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Barnes-type Peters of the first kind and poly-Cauchy of the first kind mixed-type polynomials

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Abstract

In this paper, by considering Barnes-type Peters polynomials of the first kind as well as poly-Cauchy polynomials of the first kind, we define and investigate the mixed-type polynomials of these polynomials. From the properties of Sheffer sequences of these polynomials arising from umbral calculus, we derive new and interesting identities.

1 Introduction

In this paper, we consider the polynomials

$$s_n^{(k)}(x) = s_n^{(k)}(x|\lambda;\mu) = s_n^{(k)}(x|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r)$$

called the Barnes-type Peters of the first kind and poly-Cauchy of the first kind mixed-type polynomials, whose generating function is given by

$$\prod_{j=1}^{r} \left(1 + (1+t)^{\lambda_j}\right)^{-\mu_j} \mathrm{Lif}_k\left(\ln(1+t)\right)(1+t)^x = \sum_{n=0}^{\infty} s_n^{(k)}(x|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r)\frac{t^n}{n!}, \quad (1)$$

where $\lambda_1,\ldots,\lambda_r,\mu_1,\ldots,\mu_r \in \mathbb{C}$ with $\lambda_1,\ldots,\lambda_r \neq 0$. Here, $\mathrm{Lif}_k(x)$ $(k \in \mathbb{Z})$ is the polyfactorial function ([6]) defined by

$$\mathrm{Lif}_k(x) = \sum_{m=0}^{\infty} \frac{x^m}{m!(m+1)^k}.$$

When $x = 0$, $s_n^{(k)} = s_n^{(k)}(0) = s_n^{(k)}(0|\lambda;\mu) = s_n^{(k)}(0;\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r)$ is called the the Barnes-type Peters of the first kind and poly-Cauchy of the first kind mixed-type number.

Recall that the Barnes-type Peters polynomials of the first kind, denoted by $s_n(x|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r)$, are given by the generating function as

$$\prod_{j=1}^{r} \left(1 + (1+t)^{\lambda_j}\right)^{-\mu_j}(1+t)^x = \sum_{n=0}^{\infty} s_n(x|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r)\frac{t^n}{n!}.$$

If $r = 1$, then $s_n(x|\lambda;\mu)$ are the Peters polynomials of the first kind. Peters polynomials were mentioned in [9, p.128] and have been investigated in e.g. [5].

The poly-Cauchy polynomials of the first kind, denoted by $c_n^{(k)}(x)$ ([3, 7]), are given by the generating function as

$$\mathrm{Lif}_k\left(\ln(1+t)\right)(1+t)^x = \sum_{n=0}^{\infty} c_n^{(k)}(-x)\frac{t^n}{n!}.$$

The generalized Barnes-type Euler polynomials $E_n(x|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r)$ are defined by the generating function

$$\prod_{j=1}^{r} \left(\frac{2}{1+e^{\lambda_j t}}\right)^{\mu_j} e^{xt} = \sum_{n=0}^{\infty} E_n(x|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r)\frac{t^n}{n!}.$$

If $\mu_1 = \cdots = \mu_r = 1$, then $E_n(x|\lambda_1,\ldots,\lambda_r) = E_n(x|\lambda_1,\ldots,\lambda_r;1,\ldots,1)$ are called the Barnes-type Euler polynomials. If further $\lambda_1 = \cdots = \lambda_r = 1$, then $E_n^{(r)}(x) = E_n(x|1,\ldots,1;1,\ldots,1)$ are called the Euler polynomials of order r.

In this paper, by considering Barnes-type Peters polynomials of the first kind as well as poly-Cauchy polynomials of the first kind, we define and investigate the mixed-type polynomials of these polynomials. From the properties of Sheffer sequences of these polynomials arising from umbral calculus, we derive new and interesting identities.

2

2   Umbral calculus

Let $\mathbb{C}$ be the complex number field and let $\mathcal{F}$ be the set of all formal power series in the variable t:

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \,\middle|\, a_k \in \mathbb{C} \right\}. \tag{2}$$

Let $\mathbb{P} = \mathbb{C}[x]$ and let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. $\langle L|p(x)\rangle$ is the action of the linear functional L on the polynomial $p(x)$, and we recall that the vector space operations on $\mathbb{P}^*$ are defined by $\langle L + M|p(x)\rangle = \langle L|p(x)\rangle + \langle M|p(x)\rangle$, $\langle cL|p(x)\rangle = c\langle L|p(x)\rangle$, where c is a complex constant in $\mathbb{C}$. For $f(t) \in \mathcal{F}$, let us define the linear functional on $\mathbb{P}$ by setting

$$\langle f(t)|x^n\rangle = a_n, \quad (n \geq 0). \tag{3}$$

In particular,

$$\langle t^k|x^n\rangle = n!\delta_{n,k} \quad (n, k \geq 0), \tag{4}$$

where $\delta_{n,k}$ is the Kronecker's symbol.

For $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L|x^k\rangle}{k!} t^k$, we have $\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle$. That is, $L = f_L(t)$. The map $L \mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ denotes both the algebra of formal power series in t and the vector space of all linear functionals on $\mathbb{P}$, and so an element $f(t)$ of $\mathcal{F}$ will be thought of as both a formal power series and a linear functional. We call $\mathcal{F}$ the *umbral algebra* and the *umbral calculus* is the study of umbral algebra. The order $O\big(f(t)\big)$ of a power series $f(t)(\neq 0)$ is the smallest integer k for which the coefficient of t^k does not vanish. If $O\big(f(t)\big) = 1$, then $f(t)$ is called a *delta series*; if $O\big(f(t)\big) = 0$, then $f(t)$ is called an *invertible series*. For $f(t), g(t) \in \mathcal{F}$ with $O\big(f(t)\big) = 1$ and $O\big(g(t)\big) = 0$, there exists a unique sequence $s_n(x)$ ($\deg s_n(x) = n$) such that $\langle g(t)f(t)^k|s_n(x)\rangle = n!\delta_{n,k}$, for $n, k \geq 0$. Such a sequence $s_n(x)$ is called the *Sheffer sequence* for $\big(g(t), f(t)\big)$ which is denoted by $s_n(x) \sim \big(g(t), f(t)\big)$.

For $f(t), g(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$, we have

$$\langle f(t)g(t)|p(x)\rangle = \langle f(t)|g(t)p(x)\rangle = \langle g(t)|f(t)p(x)\rangle \tag{5}$$

and

$$f(t) = \sum_{k=0}^{\infty} \langle f(t)|x^k\rangle \frac{t^k}{k!}, \quad p(x) = \sum_{k=0}^{\infty} \langle t^k|p(x)\rangle \frac{x^k}{k!} \tag{6}$$

([9, Theorem 2.2.5]). Thus, by (6), we get

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k} \quad \text{and} \quad e^{yt} p(x) = p(x + y). \tag{7}$$

Sheffer sequences are characterized in the generating function ([9, Theorem 2.3.4]).

Lemma 1 *The sequence $s_n(x)$ is Sheffer for $\big(g(t), f(t)\big)$ if and only if*

$$\frac{1}{g\big(\bar{f}(t)\big)} e^{y\bar{f}(t)} = \sum_{k=0}^{\infty} \frac{s_k(y)}{k!} t^k \quad (y \in \mathbb{C}),$$

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where $\bar{f}(t)$ is the compositional inverse of $f(t)$.

For $s_n(x) \sim \big(g(t), f(t)\big)$, we have the following equations ([9, Theorem 2.3.7, Theorem 2.3.5, Theorem 2.3.9]):

$$f(t)s_n(x) = ns_{n-1}(x) \quad (n \geq 0), \tag{8}$$

$$s_n(x) = \sum_{j=0}^{n} \frac{1}{j!} \left\langle g\big(\bar{f}(t)\big)^{-1}\bar{f}(t)^j | x^n \right\rangle x^j, \tag{9}$$

$$s_n(x + y) = \sum_{j=0}^{n} \binom{n}{j} s_j(x)p_{n-j}(y), \tag{10}$$

where $p_n(x) = g(t)s_n(x)$.

Assume that $p_n(x) \sim \big(1, f(t)\big)$ and $q_n(x) \sim \big(1, g(t)\big)$. Then the transfer formula ([9, Corollary 3.8.2]) is given by

$$q_n(x) = x \left(\frac{f(t)}{g(t)}\right)^n x^{-1} p_n(x) \quad (n \geq 1).$$

For $s_n(x) \sim \big(g(t), f(t)\big)$ and $r_n(x) \sim \big(h(t), l(t)\big)$, assume that

$$s_n(x) = \sum_{m=0}^{n} C_{n,m} r_m(x) \quad (n \geq 0).$$

Then we have ([9, p.132])

$$C_{n,m} = \frac{1}{m!} \left\langle \frac{h\big(\bar{f}(t)\big)}{g\big(\bar{f}(t)\big)} l\big(\bar{f}(t)\big)^m \bigg| x^n \right\rangle. \tag{11}$$

3 Main results

From the definition (1), $s_n^{(k)}(x|\lambda_1, \ldots, \lambda_r; \mu_1, \ldots, \mu_r)$ is the Sheffer sequence for the pair

$$g(t) = \prod_{j=1}^{r}(1 + e^{\lambda_j t})^{\mu_j} \frac{1}{\text{Lif}_k(t)} \quad \text{and} \quad f(t) = e^t - 1.$$

So,

$$s_n^{(k)}(x|\lambda_1, \ldots, \lambda_r; \mu_1, \ldots, \mu_r) \sim \left(\prod_{j=1}^{r}(1 + e^{\lambda_j t})^{\mu_j} \frac{1}{\text{Lif}_k(t)}, e^t - 1\right). \tag{12}$$

4

3.1 Explicit expressions

Let $(n)_j = n(n-1)\cdots(n-j+1)$ $(j \geq 1)$ with $(n)_0 = 1$. The (signed) Stirling numbers of the first kind $S_1(n, m)$ are defined by

$$(x)_n = \sum_{m=0}^{n} S_1(n, m) x^m.$$

Theorem 1

$$s_n^{(k)}(x|\lambda_1, \ldots, \lambda_r; \mu_1, \ldots, \mu_r)$$

$$= 2^{-\sum_{j=1}^{r} \mu_j} \sum_{m=0}^{n} S_1(n, m) \sum_{l=0}^{m} \frac{\binom{m}{l}}{(l+1)^k} E_{m-l}(x|\lambda_1, \ldots, \lambda_r; \mu_1, \ldots, \mu_r) \tag{13}$$

$$= \sum_{j=0}^{n} \sum_{l=0}^{n-j} \binom{n}{l} S_1(n-l, j) s_l^{(k)} x^j \tag{14}$$

$$= \sum_{j=0}^{n} \sum_{l=0}^{n-j} \sum_{i=0}^{l} \binom{n}{l} \binom{l}{i} S_1(n-l, j) c_i^{(k)} s_{l-i}(\lambda_1, \ldots, \lambda_r; \mu_1, \ldots, \mu_r) x^j \tag{15}$$

$$= \sum_{l=0}^{n} \binom{n}{l} s_{n-l}(\lambda_1, \ldots, \lambda_r; \mu_1, \ldots, \mu_r) c_l^{(k)}(-x), \tag{16}$$

$$= \sum_{l=0}^{n} \binom{n}{l} s_l(x|\lambda_1, \ldots, \lambda_r; \mu_1, \ldots, \mu_r) c_{n-l}^{(k)}. \tag{17}$$

Proof. Since

$$\prod_{j=1}^{r} (1 + e^{\lambda_j t})^{\mu_j} \frac{1}{\mathrm{Lif}_k(t)} s_n^{(k)}(x|\lambda_1, \ldots, \lambda_r; \mu_1, \ldots, \mu_r) \sim (1, e^t - 1) \tag{18}$$

and

$$(x)_n \sim (1, e^t - 1), \tag{19}$$

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we have

$$s_n^{(k)}(x|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r) = s_n^{(k)}(x)$$

$$= \prod_{j=1}^{r}(1+e^{\lambda_j t})^{-\mu_j}\operatorname{Lif}_k(t)(x)_n$$

$$= \sum_{m=0}^{n} S_1(n,m) \prod_{j=1}^{r}(1+e^{\lambda_j t})^{-\mu_j}\operatorname{Lif}_k(t)x^m$$

$$= \sum_{m=0}^{n} S_1(n,m) \prod_{j=1}^{r}(1+e^{\lambda_j t})^{-\mu_j} \sum_{l=0}^{m}\frac{t^l}{l!(l+1)^k}x^m$$

$$= \sum_{m=0}^{n} S_1(n,m) \prod_{j=1}^{r}(1+e^{\lambda_j t})^{-\mu_j} \sum_{l=0}^{m}\frac{(m)_l}{l!(l+1)^k}x^{m-l}$$

$$= \sum_{m=0}^{n} S_1(n,m) \sum_{l=0}^{m}\frac{\binom{m}{l}}{(l+1)^k} \prod_{j=1}^{r}(1+e^{\lambda_j t})^{-\mu_j}x^{m-l}$$

$$= 2^{-\sum_{j=1}^{r}\mu_j} \sum_{m=0}^{n} S_1(n,m) \sum_{l=0}^{m}\frac{\binom{m}{l}}{(l+1)^k} \prod_{j=1}^{r}\left(\frac{2}{1+e^{\lambda_j t}}\right)^{\mu_j}x^{m-l}$$

$$= 2^{-\sum_{j=1}^{r}\mu_j} \sum_{m=0}^{n} S_1(n,m) \sum_{l=0}^{m}\frac{\binom{m}{l}}{(l+1)^k}E_{m-l}(x|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r).$$

So, we get (13).
By (9) with (12), we get

$$\left\langle g(\bar{f}(t))^{-1}\bar{f}(t)^j|x^n\right\rangle$$

$$= \left\langle \prod_{j=1}^{r}(1+(1+t)^{\lambda_j})^{-\mu_j}\operatorname{Lif}_k\big(\ln(1+t)\big)\big(\ln(1+t)\big)^j\Big|x^n\right\rangle$$

$$= \left\langle \prod_{j=1}^{r}(1+(1+t)^{\lambda_j})^{-\mu_j}\operatorname{Lif}_k\big(\ln(1+t)\big)\Big|j!\sum_{l=j}^{\infty}S_1(l,j)\frac{t^l}{l!}x^n\right\rangle$$

$$= j!\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)\left\langle \prod_{j=1}^{r}(1+(1+t)^{\lambda_j})^{-\mu_j}\operatorname{Lif}_k\big(\ln(1+t)\big)\Big|x^{n-l}\right\rangle$$

$$= j!\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)\left\langle \sum_{i=0}^{\infty}s_i^{(k)}\frac{t^i}{i!}\Big|x^{n-l}\right\rangle$$

$$= j!\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)s_{n-l}^{(k)}.$$

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On the other hand,

$$
\left\langle g\big(\bar{f}(t)\big)^{-1}\bar{f}(t)^j \big| x^n \right\rangle
$$

$$
= j! \sum_{l=j}^{n} \binom{n}{l} S_1(l,j) \left\langle \prod_{j=1}^{r} \left(1+(1+t)^{\lambda_j}\right)^{-\mu_j} \Big| \mathrm{Lif}_k\big(\ln(1+t)\big) x^{n-l} \right\rangle
$$

$$
= j! \sum_{l=j}^{n} \binom{n}{l} S_1(l,j) \sum_{i=0}^{n-l} \binom{n-l}{i} c_i^{(k)} \left\langle \prod_{j=1}^{r} \left(1+(1+t)^{\lambda_j}\right)^{-\mu_j} \Big| x^{n-l-i} \right\rangle
$$

$$
= j! \sum_{l=j}^{n} \binom{n}{l} S_1(l,j) \sum_{i=0}^{n-l} \binom{n-l}{i} c_i^{(k)} \left\langle \sum_{m=0}^{\infty} s_m(\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r) \frac{t^m}{m!} \Big| x^{n-l-i} \right\rangle
$$

$$
= j! \sum_{l=j}^{n} \sum_{i=0}^{n-l} \binom{n}{l} \binom{n-l}{i} S_1(l,j) c_i^{(k)} s_{n-l-i}(\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r).
$$

Thus, we obtain

$$
s_n^{(k)}(x) = \sum_{j=0}^{n} \sum_{l=0}^{n-j} \binom{n}{l} S_1(n-l,j) s_l^{(k)} x^j
$$

$$
= \sum_{j=0}^{n} \sum_{l=0}^{n-j} \sum_{i=0}^{l} \binom{n}{l} \binom{l}{i} S_1(n-l,j) c_i^{(k)} s_{l-i}(\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r) x^j,
$$

which are the identities (14) and (15).

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Next,

$$
\begin{aligned}
s_n^{(k)}(y|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r) &= \left\langle \sum_{i=0}^{\infty} s_i^{(k)}(y|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r)\frac{t^i}{i!}\Big|x^n\right\rangle \\
&= \left\langle \prod_{j=1}^{r}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\mathrm{Lif}_k\big(\ln(1+t)\big)(1+t)^y\Big|x^n\right\rangle \\
&= \left\langle \prod_{j=1}^{r}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\Big|\mathrm{Lif}_k\big(\ln(1+t)\big)(1+t)^y x^n\right\rangle \\
&= \left\langle \prod_{j=1}^{r}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\Big|\sum_{l=0}^{\infty}c_l^{(k)}(-y)\frac{t^l}{l!}x^n\right\rangle \\
&= \sum_{l=0}^{n}\binom{n}{l}c_l^{(k)}(-y)\left\langle \prod_{j=1}^{r}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\Big|x^{n-l}\right\rangle \\
&= \sum_{l=0}^{n}\binom{n}{l}c_l^{(k)}(-y)\left\langle \sum_{i=0}^{\infty}s_i(\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r)\frac{t^i}{i!}\Big|x^{n-l}\right\rangle \\
&= \sum_{l=0}^{n}\binom{n}{l}c_l^{(k)}(-y)s_{n-l}(\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r).
\end{aligned}
$$

Thus, we obtain (16).
Finally, we obtain that

$$
\begin{aligned}
s_n^{(k)}(y|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r) &= \left\langle \sum_{i=0}^{\infty} s_i^{(k)}(y|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r)\frac{t^i}{i!}\Big|x^n\right\rangle \\
&= \left\langle \prod_{j=1}^{r}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\mathrm{Lif}_k\big(\ln(1+t)\big)(1+t)^y\Big|x^n\right\rangle \\
&= \left\langle \mathrm{Lif}_k\big(\ln(1+t)\big)\Big|\prod_{j=1}^{r}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}(1+t)^y x^n\right\rangle \\
&= \left\langle \mathrm{Lif}_k\big(\ln(1+t)\big)\Big|\sum_{l=0}^{\infty}s_l(y|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r)\frac{t^l}{l!}x^n\right\rangle \\
&= \sum_{l=0}^{n}s_l(y|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r)\binom{n}{l}\left\langle \mathrm{Lif}_k\big(\ln(1+t)\big)\Big|x^{n-l}\right\rangle \\
&= \sum_{l=0}^{n}s_l(y|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r)\binom{n}{l}\left\langle \sum_{i=0}^{\infty}c_i^{(k)}\frac{t^i}{i!}\Big|x^{n-l}\right\rangle \\
&= \sum_{l=0}^{n}\binom{n}{l}s_l(y|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r)c_{n-l}^{(k)}.
\end{aligned}
$$

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Thus, we get the identity (17). ∎

3.2 Sheffer identity

Theorem 2

$$s_n^{(k)}(x+y|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r) = \sum_{j=0}^{n} \binom{n}{j} s_j^{(k)}(x|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r)(y)_{n-j}. \quad (20)$$

Proof. By (12) with

$$p_n(x) = \prod_{j=1}^{r}(1+e^{\lambda_j t})^{\mu_j} \frac{1}{\mathrm{Lif}_k(t)} s_n(x|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r)$$
$$= (x)_n \sim (1, e^t - 1),$$

using (10), we have (20). ∎

3.3 Difference relations

Theorem 3

$$s_n^{(k)}(x+1|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r) - s_n^{(k)}(x|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r)$$
$$= ns_{n-1}^{(k)}(x|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r). \quad (21)$$

Proof. By (8) with (12), we get

$$(e^t-1)s_n^{(k)}(x|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r) = ns_{n-1}^{(k)}(x|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r).$$

By (7), we have (21). ∎

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3.4 Recurrence

Theorem 4

$$
s_{n+1}^{(k)}(x|\lambda_1, \ldots, \lambda_r; \mu_1, \ldots, \mu_r)
$$

$$
= x s_n^{(k)}(x-1|\lambda_1, \ldots, \lambda_r; \mu_1, \ldots, \mu_r)
$$

$$
- 2^{-1-\sum_{j=1}^r \mu_j} \sum_{m=0}^n \sum_{l=0}^m \sum_{i=1}^r S_1(n,m) \frac{\binom{m}{l}}{(l+1)^k} \mu_i \lambda_i E_{m-l}(x+\lambda_i - 1|\lambda; \mu + e_i)
$$

$$
+ 2^{-\sum_{j=1}^r \mu_j} \sum_{m=0}^n \sum_{l=0}^m \frac{\binom{m}{l}}{(m+2-l)^k} S_1(n,m) E_l(x-1|\lambda; \mu) \tag{22}
$$

$$
= x s_n^{(k)}(x-1|\lambda_1, \ldots, \lambda_r; \mu_1, \ldots, \mu_r)
$$

$$
- \frac{1}{2} \sum_{j=0}^n \sum_{l=0}^{n-j} \sum_{i=1}^r \binom{n}{l} \mu_i \lambda_i^{j+1} S_1(n-l,j) s_l^{(k)} E_j\left(\frac{x+\lambda_i - 1}{\lambda_i} \right)
$$

$$
+ 2^{-\sum_{j=1}^r \mu_j} \sum_{m=0}^n \sum_{l=0}^m \frac{\binom{m}{l}}{(m+2-l)^k} S_1(n,m) E_l(x-1|\lambda; \mu). \tag{23}
$$

Proof. By applying

$$
s_{n+1}(x) = \left(x - \frac{g'(t)}{g(t)} \right) \frac{1}{f'(t)} s_n(x) \tag{24}
$$

([9, Corollary 3.7.2]) with (12), we get

$$
s_{n+1}^{(k)}(x|\lambda_1, \ldots, \lambda_r; \mu_1, \ldots, \mu_r)
$$

$$
= x s_n^{(k)}(x-1|\lambda_1, \ldots, \lambda_r; \mu_1, \ldots, \mu_r) - e^{-t} \frac{g'(t)}{g(t)} s_n^{(k)}(x|\lambda_1, \ldots, \lambda_r; \mu_1, \ldots, \mu_r).
$$

Since

$$
\frac{g'(t)}{g(t)} = \big(\ln g(t) \big)'
$$

$$
= \left(\sum_{i=1}^r \mu_i \ln(1+e^{\lambda_i t}) - \ln \operatorname{Lif}_k(t) \right)'
$$

$$
= \sum_{i=1}^r \frac{\mu_i \lambda_i e^{\lambda_i t}}{1+e^{\lambda_i t}} - \frac{\operatorname{Lif}_k'(t)}{\operatorname{Lif}_k(t)},
$$

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by (13), we have

$$
\frac{g'(t)}{g(t)} s_n^{(k)}(x)
$$

$$
= \left(\sum_{i=1}^{r} \frac{\mu_i \lambda_i e^{\lambda_i t}}{1 + e^{\lambda_i t}} - \frac{\mathrm{Lif}_k'(t)}{\mathrm{Lif}_k(t)} \right) s_n^{(k)}(x)
$$

$$
= 2^{-1-\sum_{j=1}^{r} \mu_j} \sum_{m=0}^{n} \sum_{l=0}^{m} S_1(n,m) \frac{\binom{m}{l}}{(l+1)^k} \sum_{i=1}^{r} \mu_i \lambda_i e^{\lambda_i t} \frac{2}{1 + e^{\lambda_i t}} \prod_{j=1}^{r} \left(\frac{2}{1 + e^{\lambda_j t}} \right)^{\mu_j} x^{m-l}
$$

$$
- \sum_{m=0}^{n} S_1(n,m) \prod_{j=1}^{r} (1 + e^{\lambda_j t})^{-\mu_j} \mathrm{Lif}_k'(t) x^m . \tag{25}
$$

The first term in (25) is

$$
2^{-1-\sum_{j=1}^{r} \mu_j} \sum_{m=0}^{n} \sum_{l=0}^{m} \sum_{i=1}^{r} S_1(n,m) \frac{\binom{m}{l}}{(l+1)^k} \mu_i \lambda_i E_{m-l}(x + \lambda_i | \lambda; \mu + e_i) ,
$$

where $\lambda = (\lambda_1, \ldots, \lambda_r)$, $\mu = (\mu_1, \ldots, \mu_r)$ and $e_i = (\underbrace{0, \ldots, 0}_{i-1}, 1, \underbrace{0, \ldots, 0}_{r-i})$ $(i = 1, 2, \ldots, r)$.

Since

$$
\mathrm{Lif}_{k-1}(t) - \mathrm{Lif}_k(t) = \left(\frac{1}{2^{k-1}} - \frac{1}{2^k} \right) t + \cdots ,
$$

the second term in (25) is

$$
2^{-\sum_{j=1}^{r} \mu_j} \sum_{m=0}^{n} S_1(n,m) \mathrm{Lif}_k'(t) \prod_{j=1}^{r} \left(\frac{2}{1 + e^{\lambda_j t}} \right)^{\mu_j} x^m
$$

$$
= 2^{-\sum_{j=1}^{r} \mu_j} \sum_{m=0}^{n} S_1(n,m) \frac{\mathrm{Lif}_{k-1}(t) - \mathrm{Lif}_k(t)}{t} E_m(x | \lambda; \mu)
$$

$$
= 2^{-\sum_{j=1}^{r} \mu_j} \sum_{m=0}^{n} S_1(n,m) \left(\mathrm{Lif}_{k-1}(t) - \mathrm{Lif}_k(t) \right) \frac{E_{m+1}(x | \lambda; \mu)}{m+1}
$$

$$
= 2^{-\sum_{j=1}^{r} \mu_j} \sum_{m=0}^{n} \frac{S_1(n,m)}{m+1} \left(\sum_{l=0}^{m+1} \frac{t^l}{l!(l+1)^{k-1}} E_{m+1}(x | \lambda; \mu) - \sum_{l=0}^{m+1} \frac{t^l}{l!(l+1)^k} E_{m+1}(x | \lambda; \mu) \right)
$$

$$
= 2^{-\sum_{j=1}^{r} \mu_j} \sum_{m=0}^{n} \frac{S_1(n,m)}{m+1} \left(\sum_{l=0}^{m+1} \frac{\binom{m+1}{l}}{(l+1)^{k-1}} E_{m+1-l}(x | \lambda; \mu) - \sum_{l=0}^{m+1} \frac{\binom{m+1}{l}}{(l+1)^k} E_{m+1-l}(x | \lambda; \mu) \right)
$$

$$
= 2^{-\sum_{j=1}^{r} \mu_j} \sum_{m=0}^{n} \frac{S_1(n,m)}{m+1} \sum_{l=1}^{m+1} \frac{\binom{m+1}{l} l}{(l+1)^k} E_{m+1-l}(x | \lambda; \mu)
$$

$$
= 2^{-\sum_{j=1}^{r} \mu_j} \sum_{m=0}^{n} S_1(n,m) \sum_{l=0}^{m} \frac{\binom{m}{l}}{(m+2-l)^k} E_l(x | \lambda; \mu) .
$$

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Thus, we obtain

$$
s_{n+1}^{(k)}(x) = x s_n^{(k)}(x-1)
$$
$$
- 2^{-1-\sum_{j=1}^r \mu_j} \sum_{m=0}^n \sum_{l=0}^m \sum_{i=1}^r S_1(n,m) \frac{\binom{m}{l}}{(l+1)^k} \mu_i \lambda_i E_{m-l}(x+\lambda_i-1|\lambda;\mu+e_i)
$$
$$
+ 2^{-\sum_{j=1}^r \mu_j} \sum_{m=0}^n \sum_{l=0}^m \frac{\binom{m}{l}}{(m+2-l)^k} S_1(n,m) E_l(x-1|\lambda;\mu),
$$

which is (22).

On the other hand, by (14) with (22), we have

$$
\frac{g'(t)}{g(t)} s_n^{(k)}(x)
$$
$$
= \left(\sum_{i=1}^r \frac{\mu_i \lambda_i e^{\lambda_i t}}{1+e^{\lambda_i t}} - \frac{\mathrm{Lif}_k'(t)}{\mathrm{Lif}_k(t)} \right) s_n^{(k)}(x)
$$
$$
= \frac{1}{2} \sum_{i=1}^r \mu_i \lambda_i e^{\lambda_i t} \frac{2}{1+e^{\lambda_i t}} \sum_{j=0}^n \sum_{l=0}^{n-j} \binom{n}{l} S_1(n-l,j) s_l^{(k)} x^j
$$
$$
- 2^{-\sum_{j=1}^r \mu_j} \sum_{m=0}^n \sum_{l=0}^m \frac{\binom{m}{l}}{(m+2-l)^k} S_1(n,m) E_l(x|\lambda;\mu). \tag{26}
$$

The first term in (26) is

$$
\frac{1}{2} \sum_{j=0}^n \sum_{l=0}^{n-j} \binom{n}{l} S_1(n-l,j) s_l^{(k)} \sum_{i=1}^r \mu_i \lambda_i e^{\lambda_i t} \frac{2}{1+e^{\lambda_i t}} x^j
$$
$$
= \frac{1}{2} \sum_{j=0}^n \sum_{l=0}^{n-j} \binom{n}{l} S_1(n-l,j) s_l^{(k)} \sum_{i=1}^r \mu_i \lambda_i e^{\lambda_i t} \lambda_i^j E_j\left(\frac{x}{\lambda_i}\right)
$$
$$
= \frac{1}{2} \sum_{j=0}^n \sum_{l=0}^{n-j} \binom{n}{l} S_1(n-l,j) s_l^{(k)} \sum_{i=1}^r \mu_i \lambda_i^{j+1} E_j\left(\frac{x+\lambda_i}{\lambda_i}\right).
$$

Thus, we obtain

$$
s_{n+1}^{(k)}(x) = x s_n^{(k)}(x-1)
$$
$$
- \frac{1}{2} \sum_{j=0}^n \sum_{l=0}^{n-j} \sum_{i=1}^r \binom{n}{l} \mu_i \lambda_i^{j+1} S_1(n-l,j) s_l^{(k)} E_j\left(\frac{x+\lambda_i-1}{\lambda_i}\right)
$$
$$
+ 2^{-\sum_{j=1}^r \mu_j} \sum_{m=0}^n \sum_{l=0}^m \frac{\binom{m}{l}}{(m+2-l)^k} S_1(n,m) E_l(x-1|\lambda;\mu).
$$

which is (23). ∎

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3.5 Differentiation

Theorem 5

$$\frac{d}{dx}s_n^{(k)}(x|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r) = n!\sum_{l=0}^{n-1}\frac{(-1)^{n-l-1}}{l!(n-l)}s_l^{(k)}(x|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r). \quad (27)$$

Proof. We shall use

$$\frac{d}{dx}s_n(x) = \sum_{l=0}^{n-1}\binom{n}{l}\left\langle\bar{f}(t)|x^{n-l}\right\rangle s_l(x)$$

(*Cf.* [9, Theorem 2.3.12]). Since

$$\left\langle\bar{f}(t)|x^{n-l}\right\rangle = \left\langle\ln(1+t)|x^{n-l}\right\rangle$$

$$= \left\langle\sum_{m=1}^{\infty}\frac{(-1)^{m-1}t^m}{m}\Big|x^{n-l}\right\rangle$$

$$= \sum_{m=1}^{n-l}\frac{(-1)^{m-1}}{m}\left\langle t^m|x^{n-l}\right\rangle$$

$$= \sum_{m=1}^{n-l}\frac{(-1)^{m-1}}{m}(n-l)!\delta_{m,n-l}$$

$$= (-1)^{n-l-1}(n-l-1)!,$$

with (12), we have

$$\frac{d}{dx}s_n^{(k)}(x|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r)$$

$$= \sum_{l=0}^{n-1}\binom{n}{l}(-1)^{n-l-1}(n-l-1)!s_l^{(k)}(x|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r)$$

$$= n!\sum_{l=0}^{n-1}\frac{(-1)^{n-l-1}}{l!(n-l)}s_l^{(k)}(x|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r),$$

which is the identity (27). ∎

3.6 A more relation

The classical Cauchy numbers c_n are defined by

$$\frac{t}{\ln(1+t)} = \sum_{n=0}^{\infty}c_n\frac{t^n}{n!}$$

(see e.g. [2, 6]).

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Theorem 6 *For $n \geq 1$, we have*

$$s_n^{(k)}(x|\lambda_1, \ldots, \lambda_r; \mu_1, \ldots, \mu_r)$$

$$= x s_{n-1}^{(k)}(x - 1|\lambda_1, \ldots, \lambda_r; \mu_1, \ldots, \mu_r) + \frac{1}{n} \sum_{l=1}^{n} \binom{n}{l} c_{n-l} \left(s_l^{(k-1)}(x - 1) - s_l^{(k)}(x - 1) \right)$$

$$- \sum_{i=1}^{r} \mu_i \lambda_i s_{n-1}^{(k)}(x + \lambda_i - 1|\lambda; \mu + e_i). \tag{28}$$

Proof. For $n \geq 1$, we have

$$s_n^{(k)}(y|\lambda_1, \ldots, \lambda_r; \mu_1, \ldots, \mu_r)$$

$$= \left\langle \sum_{l=0}^{\infty} s_l^{(k)}(y|\lambda_1, \ldots, \lambda_r; \mu_1, \ldots, \mu_r) \frac{t^l}{l!} \middle| x^n \right\rangle$$

$$= \left\langle \prod_{j=1}^{r} \left(1 + (1 + t)^{\lambda_j}\right)^{-\mu_j} \mathrm{Lif}_k\left(\ln(1 + t)\right)(1 + t)^y \middle| x^n \right\rangle$$

$$= \left\langle \partial_t \left(\prod_{j=1}^{r} \left(1 + (1 + t)^{\lambda_j}\right)^{-\mu_j} \mathrm{Lif}_k\left(\ln(1 + t)\right)(1 + t)^y \right) \middle| x^{n-1} \right\rangle$$

$$= \left\langle \left(\partial_t \prod_{j=1}^{r} \left(1 + (1 + t)^{\lambda_j}\right)^{-\mu_j} \right) \mathrm{Lif}_k\left(\ln(1 + t)\right)(1 + t)^y \middle| x^{n-1} \right\rangle$$

$$+ \left\langle \prod_{j=1}^{r} \left(1 + (1 + t)^{\lambda_j}\right)^{-\mu_j} \left(\partial_t \mathrm{Lif}_k\left(\ln(1 + t)\right) \right)(1 + t)^y \middle| x^{n-1} \right\rangle$$

$$+ \left\langle \prod_{j=1}^{r} \left(1 + (1 + t)^{\lambda_j}\right)^{-\mu_j} \mathrm{Lif}_k\left(\ln(1 + t)\right) \left(\partial_t (1 + t)^y \right) \middle| x^{n-1} \right\rangle.$$

The third term is

$$y \left\langle \prod_{j=1}^{r} \left(1 + (1 + t)^{\lambda_j}\right)^{-\mu_j} \mathrm{Lif}_k\left(\ln(1 + t)\right)(1 + t)^{y-1} \middle| x^{n-1} \right\rangle$$

$$= y s_{n-1}^{(k)}(y - 1|\lambda_1, \ldots, \lambda_r; \mu_1, \ldots, \mu_r).$$

Since

$$\mathrm{Lif}_{k-1}\left(\ln(1 + t)\right) - \mathrm{Lif}_k\left(\ln(1 + t)\right) = \left(\frac{1}{2^{k-1}} - \frac{1}{2^k} \right) t + \cdots,$$

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the second term is

$$\left\langle \prod_{j=1}^{r}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\frac{\mathrm{Lif}_{k-1}\big(\ln(1+t)\big)-\mathrm{Lif}_k\big(\ln(1+t)\big)}{(1+t)\ln(1+t)}(1+t)^{y}\Big|x^{n-1}\right\rangle$$

$$=\left\langle \prod_{j=1}^{r}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\right.$$

$$\left.\frac{\mathrm{Lif}_{k-1}\big(\ln(1+t)\big)-\mathrm{Lif}_k\big(\ln(1+t)\big)}{t}(1+t)^{y-1}\Big|\frac{t}{\ln(1+t)}x^{n-1}\right\rangle$$

$$=\left\langle \prod_{j=1}^{r}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\right.$$

$$\left.\frac{\mathrm{Lif}_{k-1}\big(\ln(1+t)\big)-\mathrm{Lif}_k\big(\ln(1+t)\big)}{t}(1+t)^{y-1}\Big|\sum_{l=0}^{\infty}c_l\frac{t^l}{l!}x^{n-1}\right\rangle$$

$$=\sum_{l=0}^{n-1}\binom{n-1}{l}c_l\left\langle \prod_{j=1}^{r}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\right.$$

$$\left.(1+t)^{y-1}\Big|\frac{\mathrm{Lif}_{k-1}\big(\ln(1+t)\big)-\mathrm{Lif}_k\big(\ln(1+t)\big)}{t}x^{n-1-l}\right\rangle$$

$$=\sum_{l=0}^{n-1}\frac{1}{n-l}\binom{n-1}{l}c_l\left\langle \prod_{j=1}^{r}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\right.$$

$$\left.(1+t)^{y-1}\Big|\left(\mathrm{Lif}_{k-1}\big(\ln(1+t)\big)-\mathrm{Lif}_k\big(\ln(1+t)\big)\right)x^{n-l}\right\rangle$$

$$=\sum_{l=0}^{n-1}\frac{1}{n-l}\binom{n}{l}c_l\left(\left\langle \prod_{j=1}^{r}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\mathrm{Lif}_{k-1}\big(\ln(1+t)\big)(1+t)^{y-1}\Big|x^{n-l}\right\rangle\right.$$

$$\left.-\left\langle \prod_{j=1}^{r}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\mathrm{Lif}_k\big(\ln(1+t)\big)(1+t)^{y-1}\Big|x^{n-l}\right\rangle\right)$$

$$=\frac{1}{n}\sum_{l=0}^{n-1}\binom{n}{l}c_l\big(s_{n-l}^{(k-1)}(y-1|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r)-s_{n-l}^{(k)}(y-1|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r)\big)$$

$$=\frac{1}{n}\sum_{l=1}^{n}\binom{n}{l}c_{n-l}\big(s_l^{(k-1)}(y-1|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r)-s_l^{(k)}(y-1|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r)\big).$$

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Since

$$
\partial_t \prod_{j=1}^{r} \left(1 + (1+t)^{\lambda_j}\right)^{-\mu_j}
$$

$$
= -\sum_{i=1}^{r} \mu_i \lambda_i (1+t)^{\lambda_i - 1} \left(1 + (1+t)\lambda_i\right)^{-1} \prod_{j=1}^{r} \left(1 + (1+t)^{\lambda_j}\right)^{-\mu_j},
$$

the first term is

$$
-\sum_{i=1}^{r} \mu_i \lambda_i \left\langle \left(1 + (1+t)\lambda_i\right)^{-1} \prod_{j=1}^{r} \left(1 + (1+t)^{\lambda_j}\right)^{-\mu_j} \mathrm{Lif}_k \left(\ln(1+t)\right)(1+t)^{y+\lambda_i - 1} \Big| x^{n-1} \right\rangle
$$

$$
= -\sum_{i=1}^{r} \mu_i \lambda_i s_{n-1}^{(k)}(y + \lambda_i - 1 | \lambda; \mu + e_i).
$$

Therefore, we obtain

$$
s_n^{(k)}(x | \lambda_1, \ldots, \lambda_r; \mu_1, \ldots, \mu_r)
$$

$$
= x s_{n-1}^{(k)}(x - 1 | \lambda_1, \ldots, \lambda_r; \mu_1, \ldots, \mu_r) + \frac{1}{n} \sum_{l=1}^{n} \binom{n}{l} c_{n-l} \left(s_l^{(k-1)}(x-1) - s_l^{(k)}(x-1)\right)
$$

$$
- \sum_{i=1}^{r} \mu_i \lambda_i s_{n-1}^{(k)}(x + \lambda_i - 1 | \lambda; \mu + e_i),
$$

which is the identity (28). ∎

3.7 A relation including the Stirling numbers of the first kind

Theorem 7 *For $n - 1 \geq m \geq 1$, we have*

$$
m \sum_{l=0}^{n-m} \binom{n}{l} S_1(n-l, m) s_l^{(k)}(\lambda_1, \ldots, \lambda_r; \mu_1, \ldots, \mu_r)
$$

$$
= (m-1) \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-l-1, m-1) s_l^{(k)}(-1 | \lambda_1, \ldots, \lambda_r; \mu_1, \ldots, \mu_r)
$$

$$
+ \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-l-1, m-1) s_l^{(k-1)}(-1 | \lambda_1, \ldots, \lambda_r; \mu_1, \ldots, \mu_r)
$$

$$
- m \sum_{l=0}^{n-m-1} \sum_{i=1}^{r} \binom{n-1}{l} S_1(n-l-1, m) \mu_i \lambda_i s_l^{(k)}(\lambda_i - 1 | \lambda; \mu + e_i). \tag{29}
$$

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Proof. We shall compute

$$\left\langle \prod_{j=1}^{r} \left(1 + (1+t)^{\lambda_j}\right)^{-\mu_j} \mathrm{Lif}_k\big(\ln(1+t)\big) \big(\ln(1+t)\big)^m \Big| x^n \right\rangle$$

in two different ways. On the one hand, it is equal to

$$\left\langle \prod_{j=1}^{r} \left(1 + (1+t)^{\lambda_j}\right)^{-\mu_j} \mathrm{Lif}_k\big(\ln(1+t)\big) \Big| \big(\ln(1+t)\big)^m x^n \right\rangle$$

$$= \left\langle \prod_{j=1}^{r} \left(1 + (1+t)^{\lambda_j}\right)^{-\mu_j} \mathrm{Lif}_k\big(\ln(1+t)\big) \Big| m! \sum_{l=m}^{\infty} S_1(l,m)\frac{t^l}{l!} x^n \right\rangle$$

$$= m! \sum_{l=m}^{n} \binom{n}{l} S_1(l,m) \left\langle \prod_{j=1}^{r} \left(1 + (1+t)^{\lambda_j}\right)^{-\mu_j} \mathrm{Lif}_k\big(\ln(1+t)\big) \Big| x^{n-l} \right\rangle$$

$$= m! \sum_{l=m}^{n} \binom{n}{l} S_1(l,m) \left\langle \sum_{i=0}^{\infty} s_i^{(k)}(\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r)\frac{t^i}{i!} \Big| x^{n-l} \right\rangle$$

$$= m! \sum_{l=m}^{n} \binom{n}{l} S_1(l,m) s_{n-l}^{(k)}(\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r)$$

$$= m! \sum_{l=0}^{n-m} \binom{n}{l} S_1(n-l,m) s_l^{(k)}(\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r).$$

On the other hand, it is equal to

$$\left\langle \partial_t \left(\prod_{j=1}^{r} \left(1 + (1+t)^{\lambda_j}\right)^{-\mu_j} \mathrm{Lif}_k\big(\ln(1+t)\big) \big(\ln(1+t)\big)^m \right) \Big| x^{n-1} \right\rangle$$

$$= \left\langle \left(\partial_t \prod_{j=1}^{r} \left(1 + (1+t)^{\lambda_j}\right)^{-\mu_j} \right) \mathrm{Lif}_k\big(\ln(1+t)\big) \big(\ln(1+t)\big)^m \Big| x^{n-1} \right\rangle$$

$$+ \left\langle \prod_{j=1}^{r} \left(1 + (1+t)^{\lambda_j}\right)^{-\mu_j} \left(\partial_t \mathrm{Lif}_k\big(\ln(1+t)\big)\right) \big(\ln(1+t)\big)^m \Big| x^{n-1} \right\rangle$$

$$+ \left\langle \prod_{j=1}^{r} \left(1 + (1+t)^{\lambda_j}\right)^{-\mu_j} \mathrm{Lif}_k\big(\ln(1+t)\big) \left(\partial_t \big(\ln(1+t)\big)^m\right) \Big| x^{n-1} \right\rangle. \tag{30}$$

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The third term of (30) is equal to

$$
m \left\langle \prod_{j=1}^{r} \left(1 + (1+t)^{\lambda_j}\right)^{-\mu_j} \mathrm{Lif}_k\big(\ln(1+t)\big)(1+t)^{-1} \,\middle|\, \big(\ln(1+t)\big)^{m-1} x^{n-1} \right\rangle
$$

$$
= m \left\langle \prod_{j=1}^{r} \left(1 + (1+t)^{\lambda_j}\right)^{-\mu_j} \mathrm{Lif}_k\big(\ln(1+t)\big)(1+t)^{-1} \,\right|
$$

$$
(m-1)! \sum_{l=m-1}^{\infty} S_1(l, m-1) \frac{t^l}{l!} x^{n-1} \right\rangle
$$

$$
= m! \sum_{l=m-1}^{n-1} \binom{n-1}{l} S_1(l, m-1)
$$

$$
\times \left\langle \prod_{j=1}^{r} \left(1 + (1+t)^{\lambda_j}\right)^{-\mu_j} \mathrm{Lif}_k\big(\ln(1+t)\big)(1+t)^{-1} \,\middle|\, x^{n-1-l} \right\rangle
$$

$$
= m! \sum_{l=m-1}^{n-1} \binom{n-1}{l} S_1(l, m-1) s_{n-1-l}^{(k)}(-1|\lambda_1, \ldots, \lambda_r; \mu_1, \ldots, \mu_r)
$$

$$
= m! \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-l-1, m-1) s_l^{(k)}(-1|\lambda_1, \ldots, \lambda_r; \mu_1, \ldots, \mu_r).
$$

The second term of (30) is equal to

$$
\left\langle \prod_{j=1}^{r} \left(1 + (1+t)^{\lambda_j}\right)^{-\mu_j} \left(\frac{\mathrm{Lif}_{k-1}\big(\ln(1+t)\big) - \mathrm{Lif}_k\big(\ln(1+t)\big)}{(1+t)\ln(1+t)} \right) \big(\ln(1+t)\big)^m \,\middle|\, x^{n-1} \right\rangle
$$

$$
= \left\langle \prod_{j=1}^{r} \left(1 + (1+t)^{\lambda_j}\right)^{-\mu_j} \mathrm{Lif}_{k-1}\big(\ln(1+t)\big)(1+t)^{-1} \,\middle|\, \big(\ln(1+t)\big)^{m-1} x^{n-1} \right\rangle
$$

$$
- \left\langle \prod_{j=1}^{r} \left(1 + (1+t)^{\lambda_j}\right)^{-\mu_j} \mathrm{Lif}_k\big(\ln(1+t)\big)(1+t)^{-1} \,\middle|\, \big(\ln(1+t)\big)^{m-1} x^{n-1} \right\rangle
$$

$$
= (m-1)! \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-l-1, m-1) s_l^{(k-1)}(-1|\lambda_1, \ldots, \lambda_r; \mu_1, \ldots, \mu_r)
$$

$$
- (m-1)! \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-l-1, m-1) s_l^{(k)}(-1|\lambda_1, \ldots, \lambda_r; \mu_1, \ldots, \mu_r).
$$

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The first term of (30) is equal to

$$
\left\langle \left(\partial_t \prod_{j=1}^{r} \left(1 + (1+t)^{\lambda_j}\right)^{-\mu_j} \right) \mathrm{Lif}_k\big(\ln(1+t)\big)\big(\ln(1+t)\big)^m \Big| x^{n-1} \right\rangle
$$

$$
= -\sum_{i=1}^{r} \mu_i \lambda_i \left\langle \left(1 + (1+t)^{\lambda_i}\right)^{-1} \prod_{j=1}^{r} \left(1 + (1+t)^{\lambda_i}\right)^{-\mu_j} \right.
$$
$$
\left. \mathrm{Lif}_k\big(\ln(1+t)\big)(1+t)^{\lambda_i - 1} \Big| \big(\ln(1+t)\big)^m x^{n-1} \right\rangle
$$

$$
= -\sum_{i=1}^{r} \mu_i \lambda_i \left\langle \left(1 + (1+t)^{\lambda_i}\right)^{-1} \prod_{j=1}^{r} \left(1 + (1+t)^{\lambda_i}\right)^{-\mu_j} \right.
$$
$$
\left. \mathrm{Lif}_k\big(\ln(1+t)\big)(1+t)^{\lambda_i - 1} \Big| m! \sum_{l=m}^{\infty} S_1(l,m) \frac{t^l}{l!} x^{n-1} \right\rangle
$$

$$
= -m! \sum_{i=1}^{r} \mu_i \lambda_i \sum_{l=m}^{n-1} \binom{n-1}{l} S_1(l,m)
$$
$$
\times \left\langle \left(1 + (1+t)^{\lambda_i}\right)^{-1} \prod_{j=1}^{r} \left(1 + (1+t)^{\lambda_i}\right)^{-\mu_j} \mathrm{Lif}_k\big(\ln(1+t)\big)(1+t)^{\lambda_i - 1} \Big| x^{n-1-l} \right\rangle
$$

$$
= -m! \sum_{i=1}^{r} \mu_i \lambda_i \sum_{l=m}^{n-1} \binom{n-1}{l} S_1(l,m) s_{n-1-l}^{(k)}(\lambda_i - 1 | \lambda; \mu + e_i)
$$

$$
= -m! \sum_{l=0}^{n-m-1} \sum_{i=1}^{r} \binom{n-1}{l} S_1(n-1-l,m) \mu_i \lambda_i s_l^{(k)}(\lambda_i - 1 | \lambda; \mu + e_i).
$$

Therefore, we get, for $n-1 \geq m \geq 1$,

$$
m! \sum_{l=0}^{n-m} \binom{n}{l} S_1(n-l,m) s_l^{(k)}(\lambda_1, \ldots, \lambda_r; \mu_1, \ldots, \mu_r)
$$

$$
= m! \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-l-1, m-1) s_l^{(k)}(-1)
$$

$$
+ (m-1)! \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-l-1, m-1) s_l^{(k-1)}(-1)
$$

$$
- (m-1)! \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-l-1, m-1) s_l^{(k)}(-1)
$$

$$
- m! \sum_{l=0}^{n-m-1} \sum_{i=1}^{r} \binom{n-1}{l} S_1(n-l-1, m) \mu_i \lambda_i s_l^{(k)}(\lambda_i - 1 | \lambda; \mu + e_i).
$$

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Dividing both sides by $(m-1)!$, we obtain for $n-1 \geq m \geq 1$

$$m \sum_{l=0}^{n-m} \binom{n}{l} S_1(n-l,m) s_l^{(k)}(\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r)$$

$$= (m-1) \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-l-1,m-1) s_l^{(k)}(-1|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r)$$

$$+ \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-l-1,m-1) s_l^{(k-1)}(-1|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r)$$

$$- m \sum_{l=0}^{n-m-1} \sum_{i=1}^{r} \binom{n-1}{l} S_1(n-l-1,m) \mu_i \lambda_i s_l^{(k)}(\lambda_i - 1|\lambda;\mu+e_i) .$$

Thus, we get (29). ∎

3.8 A relation with the falling factorials

Theorem 8

$$s_n^{(k)}(x|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r) = \sum_{m=0}^{n} \binom{n}{m} s_{n-m}^{(k)} (x)_m . \tag{31}$$

Proof. For (12) and (19), assume that $s_n^{(k)}(x|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r) = \sum_{m=0}^{n} C_{n,m}(x)_m$. By (11), we have

$$C_{n,m} = \frac{1}{m!} \left\langle \frac{1}{\prod_{j=1}^{r}\left(1+e^{\lambda_j \ln(1+t)}\right)^{\mu_j}} \mathrm{Lif}_k\big(\ln(1+t)\big) t^m \Big| x^n \right\rangle$$

$$= \frac{1}{m!} \left\langle \prod_{j=1}^{r}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j} \mathrm{Lif}_k\big(\ln(1+t)\big) \Big| t^m x^n \right\rangle$$

$$= \binom{n}{m} \left\langle \prod_{j=1}^{r}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j} \mathrm{Lif}_k\big(\ln(1+t)\big) \Big| x^{n-m} \right\rangle$$

$$= \binom{n}{m} s_{n-m}^{(k)} .$$

Thus, we get the identity (31). ∎

3.9 A relation with higher-order Frobenius-Euler polynomials

For $\alpha \in \mathbb{C}$ with $\alpha \neq 1$, the Frobenius-Euler polynomials of order r, $H_n^{(r)}(x|\alpha)$ are defined by the generating function

$$\left(\frac{1-\alpha}{e^t-\alpha}\right)^r e^{xt} = \sum_{n=0}^{\infty} H_n^{(r)}(x|\alpha)\frac{t^n}{n!}$$

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(see e.g. [4]).

Theorem 9

$$
s_n^{(k)}(x|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r) = \sum_{m=0}^{n} \left(\sum_{j=0}^{n-m}\sum_{l=0}^{n-m-j} \binom{s}{j}\binom{n-j}{l}(n)_j \right.
$$
$$
\left. \times (1-\alpha)^{-j} S_1(n-j-l,m) s_l^{(k)} \right) H_m^{(s)}(x|\alpha). \tag{32}
$$

Proof. For (12) and

$$
H_n^{(s)}(x|\alpha) \sim \left(\left(\frac{e^t-\alpha}{1-\alpha} \right)^s, t \right), \tag{33}
$$

assume that $s_n^{(k)}(x|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r) = \sum_{m=0}^{n} C_{n,m} H_m^{(s)}(x|\alpha)$. By (11), similarly to the proof of (29), we have

$$
C_{n,m} = \frac{1}{m!} \left\langle \frac{\left(\frac{e^{\ln(1+t)}-\alpha}{1-\alpha} \right)^s}{\prod_{j=1}^{r}\left(1+e^{\lambda_j \ln(1+t)}\right)^{\mu_j}} \mathrm{Lif}_k\big(\ln(1+t)\big)\big(\ln(1+t)\big)^m \bigg| x^n \right\rangle
$$

$$
= \frac{1}{m!(1-\alpha)^s} \left\langle \prod_{j=1}^{r}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j} \mathrm{Lif}_k\big(\ln(1+t)\big)\big(\ln(1+t)\big)^m (1-\alpha+t)^s \bigg| x^n \right\rangle
$$

$$
= \frac{1}{m!(1-\alpha)^s}
$$
$$
\times \left\langle \prod_{j=1}^{r}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j} \mathrm{Lif}_k\big(\ln(1+t)\big)\big(\ln(1+t)\big)^m \bigg| \sum_{i=0}^{\min\{s,n\}} \binom{s}{i}(1-\alpha)^{s-i}t^i x^n \right\rangle
$$

$$
= \frac{1}{m!(1-\alpha)^s} \sum_{i=0}^{n-m} \binom{s}{i}(1-\alpha)^{s-i}(n)_i
$$
$$
\times \left\langle \prod_{j=1}^{r}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j} \mathrm{Lif}_k\big(\ln(1+t)\big)\big(\ln(1+t)\big)^m \bigg| x^{n-i} \right\rangle
$$

$$
= \frac{1}{m!(1-\alpha)^s} \sum_{i=0}^{n-m} \binom{s}{i}(1-\alpha)^{s-i}(n)_i \sum_{l=0}^{n-m-i} m! \binom{n-i}{l} S_1(n-i-l,m) s_l^{(k)}
$$

$$
= \sum_{i=0}^{n-m}\sum_{l=0}^{n-m-i} \binom{s}{i}\binom{n-i}{l}(n)_i(1-\alpha)^{-i} S_1(n-i-l,m) s_l^{(k)}.
$$

Thus, we get the identity (32). ∎

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3.10 A relation with higher-order Bernoulli polynomials

Bernoulli polynomials $\mathfrak{B}_n^{(r)}(x)$ of order r are defined by

$$\left(\frac{t}{e^t-1}\right)^r e^{xt} = \sum_{n=0}^{\infty} \frac{\mathfrak{B}_n^{(r)}(x)}{n!} t^n$$

(see e.g. [9, Section 2.2]). In addition, Cauchy numbers of the first kind $\mathfrak{C}_n^{(r)}$ of order r are defined by

$$\left(\frac{t}{\ln(1+t)}\right)^r = \sum_{n=0}^{\infty} \frac{\mathfrak{C}_n^{(r)}}{n!} t^n$$

(see e.g. [1, (2.1)], [8, (6)]).

Theorem 10

$$s_n^{(k)}(x|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r)$$
$$= \sum_{m=0}^{n} \left(\sum_{i=0}^{n-m} \sum_{l=0}^{n-m-i} \binom{n}{i}\binom{n-i}{l} \mathfrak{C}_i^{(s)} S_1(n-i-l,m) s_l^{(k)}\right) \mathfrak{B}_m^{(s)}(x). \qquad (34)$$

Proof. For (12) and

$$\mathfrak{B}_n^{(s)}(x) \sim \left(\left(\frac{e^t-1}{t}\right)^s, t\right), \qquad (35)$$

assume that $s_n^{(k)}(x|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r) = \sum_{m=0}^{n} C_{n,m} \mathfrak{B}_m^{(s)}(x)$. By (11), similarly to the proof of (29), we have

$$C_{n,m} = \frac{1}{m!} \left\langle \frac{\left(\frac{e^{\ln(1+t)}-1}{\ln(1+t)}\right)^s}{\prod_{j=1}^{r}(1+e^{\lambda_j \ln(1+t)})^{\mu_j}} \mathrm{Lif}_k\big(\ln(1+t)\big)\big(\ln(1+t)\big)^m \Big| x^n \right\rangle$$

$$= \frac{1}{m!} \left\langle \prod_{j=1}^{r}\big(1+(1+t)^{\lambda_j}\big)^{-\mu_j} \mathrm{Lif}_k\big(\ln(1+t)\big)\big(\ln(1+t)\big)^m \Big| \left(\frac{t}{\ln(1+t)}\right)^s x^n \right\rangle$$

$$= \frac{1}{m!} \left\langle \prod_{j=1}^{r}\big(1+(1+t)^{\lambda_j}\big)^{-\mu_j} \mathrm{Lif}_k\big(\ln(1+t)\big)\big(\ln(1+t)\big)^m \Big| \sum_{i=0}^{\infty} \mathfrak{C}_i^{(s)} \frac{t^i}{i!} x^n \right\rangle$$

$$= \frac{1}{m!} \sum_{i=0}^{n-m} \mathfrak{C}_i^{(s)} \binom{n}{i} \left\langle \prod_{j=1}^{r}\big(1+(1+t)^{\lambda_j}\big)^{-\mu_j} \mathrm{Lif}_k\big(\ln(1+t)\big)\big(\ln(1+t)\big)^m \Big| x^{n-i} \right\rangle$$

$$= \frac{1}{m!} \sum_{i=0}^{n-m} \mathfrak{C}_i^{(s)} \binom{n}{i} \sum_{l=0}^{n-m-i} m! \binom{n-i}{l} S_1(n-i-l,m) s_l^{(k)}$$

$$= \sum_{i=0}^{n-m} \sum_{l=0}^{n-m-i} \binom{n}{i}\binom{n-i}{l} \mathfrak{C}_i^{(s)} S_1(n-i-l,m) s_l^{(k)}.$$

Thus, we get the identity (34). ∎

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On the hyperstablity of a functional equation in commutative groups

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Abstract. Using the fixed point method, we prove the hyperstablity of the functional equation

$$f(ax + by) = \frac{(a+b)}{2}f(x+y) + \frac{(a-b)}{2}f(x-y),$$

where a, b are different integers greater than 1, in the class of functions from a commutative group into a commutative complete metric group.

1. **Introduction and preliminaries**

The stability problem of functional equations originated from a question of Ulam [33] concerning the stability of group homomorphisms. Hyers [19] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [4] for additive mappings and by Rassias [29] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [17] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [6, 12, 13, 14, 15, 16, 20, 21, 25, 27, 28, 30, 31, 34]).

We say a functional equation $\mathfrak{D}$ is *hyperstable* if any function f satisfying the equation $\mathfrak{D}$ approximately is a true solution of $\mathfrak{D}$. It seems that the first hyperstability result was published in [6] and concerned the ring homomorphisms. However, The term *hyperstability* has been used for the first time in [24]. Quite often the hyperstability is confused with superstability, which admits also bounded functions. Numerous papers on this subject have been published and we refer to [1, 2, 3, 5, 7, 8, 9, 11, 18, 24, 26, 32].

Throughout this paper, we will denote the set of natural numbers by $\mathbb{N}$, the set of integers by $\mathbb{Z}$ and the set of real numbers by $\mathbb{R}$. Let $\mathbb{N}_+$ be the set of positive integers. By $\mathbb{N}_m$, $m \in \mathbb{N}_+$, we will denote the set of all integers greater than or equal to m. Let $\mathbb{R}_0 = [0, \infty)$ the set of nonnegative real numbers and $\mathbb{R}_+ = (0, \infty)$ the set of positive real numbers. We write B^A to mean " the family of all functions mapping from a nonempty set A into a nonempty set B ".

Definition 1.1. Let X be a nonempty set, (Y, d) be a metric space, $\varepsilon \in \mathbb{R}_0^{X^n}$ and $\mathcal{F}_1$, $\mathcal{F}_2$ be operators mapping from a nonempty set $\mathcal{D} \subset Y^X$ into Y^{X^n}. We say that the operator equation

$$\mathcal{F}_1\varphi(x_1, \ldots, x_n) = \mathcal{F}_2\varphi(x_1, \ldots, x_n), \quad (x_1, \ldots, x_n \in X) \tag{1.1}$$

---

is ε-hyperstable provided that every $\varphi_0 \in \mathcal{D}$ which satisfies

$$d\left(\mathcal{F}_1\varphi_0(x_1, \ldots, x_n), \mathcal{F}_2\varphi_0(x_1, \ldots, x_n)\right) \leq \varepsilon(x_1, \ldots, x_n), \quad (x_1, \ldots, x_n \in X)$$

fulfills the equation (1.1).

In 2011, Kenary [22] introduced and proved the Hyers-Ulam stability for the following functional equation

$$f(ax + by) = \frac{(a+b)}{2}f(x+y) + \frac{(a-b)}{2}f(x-y) \tag{1.2}$$

in non-Archimedean normed spaces and in random normed spaces, where m, n are different integers greater than 1. In 2011, Kenary, Jang and Park [23] proved the Hyers-Ulam stability of (1.2) in various normed spaces by using the fixed point method.

In this paper, using the fixed point method derived from [10, Theorem 1], we prove the hyperstability of (1.2) in the class of functions from a commutative group into a commutative complete metric group.

Before proceeding to the main results, we state the following theorem which is useful for our purpose.

Theorem 1.2. ([10, Theorem 1]) *Let X be a nonempty set, (Y, d) a complete metric space, $f_1, \ldots, f_s \colon X \to X$ and $L_1, \ldots, L_s \colon X \to \mathbb{R}_0$ be given mappings. Let $\Lambda \colon \mathbb{R}_0^X \to \mathbb{R}_0^X$ be a linear operator defined by*

$$\Lambda\delta(x) := \sum_{i=1}^{s} L_i(x)\delta(f_i(x)), \tag{1.3}$$

for $\delta \in \mathbb{R}_0^X$ and $x \in X$. If $\mathcal{T} \colon Y^X \to Y^X$ is an operator satisfying the inequality

$$d\left(\mathcal{T}\xi(x), \mathcal{T}\mu(x)\right) \leq \sum_{i=1}^{s} L_i(x)d\left(\xi(f_i(x)), \mu(f_i(x))\right), \quad \xi, \mu \in Y^X, x \in X,$$

and a function $\varepsilon \colon X \to \mathbb{R}_0$ and a mapping $\varphi \colon X \to Y$ satisfy

$$d\left(\mathcal{T}\varphi(x), \varphi(x)\right) \leq \varepsilon(x), \quad (x \in X),$$

$$\varepsilon^*(x) := \sum_{k=0}^{\infty} \Lambda^k \varepsilon(x) < \infty, \quad (x \in X),$$

then for every $x \in X$, the limit

$$\psi(x) := \lim_{n \to \infty} \mathcal{T}^n\varphi(x),$$

exists and the function $\psi \in Y^X$ so defined is a unique fixed point of $\mathcal{T}$ with

$$d\left(\varphi(x), \psi(x)\right) \leq \varepsilon^*(x), \quad (x \in X).$$

2. **Main results**

Suppose $(G, +)$ and $(H, +)$ are abelian groups, and d is a metric on H such that

(i): d is invariant with respect to $+$, that is,

$$d(u + w, v + w) = d(u, v), \quad (u, v, w \in H);$$

(ii): (H, d) is a complete metric space.

We will denote by $Aut(G)$ the family of all automorphisms of G. Moreover, for each $u \colon G \to G$ we write $ux := u(x)$ for $x \in G$ and we define u' by $u'x := x - ux$ for $x \in G$.
Let

$$l(G) := \big\{ u \in Aut(G) : (u' - u), (au' + bu) \in Aut(G),$$

$$\alpha_u := \frac{2}{a+b}\lambda(au' + bu) + \frac{|a-b|}{a+b}\lambda(u' - u) < 1 \big\} \neq \varnothing, \tag{2.1}$$

where

$$\lambda(u) := \inf \big\{ t \in \mathbb{R}_0 : \varepsilon(ux, uy) \leq t\varepsilon(x, y), \quad \forall x, y \in G \big\}$$

for $u \in Aut(G)$ and $\varepsilon \colon G^2 \to \mathbb{R}_0$. The following theorem is a result concerning the hyperstability of the functional equation (1.2).

Theorem 2.1. *Let $f : G \to H$ be a mapping satisfying the inequality*

$$d\left(f(x + y), \frac{2}{a+b}f(ax + by) - \frac{a-b}{a+b}f(x - y) \right) \leq \varepsilon(x, y) \tag{2.2}$$

for all $x, y \in G$, where $\varepsilon \colon G^2 \to \mathbb{R}_0$ is an arbitrary function. Assume that there exists a nonempty subset $\mathcal{U} \subset l(G)$ such that

$$u \circ v = v \circ u, \quad \forall u, v \in \mathcal{U},$$

and

$$\begin{aligned} &\inf \big\{ \varepsilon(u'x, ux) : u \in \mathcal{U} \big\} = 0, \quad \forall x \in G, \\ &\sup \big\{ \alpha_u : u \in \mathcal{U} \big\} < 1, \end{aligned} \tag{2.3}$$

then f is a solution of (1.2) on G.

Proof. Let us fix $u \in \mathcal{U}$. Replacing x with $u'x$ and y with ux in (2.2), we get

$$d\left(f(x), \frac{2}{a+b}f\big((au' + bu)x\big) - \frac{a-b}{a+b}f\big((u' - u)x\big) \right) \leq \varepsilon(u'x, ux) := \varepsilon_u(x) \tag{2.4}$$

for all $x \in G$. We define the operators $\mathcal{T}_u \colon H^G \to H^G$ and $\Lambda_u \colon \mathbb{R}_0^G \to \mathbb{R}_0^G$ by

$$\mathcal{T}_u \xi(x) := \frac{2}{a+b}\xi\big((au' + bu)x\big) - \frac{a-b}{a+b}\xi\big((u' - u)x\big), \tag{2.5}$$

$$\Lambda_u \delta(x) := \frac{2}{a+b}\delta\big((au' + bu)x\big) + \frac{|a-b|}{a+b}\delta\big((u' - u)x\big)$$

for all $x \in G$, $\xi \in H^G$ and $\delta \in \mathbb{R}_0^G$. Then (2.4) becomes

$$d\big(f(x), \mathcal{T}_u f(x) \big) \leq \varepsilon_u(x)$$

for all $x \in G$.

The operator $\Lambda_u : \mathbb{R}_0^G \to \mathbb{R}_0^G$ has the form given by (1.3) with $s = 2$ and $f_1(x) = (au' + bu)x$, $f_2(x) = (u' - u)x$, $L_1(x) = \frac{2}{a+b}$, $L_2(x) = \frac{|a-b|}{a+b}$ for all $x \in G$.
Further,

$$d\left(\mathcal{T}_u\xi(x), \mathcal{T}_u\mu(x)\right) = d\left(\frac{2}{a+b}\xi\Big((au'+bu)x\Big) - \frac{a-b}{a+b}\xi\Big((u'-u)x\Big),\right.$$
$$\left.\frac{2}{a+b}\mu\Big((au'+bu)x\Big) - \frac{a-b}{a+b}\mu\Big((u'-u)x\Big)\right)$$
$$\leq \frac{2}{a+b}d\Big(\xi((au'+bu)x), \mu((au'+bu)x)\Big) + \frac{|a-b|}{a+b}d\Big(\xi((u'-u)x), \mu((u'-u)x)\Big)$$

for all $x \in G$ and $\xi, \mu \in H^G$.

