Volume 20, Number 4 April 2016
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of Computational Analysis and Applications

EUDOXUS PRESS,LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE

SCOPE OF THE JOURNAL

An international publication of Eudoxus Press, LLC
(fourteen times annually)

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Journal of Computational Analysis and Applications

An international publication of Eudoxus Press, LLC, of TN.

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Modelling by Shepard-type curves and surfaces

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Dedicated to the memory of Prof. D.D. Stancu

Abstract

First parametric curves of Shepard-type are studied, which overcome some of the original Shepard operator's drawbacks, have some advantages with respect to the Bézier case and are optimal in some sense. Bounds for the deviation and approximation results for Shepard-type operators faster converging than the original one are proved. As an application a weighted progressive iterative approximation technique interesting in CAGD and an extension to tensor product surfaces case are given.

Key-words: Shepard-type operators; deviation; weighted progressive iterative approximation; tensor product surfaces.

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1 Introduction

Shepard-type operators are rational operators of interpolatory type widely used in classical Approximation Theory and in scatter data interpolation problems and they allow approximation results not possible by polynomials. However they suffer of drawback of flat spots phenomenon, which makes them unsuitable for CAGD.

The purpose of the present paper is to study a new class of Shepard-type curves $S_{n,\lambda}$ overcoming the above drawback (Section 2). The paper is organized as follows. In Section 3 Theorem 1 gives an estimation of the maximal distance between $S_{n,\lambda}[P]$ and the control polygon P in terms of the maximal absolute first order difference of the control points. In Section 4 we construct a sequence of Shepard-type operators based on $S_{n,\lambda}$ converging to the global Shepard-type interpolating operator and in Theorems 2, 3, 4, 5 and 6 we give convergence results and approximation error estimates. The results are applied in CAGD to study the weighted progressive iterative approximation (WPIA in short) technique in Theorems 7 and 8. The key idea is to iteratively change the control points of the active curve to deform towards the target shape represented by the point data. So by adjusting the control points of $S_{n,\lambda}$ curves and by using a weight, the WPIA process generates sequences of curves converging to the global Shepard-type interpolating curve at the original control points. Moreover an optimal value of the weight giving the fastest convergence rate is shown in Theorem 7. Based on such format, data points can be adaptively fit. Finally in Section 5 the results are extended to tensor product surfaces. The proofs of main results are in Section 6. The demonstration techniques are based on direct estimates of $S_{n,\lambda}$ operator and preliminary Lemmas on the eigenstructure of $S_{n,\lambda}$ operator interesting in themselves.

Near-interpolating curves $S_{n,\lambda}$ have some advantages with respect to Bézier curves: the parameter λ can be used as a shape control tool to draw a pencil of curves and choose the desired shape; $S_{n,\lambda}$ curves have pseudo-local control property against the global behaviour of Bézier curves; the deviation between $S_{n,\lambda}$ function and its control polygon is smaller than for the Bézier case (see Section 3); the corresponding weighted progressive iterative approximation process has faster rate of convergence than for Bézier case (Section 4); these advantages extend to the surfaces case (Section 5). Numerical experiments are also shown, verifying our theoretical analysis.

2 Near-interpolating curves

Let $A_n(t) = [A_{n,0}(t), A_{n,1}(t), \ldots, A_{n,n}(t)]^T$, where

$$A_{n,i}(t) = \frac{1/\left((t-t_i)^s + \lambda\right)}{\sum_{i=0}^{n} 1/\left((t-t_i)^s + \lambda\right)}, \tag{1}$$

for $0 \leq i \leq n$, $n \in \mathbb{N}$, $t \in [0,1]$, $t_i = i/n$, $i = 0, \ldots, n$, s even > 2 and $0 < n^s\lambda \leq 1/(6\zeta(s))$, with ζ being the zeta Riemann function.

In the following Lemma 5 we will show that $A_{n,i}(t)$, $0 \le i \le n$, form a basis generating a subspace S of rational functions of degree (sn, sn), with

$$0 \le A_{n,i}(t) \le 1, \ i = 0, \dots, n, \ \sum_{i=0}^{n} A_{n,i}(t) = 1. \tag{2}$$

Hence in the following the functions $A_{n,i}$, $i = 0, \dots, n$, are called blending functions. Given the blending functions $A_{n,i}(t)$ defined by (1) and a control polygon $P = [P_0, P_1, \dots, P_n]^T$, $P_i \in \mathbb{R}^d$, $i = 0, \dots, n$, $d \ge 2$, introduce the near-interpolating parametric Shepard-type curve $S_{n,\lambda}[P, t]$ defined by

$$S_{n,\lambda}[P, t] = \sum_{i=0}^{n} A_{n,i}(t) P_i = A_n(t) P. \tag{3}$$

Hence by (1)–(3) it is easy to check that $S_{n,\lambda}[P, t]$:

- is a rational curve of degree (sn, sn);
- it reproduces points;
- it is symmetric (i.e., $S_{n,\lambda}[P, 1-t] = S_{n,\lambda}[\tilde{P}, t], \ \tilde{P} = [P_n, \dots, P_0], \ \forall t \in [0,1]$);
- it is smooth;
- it is nondegenerate;
- it lies in the convex hull of the control polygon P;
- it satisfies the pseudo-local control property (indeed each function $A_{n,j}(t)$, $0 \le j \le n$, attains its maximum value close to 1 at $t = t_j$ and is very small for $|t - t_j| > 1/n$, in other words the point P_j influences strongly the shape of the curve in a neighborhood of $t = t_j$);
- it interpolates at the control points, as λ tends to 0 (see the following remark on Balazs-Shepard operator);
- it satisfies the degree elevation-type formula

$$\overline{S}_{n+1,\lambda}[P \cup \overline{P}, t] = \frac{S_{n,\lambda}[P, t] D_n(t)}{\overline{D}_{n+1}(t)} + \frac{\overline{P} / \left((t - \bar{t})^s + \lambda\right)}{\overline{D}_{n+1}(t)}, \ \bar{t} \ne t_k, \ k = 0, \dots, n,$$

with

$$\overline{S}_{n+1,\lambda}[P \cup \overline{P}, t] = \frac{\sum_{k=0}^{n} P_i / ((t - t_i)^s + \lambda) + \overline{P} / \left((t - \bar{t})^s + \lambda\right)}{\sum_{k=0}^{n} 1 / ((t - t_i)^s + \lambda) + 1 / \left((t - \bar{t})^s + \lambda\right)},$$

$$D_n(t) = \sum_{k=0}^{n} \frac{1}{(t - t_i)^s + \lambda},$$

$$\overline{D}_{n+1}(t) = \sum_{k=0}^{n} \frac{1}{(t - t_i)^s + \lambda} + \frac{1}{(t - \bar{t})^s + \lambda}.$$

By the above remarks $S_{n,\lambda}[P, t]$ can be considered a parametric curve approximating the control polygon P.

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In the nonparametric case, i.e. $P_i = (t_i, f(t_i))$, with f a continuous function on $[0,1]$, operators similar to (3) were studied to approximate surface data [1] or noisy values [8]. When λ tends to 0, then $S_{n,\lambda}$ tends to the well-known Balazs-Shepard operator [14]

$$S_n(f,t) = \frac{\sum_{i=0}^{n} f(t_i)/(t-t_i)^s}{\sum_{i=0}^{n} 1/(t-t_i)^s}, \; s \text{ even } \geq 2,$$

interpolating f at t_i, $i = 0, \ldots, n$. Such operators are extensively used in applicative problems involving scattered data interpolation and they have been subject of several papers proving approximation results not possible by polynomials [2, 5, 6, 7].

It is easy to see that the presence of parameter λ in (3) makes the structure of $S_{n,\lambda}$ not far from the simple S_n and analogous convergence results and error estimates can be proved as in [5, 7, 12].

For example if $\| \, \|$ denotes the usual supremum norm on $[0,1]$ and $\omega(f)$ the modulus of continuity of f, then working as in [6, 7, 12]

$$\|f - S_{n,\lambda}(f)\| \leq \text{const } \omega\left(f; \frac{1}{n}\right). \tag{4}$$

On the other hand the choice $0 < \lambda n^s \leq 1/(6\zeta(s))$, makes $S_{n,\lambda}[P,t]$ a curve near-interpolating the control polygon, overcoming the flat spots drawback affecting the original Shepard operator. If $\lambda \to \infty$, then $S_{n,\lambda}[P,t]$ tends to the arithmetic mean of P_i, $i = 0, \ldots, n$.

Here for the sake of simplicity and in analogy to the Bernstein-Bézier case we assumed that the knots are uniformly spaced.

3 Bounds for the deviation

In this Section we give an answer to the question if near-interpolating curves are good curves, in the sense that they do not deviate too much from the data points polygon. This problem is interesting in many CAGD applications, like intersection testing, creating tolerance envelopes, rendering or design (see, e.g., [11, 13]). The following theorem estimates the maximum distance between $S_{n,\lambda}[P,t]$ and its control polygon in terms of the maximal absolute first order difference of the control points of P. The implication of this result is to give a finer localization for the function than by standard convex-hull or mini-max bound. To this end following [11, 13] we view the polygon $P = [P_0, P_1, \ldots, P_n]$, $P_i \in \mathbb{R}$, $i = 0, \ldots, n$, as the piecewise linear function $p(t)$ given by

$$p(t) = P_i + n(t-t_i)(P_{i+1} - P_i), \quad t_i \leq t \leq t_{i+1}, \; i = 0, \ldots, n-1.$$

Then let $e(t) = e_n(t) = |S_{n,\lambda}[P,t] - p(t)|$, with $S_{n,\lambda}[P,t]$ the univariate near-interpolating Shepard-type function and $\Delta P = \max_{0 \leq i \leq n-1} |P_{i+1} - P_i|$. We have

Theorem 1 *For every $t \in [0,1]$*

$$e(t) \leq \Delta P C_{n,\lambda}(t), \tag{5}$$

where

$$C_{n,\lambda}(t) = a_j(t) + a_j^-(t) + a_j^+(t),$$

when $t \in [t_j, t_{j+1}]$ for some $0 \leq j \leq n-1$, and

$$a_j(t) := \frac{1}{D}\begin{cases} \left|\frac{n(t-t_j)}{(t-t_j)^s+\lambda} + \frac{n(t-t_j)-1}{(t_{j+1}-t)^s+\lambda}\right|, & \text{if } |t-t_j| \leq \frac{1}{2n}, \\ \left|\frac{n(t_{j+1}-t)}{(t_{j+1}-t)^s+\lambda} + \frac{n(t_{j+1}-t)-1}{(t_j-t)^s+\lambda}\right|, & \text{if } |t_{j+1}-t| \leq \frac{1}{2n}, \end{cases}$$

$$a_j^-(t) := \frac{1}{D}\begin{cases} \sum_{k=0}^{j-1} \frac{j-k+n(t-t_j)}{(t-t_k)^s+\lambda}, & \text{if } |t-t_j| \leq \frac{1}{2n}, \\ \sum_{i=0}^{j-1} \frac{j+1-i-n(t_{j+1}-t)}{(t-t_i)^s+\lambda}, & \text{if } |t_{j+1}-t| \leq \frac{1}{2n}, \end{cases}$$

$$a_j^+(t) := \frac{1}{D}\begin{cases} \sum_{k=j+2}^{n} \frac{k-j-n(t-t_j)}{(t-t_k)^s+\lambda}, & \text{if } |t-t_j| \leq \frac{1}{2n}, \\ \sum_{i=j+2}^{n} \frac{i-j-1+n(t_{j+1}-t)}{(t-t_i)^s+\lambda}, & \text{if } |t_{j+1}-t| \leq \frac{1}{2n}, \end{cases}$$

with

$$D := D_n(t) := \sum_{i=0}^{n} \frac{1}{(t-t_i)^s+\lambda}.$$

Remarks. Pointwise estimate (5) corresponds to the fact that for λ vanishing $e(t)$ goes to 0 at t_i, $i = 0, \ldots, n$, as expected since $S_{n,\lambda}[P]$ approaches original interpolating Shepard function. Compare [13, p. 582] for an analogous result for Bernstein-Bézier polynomials.

By Theorem 1, if n and λ are fixed, we find a domain where $S_{n,\lambda}[P,t]$ lies. When n is fixed and λ goes to 0, $C_{n,\lambda}(t)$ tends to $C_{n,0}(t)$, where $C_{n,0}(t)$ corresponds to the function bounding the deviation for the original Shepard function. By numerical tests we found $C_{n,0}(t) \leq 0.28$, $t \in [0,1]$, $n \leq 50$. On the other hand the choice of λ very close to 0 is to be avoided, because the corresponding $S_{n,\lambda}$ function is very close to the original Shepard function giving a very bad approximation near the control points because of the flat spots phenomenon.

Then we have implemented the following procedure in a Matlab environment: given $n+1$ control points, we draw $S_{n,\lambda}$ functions for various λ, we choose that value $\overline{\lambda}$ giving a satisfactory shape and by Theorem 1 we draw a region where the corresponding function $S_{n,\overline{\lambda}}$ lies.

The following examples refer to the cases presented in [11, p. 629], where the deviation of a Bernstein-Bézier polynomial from its control polygon was studied. Here the control polygon P is represented by a continuous line, the near-interpolating Shepard-type function by a dashed line, the bounding region is shaded. We have chosen $s = 4$ (indeed also an even $s > 4$ does not give real improvements, on the other hand the choice $s = 2$ gives worse results, as expected from the theoretical results (cfr. [7])). From our experiments it follows that by a proper choice of λ the bound for the deviation of $S_{n,\lambda}$ functions is smaller than for the Bézier case (cfr. [11, Theorem 4.2, p. 621]).

3.1 Example 1

Here $P_0 = 0$, $P_1 = 1$, $P_2 = 1$, $P_3 = 0$. The choice $\lambda = 2 \times 10^{-3}$ gives a satisfactory shape for the corresponding near-interpolating function and by the estimate in Theorem 1 we get the corresponding bounding region clipped with the min-max bound (see Fig. 1). We have $\max_{t\in[0,1]} C_{n,\lambda}(t) = 0.2319$, $\Delta P = 1$; so by (5) $\max_{t\in[0,1]} e(t) \leq 0.232$.

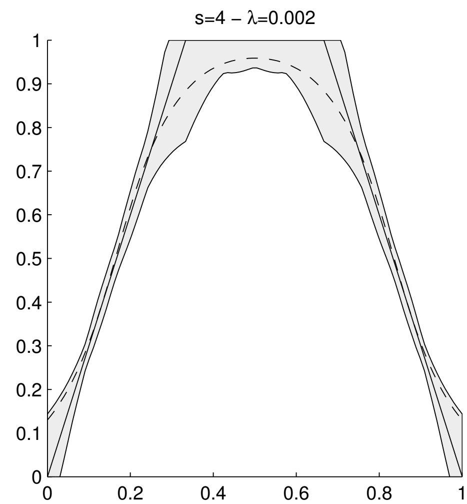

Figure 1: Deviation for Example 1.

3.2 Example 2

Here $P_0 = 0$, $P_1 = 1$, $P_2 = -1$, $P_3 = 0$. We have chosen $\lambda = 1.5 \times 10^{-3}$ and the corresponding bounding region clipped with the min-max bound is drawn in Fig. 2. We have $\max_{t\in[0,1]} C_{n,\lambda}(t) = 0.1893$, $\Delta P = 2$; so by (5) $\max_{t\in[0,1]} e(t) \leq 0.379$.

3.3 Example 3

Here $P_0 = 0$, $P_1 = 1$, $P_2 = 2$, $P_3 = 3$, $P_4 = 2$, $P_5 = 1$. We have chosen $\lambda = 3 \times 10^{-4}$ and the corresponding bounding region clipped with the min-max bound is drawn in Fig. 3. Here $\max_{t\in[0,1]} C_{n,\lambda}(t) = 0.274$, $\Delta P = 1$; so by (5) $\max_{t\in[0,1]} e(t) \leq 0.274$.

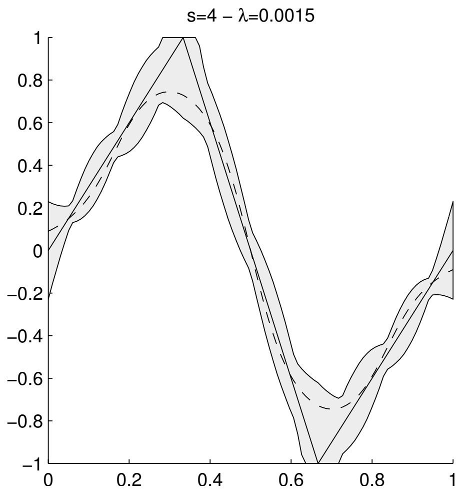

Figure 2: Deviation for Example 2.

4 Weighted progressive iterative approximation

In this Section we study the WPIA property of $S_{n,\lambda}$ curves defined in Section 2. Consider the nonparametric case, i.e.,

$$S_{n,\lambda}(f;t) = \sum_{i=0}^{n} A_{i,n}(t) f(t_i) = A_n(t)\overline{f},$$

with $f \in C([0,1])$ and $\overline{f} = [f(t_0), f(t_1), \dots, f(t_n)]^T$. From (2) it follows that $S_{n,\lambda}$ preserves constants.

Introduce the global interpolating Shepard-type operator defined by

$$G_{n,\lambda}(f;t) = \sum_{i=0}^{n} A_{n,i}(t) f_i^G = A_n(t)\overline{f}^G, \quad \overline{f}^G = \left[f_0^G, f_1^G, \dots, f_n^G\right]^T, \tag{6}$$

with

$$G_{n,\lambda}(f;t_i) = f(t_i), \; i = 0, \dots, n. \tag{7}$$

In other words the values f_i^G, $i = 0, \dots, n$, are determined by solving the linear system (7), guaranteeing for the Shepard-type operator $G_{n,\lambda}$ the interpolation condition at the given values $f(t_i)$ and overcoming the flat spot phenomenon of the original interpolating Shepard operator.

If $n^s\lambda = o(1)$, then $G_{n,\lambda}$ tends to the original Shepard operator and f_i^G tend to $f(t_i)$, $i = 0, \dots, n$.

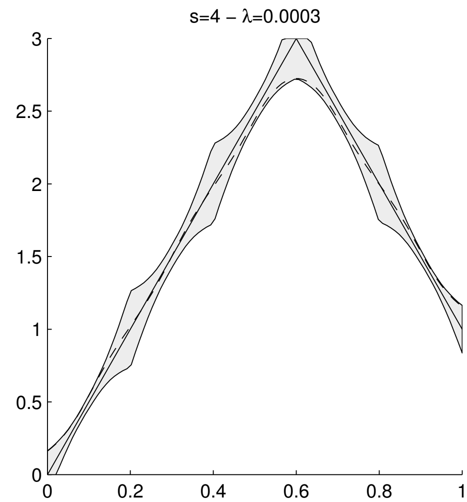

Figure 3: Deviation for Example 3.

We remark that the system (7) can be written as $B\overline{f}^G = \overline{f}$, where

$$B = \begin{pmatrix} A_{n,0}(t_0) & A_{n,1}(t_0) & \cdots & A_{n,n}(t_0) \\ A_{n,0}(t_1) & A_{n,1}(t_1) & \cdots & A_{n,n}(t_1) \\ \vdots & \vdots & \ddots & \vdots \\ A_{n,0}(t_n) & A_{n,1}(t_n) & \cdots & A_{n,n}(t_n) \end{pmatrix} \tag{8}$$

is the collocation matrix of the basis $A_{n,i}$, $i = 0, \ldots, n$.

We observe that B is a symmetric, centrosymmetric and positive stochastic matrix.

Remark. If $n^s\lambda = o(1)$, then $G_{n,\lambda}$ tends to the original Shepard operator, therefore B tends to the identity matrix, consequently all the eigenvalues of B tend to 1.

We denote by $M^k = M \ldots M$ k times the k-th iterate of M operator.

It is easy to deduce from (4) that for any fixed m

$$\left\|f - S^m_{n,\lambda}(f)\right\| \le \text{const } \omega\left(f; \frac{1}{n}\right).$$

Moreover if $\lambda^{(n)}_{n-1} < 1$ denotes the second largest eigenvalue of B, then

Theorem 2 *For any $m \in \mathbb{N}$ and $t \in [0,1]$,*

$$\left|f(t_0) - S^m_{n,\lambda}(f;t)\right| \le \|f\| \left(\lambda^{(n)}_{n-1}\right)^{m-1}.$$

Hence

$$\lim_m S^m_{n,\lambda}(f;t) = f(t_0). \tag{9}$$

Remark. It is well-known that an analogous result to (9) holds for Bernstein-type operators (see, e.g., [3]).

From (9) we deduce that we have to smartly combine the iterates of $S_{n,\lambda}$ operator to improve the approximation by such operator.

To this aim for every $f \in C([0,1])$ we construct the sequence of rational functions of degree (sn, sn) $\left\{\tilde{S}_m(f;t)\right\}_{m=0}^{\infty}$ defined by

$$\tilde{S}_m(f;t) = A_n(t)\left[I + w\sum_{i=1}^{m}(I - wB)^{i-1}(I - B)\right]\overline{f}, \tag{10}$$

with $0 < w$ a fixed parameter that can take as any possible value as long as it can guarantee the convergence of the above rational functions (see later). It is easy to see that

$$\begin{aligned}\tilde{S}_0(f;t) =& S_{n,\lambda}(f;t)\\ \tilde{S}_1(f;t) =& S_{n,\lambda}(f;t) + wS_{n,\lambda}(f - S_{n,\lambda}(f);t)\\ =&(1+w)S_{n,\lambda}(f;t) - wS_{n,\lambda}^2(f;t)\end{aligned}$$

and

$$\tilde{S}_2(f;t) = (1+2w)S_{n,\lambda}(f;t) - w(2+w)S_{n,\lambda}^2(f;t) + w^2S_{n,\lambda}^3(f;t).$$

Note that in general $\tilde{S}_m$ is not a positive operator.

In particular if $w = 1$, then

$$\tilde{S}_m(f;t) = A_n(t)\left[\sum_{i=0}^{m}(I - B)^i\right]\overline{f}. \tag{11}$$

Cfr. [10, 15] for an analogous operator based on Bernstein polynomials.

We note that since the matrix B is symmetric and centrosymmetric, the number of operations to compute the quantity inside brackets in (10) is reduced by 1/4, that is $(m-1)n^3/2 + O(n^2)$.

In the sequel C will denote a positive constant which may assume different values even in the same formula.

The following Theorem 3 gives a motivation for the construction of $\tilde{S}_m$, i.e. it shows that $\tilde{S}_m(f)$ approximates f at the knots t_i, $i = 0, \ldots, n$, better than original $S_{n,\lambda}(f)$, in other words the loss of positivity is compensated by a better degree of approximation at the knots. Indeed if $\lambda_0^{(n)}$ denotes the smallest eigenvalue of B, then

Theorem 3 *Let $\rho(I - wB) < 1$. Then for any $f \in C([0,1])$ and for any fixed $m > 0$*

$$\left|f(t_i) - \tilde{S}_m(f;t_i)\right| < 2\|f\|\left(1 - w\lambda_0^{(n)})^m(1 - \lambda_0^{(n)}\right),\ i = 0, \ldots, n. \tag{12}$$

In particular if $w = 1$, then

$$\left\|f(t_i) - \tilde{S}_m(f;t_i)\right\| < 2\|f\|\left(1 - \lambda_0^{(n)}\right)^{m+1}\ \ i = 0, \ldots, n. \tag{13}$$

Now we examine the behaviour of $\tilde{S}_m$ for n fixed and $m \to \infty$.

Theorem 4 *Let $\rho(I - wB) < 1$. If $f \in C([0,1])$, then for every $t \in [0,1]$ and n*

$$\tilde{S}_\infty(f;t) := \lim_{m\to\infty} \tilde{S}_m(f;t) = G_{n,\lambda}(f;t) = A_n(t)\overline{f}^G = A_n(t)B^{-1}\overline{f}. \tag{14}$$

In particular if $w < 2$, then (14) holds true for every $t \in [0,1]$ and n.

Moreover

$$\left\| f - \tilde{S}_\infty(f) \right\| \le C\omega\left(f; \frac{1}{n}\right). \tag{15}$$

Furthermore if $f \in C^{n+1}([0,1])$

$$\left| f(t) - \tilde{S}_\infty(f;t) \right| \le \frac{|(t-t_0)(t-t_1)\cdots(t-t_n)|}{(n-1)!} M, \tag{16}$$

with

$$M = \max_{0\le t\le 1} \left| f^{(n+1)}(t) - \tilde{S}_\infty^{(n+1)}(f;t) \right|.$$

Theorem 4 says that by the sequence (10) we can reach the global interpolating operator (6) without solving the linear system (7). In other words the sequence $\{\tilde{S}_m\}_m$ continuously links $S_{n,\lambda}$ operator to $G_{n,\lambda}$ operator.

Compare with [15] for an analogous result for Bernstein operator.

Moreover we give an estimate of approximation error of $\tilde{S}_\infty$ by $\tilde{S}_m$.

Theorem 5 *Let $\rho(I - wB) < 1$. For any $f \in C([0,1])$*

$$\left\| \tilde{S}_\infty(f) - \tilde{S}_m(f) \right\| < 2\|f\| \left(1 - w\lambda_0^{(n)}\right)^m \left(1 - \lambda_0^{(n)}\right). \tag{17}$$

The fastest rate is attained when $w = 2/\left(1 + \lambda_0^{(n)}\right)$, therefore

$$\left\| \tilde{S}_\infty(f) - \tilde{S}_m(f) \right\| < 2\|f\| \left(\frac{1 - \lambda_0^{(n)}}{1 + \lambda_0^{(n)}}\right)^m \left(1 - \lambda_0^{(n)}\right). \tag{18}$$

In addition

Theorem 6 *Let $w = 1$. Then*

$$\left\| \tilde{S}_\infty(f) - \tilde{S}_m(f) \right\| < \|f\| \frac{1}{2^m}. \tag{19}$$

If $n^s\lambda = o(1)$, then

$$\left\| \tilde{S}_\infty(f) - \tilde{S}_m(f) \right\| = o(1)^{m+1}. \tag{20}$$

Remark. From Theorem 6 we deduce that the rate of convergence of $\tilde{S}_m$ to $\tilde{S}_\infty$ is faster than in the analogous Bernstein case [15].

The above results find application in CAGD to construct sequences of curves based on $S_{n,\lambda}$ operator converging to the global interpolating Shepard-type curve based on $G_{n,\lambda}$ operator. Let us see in detail the WPIA process.

Given the control polygon $P = [P_0, \ldots, P_n]^T$ and the basis $A_{n,i}(t)$, $i = 0, \ldots, n$, defined by (1), we can generate the initial curve

$$\gamma_w^0(t) = \sum_{i=0}^{n} A_{n,i}(t) P_i^0 = S_{n,\lambda}[P, t],$$

with $P_i^0 = P_i$, $i = 0, \ldots, n$. Then we calculate the remaining curves of the sequence $\gamma_w^{k+1}(t)$, for $k \geq 0$ as follows

$$\gamma_w^{k+1}(t) = \sum_{i=0}^{n} P_i^{k+1} A_{n,i}(t), \tag{21}$$

with

$$P_i^{k+1} = P_i^k + w\overline{\Delta}_i^k,$$

and $\overline{\Delta}_i^k$ the adjusting vectors given by

$$\overline{\Delta}_i^k = P_i - \gamma_w^k(t_i), \ i = 0, 1, \ldots, n, \tag{22}$$

in other words we multiply all the adjusting vectors by a common weight w. Then the iterative process can be written in matrix form as follows:

$$\begin{aligned} \left[\overline{\Delta}_0^k, \overline{\Delta}_1^k, \ldots, \overline{\Delta}_n^k\right]^T &= (I - wB)\left[\overline{\Delta}_0^{k-1}, \overline{\Delta}_1^{k-1}, \ldots, \overline{\Delta}_n^{k-1}\right]^T \\ &= (I - wB)^k \left[\overline{\Delta}_0^0, \overline{\Delta}_1^0, \ldots, \overline{\Delta}_n^0\right]^T. \end{aligned} \tag{23}$$

The weight w in (23) can be taken as any possible value, as long as it can guarantee the convergence of the above iterative process.

Remark. In the not-parametric case curves γ_w^k correspond to the rational functions defined by (10).

Now we show how to determine the value of w to obtain the fastest convergence rate. We say that γ_w^0 curve satisfies the WPIA property iff $\lim_k \gamma_w^k(t_i) = P_i$, $i = 0, \ldots, n$.

We have

Theorem 7 *If $\rho(I - wB) < 1$, curve γ_w^0 satisfies the WPIA property. In particular if $w < 2$, curve γ_w^0 satisfies WPIA property. Moreover the WPIA process has the fastest convergence rate when*

$$w = \frac{2}{1 + \lambda_0^{(n)}} \tag{24}$$

and in such case

$$\rho(I - wB) = \frac{1 - \lambda_0^{(n)}}{1 + \lambda_0^{(n)}}.$$

Remarks. WPIA property makes possible to construct a sequence of control polygons converging to the control polygon of an interpolating curve of Shepard-type. Moreover the parameter k can be used as shape parameter in order to model different shapes, obtaining as extreme cases the Shepard-type curve and the global interpolating Shepard-type curve. By choosing an optimal value of the weight w, Theorem 7 shows that the weighted PIA shares the progressive iterative approximation property and has the fastest convergence rate. As remarked before the rate is faster than for the Bézier case (cfr. [9]).

We observe that here firstly the convergence results for WPIA property are deduced from the analogous approximation results in the nonparametric case (cfr. [9]).

If $n^s\lambda = o(1)$, from the remark to (8) it follows that the value of w in (24) approaches 1 from above. Therefore we call the WPIA process for $w = 1$ (see following PIA technique) "quasi" optimal.

If $w = 1$, then the corresponding progressive iterative approximation process (called PIA in short) is given by

$$\gamma^0(t) = \sum_{i=0}^{n} A_{n,i}(t)P_i^0, \quad \gamma^{k+1}(t) = \sum_{i=0}^{n} P_i^{k+1}A_{n,i}(t), \quad k \geq 0, \tag{25}$$

where $P_i^0 = P_i$ for all $i = 0, \dots, n$, and $\tilde{\Delta}_i^k$ are the adjusting vectors given by

$$P_i^{k+1} = P_i^k + \tilde{\Delta}_i^k, \ \tilde{\Delta}_i^k = P_i - \gamma^k(t_i), \ i = 0, 1, \dots, n.$$

Then the iterative process can be written in matrix form as follows:

$$\begin{aligned}\left[\tilde{\Delta}_0^k, \tilde{\Delta}_1^k, \dots, \tilde{\Delta}_n^k\right]^T &= (I - B)\left[\tilde{\Delta}_0^{k-1}, \tilde{\Delta}_1^{k-1}, \dots, \tilde{\Delta}_n^{k-1}\right]^T \\ &= (I - B)^k \left[\tilde{\Delta}_0^0, \tilde{\Delta}_1^0, \dots, \tilde{\Delta}_n^0\right]^T.\end{aligned}$$

Remark. Curves γ^k in the not-parametric case correspond to rational functions in (11).

We say that γ^0 curve satisfies the PIA property iff $\lim_k \gamma^k(t_i) = P_i$, $i = 0, \dots, n$.

We have

Theorem 8 *Curve γ^0 satisfies the PIA property. Moreover under the assumptions of Theorem 6, the rate of convergence is faster than in Bézier case.*

Remarks. Based on PIA format, we can design an adaptive fitting method to fit data points, by adjusting the control points corresponding to these data points, if fitting precision is above a predefined threshold.

We observe that the above PIA and weighted PIA processes can be interpreted in terms of classical iterative methods for linear systems; indeed PIA and weighted PIA iterations correspond to classical Richardson and classical modified Richardson method, respectively (compare [4] for an analogous revisitation for Bézier curves).

4.1 Example

Consider a helix of radius 5 given by (cfr. [9])

$$(x(t), y(t), z(t)) = (5\cos t, 5\sin t, t),\ t \in [0, 6\pi].$$

A sequence of 19 control points is sampled from the helix as

$$(x(s_i), y(s_i), z(s_i)),\ s_i = i\frac{\pi}{3},\ i = 0, 1, \ldots, 18. \tag{26}$$

Starting with these control points we fit the helix by two sequences of curves generated by the WPIA process defined by (21), (22) and (24) with $s = 4$, $\lambda = 4 \times 10^{-6}$ and $w \simeq 1.57$, and by the PIA process defined by (25) with the same value of $\lambda = 4 \times 10^{-6}$, respectively. Figures 4 and 5 show the starting, the second and the fourth curves of such sequences for WPIA and PIA, respectively, and star symbol denotes the control points given in (26). Note that the results in Fig. 5 are very similar to Fig. 4, as expected from the remark to (24). The fitting errors of the above WPIA and PIA processes (the maximal Euclidean norm of the corresponding adjusting vectors of such curves) are shown in Fig. 6 for the first 40 iterations. Figure 6 shows that the WPIA process reaches a faster rate of convergence than PIA, as expected from Theorem 7.

5 Modelling by Shepard-type surfaces

Letting $P_{i,j} \in \mathbb{R}^3$, $i = 0, \ldots, m$, $j = 0, \ldots, n$, $m, n \in \mathbb{N}$, be the vertices of the control net P, introduce tensor product near-interpolating surface $S_{m,n,\lambda,\mu}[P, u, v]$ defined by

$$\begin{aligned} S_{m,n,\lambda,\mu}[P, u, v] &= \sum_{i=0}^{m}\sum_{j=0}^{n} P_{i,j} A_{m,i}(u) A_{n,j}(v) \\ &= A_m^T(u) P A_n(v) \end{aligned} \tag{27}$$

with $0 < m^s\lambda < 1/(2\zeta(s))$, $0 < n^s\mu < 1/(2\zeta(s))$, $(u, v) \in [0, 1]^2$, $t_i = t_{i,m} = i/m$, $y_j = y_{j,n} = j/n$, $P = (P_{i,j})_{\substack{i=0,\ldots,m \\ j=0,\ldots,n}}$.

Since

$$\sum_{i=0}^{m}\sum_{j=0}^{n} A_{m,i}(n) A_{n,j}(v) = 1,\ 0 \le A_{m,i}(u) A_{n,j}(v) \le 1,$$

it follows from (27) that $S_{m,n,\lambda,\mu}[P, u, v]$ is a rational surface of degree (sm, sm) with respect to u and (sn, sn) with respect to v, lying in the convex hull of the control net P.

If λ and μ tend to 0, then $S_{m,n,\lambda,\mu}[P, t_{i,m}, t_{j,n}] \to P_{i,j}$, $0 \le i \le m$, $0 \le j \le n$, hence analogously to the curve case, $S_{m,n,\lambda,\mu}[P]$ may be considered a near-interpolating surface, overcoming the flat spots drawback occurring for original

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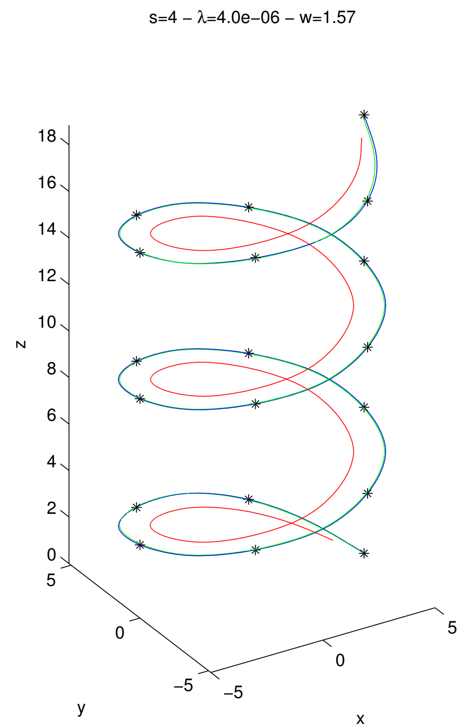

Figure 4: Weighted progressive iterative approximation at the starting, second and fourth iteration

Shepard surfaces. As in the curve case, here the parameters λ and μ may be used as shape control tools to model the form of an object.

In Fig. 7 we use $S_{m,n,\lambda,\mu}$ with $s = 4$ and $\lambda = \mu = 10^{-6}$ to model the shape of the Vesuvius, a volcano near Napoli (Italy), based on 650×380 control vertices.

The results of Section 3 readily generalize to the tensor product setting. Indeed, if $h(u, v)$ denotes the piecewise bilinear function corresponding to the control net P and

$$D(u, v) := S_{m,n,\mu,\lambda}[P, u, v] - h(u, v)$$

denotes the deviation of the Shepard surface $S_{m,n,\lambda,\mu}$ from h, then we can follow [13, Section 3, p. 584] and by Theorem 1 we can bound $D(u, v)$ by directional first forward differences. We omit details.

Also the WPIA process of Section 4 can be easily extended to the tensor

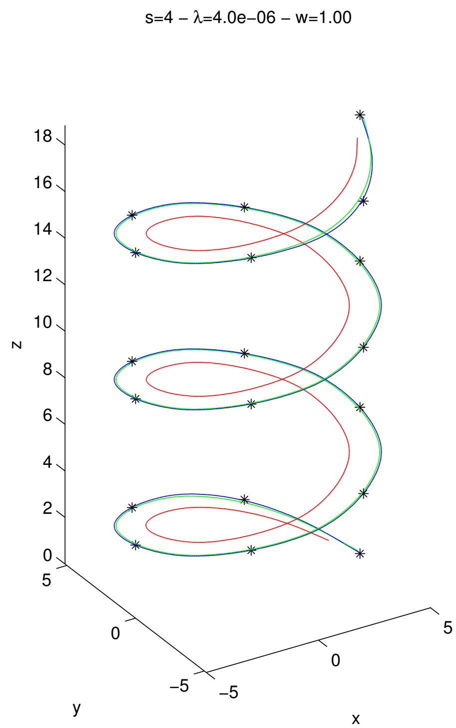

Figure 5: Progressive iterative approximation at the starting, second and fourth iteration

product surfaces. We can generate the initial surface

$$S^0(x,y)=\sum_{i=0}^{m}\sum_{j=0}^{n}P^0_{ij}A_{n,i}(x)A_{m,j}(y)$$

with $P^0_{ij}=P_{i,j}$ for all $i=0,1,\ldots,m$ and $j=0,1,\ldots,n$. Then the remaining surfaces of the sequence $S^{k+1}(x,y)$ for $k\geq 0$ can be calculated as follows

$$S^{k+1}(x,y)=\sum_{i=0}^{m}\sum_{j=0}^{n}P^{k+1}_{i,j}A_{m,i}(x)A_{n,j}(y),$$

where

$$P^{k+1}_{i,j}=P^k_{i,j}+w\Delta^k_{i,j},\Delta_{i,j}=P_{i,j}-S^k(t_i,y_j),i=0,\ldots,m,j=0,\ldots,n.$$

The iterative process can be written in matrix form

$$\Delta^k=(I-wB)\Delta^{k-1}=(I-wB)^k\Delta^0,$$

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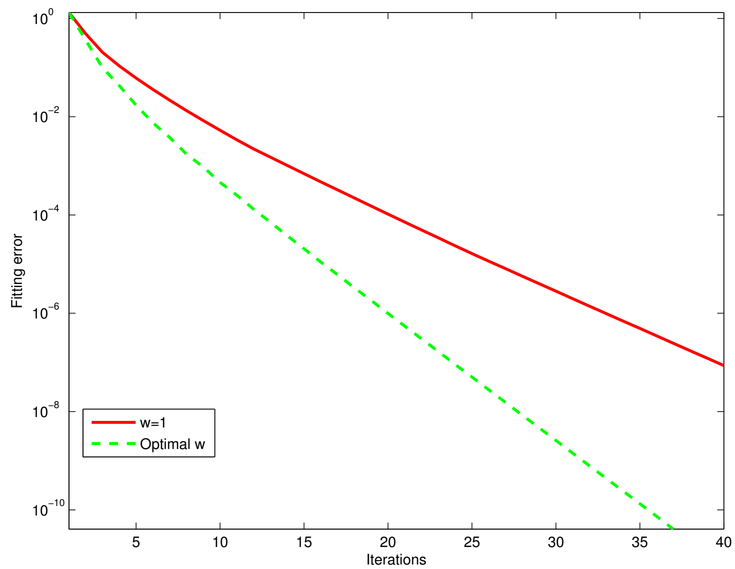

Figure 6: Fitting error vs. iterations

where

$$\Delta^j = \left[\Delta^j_{0,0}, \Delta^j_{0,1}, \ldots, \Delta^j_{0,n}, \Delta^j_{1,0}, \Delta^j_{1,1}, \ldots, \Delta^j_{1,n}, \ldots, \Delta^j_{m,0}, \Delta^j_{m,1}, \ldots \Delta^j_{m,n},\right], \; j = 0, \ldots, k,$$

I is the identity matrix of order $(m+1)(n+1)$ and $B = B_1 \otimes B_2$ is the Kronecker product of two collocation matrices

$$B_1 = A_{m,j}(t_i)^{i=0,\ldots,m}_{j=0,\ldots,m}, \;\; B_2 = A_{n,j}(y_i)^{i=0,\ldots,n}_{j=0,\ldots,n}.$$

Thus we get the surface sequence $S^k(x, y)$, $k = 0, \ldots$. If $\lim_{k\to\infty} S^k(x_i, y_j) = P_{i,j}$, $i = 0, \ldots, m$, $j = 0, \ldots, n$ then we say that the initial surface S^0 has the weighted progressive iteration approximation (WPIA in short) property.

We have

Theorem 9 *The surface S^0 has WPIA property if $\rho(I - wB) < 1$. Moreover the weighted PIA approximation has the fastest convergence rate when*

$$w = \frac{2}{1 + \lambda_m(B_1)\mu_n(B_2)},$$

where $\lambda_m(B_1)$ and $\mu_n(B_2)$ are the smallest eigenvalues of B_1 and B_2, respectively, and in such case

$$\rho(I - wB) = \frac{1 - \lambda_m(B_1)\mu_n(B_2)}{1 + \lambda_m(B_1)\mu_n(B_2)}.$$

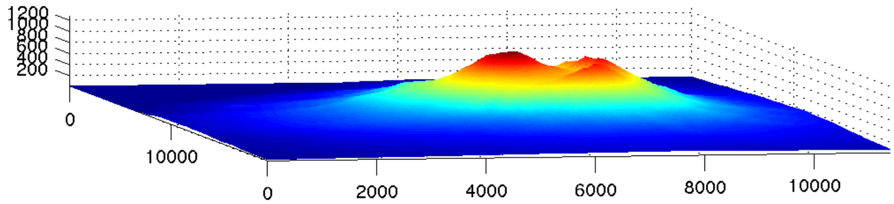

Figure 7: Modelled shape of Vesuvius volcano.

Remarks. Obviously if $w = 1$ then from Theorem 9 we prove that S^0 has the PIA property (cfr. Theorem 8). Finally, as remarked after Theorem 6, the convergence rate is faster than for the Bézier surfaces case [9].

6 Proofs of main results

6.1 Proof of Theorem 1

Let $t \in [t_j, t_{j+1}]$, for some $0 \le j \le n-1$.
Case 1. $|t - t_j| = \min_{i=0,\dots,n-1} |t - t_i| \le 1/(2n)$.
It follows that

$$\begin{aligned} e(t) &= \left| P_j + n(t-t_j)(P_{j+1} - P_j) - \frac{1}{D}\sum_{i=0}^{n} \frac{P_i}{(t-t_i)^s + \lambda} \right| \\ &\le \frac{1}{D}\left| \frac{n(t-t_j)(P_{j+1}-P_j)}{(t-t_j)^s+\lambda} + \frac{P_j + n(t-t_j)(P_{j+1}-P_j) - P_{j+1}}{(t-t_{j+1})^s + \lambda} \right| \\ &\quad + \frac{1}{D}\left\{ \sum_{i=0}^{j-1} + \sum_{i=j+2}^{n} \right\} \frac{|P_j + n(t-t_j)(P_{j+1}-P_j) - P_i|}{(t-t_i)^s+\lambda} \\ &:= A_j + A_j^- + A_j^+. \end{aligned}$$

Clearly

$$A_j \le \frac{1}{D}|P_{j+1} - P_j| \left| \frac{n(t-t_j)}{(t-t_j)^s+\lambda} + \frac{n(t-t_j)-1}{(t_{j+1}-t)^s+\lambda} \right|.$$

Moreover we can prove that

$$A_j^- \le \frac{1}{D}\Delta P \sum_{k=0}^{j-1} \frac{|j-k+n(t-t_j)|}{(t-t_k)^s+\lambda}.$$

and similarly

$$A_j^+ \leq \frac{1}{D}\Delta P \sum_{k=j+2}^{n} \frac{|k-j-n(t-t_j)|}{(t-t_k)^s+\lambda}.$$

Case 2. $|t_{j+1}-t| \leq 1/(2n)$.
Since $p(t) = P_{j+1} + n(t_{j+1}-t)(P_j - P_{j+1})$

$$\begin{aligned} e(t) &= \left| P_{j+1} + n(t_{j+1}-t)(P_j - P_{j+1}) - \frac{1}{D}\sum_{i=0}^{n} \frac{P_i}{(t-t_i)^s+\lambda} \right| \\ &\leq \frac{1}{D}\left| n\frac{(t_{j+1}-t)(P_j-P_{j+1})}{(t-t_{j+1})^s+\lambda} + \frac{(n(t_{j+1}-t)-1)\,(P_j-P_{j+1})}{(t_j-t)^s+\lambda} \right| \\ &\quad + \frac{1}{D}\left\{ \sum_{i=0}^{j-1} + \sum_{i=j+2}^{n} \right\} \left| \frac{P_{j+1}+n(t_{j+1}-t)(P_j-P_{j+1})-P_i}{(t-t_i)^s+\lambda} \right| \\ &:= B_j + B_j^- + B_j^+. \end{aligned}$$

Again, clearly

$$B_j \leq \frac{1}{D}\left|P_{j+1}-P_j\right| \left| \frac{n(t_{j+1}-t)}{(t-t_{j+1})^s+\lambda} + \frac{n(t_{j+1}-t)-1}{(t_j-t)^s+\lambda} \right|.$$

Moreover

$$B_j^- \leq \frac{1}{D}\Delta P \sum_{i=0}^{j-1} \frac{j+1-i-n(t_{j+1}-t)}{(t-t_i)^s+\lambda},$$

and

$$B_j^+ \leq \frac{1}{D}\Delta P \sum_{i=j+2}^{n} \frac{i-j-1+n(t_{j+1}-t)}{(t-t_i)^s+\lambda}.$$

And the statement follows. □

The proofs of results of Section 4 are based on some preliminary lemmas interesting in themselves.

In the following denote by $\lambda_i^{(n)} = \lambda_i$, $i = 0,\ldots,n$, the $n+1$ eigenvalues of B, which are sorted in increasing order, i.e., $\lambda_0^{(n)} \leq \lambda_1^{(n)} \leq \cdots \leq \lambda_n^{(n)}$, and denote by $\rho(B)$ its spectral radius, which is the maximal absolute value of its eigenvalues, i.e., $\rho(B) = \max\left\{\left|\lambda_0^{(n)}\right|, \left|\lambda_n^{(n)}\right|\right\}$.

Lemma 1 *For $i = 0,\ldots,n$, we have $1/2 < \lambda_i^{(n)} \leq 1$.*

Proof. We observe that B is a symmetric positive stochastic matrix. If $n^s\lambda \leq 1/(6\zeta(s))$, then by Gerschgorin's theorem and from a well-known result for such matrices $\lambda_n^{(n)} = 1$ and $1/2 < \lambda_i^{(n)} < 1$, $i = 0,\ldots,n-1$. And the assertion follows. □

Remark. Lemma 1 implies that $||I-B|| = \rho(I-B) < 1$, where $\rho(I-B)$ is the spectral radius of the matrix $I-B$, with I the identity matrix.

If $\hat{f}_k(t)$ is the eigenfunction of $S_{n,\lambda}$ corresponding to $\lambda_k^{(n)}$, then

$$S_{n,\lambda}(\hat{f}_k;t) = \lambda_k^{(n)} \hat{f}_k(t). \tag{28}$$

Therefore

Lemma 2 *The eigenvalues of B coincide with the eigenvalues of $S_{n,\lambda}$.*

Proof. In the representation (28) we set $t = 0, 1/n, 2/n, \dots, 1$ and obtain

$$B\hat{\overline{f}}_k = \lambda_k^{(n)} I \hat{\overline{f}}_k,$$

where $\hat{\overline{f}}_k = \left[\hat{f}_k(0), \hat{f}_k(1/n), \dots, \hat{f}_k(1)\right]^T$. The last equation implies that

$$\det\left(B - \lambda_k^{(n)} I\right) = 0,$$

i.e., $\lambda_k^{(n)}$ is an eigenvalue of B. $\square$

The next statement is well-known from the theory of linear algebra (cfr. [15]).

Lemma 3 *If T is a square matrix with $\rho(T) < 1$, then the matrix $I - T$ is invertible and we have*

$$(I - T)^{-1} = I + T + T^2 + \cdots .$$

If we set $T = I - B$ in the last formula we obtain

Lemma 4 *The matrix B is invertible and we have*

$$B^{-1} = I + (I - B) + (I - B)^2 + \cdots = \sum_{k=0}^{\infty} (I - B)^k.$$

Lemma 4 shows that the matrix B is nonsingular and gives a useful representation of the inverse matrix B^{-1}.

Lemma 5 *The functions $A_{n,i}$, $i = 0, \dots, n$, form a basis generating a subspace of rational functions of degree (sn, sn).*

Proof. Assume that $A_{n,i}$ is linearly dependent. Consequently there exist $\alpha_0, \alpha_1, \dots, \alpha_n$, at least one of which different from 0, such that

$$h_n(x) := \alpha_0 A_{n,0}(x) + \alpha_1 A_{n,1}(x) + \ldots + \alpha_n A_{n,n}(x) = 0$$

for all $x \in [0, 1]$. In particular $h_n(x) = 0$ at $x = 0, 1/n, 2/n, \dots, 1$. This implies that the rows of B are linearly dependent. But this is impossible because by Lemma 4 we know that the matrix B is invertible, hence its rows are linearly independent. So $A_{n,i}$ are linearly independent. $\square$

We end our study on B with the following observation.

Lemma 6 *All rows of B and B^{-1}sum to 1.*

Proof. We know that B is stochastic. On the other hand from (6)-(8) if $\overline{f} = [1,\ldots,1]^T$, then $\overline{f}^G = [1,\ldots,1]^T$, which implies the rows of B^{-1} sum to 1. □

Lemma 7 *Let $\rho(I-wB) < 1$. Then the sequence of rational operators $\left\{\tilde{S}_m\right\}_{m=0}^{\infty}$ uniformly tends to its limiting operator $\tilde{S}_\infty$ on $[0,1]$, as $m \to \infty$ and*

$$\tilde{S}_\infty(f;t) = A_n(t)B^{-1}\overline{f}. \tag{29}$$

Proof. The representation (10) implies

$$\tilde{S}_\infty(f;t) - \tilde{S}_m(f;t) = wA_n(t)\left[\sum_{i=m+1}^{\infty}(I-wB)^{i-1}(I-B)\right]\overline{f}. \tag{30}$$

From the assumption $\rho(I-wB) < 1$ by Lemma 3 we know that the power series $I + (I-wB) + (I-wB)^2 \ldots$ is convergent and this implies that the matrix

$$\sum_{i=m+1}^{\infty}(I-wB)^{i-1} \to 0,$$

as $n \to \infty$. From (10) and Lemma 3 we deduce

$$\tilde{S}_\infty(f;t) = A_n(t)(I + w(I-B)w^{-1}B^{-1})\overline{f},$$

that is (29). □

6.2 Proof of Theorem 2

Letting

$$f(t_0) = \bar{I}\bar{f},\ \bar{I} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \vdots & 0 \end{pmatrix},$$

it results $\forall t \in [0,1]$

$$\left|f(t_0) - S_{n,\lambda}^m(f;t)\right| \le A_n(t)\left[\bar{I} - B^{m-1}\right]\bar{f}.$$

On the other hand

$$\left\|A_n(t)\left[\bar{I} - B^{m-1}\right]\right\| \le \left\|\bar{I} - B^{m-1}\right\| = \rho\left(\bar{I} - B^{m-1}\right) \le \left(\lambda_{n-1}^{(n)}\right)^{m-1}.$$

Since by Lemma 1 $\lambda_{n-1}^{(n)} < 1$, the assertion follows. □

6.3 Proof of Theorem 3

From (29) and (30) and Lemmas 1 and 3

$$\begin{aligned}\left|\tilde{S}_\infty(f;t_i)-\tilde{S}_m(f;t_i)\right| &= \left|f(t_i)-\tilde{S}_m(f;t_i)\right| \\ &\le w\left\|B(I-B)\sum_{i=m+1}^{\infty}(I-wB)^{i-1}\right\|\|f\| \\ &= w\left\|B(I-B)(I-wB)^m\sum_{i=0}^{\infty}(I-wB)^i\right\|\|f\| \\ &= w\left\|B(I-B)(I-wB)^m w^{-1}B^{-1}\right\|\|f\| \\ &\le ||f||\left(1-\lambda_0^{(n)}\right)\left(1-w\lambda_0^{(n)}\right)^m\left(\lambda_0^{(n)}\right)^{-1} \\ &< 2||f||\left(1-\lambda_0^{(n)}\right)\left(1-w\lambda_0^{(n)}\right)^m.\end{aligned}$$

In particular if $w=1$, we deduce (13). □

6.4 Proof of Theorem 4

For a given function f let us denote by $\overline{S}_{n,\lambda}(f)$ the vector

$$\overline{S}_{n,\lambda}(f)=\left[S_{n,\lambda}(f;0),S_{n,\lambda}\left(f;\frac{1}{n}\right),\ldots,S_{n,\lambda}(f;1)\right]^T.$$

Obviously

$$\overline{S}_{n,\lambda}(f)=B\overline{f}.$$

In a similar way if we denote by

$$\overline{S}_\infty(f)=\left[\tilde{S}_\infty(f;0),\tilde{S}_\infty\left(f;\frac{1}{n}\right),\ldots,\tilde{S}_\infty(f;1)\right]^T,$$

then (29) implies

$$\overline{S}_\infty(f)=BB^{-1}\overline{f}=\overline{f},$$

that is $\tilde{S}_\infty(f)$ is the rational function of type $S_{n,\lambda}$ interpolating f at the knots $0,1/n,2/n,\ldots,1$, i.e., $\tilde{S}_\infty=G_{n,\lambda}$. Finally if $0<w<2$, then $\rho(I-wB)=(1-w\lambda_0^{(n)})<1$ and we can follow the first part of the proof.

Now we prove (15). From (4) and Lemmas 1, 6 and 7

$$\left|f(t)-\tilde{S}_\infty(f;t)\right|\le|f(t)-S_{n,\lambda}(f;t)|+\left|S_{n,\lambda}(f;t)-\tilde{S}_\infty(f;t)\right|$$

$$\le C\omega\left(f;\frac{1}{n}\right)+A_n(t)B^{-1}\left|B\overline{f}-\overline{f}\right|\le C\omega\left(f;\frac{1}{n}\right).$$

Working as in the proof of the well-known pointwise estimate of Lagrange interpolating polynomial, we get (16). The proof of Theorem 4 is completed. □

6.5 Proof of Theorem 5

From (30) and Lemma 1 working as in the proof of Theorem 3 we have

$$\begin{aligned}\left\|\tilde{S}_\infty(f) - \tilde{S}_m(f)\right\| &\le w\,\|f\|\,\|A_n\| \left\|(I-B)\sum_{i=m+1}^{\infty}(I-wB)^{i-1}\right\| \\ &\le \|f\| \left(1-\lambda_0^{(n)}\right)\left(1-w\lambda_0^{(n)}\right)^m \left(\lambda_0^{(n)}\right)^{-1},\end{aligned}$$

that is (17). Working as in [9] the assertion follows. □

6.6 Proof of Theorem 6

We know that

$$A_{n,j}(t_j) = 1 - \sum_{i\neq j} A_{n,i}(t_j),\ \ j = 0,1,\ldots,n,$$

hence by Gerschgorin's theorem for some $j = 0,1,\ldots,n$,

$$\begin{aligned}1-\lambda_0^{(n)} &\le 2\sum_{i\neq j} A_{n,i}(t_j) \\ &= 2\frac{\sum_{i\neq j} n^s/\left((i-j)^s + n^s\lambda\right)}{1/\lambda + \sum_{i\neq j} n^s/\left((i-j)^s+n^s\lambda\right)} \\ &= 2n^s\lambda \frac{\sum_{i\neq j} 1/\left((i-j)^s+n^s\lambda\right)}{1+n^s\lambda\sum_{i\neq j} 1/\left((i-j)^s+n^s\lambda\right)}\end{aligned} \tag{31}$$

for some j.

If $n^s\lambda = o(1)$, then by (31) one gets $1-\lambda_0^{(n)} = o(1)$ and by Theorem 5 (20) follows. Since $1-\lambda_0^{(n)} < 1/2$ (see Lemma 1) by Theorem 5 (19) follows. □

6.7 Proof of Theorem 7

WPIA property follows from the remark to (23) and Theorem 3. From Theorem 5 we deduce the assertion. □

6.8 Proof of Theorem 8

From Theorem 7 and the remark to Lemma 1 we immediately deduce the PIA property for γ^0 curves. By Theorem 6 the assertion follows. □

6.9 Proof of Theorem 9

We can work as in [9]. □

7 Conclusions

Near-interpolating parametric curves of Shepard-type are introduced and studied. Such curves, overcoming the flat spots drawback of original Shepard curves, present some advantages with respect to Bézier curves, like the shape control parameter λ, the pseudo-local control property, a smaller deviation from the control polygon, a faster rate of convergence of weighted progressive iterative approximation process and higher flexibility to model objects. Theorem 1 gives a bound of the maximal distance between such curves and their control polygon in terms of the maximal absolute first order difference of the control points. Such a bound gives a strong localization of the curves, useful in some CAGD problems.

New operators of Shepard-type faster converging than in the analogous Bernstein-Bézier case are studied and approximation results are established in Theorems 2–6. The results are applied to get a weighted progressive iterative approximation algorithm in (21)–(22) and Theorems 7–8 to reach the global interpolating Shepard-type curve without solving a linear system. Such technique gives a straightward and intuitive algorithm to generate sequences of curves with finer and finer precision for data point fitting, of interest in Computer-Aided-Modelling.

Analogously tensor product near-interpolation surfaces of Shepard-type are introduced and similar results are presented to model surfaces. Further research is necessary to address questions on the optimal choice in some sense of the parameter λ, on the optimal choice for the knots instead of equidistant nodes, on sharp bound in the estimate of the deviation and on surfaces extension to triangular patches.

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Hesitant fuzzy set theory applied to BCK/BCI-algebras

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Abstract. The notions of hesitant fuzzy subalgebras and hesitant fuzzy ideals of BCK/BCI-algebras are introduced, and related properties are investigated. Characterizations of hesitant fuzzy subalgebras and hesitant fuzzy ideals of BCK/BCI-algebras are discussed. Given a special set, conditions for this set to be a hesitant fuzzy ideal are provided.

1. Introduction

The notions of Atanassov's intuitionistic fuzzy sets, type 2 fuzzy sets and fuzzy multisets etc. are a generalization of fuzzy sets. As another generalization of fuzzy sets, Torra [6] introduced the notion of hesitant fuzzy sets which are a very useful to express peoples hesitancy in daily life. The hesitant fuzzy set is a very useful tool to deal with uncertainty, which can be accurately and perfectly described in terms of the opinions of decision makers. Xu and Xia [11] proposed a variety of distance measures for hesitant fuzzy sets, based on which the corresponding similarity measures can be obtained. They investigated the connections of the aforementioned distance measures and further develop a number of hesitant ordered weighted distance measures and hesitant ordered weighted similarity measures. Also, hesitant fuzzy set theory is used in decision making problem etc. (see [5, 8, 9, 10, 12]), and is applied to residuated lattices and MTL-algebras (see [2, 4]).

In this paper, we introduce the notions of hesitant fuzzy subalgebras and hesitant fuzzy ideals of BCK/BCI-algebras, and investigate their relations and properties. We consider characterizations of hesitant fuzzy subalgebras and hesitant fuzzy ideals of BCK/BCI-algebras. Given a special set, we provide conditions for this set to be a hesitant fuzzy ideal.

2. Preliminaries

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers.

An algebra $(X;*,0)$ of type $(2,0)$ is called a *BCI-algebra* if it satisfies the following conditions:

(I) $(\forall x,y,z \in X)$ $(((x*y)*(x*z))*(z*y)=0)$,

(II) $(\forall x,y \in X)$ $((x*(x*y))*y=0)$,

(III) $(\forall x \in X)$ $(x*x=0)$,

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[0]**2010 Mathematics Subject Classification**: 06F35; 03G25; 03E72.

[0]**Keywords**: hesitant fuzzy subalgebras; hesitant fuzzy ideals.

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(IV) $(\forall x, y \in X)$ $(x * y = 0, y * x = 0 \Rightarrow x = y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)$ $(0 * x = 0)$,

then X is called a *BCK-algebra*. A BCK-algebra X is said to be *positive implicative* if it satisfies:

$$(\forall x, y, z \in X)\,((x * y) * z = (x * z) * (y * z))\,. \tag{2.1}$$

A BCK-algebra X is said to be *implicative* if it satisfies:

$$(\forall x, y \in X)\,(x = x * (y * x))\,. \tag{2.2}$$

Any BCK/BCI-algebra X satisfies the following conditions:

$$(\forall x \in X)\,(x * 0 = x)\,, \tag{2.3}$$

$$(\forall x, y, z \in X)\,(x \leq y \;\Rightarrow\; x * z \leq y * z,\; z * y \leq z * x)\,, \tag{2.4}$$

$$(\forall x, y, z \in X)\,((x * y) * z = (x * z) * y)\,, \tag{2.5}$$

$$(\forall x, y, z \in X)\,((x * z) * (y * z) \leq x * y) \tag{2.6}$$

where $x \leq y$ if and only if $x * y = 0$. A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. A subset A of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies:

(b1) $0 \in A$.

(b2) $(\forall x \in X)(\forall y \in A)(x * y \in A \;\Rightarrow\; x \in A)$.

We refer the reader to the books [1, 3] for further information regarding BCK/BCI-algebras.

3. Hesitant fuzzy subalgebras/ideals

Definition 3.1 ([6, 7])**.** Let E be a reference set. A *hesitant fuzzy set* on E is defined in terms of a function that when applied to E returns a subset of $[0, 1]$, which can be viewed as the following mathematical representation:

$$H_E := \{(e, h_E(e)) \mid e \in E\}$$

where $h_E : E \to \mathscr{P}([0, 1])$.

In what follows, we take a BCK/BCI-algebra X as a reference set unless otherwise specified.

Definition 3.2. Given a non-empty subset A of X, a hesitant fuzzy set

$$H_X := \{(x, h_X(x)) \mid x \in X\}$$

on X satisfying the following condition:

$$h_X(x) = \emptyset \text{ for all } x \notin A \tag{3.1}$$

is called a *hesitant fuzzy set related to* A (briefly, A*-hesitant fuzzy set*) on X, and is represented by $H_A := \{(x, h_A(x)) \mid x \in X\}$, where h_A is a mapping from X to $\mathscr{P}([0, 1])$ with $h_A(x) = \emptyset$ for all $x \notin A$.

