \nabla^2 f_3(\bar{x})p)) + \rho_3 d^2(x, \bar{x}) = -11x^2. \qquad (III)$$

The above inequalities show that $f = (f_1, f_2, f_3)$ is second order (F, α, ρ, d)-V-convex for all $p \in R$ at $\bar{x}$.

If $\alpha_1(x, \bar{x}) = \alpha_2(x, \bar{x}) = \alpha_3(x, \bar{x}) = 2$, then Inequality (II) does not hold. If $\alpha_1(x, \bar{x}) = \alpha_2(x, \bar{x}) = \alpha_3(x, \bar{x}) = 4$, then Inequality (I) is not satisfied. Similarly, if $\alpha_1(x, \bar{x}) = \alpha_2(x, \bar{x}) = \alpha_3(x, \bar{x}) = 12$, then Inequality (I) is not satisfied. Hence, $f = (f_1, f_2, f_3)$ is not second order (F, α, ρ, d)-convex [3] for all $p \in \mathbb{R}$ at $\bar{x}$.

Let $\alpha_1(x, \bar{x}) = \alpha_2(x, \bar{x}) = \alpha_3(x, \bar{x}) = 1$. Then Inequality (II) does not hold. Therefore, $f = (f_1, f_2, f_3)$ is not second order (F, ρ)-convex [19] for all $p \in \mathbb{R}$ at $\bar{x}$.

Let $\rho_1 = \rho_2 = \rho_3 = 0$. Then Inequalities (I) and (II) are not satisfied. Hence $f = (f_1, f_2, f_3)$ is not second order V-invex [15] for all $p \in \mathbb{R}$ at $\bar{x}$.

Definition 10. Function $f : X \to \mathbb{R}^k$ is said to be (strictly) second order $(F, \tilde{\alpha}, \tilde{\rho}, d)$-V-quasiconvex at $\bar{x} \in X$, if there exist functions $\tilde{\alpha}_i : X \times X \to \mathbb{R}_+ \setminus \{0\}, i \in K,\ d : X \times X \to \mathbb{R}$ and $\tilde{\rho} \in \mathbb{R}$ such that

$$\sum_{i=1}^{k} \tilde{\alpha}_i(x, \bar{x}) f_i(x) \leqq \sum_{i=1}^{k} \tilde{\alpha}_i(x, \bar{x}) f_i(\bar{x}) - \frac{1}{2}p^T\nabla^2 \sum_{i=1}^{k} \tilde{\alpha}_i(x, \bar{x}) f_i(\bar{x})p$$

$$\Rightarrow F(x, \bar{x}; \sum_{i=1}^{k}(\nabla f_i(\bar{x}) + \nabla^2 f_i(\bar{x})p)) + \tilde{\rho} d^2(x, \bar{x})\ (<) \leqq\ 0$$

holds for all $p \in \mathbb{R}^n$ and for all $x \in X$.

Definition 11. Function $f : X \to \mathbb{R}^k$ is said to be (strictly) second order $(F, \bar{\alpha}, \bar{\rho}, d)$-V-pseudoconvex at $\bar{x} \in X$, if there exist functions $\bar{\alpha}_i : X \times X \to \mathbb{R}_+ \setminus \{0\}, i \in K,\ d : X \times X \to \mathbb{R}$ and $\bar{\rho} \in \mathbb{R}$ such that

$$F(x, \bar{x}; \sum_{i=1}^{k}(\nabla f_i(\bar{x}) + \nabla^2 f_i(\bar{x})p)) + \bar{\rho} d^2(x, \bar{x}) \geqq 0$$

$$\Rightarrow \sum_{i=1}^{k} \bar{\alpha}_i(x,\bar{x}) f_i(x)\ (>) \geqq \ \sum_{i=1}^{k} \bar{\alpha}_i(x,\bar{x}) f_i(\bar{x}) - \tfrac{1}{2} p^T \nabla^2 \sum_{i=1}^{k} \bar{\alpha}_i(x,\bar{x}) f_i(\bar{x}) p$$

holds for all $p \in \mathbb{R}^n$ and for all $x \in X$.

Remark 2. By using the sublinearity of F, one can see from the above definitions that a second order (F,α,ρ,d)-V-convex function is both second order $(F,\bar{\alpha},\bar{\rho},d)$-V-pseudoconvex (with $\bar{\alpha}_i = \dfrac{1}{\alpha_i}$, $i \in K$ and $\bar{\rho} = \sum\limits_{i=1}^{k} \dfrac{1}{\alpha_i(x,\bar{x})}\rho_i$) and second order $(F,\tilde{\alpha},\tilde{\rho},d)$-V-quasiconvex (with $\tilde{\alpha}_i = \dfrac{1}{\alpha_i}$, $i \in K$ and $\tilde{\rho} = \sum\limits_{i=1}^{k} \dfrac{1}{\alpha_i(x,\bar{x})}\rho_i$). Obviously, the converse is not necessarily true.

Following Kuhn-Tucker theorem will be needed in the sequel:

Proposition 1 [12]. Let $\bar{x}$ be a weakly efficient solution of (P) at which the Kuhn-Tucker constraint qualification is satisfied. Then there exist $\lambda \in \mathbb{R}^k$ and $u \in \mathbb{R}^m$ such that

$$\sum_{i=1}^{k} \nabla \lambda_i f_i(\bar{x}) + \sum_{j=1}^{m} \nabla u_j g_j(\bar{x}) = 0,$$

$$\sum_{j=1}^{m} u_j g_j(\bar{x}) = 0,$$

$$\lambda \geqq 0,\ \sum_{i=1}^{k} \lambda_i = 1,\ u \geqq 0.$$

3. Mangasarian type duality

In this section, we consider the following second order Mangasarian type dual for (P) and discuss duality results.

(SD) Maximize $\big(f_1(y) + u^T g(y) - \tfrac{1}{2} p^T \nabla^2 (f_1(y) + u^T g(y)) p,$

$$\ldots, f_k(y) + u^T g(y) - \tfrac{1}{2} p^T \nabla^2 (f_k(y) + u^T g(y)) p\big)$$

subject to

$$\sum_{i=1}^{k} (\nabla \lambda_i f_i(y) + \nabla^2 \lambda_i f_i(y) p) + \sum_{j=1}^{m} (\nabla u_j g_j(y) + \nabla^2 u_j g_j(y) p) = 0, \tag{1}$$

$$\lambda \geqq 0, \tag{2}$$

$$\sum_{i=1}^{k} \lambda_i = 1, \tag{3}$$

$$u \geqq 0. \tag{4}$$

Let Q be the set of all feasible solutions of (SD).

Theorem 1 (Weak duality)**.** Suppose that for all $x \in S$ and $(y, u, \lambda, p) \in Q$,

(i) f is second order (F, α, ρ, d)-V-convex at y, and g is second order $(F, \hat{\alpha}, \hat{\rho}, d)$-V-convex at y;

(ii) $\sum_{i=1}^{k} \dfrac{\lambda_i}{\alpha_i(x, y)} = 1$ and $\hat{\alpha}_j(x, y) = 1,\ j \in M$; and

(iii) $\sum_{i=1}^{k} \dfrac{\lambda_i \rho_i}{\alpha_i(x, y)} + \sum_{j=1}^{m} u_j \hat{\rho}_j \geqq 0.$

Then

$$f_i(x) \nless f_i(y) + \sum_{j=1}^{m} u_j g_j(y) - \frac{1}{2} p^T \nabla^2 (f_i(y) + \sum_{j=1}^{m} u_j g_j(y))p,\ i \in K. \tag{5}$$

Proof. Suppose contrary to the result that (5) cannot hold, i.e.,

$$f_i(x) < f_i(y) + \sum_{j=1}^{m} u_j g_j(y) - \frac{1}{2} p^T \nabla^2 (f_i(y) + \sum_{j=1}^{m} u_j g_j(y))p,\ i \in K,$$

which on using (2), (3), $\alpha_i(x, y) > 0,\ i \in K$ and hypothesis (ii) becomes

$$\sum_{i=1}^{k} \frac{\lambda_i f_i(x)}{\alpha_i(x, y)} < \sum_{i=1}^{k} \frac{\lambda_i f_i(y)}{\alpha_i(x, y)} + \sum_{j=1}^{m} u_j g_j(y) - \frac{1}{2} p^T \nabla^2 \left\{ \sum_{i=1}^{k} \frac{\lambda_i f_i(y)}{\alpha_i(x, y)} + \sum_{j=1}^{m} u_j g_j(y) \right\} p,$$

or

$$\sum_{i=1}^{k} \frac{\lambda_i f_i(x)}{\alpha_i(x, y)} - \sum_{i=1}^{k} \frac{\lambda_i f_i(y)}{\alpha_i(x, y)} - \sum_{j=1}^{m} u_j g_j(y) + \frac{1}{2} p^T \nabla^2 \left\{ \sum_{i=1}^{k} \frac{\lambda_i f_i(y)}{\alpha_i(x, y)} + \sum_{j=1}^{m} u_j g_j(y) \right\} p < 0. \tag{6}$$

According to hypothesis (i), it follows that

$$f_i(x) - f_i(y) + \frac{1}{2} p^T \nabla^2 f_i(y) p \geqq F(x, y; \alpha_i(x, y)(\nabla f_i(y) + \nabla^2 f_i(y) p)) + \rho_i d^2(x, y)$$

and

$$g_j(x) - g_j(y) + \frac{1}{2} p^T \nabla^2 g_j(y) p \geqq F(x, y; \hat{\alpha}_j(x, y)(\nabla g_j(y) + \nabla^2 g_j(y) p)) + \hat{\rho}_j d^2(x, y).$$

On multiplying the first inequality by $\dfrac{\lambda_i}{\alpha_i(x, y)} \geqq 0,\ i \in K$ and second by $u_j \geqq 0$, with $\hat{\alpha}_j(x, y) = 1,\ j \in M$, then summing over i and j respectively, and on using the sublinearity of F, we have

$$\sum_{i=1}^{k}\frac{\lambda_i f_i(x)}{\alpha_i(x,y)}+\sum_{j=1}^{m}u_jg_j(x)-\sum_{i=1}^{k}\frac{\lambda_i f_i(y)}{\alpha_i(x,y)}-\sum_{j=1}^{m}u_jg_j(y)$$

$$+\frac{1}{2}p^T\nabla^2\left\{\sum_{i=1}^{k}\frac{\lambda_i f_i(y)}{\alpha_i(x,y)}+\sum_{j=1}^{m}u_jg_j(y)\right\}p$$

$$\geqq F\left(x,y;\sum_{i=1}^{k}(\nabla\lambda_i f_i(y)+\nabla^2\lambda_i f_i(y)p)\right)+F\left(x,y;\sum_{j=1}^{m}(\nabla u_jg_j(y)+\nabla^2u_jg_j(y)p)\right)$$

$$+\sum_{i=1}^{k}\frac{\lambda_i\rho_i d^2(x,y)}{\alpha_i(x,y)}+\sum_{j=1}^{m}u_j\hat{\rho}_j d^2(x,y). \quad (7)$$

The relations (1), (7) and the sublinearity of F yield

$$\sum_{i=1}^{k}\frac{\lambda_i f_i(x)}{\alpha_i(x,y)}+\sum_{j=1}^{m}u_jg_j(x)-\sum_{i=1}^{k}\frac{\lambda_i f_i(y)}{\alpha_i(x,y)}-\sum_{j=1}^{m}u_jg_j(y)$$

$$+\frac{1}{2}p^T\nabla^2\left\{\sum_{i=1}^{k}\frac{\lambda_i f_i(y)}{\alpha_i(x,y)}+\sum_{j=1}^{m}u_jg_j(y)\right\}p$$

$$\geqq\left(\sum_{i=1}^{k}\frac{\lambda_i\rho_i}{\alpha_i(x,y)}+\sum_{j=1}^{m}u_j\hat{\rho}_j\right)d^2(x,y),$$

which by virtue of hypothesis (iii) gives

$$\sum_{i=1}^{k}\frac{\lambda_i f_i(x)}{\alpha_i(x,y)}+\sum_{j=1}^{m}u_jg_j(x)-\sum_{i=1}^{k}\frac{\lambda_i f_i(y)}{\alpha_i(x,y)}-\sum_{j=1}^{m}u_jg_j(y)$$

$$+\frac{1}{2}p^T\nabla^2\left\{\sum_{i=1}^{k}\frac{\lambda_i f_i(y)}{\alpha_i(x,y)}+\sum_{j=1}^{m}u_jg_j(y)\right\}p\geqq 0.$$

By $u\geqq 0$ and $g(x)\leqq 0$, it follows that

$$\sum_{i=1}^{k}\frac{\lambda_i f_i(x)}{\alpha_i(x,y)}-\sum_{i=1}^{k}\frac{\lambda_i f_i(y)}{\alpha_i(x,y)}-\sum_{j=1}^{m}u_jg_j(y)+\frac{1}{2}p^T\nabla^2\left\{\sum_{i=1}^{k}\frac{\lambda_i f_i(y)}{\alpha_i(x,y)}+\sum_{j=1}^{m}u_jg_j(y)\right\}p\geqq 0,$$

a contradiction to (6). This completes the proof. □

Theorem 2 (Strong duality)**.** Let $\bar{x}$ be a weakly efficient solution of (P) at which the Kuhn-Tucker constraint qualification is satisfied. Then there exist $\bar{\lambda}\in\mathbb{R}^k$, $\bar{u}\in\mathbb{R}^m$ and $\bar{p}\in\mathbb{R}^n$ such that $(\bar{x},\bar{u},\bar{\lambda},\bar{p}=0)\in Q$ and the corresponding objective values of

(P) and (SD) are equal. If, in addition, the hypotheses of weak duality (Theorem 1) hold, then $(\bar{x}, \bar{u}, \bar{\lambda}, \bar{p} = 0)$ is a weakly efficient solution of (SD).

Proof. Since $\bar{x}$ is a weakly efficient solution of (P) at which the Kuhn-Tucker constraint qualification is satisfied, from Proposition 1, there exist $\bar{\lambda} \in \mathbb{R}^k$ and $\bar{u} \in \mathbb{R}^m$ such that

$$\sum_{i=1}^{k} \nabla \bar{\lambda}_i f_i(\bar{x}) + \sum_{j=1}^{m} \nabla \bar{u}_j g_j(\bar{x}) = 0,$$

$$\sum_{j=1}^{m} \bar{u}_j g_j(\bar{x}) = 0,$$

$$\bar{\lambda} \geqq 0, \ \sum_{i=1}^{k} \bar{\lambda}_i = 1, \ \bar{u} \geqq 0.$$

Therefore, $(\bar{x}, \bar{u}, \bar{\lambda}, \bar{p} = 0) \in Q$ and the corresponding objective values of (P) and (SD) are equal. Weak efficiency of $(\bar{x}, \bar{u}, \bar{\lambda}, \bar{p} = 0)$ thus follows from weak duality (Theorem 1). □

Theorem 3 (Strict converse duality)**.** Let $\bar{x} \in S$ and $(\bar{y}, \bar{u}, \bar{\lambda}, \bar{p}) \in Q$ such that

(*i*) $\sum_{i=1}^{k} \bar{\lambda}_i f_i(\bar{x}) \leqq \sum_{i=1}^{k} \bar{\lambda}_i f_i(\bar{y}) + \sum_{j=1}^{m} \bar{u}_j g_j(\bar{y}) - \frac{1}{2}\bar{p}^T \nabla^2 \left\{ \sum_{i=1}^{k} \bar{\lambda}_i f_i(\bar{y}) + \sum_{j=1}^{m} \bar{u}_j g_j(\bar{y}) \right\} \bar{p};$

(*ii*) f is strictly second order (F, α, ρ, d)-V-convex at $\bar{y}$ with $\alpha_i(\bar{x}, \bar{y}) = 1, \ i \in K$ and g is second order $(F, \hat{\alpha}, \hat{\rho}, d)$-V-convex at $\bar{y}$ with $\hat{\alpha}_j(\bar{x}, \bar{y}) = 1, \ j \in M$; and

(*iii*) $\sum_{i=1}^{k} \bar{\lambda}_i \rho_i + \sum_{j=1}^{m} \bar{u}_j \hat{\rho}_j \geqq 0.$

Then $\bar{x} = \bar{y}$.

Proof. We assume that $\bar{x} \neq \bar{y}$, and exhibit a contradiction. Using (2)-(4), hypothesis (*ii*), and the sublinearity of F, we obtain

$$\sum_{i=1}^{k} \bar{\lambda}_i f_i(\bar{x}) - \sum_{i=1}^{k} \bar{\lambda}_i f_i(\bar{y}) + \frac{1}{2}\bar{p}^T \nabla^2 \sum_{i=1}^{k} \bar{\lambda}_i f_i(\bar{y}) \bar{p} > F\left(\bar{x}, \bar{y}; \sum_{i=1}^{k} (\nabla \bar{\lambda}_i f_i(\bar{y}) + \nabla^2 \bar{\lambda}_i f_i(\bar{y}) \bar{p})\right) + \sum_{i=1}^{k} \bar{\lambda}_i \rho_i d^2(\bar{x}, \bar{y})$$

and

$$\sum_{j=1}^{m} \bar{u}_j g_j(\bar{x}) - \sum_{j=1}^{m} \bar{u}_j g_j(\bar{y}) + \frac{1}{2}\bar{p}^T \nabla^2 \sum_{j=1}^{m} \bar{u}_j g_j(\bar{y}) \bar{p} \geqq F\left(\bar{x}, \bar{y}; \sum_{j=1}^{m} (\nabla \bar{u}_j g_j(\bar{y}) + \nabla^2 \bar{u}_j g_j(\bar{y}) \bar{p})\right)$$

$$+\sum_{j=1}^{m}\bar{u}_j\hat{\rho}_j d^2(\bar{x},\bar{y}).$$

Adding these inequalities, we get

$$\sum_{i=1}^{k}\bar{\lambda}_i f_i(\bar{x})+\sum_{j=1}^{m}\bar{u}_j g_j(\bar{x})-\sum_{i=1}^{k}\bar{\lambda}_i f_i(\bar{y})-\sum_{j=1}^{m}\bar{u}_j g_j(\bar{y})+\frac{1}{2}\bar{p}^T\nabla^2\left\{\sum_{i=1}^{k}\bar{\lambda}_i f_i(\bar{y})+\sum_{j=1}^{m}\bar{u}_j g_j(\bar{y})\right\}\bar{p}$$

$$> F\left(\bar{x},\bar{y};\sum_{i=1}^{k}(\nabla\bar{\lambda}_i f_i(\bar{y})+\nabla^2\bar{\lambda}_i f_i(\bar{y})\bar{p})\right)+F\left(\bar{x},\bar{y};\sum_{j=1}^{m}(\nabla\bar{u}_j g_j(\bar{y})+\nabla^2\bar{u}_j g_j(\bar{y})\bar{p})\right)$$

$$+\sum_{i=1}^{k}\bar{\lambda}_i\rho_i d^2(\bar{x},\bar{y})+\sum_{j=1}^{m}\bar{u}_j\hat{\rho}_j d^2(\bar{x},\bar{y})$$

$$\geqq F\left(\bar{x},\bar{y};\sum_{i=1}^{k}(\nabla\bar{\lambda}_i f_i(\bar{y})+\nabla^2\bar{\lambda}_i f_i(\bar{y})\bar{p})+\sum_{j=1}^{m}(\nabla\bar{u}_j g_j(\bar{y})+\nabla^2\bar{u}_j g_j(\bar{y})\bar{p})\right)$$

$$+\sum_{i=1}^{k}\bar{\lambda}_i\rho_i d^2(\bar{x},\bar{y})+\sum_{j=1}^{m}\bar{u}_j\hat{\rho}_j d^2(\bar{x},\bar{y}) \text{ (by the sublinearity of } F\text{)},$$

which on using (1) and $F(\bar{x},\bar{y};0)=0$ gives

$$\sum_{i=1}^{k}\bar{\lambda}_i f_i(\bar{x})+\sum_{j=1}^{m}\bar{u}_j g_j(\bar{x})-\sum_{i=1}^{k}\bar{\lambda}_i f_i(\bar{y})-\sum_{j=1}^{m}\bar{u}_j g_j(\bar{y})+\frac{1}{2}\bar{p}^T\nabla^2\left\{\sum_{i=1}^{k}\bar{\lambda}_i f_i(\bar{y})+\sum_{j=1}^{m}\bar{u}_j g_j(\bar{y})\right\}\bar{p}$$

$$>\left(\sum_{i=1}^{k}\bar{\lambda}_i\rho_i+\sum_{j=1}^{m}\bar{u}_j\hat{\rho}_j\right)d^2(\bar{x},\bar{y}).$$

This inequality along with hypothesis (iii), $\bar{u}\geqq 0$ and $g(\bar{x})\leqq 0$ yields

$$\sum_{i=1}^{k}\bar{\lambda}_i f_i(\bar{x})-\sum_{i=1}^{k}\bar{\lambda}_i f_i(\bar{y})-\sum_{j=1}^{m}\bar{u}_j g_j(\bar{y})+\frac{1}{2}\bar{p}^T\nabla^2\left\{\sum_{i=1}^{k}\bar{\lambda}_i f_i(\bar{y})+\sum_{j=1}^{m}\bar{u}_j g_j(\bar{y})\right\}\bar{p}>0,$$

a contradiction to hypothesis (i). Hence, $\bar{x}=\bar{y}$. □

4. Mond-Weir type duality

In this section, we present the following Mond-Weir [13] type dual associated to (P):

(MD) Maximize $(f_1(y)-\frac{1}{2}p^T\nabla^2 f_1(y)p,\ldots,f_k(y)-\frac{1}{2}p^T\nabla^2 f_k(y)p)$

subject to

$$\sum_{i=1}^{k}(\nabla \lambda_i f_i(y) + \nabla^2 \lambda_i f_i(y)p) + \sum_{j=1}^{m}(\nabla u_j g_j(y) + \nabla^2 u_j g_j(y)p) = 0, \tag{8}$$

$$u_j g_j(y) - \frac{1}{2}p^T \nabla^2 u_j g_j(y)p \geqq 0,\ j \in M, \tag{9}$$

$$\lambda \geqq 0, \tag{10}$$

$$\sum_{i=1}^{k} \lambda_i = 1, \tag{11}$$

$$u \geqq 0. \tag{12}$$

Let U be the set of all feasible solutions of (MD). In this section and in Section 5, f^λ denotes the vector $(\lambda_1 f_1, \lambda_2 f_2, \ldots, \lambda_k f_k)$ and g^u denotes the vector $(u_1 g_1, u_2 g_2, \ldots, u_m g_m)$.

Theorem 4 (Weak duality)**.** Suppose that for all $x \in S$ and $(y, u, \lambda, p) \in U$,

- (i) f^λ is second order $(F, \bar{\alpha}, \bar{\rho}, d)$-V-pseudoconvex at y, and g^u is second order $(F, \tilde{\alpha}, \tilde{\rho}, d)$-V-quasiconvex at y; and
- (ii) $\bar{\rho} + \tilde{\rho} \geqq 0$.

Then

$$f_i(x) \not< f_i(y) - \frac{1}{2}p^T \nabla^2 f_i(y)p,\ i \in K. \tag{13}$$

Proof. Since $x \in S$ and $(y, u, \lambda, p) \in U$, we have

$$u_j g_j(x) \leqq 0 \leqq u_j g_j(y) - \frac{1}{2}p^T \nabla^2 u_j g_j(y)p,\ j \in M.$$

As $\tilde{\alpha}_j(x, y) > 0,\ j \in M$, we get

$$\sum_{j=1}^{m} \tilde{\alpha}_j(x, y) u_j g_j(x) \leqq \sum_{j=1}^{m} \tilde{\alpha}_j(x, y) u_j g_j(y) - \frac{1}{2}p^T \nabla^2 \sum_{j=1}^{m} \tilde{\alpha}_j(x, y) u_j g_j(y)p.$$

Using second order $(F, \tilde{\alpha}, \tilde{\rho}, d)$-V-quasiconvexity of g^u at y, we obtain

$$F\left(x, y; \sum_{j=1}^{m}(\nabla u_j g_j(y) + \nabla^2 u_j g_j(y)p)\right) + \tilde{\rho} d^2(x, y) \leqq 0. \tag{14}$$

The equation (8) along with the sublinearity of F gives

$$F\left(x, y; \sum_{i=1}^{k}(\nabla\lambda_i f_i(y) + \nabla^2\lambda_i f_i(y)p)\right) + F\left(x, y; \sum_{j=1}^{m}(\nabla u_j g_j(y) + \nabla^2 u_j g_j(y)p)\right)$$

$$\geqq F\left(x, y; \sum_{i=1}^{k}(\nabla\lambda_i f_i(y) + \nabla^2\lambda_i f_i(y)p) + \sum_{j=1}^{m}(\nabla u_j g_j(y) + \nabla^2 u_j g_j(y)p)\right) = 0. \quad (15)$$

Inequalities (14), (15) and hypothesis (ii) imply

$$F\left(x, y; \sum_{i=1}^{k}(\nabla\lambda_i f_i(y) + \nabla^2\lambda_i f_i(y)p)\right) + \bar{\rho}d^2(x, y) \geqq 0,$$

which by second order $(F, \bar{\alpha}, \bar{\rho}, d)$-V-pseudoconvexity of f^λ at y yields

$$\sum_{i=1}^{k}\bar{\alpha}_i(x, y)\lambda_i f_i(x) \geqq \sum_{i=1}^{k}\bar{\alpha}_i(x, y)\lambda_i f_i(y) - \frac{1}{2}p^T\nabla^2\sum_{i=1}^{k}\bar{\alpha}_i(x, y)\lambda_i f_i(y)p. \quad (16)$$

Now suppose contrary to (13), i.e.,

$$f_i(x) < f_i(y) - \frac{1}{2}p^T\nabla^2 f_i(y)p, \ i \in K.$$

Using $\lambda \geqq 0, \ \sum_{i=1}^{k}\lambda_i = 1$, and $\bar{\alpha}_i(x, y) > 0, \ i \in K$, we get

$$\sum_{i=1}^{k}\bar{\alpha}_i(x, y)\lambda_i f_i(x) < \sum_{i=1}^{k}\bar{\alpha}_i(x, y)\lambda_i f_i(y) - \frac{1}{2}p^T\nabla^2\sum_{i=1}^{k}\bar{\alpha}_i(x, y)\lambda_i f_i(y)p,$$

a contradiction to (16). Hence the theorem. □

The proof of the following weak duality theorem is similar to that of Theorem 4, and hence is omitted.

Theorem 5 (Weak duality). Suppose that for all $x \in S$ and $(y, u, \lambda, p) \in U$,

(i) f^λ is second order $(F, \tilde{\alpha}, \tilde{\rho}, d)$-V-quasiconvex at y, and g^u is strictly second order $(F, \bar{\alpha}, \bar{\rho}, d)$-V-pseudoconvex at y; and

(ii) $\tilde{\rho} + \bar{\rho} \geqq 0$.

Then

$$f_i(x) \nless f_i(y) - \frac{1}{2}p^T\nabla^2 f_i(y)p, \ i \in K.$$

Since the proof of the strong duality theorem follows on the similar lines of Theorem 2, we just state the theorem but omit the details of the proof.

Theorem 6 (Strong duality)**.** Let $\bar{x}$ be a weakly efficient solution of (P) at which the Kuhn-Tucker constraint qualification is satisfied. Then there exist $\bar{\lambda} \in \mathbb{R}^k$, $\bar{u} \in \mathbb{R}^m$ and $\bar{p} \in \mathbb{R}^n$ such that $(\bar{x}, \bar{u}, \bar{\lambda}, \bar{p} = 0) \in U$ and the corresponding objective values of (P) and (MD) are equal. If, in addition, the hypotheses of weak duality (Theorem 4 or 5) hold, then $(\bar{x}, \bar{u}, \bar{\lambda}, \bar{p} = 0)$ is a weakly efficient solution of (MD).

Theorem 7 (Strict converse duality)**.** Let $\bar{x} \in S$ and $(\bar{y}, \bar{u}, \bar{\lambda}, \bar{p}) \in U$ such that

(i) $\sum\limits_{i=1}^{k} \bar{\lambda}_i f_i(\bar{x}) \leqq \sum\limits_{i=1}^{k} \bar{\lambda}_i f_i(\bar{y}) - \frac{1}{2}\bar{p}^T \nabla^2 \sum\limits_{i=1}^{k} \bar{\lambda}_i f_i(\bar{y})\bar{p}$;

(ii) $f^{\bar{\lambda}}$ is strictly second order $(F, \bar{\alpha}, \bar{\rho}, d)$-V-pseudoconvex at $\bar{y}$ with $\bar{\alpha}_i(\bar{x}, \bar{y}) = \delta(\bar{x}, \bar{y})$, $i \in K$ and $g^{\bar{u}}$ is second order $(F, \tilde{\alpha}, \tilde{\rho}, d)$-V-quasiconvex at $\bar{y}$; and

(iii) $\bar{\rho} + \tilde{\rho} \geqq 0$.

Then $\bar{x} = \bar{y}$.

Proof. We assume that $\bar{x} \neq \bar{y}$ and exhibit a contradiction. Since $\bar{x} \in S$ and $(\bar{y}, \bar{u}, \bar{\lambda}, \bar{p}) \in U$, we have

$$\bar{u}_j g_j(\bar{x}) \leqq 0 \leqq \bar{u}_j g_j(\bar{y}) - \frac{1}{2}\bar{p}^T \nabla^2 \bar{u}_j g_j(\bar{y})\bar{p}, \ j \in M.$$

By $\tilde{\alpha}_j(\bar{x}, \bar{y}) > 0$, $j \in M$, it follows that

$$\sum_{j=1}^{m} \tilde{\alpha}_j(\bar{x}, \bar{y})\bar{u}_j g_j(\bar{x}) \leqq \sum_{j=1}^{m} \tilde{\alpha}_j(\bar{x}, \bar{y})\bar{u}_j g_j(\bar{y}) - \tfrac{1}{2}\bar{p}^T \nabla^2 \sum_{j=1}^{m} \tilde{\alpha}_j(\bar{x}, \bar{y})\bar{u}_j g_j(\bar{y})\bar{p}.$$

On using second order $(F, \tilde{\alpha}, \tilde{\rho}, d)$-V-quasiconvexity of $g^{\bar{u}}$ at $\bar{y}$, we get

$$F(\bar{x}, \bar{y}; \sum_{j=1}^{m} (\nabla \bar{u}_j g_j(\bar{y}) + \nabla^2 \bar{u}_j g_j(\bar{y})\bar{p})) + \tilde{\rho} d^2(\bar{x}, \bar{y}) \leqq 0. \tag{17}$$

Now from (8), (17), hypothesis (iii) and the sublinearity of F, we obtain

$$F(\bar{x}, \bar{y}; \sum_{i=1}^{k} (\nabla \bar{\lambda}_i f_i(\bar{y}) + \nabla^2 \bar{\lambda}_i f_i(\bar{y})\bar{p})) + \bar{\rho} d^2(\bar{x}, \bar{y}) \geqq 0.$$

The strict second order $(F, \bar{\alpha}, \bar{\rho}, d)$-V-pseudoconvexity of $f^{\bar{\lambda}}$ at $\bar{y}$ yields

$$\sum_{i=1}^{k} \bar{\alpha}_i(\bar{x}, \bar{y})\bar{\lambda}_i f_i(\bar{x}) > \sum_{i=1}^{k} \bar{\alpha}_i(\bar{x}, \bar{y})\bar{\lambda}_i f_i(\bar{y}) - \tfrac{1}{2}\bar{p}^T \nabla^2 \sum_{i=1}^{k} \bar{\alpha}_i(\bar{x}, \bar{y})\bar{\lambda}_i f_i(\bar{y})\bar{p}.$$

Since $\bar{\alpha}_i(\bar{x}, \bar{y}) = \delta(\bar{x}, \bar{y}),\ i \in K$, we have

$$\sum_{i=1}^{k} \bar{\lambda}_i f_i(\bar{x}) > \sum_{i=1}^{k} \bar{\lambda}_i f_i(\bar{y}) - \frac{1}{2}\bar{p}^T \nabla^2 \sum_{i=1}^{k} \bar{\lambda}_i f_i(\bar{y})\bar{p},$$

a contradiction to hypothesis (i). Hence, $\bar{x} = \bar{y}$. □

5. General Mond-Weir type duality

For (P), we present the following second order general Mond-Weir type dual:

(GD) Maximize $\Big(f_1(y) + \sum_{j \in J_\circ} u_j g_j(y) - \frac{1}{2} p^T \nabla^2 (f_1(y) + \sum_{j \in J_\circ} u_j g_j(y))p,$

$$\ldots, f_k(y) + \sum_{j \in J_\circ} u_j g_j(y) - \tfrac{1}{2} p^T \nabla^2 (f_k(y) + \sum_{j \in J_\circ} u_j g_j(y))p\Big)$$

subject to

$$\sum_{i=1}^{k} (\nabla \lambda_i f_i(y) + \nabla^2 \lambda_i f_i(y) p) + \sum_{j=1}^{m} (\nabla u_j g_j(y) + \nabla^2 u_j g_j(y) p) = 0, \tag{18}$$

$$u_j g_j(y) - \frac{1}{2} p^T \nabla^2 u_j g_j(y) p \geqq 0,\ j \in J_\beta,\ \beta = 1, 2, \ldots, r, \tag{19}$$

$$\lambda \geqq 0, \tag{20}$$

$$\sum_{i=1}^{k} \lambda_i = 1, \tag{21}$$

$$u \geqq 0, \tag{22}$$

where $J_\beta \subseteq M,\ \beta = 0, 1, 2, \ldots, r$ with $\bigcup_{\beta=0}^{r} J_\beta = M$ and $J_\beta \cap J_\gamma = \emptyset$, if $\beta \neq \gamma$.

Remark 3. Let $J_\beta = \emptyset$. Then the dual (GD) reduces to Mangasarian type dual considered in Section 3. If $J_\circ = \emptyset$, then (GD) becomes Mond-Weir type dual discussed in Section 4.

Let Y be the set of all feasible solutions of (GD).

Theorem 8 (Weak duality)**.** Suppose that for all $x \in S$ and $(y, u, \lambda, p) \in Y$,

(i) $(\lambda_i f_i + u_{J_\circ} g_{J_\circ})_{i \in K}$ is second order $(F, \bar{\alpha}, \bar{\rho}, d)$-V-pseudoconvex at y, and $(u_j g_j)_{j \in J_\beta}$, $\beta = 1, 2, \ldots, r$ is second order $(F, \tilde{\alpha}, \tilde{\rho}, d)$-V-quasiconvex at y; and

(ii) $\bar{\rho} + \sum_{\beta=1}^{r} \tilde{\rho}_\beta \geqq 0$.

Then

$$f_i(x) \nless f_i(y) + \sum_{j\in J_\circ} u_j g_j(y) - \frac{1}{2}p^T\nabla^2\{f_i(y) + \sum_{j\in J_\circ} u_j g_j(y)\}p,\ i \in K. \tag{23}$$

Proof. Since $x \in S$ and $(y, u, \lambda, p) \in Y$, we have

$$u_j g_j(x) \leqq 0 \leqq u_j g_j(y) - \frac{1}{2}p^T\nabla^2 u_j g_j(y)p,\ j \in J_\beta,\ \beta = 1, 2, \ldots, r.$$

As $\tilde{\alpha}_j(x, y) > 0,\ j \in J_\beta$, we get

$$\sum_{j\in J_\beta} \tilde{\alpha}_j(x,y)u_j g_j(x) \leqq \sum_{j\in J_\beta} \tilde{\alpha}_j(x,y)u_j g_j(y) - \tfrac{1}{2}p^T\nabla^2 \sum_{j\in J_\beta} \tilde{\alpha}_j(x,y)u_j g_j(y)p,\ \beta = 1, 2, \ldots, r.$$

The second order $(F, \tilde{\alpha}, \tilde{\rho}, d)$-V-quasiconvexity of $(u_j g_j)_{j\in J_\beta}, \beta = 1, 2, \ldots, r$ at y implies

$$F\left(x, y; \sum_{j\in J_\beta} (\nabla u_j g_j(y) + \nabla^2 u_j g_j(y)p)\right) + \tilde{\rho}_\beta d^2(x, y) \leqq 0,\ \beta = 1, 2, \ldots, r. \tag{24}$$

Inequality (24) along with (18), hypothesis (ii) and the sublinearity of F yields

$$F\left(x, y; \sum_{i=1}^{k}(\nabla\lambda_i f_i(y) + \nabla^2\lambda_i f_i(y)p) + \sum_{j\in J_\circ} (\nabla u_j g_j(y) + \nabla^2 u_j g_j(y)p)\right) + \bar{\rho}d^2(x, y) \geqq 0.$$

On using second order $(F, \bar{\alpha}, \bar{\rho}, d)$-V-pseudoconvexity of $(\lambda_i f_i + u_{J_\circ} g_{J_\circ})_{i\in K}$ at y, we obtain

$$\sum_{i=1}^{k} \bar{\alpha}_i(x, y)\left(\lambda_i f_i(x) + \sum_{j\in J_\circ} u_j g_j(x)\right) \geqq \sum_{i=1}^{k} \bar{\alpha}_i(x, y)\left(\lambda_i f_i(y) + \sum_{j\in J_\circ} u_j g_j(y)\right) - \tfrac{1}{2}p^T\nabla^2\left(\sum_{i=1}^{k} \bar{\alpha}_i(x, y)(\lambda_i f_i(y) + \sum_{j\in J_\circ} u_j g_j(y))\right)p. \tag{25}$$

Now, suppose contrary to the result that (23) cannot hold, then by $u \geqq 0$ and $g(x) \leqq 0$, it follows that

$$f_i(x) + \sum_{j\in J_\circ} u_j g_j(x) < f_i(y) + \sum_{j\in J_\circ} u_j g_j(y) - \tfrac{1}{2}p^T\nabla^2\{f_i(y) + \sum_{j\in J_\circ} u_j g_j(y)\}p,\ i \in K.$$

Using (20), (21), $\bar{\alpha}_i(x, y) > 0,\ i \in K$ and summing over i, we get

$$\sum_{i=1}^{k} \bar{\alpha}_i(x, y)(\lambda_i f_i(x) + \sum_{j\in J_\circ} u_j g_j(x)) < \sum_{i=1}^{k} \bar{\alpha}_i(x, y)(\lambda_i f_i(y) + \sum_{j\in J_\circ} u_j g_j(y))$$

$$-\tfrac{1}{2}p^T\nabla^2\{\sum_{i=1}^{k}\bar{\alpha}_i(x,y)(\lambda_i f_i(y)+\sum_{j\in J_\circ}u_jg_j(y))\}p,$$

a contradiction to (25). This completes the proof. □

The proof of the following strong duality theorem follows on the lines of Theorem 2 and hence being omitted.

Theorem 9 (Strong duality)**.** Let $\bar{x}$ be a weakly efficient solution of (P) at which the Kuhn-Tucker constraint qualification is satisfied. Then there exist $\bar{\lambda}\in\mathbb{R}^k$, $\bar{u}\in\mathbb{R}^m$ and $\bar{p}\in\mathbb{R}^n$ such that $(\bar{x},\bar{u},\bar{\lambda},\bar{p}=0)\in Y$ and the corresponding objective values of (P) and (GD) are equal. If, in addition, the hypotheses of weak duality (Theorem 8) hold, then $(\bar{x},\bar{u},\bar{\lambda},\bar{p}=0)$ is a weakly efficient solution of (GD).

Theorem 10 (Strict converse duality)**.** Let $\bar{x}\in S$ and $(\bar{y},\bar{u},\bar{\lambda},\bar{p})\in Y$ such that

(*i*) $\sum_{i=1}^{k}\bar{\lambda}_if_i(\bar{x})\leqq\sum_{i=1}^{k}\bar{\lambda}_if_i(\bar{y})+\sum_{j\in J_\circ}\bar{u}_jg_j(\bar{y})-\tfrac{1}{2}\bar{p}^T\nabla^2(f_i(\bar{y})+\sum_{j\in J_\circ}\bar{u}_jg_j(\bar{y}))\bar{p}$;

(*ii*) $(\bar{\lambda}_if_i+\bar{u}_{J_\circ}^Tg_{J_\circ})_{i\in K}$ is strictly second order $(F,\bar{\alpha},\bar{\rho},d)$-V-pseudoconvex at $\bar{y}$ with $\bar{\alpha}_i(\bar{x},\bar{y})=\delta(\bar{x},\bar{y}),\ i\in K$ and $(\bar{u}_jg_j)_{j\in J_\beta},\ \beta=1,2,\ldots,r$ is second order $(F,\tilde{\alpha},\tilde{\rho},d)$-V-quasiconvex at $\bar{y}$; and

(*iii*) $\bar{\rho}+\sum_{\beta=1}^{r}\tilde{\rho}_\beta\geqq 0$.

Then $\bar{x}=\bar{y}$.

Proof. We assume that $\bar{x}\neq\bar{y}$ and exhibit a contradiction. Since $\bar{x}\in S$ and $(\bar{y},\bar{u},\bar{\lambda},\bar{p})\in Y$, we have

$$\bar{u}_jg_j(\bar{x})\leqq 0\leqq\bar{u}_jg_j(\bar{y})-\frac{1}{2}\bar{p}^T\nabla^2\bar{u}_jg_j(\bar{y})\bar{p},\ j\in J_\beta,\ \beta=1,2,\ldots,r.$$

As $\tilde{\alpha}_j(\bar{x},\bar{y})>0,\ j\in J_\beta$, it follows that

$$\sum_{j\in J_\beta}\tilde{\alpha}_j(\bar{x},\bar{y})\bar{u}_jg_j(\bar{x})\leqq\sum_{j\in J_\beta}\tilde{\alpha}_j(\bar{x},\bar{y})\bar{u}_jg_j(\bar{y})-\tfrac{1}{2}\bar{p}^T\nabla^2\sum_{j\in J_\beta}\tilde{\alpha}_j(\bar{x},\bar{y})\bar{u}_jg_j(\bar{y})\bar{p},\ \beta=1,2,\ldots,r.$$

The second order $(F,\tilde{\alpha},\tilde{\rho},d)$-V-quasiconvexity of $(\bar{u}_jg_j)_{j\in J_\beta},\ \beta=1,2,\ldots,r$ at $\bar{y}$ gives

$$F\left(\bar{x},\bar{y};\sum_{j\in J_\beta}(\nabla\bar{u}_jg_j(\bar{y})+\nabla^2\bar{u}_jg_j(\bar{y})\bar{p})\right)+\tilde{\rho}_\beta d^2(\bar{x},\bar{y})\leqq 0. \tag{26}$$

The inequality (26) along with (18), hypothesis (*iii*) and the sublinearity of F yields

$$F\left(\bar{x},\bar{y};\sum_{i=1}^{k}(\nabla\bar{\lambda}_if_i(\bar{y})+\nabla^2\bar{\lambda}_if_i(\bar{y})\bar{p})+\sum_{j\in J_\circ}(\nabla\bar{u}_jg_j(\bar{y})+\nabla^2\bar{u}_jg_j(\bar{y})\bar{p})\right)$$
$$+\bar{\rho}d^2(\bar{x},\bar{y})\geqq 0.$$

On using strict second order $(F, \bar{\alpha}, \bar{\rho}, d)$-V-pseudoconvexity of $(\bar{\lambda}_i f_i + \bar{u}_{J_\circ}^T g_{J_\circ})_{i \in K}$ at $\bar{y}$, we obtain

$$\sum_{i=1}^{k} \bar{\alpha}_i(\bar{x}, \bar{y}) \left(\bar{\lambda}_i f_i(\bar{x}) + \sum_{j \in J_\circ} \bar{u}_j g_j(\bar{x}) \right) > \sum_{i=1}^{k} \bar{\alpha}_i(\bar{x}, \bar{y}) \left(\bar{\lambda}_i f_i(\bar{y}) + \sum_{j \in J_\circ} \bar{u}_j g_j(\bar{y}) \right)$$
$$-\frac{1}{2}\bar{p}^T \nabla^2 \left(\sum_{i=1}^{k} \bar{\alpha}_i(\bar{x}, \bar{y})(\bar{\lambda}_i f_i(\bar{y}) + \sum_{j \in J_\circ} \bar{u}_j g_j(\bar{y})) \right) \bar{p},$$

which by the feasibility of $\bar{x}$ for (P) gives

$$\sum_{i=1}^{k} \bar{\alpha}_i(\bar{x}, \bar{y}) \bar{\lambda}_i f_i(\bar{x}) > \sum_{i=1}^{k} \bar{\alpha}_i(\bar{x}, \bar{y}) \left(\bar{\lambda}_i f_i(\bar{y}) + \sum_{j \in J_\circ} \bar{u}_j g_j(\bar{y}) \right)$$
$$-\frac{1}{2}\bar{p}^T \nabla^2 \left(\sum_{i=1}^{k} \bar{\alpha}_i(\bar{x}, \bar{y})(\bar{\lambda}_i f_i(\bar{y}) + \sum_{j \in J_\circ} \bar{u}_j g_j(\bar{y})) \right) \bar{p}.$$

Since $\bar{\alpha}_i(\bar{x}, \bar{y}) = \delta(\bar{x}, \bar{y}),\ i \in K$, we obtain

$$\sum_{i=1}^{k} \bar{\lambda}_i f_i(\bar{x}) > \sum_{i=1}^{k} \bar{\lambda}_i f_i(\bar{y}) + \sum_{j \in J_\circ} \bar{u}_j g_j(\bar{y}) - \frac{1}{2}\bar{p}^T \nabla^2 \left(\sum_{i=1}^{k} \bar{\lambda}_i f_i(\bar{y}) + \sum_{j \in J_\circ} \bar{u}_j g_j(\bar{y}) \right) \bar{p},$$

a contradiction to hypothesis (i). Hence, $\bar{x} = \bar{y}$. □

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ON A FIFTH-ORDER DIFFERENCE EQUATION

STEVO STEVIĆ*, JOSEF DIBLÍK, BRATISLAV IRIČANIN, AND ZDENĚK ŠMARDA

Abstract. We investigate the following difference equation

$$x_n = \frac{x_{n-3}x_{n-4}x_{n-5}}{x_{n-1}x_{n-2}(a_n + b_n x_{n-3}x_{n-4}x_{n-5})}, \quad n \in \mathbb{N}_0,$$

where $(a_n)_{n\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0}$ are two real sequences and the initial values $x_{-5}, \dots, x_{-1}$ are real numbers. The case when the sequences $(a_n)_{n\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0}$ are constant is thoroughly studied. Our results considerably extend some results in the recent literature.

1. Introduction

There has been a great recent interest in nonlinear difference equations and systems of difference equations (see, for example, [1]-[6], [8]-[14], [18]-[43] and the references therein), and, among them, some renewed interest in the difference equations and systems which can be solved in closed form (see, for example, [1]-[4], [6], [8], [19], [22], [23], [26], [27], [29]-[37], [39]-[43] and the related references therein). For some classical methods for solving difference equations and systems see, for example, [7], [16] and [17]. Many of the papers in the theory deal with difference equations and systems which can be regarded as perturbations of solvable ones (see, for example, [25] and [38]), so that their solutions are frequently compared with the solutions of the solvable ones, or are connected with some other solvable equations as it is the case in [21], [25] and [38]. This fact also shows the importance of solvable difference equations and systems.

Among the papers in the area there are some which present formulas of some particular difference equations and/or systems of difference equations which are almost always proved by induction, but do not give any theoretical explanation related to the presented formulas and how the equations/systems can be solved. Paper [22] by S. Stević, in which a natural explanation is given for the formula presented in [8], motivated numerous authors to re-attract their interest in difference equations which can be solved in closed form. Various other explanations and extensions of some results in the literature can be also found in papers [26], [36] and [41].

It is said that the difference equation

$$x_n = f(x_{n-1}, \dots, x_{n-k}), \quad n \in \mathbb{N}_0,$$

where $k \in \mathbb{N}$, is solvable in closed form if every solution can be written in terms of the initial values $x_{-k}, \dots, x_{-1}$ and index n only.

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Paper [43] is one of the papers of above mentioned type. Namely, formulas for solutions of the next four difference equations

$$x_{n+1} = \frac{x_{n-2}x_{n-3}x_{n-4}}{x_nx_{n-1}(1 + x_{n-2}x_{n-3}x_{n-4})}, \quad n \in \mathbb{N}_0, \tag{1}$$

$$x_{n+1} = \frac{x_{n-2}x_{n-3}x_{n-4}}{x_nx_{n-1}(1 - x_{n-2}x_{n-3}x_{n-4})}, \quad n \in \mathbb{N}_0, \tag{2}$$

$$x_{n+1} = \frac{x_{n-2}x_{n-3}x_{n-4}}{x_nx_{n-1}(-1 + x_{n-2}x_{n-3}x_{n-4})}, \quad n \in \mathbb{N}_0, \tag{3}$$

$$x_{n+1} = \frac{x_{n-2}x_{n-3}x_{n-4}}{x_nx_{n-1}(-1 - x_{n-2}x_{n-3}x_{n-4})}, \quad n \in \mathbb{N}_0, \tag{4}$$

are presented in [43] and for some of them are given sketches of the inductive proofs, but there are no theoretical explanations for the formulas.

A natural problem is to extend the results in [43] and give theoretical explanations for formulas presented therein.

Here, we will study the next difference equation

$$x_n = \frac{x_{n-3}x_{n-4}x_{n-5}}{x_{n-1}x_{n-2}(a_n + b_nx_{n-3}x_{n-4}x_{n-5})}, \quad n \in \mathbb{N}_0, \tag{5}$$

where $(a_n)_{n\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0}$ are real sequences and the initial values $x_{-5}, \dots, x_{-1}$ are real numbers, which is a natural extension of equations (1)-(4) (we shifted the indices backward for one, since the equation in this form as well as the result might look clearer).

To deal with equation (5) we essentially use the idea in [22], later exploited in numerous papers, where a suitable change of variables is used so that the equation therein is transformed into a solvable difference equation (see, for example, [1], [2], [4], [19], [27], [29]-[31], [33]-[37], [39]-[42]).

Solution $(x_n)_{n\geq -s}$, of the difference equation

$$x_n = f(x_{n-1}, \dots, x_{n-s}), \quad n \in \mathbb{N}_0, \tag{6}$$

where $f : \mathbb{R}^s \to \mathbb{R}$, $s \in \mathbb{N}$, is called eventually periodic with period p, if there is an $n_1 \geq -s$ such that

$$x_{n+p} = x_n, \quad \text{for} \quad n \geq n_1.$$

It is called periodic with period p, if $n_1 = -s$. For some results in this area see, e.g. [5, 9, 10, 11, 12, 13, 14, 15, 18, 20, 24, 28] and the references therein.

Throughout the paper we use the following standard conventions

$$\sum_{j=k}^{l} a_j = 0, \quad \text{when} \quad k > l,$$

and

$$\prod_{j=k}^{k-1} b_j = 1,$$

where k and l are integers.

2. Formulas for well-defined solutions of equation (5)

Assume that $(x_n)_{n\geq -5}$ is a solution of equation (5). If $x_{-5} = 0$ or $x_{-4} = 0$ or $x_{-3} = 0$ and $x_{-2} \neq 0 \neq x_{-1}$, then from (5) we see that x_0 is or not defined (if $a_0 = 0$) or $x_0 = 0$, and consequently x_1 is not defined. If $x_{-2} = 0$ or $x_{-1} = 0$, then

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from (5) we see that x_0 is not defined. This means that if one of the initial values x_{-j}, $j \in \{1, \dots, 5\}$ is equal to zero, then such a solution is not defined.

Now assume that $x_{n_0} = 0$ for some $n_0 \in \mathbb{N}_0$, that x_i are defined when $-5 \le i \le n_0$, and that n_0 is the smallest index for which a member of the solution is equal to zero. Then, from (5) we see that $x_{n_0-3} = 0$ or $x_{n_0-4} = 0$ or $x_{n_0-5} = 0$, which along with (5) would imply respectively that x_{n_0-2} if $n_0 \ge 2$ is not defined, or x_{n_0-3} if $n_0 \ge 3$ is not defined, or x_{n_0-4} if $n_0 \ge 4$ is not defined, which would be a contradiction with the fact that x_n are defined for $-5 \le n \le n_0$.

If $n_0 = 2$, then we have that $x_{-1} = 0$ or $x_{-2} = 0$ or $x_{-3} = 0$, if $n_0 = 1$, then we have that $x_{-2} = 0$ or $x_{-3} = 0$ or $x_{-4} = 0$, while if $n_0 = 0$, then we have that $x_{-3} = 0$ or $x_{-4} = 0$ or $x_{-5} = 0$. So, in these three cases we have that at least one of the initial values is equal to zero, and consequently by previous considerations such solutions are not defined.

If $n_0 = 3$, then from (5) we have that $x_0 = 0$ or $x_{-1} = 0$ or $x_{-2} = 0$. If $x_{-1} = 0$ or $x_{-2} = 0$, then x_0 is not defined, while the case $x_0 = 0$ has been previously considered.

If $n_0 = 4$, then from (5) we have that $x_1 = 0$ or $x_0 = 0$ or $x_{-1} = 0$. If $x_{-1} = 0$, then x_0 is not defined, while the cases $x_1 = 0$ or $x_0 = 0$ have been previously considered. Thus, according to all above mentioned such solutions are not defined.

Hence of some interest are solutions for which

$$x_{-j} \ne 0, \quad j \in \{1, \dots, 5\},$$

since for them it must be

$$x_n \ne 0, \qquad n \ge -5. \tag{7}$$

Now assume that $(x_n)_{n \ge -5}$ is a well-defined solution of equation (5). By previous considerations we have that (7) holds, so that for every well-defined solution we can use the following change of variables

$$y_n = \frac{1}{x_n x_{n-1} x_{n-2}}, \quad n \ge -3, \tag{8}$$

which transforms equation (5) into the following linear third-order difference equation

$$y_n = a_n y_{n-3} + b_n, \quad n \in \mathbb{N}_0. \tag{9}$$

Now note that every integer $n \ge -3$ can be written in the following form $n = 3m + i$, for some $m \ge -1$ and $i \in \{0, 1, 2\}$. Hence, equation (9) can be written in the next form

$$y_{3m+i} = a_{3m+i} y_{3(m-1)+i} + b_{3m+i}, \quad m \in \mathbb{N}_0, \tag{10}$$

where $i \in \{0, 1, 2\}$.

This means that the sequences $(y_{3m+i})_{m \ge -1}$, $i \in \{0, 1, 2\}$, are solutions of the following three linear first order difference equations

$$z_m = a_{3m+i} z_{m-1} + b_{3m+i}, \quad m \in \mathbb{N}_0, \tag{11}$$

$i \in \{0, 1, 2\}$.

The linear first order difference equation is solved in closed form and by using well-known formula for its solution we have that

$$y_{3m+i} = y_{i-3} \prod_{j=0}^{m} a_{3j+i} + \sum_{l=0}^{m} b_{3l+i} \prod_{j=l+1}^{m} a_{3j+i}, \quad m \in \mathbb{N}_0, \tag{12}$$

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$i \in \{0,1,2\}$. This formula can be easily obtained, for example, if we multiply the next equalities

$$y_{3l+i} = a_{3l+i}y_{3(l-1)+i} + b_{3l+i},$$

by $\prod_{j=l+1}^{m} a_{3j+i}$, $l = 0,1,\dots,m$, and sum up such obtained equalities ([17]).

Now we find formulas for well-defined solutions of equation (5). From (8) with $n = 3m+i$, we have that

$$\begin{aligned} x_{3m+i} &= \frac{1}{y_{3m+i}x_{3m+i-1}x_{3m+i-2}} \\ &= \frac{x_{3m+i-3}}{y_{3m+i}x_{3m+i-1}x_{3m+i-2}x_{3m+i-3}} = \frac{y_{3m+i-1}}{y_{3m+i}}x_{3(m-1)+i}. \end{aligned} \tag{13}$$

By repeating use of (13) we obtain

$$x_{3m} = x_{-3}\prod_{s=0}^{m}\frac{y_{3s-1}}{y_{3s}}, \quad m \ge -1, \tag{14}$$

and

$$x_{3m+i} = x_{i-6}\prod_{s=-1}^{m}\frac{y_{3s+i-1}}{y_{3s+i}}, \quad m \ge -1, \tag{15}$$

for $i \in \{1,2\}$.

Using formula (12) in (14) and (15) we obtain formulas for general solution of equation (5)

$$\begin{aligned} x_{3m} &= x_{-3}\prod_{s=0}^{m}\frac{y_{3(s-1)+2}}{y_{3s}} \\ &= x_{-3}\prod_{s=0}^{m}\frac{y_{-1}\prod_{j=0}^{s-1}a_{3j+2} + \sum_{l=0}^{s-1}b_{3l+2}\prod_{j=l+1}^{s-1}a_{3j+2}}{y_{-3}\prod_{j=0}^{s}a_{3j} + \sum_{l=0}^{s}b_{3l}\prod_{j=l+1}^{s}a_{3j}} \\ &= x_{-3}\prod_{s=0}^{m}\frac{(x_{-1}x_{-2}x_{-3})^{-1}\prod_{j=0}^{s-1}a_{3j+2} + \sum_{l=0}^{s-1}b_{3l+2}\prod_{j=l+1}^{s-1}a_{3j+2}}{(x_{-3}x_{-4}x_{-5})^{-1}\prod_{j=0}^{s}a_{3j} + \sum_{l=0}^{s}b_{3l}\prod_{j=l+1}^{s}a_{3j}}, \end{aligned} \tag{16}$$

$m \ge -1$, and

$$\begin{aligned} x_{3m+i} &= x_{i-6}\prod_{s=-1}^{m}\frac{y_{3s+i-1}}{y_{3s+i}} \\ &= x_{i-6}\prod_{s=-1}^{m}\frac{y_{i-4}\prod_{j=0}^{s}a_{3j+i-1} + \sum_{l=0}^{s}b_{3l+i-1}\prod_{j=l+1}^{s}a_{3j+i-1}}{y_{i-3}\prod_{j=0}^{s}a_{3j+i} + \sum_{l=0}^{s}b_{3l+i}\prod_{j=l+1}^{s}a_{3j+i}} \\ &= x_{i-6}\prod_{s=-1}^{m}\frac{(x_{i-4}x_{i-5}x_{i-6})^{-1}\prod_{j=0}^{s}a_{3j+i-1} + \sum_{l=0}^{s}b_{3l+i-1}\prod_{j=l+1}^{s}a_{3j+i-1}}{(x_{i-3}x_{i-4}x_{i-5})^{-1}\prod_{j=0}^{s}a_{3j+i} + \sum_{l=0}^{s}b_{3l+i}\prod_{j=l+1}^{s}a_{3j+i}}, \end{aligned} \tag{17}$$

for $m \ge -1$ and for $i \in \{1,2\}$.

3. Case when sequences a_n and b_n are constant

In this section we consider the case when the sequences $(a_n)_{n\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0}$ are constant, that is, when

$$a_n = a, \qquad b_n = b, \qquad n \in \mathbb{N}_0.$$

In this case equation (5) becomes

$$x_n = \frac{x_{n-3}x_{n-4}x_{n-5}}{x_{n-1}x_{n-2}(a + bx_{n-3}x_{n-4}x_{n-5})}, \quad n \in \mathbb{N}_0. \tag{18}$$

By using formulas (16) and (17) in this case we obtain

$$\begin{aligned} x_{3m} &= x_{-3}\prod_{s=0}^{m}\frac{(x_{-1}x_{-2}x_{-3})^{-1}\prod_{j=0}^{s-1}a + \sum_{l=0}^{s-1}b\prod_{j=l+1}^{s-1}a}{(x_{-3}x_{-4}x_{-5})^{-1}\prod_{j=0}^{s}a + \sum_{l=0}^{s}b\prod_{j=l+1}^{s}a} \\ &= x_{-3}\prod_{s=0}^{m}\frac{(x_{-1}x_{-2}x_{-3})^{-1}a^{s} + b\sum_{l=0}^{s-1}a^{s-l-1}}{(x_{-3}x_{-4}x_{-5})^{-1}a^{s+1} + b\sum_{l=0}^{s}a^{s-l}}, \end{aligned} \tag{19}$$

$m \ge -1$, and

$$\begin{aligned} x_{3m+i} &= x_{i-6}\prod_{s=-1}^{m}\frac{(x_{i-4}x_{i-5}x_{i-6})^{-1}\prod_{j=0}^{s}a + \sum_{l=0}^{s}b\prod_{j=l+1}^{s}a}{(x_{i-3}x_{i-4}x_{i-5})^{-1}\prod_{j=0}^{s}a + \sum_{l=0}^{s}b\prod_{j=l+1}^{s}a} \\ &= x_{i-6}\prod_{s=-1}^{m}\frac{(x_{i-4}x_{i-5}x_{i-6})^{-1}a^{s+1} + b\sum_{l=0}^{s}a^{s-l}}{(x_{i-3}x_{i-4}x_{i-5})^{-1}a^{s+1} + b\sum_{l=0}^{s}a^{s-l}}, \end{aligned} \tag{20}$$

for $m \ge -1$ and for $i \in \{1, 2\}$.

We have now two cases.

3.1. Case $a \neq 1$. In this case formulas (19) and (20) become

$$\begin{aligned} x_{3m} &= x_{-3}\prod_{s=0}^{m}\frac{(x_{-1}x_{-2}x_{-3})^{-1}a^{s} + b\sum_{l=0}^{s-1}a^{s-l-1}}{(x_{-3}x_{-4}x_{-5})^{-1}a^{s+1} + b\sum_{l=0}^{s}a^{s-l}} \\ &= x_{-3}\prod_{s=0}^{m}\frac{(x_{-1}x_{-2}x_{-3})^{-1}a^{s}(1-a) + b(1-a^{s})}{(x_{-3}x_{-4}x_{-5})^{-1}a^{s+1}(1-a) + b(1-a^{s+1})} \\ &= x_{-3}\prod_{s=0}^{m}\frac{a^{s}((1-a)(x_{-1}x_{-2}x_{-3})^{-1} - b) + b}{a^{s+1}((1-a)(x_{-3}x_{-4}x_{-5})^{-1} - b) + b}, \end{aligned} \tag{21}$$

$m \ge -1$, and

$$\begin{aligned} x_{3m+i} &= x_{i-6}\prod_{s=-1}^{m}\frac{(x_{i-4}x_{i-5}x_{i-6})^{-1}a^{s+1} + b\sum_{l=0}^{s}a^{s-l}}{(x_{i-3}x_{i-4}x_{i-5})^{-1}a^{s+1} + b\sum_{l=0}^{s}a^{s-l}} \\ &= x_{i-6}\prod_{s=-1}^{m}\frac{(x_{i-4}x_{i-5}x_{i-6})^{-1}a^{s+1}(1-a) + b(1-a^{s+1})}{(x_{i-3}x_{i-4}x_{i-5})^{-1}a^{s+1}(1-a) + b(1-a^{s+1})} \\ &= x_{i-6}\prod_{s=-1}^{m}\frac{a^{s+1}((1-a)(x_{i-4}x_{i-5}x_{i-6})^{-1} - b) + b}{a^{s+1}((1-a)(x_{i-3}x_{i-4}x_{i-5})^{-1} - b) + b}, \end{aligned} \tag{22}$$

for $m \ge -1$ and for $i \in \{1, 2\}$.