Note that, in view of the definition of $\lambda(u)$,

$$\varepsilon(ux, uy) \leq \lambda(u)\varepsilon(x, y), \qquad x, y \in G,$$

So it is easy to show by induction on k that

$$\Lambda_u^k \varepsilon_u(x) \leq \alpha_u^k \varepsilon(u'x, ux),$$

for all $x \in G$, where

$$\alpha_u = \left(\frac{2}{a+b}\lambda(au'+bu) + \frac{|a-b|}{a+b}\lambda(u'-u)\right).$$

Hence

$$\varepsilon^*(x) := \sum_{k=0}^{\infty} \Lambda_u^k \varepsilon_u(x) \leq \varepsilon(u'x, ux) \sum_{k=0}^{\infty} \alpha_u^k = \frac{\varepsilon(u'x, ux)}{1 - \alpha_u} < \infty$$

for all $x \in G$. By Theorem 1.2, there exists a unique solution $F_u : G \to H$ of the equation

$$F_u(x) = \frac{2}{a+b}F_u\Big((au'+bu)x\Big) - \frac{a-b}{a+b}F_u\Big((u'-u)x\Big)$$

for all $x \in G$, which is a fixed point of $\mathcal{T}_u$ such that

$$d\Big(F_u(x), f(x)\Big) \leq \frac{\varepsilon(u'x, ux)}{1 - \alpha_u}$$

for all $x \in G$. Moreover,

$$F_u(x) = \lim_{k \to \infty} \mathcal{T}_u^k f(x)$$

for all $x \in G$.

To prove that F_u satisfies the functional equation (1.2) on G, just prove the following inequality

$$d\left(\mathcal{T}_u^n f(x+y), \frac{2}{a+b}\mathcal{T}_u^n f(ax+by) - \frac{a-b}{a+b}\mathcal{T}_u^n f(x-y)\right) \leq \alpha_u^n \varepsilon(x, y) \tag{2.6}$$

for all $x, y \in G$, and $n \in \mathbb{N}$.

Indeed, if $n = 0$ then (2.6) is simply (2.2). So, take $n \in \mathbb{N}_+$ and suppose that (2.6) holds for n

and $x, y \in G$. Then, by using (2.5) and the triangle inequality, we have

$$d\left(\mathcal{T}_u^{n+1} f(x+y), \frac{2}{a+b}\mathcal{T}_u^{n+1} f(ax+by) - \frac{a-b}{a+b}\mathcal{T}_u^{n+1} f(x-y)\right)$$

$$= d\left(\frac{2}{a+b}\mathcal{T}_u^n f\Big((au'+bu)(x+y)\Big) - \frac{a-b}{a+b}T_u^n f\Big((u'-u)(x+y)\Big),\right.$$

$$\left.(\frac{2}{a+b})^2\mathcal{T}_u^n f\Big((au'+bu)(ax+by)\Big) - (\frac{2}{a+b})(\frac{a-b}{a+b})T_u^n f\Big((u'-u)(ax+by)\Big)\right.$$

$$\left.-(\frac{2}{a+b})(\frac{a-b}{a+b})T_u^n f\Big((au'+bu)(x-y)\Big) + (\frac{a-b}{a+b})^2 T_u^n f\Big((u'-u)(x-y)\Big)\right)$$

$$\leq \frac{2}{a+b} d\left(\mathcal{T}_u^n f\Big((au'+bu)(x+y)\Big),\right.$$

$$\left.\frac{2}{a+b}\mathcal{T}_u^n f\Big((au'+bu)(ax+by)\Big) - \frac{a-b}{a+b}T_u^n f\Big((au'+bu)(x-y)\Big)\right)$$

$$+\frac{|a-b|}{a+b} d\left(\mathcal{T}_u^n f\Big((u'-u)(x+y)\Big),\right.$$

$$\left.\frac{2}{a+b}\mathcal{T}_u^n f\Big((u'-u)(ax+by)\Big) - \frac{a-b}{a+b}T_u^n f\Big((u'-u)(x-y)\Big)\right)$$

$$\leq \alpha_u^n \left(\frac{2}{a+b}\varepsilon\Big((au'+bu)x, (au'+bu)y\Big) + \frac{|a-b|}{a+b}\varepsilon\Big((u'-u)x, (u'-u)y\Big)\right)$$

$$\leq \varepsilon(x,y)\alpha_u^n \left(\frac{2}{a+b}\lambda(au'+bu) + \frac{|a-b|}{a+b}\lambda(u'-u)\right)$$

$$= \alpha_u^{n+1}\varepsilon(x,y).$$

By induction, we have shown that (2.6) holds for all $x, y \in G$. Letting $n \to \infty$ in (2.6), we get

$$F_u(ax+by) = \frac{a+b}{2}F_u(x+y) + \frac{a-b}{2}F_u(x-y)$$

for all $x, y \in G$. Thus, we have proved that for every $u \in \mathcal{U}$ there exists a function $F_u : G \to H$ which is a solution of the functional equation (1.2) on G and satisfies

$$d\Big(f(x), F_u(x)\Big) \leq \frac{\varepsilon(u'x, ux)}{1-\alpha_u}$$

for all $x \in G$. By (2.3), we get

$$d\Big(f(x), F_u(x)\Big) \quad \leq \frac{\inf_{u \in \mathcal{U}} \varepsilon(u'x, ux)}{1 - \sup_{u \in \mathcal{U}} \alpha_u}$$
$$= 0$$

for all $x \in G$. This means that $F_u(x) = f(x)$ for all $x \in G$ and $u \in \mathcal{U}$, and hence

$$f(ax+by) = \frac{a+b}{2}f(x+y) + \frac{a-b}{2}f(x-y)$$

for all $x, y \in G$, which implies that f satisfies the functional equation (1.2) on G. $\qquad\square$

In the next theorem, we will study the hyperstablity of the functional equation (1.2) on G without 0 the neutral element, because of the reason that one can easily deduce some applications.

Theorem 2.2. *Let $f : G \to H$ be a mapping satisfying the inequality*

$$d\left(f(x+y), \frac{2}{a+b}f(ax+by) - \frac{a-b}{a+b}f(x-y)\right) \leq \varepsilon(x,y)$$

for all $x, y \in G\backslash\{0\}$, where 0 is the neutral element of the group $(G, +)$ and $\varepsilon : (G\backslash\{0\})^2 \to \mathbb{R}_0$ is an arbitrary function. Assume that there exists a nonempty subset $\mathcal{U} \subset l(G)$ such that

$$u \circ v = v \circ u \qquad (u, v \in \mathcal{U}),$$

and

$$
\begin{aligned}
&\inf\left\{\varepsilon(u'x, ux) : u \in \mathcal{U}\right\} = 0, \quad \forall x \in G\backslash\{0\}, \\
&\sup\left\{\alpha_u : u \in \mathcal{U}\right\} < 1,
\end{aligned}
\tag{2.7}
$$

then f is a solution of the functional equation (1.2) on $G\backslash\{0\}$.

Proof. The proof is the same as in the proof of Theorem 2.1. $\qquad\square$

3. Some consequences

From Theorem 2.2, we can obtain the following corollaries as natural results.

Corollary 3.1. *Let E and F be a normed space and a Banach space, respectively. Assume that X is a subgroup of the group $(E, +)$, $p < 0$, $q < 0$ and $c \geq 0$. If $f : X \to F$ satisfies*

$$\left\| f(ax+by) - \frac{a+b}{2}f(x+y) - \frac{a-b}{2}f(x-y) \right\| \leq c\left(\|x\|^p + \|y\|^q\right) \tag{3.1}$$

for all $x, y \in X\backslash\{0\}$, then f satisfies the functional equation (1.2) on $X\backslash\{0\}$.

Proof. The proof follows from Theorem 2.2 by taking

$$\varepsilon(x,y) = c\left(\|x\|^p + \|y\|^q\right), \quad x, y \in X\backslash\{0\},$$

with some real numbers $c \geq 0$, $p < 0$, $q < 0$ and $d(x,y) = \|x - y\|$. For each $m \in \mathbb{N}_+$ define $u_m : X\backslash\{0\} \to X\backslash\{0\}$ by $u_m x := -mx$ and $u'_m : X\backslash\{0\} \to X\backslash\{0\}$ by $u'_m x := (1+m)x$. Then

$$
\begin{aligned}
\varepsilon(u_m x, u_k y) &= \varepsilon(-mx, -ky) = c\left(\|-mx\|^p + \|-ky\|^q\right) \\
&= cm^p\|x\|^p + ck^q\|y\|^q \leq (m^p + k^q)\,c\left(\|x\|^p + \|y\|^q\right) \\
&= (m^p + k^q)\,\varepsilon(x,y)
\end{aligned}
$$

for all $x \in X \backslash \{0\}$, $k, m \in \mathbb{N}_+$. Hence

$$\lim_{m\to\infty} \varepsilon(u'_m x, u_m y) \quad \leq \lim_{m\to\infty}\left((1+m)^p + m^q\right)\varepsilon(x,y) = 0$$

for all $x, y \in X\backslash\{0\}$. Then (2.7) is valid with $\lambda(u_m) = m^p + m^q$ for $m \in \mathbb{N}_+$, and there exists $n_0 \in \mathbb{N}_+$ such that

$$\frac{2}{a+b}\left(\left|a+(a-b)m\right|^p + \left|a+(a-b)m\right|^q\right) + \frac{|a-b|}{a+b}\left((2m+1)^p + (2m+1)^q\right) < 1 \; (m \geq n_0).$$

So it easily seen that (2.1) is fulfilled with

$$\mathcal{U} := \left\{u_m \in Aut\, X : m \in \mathbb{N}_{n_0}\right\}.$$

Therefore, by Theorem 2.2, every $f : X \to F$ satisfying (3.1) is a solution of the functional equation (1.2) on $X\backslash\{0\}$. $\qquad\square$

Corollary 3.2. *Let E and F be a normed space and a Banach space, respectively. Assume that X is a subgroup of the group $(E, +)$, $p, q \in \mathbb{R}$, $p + q < 0$ and $c \geq 0$. If $f : X \to F$ satisfies*

$$\left\| f(ax + by) - \frac{a+b}{2} f(x+y) - \frac{a-b}{2} f(x-y) \right\| \leq c \, \|x\|^p \, \|y\|^q$$

for all $x, y \in X \backslash \{0\}$, then f satisfies the functional equation (1.2) on $X \backslash \{0\}$.

Proof. It is easily seen that the function ε given by

$$\varepsilon(x, y) = c \, \|x\|^p \, \|y\|^q \quad x, y \in X \backslash \{0\},$$

satisfies (2.7), since

$$\varepsilon(mx, ky) = c \, \|mx\|^p \, \|ky\|^q = c \, |m|^p \, |k|^q \, \|x\|^p \, \|y\|^q = |m|^p \, |k|^q \, \varepsilon(x, y)$$

for all $x, y \in X \backslash \{0\}$, $k, m \in \mathbb{Z}$, and $km \neq 0$.

The remainder of the proof is similar to the proof of Corollary 3.1. $\qquad\square$

By an analogous conclusion, the function ε given by

$$\varepsilon(x, y) = c \left(\|x\|^p + \|y\|^q + \|x\|^p \, \|y\|^q \right), \quad x, y \in X \backslash \{0\},$$

satisfies (2.7), since

$$
\begin{aligned}
\varepsilon(mx, ky) &= c \left(\|mx\|^p + \|ky\|^q + \|mx\|^p \, \|ky\|^q \right) \\
&= c \left(|m|^p \, \|x\|^p + |k|^q \, \|y\|^p + |m|^p \, |k|^q \, \|x\|^p \, \|y\|^q \right) \\
&\leq \left(|m|^p + |k|^q + |m|^p \, |k|^q \right) \varepsilon(x, y)
\end{aligned}
$$

for all $x, y \in X \backslash \{0\}$, $k, m \in \mathbb{Z}$, and $km \neq 0$. So we have the following corollary.

Corollary 3.3. *Let E and F be a normed space and a Banach space, respectively. Assume that X is a subgroup of the group $(E, +)$, and $p < 0$, $q < 0$, $p + q < 0$ and $c \geq 0$. If $f : X \to F$ satisfies*

$$\left\| f(ax + by) - \frac{a+b}{2} f(x+y) - \frac{a-b}{2} f(x-y) \right\| \leq c \left(\|x\|^p + \|y\|^q + \|x\|^p \, \|y\|^q \right)$$

for all $x, y \in X \backslash \{0\}$, then f satisfies the functional equation (1.2) on $X \backslash \{0\}$.

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A fractional finite difference inclusion

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Abstract. In this manuscript we investigated the fractional finite difference inclusion $\Delta_{\mu-2}^{\mu}x(t) \in F(t,x(t),\Delta x(t))$ via the boundary conditions $\Delta x(b+\mu) = A$ and $x(\mu-2) = B$, where $1 < \mu \le 2$, $A, B \in \mathbb{R}$ and $F : \mathbb{N}_{\mu-2}^{b+\mu+2} \times \mathbb{R} \times \mathbb{R} \to 2^{\mathbb{R}}$ is a compact valued multifunction.
Keywords: Fixed point of multifunction, Fractional finite difference inclusion, Hausdorff metric.

1 Introduction

We recall that there are several published works on the existence of solutions for some fractional finite difference equations. In [1, 3, 7] the boundary value problems for discrete fractional equations were explain in detail. In [4] the discrete nabla fractional Taylor formulae were described together with some well known inequalities. In [8] the authors investigated a k-dimensional system of fractional finite difference equations. The papers [11, 12, 13, 14] reported several results on discrete boundary value problem and existence results for fractional difference equations. Further results can be seen in [17, 19, 20, 21] and the references therein. The readers can find more details about elementary notions and definitions of fractional finite difference equations in [5, 6, 10, 15] and [16].

In [2] it was proved the existence of solutions for nonlinear fractional q-difference inclusions involving convex as well as non-convex valued maps with nonlocal Robin (separated) conditions. However,in the of our knowledge there is no research on fractional finite difference inclusions so far. Here, we give a motivation about the importance of studying the fractional finite difference inclusions. For example, consider the fractional finite difference equation $\Delta^{\mu}y(t) = h(t+\mu-2, y(t+\mu-2))$ via the boundary conditions $y(\mu-3) = 0$, $\Delta^{\alpha}y(\mu-1-\alpha) = 0$ and $\Delta^{\beta}y(\mu+b+1-\beta) = 0$, where $t \in \mathbb{N}_0^{b+3}$, $b \in \mathbb{N}_0$, $2 < \mu \le 3$, $1 < \alpha \le 2$, $0 < \beta \le 1$ and $h : \mathbb{N}_{\mu-2}^{b+\mu+1} \times \mathbb{R} \to \mathbb{R}$ is a map. Define the compact valued multifunction $T_h : \mathbb{N}_{\mu-2}^{b+\mu+1} \times \mathbb{R} \to 2^{\mathbb{R}}$ by $T_h(t) = \{h(t+\mu-2, y(t+\mu-2))\}$. It is easy to check that each solution of the fractional finite difference equation $\Delta^{\mu}y(t) = h(t+\mu-2, y(t+\mu-2))$ is a solution for the fractional finite difference inclusion $\Delta^{\mu}y(t) \in T_h(t)$. Thus, studying the fractional finite difference inclusions needs more potential mathematical abilities. In this paper,

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we provide some preliminaries to investigate the existence of solution for a fractional finite difference inclusion.

We recall some basic definitions utilized in the rest of the manuscript. The Gamma function is defined by $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$ for the complex numbers z in which the real part of z is positive (see [18]). Now, we define $t^{\underline{\mu}} := \frac{\Gamma(t+1)}{\Gamma(t+1-\mu)}$ for all $t, \mu \in \mathbb{R}$ whenever the right-hand side is defined (see [7] and [16]). In this paper, we use the notations $\mathbb{N}_p = \{p, p+1, p+2, \ldots\}$ for all $p \in \mathbb{R}$ and $\mathbb{N}_p^q = \{p, p+1, p+2, \ldots, q\}$ for all real numbers p and q whenever $q - p$ is a natural number. Let $\mu > 0$ with $m - 1 < \mu < m$ for some natural number m. The μ-th fractional sum of f based at a is defined by $\Delta_a^{-\mu} f(t) = \frac{1}{\Gamma(\mu)} \sum_{k=a}^{t-\mu} (t - \sigma(k))^{\underline{\mu-1}} f(k)$ for all $t \in \mathbb{N}_{a+\mu}$, where $\sigma(k) = k + 1$ is the forward jump operator (see [3, 8]). Similarly, we define $\Delta_a^{\mu} f(t) = \frac{1}{\Gamma(-\mu)} \sum_{k=a}^{t+\mu} (t - \sigma(k))^{\underline{-\mu-1}} f(k)$ for all $t \in \mathbb{N}_{a+m-\mu}$ (see [7, 16]). Note that, the domain of $\Delta_a^r f$ is $\mathbb{N}_{a+m-r}$ for $r > 0$ and $\mathbb{N}_{a-r}$ for $r < 0$. Also, for the natural number $\mu = m$, we have the known formula $\Delta_a^{\mu} f(t) = \Delta^m f(t) = \sum_{i=0}^m (-1)^i \binom{m}{i} f(t + m - i)$ (see [7, 16]). Finally, set the trivial sum $\Delta_a^0 f(t) = f(t)$ for all $t \in \mathbb{N}_a$.

Lemma 1.1. *[7] Let $m \geq 1$, $m - 1 < \mu \leq m$ and $h : \mathbb{N}_{\mu-2}^{b+\mu+2} \to \mathbb{R}$ be a map. The general solution of the equation $\Delta_{\mu-2}^{\mu} y(t) = h(t)$ is given by*

$$y(t) = \sum_{i=1}^m c_i t^{\underline{\mu-i}} + \frac{1}{\Gamma(\mu)} \sum_{s=0}^{t-\mu} (t - \sigma(s))^{\underline{\mu-1}} h(s)$$

for all $t \in \mathbb{N}_{\mu-2}$, where $c_1, \cdots, c_m$ are arbitrary constants.

Let $(\mathcal{X}, d)$ be a metric space. The Hausdorff metric $H_d : 2^{\mathcal{X}} \times 2^{\mathcal{X}} \to [0, \infty)$ is defined by

$$H_d(C, E) = \max\{\sup_{c \in C} d(c, E), \sup_{e \in E} d(C, e)\},$$

where $d(C, e) = \inf_{c \in C} d(c, e)$ ([18]). We denote the set of compact subsets of $\mathcal{X}$ by $P_{cp}(\mathcal{X})$ and the set of closed subsets of $\mathcal{X}$ by $C(\mathcal{X})$. Let $T : \mathcal{X} \to 2^{\mathcal{X}}$ be a multifunction. An element $x \in \mathcal{X}$ is called a fixed point of T whenever $x \in Tx$ ([18]). A multifunction $T : \mathcal{X} \to C(\mathcal{X})$ is called a contraction whenever there exists $\lambda \in (0, 1)$ such that $H_d(T(x), T(y)) \leq \lambda d(x, y)$ for all $x, y \in \mathcal{X}$ ([18]). In 1970, Covitz and Nadler proved next theorem ([9]).

Theorem 1.2. *Each closed valued contraction multifunction on a complete metric space has a fixed point.*

2 Main result

Now, we are ready to investigate the existence of solutions for fractional finite difference inclusion

$$\Delta_{\mu-2}^{\mu} x(t) \in F(t, x(t), \Delta x(t)) \tag{1}$$

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via the boundary conditions $\Delta x(b + \mu) = A$ and $x(\mu - 2) = B$, where $x : \mathbb{N}_{\mu-2}^{b+\mu+2} \to \mathbb{R}$ is a map, $1 < \mu \leq 2$ and $F : \mathbb{N}_{\mu-2}^{b+\mu+2} \times \mathbb{R} \times \mathbb{R} \to 2^{\mathbb{R}}$ is a compact valued multifunction.

Lemma 2.1. *Let* $1 < \mu \leq 2$ *and* $x : \mathbb{N}_{\mu-2}^{b+\mu+2} \to \mathbb{R}$ *and* $y : \mathbb{N}_0^{b+2} \to \mathbb{R}$ *be two maps. Then* x_0 *is a solution of the fractional finite difference equation* $\Delta_{\mu-2}^{\mu} x(t) = y(t)$ *via the boundary conditions* $\Delta x(b + \mu) = A$ *and* $x(\mu - 2) = B$ *if and only if* x_0 *is a solution of the fractional sum equation* $x(t) = \varphi(t) + \sum_{s=0}^{b+1} G(t, s)y(s)$, *where* $\varphi(t) = \left[\frac{A}{(\mu-1)(\mu+b)^{\underline{\mu-2}}} - \frac{B(\mu-2)}{(b+3)\Gamma(\mu)} \right] t^{\underline{\mu-1}} + \frac{B}{\Gamma(\mu-1)} t^{\underline{\mu-2}}$, $G(t, s) = -\frac{t^{\underline{\mu-1}}(b + \mu - \sigma(s))^{\underline{\mu-2}}}{(\mu+b)^{\underline{\mu-2}}} + (t - \sigma(s))^{\underline{\mu-1}}$ *whenever* $0 \leq s \leq t - \mu \leq b + 1$ *and* $G(t, s) = -\frac{t^{\underline{\mu-1}}(b + \mu - \sigma(s))^{\underline{\mu-2}}}{(\mu+b)^{\underline{\mu-2}}}$ *whenever* $0 \leq t - \mu < s \leq b + 1$.

Proof. Let x_0 be a solution of the equation $\Delta_{\mu-2}^{\mu} x(t) = y(t)$ via the boundary conditions $\Delta x(b + \mu) = A$ and $x(\mu - 2) = B$. By using Lemma 1.1, we get

$$x_0(t) = c_1 t^{\underline{\mu-1}} + c_2 t^{\underline{\mu-2}} + \frac{1}{\Gamma(\mu)} \sum_{s=0}^{t-\mu} (t - \sigma(s))^{\underline{\mu-1}} y(s)$$

where $c_1, c_2 \in \mathbb{R}$. By using the boundary condition $x_0(\mu - 2) = B$, we obtain $c_2 = \frac{B}{\Gamma(\mu-1)}$. Since $\Delta x_0(t) = c_1(\mu - 1)t^{\underline{\mu-2}} + c_2(\mu - 2)t^{\underline{\mu-3}} + \frac{1}{\Gamma(\mu-1)} \sum_{s=0}^{t-\mu+1} (t - \sigma(s))^{\underline{\mu-2}} y(s)$, by using the boundary condition $\Delta x_0(b + \mu) = A$ we get

$$c_1 = \frac{A}{(\mu - 1)(\mu + b)^{\underline{\mu-2}}} - \frac{B(\mu - 2)}{(b + 3)\Gamma(\mu)} - \frac{1}{(\mu + b)^{\underline{\mu-2}}\Gamma(\mu)} \sum_{s=0}^{b+1} (b + \mu - \sigma(s))^{\underline{\mu-2}} y(s).$$

Hence,

$$x_0(t) = \left[\frac{A}{(\mu - 1)(\mu + b)^{\underline{\mu-2}}} - \frac{B(\mu - 2)}{(b + 3)\Gamma(\mu)} \right] t^{\underline{\mu-1}} + \frac{B}{\Gamma(\mu - 1)} t^{\underline{\mu-2}}$$

$$-\frac{t^{\underline{\mu-1}}}{(\mu + b)^{\underline{\mu-2}}\Gamma(\mu)} \sum_{s=0}^{b+1} (b + \mu - \sigma(s))^{\underline{\mu-2}} y(s) + \frac{1}{\Gamma(\mu)} \sum_{s=0}^{t-\mu} (t - \sigma(s))^{\underline{\mu-1}} y(s) = \varphi(t) + \sum_{s=0}^{b+1} G(t, s)y(s).$$

Now let x_0 be a solution of the equation $x(t) = \varphi(t) + \sum_{s=0}^{b+1} G(s, t)y(s)$. Then, we have

$$x_0(t) = \left[\frac{A}{(\mu - 1)(\mu + b)^{\underline{\mu-2}}} - \frac{B(\mu - 2)}{(b + 3)\Gamma(\mu)} \right] t^{\underline{\mu-1}} + \frac{B}{\Gamma(\mu - 1)} t^{\underline{\mu-2}}$$

$$-\frac{t^{\underline{\mu-1}}}{(\mu + b)^{\underline{\mu-2}}\Gamma(\mu)} \sum_{s=0}^{b+1} (b + \mu - \sigma(s))^{\underline{\mu-2}} y(s) + \frac{1}{\Gamma(\mu)} \sum_{s=0}^{t-\mu} (t - \sigma(s))^{\underline{\mu-1}} y(s).$$

Since $(\mu - 2)^{\underline{\mu-1}} = 0$, $(\mu - 2)^{\underline{\mu-2}} = \Gamma(\mu - 1)$ and $\sum_{s=0}^{-2} (\mu - 2 - \sigma(s))^{\underline{\mu-1}} y(s) = 0$, we get $x_0(\mu - 2) = B$. A simple calculation shows that $\Delta x_0(\mu + b) = A$. On the other hand, it is easy to check that $x_0(t) = c_1 t^{\underline{\mu-1}} + c_2 t^{\underline{\mu-2}} + \frac{1}{\Gamma(\mu)} \sum_{s=0}^{t-\mu} (t - \sigma(s))^{\underline{\mu-1}} y(s)$ is a solution for the equation $\Delta_{\mu-2}^{\mu} x(t) = y(t)$ and so $\Delta^{\mu} x_0(t) = y(t)$. This completes the proof. $\square$

A function $x : \mathbb{N}_{\mu-2}^{b+\mu+2} \to \mathbb{R}$ is a solution for the problem (1) whenever it satisfies the boundary conditions and there exists a function $y : \mathbb{N}_0^{b+2} \to \mathbb{R}$ such that $y(t) \in F(t, x(t), \Delta x(t))$ for all $t \in \mathbb{N}_0^{b+2}$ and

$$x(t) = \Big[\frac{A}{(\mu-1)(\mu+b)^{\underline{\mu-2}}} - \frac{B(\mu-2)}{(b+3)\Gamma(\mu)}\Big]t^{\underline{\mu-1}} + \frac{B}{\Gamma(\mu-1)}t^{\underline{\mu-2}}$$

$$-\frac{t^{\underline{\mu-1}}}{(\mu+b)^{\underline{\mu-2}}\Gamma(\mu)}\sum_{s=0}^{b+1}(b+\mu-\sigma(s))^{\underline{\mu-2}}y(s) + \frac{1}{\Gamma(\mu)}\sum_{s=0}^{t-\mu}(t-\sigma(s))^{\underline{\mu-1}}y(s).$$

Let $\mathcal{X}$ be the set of all functions $x : \mathbb{N}_{\mu-2}^{b+\mu+2} \to \mathbb{R}$ endowed with the norm

$$\|x\| = \max_{t \in \mathbb{N}_{\mu-2}^{b+\mu+2}} |x(t)| + \max_{t \in \mathbb{N}_{\mu-2}^{b+\mu+2}} |\Delta x(t)|.$$

Suppose that $\{x_n\}$ is a Cauchy sequence in $\mathcal{X}$. For each $\epsilon > 0$, choose a natural number N such that $\|x_n - x_m\| < \epsilon$ for all $m, n > N$. Hence, $\max_{t \in \mathbb{N}_{\mu-2}^{b+\mu+2}} |x_n(t) - x_m(t)| < \epsilon$ and $\max_{t \in \mathbb{N}_{\mu-2}^{b+\mu+2}} |\Delta x_n(t) - \Delta x_m(t)| < \epsilon$. Choose $x(t), z(t) \in \mathbb{R}$ such that $x_n(t) \to x(t)$ and $\Delta x_n(t) \to z(t)$ for all $t \in \mathbb{N}_{\mu-2}^{b+\mu+2}$. Note that, $\Delta x_n(t) = x_n(t+1) - x_n(t)$ for all n and so $\Delta x(t) = x(t+1) - x(t) = z(t)$. This implies that there exists a natural number M such that $|x_n(t) - x(t)| < \frac{\epsilon}{2}$ and $|\Delta x_n(t) - \Delta x(t)| < \frac{\epsilon}{2}$ for all $t \in \mathbb{N}_{\mu-2}^{b+\mu+2}$ and $n > M$. Thus,

$$\|x_n - x\| = \max_{t \in \mathbb{N}_{\mu-2}^{b+\mu+2}} |x_n(t) - x(t)| + \max_{t \in \mathbb{N}_{\mu-2}^{b+\mu+2}} |\Delta x_n(t) - \Delta x(t)| < \epsilon.$$

This shows that $(\mathcal{X}, \|.\|)$ is a Banach space. For each $x \in \mathcal{X}$, put

$$S_{F,x} = \{y : \mathbb{N}_0^{b+2} \to \mathbb{R} : \ y(t) \in F(t, x(t), \Delta x(t)) \text{ for all } t \in \mathbb{N}_0^{b+2}\}.$$

Note that, the selection principle implies that $S_{F,x}$ is nonempty because $F(t, x(t), \Delta x(t)) \neq \emptyset$. Put

$$G_1 = \max_{t \in \mathbb{N}_{\mu-2}^{\mu+b+2}} \frac{1}{(\mu+b)^{\underline{\mu-2}}\Gamma(\mu)} \sum_{s=0}^{b+2} \Big[(b+\mu-\sigma(s))^{\underline{\mu-2}}t^{\underline{\mu-1}} + (\mu+b)^{\underline{\mu-2}}(t-\sigma(s))^{\underline{\mu-1}}\Big]$$

and

$$G_2 = \max_{t \in \mathbb{N}_{\mu-2}^{\mu+b+2}} \frac{1}{(\mu+b)^{\underline{\mu-2}}\Gamma(\mu-1)} \sum_{s=0}^{b+2} \Big[(b+\mu-\sigma(s))^{\underline{\mu-2}}t^{\underline{\mu-2}} + (\mu+b)^{\underline{\mu-2}}(t-\sigma(s))^{\underline{\mu-2}}\Big].$$

Since every Green function is bounded and the summations are finite, G_1 and G_2 are real numbers.

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Theorem 2.2. *Let $g : \mathbb{N}_{\mu-2}^{b+\mu+2} \to \mathbb{R}$ be a map such that $0 < \max_{t \in \mathbb{N}_{\mu-2}^{b+\mu+2}} |g(t)|(G_1 + G_2) < 1$. Suppose that $F : \mathbb{N}_{\mu-2}^{b+\mu+2} \times \mathbb{R} \times \mathbb{R} \to P_{cp}(\mathbb{R})$ is a multifunction such that*

$$H_d(F(t, x_1, x_2), F(t, z_1, z_2)) \le g(t)(|x_1 - z_1| + |x_2 - z_2|) \tag{2}$$

for all $t \in \mathbb{N}_{\mu-2}^{b+\mu+2}$ and $x_1, x_2, z_1, z_2 \in \mathbb{R}$. Then the inclusion problem (1) has a solution.

Proof. Choose $y \in S_{F,x}$. Define $h \in \mathcal{X}$ by

$$h(t) = \left[\frac{A}{(\mu-1)(\mu+b)^{\underline{\mu-2}}} - \frac{B(\mu-2)}{(b+3)\Gamma(\mu)}\right] t^{\underline{\mu-1}} + \frac{B}{\Gamma(\mu-1)} t^{\underline{\mu-2}}$$

$$-\frac{t^{\underline{\mu-1}}}{(\mu+b)^{\underline{\mu-2}}\Gamma(\mu)} \sum_{s=0}^{b+1}(b+\mu-\sigma(s))^{\underline{\mu-2}}y(s) + \frac{1}{\Gamma(\mu)} \sum_{s=0}^{t-\mu}(t-\sigma(s))^{\underline{\mu-1}}y(s)$$

for all $t \in \mathbb{N}_{\mu-2}^{\mu+b+2}$. This shows that the set

$$\left\{h \in \mathcal{X} : \text{ there exists } y \in S_{F,x} \text{ such that } h(t) = w(t) \text{ for all } t \in \mathbb{N}_{\mu-2}^{\mu+b+2}\right\}$$

is nonempty, where

$$w(t) = \left[\frac{A}{(\mu-1)(\mu+b)^{\underline{\mu-2}}} - \frac{B(\mu-2)}{(b+3)\Gamma(\mu)}\right] t^{\underline{\mu-1}} + \frac{B}{\Gamma(\mu-1)} t^{\underline{\mu-2}}$$

$$-\frac{t^{\underline{\mu-1}}}{(\mu+b)^{\underline{\mu-2}}\Gamma(\mu)} \sum_{s=0}^{b+1}(b+\mu-\sigma(s))^{\underline{\mu-2}}y(s) + \frac{1}{\Gamma(\mu)} \sum_{s=0}^{t-\mu}(t-\sigma(s))^{\underline{\mu-1}}y(s).$$

Now, define the multifunction $T : \mathcal{X} \to 2^{\mathcal{X}}$ by

$$T(x) = \left\{h \in \mathcal{X} : \text{ there exists } y \in S_{F,x} \text{ such that } h(t) = w(t) \text{ for all } t \in \mathbb{N}_{\mu-2}^{\mu+b+2}\right\}.$$

First, we show that $T(x)$ is a closed subset of $\mathcal{X}$ for all $x \in \mathcal{X}$. Let $x \in \mathcal{X}$ and $\{u_n\}_{n\geq 1}$ be a sequence in $T(x)$ with $u_n \to u$. Choose $y_n \in S_{F,x}$ such that

$$u_n(t) = \left[\frac{A}{(\mu-1)(\mu+b)^{\underline{\mu-2}}} - \frac{B(\mu-2)}{(b+3)\Gamma(\mu)}\right] t^{\underline{\mu-1}} + \frac{B}{\Gamma(\mu-1)} t^{\underline{\mu-2}}$$

$$-\frac{t^{\underline{\mu-1}}}{(\mu+b)^{\underline{\mu-2}}\Gamma(\mu)} \sum_{s=0}^{b+1}(b+\mu-\sigma(s))^{\underline{\mu-2}}y_n(s) + \frac{1}{\Gamma(\mu)} \sum_{s=0}^{t-\mu}(t-\sigma(s))^{\underline{\mu-1}}y_n(s)$$

for all $t \in \mathbb{N}_{\mu-2}^{b+\mu+2}$ and $n \geq 1$. Since F has compact values, $\{y_n\}_{n\geq 1}$ has a subsequence which converges to some $y : \mathbb{N}_0^{b+2} \to \mathbb{R}$. We denote this subsequence again by $\{y_n\}_{n\geq 1}$. It is easy to check that $y \in S_{F,x}$ and

$$u_n(t) \to u(t) = \left[\frac{A}{(\mu-1)(\mu+b)^{\underline{\mu-2}}} - \frac{B(\mu-2)}{(b+3)\Gamma(\mu)}\right] t^{\underline{\mu-1}} + \frac{B}{\Gamma(\mu-1)} t^{\mu-2}$$

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$$-\frac{t^{\underline{\mu-1}}}{(\mu+b)^{\underline{\mu-2}}\Gamma(\mu)}\sum_{s=0}^{b+1}(b+\mu-\sigma(s))^{\underline{\mu-2}}y(s)+\frac{1}{\Gamma(\mu)}\sum_{s=0}^{t-\mu}(t-\sigma(s))^{\underline{\mu-1}}y(s)$$

for all $t\in\mathbb{N}_{\mu-2}^{b+\mu+2}$. This implies that $u\in T(x)$ and so the multifunction T has closed values. Now, we show that T is a contraction multifunction with the constant

$$\lambda=\max_{t\in\mathbb{N}_{\mu-2}^{b+\mu+2}}|g(t)|(G_1+G_2)<1.$$

Let $x,z\in\mathcal{X}$, $h_1\in T(x)$ and $h_2\in T(z)$. Choose $y_1\in S_{F,x}$ and $y_2\in S_{F,z}$ such that

$$h_1(t)=\Big[\frac{A}{(\mu-1)(\mu+b)^{\underline{\mu-2}}}-\frac{B(\mu-2)}{(b+3)\Gamma(\mu)}\Big]t^{\underline{\mu-1}}+\frac{B}{\Gamma(\mu-1)}t^{\underline{\mu-2}}$$

$$-\frac{t^{\underline{\mu-1}}}{(\mu+b)^{\underline{\mu-2}}\Gamma(\mu)}\sum_{s=0}^{b+1}(b+\mu-\sigma(s))^{\underline{\mu-2}}y_1(s)+\frac{1}{\Gamma(\mu)}\sum_{s=0}^{t-\mu}(t-\sigma(s))^{\underline{\mu-1}}y_1(s)$$

and

$$h_2(t)=\Big[\frac{A}{(\mu-1)(\mu+b)^{\underline{\mu-2}}}-\frac{B(\mu-2)}{(b+3)\Gamma(\mu)}\Big]t^{\underline{\mu-1}}+\frac{B}{\Gamma(\mu-1)}t^{\underline{\mu-2}}$$

$$-\frac{t^{\underline{\mu-1}}}{(\mu+b)^{\underline{\mu-2}}\Gamma(\mu)}\sum_{s=0}^{b+1}(b+\mu-\sigma(s))^{\underline{\mu-2}}y_2(s)+\frac{1}{\Gamma(\mu)}\sum_{s=0}^{t-\mu}(t-\sigma(s))^{\underline{\mu-1}}y_2(s)$$

for all $t\in\mathbb{N}_{\mu-2}^{b+\mu+2}$. Since

$$H_d(F(t,x(t),\Delta x(t)),F(t,z(t),\Delta z(t)))\le g(t)(|x(t)-z(t)|+|\Delta x(t)-\Delta z(t)|),$$

we get $|y_1(t)-y_2(t)|\le g(t)(|x(t)-z(t)|+|\Delta x(t)-\Delta z(t)|)$ for all $t\in\mathbb{N}_0^{b+2}$. Hence,

$$|h_1(t)-h_2(t)|\le\sum_{s=0}^{b+1}\frac{(b+\mu-\sigma(s))^{\underline{\mu-2}}t^{\underline{\mu-1}}}{\Gamma(\mu)(\mu+b)^{\underline{\mu-2}}}|y_1(s)-y_2(s)|+\sum_{s=0}^{t-\mu}\frac{(t-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)}|y_1(s)-y_2(s)|.$$

Since $\sum_{s=t-\mu+1}^{b+2}\frac{(t-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)}=0$ and $\frac{(b+\mu-\sigma(b+2))^{\underline{\mu-2}}}{\Gamma(\mu)(\mu+b)^{\underline{\mu-2}}}=0$, we obtain

$$|h_1(t)-h_2(t)|\le\sum_{s=0}^{b+2}\frac{(b+\mu-\sigma(s))^{\underline{\mu-2}}t^{\underline{\mu-1}}+(\mu+b)^{\underline{\mu-2}}(t-\sigma(s))^{\underline{\mu-1}}}{(\mu+b)^{\underline{\mu-2}}\Gamma(\mu)}|y_1(s)-y_2(s)|$$

and so

$$\max_{t\in\mathbb{N}_{\mu-2}^{\mu+b+2}}|h_1(t)-h_2(t)|\le G_1\max_{t\in\mathbb{N}_{\mu-2}^{\mu+b+2}}|g(t)|(\max_{t\in\mathbb{N}_{\mu-2}^{\mu+b+2}}|x(t)-z(t)|+\max_{t\in\mathbb{N}_{\mu-2}^{\mu+b+2}}|\Delta x(t)-\Delta z(t)|)$$

$$=G_1\max_{t\in\mathbb{N}_{\mu-2}^{\mu+b+2}}|g(t)|\|x-z\|.$$

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Since

$$\Delta h_1(t) = \left[\frac{A}{(\mu+b)^{\underline{\mu-2}}} - \frac{B(\mu-2)}{(b+3)\Gamma(\mu-1)}\right]t^{\underline{\mu-2}} + \frac{B}{\Gamma(\mu-2)}t^{\underline{\mu-3}}$$

$$-\frac{t^{\underline{\mu-2}}}{(\mu+b)^{\underline{\mu-2}}\Gamma(\mu-1)}\sum_{s=0}^{b+1}(b+\mu-\sigma(s))^{\underline{\mu-2}}y_1(s) + \frac{1}{\Gamma(\mu-1)}\sum_{s=0}^{t-\mu+1}(t-\sigma(s))^{\underline{\mu-2}}y_1(s),$$

we get

$$|\Delta h_1(t) - \Delta h_2(t)| \leq \sum_{s=0}^{b+2}\frac{(b+\mu-\sigma(s))^{\underline{\mu-2}}t^{\underline{\mu-2}} + (\mu+b)^{\underline{\mu-2}}(t-\sigma(s))^{\underline{\mu-2}}}{(\mu+b)^{\underline{\mu-2}}\Gamma(\mu-1)}|y_1(s) - y_2(s)|$$

and so

$$\max_{t\in\mathbb{N}_{\mu-2}^{\mu+b+2}}|\Delta h_1(t) - \Delta h_2(t)| \leq G_2 \max_{t\in\mathbb{N}_{\mu-2}^{\mu+b+2}}|g(t)|(\max_{t\in\mathbb{N}_{\mu-2}^{\mu+b+2}}|x(t) - z(t)| + \max_{t\in\mathbb{N}_{\mu-2}^{\mu+b+2}}|\Delta x(t) - \Delta z(t)|)$$

$$= G_2 \max_{t\in\mathbb{N}_{\mu-2}^{\mu+b+2}}|g(t)|\|x - z\|.$$

Hence, $\|h_1 - h_2\| \leq \max_{t\in\mathbb{N}_{\mu-2}^{b+\mu+2}}|g(t)|(G_1 + G_2)\|x - z\| = \lambda\|x - z\|$ for all $x, z \in \mathcal{X}$, $h_1 \in T(x)$ and $h_2 \in T(z)$. This implies that $H_d(T(x), T(z)) \leq \lambda\|x - z\|$ for all $x, z \in \mathcal{X}$ and so the multifunction T is a contraction with closed values. Now by using Lemma 1.2, T has a fixed point which is a solution for the inclusion problem (1). $\square$

Now, we present an example to illustrate the problem.

Example 2.1. *Consider the fractional finite difference inclusion*

$$\Delta_{-0.5}^{1.5}x(t) \in \left[0, e^t + \frac{\sin(x(t))}{e^7} + 8t^2 - \frac{|\Delta x(t)|}{e^7 + |\Delta x(t)|}\right] \tag{3}$$

via the boundary conditions $\Delta x(6.5) = 5$ and $x(-0.5) = -15$. Put $\mu = 1.5$, $b = 5$, $A = 5$, $B = -15$ and $F(t, x_1, x_2) = \left[0, e^t + \frac{\sin x_1}{e^7} + 8t^2 - \frac{|x_2|}{e^7+|x_2|}\right]$ for all $t \in \mathbb{N}_{-0.5}^{8.5}$ and $x_1, x_2 \in \mathbb{R}$. Note that, $e^t + \frac{\sin x_1}{e^7} + 8t^2 - \frac{|x_2|}{e^7+|x_2|} > 0$ for all $t \in \mathbb{N}_{-0.5}^{8.5}$ and $x_1, x_2 \in \mathbb{R}$ and so F is a compact valued multifunction on $\mathbb{N}_{-0.5}^{8.5} \times \mathbb{R} \times \mathbb{R}$. Define $g(t) = \frac{1}{165|t|}$. Note that $\max_{t\in\mathbb{N}_{-0.5}^{8.5}}|g(t)| = \frac{2}{165}$,

$$G_1 = \max_{t\in\mathbb{N}_{-0.5}^{8.5}}\frac{1}{(6.5)^{\underline{-0.5}}\Gamma(1.5)}\sum_{s=0}^{7}(5.5-s)^{\underline{-0.5}}t^{\underline{0.5}} + (6.5)^{\underline{-0.5}}(t-\sigma(s))^{\underline{0.5}} \simeq 66.8457$$

and

$$G_2 = \max_{t\in\mathbb{N}_{-0.5}^{8.5}}\frac{1}{(6.5)^{\underline{-0.5}}\Gamma(0.5)}\sum_{s=0}^{7}(5.5-s)^{\underline{-0.5}}t^{\underline{-0.5}} + (6.5)^{\underline{-0.5}}(t-\sigma(s))^{\underline{-0.5}} \simeq 15.3947.$$

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Also, $\lambda = \frac{2}{165}(66.8457 + 15.3947) < 1$. *On the other hand, we have*

$$\sup_{b \in F(t, y_1, y_2)} d(F(t, x_1, x_2), b) = \frac{\sin y_1}{e^7} - \frac{|y_2|}{e^7 + |y_2|} - \frac{\sin x_1}{e^7} + \frac{|x_2|}{e^7 + |x_2|}$$

whenever $\frac{\sin y_1}{e^7} - \frac{|y_2|}{e^7+|y_2|} > \frac{\sin x_1}{e^7} - \frac{|x_2|}{e^7+|x_2|}$ *and* $\sup_{b \in F(t, y_1, y_2)} d(F(t, x_1, x_2), b) = 0$ *otherwise. This implies that*

$$H_d(F(t, x_1, x_2), F(t, y_1, y_2)) \le \left| \frac{\sin y_1}{e^7} - \frac{|y_2|}{e^7 + |y_2|} - \frac{\sin x_1}{e^7} + \frac{|x_2|}{e^7 + |x_2|} \right|$$

$$\le \frac{1}{e^7} |\sin y_1 - \sin x_1| + \frac{1}{e^7} |y_2 - x_2| \le g(t)(|x_1 - y_1| + |x_2 - y_2|)$$

for all $t \in \mathbb{N}_{-0.5}^{8.5}$ *and* $x_1, x_2, y_1, y_2 \in \mathbb{R}$. *Now by using Theorem 2.2, the problem (3) has at least one solution.*

Remark 2.1. *The values A and B are arbitrary constants in the problem. In particular, the problem (3) has a solution via the boundary conditions* $\Delta x(6.5) = A$ *and* $x(-0.5) = B$ *for all* $A, B \in \mathbb{R}$.

Acknowledgments

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9

Some implicit properties of the second kind Bernoulli polynomials of order α

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Abstract In this paper we define the second kind Bernoulli polynomials of order α in the complex plane and find some interesting properties, symmetric identities of this polynomials. We also derive some relations between the second kind Bernoulli polynomials of order α, Euler polynomials of the second kind of order α, the stirling numbers and other polynomials.

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1. Introduction

The Bernoulli, Euler and Genocchi polynomials have been a subject of investigations and have been used in various branches of mathmatics such as theory of numbers, calculus of finite differences, combinatorial analysis, p-adic analytic number theory etc. These polynomials which are divided the first kind or the second kind have been researched by many mathematicians. Those polynomials have been extended and generalized in various direction, in particular, Bernoulli polynomials have been discussed by C. S. Ryoo, Q.-M. Luo, H. M. Srivastava, and T. Kim, etc(see [1-23]).

In this paper we use the following notations. $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}$ denotes the ring of integers, $\mathbb{Q}$ denotes the fild of rational numbers, $\mathbb{C}$ denotes the set of complex numbers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

The second kind Bernoulli numbers $\widetilde{B}_n$ and polynomials $\widetilde{B}_n(x)$ are by means of the

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following generating function (see [16-18]):

$$\sum_{n=0}^{\infty} \widetilde{B}_n \frac{t^n}{n!} = \frac{te^t}{e^{2t} - 1} \quad (|t| < \pi),$$

$$\sum_{n=0}^{\infty} \widetilde{B}_n(x) \frac{t^n}{n!} = \left(\frac{te^t}{e^{2t} - 1}\right) e^{tx} \quad (|t| < \pi),$$

(1.1)

respectively. From the above generating function, we can easily see that

$$\widetilde{B}_n := \widetilde{B}_n(0).$$

Also, the Euler polynomials $\widetilde{E}_n(x)$ of second kinds are given by

$$\sum_{n=0}^{\infty} \widetilde{E}_n(x) \frac{t^n}{n!} = \left(\frac{2e^t}{e^{2t} + 1}\right) e^{tx} \quad (|t| < \frac{\pi}{2}).$$

(1.2)

In [7], they expanded the Euler polynomials using a real or complex parameter α and introduced the Euler numbers and polynomials of the second kind of order α as follows:

$$\sum_{n=0}^{\infty} \widetilde{E}_n^{(\alpha)} \frac{t^n}{n!} = \left(\frac{2e^t}{e^{2t} + 1}\right)^{\alpha} \quad (|t| < \frac{\pi}{2}),$$

$$\sum_{n=0}^{\infty} \widetilde{E}_n^{(\alpha)}(x) \frac{t^n}{n!} = \left(\frac{2e^t}{e^{2t} + 1}\right)^{\alpha} e^{tx} \quad (|t| < \frac{\pi}{2}).$$

(1.3)

We usually define the central factorial numbers by the following expansion formula.

$$\sum_{k=0}^{n} T(n,k)x(x - 1^2)(x - 2^2) \cdots (x - (k-1)^2) = x^n.$$

(1.4)

or by the generating function

$$(2k)! \sum_{n=k}^{\infty} T(n,k) \frac{x^{2n}}{(2n)!} = (e^x + e^{-x} - 2)^k.$$

(1.5)

By using (1.4) and (1.5), we can find some properties of the central factorial numbers $T(n,k)$ (see [8,23]).

The stirling numbers of the second kind are the number of ways to partition a set of n objects into k non-empty subsets and are denoted by $S_2(n,k)$. The Stirling numbers of each kind according to the parameters n, k can form mutually inverse triangular matrices. The Stirling numbers of the second kind $S_2(n,k)$ can be defined by means of

$$x^n = \sum_{k=0}^{n} S_2(n,k)x(x - 1)(x - 2) \cdots (x - n + 1),$$

(1.6)

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or by the generating function

$$(e^x - 1)^k = k! \sum_{n=k}^{\infty} S_2(n,k) \frac{x^n}{n!}. \tag{1.7}$$

It follows from (1.6) or (1.7) that

$$S_2(n,k) = S_2(n-1,k-1) + kS_2(n-1,k),$$

with $S_2(n,0) = 0(n > 0)$, $S_2(n,n) = 1$, $S_2(n,1) = 1(n > 0)$, $S_2(n,k) = 0(k > n$ or $k < 0)$ (see [4,10,22-23]).

Associated Stirling numbers of the second kind $b(n,k)$ are defined by

$$(e^x - 1 - x)^k = k! \sum_{n=2k}^{\infty} b(n,k) \frac{x^n}{n!}. \tag{1.8}$$

It follows from (1.8) that

$$b(n,k) = (n-1)b(n-2,k-1) + kb(n-1,k),$$

with $b(n,0) = 0(n > 0)$, $b(0,0) = 1$, $b(n,1) = 1(n > 1)$, $b(n,k) = 0(2k > n$ or $k < 0)$(see [3-4, 10-11]).

$S_n(x)$ is the stirling polynomials, defined by

$$\sum_{n=0}^{\infty} \frac{S_n(x)}{n!} t^n = \left(\frac{t}{1 - e^{-t}} \right)^{x+1} \tag{1.9}$$

The stirling polynomials are related to the stirling numbers of the first kind $S_1(n,m)$ by

$$S_n(m) = \frac{(-1)^n}{\binom{m}{n}} S_1(m+1, m-n+1),$$

where $\binom{m}{n}$ is a binomial coefficient and m is an integer with $m \geq n$(see [19]).

This paper is organized as follows. In section 2, we construct the second kind Bernoulli polynomials of order α in the complex plane. We also study some interesting and basic properties of their polynomials. In section 3, we find some relations between the second kind Bernoulli polynomials of order α and Euler polynomials of the second kind of order α, the stirling numbers and other polynomials. We also define the power sum polynomials in oder to derive some symmetric identities about the second kind Bernoulli polynomials of order α in the complex plane.

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2. Some properties of the second kind Bernoulli polynomials of oder α

In this section, we construct the second kind Bernoulli polynomials of order α in the complex plane. We get some basic propertites of their polynomials.

Definition 2.1. For a real or complex parameter α, the second kind Bernoulli numbers $\widetilde{B}_n^{(\alpha)}$ and the second kind Bernoulli polynomials $\widetilde{B}_n^{(\alpha)}(x)$, each of degree n in x as well as in α, are defined by means of the following generating functions:

$$\sum_{n=0}^{\infty} \widetilde{B}_n^{(\alpha)} \frac{t^n}{n!} = \left(\frac{te^t}{e^{2t}-1} \right)^{\alpha},$$

and

$$\sum_{n=0}^{\infty} \widetilde{B}_n^{(\alpha)}(x) \frac{t^n}{n!} = \left(\frac{te^t}{e^{2t}-1} \right)^{\alpha} e^{tx}, \quad (|t| < \pi; \ 1^{\alpha} := 1),$$

respectively. Note that

$$\widetilde{B}_n^{(\alpha)}(0) := \widetilde{B}_n^{(\alpha)}, \quad \widetilde{B}_n^{(1)} = \widetilde{B}_n(x), \quad \widetilde{B}_n^{(0)}(x) = x^n, \quad (n \in \mathbb{N}_0).$$

From Definition 2,1, we can find the above polynomials by using computer.

n	$\widetilde{B}_n$	$\widetilde{B}_n^{(\alpha)}$	$\widetilde{B}_n^{(\alpha)}(x)$
$n=0$	$\dfrac{1}{2}$	$\dfrac{1}{2^\alpha}$	$\dfrac{1}{2^\alpha}$
$n=1$	0	0	$\dfrac{1}{2^\alpha}x$
$n=2$	$-\dfrac{1}{6}$	$\dfrac{1}{2^\alpha}\left(-\dfrac{1}{3}\alpha\right)$	$\dfrac{1}{2^\alpha}\left(-\dfrac{1}{3}\alpha+x^2\right)$
$n=3$	0	0	$\dfrac{1}{2^\alpha}(-\alpha x+x^3)$
$n=4$	$\dfrac{7}{30}$	$\dfrac{1}{2^\alpha}\left(\dfrac{1}{270}\alpha+\dfrac{1}{27}\alpha^2\right)$	$\dfrac{1}{2^\alpha}\left(\dfrac{2}{15}\alpha+\dfrac{1}{3}\alpha^2-2\alpha x^2+x^4\right)$
$n=5$	0	0	$\dfrac{1}{2^\alpha}\left(\left(\dfrac{2}{3}\alpha+\dfrac{5}{3}\alpha^2\right)x-\dfrac{10}{3}\alpha x^3+x^5\right)$
$n=6$	$-\dfrac{31}{42}$	$\dfrac{1}{2^\alpha}\left(-\dfrac{16}{63}\alpha-\dfrac{6}{9}\alpha^2-\dfrac{5}{9}\alpha^3\right)$	$\dfrac{1}{2^\alpha}\left(-\dfrac{16}{63}\alpha+\dfrac{10}{9}\alpha^2-\dfrac{16}{9}\alpha^2-\dfrac{5}{9}\alpha^3+(2\alpha+5\alpha^2)x^2-5\alpha x^4+x^6\right)$
...	...	...	...

Table 1: The comparison of $\widetilde{B}_n$, $\widetilde{B}_n^{(\alpha)}$ and $\widetilde{B}_n^{(\alpha)}(x)$

From now on, we will find the basic propertities of the second kind Bernoulli polynomials of order α from Definition 2.1.

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Theorem 2.2. Let $\alpha, \beta \in \mathbb{C}$. Then we find that

$$\widetilde{B}_n^{(\alpha+\beta)}(x+y) = \sum_{k=0}^n \binom{n}{k} \widetilde{B}_k^{(\alpha)}(x) \widetilde{B}_{n-k}^{(\beta)}(y).$$

Proof. From Definition 2.1, we obtain as the following equation by the binomial theorem.

$$\sum_{n=0}^\infty \widetilde{B}_n^{(\alpha+\beta)}(x+y) \frac{t^n}{n!} = \left(\frac{te^t}{e^{2t}-1}\right)^{\alpha+\beta} e^{t(x+y)}$$

$$= \sum_{n=0}^\infty \widetilde{B}_n^{(\alpha)}(x) \frac{t^n}{n!} \sum_{n=0}^\infty \widetilde{B}_n^{(\beta)}(y) \frac{t^n}{n!}$$

$$= \sum_{n=0}^\infty \sum_{k=0}^n \binom{n}{k} \widetilde{B}_k^{(\alpha)}(x) \widetilde{B}_{n-k}^{(\beta)}(y) \frac{t^n}{n!}.$$

Comparing the both sides of $\frac{t^n}{n!}$, we complete the proof.

$\square$

Corollary 2.3. Let $\alpha, \beta \in \mathbb{C}$. From Theorem 2.2, we can see that
(i) If we suppose $\beta = 0$, then

$$\widetilde{B}_n^{(\alpha)}(x+y) = \sum_{k=0}^n \binom{n}{k} \widetilde{B}_k^{(\alpha)}(x) y^{n-k} \quad = \sum_{k=0}^n \binom{n}{k} \widetilde{B}_k^{(\alpha)}(y) x^{n-k}.$$

(ii) If we suppoes $\beta = 0$ and $y = 0$, then

$$\widetilde{B}_n^{(\alpha)}(x) = \sum_{k=0}^n \binom{n}{k} \widetilde{B}_k^{(\alpha)} x^{n-k} \quad = \sum_{k=0}^n \binom{n}{k} \widetilde{B}_k^{(\alpha)} x^k.$$

Theorem 2.4. Let $\alpha, x \in \mathbb{C}$ and $n \in \mathbb{N}_0$. Then we get that

$$\widetilde{B}_n^{(\alpha)}(-x) = (-1)^n \widetilde{B}_n^{(\alpha)}(x).$$

Proof. Substituting $-x$ instead of x we get the following equation.

$$\sum_{n=0}^\infty \widetilde{B}_n^{(\alpha)}(-x) \frac{t^n}{n!} = \left(\frac{te^t}{e^{2t}-1}\right)^\alpha e^{-tx}$$

$$= \left(\frac{-te^{-t}}{e^{-2t}-1}\right)^\alpha e^{-tx}$$

$$= \sum_{n=0}^\infty (-1)^n \widetilde{B}_n^{(\alpha)}(x) \frac{t^n}{n!}.$$

Hence the proof of Theorem 2.4. is clear.

<center>5</center>

□

Theorem 2.5. Let $\alpha \in \mathbb{C}$ and $n \in \mathbb{N}_0$. Then we have

$$\widetilde{B}_n^{(\alpha)}(x+2) - \widetilde{B}^{(\alpha)}(x) = n\widetilde{B}_n^{(\alpha-1)}(x+1) = \sum_{k=0}^{n} \binom{n}{k} k\widetilde{B}_{n-k}^{(\alpha-1)}(x).$$

Proof. By using Definition 2.1, we get that

$$\widetilde{B}_n^{(\alpha)}(x+2) - \widetilde{B}^{(\alpha)}(x)\frac{t^n}{n!} = \left(\frac{te^t}{e^{2t}-1}\right)^{\alpha} e^{t(x+2)} - \left(\frac{te^t}{e^{2t}-1}\right)^{\alpha} e^{tx}$$

$$= \left(\frac{te^t}{e^{2t}-1}\right)^{\alpha-1} te^{t(x+1)}.$$

We can find out the following equation.

$$\left(\frac{te^t}{e^{2t}-1}\right)^{\alpha-1} te^{t(x+1)} = \sum_{n=0}^{\infty} \widetilde{B}_{n-1}^{(\alpha-1)}(x)\frac{t^n}{n!} \sum_{n=0}^{\infty} n\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k} k\widetilde{B}_{n-k}^{(\alpha-1)}(x)\frac{t^n}{n!}.$$

We also find the other equation as the follows:

$$\left(\frac{te^t}{e^{2t}-1}\right)^{\alpha-1} te^{t(x+1)} = \sum_{n=0}^{\infty} \widetilde{B}_n^{(\alpha-1)}(x+1)\frac{t^{n+1}}{n!} = \sum_{n=0}^{\infty} n\widetilde{B}_{n-1}^{(\alpha-1)}(x+1)\frac{t^n}{n!}.$$

Hence we terminate the proof of Theorem 2.5.

□

Theorem 2.6. Let $\alpha \in \mathbb{C}$ and $n \in \mathbb{N}_0$. Then we derive that

(i) $\displaystyle\sum_{k=0}^{n-1} \binom{n-1}{k} \widetilde{B}_k^{(\alpha-1)}(x) = \frac{1}{n}\left[\widetilde{B}_n^{(\alpha)}(x+2) - \widetilde{B}_n^{(\alpha)}(x)\right].$

(ii) $\displaystyle\widetilde{B}_n^{(\alpha-1)}(x+1) = \frac{1}{n+1}\left[\sum_{k=0}^{n+1} 2^{n+1-k}\binom{n+1}{k}\widetilde{B}_k^{(\alpha)}(x) - \widetilde{B}_{n+1}^{(\alpha)}(x)\right].$

(iii) $\displaystyle\widetilde{B}_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n}{k}\left[2^{n-k}\widetilde{B}_k^{(\alpha)}(x) - k\widetilde{B}_{n-k}^{(\alpha-1)}(x)\right].$

Proof. (i) Using Corollary 2.3.(i), we can see that

$$\widetilde{B}_n^{(\alpha)}(x+1) = \sum_{k=0}^{n} \binom{n}{k}\widetilde{B}_k^{(\alpha)}(x), \tag{2.1}$$

and

$$\widetilde{B}_n^{(\alpha)}(x+2) = \sum_{k=0}^{n} \binom{n}{k} 2^{n-k}\widetilde{B}_k^{(\alpha)}(x). \tag{2.2}$$

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If we combine Theorem 2.5 and (2.1), then we find that

$$\widetilde{B}_n^{(\alpha)}(x+2) - \widetilde{B}_n^{(\alpha)}(x) = n \sum_{k=0}^{n-1} \binom{n-1}{k} \widetilde{B}_k^{(\alpha-1)}(x).$$

Therefore, we can see Theorem 2.6.(i).

(ii) From (2.2) we can transform Theorem 2.5 as following equation:

$$\sum_{k=0}^{n} \binom{n}{k} 2^{n-k} \widetilde{B}_k^{(\alpha)}(x) - \widetilde{B}_n^{(\alpha)}(x) = n \widetilde{B}_{n-1}^{(\alpha-1)}(x+1).$$

Hence we have

$$\widetilde{B}_n^{(\alpha-1)}(x+1) = \frac{1}{n+1} \left[\sum_{k=0}^{n+1} 2^{n+1-k} \binom{n+1}{k} \widetilde{B}_k^{(\alpha)}(x) - \widetilde{B}_{n+1}^{(\alpha)}(x) \right].$$

(iii) This proof is very similar to the proof of (ii). We find the below equation from Theorem 2.5 and (2.2).

$$\sum_{k=0}^{n} \binom{n}{k} 2^{n-k} \widetilde{B}_k^{(\alpha)}(x) - \widetilde{B}_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n}{k} k \widetilde{B}_{n-k}^{(\alpha-1)}(x).$$

From the above equation we know that

$$\widetilde{B}_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n}{k} \left[2^{n-k} \widetilde{B}_k^{(\alpha)}(x) - k \widetilde{B}_{n-k}^{(\alpha-1)}(x) \right].$$

Thus we wind up the proof of Theorem 2.6.

$\square$

Corollary 2.7. Let n be non-negative integer. From Theorem 2.4.(ii), we can see that

$$x^n = \frac{1}{n+1} \sum_{k=0}^{n+1} 2^{n+1-k} \binom{n+1}{k} \widetilde{B}_k(x-1).$$

Theorem 2.8. For w_1, w_2 be positive integers and $\alpha \in \mathbb{C}$, we have

$$\sum_{n=0}^{l} \binom{l}{n} \sum_{i=0}^{w_2-1} \sum_{j=0}^{w_1-1} w_1^{l-n} w_2^n \widetilde{B}_{l-n}^{(\alpha)}(w_2 x + 2j\frac{w_2}{w_1}) \widetilde{B}_n^{(\alpha)}(w_1 y + 2i\frac{w_1}{w_2})$$

$$= \sum_{n=0}^{l} \binom{l}{n} \sum_{i=0}^{w_1-1} \sum_{j=0}^{w_2-1} w_2^{l-n} w_1^n \widetilde{B}_{l-n}^{(\alpha)}(w_1 x + 2j\frac{w_1}{w_2}) \widetilde{B}_n^{(\alpha)}(w_2 y + 2i\frac{w_2}{w_1}).$$

Proof. We can consider that

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$$A(t) := \frac{(e^{2w_1 w_2 t} - 1)^2 (w_1 w_2 t^2 e^{t(w_1 + w_2)})^\alpha e^{w_1 w_2 t(x+y)}}{(e^{2w_1 t} - 1)^{\alpha+1}(e^{2w_2 t} - 1)^{\alpha+1}}.$$

Then we are able to express $A(t)$ as follows:

$$\begin{aligned}
A(t) :&= \left(\frac{w_1 t e^{w_1 t}}{e^{2w_1 t} - 1}\right)^\alpha e^{w_1 w_2 t x} \sum_{i=0}^{w_2-1} e^{2i w_1 t} \left(\frac{w_2 t e^{w_2 t}}{e^{2w_2 t} - 1}\right)^\alpha e^{w_1 w_2 t y} \sum_{j=0}^{w_1-1} e^{2j w_2 t} \\
&= \sum_{l=0}^{\infty}\sum_{n=0}^{l} \binom{l}{n} \sum_{i=0}^{w_2-1}\sum_{j=0}^{w_1-1} w_1^{l-n} w_2^n \widetilde{B}_{l-n}^{(\alpha)}(w_2 x + 2j\frac{w_2}{w_1})\widetilde{B}_n^{(\alpha)}(w_1 y + 2i\frac{w_1}{w_2})\frac{t^l}{l!}.
\end{aligned} \tag{2.3}$$

We also represent $A(t)$ as the following form.

$$\begin{aligned}
A(t) :&= \left(\frac{w_2 t e^{w_2 t}}{e^{2w_2 t} - 1}\right)^\alpha e^{w_1 w_2 t x} \sum_{i=0}^{w_1-1} e^{2i w_2 t} \left(\frac{w_1 t e^{w_1 t}}{e^{2w_1 t} - 1}\right)^\alpha e^{w_1 w_2 t y} \sum_{j=0}^{w_2-1} e^{2j w_1 t} \\
&= \sum_{l=0}^{\infty}\sum_{n=0}^{l} \binom{l}{n} \sum_{i=0}^{w_1-1}\sum_{j=0}^{w_2-1} w_2^{l-n} w_1^n \widetilde{B}_{l-n}^{(\alpha)}(w_1 x + 2j\frac{w_1}{w_2})\widetilde{B}_n^{(\alpha)}(w_2 y + 2i\frac{w_2}{w_1})\frac{t^l)}{l!}.
\end{aligned} \tag{2.4}$$

By using the coefficient comparison in (2.3) and (2.4), we proved Theorem 2.8.

$$\square$$

Corollary 2.9. By substituting $w_1 = 1$ from Theorem 2.8, we easily see that

$$\sum_{n=0}^{l} \binom{l}{n} \sum_{i=0}^{w_2-1} w_2^n \widetilde{B}_{l-n}^{(\alpha)}(w_2 x)\widetilde{B}_n^{(\alpha)}(y + \frac{2i}{w_2})$$

$$= \sum_{n=0}^{l} \binom{l}{n} \sum_{j=0}^{w_2-1} w_2^{l-n} \widetilde{B}_n^{(\alpha)}(w_2 y)\widetilde{B}_{l-n}^{(\alpha)}(x + \frac{2j}{w_2}).$$

Theorem 2.10. For w_1, w_2 be positive integers, $\alpha \in \mathbb{C}$ and $n \in \mathbb{N}$, we obtain

$$\sum_{l=0}^{n} \binom{n}{l} l w_1^{n+1-l} w_2^{l-1} \widetilde{B}_{n-l}^{(\alpha-1)}(w_2 x) \sum_{i=0}^{w_2-1} \widetilde{B}_{l-1}^{(\alpha)}(w_1 y + 2i\frac{w_1}{w_2} + \frac{w_1}{w_2})$$

$$= \sum_{l=0}^{n} \binom{n}{l} l w_2^{n+1-l} w_1^{l-1} \widetilde{B}_{n-l}^{(\alpha-1)}(w_1 x) \sum_{j=0}^{w_1-1} \widetilde{B}_{l-1}^{(\alpha)}(w_2 y + 2j\frac{w_2}{w_1} + \frac{w_2}{w_1}).$$

Proof. Assume that

$$B(t) := \frac{(e^{2w_1 w_2 t} - 1)(w_1 w_2 t^2 e^{t(w_1 + w_2)})^\alpha e^{w_1 w_2 t(x+y)}}{(e^{2w_1 t} - 1)^\alpha (e^{2w_2 t} - 1)^\alpha}.$$

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Then we express $B(t)$ as follows:

$$
\begin{aligned}
B(t) :=& \left(\frac{w_1 t e^{w_1 t}}{e^{2w_1 t}-1}\right)^{\alpha-1} e^{w_1 w_2 t x} w_1 t e^{w_1 t} \sum_{i=0}^{w_2-1} e^{2iw_1 t} \left(\frac{w_2 t e^{w_2 t}}{e^{2w_2 t}-1}\right)^{\alpha} e^{w_1 w_2 t y} \\
=& \sum_{n=1}^{\infty} \sum_{l=0}^{n} \binom{n}{l} l \widetilde{B}_{n-l}^{(\alpha-1)}(w_2 x) w_1^{n+1-l} w_2^{l-1} \sum_{i=0}^{w_2-1} \widetilde{B}_{l-1}^{(\alpha)}\left(w_1 y + 2i\frac{w_1}{w_2} + \frac{w_1}{w_2}\right) \frac{t^n}{n!}.
\end{aligned}
\tag{2.5}
$$

We can also represent $B(t)$ as the following equation.

$$
\begin{aligned}
B(t) :=& \left(\frac{w_2 t e^{w_2 t}}{e^{2w_2 t}-1}\right)^{\alpha-1} e^{w_1 w_2 t x} w_2 t e^{w_2 t} \sum_{j=0}^{w_1-1} e^{2iw_2 t} \left(\frac{w_1 t e^{w_1 t}}{e^{2w_1 t}-1}\right)^{\alpha} e^{w_1 w_2 t y} \\
=& \sum_{n=1}^{\infty} \sum_{l=0}^{n} \binom{n}{l} l \widetilde{B}_{n-l}^{(\alpha-1)}(w_1 x) w_2^{n+1-l} w_1^{l-1} \sum_{j=0}^{w_1-1} \widetilde{B}_{l-1}^{(\alpha)}\left(w_2 y + 2j\frac{w_2}{w_1} + \frac{w_2}{w_1}\right) \frac{t^n}{n!}.
\end{aligned}
\tag{2.6}
$$

We get the following symmetric equation from (2.5) and (2.6).

$$
\begin{aligned}
& \sum_{l=0}^{n} \binom{n}{l} l w_1^{n+1-l} w_2^{l-1} \widetilde{B}_{n-l}^{(\alpha-1)}(w_2 x) \sum_{i=0}^{w_2-1} \widetilde{B}_{l-1}^{(\alpha)}\left(w_1 y + 2i\frac{w_1}{w_2} + \frac{w_1}{w_2}\right) \\
=& \sum_{l=0}^{n} \binom{n}{l} l w_2^{n+1-l} w_1^{l-1} \widetilde{B}_{n-l}^{(\alpha-1)}(w_1 x) \sum_{j=0}^{w_1-1} \widetilde{B}_{l-1}^{(\alpha)}\left(w_2 y + 2j\frac{w_2}{w_1} + \frac{w_2}{w_1}\right).
\end{aligned}
$$

Thus, we complete the proof of Theorem 2.10.

