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Dartmouth College
8000 Cummings Hall,
Hanover, NH 03755-8000
603-646-3843 (X 3546 Secr.)
e-mail:george.cybenko@dartmouth.edu
Approximation Theory and Neural
Networks

Sever S. Dragomir
School of Computer Science and
Mathematics, Victoria University,
PO Box 14428,
Melbourne City,
MC 8001, AUSTRALIA
Tel. +61 3 9688 4437
Fax +61 3 9688 4050
sever.dragomir@vu.edu.au
Inequalities, Functional Analysis,
Numerical Analysis, Approximations,
Information Theory, Stochastics.

Oktay Duman
TOBB University of Economics and
Technology,
Department of Mathematics, TR-
06530,
Ankara, Turkey,
oduman@etu.edu.tr
Classical Approximation Theory,
Summability Theory, Statistical
Convergence and its Applications

Saber N. Elaydi
Department Of Mathematics
Trinity University
715 Stadium Dr.
San Antonio, TX 78212-7200
210-736-8246
e-mail: selaydi@trinity.edu
Ordinary Differential Equations,
Difference Equations

Christodoulos A. Floudas
Department of Chemical Engineering
Princeton University
Princeton,NJ 08544-5263
609-258-4595(x4619 assistant)
e-mail: floudas@titan.princeton.edu
Optimization Theory&Applications,
Global Optimization

J .A. Goldstein
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152
901-678-3130
jgoldste@memphis.edu

Partial Differential Equations,
Semigroups of Operators

H. H. Gonska
Department of Mathematics
University of Duisburg
Duisburg, D-47048
Germany
011-49-203-379-3542
e-mail: heiner.gonska@uni-due.de
Approximation Theory, Computer
Aided Geometric Design

John R. Graef
Department of Mathematics
University of Tennessee at
Chattanooga
Chattanooga, TN 37304 USA
John-Graef@utc.edu
Ordinary and functional
differential equations, difference
equations, impulsive systems,
differential inclusions, dynamic
equations on time scales, control
theory and their applications

Weimin Han
Department of Mathematics
University of Iowa
Iowa City, IA 52242-1419
319-335-0770
e-mail: whan@math.uiowa.edu
Numerical analysis, Finite element
method, Numerical PDE, Variational
inequalities, Computational
mechanics

Tian-Xiao He
Department of Mathematics and
Computer Science
P.O. Box 2900, Illinois Wesleyan
University
Bloomington, IL 61702-2900, USA
Tel (309)556-3089
Fax (309)556-3864
the@iwu.edu
Approximations, Wavelet,
Integration Theory, Numerical
Analysis, Analytic Combinatorics

Margareta Heilmann
Faculty of Mathematics and Natural
Sciences, University of Wuppertal
Gaußstraße 20
D-42119 Wuppertal, Germany,
heilmann@math.uni-wuppertal.de

Approximation Theory (Positive
Linear Operators)

Xing-Biao Hu
Institute of Computational
Mathematics
AMSS, Chinese Academy of Sciences
Beijing, 100190, CHINA
hxb@lsec.cc.ac.cn
Computational Mathematics

Jong Kyu Kim
Department of Mathematics
Kyungnam University
Masan Kyungnam,631-701,Korea
Tel 82-(55)-249-2211
Fax 82-(55)-243-8609
jongkyuk@kyungnam.ac.kr
Nonlinear Functional Analysis,
Variational Inequalities, Nonlinear
Ergodic Theory, ODE, PDE,
Functional Equations.

Robert Kozma
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152, USA
rkozma@memphis.edu
Neural Networks, Reproducing Kernel
Hilbert Spaces,
Neural Percolation Theory

Mustafa Kulenovic
Department of Mathematics
University of Rhode Island
Kingston, RI 02881,USA
kulenm@math.uri.edu
Differential and Difference
Equations

Irena Lasiecka
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152
PDE, Control Theory, Functional
Analysis, lasiecka@memphis.edu

Burkhard Lenze
Fachbereich Informatik
Fachhochschule Dortmund
University of Applied Sciences
Postfach 105018
D-44047 Dortmund, Germany
e-mail: lenze@fh-dortmund.de
Real Networks, Fourier Analysis,
Approximation Theory

Hrushikesh N. Mhaskar
Department Of Mathematics
California State University
Los Angeles, CA 90032
626-914-7002
e-mail: hmhaska@gmail.com
Orthogonal Polynomials,
Approximation Theory, Splines,
Wavelets, Neural Networks

Ram N. Mohapatra
Department of Mathematics
University of Central Florida
Orlando, FL 32816-1364
tel.407-823-5080
ram.mohapatra@ucf.edu
Real and Complex Analysis,
Approximation Th., Fourier
Analysis, Fuzzy Sets and Systems

Gaston M. N'Guerekata
Department of Mathematics
Morgan State University
Baltimore, MD 21251, USA
tel: 1-443-885-4373
Fax 1-443-885-8216
Gaston.N'Guerekata@morgan.edu
nguerekata@aol.com
Nonlinear Evolution Equations,
Abstract Harmonic Analysis,
Fractional Differential Equations,
Almost Periodicity & Almost
Automorphy

M.Zuhair Nashed
Department Of Mathematics
University of Central Florida
PO Box 161364
Orlando, FL 32816-1364
e-mail: znashed@mail.ucf.edu
Inverse and Ill-Posed problems,
Numerical Functional Analysis,
Integral Equations, Optimization,
Signal Analysis

Mubenga N. Nkashama
Department OF Mathematics
University of Alabama at Birmingham
Birmingham, AL 35294-1170
205-934-2154
e-mail: nkashama@math.uab.edu
Ordinary Differential Equations,
Partial Differential Equations

Vassilis Papanicolaou
Department of Mathematics

National Technical University of
Athens
Zografou campus, 157 80
Athens, Greece
tel:: +30(210) 772 1722
Fax   +30(210) 772 1775
papanico@math.ntua.gr
Partial Differential Equations,
Probability

Choonkil Park
Department of Mathematics
Hanyang University
Seoul 133-791
S. Korea, baak@hanyang.ac.kr
Functional Equations

Svetlozar (Zari) Rachev,
Professor of Finance, College of
Business, and Director of
Quantitative Finance Program,
Department of Applied Mathematics &
Statistics
Stonybrook University
312 Harriman Hall, Stony Brook, NY
11794-3775
tel: +1-631-632-1998,
svetlozar.rachev@stonybrook.edu

Alexander G. Ramm
Mathematics Department
Kansas State University
Manhattan, KS 66506-2602
e-mail: ramm@math.ksu.edu
Inverse and Ill-posed Problems,
Scattering Theory, Operator Theory,
Theoretical Numerical Analysis,
Wave Propagation, Signal Processing
and Tomography

Tomasz Rychlik
Polish Academy of Sciences
Instytut Matematyczny PAN
00-956 Warszawa, skr. poczt. 21
ul. Śniadeckich 8
Poland
trychlik@impan.pl
Mathematical Statistics,
Probabilistic Inequalities

Boris Shekhtman
Department of Mathematics
University of South Florida
Tampa, FL 33620, USA
Tel 813-974-9710
shekhtma@usf.edu

Approximation Theory, Banach
spaces, Classical Analysis

T. E. Simos
Department of Computer
Science and Technology
Faculty of Sciences and Technology
University of Peloponnese
GR-221 00 Tripolis, Greece
Postal Address:
26 Menelaou St.
Anfithea - Paleon Faliron
GR-175 64 Athens, Greece
tsimos@mail.ariadne-t.gr
Numerical Analysis

H. M. Srivastava
Department of Mathematics and
Statistics
University of Victoria
Victoria, British Columbia V8W 3R4
Canada
tel.250-472-5313; office,250-477-
6960 home, fax 250-721-8962
harimsri@math.uvic.ca
Real and Complex Analysis,
Fractional Calculus and Appl.,
Integral Equations and Transforms,
Higher Transcendental Functions and
Appl.,q-Series and q-Polynomials,
Analytic Number Th.

I. P. Stavroulakis
Department of Mathematics
University of Ioannina
451-10 Ioannina, Greece
ipstav@cc.uoi.gr
Differential Equations
Phone +3-065-109-8283

Manfred Tasche
Department of Mathematics
University of Rostock
D-18051 Rostock, Germany
manfred.tasche@mathematik.uni-
rostock.de
Numerical Fourier Analysis, Fourier
Analysis, Harmonic Analysis, Signal
Analysis, Spectral Methods,
Wavelets, Splines, Approximation
Theory

Roberto Triggiani
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152
PDE, Control Theory, Functional

Analysis, rtrggani@memphis.edu

Juan J. Trujillo
University of La Laguna
Departamento de Analisis Matematico
C/Astr.Fco.Sanchez s/n
38271. LaLaguna. Tenerife.
SPAIN
Tel/Fax 34-922-318209
Juan.Trujillo@ull.es
Fractional: Differential Equations-
Operators-Fourier Transforms,
Special functions, Approximations,
and Applications

Ram Verma
International Publications
1200 Dallas Drive #824 Denton,
TX 76205, USA
Verma99@msn.com
Applied Nonlinear Analysis,
Numerical Analysis, Variational
Inequalities, Optimization Theory,
Computational Mathematics, Operator
Theory

Xiang Ming Yu
Department of Mathematical Sciences
Southwest Missouri State University
Springfield, MO 65804-0094
417-836-5931
xmy944f@missouristate.edu
Classical Approximation Theory,
Wavelets

Lotfi A. Zadeh
Professor in the Graduate School
and Director, Computer Initiative,
Soft Computing (BISC)
Computer Science Division
University of California at
Berkeley
Berkeley, CA 94720
Office: 510-642-4959
Sec: 510-642-8271
Home: 510-526-2569
FAX: 510-642-1712
zadeh@cs.berkeley.edu
Fuzzyness, Artificial Intelligence,
Natural language processing, Fuzzy
logic

Richard A. Zalik
Department of Mathematics
Auburn University
Auburn University, AL 36849-5310
USA.
Tel 334-844-6557 office
       678-642-8703 home
Fax 334-844-6555
zalik@auburn.edu
Approximation Theory, Chebychev
Systems, Wavelet Theory

Ahmed I. Zayed
Department of Mathematical Sciences
DePaul University
2320 N. Kenmore Ave.
Chicago, IL 60614-3250
773-325-7808
e-mail: azayed@condor.depaul.edu
Shannon sampling theory, Harmonic
analysis and wavelets, Special
functions and orthogonal
polynomials, Integral transforms

Ding-Xuan Zhou
Department Of Mathematics
City University of Hong Kong
83 Tat Chee Avenue
Kowloon, Hong Kong
852-2788 9708,Fax:852-2788 8561
e-mail: mazhou@cityu.edu.hk
Approximation Theory, Spline
functions, Wavelets

Xin-long Zhou
Fachbereich Mathematik, Fachgebiet
Informatik
Gerhard-Mercator-Universitat
Duisburg
Lotharstr.65, D-47048 Duisburg,
Germany
e-mail:Xzhou@informatik.uni-
duisburg.de
Fourier Analysis, Computer-Aided
Geometric Design, Computational
Complexity, Multivariate
Approximation Theory, Approximation
and Interpolation Theory

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Journal of Computational Analysis and Applications

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Editor in Chief: George Anastassiou

Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152-3240, U.S.A.

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Prof.George A. Anastassiou
Department of Mathematical Sciences
The University of Memphis
Memphis,TN 38152, USA.
Tel. 901.678.3144
e-mail: ganastss@memphis.edu

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FIXED POINTS IN TOPOLOGICAL VECTOR SPACE(tvs)VALUED CONE METRIC SPACES

Muhammd Arshad(marshad_zia@yahoo.com)
Department of mathematics, International Islamic University,
H-10, Islamabad-44000, Pakistan.

Abstract: We use the notion of topological vector space valued cone metric space and generalized a common fixed point theorem of a pair of mappings satisfying a generalized contractive type condition. Our results extend some well-known recent results in the literature.

1 Introduction and Preliminaries

Many authors [1, 3, 4, 6, 17, 10, 11, 12, 13, 14, 15, 16, 17, 18, 21] studied fixed points results of mappings satisfying contractive type condition in Banach space valued cone metric spaces. The class of tvs-cone metric spaces is bigger than the class of cone metric spaces studied in [2, 7, 8, 19, 20]. Recently Azam et al. [5] obtain common fixed points of mappings satisfying a generalized contractive type condition in tvs-cone metric spaces. In this paper we continue these investigations to generalize the results in [1, 10].

Let (E, τ) be always a topological vector space (tvs) and P a subset of E. Then, P is called a cone whenever
(i) P is closed, non-empty and $P \neq \{0\}$,
(ii) $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers a, b,
(iii) $P \cap (-P) = \{0\}$.

For a given cone $P \subseteq E$, we can define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. $x < y$ will stand for $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in int P$, where $int P$ denotes the interior of P.

Definition 1 *Let X be a non-empty set. Suppose the mapping $d : X \times X \to E$ satisfies*
(d$_1$) $0 \preceq d(x, y)$ *for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$,*
(d$_2$) $d(x, y) = d(y, x)$ *for all $x, y \in X$,*
(d$_3$) $d(x, y) \preceq d(x, z) + d(z, y)$ *for all $x, y, z \in X$.*
Then d is called a topological vector space-valued cone metric on X and (X, d) is called a topological vector space-valued cone metric space.

1

If E is a real Banach space then (X, d) is called (Banach space valued) cone metric space [1, 6, 17, 10, 21]

Definition 2 *[7] Let (X, d) be a tvs-cone metric space, $x \in X$ and $\{x_n\}_{n \geq 1}$ a sequence in X. Then*
(i) $\{x_n\}_{n \geq 1}$ converges to x whenever for every $c \in E$ with $0 \ll c$ there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$. We denote this by $\lim_{n \to \infty} x_n = x$ or $x_n \to x$.
(ii) $\{x_n\}_{n \geq 1}$ is a Cauchy sequence whenever for every $c \in E$ with $0 \ll c$ there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.
(iii) (X, d) is a complete cone metric space if every Cauchy sequence is convergent.

Lemma 3 *[7] Let (X, d) be a tvs-cone metric space, P be a cone. Let $\{x_n\}$ be a sequence in X and $\{a_n\}$ be a sequence in P converging to $\mathbf{0}$. If $d(x_n, x_m) \preceq a_n$ for every $n \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence.*

The fixed point theorems and other results, in the case of cone metric spaces with non-normal solid cones, cannot be proved by reducing to metric spaces. Further, the vector valued function cone metric is not continuous in the general case.

Remark 4 *[7] Let A, B, C, D, E be non negative real numbers with $A + B + C + D + E < 1$, $B = C$ or $D = E$. If $\lambda = (A + B + D)(1 - C - D)^{-1}$ and $\mu = (A + C + E)(1 - B - E)^{-1}$, then $\lambda \mu < 1$.*

2 Common Fixed Points

The following theorem improves/generalizes the results in [1, 7].

Theorem 5 *Let (X, d) be a complete topological vector space-valued cone metric space, P be a cone and m, n be positive integers. If mappings $F, G : X \to X$ satisfies:*

$$d(Fx, Gy) \preceq A \, d(x, y) + B \, d(x, Fx) + Cd(y, Gy) + D \, d(x, Gy) + E \, d(y, Fx) \quad (2.1)$$

for all $x, y \in X$, where A, B, C, D, E are non negative real numbers with $A + B + C + D + E < 1$, $B = C$ or $D = E$. Then F and G have a unique common fixed point.