3.2. **Case** $a = 1$**.** In this case formulas (19) and (20) become

$$x_{3m} = x_{-3} \prod_{s=0}^{m} \frac{(x_{-1}x_{-2}x_{-3})^{-1} + bs}{(x_{-3}x_{-4}x_{-5})^{-1} + b(s+1)}, \tag{23}$$

$m \geq -1$, and

$$x_{3m+i} = x_{i-6} \prod_{s=-1}^{m} \frac{(x_{i-4}x_{i-5}x_{i-6})^{-1} + b(s+1)}{(x_{i-3}x_{i-4}x_{i-5})^{-1} + b(s+1)}, \tag{24}$$

for $m \geq -1$ and for $i \in \{1, 2\}$.

3.3. **Asymptotic behavior of solutions of equation** (18)**.** Here we study the asymptotic behavior of well-defined solutions of equation (18). Prior to stating and proving our results we introduce some quantities which will be used in the statements of the results.

Set

$$L_0 = \frac{(1-a)(x_{-1}x_{-2}x_{-3})^{-1} - b}{a((1-a)(x_{-3}x_{-4}x_{-5})^{-1} - b)},$$

$$L_i = \frac{(1-a)(x_{i-4}x_{i-5}x_{i-6})^{-1} - b}{(1-a)(x_{i-3}x_{i-4}x_{i-5})^{-1} - b},$$

where $i = 1, 2$, and set

$$K_0 = \frac{1}{x_{-1}x_{-2}x_{-3}b} - \frac{1}{x_{-3}x_{-4}x_{-5}b} - 1,$$

$$K_i = \frac{1}{b}\left(\frac{1}{x_{i-4}x_{i-5}x_{i-6}} - \frac{1}{x_{i-3}x_{i-4}x_{i-5}}\right),$$

where $i = 1, 2$.

Our first result considers the case $|a| > 1$, $b \neq 0$.

Theorem 1. *Assume that $|a| > 1$, $b \neq 0$, and $(x_n)_{n \geq -5}$ is a well-defined solution of equation* (18)*. Then the following statements are true.*

(a) If $|L_0| < 1$, then $x_{3m} \to 0$ as $m \to +\infty$.
(b) If $|L_0| > 1$, then $|x_{3m}| \to +\infty$ as $m \to +\infty$.
(c) If $L_0 = 1$, then the sequence $(x_{3m})_{m \geq -1}$ is constant.
(d) If $L_0 = -1$, then the sequences $(x_{6m})_{m \in \mathbb{N}_0}$ and $(x_{6m+3})_{m \geq -1}$ are convergent.
(e) If $|L_i| < 1$, for some $i \in \{1, 2\}$, then $x_{3m+i} \to 0$ as $m \to +\infty$.
(f) If $|L_i| > 1$, for some $i \in \{1, 2\}$, then $|x_{3m+i}| \to +\infty$ as $m \to +\infty$.
(g) If $L_i = 1$, for some $i \in \{1, 2\}$, then the sequence $(x_{3m+i})_{m \geq -2}$ is constant.
(h) If $L_i = -1$, for some $i \in \{1, 2\}$, then the sequences $(x_{6m+i})_{m \geq -1}$ and $(x_{6m+3+i})_{m \geq -1}$ are convergent.

Proof. (a), (b) Let

$$p_s = \frac{a^s((1-a)(x_{-1}x_{-2}x_{-3})^{-1} - b) + b}{a^{s+1}((1-a)(x_{-3}x_{-4}x_{-5})^{-1} - b) + b}. \tag{25}$$

Then we have

$$\lim_{s \to +\infty} p_s = \lim_{s \to +\infty} \frac{((1-a)(x_{-1}x_{-2}x_{-3})^{-1} - b)/a + \frac{b}{a^{s+1}}}{(1-a)(x_{-3}x_{-4}x_{-5})^{-1} - b + \frac{b}{a^{s+1}}} = L_0. \tag{26}$$

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From (21) and (26), these two statements easily follow.

(c) In this case, we have that

$$p_s = \frac{((1-a)(x_{-1}x_{-2}x_{-3})^{-1} - b)/a + \frac{b}{a^{s+1}}}{(1-a)(x_{-3}x_{-4}x_{-5})^{-1} - b + \frac{b}{a^{s+1}}} = 1, \quad s \in \mathbb{N}_0. \tag{27}$$

From (21) and (27) the result easily follows.

(d) Since $L_0 = -1$, and by using the asymptotic relation

$$\frac{1}{1+x} = 1 - x + O(x^2), \tag{28}$$

when x is close to the origin, we have that

$$\begin{aligned} p_s &= -\frac{(1-a)(x_{-3}x_{-4}x_{-5})^{-1} - b - \frac{b}{a^{s+1}}}{(1-a)(x_{-3}x_{-4}x_{-5})^{-1} - b + \frac{b}{a^{s+1}}} \\ &= -\left(1 - \frac{b}{a^{s+1}((1-a)(x_{-3}x_{-4}x_{-5})^{-1} - b)}\right) \\ &\quad \times \left(1 - \frac{b}{a^{s+1}((1-a)(x_{-3}x_{-4}x_{-5})^{-1} - b)} + O\Big(\frac{1}{a^{2s}}\Big)\right) \\ &= -\left(1 + O\Big(\frac{1}{a^s}\Big)\right), \end{aligned} \tag{29}$$

for large enough s. From (21), (29), the assumption $|a| > 1$, and by a known criterion for the convergence of products the result easily follows.

(e), (f) Let

$$q_s^i = \frac{a^{s+1}((1-a)(x_{i-4}x_{i-5}x_{i-6})^{-1} - b) + b}{a^{s+1}((1-a)(x_{i-3}x_{i-4}x_{i-5})^{-1} - b) + b}, \quad i = 1, 2. \tag{30}$$

Then we have

$$\lim_{s\to+\infty} q_s^i = \lim_{s\to+\infty} \frac{(1-a)(x_{i-4}x_{i-5}x_{i-6})^{-1} - b + \frac{b}{a^{s+1}}}{(1-a)(x_{i-3}x_{i-4}x_{i-5})^{-1} - b + \frac{b}{a^{s+1}}} = L_i, \quad i = 1, 2. \tag{31}$$

From (22) and (31), these two statements easily follow.

(g) In this case we have that

$$q_s^i = \frac{(1-a)(x_{i-4}x_{i-5}x_{i-6})^{-1} - b + \frac{b}{a^{s+1}}}{(1-a)(x_{i-3}x_{i-4}x_{i-5})^{-1} - b + \frac{b}{a^{s+1}}} = 1, \quad s \in \mathbb{N}_0. \tag{32}$$

From (22) and (32) the result easily follows.

(h) Since $L_i = -1$ and by using (28), we have that

$$\begin{aligned} q_s^i &= -\frac{(1-a)(x_{i-3}x_{i-4}x_{i-5})^{-1} - b - \frac{b}{a^{s+1}}}{(1-a)(x_{i-3}x_{i-4}x_{i-5})^{-1} - b + \frac{b}{a^{s+1}}} \\ &= -\left(1 - \frac{b}{a^{s+1}((1-a)(x_{i-3}x_{i-4}x_{i-5})^{-1} - b)}\right) \\ &\quad \times \left(1 - \frac{b}{a^{s+1}((1-a)(x_{i-3}x_{i-4}x_{i-5})^{-1} - b)} + O\Big(\frac{1}{a^{2s}}\Big)\right) \\ &= -\left(1 + O\Big(\frac{1}{a^s}\Big)\right), \end{aligned} \tag{33}$$

for large enough s. From (22), (33), the assumption $|a| > 1$, and by a known criterion for the convergence of products the result easily follows. □

Now we consider the case $|a| < 1$, $b \neq 0$.

Theorem 2. *Assume that $|a| < 1$, $b \neq 0$, and $(x_n)_{n\geq -5}$ is a well-defined solution of equation* (18). *Then the sequences $(x_{3m+i})_{m\geq -1}$, $i \in \{0,1,2\}$, are convergent.*

Proof. From (25) and (28), we have

$$\begin{aligned} p_s =& \frac{1 + a^s((1-a)(x_{-1}x_{-2}x_{-3})^{-1} - b)/b}{1 + a^{s+1}((1-a)(x_{-3}x_{-4}x_{-5})^{-1} - b)/b} \\ =&(1 + a^s((1-a)(x_{-1}x_{-2}x_{-3})^{-1} - b)/b) \\ &\times (1 - a^{s+1}((1-a)(x_{-3}x_{-4}x_{-5})^{-1} - b)/b + O(a^{2s})) \\ =&1 + O(a^s), \end{aligned} \tag{34}$$

for large enough s, while from (30) and by using (28), we have

$$\begin{aligned} q_s^i =& \frac{1 + a^{s+1}((1-a)(x_{i-4}x_{i-5}x_{i-6})^{-1} - b)/b}{1 + a^{s+1}((1-a)(x_{i-3}x_{i-4}x_{i-5})^{-1} - b)/b} \\ =&(1 + a^{s+1}((1-a)(x_{i-4}x_{i-5}x_{i-6})^{-1} - b)/b) \\ &\times (1 - a^{s+1}((1-a)(x_{i-3}x_{i-4}x_{i-5})^{-1} - b)/b + O(a^{2s})) \\ =&1 + O(a^s), \end{aligned} \tag{35}$$

for large enough s and $i \in \{1,2\}$. From (21), (22), (34), (35), the assumption $|a| < 1$, and by a known criterion for the convergence of products we have that the sequences $(x_{3m+i})_{m\geq -1}$, $i \in \{0,1,2\}$, are convergent. $\square$

Now we consider the case $a = 1$, $b \neq 0$.

Theorem 3. *Assume that $a = 1$, $b \neq 0$, and $(x_n)_{n\geq -5}$ is a well-defined solution of equation* (18). *Then the following statements are true.*

(a) If $a = 1$ and $K_0 < 0$, then $x_{3m} \to 0$ as $m \to +\infty$.
(b) If $a = 1$ and $K_0 > 0$, then $|x_{3m}| \to +\infty$ as $m \to +\infty$.
(c) If $a = 1$ and $K_0 = 0$, then the sequence $(x_{3m})_{m\geq -1}$ is constant.
(d) If $a = 1$ and $K_i < 0$, for some $i \in \{1,2\}$, then $x_{3m+i} \to 0$ as $m \to +\infty$.
(e) If $a = 1$ and $K_i > 0$, for some $i \in \{1,2\}$, then $|x_{3m+i}| \to +\infty$ as $m \to +\infty$.
(f) If $a = 1$ and $K_i = 0$, for some $i \in \{1,2\}$, then the sequence $(x_{3m+i})_{m\geq -1}$ is constant.

Proof. (a), (b) Let

$$r_s = \frac{(x_{-1}x_{-2}x_{-3})^{-1} + bs}{(x_{-3}x_{-4}x_{-5})^{-1} + b + bs}. \tag{36}$$

From (36) and by using (28) we have that

$$\begin{aligned} r_s =& \frac{1 + \frac{1}{bx_{-1}x_{-2}x_{-3}s}}{1 + \frac{(bx_{-3}x_{-4}x_{-5})^{-1}+1}{s}} \\ =&\Big(1 + \frac{1}{bx_{-1}x_{-2}x_{-3}s}\Big)\Big(1 - \frac{(bx_{-3}x_{-4}x_{-5})^{-1}+1}{s} + O\Big(\frac{1}{s^2}\Big)\Big) \\ =&1 + \frac{K_0}{s} + O\Big(\frac{1}{s^2}\Big), \end{aligned} \tag{37}$$

for large enough s. From (23) and (37), and known criteria for the convergence of products, these two statements easily follow.

(c) Note that in this case $r_s = 1$, $s \in \mathbb{N}_0$, from which the result follows.

(d), (e) Let

$$r_s^i = \frac{(x_{i-4}x_{i-5}x_{i-6})^{-1} + b + bs}{(x_{i-3}x_{i-4}x_{i-5})^{-1} + b + bs}, \quad i = 1, 2. \tag{38}$$

From (38) and by using (28) we have that

$$\begin{aligned} r_s^i =& \frac{1 + \frac{(bx_{i-4}x_{i-5}x_{i-6})^{-1}+1}{s}}{1 + \frac{(bx_{i-3}x_{i-4}x_{i-5})^{-1}+1}{s}} \\ =& \Big(1 + \frac{(bx_{i-4}x_{i-5}x_{i-6})^{-1}+1}{s}\Big)\Big(1 - \frac{(bx_{i-3}x_{i-4}x_{i-5})^{-1}+1}{s} + O\Big(\frac{1}{s^2}\Big)\Big) \\ =& 1 + \frac{K_i}{s} + O\Big(\frac{1}{s^2}\Big), \end{aligned} \tag{39}$$

for large enough s and $i \in \{1, 2\}$. From (24) and (39), and known criteria for the convergence of products, these two statements easily follow.

(f) Note that in this case $r_s^i = 1$, $i = 1, 2$, $s \in \mathbb{N}_0$, from which the result follows. □

If $a = -1$ and $b \neq 0$, then from (21) and (22), we have that

$$x_{3m} = x_{-3} \prod_{s=0}^{m} \frac{(-1)^s(2(x_{-1}x_{-2}x_{-3})^{-1} - b) + b}{(-1)^{s+1}(2(x_{-3}x_{-4}x_{-5})^{-1} - b) + b}, \quad m \geq -1, \tag{40}$$

and

$$x_{3m+i} = x_{i-6} \prod_{s=-1}^{m} \frac{(-1)^{s+1}(2(x_{i-4}x_{i-5}x_{i-6})^{-1} - b) + b}{(-1)^{s+1}(2(x_{i-3}x_{i-4}x_{i-5})^{-1} - b) + b}, \quad m \geq -1, \tag{41}$$

for some $i \in \{1, 2\}$.

Hence, by using formulas (40) and (41) we have that

$$x_{6m} = x_{-3} \frac{(x_{-1}x_{-2}x_{-3})^{-1}}{b - (x_{-3}x_{-4}x_{-5})^{-1}} \left(\frac{(x_{-1}x_{-2}x_{-3})^{-1}(b - (x_{-1}x_{-2}x_{-3})^{-1})}{(x_{-3}x_{-4}x_{-5})^{-1}(b - (x_{-3}x_{-4}x_{-5})^{-1})}\right)^m, \tag{42}$$

$$x_{6m+3} = x_{-3} \left(\frac{(x_{-1}x_{-2}x_{-3})^{-1}(b - (x_{-1}x_{-2}x_{-3})^{-1})}{(x_{-3}x_{-4}x_{-5})^{-1}(b - (x_{-3}x_{-4}x_{-5})^{-1})}\right)^{m+1}, \tag{43}$$

$$x_{6m+i} = x_{i-6} \left(\frac{(x_{i-4}x_{i-5}x_{i-6})^{-1}(b - (x_{i-4}x_{i-5}x_{i-6})^{-1})}{(x_{i-3}x_{i-4}x_{i-5})^{-1}(b - (x_{i-3}x_{i-4}x_{i-5})^{-1})}\right)^{m+1}, \tag{44}$$

$$x_{6m+3+i} = x_{i-3} \left(\frac{(x_{i-4}x_{i-5}x_{i-6})^{-1}(b - (x_{i-4}x_{i-5}x_{i-6})^{-1})}{(x_{i-3}x_{i-4}x_{i-5})^{-1}(b - (x_{i-3}x_{i-4}x_{i-5})^{-1})}\right)^{m+1}, \tag{45}$$

for $i \in \{1, 2\}$.

From (42)-(45) it is not difficult to describe the asymptotic behavior of well-defined solutions of equation (18) for the case $a = -1$, in terms of the quantities

$$N_0 := \frac{(x_{-1}x_{-2}x_{-3})^{-1}(b - (x_{-1}x_{-2}x_{-3})^{-1})}{(x_{-3}x_{-4}x_{-5})^{-1}(b - (x_{-3}x_{-4}x_{-5})^{-1})}$$

and

$$N_i := \frac{(x_{i-4}x_{i-5}x_{i-6})^{-1}(b-(x_{i-4}x_{i-5}x_{i-6})^{-1})}{(x_{i-3}x_{i-4}x_{i-5})^{-1}(b-(x_{i-3}x_{i-4}x_{i-5})^{-1})}, \quad i=1,2.$$

Namely, it is easy to see that the following result holds.

Theorem 4. *Assume that $a=-1$, $b\neq 0$, and $(x_n)_{n\geq -5}$ is a well-defined solution of equation* (18)*. Then the following statements are true.*

(a) If $|N_i|<1$ for some $i\in\{0,1,2\}$, then $x_{6m+3j+i}\to 0$, $j=0,1$, as $m\to+\infty$.
(b) If $|N_i|>1$ for some $i\in\{0,1,2\}$, then $|x_{6m+3j+i}|\to+\infty$, $j=0,1$, as $m\to+\infty$.
(c) If $N_i=1$ for some $i\in\{0,1,2\}$, then the sequences $(x_{6m+3j+i})_{m\geq -1}$, $j=0,1$, are constant.
(d) If $N_i=-1$ for some $i\in\{0,1,2\}$, then the sequences $(x_{6m+3j+i})_{m\geq -1}$, $j=0,1$, are two-periodic.

Now we consider the case $a\neq 0$, $b=0$. In this case equation (18) becomes

$$x_n = \frac{x_{n-3}x_{n-4}x_{n-5}}{x_{n-1}x_{n-2}a}, \quad n\in\mathbb{N}_0, \tag{46}$$

and formulas (21)-(24) also hold. Hence for $a\in\mathbb{R}\setminus\{0\}$, we have

$$x_{3m} = x_{-3}\left(\frac{x_{-4}x_{-5}}{ax_{-1}x_{-2}}\right)^{m+1}, \tag{47}$$

$m\geq -1$, and

$$x_{3m+i} = x_{i-3}\left(\frac{x_{i-3}}{x_{i-6}}\right)^{m+1}, \tag{48}$$

for $m\geq -1$ and for $i\in\{1,2\}$.

Let

$$L_3 := \frac{x_{-4}x_{-5}}{ax_{-1}x_{-2}} \quad \text{and} \quad L_{3+i} := \frac{x_{i-3}}{x_{i-6}}, \quad i\in\{1,2\}.$$

By using formulas (47) and (48) it is easy to see that the following result holds. We omit the proof.

Theorem 5. *Assume that $a\neq 0$, $b=0$, and $(x_n)_{n\geq -5}$ is a well-defined solution of equation* (18)*. Then the following statements are true.*

(a) If $|L_{3+i}|<1$, for some $i\in\{0,1,2\}$, then $x_{3m+i}\to 0$ as $m\to+\infty$.
(b) If $|L_{3+i}|>1$, for some $i\in\{0,1,2\}$, then $|x_{3m+i}|\to\infty$ as $m\to+\infty$.
(c) If $L_{3+i}=1$, for some $i\in\{0,1,2\}$, then the sequence $(x_{3m+i})_{m\geq -2}$ is constant.
(d) If $L_{3+i}=-1$, for some $i\in\{0,1,2\}$, then the sequence $(x_{3m+i})_{m\geq -2}$ is two-periodic.

4. Domain of undefinable solutions of equation (5)

In Section 2 we proved that solutions of equation (5) for which is $x_{-j} = 0$ for some $j \in \{1, 2, 3, 4, 5\}$ are not defined. The set of all such initial values is characterized here.

Definition 1. ([32]) Consider the difference equation

$$x_n = f(x_{n-1}, \dots, x_{n-s}, n), \quad n \in \mathbb{N}_0, \tag{49}$$

where $s \in \mathbb{N}$, and $x_{-i} \in \mathbb{R}$, $i = \overline{1, s}$. The string of numbers $x_{-s}, \dots, x_{-1}, x_0, \dots, x_{n_0}$ where $n_0 \geq -1$, is called an *undefined solution* of equation (49) if

$$x_j = f(x_{j-1}, \dots, x_{j-s}, j)$$

for $0 \leq j < n_0 + 1$, and x_{n_0+1} is not defined number, that is, the quantity $f(x_{n_0}, \dots, x_{n_0-s+1}, n_0 + 1)$ is not defined.

The set of all initial values $x_{-s}, \dots, x_{-1}$ which generate undefined solutions of equation (49) is called *domain of undefinable solutions* of the equation.

The next result characterizes the domain of undefinable solutions of equation (5) for the case $a_n \neq 0$, $b_n \neq 0$, $n \in \mathbb{N}_0$.

Theorem 6. *Assume that $a_n \neq 0$, $b_n \neq 0$, $n \in \mathbb{N}_0$. Then the domain of undefinable solutions of equation (5) is the following set*

$$\mathcal{U} = \bigcup_{m \in \mathbb{N}_0} \bigcup_{i=0}^{2} \Big\{ (x_{-5}, \dots, x_{-1}) \in \mathbb{R}^5 : x_{i-3}x_{i-4}x_{i-5} = \frac{1}{c_m}, \ \textit{when} \ c_m := -\sum_{j=0}^{m} \frac{b_{3j+i}}{a_{3j+i}} \prod_{l=0}^{j-1} \frac{1}{a_{3l+i}} \neq 0 \Big\}$$
$$\bigcup \bigcup_{j=1}^{5} \Big\{ (x_{-5}, \dots, x_{-1}) \in \mathbb{R}^5 : x_{-j} = 0 \Big\}. \tag{50}$$

Proof. As we have already mentioned the set

$$\bigcup_{j=1}^{5} \Big\{ (x_{-5}, \dots, x_{-1}) \in \mathbb{R}^5 : x_{-j} = 0 \Big\},$$

belongs to the domain of undefinable solutions of equation (5).

Now we will consider the case when $x_{-j} \neq 0$, $j = \overline{1, 5}$ (i.e. $x_n \neq 0$ for every $n \geq -5$). Such a solution $(x_n)_{n \geq -5}$ is not defined if

$$x_{n-3}x_{n-4}x_{n-5} = -\frac{a_n}{b_n} \tag{51}$$

for some $n \in \mathbb{N}_0$.

Since the change of variables (8) implies that equation (5) is transformed to the equations in (10), this along with (51) implies that the solution is not defined if

$$y_{3(m-1)+i} = -\frac{b_{3m+i}}{a_{3m+i}} \tag{52}$$

for some $m \in \mathbb{N}_0$ and $i \in \{0, 1, 2\}$.

Set

$$f_{3m+i}(t) := a_{3m+i}t + b_{3m+i}, \quad m \in \mathbb{N}_0, \ i \in \{0, 1, 2\}.$$

Then $f^{-1}_{3m+i}(t) = (t - b_{3m+i})/a_{3m+i}$, $m \in \mathbb{N}_0$, $i \in \{0, 1, 2\}$, so that

$$f^{-1}_{3m+i}(0) = -\frac{b_{3m+i}}{a_{3m+i}}, \quad m \in \mathbb{N}_0, \ i \in \{0, 1, 2\}. \tag{53}$$

Now write equations in (10) as

$$y_{3m+i} = f_{3m+i}(y_{3(m-1)+i}), \quad m \in \mathbb{N}_0,$$

for $i \in \{0, 1, 2\}$.

Then, we have

$$y_{3m+i} = f_{3m+i} \circ f_{3(m-1)+i} \circ \cdots \circ f_i(y_{i-3}), \quad m \in \mathbb{N}_0,\ i \in \{0, 1, 2\}. \tag{54}$$

From (53) and (54) we have that (52) holds for some $m \in \mathbb{N}_0$, $i \in \{0, 1, 2\}$, if and only if

$$y_{i-3} = f_i^{-1} \circ \cdots \circ f_{3m+i}^{-1}(0),$$

that is,

$$y_{i-3} = -\sum_{j=0}^{m} \frac{b_{3j+i}}{a_{3j+i}} \prod_{l=0}^{j-1} \frac{1}{a_{3l+i}},$$

for some $m \in \mathbb{N}_0$ and $i \in \{0, 1, 2\}$, which along with the relations

$$y_{i-3} = \frac{1}{x_{i-3}x_{i-4}x_{i-5}}, \quad i \in \{0, 1, 2\},$$

implies that the first union in (50) belongs to the domain of undefinable solutions too, as desired. □

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MODIFIED THREE-STEP ITERATIVE SCHEMES FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN UNIFORMLY CONVEX METRIC SPACES

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Abstract. We prove the existence of the common fixed point for three asymptotically nonexpensive mappings defined on a $\mathcal{A}$-uniformly convex metric space. A three-step iterative scheme is constructed which converges to the common fixed point. We also generalize the results of several authors.

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1. Introduction

It is well known that the parallelogram law distinguishes the Hilbert spaces from the general Banach spaces. Recently many authors have introduced the idea for solving problems in Banach spaces by establishing identities and inequalities analogous to the parallelogram law (see for example [9, 20]).

In 1965, the Banach contraction principle was extended to nonexpansive mappings by Browder [3], Goehde [7] and Kirk [10]. In [10], Kirk proved that

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there exists a k-Lipschitzian map which has no fixed point. Goebel and Kirk [6] introduced the notion of asymptotically nonexpansive mappings and obtained a generalization of the results obtained in [3, 7, 10]. Afterwards Takahahashi [20] introduced the notion of convexity in metric spaces. Subsequently, Ćirić [5], Guay et al. [8], Shimizu and Takahashi [15] and many other authors have studied fixed point theorems on convex metric spaces. Shimizu and Takahashi [16] introduced the concept of uniform convexity in convex metric spaces and studied its properties.

Definition 1.1. ([18]) Let (X, d) be a metric space. A mapping $W : X \times X \times [0,1] \to X$ is said to be a *convex structure* on X if for each $(x, y, \lambda) \in X \times X \times [0,1]$ and $u \in X$,

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda) d(u, y).$$

The metric space X together with W is called a *convex metric space.*

Definition 1.2. Let X be a convex metric space. A nonempty subset A of X is said to be *convex* if $W(x, y, \lambda) \in A$ whenever $(x, y, \lambda) \in A \times A \times [0,1]$.

Takahashi [18] has shown that open spheres $B(x, r) = \{y \in X : d(x, y) < r\}$ and closed spheres $B(x, r) = \{y \in X : d(x, y) \leq r\}$ are convex. All normed spaces and their convex subsets are convex metric spaces. But there are many examples of convex metric spaces which are not embedded in any normed space (see Takahashi [18]).

Recently, Beg [1] introduced and studied the notion of 2-uniformly convex metric spaces.

Definition 1.3. ([1]) A convex metric space X is said to have *property* (B) if it satisfies $d(W(x, a, \alpha), W(y, a, \alpha)) = \alpha d(x, y)$. Taking $x = a$, property (B) implies $\alpha d(x, y) = W(y, a, \alpha)$.

Definition 1.4. ([1]) A convex complete metric space X is said to be *uniformly convex* if for all $x, y, a \in X$,

$$\begin{aligned} &[d\left(a, W\left(x, y, 1/2\right)\right)]^2 \\ &\leq \frac{1}{2}\left(1 - \delta\left(\frac{d(x, y)}{\max\left\{d(a, x), d(a, y)\right\}}\right)\right)\left([d(a, x)]^2 + [d(a, y)]^2\right), \end{aligned}$$

where the function δ is a strictly increasing function on the set of strictly positive numbers and $\delta(0) = 0$.

Remark 1.5. ([1]) Uniformly convex Banach spaces are uniformly convex metric spaces.

Definition 1.6. ([1]) A uniformly convex metric space X is said to be *2-uniformly convex* if there exists a constant $c > 0$ such that $\delta(\epsilon) \geq c\epsilon^2$.

Definition 1.7. (1) Let A be a nonempty subset of a metric space X. A mapping $T : A \to A$ is said to be *asymptotically nonexpansive* [6] if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\sum_{n=1}^{\infty}(k_n - 1) < +\infty$ such that

$$d(T^n x, T^n y) \leq k_n d(x, y) \quad \text{for all } x, y \in A,\ n \geq 1.$$

(2) T is said to be *uniformly L-Lipschitzian* with a Lipschitzian constant $L \geq 1$, i.e., there exists a constant $L \geq 1$ such that

$$d(T^n x, T^n y) \leq L d(x, y) \quad \text{for all } x, y \in A,\ n \geq 1.$$

This is a class of mapping introduced by Goebel and Kirk [6], where it is shown that if A is a nonempty bounded closed convex subset of a uniformly convex Banach space and $T : A \to A$ is asymptotically nonexpansive, then T has a fixed point and, moreover, the set $F(T)$ of fixed points of T is closed and convex.

Remark 1.8. As an application of the Lagrange mean value theorem, we can see that

$$t^q - 1 \leq q t^q (t - 1)$$

for $t \geq 1$ and $q > 1$. This together with $\sum_{n=1}^{\infty}(k_n - 1) < +\infty$ implies that $\sum_{n=1}^{\infty}(k_n^q - 1) < +\infty$.

Theorem 1.9. ([16, Theorem 1]) *If (X, d) is uniformly convex complete metric space then every decreasing sequence of nonempty closed bounded convex subsets of X has nonempty intersection.*

Definition 1.10. Let (X, d) be a metric space and Y be a topological space. A mapping $T : X \to X$ is said to be *completely continuous* if the image of each bounded set in X is contained in a compact subset of Y.

In [1, 2], Beg proved the following remarkable results.

Theorem 1.11. *Let (X, d) be a uniformly convex metric space having property (B). Then X is 2-unformly convex if and only if there exists a number $c > 0$ such that*

$$2\left[d\left(a, W\left(x, y, 1/2\right)\right)\right]^2 + c\left[d(x, y)\right]^2 \leq \left[d(a, x)\right]^2 + \left[d(a, y)\right]^2 \qquad (1.1)$$

for all $a, x, y \in X$.

Theorem 1.12. *Let A be a nonempty closed bounded convex subset of a uniformly convex complete metric space (X, d) and $T : A \to A$ be an asymptotically nonexpansive mapping. Then T has a fixed point.*

Theorem 1.13. *Let (X, d) be a convex metric space and A be a nonempty convex subset of X. Let $L > 0$ and $T : A \to A$ be uniformly L-Lipschitzian. For $x_1 \in A$. Define*

$$y_n = W(T^n x_n, x_n, 1/2), \quad x_{n+1} = W(T^n y_n, x_n, 1/2)$$

and set $c_n = d(T^n x_n, x_n)$ for all $n \in \mathbb{N}$. Then

$$d(x_n, Tx_n) \leq c_n + c_{n-1}(L + 3L^2 + 2L^3)$$

for all $n \in \mathbb{N}$.

Theorem 1.14. *Let (X, d) be a 2-uniformly convex metric space having property (B), A be a nonempty closed bounded convex subset of X and $T : A \to A$ be asymptotically nonexpansive with sequence $\{k_n\} \in [1, +\infty)^{\mathbb{N}}$ and $\sum_{n=1}^{\infty} (k_n^2 - 1) < +\infty$. Let $x_1 \in A$ and $x_{n+1} = W(T^n x_n, x_n, 1/2)$ for all $n \in \mathbb{N}$.*
Then $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Theorem 1.15. *Let (X, d) be 2-uniformly convex metric space having property (B), A be a nonempty closed bounded convex subset of X and $T : A \to A$ be completely continuous asymptotically nonexpansive mapping with sequence $\{k_n\} \in [1, +\infty)^{\mathbb{N}}$ and $\sum_{n=1}^{\infty} (k_n^2 - 1) < +\infty$.*

Let $x_1 \in A$ and $x_{n+1} = W(T^n x_n, x_n, \frac{1}{7}2)$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ converges to some fixed point of T.

In [12], Rafiq introduced the notion of $\mathcal{A}$-uniformly convex metric space defined as follows:

Definition 1.16. A convex complete metric space X is said to be $\mathcal{A}$-*uniformly convex* if for all $x, y, a \in X$,

$$\begin{aligned}[d\left(a, W\left(x, y, \lambda\right)\right)]^2 \leq \max\{\lambda, 1-\lambda\} &\left(1 - \delta\left(\frac{d(x,y)}{\max\left\{d(a,x), d(a,y)\right\}}\right)\right) \\ &\times \left([d(a,x)]^2 + [d(a,y)]^2\right),\end{aligned} \tag{1.2}$$

where the function δ is a strictly increasing function on the set of strictly positive numbers and $\delta(0) = 0$.

Remark 1.17. 1. The inequality (1.2) can be easily proved in a Hilbert space H using the well known identity [9]

$$\|\lambda x+(1-\lambda)y\|^2=\lambda\|x\|^2+(1-\lambda)\|y\|^2-\lambda(1-\lambda)\|x-y\|^2$$

for all $x,y\in H$.

2. For $\lambda=1/2$ in (1.2), we get the inequality (1.1).

3. Uniformly convex Banach spaces are $\mathcal{A}$-uniformly convex metric spaces.

Definition 1.18. ([12]) The $\mathcal{A}$-uniformly convex metric space X is said to be $(2,\mathcal{A})$-*uniformly convex* if there exists a constant $c>0$ such that $\delta(\epsilon)\geq c\epsilon^2$.

The purpose of this paper is to generalize the results of [2, 4, 6, 11, 13, 14, 17, 19, 21] and construct a three-step iterative scheme, convergent to the common fixed point, for three asymptotically nonexpansive mappings defined on a $\mathcal{A}$-uniformly convex metric space.

2. Main Results

In the sequel, we will need the following results.

The following lemma is now well known.

Lemma 2.1. *Let $\{a_n\}$ and $\{b_n\}$ be sequences of non-negative real numbers such that $a_{n+1}\leq(1+b_n)a_n$ for all $n\geq 1$ and $\sum_{n=1}^{\infty}b_n<\infty$. Then $\lim_{n\to\infty}a_n$ exists.*

Theorem 2.2. ([2, 12]) *Let A be a nonempty closed convex subset of a uniformly convex complete metric space (X,d) and $T:A\to A$ be an asymptotically nonexpansive mapping. Then the set $F(T)$ of fixed points of T is closed and convex.*

Theorem 2.3. ([12]) *Let (X,d) be a $\mathcal{A}$-uniformly convex metric space. Then X is $(2,A)$-uniformly convex if and only if there exists a number $c>0$ such that for all a, x, y in X and $\lambda\in[0,1]$,*

$$\begin{aligned}&[d(a,W(x,y,\lambda))]^2\\&\leq\max\{\lambda,1-\lambda\}\left[[d(a,x)]^2+[d(a,y)]^2-c\,[d(x,y)]^2\right].\end{aligned}\tag{2.1}$$

Theorem 2.4. *Let (X,d) be a $(2,\mathcal{A})$-uniformly convex metric space, A be a nonempty closed convex subset of X and $T,S,H:A\to A$ be asymptotically nonexpansive with sequence $\{k_n\}\in[1,+\infty)^{\mathbb{N}}$ and $\sum_{n=1}^{\infty}(k_n-1)<+\infty$. Let $x_1\in A$ and $x_{n+1}=W(T^ny_n,x_n,\alpha_n)$, $y_n=W(S^nz_n,x_n,\beta_n)$, $z_n=$*

$W(H^n x_n, x_n, \beta_n)$ for all $n \in \mathbb{N}$, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are the real sequences in $[0,1]$ satisfying $\alpha_n, \beta_n, \gamma_n \in [\delta, 1-\delta]$ for some $\delta \in (0,1)$. Then $\lim_{n\to\infty} d(T^n y_n, x_n) = 0 = \lim_{n\to\infty} d(S^n z_n, x_n) = \lim_{n\to\infty} d(H^n x_n, x_n)$.

Proof. Since T, S and H are asymptotically nonexpensive, so each mapping possesses a fixed point $p \in A$ by Theorem 1.12. Let $p \in F(T) \cap F(S) \cap F(H)$.

Claim. $\{x_n\}$ is bounded.

For this claim, we compute as follows:

$$\begin{aligned} d(p, x_{n+1}) &= d(p, W(T^n y_n, x_n, \alpha_n)) \\ &\le \alpha_n d(p, T^n y_n) + (1-\alpha_n) d(p, x_n) \\ &= \alpha_n d(T^n p, T^n y_n) + (1-\alpha_n) d(p, x_n) \\ &\le \alpha_n k_n d(p, y_n) + (1-\alpha_n) d(p, x_n), \end{aligned} \tag{2.2}$$

$$\begin{aligned} d(p, y_n) &= d(p, W(S^n z_n, x_n, \beta_n)) \\ &\le \beta_n d(p, S^n z_n) + (1-\beta_n) d(p, x_n) \\ &= \beta_n d(S^n p, S^n z_n) + (1-\beta_n) d(p, x_n) \\ &\le \beta_n k_n d(p, z_n) + (1-\beta_n) d(p, x_n), \end{aligned} \tag{2.3}$$

$$\begin{aligned} d(p, z_n) &= d(p, W(H^n x_n, x_n, \beta_n)) \\ &\le \beta_n d(p, H^n x_n) + (1-\beta_n) d(p, x_n) \\ &= \beta_n k_n d(H^n p, H^n x_n) + (1-\beta_n) d(p, x_n) \\ &\le \beta_n k_n d(p, x_n) + (1-\beta_n) d(p, x_n) \\ &= [1 + (k_n - 1)\beta_n] d(p, x_n) \\ &\le k_n d(p, x_n). \end{aligned} \tag{2.4}$$

Substituting (2.4) in (2.3) gives

$$\begin{aligned} d(p, y_n) &\le \left[1 + \left(k_n^2 - 1\right)\beta_n\right] d(p, x_n) \\ &\le k_n^2 d(p, x_n). \end{aligned} \tag{2.5}$$

From (2.5) in (2.2), we get

$$\begin{aligned} d(p, x_{n+1}) &\le \left[1 + \left(k_n^3 - 1\right)\alpha_n\right] d(p, x_n) \\ &\le \left[1 + \left(k_n^3 - 1\right)\right] d(p, x_n), \end{aligned}$$

which implies that $\lim_{n\to\infty} d(p, x_n)$ exists and $\{x_n\}$ is bounded.

Let $M=\sup_{n\geq 0} d(p,x_n)$. Now with the help of (2.1), we have

$$\begin{aligned}
&[d(p,x_{n+1})]^2\\
&= [d(p,W(T^n y_n,x_n,\alpha_n))]^2\\
&\leq \max\{\alpha_n,1-\alpha_n\}\left[\,[d(p,T^n y_n)]^2+[d(p,x_n)]^2-c\,[d(T^n y_n,x_n)]^2\,\right]\\
&\leq [d(p,T^n y_n)]^2+[d(p,x_n)]^2-\max\{\alpha_n,1-\alpha_n\}c\,[d(T^n y_n,x_n)]^2\\
&= [d(T^n p,T^n y_n)]^2+[d(p,x_n)]^2-\max\{\alpha_n,1-\alpha_n\}c\,[d(T^n y_n,x_n)]^2\\
&\leq k_n^2[d(p,y_n)]^2+[d(p,x_n)]^2-\max\{\alpha_n,1-\alpha_n\}c\,[d(T^n y_n,x_n)]^2,
\end{aligned}\tag{2.6}$$

$$\begin{aligned}
&[d(p,y_n)]^2\\
&= [d(p,W(S^n z_n,x_n,\beta_n))]^2\\
&\leq \max\{\beta_n,1-\beta_n\}\left[\,[d(p,S^n z_n)]^2+[d(p,x_n)]^2-c\,[d(S^n z_n,x_n)]^2\,\right]\\
&\leq [d(p,S^n z_n)]^2+[d(p,x_n)]^2-\max\{\beta_n,1-\beta_n\}c\,[d(S^n z_n,x_n)]^2\\
&= [d(S^n p,S^n z_n)]^2+[d(p,x_n)]^2-\max\{\beta_n,1-\beta_n\}c\,[d(S^n z_n,x_n)]^2\\
&\leq k_n^2[d(p,z_n)]^2+[d(p,x_n)]^2-\max\{\beta_n,1-\beta_n\}c\,[d(S^n z_n,x_n)]^2,
\end{aligned}\tag{2.7}$$

$$\begin{aligned}
&[d(p,z_n)]^2\\
&= [d(p,W(H^n x_n,x_n,\gamma_n))]^2\\
&\leq \max\{\gamma_n,1-\gamma_n\}\left[[d(p,H^n x_n)]^2+[d(p,x_n)]^2-c\,[d(H^n x_n,x_n)]^2\right]\\
&\leq [d(p,H^n x_n)]^2+[d(p,x_n)]^2-\max\{\gamma_n,1-\gamma_n\}c\,[d(H^n x_n,x_n)]^2\\
&= [d(H^n p,H^n x_n)]^2+[d(p,x_n)]^2-\max\{\gamma_n,1-\gamma_n\}c\,[d(H^n x_n,x_n)]^2\\
&\leq k_n^2[d(p,x_n)]^2+[d(p,x_n)]^2-\max\{\gamma_n,1-\gamma_n\}c\,[d(H^n x_n,x_n)]^2\\
&= (1+k_n^2)\,[d(p,x_n)]^2-\max\{\gamma_n,1-\gamma_n\}c\,[d(H^n x_n,x_n)]^2.
\end{aligned}\tag{2.8}$$

Substituting (2.8) in (2.7), we get

$$\begin{aligned}
[d(p,y_n)]^2 \leq{}& (1+k_n^2+k_n^4)\,[d(p,x_n)]^2\\
&-k_n^2\max\{\gamma_n,1-\gamma_n\}c\,[d(H^n x_n,x_n)]^2\\
&-\max\{\beta_n,1-\beta_n\}c\,[d(S^n z_n,x_n)]^2,
\end{aligned}\tag{2.9}$$

and by (2.6), we obtain

$$\begin{aligned}
&[d(p, x_{n+1})]^2 \\
&\le (1+k_n^2+k_n^4+k_n^6)\,[d(p,x_n)]^2 - \max\{\alpha_n, 1-\alpha_n\}c\,[d(T^n y_n, x_n)]^2 \\
&\quad - k_n^2 \max\{\beta_n, 1-\beta_n\}c\,[d(S^n z_n, x_n)]^2 \\
&\quad - k_n^4 \max\{\gamma_n, 1-\gamma_n\}c\,[d(H^n x_n, x_n)]^2 \qquad (2.10)\\
&\le [1+(k_n^7-1)]\,[d(p,x_n)]^2 - \max\{\alpha_n, 1-\alpha_n\}c\,[d(T^n y_n, x_n)]^2 \\
&\quad - k_n^2 \max\{\beta_n, 1-\beta_n\}c\,[d(S^n z_n, x_n)]^2 \\
&\quad - k_n^4 \max\{\gamma_n, 1-\gamma_n\}c\,[d(H^n x_n, x_n)]^2 .
\end{aligned}$$

With the help of condition $\alpha_n, \beta_n, \gamma_n \in [\delta, 1-\delta]$ for some $\delta \in (0,1)$, it can be easily seen that

$$\max\{\alpha_n, 1-\alpha_n\}, \quad \max\{\beta_n, 1-\beta_n\}, \quad \max\{\gamma_n, 1-\gamma_n\} \ge \delta. \qquad (2.11)$$

Using (2.11) in (2.10) and by $k_n \ge 1$, we obtain

$$\begin{aligned}
[d(p, x_{n+1})]^2 \le{}& [1+(k_n^7-1)]\,[d(p,x_n)]^2 \\
&- \delta c\Big([d(T^n y_n, x_n)]^2 + [d(S^n z_n, x_n)]^2 + [d(H^n x_n, x_n)]^2 \Big),
\end{aligned}$$

which implies that

$$\begin{aligned}
&\delta c\left([d(T^n y_n, x_n)]^2 + [d(S^n z_n, x_n)]^2 + [d(H^n x_n, x_n)]^2\right) \\
&\le [d(p,x_n)]^2 - [d(p, x_{n+1})]^2 + M^2(k_n^7-1).
\end{aligned}$$

Thus

$$\begin{aligned}
&\delta c\left(\sum_{j=1}^m \left[d(T^j y_j, x_j)\right]^2 + \sum_{j=1}^m \left[d(H^j x_j, x_j)\right]^2 + \sum_{j=1}^m \left[d(S^j z_j, x_j)\right]^2 \right) \\
&\le \frac{M^2}{2} \sum_{j=1}^m \left(k_j^7 - 1\right) + \sum_{j=1}^m \Big([d(p,x_j)]^2 - [d(p,x_{j+1})]^2 \Big).
\end{aligned}$$

Hence

$$\sum_{j=1}^\infty \left[d(T^j y_j, x_j)\right]^2 < +\infty,$$

$$\sum_{j=1}^\infty \left[d(S^j z_j, x_j)\right]^2 < +\infty,$$

and

$$\sum_{j=1}^{\infty} \left[d(H^j x_j, x_j)\right]^2 < +\infty.$$

It implies that

$$\lim_{n\to\infty} d(T^n y_n, x_n) = 0,$$
$$\lim_{n\to\infty} d(S^n z_n, x_n) = 0,$$

and

$$\lim_{n\to\infty} d(H^n x_n, x_n) = 0.$$

This completes the proof. □

Theorem 2.5. *Let (X, d) be a $(2, \mathcal{A})$-uniformly convex metric space, A be a nonempty closed convex subset of X and $T, S, H : A \to A$ be asymptotically nonexpansive with sequence $\{k_n\} \in [1, +\infty)^{\mathbb{N}}$ and $\sum_{n=1} \infty (k_n - 1) < +\infty$. Further let H be completely continuous and T and S are continuous. Let $x_1 \in A$ and $x_{n+1} = W(T^n y_n, x_n, \alpha_n)$, $y_n = W(S^n z_n, x_n, \beta_n)$, $z_n = W(H^n x_n, x_n, \beta_n)$ for all $n \in \mathbb{N}$, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are the real sequences in $[0, 1]$ satisfying $\alpha_n, \beta_n, \gamma_n \in [\delta, 1 - \delta]$ for some $\delta \in (0, 1)$. Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge to the common fixed point of T, S and H.*

Proof. Consider

$$\begin{aligned}
d(x_{n+1}, H^n x_{n+1}) &\le d(x_{n+1}, x_n) + d(x_n, H^n x_n) + d(H^n x_n, H^n x_{n+1}) \\
&\le d(x_{n+1}, x_n) + d(x_n, H^n x_n) + k_n d(x_n, x_{n+1}) \\
&= (1 + k_n) d(x_{n+1}, x_n) + d(x_n, H^n x_n) \\
&= (1 + k_n) d(W(T^n y_n, x_n, 1/2), x_n) + d(x_n, H^n x_n) \\
&= (1 + k_n)\, \alpha_n d(T^n y_n, x_n) + d(x_n, H^n x_n) \\
&\to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Thus

$$\begin{aligned}
d(x_{n+1}, H x_{n+1}) &\le d(x_{n+1}, H^{n+1} x_{n+1}) + d(H^{n+1} x_{n+1}, H x_{n+1}) \\
&\le d(x_{n+1}, H^{n+1} x_{n+1}) + k_1 d(x_{n+1}, H^n x_{n+1}) \\
&\to 0 \quad \text{as } n \to \infty,
\end{aligned}$$

which implies that

$$\lim_{n\to\infty} d(x_n, H x_n) = 0.$$

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Since H is completely continuous and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{Hx_{n_k}\}$ converges.

Therefore from $\lim_{n\to\infty} d(x_n, Hx_n) = 0$, $\{x_{n_k}\}$ converges. Let $\lim_{n\to\infty} x_{n_k} = p$. It follows from the continuity of H and $\lim_{n\to\infty} d(x_n, Hx_n) = 0$ that $p = Hp$. We know that $\lim_{n\to\infty} d(p, x_n)$ exists. But $\lim_{n\to\infty} d(p, x_{n_k}) = 0$. This implies $\lim_{n\to\infty} d(p, x_n) = 0$, i.e., $\lim_{n\to\infty} x_n = p$. Since

$$\begin{aligned} d(x_n, z_n) &= d(x_n, W(H^n x_n, x_n, \gamma_n)) = \gamma_n d(x_n, H^n x_n) \\ &\to 0 \quad \text{as } n \to \infty, \end{aligned}$$

and

$$\begin{aligned} d(x_n, y_n) &= d(x_n, W(S^n z_n, x_n, \beta_n)) = \beta_n d(x_n, S^n z_n) \\ &\to 0 \quad \text{as } n \to \infty, \end{aligned}$$

so $\lim_{n\to\infty} z_n = p = \lim_{n\to\infty} y_n$.

The following estimates hold:

$$\begin{aligned} d(x_n, x_{n+1}) &= d(x_n, W(T^n y_n, x_n, \alpha_n)) = \alpha_n d(x_n, T^n y_n) \\ &\to 0 \quad \text{as } n \to \infty, \end{aligned}$$

$$\begin{aligned} d(y_n, y_{n-1}) &\le d(y_n, x_n) + d(x_n, x_{n-1}) + d(x_{n-1}, y_{n-1}) \\ &\to 0 \quad \text{as } n \to \infty, \end{aligned}$$

$$\begin{aligned} d(z_n, z_{n-1}) &\le d(z_n, x_n) + d(x_n, x_{n-1}) + d(x_{n-1}, z_{n-1}) \\ &\to 0 \quad \text{as } n \to \infty, \end{aligned}$$

$$\begin{aligned} d(z_n, S^n z_n) &\le d(z_n, x_n) + d(x_n, S^n z_n) \\ &\to 0 \quad \text{as } n \to \infty, \end{aligned}$$

$$\begin{aligned} d(y_n, T^n y_n) &\le d(y_n, x_n) + d(x_n, T^n y_n) \\ &\to 0 \quad \text{as } n \to \infty, \end{aligned}$$

$$\begin{aligned} & d(z_n, S^{n-1} z_n) \\ & \le d(z_n, z_{n-1}) + d(z_{n-1}, S^{n-1} z_{n-1}) + d(S^{n-1} z_{n-1}, S^{n-1} z_n) \\ & \le d(z_n, z_{n-1}) + d(z_{n-1}, S^{n-1} z_{n-1}) + k_{n-1} d(z_{n-1}, z_n) \\ & = (1 + k_{n-1}) d(z_n, z_{n-1}) + d(z_{n-1}, S^{n-1} z_{n-1}) \\ & \to 0 \quad \text{as } n \to \infty, \end{aligned} \tag{2.12}$$

$$\begin{aligned} d(z_n, Sz_n) &\le d(z_n, S^n z_n) + d(S^n z_n, Sz_n) \\ &\le d(z_n, S^n z_n) + k_1 d(S^{n-1} z_n, z_n) \\ &\to 0 \quad \text{as } n \to \infty, \end{aligned} \tag{2.13}$$

$$\begin{aligned} &d(y_n, T^{n-1} y_n) \\ &\le d(y_n, y_{n-1}) + d(y_{n-1}, T^{n-1} y_{n-1}) + d(T^{n-1} y_{n-1}, T^{n-1} y_n) \\ &\le d(y_n, y_{n-1}) + d(y_{n-1}, T^{n-1} y_{n-1}) + k_{n-1} d(y_n, y_{n-1}) \\ &= (1 + k_{n-1}) d(y_n, y_{n-1}) + d(y_{n-1}, T^{n-1} y_{n-1}) \\ &\to 0 \quad \text{as } n \to \infty, \end{aligned} \tag{2.14}$$

$$\begin{aligned} d(y_n, Ty_n) &\le d(y_n, T^n y_n) + d(T^n y_n, Ty_n) \\ &\le d(y_n, T^n y_n) + k_1 d(T^{n-1} y_n, y_n) \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{2.15}$$

Now according to the continuity of T and S and by using (2.15) and (2.13), we obtain $Tp = p = Sp$. Thus p is the common fixed point of T, S and H. This completes the proof. □

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ON IDENTITIES BETWEEN SUMS OF EULER NUMBERS AND GENOCCHI NUMBERS OF HIGHER ORDER

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Abstract. In this paper we consider differential equations which are closely related to the generating functions of Euler numbers. By using the same method of Kim's calculation in Kim [24,25], we derive identities involving Euler numbers arising from differential equations. In particular, we derive some new identities between the sums of Euler numbers and Genocchi numbers of higher order.

1. Introduction

We consider the Euler numbers defined by the generating function as follows(see [4,8,17,21-25]):

$$E(t) = \frac{2}{e^t + 1} = \sum_{k=0}^{\infty} E_k \frac{t^k}{k!} \tag{1}$$

and the Genocchi numbers defined by the generating function as follows(see [2,3,5,7,9-16,18,20,26-30,32,33,35,36]):

$$G(t) = \frac{2t}{e^t + 1} = \sum_{k=0}^{\infty} G_k \frac{t^k}{k!}. \tag{2}$$

Kim(2012) derived some new identities between the sums of products of Frobenius-Euler polynomials and Frobenius-Euler polynomials of higher order (see[1,6,31,34,37,38]).

In this paper we derive differential equations which are closely related to the generating function of Euler numbers. By using these differential equations, we derive some identities between the sums of products of Euler numbers and Euler numbers of higher order. In particular, we obtain some identities between the sums of Euler numbers and Genocchi numbers of higher order.

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2. Computation of sums of the products of Euler numbers

In this section we assume that

$$F = F(t) = \frac{1}{e^t+1} \quad \text{and} \quad F^N = \underbrace{F \times \cdots \times F}_{N-\text{times}} \tag{3}$$

for $N \in \mathbb{N}$. Thus, by (3), we get

$$\begin{aligned} F(t)^{(1)} = \frac{dF(t)}{dt} &= \frac{-e^t}{(e^t+1)^2} \\ &= \frac{-1-e^{-t}}{(e^t+1)^2} + \frac{1}{(e^t+1)^2} \\ &= -F + F^2. \end{aligned} \tag{4}$$

Let us consider the derivative of (4) with respect to t as follows:

$$2FF^{(1)} = F^{(1)} + F^{(2)}. \tag{5}$$

Thus, by (5) and (3), we get

$$2!F^3 - 2F^2 = F^{(1)} + F^{(2)}. \tag{6}$$

From (6), we note that

$$\begin{aligned} 2!F^3 &= 2F^2 + F^{(1)} + F^{(2)} \\ &= 2(F + F^{(1)}) + F^{(1)} + F^{(2)} \\ &= 2F + 3F^{(1)} + F^{(2)}, \end{aligned} \tag{7}$$

where $F^{(2)} = \frac{d^2F}{dt^2}$. Thus, by the derivative of (5) with respect to t, we get

$$2!3F^2F^{(1)} = 2F^{(1)} + 3F^{(2)} + F^{(3)}. \tag{8}$$

By (8), we see that

$$2!3F^2(-F + F^2) = 2F^{(1)} + 3F^{(2)} + F^{(3)} \tag{9}$$

and

$$F^{(1)} = -F + F^2. \tag{10}$$

By (9), we see that

$$\begin{aligned} 3!(-1)^3F^4 &= 6F^3 + 2F^{(1)} + 3F^{(2)} + F^{(3)} \\ &= 6(F + \frac{3}{2}F^{(1)} + \frac{1}{2}F^{(2)}) + 2F^{(1)} + 3F^{(2)} + F^{(3)} \\ &= 6F + 11F^{(1)} + 6F^{(2)} + F^{(3)}. \end{aligned} \tag{11}$$

Continuing this process, we get

$$(N-1)!F^N = \sum_{k=0}^{N-1} a_K(N)F^{(k)} \tag{12}$$

where $F^{(k)} = \frac{d^kF}{dt^k}$ and $N \in \mathbb{N}$.

Now we try to find the coefficients $a_k(N)$ in (12). By (12), we differentiate the both sides of (12) as follows:

$$(N-1)!NF^{N-1}F^{(1)} = \sum_{k=0}^{N-1} a_k(N)F^{(k+1)}$$

$$= \sum_{k=1}^{N} a_{k-1}(N)F^{(k)}. \tag{13}$$

By (7), we get

$$\begin{aligned}(N-1)!NF^{N-1}F^{(1)} &= N!F^N(-F+F^2)\\ &= N!(-F^N+F^{N+1})\\ &= -N!F^N+N!F^{N+1}.\end{aligned} \tag{14}$$

By (13) and (14), we get

$$\begin{aligned}N!F^{N+1} &= N!F^N+\sum_{k=1}^{N} a_{k-1}(N)F^{(k)}\\ &= N(N-1)!F^N+\sum_{k=1}^{N} a_{k-1}(N)F^{(k)}\\ &= N\sum_{k=0}^{N-1} a_k(N)F^{(k)}+\sum_{k=1}^{N} a_{k-1}(N)F^{(k)}.\end{aligned} \tag{15}$$

In (12), replacing N by $N+1$, we have

$$N!F^{N+1}=\sum_{k=0}^{N} a_k(N+1)F^{(k)}. \tag{16}$$

By (15) and (16), we get

$$\sum_{k=0}^{N} a_k(N+1)F^{(k)} = N\sum_{k=0}^{N-1} a_k(N)F^{(k)}+\sum_{k=1}^{N} a_{k-1}(N)F^{(k)}. \tag{17}$$

By comparing coefficients on the both sides of (17), we have the followings:

$$a_0(N+1)=Na_0(N) \text{ and } a_N(N+1)=a_{N-1}(N). \tag{18}$$

For $1\le k\le n-1$, we have

$$a_k(N+1)=Na_k(N)=a_{k-1}(N) \tag{19}$$

where $a_k(N)=0$ for $k\ge N$ or $k<0$. From (19), we note that

$$\begin{aligned}a_0(N+1) &= Na_0(N)=N(N-1)a_0(N-1)\\ &= \cdots = N(N-1)\cdots 2a_0(2).\end{aligned} \tag{20}$$

By comparing coefficients on the both sides of (15) with $N=1$,

$$\begin{aligned}F+F^{(1)} &= 1!F^2=\sum_{k=0}^{1} a_k(2)F^{(k)}\\ &= a_0(2)F+a_1(2)F^{(1)}.\end{aligned} \tag{21}$$

Thus, by (21), we get

$$a_0(2)=1 \text{ and } a_1(2)=1. \tag{22}$$

Finally, we derive the values of $a_k(N)$ in (12) from (19).

Let us consider the following two variable function with variables t,s:

$$g(t,s)=\sum_{N\ge 1}\sum_{0\le k\le N-1} a_k(N)\frac{t^N s^k}{N!}, \text{ where } |t|<1. \tag{23}$$

Then, Kim [25] derived the followings:

$$g(t,s)=\sum_{N\geq 1}\sum_{0\leq k\leq N-1}\frac{N!}{(k+1)!}\sum_{l_1+\cdots+l_{k+1}=N}\frac{1}{l_1\cdots l_{k+1}}\frac{t^N}{N1}s^k. \tag{24}$$

By (23) and (24), we get

$$a_k(N)=\frac{N!}{(k+1)!}\sum_{l_1+\cdots+l_{k+1}=N}\frac{1}{l_1\cdots l_{k+1}}. \tag{25}$$

Therefore, by (12) and (25), we obtain the following theorem.

Theorem 2.1. *For $n\in\mathbb{N}$, let us consider the following differential equation with respect to t:*

$$F^N(t)=N\sum_{k=0}^{N-1}\frac{1}{(k+1)!}\sum_{l_1+\cdots+l_{k+1}=N}\frac{1}{l_1\cdots l_{k+1}}F^{(k)}(t) \tag{26}$$

where $F^{(k)}(t)=\frac{d^kF(t)}{dt^k}$ and $F^N(t)=\underbrace{F(t)\times\cdots\times F(t)}_{N-times}$. Them $F(t)=\frac{1}{e^t+1}$ is a solution of (26).

We assume that

$$\underbrace{\left(\frac{2}{e^t+1}\right)\cdots\left(\frac{2}{e^t+1}\right)}_{N-\text{times}}=\sum_{n=0}^{\infty}E_n^{(N)}\frac{t^n}{n!} \tag{27}$$

where $E_n^{(N)}$ are called the n-th Euler numbers of order N. By (3) and (27), we get

$$\begin{aligned}F^N(t) &= \underbrace{\left(\frac{1}{e^t+1}\right)\cdots\left(\frac{1}{e^t+1}\right)}_{N-\text{times}}\\ &= \frac{1}{2^N}\underbrace{\left(\frac{2}{e^t+1}\right)\cdots\left(\frac{2}{e^t+1}\right)}_{N-\text{times}}\\ &= \frac{1}{2^N}\sum_{n=0}^{\infty}E_n^{(N)}\frac{t^n}{n!}\end{aligned} \tag{28}$$

and

$$\begin{aligned}F(t) &= \frac{1}{2^N}\left(\frac{2}{e^t+1}\right)\\ &= \frac{1}{2}\sum_{l=0}^{\infty}E_l\frac{t^l}{l!}.\end{aligned} \tag{29}$$

From (29), we note that

$$\begin{aligned}F(t^{(k)}) &= \frac{d^kF(t)}{dt^k}\\ &= \frac{1}{2}\sum_{l=0}^{\infty}E_{l+k}\frac{t^l}{l!}.\end{aligned} \tag{30}$$

Therefore, by (26), (29), and (30), we obtain the following theorem.

Theorem 2.2. *For $N \in \mathbb{N}$, $n \in \mathbb{Z}_+$, we have*

$$E_n^{(N)} = 2^N N \sum_{k=0}^{N-1} \frac{1}{(k+1)!} \sum_{l_1+\cdots+l_{k+1}=N} \frac{E_{n+k}}{l_1 \cdots l_{k+1}}. \tag{31}$$

From (28), we can derive the following equation:

$$\begin{aligned}
\sum_{n=0}^{\infty} E_n^{(N)} \frac{t^n}{n!} &= \underbrace{\left(\frac{2}{e^t+1}\right) \cdots \left(\frac{2}{e^t+1}\right)}_{N-\text{times}} \\
&= \underbrace{\left(\sum_{l_1=0}^{\infty} E_{l_1} \frac{t^{l_1}}{l_1}\right) \cdots \left(\sum_{l_N=0}^{\infty} E_{l_N} \frac{t^{l_N}}{l_N}\right)}_{N-\text{times}} \\
&= \sum_{n=0}^{\infty} \sum_{l_1+\cdots+l_N=n} \left(\frac{E_{l_1} \cdots E_{l_N} n!}{l_1! \cdots l_N!}\right) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{l_1+\cdots+l_N=n} \binom{n}{l_1 \cdots l_N} E_{l_1} \cdots E_{l_N} \right) \frac{t^n}{n!}. \qquad (32)
\end{aligned}$$

Therefore, by (31) and (32), we obtain the following corollary.