$\square$

Corollary 2.11. When $w_1 = 1$ from Theorem 2.10, we can see that

$$
\begin{aligned}
& \sum_{l=0}^{n} \binom{n}{l} l w_2^{l-1} \widetilde{B}_{n-l}^{(\alpha-1)}(w_2 x) \sum_{i=0}^{w_2-1} \widetilde{B}_{l-1}^{(\alpha)}\left(y + \frac{2i}{w_2} + \frac{1}{w_2}\right) \\
=& \sum_{l=0}^{n} \binom{n}{l} l w_2^{n+1-l} \widetilde{B}_{n-l}^{(\alpha-1)}(x) \widetilde{B}_{l-1}^{(\alpha)}(w_2 y + w_2).
\end{aligned}
$$

3. Some relations of the second kind Bernoulli polynomials of order α, the various kinds of numbers and polynomials

In this section, we investigate some interesting relations between the second kind Bernoulli polynomials of order α and the Euler polynomials of the second kind of order α. We find some relations between the second kind Bernoulli polynomials of order α, stirling numbers, the associated stirling numbers of the second kind and central factorial numbers. We also derive the analogue of Srivastava-Pintér addition theorem(see [14,20]).

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Theorem 3.1. Let $n \in \mathbb{N}_0$ and $\alpha \in \mathbb{C}$. Then one has

$$\widetilde{B}_n^{(\alpha)}(x) = \left(\frac{1}{2}\right)^n \sum_{k=0}^n \binom{n}{k} \widetilde{B}_{n-k}^{(\alpha)}(x)\widetilde{E}_k^{(\alpha)}(x) = \left(\frac{1}{2}\right)^n \sum_{k=0}^n \binom{n}{k} \widetilde{B}_{n-k}^{(\alpha)}\widetilde{E}_k^{(\alpha)}(2x).$$

Proof. We can find the following equation by using generating function.

$$\sum_{n=0}^\infty \widetilde{B}_n^{(\alpha)}(x)\frac{t^n}{n!} = \left(\frac{te^t}{e^{2t}-1}\right)^\alpha e^{tx} = \left(\frac{te^{\frac{1}{2}t}}{e^t-1}\right)^\alpha \left(\frac{e^{\frac{1}{2}t}}{e^t+1}\right)^\alpha e^{tx}.$$

From the above equation we can combine the Euler polynomials of the second kind of oder α, $\widetilde{E}_n^{(\alpha)}$, by using Cauchy product as follows:.

$$\sum_{n=0}^\infty \widetilde{B}_n^{(\alpha)}(x)\frac{t^n}{n!} = \sum_{n=0}^\infty \sum_{k=0}^n \binom{n}{k}\left(\frac{1}{2}\right)^n \widetilde{B}_{n-k}^{(\alpha)}\widetilde{E}_k^{(\alpha)}(2x)\frac{t^n}{n!}.$$

Hence, the proof of Theorem 3.1 is clear.

$\square$

From simple transformation, we are able to represent to the following equation from Theorem 3.1.

$$2^n \widetilde{B}_n^{(\alpha)}(\frac{x}{2}) = \sum_{k=0}^n \binom{n}{k}\widetilde{B}_{n-k}^{(\alpha)}(\frac{x}{2})\widetilde{E}_k^{(\alpha)}(\frac{x}{2}) = \sum_{k=0}^n \binom{n}{k}\widetilde{B}_{n-k}^{(\alpha)}\widetilde{E}_k^{(\alpha)}(x).$$

By using addition theorem of the Euler polynomials of the second kind of order α, we find the analogue of Srivastava-Pintér addition theorem as the following theorem 3.2.

Theorem 3.2. Let $\alpha \in \mathbb{C}$ and $n \in \mathbb{N}_0$. Then we obtain that

$$\widetilde{B}_n^{(\alpha)}(x+y) = \frac{1}{2}\sum_{k=0}^n \binom{n}{k}\left[\widetilde{B}_k^{(\alpha)}(y+2) + \widetilde{B}_k^{(\alpha)}(y)\right]\widetilde{E}_{n-k}(x-1).$$

Proof. In [6], we can make the following equation by using simple calculation.

$$x^n = \frac{1}{2}\left[\sum_{k=0}^n \binom{n}{k}2^{n-k}\widetilde{E}_k(x-1) + \widetilde{E}_n(x-1)\right]. \tag{3.1}$$

Substituting (3.1) in Corollary 2.3.(i), inverting the order of summation and using the elementary combinatorial, we get

$$\widetilde{B}_n^{(\alpha)}(x+y) = \sum_{k=0}^n \binom{n}{k}\widetilde{B}_k^{(\alpha)}(y)\left[\frac{1}{2}\left\{\sum_{j=0}^{n-k}\binom{n-k}{j}2^{n-k-j}\widetilde{E}_j(x-1) + \widetilde{E}_{n-k}(x-1)\right\}\right]$$

$$= \frac{1}{2}\left[\sum_{j=0}^n \binom{n}{j}\widetilde{E}_j(x-1)\sum_{k=0}^{n-j}\binom{n-j}{k}2^{n-k-j}\widetilde{B}_k^{(\alpha)}(y) + \sum_{k=0}^n \binom{n}{k}\widetilde{B}_k^{(\alpha)}(y)\widetilde{E}_{n-k}(x-1)\right].$$

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Note that

$$\widetilde{B}_{n-j}^{(\alpha)}(y+2) = \sum_{k=0}^{n-j} \binom{n-j}{k} 2^{n-k-j} \widetilde{B}_k^{(\alpha)}(y).$$

By applying the above form, we derive that

$$\widetilde{B}_n^{(\alpha)}(x+y) = \frac{1}{2} \sum_{k=0}^{n} \binom{n}{k} \left[\widetilde{B}_k^{(\alpha)}(y+2) + \widetilde{B}_k^{(\alpha)}(y) \right] \widetilde{E}_{n-k}(x-1).$$

Hence, we complete the proof of Theorem 3.2.

$\square$

Corollary 3.3. When $\alpha = 1$ from Theorem 3.2, we see that

$$\widetilde{B}_n(x+y) = \frac{1}{2} \sum_{k=0}^{n} \binom{n}{k} \left[\widetilde{B}_k(y+2) + \widetilde{B}_k(y) \right] \widetilde{E}_{n-k}(x-1).$$

Corollary 3.4. By setting $y = 0$ from Theorem 3.2, we have

$$\widetilde{B}_n^{(\alpha)}(x) = \frac{1}{2} \sum_{k=0}^{n} \binom{n}{k} \left[\widetilde{B}_k^{(\alpha)}(2) + \widetilde{B}_k^{(\alpha)} \right] \widetilde{E}_{n-k}(x-1).$$

Theorem 3.5. Let $n \in \mathbb{N}_0$ and $m \in \mathbb{N}$. Then we get

$$\widetilde{B}_n^{(m)} = \frac{n!}{2^m} \sum_{\substack{v_1, \cdots, v_m = 0 \\ v_1 + \cdots + v_m = n}} \frac{\left(\sum_{k=0}^{v_1} \binom{v_1}{k} \left(\widetilde{B}_{v_1-k}(1)\widetilde{E}_k + \widetilde{B}_{v_1-k}\widetilde{E}_k(-1) \right) \right) \cdots}{v_1 \cdots}$$

$$\times \frac{\left(\sum_{k=0}^{v_m} \binom{v_m}{k} \left(\widetilde{B}_{v_m-k}(1)\widetilde{E}_k + \widetilde{B}_{v_m-k}\widetilde{E}_k(-1) \right) \right)}{v_m!}.$$

Proof. Suppose that $\alpha = 1$. That is,

$$\sum_{n=0}^{\infty} \widetilde{B}_n \frac{t^n}{n!} = \left(\frac{te^t}{e^{2t} - 1} \right) = \frac{te^{3t}}{(e^{2t}-1)(e^{2t}+1)} + \frac{te^t}{(e^{2t}-1)(e^{2t}+1)}$$

$$= \sum_{n=0}^{\infty} \widetilde{B}_n(1) \sum_{n=0}^{\infty} \frac{1}{2} \widetilde{E}_n \frac{t^n}{n!} + \sum_{n=0}^{\infty} \widetilde{B}_n \sum_{n=0}^{\infty} \frac{1}{2} \widetilde{E}_n(-1) \frac{t^n}{n!}$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \left(\widetilde{B}_{n-k}(1)\widetilde{E}_k + \widetilde{B}_{n-k}\widetilde{E}_k(-1) \right) \frac{t^n}{n!}.$$

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Consider that $\alpha = 2$. Then we have

$$
\sum_{n=0}^{\infty} \widetilde{B}_n^{(2)} \frac{t^n}{n!} = \left(\frac{1}{2} \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{n} \binom{n}{k} \left(\widetilde{B}_{n-k}(1)\widetilde{E}_k + \widetilde{B}_{n-k}\widetilde{E}_k(-1) \right) \right\} \frac{t^n}{n!} \right)
$$

$$
\times \left(\frac{1}{2} \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{n} \binom{n}{k} \left(\widetilde{B}_{n-k}(1)\widetilde{E}_k + \widetilde{B}_{n-k}\widetilde{E}_k(-1) \right) \right\} \frac{t^n}{n!} \right)
$$

$$
= \frac{1}{2^2} \sum_{n=0}^{\infty} \sum_{\substack{v_1,v_2=0 \\ v_1+v_2=2}} \frac{\sum_{k=0}^{v_1} \binom{v_1}{k} \left(\widetilde{B}_{v_1-k}(1)\widetilde{E}_k + \widetilde{B}_{v_1-k}\widetilde{E}_k(-1) \right)}{v_1!}
$$

$$
\times \frac{\sum_{k=0}^{v_2} \binom{v_2}{k} \left(\widetilde{B}_{v_2-k}(1)\widetilde{E}_k + \widetilde{B}_{v_2-k}\widetilde{E}_k(-1) \right)}{v_2!}.
$$

Therefore, we can derive the below equation when $\alpha = m$.

$$
\widetilde{B}_n^{(m)} = \frac{n!}{2^m} \sum_{\substack{v_1,\cdots,v_m=0 \\ v_1+\cdots+v_m=n}} \frac{\left(\sum_{k=0}^{v_1} \binom{v_1}{k} \left(\widetilde{B}_{v_1-k}(1)\widetilde{E}_k + \widetilde{B}_{v_1-k}\widetilde{E}_k(-1) \right) \right) \cdots}{v_1 \cdots}
$$

$$
\times \frac{\left(\sum_{k=0}^{v_m} \binom{v_m}{k} \left(\widetilde{B}_{v_m-k}(1)\widetilde{E}_k + \widetilde{B}_{v_m-k}\widetilde{E}_k(-1) \right) \right)}{v_m!}.
$$

$\square$

Theorem 3.6. Let $n \in \mathbb{N}$ and $\alpha \in \mathbb{C}$. Then we have

$$
\widetilde{B}_n^{(\alpha)}(x+y) = \sum_{j=0}^{n} \binom{n}{j} \sum_{k=0}^{n-j} \binom{x}{k} k! \widetilde{B}_j^{(\alpha)}(y) S_2(n-j,k),
$$

where $S_2(n-j,k)$ is the stirling numbers of the second kind.

Proof. From Definition which is the stirling numbers of the second kind we can find this theorem. Note that

$$
x^n = \sum_{k=0}^{n} \binom{x}{k} k! S_2(n,k),
$$

where $S_2(n,k)$ are the stirling numbers of the second kind.
By combining the above equation and Corollary 2.3.(i), we can easily see that

$$
\widetilde{B}_n^{(\alpha)}(x+y) = \sum_{j=0}^{n} \binom{n}{j} \sum_{k=0}^{n-j} \binom{x}{k} k! \widetilde{B}_j^{(\alpha)}(y) S_2(n-j,k),
$$

Therefore, we complete the proof.

$\square$

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Theorem 3.7. Let $n \in \mathbb{N}_0$ and $\alpha \in \mathbb{C}$. Then we derive

$$\widetilde{B}_n^{(\alpha)} = \sum_{k=0}^{n} \binom{n}{k} 2^{n-\alpha-k} \sum_{j=0}^{n-k} \frac{(-1)^j}{\binom{n+j-k}{j}} \binom{\alpha+j-1}{j} b(n-k+j,j) \alpha^k,$$

where $b(n - k + j, j)$ is the associated stirling numbers of the second kind.

Proof. By using the associated stirling numbers of the second kind, we find that

$$\sum_{n=0}^{\infty} \widetilde{B}_n^{(\alpha)} \frac{t^n}{n!} = \left(\frac{te^t}{e^{2t}-1} \right)^{\alpha} = \frac{1}{2^{\alpha}} \left(\frac{1}{1+\frac{1}{2t}(e^{2t}-2t-1)} \right)^{\alpha} e^{\alpha t}$$

$$= \frac{1}{2^{\alpha}} \sum_{j=0}^{\infty} \binom{\alpha+j-1}{j} \left(-\frac{1}{2} \right)^j (e^{2t}-2t-1)^j t^{-j} e^{\alpha t}$$

$$= \frac{1}{2^{\alpha}} \sum_{j=0}^{\infty} (-1)^j \binom{\alpha+j-1}{j} j! \sum_{n=j}^{\infty} 2^n b(n+j,j) \frac{t^n}{(n+j)!} e^{\alpha t}$$

$$= \frac{1}{2^{\alpha}} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} \sum_{j=0}^{n-k} \frac{(-1)^j}{\binom{n+j-k}{j}} \binom{\alpha+j-1}{j} b(n+j-k,j) \alpha^k \frac{t^n}{n!}.$$

Therefore we express $\widetilde{B}_n^{(\alpha)}$ as follows.

$$\widetilde{B}_n^{(\alpha)} = \sum_{k=0}^{n} \binom{n}{k} 2^{n-\alpha-k} \sum_{j=0}^{n-k} \frac{(-1)^j}{\binom{n+j-k}{j}} \binom{\alpha+j-1}{j} b(n-k+j,j) \alpha^k.$$

$\square$

Theorem 3.8. Let $n \in \mathbb{N}_0$ and $\alpha \in \mathbb{C}$. Then we obtain

$$\widetilde{B}_n^{(\alpha)}(x) = \sum_{l=0}^{n} \sum_{k=0}^{n-l} \binom{n-k}{l} 2^{n-\alpha-l-k} \alpha^k \sum_{j=0}^{n-l-k} \binom{n+j}{k} \frac{(-1)^j}{\binom{n+j}{j}} \binom{\alpha+j-1}{j} b(n+j-l-k,j) x^l.$$

Proof. By using the associated stirling numbers of the second kind, we express the polynomials $\widetilde{B}_n^{(\alpha)}(x)$ as follows.

$$\sum_{n=0}^{\infty} \widetilde{B}_n^{(\alpha)}(x) \frac{t^n}{n!} = \left(\frac{te^t}{e^{2t}-1} \right)^{\alpha} e^{tx} = \sum_{n=0}^{\infty} \widetilde{B}_n^{(\alpha)} \frac{t^n}{n!} \sum_{n=0}^{\infty} x^n \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \sum_{l=0}^{n} \sum_{k=0}^{n-l} \binom{n-k}{l} 2^{n-\alpha-l-k} \alpha^k \sum_{j=0}^{n-l-k} \binom{n+j}{k} \frac{(-1)^j}{\binom{n+j}{j}} \binom{\alpha+j-1}{j}$$

$$\times b(n+j-l-k,j) x^l \frac{t^n}{n!}.$$

From the above equation, we have the Theorem 3.8 as follows:

$$\widetilde{B}_n^{(\alpha)}(x) = \sum_{l=0}^{n}\sum_{k=0}^{n-l}\binom{n-k}{l}2^{n-\alpha-l-k}\alpha^k \sum_{j=0}^{n-l-k}\binom{n+j}{k}\frac{(-1)^j}{\binom{n+j}{j}}\binom{\alpha+j-1}{j}b(n+j-l-k,j)x^l.$$

Thus, we complete the proof of Theorem 3.8.

$\square$

Theorem 3.9. Let $n \in \mathbb{N}_0$ and $\alpha \in \mathbb{C}$. Then we obtain that

$$\left(\widetilde{B} + \widetilde{B}^{(\alpha)}(1+\alpha)\right)^n = \frac{2^n S_n(\alpha)}{2^{\alpha+1}},$$

where $S_n(\alpha)$ is the stirling polynomials.

Proof. Consider that $\widetilde{B}^n = \widetilde{B}_n$. Then we get the following equation from binomial operation and definition of stirling polynomials.

$$2^{\alpha+1}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n\left(\widetilde{B}+\widetilde{B}^{(\alpha)}(1+\alpha)\right)^n\frac{t^n}{n!} = 2\sum_{n=0}^{\infty}\widetilde{B}_n\frac{\left(\frac{1}{2}t\right)^n}{n!}2^{\alpha}\sum_{n=0}^{\infty}\widetilde{B}_n^{(\alpha)}(1+\alpha)\frac{\left(\frac{1}{2}t\right)^n}{n!}$$

$$= \left(\frac{te^t}{e^t-1}\right)^{\alpha+1} = \sum_{n=0}^{\infty}S_n(\alpha)\frac{t^n}{n!}.$$

Hence we can finish the proof.

$\square$

Theorem 3.10. Let $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$. Then we derive that

$$\binom{n}{2k}\widetilde{B}_{n-2k}^{(-2k)}(x) = \sum_{l=0}^{n-2k}2^{2k+l}T(k+l,k)x^{n-(2k+l)},$$

where $T(n,k)$ is the central factorial numbers.

Proof. Consider that

$$\sum_{n=k}^{\infty}T(n,k)\frac{t^{2n}}{(2n)!} = \frac{(e^t-1)^{2k}}{(2k)!e^{tk}}.$$

In oder to make relation of the second kind Bernoulli polynomials and the central factorial numbers, we can express as the follows:

$$\frac{1}{4^k(2k)!}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n\widetilde{B}_n^{(-2k)}(x)\frac{t^{2k+n}}{n!} = \sum_{n=k}^{\infty}T(n,k)\frac{t^{2n}}{(2n)!}\sum_{n=0}^{\infty}\left(\frac{1}{2}x\right)^n\frac{t^n}{n!}.$$

From the above equation, we obtain the below equation by using some calculation and comparing the both sides of $\frac{t^{2k+n}}{(2k+n)!}$.

$$\binom{n}{2k}\left(\frac{1}{2}\right)^n\widetilde{B}_{n-2k}^{(-2k)}(x) = \sum_{l=0}^{n-2k}\frac{T(k+l,k)}{2^{n-2k-l}}x^{n-2k-l}.$$

Therefore the Theorem 3.10 is clear.

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$\square$

Let $l \in \mathbb{N}_0$ and n be non-negative integers. Then we can define $\widetilde{P}_l(n)$ as follows:

$$\widetilde{P}_l(n) = \widetilde{B}_l(2n) - \widetilde{B}_l = l \sum_{i=0}^{n-1} (1+2i)^{l-1}.$$

where $\widetilde{P}_l(n)$ is the symmetric power sum polynomials.

Theorem 3.11. For w_1, w_2 be positive integers and $\alpha \in \mathbb{C}$, we get

$$\sum_{l=0}^{n} \binom{n}{l} \widetilde{B}_{n-l}^{(\alpha-1)}(w_1 y) \sum_{s=0}^{l} \binom{l}{s} w_1^{l-s} w_2^{n-l+s} \widetilde{B}_{l-s}^{(\alpha)}(w_2 x) \widetilde{P}_s(w_1)$$

$$= \sum_{l=0}^{n} \binom{n}{l} \widetilde{B}_{n-l}^{(\alpha-1)}(w_2 y) \sum_{s=0}^{l} \binom{l}{s} w_2^{l-s} w_1^{n-l+s} \widetilde{B}_{l-s}^{(\alpha)}(w_1 x) \widetilde{P}_s(w_2).$$

Proof. Where

$$C(t) := \frac{(e^{2w_1 w_2 t} - 1)(w_1 w_2 t^2 e^{t(w_1+w_2)})^\alpha e^{w_1 w_2 t(x+y)}}{(e^{2w_1 t} - 1)^\alpha (e^{2w_2 t} - 1)^\alpha},$$

say that

$$C(t) := \left(\frac{w_1 t e^{w_1 t}}{e^{2w_1 t} - 1} \right)^\alpha e^{w_1 w_2 t x} \sum_{i=0}^{w_1-1} w_2 t e^{(1+2i)w_2 t} \left(\frac{w_2 t e^{w_2 t}}{e^{2w_2 t} - 1} \right)^{\alpha-1} e^{w_1 w_2 t y}$$

$$= \sum_{n=0}^{\infty} \widetilde{B}_n^{(\alpha)}(w_2 x) \frac{(w_1 t)^n}{n!} \sum_{s=0}^{\infty} \widetilde{P}_s(w_1) \frac{(w_2 t)^s}{s!} \sum_{n=0}^{\infty} \widetilde{B}_n^{(\alpha-1)}(w_1 y) \frac{(w_2 t)^n}{n!} \qquad (3.2)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l} \widetilde{B}_{n-l}^{(\alpha-1)}(w_1 y) \sum_{s=0}^{l} \binom{l}{s} w_1^{l-s} w_2^{n-l+s} \widetilde{P}_s(w_1) \widetilde{B}_{l-s}^{(\alpha)}(w_2 x) \right) \frac{t^n}{n!},$$

which is true. $A(t)$ get transformed in the other following equation.

$$C(t) := \left(\frac{w_2 t e^{w_2 t}}{e^{2w_2 t} - 1} \right)^\alpha e^{w_1 w_2 t x} \sum_{i=0}^{w_2-1} 2w_1 t e^{(1+2i)w_1 t} \left(\frac{w_1 t e^{w_1 t}}{e^{2w_1 t} - 1} \right)^{\alpha-1} e^{w_1 w_2 t y}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l} \widetilde{B}_{n-l}^{(\alpha-1)}(w_2 y) \sum_{s=0}^{l} \binom{l}{s} w_2^{l-s} w_1^{n-l+s} \widetilde{P}_s(w_2) \widetilde{B}_{l-s}^{(\alpha)}(w_1 x) \right) \frac{t^n}{n!}. \qquad (3.3)$$

Comparing the coefficients of $\frac{t^n}{n!}$ in (3.2) and (3.3), we get the equation as:

$$\sum_{l=0}^{n} \binom{n}{l} \widetilde{B}_{n-l}^{(\alpha-1)}(w_1 y) \sum_{s=0}^{l} \binom{l}{s} w_1^{l-s} w_2^{n-l+s} \widetilde{B}_{l-s}^{(\alpha)}(w_2 x) \widetilde{P}_s(w_1)$$

$$= \sum_{l=0}^{n} \binom{n}{l} \widetilde{B}_{n-l}^{(\alpha-1)}(w_2 y) \sum_{s=0}^{l} \binom{l}{s} w_2^{l-s} w_1^{n-l+s} \widetilde{B}_{l-s}^{(\alpha)}(w_1 x) \widetilde{P}_s(w_2).$$

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Therefore we complete the proof.

$\square$

By using Theorem 3.11, we also get symmetric property of the second kind Bernoulli polynomials of order α

Corollary 3.12. By substituting $w_1 = 1$ from Theorem 3.11, we have

$$\sum_{l=0}^{n} \binom{n}{l} \widetilde{B}_{n-l}^{(\alpha-1)}(y) \sum_{s=0}^{l} s \binom{l}{s} w_2^{n-l+s} \widetilde{B}_{l-s}^{(\alpha)}(w_2 x)$$
$$= \sum_{l=0}^{n} \binom{n}{l} \widetilde{B}_{n-l}^{(\alpha-1)}(w_2 y) \sum_{s=0}^{l} \binom{l}{s} w_2^{l-s} \widetilde{B}_{l-s}^{(\alpha)}(x) \widetilde{P}_s(w_2).$$

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An oscillation of the solution for a nonlinear second-order stochastic differential equation

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Abstract. In this paper, we study the oscillatory properties for asymptotic behaviors of solutions of a class of nonlinear second-order stochastic Itô equations. Meanwhile, we investigate existence of zeros of its solutions with probability 1. Sufficient conditions for the oscillation and nonoscillation of solutions are obtained on the half-line $[t_0, \infty)$ for every $t_0 > 0$.

Keywords: Oscillation, Stochastic differential equations, Zeros of solutions, Wiener process, Itô integral

AMS Subject Classification: 60H10; 60H25; 34K11

1   Introduction

During the past few decades, stochastic differential equations (SDEs) are becoming increasingly important as models of stochastic phenomena that play a prominent role in a diverse range of application areas, including mathematical modeling in engineering and physics, geophysical sciences, stochastic control, mechanics, environmental processes, mathematical biology, molecular dynamics for chemistry, epidemiology, economic modeling, industrial mathematics and mathematical finance [1-10]. Indeed, these models can be stochastic for different reasons. Therefore, numerous studies have been performed to understanding their dynamical behaviors, particularly in relation to problems of the specification of the stochastic processes governing the behaviors of an underlying quantity, as well as fundamental microscopic laws generate stochastic behaviors in the case of coarse-graining and modeling error and so on [11-16]. However, a complete understanding of SDEs theory requires familiarity with advanced probability and stochastic processes, whereas solutions of such models are themself stochastic processes.

Further, in particular, second-order differential equations with random coefficients have found wide variety applications in branches of science. Typically, they are mathematical models of objects under the influence of random forces such that the presence of infinite set of zeros of solutions for these equations indicates that the evolution of investigated objects is oscillatory. Recently, research work about oscillation phenomenon occupies an important place in differential stochastic theory due to the sensitivity of stochastic forces and behaviors. Moreover, the stochastic theory for these equations, as well as the theory of oscillatory solutions of deterministic equations have been studied extensively and are well-developed. Oscillation and nonoscillatition conditions for both linear and nonlinear differential equations, difference equations and delay equations have been investigated in [17-22]. The oscillating properties for solutions of difference equations can be found in the excellent monograph of Agarwal et al. [23]. Besides, the authors in the monograph [24] were devoted to the problem of relationship between oscillation behavior of solutions for differential equations and the corresponding difference equations. On the contrary, the theory of the oscillation of stochastic system is not well-developed.

Incidentally, Mao in [25] considered the stochastic equation of the following form

$$\ddot{x} + kx = h\dot{W}(t),$$

where $\dot{W}(t)$ is a Wiener process which is nowhere differentiable. It was proved that the solution with initial values $x(0) = 1$, $\dot{x}(0) = 0$ has infinitely many zeros, all simple, on each half-line $[t_0, \infty)$ for every $t_0 \geq 0$. The first two moments of the first zero were estimated.

In contrast, the more general equation of the form

$$\ddot{x} + k(t, x, \dot{x}) = h\dot{W}(t),$$

was studied in [26]. The author there demonstrated that this equation has infinitely many zeros with probability 1. Consequently, the explicit upper and lower estimates for the expected values of these zeros were obtained. However, the Itô stochastic equations of the form

$$\ddot{x} + (p(t) + q(t)\dot{W}(t))x = 0$$

was considered by method of asymptotic equivalence in [27,28], whereas the oscillation of solutions was analyzed. In the monographs [29,30], the oscillatory properties of solutions for both linear and nonlinear stochastic delay differential equations with multiplicative noise are given. It was shown that such noise induces an oscillation in solutions. Besides, the oscillation of solutions of first order nonlinear stochastic difference equations is investigated in [31].

The purpose of this paper is to study an asymptotic behavior, as $t \to \infty$, of solutions of a second order stochastic Itô equation. Meanwhile, we investigate existence of zeros of its solutions with probability 1. In the sequel, unless otherwise specified, we say that a solution is oscillatory if it has infinitely many zeros with probability 1 on the half-line $[0, \infty)$. A solution which is not oscillatory is called nonoscillatory.

This paper is organized in five sections including the introduction. In the next section, we present some necessary definitions and preliminary results that will be used in this work. In the same time, statement of a second order SDEs is introduced. In Section 3, the discussion of a solution for linear case of second-order SDEs is presented, as well as the conditions of nonoscillatory behavior of its solutions for nonlinear case of SDEs are constructed. Finally, the conclusions are drawn in Section 4.

2   Statement of the problem and auxiliary results

The material in this section is basic in some sense. For the reader's convenience, we present some necessary definitions and auxiliary results related to the SDEs theory that will be used in the remainder of this paper.

Let us consider a nonlinear second-order stochastic equation of the following form

$$\ddot{x} + p(t, x, \dot{x}) + q(t, x, \dot{x})\dot{W}(t) = 0, t \geq 0. \tag{1}$$

While the corresponding system of stochastic Itô equations will be written as

$$\begin{aligned} dx_1 &= x_2 dt, \\ dx_2 &= -p(t, x_1, x_2)dt - q(t, x_1, x_2)dW(t), \end{aligned} \tag{2}$$

where $x \in R^1, t \geq 0, W(t)$ is a standard Wiener process defined on the probability space (Ω, F, P), $\{F_t, t \geq 0\}$ is the family of σ-algebras adapted to $W(t)$, and the functions $p(t, x_1, x_2)$ and $q(t, x_1, x_2)$ are continuous with respect to $x_1, x_2 \in R^1$ for $t \geq 0$, as well as satisfy the Lipschitz condition with respect to x_1, x_2 together with linear growth condition. Without loss of generality, we assume that $p(t, 0, 0) = q(t, 0, 0) = 0$.

It should be noted that the presence of stochastic in equation (1) causes new difficulties in studying the oscillation of solutions. In this regard, we mention here the following remark: Firstly, solutions of equation (1) are random processes, so their zeros are random variables with certain properties. As a consequence, we need to introduce a new definition of zero which is different of the deterministic case $(q = 0)$. Secondly, from the Strook-Varadham support theorem, it follows that solutions of equation (1) can be nonoscillatory on finite intervals. Therefore, the oscillatory solutions should be considered only on infinite intervals. Thirdly, since solutions of equation (1) have only first derivative, so we can not use a second derivative to apply the concavity property of the solution between two successive zeros. It is well known that this method is used in the deterministic case.

Nevertheless, system (2) is a particular case of general second-order system, it would seem that this simplifies its investigation, as well it is the system with a degenerate diffusion that completes its investigation by probability methods. Subsequently, under the above assumptions of equation (1) and the corresponding system (2), we assume that the solution $\bar{x}(t) = (x_1(t), x_2(t))$ of system (2) subject to the initial condition $\bar{x}(t_0) = \bar{x}_0$ satisfy all necessary requirements of the existence of a unique solution for $t \geq t_0$, whereas $\bar{x}_0$ is an $F_{t_0}-$measurable random variable. In addition, the process $\bar{x}(t)$ will never reach the origin $(0,0)$, for more details see Lemma 2.3 in [32]. In our notation, let $x_1(t) = x(t)$. Throughout this paper, a solution $x(t)$ of equation (1) is called a nontrivial solution if it satisfy the following condition

$$P\{x(t) = 0, \ t > t_0\} = 0$$

On the other hand, for any nontrivial solution $x(t)$ of equation (1), where $t \geq t_0 \geq 0$, the random variable τ_1 can be defined as follows

$$\tau_1 = \begin{cases} \inf\{t > t_0 \mid x_1(t) = 0\}, & \text{if } \{t > t_0 \mid x_1(t) = 0\} \neq \varnothing, \\ \infty, & \text{otherwise.} \end{cases} \tag{3}$$

Now, we will introduce the definition of zeros of a solution $x(t)$ on the half-line $t > 0$.

Definition 2.1 The random variable τ_1 is called the first zero of a solution $x(t)$ on the interval $t \geq t_0$, if $\tau_1 < \infty$ with probability 1.

In consequence, one can define another random variable τ_2 as follows

$$\tau_2 = \begin{cases} \inf\{t > \tau_1 \mid x_1(t) = 0\}, & \text{if } \{t > \tau_1 \mid x_1(t) = 0\} \neq \varnothing, \\ \infty, & \text{otherwise.} \end{cases} \tag{4}$$

Here, the random variable τ_2 is called the second zero of a solution $x(t)$ on the interval $t \geq t_0$, if $\tau_2 < \infty$ with probability 1.

Correspondingly, one can define by induction a sequence of zeros $\{\tau_n\}$ of a solution $x(t)$ on the interval $t \geq t_0$. Particularly, if $t_0 = 0$. Then, we deal with zeros on the half-line $t > 0$.

Definition 2.2 A nontrivial solution $x(t)$ of equation (1) is called oscillatory on the half-line $t > 0$, if it has infinitely many zeros there. Otherwise, it is called nonoscillatory.

3 Main results and behavior solutions of the SDEs

In this section, some definitions and results are briefly reviewed to establish and generalize the results to the main equation in this work. Meanwhile, we study the behavior of the zeros of solutions for a class of second-order SDEs subject to some initial conditions, as well as we detect the conditions of nonoscillatory behavior of its solutions.

3.1 Linear stochastic Itô equation

Consider the following equation

$$\ddot{x} + x = f(t)\dot{W}(t), \tag{5}$$

subject to the initial conditions

$$x(0) = 1, \dot{x}(0) = 0, \tag{6}$$

where $f(t)$ is a nonrandom function defined on $t \geq 0$ such that a stochastic Itô integral

$$\int_0^t f(s)dW(s)$$

is defined for any $t > 0$.

Note that equation (5) is special case of equation (1), so it satisfies all arguments mentioned in the previous part of our work. Further, we give the following theorem regarding to study the behavior of the zeros of solutions for equation (5) with initial conditions (6).

Theorem 3.1 Assume that $f(t)$ satisfies the following conditions:

1. $f(t)$ is differentiable function for $t \geq 0$ such that $f(0) \geq 0$,

2. $(\sin(t - s)f(s))' \leq 0$ for $0 \leq s \leq t \leq \frac{\pi}{2}$.

Then, the solution of equation (5) subject to initial conditions (6) oscillates on the half-line $t \geq 0$. Besides, a mathematical expectation τ_1 of the first zero satisfies the estimation

$$E\tau_1 \geq 2t^\star \Phi(\frac{1}{\sqrt{t^\star}}), \tag{7}$$

where t^* is the solution of the equation

$$f(0) = \cot(t) \tag{8}$$

on $[0, \frac{\pi}{2}]$ and $\Phi(z) = \frac{1}{2\pi} \int_0^z e^{\frac{-u^2}{2}} du$.

Proof. From Itô formula, the representation of the solution of (5) with initial conditions (6) is given by

$$x(t) = \cos(t) + \int_0^t f(s)\sin(t - s)dW(s). \tag{9}$$

Which implies that

$$x(t) = \cos(t) + \sin(t) \int_0^t f(s)\cos(s)dW(s) - \cos(t) \int_0^t f(s)\sin(s)dW(s).$$

Accordingly, the process $x(t)$ can be written as

$$x(t) = \cos(t) + \bar{W}_1(p(t))\sin(t) + \bar{W}_2(q(t))\cos(t),$$

where

$$p(t) = \int_0^t f^2(s)\cos^2(s)ds, \quad q(t) = \int_0^t f^2(s)\sin^2(s)ds,$$

and $\bar{W}_1$, $\bar{W}_2$ are Wiener processes.

In contrast, if we consider $x(t)$ at the times $t_m = (2m + \frac{1}{2})\pi$ for $m = 1, 2, 3, ...$, and define a sequence $\{Y_m\}$ by $Y_m = x((2m + \frac{1}{2})\pi) - x((2(m - 1) + \frac{1}{2})\pi)$, whereas $x((2m + \frac{1}{2})\pi) = \bar{W}_1 \left(\int_0^{(2m+\frac{1}{2})\pi} \cos^2(s)f^2(s)ds \right)$.

Then, $Y_0 = \bar{W}(\frac{\pi}{4})$, $Y_1 = \bar{W}_1(\frac{5\pi}{4}) - \bar{W}_1(\frac{\pi}{4}), ...$, is a sequence of random variables with mean zero and variance

$$\int\limits_{(2(m-1)+\frac{1}{2})\pi}^{(2m+\frac{1}{2})\pi} \cos^2(s)f^2(s)ds.$$

Here, it is worth to mention that

$$x((2m + \frac{1}{2})\pi) = Y_0 + Y_1 + ... + Y_m. \tag{10}$$

By the familiar theorems on the limits of sums of independent random variables (e.g. the law of the iterated logarithm), it follows that the sequence $\{x((2m+\frac{1}{2})\pi)\}$ has infinitely many switches of sign. Since $x(t)$ is continuous on $[0, \infty)$, so it has infinitely many zeros on $[0, \infty)$. Therefore, it oscillates on $[0, \infty)$.

Now, let us prove the estimation (7) for the first zero of the oscillation. By applying the integration-by-parts formula to (9), we obtain

$$x(t) = \cos(t) - \int\limits_0^t (\sin(t-s)f(s))'dW(s) \geq \cos(t) + \int\limits_0^t (\sin(t-s)f(s))'ds, \tag{11}$$

for $\omega \in \Omega$, where $W(t) \geq -1$ and $0 \leq s \leq t \leq \frac{\pi}{2}$.

From properties of a Wiener process, it follows that

$$P\left\{\omega \mid \max_{t \in [0,T]} W(t) > -1\right\} = 2\Phi(\frac{1}{2\sqrt{T}}), \tag{12}$$

where $\Phi(z) = \frac{1}{2\sqrt{2\pi}} \int\limits_0^z e^{\frac{-u^2}{2}} du$.

As a result, from equation (11), we obtain the estimate

$$x(t) \geq \cos(t) - f(0)\sin(t) > 0, \tag{13}$$

for $t \in [0, t^*)$, where t^* is solution of equation (8).

Hence, from equations (12) and (13), we have

$$P\left\{\tau \geq \cot^{-1}(f(0))\right\} \geq 2\Phi\left(\frac{1}{\sqrt{\cot^{-1}(f(0))}}\right), \tag{14}$$

for the first zero τ_1. By using Chebyshev's inequality, it yields that $E\tau_1 \geq 2t^*\Phi(\frac{1}{\sqrt{t^*}})$. The proof is complete. ■

3.2 Nonlinear stochastic Itô equation

Let $P(a, n) = \{x \in R^m \mid (x - a, n) \geq 0\}$, where $a, n \in R^m$, and $(\cdot, \cdot)$ is the usual scalar product. Thus, a polyhedron is any set of the form

$$\bigcap_{\alpha \in I} P(a_\alpha, n_\alpha), \tag{15}$$

where $I = \{1, ..., N\}$ is a finite subset of N.

Now, consider a system of SDEs

$$dx = f(t, x)dt + g(t, x)dW(t) \tag{16}$$

where $x \in R^m$, $f : [0, \infty) \times R^m \to R^m$, $g = [g_{ij}] : [0, \infty] \times R^m \to R^m \times R^r$ are mappings, and $W(t)$ is an $r-$dimensional Wiener process.

Definition 3.1 A set $K \in R^m$ is said to be stochastically invariant for system (16), if for any $x(0) \in K$ and every solution $x(t)$ of equation (1), then $P\{x(t) \in K, t > 0\} = 1$.

The next theorem states conditions of an invariance of the set (15) for system (16).

Theorem 3.2 [33] Let $K = \bigcap_{\alpha \in I} P(a_\alpha, n_\alpha)$ be a polyhedron in R^m. Suppose that the coefficients $f(t, x)$ and $g(t, x)$ of system (16) are defined for $t \geq 0$, $x \in R^m$, and satisfy the following conditions:

1. for each $T > 0$, there exists a constant $Kt_T > 0$ such that for all $x \in K$ and $t \in [0, T)$,

$$\|f(t, x)\|^2 + \|g(t, x)\|^2 \leq K_T(1 + |x|^2);$$

2. for all $T > 0$, $x \in K$, $y \in K$ and $t \in [0, T)$,

$$\|f(t, x) - f(t, y)\| + \|g(t, x) - g(t, y)\| \leq K_T |x - y|;$$

3. for each $x \in K$, the functions $f(\cdot, x)$ and $g(\cdot, x)$, defined for $t \geq 0$, are continuous.

Then, the set K is invariant for the system (16) if and only if the following condition holds:

(a) for all $\alpha \in I$ and $x \in K$ such that $(x - a_\alpha, n_\alpha) = 0$, we have

$$(f(t, x), n_\alpha) \geq 0 \text{ and } (g_j(t, x), n_\alpha) = 0,$$

where $t \geq 0$, $j = \overline{1, r}$, and g_j is the $j - th$ column of the matrix $g = [g_{ij}]$.

Now, we use the above theorem to find the conditions of nonoscillatory behavior of the solutions of equation (1). As well, we state the following theorem:

Theorem 3.3 Suppose that the functions p and q in equation (1) satisfy the conditions (1)-(3) of Theorem 3.2. Moreover, if

(1) $p(t, x_1, 0) \geq 0$, $x_1 < 0$, $t \geq 0$;

(2) $p(t, x_1, 0) \leq 0$, $x_1 > 0$, $t \geq 0$; $\qquad\qquad$ (17)

(3) $q(t, x_1, 0) = 0$, $t \geq 0, x_1 \in R^1$.

Then, all solutions of equation (1) with nonrandom initial values such that $x(0) > 0$, $\dot{x}(0) \geq 0$ or $x(0) < 0$, $\dot{x}(0) \leq 0$ are not oscillate on the half-line $[0, \infty)$.

Proof. We consider any solution of equation (1) with initial values $x(0) > 0$, $\dot{x}(0) \geq 0$. It corresponds to the solution (x_1, x_2) of system (2) with initial values $x_1(0) > 0$, $x_2(0) \geq 0$. Obviously, there exists $\epsilon > 0$ such that $0 < \epsilon \leq x_1(0)$.

Let M be a set such that $M = \{(x_1, x_2) \mid x_1 \geq \epsilon, \ x_2 \geq 0\}$. It is a polyhedron, if we set $a_1 = \epsilon$, $n_1 = l_1 = (1, 0)^T$, $a_2 = 0$, $n_2 = l_2 = (0, 1)^T$. Then, $M = \bigcap_{\alpha \in I} P(a_\alpha, n_\alpha)$, where $I = \{1, 2\}$. Consequently, the boundaries of this polyhedron are lines

$$\gamma_1 = \{(x_1, x_2) \mid x_1 = \epsilon, \ x_2 \geq 0\},$$

$$\gamma_2 = \{(x_1, x_2) \mid x_1 \geq \epsilon, \ x_2 = 0\}.$$

$\qquad\qquad$

Therefore, by using Theorem 3.2, the functions f and q have the form

$$f(t, x_1, x_2) = (x_2, -p(t, x_1, x_2))^T,$$

$$g(t, x_1, x_2) = (0, -q(t, x_1, x_2))^T,$$

Next, we verify the conditions of Theorem 3.2. On the boundary γ_1, we have

$$(f, n_1) = (f, l_1) = x_2 \geq 0, \text{ and } (g, n_1) = (g, l_1) = 0.$$

From condition (3) of equation (17), we have

$$(f, n_2) = (f, l_2) = -p(t, x_1, 0) \geq 0,$$

and

$$(g, n_2) = (g, l_2) = -q(t, x_1, 0) = 0.$$

on the boundary γ_2.

Again from Theorem 3.2, it follows that the set M is the invariant set for the solutions of system (2). Thus, the curve $(x_1(t), x_2(t))$ does not intersect with probability 1 the line $x_1 = 0$. This means that the solutions of equation (1) with initial values $x(0) > 0$, $\dot{x}(0) \geq 0$ do not oscillate.

It remains to consider the case with initial values $x(0) < 0$, $\dot{x}(0) \leq 0$. We only introduce the polyhedron $M_1 = \{(x_1, x_2) \mid x_1 \leq -\epsilon, \ x_2 \leq 0\}$ instead the set M. Hence, the proof is complete. $\blacksquare$

4   Concluding remarks

The use of SDEs is a natural way to model real-world phenomena under stochastic processes. In this paper, we study the qualitative behavior of nonlinear second order stochastic differential equations. Interest focuses on solutions of such equations which are oscillatory. A nontrivial solution is called oscillatory if it has infinitely many zeros with probability 1 on half-line. Otherwise, it is called nonoscillatory. The sufficient conditions for the oscillation and nonoscillation of solutions are obtained.

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Fixed point theorems and T-stability of Picard iteration for generalized Lipschitz mappings in cone metric spaces over Banach algebras

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Abstract: In this paper, we obtain the existence of non-normal solid cone and some fixed point theorems for generalized Lipschitz contractive mappings in cone metric spaces over Banach algebras. Our results greatly generalize the main work by Xu and Radenović (Fixed Point Theory and Applications, 2014, 2014: 102). Moreover, we verify the P property and T-stability of Picard's iteration. Further, we give an example to illustrate that our works are never a copy of metric results in the literature.

1 Introduction

Since Huang and Zhang [1] introduced the concept of cone metric space, many scholars have focused on fixed point theorems in such spaces. There are lots of works on fixed point results in the setting of cone metric spaces (see [2-6]). It is said that [1] is well-known as a result of the fact that cone metric spaces generalize metric spaces and expands the famous Banach contraction principle. But recently, it had not yet been a hot topic since some authors appealed to the equivalence of some metric and cone metric fixed point results

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(see [7-12]). Owing to these reasons, people set out to lose interest in studying fixed point theorems in cone metric spaces. However, the present situation has gone better since, very recently, Liu and Xu [13] introduced the concept of cone metric space over Banach algebra and obtained some fixed point theorems in normal cone metric spaces over Banach algebras. Moreover, they gave an example to illustrate that the non-equivalence of versions of fixed point theorems between cone metric spaces over Banach algebras and (general) metric spaces (in usual sense), which shows that it is essentially necessary to investigate fixed points in cone metric spaces over Banach algebras. Lately, Xu and Radenović [15] delete the normality of cones and greatly generalize the main results of [13]. Throughout this paper, we obtain the existence of non-normal solid cone for generalized Lipschitz mappings in cone metric spaces over Banach algebras. Moreover, we present some fixed point theorems for such mappings in such setting by omitting the assumptions of normalities of cones. Our theorems include the main results of [13] and [15]. Furthermore, we consider the mapping's P property and T-stability of Picard's iteration. Our results greatly unite and extend the main work of [13-15] and [17-21]. In addition, we give an example to illustrate our results in cone metric spaces over Banach algebras are never equivalent to the counterpart of metric spaces.

Let $\mathcal{A}$ be a Banach algebra with a unit e, and θ the zero element of $\mathcal{A}$. A nonempty closed convex subset K of $\mathcal{A}$ is called a cone if $\{\theta, e\} \subset K$, $K^2 = KK \subset K$, $K \cap (-K) = \{\theta\}$ and $\lambda K + \mu K \subset K$ for all $\lambda, \mu \geq 0$. On this basis, we define a partial ordering $\preceq$ with respect to K by $x \preceq y$ if and only if $y - x \in K$. We shall write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$, while $x \ll y$ will indicate that $y - x \in \text{int} K$, where $\text{int} K$ stands for the interior of K. If $\text{int} K \neq \emptyset$, then K is said to be a solid cone. Write $\|\cdot\|$ as the norm on $\mathcal{A}$. A cone K is called normal if there is a number $M > 0$ such that for all $x, y \in \mathcal{A}$, $\theta \preceq x \preceq y$ implies $\|x\| \leq M\|y\|$. The least positive number satisfying above is called the normal constant of K.

In the sequel we always suppose that $\mathcal{A}$ is a Banach algebra with a unit e, K is a solid cone in $\mathcal{A}$, and $\preceq$ is a partial ordering with respect to K.

Definition 1.1.([13]) Let X be a nonempty set and $\mathcal{A}$ a Banach algebra. Suppose that the mapping $d : X \times X \to \mathcal{A}$ satisfies:

(i) $\theta \prec d(x, y)$ for all $x, y \in X$ with $x \neq y$ and $d(x, y) = \theta$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

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(iii) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space over Banach algebra.

Definition 1.2.([15])   Let (X, d) be a cone metric space, $x \in X$ and $\{x_n\}$ a sequence in X. Then

(i) $\{x_n\}$ converges to x whenever for every $c \gg \theta$ there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$. We denote this by $\lim_{n \to \infty} x_n = x$ or $x_n \to x$ $(n \to \infty)$.

(ii) $\{x_n\}$ is a Cauchy sequence whenever for each $c \gg \theta$ there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.

(iii) (X, d) is a complete cone metric space if every Cauchy sequence is convergent.

Definition 1.3.   Let (X, d) be a cone metric space, $\{y_n\}$ a sequence in X and T a self-map of X. Let x_0 be a point of X, $x_{n+1} = Tx_n$ a Picard's iteration in X. The iteration procedure $x_{n+1} = Tx_n$ is said to be T-stable with respect to T if $\{x_n\}$ converges to a fixed point q of T, and for each $c \gg \theta$, there exists a natural number N such that $d(y_{n+1}, Ty_n) \ll c$ for all $n > N$, then $\lim_{n \to \infty} y_n = q$.

Remark 1.4.   Comparing Definition 2.1 of [18] and Definition 1.3, we find that, the conditions of the former are stronger than the latter. Actually, if $\lim_{n \to \infty} d(y_{n+1}, Ty_n) = \theta$, then we must have that for each $c \gg \theta$ there exists an integer $N > 0$ such that $d(y_{n+1}, Ty_n) \ll c$ for all $n > N$. But the contrary is not true (see [6]).

Lemma 1.5.([6])   Let $u, v, w \in \mathcal{A}$. If $u \preceq v$ and $v \ll w$, then $u \ll w$.

Lemma 1.6.([6])   Let $\mathcal{A}$ be a Banach algebra and $\{a_n\}$ a sequence in $\mathcal{A}$. If $a_n \to \theta$ $(n \to \infty)$, then for any $c \gg \theta$, there exists N such that for all $n > N$, one has $a_n \ll c$.

Lemma 1.7.([16])   Let $\mathcal{A}$ be a Banach algebra with a unit e, $x \in \mathcal{A}$, then the limit $\lim_{n \to \infty} \|x^n\|^{\frac{1}{n}}$ exists and the spectral radius $\rho(x)$ satisfies

$$\rho(x) = \lim_{n \to \infty} \|x^n\|^{\frac{1}{n}} = \inf \|x^n\|^{\frac{1}{n}}.$$

If $\rho(x) < |\lambda|$, then $\lambda e - x$ is invertible in $\mathcal{A}$, moreover,

$$(\lambda e - x)^{-1} = \sum_{i=0}^{\infty} \frac{x^i}{\lambda^{i+1}},$$

where λ is a complex constant.

Lemma 1.8.([16])   Let $\mathcal{A}$ be a Banach algebra with a unit e, $a, b \in \mathcal{A}$. If a commutes with b, then

$$\rho(a + b) \leq \rho(a) + \rho(b), \qquad \rho(ab) \leq \rho(a)\rho(b).$$

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2 Main results

In this section we give some basic but important properties, which will be used constantly in the sequel. Moreover, we introduce a class of contractive mappings with some generalized Lipschitz constants and prove the existence of non-normal solid cone and several fixed point theorems based on them without the assumption of normalities of cones. In addition, we obtain the fixed point periodic property and T-stability of Picard's iteration. All results greatly generalize the main assertions of [13-15] and [17-21]. Further, we display an example to illustrate the applications. In the end, we give another example to claim that our results in the setting of cone metric spaces over Banach algebras are never equivalent to those in usual metric spaces.

Lemma 2.1. Let $\mathcal{A}$ be a Banach algebra and $k \in \mathcal{A}$. If $\rho(k) < 1$, then $\lim\limits_{n \to \infty} \|k^n\| = 0$.

Proof. Since $\rho(k) = \lim\limits_{n \to \infty} \|k^n\|^{\frac{1}{n}} < 1$, then there exists $\alpha > 0$ such that $\lim\limits_{n \to \infty} \|k^n\|^{\frac{1}{n}} < \alpha < 1$. Letting n be big enough, we obtain $\|k^n\|^{\frac{1}{n}} \le \alpha$, then $\|k^n\| \le \alpha^n \to 0$ $(n \to \infty)$. Hence $\|k^n\| \to 0$ $(n \to \infty)$. $\qquad \square$

Lemma 2.2. Let $\mathcal{A}$ be a Banach algebra with a unit e, $\{x_n\}$ a sequence in $\mathcal{A}$. If x_n converges to x in $\mathcal{A}$, and for any $n \ge 1$, x_n commutes with x, then $\rho(x_n) \to \rho(x)$ as $n \to \infty$.

Proof. Since x_n commutes with x, then it follows by Lemma 1.8 that

$$\rho(x_n) \le \rho(x_n - x) + \rho(x) \Rightarrow \rho(x_n) - \rho(x) \le \rho(x_n - x),$$
$$\rho(x) \le \rho(x - x_n) + \rho(x_n) \Rightarrow \rho(x) - \rho(x_n) \le \rho(x - x_n),$$

thus

$$|\rho(x_n) - \rho(x)| \le \rho(x_n - x) \le \|x_n - x\| \Rightarrow \rho(x_n) \to \rho(x)(n \to \infty).$$

$\qquad \square$

Lemma 2.3. Let $\mathcal{A}$ be a Banach algebra with a unit e and K be a solid cone in $\mathcal{A}$. Let $\{a_n\}$ and $\{c_n\}$ be two sequences in K satisfying the following inequality:

$$a_{n+1} \preceq ha_n + c_n, \tag{2.1}$$

where $h \in K$ and $\rho(h) < 1$. If for each $c \gg \theta$, there exists N such that $c_n \ll c$ for all $n > N$, then $a_n \ll c$ $(n > N)$.

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Proof. By virtue of $\rho(h) < 1$, it follows by Lemma 1.7, $e - h$ is invertible and $(e-h)^{-1} = \sum_{i=0}^{\infty} h^i$. Moreover, by Lemma 2.1, it establishes $\|h^n\| \to 0$ $(n \to \infty)$. Assume $c \gg \theta$ be arbitrary. Then there exists N_1 such that for all $n > N_1$, we have

$$c_n \ll \frac{(e-h)c}{2}. \tag{2.2}$$

Since

$$\|h^{n-N_1} a_{N_1+1}\| \leq \|h^{n-N_1}\| \|a_{N_1+1}\| \to 0 \quad (n \to \infty),$$

thus there is N_2 such that for all $n > N_2$, it satisfies

$$h^{n-N_1} a_{N_1+1} \ll \frac{c}{2}. \tag{2.3}$$

Put $N = \max\{N_1, N_2\}$, then for all $n > N$, both (2.2) and (2.3) are satisfied. Taking advantage of (2.1), we speculate that

$$a_{n+1} - ha_n \preceq c_n,$$
$$ha_n - h^2 a_{n-1} \preceq hc_{n-1},$$
$$h^2 a_{n-1} - h^3 a_{n-2} \preceq h^2 c_{n-2},$$
$$\cdots\cdots\cdots$$
$$h^{n-N_1-1} a_{N_1+2} - h^{n-N_1} a_{N_1+1} \preceq h^{n-N_1-1} c_{N_1+1}.$$

Conbine with the above terms, for all $n > N$, it follows that

$$a_{n+1} \preceq h^{n-N_1} a_{N_1+1} + c_n + hc_{n-1} + h^2 c_{n-2} + \cdots + h^{n-N_1-1} c_{N_1+1}$$
$$\ll \frac{c}{2} + (e + h + h^2 + \cdots + h^{n-N_1-1}) \cdot \frac{(e-h)c}{2}$$
$$\preceq \frac{c}{2} + (e-h)^{-1} \cdot \frac{(e-h)c}{2} = c.$$

$\square$

Remark 2.4. Lemma 2.3 greatly generalizes Lemma 1 of [17] and Lemma 1.5 of [18]. Virtually, we delete the normality of K. Moreover, our conditions are weaker than them. Indeed, if $a_n \to \theta$ $(n \to \infty)$, then $a_n \ll c$ $(n > N)$. But the converse is not true (see [6]). Further, if $\|k\| < 1$, it is natural that $\rho(k) < 1$. Yet, the converse is not true.

Theorem 2.5. Let (X, d) be a cone metric space over Banach algebra $\mathcal{A}$ and K be a solid cone in $\mathcal{A}$. Suppose that the mapping $T : X \to X$ satisfies the following contractive condition:

$$d(Tx, Ty) \preceq k_1 d(x, y) + k_2 d(x, Tx) + k_3 d(y, Ty) + k_4 d(x, Ty) + k_5 d(y, Tx),$$

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for all $x, y \in X$, where $k_i \in K (i = 1, \ldots, 5)$ are generalized Lipschitz constants with $\rho(k_1) + \rho(k_2 + k_3 + k_4 + k_5) < 1$. If k_1 commutes with $k_2 + k_3 + k_4 + k_5$, then there exists a sequence $\{x_n\}$ in X such that it is a Cauchy sequence. Moreover, if $\{d(x_n, y_n)\}$ converges to some non-zero element in $\mathcal{A}$ for any two different Cauchy sequence $\{x_n\}$ and $\{y_n\}$, then K is a non-normal cone.

Proof. Fix $x_0 \in X$ and set $x_{n+1} = Tx_n = T^{n+1}x_0$. Then we have

$$
\begin{aligned}
d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\
&\preceq k_1 d(x_n, x_{n-1}) + k_2 d(x_n, Tx_n) + k_3 d(x_{n-1}, Tx_{n-1}) \\
&\quad + k_4 d(x_n, Tx_{n-1}) + k_5 d(x_{n-1}, Tx_n) \\
&= k_1 d(x_n, x_{n-1}) + k_2 d(x_n, x_{n+1}) + k_3 d(x_{n-1}, x_n) \\
&\quad + k_5 d(x_{n-1}, x_{n+1}) \\
&\preceq (k_1 + k_3 + k_5) d(x_n, x_{n-1}) + (k_2 + k_5) d(x_{n+1}, x_n). \qquad (2.4)
\end{aligned}
$$

We also have

$$
\begin{aligned}
d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) = d(Tx_{n-1}, Tx_n) \\
&\preceq k_1 d(x_{n-1}, x_n) + k_2 d(x_{n-1}, Tx_{n-1}) + k_3 d(x_n, Tx_n) \\
&\quad + k_4 d(x_{n-1}, Tx_n) + k_5 d(x_n, Tx_{n-1}) \\
&= k_1 d(x_n, x_{n-1}) + k_2 d(x_{n-1}, x_n) + k_3 d(x_n, x_{n+1}) \\
&\quad + k_4 d(x_{n-1}, x_{n+1}) \\
&\preceq (k_1 + k_2 + k_4) d(x_n, x_{n-1}) + (k_3 + k_4) d(x_{n+1}, x_n). \qquad (2.5)
\end{aligned}
$$

Add up (2.4) and (2.5) yields that

$$
\begin{aligned}
2d(x_{n+1}, x_n) &\preceq (2k_1 + k_2 + k_3 + k_4 + k_5) d(x_n, x_{n-1}) \\
&\quad + (k_2 + k_3 + k_4 + k_5) d(x_{n+1}, x_n),
\end{aligned}
$$

which establishes that

$$
(2e - k_2 - k_3 - k_4 - k_5) d(x_{n+1}, x_n) \preceq (2k_1 + k_2 + k_3 + k_4 + k_5) d(x_n, x_{n-1}).
$$

Put $k = k_2 + k_3 + k_4 + k_5$, then

$$
(2e - k) d(x_{n+1}, x_n) \preceq (2k_1 + k) d(x_n, x_{n-1}). \qquad (2.6)
$$

6

Since $\rho(k) \leq \rho(k_1) + \rho(k) < 1 < 2$, then by Lemma 1.7 it follows that $2e - k$ is invertible. Furthermore,

$$(2e - k)^{-1} = \sum_{i=0}^{\infty} \frac{k^i}{2^{i+1}}.$$

By multiplying in both sides of (2.6) by $(2e - k)^{-1}$, we arrive at

$$d(x_{n+1}, x_n) \preceq (2e - k)^{-1}(2k_1 + k)d(x_n, x_{n-1}). \tag{2.7}$$

Denote $h = (2e - k)^{-1}(2k_1 + k)$, then by (2.7) we get

$$d(x_{n+1}, x_n) \preceq hd(x_n, x_{n-1}) \preceq \cdots \preceq h^n d(x_1, x_0).$$

By Lemma 1.8 we conclude that

$$\rho\Big(\sum_{i=0}^{n} \frac{k^i}{2^{i+1}} \Big) \leqslant \sum_{i=0}^{n} \rho\Big(\frac{k^i}{2^{i+1}} \Big) \leqslant \sum_{i=0}^{n} \frac{[\rho(k)]^i}{2^{i+1}},$$

which implies by Lemma 2.2 that

$$\rho\Big(\sum_{i=0}^{\infty} \frac{k^i}{2^{i+1}} \Big) \leqslant \sum_{i=0}^{\infty} \frac{[\rho(k)]^i}{2^{i+1}}.$$

Since k_1 commutes with k, it follows that

$$(2e - k)^{-1}(2k_1 + k) = \Big(\sum_{i=0}^{\infty} \frac{k^i}{2^{i+1}} \Big)(2k_1 + k) = 2\Big(\sum_{i=0}^{\infty} \frac{k^i}{2^{i+1}} \Big)k_1 + \sum_{i=0}^{\infty} \frac{k^{i+1}}{2^{i+1}}$$

$$= 2k_1\Big(\sum_{i=0}^{\infty} \frac{k^i}{2^{i+1}} \Big) + k\sum_{i=0}^{\infty} \frac{k^i}{2^{i+1}} = (2k_1 + k)\Big(\sum_{i=0}^{\infty} \frac{k^i}{2^{i+1}} \Big) = (2k_1 + k)(2e - k)^{-1},$$

that is to say, $(2e - k)^{-1}$ commutes with $2k_1 + k$. Then by Lemma 1.8 we gain

$$\rho(h) = \rho\big((2e - k)^{-1}(2k_1 + k)\big) \leq \rho\big((2e - k)^{-1}\big)\rho(2k_1 + k)$$

$$\leq \rho\Big(\sum_{i=0}^{\infty} \frac{k^i}{2^{i+1}} \Big)[2\rho(k_1) + \rho(k)] \leq \Big(\sum_{i=0}^{\infty} \frac{[\rho(k)]^i}{2^{i+1}} \Big)[2\rho(k_1) + \rho(k)]$$

$$= \frac{1}{2 - \rho(k)}[2\rho(k_1) + \rho(k)] < 1,$$

which establishes that $e - h$ is invertible and $\|h^m\| \to 0$ $(m \to \infty)$. Thus for all $n > m$,

$$d(x_n, x_m) \preceq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m)$$

$$\preceq (h^{n-1} + h^{n-2} + \cdots + h^m)d(x_1, x_0)$$

$$= (h^{n-m-1} + h^{n-m-2} + \cdots + h + e)h^m d(x_1, x_0)$$

$$\preceq \Big(\sum_{i=0}^{\infty} h^i \Big)h^m d(x_1, x_0)$$

$$= (e - h)^{-1}h^m d(x_1, x_0).$$

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Owing to

$$\|(e-h)^{-1}h^m d(x_1, x_0)\| \leq \|(e-h)^{-1}\|\|h^m\|\|d(x_1, x_0)\| \to 0 \ (m \to \infty),$$

we have $(e-h)^{-1}h^m d(x_1, x_0) \to \theta \ (m \to \infty)$. So by using Lemma 1.5 and 1.6, we easily see that $\{x_n\}$ is a Cauchy sequence in X.

Now we take $y_0 \in X$ such that $y_0 \neq x_0$. Using the same method as the above mentioned, we can show that $\{y_n\}$ is also a Cauchy sequence if $y_{n+1} = Ty_n = T^{n+1}y_0$. In the following we suppose the contrary, that is, K is normal. We shall prove that $\{d(x_n, y_n)\}$ is convergent in $(\mathcal{A}, \|\cdot\|)$ if K is a normal cone with normal constant M. In fact, in view of the completeness of $(\mathcal{A}, \|\cdot\|)$, it will be enough to show that the sequence $\{d(x_n, y_n)\}$ is a Cauchy sequence. To this end, let $\varepsilon > 0$ and choose $c \gg \theta$ and $\|c\| < \frac{\varepsilon}{4M+2}$. Since $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences, there is N such that $d(x_n, x_m) \ll c$ and $d(y_n, y_m) \ll c$ for all $n, m > N$. It is clear that

$$d(x_n, y_n) \preceq d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n) \preceq d(x_m, y_m) + 2c, \tag{2.8}$$

$$d(x_m, y_m) \preceq d(x_m, x_n) + d(x_n, y_n) + d(y_n, y_m) \preceq d(x_n, y_n) + 2c. \tag{2.9}$$

It follows immediately from (2.8) and (2.9) that

$$\theta \preceq d(x_m, y_m) + 2c - d(x_n, y_n) \preceq d(x_n, y_n) + 2c + 2c - d(x_n, y_n) = 4c. \tag{2.10}$$

By virtue of the normality of K, (2.10) means that

$$\|d(x_m, y_m) + 2c - d(x_n, y_n)\| \leq 4M\|c\|.$$

Hence, it ensures us that

$$\|d(x_m, y_m) - d(x_n, y_n)\| \leq \|d(x_m, y_m) + 2c - d(x_n, y_n)\| + \|2c\| \leq (4M+2)\|c\| < \varepsilon,$$

which implies that $\{d(x_n, y_n)\}$ is Cauchy and hence convergent.

Next, put $\lim\limits_{n\to\infty} d(x_n, y_n) = a$, it is evident that $\theta \preceq a$. Finally, we claim that $a = \theta$. Actually, if there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = y_{n_0}$, the claim is clear. Without loss of

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generality, we suppose that $x_n \neq y_n$ for all $n \in \mathbb{N}$. Notice that

$$
\begin{aligned}
d(x_{n+1}, y_{n+1}) &= d(Tx_n, Ty_n) \\
&\preceq k_1 d(x_n, y_n) + k_2 d(x_n, Tx_n) + k_3 d(y_n, Ty_n) \\
&\quad + k_4 d(x_n, Ty_n) + k_5 d(y_n, Tx_n) \\
&= k_1 d(x_n, y_n) + k_2 d(x_n, x_{n+1}) + k_3 d(y_n, y_{n+1}) \\
&\quad + k_4 d(x_n, y_{n+1}) + k_5 d(y_n, x_{n+1}) \\
&\preceq (k_1 + k_4 + k_5) d(x_n, y_n) + (k_2 + k_5) d(x_n, x_{n+1}) + (k_3 + k_4) d(y_n, y_{n+1}).
\end{aligned}
$$

Taking the limit as $n \to \infty$, we obtain that

$$
a \preceq (k_1 + k_4 + k_5) a.
$$

Set $\lambda = k_1 + k_4 + k_5$, then it follows that

$$
a \preceq \lambda a \preceq \cdots \preceq \lambda^n a.
$$

Because $\lambda \preceq k_1 + k$ leads to $\lambda^n \preceq (k_1 + k)^n$, moreover, by Lemma 2.1, $\rho(k_1 + k) \leq \rho(k_1) + \rho(k) < 1$ leads to $(k_1 + k)^n \to \theta \ (n \to \infty)$, we claim that, for each $c \gg \theta$, there exists $n_0(c)$ such that $\lambda^n \ll c$ such that for all $n > n_0(c)$. Consequently, $a = \theta$, a contradiction. $\qquad \square$

It is clear that if T is a map which has a fixed point u, then u is also a fixed point of T^n for each $n \in \mathbb{N}$. It is well known that the converse is not true. If a map T satisfies $F(T) = F(T^n)$ for each $n \in \mathbb{N}$, where $F(T)$ stands for the set of all fixed points of T, then it is said to have a property P (see [19-21]). The following results are generalizations of the corresponding results in metric and cone metric spaces (see [20-21]). It will be deduced also without using normality of the cone.

Theorem 2.6. Let (X, d) be a complete cone metric space over Banach algebra $\mathcal{A}$ and K be a solid cone in $\mathcal{A}$. Let $T : X \to X$ be a mapping such that $F(T) \neq \emptyset$ and that

$$
d(Tx, T^2 x) \preceq k d(x, Tx) \tag{2.11}
$$

for all $x \in X$, where $k \in K$ is a generalized Lipschitz constant with $\rho(k) < 1$. Then T has a property P.