Proof. For $x_0 \in X$ and $k \geq 0$, define

$$\begin{aligned} x_{2k+1} &= Fx_{2k} \\ x_{2k+2} &= Gx_{2k+1}. \end{aligned}$$

Then,

$$
\begin{aligned}
d(x_{2k+1}, x_{2k+2}) \ &= \ d(Fx_{2k}, Gx_{2k+1}) \\
&\preceq \ Ad(x_{2k}, x_{2k+1}) + Bd(x_{2k}, Fx_{2k}) + Cd(x_{2k+1}, Gx_{2k+1}) \\
&\quad + Dd(x_{2k}, Gx_{2k+1}) + Ed(x_{2k+1}, Fx_{2k}) \\
&\preceq \ [A + B] \ d(x_{2k}, x_{2k+1}) + Cd(x_{2k+1}, x_{2k+2}) + D \ d(x_{2k}, x_{2k+2}) \\
&\preceq \ [A + B + D] \ d(x_{2k}, x_{2k+1}) + [C + D] \ d(x_{2k+1}, x_{2k+2}).
\end{aligned}
$$

It implies that

$$
[1 - C - D]d(x_{2k+1}, x_{2k+2}) \preceq [A + B + D] \ d(x_{2k}, x_{2k+1}).
$$

That is,

$$
d(x_{2k+1}, x_{2k+2}) \preceq \lambda \, d(x_{2k}, \ x_{2k+1}),
$$

where $\lambda = \dfrac{A + B + D}{1 - C - D}$. Similarly,

$$
\begin{aligned}
d(x_{2k+2}, x_{2k+3}) \ &= \ d(Fx_{2k+2}, Gx_{2k+1}) \\
&\preceq \ Ad(x_{2k+2}, x_{2k+1}) + B \ d(x_{2k+2}, Fx_{2k+2}) + Cd(x_{2k+1}, Gx_{2k+1}) \\
&\quad + Dd(x_{2k+2}, Gx_{2k+1}) + E \ d(x_{2k+1}, Fx_{2k+2}) \\
&\preceq \ A \ d(x_{2k+2}, x_{2k+1}) + B \ d(x_{2k+2}, \ x_{2k+3}) + Cd(x_{2k+1}, \ x_{2k+2}) \\
&\quad + D \ d(x_{2k+2}, \ x_{2k+2}) + E \ d(x_{2k+1}, \ x_{2k+3}) \\
&\preceq \ [A + C + E] \ d(x_{2k+1}, \ x_{2k+2}) + [B + E] \ d(x_{2k+2}, x_{2k+3}),
\end{aligned}
$$

which implies

$$
d(x_{2k+2}, x_{2k+3}) \preceq \mu \, d(x_{2k+1}, \ x_{2k+2})
$$

with $\mu = \dfrac{A + C + E}{1 - B - E}$. Now by induction, we obtain for each $k = 0, 1, 2, \ldots$

$$
\begin{aligned}
d(x_{2k+1}, x_{2k+2}) \ &\preceq \ \lambda \, d(x_{2k}, x_{2k+1}) \\
&\preceq \ (\mu) \, d(x_{2k-1}, x_{2k}) \\
&\preceq \ \lambda(\lambda\mu) \, d(x_{2k-2}, \ x_{2k-1}) \\
&\preceq \ \cdots \preceq \lambda(\lambda\mu)^k d(x_0, \ x_1)
\end{aligned}
$$

and

$$
\begin{aligned}
d(x_{2k+2}, x_{2k+3}) \ &\preceq \ \mu \, d(x_{2k+1}, \ x_{2k+2}) \\
&\preceq \ \cdots \preceq (\lambda\mu)^{k+1} \, d(x_0, \ x_1).
\end{aligned}
$$

3

For $p < q$ and by Remark 1.4, we have

$$
\begin{aligned}
d(x_{2p+1},\ x_{2q+1}) \ \preceq\ & d(x_{2p+1},\ x_{2p+2}) + d(x_{2p+2},\ x_{2p+3}) + d(x_{2p+3},\ x_{2p+4}) \\
& + \cdots + d(x_{2q},\ x_{2q+1}) \\
\preceq\ & \left[\lambda \sum_{i=p}^{q-1} (\lambda\mu)^i + \sum_{i=p+1}^{q} (\lambda\mu)^i \right] d(x_0, x_1) \\
\preceq\ & \left[\frac{\lambda(\lambda\mu)^p}{1 - \lambda\mu} + \frac{(\lambda\mu)^{p+1}}{1 - \lambda\mu} \right] d(x_0, x_1) \\
\preceq\ & (1 + \lambda) \left[\frac{(\lambda\mu)^p}{1 - \lambda\mu} \right] d(x_0, x_1).
\end{aligned}
$$

In analogous way, we deduce

$$
d(x_{2p},\ x_{2q+1}) \preceq (1 + \lambda) \left[\frac{(\lambda\mu)^p}{1 - \lambda\mu} \right] d(x_0, x_1),
$$

$$
d(x_{2p},\ x_{2q}) \preceq (1 + \lambda) \left[\frac{(\lambda\mu)^p}{1 - \lambda\mu} \right] d(x_0, x_1)
$$

and

$$
d(x_{2p+1},\ x_{2q}) \preceq (1 + \lambda) \left[\frac{(\lambda\mu)^p}{1 - \lambda\mu} \right] d(x_0, x_1).
$$

Hence, for $0 < n < m$

$$
d(x_n, x_m) \preceq a_n
$$

where $a_n = (1 + \lambda) \left[\dfrac{(\lambda\mu)^p}{1 - \lambda\mu} \right] d(x_0, x_1)$ with p the integer part of $n/2$. Fix $\mathbf{0} \ll c$ and choose a symmetric neighborhood V of $\mathbf{0}$ such that $c + V \subseteq \mathrm{int}P$. Since $a_n \to \mathbf{0}$ as $n \to \infty$, by Lemma 1.3, we deduce that $\{x_n\}$ is a Cauchy sequence. Since X is a complete, there exist $u \in X$ such that $x_n \to u$. Fix $\mathbf{0} \ll c$ and choose $n_0 \in \mathbb{N}$ be such that

$$
d(u, x_{2n}) \ll \frac{c}{3K}, \quad d(x_{2n-1}, x_{2n}) \ll \frac{c}{3K}, \quad d(u, x_{2n-1}) \ll \frac{c}{3K}
$$

for all $n \geq n_0$, where

$$
K = \max \left\{ \frac{1 + D}{1 - B - E},\ \frac{A + E}{1 - B - E},\ \frac{C}{1 - B - E} \right\}.
$$

4

Now,

$$
\begin{aligned}
d(u, Fu) \;\preceq\;& d(u, x_{2n}) + d(x_{2n}, Fu) \\
\preceq\;& d(u, x_{2n}) + d(Gx_{2n-1}, Fu) \\
\preceq\;& d(u, x_{2n}) + A\, d(u, x_{2n-1}) + B\, d(u, Fu) + Cd(x_{2n-1}, Gx_{2n-1}) \\
& + D\, d(u, Gx_{2n-1}) + E\, d(x_{2n-1}, Fu) \\
\preceq\;& d(u, x_{2n}) + A\, d(u, x_{2n-1}) + B\, d(u, Fu) + Cd(x_{2n-1}, x_{2n}) \\
& + D\, d(u, x_{2n}) + E\, d(x_{2n-1}, u) + E\, d(u, Fu)] \\
\preceq\;& (1 + D)\, d(u, x_{2n}) + (A + E)\, d(u, x_{2n-1}) + Cd(x_{2n-1}, x_{2n}) \\
& + (B + E)\, d(u, Fu).
\end{aligned}
$$

So,

$$
\begin{aligned}
d(u, Fu) \;\preceq\;& K\, d(u, x_{2n}) + K\, d(u, x_{2n-1}) + K\, d(x_{2n-1}, x_{2n}) \\
\ll\;& \frac{c}{3} + \frac{c}{3} + \frac{c}{3} = c
\end{aligned}
$$

Hence

$$
d(u, Fu) \ll \frac{c}{p}
$$

for every $p \in \mathbb{N}$. From

$$
\frac{c}{p} - d(u, Fu) \in \mathrm{int}P,
$$

being P closed, as $p \to \infty$, we deduce $-d(u, Fu) \in P$ and so $d(u, Fu) = \mathbf{0}$. This implies that $u = Fu$. Similarly, by using the inequality,

$$
d(u, Gu) \preceq d(u, x_{2n+1}) + d(x_{2n+1}, Gu),
$$

we can show that $u = Gu$, which in turn implies that u is a common fixed point of F, G and, that is

$$
u = Fu = Gu.
$$

For uniqueness, assume that there exists another point u^* in X such that

$$
u^* = Tu^* = Gu^*
$$

for some u^* in X. From

$$
\begin{aligned}
d(u, u^*) \;=\;& d(Fu, Gu^*) \\
\preceq\;& Ad(u, u^*) + Bd(u, Fu) + Cd(u^*, Gu^*) \\
& + Dd(u, Gu^*) + Ed(u^*, Fu) \\
\preceq\;& Ad(u, u^*) + Bd(u, u) + Cd(u^*, u^*) \\
& + D\, d(u, u^*) + Ed(u, u^*) \\
\preceq\;& (A + D + E)d(u, u^*),
\end{aligned}
$$

we obtain that $u^* = u$. ∎

By substituting $D = E = 0$ in the Theorem 2.1, we obtain the following result.

5

Corollary 6 *Let (X, d) be a complete topological vector space-valued cone metric space, P be a cone and m, n be positive integers. If mappings $F, G : X \to X$ satisfies:*

$$d(Fx, Gy) \preceq A \ d(x, y) + B \ d(x, Fx) + Cd(y, Gy) \qquad (2.2)$$

for all $x, y \in X$, where A, B, C are non negative real numbers with $A + B + C < 1$. Then F and G have a unique common fixed point.

By substituting $B = C = 0$ in the Theorem 2.1, we obtain the following result.

Corollary 7 *Let (X, d) be a complete topological vector space-valued cone metric space, P be a cone and m, n be positive integers. If mappings $F, G : X \to X$ satisfies:*

$$d(Fx, Gy) \preceq A \ d(x, y) + D \ d(x, Gy) + E \ d(y, Fx) \qquad (2.3)$$

for all $x, y \in X$, where A, D, E are non negative real numbers with $A + D + E < 1$. Then F and G have a unique common fixed point.

By substituting $F = T^m$, $G = T^n$ in the Theorem 2.1, we obtain the following result.

Corollary 8 *[7] Let (X, d) be a complete topological vector space-valued cone metric space, P be a cone and m, n be positive integers. If a mapping $T : X \to X$ satisfies:*

$$d(T^m x, T^n y) \preceq A \ d(x, y) + B \ d(x, T^m x) + Cd(y, T^n y) + D \ d(x, T^n y) + E \ d(y, T^m x) \qquad (2.4)$$

for all $x, y \in X$, where A, B, C, D, E are non negative real numbers with $A + B + C + D + E < 1$, $B = C$ or $D = E$. Then T has a unique fixed point.

Corollary 9 *[1] Let (X, d) be a complete Banach space-valued cone metric space, P be a cone. If a mapping $F, G : X \to X$ satisfies:*

$$d(Fx, Gy) \preceq pd(x, y) + \ q \left[d(x, Fx) + d(y, Gy) \right] + \ r \left[d(x, Gy) + E \ d(y, Fx) \right] \qquad (2.5)$$

for all $x, y \in X$, where p, q, r are non negative real numbers with $p + 2q + 2r < 1$. Then F and G have a unique common fixed point.

3 Multivalued Fixed point results in tvs-valued cone metric spaces

In the sequel, let $\mathbb{E}$ be a locally convex Hausdorff tvs with its zero vector θ, P be a proper, closed and convex pointed cone in $\mathbb{E}$ with $int \ P \neq \emptyset$ and $\preccurlyeq$ denotes the induced partial ordering with respect to P.

6

According to [5] let (X, d) be a tvs-valued cone metric space with a solid cone P and $CB(X)$ be a collection of nonempty closed and bounded subsets of X. Let $T : X \to CB(X)$ be a multi-valued mapping. For any $x \in X$, $A \in CB(X)$, define a set $W_x(A)$ as follows:

$$W_x(A) = \{d(x, a) : a \in A\}.$$

Thus, for any $x, y \in X$, we have

$$W_x(Ty) = \{d(x, u) : u \in Ty\}.$$

Definition 10 *[9] Let (X, d) be a cone metric space with the solid cone P. A multi-valued mapping $S : X \to 2^{\mathbb{E}}$ is said to be bounded from below if, for any $x \in X$, there exists $z(x) \in \mathbb{E}$ such that*

$$Sx - z(x) \subset P.$$

Definition 11 *[9] Let (X, d) be a cone metric space with the solid cone P. A cone P is said to be complete if, for any bounded from above and nonempty subset A of $\mathbb{E}$, $\sup A$ exists in $\mathbb{E}$. Equivalently, a cone P is complete if, for any bounded from below and nonempty subset A of $\mathbb{E}$, $\inf A$ exists in $\mathbb{E}$.*

Definition 12 *[5] Let (X, d) be a tvs-valued cone metric space with the solid cone P. A multi-valued mapping $T : X \to CB(X)$ is said to have the lower bound property (l.b. property) on X if, for any $x \in X$, the multi-valued mapping $S_x : X \to 2^{\mathbb{E}}$ defined by*

$$S_x(y) = W_x(Ty)$$

is bounded from below, that is, for any $x, y \in X$, there exists an element $\ell_x(Ty) \in \mathbb{E}$ such that

$$W_x(Ty) - \ell_x(Ty) \subset P.$$

$\ell_x(Ty)$ *is called the lower bound of T associated with (x, y).*

Definition 13 *[5] Let (X, d) be a tvs-valued cone metric space with the solid cone P. A multi-valued mapping $T : X \to CB(X)$ is said to have the greatest lower bound property (for short, g.l.b. property) on X if the greatest lower bound of $W_x(Ty)$ exists in $\mathbb{E}$ for all $x, y \in X$. We denote $d(x, Ty)$ by the greatest lower bound of $W_x(Ty)$, that is,*

$$d(x, Ty) = \inf\{d(x, u) : u \in Ty\}.$$

According to [20], we denote

$$s(p) = \{q \in \mathbb{E} : p \preccurlyeq q\}$$

for all $q \in \mathbb{E}$ and

$$s(a, B) = \bigcup_{b \in B} s(d(a, b)) = \bigcup_{b \in B} \{x \in \mathbb{E} : \ d(a, b) \preccurlyeq x\}$$

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for all $a \in X$ and $B \in CB(X)$. For any $A, B \in CB(X)$, we denote

$$s(A, B) = \left(\underset{a \in A}{\cap} \, s(a, B) \right) \cap \left(\underset{b \in B}{\cap} \, s(b, A) \right).$$

Remark 14 *[20] Let (X, d) be a tvs-valued cone metric space. If $\mathbb{E} = R$ and $P = [0, +\infty)$, then (X, d) is a metric space. Moreover, for any $A, B \in CB(X)$, $H(A, B) = \inf s(A, B)$ is the Hausdorff distance induced by d.*

Now we present the following theorem regarding the common fixed point of multivalued mapping with g.l.b property.

Theorem 15 *Let (X, d) be a complete tvs-valued cone metric space with the solid (normal or non-normal) cone P and let $S, T : X \longrightarrow CB(X)$ be multivalued mappings with g.l.b property such that*

$$A \, d(x, y) + B \, d(x, Sx) + Cd(y, Ty) + Dd(x, Ty) + Ed(y, Sx)) \in s(Sx, Ty) \quad (2.6)$$

ffor all $x, y \in X$, where A, B, C, D, E are non negative real numbers with $A + B + C + D + E < 1$. Then S and T have common fixed point.
Proof. Let x_0 be an arbitrary point in X and $x_1 \in Sx_0$. From (2.6), we have

$$Ad(x_0, x_1) + B(x_0, Sx_0) + Cd(x_1, Tx_1) + Dd(x_0, Tx_1) + Ed(x_1, Sx_0) \in s(Sx_0, Tx_1).$$

This implies that

$$Ad(x_0, x_1) + B(x_0, Sx_0) + Cd(x_1, Tx_1) + Dd(x_0, Tx_1) + Ed(x_1, Sx_0) \in \left(\underset{x \in Sx_0}{\cap} s(x, Tx_1) \right)$$

and

$$Ad(x_0, x_1) + B(x_0, Sx_0) + Cd(x_1, Tx_1) + Dd(x_0, Tx_1) + Ed(x_1, Sx_0) \in s(x, Tx_1) \; for \; all \; x \in Sx_0.$$

Since $x_1 \in Sx_0$, so we have

$$Ad(x_0, x_1) + B(x_0, Sx_0) + Cd(x_1, Tx_1) + Dd(x_0, Tx_1) + Ed(x_1, Sx_0) \in s(x_1, Tx_1)$$

and

$$Ad(x_0, x_1) + B(x_0, Sx_0) + Cd(x_1, Tx_1) + Dd(x_0, Tx_1) + Ed(x_1, Sx_0) \in s(x_1, Tx_1) = \underset{x \in Tx_1}{\cup} s(d(x_1, x)).$$

So there exists some $x_2 \in Tx_1$, such that

$$Ad(x_0, x_1) + B(x_0, Sx_0) + Cd(x_1, Tx_1) + Dd(x_0, Tx_1) + Ed(x_1, Sx_0) \in s(d(x_1, x_2)).$$

That is

$$d(x_1, x_2) \preceq Ad(x_0, x_1) + B(x_0, Sx_0) + Cd(x_1, Tx_1) + Dd(x_0, Tx_1) + Ed(x_1, Sx_0).$$

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By using the greatest lower bound property (g.l.b property) of S and T, we get

$$d(x_1, x_2) \preceq Ad(x_0, x_1) + B(x_0, x_1) + Cd(x_1, x_2) + Dd(x_0, x_2) + Ed(x_1, x_1),$$

which implies that

$$d(x_1, x_2) \preceq (A + B + D)d(x_0, x_1) + (C + D)d(x_1, x_2)$$

which further implies that

$$d(x_1, x_2) \preceq \frac{A + B + D}{1 - C - D} d(x_0, x_1).$$

Similarly from (2.6), we get

$$Ad(x_1, x_2) + B(x_2, Sx_2) + Cd(x_1, Tx_1) + Dd(x_2, Tx_1) + Ed(x_1, Sx_2) \in s(Tx_1, Sx_2).$$

This implies that

$$Ad(x_1, x_2) + B(x_2, Sx_2) + Cd(x_1, Tx_1) + Dd(x_2, Tx_1) + Ed(x_1, Sx_2) \in \left(\bigcap_{x \in Tx_1} s(x, Sx_2) \right)$$

and

$$Ad(x_1, x_2) + B(x_2, Sx_2) + Cd(x_1, Tx_1) + Dd(x_2, Tx_1) + Ed(x_1, Sx_2) \in s(x, Sx_2) \text{ for all } x \in Tx_1.$$

Since $x_2 \in Tx_1$, so we have

$$Ad(x_1, x_2) + B(x_2, Sx_2) + Cd(x_1, Tx_1) + Dd(x_2, Tx_1) + Ed(x_1, Sx_2) \in s(x_2, Sx_2)$$

and

$$Ad(x_1, x_2) + B(x_2, Sx_2) + Cd(x_1, Tx_1) + Dd(x_2, Tx_1) + Ed(x_1, Sx_2) \in s(x_2, Sx_2) = \bigcup_{x \in Sx_2} s(d(x_2, x)).$$

So there exists some $x_3 \in Sx_2$, such that

$$Ad(x_1, x_2) + B(x_2, Sx_2) + Cd(x_1, Tx_1) + Dd(x_2, Tx_1) + Ed(x_1, Sx_2) \in s(d(x_2, x_3)).$$

That is

$$d(x_2, x_3) \preceq Ad(x_1, x_2) + B(x_2, Sx_2) + Cd(x_1, Tx_1) + Dd(x_2, Tx_1) + Ed(x_1, Sx_2).$$