Corollary 2.3. *For $N \in \mathbb{N}$, $n \in \mathbb{Z}_+$, we have*

$$\sum_{l_1+\cdots+l_N=n} \binom{n}{l_1 \cdots l_N} E_{l_1} \cdots E_{l_N} = 2^N N \sum_{k=0}^{N-1} \frac{1}{(k+1)!} \sum_{l_1+\cdots+l_{k+1}=N} \frac{E_{n+k}}{l_1 \cdots l_{k+1}}. \tag{33}$$

3. Identities between sums of Euler numbers and Genocchi numbers of higher order

In this section we assume that

$$G^N = G^N(t) \underbrace{\left(\frac{2t}{e^t+1}\right) \cdots \left(\frac{2t}{e^t+1}\right)}_{N-\text{times}} = \sum_{n=0}^{\infty} G_n^{(N)} \frac{t^n}{n!} \tag{34}$$

where $G_n^{(N)}$ are called the n-th Genocchi numbers of order N. We note that

$$\frac{d^k}{dt^k} \left(\frac{2t}{e^t+1}\right) |_{t=0} = G_k \tag{35}$$

for $k \in \mathbb{N}$. By (35), we obtain the following equation:

$$G_k \quad = \quad \frac{d^k}{dt^k} \left(\frac{2t}{e^t+1}\right) |_{t=0}$$

$$\begin{aligned} &= \sum_{l=0}^{k}\binom{k}{l}(2t)^l F^{(k-l)} \\ &= F^{(k)}+2kF^{(k-1)}. \end{aligned} \tag{36}$$

By (30) and (36), we get

$$\begin{aligned} G &= \sum_{k=0}^{\infty} G_k \frac{t^k}{k!} \\ &= \sum_{k=0}^{\infty} F^{(k)}\frac{t^k}{k!} + 2\sum_{k=0}^{\infty} kF^{(k-1)}\frac{t^k}{k!} \\ &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} E_{l+k}\frac{t^l}{l!}\frac{t^k}{k!} + 2\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} kE_{l+k-1}\frac{t^l}{l!}\frac{t^k}{k!} \\ &= \sum_{k=0}^{\infty}\left[\sum_{l=0}^{\infty}\left(E_{l+k}+2kE_{l+k-1}\right)\frac{t^l}{l!}\right]\frac{t^k}{k!}. \end{aligned} \tag{37}$$

Therefore, by (37), we obtain the following theorem which is the identities between the sums of Euler numbers and Genocchi numbers.

Theorem 3.1. *For $k \in \mathbb{Z}_+$, we have*

$$G_k = \sum_{l=0}^{\infty}\left(E_{l+k}+2kE_{l+k-1}\right)\frac{t^l}{l!}. \tag{38}$$

From (38), we easily see that

$$\begin{aligned} G_k^{(N)} &= \left(F^{(k)}+2kF^{(k-1)}\right)^{(N)} \\ &= F^{(N+k)}+2kF^{(N+k-1)}. \end{aligned} \tag{39}$$

By (39), we get

$$\begin{aligned} G^N &= \sum_{k=0}^{\infty} G_k^{(N)}\frac{t^k}{k!} \\ &= \sum_{k=0}^{\infty}\left(F^{(N+k)}+2kF^{(N+k-1)}\right)\frac{t^k}{k!} \\ &= \sum_{k=0}^{\infty}\left(\sum_{l=0}^{\infty} E_{N+l+k}\frac{t^l}{l!}\right)\frac{t^k}{k!} + 2\sum_{k=0}^{\infty}\left(\sum_{l=0}^{\infty} E_{N+l+k-1}\frac{t^l}{l!}\right)\frac{t^k}{k!} \\ &= \sum_{k=0}^{\infty}\left(\sum_{l=0}^{\infty} E_{N+l+k}+2kE_{N+l+k-1}\frac{t^l}{l!}\right)\frac{t^k}{k!}. \end{aligned} \tag{40}$$

Therefore, by (40), we obtain the following theorem which is the identities between the sums of Euler numbers and Genocchi numbers of higher order.

Theorem 3.2. *For $N \in \mathbb{N}$, $k \in \mathbb{Z}_+$, we have*

$$G_k^{(N)} = \sum_{l=0}^{\infty}\left(E_{l+k}+2kE_{l+k-1}\right)\frac{t^l}{l!}. \tag{41}$$

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An algorithm for multi-attribute decision making based on soft rough sets *

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Abstract: Based on soft rough sets, some new concepts such as soft decision systems, soft relative positive regions, relative reduction in soft decision systems and conditional significance relative to decision partition soft sets are proposed. The multi-attribute decision rule in soft decision systems is presented. An algorithm of multi-attribute decision making based on soft rough sets is given.

Keywords: Soft rough set; Partition soft set; Soft decision system; Relative reduction; Decision making; Decision rule

1 Introduction

In 1999, Molodtsov [9] initiated soft set theory as a new mathematical tool for dealing with uncertainties which classical mathematical tools cannot handle. Recently, there has been a rapid growth of interest in soft set theory. Many efforts have been devoted to further generalizations and extensions of soft sets. Recently there has been a rapid growth of interest in soft set theory and its applications. Many efforts have been devoted to further generalizations and extensions of soft sets. Maji et al. [11] defined fuzzy soft sets, combining soft sets with fuzzy sets. Maji et al. [12] reported a detailed theoretical study on soft sets, with emphasis on the algebraic operations. Jiang et al. [7] extended soft sets with description logics. Aktas et al. [1] initiated the notion of soft groups, extending fuzzy groups. Feng et al. [2, 5] investigated the relationships among soft sets, rough sets and fuzzy sets.

Applications of soft set theory in decision making problems was initiated in [10]. To address fuzzy soft set based decision making problems, Roy et al. [14] presented a novel method of object recognition from an imprecise multi-observer data. Using level soft sets, Feng et al. [3] proposed an adjustable approach to fuzzy soft set based decision making. This approach was further investigated in [4, 8]. Although soft sets have been applied by several authors to the study of

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decision making under uncertainty, it seems that soft set based group decision making has not been discussed yet in the literature. Thus the present study can be seen as a first attempt toward the possible application of soft rough approximations in multi-attribute decision making problems under uncertainty.

The purpose of this paper is to give a method for multi-attribute decision making applying soft rough sets.

2 Preliminaries

Throughout this paper, U denotes an initial universe, E denotes the set of all possible attributes or parameters, 2^U denotes the family of all subsets of U and $|\,.\,|$ is the cardinality of a set. We only consider the case where both U and E are nonempty finite sets.

2.1 Soft sets

Definition 2.1 ([9]). *Let $A \subseteq E$. A pair (f, A) is called a soft set over U, if f is a mapping defined by $f : A \to 2^U$.*

In other words, a soft set over U is a parameterized family of subsets of the U. For $e \in A$, $f(e)$ may be considered as the set of e-approximate elements of (f, A).

Definition 2.2 ([12]). *Let (f, A) and (g, B) be two soft sets over U.*

(1) (f, A) is called a soft subset of (g, B), if $A \subseteq D$ and $f(e) = g(e)$ for each $e \in A$. We denote it by $(f, A) \tilde{\subset} (g, B)$.

(2) (f, A) and (g, B) are called soft equal, if $A = B$ and $f(e) = g(e)$ for each $e \in A$. We denote it by $(f, A) = (g, B)$.

Obviously, $(f, A) = (g, B)$ if and only if $(f, A) \tilde{\subset} (g, B)$ and $(f, A) \tilde{\supset} (g, B)$.

Definition 2.3 ([12]). *Let (f, A), (g, B) and (h, C) be soft sets over U.*

(1) (h, C) is called the intersection of (f, A) and (g, B), if $C = A \cap B$ and $h(e) = f(e) \cap g(e)$ for each $e \in C$. We denote (h, C) by $(f, A) \tilde{\cap} (g, B)$.

(2) (h, C) is called the union of (f, A) and (g, B), if $C = A \cup B$ and

$$h(e) = \begin{cases} f(e), & e \in A - B, \\ f(e) \cup g(e), & e \in A \cap B, \\ g(e), & e \in B - A. \end{cases}$$

We denote (h, C) by $(f, A) \tilde{\cup} (g, B)$.

Definition 2.4 ([12]). *Let (f, A) and (g, B) be two soft sets over U. (f, A) AND (g, B) denoted by $(f, A) \wedge (g, B)$ is defined by $(f, A) \wedge (g, B) = (h, A \times B)$, where $h(a, b) = f(a) \cap g(b)$ for each $(a, b) \in A \times B$.*

Definition 2.5 ([5]). *A soft set (f, A) over U is called a partition soft set if $\{f(e)|\ e \in A\}$ forms a partition of U.*

Definition 2.6 ([6]). *Let $A \subseteq E$. Let (f, A) be soft sets over U. (f, A) is called a bijective soft set, if f is a mapping $f : A \to 2^U$ such that*

(1) $\bigcup_{e \in A} f(e) = U$.

(2) *For $e_i, e_j \in A$ and $e_i \neq e_j$, $f(e_i) \cap f(e_j) = \emptyset$.*

In other words, suppose $\mathcal{B} = \{f(e_i)|\ e_i \in A, \ \ \leq 1 \leq i \leq n\} \subseteq 2^U$. From Definition 2.7, the mapping $f : A \to 2^U$ can be transformed to the mapping $f : A \to \mathcal{B}$, which is a bijective function, namely, for every $X \in \mathcal{B}$, there is exactly one attribute $e \in A$ such that $f(e) = X$ and no unmapped element remains in both A and $\mathcal{B}$.

Proposition 2.7. *Let (f, A) be a bijective soft set U and let (g, B) be a null soft set over U. $(h, C) = (f, A) \ \widetilde{\cup} \ (g, B)$ is a bijective soft set.*

Proposition 2.8. *Let (f, A) and (g, B) be two bijective soft sets over U. Then $(h, A \times B) = (f, A) \wedge (g, B)$ is also a bijective soft set.*

2.2 Soft rough sets

Definition 2.9 ([5]). *Let (f, A) be a soft set over U and $X \subseteq U$. Then the pair $P = (U, (f, A))$ is called a soft approximation space. Based on the soft approximation space P, we define the following two operations*

$$\underline{apr}_P X = \{x \in U|\ \exists\ e \in A \ s.t. \ x \in f(e) \subseteq X\},$$

$$\overline{apr}_P X = \{x \in U|\ \exists\ e \in A \ s.t. \ x \in f(e), f(e) \cap X \neq \emptyset\}.$$

$\underline{apr}_P(X)$ and $\overline{apr}_P(X)$ are called the soft P-lower approximation and the soft P-upper approximation of X, respectively. In general, we refer to the pair $(\underline{apr}_P(X), \overline{apr}_P(X))$ as the soft rough set of X with respect to P. Moreover, the sets

$Pos_P(X) = \underline{apr}_P(X)$, $Neg_P(X) = U - \overline{apr}_P(X)$,

$Bnd_P(X) = \overline{apr}_P(X) - \underline{apr}_P(X)$

are called the soft P-positive region, the soft P-negative region and the soft P-boundary region of X, respectively. X is said to be soft a soft P-definable set if $\underline{apr}_P(X) = \overline{apr}_P(X)$; otherwise, X is called a soft P-rough set.

From the analogy with Pawlak rough sets, we also have the following interpretation of above concepts.

(1) $x \in Pos_P(X) = \underline{apr}_P(X)$ means that x surely belongs to X with respect to P;

(2) $x \in \overline{apr}_P(X)$ means that x possibly belongs to X with respect to P;

(3) $x \in Neg_P(X)$ means that x surely does not belong to X with respect to P.

Clearly, $\underline{apr}_P(X)$ and $\overline{apr}_P(X)$ can be expressed equivalently as:

$\underline{apr}_P(X) = \cup_{e \in A}\{f(e)|\ f(e) \subseteq X\}$, $\overline{apr}_P(X) = \cup_{e \in A}\{f(e)|\ f(e) \cap X \neq \emptyset\}$.

3 Soft decision systems

It is worth nothing that information systems and soft sets are closely related. Given a soft set (f, A) over U. (f, A) could induce an information system in a natural way. In the section, we mainly discuss soft decision systems.

Let $(f_i, C_i)(i = 1, 2, \cdots, n)$ be bijective soft sets over U where $C_i \cap C_j = \emptyset$ for $i \neq j$. Denote

$$(f, C) = \widetilde{\cup}_{i=1}^{n}(f_i, C_i), \quad (\varphi, K) = \bigwedge_{i=1}^{n}(f_i, C_i).$$

where $C = \bigcup_{i=1}^{n} C_i$ and $K = C_1 \times C_2 \times \ldots \times C_n$.

3.1 Soft relative positive regions

Definition 3.1. *Let (f, A) and (g, B) be two soft sets over U. Then the soft positive region of (f, A) to (g, B) is defined as follows*

$$Pos_{(f,A)}\ (g, B) = \bigcup_{b \in B} \underline{apr}_P g(b) = \bigcup_{b \in B} \{x \in U |\ \exists\ e \in A\ s.t.\ x \in f(e) \subseteq g(b)\},$$

where $P = (U, (f, A))$.

Example 3.2. *Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$ be a common universe, $A = \{e_1, e_2, e_3, e_4\}$ and $B = \{b_1, b_2\}$ be two attribute sets. Suppose that (f, A) and (g, B) are two soft sets over U.*
The mapping of (f, A) is given below:
$f(e_1) = \{x_1, x_2\}, \quad f(e_2) = \{x_4, x_5, x_6\}, \quad f(e_3) = \{x_3, x_7\}.$
The mapping of (g, B) is given below:
$g(b_1) = \{x_1, x_2, x_3\}, \quad g(b_2) = \{x_4, x_5, x_6, x_7\}.$
Then $\underline{apr}_{(U,(f,A))}\ g(b_1) = \{x_1, x_2\}, \quad \underline{apr}_{(U,(f,A))}\ g(b_2) = \{x_4, x_5, x_6\}.$
So $Pos_{(f,A)}\ (g, B) = \{x_1, x_2, x_4, x_5, x_6\}.$

Definition 3.3. *Let $(f_i, C_i)(i = 1, 2, \cdots, n)$ be bijective soft sets over U where $C_i \cap C_j = \emptyset$ for $i \neq j$. Let (g, D) be a partition soft set over U where $C \cap D = \emptyset$. Then the triple $(U, (f, C), (g, D))$ is called a soft decision system, (f, C) is called the condition bijective soft set and (g, D) is called the decision partition soft set.*

Accordingly, in a soft decision system $(U, (f, C), (g, D))$, we have

$$Pos_{(\varphi,K)}\ (g, D) = \bigcup_{d \in D} \underline{apr}_P g(d) = \bigcup_{d \in D} \{x \in U |\ \exists\ e \in K\ s.t.\ x \in \varphi(e) \subseteq g(d)\},$$

where $K = C_1 \times C_2 \times \ldots \times C_n$ and $P = (U, (\varphi, K))$. We call it soft relative positive regions of soft decision systems.

For a given soft decision system, we always consider $Pos_{(\varphi,K)}\ (g, D) \neq \emptyset$.

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Example 3.4. *Suppose that* $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ *is a common universe, which is a set of six shops.* $C = \bigcup_{i=1}^{3} C_i$ *denotes the attribute set where* C_1 *stands for the empowerment of sales personnel,* C_2 *stands for the perceived quality of goods, and* C_3 *stands for the high traffic location, respectively. The value sets of these attributes are* $C_1 = \{high,\ medium,\ low\}$, $C_2 = \{good,\ average\}$ *and* $C_3 = \{no,\ yes\}$, *respectively. And* $D = \{profit, loss\}$ *describes shop profit or loss. Suppose that the six shops are characterized by the condition bijective soft set* $\widetilde{\cup}_{i=1}^{3}(f_i, C_i)$, *and the management benefit of shop is characterized by the decision partition soft set* (g, D).

The mapping of each bijective soft set over U *is defined as follows:*

$f_1(high) = \{x_1, x_6\}$, $f_1(medium) = \{x_2, x_3, x_5\}$, $f_1(low) = \{x_4\}$,

$f_2(good) = \{x_1, x_2, x_3\}$, $f_2(average) = \{x_4, x_5, x_6\}$,

$f_3(no) = \{x_1, x_2, x_3, x_4\}$, $f_3(yes) = \{x_5, x_6\}$.

The mapping of the decision partition soft set over U *is defined as follows:*

$g(profit) = \{x_1, x_3, x_6\}$, $g(loss) = \{x_2, x_4, x_5\}$.

Then we can view each bijective soft set (f_i, C_i) *as a collection of approximations as follows:*

$(f_1, C_1) = \{high = \{x_1, x_6\}, medium = \{x_2, x_3, x_5\}, low = \{x_4\}\}$,

$(f_2, C_2) = \{good = \{x_1, x_2, x_3\}, average = \{x_4, x_5, x_6\}\}$,

$(f_3, C_3) = \{no = \{x_1, x_2, x_3, x_4\}, yes = \{x_5, x_6\}\}$.

Similarly, $(g, D) = \{profit = \{x_1, x_3, x_6\}, loss = \{x_2, x_4, x_5\}\}$.

Denote

$$(f, C) = \widetilde{\cup}_{i=1}^{3}(f_i, C_i), \qquad (\varphi, K) = \bigwedge_{i=1}^{3}(f_i, C_i),$$

where $C = \bigcup_{i=1}^{3} C_i$ *and* $K = C_1 \times C_2 \times C_3$.

Let $e_i \in K$, *then*

e_1 *=high and good and no,* e_2 *=medium and good and no,*

e_3 *=low and average and no,* e_4 *=medium and average and yes,*

e_5 *=high and average and yes.*

$\varphi(e_1) = \{x_1\}$, $\varphi(e_2) = \{x_2, x_3\}$, $\varphi(e_3) = \{x_4\}$,

$\varphi(e_4) = \{x_5\}$, $\varphi(e_5) = \{x_6\}$.

Besides, we have the tabular form of (φ, K) *given in Table 2.*

Table 1: Tabular representation of (φ, K)

	e_1	e_2	e_3	e_4	e_5
x_1	1	0	0	0	0
x_2	0	1	0	0	0
x_3	0	1	0	0	0
x_4	0	0	1	0	0
x_5	0	0	0	1	0
x_6	0	0	0	0	1

So $(U, (f, C), (g, D))$ *is a soft decision system on how to choose profitable shops. Thus*

$$\underline{apr}_{(U,(f,C))}\ g(profit) = \{x_1, x_6\}, \quad \underline{apr}_{(U,(f,C))}\ g(loss) = \{x_4, x_5\}.$$

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Therefore $Pos_{(\varphi,K)}\ (g,D) = \{x_1, x_4, x_5, x_6\}$.

3.2 Relative reduction in soft decision systems

Definition 3.5. *Let* $(U,(f,C),(g,D))$ *be a soft decision system and let* $1 \le j \le n$. *Then*

(1) (f_j, C_j) *is called a soft dispensable set of* (f,C) *relative to* (g,D), *if* $Pos_{(\varphi,K)}\ (g,D) = Pos_{(\psi,Q)}\ (g,D)$, *where* $(\psi,Q) = \bigwedge_{i=1, i\neq j}^{n} (f_i, C_i)$. *Otherwise,* (f_j, C_j) *is called a soft indispensable set of* (f,C) *relative to* (g,D).

(2) (f,C) *is called a soft independent set relative to* (g,D), *if every soft bijective set* (f_i, C_i) *of* (f,C) *is a soft indispensable set relative to* (g,D). *Otherwise,* (f,C) *is called a soft dependent set relative to* (g,D).

(3) *The unit set of all the soft indispensable set of* (f,C) *relative to* (g,D) *is called the core of* (f,C) *relative to* (g,D), *denoted by* $core((f,C),(g,D))$.

Definition 3.6. *Let* $(U,(f,C),(g,D))$ *be a soft decision system. Let* $k = 1,2,\ldots,m$ *and* $1 \le j_k \le n$, *denote*

$$(f', C') = \widetilde{\cup}_{k=1}^{m} (f_{j_k}, C_{j_k}) \text{ and } (\varphi', K') = \bigwedge_{k=1}^{m} (f_{j_k}, C_{j_k}).$$

(f', C') *is called a relative reduction in* $(U,(f,C),(g,D))$, *if*
(1) $Pos_{(\varphi,K)}\ (g,D) = Pos_{(\varphi',K')}\ (g,D)$,
(2) (f', C') *is a soft independent set relative to* (g,D).

Example 3.7. *In Example 3.4, denote*

$(\varphi_1, K_1) = (f_1, C_1) \wedge (f_2, C_2)$, $(\varphi_2, K_2) = (f_1, C_1) \wedge (f_3, C_3)$,
$(\varphi_3, K_3) = (f_2, C_2) \wedge (f_3, C_3)$.

We have

$Pos_{(\varphi_1,K_1)}\ (g,D) = Pos_{(\varphi_2,K_2)}\ (g,D) = Pos_{(\varphi,K)}\ (g,D) = \{x_1, x_4, x_5, x_6\}$,
$Pos_{(\varphi_3,K_3)}\ (g,D) = \{x_4\}$.

But

$Pos_{(f_1,C_1)}\ (g,D) = \{x_1, x_4, x_6\} \neq Pos_{(\varphi,K)}\ (g,D)$,
$Pos_{(f_3,C_3)}\ (g,D) = \emptyset \neq Pos_{(\varphi,K)}\ (g,D)$.

So

$(f_1, C_1) \cup (f_2, C_2)$ *and* $(f_1, C_1) \cup (f_3, C_3)$ *are both relative reductions in* $(U,(f,C),(g,D))$.

3.3 Dependent degree of decision partition soft sets

Definition 3.8. *Let* (f,A) *and* (g,B) *be two soft sets over* U. (f,A) *is said to depend on* (g,B) *to a degree* k $(0 \le k \le 1)$, *denoted* $(f,A) \Rightarrow_k (g,B)$, *if*

$$k = \gamma((f,A),(g,B)) = \frac{\mid Pos_{(f,A)}\ (g,B) \mid}{\mid U \mid}.$$

Accordingly, in a soft decision system $(U, (f, C), (g, D))$, we have

$$k = \gamma((\varphi, K),\ (g, D)) = \frac{\mid Pos_{(\varphi,K)}\ (g, D) \mid}{\mid U \mid}.$$

We call it the dependent degree of decision partition soft sets upon condition bijective soft sets. It characters a degree of condition bijective soft sets in classifying decision partition soft sets. Obviously, we have $0 \leq k \leq 1$.

If $k = 1$, then (g, D) is completely dependent on (f, C).

If $k = 0$, then (g, D) is completely independent on (f, C).

Example 3.9. *In example 3.4, the dependent degree of the decision partition soft set (g, D) upon the condition bijective soft set $(f, C) = \widetilde{\cup}_{i=1}^{3}(f_i, C_i)$:*

$$k = \gamma(\bigwedge_{i=1}^{3}(f_i, C_i), (g, D)) = \frac{|\{x_1, x_4, x_5, x_6\}|}{|U|} = \frac{4}{6} = \frac{2}{3}$$

Proposition 3.10. *Let $(U, (f, C), (g, D))$ be a soft decision system. Let $m, n \in N$ and $m < n$. Then*

$$\gamma(\bigwedge_{i=1}^{m}(f_i, C_i), (g, D)) \leq \gamma(\bigwedge_{i=1}^{n}(f_i, C_i), (g, D)).$$

Proof. Since we have

$$\begin{aligned}
\gamma((\varphi, K), (g, D)) &= \frac{\mid Pos_{(\varphi,K)}(g, D) \mid}{\mid U \mid} = \frac{\mid \bigcup_{d\in D} \underline{apr}_P g(d) \mid}{\mid U \mid} \\
&= \frac{\mid \bigcup_{d\in D} \{x \in U \mid\ \exists\ e \in K\ s.t.\ x \in \varphi(e) \subseteq g(d)\} \mid}{\mid U \mid},
\end{aligned}$$

$$\begin{aligned}
\gamma((\varphi', K'), (g, D)) &= \frac{\mid Pos_{(\varphi',K')}(g, D) \mid}{\mid U \mid} = \frac{\mid \bigcup_{d\in D} \underline{apr}_{P'} g(d) \mid}{\mid U \mid} \\
&= \frac{\mid \bigcup_{d\in D} \{x \in U \mid\ \exists\ e \in K'\ s.t.\ x \in \varphi'(e) \subseteq g(d)\} \mid}{\mid U \mid},
\end{aligned}$$

where $K = \bigwedge_{i=1}^{n} C_i,\ K' = \bigwedge_{i=1}^{m} C_i,\ P = (U, (f, C))\ and\ P' = (U, (f', C'))$.

By Definition 2.6, for any $(c_1, c_2, \cdots, c_n) \in C_1 \times C_2 \times \cdots \times C_n$, we have

$$\varphi(c_1, c_2, \cdots, c_n) = f_1(c_1) \cap f_2(c_2) \cap \cdots \cap f_m(c_m) \cap \cdots \cap f_n(c_n).$$

Moreover, for any $(c_1, c_2, \cdots, c_m) \in C_1 \times C_2 \times \cdots \times C_m$, we also have

$$\varphi'(c_1, c_2, \cdots, c_m) = f_1(c_1) \cap f_2(c_2) \cap \cdots \cap f_m(c_m).$$

For $m, n \in N$ and $m < n$, $\underline{apr}_{P'} g(d) \subseteq \underline{apr}_P g(d)$.
So

$$\bigcup_{d \in D} \underline{apr}_{P'} g(d) \subseteq \bigcup_{d \in D} \underline{apr}_P g(d).$$

Hence

$$\gamma(\bigwedge_{i=1}^{m} (f_i, C_i), (g, D))\} \leq \gamma(\bigwedge_{i=1}^{n} (f_i, C_i), (g, D)).$$

□

In other words, condition bijective soft sets can explain the most detailed classification of decision partition soft sets. And deleting some condition bijective soft sets can lose some information about decision partition soft sets. Thus, more information (more condition bijective soft sets) can result in bigger dependent degree of decision partition soft sets.

3.4 Conditional significance relative to decision partition soft sets

Definition 3.11. *Let $(U, (f, C), (g, D))$ be a soft decision system and $1 \leq j \leq n$. The conditional significance of (f_j, C_j) in (f, C) relative to (g, D) is denoted and defined as follows*

$$s((f_j, C_j), (f, C), (g, D)) = \gamma(\bigwedge_{i=1}^{n} (f_i, C_i), (g, D)) \ - \ \gamma(\bigwedge_{i=1, i \neq j}^{n} (f_i, C_i), (g, D)).$$

This definition indicates the decrease of the dependent degree of decision partition soft sets when deleting one bijective soft set (f_j, C_j) from (f, C). The following results are easily obtained from the above definitions.

Proposition 3.12. *Let $(U, (f, C), (g, D))$ be a soft decision system and $1 \leq j \leq n$.*

(1) $0 \leq s((f_j, C_j), (f, C), (g, D)) \leq 1$,
(2)(f_j, C_j) *is a soft indispensable set of (f, C) to (g, D) if and only if*

$$s((f_j, C_j), (f, C), (g, D)) > 0,$$

(3) $core((f, C), (g, D)) = \widetilde{\cup}\{(f_j, C_j) \mid s((f_j, C_j), (f, C), (g, D)) > 0, j = 1, 2, \cdots, n\}$.

Theorem 3.13. *Let $(U, (f, C), (g, D))$ be a soft decision system. Let $k = 1, 2, \ldots, m$ and $1 \leq j_k \leq n$, denote*

$$(f', C') = \widetilde{\cup}_{k=1}^{m} (f_{j_k}, C_{j_k}) \text{ and } (\varphi', K') = \bigwedge_{k=1}^{m} (f_{j_k}, C_{j_k}),$$

where $C' = \bigcup_{j_k=1}^{m} C_{j_k}$ and $K' = C_{j1} \times C_{j2} \times \ldots \times C_{jm}$.
If $\gamma((\varphi', K'), (g, D)) = \gamma((\varphi, K), (g, D))$ and $s((f_j, C_j), (f', C'), (g, D)) > 0$, then (f', C') is a relative reduction of $(U, (f, C), (g, D))$.

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3.5 The multi-attribute decision rule in soft decision systems

Definition 3.14. *Let $(U,(f,C),(g,D))$ be a soft decision system. Let $e \in K = C_1 \times C_2 \times \ldots \times C_n,\ d \in D$. The soft rough membership function of $\varphi(e)$ relative to $g(d)$ is denoted and defined as follows*

$$\xi(\varphi(e), g(d)) = \frac{|\varphi(e) \cap g(d)|}{|\varphi(e)|}.$$

Definition 3.15. *Let $(U,(f,C),(g,D))$ be soft decision system. Let $k = 1,2,\ldots,m$ and $1 \le j_k \le n$, denote*

$$(f',C') = \widetilde{\cup}_{k=1}^{m}(f_{j_k}, C_{j_k}) \text{ and } (\varphi', K') = \bigwedge_{k=1}^{m}(f_{j_k}, C_{j_k}),$$

where $C = \bigcup_{j_k=1}^{m} C_{j_k}$ and $K' = C_{j1} \times C_{j2} \times \ldots \times C_{jm}$. Let (f',C') be a relative reduction of $(U,(f,C),(g,D))$. We call

$$\textit{If } e, \ \textit{then } d \ (\xi(\varphi'(e), g(d)))$$

the multi-attribute decision rule by induced (f',C') in $(U,(f,C),(g,D))$, where $e \in K',\ d \in D$ and $\xi(\varphi'(e), g(d))$ denotes the soft rough membership function of $\varphi'(e)$ relative to $g(d)$, which expresses the support degree of rules.

4 An algorithm for multi-attribute decision making based on soft rough sets

Based on above definitions and results, we will give an algorithm for the multi-attribute decision rule.

Algorithms:

Step 1. Construct a soft decision system $(U,(f,C),(g,D))$.

Step 2. Calculate the dependent degree of (g,D) upon $\bigwedge_{i=1,i\neq j}^{n}(f_i,C_i)$ $(j = 0,1,2,\ldots,n)$.

Step 3. Calculate each conditional significance of (f_j,C_j) in (f,C) relative to (g,D) by Definition 3.11.

Step 4. Find $\mathrm{core}((f,C),(g,D))$ by Proposition 3.12.

Step 5. Find relative reductions in $(U,(f,C),(g,D))$ by Theorem 3.13.

(1) If $\gamma(\mathrm{core}((f,C),(g,D)),(g,D)) = \gamma((f,C),(g,D))$, then $\mathrm{core}((f,C),(g,D))$ is a relative reduction in $(U,(f,C),(g,D))$. In this case, the process stops.

Otherwise, it continues (2).

(2) Denote

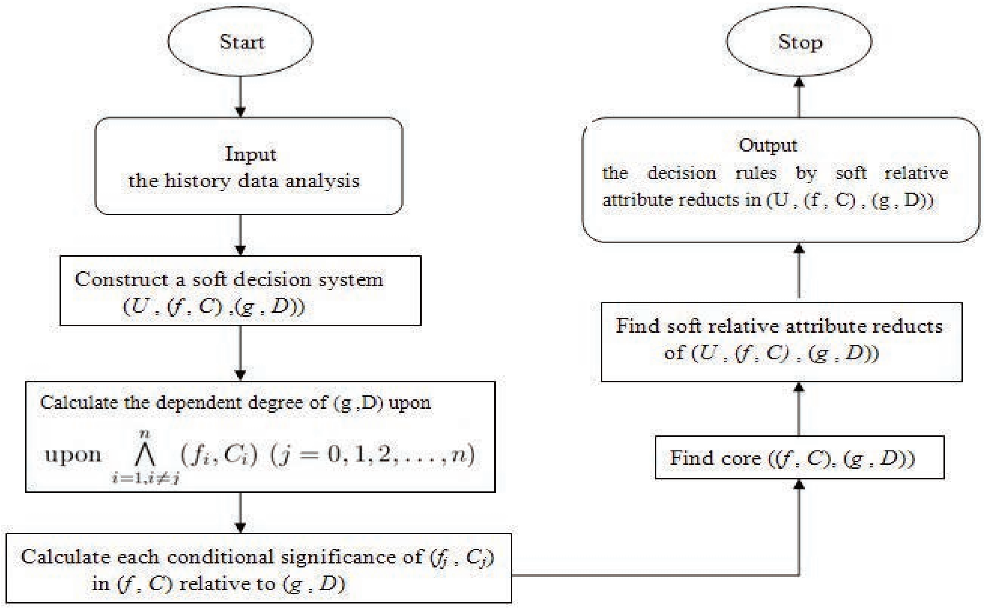
$\mathrm{core}((f,C),(g,D)) = \bigwedge_{k=1}^{m}(f_{j_k},C_{j_k})$, where $k = 1,2,\ldots,m$ and $1 \le j_k \le n$.

(a) Calculate the conditional significance of each bijective soft set (f_i, C_i) $(i \neq j_k)$ about $\widetilde{\cup}_{k=1}^{m}(f_{j_k}, C_{j_k})$ relative to (g, D) by Definition 3.11.

(b) Select (f_i, C_i) with maximal conditional significance one by one. If there are many soft sets with the same maximal significant, we choose the attribute set containing the most elements. So core$((f, C), (g, D))\widetilde{\cup}(f_i, C_i)$ is a relative reduction in $(U, (f, C), (g, D))$.

Step 6. Obtain decision rules by relative reductions in the soft decision system $(U, (f, C), (g, D))$.(Fig.1)

An algorithm



5 Conclusions

This method is based on cases of library history data analysis, then we can find the useful information. The multi-attribute decision rule and the support degree of rules provides scientific objective basis. This method reduces the search domain and hence does a more efficient retrieval than the existing methods. Therefore, the new evaluation method can help users to decide the component adapter scheme and reduce pressure and subjectivity in the component reuse process adapter decision-making.

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Fixed point results for modular ultrametric spaces

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Abstract

In this study, we define the notion of modular ultrametric space. We present a fixed point theorem in modular spherically complete ultrametric space, and prove coincidence point theorem for three self maps in a modular spherically complete ultrametric space.

1 Introduction

Fixed point theory is a developing field of mathematics with various applications to engineering, applied mathematics, some disciplines of sciences, etc. Fixed point theorems play a key role in this theory. Under certain conditions, we get some results related to a self map on any set, which allows one or more fixed points by means of them.

Ultrametric space is a kind of metric space but it has the strong triangle inequality, i.e.,

$$d(x, y) < \max\{d(x, z), d(z, y)\}.$$

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This metric is also known non-Archimedean metric. The notion of ultrametric is utilized outside mathematics. For example, ultrametric distances are tools of taxonomy and phylogenetic tree. The notion of ultrametric space was introduced by Van Rooij [32]. Gajic [13] proved a fixed point theorem for a class of generalized contractive mapping on ultrametric space. Rao et al. [30] introduced two coincidence point theorems for three and four self maps in a spherically complete ultrametric space. Some fixed point results on ultrametric spaces were given by Kirk and Shahzad [17]. There are also some studies in [10, 21].

Modular space was appeared by Nakano [24] in 1950. Many authors [19, 20, 25, 26, 27, 28, 29] gave some remarks on modular spaces. The concept of a modular metric space more general than a metric space was presented by Chistyakov [6]. He also developed the theory of modular metric spaces in [7, 8]. Chaipunya et al. [5] showed the existence of fixed point and uniqueness of quasi-contractive mappings in modular metric spaces. Azadifar et al. [4] proved the existence and uniqueness of a common fixed point of compatible mappings of integral type in modular metric spaces. Hussain and Salimi [14] investigated the existence of fixed points of generalized α-admissible modular contractive mappings in modular metric spaces. Kilinc and Alaca [15] defined (ε, k)-uniformly locally contractive mappings and η-chainable concept and proved a fixed point theorem for these concepts in complete modular metric spaces. Many studies were done in [1, 2, 3, 9, 11, 12, 16, 18, 22, 31, 33, 34].

In this paper, we first introduce the notion of modular ultrametric space. We give some fixed point theorems in a modular spherically complete ultrametric space.

2 Preliminaries

Definition 2.1. [27]. A modular on a real linear space X is a functional $\rho : X \longrightarrow [0, \infty]$ satisfying the following statements:

(A1) $\rho(0) = 0$;

(A2) If $x \in X$ and $\rho(\alpha x) = 0$ for all positive real numbers α, then $x = 0$;

(A3) $\rho(-x) = \rho(x)$ for all $x \in X$;

(A4) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and $x, y \in X$.

Let X be a nonempty set and $\lambda \in (0, \infty)$. We indicate that the function

$$\omega : (0, \infty) \times X \times X \to [0, \infty]$$

is denoted by $\omega_\lambda(x, y) = \omega(\lambda, x, y)$ for all $\lambda > 0$ and $x, y \in X$.

Definition 2.2. [7]. Let X be a nonempty set. The function

$$\omega : (0, \infty) \times X \times X \to [0, \infty]$$

is called a metric modular on X if, for all $x, y, z \in X$, the following conditions hold:

(i) $\omega_\lambda(x, y) = 0$ for all $\lambda > 0 \iff x = y$;

(ii) $\omega_\lambda(x, y) = \omega_\lambda(y, x)$ for all $\lambda > 0$;

(iii) $\omega_{\lambda+\mu}(x, y) \leq \omega_\lambda(x, z) + \omega_\mu(z, y)$ for all $\lambda, \mu > 0$.

Let's recall the definitions of two sets X_ω and X_ω^* [7]:

$$X_\omega \equiv X_\omega(x_0) = \{x \in X : \omega_\lambda(x, x_0) \to 0 \ \text{ as } \ \lambda \to \infty\}$$

and

$$X_\omega^* \equiv X_\omega^*(x_0) = \{x \in X : \exists \lambda = \lambda(x) > 0 \ \text{ such that } \ \omega_\lambda(x, x_0) < \infty\}.$$

Definition 2.3. [32]. Let (X, d) be a metric space. If the metric d satisfies the strong triangle inequality

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}$$

for all $x, y, z \in X$, then it is called *ultrametric* on X. The pair (X, d) is said to be *ultrametric space.*

3 Modular ultrametric spaces

In this section, we first give some new definitions.

Definition 3.1. Let (X, ω) be a modular metric space. If ω satisfies the strong triangle inequality

$$\omega_\lambda(x, y) \leq \max\{\omega_\lambda(x, z), \omega_\lambda(z, y)\}$$

for all $x, y, z \in X$, then it is called a *modular ultrametric* on X.

Definition 3.2. Let (X, ω) be a modular ultrametric space. For $r > 0$ and $x \in X_\omega$, we define the open sphere $B_\omega(x, r)$ and the closed sphere $B_\omega[x, r]$ with centre x and radius r as follows:

$$\begin{aligned} B_\omega(x, r) =& \{y \in X_\omega : \omega_\lambda(x, y) < r\} \\ B_\omega[x, r] =& \{y \in X_\omega : \omega_\lambda(x, y) \leq r\}. \end{aligned}$$

Definition 3.3. The modular ultrametric space X_ω^* is called a *modular spherically complete ultrametric space* if every nest of balls has a nonempty intersection.

Theorem 3.4. *Let X_ω^* be a modular spherically complete ultrametric space. Assume that there exists an element $x = x(\lambda) \in X_\omega^*$ such that $\omega_\lambda(x, Tx) < \infty$. If $T : X_\omega^* \to X_\omega^*$ is a map such that for every $x, y \in X_\omega^*$, $x \neq y$,*

$$\omega_\lambda(Tx, Ty) < \max\{\omega_\lambda(x, Tx), \omega_\lambda(x, y), \omega_\lambda(y, Ty)\}, \tag{3.1}$$

then T has a unique fixed point.

Proof. Let $B_a = B_\omega[a, \omega_\lambda(a, Ta)]$ be the closed sphere centered at a with the radius $\omega_\lambda(a, Ta)$ and let $\mathcal{A}$ be the collection of these spheres for all $a \in X_\omega^*$.

It is clear that the relation

$$B_a \leq B_b \quad \Leftrightarrow \quad B_b \subseteq B_a$$

is a partial order on $\mathcal{A}$.

Now we pay attention to a totally ordered subfamily $\mathcal{A}_1$ of $\mathcal{A}$. Since X_ω^* is modular spherically complete, we have

$$\bigcap_{B_a \in \mathcal{A}_1} B_a = B \neq \emptyset.$$

Let $b \in B$, $b_a \in \mathcal{A}_1$ and $x \in B_b$. Then we get the following:

$$\omega_\lambda(x, b) \leq \omega_\lambda(b, Tb) \leq \max\{\omega_\lambda(b, a), \omega_\lambda(a, Ta), \omega_\lambda(Ta, Tb)\}$$

$$= \max\{\omega_\lambda(a, Ta), \omega_\lambda(Ta, Tb)\}. \tag{3.2}$$

The case $\omega_\lambda(Ta, Tb) \leq \omega_\lambda(a, Ta)$ implies that

$$\omega_\lambda(x, b) \leq \omega_\lambda(a, Ta).$$

In case of $\omega_\lambda(Ta, Tb) > \omega_\lambda(a, Ta)$, it follows from (3.2) that

$$\begin{aligned}\omega_\lambda(x, b) \leq \omega_\lambda(b, Tb) \leq &\omega_\lambda(Ta, Tb) < \max\{\omega_\lambda(a, Ta), \omega_\lambda(a, b), \omega_\lambda(b, Tb)\}\\ &= \max\{\omega_\lambda(a, Ta), \omega_\lambda(b, Tb)\}.\end{aligned}$$

By $\omega_\lambda(b, Tb) \leq \omega_\lambda(a, Ta)$, we have

$$\omega_\lambda(x, b) \leq \omega_\lambda(a, Ta)$$

and $\omega_\lambda(b, Tb) > \omega_\lambda(a, Ta)$ shows that $\omega_\lambda(b, Tb) < \omega_\lambda(b, Tb)$, which is a contradiction. So $\omega_\lambda(x, b) \leq \omega_\lambda(a, Ta)$ for $x \in B_b$.

Since we have

$$\omega_\lambda(x, a) \leq \omega_\lambda(a, Ta),$$

$x \in B_a$ and $B_b \subseteq B_a$ for any $B_a \in \mathcal{A}_1$. Thus B_b is the upper bound for the family $\mathcal{A}$. From Zorn's Lemma, we conclude that $\mathcal{A}$ has a maximal element B_z where $z \in X^*_\omega$.

Now we prove $z = Tz$. Suppose that $z \neq Tz$. The inequality (3.1) implies that

$$\omega_\lambda(Tz, T(Tz)) < \omega_\lambda(z, Tz).$$

If $y \in B_{Tz}$, then $\omega_\lambda(y, Tz) \leq \omega_\lambda(Tz, T(Tz)) < \omega_\lambda(z, Tz)$. In this case, we get

$$\omega_\lambda(y, z) \leq \max\{\omega_\lambda(y, Tz), \omega_\lambda(Tz, z)\} = \omega_\lambda(Tz, z),$$

i.e., $y \in B_z$ and $B_{Tz} \subseteq B_z$. Moreover, $z \notin B_{Tz}$ since

$$\omega_\lambda(z, Tz) > \omega_\lambda(Tz, T(Tz)).$$

As a consequence, we have $B_{Tz} \subsetneq B_z$ but it contradicts to the maximality of B_z. Hence we have $z = Tz$.

It only remains to show the uniqueness. For this purpose, we take u as a different fixed point. For $u \neq z$, we have

$$\omega_\lambda(z, u) = \omega_\lambda(Tz, Tu) < \max\{\omega_\lambda(Tz, z), \omega_\lambda(z, u), \omega_\lambda(u, Tu)\} = \omega_\lambda(z, u)$$

which is a contradiction. This completes the proof. □

Theorem 3.5. *Let X^*_ω be a modular ultrametric space, and let $f, S, T : X^*_\omega \to X^*_\omega$ be maps satisfying*

(1) $f(X^*_\omega)$ is modular spherically complete,

(2) $\omega_\lambda(Sx,Ty) < \max\{\omega_\lambda(fx,fy),\omega_\lambda(fx,Sx),\omega_\lambda(fy,Ty)\}$ *for* $x,y \in X_\omega^*$, $x \neq y$,

(3) $fS = Sf,\ fT = Tf,\ ST = TS$,

(4) $S(X_\omega^*) \subseteq f(X_\omega^*),\ T(X_\omega^*) \subseteq f(X_\omega^*)$.

Then either $fw = Sw$ *or* $fw = Tw$ *for some* $w \in X_\omega^*$.

Proof. For $a \in X_\omega^*$, let $B_a = [fa, \max\{\omega_\lambda(fa,Sa),\omega_\lambda(fa,Ta)\}]$ denote the closed sphere centered at fa with the radius $\max\{\omega_\lambda(fa,Sa),\omega_\lambda(fa,Ta)\}$. Let A be the collection of all the spheres for all $a \in f(X_\omega^*)$. We state that the relation $B_a \leq B_b$ iff $B_b \subseteq B_a$ is a partial order on A. For a totally ordered subfamily A_1 of A, since $f(X_\omega^*)$ is modular spherically complete, we have

$$\bigcap_{B_a \in A_1} B_a = B \neq \emptyset.$$

Let $fb \in B$ where $b \in f(X_\omega^*)$ and $B_a \in A_1$. Then we have $fb \in B_a$ and so

$$\omega_\lambda(fb,fa) \leq \max\{\omega_\lambda(fa,Sa),\omega_\lambda(fa,Ta)\}. \tag{3.3}$$

If $a = b$, then $B_a = B_b$. Assume that $a \neq b$ and $x \in B_b$. It follows from the condition (2) and (3.3) that

$$\begin{aligned}
\omega_\lambda(x,fb) &\leq \max\{\omega_\lambda(fb,Sb),\omega_\lambda(fb,Tb)\}\\
&\leq \max\{\omega_\lambda(fb,fa),\omega_\lambda(fa,Ta),\omega_\lambda(Ta,Sb),\\
&\qquad\quad \omega_\lambda(fb,fa),\omega_\lambda(fa,Sa),\omega_\lambda(Sa,Tb)\}\\
&< \max\{\omega_\lambda(fb,fa),\omega_\lambda(fa,Ta),\omega_\lambda(fa,Sa),\\
&\qquad\quad \max\{\omega_\lambda(fb,fa),\omega_\lambda(fb,Sb),\omega_\lambda(fa,Ta)\},\\
&\qquad\quad \max\{\omega_\lambda(fa,fb),\omega_\lambda(fa,Sa),\omega_\lambda(fb,Tb)\}\}\\
&= \max\{\omega_\lambda(fa,Sa),\omega_\lambda(fa,Ta)\}.
\end{aligned}$$

Thus

$$\omega_\lambda(x,fb) < \max\{\omega_\lambda(fa,Sa),\omega_\lambda(fa,Ta)\}. \tag{3.4}$$

From (3.3) and (3.4), we get

$$\begin{aligned}
\omega_\lambda(x,fa) &\leq \max\{\omega_\lambda(x,fb),\omega_\lambda(fb,fa)\}\\
&\leq \max\{\omega_\lambda(fa,Sa),\omega_\lambda(fa,Ta)\}.
\end{aligned}$$

Therefore, $x \in B_a$. We have also $B_b \subseteq B_a$ for any $B_a \in A_1$ and B_b is an upper bound in A for the family A_1. By Zorn's Lemma, there is a maximal element, denoted by B_z, in A, where $z \in F(X_\omega^*)$. There exists an element $w \in X_\omega^*$ such that $z = fw$.

Suppose $fw \neq Sw$ and $fw \neq Tw$. Since $fS = Sf$,

$$\begin{aligned}
\omega_\lambda(Sfw,TSw) &< \max\{\omega_\lambda(f^2w,fSw),\omega_\lambda(f^2w,Sfw),\omega_\lambda(fSw,TSw)\}\\
&= \omega_\lambda(f^2w,fSw).
\end{aligned} \tag{3.5}$$

Since $fT = Tf$,

$$\begin{aligned}
\omega_\lambda(STw,Tfw) &< \max\{\omega_\lambda(fTw,f^2w),\omega_\lambda(fTw,STw),\omega_\lambda(f^2w,Tfw)\}\\
&= \omega_\lambda(f^2w,fTw)
\end{aligned} \tag{3.6}$$

Since $ST = TS$, it follows from (3.5) and (3.6) that

$$\begin{aligned}
\omega_\lambda(Sfw, S^2w) &\leq \max\{\omega_\lambda(Sfw, TSw), \omega_\lambda(TSw, Tfw), \omega_\lambda(Tfw, S^2w)\} \\
&< \max\{\omega_\lambda(f^2w, fSw), \omega_\lambda(f^2w, fTw), \\
&\qquad \max\{\omega_\lambda(fSw, f^2w), \omega_\lambda(fSw, S^2w), \omega_\lambda(f^2w, Tfw)\}\} \\
&= \max\{\omega_\lambda(f^2w, fSw), \omega_\lambda(f^2w, fTw)\}.
\end{aligned} \tag{3.7}$$

From (3.5) and (3.7), we have

$$\begin{aligned}
&\max\{\omega_\lambda(Sfw, TSw), \omega_\lambda(Sfw, S^2w)\} \\
&\qquad < \max\{\omega_\lambda(f^2w, fSw), \omega_\lambda(f^2w, fTw)\}.
\end{aligned} \tag{3.8}$$

By (3.5) and (3.6),

$$\begin{aligned}
\omega_\lambda(Tfw, T^2w) &\leq \max\{\omega_\lambda(Tfw, TSw), \omega_\lambda(TSw, Sfw), \omega_\lambda(Sfw, T^2w)\} \\
&< \max\{\omega_\lambda(f^2w, fTw), \omega_\lambda(f^2w, fSw), \\
&\qquad \max\{\omega_\lambda(f^2w, fTw), \omega_\lambda(f^2w, Sfw), \omega_\lambda(fTw, T^2w)\}\} \\
&= \max\{\omega_\lambda(f^2w, fTw), \omega_\lambda(f^2w, fSw)\}.
\end{aligned} \tag{3.9}$$

From (3.6) and (3.9), we have

$$\begin{aligned}
&\max\{\omega_\lambda(STw, Tfw), \omega_\lambda(Tfw, T^2w)\} \\
&\qquad < \max\{\omega_\lambda(f^2w, fTw), \omega_\lambda(f^2w, fSw)\}.
\end{aligned} \tag{3.10}$$

If $\max\{\omega_\lambda(f^2w, fTw), \omega_\lambda(f^2w, fSw)\} = \omega_\lambda(f^2w, fSw)$, then from (3.8), we have

$$\max\{\omega_\lambda(Sfw, TSw), \omega_\lambda(Sfw, S^2w)\} < \omega_\lambda(f^2w, fSw)$$

which gives $f^2w \notin B_{Sw}$. Hence $fz \notin B_{Sw}$. But $fz \in B_z$. Hence $B_z \nsubseteq B_{Sw}$. It is a contradiction to the maximality of B_z in A, since $Sw \in S(X^*_\omega) \subseteq f(X^*_\omega)$. If

$$\max\{\omega_\lambda(f^2w, fTw), \omega_\lambda(f^2w, fSw)\} = \omega_\lambda(f^2w, fTw),$$

then from (3.10), $\max\{\omega_\lambda(STw, Tfw), \omega_\lambda(Tfw, T^2w)\} < \omega_\lambda(f^2w, fTw)$ which gives $f^2w \notin B_{Tw}$. Hence $fz \notin B_{Tw}$. Since $fz \in B_z$, we get $B_z \nsubseteq B_{Tw}$. It contradicts to the maximality of B_z in A, since $Tw \in T(X^*_\omega) \subseteq f(X^*_\omega)$. As a result, either $fw = Sw$ or $fw = Tw$. □

Proposition 3.6. *Let X^*_ω be a modular spherically complete ultrametric space and let*

$$f, T : X^*_\omega \to X^*_\omega$$

be maps satisfying $T(X^*_\omega) \subseteq f(X^*_\omega)$ and

$$\omega_\lambda(Tx, Ty) < \max\{\omega_\lambda(fx, fy), \omega_\lambda(fx, Tx), \omega_\lambda(fy, Ty)\} \tag{3.11}$$

for all $x, y \in X^*_\omega$, with $x \neq y$. Then there exists $z \in X^*_\omega$ such that $fz = Tz$. Moreover, if f and T are coincidentally commuting at z, then z is a unique common fixed point of f and T.

Proof. Let $B_a = [fa, \omega_\lambda(fa, Ta)]$ represent the closed sphere centered at fa with radius $\omega_\lambda(fa, Ta)$ and let A be the collection of these spheres for all $a \in X$. By the same reasoning as in Theorem 3.5, we conclude that A has a maximal element B_z for an element $z \in X^*_\omega$.

Let's assume $fz \neq Tz$. Since $Tz \in T(X) \subseteq f(X)$, there exists $w \in X^*_\omega$ such that $Tz = fw$. It is clear that $w \neq z$. From (3.11), we have

$$\begin{aligned}\omega_\lambda(fw, Tw) =& \omega_\lambda(Tz, Tw)\\ &< \max\{\omega_\lambda(fz, fw), \omega_\lambda(fz, Tz), \omega_\lambda(fw, Tw)\}\\ &= \omega_\lambda(fz, fw).\end{aligned}$$

Thus $fz \notin B_w$ and $B_z \nsubseteq B_w$. This contradicts to the maximality of B_z. So $fz = Tz$.

On the other hand, we suppose that f and T are coincidentally commuting at z. Then

$$f^2z = f(fz) = fTz = Tfz = T(Tz) = T^2z.$$

Suppose $fz \neq z$. By (3.11), we conclude that

$$\begin{aligned}\omega_\lambda(Tfz, Tz) &< \max\{\omega_\lambda(f^2z, fz), \omega_\lambda(f^2z, Tfz), \omega_\lambda(fz, Tz)\}\\ &= \omega_\lambda(Tfz, Tz),\end{aligned}$$

which is a contradiction. Thus $z = fz = Tz$.

Now we show the uniqueness. Let u be a different fixed point. For $u \neq z$, we have

$$\begin{aligned}\omega_\lambda(z, u) &= \omega_\lambda(Tz, Tu) < \max\{\omega_\lambda(fz, fu), \omega_\lambda(fz, Tz), \omega_\lambda(fu, Tu)\}\\ &= \omega_\lambda(z, u),\end{aligned}$$

which is a contradiction. As a consequence, we have the required result. □

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On the backward difference scheme for a class of SIRS epidemic models with nonlinear incidence

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Abstract. In this paper, we construct a backward difference scheme for a class of SIRS epidemic models with nonlinear incidence rate $\beta f(S)g(I)$ and vaccination in susceptible. The dynamical properties of the scheme are investigated. By using the inductive method and the linearization method of difference equations, the positivity and the boundedness of solutions, the existence and local stability of equilibria are obtained. By constructing new discrete type Lyapunov functions, under the conditions which functions $f(S)$ and $g(I)$ satisfy assumptions $(H_1)-(H_3)$, the global stability of the equilibria is obtained. That is, the disease-free equilibrium is globally asymptotically stable if basic reproduction number $\mathscr{R}_0 \leq 1$, and the endemic equilibrium is globally asymptotically stable if $\mathscr{R}_0 > 1$.

Keywords: discrete SIRS epidemic model; backward difference scheme; nonlinear incidence; local and global stability; discrete Lyapunov function.

1. Introduction

As we well known, for some practical purposes, especially the numerical computing, it is often necessary to discretize the continuous-time model to a corresponding discrete difference scheme, that is discrete dynamical model.

In recent years, aim at the continuous-time SIR and SIRS epidemic models, the various discrete dynamical models are constructed, and the dynamical properties of these models are studied in many articles, for example, see [1-22] and the reference therein. Many important results have been established. These results focus on: the computation of the basic reproduction number, the local and global stability of the disease-free equilibrium and endemic equilibrium, the permanence, persistence and extinction of the disease, the bifurcation and chaos phenomena, etc.

Particularly, we see that, in [1,2], the authors studied a class of discrete SIRS epidemic models with time delays and bilinear incidence derived from corresponding continuous models by applying the nonstandard finite difference scheme (See [23-26]), and the sufficient conditions on the global asymptotic stability of the disease-free equilibrium and the permanence of the disease are established. In [3], the authors studied a discrete SIRS epidemic model with bilinear incidence derived from corresponding continuous model by applying the backward difference scheme, and the sufficient and necessary conditions on the global asymptotic stability of the disease-free equilibrium and endemic equilibrium are established. In [4], the authors discussed a class of discrete SIRS epidemic models with general nonlinear incidence derived from corresponding continuous model by applying the forward d-ifference scheme, and the sufficient conditions for the existence and local stability of the disease-free equilibrium and endemic equilibrium are obtained. In [5], the authors discussed a class of discrete SIRS epidemic models with standard incidence discretized from corresponding continuous model by applying the forward difference scheme, and the sufficient condition for the global stability of the endemic equilibrium is established.

However, from above articles we easily see that the studies on the backward difference scheme for SIRS epidemic models with nonlinear incidence are few. In this paper, we construct a backward difference scheme for a class of continuous-time SIRS epidemic models with nonlinear incidence $\beta f(S)g(I)$ and vaccination in susceptible. We will study the dynamical properties, especially the global asymptotic stability of the disease-free equilibrium and endemic equilibrium for this discrete model. Firstly, the basic properties of the model, such as, the positivity and the boundedness of solutions, the existence and local stability of equilibria are discussed by using the inductive method and the linearization method of difference equations. Further, by constructing new discrete type Lyapunov functions which is different from those given in [3] and using the theory of stability of difference equations, we will establish the global asymptotic stability of equilibria under the assumptions $(H_1)-(H_3)$ (see

Section 2). That is, when assumptions $(H_1)-(H_3)$ hold, the disease-free equilibrium is globally asymptotically stable if and only if basic reproduction number $\mathscr{R}_0 \leq 1$, and the endemic equilibrium is globally asymptotically stable if and only if $\mathscr{R}_0 > 1$.

The organization of this paper is as follows. In the second section we firstly introduce a backward difference scheme, that is discrete dynamical model, for SIRS epidemic models with nonlinear incidence, and further give some basic assumptions. In the third section the results on the positivity and boundedness of solutions, the existence and local stability of equilibria for the model are stated and proved. In the fourth section we will state and prove the theorems on the global asymptotic stability of the disease-free equilibrium and endemic equilibrium for the model. Lastly, in the fifth section we will give a conclusion.

2. Model description

We consider the following continuous SIRS epidemic model with nonlinear incidence and vaccination in susceptible

$$
\begin{aligned}
\frac{dS}{dt} &= A - \beta f(S)g(I) - d_1 S - \eta S + \delta R, \\
\frac{dI}{dt} &= \beta f(S)g(I) - d_2 I - \gamma I, \\
\frac{dR}{dt} &= \eta S + \gamma I - d_3 R - \delta R,
\end{aligned} \tag{1}
$$

where $S(t), I(t)$ and $R(t)$ denote the numbers of susceptible, infected and recovered individuals at time t, respectively. A is the recruitment rate of the total population, d_1, d_2, and d_3 represent the death rate of susceptible, infected and recovered individuals, respectively. Particularly, death rate d_2 includes the natural death rate and the disease-related death rate of the infected individuals. δ is the rate at which recovered individuals lose immunity and return to the susceptible class. γ is the natural recovery rate of the infective individuals, β is the proportionality constant. $f(S)$ and $g(I)$ are continuous functions defined on $[0, \infty)$. The transmission of the infection is governed by a nonlinear incidence rate $\beta f(S)g(I)$. In this paper, we always assume that δ is nonnegative constant, and $A, d_1, d_2, d_3, \beta, \gamma$ are positive constants.

Now, we use the backward difference scheme to discretize model (1). Let $h > 0$ be the time step size. Since

$$
\frac{dS(t)}{dt} = \lim_{h\to 0} \frac{S(t+h) - S(t)}{h}, \quad \frac{dI(t)}{dt} = \lim_{h\to 0} \frac{I(t+h) - I(t)}{h},
$$

$$\frac{dR(t)}{dt} = \lim_{h\to 0} \frac{R(t+h)-R(t)}{h}.$$

$$\lim_{h\to 0}(A - \beta f(S(t+h))g(I(t+h)) - (d_1+\eta)S(t+h) + \delta R(t+h))$$
$$= A - \beta f(S(t))g(I(t)) - (d_1+\eta)S(t) + \delta R(t),$$
$$\lim_{h\to 0}(\beta f(S(t+h))g(I(t+h)) - (d_2+\gamma)I(t+h))$$
$$= \beta f(S(t))g(I(t)) - (d_2+\gamma)I(t)$$

and

$$\lim_{h\to 0}(\eta S(t+h) + \gamma I(t+h) - (d_3+\delta)R(t+h))$$
$$= \eta S(t) + \gamma I(t) - (d_3+\delta)R(t),$$

we can assume from model (1) for any $h > 0$

$$\begin{aligned} \frac{S(t+h)-S(t)}{h} &= A - \beta f(S(t+h))g(I(t+h)) \\ &\quad -(d_1+\eta)S(t+h) + \delta R(t+h), \\ \frac{I(t+h)-I(t)}{h} &= \beta f(S(t+h))g(I(t+h)) - (d_2+\gamma)I(t+h), \\ \frac{R(t+h)-R(t)}{h} &= \eta S(t+h) + \gamma I(t+h) - (d_3+\delta)R(t+h). \end{aligned} \tag{2}$$

Denote $t = n$, $t+h = n+1$, $S(t) = S_n$, $I(t) = I_n$, $R(t) = R_n$, $S(t+h) = S_{n+1}$, $I(t+h) = I_{n+1}$ and $R(t+h) = R_{n+1}$, then from (2) we further obtain the following discrete SIRS epidemic model with nonlinear incidence and vaccination in susceptible

$$\begin{aligned} S_{n+1} - S_n &= h[A - \beta f(S_{n+1})g(I_{n+1}) - (d_1+\eta)S_{n+1} + \delta R_{n+1}], \\ I_{n+1} - I_n &= h[\beta f(S_{n+1})g(I_{n+1}) - (d_2+\gamma)I_{n+1}], \\ R_{n+1} - R_n &= h[\eta S_{n+1} + \gamma I_{n+1} - (d_3+\delta)R_{n+1}]. \end{aligned} \tag{3}$$

In this paper, our main aim namely is to investigate the dynamical properties of model (3). The initial condition for model (3) is given in the following form

$$S_0 > 0,\ I_0 > 0,\ R_0 \geq 0. \tag{4}$$

For model (3) we firstly introduce the following assumption.

(H_1) Functions $f(S)$ and $g(I)$ are continuously differentiable and monotone increasing on R, $f(0) = g(0) = 0$, $\frac{I}{g(I)}$ is monotone increasing on $(0, +\infty)$ and $g'(0) > 0$.

Remark 1. It is obvious that assumption (H_1) is basic for model (3). In fact, when $f(S) = S^p$ (where $0 < p \leq 1$) or $f(S) = \frac{S}{1+\alpha S}$ and $g(I) = \frac{I}{1+\omega I}$, then assumption (H_1) naturally holds.

Remark 2. If function $g(I)$ satisfies that second order derivative $g''(I)$ exists and $g''(I) \leq 0$ for all $I \in [0, \infty)$, then we can easily prove that $\frac{I}{g(I)}$ is monotone increasing on $I \in (0, +\infty)$.

Define a function $F(u, v)$ as follows. For any $u, v \in R$, if $u \neq v$ then $F(u, v) = \frac{f(u)-f(v)}{u-v}$, and if $u = v$ then $F(u, u) = f'(u)$. In order to obtain the global asymptotic stability of equilibria of model (3), we need to further introduce the following assumption.

(H_2) $(d_1 + d_3)\gamma - (d_2 + d_3)\eta > 0$.

(H_3) There are positive constants K_1 and K_3 such that

$$4d_1[d_3 + k_4(d_3 + \delta)] > (d_1 + d_3 - k_4\eta - K_1\delta F(u, v))^2$$

and

$$4K_1(d_1 + \eta)d_2F(u, v) > (d_1 + d_2 - K_3\beta F(u, v))^2$$

for any $0 < u, v \leq \frac{A}{d}$ with $u \neq v$, where $k_4 = \frac{d_2+d_3}{\gamma}$ and $d = \min\{d_1, d_2, d_3\}$.