Proof. We will always assume that $n > 1$, since the statement for $n = 1$ is trivial. Let $z \in F(T^n)$. By the hypotheses, it is clear that

$$d(z, Tz) = d(TT^{n-1}z, T^2T^{n-1}z) \preceq kd(T^{n-1}z, T^n z) = kd(TT^{n-2}z, T^2T^{n-2}z)$$

$$\preceq k^2 d(T^{n-2}z, T^{n-1}z) \preceq \cdots \preceq k^n d(z, Tz).$$

On account of $\rho(k) < 1$, it follows by Lemma 2.1 that $\|k^n\| \to 0$ $(n \to \infty)$. Thus $\|k^n d(z, Tz)\| \leq \|k^n\| \|d(z, Tz)\| \to 0$ $(n \to \infty)$. Hence $d(z, Tz) = \theta$, that is., $Tz = z$. □

Theorem 2.7. Let (X, d) be a complete cone metric space over Banach algebra $\mathcal{A}$ and K be a solid cone in $\mathcal{A}$. Suppose that the mapping $T : X \to X$ satisfies the following contractive condition:

$$d(Tx, Ty) \preceq k_1 d(x, y) + k_2 d(x, Tx) + k_3 d(y, Ty) + k_4 d(x, Ty) + k_5 d(y, Tx),$$

for all $x, y \in X$, where $k_i \in K (i = 1, \ldots, 5)$ are generalized Lipschitz constants with $\rho(k_1) + \rho(k_2 + k_3 + k_4 + k_5) < 1$. If k_1 commutes with $k_2 + k_3 + k_4 + k_5$, then T has a unique fixed point in X. Moreover, for arbitrary $x \in X$, iterative sequence $\{T^n x\}$ converges to the fixed point. Further, T has a property P.

Proof. By using Theorem 2.5, we obtain $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. We shall prove x^* is the fixed point of T. To this end, on the one hand, we have

$$
\begin{aligned}
d(Tx^*, x^*) &\preceq d(Tx^*, Tx_n) + d(Tx_n, x^*) \\
&\preceq k_1 d(x^*, x_n) + k_2 d(x^*, Tx^*) + k_3 d(x_n, Tx_n) \\
&\quad + k_4 d(x^*, Tx_n) + k_5 d(x_n, Tx^*) + d(Tx_n, x^*) \\
&= k_1 d(x^*, x_n) + k_2 d(x^*, Tx^*) + k_3 d(x_n, x_{n+1}) \\
&\quad + (e + k_4) d(x^*, x_{n+1}) + k_5 d(x_n, Tx^*) \\
&\preceq (k_1 + k_3 + k_5) d(x_n, x^*) + (k_2 + k_5) d(Tx^*, x^*) \\
&\quad + (e + k_3 + k_4) d(x_{n+1}, x^*),
\end{aligned}
$$

which implies that

$$(e - k_2 - k_5) d(Tx^*, x^*) \preceq (k_1 + k_3 + k_5) d(x_n, x^*) + (e + k_3 + k_4) d(x_{n+1}, x^*). \quad (2.12)$$

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On the other hand, we obtain

$$
\begin{aligned}
d(Tx^*, x^*) &\preceq d(Tx_n, Tx^*) + d(Tx_n, x^*) \\
&\preceq k_1 d(x_n, x^*) + k_2 d(x_n, Tx_n) + k_3 d(x^*, Tx^*) \\
&\quad + k_4 d(x_n, Tx^*) + k_5 d(x^*, Tx_n) + d(Tx_n, x^*) \\
&= k_1 d(x_n, x^*) + k_2 d(x_n, x_{n+1}) + k_3 d(x^*, Tx^*) \\
&\quad + (e + k_5) d(x^*, x_{n+1}) + k_4 d(x_n, Tx^*) \\
&\preceq (k_1 + k_2 + k_4) d(x_n, x^*) + (k_3 + k_4) d(Tx^*, x^*) \\
&\quad + (e + k_2 + k_5) d(x_{n+1}, x^*),
\end{aligned}
$$

which means that

$$
(e - k_3 - k_4) d(Tx^*, x^*) \preceq (k_1 + k_2 + k_4) d(x_n, x^*) + (e + k_2 + k_5) d(x_{n+1}, x^*). \tag{2.13}
$$

Combining (2.12) and (2.13) yields that

$$
\begin{aligned}
(2e - k_2 - k_3 - k_4 - k_5) &d(Tx^*, x^*) \\
&\preceq (2k_1 + k_2 + k_3 + k_4 + k_5) d(x_n, x^*) \\
&\quad + (2e + k_2 + k_3 + k_4 + k_5) d(x_{n+1}, x^*).
\end{aligned}
$$

Denote $k = k_2 + k_3 + k_4 + k_5$, then

$$
(2e - k) d(Tx^*, x^*) \preceq (2k_1 + k) d(x_n, x^*) + (2e + k) d(x_{n+1}, x^*).
$$

Consequently, we deduce that

$$
\begin{aligned}
d(Tx^*, x^*) &\preceq (2e - k)^{-1} (2k_1 + k) d(x_n, x^*) \\
&\quad + (2e - k)^{-1} (2e + k) d(x_{n+1}, x^*).
\end{aligned}
$$

In view of $x_n \to x^*$ $(n \to \infty)$, then for each $\frac{c}{m} \gg \theta$ $(m = 1, 2, \ldots)$, there exists N_m such that for all $n > N_m$, one has $d(x_n, x^*) \ll \frac{c}{m}$. Hence we speculate that

$$
\begin{aligned}
d(Tx^*, x^*) &\preceq (2e - k)^{-1} (2k_1 + k) d(x_n, x^*) + (2e - k)^{-1} (2e + k) d(x_{n+1}, x^*) \\
&\ll [(2e - k)^{-1} (2k_1 + k) + (2e - k)^{-1} (2e + k)] \frac{c}{m} \to \theta \ (m \to \infty),
\end{aligned}
$$

which follows that $Tx^* = x^*$.

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In the following we shall show the fixed point is unique. Actually, if there is another fixed point y^*, then

$$d(x^*, y^*) \preceq k_1 d(x^*, y^*) + k_2 d(x^*, Tx^*) + k_3 d(y^*, Ty^*)$$
$$+ k_4 d(x^*, Ty^*) + k_5 d(y^*, Tx^*)$$
$$= (k_1 + k_4 + k_5) d(x^*, y^*).$$

Set $\lambda = k_1 + k_4 + k_5$, then it follows that

$$d(x^*, y^*) \preceq \lambda d(x^*, y^*) \preceq \cdots \preceq \lambda^n d(x^*, y^*).$$

By the proof of Theorem 2.5, we claim that, for each $c \gg \theta$, there exists $n_0(c)$ such that $\lambda^n \ll c$ such that for all $n > n_0(c)$. Consequently, $d(x^*, y^*) = \theta$, that is, $x^* = y^*$.

Finally, we shall prove that T has a property P. We have to show that the mapping T satisfies the condition (2.11). Indeed, firstly we have

$$d(Tx, T^2x) = d(Tx, TTx)$$
$$\preceq k_1 d(x, Tx) + k_2 d(x, Tx) + k_3 d(Tx, T^2x)$$
$$+ k_4 d(x, T^2x) + k_5 d(Tx, Tx)$$
$$\preceq k_1 d(x, Tx) + k_2 d(x, Tx) + k_3 d(Tx, T^2x)$$
$$+ k_4 d(x, Tx) + k_4 d(Tx, T^2x),$$

that is,

$$(e - k_3 - k_4) d(Tx, T^2x) \preceq (k_1 + k_2 + k_4) d(x, Tx). \tag{2.14}$$

Also, we have

$$d(Tx, T^2x) = d(TTx, Tx)$$
$$\preceq k_1 d(Tx, x) + k_2 d(Tx, T^2x) + k_3 d(x, Tx)$$
$$+ k_4 d(Tx, Tx) + k_5 d(x, T^2x)$$
$$\preceq k_1 d(x, Tx) + k_2 d(Tx, T^2x) + k_3 d(x, Tx)$$
$$+ k_5 d(x, Tx) + k_5 d(Tx, T^2x),$$

that is,

$$(e - k_2 - k_5) d(Tx, T^2x) \preceq (k_1 + k_3 + k_5) d(x, Tx). \tag{2.15}$$

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Adding (2.14) and (2.15) we obtain

$$(2e - k)d(Tx, T^2x) \preceq (2k_1 + k)d(x, Tx).$$

According to the above proof, we demonstrate that

$$d(Tx, T^2x) \preceq hd(x, Tx),$$

where $h = (2e - k)^{-1}(2k_1 + k)$ and $\rho(h) < 1$. Therefore, by Theorem 2.6, T has a property P. $\qquad\square$

Theorem 2.8. Let (X, d) be a complete cone metric space over Banach algebra $\mathcal{A}$ and K be a solid cone in $\mathcal{A}$. Suppose that the mapping $T : X \to X$ satisfies the following contractive condition:

$$d(Tx, Ty) \preceq k_1 d(x, y) + k_2 d(x, Tx) + k_3 d(y, Ty) + k_4 d(x, Ty) + k_5 d(y, Tx),$$

for all $x, y \in X$, where $k_i \in K(i = 1, \dots, 5)$ are generalized Lipschitz constants with $\rho(k_1) + \rho(k_2 + k_3 + k_4 + k_5) < 1$. If k_1 commutes with $k_2 + k_3 + k_4 + k_5$, then Picard's iteration is T-stable.

Proof. By utilizing Theorem 2.7, we obtain that T has a unique fixed point q in X. Assume that $\{y_n\} \subset X$ satisfies the following condition: for each $c \gg \theta$, there exists N such that for all $n > N$, $d(y_{n+1}, Ty_n) \ll c$. Firstly we have

$$
\begin{aligned}
d(Ty_n, q) &= d(Ty_n, Tq) \\
&\preceq k_1 d(y_n, q) + k_2 d(y_n, Ty_n) + k_3 d(q, Tq) \\
&\quad + k_4 d(y_n, Tq) + k_5 d(q, Ty_n) \\
&= k_1 d(y_n, q) + k_2 d(y_n, Ty_n) + k_4 d(y_n, q) + k_5 d(q, Ty_n) \\
&\preceq (k_1 + k_4)d(y_n, q) + k_2[d(y_n, q) + d(q, Ty_n)] + k_5 d(q, Ty_n) \\
&= (k_1 + k_2 + k_4)d(y_n, q) + (k_2 + k_5)d(q, Ty_n). \qquad (2.16)
\end{aligned}
$$

Secondly, we arrive at

$$
\begin{aligned}
d(Ty_n, q) &= d(q, Ty_n) = d(Tq, Ty_n) \\
&\preceq k_1 d(q, y_n) + k_2 d(q, Tq) + k_3 d(y_n, Ty_n) \\
&\quad + k_4 d(q, Ty_n) + k_5 d(y_n, Tq) \\
&= k_1 d(y_n, q) + k_3 d(y_n, Ty_n) + k_4 d(q, Ty_n) + k_5 d(y_n, q) \\
&\preceq (k_1 + k_5)d(y_n, q) + k_3[d(y_n, q) + d(q, Ty_n)] + k_4 d(q, Ty_n) \\
&= (k_1 + k_3 + k_5)d(y_n, q) + (k_3 + k_4)d(q, Ty_n). \qquad (2.17)
\end{aligned}
$$

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Adding up (2.16) and (2.17) yields that

$$2d(Ty_n, q) \preceq (2k_1 + k_2 + k_3 + k_4 + k_5)d(y_n, q) + (k_2 + k_3 + k_4 + k_5)d(q, Ty_n),$$

Denote $k = k_2 + k_3 + k_4 + k_5$, then we get

$$(2e - k)d(Ty_n, q) \preceq (2k_1 + k)d(y_n, q).$$

Based on the proof of Theorem 2.5, it is not hard to verify that

$$d(Ty_n, q) \preceq hd(y_n, q),$$

where $h = (2e - k)^{-1}(2k_1 + k)$ and $\rho(h) < 1$.

Setting $a_n = d(y_n, q)$ and $c_n = d(y_{n+1}, Ty_n)$, we claim that

$$a_{n+1} = d(y_{n+1}, q) \preceq d(y_{n+1}, Ty_n) + d(Ty_n, q) \preceq c_n + ha_n.$$

If for each $c \gg \theta$, there exists N such that for all $n > N$, $c_n = d(y_{n+1}, Ty_n) \ll c$. Then, making full use of Lemma 2.3, we get $a_n = d(y_n, q) \ll c$, which leads to $y_n \to q$ as $n \to \infty$. That is to say, the Picard's iteration is T-stable.

Corollary 2.9. Let (X, d) be a complete cone metric space over Banach algebra $\mathcal{A}$ and K be a solid cone in $\mathcal{A}$. Suppose that the mapping $T : X \to X$ satisfies the following contractive condition:

$$d(Tx, Ty) \preceq kd(x, y),$$

for all $x, y \in X$, where $k \in K$ is a generalized Lipschitz constant with $\rho(k) < 1$. Then T has a unique fixed point in X. Moreover, for any $x \in X$, iterative sequence $\{T^n x\}$ converges to the fixed point. Further, T has a property P and Picard's iteration is T-stable.

Corollary 2.10. Let (X, d) be a complete cone metric space over Banach algebra $\mathcal{A}$ and K be a solid cone in $\mathcal{A}$. Suppose that the mapping $T : X \to X$ satisfies the following contractive condition:

$$d(Tx, Ty) \preceq \frac{k}{2}[d(x, Tx) + d(y, Ty)],$$

for all $x, y \in X$, where $k \in K$ is a generalized Lipschitz constant with $\rho(k) < 1$. Then T has a unique fixed point in X. Moreover, for every $x \in X$, iterative sequence $\{T^n x\}$ converges to the fixed point. Further, T has a property P and Picard's iteration is T-stable.

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Corollary 2.11. Let (X, d) be a complete cone metric space over Banach algebra $\mathcal{A}$ and K be a solid cone in $\mathcal{A}$. Suppose that the mapping $T : X \to X$ satisfies the following contractive condition:

$$d(Tx, Ty) \preceq \frac{k}{2}[d(x, Ty) + d(y, Tx)],$$

for all $x, y \in X$, where $k \in K$ is a generalized Lipschitz constant with $\rho(k) < 1$. Then T has a unique fixed point in X. Moreover, for each $x \in X$, iterative sequence $\{T^n x\}$ converges to the fixed point. Further, T has a property P and Picard's iteration is T-stable.

Remark 2.12. Throughout the conclusions above, we focus on fixed point theorems in cone metric spaces over Banach algebras instead of the theorems only in cone metric spaces. All the coefficients are vector elements and the multiplications such as $kd(x, y)$ are vector multiplications instead of usual scalar ones, which may bring us more convenience in applications.

Remark 2.13. In our results such as Corollary 2.9, we only suppose the spectral radius of k is less than 1, while $\|k\| < 1$ is not assumed. Generally speaking, it is meaningful since by Remark 2.4, the condition $\rho(k) < 1$ is weaker than that $\|k\| < 1$.

Remark 2.14. Compared with the main results of [13] and [15], our main results in this paper deal not only with the fixed point theorems for generalized Lipschitz mappings, but also with P property and T-stability of Picard's iteration, all in the setting of cone metric spaces under the condition that the underlying cones are solid without assumption of normality. These results may be more valuable to put into use since the cones discussed are not necessarily normal under ordinary conditions. Therefore, it is an interesting thing to discuss the fixed point results in cone metric spaces over Banach algebras without the assumption that the underlying cones are normal. The following examples show that our main results will be very useful.

Example 2.15. Let $\mathcal{A} = C_{\mathbb{R}}^1[0, 1]$ and define a norm on $\mathcal{A}$ by $\|x\| = \|x\|_\infty + \|x'\|_\infty$. Define multiplication in $\mathcal{A}$ as just pointwise multiplication. Then $\mathcal{A}$ is a real Banach algebra with a unit $e = 1$ ($e(t) = 1$ for all $t \in [0, 1]$). The set $K = \{x \in \mathcal{A} : x(t) \geq 0 \text{ for all } t \in [0, 1]\}$ is a cone in $\mathcal{A}$. Moreover, K is a non-normal solid cone (see [6]). Let $X = \{a, b, c\}$. Define $d : X \times X$ by $d(a, b)(t) = d(b, a)(t) = e^t$, $d(b, c)(t) = d(c, b)(t) = 2e^t$, $d(c, a)(t) = d(a, c)(t) = 3e^t$ and $d(x, x)(t) = \theta$ for all $t \in [0, 1]$ and each $x \in X$. We have that (X, d) is a solid cone metric space over Banach algebra $\mathcal{A}$. Further, let $T : X \to X$ be a mapping defined with $Ta = Tb = b$, $Tc = a$ and let $k_1, k_2, k_3, k_4, k_5 \in K$ defined with $k_1(t) = \frac{1}{3}t + \frac{1}{2}$, $k_2(t) = k_3(t) = k_4(t) = k_5(t) = \frac{1}{25}$ for all $t \in [0, 1]$. By careful calculations one can get

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that T is not a Banach contraction and all the conditions of Theorems 2.7 are fulfilled. The point $x = b$ is the unique fixed point of T. By using Theorem 2.7 and Theorem 2.8, we can also conclude that T has a P property and Picard's iteration is T-stable.

Example 2.16. Let $\mathcal{A} = \mathbb{R}^2$ and the norm be $\|(x_1, x_2)\| = |x_1| + |x_2|$. Define the multiplication by

$$xy = (x_1, x_2)(y_1, y_2) = (x_1 y_1, x_1 y_2 + x_2 y_1),$$

where $x = (x_1, x_2), y = (y_1, y_2) \in \mathcal{A}$. Then $\mathcal{A}$ is a Banach algebra with a unit $e = (1, 0)$. Taking $X = [0, 0.55] \times (-\infty, +\infty)$, $K = \{(x_1, x_2) \in \mathcal{A} : x_1, x_2 \geq 0\}$ and

$$d(x, y) = (|x_1 - y_1|, |x_2 - y_2|) \text{ for all } x = (x_1, x_2), y = (y_1, y_2) \in X,$$

we claim that (X, d) is a cone metric space over $\mathcal{A}$ and K is a normal solid cone with normal constant $M = 1$.

Define a mapping $T : X \to X$ as

$$Tx = T(x_1, x_2) = \left(\frac{1}{2} \left(\cos \frac{x_1}{2} - |x_1 - \frac{1}{2}| \right), \arctan(1 + |x_2|) + \ln(x_1 + 1) \right).$$

By using mean value theorem of differentials, it follows that

$$
\begin{aligned}
d(Tx, Ty) &= d(T(x_1, x_2), T(y_1, y_2)) \\
&= \left(\left| \frac{1}{2} \left(\cos \frac{x_1}{2} - \cos \frac{y_1}{2} - |x_1 - \frac{1}{2}| + |y_1 - \frac{1}{2}| \right) \right|, \right. \\
&\qquad \left. \left| \arctan(1 + |x_2|) - \arctan(1 + |y_2|) + \ln(x_1 + 1) - \ln(y_1 + 1) \right| \right) \\
&\preceq \left(\left| \frac{x_1 + y_1}{4} \right| \left| \frac{x_1 - y_1}{4} \right| + \frac{1}{2} |x_1 - y_1|, \frac{1}{2} |x_2 - y_2| + |x_1 - y_1| \right) \\
&\preceq \left(\frac{5}{8}, 1 \right) (|x_1 - y_1|, |x_2 - y_2|) \\
&= \left(\frac{5}{8}, 1 \right) d(x, y)
\end{aligned}
$$

for all $x, y \in X$. Put $k = (\frac{5}{8}, 1)$. Simple calculations show that all conditions of Corollary 2.9 are satisfied. Thus by Corollary 2.9, T has a unique fixed point in X. Further, T has a property P and Picard's iteration is T-stable.

The following statement indicates our fixed point results in cone metric space over Banach algebra $\mathcal{A}$ are not equivalent to those in metric spaces. In order to end this, put

$$d_1(x, y) = \inf_{\{u \in P : d(x,y) \preceq u\}} \|u\|, \qquad d_2(x, y) = \inf\{r \in \mathbb{R} : d(x, y) \preceq re\},$$

where $x, y \in X$ and $e = (e_1, e_2) \in \text{int}K$. Then by Theorem 2.2 of [10], d_1 and d_2 are both equivalent metrics. Hence we need to consider only one of them. Let us refer to the

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metric d_2. We shall prove our conclusions are not equivalent to the well-known Banach contraction principle, which means Theorem 2.4 of [8] does not hold in the setting of cone metric spaces over Banach algebras. As a matter of fact, taking $x' = (\frac{1}{2}, 0)$, $y' = (0, 0)$, $e = (1, \frac{1}{2})$, we have

$$
\begin{aligned}
d_2(Tx', Ty') &= \inf \left\{ r \in \mathbb{R} : \left(\frac{1}{2} \cos \frac{1}{4} - \frac{1}{4}, \ln \frac{3}{2} \right) \preceq r \left(1, \frac{1}{2} \right) \right\} \\
&= \max \left\{ \frac{1}{2} \cos \frac{1}{4} - \frac{1}{4}, 2 \ln \frac{3}{2} \right\} = 2 \ln \frac{3}{2} \\
&\geq \frac{1}{2} = d_2(x', y'),
\end{aligned}
$$

which implies that there does not exist $\lambda \in [0, 1)$ such that

$$
d_2(Tx, Ty) \leq \lambda d_2(x, y)
$$

for all $x, y \in X$. Thus it does not satisfy the contractive condition of Banach contraction principle. That is to say, Theorem 2.4 of [8] is unsuitable for cone metric spaces over Banach algebras.

Remark 2.17. Since the contractive mapping in Example 2.15 is generalized Lipschitz mapping, we are easy to make a conclusion that Corollary 2.1 in [5] cannot cope with Example 2.15, which infers that the main results in the setting of cone metric spaces over Banach algebras are very meaningful.

Remark 2.18. In Example 2.16, we are not hard to see that the main results in this paper are indeed more different from the standard results of cone metric spaces presented in the literature. Also, Example 2.16 shows that cone metric space over Banach algebra do be a real generalization of metric space even if some works with the assumption of normal cone.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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Int-soft filters of MTL-algebras

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Abstract. The notions of (Boolean) int-soft filters in MTL-algebras are introduced, and several properties are investigated. Characterizations of (Boolean) int-soft filters are discussed, and a condition for an int-soft filter to be Boolean is provided. The extension property for a Boolean int-soft filter is constructed, and the least int-soft filter containing a given soft set is established.

1. Introduction

The logic MTL, Monoidal t-norm based logic, was introduced by Esteva and Godo in [3]. This logic is very interesting from many points of view. From the logic point of view, it can be regarded as a weak system of Fuzzy Logic. Indeed, it arises from Hájek's Basic Logic BL [4] by replacing the axiom

$$(A\hat{\wedge}(A \to B)) \leftrightarrow (A \wedge B)$$

by the weaker axiom

$$(A\hat{\wedge}(A \to B)) \to (A \wedge B).$$

In connection with the logic MTL, Esteva and Godo [3] introduced a new algebra, called a *MTL-algebra*, and studied several basic properties. They also introduced the notion of (prime) filters in MTL-algebras. Vetterlein [8] studied MTL-algebras arising from partially ordered groups. Borzooei, Khosravi Shoar and Americ [1] discussed some types of filters in MTL-algebras. Morton and van Alten [6] considered the algebraic semantics of the monoidal t-norm logic(MTL) with unary operations (modalities).

In this paper, we introduce the notion of (Boolean) int-soft filters in MTL-algebras, and investigate several properties. We discuss characterizations of (Boolean) int-soft filters, and provide a condition for an int-soft filter to be Boolean. We establish the extension property for a Boolean int-soft filter. We also construct the least int-soft filter containing a given soft set.

2. Preliminaries

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By a *residuated lattice* we shall mean a lattice $L = (L, \leq, \wedge, \vee, \odot, \rightarrow, 0, 1)$ containing the least element 0 and the largest element 1, and endowed with two binary operations $\odot$ (called *product*) and $\rightarrow$ (called *residuum*) such that

- $\odot$ is associative, commutative and isotone.
- $(\forall x \in L)\,(x \odot 1 = x)$.
- The Galois correspondence holds, that is,

$$(\forall x, y, z \in L)\,(x \odot y \leq z \iff x \leq y \rightarrow z).$$

In a residuated lattice, the following are true (see [7]):

(a1) $x \leq y \iff x \rightarrow y = 1$.
(a2) $0 \rightarrow x = 1$, $1 \rightarrow x = x$, $x \rightarrow (y \rightarrow x) = 1$.
(a3) $y \leq (y \rightarrow x) \rightarrow x$.
(a4) $x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z = y \rightarrow (x \rightarrow z)$.
(a5) $x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y)$, $\quad x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$.
(a6) $y \leq x \Rightarrow x \rightarrow z \leq y \rightarrow z$, $z \rightarrow y \leq z \rightarrow x$.
(a7) $\left(\bigvee\limits_{i \in \Gamma} y_i \right) \rightarrow x = \bigwedge\limits_{i \in \Gamma} (y_i \rightarrow x)$.

We define $x^* = \bigvee \{ y \in L \mid x \odot y = 0 \}$, equivalently, $x^* = x \rightarrow 0$. Then

(a8) $0^* = 1$, $\quad 1^* = 0$, $\quad x \leq x^{**}$, and $x^* = x^{***}$.

Based on the Hájek's results [4], Axioms of MTL and Formulas which are provable in MTL, Esteva and Godo [3] defined the algebras, so called MTL-algebras, corresponding to the MTL-logic in the following way.

Definition 2.1. An *MTL-algebra* is a residuated lattice $L = (L, \leq, \wedge, \vee, \odot, \rightarrow, 0, 1)$ satisfying the pre-linearity equation:

$$(x \rightarrow y) \vee (y \rightarrow x) = 1.$$

In an MTL-algebra, the following are true:

(a9) $x \rightarrow (y \vee z) = (x \rightarrow y) \vee (x \rightarrow z)$.
(a10) $x \odot y \leq x \wedge y$.

Definition 2.2 ([3])**.** Let L be an MTL-algebra. A nonempty subset F of L is called a *filter* of L if it satisfies

(b1) $(\forall x, y \in F)\,(x \odot y \in F)$.
(b2) $(\forall x \in F)\,(\forall y \in L)\,(x \leq y \Rightarrow y \in F)$.

Since $\wedge$ is not definable from $\odot$ and $\rightarrow$ in a MTL-algebra, one could consider that the further condition

(b3) $(\forall x, y \in F)\,(x \wedge y \in F)$

should be also required for a filter. However the condition (b3) is indeed redundant because it is a consequence of conditions (b1) and (b2). Namely, since $x \odot y \leq x \wedge y$, if $x, y \in F$ then $x \odot y \in F$ and thus $x \wedge y \in F$ as well.

Proposition 2.3. *A nonempty subset F of an MTL-algebra L is a filter of L if and only if it satisfies:*

(b4) $1 \in F$.
(b5) $(\forall x \in F) \, (\forall y \in L) \, (x \to y \in F \Rightarrow y \in F)$.

A soft set theory is introduced by Molodtsov [5], and Çağman et al. [2] provided new definitions and various results on soft set theory.

Let $\mathscr{P}(U)$ denote the power set of an initial universe set U and $A, B, C, \cdots \subseteq E$ where E is a set of parameters.

Definition 2.4 ([2, 5]). A *soft set* $\left(\tilde{f}, A \right)$ over U is defined to be the set of ordered pairs

$$\left(\tilde{f}, A \right) := \left\{ \left(x, \tilde{f}(x) \right) : x \in E, \, \tilde{f}(x) \in \mathscr{P}(U) \right\},$$

where $\tilde{f} : E \to \mathscr{P}(U)$ such that $\tilde{f}(x) = \emptyset$ if $x \notin A$.

For a soft set $\left(\tilde{f}, L \right)$ over U, the set

$$i_L \left(\tilde{f}; \gamma \right) = \left\{ x \in L \mid \gamma \subseteq \tilde{f}(x) \right\}$$

is called the γ-*inclusive set* of $\left(\tilde{f}, L \right)$.

3. Int-soft filters

In what follows, we take an MTL-algebra L as a set of parameters.

Definition 3.1. A soft set $\left(\tilde{f}, L \right)$ over U is called an *int-soft filter* of L if it satisfies:

$$(\forall x, y \in L) \left(\tilde{f}(x \odot y) \supseteq \tilde{f}(x) \cap \tilde{f}(y) \right). \tag{3.1}$$

$$(\forall x, y \in L) \left(x \leq y \Rightarrow \tilde{f}(x) \subseteq \tilde{f}(y) \right). \tag{3.2}$$

Example 3.2. Let $L = [0, 1]$ and define a product $\odot$ and a residuum $\to$ on L as follows:

$$x \odot y := \begin{cases} x \wedge y & \text{if } x + y > \frac{1}{2}, \\ 0 & \text{otherwise} \end{cases} \qquad x \to y := \begin{cases} 1 & \text{if } x \leq y, \\ (0.5 - x) \vee y & \text{if } x > y \end{cases}$$

for all $x, y \in L$. Then L is an MTL-algebra. Let $\left(\tilde{f}, L \right)$ be a soft set over U in which

$$\tilde{f}(x) := \begin{cases} \alpha & \text{if } x \in (0.5, 1], \\ \beta & \text{otherwise,} \end{cases}$$

where $\alpha \supseteq \beta$ in $\mathscr{P}(U)$. Then it is routine to verify that $\left(\tilde{f}, L \right)$ is an int-soft filter of L.

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We provide characterizations of an int-soft filter.

Theorem 3.3. *A soft set* $\left(\tilde{f}, L\right)$ *over* U *is an int-soft filter of* L *if and only if it satisfies:*

$$(\forall x \in L) \left(\tilde{f}(1) \supseteq \tilde{f}(x)\right), \tag{3.3}$$

$$(\forall x, y \in L) \left(\tilde{f}(y) \supseteq \tilde{f}(x) \cap \tilde{f}(x \rightarrow y)\right). \tag{3.4}$$

Proof. Assume that $\left(\tilde{f}, L\right)$ is an int-soft filter of L. Since $x \leq 1$ for all $x \in L$, it follows from (3.2) that $\tilde{f}(x) \subseteq \tilde{f}(1)$ for all $x \in L$. Since $x \leq (x \rightarrow y) \rightarrow y$, we have $x \odot (x \rightarrow y) \leq y$ for all $x, y \in L$ by the Galois correspondence. It follows from (3.2) and (3.1) that

$$\tilde{f}(y) \supseteq \tilde{f}(x \odot (x \rightarrow y)) \supseteq \tilde{f}(x) \cap \tilde{f}(x \rightarrow y)$$

for all $x, y \in L$.

Conversely, let $\left(\tilde{f}, L\right)$ be a soft set over U which satisfy two conditions (3.3) and (3.4). Let $x, y \in L$ be such that $x \leq y$. Then $x \rightarrow y = 1$, and so

$$\tilde{f}(y) \supseteq \tilde{f}(x) \cap \tilde{f}(x \rightarrow y) = \tilde{f}(x) \cap \tilde{f}(1) = \tilde{f}(x),$$

for all $x \in L$. This proves (3.2). Using (a4), we know that

$$x \rightarrow (y \rightarrow (x \odot y)) = (x \odot y) \rightarrow (x \odot y) = 1.$$

Using (3.3) and (3.4), we have

$$
\begin{aligned}
\tilde{f}(x \odot y) &\supseteq \tilde{f}(y) \cap \tilde{f}(y \rightarrow (x \odot y)) \\
&\supseteq \tilde{f}(y) \cap \left(\tilde{f}(x) \cap \tilde{f}(x \rightarrow (y \rightarrow (x \odot y)))\right) \\
&= \tilde{f}(y) \cap \left(\tilde{f}(x) \cap \tilde{f}(1)\right) \\
&= \tilde{f}(x) \cap \tilde{f}(y)
\end{aligned}
$$

for all $x, y \in L$. Therefore $\left(\tilde{f}, L\right)$ is an int-soft filter of L. $\square$

Theorem 3.4. *A soft set* $\left(\tilde{f}, L\right)$ *over* U *is an int-soft filter of* L *if and only if it satisfies:*

$$(\forall a, b, c \in L) \left(a \leq b \rightarrow c \;\Rightarrow\; \tilde{f}(c) \supseteq \tilde{f}(a) \cap \tilde{f}(b)\right). \tag{3.5}$$

Proof. Assume that $\left(\tilde{f}, L\right)$ is an int-soft filter of L. Let $a, b, c \in L$ be such that $a \leq b \rightarrow c$. Then $\tilde{f}(a) \subseteq \tilde{f}(b \rightarrow c)$ by (3.2), and so

$$\tilde{f}(c) \supseteq \tilde{f}(b) \cap \tilde{f}(b \rightarrow c) \supseteq \tilde{f}(b) \cap \tilde{f}(a).$$

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Conversely, let $\left(\tilde{f}, L\right)$ be a soft set over U satisfying (3.5). Since $x \leq x \to 1$ for all $x \in L$, it follows from (3.5) that

$$\tilde{f}(1) \supseteq \tilde{f}(x) \cap \tilde{f}(x) = \tilde{f}(x)$$

for all $x \in L$. Since $x \to y \leq x \to y$ for all $x, y \in L$, we have

$$\tilde{f}(y) \supseteq \tilde{f}(x) \cap \tilde{f}(x \to y)$$

for all $x, y \in L$. Therefore $\left(\tilde{f}, L\right)$ is an int-soft filter of L. $\square$

Corollary 3.5. *A soft set* $\left(\tilde{f}, L\right)$ *over* U *is an int-soft filter of* L *if and only if it satisfies the following assertion:*

$$\tilde{f}(x) \supseteq \bigcap_{k=1}^{n} \tilde{f}(a_k) \tag{3.6}$$

whenever $a_n \to (\cdots \to (a_2 \to (a_1 \to x)) \cdots) = 1$ *for every* $a_1, a_2, \cdots, a_n \in L$.

Proof. It is by induction. $\square$

Theorem 3.6. *For a filter* F *of* L *and* $a \in L$, *let* $\left(\tilde{f}, L\right)$ *be a soft set over* U *defined by*

$$\tilde{f}(x) := \begin{cases} \gamma_1 & \text{if } x \in \{z \in L \mid a \vee z \in F\}, \\ \gamma_2 & \text{otherwise,} \end{cases}$$

for all $x \in L$ *where* $\gamma_2 \subsetneq \gamma_1$ *in* $\mathscr{P}(U)$. *Then* $\left(\tilde{f}, L\right)$ *is an int-soft filter of* L.

Proof. Since $a \vee 1 \in F$, we have $1 \in \{z \in L \mid a \vee z \in F\}$ and so $\tilde{f}(1) = \gamma_1 \supseteq \tilde{f}(x)$ for all $x \in L$. Now if $y \in \{z \in L \mid a \vee z \in F\}$, then clearly

$$\tilde{f}(y) = \gamma_1 \supseteq \tilde{f}(x) \cap \tilde{f}(x \to y).$$

Suppose that $y \notin \{z \in L \mid a \vee z \in F\}$. Then at least one of x and $x \to y$ does not belong to $\{z \in L \mid a \vee z \in F\}$. Hence

$$\tilde{f}(y) = \gamma_2 = \tilde{f}(x) \cap \tilde{f}(x \to y),$$

and therefore $\left(\tilde{f}, L\right)$ is an int-soft filter of L. $\square$

Theorem 3.7. *A soft set* $\left(\tilde{f}, L\right)$ *over* U *is an int-soft filter of* L *if and only if the nonempty* γ-*inclusive set* $i_L\left(\tilde{f}; \gamma\right)$ *is a filter of* L *for all* $\gamma \in \mathscr{P}(U)$.

Proof. Assume that the nonempty γ-inclusive set $i_L\left(\tilde{f}; \gamma\right)$ is a filter of L for all $\gamma \in \mathscr{P}(U)$. For any $x \in L$, let $\tilde{f}(x) = \gamma$. Then $x \in i_L\left(\tilde{f}; \gamma\right)$. Since $i_L\left(\tilde{f}; \gamma\right)$ is a filter of L, we have $1 \in i_L\left(\tilde{f}; \gamma\right)$ and so $\tilde{f}(x) = \gamma \subseteq \tilde{f}(1)$. For any $x, y \in L$, let $\tilde{f}(x \to y) \cap \tilde{f}(x) = \gamma$. Then $x \to y \in i_L\left(\tilde{f}; \gamma\right)$

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and $x \in i_L\left(\tilde{f}; \gamma\right)$. It follows from (b5) that $y \in i_L\left(\tilde{f}; \gamma\right)$. Hence $\tilde{f}(y) \supseteq \gamma = \tilde{f}(x \to y) \cap \tilde{f}(x)$. Therefore $\left(\tilde{f}, L\right)$ is an int-soft filter of L by Theorem 3.3.

Conversely, suppose that $\left(\tilde{f}, L\right)$ is an int-soft filter of L. Let $\gamma \in \mathscr{P}(U)$ be such that $i_L\left(\tilde{f}; \gamma\right) \neq \emptyset$. Then there exists $a \in i_L\left(\tilde{f}; \gamma\right)$, and so $\gamma \subseteq \tilde{f}(a)$. It follows from (3.3) that $\gamma \subseteq \tilde{f}(a) \subseteq \tilde{f}(1)$. Thus $1 \in i_L\left(\tilde{f}; \gamma\right)$. Let $x, y \in L$ be such that $x \to y \in i_L\left(\tilde{f}; \gamma\right)$ and $x \in i_L\left(\tilde{f}; \gamma\right)$. Then $\gamma \subseteq \tilde{f}(x \to y)$ and $\gamma \subseteq \tilde{f}(x)$. It follows from (3.4) that

$$\gamma \subseteq \tilde{f}(x \to y) \cap \tilde{f}(x) \subseteq \tilde{f}(y),$$

that is, $y \in i_L\left(\tilde{f}; \gamma\right)$. Thus $i_L\left(\tilde{f}; \gamma\right) (\neq \emptyset)$ is a filter of L by Proposition 2.3. $\qquad \square$

Theorem 3.8. If $\left(\tilde{f}, L\right)$ is an int-soft filter of L, then the set

$$\Omega_a := \{x \in L \mid \tilde{f}(x) \supseteq \tilde{f}(a)\}$$

is a filter of L for every $a \in L$.

Proof. Since $\tilde{f}(1) \supseteq \tilde{f}(x)$ for all $x \in L$. we have $1 \in \Omega_a$. Let $x, y \in L$ be such that $x \in \Omega_a$ and $x \to y \in \Omega_a$. Then $\tilde{f}(x) \supseteq \tilde{f}(a)$ and $\tilde{f}(x \to y) \supseteq \tilde{f}(a)$. Since $\tilde{f}$ is an int-soft filter of L, it follows from (3.4) that

$$\tilde{f}(y) \supseteq \tilde{f}(x) \cap \tilde{f}(x \to y) \supseteq \tilde{f}(a)$$

so that $y \in \Omega_a$. Hence Ω_a is a filter of L. $\qquad \square$

Theorem 3.9. Let $a \in L$ and let $\left(\tilde{f}, L\right)$ be a soft set over U. Then

(1) If Ω_a is a filter of L, then $\left(\tilde{f}, L\right)$ satisfies the following implication:

$$(\forall x, y \in L)\,(\tilde{f}(a) \subseteq \tilde{f}(x \to y) \cap \tilde{f}(x) \implies \tilde{f}(a) \subseteq \tilde{f}(y)). \tag{3.7}$$

(2) If $\left(\tilde{f}, L\right)$ satisfies (3.3) and (3.7), then Ω_a is a filter of L.

Proof. (1) Assume that Ω_a is a filter of L. Let $x, y \in L$ be such that

$$\tilde{f}(a) \subseteq \tilde{f}(x \to y) \cap \tilde{f}(x).$$

Then $x \to y \in \Omega_a$ and $x \in \Omega_a$. Using (b5), we have $y \in \Omega_a$ and so $\tilde{f}(y) \supseteq \tilde{f}(a)$.
(2) Suppose that $\tilde{f}$ satisfies (3.3) and (3.7). From (3.3) it follows that $1 \in \Omega_a$. Let $x, y \in L$ be such that $x \in \Omega_a$ and $x \to y \in \Omega_a$. Then $\tilde{f}(a) \subseteq \tilde{f}(x)$ and $\tilde{f}(a) \subseteq \tilde{f}(x \to y)$, which imply that $\tilde{f}(a) \subseteq \tilde{f}(x) \cap \tilde{f}(x \to y)$. Thus $\tilde{f}(a) \subseteq \tilde{f}(y)$ by (3.7), and so $y \in \Omega_a$. Therefore Ω_a is a filter of L. $\qquad \square$

Proposition 3.10. Let $\left(\tilde{f}, L\right)$ be an int-soft filter of L. Then the following are equivalent:

(1) $(\forall x, y, z \in L)\,\left(\tilde{f}(x \to z) \supseteq \tilde{f}(x \to (y \to z)) \cap \tilde{f}(x \to y)\right).$

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(2) $(\forall x, y \in L)\left(\tilde{f}(x \to y) \supseteq \tilde{f}(x \to (x \to y))\right)$.

(3) $(\forall x, y, z \in L)\left(\tilde{f}((x \to y) \to (x \to z)) \supseteq \tilde{f}(x \to (y \to z))\right)$.

Proof. $(1) \Rightarrow (2)$. Suppose that $\left(\tilde{f}, L\right)$ satisfies the condition (1). Taking $z = y$ and $y = x$ in (1) and using (3.3), we have

$$
\begin{aligned}
\tilde{f}(x \to y) \supseteq \tilde{f}(x \to (x \to y)) &\cap \tilde{f}(x \to x) \\
&= \tilde{f}(x \to (x \to y)) \cap \tilde{f}(1) \\
&= \tilde{f}(x \to (x \to y))
\end{aligned}
$$

for all $x, y, z \in L$.

$(2) \Rightarrow (3)$. Suppose that $\left(\tilde{f}, L\right)$ satisfies the condition (2) and let $x, y, z \in L$. Since

$$
x \to (y \to z) \leq x \to ((x \to y) \to (x \to z)),
$$

it follows that

$$
\begin{aligned}
\tilde{f}((x \to y) \to (x \to z)) &= \tilde{f}(x \to ((x \to y) \to z)) \\
&\supseteq \tilde{f}(x \to (x \to ((x \to y) \to z))) \\
&= \tilde{f}(x \to ((x \to y) \to (x \to z))) \\
&\supseteq \tilde{f}(x \to (y \to z)).
\end{aligned}
$$

$(3) \Rightarrow (1)$. If $\left(\tilde{f}, L\right)$ satisfies the condition (3), then

$$
\begin{aligned}
\tilde{f}(x \to y) &\supseteq \tilde{f}((x \to y) \to (x \to z)) \cap \tilde{f}(x \to y) \\
&\supseteq \tilde{f}(x \to (y \to z)) \cap \tilde{f}(x \to y).
\end{aligned}
$$

This completes the proof. $\qquad\square$

Theorem 3.11. *For a fixed element $a \in L$, let $\left(\tilde{f}_a, L\right)$ be a soft set over U defined by*

$$
\tilde{f}_a(x) := \begin{cases} \gamma_1 & \text{if } a \leq x, \\ \gamma_2 & \text{otherwise,} \end{cases}
$$

where $\gamma_1 \supsetneq \gamma_2$ in $\mathscr{P}(U)$. Then $\left(\tilde{f}_a, L\right)$ is an int-soft filter of L if and only if it satisfies the following implication:

$$
(\forall x, y \in L)\, (a \leq y \to x, \ a \leq y \Rightarrow a \leq x). \tag{3.8}
$$

Proof. Assume that $\left(\tilde{f}_a, L\right)$ is an int-soft filter of L and let $x, y \in L$ be such that $a \leq y \to x$ and $a \leq y$. Then $\tilde{f}_a(y \to x) = \gamma_1 = \tilde{f}_a(y)$, and thus

$$
\tilde{f}_a(x) \supseteq \tilde{f}_a(y \to x) \cap \tilde{f}_a(y) = \gamma_1
$$

which implies that $\tilde{f}_a(x) = \gamma_1$ and so $a \leq x$.

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Conversely, suppose that (3.8) is valid. Note that $i_L\left(\tilde{f}_a; \gamma_2\right) = L$ and $i_L\left(\tilde{f}_a; \gamma_1\right) = \{x \in L \mid a \leq x\}$. Obviously $1 \in i_L\left(\tilde{f}_a; \gamma_1\right)$. Let $x, y \in L$ be such that $x \in i_L\left(\tilde{f}_a; \gamma_1\right)$ and $x \to y \in i_L\left(\tilde{f}_a; \gamma_1\right)$. Then $a \leq x$ and $a \leq x \to y$, which imply from (3.8) that $a \leq y$, that is, $y \in i_L\left(\tilde{f}_a; \gamma_1\right)$. Hence $i_L\left(\tilde{f}_a; \gamma_1\right)$ is a filter of L. Using Theorem 3.7, we know that $\left(\tilde{f}_a, L\right)$ is an int-soft filter of L. $\qquad\square$

Definition 3.12. An int-soft filter $\left(\tilde{f}, L\right)$ of L is said to be *Boolean* if it satisfies the following identity

$$(\forall x \in L)\left(\tilde{f}(x \vee x^*) = \tilde{f}(1)\right). \tag{3.9}$$

Proposition 3.13. *Every Boolean int-soft filter $\left(\tilde{f}, L\right)$ of L satisfies the following inclusion:*

$$(\forall x, y, z \in L)\left(\tilde{f}(x \to z) \supseteq \tilde{f}(x \to (z^* \to y)) \cap \tilde{f}(y \to z)\right). \tag{3.10}$$

Proof. Using (a5), we have

$$y \to z \leq (z^* \to y) \to (z^* \to z) \leq (x \to (z^* \to y)) \to (x \to (z^* \to z)).$$

It follows from (3.2) that

$$\tilde{f}(y \to z) \subseteq \tilde{f}((x \to (z^* \to y)) \to (x \to (z^* \to z)))$$

so from (3.4) that

$$\tilde{f}(x \to (z^* \to z)) \supseteq \tilde{f}(x \to (z^* \to y)) \cap \tilde{f}((x \to (z^* \to y)) \to (x \to (z^* \to z)))$$
$$\supseteq \tilde{f}(x \to (z^* \to y)) \cap \tilde{f}(y \to z).$$

Since

$$z^* \vee z = ((z^* \to z) \to z) \wedge ((z \to z^*) \to z^*) \leq (z^* \to z) \to z,$$

we have $\tilde{f}((z^* \to z) \to z) \supseteq \tilde{f}(z^* \vee z) = \tilde{f}(1)$. Since

$$x \to (z^* \to z) \leq ((z^* \to z) \to z) \to (x \to z),$$

it follows from (3.2) that

$$\tilde{f}(x \to (z^* \to z)) \subseteq \tilde{f}(((z^* \to z) \to z) \to (x \to z)).$$

Thus

$$\tilde{f}(x \to z) \supseteq \tilde{f}((z^* \to z) \to z) \cap \tilde{f}(((z^* \to z) \to z) \to (x \to z))$$
$$\supseteq \tilde{f}(1) \cap \tilde{f}(x \to (z^* \to z))$$
$$= \tilde{f}(x \to (z^* \to z))$$
$$\supseteq \tilde{f}(x \to (z^* \to y)) \cap \tilde{f}(y \to z).$$

This completes the proof. $\qquad\square$

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We provide a condition for an int-soft filter to be Boolean.

Proposition 3.14. *If an int-soft filter* $\left(\tilde{f}, L\right)$ *of L satisfies the following inclusion*

$$(\forall x, y \in L)\left(\tilde{f}(x) \supseteq \tilde{f}((x \to y) \to x)\right),\qquad (3.11)$$

then it is Boolean.

Proof. Using (a2), (a4) and (a5), we have

$$
\begin{aligned}
1 = x &\to ((x^* \to x) \to x) \\
&\le ((x^* \to x) \to x)^* \to x^* \\
&\le (x^* \to x) \to (((x^* \to x) \to x)^* \to x) \\
&= ((x^* \to x) \to x)^* \to ((x^* \to x) \to x) \\
&= (((x^* \to x) \to x) \to 0) \to ((x^* \to x) \to x).
\end{aligned}
$$

It follows from (3.2), (3.3) and (3.11) that

$$\tilde{f}((x^* \to x) \to x) \supseteq \tilde{f}((((x^* \to x) \to x) \to 0) \to ((x^* \to x) \to x)) = \tilde{f}(1).$$

Using (a7) and (a9), since

$$
\begin{aligned}
(x^* \to x) \to x &\le ((x^* \to x) \to x) \vee ((x^* \to x) \to x^*) \\
&= (x^* \to x) \to (x \vee x^*) \\
&= (1 \wedge (x^* \to x)) \to (x \vee x^*) \\
&= ((x \to x) \wedge (x^* \to x)) \to (x \vee x^*) \\
&= ((x \vee x^*) \to x) \to (x \vee x^*),
\end{aligned}
$$

we get

$$
\begin{aligned}
\tilde{f}(1) = \tilde{f}((x^* \to x) \to x) \\
\subseteq \tilde{f}(((x \vee x^*) \to x) \to (x \vee x^*)) \\
\subseteq \tilde{f}(x \vee x^*),
\end{aligned}
$$

and so $\tilde{f}(x \vee x^*) = \tilde{f}(1)$. Therefore $\left(\tilde{f}, L\right)$ is Boolean. $\qquad\square$

Proposition 3.15. *If an int-soft filter* $\left(\tilde{f}, L\right)$ *of L satisfies the condition* (3.10), *then it satisfies the condition* (3.11).

Proof. Since $(x \to y) \to x \le x^* \to x$, it follows from (3.2) that

$$
\begin{aligned}
\tilde{f}(x) = \tilde{f}(1 \to x) \\
\supseteq \tilde{f}(1 \to (x^* \to x^*)) \cap \tilde{f}(x^* \to x) \\
\supseteq \tilde{f}(1) \cap \tilde{f}((x \to y) \to x) \\
= \tilde{f}((x \to y) \to x).
\end{aligned}
$$

Hence $\left(\tilde{f}, L\right)$ satisfies the condition (3.11). □

Proposition 3.16. *If an int-soft filter* $\left(\tilde{f}, L\right)$ *of L satisfies* (3.11), *then it satisfies the following inclusion:*

$$(\forall x, y, z \in L) \left(\tilde{f}(x \to z) \supseteq \tilde{f}(x \to (y \to z)) \cap \tilde{f}(x \to y) \right). \tag{3.12}$$

Proof. Since $x \to (y \to z) = y \to (x \to z) \le (x \to y) \to (x \to (x \to z))$, it follows from (3.2) that

$$\tilde{f}(x \to (y \to z)) \subseteq \tilde{f}((x \to y) \to (x \to (x \to z)))$$

so from (3.4) that

$$\tilde{f}(x \to (x \to z)) \supseteq \tilde{f}(x \to y) \cap \tilde{f}((x \to y) \to (x \to (x \to z)))$$
$$\supseteq \tilde{f}(x \to y) \cap \tilde{f}(x \to (y \to z)).$$

Since

$$x \to (x \to z) \le x \to (((x \to z) \to z) \to z) = ((x \to z) \to z) \to (x \to z),$$

we have

$$\tilde{f}(x \to z) \supseteq \tilde{f}(((x \to z) \to z) \to (x \to z))$$
$$\supseteq \tilde{f}(x \to (x \to z))$$
$$\supseteq \tilde{f}(x \to y) \cap \tilde{f}(x \to (y \to z))$$

by using (3.2) and (3.10). This completes the proof. □

Proposition 3.17. *Every Boolean int-soft filter* $\left(\tilde{f}, L\right)$ *of L satisfies the following inclusion:*

$$(\forall x, y, z \in L) \left(\tilde{f}(x \to z) \supseteq \tilde{f}(x \to (y \to z)) \cap \tilde{f}(x \to y) \right).$$

Proof. Note that $x \to (y \to z) = y \to (x \to z) \le (x \to y) \to (x \to (x \to z))$ and

$$x \to (x \to z) \le x \to (((x \to z) \to z) \to z) = ((x \to z) \to z) \to (x \to z)$$

for all $x, y, z \in L$. It follows from (3.2), (3.4), and Propositions 3.13 and 3.14 that

$$\tilde{f}(x \to z) \supseteq \tilde{f}(((x \to z) \to z) \to (x \to z))$$
$$\supseteq \tilde{f}(x \to (x \to z))$$
$$\supseteq \tilde{f}(x \to y) \cap \tilde{f}((x \to y) \to (x \to (x \to z)))$$
$$\supseteq \tilde{f}(x \to y) \cap \tilde{f}(x \to (y \to z)).$$

This completes the proof. □

Combining Propositions 3.13, 3.14 and 3.15, we have a characterization of a Boolean int-soft filter.

Theorem 3.18. *Let* $\left(\tilde{f}, L\right)$ *be an int-soft filter of L. Then the following assertions are equivalent:*

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(1) $\left(\tilde{f}, L\right)$ is Boolean.

(2) $\left(\tilde{f}, L\right)$ satisfies the condition (3.10).

(3) $\left(\tilde{f}, L\right)$ satisfies the condition (3.11).

Proposition 3.19. Every Boolean int-soft filter $\left(\tilde{f}, L\right)$ of L satisfies:

$$(\forall x, y \in L) \left(\tilde{f}(x \to y) \subseteq \tilde{f}(((y \to x) \to x) \to y)\right). \tag{3.13}$$

Proof. Let $\left(\tilde{f}, L\right)$ be a Boolean int-soft filter of L. Since $y \leq ((y \to x) \to x) \to y$, we have

$$(((y \to x) \to x) \to y) \to x \leq y \to x \tag{3.14}$$

by (a6). Using (a4), (a5), (a6) and (3.14), we get

$$
\begin{aligned}
x \to y &\leq ((y \to x) \to x) \to ((y \to x) \to y) \\
&= (y \to x) \to (((y \to x) \to x) \to y) \\
&\leq ((((y \to x) \to x) \to y) \to x) \to (((y \to x) \to x) \to y)
\end{aligned}
$$

and so

$$
\begin{aligned}
\tilde{f}(((y \to x) \to x) \to y) &\supseteq \tilde{f}(((((y \to x) \to x) \to y) \to x) \to (((y \to x) \to x) \to y)) \\
&\supseteq \tilde{f}(x \to y)
\end{aligned}
$$

by Theorem 3.18(3) and (3.2). $\qquad \square$

Theorem 3.20. (Extension Property for Boolean int-soft filter) *Let* $\left(\tilde{f}, L\right)$ *and* $(\tilde{g}, L)$ *be two int-soft filters of* L *such that* $\tilde{f}(1) = \tilde{g}(1)$ *and* $\tilde{f}(x) \subseteq \tilde{g}(x)$ *for all* $x \in L$. *If* $\left(\tilde{f}, L\right)$ *is Boolean, then so is* $(\tilde{g}, L)$.

Proof. Assume that $\left(\tilde{f}, L\right)$ is a Boolean int-soft filter of L. Then $\tilde{f}(x \vee x^*) = \tilde{f}(1)$ for all $x \in L$. It follows from the hypothesis that

$$\tilde{g}(x \vee x^*) \supseteq \tilde{f}(x \vee x^*) = \tilde{f}(1) = \tilde{g}(1). \tag{3.15}$$

Combining (3.15) and (3.3), we have $\tilde{g}(x \vee x^*) = \tilde{g}(1)$ for all $x \in L$. Hence $(\tilde{g}, L)$ is a Boolean int-soft filter of L. $\qquad \square$

For any soft set $\left(\tilde{f}, L\right)$ over U, let $(\tilde{g}, L)$ be a soft set over U in which

$$\tilde{g}(x) := \bigcup \left\{ \bigcap_{k=1}^{n} \tilde{f}(a_n) \;\middle|\; \begin{array}{l} a_n \to (\cdots \to (a_2 \to (a_1 \to x)) \cdots) = 1, \\ a_1, a_2, \cdots, a_n \in L \end{array} \right\} \tag{3.16}$$

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for all $x \in L$. Let $a, b, x \in L$ be such that $a \leq b \to x$. Take $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_m \in L$ such that

$$a_n \to (\cdots \to (a_2 \to (a_1 \to a)) \cdots) = 1,$$
$$b_m \to (\cdots \to (b_2 \to (b_1 \to b)) \cdots) = 1,$$
$$\tilde{g}(a) = \bigcap_{k=1}^{n} \tilde{f}(a_k),$$
$$\tilde{g}(b) = \bigcap_{j=1}^{m} \tilde{f}(b_j).$$

Then

$$b_m \to (\cdots \to (b_1 \to (a_n \to (\cdots \to (a_2 \to (a_1 \to x)) \cdots)))) \cdots) = 1,$$

and so

$$\tilde{g}(x) \supseteq \tilde{f}(a_1) \cap \tilde{f}(a_2) \cap \cdots \cap \tilde{f}(a_n) \cap \tilde{f}(b_1) \cap \tilde{f}(b_2) \cap \cdots \tilde{f}(b_m)$$
$$= \left(\bigcap_{k=1}^{n} \tilde{f}(a_k) \right) \cap \left(\bigcap_{j=1}^{m} \tilde{f}(b_j) \right)$$
$$= \tilde{g}(a) \cap \tilde{g}(b).$$

Hence $(\tilde{g}, L)$ is an int-soft filter of L by Theorem 3.4. Since $x \to x = 1$ for all $x \in L$, we have $\tilde{f}(x) \subseteq \tilde{g}(x)$ for all $x \in L$. Thus $(\tilde{g}, L)$ contains $\left(\tilde{f}, L \right)$. Let $\left(\tilde{h}, L \right)$ be an int-soft filter of L that contains $\left(\tilde{f}, L \right)$. Then

$$\tilde{g}(x) = \bigcup \left\{ \bigcap_{k=1}^{n} \tilde{f}(a_n) \; \middle| \; \begin{array}{c} a_n \to (\cdots \to (a_2 \to (a_1 \to x)) \cdots) = 1, \\ a_1, a_2, \cdots, a_n \in L \end{array} \right\}$$
$$\subseteq \bigcup \left\{ \bigcap_{k=1}^{n} \tilde{h}(a_n) \; \middle| \; \begin{array}{c} a_n \to (\cdots \to (a_2 \to (a_1 \to x)) \cdots) = 1, \\ a_1, a_2, \cdots, a_n \in L \end{array} \right\}$$
$$\subseteq \bigcup \tilde{h}(x) = \tilde{h}(x)$$

by Corollary 3.5, that is, $(\tilde{g}, L)$ is contained in $\left(\tilde{h}, L \right)$.

We summarize this as follows:

Theorem 3.21. *For any soft set* $\left(\tilde{f}, L \right)$ *over* U, *the soft set* $(\tilde{g}, L)$ *over* U *in which*

$$\tilde{g}(x) := \bigcup \left\{ \bigcap_{k=1}^{n} \tilde{f}(a_n) \; \middle| \; \begin{array}{c} a_n \to (\cdots \to (a_2 \to (a_1 \to x)) \cdots) = 1, \\ a_1, a_2, \cdots, a_n \in L \end{array} \right\}$$

for all $x \in L$ *is the least int-soft filter of* L *that contains* $\left(\tilde{f}, L \right)$.

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Conclusion

Based on the soft set theory, we have introduced the notion of (Boolean) int-soft filters in MTL-algebras, and have investigated several properties. We have discussed characterizations of (Boolean) int-soft filters, and have provided a condition for an int-soft filter to be Boolean. We have established the extension property for a Boolean int-soft filter. We have also constructed the least int-soft filter containing a given soft set. Future research will focus on applying the notions and contents to other types of filters in related algebraic structures, and on studying it again by using Boolean algebra instead of $\mathcal{P}(U)$.

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Convergence Analysis of New Iterative Approximating Schemes with Errors for Total Asymptotically Nonexpansive Mappings in Hyperbolic Spaces

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Abstract. The purpose of this paper is to introduce the concept of total asymptotically nonexpansive mappings and to prove some Δ-convergence theorems of iteration processes with errors to approximating a common fixed point for four total asymptotically nonexpansive mappings in hyperbolic spaces. The results presented in the paper extend and improve some recent results announced in the current literature.

Key Words and Phrases. New iterative approximations with errors, asymptotically nonexpansive mapping, total asymptotically nonexpansive mapping, common fixed point, convergence analysis.

AMS Subject Classification. 47H09, 47H10, 54E70.

1 Introduction and preliminaries

Most of the problems in various disciplines of science are nonlinear in nature, whereas fixed point theory proposed in the setting of normed linear spaces or Banach spaces majorly depends on the linear structure of the underlying spaces. A nonlinear framework for fixed point theory is a metric space embedded with a 'convex structure'. The class of hyperbolic spaces, nonlinear in nature, is a general abstract theoretic setting with rich geometrical structure for metric fixed point theory. The study of hyperbolic spaces has been largely motivated and dominated by questions about hyperbolic groups, one of the main objects of study in geometric group theory.

Throughout this paper, we work in the setting of hyperbolic spaces due to Kohlenbach [1], defined below, which is more restrictive than the hyperbolic type introduced in [2] and more general than the concept of hyperbolic space in [3].

A hyperbolic spaces is a metric space (X, d) together with a mapping $W : X^2 \times [0, 1] \to X$ satisfying

(i) $d(u, W(x, y, \alpha)) \leq \alpha d(u, x) + (1 - \alpha)d(u, y)$;
(ii) $d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta|d(x, y)$;
(iii) $W(x, y, \alpha) = W(y, x, (1 - \alpha))$;
(iv) $d(W(x, z, \alpha), W(y, w, \alpha)) \leq \alpha d(x, y) + (1 - \alpha)d(z, w)$

for all $u, x, y, z, w \in X$ and $\alpha, \beta \in [0, 1]$ (see also [4]). A nonempty subset K of a hyperbolic space X is convex if $W(x, y, \alpha) \in K$ for all $x, y \in K$ and $\alpha \in [0, 1]$. The class of hyperbolic spaces contains normed spaces and convex subsets thereof, the Hilbert open unit ball equipped with the hyperbolic metric [5], Hadamard manifolds as well as CAT(0) spaces in the sense of Gromov (see [6]).

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A hyperbolic space is uniformly convex [7, 8] if for any $r > 0$ and $\epsilon \in (0, 2]$ there exists $\delta \in (0, 1]$ such that for all $u, x, y \in X$, we have

$$d(W(x, y, \frac{1}{2}), u) \leq (1 - \delta)r,$$

provided $d(x, u) \leq r$, $d(y, u) \leq r$ and $d(x, y) \geq \epsilon r$.

A map $\eta : (0, +\infty) \times (0, 2] \to (0, 1]$, which provides such $\delta = \eta(r, \epsilon)$ for given $r > 0$ and $\epsilon \in (0, 2]$, is known as a modulus of uniform convexity of X. We call η monotone if it decreases with r (for fixed ϵ), i.e., $\forall \epsilon > 0$, $\forall r_2 \geq r_1 > 0$ $(\eta(r_2, \epsilon) \leq \eta(r_1, \epsilon))$.

In the sequel, let (X, d) be a metric space, and let K be a nonempty subset of X. We shall denote the fixed point set of a mapping T by $F(T) = \{x \in K : Tx = x\}$.

A mapping $T : K \to K$ is said to be nonexpansive if

$$d(Tx, Ty) \leq d(x, y), \quad \forall x, y \in K.$$

A mapping $T : K \to K$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [0, +\infty)$ with $k_n \to 0$ such that

$$d(T^n x, T^n y) \leq (1 + k_n)d(x, y), \quad \forall n \geq 1, x, y \in K.$$

A mapping $T : K \to K$ is said to be uniformly L-Lipschitzian if there exists a constant $L > 0$ such that
$$d(T^n x, T^n y) \leq Ld(x, y), \quad \forall n \geq 1, x, y \in K.$$

Definition 1.1 A mapping $T : K \to K$ is said to be $(\{\mu_n\}, \{\xi_n\}, \rho)$-total asymptotically nonexpansive if there exist nonnegative sequences $\{\mu_n\}$, $\{\xi_n\}$ with $\mu_n \to 0$, $\xi_n \to 0$ and a strictly increasing continuous function $\rho : [0, +\infty) \to [0, +\infty)$ with $\rho(0) = 0$ such that

$$d(T^n x, T^n y) \leq d(x, y) + \mu_n \rho\big(d(x, y)\big) + \xi_n, \quad \forall n \geq 1, x, y \in K.$$

Remark 1.1 From the definitions, it is to know that each nonexpansive mapping is an asymptotically nonexpansive mapping with a sequence $\{k_n = 0\}$, and each asymptotically nonexpansive mapping is a $(\{\mu_n\}, \{\xi_n\}, \rho)$-total asymptotically nonexpansive mapping with $\xi_n = 0$, and $\rho(t) = t$, $t \geq 0$.

The existence of fixed points of various nonlinear mappings has relevant applications in many branches of nonlinear analysis and topology. On the other hand, there are certain situations where it is difficult to derive conditions for the existence of fixed points for certain types of nonlinear mappings. It is worth to mention that fixed point theory for nonexpansive mappings, a limit case of a contraction mapping when the Lipschitz constant is allowed to be 1, requires tools far beyond metric fixed point theory. Iteration schemas are the only main tool for analysis of generalized nonexpansive mappings. Fixed point theory has a computational flavor as one can define effective iteration schemas for the computation of fixed points of various nonlinear mappings. The problem of finding a common fixed point of some nonlinear mappings acting on a nonempty convex domain often arises in applied mathematics.

On the other hand, Zhao et al. [9] introduced a mixed type iteration for total asymptotically nonexpansive mappings in hyperbolic spaces, and prove some Δ-convergence theorems for the iteration process approximating to a common fixed point; Zhao et al. [10] consider convergence theorems for total asymptotically nonexpansive mappings in hyperbolic spaces. Furthermore, Fukhar-ud-din and Kalsoom [11] extended iterative process with errors to asymptotically nonexpansive mappings in hyperbolic spaces, and obtained some convergence results.

Motivated and inspired by the above works, the purpose of this paper is to introduce the concepts of total asymptotically nonexpansive mappings and to prove some Δ-convergence theorems of iteration process with errors for approximating a common fixed point of four total asymptotically nonexpansive mappings in hyperbolic spaces. The results presented in the paper extend and improve some recent results given in [9-25].

<div align="center">2</div>

2   Preliminaries

In order to define the concept of Δ-convergence in the general setup of hyperbolic spaces, we first collect some basic concepts.

Let $\{x_n\}$ be a bounded sequence in a hyperbolic space X. For $x \in X$, we define a continuous functional $r(\cdot, \{x_n\}) : X \to [0, +\infty)$ by

$$r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}.$$

The asymptotic center $A_K(\{x_n\})$ of a bounded sequence $\{x_n\}$ with respect to $K \subset X$ is the set

$$A_K(\{x_n\}) = \{x \in X : r(x, \{x_n\}) \le r(y, \{x_n\}), \forall y \in K\}.$$

This is the set of minimizers of the functional $r(\cdot, \{x_n\})$. If the asymptotic center is taken with respect to X, then it is simply denoted by $A(\{x_n\})$. It is known that uniformly convex Banach spaces and CAT(0) spaces enjoy the property that 'bounded sequences have unique asymptotic centers with respect to closed convex subsets'. The following lemma is due to Leustean [26] and ensures that this property also holds in a complete uniformly convex hyperbolic space.

Lemma 2.1 ([26]) Let (X, d, W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity. Then every bounded sequence $\{x_n\}$ in X has a unique asymptotic center with respect to any nonempty closed convex subset K of X.

Recall that a sequence $\{x_n\}$ in X is said to Δ-converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta\text{-lim}_{n\to\infty} x_n = x$ and call x the Δ-limit of $\{x_n\}$.

Lemma 2.2 ([27]) Let $\{a_n\}$, $\{b_n\}$ and $\{\omega_n\}$ be sequences of nonnegative real numbers satisfying

$$a_{n+1} \le (1 + \omega_n)a_n + b_n, \quad \forall n \ge 1.$$

If $\sum_{n=1}^{\infty} \omega_n < +\infty$ and $\sum_{n=1}^{\infty} b_n < +\infty$, then $\lim_{n\to\infty} a_n$ exists. If there exists a subsequence $\{a_{n_i}\} \subset \{a_n\}$ such that $a_{n_i} \to 0$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.3 ([17]) Let (X, d, W) be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x \in X$ and $\{\alpha_n\}$ be a sequence in $[a, b]$ for some $a, b \in (0, 1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that

$$\limsup_{n\to\infty} d(x_n, x) \le c, \limsup_{n\to\infty} d(y_n, x) \le c,$$
$$\lim_{n\to\infty} d(W(x_n, y_n, \alpha_n), x) = c$$

for some $c \ge 0$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Lemma 2.4 [17] Let K be a nonempty closed convex subset of uniformly convex hyperbolic space and $\{x_n\}$ be a bounded sequence in K such that $A(\{x_n\}) = \{y\}$ and $r(\{x_n\}) = \zeta$. If $\{y_m\}$ is another sequence in K such that $\lim_{m\to\infty} r(y_m, \{x_n\}) = \zeta$, then $\lim_{m\to\infty} y_m = y$.

3   Main results

In this section, we give our main results.