By using the greatest lower bound property (g.l.b property) of S and T, we get

$$d(x_2, x_3) \preceq Ad(x_1, x_2) + B(x_2, x_3) + Cd(x_1, x_2) + Dd(x_2, x_2) + Ed(x_1, x_3).$$

which implies that

$$d(x_2, x_3) \preceq (A + C + E)d(x_1, x_2) + (B + E)(x_2, x_3).$$

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This further implies

$$d(x_2, x_3) \preceq \frac{A + C + E}{1 - B - E} d(x_1, x_2).$$

Let $\delta = \max\{\frac{A+B+D}{1-C-D}, \frac{A+C+E}{1-B-E}\}$. Then $\delta < 1$. Thus inductively, one can easily construct a sequence $\{x_n\}$ in X such that

$$x_{2n+1} \in Sx_{2n}, \quad x_{2n+2} \in Tx_{2n+1}$$

and

$$d(x_{2n}, x_{2n+1}) \preccurlyeq \delta d(x_{2n-1}, x_{2n}).$$

for each $n \geq 0$. We assume that $x_n \neq x_{n+1}$ for each $n \geq 0$. Otherwise, there exists n such that $x_{2n} = x_{2n+1}$. Then $x_{2n} \in Sx_{2n}$ and x_{2n} is a fixed point of S and hence a fixed point of T. Similarly, if $x_{2n+1} = x_{2n+2}$ for some n, then x_{2n+1} is a common fixed point of T and S. Similarly, one can show that

$$d(x_{2n+1}, x_{2n+2}) \preccurlyeq \delta d(x_{2n}, x_{2n+1}).$$

Thus we have

$$d(x_n, x_{n+1}) \preccurlyeq \delta d(x_{n-1}, x_n) \preccurlyeq \delta^2 d(x_{n-2}, x_{n-1}) \preccurlyeq \cdots \preccurlyeq \delta^n d(x_0, x_1)$$

for each $n \geq 0$. Now, for any $m > n$, consider

$$
\begin{aligned}
d(x_m, \ x_n) \ &\preccurlyeq \ d(x_n, \ x_{n+1}) + d(x_{n+1}, \ x_{n+2}) + \cdots + d(x_{m-1}, \ x_m) \\
&\preccurlyeq \ \left[\delta^n + \delta^{n+1} + \cdots + \delta^{m-1}\right] d(x_0, x_1) \\
&\preccurlyeq \ \left[\frac{\delta^n}{1-\delta}\right] d(x_0, x_1).
\end{aligned}
$$

Let $\theta \ll c$ be given and choose a symmetric neighborhood V of θ such that $c + V \subseteq \mathrm{int} P$. Also, choose a natural number k_1 such that $\left[\frac{\delta^n}{1-\delta}\right] d(x_0, x_1) \in V$ for all $n \geq k_1$. Then $\frac{\delta^n}{1-\delta} d(x_1, x_0) \ll c$ for all $n \geq k_1$. Thus we have

$$d(x_m, \ x_n) \preccurlyeq \left[\frac{\delta^n}{1-\delta}\right] d(x_0, x_1) \ll c$$

for all $m > n$. Therefore, $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $\nu \in X$ such that $x_n \to \nu$. Choose a natural number k_2 such that

$$\frac{1+E}{1-C} d(\nu, x_{2n+1}) \ll \frac{c}{3}, \quad \frac{A}{1-C} d(x_{2n}, \nu) \ll \frac{c}{3} \text{ and } \frac{B}{1-C} d(x_{2n}, x_{2n}) \ll \frac{c}{3} \tag{2.7}$$

for all $n \geq k_2$. Then, for all $n \geq k_2$, we have

$$Ad(x_{2n}, \nu) + Bd(x_{2n}, Sx_{2n}) + Cd(\nu, T\nu) + Dd(x_{2n}, T\nu) + Ed(\nu, Sx_{2n}) \in s(Sx_{2n}, T\nu).$$

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This implies that

$$Ad(x_{2n},v)+Bd(x_{2n},Sx_{2n})+Cd(v,Tv)+Dd(x_{2n},T\nu)+Ed(\nu,Sx_{2n}) \in \left(\underset{x \in Sx_{2n}}{\cap} s\,(x,Tv) \right)$$

and we have

$$Ad(x_{2n},v)+Bd(x_{2n},Sx_{2n})+Cd(v,Tv)+Dd(x_{2n},T\nu)+Ed(\nu,Sx_{2n}) \in s\,(x,Tv) \ \ for\ all\ x \in Sx_{2n}.$$

Since $x_{2n+1} \in Sx_{2n}$, so we have

$$Ad(x_{2n},v)+Bd(x_{2n},Sx_{2n})+Cd(v,Tv)+Dd(x_{2n},T\nu)+Ed(\nu,Sx_{2n}) \in s\,\left(x_{2n+1},Tv\right).$$

By definition, we obtain

$$Ad(x_{2n},v)+Bd(x_{2n},Sx_{2n})+Cd(v,Tv)+Dd(x_{2n},T\nu)+Ed(\nu,Sx_{2n}) \in s\,\left(x_{2n+1},Tv\right) = \underset{u' \in Tu}{\cup} s\,\left(d\left(x_{2n+1},u'\right)\right).$$

There exists some $\nu_n \in Tv$ such that

$$Ad(x_{2n},v)+Bd(x_{2n},Sx_{2n})+Cd(v,Tv)+Dd(x_{2n},T\nu)+Ed(\nu,Sx_{2n}) \in s\,\left(x_{2n+1},Tv\right) \in s\,\left(d(x_{2n+1},\nu_n)\right),$$

that is

$$d(x_{2n+1},\nu_n) \preceq Ad(x_{2n},v)+Bd(x_{2n},Sx_{2n})+Cd(v,Tv)+Dd(x_{2n},T\nu)+Ed(\nu,Sx_{2n}).$$

By using the greatest lower bound property (g.l.b property) of S and T, we have

$$d(x_{2n+1},v_n) \preceq Ad(x_{2n},v)+Bd(x_{2n},x_{2n})+Cd(v,\nu_n)+Dd(x_{2n},\nu_n)+Ed(\nu,x_{2n+1}).$$

Now by using the triangular inequality, we get

$$d\left(x_{2n+1},\nu_n\right) \preceq Ad(x_{2n},v)+Bd(x_{2n},x_{2n+1})+Cd(v,x_{2n+1})+Dd(x_{2n},\nu_n)+Ed(\nu,x_{2n+1})$$

and it follows that

$$d\left(x_{2n+1},\nu_n\right) \preceq \frac{A}{1-C}d(x_{2n},v) + \frac{B}{1-C}d(x_{2n},x_{2n})) + \frac{C+E}{1-C}d(v,x_{2n+1}).$$

By using again triangular inequality, we get

$$
\begin{aligned}
d(\nu,\nu_n) \quad &\preceq \quad d(\nu,x_{2n+1}) + d(x_{2n+1},\nu_n) \\
&\preceq \quad d(\nu,x_{2n+1}) + \frac{A}{1-C}d(x_{2n},v) + \frac{B}{1-C}d(x_{2n},x_{2n})) + \frac{C+E}{1-C}d(\nu,x_{2n+1}) \\
&\preceq \quad \frac{1+E}{1-C}d(\nu,x_{2n+1}) + \frac{A}{1-C}d(x_{2n},v) + \frac{B}{1-C}d(x_{2n},x_{2n}) \\
&\ll \quad \frac{c}{3} + \frac{c}{3} + \frac{c}{3} = c
\end{aligned}
$$

Thus, we get

$$d(v,v_n) \ll \frac{c}{m}$$

for all $m \geq 1$ and so $\frac{c}{m} - d(v,v_n) \in P$ for all $m \geq 1$. Since $\frac{c}{m} \to \theta$ as $m \to \infty$ and P is closed, it follows that $-d(v,v_n) \in P$. But $d(v,v_n) \in P$. Therefore, $d(v,v_n) = \theta$ and $v_n \to v \in Tv$, since Tv is closed. This implies that v is a common point of S and T. This completes the proof. $\blacksquare$

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Corollary 16 *[5] Let (X, d) be a complete tvs-valued cone metric space with the solid (normal or non-normal) cone P and let $S, T : X \longrightarrow CB(X)$ be multivalued mappings with g.l.b property such that*

$$B\ d(x, Sx) + Cd(y, Ty) \in s\,(Sx, Ty)$$

ffor all $x, y \in X$, where B, C are non negative real numbers with $B + C < 1$. Then S and T have common fixed point.

Theorem 17 *[5] Let (X, d) be a complete tvs-valued cone metric space with the solid (normal or non-normal) cone P and let $S, T : X \longrightarrow CB(X)$ be multivalued mappings with g.l.b property such that*

$$Dd(x, Ty) + Ed(y, Sx)) \in s\,(Sx, Ty)$$

ffor all $x, y \in X$, where D, E are non negative real numbers with $D + E < 1$. Then S and T have common fixed point.

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13

ON THE TWISTED q-CHANGHEE POLYNOMIALS OF HIGHER ORDER

JIN-WOO PARK

ABSTRACT. The q-Changhee polynomials and numbers are introduced by T. Kim et al in [3]. Some interesting properties of those polynomials are derived from umbral calculus (see [4]). In this paper, we consider Witt-type formula for the n-th twisted q-Changhee numbers and polynomials of higher order and derive some new interesting identities and properties of those polynomials and numbers from the Witt-type formula which are related to special polynomials and numbers.

1. INTRODUCTION

Let p be an odd prime number. $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic numbers and the completion of algebraic closure of $\mathbb{Q}_p$. The p-adic norm $|\cdot|_p$ is normalized by $|p|_p = \frac{1}{p}$. Let $C(\mathbb{Z}_p)$ be the space of continuous functions on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, the *fermionic p-adic integral on $\mathbb{Z}_p$* is defined by T.Kim to be

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N \to \infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N - 1} f(x)(-q)^x, \text{ (see [6, 7, 9]).} \quad (1.1)$$

Let $f_1(x) = f(x+1)$. Then, by (1.1), we get

$$qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0), \text{ (see [6, 7]).} \quad (1.2)$$

By (1.2), we easily see that

$$q^n I_{-q} + (-1)^{n-1} I_{-q} = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} f(l), \quad (1.3)$$

where $f_n(x) = f(x+n)$ and $n \geq 0$.

It is well known that the *twisted q-Euler polynomials* are defined by the generating function to be

$$\frac{[2]_q}{1 + q\varepsilon e^t} e^{xt} = \sum_{n=0}^{\infty} E_{n,\varepsilon,q}(x) \frac{t^n}{n!}, \text{ (see [13]).} \quad (1.4)$$

When $x = 0$, $E_{n,\varepsilon,q} = E_{n,\varepsilon,q}(0)$ are called the $n-th$ *twisted q-Euler numbers*. For $\varepsilon = 1$, $E_{n,1,q}(x) = E_{n,q}(x)$ are the n-th q-Euler polynomials, and $x = 0$, $E_{n,1,q}(0) = E_{n,q}(0)$ are the n-th q-Euler numbers.

Indeed, we note that $E_{n,1,q}(x) = H_n(x| - q)$, where $H_n(x|\lambda)$ are the Frobenius-Euler polynomials which are defined by the generating function to be

$$\frac{1-\lambda}{e^t - \lambda}e^{tx} = \sum_{n=0}^{\infty} H_n(x|\lambda)\frac{t^n}{n!}, \text{ (see [1])}.$$

Recently, the q-Changhee polynomials are defined by the generating function to be

$$\frac{[2]_q}{1+q^{\varepsilon t}}(1+t)^x = \sum_{n=0}^{\infty} Ch_{n,q}(x)\frac{t^n}{n!}, \text{ (see [10])}. \tag{1.5}$$

When $x = 0$, $Ch_{n,\varepsilon,q} = Ch_{n,\varepsilon,q}(0)$ are called the q-*Changhee numbers*, (see [3]). The Stirling number of the first kind is defined by

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l)x^l, \text{ (see [3])}. \tag{1.6}$$

The q-Changhee numbers and polynomials are introduced by T. Kim et. al. in [3], and found interesting identities in [5, 8, 11, 12]. In this paper, we consider the twisted q-Changhee numbers and polynomials of order k which are derived from the multivariate fermionic p-adic q-integral of higher order on $\mathbb{Z}_p$, and give some relationship between twisted q-Changhee polynomials and numbers of higher-order and special polynomials and numbers.

2. Twisted q-Changhee numbers and polynomials of higher-order

For $n \in \mathbb{N}$, let T_p be the p-adic locally constant space defined by

$$T_p = \bigcup_{n \geq 1} C_{p^n} = \lim_{n \to \infty} C_{p^n},$$

where $C_{p^n} = \left\{ \omega | \omega^{p^n} = 1 \right\}$ is the cyclic group of order p^n.

For $\varepsilon \in T_p$, let us take $f(x) = (1 + \varepsilon t)^x$ for $|t|_p < p^{-\frac{1}{p-1}}$. Then by (1.2), we get

$$\int_{\mathbb{Z}_p} (1+\varepsilon t)^x d\mu_{-q}(x) = \frac{[2]_q}{q\varepsilon t + [2]_q} = \sum_{n=0}^{\infty} Ch_{n,\varepsilon,q}\frac{t^n}{n!} \tag{2.1}$$

where $Ch_{n,\varepsilon,q}$ are called the n-*th twisted q-Changhee numbers*.

From (2.1), we can derive the following equation:

$$\int_{\mathbb{Z}_p} (1+\varepsilon t)^{x+y} d\mu_{-q}(y) = \frac{[2]_q}{q\varepsilon t + [2]_q}(1+\varepsilon t)^x = \sum_{n=0}^{\infty} Ch_{n,\varepsilon,q}(x)\frac{t^n}{n!}, \tag{2.2}$$

where $Ch_{n,\varepsilon,q}(x)$ are called the n-*th twisted q-Changhee polynomials*. Note that $Ch_{n,\varepsilon,q}(0) = Ch_{n,\varepsilon,q}$ are n-th twisted q-Changhee numbers.

Since

$$\begin{aligned}
\int_{\mathbb{Z}_p} (1+\varepsilon t)^{x+y} d\mu_{-q}(y) &= \sum_{n=0}^{\infty} \varepsilon^n \int_{\mathbb{Z}_p} \binom{x+y}{n} d\mu_{-q}(y)t^n \\
&= \sum_{n=0}^{\infty} \varepsilon^n \int_{\mathbb{Z}_p} (x+y)_n d\mu_{-q}(y)\frac{t^n}{n!},
\end{aligned} \tag{2.3}$$

by (2.2) and (2.3), we obtained the following theorem.

Theorem 2.1. *For $n \geq 0$, we have*

$$Ch_{n,\varepsilon,q}(x) = \varepsilon^n \int_{\mathbb{Z}_p} (x+y)_n d\mu_{-q}(y).$$

From (2.1), we note that

$$\sum_{n=0}^{\infty} \varepsilon^n \int_{\mathbb{Z}_p} \binom{x}{n} d\mu_{-q}(x) t^n = \frac{[2]_q}{q\varepsilon t + [2]_q} = \sum_{n=0}^{\infty} \left(-\frac{q\varepsilon}{[2]_q} \right)^n t^n. \qquad (2.4)$$

Thus, by comparing the coefficients on the both sides, we obtain the following theorem.

Theorem 2.2. *For $n \geq 0$, we have*

$$\int_{\mathbb{Z}_p} \binom{x}{n} d\mu_{-q}(x) = \left(-\frac{q}{[2]_q} \right)^n.$$

Replacing t by $\frac{e^t - 1}{\varepsilon}$ in (2.2), we get

$$\sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!} = \frac{[2]_q}{qe^t - 1} e^{xt} = \sum_{n=0}^{\infty} Ch_{n,\varepsilon,q}(x) \frac{1}{n!} \left(\frac{e^t - 1}{\varepsilon} \right)^n, \qquad (2.5)$$

where $E_{n,q}$ is the n-th q-Euler polynomials and

$$\begin{aligned}
\sum_{n=0}^{\infty} Ch_{n,\varepsilon,q}(x) \frac{1}{n!} \left(\frac{e^t - 1}{\varepsilon} \right)^n &= \sum_{n=0}^{\infty} Ch_{n,\varepsilon,q}(x) \frac{1}{n!} \varepsilon^{-n} n! \left(\sum_{m=n}^{\infty} S_2(m,n) \frac{t^m}{m!} \right) \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{m} Ch_{n,\varepsilon,q}(x) S_2(m,n) \varepsilon^{-n} \frac{t^m}{m!},
\end{aligned} \qquad (2.6)$$

where $S_2(m,n)$ is the Striling number of the second kind.

By comparing the coefficients on the both sides of (2.5) and (2.6), we obtain the following theorem.

Theorem 2.3. *For $n \geq 0$, we have*

$$E_{n,q}(x) = \sum_{m=0}^{n} Ch_{m,\varepsilon,q}(x) S_2(n,m) \varepsilon^{-m}.$$

By Theorem 2.1, we easily get

$$\begin{aligned}
Ch_{n,\varepsilon,q}(x) &= \varepsilon^n \int_{\mathbb{Z}_p} (x+y)_n d\mu_{-q}(y) \\
&= \varepsilon^n \sum_{l=0}^{n} S_1(n,l) \int_{\mathbb{Z}_p} (x+y)^l d\mu_{-q}(y) = \varepsilon^n \sum_{l=0}^{n} S_1(n,l) E_{l,q}(x).
\end{aligned} \qquad (2.7)$$

Therefore, by (2.7), we obtain the following theorem.

Theorem 2.4. *For $n \geq 0$, we have*

$$Ch_{n,\varepsilon,q}(x) = \varepsilon^n \sum_{l=0}^{n} S_1(n,l) E_{l,q}(x).$$

where $S_1(n,l)$ is the Stirling number of the first kind.