Remark 3. When $f(S) \equiv S$, we have $F(u, v) \equiv 1$. Choosing positive constants $K_1 = \frac{d_1+d_3-k_4\eta}{\delta}$ and $K_2 = \frac{d_1+d_2}{\beta}$, then assumption (H_3) naturally holds.

Remark 4. When $f(S) = \frac{S}{1+\omega S}$, we have that there is a $\xi = \xi(u, v) \in (0, \frac{A}{d})$ such that $F(u, v) = \frac{1}{(1+\omega\xi)^2}$. Obviously,

$$\frac{1}{(1 + \omega\frac{A}{d})^2} \leq F(u, v) = \frac{1}{(1 + \omega\xi)^2} \leq 1$$

for all $u, v \in (0, \frac{A}{d}]$ with $u \neq v$. Choose positive constants

$$K_1 = \frac{2(d_1 + d_3 - k_4\eta)(1 + \omega\frac{A}{d})^2}{\delta(1 + (1 + \omega\frac{A}{d})^2)}, \quad K_3 = \frac{2(d_1 + d_2)(1 + \omega\frac{A}{d})^2}{\beta(1 + (1 + \omega\frac{A}{d})^2)},$$

then we can easily obtain that

$$(d_1 + d_3 - k_4\eta - K_1\delta F(u, v))^2 \leq (1 - \frac{2(1 + \omega\frac{A}{d})^2}{1 + (1 + \omega\frac{A}{d})^2})^2(d_1 + d_3 - k_4\eta)^2,$$

$$(d_1 + d_2 - K_3\beta F(u, v))^2 \leq (1 - \frac{2(1 + \omega\frac{A}{d})^2}{1 + (1 + \omega\frac{A}{d})^2})^2(d_1 + d_2)^2$$

and

$$4K_1(d_1 + \eta)d_2F(u, v) \geq 8(d_1 + \eta)d_2\frac{d_1 + d_3 - k_4\eta}{\delta(1 + (1 + \omega\frac{A}{d})^2)}$$

for all $u, v \in (0, \frac{A}{d}]$ with $u \neq v$.

Therefore, for $f(S)=\frac{S}{1+\omega S}$, if we assume that the following conditions hold

$$4d_1(d_3+k_4(d_3+\delta))>(1-\frac{2(1+\omega\frac{A}{d})^2}{1+(1+\omega\frac{A}{d})^2})^2(d_1+d_3-k_4\eta)^2 \tag{5}$$

and

$$8(d_1+\eta)d_2\frac{d_1+d_3-k_4\eta}{\delta(1+(1+\omega\frac{A}{d})^2)}>(1-\frac{2(1+\omega\frac{A}{d})^2}{1+(1+\omega\frac{A}{d})^2})^2(d_1+d_2)^2, \tag{6}$$

then it can be easily proved that assumption (H_3) holds.

Particularly, when $\omega=0$, that is $f(S)\equiv S$, we easily see that conditions (5) and (6) naturally hold.

3. Basic properties

Firstly, on the existence of positive solutions with initial condition (4) and the boundedness of all solutions of model (3), we have the following results.

Theorem 1. *Suppose that (H_1) holds. Then model (3) has a unique positive solution (S_n,I_n,R_n) for all $n\geq 0$ with initial condition (4), and*

$$\limsup_{n\to\infty}(S_n+I_n+R_n)\leq\frac{A}{d},$$

where $d=\min\{d_1,d_2,d_3\}$.

Proof: When $n=0$, from model (3) we have

$$\begin{aligned}(1+h(d_1+\eta))S_1&=S_0+h[A+\delta R_1-\beta f(S_1)g(I_1)],\\(1+h(d_2+\gamma))I_1&=I_0+h\beta f(S_1)g(I_1),\\(1+h(d_3+\delta))R_1&=R_0+h[\eta S_1+\gamma I_1].\end{aligned} \tag{7}$$

Solving S_1 from (7), we obtain

$$\begin{aligned}S_1=&\ \frac{1}{1+h(d_1+\eta)}[S_0+h(A+\delta R_1)+I_0-(1+h(d_2+\gamma))I_1]\\=&\ \frac{1}{1+h(d_1+\eta)}[S_0+I_0+h(A+\frac{\delta R_0}{1+h(d_3+\delta)})\\&-(1+h(d_2+\gamma)-\frac{h^2\gamma\delta}{1+h(d_3+\delta)})I_1+\frac{h^2\delta\eta}{1+h(d_3+\delta)}S_1].\end{aligned}$$

Therefore, (7) is equivalent to

$$\begin{aligned}S_1=&\ a^{-1}\frac{1}{1+h(d_1+\eta)}[S_0+I_0+h(A+\frac{\delta R_0}{1+h(d_3+\delta)}\\&-(1+h(d_2+\gamma)-\frac{h^2\gamma\delta}{1+h(d_3+\delta)})I_1],\\I_1=&\ \frac{I_0+h\beta f(S_1)g(I_1)}{1+h(d_2+\gamma)},\quad R_1=\frac{R_0+h\gamma I_1}{1+h(d_3+\delta)},\end{aligned} \tag{8}$$

where

$$a = 1 - \frac{h^2\delta\eta}{(1+h(d_1+\eta))(1+h(d_3+\delta))}.$$

Obviously, $a > 0$, $1 + h(d_2+\gamma) - \frac{h^2\gamma\delta}{1+h(d_3+\delta)} > 0$. Let

$$\bar{I}_1 = \frac{S_0 + I_0 + h(A + \frac{\delta R_0}{1+h(d_3+\delta)})}{1 + h(d_2+\gamma) - \frac{h^2\gamma\delta}{1+h(d_3+\delta)}},$$

then from (8) we have $S_1 > 0$ when $0 < I_1 < \bar{I}_1$, $S_1 < 0$ when $I_1 > \bar{I}_1$, and $S_1 = 0$ when $I_1 = \bar{I}_1$.

Let

$$\Psi^*(I_1) \triangleq \frac{I_1}{g(I_1)} - \frac{1}{1+h(d_2+\gamma)}(\frac{I_0}{g(I_1)} + h\beta f(S_1))$$

with

$$\begin{aligned} S_1 = \ & a^{-1}\frac{1}{1+h(d_1+\eta)}[S_0 + I_0 + h(A + \frac{\delta R_0}{1+h(d_3+\delta)}) \\ & -(1 + h(d_2+\gamma) - \frac{h^2\gamma\delta}{1+h(d_3+\delta)})I_1]. \end{aligned}$$

Then, from (8) we have $\Psi^*(I_1) = 0$. Under assumption (H_1), we obtain that $\Psi^*(I_1)$ is monotonically increasing for $I_1 > 0$ and $\lim_{I_1\to 0}\Psi^*(I_1) = -\infty$. On the other hand, when $I_1 = \bar{I}_1$ we have $f(S_1) = f(0) = 0$ and hence,

$$\begin{aligned} \Psi^*(\bar{I}_1) = \ & \frac{\bar{I}_1}{g(\bar{I}_1)} - \frac{1}{1+h(d_2+\gamma)}\frac{I_0}{g(\bar{I}_1)} \\ = \ & \frac{1}{g(\bar{I}_1)}(\bar{I}_1 - \frac{I_0}{1+h(d_2+\gamma)}) > 0. \end{aligned}$$

Therefore, $\Psi^*(I_1) = 0$ has a unique positive solution $y^* \in (0, \bar{I}_1)$. That is,

$$y^* = \frac{1}{1+h(d_2+\gamma)}(I_0 - h\beta f(S_1)g(y^*)).$$

Now, we show that y^* is the unique solution of $\Psi^*(I_1) = 0$ on $(0, \infty)$. Otherwise, there is a $y' \in [\bar{I}_1, \infty)$ such that $\Psi^*(y') = 0$. Since $y' \geq \bar{I}_1$, we have that $S_1 \leq 0$ when $I_1 = y'$. From (H_1), we have $f(S) \leq 0$ for any $S \leq 0$. Hence, from $\Psi^*(y') = 0$ we further have $y' \leq \frac{I_0}{1+h(d_2+\gamma)}$. On the other hand, since $\bar{I}_1 > \frac{I_0}{1+h(d_2+\gamma)}$, we obtain $y' > \frac{I_0}{1+h(d_2+\gamma)}$, which leads to a contradiction.

Therefore, we have $I_1 = y^* > 0$. Again from (8), we further also have $S_1 > 0$ and $R_1 > 0$. This shows that from (7) we can obtain a unique positive solution (S_1, I_1, R_1).

When $n = 1$, a similarly argument as in above, we can obtain a unique positive solution (S_2, I_2, R_2) satisfying model (3) at $n = 1$. By using the induction, we finally obtain a unique positive solution (S_n, I_n, R_n) for all $n > 0$ satisfying model (3).

Let $N_n = S_n + I_n + R_n$, then from model (3) we have

$$N_n = N_{n-1} + h(A - d_1 S_n - d_2 I_n - d_3 R_n) \leq N_{n-1} + h(A - dN_n).$$

Hence,

$$N_n \leq \frac{hA + N_{n-1}}{1 + hd}.$$

By using iteration method, we obtain

$$\begin{aligned} N_n \leq & \frac{hA}{1+hd} + \frac{hA}{(1+hd)^2} + \cdots + \frac{hA}{(1+hd)^n} + \frac{N_0}{(1+hd)^n} \\ = & \frac{A}{d}[1 - \frac{1}{(1+hd)^n}] + \frac{N_0}{(1+hd)^n}. \end{aligned}$$

Therefore, it holds that $\limsup_{n\to+\infty} N_n \leq \frac{A}{d}$. This completes the proof.

The basic reproduction number for model (3) can be defined by

$$\mathscr{R}_0 = \frac{\beta f(S^0) g'(0)}{d_2 + \gamma},$$

where $S^0 = \frac{A(d_3+\delta)}{d_1(d_3+\delta)+d_3\eta}$. On the existence of equilibria of model (3), we have the following result.

Theorem 2. *Suppose that (H_1) holds.*

(1). If $\mathscr{R}_0 \leq 1$, then model (3) has only a unique disease-free equilibrium $E^0 = (S^0, 0, R^0)$, where S^0 is given in the above and $R^0 = \frac{\eta A}{d_1(d_3+\delta)+d_3\eta}$.

(2). If $\mathscr{R}_0 > 1$, then model (3) has a unique endemic equilibrium $E^* = (S^*, I^*, R^*)$, except for disease-free equilibrium E^0.

Proof: We know that any equilibrium $E = (S, I, R)$ of model (3) satisfies the following equation

$$\begin{aligned} & A - \beta f(S) g(I) - (d_1 + \eta) S + \delta R = 0, \\ & \beta f(S) g(I) - (d_2 + \gamma) I = 0, \\ & \eta S + \gamma I - (d_3 + \delta) R = 0. \end{aligned} \tag{9}$$

Firstly, when $I = 0$, we have

$$A - (d_1 + \eta) S + \delta R = 0, \quad \eta S - (d_3 + \delta) R = 0.$$

From this, we directly obtain disease-free equilibrium $E^0 = (S^0, 0, R^0)$.

Second, when $I > 0$, from equation (9), we obtain

$$R = \frac{\eta S + \gamma I}{\delta + d_3}, \quad S = S^0 - \frac{d_2(\delta + d_3) + d_3\gamma}{d_1(\delta + d_3) + d_3\eta} I.$$

Substituting S into the second equation of (9) and we have

$$\beta f(S^0 - \frac{d_2(\delta + d_3) + d_3\gamma}{d_1(\delta + d_3) + d_3\eta} I) g(I) - (d_2 + \gamma) I = 0.$$

Let

$$H(I) = \beta f(S^0 - \frac{d_2(\delta + d_3) + d_3\gamma}{d_1(\delta + d_3) + d_3\eta} I) \frac{g(I)}{I} - (d_2 + \gamma).$$

By assumption (H_1), $H(I)$ is strictly monotone decreasing on $(0, +\infty)$ and satisfies

$$\lim_{I \to 0^+} H(I) = \beta f(S^0) g'(0) - (d_2 + \gamma) = (d_2 + \gamma)(\mathscr{R}_0 - 1)$$

and we also have $H(\bar{I}) = -(d_2 + \gamma) < 0$, where $\bar{I} = \frac{S^0(d_1(d_3+\delta)+d_3\eta)}{d_2(d_3+\delta)+d_3\gamma}$.

When $\mathscr{R}_0 \leq 1$, we have $\lim_{I \to 0^+} H(I) \leq 0$. Consequently, there is not any $I^* > 0$ such that $H(I^*) = 0$. Therefore, model (3) only has a unique disease-free equilibrium E_0.

When $\mathscr{R}_0 > 1$, we have $\lim_{I \to 0^+} H(I) > 0$. Therefore, there exists a unique $I^* \in (0, \bar{I})$ such that $H(I^*) = 0$. Furthermore, we have $S^* = S^0 - \frac{d_2(\delta+d_3)+d_3\gamma}{d_1(\delta+d_3)+d_3\eta} I^* > 0$ and $R^* = \frac{\eta S^* + \gamma I^*}{\delta + d_3} > 0$. This implies that model (3) has a unique endemic equilibrium $E^* = (S^*, I^*, R^*)$. This completes the proof.

Further, on the local stability of equilibria of model (3), we have the following result.

Theorem 3. *Suppose that (H_1) holds.*

(1). When $\mathscr{R}_0 < 1$, then disease-free equilibrium E^0 of model (3) is locally asymptotically stable.

(2). When $\mathscr{R}_0 > 1$, then disease-free equilibrium E^0 of model (3) is unstable, and endemic equilibrium E^* is locally asymptotically stable.

Proof: Calculating the linearization system of model (3) at equilibrium E^0, we have

$$\begin{aligned} u_{n+1} - u_n = & \ h[\beta f(S^0) g'(0) v_{n+1} - (d_1 + \eta) u_{n+1} + \delta w_{n+1}], \\ v_{n+1} - v_n = & \ h[\beta f(S^0) g'(0) v_{n+1} - (d_2 + \gamma) v_{n+1}], \\ w_{n+1} - w_n = & \ h[\eta u_{n+1} + \gamma v_{n+1} - (d_3 + \delta) w_{n+1}]. \end{aligned} \tag{10}$$

From the second equation of system (10), we have

$$v_{n+1} = \frac{v_n}{1 + h[d_2 + \gamma - \beta f(S^0) g'(0)]}. \tag{11}$$

When $\mathcal{R}_0 < 1$, we obtain

$$0 < \frac{1}{1 + h[d_2 + \gamma - \beta f(S^0)g'(0)]} = \frac{1}{1 + h(d_2 + \gamma)(1 - \mathcal{R}_0)} < 1.$$

Therefore, from (11) we have $\lim_{n\to\infty} v_n = 0$. From the first and third equations of system (10) we have

$$\begin{pmatrix} u_{n+1} \\ w_{n+1} \end{pmatrix} = A^{-1} \begin{pmatrix} u_n \\ w_n \end{pmatrix} + A^{-1} \begin{pmatrix} -h\beta f(S^0)g'(0) \\ h\gamma \end{pmatrix} v_{n+1}, \tag{12}$$

where

$$A = \begin{pmatrix} 1 + h[d_1 + \eta] & -h\delta \\ -h\eta & 1 + h[d_3 + \delta] \end{pmatrix}.$$

It is clear that two eigenvalues λ_i $(i = 1, 2)$ of matrix A satisfy $|\lambda_i| > 1$. Hence, norms of two eigenvalues of A^{-1} are less than one. Since $\lim_{n\to\infty} v_n = 0$, from (12) we can obtain $\lim_{n\to\infty} u_n = 0$ and $\lim_{n\to\infty} w_n = 0$. This shows that equilibrium E^0 of model (3) is locally stable.

When $\mathscr{R}_0 > 1$, since $|\frac{1}{1+h[d_2+\gamma-\beta f(S^0)g'(0)]}| > 1$, from (11) we obtain $\lim_{n\to\infty} v_n = \infty$. Therefore, E^0 is unstable.

Calculating the linearization system of model (3) at equilibrium E^*, we have

$$\begin{aligned} u_{n+1} = {} & u_n - h[\beta f'(S^*)g(I^*)u_{n+1} - \beta f(S^*)g'(I^*)v_{n+1} \\ & -(d_1 + \eta)u_{n+1} + \delta w_{n+1}], \\ v_{n+1} = {} & v_n + h[\beta f'(S^*)g(I^*)u_{n+1} + \beta f(S^*)g'(I^*)v_{n+1} - (d_2 + \gamma)v_{n+1}], \\ w_{n+1} = {} & w_n + h[\eta u_{n+1} + \gamma v_{n+1} - (d_3 + \delta)w_{n+1}]. \end{aligned} \tag{13}$$

Let

$$A = \begin{pmatrix} 1 + h[\beta f'(S^*)g(I^*) + d_1 + \eta] & h\beta f(S^*)g'(I^*) & -h\delta \\ -h\beta f'(S^*)g(I^*) & 1 - h[\beta f(S^*)g'(I^*) - d_2 - \gamma] & 0 \\ -h\eta & -h\gamma & 1 + h[d_3 + \delta] \end{pmatrix}$$

and $X_n = (u_n, v_n, w_n)^T$, then equation (13) can be rewrote into

$$X_{n+1} = A^{-1}X_n. \tag{14}$$

It is clear that if all eigenvalues λ of matrix $-A$ satisfy $|\lambda| > 1$, then all eigenvalues σ of matrix A^{-1} will satisfy $|\sigma| < 1$. The characteristic equation of $-A$ is $|\lambda E + A| = 0$, where E is the unit matrix. Let $r = \frac{\lambda+1}{h}$, then we easily obtain

$$|\lambda E + A| = r^3 + ar^2 + br + c,$$

where

$$a = d_1 + \eta + \beta f'(S^*)g(I^*) + d_3 + \delta + d_2 + \gamma - \beta f(S^*)g'(I^*),$$

$$\begin{aligned} b = \ & d_1(d_3+\delta) + d_3\eta + (d_3+\delta)\beta f'(S^*)g(I^*) + (d_2+\gamma)\beta f'(S^*)g(I^*) \\ & (d_3+\delta+d_1+\eta)(d_2+\gamma-\beta f(S^*)g'(I^*)) \end{aligned}$$

and

$$c = (d_1(d_3+\delta)+d_3\eta)(d_2+\gamma-\beta f(S^*)g'(I^*)) + (d_2(d_3+\delta)+d_3\gamma)\beta f'(S^*)g(I^*).$$

From assumption (H_1), we easily obtain $\frac{g(I)}{I} - g'(I) \geq 0$ for all $I > 0$. Since $\beta f(S^*)g(I^*) - (d_2+\gamma)I^* = 0$, we obtain $d_2+\gamma-\beta f(S^*)g'(I^*) \geq 0$. Hence, we have $a > 0$, $b > 0$ and $c > 0$. By calculating, we further obtain

$$\begin{aligned} ab - c = \ & (d_1+\eta+d_3+\delta)[(d_2+\gamma-\beta f(S^*)g'(I^*))^2 \\ & +(d_3+\delta)(d_1+\beta f'(S^*)g(I^*)) + d_3\eta] \\ & +(d_2+\gamma-\beta f(S^*)g'(I^*))[(d_3+\delta+d_1+\eta)^2 \\ & +\beta f'(S^*)g(I^*)(2(d_3+\delta)+d_2+\gamma+d_1+\eta)] \\ & +\beta f'(S^*)g(I^*)[(d_1+\eta)(d_2+\gamma)+d_1(d_3+\delta)+\gamma\delta] \\ & +(\beta f'(S^*)g(I^*))^2(d_2+\gamma+d_3+\delta) > 0. \end{aligned}$$

Therefore, by the Routh-Hurwitz criterion all roots of equation

$$r^3 + ar^2 + br + c = 0$$

have the negative real parts. Since $\lambda = hr - 1$, we further obtain that all eigenvalues λ of matrix $-A$ satisfy $|\lambda| > 1$. Therefore, the zero solution $X = 0$ of equation (14) is asymptotically stable. This shows that equilibrium E^* is locally asymptotically stable. This completes the proof.

Remark 5. From Theorems 3 we directly see that assumptions (H_2) and (H_3) only are used to obtain the global asymptotic stability of equilibria of model (3).

Remark 6. From the results obtained in this section, we easily see that the backward difference scheme, that is discrete dynamical model (3), for a class of SIRS epidemic models (1) with nonlinear incidence is provided for us with excellent properties in the local stability of equilibria and the permanence of disease. These properties nearly are same to corresponding continuous-time model (1).

4. The global stability

Now, we study the stability of equilibria of model (3). Firstly, on the global stability of disease-free equilibrium E^0, we have the following result:

Theorem 4. *Suppose that $(H_1)-(H_3)$ hold. If $\mathscr{R}_0 \le 1$, then for any time step size $h > 0$ disease-free equilibrium E^0 of model (3) is globally asymptotically stable.*

Proof: Model (3) can be rewritten as the following form

$$\begin{aligned} S_{n+1} - S_n = &\ h[-(d_1+\eta)(S_{n+1}-S^0) - \beta g(I_{n+1})(f(S_{n+1}) - f(S^0)) \\ &+\delta(R_{n+1} - R^0) - \beta f(S^0) g(I_{n+1})], \\ I_{n+1} - I_n = &\ h[\beta g(I_{n+1})(f(S_{n+1}) - f(S^0)) \\ &-(d_2+\gamma) I_{n+1} + \beta f(S^0) g(I_{n+1})], \\ R_{n+1} - R_n = &\ h[\eta(S_{n+1} - S^0) + \gamma I_{n+1} - (d_3+\delta)(R_{n+1} - R^0)]. \end{aligned} \tag{15}$$

We consider the following Lyapunov function

$$\begin{aligned} W_n = &\ \frac{1}{2}(S_n - S^0 + I_n + R_n - R^0)^2 + k_1 \int_{S^0}^{S_n} (f(\tau) - f(S^0)) d\tau \\ &+(k_2+k_3) I_n + \frac{k_4}{2}(R_n - R^0)^2, \end{aligned}$$

where $k_i\ (i = 1, 2, 3, 4)$ are positive constants which will be determined in the following. Calculating difference of W_n along solutions of equation (15), by assumption (H_1) we have

$$\begin{aligned} W_{n+1} - W_n = &\ k_1 \int_{S_n}^{S_{n+1}} (f(\tau) - f(S^0)) d\tau + (k_2+k_3)(I_{n+1} - I_n) \\ &+\frac{k_4}{2}[(R_{n+1} - R^0)^2 - (R_n - R^0)^2] \\ &+\frac{1}{2}[(S_{n+1} - S^0 + I_{n+1} + R_{n+1} - R^0)^2 \\ &-(S_n - S^0 + I_n + R_n - R^0)^2] \\ = &\ k_1(S_{n+1} - S_n)(f(S_{n+1}) - f(S^0)) + (k_2+k_3)(I_{n+1} - I_n) \\ &+\frac{k_4}{2}[(R_{n+1} - R_n)(R_n - R_{n+1} + 2(R_{n+1} - R^0)] \\ &+\frac{1}{2}[(S_{n+1} - S_n + I_{n+1} - I_n + R_{n+1} - R_n) \\ &\times(S_n - S_{n+1} + 2(S_{n+1} - S^0) + I_n - I_{n+1} \\ &+2I_{n+1} + R_n - R_{n+1} + 2(R_{n+1} - R^0)] \\ \le &\ k_1(S_{n+1} - S_n)(f(S_{n+1}) - f(S^0)) + (k_2+k_3)(I_{n+1} - I_n) \\ &+k_4(R_{n+1} - R_n)(R_{n+1} - R^0) + (S_{n+1} - S^0 + I_{n+1} + R_{n+1} - R^0) \\ &\times(S_{n+1} - S_n + I_{n+1} - I_n + R_{n+1} - R_n) \\ = &\ k_1 h[-(d_1+\eta)(S_{n+1} - S^0) - \beta g(I_{n+1})(f(S_{n+1}) - f(S^0)) \end{aligned}$$

$$
\begin{aligned}
&+\delta(R_{n+1}-R^0)-\beta f(S^0)g(I_{n+1})](f(S_{n+1})-f(S^0))\\
&+k_2h[\beta g(I_{n+1})(f(S_{n+1})-f(S^0))-(d_2+\gamma)I_{n+1}\\
&+\beta f(S^0)g(I_{n+1})]+k_3h(\beta f(S_{n+1})g(I_{n+1})-(d_2+\gamma)I_{n+1})\\
&+k_4h[\eta(S_{n+1}-S^0)+\gamma I_{n+1}-(d_3+\delta)(R_{n+1}-R^0)]\\
&\times(R_{n+1}-R^0)+h[-d_1(S_{n+1}-S^0)-d_2I_{n+1}\\
&-d_3(R_{n+1}-R^0)]\times(S_{n+1}-S^0+I_{n+1}+R_{n+1}-R^0)
\end{aligned}
$$

Since $\mathscr{R}_0=\frac{\beta f(S^0)g'(0)}{d_2+\gamma}\le 1$, we have $\beta f(S^0)g'(0)\le d_2+\gamma$. Under assumption (H_1), we have

$$\frac{g(I_{n+1})}{I_{n+1}}\le\lim_{I\to 0^+}\frac{g(I)}{I}=g'(0).$$

Choosing constants $k_2=k_1f(S^0)$ and $k_4=\frac{d_2+d_3}{\gamma}$, we further have

$$
\begin{aligned}
W_{n+1}-W_n\le\ & -k_1h\beta g(I_{n+1})(f(S_{n+1})-f(S^0))^2-d_1h(S_{n+1}-S^0)^2\\
&-d_2hI_{n+1}^2-k_4h(d_3+\delta)(R_{n+1}-R^0)^2\\
&-d_3h(R_{n+1}-R^0)^2-h(d_1+d_2)(S_{n+1}-S^0)I_{n+1}\\
&-h(d_1+d_3-k_4\eta)(S_{n+1}-S^0)(R_{n+1}-R^0)\\
&-(d_1+\eta)k_1h[f(S_{n+1})-f(S^0)](S_{n+1}-S^0)\\
&+k_1h\delta(f(S_{n+1})-f(S^0))(R_{n+1}-R^0)\\
&+k_2h\beta f(S^0)I_{n+1}[\frac{g(I_{n+1})}{I_{n+1}}-g'(0)]\\
&+k_3h\beta I_{n+1}[f(S_{n+1})(\frac{g(I_{n+1})}{I_{n+1}}-g'(0))+g'(0)(f(S_{n+1})-f(S^0))]\\
\le\ & -d_1h(S_{n+1}-S^0)^2-d_2hI_{n+1}^2-(k_4(d_3+\delta)+d_3)h(R_{n+1}-R^0)^2\\
&+k_1h\delta(f(S_{n+1})-f(S^0))(R_{n+1}-R^0)\\
&-(d_1+\eta)k_1h[f(S_{n+1})-f(S^0)](S_{n+1}-S^0)\\
&-(d_1+d_2)h(S_{n+1}-S^0)I_{n+1}+k_3h\beta g'(0)I_{n+1}(f(S_{n+1})-f(S^0))\\
&-(d_1+d_3-k_4\eta)h(S_{n+1}-S^0)(R_{n+1}-R^0)\\
=\ & -d_1h(S_{n+1}-S^0)^2-d_2hI_{n+1}^2-(k_4(d_3+\delta)+d_3)h(R_{n+1}-R^0)^2\\
&-(d_1+d_3-k_4\eta)h(S_{n+1}-S^0)(R_{n+1}-R^0)\\
&-(d_1+d_2)h(S_{n+1}-S^0)I_{n+1}\\
&-(d_1+\eta)k_1h[\frac{f(S_{n+1})-f(S^0)}{S_{n+1}-S^0}](S_{n+1}-S^0)^2\\
&+k_1h\delta(\frac{f(S_{n+1})-f(S^0)}{S_{n+1}-S^0})(S_{n+1}-S^0)(R_{n+1}-R^0)
\end{aligned}
$$

$$\begin{aligned}
&+k_3h\beta g'(0)I_{n+1}(S_{n+1}-S^0)(\frac{f(S_{n+1})-f(S^0)}{S_{n+1}-S^0})\\
= \ &-h[(S_{n+1}-S^0, I_{n+1})P(S_{n+1}-S^0, I_{n+1})^T\\
&+(S_{n+1}-S^0, R_{n+1}-R^0)Q(S_{n+1}-S^0, R_{n+1}-R^0)^T],
\end{aligned}$$

where

$$P=\begin{pmatrix} k_1(d_1+\eta)F(S_{n+1},S^0) & p_{12} \\ p_{12} & d_2, \end{pmatrix}, \quad Q=\begin{pmatrix} d_1 & q_{12} \\ q_{12} & [d_3+k_4(d_3+\delta)] \end{pmatrix}$$

with

$$p_{12}=\frac{1}{2}(d_1+d_2-k_3\beta g'(0)F(S_{n+1},S^0)),$$
$$q_{12}=\frac{1}{2}(d_1+d_3-k_4\eta-k_1\delta F(S_{n+1},S^0)).$$

Further, we choose $k_1=K_1$ and $k_3=\frac{K_3}{g'(0)}$, then assumption (H_3) implies that matrices P and Q are positive definite. This implies that

$$W_{n+1}-W_n<0 \quad \text{for all} \quad (S_n, I_n, R_n)\neq(S^0, 0, R^0).$$

By the Lyapunov's theorems on the global asymptotical stability for difference e-quations [28], we obtain that disease-free equilibrium E^0 is globally asymptotically stable. This completes the proof.

Remark 7. In articles [1-3,5], the authors studied the global properties of solutions for the various discrete difference scheme, such as the nonstandard finite difference scheme, backward difference scheme and forward difference scheme, for continuous-time SIRS epidemic models. The condition that the death rate (d_1) of susceptible is less than or equal to the death rate (d_2) of infected and the death rate (d_3) of recovered, that is, $d_1\leq\min\{d_2,d_3\}$ is required. Therefore, the global asymptotic stability of the disease-free equilibrium can be established only when the basic reproduction number $\mathscr{R}_0\leq 1$, except for some basic assumptions, for example, such as assumption (H_1) for model (3).

However, in this paper we do not require the condition $d_1\leq\min\{d_2,d_3\}$ for model (3). Therefore, in order to obtain the global stability of the disease-free equilibrium of model (3), a new Lyapunov function is constructed and the assumption (H_3) is introduced.

On the global stability of the endemic equilibrium E^*, we have the following result.

Theorem 5. *Suppose that $(H_1)-(H_3)$ hold. If $\mathscr{R}_0>1$, then for any time step size $h>0$ endemic equilibrium E^* of model (3) is globally asymptotically stable.*

Proof: The model (3) can be rewritten as the following form:

$$\begin{aligned}
S_{n+1}-S_n= \;& h[-\beta g(I_{n+1})[f(S_{n+1})-f(S^*)]-(d_1+\eta)(S_{n+1}-S^*)\\
&+\delta(R_{n+1}-R^*)-\beta f(S^*)(g(I_{n+1})-g(I^*))],\\
I_{n+1}-I_n= \;& h[\beta g(I_{n+1})(f(S_{n+1})-f(S^*))-(d_2+\gamma)(I_{n+1}-I^*)\\
&+\beta f(S^*)(g(I_{n+1})-g(I^*))],\\
R_{n+1}-R_n= \;& h[\eta(S_{n+1}-S^*)+\gamma(I_{n+1}-I^*)-(d_3+\delta)(R_{n+1}-R^*)].
\end{aligned} \tag{16}$$

Since $(d_2+\gamma)I^*=\beta f(S^*)g(I^*)$, we also have

$$\begin{aligned}
I_{n+1}-I_n= \;& h[\beta f(S_{n+1})g(I_{n+1})-(d_2+\gamma)I_{n+1}]\\
= \;& \beta hI_{n+1}[f(S_{n+1})\frac{g(I_{n+1})}{I_{n+1}}-f(S^*)\frac{g(I^*)}{I^*}]\\
= \;& \beta hI_{n+1}[f(S_{n+1})(\frac{g(I_{n+1})}{I_{n+1}}-\frac{g(I^*)}{I^*})+\frac{g(I^*)}{I^*}(f(S_{n+1})-f(S^*))]
\end{aligned} \tag{17}$$

and

$$\begin{aligned}
&S_{n+1}-S_n+I_{n+1}-I_n+R_{n+1}-R_n\\
&=\;-d_1(S_{n+1}-S^*)-d_2(I_{n+1}-I^*)-d_3(R_{n+1}-R^*).
\end{aligned} \tag{18}$$

We consider the following Lyapunov function

$$\begin{aligned}
V_n= \;& \frac{1}{2}(S_n-S^*+I_n-I^*+R_n-R^*)^2+k_1\int_{S^*}^{S_n}(f(\tau)-f(S^*))d\tau\\
&+k_2\int_{I^*}^{I_n}\frac{g(\tau)-g(I^*)}{g(\tau)}d\tau+k_3(I_n-I^*-I^*\ln\frac{I_n}{I^*})+\frac{k_4}{2}(R_n-R^*)^2,
\end{aligned}$$

where $k_i\,(i=1,2,3,4)$ are positive constants which will be determined in the following. Calculating difference of V_n along equation (16), then by (17) and (18) we have

$$\begin{aligned}
V_{n+1}-V_n= \;& k_1[\int_{S_n}^{S_{n+1}}(f(\tau)-f(S^*))d\tau]+k_2\int_{I_n}^{I_{n+1}}\frac{g(\tau)-g(I^*)}{g(\tau)}d\tau\\
&+k_3(I_{n+1}-I_n-I^*\ln\frac{I_{n+1}}{I_n})+\frac{k_4}{2}[(R_{n+1}-R^*)^2-(R_n-R^*)^2]\\
&+\frac{1}{2}[(S_{n+1}-S^*+I_{n+1}-I^*+R_{n+1}-R^*)^2\\
&-(S_n-S^*+I_n-I^*+R_n-R^*)^2]\\
\leq \;& k_1(S_{n+1}-S_n)(f(S_{n+1})-f(S^*))\\
&+k_2(I_{n+1}-I_n)(\frac{g(I_{n+1})-g(I^*)}{g(I_{n+1})})+k_3(I_{n+1}-I_n)\frac{I_{n+1}-I^*}{I_{n+1}}\\
&+\frac{k_4}{2}(R_{n+1}-R_n)(R_n-R_{n+1}+2(R_{n+1}-R^*))\\
&+\frac{1}{2}(S_{n+1}-S_n+I_{n+1}-I_n+R_{n+1}-R_n)
\end{aligned}$$

$$
\begin{aligned}
&\times(S_n - S_{n+1} + 2(S_{n+1} - S^*) + I_n - I_{n+1} \\
&+2(I_{n+1} - I^*) + R_n - R_{n+1} + 2(R_{n+1} - R^*)) \\
\leq\ & k_1(S_{n+1} - S_n)(f(S_{n+1}) - f(S^*)) + k_4(R_{n+1} - R_n)(R_{n+1} - R^*) \\
&+k_2(I_{n+1} - I_n)(\frac{g(I_{n+1}) - g(I^*)}{g(I_{n+1})}) + k_3(I_{n+1} - I_n)\frac{I_{n+1} - I^*}{I_{n+1}} \\
&+(S_{n+1} - S_n + I_{n+1} - I_n + R_{n+1} - R_n) \\
&\times(S_{n+1} - S^* + I_{n+1} - I^* + R_{n+1} - R^*) \\
=\ & k_1 h[-\beta g(I_{n+1})[f(S_{n+1}) - f(S^*)] - (d_1 + \eta)(S_{n+1} - S^*) + \delta(R_{n+1} - R^*) \\
&-\beta f(S^*)(g(I_{n+1}) - g(I^*))](f(S_{n+1}) - f(S^*)) \\
&+k_2 h[\beta g(I_{n+1})(f(S_{n+1}) - f(S^*)) - (d_2 + \gamma)(I_{n+1} - I^*) \\
&+\beta f(S^*)(g(I_{n+1}) - g(I^*))](\frac{g(I_{n+1}) - g(I^*)}{g(I_{n+1})}) \\
&+k_3 h[\beta f(S_{n+1})(\frac{g(I_{n+1})}{I_{n+1}} - \frac{g(I^*)}{I^*}) + \beta\frac{g(I^*)}{I^*}(f(S_{n+1}) - f(S^*))](I_{n+1} - I^*) \\
&+k_4 h[\eta(S_{n+1} - S^*) + \gamma(I_{n+1} - I^*) - (d_3 + \delta)(R_{n+1} - R^*)](R_{n+1} - R^*) \\
&-h(d_1(S_{n+1} - S^*) + d_2(I_{n+1} - I^*) + d_3(R_{n+1} - R^*)) \\
&\times(S_{n+1} - S^* + I_{n+1} - I^* + R_{n+1} - R^*)
\end{aligned}
$$

Choosing constants $k_2 = k_1 f(S^*)$ and $k_4 = \frac{d_2+d_3}{\gamma}$, we further have

$$
\begin{aligned}
V_{n+1} - V_n \leq\ & -d_1 h(S_{n+1} - S^*)^2 - d_2 h(I_{n+1} - I^*)^2 \\
&-h[k_4(d_3 + \delta) + d_3](R_{n+1} - R^*)^2 - (d_1 + d_2)h(S_{n+1} - S^*)(I_{n+1} - I^*) \\
&-(d_1 + d_3 - k_4\eta)h(R_{n+1} - R^*)(S_{n+1} - S^*) \\
&-k_1(d_1 + \eta)h(S_{n+1} - S^*)(f(S_{n+1}) - f(S^*)) \\
&+k_1 h\delta(R_{n+1} - R^*)(f(S_{n+1}) - f(S^*)) \\
&+k_2 h\frac{g(I_{n+1}) - g(I^*)}{g(I_{n+1})}[\beta f(S^*)(g(I_{n+1})) - g(I^*)) - (d_2 + \gamma)(I_{n+1} - I^*)] \\
&+k_3 h\beta f(S_{n+1})(I_{n+1} - I^*)(\frac{g(I_{n+1})}{I_{n+1}} - \frac{g(I^*)}{I^*}) \\
&+k_3 h\beta\frac{g(I^*)}{I^*}(I_{n+1} - I^*)(f(S_{n+1}) - f(S^*)).
\end{aligned}
$$

From assumption (H_1) and $d_2 + \gamma = \beta f(S^*)\frac{g(I^*)}{I^*}$, we have

$$
\begin{aligned}
& k_2\frac{g(I_{n+1}) - g(I^*)}{g(I_{n+1})}[\beta f(S^*)(g(I_{n+1}) - g(I^*)) - (d_2 + \gamma)(I_{n+1} - I^*)] \\
=\ & k_2\frac{g(I_{n+1}) - g(I^*)}{g(I_{n+1})}[\beta f(S^*)g(I_{n+1}) - (d_2 + \gamma)I_{n+1}] \\
=\ & \frac{k_2\beta f(S^*)I_{n+1}}{g(I_{n+1})}(g(I_{n+1}) - g(I^*))[\frac{g(I_{n+1})}{I_{n+1}} - \frac{g(I^*)}{I^*}] \leq 0
\end{aligned}
$$

and

$$k_3\beta f(S_{n+1})(I_{n+1}-I^*)(\frac{g(I_{n+1})}{I_{n+1}}-\frac{g(I^*)}{I^*})\leq 0.$$

Hence,

$$\begin{aligned}
V_{n+1}-V_n \leq\ & -d_1h(S_{n+1}-S^*)^2-d_2h(I_{n+1}-I^*)^2\\
& -h[k_4(d_3+\delta)+d_3](R_{n+1}-R^*)^2-(d_1+d_2)h(S_{n+1}-S^*)(I_{n+1}-I^*)\\
& -(d_1+d_3-k_4\eta)h(R_{n+1}-R^*)(S_{n+1}-S^*)\\
& -k_1(d_1+\eta)h(S_{n+1}-S^*)(f(S_{n+1})-f(S^*))\\
& +k_1h\delta(R_{n+1}-R^*)(f(S_{n+1})-f(S^*))\\
& +k_3h\beta\frac{g(I^*)}{I^*}(I_{n+1}-I^*)(f(S_{n+1})-f(S^*))\\
=\ & -d_1h(S_{n+1}-S^*)^2-d_2h(I_{n+1}-I^*)^2\\
& -h[k_4(d_3+\delta)+d_3](R_{n+1}-R^*)^2-(d_1+d_2)h(S_{n+1}-S^*)(I_{n+1}-I^*)\\
& -(d_1+d_3-k_4\eta)h(R_{n+1}-R^*)(S_{n+1}-S^*)\\
& -k_1(d_1+\eta)h(S_{n+1}-S^*)^2\frac{f(S_{n+1})-f(S^*)}{S_{n+1}-S^*}\\
& +k_1h\delta(R_{n+1}-R^*)(S_{n+1}-S^*)\frac{f(S_{n+1})-f(S^*)}{S_{n+1}-S^*}\\
& +k_3h\beta\frac{g(I^*)}{I^*}(I_{n+1}-I^*)(S_{n+1}-S^*)\frac{f(S_{n+1})-f(S^*)}{S_{n+1}-S^*}\\
=\ & -h[(S_{n+1}-S^*,I_{n+1}-I^*)P(S_{n+1}-S^*,I_{n+1}-I^*)^T\\
& +(S_{n+1}-S^*,R_{n+1}-R^*)Q(S_{n+1}-S^*,R_{n+1}-R^*)^T],
\end{aligned}$$

where

$$P=\begin{pmatrix} k_1(d_1+\eta)F(S_{n+1},S^*) & p_{12}\\ p_{12} & d_2\end{pmatrix},\quad Q=\begin{pmatrix} d_1 & q_{12}\\ q_{12} & [d_3+k_4(d_3+\delta)]\end{pmatrix}$$

with

$$p_{12}=\frac{1}{2}(d_1+d_2-k_3\beta\frac{g(I^*)}{I^*}F(S_{n+1},S^*)),$$
$$q_{12}=\frac{1}{2}(d_1+d_3-k_4\eta-k_1\delta F(S_{n+1},S^*)).$$

Further, we choose $k_1=K_1$ and $k_3=K_3\frac{I^*}{g(I^*)}$, then assumption (H_3) implies that matrices P and Q are positive definite. This implies that

$$V_{n+1}-V_n<0 \quad \text{for all} \quad (S_n,I_n,R_n)\neq(S^*,I^*,R^*).$$

By the Lyapunov's theorems on the globally asymptotical stability for difference equations [28], we directly obtained that the endemic equilibrium E^* is globally asymptotically stable. This completes the proof.

Remark 8. From the above discussion we immediately see that constant $\mathscr{R}_0$ is the basic reproduction number of model (3) and it can completely determine the global asymptotic stability of model (3).

Remark 9. From the above discussions we easily see that assumption (H_2) only is used to ensure the positivity of constant K_1. When $\eta = 0$, that is, there is not vaccination in susceptible, then assumption (H_2) naturally holds.

As consequences of Theorems 4 and 5, combining Remarks 3 and 4 we have the following corollaries.

Corollary 1. *Assume that in model (3) $f(S) \equiv S$ and (H_1) and (H_2) hold.*

(1). If $\mathscr{R}_0 \leq 1$, then disease-free equilibrium E^0 of model (3) is globally asymptotically stable.

(2). If $\mathscr{R}_0 > 1$, then endemic equilibrium E^* of model (3) is globally asymptotically stable.

Corollary 2. *Assume that in model (3) $f(S) \equiv \frac{S}{1+\omega S}$, (H_1) and (H_2), and conditions (5) and (6) hold.*

(1). If $\mathscr{R}_0 \leq 1$, then disease-free equilibrium E^0 of model (3) is globally asymptotically stable.

(2). If $\mathscr{R}_0 > 1$, then endemic equilibrium E^* of model (3) is globally asymptotically stable.

Remark 10. In [3], the following backward difference scheme for SIRS epidemic model with the bilinear incidence is studied

$$\begin{aligned} S_{n+1} = &\quad S_n + B - \mu_1 S_{n+1} - \beta S_{n+1} I_{n+1} + \delta R_{n+1} \\ I_{n+1} = &\quad I_n + \beta S_{n+1} I_{n+1} - (\mu_2 + \gamma) I_{n+1} \\ R_{n+1} = &\quad R_n + \gamma I_{n+1} - (\mu_3 + \delta) R_{n+1}. \end{aligned} \tag{19}$$

The condition $\mu_1 \leq \min\{\mu_2, \mu_3\}$ is required. By constructing the discrete Lyapunov functions $U_\delta^{E^0}$ and $U_\delta^{E^*}$ (see the proof of Theorem 2.1 in [3]), the authors established that if the basic reproduction number $\mathscr{R}_0 \leq 1$, then disease-free equilibrium E^0 of model (19) is globally asymptotically stable, and if $\mathscr{R}_0 > 1$, then endemic equilibrium E^* of model (19) is globally asymptotically stable.

By computing, we easily see that the Lyapunov functions $U_\delta^{E^0}$ and $U_\delta^{E^*}$ are not applicable for model (3). Therefore, in this paper we construct a class of new Lyapunov functions to study the global asymptotic stability of model (3).

Furthermore, we also see that above Corollary 1 is an extension of the main results given in [3] in the nonlinear incidence case.

Remark 11. Analyzing the conditions and results given in Corollary 1 and Theorems 3 given in Section 3, we can propose an important and interesting open problem for general model (3): whether only when assumption (H_1) holds, we can obtain that the disease-free equilibrium is globally asymptotically stable if and only if $\mathscr{R}_0 \leq 1$, and the endemic equilibrium is globally asymptotically stable if and only if $\mathscr{R}_0 > 1$.

5. Conclusion

In [4], the dynamical properties of the forward difference scheme for a class of SIRS epidemic models with general nonlinear incidence are investigated. It is shown that when step size h is small enough the disease-free equilibrium and endemic equiloibrium are local asymptotically stable, and along step size h increase, the scheme will occur the bifurcation phenomena.

In this paper, the dynamical properties of the backward difference scheme for a class of SIRS epidemic models with nonlinear incidence $\beta f(S)g(I)$ are investigated. From the main results obtained in this paper, we see that the backward difference scheme, that is discrete dynamical model (3), is provided for us with excellent dynamical properties for any step size h in the local and global stability of equilibria. These properties nearly are same to corresponding continuous-time model (1).

Furthermore, we also see that the results on the global asymptotic stability of the endemic equilibrium for the backward difference scheme for SIRS epidemic model with bilinear incidence obtained in [3] are directly extended. By constructing new discrete Lyapunov functions we established the sufficient and necessary conditions on the global asymptotic stability of the disease-free equilibrium and endemic equilibrium for a class of discrete SIRS epidemic models with general nonlinear incidence $\beta f(S)g(I)$, vaccination in susceptible and different death rates d_1, d_2 and d_3. That is, under assumptions $(H_1) - (H_3)$, the disease-free equilibrium is globally asymptotically stable if and only if basic reproduction number $\mathscr{R}_0 \leq 1$, and the endemic equilibrium is globally asymptotically stable if and only if $\mathscr{R}_0 > 1$.

However, we also see that assumption (H_3) is very strong. For the local stability of the disease-free equilibrium and endemic equilibrium for model (3), the assumption (H_3) is not required. Therefore, an interesting and important open problem is whether the assumption (H_3) can be weakened in the studies of the global stability of equilibria of model (3).

On the other hand, we know that there is the nonstandard difference scheme to discretize continuous-time model (1) with nonlinear incidence. For the the nonstandard difference scheme of model (1) whether we also can establish the same results, like in this paper, still is an interesting open problem.

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Bounds for the largest eigenvalue of nonnegative tensors

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Abstract

In this paper, we establish some eigenvalue properties of nonnegative tensors. We derive new bounds for the largest eigenvalue (Z-eigenvalue, H-eigenvalue, and B-eigenvalue) of nonnegative tensors. Numerical examples show the efficiency of these bounds.

Key words: Nonnegative tensor; Spectral radius; Eigenvalue; Bound
AMSC (2010): 15A18; 15A69; 65F15; 65F10

1 Introduction

Eigenvalue problems of higher order tensors have become an important topic of study in a new applied mathematics branch, numerical multilinear algebra, and they have a wide range of practical applications [2, 5, 6, 7, 8, 9, 10, 17].

First, we recall some definitions on tensors. Let $\mathbb{R}$ be the real field. An m-th order n dimensional square tensor $\mathcal{A}$ consists of n^m entries in $\mathbb{R}$, which is defined as follows:

$$\mathcal{A} = (\mathcal{A}_{i_1 i_2 \cdots i_m}), \quad \mathcal{A}_{i_1 i_2 \cdots i_m} \in \mathbb{R}, \quad 1 \leq i_1, i_2, \cdots i_m \leq n.$$

$\mathcal{A}$ is called nonnegative if $\mathcal{A}_{i_1 i_2 \cdots i_m} \geq 0$. To an n-vector x, real or complex, we define the n-vector:

$$\mathcal{A}x^{m-1} = \left(\sum_{i_2, \cdots, i_m = 1}^{n} a_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_n} \right)_{1 \leq i \leq n},$$

$$x^{[m-1]} = (x_i^{m-1})_{1 \leq i \leq n}.$$

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In this paper, we continue this research on the eigenvalue problems for tensors. In section 2, bounds for the largest Z-eigenvalue are obtained, and proved to be tighter than that in Corollary 4.5 in [16]. In section 3, bounds for the largest H-eigenvalue are given. Moreover, the upper bound for the largest B-eigenvalue is presented in section 4.

2 Notation and preliminaries.

The following two definitions were first introduced and studied by Qi and Lim [4, 11].

Definition 2.1. *Let $\mathcal{A}$ be an m-order and n-dimensional tensor. A pair $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})$ is called an eigenvalue-eigenvector (or simply eigenpair) of $\mathcal{A}$ if they satisfy the equation*

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}.$$

We call (λ, x) an H-eigenpair if they are both real.

Definition 2.2. *Let $\mathcal{A}$ be an m-order and n-dimensional tensor. A pair $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})$ is called an E-eigenvalue and E-eigenvector (or simply E-eigenpair) of $\mathcal{A}$ if they satisfy the equation*

$$\begin{cases} \mathcal{A}x^{m-1} = \lambda x, \\ x^T x = 1. \end{cases} \tag{1}$$

We call (λ, x) an Z-eigenpair if they are both real.

Recently, Chang et al. [1, 2] generalized the notion of eigenvalues of higher order tensors to tensor pairs (or tensor pencils).

Definition 2.3. *Let $\mathcal{A}$, $\mathcal{B}$ be two m-order and n-dimensional tensors. A pair $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})$ is called an B-eigenvalue and B-eigenvector of $\mathcal{A}$ relative to $\mathcal{B}$ if they satisfy the equation*

$$\mathcal{A}x^{m-1} = \lambda \mathcal{B}x^{[m-1]}.$$

The following definition for irreducibility has been introduced in [1, 11].

Definition 2.4. *The tensor $\mathcal{A}$ is called reducible if there exists a nonempty proper index subset $\mathbb{J} \subset \{1, 2, \cdots, n\}$ such that $a_{i_1,i_2,\cdots,i_m} = 0, \ \forall i_1 \in \mathbb{J}, \ \forall i_2, \cdots, i_m \notin \mathbb{J}$. If $\mathcal{A}$ is not reducible, then we call $\mathcal{A}$ to be irreducible.*

In this paper, let $N = \{1, 2, \ldots, n\}$, we define the ith row sum of $\mathcal{A}$ as $R_i(\mathcal{A}) = \sum\limits_{i_2,\cdots,i_m=1}^{n} a_{ii_2\cdots i_m}$, and denote the largest and the smallest row sums of $\mathcal{A}$ by

$$R_{\max}(\mathcal{A}) = \max_{i=1,\cdots,n} R_i(\mathcal{A}), \ R_{\min}(\mathcal{A}) = \min_{i=1,\cdots,n} R_i(\mathcal{A}).$$

Furthermore, a real tensor of order m dimension n is called the unit tensor, if its entries are $\delta_{i_1\ldots i_m}$ for $i_1, \ldots, i_m \in N$, where

$$\delta_{i_1\ldots i_m} = \begin{cases} 1, & \text{if } i_1 = \ldots = i_m \\ 0, & \text{otherwise.} \end{cases}$$

And we define

$$r_i(\mathcal{A}) = \sum_{\delta_{ii_2\ldots i_m}=0} a_{ii_2\ldots i_m}, \quad r_i^j(\mathcal{A}) = \sum_{\substack{\delta_{ii_2\ldots i_m}=0, \\ \delta_{ji_2\ldots i_m}=0}} a_{ii_2\ldots i_m} = r_i(\mathcal{A}) - a_{ij\ldots j}.$$

3 Bounds for the largest Z-eigenvalue.

First, we list some results about the largest Z-eigenvalue of tensors.

Definition 3.1. *Let $\mathcal{A}$ be an m-order and n-dimensional tensor. We define $\sigma(\mathcal{A})$ the Z-spectrum of $\mathcal{A}$ by the set of all Z-eigenvalues of $\mathcal{A}$. Assume $\sigma(\mathcal{A}) \neq \emptyset$, then the Z-spectral radius of $\mathcal{A}$ is denoted by*

$$\rho(\mathcal{A}) = \max\{|\lambda| : \lambda \in \sigma(\mathcal{A})\}.$$

In [3], Chang, Pearson, and Zhang gave the following bounds for the Z-eigenvalues of an m-order n-dimensional tensor $\mathcal{A}$.

Lemma 3.2. *(Proposition 3.3 in [3]) Let $\mathcal{A}$ be an m-order and n-dimensional tensor. Then*

$$\rho(\mathcal{A}) \leq \sqrt{n} \max_{i \in N} \sum_{i_2,\ldots,i_m=1}^{n} \left|a_{ii_2\ldots i_m}\right|. \tag{2}$$

For the positively homogeneous operators, Song and Qi [16] studied the relationship between the Gelfand formula and the spectral radius as well as the upper bound of the spectral radius. From Corollary 4.5 in [16], we can get the following Lemma:

Lemma 3.3. *(Corollary 4.5 in [16]) Let $\mathcal{A}$ be an m-order and n-dimensional tensor. Then*

$$\rho(\mathcal{A}) \leq \max_{i \in N} \sum_{i_2,\ldots,i_m=1}^{n} \left|a_{ii_2\ldots i_m}\right|. \tag{3}$$

Obviously, the bound in (3) is better than the bound in (2). Here, we give another proof of Lemma 3.3, which is very simple.

Proof. Suppose that λ is an Z-eigenvalue of $\mathcal{A}$ with eigenvector x. Assume that

$$|x_i| = \max_{j \in N} |x_j|.$$

Consider the i-th equation of (1). We have

$$\lambda x_i = \sum_{i_2,\cdots,i_m=1}^{n} a_{ii_2\cdots i_m} x_{i_2} \ldots x_{i_m}.$$

By $|x_i| \leq \sqrt[m-1]{|x_i|} \leq 1$, we can get

$$|\lambda| \leq \sum_{i_2,\cdots,i_m=1}^{n} |a_{ii_2\cdots i_m}| \left|\frac{x_{i_2}}{\sqrt[m-1]{|x_i|}}\right| \ldots \left|\frac{x_{i_m}}{\sqrt[m-1]{|x_i|}}\right| \leq \sum_{i_2,\ldots,i_m=1}^{n} \left|a_{ii_2\ldots i_m}\right|.$$

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Thus, we complete the proof. □

Note that λ and x may be non-real here.

A tensor $\mathcal{A}$ is called *weakly symmetric* if the associated homogeneous polynomial $\mathcal{A}x^m$ satisfies

$$\nabla \mathcal{A}x^m = m\mathcal{A}x^{m-1}.$$

This concept was first introduced and used by Chang, Pearson and Zhang [3] for studying the properties of Z-eigenvalue of nonnegative tensors and presented the following Perron-Frobenius Theorem for the Z-eigenvalue of nonnegative tensors, which was later reproved as Lemma 4.7 by Song and Qi in [16], using a different technique.

Lemma 3.4. *Suppose that m-order n-dimensional tensor $\mathcal{A}$ is weakly symmetric, nonnegative and irreducible. Then $\rho(\mathcal{A})$ is a positive Z-eigenvalue with a positive Z-eigenvector.*

Based on the above Lemma, we give the main result of this section.

Theorem 3.5. *Suppose that m-order n-dimensional tensor $\mathcal{A}$ is weakly symmetric, nonnegative and irreducible. Then*

$$\rho(\mathcal{A}) \leq \max_{i,j\in N, j\neq i} \frac{1}{2}\{a_{i\dots i} + a_{j\dots j} + r_i^j(\mathcal{A}) + \Delta_{i,j}^{\frac{1}{2}}(\mathcal{A})\},$$

where

$$\Delta_{i,j}(\mathcal{A}) = (a_{i\dots i} - a_{j\dots j} + r_i^j(\mathcal{A}))^2 + 4a_{ij\dots j}r_j(\mathcal{A}).$$

Proof. Let $x = (x_1, \dots, x_n)^T$ be an Z-eigenvector of $\mathcal{A}$ corresponding to $\rho(\mathcal{A})$, that is,

$$\mathcal{A}x^{m-1} = \rho(\mathcal{A})x, \tag{4}$$

Let

$$x_t \geq x_s \geq \max\{x_k, \ \ k = 1, \dots, n, \ \ k \neq t, s\}.$$

Obviously, by Lemma 3.4, we have $x_t > 0$, $x_s > 0$. From Corollary 4.10 in [3], we have

$$\rho(\mathcal{A}) - a_{i\dots i} \geq 0, \ \ i = 1, \dots, n.$$

Consider the equation of (1), by $x_t^{m-1} \leq x_t$, $x_s^{m-1} \leq x_s$, we can get

$$\begin{aligned}
(\rho(\mathcal{A}) - a_{t\dots t})x_t &= \sum_{\substack{\delta_{ti_2\dots i_m}=0,\\ \delta_{si_2\dots i_m}=0}} a_{ti_2\dots i_m}x_{i_2}\dots x_{i_m} + a_{ts\dots s}x_s^{m-1} + a_{t\dots t}(x_t^{m-1} - x_t)\\
&\leq \sum_{\substack{\delta_{ti_2\dots i_m}=0,\\ \delta_{si_2\dots i_m}=0}} a_{ti_2\dots i_m}x_t^{m-1} + a_{ts\dots s}x_s^{m-1}\\
&\leq r_t^s(\mathcal{A})x_t + a_{ts\dots s}x_s, \qquad (5)
\end{aligned}$$

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equivalently,

$$(\rho(\mathcal{A}) - a_{t\ldots t} - r_t^s(\mathcal{A}))x_t \leq a_{ts\ldots s}x_s.$$

Moreover, from equality (1), we similarly get

$$\begin{aligned}(\rho(\mathcal{A}) - a_{s\ldots s})x_s &= \sum_{\delta_{si_2\ldots i_m}=0} a_{si_2\ldots i_m}x_{i_2}\ldots x_{i_m} + a_{s\ldots s}(x_s^{m-1} - x_s)\\ &\leq r_s(\mathcal{A})x_t^{m-1} + a_{s\ldots s}(x_s^{m-1} - x_s)\\ &\leq r_s(\mathcal{A})x_t. \qquad (6)\end{aligned}$$

Multiplying equation (5) and (6), we get

$$(\rho(\mathcal{A}) - a_{t\ldots t} - r_t^s(\mathcal{A}))(\rho(\mathcal{A}) - a_{s\ldots s}) \leq a_{ts\ldots s}r_s(\mathcal{A}).$$

Then, solving for $\rho(\mathcal{A})$,

$$\rho(\mathcal{A}) \leq \frac{1}{2}\{a_{i\ldots i} + a_{j\ldots j} + r_i^j(\mathcal{A}) + \Delta_{i,j}^{\frac{1}{2}}(\mathcal{A})\} \leq \max_{i,j\in N, j\neq i} \frac{1}{2}\{a_{i\ldots i} + a_{j\ldots j} + r_i^j(\mathcal{A}) + \Delta_{i,j}^{\frac{1}{2}}(\mathcal{A})\}.$$

Thus, we complete the proof. □

From Theorem 3.5 in [12], we know that

$$\max_{i,j\in N, j\neq i} \frac{1}{2}\{a_{i\ldots i} + a_{j\ldots j} + r_i^j(\mathcal{A}) + \Delta_{i,j}^{\frac{1}{2}}(\mathcal{A})\} \leq \max_{i\in N} \sum\nolimits_{i_2,\ldots,i_m=1}^{n} a_{ii_2\ldots i_m},$$

that is to say, our new bound in Theorem 3.5 is always better than the bound in Lemma 3.3. We now show the efficiency of the upper bound in Theorem 3.5 by the following example which was introduced in [3].

Example 3.1. Consider the tensor $\mathcal{A} = (a_{ijkl})$ of order 4 dimension 2 with entries defined as follows:

$$a_{1111} = \frac{1}{2}, \ \ a_{2222} = 3, \ \ \text{and} \ \ a_{ijkl} = \frac{1}{3} \ \ \text{elsewhere}.$$

By Lemma 3.2, we have

$$\rho(\mathcal{A}) \leq 10.6666.$$

By Lemma 3.3, we have

$$\rho(\mathcal{A}) \leq 5.3333.$$

By Theorem 3.5, we have

$$\rho(\mathcal{A}) \leq 5.1667.$$

In fact, $\rho(\mathcal{A}) = 3.1092$. Hence, the bound in Theorem 3.5 is tight and sharper.

4 Bounds for the largest H-eigenvalue

In this section, we give the lower bound and the upper bound for the largest H-eigenvalue of an m-order n-dimensional nonnegative tensor $\mathcal{A}$.

Definition 4.1. *Let $\mathcal{A}$ be an m-order and n-dimensional tensor. We define the H-spectrum of $\mathcal{A}$, denoted $H(\mathcal{A})$ to be the set of all H-eigenvalues of $\mathcal{A}$. Assume $H(\mathcal{A}) \neq 0$, then the H-spectral radius of $\mathcal{A}$, denoted $\mu(\mathcal{A})$, is defined as* $\mu(\mathcal{A}) = \max\{|\lambda| : \lambda \in H(\mathcal{A})\}$.

First, we introduce some results for H-eigenvalue of nonnegative tensors [1, 13, 14], which are generalized from nonnegative matrices.

Theorem 4.2. *If $\mathcal{A}$ is irreducible and nonnegative, then there exists a number $\mu(\mathcal{A}) > 0$ and a vector $x_0 > 0$, such that $\mathcal{A}x_0^{m-1} = \mu(\mathcal{A})x_0^{[m-1]}$. Moreover, if λ is an eigenvalue of $\mathcal{A}$, then $|\lambda| \leq \mu(\mathcal{A})$.*

Lemma 4.3. *(Lemma 5.2 in [13]) Let $\mathcal{A}$ be an m-order and n-dimensional nonnegative tensor. Then*

$$R_{\min}(\mathcal{A}) \leq \mu(\mathcal{A}) \leq R_{\max}(\mathcal{A}). \tag{7}$$

According to some eigenvalue inclusion theorems, Li, Li and Kong [12] obtained the following upper bound for the spectral radius of a nonnegative tensor, which is sharper than the upper bound in Lemma 4.3.