Theorem 3.1 Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let $T_i : K \to K, i = 1, 2$, be a uniformly L_i-Lipschitzian and $(\{\mu_n^i\}, \{\xi_n^i\}, \rho^i)$-total asymptotically nonexpansive mapping with $\{\mu_n^i\}$ and $\{\xi_n^i\}$ satisfying $\lim_{n\to\infty} \mu_n^i = 0$, $\lim_{n\to\infty} \xi_n^i = 0$ and a strictly increasing continuous function $\rho^i : [0, +\infty) \to [0, +\infty)$ with $\rho^i(0) = 0, i = 1, 2$, let $S_i : K \to K, i = 1, 2$, be a uniformly $\hat{L}_i$-Lipschitzian and $(\{\hat{\mu}_n^i\}, \{\hat{\xi}_n^i\}, \hat{\rho}^i)$-total asymptotically nonexpansive mapping with $\{\hat{\mu}_n^i\}$

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and $\{\hat{\xi}_n^i\}$ satisfying $\lim_{n\to\infty} \hat{\mu}_n^i = 0$, $\lim_{n\to\infty} \hat{\xi}_n^i = 0$ and a strictly increasing continuous function $\hat{\rho}^i : [0, +\infty) \to [0, +\infty)$ with $\hat{\rho}^i(0) = 0, i = 1, 2$. Assume that $\mathcal{F} := \bigcap_{i=1}^2 (F(T_i) \cap F(S_i)) \neq \phi$, and for arbitrarily chosen $x_1 \in K$, a new iterative approximating scheme $\{x_n\}$ with errors is defined as follows:

$$x_{n+1} = W(S_1^n x_n, W(T_1^n y_n, u_n, \theta_{n_1}), \alpha_n),$$
$$y_n = W(S_2^n x_n, W(T_2^n x_n, v_n, \theta_{n_2}), \beta_n), \tag{3.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$, $\{\zeta_n\}$, $\{\lambda_n\}$ are sequences in $[0, 1]$ and $\{u_n\}$, $\{v_n\}$ are bounded sequences in K and $\theta_{n_1} = 1 - \frac{\gamma_n}{1-\alpha_n} = \frac{\delta_n}{1-\alpha_n}$, $\theta_{n_2} = 1 - \frac{\lambda_n}{1-\beta_n} = \frac{\zeta_n}{1-\beta_n}$. Let $\{\mu_n^i\}$, $\{\xi_n^i\}$, ρ^i, $\{\hat{\mu}_n^i\}$, $\{\hat{\xi}_n^i\}$, $\hat{\rho}^i$, $i = 1, 2$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$, $\{\zeta_n\}$ and $\{\lambda_n\}$ satisfy the following conditions:

(i) $\sum_{n=1}^\infty \mu_n^i < +\infty$, $\sum_{n=1}^\infty \xi_n^i < +\infty$, $\sum_{n=1}^\infty \hat{\xi}_n^i < +\infty$, $\sum_{n=1}^\infty \gamma_n < +\infty$, $\sum_{n=1}^\infty \lambda_n < +\infty$, $i = 1, 2$;

(ii) There exist constants $a, b \in (0, 1)$ such that $\{\alpha_n\} \subset [a, b]$, $\{\beta_n\} \subset [a, b]$, $\{\delta_n\} \subset [a, b]$, $\{\zeta_n\} \subset [a, b]$ and $\lim_{n\to\infty} \alpha_n = \alpha \in [a, b]$;

(iii) There exist a constant $M^* > 0$ such that $\rho^i(r) \leq M^* r$ and $\hat{\rho}^i(r) \leq M^* r$, $r > 0, i = 1, 2$;

(iv) $d(x, y) \leq d(S_i x, y)$ for all $x, y \in K$ and $i = 1, 2$.

Then the iterative sequence $\{x_n\}$ defined by (3.1) Δ-converges to a common fixed point of $\mathcal{F} := \bigcap_{i=1}^2 (F(T_i) \cap F(S_i))$.

Proof. Set $L = \max\{L_i, \hat{L}_i, i = 1, 2\}$, $\mu_n = \max\{\mu_n^i, \hat{\mu}_n^i, i = 1, 2\}$, and $\xi_n = \max\{\xi_n^i, \hat{\xi}_n^i, i = 1, 2\}$, $\rho = \max\{\rho^i, \hat{\rho}^i, i = 1, 2\}$. By condition (i), we know that $\sum_{n=1}^\infty \mu_n < +\infty$, $\sum_{n=1}^\infty \xi_n < +\infty$. The proof of Theorem 3.1 is divided into three steps.

Step 1. We first prove that $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in \mathcal{F}$.

For any given $p \in \mathcal{F}$, since T_i and $S_i, i = 1, 2$, are total asymptotically nonexpansive mappings, by condition (iii) and (3.1), we have

$$\begin{aligned}
&d(x_{n+1}, p) \\
&= d(W(S_1^n x_n, W(T_1^n y_n, u_n, \theta_{n_1}), \alpha_n), p) \\
&\leq \alpha_n d(S_1^n x_n, p) + (1 - \alpha_n) d(W(T_1^n y_n, u_n, \theta_{n_1}), p) \\
&\leq \alpha_n d(S_1^n x_n, p) + (1 - \alpha_n)\big(\theta_{n_1} d(T_1^n y_n, p) + (1 - \theta_{n_1}) d(u_n, p)\big) \\
&\leq \alpha_n d(S_1^n x_n, p) + \delta_n d(T_1^n y_n, p) + \gamma_n d(u_n, p) \\
&\leq \alpha_n[d(x_n, p) + \mu_n \rho\big(d(x_n, p)\big) + \xi_n] \\
&\quad + \delta_n[d(y_n, p) + \mu_n \rho\big(d(y_n, p)\big) + \xi_n] + \gamma_n d(u_n, p) \\
&\leq \alpha_n[(1 + \mu_n M^*) d(x_n, p) + \xi_n] \\
&\quad + \delta_n[(1 + \mu_n M^*) d(y_n, p) + \xi_n] + \gamma_n d(u_n, p) \\
&\leq \alpha_n(1 + \mu_n M^*) d(x_n, p) + \delta_n(1 + \mu_n M^*) d(y_n, p) \\
&\quad + \gamma_n d(u_n, p) + (\alpha_n + \delta_n)\xi_n, \tag{3.2}
\end{aligned}$$

where

$$\begin{aligned}
d(y_n, p) \quad &= d(W(S_2^n x_n, W(T_2^n x_n, v_n, \theta_{n_2}), \beta_n), p) \\
&\leq \beta_n d(S_2^n x_n, p) + (1 - \beta_n) d(W(T_2^n x_n, v_n, \theta_{n_2}), p) \\
&\leq \beta_n d(S_2^n x_n, p) + (1 - \beta_n)[\theta_{n_2} d(T_2^n x_n, p) + (1 - \theta_{n_2}) d(v_n, p)] \\
&\leq \beta_n d(S_2^n x_n, p) + \zeta_n d(T_2^n x_n, p) + \lambda_n d(v_n, p) \\
&\leq \beta_n[d(x_n, p) + \mu_n \rho\big(d(x_n, p)\big) + \xi_n] \\
&\quad + \zeta_n[d(x_n, p) + \mu_n \rho\big(d(x_n, p)\big) + \xi_n] + \lambda_n d(v_n, p) \\
&\leq (\beta_n + \zeta_n)[(1 + \mu_n M^*) d(x_n, p) + \xi_n] + \lambda_n d(v_n, p) \\
&= (\beta_n + \zeta_n)(1 + \mu_n M^*) d(x_n, p) + \lambda_n d(v_n, p) + (\beta_n + \zeta_n)\xi_n \\
&\leq (1 + \mu_n M^*) d(x_n, p) + \lambda_n d(v_n, p) + \xi_n. \tag{3.3}
\end{aligned}$$

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Substituting (3.3) into (3.2) and simplifying it, we have

$$
\begin{aligned}
d(x_{n+1}, p) \quad &\leq \delta_n(1 + \mu_n M^*)[(1 + \mu_n M^*)d(x_n, p) + \lambda_n d(v_n, p) + \xi_n] \\
&\quad + \alpha_n(1 + \mu_n M^*)d(x_n, p) + \gamma_n d(u_n, p) + (\alpha_n + \delta_n)\xi_n \\
&= (\alpha_n + \delta_n + \alpha_n \mu_n M^* + 2\delta_n \mu_n M^* + \delta_n \mu_n^2 M^{*2})d(x_n, p) \\
&\quad + \gamma_n d(u_n, p) + \lambda_n \delta_n(1 + \mu_n M^*)d(v_n, p) \\
&\quad + [\alpha_n + \delta_n + \delta_n(1 + \mu_n M^*)]\xi_n \\
&\leq [1 + \mu_n M^*(\alpha_n + 2\delta_n + \delta_n \mu_n M^*)]d(x_n, p) \\
&\quad + \gamma_n d(u_n, p) + \lambda_n \delta_n(1 + \mu_n M^*)d(v_n, p) \\
&\quad + [1 + \delta_n(1 + \mu_n M^*)]\xi_n \\
&= (1 + \omega_n)d(x_n, p) + b_n,
\end{aligned} \tag{3.4}
$$

where $\omega_n = \mu_n M^*(\alpha_n + 2\delta_n + \delta_n \mu_n M^*)$, $b_n = \gamma_n d(u_n, p) + \lambda_n \delta_n(1 + \mu_n M^*)d(v_n, p) + [1 + \delta_n(1 + \mu_n M^*)]\xi_n$. Since $\sum_{n=1}^{\infty} \mu_n < +\infty$, $\sum_{n=1}^{\infty} \xi_n < +\infty$ and condition (i),(ii), and $\{u_n\}$, $\{v_n\}$ are bounded sequences in K, it follows from Lemma 2.2 that $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in \mathcal{F}$.

Step 2. We show that

$$
\lim_{n\to\infty} d(x_n, T_i x_n) = 0, \qquad \lim_{n\to\infty} d(x_n, S_i x_n) = 0, \quad i = 1, 2. \tag{3.5}
$$

For each $p \in \mathcal{F}$, from the proof of Step 1, we know that $\lim_{n\to\infty} d(x_n, p)$ exists. We may assume that $\lim_{n\to\infty} d(x_n, p) = c \geq 0$. If $c = 0$, then the conclusion is trivial. Next, we deal with the case $c > 0$. From (3.3), we have

$$
d(y_n, p) \leq (1 + \mu_n M^*)d(x_n, p) + \lambda_n d(v_n, p) + \xi_n. \tag{3.6}
$$

Taking $\limsup$ on both sides in (3.6), we have

$$
\limsup_{n\to\infty} d(y_n, p) \leq c. \tag{3.7}
$$

In addition, since

$$
d(T_1^n y_n, p) \leq d(y_n, p) + \mu_n \rho\big(d(y_n, p)\big) + \xi_n \leq (1 + \mu_n M^*)d(y_n, p) + \xi_n
$$

and

$$
d(S_1^n x_n, p) \leq (1 + \mu_n M^*)d(x_n, p) + \xi_n,
$$

we have

$$
\limsup_{n\to\infty} d(T_1^n y_n, p) \leq c \tag{3.8}
$$

and

$$
\limsup_{n\to\infty} d(S_1^n x_n, p) \leq c. \tag{3.9}
$$

Also

$$
\begin{aligned}
&d(W(T_1^n y_n, u_n, \theta_{n_1}), p) \\
&\leq \theta_{n_1} d(T_1^n y_n, p) + (1 - \theta_{n_1})d(u_n, p) \\
&= \frac{\delta_n}{1 - \alpha_n} d(T_1^n y_n, p) + \frac{\gamma_n}{1 - \alpha_n} d(u_n, p) \\
&\leq d(T_1^n y_n, p) + \frac{\gamma_n}{1 - b} d(u_n, p).
\end{aligned} \tag{3.10}
$$

Since $\sum_{n=1}^{\infty} \gamma_n < +\infty$, $\lim_{n\to\infty} \gamma_n = 0$, by boundedness of $\{u_n\}$ in K and (3.8), taking $\limsup$ on both sides in (3.10), we have

$$
\limsup_{n\to\infty} d(W(T_1^n y_n, u_n, \theta_{n_1}), p) \leq c. \tag{3.11}
$$

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By $\lim_{n\to\infty} d(x_{n+1}, p) = c$, it is easy to prove that

$$\lim_{n\to\infty} d(W(S_1^n x_n, W(T_1^n y_n, u_n, \theta_{n_1}), \alpha_n), p) = c. \tag{3.12}$$

It follows from (3.9), (3.11), (3.12) and Lemma 2.3 that

$$\lim_{n\to\infty} d(S_1^n x_n, W(T_1^n y_n, u_n, \theta_{n_1})) = 0. \tag{3.13}$$

Since

$$
\begin{aligned}
d(x_{n+1}, p) \quad &= d(W(S_1^n x_n, W(T_1^n y_n, u_n, \theta_{n_1}), \alpha_n), p) \\
&\le d(S_1^n x_n, p) + d(S_1^n x_n, x_{n+1}) \\
&\le d(S_1^n x_n, p) + (1 - \alpha_n) d(S_1^n x_n, W(T_1^n y_n, u_n, \theta_{n_1})),
\end{aligned}
$$

with the help of (3.13), we have

$$\liminf_{n\to\infty} d(S_1^n x_n, p) \ge c.$$

Combined with (3.9), it yields that

$$\lim_{n\to\infty} d(S_1^n x_n, p) = c. \tag{3.14}$$

Since

$$
\begin{aligned}
d(x_{n+1}, p) \quad &\le \alpha_n d(S_1^n x_n, p) + \delta_n d(T_1^n y_n, p) + \gamma_n d(u_n, p) \\
&\le \alpha_n d(S_1^n x_n, p) + \delta_n [d(y_n, p) + \mu_n \rho\big(d(y_n, p)\big) + \xi_n] \\
&\quad + \gamma_n d(u_n, p) \\
&\le \alpha_n d(S_1^n x_n, p) + \delta_n (1 + \mu_n M^*) d(y_n, p) + \gamma_n d(u_n, p) + \delta_n \xi_n \\
&\le \alpha_n d(S_1^n x_n, p) + (1 - \alpha_n)(1 + \mu_n M^*) d(y_n, p) \\
&\quad + \gamma_n d(u_n, p) + (1 - \alpha_n) \xi_n,
\end{aligned}
$$

we get

$$
\begin{aligned}
&\frac{d(x_{n+1}, p) - \alpha_n d(S_1^n x_n, p)}{1 - \alpha_n} \\
&\le (1 + \mu_n M^*) d(y_n, p) + \frac{\gamma_n}{1 - \alpha_n} d(u_n, p) + \xi_n.
\end{aligned}
$$

By condition (ii), (3.12) and (3.14), we have

$$\liminf_{n\to\infty} d(y_n, p) \ge c.$$

Combined with (3.7), it yields that

$$\lim_{n\to\infty} d(y_n, p) = c. \tag{3.15}$$

By the same method and (3.15), we can also prove that

$$\lim_{n\to\infty} d(S_2^n x_n, W(T_2^n x_n, v_n, \theta_{n_2})) = 0. \tag{3.16}$$

It follows from virtue of condition (iv), (3.13), and (3.16) that

$$\lim_{n\to\infty} d(x_n, W(T_1^n y_n, u_n, \theta_{n_1})) \le \lim_{n\to\infty} d(S_1^n x_n, W(T_1^n y_n, u_n, \theta_{n_1})) = 0, \tag{3.17}$$

and

$$\lim_{n\to\infty} d(x_n, W(T_2^n x_n, v_n, \theta_{n_2})) \le \lim_{n\to\infty} d(S_2^n x_n, W(T_2^n x_n, v_n, \theta_{n_2})) = 0. \tag{3.18}$$

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From (3.1) and (3.16), we have

$$
\begin{aligned}
d(y_n, S_2^n x_n) \quad &= d(W(S_2^n x_n, W(T_2^n x_n, v_n, \theta_{n_2}), \beta_n), S_2^n x_n) \\
&\leq (1 - \beta_n) d(S_2^n x_n, W(T_2^n x_n, v_n, \theta_{n_2})) \to 0 \\
&(\text{as} \quad n \to \infty)
\end{aligned} \tag{3.19}
$$

and

$$
d(x_n, y_n) = d(x_n, W(T_2^n x_n, v_n, \theta_{n_2})) + d(S_2^n x_n, W(T_2^n x_n, v_n, \theta_{n_2})) + d(S_2^n x_n, y_n).
$$

It follows from (3.16), (3.18) and (3.19) that

$$
\lim_{n \to \infty} d(x_n, y_n) = 0. \tag{3.20}
$$

This together with (3.17) implies that

$$
\begin{aligned}
&d(x_n, W(T_1^n x_n, u_n, \theta_{n_1})) \\
&\leq d(x_n, W(T_1^n y_n, u_n, \theta_{n_1})) + d(W(T_1^n y_n, u_n, \theta_{n_1}), W(T_1^n x_n, u_n, \theta_{n_1})) \\
&\leq d(x_n, W(T_1^n y_n, u_n, \theta_{n_1})) + \theta_{n_1} d(T_1^n y_n, T_1^n x_n) \\
&\leq d(x_n, W(T_1^n y_n, u_n, \theta_{n_1})) + \frac{\delta_n}{1 - \alpha_n} L d(y_n, x_n) \to 0 \quad (n \to \infty).
\end{aligned} \tag{3.21}
$$

On the other hand, from (3.13) and (3.20), we have

$$
\begin{aligned}
&d(S_1^n x_n, W(T_1^n x_n, u_n, \theta_{n_1})) \\
&\leq d(S_1^n x_n, W(T_1^n y_n, u_n, \theta_{n_1})) + d(W(T_1^n y_n, u_n, \theta_{n_1}), W(T_1^n x_n, u_n, \theta_{n_1})) \\
&\leq d(S_1^n x_n, W(T_1^n y_n, u_n, \theta_{n_1})) + \theta_{n_1} d(T_1^n y_n, T_1^n x_n) \\
&\leq d(S_1^n x_n, W(T_1^n y_n, u_n, \theta_{n_1})) + \frac{\delta_n}{1 - \alpha_n} L d(y_n, x_n) \to 0 \quad (n \to \infty).
\end{aligned} \tag{3.22}
$$

From (3.21) and (3.22), we have that

$$
\begin{aligned}
d(S_1^n x_n, x_n) \quad &\leq d(S_1^n x_n, W(T_1^n x_n, u_n, \theta_{n_1})) \\
&+ d(W(T_1^n x_n, u_n, \theta_{n_1}), x_n) \to 0 \quad (n \to \infty).
\end{aligned} \tag{3.23}
$$

In addition, since

$$
\begin{aligned}
d(x_{n+1}, x_n) \quad &= d(W(S_1^n x_n, W(T_1^n y_n, u_n, \theta_{n_1}), \alpha_n), x_n) \\
&\leq \alpha_n d(S_1^n x_n, x_n) + (1 - \alpha_n) d(W(T_1^n y_n, u_n, \theta_{n_1}), x_n),
\end{aligned}
$$

it follows from (3.17) and (3.23) that

$$
\lim_{n \to \infty} d(x_{n+1}, x_n) = 0. \tag{3.24}
$$

Observe that

$$
\begin{aligned}
d(x_n, T_1^n x_n) \quad &\leq d(x_n, W(T_1^n x_n, u_n, \theta_{n_1})) + d(W(T_1^n x_n, u_n, \theta_{n_1}), T_1^n x_n) \\
&\leq d(x_n, W(T_1^n x_n, u_n, \theta_{n_1})) + (1 - \theta_{n_1}) d(T_1^n x_n, u_n) \\
&\leq d(x_n, W(T_1^n x_n, u_n, \theta_{n_1})) \\
&\quad + \frac{\gamma_n}{1 - \alpha_n} [d(T_1^n x_n, x_n) + d(x_n, p) + d(u_n, p)],
\end{aligned}
$$

then

$$
\begin{aligned}
d(x_n, T_1^n x_n) \quad &\leq \frac{1 - \alpha_n}{1 - \alpha_n - \gamma_n} d(x_n, W(T_1^n x_n, u_n, \theta_{n_1})) \\
&\quad + \frac{\gamma_n}{1 - \alpha_n - \gamma_n} [d(x_n, p) + d(u_n, p)].
\end{aligned} \tag{3.25}
$$

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By boundedness of $\{u_n\}$ in K and condition (i), (ii) and $\lim_{n \to \infty} d(x_n, p)$ exists and (3.21), (3.25), we have

$$\lim_{n \to \infty} d(x_n, T_1^n x_n) = 0. \tag{3.26}$$

Similarly, we can also prove that

$$\lim_{n \to \infty} d(x_n, T_2^n x_n) = 0. \tag{3.27}$$

For all $i = 1, 2$, now we know

$$
\begin{aligned}
d(x_n, T_i x_n) \quad &\leq d(x_n, x_{n+1}) + d(x_{n+1}, T_i^{n+1} x_{n+1}) \\
&\quad + d(T_i^{n+1} x_{n+1}, T_i^{n+1} x_n) + d(T_i^{n+1} x_n, T_i x_n) \\
&\leq (1 + L) d(x_n, x_{n+1}) + d(x_{n+1}, T_i^{n+1} x_{n+1}) + L d(T_i^n x_n, x_n).
\end{aligned}
$$

It follows from (3.24), (3.26) and (3.27) that

$$\lim_{n \to \infty} d(x_n, T_i x_n) = 0, \quad i = 1, 2.$$

By virtue of condition (iv), i.e.,

$$d(S_1 x_n,, W(T_1^n x_n, u_n, \theta_{n_1})) \leq d(S_1^n x_n,, W(T_1^n x_n, u_n, \theta_{n_1})),$$

we have

$$
\begin{aligned}
d(x_n, S_1 x_n) \quad &\leq d(x_n,, W(T_1^n x_n, u_n, \theta_{n_1})) + d(S_1 x_n,, W(T_1^n x_n, u_n, \theta_{n_1})) \\
&\leq d(x_n,, W(T_1^n x_n, u_n, \theta_{n_1})) + d(S_1^n x_n,, W(T_1^n x_n, u_n, \theta_{n_1})),
\end{aligned}
$$

from (3.21) and (3.22), which implies that

$$\lim_{n \to \infty} d(x_n, S_1 x_n) = 0.$$

By the same method, we can also prove that

$$\lim_{n \to \infty} d(x_n, S_2 x_n) = 0.$$

Step 3. We shall prove that the sequence $\{x_n\}$ Δ-converges to a common fixed point of $\mathcal{F} := \bigcap_{i=1}^{2} (F(T_i) \cap F(S_i))$.

In fact, for each $p \in F$, $\lim_{n \to \infty} d(x_n, p)$ exist. This implies that the sequence $\{d(x_n, p)\}$ is bounded, so is the sequence $\{x_n\}$. Hence, by virtue of Lemma 2.1, $\{x_n\}$ has a unique asymptotic center $A_K(\{x_n\}) = \{x\}$.

Let $\{u_n\}$ be any subsequence of $\{x_n\}$ with $A_K(\{u_n\}) = \{u\}$. It follows from (3.5) that

$$\lim_{n \to \infty} d(u_n, T_i u_n) = 0. \tag{3.28}$$

Next, we show that $u \in F(T_i)$, for all $i = 1, 2$. For this, we define a sequence $\{z_n^i\}$ in K by $z_m^i = T_i^m u$, for all $i = 1, 2$. So we calculate

$$
\begin{aligned}
&d(z_m^i, u_n) \\
&\leq d(T_i^m u, T_i^m u_n) + d(T_i^m u_n, T_i^{m-1} u_n) + \cdots + d(T_i u_n, u_n) \\
&\leq d(u, u_n) + \mu_m \rho\big(d(u, u_n)\big) + \xi_m + \sum_{k=1}^{m} d(T_i^k u_n, T_i^{k-1} u_n) \\
&\leq (1 + \mu_m M^*) d(u, u_n) + \xi_m + \sum_{k=1}^{m} d(T_i^k u_n, T_i^{k-1} u_n). \tag{3.29}
\end{aligned}
$$

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Since T_i is uniformly L-Lipschitzian, it follows from (3.29) that

$$d(z_m^i, u_n) \leq (1 + \mu_m M^*) d(u, u_n) + \xi_m + mLd(T_i u_n, u_n).$$

Taking $\limsup$ on both sides of the above estimate and using (3.28), we have

$$
\begin{aligned}
r(z_m^i, \{u_n\}) \quad &= \limsup_{n \to \infty} d(z_m^i, u_n) \\
&\leq (1 + \mu_m M^*) \limsup_{n \to \infty} d(u, u_n) + \xi_m \\
&= (1 + \mu_m M^*) r(u, \{u_n\}) + \xi_m,
\end{aligned}
$$

and so

$$\limsup_{m \to \infty} r(z_m^i, \{u_n\}) \leq r(u, \{u_n\}).$$

Based on $A_K(\{u_n\}) = \{u\}$ and the definition of asymptotic center $A_K(\{u_n\})$ of a bounded sequence $\{u_n\}$ with respect to $K \subset X$, we have

$$r(u, \{u_n\}) \leq r(y, \{u_n\}), \quad \forall y \in K.$$

This implies that

$$\liminf_{m \to \infty} r(z_m^i, \{u_n\}) \geq r(u, \{u_n\}).$$

Hence, we have

$$\lim_{m \to \infty} r(z_m^i, \{u_n\}) = r(u, \{u_n\}).$$

It follows from Lemma 2.4 that $\lim_{m \to \infty} z_m^i = u$, namely, $\lim_{m \to \infty} T_i^m u = u$. As T_i is uniformly continuous, so that $T_i u = T_i(\lim_{m \to \infty} T_i^m u) = \lim_{m \to \infty} T_i^{m+1} u = u$. That is, $u \in F(T_i)$. Similarly, we also can show that $u \in F(S_i)$, for all $i = 1, 2$. Hence, u is the common fixed point of T_i and S_i, for all $i = 1, 2$. And we want to show $x = u$, suppose $x \neq u$, by the uniqueness of asymptotic centers,

$$
\begin{aligned}
\limsup_{n \to \infty} d(u_n, u) \quad &< \limsup_{n \to \infty} d(u_n, x) \\
&\leq \limsup_{n \to \infty} d(x_n, x) \\
&< \limsup_{n \to \infty} d(x_n, u) \\
&= \limsup_{n \to \infty} d(u_n, u),
\end{aligned}
$$

a contradiction. Thus we have $x = u$. Since $\{u_n\}$ is an arbitrary subsequence of $\{x_n\}$, $A(\{u_n\}) = \{x\}$ for all subsequence $\{u_n\}$ of $\{x_n\}$. This proves that $\{x_n\}$ Δ-converges to a common fixed point of $\mathcal{F} := \bigcap_{i=1}^2 (F(T_i) \cap F(S_i))$. This completes the proof. $\square$

The following theorem can be obtained from Theorem 3.1 immediately.

Theorem 3.2 Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let $T_i : K \to K, i = 1, 2$, be a uniformly L_i-Lipschitzian and $(\{\mu_n^i\}, \{\xi_n^i\}, \rho^i)$-total asymptotically nonexpansive mapping with $\{\mu_n^i\}$ and $\{\xi_n^i\}$ satisfying $\lim_{n \to \infty} \mu_n^i = 0$, $\lim_{n \to \infty} \xi_n^i = 0$ and a strictly increasing continuous function $\rho^i :$ $[0, +\infty) \to [0, +\infty)$ with $\rho^i(0) = 0, i = 1, 2$, let $S_i : K \to K, i = 1, 2$, be a uniformly $\hat{L}_i$-Lipschitzian and asymptotically nonexpansive mapping with $\{k_n^i\} \subset [0, +\infty)$ satisfying $\lim_{n \to \infty} k_n^i = 0$. Assume that $\mathcal{F} := \bigcap_{i=1}^2 (F(T_i) \cap F(S_i)) \neq \phi$, and for arbitrarily chosen $x_1 \in K$, $\{x_n\}$ is defined as follows:

$$
\begin{aligned}
x_{n+1} &= W(S_1^n x_n, W(T_1^n y_n, u_n, \theta_{n_1}), \alpha_n), \\
y_n &= W(S_2^n x_n, W(T_2^n x_n, v_n, \theta_{n_2}), \beta_n),
\end{aligned}
\tag{3.30}
$$

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where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$, $\{\zeta_n\}$, $\{\lambda_n\}$ are sequences in $[0,1]$ and $\{u_n\}$, $\{v_n\}$ are bounded sequences in K and $\theta_{n_1} = 1 - \frac{\gamma_n}{1-\alpha_n} = \frac{\delta_n}{1-\alpha_n}$, $\theta_{n_2} = 1 - \frac{\lambda_n}{1-\beta_n} = \frac{\zeta_n}{1-\beta_n}$. Let $\{\mu_n^i\}$, $\{\xi_n^i\}$, ρ^i, $\{k_n^i\}$, $i = 1, 2$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$, $\{\zeta_n\}$ and $\{\lambda_n\}$ satisfy the following conditions:

(i) $\sum_{n=1}^{\infty} \mu_n^i < +\infty$, $\sum_{n=1}^{\infty} \xi_n^i < +\infty$, $\sum_{n=1}^{\infty} k_n^i < +\infty$, $\sum_{n=1}^{\infty} \gamma_n < +\infty$, $\sum_{n=1}^{\infty} \lambda_n < +\infty$, $i = 1, 2$;

(ii) There exist constants $a, b \in (0, 1)$ such that $\{\alpha_n\} \subset [a, b]$, $\{\beta_n\} \subset [a, b]$, $\{\delta_n\} \subset [a, b]$, $\{\zeta_n\} \subset [a, b]$ and $\lim_{n \to \infty} \alpha_n = \alpha \in [a, b]$;

(iii) There exist a constant $M^* > 0$ such that $\rho^i(r) \le M^* r$, $r > 0, i = 1, 2$;

(iv) $d(x, y) \le d(S_i x, y)$ for all $x, y \in K$ and $i = 1, 2$.

Then the sequence $\{x_n\}$ defined by (3.30) Δ-converges to a common fixed point of $\mathcal{F} := \bigcap_{i=1}^{2}(F(T_i) \cap F(S_i))$.

Proof. Take $\hat{\rho}^i(t) = t$, $t \ge 0$, $\hat{\xi}_n^i = 0$, $\hat{\mu}_n^i = k_n^i$, $i = 1, 2$, in Theorem 3.1. Since all the conditions in Theorem 3.1 are satisfied, it follows from Theorem 3.1 that the sequence $\{x_n\}$ Δ-converges to a common fixed point of $\mathcal{F} := \bigcap_{i=1}^{2}(F(T_i) \cap F(S_i))$. This completes the proof of Theorem 3.2. $\square$

Theorem 3.3 Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let $T_i : K \to K, i = 1, 2$, be a uniformly L_i-Lipschitzian and $(\{\mu_n^i\}, \{\xi_n^i\}, \rho^i)$-total asymptotically nonexpansive mapping with $\{\mu_n^i\}$ and $\{\xi_n^i\}$ satisfying $\lim_{n \to \infty} \mu_n^i = 0$, $\lim_{n \to \infty} \xi_n^i = 0$ and a strictly increasing continuous function $\rho^i : [0, +\infty) \to [0, +\infty)$ with $\rho^i(0) = 0, i = 1, 2$, let $S_i : K \to K, i = 1, 2$, be a uniformly $\hat{L}_i$-Lipschitzian and asymptotically nonexpansive mapping with $\{k_n^i\} \subset [0, +\infty)$ satisfying $\lim_{n \to \infty} k_n^i = 0$. Assume that $\mathcal{F} := \bigcap_{i=1}^{2}(F(T_i) \cap F(S_i)) \ne \phi$, and for arbitrarily chosen $x_1 \in K$, $\{x_n\}$ is defined as follows:

$$x_{n+1} = W(S_1^n x_n, T_1^n y_n, \alpha_n),$$
$$y_n = W(S_2^n x_n, T_2^n x_n, \beta_n), \tag{3.31}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$. Let $\{\mu_n^i\}$, $\{\xi_n^i\}$, ρ^i, $\{k_n^i\}$, $i = 1, 2$, $\{\alpha_n\}$, $\{\beta_n\}$ satisfy the following conditions:

(i) $\sum_{n=1}^{\infty} \mu_n^i < +\infty$, $\sum_{n=1}^{\infty} \xi_n^i < +\infty$, $\sum_{n=1}^{\infty} k_n^i < +\infty$;

(ii) There exist constants $a, b \in (0, 1)$ such that $\{\alpha_n\} \subset [a, b]$, $\{\beta_n\} \subset [a, b]$, $\lim_{n \to \infty} \alpha_n = \alpha \in [a, b]$;

(iii) There exist a constant $M^* > 0$ such that $\rho^i(r) \le M^* r$, $r > 0, i = 1, 2$;

(iv) $d(x, y) \le d(S_i x, y)$ for all $x, y \in K$ and $i = 1, 2$.

Then the sequence $\{x_n\}$ defined by (3.31) Δ-converges to a common fixed point of $\mathcal{F} := \bigcap_{i=1}^{2}(F(T_i) \cap F(S_i))$.

Proof. Take $\hat{\rho}^i(t) = t$, $t \ge 0$, $\hat{\xi}_n^i = 0$, $\hat{\mu}_n^i = k_n^i$, $i = 1, 2$ and $\gamma_n \equiv \lambda_n \equiv 0$ in Theorem 3.1. Since all the conditions in Theorem 3.1 are satisfied, it follows from Theorem 3.1 that the sequence $\{x_n\}$ Δ-converges to a common fixed point of $\mathcal{F} := \bigcap_{i=1}^{2}(F(T_i) \cap F(S_i))$. This completes the proof. $\square$

Theorem 3.4 Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let $T_i : K \to K, i = 1, 2$, be a uniformly L_i-Lipschitzian and $(\{\mu_n^i\}, \{\xi_n^i\}, \rho^i)$-total asymptotically nonexpansive mapping with $\{\mu_n^i\}$ and $\{\xi_n^i\}$ satisfying $\lim_{n \to \infty} \mu_n^i = 0$, $\lim_{n \to \infty} \xi_n^i = 0$ and a strictly increasing continuous function $\rho^i : [0, +\infty) \to [0, +\infty)$ with $\rho^i(0) = 0, i = 1, 2$, Suppose that $\mathcal{F} := \bigcap_{i=1}^{2} F(T_i) \ne \phi$, and for arbitrarily chosen $x_1 \in K$, $\{x_n\}$ is defined as follows:

$$x_{n+1} = W(x_n, T_1^n y_n, \alpha_n),$$
$$y_n = W(x_n, T_2^n x_n, \beta_n), \tag{3.32}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$. Let $\{\mu_n^i\}$, $\{\xi_n^i\}$, ρ^i, $i = 1, 2$, $\{\alpha_n\}$, $\{\beta_n\}$ satisfy the following conditions:

(i) $\sum_{n=1}^{\infty} \mu_n^i < +\infty$, $\sum_{n=1}^{\infty} \xi_n^i < +\infty$;

(ii) There exist constants $a, b \in (0, 1)$ such that $\{\alpha_n\} \subset [a, b]$, $\{\beta_n\} \subset [a, b]$;

(iii) There exist a constant $M^* > 0$ such that $\rho^i(r) \le M^* r$, $r > 0, i = 1, 2$.

Then the sequence $\{x_n\}$ defined by (3.32) Δ-converges to a common fixed point of $\mathcal{F} := \bigcap_{i=1}^{2} F(T_i)$.

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Proof. Take $\gamma_n \equiv \lambda_n \equiv 0$ and $S_i = I, i = 1, 2$ in Theorem 3.1. Since all the conditions in Theorem 3.1 are satisfied, it follows from Theorem 3.1 that the sequence $\{x_n\}$ Δ-converges to a common fixed point of $\mathcal{F} := \bigcap_{i=1}^{2} F(T_i)$. □

Remark 3.1 The results of Theorems 3.3 and 3.4 improve the corresponding results in Theorem 2.1 of [9] and Theorem 7 of [10], respectively.

Acknowledgements

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12

SOME OSTROWSKI TYPE INTEGRAL INEQUALITIES FOR DOUBLE INTEGRAL ON TIME SCALES

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ABSTRACT. Weighted montgomery identity on time scales for functions of two variables is established. Corresponding discrete and continuous versions of montgomery identities for functions of two variables are obtained. By using the obtained weighted montgomery identity on time scales, an Ostrowski type inequality for double integrals on time scales is pointed out as well.

1. INTRODUCTION AND PRELIMINARY RESULTS

The Ostrowski type inequality, which was originally presented by Ostrowski in [14], can be used to estimate the absolute deviation of a function from its integral mean.

In [6], Bohner and Matthews derived the Montgomery identity on time scales and established the following Ostrowski inequality on time scales, which unifies and extends corresponding discrete [7], continuous [13] and other cases.

Theorem 1. *Let* $a, b, s, t \in \mathbb{T}$ *with* $a < b$ $f : [a, b] \to \mathbb{R}$ *be a differentiable function with the property that,* $M = \sup_{a < t < b} |f\Delta(t)| < \infty$*, induces*

$$\left| f(t) - \frac{1}{(b-a)} \int_a^b f(\sigma(s))\Delta s \right| \leq \frac{M}{b-a} + (h_2(t,a) + h_2(t,b)) \qquad (1.1)$$

is the best possible in the sense that rightside cannot be replaced by a smaller quantity. Where $h(.,.)$ *is defined by definition (6) below.*

2. TIME SCALE ESSENTIALS

During the past decades, with the development of the theory of differential and integral equations as well as difference equations, a lot of integral and difference inequalities have

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been discovered e.g., and the references therein, which play an important role in the re-
search of boundedness, global existence, stability of solutions of differential and integral
equations as well as difference equations. On the other hand, Hilger initiated the theory
of time scales as a theory capable to contain both difference and differential calculus in a
consistent way. Since then many authors have expounded on various aspects of the theory
of dynamic equations on time scales including various inequalities on time scales. The
existence of a derivative at a point in a time scale depends on the type of the point itself,
because time scale may not be connected. Points are classified according to two major
operators, the forward jump operators and the backward jump operators.

Definition 1. *For an arbitrary time scale T, the forward jump operator, $\sigma : \mathbb{T} \to \mathbb{T}$, is
defined to be*

$$\sigma(t) := inf\{s \in \mathbb{T} : s > t\}$$

If the set of all points in T that are larger than t is empty, then $inf\phi = sup\mathbb{T}$.
If $\mathbb{T}$ has a maximum t, then $\sigma(t) = t$.
The backward jump operator, $\rho : \mathbb{T} \to \mathbb{T}$, is

$$\rho(t) := sup\{s \in \mathbb{T} : s < t\}$$

If the set of all points in $\mathbb{T}$ that are less than t is empty, then $sup\phi = inf\mathbb{T}$.
If T has a minimum t, then $\rho(t) = t$.

Definition 2. *Using these two operators for $t \in \mathbb{T}$, the point can be classified in the
following manner, t is right-scattered if $\sigma(t) > t$, t is right-dense if $\sigma(t) = t$, t is left-
scattered if $\rho(t) < t$, t is left-dense if $\rho(t) = t$, t is isolated if it is left- and right-scattered:
$\rho(t) < t < \sigma(t)$ and t is dense if it is both left- and right-dense $\rho(t) = t = \sigma(t)$ for $t \in \mathbb{T}$.*

Definition 3. *The graininess function, $\mu : T \to [0, \infty)$, is defined to be*

$$\mu(t) = \sigma(t) - t$$

*The graininess function essentially describes the step size between two consecutive points
in $\mathbb{T}$. Oftentimes the differences in results obtained from discrete and continuous calculus
stem from the different value of the graininess function evaluated at a given point t.*

Definition 4. *The derivative in time scale calculus, called the delta derivative, determines
the rates of forward change over a time scale. For a function $f : \mathbb{T} \to R$ and $t \in \mathbb{T}^k$, the*

delta derivative of f at t, $f^\Delta(t)$, is defined to be the number, when it exists, where for any given $\epsilon > 0$, there exists a neighborhood U of t such that

$$\left|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]\right| \leq \epsilon|\sigma(t) - s| \tag{2.2}$$

is true for all $s \in U$. Here, $\mathbb{T}^k = \mathbb{T} \setminus \{m\}$ when $\mathbb{T}$ has a left-scattered maximum m; otherwise, $\mathbb{T}^k = \mathbb{T}[1]$. Since the delta derivative definition involves the forward jump operator, if the time scale has a left scattered maximum m, then one cannot jump past this point. Therefore, this point is removed from the set of points used to determine the delta derivative. However, if the time scale does not contain such a left-scattered maximum, then $\mathbb{T}^k$ is equivalent to the time scale.

Take $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t$, $\mu(t) = 0, f^\Delta = f'$ is the derivative used in standard calculus. If $\mathbb{T} = \mathbb{Z}$, $\sigma(t) = t + 1$, $\mu(t) = 1$, $f^\Delta = \Delta f$ is the forward difference operator used in difference equations.

Theorem 2 (Properties of the Delta Derivative). *$f : \mathbb{T} \to R$ be a function and $t \in \mathbb{T}^k$ as defined above. For such a function, the following properties hold:*

(1) *If f is delta differentiable at t, then f is continuous at t.*

(2) *If t is right-scattered and f is continuous at t, then the delta derivative of f, f^Δ, is defined as follows*

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}$$

(3) *If t is right-dense, then the delta derivative at t is as follows (if and only if the limit exists as a finite number)*

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

(4) *If f is delta differentiable at t, then*

$$f(\sigma(t)) = f(t) + \mu(t)\, f^\Delta(t)$$

(5) *If $\mathbb{T} = \mathbb{R}$, then the delta derivative is $f'(t)$ from continuous calculus.*

(6) *If $\mathbb{T} = \mathbb{Z}$, then the delta derivative is the forward difference, Δf, from discrete calculus.*

Definition 5. *If $F^\Delta(t) = f(t)$ for all $t \in \mathbb{T}^k$, then $F(t)$ is said to be anti-derivative of $f(t)$ and $f(t)$ is said to be delta integrable provided that $f(t)$ is rd-continuous. The cauchy integral of $f(t)$ is defined by $\int_r^s f(t)\Delta(t) = F(s) - F(r)$.*

Theorem 3. *Let f, g be rd-continuous, $a, b, c \in \mathbb{T}$ and $\alpha, \beta \in \mathbb{R}$, then*

(1) $\int_a^b (\alpha f(t) + \beta g(t)) \triangle t = \alpha \int_a^b f(t) \triangle t + \beta \int_a^b g(t) \triangle t$

(2) $\int_a^b f(t) \Delta t = - \int_b^a f(t) \Delta t$

(3) $\int_a^c f(t) \Delta t = \int_a^b f(t) \Delta t + \int_b^c f(t) \Delta t$

(4) $\int_a^b f(t) \, g^\Delta(t) \, \Delta t = f(b) \, g(b) - f(a) \, g(a) - \int_a^b f^\Delta(t) \, g(\sigma(t)) \Delta t$

Theorem 4. *If f is Δ-integrable on $[a, b]$, then so is $|f|$, and*

$$\left| \int_a^b f(t) \Delta t \right| \leq \int_a^b |f(t)| \Delta t$$

Let $\mathbb{T}_1$, $\mathbb{T}_2$ be two time scales. Let σ_i, ρ_i and Δ_i be the forward jump operator, the backward jump operator and the delta differentiation, respectively on $\mathbb{T}_i$, for $i = 1, 2$. Let $a, b \in \mathbb{T}_1$, $c, d \in \mathbb{T}_2$, with $a < b$, $c < d$. $[a, b)$ and $[c, d)$ are the half-closed bounded intervals in $\mathbb{T}_1$ and $\mathbb{T}_2$ respectively, and a "rectangle" in $\mathbb{T}_1 \times \mathbb{T}_2$ by

$$\mathbb{R} = [a, b) \times [c, d) = \{(t_1, t_2) : t_1 \in [a, b), t_2 \in [c, d)\}$$

Let f be a real valued function on $\mathbb{T}_1 \times \mathbb{T}_2$. This function f is said to be rd-continuous in t_2 if $a_1 \in \mathbb{T}_1$, then function f is real valued function on $\mathbb{T}_1 \times \mathbb{T}_2$, this function f is said to be rd-continuous in t_2 if $a_1 \in \mathbb{T}_1$, then $f(a_1, t_2)$ is rd-continuous on $\mathbb{T}_2$. CC_{rd} denotes the set of functions $f(a_1, t_2)$ on $\mathbb{T}_1 \times \mathbb{T}_2$, having the properties:

(1) f is rd-continuous in t_1 and t_2.

(2) If $(x_1, x_2) \in \mathbb{T}_1 \times \mathbb{T}_2$ with x_1 right dense and x_2 right dense, then f is continuous at (x_1, x_2).

Definition 6. *Let $g_k, h_k : \mathbb{T}^2 \to \mathbb{R}$, $k \in \mathbb{N}_0$ be defined by $g_0(t, s) = h_0(t, s) = 1$ for all $s, t \in \mathbb{T}$ and then recursively by*

$$g_{k+1}(t, s) = \int_s^t g_k(\sigma(\tau), s) \, \Delta \tau \, \forall s, t \in \mathbb{T} \tag{2.3}$$

$$h_{k+1}(t, s) = \int_s^t h_k(\sigma(\tau), s) \Delta \tau \, \forall s, t \in \mathbb{T} \tag{2.4}$$

3. MAIN RESULTS

Lemma 1 (Weighted Montgomery Identity on Time Scales). *Let $g : [a, b] \to [0, \infty)$, $G : [c, d] \to [0, \infty)$ be rd-continuous and positive and $h : [a, b] \to \mathbb{R}$, $H : [c, d] \to \mathbb{R}$ be invertible and differentiable, such that $g(t_1) = h^{\Delta_1}(t_1)$ on $[a, b]$ and $G(t_2) = H^{\Delta_2}(t_2)$. Let $a, b, s_1, t_1 \in \mathbb{T}_1$, $c, d, s_2, t_2 \in \mathbb{T}_2$ with $a < b, c < d$,*

$A = h^{-1}\left(\left(1-\frac{\alpha}{2}\right)h(a) + \frac{\alpha}{2}h(b)\right),\ B = h^{-1}\left(\left(1-\frac{\alpha}{2}\right)h(b) + \frac{\alpha}{2}h(a)\right),$

$C = H^{-1}\left(\left(1-\frac{\beta}{2}\right)G(c) + \frac{\beta}{2}H(d)\right),\ D = H^{-1}\left(\left(1-\frac{\beta}{2}\right)G(d) + \frac{\beta}{2}H(c)\right)$

and

$f : [a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2} \to \mathbb{R}$ *is $\Delta_1\Delta_2$ differentiable. Then for all $s_1 \in [A,B], s_2 \in [C,D]$,*

$0 \leq \alpha, \beta \leq 1$, *we have*

$$\int_a^b \int_c^d W(t_1,t_2,s_1,s_2)\frac{\partial^2 f(s_1,s_2)}{\Delta_1 s_1 \Delta_2 s_2}\Delta_1 s_1 \Delta_2 s_2$$

$$= \left(\frac{\beta}{2}H(d) - \frac{\beta}{2}H(c)\right)\left\{h(b) - \left(\left(1-\frac{\alpha}{2}\right)h(b) + \frac{\alpha}{2}h(a)\right)\right\}(f(b,c) - f(a,d) + f(b,d))$$

$$+ \left(\frac{\alpha}{2}h(a) + \frac{\alpha}{2}h(b)\right)\{(1-\beta)H(c) + (\beta-1)H(d)\}(f(a,t_2) + f(b,t_2))$$

$$+ \left(\frac{\beta}{2}H(c) + \frac{\beta}{2}H(d)\right)\{(1-\alpha)h(b) + (\alpha-1)h(a)\}(f(t_1,c) - f(t_1,d))$$

$$+ ((1-\alpha)h(b) + (\alpha-1)h(a))\,((1-\beta)H(d) + (\beta-1)H(c))\,f(t_1,t_2)$$

$$+ \left[H(c) - \left(\left(1-\frac{\beta}{2}\right)H(c) + \frac{\beta}{2}H(d)\right)\right]\int_a^b h'(s_1)f(\sigma(s_1),c)\Delta_1 s_1$$

$$- \left[H(t_2) - \left(\left(1-\frac{\beta}{2}\right)H(c) + \frac{\beta}{2}H(d)\right)\right]\int_a^b h'(s_1)f(\sigma(s_1),t_2)\Delta_1 s_1$$

$$- \int_c^d H'(s_2)\,((1-\alpha)h(a) + (\alpha-1)h(b)))\,f(t_1,\sigma(s_2))\Delta_2 s_2$$

$$- \int_c^d H'(s_2)\left\{\left(h(a) - \left(\left(1-\frac{\alpha}{2}\right)h(a) + \frac{\alpha}{2}h(b)\right)\right)\right\}f(a,\sigma(s_2))\Delta_2 s_2$$

$$- \int_c^d H'(s_2)\left\{\left(h(b) - \left(\left(1-\frac{\alpha}{2}\right)h(b) + \frac{\alpha}{2}h(a)\right)\right)\right\}f(b,\sigma(s_2))\Delta_2 s_2$$

$$+ \int_c^d H'(s_2)\left\{\int_a^b h'(s_1)f(\sigma(s_1),\sigma(s_2))\Delta_1 s_1\right\}\Delta_2(t_2)$$

where

$$W_1(t_1,t_2,s_1,s_2) = \begin{cases} h(s_1) - \left((1-\frac{\alpha}{2})h(a) + \frac{\alpha}{2}h(b)\right), & s_1 \in [a,t_1) \\ h(s_1) - \left(\frac{\alpha}{2}h(a) + (1-\frac{\alpha}{2})h(b)\right), & s_1 \in [t_1,b] \\ H(s_2) - \left((1-\frac{\beta}{2})H(c) + \frac{\beta}{2}H(d)\right), & s_2 \in [c,t_2) \\ H(s_2) - \left(\frac{\beta}{2}H(c) + (1-\frac{\beta}{2})H(d)\right), & s_2 \in [t_2,d] \end{cases}$$

Proof. Let's start with

$$\int_a^b \int_c^d W(t_1,t_2,s_1,s_2)\frac{\partial^2 f(s_1,s_2)}{\Delta_1 s_1 \Delta_2 s_2}\Delta_1 s_1 \Delta_2 s_2$$

$$
= \int_a^{t_1}\int_c^{t_2}\left\{h(s_1)-\left(\left(1-\frac{\alpha}{2}\right)h(a)+\frac{\alpha}{2}h(b)\right)\right\}\left\{H(s_2)-\left(\left(1-\frac{\beta}{2}\right)H(c)+\frac{\beta}{2}H(d)\right)\right\}
$$

$$
\frac{\partial^2 f(s_1,s_2)}{\Delta_1 s_1 \Delta_2 s_2}\Delta_1 s_1 \Delta_2 s_2 + \int_a^{t_1}\int_{t_2}^d \left\{h(s_1)-\left(\left(1-\frac{\alpha}{2}\right)h(a)+\frac{\alpha}{2}h(b)\right)\right\}
$$

$$
\left\{H(s_2)-\left(\left(1-\frac{\beta}{2}\right)H(d)+\frac{\beta}{2}H(c)\right)\right\}\frac{\partial^2 f(s_1,s_2)}{\Delta_1 s_1 \Delta_2 s_2}\Delta_1 s_1 \Delta_2 s_2
$$

$$
+\int_{t_1}^b\int_c^{t_2}\left\{h(s_1)-\left(\left(1-\frac{\alpha}{2}\right)h(b)+\frac{\alpha}{2}h(a)\right)\right\}\left\{H(s_2)-\left(\left(1-\frac{\beta}{2}\right)H(c)+\frac{\beta}{2}H(d)\right)\right\}
$$

$$
\frac{\partial^2 f(s_1,s_2)}{\Delta_1 s_1 \Delta_2 s_2}\Delta_1 s_1 \Delta_2 s_2 + \int_{t_1}^b\int_{t_2}^d \left\{h(s_1)-\left(\left(1-\frac{\alpha}{2}\right)h(b)+\frac{\alpha}{2}h(a)\right)\right\}\left\{H(s_2)-\right.
$$

$$
\left.\left(\left(1-\frac{\beta}{2}\right)H(d)+\frac{\beta}{2}H(c)\right)\right\}\frac{\partial^2 f(s_1,s_2)}{\Delta_1 s_1 \Delta_2 s_2}\Delta_1 s_1 \Delta_2 s_2.
$$

Now

$$
= \int_a^{t_1}\left\{h(s_1)-\left(\left(1-\frac{\alpha}{2}\right)h(a)+\frac{\alpha}{2}h(b)\right)\right\}\left[\left\{H(t_2)-\left(\left(1-\frac{\beta}{2}\right)H(c)+\frac{\beta}{2}H(d)\right)\right\}\right.
$$

$$
\frac{\partial f(s_1,t_2)}{\Delta_1 s_1}-\left\{H(c)-\left(\left(1-\frac{\beta}{2}\right)H(c)+\frac{\beta}{2}H(d)\right)\right\}\frac{\partial f(s_1,c)}{\Delta_1 s_1}-\int_c^{t_2}H'(s_2)\frac{\partial f(s_1,\sigma(s_2))}{\Delta_1 s_1}\right]\Delta_2 s_2
$$

$$
+\int_a^{t_1}\left\{h(s_1)-\left(\left(1-\frac{\alpha}{2}\right)h(a)+\frac{\alpha}{2}h(b)\right)\right\}\left[\left\{H(d)-\left(\left(1-\frac{\beta}{2}\right)H(d)+\frac{\beta}{2}H(c)\right)\right\}\right.
$$

$$
\frac{\partial f(s_1,d)}{\Delta_1 s_1}-\left\{H(t_2)-\left(\left(1-\frac{\beta}{2}\right)H(d)+\frac{\beta}{2}H(c)\right)\right\}\frac{\partial f(s_1,t_2)}{\Delta_1 s_1}-\int_{t_2}^d H'(s_2)\frac{\partial f(s_1,\sigma(s_2))}{\Delta_1 s_1}\right]\Delta_2 s_2
$$

$$
+\int_{t_1}^b\left\{h(s_1)-\left(\left(1-\frac{\alpha}{2}\right)h(b)+\frac{\alpha}{2}h(a)\right)\right\}\left[\left\{H(t_2)-\left(\left(1-\frac{\beta}{2}\right)H(c)+\frac{\beta}{2}H(d)\right)\right\}\right.
$$

$$
\frac{\partial f(s_1,t_2)}{\Delta_1 s_1}-\left\{H(c)-\left(\left(1-\frac{\beta}{2}\right)H(c)+\frac{\beta}{2}H(d)\right)\right\}\frac{\partial f(s_1,c)}{\Delta_1 s_1}-\int_c^{t_2}H'(s_2)\frac{\partial f(s_1,\sigma(s_2))}{\Delta_1 s_1}\right]\Delta_2 s_2
$$

$$
+\int_{t_1}^b\left\{h(s_1)-\left(\left(1-\frac{\alpha}{2}\right)h(b)+\frac{\alpha}{2}h(a)\right)\right\}\left[\left\{H(d)-\left(\left(1-\frac{\beta}{2}\right)H(d)+\frac{\beta}{2}H(c)\right)\right\}\right.
$$

$$
\frac{\partial f(s_1,t_2)}{\Delta_1 s_1}-\left\{H(t_2)-\left(\left(1-\frac{\beta}{2}\right)H(d)+\frac{\beta}{2}H(c)\right)\right\}\frac{\partial f(s_1,c)}{\Delta_1 s_1}-\int_c^{t_2}H'(s_2)\frac{\partial f(s_1,\sigma(s_2))}{\Delta_1 s_1}\right]\Delta_2 s_2
$$

and

$$
= \left[H(t_2)-\left(\left(1-\frac{\beta}{2}\right)H(c)+\frac{\beta}{2}H(d)\right)\right]\left\{\left[h(t_1)-\left(1-\frac{\alpha}{2}\right)h(a)-\frac{\alpha}{2}h(b)\right]f(t_1,t_2)\right.
$$

$$
\left.-\left[h(a)-\left(1-\frac{\alpha}{2}\right)h(a)-\frac{\alpha}{2}h(b)\right]f(a,t_2)-\int_a^{t_1}h'(s_1)f(\sigma(s_1),t_2)\Delta_1 s_1\right\}
$$

$$- \left[H(c) - \left(\left(1 - \frac{\beta}{2} \right) H(c) + \frac{\beta}{2} H(d) \right) \right] \left\{ \left[h(t_1) - \left(1 - \frac{\alpha}{2} \right) h(a) - \frac{\alpha}{2} h(b) \right] f(t_1, c) \right.$$

$$- \left[h(a) - \left(1 - \frac{\alpha}{2} \right) h(a) - \frac{\alpha}{2} h(b) \right] f(a, c) - \int_a^{t_1} h'(s_1) f(\sigma(s_1), c) \, \Delta_1 s_1 \right\}$$

$$- \int_c^{t_2} H'(s_2) \left\{ \left[h(t_1) - \left(1 - \frac{\alpha}{2} \right) h(a) - \frac{\alpha}{2} h(b) \right] f(t_1, \sigma(s_2)) - \left[h(a) - \left(1 - \frac{\alpha}{2} \right) \right. \right.$$

$$\left. h(a) - \frac{\alpha}{2} h(b) \right] f(a, \sigma(s_2)) - \int_a^{t_1} h'(s_1) f(\sigma(s_1), \sigma(s_2)) \, \Delta_1 s_1 \right\} \Delta_2(s_2)$$

$$\left[H(d) - \left(1 - \frac{\beta}{2} \right) H(d) + \frac{\beta}{2} H(c) \right] \left\{ \left[h(t_1) - \left(\left(1 - \frac{\alpha}{2} \right) h(a) + \frac{\alpha}{2} h(b) \right) \right] f(t_1, d) \right.$$

$$- \left[h(a) - \left(\left(1 - \frac{\alpha}{2} \right) h(a) + \frac{\alpha}{2} h(b) \right) \right] f(a, d) - \int_a^{t_1} h'(s_1) f(\sigma(s_1), d) \, \Delta_1 s_1 \right\}$$

$$- \left[H(t_2) - \left(\left(1 - \frac{\beta}{2} \right) H(d) + \frac{\beta}{2} H(c) \right) \right] \left\{ \left[h(s_1) - \left(1 - \frac{\alpha}{2} \right) h(a) - \frac{\alpha}{2} h(b) \right] f(t_1, t_2) \right.$$

$$- \left[h(a) - \left(1 - \frac{\alpha}{2} \right) h(a) - \frac{\alpha}{2} h(b) \right] f(a, t_2) - \int_a^{t_1} h'(s_1) f(\sigma(s_1), d) \, \Delta_1 s_1 \right\}$$

$$- \int_{t_2}^{d} H'(s_2) \left\{ \left[h(t_1) - \left(1 - \frac{\alpha}{2} \right) h(a) - \frac{\alpha}{2} h(b) \right] f(t_1, \sigma(s_2)) - \left[h(a) - \left(1 - \frac{\alpha}{2} \right) h(a) \right. \right.$$

$$\left. - \frac{\alpha}{2} h(b) \right] f(a, \sigma(s_2)) - \int_a^{t_1} h'(s_1) f(\sigma(s_1), \sigma(s_2)) \, \Delta_1 s_1 \right\} \Delta_2(s_2) +$$

$$\left[H(t_2) - \left(\left(1 - \frac{\beta}{2} \right) H(c) + \frac{\beta}{2} H(d) \right) \right] \left\{ \left[h(b) - \left(1 - \frac{\alpha}{2} \right) h(b) - \frac{\alpha}{2} h(a) \right] f(b, t_2) \right.$$

$$- \left[h(t_1) - \left(1 - \frac{\alpha}{2} \right) h(b) - \frac{\alpha}{2} h(a) \right] f(t_1, t_2) - \int_{t_1}^{b} h'(s_1) f(\sigma s_1, t_2) \, \Delta_1 s_1 \right\}$$

$$- \left[H(c) - \left(\left(1 - \frac{\beta}{2} \right) H(c) + \frac{\beta}{2} H(d) \right) \right] \left\{ \left[h(t_1) - \left(1 - \frac{\alpha}{2} \right) h(b) - \frac{\alpha}{2} h(a) \right] f(b, c) \right.$$

$$- \left[h(t_1) - \left(1 - \frac{\alpha}{2} \right) h(b) - \frac{\alpha}{2} h(a) \right] f(t_1, c) - \int_{t_1}^{b} h'(s_1) f(\sigma(s_1), c) \, \Delta_1 s_1 \right\}$$

$$- \int_c^{y} h'(t) \left\{ \left(h(b) - \left(\left(1 - \frac{\alpha}{2} \right) h(b) + \frac{\alpha}{2} h(a) \right) \right) f(b, \sigma(t)) - \left[h(t_1) - \left(1 - \frac{\alpha}{2} \right) h(b) \right. \right.$$

$$\left. + \frac{\alpha}{2} h(a) \right] f(t_1, \sigma(s_2)) - \int_{t_1}^{b} h'(s_1) f(\sigma(s_1), \sigma(s_2)) \Delta_1 s \right\} \Delta_2(s_2) +$$

$$
\left[H(d) - \left(\left(1 - \frac{\beta}{2}\right) H(d) + \frac{\beta}{2} H(c) \right) \right] \left\{ \left[h(b) - \left(1 - \frac{\alpha}{2}\right) h(b) - \frac{\alpha}{2} h(a) \right] f(b,d) \right.
$$

$$
\left. - \left[h(t_1) - \left(1 - \frac{\alpha}{2}\right) h(b) - \frac{\alpha}{2} h(a) \right] f(t_1,d) - \int_{t_1}^{b} h'(s_1) f(\sigma(s_1), \sigma(S_2)) \, \Delta_1 s_1 \right\}
$$

$$
- \left[h(y) - \left(\left(1 - \frac{\beta}{2}\right) H(d) + \frac{\beta}{2} H(c) \right) \right] \left\{ \left[h(b) - \left(1 - \frac{\alpha}{2}\right) h(b) - \frac{\alpha}{2} h(a) \right] f(b,t_2) \right.
$$

$$
\left. - \left[h(t_1) - \left(1 - \frac{\alpha}{2}\right) h(b) + \frac{\alpha}{2} h(a) \right] f(t_1,t_2) - \int_{t_1}^{b} H'(s_2) f(\sigma(s_1), t_2) \, \Delta_1 s \right\}
$$

$$
- \int_{t_2}^{d} H'(s_2) \left\{ \left(h(t_1) - \left(\left(1 - \frac{\alpha}{2}\right) h(a) + \frac{\alpha}{2} h(b) \right) \right) f(t_1, \sigma(s_2)) - \left[h(a) - \left(1 - \frac{\alpha}{2}\right) h(a) \right. \right.
$$

$$
\left. \left. + \frac{\alpha}{2} h(b) \right] f(a, \sigma(t)) - \int_{a}^{t_1} h'(s_1) f(\sigma(s_1), \sigma(s_2)) \Delta_1 s_1 \right\} \Delta_2(s_2)
$$

$$
= \left(\frac{\beta}{2} H(d) - \frac{\beta}{2} H(c) \right) \left(h(b) - \left(\left(1 - \frac{\alpha}{2}\right) h(b) + \frac{\alpha}{2} h(a) \right) \right) (f(b,c) - f(a,d) + f(b,d))
$$

$$
+ \left(\frac{\alpha}{2} h(a) + \frac{\alpha}{2} h(b) \right) ((1 - \beta) H(c) + (\beta - 1) H(d)) (f(a,t_2) + f(b,t_2))
$$

$$
+ \left(\frac{\beta}{2} H(c) + \frac{\beta}{2} H(d) \right) ((1 - \alpha) h(b) + (\alpha - 1) h(a)) (f(t_1,c) - f(t_1,d))
$$

$$
+ ((1 - \alpha) h(b) + (\alpha - 1) h(a)) ((1 - \beta) H(d) + (\beta - 1) H(c)) f(t_1, t_2)
$$

$$
+ \left[H(c) - \left(\left(1 - \frac{\beta}{2}\right) H(c) + \frac{\beta}{2} H(d) \right) \right] \int_{a}^{b} h'(s_1) f(\sigma(s_1), c) \, \Delta_1 s_1
$$

$$
- \left[h(t_2) - \left(\left(1 - \frac{\beta}{2}\right) H(c) + \frac{\beta}{2} H(d) \right) \right] \int_{a}^{b} h'(s_1) f(\sigma(s_1), t_2) \, \Delta_1 s_1
$$

$$
- \int_{c}^{d} H'(s_2) ((1 - \alpha) h(a) + (\alpha - 1) h(b)) f(t_1, \sigma(s_2)) \, \Delta_2 s_2
$$

$$
+ \int_{c}^{d} H'(s_2) \left\{ \left(h(a) - \left(\left(1 - \frac{\alpha}{2}\right) h(a) + \frac{\alpha}{2} h(b) \right) \right) \right\} f(a, \sigma(s_2)) \Delta_2 s_2
$$

$$
+ \int_{c}^{d} H'(s_2) \left\{ \int_{a}^{b} h'(s_1) f(\sigma(s_1), \sigma(s_2)) \, \Delta_1 s_1 \right\} \Delta_2(s_2)
$$

$\square$

Remark 1. *When* $\mathbb{T} = \mathbb{R}$

$$
\int_{a}^{b} \int_{c}^{d} W(t_1, t_2, s_1, s_2) \frac{d^2 f(s_1, s_2)}{d_1 s_1 d_2 s_2} d_1 s_1 d_2 s_2
$$

$$
= \left(\frac{\beta}{2} H(d) - \frac{\beta}{2} H(c) \right) \left\{ h(b) - \left(\left(1 - \frac{\alpha}{2}\right) h(b) + \frac{\alpha}{2} h(a) \right) \right\} (f(b,c) - f(a,d) + f(b,d))
$$

$$
+ \left(\frac{\alpha}{2} h(a) + \frac{\alpha}{2} h(b) \right) \{ (1 - \beta) H(c) + (\beta - 1) H(d) \} (f(a, t_2) + f(b, t_2))
$$

$$
+ \left(\frac{\beta}{2} H(c) + \frac{\beta}{2} H(d) \right) \{ (1 - \alpha) h(b) + (\alpha - 1) h(a) \} (f(t_1, c) - f(t_1, d))
$$

$$
+ ((1 - \alpha) h(b) + (\alpha - 1) h(a)) ((1 - \beta) H(d) + (\beta - 1) H(c)) f(t_1, t_2)
$$

$$
+ \left[H(c) - \left(\left(1 - \frac{\beta}{2} \right) H(c) + \frac{\beta}{2} H(d) \right) \right] \int_a^b h'(s_1) f(s_1, c) d_1 s_1
$$

$$
- \left[H(t_2) - \left(\left(1 - \frac{\beta}{2} \right) H(c) + \frac{\beta}{2} H(d) \right) \right] \int_a^b h'(s_1) f(s_1, t_2) d_1 s_1
$$

$$
- \int_c^d H'(s_2) ((1 - \alpha) h(a) + (\alpha - 1) h(b)) f(t_1, s_2) d_2 s_2
$$

$$
+ \int_c^d H'(s_2) \left\{ \left(h(a) - \left(\left(1 - \frac{\alpha}{2} \right) h(a) + \frac{\alpha}{2} h(b) \right) \right) \right\} f(a, s_2) d_2 s_2
$$

$$
+ \int_c^d H'(s_2) \left\{ \int_a^b h'(s_1) f(s_1, s_2) d_1 s_1 \right\} d_2(t_2)
$$

Remark 2. $\mathbb{T} = \mathbb{Z}$

$$
\sum_{s_1=a}^{b-1} \sum_{s_2=c}^{d-1} W(t_1, t_2, s_1, s_2)
$$

$$
= \left(\frac{\beta}{2} H(d) - \frac{\beta}{2} H(c) \right) \left\{ h(b) - \left(\left(1 - \frac{\alpha}{2} \right) h(b) + \frac{\alpha}{2} h(a) \right) \right\} (f(b, c) - f(a, d) + f(b, d))
$$

$$
+ \left(\frac{\alpha}{2} h(a) + \frac{\alpha}{2} h(b) \right) \{ (1 - \beta) H(c) + (\beta - 1) H(d) \} (f(a, t_2) + f(b, t_2))
$$

$$
+ \left(\frac{\beta}{2} H(c) + \frac{\beta}{2} H(d) \right) \{ (1 - \alpha) h(b) + (\alpha - 1) h(a) \} (f(t_1, c) - f(t_1, d))
$$

$$
+ ((1 - \alpha) h(b) + (\alpha - 1) h(a)) ((1 - \beta) H(d) + (\beta - 1) H(c)) f(t_1, t_2)
$$

$$
+ \left[H(c) - \left(\left(1 - \frac{\beta}{2} \right) H(c) + \frac{\beta}{2} H(d) \right) \right] \sum_{s_1=a}^{b-1} h'(s_1) f(t_1 + 1, c)
$$

$$
- \left[H(t_2) - \left(\left(1 - \frac{\beta}{2} \right) H(c) + \frac{\beta}{2} H(d) \right) \right] \sum_{s_1=a}^{b-1} h'(s_1) f(t_1 + 1, t_2)
$$

$$
- \sum_{s_2=c}^{d-1} H'(s_2) ((1 - \alpha) h(a) + (\alpha - 1) h(b)) f(t_1, t_2 + 1)
$$

$$
+ \sum_{s_2=c}^{d-1} H'(s_2) \left\{ \left(h(a) - \left(\left(1 - \frac{\alpha}{2} \right) h(a) + \frac{\alpha}{2} h(b) \right) \right) \right\} f(a, t_2 + 1)
$$

$$
+ \sum_{s_1=a}^{b-1} \sum_{s_2=c}^{d-1} H'(s_2) \left\{ \int_a^b h'(s_1) f(t_1 + 1, t_2 + 1) \right\}
$$