In viewpoint of (2.3), the $n-th$ *twisted q-Changhee numbers of the first kind with order k* are defined by the generating function to be

$$Ch_{n,\varepsilon}^{(k)} = \varepsilon^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_k)_n d\mu_{-q}(x_1) \ldots d\mu_{-q}(x_k), \qquad (2.8)$$

where n is a positive integer.

By (2.8), we easily get

$$\sum_{n=0}^{\infty} Ch_{n,\varepsilon,q}^{(k)} \frac{t^n}{n!} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} \binom{x_1 + \cdots + x_k}{n} (\varepsilon t)^n d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k)$$

$$= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1 + \varepsilon t)^n d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k). \qquad (2.9)$$

From (2.1) and (2.9), we have

$$\sum_{n=0}^{\infty} Ch_{n,\varepsilon,q}^{(k)} \frac{t^n}{n!} = \left(\frac{[2]_q}{q\varepsilon t + [2]_q} \right)^k, \qquad (2.10)$$

and

$$\left(\frac{[2]_q}{q\varepsilon t + [2]_q} \right)^k = \sum_{n=0}^{\infty} \left(\sum_{l_1 + \cdots + l_k = n} \binom{n}{l_1, \ldots, l_k} Ch_{l_1,\varepsilon,q} \cdots Ch_{l_k,\varepsilon,q} \right) \frac{t^n}{n!}. \qquad (2.11)$$

By simple calculation. we easily see that

$$\left(\frac{[2]_q}{q\varepsilon t + [2]_q} \right)^k = \sum_{n=0}^{\infty} \left(-\frac{q}{[2]_q} \right)^n n! \varepsilon^n \binom{k + n - 1}{n} \frac{t^n}{n!}. \qquad (2.12)$$

Thus, by (2.10) and (2.12), we get

$$[2]_q^n Ch_{n,\varepsilon,q}^{(k)} = (-q)^n n! \varepsilon^n \binom{n + k - 1}{n}$$

$$= (-q)^n \varepsilon^n (k + n - 1)_n \qquad (2.13)$$

$$= (-q)^n \varepsilon^n \sum_{l=0}^{n} S_1(n, l)(k + n - 1)^l.$$

Therefore, by (2.10), (2.11) and (2.13), we obtain the following theorem.

Theorem 2.5. *For $n \geq 0$, we have*

$$[2]_q^n Ck_{n,\varepsilon,q}^{(k)} = [2]_q^n \sum_{l_1 + \cdots + l_k = n} \binom{n}{l_1, \ldots, l_k} Ch_{i_1,\varepsilon,q} \cdots Ch_{l_k,\varepsilon,q}$$

$$= (-q)^n \varepsilon^n \sum_{l=0}^{n} S_1(n, l)(k + n - 1)^l.$$

From (2.8), we have

$$Ch_{n,\varepsilon,q}^{(k)} = \varepsilon^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_k)_n d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k)$$

$$= \varepsilon^n \sum_{l=0}^{n} S_1(n, l) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_k)^l d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k). \qquad (2.14)$$

Now, we observe that

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(x_1+\cdots+x_k)t} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k) = \left(\frac{[2]_q}{qe^t+1}\right)^k = \sum_{n=0}^{\infty} E_{n,q}^{(k)} \frac{t^n}{n!}, \quad (2.15)$$

where $E_{n,q}^{(k)}$ are the q-Euler numbers of order k.

From (2.14) and (2.15), we obtain the following theorem.

Theorem 2.6. *For $n \geq 0$, we have*

$$Ch_{n,\varepsilon,q}^{(k)} = \varepsilon^n \sum_{l=0}^{n} S_1(n,l) E_{l,q}^{(k)}.$$

Replacing t by $\frac{e^t-1}{\varepsilon}$, we get

$$\sum_{n=0}^{\infty} Ch_{n,\varepsilon,q}^{(k)} \frac{1}{n!} \left(\frac{e^t-1}{\varepsilon}\right)^n = \left(\frac{[2]_q}{qe^t+1}\right)^k = \sum_{n=0}^{\infty} E_{n,q}^{(k)} \frac{t^n}{n!}, \quad (2.16)$$

and

$$\sum_{n=0}^{\infty} Ch_{n,\varepsilon,q}^{(k)} \frac{1}{n!} \left(\frac{e^t-1}{\varepsilon}\right)^n = \sum_{m=0}^{\infty} \left(\sum_{n=0}^{m} \varepsilon^{-n} Ch_{n,\varepsilon,q}^{(k)} S_2(m,n)\right) \frac{t^m}{m!}. \quad (2.17)$$

Thus, by (2.16) and (2.17), we obtain the following theorem.

Theorem 2.7. *For $n \geq 0$, we have*

$$E_{n,q}^{(k)} = \sum_{m=0}^{n} \varepsilon^{-m} Ch_{m,\varepsilon,q}^{(k)} S_2(n,m).$$

Now we define the *twisted q-Changhee polynomials of the first kind with order k* as follows:

$$Ch_{n,\varepsilon,q}^{(k)}(x) = \varepsilon^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1+\cdots+x_k+x)_n d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k), \quad (2.18)$$

where $n \geq 0$ and $k \in \mathbb{N}$.

From (2.18), we can derive the generating function of the twisted q-Changhee polynomials as follows:

$$\sum_{n=0}^{\infty} Ch_{n,\varepsilon,q}^{(k)}(x) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+\varepsilon t)^{x_1+\cdots+x_k+x} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k)$$

$$= \left(\frac{[2]_q}{q\varepsilon t + [2]_q}\right)^k (1+\varepsilon t)^x. \quad (2.19)$$

It is easy to show that

$$\left(\frac{[2]_q}{q\varepsilon t + [2]_q}\right)^k (1+\varepsilon t)^x = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \varepsilon^m \binom{n}{m} (x)_m Ch_{n-m,\varepsilon,q}^{(k)}\right) \frac{t^n}{n!}. \quad (2.20)$$

By (2.20), we get

$$Ch_{n,\varepsilon,q}^{(k)}(x) = \sum_{m=0}^{n} \varepsilon^m \binom{x}{m} \frac{n!}{(n-m)!} Ch_{n-m,\varepsilon,q}^{(k)}$$

$$= \sum_{m=0}^{n} \varepsilon^{n-m} \binom{x}{n-m} \frac{n!}{m!} Ch_{m,\varepsilon,q}^{(k)}. \quad (2.21)$$

From (2.18), we have

$$
\begin{aligned}
Ch_{n,\varepsilon,q}^{(k)}(x) &= \varepsilon^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_k + x)_n d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k) \\
&= \varepsilon^n \sum_{l=0}^{n} S_1(n,l) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_k + x)^l d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k) \\
&= \varepsilon^n \sum_{l=0}^{n} S_1(n,l) E_{l,q}^{(k)}(x).
\end{aligned}
$$

$$
\text{(2.22)}
$$

Hence, by (2.22), we obtain the following theorem.

Theorem 2.8. *For $n \geq 0$, we have*

$$
Ch_{n,\varepsilon,q}^{(k)}(x) = \sum_{m=0}^{n} \varepsilon^m \binom{x}{n-m} \frac{n!}{m!} Ch_{m,\varepsilon,q}^{(k)} = \varepsilon^n \sum_{l=0}^{n} S_1(n,l) E_{l,q}^{(k)}(x).
$$

where $E_{l,q}^{(k)}$ are the q-Euler polynomials of order k.

Now, we consider the *twisted q-Changhee polynomials of second kind with order k* as follows:

$$
\widehat{Ch}_{n,\varepsilon,q}^{(k)}(x) = \varepsilon^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (-x_1 - \cdots - x_k + x)_n d\mu_{-q}(x_1) \cdots d\mu_{-q}(x)_k. \quad \text{(2.23)}
$$

By (2.23), we have

$$
\begin{aligned}
\sum_{n=0}^{\infty} \widehat{Ch}_{n,\varepsilon,q}^{(k)}(x) \frac{t^n}{n!} &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+\varepsilon t)^{-x_1-\cdots-x_k+x} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k) \\
&= \left(\frac{[2]_q}{\varepsilon t + [2]_q} \right)^k (1+\varepsilon t)^{k+x},
\end{aligned}
$$

$$
\text{(2.24)}
$$

where k is positive integer.

Hence,

$$
\begin{aligned}
&\widehat{Ch}_{n,\varepsilon,q}^{(k)}(x) \\
&= \varepsilon^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (-x_1 - \cdots - x_k + x)_n d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k) \\
&= \varepsilon^n \sum_{l=0}^{n} S_1(n,l)(-1)^l \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_k - x)^l d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k) \\
&= \varepsilon^n \sum_{l=0}^{n} S_1(n,l)(-1)^l E_{l,q}^{(k)}(-x).
\end{aligned}
$$

$$
\text{(2.25)}
$$

Therefor, by (2.25), we obtain the following theorem.

Theorem 2.9. *For $n \geq 0$, we have*

$$
\widehat{Ch}_{n,\varepsilon,q}^{(k)}(x) = \varepsilon^n \sum_{l=0}^{n} S_1(n,l)(-1)^l E_{l,q}^{(k)}(-x).
$$

Now, we consider the n-th twisted q-Changhee polynomials of the first kind relate to n-th twisted q-Changhee polynomials of second kind.

$$\frac{(-1)^n \widehat{Ch}_{n,\varepsilon,q}^{(k)}(x)}{n!}$$

$$=(-1)^n \varepsilon^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \binom{-x_1 - \cdots - x_k + x}{n} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k)$$

$$=\varepsilon^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \binom{x_1 + \cdots + x_k - x + n - 1}{n} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k)$$

$$=\varepsilon^n \sum_{m=0}^{\infty} \binom{n-1}{n-m} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \binom{x_1 + \cdots + x_k - x}{m} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k)$$

$$=\varepsilon^n \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{\varepsilon^{-m}}{m!} m! \varepsilon^m \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \binom{x_1 + \cdots + x_k - x}{m} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k)$$

$$=\sum_{m=1}^{n} \binom{n-1}{m-1} \varepsilon^{n-m} \frac{Ch_{m,\varepsilon,q}^{(k)}(-x)}{m!}.$$

$$(2.26)$$

By (2.26) and proceeding similar to (2.26), we have the following theorem.

Theorem 2.10. *For $n \geq 0$, we have*

$$\frac{(-1)^n \widehat{Ch}_{n,\varepsilon,q}^{(k)}(x)}{n!} = \sum_{m=1}^{n} \binom{n-1}{m-1} \varepsilon^{n-m} \frac{Ch_{m,\varepsilon,q}^{(k)}(-x)}{m!},$$

and

$$\frac{(-1)^n Ch_{n,\varepsilon,q}^{(k)}(x)}{n!} = \sum_{m=1}^{n} \binom{n-1}{m-1} \varepsilon^{n-m} \frac{\widehat{Ch}_{m,\varepsilon,q}^{(k)}(-x)}{m!},$$

By (2.25),

$$\widehat{Ch}_{n,\varepsilon,q}(x)$$

$$=\varepsilon^n \sum_{l=0}^{n} S_1(n,l)(-1)^l \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_k - x)^l d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k)$$

$$=\varepsilon^n \sum_{l=0}^{n} S_1(n,l) \sum_{m=0}^{l} (-1)^{l+m} \binom{l}{m} E_{l-m}^{(k)} x^m,$$

and thus we obtain the following theorem.

Theorem 2.11. *For $n \geq 0$, we have*

$$\widehat{Ch}_{n,\varepsilon,q}(x) = \varepsilon^n \sum_{l=0}^{n} \sum_{m=0}^{l} (-1)^{l+m} \binom{l}{m} S_1(n,l) E_{l-m}^{(k)} x^m.$$

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Department of Mathematics Education, Daegu University, Jillyang, Gyeongsan, Gyeongbuk 712-714, Republic of Korea.

   E-mail address: `a0417001@knu.ac.kr`

SOME SYMMETRY IDENTITIES FOR THE (h,q)-BERNOULLI POLYNOMIALS UNDER THE THIRD DIHEDRAL GROUP D_3 ARISING FROM q-VOLKENBORN INTEGRAL ON $\mathbb{Z}_p$

S.-H. RIM, T. G. KIM, S. H. LEE

ABSTRACT. In this paper, we give some new identities of symmetry for the (h,q)-Bernoulli polynomials arising from q-Volkenborn integral on $\mathbb{Z}_p$.

1. INTRODUCTION

let p be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = 1/p$ and let q be an indeterminate in $\mathbb{C}_p$ with $|1-q|_p < p^{-\frac{1}{p-1}}$. The q-extension of x is defined by $[x]_q = \frac{1-q^x}{1-q}$. Note that $\lim_{q\to 1}[x]_q = x$. Suppose that f is a uniformly differentiable function on $\mathbb{Z}_p$. Then the p-adic q-Vollenborn integral is defined by Kim to be

$$
I_q(f) = \int_{\mathbb{Z}_p} f(x)d\mu_q(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)\mu_q(x + p^N\mathbb{Z}_p)
$$
(1)
$$
= \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x)q^x.
$$

As is well known, Carlitz's q-Bernoulli numbers are defined by

$$
\beta_{0,q} = 1, \quad q(q\beta+1)^n - \beta_{n,q} = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1, \end{cases}
$$

with the usual convention about replacing β_q^n by $\beta_{n,q}$ (see [1,8,10]).

The q-Bernoulli polynomials are given by

$$
\beta_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} [x]_q^{n-l} q^{lx} \beta_{l,q}
$$

$$
= \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{lx} \frac{l+1}{[l+1]_q}, \quad \text{(see [10])}.
$$

In 1999, Kim gave the formula which is given by

$$
\beta_{n,q}(x) = \int_{\mathbb{Z}_p} [x+y]_q^n d\mu_q(x), \quad (n \in \mathbb{N} \cup \{0\},) \quad \text{(see [1-15])}.
$$

1

For $h \in \mathbb{Z}$, we consider (h, q)-Bernoulli polynomials as follows:

$$
\beta_{n,q}^{(h)}(x) = \int_{\mathbb{Z}_p} q^{(h-1)x}[x+y]_q^n d\mu_q(x), \quad (n \in \mathbb{Z}_{\geq 0})
$$

(2)

$$
= \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} q^{lx}(-1)^l \frac{h+1}{[h+l]_q}, \quad (\text{see } [8,10]).
$$

When $x = 0$, $\beta_{n,q}^{(h)} = \beta_{n,q}^{(h)}(0)$ are called the (h, q)-Bernoulli numbers.

In this paper, we consider the symmetric identities for the (h, q)-Bernoulli polynomials under the third Dihedral group D_3 which are derive from p-adic q-Volkenborn integral on $\mathbb{Z}_p$.

2. Symmetric identities for the (h, q)-Bernoulli polynomials

Let w_1, w_2, w_3 be positive integers. Then we observe that

(3)

$$
\int_{\mathbb{Z}_p} q^{(h-1)w_2w_3y} e^{[w_2w_3y+w_1w_2w_3x+w_1w_3i+w_1w_2j]_q t} d\mu_{q^{w_2w_3}}(y)
$$

$$
= \lim_{N \to \infty} \frac{1}{[p^N]_{q^{w_2w_3}}} \sum_{y=0}^{p^N-1} q^{(h-1)w_2w_3y} e^{[w_2w_3y+w_1w_2w_3x+w_1w_3i+w_1w_2j]_q t} q^{w_2w_3y}
$$

$$
= \lim_{N \to \infty} \frac{1}{[w_1p^N]_{q^{w_2w_3}}} \sum_{k=0}^{w_1-1} \sum_{y=0}^{p^N-1} q^{hw_2w_3(k+w_1y)} e^{[w_2w_3(k+w_1y)+w_1w_2w_3x+w_1w_3i+w_1w_2j]_q t}.
$$

By (3), we get

$$
\frac{1}{[w_2w_3]_q} \sum_{i=0}^{w_2-1} \sum_{j=0}^{w_3-1} q^{(w_1w_3i+w_1w_2j)h}
$$

$$
\times \int_{\mathbb{Z}_p} q^{(h-1)w_2w_3y} e^{[w_2w_3y+w_1w_2w_3x+w_1w_3i+w_1w_2j]_q t} d\mu_{q^{w_2w_3}}(y)
$$

(4)

$$
= \lim_{N \to \infty} \frac{1}{[w_1w_2w_3p^N]_q} \sum_{i=0}^{w_2-1} \sum_{j=0}^{w_3-1} \sum_{k=0}^{w_1-1} q^{h(w_1w_3i+w_1w_2j+w_2w_3k)+hw_1w_2w_3y}
$$

$$
\times e^{[w_2w_3(k+w_1y)+w_1w_2w_3x+w_1w_3i+w_1w_2j]_q t}.
$$

From (4), we note that the expression is invariant under any permutation of w_1, w_2, w_3 in third Dihedral group D_3. Therefore, by (4), we obtain the following theorem.

Theorem 2.1. *Let w_1, w_2, w_3 be positive integers. Then, the following expressions*

$$
\frac{1}{[w_{\sigma(2)}w_{\sigma(3)}]_q} \sum_{i=0}^{w_{\sigma(2)}-1} \sum_{j=0}^{w_{\sigma(3)}-1} q^{h(w_{\sigma(1)}w_{\sigma(3)}i+w_{\sigma(1)}w_{\sigma(2)}j)}
$$

$$
\times \int_{\mathbb{Z}_p} q^{(h-1)w_{\sigma(2)}w_{\sigma(3)}y} e^{[w_{\sigma(2)}w_{\sigma(3)}y+w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}x+w_{\sigma(1)}w_{\sigma(3)}i+w_{\sigma(1)}w_{\sigma(2)}j]_q t} d\mu_{q^{w_{\sigma(2)}w_{\sigma(3)}}}(y)
$$

are the same for any $\sigma \in D_3$.