Definition 3.3. Given a non-empty subset (subalgebra as much as possible) A of X, let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be an A-hesitant fuzzy set on X. Then $H_A := \{(x, h_A(x)) \mid x \in X\}$ is called a *hesitant fuzzy subalgebra of X related to A* (briefly, *A-hesitant fuzzy subalgebra* of X) if it satisfies the following condition:

$$(\forall x, y \in A)\,(h_A(x * y) \supseteq h_A(x) \cap h_A(y))\,. \tag{3.2}$$

An A-hesitant fuzzy subalgebra of X with $A = X$ is called a *hesitant fuzzy subalgebra* of X.

Example 3.4. Let $X = \{0, 1, a, b, c\}$ be a BCI-algebra with the following Cayley table:

$*$	0	1	a	b	c
0	0	0	c	b	a
1	1	0	c	b	a
a	a	a	0	c	b
b	b	b	a	0	c
c	c	c	b	a	0

For a subalgebra $A = \{0, a, b, c\}$ of X, let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be an A-hesitant fuzzy set on X defined by

$$h_A(x) = \begin{cases} [0,1] & \text{if } o(x) = 1, \\ \{k \in [0,1] \mid k \le \frac{1}{r}\} & \text{if } o(x) = r \neq 1 \end{cases}$$

where $o(x) = \min\{n \in \mathbb{N} \mid 0 * x^n = 0\}$. Then

$$H_A = \{(0, [0,1]), (1, \emptyset), (a, [0, \tfrac{1}{4}]), (b, [0, \tfrac{1}{2}]), (c, [0, \tfrac{1}{4}])\}$$

and it is an A-hesitant fuzzy subalgebra of X.

Theorem 3.5. *Let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be an A-hesitant fuzzy set on X where A is a non-empty subset (subalgebra as much as possible) of X. Then the following are equivalent:*

(1) *$H_A := \{(x, h_A(x)) \mid x \in X\}$ is an A-hesitant fuzzy subalgebra of X.*
(2) *For any $\varepsilon \in \mathscr{P}([0,1])$, the set $H_A(\varepsilon) := \{x \in X \mid h_A(x) \supseteq \varepsilon\}$ is a subalgebra of X whenever it is non-empty.*

The set $H_A(\varepsilon) := \{x \in X \mid h_A(x) \supseteq \varepsilon\}$ is called a *hesitant fuzzy ε-level set* of $H_A := \{(x, h_A(x)) \mid x \in X\}$.

Proof. Assume that $H_A := \{(x, h_A(x)) \mid x \in X\}$ is an A-hesitant fuzzy subalgebra of X. Let $\varepsilon \in \mathscr{P}([0,1])$ be such that $H_A(\varepsilon) := \{x \in X \mid h_A(x) \supseteq \varepsilon\} \neq \emptyset$. If $x, y \in H_A(\varepsilon)$, then $h_A(x) \supseteq \varepsilon$ and $h_A(y) \supseteq \varepsilon$. It follows from (3.2) that

$$h_A(x * y) \supseteq h_A(x) \cap h_A(y) \supseteq \varepsilon$$

and so that $x * y \in H_A(\varepsilon)$. Therefore $H_A(\varepsilon) := \{x \in X \mid h_A(x) \supseteq \varepsilon\}$ is a subalgebra of X for all $\varepsilon \in \mathscr{P}([0,1])$ whenever it is non-empty.

Conversely, suppose that the non-empty hesitant fuzzy ε-level set of $H_A := \{(x, h_A(x)) \mid x \in X\}$ is a subalgebra of X for all $\varepsilon \in \mathscr{P}([0,1])$. For any $x, y \in A$, let $h(x) = \varepsilon_x$ and $h(y) = \varepsilon_y$. Take

$\varepsilon = \varepsilon_x \cap \varepsilon_y$. Then $x, y \in H_A(\varepsilon)$, and so $x * y \in H_A(\varepsilon)$. Hence $h_A(x * y) \supseteq \varepsilon = \varepsilon_x \cap \varepsilon_y = h(x) \cap h(y)$. Therefore $H_A := \{(x, h_A(x)) \mid x \in X\}$ is an A-hesitant fuzzy subalgebra of X. □

Definition 3.6. Given a non-empty subset (subalgebra as much as possible) A of X, an A-hesitant fuzzy set $H_A := \{(x, h_A(x)) \mid x \in X\}$ on X is called a *hesitant fuzzy ideal of X related to A* (briefly, *A-hesitant fuzzy ideal* of X) if it satisfies:

$$(\forall x, y \in A)\,(h_A(x * y) \cap h_A(y) \subseteq h_A(x) \subseteq h_A(0))\,. \tag{3.3}$$

An A-hesitant fuzzy ideal of X with $A = X$ is called a *hesitant fuzzy ideal* of X.

Example 3.7. (1) Let $X = \{0, a, b, c\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	a	0

For a subalgebra $A = \{0, a, b\}$ of X, let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by

$$H_A = \left\{(0, [\tfrac{1}{4}, \tfrac{3}{4}]), (a, (\tfrac{1}{4}, \tfrac{1}{2})), (b, [\tfrac{1}{2}, \tfrac{3}{4})), (c, \emptyset)\right\}.$$

Then $H_A := \{(x, h_A(x)) \mid x \in X\}$ is an A-hesitant fuzzy ideal of X.
(2) Let $X = \{0, 1, a, b, c\}$ be a BCI-algebra with the following Cayley table:

$*$	0	1	a	b	c
0	0	0	a	b	c
1	1	0	a	b	c
a	a	a	0	c	b
b	b	b	c	0	a
c	c	c	b	a	0

Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by

$$H_X = \{(0, [0, 1)), (1, [0.2, 0.7]), (a, (0.2, 0.3]), (b, \{0.4, 0.5\}), (c, [0.6, 0.7))\}.$$

It is routine to verify that $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy ideal of X.

Theorem 3.8. *Let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be an A-hesitant fuzzy set on X where A is a non-empty subset (subalgebra as much as possible) of X. Then the following are equivalent:*

(1) *$H_A := \{(x, h_A(x)) \mid x \in X\}$ is an A-hesitant fuzzy ideal of X.*
(2) *The non-empty hesitant fuzzy ε-level set of $H_A := \{(x, h_A(x)) \mid x \in X\}$ is an ideal of X for all $\varepsilon \in \mathscr{P}([0, 1])$.*

Proof. Suppose that $H_A := \{(x, h_A(x)) \mid x \in X\}$ is an A-hesitant fuzzy ideal of X. Let $x, y \in X$ and $\varepsilon \in \mathscr{P}([0,1])$ be such that $x * y \in H_A(\varepsilon)$ and $y \in H_A(\varepsilon)$. Then $h_A(x*y) \supseteq \varepsilon$ and $h_A(y) \supseteq \varepsilon$. It follows from (3.3) that

$$h_A(0) \supseteq h_A(x) \supseteq h_A(x*y) \cap h_A(y) \supseteq \varepsilon.$$

Hence $0 \in H_A(\varepsilon)$ and $x \in H_A(\varepsilon)$, and therefore $H_A(\varepsilon)$ is an ideal of X.

Conversely, assume that the second condition is valid. For any $x \in X$, let $h_A(x) = \varepsilon_x$. Then $x \in H_A(\varepsilon_x)$. Since $H_A(\varepsilon_x)$ is an ideal of X, we have $0 \in H_A(\varepsilon_x)$ and so $h_A(x) = \varepsilon_x \subseteq h_A(0)$. For any $x, y \in A$, let $h_A(x*y) = \varepsilon_{x*y}$ and $h_A(y) = \varepsilon_y$. If we take $\varepsilon = \varepsilon_{x*y} \cap \varepsilon_y$, then $x * y \in H_A(\varepsilon)$ and $y \in H_A(\varepsilon)$ which imply that $x \in H_A(\varepsilon)$ Thus $h_A(x) \supseteq \varepsilon = \varepsilon_{x*y} \cap \varepsilon_y = h_A(x*y) \cap h_A(y)$. Therefore $H_A := \{(x, h_A(x)) \mid x \in X\}$ is an A-hesitant fuzzy ideal of X. □

Proposition 3.9. *Let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be an A-hesitant fuzzy ideal of X where A is a subalgebra of X. Then the following assertions are valid.*

(1) $(\forall x, y \in A)\,(x \leq y \;\Rightarrow\; h_A(x) \supseteq h_A(y))$.
(2) $(\forall x, y, z \in A)\,(x * y \leq z \;\Rightarrow\; h_A(x) \supseteq h_A(y) \cap h_A(z))$.

Proof. (1) Let $x, y \in A$ be such that $x \leq y$. Then $0 = x * y \in A$, and so

$$h_A(y) = h_A(0) \cap h_A(y) = h_A(x*y) \cap h_A(y) \subseteq h_A(x)$$

by (3.3).
(2) Let $x, y, z \in A$ be such that $x * y \leq z$. Then

$$h_A(z) = h_A(0) \cap h_A(z) = h_A((x*y)*z) \cap h_A(z) \subseteq h_A(x*y).$$

It follows that $h_A(y) \cap h_A(z) \subseteq h_A(x*y) \cap h_A(y) \subseteq h_A(x)$. □

Corollary 3.10. *Every hesitant fuzzy ideal $H_X := \{(x, h_X(x)) \mid x \in X\}$ of X satisfies the following assertions.*

(1) $h_X(y) \subseteq h_X(x)$ *for all* $x, y \in X$ *with* $x \leq y$.
(2) $h_X(y) \cap h_X(z) \subseteq h_X(x)$ *for all* $x, y, z \in X$ *with* $x * y \leq z$.

The following corollary is easily proved by induction.

Corollary 3.11. *Given a subalgebra A of X, every A-hesitant fuzzy ideal $H_A := \{(x, h_A(x)) \mid x \in X\}$ of X satisfies the following condition:*

$$(\cdots(x * a_1) * \cdots) * a_n = 0 \;\Rightarrow\; \bigcap_{k=1}^{n} h_A(a_k) \subseteq h_A(x) \tag{3.4}$$

for all $x, a_1, a_2, \cdots, a_n \in X$.

Proposition 3.12. *Let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be an A-hesitant fuzzy ideal of X where A is a subalgebra of X. Then the following assertions are equivalent.*

(1) $(\forall x, y \in A)\,(h_A((x*y)*y) \subseteq h_A(x*y))$.

(2) $(\forall x, y, z \in A)\,(h_A((x * y) * z) \subseteq h_A((x * z) * (y * z)))$.

Proof. Assume that (1) is valid and let $x, y, z \in A$. Since

$$((x * (y * z)) * z) * z = ((x * z) * (y * z)) * z \leq (x * y) * z,$$

it follows from Proposition 3.9(1), (1) and (2.5) that

$$\begin{aligned} h_A((x * y) * z) &\subseteq h_A(((x * (y * z)) * z) * z) \\ &\subseteq h_A((x * (y * z)) * z) \\ &= h_A((x * z) * (y * z)). \end{aligned}$$

Conversely, suppose that (2) holds. If we put $z := y$ in (2), then

$$h_A((x * y) * y) \subseteq h_A((x * y) * (y * y)) = h_A((x * y) * 0) = h_A(x * y)$$

which proves (1). □

Theorem 3.13. *For a subalgebra A of a BCK-algebra X, every A-hesitant fuzzy ideal is an A-hesitant fuzzy subalgebra.*

Proof. Let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be an A-hesitant fuzzy ideal of X. Then

$$\begin{aligned} h_A(x * y) &\supseteq h_A\left((x * y) * x\right) \cap h_A(x) = h_A\left((x * x) * y\right) \cap h_A(x) \\ &= h_A(0 * y) \cap h_A(x) = h_A(0) \cap h_A(x) \supseteq h_A(x) \cap h_A(y) \end{aligned}$$

by (3.3), (2.5), (III) and (V). Hence $H_A := \{(x, h_A(x)) \mid x \in X\}$ is an A-hesitant fuzzy subalgebra of X. □

The following example shows that the converse of Theorem 3.13 is not true in general.

Example 3.14. Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	0
b	b	b	0	0	0
c	c	c	c	0	0
d	d	c	c	a	0

Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by

$$H_X = \left\{(0, [0, 1]), (a, (0, \tfrac{1}{4})), (b, (0, \tfrac{1}{2})), (c, [\tfrac{1}{4}, \tfrac{3}{4})), (d, (\tfrac{6}{8}, \tfrac{7}{8}))\right\}.$$

Then $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy subalgebra of X, but not a hesitant fuzzy ideal of X since $h_X(d * b) \cap h_X(b) = [\frac{1}{4}, \frac{1}{2}) \nsubseteq (\frac{6}{8}, \frac{7}{8}) = h_X(d)$.

In a BCI-algebra X, Theorem 3.13 is not true in general as seen in the following example.

Example 3.15. Let $(Y, *, 0)$ be a BCI-algebra and $(\mathbb{Z}, +, 0)$ an additive group of integers. Let $(\mathbb{Z}, -, 0)$ be the adjoint BCI-algebra of $(\mathbb{Z}, +, 0)$ and let $X := Y \times \mathbb{Z}$. Then $(X, \otimes, (0, 0))$ is a BCI-algebra where the operation $\otimes$ is given by

$$(\forall (x, m), (y, n) \in X) \left((x, m) \otimes (y, n) = (x * y, m - n)\right).$$

For a subset $A := Y \times \mathbb{N}_0$ of X where $\mathbb{N}_0$ is the set of nonnegative integers, let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on X in which h_X is given as follows:

$$h_X : X \to \mathscr{P}([0,1]),\ x \mapsto \begin{cases} [\frac{1}{2}, 1] & \text{if } x \in A, \\ [\frac{1}{3}, 1] & \text{otherwise.} \end{cases}$$

Then $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy ideal of X. But it is not a hesitant fuzzy subalgebra of X since

$$h_X((0, 0) \otimes (0, 1)) = h_X(0, -1) = [\tfrac{1}{3}, 1] \not\supseteq [\tfrac{1}{2}, 1] = h_X(0, 0) \cap h_X(0, 1).$$

Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on X. For any $a, b \in X$ and $n \in \mathbb{N}$, let

$$h_X[b; a^n] := \{x \in X \mid h_X((x * b) * a^n) = h_X(0)\}$$

where $(x * b) * a^n = ((\cdots((x * b) * a) * a) * \cdots) * a$ in which a appears n-times. Obviously, $a, b, 0 \in h_X[b; a^n]$.

Proposition 3.16. *Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on X such that $h_X(x) \subseteq h_X(0)$ and $h_X(x * y) = h_X(x) \cup h_X(y)$ for all $x, y \in X$. For any $a, b \in X$ and $n \in \mathbb{N}$, if $x \in h_X[b; a^n]$ then $x * y \in h_X[b; a^n]$ for all $y \in X$.*

Proof. Let $x \in h_X[b; a^n]$. Then $h_X((x * b) * a^n) = h_X(0)$, and thus

$$\begin{aligned} h_X(((x * y) * b) * a^n) &= h_X(((x * b) * y) * a^n) \\ &= h_X(((x * b) * a^n) * y) \\ &= h_X((x * b) * a^n) \cup h_X(y) \\ &= h_X(0) \cup h_X(y) = h_X(0) \end{aligned}$$

for all $y \in X$. Hence $x * y \in h_X[b; a^n]$ for all $y \in X$. □

Proposition 3.17. *Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on a BCK-algebra X. If an element $a \in X$ satisfies:*

$$(\forall x \in X)\,(x \leq a), \tag{3.5}$$

then $h_X[b; a^n] = X = h_X[a; b^n]$ for all $b \in X$ and $n \in \mathbb{N}$.

Proof. Let $b, x \in X$ and $n \in \mathbb{N}$. Then

$$\begin{aligned} h_X((x * b) * a^n) &= h_X(((x * b) * a) * a^{n-1}) \\ &= h_X(((x * a) * b) * a^{n-1}) \\ &= h_X((0 * b) * a^{n-1}) \\ &= h_X(0) \end{aligned}$$

by (2.5), (3.5) and (V), and so $x \in h_X[b; a^n]$, which shows that $h_X[b; a^n] = X$. Similarly $h_X[a; b^n] = X$. □

Corollary 3.18. *If $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy set on a bounded BCK-algebra X, then $h_X[b; u^n] = X = h_X[u; b^n]$ for all $b \in X$ and $n \in \mathbb{N}$ where u is the unit of X.*

Proposition 3.19. *Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on X such that*

$$(\forall x, y \in X)\,(x \leq y \;\Rightarrow\; h_X(x) \supseteq h_X(y))\,. \tag{3.6}$$

If $b \leq c$ in X, then $h_X[b; a^n] \subseteq h_X[c; a^n]$ for all $a \in X$ and $n \in \mathbb{N}$.

Proof. Let $b, c \in X$ be such that $b \leq c$. For any $a \in X$ and $n \in \mathbb{N}$, if $x \in h_X[b; a^n]$ then

$$\begin{aligned} h_X(0) &= h_X((x * b) * a^n) = h_X((x * a^n) * b) \\ &\subseteq h_X((x * a^n) * c) = h_X((x * c) * a^n) \end{aligned}$$

by (2.4) and (3.6), and so $h_X((x * c) * a^n) = h_X(0)$. Thus $x \in h_X[c; a^n]$, and therefore $h_X[b; a^n] \subseteq h_X[c; a^n]$ for all $a \in X$ and $n \in \mathbb{N}$. □

Corollary 3.20. *If $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy ideal of X, then $h_X[b; a^n] \subseteq h_X[c; a^n]$ for all $n \in \mathbb{N}$ and $a, b, c \in X$ with $b \leq c$.*

The following example shows that there exists a hesitant fuzzy set $H_X := \{(x, h_X(x)) \mid x \in X\}$ on X such that the set $h_X[b; a^n]$ is not an ideal of X for some $a, b \in X$ and $n \in \mathbb{N}$.

Example 3.21. Let $X = \{0, a, b, c\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by

$$H_X = \left\{(0, [0, \tfrac{1}{2}]), (a, [0, \tfrac{1}{3}]), (b, [0, \tfrac{1}{3}]), (c, [0, \tfrac{1}{3}])\right\}.$$

Then $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy set (moreover, hesitant fuzzy ideal) on X and $h_X[a; c] = \{x \in X \mid h_X((x * a) * c) = h_X(0)\} = \{0, a, c\}$ which is not an ideal of X since $b * a = a \in h_X[a; c]$ but $b \notin h_X[a; c]$.

We now consider conditions for a set $h_X[b; a^n]$ to be an ideal of X.

Theorem 3.22. *Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on a positive implicative BCK-algebra X in which h_X is injective. Then $h_X[b; a^n]$ is an ideal of X for all $a, b \in X$ and $n \in \mathbb{N}$.*

Proof. Let $a, b, x, y \in X$ and $n \in \mathbb{N}$ be such that $x * y \in h_X[b; a^n]$ and $y \in h_X[b; a^n]$. Then $h_X((y * b) * a^n) = h_X(0)$, which implies that $(y * b) * a^n = 0$ since h_X is injective. Hence

$$\begin{aligned}
h_X(0) &= h_X(((x * y) * b) * a^n) \\
&= h_X((((x * y) * b) * a) * a^{n-1}) \\
&= h_X((((x * b) * (y * b)) * a) * a^{n-1}) \\
&= h_X(((((x * b) * a) * ((y * b) * a)) * a) * a^{n-2}) \\
&= \cdots \\
&= h_X(((x * b) * a^n) * ((y * b) * a^n)) \\
&= h_X(((x * b) * a^n) * 0) \\
&= h_X((x * b) * a^n),
\end{aligned}$$

which shows that $x \in h_X[b; a^n]$. Therefore $h_X[b; a^n]$ is an ideal of X for all $a, b \in X$ and $n \in \mathbb{N}$. □

Theorem 3.23. *Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on a positive implicative BCK-algebra X such that*

$$(\forall x, y \in X)\,(h_X(x) \subseteq h_X(0),\ h_X(x * y) = h_X(x) \cap h_X(y)). \tag{3.7}$$

Then $h_X[b; a^n]$ is an ideal of X for all $a, b \in X$ and $n \in \mathbb{N}$.

Proof. Let $a, b, x, y \in X$ and $n \in \mathbb{N}$ be such that $x * y \in h_X[b; a^n]$ and $y \in h_X[b; a^n]$. Then $h_X((y * b) * a^n) = h_X(0)$, which implies from (3.7) that

$$\begin{aligned}
h_X(0) &= h_X(((x * y) * b) * a^n) \\
&= h_X((((x * y) * b) * a) * a^{n-1}) \\
&= h_X((((x * b) * (y * b)) * a) * a^{n-1}) \\
&= h_X(((((x * b) * a) * ((y * b) * a)) * a) * a^{n-2}) \\
&= \cdots \\
&= h_X(((x * b) * a^n) * ((y * b) * a^n)) \\
&= h_X((x * b) * a^n) \cap h_X((y * b) * a^n) \\
&= h_X((x * b) * a^n) \cap h_X(0) \\
&= h_X((x * b) * a^n).
\end{aligned}$$

Hence $x \in h_X[b; a^n]$, and therefore $h_X[b; a^n]$ is an ideal of X for all $a, b \in X$ and $n \in \mathbb{N}$. □

Since every implicative BCK-algebra is positive implicative, we have the following corollary.

Corollary 3.24. *Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on an implicative BCK-algebra X in which the condition* (3.7) *is valid or h_X is injective. Then $h_X[b; a^n]$ is an ideal of X for all $a, b \in X$ and $n \in \mathbb{N}$.*

Corollary 3.25. *Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on a BCK-algebra X, where X satisfies any one of the following conditions:*

$$(\forall x, y \in X)\,(x * y = (x * y) * y)\,, \tag{3.8}$$

$$(\forall x, y \in X)\,((x * (x * y)) * (y * x) = x * (x * (y * (y * x))))\,, \tag{3.9}$$

$$(\forall x, y \in X)\,(x * y = (x * y) * (x * (x * y)))\,, \tag{3.10}$$

$$(\forall x, y \in X)\,(x * (x * y) = (x * (x * y)) * (x * y))\,, \tag{3.11}$$

$$(\forall x, y \in X)\,((x * (x * y)) * (y * x) = (y * (y * x)) * (x * y))\,. \tag{3.12}$$

If $H_X := \{(x, h_X(x)) \mid x \in X\}$ satisfies the condition (3.7) or h_X is injective, then $h_X[b; a^n]$ is an ideal of X for all $a, b \in X$ and $n \in \mathbb{N}$.

Proposition 3.26. *Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on X in which h_X is injective. If J is an ideal of X, then the following assertion is valid.*

$$(\forall a, b \in J)\,(\forall n \in \mathbb{N})\,(h_X[b; a^n] \subseteq J)\,. \tag{3.13}$$

Proof. For any $a, b \in J$ and $n \in \mathbb{N}$, let $x \in h_X[b; a^n]$. Then

$$h_X(((x * b) * a^{n-1}) * a) = h_X((x * b) * a^n) = h_X(0)$$

and so $((x * b) * a^{n-1}) * a = 0 \in J$ because h_X is injective. Since J is an ideal of X, it follows from (b2) that $(x * b) * a^{n-1} \in J$. Continuing this process, we have $x * b \in J$ and thus $x \in J$. Therefore $h_X[b; a^n] \subseteq J$ for all $a, b \in J$ and $n \in \mathbb{N}$. □

Theorem 3.27. *Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on a BCK-algebra X. For any subset J of X, if the condition (3.13) holds, then J is an ideal of X.*

Proof. Suppose that the condition (3.13) is valid. Not that $0 \in h_X[b; a^n] \subseteq J$. Let $x, y \in X$ be such that $x * y \in J$ and $y \in J$. Taking $b := x * y$ implies that

$$\begin{aligned} h_X((x * b) * y^n) &= h_X((x * (x * y)) * y^n) \\ &= h_X(((x * (x * y)) * y) * y^{n-1}) \\ &= h_X(((x * y) * (x * y)) * y^{n-1}) \\ &= h_X(0 * y^{n-1}) = h_X(0), \end{aligned}$$

and so $x \in h_X[b; y^n] \subseteq J$ with $b = x * y$. Therefore J is an ideal of X. □

Theorem 3.28. *If $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy ideal of X, then the set*

$$H_a := \{x \in X \mid h_X(a) \subseteq h_X(x)\}$$

is an ideal of X for all $a \in X$.

Proof. Let $x, y \in X$ be such that $x * y \in H_a$ and $y \in H_a$. Then $h_X(a) \subseteq h_X(x * y)$ and $h_X(a) \subseteq h_X(y)$. It follows from (3.3) that

$$h_X(a) \subseteq h_X(x * y) \cap h_X(y) \subseteq h_X(x) \subseteq h_X(0)$$

and so that $0 \in H_a$ and $x \in H_a$. Therefore H_a is an ideal of X for all $a \in X$. □

Theorem 3.29. *Let $a \in X$ and let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on X. Then*

(1) *If H_a is an ideal of X, then $H_X := \{(x, h_X(x)) \mid x \in X\}$ satisfies:*

$$(\forall x, y \in X) \left(h_X(a) \subseteq h_X(x * y) \cap h_X(y) \;\Rightarrow\; h_X(a) \subseteq h_X(x) \right). \tag{3.14}$$

(2) *If $H_X := \{(x, h_X(x)) \mid x \in X\}$ satisfies the condition (3.14) and $h_X(x) \subseteq h_X(0)$ for all $x \in X$, then H_a is an ideal of X.*

Proof. (1) Assume that H_a is an ideal of X and let $x, y \in X$ be such that $h_X(a) \subseteq h_X(x*y) \cap h_X(y)$. Then $x * y \in H_a$ and $y \in H_a$, which imply that $x \in H_a$, that is, $h_X(a) \subseteq h_X(x)$.
(2) Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on X in which the condition (3.14) holds and $h_X(x) \subseteq h_X(0)$ for all $x \in X$. Then $0 \in H_a$. Let $x, y \in X$ be such that $x * y \in H_a$ and $y \in H_a$. Then $h_X(a) \subseteq h_X(x * y)$ and $h_X(a) \subseteq h_X(y)$, and so $h_X(a) \subseteq h_X(x * y) \cap h_X(y)$. It follows from (3.14) that $h_X(a) \subseteq h_X(x)$, that is, $x \in H_a$. Therefore H_a is an ideal of X. □

4. Conclusions

We have introduced the notions of hesitant fuzzy subalgerbas and hesitant fuzzy ideals of BCK/BCI-algebras, and have investigated their relations and properties. We have considered characterizations of hesitant fuzzy subalgerbas and hesitant fuzzy ideals of BCK/BCI-algebras. Given a special set, we have provided conditions for this set to be a hesitant fuzzy ideal. Future research will focus on applying the notions/contents to other types of ideals in BCK/BCI-algebras and related algebraic structures.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 20, NO.4, 2016, COPYRIGHT 2016 EUDOXUS PRESS, LLC

Fractional q-integrodifference equations and inclusions with nonlocal fractional q-integral conditions

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Abstract

In this paper, we study a class of fractional q-integrodifference equations with nonlocal fractional q-integral boundary conditions which have different quantum numbers. Some new existence and uniqueness results are obtained by using fixed point theorems. Both cases the single- and multi-valued are considered. Some examples illustrating our results are also presented.

Keywords: fractional q-difference equation; boundary value problem; existence; fixed point theorems
2010 Mathematics Subject Classifications: 34A08; 34B18; 39A13.

1 Introduction

In this paper, we will study the existence and uniqueness of solutions of a class of fractional q-integrodifference equations with nonlocal fractional q-integral conditions which have different quantum numbers. In the first part, we deal with the following nonlocal fractional q-integral boundary value problem of nonlinear fractional q-integrodifference equation

$$\begin{cases} {}^cD_q^{\alpha}x(t) = f(t, x(t), I_z^{\delta}x(t)), \quad t \in (0, T), \\ x(\zeta) = g(x), \qquad \lambda I_p^{\beta}x(\eta) = I_r^{\gamma}x(\xi), \quad 0 < \zeta < \eta < \xi < T, \end{cases} \tag{1.1}$$

where $0 < p, q, r, z < 1$, $1 < \alpha \le 2$, $\beta, \gamma, \delta > 0$, $\lambda \in \mathbb{R}$ are given constants, D_q^{α} is the fractional q-derivative of Caputo type of order α, I_{ϕ}^{ψ} is the fractional ϕ-integral of order ψ with $\phi = p, r, z$ and $\psi = \beta, \gamma, \delta$, $f : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : C([0, T], \mathbb{R}) \to \mathbb{R}$ are continuous functions.

The study of q-difference equations, initiated by Jackson [20, 21], Carmichael [12], Mason [24] and Adams [1] in the first quarter of 20th century, has been developed over the years, for instance, see [14, 17, 22]. In recent years, the topic has attracted the attention of several researchers and a variety of new results can be found in the papers [3], [4], [5], [6], [7], [8], [13], [15], [16], [18], [19], [23].

The case $\zeta = 0, g = 0$ was studied recently in [2], where existence and uniqueness results are proved by applying Banach's contraction principle, Krasnoselskii's fixed point theorem and Leray-Schauder's Nonlinear Alternative.

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Nonlocal conditions were initiated by Bitsadze [9]. As remarked by Byszewski [10, 11], the nonlocal condition can be more useful than the standard initial condition to describe some physical phenomena. For example, $g(x)$ may be given by $g(x) = \sum_{i=1}^{p} c_i x(t_i)$ where $c_i, i = 1, \ldots, p$, are given constants and $0 < t_1 < \ldots < t_p \leq T$.

In Section 3 we give some sufficient conditions for the existence and uniqueness of solutions and for the existence of at least one solution of problem (1.1). The first result is based on Banach's contraction principle and the second on a fixed point theorem due to D. O'Regan [25]. Concrete examples are also provided to illustrate the possible applications of the established analytical results.

In the second part we consider the multi-valued analogue of problem (1.1) given by

$$\begin{cases} {}^cD_q^{\alpha} x(t) \in F(t, x(t), I_z^{\delta} x(t)), \quad t \in (0, T), \\ x(\zeta) = g(x), \qquad \lambda I_p^{\beta} x(\eta) = I_r^{\gamma} x(\xi), \quad 0 < \zeta < \eta < \xi < T, \end{cases} \tag{1.2}$$

where $F : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalue map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subset of $\mathbb{R}$.

We give an existence result for the problem (1.2) in the case when the right hand side is convex valued by using the Nonlinear Alternative for contractive maps.

Note that there are four different quantum numbers and the boundary condition of (1.1) does not contain the value of unknown function x at the right-side of boundary point $t = T$. *One may interpret the q-integral boundary condition in (1.1) as the q-integrals with different quantum numbers are related through a real number λ.*

The paper is organized as follows: In Section 2, for the convenience of the reader, we cite some definitions and fundamental results on q-calculus as well as the fractional q-calculus. An auxiliary lemma, needed in the proofs of our main results is presented in Section 3. In Section 4 we prove our main results for single-valued case and in Section 5 we prove our main results for multi-valued case.

2 Preliminaries

To make this paper self-contained, below we recall some known facts on fractional q-calculus. The presentation here can be found in, for example, [7], [16], [26].

For $q \in (0, 1)$, define

$$[a]_q = \frac{1 - q^a}{1 - q}, \quad a \in \mathbb{R}. \tag{2.1}$$

The q-analogue of the power function $(1 - b)^k$ with $k \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$ is

$$(1 - b)^{(0)} = 1, \quad (1 - b)^{(k)} = \prod_{i=0}^{k-1} (1 - bq^i), \quad k \in \mathbb{N}, \quad b \in \mathbb{R}. \tag{2.2}$$

More generally, if $\gamma \in \mathbb{R}$, then

$$(1 - b)^{(\gamma)} = \prod_{i=0}^{\infty} \frac{1 - bq^i}{1 - bq^{\gamma+i}}. \tag{2.3}$$

We use the notation $0^{(\gamma)} = 0$ for $\gamma > 0$. The q-gamma function is defined by

$$\Gamma_q(x) = \frac{(1 - q)^{(x-1)}}{(1 - q)^{x-1}}, \quad x \in \mathbb{R} \setminus \{0, -1, -2, \ldots\}. \tag{2.4}$$

Obviously, $\Gamma_q(x + 1) = [x]_q \Gamma_q(x)$.

The q-derivative of a function h is defined by

$$(D_q h)(x) = \frac{h(x) - h(qx)}{(1 - q)x} \quad \text{for} \quad x \neq 0 \quad \text{and} \quad (D_q h)(0) = \lim_{x \to 0} (D_q h)(x), \tag{2.5}$$

and q-derivatives of higher order are given by

$$(D_q^0 h)(x) = h(x) \quad \text{and} \quad (D_q^k h)(x) = D_q(D_q^{k-1} h)(x), \quad k \in \mathbb{N}. \tag{2.6}$$

The q-integral of a function h defined on the interval $[0, b]$ is given by

$$(I_q h)(x) = \int_0^x h(s)d_qs = x(1-q)\sum_{i=0}^{\infty} h(xq^i)q^i, \quad x \in [0, b]. \tag{2.7}$$

If $a \in [0, b]$ and h is defined in the interval $[0, b]$, then its integral from a to b is defined by

$$\int_a^b h(s)d_qs = \int_0^b h(s)d_qs - \int_0^a h(s)d_qs. \tag{2.8}$$

Similar to derivatives, an operator I_q^k is given by

$$(I_q^0 h)(x) = h(x) \text{ and } (I_q^k h)(x) = I_q(I_q^{k-1}h)(x), \quad k \in \mathbb{N}. \tag{2.9}$$

The fundamental theorem of calculus applies to these operators D_q and I_q, i.e.,

$$(D_q I_q h)(x) = h(x), \tag{2.10}$$

and if h is continuous at $x = 0$, then

$$(I_q D_q h)(x) = h(x) - h(0). \tag{2.11}$$

Definition 2.1 *Let $\nu \geq 0$ and h be a function defined on $[0, T]$. The fractional q-integral of Riemann-Liouville type is given by $(I_q^0 h)(x) = h(x)$ and*

$$(I_q^\nu h)(x) = \frac{1}{\Gamma_q(\nu)}\int_0^x (x-qs)^{(\nu-1)}h(s)d_qs, \quad \nu > 0, \quad x \in [0, T]. \tag{2.12}$$

Definition 2.2 *The fractional q-derivative of the Riemann-Liouville type of order $\nu \geq 0$ is defined by $(D_q^0 h)(x) = h(x)$ and*

$$(D_q^\nu h)(x) = (D_q^{[\nu]} I_q^{[\nu]-\nu} h)(x), \quad \nu > 0, \tag{2.13}$$

where $[\nu]$ is the smallest integer greater than or equal to ν.

Definition 2.3 *The fractional q-derivative of the Caputo type of order $\alpha \geq 0$ is defined by*

$$({}^cD_q^\alpha h)(x) = (I_q^{[\alpha]-\alpha} D_q^{[\alpha]} h)(x), \quad \alpha > 0, \tag{2.14}$$

provided that $D_q^{[\alpha]}h(x)$ exists on $[0, T]$.

Lemma 2.4 *[27] Let $\alpha, \beta \geq 0$ and f be a function defined in $[0, T]$. Then, the following formulas hold:*

(1) $(I_q^\beta I_q^\alpha f)(x) = (I_q^{\alpha+\beta} f)(x)$,

(2) $(D_q^\alpha I_q^\alpha f)(x) = f(x)$.

Lemma 2.5 *[28] Let $\alpha > 0$ and ν be a positive integer. Then, the following equality holds:*

$$(I_q^\alpha D_q^\nu f)(x) = (D_q^\nu I_q^\alpha f)(x) - \sum_{k=0}^{\nu-1} \frac{x^{\alpha-\nu+k}}{\Gamma_q(\alpha+k-\nu+1)}(D_q^k f)(0). \tag{2.15}$$

Lemma 2.6 *[29] Let $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$ and $a < x$. Then, the following formula is valid*

$$(I_q^{\alpha\, c}D_q^\alpha f)(x) = f(x) - \sum_{k=0}^{[\alpha]-1} \frac{(D_q^k f)(a)}{\Gamma_q(k+1)}(x-a)^{(k)}. \tag{2.16}$$

Definition 2.7 *For any $m, n > 0$,*

$$B_q(m,n) = \int_0^1 u^{(m-1)}(1-qu)^{(n-1)} d_q u \tag{2.17}$$

is called the q-beta function.

The expression of q-beta function in terms of the q-gamma function can be written as

$$B_q(m,n) = \frac{\Gamma_q(m)\Gamma_q(n)}{\Gamma_q(m+n)}.$$

Lemma 2.8 *[2] Let $\alpha, \beta, \gamma > 0$ and $0 < p, q, r < 1$. Then we have*

$$\begin{aligned} &\int_0^\eta \int_0^x \int_0^y (\eta - px)^{(\alpha-1)}(x-qy)^{(\beta-1)}(y-rz)^{(\gamma-1)} d_r z d_q y d_p x \\ &\qquad = \frac{1}{[\gamma]_r} B_p(\alpha, \beta+\gamma+1) B_q(\beta, \gamma+1)\eta^{\alpha+\beta+\gamma}. \end{aligned} \tag{2.18}$$

3 An auxiliary lemma

Lemma 3.1 *Let $\beta, \gamma > 0$, $\lambda \in \mathbb{R}$ and $0 < p, q, r < 1$. Then, for $y \in C([0,T], \mathbb{R})$, the unique solution of boundary value problem*

$${}^cD_q^\alpha x(t) = y(t), \quad t \in (0,T), \quad 1 < \alpha \le 2, \tag{3.1}$$

subject to the nonlocal fractional condition

$$x(\zeta) = g(x), \qquad \lambda I_p^\beta x(\eta) = I_r^\gamma x(\xi), \quad 0 < \zeta < \eta < \xi < T, \tag{3.2}$$

is given by

$$\begin{aligned} x(t) \ = \ & \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} y(s) d_q s \\ & - \frac{t-\zeta}{D\Gamma_q(\alpha)} \left\{ \frac{1}{\Gamma_r(\gamma)} \int_0^\xi \int_0^s (\xi - rs)^{(\gamma-1)}(s-qu)^{(\alpha-1)} y(u) d_q u d_r s \right. \\ & \left. - \frac{\lambda}{\Gamma_p(\beta)} \int_0^\eta \int_0^s (\eta - ps)^{(\beta-1)}(s-qu)^{(\alpha-1)} y(u) d_q u d_p s \right\} \\ & - \frac{t\Psi - \Omega}{D\Gamma_q(\alpha)} \int_0^\zeta (\zeta - qs)^{\alpha-1} y(s) d_q s + \frac{t\Psi - \Omega}{D} g(x), \end{aligned} \tag{3.3}$$

where

$$D = \zeta\Psi - \Omega \neq 0, \tag{3.4}$$

and

$$\Omega = \frac{\lambda \eta^{\beta+1}}{\Gamma_p(\beta+2)} - \frac{\xi^{\gamma+1}}{\Gamma_r(\gamma+2)}, \qquad \Psi = \frac{\lambda\eta^\beta}{\Gamma_p(\beta+1)} - \frac{\xi^\gamma}{\Gamma_r(\gamma+1)}. \tag{3.5}$$

Proof. From $1 < \alpha \le 2$, we let $n = 2$. Using the Definition 2.3 and Lemma 2.4, the equation (3.1) can be expressed as

$$(I_q^\alpha I_q^{[\alpha]-\alpha} D_q^{[\alpha]} x)(t) = (I_q^\alpha y)(t).$$

From Lemma 2.6, we have

$$x(t) = c_1 t + c_2 + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} y(s) d_q s \tag{3.6}$$

for some constants $c_1, c_2 \in \mathbb{R}$. It follows from the first condition of (3.2) that

$$c_1 \zeta + c_2 = g(x) - \int_0^\zeta \frac{(\zeta-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} y(s) d_q s. \tag{3.7}$$

Applying the Riemann-Liouville fractional p-integral of order $\beta > 0$ for (3.6) we have

$$\begin{aligned} I_p^\beta x(t) &= \int_0^t \frac{(t-ps)^{(\beta-1)}}{\Gamma_p(\beta)} \left(c_1 s + c_2 + \int_0^s \frac{(s-qu)^{(\alpha-1)}}{\Gamma_q(\alpha)} y(u) d_q u \right) d_p s \\ &= \frac{c_1}{\Gamma_p(\beta)} \int_0^t (t-ps)^{(\beta-1)} s d_p s + \frac{c_2}{\Gamma_p(\beta)} \int_0^t (t-ps)^{(\beta-1)} d_p s \\ &\quad + \frac{1}{\Gamma_p(\beta)\Gamma_q(\alpha)} \int_0^t \int_0^s (t-ps)^{(\beta-1)} (s-qu)^{(\alpha-1)} y(u) d_q u d_p s \\ &= c_1 \frac{t^{\beta+1}}{\Gamma_p(\beta+2)} + c_2 \frac{t^\beta}{\Gamma_p(\beta+1)} \\ &\quad + \frac{1}{\Gamma_p(\beta)\Gamma_q(\alpha)} \int_0^t \int_0^s (t-ps)^{(\beta-1)} (s-qu)^{(\alpha-1)} y(u) d_q u d_p s, \end{aligned}$$

since

$$\begin{aligned} \int_0^t (t-ps)^{(\beta-1)} \, s \, d_p s &= (1-q)t \sum_{n=0}^\infty q^n (t-qtq^n)^{(\beta-1)} t q^n \\ &= (1-q) t^{\beta+1} \sum_{n=0}^\infty q^n (1-qq^n)^{(\beta-1)} q^n \\ &= t^{\beta+1} \int_0^1 (1-qs)^{(\beta-1)} s d_p s = t^{\beta+1} B_p(\beta, 2) = t^{\beta+1} \frac{\Gamma_p(\beta)}{\Gamma_p(\beta+2)}, \end{aligned}$$

with $\Gamma_p(2) = 1$.

In particular, we have

$$\begin{aligned} I_p^\beta x(\eta) &= c_1 \frac{\eta^{\beta+1}}{\Gamma_p(\beta+2)} + c_2 \frac{\eta^\beta}{\Gamma_p(\beta+1)} \\ &\quad + \frac{1}{\Gamma_p(\beta)\Gamma_q(\alpha)} \int_0^\eta \int_0^s (\eta-ps)^{(\beta-1)} (s-qu)^{(\alpha-1)} y(u) d_q u d_p s. \end{aligned} \tag{3.8}$$

Using the Riemann-Liouville fractional r-integral of order $\gamma > 0$ and repeating the above process, we get

$$\begin{aligned} I_r^\gamma x(\xi) &= c_1 \frac{\xi^{\gamma+1}}{\Gamma_r(\gamma+2)} + c_2 \frac{\xi^\gamma}{\Gamma_r(\gamma+1)} \\ &\quad + \frac{1}{\Gamma_r(\gamma)\Gamma_q(\alpha)} \int_0^\xi \int_0^s (\xi-rs)^{(\gamma-1)} (s-qu)^{(\alpha-1)} y(u) d_q u d_r s. \end{aligned} \tag{3.9}$$

The second nonlocal condition of (3.2) implies

$$c_1 \Omega + c_2 \Psi = \frac{1}{\Gamma_r(\gamma)\Gamma_q(\alpha)} \int_0^\xi \int_0^s (\xi-rs)^{(\gamma-1)} (s-qu)^{(\alpha-1)} y(u) d_q u d_r s$$

$$-\frac{\lambda}{\Gamma_p(\beta)\Gamma_q(\alpha)}\int_0^\eta\int_0^s(\eta-ps)^{(\beta-1)}(s-qu)^{(\alpha-1)}y(u)d_qud_ps. \tag{3.10}$$

By solving the system of equations (3.7), (3.10) we find

$$\begin{aligned} c_1 &= \frac{\Psi}{D}\left(g(x)-\int_0^\zeta\frac{(\zeta-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}y(s)d_qs\right) \\ &\quad -\frac{1}{D\Gamma_r(\gamma)\Gamma_q(\alpha)}\int_0^\xi\int_0^s(\xi-rs)^{(\gamma-1)}(s-qu)^{(\alpha-1)}y(u)d_qud_rs \\ &\quad +\frac{\lambda}{D\Gamma_p(\beta)\Gamma_q(\alpha)}\int_0^\eta\int_0^s(\eta-ps)^{(\beta-1)}(s-qu)^{(\alpha-1)}y(u)d_qud_ps, \end{aligned}$$

and

$$\begin{aligned} c_2 &= -\frac{\Omega}{D}\left(g(x)-\int_0^\zeta\frac{(\zeta-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}y(s)d_qs\right) \\ &\quad +\frac{\zeta}{D\Gamma_r(\gamma)\Gamma_q(\alpha)}\int_0^\xi\int_0^s(\xi-rs)^{(\gamma-1)}(s-qu)^{(\alpha-1)}y(u)d_qud_rs \\ &\quad -\frac{\lambda\zeta}{D\Gamma_p(\beta)\Gamma_q(\alpha)}\int_0^\eta\int_0^s(\eta-ps)^{(\beta-1)}(s-qu)^{(\alpha-1)}y(u)d_qud_ps. \end{aligned}$$

Substituting the values of c_1 and c_2 in (3.6), we get the desired result in (3.3). □

4 Existence results for single-valued problem (1.1)

In this section, we denote by $\mathcal{C}=C([0,T],\mathbb{R})$ the Banach space of all continuous functions from $[0,T]$ to $\mathbb{R}$ endowed with the norm defined by $\|x\|=\sup_{t\in[0,T]}|x(t)|$. In view of Lemma 3.1, we define an operator $\mathcal{P}:\mathcal{C}\to\mathcal{C}$ by

$$\begin{aligned} (\mathcal{P}x)(t) &= \int_0^t\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}f(s,x(s),I_z^\delta x(s))d_qs \\ &\quad -\frac{t-\zeta}{D\Gamma_q(\alpha)}\left\{\frac{1}{\Gamma_r(\gamma)}\int_0^\xi\int_0^s(\xi-rs)^{(\gamma-1)}(s-qu)^{(\alpha-1)}f(u,x(u),I_z^\delta x(u))d_qud_rs\right. \\ &\quad \left.-\frac{\lambda}{\Gamma_p(\beta)}\int_0^\eta\int_0^s(\eta-ps)^{(\beta-1)}(s-qu)^{(\alpha-1)}f(u,x(u),I_z^\delta x(u))d_qud_ps\right\} \\ &\quad -\frac{t\Psi-\Omega}{D\Gamma_q(\alpha)}\int_0^\zeta(\zeta-qs)^{\alpha-1}f(s,x(s),I_z^\delta x(s))d_qs+\frac{t\Psi-\Omega}{D}g(x), \end{aligned} \tag{4.1}$$

where $D\neq0$ is defined by (3.4) and Ω,Ψ are defined by (3.5). It should be noticed that problem (1.1) has solutions if and only if the operator $\mathcal{P}$ has fixed points.

Let us define $\mathcal{P}_{1,2}:\mathcal{C}\to\mathcal{C}$ by

$$\begin{aligned} (\mathcal{P}_1x)(t) &= \int_0^t\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}f(s,x(s),I_z^\delta x(s))d_qs \\ &\quad -\frac{t-\zeta}{D\Gamma_q(\alpha)}\left\{\frac{1}{\Gamma_r(\gamma)}\int_0^\xi\int_0^s(\xi-rs)^{(\gamma-1)}(s-qu)^{(\alpha-1)}f(u,x(u),I_z^\delta x(u))d_qud_rs\right. \\ &\quad \left.-\frac{\lambda}{\Gamma_p(\beta)}\int_0^\eta\int_0^s(\eta-ps)^{(\beta-1)}(s-qu)^{(\alpha-1)}f(u,x(u),I_z^\delta x(u))d_qud_ps\right\} \\ &\quad -\frac{t\Psi-\Omega}{D\Gamma_q(\alpha)}\int_0^\zeta(\zeta-qs)^{\alpha-1}f(s,x(s),I_z^\delta x(s))d_qs, \qquad t\in[0,T], \end{aligned} \tag{4.2}$$

and

$$(\mathcal{P}_2x)(t) = \frac{t\Psi - \Omega}{D} g(x), \qquad t \in [0,T]. \tag{4.3}$$

Clearly

$$(\mathcal{P}x)(t) = (\mathcal{P}_1x)(t) + (\mathcal{P}_2x)(t), \qquad t \in [0,T]. \tag{4.4}$$

For convenience we introduce the notations:

$$\begin{aligned} p_0 : \quad &= \frac{T^\alpha}{\Gamma_q(\alpha+1)} + \frac{T+\zeta}{|D|\Gamma_r(\gamma)} \frac{\xi^{\alpha+\gamma} B_r(\gamma, \alpha+1)}{\Gamma_q(\alpha+1)} \\ &+ \frac{|\lambda|(T+\zeta)}{|D|\Gamma_p(\beta)} \frac{\eta^{\alpha+\beta} B_p(\beta, \alpha+1)}{\Gamma_q(\alpha+1)} + \frac{T|\Psi| + |\Omega|}{D\Gamma_q(\alpha)} \frac{\zeta^\alpha}{\Gamma_q(\alpha+1)}, \end{aligned} \tag{4.5}$$

$$\begin{aligned} q_0 : \quad &= \frac{T^{\alpha+\delta} B_q(\alpha, \delta+1)}{\Gamma_q(\alpha)\Gamma_z(\delta+1)} + \frac{T+\zeta}{|D|\Gamma_r(\gamma)} \frac{\xi^{\alpha+\gamma+\delta} B_q(\alpha, \delta+1) B_r(\gamma, \alpha+\delta+1)}{\Gamma_q(\alpha)\Gamma_z(\delta+1)} \\ &+ \frac{|\lambda|(T+\zeta)}{|D|\Gamma_p(\beta)} \frac{\eta^{\alpha+\beta+\delta} B_q(\alpha, \delta+1) B_p(\beta, \alpha+\delta+1)}{\Gamma_q(\alpha)\Gamma_z(\delta+1)} \\ &+ \frac{T|\Psi| + |\Omega|}{D\Gamma_q(\alpha)} \frac{\zeta^{\alpha+\delta} B_q(\alpha, \delta+1)}{\Gamma_q(\alpha)\Gamma_z(\delta+1)}, \end{aligned} \tag{4.6}$$

and

$$k_0 := \frac{T|\Psi| + |\Omega|}{|D|}. \tag{4.7}$$

Theorem 4.1 *Let $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that*

(A_1) $|f(t, w_1, w_2) - f(t, \bar{w}_1, \bar{w}_2)| \le L_1|w_1 - \bar{w}_1| + L_2|w_2 - \bar{w}_2|, \forall t \in [0,T],\ L_1 > 0,\ L_2 > 0,\ w_1, \bar{w}_1, w_2, \bar{w}_2 \in \mathbb{R}$*;*

(A_2) $g : C([0,T], \mathbb{R}) \to \mathbb{R}$ *is a continuous function satisfying the condition:*

$$|g(u) - g(v)| \le \ell \|u - v\|, \ \ \ell < k_0^{-1}, \quad \forall\ u, v \in C([0,T], \mathbb{R}),\ \ell > 0;$$

(A_3) $\kappa := L_1 p_0 + L_2 q_0 + \ell\, k_0 < 1$.

Then the boundary value problem (1.1) *has a unique solution.*

Proof. For $x, y \in \mathcal{C}$ and for each $t \in [0,T]$, from the definition of $\mathcal{P}$, assumptions (A_1), (A_2) and Lemma 2.8, we obtain

$$\begin{aligned} |(\mathcal{P}x)(t) - (\mathcal{P}y)(t)| \quad \le \quad & \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s, x(s), I_z^\delta x(s)) - f(s, y(s), I_z^\delta y(s))| d_q s \\ & + \frac{T+\zeta}{|D|\Gamma_q(\alpha)} \Bigg\{ \frac{1}{\Gamma_r(\gamma)} \int_0^\xi \int_0^s (\xi - rs)^{(\gamma-1)} (s-qu)^{(\alpha-1)} \times \\ & \times |f(u, x(u), I_z^\delta x(u)) - f(u, y(u), I_z^\delta y(u))| d_q u d_p s \\ & + \frac{|\lambda|}{\Gamma_p(\beta)} \int_0^\eta \int_0^s (\eta - ps)^{(\beta-1)} (s-qu)^{(\alpha-1)} \times \\ & \times |f(u, x(u), I_z^\delta x(u)) - f(u, y(u), I_z^\delta y(u))| d_q u d_p s \Bigg\} \\ & + \frac{T|\Psi| + |\Omega|}{D\Gamma_q(\alpha)} \int_0^\zeta (\zeta - qs)^{\alpha-1} |f(s, x(s), I_z^\delta x(s)) - f(s, y(s), I_z^\delta y(s))| d_q s \\ & + \frac{T|\Psi| + |\Omega|}{|D|} |g(x) - g(y)| \end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\left(L_1\|x-y\|+L_2\|x-y\|\int_0^s \frac{(s-zv)^{(\delta-1)}}{\Gamma_z(\delta)}d_zv\right)d_qs\\
&\quad+\frac{T+\zeta}{|D|\Gamma_r(\gamma)\Gamma_q(\alpha)}\int_0^\xi\int_0^s(\xi-rs)^{(\gamma-1)}(s-qu)^{(\alpha-1)}\\
&\quad\times\left(L_1\|x-y\|+L_2\|x-y\|\int_0^u \frac{(u-zv)^{(\delta-1)}}{\Gamma_z(\delta)}d_zv\right)d_qud_rs\\
&\quad+\frac{|\lambda|(T+\zeta)}{|D|\Gamma_p(\beta)\Gamma_q(\alpha)}\int_0^\eta\int_0^s(\eta-ps)^{(\beta-1)}(s-qu)^{(\alpha-1)}\\
&\quad\times\left(L_1\|x-y\|+L_2\|x-y\|\int_0^u \frac{(u-zv)^{(\delta-1)}}{\Gamma_z(\delta)}d_zv\right)d_qud_ps\\
&\quad+\frac{T|\Psi|+|\Omega|}{D\Gamma_q(\alpha)}\int_0^\zeta(\zeta-qs)^{\alpha-1}\left(L_1\|x-y\|+L_2\|x-y\|\int_0^s \frac{(s-zv)^{(\delta-1)}}{\Gamma_z(\delta)}d_zv\right)d_qs\\
&\quad+\frac{T|\Psi|+|\Omega|}{|D|}|g(x)-g(y)|\\
&\leq \|x-y\|\left\{\frac{T^\alpha}{\Gamma_q(\alpha+1)}L_1+\frac{T^{\alpha+\delta}B_q(\alpha,\delta+1)}{\Gamma_q(\alpha)\Gamma_z(\delta+1)}L_2\right.\\
&\quad+\frac{T+\zeta}{|D|\Gamma_r(\gamma)}\left[\frac{\xi^{\alpha+\gamma}B_r(\gamma,\alpha+1)L_1}{\Gamma_q(\alpha+1)}+\frac{\xi^{\alpha+\gamma+\delta}B_q(\alpha,\delta+1)B_r(\gamma,\alpha+\delta+1)L_2}{\Gamma_q(\alpha)\Gamma_z(\delta+1)}\right]\\
&\quad+\frac{|\lambda|(T+\zeta)}{|D|\Gamma_p(\beta)}\left[\frac{\eta^{\alpha+\beta}B_p(\beta,\alpha+1)L_1}{\Gamma_q(\alpha+1)}+\frac{\eta^{\alpha+\beta+\delta}B_q(\alpha,\delta+1)B_p(\beta,\alpha+\delta+1)L_2}{\Gamma_q(\alpha)\Gamma_z(\delta+1)}\right]\\
&\quad\left.+\frac{T|\Psi|+|\Omega|}{|D|\Gamma_q(\alpha)}\left[\frac{\zeta^\alpha}{\Gamma_q(\alpha+1)}L_1+\frac{\zeta^{\alpha+\delta}B_q(\alpha,\delta+1)}{\Gamma_q(\alpha)\Gamma_z(\delta+1)}L_2\right]\right\}+\frac{T|\Psi|+|\Omega|}{|D|}\ell\|x-y\|\\
&= (L_1p_0+L_2q_0+k_0\ell)\|x-y\|.
\end{aligned}
$$

Hence

$$\|(\mathcal{P}x)-(\mathcal{P}y)\| \leq \kappa\|x-y\|.$$

As $\kappa<1$ by (A_3), the operator $\mathcal{P}$ is a contraction map from the Banach space $\mathcal{C}$ into itself. Hence the conclusion of the theorem follows by the contraction mapping principle (Banach fixed point theorem). □

Example 4.2 *Consider the following nonlocal problem of q-integrodifference equation*

$$
\begin{cases}
{}^cD_{\frac{1}{2}}^{\frac{3}{2}}x(t)=\dfrac{e^{-3t}}{2(t+\sqrt{5})^2}\cdot\dfrac{|x|}{1+|x|}+\dfrac{1}{2}I_{\frac{3}{8}}^{\frac{5}{2}}x(t)+\dfrac{3}{2}, \quad t\in(0,1/2),\\
x\left(\dfrac{1}{8}\right)=\dfrac{1}{15}x\left(\dfrac{3}{8}\right)+\dfrac{2}{3}, \quad \dfrac{2}{5}I_{\frac{2}{5}}^{\frac{7}{3}}x\left(\dfrac{1}{4}\right)=I_{\frac{2}{3}}^{\frac{5}{4}}x\left(\dfrac{1}{3}\right).
\end{cases} \tag{4.8}
$$

Here, $\alpha=3/2$, $q=1/2$, $\delta=5/2$, $z=3/8$, $T=1/2$, $\zeta=1/8$, $\lambda=2/5$, $\beta=7/3$, $p=2/5$, $\eta=1/4$, $\gamma=5/4$, $r=2/3$, $\xi=1/3$, $g(x)=(1/15)x+(2/3)$ and $f(t,x,I_z^\delta x)=(e^{-3t}|x|)/(2(t+\sqrt{5})^2(1+|x|))+(1/2)I_{3/8}^{5/2}x+(3/2)$. By using the Maple program, we find that $\Omega=-0.04119212$, $\Psi=-0.22035718$, $D=0.01364747$, $p_0=1.57981377$, $q_0=0.02586708$ and $k_0=11.09148475$.

As $|f(t,w_1,w_2)-f(t,\bar{w}_1,\bar{w}_2)|\leq(1/10)|w_1-\bar{w}_1|+(1/2)|w_2-\bar{w}_2|$ and $|g(x)-g(y)|\leq(1/15)|x-y|$, therefore, (A_1) and (A_2) are satisfied with $L_1=1/10$, $L_2=1/2$ and $\ell=1/15<1/11.09148475=k_0^{-1}$, respectively. Hence $\kappa=L_1p_0+L_2q_0+\ell k_0=0.91034723<1$. By the conclusion of Theorem 4.1, the nonlocal problem (4.8) has a unique solution on $[0,1/2]$.

Our next existence result relies on a fixed point theorem due to O'Regan in [25].

Lemma 4.3 *Denote by U an open set in a closed, convex set C of a Banach space E. Assume $0 \in U$. Also assume that $F(\bar{U})$ is bounded and that $F : \bar{U} \to C$ is given by $F = F_1 + F_2$, in which $F_1 : \bar{U} \to E$ is continuous and completely continuous and $F_2 : \bar{U} \to E$ is a nonlinear contraction (i.e., there exists a nonnegative nondecreasing function $\phi : [0, \infty) \to [0, \infty)$ satisfying $\phi(z) < z$ for $z > 0$, such that $\|F_2(x) - F_2(y)\| \le \phi(\|x - y\|)$ for all $x, y \in \bar{U}$). Then, either*

(C1) F has a fixed point $u \in \bar{U}$; or

(C2) there exist a point $u \in \partial U$ and $\lambda \in (0, 1)$ with $u = \lambda F(u)$, where $\bar{U}$ and ∂U, respectively, represent the closure and boundary of U.

Let

$$\Omega_r = \{x \in C([0, T], \mathbb{R}) : \|x\| < r\},$$

and denote the maximum number by

$$M_r = \max\{|f(t, x, y)| : (t, x, y) \in [0, T] \times [-r, r] \times [-r, r]\}.$$

Theorem 4.4 *Let $f : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that (A_2) holds. In addition we assume that*

(A_4) $g(0) = 0$;

(A_5) *there exist a continuous nondecreasing function $\psi : [0, \infty) \to (0, \infty)$ and a function $p \in C([0, T], \mathbb{R}^+)$ such that*

$$|f(t, w_1, w_2)| \le p(t)\psi(|w_1|) + |w_2| \quad \text{for each } (t, w_1, w_2) \in [0, T] \times \mathbb{R}^2;$$

(A_6) $\sup\limits_{r \in (0,\infty)} \dfrac{r}{p_0 \psi(r)\|p\|} > \dfrac{1}{1 - k_0 \ell - q_0}$, *where p_0, q_0 and k_0 are defined in (4.5), (4.6) and (4.7), respectively, and $k_0 \ell + q_0 < 1$.*

Then the boundary value problem (1.1) *has at least one solution on* $[0, T]$.

Proof. By the assumption (A_6), there exists a number $r_0 > 0$ such that

$$\frac{r_0}{p_0 \psi(r_0)\|p\|} > \frac{1}{1 - k_0 \ell - q_0}. \tag{4.9}$$

We shall show that the operators $\mathcal{P}_1$ and $\mathcal{P}_2$ defined by (4.2) and (4.3), respectively, satisfy all the conditions of Lemma 4.3.