Remark 3. *By taking $h(s_1) = s_1$, $H(s_2) = s_2$, we obtain*

$$\int_a^b \int_c^d W(t_1, t_2, s_1, s_2) \frac{\partial^2 f(s_1, s_2)}{\Delta_1 s_1 \Delta_2 s_2} \Delta_1 s_1 \Delta_2 s_2 = (1-\alpha)(1-\beta)(b-a)(d-c)f(t_1, t_2)$$

$$+ (b-a)(d-c)\left\{(1-\beta)\frac{\alpha}{2}[f(a, t_2) + f(b, t_2)] + \frac{\beta}{2}(1-\alpha)[f(t_1, c) + f(t_1, d)]\right\}$$

$$+ \frac{\alpha\beta}{4}(b-a)(d-c)\{f(b, c) + f(a, d) + f(b, d)\} - \frac{\beta}{2}(d-c)\int_a^b \{f(\sigma(s_1), c)$$

$$+ f(\sigma(s_1), d)\}\Delta_1 s_1 - \frac{\alpha}{2}(b-a)\int_c^d \{f(a, \sigma(t)) + f(b, \sigma(t))\}\Delta_2 t_2$$

$$+ \int_a^b \int_c^d f(\sigma(s_1), \sigma(s_2))\Delta_1 s_1 \Delta_2 s_2$$

Theorem 5. *Under the conditions of Lemma, if $f^{\Delta_1 \Delta_2} \in L_2((a, b)_{\mathbb{T}_1} \times (c, d)_{\mathbb{T}_2})$, with $h(s_1) = s_1$, $H(s_2) = s_2$, then we have*

$$(1-\alpha)(1-\beta)(b-a)(d-c)f(t_1, t_2) + (b-a)(d-c)\left[(1-\beta)\frac{\alpha}{2}[f(a, t_2) + f(b, t_2)]\right.$$

$$\left.+ \frac{\beta}{2}(1-\alpha)\{f(t_1, c) + f(t_1, d)\}\right] + \frac{\alpha}{2}\frac{\beta}{2}(b-a)(d-c)\{f(b, c) + f(a, d) + f(b, d)\}$$

$$- \frac{\beta}{2}(d-c)\int_a^b \{f(\sigma(s_1), c) + f(\sigma(s_1), d)\}\Delta_1 s_1 - \frac{\alpha}{2}(b-a)\int_c^d \{f(a, \sigma(t))$$

$$+ f(b, \sigma(t))\}\Delta_2 t_2 + \int_a^b \int_c^d f(\sigma(s_1), \sigma(s_2))\Delta_1 s_1 \Delta_2 s_2 - \frac{[f(b, d) - f(a, d) - f(b, c) + f(a, c)]}{(b-a)^2(d-c)^2}$$

$$\times\left[h_2\left(t_1, a+\alpha\frac{b-a}{2}\right) - h_2\left(a, a+\alpha\frac{b-a}{2}\right) + h_2\left(b, b-\alpha\frac{b-a}{2}\right) - h_2\left(t_1, b-\alpha\frac{b-a}{2}\right)\right]$$

$$\times\left[h_2\left(t_2, c+\alpha\frac{d-c}{2}\right) - h_2\left(c, c+\alpha\frac{d-c}{2}\right) + h_2\left(d, d-\alpha\frac{d-c}{2}\right) - h_2\left(t_2, d-\alpha\frac{d-c}{2}\right)\right]$$

$$\leq \left[\left\{\frac{b^3-a^3}{3} - 2\left(a+\alpha\frac{b-a}{2}\right)\left(h_2\left(t_1, a+\alpha\frac{b-a}{2}\right)\right) - h_2\left(a, a+\alpha\frac{b-a}{2}\right) - \left(a+\alpha\frac{b-a}{2}\right)^2(t_1-a)\right\}\right]$$

$$- 2\left(b-\alpha\frac{b-a}{2}\right)\left(\left(h_2\left(b, b-\alpha\frac{b-a}{2}\right) - h_2\left(t_1, b-\alpha\frac{b-a}{2}\right)\right) - \left(b-\alpha\frac{b-a}{2}\right)^2(b-t_1)\right)$$

$$\times\left[\left\{\frac{d^3-c^3}{3} - 2\left(c+\alpha\frac{d-c}{2}\right)\left(h_2\left(t_2, c+\alpha\frac{d-c}{2}\right)\right) - h_2\left(c, c+\alpha\frac{d-c}{2}\right) - \left(c+\alpha\frac{d-c}{2}\right)^2(t_2-c)\right\}\right]$$

$$- 2\left(d-\alpha\frac{d-c}{2}\right)\left(\left(h_2\left(d, d-\alpha\frac{d-c}{2}\right) - h_2\left(t_2, d-\alpha\frac{d-c}{2}\right)\right) - \left(d-\alpha\frac{d-c}{2}\right)^2(d-t_2)\right)$$

$$- \frac{1}{(b-a)(d-c)}\left[\left(\left(h_2\left(t_1, a+\alpha\frac{b-a}{2}\right)\right) - h_2\left(a, a+\alpha\frac{b-a}{2}\right) + \left(\left(h_2\left(b, b-\alpha\frac{b-a}{2}\right)\right)\right.\right.$$

$$-h_2\left(t_1, b-\alpha\frac{b-a}{2}\right)\right)\right]\right)\left[\left(\left(h_2\left(t_2, c+\alpha\frac{d-c}{2}\right)\right)-h_2\left(c, c+\alpha\frac{d-c}{2}\right)\right.$$

$$+\left(\left(h_2\left(d, d-\alpha\frac{d-c}{2}\right)\right)-h_2\left(t_2, d-\alpha\frac{d-c}{2}\right)\right)\right)\right]\sqrt{T\left(f^{\Delta_1\Delta_2}\right)}.$$

where

$$T(f)=\int_a^b\int_c^d f^2(s_1,s_2)\Delta_2 s_2\Delta_1 s_1-\frac{1}{(b-a)(d-c)}\left(\int_a^b\int_c^d f(s_1,s_2)\Delta_2 s_2\Delta_1 s_1\right)^2$$

Proof. From the definition of $W(t_1,t_2,s_1,s_2)$, and taking $h(s_1)=s_1$, $H(s_2)=s_2$, we obtain

$$\int_a^b\int_c^d W(t_1,t_2,s_1,s_2)\Delta_2 s_2\Delta_1 s_1=\int_a^b W_1(t_1,s_1)\Delta_1 s_1\int_c^d W_2(t_2,s_2)\Delta_2 s_2$$

$$\left[\int_a^b\left\{s_1-\left(a+\alpha\frac{b-a}{2}\right)\right\}\Delta_1 s_1+\int_{t_1}^b\left\{s_1-\left(b-\alpha\frac{b-a}{2}\right)\right\}\Delta_1 s_1\right]$$

$$\times\left[\int_c^{t_2}\left\{s_2-\left(c+\alpha\frac{d-c}{2}\right)\right\}\Delta_2 s_2+\int_{t_2}^d\left\{s_2-\left(d-\alpha\frac{d-c}{2}\right)\right\}\Delta_2 s_2\right]$$

$$=\left[h_2\left(t_1, a+\alpha\frac{b-a}{2}\right)-h_2\left(a, a+\alpha\frac{b-a}{2}\right)+h_2\left(b, b-\alpha\frac{b-a}{2}\right)-h_2\left(t_1, b-\alpha\frac{b-a}{2}\right)\right]$$

$$\times\left[h_2\left(t_2, c+\alpha\frac{d-c}{2}\right)-h_2\left(c, c+\alpha\frac{d-c}{2}\right)+h_2\left(d, d-\alpha\frac{d-c}{2}\right)-h_2\left(t_2, d-\alpha\frac{d-c}{2}\right)\right]$$

and

$$\int_a^b\int_c^d W^2(t_1,t_2,s_1,s_2)\Delta_2 s_2\Delta_1 s_1=\int_a^b W_1^2(t_1,s_1)\Delta_1 s\int_c^d W_2^2(t_2,s_2)\Delta_2 s_2$$

$$=\left\{\int_a^{t_1}\left[s_1-\left(a+\alpha\frac{b-a}{2}\right)\right]^2\Delta_1 s_1+\int_{t_1}^b\left[s_1-\left(b-\alpha\frac{b-a}{2}\right)\right]^2\right\}\Delta_1 s_1$$

$$\times\left\{\int_c^{t_2}\left[s_2-\left(c+\alpha\frac{d-c}{2}\right)\right]^2\Delta_2 s_2+\int_{t_2}^d\left[s_2-\left(d-\alpha\frac{d-c}{2}\right)\right]^2\right\}\Delta_2 s_2$$

$$=\left[\int_a^{t_1}\left\{s_1^2-2\left(a+\alpha\frac{b-a}{2}\right)\left(s_1-\left(a+\alpha\frac{b-a}{2}\right)-\left(a+\alpha\frac{b-a}{2}\right)^2\right)\right\}\Delta_1 s_1\right.$$

$$+\int_{t_1}^b\left\{s_1^2-2\left(b-\alpha\frac{b-a}{2}\right)\left(s_1-\left(b-\alpha\frac{b-a}{2}\right)-\left(b-\alpha\frac{b-a}{2}\right)^2\right)\right\}\Delta_1 s_1\right]$$

$$\times\left[\int_c^{t_2}\left\{s_2^2-2\left(c+\alpha\frac{d-c}{2}\right)\left(s_2-\left(c+\alpha\frac{d-c}{2}\right)-\left(c+\alpha\frac{d-c}{2}\right)^2\right)\right\}\Delta_2 s_2\right.$$

$$+\int_{t_2}^d\left\{s_2^2-2\left(d-\alpha\frac{d-c}{2}\right)\left(s_2-\left(d-\alpha\frac{d-c}{2}\right)-\left(d-\alpha\frac{d-c}{2}\right)^2\right)\right\}\Delta_2 s_2\right]$$

$$\leq \left[\int_a^{t_1}\left\{\frac{s_1^2+s_1\sigma(s_1)+(\sigma(s_1))^2}{3}-2\left(a+\alpha\frac{b-a}{2}\right)\left(s_1-\left(a+\alpha\frac{b-a}{2}\right)-\left(a+\alpha\frac{b-a}{2}\right)^2\right)\Delta_1 s_1\right\}\right]$$

$$\left[\int_{t_1}^{b}\left\{\frac{s^2+s_1\sigma(s_1)+(\sigma(s_1))^2}{3}-2\left(b-\alpha\frac{b-a}{2}\right)\left(s_1-\left(b-\alpha\frac{b-a}{2}\right)-\left(b-\alpha\frac{b-a}{2}\right)^2\right)\Delta_1 s_1\right\}\right]$$

$$\left[\int_{c}^{t_2}\left\{\frac{s_2^2+s_2\sigma(s_2)+(\sigma(s_2))^2}{3}-2\left(c+\alpha\frac{d-c}{2}\right)\left(s_2-\left(c+\alpha\frac{d-c}{2}\right)-\left(c+\alpha\frac{d-c}{2}\right)^2\right)\Delta_2 s_2\right\}\right]$$

$$\left[\int_{y}^{d}\left\{\frac{s_2^2+s_2\sigma(s_2)+(\sigma(s_2))^2}{3}-2\left(d-\alpha\frac{d-c}{2}\right)\left(s_2-\left(d-\alpha\frac{d-c}{2}\right)-\left(d-\alpha\frac{d-c}{2}\right)^2\right)\Delta_2 s_2\right\}\right]$$

$$=\frac{t_1^{\,3}-a^3}{3}-2\left(a+\alpha\frac{b-a}{2}\right)\left[h_2\left(t_1,a+\alpha\frac{b-a}{2}\right)-h_2\left(a,a+\alpha\frac{b-a}{2}\right)\right]-\left(a+\alpha\frac{b-a}{2}\right)^2(t_1-a)$$

$$+\frac{b^3-t_1^{\,3}}{3}-2\left(b-\alpha\frac{b-a}{2}\right)\left[h_2\left(b,b-\alpha\frac{b-a}{2}\right)-h_2\left(t_1,b-\alpha\frac{b-a}{2}\right)\right]-\left(b-\alpha\frac{b-a}{2}\right)^2(b-t_1)$$

$$+\frac{t_2^{\,3}-c^3}{3}-2\left(c+\alpha\frac{d-c}{2}\right)\left[h_2\left(t_2,c+\alpha\frac{d-c}{2}\right)-h_2\left(c,c+\alpha\frac{d-c}{2}\right)\right]-\left(c+\alpha\frac{d-c}{2}\right)^2(t_2-c)$$

$$+\frac{d^3-t_2^{3}}{3}-2\left(d-\alpha\frac{d-c}{2}\right)\left[h_2\left(d,d-\alpha\frac{d-c}{2}\right)-h_2\left(t_2,d-\alpha\frac{d-c}{2}\right)\right]-\left(d-\alpha\frac{d-c}{2}\right)^2(d-t_2)$$

Furthermore, we have

$$\int_a^b\int_c^d\left[W(t_1,t_2,s_1,s_2)-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d W(t_1,t_2,s_1,s_2)\Delta_2 s_2\Delta_1 s_1\right]\Delta_2 s_2\Delta_1 s_1$$

$$\times\left[\frac{\partial^2 f(s_1,s_2)}{\Delta_1 s_1\Delta_2 s_2}-\frac{(1}{(b-a)(d-c)}\int_a^b\int_c^d\frac{\partial^2 f(s_1,s_2)}{\Delta_1 s_1\Delta_2 s_2}\right]\Delta_2 s_2\Delta_1 s_1$$

$$\int_a^b\int_c^d W(t_1,t_2,s_1,s_2)\frac{\partial^2 f(s_1,s_2)}{\Delta_1 s_1\Delta_2 s_2}\Delta_2 s_2\Delta_1 s_1-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d W(t_1,t_2,s_1,s_2)\Delta_2 s_2\Delta_1 s_1$$

$$\times\int_a^b\int_c^d\frac{\partial^2 f(s_1,s_2)}{\Delta_1 s_1\Delta_2 s_2}\Delta_2 s_2\Delta_1 s_1$$

$$=\int_a^b\int_c^d W(t_1,t_2,s_1,s_2)\frac{\partial^2 f(s_1,s_2)}{\Delta_1 s_1\Delta_2 s_2}\Delta_2 s_2\Delta_1 s_1-\frac{[f(b,d)-f(a,d)-f(b,c)+f(a,c)]}{(b-a)(d-c)}$$

$$\times\int_a^b\int_c^d W(t_1,t_2,s_1,s_2)\Delta_2 s_2\Delta_1 s_1$$

On the other hand

$$\int_a^b\int_c^d\left[W(t_1,t_2,s_1,s_2)-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d W(t_1,t_2,s_1,s_2)\Delta_2 s_2\Delta_1 s_1\right]\Delta_2 s_2\Delta_1 s_1$$

$$\times\left[\frac{\partial^2 f(s_1,s_2)}{\Delta_1 s_1\Delta_2 s_2}-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d\frac{\partial^2 f(s_1,s_2)}{\Delta_1 s_1\Delta_2 s_2}\right]\Delta_2 s_2\Delta_1 s_1$$

$$\leq \left\| W(t_1, t_2, ...) - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d W(t_1, t_2, s_1, s_2) \Delta_2 s_2 \Delta_1 s_1 \right\|_2$$

$$\times \left\| \frac{\partial^2 f(.,.)}{\Delta_1 s \Delta_2 t} - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d \frac{\partial^2 f(s_1, s_2)}{\Delta_1 s_1 \Delta_2 s_2} \right\|_2$$

$$= \left[\int_a^b \int_c^d W^2(t_1, t_2, s_1, s_2) \Delta_2 s_2 \Delta_1 s_1 - \frac{1}{(b-a)(d-c)} \left(\int_a^b \int_c^d W(t_1, t_2, s_1, s_2) \Delta_2 s_2 \Delta_1 s_1 \right)^2 \right]^{\frac{1}{2}}$$

$$\times \left[\int_a^b \int_c^d \left(\frac{\partial^2 f(s_1, s_2)}{\Delta_1 s_1 \Delta_2 s_2} \right)^2 - \frac{1}{(b-a)(d-c)} \left(\int_a^b \int_c^d \frac{\partial^2 f(s_1, s_2)}{\Delta_1 s \Delta_2 s_2} \right)^2 \right]^{\frac{1}{2}}$$

$$\times \left[\int_a^b \int_c^d W^2(t_1, t_2, s_1, s_2) \Delta_2 s_2 \Delta_1 s_1 - \frac{1}{(b-a)(d-c)} \left(\int_a^b \int_c^d W(t_1, t_2, s_1, s_2) \Delta_2 s_2 \Delta_1 s_1 \right)^2 \right]^{\frac{1}{2}}$$

$$\times \sqrt{T(f^{\Delta_1 \Delta_2})}.$$

$\square$

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The Henstock-Stieltjes integral for fuzzy-number-valued functions on a infinite interval[†]

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Abstract: In this paper, the Henstock-Stieltjes integral for fuzzy-number-valued functions on a infinite interval which is an extension of the usual fuzzy Riemann-Stieltjes integral on infinite interval is firstly defined and discussed. Several necessary and sufficient conditions of the integrability for fuzzy-number-valued functions are given by means of the Henstock-Stieltjes integral of real-valued functions on infinite interval and Henstock integral of fuzzy-number-valued functions on infinite interval.
Keywords: Fuzzy numbers; Fuzzy Henstock integral; Stieltjes integral
AMS subject classifications. 26E50; 28E10.

1 Introduction

Recently, in order to complete the theory of fuzzy integrals and to meet the solving need of the fuzzy differential equations [1-3], fuzzy integrals of fuzzy-number-valued functions have been studied by many authors from different points of views, including Nanda [4], Wu et al. [5] and other authors [6-9]. As an extension for Riemann integral and Lebesgue integral, the Stieltjes integral plays an important role in probability theory, stochastic processes, physics, econometrics, biometrics and numerical analysis[10-13] in the Mathematics analysis. In fact, the establishment of the Stieltjes integral was related to the moment of inertia in physics [14]. Until 1909, Riesz presented a general expression for the linear functional of the space of the continuous functions in a finite interval by Stieltjes integral [15]. After Riesz' work, people find that the Stieltjes integral is a powerful tool in several branches of mathematics. In the fuzzy analysis, in 1968, Zadeh defined the probability measure of a fuzzy event by using the Lebesgue-Stieltjes integral of the membership function [16]. It is well known that the notion of the Stieltjes integral for fuzzy-number-valued functions was originally proposed by Nanda [4] in 1989. In 1998, Wu [17] discussed and defined the concept of fuzzy Riemann-Stieltjes integral by means of the representation theorem of fuzzy-number-valued functions, whose membership function could be obtained by solving a nonlinear programming problem, but it is difficult to calculate and extend to the higher-dimensional space. In 2006, Ren et al. introduced the concept of two kinds of fuzzy Riemann-Stieltjes integral for fuzzy-number-valued functions [18,19] and showed that a continuous fuzzy-number-valued function was fuzzy Riemann-Stieltjes integrable with respect to a real-valued increasing function. To overcome the limitations of the existing studies and to characterize continuous linear functionals on the space of Henstock integrable fuzzy-number-valued functions, in 2014, the concept of the Henstock-Stieltjes integral for fuzzy-number-valued functions is defined and discussed, and some useful results for this integral are shown [20].

The expectations of fuzzy random variables were investigated by M. L. Puri and D. A. Ralescu in 1986 [21]. It well known that the notion of a fuzzy random variable as a fuzzy-number-valued function and the expectation $E(X)$ of a fuzzy random variable X was defined by a fuzzy integral $E(X) = \int X$ or set-valued integral of X_λ [21]. In 2007, the concept of the fuzzy Henstock integral on infinite interval is proposed and discussed in order to solve the expectation $E(X)$ of a fuzzy random variable X which distribution function has some kinds of discontinuity or non-integrability [7]. In this paper, the Henstock-Stieltjes integral for fuzzy-number-valued functions on infinite interval which is an extension of the usual fuzzy Riemann-Stieltjes integral on infinite interval is firstly defined and discussed. Several necessary and sufficient conditions of the integrability for fuzzy-number-valued functions are given by means of the

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Henstock-Stieltjes integral of real-valued functions on infinite interval and Henstock integral of fuzzy-number-valued functions on infinite interval.

2 Preliminaries

Fuzzy set $\tilde{u} \in E^1$ is called a fuzzy number if $\tilde{u}$ is a normal, convex fuzzy set, upper semi-continuous and supp $u = \overline{\{x \in \mathbb{R} \mid u(x) > 0\}}$ is compact. Here $\bar{A}$ denotes the closure of A. We use E^1 to denote the fuzzy number space [22].

Let $\tilde{u}, \tilde{v} \in E^1, k \in \mathbb{R}$, the addition and scalar multiplication are defined by

$$[\tilde{u} + \tilde{v}]_\lambda = [\tilde{u}]_\lambda + [\tilde{v}]_\lambda, \qquad [k\tilde{u}]_\lambda = k[\tilde{u}]_\lambda,$$

respectively, where $[\tilde{u}]_\lambda = \{x : u(x) \geqslant \lambda\} = [u_\lambda^-, u_\lambda^+]$, for any $\lambda \in [0, 1]$.

We use the Hausdorff distance between fuzzy numbers given by $D : E^1 \times E^1 \to [0, +\infty)$ as follows [22]:

$$D(\tilde{u}, \tilde{v}) = \sup_{\lambda \in [0,1]} d([\tilde{u}]_\lambda, [\tilde{v}]_\lambda) = \sup_{\lambda \in [0,1]} \max\{|u_\lambda^- - v_\lambda^-|, |u_\lambda^+ - v_\lambda^+|\},$$

where d is the Hausdorff metric. $D(\tilde{u}, \tilde{v})$ is called the distance between $\tilde{u}$ and $\tilde{v}$.

Lemma 2.1 [22]. If $\tilde{u} \in E^1$, then
(1) $[\tilde{u}]_\lambda$ is non-empty bounded closed interval for all $\lambda \in [0, 1]$;
(2) $[\tilde{u}]_{\lambda_1} \supset [\tilde{u}]_{\lambda_2}$ for any $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant 1$;
(3) for any $\{\lambda_n\}$ converging increasingly to $\lambda \in (0, 1]$,

$$\bigcap_{n=1}^{\infty} [\tilde{u}]_{\lambda_n} = [\tilde{u}]_\lambda.$$

Conversely, if for all $\lambda \in [0, 1]$, there exists $A_\lambda \subset \mathbb{R}$ satisfying $(1) \sim (3)$, then there exists a unique $\tilde{u} \in E^1$ such that $[\tilde{u}]_\lambda = A_\lambda, \lambda \in (0, 1]$, and $[\tilde{u}]_0 = \overline{\bigcup_{\lambda \in (0,1]} [\tilde{u}]_\lambda} \subset A_0$.

Definition 2.1 [7, 20, 23]. $\overline{R}$ denote the generalized real line, for $\tilde{f}$ defined on $[a, +\infty]$, we define $\tilde{f}(+\infty) = \tilde{0}$, and $\tilde{0} \cdot (+\infty) = \tilde{0}$.

Let $\delta : [a, +\infty] \to R^+$ be a positive real function. A division $P = \{[x_{i-1}, x_i]; \xi_i\}$ is said to be δ-fine, if the following conditions are satisfied:
$(1) a = x_0 < x_1 < ... < x_{n-1} = b < x_n = +\infty$;
$(2) \xi_i \in [x_{i-1}, x_i] \subset O(\xi_i), i = 1, 2, ..., n$;
where $O(\xi_i) = (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ for $i = 1, 2, ..., n-1$, and $O(\xi_n) = [b, +\infty)$.

For brevity, we write $T = \{[u, v]; \xi\}$, where $[u, v]$ denotes a typical interval in T and ξ is the associated point of $[u, v]$.

Definition 2.2. Let $\alpha : [a, +\infty] \to \mathbb{R}$ be an increasing function. A function $f : [a, +\infty] \to \mathbb{R}$ is Henstock-Stieltjes integrable with respect to α on $[a, +\infty]$ if there exists a real number I such that for every $\varepsilon > 0$, there is a function $\delta(x) > 0$ on $[a, +\infty]$ such that for any δ-fine division $T = \{[x_{i-1}, x_i]; \xi_i\}_{i=1}^n$, we have

$$|\sum_{i=1}^{n} f(\xi_i)[\alpha(x_i) - \alpha(x_{i-1})] - I| < \varepsilon.$$

As usual, we write $(HS) \int_a^{+\infty} f(x) \mathrm{d}\alpha = I$ and $(f, \alpha) \in HS[a, +\infty]$.

Recall, also, that a function $\tilde{f} : [a, b] \to E^1$ is said to be bounded if there exists $M \in \mathbb{R}$ such that $\|\tilde{f}(x)\| = D(\tilde{f}(x), \tilde{0}) \leqslant M$ for any $x \in [a, b]$. Notice that here $\|\tilde{f}(x_0)\|$ does not stand for the norm of E^1.

3 The fuzzy Henstock-Stieltjes integral on infinite interval and its properties

In this section we shall give the definition of the Henstock-Stieltjes integral for fuzzy-number-valued functions on a infinite interval.

Definition 3.1. Let $\alpha : [a, +\infty] \to \mathbb{R}$ be an increasing function. A fuzzy-number-valued function $\tilde{f}(x)$ is said to be fuzzy Henstock-Stieltjes integrable with respect to α on $[a, +\infty]$ if there exists a fuzzy

number $\tilde{H} \in E^1$ such that for every $\varepsilon > 0$, there is a function $\delta(x) > 0$ on $[a, +\infty]$ such that for any δ-fine division $T = \{[x_{i-1}, x_i]; \xi_i\}_{i=1}^n$, we have

$$D(\sum_{i=1}^{n} \tilde{f}(\xi_i)[\alpha(x_i) - \alpha(x_{i-1})], \tilde{H}) < \varepsilon.$$

We write $(FHS) \int_a^{+\infty} \tilde{f}(x)\mathrm{d}\alpha = \tilde{H}$ and $(\tilde{f}, \alpha) \in FHS[a, +\infty]$.

The definition of $\tilde{f} \in FHS(-\infty, a]$ is similar. Naturally, we define $\tilde{f} \in FHS(-\infty, +\infty)$ iff $\tilde{f} \in FHS(-\infty, a]$ and $\tilde{f} \in FHS[a, +\infty)$, and furthermore

$$(FHS) \int_{-\infty}^{+\infty} \tilde{f}(x)dd\alpha = (FHS) \int_{-\infty}^{a} \tilde{f}(x)dd\alpha + (FHS) \int_{a}^{+\infty} \tilde{f}(x)dd\alpha.$$

Remark 3.1. It is clear, if $\tilde{f}(x)$ is a real-valued function then Definition 3.1 implies the definition of (HS) integral on infinite interval introduced by [20]; if $\alpha(x) = x$, then Definition 3.1 implies the definition of (FH) integral introduced by Gong et al. [7].

Remark 3.2. From the definition of the fuzzy Henstock-Stieltjes integral and the fact that (E^1, D) is a complete metric space, we can easily obtain the following conclusions.

Theorem 3.1. Let $\alpha : [a, +\infty] \to \mathbb{R}$ be an increasing function. A fuzzy-number-valued function $\tilde{f}$ is fuzzy Henstock-Stieltjes integrable with respect to α on $[a, +\infty]$ if and only if for every $\epsilon > 0$, there is a function $\delta(x) > 0$ on $[a, +\infty]$ such that for any δ-fine division $T = \{[u, v]; \xi\}$ and $T' = \{[u', v']; \xi'\}$, we have

$$D(\sum_T \tilde{f}(\xi)[\alpha(v) - \alpha(u)], \sum_{T'} \tilde{f}(\xi')[\alpha(v') - \alpha(u')]) < \varepsilon.$$

Theorem 3.2. Let $\alpha : [a, +\infty] \to \mathbb{R}$ be an increasing function and let $\tilde{f} : [a, +\infty] \to E^1$. Then the following statements are equivalent:

(1) $(\tilde{f}, \alpha) \in FHS[a, +\infty]$ and $(FHS) \int_a^{+\infty} \tilde{f}(x)\mathrm{d}\alpha = \tilde{A}$;

(2) for any $\lambda \in [0, 1]$, f_λ^- and f_λ^+ are Henstock-Stieltjes integrable with respect to α on $[a, +\infty]$ for any $\lambda \in [0, 1]$ uniformly ($\delta(x)$ is independent of $\lambda \in [0, 1]$), and

$$[(FHS) \int_a^{+\infty} \tilde{f}(x)\mathrm{d}\alpha]_\lambda = [(HS) \int_a^{+\infty} f_\lambda^-(x)\mathrm{d}\alpha, (HS) \int_0^{+\infty} f_\lambda^+(x)\mathrm{d}\alpha].$$

(3) For any $b > a$, $\tilde{f} \in FH[a, b]$, $\lim_{b \to +\infty} \int_a^b \tilde{f}(x)\mathrm{d}\alpha$ as a fuzzy number exists and

$$\lim_{b \to +\infty} \int_a^b \tilde{f}(x)\mathrm{d}\alpha = \int_a^{+\infty} \tilde{f}(x)\mathrm{d}\alpha.$$

Proof. First, we prove that (1) is equivalent to (2).

(1) *implies* (2): If $\int_a^{+\infty} \tilde{f}(x)\mathrm{d}\alpha = \tilde{A}$, then given $\varepsilon > 0$, there exists a positive-valued function $\delta(x)$ on $[a, +\infty]$ such that for any $\delta-$fine division of $[a, +\infty]$: $T = \{[x_{i-1}, x_i]; \xi_i\}$, we have

$$D(\sum_i \tilde{f}(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})), \tilde{A}) < \varepsilon,$$

i.e.

$$\sup_{\lambda \in [0,1]} \max\{|[\sum_i \tilde{f}(\xi_i)(\alpha(x_i) - \alpha(x_{i-1}))]_\lambda^- - A_\lambda^-|, |[\sum_i \tilde{f}(\xi_i)(\alpha(x_i) - \alpha(x_{i-1}))]_\lambda^+ - A_\lambda^+|\} < \varepsilon,$$

so for any $\lambda \in [0, 1]$ and any $\delta-$fine division $T = \{[x_{i-1}, x_i]; \xi_i\}$, we have

$$|[\sum_i \tilde{f}(\xi_i)(\alpha(x_i) - \alpha(x_{i-1}))]_\lambda^- - A_\lambda^-| = |\sum_i f_\lambda^-(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})) - A_\lambda^-| < \varepsilon,$$

$$|[\sum_i \tilde{f}(\xi_i)(\alpha(x_i) - \alpha(x_{i-1}))]_\lambda^+ - A_\lambda^+| = |\sum_i f_\lambda^+(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})) - A_\lambda^+| < \varepsilon.$$

Thus, f_λ^- and f_λ^+ are Henstock integrable uniformly with respect to $\lambda \in [0,1]$ on $[a, +\infty)$, and

$$A_\lambda = [\int_a^{+\infty} f_\lambda^-(x)\mathrm{d}\alpha, \int_a^{+\infty} f_\lambda^+(x)\mathrm{d}\alpha].$$

Conversely, since f_λ^- and f_λ^+ are Henstock integrable uniformly with respect to $\lambda \in [0,1]$ on $[a, +\infty)$, then given $\varepsilon > 0$, there exists a positive-valued function $\delta(x)$ on $[a, +\infty]$ such that for any $\delta-$fine division $T = \{[x_{i-1}, x_i]; \xi_i\}$, and for any $\lambda \in [0,1]$, we have

$$|\sum_i f_\lambda^-(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})) - A_\lambda^-| < \varepsilon, \tag{0.1}$$

$$|\sum_i f_\lambda^+(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})) - A_\lambda^+| < \varepsilon. \tag{0.2}$$

We can prove that the class of closed intervals $\{[A_\lambda^-, A_\lambda^+] : \lambda \in [0,1]\}$ determines a fuzzy number. In fact, $[A_\lambda^-, A_\lambda^+]$ satisfies the conditions of lemma 2.1.
 (1) Since $f_\lambda^-(x) \leq f_\lambda^+(x), \lambda \in [0,1]$, then $A_\lambda^- \leq A_\lambda^+$, i.e. $[A_\lambda^-, A_\lambda^+]$ is a closed interval, $\lambda \in [0,1]$,
 (2) For any $0 \leq \lambda_1 \leq \lambda_2 \leq 1$,

$$f_{\lambda_1}^-(x) \leq f_{\lambda_2}^-(x) \leq f_{\lambda_2}^+(x) \leq f_{\lambda_1}^+(x).$$

This implies

$$\int_a^{+\infty} f_{\lambda_1}^-(x)\mathrm{d}\alpha \leq \int_a^{+\infty} f_{\lambda_2}^-(x)\mathrm{d}\alpha \leq \int_a^{+\infty} f_{\lambda_2}^+(x)\mathrm{d}\alpha \leq \int_a^{+\infty} f_{\lambda_1}^+(x)\mathrm{d}\alpha.$$

That is, $[A_{\lambda_1}^-, A_{\lambda_1}^+] \supset [A_{\lambda_2}^-, A_{\lambda_2}^+]$.
 (3) For any $\{\lambda_n\}$ increasingly converging to $\lambda \in (0,1]$,

$$\bigcap_{n=1}^{\infty} [\tilde{f}(x)]_{\lambda_n} = [\tilde{f}(x)]_\lambda,$$

i.e.

$$\bigcap_{n=1}^{\infty} [f_{\lambda_n}^-(x), f_{\lambda_n}^+(x)] = [f_\lambda^-(x), f_\lambda^+(x)].$$

That is

$$\lim_{n\to\infty} f_{\lambda_n}^-(x) = f_\lambda^-(x), \lim_{n\to\infty} f_{\lambda_n}^+(x) = f_\lambda^+(x).$$

Note that

$$f_0^-(x) \leq f_{\lambda_n}^-(x) \leq f_1^-(x), f_1^+(x) \leq f_{\lambda_n}^+(x) \leq f_0^+(x).$$

This implies

$$0 \leq f_{\lambda_n}^-(x) - f_0^-(x) \leq f_1^-(x) - f_0^-(x), 0 \leq f_{\lambda_n}^+(x) - f_1^+(x) \leq f_0^+(x) - f_1^+(x).$$

By the non-negativeness and Henstock integrability of $f_1^- - f_0^-, f_0^+ - f_1^+$, we know that $f_1^- - f_0^-, f_0^+ - f_1^+$ are Lebesgue integrable (refer to [9]). Hence $f_{\lambda_n}^-(x) - f_0^-(x), f_{\lambda_n}^+(x) - f_1^+(x)$ are Lebesgue integrable, and

$$\lim_{n\to\infty} \int_a^{+\infty} (f_{\lambda_n}^-(x) - f_0^-(x))\mathrm{d}\alpha = \int_a^{+\infty} (f_\lambda^-(x) - f_0^-(x))\mathrm{d}\alpha,$$

$$\lim_{n\to\infty} \int_a^{+\infty} (f_{\lambda_n}^+(x) - f_1^+(x))\mathrm{d}\alpha = \int_a^{+\infty} (f_\lambda^+(x) - f_1^+(x))\mathrm{d}\alpha.$$

That is

$$\lim_{n\to\infty} \int_a^{+\infty} f_{\lambda_n}^-(x)\mathrm{d}\alpha = \int_a^{+\infty} f_\lambda^-(x)\mathrm{d}\alpha, \lim_{n\to\infty} \int_a^{+\infty} f_{\lambda_n}^+(x)\mathrm{d}\alpha = \int_a^{+\infty} f_\lambda^+(x).$$

Thus, $\bigcap_{n=1}^{\infty} [A_{\lambda_n}^-, A_{\lambda_n}^+] = [A_\lambda^-, A_\lambda^+]$.

Combining the inequality (1) and (2) we obtain

$$D(\sum_i (\alpha(x_i) - \alpha(x_{i-1}))\tilde{f}(\xi_i), \tilde{A}) < \varepsilon,$$

i.e.

$$\tilde{f} \in FH[a, +\infty), \int_a^{+\infty} \tilde{f}(x)\mathrm{d}\alpha = \tilde{A}.$$

We'll prove that (i) is equivalent to (iii) as follows.

(1) *implies* (3): Let $\varepsilon > 0$. Suppose $\tilde{f} \in FH[a, +\infty)$. There exists a positive-valued function δ on $[a, +\infty]$ such that

$$D(\sum_i \tilde{f}(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})), \int_a^{+\infty} \tilde{f}) < \varepsilon.$$

for any $\delta-$fine division of $[a, +\infty] : T = \{[x_{i-1}, x_i]; \xi_i\}_{i=1}^n$. On the other hand, by the Cuachy Rule about $\tilde{f} \in FH[a, b]$ (refer to Th 2.3 of [24]), then $\tilde{f} \in FH[a, b]$ for any $b > a$. There is a positive-valued function δ_1 on $[a, b]$ such that for any δ_1-fine division of $[a, b] : T = \{[x_{i-1}, x_i]; \xi_i\}_{i=1}^n$, we have

$$D(\sum_i \tilde{f}(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})), \int_a^b \tilde{f}) < \varepsilon.$$

We may assume that $\delta_1 \leq \delta$ for any $\xi \in [a, b]$. Then

$$D(\int_a^{+\infty} \tilde{f}, \int_a^b \tilde{f})$$

$$\leq D(\sum_i \tilde{f}(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})), \int_a^{+\infty} \tilde{f}) + D(\sum_i \tilde{f}(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})), \int_a^b \tilde{f}) + D(\tilde{f}(+\infty)\mu([b, +\infty))$$

$$< \varepsilon + \varepsilon = 2\varepsilon.$$

Hence

$$\lim_{b \to +\infty} \int_a^b \tilde{f}(x)\mathrm{d}\alpha = \int_a^{+\infty} \tilde{f}(x)\mathrm{d}\alpha.$$

(3) *implies* (1): Let $\varepsilon > 0$. Choose a sequence $a = b_0 < b_1 < b_2 < ...., b_k \uparrow +\infty$. Since $\tilde{f} \in FH[b_{k-1}, b_k], k = 1, 2, 3, ...$, there exist δ_k such that

$$D(\sum_{[b_{k-1}, b_k]} \tilde{f}(\xi)(v - u), \int_{b_{k-1}}^{b_k} \tilde{f}) < \varepsilon/2^{k+2}.$$

for any δ_k-fine division on $[b_{k-1}, b_k], k = 1, 2, 3, ...$ Suppose $\lim_{b \to +\infty} \int_a^b \tilde{f}(x)\mathrm{d}\alpha = \tilde{A}$. Choose N such that $b > b_N$ which implies $D(\int_a^b \tilde{f}(x)\mathrm{d}\alpha, \tilde{A}) < \varepsilon/2$.

Define

$$\delta(\xi) = \begin{cases} \delta_1(\xi), & \xi \in [b_0, b_1), \\ \delta_k(\xi), & \xi \in (b_{k-1}, b_k), k = 1, 2, 3, ..., \\ min(\delta_k(b_k), \delta_{k+1}(b_k)) & \xi = b_k, k = 1, 2, 3, ... \end{cases}$$

For any $\delta-$fine division $P = \{[x_{i-1}, x_i]; \xi_i\}$ satisfies $i = 1, 2, ..., n$, $O(\xi_n) = [b, +\infty)$ and $b > b_N$, we have

$$D(\sum_i \tilde{f}(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})), \tilde{A})$$

$$\leq D(\int_a^b \tilde{f}, \tilde{A}) + D(\int_a^b \tilde{f}, \sum_i \tilde{f}(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})))$$

$$< \varepsilon/2 + D(\int_a^b \tilde{f}, \sum_{i=1}^{n-1} \tilde{f}(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})) + \tilde{f}(+\infty)\mu([b, +\infty))$$

$$\leq \varepsilon/2 + \sum_{k=1}^{+\infty} \varepsilon/2^{k+2} = 2\varepsilon$$

Hence, $\tilde{f} \in FH[a, +\infty)$ and

$$\lim_{b \to +\infty} \int_a^b \tilde{f}(x)\mathrm{d}\alpha = \int_a^{+\infty} \tilde{f}(x)dx.$$

The proof is complete.

Theorem 3.3. Let $\alpha : [a, +\infty) \to \mathbb{R}$ be an increasing function such that $\alpha \in C^1[a, +\infty)$ and $\tilde{f} : [a, +\infty) \to E^1$ be a bounded fuzzy-number-valued function. Then $\tilde{f}$ is fuzzy Henstock-Stieltjes integrable with respect to α on $[a, +\infty)$ if and only if $\tilde{f}\alpha'$ is fuzzy Henstock integrable on $[a, +\infty)$. Furthermore, we have

$$(FHS) \int_a^{+\infty} \tilde{f}(x)\mathrm{d}\alpha = (FH) \int_a^{+\infty} \tilde{f}(x)\alpha'(x)\mathrm{d}x,$$

where (FH) integral denotes the fuzzy Henstock integral introduced by Wu et al. [5].

Proof. Since $\tilde{f} : [a, +\infty) \to E^1$ is bounded on $[a, +\infty)$, $\sup_{x \in [a, +\infty)} D(\tilde{f}(x), \tilde{0})$ exists. Continuity of α' on $[a, b]$ implies uniform continuity on $[a, b]$ for any $b > a$. Hence, for each $\epsilon > 0$, there exists $\eta > 0$ such that

$$|\alpha'(x) - \alpha'(y)| < \frac{\epsilon}{3 \sup_{x \in [a, +\infty)} D(\tilde{f}(x), \tilde{0}) \cdot (b - a)}$$

for any $x, y \in [a, b]$ satisfying $|x - y| < \eta$. Choose a positive-valued function $\delta_1(x)$ on $[a, b]$ with $\delta_1(x) < \eta$ for all $x \in [a, b]$. Let $T = \{[x_{i-1}, x_i]; \xi_i\}_{i=1}^n$ be a δ_1-fine division on $[a, b]$, then by Lagrange mean value theorem, there exists $\bar{x}_i \in [x_{i-1}, x_i]$ such that

$$\alpha(x_i) - \alpha(x_{i-1}) = \alpha'(\bar{x}_i)(\alpha(x_i) - \alpha(x_{i-1})), \qquad (1 \leqslant i \leqslant n).$$

Since $|\bar{x}_i - x_i| \leqslant \delta_1(x_i) < \eta$ for $1 \leqslant i \leqslant n$, we have

$$|\alpha'(\bar{x}_i) - \alpha'(x_i)| < \frac{\epsilon}{3 \sup_{x \in [a, +\infty)} D(\tilde{f}(x), \tilde{0}) \cdot (b - a)}$$

for $1 \leqslant i \leqslant n$. Hence, for any δ_1-fine division $T = \{[x_{i-1}, x_i]; \xi_i\}_{i=1}^n$ on $[a, b]$, we have

$$D(\sum_{i=1}^n \tilde{f}(\xi_i)[\alpha(x_i) - \alpha(x_{i-1})], \sum_{i=1}^n \tilde{f}(\xi_i)\alpha'(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})))$$

$$= D(\sum_{i=1}^n \tilde{f}(\xi_i)\alpha'(\bar{x}_i)(\alpha(x_i) - \alpha(x_{i-1})), \sum_{i=1}^n \tilde{f}(\xi_i)\alpha'(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})))$$

$$\leqslant \sum_{i=1}^n D(\tilde{f}(\xi_i)\alpha'(\bar{x}_i)(\alpha(x_i) - \alpha(x_{i-1})), \tilde{f}(\xi_i)\alpha'(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})))$$

$$= \sum_{i=1}^{n} \sup_{\lambda \in [0,1]} \max\{|f_\lambda^-(\xi_i)[\alpha'(\bar{x}_i) - \alpha'(\xi_i)]|, |f_\lambda^+(\xi_i)[\alpha'(\bar{x}_i) - \alpha'(\xi_i)]|\}(\alpha(x_i) - \alpha(x_{i-1}))$$

$$\leqslant \sum_{i=1}^{n} |\alpha'(\bar{x}_i) - \alpha'(\xi_i)|(\alpha(x_i) - \alpha(x_{i-1})) \sup_{\lambda \in [0,1]} \max\{|f_\lambda^-(\xi_i)|, |f_\lambda^+(\xi_i)|\}$$

$$\leqslant (b - a) \cdot \frac{\epsilon}{3 \sup_{x \in [a, +\infty)} D(\tilde{f}(x), \tilde{0}) \cdot (b - a)} \cdot \sup_{x \in [a, +\infty)} D(\tilde{f}(x), \tilde{0})$$

$$< \frac{\epsilon}{3}. \qquad\qquad (*)$$

On the other hand, since $\tilde{f}$ is fuzzy Henstock-Stieltjes integrable with respect to α on $[a, +\infty)$, by Theorem 3.1, there is a function $\delta_2(x) > 0$ such that for any δ_2-fine division $T = \{[u, v]; \xi\}$ and $T' = \{[u', v']; \xi'\}$ on $[a, b]$, we have

$$D(\sum_T \tilde{f}(\xi)[\alpha(v) - \alpha(u)], \sum_{T'} \tilde{f}(\xi')[\alpha(v') - \alpha(u')]) < \frac{\varepsilon}{3}.$$

Define $\delta(x)$ on $[a, b]$ by $\delta(x) = \min\{\delta_1(x), \delta_2(x)\}$. Then for any δ-fine division $T = \{[u, v]; \xi\}$ and $T' = \{[u', v']; \xi'\}$ on $[a, b]$, we have

$$D(\sum_T \tilde{f}(\xi)\alpha'(\xi)(v - u), \sum_{T'} \tilde{f}(\xi')\alpha'(\xi')(v' - u'))$$

$$\leqslant D(\sum_T \tilde{f}(\xi)\alpha'(\xi)(v - u), \sum_T \tilde{f}(\xi)[\alpha(v) - \alpha(u)])$$

$$+ D(\sum_T \tilde{f}(\xi)[\alpha(v) - \alpha(u)], \sum_{T'} \tilde{f}(\xi')[\alpha(v') - \alpha(u')])$$

$$+ D(\sum_{T'} \tilde{f}(\xi')[\alpha(v') - \alpha(u')], \sum_{T'} \tilde{f}(\xi')\alpha'(\xi')(v' - u'))$$

$$< \epsilon.$$

Hence, $\tilde{f}\alpha'$ is Henstock integrable on $[a, b]$ for any $[a, b]$ by Theorem 2.3 of [5], and by above formula (*), we know that

$$(FHS) \int_a^b \tilde{f}(x)\mathrm{d}\alpha = (FH) \int_a^b \tilde{f}(x)\alpha'(x)\mathrm{d}x.$$

Applied Theorem 3.1, $\tilde{f}\alpha'$ is Henstock integrable on $[a, +\infty)$.

Conversely, if $\tilde{f}\alpha'$ is Henstock integrable on $[a, +\infty)$, then by Theorem 2.3 of [5], for each $\epsilon > 0$, there is a function $\delta_3(x) > 0$ such that for any δ_3-fine division $T = \{[u, v]; \xi\}$ and $T' = \{[u', v']; \xi'\}$, we have

$$D(\sum_T \tilde{f}(\xi)\alpha'(\xi)(v - u), \sum_{T'} \tilde{f}(\xi')\alpha'(\xi')(v' - u')) < \frac{\varepsilon}{3}.$$

Define $\delta(x)$ on $[a, +\infty)$ by $\delta(x) = \min\{\delta_1(x), \delta_3(x)\}$. Then for any δ-fine division $T = \{[u, v]; \xi\}$ and $T' = \{[u', v']; \xi'\}$, we have

$$D(\sum_T \tilde{f}(\xi)[\alpha(v) - \alpha(u)], \sum_{T'} \tilde{f}(\xi')[\alpha(v') - \alpha(u')])$$

$$\leqslant D(\sum_T \tilde{f}(\xi)[\alpha(v) - \alpha(u)], \sum_T \tilde{f}(\xi)\alpha'(\xi)(v - u))$$

$$+ D(\sum_T \tilde{f}(\xi)\alpha'(\xi)(v - u), \sum_{T'} \tilde{f}(\xi')\alpha'(\xi')(v' - u'))$$

$$+ D(\sum_{T'} \tilde{f}(\xi')\alpha'(\xi')(v' - u'), \sum_{T'} \tilde{f}(\xi')[\alpha(v') - \alpha(u')])$$

$$< \epsilon.$$

Hence, $\tilde{f}$ is fuzzy Henstock-Stieltjes integrable with respect to α on $[a, +\infty)$.

In the following part, we will prove the equation $(FHS)\int_a^{+\infty} \tilde{f}d\alpha = (FH)\int_a^{+\infty} \tilde{f}\alpha'dx$. For any division $T : a = x_0 < x_1 < x_2 <, \cdots, < x_n = b$, according to the Lagrange mean value theorem, we have

$$\alpha(x_i) - \alpha(x_{i-1}) = \alpha'(\bar{x}_i)(\alpha(x_i) - \alpha(x_{i-1})), \qquad (x_{i-1} < \bar{x}_i < x_i).$$

This implies

$$\sum_{i=1}^{n} \tilde{f}(\bar{x}_i)[\alpha(x_i) - \alpha(x_{i-1})] = \sum_{i=1}^{n} \tilde{f}(\bar{x}_i)\alpha'(\bar{x}_i)(\alpha(x_i) - \alpha(x_{i-1})).$$

That is

$$(FHS)\int_a^{+\infty} \tilde{f}(x)d\alpha = (FH)\int_a^{+\infty} \tilde{f}(x)\alpha'(x)dx.$$

The proof is complete.

Theorem 3.4. Let $\alpha : [a, +\infty) \to \mathbb{R}$ be an increasing function such that $\alpha \in C^1[a, +\infty)$, $|\alpha'(x)| \leqslant M$ and $\tilde{f} = \tilde{0}$ a.e. on $[a, +\infty)$ (i.e. $\tilde{f}(x) = \tilde{0}$ on $[a, +\infty)$ except a Lebesgue zero measure set). Then $(\tilde{f}, \alpha) \in FHS[a, +\infty)$ and $(FHS)\int_a^{+\infty} \tilde{f}(x)d\alpha = \tilde{0}$.

Proof. Since $\alpha(x) \in C^1[a, +\infty)$, and $|\alpha'(x)| \leqslant M$ for all $x \in [a, +\infty)$. By Lagrange mean value theorem, there exists $\xi \in [x_{i-1}, x_i]$ such that

$$\alpha(x_i) - \alpha(x_{i-1}) = \alpha'(\xi)(\alpha(x_i) - \alpha(x_{i-1})) \leqslant M(\alpha(x_i) - \alpha(x_{i-1})).$$

Let $S = \{x | \tilde{f}(x) \neq \tilde{0}\}$ and for each positive integer n, set $S = \bigcup S_n \subset [a, +\infty)$, where $S_n = \{x | n-1 < D(\tilde{f}(x), \tilde{0}) \leqslant n\}, n = 1, 2, 3, \cdots$. For every $\epsilon > 0$ and a positive integer n, choose an open set G_n such that $S_n \subset G_n$ and $\mu(G_n) < \frac{\epsilon}{nM2^n}$. Define $\delta(x)$ on $[a, +\infty)$ by

$$\delta(x) = \begin{cases} 1, & x \in [a, +\infty) \backslash S, \\ \delta(x), & \text{such that } (x - \delta(x), x + \delta(x)) \subset G_n, x \in S_n, n = 1, 2, \cdots. \end{cases}$$

For any δ-fine division $T = \{[x_{i-1}, x_i]; \xi_i\}$, we have

$$D(\int_a^{+\infty} \tilde{f}(x)d\alpha, \tilde{0}) = D(\sum_{\xi_i \in S} \tilde{f}(\xi_i)[\alpha(x_i) - \alpha(x_{i-1})] + \sum_{\xi_i \in [a,+\infty)\backslash S} \tilde{f}(\xi_i)[\alpha(x_i) - \alpha(x_{i-1})], \tilde{0})$$

$$= D(\sum_{\xi_i \in S} \tilde{f}(\xi_i)[\alpha(x_i) - \alpha(x_{i-1})], \tilde{0}) \leqslant \sum_{\xi_i \in S} D(\tilde{f}(\xi_i)[\alpha(x_i) - \alpha(x_{i-1})], \tilde{0})$$

$$\leqslant M \sum_{i=1}^{\infty} \sum_{\xi_i \in S_i} D(\tilde{f}(\xi_i), \tilde{0})(\alpha(x_i) - \alpha(x_{i-1})) < M \sum_{i=1}^{\infty} i \cdot \frac{\varepsilon}{iM2^i}$$

$$= \varepsilon.$$

The proof is complete.

Remark 3.3. Let $\alpha : [a, +\infty) \to \mathbb{R}$ be an increasing function and $\alpha \in C^1[a, +\infty)$, and $|\alpha'(x)| \leqslant M$. If $\tilde{f}(x) = \tilde{g}(x)$ a.e. on $[a, +\infty)$ and $(\tilde{f}, \alpha) \in FHS[a, +\infty)$, then $(\tilde{g}, \alpha) \in FHS[a, +\infty)$ and

$$(FHS)\int_a^{+\infty} \tilde{f}(x)d\alpha = (FHS)\int_a^{+\infty} \tilde{g}(x)d\alpha.$$

Using Theorem 3.4, naturally, we have the following conclusion.

Theorem 3.5. Let $\alpha : [a, +\infty) \to \mathbb{R}$ be an increasing function. If $\tilde{f}(x) = \tilde{0}$ a.e.s. on $[a, +\infty)$ (i.e. $\tilde{f}(x) = \tilde{0}$ on $[a, +\infty)$ except a α-Lebesgue-Stieltjes zero measure set), then $(\tilde{f}, \alpha) \in FHS[a, +\infty)$ and $(FHS)\int_a^{+\infty} \tilde{f}(x)d\alpha = \tilde{0}$.

4 Conclusion

The aim of this paper is attempt to extend the theory of the fuzzy Henstock-Stieltjes integral on a infinite interval, we firstly define and discuss the Henstock-Stieltjes integral for fuzzy-number-valued functions on a infinite interval. On the other hand, the integrability of the fuzzy Henstock-Stieltjes integral on a infinite interval are also shown and discussed. In the future, we shall consider the continuity and the differentiability of the primitive for the fuzzy Henstock-Stieltjes integral on a infinite interval, the quadrature rules for the fuzzy Henstock-Stieltjes integral on a infinite interval, the convergence theorems for sequences of the fuzzy Henstock-Stieltjes integrable functions on a infinite interval, and so on.

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New weighted q-Čebyšev-Grüss type inequalities for double integrals

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Abstract: In this paper, we establish the weighted double q-integrals Montgomery identity for functions of two independent variables, then obtain weighted q-Čebyšev-Grüss type inequalities for double integrals. Furthermore, weighted q-Ostrowski type inequalities for double integrals are also given.

Keywords: Čebyšev-Grüss type inequalities; Ostrowski type inequalities; Montgomery identity; double q-integrals

2010 Mathematics Subject Classification: 34A08; 26D10; 26D15.

1 Introduction and preliminaries

1882, P.L. Čebyšev [7] prove that, if $f', g' \in L_\infty[a,b]$, then

$$|T(f,g)| \leq \frac{1}{12}(b-a)^2 \|f'\|_\infty \|g'\|_\infty, \tag{1.1}$$

where for two functions $f, g : [a,b] \to \mathbb{R}$, the functional

$$T(f,g) = \frac{1}{b-a}\int_a^b f(x)g(x)dx - \left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right), \tag{1.2}$$

and $\|\cdot\|_\infty$ denotes the norm in $L_\infty[a,b]$ defined as $\|f\|_\infty = ess \sup\limits_{t\in[a,b]} |f(t)|$.

In 1935, G. Grüss [13] showed that

$$|T(f,g)| \leq \frac{1}{4}(M-m)(N-n), \tag{1.3}$$

provided m, M, n and N are real numbers satisfying the conditions,

$$m \leq f(x) \leq M, \quad n \leq g(x) \leq N, \tag{1.4}$$

for all $x \in [a,b]$, where $T(f,g)$ is as defined by (1.2).

In 1938, Ostrowski [19] proved the following integral inequality:

Let $f : I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, be a mapping that is differentiable in the interior of I ($IntI$), and let $a, b \in IntI$, $a < b$. If $|f'(t)| \leq M$, $\forall t \in (a,b)$, then,

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{(x-\frac{a+b}{2})^2}{(b-a)^2} \right](b-a)M, \tag{1.5}$$

for all $x \in [a,b]$.

During the past few years, many researchers have given considerable attention to the above results and various generalizations, extensions and variants of these inequalities (1.1), (1.3) and (1.5) have appeared in the literature, see [1, 2, 3, 6, 8, 9, 11, 12, 16, 17, 18, 20, 21, 22] and the references cited therein. Find new

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inequalities in the multidimensional cases still an interesting problem. In [4, 10], the authors proved the double integrals Montgomery identity:

$$f(x,y) = \frac{1}{b-a}\int_a^b f(t,y)dt + \frac{1}{d-c}\int_c^d f(x,s)ds - \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dtds$$
$$+ \int_a^b\int_c^d P(x,t)Q(y,s)\frac{\partial^2 f(t,s)}{\partial t\partial s}dtds, \quad (1.6)$$

where $f : [a,b] \times [c,d] \to \mathbb{R}$ is differentiable, the derivative $\frac{\partial^2 f(t,s)}{\partial t\partial s}$ is integrable on $[a,b] \times [c,d]$, and the Peano kernels $P(x,t)$ and $Q(y,x)$ are defined by

$$P(x,t) = \begin{cases} \frac{t-a}{b-a}, & a \le t \le x, \\ \frac{t-b}{b-a}, & x \le t \le b \end{cases} \quad \text{and} \quad Q(y,s) = \begin{cases} \frac{s-c}{d-c}, & c \le s \le y, \\ \frac{s-d}{d-c}, & y \le s \le d. \end{cases}$$

Furthermore, Guezane-Lakoud and Aissaoui [14] established new extension of the weighted Montgomery identity (1.6) for functions of two independent variables, then obtained new Čebyšev type inequalities.

For the sake of convenience, some definitions and propositions are cited on q-integral as follows. Some details see [5, 15].

In what follows, q is a real number satisfying $0 < q < 1$.

Definition 1.1 ([5])**.** For an arbitrary function $f(x)$, the q-differential is defined by $(d_q f)(x) = f(qx) - f(x)$. In particular, $d_q x = (q-1)x$. q-derivative is defined by

$$(D_q f)(x) = \frac{d_q f(x)}{d_q x} = \frac{f(qx) - f(x)}{(q-1)x}, \quad (D_q f)(0) = \lim_{x\to 0}(D_q f)(x).$$

Clearly, if $f(x)$ is differentiable, then $\lim_{q\to 1^-}(D_q f)(x) = \frac{df(x)}{dx}$. And q-derivatives of higher order by

$$(D_q^0 f)(x) = f(x) \quad \text{and} \quad (D_q^n f)(x) = D_q(D_q^{n-1}f)(x), \quad n \in \mathbb{N}.$$

Definition 1.2 ([5])**.** Suppose $0 < a < b$. The definite q-integral is defined as

$$(I_q f)(x) = \int_0^x f(t)d_q t = x(1-q)\sum_{n=0}^{\infty} f(xq^n)q^n, \quad x \in [0,b]. \quad (1.7)$$

and

$$\int_a^b f(x)d_q x = \int_0^b f(x)d_q x - \int_0^a f(x)d_q x. \quad (1.8)$$

Similarly as done for derivatives, an operator I_q^n can be defined, namely,

$$(I_q^0 f)(x) = f(x) \quad \text{and} \quad (I_q^n f)(x) = I_q(I_q^{n-1}f)(x), \quad n \in \mathbb{N}.$$

The definite q-integral defined above is too general for our purpose of studying inequalities. For example, if $f(x) \ge 0$, it is not necessarily true that $\int_a^b f(x)d_q x \ge 0$.

From now on, we will use a special type of the definite q-integral, which we will call the restricted definite q-integral. Throughout all the paper, we will use the following notations:

$$c_j = bq^j, \quad \text{for} \quad j \in \{0, 1, \cdots, n\}, \quad a = c_n = bq^n.$$

Definition 1.3 ([5])**.** Let $0 < q < 1$, $b > 0$, and $n \in \mathbb{Z}^+$. The restricted q-integral is defined as $\int_{bq^n}^b f(x)d_q x$.

The following formula readily follows from (1.7) and (1.8):

$$\int_a^b f(x)d_qx = \int_{bq^n}^b f(x)d_qx = (1-q)b\sum_{j=0}^{n-1}q^j f(bq^j) = (1-q)\sum_{j=0}^{n-1}c_j f(c_j).$$

Note that the restricted integral $\int_a^b f(x)d_qx$ is just a finite sum, so no questions about convergency arise. It is easy to check that

$$\int_a^b D_q f d_q x = f(b) - f(a).$$

Obviously, if $f(x) \geq g(x)$ on $[a,b]$, then $\int_a^b f(x)d_qx \geq \int_a^b g(x)d_qx$. If $0 < k < n$, then

$$\int_a^b f(x)d_qx = \int_a^{c_k} f(x)d_qx + \int_{c_k}^b f(x)d_qx.$$

The following is the formula for the q-integration by parts:

$$\int_a^b f(x)(D_q g)(x)d_qx = [f(x)g(x)]_a^b - \int_a^b g(qx)(D_q f)(x)d_qx.$$

Cauciman [15] gave q-integral Grüss's inequality as follows: *Assume that (1.4) holds, then*

$$\left| \frac{1}{b-a}\int_a^b f(x)g(x)d_qx - \left(\frac{1}{b-a}\int_a^b f(x)d_qx\right)\left(\frac{1}{b-a}\int_a^b g(x)d_qx\right) \right| \leq \frac{1}{4}(M-m)(N-n).$$

Assume that $w : [a,b] \to [0,\infty)$ satisfying $\int_a^b w(x)d_qx = 1$. Set $W(t) = \int_a^t w(x)d_qx$ for $t \in [a,b]$, $W(t) = 0$ for $t < a$, and $W(t) = 1$ for $t > b$. We give weighted q-integral Peano kernel $P_w(x,t)$ defined by

$$P_w(x,t) = \left\{ \begin{array}{ll} W(t), & a \leq t \leq x, \\ W(t) - 1, & x \leq t \leq b. \end{array} \right.$$

Then the following weighted q-integral Montgomery identity holds: (see [23])

$$f(x) = \int_a^b w(t)f(qt)d_qt + \int_a^b P_w(x,t)(D_q f)(t)d_qt.$$

In 2011, Yang [23] obtained the following inequalities:

$$|T(w,f,g)| \leq \|D_q f\|\|D_q g\| \int_a^b w(x)H^2(qx)d_qx,$$

and

$$|T(w,f,g)| \leq \frac{1}{2}\int_a^b w(x)[|g(qx)|\|D_q f\| + |f(qx)|\|D_q g\|]H(qx)d_qx,$$

where $\|\cdot\|$ as $\|h\| = \sup_{t\in[a,b]}|h(t)|$ for $h \in C[a,b]$,

$$T(w,f,g) = \int_a^b w(x)f(qx)g(qx)d_qx - \left(\int_a^b w(x)f(qx)d_qx\right)\left(\int_a^b w(x)g(qx)d_qx\right),$$

and

$$H(x) = \int_a^b |P_w(x,t)|d_qt$$

for all $x \in [a,b]$.

Motivated by the results mentioned above, by using weighted q-integral Montgomery identity for functions of two independent variables, we establish some new weighted q-Čebyšev type inequalities for double integrals. Furthermore, weighted q-Ostrowski type inequalities for double integrals are also given.

2   Weighted q-Čebyšev type inequalities for double integrals

Assume that $w : [a,b] \to \mathbb{R}_0 = [0,\infty)$ and $u : [c,d] \to \mathbb{R}_0$ satisfying $\int_a^b w(x)d_{q_1}x = \int_c^d u(y)d_{q_2}y = 1$, where $0 < q_1, q_2 < 1$. Set $W(t) = \int_a^t w(x)d_{q_1}x$ for $t \in [a,b]$ and $U(s) = \int_a^s u(y)d_{q_2}y$ for $s \in [c,d]$, so we have $W(a) = U(c) = 0$ and $W(b) = U(b) = 1$. We give the following weighted q-integral Peano kernels $P_w(x,t)$ and $Q_u(y,t)$ defined by

$$P_w(x,t) = \begin{cases} W(t), & a \le t \le x, \\ W(t) - 1, & x \le t \le b, \end{cases} \quad \text{and} \quad Q_u(y,s) = \begin{cases} U(s), & c \le s \le y, \\ U(s) - 1, & y \le s \le d. \end{cases} \tag{2.1}$$

We use the following notations to simplify details of the presentation. Let $\frac{\partial f(t,s)}{d_{q_1}t}$ and $\frac{\partial f(t,s)}{d_{q_2}s}$ be partial q-derivative on t and s, respectively. For some suitable functions $w : [a,b] \to \mathbb{R}_0$, $u : [c,d] \to \mathbb{R}_0$ and $f, g : \Omega = [a,b] \times [c,d] \to \mathbb{R}$, we set

$$T(w,u,f,g) = \int_a^b \int_c^d w(x)u(y)f(q_1x,q_2y)g(q_1x,q_2y)d_{q_1}xd_{q_2}y$$

$$- \int_a^b \int_c^d w(x)u(y)g(q_1x,q_2y)\left(\int_a^b w(t)f(q_1t,q_2y)d_{q_1}t\right)d_{q_1}xd_{q_2}y$$

$$- \int_a^b \int_c^d w(x)u(y)g(q_1x,q_2y)\left(\int_c^d u(s)f(q_1x,q_2s)d_{q_2}s\right)d_{q_1}xd_{q_2}y$$

$$+ \left(\int_a^b \int_c^d w(x)u(y)f(q_1x,q_2y)d_{q_1}xd_{q_2}y\right)\left(\int_a^b \int_c^d w(x)u(y)g(q_1x,q_2y)d_{q_1}xd_{q_2}y\right)$$

and define $\|\cdot\|$ as $\|h\| = \sup_{(t,s)\in\Omega}|h(t,s)|$ for $h \in C(\Omega,\mathbb{R})$.

Theorem 2.1. *Let $f : \Omega \to \mathbb{R}$, $w : [a,b] \to \mathbb{R}_0$ and $u : [c,d] \to \mathbb{R}_0$ satisfying $\int_a^b w(x)d_{q_1}x = \int_c^d u(y)d_{q_2}y = 1$, then*

$$f(x,y) = \int_a^b w(t)f(q_1t,y)d_{q_1}t + \int_c^d u(s)f(x,q_2s)d_{q_2}s - \int_a^b \int_c^d w(t)u(s)f(q_1t,q_2s)d_{q_1}td_{q_2}s$$

$$+ \int_a^b \int_c^d P_w(x,t)Q_u(y,s)\frac{\partial f(t,s)}{d_{q_1}td_{q_2}s}d_{q_1}td_{q_2}s, \tag{2.2}$$

for $(x,y) \in \Omega$, where the weighted q-integral Peano kernels $P_w(x,t)$ and $Q_u(s,y)$ are defined by (2.1).