Now, we note that

(5)

$$[w_2 w_3 y + w_1 w_2 w_3 x + w_1 w_3 i + w_1 w_2 j]_q = [w_2 w_3]_q \left[y + w_1 x + \frac{w_1}{w_2} i + \frac{w_1}{w_3} j \right]_{q^{w_2 w_3}}$$

Therefore, by (2), Theorem 1 and (5), we obtain the following theorem.

Theorem 2.2. *For w_1, w_2, $w_3 \in \mathbb{N}$, the following expressions*

$$[w_{\sigma(2)} w_{\sigma(3)}]_q^{n-1} \sum_{i=0}^{w_{\sigma(2)}-1} \sum_{j=0}^{w_{\sigma(3)}-1} q^{h(w_{\sigma(1)} w_{\sigma(3)} i + w_{\sigma(1)} w_{\sigma(2)} j)}$$

$$\times \beta_{n,q^{w_{\sigma(2)} w_{\sigma(3)}}}^{(h)} \left(w_{\sigma(1)} x + \frac{w_{\sigma(1)}}{w_{\sigma(2)}} i + \frac{w_{\sigma(1)}}{w_{\sigma(3)}} j \right)$$

are the same for any $\sigma \in D_3$.

It is not difficult to show that

(6)

$$\left[y + w_1 x + \frac{w_1}{w_2} i + \frac{w_1}{w_3} j \right]_{q^{w_2 w_3}} = \frac{1 - q^{w_1 w_3 i + w_1 w_2 j}}{1 - q^{w_2 w_3}} + q^{w_1 w_3 i + w_1 w_2 j}[y + w_1 x]_{q^{w_2 w_3}}$$

$$= \frac{[w_1]_q}{[w_2 w_3]_q} [w_3 i + w_2 j]_{q^{w_1}} + q^{w_1 w_3 i + w_1 w_2 j}[y + w_1 x]_{q^{w_2 w_3}}$$

From (6), we have

(7)
$$\int_{\mathbb{Z}_p} \left[y + w_1 x + \frac{w_1}{w_2} i + \frac{w_1}{w_3} j \right]_{q^{w_2 w_3}}^n q^{(h-1) w_2 w_3 y} d\mu_{q^{w_2 w_3}}(y)$$

$$= \sum_{k=0}^n \binom{n}{k} \left(\frac{[w_1]_q}{[w_2 w_3]_q} \right)^{n-k} [w_3 i + w_2 j]_{q^{w_1}}^{n-k} q^{k(w_1 w_3 i + w_1 w_2 j)} \beta_{k,q^{w_2 w_3}}^{(h)}(w_1 x).$$

Thus, by Theorem 2 and (7), we get

(8)

$$[w_2 w_3]_q^{n-1} \sum_{i=0}^{w_2-1} \sum_{j=0}^{w_3-1} q^{h(w_1 w_3 i + w_1 w_2 j)} \int_{\mathbb{Z}_p} q^{(h-1) w_2 w_3 y} \left[y + w_1 x + \frac{w_1}{w_2} i + \frac{w_1}{w_3} j \right]_{q^{w_2 w_3}}^n d\mu_{q^{w_2 w_3}}(y)$$

$$= \sum_{k=0}^n \binom{n}{k} [w_2 w_3]_q^{k-1} [w_1]_q^{n-k} \beta_{k,q^{w_2 w_3}}^{(h)}(w_1 x) \sum_{i=0}^{w_2-1} \sum_{j=0}^{w_3-1} q^{(k+h)(w_1 w_3 i + w_1 w_2 j)} [w_3 i + w_2 j]_{q^{w_1}}^{n-k}$$

$$= \sum_{k=0}^n \binom{n}{k} [w_2 w_3]_q^{k-1} [w_1]_q^{n-k} \beta_{k,q^{w_2 w_3}}^{(h)}(w_1 x) \, T_{n,q^{w_1}}^{(h)}(w_2, w_3|k),$$

where

(9) $$T_{n,q}^{(h)}(w_1, w_2|k) = \sum_{i=0}^{w_1-1} \sum_{j=0}^{w_2-1} q^{(k+h)(w_2 i + w_1 j)} [w_2 i + w_1 j]_q^{n-k}.$$

As this expression is invariant under the third Dihedral group D_3, we have the following theorem.

Theorem 2.3. *For $n \geq 0$, w_1, w_2, $w_3 \in \mathbb{N}$, the following expressions*

$$\sum_{k=0}^{n} \binom{n}{k} [w_{\sigma(2)} w_{\sigma(3)}]_q^{k-1} [w_{\sigma(1)}]_q^{n-k} \beta_{k,q^{w_{\sigma(2)} w_{\sigma(3)}}}^{(h)} (w_{\sigma(1)} x) \; T_{n,q^{w_{\sigma(1)}}}^{(h)} (w_{\sigma(2)}, w_{\sigma(3)} | k)$$

are all the same for any $\sigma \in D_3$.

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Seog-Hoon Rim

Department of Mathematics Education,
Kyungpook National University, Tagegu 702-701, S. Korea
E-mail: shrim@knu.ac.kr

Tae Gyun Kim

Jangjeon Research Institute for Mathematics and Physics, Hapcheon 678-800, S. Korea
Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea
E-mail: tgkim2013@hotmail.com

Sang Hun Lee

Division of General Education,
Kwangwoon University, Seoul 139-701, Republic of Korea
E-mail: leesh58@kw.ac.kr

SOME IDENTITIES OF BELL POLYNOMIALS ASSOCIATED WITH p-ADIC INTEGRAL ON $\mathbb{Z}_p$

SEOG-HOON RIM, HONG KYUNG PAK, J.K. KWON, AND TAE GYUN KIM

ABSTRACT. In this paper, we investigate some identities of Bell polynomials associated with special polynomials which are derived from p-adic integral on $\mathbb{Z}_p$.

1. INTRODUCTION

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p, \mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = \frac{1}{p}$. Let q be an indeterminate in $\mathbb{C}_p$ with $|1-q|_p < p^{-\frac{1}{p-1}}$ and let the q-extension of number x is defined as $[x]_q = \frac{1-q^x}{1-q}$. The Euler polynomials of order r are defined by the generating function to be

$$\left(\frac{2}{e^t+1}\right)^r e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x) \frac{t^n}{n!}, \quad \text{(see } [1-18])$$

and the higher-order Bernoulli polynomials of order r are given by

$$\left(\frac{t}{e^t-1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x) \frac{t^n}{n!}, \quad \text{(see } [9-10]).$$

When $x = 0$, $B_n^{(r)} = B_n^{(r)}(0)$, $E_n^{(r)} = E_n^{(r)}(0)$ are called higher-order Bernoulli numbers and Euler numbers.

Let $f(x)$ be a uniformly continuous function on $\mathbb{Z}_p$. Then the bosonic p-adic integral on $\mathbb{Z}_p$ is defined by

$$(1) \qquad \int_{\mathbb{Z}_p} f(x) d\mu_0(x) = \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \quad \text{(see } [12]),$$

and the fermieuic p-adic integral on $\mathbb{Z}_p$ is given by

$$(2) \qquad \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=-1}^{p^N-1} f(x)(-1)^x, \quad \text{(see } [12]).$$

Thus, we have

$$(3) \qquad \int_{\mathbb{Z}_p} f(x+1) d\mu_0(x) - \int_{\mathbb{Z}_p} f(x) d\mu_0(x) = f'(0),$$

and

$$(4) \qquad \int_{\mathbb{Z}_p} f(x+1) d\mu_{-1}(x) + \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = 2f(0).$$

1

As is well know, the higher-order Changhee polynomials are given by

$$(5) \qquad \left(\frac{2}{t+2}\right)^r (1+t)^x = \sum_{n=0}^{\infty} Ch_n^{(r)}(x)\frac{t^n}{n!}, \quad (\text{see } [11-15]),$$

and the higher-order Daehee polynomials are defined by the generating function to be

$$(6) \qquad \left(\frac{\log(1+t)}{t}\right)^r (1+t)^x = \sum_{n=0}^{\infty} D_n^{(r)}(x)\frac{t^n}{n!}, \quad (\text{see } [11-15]).$$

When $x = 0$, $Ch_n^{(r)} = Ch_n^{(r)}(0)$ and $D_n^{(r)} = D_n^{(r)}(0)$ are called the Changhee numbers and the Daehee numbers with order r.

Finally, we introduce the Bell polynomials which are given by the generating function to be

$$(7) \qquad e^{(e^t-1)x} = \sum_{n=0}^{\infty} Bel_n(x)\frac{t^n}{n!}, \quad (\text{see } [4, 14, 16]).$$

The purpose of this paper is to given some identities of Bell polynomials associated with special polynomials arising from p-adic integral on $\mathbb{Z}_p$.

2. Some identities of Bell polynomials

From (2), we note that

$$(8) \qquad \begin{aligned} &\int_{\mathbb{Z}_p} e^{(e^t-1)(x+y)} d\mu_0(y) \\ &= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} (x+y)^n d\mu_0(y)\frac{(e^t-1)^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} B_k(x)S_2(n,k)\right)\frac{t^n}{n!}, \end{aligned}$$

where $S_2(n,k)$ is the Stirling number of the second kind. On the other hand,

$$(9) \qquad \int_{\mathbb{Z}_p} e^{(e^t-1)(x+y)} d\mu_0(y) = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} Bel_n(x+y)d\mu_0(y)\frac{t^n}{n!}.$$

Thus, by (8) and (9), we get

$$(10) \qquad \int_{\mathbb{Z}_p} Bel_n(x+y)d\mu_0(y) = \sum_{k=0}^{n} B_k(x)S_2(n,k).$$

By the same method as (10), we get

$$(11) \qquad \int_{\mathbb{Z}_p} Bel_n(x+y)d\mu_{-1}(y) = \sum_{k=0}^{n} E_k(x)S_2(n,k).$$

Note that

$$
\text{(12)} \quad \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+t)^{(x_1+\cdots+x_r+x)} d\mu_0(x_1) \cdots d\mu_0(x_r) = \left(\frac{\log(1+t)}{t}\right)^r (1+t)^x
$$

$$
= \sum_{n=0}^{\infty} D_n^{(r)}(x) \frac{t^n}{n!}.
$$

By replacing t by $e^{e^t-1} - 1$, we get

$$
\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(e^t-1)(x_1+\cdots+x_r+x)} d\mu_0(x_1) \cdots d\mu_0(x_r)
$$

$$
= \left(\frac{e^t-1}{e^{e^t-1}-1}\right)^r e^{(e^t-1)x} = \left(\sum_{l=0}^{\infty} B_l^{(r)} \frac{(e^t-1)^l}{l!}\right)\left(\sum_{m=0}^{\infty} Bel_m(x) \frac{t^m}{m!}\right)
$$

$$
= \left(\sum_{l=0}^{\infty} B_l^{(r)} \sum_{k=l}^{\infty} S_2(k,l) \frac{t^k}{k!}\right)\left(\sum_{m=0}^{\infty} Bel_m(x) \frac{t^m}{m!}\right)
$$

$$
\text{(13)} \quad = \left(\sum_{k=0}^{\infty} \left(\sum_{l=0}^{k} B_l^{(r)} S_2(k,l)\right) \frac{t^k}{k!}\right)\left(\sum_{m=0}^{\infty} Bel_m(x) \frac{t^m}{m!}\right)
$$

$$
= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \frac{Bel_m(x)n!}{m!(n-m)!} \sum_{l=0}^{n-m} B_l^{(r)} S_2(n-m,l)\right) \frac{t^n}{n!}
$$

$$
= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} Bel_m(x) \sum_{l=0}^{n-m} B_l^{(r)} S_2(n-m,l)\right) \frac{t^n}{n!}.
$$

On the other hand,

$$
\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(e^t-1)(x_1+\cdots+x_r+x)} d\mu_0(x_1) \cdots d\mu_0(x_r)
$$

$$
\text{(14)} \quad = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} Bel_n(x_1+\cdots+x_r+x) d\mu_0(x_1) \cdots d\mu_0(x_r) \frac{t^n}{n!}.
$$

Therefore, we obtain the following theorem.

Theorem 1. *For $n \geq 0$, we have*

$$
\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} Bel_n(x_1+\cdots+x_r+x) d\mu_0(x_1) \cdots d\mu_0(x_r)
$$

$$
= \sum_{m=0}^{n} \binom{n}{m} Bel_m(x) \sum_{l=0}^{n-m} B_l^{(r)} S_2(n-m,l).
$$

From (12), we note that

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(e^t-1)(x_1+\cdots+x_r+x)} d\mu_0(x_1) \cdots d\mu_0(x_r)$$

$$= \sum_{n=0}^{\infty} D_n^{(r)}(x) \frac{1}{n!} \left(e^{(e^t-1)} - 1 \right)^n = \sum_{k=0}^{\infty} D_k^{(r)}(x) \sum_{m=k}^{\infty} S_2(m,k) \frac{(e^t-1)^m}{m!}$$

(15)
$$= \sum_{m=0}^{\infty} \sum_{k=0}^{m} D_k^{(r)}(x) S_2(m,k) \frac{1}{m!} (e^t-1)^m$$

$$= \sum_{m=0}^{\infty} \sum_{k=0}^{m} D_k^{(r)}(x) S_2(m,k) \sum_{n=m}^{\infty} S_2(n,m) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{n} \sum_{k=0}^{m} D_k^{(r)}(x) S_2(m,k) S_2(n,m) \right\} \frac{t^n}{n!}.$$

Therefore, by Theorem 1 and (15), we obtain the following theorem.

Theorem 2. *For $n \geq 0$, we have*

$$\sum_{m=0}^{n} \binom{n}{m} Bel_m(x) \sum_{l=0}^{n-m} B_l^{(r)} S_2(n-m,l)$$

$$= \sum_{m=0}^{n} \sum_{k=0}^{m} D_k^{(r)}(x) S_2(m,k) S_2(n,m).$$

From (7), we note that

$$e^{xt} = \sum_{m=0}^{\infty} Bel_m(x) \frac{1}{m!} \left(\log(1+t) \right)^m$$

(16)
$$= \sum_{m=0}^{\infty} Bel_m(x) \sum_{n=m}^{\infty} S_1(n,m) \frac{t^m}{m!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} Bel_m(x) S_1(n,m) \right) \frac{t^n}{n!},$$

where $S_1(n,m)$ is the Stirling number of the first kind.

Therefore, by (16), we obtain the following theorem.

Theorem 3. *For $n \geq 0$, we have*

$$x^n = \sum_{m=0}^{n} Bel_m(x) S_1(n,m).$$

It is easy to show that

(17)
$$\int_{\mathbb{Z}_p} e^{xt} d\mu_0(x) = \frac{t}{e^t-1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$

Thus, by (17), we have

$$\int_{\mathbb{Z}_p} x^n d\mu_0(x) = B_n, \quad (n \geq 0).$$

From Theorem 3, we can derive the following equation:

$$(18) \qquad B_n = \int_{\mathbb{Z}_p} x^n d\mu_0(x) = \sum_{m=0}^{n} S_1(n, m) \int_{\mathbb{Z}_p} Bel_m(x) d\mu_0(x), \quad (n \geq 0).$$

Therefore, by (10) and (18), we obtain the following theorem.

Theorem 4. *For $n \geq 0$, we have*

$$B_n = \sum_{m=0}^{n} \sum_{k=0}^{m} S_1(n, m) S_2(m, k) B_k.$$

It is not difficult to show that

$$(19) \qquad \int_{\mathbb{Z}_p} e^{xt} d\mu_{-1}(x) = \frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}.$$

Thus, by (19), we get

$$(20) \qquad \int_{\mathbb{Z}_p} x^n d\mu_{-1}(x) = E_n, \quad (n \geq 0).$$

From Theorem 3 and (20), we have

$$(21) \qquad E_n = \int_{\mathbb{Z}_p} x^n d\mu_{-1}(x) = \sum_{m=0}^{n} S_1(n, m) \int_{\mathbb{Z}_p} Bel_m(x) d\mu_{-1}(x).$$

Therefore, by (11) and (21), we obtain the following theorem.

Theorem 5. *For $n \geq 0$, we have*

$$E_n = \sum_{m=0}^{n} \sum_{k=0}^{m} S_1(n, m) S_2(m, k) E_k.$$

Now, we consider the following equation.

$$(22) \qquad \begin{aligned} e^{(x+x_1+\cdots+x_r)t} &= \sum_{m=0}^{\infty} Bel_m(x_1 + \cdots + x_r + x) \frac{(\log(1 + t))^m}{m!} \\ &= \sum_{m=0}^{\infty} Bel_m(x_1 + \cdots + x_r + x) \sum_{n=m}^{\infty} S_1(n, m) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} Bel_m(x_1 + \cdots + x_r + x) S_1(n, m) \right) \frac{t^n}{n!}. \end{aligned}$$

Thus, by (22), we have the following theorem.