Step 1. *The operator $\mathcal{P}_1$ is continuous and completely continuous.* We first show that $\mathcal{P}_1(\bar{\Omega}_{r_0})$ is bounded. For any $x \in \bar{\Omega}_{r_0}$, we have

$$\begin{aligned}
\|\mathcal{P}_1 x\| \le\ & \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s, x(s), I_z^\delta x(s))| d_q s \\
& + \frac{T + \zeta}{|D|\Gamma_r(\gamma)\Gamma_q(\alpha)} \int_0^\xi \int_0^s (\xi - rs)^{(\gamma-1)} (s - qu)^{(\alpha-1)} |f(u, x(u), I_z^\delta x(u))| d_q u d_r s \\
& + \frac{|\lambda|(T + \zeta)}{|D|\Gamma_p(\beta)\Gamma_q(\alpha)} \int_0^\eta \int_0^s (\eta - ps)^{(\beta-1)} (s - qu)^{(\alpha-1)} |f(u, x(u), I_z^\delta x(u))| d_q u d_p s \\
& + \frac{T|\Psi| + |\Omega|}{D\Gamma_q(\alpha)} \int_0^\zeta (\zeta - qs)^{\alpha-1} |f(s, x(s), I_z^\delta x(s))| d_q s \\
\le\ & \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \left(p(s)\psi(\|x\|) + \|x\| \int_0^s \frac{(s - zv)^{(\delta-1)}}{\Gamma_z(\delta)} d_z v \right) d_q s \\
& + \frac{T + \zeta}{|D|\Gamma_r(\gamma)\Gamma_q(\alpha)} \int_0^\xi \int_0^s (\xi - rs)^{(\gamma-1)} (s - qu)^{(\alpha-1)}
\end{aligned}$$

$$
\begin{aligned}
&\times\left(p(u)\psi(\|x\|)+\|x\|\int_0^u\frac{(u-zv)^{(\delta-1)}}{\Gamma_z(\delta)}d_zv\right)d_qud_rs\\
&+\frac{|\lambda|(T+\zeta)}{|D|\Gamma_p(\beta)\Gamma_q(\alpha)}\int_0^\eta\int_0^s(\eta-ps)^{(\beta-1)}(s-qu)^{(\alpha-1)}\\
&\times\left(p(u)\psi(\|x\|)+\|x\|\int_0^u\frac{(u-zv)^{(\delta-1)}}{\Gamma_z(\delta)}d_zv\right)d_qud_ps\\
&+\frac{T|\Psi|+|\Omega|}{D\Gamma_q(\alpha)}\int_0^\zeta(\zeta-qs)^{\alpha-1}\left(p(s)\psi(\|x\|)+\|x\|\int_0^s\frac{(s-zv)^{(\delta-1)}}{\Gamma_z(\delta)}d_zv\right)d_qs\\
\le\;&\frac{\|p\|\psi(r_0)T^\alpha}{\Gamma_q(\alpha+1)}+\frac{r_0T^{\alpha+\delta}B_q(\alpha,\delta+1)}{\Gamma_q(\alpha)\Gamma_z(\delta+1)}\\
&\frac{T+\zeta}{|D|\Gamma_r(\gamma)}\left(\frac{\xi^{\alpha+\gamma}B_r(\gamma,\alpha+1)\|p\|\psi(r_0)}{\Gamma_q(\alpha+1)}+\frac{r_0\xi^{\alpha+\gamma+\delta}B_q(\alpha,\delta+1)B_r(\gamma,\alpha+\delta+1)}{\Gamma_q(\alpha)\Gamma_z(\delta+1)}\right)\\
&\frac{|\lambda|(T+\zeta)}{|D|\Gamma_p(\beta)}\left(\frac{\eta^{\alpha+\beta}B_p(\beta,\alpha+1)\|p\|\psi(r_0)}{\Gamma_q(\alpha+1)}+\frac{r_0\eta^{\alpha+\beta+\delta}B_q(\alpha,\delta+1)B_p(\beta,\alpha+\delta+1)}{\Gamma_q(\alpha)\Gamma_z(\delta+1)}\right)\\
&+\frac{T|\Psi|+|\Omega|}{D\Gamma_q(\alpha)}\left(\frac{\|p\|\psi(r_0)\zeta^\alpha}{\Gamma_q(\alpha+1)}+\frac{r_0\zeta^{\alpha+\delta}B_q(\alpha,\delta+1)}{\Gamma_q(\alpha)\Gamma_z(\delta+1)}\right)\\
=\;&\|p\|\psi(r_0)p_0+r_0q_0:=G.
\end{aligned}
$$

Thus the operator $\mathcal{P}_1(\bar{\Omega}_{r_0})$ is uniformly bounded. For any $t_1,t_2\in[0,T],t_1<t_2$, we have

$$
\begin{aligned}
&|(\mathcal{P}_1x)(t_2)-(\mathcal{P}_1x)(t_1)|\\
\le\;&\Bigg|\int_0^{t_2}\frac{(t_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|f(s,x(s),I_z^\delta x(s))|d_qs\\
&-\int_0^{t_1}\frac{(t_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|f(s,x(s),I_z^\delta x(s))|d_qs\Bigg|\\
&+\frac{|t_2-t_1|}{|D|\Gamma_r(\gamma)\Gamma_q(\alpha)}\int_0^\xi\int_0^s(\xi-rs)^{(\gamma-1)}(s-qu)^{(\alpha-1)}|f(u,x(u),I_z^\delta x(u))|d_qud_rs\\
&+\frac{|\lambda||t_2-t_1|}{|D|\Gamma_p(\beta)\Gamma_q(\alpha)}\int_0^\eta\int_0^s(\eta-ps)^{(\beta-1)}(s-qu)^{(\alpha-1)}|f(u,x(u),I_z^\delta x(u))|d_qud_ps\\
&+\frac{|\Psi||t_2-t_1|}{|D|\Gamma_q(\alpha)}\int_0^\zeta(\zeta-qs)^{\alpha-1}|f(s,x(s),I_z^\delta x(s))|d_qs\\
\le\;&\int_0^{t_2}\frac{|(t_2-qs)^{(\alpha-1)}-(t_1-qs)^{(\alpha-1)}|}{\Gamma_q(\alpha)}\left(p(s)\psi(\|x\|)+\|x\|\int_0^s\frac{(s-zv)^{(\delta-1)}}{\Gamma_z(\delta)}d_zv\right)d_qs\\
&+\int_{t_1}^{t_2}\frac{(t_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\left(p(s)\psi(\|x\|)+\|x\|\int_0^s\frac{(s-zv)^{(\delta-1)}}{\Gamma_z(\delta)}d_zv\right)d_qs\\
&+\frac{|t_2-t_1|}{|D|\Gamma_r(\gamma)\Gamma_q(\alpha)}\int_0^\xi\int_0^s(\xi-rs)^{(\gamma-1)}(s-qu)^{(\alpha-1)}\\
&\times\left(p(u)\psi(\|x\|)+\|x\|\int_0^u\frac{(u-zv)^{(\delta-1)}}{\Gamma_z(\delta)}d_zv\right)d_qud_rs\\
&+\frac{|\lambda||t_2-t_1|}{|D|\Gamma_p(\beta)\Gamma_q(\alpha)}\int_0^\eta\int_0^s(\eta-ps)^{(\beta-1)}(s-qu)^{(\alpha-1)}\\
&\times\left(p(u)\psi(\|x\|)+\|x\|\int_0^u\frac{(u-zv)^{(\delta-1)}}{\Gamma_z(\delta)}d_zv\right)d_qud_ps\\
&+\frac{|\Psi||t_2-t_1|}{|D|\Gamma_q(\alpha)}\left(p(s)\psi(\|x\|)+\|x\|\int_0^s\frac{(s-zv)^{(\delta-1)}}{\Gamma_z(\delta)}d_zv\right)d_qs,
\end{aligned}
$$

which is independent of x and tends to zero as $t_2-t_1\to0$. Thus, $\mathcal{P}_1$ is equicontinuous. Hence, by the

Arzelá-Ascoli Theorem, $\mathcal{P}_1(\bar{\Omega}_{r_0})$ is a relatively compact set. Now, let $x_n, y_n \subset \bar{\Omega}_{r_0}$ with $\|x_n - x\| \to 0$ and $\|y_n - y\| \to 0$. Then the limits $\|x_n(t) - x(t)\| \to 0$ and $\|y_n(t) - y(t)\| \to 0$ are uniformly valid on $[0,T]$. From the uniform continuity of $f(t,x,y)$ on the compact set $[0,T] \times [-r_0, r_0] \times [-r_0, r_0]$, it follows that $\|f(t, x_n(t), y_n(t)) - f(t, x(t), y(t))\| \to 0$ is uniformly valid on $[0,T]$. Hence $\|\mathcal{P}_1 x_n - \mathcal{P}_1 x\| \to 0$ as $n \to \infty$ which proves the continuity of $\mathcal{P}_1$. This completes the proof of Step 1.

Step 2. *The operator* $\mathcal{P}_2 : \bar{\Omega}_{r_0} \to C([0,T],\mathbb{R})$ *is contractive.* This is a consequence of (A_2). Indeed, for $x, y \in C([0,T],\mathbb{R})$, we have

$$\begin{aligned} |\mathcal{P}_2 x(t) - \mathcal{P}_2 y(t)| &= \left| \frac{t\Psi - \Omega}{D} \right| |g(x) - g(y)| \\ &\le \frac{T|\Psi| + |\Omega|}{|D|} |g(x) - g(y)|, \\ &\le k_0 \ell \|x - y\|, \end{aligned}$$

which, on taking supremum over $t \in [0,T]$, yields

$$\|\mathcal{P}_2 x - \mathcal{P}_2 y\| \le L_0 \|x - y\|, \quad L_0 = k_0 \ell < 1.$$

This shows that $\mathcal{P}_2$ is a contraction as $L_0 < 1$.

Step 3. *The set* $\mathcal{P}(\bar{\Omega}_{r_0})$ *is bounded.* The assumptions (A_2) and (A_4) imply that

$$\|\mathcal{P}_2(x)\| \le k_0 \ell r_0,$$

for any $x \in \bar{\Omega}_{r_0}$. This, with the boundedness of the set $\mathcal{P}_1(\bar{\Omega}_{r_0})$ implies that the set $\mathcal{P}(\bar{\Omega}_{r_0})$ is bounded.

Step 4. *Finally, it will be shown that the case (C2) in Lemma 4.3 does not hold.* On the contrary, we suppose that (C2) holds. Then, we have that there exist $\theta \in (0,1)$ and $x \in \partial\Omega_{r_0}$ such that $x = \theta \mathcal{P} x$. So, we have $\|x\| = r_0$ and

$$\begin{aligned} x(t) &= \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, x(s), I_z^\delta x(s)) d_q s \\ &- \frac{t - \zeta}{D\Gamma_q(\alpha)} \Bigg\{ \frac{1}{\Gamma_r(\gamma)} \int_0^\xi \int_0^s (\xi - rs)^{(\gamma-1)} (s - qu)^{(\alpha-1)} f(u, x(u), I_z^\delta x(u)) d_q u d_r s \\ &- \frac{\lambda}{\Gamma_p(\beta)} \int_0^\eta \int_0^s (\eta - ps)^{(\beta-1)} (s - qu)^{(\alpha-1)} f(u, x(u), I_z^\delta x(u)) d_q u d_p s \Bigg\} \\ &- \frac{t\Psi - \Omega}{D\Gamma_q(\alpha)} \int_0^\zeta (\zeta - qs)^{\alpha-1} f(s, x(s), I_z^\delta x(s)) d_q s + \frac{t\Psi - \Omega}{D} g(x), \quad t \in [0,T]. \end{aligned}$$

Using the assumptions $(A_4) - (A_6)$, we get

$$r_0 \le p_0 \psi(r_0) \|p\| + r_0 q_0 + k_0 \ell r_0.$$

Thus, we get a contradiction:

$$\frac{r_0}{p_0 \psi(r_0) \|p\|} \le \frac{1}{1 - k_0 \ell - q_0}.$$

Thus the operators $\mathcal{P}_1$ and $\mathcal{P}_2$ satisfy all the conditions of Lemma 4.3. Hence, the operator $\mathcal{P}$ has at least one fixed point $x \in \bar{\Omega}_{r_0}$, which is the solution of the problem (1.1). This completes the proof. □

Example 4.5 *Consider the following nonlocal problem of q-integrodifference equation*

$$\begin{cases} {}^c D_{\frac{1}{5}}^{\frac{7}{4}} x(t) = \dfrac{t^2 + 1}{35} \left(|x| + \dfrac{|x| + 1}{|x| + 2} \right) + I_{\frac{2}{7}}^{\frac{3}{4}} x, \quad t \in (0, 1/2), \\ x\left(\dfrac{1}{7}\right) = \dfrac{1}{12} \sin\left(x\left(\dfrac{1}{4}\right)\right), \quad \dfrac{1}{10} I_{\frac{1}{6}}^{\frac{4}{3}} x\left(\dfrac{3}{10}\right) = I_{\frac{1}{2}}^{\frac{4}{7}} x\left(\dfrac{2}{5}\right). \end{cases} \tag{4.10}$$

Here, $\alpha = 7/4$, $q = 1/5$, $\delta = 3/4$, $z = 2/7$, $T = 1/2$, $\zeta = 1/7$, $\lambda = 1/10$, $\beta = 4/3$, $p = 1/6$, $\eta = 3/10$, $\gamma = 4/7$, $r = 1/2$, $\xi = 2/5$, $g(x) = (1/12)\sin x$ and $f(t, x, I_z^\delta x) = ((t^2+1)/35)(|x|+((|x|+1)/(|x|+2)))+I_{2/7}^{3/4}x$. By using the Maple program, we find that $\Omega = -0.18824514$, $\Psi = -0.62150021$, $D = 0.09945940$, $p_0 = 0.92752573$, $q_0 = 0.37709650$ and $k_0 = 5.01707462$.

As $|g(x) - g(y)| \leq (1/12)|x - y|$ with $\ell = (1/12) < (1/5.01707462) = k_0^{-1}$ and $g(0) = 0$, therefore, (A_2) and (A_4) are satisfied, respectively. Since $|f(t, w_1, w_2)| = |((t^2 + 1)/35)(|w_1| + ((|w_1| + 1)/(|w_1| + 2))) + w_2| \leq ((t^2+1)/35)(w_1^2+3|w_1|+1)+|w_2|$, we choose $p(t) = (t^2+1)/35$ and $\psi(|w_1|) = w_1^2+3|w_1|+1$. We can show that

$$\sup_{r\in(0,\infty)} \frac{r}{p_0\psi(r)\|p\|} = 6.03756836 > 4.88247997 = \frac{1}{1 - k_0\ell - q_0}.$$

Therefore, by Theorem 4.4, the boundary value problem (4.10) has at least one solution on $[0, 1/2]$.

5 Existence results for multi-valued problem (1.2)

Let us recall some basic definitions on multi-valued maps [30, 31].

For a normed space $(X, \|\cdot\|)$, let $P_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, $P_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$, $P_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$, and $P_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$. A multi-valued map $G : X \to \mathcal{P}(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. The map G is bounded on bounded sets if $G(\mathbb{B}) = \cup_{x\in\mathbb{B}}G(x)$ is bounded in X for all $\mathbb{B} \in P_b(X)$ (i.e. $\sup_{x\in\mathbb{B}}\{\sup\{|y| : y \in G(x)\}\} < \infty$). G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $G(\mathcal{N}_0) \subseteq N$. G is said to be completely continuous if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in P_b(X)$. If the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, i.e., $x_n \to x_*$, $y_n \to y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$. G has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $FixG$. A multivalued map $G : [0; 1] \to P_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function

$$t \longmapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$

is measurable.

Definition 5.1 *A function $x \in AC^1([0, T], \mathbb{R})$ is a solution of the problem (1.2) if $x(\zeta) = g(x)$, $\lambda I_p^\beta x(\eta) = I_r^\gamma x(\xi)$, and there exists a function $f \in L^1([0, T], \mathbb{R})$ such that $f(t) \in F(t, x(t), I_z^\delta x(t))$ a.e. on $[0, T]$ and*

$$\begin{aligned} x(t) \; &= \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s) d_q s \\ &-\frac{t-\zeta}{D\Gamma_q(\alpha)}\left\{ \frac{1}{\Gamma_r(\gamma)} \int_0^\xi \int_0^s (\xi - rs)^{(\gamma-1)}(s-qu)^{(\alpha-1)} f(u) d_q u d_r s \right. \\ &\left. -\frac{\lambda}{\Gamma_p(\beta)} \int_0^\eta \int_0^s (\eta - ps)^{(\beta-1)}(s-qu)^{(\alpha-1)} f(u) d_q u d_p s \right\} \\ &-\frac{t\Psi - \Omega}{D\Gamma_q(\alpha)} \int_0^\zeta (\zeta - qs)^{\alpha-1} f(s) d_q s + \frac{t\Psi - \Omega}{D} g(x). \end{aligned} \tag{5.1}$$

Here $AC^1([0, T], \mathbb{R})$ will denote the space of functions $x : [0, T] \to \mathbb{R}$ that are absolutely continuous and whose first derivative is absolutely continuous.

Definition 5.2 *A multivalued map $F : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if*

(i) $t \longmapsto F(t, x, y)$ *is measurable for each* $x, y \in \mathbb{R}$;

(ii) $(x, y) \longmapsto F(t, x, y)$ *is upper semicontinuous for almost all* $t \in [0, T]$;

Further a Carathéodory function F is called L^1-Carathéodory if

(iii) *for each $\alpha > 0$, there exists $\varphi_\alpha \in L^1([0,T],\mathbb{R}^+)$ such that*

$$\|F(t,x,y)\| = \sup\{|v| : v \in F(t,x,y)\} \leq \varphi_\alpha(t)$$

for all $\|x\|, \|y\| \leq \alpha$ and for a.e. $t \in [0,T]$.

For each $x, y \in C([0,T],\mathbb{R})$, define the set of selections of F by

$$S_{F,x,y} := \{v \in L^1([0,T],\mathbb{R}) : v(t) \in F(t,x(t),y(t)) \text{ for a.e. } t \in [0,T]\}.$$

The following lemma will be used in the sequel.

Lemma 5.3 *([32]) Let X be a Banach space. Let $F : [0,T] \times \mathbb{R}^2 \to \mathcal{P}_{cp,c}(X)$ be an L^1- Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1([0,T],X,X)$ to $C([0,T],X,X)$. Then the operator*

$$\Theta \circ S_F : C([0,T],X,X) \to \mathcal{P}_{cp,c}(C([0,T],X,X)), \quad x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x,y})$$

is a closed graph operator in $C([0,T],X,X) \times C([0,T],X,X)$.

To prove our main result in this section, we use the following form of the Nonlinear Alternative for contractive maps [33, Corollary 3.8].

Theorem 5.4 *Let X be a Banach space, and D a bounded neighborhood of $0 \in X$. Let $Z_1 : X \to \mathcal{P}_{cp,c}(X)$ and $Z_2 : \bar{D} \to \mathcal{P}_{cp,c}(X)$ two multi-valued operators satisfying*

(a) Z_1 is contraction, and

(b) Z_2 is u.s.c and compact.

Then, if $G = Z_1 + Z_2$, either

(i) G has a fixed point in $\bar{D}$ or

(ii) there is a point $u \in \partial D$ and $\lambda \in (0,1)$ with $u \in \lambda G(u)$.

Theorem 5.5 *Assume that (A_2) holds. In addition we suppose that:*

(H_1) *$F : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ is L^1-Carathéodory multivalued map;*

(H_2) *there exists a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and a function $p \in C([0,T],\mathbb{R}^+)$ such that*

$$\|F(t,x,y)\|_{\mathcal{P}} := \sup\{|v| : v \in F(t,x,y)\} \leq p(t)\psi(\|x\|) + |y|$$

for each $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}$;

(H_3) *there exists a number $M > 0$ such that*

$$\frac{(1 - k_0\ell - q_0)M}{\psi(M)p_0\|p\|} > 1, \quad k_0\ell + q_0 < 1, \tag{5.2}$$

where p_0, q_0, k_0 are defined in (4.5), (4.6) and (4.7) respectively.

Then the boundary value problem (1.2) has at least one solution on $[0,T]$.

Proof. To transform the problem (1.2) to a fixed point, we introduce an operator $\mathcal{N}: C([0,T],\mathbb{R}) \longrightarrow \mathcal{P}(C([0,T],\mathbb{R}))$ defined by

$$\mathcal{N}(x) = \left\{ \begin{array}{l} h \in C([0,T],\mathbb{R}) : \\ h(t) = \left\{ \begin{array}{l} \displaystyle\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s) d_q s \\ \displaystyle -\frac{t-\zeta}{D\Gamma_q(\alpha)} \Bigg\{ \frac{1}{\Gamma_r(\gamma)} \int_0^\xi \int_0^s (\xi - rs)^{(\gamma-1)} (s-qu)^{(\alpha-1)} f(u) d_q u d_r s \\ \displaystyle -\frac{\lambda}{\Gamma_p(\beta)} \int_0^\eta \int_0^s (\eta - ps)^{(\beta-1)} (s-qu)^{(\alpha-1)} f(u) d_q u d_p s \Bigg\} \\ \displaystyle -\frac{t\Psi - \Omega}{D\Gamma_q(\alpha)} \int_0^\zeta (\zeta - qs)^{\alpha-1} f(s) d_q s + \frac{t\Psi - \Omega}{D} g(x), \end{array} \right. \end{array} \right\}$$

for $f \in S_{F,x}$.

Now, we define two operators $\mathcal{A} : C([0,T],\mathbb{R}) \longrightarrow C([0,T],\mathbb{R})$ by

$$\mathcal{A}x(t) = \frac{t\Psi - \Omega}{D} g(x), \tag{5.3}$$

and a multi-valued operator $\mathcal{B} : C([0,T],\mathbb{R}) \longrightarrow \mathcal{P}(C([0,T],\mathbb{R}))$ by

$$\mathcal{B}(x) = \left\{ \begin{array}{l} h \in C([0,T],\mathbb{R}) : \\ h(t) = \left\{ \begin{array}{l} \displaystyle\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s) d_q s \\ \displaystyle -\frac{t-\zeta}{D\Gamma_q(\alpha)} \Bigg\{ \frac{1}{\Gamma_r(\gamma)} \int_0^\xi \int_0^s (\xi - rs)^{(\gamma-1)} (s-qu)^{(\alpha-1)} f(u) d_q u d_r s \\ \displaystyle -\frac{\lambda}{\Gamma_p(\beta)} \int_0^\eta \int_0^s (\eta - ps)^{(\beta-1)} (s-qu)^{(\alpha-1)} f(u) d_q u d_p s \Bigg\} \\ \displaystyle -\frac{t\Psi - \Omega}{D\Gamma_q(\alpha)} \int_0^\zeta (\zeta - qs)^{\alpha-1} f(s) d_q s. \end{array} \right. \end{array} \right\} \tag{5.4}$$

Observe that $\mathcal{N} = \mathcal{A} + \mathcal{B}$. We shall show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 5.4 on $[0,T]$. The proof consists of several steps and claims.

Step 1: We show that $\mathcal{A}$ *is a contraction on* $C([0,T],\mathbb{R})$. The proof is similar to the one for the operator $\mathcal{Q}_2$ in Step 2 of Theorem 4.4.

Step 2: $\mathcal{B}$ *is compact and convex valued and it is completely continuous.* This will be established in several claims.

Claim I: $\mathcal{B}$ *maps bounded sets into bounded sets in* $C([0,T],\mathbb{R})$. Let $B_R = \{x \in C([0,T],\mathbb{R}) : \|x\| \le R\}$ be a bounded set in $C([0,T],\mathbb{R})$. Then, for each $h \in \mathcal{B}(x), x \in B_\rho$, there exists $f \in S_{F,x}$ such that

$$\begin{aligned} h(t) &= \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s) d_q s - \frac{t-\zeta}{D\Gamma_q(\alpha)} \Bigg\{ \frac{1}{\Gamma_r(\gamma)} \int_0^\xi \int_0^s (\xi - rs)^{(\gamma-1)} (s-qu)^{(\alpha-1)} f(u) d_q u d_r s \\ &\quad -\frac{\lambda}{\Gamma_p(\beta)} \int_0^\eta \int_0^s (\eta - ps)^{(\beta-1)} (s-qu)^{(\alpha-1)} f(u) d_q u d_p s \Bigg\} - \frac{t\Psi - \Omega}{D\Gamma_q(\alpha)} \int_0^\zeta (\zeta - qs)^{\alpha-1} f(s) d_q s. \end{aligned}$$

Then, for $t \in [0,T]$, we have

$$\begin{aligned} |h(t)| &\le \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s)| d_q s \\ &\quad + \frac{T+\zeta}{|D|\Gamma_q(\alpha)} \Bigg\{ \frac{1}{\Gamma_r(\gamma)} \int_0^\xi \int_0^s (\xi - rs)^{(\gamma-1)} (s-qu)^{(\alpha-1)} |f(u)| d_q u d_r s \end{aligned}$$

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$$
\begin{aligned}
&+\frac{|\lambda|(T+\zeta)}{\Gamma_p(\beta)}\int_0^\eta\int_0^s(\eta-ps)^{(\beta-1)}(s-qu)^{(\alpha-1)}|f(u)|d_qud_ps\Bigg\}\\
&+\frac{T|\Psi|+|\Omega|}{|D|\Gamma_q(\alpha)}\int_0^\zeta(\zeta-qs)^{\alpha-1}|f(s)|d_qs\\
\leq\ & \frac{\|p\|\psi(R)T^\alpha}{\Gamma_q(\alpha+1)}+\frac{RT^{\alpha+\delta}B_q(\alpha,\delta+1)}{\Gamma_q(\alpha)\Gamma_z(\delta+1)}\\
&+\frac{T+\zeta}{|D|\Gamma_r(\gamma)}\left(\frac{\xi^{\alpha+\gamma}B_r(\gamma,\alpha+1)\|p\|\psi(R)}{\Gamma_q(\alpha+1)}+\frac{R\xi^{\alpha+\gamma+\delta}B_q(\alpha,\delta+1)B_r(\gamma,\alpha+\delta+1)}{\Gamma_q(\alpha)\Gamma_z(\delta+1)}\right)\\
&+\frac{|\lambda|(T+\zeta)}{|D|\Gamma_p(\beta)}\left(\frac{\eta^{\alpha+\beta}B_p(\beta,\alpha+1)\|p\|\psi(R)}{\Gamma_q(\alpha+1)}+\frac{R\eta^{\alpha+\beta+\delta}B_q(\alpha,\delta+1)B_p(\beta,\alpha+\delta+1)}{\Gamma_q(\alpha)\Gamma_z(\delta+1)}\right)\\
&+\frac{T|\Psi|+|\Omega|}{|D|\Gamma_q(\alpha)}\left(\frac{\|p\|\psi(R)\zeta^\alpha}{\Gamma_q(\alpha+1)}+\frac{R\zeta^{\alpha+\delta}B_q(\alpha,\delta+1)}{\Gamma_q(\alpha)\Gamma_z(\delta+1)}\right).
\end{aligned}
$$

Thus,

$$\|h\|\leq\psi(R)p_0\|p\|+Rq_0.$$

Claim II: $\mathcal{B}$ *maps bounded sets into equi-continuous sets.* Let $t_1,t_2\in[0,T]$ with $t_1<t_2$ and $x\in B_R$. Then, for each $h\in\mathcal{B}(x)$, we obtain

$$
\begin{aligned}
|h(t_2)-h(t_1)| \leq\ & \left|\int_0^{t_2}\frac{(t_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}f(s)d_qs-\int_0^{t_1}\frac{(t_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}f(s)d_qs\right|\\
&+\frac{|t_2-t_1|}{|D|\Gamma_r(\gamma)\Gamma_q(\alpha)}\int_0^\xi\int_0^s(\xi-rs)^{(\gamma-1)}(s-qu)^{(\alpha-1)}|f(u)|d_qud_rs\\
&+\frac{|\lambda||t_2-t_1|}{|D|\Gamma_p(\beta)\Gamma_q(\alpha)}\int_0^\eta\int_0^s(\eta-ps)^{(\beta-1)}(s-qu)^{(\alpha-1)}|f(u)|d_qud_ps\\
&+\frac{|\Psi||t_2-t_1|}{|D|\Gamma_q(\alpha)}\int_0^\zeta(\zeta-qs)^{\alpha-1}|f(s)|d_qs\\
\leq\ & \int_0^{t_2}\frac{|(t_2-qs)^{(\alpha-1)}-(t_1-qs)^{(\alpha-1)}|}{\Gamma_q(\alpha)}\left(p(s)\psi(\|x\|)+\|x\|\int_0^s\frac{(s-zv)^{(\delta-1)}}{\Gamma_z(\delta)}d_zv\right)d_qs\\
&+\int_{t_1}^{t_2}\frac{(t_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\left(p(s)\psi(\|x\|)+\|x\|\int_0^s\frac{(s-zv)^{(\delta-1)}}{\Gamma_z(\delta)}d_zv\right)d_qs\\
&+\frac{|t_2-t_1|}{|D|\Gamma_r(\gamma)\Gamma_q(\alpha)}\int_0^\xi\int_0^s(\xi-rs)^{(\gamma-1)}(s-qu)^{(\alpha-1)}\\
&\times\left(p(u)\psi(\|x\|)+\|x\|\int_0^u\frac{(u-zv)^{(\delta-1)}}{\Gamma_z(\delta)}d_zv\right)d_qud_rs\\
&+\frac{|\lambda||t_2-t_1|}{|D|\Gamma_p(\beta)\Gamma_q(\alpha)}\int_0^\eta\int_0^s(\eta-ps)^{(\beta-1)}(s-qu)^{(\alpha-1)}\\
&\times\left(p(u)\psi(\|x\|)+\|x\|\int_0^u\frac{(u-zv)^{(\delta-1)}}{\Gamma_z(\delta)}d_zv\right)d_qud_ps\\
&+\frac{|\Psi||t_2-t_1|}{|D|\Gamma_q(\alpha)}\int_0^\zeta(\zeta-qs)^{\alpha-1}\left(p(s)\psi(\|x\|)+\|x\|\int_0^s\frac{(s-zv)^{(\delta-1)}}{\Gamma_z(\delta)}d_zv\right)d_qs.
\end{aligned}
$$

Obviously the right hand side of the above inequality tends to zero independently of $x\in B_\rho$ as $t_2-t_1\to 0$. Therefore it follows by the Ascoli-Arzelá theorem that $\mathcal{B}:C([0,T],\mathbb{R})\to\mathcal{P}(C([0,T],\mathbb{R}))$ is completely continuous.
Claim III: $\mathcal{B}$ *has a closed graph.* Let $x_n\to x_*,h_n\in\mathcal{B}(x_n)$ and $h_n\to h_*$. Then we need to show that

$h_* \in \mathcal{B}(x_*)$. Associated with $h_n \in \mathcal{B}(x_n)$, there exists $f_n \in S_{F,x_n}$ such that for each $t \in [0,T]$,

$$\begin{aligned} h_n(t) &= \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f_n(s)d_qs - \frac{t-\zeta}{D\Gamma_q(\alpha)}\left\{\frac{1}{\Gamma_r(\gamma)}\int_0^\xi\int_0^s(\xi-rs)^{(\gamma-1)}(s-qu)^{(\alpha-1)}f_n(u)d_qud_rs\right. \\ &\left. -\frac{\lambda}{\Gamma_p(\beta)}\int_0^\eta\int_0^s(\eta-ps)^{(\beta-1)}(s-qu)^{(\alpha-1)}f_n(u)d_qud_ps\right\} - \frac{t\Psi-\Omega}{D\Gamma_q(\alpha)}\int_0^\zeta(\zeta-qs)^{\alpha-1}f_n(s)d_qs. \end{aligned}$$

Thus it suffices to show that there exists $f_* \in S_{F,x_*}$ such that for each $t \in [0,T]$,

$$\begin{aligned} h_*(t) &= \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f_*(s)d_qs - \frac{t-\zeta}{D\Gamma_q(\alpha)}\left\{\frac{1}{\Gamma_r(\gamma)}\int_0^\xi\int_0^s(\xi-rs)^{(\gamma-1)}(s-qu)^{(\alpha-1)}f_*(u)d_qud_rs\right. \\ &\left. -\frac{\lambda}{\Gamma_p(\beta)}\int_0^\eta\int_0^s(\eta-ps)^{(\beta-1)}(s-qu)^{(\alpha-1)}f_*(u)d_qud_ps\right\} - \frac{t\Psi-\Omega}{D\Gamma_q(\alpha)}\int_0^\zeta(\zeta-qs)^{\alpha-1}f_*(s)d_qs. \end{aligned}$$

Let us consider the linear operator $\Theta : L^1([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$ given by

$$\begin{aligned} f \mapsto \Theta(f)(t) &= \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s)d_qs - \frac{t-\zeta}{D\Gamma_q(\alpha)}\left\{\frac{1}{\Gamma_r(\gamma)}\int_0^\xi\int_0^s(\xi-rs)^{(\gamma-1)}(s-qu)^{(\alpha-1)}f(u)d_qud_rs\right. \\ &\left. -\frac{\lambda}{\Gamma_p(\beta)}\int_0^\eta\int_0^s(\eta-ps)^{(\beta-1)}(s-qu)^{(\alpha-1)}f(u)d_qud_ps\right\} - \frac{t\Psi-\Omega}{D\Gamma_q(\alpha)}\int_0^\zeta(\zeta-qs)^{\alpha-1}f(s)d_qs. \end{aligned}$$

Observe that

$$\begin{aligned} \|h_n(t)-h_*(t)\| &\le \left\| \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}(f_n(s)-f_*(s))d_qs \right. \\ &\quad -\frac{t-\zeta}{D\Gamma_q(\alpha)}\left\{\frac{1}{\Gamma_r(\gamma)}\int_0^\xi\int_0^s(\xi-rs)^{(\gamma-1)}(s-qu)^{(\alpha-1)}(f_n(u)-f_*(u))d_qud_rs\right. \\ &\quad \left. -\frac{\lambda}{\Gamma_p(\beta)}\int_0^\eta\int_0^s(\eta-ps)^{(\beta-1)}(s-qu)^{(\alpha-1)}(f_n(u)-f_*(u))d_qud_ps\right\} \\ &\quad \left. -\frac{t\Psi-\Omega}{D\Gamma_q(\alpha)}\int_0^\zeta(\zeta-qs)^{\alpha-1}(f_n(s)-f_*(s))d_qs \right\| \to 0, \end{aligned}$$

as $n \to \infty$. Thus, it follows by Lemma 5.3 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, therefore, we have

$$\begin{aligned} h_*(t) &= \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f_*(s)d_qs - \frac{t-\zeta}{D\Gamma_q(\alpha)}\left\{\frac{1}{\Gamma_r(\gamma)}\int_0^\xi\int_0^s(\xi-rs)^{(\gamma-1)}(s-qu)^{(\alpha-1)}f_*(u)d_qud_rs\right. \\ &\left. -\frac{\lambda}{\Gamma_p(\beta)}\int_0^\eta\int_0^s(\eta-ps)^{(\beta-1)}(s-qu)^{(\alpha-1)}f_*(u)d_qud_ps\right\} - \frac{t\Psi-\Omega}{D\Gamma_q(\alpha)}\int_0^\zeta(\zeta-qs)^{\alpha-1}f_*(s)d_qs \end{aligned}$$

for some $f_* \in S_{F,x_*}$. Hence $\mathcal{B}$ has a closed graph (and therefore has closed values). In consequence, the operator $\mathcal{B}$ is compact valued.

Thus the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 5.4 and hence its conclusion implies either condition (i) or condition (ii) holds. We show that the conclusion (ii) is not possible. If $x \in \theta\mathcal{A}(x) + \theta\mathcal{B}(x)$ for $\theta \in (0,1)$, then there exists $f \in S_{F,x}$ such that

$$x(t) = \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s)d_qs - \frac{t-\zeta}{D\Gamma_q(\alpha)}\left\{\frac{1}{\Gamma_r(\gamma)}\int_0^\xi\int_0^s(\xi-rs)^{(\gamma-1)}(s-qu)^{(\alpha-1)}f(u)d_qud_rs\right.$$

$$-\frac{\lambda}{\Gamma_p(\beta)}\int_0^\eta\int_0^s(\eta-ps)^{(\beta-1)}(s-qu)^{(\alpha-1)}f(u)d_qud_ps\Bigg\}$$
$$-\frac{t\Psi-\Omega}{D\Gamma_q(\alpha)}\int_0^\zeta(\zeta-qs)^{\alpha-1}f(s)d_qs+\frac{t\Psi-\Omega}{D}g(x),\quad t\in[0,T].$$

Following the method for proof of Claim I, we can obtain

$$\|x\|\le\psi(\|x\|)p_0\|p\|+q_0\|x\|+k_0\ell\|x\|. \tag{5.5}$$

If condition (ii) of Theorem 5.4 holds, then there exists $\theta\in(0,1)$ and $x\in\partial B_r$ with $x=\theta\mathcal{N}(x)$. Then, x is a solution of (1.2) with $\|x\|=M$. Now, by the inequality (5.5), we get

$$\frac{(1-k_0\ell-q_0)M}{\psi(M)p_0\|p\|}\le 1$$

which contradicts (5.2). Hence, $\mathcal{N}$ has a fixed point in $[0,T]$ by Theorem 5.4, and consequently the problem (1.2) has a solution. This completes the proof. □

Example 5.6 *Consider the following nonlocal problem of q-integrodifference inclusion*

$$\begin{cases} {}^cD^{\frac{5}{3}}_{\frac{2}{9}}x(t)\in F\left(t,x,I^{\frac{5}{6}}_{\frac{3}{7}}x\right), & t\in(0,1/2),\\ x\left(\frac{1}{6}\right)=\dfrac{|x(1/16)|}{60(1+|x(1/16)|)}, & \dfrac{2}{3}I^{\frac{7}{4}}_{\frac{1}{3}}x\left(\frac{1}{5}\right)=I^{\frac{4}{3}}_{\frac{3}{5}}x\left(\frac{3}{8}\right), \end{cases} \tag{5.6}$$

where $F:[0,1/2]\times\mathbb{R}^2\to\mathcal{P}(\mathbb{R})$ *is a multivalued map given by*

$$x\to F(t,x,I^{\frac{5}{6}}_{\frac{3}{7}}x)=\left[\frac{t|x|(1+\cos^2 4x)}{12(1+|x|)}+I^{\frac{5}{6}}_{\frac{3}{7}}x,\frac{(t+1)(|x|+1)e^{-3x^2}}{16(1+\sin^2 2x)}+I^{\frac{5}{6}}_{\frac{3}{7}}x\right].$$

Here, $\alpha=5/3$, $q=2/9$, $\delta=5/6$, $z=3/7$, $T=1/2$, $\zeta=1/6$, $\lambda=2/3$, $\beta=7/4$, $p=1/3$, $\eta=1/5$, $\gamma=4/3$, $r=3/5$, $\xi=3/8$, $g(x)=(1/60)(|x|/(1+|x|))$. By using the Maple program, we find that $\Omega=-0.04690826$, $\Psi=-0.20641547$, $D=0.01250568$, $p_0=1.95003166$, $q_0=0.58637355$ and $k_0=12.00382078$.

As $|g(x)-g(y)|\le(1/60)|x-y|$, therefore, (A_2) is satisfied with $\ell=(1/60)<(1/12.00382078)=k_0^{-1}$. For $f\in F$ and $x,y\in\mathbb{R}$, we have

$$|f|\le\max\left(\frac{t|x|(1+\cos^2 4x)}{12(1+|x|)}+y,\frac{(t+1)(|x|+1)e^{-3x^2}}{16(1+\sin^2 2x)}+y\right)\le\frac{t+1}{16}(|x|+1)+|y|.$$

Thus

$$\|F(t,x,y)\|_{\mathcal{P}}:=\sup\{|v|\ :\ v\in F(t,x,y)\}\le p(t)\psi(\|x\|)+|y|,\quad x,y\in\mathbb{R},$$

with $p(t)=(t+1)/16$ and $\psi(\|x\|)=\|x\|+1$. By computing directly, we found that there exists a constant $M>5.94574011$ such that

$$\frac{(1-k_0\ell-q_0)M}{\psi(M)p_0\|p\|}>1.$$

Clearly, all the conditions of Theorem 5.5 are satisfied. Hence, the nonlocal boundary value problem (5.6) has at least one solution on $[0,1/2]$.

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On the Solvability of a System of Multi-Point Second Order Boundary Value Problem

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Abstract

In this paper, by using Leray-Schauder fixed point theorem, we obtain existence of at least one symmetric solution for a multi-point second order boundary value problem. We give an example to demonstrate our main result.

2010 Mathematics Subject Classification: 34B10, 34B18, 39A10.
Keywords: boundary value problem, existence of solution, fixed point method, system.

1 Introduction

The study of multi-point boundary value problems for linear second-order ordinary differential equations were initiated by Il'in and Moiseev [7]. Motivated by the study of Il'in and Moiseev [7], Gupta [4] studied nonlinear three-point boundary value problems for nonlinear ordinary differential equations. Since then, the more general nonlinear multi-point boundary value problems have been studied by many authors. We refer the reader to [5, 8, 10, 13, 16].

On the other hand, the existence of symmetric positive solutions of second order boundary value problems have been studied by some authors, see [1, 6]. Most of the study of symmetric positive solution is limited to Dirichlet boundary value problem, Sturm-Liouville boundary value problem and Neumann boundary value problem. However, there is not so much work on symmetric positive solutions for second-order m-point boundary value problems see [2, 9, 12].

Young and Tisdell [14], studied the following singular boundary value problem (BVP)

$$\left\{ \frac{1}{p}(py^{'})^{'} = qf(t,y), \quad t \in (0,T), \right.$$

coupled with various forms of the following boundary conditions:

$$\begin{cases} -\alpha y(0) + \beta \lim\limits_{t\to 0^+} p(t)y^{'}(t) = c, \\ \gamma y(T) + \delta \lim\limits_{t\to T^-} p(t)y^{'}(t) = d. \end{cases}$$

They established the existence of solutions to second-order singular boundary value problems by using Leray-Schauder fixed point theorem.

We notice that a type of symmetric problem has received much attention, for instance,[5, 8, 15], and the references therein. Jiang et al. [8] studied the following a singular system

$$\begin{cases} -x^{''}(t) = a_1(t)f(t,x(t),y(t)), \quad t \in (0,1), \\ -y^{''}(t) = a_2(t)g(t,x(t),y(t)), \quad t \in (0,1), \\ x(0) = \sum\limits_{i=1}^{m} \alpha_i y(\xi_i), \ x(1) = \sum\limits_{i=1}^{m} \alpha_i y(\widetilde{\xi_i}), \\ y(0) = \sum\limits_{i=1}^{m} \beta_i x(\eta_i), \ y(1) = \sum\limits_{i=1}^{m} \beta_i x(\widetilde{\eta_i}). \end{cases}$$

By using a fixed point theorem in a cone, they obtained at least one or two symmetric positive solutions.

Motivated by this results mentioned above, in this paper, we consider the following second order multi-point boundary value problem

$$\begin{cases} (p(t)u^{'}(t))^{'} = f(t, u(t), u^{'}(t)), \quad t \in (a, b), \\ u(a) = \sum_{i=1}^{m-2} \alpha_i p(\eta_i) u^{'}(\eta_i), u(b) = \sum_{i=1}^{m-2} \alpha_i p(\xi_i) u^{'}(\xi_i). \end{cases} \tag{1.1}$$

Throughout this paper we assume that following conditions hold:

$(C1)$ $f \in \mathcal{C}([a, b] \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ and $f(t, 0, 0) \not\equiv 0$ for $t \in [a, b]$, $p \in \mathcal{C}'([a, b], \mathbb{R})$ with $p > 0$ on (a, b) and $\int_a^b \frac{ds}{p(s)} < +\infty$ and f is symmetric function on $[a, b]$ such that $f(b + a - t, u, v) = f(t, u, v)$, and p antisymmetric function on $[a, b]$ such that $p(b + a - t) = -p(t)$ and f is an even function in v, i.e., $f(t, u, v) = f(t, u, -v)$.

$(C2)$ $\alpha, \beta \in [0, \infty)$, $\alpha_i \in [0, \infty)$, $\xi_i, \eta_i \in (a, b)$ such that $\xi_i = b + a - \eta_i$ for $i \in \{1, 2, ..., m - 2\}$.

By using Leray-Schauder fixed point theorem, we get the existence of symmetric solution for the BVP (1.1).

This paper contains three sections besides the Introduction. In Section 2, we present some necessary preliminaries that will be used to prove our main result. In Section 3, we obtain the existence of at least one symmetric solution for the BVP (1.1). Finally we give an example to illustrate our result in Section 4.

To the best of our knowledge, there is no earlier literature studying this problem. This paper attempts to fill part of this gap in the literatures.

As for notation, if y, $z \in \mathbb{R}$, then $\langle y, z\rangle$ denotes their usual inner product and $\|z\|$ denotes the Euclidean norm of z. We adopt the standart norm for elements u of $\mathcal{C}^{'}([a, b], \mathbb{R}^n)$, namely

$$||u(t)||_0 := \max\{\max_{t\in[a,b]} \|u(t)\|, \max_{t\in[a,b]} \|u^{'}(t)\|\}.$$

For all $t \in (a, b)$, we have

$$\begin{aligned} \langle u(t), p(t)u'(t)\rangle^{'} &= \langle u(t), (p(t)u'(t))^{'}\rangle + \langle u^{'}(t), p(t)u'(t)\rangle \\ &= \langle u(t), f(t, u(t), u'(t))\rangle + p(t)\|u^{'}(t)\|^2. \end{aligned}$$

The above identity will be needed in the proof of our main result and our technique is based on a priori bound. We refer the reader to the papers [11, 14] and the references therein.

2 Preliminaries

In this section, we will employ several lemmas to prove the main result in this paper. When $n = 1$, (1.1) reduces to the scaler equation.

Lemma 2.1 *Suppose the condition* $D = -\int_a^b \frac{1}{p(s)} ds \neq 0$ *hold. Then, for* $h \in \mathcal{C}([a, b], \mathbb{R}^n)$*, and symmetric on* $[a, b]$*, the BVP*

$$\begin{cases} (p(t)u^{'}(t))^{'} = h(t), \quad t \in (a, b), \\ u(a) = \sum_{i=1}^{m-2} \alpha_i p(\eta_i) u^{'}(\eta_i), u(b) = \sum_{i=1}^{m-2} \alpha_i p(\xi_i) u^{'}(\xi_i), \end{cases} \tag{2.1}$$

has a unique solution u

$$u(t) = \int_a^b G(t, s)h(s)ds + \sum_{i=1}^{m-2} \alpha_i \int_a^b {G_t}^{[1]}(\eta_i, s)h(s)ds,$$

where ${G_t}^{[1]}(t, s) = p(t){G_t}^{'}(t, s)$ *and* $G(t, s)$ *be the Green's function for* (2.1) *is given by*

$$G(t, s) = \frac{1}{D} \begin{cases} \theta(t)\varphi(s), & a \le t \le s \le b, \\ \theta(s)\varphi(t), & a \le s \le t \le b, \end{cases} \tag{2.2}$$

where $\theta(t)$ and $\varphi(t)$ are given by

$$\theta(t) = \int_a^t \frac{1}{p(\tau)} d\tau, \tag{2.3}$$

$$\varphi(t) = \int_t^b \frac{1}{p(\tau)} d\tau, \tag{2.4}$$

respectively.

Proof. $u(t) = \int_a^b G(t,s)h(s)ds + \sum_{i=1}^{m-2} \alpha_i \int_a^b {G_t}^{[1]}(\eta_i, s)h(s)ds$ be a solution of (2.1), then we have that

$$\begin{aligned} u(t) \quad &= \quad \frac{1}{D}\int_a^t \theta(s)\varphi(t)h(s)ds + \frac{1}{D}\int_t^b \theta(t)\varphi(s)h(s)ds \\ &+ \quad \sum_{i=1}^{m-2} \alpha_i \int_a^b {G_t}^{[1]}(\eta_i, s)h(s)ds, \end{aligned}$$

$$\begin{aligned} p(t)u^{'}(t) \quad &= \quad p(t)\varphi^{'}(t)\int_a^t \frac{1}{D}\theta(s)h(s)ds + p(t)\theta^{'}(t)\int_t^b \frac{1}{D}\varphi(s)h(s)ds \\ &= \quad \int_a^b {G_t}^{[1]}(t,s)h(s)ds, \end{aligned}$$

and

$$\begin{aligned} (p(t)u^{'}(t))^{'} \quad &= \quad (p(t)\varphi^{'}(t))^{'}\int_a^t \frac{1}{D}\theta(s)h(s)ds + p(t)\varphi^{'}(t)\frac{1}{D}\theta(t)h(t) \\ &+ \quad (p(t)\theta^{'}(t))^{'}\int_t^b \frac{1}{D}\varphi(s)h(s)ds - p(t)\theta^{'}(t)\frac{1}{D}\varphi(t)h(t) \\ &= \quad p(t)\varphi^{'}(t)\frac{1}{D}\theta(t)h(t) - p(t)\theta^{'}(t)\frac{1}{D}\varphi(t)h(t) \\ &= \quad \frac{p(t)}{D}\left[\varphi^{'}(t)\theta(t) - \theta^{'}(t)\varphi(t)\right]h(t) \\ &= \quad h(t). \end{aligned}$$

Since

$$\begin{aligned} u(a) \quad &= \quad \frac{1}{D}\int_a^b \theta(a)\varphi(s)h(s)ds + \sum_{i=1}^{m-2} \alpha_i \int_a^b {G_t}^{[1]}(\eta_i, s)h(s)ds \\ &= \quad \sum_{i=1}^{m-2} \alpha_i \int_a^b {G_t}^{[1]}(\eta_i, s)h(s)ds \\ &= \quad \sum_{i=1}^{m-2} \alpha_i p(\eta_i)u^{'}(\eta_i), \end{aligned}$$

and

$$\begin{aligned}
u(b) &= \frac{1}{D}\int_a^b \theta(s)\varphi(b)h(s)ds + \sum_{i=1}^{m-2}\alpha_i \int_a^b {G_t}^{[1]}(\eta_i, s)h(s)ds \\
&= \sum_{i=1}^{m-2}\alpha_i \int_a^b {G_t}^{[1]}(\eta_i, s)h(s)ds \\
&= \sum_{i=1}^{m-2}\alpha_i p(\eta_i)\varphi^{'}(\eta_i)\int_a^{\eta_i}\frac{1}{D}\theta(s)h(s)ds + \sum_{i=1}^{m-2}\alpha_i p(\eta_i)\theta^{'}(\eta_i)\int_{\eta_i}^b \frac{1}{D}\varphi(s)h(s)ds \\
&= -\sum_{i=1}^{m-2}\alpha_i \int_a^{\eta_i}\frac{1}{D}\theta(s)h(s)ds + \sum_{i=1}^{m-2}\alpha_i \int_{\eta_i}^b \frac{1}{D}\varphi(s)h(s)ds \\
&= -\sum_{i=1}^{m-2}\alpha_i \int_{b+a-\eta_i}^b \frac{1}{D}\theta(b+a-s)h(b+a-s)d(b+a-s) \\
&\quad + \sum_{i=1}^{m-2}\alpha_i \int_a^{b+a-\eta_i}\frac{1}{D}\varphi(b+a-s)h(b+a-s)d(b+a-s) \\
&= -\sum_{i=1}^{m-2}\alpha_i \int_{\xi_i}^b \frac{1}{D}(-\varphi(s))h(s)ds + \sum_{i=1}^{m-2}\alpha_i \int_a^{\xi_i}\frac{1}{D}(-\theta(s))h(s)ds \\
&= \sum_{i=1}^{m-2}\alpha_i p(\xi_i)u^{'}(\xi_i).
\end{aligned}$$

We are able to obtain the boundary value conditions. □

Lemma 2.2 *For $t, s \in [a, b]$, we have $G(b+a-t, b+a-s) = G(t, s)$.*

Proof. In fact, if $t \le s$, then $1-t \ge 1-s$. In view of (2.2) and the assumption (C1), we get

$$\begin{aligned}
G(b+a-t, b+a-s) &= \frac{1}{D}\left(\int_a^{b+a-s}\frac{1}{p(\tau)}d\tau\right)\left(\int_{b+a-t}^b \frac{1}{p(\tau)}d\tau\right) \\
&= \frac{1}{D}\left(\int_b^s \frac{1}{p(b+a-\tau)}d(b+a-\tau)\right)\left(\int_t^a \frac{1}{p(b+a-\tau)}d(b+a-\tau)\right) \\
&= \frac{1}{D}\left(\int_s^b \frac{1}{p(\tau)}d\tau\right)\left(\int_a^t \frac{1}{p(\tau)}d\tau\right) \\
&= G(t, s),\ a \le t \le s \le b.
\end{aligned}$$

Similarly, we can prove that $G(b+a-t, b+a-s) = G(t, s),\ a \le s \le t \le b$. We have $G(b+a-t, b+a-s) = G(t, s)$ for all $(t, s) \in [a, b] \times [a, b]$, i.e., G(t,s) is symmetric function on $[a, b] \times [a, b]$. □

Lemma 2.3 *For $t, s \in [a, b]$, we have* $\max\limits_{(t,s)\in[a,b]\times[a,b]}|G(t,s)| \le \int_a^b \frac{1}{p(t)}dt \max\limits_{(t,s)\in[a,b]\times[a,b]}|{G_t}^{[1]}(t,s)|$.

Proof. We apply (2.2), we get that for $t \in [a, b]$

$$\frac{G(t,s)}{{G_t}^{[1]}(t,s)} = \begin{cases} \dfrac{\theta(t)}{p(t)\theta'(t)}, & a \le t \le s \le b, \\ \dfrac{\varphi(t)}{p(t)\varphi'(t)}, & a \le s \le t \le b, \end{cases} = \begin{cases} \theta(t), & a \le t \le s \le b, \\ -\varphi(t), & a \le s \le t \le b, \end{cases} \tag{2.5}$$

Then we obtain that from (2.5), $\max_{(t,s)\in[a,b]\times[a,b]}|G(t,s)| \le \int_a^b \frac{1}{p(t)}dt \max\limits_{(t,s)\in[a,b]\times[a,b]}|{G_t}^{[1]}(t,s)|$ for $t \in [a, b]$. □

Let $\mathcal{B} = \mathcal{C}^{'}([a,b];\mathcal{R}^n)$ then $\mathcal{B}$ is a Banach space with $||u(t)||_0 := \max\{\max_{t\in[a,b]} \|u(t)\|, \max_{t\in[a,b]} \|u^{'}(t)\|\}$, and define a cone $P \subset \mathcal{B}$ by

$$P = \{u \in \mathcal{B} : u(t) \text{ } is \text{ } symmetric \text{ } on \text{ [a,b]}\}.$$

We define the integral operator $T : P \to \mathcal{B}$ by

$$Tu(t) = \int_a^b G(t,s)f(s,u(s),u'(s))ds + \frac{1}{\alpha}\sum_{i=1}^{m-2}\alpha_i\int_a^b {G_t}^{[1]}(\eta_i,s)f(s,u(s),u'(s))ds,$$

where $G(t,s)$ is given by (2.2) and ${G_t}^{[1]}(t,s) = p(t){G_t}^{'}(t,s)$.

Lemma 2.4 *Let $(C1)$-$(C2)$ hold. Then $T : P \to P$ is completely continuous.*

Proof. For all $u \in P$, using that $f(t,u(t),u'(t))$ is symmetric on [a,b], and $p(t)$ is antisymmetric on [a,b] and by Lemma 2.2, we have

$$\begin{aligned}
Tu(b+a-t) &= \int_a^b G(b+a-t,s)f(s,u(s),u'(s))ds + \frac{1}{\alpha}\sum_{i=1}^{m-2}\alpha_i\int_a^b {G_t}^{[1]}(\eta_i,s)f(s,u(s),u'(s))ds \\
&= \int_b^a G(b+a-t,b+a-s)f(b+a-s,u(b+a-s),u'(b+a-s))d(b+a-s) \\
&+ \frac{1}{\alpha}\sum_{i=1}^{m-2}\alpha_i\int_a^b {G_t}^{[1]}(\eta_i,s)f(s,u(s),u'(s))ds \\
&= \int_a^b G(t,s)f(s,u(s),u'(s))d(s) + \frac{1}{\alpha}\sum_{i=1}^{m-2}\alpha_i\int_a^b {G_t}^{[1]}(\eta_i,s)f(s,u(s),u'(s))ds \\
&= Tu(t),
\end{aligned}$$

for every $t \in [a,b]$. This implies that $Tu(t)$ is symmetric on $[a,b]$. So, $Tu \in P$ and then $Tu \subset P$. Next, by standard methods and Arzela-Ascoli theorem, one can easily prove that operator T is completely continuous. □

3 Main result

In this section, we discuss the existence of at least one symmetric solution for the problem (1.1). The following fixed point theorem is fundamental and important to the proof of our main result.

Lemma 3.1 *(See[3]). Let B be a Banach space with $P \subseteq B$ closed and convex. Assume U is a open subset of P with $0 \in U$ and $T : \overline{U} \to P$ is a continuous and compact map. Then either*

(i) T has a fixed point in $\overline{U}$, or

(ii) there exists $u \in \partial U$ and $\lambda \in (0,1)$ such that $u = \lambda Tu$.

Lemma 3.2 *Let $f \in \mathcal{C}([a,b]\times\mathbb{R}^n\times\mathbb{R}^n,\mathbb{R}^n)$ and let $(C1)$-$(C2)$ hold with*

$$G_0 := \max_{(t,s)\in[a,b]\times[a,b]} \mid {G_t}^{[1]}(t,s) \mid \left(\int_a^b \frac{1}{p(t)}dt + \frac{1}{\alpha}\sum_{i=1}^{m-2}\alpha_i\right),$$

$$G_1 := \max_{(t,s)\in[a,b]\times[a,b]} \mid G_t^{'}(t,s) \mid,$$

$$M := \max\{G_0 W(b-a), G_1 W(b-a)\}.$$

If there exist non-negative constant V and W such that

$$||f(t,u,v)|| \le V\left[\langle u(t), f(t,u(t),v(t))\rangle + p(t)||v||^2\right] + W, \tag{3.1}$$

for all $(t,u,v) \in [a,b] \times \mathbb{R}^n \times \mathbb{R}^n$. *Then all solutions* $u = u(t)$ *to the BVP* (1.1) *satisfy*

$$||u(t)||_0 \leq M,$$

for all $t \in [a,b]$.

Proof. Let $u = u(t) \in$ P be any solution to the BVP (1.1). From Lemma 2.1, we get

$$u(t) = \int_a^b G(t,s) f(s,u(s),u'(s))ds + \frac{1}{\alpha}\sum_{i=1}^{m-2}\alpha_i \int_a^b {G_t}^{[1]}(\eta_i,s) f(s,u(s),u'(s))ds.$$

Then

$$\begin{aligned}
\|u(t)\| &\leq \max_{(t,s)\in[a,b]\times[a,b]} \mid G(t,s) \mid \int_a^b ||f(s,u(s),u'(s))||ds \\
&+ \max_{(t,s)\in[a,b]\times[a,b]} \mid {G_t}^{[1]}(t,s) \mid \frac{1}{\alpha}\sum_{i=1}^{m-2}\alpha_i \int_a^b ||f(s,u(s),u'(s))||ds
\end{aligned}$$

$$\begin{aligned}
&\leq \max_{(t,s)\in[a,b]\times[a,b]} \mid {G_t}^{[1]}(t,s) \mid \left(\int_a^b \frac{1}{p(t)}dt + \frac{1}{\alpha}\sum_{i=1}^{m-2}\alpha_i\right) \int_a^b ||f(s,u(s),u'(s))||ds \\
&\leq G_0 \int_a^b \big(V\left[\langle u(s), f(s,u(s),u'(s))\rangle + p(s)||u'(s)||^2\right] + W\big)\, ds \\
&\leq G_0 V \int_a^b \left[\langle u(s), f(s,u(s),u'(s))\rangle + p(s)||u'(s)||^2\right] ds + G_0 W(b-a) \\
&\leq G_0 V \int_a^b \langle u(t), p(t)u'(t)\rangle^{'} ds + G_0 W(b-a) \\
&\leq G_0 V\left[\langle u(b), p(b)u'(b)\rangle - \langle u(a), p(a)u'(a)\rangle\right] + G_0 W(b-a) \\
&\leq G_0 V\left[\langle u(b), p(b)u'(b) - p(b)u'(b)\rangle\right] + G_0 W(b-a) \\
&\leq G_0 W(b-a),
\end{aligned}$$

and

$$u^{'}(t) = \int_a^b {G_t}^{'}(t,s) f(s,u(s),u'(s))ds,$$

$$\begin{aligned}
\|u^{'}(t)\| &\leq \max_{(t,s)\in[a,b]\times[a,b]} \mid {G_t}^{'}(t,s) \mid \int_a^b ||f(s,u(s),u'(s))||ds, \\
&\leq G_1 \int_a^b \big(V\left[\langle u(s), f(s,u(s),u'(s))\rangle + p(s)||u'(s)||^2\right] + W\big)\, ds, \\
&\leq G_1 W(b-a).
\end{aligned}$$

We have

$$\begin{aligned}
||u(t)||_0 &:= \max\{\max_{t\in[a,b]} \|u(t)\|, \max_{t\in[a,b]} \|u^{'}(t)\|\}, \\
&\leq \max\{G_0 W(b-a), G_1 W(b-a)\}, \\
&\leq M.
\end{aligned}$$

Thus our claimed a priori bound has been obtained. □

Theorem 3.1 *Under the condition of Lemma* 3.2, *the BVP* (1.1) *has at least one symmetric solution.*

Proof. Let u be a possible solution of (1.1). Lemma 3.2 implies that $||u(t)||_0 \leq M$ for all $t \in [a,b]$. We now apply the priori bound result to has the existence of solution.

Let $\Omega := \{u \in P : ||u(t)||_0 < M+1\}$. By Lemma 2.4, we know that the operator $T : P \to P$ is completely continuous. Since all possible solutions of (1.1) satisfy $||u(t)||_0 \leq M$. It follows that there isn't u in $\partial\Omega$ and $\lambda \in (0,1)$ such that $u = \lambda T u$. We conclude that (ii) of Lemma 3.1 does not hold. Therefore, the operator T has a fixed point in $\overline{U}$, which is a symmetric solution of (1.1). □

4 Example

Example 4.1 We consider the following multi-point second order boundary value problem with $n = 2$, $m = 3$,

$$\begin{cases} (p(t)u^{'}(t))^{'} = f(t, u(t), u^{'}(t)), \quad t \in (0,1), \\ u(0) = \frac{1}{2}p\left(\frac{1}{4}\right)u^{'}\left(\frac{1}{4}\right), u(1) = \frac{1}{2}p\left(\frac{3}{4}\right)u^{'}\left(\frac{3}{4}\right), \end{cases} \tag{4.1}$$

where $p(t)$ and $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}^2$ are given by

$$\begin{aligned} p(t) &= \left(\frac{1}{2} - t\right), \\ f(t,u) &= (f_1(t,u_1,u_2), f_2(t,u_1,u_2)) \\ &= (t(1-t)u_1 + u_1^3u_2^2, t(1-t)u_2 + u_2^3u_1^2), \end{aligned}$$

and f not depending on $u^{'}$. It is easy to see that $f(t,u) = f(1-t,u)$ and $p(t) = -p(1-t)$. We claim that the above f satisfies the condition of Lemma 3.2, (3.1). Note that for all $(t,u) \in [0,1] \times \mathbb{R}^2$ we have

$$\begin{aligned} ||f(t,u)|| &\leq |f_1(t,u_1,u_2)| + |f_2(t,u_1,u_2)| \\ &\leq t(1-t)|u_1| + |u_1|^3u_2^2 + t(1-t)|u_2| + |u_2|^3u_1^2. \end{aligned}$$

If we choose $V = 1$ and $W = 4$ then, for all $(t, u_1, u_2) \in [0,1] \times \mathbb{R}^2$,

$$\begin{aligned} V\langle u(t), f(t,u(t))\rangle + W &= \left[t(1-t)u_1^2 + u_1^4u_2^2 + t(1-t)u_2^2 + u_2^4u_1^2\right] + 4 \\ &\geq \left[t(1-t)|u_1| - 1 + |u_1^3|u_2^2 - 1 + t(1-t)|u_2| - 1 + |u_2|^3u_1^2 - 1\right] + 4 \\ &\geq t(1-t)|u_1| + |u_1^3|u_2^2 + t(1-t)|u_2| + |u_2|^3u_1^2 \\ &\geq ||f(t,u)||. \end{aligned}$$

So, $f(t,u)$ satisfies Lemma 3.2. Then, Theorem 3.1 hold. The BVP (1.1) has at least one symmetric solution. □

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 20, NO.4, 2016, COPYRIGHT 2016 EUDOXUS PRESS, LLC

Cascadic Multigrid Method for The Elliptic Monge-Ampère Equation☆

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Abstract

The elliptic Monge-Ampère (M-A) equation is a fully nonlinear partial differential equation, which originated in geometric surface theory and has been widely applied in dynamic meteorology, elasticity, geometric optics, image processing and others. The numerical solution of the elliptic Monge-Ampère equation has been a subject of increasing interest recently. In this paper, the cascadic multigrid method (CMG) is used to solve numerically the M-A equation. Before the application of CMG method, an equivalent form of M-A equation is given. On each successive refinement level, weak formulation of this equivalent form can be written and finite element methods can be used successfully. We analyze the convergence and computational complexity for the cascadic multigrid method. And we find that the CMG method is optimal with respect to the energy norm. Finally, numerical experiments confirm the efficiency and robustness of CMG method.