Proof. According to the weighted q-integral Peano kernels $P_w(x,t)$ and $Q_u(s,y)$ and the proof of Theorem 1 in [23], we obtain

$$\int_a^b \int_c^d P_w(x,t)Q_u(y,s)\frac{\partial f(t,s)}{d_{q_1}td_{q_2}s}d_{q_1}td_{q_2}s = \int_a^b P_w(x,t)\left(\int_c^d Q_u(y,s)\frac{\partial f(t,s)}{d_{q_1}td_{q_2}s}d_{q_2}s\right)d_{q_1}t$$

$$= \int_a^b P_w(x,t)\left(\frac{\partial f(t,y)}{d_{q_1}t} - \int_c^d u(s)\frac{\partial f(t,q_2s)}{d_{q_1}t}d_{q_2}s\right)d_{q_1}t$$

$$= \int_a^b P_w(x,t)\frac{\partial f(t,y)}{d_{q_1}t}d_{q_1}t - \int_c^d u(s)\left(\int_a^b P_w(x,t)\frac{\partial f(t,q_2s)}{d_{q_1}t}d_{q_1}t\right)d_{q_2}s$$

$$= \left(f(x,y) - \int_a^b w(t)f(q_1t,y)d_{q_1}t\right) - \int_c^d u(s)\left(f(x,q_2s) - \int_a^b w(t)f(q_1t,q_2s)d_{q_1}t\right)d_{q_2}s$$

$$= f(x,y) - \int_a^b w(t)f(q_1t,y)d_{q_1}t - \int_c^d u(s)f(x,q_2s)d_{q_2}s + \int_a^b \int_c^d w(t)u(s)f(q_1t,q_2s)d_{q_1}td_{q_2}s.$$

Thus we have

$$f(x,y) = \int_a^b w(t)f(q_1t,y)d_{q_1}t + \int_c^d u(s)f(t,q_2s)d_{q_2}s - \int_a^b \int_c^d w(t)u(s)f(q_1t,q_2s)d_{q_1}td_{q_2}s$$
$$+ \int_a^b \int_c^d P_w(x,t)Q_u(y,s)\frac{\partial f(t,s)}{d_{q_1}td_{q_2}s}d_{q_1}td_{q_2}s,$$

and this completes the proof. $\square$

Theorem 2.2. *Let* $f,g : \Omega \to \mathbb{R}$, $w : [a,b] \to \mathbb{R}_0$ *and* $u : [c,d] \to \mathbb{R}_0$ *satisfying* $\int_a^b w(x)d_{q_1}x = \int_c^d u(y)d_{q_2}y = 1$, *then*

$$|T(w,u,f,g)| \leq \left\|\frac{\partial f(t,s)}{d_{q_1}td_{q_2}s}\right\| \left\|\frac{\partial g(t,s)}{d_{q_1}td_{q_2}s}\right\| \int_a^b \int_c^d w(x)u(y)H^2(q_1x,q_2y)d_{q_1}xd_{q_2}y, \tag{2.3}$$

where

$$H(x,y) = \int_a^b \int_c^d |P_w(x,t)Q_u(y,s)|d_{q_1}td_{q_2}s.$$

Proof. Since the functions f and g satisfy the hypothesis of Theorem 2.1, the following identities hold:

$$f(x,y) = \int_a^b w(t)f(q_1t,y)d_{q_1}t + \int_c^d u(s)f(t,q_2s)d_{q_2}s - \int_a^b \int_c^d w(t)u(s)f(q_1t,q_2s)d_{q_1}td_{q_2}s$$
$$+ \int_a^b \int_c^d P_w(x,t)Q_u(y,s)\frac{\partial f(t,s)}{d_{q_1}td_{q_2}s}d_{q_1}td_{q_2}s, \tag{2.4}$$

and

$$g(x,y) = \int_a^b w(t)g(q_1t,y)d_{q_1}t + \int_c^d u(s)g(x,q_2s)d_{q_2}s - \int_a^b \int_c^d w(t)u(s)g(q_1t,q_2s)d_{q_1}td_{q_2}s$$
$$+ \int_a^b \int_c^d P_w(x,t)Q_u(y,s)\frac{\partial g(t,s)}{d_{q_1}td_{q_2}s}d_{q_1}td_{q_2}s. \tag{2.5}$$

Due to the above two inequalities (2.4) and (2.5), we have

$$f(q_1x,q_2y) = \int_a^b w(t)f(q_1t,q_2y)d_{q_1}t + \int_c^d u(s)f(q_1x,q_2s)d_{q_2}s - \int_a^b \int_c^d w(t)u(s)f(q_1t,q_2s)d_{q_1}td_{q_2}s$$
$$+ \int_a^b \int_c^d P_w(q_1x,t)Q_u(q_2y,s)\frac{\partial f(t,s)}{d_{q_1}td_{q_2}s}d_{q_1}td_{q_2}s, \tag{2.6}$$

and

$$g(q_1x,q_2y) = \int_a^b w(t)g(q_1t,q_2y)d_{q_1}t + \int_c^d u(s)g(q_1x,q_2s)d_{q_2}s - \int_a^b \int_c^d w(t)u(s)g(q_1t,q_2s)d_{q_1}td_{q_2}s$$
$$+ \int_a^b \int_c^d P_w(q_1x,t)Q_u(q_2y,s)\frac{\partial g(t,s)}{d_{q_1}td_{q_2}s}d_{q_1}td_{q_2}s. \tag{2.7}$$

Multiplying (2.6) by (2.7), we obtain

$$\left(f(q_1x,q_2y) - \int_a^b w(t)f(q_1t,q_2y)d_{q_1}t - \int_c^d u(s)f(q_1x,q_2s)d_{q_2}s + \int_a^b \int_c^d w(t)u(s)f(q_1t,q_2s)d_{q_1}td_{q_2}s\right)$$
$$\times \left(g(q_1x,q_2y) - \int_a^b w(t)g(q_1t,q_2y)d_{q_1}t - \int_c^d u(s)g(q_1x,q_2s)d_{q_2}s + \int_a^b \int_c^d w(t)u(s)g(q_1t,q_2s)d_{q_1}td_{q_2}s\right)$$
$$= \left(\int_a^b \int_c^d P_w(q_1x,t)Q_u(q_2y,s)\frac{\partial f(t,s)}{d_{q_1}td_{q_2}s}d_{q_1}td_{q_2}s\right)\left(\int_a^b \int_c^d P_w(q_1x,t)Q_u(q_2y,s)\frac{\partial g(t,s)}{d_{q_1}td_{q_2}s}d_{q_1}td_{q_2}s\right).$$

Consequently,

$$
f(q_1x, q_2y)g(q_1x, q_2y) - f(q_1x, q_2y)\int_a^b w(t)g(q_1t, q_2y)d_{q_1}t - f(q_1x, q_2y)\int_c^d u(s)g(q_1x, q_2s)d_{q_2}s
$$

$$
+ f(q_1x, q_2y)\int_a^b\int_c^d w(t)u(s)g(q_1t, q_2s)d_{q_1}td_{q_2}s - g(q_1x, q_2y)\int_a^b w(t)f(q_1t, q_2y)d_{q_1}t
$$

$$
+ \int_a^b w(t)f(q_1t, q_2y)d_{q_1}t\int_a^b w(t)g(q_1t, q_2y)d_{q_1}t + \int_a^b w(t)f(q_1t, q_2y)d_{q_1}t\int_c^d u(s)g(q_1x, q_2s)d_{q_2}s
$$

$$
- \int_a^b w(t)f(q_1t, q_2y)d_{q_1}t\int_a^b\int_c^d w(t)u(s)g(q_1t, q_2s)d_{q_1}td_{q_2}s - g(q_1x, q_2y)\int_c^d u(s)f(q_1x, q_2s)d_{q_2}s
$$

$$
+ \int_c^d u(s)f(q_1x, q_2s)d_{q_2}s\int_a^b w(t)g(q_1t, q_2y)d_{q_1}t + \int_c^d u(s)f(q_1x, q_2s)d_{q_2}s\int_c^d u(s)g(q_1x, q_2s)d_{q_2}s
$$

$$
- \int_c^d u(s)f(q_1x, q_2s)d_{q_2}s\int_a^b\int_c^d w(t)u(s)g(q_1t, q_2s)d_{q_1}td_{q_2}s + g(q_1x, q_2y)\int_a^b\int_c^d w(t)u(s)f(q_1t, q_2s)d_{q_1}td_{q_2}s
$$

$$
- \int_a^b w(t)g(q_1t, q_2y)d_{q_1}t\int_a^b\int_c^d w(t)u(s)f(q_1t, q_2s)d_{q_1}td_{q_2}s - \int_a^b\int_c^d w(t)u(s)f(q_1t, q_2s)d_{q_1}td_{q_2}s
$$

$$
\times \int_c^d u(s)g(q_1x, q_2s)d_{q_2}s + \int_a^b\int_c^d w(t)u(s)f(q_1t, q_2s)d_{q_1}td_{q_2}s\int_a^b\int_c^d w(t)u(s)g(q_1t, q_2s)d_{q_1}td_{q_2}s
$$

$$
= \left(\int_a^b\int_c^d P_w(q_1x, t)Q_u(q_2y, s)\frac{\partial f(t, s)}{d_{q_1}td_{q_2}s}d_{q_1}td_{q_2}s\right)\left(\int_a^b\int_c^d P_w(q_1x, t)Q_u(q_2y, s)\frac{\partial g(t, s)}{d_{q_1}td_{q_2}s}d_{q_1}td_{q_2}s\right) \quad (2.8)
$$

Multiplying both sides of (2.8) by $w(x)u(y)$, then q-integrating the resultant identity over Ω, we get

$$
T(w, u, f, g) = \int_a^b\int_c^d w(x)u(y)\left[\left(\int_a^b\int_c^d P_w(q_1x, t)Q_u(q_2y, s)\frac{\partial f(t, s)}{d_{q_1}td_{q_2}s}d_{q_1}td_{q_2}s\right)\right.
$$

$$
\left.\times\left(\int_a^b\int_c^d P_w(q_1x, t)Q_u(q_2y, s)\frac{\partial g(t, s)}{d_{q_1}td_{q_2}s}d_{q_1}td_{q_2}s\right)\right]d_{q_1}xd_{q_2}y.
$$

Finally, using the properties of modulus we observe that

$$
|T(w, u, f, g)| \leq \left\|\frac{\partial f(t, s)}{d_{q_1}td_{q_2}s}\right\|\left\|\frac{\partial g(t, s)}{d_{q_1}td_{q_2}s}\right\|\int_a^b\int_c^d w(x)u(y)\left[\left(\int_a^b\int_c^d |P_w(q_1x, t)Q_u(q_2y, s)|d_{q_1}td_{q_2}s\right)\right.
$$

$$
\left.\times\left(\int_a^b\int_c^d |P_w(q_1x, t)Q_u(q_2y, s)|d_{q_1}td_{q_2}s\right)\right]d_{q_1}xd_{q_2}y
$$

$$
= \left\|\frac{\partial f(t, s)}{d_{q_1}td_{q_2}s}\right\|\left\|\frac{\partial g(t, s)}{d_{q_1}td_{q_2}s}\right\|\int_a^b\int_c^d w(x)u(y)H^2(q_1x, q_2y)d_{q_1}xd_{q_2}y.
$$

This completes the proof of Theorem 2.2. $\qquad\square$

Theorem 2.3. *Let* $f, g : \Omega \to \mathbb{R}$, $w : [a, b] \to \mathbb{R}_0$ *and* $u : [c, d] \to \mathbb{R}_0$ *satisfying* $\int_a^b w(x)d_{q_1}x = \int_c^d u(y)d_{q_2}y = 1$, *then*

$$
|T(w, u, f, g)| \leq \frac{1}{2}\int_a^b\int_c^d w(x)u(y)
$$

$$
\times\left(|g(q_1x, q_2y)|\left\|\frac{\partial f(t, s)}{d_{q_1}td_{q_2}s}\right\| + |f(q_1x, q_2y)|\left\|\frac{\partial g(t, s)}{d_{q_1}td_{q_2}s}\right\|\right)H(q_1x, q_2y)d_{q_1}xd_{q_2}y, \quad (2.9)
$$

where $H(x, y)$ *is defined in Theorem 2.2.*

Proof. Multiplying both sides of (2.6) and (2.7) by $w(x)u(y)g(q_1x, q_2y)$ and $w(x)u(y)f(q_1x, q_2y)$, adding the resulting identities and rewriting, we have

$$
w(x)u(y)f(q_1x, q_2y)g(q_1x, q_2y)
$$

$$
= \frac{1}{2} \left(w(x)u(y)g(q_1x, q_2y) \int_a^b w(t)f(q_1t, q_2y)d_{q_1}t + w(x)u(y)g(q_1x, q_2y) \int_c^d u(s)f(q_1x, q_2s)d_{q_2}s \right.
$$

$$
- w(x)u(y)g(q_1x, q_2y) \int_a^b \int_c^d w(t)u(s)f(q_1t, q_2s)d_{q_1}td_{q_2}s + w(x)u(y)f(q_1x, q_2y) \int_a^b w(t)g(q_1t, q_2y)d_{q_1}t
$$

$$
\left. + w(x)u(y)f(q_1x, q_2y) \int_c^d u(s)g(q_1x, q_2s)d_{q_2}s - w(x)u(y)f(q_1x, q_2y) \int_a^b \int_c^d w(t)u(s)g(q_1t, q_2s)d_{q_1}td_{q_2}s \right)
$$

$$
+ \frac{1}{2} \left(w(x)u(y)g(q_1x, q_2y) \int_a^b \int_c^d P_w(q_1x, t)Q_u(q_2y, s) \frac{\partial f(t, s)}{d_{q_1}td_{q_2}s} d_{q_1}td_{q_2}s \right.
$$

$$
\left. + w(x)u(y)f(q_1x, q_2y) \int_a^b \int_c^d P_w(q_1x, t)Q_u(q_2y, s) \frac{\partial g(t, s)}{d_{q_1}td_{q_2}s} d_{q_1}td_{q_2}s \right). \quad (2.10)
$$

Q-integrating both sides of (2.10) with respect to x from a to b and y from c to d and rewriting we have

$$
T(w, uf, g) = \frac{1}{2} \left(\int_a^b \int_c^d w(x)u(y)g(q_1x, q_2y) \left(\int_a^b \int_c^d P_w(q_1x, t)Q_u(q_2y, s) \frac{\partial f(t, s)}{d_{q_1}td_{q_2}s} d_{q_1}td_{q_2}s \right) d_{q_1}xd_{q_2}y \right.
$$

$$
\left. + \int_a^b \int_c^d w(x)u(y)f(q_1x, q_2y) \left(\int_a^b \int_c^d P_w(q_1x, t)Q_u(q_2y, s) \frac{\partial g(t, s)}{d_{q_1}td_{q_2}s} d_{q_1}td_{q_2}s \right) d_{q_1}xd_{q_2}y \right). \quad (2.11)
$$

Finally, from (2.11) and using the properties of modulus we observe that

$$
T(w, uf, g) \leq \frac{1}{2} \left(\int_a^b \int_c^d w(x)u(y)|g(q_1x, q_2y)| \left(\int_a^b \int_c^d |P_w(q_1x, t)Q_u(q_2y, s)| \left| \frac{\partial f(t, s)}{d_{q_1}td_{q_2}s} \right| d_{q_1}td_{q_2}s \right) d_{q_1}xd_{q_2}y \right.
$$

$$
\left. + \int_a^b \int_c^d w(x)u(y)|f(q_1x, q_2y)| \left(\int_a^b \int_c^d |P_w(q_1x, t)Q_u(q_2y, s)| \left| \frac{\partial g(t, s)}{d_{q_1}td_{q_2}s} \right| d_{q_1}td_{q_2}s \right) d_{q_1}xd_{q_2}y \right)
$$

$$
\leq \frac{1}{2} \int_a^b \int_c^d w(x)u(y) \left(|g(q_1x, q_2y)| \left\| \frac{\partial f(t, s)}{d_{q_1}td_{q_2}s} \right\| + |f(q_1x, q_2y)| \left\| \frac{\partial g(t, s)}{d_{q_1}td_{q_2}s} \right\| \right)
$$

$$
\times \left(\int_a^b \int_c^d |P_w(q_1x, t)Q_u(q_2y, s)d_{q_1}td_{q_2}s \right) d_{q_1}xd_{q_2}y
$$

$$
= \frac{1}{2} \int_a^b \int_c^d w(x)u(y) \left(|g(q_1x, q_2y)| \left\| \frac{\partial f(t, s)}{d_{q_1}td_{q_2}s} \right\| + |f(q_1x, q_2y)| \left\| \frac{\partial g(t, s)}{d_{q_1}td_{q_2}s} \right\| \right) H(q_1x, q_2y)d_{q_1}xd_{q_2}y.
$$

This completes the proof of Theorem 2.3. □

Remark 2.4. If $q_1, q_2 \to 1^-$, by Definitions 1.2 and 1.2, the partial q-derivative and double q-integrals are the usual partial derivative and double integrals, so Theorems 2.2 and 2.3 are reduced to Theorems 3 and 4 in [14].

Theorem 2.5. *Let* $f, g : \Omega \to \mathbb{R}$, $w : [a, b] \to \mathbb{R}_0$ *and* $u : [c, d] \to \mathbb{R}_0$ *satisfying* $\int_a^b w(x)d_{q_1}x = \int_c^d u(y)d_{q_2}y = 1$, *then*

$$
|T(w, u, f, g)| \leq \|g(q_1x, q_2y)\| \left\| \frac{\partial f(t, s)}{d_{q_1}td_{q_2}s} \right\| \int_a^b \int_c^d w(x)u(y)H(q_1x, q_2y)d_{q_1}xd_{q_2}y, \quad (2.12)
$$

and

$$|T(w,u,f,g)| \leq \|f(q_1x, q_2y)\| \left\| \frac{\partial g(t,s)}{d_{q_1}t d_{q_2}s} \right\| \int_a^b \int_c^d w(x)u(y)H(q_1x, q_2y)d_{q_1}x d_{q_2}y, \tag{2.13}$$

where $H(x,y)$ is defined in Theorem 2.2.

Proof. We prove only (2.12), since the proof of (2.13) is similar. The identity (2.6) shows that

$$f(q_1x, q_2y) = \int_a^b w(t)f(q_1t, q_2y)d_{q_1}t + \int_c^d u(s)f(q_1x, q_2s)d_{q_2}s - \int_a^b \int_c^d w(t)u(s)f(q_1t, q_2s)d_{q_1}t d_{q_2}s$$

$$+ \int_a^b \int_c^d P_w(q_1x, t)Q_u(q_2y, s)\frac{\partial f(t,s)}{d_{q_1}t d_{q_2}s}d_{q_1}t d_{q_2}s, \tag{2.14}$$

for $(x,y) \in \Omega$.

Now, if we multiply (2.14) by $w(x)u(y)g(q_1x, q_2y)$ and q-integrate over $(x,y) \in \Omega$, we deduce

$$\int_a^b \int_c^d w(x)u(y)f(q_1x, q_2y)g(q_1x, q_2y)d_{q_1}x d_{q_2}y$$

$$= \int_a^b \int_c^d w(x)u(y)g(q_1x, q_2y)\left(\int_a^b w(t)f(q_1t, q_2y)d_{q_1}t\right)d_{q_1}x d_{q_2}y$$

$$+ \int_a^b \int_c^d w(x)u(y)g(q_1x, q_2y)\left(\int_c^d u(s)f(q_1x, q_2s)d_{q_2}s\right)d_{q_1}x d_{q_2}y$$

$$- \int_a^b \int_c^d w(x)u(y)g(q_1x, q_2y)d_{q_1}x d_{q_2}y \int_a^b \int_c^d w(t)u(s)f(q_1t, q_2s)d_{q_1}t d_{q_2}s$$

$$+ \int_a^b \int_c^d w(x)u(y)g(q_1x, q_2y)\left(\int_a^b \int_c^d P_w(q_1x, t)Q_u(q_2y, s)\frac{\partial f(t,s)}{d_{q_1}t d_{q_2}s}d_{q_1}t d_{q_2}s\right)d_{q_1}x d_{q_2}y,$$

which provides another representation for the functional $T(w,u,f,g)$ namely,

$$T(w,u,f,g) = \int_a^b \int_c^d w(x)u(y)g(q_1x, q_2y)\left(\int_a^b \int_c^d P_w(q_1x, t)Q_u(q_2y, s)\frac{\partial f(t,s)}{d_{q_1}t d_{q_2}s}d_{q_1}t d_{q_2}s\right)d_{q_1}x d_{q_2}y, \tag{2.15}$$

From (2.15) and using modules properties, it yields

$$|T(w,u,f,g)| \leq \int_a^b \int_c^d w(x)u(y)|g(q_1x, q_2y)|\left(\int_a^b \int_c^d |P_w(q_1x, t)Q_u(q_2y, s)|\left|\frac{\partial f(t,s)}{d_{q_1}t d_{q_2}s}\right|d_{q_1}t d_{q_2}s\right)d_{q_1}x d_{q_2}y$$

$$\leq \|g(q_1x, q_2y)\|\left\|\frac{\partial f(t,s)}{d_{q_1}t d_{q_2}s}\right\|\int_a^b \int_c^d w(x)u(y)\left(\int_a^b \int_c^d |P_w(q_1x, t)Q_u(q_2y, s)|d_{q_1}t d_{q_2}s\right)d_{q_1}x d_{q_2}y$$

$$= \|g(q_1x, q_2y)\|\left\|\frac{\partial f(t,s)}{d_{q_1}t d_{q_2}s}\right\|\int_a^b \int_c^d w(x)u(y)H(q_1x, q_2y)d_{q_1}x d_{q_2}y.$$

This completes the proof of Theorem 2.5. $\qquad\square$

3 Weighted q-Ostrowski type inequalities for double integrals

For some given functions $f, g : \Omega \to \mathbb{R}$ and $w : [a, b] \to \mathbb{R}_0$ and $u : [c, d] \to \mathbb{R}_0$ satisfying $\int_a^b w(x)d_{q_1}x = \int_c^d u(y)d_{q_2}y = 1$,

$$S(w, u, f, g) = f(q_1x, q_2y)g(q_1x, q_2y) - \frac{1}{2}\left(f(q_1x, q_2y)\int_a^b w(t)g(q_1t, q_2y)d_{q_1}t + \int_c^d u(s)g(q_1x, q_2s)d_{q_2}s \right.$$

$$\times f(q_1x, q_2y) + g(q_1x, q_2y)\int_a^b w(t)f(q_1t, q_2y)d_{q_1}t + g(q_1x, q_2y)\int_c^d u(s)f(q_1x, q_2s)d_{q_2}s$$

$$\left. -f(q_1x, q_2y)\int_a^b\int_c^d w(t)u(s)g(q_1t, q_2s)d_{q_1}td_{q_2}s - g(q_1x, q_2y)\int_a^b\int_c^d w(t)u(s)f(q_1t, q_2s)d_{q_1}td_{q_2}s \right).$$

Theorem 3.1. *Let $f, g : \Omega \to \mathbb{R}$, $w : [a, b] \to \mathbb{R}_0$ and $u : [c, d] \to \mathbb{R}_0$ satisfying $\int_a^b w(x)d_{q_1}x = \int_c^d u(y)d_{q_2}y = 1$, then*

$$|S(w, u, f, g)| \leq \frac{1}{2}\left(|f(q_1x, q_2y)|\left\|\frac{\partial g(t, s)}{d_{q_1}td_{q_2}s}\right\| + |g(q_1x, q_2y)|\left\|\frac{\partial f(t, s)}{d_{q_1}td_{q_2}s}\right\| \right) H(q_1x, q_2y), \qquad (3.1)$$

and

$$|T(w, u, f, g)| \leq \frac{1}{2}\left(|f(q_1x, q_2y)|\left\|\frac{\partial g(t, s)}{d_{q_1}td_{q_2}s}\right\| + |g(q_1x, q_2y)|\left\|\frac{\partial f(t, s)}{d_{q_1}td_{q_2}s}\right\| \right) \int_a^b\int_c^d w(x)u(y)H(q_1x, q_2y)d_{q_1}xd_{q_2}y.$$
$$(3.2)$$

where $H(x, y)$ is defined in Theorem 2.2.

Proof. Multiplying both sides of (2.6) and (2.7) by $g(q_1x, q_2y)$ and $f(q_1x, q_2y)$, adding the resulting identities and rewriting we have

$$f(q_1x, q_2y)g(q_1x, q_2y) = \frac{1}{2}\left(f(q_1x, q_2y)\int_a^b w(t)g(q_1t, q_2y)d_{q_1}t + f(q_1x, q_2y)\int_c^d u(s)g(q_1x, q_2s)d_{q_2}s \right.$$

$$+ g(q_1x, q_2y)\int_a^b w(t)f(q_1t, q_2y)d_{q_1}t + g(q_1x, q_2y)\int_c^d u(s)f(q_1x, q_2s)d_{q_2}s$$

$$\left. -f(q_1x, q_2y)\int_a^b\int_c^d w(t)u(s)g(q_1t, q_2s)d_{q_1}td_{q_2}s - g(q_1x, q_2y)\int_a^b\int_c^d w(t)u(s)f(q_1t, q_2s)d_{q_1}td_{q_2}s \right)$$

$$+ \frac{1}{2}\left(f(q_1x, q_2y)\int_a^b\int_c^d P_w(q_1x, t)Q_u(q_2y, s)\frac{\partial g(t, s)}{d_{q_1}td_{q_2}s}d_{q_1}td_{q_2}s \right.$$

$$\left. + g(q_1x, q_2y)\int_a^b\int_c^d P_w(q_1x, t)Q_u(q_2y, s)\frac{\partial f(t, s)}{d_{q_1}td_{q_2}s}d_{q_1}td_{q_2}s \right), \quad (3.3)$$

which implies

$$S(w, u, f, g) = \frac{1}{2}\left(f(q_1x, q_2y)\int_a^b\int_c^d P_w(q_1x, t)Q_u(q_2y, s)\frac{\partial g(t, s)}{d_{q_1}td_{q_2}s}d_{q_1}td_{q_2}s \right.$$

$$\left. + g(q_1x, q_2y)\int_a^b\int_c^d P_w(q_1x, t)Q_u(q_2y, s)\frac{\partial f(t, s)}{d_{q_1}td_{q_2}s}d_{q_1}td_{q_2}s \right).$$

We observe

$$|S(w, u, f, g)| \leq \frac{1}{2}\left(|f(q_1x, q_2y)|\left\|\frac{\partial g(t, s)}{d_{q_1}td_{q_2}s}\right\| + |g(q_1x, q_2y)|\left\|\frac{\partial f(t, s)}{d_{q_1}td_{q_2}s}\right\| \right) H(q_1x, q_2y).$$

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Multiplying both sides of (3.3) by $w(x)u(y)$ and q-integrate over $(x, y) \in \Omega$, we deduce

$$T(w, u, f, g) = \frac{1}{2} \left(\int_a^b \int_c^d w(x)u(y)f(q_1x, q_2y) \left(\int_a^b \int_c^d P_w(q_1x, t)Q_u(q_2y, s) \frac{\partial g(t, s)}{d_{q_1}t d_{q_2}s} d_{q_1}t d_{q_2}s \right) d_{q_1}x d_{q_2}y \right.$$

$$\left. + \int_a^b \int_c^d w(x)u(y)g(q_1x, q_2y) \left(\int_a^b \int_c^d P_w(q_1x, t)Q_u(q_2y, s) \frac{\partial f(t, s)}{d_{q_1}t d_{q_2}s} d_{q_1}t d_{q_2}s \right) d_{q_1}x d_{q_2}y \right).$$

We observe

$$|T(w, u, f, g)| \leq \frac{1}{2} \left(|f(q_1x, q_2y)| \left\| \frac{\partial g(t, s)}{d_{q_1}t d_{q_2}s} \right\| + |g(q_1x, q_2y)| \left\| \frac{\partial f(t, s)}{d_{q_1}t d_{q_2}s} \right\| \right) \int_a^b \int_c^d w(x)u(y)H(q_1x, q_2y)d_{q_1}x d_{q_2}y.$$

This completes the proof of Theorem 3.1. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Let $g(x, y) = 1$, we have the following corollary.

Corollary 3.2. *Let $f : \Omega \to \mathbb{R}$, $w : [a, b] \to \mathbb{R}_0$ and $u : [c, d] \to \mathbb{R}_0$ satisfying $\int_a^b w(x)d_{q_1}x = \int_c^d u(y)d_{q_2}y = 1$, then*

$$\left| f(x, y) - \int_a^b w(t)f(q_1t, y)d_{q_1}t - \int_c^d u(s)f(x, q_2s)d_{q_2}s + \int_a^b \int_c^d w(t)u(s)f(q_1t, q_2s)d_{q_1}t d_{q_2}s \right|$$

$$\leq H(x, y) \left\| \frac{\partial f(t, s)}{d_{q_1}t d_{q_2}s} \right\|,$$

where $H(x, y)$ is defined in Theorem 2.2, and especially, let $w(x) = \frac{1}{b-a}$ and $u(y) = \frac{1}{d-c}$, we get

$$\left| f(x, y) - \int_a^b w(t)f(q_1t, y)d_{q_1}t - \int_c^d u(s)f(x, q_2s)d_{q_2}s + \int_a^b \int_c^d w(t)u(s)f(q_1t, q_2s)d_{q_1}t d_{q_2}s \right|$$

$$\leq \frac{1}{4(q_1 + 1)(q_2 + 1)} \left((b - a)^2 + 4 \left(x - \frac{a+b}{2} \right)^2 \right) \left((d - c)^2 + 4 \left(y - \frac{c+d}{2} \right)^2 \right) \left\| \frac{\partial f(t, s)}{d_{q_1}t d_{q_2}s} \right\|. \quad (3.4)$$

Remark 3.3. If $q_1, q_2 \to 1^-$, the inequality (3.4) are reduced to the main result in [4].

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QUADRATIC ρ-FUNCTIONAL INEQUALITIES IN NORMED SPACES

IKAN CHOI, SUNGHOON KIM*, GEORGE A. ANASTASSIOU, AND CHOONKIL PARK*

ABSTRACT. In this paper, we solve the quadratic ρ-functional inequalities

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \tag{0.1}$$
$$\leq \left\| \rho \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \right) \right\|,$$

where ρ is a number with $|\rho| < 1$ and

$$\left\| 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \right\| \tag{0.2}$$
$$\leq \|\rho(f(x+y) + f(x-y) - 2f(x) - 2f(y))\|,$$

where ρ is a number with $|\rho| < \frac{1}{2}$. Using the fixed point method, we prove the Hyers-Ulam stability of the quadratic ρ-functional inequalities (0.1) and (0.2) in normed spaces.

1. INTRODUCTION AND PRELIMINARIES

The stability problem of functional equations originated from a question of Ulam [29] concerning the stability of group homomorphisms.

The functional equation

$$f(x+y) = f(x) + f(y)$$

is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [14] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [20] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [11] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. A Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [23] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [6] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [7] proved the Hyers-Ulam stability of the quadratic functional equation. The functional equation $f\left(\frac{x+y}{2}\right) = \frac{1}{2}f(x) + \frac{1}{2}f(y)$ is called the *Jensen type quadratic functional equation*. The stability problems of several functional equations

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have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1, 3, 16, 17, 21, 22, 25, 26, 27, 28, 30, 31]).

In [12], Gilányi showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(xy^{-1})\| \le \|f(xy)\| \tag{1.1}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(xy) + f(xy^{-1}).$$

See also [10, 24]. Gilányi [13] and Fechner [9] proved the Hyers-Ulam stability of the functional inequality (1.1). Park, Cho and Han [18] proved the Hyers-Ulam stability of additive functional inequalities.

Lemma 1.1. (Banach fixed-point theorem) *Let (S, d) be a complete metric space and let $T : S \to S$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in S$, there exists a positive integer n_0 such that*
   (1) $d(T^n x, T^{n+1} x) < \infty, \qquad \forall n \ge n_0$;
   (2) *the sequence $\{T^n x\}$ converges to a fixed point y^* of T;*
   (3) *y^* is the unique fixed point of T in the set $Y = \{y \in S \mid d(T^{n_0} x, y) < \infty\}$;*
   (4) *$d(y, y^*) \le \frac{1}{1-\alpha} d(y, Ty)$ for all $y \in Y$.*

Since we defined the metric d as generalized metric in order to use this lemma in the proof of the problem we extend the lemma.

Lemma 1.2. ([8]) *Let (S, d) be a complete generalized metric space and let $J : S \to S$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in S$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that
   (1) $d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \ge n_0$;
   (2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
   (3) *y^* is the unique fixed point of J in the set $Y = \{y \in S \mid d(J^{n_0} x, y) < \infty\}$;*
   (4) *$d(y, y^*) \le \frac{1}{1-\alpha} d(y, Jy)$ for all $y \in Y$.*

In 1996, Isac and Rassias [15] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [4, 5, 19]).

In Section 2, we solve the quadratic ρ-functional inequality (0.1) and prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (0.1) in normed spaces.

In Section 3, we solve the quadratic ρ-functional inequality (0.2) and prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (0.2) in normed spaces.

Throughout this paper, assume that X is a normed space and Y is a Banach space.

2. Hyers-Ulam stability of the quadratic ρ-functional inequality (0.1): a fixed point approach

In this section, assume that $|\rho| < 1$.

We solve the quadratic ρ-functional inequality (0.1) in normed spaces.

Lemma 2.1. *A mapping $f : X \to Y$ satisfies*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \tag{2.1}$$
$$\leq \left\| \rho \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \right) \right\|$$

for all $x, y \in X$ if and only if $f : X \to Y$ is quadratic.

Proof. Assume that $f : X \to Y$ satisfies (2.1).

Letting $x = y = 0$ in (2.1), we get $\|2f(0)\| \leq |\rho|\|2f(0)\|$. So $f(0) = 0$.

Letting $y = x$ in (2.1), we get $\|f(2x) - 4f(x)\| \leq 0$ and so $f(2x) = 4f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{4}f(x) \tag{2.2}$$

for all $x \in X$.

It follows from (2.1) and (2.2) that

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\|$$
$$\leq \left\| \rho \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \right) \right\|$$
$$= \frac{|\rho|}{2}\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\|$$

and so

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in X$.

The converse is obviously true. $\square$

We prove the Hyers-Ulam stability of the quadratic ρ-functional inequalty (0.1) in Banach spaces.

Theorem 2.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(a, b) \leq 4\alpha\varphi\left(\frac{a}{2}, \frac{b}{2}\right) \tag{2.3}$$

for all $a, b \in X$. Let $f : X \to Y$ be a mapping satisfying

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \tag{2.4}$$
$$\leq \left\| \rho \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \right) \right\| + \varphi(x, y)$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{1}{4 - 4\alpha}\varphi(x, x)$$

for all $x \in X$.

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Proof. Consider the set

$$S := \{h : X \to Y\}$$

and let d be the generalized metric on S:

$$d(g,h) := \inf\{\mu \in \mathbb{R}_+ : \|g - h\| \le \mu\varphi(x,x), x \in S\}$$

It is easy to show that (S,d) is complete. Let J be the linear mapping from S to S such that

$$Jg(x) := \frac{1}{4}g(2x) \tag{2.5}$$

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then from (2.3) and (2.5), we get

$$\|Jg(x) - Jh(x)\| = \left\|\frac{1}{4}g(2x) - \frac{1}{4}h(2x)\right\| \le \frac{1}{4}\varepsilon\varphi(2x,2x) \le \alpha\varepsilon\varphi(x,x)$$

This means $d(Jg, Jh) \le \alpha d(g,h)$.

So the function $J : S \to S$ is a contractive mapping such that

$$d(Jg, Jh) \le \alpha d(g,h)$$

for $0 \le \alpha < 1$.

Letting $y = x$ in (2.4), we get

$$\|f(2x) - 4f(x)\| \le \varphi(x,x)$$

and so

$$\|f(x) - Jf(x)\| \le \frac{1}{4}\varphi(x,x)$$

for all $x \in X$. Thus we get $d(f, Jf) \le \frac{1}{4}$.

By Lemma 1.2, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q(2a) = 4Q(a) \tag{2.6}$$

for all $a \in X$. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (2.6) such that there exists a $\mu \in (0,\infty)$ satisfying

$$\|f(a) - Q(a)\| \le \mu\varphi(a,a)$$

for all $a \in X$;

(2) $d(J^l f, Q) \to 0$ as $l \to \infty$. This implies the equality

$$\lim_{l \to \infty} \frac{1}{4^l}f\left(2^l a\right) = Q(a)$$

for all $a \in X$;

(3) $d(f, Q) \le \frac{1}{1-\alpha}d(f, Jf)$, which implies the inequality

$$d(f, Q) \le \frac{1}{4 - 4\alpha}.$$

So

$$\|f(a) - Q(a)\| \le \frac{1}{4 - 4\alpha} \varphi(a, a)$$

for all $a \in X$.

Then

$$\|Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y)\|$$
$$= \lim_{l\to\infty} \left\| \frac{1}{4^l}(f(2^l(x+y)) + f(2^l(x-y)) - 2f(2^lx) - 2f(2^ly)) \right\|$$
$$\le \lim_{l\to\infty} \left\| \frac{\rho}{4^l}(2f(2^{l-1}(x+y)) + 2f(2^{l-1}(x-y)) - f(2^lx) - f(2^ly)) \right\|$$
$$\quad + \lim_{l\to\infty} \frac{1}{4^l} \varphi(2^lx, 2^ly)$$
$$= \left\| \rho\left(2Q\left(\frac{x+y}{2}\right) + 2Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y) \right) \right\|$$

for all $x, y \in X$. Hence

$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y)$$

for all x, y. So $Q : X \to Y$ is quadratic. $\qquad\square$

Remark 2.3. We could prove the same statement with the same manner in spite of replacing the condition $\varphi(a, b) \le 4\alpha\varphi\left(\frac{a}{2}, \frac{b}{2}\right)$ into $\varphi(a, b) \le \frac{1}{4}\alpha\varphi(2a, 2b)$ by defining J such that $Jg(x) = 4g(\frac{x}{2})$ instead of $Jg(x) = \frac{1}{4}g(2x)$. It could be also applied to Theorem 3.2.

Corollary 2.4. *Let $r \ne 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\|$$
$$\le \left\| \rho\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \right) \right\| + \theta(\|x\|^r + \|y\|^r)$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{2\theta}{|4 - 2^r|} \|x\|^r$$

for all $x \in X$.

3. Hyers-Ulam stability of the quadratic ρ-functional inequality (0.2): a fixed point approach

In this section, assume that $|\rho| < \frac{1}{2}$.

We solve the quadratic ρ-functional inequality (0.2) in normed spaces.

Lemma 3.1. *A mapping $f : X \to Y$ satisfies*

$$\left\| 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \right\| \qquad (3.1)$$
$$\le \|\rho(f(x+y) + f(x-y) - 2f(x) - 2f(y))\|$$

for all $x, y \in X$ if and only if $f : X \to Y$ is quadratic.

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Proof. Assume that $f : X \to Y$ satisfies (3.1).

Letting $x = y = 0$ in (3.1), we get $\|2f(0)\| \le |\rho|\|2f(0)\|$. So $f(0) = 0$.

Letting $y = 0$ in (3.1), we get

$$\left\| 4f\left(\frac{x}{2}\right) - f(x) \right\| \le 0 \tag{3.2}$$

and so $f\left(\frac{x}{2}\right) = \frac{1}{4}f(x)$ for all $x \in X$.

It follows from (3.1) and (3.2) that

$$\frac{1}{2}\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\|$$
$$= \left\| 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \right\|$$
$$\le |\rho|\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\|$$

and so

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in X$.

The converse is obviously true. $\square$

We prove the Hyers-Ulam stability of the quadratic ρ-functional inequalty (0.2) in Banach spaces.

Theorem 3.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(a, b) \le 4\alpha\varphi\left(\frac{a}{2}, \frac{b}{2}\right)$$

for all $a, b \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$\left\| 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \right\|$$
$$\le \|\rho(f(x+y) + f(x-y) - 2f(x) - 2f(y))\| + \varphi(x, y)$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{1}{4 - 4\alpha}\varphi(x, 0)$$

for all $x \in X$.

Proof. Consider the set

$$S := \{h : X \to Y\}$$

and let d be the generalized metric on S:

$$d(g, h) := \inf\{\mu \in \mathbb{R}_+ : \|g - h\| \le \mu\varphi(x, 0), x \in S\}$$

It is easy to show that (S, d) is complete. Let J be the linear mapping from S to S such that

$$Jg(x) := \frac{1}{4}g(2x)$$

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Let $Q : X \to Y$ be defined as in the proof of Theorem 2.2. Then

$$\left\| 2Q\left(\frac{x+y}{2}\right) + 2Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y) \right\|$$

$$= \lim_{l\to\infty} \left\| \frac{1}{4^l}(2f(2^{l-1}(x+y)) + 2f(2^{l-1}(x-y)) - f(2^l x) - f(2^l y)) \right\|$$

$$\leq \lim_{l\to\infty} \left\| \frac{\rho}{4^l}(f(2^l(x+y)) + f(2^l(x-y)) - 2f(2^l x) - 2f(2^l y)) \right\|$$

$$+ \lim_{l\to\infty} \frac{1}{4^l}\varphi(2^l x, 2^l y)$$

$$= \|\rho(Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y))\|$$

for all $x, y \in X$. Hence

$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y)$$

for all x, y. So $Q : X \to Y$ is quadratic. $\qquad\square$

Corollary 3.3. *Let $r \neq 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\left\| 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \right\|$$

$$\leq \|\rho(f(x+y) + f(x-y) - 2f(x) - 2f(y))\| + \theta(\|x\|^r + \|y\|^r)$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{\theta}{|4 - 2^r|}\|x\|^r$$

for all $x \in X$.

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Solution of the Ulam stability problem for quartic (a, b)-functional equations

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Abstract. The "oldest quartic" functional equation was introduced and solved by the author of this paper (see: Glas. Mat. Ser. III 34 (54) (1999), no. 2, 243-252) which is of the form:

$$f(x + 2y) + f(x - 2y) = 4[f(x + y) + f(x - y)] - 6f(x) + 24f(y).$$

Interesting results have been achieved by S.A. Mohiuddine et al., since 2009. In this paper, we are introducing new quartic functional equations, and establish fundamental formulas for the general solution of such functional equations and for "Ulam stability" of pertinent quartic functional inequalities.

Keywords and phrases: Quartic functional equations and inequalities; Various normed spaces; Ulam stability.

AMS subject classification (2000): 39B.

1. INTRODUCTION

In 1940 S. M. Ulam [1] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following famous "_stability Ulam question_":

We are given a group G and a metric group G' with metric $\rho(.,.)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies $\rho(f(xy), f(x)f(y)) < \delta$ for all x, y in G, then a homomorphism $h : G \to G'$ exists with $\rho(f(x), h(x)) < \epsilon$ for all $x \in G$?

By now an affirmative answer has been given in several cases, and some interesting variations of the problem have also been investigated. We shall call such an $f : G \to G'$ an _approximate homomorphism_.

In 1941 D. H. Hyers [2] considered the case of approximately additive mappings $f : E \to E'$ where E and E' are Banach spaces and f satisfies the following _Hyers' inequality_

$$\left\| f(x + y) - f(x) - f(y) \right\| \le \epsilon$$

for all $x, y \in E$. It was shown that the limit

$$L(x) = \lim_{n \to \infty} 2^{-n} f(2^n x)$$

exists for all $x \in E$ and that $L : E \to E'$ is the unique additive mapping satisfying $\left\| f(x) - L(x) \right\| \le \epsilon$.

No continuity conditions are required for this result, but if $f(tx)$ is continuous in the real variable t for each fixed x, then L is linear, and if f is continuous at a single point of E then $L : E \to E'$ is also continuous.

In 1982-1994, a generalization of this result was proved by the author J. M. Rassias [3–7], as follows. He introduced the following _weaker condition_ (_or weaker inequality or the generalized Cauchy inequality_)

$$\left\| f(x + y) - [f(x) + f(y)] \right\| \le \theta \|x\|^p \|y\|^q$$

for all x, y in E, controlled by (or involving) a product of different powers of norms, where $\theta \ge 0$ and real $p, q : r = p + q \ne 1$, and retained the condition of continuity of $f(tx)$ in t for fixed x. Besides he investigated that it is possible to replace ϵ in the above _Hyers' inequality_, by a non-negative real-valued function such that the pertinent series converges and other conditions hold and still obtain stability results. In all the cases investigated in these results, the approach to the existence question was to prove

asymptotic type formulas: $L(x) = \lim_{n \to \infty} 2^{-n} f(2^n x)$; $L(x) = \lim_{n \to \infty} 2^n f(2^{-n} x)$.

Theorem (J. M. Rassias:1982-1994). Let X be a real normed linear space and let Y be a real complete normed linear space. Assume in addition that $f : X \to Y$ is an approximately additive mapping for which there exist constants $\theta \geq 0$ and $p, q \in \mathbb{R}$ such that $r = p + q \neq 1$ and f satisfies the "*generalized Cauchy inequality*"

$$\left\| f(x+y) - [f(x) + f(y)] \right\| \leq \theta \|x\|^p \|y\|^q$$

for all $x, y \in X$. Then there exists a unique additive mapping $L : X \to Y$ satisfying

$$\left\| f(x) - L(x) \right\| \leq \frac{\theta}{|2^r - 2|} \|x\|^r$$

for all $x \in X$. If in addition $f : X \to Y$ is a mapping such that the transformation $t \to f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then L is an $\mathbb{R}$-linear mapping.

In 1940, Ulam, S. M. [1] proposed the "Ulam stability problem": *When does a linear transformation near an "approximately linear" transformation exist?* Since then, many specialists on this "famous Ulam problem", have investigated interesting functional equations, for instance: D. H. Hyers [2], in 1941; T. Aoki [8], in 1950; T. M. Rassias [9], in 1978; Z. Gajda [10], in 1991; T. M. Rassias and P. Šemrl [11], in 1992; P. Găvruta [12], in 1994; S.-M. Jung [13], in 1998; K. W. Jun and H. M. Kim [14], in 2002; R. P. Agarwal et al. [15], in 2003, and others. Interesting Ulam-Hyers stability results have been established by S. A. Mohiuddine et al. ([16–19]). The "oldest quartic" functional equation was introduced and solved by the author of this paper, [20], which is of the form:

$$f(x + 2y) + f(x - 2y) = 4[f(x + y) + f(x - y)] - 6f(x) + 24f(y).$$

Since then various quartic equations have been proposed and solved by a number of experts in the area of functional equations and inequalities. For more details on these concepts, one can be referred to [21–30]. For further research in various normed spaces, we are introducing new quartic functional equations, and establish fundamental formulas for the general solution of such functional equations and for "Ulam stability" of pertinent quartic functional inequalities.

2. ON (a, b)-QUARTIC FUNCTIONAL EQUATIONS

1. Stability of General a-Quartic Functional Equation

$$2\big[f(ax + y) + f(x + ay)\big] + a(a - 1)^2 f(x - y) = 2(a^2 - 1)^2\big[f(x) + f(y)\big] + a(a + 1)^2 f(x + y) \quad (1.1)$$

where $a \neq 0, a \neq \pm 1$.

Replacing $x = y = 0$ in (1.1) one gets $4a^2(1 - a^2)f(0) = 0$, or

$$f(0) = 0. \quad (1.2)$$

Similarly, substituting $x = x, y = 0$ in (1.1), we obtain

$$f(ax) = a^4 f(x) + (a^2 - 1)^2 f(0) = a^4 f(x) \quad (1.3)$$

Also assuming $f(2x) = 16f(x)$, replacing $x = x, y = x$ in (1.1), and setting $k = a + 1 \neq 0, \pm 1$, one obtains

$$4f(kx) + a(a - 1)^2 f(0) = 4(a^2 - 1)^2 f(x) + a(a + 1)^2 f(2x), \quad (1.4)$$

or

$$f(kx) + \frac{1}{4}a(a - 1)^2 f(0) = (a^2 - 1)^2 f(x) + 4a(a + 1)^2 f(x) = k^4 f(x), \text{ or}$$

$$f(kx) = k^4 f(x) - \frac{1}{4}a(a - 1)^2 f(0) = k^4 f(x). \quad (1.5)$$

Without assuming $f(2x) = 16f(x)$, replacing $x = x, y = -x$ in (1.1), and setting $l = a - 1 \neq 0, \pm 1$, one obtains

$$2\big[f(lx) + f(-lx)\big] + a(a-1)^2 f(2x) = 2(a^2 - 1)^2\big[f(x) + f(-x)\big] + a(a+1)^2 f(0) \qquad (1.6)$$

Placing $-x$ on x in (1.6), and then subtracting the new equation from (1.6), we get $a(a-1)^2[f(2x) - f(-2x)] = 0$. Letting $x/2$ on x, we find that f is an "*even function*", such that $f(-x) = f(x)$. Thus from (1.6), we obtain

$$4f(lx) + a(a-1)^2 f(2x) = 4(a^2 - 1)^2 f(x) + a(a+1)^2 f(0). \qquad (1.7)$$

Assuming $f(2x) = 16f(x)$, we get

$$f(lx) = l^4 f(x) + \frac{1}{4}a(a+1)^2 f(0) = l^4 f(x).$$

Without assuming $f(2x) = 16f(x)$, subtracting (1.7) from (1.4), we obtain from (1.2) that

$$f(kx) - f(lx) = \frac{1}{2}a(a^2 + 1)f(2x) = \left[\left(\frac{k}{2}\right)^4 - \left(\frac{l}{2}\right)^4\right]f(2x).$$

Replacing $x \to x/2$, we get that

$$\text{``}f\left(\frac{k}{2}x\right) = \left(\frac{k}{2}\right)^4 f(x) \quad \text{if and only if} \quad f\left(\frac{l}{2}x\right) = \left(\frac{l}{2}\right)^4 f(x)\text{''}. \qquad (1.8)$$

Employing the "quartic mean", we have equivalently that

$$\text{``}\overline{f}_{\frac{k}{2}}(x) = \frac{f\left(\frac{k}{2}x\right)}{\left(\frac{k}{2}\right)^4} = f(x) \ \ \text{iff} \ \ \overline{f}_{\frac{l}{2}}(x) = \frac{f\left(\frac{l}{2}x\right)}{\left(\frac{l}{2}\right)^4} = f(x)\text{''}.$$

Let X be a real normed linear space and let Y be a real complete normed linear space. Assume $f : X \to Y$, satisfying the following general a-quartic functional inequality

$$\left\|2\big[f(ax + y) + f(x + ay)\big] + a(a-1)^2 f(x-y) - 2(a^2 - 1)^2\big[f(x) + f(y)\big] - a(a+1)^2 f(x+y)\right\| \leq c \quad (1.9)$$

where $a \neq 0, a \neq \pm 1$. Replacing $x = y = 0$ in (1.9), one gets

$$\left\|f(0)\right\| \leq c/4a^2|a^2 - 1|. \qquad (1.10)$$

Substituting $x = x, y = 0$ in (1.9), and employing the triangle inequality, we obtain:

$$\left\|f(ax) - a^4 f(x)\right\| \leq c_2 = \frac{c_1}{2} = \frac{2a^2 + |a^2 - 1|}{4a^2}c.$$

Note that

$$c_1 = \frac{2a^2 + |a^2 - 1|}{2a^2}c = \begin{cases} \frac{3a^2 - 1}{2a^2}c & \text{if } |a| > 1 \\[2mm] \frac{a^2 + 1}{2a^2}c & \text{if } |a| < 1; a \neq 0. \end{cases}$$

Therefore

$$\left\|f(ax) - a^4 f(x)\right\| \leq c_2 = \frac{c_1}{2} = \frac{3a^2 - 1}{4a^2}c, \quad \text{if } |a| > 1,$$

and

$$\left\|f(ax) - a^4 f(x)\right\| \leq c_3 = \frac{c_1}{2} = \frac{a^2 + 1}{4a^2}c, \quad \text{if } |a| < 1; a \neq 0.$$

Therefore we get

$$\left\|f(x) - a^{-4} f(ax)\right\| \leq a^{-4}c_2 = \frac{3a^2 - 1}{4a^6}c, \quad \text{if } |a| > 1,$$

and

$$\left\|f(x) - a^4 f(a^{-1}x)\right\| \leq c_3 = \frac{a^2 + 1}{4a^2}c, \quad \text{if } |a| < 1; a \neq 0.$$

Thus we can easily obtain the following general inequality

$$\left\|f(x) - a^{-4n}f(a^n x)\right\| \le \frac{3a^2 - 1}{4a^2(a^4 - 1)}(1 - a^{-4n})c, \quad \text{if } |a| > 1. \tag{1.11}$$

In fact,

$$
\begin{aligned}
\left\|f(x) - a^{-4n}f(a^n x)\right\| &\le \left\|f(x) - a^{-4}f(a^1 x)\right\| + a^{-4}\left\|f(ax) - a^{-4}f(a^2 x)\right\| + \dots \\
&\quad \dots + a^{-4(n-2)}\left\|f(a^{n-2}x) - a^{-4}f(a^{n-1}x)\right\| \\
&\quad + a^{-4(n-1)}\left\|f(a^{n-1}x) - a^{-4}f(a^n x)\right\| \\
&\le \left(1 + a^{-4} + a^{-8} + \dots + a^{-4(n-2)} + a^{-4(n-1)}\right)\frac{3a^2 - 1}{4a^6}c \\
&\le \frac{3a^2 - 1}{4a^2(a^4 - 1)}(1 - a^{-4n})c, \quad \text{if } |a| > 1.
\end{aligned} \tag{1.12}
$$

Also,

$$
\begin{aligned}
\left\|f(x) - \alpha^{4n}f(\alpha^{-n}x)\right\| &\le \left\|f(x) - \alpha^4 f(\alpha^{-1}x)\right\| + \alpha^4\left\|f(\alpha^{-1}x) - \alpha^4 f(\alpha^{-2}x)\right\| \\
&\quad + \dots + \alpha^{4(n-1)}\left\|f(\alpha^{-(n-1)}x) - \alpha^4 f(\alpha^{-n}x)\right\| \\
&\le \left(1 + \alpha^4 + \dots + \alpha^{4(n-1)}\right)\frac{\alpha^2 + 1}{4\alpha^2}c \\
&\le \frac{\alpha^2 + 1}{4\alpha^2}\frac{1}{1 - \alpha^4}(1 - \alpha^{4n})c, \quad \text{if } |\alpha| < 1.
\end{aligned}
$$

The "*alternative*" general inequality for $|\alpha| < 1$; $\alpha \ne 0$ is similarly established.

Note 1. (i) Assume $|\alpha| > 1$, and denote

$$Q_n : Q_n(x) = \alpha^{-4n}f(\alpha^n x).$$

Claim that sequence $\{Q_n\}$, $|\alpha| > 1$ is a Cauchy sequence.
     In fact, if $m > n > 0$, then

$$
\begin{aligned}
0 \le \left\|Q_n(x) - Q_m(x)\right\| &= \left\|\alpha^{-4n}f(\alpha^n x) - \alpha^{-4m}f(\alpha^m x)\right\| \\
&= |\alpha|^{-4n}\left\|f(\alpha^n x) - \alpha^{-4(m-n)}f(\alpha^{m-n}.\alpha^n x)\right\| \\
&\le |\alpha|^{-4n}\frac{3\alpha^2 - 1}{4\alpha^2(\alpha^4 - 1)}\left(1 - \alpha^{-4(m-n)}\right)c \\
&= \frac{3\alpha^2 - 1}{4\alpha^2(\alpha^4 - 1)}\left(|\alpha|^{-4n} - |\alpha|^{-4n}\alpha^{-4(m-n)}\right)c \\
&\to 0, \quad \text{as } n \to \infty \text{ (and } m \to \infty).
\end{aligned}
$$

Thus $\{Q_n\}$, $|\alpha| > 1$, is Cauchy sequence.
     Similarly, if $|\alpha| < 1$, $\alpha \ne 0$, one proves that

$$\{Q_n\}, |\alpha| < 1, \alpha \ne 0$$

is Cauchy sequence, as well.

(ii) Claim the quarticness of

$$Q : Q(x) = \lim_{n \to \infty} Q_n(x) = \lim_{n \to \infty} \alpha^{-4n}f(\alpha^n x), \quad \alpha > 1.$$

In fact, replacing $x \to \alpha^n x$, $y \to \alpha^n y$ in the α-quartic functional inequality (1.9) and then multiplying by $|\alpha|^{-4n}$, and taking limit $n \to \infty$, we obtain

$$
\begin{aligned}
0 &\le \left\|2[Q(\alpha x + y) + Q(x + \alpha y)] + \alpha(\alpha - 1)^2 Q(x - y)\right. \\
&\quad \left. -2(\alpha^2 - 1)^2[Q(x) + Q(y)] - \alpha(\alpha + 1)^2 Q(x + y)\right\| \\
&\le |\alpha|^{-4n}c \to 0, \quad n \to \infty.
\end{aligned}
$$

Thus

$$2[Q(\alpha x + y) + Q(x + \alpha y)] + \alpha(\alpha - 1)^2 Q(x - y) = 2(\alpha^2 - 1)^2[Q(x) + Q(y)] + \alpha(\alpha + 1)^2 Q(x + y)$$

leading to (1.1) and thus the quarticness of $Q : |\alpha| > 1$.

Similarly, we prove that Q is quartic for $|\alpha| < 1$, $\alpha \neq 0$. Thus the existence of Q is complete.

If

$$Q(x) = \lim_{n \to \infty} Q_n(x) = \lim_{n \to \infty} \begin{cases} a^{-4n} f(a^n x) & \text{if } |a| > 1 \\ \\ a^{4n} f(a^{-n} x) & \text{if } |a| < 1; a \neq 0 \end{cases}, \tag{1.13}$$

then

$$\|f(x) - Q(x)\| \leq \frac{2a^2 + |a^2 - 1|}{4a^2} c \cdot \begin{cases} \frac{1}{a^4 - 1} & \text{if } |a| > 1 \\ \\ \frac{1}{1 - a^4} & \text{if } |a| < 1; a \neq 0. \end{cases} \tag{1.14}$$

Note 2. Claim the uniqueness of

$$Q : Q(x) = \lim_{n \to \infty} Q_n(x) = \lim_{n \to \infty} \alpha^{-4n} f(\alpha^n x), \ \alpha > 1.$$

In fact, if there is another quartic mapping Q' satisfying (1.14), then

$$\begin{aligned} 0 &\leq \|Q(x) - Q'(x)\| \\ &\leq \|Q(x) - f(x)\| + \|f(x) - Q'(x)\|, \end{aligned}$$

or

$$\begin{aligned} 0 &\leq \|Q(x) - Q'(x)\| \\ &= |\alpha|^{-4n} \|Q(\alpha^n x) - Q'(\alpha^n x)\| \\ &= \|\alpha^{-4n} Q(\alpha^n x) - \alpha^{-4n} Q'(\alpha^n x)\| \\ &\leq \|\alpha^{-4n} Q(\alpha^n x) - \alpha^{-4n} f(\alpha^n x)\| + \|\alpha^{-4n} f(\alpha^n x) - \alpha^{-4n} Q'(\alpha^n x)\| \\ &= |\alpha|^{-4n} \big\{ \|Q(\alpha^n x) - f(\alpha^n x)\| + \|f(\alpha^n x) - Q'(\alpha^n x)\| \big\} \\ &\leq \frac{2\alpha^2 + |\alpha^2 - 1|}{2\alpha^2} |\alpha|^{-4n} c \to 0, \ \ n \to \infty \end{aligned}$$

or

$$Q(x) = Q'(x),$$

proving uniqueness of $Q : |\alpha| > 1$. Similarly, one proves uniqueness of $Q : |\alpha| < 1$, $\alpha \neq 0$.

Theorem 1.1. Let X be a normed space and Y be a Banach space. If $f : X \to Y$ is a mapping satisfying (1.9), then there exists a unique quartic mapping $Q : X \to Y$, satisfying inequality (1.14).

If $f(0) = 0$, then $\|f(x) - Q(x)\| \leq c / 2|1 - a^4|$, for $\forall a \neq 0; \pm 1$.

2. Stability of General (a, b)-Quartic Functional Equation

$$2\big[f(ax + by) + f(bx + ay)\big] + ab(a - b)^2 f(x - y) = 2(a^2 - b^2)^2\big[f(x) + f(y)\big] + ab(a + b)^2 f(x + y), \tag{2.1}$$

where $a \neq \pm b$, $a, b \neq 0, \pm 1$, $k = a + b \neq 1$, $l = a - b \neq 1$, and $a^4 + b^4 - a^2 b^2 - 1 \neq 0$.

Replacing $x = y = 0$ in this equation, one gets

$$4(a^4 + b^4 - a^2 b^2 - 1) f(0) = 0, \text{ or}$$

$$f(0) = 0 \tag{2.2}$$

Similarly substituting $x = x, y = 0$ in (2.1), we obtain

$$f(ax) + f(bx) = (a^4 + b^4)f(x) + (a^2 - b^2)^2 f(0) = (a^4 + b^4)f(x). \tag{2.3}$$

From (2.3), we observe that

$$\text{``}f(ax) = a^4 f(x) \quad \text{if and only if} \quad f(bx) = b^4 f(x).\text{''} \tag{2.4}$$

Let us introduce the following "(a, b)-quartic functional mean"

$$\overline{f_{(a,b)}}(x) = \frac{f(ax) + f(bx)}{a^4 + b^4}. \tag{2.5}$$

From (2.3) and (2.5), we find the quartic functional mean equation

$$\overline{f_{a,b)}}(x) = f(x) \tag{2.6}$$

From (2.4)-(2.5)-(2.6), one establishes

$$\text{``}\overline{f_{(a,0)}}(x) = \frac{f(ax)}{a^4} = f(x) \ \text{ iff } \ \overline{f_{(0,b)}}(x) = \frac{f(bx)}{b^4} = f(x).\text{''}$$

Also assuming $f(2x) = 16f(x)$, replacing $x = x, y = x$ in (2.1) and setting $k = a + b \neq 0, \pm 1$, one obtains

$$4f(k) + ab(a - b)^2 f(0) = 4(a^2 - b^2)^2 f(x) + ab(a + b)^2 f(2x), \tag{2.8}$$

or

$$f(k) + \frac{1}{4}ab(a - b)^2 f(0) = (a^2 - b^2)^2 f(x) + 4ab(a + b)^2 f(x) = k^4 f(x),$$

or

$$f(k) = k^4 f(x) - \frac{1}{4}ab(a - b)^2 f(0) = k^4 f(x). \tag{2.9}$$

Without assuming $f(2x) = 16f(x)$, replacing $x = x, y = -x$ in (2.1) and setting $l = a - b \neq 0, \pm 1$ with $a \neq \pm b$, one obtains

$$2\big[f(lx) + f(-lx)\big] + ab(a - b)^2 f(2x) = 2(a^2 - b^2)^2 \big[f(x) + f(-x)\big] + ab(a + b)^2 f(0) \tag{2.10}$$

Placing $-x$ on x in (2.10), and then subtracting the new equation from (2.10), we get $ab(a - b)^2 \big[f(2x) - f(-2x)\big] = 0$. Letting $x/2$ on x, we find that f is an "*even function*", such that $f(-x) = f(x)$. Thus from (2.10), we obtain

$$4f(lx) + ab(a - b)^2 f(2x) = 4(a^2 - b^2)^2 f(x) + ab(a + b)^2 f(0). \tag{2.11}$$

Assuming $f(2x) = 16f(x)$, we get

$$f(lx) = l^4 f(x) + \frac{1}{4}ab(a + b)^2 f(0) = l^4 f(x).$$

Without assuming $f(2x) = 16f(x)$, subtracting (2.11) from (2.8), we obtain from (2.2) that

$$f(kx) - f(lx) = \frac{1}{2}ab(a^2 + b^2)f(2x) = \left[\left(\frac{k}{2}\right)^4 - \left(\frac{l}{2}\right)^4\right]f(2x).$$

Replacing $x \to x/2$, we obtain that

$$f\left(\frac{k}{2}x\right) - \left(\frac{k}{2}\right)^4 f(x) = f\left(\frac{l}{2}x\right) - \left(\frac{l}{2}\right)^4 f(x).$$

Therefore, we observe that

$$\text{``}f\left(\frac{k}{2}x\right) = \left(\frac{k}{2}\right)^4 f(x) \quad \text{if and only if} \quad f\left(\frac{l}{2}x\right) = \left(\frac{l}{2}\right)^4 f(x)\text{''}. \tag{2.12}$$

Employing the "quartic mean", we have equivalently that

$$\text{``} \overline{f}_{\frac{k}{2}}(x) \frac{f\left(\frac{k}{2}x\right)}{\left(\frac{k}{2}\right)^4} = f(x) \ \text{ iff } \ \overline{f}_{\frac{l}{2}}(x) = \frac{f\left(\frac{l}{2}x\right)}{\left(\frac{l}{2}\right)^4} = f(x)\text{''}.$$

Let X be a real normed linear space and let Y be a real complete normed linear space. Assume $f : X \to Y$, satisfying the following general (a, b)-quartic functional inequality

$$\left\| 2[f(ax+by)+f(bx+ay)]+ab(a-b)^2 f(x-y)-2(a^2-b^2)^2[f(x)+f(y)]-ab(a+b)^2 f(x+y) \right\| \le c, \quad (2.13)$$

where $a, b \neq 0; a, b \neq \pm 1$. Replacing $x = y = 0$ in (2.13), one gets

$$\|f(0)\| \le c/4|a^4 + b^4 - a^2b^2 - 1|. \quad (2.14)$$

Substituting $x = x, y = 0$ in (2.13), and employing the triangle inequality and (2.14), we obtain

$$\|f(ax) + f(bx) - (a^4 + b^4)f(x)\| \le \frac{c}{2} + (a^2 - b^2)^2|f(0)|,$$

or

$$\|f(ax) + f(bx) - (a^4 + b^4)f(x)\| \le c\frac{(a^2 - b^2)^2 + 2|a^4 + b^4 - a^2b^2 - 1|}{4|a^4 + b^4 - a^2b^2 - 1|}. \quad (2.15)$$

Substituting $x = x, y = x$ in (2.13), and employing the triangle inequality and (2.14), as well as the following hypothesis

$$\|f(2x) - 16f(x)\| \le c_1 \ (\ge 0), \quad (2.16)$$

and denoting $k = a + b \neq 0, \pm 1$, we obtain:

$$\|f(kx) - k^4 f(x)\| \le \frac{1}{4}\left(c + ab(a-b)^2|f(0)| + ab(a+b)^2 c_1\right),$$

or

$$\|f(kx) - k^4 f(x)\| \le c_2 = \frac{1}{4}\left(\frac{ab(a-b)^2 + 4|a^4 + b^4 - a^2b^2 - 1|}{4|a^4 + b^4 - a^2b^2 - 1|}c + ab(a+b)^2 c_1\right), \quad (2.17)$$

or

$$\|f(x) - k^4 f(kx)\| \le k^{-4} c_2 \quad (2.18)$$

if $|k| > 1$.

Thus we easily obtain, the following general inequality:

$$\|f(x) - k^{-4n} f(k^n x)\| \le \frac{1}{k^4 - 1}(1 - k^{-4n})c_2, \ \text{ if } \ |k| > 1. \quad (2.19)$$

In fact,

$$
\begin{aligned}
\left\| f(x) - k^{-4n} f(k^n x) \right\| &\le \left\| f(x) - k^{-4} f(k^1 x) \right\| + k^{-4}\left\| f(k^1 x) - k^{-4} f(k^2 x) \right\| + \cdots \\
&\quad \cdots + k^{-4(n-2)}\left\| f(k^{n-2} x) - k^{-4} f(k^{n-1} x) \right\| \\
&\quad + k^{-4(n-1)}\left\| f(k^{n-1} x) - k^4 f(k^n x) \right\| \quad (2.20) \\
&\le k^{-4}\left(1 + k^{-4} + k^{-8} + \cdots + k^{-4(n-2)} + k^{-4(n-1)}\right)c_2 \\
&= \frac{k^{-4}}{1 - k^{-4}}(1 - k^{-4n})c_2 = \frac{1}{k^4 - 1}(1 - k^{-4n})c_2, \ \text{ if } \ |k| > 1.
\end{aligned}
$$

The "*alternative*" general inequality is similarly established, as follows

$$\|f(x) - k^{4n} f(k^{-n} x)\| \le \frac{1}{1 - k^4}(1 - k^{4n})c_2, \ \text{ if } \ |k| < 1; k \neq 0. \quad (2.21)$$

In fact,

$$
\begin{aligned}
\left\| f(x) - k^{4n} f(k^{-n}x) \right\| &\leq \left\| f(x) - k^4 f(k^{-1}x) \right\| + k^4 \left\| f(k^{-1}x) - k^4 f(k^{-2}x) \right\| + \dots \\
&\quad \dots + k^{4(n-2)} \left\| f(k^{-(n-2)}x) - k^4 f(k^{-(n-1)}x) \right\| \\
&\quad + k^{4(n-1)} \left\| f(k^{-(n-1)}x) - k^4 f(k^{-n}x) \right\| \\
&\leq (1 + k^4 + k^8 + \dots + k^{4(n-2)} + k^{4(n-1)}) c_2 \\
&\leq \frac{1}{1-k^4}(1 - k^{4n}) c_2, \quad \text{if } |k| < 1; k \neq 0.
\end{aligned}
$$
(2.22)

If we denote

$$
Q(x) = \lim_{n \to \infty} Q_n(x) = \lim_{n \to \infty}
\begin{cases}
k^{-4n} f(k^n x) & \text{if } |k| > 1 \\
k^{4n} f(k^{-n}x) & \text{if } |k| < 1; k \neq 0.
\end{cases}
$$

It follows

$$
|f(x) - Q(x)| \leq c_2 \cdot
\begin{cases}
\frac{1}{k^4-1} & \text{if } |k| > 1 \\
\frac{1}{1-k^4} & \text{if } |k| < 1; k \neq 0.
\end{cases}
$$
(2.23)

Note 3. Following Notes 1-2, we establish the existence and uniqueness of the quartic mapping Q.

If $f(0) = 0$, and $f(2x) = 16f(x)$, then

$$
\|f(x) - Q(x)\| \leq \frac{c}{2|1-k^4|}, \quad \text{for } \forall k \neq 0; \pm 1.
$$
(2.24)

Theorem 2.1. Let X be normed space and Y a Banach space. If $f : X \to Y$ is a mapping satisfying (2.13) then there exists a unique quartic mapping $Q : X \to Y$, satisfying inequality (2.23).

If $f(0) = 0$, and $f(2x) = 16f(x)$, then

$$
\|f(x) - Q(x)\| \leq \frac{c}{2|1-k^4|}, \quad \text{for } \forall k = a + b \neq 0; \pm 1.
$$

3. General Alternative a-Quartic Functional Equation

$$
\begin{aligned}
2f(ax + y) + f(x + ay) + f(ax - y) \\
= \frac{a}{2}\left[(a^2 + 4a + 1)f(x + y) - (a^2 - 4a + 1)f(x - y)\right] \\
+ (3a^4 - 4a^2 + 1)f(x) + (a^4 - 4a^2 + 3)f(y),
\end{aligned}
$$
(3.1)

where $a \neq 0, a \neq \pm 1$.

Replacing $x = y = 0$ in (3.1), one gets

$$
4a^2(a^2 - 1)f(0) = 0,
$$

or

$$
f(0) = 0.
$$
(3.2)

Substituting $x = x$, $y = 0$ in (3.1), we obtain

$$
f(ax) = a^4 f(x) + \frac{1}{3}(a^4 - 4a^2 + 3)f(0) = a^4 f(x).
$$
(3.3)

Let X be a real normed linear space and let Y be a real complete normed linear space. Assume $f : X \to Y$, satisfying the following general alternative a-quartic functional inequality

$$
\left\| 2f(ax + y) + f(x + ay) + f(ax - y) \right.
$$

$$
- \frac{a}{2}\left[(a^2 + 4a + 1)f(x + y) - (a^2 - 4a + 1)f(x - y)\right]
$$

$$\left\| -(3a^4 - 4a^2 + 1)f(x) - (a^4 - 4a^2 + 3)f(y) \right\| \leq c, \tag{3.4}$$

where $a \neq 0, a \neq \pm 1$. Replacing $x = y = 0$ in (3.4), one gets

$$4a^2 |a^2 - 1| \|f(0)\| \leq c,$$

or

$$\|f(0)\| \leq \frac{c}{a^2 |a^2 - 1|}. \tag{3.5}$$

Substituting $x = x, y = 0$ in (3.4) and employing (3.5), we obtain

$$\left\| f(x) - a^{-4}f(ax) \right\| \leq \frac{3a^2 + |a^2 - 3|}{3a^2} a^{-4}c = a^{-4}c_1. \tag{3.6}$$

Thus

$$\left\| f(x) - a^{-4n}f(a^n x) \right\| \leq \frac{1}{a^4 - 1}\left(1 - a^{-4(n+1)}\right)c_1. \tag{3.7}$$

Similarly, we obtain

$$\left\| f(x) - a^4 f(a^{-1}x) \right\| \leq \frac{3a^2 + |a^2 - 3|}{3a^2}c = c_1. \tag{3.8}$$

Thus

$$\left\| f(x) - a^{4n}f(a^{-n}x) \right\| \leq \frac{1}{1 - a^4}\left(1 - a^{4(n+1)}\right)c_1. \tag{3.9}$$

If

$$Q(x) = \lim_{n \to \infty} Q_n(x) = \lim_{n \to \infty}
\begin{cases}
a^{-4n}f(a^n x) & \text{if } |a| > 1 \\[2mm]
a^{4n}f(a^{-n}x) & \text{if } |a| < 1; a \neq 0
\end{cases}, \text{ then}$$

$$\left\| f(x) - Q(x) \right\| \leq \frac{3a^2 + |a^2 - 3|}{3a^2}c \cdot
\begin{cases}
\frac{1}{a^4 - 1} & \text{if } |a| > 1 \\[2mm]
\frac{1}{1 - a^4} & \text{if } |a| < 1; a \neq 0.
\end{cases} \tag{3.10}$$

Note 4. Following Notes 1-2, we establish existence and uniqueness of Q.