Lemma 4.4. *(Theorem 3.3 in [12]) Suppose that m-order n-dimensional tensor $\mathcal{A}$ is nonnegative. Then*

$$\mu(\mathcal{A}) \leq \max_{i,j\in N, j\neq i} \frac{1}{2}\left\{a_{i\ldots i} + a_{j\ldots j} + r_i^j(\mathcal{A}) + \Delta_{i,j}^{\frac{1}{2}}(\mathcal{A})\right\},$$

where

$$\Delta_{i,j}(\mathcal{A}) = \left(a_{i\ldots i} - a_{j\ldots j} + r_i^j(\mathcal{A})\right)^2 + 4a_{ij\ldots j}r_j(\mathcal{A}).$$

In the following Theorem, we give new bounds for the spectral radius of a nonnegative tensor.

Theorem 4.5. *Suppose that m-order n-dimensional tensor $\mathcal{A}$ is nonnegative. Then*

$$\min_{i\in N}\max_{j\in N, j\neq i} \frac{1}{2}\left\{a_{i\ldots i} + a_{j\ldots j} + r_i^j(\mathcal{A}) + \Delta_{i,j}^{\frac{1}{2}}(\mathcal{A})\right\} \leq \mu(\mathcal{A}) \leq \max_{i\in N}\min_{j\in N, j\neq i} \frac{1}{2}\left\{a_{i\ldots i} + a_{j\ldots j} + r_i^j(\mathcal{A}) + \Delta_{i,j}^{\frac{1}{2}}(\mathcal{A})\right\},$$

where

$$\Delta_{i,j}(\mathcal{A}) = \left(a_{i\ldots i} - a_{j\ldots j} + r_i^j(\mathcal{A})\right)^2 + 4a_{ij\ldots j}r_j(\mathcal{A}).$$

Proof. First, we assume that tensor $\mathcal{A}$ is strictly positive and let x be the unique positive eigenvector corresponding to $\mu(\mathcal{A})$, i.e.

$$\mathcal{A}x^{m-1} = \mu(\mathcal{A})x^{[m-1]}.$$

Assume $0 < x_t = \max_{i \in N} x_i$, then, for any $s \neq t$, we can get

$$(\mu(\mathcal{A}) - a_{t\ldots t})x_t^{m-1} - a_{ts\ldots s}x_s^{m-1} = \sum_{\substack{\delta_{ti_2\ldots i_m}=0,\\ \delta_{si_2\ldots i_m}=0}} a_{ti_2\ldots i_m} x_{i_2} \ldots x_{i_m},$$

$$(\mu(\mathcal{A}) - a_{s\ldots s})x_s^{m-1} - a_{st\ldots t}x_t^{m-1} = \sum_{\substack{\delta_{ti_2\ldots i_m}=0,\\ \delta_{si_2\ldots i_m}=0}} a_{si_2\ldots i_m} x_{i_2} \ldots x_{i_m}.$$

Solving for x_t we obtain

$$\begin{aligned}((\mu(\mathcal{A}) - a_{s\ldots s})(\mu(\mathcal{A}) - a_{t\ldots t}) - a_{st\ldots t}a_{ts\ldots s})\, x_t^{m-1} &= (\mu(\mathcal{A}) - a_{s\ldots s}) \sum_{\substack{\delta_{ti_2\ldots i_m}=0,\\ \delta_{si_2\ldots i_m}=0}} a_{ti_2\ldots i_m} x_{i_2} \ldots x_{i_m} \\ &\quad + a_{ts\ldots s} \sum_{\substack{\delta_{ti_2\ldots i_m}=0,\\ \delta_{si_2\ldots i_m}=0}} a_{si_2\ldots i_m} x_{i_2} \ldots x_{i_m}. \qquad (8)\end{aligned}$$

Recalling that $0 < x_t = \max_{i \in N} x_i$, we have

$$\begin{aligned}(\mu(\mathcal{A}) - a_{s\ldots s})(\mu(\mathcal{A}) - a_{t\ldots t}) - a_{st\ldots t}a_{ts\ldots s} &= (\mu(\mathcal{A}) - a_{s\ldots s}) \sum_{\substack{\delta_{ti_2\ldots i_m}=0,\\ \delta_{si_2\ldots i_m}=0}} a_{ti_2\ldots i_m} \frac{x_{i_2}}{x_s} \ldots \frac{x_{i_m}}{x_s} \\ &\quad + a_{ts\ldots s} \sum_{\substack{\delta_{ti_2\ldots i_m}=0,\\ \delta_{si_2\ldots i_m}=0}} a_{si_2\ldots i_m} \frac{x_{i_2}}{x_s} \ldots \frac{x_{i_m}}{x_s} \\ &\leq (\mu(\mathcal{A}) - a_{s\ldots s}) r_t^s(\mathcal{A}) + a_{ts\ldots s} r_s^t(\mathcal{A}). \qquad (9)\end{aligned}$$

Therefore

$$\mu(\mathcal{A}) \leq \frac{1}{2}\left\{a_{t\ldots t} + a_{s\ldots s} + r_t^s(\mathcal{A}) + \Delta_{t,s}^{\frac{1}{2}}(\mathcal{A})\right\}.$$

This must be true for every $s \neq t$, then, we get

$$\mu(\mathcal{A}) \leq \min_{j \in N, j \neq t} \frac{1}{2}\left\{a_{t\ldots t} + a_{j\ldots j} + r_t^j(\mathcal{A}) + \Delta_{t,j}^{\frac{1}{2}}(\mathcal{A})\right\}.$$

And this could be true for any $t \in N$, that is

$$\mu(\mathcal{A}) \leq \max_{i \in N} \min_{j \in N, j \neq i} \frac{1}{2}\left\{a_{i\ldots i} + a_{j\ldots j} + r_i^j(\mathcal{A}) + \Delta_{i,j}^{\frac{1}{2}}(\mathcal{A})\right\}.$$

Similarly, assume $0 < x_T = \min_{i \in N} x_i$, we can get

$$\mu(\mathcal{A}) \geq \min_{i \in N} \max_{j \in N, j \neq i} \frac{1}{2}\left\{a_{i\ldots i} + a_{j\ldots j} + r_i^j(\mathcal{A}) + \Delta_{i,j}^{\frac{1}{2}}(\mathcal{A})\right\}.$$

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If the tensor $\mathcal{A}$ is not strictly positive, we denote by $\mathcal{D} = (d_{i_1 \dots i_m})$ the m-order n-dimensional tensor with $d_{i_1 \dots i_m} = 1$, for all $i_1 \in N, \dots, i_m \in N$. Hence, $\mathcal{A} + t\mathcal{D}$ is strictly positive for any chosen positive real number t, and then letting $t \to 0$, the result follows by continuity. □

From the proof of the Theorem 3.5 in [12], we can get the following result:

$$\begin{aligned} \max_{i\in N} \min_{j\in N, j\neq i} \tfrac{1}{2}\left\{a_{i\dots i} + a_{j\dots j} + r_i^j(\mathcal{A}) + \Delta_{i,j}^{\frac{1}{2}}(\mathcal{A})\right\} &\leq \max_{i,j\in N, j\neq i} \tfrac{1}{2}\left\{a_{i\dots i} + a_{j\dots j} + r_i^j(\mathcal{A}) + \Delta_{i,j}^{\frac{1}{2}}(\mathcal{A})\right\} \\ &\leq R, \end{aligned}$$

where

$$\Delta_{i,j}(\mathcal{A}) = \left(a_{i\dots i} - a_{j\dots j} + r_i^j(\mathcal{A})\right)^2 + 4a_{ij\dots j} r_j(\mathcal{A}).$$

We now compare the lower bound in Theorems 4.5 with that in Lemma 4.3.

Theorem 4.6. *Suppose that m-order n-dimensional tensor $\mathcal{A}$ is nonnegative. Then*

$$R_{\min}(\mathcal{A}) \leq \min_{i\in N} \max_{j\in N, j\neq i} \frac{1}{2}\left\{a_{i\dots i} + a_{j\dots j} + r_i^j(\mathcal{A}) + \Delta_{i,j}^{\frac{1}{2}}(\mathcal{A})\right\},$$

where

$$\Delta_{i,j}(\mathcal{A}) = \left(a_{i\dots i} - a_{j\dots j} + r_i^j(\mathcal{A})\right)^2 + 4a_{ij\dots j} r_j(\mathcal{A}).$$

Proof. First, we assume that tensor $\mathcal{A}$ is strictly positive. Equivalently, we will prove that, if

$$\mu(\mathcal{A}) \geq \min_{i\in N} \max_{j\in N, j\neq i} \frac{1}{2}\left\{a_{i\dots i} + a_{j\dots j} + r_i^j(\mathcal{A}) + \Delta_{i,j}^{\frac{1}{2}}(\mathcal{A})\right\}.$$

Then, we can get

$$\mu(\mathcal{A}) \geq R_{\min}(\mathcal{A}).$$

If $\mu(\mathcal{A})$ satisfies the lower bound in the Theorem 4.5 and the matrix is positive, similar to the proof of Theorem 4.5, if assume $0 < x_T = \min_{i\in N} x_i$, for any $s \neq T$, we can get

$$(\mu(\mathcal{A}) - a_{s\dots s})(\mu(\mathcal{A}) - a_{T\dots T}) \geq (\mu(\mathcal{A}) - a_{s\dots s}) r_T^s(\mathcal{A}) + a_{Ts\dots s} r_s(\mathcal{A}).$$

If we assumed that $\mu(\mathcal{A}) \leq R_s(\mathcal{A})$, then we have that $\mu(\mathcal{A}) - a_{s\dots s} \leq r_s(\mathcal{A})$. So,

$$(\mu(\mathcal{A}) - a_{s\dots s})(\mu(\mathcal{A}) - a_{T\dots T} - r_T^s(\mathcal{A}) \geq a_{Ts\dots s} r_s(\mathcal{A}) \geq a_{Ts\dots s}(\mu(\mathcal{A}) - a_{s\dots s}),$$

that is

$$(\mu(\mathcal{A}) - a_{s\dots s})(\mu(\mathcal{A}) - a_{T\dots T} - r_T^s(\mathcal{A}) - a_{Ts\dots s}) \geq 0.$$

From Lemma 3.2 in [12], we know $\mu(\mathcal{A}) - a_{s\dots s} \geq 0$, then, we obtain

$$\mu(\mathcal{A}) - a_{T\dots T} - r_T^s(\mathcal{A}) - a_{Ts\dots s} \geq 0,$$

that is

$$\mu(\mathcal{A}) \geq R_T(\mathcal{A}) \geq R_{\min}(\mathcal{A}).$$

If the tensor $\mathcal{A}$ is not strictly positive, we denote by $\mathcal{D} = (d_{i_1\ldots i_m})$ the m-order n-dimensional tensor with $d_{i_1\ldots i_m} = 1$, for all $i_1 \in N, \ldots, i_m \in N$. Hence, $\mathcal{A} + t\mathcal{D}$ is strictly positive for any chosen positive real number t, and then letting $t \to 0$, the result follows by continuity. □

We now show the efficiency of the bounds in Theorem 4.5 by the following example.

Example 4.1. Consider the tensor $\mathcal{A} = (a_{ijkl})$ of order 4 dimension 3 with entries defined as follows:

$$a_{1111} = 1,\ \ a_{1222} = 1,\ \ a_{1333} = 1,$$

$$a_{2111} = 2,\ \ a_{2222} = 2,\ \ a_{2333} = 2,$$

$$a_{3111} = 3,\ \ a_{3222} = 3,\ \ a_{3333} = 3,$$

and $a_{ijkl} = 0$ elsewhere. By Lemma 4.3, we have

$$3 \leq \mu(\mathcal{A}) \leq 9.$$

By Lemma 4.4, we have

$$\mu(\mathcal{A}) \leq 8.$$

By Theorem 4.5, we have

$$5 \leq \mu(\mathcal{A}) \leq 7.$$

In fact, $\mu(\mathcal{A}) = 6$. Hence, the bound in Theorem 4.5 is tight and sharper.

5 Bounds for the largest B-eigenvalue

In this section, we focus our attention on the largest B-eigenvalue of a m-order n-dimensional nonnegative tensor $\mathcal{A}$ relative to $\mathcal{B}$.

Definition 5.1. *Let $\mathcal{A}$, $\mathcal{B}$ be two m-order and n-dimensional tensors. We define the B-spectrum of $\mathcal{A}$ relative to $\mathcal{B}$, denoted $T(\mathcal{A})$ to be the set of all B-eigenvalues of $\mathcal{A}$ relative to $\mathcal{B}$. Assume $T(\mathcal{A}) \neq 0$, then the B-spectral radius of $\mathcal{A}$, denoted $\nu(\mathcal{A})$, is defined as $\nu(\mathcal{A}) = \max\{|\lambda| : \lambda \in T(\mathcal{A})\}$.*

For an m-order n-dimensional tensor $\mathcal{A}$, let

$$F_{\mathcal{A}} = \left(\mathcal{A}x^{m-1}\right)^{[\frac{1}{m-1}]},$$

Song and Qi [15] proved the Perron-Frobenius property for nonnegative tensor pairs $(\mathcal{A}, \mathcal{B})$ without the requirement of the tensor inversion.

Lemma 5.2. *(Corollary 4.2 in [15]) Let $\mathcal{A}$, $\mathcal{B}$ be two weakly irreducible and nonnegative tensors with order m and dimension n and $F_{\mathcal{A}}F_{\mathcal{B}} = F_{\mathcal{B}}F_{\mathcal{A}}$. If $\exists x > 0$ such*

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that $\mathcal{B}x^{m-1} \geq x^{[m-1]}$*, then* $\mathcal{A}$ *has a unique positive B-eigenvalue with a corresponding positive B-eigenvector.*

Based on the above Lemma, we give the main results of this section.

Theorem 5.3. *Under the conditions of Lemma 5.2 and* $b_{i\dots i} > 0$ *for all* $i \in N$*. Then*

$$\nu(\mathcal{A}) \leq \max_{i\in N} \frac{R_i(\mathcal{A})}{b_{i\dots i}}.$$

Proof. Let x be the unique positive eigenvector corresponding to $\nu(\mathcal{A})$, i.e.

$$\mathcal{A}x^{m-1} = \nu(\mathcal{A})\mathcal{B}x^{m-1}.$$

Assume $0 < x_t = \max_{i\in N} x_i$, then, from the i-th equation of the above equation, we can get

$$\sum_{\delta_{ti_2\dots i_m}=0} a_{ti_2\dots i_m} x_{i_2}\dots x_{i_m} + a_{t\dots t}x_t^{m-1} = \nu(\mathcal{A})\left(\sum_{\delta_{ti_2\dots i_m}=0} b_{ti_2\dots i_m} x_{i_2}\dots x_{i_m} + b_{t\dots t}x_t^{m-1}\right),$$

Dividing both sides by x_t^{m-1} and rearranging yields

$$\nu(\mathcal{A})\left(\sum_{\delta_{ti_2\dots i_m}=0} b_{ti_2\dots i_m} \frac{x_{i_2}}{x_t}\dots\frac{x_{i_m}}{x_t} + b_{t\dots t}\right) - a_{t\dots t} = \sum_{\delta_{ti_2\dots i_m}=0} a_{ti_2\dots i_m} \frac{x_{i_2}}{x_t}\dots\frac{x_{i_m}}{x_t} \leq r_t(\mathcal{A}).$$

Hence,

$$\nu(\mathcal{A}) \leq \frac{R_t(\mathcal{A})}{\sum_{\delta_{ti_2\dots i_m}=0} b_{ti_2\dots i_m}\frac{x_{i_2}}{x_t}\dots\frac{x_{i_m}}{x_t} + b_{t\dots t}} \leq \frac{R_t(\mathcal{A})}{b_{t\dots t}}.$$

Thus, we complete the proof. □

If $\exists x > 0$ such that $\mathcal{B}x^{m-1} \geq x^{[m-1]}$ and suppose that m-order n-dimensional tensor $\mathcal{B}$ is nonnegative and diagonal, we can get

$$b_{i\dots i} \geq 1,$$

for all $i \in N$. Similar to the proof of Theorem 4.5, we can get some new bounds for $\nu(\mathcal{A})$, including the upper bound and the lower bound.

Theorem 5.4. *Under the conditions of Lemma 5.2 and suppose that m-order n-dimensional tensor* $\mathcal{B}$ *is nonnegative and diagonal. Then*

$$\begin{aligned}\min_{i\in N}\max_{j\in N, j\neq i} \frac{a_{i\dots i}b_{j\dots j}+a_{j\dots j}b_{i\dots i}+b_{j\dots j}r_i^j(\mathcal{A})+\Delta_{i,j}^{\frac{1}{2}}(\mathcal{A})}{2b_{i\dots i}b_{j\dots j}} &\leq \nu(\mathcal{A}) \\ &\leq \max_{i\in N}\min_{j\in N, j\neq i} \frac{a_{i\dots i}b_{j\dots j}+a_{j\dots j}b_{i\dots i}+b_{j\dots j}r_i^j(\mathcal{A})+\Delta_{i,j}^{\frac{1}{2}}(\mathcal{A})}{2b_{i\dots i}b_{j\dots j}},\end{aligned}$$

where

$$\Delta_{i,j}(\mathcal{A}) = \left(a_{i\ldots i}b_{j\ldots j} - a_{j\ldots j}b_{i\ldots i} + b_{j\ldots j}r_i^j(\mathcal{A})\right)^2 + 4a_{ij\ldots j}b_{i\ldots i}b_{j\ldots j}r_j(\mathcal{A}).$$

If $b_{i\ldots i} = 1$ for all $i \in N$, then, the results in Theorem 5.4 reduce to the result in Theorem 4.5.

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A note on fractional neutral integro-differential inclusions with state-dependent delay in Banach spaces

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Abstract

We have applied different fixed point theorems to examine the existence results for fractional neutral integro-differential inclusions (FNIDI) with state-dependent delay (SDD) in Banach spaces. We tend to conjointly discuss the cases once the multivalued nonlinear term takes convex values further as nonconvex values. An example is offered to demonstrate the obtained results.

Keywords: Fractional order differential equations, state-dependent delay, multivalued map, fixed point theorem, Banach spaces, semigroup theory.

2010 Mathematics Subject Classification: 26A33, 34A08, 35R12, 34A60, 34G20, 34K05, 45J05.

1 Introduction

The aim of the manuscript is to investigate the existence of mild solutions for neutral integro-differential inclusions of fractional order as given below

$$\frac{d}{dt}\big[x(t) - \mathscr{G}(t, x_{\varrho(t,x_t)})\big] \in \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} \mathscr{A}\big[x(s) - \mathscr{G}(s, x_{\varrho(s,x_s)})\big]ds + \mathscr{F}(t, x_{\varrho(t,x_t)}), \quad \text{a.e.} \quad t \in \mathscr{I} = [0, b], \tag{1.1}$$

$$x_0 = \varsigma \in \mathscr{B}, \tag{1.2}$$

such that $1 < \alpha < 2$ and $\mathscr{A} : D(\mathscr{A}) \subset \mathbb{X} \to \mathbb{X}$ denotes a linear densely defined operator of sectorial type on a complex Banach space $(\mathbb{X}, |\cdot|)$, the convolution integral within the equation is understood because the Riemann-Liouville fractional integral (see [4]) and $\mathscr{F} : \mathscr{I} \times \mathscr{B} \to \mathscr{P}(\mathbb{X})$ represents a multivalued map ($\mathscr{P}(\mathbb{X})$ is the family of nonempty subsets of $\mathbb{X}$), $\mathscr{G} : \mathscr{I} \times \mathscr{B} \to \mathbb{X}$, and $\varrho : \mathscr{I} \times \mathscr{B} \to (-\infty, b]$ are apposite functions, and $\mathscr{B}$ is a theoretical phase space axioms outlined in Preliminaries.

We recall that for any continuous function x defined on $(-\infty, b]$ and for any $t \geq 0$, we designate by x_t the part of $\mathscr{B}$ defined by $x_t(\theta) = x(t+\theta)$ for $\theta \leq 0$. Here $x_t(\cdot)$ speaks to the historical backdrop of the state from every $\theta \in (-\infty, 0]$ likely the current time t.

Fractional differential equations have picked up hefty grandness as a final result of their exertion in numerous field of science and engineering. In the latest years, there has been a major growth in differential systems

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comprising fractional derivatives, e.g. the monographs of Abbas et al. [5], Baleanu et al. [6], Podlubny [7], Diethelm [8], Kilbas et al. [9], Tarasov [10] and Anastassiou [11], and the papers [12, 13, 14, 15, 16, 17, 18, 19], and the references cited therein.

As it is known, a delay differential equation (DDE) may be a special sort of functional differential equation (FDE). FDEs with SDD seem often in applications as models of equations and for this intention the report of this kind of equations received nice care in latest years. For points of interest, we recommend the reader to check the papers by by Abada et al. [20], Ait Dads et al. [21], Anguraj et al. [22], Benchohra et al. [23], Cuevas et al. [24], Hernandez et al. [25, 26], Mallika Arjunan et al. [27] and Yan et al. [28].

In the situation where $\mathscr{F}$ is either a single or a multivalued map, the problem (1.1)-(1.2) with $\mathscr{G} = 0$ was investigated on a compact interval in Agarwal et al. [29], Benchohra et al. [30, 31]. On unbounded interval when $\mathscr{F}$ is a single map, the problem (1.1)-(1.2) with $\mathscr{G} = 0$ was discussed by Benchohra et al. [32]. According to the knowledge of the authors , there is no work on the existence results for FNIDI with SDD in Banach spaces, which is communicated in the structure (1.1)-(1.2). Roused by this thought, in this paper, we concentrate on this problem, which is common generalizations of the idea of mild solution for fractional neutral equations well known in the theory of integer order systems.

This manuscript has the following structure. In section 2, we show some preliminaries and lemmas to be utilized to demonstrate our primary results. In section 3, we show two results for the problem (1.1)-(1.2) when the right-hand side is convex valued. The principal result is focused on a fixed point theorem of Bohnenblust-Karlin [1], and the second one on the nonlinear alternative of Leray-Schauder type [2]. The final existence result is given for a nonconvex valued right-hand side by utilizing a fixed point theorem for contraction multivalued maps thanks to Covitz and Nadler [3]. An application is presented in Section 4.

2 Preliminaries

Let $C(\mathscr{I}, \mathbb{X})$ be the Banach space of all continuous functions from $\mathscr{I}$ into $\mathbb{X}$ with the norm

$$\|x\|_\infty = \sup\{|x(t)| : t \in \mathscr{I}\}.$$

Let $B(\mathbb{X})$ signifies the Banach space of all bounded linear operators from $\mathbb{X}$ into $\mathbb{X}$.

A measurable function $x : \mathscr{I} \to \mathbb{X}$ is Bochner integrable if and only if $|x|$ is Lebesgue integrable. (For extra insights about Bochner integral, see Yosida [33]).

Let $L^1(\mathscr{I}, \mathbb{X})$ signify the Banach space of all continuous functions $x : \mathscr{I} \to \mathbb{X}$ which are Bochner integrable and have norm

$$\|x\|_{L^1} = \int_0^b |x(t)|dt \quad \text{for all} \quad x \in L^1(\mathscr{I}, \mathbb{X}).$$

We expect that the phase space $(\mathscr{B}, \|\cdot\|_{\mathscr{B}})$ is a semi-normed linear space of functions mapping $(-\infty, 0]$ into $\mathbb{X}$, and fulfilling the subsequent elementary adages as a result of Hale and Kato (see more details in [34, 35]).

(P_1) If $x : (-\infty, b] \to \mathbb{X}, b > 0$, is continuous on $\mathscr{I}$ and $x_0 \in \mathscr{B}$, then for every $t \in \mathscr{I}$ the accompanying conditions hold:

(i) x_t is in $\mathscr{B}$;

(ii) $|x(t)| \leq H\|x_t\|_{\mathscr{B}}$;

(iii) $\|x_t\|_{\mathscr{B}} \leq \mathscr{D}_1(t)\sup\{|x(s)| : 0 \leq s \leq t\} + \mathscr{D}_2(t)\|x_0\|_{\mathscr{B}}$, where $H > 0$ is a constant and $\mathscr{D}_1(\cdot) : [0, +\infty) \to [0, +\infty)$ is continuous, $\mathscr{D}_2(\cdot) : [0, +\infty) \to [0, +\infty)$ is locally bounded, and $\mathscr{D}_1, \mathscr{D}_2$ are independent of $x(\cdot)$.

(P_2) For the function $x(\cdot)$ in (P_1), x_t is a $\mathscr{B}$-valued continuous function on $\mathscr{I}$.

(P_3) The space $\mathscr{B}$ is complete.

Designate $\mathscr{D}_1^* = \sup\{\mathscr{D}_1(t) : t \in \mathscr{I}\}$ and $\mathscr{D}_2^* = \sup\{\mathscr{D}_2(t) : t \in \mathscr{I}\}$.
Now, we briefly review some known results from the solution operator. The Laplace transformation of a function $f \in L^1_{loc}(\mathbb{R}^+, \mathbb{X})$ is defined by

$$\mathcal{L}(f)(\lambda) = \widehat{f}(\lambda) = \int_0^\infty e^{-\lambda t} f(t)dt, \quad \mathrm{Re}(\lambda) > \omega,$$

if the integral is definitely convergent for $Re(\lambda) > \omega$. We mention the subsequent definition [4].

Definition 2.1. *Let $\mathscr{A} : D(\mathscr{A}) \subset \mathbb{X} \to \mathbb{X}$ be a closed and linear operator on a Banach space $\mathbb{X}$. We call $\mathscr{A}$ is the generator of a solution if there exist $\omega \in \mathbb{R}$ and a strongly continuous function $S_\alpha : \mathbb{R}^+ \to B(\mathbb{X})$ such that*

$$\{\lambda^\alpha : \quad Re\,(\lambda) > \omega\} \subset \rho(\mathscr{A}),$$

and

$$\lambda^{\alpha-1}(\lambda^\alpha - \mathscr{A})^{-1}x = \int_0^\infty e^{-\lambda t} S_\alpha(t)xdt, \quad Re\,\lambda > \omega, \quad x \in \mathbb{X}.$$

In this case, $S_\alpha(t)$ is called operator function created by $\mathscr{A}$. The idea of a solution operator, as characterized above, is nearly identified with the ones of a resolvent family [37]. Having in mind the uniqueness of the Laplace transform, in the fringe case $\alpha = 1$, the family $S_\alpha(t)$ relates to a strongly continuous semigroup (see Pazy [38]), while in the case $\alpha = 2$ a solution operator relates to the idea of a cosine family; see [39]. The subsequent result is an immediate outcome of [40, Proposition 3.1 and Lemma 2.2].

Proposition 2.1. *Let $S_\alpha(t)$ be a solution operator on $\mathbb{X}$ with generator $\mathscr{A}$. Then, we have*

(a) $S_\alpha(t)$ is strongly continuous for $t \geq 0$ and $S(0) = I$;

(b) $S_\alpha(t)D(\mathscr{A}) \subset D(\mathscr{A})$ and $\mathscr{A}S_\alpha(t)x = S_\alpha(t)\mathscr{A}x$ for all $x \in D(\mathscr{A})$, $t \geq 0$;

(c) For every $x \in D(\mathscr{A})$ and $t \geq 0$,

$$S_\alpha(t)x = x + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \mathscr{A}S_\alpha(s)xds.$$

(d) Let $x \in D(\mathscr{A})$. Then $\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} S_\alpha(s)xds \in D(\mathscr{A})$ and

$$S_\alpha(t)x = x + \mathscr{A}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} S_\alpha(s)xds.$$

Definition 2.2. *A solution operator $\{S_\alpha(t)\}_{t>0}$ is called uniformly continuous if*

$$\lim_{t\to s} \|S_\alpha(t) - S_\alpha(s)\|_{B(\mathbb{X})} = 0.$$

Before we finish this section, we review some known results from multivalued analysis that we will apply in the spin-off. We recall that

$$\mathscr{P}(\mathbb{X}) = \{Y \subset \mathbb{X} : Y \neq \emptyset\}, \mathscr{P}_{cl}(\mathbb{X}) = \{Y \in \mathscr{P}(\mathbb{X}) : Y \text{ closed}\}, \quad \mathscr{P}_b(\mathbb{X}) = \{Y \in \mathscr{P}(\mathbb{X}) : Y \text{ bounded }\},$$
$$\mathscr{P}_{cp}(\mathbb{X}) = \{Y \in \mathscr{P}(\mathbb{X}) : Y \text{ compact}\},$$
$$\mathscr{P}_{cp,c}(\mathbb{X}) = \{Y \in \mathscr{P}(\mathbb{X}) : Y \text{ compact and convex}\}.$$

Remark 2.1. *For extra points of interest on this, we suggest the reader to [13].*

Definition 2.3. *The multivalued map $\mathscr{F} : \mathscr{I} \times \mathscr{B} \to \mathscr{P}(\mathbb{X})$ is said to be Carathéodory if*

(i) $t \mapsto \mathscr{F}(t, u)$ *is measurable for each* $u \in \mathscr{B}$*;*

(ii) $u \mapsto \mathscr{F}(t, u)$ *is upper semicontinuous for almost all* $t \in \mathscr{I}$*.*

Let $S_{\mathscr{F},x}$ be a set characterized by

$$S_{\mathscr{F},x} = \{v \in L^1(\mathscr{I}, \mathbb{X}) : v(t) \in \mathscr{F}(t, x_{\varrho(t,x_t)}) \text{ a.e. } t \in \mathscr{I}\}.$$

Definition 2.4. *A multivalued operator $\Upsilon : \mathbb{X} \to \mathscr{P}_{cl}(\mathbb{X})$ is called:*

(a) Λ*-Lipschitz if there exists* $\Lambda > 0$ *such that*

$$H_d(\Upsilon(x), \Upsilon(\overline{x})) \leq \Lambda\, d(x, \overline{x}) \quad \textit{for all} \quad x, \overline{x} \in \mathbb{X};$$

(b) a contraction if it is Λ*-Lipschitz with* $\Lambda < 1$*.*

Presently, we express the accompanying lemmas which are important to make our primary result.

Lemma 2.1 ([41]). *Let $\mathbb{X}$ be a Banach space. Let $\mathscr{F} : \mathscr{I} \times \mathscr{B} \to \mathscr{P}_{cp,c}(\mathbb{X})$ be an L^1-Carathéodory multivalued map and let Ψ be a linear continuous mapping from $L^1(\mathscr{I}, \mathbb{X})$ to $C(\mathscr{I}, \mathbb{X})$, then the operator*

$$\begin{aligned} \Psi \circ S_F : C(\mathscr{I}, \mathbb{X}) &\to \mathscr{P}_{cp,c}(C(\mathscr{I}, \mathbb{X})), \\ x &\mapsto (\Psi \circ S_{\mathscr{F}})(x) := \Psi(S_{\mathscr{F},x}) \end{aligned}$$

has a closed graph operator in $C(\mathscr{I}, \mathbb{X}) \times C(\mathscr{I}, \mathbb{X})$.

Lemma 2.2 (Bohnenblust-Karlin's [1]). *Let $\mathbb{X}$ be a Banach space and $D \in \mathscr{P}_{cl,c}(\mathbb{X})$. Suppose that the operator $G : D \to \mathscr{P}_{cl,c}(D)$ is upper semicontinuous and the set $G(D)$ is relatively compact in $\mathbb{X}$. Then G has a fixed point in D.*

Lemma 2.3 (Covitz and Nadler [3]). *Let $(\mathbb{X}, d)$ be a complete metric space. If $\Upsilon : \mathbb{X} \to \mathscr{P}_{cl}(\mathbb{X})$ is a contraction, then Fix $\Upsilon \neq \emptyset$.*

For more details on multivalued maps see the books of Graef et al. [42] and Castaing et al. [43].

3 Existence results

We demonstrate below the existence of solutions for the problem (1.1)-(1.2). To start with, we delineate the mild solution for the problem (1.1)-(1.2).

Definition 3.1. *We affirm that the function $x : (-\infty, b] \to \mathbb{X}$ is a mild solution of (1.1)-(1.2) if $x(t) = \varsigma(t)$ for all $t \leq 0$, the constraint of $x(\cdot)$ to the interval $[0, b]$ is continuous and there exists $v(\cdot) \in L^1(\mathscr{I}, \mathbb{X})$, such that $v(t) \in \mathscr{F}(t, x_{\varrho(t,x_t)})$ a.e. $t \in [0, b]$, and x fulfills the consecutive integral equation:*

$$x(t) = S_\alpha(t)\big[\varsigma(0) - \mathscr{G}(0, \varsigma(0))\big] + \mathscr{G}(t, x_{\varrho(t,x_t)}) + \int_0^t S_\alpha(t-s)v(s)ds \quad \textit{for each} \quad t \in \mathscr{I}. \tag{3.1}$$

Let us set

$$\mathcal{R}(\varrho^-) = \{\varrho(s, \varsigma) : (s, \varsigma) \in \mathscr{I} \times \mathscr{B},\ \varrho(s, \varsigma) \leq 0\}.$$

We generally expect that $\varrho : \mathscr{I} \times \mathscr{B} \to (-\infty, b]$ is continuous. Moreover, we suppose:

(H_ς) The function $t \to \varsigma_t$ is continuous from $\mathcal{R}(\varrho^-)$ into $\mathscr{B}$ and there exists a continuous and bounded function $L^\varsigma : \mathcal{R}(\varrho^-) \to (0, \infty)$ such that

$$\|\varsigma_t\|_{\mathscr{B}} \leq L^\varsigma(t)\|\varsigma\|_{\mathscr{B}} \quad \text{for every} \quad t \in \mathcal{R}(\varrho^-).$$

Lemma 3.4. *[26, Lemma 3.1] If $x : (-\infty, b] \to \mathbb{X}$ is a function such that $x_0 = \varsigma$, then*

$$\|x_s\|_{\mathscr{B}} \leq (\mathscr{D}_2^* + L^\varsigma)\|\varsigma\|_{\mathscr{B}} + \mathscr{D}_1^* \sup\{|x(\theta)| : \theta \in [0, \max\{0, s\}]\}, \quad s \in \mathcal{R}(\varrho^-) \cup \mathscr{I},$$

where $L^\varsigma = \sup\limits_{t \in \mathcal{R}(\varrho^-)} L^\varsigma(t)$.

3.1 Existence results: The convex case

In this section, we are dealing with the existence results for the structure (1.1)-(1.2). We expect that $\mathscr{F}$ is a compact and convex valued multivalued map and we apply Lemma 2.2 to build our first result. Thus, we have:

(H1) The solution operator $S_\alpha(t)_{t \in \mathscr{I}}$ is compact for $t > 0$, and there is $M > 0$ such that

$$\|S_\alpha(t)\|_{B(\mathbb{X})} \leq M, \quad \text{for each} \quad t \in \mathscr{I}.$$

(H2) The multivalued map $\mathscr{F} : \mathscr{I} \times \mathscr{B} \to \mathscr{P}_{cp,c}(\mathbb{X})$ is Carathéodory.

(H3) There exists a continuous function $k : \mathscr{I} \to \mathbb{R}^+$ such that

$$|\mathscr{F}(t, u) - \mathscr{F}(t, v)| \leq k(t)\|u - v\|_{\mathscr{B}}, \quad t \in \mathscr{I},\ u, v, \in \mathscr{B},$$

and

$$k^* = \sup_{t \in \mathscr{I}} \int_0^t k(s)ds < \infty.$$

(H4) The function $t \to \mathscr{F}(t, 0) = \mathscr{F}_0 \in L^1(\mathscr{I}, \mathbb{R}^+)$ with $\mathscr{F}^* = \|\mathscr{F}_0\|_{L^1}$.

(H5) The function $\mathscr{G}(t, \cdot)$ is continuous on $\mathscr{I}$ and there exists a constant $K_{\mathscr{G}} > 0$ such that

$$|\mathscr{G}(t, u) - \mathscr{G}(t, v)| \leq K_{\mathscr{G}}\|u - v\|_{\mathscr{B}}, \quad \text{for each} \quad u, v \in \mathscr{B},$$

and

$$\mathscr{G}^* = \sup_{t \in \mathscr{I}} |\mathscr{G}(t, 0)| < \infty.$$

(H6) For each $t \in \mathscr{I}$ and any bounded set $\mathcal{V} \subset \mathscr{B}$, the set $\{\mathscr{F}(t,u), \mathscr{G}(t,u) : u \in B\}$ is relatively compact in $\mathbb{X}$.

(H7) For any bounded set $\mathcal{V} \subset \mathscr{B}$, the function $\{t \to \mathscr{G}(t, x_{\varrho(t,x_t)}) : x \in \mathcal{V}\}$ is equicontinuous on $\mathscr{I}$.

Theorem 3.1. *Assume that (H1)-(H7) and (H_ς) hold. Then, the problem (1.1)-(1.2) has a mild solution on $(-\infty, b]$ provided that*

$$\left[\mathscr{D}_1^*(K_{\mathscr{G}} + Mk^*)\right] < 1. \tag{3.2}$$

Proof. We will transmute the structure (1.1)-(1.2) into a fixed point problem. We conceive the set

$$\mathcal{V} = \{x : (-\infty, b] \to \mathbb{X} : x|_{\mathscr{I}} \quad \text{is continuous and} \quad x_0 \in \mathscr{B}\},$$

where $x|_{\mathscr{I}}$ is the constraint of x to the real compact interval on $\mathscr{I}$. Recognize the multivalued operator $\Upsilon : \mathcal{V} \to \mathscr{P}(\mathcal{V})$ defined by $\Upsilon(h) = \{h \in \mathcal{V}\}$ with

$$h(t) = \begin{cases} \varsigma(t), & t \le 0; \\ S_\alpha(t)\left[\varsigma(0) - \mathscr{G}(0, \varsigma(0))\right] + \mathscr{G}(t, x_{\varrho(t,x_t)}) + \int_0^t S_\alpha(t-s)v(s)ds, & t \in \mathscr{I}, \end{cases}$$

where $v \in S_{\mathscr{F}, x_{\varrho(s,x_s)}}$. For $\varsigma \in \mathscr{B}$, we express the function $y(\cdot) : (-\infty, b] \to \mathbb{X}$ by

$$y(t) = \begin{cases} \varsigma(t), & t \le 0; \\ S_\alpha(t)\varsigma(0), & t \in \mathscr{I}, \end{cases}$$

then $y_0 = \varsigma$. For every function $z \in \mathcal{V}$ with $z_0 = 0$, we designate by $\overline{z}$ the function clear by

$$\overline{z}(t) = \begin{cases} 0, & t \le 0; \\ z(t), & t \in \mathscr{I}. \end{cases}$$

If $x(\cdot)$ fulfills (3.1), we are able to decompose it as $x(t) = z(t) + y(t)$, $t \in \mathscr{I}$, which suggests $x_t = z_t + y_t$, for each $t \in \mathscr{I}$ and also the function $z(\cdot)$ fulfills

$$z(t) = \mathscr{G}(t, z_{\rho(t,z_t+y_t)} + y_{\rho(t,z_t+y_t)}) - S_\alpha(t)\mathscr{G}(0, \varsigma(0)) + \int_0^t S_\alpha(t-s)v(s)ds, \quad t \in \mathscr{I},$$

where $v(s) \in S_{\mathscr{F}, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}}$.

Let $\mathcal{V}_b^0 = \{z \in \mathcal{V} : z(0) = 0 \in \mathscr{B}\}$. For some $z \in \mathcal{V}_b^0$, we have

$$\|z\|_b = \sup_{t \in \mathscr{I}} \|z(t)\| + \|z_0\|_{\mathscr{B}} = \sup_{t \in \mathscr{I}} \|z(t)\|.$$

Thus $\mathcal{V}_b^0$ is a Banach space with the norm $\|\cdot\|_b$. We delimit the operator $\overline{\Upsilon} : \mathcal{V}_b^0 \to \mathscr{P}(\mathcal{V}_b^0)$ by $\overline{\Upsilon}(z) = \{h \in \mathcal{V}_b^0\}$ with

$$h(t) = \mathscr{G}(t, z_{\rho(t,z_t+y_t)} + y_{\rho(t,z_t+y_t)}) - S_\alpha(t)\mathscr{G}(0, \varsigma(0)) + \int_0^t S_\alpha(t-s)v(s)ds, \quad t \in J,$$

where $v(s) \in S_{\mathscr{F}, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}}$.

We recall that the operator Υ has a fixed point if and only if $\overline{\Upsilon}$ has a fixed point. Thus, let us demonstrate that $\overline{\Upsilon}$ has a fixed point. Let

$$B_r = \{z \in \mathcal{V}_b^0 : z(0) = 0, \|z\|_b \le r\},$$

where r is any fixed real number. It is perfect that B_r is a closed, convex, bounded set in $\mathcal{V}_b^0$.

Remark 3.1. *By hypotheses (H3)-(H5) we obtain:*

(i)

$$\begin{aligned} &M\int_0^t |\mathscr{F}(s, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}) - \mathscr{F}(s,0) + \mathscr{F}(s,0)|ds \\ &\le M\int_0^t k(s)\|z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}\|_{\mathscr{B}}ds + M\int_0^t |\mathscr{F}(s,0)|ds. \end{aligned} \tag{3.3}$$

Since

$$\begin{aligned} \|z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}\|_{\mathscr{B}} &\le \|z_{\varrho(s,z_s+y_s)}\|_{\mathscr{B}} + \|y_{\varrho(s,z_s+y_s)}\|_{\mathscr{B}} \\ &\le \mathscr{D}_1^*|z(s)| + (\mathscr{D}_2^* + L^{\varsigma})\|z_0\|_{\mathscr{B}} + \mathscr{D}_1^*|y(s)| + (\mathscr{D}_2^* + L^{\varsigma})\|\varsigma\|_{\mathscr{B}} \\ &\le \mathscr{D}_1^*|z(s)| + (\mathscr{D}_2^* + L^{\varsigma} + \mathscr{D}_1^* MH)\|\varsigma\|_{\mathscr{B}} \\ &\le \mathscr{D}_1^*|z(s)| + C_1, \end{aligned}$$

where $C_1 = (\mathscr{D}_2^* + L^{\varsigma} + \mathscr{D}_1^* MH)\|\varsigma\|_{\mathscr{B}}$. *Then (3.3) becomes*

$$\begin{aligned} &M\int_0^t |\mathscr{F}(s, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}) - \mathscr{F}(s,0) + \mathscr{F}(s,0)|ds \\ &\le M\int_0^t k(s)\|z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}\|_{\mathscr{B}}ds + M\int_0^t |\mathscr{F}(s,0)|ds \\ &\le M\int_0^t k(s)\big[\mathscr{D}_1^*|z(s)| + C_1\big]ds + M\mathscr{F}^* \\ &\le M\mathscr{D}_1^* r k^* + MC_1 k^* + M\mathscr{F}^*. \end{aligned}$$

(ii)

$$\begin{aligned} &|\mathscr{G}(t, z_{\rho(t,z_t+y_t)} + y_{\rho(t,z_t+y_t)}) - \mathscr{G}(t,0) + \mathscr{G}(t,0)| \\ &\le K_{\mathscr{G}}\|z_{\rho(t,z_t+y_t)} + y_{\rho(t,z_t+y_t)}\|_{\mathscr{B}} + \mathscr{G}^* \\ &\le K_{\mathscr{G}}\mathscr{D}_1^* r + K_{\mathscr{G}} C_1 + \mathscr{G}^*. \end{aligned}$$

(iii)

$$\begin{aligned} &\int_0^{\eta_1} \|S_\alpha(\eta_2 - s) - S_\alpha(\eta_1 - s)\|_{B(\mathbb{X})}|\mathscr{F}(s, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}) - \mathscr{F}(s,0)|ds \\ &\quad + \int_0^{\eta_1} \|S_\alpha(\eta_2 - s) - S_\alpha(\eta_1 - s)\|_{B(\mathbb{X})}|\mathscr{F}(s,0)|ds \\ &\quad + \int_{\eta_1}^{\eta_2} \|S_\alpha(\eta_2 - s)\|_{B(\mathbb{X})}|\mathscr{F}(s, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}) - \mathscr{F}(s,0)|ds \\ &\quad + \int_{\eta_1}^{\eta_2} \|S_\alpha(\eta_2 - s)\|_{B(\mathbb{X})}|\mathscr{F}(s,0)|ds \\ &\le \mathscr{D}_1^* r\int_0^{\eta_1} \|S_\alpha(\eta_2 - s) - S_\alpha(\eta_1 - s)\|_{B(\mathbb{X})}k(s)ds \\ &\quad + C_1\int_0^{\eta_1} \|S_\alpha(\eta_2 - s) - S_\alpha(\eta_1 - s)\|_{B(\mathbb{X})}k(s)ds \\ &\quad + \int_0^{\eta_1} \|S_\alpha(\eta_2 - s) - S_\alpha(\eta_1 - s)\|_{B(\mathbb{X})}|\mathscr{F}(s,0)|ds \end{aligned}$$

$$\begin{aligned}
&+ \mathscr{D}_1^* r \int_{\eta_1}^{\eta_2} \|S_\alpha(\eta_2 - s)\|_{B(\mathbb{X})} k(s) ds \\
&+ C_1 \int_{\eta_1}^{\eta_2} \|S_\alpha(\eta_2 - s)\|_{B(\mathbb{X})} k(s) ds \\
&+ \int_{\eta_1}^{\eta_2} \|S_\alpha(\eta_2 - s)\|_{B(\mathbb{X})} |\mathscr{F}(s, 0)| ds.
\end{aligned}$$

Presently, we might demonstrate that $\overline{\Upsilon}$ fulfills all the assumptions of Bohnenblust-Karlin's theorem. For better comprehensibility, we break the verification into succession of steps.

Step 1: $\overline{\Upsilon}(z)$ is convex for each $z \in \mathcal{V}_b^0$.

In fact, if h_1 and h_2 have a place with $\overline{\Upsilon}(z)$, then there exists $v_1, v_2 \in S_{\mathscr{F}, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}}$ such that, for $t \in \mathscr{I}$, we have

$$h_i(t) = \mathscr{G}(t, z_{\varrho(t,z_t+y_t)} + y_{\varrho(t,z_t+y_t)}) - S_\alpha(t)\mathscr{G}(0, \varsigma(0)) + \int_0^t S_\alpha(t-s) v_i(s) ds, \quad i = 1, 2.$$

Let $0 \leq d \leq 1$. Then, for every $t \in \mathscr{I}$, we have

$$(\lambda h_1 + (1-\lambda)h_2)(t) = \mathscr{G}(t, z_{\varrho(t,z_t+y_t)} + y_{\varrho(t,z_t+y_t)}) - S_\alpha(t)\mathscr{G}(0, \varsigma(0)) + \int_0^t S_\alpha(t-s)\big[\lambda v_1(s) + (1-\lambda) v_2(s)\big] ds.$$

Since $\mathscr{F}$ has convex values, $S_{\mathscr{F}, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}}$ is convex, we see that $(\lambda h_1 + (1-\lambda)h_2) \in \overline{\Upsilon}(z)$.

Step 2: $\overline{\Upsilon}(B_r) \subseteq B_r$ for some $r > 0$.

We assert that there exists a positive number r such that $\overline{\Upsilon}(b_r) \subseteq B_r$. On the off chance that it is not true, then for every positive number r, there exists a function $z_r \in B_r$ and $h \in \overline{\Upsilon}(z_r)$ such that $|h(t)| > r$ for some $t \in \mathscr{I}$. Then from Remark 3.1, we have

$$\begin{aligned}
r < |h(t)| &\leq |\mathscr{G}(t, z_{\varrho(t,z_t+y_t)} + y_{\varrho(t,z_t+y_t)}) - \mathscr{G}(t,0) + \mathscr{G}(t,0)| + \|S_\alpha(t)\|_{B(\mathbb{X})} |\mathscr{G}(0, \varsigma(0))| \\
&\quad + \int_0^t \|S_\alpha(t-s)\|_{B(\mathbb{X})} |\mathscr{F}(s, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}) - \mathscr{F}(s,0) + \mathscr{F}(s,0)| ds \\
&\leq K_{\mathscr{G}} \mathscr{D}_1^* r + K_{\mathscr{G}} C_1 + \mathscr{G}^* + M|\mathscr{G}(0, \varsigma(0))| + M k^* \mathscr{D}_1^* r + M C_1 k^* + M \mathscr{F}^* \\
&\leq K_{\mathscr{G}} \mathscr{D}_1^* r + M k^* \mathscr{D}_1^* r + C_2,
\end{aligned}$$

where $C_2 = K_{\mathscr{G}} C_1 + \mathscr{G}^* + M|\mathscr{G}(0, \varsigma(0))| + M C_1 k^* + M \mathscr{F}^*$ is independent of r. Dividing both sides by r and taking the lower limit, we have

$$\Big[\mathscr{D}_1^* (K_{\mathscr{G}} + M k^*)\Big] \geq 1.$$

This contradicts to (3.2). Hence for some positive number r, $\overline{\Upsilon}(B_r) \subseteq B_r$.

Step 3: $\overline{\Upsilon}(B_r)$ is relatively compact.

We know that B_r is bounded and $\overline{\Upsilon}(B_r) \subseteq B_r$, it is clear that $\overline{\Upsilon}(B_r)$ is bounded. It remains to show that $\overline{\Upsilon}(B_r)$ is equicontinuous.

Let $\eta_1, \eta_2 \in \mathscr{I}$ with $\eta_1 < \eta_2$ and $z \in \overline{\Upsilon}(B_r)$. Then from the remark 3.1 (iii), we have

$$\begin{aligned}
|h(\eta_2) - h(\eta_1)| &\leq |\mathscr{G}(\eta_2, z_{\varrho(\eta_2, z_{\eta_2} + y_{\eta_2})} + y_{\varrho(\eta_2, z_{\eta_2} + y_{\eta_2})}) - \mathscr{G}(\eta_1, z_{\varrho(\eta_1, z_{\eta_1} + y_{\eta_1})} + y_{\varrho(\eta_1, z_{\eta_1} + y_{\eta_1})})| \\
&\quad + \|S_\alpha(\eta_2) - S_\alpha(\eta_1)\|_{B(\mathbb{X})} |\mathscr{G}(0, \varsigma(0))| \\
&\quad + \int_0^{\eta_1} \|S_\alpha(\eta_2 - s) - S_\alpha(\eta_1 - s)\|_{B(\mathbb{X})} |\mathscr{F}(s, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)})| ds \\
&\quad + \int_{\eta_1}^{\eta_2} \|S_\alpha(\eta_2 - s)\|_{B(\mathbb{X})} |\mathscr{F}(s, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)})| ds
\end{aligned}$$

$$\begin{aligned}
&\le |\mathscr{G}(\eta_2, z_{\varrho(\eta_2,z_{\eta_2}+y_{\eta_2})} + y_{\varrho(\eta_2,z_{\eta_2}+y_{\eta_2})}) - \mathscr{G}(\eta_1, z_{\varrho(\eta_1,z_{\eta_1}+y_{\eta_1})} + y_{\varrho(\eta_1,z_{\eta_1}+y_{\eta_1})})| \\
&\quad + \|S_\alpha(\eta_2) - S_\alpha(\eta_1)\|_{B(\mathbb{X})}|\mathscr{G}(0,\varsigma(0))| \\
&\quad + \mathscr{D}_1^* r \int_0^{\eta_1} \|S_\alpha(\eta_2 - s) - S_\alpha(\eta_1 - s)\|_{B(\mathbb{X})} k(s)ds \\
&\quad + C_1 \int_0^{\eta_1} \|S_\alpha(\eta_2 - s) - S_\alpha(\eta_1 - s)\|_{B(\mathbb{X})} k(s)ds \\
&\quad + \int_0^{\eta_1} \|S_\alpha(\eta_2 - s) - S_\alpha(\eta_1 - s)\|_{B(\mathbb{X})} |\mathscr{F}(s,0)|ds \\
&\quad + \mathscr{D}_1^* r \int_{\eta_1}^{\eta_2} \|S_\alpha(\eta_2 - s)\|_{B(\mathbb{X})} k(s)ds \\
&\quad + C_1 \int_{\eta_1}^{\eta_2} \|S_\alpha(\eta_2 - s)\|_{B(\mathbb{X})} k(s)ds \\
&\quad + \int_{\eta_1}^{\eta_2} \|S_\alpha(\eta_2 - s)\|_{B(\mathbb{X})} |\mathscr{F}(s,0)|ds.
\end{aligned}$$

At the point when $\eta_2 \to \eta_1$, the right-hand side of the overhead inequality has a tendency to zero, subsequent to by (H7) and $S_\alpha(t)$ is uniformly continuous, this demonstrates the equicontinuity. As a result of Steps 1-3, together with the Arzela-Ascoli's theorem, we conclude that the operator $\overline{\Upsilon}$ is completely continuous.

Step 4: $\overline{\Upsilon}$ has a closed graph.

Suppose that $z^n \to z^*,\ h_n \in \overline{\Upsilon}(z^n)$ with $h_n \to h_*$. We claim that $h_* \in \overline{\Upsilon}(z^*)$. In fact, assumption $h_n \in \overline{\Upsilon}(z^n)$ suggests that there exists $v_n \in S_{\mathscr{F}, z^n_{\varrho(s,z^n_s+y_s)} + y_{\varrho(s,z^n_s+y_s)}}$ such that, for every $t \in \mathscr{I}$,

$$h_n(t) = \mathscr{G}(t, z^n_{\varrho(t,z^n_t+y_t)} + y_{\varrho(t,z^n_t+y_t)}) - S_\alpha(t)\mathscr{G}(0,\varsigma(0)) + \int_0^t S_\alpha(t-s)v_n(s)ds.$$

We must demonstrate that there exists $v_* \in S_{\mathscr{F}, z^*_{\varrho(s,z^*_s+y_s)} + y_{\varrho(s,z^*_s+y_s)}}$ such that, for each $t \in \mathscr{I}$,

$$h_*(t) = \mathscr{G}(t, z^*_{\varrho(t,z^*_t+y_t)} + y_{\varrho(t,z^*_t+y_t)}) - S_\alpha(t)\mathscr{G}(0,\varsigma(0)) + \int_0^t S_\alpha(t-s)v_*(s)ds.$$

Set

$$\Theta_n(t) = h_n(t) - \mathscr{G}(t, z^n_{\varrho(t,z^n_t+y_t)} + y_{\varrho(t,z^n_t+y_t)}) - S_\alpha(t)\mathscr{G}(0,\varsigma(0)) + \int_0^t S_\alpha(t-s)v_n(s)ds,$$
$$\Theta_*(t) = h_*(t) - G(t, z^*_{\varrho(t,z^*_t+y_t)} + y_{\varrho(t,z^*_t+y_t)}) - S_\alpha(t)\mathscr{G}(0,\varsigma(0)) + \int_0^t S_\alpha(t-s)v_*(s)ds.$$

We have, for every $t \in \mathscr{I}$,

$$\|\Theta_n(t) - \Theta_*(t)\| \to 0 \quad \text{as} \quad n \to \infty.$$

Recognize the linear continuous operator $\Psi : L^1(\mathscr{I}, \mathbb{X}) \to \mathcal{V}_b^0$ defined by

$$\Psi(v)(t) = \int_0^t S_\alpha(t-s)v(s)ds.$$

From Lemma 2.1 and the definition of Ψ, it follows that $\Psi \circ S_{\mathscr{F}}$ is a closed graph operator, and for every $t \in \mathscr{I}$,

$$\Theta_n(t) \in \Psi(S_{\mathscr{F}, z^n_{\varrho(s,z^n_s+y_s)} + y_{\varrho(s,z^n_s+y_s)}}).$$

Since $z^n \to z^*$ and $\Psi \circ S_{\mathscr{F}}$ is a closed graph operator, then there exists $v_* \in S_{\mathscr{F}, z^*_{\varrho(s,z^*_s+y_s)} + y_{\varrho(s,z^*_s+y_s)}}$ such that, for each $t \in \mathscr{I}$,

$$h_*(t) - \mathscr{G}(t, z^*_{\varrho(t,z^*_t+y_t)} + y_{\varrho(t,z^*_t+y_t)}) + S_\alpha(t)\mathscr{G}(0,\varsigma(0)) = \int_0^t S_\alpha(t-s)v_*(s)ds.$$

Hence $h_* \in \overline{\Upsilon}(z^*)$.

As a result of Lemma 2.2, we find that $\overline{\Upsilon}$ has a fixed point z on the interval $(-\infty, b]$. Along these lines, $x = \overline{z} + y$ is a fixed point of the operator Υ which is the mild solution of the structure (1.1)-(1.2). $\square$

Our next result is focused on the Leray-Schauder's alternative fixed point theorem [2]. In order to utilize this theorem, we require the subsequent further hypothesis:

$(H3^*)$ There exists a function $\vartheta \in L^1(\mathscr{I}, \mathbb{R}^+)$ and a continuous non-decreasing function $\Omega : \mathbb{R}^+ \to (0, \infty)$ such that

$$|\mathscr{F}(t, u)| \leq \vartheta(t)\Omega(\|u\|_{\mathscr{B}}) \quad \text{for a.e.} \quad t \in \mathscr{I} \quad \text{and each} \quad u \in \mathscr{B}.$$

If $\mu = 1 - \mathscr{D}_1^* K_{\mathscr{G}} > 0$ and

$$\frac{\mathscr{D}_1^* M}{\mu} \int_0^b \vartheta(s) ds < \int_C^\infty \frac{ds}{\Omega(s)},$$

where $C = C_1 + \frac{\mathscr{D}_1^*}{\mu}\Big[M|\mathscr{G}(0, \varsigma(0))| + K_{\mathscr{G}} C_1 + \mathscr{G}^*\Big]$.

Theorem 3.2. *Assume that (H1), (H2), $(H3^*)$ and (H5)-(H8) are fulfilled. Then, the problem (1.1)-(1.2) has a mild solution on $(-\infty, b]$.*

Proof. Let z be solutions of the inclusion $z \in \lambda \Upsilon(z)$, for any $\lambda \in (0, 1)$, then there exists $v \in S_{F, z_{\varrho(s, z_s + y_s)} + y_{\varrho(s, z_s + y_s)}}$ such that

$$\begin{aligned}
|z(t)| &\leq \|S_\alpha(t)\|_{B(\mathbb{X})} |\mathscr{G}(0, \varsigma(0))| + |\mathscr{G}(t, z_{\varrho(t, z_t + y_t)} + y_{\varrho(t, z_t + y_t)}) - \mathscr{G}(t, 0) + \mathscr{G}(t, 0)| \\
&\quad + \int_0^t \|S_\alpha(t - s)\|_{B(\mathbb{X})} |\mathscr{F}(s, z_{\varrho(s, z_s + y_s)} + y_{\varrho(s, z_s + y_s)})| ds \\
&\leq M|\mathscr{G}(0, \varsigma(0))| + K_{\mathscr{G}} \|z_{\varrho(t, z_t + y_t)} + y_{\varrho(t, z_t + y_t)}\|_{\mathscr{B}} + \mathscr{G}^* \\
&\quad + M \int_0^t \vartheta(s) \Omega(\|z_{\varrho(s, z_s + y_s)} + y_{\varrho(s, z_s + y_s)}\|_{\mathscr{B}}) ds.
\end{aligned}$$

From the remark 3.1, we have

$$\begin{aligned}
|z(t)| &\leq M|\mathscr{G}(0, \varsigma(0))| + K_{\mathscr{G}} C_1 + K_{\mathscr{G}} \mathscr{D}_1^* |z(t)| + \mathscr{G}^* + M \int_0^t \vartheta(s) \Omega(\mathscr{D}_1^* |z(s)| + C_1) ds \\
&\leq \frac{1}{\mu}\Big[M|\mathscr{G}(0, \varsigma(0))| + K_{\mathscr{G}} C_1 + \mathscr{G}^*\Big] + \frac{M}{\mu} \int_0^t \vartheta(s) \Omega(\mathscr{D}_1^* |z(s)| + C_1) ds.
\end{aligned}$$

Then

$$\mathscr{D}_1^* |z(t)| + C_1 \leq C_1 + \frac{\mathscr{D}_1^*}{\mu}\Big[M|\mathscr{G}(0, \varsigma(0))| + K_{\mathscr{G}} C_1 + \mathscr{G}^*\Big] + \frac{\mathscr{D}_1^* M}{\mu} \int_0^t \vartheta(s) \Omega(\mathscr{D}_1^* |z(s)| + C_1) ds.$$

We conceive the function β characterized by

$$\beta(t) = \sup\{\mathscr{D}_1^* |z(s)| + C_1 : 0 \leq s \leq b\}, \quad t \in \mathscr{I}.$$

Let $t^* \in [0, t]$ be such that $\beta(t) = \mathscr{D}_1^* |z(t^*)| + C_1 \|\varsigma\|_{\mathscr{B}}$. By the aforementioned inequality, we sustain

$$\beta(t) \leq C_1 + \frac{\mathscr{D}_1^*}{\mu}\Big[M|\mathscr{G}(0, \varsigma(0))| + K_{\mathscr{G}} C_1 + \mathscr{G}^*\Big] + \frac{\mathscr{D}_1^* M}{\mu} \int_0^t \vartheta(s) \Omega(\beta(s)) ds.$$

Let us occupy the right-hand side of the overhead inequality as $v(t)$, for all $t \in \mathscr{I}$. Then, we sustain

$$\beta(t) \leq v(t), \quad \text{for all} \quad t \in \mathscr{I}.$$

From the meaning of v, we obtain

$$v(0) = C_1 + \frac{\mathscr{D}_1^*}{\mu}\Big[M|\mathscr{G}(0,\varsigma(0))| + K_{\mathscr{G}}C_1 + \mathscr{G}^*\Big] = C$$

and

$$v'(t) = \frac{\mathscr{D}_1^* M}{\mu}\vartheta(t)\Omega(\beta(t)), \quad \text{a.e.} \quad t \in \mathscr{I}.$$

Applying the non-decreasing character of Ω, we conclude

$$v'(t) \leq \frac{\mathscr{D}_1^* M}{\mu}\vartheta(t)\Omega(v(t)), \quad \text{a.e.} \quad t \in \mathscr{I},$$

and hence

$$\int_{v(0)=C}^{v(t)} \frac{ds}{\Omega(s)} \leq \frac{\mathscr{D}_1^* M}{\mu}\int_0^t \vartheta(s)ds \leq \frac{\mathscr{D}_1^* M}{\mu}\int_0^b \vartheta(s)ds < \int_C^\infty \frac{ds}{\Omega(s)}.$$

In this manner, for each $t \in \mathscr{I}$, there exists a constant Λ_* such that $v(t) \leq \Lambda_*$ and henceforth $\beta(t) \leq \Lambda_*$. Since $\|z\|_{\mathscr{B}} \leq \beta(t)$, we have $\|z\|_{\mathscr{B}} \leq \Lambda_*$. Set

$$\mathcal{U} = \{z \in \mathcal{V}_b^0 : \|z\|_\infty < \Lambda_* + 1\}.$$

From Theorem 3.1, we realize that the operator $\overline{\Upsilon} : \overline{\mathcal{U}} \to \overline{\Upsilon}(z)$ is completely continuous. Besides, from the decision of $\mathcal{U}$, there is no $z \in \partial\mathcal{U}$ such that $z = \lambda\overline{\Upsilon}(z)$, for $\lambda \in (0,1)$. As an outcome of the nonlinear alternative of Leray-Schauder type [2], we conclude that Υ has a fixed point z in $\mathcal{U}$, then the structure (1.1)-(1.2) has at least one mild solution on $(-\infty, b]$. □

3.2 Existence results: Nonconvex case

The next step is to demonstrate the existence results for the structure (1.1)-(1.2). Our result is focused around the Lemma 2.3.