Keywords: Cascadic multigrid, Finite element methods, Interpolation, Monge-Ampère equation
2000 MSC: 65F10, 65N30, 65N55, 35J60

1. Introduction

In this paper, we will introduce a cascadic multirid method for the fully nonlinear elliptic partial differential equation

$$\det(D^2u(z)) = f(z), \tag{1.1}$$

where z has n independent variables, and D^2u is the Hessian of the function u. When restricting it to domains $\Omega \subset \mathbb{R}^2$, we can rewrite the equation as

$$(u_{xx}u_{yy} - u_{xy}^2)(x, y) = f(x, y). \tag{1.2}$$

☆This work was supported by the Science Foundations of Ningxia University (No.ZR1413).
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The equation comes with Dirichlet boundary conditons

$$u(x, y) = g(x, y), \quad \text{on} \quad \partial\Omega \tag{1.3}$$

and the additional convexity constraint

$$u(x, y) \quad \text{is convex,} \tag{1.4}$$

which is required for the equation to be elliptic. Here $\Omega \subset \mathbb{R}^2$ is a bounded domain with boundary $\partial\Omega$ and $f : \Omega \to \mathbb{R}$ is a non-negative function. The equation (1.2) (along with boundary conditions and convexity constraint) is called the elliptic Monge-Ampère equation.

The elliptic M-A equation is a fully nonlinear partial differential equation, which originated in geometric surface theory and has been widely applied in dynamic meteorology, elasticity, geometric optics, image processing and others [17, 18, 25, 26]. And the numerical solution of the elliptic M-A equation has been a subject of increasing interest recently. The early paper introducing the numerical solution of the elliptic M-A equation is written by Oliker and Prussner [24]. They presented a discretization that converges to the generalized solution in two dimensions. Subsequently, other excellent methods also are used to solve the M-A equation. For the finite difference methods, it has been proved that consistency, stability and monotonicity are the convergence criterion for (1.2)-(1.4) in [2]. But narrow stencils won't be monotone and even consistent, and establishing a wide scheme requires more works [3, 12, 13, 23]. The implementation and the convergence theory of finite element methods for the elliptic M-A equation is less understood. Feng et al. [11] considered a fourth order problem by adding a small multiple of the biharmonic operator to (1.1). Brenner et al. [5] introduced C^0 penalty methods for the elliptic M-A equation. The crux of both methods is putting the essential boundary condition (1.3) into a weak form by addition and penalization techniques respectively. However, it will become very difficult to prove the solvability and convergence for new perturbed problems. Dean and Glowinski [7, 8, 9, 10, 14, 15] presented an augmented Lagrange multiplier method and a least squares method for the elliptic M-A equation. The convergence of these methods still remains an open problem. Liu and He [21] introduced meshfree method for the elliptic M-A equation. Although meshfree method is easy to implement, its convergence theory still remains open. Consequently, developing efficient discretizations still is challenging for the elliptic M-A equation.

In this paper, we will solve numerically the M-A equation by cascadic multigrid method. Firstly, we will provide an equivalent form of M-A equation and write its variational form. Then, the M-A equation can be solved by finite element (FE) methods based on successive refinement levels and interpolation techniques. We will analyze the convergence and computational complexity of CMG method.

The rest of paper is organized as follows. In Section 2, we design the cascadic multigrid algorithm for M-A equation. The convergence of CMG method will be proved in Section 3. In Section 4, some numerical experiments are presented to demonstrate the effectiveness of CMG method.

2. CMG method for M-A equation

We know that a function u is convex is equivalent to

$$u_{xx} \geq 0, \quad u_{yy} \geq 0, \quad u_{xx}u_{yy} - u_{xy}^2 \geq 0. \tag{2.1}$$

And we have

$$(\triangle u)^2 = (u_{xx} + u_{yy})^2 = u_{xx}^2 + u_{yy}^2 + 2u_{xx}u_{yy}, \tag{2.2}$$

or

$$u_{xx}u_{yy} = \frac{1}{2}((\triangle u)^2 - u_{xx}^2 - u_{yy}^2). \tag{2.3}$$

By substituting (2.3) into (1.2), we have

$$(\triangle u)^2 - u_{xx}^2 - u_{yy}^2 - 2u_{xy}^2 = 2f(x, y). \tag{2.4}$$

According to (2.1), we can choose a convex solution

$$\triangle u = \sqrt{2f(x, y) + u_{xx}^2 + u_{yy}^2 + 2u_{xy}^2}. \tag{2.5}$$

So, the equation (1.2)-(1.4) can be rewritten as

$$\triangle u = \sqrt{2f(x, y) + u_{xx}^2 + u_{yy}^2 + 2u_{xy}^2}, \quad \text{in} \quad \Omega \tag{2.6}$$

$$u = g(x, y). \quad \text{on} \quad \partial\Omega \tag{2.7}$$

Following, we design the CMG algorithm for equation (2.6)-(2.7).

2.1. Algorithm

Let $j = 0, 1, \cdot \cdot \cdot, L$ be a sequence of grid levels, where $j = 0$ denotes the coarsest grid and $j = L$ is the finest grid level. Corresponding to the sequence of grid levels, we have a nested family of partition $(\mathcal{T}_j)_{j=0}^L$ (triangles or rectangles) and the spaces of linear finite elements $(X_j)_{j=0}^L$.

Then we can design the CMG algorithm as following:
(1) First, the following equation is discretized by FE method on level $j = 0$.

$$\triangle u = \sqrt{2f(x, y)}, \quad \text{in} \quad \Omega \tag{2.8}$$

$$u = g(x, y). \quad \text{on} \quad \partial\Omega \tag{2.9}$$

Then the resulting discrete system is solved by m_0 smoothing steps with any initial value (or by direct methods).
(2) For $j = 1, \cdot \cdot \cdot, L$

First, obtaining the approximation of $u, u_{xx}, u_{yy}, u_{xy}$ on j level. This can be finished by interpolation techniques. We remark them as u^j, u_{xx}^j, u_{yy}^j and u_{xy}^j respectively. Then following equation can be discretized by FE method.

$$\triangle u = \sqrt{2f(x, y) + (u_{xx}^j)^2 + (u_{yy}^j)^2 + 2(u_{xy}^j)^2}, \quad \text{in} \quad \Omega \tag{2.10}$$

$$u = g(x,y). \quad \text{on} \quad \partial\Omega \tag{2.11}$$

With initial value u^j, the discrete system is solved by m_j smoothing steps.
Remarks: (1) The linear finite elements spaces can be defined by

$$X_j = \{u \in C(\bar{\Omega}) : u|_e \in P_1(e) \quad \forall e \in \mathcal{T}_j, \quad u|_{\partial\Omega} = 0\},$$

where $P_1(e)$ denotes the linear polynomial on the triangle or rectangle e. Clearly, we have

$$X_0 \subset X_1 \subset \cdots \subset X_L \subset H_0^1(\Omega).$$

(2) The equation (2.6)-(2.7) is inhomogeneous. We let u be a function which coincides with g on the boundary of Ω. That is to say, there exists $Rg \in H^1(\Omega)$ with $Rg|_{\partial\Omega} = g$ and $\|Rg\|_1 \leq C\|g\|_{H^{1/2}(\Omega)}$. Then we can write the variational form of (2.10)-(2.11) as following:

Let $w_j = u_j - Rg$, find $w_j \in H_0^1(\Omega)$ such that

$$a(w_j, v_j) = F(v_j), \quad v_j \in X_j, \tag{2.12}$$

where

$$a(w_j, v_j) = \int_\Omega \nabla w_j \cdot \nabla v_j,$$

$$F(v_j) = \int_\Omega -v_j \sqrt{2f(x,y) + (u_{xx}^j)^2 + (u_{yy}^j)^2 + 2(u_{xy}^j)^2} - \int_\Omega \nabla Rg \cdot \nabla v_j.$$

2.2. Approximate to u, u_{xx}, u_{yy}, u_{xy}

Indeed, constructing effective interpolation is challenging in multigrid methods. [1, 19, 20] designed kinds of interpolation operators to transfer error residual from coarse grid to fine grid. But it is missing to construct the approximation of derivative functions, especially higher derivative functions. How to construct the approximation of u_{xx}, u_{yy} and u_{xy} on j level through information on $j-1$ level? For simplicity, in this subsection we only consider nested uniform rectangle grids. In future work, we will construct the approximation of higher derivative functions on other finite elements.

In Figure 1, we show two grid levels $j-1$ and j. Here, blue lines denote grid on $j-1$ level and red lines denote uniform refinement. And all nodes on j level are divided into cases (b)-(f). We want to approximate the values of u, u_{xx}, u_{yy} and u_{xy} at each black node for each case.

Let the meshsizes be H for $j-1$ level and h for j level. u^H denotes value of u restricted on $j-1$ level and u^h denotes the value on j level. Then applying Taylor expansion we have following expressions.

For case (b),

$$u_7^h = u_{7,}^H$$

$$(u_{xx}^h)_7 = \frac{1}{4h^2}(u_5^H + u_9^H - 2u_7^H),$$

$$(u_{yy}^h)_7 = \frac{1}{4h^2}(u_6^H + u_8^H - 2u_7^H),$$

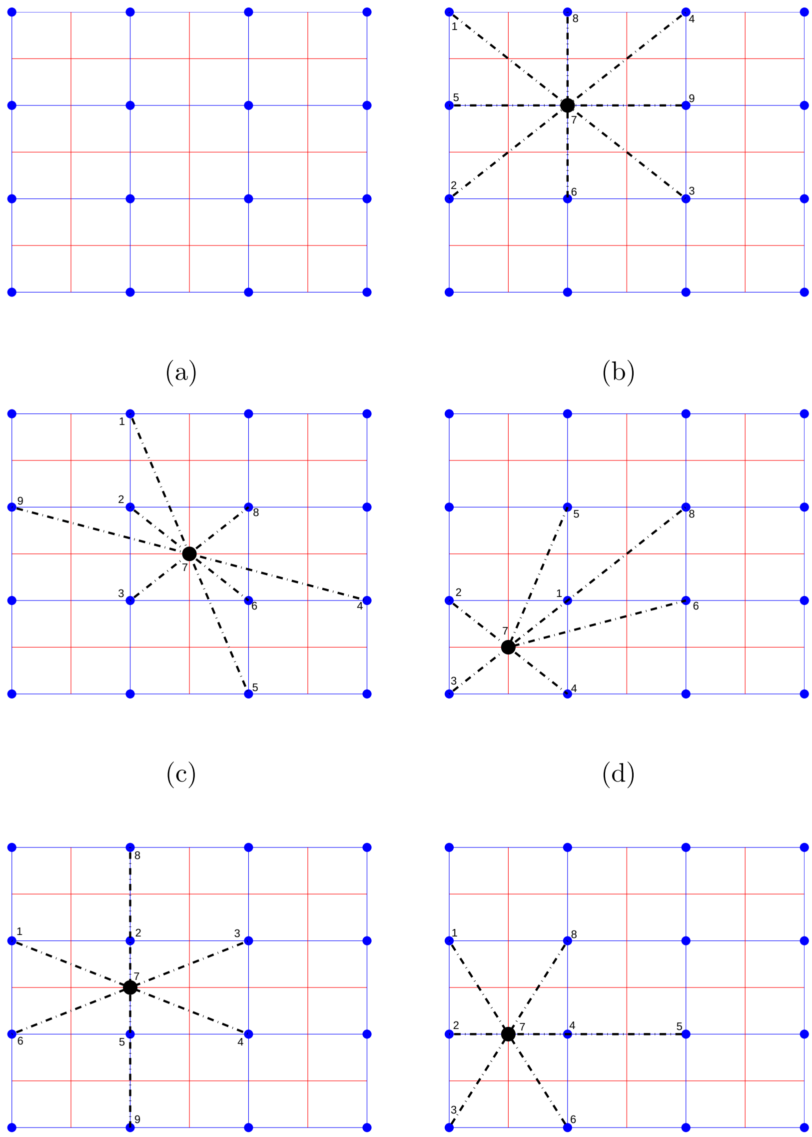

Figure 1: Two grid levels: $j-1$ and j.

$$(u_{xy}^h)_7 = \frac{1}{16h^2}(u_2^H + u_4^H - u_1^H - u_3^H).$$

For case (c),

$$u_7^h = \frac{1}{4}(u_2^H + u_3^H + u_6^H + u_8^H),$$

$$(u_{xx}^h)_7 = \frac{1}{8h^2}(u_9^H + u_4^H + u_3^H + u_8^H - 2u_2^H - 2u_6^H),$$

$$(u_{yy}^h)_7 = \frac{1}{8h^2}(u_1^H + u_5^H + u_3^H + u_8^H - 2u_2^H - 2u_6^H),$$

$$(u_{xy}^h)_7 = \frac{1}{4h^2}(u_3^H + u_8^H - u_2^H - u_6^H).$$

For case (d),

$$u_7^h = \frac{1}{4}(u_1^H + u_2^H + u_3^H + u_4^H),$$

$$(u_{xx}^h)_7 = \frac{1}{4h^2}(u_8^H + 2u_2^H + u_4^H - u_3^H - 2u_1^H - u_5^H),$$

$$(u_{yy}^h)_7 = \frac{1}{4h^2}(u_8^H + 2u_4^H + u_2^H - u_3^H - 2u_1^H - u_6^H),$$

$$(u_{xy}^h)_7 = \frac{1}{4h^2}(u_1^H + u_3^H - u_2^H - u_4^H).$$

For case (e),

$$u_7^h = \frac{1}{2}(u_2^H + u_5^H),$$

$$(u_{xx}^h)_7 = \frac{1}{8h^2}(u_1^H + u_3^H + u_4^H + u_6^H - 2u_2^H - 2u_5^H),$$

$$(u_{yy}^h)_7 = \frac{1}{8h^2}(u_8^H + u_9^H - u_2^H - u_5^H),$$

$$(u_{xy}^h)_7 = \frac{1}{8h^2}(u_3^H + u_6^H - u_1^H - u_4^H).$$

For case (f),

$$u_7^h = \frac{1}{2}(u_2^H + u_4^H),$$

$$(u_{xx}^h)_7 = \frac{1}{4h^2}(u_2^H + u_5^H - 2u_4^H),$$

$$(u_{yy}^h)_7 = \frac{1}{8h^2}(u_1^H + u_3^H + u_6^H + u_8^H - 2u_2^H - 2u_4^H),$$

$$(u_{xy}^h)_7 = \frac{1}{8h^2}(u_3^H + u_8^H - u_1^H - u_6^H).$$

3. Analysis of convergence

In this section, we prove the convergence of CMG method which is introduced in above section. C denote positive constants in this section. First, following Lemma provides the existence and uniqueness theory for variational problem (2.12).

Lemma 3.1. *Let M-A equation (1.2)-(1.4) be uniquely solvable for all $f \in L^2(\Omega)$. If there exists $Rg \in H^1(\Omega)$ with $Rg|_{\partial\Omega} = g$ and $\|Rg\|_1 \le C\|g\|_{H^{1/2}(\Omega)}$, then variational problem (2.12) has a unique solution $u_j = w_j + Rg$.*

PROOF. We have

$$a(w_j, v_j) = (\nabla w_j, \nabla v_j) \le \|w_j\|_1 \|v_j\|_1,$$

and

$$a(v_j, v_j) = |v_j|_1^2 \ge C\|v_j\|_1^2 \quad for \quad v_j \in X_j.$$

In addition,

$$\begin{aligned}
|F(v_j)| &\le \|\sqrt{2f(x,y) + (u_{xx}^j)^2 + (u_{yy}^j)^2 + 2(u_{xy}^j)^2}\|_0 \|v_j\|_0 + \|\nabla Rg\|_0 \|\nabla v_j\|_0 \\
&\le \left(\|\sqrt{2f(x,y) + (u_{xx}^j)^2 + (u_{yy}^j)^2 + 2(u_{xy}^j)^2}\|_0 + \|Rg\|_1\right) \|v_j\|_1 \\
&\le C\left(\|\sqrt{2f(x,y) + (u_{xx}^j)^2 + (u_{yy}^j)^2 + 2(u_{xy}^j)^2}\|_0 + \|g\|_{H^{1/2}(\Omega)}\right) \|v_j\|_1.
\end{aligned}$$

F is bounded because u_{xx}^j, u_{yy}^j and u_{xy}^j are known. According to Lax-Milgram Lemma, variational problem has a unique solution.

The variational form of (2.6)-(2.7) can be given by (3.1).
Find $\omega = u - Rg$, $\omega \in H_0^1(\Omega)$, such that

$$a(w, v) = \int_\Omega -v\sqrt{2f(x,y) + u_{xx}^2 + u_{yy}^2 + 2u_{xy}^2} - \int_\Omega \nabla Rg \cdot \nabla v \tag{3.1}$$

for all $v \in H_0^1(\Omega)$.

Hence, for all $v \in X_j$ we have following error equation:

$$a(u_j - u, v) =$$

$$\left(\sqrt{2f(x,y) + u_{xx}^2 + u_{yy}^2 + 2u_{xy}^2} - \sqrt{2f(x,y) + (u_{xx}^j)^2 + (u_{yy}^j)^2 + 2(u_{xy}^j)^2}, v\right).$$

Lemma 3.2. *Assume M-A equation (2.6)-(2.7) has unique solution $u \in H^2(\Omega)$ for $f \in L^2(\Omega)$, and u_j is finite element solution of equation (2.10)-(2.11). Then we have*

$$|u_j - u|_1 \le Ch_j\|u\|_2,$$

where, h_j is the meshsize on level j.

PROOF. Step (1) We prove $a(u_j-u,v)\le Ch_j\|v\|_1$ for all $v\in X_j$.

$$\begin{aligned}
&a(u_j-u,v)\\
\le\ &\|\sqrt{2f(x,y)+u_{xx}^2+u_{yy}^2+2u_{xy}^2}-\sqrt{2f(x,y)+(u_{xx}^j)^2+(u_{yy}^j)^2+2(u_{xy}^j)^2}\|_0\|v\|_0.
\end{aligned}$$

We have

$$\begin{aligned}
&\|\sqrt{2f(x,y)+u_{xx}^2+u_{yy}^2+2u_{xy}^2}-\sqrt{2f(x,y)+(u_{xx}^j)^2+(u_{yy}^j)^2+2(u_{xy}^j)^2}\|_0\\
=\ &\frac{\|(u_{xx}+u_{xx}^j)(u_{xx}-u_{xx}^j)+(u_{yy}+u_{yy}^j)(u_{yy}-u_{yy}^j)+2(u_{xy}+u_{xy}^j)(u_{xy}-u_{xy}^j)\|_0}{\|\sqrt{2f(x,y)+u_{xx}^2+u_{yy}^2+2u_{xy}^2}+\sqrt{2f(x,y)+(u_{xx}^j)^2+(u_{yy}^j)^2+2(u_{xy}^j)^2}\|_0}\\
<\ &\frac{\|u_{xx}+u_{xx}^j\|_0\|u_{xx}-u_{xx}^j\|_0+\|u_{yy}+u_{yy}^j\|_0\|u_{yy}-u_{yy}^j\|_0+2\|u_{xy}+u_{xy}^j\|_0\|u_{xy}-u_{xy}^j\|_0}{\|\sqrt{u_{xx}^2+u_{yy}^2+2u_{xy}^2}+\sqrt{(u_{xx}^j)^2+(u_{yy}^j)^2+2(u_{xy}^j)^2}\|_0}.
\end{aligned}$$

By Cauchy inequality

$$\|u_{xx}+u_{xx}^j\|_0\|u_{xx}-u_{xx}^j\|_0+\|u_{yy}+u_{yy}^j\|_0\|u_{yy}-u_{yy}^j\|_0+2\|u_{xy}+u_{xy}^j\|_0\|u_{xy}-u_{xy}^j\|_0\le I_+I_-$$

$$\begin{aligned}
I_+&=(\|u_{xx}+u_{xx}^j\|_0^2+\|u_{yy}+u_{yy}^j\|_0^2+2\|u_{xy}+u_{xy}^j\|_0^2)^{\frac{1}{2}},\\
I_-&=(\|u_{xx}-u_{xx}^j\|_0^2+\|u_{yy}-u_{yy}^j\|_0^2+2\|u_{xy}-u_{xy}^j\|_0^2)^{\frac{1}{2}}.
\end{aligned}$$

By Minkowski inequality

$$\begin{aligned}
&\|\sqrt{u_{xx}^2+u_{yy}^2+2u_{xy}^2}+\sqrt{(u_{xx}^j)^2+(u_{yy}^j)^2+2(u_{xy}^j)^2}\|_0\\
>\ &\|\sqrt{(u_{xx}+u_{xx}^j)^2+(u_{yy}+u_{yy}^j)^2+2(u_{xy}+u_{xy}^j)^2}\|_0.
\end{aligned}$$

Then we have

$$a(u_j-u,v)\le I_-\|v\|_1.$$

By the construction of u_{xx}^j, we have

$$|u_{xx}(k)-u_{xx}^j(k)|\le Ch_j,$$

k denote all nodes on j level. Similar estimate can be found for u_{yy}^j and u_{xy}^j. Hence, $a(u_j-u,v)\le Ch_j\|v\|_1$ for all $v\in X_j$.

Step (2) H^1 semi norm of error.
Let $v_j\in X_j$, with the fact $u_j-u\in H_0^1(\Omega)$ we have

$$\begin{aligned}
|u_j-u|_1^2 &= a(u_j-u,u_j-u)\\
&= a(u_j-u,u_j-v_j)+a(u_j-u,v_j-u)\\
&\le C_1h_j\|u_j-v_j\|_1+C_2|u_j-u|_1\|v_j-u\|_1\\
&\le C_1h_j|u_j-u|_1+(C_1h_j+C_2|u_j-u|_1)\|v_j-u\|_1.
\end{aligned}$$

If $C_2|u_j - u|_1 \leq C_1 h_j$, then we have the assertion directly. Otherwise,

$$|u_j - u|_1^2 \quad \leq \quad C_1 h_j |u_j - u|_1 + C_3 |u_j - u|_1 \|v_j - u\|_1.$$

After dividing by $|u_j - u|_1$, we get

$$|u_j - u|_1 \quad \leq \quad C_1 h_j + C_3 \inf_{v_j \in X_j} \|v_j - u\|_1.$$

Then, the result follows immediately from interpolation error estimate. This completes the proof.

In addition, we have following approximation property by the Aubin-Nitsche technique.

Lemma 3.3. *If u_{j-1} and u_j are the finite element solutions on levels $j-1$ and j respectively, then we have*

$$\|u_j - u_{j-1}\|_0 \leq C h_j |u_j - u_{j-1}|_1, \quad j = 1, \cdots, L. \tag{3.2}$$

Denoting the cascadic procedure on j level by operator P_{j,m_j}, CMG algorithm can be rewritten as:

(1) $u_0^* = u_0$ (by direct methods)

(2) $j = 1, \cdots, L: \qquad u_j^* = P_{j,m_j} u_{j-1}^*$.

Here P_{j,m_j} contains two processes: interpolation and smoothing (m_j steps). The smoothing can be several basic iteration methods such as Gauss-Seidel and SSOR.

As in [4], we consider following type of basic iterations started with $u_j^0 \in X_j$:

$$u_j - P_{j,m_j} u_j^0 = R_{j,m_j}(u_j - u_j^0). \tag{3.3}$$

We are accustomed to call the basic iteration an energy reducing smoother, if it satisfies following smoothing properties:

$$|R_{j,m_j} v_j|_1 \leq C \frac{h_j^{-1}}{m_j^r} \|v_j\|_0, \tag{3.4}$$

$$|R_{j,m_j} v_j|_1 \leq |v_j|_1, \tag{3.5}$$

for $\forall v_j \in X_j$ with parameter $0 < r \leq 1$. Here, m_j is the number of steps of smoothing applied on level j.

Lemma 3.4. *The symmetric Gauss-Seidel, SSOR and the damped Jacobi iteration satisfy (3.4)-(3.5) with $r = \frac{1}{2}$.*

Proof. See [16].

Then, general algebraic error of CMG method can be estimated by following Theorem.

Theorem 3.1. *If the basic iteration is an energy reducing smoother in CMG method, then the algebraic error can be estimated by*

$$|u_L - u_L^*|_1 \leq C \sum_{j=1}^{L} \frac{h_j}{m_j^r} \|u\|_2.$$

PROOF.

$$\begin{aligned} |u_L - u_L^*|_1 &= |u_L - P_{L,m_L} u_{L-1}^*|_1 \\ &= |R_{L,m_L}(u_L - u_{L-1}^*)|_1 \\ &\leq |R_{L,m_L}(u_L - u_{L-1})|_1 + |R_{L,m_L}(u_{L-1} - u_{L-1}^*)|_1 \\ &\leq C \frac{h_L^{-1}}{m_L^r} \|u_L - u_{L-1}\|_0 + |u_{L-1} - u_{L-1}^*|_1. \end{aligned}$$

By Lemma 3.3 and Lemma 3.2,

$$\begin{aligned} |u_L - u_L^*|_1 &\leq C \frac{1}{m_L^r} |u_L - u_{L-1}|_1 + |u_{L-1} - u_{L-1}^*|_1 \\ &\leq C \sum_{j=1}^{L} \frac{1}{m_j^r} |u_j - u_{j-1}|_1 \\ &\leq C \sum_{j=1}^{L} \frac{1}{m_j^r} (|u_j - u|_1 + |u - u_{j-1}|_1) \\ &\leq C \sum_{j=1}^{L} \frac{h_j}{m_j^r} \|u\|_2. \end{aligned}$$

Let the basic iteration in CMG method be one of energy reducing smoothers with $r = \frac{1}{2}$. In order to ensure optimal accuracy, we select parameters in accordance with the following method.

(i) Meshsize h_j satisfy

$$\frac{2^{L-j} h_L}{C} \leq h_j \leq C 2^{L-j} h_L.$$

(ii) Number of smoothing steps m_j satisfy

$$m_j = \lceil 4^{L-j} m_L \rceil, \quad m_L = \lceil m_* L^2 \rceil.$$

Theorem 3.2. *When select parameters in accordance with (i)-(ii), we have algebraic error*

$$|u_L - u_L^*|_1 \leq C \frac{h_L}{m_*^{1/2}} \|u\|_2,$$

and complexity

$$\sum_{j=1}^{L} m_j n_j \leq C m_* n_L (1 + \log n_L)^3,$$

where $n_j = \dim X_j$.

Proof. The conclusion follows directly by Lemma 3.4 and Theorem 3.1.

Comparing Lemma 3.2 with Theorem 3.2, we have

$$|u_L - u|_1 \approx |u_L - u_L^*|_1 = O(h_L),$$

and

$$\text{amount of work} = O(n_L).$$

Consequently, the CMG algorithm is optimal with respect to the energy norm.

4. Numerical experiments

In this section, to demonstrate the effectiveness of the CMG method, we present some numerical experiments. We consider solving the M-A equation (1.2)-(1.4) using CMG method in the domain $[0,1]^2$ with exact solution $u = \exp(\frac{x^2+y^2}{2})$. On each grid level, the given equation is discretized by Q_1 finite element methods. We let $m_* = 10$, and choose SSOR as the basic iteration.

Table 1 shows the relative error of H^1 semi-norm, L^2 norm and Max norm on six levels respectively. We observe that the CMG algorithm is fast and robust. The numerical results coincide with Theorem 3.2.

Figure 2 shows the error $u - u_j^*$ on these six levels.

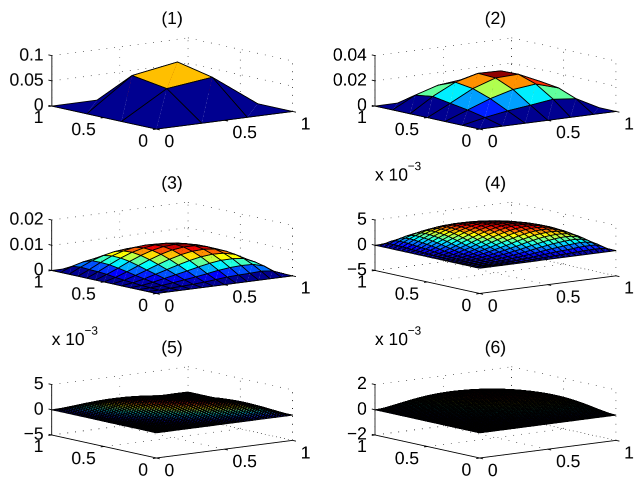

Figure 2: The error on each level.

Table 1: Relative error on $j = 0, \cdots, 5$ levels. CPU time= 240.17 seconds.

j	h_j	$\frac{\lvert u-u_j^*\rvert_1}{\lvert u\rvert_1}$	$\frac{\lVert u-u_j^*\rVert_0}{\lVert u\rVert_0}$	$\frac{\lVert u-u_j^*\rVert_\infty}{\lVert u\rVert_\infty}$
0	1/3	8.35E-02	2.06E-02	2.92E-02
1	1/6	3.95E-02	7.53E-03	9.20E-03
2	1/12	1.96E-02	3.19E-03	3.72E-03
3	1/24	1.01E-02	1.49E-03	1.65E-03
4	1/48	5.27E-03	7.53E-04	8.15E-04
5	1/96	2.97E-03	5.17E-04	5.62E-04

5. Conclusions

The elliptic Monge-Ampère equation is a fully nonlinear partial differential equation which has a wide range of applications. In this paper, we provided an cascadic multigrid method for the M-A equation. We proved the convergence of CMG method. And we found that the CMG method is optimal with respect to the energy norm (H^1 semi-norm). Finally, some numerical experiments were presented to demonstrate the the efficiency and robustness of CMG method. But we still have a lot of work to do in future work. For example, how to construct the approximation of derivative functions (especially higher derivative functions) on unstructured or adaptive grid levels is not understood.

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Iterative algorithms for zeros of accretive operators and fixed points of nonexpansive mappings in Banach spaces

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Abstract

In this paper, we introduce two new iterative algorithms (one implicit and one explicit) for finding a common point of the set of zeros of an accretive operator and the set of fixed points of a nonexpansive mapping in a real uniformly convex Banach space having a uniformly Gâteaux differentiable norm. Then under suitable control conditions, we establish strong convergence of sequence generated by proposed algorithm to a common point of above two sets, which is a solution of a ceratin variational inequality. The main theorems develop and complement some well-known results in the literature.

MSC: 47H06, 47H09, 47H10, 47J25, 49M05, 65J15.

Key words: Iterative algorithm; Accretive operator; Resolvent; Zeros; Nonexpansive mappings; Fixed points; Variational inequalities;

1. Introduction

Let E be a real Banach space with norm $\|\cdot\|$ and the dual space E^*. The value of $x^* \in E^*$ at $y \in E$ is denoted by $\langle y, x^* \rangle$ and the normalized duality mapping $\mathcal{J}$ from E into 2^{E^*} is defined by

$$\mathcal{J}(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|\|x^*\|, \ \|x\| = \|x^*\|\}, \quad \forall x \in E.$$

Recall that a (possibly multivalued) operator $A \subset E \times E$ with the domain $D(A)$ and the range $R(A)$ in E is *accretive* if , for each $x_i \in D(A)$ and $y_i \in Ax_i$ $(i = 1,\ 2)$, there exists a $j \in \mathcal{J}(x_1 - x_2)$ such that $\langle y_1 - y_2, j \rangle \geq 0$. (Here $\mathcal{J}$ is the normalized duality mapping.) In a Hilbert space, an accretive operator is also called monotone operator.

Interest in accretive operators stems mainly from their firm connection with evolution equations. It is well-known that many physically significant problems can be modeled by initial-value problems of the form

$$\frac{dx(t)}{dt} + Ax(t) \ni 0, \quad x(0) = x_0, \tag{1.1}$$

where A is an an accretive operator in a ceratin Banach space. Typical examples where such evolution equations occurs can be found in the heat, wave, or Schrodinger equations. If in (1.1), $x(t)$ is independent of t, then (1.1) reduces $Az \ni 0$ whose solutions correspond to the equilibrium points of system (1.1). Consequently, the iterative algorithms of Halpern type, Mann type, and Rockafellar type have extensively been studied over the last forty years for constructions of zeros of accretive operators (see, e,g., [1–17] and the references therein). As an original one, the following iterative algorithm in Hilbert spaces or Banach spaces was considered by many authors: for resolvent J_{r_n} of m-accretive operator A,

$$x_{n+1} = J_{r_n} x_n, \quad \forall n \geq 0,$$

where the initial guess $x_0 \in E$ is chosen arbitrarily (see, e.g., [4,5,12] and the references therein). In particular, in order to find a zero of a monotone operator A, Rockafellar [13] introduced a powerful and successful algorithm which is recognized as Rockafellar proximal point algorithm in Hilbert space H : for any initial point $x_0 \in H$, a sequence $\{x_n\}$ is generated by

$$x_{n+1} = J_{r_n}(x_n + e_n), \quad \forall n \geq 0,$$

where $J_r = (I + rA)^{-1}$, for $r > 0$, is the resolvent of A and $\{e_n\}$ is an error sequence in H.

Xu [18] in 2006 and Song and Yang [19] in 2009 obtained the strong convergence of the regularization method for Rockafellar's proximal point algorithm in a Hilbert space H: for any initial point $x_0 \in H$

$$x_{n+1} = J_{r_n}(\alpha_n u + (1 - \alpha_n)x_n + e_n), \quad \forall n \geq 0,$$

where $\{\alpha_n\} \subset (0, 1)$, $\{e_n\} \subset H$ and $\{r_n\} \subset (0, \infty)$.

On the other hand, in 2011, He *et al.* [20] studied the following iterative algorithm for finding a common point of the set of zeros of accretive operator A such that $A^{-1}0 \neq \emptyset$ and $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA)$ and the set of fixed points of a nonexpansive mapping S in a real reflexive Banach space E having a weakly sequentially continuous duality mapping:

$$\begin{cases} x_0 = x \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S J_{r_n} x_n, \quad \forall n \geq 0, \end{cases} \tag{1.2}$$

where $\{\alpha_n\}$ and $\{\beta_n\} \subset [0, 1]$, $\lim_{n\to\infty} r_n = r$ and $f : C \to C$ is a contractive mapping. Under the suitable conditions $\{\alpha_n\}$ and $\{\beta_n\}$, they also showed that the sequence $\{x_n\}$ generated by (1.2) converges strongly to a common point in $F(S) \cap A^{-1}0$, which is a solution of a certain variational inequality.

Inspired and motivated by the above-mentioned results, in this paper, we introduce new implicit and explicit algorithms for finding a common point of the set of zeros of accretive operator A and the set of fixed points of a nonexpansive mapping S in a real uniformly convex Banach space E having a uniformly Gâteaux differentiable norm. Under suitable control conditions, we prove that the sequence generated by proposed iterative algorithm converge strongly to a common point in $A^{-1}0 \cap F(S)$, which is a solution of a certain variational inequality. The main results develop and supplement the corresponding results of He *et al.* [20] as well as Xu [18] and Song and Yang [19] and the reference therein.

2. Preliminaries and Lemmas

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be its dual. Let C be a nonempty subset of E. The value of $f \in E^*$ at $x \in E$ will be denoted by $\langle x, f\rangle$. When $\{x_n\}$ is a sequence in E, then $x_n \to x$ ($x_n \rightharpoonup x$) will denote strong (weak) convergence of the sequence $\{x_n\}$ to x. For the mapping $S : C \to C$, $F(S)$ will denote the set of fixed point of S; that is, $F(S) = \{x \in C : Sx = x\}$.

A Banach space E is said to be *uniformly convex* if for all $\varepsilon \in [0, 2]$, there exists $\delta_\varepsilon > 0$ such that

$$\|x\| = \|y\| = 1 \text{ implies } \frac{\|x + y\|}{2} < 1 - \delta_\varepsilon \text{ whenever } \|x - y\| \geq \varepsilon.$$

Let $l > 1$ and $M > 0$ be two fixed real numbers. Then a Banach space is uniformly convex if and only if there exists a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that

$$\|\lambda x + (1-\lambda)y\|^l \leq \lambda\|x\|^l + (1-\lambda)\|y\|^l - \omega(\lambda)g(\|x - y\|), \tag{2.1}$$

for all $x,\ y \in B_M(0) = \{x \in E : \|x\| \leq M\}$, where $\omega(\lambda) = \lambda^l(1-\lambda) + \lambda(1-\lambda)^l$. For more detail, see Xu [21].

The norm of E is said to be *Gâteaux differentiable* if

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.2}$$

exists for each $x,\ y$ in its unit sphere $U = \{x \in E : \|x\| = 1\}$. Such an E is said to be *smooth* Banach space. The norm is said to be *uniformly Gâteaux differentiable* if for $y \in U$, the limit is attained uniformly for $x \in U$. The space E is said to have a *uniformly Fréchet differentiable norm* (and E is said to be *uniformly smooth*) if the limit in (2.2) is attained uniformly for $(x, y) \in U \times U$. It is known that E is smooth if and only if the normalized duality mapping $\mathcal{J}$ is single-valued. Also, it is well-known that if E has a uniformly Gâteaux differentiable norm, $\mathcal{J}$ is norm to weak* uniformly continuous on each bounded subsets of E. The following property of the normalized duality mapping $\mathcal{J}$ is well-known: $\mathcal{J}(-x) = -\mathcal{J}(x)$ for all $x \in E$ ([22]).

An accretive operator A is said to satisfy *the range condition* if $\overline{D(A)} \subset R(I + rA)$ for all $r > 0$, where I is an identity operator of E and $\overline{D(A)}$ denotes the closure of the domain $D(A)$ of A. An accretive operator A is called *m-accretive* if $R(I + rA) = E$ for each $r > 0$. If A is an accretive operator which satisfies the range condition, then we can define, for each $r > 0$ a mapping $J_r : R(I + rA) \to D(A)$ defined by $J_r = (I + rA)^{-1}$, which is called the *resolvent* of A. We know that J_r is nonexpansive (i.e., $\|J_r x - J_r y\| \leq \|x - y\|,\ \ \forall x, y \in R(I + rA)$) and $A^{-1}0 = F(J_r) = \{x \in D(J_r) : J_r x = x\}$ for all $r > 0$. For these facts, see [22].

We need the following lemmas for the proof of our main results. We refer to [22] for Lemma 2.1, Lemma 2.2, and Lemma 2.3.

Lemma 2.1. *If E be a real smooth Banach space, then one has*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, \mathcal{J}(x+y)\rangle, \quad \forall x,\ y \in E,$$

where $\mathcal{J}$ is the normalized duality mapping of E.

Lemma 2.2 (The Resolvent Identity)**.** *For $\lambda > 0$, $\mu > 0$ and $x \in H$,*

$$J_\lambda x = J_\mu\left(\frac{\mu}{\lambda} + \left(1 - \frac{\mu}{\lambda}\right)J_\lambda x\right).$$

Lemma 2.3. *Let E be a real Banach space having a uniformly Gâteaux differentiable norm, let C be a nonempty closed convex subset of E, and let $\{y_n\}$ be a bounded sequence in E. Let* LIM *be a Banach limit and $q \in C$. Then*

$$\mathrm{LIM}\|y_n - q\|^2 = \min_{x \in C} \mathrm{LIM}\|x_n - x\|^2$$

if and only if

$$\mathrm{LIM}\langle x - q, \mathcal{J}(y_n - q)\rangle \leq 0, \quad \forall x \in C,$$

where $\mathcal{J}$ is the normalized duality mapping of E.

The following lemma is given in [23].

Lemma 2.4 ([23])**.** *Let $\{s_n\}$ be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \leq (1-\lambda_n)s_n + \lambda_n\delta_n + \gamma_n, \quad \forall n \geq 0,$$

where $\{\lambda_n\}$, $\{\delta_n\}$ and $\{\gamma_n\}$ satisfy the following conditions:

(i) $\{\lambda_n\} \subset [0,1]$ *and* $\sum_{n=0}^{\infty} \lambda_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=0}^{\infty} \lambda_n|\delta_n| < \infty$;

(iii) $\gamma_n \geq 0$ $(n \geq 0)$, $\sum_{n=0}^{\infty} \gamma_n < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Finally, we will use the next lemma which is of fundamental importance for our proof.

Lemma 2.5 ([24])**.** *Let $\{s_n\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{s_{n_i}\}$ of $\{s_n\}$ such that $s_{n_i} < s_{n_i+1}$ for all $i \geq 0$. For every $n \geq n_0$, define the sequence of integers $\{\tau(n)\}$ by*

$$\tau(n) := \max\{k \leq n : s_k < s_{k+1}\}.$$

Then $\{\tau(n)\}_{n \geq n_0}$ is a nondecreasing sequence verifying

$$\lim_{n\to\infty} \tau(n) = \infty,$$

and, for all $n \geq n_0$, the following two estimates hold:

$$s_{\tau(n)} \leq s_{\tau(n)+1}, \quad s_n \leq s_{\tau(n)+1}.$$

3. Iterative algorithms

Throughout the rest of this paper, we always assume the following:

- E is a real Banach space;
- $\mathcal{J}$ is the normalized duality mapping of E;
- C is a nonempty closed convex subset of E;
- $A \subset E \times E$ is an accretive operator in E such that $A^{-1}0 \neq \emptyset$ and $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA)$;
- J_r is the resolvent of A for each $r > 0$;
- $S : C \to C$ is a nonexpansive mapping with $F(S) \cap A^{-1}0 \neq \emptyset$;
- $f : C \to C$ is a contractive mapping with a constant $k \in (0, 1)$.

In this section, we introduce the following algorithm that generates a net $\{x_t\}_{t\in(0,1)}$ in an implicit way:

$$x_t = J_r(tfx_t + (1 - t)Sx_t). \tag{3.1}$$

We prove strong convergence of $\{x_t\}$ as $t \to 0$ to a point q in $A^{-1}0 \cap F(S)$ which is a solution of the following variational inequality:

$$\langle (I - f)q, \mathcal{J}(q - p)\rangle \geq 0, \quad \forall p \in A^{-1}0 \cap F(S). \tag{3.2}$$

We also propose the following algorithm which generates a sequence in an explicit way:

$$x_{n+1} = J_{r_n}(\alpha_n fx_n + (1 - \alpha_n)Sx_n), \quad \forall n \geq 0, \tag{3.3}$$

where $\{\alpha_n\} \subset (0, 1)$, $\{r_n\} \subset (0, \infty)$ and $x_0 \in C$ is an arbitrary initial guess, and establish the strong convergence of this sequence to a point q in $A^{-1}0\cap F(S)$, which is also a solution of the variational inequality (3.2).

3.1. Strong convergence of the implicit algorithm

Now, for $t \in (0, 1)$, consider a mapping $Q_t : C \to C$ defined by

$$Q_t x = J_r(tfx + (1 - t)Sx), \quad \forall x \in C.$$

It is easy to see that Q_t is a contractive mapping with constant $1 - (1 - k)t$. Indeed, we have

$$\begin{aligned}\|Q_t x - Q_t y\| &\leq t\|fx - fy\| + \|(1 - t)Sx - (1 - t)Sy\| \\ &\leq tk\|x - y\| + (1 - t)\|x - y\| \\ &= (1 - (1 - k)t)\|x - y\|.\end{aligned}$$

Hence Q_t has a unique fixed point, denoted x_t, which uniquely solves the fixed point equation (3.1).

We summary the basic properties of $\{x_t\}$ and $\{y_t\}$, where $y_t = tfx_t + (1-t)Sx_t$ for $t \in (0,1)$.

Proposition 3.1. *Let E be a uniformly convex Banach space. Let the net $\{x_t\}$ be defined by* (3.1)*, and let $\{y_t\}$ be a net defined by $y_t = tfx_t + (1-t)Sx_t$ for $t \in (0,1)$. Then*

(1) *$\{x_t\}$ and $\{y_t\}$ are bounded for $t \in (0,1)$;*

(2) *x_t defines a continuous path from $(0,1)$ in C and so does y_t;*

(3) $\lim_{t\to 0} \|y_t - Sx_t\| = 0$*;*

(4) $\lim_{t\to 0} \|y_t - J_r y_t\| = 0$*;*

(5) $\lim_{t\to 0} \|x_t - y_t\| = 0$*;*

(6) $\lim_{t\to 0} \|y_t - Sy_t\| = 0$*;*

Proof. (1) Let $p \in F(S) \cap A^{-1}0$. Observing $p = Sp = J_r p$, we have

$$\begin{aligned}
\|x_t - p\| &= \|J_r(tfx_t + (1-t)Sx_t) - J_r p\| = \|J_r y_t - J_r p\| \\
&\le \|y_t - p\| \\
&= \|t(fx_t - fp) + t(fp - p) + (1-t)(Sx_t - Sp)\| \\
&\le tk\|x_t - p\| + t\|fp - p\| + (1-t)\|x_t - p\|.
\end{aligned}$$

So, it follows that

$$\|x_t - p\| \le \frac{\|fp - p\|}{1-k} \text{ and } \|y_t - p\| \le \frac{\|fp - p\|}{1-k}.$$

Hence $\{x_t\}$ and $\{y_t\}$ are bounded and so are $\{fx_t\}$, $\{Sx_t\}$, $\{Sy_t\}$, and $\{J_r y_t\}$.

(2) Let $t,\ t_0 \in (0,1)$ and calculate

$$\begin{aligned}
\|x_t - x_{t_0}\| &= \|J_r(tfx_t + (1-t)Sx_t) - J_r(t_0 f x_{t_0} + (1-t_0)Sx_{t_0})\| \\
&\le \|(t-t_0)fx_t + t_0(fx_t - fx_{t_0}) \\
&\qquad - (t-t_0)Sx_t + (1-t_0)Sx_t - (1-t_0)J_r x_{t_0}\| \\
&\le |t-t_0|\|fx_t\| + t_0 k\|x_t - x_{t_0}\| \\
&\qquad + |t-t_0|\|Sx_t\| + (1-t_0)\|x_t - x_{t_0}\|.
\end{aligned}$$

It follows that

$$\|x_t - x_{t_0}\| \le \frac{\|fx_t\| + \|Sx_t\|}{t_0(1-k)}|t-t_0|.$$

This show that x_t is locally Lipschitzian and hence continuous. Also we have

$$\|y_t - y_{t_0}\| \le \frac{\|fx_t\| + \|Sx_t\|}{t_0(1-k)}|t-t_0|,$$

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and hence y_t is a continuous path.

(3) By the boundedness of $\{fx_t\}$ and $\{Sx_t\}$ in (1), we have

$$\begin{aligned}\|y_t - Sx_t\| &= \|tfx_t + (1-t)Sx_t - Sx_t\|\\ &\le t\|fx_t - Sx_t\| \to 0 \quad \text{as} \quad t \to 0.\end{aligned}$$

(4) Let $p \in A^{-1}0 \cap F(S)$. Then, it follows from Lemma 2.2 (Resolvent Identity) that

$$J_r y_t = J_{\frac{r}{2}}\left(\frac{1}{2}y_t + \frac{1}{2}J_r y_t\right).$$

Then we have

$$\|J_r y_t - p\| = \left\|J_{\frac{r}{2}}\left(\frac{1}{2}y_t + \frac{1}{2}J_r y_t\right) - p\right\| \le \left\|\frac{1}{2}\left(y_t - p\right) + \frac{1}{2}\left(J_r y_t - p\right)\right\|.$$

By the inequality (2.1) ($l = 2, \lambda = \frac{1}{2}$), we obtain that

$$\begin{aligned}\|J_r y_t - p\|^2 &\le \left\|J_{\frac{r}{2}}\left(\frac{1}{2}y_t + \frac{1}{2}J_r y_t\right) - p\right\|^2\\ &\le \left\|\frac{1}{2}\left(y_t - p\right) + \frac{1}{2}\left(J_r y_t - p\right)\right\|^2\\ &\le \frac{1}{2}\|y_t - p\|^2 + \frac{1}{2}\|J_r y_t - p\|^2 - \frac{1}{4}g(\|y_t - J_r y_t\|)\\ &\le \frac{1}{2}\|y_t - p\|^2 + \frac{1}{2}\|y_t - p\|^2 - \frac{1}{4}g(\|y_t - J_r y_t\|)\\ &= \|y_t - p\|^2 - \frac{1}{4}g(\|y_t - J_r y_t\|)\end{aligned} \tag{3.4}$$

Thus, from (3.1), the convexity of the real function $\psi(t) = t^2$ ($t \in (-\infty, \infty)$) and the inequality (3.4), we have

$$\begin{aligned}\|x_t - p\|^2 &= \|J_r y_t - p\|^2\\ &\le \|y_t - p\|^2 - \frac{1}{4}g(\|y_t - J_r y_t\|)\\ &= \|t(fx_t - p) + (1-t)(Sx_t - p)\|^2 - \frac{1}{4}g(\|y_t - J_r y_t\|)\\ &\le t\|fx_t - p\|^2 + (1-t)\|x_t - p\|^2 - \frac{1}{4}g(\|y_t - J_r y_t\|)\end{aligned}$$

and hence

$$\frac{1}{4}g(\|y_t - J_r y_t\|)) \le t(\|fx_t - p\|^2 - \|x_t - p\|^2).$$

By boundedness of $\{fx_t\}$ and $\{x_t\}$, letting $t \to 0$ yields

$$\lim_{t\to 0} g(\|y_t - J_r y_t\|) = 0.$$

Thus, from the property of the function g in (2.1), it follows that

$$\lim_{t\to 0} \|y_t - J_r y_t\| = 0.$$

(5) By (4), we have

$$\|x_t - y_t\| \le \|x_t - J_r y_t\| + \|J_r y_t - y_t\| = \|J_r y_t - y_t\| \to 0 \quad (t \to 0).$$

(6) By (3) and (5), we have

$$\begin{aligned}\|y_t - Sy_t\| &\le \|y_t - Sx_t\| + \|Sx_t - Sy_t\| \\ &\le \|y_t - Sx_t\| + \|x_t - y_t\| \to 0 \quad (t \to 0). \qquad \square\end{aligned}$$

We establish the strong convergence of the net $\{x_t\}$ as $t \to 0$, which guarantees the existence of solutions of the variational inequality (3.2).

Theorem 3.2. Let E be a uniformly convex Banach space having a uniformly Gâteaux differentiable norm. Let $\{x_t\}$ be a net defined by (3.1), and let $\{y_t\}$ be a net defined by $y_t = tfx_t + (1-t)Sx_t$ for $t \in (0,1)$. Then the nets $\{x_t\}$ and $\{y_t\}$ converge strongly to a point $q \in A^{-1}0 \cap F(S)$ as $t \to 0$, which is the unique solution of the variational inequality (3.2).

Proof. By (1) in Proposition 3.1, we see that $\{x_t\}$ and $\{y_t\}$ are bounded. Assume $t_n \to 0$. Set $x_n := x_{t_n}$ and $y_n := y_{t_n}$. We use the so-called optimization method (see [25]). Define $\phi : C \to \mathbb{R}$ by

$$\phi(x) = \mathrm{LIM}\|y_n - x\|^2, \quad x \in C,$$

where LIM is a Banach limit on l^∞. Since ϕ is continuous and convex, $\phi(z) \to \infty$ as $\|z\| \to \infty$ and E is reflexive, ϕ attain its infimum over C. Let

$$K = \{x \in C : \phi(x) = \min_{x \in C} \mathrm{LIM}\|y_n - x\|^2\}.$$

It is easily seen that K is a nonempty closed convex bounded subset of E. Moreover, K is invariant under J_r. Indeed, since $\|y_t - J_r y_t\| \to 0$ by (4) in Proposition 3.1, it follows that for each $z \in K$

$$\phi(J_r z) = \mathrm{LIM}\|y_n - J_r z\|^2 = \mathrm{LIM}\|J_r y_n - J_r z\|^2 \le \mathrm{LIM}\|y_n - z\|^2 = \phi(z),$$

so that $J_r K \subset K$. By the fixed point property for nonexpansive mappings of a uniformly convex Banach space E (cf. Theorem 5.1 in [26]), J_r has a fixed point, say q, in K. Also by (6) in Proposition 3.1, K is invariant under S, that is, $SK \subset K$. Also S has a fixed point $\widehat{q}$ in K. By uniform convexity of E, we have $q = \widehat{q}$ (cf. Theorem 2.9.11 in [22]) and hence $q \in A^{-1}0 \cap F(S)$. By Lemma 2.3, we obtain

$$\mathrm{LIM}\langle x - q, \mathcal{J}(y_n - q)\rangle \le 0, \quad \forall x \in C. \tag{3.5}$$

Since

$$y_t - q = t(fx_t - q) + (1-t)(Sx_t - q),$$

we have

$$\begin{aligned}\|y_t - q\|^2 &= t\langle fx_t - q, \mathcal{J}(y_t - q)\rangle + (1-t)\langle Sx_t - q, \mathcal{J}(y_t - q)\rangle \\ &\le t\langle fx_t - q, \mathcal{J}(y_t - q)\rangle + (1-t)\|x_t - q\|\|y_t - q\| \\ &\le t\langle fx_t - q, \mathcal{J}(y_t - q)\rangle + (1-t)\|y_t - q\|^2.\end{aligned}$$

Hence

$$\begin{aligned}\|y_t - q\|^2 &\le \langle fx_t - q, \mathcal{J}(y_t - q)\rangle \\ &= \langle fx_t - x, \mathcal{J}(y_t - q)\rangle + \langle x - q, \mathcal{J}(y_t - q)\rangle.\end{aligned} \tag{3.6}$$

Thus, by (3.5), for $x \in C$

$$\begin{aligned}\mathrm{LIM}\|y_t - q\|^2 &\le \mathrm{LIM}\langle fx_n - x, \mathcal{J}(y_n - q)\rangle + \mathrm{LIM}\langle x - q, \mathcal{J}(y_n - q)\rangle \\ &= \mathrm{LIM}\langle fx_n - x, \mathcal{J}(y_n - q)\rangle \\ &\le \mathrm{LIM}\|fx_n - x\|\|y_n - q\|.\end{aligned}$$

In particular,

$$\begin{aligned}\mathrm{LIM}\|y_n - q\|^2 &\le \mathrm{LIM}\|fx_n - fq\|\|y_n - q\| \\ &\le k\mathrm{LIM}\|x_n - q\|\|y_n - q\| \le k\mathrm{LIM}\|y_n - q\|^2.\end{aligned}$$

Hence

$$\mathrm{LIM}\|y_n - q\|^2 = 0,$$

and there exists a subsequence which is still denoted by $\{y_n\}$ such that $y_n \to q$.

Now assume that there exists another subsequence $\{y_m\}$ of $\{y_t\}$ such that $y_m \to \overline{q} \in A^{-1}0 \cap F(S)$. Then, by (5) in Proposition 3.1, $x_m \to \overline{q}$. So, it follows from (3.6) that

$$\|\overline{q} - q\|^2 \le \langle f\overline{q} - q, \mathcal{J}(\overline{q} - q)\rangle. \tag{3.7}$$

Interchanging $\overline{q}$ and q, we obtain

$$\|q - \overline{q}\|^2 \le \langle fq - \overline{q}, \mathcal{J}(q - \overline{q})\rangle. \tag{3.8}$$

Adding up (3.7) and (3.8) yields

$$2\|\overline{q} - q\|^2 \le \langle f\overline{q} - fq, \mathcal{J}(\overline{q} - q)\rangle + \langle \overline{q} - q, \mathcal{J}(\overline{q} - q)\rangle \le (1 + k)\|\overline{q} - q\|^2.$$

Since $k \in (0, 1)$, this implies that $\overline{q} = q$. Hence $y_t \to q$ as $t \to 0$ and by (5) in Proposition 3.1, also $x_t \to q$ as $t \to 0$.

Finally, we show that q is the unique solution of the variational inequality (3.2). To this end, noting

$$\begin{aligned}y_t - fx_t &= -\frac{1-t}{t}(y_t - Sx_t) \\ &= -\frac{1-t}{t}(x_t - Sx_t) + \left(1 - \frac{1}{t}\right)(y_t - x_t),\end{aligned}$$

and $\langle (I-S)x_t - (I-S)p, \mathcal{J}(x_t - p)\rangle \ge 0$ by nonexpansivity of S, we have for $p \in A^{-1}0 \cap F(S)$,

$$\begin{aligned}\langle y_t - fx_t, \mathcal{J}(x_t - p)\rangle &= -\frac{1-t}{t}\langle (I - S)x_t - (I - S)p, \mathcal{J}(x_t - p)\rangle \\ &\quad + \left(1 - \frac{1}{t}\right)\langle y_t - x_t, \mathcal{J}(x_t - p)\rangle \\ &\le \left(1 - \frac{1}{t}\right)\|y_t - x_t\|\|x_t - p\| \\ &\le \|y_t - x_t\|\|x_t - q\|.\end{aligned}$$

Since $x_t,\ y_t \to q$ and $fx_t \to fq$ as $t \to 0$, by (5) in Proposition 3.1, letting $t \to 0$ yields

$$\langle (I-f)q, \mathcal{J}(q-p)\rangle \le 0.$$

This implies that q is a solution of the variational inequality (3.2). If $\widetilde{q} \in A^{-1}0 \cap F(S)$ is other solution of the variational inequality (3.2), then

$$\langle (I-f)\widetilde{q}, \mathcal{J}(\widetilde{q}-q)\rangle \le 0. \tag{3.9}$$

Interchanging $\overline{q}$ and q, we obtain

$$\langle (I-f)q, \mathcal{J}(q-\widetilde{q})\rangle \le 0. \tag{3.10}$$

Adding up (3.9) and (3.10) yields

$$(1-k)\|\widetilde{q}-q\|^2 \le 0.$$

That is, $q = \widetilde{q}$. Hence q is the unique solution of the variational inequality (3.2). This completes the proof. $\square$

Corollary 3.3. Let E be a uniformly convex and uniformly smooth Banach space. Let $\{x_t\}$ be a net defined by (3.1), and let $\{y_t\}$ be a net defined by $y_t = tfx_t + (1-t)Sx_t$ for $t \in (0,1)$. Then the nets $\{x_t\}$ and $\{y_t\}$ converge strongly to a point $q \in A^{-1}0 \cap F(S)$ as $t \to 0$, which is the unique solution of the variational inequality (3.2).

3.2. Strong convergence of the explicit algorithm

Now, using Theorem 3.2, we show the strong convergence of the sequence generated by the explicit algorithm (3.3) to a point $q \in A^{-1}0 \cap F(S)$, which is the unique solution of the variational inequality (3.2).

Theorem 3.4. *Let E be a uniformly convex Banach space having a uniformly Gâteaux differentiable norm. Let $\{\alpha_n\} \in (0,1)$ and $\{r_n\} \subset (0,\infty)$ satisfy the conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(C3) $|\alpha_{n+1} - \alpha_n| \le o(\alpha_{n+1}) + \sigma_n,\quad \sum_{n=0}^{\infty} \sigma_n < \infty$ *(the perturbed control condition);*

(C4) $r_n \ge \varepsilon > 0$ *for* $n \ge 0$ *and* $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty$.

Let $x_0 = x \in C$ be chosen arbitrarily, and let $\{x_n\}$ be a sequence generated by

$$x_{n+1} = J_{r_n}(\alpha_n fx_n + (1-\alpha_n)Sx_n), \quad \forall n \ge 0. \tag{3.11}$$

Let $\{y_n\}$ be a sequence defined by $y_n = \alpha_n fx_n + (1-\alpha_n)Sx_n$. Then $\{x_n\}$ and $\{y_n\}$ converge strongly to $q \in A^{-1}0 \cap F(S)$, where q is the unique solution of the variational inequality (3.2).

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Proof. First, we note that by Theorem 3.2, there exists the unique solution q of the variational inequality

$$\langle (I-f)q, \mathcal{J}(q-p)\rangle \le 0, \quad \forall p \in A^{-1}0 \cap F(S),$$

where $q = \lim_{t\to 0} x_t = \lim_{t\to 0} y_t$ being defined by $x_t = J_r(tfx_t + (1-t)Sx_t)$ and $y_t = tfx_t + (1-t)Sx_t$ for $0 < t < 1$, respectively.

We divide the proof into several steps.

Step 1. We show that $\|x_n - p\| \le \max\{\|x_0 - p\|, \frac{1}{1-k}\|fp - p\|\}$ for all $n \ge 0$ and all $p \in A^{-1}0 \cap F(S)$, and so $\{x_n\}$, $\{y_n\}$, $\{J_{r_n}x_n\}$, $\{Sx_n\}$, $\{J_{r_n}y_n\}$, $\{Sy_n\}$ and $\{fx_n\}$ are bounded. Indeed, let $p \in A^{-1}0 \cap F(S)$. From $A^{-1}0 = F(J_r)$ for each $r > 0$, we know $p = Sp = J_{r_n}p$. Then we have

$$\begin{aligned}
\|x_{n+1} - p\| \le \|y_n - p\| &= \|\alpha_n(fx_n - p) + (1-\alpha_n)(Sx_n - Sp)\| \\
&\le \alpha_n\|fx_n - p\| + (1-\alpha_n)\|x_n - p\| \\
&\le \alpha_n(\|fx_n - fp\| + \|fp - p\|) + (1-\alpha_n)\|x_n - p\| \\
&\le \alpha_n k\|x_n - p\| + \alpha_n\|fp - p\| + (1-\alpha_n)\|x_n - p\| \\
&= (1-(1-k)\alpha_n)\|x_n - p\| + (1-k)\alpha_n\frac{\|fp - p\|}{1-k} \\
&\le \max\left\{\|x_n - p\|, \frac{1}{1-k}\|f(p) - p\|\right\}.
\end{aligned}$$

Using an induction, we obtain

$$\|x_n - p\| \le \max\left\{\|x_0 - p\|, \frac{1}{1-k}\|fp - p\|\right\}.$$

Hence $\{x_n\}$ is bounded. Also for $p \in A^{-1}0 \cap F(S)$, we get

$$\begin{aligned}
\|y_n - p\| &\le \alpha_n\|fx_n - fp\| + (1-\alpha_n)\|Sx_n - Sp\| + \alpha_n\|fp - p\| \\
&\le \alpha_n k\|x_n - p\| + (1-\alpha_n)\|x_n - p\| + \alpha_n\|fp - p\| \\
&= (1-(1-k)\alpha_n)\|x_n - p\| + (1-k)\alpha_n\frac{\|fp - p\|}{1-k} \\
&\le \max\left\{\|x_n - p\|, \frac{\|fp - p\|}{1-k}\right\},
\end{aligned}$$

and so $\{y_n\}$ is bounded, and so are $\{y_n\}$, $\{J_{r_n}y_n\}$, $\{Sx_n\}$, $\{Sy_n\}$ and $\{fx_n\}$. Moreover, it follows from condition (C1) that

$$\|y_n - Sx_n\| = \alpha_n\|fx_n - Sx_n\| \le \alpha_n(\|fx_n\| + \|Sx_n\|) \to 0 \quad (n \to \infty). \tag{3.12}$$

Step 2. We show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. First, from Lemma 2.2 (Resolvent

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identity), we observe that

$$\begin{aligned}
&\|J_{r_n}y_n - J_{r_{n-1}}y_{n-1}\| \\
&= \left\| J_{r_{n-1}}\left(\frac{r_{n-1}}{r_n}y_n + \left(1-\frac{r_{n-1}}{r_n}\right)J_{r_n}y_n\right) - J_{r_{n-1}}y_{n-1}\right\| \\
&\le \left\| \frac{r_{n-1}}{r_n}y_n + \left(1-\frac{r_{n-1}}{r_n}\right)J_{r_n}y_n\right) - y_{n-1}\right\| \\
&\le \|y_n - y_{n-1}\| + \left|1-\frac{r_{n-1}}{r_n}\right|(\|y_n - y_{n-1}\| + \|J_{r_n}y_n - y_{n-1}\|) \\
&\le \|y_n - y_{n-1}\| + \left|\frac{r_n - r_{n-1}}{\varepsilon}\right|M_1,
\end{aligned} \tag{3.13}$$

where $M_1 = \sup_{n\ge 0}\{\|J_{r_n}y_n - y_{n-1}\| + \|y_n - y_{n-1}\|\}$. Since

$$\begin{cases} y_n = \alpha_n f(x_n) + (1-\alpha_n)Sx_n, \\ y_{n-1} = \alpha_{n-1}f(x_{n-1}) + (1-\alpha_{n-1})Sx_{n-1}, \quad \forall n \ge 1, \end{cases}$$

by (3.13), we have for $n \ge 1$,

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|J_{r_n}y_n - J_{r_{n-1}}y_{n-1}\| \le \ \|y_n - y_{n-1}\| + \left|\frac{r_n - r_{n-1}}{\varepsilon}\right|M_1 \\
&= \|(1-\alpha_n)(Sx_n - Sx_{n-1}) + \alpha_n(fx_n - fx_{n-1}) \\
&\quad + (\alpha_n - \alpha_{n-1})(fx_{n-1} - Sx_{n-1})\| + \left|\frac{r_n - r_{n-1}}{\varepsilon}\right|M_1 \\
&\le (1-\alpha_n)\|x_n - x_{n-1}\| + k\alpha_n\|x_n - x_{n-1}\| \\
&\quad + |\alpha_n - \alpha_{n-1}|M_2 + \left|1-\frac{r_{n-1}}{r_n}\right|M_1 \\
&\le (1-(1-k)\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|M_2 + \left|\frac{r_n - r_{n-1}}{\varepsilon}\right|M_1,
\end{aligned} \tag{3.14}$$

where $M_2 = \sup\{\|f(x_n) - Sx_n\| : n \ge 0\}$. Thus, by (C3) we have

$$\|x_{n+1} - x_n\| \le (1-(1-k)\alpha_n)\|x_n - x_{n-1}\| + M_2(o(\alpha_n) + \sigma_{n-1}) + M_1\left|\frac{r_n - r_{n-1}}{\varepsilon}\right|.$$

In (3.14), by taking $s_{n+1} = \|x_{n+1} - x_n\|$, $\lambda_n = (1-k)\alpha_n$, $\lambda_n\delta_n = M_2 o(\alpha_n)$ and

$$\gamma_n = M_1\left|\frac{r_n - r_{n-1}}{\varepsilon}\right| + M_2\sigma_{n-1},$$

we have

$$s_{n+1} \le (1-\lambda_n)s_n + \lambda_n\delta_n + \gamma_n.$$

Hence, by the conditions (C1), (C2), (C3), (C4) and Lemma 2.4, we obtain

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0.$$

Now, in order to prove that $\lim_{n\to\infty}\|x_n - q\| = 0$, we consider two possible cases as in [8] and [16].