If $f(0) = 0$, then $\left\| f(x) - Q(x) \right\| \leq \frac{c}{2|1 - a^4|}$, for $\forall a \neq 0; \pm 1$.

Theorem 3.1. Let X be a normed space and Y a Banach space. If $f : X \to Y$ is a mapping satisfying (3.1), then there exists a unique quartic mapping $Q : X \to Y$, satisfying inequality (3.4).
   If $f(0) = 0$, then $\left\| f(x) - Q(x) \right\| \leq c \big/ 2|1 - a^4|$, for $\forall a \neq 0; \pm 1$.

OPEN RESEARCH PROBLEMS

OPEN PROBLEM A.
   Employing both the "*Hyers' direct method*" and the "*fixed point method*", it is still "*open*", the investigation of "*generalized Ulam stabilities*" and "*generalized Ulam superstabilities*" of these quartic functional equations in various normed spaces, domains and groups such as in
1. Banach spaces;
2. Banach algebras; $C*$-algebras;
3. N-multi-Banach spaces; multi-Banach spaces;
4. Multi-normed spaces;
5. Quasi-Banach spaces;
6. Quasi-β[beta]-normed spaces;
7. Non-Archimedean normed spaces;
8. Fuzzy normed spaces;

9. Quasi fuzzy normed spaces;
10. Non- Archimedean fuzzy normed spaces;
11. Intuitionistic normed spaces;
12. Random normed spaces; and probabilistic normed spaces;
13. Non-Archimedean RN[Random Normed]-spaces;
14. Intuitionistic random normed spaces;
15. Intuitionistic fuzzy normed spaces;
16. intuitionistic fuzzy Banach algebras;
17. Intuitionistic Non-Archimedean fuzzy normed spaces;
18. Menger normed spaces;
19. Menger probabilistic normed spaces;
20. Non-Archimedean Menger normed spaces;
21. Intuitionistic Menger normed spaces;
22. L-non-Archimedean- fuzzy Euclidean normed spaces;
23. F-spaces; Fréchet spaces;
24. Banach modules;
25. Distributions and Hyperfunctions;
   as well as, on:
26. Restricted domains;
27. Heisenberg groups.

OPEN PROBLEM B.

28. Exploiting the elementary "*M. Hosszu's method*", due to M. Hosszu (see: "On the Fréchet's functional equation"", Bull. Inst. Politech. Iasi 10, (1964), 1-2, 27-28), determine the general solution and the Ulam stability of each one of these quartic functional equations. The advantage of this method is that we do not assume any regularity conditions on the unknown function f. See also the paper of the authors Xu et al.: [31].

29. Employing two "*L. Szekelyhidi's fundamental results*", due to L. Szekelyhidi (see: "Convolution type functional equation on topological abelian groups", World Scientific, Singapore, 1991), determine the general solution and the Ulam stability of each one of these quartic functional equations in certain types of groups, such as, "commutative groups". See: [31].

OPEN PROBLEM C.

30. Exploiting the elementary "*M. Hosszu's method*", due to M. Hosszu (see: "On the Fréchet's functional equation", Bull. Inst. Politech. Iasi 10, (1964), 1-2, 27-28), determine the general solution and the Ulam stability of each one of these pertinent "*Pexider quartic*" functional equations "with or without *involution*" [32]. We do not assume any regularity conditions on the unknown functions. For "*Pexider quartic*" equations [31]:

(1) $$f(ax + y) + f(x + ay) = g(x + y) + \bar{g}(x - y) + h(x) + \bar{h}(y),$$

(2) $$f(ax + by) + f(bx + ay) = g(x + y) + \bar{g}(x - y) + h(x) + \bar{h}(y),$$

with fixed integers $a, b \neq 0, \pm 1$ and unkown functions $f, \bar{f}, g, \bar{g}, h, \bar{h}$. See also the papers of the author, et al.: ([31, 33]).

31. Employing two "*L. Szekelyhidi's fundamental results*", due to L. Szekelyhidi (see: "Convolution type functional equation on topological abelian groups", World Scientific, Singapore, 1991), determine the general solution and the Ulam stability of each one of these pertinent "*Pexider quartic*" functional equations with or without "*involution*" in certain types of groups, such as, "commutative groups". See: [31, 33].

32. Investigate Ulam-Hyers stabilities of pertinent "*quartic derivations*" from a Banach algebra into its Banach modules. See: ([34, 35]).

33. Establish the solution and stabilities of *"conditionally"* quartic functional equations, for instance: $\|x\| = \|y\|$.

34. Prove stabilities of *"orthogonally"* quartic functional equations, *"in the sense of J. Rätz"* [36]: *"$x \perp y \iff \langle x, y \rangle = 0$".*

35. Work on stabilities of *"quartic-like"* functional equations, such as:

1.
$$2\big[f_1(ax+y) + f_2\big(x + ay + \sigma(y)\big)\big] + a(a-1)^2 g_1(x-y)$$

$$= 2(a^2-1)^2\big[h_1(x) + h_2(y)\big] + a(a+1)^2 g_2(x+y),$$

$\sigma = \sigma(y)$ is *"involution"*: $\sigma(x+y) = \sigma(x) + \sigma(y)$; $\sigma(\sigma(x)) = x$.

2.
$$2f(ax + y + \rho) + f(x + ay + \tau) + f(ax - y)$$

$$= \frac{a}{2}\big[(a^2 + 4a + 1)f(x+y) - (a^2 - 4a + 1)f(x-y)\big]$$

$$+(3a^4 - 4a^2 + 1)f(x) + (a^4 - 4a^2 + 3)f(y).$$

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Existence of positive solutions for summation boundary value problem for a fourth-order difference equations

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Abstract

In this paper, we study the existence of positive solutions to the difference-summation boundary value problem

$$\Delta^4 u(t-2) + a(t)f(u) = 0, \qquad t \in \{2, 3, ..., T\},$$

$$u(0) = \Delta u(0) = \Delta^2 u(0) = 0, \qquad u(T+2) = \alpha \sum_{s=4}^{\eta} u(s),$$

where f is continuous, $T \geq 5$ is a fixed positive integer, $\eta \in \{4, 5, ..., T-1\}$, $0 < \alpha < \frac{4T(T+1)(T+2)}{(\eta-3)(\eta+2)(\eta^2-\eta+4)}$. We show the existence of at least one positive solution if f is either superlinear or sublinear by applying Guo–Krasnoselskii fixed point theorem in cones.

Keywords : Positive solution; Boundary value problem; Fixed point theorem; Cone
2010 Mathematics Subject Classification: 39A15, 34B15

1 Introduction

The existence of solutions for boundary value problems of difference equations has received much attention. For example, see [4-15] and the references therein.

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Liang *et al.* in [4] considered the fourth-order boundary value problem of the form

$$\begin{cases} \Delta^4 x(t-2) + a(t)f(x) = 0, & t \in \{2,3,...,T\}, \\ x(0) = x(T+2) = 0, & \Delta^2 x(0) = \Delta^2 x(T) = 0, \end{cases}$$

where $T > 2$. Existence and uniqueness of solutions are obtained by a fixed point theorem.

Ma *et al.* in [5] considered the fourth-order boundary value problem of the form

$$\begin{cases} \Delta^4 u(t-2) - \lambda f(t, u(t)) = 0, & t \in \{2,3,...,T\}, \\ u(1) = u(T+1) = \Delta^2 u(0) = \Delta^2 u(T) = 0, \end{cases}$$

where λ is a parameter, $T > 5$. Existence and uniqueness of solutions are obtained by the theory of fixed-point index in cones.

In this paper, we consider the existence of positive solutions to the equation

$$\Delta^4 u(t-2) + a(t)f(u) = 0, \qquad t \in \{2,3,...,T\}, \tag{1.1}$$

with summation boundary condition

$$u(0) = \Delta u(0) = \Delta^2 u(0) = 0, \qquad u(T+2) = \alpha \sum_{s=4}^{\eta} u(s), \tag{1.2}$$

where f is continuous.

The aim of this paper is to give some results for existence of positive solutions to (1.1)-(1.2).

Let $\mathbb{N}$ be the nonnegative integer, we let $\mathbb{N}_{i,j} = \{k \in \mathbb{N} | \ i \leq k \leq j\}$ and $\mathbb{N}_p = \mathbb{N}_{0,p}$. By the positive solution of (1.1)-(1.2) we mean that a function $u(t):$ $\mathbb{N}_{T+2} \to [0, \infty)$ and satisfies the problem (1.1)-(1.2).

Throughout this paper, we suppose the following conditions hold:
(H1) $T \geq 5$ is a fixed positive integer, $\eta \in \{4,5,...,T-1\}$, constant $\alpha > 0$ such that $0 < \alpha < \frac{4T(T+1)(T+2)}{(\eta-3)(\eta+2)(\eta^2-\eta+4)}$.
(H2) $f \in C([0,\infty),[0,\infty))$, f is either superlinear or sublinear. Set

$$f_0 = \lim_{u \to 0^+} \frac{f(u)}{u}, \qquad f_\infty = \lim_{u \to \infty} \frac{f(u)}{u}.$$

Then $f_0 = 0$ and $f_\infty = \infty$ correspond to the superlinear case, and $f_0 = \infty$ and $f_\infty = 0$ correspond to the sublinear case.
(H3) $a \in C(\mathbb{N}_{2,T},[0,\infty))$, a is not identical zero.

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The proof of the main theorem is based upon an application of the following Guo-Krasnoselskii's fixed point theorem in a cone.

Theorem 1.1. *Let E be a Banach space, and let $K \subset E$ be a cone. Assume Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let*

$$A : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \longrightarrow K$$

be a completely continuous operator such that
(i) $\|Au\| \leqslant \|u\|$, $u \in K \cap \partial\Omega_1$, and $\|Au\| \geqslant \|u\|$, $u \in K \cap \partial\Omega_2$; or
(ii) $\|Au\| \geqslant \|u\|$, $u \in K \cap \partial\Omega_1$, and $\|Au\| \leqslant \|u\|$, $u \in K \cap \partial\Omega_2$.
Then A has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

2 Preliminaries

We now state and prove several lemmas before stating our main results.

Lemma 2.1. *Suppose that $y(t) \in C(\mathbb{N}_{2,T})$ and $y(t) > 0$. Then the linear boundary value problem*

$$\Delta^4 u(t-2) + y(t) = 0, \qquad t \in \mathbb{N}_{2,T}, \tag{2.1}$$

$$u(0) = \Delta u(0) = \Delta^2 u(0) = 0, \qquad u(T+2) = \alpha \sum_{s=4}^{\eta} u(s), \tag{2.2}$$

has a unique solution

$$u(t) = \frac{t^{\underline{3}}}{6\Lambda}\left[\sum_{s=2}^{T}(T-s+3)^{\underline{3}}y(s) - \frac{\alpha}{4}\sum_{s=2}^{\eta-2}(\eta-s+2)^{\underline{4}}y(s)\right]$$
$$- \frac{1}{6}\sum_{s=2}^{t-2}(t-s+1)^{\underline{3}}y(s), \tag{2.3}$$

where

$$\Lambda := T(T+1)(T+2) - \frac{\alpha}{4}(\eta-3)(\eta+2)(\eta^2-\eta+4). \tag{2.4}$$

Proof. In fact, if $u(t)$ is a solution of problem (2.1), by the discrete Taylor expansion formula, we have

$$u(t) = C_1 t^{\underline{3}} + C_2 t^{\underline{2}} + C_3 t^{\underline{1}} + C_4 - \frac{1}{6}\sum_{s=0}^{t-4}(t-s-1)^{\underline{3}}y(s+2), \quad t \in \mathbb{N}_{T+2}.$$

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Applying the first boundary condition $u(0) = \Delta u(0) = \Delta^2 u(0) = 0$ in (2.2), we obtain

$$C_2 = C_3 = C_4 = 0.$$

So,

$$u(t) = C_1 t^{\underline{3}} - \frac{1}{6} \sum_{s=0}^{t-4} (t - s - 1)^{\underline{3}} y(s + 2), \tag{2.5}$$

From (2.5) and the second boundary condition in (2.2) implies

$$\alpha \sum_{s=4}^{\eta} u(s) = \alpha C_1 \sum_{s=4}^{\eta} (s)^{\underline{3}} - \frac{\alpha}{6} \sum_{s=4}^{\eta} \sum_{\xi=0}^{s-4} (s - \xi - 1)^{\underline{3}} y(\xi + 2)$$

$$= \alpha C_1 \sum_{s=0}^{\eta-4} (s + 4)^{\underline{3}} - \frac{\alpha}{6} \sum_{s=0}^{\eta-4} \sum_{\xi=0}^{\eta-s-4} (\xi + 3)^{\underline{3}} y(s + 2)$$

$$= C_1 (T + 2)^{\underline{3}} - \frac{1}{6} \sum_{s=0}^{T-2} (T - s + 1)^{\underline{3}} y(s + 2).$$

Solving the above equation for a constant C_1, we get

$$C_1 = \frac{1}{6\Lambda} \sum_{s=0}^{T-2} (T - s + 1)^{\underline{3}} y(s + 2) - \frac{\alpha}{6\Lambda} \sum_{s=0}^{\eta-4} \sum_{\xi=0}^{\eta-s-4} (\xi + 3)^{\underline{3}} y(s + 2)$$

where Λ is defined by (2.4)

Therefore, (2.1)-(2.2) has a unique solution

$$u(t) = \frac{t^{\underline{3}}}{6\Lambda} \left[\sum_{s=2}^{T} (T - s + 3)^{\underline{3}} y(s) - \frac{\alpha}{4} \sum_{s=2}^{\eta-2} (\eta - s + 2)^{\underline{4}} y(s) \right]$$

$$- \frac{1}{6} \sum_{s=2}^{t-2} (t - s + 1)^{\underline{3}} y(s).$$

<div align="right">□</div>

Lemma 2.2. *The function*

$$G(t, s) = \frac{1}{6\Lambda} \begin{cases} -\Lambda(t - s + 1)^{\underline{3}} + t^{\underline{3}}(T - s + 3)^{\underline{3}} - \frac{\alpha t^{\underline{3}}}{4}(\eta - s + 2)^{\underline{4}}, & s \in \mathbb{N}_{2,t-2} \cap \mathbb{N}_{2,\eta-2} \\ -\Lambda(t - s + 1)^{\underline{3}} + t^{\underline{3}}(T - s + 3)^{\underline{3}}, & s \in \mathbb{N}_{\eta-1,t-2} \\ t^{\underline{3}}(T - s + 3)^{\underline{3}} - \frac{\alpha t^{\underline{3}}}{4}(\eta - s + 2)^{\underline{4}}, & s \in \mathbb{N}_{t-1,\eta-2} \\ t^{\underline{3}}(T - s + 3)^{\underline{3}}, & s \in \mathbb{N}_{t-1,T} \cap \mathbb{N}_{\eta-1,T} \end{cases} \tag{2.6}$$

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where Λ is defined by (2.4), is the Green's function of the problem

$$-\Delta^4 u(t-2) = 0, \qquad t \in \mathbb{N}_{2,T},$$

$$u(0) = \Delta u(0) = \Delta^2 u(0) = 0, \qquad u(T+2) = \alpha \sum_{s=4}^{\eta} u(s). \qquad (2.7)$$

Proof. Suppose $t < \eta$. The unique solution of problem (2.1)-(2.2) can be written

$$u(t) = -\frac{1}{6} \sum_{s=2}^{t-2} (t-s+1)^{\underline{3}} y(s)$$

$$+ \frac{t^{\underline{3}}}{6\Lambda} \left[\sum_{s=2}^{t-2} (T-s+3)^{\underline{3}} y(s) + \sum_{s=t-1}^{\eta-2} (T-s+3)^{\underline{3}} y(s) + \sum_{s=\eta-1}^{T} (T-s+3)^{\underline{3}} y(s) \right]$$

$$- \frac{\alpha t^{\underline{3}}}{24\Lambda} \left[\sum_{s=2}^{t-2} (\eta-s+2)^{\underline{4}} y(s) + \sum_{s=t-1}^{\eta-2} (\eta-s+2)^{\underline{4}} y(s) \right]$$

$$= \frac{1}{6\Lambda} \sum_{s=2}^{t-2} \left[-\Lambda(t-s+1)^{\underline{3}} + t^{\underline{3}}(T-s+3)^{\underline{3}} - \frac{\alpha t^{\underline{3}}}{4}(\eta-s+2)^{\underline{4}} \right] y(s)$$

$$+ \frac{1}{6\Lambda} \sum_{s=t-1}^{\eta-2} \left[t^{\underline{3}}(T-s+3)^{\underline{3}} - \frac{\alpha t^{\underline{3}}}{4}(\eta-s+2)^{\underline{4}} \right] y(s)$$

$$+ \frac{1}{6\Lambda} \sum_{s=\eta-1}^{T} t^{\underline{3}}(T-s+3)^{\underline{3}} y(s)$$

$$= \sum_{s=2}^{T} G(t,s) y(s).$$

Suppose $t \geq \eta$. The unique solution of problem (2.1)-(2.2) can be written

$$u(t) = -\frac{1}{6} \left[\sum_{s=2}^{\eta-2} (t-s+1)^{\underline{3}} y(s) + \sum_{s=\eta-1}^{t-2} (t-s+1)^{\underline{3}} y(s) \right]$$

$$+ \frac{t^{\underline{3}}}{6\Lambda} \left[\sum_{s=2}^{\eta-2} (T-s+3)^{\underline{3}} y(s) + \sum_{s=\eta-1}^{t-2} (T-s+3)^{\underline{3}} y(s) + \sum_{s=t-1}^{T} (T-s+3)^{\underline{3}} y(s) \right]$$

$$- \frac{\alpha t^{\underline{3}}}{24\Lambda} \sum_{s=2}^{\eta-2} (\eta-s+2)^{\underline{4}} y(s)$$

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$$= \frac{1}{6\Lambda} \sum_{s=2}^{\eta-2} \left[-\Lambda(t-s+1)^{\underline{3}} + t^{\underline{3}}(T-s+3)^{\underline{3}} - \frac{\alpha t^{\underline{3}}}{4}(\eta-s+2)^{\underline{4}} \right] y(s)$$

$$+ \frac{1}{6\Lambda} \sum_{s=\eta-1}^{t-2} \left[-\Lambda(t-s+1)^{\underline{3}} + t^{\underline{3}}(T-s+3)^{\underline{3}} \right] y(s)$$

$$+ \frac{1}{6\Lambda} \sum_{s=t-1}^{T} t^{\underline{3}}(T-s+3)^{\underline{3}} y(s)$$

$$= \sum_{s=2}^{T} G(t,s)y(s).$$

Then the unique solution of problem (2.1)-(2.2) can be written as $u(t) = \sum_{s=2}^{T} G(t,s)y(s)$. The proof is complete. $\square$

We observe that the condition $0 < \alpha < \frac{4T(T+1)(T+2)}{(\eta-3)(\eta+2)(\eta^2-\eta+4)}$. implies $G(t,s)$ is positive on $\mathbb{N}_{2,T} \times \mathbb{N}_{2,T}$.

Let
$$M_1 = \min \left\{ \frac{G(t,s)}{G(t,t)} : t \in \mathbb{N}_{2,T}, s \in \mathbb{N}_{2,T} \right\} \tag{2.8}$$

$$M_2 = \max \left\{ \frac{G(t,s)}{G(t,t)} : t \in \mathbb{N}_{T+2}, s \in \mathbb{N}_{2,T} \right\} \tag{2.9}$$

Lemma 2.3. *Let* $(t,s) \in \mathbb{N}_{2,T} \times \mathbb{N}_{2,T}$. *Then we have*

$$G(t,s) \geq M_1 G(t,t.) \tag{2.10}$$

where $0 < M_1 < 1$ *is a constant given by*

$$M_1 = \min \left\{ \frac{4(T-\eta+7)^{\underline{3}} - 4\Lambda\left(\frac{\eta-5}{\eta-2}\right) - \alpha\eta^{\underline{4}}}{4(T+1)^{\underline{3}} - 24\alpha}, \frac{4(T-\eta+5)^{\underline{3}} - \alpha(\eta+1)^{\underline{4}}}{4(T+1)^{\underline{3}} - 24\alpha}, \right.$$

$$\frac{6}{4(T+1)^{\underline{3}} - 24\alpha}, \frac{4(T-\eta-5)^{\underline{3}} - 4\Lambda\left(\frac{T-3}{T}\right) - \alpha\eta^{\underline{4}}}{4(T-\eta+4)^{\underline{3}}},$$

$$\left. \frac{60(\eta-1)^{\underline{3}} - \Lambda(T-\eta+2)^{\underline{3}}}{T^{\underline{3}}(T-\eta+4)^{\underline{3}}}, \frac{6}{(T-\eta+4)^{\underline{3}}} \right\} \tag{2.11}$$

Proof. In order that 2.10 holds, it is sufficient that M_1 satisfies

$$M_1 \leq \min_{(t,s)\in\mathbb{N}_{2,T}\times\mathbb{N}_{2,T}} \frac{G(t,s)}{G(t,t)}. \tag{2.12}$$

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Then we may choose

$$M_1 \leq \min \left\{ \min_{(t,s)\in\mathbb{N}_{2,\eta-2}\times\mathbb{N}_{2,T}} \frac{G(t,s)}{G(t,t)}, \min_{(t,s)\in\mathbb{N}_{\eta-1,T}\times\mathbb{N}_{2,T}} \frac{G(t,s)}{G(t,t)} \right\}. \tag{2.13}$$

since

$$\min_{(t,s)\in\mathbb{N}_{2,\eta-2}\times\mathbb{N}_{2,T}} \frac{G(t,s)}{G(t,t)}$$

$$= \min_{t\in\mathbb{N}_{2,\eta-2}} \left\{ \min_{s\in\mathbb{N}_{2,t-2}} \frac{-\Lambda(t-s+1)^{\underline{3}}+t^{\underline{3}}(T-s+3)^{\underline{3}}-\frac{\alpha t^{\underline{3}}}{4}(\eta-s+2)^{\underline{4}}}{t^{\underline{3}}(T-t+3)^{\underline{3}}-\frac{\alpha t^{\underline{3}}}{4}(\eta-t+2)^{\underline{4}}}, \right.$$

$$\min_{s\in\mathbb{N}_{t-1,\eta-2}} \frac{t^{\underline{3}}(T-s+3)^{\underline{3}}-\frac{\alpha t^{\underline{3}}}{4}(\eta-s+2)^{\underline{4}}}{t^{\underline{3}}(T-t+3)^{\underline{3}}-\frac{\alpha t^{\underline{3}}}{4}(\eta-t+2)^{\underline{4}}},$$

$$\left. \min_{s\in\mathbb{N}_{\eta-1,T}} \frac{t^{\underline{3}}(T-s+3)^{\underline{3}}}{t^{\underline{3}}(T-t+3)^{\underline{3}}-\frac{\alpha t^{\underline{3}}}{4}(\eta-t+2)^{\underline{4}}} \right\}$$

$$\geq \min_{t\in\mathbb{N}_{2,\eta-2}} \left\{ \frac{-\Lambda(t-1)^{\underline{3}}+t^{\underline{3}}(T-t+5)^{\underline{3}}-\frac{\alpha t^{\underline{3}}}{4}\eta^{\underline{4}}}{t^{\underline{3}}(T-t+3)^{\underline{3}}-\frac{\alpha t^{\underline{3}}}{4}(\eta-t+2)^{\underline{4}}}, \frac{(T-\eta+5)^{\underline{3}}-\frac{\alpha}{4}(\eta-t+3)^{\underline{4}}}{(T-t+3)^{\underline{3}}-\frac{\alpha}{4}(\eta-t+2)^{\underline{4}}}, \right.$$

$$\left. \frac{3^{\underline{3}}}{(T-t+3)^{\underline{3}}-\frac{\alpha}{4}(\eta-t+2)^{\underline{4}}} \right\}$$

$$\geq \min \left\{ \frac{-\Lambda\left(1-\frac{3}{\eta-2}\right)+(T-\eta+7)^{\underline{3}}-\frac{\alpha}{4}(\eta)^{\underline{4}}}{(T+1)^{\underline{3}}-\frac{\alpha 4^{\underline{4}}}{4}}, \frac{(T-\eta+5)^{\underline{3}}-\frac{\alpha}{4}(\eta+1)^{\underline{4}}}{(T+1)^{\underline{3}}-\frac{\alpha 4^{\underline{4}}}{4}}, \frac{6}{(T+1)^{\underline{3}}-\frac{\alpha 4^{\underline{4}}}{4}} \right\}$$

$$= \min \left\{ \frac{4(T-\eta+7)^{\underline{3}}-4\Lambda\left(\frac{\eta-5}{\eta-2}\right)-\alpha\eta^{\underline{4}}}{4(T+1)^{\underline{3}}-24\alpha}, \frac{4(T-\eta+5)^{\underline{3}}-\alpha(\eta+1)^{\underline{4}}}{4(T+1)^{\underline{3}}-24\alpha}, \frac{6}{4(T+1)^{\underline{3}}-24\alpha} \right\} \tag{2.14}$$

Similarly, we get

$$\min_{(t,s)\in\mathbb{N}_{\eta-1,T}\times\mathbb{N}_{2,T}} \frac{G(t,s)}{G(t,t)}$$

$$\geq \min \left\{ \frac{4(T-\eta-5)^{\underline{3}}-4\Lambda\left(\frac{T-3}{T}\right)-\alpha\eta^{\underline{4}}}{4(T-\eta+4)^{\underline{3}}}, \frac{60(\eta-1)^{\underline{3}}-\Lambda(T-\eta+2)^{\underline{3}}}{T^{\underline{3}}(T-\eta+4)^{\underline{3}}}, \frac{6}{(T-\eta+4)^{\underline{3}}} \right\} \tag{2.15}$$

The (2.11) is immediate from (2.14)-(2.15)   □

Lemma 2.4. *Let* $(t,s) \in \mathbb{N}_{T+2} \times \mathbb{N}_{2,T}$. *Then we have*

$$G(t,s) \leq M_2 G(t,t.) \tag{2.16}$$

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where $M_2 \geq 1$ is a constant given by

$$M_2 = \max\left\{ \frac{24\Lambda + (\eta-2)^{\underline{3}}(T+1)^{\underline{3}}}{\alpha(\eta-2)^{\underline{3}}\eta^{\underline{4}}}, \frac{4(T-\eta+6)^{\underline{3}} - 24\alpha}{4(T-\eta+6)^{\underline{3}} - \alpha\eta^{\underline{4}}}, \frac{4(T-\eta+4)^{\underline{3}}}{4(T-\eta+5)^{\underline{3}} - \alpha\eta^{\underline{4}}}, \right.$$
$$\left. \frac{4T^{\underline{3}} - 24\alpha(\eta-1)^{\underline{3}}}{6(\eta-1)^{\underline{3}}}, \frac{T^{\underline{3}}(T-\eta+4)^{\underline{3}} - 6\Lambda}{6(\eta-1)^{\underline{3}}}, \frac{(T-\eta+5)^{\underline{3}}}{6} \right\} \qquad (2.17)$$

Proof. For $t = 0, 1$, from (2.6) we get

$$G(0,s) = G(0,0) = 0; \quad G(1,s) = G(1,1) = 0.$$

Then we may choose $M_2 = 1$. For $t \in \mathbb{N}_{2,T}$, if 2.16 holds, it is sufficient that M_2 satisfies

$$M_2 \geq \max_{(t,s)\in\mathbb{N}_{2,T}\times\mathbb{N}_{2,T}} \frac{G(t,s)}{G(t,t)}. \qquad (2.18)$$

Then we may choose

$$M_2 \geq \max\left\{ \max_{(t,s)\in\mathbb{N}_{2,\eta-2}\times\mathbb{N}_{2,T}} \frac{G(t,s)}{G(t,t)}, \max_{(t,s)\in\mathbb{N}_{\eta-1,T}\times\mathbb{N}_{2,T}} \frac{G(t,s)}{G(t,t)} \right\}. \qquad (2.19)$$

since

$$\max_{(t,s)\in\mathbb{N}_{2,\eta-2}\times\mathbb{N}_{2,T}} \frac{G(t,s)}{G(t,t)}$$

$$= \max_{t\in\mathbb{N}_{2,\eta-2}}\left\{ \max_{s\in\mathbb{N}_{2,t-2}} \frac{-\Lambda(t-s+1)^{\underline{3}} + t^{\underline{3}}(T-s+3)^{\underline{3}} - \frac{\alpha t^{\underline{3}}}{4}(\eta-s+2)^{\underline{4}}}{t^{\underline{3}}(T-t+3)^{\underline{3}} - \frac{\alpha t^{\underline{3}}}{4}(\eta-t+2)^{\underline{4}}}, \right.$$

$$\max_{s\in\mathbb{N}_{t-1,\eta-2}} \frac{t^{\underline{3}}(T-s+3)^{\underline{3}} - \frac{\alpha t^{\underline{3}}}{4}(\eta-s+2)^{\underline{4}}}{t^{\underline{3}}(T-t+3)^{\underline{3}} - \frac{\alpha t^{\underline{3}}}{4}(\eta-t+2)^{\underline{4}}},$$

$$\left. \max_{s\in\mathbb{N}_{\eta-1,T}} \frac{t^{\underline{3}}(T-s+3)^{\underline{3}}}{t^{\underline{3}}(T-t+3)^{\underline{3}} - \frac{\alpha t^{\underline{3}}}{4}(\eta-t+2)^{\underline{4}}} \right\}$$

$$\leq \max_{t\in\mathbb{N}_{2,\eta-2}}\left\{ \frac{-\Lambda 3^{\underline{3}} + t^{\underline{3}}(T+1)^{\underline{3}} - \frac{\alpha t^{\underline{3}}}{4}(\eta-t+4)^{\underline{4}}}{t^{\underline{3}}(T-t+3)^{\underline{3}} - \frac{\alpha t^{\underline{3}}}{4}(\eta-t+2)^{\underline{4}}}, \frac{(T-t+4)^{\underline{3}} - \frac{\alpha 4^{\underline{4}}}{4}}{(T-t+3)^{\underline{3}} - \frac{\alpha}{4}(\eta-t+2)^{\underline{4}}}, \right.$$

$$\left. \frac{(T-\eta+4)^{\underline{3}}}{(T-t+3)^{\underline{3}} - \frac{\alpha}{4}\eta^{\underline{4}}} \right\}$$

$$\leq \max\left\{ \frac{24\Lambda + (\eta-2)^{\underline{3}}(T+1)^{\underline{3}}}{\alpha(\eta-2)^{\underline{3}}\eta^{\underline{4}}}, \frac{4(T-\eta+6)^{\underline{3}} - 24\alpha}{4(T-\eta+6)^{\underline{3}} - \alpha\eta^{\underline{4}}}, \frac{4(T-\eta+4)^{\underline{3}}}{4(T-\eta+5)^{\underline{3}} - \alpha\eta^{\underline{4}}} \right\}$$
$$(2.20)$$

Similarly, we get

$$\max_{(t,s)\in\mathbb{N}_{\eta-1,T}\times\mathbb{N}_{2,T}} \frac{G(t,s)}{G(t,t)}$$

$$\leq \max\left\{ \frac{4T^{\underline{3}} - 24\alpha(\eta-1)^{\underline{3}}}{6(\eta-1)^{\underline{3}}}, \frac{T^{\underline{3}}(T-\eta+4)^{\underline{3}} - 6\Lambda}{6(\eta-1)^{\underline{3}}}, \frac{(T-\eta+5)^{\underline{3}}}{6} \right\} \qquad (2.21)$$

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For $t = T + 1, T + 2$ from (2.6) we get

$$G(T + 1, s) < \frac{1}{6\Lambda} \left[-\Lambda(T - s + 2)^{\underline{3}} + (T + 1)^{\underline{3}} \left((T - s + 3)^{\underline{3}} - \frac{\alpha}{4}(\eta - s + 2)^{\underline{4}} \right) \right]$$
$$= -\frac{\alpha}{24\Lambda}(T + 1)^{\underline{3}}(\eta - s + 2)^{\underline{4}}$$
$$< -\frac{\alpha}{24\Lambda}(T + 1)^{\underline{3}}(\eta - T + 1)^{\underline{4}}$$
$$= G(T + 1, T + 1),$$

$$G(T + 2, s) < \frac{1}{6\Lambda} \left[-\Lambda(T - s + 3)^{\underline{3}} + (T + 2)^{\underline{3}} \left((T - s + 3)^{\underline{3}} - \frac{\alpha}{4}(\eta - s + 2)^{\underline{4}} \right) \right]$$
$$= \frac{1}{6\Lambda} \left[(T - s + 2)^{\underline{3}} \left(\frac{T - s + 3}{T - s} \right) ((T + 2)^{\underline{3}} - \Lambda) - \frac{\alpha}{4}(\eta - s + 2)^{\underline{4}} \right]$$
$$= -\frac{\alpha}{24\Lambda}(T + 2)^{\underline{3}}(\eta - s + 2)^{\underline{4}}$$
$$< -\frac{\alpha}{24\Lambda}(T + 2)^{\underline{3}}(\eta - T)^{\underline{4}}$$
$$= G(T + 2, T + 2).$$

Then we choose $M_2 = 1$. So (2.17) is immediate from (2.20)-(2.21).   □

3   Main Results

Now we are in the position to establish the main result.

Theorem 3.1. *Assume $(H1)$ - $(H3)$ hold. Then the problem (1.1)-(1.2) has at least one positive solution.*

Proof. In the following, we denote

$$m = \min_{t \in \mathbb{N}_{\eta-1,T}} G(t,t), \quad M = \max_{t \in \mathbb{N}_{T+2}} G(t,t).$$

Then $0 < m < M$.

Let E be the Banach's space defined by $E = \{u : \mathbb{N}_{T+2} \to R\}$. Define

$$K = \{u \in E : u \geqslant 0, t \in \mathbb{N}_{T+2} \text{ and } \min_{t \in \mathbb{N}_{2,T}} u(t) \geq \sigma \parallel u \parallel\}.$$

where $\sigma = \frac{M_1 m}{M_2 M} \in (0,1), \parallel u \parallel = \max_{t \in \mathbb{N}_{T+2}} \mid u(t) \mid$. It is obvious that K is a cone in E.

We define the operator $A : K \to E$ by

$$(Au)(t) = \sum_{s=2}^{T} G(t,s)a(s)f(u(s)), t \in \mathbb{N}_{T+2}.$$

It is clear that problem (1.1)-(1.2) has a solution u if and only if $u \in K$ is a fixed point of operator A. We shall now show that the operator A maps K to itself. For this, let $u \in K$, from $(H_2) - (H_3)$, we get

$$(Au)(t) = \sum_{s=1}^{T} G(t,s)a(s)f(u(s)) \geq 0, t \in \mathbb{N}_{T+2}. \tag{3.1}$$

from (2.9), we obtain

$$(Au)(t) = \sum_{s=2}^{T} G(t,s)a(s)f(u(s)) \leq M_2 \sum_{s=2}^{T} G(t,t)a(s)f(u(s))$$

$$\leq M_2 M \sum_{s=2}^{T} a(s)f(u(s)), \quad t \in \mathbb{N}_{T+2}.$$

Therefore

$$\| Au \| \leq M_2 M \sum_{s=2}^{T} a(s)f(u(s)). \tag{3.2}$$

Now from $(H_2), (H_3)$, (2.8) and (3.2), for $t \in \mathbb{N}_{\eta,T}$, we have

$$(Au)(t) \geq M_1 \sum_{s=2}^{T} G(t,t)a(s)f(u(s)) \geq M_1 m \sum_{s=2}^{T} a(s)f(u(s))$$

$$\geq \frac{M_1 m}{M_2 M} \| Au \| = \sigma \| u \| .$$

Then

$$\min_{t \in \mathbb{N}_{\eta,T}} (Au)(t) \geq \sigma \| u \| . \tag{3.3}$$

From (3.1)-(3.2), we obtain $Au \in K$, Hence $A(K) \subseteq K$. So $F : k \to K$ is completely continuous.

Superlinear case. $f_0 = 0$ and $f_\infty = \infty$. Since $f_0 = 0$, we may choose $H_1 > 0$ so that $f(u) \leqslant \epsilon_1 u$, for $0 < u \leqslant H_1$, where $\epsilon_1 > 0$ satisfies

$$\epsilon_1 M_2 M \sum_{s=2}^{T} a(s) \leq 1. \tag{3.4}$$

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Thus, if we let

$$\Omega_1 = \{u \in E : \|u\| < H_1\},$$

then for $u \in K \cap \partial\Omega_1$, we get

$$(Au)(t) \leq M_2 \sum_{s=2}^{T} G(t,t)a(s)f(u(s)) \leq \epsilon_1 M_2 M \sum_{s=2}^{T} a(s)u(s)$$

$$\leq \epsilon_1 M_2 M \sum_{s=2}^{T} a(s)\|u\| \leq \|u\|.$$

Thus $\|Au\| \leq \|u\|$, $u \in K \cap \partial\Omega_1$.

Further, since $f_\infty = \infty$, there exists $\widehat{H}_2 > 0$ such that $f(u) \geq \epsilon_2 u$, for $u \geq \widehat{H}_2$, where $\epsilon_2 > 0$ satisfies

$$\epsilon_2 M_1 \sigma \sum_{s=\eta-1}^{T} G(\eta-1, \eta-1)a(s) \geq 1. \tag{3.5}$$

Let $H_2 = \max\{2H_1, \frac{\widehat{H}_2}{\sigma}\}$ and $\Omega_2 = \{u \in E : \|u\| < H_2\}$. Then $u \in K \cap \partial\Omega_2$ implies

$$\min_{t \in \mathbb{N}_{\eta-1,T}} u(t) \geq \sigma\|u\| \geq \widehat{H}_2.$$

Applying (2.8) and (3.5), we get

$$(Au)(\eta-1) = M_1 \sum_{s=2}^{T} G(\eta-1,s)a(s)f(u(s)) \geq M_1 \sum_{s=\eta-1}^{T} G(\eta-1,\eta-1)a(s)f(u(s))$$

$$\geq \varepsilon_2 M_1 \sum_{s=\eta-1}^{T} G(\eta-1,\eta-1)a(s)y(s) \geq \varepsilon_2 M_1 \sigma \sum_{s=\eta-1}^{T} G(\eta-1,\eta-1)a(s)\|u\|$$

$$\geq \|u\|.$$

Hence, $\|Au\| \geq \|u\|$, $u \in K \cap \partial\Omega_2$. By the first part of Theorem 1.1, A has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$ such that $H_1 \leqslant \|u\| \leqslant H_2$.

Sublinear case. $f_0 = \infty$ and $f_\infty = 0$. Since $f_0 = \infty$, choose $H_3 > 0$ such that $f(u) \geqslant \epsilon_3 u$ for $0 < u \leqslant H_3$, where $\varepsilon_3 > 0$ satisfies

$$\epsilon_3 M_1 \sigma \sum_{s=\eta-1}^{T} G(\eta-1, \eta-1)a(s) \geqslant 1. \tag{3.6}$$

Let

$$\Omega_3 = \{u \in E : \|u\| < H_3\},$$

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then for $u \in K \cap \partial\Omega_3$, we get

$$(Au)(\eta - 1) \geq M_1 \sum_{s=\eta-1}^{T} G(\eta - 1, \eta - 1)a(s)f(u(s)) \geq \epsilon_3 M_1 \sum_{s=\eta-1}^{T} G(\eta - 1, \eta - 1)a(s)y(s)$$

$$\geq \varepsilon_3 M_1 \sigma \sum_{s=\eta-1}^{T} G(\eta - 1, \eta - 1)a(s)\|u\| \geq \|u\|.$$

Thus, $\|Au\| \geqslant \|u\|$, $u \in K \cap \partial\Omega_3$.

Now, since $f_\infty = 0$, there exists $\widehat{H}_4 > 0$ so that $f(u) \leqslant \epsilon_4 u$ for $u \geqslant \widehat{H}_4$, where $\epsilon_4 > 0$ satisfies

$$\epsilon_4 M_2 M \sum_{s=\eta-1}^{T} a(s) \geqslant 1. \qquad (3.7)$$

Subcase 1. Suppose f is bounded, $f(u) \leq L$ for all $u \in [0, \infty)$ for some $L > 0$. Let $H_4 = \max\{2H_3, LM_2M \sum_{s=1}^{T} a(s)\}$.

Then for $u \in K$ and $\|u\| = H_4$, we get

$$(Au)(\eta) \leq M_2 \sum_{s=2}^{T} G(t, t)a(s)f(u(s)) \leq LM_2M \sum_{s=2}^{T} a(s)$$

$$\leq H_4 = \|u\|$$

Thus $(Au)(t) \leq \|u\|$.

Subcase 2. Suppose f is unbounded, there exist $H_4 > \max\{2H_3, \frac{\widehat{H}_4}{\sigma}\}$ such that $f(u) \leq f(H_4)$ for all $0 < u \leq H_4$. Then for $u \in K$ with $\|u\| = H_4$ from (2.9) and (3.7), we have

$$(Au)(t) \leq M_2 \sum_{s=2}^{T} G(t, t)a(s)f(u(s)) \leq M_2M \sum_{s=2}^{T} a(s)f(H_4)$$

$$\leq \epsilon_4 M_2 M \sum_{s=2}^{T} a(s)H_4 \leq H_4 = \|u\|.$$

Thus in both cases, we may put $\Omega_4 = \{u \in E : \|u\| < H_4\}$. Then

$$\|Au\| \leqslant \|u\|, u \in K \cap \partial\Omega_4.$$

By the second part of Theorem 1.1, A has a fixed point u in $K \cap (\overline{\Omega}_4 \setminus \Omega_3)$, such that $H_3 \leqslant \|u\| \leqslant H_4$. This completes the sublinear part of the theorem. Therefore, the problem (1.1)-(1.2) has at least one positive solution. $\qquad \square$

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4   Some examples

In this section, in order to illustrate our result, we consider some examples.

Example 4.1   Consider the BVP

$$\Delta^4 u(t-2) + t^2 u^k = 0, \qquad t \in N_{2,6}, \tag{4.1}$$

$$u(0) = \Delta u(0) = \Delta^2 u(0) = 0, \qquad u(8) = \frac{2}{3}\sum_{s=4}^{5} u(s). \tag{4.2}$$

Set $\alpha = \frac{2}{3}$, $\eta = 5$, $T = 6$, $a(t) = t^2$, $f(u) = u^k$.
We can show that

$$T(T+1)(T+2) - \frac{\alpha}{4}(\eta - 3)(\eta + 2)(\eta^2 - \eta + 4) = 280 > 0.$$

Case I : $k \in (1, \infty)$. In this case, $f_0 = 0$, $f_\infty = \infty$ and (i) of theorem 3.1 holds. Then BVP (4.1)-(4.2) has at least one positive solution.

Case II : $k \in (0, 1)$. In this case, $f_0 = \infty$, $f_\infty = 0$ and (ii) of theorem 3.1 holds. Then BVP (4.1)-(4.2) has at least one positive solution.

Example 4.2   Consider the BVP

$$\Delta^4 u(t-2) + e^t t^e \left(\frac{\pi \sin u + 2\cos u}{u^2}\right) = 0, \qquad t \in N_{2,8}, \tag{4.3}$$

$$u(0) = \Delta u(0) = \Delta^2 u(0) = 0, \qquad u(10) = \frac{1}{3}\sum_{s=4}^{6} u(s), \tag{4.4}$$

Set $\alpha = \frac{1}{3}$, $\eta = 6$, $T = 8$, $a(t) = e^t t^e$, $f(u) = \frac{\pi \sin u + 2\cos u}{u^2}$.
We can show that

$$T(T+1)(T+2) - \frac{\alpha}{4}(\eta - 3)(\eta + 2)(\eta^2 - \eta + 4) = 596 > 0.$$

Through a simple calculation we can get $f_0 = \infty$, $f_\infty = 0$. Thus, by (ii) of theorem 3.1, we can get BVP (4.3)-(4.4) has at least one positive solution.

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Similarity measure between generalized intuitionistic fuzzy sets and its application to pattern recognition

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Abstract

This paper presents new methods for measuring similarity between generalized intuitionistic fuzzy sets (GIFSs) and its application to pattern recognition. Firstly, the geometrical interpretation of GIFSs is carefully reviewed and then the results of the interpretation is utilized to generate new methods for measuring similarity in order to calculate the degree of similarity between GIFSs. Numerical example is given to illustrate the application of the proposed similarity measures. Finally, we also use the proposed similarity measures to characterize the similarity between linguistic variables.

1 Introduction

As a generalization of fuzzy sets, intuitionistic fuzzy sets (IFSs) were presented by Atanassov [1, 2, 3]. Since IFSs can present the degrees of membership and non-membership with a degree of hesitancy, the knowledge and semantic representation become more meaningful and applicable. These IFSs have been widely studied and applied in various areas, such as logic programming [4], decision making [7, 22], pattern recognition [12, 14, 15, 16, 19, 26] and medical diagnosis

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[9, 25], and seem to have more popular than fuzzy sets technology. Mondal and Samanta [18] introduced generalized intuitionistic fuzzy sets (GIFSs) as a generalization of IFSs and studied their basic properties. Park et al. [20] proposed a method to calculate the correlation coefficient of GIFSs. There is a little investigation on GIFSs. Similarity assessment plays a fundamental and important role in inference and approximate reasoning in all applications of intuitionistic fuzzy logic [4]. For different purposes different similarity measures should be used. Based on the importance of the problem, the effectiveness and properties of the different similarity measures for IFSs have been compared and examined by many researchers (e.g. Hung and Yang [12], Li and Cheng [14], Li et al. [15], Liang and Shi [16], Mitchell [19], Szmidt and Baldwin [23]). The analysis of similarity is also a fundamental issue while employing GIFSs. Recently, Park et al. [21] proposed and applied similarity measure to compare generalized intuitionistic fuzzy preferences given by individuals (experts) and evaluated an extent of a group agreement. In this paper, we propose new similarity measures based on the geometrical representation for GIFS. The proposed similarity measures depend on the triplet, membership degree, nonmembership degree, and hesitation margin. This paper proves that the proposed similarity measures satisfy the properties of axiomatic definition for similarity measures. Numerical example is given to illustrate the application of the developed similarity measures. Furthermore, we use the proposed similarity measures to characterize the similarity between linguistic variables.

2 Brief introduction of GIFSs

In the following, we firstly recall basic notions and definitions of GIFSs which can be found in [18].

Let X be the universe of discourse. A generalized intuitionistic fuzzy set (GIFS) A in X is an object having the form

$$A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\} \tag{1}$$

where $\mu_A, \nu_A : X \to [0,1]$ denote membership function and non-membership function, respectively, of A and satisfy $\min\{\mu_A(x), \nu_A(x)\} \leq 0.5$ for all $x \in X$. Let GIFS(X) denote the set of all GIFSs in X.

For an IFS $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$, it is observed that $\mu_A(x) + \nu_A(x) \leq 1$ implies $\min\{\mu_A(x), \nu_A(x)\} \leq 0.5$ for each $x \in X$. Thus, every IFS is GIFS.

For each GIFS A in X, we call

$$\phi_A(x) = 1 - \mu_A(x) - \nu_A(x) \tag{2}$$

a generalized intuitionistic fuzzy index (or a hesitation margin) of x in A and it expresses a lack/excess of knowledge of whether x belongs to A or not. (see, [20]). It is obvious that $-0.5 \leq \phi_A(x) \leq 1$ for each $x \in X$.

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Having in mind that for each element x belonging to a GIFS A, the values of membership, non-membership and generalized intuitionistic fuzzy index add up to one, i.e.

$$\mu_A(x) + \nu_A(x) + \phi_A(x) = 1 \tag{3}$$

and that each of the membership and non-membership are from $[0, 1]$ and the generalized intuitionistic fuzzy index is from $[-0.5, 1]$, we can imagine a cuboid (Figure 1) inside which there is a polygon $ADBEGF$ where the above equation is fulfilled. In other words, the polygon $ADBEGF$ represents a surface where coordinates of any element belonging to a GIFS can be represented. Each point belonging to the polygon $ADBEGF$ is described via three coordinates: (μ, ν, ϕ). Points A and B represent crisp elements. Point $A(1, 0, 0)$ represents elements fully belonging to a GIFS as $\mu = 1$. Point $B(0, 1, 0)$ represents elements fully not belonging to a GIFS as $\nu = 1$. Point $D(0, 0, 1)$ represents element about which we are not able to say if they belong or not belong to a GIFS (generalized intuitionistic fuzzy index $\phi = 1$). Point $E(1, 0.5, -0.5)$ represents element about which we can say to belong to a GIFS ($\phi = -0.5$). Point $F(0.5, 1, -0.5)$ represents element about which we can say to not belong to a GIFS ($\phi = -0.5$). Such an interpretation is intuitively appealing and provides means for the representation of many aspects of imperfect information. Segment AB (where $\phi = 0$) represents elements belonging to classical fuzzy sets ($\mu + \nu = 1$). Triangle ADB (where $0 \leq \phi \leq 1$) represents elements belonging to IFSs ($0 \leq \mu + \nu \leq 1$). Any other combination of the values characterizing a GIFS can be represented inside the triangles AGF and BEG. In other words, each element belonging to a GIFS can be represented as a point (μ, ν, ϕ) belonging to the polygon $ADBEGF$ (cf. Figure 1).
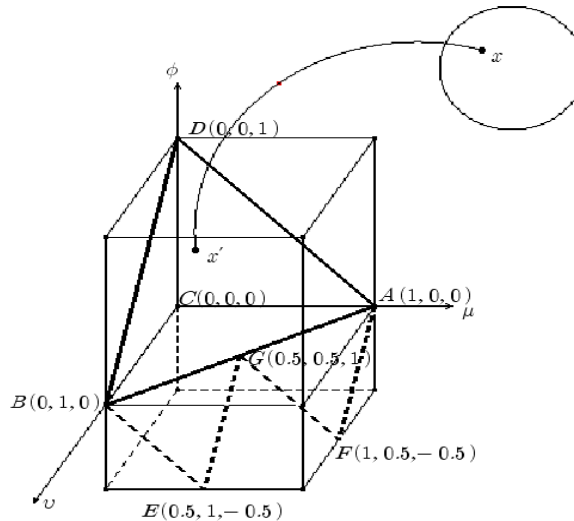

Figure 1: A geometrical interpretation of a GIFS

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It is worth mentioning that the geometrical interpretation is directly related to the definition of a GIFS, and it does not need any additional assumptions. By employing the above geometrical representation, a GIFS A can be expressed as

$$A = \{(\mu_A(x), \nu_A(x), \phi_A(x)) \mid x \in X\}. \tag{4}$$

Therefore, this representation of a GIFS will be a point of departure for considering the our method in calculating the degree of similarity between GIFSs.

For $A, B \in \text{GIFS}(X)$, Mondal and Samanta [18] defined the notion of containment as follows:

$$A \subseteq B \iff \mu_A(x) \leq \mu_B(x) \text{ and } \nu_A(x) \geq \nu_B(x) \ \forall x \in X. \tag{5}$$

As above-mentioned, we can not omit the third parameter (hesitancy degree) in the representation of GIFSs and then redefine the notion of containment as follows:

$$A \subseteq B \iff \mu_A(x) \leq \mu_B(x), \ \nu_A(x) \geq \nu_B(x) \text{ and } \phi_A(x) \geq \phi_B(x) \ \forall x \in X. \tag{6}$$

Definition 2.1 Let $S : \text{GIFS}(X) \times \text{GIFS}(X) \to [0,1]$ be a mapping. $S(A, B)$ is said to be the degree of similarity between $A \in \text{GIFS}(X)$ and $B \in \text{GIFS}(X)$ if $S(A, B)$ satisfies the properties (SP1)-(SP4):
(SP1) $0 \leq S(A, B) \leq 1$;
(SP2) $S(A, B) = 1$ if and only if $A = B$;
(SP3) $S(A, B) = S(B, A)$;
(SP4) $S(A, C) \leq S(A, B)$ and $S(A, C) \leq S(B, C)$ if $A \subseteq B \subseteq C$, $A, B, C \in \text{GIFS}(X)$.

Definition 2.2 Let $D : \text{GIFS}(X) \times \text{GIFS}(X) \to [0,1]$ be a mapping. $D(A, B)$ is called a distance $A \in \text{GIFS}(X)$ and $B \in \text{GIFS}(X)$ if $D(A, B)$ satisfies the properties (DP1)-(DP4):
(DP1) $0 \leq D(A, B) \leq 1$;
(DP2) $D(A, B) = 0$ if and only if $A = B$;
(DP3) $D(A, B) = D(B, A)$;
(DP4) $D(A, B) \leq D(A, C)$ and $D(B, C) \leq D(A, C)$ if $A \subseteq B \subseteq C$, $A, B, C \in \text{GIFS}(X)$.

Because distance and similarity measures are complementary concepts, similarity measures can be used to define distance measures and vice verses.

3 New similarity measures between GIFSs

In this section, we take into account three parameters describing GIFSs to propose a new similarity measures between GIFSs based on the geometrical representation of GIFSs.

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Let $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$ and $B = \{(x, \mu_B(x), \nu_B(x)) \mid x \in X\}$ be two GIFSs in $X = \{x_1, x_2, \ldots, x_n\}$. We propose a new similarity measure:

$$S_g(A, B)$$

$$= 1 - \frac{1}{n}\sum_{i=1}^{n}\left(\frac{|\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)| + |\phi_A(x_i) - \phi_B(x_i)|}{4}\right.$$

$$\left. + \frac{\max(|\mu_A(x_i) - \mu_B(x_i)|, |\nu_A(x_i) - \nu_B(x_i)|, |\phi_A(x_i) - \phi_B(x_i)|)}{2}\right), \quad (7)$$

where $\phi_A(x_i)$ and $\phi_B(x_i)$ are, respectively, the hesitancy degree of the element $x_i \in X$ to the sets A and B.

Theorem 3.1 $S_g(A, B)$ *is the similarity measure between two GIFSs A and B.*

Proof For the sake of simplicity, IFSs A and B are denoted by $A = \{(\mu_A(x_i), \nu_A(x_i), \phi(x_i)) \mid x_i \in X\}$ and $B = \{(\mu_B(x_i), \nu_B(x_i), \phi_B(x_i)) \mid x_i \in X\}$, respectively. Obviously, $S_g(A, B)$ satisfies (SP1) and (SP3) of Definition 1. We only need to prove that $S_g(A, B)$ satisfies (SP2) and (SP4).

(SP2): From (6), we have

$$S_g(A, B) = 1$$

$$\Leftrightarrow \mu_A(x_i) = \mu_B(x_i), \nu_A(x_i) = \nu_B(x_i), \phi_A(x_i) = \phi_B(x_i), \forall x_i \in X$$

$$\Leftrightarrow A = B.$$

(SP4): For any IFS $C = \{(\mu_C(x_i), \nu_C(x_i), \phi_C(x_i)) \mid x_i \in X\}$, if $A \subseteq B \subseteq C$, then we have

$$S_g(A, C) = 1 - \frac{1}{n}\sum_{i=1}^{n}\left(\frac{|\mu_A(x_i) - \mu_C(x_i)| + |\nu_A(x_i) - \nu_C(x_i)| + |\phi_A(x_i) - \phi_C(x_i)|}{4}\right.$$

$$\left. + \frac{\max(|\mu_A(x_i) - \mu_C(x_i)|, |\nu_A(x_i) - \nu_C(x_i)|, |\phi_A(x_i) - \phi_C(x_i)|)}{2}\right).$$

It is easy to see that

$$|\mu_A(x_i) - \mu_C(x_i)| \geq |\mu_A(x_i) - \mu_B(x_i)|, |\nu_A(x_i) - \nu_C(x_i)| \geq |\nu_A(x_i) - \nu_B(x_i)|,$$
$$|\phi_A(x_i) - \phi_C(x_i)| \geq |\phi_A(x_i) - \phi_B(x_i)|.$$

So we have

$$\frac{|\mu_A(x_i) - \mu_C(x_i)| + |\nu_A(x_i) - \nu_C(x_i)| + |\phi_A(x_i) - \phi_C(x_i)|}{4}$$

$$+ \frac{\max(|\mu_A(x_i) - \mu_C(x_i)|, |\nu_A(x_i) - \nu_C(x_i)|, |\phi_A(x_i) - \phi_C(x_i)|)}{2}$$

$$\geq \frac{|\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)| + |\phi_A(x_i) - \phi_B(x_i)|}{4}$$

$$+ \frac{\max(|\mu_A(x_i) - \mu_B(x_i)|, |\nu_A(x_i) - \nu_B(x_i)|, |\phi_A(x_i) - \phi_B(x_i)|)}{2}$$

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and thus we get $S_g(A, C) \leq S_g(A, B)$. By the same reason, we can get $S_g(A, C) \leq S_g(B, C)$.

However, the elements in the universe may have different importance in pattern recognition. We should consider the weight of the elements so that we can obtain more reasonable results in pattern recognition.

Assume that the weight of x_i in X is w_i, where $w_i \in [0, 1]$ $(i = 1, 2, \ldots, n)$ and $\sum_{i=1}^{n} w_i = 1$. The similarity measure between GIFSs A and B can be obtained by the following form:

$$
\begin{aligned}
&S_{gw}(A, B) \\
&= 1 - \sum_{i=1}^{n} w_i \left(\frac{|\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)| + |\phi_A(x_i) - \phi_B(x_i)|}{4} \right. \\
&\qquad \left. + \frac{\max(|\mu_A(x_i) - \mu_B(x_i)|, |\nu_A(x_i) - \nu_B(x_i)|, |\phi_A(x_i) - \phi_B(x_i)|)}{2} \right).
\end{aligned} \tag{8}
$$

Likewise, for $S_{gw}(A, B)$, the following theorem holds.

Theorem 3.2 $S_{gw}(A, B)$ *is the similarity measure between two GIFSs A and B.*

Proof The proof is similar to that of Theorem 3.1.

Remark 3.3 Obviously, if $w_i = 1/n$ $(i = 1, 2, \ldots, n)$, (8) becomes (7). So, (7) is a special case of (8).

Now, we propose another new similarity measure between GIFSs $A = \{(x, \mu_A(x), \nu_A(x))|x \in X\}$ and $B = \{(x, \mu_B(x), \nu_B(x))|x \in X\}$ in $X = \{x_1, x_2, \ldots, x_n\}$ as follows:

Let $\varphi_{\mu_{AB}}(i) = |\mu_A(x_i) - \mu_B(x_i)|/2, \varphi_{\nu_{AB}}(i) = |\nu_A(x_i) - \nu_B(x_i)|/2, \varphi_{\phi_{AB}}(i) = |\phi_A(x_i) - \phi_B(x_i)|/2$ and $x_i \in X$. Then

$$
S_d^p(A, B) = 1 - \frac{1}{\sqrt[p]{n}} \sqrt[p]{\sum_{i=1}^{n} (\varphi_{\mu_{AB}}(i) + \varphi_{\nu_{AB}}(i) + \varphi_{\pi_{AB}}(i))^p}, \tag{9}
$$

where $1 \leq p < \infty$.

Theorem 3.4 $S_d^p(A, B)$ *is the similarity measure between two GIFSs A and B.*

Proof Obviously, $S_d^p(A, B)$ satisfies (SP1) and (SP3). As to (SP2) and (SP4), we give the following proof.

(SP2): From (8), we have

$$
\begin{aligned}
S_d^p(A, B) &= 1 \\
&\Leftrightarrow \varphi_{\mu_{AB}}(i) = 0, \varphi_{\nu_{AB}}(i) = 0, \varphi_{\phi_{AB}}(i) = 0, \ \forall i = 1, \ldots, n \\
&\Leftrightarrow \mu_A(x_i) = \mu_B(x_i), \nu_A(x_i) = \nu_B(x_i), \phi_A(x_i) = \phi_B(x_i), \ \forall x_i \in X \\
&\Leftrightarrow A = B.
\end{aligned}
$$

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(SP4): Since $A \subseteq B \subseteq C$, we have $\mu_A(x_i) \leq \mu_B(x_i) \leq \mu_C(x_i)$, $\nu_A(x_i) \geq \nu_B(x_i) \geq \nu_C(x_i)$ and $\phi_A(x_i) \geq \phi_B(x_i) \geq \phi_C(x_i)$ for any $x_i \in X$. Then we have

$$\varphi_{\mu_{AB}}(i) + \varphi_{\nu_{AB}}(i) + \varphi_{\phi_{AB}}(i)$$
$$= |\mu_A(x_i) - \mu_B(x_i)|/2 + |\nu_A(x_i) - \nu_B(x_i)|/2 + |\phi_A(x_i) - \phi_B(x_i)|/2$$
$$\leq |\mu_A(x_i) - \mu_C(x_i)|/2 + |\nu_A(x_i) - \nu_C(x_i)|/2 + |\phi_A(x_i) - \phi_C(x_i)|/2$$
$$= \varphi_{\mu_{AC}}(i) + \varphi_{\nu_{AC}}(i) + \varphi_{\phi_{AC}}(i).$$

So we have

$$\sqrt[p]{\sum_{i=1}^{n}(\varphi_{\mu_{AC}}(i) + \varphi_{\nu_{AC}}(i) + \varphi_{\phi_{AC}}(i))^p} \geq \sqrt[p]{\sum_{i=1}^{n}(\varphi_{\mu_{AB}}(i) + \varphi_{\nu_{AB}}(i) + \varphi_{\phi_{AB}}(i))^p}.$$

Therefore, $S_d^p(A, C) \leq S_d^p(A, B)$. In the similar way, it is easy to prove $S_d^p(A, C) \leq S_d^p(B, C)$.

Similar to (8), considering the weight w_i of $x_i \in X$, the similarity measure of GIFSs A and B can be obtained as following form.

$$S_{dw}^p(A, B) = 1 - \sqrt[p]{\sum_{i=1}^{n} w_i(\varphi_{\mu_{AB}}(i) + \varphi_{\nu_{AB}}(i) + \varphi_{\phi_{AB}}(i))^p}, \tag{10}$$

where $1 \leq p < \infty$.

Likewise, for $S_{dw}^p(A, B)$, the following theorem holds.

Theorem 3.5 $S_{dw}^p(A, B)$ *is the similarity measure between two GIFSs A and B.*

Proof The proof is similar to that of Theorem 3.4.

4 An application to pattern recognition problem

Assume that a question related to pattern recognition is given using GIFSs.

Assume that there exist m patterns which are represented by GIFSs $A_i = \{(x_i, \mu_A(x_i), \nu_A(x_i)) \mid x_i \in X\}$ $(i = 1, 2, \ldots, m)$, where $X = \{x_1, x_2, \ldots, x_n\}$. Suppose that there be a sample to be recognized which is represented by GIFS $B = \{(x_i, \mu_B(x_i), \nu_B(x_i)) \mid x_i \in X\}$. Set

$$S(A_{i_0}, B) = \max_{1 \leq i \leq n} \{S(A_i, B)\}, \tag{11}$$

where $S(A_i, B)$ is the similarity measure between A_i and B $(i = 1, 2, \ldots, n)$ given by (8) or (10).

According to the principle of the maximum degree of similarity between GIFSs, it can be decided that the sample B belongs to some pattern A_{i_0}.

Example 4.1 Assume that there are three patterns denoted with GIFSs in $X = \{x_1, x_2, x_3\}$. Three patterns A_1, A_2 and A_3 are denoted as follows:

$$
\begin{aligned}
A_1 &= \{(x_1, 0.6, 0.3), (x_2, 0.8, 0.3), (x_3, 0.7, 0.4)\}; \\
A_2 &= \{(x_1, 0.5, 0.6), (x_2, 0.5, 0.4), (x_3, 0.7, 0.5)\}; \\
A_3 &= \{(x_1, 0.6, 0.5), (x_2, 0.7, 0.4), (x_3, 0.8, 0.4)\}.
\end{aligned}
$$

Assume that a sample $B = \{(x_1, 0.6, 0.3), (x_2, 0.7, 0.5), (x_3, 0.7, 0.5)\}$ is given. Given three kinds of mineral fields, each is featured by the content of three minerals and contains one kind of typical hybrid minerals. The three kinds of typical hybrid minerals are represented by GIFSs A_1, A_2 and A_3 in X, respectively. Given another kind of hybrid mineral B, to which field does this kind of mineral B most probably belong to ?

For convenience, assume that the weight w_i of x_i in X are equal and $p = 2$. By (8) and (10), we have

$$
\begin{aligned}
S_g(A_1, B) = 0.900, \ S_g(A_2, B) = 0.800, \ S_g(A_3, B) = 0.866; \\
S_d^2(A_1, B) = 0.971, \ S_d^2(A_2, B) = 0.755, \ S_d^2(A_3, B) = 0.859.
\end{aligned}
$$

From this data, the proposed similarity measures S_g and S_d^p show the same classification according to the principle of the maximum degree of similarity between GIFSs. That is, the sample B belongs to the pattern A_1.

The results of above example indicates the proposed similarity measure to be good in pattern recognition problems. In the following example, the proposed similarity measure is used to characterize the similarity between linguistic variables.

Example 4.2 Let $F = \{(x, \mu_F(x), \nu_F(x)) : x \in X\}$ be a GIFS in X. For any positive real number n, We define the GIFS F^n as follows:

$$
F^n = \{(x, (\mu_F(x))^n, 1 - (1 - \nu_F(x))^n) \mid x \in X\}.
$$

Using the above operation, we also define the concentration and dilation of F as follows:

- concentration: $\text{CON}(F) = F^2$;

- dilation: $\text{DIL}(F) = F^{1/2}$.

Like the fuzzy sets, $\text{CON}(F)$ and $\text{DIL}(F)$ may be treated as "very (F)" and "more or less (F)", respectively.

In the next, we consider a GIFS F in $X = \{6, 7, 8, 9, 10\}$ defined by

$$
F = \{\{(6, 0.2, 0.9), (7, 0.4, 0.7), (8, 0.7, 0.4), (9, 0.9, 0.1), (10, 1, 0)\}.
$$

With taking into account the characterization of linguistic variables, we regard F as "LARGE" in X. Using the operations of concentration and dilation

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- $F^{1/2}$ may be treated as "More or less LARGE",

- F^2 may be treated as "Very LARGE",

- F^4 may be treated as "Very very LARGE".

The proposed similarity measure is utilized to calculate the degree of similarity between these GIFSs. The results are summarized in Table 1. In Table 1, L., V.L., V.V.L. and M.L.L. denote LARGE, Very LARGE, Very very LARGE and More or less LARGE, respectively.

Table 1: The values calculated by the proposed similarity measure S_d^1

S_d^1	M.L.L.	L.	V.L.	V.V.L.
M.L.L.	1	0.8562	0.7134	0.6022
L.	0.8562	1	0.8540	0.7428
V.L.	0.7134	0.8540	1	0.8848
V.V.L.	0.6022	0.7428	0.8848	1

From the viewpoint of mathematical operations, the similarities between the above GIFSs require the following conditions:

$$S(\text{M.L.L.}, \text{L.}) > S(\text{M.L.L.}, \text{V.L.}) > S(\text{M.L.L.}, \text{V.V.L.}), \qquad (12)$$

$$S(\text{L.}, \text{M.L.L.}) > S(\text{L.}, \text{V.L.}) > S(\text{L.}, \text{V.V.L.}), \qquad (13)$$

$$S(\text{V.L.}, \text{V.V.L.}) > S(\text{V.L.}, \text{L.}) > S(\text{V.L.}, \text{M.L.L.}), \qquad (14)$$

$$S(\text{V.V.L.}, \text{V.L.}) > S(\text{V.V.L.}, \text{L.}) > S(\text{V.V.L.}, \text{M.L.L.}). \qquad (15)$$

From Table 1, it can be seen that the proposed similarity measure S_d^1 satisfies the requirements (12)-(15). Therefore, the proposed similarity measure S_d^1 is reliable in applications with compound linguistic variables.

5   Conclusions

We apply the principle of maximum degree of similarity measures between GIFSs to solve the pattern recognition problem. Based on the geometrical interpretation of GIFSs, we take into account three parameters (membership, non-membership, hesitation margin) to propose a similarity measure for calculating the degree of similarity between GIFSs. Numerical example is given to illustrate the application of the developed similarity measures. Furthermore, we use the proposed similarity measures to characterize the similarity between linguistic variables.

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