Theorem 6. *For $n \geq 0$, we have*

$$(x + x_1 + \cdots + x_r)^n = \sum_{m=0}^{n} Bel_m(x_1 + \cdots + x_r + x) S_1(n, m).$$

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From (4), we can easily derive the following equation:

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(x_1+\cdots+x_r+x)t} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) = \left(\frac{2}{e^t+1}\right)^r e^{xt}$$

(23)
$$= \sum_{n=0}^{\infty} E_n^{(r)}(x)\frac{t^n}{n!}.$$

Thus, by (23), we get

(24) $$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_r + x)^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) = E_n^{(r)}(x).$$

By (3), we easily get

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(x_1+\cdots+x_r+x)t} d\mu_0(x_1) \cdots d\mu_0(x_r) = \left(\frac{t}{e^t-1}\right)^r e^{xt}$$

(25)
$$= \sum_{n=0}^{\infty} B_n^{(r)}(x)\frac{t^n}{n!}.$$

From (25), we have

(26) $$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_r + x)^n d\mu_0(x_1) \cdots d\mu_0(x_r) = B_n^{(r)}(x).$$

From Theorem 6, (24) and (26), we have

(27) $$B_n^{(r)}(x) = \sum_{m=0}^{n} S_1(n,m) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} Bel_m(x+x_1+\cdots+x_r) d\mu_0(x_1) \cdots d\mu_0(x_r)$$

and
(28)

$$E_n^{(r)}(x) = \sum_{m=0}^{n} S_1(n,m) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} Bel_m(x+x_1+\cdots+x_r) d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r).$$

Now, we observe that

$$\sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} Bel_n(x+x_1+\cdots+x_r) d\mu_0(x_1) \cdots d\mu_0(x_r)\frac{t^n}{n!}$$

$$= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(e^t-1)(x_1+\cdots+x_r+x)t} d\mu_0(x_1) \cdots d\mu_0(x_r)$$

(29)
$$= \sum_{m=0}^{\infty} B_m^{(r)}(x)\frac{1}{m!}\left(e^t-1\right)^m$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} B_m^{(r)}(x)S_2(n,m)\right)\frac{t^n}{n!}.$$

Thus, by (29), we get

(30) $$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} Bel_n(x_1+\cdots+x_r+x) d\mu_0(x_1) \cdots d\mu_0(x_r) = \sum_{m=0}^{n} B_m^{(r)}(x)S_2(n,m).$$

Therefore, by (27) and (30), we obtain the following theorem.

Theorem 7. *For $n \geq 0$, we have*

$$B_n^{(r)}(x) = \sum_{m=0}^{n} \sum_{k=0}^{m} S_1(n,m) S_2(m,k) B_k^{(r)}(x).$$

By the same method of (29), we get

$$\sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} Bel_n(x + x_1 + \cdots + x_r) d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \frac{t^n}{n!}$$

(31)
$$= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(e^t-1)(x_1+\cdots+x_r+x)t} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} E_m^{(r)}(x) S_2(n,m) \right) \frac{t^n}{n!}.$$

From (31), we have

(32) $$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} Bel_n(x_1 + \cdots + x_r + x) d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) = \sum_{m=0}^{n} E_m^{(r)}(x) S_2(n,m).$$

Therefore, by Theorem 6 and (32), we obtain the following theorem.

Theorem 8. *For $n \geq 0$, we have*

$$E_n^{(r)}(x) = \sum_{m=0}^{n} \sum_{k=0}^{m} S_1(n,m) S_2(m,k) E_k^{(r)}(x).$$

From (4), we have

(33)
$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+t)^{(x_1+\cdots+x_r+x)} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r)$$

$$= \left(\frac{2}{1+t} \right)^r (1+t)^x = \sum_{n=0}^{\infty} Ch_n^{(r)}(x) \frac{t^n}{n!}.$$

By replacing t by $e^{(e^t-1)} - 1$, we get

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(e^t-1)(x_1+\cdots+x_r+x)} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r)$$

$$= \sum_{m=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_r + x) d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \frac{1}{m!} (e^t - 1)^m$$

(34)
$$= \sum_{m=0}^{\infty} E_m^{(r)}(x) \frac{1}{m!} (e^t - 1)^m$$

$$= \sum_{m=0}^{\infty} E_m^{(r)}(x) \sum_{n=m}^{\infty} S_2(n,m) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} E_m^{(r)}(x) S_2(n,m) \right) \frac{t^n}{n!},$$

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and

$$2^r e^{-(e^t-1)r} e^{(e^t-1)x} = 2^r \Big(\sum_{l=0}^{\infty} Bel_l(-r) \frac{t^l}{l!} \Big) \Big(\sum_{m=0}^{\infty} Bel_m(x) \frac{t^m}{m!} \Big)$$

(35)
$$= 2^r \sum_{n=0}^{\infty} \Big(\sum_{m=0}^{n} \frac{Bel_m(x) Bel_{n-m}(-r) n!}{m!(n-m)!} \Big) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \Big(2^r \sum_{m=0}^{n} \binom{n}{m} Bel_m(x) Bel_{n-m}(-r) \Big) \frac{t^n}{n!}.$$

Therefore, by (33),(34) and (35), we obtain the following theorem.

Theorem 9. *For $n \geq 0$, we have*

$$\sum_{m=0}^{n} E_m^{(r)}(x) S_2(n,m) = 2^r \sum_{m=0}^{n} \binom{n}{m} Bel_m(x) Bel_{n-m}(-r).$$

Now, we observe that

$$\sum_{m=0}^{\infty} Ch_m^{(r)}(x) \frac{1}{m!} \Big(e^{(e^t-1)} - 1 \Big)^m = \sum_{m=0}^{\infty} Ch_m^{(r)}(x) \sum_{k=m}^{\infty} S_2(k,m) \frac{(e^t-1)^k}{k!}$$

$$= \sum_{k=0}^{\infty} \sum_{m=0}^{k} Ch_m^{(r)}(x) S_2(k,m) \frac{1}{k!} (e^t-1)^k$$

(36)
$$= \sum_{k=0}^{\infty} \sum_{m=0}^{k} Ch_m^{(r)}(x) S_2(k,m) \sum_{n=k}^{\infty} S_2(n,k) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} \sum_{m=0}^{k} Ch_m^{(r)}(x) S_2(k,m) S_2(n,k) \Big) \frac{t^n}{n!}.$$

Therefore, by (33), (34) and (36), we obtain the following theorem.

Theorem 10. *For $n \geq 0$, we have*

$$\sum_{m=0}^{n} E_m^{(r)}(x) S_2(n,m) = \sum_{k=0}^{n} \sum_{m=0}^{k} Ch_m^{(r)}(x) S_2(k,m), S_2(n,k).$$

From (4), we have

$$
\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(e^t-1)(x_1+\cdots+x_r+x)} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r)
$$

$$
= \Big(\frac{2}{e^{e^t-1}+1}\Big)^r e^{(e^t-1)x}
$$

$$
= \Big(\sum_{m=0}^{\infty} E_m^{(r)} \frac{(e^t-1)^m}{m!}\Big)\Big(\sum_{l=0}^{\infty} Bel_l(x)\frac{t^l}{l!}\Big)
$$

$$
(37) \qquad = \Big(\sum_{m=0}^{\infty} E_m^{(r)} \sum_{k=m}^{\infty} S_2(k,m)\frac{t^k}{k!}\Big)\Big(\sum_{l=0}^{\infty} Bel_l(x)\frac{t^l}{l!}\Big)
$$

$$
= \Big(\sum_{k=0}^{\infty}\Big(\sum_{m=0}^{k} E_m^{(r)} S_2(k,m)\Big)\frac{t^k}{k!}\Big)\Big(\sum_{l=0}^{\infty} Bel_l(x)\frac{t^l}{l!}\Big)
$$

$$
= \sum_{n=0}^{\infty}\Big\{\sum_{k=0}^{n}\sum_{m=0}^{k} E_m^{(r)}(x)S_2(k,m)Bel_{n-k}(x)\frac{n!}{k!(n-k)!}\Big\}\frac{t^n}{n!}
$$

$$
= \sum_{n=0}^{\infty}\Big\{\sum_{k=0}^{n}\binom{n}{k}\sum_{m=0}^{k} E_m^{(r)}(x)S_2(k,m)Bel_{n-k}(x)\Big\}\frac{t^n}{n!}
$$

Therefore, by (34) and (37), we obtain the following theorem.

Theorem 11. *For $n \geq 0$, we have*

$$
\sum_{k=0}^{n} E_k^{(r)}(x)S_2(n,k) = \sum_{k=0}^{n}\binom{n}{k}\sum_{m=0}^{k} E_m^{(r)}(x)S_2(k,m)Bel_{n-k}(x).
$$

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DEPARTMENT OF MATHEMATICS, KYUNGPOOK NATIONAL UNIVERSITY, TAEGU 702-701, S. KOREA
    E-mail address: shrim@knu.ac.kr

DEPARTMENT OF COMPUTER SCIENCE, DAEGU HAANY UNIVERSITY, KYUNGSAN 712-715, S. KOREA
    E-mail address: hkpak@dhu.ac.kr

DEPARTMENT OF MATHEMATICS, KYUNGPOOK NATIONAL UNIVERSITY, TAEGU 702-701, S. KOREA
    E-mail address: mathkjk26@hanmail.net

JANGJEON RESEARCH INSTITUTE FOR MATHEMATICS AND PHYSICS, HAPCHEON 678-800, S. KOREA, DEPARTMENT OF MATHEMATICS, KWANGWOON UNIVERSITY, SEOUL 139-701, S. KOREA
    E-mail address: tgkim2013@hotmail.com

ON A PRODUCT-TYPE OPERATOR FROM WEIGHTED BERGMAN-NEVANLINNA SPACES TO WEIGHTED ZYGMUND SPACES ON THE UNIT DISK

ZHI JIE JIANG, HONG BIN BAI, AND ZUO AN LI

ABSTRACT. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk, φ an analytic self-mapping of $\mathbb{D}$ and ψ an analytic function in $\mathbb{D}$. Let $\mathcal{D}$ be the differentiation operator and $W_{\varphi,\psi}$ the weighted composition operator. The boundedness and compactness of the product-type operator $W_{\varphi,\psi}\mathcal{D}$ from weighted Bergman-Nevanlinna spaces to weighted Zygmund spaces on $\mathbb{D}$ are characterized.

1. INTRODUCTION

Let $\mathbb{C}$ be the complex plane, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ the open unit disk in $\mathbb{C}$, $H(\mathbb{D})$ the class of all holomorphic functions on $\mathbb{D}$, φ a holomorphic self-mapping of $\mathbb{D}$ and $\psi \in H(\mathbb{D})$. *Weighted composition operator $W_{\varphi,\psi}$ on $H(\mathbb{D})$ is defined by*

$$W_{\varphi,\psi} f(z) = \psi(z) \cdot f(\varphi(z)), \ z \in \mathbb{D}.$$

If $\psi \equiv 1$ the operator is reduced to, so called, *composition operator* and usually denote by C_φ. If $\varphi(z) = z$, it is reduced to, so called, *multiplication operator* and usually denote by M_ψ. Standard problem is to provide function theoretic characterizations when φ and ψ induce a bounded or compact weighted composition operator. Weighted composition operators between various spaces of holomorphic functions on different domains have been studied by numerous authors, see, e.g., $[1, 2, 8, 9, 11, 13\text{--}17, 19, 21, 23, 28, 34, 35, 45, 49, 50, 53]$ and the references therein.

Let $\mathcal{D}$ be the differentiation operator on $H(\mathbb{D})$, that is,

$$\mathcal{D}f(z) = f'(z), \ z \in \mathbb{D}.$$

The product-type operator $C_\varphi \mathcal{D}$ has been studied, for example, in $[4, 18, 20, 25, 26, 29, 41, 44, 46]$. In [31] Sharma has studied the following operators from Bergman-Nevanlinna spaces to Bloch-type spaces:

$$M_\psi C_\varphi \mathcal{D} f(z) = \psi(z) f'(\varphi(z)),$$

$$M_\psi \mathcal{D} C_\varphi f(z) = \psi(z) \varphi'(z) f'(\varphi(z)),$$

$$C_\varphi M_\psi \mathcal{D} f(z) = \psi(\varphi(z)) f'(\varphi(z)),$$

and

$$C_\varphi \mathcal{D} M_\psi f(z) = \psi'(\varphi(z)) f(\varphi(z)) + \psi(\varphi(z)) f'(\varphi(z)),$$

1

for $z \in \mathbb{D}$ and $f \in H(\mathbb{D})$. These operators on weighted Bergman spaces, were also studied in [51] and [52] by Stević, Sharma and Bhat. If we consider the *product-type operator* $W_{\varphi,\psi}\mathcal{D}$, then it is clear that

$$M_\psi C_\varphi \mathcal{D} = W_{\varphi,\psi}\mathcal{D}, \quad M_\psi \mathcal{D} C_\varphi = W_{\varphi,\psi\cdot\varphi'}\mathcal{D},$$

$$C_\varphi M_\psi \mathcal{D} = W_{\varphi,\psi\circ\varphi}\mathcal{D} \quad \text{and} \quad C_\varphi \mathcal{D} M_\psi = W_{\varphi,\psi'\circ\varphi} + W_{\varphi,\psi\circ\varphi}\mathcal{D}.$$

Quite recently, the present author has considered operator $W_{\varphi,\psi}\mathcal{D}$ from weighted Bergman spaces to weighted Zygmund spaces in [10]. This paper is devoted to characterizing the boundedness and compactness of operator $W_{\varphi,\psi}\mathcal{D}$ from weighted Bergman-Nevanlinna spaces to weighted Zygmund spaces. It can be regarded as a continuation of the investigation of operators from weighted Bergman-Nevanlinna spaces to other spaces (see. e.g., [12] and [30]).

Next we introduce the needed spaces and some facts. Let $dA(z) = \frac{1}{\pi}dxdy$ be the normalized Lebesgue measure on $\mathbb{D}$. For $\alpha > -1$, let $dA_\alpha(z) = (\alpha + 1)(1-|z|^2)^\alpha dA(z)$ be the weighted Lebesgue measure on $\mathbb{D}$. The *weighted Bergman-Nevanlinna space* $\mathcal{A}_{\log}^\alpha$ on $\mathbb{D}$ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{A}_{\log}^\alpha} = \int_{\mathbb{D}} \log(1 + |f(z)|)dA_\alpha(z) < \infty.$$

It is a Fréchet space with the translation invariant metric

$$d(f,g) = \|f - g\|_{\mathcal{A}_{\log}^\alpha}.$$

For some details of this space, see, e.g., [6], [7], [47] and [54].

For $\beta > 0$, the *weighted-type* $\mathcal{A}_\beta$ consists of all $f \in H(\mathbb{D})$ such that

$$\sup_{z\in\mathbb{D}}(1 - |z|^2)^\beta|f(z)| < \infty.$$

This space is a non-separable Banach space with the norm defined by

$$\|f\|_{\mathcal{A}_\beta} = \sup_{z\in\mathbb{D}}(1 - |z|^2)^\beta|f(z)|.$$

The closure of the set of polynomials in $\mathcal{A}_\beta$ is denoted by $\mathcal{A}_{\beta,0}$, which is a separable Banach space and consists exactly of those functions f in $\mathcal{A}_\beta$ satisfying the next condition

$$\lim_{|z|\to 1^-}(1 - |z|^2)^\beta|f(z)| = 0.$$

For $\beta > 0$, the *weighted Bloch space* is defined by

$$\mathcal{B}_\beta = \left\{ f \in H(\mathbb{D}) : \sup_{z\in\mathbb{D}}(1 - |z|^2)^\beta|f'(z)| < \infty \right\}.$$

Under the norm

$$\|f\|_{\mathcal{B}_\beta} = |f(0)| + \sup_{z\in\mathbb{D}}(1 - |z|^2)^\beta|f'(z)|,$$

it is a Banach space. For more detail on the space, see, e.g. [55]. The closure of the set of polynomials in $\mathcal{B}_\beta$ is called the *little weighted Bloch space* and is denoted by $\mathcal{B}_{\beta,0}$. For a good source for such spaces, we refer to [55].

For $\beta > 0$, the *weighted Zygmund space* $\mathcal{Z}_\beta$ consists of all $f \in H(\mathbb{D})$ such that

$$\sup_{z\in\mathbb{D}}(1 - |z|^2)^\beta|f''(z)| < \infty.$$

It is a Banach space with the norm

$$\|f\|_{\mathcal{Z}_\beta} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}}(1 - |z|^2)^\beta|f''(z)|.$$

The *little weighted Zygmund space* $\mathcal{Z}_{\beta,0}$ consists those functions f in $\mathcal{Z}_\beta$ satisfying

$$\lim_{|z|\to 1^-}(1-|z|^2)^\beta|f''(z)|=0,$$

and it is a closed subspace of the weighted Zygmund space.

For weighted-type spaces, weighted Bloch spaces and weighted Zygmund spaces on the unit disk, the upper half plane, the unit ball, the unit polydisk and some operators, see, e.g. $[5, 11, 16, 22–24, 27, 28, 32, 33, 36–40, 42, 43, 48]$ and the references therein.

Since the weighted Bergman-Nevanlinna space is a Fréchet space and not a Banach space, it is necessary to introduce several definitions needed in this paper. Let X and Y be topological vector spaces whose topologies are given by translation invariant metrics d_X and d_Y, respectively, and let $L : X \to Y$ be a linear operator. It is said that L is *metrically bounded* if there exists a positive constant K such that $d_Y(Lf,0) \leq Kd_X(f,0)$ for all $f \in X$. When X and Y are Banach spaces, the metrical boundedness coincides with the usual definition of bounded operators between Banach spaces. Recall that $L : X \to Y$ is *metrically compact* if it maps bounded sets into relatively compact sets. When X and Y are Banach spaces, the metrical compactness coincides with the usual definition of compact operators between Banach spaces. When $X = \mathcal{A}_{\log}^\alpha$ and Y is a Banach space, we define

$$\|L\|_{\mathcal{A}_{\log}^\alpha \to Y} = \sup_{\|f\|_{\mathcal{A}_{\log}^\alpha}\leq 1}\|Lf\|_Y,$$

and we often write $\|L\|_{\mathcal{A}_{\log}^\alpha \to Y}$ by $\|L\|$.

Throughout this paper, an operator is bounded (respectively, compact), if it is metrically bounded (respectively, metrically compact). Constants are denoted by C, they are positive and may differ from one occurrence to the next. The notation $a \asymp b$ means that there exists a positive constant C such that $a/C \leq b \leq Ca$.

2. The operator $W_{\varphi,\psi}\mathcal{D} : \mathcal{A}_{\log}^\alpha \to \mathcal{Z}_\beta \ (\mathcal{Z}_{\beta,0})$

Our first lemma characterizes the compactness in terms of sequential convergence. Since the proof is standard, it is omitted (see, e.g., Proposition 3.11 in [3]).