Case 1. Assume that $\{\|x_n - q\|\}$ is a monotone sequence. In other words, for n_0 large enough, $\{\|x_n - q\|\}$ is either nondecreasing or nonincreasing. Hence $\{\|x_n - q\|\}$ converges (since $\{\|x_n - q\|\}$ is bounded).

Step 3. We show that $\lim_{n\to\infty} \|y_n - J_{r_n}y_n\| = 0$. First, from Lemma 2.2 (Resolvent Identity), we know that

$$J_{r_n}y_t = J_{\frac{r_n}{2}}\left(\frac{1}{2}y_n + \frac{1}{2}J_{r_n}y_n\right).$$

Then we have

$$\|J_{r_n}y_n - q\| = \left\|J_{\frac{r}{2}}\left(\frac{1}{2}y_n + \frac{1}{2}J_{r_n}y_n\right) - q\right\| \leq \left\|\frac{1}{2}\left(y_n - q\right) + \frac{1}{2}\left(J_{r_n}y_n - q\right)\right\|.$$

By the inequality (2.1) ($l = 2, \lambda = \frac{1}{2}$), we obtain that

$$\begin{aligned} \|J_{r_n}y_n - q\|^2 &\leq \left\|J_{\frac{r_n}{2}}\left(\frac{1}{2}y_n + \frac{1}{2}J_{r_n}y_n\right) - q\right\|^2 \\ &\leq \left\|\frac{1}{2}\left(y_n - q\right) + \frac{1}{2}\left(J_{r_n}y_n - q\right)\right\|^2 \\ &\leq \frac{1}{2}\|y_n - q\|^2 + \frac{1}{2}\|J_{r_n}y_n - q\|^2 - \frac{1}{4}g(\|y_n - J_{r_n}y_n\|) \\ &\leq \frac{1}{2}\|y_n - q\|^2 + \frac{1}{2}\|y_n - q\|^2 - \frac{1}{4}g(\|y_n - J_{r_n}y_n\|) \\ &= \|y_n - q\|^2 - \frac{1}{4}g(\|y_n - J_{r_n}y_n\|) \end{aligned} \tag{3.15}$$

Thus, from (3.11), the convexity of the real function $\psi(t) = t^2$ ($t \in (-\infty, \infty)$) and the inequality (3.15), we have

$$\begin{aligned} \|x_{n+1} - q\|^2 &= \|J_{r_n}y_n - q\|^2 \\ &\leq \|y_n - q\|^2 - \frac{1}{4}g(\|y_n - J_{r_n}y_n\|) \\ &= \|\alpha_n(fx_n - q) + (1 - \alpha_n)(Sx_n - q)\|^2 - \frac{1}{4}g(\|y_n - J_{r_n}y_n\|) \\ &\leq \alpha_n\|fx_n - q\|^2 + (1 - \alpha_n)\|x_n - q\|^2 - \frac{1}{4}g(\|y_n - J_{r_n}y_n\|) \end{aligned}$$

and hence

$$\frac{1}{4}g(\|y_n - J_{r_n}y_n\|)) - \alpha_n\|fx_n - q\|^2 \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2.$$

Since $\{\|x_n - q\|\}$ converges, by condition (C1), we obtain

$$\lim_{n\to\infty} g(\|y_n - J_{r_n}y_n\|) = 0.$$

Thus, from the property of the function g in (2.1), it follows that

$$\lim_{n\to\infty} \|y_n - J_{r_n}y_n\| = 0.$$

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Step 4. We show that $\lim_{n\to\infty}\|x_n - y_n\| = 0$. Indeed, from Step 2 and Step 3, it follows that

$$\begin{aligned}\|x_n - y_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \\ &\le \|x_n - x_{n+1}\| + \|J_{r_n}y_n - y_n\| \to 0, \quad (n\to\infty).\end{aligned}$$

Step 5. We show that $\lim_{n\to\infty}\|y_n - Sy_n\| = 0$. In fact, by (3.12) and Step 4, we have

$$\begin{aligned}\|y_n - Sy_n\| &\le \|y_n - Sx_n\| + \|Sx_n - Sy_n\| \\ &\le \|y_n - Sx_n\| + \|x_n - y_n\| \to 0 \quad (n\to\infty).\end{aligned}$$

Step 6. We show that $\lim_{n\to\infty}\|y_n - J_r y_n\| = 0$ for $r > 0$. Indeed, from Lemma 2.2 (Resolvent identity), we obtain

$$\begin{aligned}\|J_{r_n}y_n - J_r y_n\| &= \left\|J_r\left(\frac{r}{r_n}y_n + \left(1-\frac{r}{r_n}\right)J_{r_n}y_n\right) - J_r y_n\right\| \\ &\le \left\|\left(\frac{r}{r_n}y_n + \left(1-\frac{r}{r_n}\right)J_{r_n}y_n\right) - y_n\right\| \\ &\le \left|1-\frac{r}{r_n}\right|\|y_n - J_{r_n}y_n\| \to 0 \quad (n\to\infty).\end{aligned} \tag{3.16}$$

Hence, by Step 3 and (3.16) we have

$$\|y_n - J_r y_n\| \le \|y_n - J_{r_n}y_n\| + \|J_{r_n}y_n - J_r y_n\| \to 0 \ \ (n\to\infty).$$

Step 7. We show that $\limsup_{n\to\infty}\langle (I-f)q, \mathcal{J}(q-y_n)\rangle \le 0$. To prove this, let a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ be such that

$$\limsup_{n\to\infty}\langle (I-f)q, \mathcal{J}(q-y_n)\rangle = \lim_{j\to\infty}\langle (I-f)q, \mathcal{J}(q-y_{n_j})\rangle$$

and $y_{n_j} \rightharpoonup z$ for some $z\in E$. From Step 5 and Step 6, it follows that $\lim_{j\to\infty}\|y_{n_j} - Sy_{n_j}\| = 0$ and $\lim_{j\to\infty}\|y_{n_j} - J_r y_{n_j}\| = 0$ for $r > 0$.

Now let $q = \lim_{t\to 0} x_t = \lim_{t\to 0} y_t$ where $y_t = tfx_t + (1-t)Sx_t$ and $x_t = J_r y_t$ for $r > 0$. Then we can write

$$y_t - y_{n_j} = t(fx_t - y_{n_j}) + (1-t)(Sx_t - y_{n_j})$$

and

$$\|x_t - y_{n_j}\| = \|J_r y_t - y_{n_j}\| \le \|y_t - y_{n_j}\| + \|J_r y_{n_j} - y_{n_j}\|.$$

Putting

$$a_j(t) = (1-t)^2\|Sy_{n_j} - y_{n_j}\|(2\|x_t - y_{n_j}\| + \|Sy_{n_j} - y_{n_j}\|) \to 0 \ \ (j\to\infty)$$

and

$$b_j(t) = \|J_r y_{n_j} - y_{n_j}\|(2\|y_t - y_{n_j}\| + \|J_r y_{n_j} - y_{n_j}\|) \to 0 \ \ (j\to\infty)$$

by Step 5 and Step 6, and using Lemma 2.1, we obtain

$$\begin{aligned}
\|x_t - y_{n_j}\|^2 &\le \|y_t - y_{n_j}\|^2 + b_j(t) \\
&\le (1-t)^2\|Sx_t - y_{n_j}\|^2 + 2t\langle fx_t - y_{n_j}, \mathcal{J}(y_t - y_{n_j})\rangle + b_j(t) \\
&\le (1-t)^2(\|Sx_t - Sy_{n_j}\| + \|Sy_{n_j} - y_{n_j}\|)^2 \\
&\quad + 2t\langle fx_t - x_t, \mathcal{J}(y_t - y_{n_j})\rangle + 2t\|x_t - y_{n_j}\|\|y_t - y_{n_j}\| \\
&\le (1-t)^2\|x_t - y_{n_j}\|^2 + a_j(t) + b_j(t) \\
&\quad + 2t\langle fx_t - x_t, \mathcal{J}(y_t - y_{n_j})\rangle + 2t\|x_t - y_{n_j}\|^2 + 2t\|x_t - y_{n_j}\|\|y_t - x_t\|.
\end{aligned}$$

The last inequality implies

$$\langle (I-f)x_t, \mathcal{J}(y_t - y_{n_j})\rangle \le \frac{t}{2}\|x_t - y_{n_j}\|^2 + \frac{1}{2t}(a_j(t) + b_j(t)) + \|x_t - y_t\|\|x_t - y_{n_j}\|.$$

It follows that

$$\limsup_{j\to\infty}\langle (I-f)x_t, \mathcal{J}(y_t - y_{n_j})\rangle \le \frac{t}{2}M^2 + \|x_t - y_t\|M, \tag{3.17}$$

where $M = \sup\{\|x_t - y_n\| : n \ge 0 \text{ and } t \in (0,1)\}$. Recalling (5) in Proposition 3.1, taking the $\limsup$ as $t \to 0$ in (3.17), and noticing the fact that the two limits are interchangeable due to the fact that J is uniformly continuous on bounded subsets of E from the strong topology of E to the weak* topology of E^*, we have

$$\limsup_{j\to\infty}\langle (I-f)q, \mathcal{J}(q - y_{n_j})\rangle \le 0.$$

Step 8. We show that $\lim_{n\to\infty}\|x_n - q\| = 0$. By using (3.11), we have

$$\|x_{n+1} - q\| \le \|y_n - q\| = \|\alpha_n(fx_n - q) + (1-\alpha_n)(Sx_n - q)\|.$$

Applying Lemma 2.1, we obtain

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\le \|y_n - q\|^2 \\
&\le (1-\alpha_n)^2\|Sx_n - q\|^2 + 2\alpha_n\langle fx_n - q, \mathcal{J}(y_n - q)\rangle \\
&\le (1-\alpha_n)^2\|x_n - q\|^2 + 2\alpha_n\langle fx_n - fq, \mathcal{J}(y_n - q)\rangle \\
&\quad + 2\alpha_n\langle fq - q, \mathcal{J}(y_n - q)\rangle \\
&\le (1-\alpha_n)^2\|x_n - q\|^2 + 2k\alpha_n\|x_n - q\|\|y_n - q\| \\
&\quad + 2\alpha_n\langle fq - q, \mathcal{J}(y_n - q)\rangle \\
&\le (1-\alpha_n)^2\|x_n - q\|^2 + 2k\alpha_n\|x_n - q\|^2 \\
&\quad + 2k\alpha_n\|x_n - q\|\|y_n - x_n\| + 2\alpha_n\langle fq - q, \mathcal{J}(y_n - q)\rangle.
\end{aligned}$$

It then follows that

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\le (1 - 2(1-k)\alpha_n + \alpha_n^2)\|x_n - q\|^2 \\
&\quad + 2k\alpha_n\|x_n - q\|\|y_n - x_n\| + 2\alpha_n\langle fq - q, \mathcal{J}(y_n - q)\rangle \\
&\le (1 - 2(1-k)\alpha_n)\|x_n - q\|^2 + \alpha_n^2 L^2 \\
&\quad + 2kL\alpha_n\|y_n - x_n\| + 2\alpha_n\langle (I-f)q, \mathcal{J}(q - y_n)\rangle,
\end{aligned} \tag{3.18}$$

where $L = \sup\{\|x_n - q\| : n \geq 0\}$. Put

$$\lambda_n = 2(1-k)\alpha_n \quad \text{and}$$
$$\delta_n = \frac{\alpha_n L^2}{2(1-k)} + \frac{kL}{(1-k)}\|y_n - x_n\| + \frac{1}{1-k}\langle (I-f)q, \mathcal{J}(q-y_n)\rangle.$$

From (C1), (C2), Step 4 and Step 7, it follows that have $\lambda_n \to 0$, $\sum_{n=0}^{\infty}\lambda_n = \infty$ and $\limsup_{n\to\infty}\delta_n \leq 0$. Since (3.18) reduces to

$$\|x_{n+1} - q\|^2 \leq (1-\lambda_n)\|x_n - q\|^2 + \lambda_n\delta_n,$$

from Lemma 2.4 with $\gamma_n = 0$, we conclude that $\lim_{n\to\infty}\|x_n - q\| = 0$. By Step 4, we also have $\lim_{n\to\infty} y_n = q$.

Case 2. Assume that $\{\|x_n - q\|\}$ is not a monotone sequence. Then, we can define a sequence of integers $\{\tau(n)\}$ for all $n \geq n_0$ (for some n_0 large enough) by

$$\tau(n) := \max\{k \in \mathbb{N} : k \leq n,\ \|x_k - q\| < \|x_{k+1} - q\|\}.$$

Clearly, $\{\tau(n)\}$ is a nondecreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and

$$\|x_{\tau(n)} - q\| \leq \|x_{\tau(n)+1} - q\|$$

for all $n \geq n_0$. In this case, by using the same argument as in Step 2 – Step 8 with $\{x_{\tau(n)}\}$, $\{y_{\tau(n)}\}$, $\{J_{r_{\tau(n)}} y_{\tau(n)}\}$, $\{J_r y_{\tau(n)}\}$, $\{Sx_{\tau(n)}\}$, $\{Sy_{\tau(n)}\}$, and $\{fx_{\tau(n)}\}$, we obtain the following:

Step 2′ $\lim_{n\to\infty}\|x_{\tau(n)+1} - x_{\tau(n)}\| = 0$;

Step 3′ $\lim_{n\to\infty}\|y_{\tau(n)} - J_{r_{\tau(n)}} y_{\tau(n)}\| = 0$.

Step 4′ $\lim_{n\to\infty}\|x_{\tau(n)} - y_{\tau(n)}\| = 0$.

Step 5′ $\lim_{n\to\infty}\|y_{\tau(n)} - Sy_{\tau(n)}\| = 0$.

Step 6′ $\lim_{n\to\infty}\|y_{\tau(n)} - J_r y_{\tau(n)}\| = 0$ for $r > 0$.

Step 7′ $\limsup_{n\to\infty}\langle (I-f)q, \mathcal{J}(q - y_{\tau(n)})\rangle \leq 0$.

Step 8′ $\lim_{n\to\infty}\|x_{\tau(n)} - q\| = 0$ and $\lim_{n\to\infty}\|x_{\tau(n)+1} - q\| = 0$.

Thus, from Lemma 2.5, we have

$$\|x_n - q\| \leq \|x_{\tau(n)+1} - q\|.$$

Therefore, $\lim_{n\to\infty}\|x_n - q\| = 0$. This completes the proof. □

Corollary 3.5. *Let E be a uniformly convex and uniformly smooth Banach space. Let C, A, J_{r_n}, S, and f be as in Theorem 3.4. Let $\{\alpha_n\} \in (0,1)$ and $\{r_n\} \subset (0,\infty)$ satisfy the conditions* (C1), (C2), (C3) and (C4) *in Theorem 3.4. Let $x_0 = x \in C$ be chosen arbitrarily, and let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = J_{r_n}(\alpha_n f x_n + (1-\alpha_n)Sx_n), \quad \forall n \geq 0.$$

Let $\{y_n\}$ *be a sequence defined by* $y_n = \alpha_n f x_n + (1-\alpha_n) S x_n$. *Then* $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to* $q \in A^{-1}0 \cap F(S)$, *where* q *is the unique solution of the variational inequality* (3.2).

Corollary 3.6. *Let* E, C, A, J_{r_n}, S, *and* f *be as in Theorem 3.4. Let* $\{\alpha_n\} \in (0,1)$ *and* $\{r_n\} \subset (0,\infty)$ *satisfy the conditions* (C1), (C2), (C3) and (C4) *in Theorem 3.4. Let* $x_0 = x \in C$ *be chosen arbitrarily, and let* $\{x_n\}$ *be a sequence generated by*

$$x_{n+1} = J_{r_n}(\alpha_n f x_n + (1-\alpha_n) S x_n + e_n), \quad \forall n \geq 0,$$

where $\{e_n\} \subset E$ *satisfies* $\sum_{n=0}^{\infty} \|e_n\| < \infty$ *or* $\lim_{n\to\infty} \frac{\|e_n\|}{\alpha_n} = 0$. *Let* $\{y_n\}$ *be a sequence defined by* $y_n = \alpha_n f x_n + (1-\alpha_n) S x_n + e_n$. *Then* $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to* $q \in A^{-1}0 \cap F(S)$, *where* q *is the unique solution of the variational inequality* (3.2).

Proof. Let $z_{n+1} = J_{r_n}(\alpha_n f z_n + (1-\alpha_n) S z_n)$ for $n \geq 0$. Then by Theorem 3.4, $\{z_n\}$ converges strongly to a point $q \in A^{-1}0 \cap F(S)$ where q is the unique solution of the variational inequality (3.2), and

$$\begin{aligned} \|x_{n+1} - z_{n+1}\| &\leq \|\alpha_n f x_n + (1-\alpha_n) S x_n - (\alpha_n z_n + (1-\alpha_n) S z_n + e_n)\| \\ &\leq \alpha_n \|f x_n - f z_n\| + (1-\alpha_n)\|S x_n - S z_n\| + \|e_n\| \\ &\leq (1-(1-k)\alpha_n)\|x_n - z_n\| + \|e_n\|. \end{aligned}$$

By Lemma 2.4, we obtain

$$\lim_{n\to\infty} \|x_n - z_n\| = 0,$$

and hence the desired result follows. □

Remark 3.7. (1) We point out that our iterative algorithms (3.1) and (3.3) for finding common point in the set of zeros of an accretive operator and the set of fixed points of a nonexpansive mapping are new ones different from those in the literature (see [20] and others in References). Thus Theorem 3.2 and Theorem 3.4 develop, and complement the recent corresponding results studied by many authors in this direction.

(2) If we take $fx = u$, $\forall x \in C$, as a constant function and $Sx = x$, $\forall x \in C$, as the identity mapping in Corollary 3.6, then the result extends corresponding results of Xu [18] and Song and Yang [19] in Hilbert spaces to a Banach space setting.

(3) The control condition (C3) in Theorem 3.4 can be replaced by the condition $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$; or the condition $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$, which are not comparable ([27]).

(4) The results in this paper apply to all L^p spaces, $1 < p < \infty$.

Acknowledgments

This research was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2014010491).

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 20, NO.4, 2016, COPYRIGHT 2016 EUDOXUS PRESS, LLC

Fixed points by some iterative algorithms in Banach and Hilbert spaces with some applications

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Abstract

In this paper, we apply a fixed point approach to derive an iterative method that converges to a general iteration scheme with errors in Hilbert and Banach spaces. Further, we obtain necessary and sufficient conditions for this sequence to converge to a common fixed point of two self mappings in normed spaces. Finally, we apply this iteration process to obtain a solution of a nonlinear equation.

1 Introduction

The iteration technique is a topic of great interest for a long time in the fixed point theory. During the last half of a century significant efforts have been applied to study fixed points by some iteration schemes. Indeed, such iterations based on the convergence of the sequence of iteration scheme and the kind of the mappings in each certain iteration. Therefore, this study is of a great importance for applications. This is the main motivation of the present paper and here in this paper, we prove some fixed point theorems using the so called, doubly G-iteration process with errors, then we apply this iteration to prove the existence theorem for the solution of a certain functional equation.

Let N be a normed linear space, $K \subset N$. A mapping $T : K \to K$ is said to be strongly pseudocontractive if there exists $t > 1$ such that the inequality

$$\|x - y\| \leq \|(1 + r)(x - y) - rt(Tx - Ty)\|$$

AMS: 47H10, 54H25
Key words and phrases: Mann iteration, G-iteration, Fixed points.

holds for every $x, y \in K$ and $r > 0$. A mapping U with domain $D(U)$ and range $R(U)$ in N is called accretive if the following inequality

$$\|x - y\| \leq \|x - y + s(Ux - Uy)\|$$

holds for every $x, y \in D(U)$ and for all $s > 0$. Browder [5] proved that T is pseudocontractive if and only if $(I - T)$ is accretive, where I denotes the identity operator.
Let X be a real Banach space and X^* its dual. For $1 < p < \infty$, the duality mapping $J_p : X \to 2^{X^*}$, is defined by

$$J_p(x) = \{f^* \in X^* : \langle x\ ,\ f^* \rangle = \|x\|^p, \|f^*\|^p = \|x\|^{p-1}\}, \quad x \in X,$$

where $\langle .,. \rangle$ denotes the generalized duality pairing between X and X^*. Recall that a mapping $A : X \to X$ is said to be accretive if for all $x, y\ \in D(A)$ there exists $j_p(x - y)\ \in\ J_p(x - y)$ such that

$$\langle Ax - Ay\ ,\ j_p(x - y) \rangle \geq 0,$$

and is said to be strongly accretive if $A - kI$ is accretive where $k \in (0, 1)$ is a constant and I denotes the identity operator on X. Let $S(T) = \{x^* \in D(A) : Ax^* = f\} \neq \emptyset$ denote the solution set of the equation $Ax = f$. If $\langle Ax - Ay\ ,\ j_p(x - y) \rangle \geq 0$ for all $x \in D(A)$ and $y = x^* \in S(T)$, then A is said to be quasi-accretive. The notion of strongly quasi-accretive is similarly defined.
A mapping $T : X \to X$ is said to be pseudo-contractive if for all $x, y\ \in D(T)$, there exists $j_p(x - y) \in J_p(x - y)$ such that

$$\langle (I - T)x - (I - T)y\ ,\ j_p(x - y) \rangle \geq 0.$$

Observe that T is pseudo-contractive if and only if $A = (I - T)$ is accretive. A map T is called hemicontractive if and only if $A = (I - T)$ is quasi-accretive.
Let X be a real Banach space of dimension $\dim X > 2$. The modulus of smoothness of X is defined by

$$\rho_X(\tau) = \sup \left\{ \frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau \right\}; \quad \tau > 0.$$

If $\rho_X(\tau) > 0$ for all $\tau > 0$, then X is said to be smooth. If there exists a constant $c > 0$ and a real number $1 < p < \infty$ such that $\rho_X(\tau) \leq C\tau^p$, then X is said to p-uniformly smooth Banach space, then the following geometric inequality holds (see e.g., [4, 6]):

$$\|x + y\|^p \leq \|x\|^p + p\ \langle y\ ,\ j_p(x + y) \rangle + C_p \|y\|^p, \quad x, y \in X, \tag{1}$$

for some real positive constant $C_p \geq 1$. If T is a self-mapping of a closed convex subset E of X and I the identity of X, then T is a nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \text{ for all } x, y \in E.$$

Krasnoselskii [12] proved that the sequence of iteration $\{T^n(x_0)\}$, starting from a given point $x_0 \in E$, does not converge necessarily to a fixed point of T, whereas the sequence $\{T^n{}_\lambda(x_0)\}$, where

$$T_\lambda = (1 - \lambda)I + \lambda T, \ \ 0 < \lambda < 1,$$

may converge to a fixed point of T, as shown by Krasnoselskii [12] which assumed $\lambda = \frac{1}{2}$, E compact and X uniformly convex. The above scheme has been extended by means of so-called,

Mann iterative process (see [15]), associated with T and described in the following way:
Let $x_0 \in E$ and $\{x_n\}$ be a sequence defined by

$$x_{n+1} = (1 - c_n)x_n + c_n T x_n, \tag{2}$$

for $n = 0, 1, 2, \dots,$ where
(a) $0 \leq c_n < 1, \quad n \geq 0,$
(b) $\lim\limits_{n\to\infty} c_n = 0,$
(c) $\sum\limits_{n=1}^{\infty} c_n = \infty.$
The scheme (2) has been studied by many authors (see for example [4, 11, 13, 16, 17, 20, 23, 25, 26, 27, 29]) and others. See also the work in double sequence setting [1, 2].
The scheme (2) has been extended by means of the so-called G-iteration process (see [21, 22]) associated with a single mapping T and described in the following manner:
Let $x_0 \in E$ and $\{x_n\}$ be a sequence defined by

$$x_{n+1} = (\mu_n - \lambda_n)x_n + \lambda_n T x_n + (1 - \mu_n)T x_{n-1} \text{ for } n \geq 0, \tag{3}$$

where $\{\mu_n\}$ and $\{\lambda_n\}$ satisfy
(i) $\lambda_0 = \mu_0 = 1,$
(ii) $0 < \lambda_n < 1, \ 0 \leq \mu_n \leq 1$ such that $\ \mu_n \geq \lambda_n, \, n > 0,$
(iii) $\lim\limits_{n\to\infty} \lambda_n = h > 0,$
(iv) $\lim\limits_{n\to\infty} \mu_n = 1.$
We note that when $\mu_n = 1$ for all $n \in \mathbb{N}$, the G-iteration process reduces to Mann iteration (1). It should be noted that the iteration scheme (3) is called G-iteration because its more general than Mann iteration process.
Let us compute the roots of equations numerically. That is, we would like to find numeric solutions to equations of one variable that can be written in the form $T(x) = 0$. Note that we do not put any restriction on the mapping T, only that it is a reasonably well-behaved mapping that we know how to evaluate. This kind of assumptions about the mapping turns out to be very important in numerical computation. In general, if we can place certain kinds of restrictions on the mapping, we will be able to use better and better methods to calculate its properties. A certain process using an algorithm for solving equations called fixed point iterations. In order to use fixed point iterations, we need the following information:
1. We need to know that there is a solution to the equation
2. We need to know approximately where the solution is (i.e. an approximation to the solution).
However, if the numerical method involves iteration then the root can be approximated to whatever accuracy we desire. One good way to measure the speed of the convergence is to use the ratio of the errors between successive iterations. In most cases the root must be obtained by numerical methods using a recipe or algorithm.
The idea of considering fixed point iteration procedures with errors comes from practical numerical computations. This topic of research play an important role in the stability problem of fixed point iterations. In 1995, Liu [14] initiated a study of fixed point iterations with errors. Several authors have proved some fixed point theorems for some certain iterations using several classes of mappings (see [3, 4, 9, 7, 8, 10, 18, 19, 25, 28] and others).
Now, we give the following iteration process with errors.

For $x_0 \in N$ and $n \in \mathbb{N} \cup \{0\}$, set

$$\begin{aligned} x_{n+1} &= (\mu_n - \lambda_n)x_n + \eta_n\lambda_n Tx_n + \eta_n(1-\mu_n)Tx_{n-1} \\ &+ (1-\eta_n)\lambda_n Sx_n + (1-\eta_n)(1-\mu_n)Sx_{n-1} + (1-\eta_n)(1-\mu_n)u_n \end{aligned} \tag{4}$$

where $\{\mu_n\}$ and $\{\lambda_n\}$ satisfy (i), (ii),(iii) and (iv) and $0 \leq \eta_n \leq 1$.

Remark 1.1 *It should be remarked that the above iteration scheme (4) is more general than some other scheme from literature. If $\mu_n = 1$ and $\eta_n = 1$ or $\eta_n = 0$ for all $n \in \mathbb{N}$, we obtain the Mann iteration process as defined by (2). Also, if $\eta_n = 1$ or $\eta_n = 0$, then we obtain the G-iteration process as defined by (3).*

2 Fixed Points in Hilbert Spaces

In this section, we give an implicit fixed point iterations associated with general hemicontractive mappings in Hilbert spaces. Some examples are also given to clear the role of the conditions on parameters of the defined iterations.

Definition 2.1 *Let $F(T) := \{x \in H : Tx = x\}$, $F(S) := \{x \in H : Sx = x\}$ and let K be a nonempty subset of H. Two mappings $S, T : K \to K$ are called general hemicontractive if $F(T) \cap F(S) \neq \varnothing$; and*

$$\|Tx - Sx^*\|^2 \leq \|x - Sx^*\|^2 + \|x - Tx\|^2 \text{ for all } x \in H, x^* \in F(T) \bigcap F(S).$$

In the above definition if $S = I$, where I denotes the identity mapping, we obtain the definition of hemicontractive mapping (see [24]).

Theorem 2.1 *Let K be a compact convex subset of a real Hilbert space H and $S, T : K \to K$ be continuous general hemicontractive mappings. Let $\{\alpha_n\}$ be a real sequence in [0,1] satisfying $\{\alpha_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0,1)$. For arbitrary $x_0 \in K$ and $\{v_n\}$ in K, define the sequence $\{x_n\}$ by*

$$\begin{aligned} &x_0 \in K \\ &Sx_n = \alpha_n x_{n-1} + (1-\alpha_n)(\lambda T\nu_n + (1-\lambda)S\nu_n) \end{aligned}$$

satisfying

$$\sum_{n\geq 1} \|S\nu_n - Sx_n\| < \infty.$$

Then $\{x_n\}$ converges strongly to a coincidence point of S, T.

Proof: The proof is very similar to the corresponding result in [24], by using Definition 2.1, so it will be omitted.
The next examples reveal that conditions on α_n must be imposed for the alodiality of Theorem 2.1.

Example 2.1 *Let (α_n) be a sequence in $(0,1)$ defined by $\alpha_n = 1 - \frac{5}{n+5}$. Define a sequence (x_n) in K by*

$$\begin{aligned} &x_0 = \frac{1}{5} \\ &x_n = \alpha_n x_{n-1} + (1-\alpha_n)(\lambda T\nu_n + (1-\lambda)S\nu_n). \end{aligned}$$

Let $\nu_n = \frac{1}{5}$, for all $n \geq 1$. Consider that

$$\begin{aligned} x_n &= \alpha_n x_{n-1} + (1-\alpha_n)(\lambda T\nu_n + (1-\lambda)S\nu_n) \\ &= \alpha_n x_{n-1} + 1 - \alpha_n \nu_n \\ &= \alpha_n x_{n-1} + \frac{1-\alpha_n}{5} = \left(1 - \frac{5}{n+5}\right)x_{n-1} + \frac{1}{n+5}. \end{aligned}$$

By induction, it follows that $x_n = \frac{1}{5}$ for all $n \geq 1$.

Example 2.2 *Let $X = \mathbb{R}^2$, $S, T : K \times K \to K \times K$, where $K = [-1, 1]$. Define*

$$T(x, y) = (-x, -y);\ x, y \in K.$$

Let S be the identity mapping I. Then, $(0, 0)$ is the only fixed point of and T. Let (α_n) be a sequence in $(0, 1)$. Fix $\delta > 1$ and define a sequence (α_n) in K by

$$\alpha_n = \begin{cases} (\frac{1}{n\delta}, 0) & \text{if } n \text{ is odd} \\ \\ (\frac{1}{n}, 0) & \text{if } n \text{ is even.} \end{cases}$$

Take the initial point $x_1 = (\frac{\delta-1}{\delta+1}, 0)$. Then it is shown easily by induction that $x_{2n+1} = (\frac{\delta-1}{\delta+1}, 0)$ for all $n \geq 1$. Thus, the sequence (x_n) does not converge to $(0, 0)$.

3 Convergence Theorem

In this section, it is proved that for two mappings S and T which satisfy condition (5) below, if the sequence of iteration associated with S, T as defined in (4), then it converges to a common fixed point of S and T.
The contractive condition to be used is the following:
For all $x, y \in N$,

$$\begin{aligned} &\lambda\|Tx - Ty\| + (1-\lambda)\|Sx - Sy\| \\ &\leq \lambda\Big[\alpha\|x-y\| + \beta\|x - Tx\| + \gamma\|y - Tx\| + \delta\max\{\|y - Ty\|, \|x - Ty\|\}\Big] \\ &+(1-\lambda)\Big[\alpha\|x-y\| + \beta\|x - Sx\| + \gamma\|y - Sx\| + \delta\max\{\|y - Sy\|, \|x - Sy\|\}\Big], \end{aligned} \tag{5}$$

where, $0 < \lambda < 1$ and $\alpha, \beta, \gamma \geq 0$ with $0 < \alpha + \beta + \gamma + \delta < 1$.
First of all we prove the following theorem:

Theorem 3.1 *Let K be a nonempty closed convex subset of a normed space N. Let $S, T : K \to K$ be mappings satisfying condition (5) and the following condition:*

$$S^2 = T^2 = I, \text{ where } I \text{ denotes the identity mapping.} \tag{6}$$

Let $\{x_n\}$ be the sequence of iteration as defined by (4). If the sequence $\{x_n\}$ converges to a point $z \in K$, then z is the unique common fixed point of S and T.

Proof: For each $n \geq 0$, we have

$$\begin{aligned}
&\eta_n\|x_{n+1}-Tz\|+(1-\eta_n)\|x_{n+1}-Sz\| \\
&\leq \eta_n[(\mu_n-\lambda_n)\|x_n-Tz\|+\lambda_n\|Tx_n-Tz\|+(1-\mu_n)\|Tx_{n-1}-Tz\|]+\eta_n(1-\mu_n)\|u_n\| \\
&+(1-\eta_n)[(\mu_n-\lambda_n)\|x_n-Sz\|+\lambda_n\|Sx_n-Sz\|+(1-\mu_n)\|Sx_{n-1}-Sz\|]. \qquad (7)
\end{aligned}$$

Since S and T satisfy (6), then by using (7) we have

$$\begin{aligned}
&\eta_n\|Tx_n-Tz\|+(1-\eta_n)\|Sx_n-Sz\| \\
&\leq \alpha\|x_n-z\|+\beta\Big(\eta_n\|x_n-Tx_n\|+(1-\eta_n)\|x_n-Sx_n\|\Big) \\
&+\gamma\Big(\eta_n\|z-Tx_n\|+(1-\eta_n)\|z-Sx_n\|\Big) \\
&+\delta\Big(\eta_n\max\{\|z-Tz\|,\|x_n-Tz\|\}+(1-\eta_n)\max\{\|z-Sz\|,\|x_n-Sz\|\}\Big). \qquad (8)
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\eta_n\|x_{n+1}-Tz\|+(1-\eta_n)\|x_{n+1}-Sz\| \\
&\leq (1-\mu_n)\|u_n\|+(\mu_n-\lambda_n)\Big(\eta_n\|x_n-Tz\|+(1-\eta_n)\|x_n-Sz\|\Big) \\
&+(1-\mu_n)\Big(\eta_n\|Tx_{n-1}-Tz\|+(1-\eta_n)\|Sx_{n-1}-Sz\|\Big) \\
&+\alpha\lambda_n\|x_n-z\|+\beta\lambda_n\Big(\eta_n\|x_n-Tx_n\|+(1-\eta_n)\|x_n-Sx_n\|\Big) \\
&+\gamma\lambda_n\Big(\|z-x_n\|+\eta_n\|x_n-Tx_n\|+(1-\eta_n)\|x_n-Sx_n\|\Big) \\
&+\delta\lambda_n\Big(\eta_n\max\{\|z-Tz\|,\|x_n-Tz\|\}+(1-\eta_n)\max\{\|z-Sz\|,\|x_n-Sz\|\}\Big). \qquad (9)
\end{aligned}$$

From (6), one gets

$$\begin{aligned}
&\eta_n\|x_n-Tx_n\|+(1-\eta_n)\|x_n-Sx_n\| \\
&\leq \frac{1}{\lambda_n}[\|\mu_n x_n-x_{n+1}\|]+\frac{1-\mu_n}{\lambda_n}\Big(\eta_n\|Tx_{n-1}\|+(1-\eta_n)\|Sx_{n-1}\|\Big)+\frac{1-\mu_n}{\lambda_n}\|u_n\|. \quad (10)
\end{aligned}$$

Then, $\|x_n-Tx_n\|\to 0$ and $\|x_n-Sx_n\|\to 0$ as $n\to\infty$. Substituting (10) in (9) and letting $n\to\infty$, we obtain

$$\lambda\|z-Tz\|+(1-\lambda)\|z-Sz\|\leq(1-h(1-\delta))\Big(\lambda\|z-Tz\|+(1-\lambda)\|z-Sz\|\Big).$$

Since $0\leq[1-h(1-\delta)]<1$ and $0<\lambda<1$, we obtain that

$$\lambda\|z-Tz\|+(1-\lambda)\|z-Sz\|=0.$$

Then, $Tz=z$ and $Sz=z$. Hence $Tz=Sz=z$, therefore z is a common fixed point of S and T. Now using (9), we have

$$Tz=z \text{ and hence, } Sz=z. \qquad (11)$$

It follows that

$$Sz = Tz = z$$

and z is a common fixed point of S and T.
Now to prove the uniqueness of z, let $w (w \neq z)$ be another common fixed point of S and T. Then, we have

$$\begin{aligned}
\|z-w\| &= \lambda\|T(Tz)-T(Tw)\| + (1-\lambda)\|S(Sz)-S(Tw)\| \\
&\leq \alpha\Big(\lambda\|STz - STw\| + (1-\lambda)\|TSz - TSw\|\Big) \\
&+ \beta\Big(\lambda\|STz - T^2z\| + (1-\lambda)\|TST - S^2z\|\Big) \\
&+ \gamma\Big(\lambda\|STw - T^2z\| + (1-\lambda)\|TSw - S^2z\|\Big) \\
&+ \delta\Big(\lambda \max\{\|STw - T^2w\|, \|STz - T^2w\|\} \\
&+ (1-\lambda)\max\{\|TSw - S^2w\|, \|TSz - S^2w\|\}\Big) \\
&\leq \eta\|z-w\|,
\end{aligned}$$

a contradiction, since $0 < \eta = \alpha + \gamma + \delta < 1$, then $z = w$. This completes the proof of the theorem.
Now, we give an example to discuss the validity of the hypothesis and degree of generality of the above theorem.

Example 3.1 *Let $N = \mathbb{R}^n$, the set of all n-tuples i.e., $x = (x_1, x_2, \ldots, x_n)$ of real numbers and the norm $\|x\|$ is defined by*

$$\|x\| = \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}}, \; x \in \mathbb{R}^n.$$

Further, let $K = \{x : \|x\| \leq 1, \, x \in \mathbb{R}^n\}$ and define the mappings $S, T : K \to K$ such that for arbitrary $x = (x_1, x_2, x_3, , , \ldots, x_n) \in K$,

$$Sx = (x_2, x_1, 0, 0 \ldots, 0),$$

and

$$Tx = (-x_1, -x_2, 0, 0, \ldots, 0).$$

Suppose $\{x_n\}$ be a sequence of elements of K satisfying condition (4) with $u_n = 0$, where

$$\eta_n = 1, \;\; \lambda_n = 1 - \frac{n}{2n+1} \;\; \text{and} \;\; \mu_n = \frac{1}{2} - \frac{n+3}{2n+3} \; \text{for} \;\; n \geq 0.$$

Consider, $x_1 = (0.5, 0, 0, \ldots, 0) \in K$, then it is easy to see that

$$x_2 = (0.166667, -0.333333, 0, 0, \ldots, 0) \;\; \text{and} \;\; x_3 = (0.21903, -0.2810981, 0, 0, \ldots, 0) \; \text{etc.}$$

Now it is easy to see that all conditions of Theorem 2.1 are satisfied, for instance taking $x = x_1$ and $y = x_2$ then, we have $0.4713996 \leq \alpha + \gamma + \delta < 1$, which is true, since $0 \leq \alpha + \gamma + \delta < 1$. Also, $0 = (\underbrace{0, 0, \ldots, 0}_{n})$ is the unique common fixed point of S and T.

4 An application

In this section we will apply the iteration process as defined below to find the solution of the equation $Tx = f$. For this purpose we let X be a Banach space and let $T : D(T) \subset X \to X$ and $S : D(S) \subset X \to X$ be locally Lipschitz and strongly quasi-accretive mappings. It is proved that an iteration process (4) converges strongly to the unique solution of the equation $Tx = Sx = f,\ f \in R(T) \bigcap R(S)$.

Theorem 4.1 *Let X be a real $p-$uniformly smooth Banach space. Also, suppose that the mapping $S : D(T) \subset X \to X$ and let $T : D(T) \subset X \to X$ be nonexpansive, locally Lipschitz and strongly quasi-accretive operators with open domains $D(T)$ and $D(S)$ in X such that the equation $Tx = Sx = f$ has a solution $x^* \in D(T) \bigcap D(S)$ for $f \in R(T) \bigcap R(S)$ arbitrary but fixed. Define $T_\lambda : D(T) \to X$ and $S_\lambda : D(S) \to X$ by*

$$T_\lambda x = x - \lambda(Tx - f) \ \textit{for all} \ x \in D(T)$$

and

$$S_\lambda x = x - \lambda(Sx - f) \ \textit{for all} \ x \in D(S).$$

Then there exists a neighborhood B of x^* and a real number $\lambda \in (0, 1)$ such that starting with an arbitrary $x_0 \in B$ the iteration sequence $\{x_n\}$ generated by (4) remains in B and converges strongly to x^* with convergence being at least as fast as geometric progression.

Proof: Since S, T are locally Lipschitz, there is an $r > 0$ such that T is Lipschitz on

$$B = \bar{B}_r(x_0) = \{x \in X : \|x - x^*\| \le r\} \subset D(T)$$

and S is Lipschitz on

$$B = \bar{B}_r(x_0) = \{x \in X : \|x - x^*\| \le r\} \subset D(S).$$

Let $k \in (0, 1)$ and $L > 1, p > 1$ denote the strong accretivity and Lipschitz constant of A respectively. Observe that $f = Tx^*$. Pick an arbitrary $x_0 \in B$, choose

$$h\lambda = \left(\frac{k}{L^p C_p}\right)^{\frac{1}{p-1}}$$

and generate the sequence $\{x_n\}_{n\ge0}$ as in (4). We now prove that $x_n \in B$, for all $n \ge 0$.
Suppose that $x_0 \in B$. Then

$$\begin{aligned}
&\|x_{n+1} - x^*\|^p = \|(\mu_n - \lambda_n)x_n + \lambda_n\eta_n Tx_n + (1 - \mu_n)\eta_n(Tx_{n-1} + u_n)\\
&+(1 - \eta_n)\lambda_n Sx_n + (1 - \mu_n)(1 - \eta_n)(Sx_{n-1}) - x^*\|^p\\
&= \|(\mu_n - \lambda_n)x_n + \lambda_n\eta_n[x_n - \lambda(Tx_n - f)]\\
&+(1 - \mu_n)\eta_n[x_{n-1} - \lambda(Tx_{n-1} - f) + u_n]\\
&+\lambda_n(1 - \eta_n)[x_n - \lambda(Sx_n - f) + u_n]\\
&+(1 - \mu_n)(1 - \eta_n)[x_{n-1} - \lambda(Sx_{n-1} - f) + u_n] - x^*\|^p\\
&= \|x_n - \lambda\lambda_n\bigg(\eta_n(Tx_n - Tx^*) + (1 - \eta_n)(Sx_n - Sx^*)\bigg)\\
&-(1 - \mu_n)\lambda\bigg(\eta_n(Tx_n - Tx^*) + (1 - \eta_n)(Sx_n - Sx^*)\bigg) + (1 - \mu_n)u_n - x^*\|^p\\
&= \|x_n - x^* - (\lambda\lambda_n + (1 - \mu_n)\lambda)(\eta_n(Tx_n - Tx^*) + (1 - \eta_n)(Sx_n - Sx^*)) + (1 - \mu_n)u_n\|^p.
\end{aligned}$$

Now using (1), we obtain

$$\begin{aligned}
&\|x_{n+1}-x^*\|^p \\
&\le \|x_n-x^*+(1-\mu_n)u_n\|^p \\
&+p[(\mu_n-1)\lambda-\lambda\lambda_n]\langle \eta_n(Tx_n-Tx^*)+(1-\eta_n)(Sx_n-Sx^*), J_p(x_n-x^*)\rangle \\
&+C_p[\lambda\lambda_n+(1-\mu_n)\lambda]^p\|\eta_n(Tx_n-Tx^*)+(1-\eta_n)(Sx_n-Sx^*)\|^p.
\end{aligned}$$

Since S and T are nonexpansive, then

$$\begin{aligned}
&\|x_{n+1}-x^*\|^p \\
&\le (1-ph\lambda k+L^pC_p(h\lambda)^p)\|x_n-x^*\|^p+(1-\mu_n)\,\eta(u_n,p) \\
&\le (1-(pk-L^pC_p(h\lambda)^{p-1})h\lambda)\|x_n-x^*\|^p+(1-\mu_n)\,\eta(u_n,p) \\
&= (1-(p-1)k(\frac{k}{L^pC_p})^{\frac{1}{p-1}})\|x_n-x^*\|^p+(1-\mu_n)\,\eta(\|u_n\|,p)\le r^p+(1-\mu_n)\,\eta(u_n,p),
\end{aligned}$$

where $\eta(u_n,p)$ is a function depends on u_n and p. Now, since $x_0\in B$ by choice of the initial guess, it follows by the inductive hypothesis that the sequence $\{x_n\}$ remains in B. Set

$$\delta^*=\left(1-(p-1)k(\frac{k}{L^pC_p})^{\frac{1}{p-1}}\right)^{\frac{1}{p}}$$

and observe that $\delta^*\in(0,1)$ since

$$k<\frac{LC_p^{\frac{1}{p}}}{(p-1)^{\frac{p-1}{p}}}\ ,\quad \text{where}\ \ 1<p<\infty$$

Hence, we obtain

$$\|x_n-x^*\|^p\le(\delta^*)^{np}\|x_0-x^*\|^p+(1-\mu_n)\,\eta(u_n,p),$$

since $\delta^{*np}\to 0$ as $n\to\infty$ the assertions of the theorem follows and the proof is complete.

Acknowledgements The authors would like to thank Scientific Research Deanship at Umm Al-Qura University (Project ID 43305020) for the financial support.

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A variant of second-order Arnoldi method for solving the quadratic eigenvalue problem☆

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Abstract

In this paper, we give a variant of second-order Krylov subspace $R_n(A, B; u)$ based on a pair of square matrices A and B and a vector u, which is a modification of second-order Krylov subspace presented by Bai and Su [SIAM J. Matrix Anal. Appl., 26(2005) 640-659]. Then we can compute an orthonormal basis of $R_n(A, B; u)$ by using second-order Arnoldi procedure. By applying the standard Rayleigh-Ritz orthogonal projection technique, a variant of second-order Arnoldi method (VSOAR) for solving large-scale quadratic eigenvalue problems (QEPs) has been presented. Finally, numerical experiments are given to show the efficiency of the new method.

Keywords: variant of second-order Arnoldi method (VSOAR), Krylov subspace, quadratic eigenvalue problem (QEP), Arnoldi procedure, Rayleigh-Ritz orthogonal projection
2000 MSC: 65F10, 65F50.

1. Introduction

The large-scale quadratic eigenvalue problem (QEP)

$$Q(\lambda)x = (\lambda^2 M + \lambda D + K)x = 0, \tag{1}$$

☆This work is supported by NNSF of China No. 11461046 and No. 61175127, Program of Young Scientist of Jiangxi Province, China No. 20122BCB23003, and Science Fund of Educational Department of Jiangxi Province No. GJJ14154.

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(where $M, D, K \in \mathbb{R}^{n\times n}$,$\lambda \in \mathbb{C}$ and $x \in \mathbb{R}^n$) arises in many scientific and engineering applications, see [14] for a survey. Krylov subspace methods for the solution of quadratic eigenvalue problems have been studied by many authors, such as Parlett and Chen [14], Saad [18], Mehrmann and Watkins [11], Bai and Su [3] and the references therein. A Krylov subspace-based method is often the method of choice due to its simplicity, its availability of reliable and efficient processes for generating its orthonormal basis, and the superiority of convergence [3, 7, 8, 13, 18]. Many state-of-the-art Krylov subspace methods for solving large-scale eigenvalue problems are presented in [4]. Moreover, the solution methods for quadratic eigenvalue problem are reviewed by Tisseur and Meerbergen in [23].

As well known, the generalized eigenvalue problem of the form $Ax = \lambda Bx$ can be reduced to the linear eigenvalue problem in a form such as $B^{-1}Ax = x$, explicitly or implicitly, and then a Krylov subspace-based method can be applied [3]. The quadratic eigenvalue problem (QEP) of the following form

$$(\lambda^2 M + \lambda D + K)x = 0$$

is usually processed in two stages, as recommended in most literature such as [5, 6, 8, 1], public domain packages, and proprietary software today. At the first stage, the QEP is transformed into an equivalent generalized eigenvalue problem:

$$Cy = \lambda Gy \tag{2}$$

where $y^T = [\lambda x^T, x^T]$, and C and G are in forms as follows

$$C = \begin{pmatrix} -D & -K \\ I & 0 \end{pmatrix}, G = \begin{pmatrix} M & 0 \\ 0 & I \end{pmatrix},$$

where the matrix M is assumed to be nonsingular throughout the paper. At the second stage, it transforms the generalized eigenvalue problem (2) to a standard eigenvalue problem $Ax = \lambda x$ and then a Krylov subspace-based methods can be used for this problem[2, 12, 16, 17, 19, 20, 24]. This kind of approach can take advantages of Krylov subspace-based methods, such as the fast convergence rate and the simultaneous convergence of a group of eigenvalues, but it also suffers some disadvantages, such as having to solve the generalized eigenvalue problem, which has twice the dimension of the original QEP and, more importantly, will lose the original structures of the

QEP in the process of linearization [3]. The reader is referred to [23] for a recent survey on theory, applications, and algorithms of the QEP.

For years, researchers have been studying numerical methods which can be applied to the large-scale QEP directly, such as the Jacobi-Davidson method[21, 22], Arnoldi and Lanczos-type process[9]. Instead of using the linearization technique, the QEP can be projected onto a properly chosen low-dimensional subspace to reduce to a QEP directly with matrix dimension of lower order. Unfortunately, the method is strongly dependent on the initial approximation. Then the reduced QEP problem can then be solved by a standard dense matrix technique. The method computes one eigenvalue each time with local convergence versus Krylov subspace methods in which a group of eigenvalues are approximated with global convergence. In [15], a direct Krylov-type subspace method with a generalized Arnoldi procedure has been briefly described, but the procedure presented does not compute an orthonormal basis of the desired Krylov-type subspace. In [9], Arnoldi and Lanczos-type processes are developed to construct projections of the QEP. The convergence of these methods is usually slower than a Krylov subspace method applied to the mathematically equivalent linear eigenvalue problem. Recently, a subspace approximation based method was recently presented in [10], by using the perturbation theory of the QEP. The success of the method is strongly dependent on the initial approximation, although Rayleigh quotient iteration can be used for acceleration. In [3], Bai and Su developed a projection method, named by SOAR, which not only can be applied to the QEP directly to preserve the essential structures of the QEP, but also achieves the superior global convergence behaviors of Krylov subspace methods by linearization.

Motivated by the idea of [3], we present a new and efficient variant of SOAR in this paper, denoted by VSOAR mehtod. Firstly, we introduce a variant of second-order Krylov subspace $R_n(A, B; u)$ based on a pair of square matrices A and B and a vector u . The basis vectors of the subspace are defined via a homogenous recurrence of degree 2 with coefficient matrices A and B . Then a variant of second-order Arnoldi (VSOAR) procedure is presented for generating an basis of $R_n(A, B; u)$. As an application of the VSOAR procedure, a Rayleigh-Ritz orthogonal projection technique based on $R_n(A, B; u)$ is discussed for finding the largest magnitude eigenvalue and the corresponding eigenvector of the large-scale QEP (1).

The rest of this paper is organized as follows. In Section 2, we introduce a variant of second-order Krylov subspace $R_n(A, B; u)$ and a simple VSOAR procedure for generating an orthonormal basis of subspace. In Section 3 we discuss the deflation and the convergence of the VSOAR. In Section 4, a Rayleigh-Ritz procedure for solving the QEP (1) is presented. Numerical examples are presented in Section 5 to show the efficiency of this new method.

2. A variant of second-order Krylov subspace.

In this section, we first define a variant of second-order Krylov subspace induced by a pair of matrices A and B and a vector u. Then we discuss the motivation for such a generalization.

Definition 1. *Let A and B be square matrices of order N, and let $u \neq 0$ be a vector of order N. Then the sequence*

$$r_0, r_1, r_2, ..., r_{n-1}, \cdots \tag{3}$$

where

$$\begin{cases} r_0 = u \\ r_1 = Br_0 \\ r_2 = Ar_1 \\ r_j = Ar_{j-1} + Br_{j-2}, j \geq 3 \end{cases} \tag{4}$$

is called a variant of second-order Krylov sequence based on A,B and u. The space

$$R_n(A, B; u) = span\{r_0, r_1, r_2, ...r_{n-1}\}$$

is called a variant of the n-th second-order Krylov subspace.

First, just like second-order Krylov sequence, the variant of second-order Krylov sequence also has the important characterization in terms of matrix polynomials, i.e.,

$$\begin{cases} r_0 = u, \\ r_1 = Br_0 = Bu, \\ r_2 = Ar_1 = ABu, \\ r_3 = Ar_2 + Br_1 = (A^2B + B^2)u, \\ r_4 = Ar_3 + Br_2 = (A^3B + AB^2 + BAB)u, \\ r_5 = Ar_4 + Br_3 = (A^4B + A^2B^2 + ABAB + BA^2B + B^3)u. \end{cases} \tag{5}$$

Second, we note that the subspace $R_n(A, B; u)$ generalizes the standard Krylov subspace $K_n(B; u)$ in the way that when A is a zero matrix, that is,

$$R_n(0, B; u) = K_n(B; u).$$

We now discuss the motivation for the definition of the variant of second-order Krylov subspace $R_n(A, B; u)$ in the context of solving QEP (1). Recall that the QEP (1) can transformed into an equivalent generalized eigenvalue problem (2). If one applies a Krylov subspace technique to (2), then an associated Krylov subspace would naturally be

$$K_n(H; v) = span\{v, Hv, H^2 v, ..., H^{n-1} v\}, \tag{6}$$

where v is a starting vector of length $2N$, and

$$H = G^{-1}C = \begin{pmatrix} -M^{-1}D & -M^{-1}K \\ I & 0 \end{pmatrix}.$$

Let $A = -M^{-1}D$, $B = -M^{-1}K$ and $v = [u^T, 0]^T$; then the second-order Krylov vectors $\{r_j\}$ of length N and the standard Krylov vectors $\{H^j v\}$ of length $2N$ defined in (6) have the following relation:

$$H^j v = \begin{pmatrix} r_j \\ r_{j-1} \end{pmatrix}, for j \geq 1.$$

Note that, in the second-order Krylov subspace, we first used the matrix A , i.e.,$r_0 = u, r_1 = Au$. However, in this paper the matrix B was used firstly to construct the variant of second-order Krylov subspace. If we define $v = [0, u^T]^T$, then we can derive the variant of second-order Krylov vectors $\{r_j\}$ of length N defined in (3) and the standard Krylov vectors $H^j v$ of length $2N$ defined in (6) are related as the following form:

$$\begin{cases} H^0 v = \begin{pmatrix} 0 \\ r_0 \end{pmatrix}, \\ Hv = \begin{pmatrix} r_1 \\ 0 \end{pmatrix}, \\ H^j v = \begin{pmatrix} r_j \\ r_{j-1} \end{pmatrix}, for j \geq 3. \end{cases} \tag{7}$$

Equation (7) indicates that the subspace $R_n(A, B; u)$ of R^n should be able to provide sufficient information to let us directly work with the QEP, instead of using the subspace $K_n(H; v)$ of R^{2N} for the linearized eigenvalue problem (2).

We turn to the question of how to construct an orthonormal basis q_n of $R_n(A, B; u)$. Namely

$$span\{q_1, q_2, q_3, ..., q_n\} = R_n(A, B; u) \ for \ n \geq 1.$$

The following is a procedure to implicitly apply to the sequence of variant of second-order Krylov vector r_n to generate an orthonormal basis $\{q_1, q_2, q_3, ..., q_n\}$. We call it a VSOAR (variant of second-order Arnoldi) procedure.

Algorithm 1: (VSOAR procedure)

$q_2 = Bu/\|Bu\|_2$
$p_2 = 0$
for $j = 2, 3, ..., n$
    $r = Aq_j + Bp_j$
    $s = q_j$
    for $i = 2, 3, ...j$
        $t_{ij} = q_i^T r$
        $r := r - q_i t_{ij}$
        $s := s - q_i t_{ij}$
    end for
    $t_{j+1,j} = \|r\|_2$
    if $t_{j+1,j} = 0$ **stop**
    $q_{j+1} = r/t_{j+1,j}$
    $p_{j+1} = s/t_{j+1,j}$
end for
for $i = 2, 3, ..., n$
    $t_{i1} = q_i^T u$
    $q_1 = u - q_i t_{i1}$
end for
$t_{11} = \|q_1\|_2$
$q_1 = q_1/t_{11}$

Let us recall the following SOAR procedure for generating an orthonormal basis $\{\bar{q}_1, \bar{q}_2, \bar{q}_3, ..., \bar{q}_n\}$ of the second-order Krylov subspace $G_n(A, B; u)$,where H and v are defined in (7).

Algorithm 2:[3] (SOAR procedure)

$\bar{q}_1 = u/\|u\|_2$
$\bar{p}_1 = 0$
for $j = 1, 2, 3, ..., n$
  $\bar{r} = A\bar{q}_j + B\bar{p}_j$
  $\bar{s} = \bar{q}_j$
  for $i = 1, 2, 3, ...j$
    $\bar{t}_{ij} = \bar{q}_i^T \bar{r}$
    $\bar{r} := \bar{r} - \bar{q}_i \bar{t}_{ij}$
    $\bar{s} := \bar{s} - \bar{q}_i \bar{t}_{ij}$
  end for
  $\bar{t}_{j+1,j} = \|\bar{r}\|_2$
  if $\bar{t}_{j+1,j} = 0$ **stop**
  $\bar{q}_{j+1} = \bar{r}/\bar{t}_{j+1,j}$
  $\bar{p}_{j+1} = \bar{s}/\bar{t}_{j+1,j}$
end for

If Q_n,$\bar{Q}_n$ respectively denotes the $N \times n$ matrix with column vectors $\{q_1, q_2, q_3, ..., q_n\}$ and $\{\bar{q}_1, \bar{q}_2, \bar{q}_3, ..., \bar{q}_n\}$; P_n,$\bar{P}_n$ respectively denotes the $N \times n$ matrix with column vectors $\{p_1, p_2, p_3, ..., p_n\}$ and $\{\bar{p}_1, \bar{p}_2, \bar{p}_3, ..., \bar{p}_n\}$; T_n,$\bar{T}_n$ respectively denotes the $n \times n$ upper Hessenberg matrix with nonzero entries t_{ij},$\bar{t}_{ij}$ as defined in the Algorithm 1 and Algorithm 2. In [3], the following relations hold:

$$A\bar{Q}_n + B\bar{P}_n = \bar{Q}_n \bar{T}_n + \bar{q}_{n+1} e_n^T \bar{t}_{n+1,n}, \tag{8}$$

$$\bar{Q}_n = \bar{P}_n \bar{T}_n + \bar{q}_{n+1} e_n^T \bar{t}_{n+1,n}, \tag{9}$$

with the orthonormality of the vector sequence $\{\bar{q}_1, \bar{q}_2, \bar{q}_3, ..., \bar{q}_n, \bar{q}_{n+1}\}$. Let $\tilde{T}_n$ be an $(n+1) \times n$ upper Hessenberg matrix of the form $\tilde{T}_n = \begin{pmatrix} \bar{T}_n \\ e^T \bar{t}_{n+1,n} \end{pmatrix}$. Then equations (8) and (9) can be rewritten in the compact form

$$\begin{pmatrix} A & B \\ I & 0 \end{pmatrix} \begin{pmatrix} \bar{Q}_n \\ \bar{P}_n \end{pmatrix} = \begin{pmatrix} \bar{Q}_{n+1} \\ \bar{P}_{n+1} \end{pmatrix} \tilde{T}_n. \tag{10}$$

We note that if we let Bu be an initial vector , then in Algorithm 1 lines 1-14 is an SOAR procudere which is based on matrix A and B with the initial Bu . Let $\hat{Q}_{n-1} = [q_2, q_3, q_4, ..., q_n]$, $\hat{P}_{n-1} = [p_2, p_3, p_4, ..., p_n]$,

$$\hat{T}_{n-1} = \begin{pmatrix} t_{22} & t_{23} & t_{24} & \cdots & t_{2n} \\ t_{32} & t_{33} & t_{34} & \cdots & t_{3n} \\ 0 & t_{43} & t_{44} & \cdots & t_{4n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & t_{nn} \end{pmatrix}, \tilde{T}_{n-1} = \begin{pmatrix} \hat{T}_{n-1} \\ e_n^T t_{n+1,n} \end{pmatrix}.$$

Then the following relations hold:

$$A\hat{Q}_{n-1} + B\hat{P}_{n-1} = \hat{Q}_{n-1}\hat{T}_{n-1} + q_{n+1}e_n^T t_{n+1,n}, \tag{11}$$

$$\hat{Q}_{n-1} = \hat{P}_{n-1}\hat{T}_{n-1} + q_{n+1}e_n^T t_{n+1,n}. \tag{12}$$

Lines 15-22 are one step of Arnoldi procedure, and so we can get

$$t_{11}q_1 = u - \sum_{i=2}^{n} t_{i1}q_i. \tag{13}$$

For the rest of this section, we prove that the vector sequence $\{q_1, q_2, q_3, ..., q_n\}$ indeed is an orthonormal basis of the generalized second-order Krylov subspace $R_n(A, B; u)$. First, we introduce the following theorem:

Theorem 2. *[3] If $t_{j+1,j} \neq 0$ for $j \geq 1$ in Algorithm 2, then the vector sequence $\{\bar{q}_1, \bar{q}_2, \bar{q}_3, ..., \bar{q}_n\}$ forms an orthonormal basis of the second Krylov subspace $G_n(A, B; u)$,i.e.,*

$$span\{\bar{Q}_n\} = G_n(A, B; u) \; for \; j \geq 1$$

and $\bar{q}_i^T \bar{q}_k = 0$ if $i \neq k$ and $\bar{q}_i^T \bar{q}_i = 1$ for $i, k = 1, 2, ..., n$.