Theorem 3.3. *Assume that the subsequent hypotheses hold:*

(H8) $\mathscr{F} : \mathscr{I} \times \mathscr{B} \to \mathscr{P}_{cp}(\mathbb{X})$ *has the assets that* $\mathscr{F}(\cdot, u) : \mathscr{I} \to \mathscr{P}_{cp}(\mathbb{X})$ *is measurable, for each* $u \in \mathscr{B}$.

(H9) There exists $\wp \in L^1(\mathscr{I}, \mathbb{R}^+)$ *such that*

$$H_d(\mathscr{F}(t,u), \mathscr{F}(t,\overline{u})) \leq \wp(t)\|u - \overline{u}\|_{\mathscr{B}}, \quad \textit{for every} \quad u, \overline{u} \in \mathscr{B},$$

and

$$d(0, \mathscr{F}(t,0)) \leq \wp(t) \quad \textit{a.e.} \quad t \in \mathscr{I}.$$

(H10) There exists a positive constant $L_* > 0$ *such that*

$$|\mathscr{G}(t,u) - \mathscr{G}(t,\overline{u})| \leq L_*\|u - \overline{u}\|_{\mathscr{B}}, \quad \textit{a.e} \quad t \in \mathscr{I} \quad \textit{and for all} \quad u.\overline{u} \in \mathscr{B}.$$

Then the problem (1.1)-(1.2) has at least one mild solution on $(-\infty, b]$.

Remark 3.2. *For every* $z \in \mathcal{V}_b^0$, *the set* $S_{\mathscr{F},z}$ *is nonempty, since, by (H8),* $\mathscr{F}$ *has a measurable choice [43, Theorem III.6].*

Proof. Let $\overline{\Upsilon} : \mathcal{V}_b^0 \to \mathscr{P}(\mathcal{V}_b^0)$, where $\overline{\Upsilon}$ is defined in Theorem 3.1 be solutions of the problem (1.1)-(1.2). Presently, we might demonstrate that the operator $\overline{\Upsilon}$ fulfills all the states of Lemma 2.3. For our comfort, we split up the proof into two steps:

Step 1: $\overline{\Upsilon}(z) \in \mathscr{P}_{cl}(\mathcal{V}_b^0)$ for all $z \in \mathcal{V}_b^0$.

In fact, let $(z^n)_{n\geq 0} \in \overline{\Upsilon}(z)$ be such that $z^n \to \tilde{z} \in \mathcal{V}_b^0$. Then $\tilde{z} \in \mathcal{V}_b^0$ and there exists $v_n \in S_{\mathscr{F}, z_{\varrho(s,z_s+y_s)}+y_{\varrho(s,z_s+y_s)}}$ such that, for every $t \in \mathscr{I}$,

$$z^n(t) = \mathscr{G}(t, z_{\varrho(t,z_t+y_t)} + y_{\varrho(t,z_t+y_t)}) - S_\alpha(t)\mathscr{G}(0, \varsigma(0)) + \int_0^t S_\alpha(t-s)v_n(s)ds.$$

Utilizing the way that $\mathscr{F}$ has compact values and from (H9), we may go to a subsequence if important to get that v_n converges to v in $L^1(\mathscr{I}, \mathbb{X})$ and consequently $v \in S_{\mathscr{F}, z_{\varrho(s,z_s+y_s)}+y_{\varrho(s,z_s+y_s)}}$. Then, for every $t \in \mathscr{I}$,

$$z^n(t) \to \tilde{z} = \mathscr{G}(t, z_{\varrho(t,z_t+y_t)} + y_{\varrho(t,z_t+y_t)}) - S_\alpha(t)\mathscr{G}(0, \varsigma(0)) + \int_0^t S_\alpha(t-s)v(s)ds.$$

So $\tilde{z} \in \overline{\Upsilon}(z)$.

Step 2: There exists $\Lambda < 1$ such that

$$H_d(\mathscr{F}(z), \mathscr{F}(\overline{z})) \leq \Lambda \|z - \overline{z}\|_\infty \quad \text{for all} \quad z, \overline{z} \in \mathcal{V}_b^0.$$

Let $z, \overline{z} \in \mathcal{V}_b^0$ and $h \in \overline{\Upsilon}(z)$. Then there exists $v(t) \in \mathscr{F}(t, z_{\varrho(t,z_t+y_t)} + y_{\varrho(t,z_t+y_t)})$ such that, for every $t \in \mathscr{I}$,

$$h(t) = \mathscr{G}(t, z_{\varrho(t,z_t+y_t)} + y_{\varrho(t,z_t+y_t)}) - S_\alpha(t)\mathscr{G}(0, \varsigma(0)) + \int_0^t S(t-s)v(s)ds.$$

From (H9), it takes after that

$$H_d\Big(\mathscr{F}(t, z_{\varrho(t,z_t+y_t)} + y_{\varrho(t,z_t+y_t)}), \mathscr{F}(t, \overline{z}_{\varrho(t,z_t+y_t)} + y_{\varrho(t,z_t+y_t)})\Big) \leq \wp(t)\|z_{\varrho(t,z_t+y_t)} - \overline{z}_{\varrho(t,z_t+y_t)}\|_{\mathscr{B}}.$$

Therefore, there is $w \in \mathscr{F}(t, \overline{z}_{\varrho(t,z_t+y_t)} + y_{\varrho(t,z_t+y_t)})$ such that

$$|v(t) - w| \leq \wp(t)\|z_{\varrho(t,z_t+y_t)} - \overline{z}_{\varrho(t,z_t+y_t)}\|_{\mathscr{B}}, \quad t \in \mathscr{I}.$$

Recognize $U : \mathscr{I} \to \mathscr{P}(\mathbb{X})$ specified by

$$U(t) = \{w \in \mathbb{X} : |v(t) - w| \leq \wp(t)\|z_{\varrho(t,z_t+y_t)} - \overline{z}_{\varrho(t,z_t+y_t)}\|_{\mathscr{B}}\}.$$

Since the multivalued operator $V(t) = U(t) \bigcap \mathscr{F}(t, \overline{z}_{\varrho(t,z_t+y_t)} + y_{\varrho(t,z_t+y_t)})$ is measurable [43, Proposition III.4], there exists a function $\overline{v}(t)$, which is measurable choice v. Along these lines, $\overline{v}(t) \in \mathscr{F}(t, \overline{z}_{\varrho(t,z_t+y_t)} + y_{\varrho(t,z_t+y_t)})$, and utilizing phase space axioms, for every $t \in J$, we obtain

$$\begin{aligned}|v(t) - \overline{v}(t)| &\leq \wp(t)\|z_{\varrho(t,z_t+y_t)} - \overline{z}_{\varrho(t,z_t+y_t)}\|_{\mathscr{B}}\\ &\leq \wp(t)\mathscr{D}_1^*|z(t) - \overline{z}(t)|.\end{aligned}$$

For every $t \in \mathscr{I}$, give us a chance to characterize

$$\overline{h}(t) = \mathscr{G}(t, \overline{z}_{\varrho(t,z_t+y_t)} + y_{\varrho(t,z_t+y_t)}) - S_\alpha(t)\mathscr{G}(0, \varsigma(0)) + \int_0^t S(t-s)\overline{v}(s)ds.$$

Then, for every $t \in \mathscr{I}$,

$$\begin{aligned}|h(t) - \overline{h}(t)| \leq\ & |\mathscr{G}(t, z_{\varrho(t,z_t+y_t)} + y_{\varrho(t,z_t+y_t)}) - \mathscr{G}(t, \overline{z}_{\varrho(t,z_t+y_t)} + y_{\varrho(t,z_t+y_t)})|\\ &+ \int_0^t \|S_\alpha(t-s)\|_{B(\mathbb{X})}|v(s) - \overline{v}(s)|ds\\ \leq\ & L_*\|z_{\varrho(t,z_t+y_t)} - \overline{z}_{\varrho(t,z_t+y_t)}\|_{\mathscr{B}} + M\mathscr{D}_1^*\int_0^t \wp(s)|z(s) - \overline{z}(s)|ds\\ \leq\ & L_*\mathscr{D}_1^*|z(t) - \overline{z}(t)| + \int_0^t \overline{\wp}(s)|z(s) - \overline{z}(s)|ds\end{aligned}$$

$$\begin{aligned}
&\leq \Big[L_* \mathscr{D}_1^* e^{\tau L(t)}\Big]\Big[e^{-\tau L(t)}|z(t)-\overline{z}(t)|\Big] + \int_0^t \Big[\overline{\wp}(s) e^{\tau L(s)}\Big]\Big[e^{-\tau L(s)}|z(s)-\overline{z}(s)|\Big] ds \\
&\leq L_* \mathscr{D}_1^* e^{\tau L(t)} \|z-\overline{z}\|_{\overline{\mathcal{V}}} + \|z-\overline{z}\|_{\overline{\mathcal{V}}} \int_0^t \Big[\frac{e^{\tau L(s)}}{\tau}\Big]' ds \\
&\leq \Big[L_* \mathscr{D}_1^* + \frac{1}{\tau}\Big] e^{\tau L(t)} \|z-\overline{z}\|_{\overline{\mathcal{V}}},
\end{aligned}$$

where $\tau > 0$, $L(t) = \displaystyle\int_0^t \overline{\wp}(s)ds$, $\overline{\wp}(t) = M\mathscr{D}_1^* \wp(t)$, and $\|\cdot\|_{\overline{\mathcal{V}}}$ is the Bielecki-type norm on $\mathcal{V}_b^0$ defined by $\|z\|_{\overline{\mathcal{V}}} = \sup\{e^{-\tau L(t)}\|z(t)\| : t \in \mathscr{I}\}$.

Thus, we obtain

$$\|h-\overline{h}\|_{\overline{\mathcal{V}}} \leq \Big[L_* \mathscr{D}_1^* + \frac{1}{\tau}\Big]\|z-\overline{z}\|_{\overline{\mathcal{V}}}.$$

By exchanging the parts of z and $\overline{z}$, we have

$$H_d(\overline{\Upsilon}(z), \overline{\Upsilon}(\overline{z})) \leq \Big[L_* \mathscr{D}_1^* + \frac{1}{\tau}\Big]\|z-\overline{z}\|_{\overline{\mathcal{V}}}.$$

Settling $\tau > 0$ and for $\big[L_* \mathscr{D}_1^* + \frac{1}{\tau}\big] < 1$, implies $\overline{\Upsilon}$ is a contraction, and by Lemma 2.3, it has a fixed point z, which represents a mild solution (1.1)-(1.2). $\square$

4 Application

We consider the FNIDI with SDD, namely:

$$\begin{aligned}
\frac{\partial}{\partial t}\Big[u(t,\xi) - g(t, u(t-\sigma(u(t,0)),\xi))\Big] &\in \int_t^0 \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\Big(\frac{\partial^2}{\partial \xi^2} - r\Big)\Big[u(s,\xi) - g(s, u(s-\sigma(u(s,0)),\xi))\Big] ds \\
&+ \Big[f_1(t, u(t-\sigma(u(t,0)),\xi)), f_2(t, u(t-\sigma(u(t,0)),\xi))\Big], \quad 0 \leq t \leq b, \quad 0 \leq \xi \leq \pi, \qquad (4.1)
\end{aligned}$$

$$u(t,0) = u(t,\pi) = 0, \quad t \in \mathscr{I}, \qquad (4.2)$$

$$u(\theta,\xi) = u_0(\theta,\xi), \quad \theta \in (-\infty, 0], \quad \xi \in \mathscr{J} = [0,\pi], \qquad (4.3)$$

where $1 < \alpha < 2$, $(u_0, \sigma) \in C(\mathbb{R}, [0,\infty))$, $L_\xi = \Big(\dfrac{\partial^2}{\partial \xi^2} - r\Big)$, $r > 0$ stands for the operator with respect to the special variable ξ, $f_1, f_2 : \mathscr{I} \times \mathscr{B} \to \mathbb{R}$ are measurable in t and continuous in x, and $g : \mathscr{I} \times \mathscr{B} \to \mathbb{R}$ are appropriate functions. We expect that for every $t \in \mathscr{I}$, $f_1(t,\cdot)$ is lower semicontinuous , and assume that for each $t \in \mathscr{I}$, $f_2(t,\cdot)$ is upper semicontinuous.

Consider $\mathbb{X} = L^2([0,\pi],\mathbb{R})$ and the operator $\mathscr{A} : L_\xi : D(\mathscr{A}) \subset \mathbb{X} \to \mathbb{X}$ with domain

$$D(\mathscr{A}) = \{u \in \mathbb{X} : u'' \in \mathbb{X}, \quad u(0) = u(\pi) = 0\}.$$

$\mathscr{A}$ is densely defined in $\mathbb{X}$ and is sectorial. As a result $\mathscr{A}$ represents a generator of a solution operator on $\mathbb{X}$. For the phase space, we pick $\mathscr{B} = C_\gamma = \{\varsigma \in C((-\infty,0] : \mathbb{X}) : \lim_{\theta \to -\infty} e^{\gamma\theta}\varsigma(\theta) \quad \text{exists in } \mathbb{X}\}$ invested with the norm

$$|\varsigma| = \sup_{-\infty<\theta\leq 0} e^{\gamma\theta}|\varsigma(\theta)|.$$

We note that the phase space C_γ satisfies the conditions $(P_1), (P_2)$ and (P_3). Set

$$\begin{aligned}
x(t)(\xi) &= u(t,\xi), \quad t \in \mathscr{I}, \quad \xi \in \mathscr{J}, \\
\varsigma(0)(\xi) &= u_0(\theta,\xi), \quad t \in \mathscr{I}, \quad \theta \leq 0, \\
\mathscr{G}(t,\varsigma)(\xi) &= g(t, \varsigma(0,\xi)), \quad t \in \mathscr{I}, \quad \xi \in \mathscr{J},
\end{aligned}$$

$$\mathscr{F}(t,\varsigma)(\xi)=\Big[f_1(t,\varsigma(0,\xi)),f_2(t,\varsigma(0,\xi))\Big],\quad t\in\mathscr{I},\quad \xi\in\mathscr{J},$$
$$\varrho(t,\varsigma)=t-\sigma(\varsigma(0,0)).$$

The multivalued map $\mathscr{F}$ is u.s.c. with compact convex values [43]. Hence (H1) and (H2) are satisfied.

At this stage, the existence of mild solutions can be reasoned from an immediate application of Theorem 3.2.

Theorem 4.1. *Let $\varsigma\in C_\gamma$ be such that (H_ς) holds, and let $t\to\varsigma_t$ be continuous on $\mathcal{R}(\varrho^-)$. Moreover, we assume that $(H3^*)$ is fulfilled. Thus, there exists at least one mild solution of (4.1)-(4.3).*

Corollary 4.1. *Let $\varsigma\in C_\gamma$ be continuous and bounded and assume that $(H3^*)$ holds. Thus, there exists at least one mild solution of (4.1)-(4.3) on $(-\infty,b]$.*

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NEW HERMITE–HADAMARD'S INEQUALITIES FOR PREINVEX FUNCTIONS VIA FRACTIONAL INTEGRALS

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Abstract. In this paper,we have established some Hermite− Hadamard inequalities for preinvex functions via fractional integrals and these results have some relationship with the obtained results. Application of the obtained results are given as well.

1. Introduction

The usefulness of inequalities involving convex functions is realized from the very beginning and is now widely acknowledged as one of the prime driving forces behind the development of several modern branches of mathematics and has been given considerable attention. One of the most famous inequalities for convex functions is Hermite−Hadamard's inequality, stated as [12]:

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

Both inequalities hold in the reversed direction for f to be concave.

In [19] Pearce and J. Pecaric established the following result connected with the right part of (1).

Theorem 1. *Let $f : I^0 \subseteq R \to R$ is a differentiable function on $I^0 \quad a, b \in I^0$ with $a < b$,and If $|f'|^q$ is convex function on $[a, b]$,for some fixed $q \geq 1$.then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int\limits_a^b f(x)dx\right| \leq \frac{b-a}{4}\left[\frac{|f'(a)|^q + |f'(b)|^q}{2}\right]^{\frac{1}{q}}.$$

If $|f'|^q$ is concave function on $[a, b]$, for some fixed $q \geq 1$.then

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int\limits_a^b f(x)dx\right| \leq \frac{b-a}{4}\left|f'\left(\frac{a+b}{2}\right)\right|.$$

Some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper.

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Definition 1. *Let $f \in L^1[a,b]$. The left−sided and right−sided Riemann−Liouville fractional integrals of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \qquad a < x$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \qquad x < b$$

respectively, where $\Gamma(.)$ is Gamma function and its definition is $\Gamma(\alpha) = \int_0^{\infty} e^{-u} u^{\alpha-1} du$. It is to be noted that $J_{a+}^{0} f(x) = J_{b-}^{0} f(x) = f(x)$.

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral. Properties concerning this operator can be found in [13] and for some recent results connected with fractional integral inequalities, see [8], [9], [10], [17], [22],[24]. Hermite−Hadamard Inequality has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found in [11], [15], [16], [19] and the references cited therein.

In [24] Sarikaya et. al. proved a variant of Hermite−Hadamard's inequalities in fractional integral forms as follows:

Theorem 2. *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L^1[a,b]$. If f is convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{2}$$

with $\alpha > 0$

Remark 1. *For $\alpha = 1$, inequality (2) reduces to inequality (1).*

Using the following identity Sarikaya et. al. established the following result which hold for convex functions.

Lemma 1. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b), with $a < b$ and $f' \in L[a,b]$, then the following identity holds:*

$$\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] = \frac{b-a}{2} \int_0^{1/2} \left[(1-t)^{\alpha} - t^{\alpha}\right] f'(ta+(1-t)b)dt, \tag{3}$$

Theorem 3. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b), with $a < b$. If $|f'|$ is convex function on $[a,b]$, then the following inequalities for fractional integrals holds:*

$$\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] \leq \frac{b-a}{2(\alpha+1)} \left(1 - \frac{1}{2^{\alpha}}\right) \left(|f'(a)| + |f'(b)| \right). \tag{4}$$

In recent years several extensions and generalizations have been considered for classical convexity. A significant generalization of convex functions is that of invex functions introduced by Hanson in [14]. Weir and Mond [18] introduced the concept of preinvex functions and applied it to the establishment of the sufficient

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optimality conditions and duality in nonlinear programming. Pini [20] introduced the concept of prequasiinvex as a generalization of invex functions. Later, Mohan and Neogy[18] obtained some properties of generalized preinvex functions. Noor [2]-[4] has established some Hermite-Hadamard type inequalities for preinvex and log-preinvex functions. In recent papers Barani et al. in [5] presented some estimates of the right hand side of a Hermite-Hadamard type inequality in which some preinvex functions are involved. In this paper we generalized the results in [15] and [16] for preinvex functions via fractional integrals. Let K be a closed set R^n and let $f : K \to \mathrm{R}$ and $\eta : K \times K \to \mathrm{R}$ be continuous functions. Let $x \in K$, then the set K is said to be invex at x with respect to $\eta(.,.)$,

If

$$x + t\eta(y, x) \in K, \forall x, y \in K, t \in [0, 1] .$$

K is said to be an invex set with respect to η if K is invex at each $x \in K$. The invex set K is also called a η-connected set.

Definition 2. *The function f on the invex set K is said to be preinvex with respect to η, if*

$$f(u + t\eta(v, u)) \leq +(1 - t) f(u) + tf(v), \forall u, v \in K, t \in [0, 1] .$$

The function f is said to be preconcave if and only if $-f$ is preinvex.

In the recent paper, Noor [4] has obtained the following Hermite-Hadamard inequalities for the preinvex functions:

Theorem 4. *Let $f : [a, a + \eta(b, a)] \to (0, \infty)$ be a open preinvex function on the interval of real numbers K^0 (the interior of K^0) and $a, b \in K^0$ with $a < a+\eta(b, a)$. the following inequality holds:*

$$f\left(\frac{2a + \eta(b, a)}{2}\right) \leq \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(x)dx \leq \frac{f(a) + f(b)}{2}.$$

Barani, Ghazanfari and Dragomir in [5], presented the following estimates of the right-side of a Hermite-Hadamard type inequality in which some preinvex functions are involved.

Theorem 5. *Let $K \subseteq R$ be an open invex subset with respect to $\eta : K \times K \to R$. Suppose that $f : K \to R$ is a differentiable function. If $|f'|$ is preinvex on K, then for every $a, b \in K$ with $\eta(b, a) \neq 0$ the following inequality holds:*

$$\left|\frac{f(a) + f(a + \eta(b, a))}{2} - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(x)dx\right| \leq \frac{\eta(b, a)}{8} \{|f'(a)| + |f'(b)|\} .$$

Theorem 6. *Let $K \subseteq R$ be an open invex subset with respect to $\eta : K \times K \to R$. Suppose that $f : K \to R$ is a differentiable function. Assume $p \in R$ with $p > 1$. If $|f'|^{\frac{p}{(p-1)}}$ is preinvex on K, then for every $a, b \in K$ with $\eta(b, a) \neq 0$ the following inequality holds:*

$$\left|\frac{f(a) + f(a + \eta(b, a))}{2} - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(x)dx\right| \leq \frac{\eta(b, a)}{2(1 + p)^{1/p}} \left\{\frac{|f'(a)|^{\frac{p}{(p-1)}} + |f'(b)|^{\frac{p}{(p-1)}}}{2}\right\}^{\frac{p-1}{p}} .$$

The aim of this paper is to establish left Hermite−Hadamard type inequalities for Riemann−Liouville fractional integral using the identity obtained for fractional integrals.

2. Main Results

In order to obtain our results, we modified [15, Lemma 2.1] as following:

Lemma 2. *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \to R$ and $a, b \in A$ with $a < a + \eta(b,a)$ .Suppose that $f : A \to \mathbb{R}$ be a differentiable function.If $f'$is preinvex function on A and $f' \in L[a, a+\eta(b,a)]$, then the following identity for Riemann−Liouville fractional integrals holds:*

$$f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J^{\alpha}_{a+}f(a+\eta(b,a)) + J^{\alpha}_{a+\eta(b,a)-}f(a)\right] = \frac{\eta(b,a)}{2}\sum_{k=1}^{4} I_k, \quad (5)$$

where

$$I_1 = \int_0^{1/2} t^{\alpha} f'(a+t\eta(b,a))dt, \qquad I_2 = \int_0^{1/2} (-t^{\alpha}) f'(b+t\eta(a,b))dt,$$
$$I_3 = \int_{1/2}^{1} (t^{\alpha}-1) f'(a+t\eta(b,a))dt, \quad I_4 = \int_{1/2}^{1} (1-t^{\alpha}) f'(b+t\eta(a,b))dt.$$

Proof. Integrating by parts

$$\begin{aligned} I_1 &= \int_0^{1/2} t^{\alpha} f'(a+t\eta(b,a))dt \\ &= \frac{t^{\alpha} f(a+t\eta(b,a))dt}{\eta(b,a)}\bigg|_0^{1/2} - \frac{\alpha}{\eta(b,a)}\int_0^{1/2} t^{\alpha-1} f(a+t\eta(b,a))dt \\ &= \frac{2^{-\alpha}}{\eta(b,a)} f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{\alpha}{\eta(b,a)}\int_0^{1/2} t^{\alpha-1} f(a+t\eta(b,a))dt. \end{aligned}$$

Analogously:

$$I_2 = \frac{2^{-\alpha}}{\eta(b,a)} f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{\alpha}{\eta(b,a)}\int_0^{1/2} t^{\alpha-1} f(b+t\eta(a,b))dt$$

and

$$\begin{aligned} I_3 &= \int_{1/2}^{1} (t^{\alpha}-1) f'(a+t\eta(b,a))dt \\ &= \frac{(t^{\alpha}-1) f((a+t\eta(b,a))}{\eta(b,a)}\bigg|_{1/2}^{1} - \frac{\alpha}{\eta(b,a)}\int_{1/2}^{1} t^{\alpha-1} f(a+t\eta(b,a))dt \\ &= \frac{1-2^{-\alpha}}{\eta(b,a)} f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{\alpha}{\eta(b,a)}\int_{1/2}^{1} t^{\alpha-1} f(a+t\eta(b,a))dt. \end{aligned}$$

Analogously:

$$I_4 = \frac{1-2^{-\alpha}}{\eta(b,a)} f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{\alpha}{\eta(b,a)}\int_{1/2}^{1} t^{\alpha-1} f(b+t\eta(a,b))dt$$

Adding above equalities, we get

$$\frac{2}{\eta(b,a)}f\left(\frac{2a+\eta(b,a)}{2}\right)-\frac{\alpha}{\eta(b,a)}\left[\int_0^1 t^{\alpha-1}f(a+t\eta(b,a))dt+\int_0^1 t^{\alpha-1}f((b+t\eta(a,b))dt\right]$$
$$=I_1+I_2+I_3+I_4.$$

Now making substitution $u=(a+t\eta(b,a))$, we have

$$\begin{aligned}\int_0^1 t^{\alpha-1}f(a+t\eta(b,a))dt &= \frac{1}{\eta^{\alpha}(b,a)}\int_a^{a+\eta(b,a)}(u-a)^{\alpha-1}f(u)du\\ &= \frac{\Gamma(\alpha)}{\eta^{\alpha}(b,a)}J_{b^-}^{\alpha}f(a),\end{aligned}$$

likewise

$$\int_0^{1/2} t^{\alpha-1}f(b+t\eta(a,b))=\frac{\Gamma(\alpha)}{\eta^{\alpha}(b,a)}J_{a^+}^{\alpha}f(a+\eta(b,a)),$$

which completes our proof. □

New upper bound for the left-hand side of (2) for convex functions is proposed in the following theorem.

Theorem 7. *Let $A\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta:A\times A\to R$ and $a,b\in A$ with $a<a+\eta(b,a)$. Such that $f'\in L[a,a+\eta(b,a)]$. Suppose that $f:A\to\mathbb{R}$ be a differentiable function. If $|f'|$ is preinvex function on A then the following inequality for fractional integrals holds for $0<\alpha\leq 1$:*

$$\left|f\left(\frac{2a+\eta(b,a)}{2}\right)-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J_{a^+}^{\alpha}f(a+\eta(b,a))+J_{a+\eta(b,a)^-}^{\alpha}f(a)\right]\right|$$
$$\leq\frac{\eta(b,a)}{2^{\alpha+1}(\alpha+1)}(\ |f'(a)|+|f'(b)|\ )\quad(6)$$

Proof. By using the properties of modulus on Lemma 2, we have

$$\left|f\left(\frac{2a+\eta(b,a)}{2}\right)-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J_{a^+}^{\alpha}f(a+\eta(b,a))+J_{a+\eta(b,a)^-}^{\alpha}f(a)\right]\right|\leq\frac{\eta(b,a)}{2}\sum_{k=1}^{4}|I_k|.$$

Now, using preinvexity of $|f'|$, we have

$$\begin{aligned}|I_1| &\leq \int_0^{1/2}t^{\alpha}|f'(a+t\eta(b,a))|dt\leq\int_0^{1/2}t^{\alpha}|f'(1-t)a+tb|dt\\ &\leq |f'(a)|\int_0^{1/2}t^{\alpha}(1-t)dt+|f'(b)|\int_0^{1/2}t^{\alpha+1}dt\\ &= \frac{\alpha+3}{2^{\alpha+2}(\alpha+1)(\alpha+2)}|f'(a)|+\frac{1}{2^{\alpha+2}(\alpha+2)}|f'(b)|.\end{aligned}$$

Analogously:

$$|I_2|\leq\frac{\alpha+3}{2^{\alpha+2}(\alpha+1)(\alpha+2)}|f'(b)|+\frac{1}{2^{\alpha+2}(\alpha+2)}|f'(a)|.$$

By using preinvexity on $|f'|$ and fact that for $\alpha\in(0,1]$ and $\forall\ t_1,t_2\in[0,1]$,

$$|{t_1}^{\alpha}-{t_2}^{\alpha}|\leq|t_1-t_2|^{\alpha},$$

$$
\begin{aligned}
|I_3| &\leq |f'(a)| \int_{1/2}^{1} (1-t^{\alpha})(1-t)\ dt + |f'(b)| \int_{1/2}^{1} (1-t^{\alpha})t\ dt \\
&\leq |f'(a)| \int_{1/2}^{1} (1-t)^{\alpha+1}\ dt + |f'(b)| \int_{1/2}^{1} (t-t^{\alpha+1})\ dt \\
&= \frac{1}{2^{\alpha+2}(\alpha+2)}|f'(a)| + \frac{\alpha+3}{2^{\alpha+2}(\alpha+1)(\alpha+2)}|f'(b)|,
\end{aligned}
$$

similarly

$$
|I_4| \leq \frac{1}{2^{\alpha+2}(\alpha+2)}|f'(b)| + \frac{\alpha+3}{2^{\alpha+2}(\alpha+1)(\alpha+2)}|f'(a)|,
$$

which completes the proof.

Corollary 1. *If we take $\eta(b,a) = b-a$ in Theorem 7, then inequality* (6) *becomes inequality as*

$$
\left| f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a) \right] \right| \leq \frac{(b-a)}{2^{\alpha+1}(\alpha+1)} (\ |f'(a)| + |f'(b)|\ ). \quad (7)
$$

□

Remark 2. *If we take $\alpha = 1$, in Corollary 1 then inequality* (7) *becomes inequality as obtained in* [15, *Theorem* 2.2] .

The corresponding version for powers of the absolute value of the derivative is incorporated in the following theorem.

Theorem 8. *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \to R$ and $a, b \in A$ with $a < a + \eta(b,a)$ .Such that $f' \in L[a, a+\eta(b,a)]$.Suppose that $f : A \to \mathbb{R}$ be a differentiable function. If $|f'|^{\frac{p}{p-1}}$ is preinvex function on A for some fixed $p \geq 1$ with $q = \frac{p}{p-1}$, then the following inequality for fractional integrals holds for $0 < \alpha \leq 1$:*

$$
\begin{aligned}
&\left| f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)} \left[J_{a+}^{\alpha} f(a+\eta(b,a)) + J_{a+\eta(b,a)-}^{\alpha} f(a) \right] \right| \\
&\quad \leq \frac{\eta(b,a)}{2^{\alpha+1}(\alpha p+1)^{1/p}} \left[\left(\frac{|f'(b)|^q + 3|f'(a)|^q}{4} \right)^{1/q} + \left(\frac{|f'(a)|^q + 3|f'(b)|^q}{4} \right)^{1/q} \right]. \quad (8)
\end{aligned}
$$

Proof. From Lemma 2 and using Hölder inequality with properties of modulus, we have

$$
\left| f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a+}^{\alpha} f(a+\eta(b,a)) + J_{a+\eta(b,a)-}^{\alpha} f(a) \right] \right| \leq \frac{\eta(b,a)}{2} \sum_{k=1}^{4} |I_k|.
$$

By using the convexity of $|f'|^q$, we have

$$
\begin{aligned}
|I_1| &\leq \left(\int_0^{1/2} t^{\alpha p} dt \right)^{1/p} \left(\int_0^{1/2} |f'(a+t\eta(b,a))|^q\ dt \right)^{1/q} \\
&\leq \left(\frac{1}{2^{\alpha p+1}(\alpha p+1)} \right)^{1/p} \left(|f'(a)|^q \int_0^{1/2} (1-t) dt + |f'(b)|^q \int_0^{1/2} t\ dt \right)^{1/q} \\
&= \left(\frac{1}{2^{\alpha p+1}(\alpha p+1)} \right)^{1/p} \left(\frac{|f'(b)|^q + 3|f'(a)|^q}{8} \right)^{1/q},
\end{aligned}
$$

similarly

$$|I_2| \leq \left(\frac{1}{2^{\alpha p+1}(\alpha p+1)}\right)^{1/p}\left(\frac{3|f'(b)|^q+|f'(a)|^q}{8}\right)^{1/q},$$

now

$$|I_3| \leq \left(\int_{1/2}^{1}(1-t^{\alpha})^p dt\right)^{1/p}\left(\int_{1/2}^{1}\left|f'\left(tb+(1-t)a\right)\right|^q dt\right)^{1/q}.$$

Let $\alpha \in (0,1]$ and $\forall\, t_1, t_2 \in [0,1]$,

$$|t_1{}^{\alpha} - t_2{}^{\alpha}| \leq |t_1 - t_2|^{\alpha},$$

therefore

$$\int_{1/2}^{1}(1-t^{\alpha})^p dt \leq \int_{1/2}^{1}(1-t)^{\alpha p} dt = \frac{1}{2^{\alpha p+1}(\alpha p+1)}$$

Hence

$$|I_3| \leq \left(\frac{1}{2^{\alpha p+1}(\alpha p+1)}\right)^{1/p}\left(\frac{|f'(b)|^q+3|f'(a)|^q}{8}\right)^{1/q},$$

and

$$|I_4| \leq \left(\frac{1}{2^{\alpha p+1}(\alpha p+1)}\right)^{1/p}\left(\frac{3|f'(b)|^q+|f'(a)|^q}{8}\right)^{1/q},$$

which completes the proof. ∎

Corollary 2. *.If we take $\eta(b,a) = b-a$ in Theorem 8, then inequality* (8) *becomes inequality* (2.1) *of* [16, Theorem 2.3]

$$\left|f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b) + J_{b^-}^{\alpha}f(a)\right]\right|$$
$$\leq \frac{(b-a)}{2^{\alpha+1}(\alpha p+1)^{1/p}}\left[\left(\frac{|f'(b)|^q+3|f'(a)|^q}{4}\right)^{1/q} + \left(\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{1/q}\right]. \quad (9)$$

Remark 3. *If we take $\alpha = 1$, in Corollary 2 then inequality* (9) *becomes inequality* (2.1) *of* [15, *Theorem* 2.3].

Another similar result may be extended in the following theorem.

Theorem 9. *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \to R$ and $a, b \in A$ with $a < a + \eta(b,a)$.Such that $f' \in L[a, a+\eta(b,a)]$.Suppose that $f : A \to \mathbb{R}$ be a differentiable function. If $|f'|^{\frac{p}{p-1}}$ is preinvex function on A for some fixed $p > 1$ with $q = \frac{p}{p-1}$, then the following inequality for fractional integrals holds for $\alpha > 0$:*

$$\left|f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J_{a^+}^{\alpha}f(b) + J_{b^-}^{\alpha}f(a)\right]\right| \leq \frac{\eta(b,a)}{2^{\alpha+1}(\alpha+1)}\times$$
$$\left[\left(\frac{(\alpha+3)\,|f'(b)|^q+(\alpha+1)\,|f'(a)|^q}{2(\alpha+2)}\right)^{1/q}+\left(\frac{(\alpha+3)\,|f'(a)|^q+(\alpha+1)\,|f'(b)|^q}{2(\alpha+2)}\right)^{1/q}\right]. \quad (10)$$

Proof. Using the well-known power-mean integral inequality for $q > 1$ we have

$$|I_1| \leq \left(\int_0^{1/2} t^{\alpha} dt\right)^{1-1/q}\left(\int_0^{1/2} t^{\alpha}\left|f'\left(a+t\eta(b,a)\right)\right|^q dt\right)^{1/q}$$

By preinvexity of $|f'|^q$

$$|I_1| \leq \left(\frac{1}{2^{\alpha+1}(\alpha+1)}\right)^{1-1/q} \left(\frac{\alpha+3}{2^{\alpha+2}(\alpha+1)(\alpha+2)}|f'(a)|^q + \frac{1}{2^{\alpha+2}(\alpha+2)}|f'(b)|^q\right)^{1/q}$$
$$= \frac{1}{2^{\alpha+1}(\alpha+1)}\left(\frac{(\alpha+3)|f'(a)|^q + (\alpha+1)|f'(b)|^q}{2(\alpha+2)}\right)^{1/q}$$

Analogously:

$$|I_2| \leq \frac{1}{2^{\alpha+1}(\alpha+1)}\left(\frac{(\alpha+3)|f'(b)|^q + (\alpha+1)|f'(a)|^q}{2(\alpha+2)}\right)^{1/q}$$
$$|I_3| \leq \frac{1}{2^{\alpha+1}(\alpha+1)}\left(\frac{(\alpha+3)|f'(b)|^q + (\alpha+1)|f'(a)|^q}{2(\alpha+2)}\right)^{1/q}$$

and

$$|I_4| \leq \frac{1}{2^{\alpha+1}(\alpha+1)}\left(\frac{(\alpha+3)|f'(a)|^q + (\alpha+1)|f'(b)|^q}{2(\alpha+2)}\right)^{1/q}$$

Combining all the obtained inequalities, we get desired inequality. Which completes the proof. ∎

Corollary 3. *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \to R$ and $a, b \in A$ with $a < a + \eta(b, a)$. Such that $f' \in L[a, a + \eta(b, a)]$. Suppose that $f : A \to \mathbb{R}$ be a differentiable function. If $|f'|^q$ is preinvex function on A for some fixed $q > 1$ then the following inequality for fractional integrals holds for*

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx - f\left(\frac{2a+\eta(b,a)}{2}\right)\right| \leq \frac{\eta(b,a)}{8}\left(\frac{1+2^{1/q}}{3^{1/p}}\right)[|f'(a)|+|f'(b)|], \quad (11)$$

Proof. If we take $\alpha = 1$ in Theorem 9, then inequality (10) becomes as:

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx - f\left(\frac{2a+\eta(b,a)}{2}\right)\right|$$
$$\leq \frac{\eta(b,a)}{8}\left[\left(\frac{2|f'(a)|^q + |f'(b)|^q}{3}\right)^{1/q} + \left(\frac{|f'(a)|^q + 2|f'(b)|^q}{3}\right)^{1/q}\right],$$

which can be made equivalent to (11) by using the fact:

$$\sum_{i=1}^n (a_i + b_i)^r \leq \sum_{i=1}^n {a_i}^r + \sum_{i=1}^n {b_i}^r,$$

for $0 \leq r < 1$, $a_1, a_2, ..., a_n \geq 0$ and $b_1, b_2, ..., b_n \geq 0$. ∎

Remark 4. *Inequality* (11) *is an improvement of obtained inequality as in* [16, *Theorem* 2.1].

3. *Applications to special means*

In what follows we give certain generalizations of some notions for a positive valued function of a positive variable.

Definition 3. *A function $M : R_+ \to R_+$, is called a Mean function if it has the following properties:*

(1) Homogeneity : $M(ax, ay) = aM(x, y)$, for all $a > 0$,
(2) Symmetry : $M(x, y) = M(y, x)$,
(3) Reflexivity : $M(x, x) = x$,

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(4) Monotonicity : If $x \leqslant x$ *and* $y \leqslant y'$ *then* $M(x,y) = M(x',y')$,
(5) Internality : min $\{x,y\} \leqslant M(x,y) \leqslant max\{x,y\}$.

We consider some mens for arbitrary positive real numbers a and b (see for instance [7]).

The arithmetic mean

$$A(a,b) = \frac{a+b}{2} , \qquad a,b \in \mathbf{R}$$

The geometric mean

$$G(a,b) = \sqrt{ab} , \qquad a,b \in \mathbf{R}$$

The harmonic mean

$$H(a,b) = \frac{2ab}{a+b} , \qquad a,b \in \mathbf{R}\backslash\{0\}$$

The power mean

$$P(a,b) = \left(\frac{a^r+b^r}{2}\right)^{\frac{1}{r}} , \qquad r \geqslant 1$$

.The identric mean

$$I(a,b) = \begin{cases} a & \text{if } a = b \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if } a \neq b \end{cases} , \quad a,b > 0$$

The logarithmic mean

$$L(a,b) = \begin{cases} a & \text{if } a = b \\ \frac{b-a}{lnb-lna} & \text{if } a \neq b \end{cases} ,$$

Generalized logarithmic mean

$$L_n(a,b) = \begin{cases} a & \text{if } a = b \\ \left[\frac{b^{n+1}-a^{n+1}}{(n+1)(b-a)}\right]^{\frac{1}{n}} & \text{if } a \neq b \end{cases} ,n \in \mathbf{Z}\backslash\{-1,0\}; \quad a,b > 0$$

Now, using the results of Section Main Results, some new inequalities are derived for the above means.It is well known that Lp is monotonic nondecreasing over $p \in R$,with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequality $H \leqslant G \leqslant L \leqslant I \leqslant A$.

Now letting a and b be positive real numbers such that $a < b$.Consider the function $M := M(a,b) : [a, a+\eta(b,a)] \times [a, a+\eta(b,a)] \to R^+$,which is one of the above mentioned means, therefore one can obtain variant inequalities for these means as follows:

Setting $\eta(b,a) = M(b-a)$ in $(6), (8), (10)$, one can obtain the following interesting inequalities involving means:

$$\left| f\left(\frac{2a+M(b,a)}{2}\right) - \frac{\Gamma(\alpha+1)}{2M^\alpha(b,a)}\left[J^\alpha_{a^+}f(a+M(b,a)) + J^\alpha_{a+M(b,a)-}f(a)\right]\right| \\ \leq \frac{M(b,a)}{2^{\alpha+1}(\alpha+1)}(\ |f'(a)| + |f'(b)|\ ). \quad (12)$$

$$\left|f\left(\frac{2a+M(b,a)}{2}\right)-\frac{\Gamma(\alpha+1)}{2M^{\alpha}(b,a)}\left[J^{\alpha}_{a+}f(a+M(b,a))+J^{\alpha}_{a+M(b,a)-}f(a)\right]\right|$$
$$\leq\frac{M(b,a)}{2^{\alpha+1}(\alpha p+1)^{1/p}}\left[\left(\frac{|f'(b)|^{q}+3|f'(a)|^{q}}{4}\right)^{1/q}+\left(\frac{|f'(a)|^{q}+3|f'(b)|^{q}}{4}\right)^{1/q}\right]. \quad (13)$$

$$\left|f\left(\frac{2a+M(b,a)}{2}\right)-\frac{\Gamma(\alpha+1)}{2M^{\alpha}(b,a)}\left[J^{\alpha}_{a+}f(a+M(b,a))+J^{\alpha}_{a+M(b,a)-}f(a)\right]\right|\leq$$
$$\frac{M(b,a)}{2^{\alpha+1}(\alpha+1)}\left[\left(\frac{(\alpha+3)|f'(a)|^{q}+(\alpha+1)|f'(b)|^{q}}{2(\alpha+2)}\right)^{\frac{1}{q}}+\left(\frac{(\alpha+3)|f'(b)|^{q}+(\alpha+1)|f'(a)|^{q}}{2(\alpha+2)}\right)^{\frac{1}{q}}\right]. \quad (14)$$

Proof. Letting $M=A,G,H,P,I,L,L_P$ in (12) , (13) , and (14) , we can get the required inequalities, and the details are left to interested reader. ∎

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The Borel direction and uniqueness of meromorphic function *

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Abstract

The purpose of this paper is to investigate the relationship between Borel directions and shared-set of meromorphic functions and obtain some results of meromorphic functions sharing one finite set in an angular domain containing a Borel line.

Key words: Meromorphic function; Borel direction; Uniqueness.
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1 Introduction and main results

We use $\mathbb{C}$ to denote the open complex plane, $\widehat{\mathbb{C}}(=\mathbb{C}\bigcup\{\infty\})$ to denote the extended complex plane, and $\Omega(\subset\mathbb{C})$ to denote an angular domain. We assume that the readers are familiar with the standard notations and fundamental results of Nevanlinna value distribution theory of meromorphic functions (see [7, 16]).

Let S be a set of distinct elements in $\widehat{\mathbb{C}}$ and $\Omega := \{z:\ \alpha\leq \arg z\leq\beta\}\subseteq\mathbb{C}$. Define

$$E(S,\Omega,f)=\bigcup_{a\in S}\{z\in\Omega|f_a(z)=0,\ \textit{counting multiplicities}\},$$

$$\overline{E}(S,\Omega,f)=\bigcup_{a\in S}\{z\in\Omega|f_a(z)=0,\ \textit{ignoring multiplicities}\},$$

where $f_a(z)=f(z)-a$ if $a\in\mathbb{C}$ and $f_\infty(z)=1/f(z)$.

Let f and g be two non-constant meromorphic functions in $\mathbb{C}$. We say f and g share the set S CM (counting multiplicities) in Ω if $E(S,\Omega,f)=E(S,\Omega,g)$; we say f and g share the set S IM (ignoring multiplicities) in Ω if $\overline{E}(S,\Omega,f)=\overline{E}(S,\Omega,g)$. If $S=\{a\}$, where $a\in\widehat{\mathbb{C}}$, we say f and g share the value a CM in Ω if $E(S,\Omega,f)=E(S,\Omega,g)$, and we say f and g share the value a IM in Ω if $\overline{E}(S,\Omega,f)=\overline{E}(S,\Omega,g)$. If $\Omega=\mathbb{C}$, we give the simple notation as before, $E(S,f),\overline{E}(S,f)$ and so on(see [17]).

In 1926, R. Nevanlinna (see [11]) proved his famous five-value and four-value theorems. After this very work, many investigations studied the uniqueness of meromorphic functions with shared values in the whole complex plane (see [17]). Around 2003, Zheng [18, 19] was the first to study the uniqueness of meromorphic functions sharing five values and four values in some angular domain under some condition.

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Theorem 1.1 *([18, Theorem 1.1]). Let $f(z)$ and $g(z)$ be both transcendental meromorphic functions and let $f(z)$ be of the finite lower order μ and such that for some $a \in \widehat{\mathbb{C}}$ and an integer $p \geq 0$, $\delta = \delta(a, f^{(p)}) > 0$. For q pair of real numbers α_j, β_j satisfying*

$$-\pi \leq \alpha_1 < \beta_1 \leq \alpha_2 < \beta_2 \leq \cdots \leq \alpha_q < \beta_q \leq \pi,$$

and

$$\sum_{j=1}^{q} (\alpha_{j+1} - \beta_j) < \frac{4}{\sigma} \arcsin \sqrt{\frac{\delta}{2}}, \tag{1}$$

where $\sigma = \max\{\omega, \mu\}$ and $\omega = \max\{\frac{\pi}{\beta_1 - \alpha_1}, \cdots, \frac{\pi}{\beta_q - \alpha_q}\}$, assume that $f(z)$ and $g(z)$ have five distinct IM shared values in $\Omega = \bigcup_{j=1}^{q}\{z : \alpha_j \leq \arg z \leq \beta_j\}$. If $\omega < \lambda(f)$, then $f(z) \equiv g(z)$.

After Zheng's work, there were many interesting results about the uniqueness with shared values in the angular domain, see [1, 9, 14, 15, 20].

In 2006, Lin, Mori and Tohge [9] dealt with the uniqueness problem on meromorphic functions sharing three finite sets in an angular domain and obtained the following theorems.

Theorem 1.2 *(see [9, Thereom 1]). Let $S_1 = \{\infty\}$, $S_2 = \{\omega | \omega^{n-1}(\omega + a) - b = 0\}$, $S_3 = \{0\}$, where $n(\geq 4)$ is an integer, and a, b are two nonzero constants, such that the algebraic equation $\omega^{n-1}(\omega + a) - b = 0$ has no multiple roots. Assume that f is a meromorphic function of lower order $\mu(f) \in (1/2, \infty)$ in $\widehat{\mathbb{C}}$ and $\delta := \delta(\iota, f) > 0$ for some $\iota \in \widehat{\mathbb{C}} \backslash \{0, -a\}$. Then, for each $\sigma < \infty$ with $\mu(f) \leq \sigma \leq \lambda(f)$, there exists an angular domain $\Omega = \Omega(\alpha, \beta) := \{z : \alpha < \arg z < \beta\}$ with $0 \leq \alpha < \beta \leq 2\pi$ and*

$$\beta - \alpha > \max\left\{\frac{\pi}{\sigma}, 2\pi - \frac{4}{\sigma} \arcsin \sqrt{\frac{\delta}{2}}\right\} \tag{2}$$

such that if the conditions $E(S_3, f) = E(S_3, g)$ and $E(S_j, \Omega, f) = E(S_j, \Omega, g)$ $(j = 1, 2)$ hold for a meromorphic function g of finite order or, more generally, with the growth satisfying either $\log T(r, g) = O(\log T(r, f))$ or

$$\lim_{r \to \infty} \frac{\log \log T(r, g)}{\min\{\log r, \log T(r, f)\}} = 0, \quad r \notin E_1, \tag{3}$$

where E_1 is a set of finite linear measure, then $f \equiv g$.

It is well known that Borel direction is an important singular direction for meromorphic function in the fields of complex analysis, and Borel directions played an important role in the topic of angular distribution(see [8, 12, 13]). Valiron [16] proved that every meromorphic function of finite order $\rho > 0$ has at least one Borel direction of order ρ, where the order of meromorphic function f is defined by $\rho = \rho(f) = \limsup\limits_{r \to \infty} \frac{\log T(r, f)}{\log r}$.

In 2012, Long and Wu[10] was the first to investigate the problem concerning Borel direction and shared value of meromorphic functions and obtained the following theorems.

Theorem 1.3 (see [10, Theorem 1.1]) *Let f be meromorphic function of infinite order $\rho(r)$, $g \in M(\rho(r))$, $\arg z = \theta (0 \leq \theta < 2\pi)$ be one Borel direction of $\rho(r)$ order of meromorphic function f, $a_i \in \widehat{\mathbb{C}} (i = 1, 2, 3, 4, 5)$ be five distinct complex numbers. If f and g share $a_i (i = 1, 2, 3, 4, 5)$ IM in the angular domain $\Omega(\theta - \varepsilon, \theta + \varepsilon)$ for any $\varepsilon (0 < \varepsilon < \pi)$, then $f \equiv g$.*

Definition 1.1 *[2]. Let f be a meromorphic function of infinite order, $\rho(r)$ is a real function satisfying the following conditions:*

(i) $\rho(r)$ is continuous, non-decreasing for $r \geq r_0$ and $\rho(r) \to \infty$ as $r \to \infty$;

(ii)

$$\lim_{r \to \infty} \frac{\log U(R)}{\log U(r)} = 1, \quad R = r + \frac{r}{\log U(r)},$$

where $U(r) = r^{\rho(r)}$ $(r \geq r_0)$;

(iii)

$$\limsup_{r\to\infty} \frac{\log T(r,f)}{\log U(r)} = 1.$$

Then $\rho(r)$ *is called infinite order of meromorphic function* f. *This definition is given by Xiong Qinglai[2].*

Let $\rho(r)$ be infinite order of meromorphic function f, we will denote by $M(\rho(r))$ the set of meromorphic function g satisfying $0 < \limsup_{r\to\infty} \frac{\log T(r,g)}{\rho(r)\log r} \leq 1$, that is,

$$M(\rho(r)) := \left\{ g : 0 < \limsup_{r\to\infty} \frac{\log T(r,g)}{\rho(r)\log r} \leq 1 \right\}.$$

Let $\alpha < \beta, \beta - \alpha < 2\pi, r > 0$, and $\Omega(\alpha, \beta, r) := \{z : \alpha \leq \arg z \leq \beta, 0 < |z| \leq r\}$. The definition of Borel direction of meromorphic functions f of infinite order $\rho(r)$ is defined as follows.

Definition 1.2 *[2]. Let* f *be meromorphic functions of infinite order* $\rho(r)$, *if for any* $\varepsilon (0 < \varepsilon < \pi)$, *the equality*

$$\limsup_{r\to\infty} \frac{\log n(\Omega(\theta-\varepsilon, \theta+\varepsilon, r), f = a)}{\rho(r)\log r} = 1,$$

holds for any complex number $a \in \widehat{\mathbb{C}}$, *at most except two exception, where* $n(\Omega(\theta-\varepsilon, \theta+\varepsilon, r), f = a)$ *is the counting function of zero of the function* $f - a$ *in the angular domain* $\Omega(\theta-\varepsilon, \theta+\varepsilon)$, *counting multiplicities. Then the ray* $\arg z = \theta$ *is called a Borel direction of* $\rho(r)$ *order of meromorphic function* f.

Remark 1.1 *Chuang [2] proved that every meromorphic function* f *with infinite order* $\rho(r)$ *has as least one Borel direction of infinite order* $\rho(r)$.

In this paper, we will investigate the uniqueness problem of meromorphic functions sharing one finite set in an angular domain containing a Borel line. We will mainly consider the following finite set $S = \{w \in \mathbb{A} : P_1(w) = 0\}$, where

$$P_1(w) = \frac{(n-1)(n-2)}{2} w^n - n(n-2)w^{n-1} + \frac{n(n-1)}{2} w^{n-2} - c,$$

c is a complex number satisfying $c \neq 0, 1$.

Theorem 1.4 *Let* f *be meromorphic function of infinite order* $\rho(r)$, $g \in M(\rho(r))$, $\arg z = \theta (0 \leq \theta < 2\pi)$ *be one Borel direction of* $\rho(r)$ *order of meromorphic function* f, *if* $E(S, \Omega(\theta-\varepsilon, \theta+\varepsilon), f) = E(S, \Omega(\theta-\varepsilon, \theta+\varepsilon), g)$ *and* n *is an integer* ≥ 11, *then* $f \equiv g$.

A set S is called a unique range set for meromorphic functions on the Borel direction $\arg z = \theta$, if for any two nonconstant meromorphic functions f and g the condition $E(S, \Omega(\theta-\varepsilon, \theta+\varepsilon), f) = E(S, \Omega(\theta-\varepsilon, \theta+\varepsilon), g)$ implies $f \equiv g$. We denote by $\sharp S$ the cardinality of a set S. Thus, from Theorem 1.4, we can get the following corollary

Corollary 1.1 *There exists one finite set* S *with* $\sharp S = 11$, *such that any two meromorphic functions* f *and* g *on the Borel direction, which* $f(z)$ *is meromorphic function of infinite order* $\rho(r)$, $g \in M(\rho(r))$, $\arg z = \theta (0 \leq \theta < 2\pi)$ *be one Borel direction of* $\rho(r)$ *order of meromorphic function* f, *and* $E(S, \Omega(\theta-\varepsilon, \theta+\varepsilon), f) = E(S, \Omega(\theta-\varepsilon, \theta+\varepsilon), g)$.

2 Some Lemmas

We first introduce the basic notations and definitions of meromorphic functions in an angular domain as follows(see [7, 18, 19]).

Let f be a meromorphic function on the angular domain $\Omega(\alpha,\beta)=\{z:\alpha\le\arg z\le\beta\}$ and $0<\beta-\alpha\le 2\pi$. Define

$$A_{\alpha,\beta}(r,f)=\frac{\omega}{\pi}\int_1^r(\frac{1}{t^\omega}-\frac{t^\omega}{r^{2\omega}})\{\log^+|f(te^{i\alpha})|+\log^+|f(te^{i\beta})|\}\frac{dt}{t},$$

$$B_{\alpha,\beta}(r,f)=\frac{2\omega}{\pi r^\omega}\int_\alpha^\beta \log^+|f(re^{i\theta})|\sin\omega(\theta-\alpha)d\theta,$$

$$C_{\alpha,\beta}(r,f)=2\sum_{1<|b_\mu|<r}(\frac{1}{|b_\mu|^\omega}-\frac{|b_\mu|^\omega}{r^{2\omega}})\sin\omega(\theta_\mu-\alpha),$$

$$S_{\alpha,\beta}(r,f)=A_{\alpha,\beta}(r,f)+B_{\alpha,\beta}(r,f)+C_{\alpha,\beta}(r,f),$$

where $\omega=\frac{\pi}{\beta-\alpha}$ and $b_\mu=|b_\mu|e^{i\theta_\mu}(\mu=1,2,\cdots)$ are the poles of f on $\Omega(\alpha,\beta)$ counted according to their multiplicities. $S_{\alpha,\beta}(r,f)$ is called the Nevanlinna's angular characteristic, and $C_{\alpha,\beta}(r,f)$ is called the angular counting function of the poles of f on $\Omega(\alpha,\beta)$, and $\overline{C}_{\alpha,\beta}(r,f)$ is the reduced function of $C_{\alpha,\beta}(r,f)$. Similarly, when $a\ne\infty$, we will use the notations $A_{\alpha,\beta}(r,\frac{1}{f-a})$, $B_{\alpha,\beta}(r,\frac{1}{f-a})$, $C_{\alpha,\beta}(r,\frac{1}{f-a})$, $S_{\alpha,\beta}(r,\frac{1}{f-a})$ and so on.

To prove our result, we require the following Lemmas.

Lemma 2.1 *(see [6]). Let f be a nonconstant meromorphic function on $\Omega(\alpha,\beta)$. Then for arbitrary complex number a, we have*

$$S_{\alpha,\beta}\left(r,\frac{1}{f-a}\right)=S_{\alpha,\beta}(r,f)+\varepsilon(r,a),$$

where $\varepsilon(r,a)=O(1)$ as $r\to\infty$.

Lemma 2.2 *(see [5, 19]). Suppose that f is a non-constant meromorphic function in one angular domain $\Omega(\alpha,\beta)$ with $0<\beta-\alpha\le 2\pi$, then for arbitrary q distinct $a_j\in\widehat{\mathbb{C}}(1\le j\le q)$, we have*

$$(q-2)S_{\alpha,\beta}(r,f)\le\sum_{j=1}^q\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-a_j}\right)-C^0_{\alpha,\beta}(r,\frac{1}{f'})+R_{\alpha,\beta}(r,f),$$

where $C^0_{\alpha,\beta}(r,\frac{1}{f'})$ is the counting function of the zeros of f' in Ω where f does not take anyone of the values $a_j(j=1,2,\dots,q)$, and the term $\overline{C}_{\alpha,\beta}(r,\frac{1}{f-a_j})$ will be replaced by $\overline{C}_{\alpha,\beta}(r,f)$ when some $a_j=\infty$ and

$$\begin{aligned}R_{\alpha,\beta}(r,f)=&A_{\alpha,\beta}\left(r,\frac{f'}{f}\right)+B_{\alpha,\beta}\left(r,\frac{f'}{f}\right)\\&+\sum_{j=1}^q\left\{A_{\alpha,\beta}\left(r,\frac{f'}{f-a_j}\right)+B_{\alpha,\beta}\left(r,\frac{f'}{f-a_j}\right)\right\}+O(1).\end{aligned}\qquad(4)$$

Lemma 2.3 *(see [6, P138].) Let f be a nonconstant meromorphic function in the whole complex plane $\mathbb{C}$. Given one angular domain on $\Omega(\alpha,\beta)$. Then for any $1\le r<R$, we have*

$$A_{\alpha,\beta}\left(r,\frac{f'}{f}\right)\le K\left\{\left(\frac{R}{r}\right)^\omega\int_1^R\frac{\log^+T(r,f)}{t^{1+\omega}}dt+\log^+\frac{r}{R-r}+\log\frac{R}{r}+1\right\},$$

and

$$B_{\alpha,\beta}\left(r,\frac{f'}{f}\right)\leq\frac{4\omega}{r^{\omega}}m\left(r,\frac{f'}{f}\right),$$

where $\omega=\frac{\pi}{\beta-\alpha}$ *and* K *is a positive constant not depending on* r *and* R.

Remark 2.1 *Nevanlinna conjectured that*

$$A_{\alpha,\beta}\left(r,\frac{f'}{f}\right)+B_{\alpha,\beta}\left(r,\frac{f'}{f}\right)=o\left(S_{\alpha,\beta}\left(r,f\right)\right)\tag{5}$$

when r *tends to* $+\infty$ *outside an exceptional set of finite linear measure, and he proved that* $A_{\alpha,\beta}\left(r,\frac{f'}{f}\right)+B_{\alpha,\beta}\left(r,\frac{f'}{f}\right)=O(1)$ *when the function* f *is meromorphic in* $\mathbb{C}$ *and has finite order. In 1974, Gol'dberg [5] constructed a counter-example to show that (5) is not valid.*

Remark 2.2 *From Lemma 2.2 and Lemma 2.3, we can get the following conclusion:*

$$R_{\alpha,\beta}\left(r,f\right)=\begin{cases}O(1), & f\ is\ of\ finite\ order,\\ O(\log U(r)),\ r\notin E, & f\ is\ of\ infinite\ order,\end{cases}$$

where $R_{\alpha,\beta}\left(r,f\right)$ *is stated as in (1),* $U(r)=r^{\rho(r)}$, $\rho(r)$ *is infinite order of meromorphic function* f, E *is a set of finite linear measure.*

Lemma 2.4 *(see [3]). Let* f *be meromorphic function of infinite order* $\rho(r)$. *Then the ray* $\arg z=\theta$ *is one Borel direction of* $\rho(r)$ *order of meromorphic function* f *if and only if* f *satisfies the equality*

$$\limsup_{r\to\infty}\frac{\log S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)}{\rho(r)\log r}=1,\tag{6}$$

for any $\varepsilon(0<\varepsilon<\frac{\pi}{2})$.

Similar to discuss as in [1, Lemma 1] and [17], we can get the lemma below easily.

Lemma 2.5 *Suppose that* f *is a non-constant meromorphic function with infinite order* $\rho(r)$, *the ray* $\arg z=\theta$ *is one Borel direction of* $\rho(r)$ *order of meromorphic function* f. *Let* $P(f)=a_0f^p+a_1f^{p-1}+\cdots+a_p(a_0\neq0)$ *be a polynomial of* f *with degree* p, *where the coefficients* $a_j(j=0,1,\ldots,p)$ *are constants, and let* $b_j(j=1,2,\ldots,q)$ *be* $q(q\geq p+1)$ *distinct finite complex numbers. Then for any* $\varepsilon(0<\varepsilon<\pi/2)$,

$$D_{\theta-\varepsilon,\theta+\varepsilon}\left(r,\frac{P(f)\cdot f'}{(f-b_1)(f-b_2)\cdots(f-b_q)}\right)=R_{\theta-\varepsilon,\theta+\varepsilon}(r,f),$$

where $D_{\theta-\varepsilon,\theta+\varepsilon}(r,\bullet)=A_{\theta-\varepsilon,\theta+\varepsilon}(r,\bullet)+B_{\theta-\varepsilon,\theta+\varepsilon}(r,\bullet)$.