Lemma 2.1. *Let* $\alpha > -1$, $\beta > 0$ *and* $\mathcal{Y} \in \{\mathcal{Z}_\beta, \mathcal{Z}_{\beta,0}\}$. *Then the bounded operator* $W_{\varphi,\psi}\mathcal{D} : \mathcal{A}_{\log}^\alpha \to \mathcal{Y}$ *is compact if and only if for every bounded sequence* $(f_n)_{n\in\mathbb{N}}$ *in* $\mathcal{A}_{\log}^\alpha$ *such that* $f_n \to 0$ *uniformly on every compact subset of* $\mathbb{D}$ *as* $n \to \infty$, *it follows that*

$$\lim_{n\to\infty}\|W_{\varphi,\psi}\mathcal{D}f_n\|_{\mathcal{Y}}=0.$$

The next result can be found, for example, in [54].

Lemma 2.2. *Let* $\alpha > -1$ *and* $n \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$. *Then for all* $f \in \mathcal{A}_{\log}^\alpha$ *and* $z \in \mathbb{D}$, *there exists a positive constant* C *independent of* f *such that*

$$(1-|z|^2)^n|f^{(n)}(z)| \leq \exp\frac{C\|f\|_{\mathcal{A}_{\log}^\alpha}}{(1-|z|^2)^{\alpha+2}}.$$

Now we consider the boundedness of operator $W_{\varphi,\psi}\mathcal{D} : \mathcal{A}_{\log}^\alpha \to \mathcal{Z}_\beta$.

Theorem 2.3. *Let $\alpha > -1$, $\beta > 0$, φ be an analytic self-map of $\mathbb{D}$ and $\psi \in H(\mathbb{D})$. Then for all $c > 0$, the following statements are equivalent:*

(i) The operator $W_{\varphi,\psi}\mathcal{D} : \mathcal{A}^{\alpha}_{\log} \to \mathcal{Z}_{\beta}$ is bounded.

(ii) The operator $W_{\varphi,\psi}\mathcal{D} : \mathcal{A}^{\alpha}_{\log} \to \mathcal{Z}_{\beta}$ is compact.

(iii) $\psi \in \mathcal{Z}_{\beta}$,

$$M_0 = \sup_{z \in \mathbb{D}}(1 - |z|^2)^{\beta}|\psi(z)||\varphi'(z)|^2 < \infty,$$

$$M_1 = \sup_{z \in \mathbb{D}}(1 - |z|^2)^{\beta}|\psi(z)\varphi''(z) + 2\psi'(z)\varphi'(z)| < \infty,$$

$$\lim_{\varphi(z)\to\partial\mathbb{D}} \frac{(1 - |z|^2)^{\beta}}{1 - |\varphi(z)|^2}|\psi''(z)| \exp \frac{c}{(1 - |\varphi(z)|^2)^{\alpha+2}} = 0,$$

$$\lim_{\varphi(z)\to\partial\mathbb{D}} \frac{(1 - |z|^2)^{\beta}}{(1 - |\varphi(z)|^2)^2}|\psi(z)\varphi''(z) + 2\psi'(z)\varphi'(z)| \exp \frac{c}{(1 - |\varphi(z)|^2)^{\alpha+2}} = 0,$$

and

$$\lim_{\varphi(z)\to\partial\mathbb{D}} \frac{(1 - |z|^2)^{\beta}}{(1 - |\varphi(z)|^2)^3}|\psi(z)||\varphi'(z)|^2 \exp \frac{c}{(1 - |\varphi(z)|^2)^{\alpha+2}} = 0.$$

Proof. Suppose that (i) holds. Take the functions $f(z) = z$ and $f(z) = z^2$, respectively. Since the operator $W_{\varphi,\psi}\mathcal{D} : \mathcal{A}^{\alpha}_{\log} \to \mathcal{Z}_{\beta}$ is bounded, we have

$$\sup_{z \in \mathbb{D}}(1 - |z|^2)^{\beta}|\psi''(z)| \leq \|W_{\varphi,\psi}\mathcal{D}z\|_{\mathcal{Z}_{\beta}} \leq C\|W_{\varphi,\psi}\mathcal{D}\| \qquad (1)$$

and

$$\sup_{z \in \mathbb{D}}(1 - |z|^2)^{\beta}\big|\psi''(z)\varphi(z) + 2\psi'(z)\varphi'(z) + \psi(z)\varphi''(z)\big| \leq C\|W_{\varphi,\psi}\mathcal{D}\|. \qquad (2)$$

Inequality (1) shows that $\psi \in \mathcal{Z}_{\beta}$. Also by (1) and the boundedness of φ,

$$\sup_{z \in \mathbb{D}}(1 - |z|^2)^{\beta}|\psi''(z)||\varphi(z)| < \infty. \qquad (3)$$

Then by (2), (3) and the boundedness of φ,

$$M_1 = \sup_{z \in \mathbb{D}}(1 - |z|^2)^{\beta}|\psi(z)\varphi''(z) + 2\psi'(z)\varphi'(z)| < \infty. \qquad (4)$$

Let the function $f(z) = z^3$. Then

$$\sup_{z \in \mathbb{D}}(1 - |z|^2)^{\beta}\big|\psi''(z)\varphi(z)^2 + 2\psi(z)\varphi'(z)^2 + 4\psi'(z)\varphi'(z)\varphi(z) + 2\psi(z)\varphi''(z)\varphi(z)\big|$$

$$\leq C\|W_{\varphi,\psi}\mathcal{D}\|. \qquad (5)$$

By (1), (4) and (5),

$$M_0 = \sup_{z \in \mathbb{D}}(1 - |z|^2)^{\beta}|\psi(z)||\varphi'(z)|^2 \leq C\|W_{\varphi,\psi}\mathcal{D}\| < \infty. \qquad (6)$$

For $w \in \mathbb{D}$, we choose the function

$$f_1(z) = c_1 \frac{(1 - |\varphi(w)|^2)^{\alpha+2}}{(1 - \overline{\varphi(w)}z)^{2(\alpha+2)}} + c_2 \frac{(1 - |\varphi(w)|^2)^{\alpha+4}}{(1 - \overline{\varphi(w)}z)^{2(\alpha+2)+2}}$$

$$+ c_3 \frac{(1 - |\varphi(w)|^2)^{\alpha+5}}{(1 - \overline{\varphi(w)}z)^{2(\alpha+2)+3}} - \frac{(1 - |\varphi(w)|^2)^{\alpha+6}}{(1 - \overline{\varphi(w)}z)^{2(\alpha+2)+4}}$$

where
$$c_2 = -\frac{48\alpha^3 + 460\alpha^2 + 1398\alpha + 1340}{24\alpha^3 + 214\alpha^2 + 655\alpha + 682},$$
$$c_3 = \frac{16\alpha^2 + 104\alpha + 164}{6\alpha^2 + 37\alpha + 62},$$

and
$$c_1 = 1 - c_2 - c_3.$$

We also choose the function
$$g_1(z) = \frac{2\alpha + 7}{4\alpha + 8} \frac{(1 - |\varphi(w)|^2)^{\alpha+2}}{(1 - \overline{\varphi(w)}z)^{2(\alpha+2)}} - \frac{6\alpha + 21}{4\alpha + 12} \frac{(1 - |\varphi(w)|^2)^{\alpha+4}}{(1 - \overline{\varphi(w)}z)^{2(\alpha+2)+2}}$$
$$+ \frac{(1 - |\varphi(w)|^2)^{\alpha+5}}{(1 - \overline{\varphi(w)}z)^{2(\alpha+2)+3}}.$$

For the functions f_1 and g_1, we have
$$f_1(\varphi(w)) = f_1''(\varphi(w)) = f_1'''(\varphi(w)) = 0 \tag{7}$$

and
$$g_1'(\varphi(w)) = g_1''(\varphi(w)) = 0. \tag{8}$$

Consequently, (7) and (8) make the function $f(z) = f_1(z) \exp c g_1(z)$ to satisfy
$$f''(\varphi(w)) = f'''(\varphi(w)) = 0$$

and
$$f'(\varphi(w)) = C \frac{\overline{\varphi(w)}}{(1 - |\varphi(w)|^2)^{\alpha+3}} \exp \frac{c}{(1 - |\varphi(w)|^2)^{\alpha+2}},$$

where
$$C = 2c_2 + 3c_3 - 4.$$

By the boundedness of the operator $W_{\varphi,\psi}\mathcal{D} : \mathcal{A}_{\log}^\alpha \to \mathcal{Z}_\beta$, we find
$$\frac{|\varphi(w)|(1 - |w|^2)^\beta}{(1 - |\varphi(w)|^2)^{\alpha+3}} |\psi''(w)| \exp \frac{c}{(1 - |\varphi(w)|^2)^{\alpha+2}} \leq C.$$

Thus
$$\lim_{\varphi(w) \to \partial\mathbb{D}} \frac{(1 - |w|^2)^\beta}{1 - |\varphi(w)|^2} |\psi''(w)| \exp \frac{c}{(1 - |\varphi(w)|^2)^{\alpha+2}} = 0.$$

For $w \in \mathbb{D}$, we choose the functions
$$f_2(z) = \frac{3\alpha + 8}{3\alpha + 10} \frac{(1 - |\varphi(w)|^2)^{\alpha+2}}{(1 - \overline{\varphi(w)}z)^{2(\alpha+2)}} - \frac{6\alpha + 22}{3\alpha + 10} \frac{(1 - |\varphi(w)|^2)^{\alpha+4}}{(1 - \overline{\varphi(w)}z)^{2(\alpha+2)+2}}$$
$$+ \frac{6\alpha + 24}{3\alpha + 10} \frac{(1 - |\varphi(w)|^2)^{\alpha+5}}{(1 - \overline{\varphi(w)}z)^{2(\alpha+2)+3}} - \frac{(1 - |\varphi(w)|^2)^{\alpha+6}}{(1 - \overline{\varphi(w)}z)^{2(\alpha+2)+4}},$$

and
$$g_2(z) = \frac{\alpha + 3}{\alpha + 2} \frac{(1 - |\varphi(w)|^2)^{\alpha+2}}{(1 - \overline{\varphi(w)}z)^{2(\alpha+2)}} - \frac{(1 - |\varphi(w)|^2)^{\alpha+4}}{(1 - \overline{\varphi(w)}z)^{2(\alpha+2)+2}}.$$

Then
$$f_2(\varphi(w)) = f_2'(\varphi(w)) = f_2'''(\varphi(w)) = 0 \tag{9}$$

and $g_2'(\varphi(w)) = 0$. From this and (9), for the function $g(z) = f_2(z) \exp c g_2(z)$ we have

$$g'(\varphi(w)) = g'''(\varphi(w)) = 0$$

and

$$g''(\varphi(w)) = C \frac{\overline{\varphi(w)}^2}{(1 - |\varphi(w)|^2)^{\alpha+4}} \exp \frac{c}{(1 - |\varphi(w)|^2)^{\alpha+2}},$$

where

$$C = -\frac{24\alpha + 120\alpha + 141}{3\alpha + 10}.$$

By the boundedness of $W_{\varphi,\psi}\mathcal{D} : \mathcal{A}_{\log}^\alpha \to \mathcal{Z}_\beta$,

$$\|W_{\varphi,\psi}\mathcal{D}g\|_{\mathcal{Z}_\beta} \le C\|W_{\varphi,\psi}\mathcal{D}\|,$$

and from which we obtain

$$\frac{|\varphi(w)|^2(1 - |w|^2)^\beta}{(1 - |\varphi(w)|^2)^{\alpha+4}} |\psi(w)\varphi''(w) + 2\psi'(w)\varphi'(w)| \exp \frac{c}{(1 - |\varphi(w)|^2)^{\alpha+2}} \le C.$$

This shows that

$$\lim_{\varphi(w)\to\partial\mathbb{D}} \frac{(1 - |w|^2)^\beta}{(1 - |\varphi(w)|^2)^2} |\psi(w)\varphi''(w) + 2\psi'(w)\varphi'(w)| \exp \frac{c}{(1 - |\varphi(w)|^2)^{\alpha+2}} = 0.$$

For $w \in \mathbb{D}$, we choose the functions

$$f_3(z) = \frac{1}{3} \frac{(1 - |\varphi(w)|^2)^{\alpha+2}}{(1 - \overline{\varphi(w)}z)^{2(\alpha+2)}} - 2 \frac{(1 - |\varphi(w)|^2)^{\alpha+4}}{(1 - \overline{\varphi(w)}z)^{2(\alpha+2)+2}}$$
$$+ \frac{8}{3} \frac{(1 - |\varphi(w)|^2)^{\alpha+5}}{(1 - \overline{\varphi(w)}z)^{2(\alpha+2)+3}} - \frac{(1 - |\varphi(w)|^2)^{\alpha+6}}{(1 - \overline{\varphi(w)}z)^{2(\alpha+2)+4}}$$

and

$$g_3(z) = \frac{(1 - |\varphi(w)|^2)^{\alpha+2}}{(1 - \overline{\varphi(w)}z)^{2(\alpha+2)}}.$$

From a calculation, we obtain

$$f_3(\varphi(w)) = f_3'(\varphi(w)) = f_3''(\varphi(w)) = 0. \tag{10}$$

Define the function $h(z) = f_3(z) \exp c g_3(z)$. Then by (10),

$$h'(\varphi(w)) = h''(\varphi(w)) = 0,$$

and by a direct calculation,

$$h'''(\varphi(w)) = C \frac{\overline{\varphi(w)}^3}{(1 - |\varphi(w)|^2)^{\alpha+5}} \exp \frac{c}{(1 - |\varphi(w)|^2)^{\alpha+2}},$$

where $C = -30(\alpha + 2)^2 - 8$. Since $W_{\varphi,\psi}\mathcal{D} : \mathcal{A}_{\log}^\alpha \to \mathcal{Z}_\beta$ is bounded, we have

$$\|W_{\varphi,\psi}\mathcal{D}h\|_{\mathcal{Z}_\beta} \le C\|W_{\varphi,\psi}\mathcal{D}\|,$$

and so

$$(1 - |z|^2)^\beta |(W_{\varphi,\psi}\mathcal{D}h)''(z)| \le C\|W_{\varphi,\psi}\mathcal{D}\|, \tag{11}$$

for all $z \in \mathbb{D}$. Letting $z = w$ in (11) yields to

$$\frac{(1 - |w|^2)^\beta}{(1 - |\varphi(w)|^2)^{\alpha+5}} |\psi(w)||\varphi'(w)|^2 |\varphi(w)|^3 \exp \frac{c}{(1 - |\varphi(w)|^2)^{\alpha+2}} \le C\|W_{\varphi,\psi}\mathcal{D}\|.$$

Thus

$$\frac{(1-|w|^2)^\beta}{(1-|\varphi(w)|^2)^3}|\psi(w)||\varphi'(w)|^2 \exp\frac{c}{(1-|\varphi(w)|^2)^{\alpha+2}} \le \frac{C(1-|\varphi(w)|^2)^{\alpha+2}}{|\varphi(w)|^3}. \quad (12)$$

Taking limit as $\varphi(w) \to \partial\mathbb{D}$ in (12) gives

$$\lim_{\varphi(w)\to\partial\mathbb{D}} \frac{(1-|w|^2)^\beta}{(1-|\varphi(w)|^2)^3}|\psi(w)||\varphi'(w)|^2 \exp\frac{c}{(1-|\varphi(w)|^2)^{\alpha+2}} = 0.$$

The proof of the implication is finished.