Similar to the Theorem 2 in [3], we can get the following theorem.

Theorem 3. *If $t_{j+1,j} \neq 0$ for $j \geq 1$ in Algorithm 1, then the vector sequence $\{q_1, q_2, q_3, ..., q_n\}$ forms an orthonormal basis of the variant second Krylov subspace $R_n(A, B; u)$,i.e.,*

$$span\{Q_n\} = R_n(A, B; u) \; for \; j \geq 1$$

and $q_i^T q_k = 0$ if $i \neq k$ and $q_i^T q_i = 1$ for $i, k = 1, 2, ..., n$.

Proof. From Theorem 2, we know that

$$span\{q_2, q_3, ..., q_n\} = span\{r_1, r_2, ..., r_{n-1}\}$$

and $q_i^T q_k = 0$ if $i \neq k$ and $q_i^T q_i = 1$ for $i, k = 2, 3, ..., n$. From Lines 16-22 of Algorithm 1 we can get

$$span\{q_1, q_2, q_3, ..., q_n\} = span\{r_0, r_1, r_2, ..., r_{n-1}\}$$

and $q_i^T q_k = 0$ if $i \neq k$ and $q_i^T q_i = 1$ for $i, k = 1, 2, ..., n$.

Remark: In [3], the authors discussed deflation SOAR process in detail. In fact, we can apply it directly to the VSOAR process without any modifications.

Now let us discuss the situation where breakdown occurs. According to [3], we can get the theorem as follows.

Theorem 4. *The VSOAR procedure (Algorithm 1) with matrices A and B and starting vector u breaks down at a certain step j if and only if the Arnoldi procedure with matrix H and starting vector v breaks down at the same step j.*

Proof. As the essence of VSOAR procedure with matrix A ,B and initial vector u is the SOAR procedure with matrix A ,B and initial vector Bu, adding a step for orthogonalization with u. Correspondingly, we decompose the sencond Arnoldi procedure with matrix H and initial vector v into two stages. The first stage is an Arnoldi procedure with matrix H and initial vector Hv; the second stage is orthogonal with v. The second stage of two procedures is equivalent. So we only need to prove the first stage of two procedures breaks down at the same step.

The essence of SOAR procedure with matrix A, B and initial vector Bu is solving the orthogonal basis of the following vectors:

$$r_1 = Bu, r_2 = ABu = Ar_0, r_j = Ar_{j-1} + Br_{j-2}, j \geq 3.$$

Correspondingly, the essence of Arnoldi procedure with matrix H and initial vector Hv is solving the orthogonal basis of the following vectors:

$$Hv = \begin{pmatrix} r_1 \\ 0 \end{pmatrix}, H^j v = \begin{pmatrix} r_j \\ r_{j-1} \end{pmatrix}, j \geq 2.$$

From Theorem 3, we can know that SOAR procedure with matrices A and B and starting vector Bu breaks down at a certain step j if and only if the Arnoldi procedure with matrix H and starting vector Hv breaks down at the same step j.

3. A projection method applied directly to the QEP.

In this section, we apply the variant of the second-order Krylov subspace and its orthonormal basis generated by the VSOAR procedure to develop a projection technique for solving the QEP (1). The technique is completely the same as that in [3], except for the new second-order Krylov subspace and VSOAR procedure. For completeness, we present the algorithm as follows:

Algorithm 5: (VSOAR method for solving the QEP directly)

1. Run the VSOAR procedure (Algorithm 4) with $A = -M^{-1}D$,$B = -M^{-1}K$and a starting vector u to generate an $N \times m$ orthogonal matrix Q_m whose columns span an orthonormal basis of $R_n(A, B; u)$.

2. Compute M_m,D_m,K_m as defined in (17)

3. Solve the reduced QEP (16) for (λ, g) and obtain the Ritz pairs (λ, z) , where $z = Q_m g/\|Q_m g\|_2$.

4. Test the accuracy of Ritz pairs (λ, z) as approximate eigenvalues and eigenvectors of the QEP (1) by the norms of residual vectors:

$$\|(\lambda^2 M + \lambda D + K)z\|_2 \tag{14}$$

Remark: At Step 4, we use the residual norms (14) as the accuracy assessment to indicate the errors of the approximate eigenpairs (λ, z). Also you can use the relative residual norms. For detail discussion, the reader is referred to [3].

4. Numerical examples

In this section, we present some examples to illustrate the performance of the VSOAR method for solving quadratic eigenvalue problem(QEP): $Q(\lambda)x = (\lambda^2 M + \lambda D + K)x = 0$. The numerical experiments are performed in Matlab on an Inter dual core processor (1.40GHz, 2GB RAM). All experiments of this section are with starting vector $u = [1, 1, \cdots, 1]^T$, subspace dimension

Table 1: IT and CPU for VSOAR and SOAR

	VSOAR		SOAR			VSOAR		SOAR	
n	IT	CPU	IT	CPU	n	IT	CPU	IT	CPU
400	28	0.9405	34	1.0369	450	28	1.2077	62	2.4297
500	26	1.3271	66	3.1764	550	26	1.7206	68	4.1856
600	26	2.1541	70	5.2887	700	24	2.8617	25	2.7034
800	22	3.3652	23	3.1755	1000	19	4.4574	21	4.4463

Table 2: IT and CPU for VSOAR and SOAR

	VSOAR		SOAR	
n	IT	CPU	IT	CPU
1000	95	22.7176	270	58.8461
1500	96	49.2231	1000	497.6421

$m = 10$ and stopped once the number of iterations is over 1000 or current residual norm satisfies the following condition

$$\|r_k\| = \|(\lambda^2 M + \lambda D + K)z\|_2 \leq 10^{-5}.$$

Example 1.We consider the quadratic eigenvalue problem(QEP): $Q(\lambda)x = (\lambda^2 M + \lambda D + K)x = 0$ with

$$M = 0.1 \times I, K = I,$$

and

$$D = \begin{pmatrix} 0.2 & -0.1 & & & \\ -0.1 & 0.2 & -0.1 & & \\ & \ddots & \ddots & \ddots & \\ & & -0.1 & 0.2 & -0.1 \\ & & & -0.1 & 0.1 \end{pmatrix}$$

In this example, we test the dimension (denoted as $\mathbf{n}$) of matrix with $n = 400 \sim n = 1000$. The iteration steps (denoted as IT) and the computing time (denoted as **CPU**) of both VSOAR method and SOAR method are

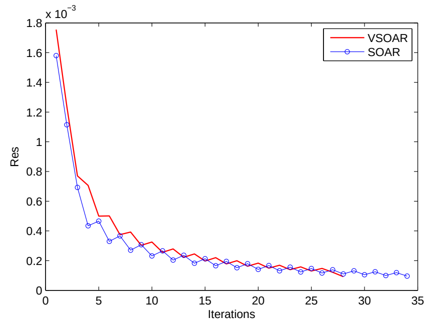

Figure 1: Example 1 for $n = 400$

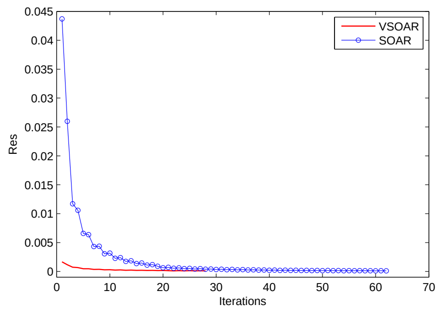

Figure 2: Example 1 for $n = 450$

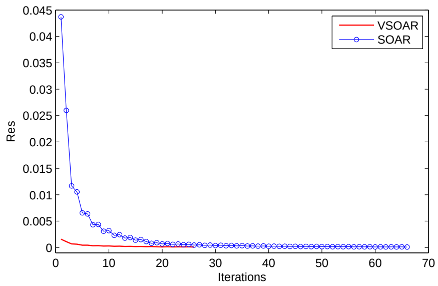

Figure 3: Example 1 for $n = 500$

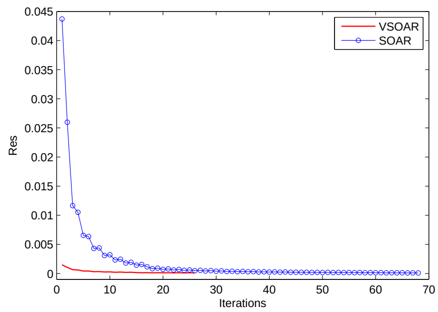

Figure 4: Example 1 for $n = 550$

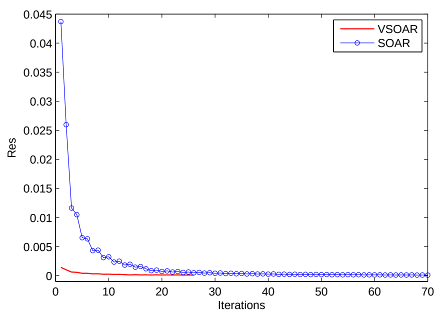

Figure 5: Example 1 for $n = 600$

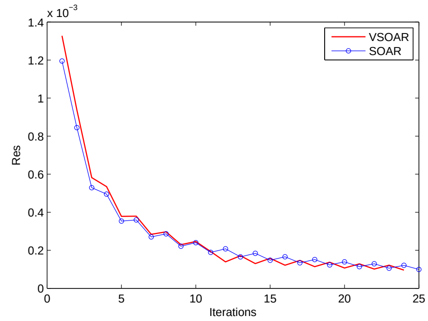

Figure 6: Example 1 for $n = 700$

listed in Table 1. The residual variation trend is showed from Figure 1 $\sim$ Figure 8.

Example 2. We consider the quadratic eigenvalue problem(QEP): $Q(\lambda)x = (\lambda^2 M + \lambda D + K)x = 0$ with

$$M = 0.1 \times I, K = I$$

$$D = \begin{pmatrix} 0 & -0.1 & & & 0.1 \\ -0.1 & 0 & -0.1 & & \\ & \ddots & \ddots & \ddots & \\ & & -0.1 & 0 & -0.1 \\ 0.1 & & & -0.1 & 0 \end{pmatrix}$$

In this example, we test the dimension of matrix $n = 1000$ and $n = 1500$. The iteration steps (denoted as IT) and the computing time (denoted as **CPU**) of both VSOAR method and SOAR method are listed in Table 2. From the Table 2 and Figure 9 and Figure 10, we know that the VSOAR is better than SOAR. Moreover, when the dimension of matrix is 1500×1500 , the SOAR will not converge.

As the above numerical experiments show, we observe that when SOAR method fails in some cases versus the VSOAR can do it well.

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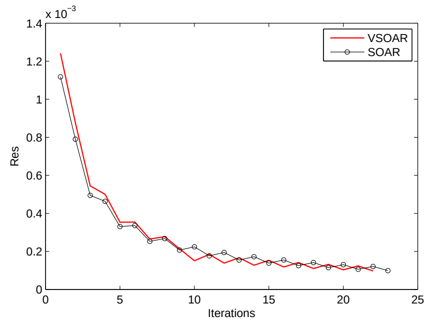

Figure 7: Example 1 for $n = 800$

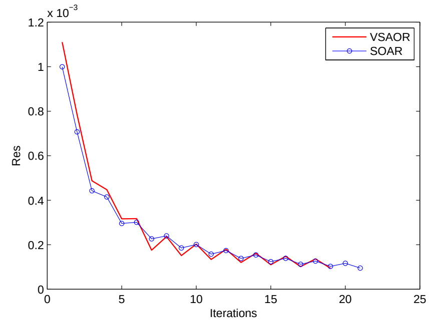

Figure 8: Example 1 for $n = 1000$

Table 3: IT and CPU for VSOAR and SOAR

	VSOAR		SOAR			VSOAR		SOAR	
n	IT	CPU	IT	CPU	n	IT	CPU	IT	CPU
300	53	0.8781	76	1.0774	500	93	4.6989	132	5.5053
800	154	20.8691	223	26.7822	1000	203	49.908	284	53.9909

Table 4: IT and CPU for VSOAR and SOAR

	VSOAR		SOAR			VSOAR		SOAR	
n	IT	CPU	IT	CPU	n	IT	CPU	IT	CPU
300	56	0.9128	80	1.1459	500	96	4.2466	138	5.4234
800	165	22.5349	230	27.4151	1000	213	46.0053	297	54.0763

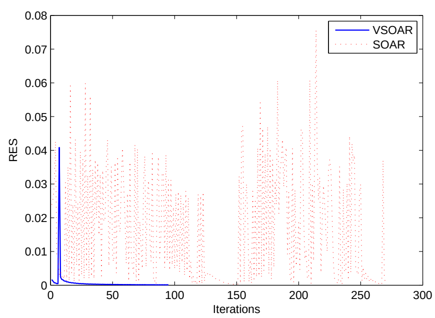

Figure 9: Example 2 for $n = 1000$

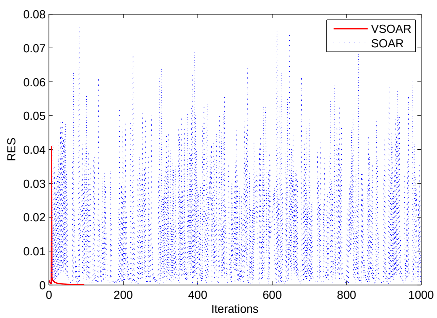

Figure 10: Example 2 for $n = 1500$

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Solvability for fractional differential inclusions with fractional nonseparated boundary conditions*

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Abstract

In this paper, a class of fractional differential inclusions with fractional non-separated (integral) boundary conditions is investigated under both convexity and non-convexity conditions on the multivalued term. Some new existence results are obtained by using standard fixed point theorems. Examples are given to illustrate the results.

Key words: *Fractional differential inclusions, boundary value problems, existence results, multivalued maps*

1 Introduction

Fractional differential equations have recently gained much importance and attention due to the fact that they have been proved to be valuable tools in the modeling of many physical phenomena [1, 2, 3]. For some recent developments on the existence results of fractional differential equations, we can refer to, for instance, [4, 5, 6, 7, 8, 9, 10, 11, 12, 13] and the references therein.

Differential inclusions arise in the mathematical modeling of certain problems in economics, optimal control, etc. and are widely studied by many authors; see [14, 15] and the references therein. For some recent works on differential inclusions of fractional order, we refer the reader to the references [4, 5, 16, 17, 18, 19, 20, 21, 22].

*Project supported by NNSF of China (Grant No.71461027), Guizhou province science and technology fund [2012]2340, lKZS[2012]10, 2015GZ36282, Hunan Provincial Natural Science Foundation of China (Grant No. 2015JJ6095) and Doctor Priming Fund Project of University of South China (Grant No. 2013XQD16).

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Motivated by the above papers, in this article, we study a new class of fractional boundary value problems, i.e., the following fractional differential inclusions with fractional non-separated boundary conditions

$$\begin{cases} {}^cD^\alpha x(t) \in F(t, x(t)),\ t \in [0, T],\ 1 < \alpha \leq 2,\ T > 0, \\ a_1x(0) + b_1x(T) = c_1, a_2({}^cD^\gamma x(0)) + b_2({}^cD^\gamma x(T)) = c_2, 0 < \gamma < 1, \end{cases} \tag{1}$$

where ${}^cD^q$ denotes the Caputo fractional derivative (see [23]) of order q, $F : [0, T] \times \mathbb{R} \to 2^{\mathbb{R}}$ is a multifunction and $a_i, b_i, c_i,\ i = 1, 2$ are real constants such that $a_1 + b_1 \neq 0$ and $b_2 \neq 0$.

Boundary value problems with integral boundary conditions constitute a very interesting and important class of problems. They include two, three, multi-point and non-local boundary value problems as special cases. Integral boundary conditions appear in the study of population dynamics and cellular systems etc. We can see the papers [24, 25], etc., for fractional differential equations with integral boundary conditions.

Along with the problem (1), we also consider the following fractional differential inclusions with fractional non-separated integral boundary conditions

$$\begin{cases} {}^cD^\alpha x(t) \in F(t, x(t)),\ t \in [0, T],\ 1 < \alpha \leq 2,\ T > 0, \\ a_1x(0) + b_1x(T) = c_1 \int_0^T g(s, x(s))ds, \\ a_2({}^cD^\gamma x(0)) + b_2({}^cD^\gamma x(T)) = c_2 \int_0^T h(s, x(s))ds,\ 0 < \gamma < 1, \end{cases} \tag{2}$$

where $g, h : [0, T] \times \mathbb{R} \to \mathbb{R}$ are given functions.

We shall give some existence results for the problems (1) and (2) when the multivalued term F is convex as well as nonconvex valued. The main tools used in this paper are nonlinear alternative of Leray and Schauder type for multivalued (single-valued) maps, a selection theorem due to Bressan and Colombo for lower semicontinuous multivalued maps with decomposable values and a fixed point theorem for multivalued contraction maps due to Covitz and Nadler. Our approachs used are standard, however their exposition in the framework of the problems (1) and (2) is new.

We remark that when $a_1 = 1$, $b_1 = 1$, $c_1 = 0$, $a_2 = 1$, $b_2 = 1$ and $c_2 = 0$, the problem (1) reduces to an anti-periodic fractional boundary value problem (see [9] with $F = f$ a given continuous function). Our results extend some results from the literature cited above and constitute a contribution to this emerging field of research. In the next section for the convenience of the reader we recall some of the main preliminary facts which we will use in this paper.

2 Preliminaries

We denote by $\mathcal{C} = C([0, T], \mathbb{R})$ the Banach space of all continuous functions from $[0, T]$ into $\mathbb{R}$ with the norm $\|x\| = \sup_{t\in[0,T]} |x(t)|$.

Let $(X, \|\cdot\|)$ be a normed space. We use the notations: $P(X) = \{Y \subseteq X : Y \neq \emptyset\}$, $P_{cl}(X) = \{Y \in P(X) : Y \text{ closed}\}$, $P_b(X) = \{Y \in P(X) : Y \text{ bounded}\}$, $P_{cp}(X) = \{Y \in P(X) : Y \text{ compact}\}$, $P_{cp,c}(X) = \{Y \in P(X) :$

Y compact, convex$\}$ and so on. Let A, $B \in P_{cl}(X)$, the Pompeiu-Hausdorff distance of A, B is defined as:

$$h(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\}.$$

A multivalued map $F : X \to P(X)$ is convex (closed) valued if $F(x)$ is convex (closed) for all $x \in X$. F is said to be completely continuous if $F(B)$ is relatively compact for every $B \in P_b(X)$. F is called upper semicontinuous on X, if for every $x_0 \in X$, the set $F(x_0)$ is a nonempty closed subset of X, and for every open set O of X containing $F(x_0)$, there exists an open neighborhood U_0 of x_0 such that $F(U_0) \subseteq O$. Equivalently, F is upper semicontinuous if the set $\{x \in X : F(x) \subseteq O\}$ is open for any open set O of X. F is called lower semicontinuous if the set $\{x \in X : F(x) \cap O \neq \emptyset\}$ is open for each open set O in X. If a multivalued map F is completely continuous with nonempty compact values, then F is upper semicontinuous if and only if F has a closed graph, i.e., if $x_n \to x_*$ and $y_n \to y_*$, then $y_n \in F(x_n)$ implies $y_* \in F(x_*)$ [26].

A multivalued map $F : [0,T] \to P_{cl}(X)$ is said to be measurable, if for every $x \in X$, the function $t \to d(x, F(t)) = \inf\{d(x,y) : y \in F(t)\}$ is measurable.

Definition 2.1. *A multivalued map $F : X \to P_{cl}(X)$ is called*
(1) γ-Lipschitz if there exists $\gamma > 0$ such that

$$h(F(x), F(y)) \leq \gamma d(x,y), \text{ for each } x, y \in X.$$

(2) a contraction if it is γ-Lipschitz with $\gamma < 1$.

Definition 2.2. *A multivalued map $F : [0,T] \times \mathbb{R} \to P(\mathbb{R})$ is said to be Carathéodory if:*
(1) $t \to F(t,x)$ is measurable for each $x \in \mathbb{R}$;
(2) $x \to F(t,x)$ is upper semicontinuous for a.e. $t \in [0,T]$.
Further, a Carathéodory function F is said to be L^1- Carathéodory if:
(3) for each $l > 0$, there exists $\varphi_l \in L^1([0,T], \mathbb{R}^+)$ such that

$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \leq \varphi_l(t)$$

for all $|x| \leq l$ and a.e. $t \in [0,T]$.

For each $x \in \mathcal{C}$, define the set of selections of F by

$$S_{F,x} = \{v \in L^1([0,T], \mathbb{R}) : v(t) \in F(t, x(t)) \text{ for a.e. } t \in [0,T]\}.$$

Lemma 2.1 (see [27]). *Let X be a Banach space. Let $F : [0,T] \times X \to P_{cp,c}(X)$ be an L^1- Carathéodory multivalued map and Γ be a linear continuous map from $L^1([0,T], X)$ to $C([0,T], X)$, then the operator*

$$\Gamma \circ S_F : C([0,T], X) \to P_{cp,c}(C([0,T], X)),\ y \mapsto (\Gamma \circ S_F)(y) = \Gamma(S_{F,y})$$

is a closed graph operator in $C([0,T], X) \times C([0,T], X)$.

Lemma 2.2 ([13]). *Let $\alpha > 0$, then the differential equation*

$$^cD^\alpha h(t) = 0$$

has solutions $h(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$ and

$$I^\alpha\, {}^cD^\alpha h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

here $c_i \in \mathbb{R}$, $i = 0, 1, 2, \cdots, n-1$, $n = [\alpha] + 1$.

Lemma 2.3. *For any $y \in C([0,T], \mathbb{R})$, the unique solution of the fractional non-separated boundary value problem*

$$\begin{cases} {}^cD^\alpha x(t) = y(t), \ t \in [0,T], \ 1 < \alpha \le 2, \\ a_1 x(0) + b_1 x(T) = c_1, a_2({}^cD^\gamma x(0)) + b_2({}^cD^\gamma x(T)) = c_2, 0 < \gamma < 1, \end{cases} \tag{3}$$

is given by

$$\begin{aligned} x(t) = &\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds - \frac{t\Gamma(2-\gamma)}{T^{1-\gamma}} \int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} y(s)ds \\ &+ \frac{t\Gamma(2-\gamma)c_2}{T^{1-\gamma}b_2} - \frac{b_1}{a_1+b_1}\Big(\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds \\ &- T^\gamma\Gamma(2-\gamma)\int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} y(s)ds\Big) \\ &- \frac{1}{a_1+b_1}\Big(\frac{b_1 c_2 T^\gamma \Gamma(2-\gamma)}{b_2} - c_1\Big). \end{aligned} \tag{4}$$

Proof. For $1 < \alpha \le 2$, by Lemma 2.2, we know that the general solution of the equation ${}^cD^\alpha x(t) = y(t)$ can be written as

$$x(t) = I^\alpha y(t) - k_1 - k_2 t = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds - k_1 - k_2 t, \tag{5}$$

where $k_1,\ k_2 \in \mathbb{R}$ are arbitrary constants. Since ${}^cD^\gamma k = 0$ (k is a constant), ${}^cD^\gamma t = \frac{t^{1-\gamma}}{\Gamma(2-\gamma)}$, ${}^cD^\gamma I^\alpha y(t) = I^{\alpha-\gamma} y(t)$ (see [23]), from (5) we have

$${}^cD^\gamma x(t) = I^{\alpha-\gamma} y(t) - \frac{k_2 t^{1-\gamma}}{\Gamma(2-\gamma)} = \int_0^t \frac{(t-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} y(s)ds - \frac{k_2 t^{1-\gamma}}{\Gamma(2-\gamma)}.$$

Using the boundary conditions, we obtain

$$a_1(-k_1) + b_1\Big(\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds - k_1 - k_2 T\Big) = c_1,$$

$$a_2 \times 0 + b_2\Big(\int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} y(s)ds - \frac{k_2 T^{1-\gamma}}{\Gamma(2-\gamma)}\Big) = c_2.$$

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Therefore we have

$$\begin{aligned} k_1 =& \frac{1}{a_1+b_1}\Big(\frac{b_1c_2T^\gamma\Gamma(2-\gamma)}{b_2}-c_1\Big)+\frac{b_1}{a_1+b_1} \\ &\times\Big(\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds-T^\gamma\Gamma(2-\gamma)\int_0^T\frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}y(s)ds\Big), \end{aligned}$$

$$k_2=\frac{\Gamma(2-\gamma)}{T^{1-\gamma}}\Big(\int_0^T\frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}y(s)ds-\frac{c_2}{b_2}\Big).$$

Substituting the values of k_1, k_2 in (5), we obtain (4). This completes the proof. □

From the proof of the above lemma, we notice that the solution (4) of the problem (3) does not depend on the parameter a_2, that is to say, the parameter a_2 is of arbitrary nature for this problem.

Definition 2.3. *A function $x\in\mathcal{C}$ is a solution of the problem (1) if it satisfies the boundary conditions in (1) and there exists a function $f\in L^1([0,T],\mathbb{R})$ such that $f(t)\in F(t,x(t))$ a.e. on $t\in[0,T]$ and*

$$\begin{aligned} x(t)=&\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)ds-\frac{t\Gamma(2-\gamma)}{T^{1-\gamma}}\int_0^T\frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}f(s)ds \\ &+\frac{t\Gamma(2-\gamma)c_2}{T^{1-\gamma}b_2}-\frac{b_1}{a_1+b_1}\Big(\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)ds \\ &-T^\gamma\Gamma(2-\gamma)\int_0^T\frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}f(s)ds\Big) \\ &-\frac{1}{a_1+b_1}\Big(\frac{b_1c_2T^\gamma\Gamma(2-\gamma)}{b_2}-c_1\Big). \end{aligned}$$

We end this section with two fixed point theorems.

Theorem 2.1 (Nonlinear alternative of Leray-Schauder type [28])**.** *Let X be a Banach space, C a closed convex subset of X, U an open subset of C with $0\in U$. Suppose that $F:\overline{U}\to P_{cp,c}(C)$ is an upper semicontinuous compact map. Then either (1) F has a fixed point in $\overline{U}$, or (2) there is a $x\in\partial U$ and $\lambda\in(0,1)$ such that $x\in\lambda F(x)$.*

Theorem 2.2 (Covitz and Nadler [29])**.** *Let (X,d) be a complete metric space. If $F:X\to P_{cl}(X)$ is a contraction, then F has a fixed point.*

3 Existence results

In this section, three existence results of the problem (1) are presented. The first one concerns with the convex valued case, and the others relate to the nonconvex valued case.

Theorem 3.1. *Suppose that the following (H1), (H2) and (H3) are satisfied.*
(H1) $F : [0,T] \times \mathbb{R} \to P_{cp,c}(\mathbb{R})$ is a Carathéodory multivalued map;
(H2) there exist $m \in L^{\infty}([0,T], \mathbb{R}^+)$ and $\varphi : [0,\infty) \to (0,\infty)$ continuous, non-decreasing such that

$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \le m(t)\varphi(|x|) \text{ for } x \in \mathbb{R}, t \in [0,T];$$

(H3) there exists a constant $M > 0$ such that

$$\frac{M}{O + \varphi(M)Q} > 1, \tag{6}$$

here

$$O = \frac{T^{\gamma}\Gamma(2-\gamma)|c_2|}{|b_2|} + \Big|\frac{b_1c_2T^{\gamma}\Gamma(2-\gamma)}{(a_1+b_1)b_2} - \frac{c_1}{a_1+b_1}\Big|,$$
$$Q = \|m\|_{L^\infty} T^{\alpha}\Big(1 + \frac{|b_1|}{|a_1+b_1|}\Big)\Big(\frac{1}{\Gamma(\alpha+1)} + \frac{\Gamma(2-\gamma)}{\Gamma(\alpha-\gamma+1)}\Big).$$

Then the boundary value problem (1) has at least one solution on $[0,T]$.

Proof. Consider the multivalued operator $N : \mathcal{C} \to P(\mathcal{C})$ defined as

$$N(x) = \{h \in \mathcal{C} : h = Sv, v \in S_{F,x}\}, \tag{7}$$

with

$$\begin{aligned}(Sv)(t) = &\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)ds - \frac{t\Gamma(2-\gamma)}{T^{1-\gamma}}\int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}v(s)ds \\ &+ \frac{t\Gamma(2-\gamma)c_2}{T^{1-\gamma}b_2} - \frac{b_1}{a_1+b_1}\Big(\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)ds \\ &- T^{\gamma}\Gamma(2-\gamma)\int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}v(s)ds\Big) \\ &- \frac{1}{a_1+b_1}\Big(\frac{b_1c_2T^{\gamma}\Gamma(2-\gamma)}{b_2} - c_1\Big).\end{aligned}$$

We put $Sv = S_1v + S_2v$ and

$$(S_1v)(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)ds, \quad (S_2v)(t) = -k_2^vt - k_1^v,$$

here k_1^v and k_2^v are constants given by

$$\begin{aligned}k_1^v = &\frac{b_1c_2T^{\gamma}\Gamma(2-\gamma)}{(a_1+b_1)b_2} - \frac{c_1}{a_1+b_1} + \frac{b_1}{a_1+b_1}\Big(\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)ds \\ &- T^{\gamma}\Gamma(2-\gamma)\int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}v(s)ds\Big),\end{aligned}$$

$$k_2^v = \frac{\Gamma(2-\gamma)}{T^{1-\gamma}}\Big(\int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}v(s)ds - \frac{c_2}{b_2}\Big).$$

Clearly, from Lemma 2.3, the fixed points of N are solutions of the problem (1). From (H1) and (H2), we have for each $x \in \mathcal{C}$, the set $S_{F,x}$ is nonempty [27]. Next we will show that N satisfies the assumptions of the nonlinear alternative of Leray-Schauder type. The proof is given in five steps.

Step 1: $N(x)$ is convex valued. Since F is convex valued, we know that $S_{F,x}$ is convex and therefore it is obvious that $N(x)$ is convex for each $x \in \mathcal{C}$.

Step 2: N maps bounded sets into bounded sets in $\mathcal{C}$. Let B_r be a bounded subset of $\mathcal{C}$ such that for any $x \in B_r$, $\|x\| \le r$. We prove that there exists a constant $l > 0$ such that for each $x \in B_r$, one has $\|h\| \le l$ for each $h \in N(x)$. Let $x \in B_r$ and $h \in N(x)$, then there exists $v \in S_{F,x}$ such that

$$h(t) = (Sv)(t) \quad \text{for} \quad t \in [0,T].$$

By simple calculations, we have

$$|(S_1v)(t)| \le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|v(s)|ds \le \varphi(r)\|m\|_{L^\infty}\frac{T^\alpha}{\Gamma(\alpha+1)},$$

$$|(S_2v)(t)| \le T|k_2^v| + |k_1^v|,$$

$$T|k_2^v| \le \varphi(r)\|m\|_{L^\infty}\frac{\Gamma(2-\gamma)T^\alpha}{\Gamma(\alpha-\gamma+1)} + \frac{T^\gamma\Gamma(2-\gamma)|c_2|}{|b_2|},$$

$$\begin{aligned}|k_1^v| \le &\frac{|b_1|}{|a_1+b_1|}\Big(\varphi(r)\|m\|_{L^\infty}\frac{T^\alpha}{\Gamma(\alpha+1)} + \varphi(r)\|m\|_{L^\infty}\frac{\Gamma(2-\gamma)T^\alpha}{\Gamma(\alpha-\gamma+1)}\Big)\\ &+ \Big|\frac{b_1c_2T^\gamma\Gamma(2-\gamma)}{(a_1+b_1)b_2} - \frac{c_1}{a_1+b_1}\Big|.\end{aligned}$$

Hence we obtain

$$\begin{aligned}\|h\| \le &\frac{T^\gamma\Gamma(2-\gamma)|c_2|}{|b_2|} + \Big|\frac{b_1c_2T^\gamma\Gamma(2-\gamma)}{(a_1+b_1)b_2} - \frac{c_1}{a_1+b_1}\Big|\\ &+ \varphi(r)\|m\|_{L^\infty}T^\alpha\Big(1+\frac{|b_1|}{|a_1+b_1|}\Big)\Big(\frac{1}{\Gamma(\alpha+1)} + \frac{\Gamma(2-\gamma)}{\Gamma(\alpha-\gamma+1)}\Big)\\ =&O + \varphi(r)Q = l.\end{aligned}$$

Step 3: N maps bounded sets into equicontinuous sets in $\mathcal{C}$. Let B_r be as in Step 2 and $0 \le t_1 < t_2 \le T$. For each $x \in B_r$ and $h \in N(x)$, there exists $v \in S_{F,x}$ such that $h(t) = (Sv)(t)$ for $t \in [0,T]$. Since we have

$$\begin{aligned}&|(S_1v)(t_2) - (S_1v)(t_1)|\\ \le&\Big|\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)ds\Big| + \Big|\int_0^{t_1}\frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)ds\Big|\\ \le&\frac{\varphi(r)\|m\|_{L^\infty}(t_2-t_1)^\alpha}{\Gamma(\alpha+1)} + \frac{\varphi(r)\|m\|_{L^\infty}\big(t_2^\alpha-(t_2-t_1)^\alpha-t_1^\alpha\big)}{\Gamma(\alpha+1)}\\ \le&\frac{\varphi(r)\|m\|_{L^\infty}(t_2^\alpha-t_1^\alpha)}{\Gamma(\alpha+1)}\end{aligned}$$

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and

$$\begin{aligned}|(S_2v)(t_2)-(S_2v)(t_1)| \le & |k_2^v|(t_2-t_1)\\ \le & \frac{\Gamma(2-\gamma)}{T^{1-\gamma}}\Big(\frac{\varphi(r)\|m\|_{L^\infty}T^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}+\frac{|c_2|}{|b_2|}\Big)(t_2-t_1),\end{aligned}$$

we deduce that

$$|h(t_2)-h(t_1)| \to 0 \text{ as } t_2 \to t_1$$

independently of $x \in B_r$ and $h \in N(x)$.

Step 4: N has a closed graph. Let $x_n \to x_*$, $h_n \in N(x_n)$ and $h_n \to h_*$, we need to show that $h_* \in N(x_*)$. Since $h_n \in N(x_n)$, there exists $v_n \in S_{F,x_n}$ such that $h_n(t) = (Sv_n)(t)$ for $t \in [0,T]$. We must prove that there exists $v_* \in S_{F,x_*}$ such that $h_*(t) = (Sv_*)(t)$ for $t \in [0,T]$.

Consider the continuous linear operator $\Gamma : L^1([0,T],\mathbb{R}) \to \mathcal{C}$ given by

$$\begin{aligned}v \to \Gamma(v)(t) = &\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)ds - \frac{t\Gamma(2-\gamma)}{T^{1-\gamma}}\int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}v(s)ds\\ &-\frac{b_1}{a_1+b_1}\Big(\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)ds - T^\gamma\Gamma(2-\gamma)\int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}v(s)ds\Big).\end{aligned}$$

And let

$$w(t) = \frac{t\Gamma(2-\gamma)c_2}{T^{1-\gamma}b_2} - \frac{1}{a_1+b_1}\Big(\frac{b_1c_2T^\gamma\Gamma(2-\gamma)}{b_2}-c_1\Big).$$

Clearly, we have $Sv = \Gamma v + w$ and

$$\Gamma(v_n)(t) = h_n(t) - w(t) \to h_*(t) - w(t) \ \text{ in } \mathcal{C}.$$

Also, by the definition of Γ, we have

$$h_n - w \in \Gamma(S_{F,x_n}).$$

It follows from Lemma 2.1 that $\Gamma\circ S_F$ is a closed graph operator. Since $x_n \to x_*$, we obtain

$$h_*(t) - w(t) = \Gamma(v_*)(t)$$

for some $v_* \in S_{F,x_*}$. This implies that $h_* \in N(x_*)$.

Step 5: A priori bounds for solutions. Let $x \in \lambda N(x)$ for some $\lambda \in (0,1)$. Then there exists $v \in S_{F,x}$ such that $x(t) = \lambda(Sv)(t)$ for $t \in [0,T]$. By the similar computations as in the step 2, we have

$$|x(t)| \le O + \varphi(\|x\|)Q \ \text{ for } \ t \in [0,T].$$

In view of (H3), there exists M such that $\|x\| \neq M$. Let us set

$$U = \{x \in \mathcal{C} : \|x\| < M\}.$$

As a consequence of Steps 1-4, together with the Arzela-Ascoli theorem, we can conclude that $N : \overline{U} \to P_{cp,c}(\mathcal{C})$ is a upper semicontinuous and completely continuous map. From the choice of U, there is no $x \in \partial U$ such that $x \in \lambda N(x)$ for some $\lambda \in (0,1)$. Hence by Theorem 2.1, we deduce that N has a fixed point $x \in \overline{U}$ which is a solution of the problem (1). This is the end of the proof. □

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Let A be a subset of $[0,T]\times\mathbb{R}$. A is said to be $\Sigma\otimes\mathcal{B}_{\mathbb{R}}$ measurable if A belongs to the σ-algebra generated by all sets of the form $J\times D$, where J is Lebesgue measurable in $[0,T]$ and D is a Borel set of $\mathbb{R}$. A subset A of $L^1([0,T],\mathbb{R})$ is said to be decomposable if for all u, $v\in A$ and $J\subseteq[0,T]$ Lebesgue measurable, then $u\chi_J+v\chi_{[0,T]-J}\in A$, where χ stands for the characteristic function.

Theorem 3.2. *Let (H2) and (H3) hold and assume:*
(H4) $F:[0,T]\times\mathbb{R}\to P_{cp}(\mathbb{R})$ is such that: (1) $(t,x)\to F(t,x)$ is $\Sigma\otimes\mathcal{B}_{\mathbb{R}}$ measurable; (2) the map $x\to F(t,x)$ is lower semicontinuous for a.e. $t\in[0,T]$. Then the problem (1) has at least one solution on $[0,T]$.

Proof. From (H2), (H4) and Lemma 4.4 of [22], the map

$$\mathcal{F}:\mathcal{C}\to P(L^1([0,T],\mathbb{R})),\quad x\to\mathcal{F}(x)=S_{F,x} \tag{8}$$

is lower semicontinuous and has nonempty closed and decomposable values. Then from a selection theorem due to Bressan and Colombo [30], there exists a continuous function $f:\mathcal{C}\to L^1([0,T],\mathbb{R})$ such that for all $x\in\mathcal{C}$, $f(x)(t)\in F(t,x(t))$ a.e. $t\in[0,T]$. Now consider the problem

$${}^cD^\alpha x(t)=f(x)(t),\ t\in[0,T] \tag{9}$$

with the boundary conditions in (1). Note that if $x\in\mathcal{C}$ is a solution of the problem (9), then x is a solution to the problem (1).

Problem (9) is then reformulated as a fixed point problem for the operator $N_1:\mathcal{C}\to\mathcal{C}$ defined by

$$N_1(x)(t)=(Sf(x))(t).$$

It can easily be shown that N_1 is continuous and completely continuous and satisfies all conditions of the Leray-Schauder nonlinear alternative for single-valued maps [28]. The proof is similar to that of Theorem 3.1, so we omit it here. This completes the proof. □

Theorem 3.3. *We assume that:*
(H5) Let $F:[0,T]\times\mathbb{R}\to P_{cp}(\mathbb{R})$ is such that: (1) the map $t\to F(t,x)$ is measurable for each $x\in\mathbb{R}$, (2) there exists $m\in L^\infty([0,T],\mathbb{R}^+)$ such that for a.e. $t\in[0,T]$ and all $x,y\in\mathbb{R}$,

$$h(F(t,x),F(t,y))\le m(t)|x-y|,\quad d(0,F(t,0))\le m(t).$$

If

$$\|m\|_{L^\infty}T^\alpha\Big(1+\frac{|b_1|}{|a_1+b_1|}\Big)\Big(\frac{1}{\Gamma(\alpha+1)}+\frac{\Gamma(2-\gamma)}{\Gamma(\alpha-\gamma+1)}\Big)<1, \tag{10}$$

then the problem (1) has at least one solution on $[0,T]$.

Proof. From (H5), for each $x\in\mathcal{C}$, the multivalued map $t\to F(t,x(t))$ is measurable and closed valued. Hence it has measurable selection (Theorem 2.2.1 [26]) and the set $S_{F,x}$ is nonempty. Let N be defined in (7). We will show that, under this situation, N satisfies the requirements of Theorem 2.2.

Step 1: For each $x \in \mathcal{C}$, $N(x) \in P_{cl}(\mathcal{C})$. Let $h_n \in N(x)$, $n \geq 1$, such that $h_n \to h$ in $\mathcal{C}$. Then $h \in \mathcal{C}$ and there exists $v_n \in S_{F,x}$, $n \geq 1$, such that

$$h_n(t) = (Sv_n)(t) \quad t \in [0,T].$$

By (H5), the sequence v_n is integrable bounded. Since F has compact values, we may pass to a subsequence if necessary to get that v_n converges to v in $L^1([0,T],\mathbb{R})$. Thus $v \in S_{F,x}$ and for each $t \in [0,T]$,

$$h_n(t) \to h(t) = (Sv)(t).$$

This means that $h \in N(x)$ and $N(x)$ is closed.

Step 2: There exists $\gamma < 1$ such that

$$h(N(x),N(y)) \leq \gamma \|x-y\| \text{ for all } x,y \in \mathcal{C}.$$

Let x, $y \in \mathcal{C}$ and $h_1 \in N(y)$, then there exists $v_1 \in S_{F,y}$ such that

$$h_1(t) = (Sv_1)(t),\ t \in [0,T].$$

From (H5)(2), we deduce

$$h(F(t,x(t)),F(t,y(t))) \leq m(t)|x(t)-y(t)|.$$

Hence, for a.e. $t \in [0,T]$, there exists $u \in F(t,x(t))$ such that

$$|v_1(t)-u| \leq m(t)|x(t)-y(t)|. \tag{11}$$

Consider the multivalued map $V : [0,T] \to P(\mathbb{R})$ given by

$$V(t) = \{u \in \mathbb{R} : |v_1(t)-u| \leq m(t)|x(t)-y(t)|\}.$$

Since $v_1(t)$, $\alpha(t) = m(t)|x(t)-y(t)|$ are measurable, Theorem III.41 in [31] implies that V is measurable. It follows from (H5) that the map $t \to F(t,x(t))$ is measurable. Hence by (11) and Proposition 2.1.43 in [26], the multivalued map $t \to V(t) \cap F(t,x(t))$ with nonempty closed values is measurable. Therefore, we can find $v_2(t) \in F(t,x(t))$ and

$$|v_1(t)-v_2(t)| \leq m(t)|x(t)-y(t)| \ \text{ for a.e. } t \in [0,T].$$

Let $h_2(t) = (Sv_2)(t)$, i.e., $h_2 \in N(x)$. Since

$$\begin{aligned}
&|h_1(t)-h_2(t)| \\
\leq& \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \big|v_1(s)-v_2(s)\big| ds \\
&+ T^\gamma \Gamma(2-\gamma) \int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} \big|v_1(s)-v_2(s)\big| ds \\
&+ \frac{|b_1|}{|a_1+b_1|} \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} \big|v_1(s)-v_2(s)\big| ds \\
&+ \frac{|b_1| T^\gamma \Gamma(2-\gamma)}{|a_1+b_1|} \int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} \big|v_1(s)-v_2(s)\big| ds \\
\leq& \|m\|_{L^\infty} T^\alpha \Big(1 + \frac{|b_1|}{|a_1+b_1|}\Big) \Big(\frac{1}{\Gamma(\alpha+1)} + \frac{\Gamma(2-\gamma)}{\Gamma(\alpha-\gamma+1)}\Big) \|x-y\|.
\end{aligned}$$

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Denote $\gamma = \|m\|_{L^\infty} T^\alpha \left(1+\frac{|b_1|}{|a_1+b_1|}\right)\left(\frac{1}{\Gamma(\alpha+1)}+\frac{\Gamma(2-\gamma)}{\Gamma(\alpha-\gamma+1)}\right)$. By using an analogous relation obtained by interchanging the roles of x and y, we get

$$h(N(x), N(y)) \leq \gamma \|x-y\|.$$

Now in view of (10), Theorem 2.2 implies that N has a fixed point which is a solution of the problem (1). This completes the proof. $\square$

4 Integral boundary problems

In this section the existence results of the problem (1) obtained above will be extended to the case of integral boundary conditions.

Lemma 4.1. *For any y, ξ, $\chi \in C([0,T],\mathbb{R})$, the unique solution of the fractional non-separated integral boundary value problem*

$$\begin{cases} {}^cD^\alpha x(t) = y(t), \ t \in [0,T], \ 1<\alpha\leq 2, \\ a_1 x(0) + b_1 x(T) = c_1 \int_0^T \xi(s)ds, \\ a_2({}^cD^\gamma x(0)) + b_2({}^cD^\gamma x(T)) = c_2 \int_0^T \chi(s)ds, \ 0<\gamma<1, \end{cases}$$

is given by

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds - \frac{t\Gamma(2-\gamma)}{T^{1-\gamma}} \int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} y(s)ds \\ &+ \frac{t\Gamma(2-\gamma)c_2}{T^{1-\gamma}b_2} \int_0^T \chi(s)ds - \frac{b_1}{a_1+b_1}\Big(\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds \\ &- T^\gamma \Gamma(2-\gamma) \int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} y(s)ds\Big) \\ &- \frac{b_1 T^\gamma \Gamma(2-\gamma)c_2}{b_2(a_1+b_1)} \int_0^T \chi(s)ds + \frac{c_1}{a_1+b_1}\int_0^T \xi(s)ds. \end{aligned}$$

To obtain the existence results of the problem (2), in view of Lemma 4.1, we define an operator $\Omega : \mathcal{C} \to P(\mathcal{C})$ as

$$\Omega(x) = \{h \in \mathcal{C} : h = Zv, v \in S_{F,x}\} \tag{12}$$

with

$$\begin{aligned} (Zv)(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds - \frac{t\Gamma(2-\gamma)}{T^{1-\gamma}} \int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} v(s)ds \\ &+ \frac{t\Gamma(2-\gamma)c_2}{T^{1-\gamma}b_2} \int_0^T h(s,x(s))ds - \frac{b_1}{a_1+b_1}\Big(\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds \\ &- T^\gamma \Gamma(2-\gamma) \int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} v(s)ds\Big) \\ &- \frac{b_1 T^\gamma \Gamma(2-\gamma)c_2}{b_2(a_1+b_1)} \int_0^T h(s,x(s))ds + \frac{c_1}{a_1+b_1}\int_0^T g(s,x(s))ds. \end{aligned}$$

Observe that if $x \in \mathcal{C}$ is a fixed point of the operator Ω, i.e., $x \in \Omega(x)$, then x is a solution of the problem (2).

From the definitions of the operators N and Ω, we know that the difference between them is very apparent, i.e., c_1, c_2 in (7) were replaced by $c_1 \int_0^T g(s, x(s))ds$ and $c_2 \int_0^T h(s, x(s))ds$ in (12). We omit the proofs of the following theorems, since they are similar to the ones obtained in Section 3.

Theorem 4.1. *Let (H1), (H2) hold and g, $h : [0,T] \times \mathbb{R} \to \mathbb{R}$ be continuous functions. Assume that:*
(A1) there exist functions m_2, $m_3 \in L^1([0,T], \mathbb{R}^+)$ and φ_2, $\varphi_3 : [0,\infty) \to (0,\infty)$ continuous and nondecreasing such that for $t \in [0,T]$, $x \in \mathbb{R}$,

$$|g(t,x)| \leq m_2(t)\varphi_2(|x|), \quad |h(t,x)| \leq m_3(t)\varphi_3(|x|);$$

(A2) there exists a constant $M > 0$ such that

$$\frac{M}{\varphi(M)Q + \varphi_3(M)\|m_3\|_{L^1}O + \frac{|c_1|}{|a_1+b_1|}\varphi_2(M)\|m_2\|_{L^1}} > 1,$$

here

$$O = \frac{T^\gamma \Gamma(2-\gamma)|c_2|}{|b_2|}\Big(1 + \frac{|b_1|}{|a_1+b_1|}\Big),$$

$$Q = \|m\|_{L^\infty} T^\alpha \Big(1 + \frac{|b_1|}{|a_1+b_1|}\Big)\Big(\frac{1}{\Gamma(\alpha+1)} + \frac{\Gamma(2-\gamma)}{\Gamma(\alpha-\gamma+1)}\Big).$$

Then the problem (2) has at least one solution on $[0,T]$.

Theorem 4.2. *Let g, $h : [0,T] \times \mathbb{R} \to \mathbb{R}$ be continuous functions. Assume (H2), (H4), (A1) and (A2) hold, then the problem (2) has at least one solution on $[0,T]$.*

Theorem 4.3. *Let $F : [0,T] \times \mathbb{R} \to P_{cp}(\mathbb{R})$ be as in Theorem 3.3. In addition, we suppose that g, $h : [0,T] \times \mathbb{R} \to \mathbb{R}$ are continuous and satisfy*

$$|g(t,x) - g(t,y)| \leq m_2(t)|x-y|,$$

$$|h(t,x) - h(t,y)| \leq m_3(t)|x-y|,$$

for each $t \in [0,T]$ and all x, $y \in \mathbb{R}$ with m_2, $m_3 \in L^1([0,T], \mathbb{R}^+)$. If

$$\|m\|_{L^\infty} T^\alpha \Big(1 + \frac{|b_1|}{|a_1+b_1|}\Big)\Big(\frac{1}{\Gamma(\alpha+1)} + \frac{\Gamma(2-\gamma)}{\Gamma(\alpha-\gamma+1)}\Big)$$
$$+ \frac{T^\gamma \Gamma(2-\gamma)|c_2|\|m_3\|_{L^1}}{|b_2|}\Big(1 + \frac{|b_1|}{|a_1+b_1|}\Big) + \frac{|c_1|\|m_2\|_{L^1}}{|a_1+b_1|} < 1,$$

then the problem (2) has at least one solution on $[0,T]$.

5 Examples

In this section, we give two examples to illustrate the results.

Example 1: Consider the fractional boundary value problem

$$\begin{cases} {}^cD^{\frac{3}{2}}x(t) \in F(t, x(t)), \ t \in [0, 1], \\ x(0) - \frac{1}{2}x(1) = 2.5, \\ 2({}^cD^{\frac{1}{2}}x(0)) + \frac{1}{3}({}^cD^{\frac{1}{2}}x(1)) = -\frac{1}{3}, \end{cases} \tag{13}$$

where $\alpha = \frac{3}{2}$, $\gamma = \frac{1}{2}$, $a_1 = 1$, $b_1 = -\frac{1}{2}$, $c_1 = 2.5$, $a_2 = 2$, $b_2 = \frac{1}{3}$, $c_2 = -\frac{1}{3}$, $T = 1$ and $F : [0, 1] \times \mathbb{R} \to P(\mathbb{R})$ is a multivalued map given by

$$F(t, x) = \{y \in \mathbb{R} : e^{-|x|} + \sin t + t^2 \le y \le 5 + \frac{|x|}{1 + x^2} + 6t^3\}.$$

In the context of this problem, we have

$$\|F(t, x)\| = \sup\{|v| : v \in F(t, x)\} \le 6 + 6t^2 \le 12, \text{ for } t \in [0, 1], x \in \mathbb{R}.$$

It is clear that F is convex compact valued and is of Carathéodory type. Let $m(t) \equiv 1$ and $\varphi(|x|) \equiv 12$, we get

$$\|F(t, x)\| = \sup\{|v| : v \in F(t, x)\} \le m(t)\varphi(|x|), \text{ for } t \in [0, 1], x \in \mathbb{R}.$$

As for the condition (6), since $O + \varphi(|x|)Q = O + 12Q$ (see O, Q in (H3)) is a constant, we can choose M large enough so that

$$\frac{M}{O + \varphi(M)Q} > 1.$$

Thus, by the conclusion of Theorem 3.1, the boundary value problem (13) has at least one solution on $[0, 1]$.

Example 2: Consider the fractional integral boundary value problem

$$\begin{cases} {}^cD^{\frac{5}{4}}x(t) \in F(t, x(t)), \ t \in [0, 1], \\ 3x(0) + \frac{1}{3}x(1) = \int_0^1 g(s, x(s))ds, \\ 2({}^cD^{\frac{1}{4}}x(0)) + 3({}^cD^{\frac{1}{4}}x(1)) = \frac{1}{4}\int_0^1 h(s, x(s))ds, \end{cases} \tag{14}$$

where $\alpha = \frac{5}{4}$, $\gamma = \frac{1}{4}$, $T = 1$, $a_1 = 3$, $b_1 = \frac{1}{3}$, $c_1 = 1$, $a_2 = 2$, $b_2 = 3$, $c_2 = \frac{1}{4}$,

$$F(t, x) = \Big[-\frac{1}{(4 + t)^2}|x| - \frac{1}{8}, -\frac{1}{10}\Big] \bigcup \Big[0, \frac{e^{-t}}{16}\sin^2 x\Big],$$

$$g(t, x) = \frac{1}{(3 + t)^2}\cos x, \ h(t, x) = \frac{|x|}{1 + |x|}.$$

From the data given above, we have

$$\sup\{|v| : v \in F(t, x)\} \le \frac{1}{8} + \frac{1}{(4 + t)^2}|x|, \ t \in [0, 1], \ x \in \mathbb{R},$$

$$h(F(t, x), F(t, y)) \le \max\{\frac{e^{-t}}{8}, \frac{1}{(4 + t)^2}\}|x - y|, \ t \in [0, 1], \ x, y \in \mathbb{R},$$

$$|g(t, x) - g(t, y)| \le \frac{1}{(3 + t)^2}|x - y|, |h(t, x) - h(t, y)| \le |x - y|, t \in [0, 1], x, y \in \mathbb{R}.$$

Then let $m_2(t) = \frac{1}{(3+t)^2}$, $m_3(t) = 1$ and $m(t) = \frac{1}{8} + \frac{e^{-t}}{8} + \frac{1}{(4+t)^2}$, we have

$$h(F(t,x), F(t,y)) \leq m(t)|x-y|, \quad d(0, F(t,0)) \leq m(t),$$

and

$$\begin{aligned}
&\|m\|_{L^\infty} T^\alpha \Big(1 + \frac{|b_1|}{|a_1+b_1|}\Big)\Big(\frac{1}{\Gamma(\alpha+1)} + \frac{\Gamma(2-\gamma)}{\Gamma(\alpha-\gamma+1)}\Big) \\
&+ \frac{T^\gamma \Gamma(2-\gamma)|c_2|\|m_3\|_{L^1}}{|b_2|}\Big(1 + \frac{|b_1|}{|a_1+b_1|}\Big) + \frac{|c_1|\|m_2\|_{L^1}}{|a_1+b_1|} \\
\leq& \frac{5}{16} \times \frac{11}{10} \times 1.8017 + \frac{0.9191}{12} \times \frac{11}{10} + \frac{3}{10} \times \frac{1}{9} \approx 0.7369 < 1.
\end{aligned}$$

Hence it follows from Theorem 4.3 that the fractional boundary value problem (14) has at least one solution on $[0,1]$.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 20, NO.4, 2016, COPYRIGHT 2016 EUDOXUS PRESS, LLC

Generalized intuitionistic fuzzy soft rough set and its application in decision making

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Abstract

Intuitionistic fuzzy set theory, soft set theory and rough set theory are mathematical tools for dealing with uncertainties and are closely related. This paper is devoted to the discussion of the combinations of intuitionistic fuzzy set, rough set and soft set. The concept of generalized intuitionistic fuzzy soft rough sets is proposed, and its properties are investigated. Furthermore, classical representations of generalized intuitionistic fuzzy soft rough approximation operators are presented. Finally, we develop an approach to generalized intuitionistic fuzzy soft rough sets based decision making and a practical example is provided to illustrate the developed approach.

Key words: Soft set; Rough set; Generalized intuitionistic fuzzy soft rough set; Decision making

1 Introduction

To solve complicated problems in economics, engineering, environmental science and social science, methods in classical mathematics are not always successful because of various types of uncertainties presented in these problems. There are several well-known theories to describe uncertainty. For instance, fuzzy set theory [1], intuitionistic fuzzy set theory [2,3], rough set theory [4,5] and other mathematical tools. But each of these theories has its inherent difficulties as pointed out in [6]. Perhaps above mentioned these theories are due to lack of parametrization tools. Theory of soft sets presented by Molodtsov [6] has enough parameters, so that it is free from inherent difficulties of above mentioned

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these theories. Soft set theory deals with uncertainty and vagueness on the one hand while on the other it has enough parametrization tools. These qualities of soft set theory make it popular among researchers and experts working in diverse areas. Soft set theory has potential applications in many different fields including the smoothness of functions, game theory, operational research, Perron integration, probability theory, and measurement theory [6, 7]. Presently, works on soft set theory are progressing rapidly. Maji et al. [8] defined several operations on soft sets and made a theoretical study on the theory of soft sets. Furthermore, based on [8], Ali et al. [9] introduced some new operations on soft sets and improved the notion of complement of soft set. They proved that certain De Morgans laws hold in soft set theory. Park et al [10] discussed some properties of equivalence soft set relations. The study of hybrid models combining soft sets with other mathematical structures is also emerging as an active research topic of soft set theory. Maji et al. [11] initiated the study on hybrid structures involving fuzzy sets and soft sets. They introduced the notion of fuzzy soft sets, which can be seen as a fuzzy generalization of soft sets. Furthermore, based on [11], Maji et al [12] modified definition of fuzzy soft sets, and presented the notion of generalized fuzzy soft sets theory. Yang et al. [13] presented the concept of the interval-valued fuzzy soft sets by combining interval-valued fuzzy set [14,15] and soft set models. By combining the concept of trapezoidal fuzzy set and soft set models, Xiao et al. [16] presented the concept of the trapezoidal fuzzy soft set which can deal with certain linguistic assessments. Yang et al. [17] presented the concept of the multi-fuzzy soft set by combining the multi-fuzzy set and soft set models, and provided its application in decision making under an imprecise environment.

The concept of rough sets, proposed by Pawlak [4, 5] as a framework for the construction of approximations of concepts, is a formal tool for modeling and processing insufficient and incomplete information. In order to handle vagueness and imprecision in the data equivalence relations play an important role in this theory. This theory has been applied successfully to solve many problems, but in daily life, it is very difficult to find an equivalence relation among the elements of a set under consideration. Therefore many authors have generalized the notion of Pawlak rough set by using non-equivalence binary relations. This has led to various other generalized rough set models [18–27].

Soft set theory, fuzzy set theory and rough set theory are all mathematical tools to deal with uncertainty. It has been found that soft set, fuzzy set and rough set are closely related concepts [28]. Feng et al. [29] provided a framework to combine fuzzy sets, rough sets and soft sets all together, which gives rise to several interesting new concepts such as rough soft sets, soft rough sets and soft rough fuzzy sets. The combination of soft set and rough set models was also discussed by some researchers [30–32].

In this paper, we devote to the discussion of the combinations of intuitionistic fuzzy set, rough set and soft set. The traditional intuitionistic fuzzy rough set [33–35] and soft set theory are two different tools to deal with uncertainty. Apparently there is no direct connection between these two theories. The major criticism on rough set theory is that it lacks parametrization tools. In order to make parametrization tools available in rough sets, we construct an intuitionistic fuzzy soft relation between the universe set U and the parameter set E in intuitionistic fuzzy soft set so that it is natural to combine intuitionistic fuzzy rough set and soft set theory. So the concept of generalized intuitionistic fuzzy soft rough sets is propose, and its some properties are discussed. Then the relationships between generalized intuitionistic fuzzy soft rough sets and the existing generalized soft rough sets are also established. We finally present an illustrative example which show that the decision making method of generalized intuitionistic fuzzy soft rough sets can be successfully applied to many problems that contain uncertainties.

The rest of this paper is organized as follows. Section 2 briefly reviews some preliminaries. In section 3, we construct the crisp soft rough approximation operators, and discuss their some interesting properties. In Section 4, an intuitionistic fuzzy soft relation is first defined by us. By combining the intuitionistic fuzzy soft relation with intuitionistic fuzzy rough sets, then the concept of generalized intuitionistic fuzzy soft rough approximation operators is presented and the properties of the generalized upper and lower intuitionistic fuzzy soft rough approximation operators are examined. Furthermore, classical representations of generalized intuitionistic fuzzy soft rough approximation operators are presented. Section 5 is devoted to studying the application of generalized intuitionistic fuzzy soft rough sets. Some conclusions and outlooks for further research are given in Section 6.

2 Preliminaries

In this section, we shall briefly recall some basic notions being used in the study.

Definition 2.1 ([36]) *Let* $L^* = \{(\mu,\nu) \in [0,1]\times[0,1] | \mu+\nu \le 1\}$ *and denote* $(\mu_1,\nu_1) \le_{L^*} (\mu_2,\nu_2) \Leftrightarrow \mu_1 \le \mu_2$ *and* $\nu_1 \ge \nu_2, \forall (\mu_1,\nu_1),(\mu_2,\nu_2) \in L^*$. *Then the pair* $(L^*, \le_{L^*})$ *is called a complete lattice. The operators* $\wedge$ *and* $\vee$ *on* $(L^*, \le_{L^*})$ *are defined as follows:for* $(\mu_1,\nu_1),(\mu_2,\nu_2) \in L^*$,

$(\mu_1,\nu_1) \wedge (\mu_2,\nu_2) = (min\{\mu_1,\mu_2\}, max\{\nu_1,\nu_2\})$,

$(\mu_1,\nu_1) \vee (\mu_2,\nu_2) = (max\{\mu_1,\mu_2\}, min\{\nu_1,\nu_2\})$,

Obviously, a complete lattice on L^* has the smallest element $0_{L^*} = (0,1)$ and the greatest element $1_{L^*} = (1,0)$. The definitions of fuzzy logical operators can be straightforwardly extended to the intuitionistic fuzzy case. The strict partial order $<_{L^*}$ is defined by

$$(\mu_1,\nu_1) <_{L^*} (\mu_2,\nu_2) \Leftrightarrow (\mu_1,\nu_1) \le_{L^*} (\mu_2,\nu_2) \ and \ (\mu_1,\nu_1) \ge_{L^*} (\mu_2,\nu_2).$$

In the following, we review the concept of the intuitionistic fuzzy set introduced by Atanassov [2,3].

Definition 2.2 ([2,3]) *Let a set* U *be fixed. An intuitionistic fuzzy (IF, for short) set* A *in* U *is an object having the form*

$A = \{< x, \mu_A(x), \gamma_A(x) > |x \in U\}$,
where $\mu_A : U \to [0,1]$, *and* $\gamma_A : U \to [0,1]$, *satisfy* $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ *for all* $x \in U$, *and* $\mu_A(x)$ *and* $\gamma_A(x)$ *are, respectively, called the degree of membership and the degree of non-membership of the element* $x \in U$ *to* A.

The family of all intuitionistic fuzzy subsets in U is denoted by $IF(U)$. The complement of an IF set A is denoted by $\sim A = \{< x, \gamma_A(x), \mu_A(x) > |x \in U\}$.

Obviously, every fuzzy set $A = \{< x, A(x) > |x \in U\} = \{< x, \mu_A(x) > |x \in U\}$ can be identified with the IF set of the form $\{< x, \mu_A(x), 1 - \mu_A(x) > |x \in U\}$ and is thus an IF set.

The basic operations on $IF(U)$ are defined as follows [2,3,37–39]: for all $A, B \in IF(U)$,
(1) $A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$ for all $x \in U$,
(2) $A = B$ iff $A \subseteq B$ and $B \subseteq A$,
(3) $A \cap B = \{< x, min\{\mu_A(x), \mu_B(x)\}, max\{\gamma_A(x), \gamma_B(x) > |x \in U\}$,
(4) $A \cup B = \{< x, max\{\mu_A(x), \mu_B(x)\}, min\{\gamma_A(x), \gamma_B(x) > |x \in U\}$.