Lemma 2.6 *Suppose that* f *is a non-constant meromorphic function with infinite order* $\rho(r)$, *the ray* $\arg z=\theta$ *is one Borel direction of* $\rho(r)$ *order of meromorphic function* f. *Then for any* $\varepsilon(0<\varepsilon<\pi/2)$, *we have*

$$C_{\theta-\varepsilon,\theta+\varepsilon}\left(r,\frac{1}{f'}\right)\leq C_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{f})+\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,f)+R_{\theta-\varepsilon,\theta+\varepsilon}(r,f)+O(1).$$

Proof: By Lemma 2.1, Lemma 2.3 and Lemma 2.4, we have

$$D_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{f})\leq D_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{f'})+D_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{f'}{f})=D_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{f'})+R_{\theta-\varepsilon,\theta+\varepsilon}(r,f),$$

then from the definition of $S(r,f)$, we have

$$S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)-C_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{f})\leq S_{\theta-\varepsilon,\theta+\varepsilon}(r,f')-C_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{f'})+R_{\theta-\varepsilon,\theta+\varepsilon}(r,f)+O(1),$$

i.e.,

$$\begin{aligned}C_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{f'})\leq &S_{\theta-\varepsilon,\theta+\varepsilon}(r,f')-S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)\\ &+C_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{f})+R_{\theta-\varepsilon,\theta+\varepsilon}(r,f)+O(1).\end{aligned}\tag{7}$$

Since

$$\begin{aligned}S_{\theta-\varepsilon,\theta+\varepsilon}(r,f')&=D_{\theta-\varepsilon,\theta+\varepsilon}(r,f')+D_{\theta-\varepsilon,\theta+\varepsilon}(r,f')\\ &\leq D_{\theta-\varepsilon,\theta+\varepsilon}(r,f)+D_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{f'}{f})+C_{\theta-\varepsilon,\theta+\varepsilon}(r,f)+\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,f)\\ &\leq S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)+\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,f)+R_{\theta-\varepsilon,\theta+\varepsilon}(r,f)+O(1),\end{aligned}\tag{8}$$

then from (7) and (8), we can get the conclusion of this lemma. □

Lemma 2.7 *Let F be transcendental entire function of infinite order $\rho(r)$, $G\in M(\rho(r))$, $\arg z=\theta(0\leq\theta<2\pi)$ be one Borel direction of $\rho(r)$ order of function f and $\Omega:=\Omega(\theta-\varepsilon,\theta+\varepsilon)$ for any $\varepsilon(0<\varepsilon<\pi)$. If F and G satisfy $E(0,\Omega,F)=E(0,\Omega,G)$ and $c_1,c_2,\ldots,c_q$ are $q(\geq 2)$ distinct non-zero complex numbers. If*

$$\limsup_{r\to\infty,r\in I}\frac{3\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,F)+\sum_{j=1}^{q}\overline{C}^{(2)}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{F-c_j})+\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{F'})}{S_{\theta-\varepsilon,\theta+\varepsilon}(r,F)}<q,\tag{9}$$

and

$$\limsup_{r\to\infty,r\in I}\frac{3\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,G)+\sum_{j=1}^{q}\overline{C}^{(2)}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{G-c_j})+\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{G'})}{S_{\theta-\varepsilon,\theta+\varepsilon}(r,G)}<q,\tag{10}$$

where $\overline{C}^{(2)}_{\theta-\varepsilon,\theta+\varepsilon}(r,\cdot)=\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\cdot)+\overline{C}^{(2}_{\theta-\varepsilon,\theta+\varepsilon}(r,\cdot)$, $\overline{C}^{(2}_{\theta-\varepsilon,\theta+\varepsilon}(r,\cdot)=\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\cdot)-C^{1)}(r,\cdot)$, $C^{1)}_{\theta-\varepsilon,\theta+\varepsilon}(r,\cdot)$ is the counting function which only counts simple zeros of the function $\cdot$ in Ω and I is some set of r of infinite linear measure, then

$$F=\frac{aG+b}{cG+d},$$

where $a,b,c,d\in\mathbb{C}$ are constants with $ad-bc\neq 0$.

Proof: Since meromorphic function F is of infinite order $\rho(r)$ and $\arg z=\theta(0\leq\theta<2\pi)$ is one Borel direction of $\rho(r)$ order of F, then we can get by Lemma 2.4 for any $\varepsilon(0<\varepsilon<\pi)$

$$\limsup_{r\to\infty}\frac{\log S_{\theta-\varepsilon,\theta+\varepsilon}(r,F)}{\rho(r)\log r}=1.\tag{11}$$

And since $G\in M(\rho(r))$, we have

$$\limsup_{r\to\infty}\frac{\log S_{\theta-\varepsilon,\theta+\varepsilon}(r,G)}{\rho(r)\log r}\leq 1.\tag{12}$$

Set $R(r)=O(\rho(r)\log r))$ as $r\to\infty,(r\notin E)$, where E is a set of finite linear measure, then from (11) and (12), we have $R(r)=o(S_{\theta-\varepsilon,\theta+\varepsilon}(r,F))=o(S_{\theta-\varepsilon,\theta+\varepsilon}(r,G))$ as $r\to\infty,r\notin E$.

Set

$$H \equiv \frac{F''}{F'} - 2\frac{F'}{F} - \left(\frac{G''}{G'} - 2\frac{G'}{G}\right). \tag{13}$$

Suppose that $H \not\equiv 0$, from Lemma 2.1 and Lemma 2.2, we have

$$D_{\theta-\varepsilon,\theta+\varepsilon}(r, H) = R(r). \tag{14}$$

Since $E(0, \Omega, F) = E(0, \Omega, G)$, and by an elementary calculation, we can conclude that if z_0 is a common simple zero of F and G in Ω, then $H(z_0) = 0$. Thus, from (13) we have

$$C^{1)}_{\theta-\varepsilon,\theta+\varepsilon}(r) \le C_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{H}) \le S_{\theta-\varepsilon,\theta+\varepsilon}(r, H) + O(1) \le C_{\theta-\varepsilon,\theta+\varepsilon}(r, H) + R(r), \tag{15}$$

where $C^{1)}_{\theta-\varepsilon,\theta+\varepsilon}(r) = C^{1)}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{F}) = C^{1)}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{G})$. The poles of H in $\Omega(\theta-\varepsilon, \theta+\varepsilon)$ can only occur at zeros of F' and G' in $\Omega(\theta-\varepsilon, \theta+\varepsilon)$ or poles of F and G in $\Omega(\theta-\varepsilon, \theta+\varepsilon)$. Moreover, H only has simple zeros in $\Omega(\theta-\varepsilon, \theta+\varepsilon)$. Hence, from (13), we have

$$\begin{aligned} C^{1)}_{\theta-\varepsilon,\theta+\varepsilon}(r) \le& \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, F) + \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, G) + \overline{C}^{0}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{F'}) + \overline{C}^{0}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{G'}) \\ &+ \sum_{j=1}^{q} \overline{C}^{(2}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{F-c_j}) + \sum_{j=1}^{q} \overline{C}^{(2}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{G-c_j}) + R(r), \end{aligned} \tag{16}$$

where $\overline{C}^{0}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{F'})$ is the reduced counting function for the zeros of F' in Ω where F does not take one of the values $0, c_1, c_2, \ldots, c_q$.

Since

$$\begin{aligned} &\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{F}) + \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{G}) \\ &\qquad = 2C^{1)}_{\theta-\varepsilon,\theta+\varepsilon}(r) + \overline{C}^{(2}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{F}) + \overline{C}^{(2}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{G}), \end{aligned} \tag{17}$$

then from (15)-(17), we have

$$\begin{aligned} &\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{F}) + \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{G}) \\ \le& 2\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, F) + 2\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, G) + 2\overline{C}^{0}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{F'}) + \\ &+ 2\overline{C}^{0}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{G'}) + \overline{C}^{(2}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{F}) + \overline{C}^{(2}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{G}) + \\ &+ 2\sum_{j=1}^{q} \overline{C}^{(2}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{F-c_j}) + 2\sum_{j=1}^{q} \overline{C}^{(2}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{G-c_j}) + R(r). \end{aligned} \tag{18}$$

By Lemma 2.2, we have

$$\begin{aligned} qS_{\theta-\varepsilon,\theta+\varepsilon}(r, F) \le& \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, F) + \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{F}) + \sum_{j=1}^{q} \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{F-c_j}) \\ &- C^{0}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{F'}) + R(r),\ \ r \notin E, \end{aligned} \tag{19}$$

$$\begin{aligned} qS_{\theta-\varepsilon,\theta+\varepsilon}(r, G) \le& \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, G) + \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{G}) + \sum_{j=1}^{q} \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{G-c_j}) \\ &- C^{0}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{G'}) + R(r),\ \ r \notin E, \end{aligned} \tag{20}$$

where E is a set of r of finite linear measure, and it needs not be the same at each occurrence. From (18)-(20), it follows for $r \notin E$,

$$\begin{aligned}
&q\{S_{\theta-\varepsilon,\theta+\varepsilon}(r,F)+S_{\theta-\varepsilon,\theta+\varepsilon}(r,G)\}\\
\le&3\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,F)+3\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,G)+\sum_{j=1}^{q}\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{F-c_j})+\sum_{j=1}^{q}\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{G-c_j})\\
&+2\sum_{j=1}^{q}\overline{C}^{(2}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{F-c_j})+2\sum_{j=1}^{q}\overline{C}^{(2}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{G-c_j})+\overline{C}^{0}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{F'})\\
&+\overline{C}^{0}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{G'})+\overline{C}^{(2}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{F})+\overline{C}^{(2}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{G})+R(r). \qquad (21)
\end{aligned}$$

Since

$$\sum_{j=1}^{q}\overline{C}^{(2}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{F-c_j})+\overline{C}^{(2}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{F})+\overline{C}^{0}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{F'})=\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{F'}), \qquad (22)$$

and

$$\sum_{j=1}^{q}\overline{C}^{(2}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{G-c_j})+\overline{C}^{(2}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{G})+\overline{C}^{0}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{G'})=\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{G'}), \qquad (23)$$

and from (21)-(23), we have for $r \notin E$,

$$\begin{aligned}
&q\{S_{\theta-\varepsilon,\theta+\varepsilon}(r,F)+S_{\theta-\varepsilon,\theta+\varepsilon}(r,G)\} \qquad (24)\\
\le&3\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,F)+3\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,G)+\sum_{j=1}^{q}\overline{C}^{(2)}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{F-c_j})+\\
&\sum_{j=1}^{q}\overline{C}^{(2)}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{G-c_j})+\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{F'})+\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{G'})+R(r).
\end{aligned}$$

From (9), (10) and (24), and since Let F be transcendental entire function of infinite order $\rho(r)$, $G \in M(\rho(r))$, $\arg z = \theta (0 \le \theta < 2\pi)$ be one Borel direction of $\rho(r)$ order of function f, we have

$$S_{\theta-\varepsilon,\theta+\varepsilon}(r,F)+S_{\theta-\varepsilon,\theta+\varepsilon}(r,G)\le o\{S_{\theta-\varepsilon,\theta+\varepsilon}(r,F)+S_{\theta-\varepsilon,\theta+\varepsilon}(r,G)\},\quad r\notin E,\ r\in I.$$

Thus, we can get a contradiction. Therefore, $H(z) \equiv 0$, that is,

$$\frac{F''}{F'}-2\frac{F'}{F}\equiv\frac{G''}{G'}-2\frac{G'}{G}.$$

For above equality, by integration, it follows that

$$F\equiv\frac{aG+b}{cG+d},$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. □

The following result can be derived from the proof of Frank-Reinders' theorem in [4].

Lemma 2.8 *Let $n \ge 6$ and*

$$P(w)=\frac{(n-1)(n-2)}{2}w^n-n(n-2)w^{n-1}+\frac{n(n-1)}{2}w^{n-2}.$$

Then $P(w)$ is a unique polynomial for transcendental meromorphic functions, i.e., for any two transcendental meromorphic functions f and g, $P(f) \equiv P(g)$ implies $f \equiv g$.

3 The Proof of Theorem 1.4

Proof: Since f is a meromorphic function of infinite order $\rho(r)$ and $\arg z = \theta(0 \leq \theta < 2\pi)$ be one Borel direction of $\rho(r)$ order of meromorphic function f, similar to (11) and (12), for any $\varepsilon(0 < \varepsilon < \pi)$, we have

$$\limsup_{r\to\infty} \frac{\log S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)}{\rho(r)\log r} = 1, \tag{25}$$

and

$$\limsup_{r\to\infty} \frac{\log S_{\theta-\varepsilon,\theta+\varepsilon}(r,g)}{\rho(r)\log r} \leq 1. \tag{26}$$

Set $R(r) = O(\rho(r)\log r))$ as $r \to \infty, r \notin E$, where E is a set of finite linear measure, then we have $R(r) = o(S_{\theta-\varepsilon,\theta+\varepsilon}(r,f))$ as $r \to \infty, r \notin E$ from (25) and (26).

From the definition of $P_1(w)$, we have $P_1(1) = 1 - c := c_1 \neq 0$, $P_1(0) = -c := c_2 \neq 0$ and

$$P_1'(w) = \frac{n(n-1)(n-2)}{2}(w-1)^2 w^{n-3}, \tag{27}$$

$$P_1(w) - c_1 = (w-1)^3 Q_1(w), \quad Q_1(1) \neq 0, \tag{28}$$

$$P_1(w) - c_2 = w^{n-2} Q_2(w), \quad Q_2(0) \neq 0, \tag{29}$$

where Q_1, Q_2 are polynomials of degree $n-3$ and 2, respectively. We also see that $Q_i(i=1,2)$ and P_1 have only simple zeros.

Let F and G be defined as $F = P_1(f)$ and $G = P_1(g)$. Since $E(S,\Omega,f) = E(S,\Omega,g)$, we have $E(0,\Omega,F) = E(0,\Omega,G)$. From (28) and (29), we have

$$\begin{aligned}
\overline{C}^{(2)}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{F-c_1}) &= \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{F-c_1}) + \overline{C}^{(2}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{F-c_1})\\
&\leq 2\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{f-1}) + \sum_{i=1}^{n-3} C_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{f-a_i})\\
&\leq (n-1)S_{\theta-\varepsilon,\theta+\varepsilon}(r,f) + R(r),
\end{aligned} \tag{30}$$

and

$$\begin{aligned}
\overline{C}^{(2)}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{F-c_2}) &= \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{F-c_2}) + \overline{C}^{(2}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{F-c_2})\\
&\leq 2\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{f}) + \sum_{j=1}^{2} C_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{f-b_j})\\
&\leq 4S_{\theta-\varepsilon,\theta+\varepsilon}(r,f) + R(r),
\end{aligned} \tag{31}$$

where $a_i(i=1,\ldots,n-3)$ and $b_j(j=1,2)$ are the zeros of $Q_1(w)$ and $Q_2(w)$ in Ω, respectively.

From (27), we have

$$\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{F'}) \leq \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{f}) + \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{f-1}) + \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{f'}). \tag{32}$$

From [6, Theorem 6.3], we have $S_{\theta-\varepsilon,\theta+\varepsilon}(r,F) = nS_{\theta-\varepsilon,\theta+\varepsilon}(r,f) + R(r)$. Thus, combining (30), (31) and (32), by Lemma 2.6, Lemma 2.7 and $n \geq 11$, we have

$$\begin{aligned}
&\limsup_{r\to\infty, r\notin E} \frac{3\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,F) + \sum_{j=1}^{2}\overline{C}^{(2)}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{F-c_j}) + \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{F'})}{S_{\theta-\varepsilon,\theta+\varepsilon}(r,F)}\\
\leq &\limsup_{r\to\infty, r\notin E} \frac{4\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,f) + (n+6)S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)}{nS_{\theta-\varepsilon,\theta+\varepsilon}(r,f)} < 2.
\end{aligned} \tag{33}$$

Similarly, we have

$$\begin{aligned} &\limsup_{r\to\infty, r\notin E} \frac{3\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,G)+\sum_{j=1}^{2}\overline{C}^{(2)}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{G-c_j})+\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{G'})}{S_{\theta-\varepsilon,\theta+\varepsilon}(r,G)} \\ \leq &\limsup_{r\to\infty, r\notin E} \frac{4\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,g)+(n+6)S_{\theta-\varepsilon,\theta+\varepsilon}(r,g)}{nS_{\theta-\varepsilon,\theta+\varepsilon}(r,g)} < 2. \end{aligned} \tag{34}$$

Thus, we have by Lemma 2.7

$$\frac{F''}{F'} - 2\frac{F'}{F} \equiv \frac{G''}{G'} - 2\frac{G'}{G}.$$

For above equality, by integration, it follows that

$$F \equiv \frac{aG+b}{cG+d},$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. Since $E(S, \Omega, f)$ is nonempty, and $E(S, \Omega, f) = E(S, \Omega, g)$, we have $b = 0, a \neq 0$. Hence

$$F \equiv \frac{aG}{cG+d} \equiv \frac{G}{AG+B}, \tag{35}$$

where $A = \frac{c}{a}$ and $B = \frac{d}{a} \neq 0$.

Two cases will be considered as follows:

Case 1: $A \neq 0$. From the definition of $P_1(w)$ and (35), we know that every zero of $P_1(g) + \frac{B}{A}$ in Ω has a multiplicity of at least n. Here, we wiil consider the three following subcases.

Subcase 1.1: $\frac{B}{A} = -c_1$. From (28), we have

$$P_1(g) + \frac{B}{A} = (g-1)^3(g-a_1)(g-a_2)\cdots(g-a_{n-3}),$$

where $a_i \neq 0, 1$ are distinct values. It follows that

$$\Theta_{\theta-\varepsilon,\theta+\varepsilon}(a_i, f) = 1 - \limsup_{r\to\infty} \frac{\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,a)}{S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)} \geq 1 - \limsup_{r\to\infty} \frac{\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,a)}{C_{\theta-\varepsilon,\theta+\varepsilon}(r,f)} \geq \frac{1}{2}.$$

We can see that it has $n-2$ values satisfying the above inequality. Thus, from Lemma 2.2 and $n \geq 11$, we can get a contradiction.

Subcase 1.2: $\frac{B}{A} = -c_2$. From (28), we have

$$P_1(g) + \frac{B}{A} = g^{n-2}(g-b_1)(g-b_2),$$

where $b_1 \neq b_2, b_i \neq 0, 1 (i = 1, 2)$. It follows that every zero of g in Ω has a multiplicity at least 2 and every zero of $g - b_i (i = 1, 2)$ in Ω has a multiplicity at least n. Then, by Lemma 2.2, we have

$$\begin{aligned} S_{\theta-\varepsilon,\theta+\varepsilon}(r,g) &\leq \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{g}) + \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{g-b_1}) + \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{g-b_2}) + R(r) \\ &\leq \frac{1}{2}C_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{g}) + \frac{1}{n}C_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{g-b_1}) + \frac{1}{n}C_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{g-b_1}) + R(r) \\ &\leq (\frac{1}{2} + \frac{2}{n})S_{\theta-\varepsilon,\theta+\varepsilon}(r,g) + R(r). \end{aligned}$$

Since $g \in M(\rho(r))$ and $n \geq 11$, we can get a contradiction.

Subcase 1.3: $\frac{B}{A} \neq -c_1, -c_2$. Similar to discuss as in Subcase 1.1 or Subcase 1.2, we can get a contradiction.

Case 2: $A = 0$. If $B \neq 1$, from (35) we have $F = \frac{G}{B}$ that is,

$$P_1(f) = \frac{1}{B}P_1(g). \tag{36}$$

From (29) and (36), we have

$$P_1(f) - \frac{c_2}{B} = \frac{1}{B}(P_1(g) - c_2) = \frac{1}{B}g^{n-2}(g - b_1)(g - b_2). \tag{37}$$

Since $\frac{c_2}{B} \neq c_2$, from (27), it follows that $P_1(f) - \frac{c_2}{B}$ at least $n-2$ distinct zeros $e_1, e_2, \ldots, e_{n-2}$. Then, we have from Lemma 2.2

$$\begin{aligned}(n-4)S_{\theta-\varepsilon,\theta+\varepsilon}(r, f) &\leq \sum_{i=1}^{n-2} \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{f - e_i}) + R(r) \\ &\leq \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{g}) + \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{g - b_1}) + \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{g - b_2}) + R(r) \\ &\leq 3S_{\theta-\varepsilon,\theta+\varepsilon}(r, g) + R(r).\end{aligned} \tag{38}$$

Since f is a meromorphic function of infinite order $\rho(r)$, $\arg z = \theta (0 \leq \theta < 2\pi)$ is one Borel direction of $\rho(r)$ order of meromorphic function f, we have from Lemma 2.4 $S(r, f) = O(\rho(r) \log r)$. Then, applying Lemma 2.7 to (36), and from (38) and $n \geq 11$, we can get a contradiction.

Thus, we have $A = 0$ and $B = 1$, that is, $P_1(f) = P_1(g)$. Note the form of $P_1(w)$, we can get that $P(f) = P(g)$. Then, by Lemma 2.8, we get $f \equiv g$.

Therefore, the proof of Theorem 1.4 is completed.

□

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Pseudo-valuations on BCH-algebras with respect to subalgebras and (closed) ideals

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Abstract. The notion of pseudo-valuations (valuations) on a BCH-algebra is introduced by using the Buşneag's model ([1, 2, 3]), and a pseudo-metric is induced by a pseudo-valuation on BCH-algebras. Conditions for a real-valued function to be an I-pseudo-valuation are provided. The fact that the binary operation in BCH-algebras is uniformly continuous is provided based on the notion of (pseudo) valuation.

1. Introduction

Buşneag [2] defined pseudo-valuation on a Hilbert algebra, and proved that every pseudo-valuation induces a pseudo metric on a Hilbert algebra. Also, Buşneag [3] provided several theorems on extensions of pseudo-valuations. Buşneag [1] introduced the notions of pseudo-valuations (valuations) on residuated lattices, and proved some theorems of extension for these (using the model of Hilbert algebras ([3])). As a generalization of BCK/BCI-algebras, Hu and Li [5, 6] introduced the notion of BCH-algebras, and Chaudhry [4] investigated several properties of BCH-algebras.

In this paper, using the Buşneag's model, we introduce the notion of pseudo-valuations (valuations) on BCH-algebras, and we induce a pseudo-metric by using a pseudo-valuation on BCH-algebras. We provide conditions for a real-valued function on a BCH-algebra X to be an I-pseudo-valuation on X. Based on the notion of (pseudo) valuation, we show that the binary operation $*$ in BCH-algebras is uniformly continuous.

2. Preliminaries

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a *BCH-algebra* if it satisfies the following conditions:

(I) $(\forall x \in X)\ (x * x = 0)$,

(II) $(\forall x, y \in X)\ (x * y = 0,\ y * x = 0 \ \Rightarrow\ x = y)$,

(III) $(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y)$.

Any BCH-algebra X satisfies the following conditions:

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Table 1. $*$-operation

$*$	0	a	b	c	d
0	0	0	0	0	d
a	a	0	0	a	d
b	b	b	0	0	d
c	c	c	c	0	d
d	d	d	d	d	0

(a1) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,
(a2) $(\forall x \in X)\ (x * 0 = 0\ \Rightarrow x = 0)$,
(a3) $(\forall x, y \in X)\ (0 * (x * y) = (0 * x) * (0 * y))$,
(a4) $(\forall x \in X)\ (x * 0 = x)$.

A BCH_0-*algebra* is a BCH-algebra X which satisfies $0 * x = 0$ for all $x \in X$.

We can define a relation $\leq$ on a BCH-algebra X by

$$(\forall x, y \in X)\ (x \leq y\ \Leftrightarrow\ x * y = 0)$$

This relation is reflexive and anti-symmetric but not transitive in general. A non-empty subset S of a BCH-algebra X is called *a subalgebra* of X if $x * y \in S$ for all $x, y \in S$.

A subset I of a BCH-algebra X is called an *ideal* if it satisfies:

(I1) $0 \in I$,
(I2) $(\forall x \in X)\ (\forall y \in I)\ (x * y \in I\ \Rightarrow\ x \in I)$.

Note that an ideal of a BCH-algebra may not be a subalgebra. An ideal I of a BCH-algebra X is said to be *closed* if $0 * x \in I$ for all $x \in I$.

3. Pseudo-valuations on BCH-algebras

In what follows let X denote a BCH-algebra unless otherwise specified.

Definition 3.1. A real-valued function ϑ on X is called a *pseudo-valuation* on X with respect to a subalgebra (briefly, S-*pseudo-valuation* on X) if it satisfies the following condition:

$$(\forall x, y \in X)\ (\vartheta(x * y) \leq \vartheta(x) + \vartheta(y)). \tag{3.1}$$

Example 3.2. Let $X = \{0, a, b, c, d\}$ be a BCH-algebra with the $*$-operation given by Table 1 (see [4]). Let ϑ be a real-valued function on X defined by

$$\vartheta = \begin{pmatrix} 0 & a & b & c & d \\ 0 & 3 & 1 & 2 & 0 \end{pmatrix}.$$

Then ϑ is an S-pseudo-valuation on X.

Proposition 3.3. *Every S-pseudo-valuation ϑ on X satisfies the following inequality:*

$$(\forall x \in X)\ (\vartheta(0 * x) \leq 3\vartheta(x)). \tag{3.2}$$

Proof. For any $x \in X$, we have

$$\begin{aligned}\vartheta(0 * x) &\leq \vartheta(0) + \vartheta(x) = \vartheta(x * x) + \vartheta(x) \\ &\leq \vartheta(x) + \vartheta(x) + \vartheta(x) = 3\vartheta(x)\end{aligned}$$

by using (I). □

Corollary 3.4. *For any S-pseudo-valuation ϑ on a BCH_0-algebra X, we have*

$$(\forall x \in X)\ (\vartheta(x) \geq 0). \tag{3.3}$$

Proof. Straightforward. □

The following example shows that Corollary 3.4 may not be true in BCH-algebras.

Example 3.5. Consider the BCH-algebra X which is given in Example 3.2. Let φ be a real-valued function on X defined by

$$\varphi = \begin{pmatrix} 0 & a & b & c & d \\ -3 & 3 & 1 & 2 & -3 \end{pmatrix}.$$

Then φ is an S-pseudo-valuation on X with negative values.

Theorem 3.6. *Let S be a subalgebra of X. For any real numbers t_1 and t_2 with $0 \leq t_1 < t_2$, let ϑ_S be a real-valued function on X defined by*

$$\vartheta_S(x) = \begin{cases} t_1 & \text{if } x \in S, \\ t_2 & \text{if } x \notin S \end{cases}$$

for all $x \in X$. Then ϑ_S is an S-pseudo-valuation on X.

Proof. Straightforward. □

Theorem 3.7. *If a real-valued function ϑ on X is an S-pseudo-valuation on X, then the set*

$$A := \{x \in X \mid \vartheta(x) \leq 0\}$$

is a subalgebra of X.

Proof. Let $x, y \in A$. Then $\vartheta(x) \leq 0$ and $\vartheta(y) \leq 0$. It follows from (3.1) that

$$\vartheta(x * y) \leq \vartheta(x) + \vartheta(y) \leq 0$$

so that $x * y \in A$. Hence A is a subalgebra of X. □

The following example shows that the converse of Theorem 3.7 may not be true, that is, there exist a BCH-algebra X and a mapping $\vartheta : X \to \mathbb{R}$ such that

(1) ϑ is not an S-pseudo-valuation on X,
(2) $A := \{x \in X \mid \vartheta(x) \leq 0\}$ is a subalgebra of X.

Table 2. $*$-operation

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	b
c	c	c	0	0

Example 3.8. Consider the BCH-algebra X which is given in Example 3.2. Let φ be a real-valued function on X defined by

$$\varphi = \begin{pmatrix} 0 & a & b & c & d \\ -3 & 1 & -5 & 4 & -3 \end{pmatrix}.$$

Then φ is not an S-pseudo-valuation on X since

$$\varphi(c * b) = \varphi(c) = 4 \not\leq -1 = 4 - 5 = \varphi(c) + \varphi(b).$$

But $A := \{x \in X \mid \varphi(x) \leq 0\} = \{0, b, d\}$ is a subalgebra of X.

Corollary 3.9. *If a real-valued function ϑ on a BCH_0-algebra X is an S-pseudo-valuation on X, then the set*

$$A := \{x \in X \mid \vartheta(x) = 0\}$$

is a subalgebra of X.

Definition 3.10. A real-valued function ϑ on X is called a *pseudo-valuation* on X with respect to an ideal (briefly, *I-pseudo-valuation* on X) if it satisfies the following two conditions:

(i) $\vartheta(0) = 0$,
(ii) $(\forall x, y \in X)\ (\vartheta(x) \leq \vartheta(x * y) + \vartheta(y))$.

Definition 3.11. A real-valued function ϑ on X is called a *pseudo-valuation* on X with respect to a closed ideal (briefly, *CI-pseudo-valuation* on X) if it satisfies the following condition:

$$(\forall x, y \in X)\ (\vartheta(0 * x) \leq \vartheta(x) \leq \vartheta(x * y) + \vartheta(y)).$$

If ϑ is an S-pseudo-valuation (resp. I-pseudo-valuation and CI-pseudo-valuation) on X satisfying the following condition:

$$(\forall x \in X)\ (x \neq 0 \ \Rightarrow\ \vartheta(x) \neq 0)$$

then we say that ϑ is an *S-valuation* (resp. *I-valuation* and *CI-valuation*) on X.

Example 3.12. Let $X = \{0, a, b, c\}$ be a BCH-algebra with the $*$-operation given by Table 2. Let ϑ be a real-valued function on X defined by

$$\vartheta = \begin{pmatrix} 0 & a & b & c \\ 0 & 1 & 3 & 2 \end{pmatrix}.$$

Then ϑ is both a CI-pseudo-valuation and an I-pseudo-valuation on X.

Theorem 3.13. *Every CI-pseudo-valuation is an S-pseudo-valuation.*

Proof. Let ϑ be a CI-pseudo-valuation on X and let $x, y \in X$. Then

$$\begin{aligned}\vartheta(x * y) &\leq \vartheta((x * y) * x) + \vartheta(x)\\ &= \vartheta((x * x) * y) + \vartheta(x)\\ &= \vartheta(0 * y) + \vartheta(x) \leq \vartheta(x) + \vartheta(y)\end{aligned}$$

Hence ϑ is an S-pseudo-valuation on X. □

The following example shows that the converse of Theorem 3.13 may not be true.

Example 3.14. Consider an S-pseudo-valuation ϑ which is given in Example 3.2. Then ϑ is not a CI-pseudo-valuation since $\vartheta(a) \not\leq \vartheta(a * b) + \vartheta(b)$.

The following example shows that a CI-pseudo-valuation may not be an I-pseudo-valuation.

Example 3.15. Consider the BCH-algebra X which is given Example 3.12. Let ϑ be a real-valued function on X defined by

$$\vartheta = \begin{pmatrix} 0 & a & b & c \\ 1 & 1 & 3 & 2 \end{pmatrix}.$$

Then ϑ is a CI-pseudo-valuation on X, but not an I-pseudo-valuation on X since $\vartheta(0) = 1 \neq 0$.

Proposition 3.16. *In a BCH_0-algebra, every CI-pseudo-valuation ϑ satisfies the following inequality:*

$$(\forall x, y)\ (\vartheta(x * y) \leq \vartheta(0) + \vartheta(x)).$$

Proof. For any $x, y \in X$, we have

$$\begin{aligned}\vartheta(x * y) &\leq \vartheta((x * y) * x) + \vartheta(x)\\ &= \vartheta((x * x) * y) + \vartheta(x)\\ &= \vartheta(0 * y) + \vartheta(x) = \vartheta(0) + \vartheta(x)\end{aligned}$$

This completes the proof. □

Theorem 3.17. *In a BCH_0-algebra, every I-pseudo-valuation is an S-pseudo-valuation.*

Proof. Let ϑ be an I-pseudo-valuation on a BCH_0-algebra X. Since

$$((x * y) * x) * y = ((x * x) * y) * y = (0 * y) * y = 0$$

for all $x, y \in X$, we have

$$\begin{aligned}0 = \vartheta(0) &= \vartheta(((x * y) * x) * y)\\ &\geq \vartheta((x * y) * x) - \vartheta(y)\\ &\geq \vartheta(x * y) - \vartheta(x) - \vartheta(y).\end{aligned}$$

Hence $\vartheta(x * y) \leq \vartheta(x) + \vartheta(y)$, and therefore ϑ is an S-pseudo-valuation on X. □

TABLE 3. $*$-operation

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	0
c	c	c	c	0

The converse of Theorem 3.17 is not true as shown by the following example.

Example 3.18. Let $X = \{0, a, b, c\}$ be a BCH_0-algebra with the $*$-operation given by Table 3. Let ϑ be a real-valued function on X defined by

$$\vartheta = \begin{pmatrix} 0 & a & b & c \\ 0 & 4 & 1 & 3 \end{pmatrix}.$$

Then ϑ is an S-pseudo-valuation , but not an I-pseudo-valuation on X since $\vartheta(a) = 4 \not\leq 1 = \vartheta(0) + \vartheta(b) = \vartheta(a * b) + \vartheta(b)$.

In general, Theorem 3.17 may not be true in a BCH-algebra as shown by the following example.

Example 3.19. Consider a commutative group $(\mathbb{R}, +, 0)$. Then $(\mathbb{R}, *, 0)$, where $x * y = x - y$, is a BCH-algebra which is not a BCH_0-algebra. Define a real-valued function ϑ on $\mathbb{R}$ by

$$\vartheta(x) = \begin{cases} 0 & \text{if } x = 0, \\ -3x + 1 & \text{otherwise} \end{cases}$$

for all $x \in \mathbb{R}$. Then ϑ is an I-pseudo-valuation on $\mathbb{R}$, but it is not an S-pseudo-valuation on $\mathbb{R}$ since $\vartheta(1 * 2) = \vartheta(-1) = 4 \not\leq -7 = \vartheta(1) + \vartheta(2)$. Moreover ϑ is not a CI-pseudo-valuation on $\mathbb{R}$.

Proposition 3.20. *For any I-pseudo-valuation ϑ on X, we have the following inequalities:*

(1) *ϑ is order preserving.*
(2) $(\forall x, y \in X)\ (\vartheta(x * y) + \vartheta(y * x) \geq 0)$.

Proof. (1) Let $x, y \in X$ be such that $x \leq y$. Then $x * y = 0$, and so

$$\vartheta(x) \leq \vartheta(x * y) + \vartheta(y) = \vartheta(0) + \vartheta(y) = \vartheta(y).$$

(2) Let $x, y \in X$. Using Definition 3.10(ii), we have $\vartheta(x*y) \geq \vartheta(x) - \vartheta(y)$ and $\vartheta(y*x) \geq \vartheta(y) - \vartheta(x)$. It follows that $\vartheta(x * y) + \vartheta(y * x) \geq 0$. □

We provide conditions for a real-valued function on X to be an I-pseudo-valuation on X.

Theorem 3.21. *If a real-valued function ϑ on X satisfies Definition* 3.10(i) *and*

$$(\forall x, y, z \in X)\ (\vartheta(((x * y) * y) * z) \geq \vartheta(x * y) - \vartheta(z)) \tag{3.4}$$

then ϑ is an I-pseudo-valuation on X.

Proof. Taking $y = 0$ in (3.4) and using (a4), we have

$$\vartheta(x * z) = \vartheta(((x * 0) * 0) * z) \geq \vartheta(x * 0) - \vartheta(z) = \vartheta(x) - \vartheta(z),$$

that is, $\vartheta(x) \leq \vartheta(x * z) + \vartheta(z)$ for all $x, z \in X$. Hence ϑ is an I-pseudo-valuation on X. $\square$

In a BCH_0-algebra X, every I-pseudo-valuation ϑ on X satisfies the inequality (3.3). We know from Example 3.19 that an I-pseudo-valuation ϑ on a BCH-algebra X does not satisfy the inequality (3.3).

Definition 3.22. An I-pseudo-valuation on a BCH-algebra X is said to be *positive* if it satisfies the inequality (3.3).

Example 3.23. Consider a CI-pseudo-valuation ϑ which is given in Example 3.12. Then ϑ is a positive I-pseudo-valuation.

Theorem 3.24. *Let ϑ be a real-valued function on X such that*

(1) $(\forall x, y \in X)\ (x \leq y \ \Rightarrow\ \vartheta(x) \leq \vartheta(y))$,
(2) $(\forall x, y \in X)\ (\vartheta(x) \leq \vartheta(x * y) + \vartheta(y))$.

Then the set

$$A := \{x \in X \mid \vartheta(x) \leq 0\} \cup \{0\}$$

is an ideal of X.

Proof. Let $x, y \in X$ be such that $x * y \in A$ and $y \in A$. If $y = 0$, then $x = x * 0 \in A$. Assume that $y \neq 0$. Then $\vartheta(y) \leq 0$. If $x * y = 0$, then $x \leq y$ which implies from (1) that $\vartheta(x) \leq \vartheta(y) \leq 0$. Hence $x \in A$. If $x * y \neq 0$, then $\vartheta(x * y) \leq 0$. Using (2), we have $\vartheta(x) \leq \vartheta(x * y) + \vartheta(y) \leq 0$ and so $x \in A$. Therefore A is an ideal of X. $\square$

Corollary 3.25. *For any I-pseudo-valuation ϑ on X, the set*

$$A := \{x \in X \mid \vartheta(x) \leq 0\}$$

is an ideal of X.

Proof. Straightforward. $\square$

The following example illustrates Corollary 3.25.

Example 3.26. Consider the I-pseudo-valuation ϑ on $\mathbb{R}$ which is described in Example 3.19. Then $A = \{x \in \mathbb{R} \mid x \geq \frac{1}{3}\} \cup \{0\}$ which is an ideal of $\mathbb{R}$.

Theorem 3.27. *If an I-pseudo-valuation ϑ on X is positive, then the set*

$$A := \{x \in X \mid \vartheta(x) = 0\}$$

is an ideal of X.

Proof. Obviously, $0 \in A$. Let $x, y \in X$ be such that $x * y \in A$ and $y \in A$. Then $\vartheta(x * y) = 0$ and $\vartheta(y) = 0$. It follows from Definition 3.10(ii) that $\vartheta(x) \leq \vartheta(x * y) + \vartheta(y) = 0$ so that $\vartheta(x) = 0$ since ϑ is positive. Hence $x \in A$, which shows that A is an ideal of X. $\square$

TABLE 4. $*$-operation

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	0
c	c	c	b	0

Combining Theorems 3.17 and 3.27, and Corollary 3.4, we have the following corollary.

Corollary 3.28. *Let X be a BCH_0-algebra. If a real-valued function ϑ on X is an I-pseudo-valuation on X, then the set*

$$A := \{x \in X \mid \vartheta(x) = 0\}$$

is an ideal of X.

Given a real-valued function ϑ on X, consider the following condition:

$$(\forall x, y, z \in X)\ (\vartheta(x * y) \leq \vartheta(x * z) + \vartheta(z * y)). \tag{3.5}$$

In general, a real-valued function ϑ on X does not satisfy the condition (3.5) as shown by the following example.

Example 3.29. Let ϑ be a real-valued function on X which is given in Example 3.12. Then ϑ does not satisfy the condition (3.5) since $\vartheta(0*b) = \vartheta(b) = 3 \not\leq 2 = 2+0 = \vartheta(c)+\vartheta(0) = \vartheta(0*c)+\vartheta(c*b)$.

Example 3.30. Let $X = \{0, a, b, c\}$ be a BCH-algebra with the $*$-operation given by Table 4. Let ϑ be a real-valued function on X defined by

$$\vartheta = \begin{pmatrix} 0 & a & b & c \\ 1 & 1 & 2 & 3 \end{pmatrix}.$$

Then ϑ satisfies the condition (3.5).

By a *pseudo-metric space* we mean an ordered pair (M, d), where M is a non-empty set and $d : M \times M \to \mathbb{R}$ is a positive function such that the following properties are satisfied: $d(x, x) = 0, d(x, y) = d(y, x)$ and $d(x, z) \leq d(x, y) + d(y, z)$ for every $x, y, z \in M$. If in the pseudo-metric space (M, d) the implication $d(x, y) = 0 \Rightarrow x = y$ holds, then (M, d) is called a *metric space*. For a real-valued function ϑ on X, define a mapping $d_\vartheta : X \times X \to \mathbb{R}$ by $d_\vartheta(x, y) = \vartheta(x * y) + \vartheta(y * x)$ for all $(x, y) \in X \times X$.

Theorem 3.31. *If a real-valued function ϑ on X is an I-pseudo-valuation on X and satisfies the condition* (3.5)*, then d_ϑ is a pseudo-metric on X, and so (X, d_ϑ) is a pseudo-metric space.*

We say d_ϑ is the I-pseudo-metric induced by an I-pseudo-valuation ϑ.

Proof. Obviously, $d_\vartheta(x,y) \geq 0$, $d_\vartheta(x,x) = 0$ and $d_\vartheta(x,y) = d_\vartheta(y,x)$ for all $x, y \in X$. Let $x, y, z \in X$. Using the condition (3.5), we have

$$\begin{aligned} d_\vartheta(x,y) + d_\vartheta(y,z) &= [\vartheta(x * y) + \vartheta(y * x)] + [\vartheta(y * z) + \vartheta(z * y)] \\ &= [\vartheta(x * y) + \vartheta(y * z)] + [\vartheta(z * y) + \vartheta(y * x)] \\ &\geq \vartheta(x * z) + \vartheta(z * x) = d_\vartheta(x,z). \end{aligned}$$

Therefore (X, d_ϑ) is a pseudo-metric space. □

Theorem 3.32. *For a real-valued function ϑ on X satisfying the condition* (3.5)*, if d_ϑ is a pseudo-metric on X, then $(X \times X, d_\vartheta^*)$ is a pseudo-metric space, where*

$$d_\vartheta^*((x,y),(a,b)) = \max\{d_\vartheta(x,a), d_\vartheta(y,b)\}$$

for all $(x,y),(a,b) \in X \times X$.

Proof. Suppose d_ϑ is a pseudo-metric on X. For any $(x,y),(a,b) \in X \times X$, we have

$$d_\vartheta^*((x,y),(x,y)) = \max\{d_\vartheta(x,x), d_\vartheta(y,y)\} = 0$$

and

$$\begin{aligned} d_\vartheta^*((x,y),(a,b)) &= \max\{d_\vartheta(x,a), d_\vartheta(y,b)\} \\ &= \max\{d_\vartheta(a,x), d_\vartheta(b,y)\} \\ &= d_\vartheta^*((a,b),(x,y)). \end{aligned}$$

Now let $(x,y),(a,b),(u,v) \in X \times X$. Then

$$\begin{aligned} & d_\vartheta^*((x,y),(u,v)) + d_\vartheta^*((u,v),(a,b)) \\ &= \max\{d_\vartheta(x,u), d_\vartheta(y,v)\} + \max\{d_\vartheta(u,a), d_\vartheta(v,b)\} \\ &\geq \max\{d_\vartheta(x,u) + d_\vartheta(u,a), d_\vartheta(y,v) + d_\vartheta(v,b)\} \\ &\geq \max\{d_\vartheta(x,a), d_\vartheta(y,b)\} \\ &= d_\vartheta^*((x,y),(a,b)). \end{aligned}$$

Therefore $(X \times X, d_\vartheta^*)$ is a pseudo-metric space. □

Corollary 3.33. *If $\vartheta : X \to \mathbb{R}$ is an I-pseudo-valuation on X and satisfies the condition* (3.5)*, then $(X \times X, d_\vartheta^*)$ is a pseudo-metric space.*

Theorem 3.34. *If $\vartheta : X \to \mathbb{R}$ is a positive I-valuation on Xsatisfying the condition* (3.5)*, then (X, d_ϑ) is a metric space.*

Proof. Suppose ϑ is a positive I-valuation on X. Then (X, d_ϑ) is a pseudo-metric space by Theorem 3.31. Let $x, y \in X$ be such that $d_\vartheta(x,y) = 0$. Then $0 = d_\vartheta(x,y) = \vartheta(x * y) + \vartheta(y * x)$, and so $\vartheta(x * y) = 0$ and $\vartheta(y * x) = 0$ since ϑ is positive. Also, since ϑ is an I-valuation on X, it follows that $x * y = 0$ and $y * x = 0$ so from (II) that $x = y$. Therefore (X, d_ϑ) is a metric space. □

Corollary 3.35. *If a real-valued function ϑ on a BCH_0-algebra X is an I-valuation and satisfies the condition* (3.5)*, then (X, d_ϑ) is a metric space.*

Theorem 3.36. *If $\vartheta : X \to \mathbb{R}$ is a positive I-valuation on X which satisfies the condition* (3.5)*, then $(X \times X, d_\vartheta^*)$ is a metric space.*

Proof. Note from Corollary 3.33 that $(X \times X, d_\vartheta^*)$ is a pseudo-metric space. Let $(x, y), (a, b) \in X \times X$ be such that $d_\vartheta^*((x, y), (a, b)) = 0$. Then

$$0 = d_\vartheta^*((x, y), (a, b)) = \max\{d_\vartheta(x, a), d_\vartheta(y, b)\},$$

and so $d_\vartheta(x, a) = 0 = d_\vartheta(y, b)$ since $d_\vartheta(x, y) \geq 0$ for all $(x, y) \in X \times X$. Hence

$$0 = d_\vartheta(x, a) = \vartheta(x * a) + \vartheta(a * x)$$

and

$$0 = d_\vartheta(y, b) = \vartheta(y * b) + \vartheta(b * y).$$

Since ϑ is positive, it follows that $\vartheta(x * a) = 0 = \vartheta(a * x)$ and $\vartheta(y * b) = 0 = \vartheta(b * y)$ so that $x * a = 0 = a * x$ and $y * b = 0 = b * y$. Using (II), we have $a = x$ and $b = y$, and so $(x, y) = (a, b)$. Therefore $(X \times X, d_\vartheta^*)$ is a metric space. □

Corollary 3.37. *If $\vartheta : X \to \mathbb{R}$ is an I-valuation on a BCH_0-algebra X satisfying the condition* (3.5)*, then $(X \times X, d_\vartheta^*)$ is a metric space.*

Proposition 3.38. *Let ϑ be an I-pseudo-valuation on X which satisfies the condition* (3.5)*. Then the I-pseudo-metric d_ϑ induced by ϑ satisfies the following inequality:*

$$d_\vartheta(x * y, a * b) \leq d_\vartheta(x * y, a * y) + d_\vartheta(a * y, a * b) \tag{3.6}$$

for all $a, b, x, y \in X$.

Proof. Using the condition (3.5), we have

$$\vartheta((x * y) * (a * b)) \leq \vartheta((x * y) * (a * y)) + \vartheta((a * y) * (a * b)),$$

$$\vartheta((a * b) * (x * y)) \leq \vartheta((a * b) * (a * y)) + \vartheta((a * y) * (x * y))$$

for all $x, y, a, b \in X$. Hence

$$\begin{aligned} d_\vartheta(x * y, a * b) &= \vartheta((x * y) * (a * b)) + \vartheta((a * b) * (x * y)) \\ &\leq [\vartheta((x * y) * (a * y)) + \vartheta((a * y) * (a * b))] \\ &\quad + [\vartheta((a * b) * (a * y)) + \vartheta((a * y) * (x * y))] \\ &= [\vartheta((x * y) * (a * y)) + \vartheta((a * y) * (x * y))] \\ &\quad + [\vartheta((a * b) * (a * y)) + \vartheta((a * y) * (a * b))] \\ &= d_\vartheta(x * y, a * y) + d_\vartheta(a * y, a * b) \end{aligned}$$

for all $x, y, a, b \in X$. □

Theorem 3.39. *Let ϑ be a positive I-valuation on X which satisfies the condition* (3.5)*. If the I-pseudo-metric d_ϑ induced by ϑ satisfies the following inequality:*

$$d_\vartheta(x, y) \geq \max\{d_\vartheta(x * a, y * a), d_\vartheta(a * x, a * y)\} \tag{3.7}$$

for all $x, y, a, b \in X$, then the operation $*$ in X is uniformly continuous.

Proof. Note that (X, d_ϑ) and $(X \times X, d^*_\vartheta)$ are metric spaces (see Theorems 3.34 and 3.36). For any $\varepsilon > 0$, if $d^*_\vartheta((x, y), (a, b)) < \frac{\varepsilon}{2}$, then $d_\vartheta(x, a) < \frac{\varepsilon}{2}$ and $d_\vartheta(y, b) < \frac{\varepsilon}{2}$. Using (3.7) and Proposition 3.38, we have

$$\begin{aligned} d_\vartheta(x * y, a * b) &\leq d_\vartheta(x * y, a * y) + d_\vartheta(a * y, a * b) \\ &\leq d_\vartheta(x, a) + d_\vartheta(y, b) < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore the operation $* : X \times X \to X$ is uniformly continuous □

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DERIVATIVES OF DECREASING FUNCTIONS WITH RESPECT TO FUZZY MEASURES

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Abstract In this paper, we consider Choquet integrals and derivatives of non-negative, continuous and decreasing functions on the non-positive real line with respect to fuzzy measures. We prove some properties of derivatives of those functions and some examples.

1. Introduction

The concept of fuzzy measure was introduced by Sugeno([7]). We note that Choquet first studied Choquet integral and T. Murofushi et al. studied Choquet integrals with respect to a fuzzy measure([2], [6]). In [8], Sugeno introduced the concept of derivatives of non-negative, continuous and incresing functions on the non-negative real line $\mathbb{R}^+ = [0, \infty)$ with respect to fuzzy measures. Choi showed some basic properties of derivatives of those functions([1]). In this paper, we consider Choquet integrals and derivatives of non-negative, continuous and decreasing functions on the non-positive real line $\mathbb{R}^- = (\infty, 0]$ with respect to fuzzy measures.

We assume that X is a nonempty set. Let $\mathcal{A}$ be the any σ-algebra of subsets of X. Then $(X, \mathcal{A})$ is called a measurable space. A fuzzy measure is a set function $\nu : \mathcal{A} \to \mathbb{R}^+$ which satisfies
(1) $\nu(\phi)$=0,
(2) $\nu(A) \leq \nu(B)$, whenever $A, B \in \mathcal{A}$ and $A \subset B$,
(3) if for every increasing sequence $\{A_n\}$ of measurable sets, then $\nu(\cup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \nu(A_n)$ and if for every decreasing sequence $\{A_n\}$ of measurable sets and $\nu(A_1) < \infty$, then $\nu(\cap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \nu(A_n)$. Recall that a function $h : X \to \mathbb{R}^+$ is said to be measurable if $\{t|h(t) > \alpha\} \in \mathcal{A}$ for all $\alpha \in (-\infty, \infty)$.

In section 2, we give Choquet integrals of non-negative, continuous and decreasing functions on $\mathbb{R}^-$ with respect to fuzzy measure according to the ideas of [8]. In section 3, we investigate some properties and examples of derivatives of those functions with respect to distorted

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Lebesgue measures. Finally, we show that existence and non-existence of derivatives depend on fuzzy measures.

2. Choquet integral of $g \in \mathcal{M}^-$

In this paper, we assume that $X = \mathbb{R}^-$. Let $\mathcal{A}$ be the smallest σ-algebra of subsets of X. Then $(X, \mathcal{A}, \nu)$ is called a fuzzy measure space.

As in [8], Choquet integral of g with respect to a fuzzy measure ν on a set A is defined by

$$(C)\int_A g(t)d\nu = \int_0^\infty \nu\left(\{t|g(t) \geq \alpha\} \cap A\right) d\alpha. \tag{2.1}$$

Let $\mathcal{M}^-$ be the set of measurable, non-negative, continuous and decreasing functions such that $g : \mathbb{R}^- \to \mathbb{R}^+$.

Definition 2.1. *([8]) Let $m : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous and increasing function $m(0) = 0$. A fuzzy measure ν_m, a distorted Lebesgue measure, is defined by*

$$\nu_m(\cdot) = m(\lambda(\cdot)), \tag{2.2}$$

where $\lambda([a, b]) = b - a$ for all $[a, b] \subset \mathbb{R}^-$.

From Definition 2.1 and (2.1), we have the following theorem([8]).

Theorem 1. *We assume that $\nu([t, t]) = 0$ for all $t \in \mathbb{R}^-$. Let $g \in \mathcal{M}^-$, then Choquet integral of g with respect to ν on $[t, 0]$ is represented as*

$$\int_0^\infty \nu(\{t|g(t) \geq \alpha\} \cap [t, 0])d\alpha = \int_t^0 \nu'([t, \tau])g(\tau)d\tau. \tag{2.3}$$

In particular, for $\nu = \nu_m$,

$$\int_0^\infty \nu(\{t|g(t) \geq \alpha\} \cap [t, 0])d\alpha = \int_t^0 m'(\tau - t)g(\tau)d\tau. \tag{2.4}$$

Proof. Let $\alpha = g(\tau), d\alpha = g'(\tau)d\tau$ and $g^{-1}(\alpha) = \tau$. By the definition of Choquet integral, we have that

$$
\begin{aligned}
&(C)\int_{[t,0]} g(\tau)d\nu(\tau) = \int_0^\infty \nu(\{t|g(t) \ge \alpha\} \cap [t,0])d\alpha \\
&= \int_0^{g(0)} \nu(\{t|g(t) \ge \alpha\} \cap [t,0])d\alpha + \int_{g(0)}^{g(t)} \nu(\{t|g(t) \ge \alpha\} \cap [t,0])d\alpha \\
&= \nu([t,0])g(0) + \int_{g(0)}^{g(t)} \nu([t, g^{-1}(\alpha)])d\alpha \\
&= \nu([t,0])g(0) + \int_0^t \nu([t,\tau])g'(\tau)d\tau \\
&= \nu([t,0])g(0) + \Big[\nu([t,\tau])g(\tau)\Big]_0^t - \int_0^t \nu'([t,\tau])g(\tau)d\tau \\
&= \nu([t,0])g(0) + \nu([t,t])g(t) - \nu([t,0])g(0) + \int_t^0 \nu'([t,\tau])g(\tau)d\tau \\
&= \int_t^0 \nu'([t,\tau])g(\tau)d\tau.
\end{aligned}
$$

For $\nu = \nu_m$, we obtain

$$(C)\int_{[t,0]} g(\tau)d\nu(\tau) = \int_t^0 \nu'([t,\tau])g(\tau)d\tau = \int_t^0 m'(\tau - t)g(\tau)d\tau.$$

□

From Theorem 1, we have the following corollary.

Corollary 2. *Let $g(t) = k$ be a constant function for all $t \in \mathbb{R}^-$ and $k \in \mathbb{R}^+$. Then*

$$\int_0^\infty \nu(\{t|g(t) \ge \alpha\} \cap [t,0])d\alpha = k\nu([t,0]).$$

In particular, for $\nu = \nu_m$,

$$\int_0^\infty \nu(\{t|g(t) \ge \alpha\} \cap [t,0])d\alpha = km(-t).$$

Proof. From (2.1) we have that

$$
\begin{aligned}
(C)\int_{[t,0]} g(\tau)d\nu(\tau) &= \int_0^\infty \nu(\{t|g(t) = k \ge \alpha\} \cap [t,0])d\alpha \\
&= \int_0^k \nu([t,0])d\alpha = k\nu([t,0]).
\end{aligned}
$$

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For $\nu = \nu_m$, we obtain that

$$(C)\int_{[t,0]} g(\tau)d\nu(\tau) = k\nu([t,0]) = km(-t).$$

$\square$

Definition 2.2. *Let f be a continuous and decreasing function with $f(0) = 0$. The derivative of f with respect to a fuzzy measure ν_m is defined as the inverse operation of Choquet integral based on (2.4) by*

$$\frac{df(t)}{d\nu_m(t)} = D_m(f) = g(t), \tag{2.5}$$

if $g(t)$ is found to be an element of $\mathcal{M}^-$.

From (2.5), let us consider a class of f's for a given $m(t)$ such that

$$\mathcal{T}_m(\mathcal{M}^-) = \Big\{ f | f(t) = \int_t^0 m'(\tau - t)g(\tau)d\tau, \quad g \in \mathcal{M}^- \Big\}. \tag{2.6}$$

3. Derivatives with respect to distorted Lebesgue measures

In this section, we discuss some properties of derivatives of continuous and decreasing functions with respect to distorted Lebesgue measures. From the conditions of $g(t)$ in (2.5), we obtain the following theorem.

Theorem 3. *$D_m(f)$ is linear for $f \in \mathcal{T}_m(\mathcal{M}^-)$ and non-negative constants.*

Proof. Let $f_1(t), f_2(t) \in \mathcal{T}_m(\mathcal{M}^-)$ and $k \in \mathbb{R}^+$. From the condition of $f_1(t)$ and $f_2(t)$, we have that

$$f_1(t) = \int_t^0 m'(\tau - t)D_m(f_1(\tau))d\tau \tag{3.1}$$

and

$$f_2(t) = \int_t^0 m'(\tau - t)D_m(f_2(\tau))d\tau. \tag{3.2}$$

Adding (3.1) and (3.2), we obtain that

$$f_1(t) + f_2(t) = \int_t^0 m'(\tau - t)\Big\{ D_m(f_1(\tau)) + D_m(f_2(\tau)) \Big\} d\tau.$$

By the definitions of $\mathcal{M}^-$ and $T_m(\mathcal{M}^-)$, we know that

$$D_m(f_1(t)) + D_m(f_2(t)) \in \mathcal{M}^- \quad \text{and} \quad f_1(t) + f_2(t) \in \mathcal{T}_m(\mathcal{M}^-).$$

From (3.1), we see that

$$kf_1(t) = \int_t^0 m'(\tau - t) k D_m(f_1(\tau)) d\tau.$$

Since $kD_m(f_1(t)) \in \mathcal{M}^-$, we have $kf_1(t) \in \mathcal{T}_m(\mathcal{M}^-)$. □

From Definition 2.2 and Theorem 3, we have the following theorem.

Theorem 4. *For $t \in \mathbb{R}^-$, we have the followings:*

(1) $\quad \dfrac{d}{d\nu_m} m(-t) = 1,$

(2) $\quad \dfrac{d}{d\nu_m}\left(n \displaystyle\int_t^0 m(\tau - t)(-\tau)^{n-1} d\tau\right) = (-t)^n, \quad n = 1, 2, \ldots$

(3) $\quad \dfrac{d}{d\nu_m}\left(m(-t) - a \displaystyle\int_t^0 m(\tau - t) e^{a\tau} d\tau\right) = e^{at}, \quad a \leq 0,$

(4) $\quad \dfrac{d}{d\nu_m}\left(\displaystyle\int_t^0 \frac{m(\tau - t)}{1 - \tau} d\tau\right) = \ln(1 - t).$

Proof. (1) From (2.4), we have that

$$\int_t^0 m'(\tau - t) d\tau = m(-t).$$

(2) By $(-t)^n \in \mathcal{M}^-$, $n = 1, 2, \ldots$, we obtain that

$$\begin{aligned}
&\int_t^0 m'(\tau - t)(-\tau)^n d\tau \\
&= \Big[m(\tau - t)(-\tau)^n\Big]_t^0 + n \int_t^0 m(\tau - t)(-\tau)^{n-1} d\tau \\
&= n \int_t^0 m(\tau - t)(-\tau)^{n-1} d\tau.
\end{aligned}$$

(3) Similarly, by $e^{at} \in \mathcal{M}^-$ for all $a \leq 0$, we have that

$$\begin{aligned}
\int_t^0 m'(\tau - t) e^{a\tau} d\tau &= \Big[m(\tau - t) e^{a\tau}\Big]_t^0 - a \int_t^0 m(\tau - t) e^{a\tau} d\tau \\
&= m(-t) - a \int_t^0 m(\tau - t) e^{a\tau} d\tau.
\end{aligned}$$

(4) Since $\ln(1 - t) \in \mathcal{M}^-$, we have

$$\begin{aligned}
\int_t^0 m'(\tau - t) \ln(1 - \tau) d\tau &= [m(\tau - t) \ln(1 - \tau)]_t^0 + \int_t^0 \frac{m(\tau - t)}{1 - \tau} d\tau \\
&= \int_t^0 \frac{m(\tau - t)}{1 - \tau} d\tau.
\end{aligned}$$

Thus (4) is proved. □

By Definition 2.1 and (2.3), we have the following remark.

Remark 1. *We assume that $g(t)$ is even function. Let $\tau - t = p$, $d\tau = dp$, and $0 \le p \le -t$. Then, we have*

$$\begin{aligned}(C)\int_{[t,0]} g(t)d\nu &= \int_t^0 \nu'([t,\tau])g(\tau)d\tau = \int_t^0 \nu'([t,\tau])g(-\tau)d\tau \\ &= \int_0^{-t} \nu'(p)g(-p-t)dp = \int_0^{\alpha} \nu'(\tau)g(\alpha-\tau)d\tau \\ &= \int_0^{\alpha} \nu'(\alpha-\tau)g(\tau)d\tau.\end{aligned}$$

In particular, for $\nu = \nu_m$,

$$(C)\int_{[t,0]} g(t)d\nu = -\int_0^{\alpha} m'(\alpha-\tau)g(\tau)d\tau.$$

From Remark 1, the Choquet integral (2.3) on $\mathbb{R}^-$ is considered as a convolution. That is, under the assumption of even function, we can apply the Laplace transformation for calculations of the Choquet integrals.

As you see (2.5) and (2.6) of Definition 2.2., to find $g(t)$ is same with the solvability of a Volterra integral equation of the first kind for given $f(t)$ and a fuzzy measure ν_m ([4]). In fact, fuzzy measures play important roles in the existence of derivatives. Now we give a theorem to explain the relation with the existence of derivatives and fuzzy measures.

Theorem 5. *(The dependence on fuzzy measures)*

Let $t \in \mathbb{R}^-$.

(1) If $\nu_m([t,0]) = m(-t) = e^{-t} - 1$, then $-t \notin \mathcal{T}_m(\mathcal{M}^-)$, that is, $\nexists D_m(-t)$.

(2) If $\nu_m([t,0]) = m(-t) = -t$, then $-t \in \mathcal{T}_m(\mathcal{M}^-)$, that is, $\exists D_m(-t)$.

Proof. Suppose that $-t \in \mathcal{T}_m(\mathcal{M}^-)$. From (2.6) of Definition 2.2., we know that

$$-t = \int_t^0 m'(\tau - t)x_1(\tau)d\tau = \int_t^0 e^{(\tau-t)}x_1(\tau)d\tau, \tag{3.3}$$

where $D_m(-t) = x_1(t) \in \mathcal{M}^-$.

But differentiating (3.3), we obtain

$$-1 = -e^{-t}\int_t^0 e^{\tau}x_1(\tau)d\tau - x_1(t) = t - x_1(t).$$

From the definition of $\mathcal{M}^-$, we obtain

$$x_1(t) = t + 1 \notin \mathcal{M}^-.$$

(1) is proved by this contradiction.

To prove (2), it is sufficient to find $x_2(t) \in \mathcal{M}^-$ such that

$$-t = \int_t^0 m'(\tau - t)x_2(\tau)d\tau = \int_t^0 x_2(\tau)d\tau. \tag{3.4}$$

Differentiating (3.4), we have

$$x_2(t) = 1 \in \mathcal{M}^-.$$

□

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Some new inequalities for the gamma function

Xiaodong Cao

Abstract

In this paper, we present some new inequalities for the gamma function. The main tools are the multiple-correction method developed in [6, 7] and a generalized Mortici's lemma.