For $(\alpha, \beta) \in L^*$, $\widehat{(\alpha, \beta)}$ denotes a constant IF set: $\widehat{(\alpha, \beta)}(x) = \{< x, \alpha, \beta > |x \in U\}$,where $\alpha, \beta \in [0,1]$ and $\alpha + \beta \leq 1$; The IF universe set is $U = 1_U = \widehat{(1,0)} = \{< x, 1, 0 > |x \in U\}$ and the IF empty set is $\emptyset = 0_U = \widehat{(0,1)} = \{< x, 0, 1 > |x \in U\}$.

Definition 2.3 ([33, 35]) *Let* $A = \{< x, \mu_A(x), \gamma_A(x) > |x \in U\} \in IF(U)$, *and* $(\alpha, \beta) \in L^*$. *The* (α, β)*-level cut set of* A, *denoted by* A_α^β , *is defined as follows:*

$$A_\alpha^\beta = \{x \in U | \mu_A(x) \geq \alpha, \gamma_A(x) \leq \beta\}.$$

$A_\alpha = \{x \in U | \mu_A(x) \geq \alpha\}$, *and* $A_{\alpha+} = \{x \in U | \mu_A(x) > \alpha\}$, *are, respectively, called the* α*-level cut set and the strong* α*-level cut set of membership generated by* A. *And* $A^\beta = \{x \in U | \gamma_A(x) \leq \beta\}$ *and* $A^{\beta+} = \{x \in U | \gamma_A(x) < \beta\}$ *are, respectively, referred to as the* β*-level cut set and the strong* β*-level cut set of non-membership generated by* A.

At the same time, other types of cut sets of the IF set A are denoted as follows:
$A_{\alpha+}^\beta = \{x \in U | \mu_A(x) > \alpha, \gamma_A(x) \leq \beta\}$, *which is called the* $(\alpha+, \beta)$*-level cut set of* A;
$A_\alpha^{\beta+} = \{x \in U | \mu_A(x) \geq \alpha, \gamma_A(x) < \beta\}$, *which is called the* $(\alpha, \beta+)$*-level cut set of* A;
$A_{\alpha+}^{\beta+} = \{x \in U | \mu_A(x) > \alpha, \gamma_A(x) < \beta\}$, *which is called the* $(\alpha+, \beta+)$*-level cut set of* A.

Theorem 2.4 ([33, 35]) *The cut sets of IF sets satisfy the following properties:* $\forall A \in IF(U), \alpha, \beta \in [0,1]$ *with* $\alpha + \beta \leq 1$,
(1) $A_\alpha^\beta = A_\alpha \cap A^\beta$,
(2) $(\sim A)_\alpha = \sim A^{\alpha+}, (\sim A)^\beta = \sim A_{\beta+}$.

Definition 2.5 ([22, 24]) *Let* U *be a nonempty and finite universe of discourse and* $R \subseteq U \times U$ *an arbitrary crisp relation on* U. *We define a set-valued function* $R_s : U \to P(U)$ *by* $R_s(x) = \{y \in U | (x, y) \in R\}, x \in U$.

The pair (U,R) is called a crisp approximation space. For any $A \subseteq U$, the upper and lower approximations of A with respect to (U,R) , denoted by $\overline{R}(A)$ and $\underline{R}(A)$, are defined, respectively, as follows:

$\overline{R}(A) = \{x \in U | R_s(x) \cap A \neq \emptyset\}$, $\underline{R}(A) = \{x \in U | R_s(x) \subseteq A\}$.

The pair $(\overline{R}(A), \underline{R}(A))$ is referred to as a crisp rough set, and $\overline{R}, \underline{R} : P(U) \to P(U)$ are, respectively, referred to as upper and lower crisp approximation operators induced from (U,R) .

3 Construction of crisp soft rough sets

In this section, we will introduce the concept of crisp soft rough sets by combining the crisp soft relation from U to E with crisp rough sets.

Definition 3.1 ([6]) *Let U be an initial universe set and E be a universe set of parameters. A pair (F,E) is called a soft set over U if $F : E \to P(U)$, where $P(U)$ is the set of all subsets of U.*

Definition 3.2 ([11]) *Let U be an initial universe set and E be a universe set of parameters. A pair (F,E) is called a fuzzy soft set over U if $F : E \to F(U)$, where $F(U)$ is the set of all fuzzy subsets of U.*

By using the concepts of the above soft set and fuzzy soft set, Cagman et al. [40, 41] introduce the definitions of crisp soft relation and fuzzy soft relation, respectively.

Definition 3.3 ([40]) *Let (F,E) be a soft set over U. Then a subset of $U \times E$ called a crisp soft relation from U to E is uniquely defined by*

$R = \{< (u,x), \mu_R(u,x) > | (u,x) \in U \times E\}$,

where $\mu_R : U \times E \to \{0,1\}$, $\mu_R(u,x) = \begin{cases} 1, & (u,x) \in R \\ 0, & (u,x) \notin R. \end{cases}$

Definition 3.4 ([41]) *Let (F,E) be a fuzzy soft set over U. Then a fuzzy subset of $U \times E$ called a fuzzy soft relation from U to E is uniquely defined by*

$R = \{< (u,x), \mu_R(u,x) > | (u,x) \in U \times E\}$,

where $\mu_R : U \times E \to [0,1]$, $\mu_R(u,x) = \mu_{F(x)}(u)$.

Based the crisp soft relation proposed by Cagman, we can construct the following crisp soft rough sets.

Definition 3.5 *Let U be an initial universe set and E be a universe set of parameters. For an arbitrary crisp soft relation R over $U \times E$, we can define a set-valued function $R_s : U \to P(E)$ by $R_s(u) = \{x \in E | (u,x) \in R\}, u \in U$.*

R is referred to as serial if for all $u \in U, R_s(u) \neq \emptyset$. The pair (U, E, R) is called a crisp soft approximation space. For any $A \subseteq E$, the upper and lower soft approximations of A with respect to (U, E, R), denoted by $\overline{R}(A)$ and $\underline{R}(A)$, are defined, respectively, as follows:

$$\overline{R}(A) = \{u \in U | R_s(u) \cap A \neq \emptyset\}, \tag{1}$$

$$\underline{R}(A) = \{u \in U | R_s(u) \subseteq A\}. \tag{2}$$

The pair $(\overline{R}(A), \underline{R}(A))$ is referred to as a crisp soft rough set, and $\overline{R}, \underline{R} : P(E) \to P(U)$ are, referred to as upper and lower crisp soft rough approximation operators, respectively.

Example 3.6 *Let U be a universal set, which is denoted by $U = \{u_1, u_2, u_3, u_4, u_5\}$. Let E be a set of parameters, where $E = \{e_1, e_2, e_3, e_4\}$. Suppose that a soft set over U is defined as follows:*

$F(e_1) = \{u_1, u_3, u_4\},\ F(e_2) = \{u_2, u_4\},\ F(e_3) = \emptyset,\ F(e_4) = U.$

Then the crisp soft relation on $U \times E$ is written by

$R = \{(u_1, e_1), (u_3, e_1), (u_4, e_1), (u_2, e_2), (u_4, e_2), (u_1, e_4), (u_2, e_4), (u_3, e_4), (u_4, e_4), (u_5, e_4)\}.$

If the set of parameter $A = \{e_2, e_3, e_4\}$, by Equations (1) and (2), we have $\underline{R}(A) = \{u_2, u_5\}$, and $\overline{R}(A) = U$.

Theorem 3.7 *Let (U, E, R) be a crisp soft approximation space. Then upper and lower crisp soft approximation operators $\overline{R}(A)$ and $\underline{R}(A)$ in Definition 3.5 satisfy the following properties: for all $A, B \in P(E)$*

(CSL1) $\underline{R}(A) =\sim \overline{R}(\sim A)$, *(CSU1)* $\overline{R}(A) =\sim \underline{R}(\sim A)$;

(CSL2) $\underline{R}(A \cap B) = \underline{R}(A) \cap \underline{R}(B)$, *(CSU2)* $\overline{R}(A \cup B) = \overline{R}(A) \cup \overline{R}(B)$;

(CSL3) $A \subseteq B \Rightarrow \underline{R}(A) \subseteq \underline{R}(B)$, *(CSU3)* $A \subseteq B \Rightarrow \overline{R}(A) \subseteq \overline{R}(B)$;

(CSL4) $\underline{R}(A \cup B) \supseteq \underline{R}(A) \cup \underline{R}(B)$, *(CSU4)* $\overline{R}(A \cap B) = \overline{R}(A) \cap \overline{R}(B)$.

Proof. The proof can be directly followed from Definition 3.5. □

From Definition 3.5, the following theorem can be easily derived.

Theorem 3.8 *Let (U, E, R) be a crisp soft approximation space, and $\overline{R}(A)$ and $\underline{R}(A)$ be the upper and lower soft approximations operators in Definition 3.5. Then*

R is serial $\Leftrightarrow \underline{R}(A) \subseteq \overline{R}(A), \forall A \in E \Leftrightarrow \underline{R}(\emptyset) = \emptyset \Leftrightarrow \overline{R}(E) = U.$

4 Construction of generalized intuitionistic fuzzy soft rough sets

In this section, inspired by the constructive method regard to generalized intuitionistic fuzzy rough sets in [35], we will present the concept of generalized intuitionistic fuzzy soft rough sets by combining the intuitionistic fuzzy soft relation from U to E with generalized intuitionistic fuzzy rough sets.

Definition 4.1 (**[42]**) *Let U be an initial universe set and E be a universe set of parameters. A pair (F, E) is called an intuitionistic fuzzy soft set over U if $F : E \to IF(U)$, where $IF(U)$ is the set of all intuitionistic fuzzy subsets of U. That is, for $\forall x \in E, F(x) = \{< u, \mu_{F(x)}(u), \gamma_{F(x)}(u) > |u \in U\} \in IF(U)$, where $\mu_{F(x)}(u) \in [0, 1]$ and $\gamma_{F(x)}(u) \in [0, 1]$ denote membership and non-membership degrees of an element u regard to intuitionistic fuzzy set $F(x)$ respectively, which satisfy the condition $\mu_{F(x)}(u) + \gamma_{F(x)}(u) \leq 1$.*

In the following, an intuitionistic fuzzy soft relation will be presented by us which is important for us to construct generalized intuitionistic fuzzy soft rough sets.

Definition 4.2 *Let (F, E) be an intuitionistic fuzzy soft set over U. Then an intuitionistic fuzzy subset of $U \times E$ called an intuitionistic fuzzy soft relation from U to E is uniquely defined by*

$R = \{< (u, x), \mu_R(u, x), \gamma_R(u, x) > |(u, x) \in U \times E\}$,

where $\mu_R : U \times E \to [0, 1]$ and $\gamma_R : U \times E \to [0, 1]$ satisfy the condition $0 \leq \mu_R(u, x) + \gamma_R(u, x) \leq 1$ for all $(u, x) \in U \times E$.

If $U = \{u_1, u_2, \cdots, u_m\}, E = \{x_1, x_2, \cdots, x_n\}$ then the intuitionistic fuzzy soft relation R from U to E can be presented by a table as in the following form

R	x_1	x_2	$\cdots$	x_n
u_1	$(\mu_R(u_1, x_1), \gamma_R(u_1, x_1))$	$(\mu_R(u_1, x_2), \gamma_R(u_1, x_2))$	$\cdots$	$(\mu_R(u_1, x_n), \gamma_R(u_1, x_n))$
u_2	$(\mu_R(u_2, x_1), \gamma_R(u_2, x_1))$	$(\mu_R(u_2, x_2), \gamma_R(u_2, x_2))$	$\cdots$	$(\mu_R(u_2, x_n), \gamma_R(u_2, x_n))$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
u_m	$(\mu_R(u_m, x_1), \gamma_R(u_m, x_1))$	$(\mu_R(u_m, x_2), \gamma_R(u_m, x_2))$	$\cdots$	$(\mu_R(u_m, x_n), \gamma_R(u_m, x_n))$

From the above form and according to the definition of IF soft relation, we could find that every IF soft set (F, E) is uniquely characterized by the IF soft relation, namely they are mutual determined. It means that an IF soft set (F, E) is formally equal to its IF soft relation. Therefore, we shall identify any IF soft set with its IF soft relation and view these two concepts as interchangeable. Now, any discussion regard to IF soft set could be converted into analysis about IF soft relation, which will bring great convenience for our future researches.

In this case, according to the definition IF soft relation, we can construct generalized intuitionistic fuzzy soft rough sets.

Definition 4.3 *Let U be an initial universe set and E be a universe set of parameters. For an arbitrary intuitionistic fuzzy soft relation R over $U \times E$, the pair (U, E, R) is called an intuitionistic fuzzy soft approximation space. For any $A \in IF(E)$, we define the upper and lower soft approximations of A with respect to (U, E, R), denoted by $\overline{R}(A)$ and $\underline{R}(A)$,* *respectively, as follows:*

$$\overline{R}(A) = \{< u, \mu_{\overline{R}(A)}(u), \gamma_{\overline{R}(A)}(u) > | u \in U\}, \tag{3}$$

$$\underline{R}(A) = \{< u, \mu_{\underline{R}(A)}(u), \gamma_{\underline{R}(A)}(u) > | u \in U\}. \tag{4}$$

where

$\mu_{\overline{R}(A)}(u) = \bigvee_{x\in E} [\mu_R(u,x) \wedge \mu_A(x)], \quad \gamma_{\overline{R}(A)}(u) = \bigwedge_{x\in E} [\gamma_R(u,x)) \vee \gamma_A(x)];$

$\mu_{\underline{R}(A)}(u) = \bigwedge_{x\in E} [\gamma_R(u,x) \vee \mu_A(x)], \quad \gamma_{\underline{R}(A)}(u) = \bigvee_{x\in E} [\mu_R(u,x) \wedge \gamma_A(x)].$

The pair $(\overline{R}(A), \underline{R}(A))$ *is referred to as a generalized IF soft rough set of* A *with respect to* (U, E, R).

By $\mu_R(u,x) + \gamma_R(u,x) \leq 1$ and $\mu_A(x) + \gamma_A(x) \leq 1$, it can be easily verified that $\overline{R}(A)$ and $\underline{R}(A) \in IF(U)$. In fact,

$$\begin{aligned}
\mu_{\overline{R}(A)}(u) + \gamma_{\overline{R}(A)}(u) &= \bigvee_{x\in E} [\mu_R(u,x) \wedge \mu_A(x)] + \bigwedge_{x\in E} [(\gamma_R(u,x)) \vee \gamma_A(x)] \\
&\leq \bigvee_{x\in E} [(1 - \gamma_R(u,x)) \wedge (1 - \gamma_A(x))] + \bigwedge_{x\in E} [(\gamma_R(u,x)) \vee \gamma_A(x)] \\
&= 1 - \bigwedge_{x\in E} [(\gamma_R(u,x)) \vee \gamma_A(x)] + \bigwedge_{x\in E} [(\gamma_R(u,x)) \vee \gamma_A(x)] \\
&= 1.
\end{aligned}$$

Hence, $\overline{R}(A) \in IF(U)$. Similarly, we can obtain $\underline{R}(A) \in IF(U)$. So we call $\overline{R}, \underline{R} : IF(E) \to IF(U)$ generalized upper and lower IF soft rough approximation operators, respectively.

Remark 4.4 *If* $\gamma_R(u,x) = 1 - \mu_R(u,x)$ *in Definition 4.2, then* R *is a fuzzy soft relation on* $U \times E$ *(see Definition 3.4), that is,* $R = \{< (u,x), \mu_R(u,x), 1 - \mu_R(u,x) > | (u,x) \in U \times E\}$. *In this case, we call* (U, E, R) *a fuzzy soft approximation space. Let* (U, E, R) *be the fuzzy soft approximation space and* $A \in IF(E)$, *then generalized IF soft rough approximation operators* $\overline{R}(A)$ *and* $\underline{R}(A)$ *in Definition 4.3 degenerate to the following forms:*

$$\overline{R}(A) = \{< u, \mu_{\overline{R}(A)}(u), \gamma_{\overline{R}(A)}(u) > | u \in U\},$$
$$\underline{R}(A) = \{< u, \mu_{\underline{R}(A)}(u), \gamma_{\underline{R}(A)}(u) > | u \in U\}.$$

where

$\mu_{\overline{R}(A)}(u) = \bigvee_{x\in E} [\mu_R(u,x) \wedge \mu_A(x)], \quad \gamma_{\overline{R}(A)}(u) = \bigwedge_{x\in E} [(1 - \mu_R(u,x)) \vee \gamma_A(x)];$

$\mu_{\underline{R}(A)}(u) = \bigwedge_{x\in E} [(1 - \mu_R(u,x)) \vee \mu_A(x)], \quad \gamma_{\underline{R}(A)}(u) = \bigvee_{x\in E} [\mu_R(u,x) \wedge \gamma_A(x)].$

In this case, the pair $(\overline{R}(A), \underline{R}(A))$ *is referred to as an IF soft rough set of* A *with respect to* (U, E, R)*. That is, generalized IF soft rough set in Definition 4.3 has included IF soft rough set.*

Remark 4.5 *Let* (U, E, R) *be a fuzzy soft approximation space. If* $A \in F(E)$*, then generalized IF soft rough approximation operators* $\overline{R}(A)$ *and* $\underline{R}(A)$ *degenerate to the following forms:*

$$\overline{R}(A) = \{< u, \mu_{\overline{R}(A)}(u) > | u \in U\}, \quad \underline{R}(A) = \{< u, \mu_{\underline{R}(A)}(u) > | u \in U\}.$$

where $\mu_{\overline{R}(A)}(u) = \bigvee\limits_{x \in E} [\mu_R(u, x) \wedge \mu_A(x)]$, $\mu_{\underline{R}(A)}(u) = \bigwedge\limits_{x \in E} [(1 - \mu_R(u, x)) \vee \mu_A(x)$.

In this case, generalized IF soft rough approximation operators $\overline{R}(A)$ *and* $\underline{R}(A)$ *are identical with the soft fuzzy rough approximation operators defined by Sun [32]. That is, generalized IF soft rough approximation operators in Definition 4.3 are an extension of the soft fuzzy rough approximation operators defined by Sun [32].*

In order to better understand the concept of generalized IF soft rough approximation operators, let us consider the following example.

Example 4.6 *Suppose that* $U = \{u_1, u_2, u_3, u_4, u_5\}$ *is the set of five houses under consideration of a decision maker to purchase. Let* E *be a parameter set, where* $E = \{e_1, e_2, e_3, e_4\}$*=\{expensive; beautiful; size; location\}. Mr. X wants to buy the house which qualifies with the parameters of* E *to the utmost extent from available houses in* U*. Assume that Mr. X describes the "attractiveness of the houses" by constructing an IF soft relation* R *from* U *to* E*. And it is presented by a table as in the following form.*

R	e_1	e_2	e_3	e_4
u_1	$(0.7, 0.2)$	$(0.6, 0.3)$	$(0.1, 0.9)$	$(0.1, 0.6)$
u_2	$(0.2, 0.6)$	$(0.6, 0.4)$	$(0.5, 0.4)$	$(0.7, 0.3)$
u_3	$(0.4, 0.6)$	$(0.8, 0.1)$	$(0.2, 0.7)$	$(0.6, 0.3)$
u_4	$(0.2, 0.5)$	$(0.3, 0.6)$	$(0.8, 0.1)$	$(0.1, 0.7)$
u_5	$(0.6, 0.4)$	$(0.5, 0.2)$	$(0.3, 0.5)$	$(0.2, 0.5)$

As a generalization of Zadeh's fuzzy set, intuitionistic fuzzy set is characterized by a membership function and a nonmembership function, and thus can depict the fuzzy character of data more detailedly and comprehensively than Zadehs fuzzy set which is only characterized by a membership function. Therefore, the characteristics of the five houses with respect to the four parameters can be represented by the intuitionistic fuzzy sets. For example, the characteristics of the house u_1 *under the parameter* e_1 *is (0.7,0.2). The values of 0.7 and 0.2 are the degrees of membership and non-membership of the house* u_1

with respect to the parameter e_1 *, respectively. In other words, house* u_1 *is expensive on the membership degree of 0.7 and it is not expensive on the non-membership degree of 0.2.*

Now suppose that Mr X gives the optimum normal decision object A *which an IF subset over the parameter set* E *as follows:*

$A = \{< e_1, 0.7, 0.1 >, < e_2, 0.3, 0.6 >, < e_3, 0.8, 0.2 >, < e_4, 0.3, 0.5 >\}$

By Equations (3) and (4), we have

$\mu_{\overline{R}(A)}(u_1) = 0.7,\ \gamma_{\overline{R}(A)}(u_1) = 0.2,\ \mu_{\overline{R}(A)}(u_2) = 0.5,\ \gamma_{\overline{R}(A)}(u_2) = 0.4,$
$\mu_{\overline{R}(A)}(u_3) = 0.4,\ \gamma_{\overline{R}(A)}(u_3) = 0.5,\ \mu_{\overline{R}(A)}(u_4) = 0.8,\ \gamma_{\overline{R}(A)}(u_4) = 0.2,$
$\mu_{\overline{R}(A)}(u_5) = 0.6,\ \gamma_{\overline{R}(A)}(u_5) = 0.4;\ \mu_{\underline{R}(A)}(u_1) = 0.3,\ \gamma_{\underline{R}(A)}(u_1) = 0.6,$
$\mu_{\underline{R}(A)}(u_2) = 0.3,\ \gamma_{\underline{R}(A)}(u_2) = 0.6,\ \mu_{\underline{R}(A)}(u_3) = 0.3,\ \gamma_{\underline{R}(A)}(u_3) = 0.6,$
$\mu_{\underline{R}(A)}(u_4) = 0.3,\ \gamma_{\underline{R}(A)}(u_4) = 0.3,\ \mu_{\underline{R}(A)}(u_5) = 0.3,\ \gamma_{\underline{R}(A)}(u_5) = 0.5.$

Thus

$$\begin{aligned}\overline{R}(A) = \{ &< u_1, 0.7, 0.2 >, < u_2, 0.5, 0.4 >, < u_3, 0.4, 0.5 >,\\ &< u_4, 0.8, 0.2 >, < u_5, 0.6, 0.4 >\}\end{aligned}$$

and

$$\begin{aligned}\underline{R}(A) = \{ &< u_1, 0.3, 0.6 >, < u_2, 0.3, 0.6 >, < u_3, 0.3, 0.6 >,\\ &< u_4, 0.3, 0.3 >, < u_5, 0.3, 0.5 >\}.\end{aligned}$$

Theorem 4.7 *Let* (U, E, R) *be an intuitionistic fuzzy soft approximation space. Then the generalized upper and lower IF soft rough approximation operators* $\overline{R}(A)$ *and* $\underline{R}(A)$ *in Definition 4.3 satisfy the following properties:* $\forall A, B \in IF(E), \forall(\alpha, \beta) \in L^*$,

(GIFSL1) $\underline{R}(A) = \sim \overline{R}(\sim A)$, *(GIFSU1)* $\overline{R}(A) = \sim \underline{R}(\sim A)$;
(GIFSL2) $\underline{R}(A \cup \widehat{(\alpha, \beta)}) = \underline{R}(A) \cup \widehat{(\alpha, \beta)}$, *(GIFSU2)* $\overline{R}(A \cap \widehat{(\alpha, \beta)}) = \overline{R}(A) \cap \widehat{(\alpha, \beta)}$;
(GIFSL3) $\underline{R}(A \cap B) = \underline{R}(A) \cap \underline{R}(B)$, *(GIFSU3)* $\overline{R}(A \cup B) = \overline{R}(A) \cup \overline{R}(B)$;
(GIFSL4) $A \subseteq B \Rightarrow \underline{R}(A) \subseteq \underline{R}(B)$, *(GIFSU4)* $A \subseteq B \Rightarrow \overline{R}(A) \subseteq \overline{R}(B)$;
(GIFSL5) $\underline{R}(A \cup B) \supseteq \underline{R}(A) \cup \underline{R}(B)$, *(GIFSU5)* $\overline{R}(A \cap B) = \overline{R}(A) \cap \overline{R}(B)$;

Proof. It can be easily followed from Definition 4.3. □

In Theorem 4.7, properties (GIFSL1) and (GIFSU1) show that the generalized upper lower IF soft rough approximation operators $\overline{R}$ and $\underline{R}$ are dual to each other.

Assume that R is an intuitionistic fuzzy soft relation from U to E, denote

$R_\alpha = \{(u, x) \in U \times E | \mu_R(u, x) \geq \alpha\}$, $R_\alpha(u) = \{x \in E | \mu_R(u, x) \geq \alpha\}$, $\alpha \in [0, 1]$,
$R_{\alpha+} = \{(u, x) \in U \times E | \mu_R(u, x) > \alpha\}$, $R_{\alpha+}(u) = \{x \in E | \mu_R(u, x) > \alpha\}, \alpha \in [0, 1)$,
$R^\alpha = \{(u, x) \in U \times E | \gamma_R(u, x) \leq \alpha\}$, $R^\alpha(u) = \{x \in E | \gamma_R(u, x) \leq \alpha\}, \alpha \in [0, 1]$,
$R^{\alpha+} = \{(u, x) \in U \times E | \gamma_R(u, x) < \alpha\}$, $R^{\alpha+}(u) = \{x \in E | \gamma_R(u, x) < \alpha\}, \alpha \in (0, 1]$.

Then R_α, $R_{\alpha+}$, R^α, and $R^{\alpha+}$ are crisp soft relations on $U \times E$.

The following Theorems 4.8 and 4.9 show that the generalized IF soft rough approximation operators can be represented by crisp soft rough approximation operators.

Theorem 4.8 *Let (U, E, R) be an intuitionistic fuzzy soft approximation space, and $A \in IF(E)$. Then the generalized upper IF soft rough approximation operator can be represented as follows: $\forall u \in U$*

(1)

$$\begin{aligned}\mu_{\overline{R}(A)}(u) &= \bigvee_{\alpha\in[0,1]} [\alpha \wedge \overline{R_\alpha}(A_\alpha)(u)] = \bigvee_{\alpha\in[0,1]} [\alpha \wedge \overline{R_\alpha}(A_{\alpha+})(u)] \\ &= \bigvee_{\alpha\in[0,1]} [\alpha \wedge \overline{R_{\alpha+}}(A_\alpha)(u)] = \bigvee_{\alpha\in[0,1]} [\alpha \wedge \overline{R_{\alpha+}}(A_{\alpha+})(u)],\end{aligned}$$

(2)

$$\begin{aligned}\gamma_{\overline{R}(A)}(u) &= \bigwedge_{\alpha\in[0,1]} [\alpha \vee \overline{R^\alpha}(A^\alpha)(u)] = \bigwedge_{\alpha\in[0,1]} [\alpha \vee \overline{R^\alpha}(A^{\alpha+})(u)] \\ &= \bigwedge_{\alpha\in[0,1]} [\alpha \vee \overline{R^{\alpha+}}(A^\alpha)(u)] = \bigwedge_{\alpha\in[0,1]} [\alpha \vee \overline{R^{\alpha+}}(A^{\alpha+})(u)]\end{aligned}$$

and moreover, for any $\alpha \in [0, 1]$,

(3) $[\overline{R}(A)]_{\alpha+} \subseteq \overline{R_{\alpha+}}(A_{\alpha+}) \subseteq \overline{R_{\alpha+}}(A_\alpha) \subseteq \overline{R_\alpha}(A_\alpha) \subseteq [\overline{R}(A)]_\alpha$,

(4) $[\overline{R}(A)]^{\alpha+} \subseteq \overline{R^{\alpha+}}(A^{\alpha+}) \subseteq \overline{R^{\alpha+}}(A^\alpha) \subseteq \overline{R^\alpha}(A^\alpha) \subseteq [\overline{R}(A)]^\alpha$.

Proof. (1) For any $u \in U$, we have that

$$\begin{aligned}\bigvee_{\alpha\in[0,1]} [\alpha \wedge \overline{R_\alpha}(A_\alpha)(u)] &= sup\{\alpha \in [0, 1] | u \in \overline{R_\alpha}(A_\alpha)\} \\ &= sup\{\alpha \in [0, 1] | R_\alpha(u) \cap A_\alpha \neq \emptyset\} \\ &= sup\{\alpha \in [0, 1] | \exists x \in E[x \in R_\alpha(u), x \in A_\alpha]\} \\ &= sup\{\alpha \in [0, 1] | \exists x \in E[\mu_R(u, x) \geq \alpha, \mu_A(x) \geq \alpha]\} \\ &= \bigvee_{x\in E} [\mu_R(u, x) \wedge \mu_A(x)] = \mu_{\overline{R}(A)}(u)\end{aligned}$$

Similarly, we can prove

$$\mu_{\overline{R}(A)}(u) = \bigvee_{\alpha\in[0,1]} [\alpha\wedge\overline{R_\alpha}(A_{\alpha+})(u)] = \bigvee_{\alpha\in[0,1]} [\alpha\wedge\overline{R_{\alpha+}}(A_\alpha)(u)] = \bigvee_{\alpha\in[0,1]} [\alpha\wedge\overline{R_{\alpha+}}(A_{\alpha+})(u)].$$

(2) By the definition of upper crisp soft rough approximation operator in Definition 3.5, we have that

$$\begin{aligned}\bigwedge_{\alpha\in[0,1]}[\alpha\vee\overline{R^{\alpha}}(A^{\alpha})(u))] &= inf\{\alpha\in[0,1]|u\in\overline{R^{\alpha}}(A^{\alpha})\}\\ &= inf\{\alpha\in[0,1]|R^{\alpha}(u)\cap A^{\alpha}\neq\emptyset\}\\ &= inf\{\alpha\in[0,1]|\exists x\in E[x\in R^{\alpha}(u),x\in A^{\alpha}]\}\\ &= inf\{\alpha\in[0,1]|\exists x\in E[\gamma_R(u,x)\leq\alpha,\gamma_A(x)\leq\alpha]\}\\ &= \bigwedge_{x\in E}[\gamma_R(u,x)\vee\gamma_A(x)]=\gamma_{\overline{R}(A)}(u).\end{aligned}$$

Likewise, we can prove that

$$\begin{aligned}\gamma_{\overline{R}(A)}(u) &= \bigwedge_{\alpha\in[0,1]}[\alpha\vee\overline{R^{\alpha}}(A^{\alpha+})(u)]=\bigwedge_{\alpha\in[0,1]}[\alpha\vee\overline{R^{\alpha+}}(A^{\alpha})(u)]\\ &= \bigwedge_{\alpha\in[0,1]}[\alpha\vee\overline{R^{\alpha+}}(A^{\alpha+})(u)].\end{aligned}$$

(3) It is easily verified that $\overline{R_{\alpha+}}(A_{\alpha+})\subseteq\overline{R_{\alpha+}}(A_{\alpha})\subseteq\overline{R_{\alpha}}(A_{\alpha})$. We only need to prove that $[\overline{R}(A)]_{\alpha+}\subseteq\overline{R_{\alpha+}}(A_{\alpha+})$ and $\overline{R_{\alpha}}(A_{\alpha})\subseteq[\overline{R}(A)]_{\alpha}$.

In fact, $\forall u\in[\overline{R}(A)]_{\alpha+}$, we have $\mu_{\overline{R}(A)}(u)>\alpha$. According to Definition 4.3, $\bigvee_{x\in E}[\mu_R(u,x)\wedge\mu_A(x)]>\alpha$ holds. Then $\exists x_0\in E$, such that $\mu_R(u,x_0)\wedge\mu_A(x_0)>\alpha$, that is, $\mu_R(u,x_0)>\alpha$ and $\mu_A(x_0)>\alpha$. Thus $x_0\in R_{\alpha+}(u)$ and $x_0\in A_{\alpha+}$. Consequently, $R_{\alpha+}(u)\cap A_{\alpha+}\neq\emptyset$. By Definition 3.5, we have $u\in\overline{R_{\alpha+}}(A_{\alpha+})$. Hence $[\overline{R}(A)]_{\alpha+}\subseteq\overline{R_{\alpha+}}(A_{\alpha+})$.

On the other hand, for any $u\in\overline{R_{\alpha}}(A_{\alpha})$, we have $\overline{R_{\alpha}}(A_{\alpha})(u)=1$. Since $\mu_{\overline{R}(A)}(u)=\bigvee_{\beta\in[0,1]}[\beta\wedge\overline{R_{\beta}}(A_{\beta})(u)]\geq\alpha\wedge\overline{R_{\alpha}}(A_{\alpha})(u)=\alpha$, we obtain $u\in[\overline{R}(A)]_{\alpha}$. Hence, $\overline{R_{\alpha}}(A_{\alpha})\subseteq[\overline{R}(A)]_{\alpha}$.

(4) Similar to the proof of (3), it can be easily verified. □

Theorem 4.9 *Let (U,E,R) be an intuitionistic fuzzy soft approximation space, and $A\in IF(E)$. Then the generalized lower IF soft rough approximation operator can be represented as follows: $\forall u\in U$*

(1)

$$\begin{aligned}\mu_{\underline{R}(A)}(u) &= \bigwedge_{\alpha\in[0,1]}[\alpha\vee(1-\underline{R^{\alpha}}(A_{\alpha+})(u))]=\bigwedge_{\alpha\in[0,1]}[\alpha\vee(1-\underline{R^{\alpha}}(A_{\alpha})(u))]\\ &= \bigwedge_{\alpha\in[0,1]}[\alpha\vee(1-\underline{R^{\alpha+}}(A_{\alpha+})(u))]=\bigwedge_{\alpha\in[0,1]}[\alpha\vee(1-\underline{R^{\alpha+}}(A_{\alpha})(u))],\end{aligned}$$

(2)

$$\begin{aligned}\gamma_{\underline{R}(A)}(u) &= \bigvee_{\alpha\in[0,1]} [\alpha \wedge (1 - \underline{R_\alpha}(A^{\alpha+})(u))] = \bigvee_{\alpha\in[0,1]} [\alpha \wedge (1 - \underline{R_\alpha}(A^{\alpha})(u))] \\ &= \bigvee_{\alpha\in[0,1]} [\alpha \wedge (1 - \underline{R_{\alpha+}}(A^{\alpha+})(u))] = \bigvee_{\alpha\in[0,1]} [\alpha \wedge (1 - \underline{R_{\alpha+}}(A^{\alpha})(u))]\end{aligned}$$

and moreover, for any $\alpha \in [0,1]$,
(3) $[\underline{R}(A)]_{\alpha+} \subseteq \underline{R^\alpha}(A_{\alpha+}) \subseteq \underline{R^{\alpha+}}(A_{\alpha+}) \subseteq \underline{R^{\alpha+}}(A_\alpha) \subseteq [\underline{R}(A)]_\alpha$,
(4) $[\underline{R}(A)]^{\alpha+} \subseteq \underline{R_\alpha}(A^{\alpha+}) \subseteq \underline{R_{\alpha+}}(A^{\alpha+}) \subseteq \underline{R_{\alpha+}}(A^\alpha) \subseteq [\underline{R}(A)]^\alpha$.

Proof. (1) and (2) According to Theorem 4.8 and 2.4, for all $u \in U$ we have

$$\begin{aligned}\mu_{\overline{R}(\sim A)}(u) &= \bigvee_{\alpha\in[0,1]} [\alpha \wedge \overline{R_\alpha}(\sim A)_\alpha(u)] = \bigvee_{\alpha\in[0,1]} [\alpha \wedge \overline{R_\alpha}(\sim A^{\alpha+})(u)] \\ &= \bigvee_{\alpha\in[0,1]} [\alpha \wedge (\sim \underline{R_\alpha}(A^{\alpha+}))(u)] = \bigvee_{\alpha\in[0,1]} [\alpha \wedge (1 - \underline{R_\alpha}(A^{\alpha+})(u))]\end{aligned}$$

and

$$\begin{aligned}\gamma_{\overline{R}(\sim A)}(u) &= \bigwedge_{\alpha\in[0,1]} [\alpha \vee \overline{R^\alpha}(\sim A)^\alpha(u)] = \bigwedge_{\alpha\in[0,1]} [\alpha \vee \overline{R^\alpha}(\sim A_{\alpha+})(u)] \\ &= \bigwedge_{\alpha\in[0,1]} [\alpha \vee (\sim \underline{R^\alpha}(A_{\alpha+}))(u)] = \bigwedge_{\alpha\in[0,1]} [\alpha \vee (1 - \underline{R^\alpha}(A_{\alpha+})(u))]\end{aligned}$$

Hence, by the duality of generalized upper and lower IF soft rough approximation operators (see Theorem 4.7), we can conclude

$\mu_{\underline{R}(A)}(u) = \gamma_{\overline{R}(\sim A)}(u) = \bigwedge_{\alpha\in[0,1]} [\alpha \vee (1 - \underline{R^\alpha}(A_{\alpha+})(u))]$,

$\gamma_{\underline{R}(A)}(u) = \mu_{\overline{R}(\sim A)}(u) = \bigvee_{\alpha\in[0,1]} [\alpha \wedge (1 - \underline{R_\alpha}(A^{\alpha+})(u))]$.

Similar to the above proof, we can obtain

$$\begin{aligned}\mu_{\underline{R}(A)}(u) &= \bigwedge_{\alpha\in[0,1]} [\alpha \vee (1 - \underline{R^\alpha}(A_\alpha)(u))] = \bigwedge_{\alpha\in[0,1]} [\alpha \vee (1 - \underline{R^{\alpha+}}(A_{\alpha+})(u))] \\ &= \bigwedge_{\alpha\in[0,1]} [\alpha \vee (1 - \underline{R^{\alpha+}}(A_\alpha)(u))],\end{aligned}$$

$$\begin{aligned}\gamma_{\underline{R}(A)}(u) &= \bigvee_{\alpha\in[0,1]} [\alpha \wedge (1 - \underline{R_\alpha}(A^\alpha)(u))] = \bigvee_{\alpha\in[0,1]} [\alpha \wedge (1 - \underline{R_{\alpha+}}(A^{\alpha+})(u))] \\ &= \bigvee_{\alpha\in[0,1]} [\alpha \wedge (1 - \underline{R_{\alpha+}}(A^\alpha)(u))].\end{aligned}$$

(3) and (4) It is similar to the proof of Theorem 4.8(3). □

5 Application of IF soft rough sets in decision making

In this section, we shall develop an approach to generalized IF soft rough sets based decision making. In the following, we will define the ring sum operation of IF sets. By the operation, an approach to generalized intuitionistic fuzzy soft rough sets based decision making will be presented.

Definition 5.1 *Let $F, G \in IF(U)$. The ring sum operation about IF sets F and G can be defined as follows:*
$F \oplus G = \{< u, \mu_F(u) + \mu_G(u) - \mu_F(u) \cdot \mu_G(u), \gamma_F(u) \cdot \gamma_G(u) > |u \in U\}$.

Let (U, E, R) be an intuitionistic fuzzy soft approximation space, where U is the universe of the discourse, E is the parameter set, and R is an intuitionistic fuzzy soft relation on $U \times E$. Then we can give an algorithm based on generalized IF soft rough sets as follows:

1. Input the intuitionistic fuzzy soft relation R from U to E, or the intuitionistic fuzzy soft set (F, E) over U.

2. Give the optimum normal decision object A which is an IF set over E, according to different needs to decision maker.

3. Compute the generalized IF soft rough approximation operators $\overline{R}(A)$ and $\underline{R}(A)$ by Equations (3) and (4).

4. Compute the choice set

$$H = \overline{R}(A) \oplus \underline{R}(A) = \{< u, \mu_{\overline{R}(A)}(u) + \mu_{\underline{R}(A)}(u) - \mu_{\overline{R}(A)}(u) \cdot \mu_{\underline{R}(A)}(u), \gamma_{\overline{R}(A)}(u) \cdot \gamma_{\underline{R}(A)}(u) > |u \in U\}.$$

5. Choose the top-level threshold value $\lambda = (\mu, \gamma) \in L^*$, where $\mu = \max\limits_{1 \leq i \leq n} \mu_H(u_i), \gamma = \min\limits_{1 \leq i \leq n} \gamma_H(u_i)$.

6. The decision is u, if IF set $H(u) \geq_{L^*} \lambda$, that is, $\mu_H(u) \geq \mu$ and $\lambda_H(u) \leq \gamma$.

In the last step of the above algorithm, one may go back to the second step and change decision object so that the final decision is only one, when there exist too many "optimal choices" to be chosen.

To illustrate the idea of algorithm given above, let us consider the example as follows.

Example 5.2 *Reconsider Example 4.6. In Example 4.6, we have computed generalized IF soft rough approximation operators $\overline{R}(A)$ and $\underline{R}(A)$ of the optimum normal decision object A. Now by using the fourth step of algorithm for generalized IF soft rough sets in decision making presented in this section, we can obtain*

$$H = \overline{R}(A) \oplus \underline{R}(A) = \{ < u_1, 0.79, 0.12 >, < u_2, 0.65, 0.24 >, < u_3, 0.58, 0.30 >, < u_4, 0.86, 0.06 >, < u_5, 0.72, 0.20 >\}.$$

Obviously, the optimal decision is u_4. Hence, Mr X will buy the house u_4.

6 Conclusion

Intuitionistic fuzzy set theory, soft set theory and rough set theory are all mathematical tools for dealing with uncertainties. This paper is devoted to the discussion of the combinations of intuitionistic fuzzy set, rough set and soft set. Based on the models presented in [35], by combining soft set theory with generalized IF rough set theory, a new soft rough set model called generalized IF soft rough set is proposed and its properties are derived. Furthermore, the relationships between generalized intuitionistic fuzzy soft rough sets with the existing generalized soft rough sets are established. Finally, a practical application based on generalized intuitionistic fuzzy soft rough sets is applied to show the validity.

Actually, there are at least two aspects in the study of rough set theory: constructive and axiomatic approaches. In further research, the axiomatization of generalized intuitionistic fuzzy soft rough approximation operators is an important and interesting issue to be addressed.

Acknowledgements

The authors would like to thank the anonymous referees for their valuable comments and suggestions. This work is supported by the National Natural Science Foundation of China (No. 11461082) and the Fundamental Research Funds for the Central Universities of Northwest University for Nationalities (No. 31920130003).

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 20, NO.4, 2016, COPYRIGHT 2016 EUDOXUS PRESS, LLC

Global Exponential Dissipativity of Static Neural Networks with Time Delay and Impulses

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Abstract

In this paper, we investigate the problem of global exponential dissipativity of static neural networks with time delay and impulses. The impulses are classified into two classes: the ones are input disturbances and the ones are stabilizing. For each type of impulses, by adopting proper Lyapunov function, sufficient conditions for global exponential dissipativity are established in terms of linear matrix inequalities (LMIs). The new sufficient conditions can explicitly reveal the influence of time delay, impulses, etc., on the dissipativity. We show that these conditions can be straightforwardly reduced to exponential stability conditions and that these stability conditions are remarkably less conservative than the existing ones. Numerical results are given to show the less conservatism of the obtained criteria compared to the existing ones.

Keywords: dissipativity, stability, static neural network, impulse, time delay, LMI.

1. Introduction.

In the past few years, neural networks (NNs) have been extensively studied duo to their applications in many areas, such as signal processing, associative memory, pattern recognition, combination optimization and so on. As reported in [1] and [2], NNs can be classified as local field neural networks and static neural networks. For both types of NNs, time delay and impulses occur unavoidably during implementation of the corresponding artificial circuits. Time delay occurs due to the finite switching speeds of the amplifiers and impulses arise from the abrupt changes in the voltages (which can affect the dynamical behaviors of the system) produced by faulty circuit elements. Research of the local field NNs with both time delay and impulses has received lots of attention in recent years, and several important and interesting sufficient conditions ensuring the existence and global exponential stability of a unique equilibrium solution are given; see, e.g., [3–7] and references therein.

Nevertheless, there is only a few theoretical results for the impulsive static NNs with time delay. As reviewed in [8–11], many neural networks exhibiting short-term memory are modeled by non-invertible networks (such as the oculomotor integrator or the head-direction system [12]) and this implies that the local field NNs and the static ones are not always equivalent. Therefore, it is necessary to pay special attention to static NNs with both time delay and impulses. Zhao

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October 16, 2014

and Wang [13] presented some stability results of the static NNs with time delay and impulses. However, the criteria presented in [13] impose strict conditions on the model parameters and the impulsive effects (for example, the impulses are assumed to be stabilizing), and thus these criteria are less applicable. Recently, the authors in [14] further investigated the problem of global exponential stability of the static NNs with time delay and impulses. The results presented in [14] were shown less conservative and more applicative than the ones given by Zhao and Wang [13].

However, all of the aforementioned results concentrate on stability of the studied NNs, that is obtaining sufficient conditions under which the trajectories starting from any initial value tend to equilibrium points. Moreover, due to the mathematical difficulties in dealing with impulses, most of these results are based on a rigorous assumption— the impulsive operator, namely $\mathcal{I}$, satisfies $\mathcal{I}(x^*) = 0$, where x^* is the equilibrium point of the interested neural networks. But, from a practical point of view, it is not always the case that the orbits of a NNs converge to equilibrium points and they can behave arbitrarily in a bounded set. Moreover, the assumption $\mathcal{I}(x^*) = 0$ is too strong, since it means that we need to know the value of the equilibrium points in advance. It is possible that there are no equilibrium points and/or $\mathcal{I}(x^*) \neq 0$ in some situations (in fact, $\mathcal{I}(x^*) \neq 0$ is more general than $\mathcal{I}(x^*) = 0$). Therefore, the concept *dissipativity* has been introduced and investigated; see, e.g., [15–22] and references therein. Dissipativity states that the trajectories of a dynamic system starting from any initial value go to a bounded set (and never go away this set) if the evolution time is sufficient long and in this set the trajectories can behave arbitrarily. Clearly, dissipativity generalizes the notion of stability. Nowadays, dissipativity has found applications in many areas, such as stability theory, chaos and synchronization theory, system norm estimation, and robust control(see [16, 19–22]).

In this paper, we consider the global exponential dissipativity of the static NNs with both time delay and impulses. According to our best knowledge, this problem has not been studied by other authors. Hence, it is our intention in this paper to tackle such an important yet challenging problem. We consider three types of impulses: the ones are input disturbances, the ones are stabilizing and the ones are "neural" type, which are neither helpful for stabilizing nor destabilizing the neural networks. Since the treatment of neutral type impulses is similar to the input disturbances ones, we divide the impulses into two classes— input disturbances impulses and stabilizing ones, and then we adopt the following guiding ideology to derive conditions of exponential dissipativity:

1. for disturbances impulses, we explore on what conditions the exponential damping rate of the used Lyapunov function overcomes the influence of impulsive disturbance;
2. for stabilizing impulses, we suppose the Lyapunov function can be growing with some exponential rate instead of assuming it must be damping (many authors require this in the stability analysis; see, e.g., [4–6, 23]), and then we explore under what conditions the effect of the stabilizing impulses can offset the growth of the Lyapunov function.

By using the above guiding ideology and some novel analysis techniques, sufficient conditions concerning the upper bound of the time delay, the magnitude of impulses, the distance between two consecutive impulsive instants (or say frequency of impulses) are derived to maintain the exponential dissipativity, when the *impulse-free* static NNs are dissipative but the impulses are input disturbances. When the impulse-free static NNs are not dissipative, sufficient conditions that utilize impulsive effects to stabilize the static NNs to be dissipative are also given.

We remark that the guiding ideology mentioned above is not our original invention, and it should be credited to Chen and Zheng [3]. In [3], this ideology is utilized to study the exponential stability of the local field NNs with impulses and time delay by using a very simple Lyapunov function—$V(t) = e^{2\gamma t}x^T(t)Px(t)$ ($\gamma > 0$ denotes the exponential convergence rate, P is a positive definite matrix), and it is shown that the obtained stability conditions are much less conservative than the existing ones. The simple form of the used Lyapunov function is very important to perform the stability analysis and for complex Lyapunov function, those analysis can not be straightforwardly generalized. On the other hand, it is a common sense that complex Lyapunov function may result in less conservative results. Therefore, it is a meaningful work to realize the excellent ideology by using more general Lyapunov functions. In our previous paper [14], we have used this ideology to study the exponential stability of the static NNs with impulses and time delay, but the utilized Lyapunov function is still the aforementioned simplest one.

In this paper, with a more general Lyapunov function, new analysis techniques are proposed to realize the ideology and benefiting from this Lyapunov function much less conservative conditions are derived. These conditions can be easily reduced to the exponential stability conditions and we show that the deduced stability conditions are also much less conservative than the existing ones. The reminder of this paper is organized as follows. In Section 2, we introduce the impulsive static NNs with time delay discussed in this paper. Some definitions and lemmas are also given in this section. In Section 3, sufficient conditions for exponential dissipativity are established in terms of LMIs. In Section 4, several numerical examples are given to show the usefulness of our results. The comparison of our results with the existing ones is the main topic. Finally, Section 5 concludes the work of this paper.

2. Problem description and preliminaries.

The impulsive static NNs with time delay can be described by the following impulsive delay differential equations:

$$\begin{cases} x'(t) = -Ax(t) + f(Cx(t-\tau(t))) + \psi(t), & t \neq t_k,\ k \in \mathbb{N}, \\ \Delta x(t) = W_k x(t^-), & t = t_k,\ k \in \mathbb{N}, \\ x(t) = \phi(t), & t \in [-\tau,\ 0], \end{cases} \tag{2.1}$$

where $x(t) = (x_1(t),\ \ldots,\ x_n(t))^T$ denotes the state variables associated with the neurons; A is a positive diagonal matrix representing the self-feedback term; $C \in \mathbb{R}^{n\times n}$ is the internal delayed connection weigh matrix; $f(Cx(t)) = (f_1(C_1x(t)),\ f_2(C_2x(t)),\ \ldots,\ f_n(C_nx(t)))^T$ is the neuron activation function, where C_i is the $i-$th row of the matrix C; $\psi(t) = (\psi_1(t),\ \ldots,\ \psi_n(t))^T$ is the external input function and each component $\psi_i(t)$ is bounded. The time delay $\tau(t)$ is bounded as $0 \leq \tau(t) \leq \tau$ and $\phi : [-\tau,\ 0] \to \mathbb{R}^n$ is a piecewise right continuous function. For the impulsive parameters, $\Delta x(t) = x(t^+) - x(t^-)$ denotes the state jumping at impulsive time instant $t = t_k$, where $x(t^+)$ and $x(t^-)$ are the right-hand and left-hand limits of the functions $x(t)$, respectively; $W_k \in \mathbb{R}^{n\times n}$ represents the abrupt change of the state at t_k; the impulsive time instants $\{t_k\}_{k=1}^{+\infty}$ satisfy $0 < t_1 < t_2 < \cdots$ and $\lim\limits_{k\to+\infty} t_k = +\infty$.

Throughout this paper, we assume that the following hypotheses are satisfied:

$$f_i(0) = 0, \quad l_i^- \leq \frac{f_i(s_1) - f_i(s_2)}{s_1 - s_2} \leq l_i^+, \quad \forall s_1,\ s_2 \in \mathbb{R} \text{ and } s_1 \neq s_2, \quad i = 1,\ 2,\ \ldots,\ n. \tag{2.2}$$

Set $L_0 = \mathrm{diag}(l_1^-,\ l_2^-,\ \ldots,\ l_n^-)$ and $L_1 = \mathrm{diag}(l_1^+,\ l_2^+,\ \ldots,\ l_n^+)$. Moreover, we will use the notation $P > 0$ (or $P < 0$) to denote that P is a symmetric and positive definite (or negative definite) matrix. If P_1, P_2 are symmetric matrices, then $P_1 > P_2$ (or $P_1 \geq P_2$) means that $P_1 - P_2$ is a positive definite (positive semi-definite) matrix. For any matrix $P \in \mathbb{R}^{n\times n}$, we use $\lambda_m(P)$ and $\lambda_M(P)$ to denote its minimal and maximal eigenvalues respectively. For any vector $z \in \mathbb{R}^n$ and matrix $P \in \mathbb{R}^{n\times n}$, $\|z\|$ denotes the Euclidean norm of z and $\|P\|$ denotes the induced norm of the matrix P, that is $\|P\| = \sqrt{\rho(P^TP)}$, where $\rho(P^TP)$ denotes the spectral radius of matrix P^TP. Moreover, for any initial function $\phi : (-\infty, 0] \to \mathbb{R}^n$ which is piecewise right continuous, we let $\|\phi\|_0 = \max_{s\in(-\infty,\,0]}\|\phi(s)\|$.

Lemma 2.1 (Berman and Plemmons [24]) *For any symmetric matrix $P \in \mathbb{R}^{n\times n}$, it holds*

$$\lambda_m(P)x^Tx \leq x^TPx \leq \lambda_M(P)x^Tx, \quad \forall x \in \mathbb{R}^n.$$

Definiton 2.1 *A neural network (2.1) is said to be global exponential dissipative system if there exists a compact set $\mathbb{S}$ in $\mathbb{R}^n$ such that $\forall\phi(t) \in \mathbb{R}^n\backslash\mathbb{S}$, there exist constants $M(\phi) > 0$ and $\gamma > 0$ such that*

$$\inf_{x(t)\in\mathbb{R}^n\backslash\mathbb{S}}\{\|x(t) - \tilde{x}\| : \tilde{x} \in \mathbb{S}\} \leq M(\phi)e^{-\gamma t}.$$

The argument γ is called dissipativity rate and the set $\mathbb{S}$ is called global exponential attractive set, where $x(t) \in \mathbb{R}^n\backslash\mathbb{S}$ means $x(t) \in \mathbb{R}^n$ but $x(t) \notin \mathbb{S}$.

Definiton 2.2 *For any function $u(t)$, we define its right-hand derivative as*

$$\mathcal{D}^+u(t) = \lim_{s\to 0^+}\frac{u(t+s) - u(t)}{s}.$$

3. Analysis of exponential dissipativity.

In this section, we analyze the exponential dissipativity of (2.1). We utilize the following Lyapunov function

$$V(t) = e^{2\gamma t}x(t)^TPx(t) + e^{2\gamma t}x^T(t-\tau)Qx(t-\tau). \tag{3.1}$$

We first provide a lemma which estimates the jump of $V(t)$ at each impulsive time instant. Note that, for $Q = 0$ this Lyapunov function reduces to $V(t) = e^{2\gamma t}x^T(t)Px(t)$, which is used in [3] and [14].

Lemma 3.1 *If there exist matrices P, $Q > 0$ and positive scalar $\mu > 0$ such that the following matrix inequalities hold:*

$$(I + W_k)^TP(I + W_k) - \mu P < 0, \quad (I + W_k)^TQ(I + W_k) - \mu Q < 0, \ \forall k \geq 1,$$

then we have for every integer $k \geq 1$ that

$$V(t_k) \leq \begin{cases} \mu V(t_k^-), & \text{if } t_k \text{ is an impulsive instant,} \\ \mu V(t_k^-) + (1-\mu)e^{2\gamma t_k}x^T(t_k - \tau)Qx(t_k - \tau), & \text{if } t_k \text{ is not an impulsive instant} \\ & \text{and } \mu < 1. \end{cases} \tag{3.2}$$

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Proof. We consider the following two cases:
(a) if $t_k - \tau = t_{k-m}$, i.e., $t_k - \tau$ is also an impulsive time instant, we have

$$\begin{aligned} V(t_k) - \mu V(t_k^-) =& e^{2\gamma t_k} x^T(t_k^-)\left[(I+W_k)^T P(I+W_k) - \mu P\right] x(t_k^-) + \\ & e^{2\gamma t_k} x^T(t_{k-m}^-)\left[(I+W_{k-m})^T Q(I+W_{k-m}) - \mu Q\right] x(t_{k-m}^-) \\ \leq & 0, \end{aligned}$$

which gives for any $\mu \geq 0$ that $V(t_k) \leq \mu V(t_k^-)$;
(b) if $t_k - \tau$ is not an impulsive time instant, we have

$$V(t_k) - \mu V(t_k^-) = e^{2\gamma k} x^T(t_k^-)\left[(I+W_k)^T P(I+W_k) - \mu P\right] x(t_k^-) + (1-\mu)e^{2\gamma t_k} x^T(t_k - \tau)Qx(t_k - \tau),$$

which gives $V(t_k) \leq \mu V(t_k^-)$ for $\mu \geq 1$ and $V(t_k) \leq \mu V(t_k^-) + (1-\mu)e^{2\gamma t_k} x^T(t_k - \tau)Qx(t_k - \tau)$ for $\mu < 1$, since $(I+W_k)^T P(I+W_k) - \mu P < 0$. ■

Because the dissipativity analysis for the case of "neutral-type" impulses (i.e., $\mu = 1$) is similar to that of input disturbances, we therefore perform the analysis by distinguishing two types of impulses: $\mu \geq 1$ and $\mu < 1$.

3.1. Disturbances impulses: $\mu \geq 1$.

When the static NNs (2.1) without impulses (i.e., $W_k = 0$, $\forall k \geq 1$) are dissipative but the impulses are input disturbances, we try to derive conditions with respect to the magnitude of the impulses and the distance between two consecutive impulsive instants, under which the NNs remain dissipative. Moreover, we want to know how fast the solutions of (2.1) converge to the attractive set.

Theorem 3.1 *Suppose there exist symmetric positive matrices* P, Q, D_1, D_2, *positive diagonal matrices* U_{ij} $(1 \leq i \leq 3,\ 1 \leq j \leq 4)$ *and scalar numbers* $\mu \geq 1$, $\gamma > 0$, $\alpha > 0$ *such that*

$$\begin{aligned} &\Psi_1 = (I+W_k)^T P(I+W_k) - \mu P < 0,\ \forall k \geq 1, \\ &\Psi_2 = (I+W_k)^T Q(I+W_k) - \mu P < 0,\ \forall k \geq 1, \\ &\Omega = \begin{bmatrix} \Omega_{11} & 0 & 0 & 0 & \Omega_{15} & 0 & \Omega_{17} & 0 \\ \star & \Omega_{22} & 0 & 0 & 0 & \Omega_{26} & 0 & \Omega_{28} \\ \star & \star & \Omega_{33} & 0 & 0 & 0 & \Omega_{37} & 0 \\ \star & \star & \star & \Omega_{44} & 0 & 0 & 0 & \Omega_{48} \\ \star & \star & \star & \star & \Omega_{55} & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \Omega_{66} & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \Omega_{77} & 0 \\ \star & \star & \star & \star & \star & \star & \star & \Omega_{88} \end{bmatrix} < 0, \end{aligned} \tag{3.3a}$$

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where $\star$ denotes the symmetric terms in a symmetric matrix and

$$\begin{aligned}
\Omega_{11} &= 2D_1 - PA - AP + \left(2\gamma + \alpha\mu + \frac{\ln\mu}{\beta}\right)P + C^T\left[2U_{11}L_1 + (L_1 - 2L_0)^T U_{21}L_1 - 2L_0^T U_{31}L_1\right]C,\\
\Omega_{15} &= -C^T U_{11} + C^T L_0^T U_{21} + C^T\left(L_0^T + L_1^T\right)U_{31},\\
\Omega_{17} &= P,\\
\Omega_{22} &= 2D_2 - QA - AQ + \left(2\gamma + \alpha\mu + \frac{\ln\mu}{\beta}\right)Q + C^T\left[2U_{12}L_1 + (L_1 - 2L_0)^T U_{22}L_1 - 2L_0^T U_{32}L_1\right]C,\\
\Omega_{26} &= -C^T U_{12} + C^T L_0^T U_{22} + C^T\left(L_0^T + L_1^T\right)U_{32},\\
\Omega_{28} &= Q,\\
\Omega_{33} &= -\alpha e^{-2\gamma\tau}P + C^T\left[2U_{13}L_1 + (L_1 - 2L_0)^T U_{23}L_1 - 2L_0^T U_{33}L_1\right]C,\\
\Omega_{37} &= -C^T U_{13} + C^T L_0^T U_{23} + C^T\left(L_0^T + L_1^T\right)U_{33},\\
\Omega_{44} &= -\alpha e^{-2\gamma\tau}Q + C^T\left[2U_{14}L_1 + (L_1 - 2L_0)^T U_{24}L_1 - 2L_0^T U_{34}L_1\right]C,\\
\Omega_{48} &= -C^T U_{14} + C^T L_0^T U_{24} + C^T\left(L_0^T + L_1^T\right)U_{34},\\
\Omega_{55} &= -U_{21} - 2U_{31},\\
\Omega_{66} &= -U_{22} - 2U_{32},\\
\Omega_{77} &= -U_{23} - 2U_{33},\\
\Omega_{88} &= -U_{24} - 2U_{34},\\
\beta &= \inf_{k\geq 1}\{t_k - t_{k-1}\},\quad t_0 = 0.
\end{aligned} \tag{3.3b}$$

Then the static NNs (2.1) *is global exponential dissipative with dissipativity rate γ, and*

$$\mathbb{S} = \left\{x : \|x\| \leq \max\left\{\frac{\sup_{t\in\mathbb{R}}\|P\psi(t)\|}{\lambda_m(D_1)},\ \frac{\sup_{t\in\mathbb{R}}\|Q\psi(t)\|}{\lambda_m(D_2)}\right\}\right\} \tag{3.4}$$

is a positive invariant and global attractive set. In particular, we have

$$\inf_{x(t)\in\mathbb{R}^n\setminus\mathbb{S}}\{\|x(t) - \tilde{x}\| : \tilde{x} \in \mathbb{S}\} < M(\phi)e^{-\gamma t}, \tag{3.5}$$

where $M(\phi) = \sqrt{\frac{\mu\lambda_1}{\lambda_0}}\|\phi\|_0,\ \lambda_1 = \max\{\lambda_M(P),\ \lambda_M(Q)\},\ \lambda_0 = \min\{\lambda_m(P),\ \lambda_m(Q)\}$.

Proof. The proof is divided into two parts.
(A) Since $\Omega < 0$, it holds for sufficient small positive number η that

$$\tilde{\Omega} = \begin{bmatrix}
\tilde{\Omega}_{11} & 0 & 0 & 0 & \Omega_{15} & 0 & \Omega_{17} & 0\\
\star & \tilde{\Omega}_{22} & 0 & 0 & 0 & \Omega_{26} & 0 & \Omega_{28}\\
\star & \star & \Omega_{33} & 0 & 0 & 0 & \Omega_{37} & 0\\
\star & \star & \star & \Omega_{44} & 0 & 0 & 0 & \Omega_{48}\\
\star & \star & \star & \star & \Omega_{55} & 0 & 0 & 0\\
\star & \star & \star & \star & \star & \Omega_{66} & 0 & 0\\
\star & \star & \star & \star & \star & \star & \Omega_{77} & 0\\
\star & \star & \star & \star & \star & \star & \star & \Omega_{88}
\end{bmatrix} < 0,$$

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where

$\tilde{\Omega}_{11} = 2D_1 + 2\gamma P - PA - AP + (\alpha\mu_1 + \mu_2)P + C^T\left[2U_{11}L_1 + (L_1 - 2L_0)^T U_{21}L_1 - 2L_0^T U_{31}L_1\right]C,$

$\tilde{\Omega}_{22} = 2D_2 + 2\gamma Q - QA - AQ + (\alpha\mu_1 + \mu_2)Q + C^T\left[2U_{12}L_1 + (L_1 - 2L_0)^T U_{22}L_1 - 2L_0^T U_{32}L_1\right]C,$

$\mu_1 = \mu + \eta, \ \ \mu_2 = \dfrac{\ln(\mu + 2\eta)}{\beta}.$

By routine calculations we get

$$\begin{aligned}
&\mathcal{D}^+V(t) + (\alpha\mu_1 + \mu_2)V(t) - \alpha V(t - \tau(t))\\
&= e^{2\gamma t}\left(2\gamma x^T(t)Px(t) + 2x^T(t)P\left[-Ax(t) + f(Cx(t - \tau(t)))\right]\right) + (\alpha\mu_1 + \mu_2)e^{2\gamma t}x^T(t)Px(t) -\\
&\quad e^{2\gamma(t-\tau(t))}\alpha x^T(t - \tau(t))Px(t - \tau(t)) + e^{2\gamma t}\left(2\gamma x^T(t - \tau)Qx(t - \tau) + 2x^T(t - \tau)Q\left[-Ax(t - \tau)\right.\right.\\
&\quad \left.\left. + f(Cx(t - \tau - \tau(t)))\right]\right) + (\alpha\mu_1 + \mu_2)e^{2\gamma t}x^T(t - \tau)Qx(t - \tau) - e^{2\gamma(t-\tau(t))}\alpha x^T(t - \tau - \tau(t))Q\\
&\quad x(t - \tau - \tau(t)) + 2e^{2\gamma t}\left(x^T(t)P\psi(t) + x^T(t - \tau)Q\psi(t - \tau)\right)\\
&\le e^{2\gamma t}\left(x^T(t)\left[2\gamma P - 2PA + (\alpha\mu_1 + \mu_2)P\right]x(t) + 2x^TPf(Cx(t - \tau(t))) - \alpha e^{-2\gamma\tau}x^T(t - \tau(t))P\right.\\
&\quad \left. x(t - \tau(t))\right) + e^{2\gamma t}\left(x^T(t - \tau)\left[2\gamma Q - 2QA + (\alpha\mu_1 + \mu_2)Q\right]x(t - \tau) + 2x^T(t - \tau)Q\right.\\
&\quad \left. f(Cx(t - \tau - \tau(t))) - \alpha e^{-2\gamma\tau}x^T(t - \tau - \tau(t))Qx(t - \tau - \tau(t))\right) +\\
&\quad 2e^{2\gamma t}\left(x^T(t)P\psi(t) + x^T(t - \tau)Q\psi(t - \tau)\right).
\end{aligned} \tag{3.6}$$

Moreover, by Lipschitz condition (2.2) and the fact that U_{ij} $(1 \le i \le 3,\ 1 \le j \le 4)$ are positive diagonal matrices, it is easy to get the following inequalities

$$\begin{aligned}
&0 \le 2e^{2\gamma t}\left[Cz_j\right]^T U_{1j}[L_1\left(Cz_j\right) - f(Cz_j)] = e^{2\gamma t}z_j^T[2C^TU_{1j}L_1C]z_j + 2e^{2\gamma t}z_j^T[-C^TU_{1j}]f(Cz_j),\\
&0 \le e^{2\gamma t}\left[(Cz_j)^T(L_1^T - L_0^T) + f^T(Cz_j) - (Cz_j)^TL_0^T\right]U_{2j}[L_1Cz_j - f(Cz_j)]\\
&\quad = e^{2\gamma t}\left(z_j^T\left[C^T(L_1 - 2L_0)^TU_{2j}L_1C\right]z_j + 2z_j^T[C^TL_0^TU_{2j}]f(Cz_j) + f^T(Cz_j)[-U_{2j}]f(Cz_j)\right),\\
&0 \le 2e^{2\gamma t}\left[f^T(Cz_j) - (Cz_j)^TL_0^T\right]U_{3j}[L_1(Cz_j) - f(Cz_j)]\\
&\quad = z_j^T(t)[-2C^TL_0^TU_{3j}L_1C]z_j + 2z_j^T[C^TL_0^TU_{3j} + C^TL_1^TU_{3j}^T]f(Cz_j) + f^T(Cz_j)[-2U_{3j}]f(Cz_j),
\end{aligned} \tag{3.7}$$

where $j = 1,\ 2,\ 3,\ 4$ and

$$z_1 = x(t),\ z_2 = x(t - \tau),\quad z_3 = x(t - \tau(t)),\ z_4 = x(t - \tau - \tau(t)). \tag{3.8}$$

Combining (3.6) and (3.7) leads to

$$\begin{aligned}
&\mathcal{D}^+V(t) + (\alpha\mu_1 + \mu_2)V(t) - \alpha V(t - \tau(t))\\
&\le 2e^{2\gamma t}\left[\left(x^T(t)P\psi(t) - x^T(t)D_1x(t)\right) + \left(x^T(t - \tau)Q\psi(t - \tau) - x^T(t - \tau)D_2x(t - \tau)\right)\right] + e^{2\gamma t}X^T\tilde{\Omega}X\\
&\le 2e^{2\gamma t}\left(\|x(t)\|\left[\|P\psi(t)\| - \lambda_m(D_1)\|x(t)\|\right] + \|x(t - \tau)\|\left[\|Q\psi(t - \tau)\| - \lambda_m(D_2)\|x(t - \tau)\|\right]\right.\\
&\quad \left. + \frac{1}{2}X^T\tilde{\Omega}X\right),
\end{aligned} \tag{3.9}$$

where $X = \left(z_1^T,\ z_2^T,\ z_3^T,\ z_4^T,\ f^T(Cz_1),\ f^T(Cz_2),\ f^T(Cz_3),\ f^T(Cz_4)\right)^T$ and z_j $(1 \le j \le 4)$ are defined by (3.8). Now, by using $\tilde{\Omega} < 0$ we have

$$\mathcal{D}^+V(t) + (\alpha\mu_1 + \mu_2)V(t) - \alpha V(t - \tau(t)) \le 0, \tag{3.10}$$

whenever $x(t) \in \mathbb{R}^n \backslash \mathbb{S}$.

(B) We next prove (3.5). To this end, we first prove the following inequality

$$V(t) < \lambda_0\zeta^2,\ t \ge -\tau, \tag{3.11}$$

where $\zeta = \sqrt{\frac{(\mu+\eta)\lambda_1}{\lambda_0}}\|\phi\|_0$ and $x(t) \in \mathbb{R}^n \backslash \mathbb{S}$.

(b1) For $t \in [-\tau,\ 0]$, by using Lemma 2.1 and $\mu_1 = \mu + \eta > 1$ we get

$$\begin{aligned} V(t) &= e^{2\gamma t}\left(x^T(t)Px(t) + x^T(t-\tau)Qx(t-\tau)\right) \\ &\le x^T(t)Px(t) + x^T(t-\tau)Qx(t-\tau) \\ &\le \lambda_1\|\phi\|_0^2 = \frac{1}{\mu_1}\lambda_0\zeta^2 < \lambda_0\zeta^2. \end{aligned} \tag{3.12}$$

(b2) We next prove $V(t) < \lambda_0\zeta^2$ for $t \in [0,\ t_1)$. If not, there exists $t \in (0,\ t_1)$ such that $V(t) \ge \lambda_0\zeta^2$. Set $\bar{t} = \inf\{t : V(t) \ge \lambda_0\zeta^2$ and $t \in [0,\ t_1)\}$. Clearly, $\bar{t} \in (0,\ t_1)$ and $V(\bar{t}) = \lambda_0\zeta^2$. Since $V(0) \le \frac{1}{\mu_1}\lambda_0\zeta^2$, there exists $\underline{t} = \sup\left\{t : V(t) \le \frac{1}{\mu_1}\lambda_0\zeta^2 \text{and } t \in [0,\ \bar{t})\right\}$. Hence, for $t \in [\underline{t},\ \bar{t}]$ we have

$$V(t) \ge \frac{1}{\mu_1}\lambda_0\zeta^2 \ \text{ and } \ V(t+\theta) \le \lambda_0\zeta^2,\ \ \theta \in [-\tau,\ 0], \tag{3.13}$$

which leads to

$$\mathcal{D}^+V(t) \le \mathcal{D}^+V(t) + \alpha(\mu_1 V(t) - V(t-\tau(t))),\ t \in [\underline{t},\ \ \bar{t}]. \tag{3.14}$$

By using (3.10) we have

$$\mathcal{D}^+V(t) + \alpha\mu_1 V(t) - \alpha V(t-\tau(t)) \le -\mu_2 V(t) \le 0, \tag{3.15}$$

and this together with (3.14) gives $\mathcal{D}^+V(t) \le 0$ for $x(t) \in \mathbb{R}^n \backslash \mathbb{S}$ and $t \in [\underline{t},\ \ \bar{t}]$. It then follows that $V(\bar{t}) \le V(\underline{t}) = \frac{1}{\mu_1}\lambda_0\zeta^2$. This is a contradiction. Therefore, $V(t) < \lambda_0\zeta^2$ for $t \in [0,\ t_1)$.

(b3) Suppose for any integer $k \ge 1$ that

$$V(t) < \lambda_0\zeta^2,\ \ t \in [-\tau,\ t_k). \tag{3.16}$$

We try to prove

$$V(t) < \lambda_0\zeta^2,\ \ t \in [t_k,\ t_{k+1}). \tag{3.17}$$

To this end, we first claim $V(t_k^-) \le \frac{1}{\mu_1}\lambda_0\zeta^2$. By the contrary, we have $V(t_k^-) > \frac{1}{\mu_1}\lambda_0\zeta^2$. For this situation, we need to consider two cases.

Case 1: $V(t) > \frac{1}{\mu_1}\lambda_0\zeta^2$ for $t \in [t_{k-1},\ t_k)$. In this case, by assumption (3.16) we have

$$V(t) > \frac{1}{\mu_1}\lambda_0\zeta^2 \ \text{ and } \ V(t+\theta) < \lambda_0\zeta^2, \ \ \theta \in [-\tau,\ 0], \tag{3.18}$$

which gives

$$\mathcal{D}^+V(t) \le \mathcal{D}^+V(t) + \alpha[\mu_1 V(t) - V(t-\tau(t))], \ t \in [t_{k-1},\ t_k). \tag{3.19}$$

Hence, by (3.10) we have $\mathcal{D}^+V(t) \le -\mu_2 V(t)$ for $x(t) \in \mathbb{R}^n \backslash \mathbb{S}$ and $t \in [t_{k-1},\ t_k)$. This gives

$$V(t) \le V(t_{k-1})e^{-\mu_2(t-t_{k-1})}, \ \ t \in [t_{k-1},\ t_k), \tag{3.20}$$

which yields

$$V(t_k^-) \le V(t_{k-1})e^{-\mu_2(t_k-t_{k-1})} \le \lambda_0\zeta^2 e^{-\mu_2\beta} = \frac{1}{\mu+2\eta}\lambda_0\zeta^2 = \frac{1}{\mu_1+\eta}\lambda_0\zeta^2 < \frac{1}{\mu_1}\lambda_0\zeta^2, \tag{3.21}$$

since $\eta > 0$. Clearly, this is a contradiction.
Case 2: There exists some $t \in [t_{k-1},\ t_k)$ such that $V(t) \le \frac{1}{\mu_1}\lambda_0\zeta^2$. Since $V(t_k^-) > \frac{1}{\mu_1}\lambda_0\xi^2$, we may set $\bar{t} = \sup\{t : V(t) \le \frac{1}{\mu_1}\lambda_0\zeta^2$ and $t \in [t_{k-1},\ t_k)\}$. Clearly, $\bar{t} \in (t_{k-1},\ t_k)$ and $V(\bar{t}) = \frac{1}{\mu_1}\lambda_0\zeta^2$. Therefore, we have

$$V(t) > \frac{1}{\mu_1}\lambda_0\zeta^2 \ \text{ and } \ V(t+\theta) \le \lambda_0\zeta^2, \ t \in [\bar{t},\ t_k), \ \ \theta \in [-\tau,\ 0], \tag{3.22}$$

which gives

$$\mathcal{D}^+V(t) \le \mathcal{D}^+V(t) + \alpha(\mu_1 V(t) - V(t-\tau(t))), \ \ t \in [\bar{t},\ t_k). \tag{3.23}$$

Hence, by (3.10) we get $\mathcal{D}^+V(t) \le 0$ for $x(t) \in \mathbb{R}^n \backslash \mathbb{S}$ and $t \in [\bar{t},\ t_k)$. This implies $V(t_k^-) \le V(\bar{t}) = \frac{1}{\mu_1}\lambda_0\zeta^2$. This again leads to contradiction.

So we have $V(t_k^-) \le \frac{1}{\mu_1}\lambda_0\zeta^2$. From Lemma 3.1 we have

$$V(t_k) \le \mu V(t_k^-) = \frac{\mu}{\mu_1}\lambda_0\zeta^2 < \lambda_0\zeta^2. \tag{3.24}$$

This together with (3.16) gives $V(t) < \lambda_0\zeta^2$, $t \in [-\tau,\ t_k]$. Now, suppose $V(t) < \lambda_0\zeta^2$ is not true for $t \in (t_k,\ t_{k+1})$. If so, we set $t^* = \inf\{t : V(t) \ge \lambda_0\zeta^2$ and $t \in [t_k,\ t_{k+1})\}$. Then $t^* \in (t_k,\ t_{k+1})$ and $V(t^*) = \lambda_0\zeta^2$. Set $\bar{t} = t_k$ if $V(t) > \frac{1}{\mu_1}\lambda_0\zeta^2$ for all $t \in [t_k,\ t^*]$; otherwise, set $\bar{t} = \sup\{t : V(t) \le \frac{1}{\mu_1}\lambda_0\zeta^2$ and $t \in [t_k,\ t^*)\}$. Therefore, for $t \in [\bar{t},\ t^*]$ we have

$$V(t) \ge \frac{1}{\mu_1}\lambda_0\zeta^2 \ \text{ and } \ V(t+\theta) \le \lambda_0\zeta^2, \ \ \theta \in [-\tau,\ 0]. \tag{3.25}$$

It then follows

$$\mathcal{D}^+V(t) \le \mathcal{D}^+V(t) + \alpha(\mu_1 V(t) - V(t-\tau(t))), \ t \in [\bar{t},\ t^*]. \tag{3.26}$$

By (3.10) we have $\mathcal{D}^+V(t) \le 0$ for $x(t) \in \mathbb{R}^n \backslash \mathbb{S}$ and $t \in [\bar{t},\ t^*]$, which implies $V(t^*) \le V(\bar{t}) < \lambda_0 \zeta^2$ (for the situation $\bar{t} = t_k$, (3.24) gives the last inequality). This obviously contradicts the fact $V(t^*) = \lambda_0 \zeta^2$.

Thus, by the method of mathematical induction, (3.17) holds for any integer k. So (3.11) is true. It therefore follows by applying Lemma 2.1 that

$$\lambda_0 e^{2\gamma t} \|x(t)\|^2 \le V(t) < \lambda_0 \zeta^2, \quad \text{when } x(t) \in \mathbb{R}^n \backslash \mathbb{S}, \tag{3.27}$$

which implies

$$\inf_{x(t)\in\mathbb{R}^n\backslash\mathbb{S}} \{\|x(t) - \tilde{x}\| : \tilde{x} \in \mathbb{S}\} \le \|x(t)\| < M(\phi) e^{-\gamma t}, \tag{3.28}$$

since the set $\mathbb{S}$ includes origin and the argument $\eta > 0$ in ζ can be sufficient small. ■

The following result gives a corollary for the case of constant delay, i.e., $\tau(t) \equiv \tau$. In this case, $x(t - \tau(t)) = x(t - \tau)$ and $x(t - \tau - \tau(t)) = x(t - 2\tau)$, and therefore the matrix Ω in (3.3a) can be compacted into an 6-block matrix.

Corollary 3.1 *Suppose $\tau(t) \equiv \tau$ and there exist symmetric positive matrices P, Q, D_1, D_2, positive diagonal matrices $U_{ij} (1 \le i,\ j \le 3)$, scalar numbers $\mu \ge 1$, $\gamma > 0$, $\alpha > 0$ such that*

$$\begin{aligned}
&\Psi_1 = (I + W_k)^T P (I + W_k) - \mu P < 0,\ \forall k \ge 1,\\
&\Psi_2 = (I + W_k)^T Q (I + W_k) - \mu Q < 0,\ \forall k \ge 1,\\
&\Omega = \begin{bmatrix} \Omega_{11} & 0 & 0 & \Omega_{14} & \Omega_{15} & 0 \\ \star & \Omega_{22} & 0 & 0 & \Omega_{25} & \Omega_{26} \\ \star & \star & \Omega_{33} & 0 & 0 & \Omega_{36} \\ \star & \star & \star & \Omega_{44} & 0 & 0 \\ \star & \star & \star & \star & \Omega_{55} & 0 \\ \star & \star & \star & \star & \star & \Omega_{66} \end{bmatrix} < 0,
\end{aligned} \tag{3.29}$$

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where

$$\begin{aligned}
\Omega_{11} &= 2D_1 + \left(2\gamma + \alpha\mu + \frac{\ln\mu}{\beta}\right)P - PA - A^T P^T + C^T[2U_{11}L_1 + (L_1 - 2L_0)^T U_{21}L_1 - 2L_0^T U_{31}L_1]C,\\
\Omega_{14} &= -C^T U_{11} + C^T L_0^T U_{21} + C^T L_0^T U_{31} + C^T L_1^T U_{31}^T,\\
\Omega_{15} &= P,\\
\Omega_{22} &= 2D_2 + \left(2\gamma + \alpha\mu + \frac{\ln\mu}{\beta}\right)Q - \alpha e^{-2\gamma\tau}P - QA - A^T Q^T + C^T[2U_{12}L_1 + (L_1 - 2L_0)^T U_{22}L_1\\
&\quad - 2L_0^T U_{32}L_1]C,\\
\Omega_{25} &= -C^T U_{12} + C^T L_0^T U_{22} + C^T L_0^T U_{32} + C^T L_1^T U_{32}^T,\\
\Omega_{26} &= Q,\\
\Omega_{33} &= -\alpha e^{-2\gamma\tau}Q + C^T[2U_{13}L_1 + (L_1 - 2L_0)^T U_{23}L_1 - 2L_0^T U_{33}L_1]C,\\
\Omega_{36} &= -C^T U_{13} + C^T L_0^T U_{23} + C^T L_0^T U_{33} + C^T L_1^T U_{33}^T,\\
\Omega_{44} &= -U_{21} - 2U_{31},\\
\Omega_{55} &= -U_{22} - 2U_{32},\\
\Omega_{66} &= -U_{23} - 2U_{33},\\
\beta &= \inf_{k\geq 1}\{t_k - t_{k-1}\},\quad t_0 = 0.
\end{aligned}\tag{3.30}$$

Then the static neural network (2.1) is global exponential dissipative with dissipativity rate γ, and in particular (3.4) holds.

Remark 3.1 *Theorem 3.1 and Corollary 3.1 require that there exists a positive lower bound β on the time instants between two adjacent impulses. This condition ensures that the impulses, which destabilize the neural networks, do not occur too frequently. When α, β, τ, γ are chosen, inequalities (3.3a) and (3.29) are linear matrix inequalities (LMIs), which can be solved numerically and very efficiently using the interior point algorithms [25].*

3.2. Stabilizing Impulses: $\mu < 1$.

In this subsection, we study the exponential dissipativity of the static neural network (2.1) for the case $\mu < 1$, i.e., the impulses are stabilizing.

Theorem 3.2 *Suppose there exist symmetric positive matrices P, Q, D_1, D_2, positive diagonal matrices $U_{ij}(1 \leq i \leq 3,\ 1 \leq j \leq 4)$ and scalar numbers $\mu < 1$, $\gamma > 0$, $\alpha > 0$ such that*

$$\begin{aligned}
&\Psi_1 = (I + W_k)^T P(I + W_k) - \mu P < 0,\ \forall k \geq 1,\\
&\Psi_2 = (I + W_k)^T Q(I + W_k) - \mu Q < 0,\ \forall k \geq 1,\\
&\Omega = \begin{bmatrix}
\Omega_{11} & 0 & 0 & 0 & \Omega_{15} & 0 & \Omega_{17} & 0\\
\star & \Omega_{22} & 0 & 0 & 0 & \Omega_{26} & 0 & \Omega_{28}\\
\star & \star & \Omega_{33} & 0 & 0 & 0 & \Omega_{37} & 0\\
\star & \star & \star & \Omega_{44} & 0 & 0 & 0 & \Omega_{48}\\
\star & \star & \star & \star & \Omega_{55} & 0 & 0 & 0\\
\star & \star & \star & \star & \star & \Omega_{66} & 0 & 0\\
\star & \star & \star & \star & \star & \star & \Omega_{77} & 0\\
\star & \star & \star & \star & \star & \star & \star & \Omega_{88}
\end{bmatrix} < 0,
\end{aligned}\tag{3.31}$$

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where

$$\begin{aligned}
\Omega_{11} &= 2D_1 - PA - AP + (2\gamma+\sigma)P + C^T\left[2U_{11}L_1 + (L_1-2L_0)^T U_{21}L_1 - 2L_0^T U_{31}L_1\right]C,\\
\Omega_{15} &= -C^T U_{11} + C^T L_0^T U_{21} + C^T\left(L_0^T + L_1^T\right)U_{31},\\
\Omega_{17} &= P,\\
\Omega_{22} &= 2D_2 - QA - AQ + (2\gamma+\sigma)Q + C^T\left[2U_{12}L_1 + (L_1-2L_0)^T U_{22}L_1 - 2L_0^T U_{32}L_1\right]C,\\
\Omega_{26} &= -C^T U_{12} + C^T L_0^T U_{22} + C^T\left(L_0^T + L_1^T\right)U_{32},\\
\Omega_{28} &= Q,\\
\Omega_{33} &= -\alpha e^{-2\gamma\tau}P + C^T\left[2U_{13}L_1 + (L_1-2L_0)^T U_{23}L_1 - 2L_0^T U_{33}L_1\right]C,\\
\Omega_{37} &= -C^T U_{13} + C^T L_0^T U_{23} + C^T\left(L_0^T + L_1^T\right)U_{33},\\
\Omega_{44} &= -\alpha e^{-2\gamma\tau}Q + C^T\left[2U_{14}L_1 + (L_1-2L_0)^T U_{24}L_1 - 2L_0^T U_{34}L_1\right]C,\\
\Omega_{48} &= -C^T U_{14} + C^T L_0^T U_{24} + C^T\left(L_0^T + L_1^T\right)U_{34},\\
\Omega_{55} &= -U_{21} - 2U_{31},\\
\Omega_{66} &= -U_{22} - 2U_{32},\\
\Omega_{77} &= -U_{23} - 2U_{33},\\
\Omega_{88} &= -U_{24} - 2U_{34},\\
\beta &= \sup_{k\geq 1}\{t_k - t_{k-1}\}, \quad t_0 = 0,\\
\sigma &= \begin{cases} \alpha, & \textit{if } 1 \leq \alpha\beta,\\ \frac{\alpha}{\mu} + \frac{\ln\mu}{\beta}, & \textit{if } \alpha\beta \leq \mu,\\ \frac{1+\ln(\alpha\beta)}{\beta}, & \textit{if } \mu < \alpha\beta < 1. \end{cases}
\end{aligned} \tag{3.32}$$

Then the static neural network (2.1) is global exponential dissipative with dissipativity rate γ, and

$$\mathbb{S} = \left\{x : \|x\| \leq \min\left\{\frac{\sup_{t\in\mathbb{R}}\|P\psi(t)\|}{\lambda_m(D_1)},\ \frac{\sup_{t\in\mathbb{R}}\|Q\psi(t)\|}{\lambda_m(D_2)}\right\}\right\} \tag{3.33}$$

is a positive invariant and global attractive set. In particular, we have

$$\inf_{x(t)\in\mathbb{R}^n\setminus\mathbb{S}}\{\|x(t)-\tilde{x}\| : \tilde{x}\in\mathbb{S}\} < M(\phi)e^{-\gamma t}, \tag{3.34}$$

where $M(\phi) = \sqrt{\frac{\mu\lambda_1}{\lambda_0}}\|\phi\|_0$, $\lambda_1 = \max\{\lambda_M(P),\ \lambda_M(Q)\}$, $\lambda_0 = \min\{\lambda_m(P),\ \lambda_m(Q)\}$.

Proof. The proof is also divided into two parts.
(A) Set $\mathcal{R}(c) = \frac{\alpha}{c} + \frac{\ln c}{\beta}$. It is easy to get $\inf\limits_{\mu\leq c<1}\mathcal{R}(c) = \sigma$ and

$$\inf_{\mu\leq c<1}\mathcal{R}(c) = \begin{cases} \mathcal{R}(1), & \text{if } \alpha\beta \geq 1,\\ \mathcal{R}(\alpha\beta), & \text{if } \mu < \alpha\beta < 1,\\ \mathcal{R}(\mu), & \text{if } \alpha\beta \leq \mu. \end{cases} \tag{3.35}$$

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Therefore, we can always choose a proper $\hat{\mu} \in [\mu, \ 1)$ such that

$$\hat{\Omega} = \begin{bmatrix} \hat{\Omega}_{11} & 0 & 0 & 0 & \Omega_{15} & 0 & \Omega_{17} & 0 \\ \star & \hat{\Omega}_{22} & 0 & 0 & 0 & \Omega_{26} & 0 & \Omega_{28} \\ \star & \star & \Omega_{33} & 0 & 0 & 0 & \Omega_{37} & 0 \\ \star & \star & \star & \Omega_{44} & 0 & 0 & 0 & \Omega_{48} \\ \star & \star & \star & \star & \Omega_{55} & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \Omega_{66} & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \Omega_{77} & 0 \\ \star & \star & \star & \star & \star & \star & \star & \Omega_{88} \end{bmatrix} < 0, \tag{3.36}$$

since $\Omega < 0$, where

$$\begin{aligned} \hat{\Omega}_{11} &= 2D_1 - PA - AP + \left(2\gamma + \frac{\alpha}{\hat{\mu}} + \frac{\ln \hat{\mu}}{\beta}\right) P + C^T \left[2U_{11}L_1 + (L_1 - 2L_0)^T U_{21} L_1 - 2L_0^T U_{31} L_1\right] C, \\ \hat{\Omega}_{22} &= 2D_2 - QA - AQ + \left(2\gamma + \frac{\alpha}{\hat{\mu}} + \frac{\ln \hat{\mu}}{\beta}\right) Q + C^T \left[2U_{12}L_1 + (L_1 - 2L_0)^T U_{22} L_1 - 2L_0^T U_{32} L_1\right] C. \end{aligned} \tag{3.37}$$

Similarly, it holds for sufficient small positive number $\eta > 0$ that

$$\tilde{\Omega} = \begin{bmatrix} \tilde{\Omega}_{11} & 0 & 0 & 0 & \Omega_{15} & 0 & \Omega_{17} & 0 \\ \star & \tilde{\Omega}_{22} & 0 & 0 & 0 & \Omega_{26} & 0 & \Omega_{28} \\ \star & \star & \Omega_{33} & 0 & 0 & 0 & \Omega_{37} & 0 \\ \star & \star & \star & \Omega_{44} & 0 & 0 & 0 & \Omega_{48} \\ \star & \star & \star & \star & \Omega_{55} & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \Omega_{66} & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \Omega_{77} & 0 \\ \star & \star & \star & \star & \star & \star & \star & \Omega_{88} \end{bmatrix} < 0, \tag{3.38}$$

since $\hat{\Omega} < 0$, where

$$\begin{aligned} \tilde{\Omega}_{11} &= 2D_1 - PA - AP + \left(2\gamma + \frac{\alpha}{\hat{\mu}} - \mu_1\right) P + C^T \left[2U_{11}L_1 + (L_1 - 2L_0)^T U_{21} L_1 - 2L_0^T U_{31} L_1\right] C, \\ \tilde{\Omega}_{22} &= 2D_2 - QA - AQ + \left(2\gamma + \frac{\alpha}{\hat{\mu}} - \mu_1\right) Q + C^T \left[2U_{12}L_1 + (L_1 - 2L_0)^T U_{22} L_1 - 2L_0^T U_{32} L_1\right] C, \\ \mu_1 &= -\frac{\ln(\hat{\mu} + \eta)}{\beta} > 0. \end{aligned} \tag{3.39}$$

By routine calculations we have

$$\begin{aligned}
&\mathcal{D}^+V(t)+\left(\frac{\alpha}{\hat{\mu}}-\mu_1\right)V(t)-\alpha V(t-\tau(t))\\
&=e^{2\gamma t}\left(2\gamma x^T(t)Px(t)+2x^T(t)P\left[-Ax(t)+f(Cx(t-\tau(t)))\right]\right)+\left(\frac{\alpha}{\hat{\mu}}-\mu_1\right)e^{2\gamma t}x^T(t)Px(t)-\\
&\quad e^{2\gamma(t-\tau(t))}\alpha x^T(t-\tau(t))Px(t-\tau(t))+e^{2\gamma t}\left(2\gamma x^T(t-\tau)Qx(t-\tau)+2x^T(t-\tau)Q\left[-Ax(t-\tau)\right.\right.\\
&\quad\left.\left.+f(Cx(t-\tau-\tau(t)))\right]\right)+\left(\frac{\alpha}{\hat{\mu}}-\mu_1\right)e^{2\gamma t}x^T(t-\tau)Qx(t-\tau)-e^{2\gamma(t-\tau(t))}\alpha x^T(t-\tau-\tau(t))\\
&\quad Qx(t-\tau-\tau(t))\\
&\le e^{2\gamma t}\left(x^T(t)\left[2\gamma P-2PA+\left(\frac{\alpha}{\hat{\mu}}-\mu_1\right)P\right]x(t)+2x^TPf(Cx(t-\tau(t)))-\alpha e^{-2\gamma\tau}x^T(t-\tau(t))\right.\\
&\quad\left.Px(t-\tau(t))\right)+e^{2\gamma t}\left(x^T(t-\tau)\left[2\gamma Q-2QA+\left(\frac{\alpha}{\hat{\mu}}-\mu_1\right)Q\right]x(t-\tau)+2x^T(t-\tau)Q\right.\\
&\quad\left.f(Cx(t-\tau-\tau(t)))-\alpha e^{-2\gamma\tau}x^T(t-\tau-\tau(t))Qx(t-\tau-\tau(t))\right).
\end{aligned}\tag{3.40}$$

It then follows by combining (3.7) and (3.40) that

$$\begin{aligned}
&\mathcal{D}^+V(t)+\left(\frac{\alpha}{\hat{\mu}}-\mu_1\right)V(t)-\alpha V(t-\tau(t))\\
&\le 2e^{2\gamma t}\left[\left(x^T(t)P\psi(t)-x^T(t)D_1x(t)\right)+\left(x^T(t-\tau)Q\psi(t-\tau)-x^T(t-\tau)D_2x(t-\tau)\right)\right]+e^{2\gamma t}X^T\tilde{\Omega}X\\
&\le 2e^{2\gamma t}\left(\|x(t)\|\left(\|P\psi(t)\|-\lambda_m(D)\|x(t)\|\right)+\frac{1}{2}X^T\tilde{\Omega}X\right),
\end{aligned}\tag{3.41}$$

where X is the same vector as used in Theorem 3.1. Therefore, we get by using $\tilde{\Omega}<0$ that

$$\mathcal{D}^+V(t)+\left(\frac{\alpha}{\hat{\mu}}-\mu_1\right)V(t)-\alpha V(t-\tau(t))\le 0,\tag{3.42}$$

when $x(t)\in\mathbb{R}^n\backslash\mathbb{S}$. This implies

$$\mathcal{D}^+V(t)+\alpha\left(\frac{1}{\hat{\mu}}V(t)-V(t-\tau(t))\right)\le\mu_1V(t).\tag{3.43}$$

(B) We next prove (3.34). To this end, it is sufficient to prove

$$V(t)<\lambda_0M^2(\phi),\quad\text{for } x(t)\in\mathbb{R}^n\backslash\mathbb{S}\text{ and } t\ge-\tau.\tag{3.44}$$

(b1) For $t\in[-\tau,\ 0]$, by using Lemma 2.1 and $\mu\le\hat{\mu}<1$ it is easy to get

$$\begin{aligned}
V(t)&=e^{2\gamma t}\left(x^T(t)Px(t)+x^T(t-\tau)Px(t-\tau)\right)\\
&\le\lambda_1\|\phi\|_0^2=\mu\lambda_0M^2(\phi)\le\hat{\mu}\lambda_0M^2(\phi)\\
&<\lambda_0M^2(\phi).
\end{aligned}\tag{3.45}$$

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(b2) We next prove $V(t) < \lambda_0 M^2(\phi)$ for $t \in [0,\ t_1)$. If not, there exists $t \in (0,\ t_1)$ such that $V(t) \geq \lambda_0 M^2(\phi)$. Set $\bar{t} = \inf\{t : V(t) \geq \lambda_0 M^2(\phi) \text{ and } t \in [0,\ t_1)\}$. Clearly, $\bar{t} \in (0,\ t_1)$ and $V(\bar{t}) = \lambda_0 M^2(\phi)$. Since $V(0) \leq \hat{\mu}\lambda_0 M^2(\phi)$, there exists $\underline{t} = \sup\left\{t : V(t) \leq \hat{\mu}\lambda_0 M^2(\phi) \text{ and } t \in [0,\ \bar{t})\right\}$. Hence, for $t \in [\underline{t},\ \bar{t}]$ we have

$$V(t) \geq \hat{\mu}\lambda_0 M^2(\phi) \ \text{ and } \ V(t+\theta) \leq \lambda_0 M^2(\phi), \quad \theta \in [-\tau,\ 0], \tag{3.46}$$

which leads to

$$\mathcal{D}^+V(t) \leq \mathcal{D}^+V(t) + \alpha\left(\frac{1}{\hat{\mu}}V(t) - V(t-\tau(t))\right), \quad t \in [\underline{t},\ \ \bar{t}]. \tag{3.47}$$

By using (3.43) we have $\mathcal{D}^+V(t) \leq \mu_1 V(t)$ for $x(t) \in \mathbb{R}^n \backslash \mathbb{S}$ and $t \in [\underline{t},\ \ \bar{t}]$ and this gives

$$V(t) \leq V(\underline{t})e^{\mu_1(t-\underline{t})}, \quad t \in [\underline{t},\ \ \bar{t}], \tag{3.48}$$

From (3.48), we know

$$V(\bar{t}) \leq V(\underline{t})e^{\mu_1(\bar{t}-\underline{t})} \leq \hat{\mu}\lambda_0 M^2(\phi)e^{\mu_1\beta} = \frac{\hat{\mu}}{\hat{\mu}+\eta}\lambda_0 M^2(\phi) < \lambda_0 M^2(\phi), \tag{3.49}$$

since $\eta > 0$. This leads to a contradiction and therefore (3.44) holds for $t \in [0,\ t_1)$.
(b3) Suppose for any integer $k \geq 1$ that

$$V(t) < \lambda_0 M^2(\phi), \ \ t \in [-\tau,\ t_k). \tag{3.50}$$

We try to prove

$$V(t) < \lambda_0 M^2(\phi), \ \ t \in [t_k,\ t_{k+1}). \tag{3.51}$$

If not, there exists $t \in [t_k,\ t_{k+1})$ such that $V(t) \geq \lambda_0 M^2(\phi)$. Set $t^* = \inf\{t : V(t) \geq \lambda_0 M^2(\phi) \text{ and } t \in [t_k,\ t_{k+1})\}$. From the assumption $V(t) < \lambda_0 M^2(\phi)$ for $t \in [-\tau,\ t_k)$ and Lemma 3.1, we have

$$\begin{aligned} V(t_k) &\leq \mu V(t_k^-) + (1-\mu)e^{2\gamma t}x^T(t_k^- - \tau)Qx(t_k^- - \tau) \\ &\leq \mu V(t_k^-) + (1-\mu)V(t_k - \tau) \\ &< \mu\lambda_0 M^2(\phi) + (1-\mu)\lambda_0 M^2(\phi) \\ &= \lambda_0 M^2(\phi), \end{aligned} \tag{3.52}$$

if $t_k - \tau$ is not an impulsive instant. For the case that $t_k - \tau$ is an impulsive instant, Lemma 3.1 gives $V(t_k) \leq \mu V(t_k^-) < \lambda_0 M^2(\phi)$ since $\mu < 1$. Therefore, for both cases we have $V(t_k) < \lambda_0 M^2(\phi)$ and this means $t^* \in (t_k,\ t_{k+1})$ and $V(t^*) = \lambda_0 M^2(\phi)$. Set $\bar{t} = \sup\{t : V(t) \leq \hat{\mu}\lambda_0 M^2(\phi) \text{ and } t \in [t_k,\ t^*)\}$, then $V(\bar{t}) = \hat{\mu}\lambda_0 M^2(\phi)$. Moreover, for $t \in [\bar{t},\ t^*]$, we have

$$\mathcal{D}^+V(t) \leq \mathcal{D}^+V(t) + \alpha\left(\frac{1}{\hat{\mu}}V(t) - V(t-\tau(t))\right). \tag{3.53}$$

Then, by using the same treatment as in **(b2)** we will arrive at $V(t^*) < \lambda_0 M^2(\phi)$, which obviously contradicts the fact $V(t^*) = \lambda_0 M^2(\phi)$.

Thus, by the method of mathematical induction, (3.51) holds for any integer k. So (3.44) is true and (3.34) follows by applying Lemma 2.1. ■

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Remark 3.2 *Since $\mu < 1$ means $\mu_1 = -\frac{\ln\hat{\mu}}{\beta} < 0$ (this is because we have assumed $\hat{\mu} \in [\mu,\ 1)$ and $\hat{\mu} + \eta < 1$), condition (3.43) implies that the static neural network (2.1) without impulses may be not dissipative in the set $\mathbb{S}$. Thus, Theorem 3.2 can be applied to design an impulsive control law to let the static neural network to be a dissipative one.*

Similar to Corollary 3.1, we can get the following result for the case $\mu < 1$ and $\tau(t) \equiv \tau$, i.e., the delay argument is a constant number.

Corollary 3.2 *Suppose $\tau(t) \equiv \tau$ and there exist symmetric positive matrices $P,\ Q,\ D_1,\ D_2,$ positive diagonal matrices $U_{ij}(1 \le i,\ j \le 3)$, scalar numbers $\mu < 1,\ \gamma > 0,\ \alpha > 0$ such that*

$$\begin{aligned}
&\Psi_1 = (I + W_k)^T P(I + W_k) - \mu P < 0,\ \forall k \ge 1,\\
&\Psi_2 = (I + W_k)^T Q(I + W_k) - \mu Q < 0,\ \forall k \ge 1,\\
&\Omega = \begin{bmatrix} \Omega_{11} & 0 & 0 & \Omega_{14} & \Omega_{15} & 0 \\ \star & \Omega_{22} & 0 & 0 & \Omega_{25} & \Omega_{26} \\ \star & \star & \Omega_{33} & 0 & 0 & \Omega_{36} \\ \star & \star & \star & \Omega_{44} & 0 & 0 \\ \star & \star & \star & \star & \Omega_{55} & 0 \\ \star & \star & \star & \star & \star & \Omega_{66} \end{bmatrix} < 0,
\end{aligned} \tag{3.54}$$

where

$$\begin{aligned}
\Omega_{11} &= 2D_1 + (2\gamma+\sigma) P - PA - A^T P^T + C^T[2U_{11}L_1 + (L_1 - 2L_0)^T U_{21} L_1 - 2L_0^T U_{31} L_1]C,\\
\Omega_{14} &= -C^T U_{11} + C^T L_0^T U_{21} + C^T L_0^T U_{31} + C^T L_1^T U_{31}^T,\\
\Omega_{15} &= P,\\
\Omega_{22} &= 2D_2 + (2\gamma+\sigma) Q - \alpha e^{-2\gamma\tau} P - QA - A^T Q^T + C^T[2U_{12}L_1 + (L_1 - 2L_0)^T U_{22} L_1\\
&\quad - 2L_0^T U_{32} L_1]C,\\
\Omega_{25} &= -C^T U_{12} + C^T L_0^T U_{22} + C^T L_0^T U_{32} + C^T L_1^T U_{32}^T,\\
\Omega_{26} &= Q,\\
\Omega_{33} &= -\alpha e^{-2\gamma\tau} Q + C^T[2U_{13}L_1 + (L_1 - 2L_0)^T U_{23} L_1 - 2L_0^T U_{33} L_1]C,\\
\Omega_{36} &= -C^T U_{13} + C^T L_0^T U_{23} + C^T L_0^T U_{33} + C^T L_1^T U_{33}^T,\\
\Omega_{44} &= -U_{21} - 2U_{31},\\
\Omega_{55} &= -U_{22} - 2U_{32},\\
\Omega_{66} &= -U_{23} - 2U_{33},\\
\beta &= \sup_{k\ge1}\{t_k - t_{k-1}\},\quad t_0 = 0,\\
\sigma &= \begin{cases} \alpha, & \text{if } 1 \le \alpha\beta,\\ \frac{\alpha}{\mu} + \frac{\ln\mu}{\beta}, & \text{if } \alpha\beta \le \mu,\\ \frac{1+\ln(\alpha\beta)}{\beta}, & \text{if } \mu < \alpha\beta < 1. \end{cases}
\end{aligned}$$

Then neural network (2.1) is global exponential dissipative with dissipativity rate γ, and in particular (3.34) holds.

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Remark 3.3 *For the case $\mu \geq 1$, the analysis in Theorem 3.1 is similar to Theorem 3.1 of [14] and Theorem 1 of [3]. This mainly benefits from the fact that $V(t_k) \leq \mu V(t_k^-)$ holds for whether $t_k - \tau$ is an impulsive instant or not. However, for stabilizing impulses, i.e., $\mu < 1$, if $Q \neq 0$ the inequality $V(t_k) \leq \mu V(t_k^-)$ which is very important for the proof of Theorem 3.3 in [14] and Theorem 5 in [3] does not hold, provided $t_k - \tau$ is not an impulsive instant. In our proof, we give a close look at this problem and find that $V(t_k) \leq \mu V(t_k^-)$ is just an interim step, while $V(t_k) < \lambda_0 M^2(\phi)$ is the final goal. In [3, 14], because $Q = 0$, i.e., $V(t) = e^{2\gamma t} x^T(t) P x(t)$, this inequality can be deduced straightforwardly as $V(t_k) \leq \mu V(t_k^-) < \lambda_0 M^2(\phi)$, since $\mu < 1$. For $Q \neq 0$, we prove this inequality by using the key relation $V(t_k) \leq \mu V(t_k^-) + (1-\mu) e^{2\gamma t} x^T(t_k^- - \tau) Q x(t_k^- - \tau)$ (see(3.52)).*

Remark 3.4 *A natural question is why we do not use other Lyapunov functions to perform the dissipativity analysis, such as $V(t) = e^{2\gamma t} x^T(t) P x(t) + \int_{-\tau}^{0} \int_{t+\theta}^{t} e^{2\gamma s} \dot{x}^T(s) Q \dot{x}(s) ds d\theta$, where $\int_{-\tau}^{0} \int_{t+\theta}^{t} e^{2\gamma s} \dot{x}^T(s) Q \dot{x}(s) ds d\theta$ (with $Q > 0$) is a commonly used component in the stability analysis of NNs with time delay (see, e.g., [10–13, 26, 27]). In such case, one may check that it is difficult to repeat the proof of Theorems 3.1 and 3.2. In particular, for $\mu \geq 1$ the inequality $V(t_k) \leq \mu V(t_k^-)$ still holds (the proof is similar to Lemma 3.1) but it is difficult to get a quadratic form (like (3.6)) of the left hand of (3.10). For the case $\mu < 1$, an inequality like $V(t_k) \leq \mu V(t_k^-) + (1-\mu) \int_{-\tau}^{0} \int_{t_k+\theta}^{t_k} e^{2\gamma s} \dot{x}^T(s) Q \dot{x}(s) ds d\theta$ still holds, and $V(t_k) < \lambda_0 M^2(\phi)$ can be proved. However, it is difficult to get a quadratic form (like (3.40)) of the left hand of (3.43). We remark that the quadratic forms shown in (3.6) and (3.40) play an important role to establish LMI criteria.*

Remark 3.5 *We note that the inequalities (3.7) are very important to derive efficient LMIs for static NNs with time delay and impulses. Without (3.7), the obtained LMIs do not contain the system matrix C in (2.1). Therefore, (3.7) is the tie between the matrix C and the obtained LMIs. This is the main difference between the analysis of the static NNs and the local field NNs.*

3.3. Application to stability analysis.

We next show that the LMI criteria given in Theorems 3.1, 3.2 and Corollaries 3.1, 3.2 can be easily reduced to exponential stability conditions. For concise, we assume in (2.1) that $\psi(t) \equiv 0$ and therefore the attractive set $\mathbb{S}$ defined by Theorems 3.1 and 3.2 shrinks to the origin $x^* = 0$. In this case, we can let $D_1 = D_2 = 0$ in (3.3a), (3.29), (3.31) and (3.54), and it is clear that the reduced LMI criteria can be regarded as exponential stability conditions. In the next section, we provide numerical results to show that the deduced stability conditions are mush less conservative than the existing ones.

Moreover, we can also consider the following static NNs

$$x'(t) = -Ax(t) + g(Bx(t)) + f(Cx(t - \tau(t))), \tag{3.55}$$

which is obviously more general than the one discussed in this paper. Similar to (3.7), by

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additionally introducing the following inequalities

$$\begin{aligned}
&0 \le 2e^{2\gamma t}\left[Bz_j\right]^T V_{1j}[H_1\left(Bz_j\right) - g(Bz_j)] = e^{2\gamma t}z_j^T[2B^T V_{1j}H_1B]z_j + 2e^{2\gamma t}z_j^T[-B^T V_{1j}]g(Bz_j),\\
&0 \le e^{2\gamma t}\left[(Bz_j)^T(H_1^T - H_0^T) + g^T(Bz_j) - (Bz_j)^T H_0^T\right] V_{2j}[H_1Bz_j - g(Bz_j)]\\
&\quad = e^{2\gamma t}\left(z_j^T\left[B^T(H_1 - 2H_0)^T V_{2j}H_1B\right]z_j + 2z_j^T[B^T H_0^T V_{2j}]g(Bz_j) + g^T(Bz_j)[-V_{2j}]g(Bz_j)\right),\\
&0 \le 2e^{2\gamma t}\left[g^T(Bz_j) - (Bz_j)^T H_0^T\right] V_{3j}[H_1(Bz_j) - g(Bz_j)]\\
&\quad = z_j^T(t)[-2B^T L_0^T V_{3j}H_1B]z_j + 2z_j^T[C^T H_0^T V_{3j} + B^T H_1^T V_{3j}^T]g(Bz_j) + g^T(Bz_j)[-2V_{3j}]g(Bz_j),
\end{aligned} \tag{3.56}$$

it can be shown that the results obtained in Theorems 3.1, 3.2 and Corollaries 3.1, 3.2 can be straightforwardly generalized to (3.55) and therefore this is a trivial difference. Here, V_{ij} ($i = 1, 2, 3,\ j = 1, 2, 3, 4$) are positive diagonal matrices, $H_0 = \mathrm{diag}(h_1^-, h_2^-, \ldots, h_n^-)$ and $H_1 = \mathrm{diag}(h_1^+, h_2^+, \ldots, h_n^+)$ which satisfy

$$g_i(0) = 0, \quad h_i^- \le \frac{g_i(s_1) - g_i(s_2)}{s_1 - s_2} \le h_i^+, \quad \forall s_1,\ s_2 \in \mathbb{R} \text{ and } s_1 \ne s_2, \quad i = 1, 2, \ldots, n. \tag{3.57}$$

4. Numerical results.

In this section, we do several numerical simulations to validate the effectiveness of our results presented in Section 3.

Example 1. Consider the following static neural network

$$\begin{cases}
\mathcal{D}^+x(t) = -\begin{bmatrix} 2.2 & 0\\ 0 & 2.8\end{bmatrix}x(t) + f\left(\begin{bmatrix} 0.75 & 0.05\\ 0.1 & 1.3\end{bmatrix}x(t-\tau)\right) + \psi(t), & t \ne t_k,\\
\Delta x(t_k) = W_k x(t_k^-), & t = t_k,\\
x(t) = \phi, & t \le 0,
\end{cases} \tag{4.1}$$

where $\psi(t) = (0.5\sin(t), 0.5\cos(t))^T$ and $f(x) = \frac{|x+1|-|x-1|}{2}$. Then, we have $L_1 = \mathrm{diag}(1,1)$ and $L_0 = \mathrm{diag}(0,0)$. We set $W_k = \mathrm{diag}(\omega,\omega)$ and the argument $\omega > 0$ can be viewed as the magnitude of the impulses. With $\omega > 0$, the linear matrix inequalities $(I + W_k)^T P(I + W_k) - \mu P < 0$ and $(I+W_k)^T Q(I+W_k) - \mu Q < 0$ imply $\mu > (1+\omega)^2 > 1$ and therefore the impulses are disturbances.

Let the exponential dissipative rate be $\gamma = 10^{-4}$ and the distance between two consecutive impulsive instants be 0.6, i.e., $t_k = t_{k-1} + 0.6$. In Table 4.1, we list the maximum impulse magnitude (denoted by $\omega_{\max}$) corresponding to different τ. The $\omega_{\max}$ is obtained by solving the LMIs in Theorem 3.1 and Corollary 3.1. For each $\omega_{\max}$, the corresponding parameter α is also given in the table.

We see from Table 4.1 that the results predicted by Corollary 3.1 is significantly less conservative than the ones predicted by Theorem 3.1, when the delay argument is a constant number. For example, if $\tau = 1$, the maximum impulse magnitude $\omega_{\max}$ obtained from Corollary 3.1 is 0.6021, while it is only 0.5463 from Theorem 3.1. Besides this, it is interesting to find in Table 4.1 that the maximum impulse magnitude $\omega_{\max}$ seems to be robust with respect to τ.

For $\tau = 1$, the dynamic behavior of impulse-free system (4.1) with different initial vector ϕ is shown in Figure 4.1 on the left, where we see that the impulse-free system is dissipative. With

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Table 4.1: $\omega_{\max}$ corresponding to different τ

Method	$\tau = 0.5$	$\tau = 1$	$\tau = 2$	$\tau = 4$	$\tau = 5$
Theorem 3.1	0.5463	0.5463	0.5462	0.5460	0.5459
α	*0.8038*	*0.8038*	*0.8036*	*0.8037*	*0.8039*
Corollary 3.1	0.6022	0.6021	0.6021	0.6020	0.6018
α	*0.6101*	*0.6102*	*0.6101*	*0.6102*	*0.6101*

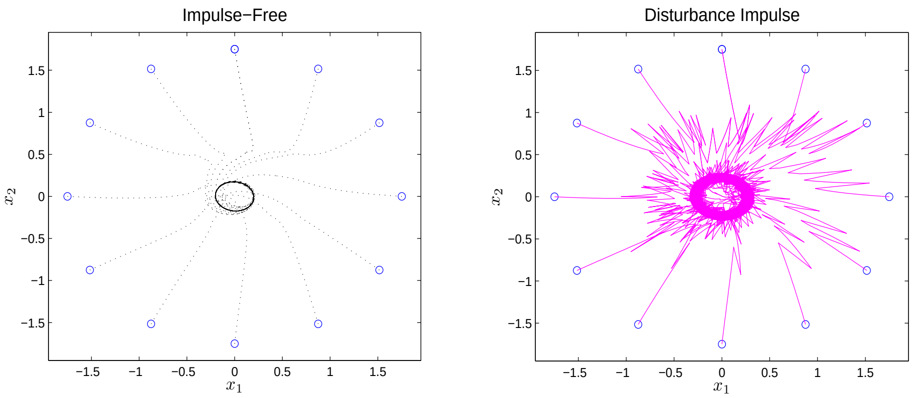

Figure 4.1: Behavior of $x(t)$ in system (4.1) without (left) and with (right) impulses. In each panel, the marker '∘' denotes the initial state of the solution trajectory.

impulse matrices $W_k = \mathrm{diag}(0.6021, 0.6021)$, impulsive system (4.1) converges to an irregular *circle*, which can be see clearly in Figure 4.1 on the right.

Example 2. We next consider the following system

$$\begin{cases} x'(t) = -\begin{bmatrix} 1e-3 & 0 \\ 0 & 2e-3 \end{bmatrix} x(t) + \begin{bmatrix} 1.08 & -2 \\ 6 & 0.92 \end{bmatrix} f(x(t-\tau(t))) + \psi(t), & t \neq t_k, \\ \Delta x(t_k) = W_k x(t_k^-), & t = t_k, \\ x(t) = \phi, & t \leq 0, \end{cases} \tag{4.2}$$

where f is the same function as we has used in Example 1, $\tau(t) = \frac{2}{1+\sin^2(10t)}$, $W_k = \mathrm{diag}(-0.08, -0.08)$ and $\psi(t) = \left(\frac{-1}{4}\sin\left(\frac{\sqrt{t}}{(1+0.28\sqrt{t})} + t\right), \frac{\cos(t)}{4}\right)^T$. Then, we know $\max\limits_{t\geq 0}\tau(t) = 2$ and the impulses are stabilizing. The simulations of impulse-free system (4.2) with 8 different initial values are shown in the left column of Figure 4.2 corresponding to three different time t, where we see clearly that the solution $x(t)$ diverges to infinity as the evolution time t increases. Let the exponential dissipative rate be $\gamma = 2e-4$. By solving the LMIs in Theorem 3.2, we get the maximal distance between two consecutive impulsive instants, $\beta_{\max} = 0.0442$. With $t_k = t_{k-1} + 0.0442$ and the same initial values, we simulate the impulsive system (4.2) and the solution trajectories are plotted in the right column of Figure 4.2. We see clearly in these three panels that each trajectory goes to

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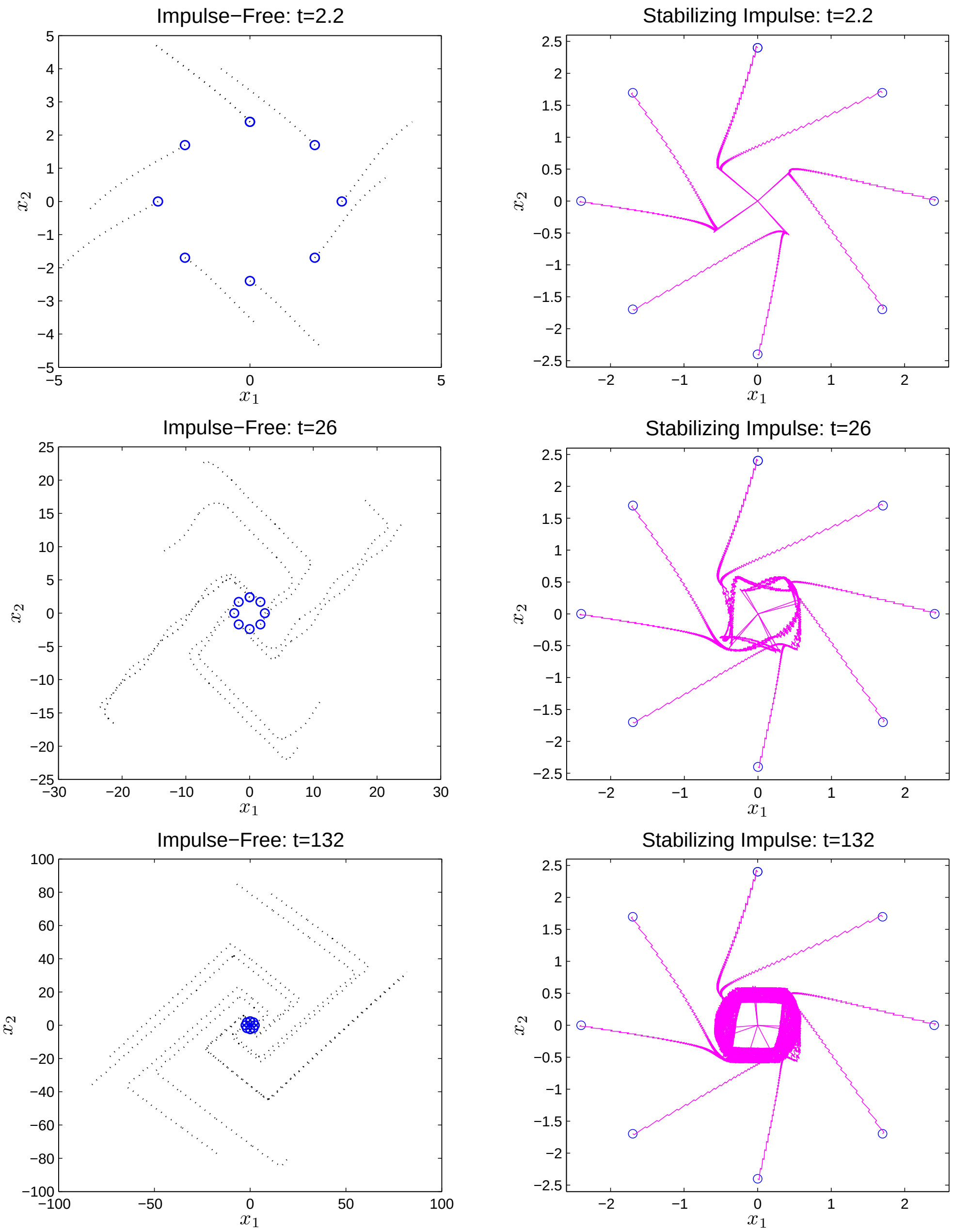

Figure 4.2: Behavior of $x(t)$ in system (4.2) without (left column) and with (right column) impulses. In each panel, the marker '$\circ$' denotes the initial state of the solution trajectory.

a bounded domain as the evolution time t increases and therefore system (4.2) with stabilizing

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impulses is really dissipative.

Example 3. Our last example is the following static neutral network

$$\begin{cases} \mathcal{D}^+ x(t) = -\begin{bmatrix} 0.2 & 0 \\ 0 & 1.3 \end{bmatrix} x(t) + f\left(\begin{bmatrix} -1.5 & -0.12 \\ -0.26 & -2.5 \end{bmatrix} x(t-\tau)\right), & t \neq t_k, \\ \Delta x(t_k) = \begin{bmatrix} -0.3 & 0 \\ 0 & -0.3 \end{bmatrix} x(t_k^-), & t = t_k, \\ x(t) = \phi, & t \leq 0, \end{cases} \tag{4.3}$$

where $f(x) = \tanh(x)$ and $t_k = t_{k-1} + \beta$. Then, we know $L_1 = \mathrm{diag}(1,1)$ and $L_0 = \mathrm{diag}(0,0)$. Clearly, the impulses are stabilizing. Since the external input function is chosen $\psi(t) \equiv 0$, we consider the global exponential stability of (4.3). The criteria given by Zhao and Wang [13] require

$$\min_{1\leq i\leq n}\left[2A_{i,i} - L_i \sum_{j=1}^{n} |C_{i,j}|\right] > \max_{1\leq i\leq n}\left[L_i \sum_{j=1}^{n} |C_{j,i}|\right] > 0.$$

However, for system (4.3), we have

$$\min_{1\leq i\leq 2}\left[2A_{i,i} - L_i \sum_{j=1}^{2} |C_{i,j}|\right] = -0.16 \text{ and } \max_{1\leq i\leq 2}\left[L_i \sum_{j=1}^{2} |C_{j,i}|\right] = 2.62.$$

Therefore, the criteria given by Zhao and Wang [13] are invalid.

Let $t_k - t_{k-1} = \beta$. The quantity β can be viewed as the distance between two consecutive impulsive instants. In Table 4.2, for different exponential convergence rate γ, we list the maximum distance (denoted by $\beta_{\max}$) by using the criterion obtained in this paper (by letting $D_1 = D_2 = 0$ in Theorem 3.2 and Corollary 3.2) and the one given in [14]. From the results listed in Table 4.2

Table 4.2: $\beta_{\max}$ corresponding to different γ

Method	$\gamma = 0.001$	$\gamma = 0.005$	$\gamma = 0.01$	$\gamma = 0.05$
Theorem 3.2	0.0616	0.0601	0.0532	0.0497
α	*0.6322*	*0.6542*	*0.6534*	*0.6904*
Corollary 3.2	0.0746	0.0702	0.0639	0.0627
α	*0.6262*	*0.6292*	*0.6356*	*0.6378*
Thm. 3.3 in [14]	0.0521	0.0489	0.0498	0.0443
α	*0.5257*	*0.5456*	*0.5578*	*0.5894*

we see that the stability criterion deduced from Corollary 3.2 is significantly less conservative than the one given in [14]. For example, for $\gamma = 0.001$, the quantity $\beta_{\max}$ predicted by Corollary 3.2 and Theorem 3.3 in [14] is 0.0746 and 0.0521 respectively, and 43.19% improvement[1] is obtained by using Corollary 3.2. For $\tau = 1.6$ and $\phi(t) = (-0.07, 0.07)^T$, we plot in Figure 4.3 on the left the solutions $x_1(t)$ and $x_2(t)$ without impulses, where we see that the solutions behave like chaos. After imposing stabilizing impulses to (4.3) with $t_k = t_{k-1} + 0.0746$, we show in Figure 4.3 on the right the behavior of the solutions and it is clear that the system is really stabilized.

[1]For two different quantities a and b, the improvement of b against a is defined by percentage $\left(100 \times \frac{b-a}{a}\right)\%$.

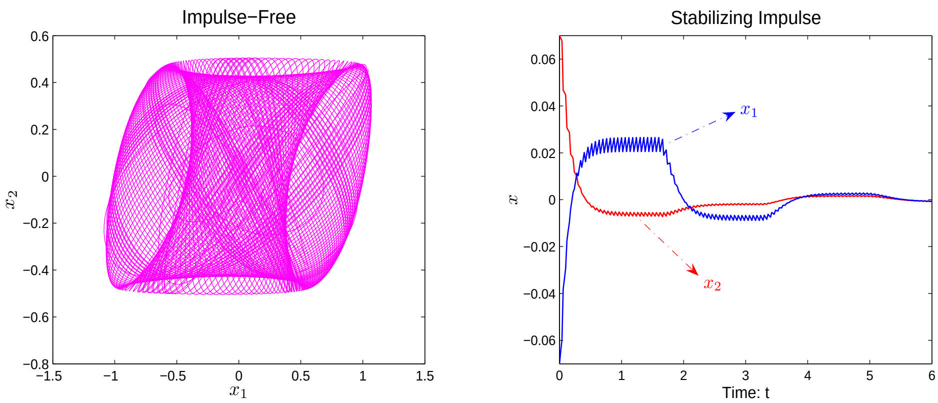

Figure 4.3: Left: chaos-like dynamic behavior of the solutions of (4.3) without impulses; Right: the solution profiles after impulsive stabilization.

5. Conclusion.

The local field neural networks and the static neural networks typically represent two fundamental models in the research of neural networks. The former has been investigated widely and deeply by many authors, while the latter has not received so much attention and systematic analysis is still rare, particularly when both the time delay and impulses are taken into account. In this paper, we propose new analysis method to study the problem of global exponential dissipativity of static neural networks with impulses and time delays. This method can be regarded as a generalization of the one used in [3] and [14], while the original method can only handle very simple Lyapunov function. Several sufficient conditions concerning global exponential dissipativity were established in terms of LMIs and therefore they can be checked efficiently via the LMI toolbox in MATLAB. Moreover, we show that the dissipativity conditions can be straightforwardly reduced to stability conditions for the static neural networks with impulses and time delay. Benefitting from the new Lyapunov function, the deduced stability conditions are much less conservative than the existing ones.

Conflict of interests

The authors declare that there is no conflict of interests regarding the publication of this article.

Acknowledgment

This work is jointly supported by Opening Fund of Artificial Intelligence Key Laboratory of Sichuan Province under Grant 2012RYJ06, the scientific research fund of Sichuan University of Science and Engineering under Grant 2012RC24, the NSF of China (11226312).

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