1 Introduction

Duo to its importance in mathematics, the problem of finding new and sharp inequalities for the gamma function and, in particular for large values of x

$$\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}dt, \quad x > 0, \tag{1.1}$$

has attracted the attention of many researchers (see [2, 3, 8, 9, 12, 14, 15, 16, 17, 18] and the references therein). Let's recall some of the classical results. Maybe one of the most well-known formula for approximation the gamma function is the Stirling's formula

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x, \quad x \to +\infty. \tag{1.2}$$

See, e.g. [1, p. 253]. The following two formulas give slightly better estimates than Stirling's formula,

$$\Gamma(x+1) \sim \sqrt{2\pi}\left(\frac{x+\frac{1}{2}}{e}\right)^{x+\frac{1}{2}}, \quad \text{(Burnside [5], 1917)}, \tag{1.3}$$

$$\Gamma(x+1) \sim \sqrt{2\pi}\left(\frac{x}{e}\right)^x\sqrt{x+\frac{1}{6}}, \quad \text{(Gosper [10], 1978)}. \tag{1.4}$$

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Ramanujian [22] proposed the claim (without proof) for the gamma function

$$\Gamma(x+1) = \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3 + 4x^2 + x + \frac{\theta_x}{30}\right)^{\frac{1}{6}}, \tag{1.5}$$

where $\theta_x \to 1$ as $x \to +\infty$ and $\frac{3}{10} < \theta_x < 1$. This open problem was solved by Karatsuba[13]. Thus (1.5) provides a more accurate estimate for the gamma function (see Sec. 2 below).

In this paper, we will continue the previous works [6, 7], and introduce a class of new approximations to improve these inequalities.

Throughout the paper, the notation $\Psi(k;x)$ denotes a polynomial of degree k in x with all coefficients non-negative, which may be different at each occurrence. Let $(a_n)_{n\ge1}$ and $(b_n)_{n\ge0}$ be two sequences of real numbers with $a_n \neq 0$ for all $n \in \mathbb{N}$. The generalized continued fraction

$$\tau = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots}} = b_0 + \frac{a_1}{b_1 +}\frac{a_2}{b_2+}\cdots = b_0 + \mathop{\mathbf{K}}_{n=0}^{\infty}\frac{a_n}{b_n}$$

is defined as the limit of the nth approximant

$$\frac{A_n}{B_n} = b_0 + \mathop{\mathbf{K}}_{k=1}^{n}\frac{a_k}{b_k}$$

as n tends to infinity. See [2, p.105].

2 A generalized Mortici's lemma

Mortici [14] established a very useful tool for measuring the rate of convergence, which says that a sequence $(x_n)_{n\ge1}$ converging to zero is the fastest possible when the difference $(x_n - x_{n+1})_{n\ge1}$ is the fastest possible. Since then, Mortici's lemma has been effectively applied in many paper such as [6, 7, 17, 18]. The following lemma is a generalization of Mortici's lemma.

Lemma 1. *If $\lim_{x\to+\infty} f(x) = 0$, and there exists the limit*

$$\lim_{x\to+\infty} x^{\lambda}\left(f(x) - f(x+1)\right) = l \in \mathbb{R}, \tag{2.1}$$

with $\lambda > 1$, then there exists the limit

$$\lim_{x\to+\infty} x^{\lambda-1} f(x) = \frac{l}{\lambda-1}. \tag{2.2}$$

Proof. It is not very difficult to prove that for $x > 2$

$$\frac{1}{(\lambda-1)x^{\lambda-1}} = \int_x^{+\infty}\frac{dt}{t^{\lambda}} \le \sum_{j=0}^{\infty}\frac{1}{(x+j)^{\lambda}} \le \int_{x-1}^{+\infty}\frac{dt}{t^{\lambda}} = \frac{1}{(\lambda-1)(x-1)^{\lambda-1}}. \tag{2.3}$$

2

For $\varepsilon > 0$, we assume that $l - \varepsilon \le x^{\lambda}\left(f(x) - f(x+1)\right) \le l + \varepsilon$ for every real number x greater than or equal to the rank $X_0 > 0$. By adding the inequalities of the form

$$(l-\varepsilon)\frac{1}{x^{\lambda}} \le f(x) - f(x+1) \le (l+\varepsilon)\frac{1}{x^{\lambda}}, \tag{2.4}$$

we get

$$(l-\varepsilon)\sum_{j=0}^{m-1}\frac{1}{(x+j)^{\lambda}} \le f(x) - f(x+m) \le (l+\varepsilon)\sum_{j=0}^{m-1}\frac{1}{(x+j)^{\lambda}} \tag{2.5}$$

for every $x \ge X_0$ and $m \ge 1$. By taking the limit as $m \to \infty$, then multiplying by $x^{\lambda-1}$, we obtain

$$(l-\varepsilon)x^{\lambda-1}\sum_{j=0}^{\infty}\frac{1}{(x+j)^{\lambda}} \le x^{\lambda-1}f(x) \le (l+\varepsilon)x^{\lambda-1}\sum_{j=0}^{\infty}\frac{1}{(x+j)^{\lambda}}. \tag{2.6}$$

It follows from (2.3) that

$$\frac{l-\varepsilon}{\lambda-1} \le x^{\lambda-1}f(x) \le \frac{l+\varepsilon}{\lambda-1}\frac{x^{\lambda-1}}{(x-1)^{\lambda-1}}. \tag{2.7}$$

Now by taking the limit as $x \to +\infty$, this completes the proof of the lemma at once. □

An example Let's consider the Ramanujan's asymptotic formula (1.5). Let the error term $E(x)$ be defined by the following relation

$$\Gamma(x+1) = \sqrt{\pi}\left(\frac{x}{e}\right)^{x}\left(8x^3 + 4x^2 + x + \frac{1}{30}\right)^{\frac{1}{6}}\left(1 + E(x)\right). \tag{2.8}$$

It follows readily from the recurrence formula $\Gamma(x+1) = x\Gamma(x)$ that

$$\begin{aligned}\ln\left(1+E(x)\right) - \ln\left(1+E(x+1)\right) = &-1 + x\ln\left(1+\frac{1}{x}\right) \\ &+ \frac{1}{6}\ln\frac{8(x+1)^3 + 4(x+1)^2 + (x+1) + \frac{1}{30}}{8x^3 + 4x^2 + x + \frac{1}{30}}.\end{aligned} \tag{2.9}$$

By using the *Mathematica* software, we expand the right-hand function in the above formula as a power series in terms of $1/x$:

$$\ln\left(1+E(x)\right) - \ln\left(1+E(x+1)\right) = \frac{11}{2880x^5} + O(\frac{1}{x^6}). \tag{2.10}$$

Thus, by Lemma 1 we have

$$\lim_{x\to+\infty} x^4\ln\left(1+E(x)\right) = \frac{11}{11520}. \tag{2.11}$$

Noting that $\lim_{u\to 0}\frac{\ln(1+u)}{u} = 1$, one get finally

$$\lim_{x\to+\infty} x^4 E(x) = \frac{11}{11520}. \tag{2.12}$$

Remark 1. Just as Motici's lemma, Lemma 1 also provides a method for finding the limit of a function as x tends to infinity.

3

3 Gosper-type inequalities

In this section, we use an example to illustrate the idea of this paper. To this end, we introduce some class of correction function $(\mathrm{MC}_k(x))_{k\ge 0}$ such that the relative error function $E_k(x)$ has the fastest possible rate of convergence, which are defined by the relations

$$\Gamma(x+1)=\sqrt{2\pi}\left(\frac{x}{e}\right)^x\sqrt{x+\frac{1}{6}+\mathrm{MC}_k(x)}\cdot\exp(E_k(x)). \tag{3.1}$$

If $\lim_{x\to+\infty}x^\mu f(x)=l\neq 0$ with constant $\mu>0$, we say that the function $f(x)$ is order $x^{-\mu}$, and write the exponent of convergence $\mu=\mu(f(x))$. Clearly if $\mu(E_k(x))=\mu_k$, we have the following asymptotic formula

$$\Gamma(x+1)=\sqrt{2\pi}\left(\frac{x}{e}\right)^x\sqrt{x+\frac{1}{6}+\mathrm{MC}_k(x)}\cdot\left(1+O(x^{-\mu_k})\right),\quad x\to+\infty. \tag{3.2}$$

Let us briefly review a so-called *multiple-correction method* presented in our previous paper [6, 7]. Actually, the *multiple-correction method* is a recursive algorithm, and one of its advantages is that by repeating correction process we always can accelerate the convergence, i.e. the sequence $(\mu(E_k(x)))_{k\ge 0}$ is a strictly increasing. The key step is to find a suitable structure of $\mathrm{MC}_k(x)$. In general, the correction function $\mathrm{MC}_k(x)$ is a finite *generalized* continued fraction (see [7] or (3.8) below) or a *hyper-power* series (see [6] or (4.7) below) in x.

It is not difficult to see that (3.1) is equivalent to

$$\ln\Gamma(x+1)=\frac{1}{2}\ln(2\pi)+x(\ln x-1)+\frac{1}{2}\ln\left(x+\mathrm{MC}_k(x)\right)+E_k(x). \tag{3.3}$$

By the recurrence formula $\Gamma(x+1)=x\Gamma(x)$, we have for $x>0$

$$E_k(x)-E_k(x+1)=-1+x\ln\left(1+\frac{1}{x}\right)+\frac{1}{2}\ln\frac{(x+1)+\frac{1}{6}+\mathrm{MC}_k(x+1)}{x+\frac{1}{6}+\mathrm{MC}_k(x)}. \tag{3.4}$$

Now by taking the initial-correction function $\mathrm{MC}_0(x)=\frac{\kappa_0}{x+\lambda_0}$ and using *Mathematica* software, we expand $E_k(x)-E_k(x+1)$ into a power series in terms of $1/x$:

$$\begin{aligned}E_0(x)-E_0(x+1)=&\frac{-\frac{1}{72}+\kappa_0}{x^3}+\frac{17-945\kappa_0-810\kappa_0\lambda_0}{540x^4}+\\&\frac{-641+33120\kappa_0-12960\kappa_0^2+43200\kappa_0\lambda_0+25920\kappa_0\lambda_0^2}{12960x^5}+O\left(\frac{1}{x^6}\right).\end{aligned} \tag{3.5}$$

The fastest possible function $E_0(x)-E_0(x+1)$ is obtained when the first two coefficients in the above formula vanish. In this case, we find $\kappa_0=\frac{1}{72},\lambda_0=\frac{31}{90}$, and

$$E_0(x)-E_0(x+1)=\frac{5929}{1166400x^5}+O\left(\frac{1}{x^6}\right). \tag{3.6}$$

By Lemma 1, we can check that

$$\lim_{x\to+\infty} x^4 E_0(x) = \frac{5929}{4665600}. \tag{3.7}$$

We continue the above correction process to successively determine the correction function $\mathrm{MC}_k(x)$ until some k^* you want. On one hand, to find the related coefficients, we often use an appropriate symbolic computations software because it's huge of computations. On the other hand, the exact expressions at each occurrence also need lot of space. Hence in this paper we omit many related details. For interesting readers, see our previous paper [6, 7]. In fact, we can prove that for $0 \le k \le 3$

$$\mathrm{MC}_k(x) = \mathop{\mathbf{K}}_{j=0}^{k} \frac{\kappa_j}{x+\lambda_j}, \tag{3.8}$$

where

$$\kappa_0 = \frac{1}{72}, \qquad \lambda_0 = \frac{31}{90},$$
$$\kappa_1 = \frac{5929}{32400}, \qquad \lambda_1 = \frac{481937}{3735270},$$
$$\kappa_2 = \frac{76899172249}{248039857296}, \qquad \lambda_2 = \frac{7745462509019287}{19149278075101482},$$
$$\kappa_3 = \frac{786873417270631211749921}{851541507731717527392144}, \qquad \lambda_3 = \frac{20983357458177516853642010672790 71}{30311088872486921466334781589254970}.$$

By Lemma 1 again, we get for some constant $C_k \neq 0$

$$\lim_{x\to+\infty} x^{2k+4} E_k(x) = C_k, \quad (k = 0, 1, 2, 3), \tag{3.9}$$

i.e. $\mu(E_k(x)) = 2k + 4$ for $k = 0, 1, 2, 3$. Thus we obtain more accurate approximation formulas:

$$\Gamma(x+1) = \sqrt{2\pi}\left(\frac{x}{e}\right)^x \sqrt{x + \frac{1}{6} + \mathrm{MC}_k(x)} \cdot \left(1 + O(x^{-(2k+4)})\right), \quad x \to +\infty. \tag{3.10}$$

It should be noted that if we rewrite $\mathrm{MC}_k(x)$ in the form of $\frac{P_r(m)}{Q_s(m)}$, where P, Q are polynomials with $r = k$ and $s = k + 1$, theoretically at least, for a large x the above formula may reduce or eliminate numerically computations compared with the previous results, see e.g. [9, 12]. This is the main advantage of the *multiple-correction method*.

The following theorem tells us how to obtain sharp inequalities.

Theorem 1. *Let* $\mathrm{MC}_k(x)$ *be defined as* (3.8). *Let* $x \ge 1$, *then we have for* $k = 0, 2$,

$$\Gamma(x+1) > \sqrt{2\pi}\left(\frac{x}{e}\right)^x \sqrt{x + \frac{1}{6} + \mathrm{MC}_k(x)}, \tag{3.11}$$

and for $k = 1, 3$,

$$\Gamma(x+1) < \sqrt{2\pi}\left(\frac{x}{e}\right)^x \sqrt{x + \frac{1}{6} + \mathrm{MC}_k(x)}. \tag{3.12}$$

Proof. We let $f_k(x) = E_k(x) - E_k(x+1)$. Clearly if $\lim_{x\to+\infty} E_k(x) = 0$, then $E_k(x) = \sum_{j=0}^{\infty} f_k(x+j)$. This transformation plays an important role in this paper (essentially, it is a difference method). Hence, in order to prove inequality $E_k(x) > 0$ (or $E_k(x) < 0$), it suffices to show that the equality $f_k(x) > 0$ (or $f_k(x) < 0$) holds under the condition $\lim_{x\to+\infty} E_k(x) = 0$. By the Stirling's formula (1.2), we can show that the condition $\lim_{x\to+\infty} E_k(x) = 0$ always holds. In what follows, we will apply this condition many times.

By using *Mathematica* software, we may prove that for $x \ge 1$

$$f_0''(x) = \frac{\Psi_1(8;x)}{x(1+x)^2(31+90x)^2(121+90x)^2(77+552x+1080x^2)^2(1709+2712x+1080x^2)^2} > 0,$$
$$f_1''(x) = -\frac{\Psi_2(13;x)(x-1)+1463\cdots9447}{x(1+x)^2)(1359251+2829648x+5976432x^2)^2\Psi_3(16;x)} < 0,$$
$$f_2''(x) = \frac{\Psi_4(20;x)}{x(1+x)^2\Psi_5(28;x)} > 0,$$
$$f_3''(x) = \frac{\Psi_6(25;x)(x-1)+17135\cdots66999}{x(1+x)^2\Psi_7(36;x)} < 0.$$

We only give the proof of inequalities in case $k = 3$, other may be proved similarly. In this case, we see that for $x \ge 1$ the inequality (3.12) is equivalent to $E_3(x) < 0$. As $\lim_{x\to+\infty} E_3(x) = 0$, it suffices to prove that $f_3(x) < 0$ for $x \ge 1$. Since $f_3'(x)$ is strictly decreasing, but $\lim_{x\to+\infty} f_3'(x) = 0$, so $f_3'(x) > 0$. Thus $f_3(x)$ is strictly increasing with $\lim_{x\to+\infty} f_3(x) = 0$, so $f_3(x) < 0$. This completes the proof of Theorem 1. □

By the *multiple-correction* method, we also find another kind of inequalities.

Theorem 2. *Let the k-th correction function* $\mathrm{MC}_k(x)$ *be defined by*

$$\mathrm{MC}_0(x) = \frac{\kappa_0}{(x+\frac{23}{90})^2+\lambda_0},$$
$$\mathrm{MC}_k(x) = \cfrac{\kappa_0}{(x+\frac{23}{90})^2+\lambda_0+} \mathop{\mathbf{K}}_{j=1}^{k} \frac{\kappa_j}{x+\lambda_j}, \quad (k \ge 1),$$

where

$$\kappa_0 = -\frac{1}{144}, \qquad \lambda_0 = \frac{4007}{21600},$$
$$\kappa_1 = \frac{4394}{637875}, \qquad \lambda_1 = \frac{130311599}{15575040},$$
$$\kappa_2 = \frac{7894414898425}{119793516544}, \qquad \lambda_2 = -\frac{265702682899837009577}{344276317894782873 60},$$
$$\kappa_3 = \frac{18975608492521061778584657 92}{771748133425325782673471473 95}, \quad \lambda_3 = \frac{303203804556162930048989281631315632448119 79}{613436431567206532574665270824029803422720 0}.$$

Then we have

$$\Gamma(x+1) < \sqrt{2\pi}\left(\frac{x}{e}\right)^x \sqrt{x+\frac{1}{6}}\left(1+\mathrm{MC}_0(x)\right), \quad x \ge 13, \tag{3.13}$$

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$$\Gamma(x+1) < \sqrt{2\pi}\left(\frac{x}{e}\right)^x \sqrt{x+\frac{1}{6}}\left(1+\mathrm{MC}_2(x)\right), \quad x \ge 6, \tag{3.14}$$

and for $k = 1, 3$,

$$\Gamma(x+1) > \sqrt{2\pi}\left(\frac{x}{e}\right)^x \sqrt{x+\frac{1}{6}}\left(1+\mathrm{MC}_k(x)\right), \quad x \ge 1. \tag{3.15}$$

Proof. Since the proof of Theorem 2 is very similar to that of Theorem 1, here we only give the outline of the proof. First, let the relative error function $E_k(x)$ be defined by

$$\Gamma(x+1) = \sqrt{2\pi}\left(\frac{x}{e}\right)^x \sqrt{x+\frac{1}{6}}\left(1+\mathrm{MC}_k(x)\right)\exp(E_k(x)). \tag{3.16}$$

Hence

$$E_k(x) - E_k(x+1) = -1 + x\ln\left(1+\frac{1}{x}\right) + \ln\frac{1+\mathrm{MC}_k(x+1)}{1+\mathrm{MC}_k(x)}. \tag{3.17}$$

By making use of *Mathematica* software and Lemma 1, we can prove

$$\mu(E_k(x)) = 2k+5, \quad (k = 0, 1, 2, 3). \tag{3.18}$$

Next we let $g_k(x) = E_k(x) - E_k(x+1)$. By using *Mathematica* software, it isn't difficult to check that

$$\begin{aligned}
g_0''(x) &= \frac{\Psi_1(14;x)(x-13) + 29707\cdots81369}{x(1+x)^2(1+6x)^2(7+6x)^2\Psi_2(16;x)} > 0, \quad x \ge 13,\\
g_1''(x) &= -\frac{\Psi_3(20;x)(x-1) + 13798\cdots89479}{x(1+x)^2(1+6x)^2(7+6x)^2\Psi_4(24;x)} < 0, \quad x \ge 1,\\
g_2''(x) &= \frac{\Psi_5(26;x)(x-6) + 97250\cdots34321}{x(1+x)^2(1+6x)^2(7+6x)^2\Psi_6(32;x)} > 0, \quad x \ge 6,\\
g_3''(x) &= -\frac{\Psi_7(32;x)(x-1) + 836559\cdots37479}{x(1+x)^2(1+6x)^2(7+6x)^2\Psi_8(40;x)} < 0, \quad x \ge 1.
\end{aligned}$$

Lastly, just as the proof of Theorem1, Theorem 2 follows from the above inequalities readily. □

4 Ramanujan-type inequalities

Theorem 3. *Let the k-th correction function* $\mathrm{MC}_k(x)$ *be defined as*

$$\mathrm{MC}_k(x) = \mathop{\mathbf{K}}_{j=0}^{k} \frac{a_j}{x+b_j}, \tag{4.1}$$

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where

$$a_0 = -\frac{11}{240}, \qquad b_0 = \frac{79}{154},$$
$$a_1 = \frac{459733}{711480}, \qquad b_1 = -\frac{1455925}{70798882},$$
$$a_2 = \frac{49600874140433}{101450127018720}, \qquad b_2 = \frac{1025910896577163509 1}{19545564575317443762},$$
$$a_3 = \frac{16908530533615252713151 1003963}{10122157915179737540319 4730976}, \quad b_3 = -\frac{61414485359080027112199200164888 34171}{20327598783892405080143667029951 7447102}.$$

Let $x \geq 1$, *then for* $k = 0, 2$,

$$\Gamma(x+1) < \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3 + 4x^2 + x + \frac{1}{30} + \mathrm{MC}_k(x)\right)^{\frac{1}{6}}, \tag{4.2}$$

and for $k = 1, 3$,

$$\Gamma(x+1) > \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3 + 4x^2 + x + \frac{1}{30} + \mathrm{MC}_k(x)\right)^{\frac{1}{6}}. \tag{4.3}$$

Proof. We define the relative error function $E_k(x)$ by the relation

$$\Gamma(x+1) = \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3 + 4x^2 + x + \frac{1}{30} + \mathrm{MC}_k(x)\right)^{\frac{1}{6}} \exp(E_k(x)). \tag{4.4}$$

Thus

$$\begin{aligned} E_k(x) - E_k(x+1) = & -1 + x\ln\left(1 + \frac{1}{x}\right) \\ & + \frac{1}{6}\ln\frac{8(x+1)^3 + 4(x+1)^2 + (x+1) + \frac{1}{30} + \mathrm{MC}_k(x+1)}{8x^3 + 4x^2 + x + \frac{1}{30} + \mathrm{MC}_k(x)}. \end{aligned} \tag{4.5}$$

By using *Mathematica* software and Lemma 1, we can check

$$\mu(E_k(x)) = 2k + 6, \quad (k = 0, 1, 2, 3). \tag{4.6}$$

We let $U_k(x) = E_k(x) - E_k(x+1)$. By making use of *Mathematica* software again, we can prove

$$U_0''(x) = \frac{\Psi_1(13; x)(x-1) + 416838558509297754261614731715717}{3x(1+x)^2(79+154x)^2(233+154x)^2\left(\Psi_{21}(3; x)(x-1) + 363565\right)^2 \Psi_{22}^2(4; x)} < 0,$$
$$U_1''(x) = \frac{\Psi_3(19; x)(x-1) + 85653\cdots 25001}{x(1+x)^2\Psi_4(28; x)} > 0,$$
$$U_2''(x) = -\frac{\Psi_5(25; x)(x-1) + 32968\cdots 13479}{x(1+x)^2\Psi_6(36; x)} < 0,$$
$$U_3''(x) = \frac{\Psi_7(31; x)(x-1) + 17145\cdots 57723}{3x(1+x)^2\Psi_8(44; x)} > 0.$$

Similar to the proof of Theorem 1, we can get the desired assertions from the above inequalities. □

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Theorem 4. *Let the first-correction function* $\mathrm{MC}_1^*(x)$ *be defined by*

$$\mathrm{MC}_1^*(x) = \frac{\kappa_0}{x+\lambda_0} + \frac{\kappa_1}{x^3+\lambda_{10}x^2+\lambda_{11}x+\lambda_{12}}, \tag{4.7}$$

where

$$\kappa_0 = -\frac{11}{240}, \qquad \lambda_0 = \frac{79}{154},$$
$$\kappa_1 = \frac{459733}{15523200}, \qquad \lambda_{10} = \frac{71181889}{70798882},$$
$$\lambda_{11} = \frac{717183502490887}{520777318696096}, \qquad \lambda_{12} = \frac{1118629052995381153799}{1958878792277282473920}.$$

Then for $x \ge 1$*, the following inequality holds true*

$$\Gamma(x+1) < \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3+4x^2+x+\frac{1}{30}+\mathrm{MC}_1^*(x)\right)^{\frac{1}{6}}. \tag{4.8}$$

Proof. Let the first-correction error function $E_1^*(x)$ be defined by

$$\Gamma(x+1) = \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3+4x^2+x+\frac{1}{30}+\mathrm{MC}_1^*(x)\right)^{\frac{1}{6}} \exp(E_1^*(x)). \tag{4.9}$$

Hence

$$\begin{aligned} E_1^*(x) - E_1^*(x+1) = & -1 + x\ln\left(1+\frac{1}{x}\right) \\ & + \frac{1}{6}\ln\frac{8(x+1)^3+4(x+1)^2+(x+1)+\frac{1}{30}+\mathrm{MC}_1^*(x+1)}{8x^3+4x^2+x+\frac{1}{30}+\mathrm{MC}_1^*(x)}. \end{aligned} \tag{4.10}$$

By using *Mathematica* software and Lemma 1, we have

$$\mu(E_1^*(x)) = 10. \tag{4.11}$$

Now we let $V(x) = E_1^*(x) - E_1^*(x+1)$. By using *Mathematica* again, we have

$$V_1''(x) = -\frac{\Psi_1(33;x)(x-1) + 96057\cdots 27429}{3x\left(\Psi_2(3;x)\right)^2 \Psi_3(12;x)\left(\Psi_4(6;x)(x-1)+2169\cdots 3461\right)^2 \Psi_5(14;x)} < 0. \tag{4.12}$$

By the same approach as the proof of Theorem 1, the inequality (4.8) follows from the (4.12). □

Remark 2. It is an interesting question whether our method may be used to obtain some sharp bounds for the ratio of the gamma functions, see e.g. [11, 19, 20, 21].

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Mathematical analysis of humoral immunity viral infection model with Hill type infection rate

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Abstract

In this paper, we propose and analyze a viral infection model with humoral immunity. The incidence rate is given by Hill type infection rate. We have derived two threshold parameters, R_0 and R_1 which completely determined the global properties of the model. By constructing suitable Lyapunov functions and applying LaSalle's invariance principle we have established the global asymptotic stability of all steady states of the model. We have proven that, if $R_0 \leq 1$, then the infection-free steady state is globally asymptotically stable (GAS), if $R_1 \leq 1 < R_0$, then the chronic-infection steady state without humoral immune response is GAS, and if $R_1 > 1$, then the chronic-infection steady state with humoral immune response is GAS.

Keywords: Virus infection; Global stability; Immune response; Lyapunov function; Hill type infection rate.

1 Introduction

In recent years, considerable attention has been paid to study the dynamical behaviors of viruses such as human immunodeficiency virus (HIV) (see e.g. [1]-[11]), hepatitis B virus (HBV) [12]-[14], hepatitis C virus (HCV) [15]-[17], human T cell leukemia (HTLV) [18] and dengue virus [19], etc. There are many benifits from mathematical models of viral infection include: (i) it provide important quantitative insights into viral dynamics in vivo, (ii) it can improve diagnosis and treatment strategies which yield to raise hopes of patients with viruses, (iii) it can be used to estimate key parameter values that control the infection process.

Nowak and Bangham [2] proposed the basic viral infection model which contains three variables x, y and v representing the populations of the uninfected target cells, infected cells and free virus particles, respectively. In [20]-[26], the basic model has been modified to take into consideration the humoral immune response. The basic model of viral infection with humoral immune response has been introduced by Murase et. al. [20] and Shifi Wang [26] as:

$$\dot{x} = \lambda - dx - \beta xv, \tag{1}$$

$$\dot{y} = \beta xv - ay, \tag{2}$$

$$\dot{v} = ky - cv - rzv, \tag{3}$$

$$\dot{z} = gzv - \mu z, \tag{4}$$

where z denotes the population of the B cells. Parameters λ, k and g represent, respectively, the rate at which new healthy cells are generated from the source within the body, the generation rate constant of free viruses produced from the infected cells and the proliferation rate constant of B cells. Parameters d, a, c and μ are the natural death rate constants of the uninfected cells, infected cells, free virus particles and B cells, respectively. Parameter β is the infection rate constant and r is the removal rate constant of the virus due to humoral immune response. All the parameters given in model (1)-(4) are positive.

In model (1)-(4), the incidence rate is supposed to be bilinear, βxv, which is based on the law of mass action. In reality, bilinear incidence rate is not accurate to describe the viral dynamics during the full course of infection. In [27], the incidence rate is given by Hill type infection rate. However, the humoral immune response has been neglected.

Our aim in this paper is to propose a viral infection model with humoral immune response and investigate its global stability analysis. The incidence rate is given by Hill type infection rate. Using Lyapunov functions, we prove that the global stability of the model is determined by two threshold parameters, the basic infection reproduction number R_0 and the humoral immune response activation number R_1. We have proven that, if $R_0 \leq 1$, then the uninfected steady state is globally asymptotically stable (GAS), if $R_1 \leq 1 < R_0$, then the infected steady state without humoral immune response is GAS, and if $R_1 > 1$, then the infected steady state with humoral immune response is GAS.

2 The model

In this section, we propose a viral infection model with humoral immune response. The incidence rate is given by a Hill type infection rate.

$$\dot{x} = \lambda - dx - \frac{\beta x^n v}{\gamma^n + x^n}, \tag{5}$$

$$\dot{y} = \frac{\beta x^n v}{\gamma^n + x^n} - ay, \tag{6}$$

$$\dot{v} = ky - cv - rvz, \tag{7}$$

$$\dot{z} = gvz - \mu z, \tag{8}$$

where γ and n are positive constants. Next, we study the properties of the solutions of the model.

2.1 Positive invariance

We note that model (5)-(8) is biologically acceptable in the sense that no population goes negative. It is straightforward to check the positive invariance of the non-negative orthant $\mathbb{R}^4_{\geq 0}$ by model (5)-(8). In the following, we show the boundednes of the solution of model (5)-(8).

Proposition 1. There exist positive numbers $L_i, i = 1, 2, 3$ such that the compact set $\Omega = \left\{(x, y, v, z) \in \mathbb{R}^4_{\geq 0} : 0 \leq x, y \leq L_1, 0 \leq v \leq L_2, 0 \leq z \leq L_3\right\}$ is positively invariant.

Proof. Let $X_1(t) = x(t) + y(t)$, then

$$\dot{X}_1 = \lambda - dx - ay \leq \lambda - s_1 X_1,$$

where $s_1 = \min\{d, a\}$. Hence $X_1(t) \leq L_1$, if $X_1(0) \leq L_1$, where $L_1 = \dfrac{\lambda}{s_1}$. Since $x(t) > 0$ and $y(t) \geq 0$, then $0 \leq x(t)$, $y(t) \leq L_1$ if $0 \leq x(0) + y(0) \leq L_1$. On the other hand, let $X_2(t) = v(t) + \frac{r}{g}z(t)$, then

$$\dot{X}_2 = ky - cv - \frac{r\mu}{g}z \leq kL_1 - s_2(v + \frac{r}{g}z) = kL_1 - s_2 X_2,$$

where $s_2 = \min\{c, \mu\}$. Hence $X_2(t) \leq L_2$, if $X_2(0) \leq L_2$, where $L_2 = \dfrac{kL_1}{s_2}$. Since $v(t) \geq 0$ and $z(t) \geq 0$, then $0 \leq v(t) \leq L_2$ and $0 \leq z(t) \leq L_3$ if $0 \leq v(0) + \frac{r}{g}z(0) \leq L_2$, where $L_3 = \frac{gL_2}{r}$.

2.2 Steady states

In this subsection, we calculate the steady states of model (5)-(8) and derive two thresholds parameters. The steady states of model (5)-(8) satisfy the following equations:

$$\lambda - dx - \frac{\beta x^n v}{\gamma^n + x^n} = 0, \tag{9}$$

$$\frac{\beta x^n v}{\gamma^n + x^n} - ay = 0, \tag{10}$$

$$ky - cv - rvz = 0, \tag{11}$$

$$(gv - \mu)z = 0. \tag{12}$$

Equation (12) has two possible solutions, $z = 0$ or $v = \mu/g$. If $z = 0$, then from Eqs. (10)-(11) we obtain

$$\frac{k\beta x^n v}{a(\gamma^n + x^n)} - cv = 0. \tag{13}$$

Equation (13) has two possibilities, $v = 0$ or $v \neq 0$. If $v = 0$, then $y = 0$ and $x = \frac{\lambda}{d}$ which leads to the uninfected steady state $E_0 = (x_0, 0, 0, 0)$, where $x_0 = \frac{\lambda}{d}$. If $v \neq 0$, then from Eqs. (9) and (13) we obtain

$$v = \frac{k}{ac}\frac{\beta x^n v}{\gamma^n + x^n} = \frac{k(\lambda - dx)}{ac} \tag{14}$$

$$\Rightarrow x = x_0 - \frac{ac}{dk}v. \tag{15}$$

Then, Eq. (13) becomes

$$\frac{k\beta\left(x_0 - \frac{ac}{dk}v\right)^n v}{a\gamma^n + a\left(x_0 - \frac{ac}{dk}v\right)^n} - cv = 0.$$

Let us define a function Ψ_1 as

$$\Psi_1(v) = \frac{k\beta\left(x_0 - \frac{ac}{dk}v\right)^n v}{a\gamma^n + a\left(x_0 - \frac{ac}{dk}v\right)^n} - cv = 0.$$

It is clear that $\Psi_1(0) = 0$, and when $v = \overline{v} = \frac{x_0 dk}{ac} > 0$, then $\Psi_1(\overline{v}) = -c\overline{v} < 0$. Since $\Psi_1(v)$ is continuous for all $v \geq 0$, then we have

$$\Psi_1'(0) = c\left(\frac{k}{ac}\frac{\beta x_0^n}{\gamma^n + x_0^n} - 1\right).$$

Therefore, if $\Psi_1'(0) > 0$ i.e. $\frac{k}{ac}\frac{\beta x_0^n}{\gamma^n + x_0^n} > 1$, then there exist a $v_1 \in (0, \overline{v})$ such that $\Psi_1(v_1) = 0$. From Eq. (11) we obtain $y_1 = \frac{c}{k}v_1 > 0$ and from Eq. (9) we define a function Ψ_2 as:

$$\Psi_2(x) = \lambda - dx - \frac{\beta x^n v_1}{\gamma^n + x^n} = 0.$$

We have $\Psi_2(0) = \lambda > 0$ and $\Psi_2(x_0) = -\frac{\beta x_0^n}{\gamma^n + x_0^n}v_1 < 0$. Since $f(x) = \frac{x^n}{\gamma^n + x^n}$ is a strictly increasing function of x, then Ψ_2 is a strictly decreasing function of x, then there exist a unique $x_1 \in (0, x_0)$ such that $\Psi_2(x_1) = 0$. It means that, an infected steady state without humoral immune response $E_1 = (x_1, y_1, v_1, 0)$ exists when $\frac{k}{ac}\frac{\beta x_0^n}{\gamma^n + x_0^n} > 1$. Now we are ready to define the basic infection reproduction number as:

$$R_0 = \frac{k}{ac}\frac{\beta x_0^n}{\gamma^n + x_0^n}.$$

The other possibility of Eq. (12) $z \neq 0$ leads to $v_2 = \dfrac{\mu}{g}$. Inserting v_2 in Eq. (9) we define a function Ψ_3 as:

$$\Psi_3(x) = \lambda - dx - \frac{\beta x^n v_2}{\gamma^n + x^n} = 0.$$

Note that Ψ_3 is a strictly decreasing function of x. Clearly, $\Psi_3(0) = \lambda > 0$ and $\Psi_3(x_0) = -\frac{\beta x_0^n v_2}{\gamma^n + x_0^n} < 0$. Thus, there exists a unique $x_2 \in (0, x_0)$ such that $\Psi_3(x_2) = 0$. It follows from Eq. (11) that,

$$y_2 = \frac{\beta x_2^n v_2}{a(\gamma^n + x_2^n)}, \quad z_2 = \frac{c}{r}\left[\frac{k}{ac}\frac{\beta x_2^n}{\gamma^n + x_2^n} - 1\right].$$

Thus $y_2 > 0$, and if $\frac{k}{acv_2}\frac{\beta x_2^n v_2}{\gamma^n + x_2^n} > 1$, then $z_2 > 0$ when . Now we define the humoral immune response activation number as:

$$R_1 = \frac{k}{ac}\frac{\beta x_2^n}{\gamma^n + x_2^n},$$

Hence, z_2 can be rewritten as $z_2 = \frac{c}{r}(R_1 - 1)$. It follows that, there exists an infected steady state with humoral immune response $E_2(x_2, y_2, v_2, z_2)$ when $R_1 > 1$. Since $x_1 < x_0$, then

$$R_1 = \frac{k}{ac}\frac{\beta x_2^n}{\gamma^n + x_2^n} < \frac{k}{ac}\frac{\beta x_0^n}{\gamma^n + x_0^n} = R_0.$$

From above we have the following result.

Lemma 1 (i) if $R_0 \le 1$, then there exists only one positive equilibrium E_0,

(ii) if $R_1 \le 1 < R_0$, then there exist two positive steady states E_0 and E_1, and

(iii) if $R_1 > 1$, then there exist three positive steady states E_0, E_1 and E_2.

3 Global stability analysis

In this section, we establish the global stability of the three steady states of system (5)-(8) employing the direct Lyapunov method and LaSalle's invariance principle.

Theorem 1. If $R_0 \le 1$, then E_0 is GAS.

Proof. Define a Lyapunov functional W_0 as follows:

$$W_0 = x - x_0 - \int_{x_0}^{x} \frac{x_0^n(\gamma^n + s^n)}{s^n(\gamma^n + x_0^n)} ds + y + \frac{a}{k}v + \frac{ar}{kg}z.$$

Calculating $\frac{dW_0}{dt}$ along the trajectories of (5)-(8) as:

$$\begin{aligned}
\frac{dW_0}{dt} &= \left(1 - \frac{x_0^n(\gamma^n + x^n)}{x^n(\gamma^n + x_0^n)}\right)\left(\lambda - dx - \frac{\beta x^n v}{\gamma^n + x^n}\right) + \frac{\beta x^n v}{\gamma^n + x^n} - ay \\
&\quad + \frac{a}{k}(ky - cv - rvz) + \frac{ar}{kg}(gvz - \mu z) \qquad (16)\\
&= \lambda\left(1 - \frac{x_0^n(\gamma^n + x^n)}{x^n(\gamma^n + x_0^n)}\right)\left(1 - \frac{x}{x_0}\right) + \frac{\beta x_0^n v}{\gamma^n + x_0^n} - \frac{ac}{k}v - \frac{ar\mu}{kg}z \\
&= \lambda\left(1 - \frac{x_0^n(\gamma^n + x^n)}{x^n(\gamma^n + x_0^n)}\right)\left(1 - \frac{x}{x_0}\right) + \frac{ac}{k}\left(\frac{k}{ac}\frac{\beta x_0^n}{(\gamma^n + x_0^n)} - 1\right)v - \frac{ar\mu}{kg}z \\
&= \frac{\lambda\gamma^n(x^n - x_0^n)(x_0 - x)}{x^n x_0(\gamma^n + x_0^n)} + \frac{ac}{k}(R_0 - 1)v - \frac{ar\mu}{kg}z. \qquad (17)
\end{aligned}$$

We have $(x^n - x_0^n)(x_0 - x) \le 0$ for all $x, n > 0$. Then if $R_0 \le 1$ then $\frac{dW_0}{dt} \le 0$ for all $x, v, z > 0$. Thus, the solutions of system (5)-(8) converge to Ω, the largest invariant subset of $\left\{\frac{dW_0}{dt} = 0\right\}$ [28]. Clearly, it follows from Eq. (17) that $\frac{dW_0}{dt} = 0$ if and only if $x = x_0$, $v = 0$ and $z = 0$. The set Ω is invariant and for any element belongs to Ω satisfies $v = 0$ and $z = 0$, then $\dot{v} = 0$. We can see from Eq. (7) that $0 = \dot{v} = ky$, and thus $y = 0$. Hence $\frac{dW_0}{dt} = 0$ if and only if $x = x_0$, $y = 0$, $v = 0$ and $z = 0$. From LaSalle's invariance principle, E_0 is GAS.

Theorem 2. If $R_1 \le 1 < R_0$, then E_1 is GAS.

Proof. We construct the following Lyapunov functional

$$W_1 = x - x_1 - \int_{x_1}^{x} \frac{x_1^n(\gamma^n + s^n)}{s^n(\gamma^n + x_1^n)} ds + y_1 H\left(\frac{y}{y_1}\right) + \frac{a}{k}v_1 H\left(\frac{v}{v_1}\right) + \frac{ar}{kg}z.$$

The time derivative of W_1 along the trajectories of (5)-(8) is given by

$$\frac{dW_1}{dt} = \left(1 - \frac{x_1^n(\gamma^n + x^n)}{x^n(\gamma^n + x_1^n)}\right)\left(\lambda - dx - \frac{\beta x^n v}{\gamma^n + x^n}\right) + \left(1 - \frac{y_1}{y}\right)\left(\frac{\beta x^n v}{\gamma^n + x^n} - ay\right) + \frac{a}{k}\left(1 - \frac{v_1}{v}\right)(ky - cv - rvz) + \frac{ar}{kg}(gvz - \mu z). \tag{18}$$

Applying $\lambda = dx_1 + \frac{\beta x_1^n v_1}{\gamma^n + x_1^n}$ and collecting terms of Eq. (18) we get

$$\frac{dW_1}{dt} = \left(1 - \frac{x_1^n(\gamma^n + x^n)}{x^n(\gamma^n + x_1^n)}\right)(dx_1 - dx) + \frac{\beta x_1^n v_1}{\gamma^n + x_1^n}\left(1 - \frac{x_1^n(\gamma^n + x^n)}{x^n(\gamma^n + x_1^n)}\right) - \frac{\beta x^n v}{\gamma^n + x^n}\frac{y_1}{y} + ay_1 - a\frac{yv_1}{v} + \frac{ac}{k}v_1 + \frac{ar}{k}v_1 z - \frac{ar\mu}{kg}z.$$

Using the equilibrium conditions for E_1, $\frac{\beta x_1^n v_1}{\gamma^n + x_1^n} = ay_1$, $cv_1 = ky_1$, we obtain

$$\begin{aligned}\frac{dW_1}{dt} &= dx_1\left(1 - \frac{x_1^n(\gamma^n + x^n)}{x^n(\gamma^n + x_1^n)}\right)\left(1 - \frac{x}{x_1}\right) + \frac{\beta x_1^n v_1}{\gamma^n + x_1^n}\left(1 - \frac{x_1^n(\gamma^n + x^n)}{x^n(\gamma^n + x_1^n)}\right) \\ &\quad - \frac{\beta x_1^n v_1}{\gamma^n + x_1^n}\frac{x^n(\gamma^n + x_1^n)vy_1}{x_1^n(\gamma^n + x^n)v_1 y} + \frac{\beta x_1^n v_1}{\gamma^n + x_1^n} - \frac{\beta x_1^n v_1}{\gamma^n + x_1^n}\frac{yv_1}{y_1 v} + \frac{\beta x_1^n v_1}{\gamma^n + x_1^n} + \frac{ar}{k}\left(v_1 - \frac{\mu}{g}\right)z \\ &= \frac{d\gamma^n(x^n - x_1^n)(x_1 - x)}{x^n(\gamma^n + x_1^n)} + \frac{ar}{k}\left(v_1 - \frac{\mu}{g}\right)z \\ &\quad + \frac{\beta x_1^n v_1}{\gamma^n + x_1^n}\left[3 - \frac{x_1^n(\gamma^n + x^n)}{x^n(\gamma^n + x_1^n)} - \frac{x^n(\gamma^n + x_1^n)vy_1}{x_1^n(\gamma^n + x^n)v_1 y} - \frac{yv_1}{y_1 v}\right].\end{aligned} \tag{19}$$

Clearly, the first term of Eq. (19) is less than or equal zero. Because the geometrical mean is less than or equal to the arithmetical mean, then the third and fourth terms of Eq. (19) are less than or equal zero. Now we show that if $R_1 \leq 1$ then $v_1 \leq \frac{\mu}{r} = v_2$. This can be achieved if we show that

$$sgn\,(x_2 - x_1) = sgn\,(v_1 - v_2) = sgn\,(R_1 - 1).$$

We have

$$\left(\frac{x_2^n}{\gamma^n + x_2^n} - \frac{x_1^n}{\gamma^n + x_1^n}\right)(x_2 - x_1) > 0, \tag{20}$$

Suppose that, $sgn\,(x_2 - x_1) = sgn\,(v_2 - v_1)$. Using the conditions of the steady states E_1 and E_2 we have

$$\begin{aligned}(\lambda - dx_2) - (\lambda - dx_1) &= \frac{\beta x_2^n v_2}{\gamma^n + x_2^n} - \frac{\beta x_1^n v_1}{\gamma^n + x_1^n} \\ &= \frac{\beta x_2^n v_2}{\gamma^n + x_2^n} - \frac{\beta x_2^n v_1}{\gamma^n + x_2^n} + \frac{\beta x_2^n v_1}{\gamma^n + x_2^n} - \frac{\beta x_1^n v_1}{\gamma^n + x_1^n} \\ &= \frac{\beta x_2^n}{\gamma^n + x_2^n}(v_2 - v_1) + \beta v_1\left(\frac{x_2^n}{\gamma^n + x_2^n} - \frac{x_1^n}{\gamma^n + x_1^n}\right),\end{aligned}$$

and from inequality (20) we get $sgn\,(x_1 - x_2) = sgn\,(x_2 - x_1)$, which leads to contradiction. Thus, $sgn\,(x_2 - x_1) = sgn\,(v_1 - v_2)$. Using the steady state conditions for E_1 we have $\frac{k}{ac}\frac{\beta x_1^n}{\gamma^n + x_1^n} = 1$, then

$$R_1 - 1 = \frac{k}{ac}\left(\frac{\beta x_2^n}{\gamma^n + x_2^n} - \frac{\beta x_1^n}{\gamma^n + x_1^n}\right).$$

From inequality (20) we get:

$$sgn\left(R_1 - 1\right) = sgn\left(v_1 - v_2\right).$$

It follows that, if $R_1 \leq 1$ then $v_1 \leq \frac{\mu}{r} = v_2$. Therefore, if $R_1 \leq 1$ then $\frac{dW_1}{dt} \leq 0$ for all $x, y, v, z > 0$, where the equality occurs at the equilibrium E_1. LaSalle's invariance principle implies the global stability of E_1.

Theorem 3. If $R_1 > 1$,then E_2 is GAS.

Proof. We construct the following Lyapunov functional

$$W_2 = x - x_2 - \int_{x_2}^{x} \frac{x_2^n(\gamma^n + s^n)}{s^n(\gamma^n + x_2^n)} ds + y_2 H\left(\frac{y}{y_2}\right) + \frac{a}{k} v_2 H\left(\frac{v}{v_2}\right) + \frac{ar}{kg} z_2 H\left(\frac{z}{z_2}\right).$$

We calculate the time derivative of W_2 along the trajectories of (5)-(8) as:

$$\begin{aligned} \frac{dW_2}{dt} &= \left(1 - \frac{x_2^n(\gamma^n + x^n)}{x^n(\gamma^n + x_2^n)}\right)\left(\lambda - dx - \frac{\beta x^n v}{\gamma^n + x^n}\right) + \left(1 - \frac{y_2}{y}\right)\left(\frac{\beta x^n v}{\gamma^n + x^n} - ay\right) \\ &+ \frac{a}{k}\left(1 - \frac{v_2}{v}\right)(ky - cv - rvz) + \frac{ar}{kg}\left(1 - \frac{z_2}{z}\right)(gvz - \mu z). \end{aligned} \tag{21}$$

Applying $\lambda = dx_2 + \frac{\beta x_2^n v_2}{\gamma^n + x_2^n}$ and collecting terms of Eq. (21) we get

$$\begin{aligned} \frac{dW_2}{dt} &= \left(1 - \frac{x_2^n(\gamma^n + x^n)}{x^n(\gamma^n + x_2^n)}\right)(dx_2 - dx) + \frac{\beta x_2^n v_2}{\gamma^n + x_2^n}\left(1 - \frac{x_2^n(\gamma^n + x^n)}{x^n(\gamma^n + x_2^n)}\right) \\ &+ \frac{\beta x_2^n v}{\gamma^n + x_2^n} - \frac{\beta x^n v}{\gamma^n + x^n}\frac{y_2}{y} + ay_2 - \frac{ac}{k}v - a\frac{yv_2}{v} \\ &+ \frac{ac}{k}v_2 + \frac{ar}{k}v_2 z - \frac{ar\mu}{kg}z - \frac{ar}{k}z_2 v + \frac{ar\mu}{kg}z_2. \end{aligned}$$

Using the equilibrium conditions for E_2

$$\frac{\beta x_2^n v_2}{\gamma^n + x_2^n} = ay_2, ky_2 = cv_2 + rv_2 z_2,\ \mu = gv_2,$$

we get

$$\begin{aligned} \frac{dW_2}{dt} &= dx_2\left(1 - \frac{x_2^n(\gamma^n + x^n)}{x^n(\gamma^n + x_2^n)}\right)\left(1 - \frac{x}{x_2}\right) + \frac{\beta x_2^n v_2}{\gamma^n + x_2^n}\left(1 - \frac{x_2^n(\gamma^n + x^n)}{x^n(\gamma^n + x_2^n)}\right) \\ &- \frac{\beta x_2^n v_2}{\gamma^n + x_2^n}\frac{x^n(\gamma^n + x_2^n)vy_2}{x_2^n(\gamma^n + x^n)v_2 y} + \frac{\beta x_2^n v_2}{\gamma^n + x_2^n} - \frac{\beta x_2^n v_2}{\gamma^n + x_2^n}\frac{yv_2}{y_2 v} + \frac{\beta x_2^n v_2}{\gamma^n + x_2^n} \\ &= \frac{d\gamma^n(x^n - x_2^n)(x_2 - x)}{x^n(\gamma^n + x_2^n)} + \frac{\beta x_2^n v_2}{\gamma^n + x_2^n}\left[3 - \frac{x_2^n(\gamma^n + x^n)}{x^n(\gamma^n + x_2^n)} - \frac{x^n(\gamma^n + x_2^n)vy_2}{x_2^n(\gamma^n + x^n)v_2 y} - \frac{yv_2}{y_2 v}\right]. \end{aligned} \tag{22}$$

Thus, if $R_1 > 1$ then x_2, y_2, v_2 and $z_2 > 0$. Clearly, we get that the first and second terms of Eq. (22) are less than or equal zero. Since the arithmetical mean is greater than or equal to the geometrical mean, then $\frac{dW_2}{dt} \leq 0$. It can be seen that, $\frac{dW_2}{dt} = 0$ if and only if $x = x_2$, $y = y_2$ and $v = v_2$. From Eq. (7), if $v = v_2$ and $y = y_2$, then $\dot{v} = 0$ and $0 = ky_2 - cv_2 - rv_2 z$, which yields $z = z_2$ and hence $\frac{dW_2}{dt}$ equal to zero at E_2. LaSalle's invariance principle implies global stability of E_2.

4 Conclusion

In this paper, we have proposed and analyzed a viral infection model with humoral immune response. The model is a four dimensional that describe the interaction between the uninfected target cells, infected cells, free virus particles and B cells. The incidence rate has been represented by Hill type infection rate. We have derived two threshold parameters, the basic reproduction number R_0 and the humoral immune response number R_1 which completely determined the basic and global properties of the viral infection model. Using Lyapunov method and applying LaSalle's invariance principle we have proven that if $R_0 \leq 1$, then the uninfected steady state is GAS, if $R_1 \leq 1 < R_0$, then the infected steady state without humoral immune response is GAS, and if $R_1 > 1$, then the infected steady state with humoral immune response is GAS.

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A Parameterized Series Representation for Apéry's Constant $\zeta(3)$

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Abstract. We prove that if $\lambda \leq 1/2$, then

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{(1-\lambda)^{n+1}} \sum_{k=1}^{n} \binom{n}{k} (-\lambda)^{n-k} \delta_k$$

with

$$\delta_k = \frac{H_k}{k^2} - \frac{1}{k}\left(\frac{\pi^2}{6} - H_k^{(2)}\right),$$

where H_k and $H_k^{(2)}$ denote the harmonic numbers and the generalized harmonic numbers of order 2, respectively.

Keywords. Apéry's constant, series representation, harmonic numbers.

2010 Mathematics Subject Classification. 11M35

1. Introduction

The famous Riemann zeta function is defined for all complex numbers s with $\Re s > 1$ by the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots . \tag{1}$$

In this note we are concerned with the special case $s = 3$, that is, with

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} = 1.20205\ldots .$$

This number is known in the literature as Apéry's constant. It is named after the Greek-French mathematician Roger Apéry (1916–1994), who proved in 1979 that $\zeta(3)$ is irrational; see [4]. A central role in his proof is played by the elegant series representation

$$\zeta(3) = \frac{5}{2} \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n^3 \binom{2n}{n}}.$$

Apéry's constant has been the subject of much attention. It appears in the solution of finding sharp bounds for the Mathieu series $\sum_{n=1}^{\infty} 2n(n^2+r^2)^{-2}$, it has applications in physics and it also occurs in the solution of probability problems; see [1], [8] and [10, A002117]. Euler, Ramanujan and numerous other researchers provided various integral and series representations for $\zeta(3)$ and related constants. We refer to Srivastava's survey paper [12] and the references therein; see also [2].

As is well-known, Euler proved that the numbers $\zeta(2n)$ $(n = 1, 2, 3, ...)$ are irrational. Thus, it is natural to ask whether the values $\zeta(2n + 1)$ $(n = 2, 3, 4, ...)$ are also irrational. This is a classical open problem. Recent progress on this subject was made by Rivoal [9], who established that infinitely many of the numbers $\zeta(2n+1)$ are irrational, and Zudilin [14], who proved that at least one of the numbers $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, $\zeta(11)$ is irrational.

It is the aim of this note to present a new singly-parameterized series representation for $\zeta(3)$ in terms of the classical harmonic numbers

$$H_k = \sum_{j=1}^{k} \frac{1}{j} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} \quad (k = 1, 2, ...).$$

and the generalized harmonic numbers of order 2

$$H_k^{(2)} = \sum_{j=1}^{k} \frac{1}{j^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} \quad (k = 1, 2, ...).$$

Our method of proof, which can be used to obtain series representations for other mathematical constants as well, is explained in detail in [3]. A key role is played by the remarkable integral representation

$$\zeta(3) = \frac{1}{2}\int_0^1 \frac{\log t\, \log(1-t)}{t\,(1-t)}dt, \tag{2}$$

which was published by Janous [7] in 2006.

2. Main Result

The following series representation for Apéry's constant is valid.

Theorem. *Let λ be a real number with $\lambda \le 1/2$. Then,*

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{(1-\lambda)^{n+1}} \sum_{k=1}^{n} \binom{n}{k} (-\lambda)^{n-k} \delta_k \tag{3}$$

with

$$\delta_k = \frac{H_k}{k^2} - \frac{1}{k}\left(\frac{\pi^2}{6} - H_k^{(2)}\right). \tag{4}$$

Proof. Let $\lambda \le 1/2$ and $0 < t < 1$. Then,

$$-1 < \frac{t-\lambda}{1-\lambda} < 1.$$

Expanding in a geometric series, we obtain

$$\frac{1}{1-t} = \frac{1}{1-\lambda} \cdot \frac{1}{1-\frac{t-\lambda}{1-\lambda}} = \frac{1}{1-\lambda}\sum_{n=0}^{\infty}\left(\frac{t-\lambda}{1-\lambda}\right)^n.$$

Since

$$(t-\lambda)^n = \sum_{k=0}^{n} \binom{n}{k} t^k (-\lambda)^{n-k}$$

(where $(-\lambda)^{n-k} = 1$ if $\lambda = n-k = 0$), we find that

$$\frac{1}{1-t} = \sum_{n=0}^{\infty} \frac{1}{(1-\lambda)^{n+1}} \sum_{k=0}^{n} \binom{n}{k} t^k (-\lambda)^{n-k}.$$

Substituting this into (2) gives

$$\begin{aligned} \zeta(3) &= \frac{1}{2}\int_0^1 \frac{\log t\, \log(1-t)}{t} \sum_{n=0}^{\infty} \frac{1}{(1-\lambda)^{n+1}} \sum_{k=0}^{n} \binom{n}{k} t^k (-\lambda)^{n-k} \\ &= \frac{1}{2}\sum_{n=0}^{\infty} \frac{1}{(1-\lambda)^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-\lambda)^{n-k}\delta_k, \end{aligned} \tag{5}$$

where

$$\delta_k = \int_0^1 t^{k-1} \log t \log(1-t) dt.$$

Here if we substitute the series

$$\log(1-t) = -\sum_{\nu=1}^{\infty} \frac{t^\nu}{\nu}$$

we obtain

$$\delta_k = -\sum_{\nu=1}^{\infty} \frac{1}{\nu} \int_0^1 t^{k+\nu-1} \log t \, dt = \sum_{\nu=1}^{\infty} \frac{1}{\nu(k+\nu)^2} \tag{6}$$

using integration by parts.

For $k = 0$ we have

$$\delta_0 = \sum_{\nu=1}^{\infty} \frac{1}{\nu^3} = \zeta(3)$$

and for $k \geq 1$ we find that

$$\delta_k = \sum_{\nu=1}^{\infty} \left(\frac{1}{k^2} \left(\frac{1}{\nu} - \frac{1}{k+\nu} \right) - \frac{1}{k} \frac{1}{(k+\nu)^2} \right) = \frac{H_k}{k^2} - \frac{\zeta(2) - H_k^{(2)}}{k}.$$

Applying (5) gives

$$\begin{aligned} \zeta(3) &= \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{(1-\lambda)^{n+1}} \left((-\lambda)^n \delta_0 + \sum_{k=1}^{n} \binom{n}{k} (-\lambda)^{n-k} \delta_k \right) \\ &= \frac{\delta_0}{2(1-\lambda)} \sum_{n=0}^{\infty} \left(\frac{-\lambda}{1-\lambda} \right)^n + \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{(1-\lambda)^{n+1}} \sum_{k=1}^{n} \binom{n}{k} (-\lambda)^{n-k} \delta_k. \end{aligned}$$

Since

$$\frac{\delta_0}{2(1-\lambda)} \sum_{n=0}^{\infty} \left(\frac{-\lambda}{1-\lambda} \right)^n = \frac{1}{2}\delta_0 = \frac{1}{2}\zeta(3),$$

and $\zeta(2) = \pi^2/6$, we conclude that (3) is valid with δ_k as given in (4). $\square$

3. Examples

We consider the cases $\lambda = 0, -1, \pm 1/2$, and $1/4$.

Example 1. The special case $\lambda = 0$ leads to the representation

$$\zeta(3) = \sum_{n=1}^{\infty} \delta_n = \sum_{n=1}^{\infty} \left(\frac{H_n}{n^2} - \frac{\zeta(2) - H_n^{(2)}}{n} \right).$$

This formula can also be proved without using the Theorem. In fact, the last expression can be written as the difference of two series whose terms all cancel, except for those in series (1) for $\zeta(3)$. Indeed,

$$\sum_{n=1}^{\infty}\delta_n = \sum_{i=1}^{\infty}\frac{H_i}{i^2} - \sum_{j=1}^{\infty}\frac{\zeta(2)-H_j^{(2)}}{j} \tag{7}$$
$$= \sum_{i=1}^{\infty}\frac{1}{i^2}\sum_{j=1}^{i}\frac{1}{j} - \sum_{j=1}^{\infty}\frac{1}{j}\sum_{i=j+1}^{\infty}\frac{1}{i^2}$$
$$= \sum_{i\geq j\geq 1}\frac{1}{i^2 j} - \sum_{i>j\geq 1}\frac{1}{i^2 j}$$
$$= \sum_{i=j\geq 1}\frac{1}{i^2 j} = \sum_{i\geq 1}\frac{1}{i^3} = \zeta(3),$$

as claimed.

If we combine this with (6) and reverse the order of summation, we get

$$\zeta(3) = \sum_{n=1}^{\infty}\delta_n = \sum_{\nu=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{\nu(n+\nu)^2} = \sum_{\nu=1}^{\infty}\frac{\zeta(2)-H_\nu^{(2)}}{\nu}.$$

Together with (7), this proves Euler's famous relation [5]

$$\sum_{i=1}^{\infty}\frac{H_n}{n^2} = 2\zeta(3).$$

Example 2. The case $\lambda = -1$ yields

$$2\zeta(3) = \sum_{n=1}^{\infty}\frac{1}{2^n}\sum_{k=1}^{n}\binom{n}{k}\delta_k.$$

This may be compared to the series

$$\frac{3}{2}\,\zeta(3) = \sum_{n=0}^{\infty}\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{(k+1)^3}$$

which in turn is the case $s = 3$ of a global series for $\zeta(s)$ due to Hasse [6] and rediscovered in [11].

Example 3. The cases $\lambda = 1/2, -1/2, 1/4$ give

$$\zeta(3) = \sum_{n=1}^{\infty}(-1)^n\sum_{k=1}^{n}(-1)^k\binom{n}{k}2^{k+1}\delta_k,$$
$$\frac{3}{2}\zeta(3) = \sum_{n=1}^{\infty}\frac{1}{3^n}\sum_{k=1}^{n}\binom{n}{k}2^k\,\delta_k,$$
$$\frac{3}{4}\zeta(3) = \sum_{n=1}^{\infty}\frac{1}{(-3)^n}\sum_{k=1}^{n}\binom{n}{k}(-4)^k\,\delta_k,$$

respectively.

4. Concluding Remarks

We conclude the paper with three remarks.

Remark 1. If we multiply both sides of (3) by $(1-\lambda)^{a+1}$ $(a \in \mathbb{R})$ and differentiate with respect to λ, then we find that

$$(a+1)\,\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{(1-\lambda)^{n+1}} \sum_{k=1}^{n} \binom{n}{k} (-\lambda)^{n-k-1} \big(n - a\lambda + (\lambda-1)k\big)\, \delta_k.$$

Applying this with $a=1$, $\lambda = -1/4$ and (3) with $\lambda=-1/4$ yields

$$\frac{5}{4}\,\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{5^n} \sum_{k=1}^{n} \binom{n}{k} 4^k (4n-5k)\, \delta_k.$$

Remark 2. Using the asymptotic formulas

$$H_k \sim \log k \quad \text{and} \quad \zeta(2) - H_k^{(2)} \sim \frac{1}{k} \quad (k \to \infty)$$

we obtain

$$\delta_k \sim \frac{\log k}{k^2} \quad (k\to\infty).$$

For $k = 1, 2, \ldots, 10$, we have the values

$$\begin{aligned}\delta_k = {} & 0.35506\ldots,\ 0.17753\ldots,\ 0.10909\ldots, 0.07487\ldots,\ 0.05506\ldots, \\ & 0.04246\ldots,\ 0.03389\ldots,\ 0.02777\ldots,\ 0.02324\ldots,\ 0.01977\ldots .\end{aligned}$$

Remark 3. Applying the series representation (6) and [13, Theorem 11d] we conclude that the sequence $\{\delta_k\}_{k=0}^{\infty}$ is not only decreasing and convex but even *completely monotonic*, that is,

$$(-1)^n\, \Delta^n \delta_k \ge 0 \quad \text{for} \quad k, n = 0, 1, 2, ...,$$

where Δ denotes the forward difference operator defined by

$$\Delta^0 \delta_k = \delta_k, \quad \Delta^n \delta_k = \Delta^{n-1}\delta_{k+1} - \Delta^{n-1}\delta_k \quad (k = 0, 1, 2, ...; \, n = 1, 2, ...).$$

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