$(iii) \Rightarrow (ii)$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{A}_{\log}^\alpha$ with $\sup_{n\in\mathbb{N}}\|f_n\|_{\mathcal{A}_{\log}^\alpha} \le M$ and $f_n \to 0$ uniformly on every compact subset of $\mathbb{D}$ as $n \to \infty$. We have that for the constant C in Lemma 2.2, for any $\varepsilon > 0$ there exits a constant $\delta \in (0,1)$ such that whenever $\delta < |\varphi(z)| < 1$, it follows that

$$\frac{(1-|z|^2)^\beta}{1-|\varphi(z)|^2}|\psi''(z)|\exp\frac{C}{(1-|\varphi(z)|^2)^{\alpha+2}} < \varepsilon,$$

$$\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^2}|\psi(z)\varphi''(z) + 2\psi'(z)\varphi'(z)|\exp\frac{C}{(1-|\varphi(z)|^2)^{\alpha+2}} < \varepsilon,$$

and

$$\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^3}|\psi(z)||\varphi'(z)|^2 \exp\frac{C}{(1-|\varphi(z)|^2)^{\alpha+2}} < \varepsilon.$$

Then by Lemma 2.2, for a fixed $\delta \in (0,1)$ we have

$$\|W_{\varphi,\psi}\mathcal{D}f_n\|_{\mathcal{Z}_\beta} = \left|(\psi \cdot f_n' \circ \varphi)(0)\right| + \left|(\psi \cdot f_n' \circ \varphi)'(0)\right| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\left|(\psi(z)f_n'(\varphi(z))''\right|$$

$$= \left|\psi(0)\right|\left|f_n'(\varphi(0))\right| + \left|\psi'(0)f_n'(\varphi(0)) + \psi(0)f_n''(\varphi(0))\varphi(0)\right|$$

$$+ \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\left|\psi''(z)f_n'(\varphi(z)) + \Big(\psi(z)\varphi''(z) + 2\psi'(z)\varphi'(z)\Big)f_n''(\varphi(z)) + \psi(z)\varphi'(z)^2f_n'''(\varphi(z))\right|$$

$$\le \Big(|\psi(0)| + |\psi'(0)|\Big)|f_n'(\varphi(0))| + |\varphi(0)||\psi(0)||f_n''(\varphi(0))| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|\psi''(z)||f_n'(\varphi(z))|$$

$$+ \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\left|\psi(z)\varphi''(z) + 2\psi'(z)\varphi'(z)\right||f_n''(\varphi(z))| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|\psi(z)||\varphi'(z)|^2|f_n'''(\varphi(z))|$$

$$\le \Big(|\psi(0)| + |\psi'(0)|\Big)|f_n'(\varphi(0))| + |\varphi(0)||\psi(0)||f_n''(\varphi(0))|$$

$$+ \sup_{|\varphi(z)|\le\delta}(1-|z|^2)^\beta|\psi''(z)||f_n'(\varphi(z))| + \sup_{\delta<|\varphi(z)|<1}(1-|z|^2)^\beta|\psi''(z)||f_n'(\varphi(z))|$$

$$+ \sup_{|\varphi(z)|\le\delta}(1-|z|^2)^\beta\left|\psi(z)\varphi''(z) + 2\psi'(z)\varphi'(z)\right||f_n''(\varphi(z))|$$

$$+ \sup_{\delta<|\varphi(z)|<1}(1-|z|^2)^\beta\left|\psi(z)\varphi''(z) + 2\psi'(z)\varphi'(z)\right||f_n''(\varphi(z))|$$

$$+ \sup_{|\varphi(z)|\le\delta}(1-|z|^2)^\beta|\psi(z)||\varphi'(z)|^2|f_n'''(\varphi(z))|$$

$$+ \sup_{\delta<|\varphi(z)|<1}(1-|z|^2)^\beta|\psi(z)||\varphi'(z)|^2|f_n'''(\varphi(z))|$$

$$\le \Big(|\psi(0)| + |\psi'(0)|\Big)|f_n'(\varphi(0))| + |\varphi(0)||\psi(0)||f_n''(\varphi(0))| + \|\psi\|_{\mathcal{Z}_\beta}\sup_{|\varphi(z)|\le\delta}\left|f_n'(\varphi(z))\right|$$

$$+ \sup_{\delta < |\varphi(z)| < 1} \frac{(1 - |z|^2)^\beta}{1 - |\varphi(z)|^2} |\psi''(z)| \exp \frac{C}{(1 - |\varphi(z)|^2)^{\alpha+2}} + M_1 \sup_{|\varphi(z)| \le \delta} |f_n''(\varphi(z))|$$

$$+ \sup_{\delta < |\varphi(z)| < 1} \frac{(1 - |z|^2)^\beta}{(1 - |\varphi(z)|^2)^2} |\psi(z)\varphi''(z) + 2\psi'(z)\varphi'(z)| \exp \frac{C}{(1 - |\varphi(z)|^2)^{\alpha+2}}$$

$$+ M_0 \sup_{|\varphi(z)| \le \delta} |f_n'''(\varphi(z))| + \sup_{\delta < |\varphi(z)| < 1} \frac{(1 - |z|^2)^\beta}{(1 - |\varphi(z)|^2)^3} |\psi(z)||\varphi'(z)|^2 \exp \frac{C}{(1 - |\varphi(z)|^2)^{\alpha+2}}.$$

By Cauchy's estimation, if $(f_n)_{n \in \mathbb{N}}$ converges to zero on each compact subset of $\mathbb{D}$, then $(f_n')_{n \in \mathbb{N}}$, $(f_n'')_{n \in \mathbb{N}}$ and $(f_n''')_{n \in \mathbb{N}}$ also do as $n \to \infty$. From this, and since both $\{z \in \mathbb{D} : |z| \le \delta\}$ and $\{0\}$ are compact subset of $\mathbb{D}$, there exists a natural number N such that whenever $n > N$, it follows that

$$\left(|\psi(0)| + |\psi'(0)| \right) |f_n'(\varphi(0))| + |\varphi(0)||\psi(0)||f_n''(\varphi(0))| < \varepsilon$$

and

$$\sup_{|\varphi(z)| \le \delta} |f_n^{(i)}(\varphi(z))| < \varepsilon,$$

where $i = 1, 2, 3$. Consequently, for all $n > N$ it follows that

$$\|W_{\varphi,\psi}\mathcal{D}f_n\|_{\mathcal{Z}_\beta} \le (4 + \|\psi\|_{\mathcal{Z}_\beta} + M_0 + M_1)\varepsilon,$$

which shows that the operator $W_{\varphi,\psi}\mathcal{D} : \mathcal{A}_{\log}^\alpha \to \mathcal{Z}_\beta$ is compact.

$(ii) \Rightarrow (i)$. This implication is obvious. The proof is finished.

Now, we consider the boundedness of operator $W_{\varphi,\psi}\mathcal{D} : \mathcal{A}_{\log}^\alpha \to \mathcal{Z}_{\beta,0}$. We first have the following result.

Lemma 2.4. *Let $\alpha > -1$, $\beta > 0$, φ be an analytic self-map of $\mathbb{D}$ and $\psi \in H(\mathbb{D})$. Then for all $c > 0$, the following statements are equivalent:*
(i)

$$\lim_{z \to \partial\mathbb{D}} \frac{(1 - |z|^2)^\beta}{1 - |\varphi(z)|^2} |\psi(z)\varphi''(z) + 2\psi'(z)\varphi'(z)| \exp \frac{c}{(1 - |\varphi(z)|^2)^{\alpha+2}} = 0.$$

(ii)

$$\lim_{\varphi(z) \to \partial\mathbb{D}} \frac{(1 - |z|^2)^\beta}{1 - |\varphi(z)|^2} |\psi(z)\varphi''(z) + 2\psi'(z)\varphi'(z)| \exp \frac{c}{(1 - |\varphi(z)|^2)^{\alpha+2}} = 0,$$

and $\psi\varphi'' + 2\psi'\varphi' \in \mathcal{A}_{\beta,0}$.

Proof. Suppose that (i) holds. Since

$$\frac{1}{1 - |\varphi(z)|^2} \exp \frac{c}{(1 - |\varphi(z)|^2)^{\alpha+2}} \ge 1$$

for all $z \in \mathbb{D}$, we have

$$(1 - |z|^2)^\beta |\psi(z)\varphi''(z) + 2\psi'(z)\varphi'(z)|$$

$$\le \frac{(1 - |z|^2)^\beta}{1 - |\varphi(z)|^2} |\psi(z)\varphi''(z) + 2\psi'(z)\varphi'(z)| \exp \frac{c}{(1 - |\varphi(z)|^2)^{\alpha+2}}$$

$$\to 0,$$

as $z \to \partial\mathbb{D}$. Hence $\psi\varphi'' + 2\psi'\varphi' \in \mathcal{A}_{\beta,0}$. Since $\varphi(z) \to \partial\mathbb{D}$ implies $z \to \partial\mathbb{D}$, it follows that the first assertion in (ii) holds.

Now suppose that (ii) holds, but (i) is not true. Then there exist constants $c_0 > 0$, $\varepsilon_0 > 0$ and a sequence $\{z_n\}$ tending to $\partial\mathbb{D}$ as $n \to \infty$ such that

$$\frac{(1 - |z_n|^2)^\beta}{1 - |\varphi(z_n)|^2}\big|\psi(z_n)\varphi''(z_n) + 2\psi'(z_n)\varphi'(z_n)\big|\exp\frac{c}{(1 - |\varphi(z_n)|^2)^{\alpha+2}} \geq \varepsilon_0. \quad (13)$$

Since $\psi\varphi'' + 2\psi'\varphi' \in \mathcal{A}_{\beta,0}$, it follows from (13) that the sequence $(z_n)_{n\in\mathbb{N}}$ has a subsequence $(z_{n_k})_{k\in\mathbb{N}}$ with $\varphi(z_{n_k}) \to \partial\mathbb{D}$. Therefore, applying $(z_{n_k})_{k\in\mathbb{N}}$ to the first assertion in (ii), we arrive a contradiction to (13), finishing the proof.

By Lemma 2.4, the following result follows similar to the proof of Theorem 2.3. Hence, the proof is omitted.

Theorem 2.5. *Let $\alpha > -1$, $\beta > 0$, φ be an analytic self-mapping of $\mathbb{D}$ and $\psi \in H(\mathbb{D})$. Then for all $c > 0$, the following statements are equivalent:*
(i) The operator $W_{\varphi,\psi}\mathcal{D} : \mathcal{A}^\alpha_{\log} \to \mathcal{Z}_{\beta,0}$ is bounded.
(ii) The operator $W_{\varphi,\psi}\mathcal{D} : \mathcal{A}^\alpha_{\log} \to \mathcal{Z}_{\beta,0}$ is compact.
(iii)

$$\psi'', \ \psi\varphi'^2, \ \psi\varphi'' + 2\psi'\varphi' \in \mathcal{A}_{\beta,0},$$

$$\lim_{\varphi(z)\to\partial\mathbb{D}} \frac{(1 - |z|^2)^\beta}{1 - |\varphi(z)|^2}|\psi''(z)|\exp\frac{c}{(1 - |\varphi(z)|^2)^{\alpha+2}} = 0,$$

$$\lim_{\varphi(z)\to\partial\mathbb{D}} \frac{(1 - |z|^2)^\beta}{(1 - |\varphi(z)|^2)^3}|\psi(z)||\varphi'(z)|^2\exp\frac{c}{(1 - |\varphi(z)|^2)^{\alpha+2}} = 0,$$

and

$$\lim_{\varphi(z)\to\partial\mathbb{D}} \frac{(1 - |z|^2)^\beta}{(1 - |\varphi(z)|^2)^2}\big|\psi(z)\varphi''(z) + 2\psi'(z)\varphi'(z)\big|\exp\frac{c}{(1 - |\varphi(z)|^2)^{\alpha+2}} = 0.$$

(iv)

$$\lim_{z\to\partial\mathbb{D}} \frac{(1 - |z|^2)^\beta}{1 - |\varphi(z)|^2}|\psi''(z)|\exp\frac{c}{(1 - |\varphi(z)|^2)^{\alpha+2}} = 0,$$

$$\lim_{z\to\partial\mathbb{D}} \frac{(1 - |z|^2)^\beta}{(1 - |\varphi(z)|^2)^3}|\psi(z)||\varphi'(z)|^2\exp\frac{c}{(1 - |\varphi(z)|^2)^{\alpha+2}} = 0,$$

and

$$\lim_{z\to\partial\mathbb{D}} \frac{(1 - |z|^2)^\beta}{(1 - |\varphi(z)|^2)^2}\big|\psi(z)\varphi''(z) + 2\psi'(z)\varphi'(z)\big|\exp\frac{c}{(1 - |\varphi(z)|^2)^{\alpha+2}} = 0.$$

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Zhi jie Jiang, Institute of Nonlinear Science and Engineering Computing, Sichuan University of Science and Engineering, Zigong, Sichuan, 643000, P. R. China
  E-mail address: matjzj@126.com

Hong bin Bai, School of Science, Sichuan University of Science and Engineering, Zigong, Sichuan, 643000, P. R. China
  E-mail address: bhb@suse.edu.com

Zuo an Li, School of Computer Science, Sichuan University of Science and Engineering, Zigong, Sichuan, 643000, P. R. China
  E-mail address: Lizuoan@suse.edu.com

Hesitant fuzzy Maclaurin symmetric mean operators and their application in multiple attribute decision making

Wu Li[a], Xiaoqiang Zhou[b,*], Guanqi Guo[a]

[a] School of Information and Communication Engineering, Hunan Institute of Science and Technology

Yueyang, 414006, P.R.China

[b] School of Computer, Hunan Institute of Science and Technology

Yueyang, 414006, P.R.China

Abstract: The Maclaurin symmetric mean (MSM), originally introduced by Maclaurin, can capture the interrelationship among the multi-input arguments. It plays an important role in many multiple attribute decision making problems. In this paper, we first extend MSM operator to deal with hesitant fuzzy information and propose some new hesitant fuzzy aggregation operators, such as the hesitant fuzzy Maclaurin symmetric mean ($HFMSM$) and the weighted hesitant fuzzy Maclaurin symmetric mean ($WHFMSM$). Then, we further investigate some desirable properties and special cases of those operators in detail. Finally, we develop an approach to hesitant fuzzy multiple attribute decision making problems based on the proposed operators. A practical example is given to illustrate the practicality and effectiveness of the proposed method.

Keywords: fuzzy set; hesitant fuzzy set; aggregation operator; Maclaurin symmetric mean; multiple attribute decision making

1 Introduction

Multiple attribute decision making is one of the most significant human activities in many fields including social science, economics, medical science, engineering, environmental science and so on. The purpose of a decision making is to find a desirable solution from a finite alternatives. In order to obtain a desirable solution, the decision information provided by decision makers always need to be aggregated into an overall one by using a proper aggregation technique. Therefore, the research on information aggregation method is an important topic in multiple attribute decision making. In the past few decades, a variety of aggregation operators have been developed and applied to multiple attribute decision making with different decision information, such as accurate numbers, fuzzy numbers, intuitionistic fuzzy numbers, trapezoidal fuzzy numbers and so on [1–4].

Recently, Torra introduced the hesitant fuzzy set (HFS) [5], which allows membership degree to have a set of possible values. Therefore, it is an efficient tool in the situation where the evaluation of an alternative under each attribute is represented by several possible values. Since its appearance, HFS has attracted more and more attention from researchers [6–8]. Hesitant fuzzy information aggregation has become a hot topic in the hesitant fuzzy set theory and lots of hesitant fuzzy aggregation operators have been developed [9–17]. For example, Xia and Xu [11] first presented some hesitant fuzzy operational laws, based on which they proposed a series of aggregation operators, such as hesitant fuzzy weighted averaging ($HFWA$) operator, hesitant fuzzy weighted geometric ($HFWG$) operator and so on. Xia et al. [17] developed some confidence induced aggregation operators for hesitant fuzzy information. Xia et al. [12] gave several series of hesitant fuzzy aggregation operators with the help of quasi-arithmetic means. Wei [10] explored some hesitant fuzzy prioritized aggregation operators and applied them to hesitant fuzzy decision making problems in which the attributes are in different priority levels. Zhang [14] extended the power aggregation operator to the hesitant fuzzy power aggregation operators, whose weighting vectors depend upon the input arguments and allow values being aggregated to support and reinforce each other. Zhu et al. [16] extended Bonferroni mean to deal with hesitant fuzzy information and get the hesitant fuzzy Bonferroni mean operator. By combining

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*Corresponding author.   Tel: +86 13789003995. E-mail address: zxq0923@163.com, liwu0817@163.com.

Mailing address: School of Computer, Hunan Institute of Science and Technology, Yueyang, Hunan, 414006, P.R.China

the Bonferroni mean and the geometric mean, Zhu et al. [15] further investigated the geometric Bonferroni mean under hesitant fuzzy environment.

The Maclaurin symmetric mean (MSM) was originally proposed by Maclaurin [18] and many important results on the MSM have been obtained [19–22]. It is worth noting that the MSM has desirable properties capturing the interrelationships among multi-input arguments. The BM also can capture the interrelationships among arguments, but it only reflect the interrelationships between two arguments whereas the MSM can reflect the interrelationships among multi-input arguments. Furthermore, for the same collection of arguments, the MSM is monotonically decreasing with respect to the parameter, which make the decision makers can select easily the parameter value according to their risk preferences in decision making progress. Therefore, the MSM is more flexible and robust such that it is more adequate to solve multiple attribute decision making problem where the attributes are independent. So far, the MSM has been used successful to deal with not only the crisp values but also the intuitionistic fuzzy values [23]. But we have not seen any work based on the MSM for aggregating hesitant fuzzy information. Thus, it is meaningful to use the MSM to develop the aggregation techniques under hesitant fuzzy environment. In this paper, motivated by Qin [23], we develop some new hesitant fuzzy aggregation operators based on the MSM, and apply them to multiple attribute decision making under hesitant fuzzy environment.

The rest of this paper is organized as follows. In Section 2, we review the notions of HFS and the MSM. In Section 3, we introduce the hesitant fuzzy Maclaurin symmetric mean ($HFMSM$) operator and discuss some desirable properties and special cases of the proposed operator. In Section 4, we further develop the weighted forms of the previous operator and apply them to hesitant fuzzy decision making. Finally, conclusions are stated in Section 5.

2   Preliminaries

In this section, we recall briefly the necessary notations on HFS and MSM. We also present the dual Maclaurin symmetric mean based on the MSM.

2.1   Hesitant fuzzy set

Torra and Narukawa [5] extended the fuzzy set to the hesitant fuzzy set (HFS), shown as follows:

Definition 2.1. *Let X be a reference set, an HFS on X is in terms of a function that when applied to X returns a subset of $[0,1]$.*

To be easily understood, Xia and Xu [11] expressed the HFS by mathematical symbol

$$H = \left\{ \frac{h_H(x)}{x} | x \in X \right\},$$

where $h_H(x)$ is a set of some values in $[0,1]$, denoting the possible membership degrees of the element $x \in X$ to the set H. For convenience, Xu and Xia [7] called $h_H(x)$ an hesitant fuzzy element (HFE).

Let h_1 and h_2 be $HFEs$, the union, intersection and complement of them are definded by Torra and Narukawa [5] as:

(1) $h_1 \cup h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} max\{\gamma_1, \gamma_2\}$;

(2) $h_1 \cap h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} min\{\gamma_1, \gamma_2\}$;

(3) $h_1^c = \cup_{\gamma_1 \in h_1}\{1 - \gamma_1\}$.

Let $\alpha > 0$, h_1 and h_2 be two $HFEs$, Xu and Xia [11] defined some operations on the $HFEs$ h_1 and h_2 as follows:

(5) $h_1 \oplus h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1 + \gamma_2 - \gamma_1\gamma_2\}$

(6) $h_1 \otimes h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1\gamma_2\}$

(7) $\alpha h = \cup_{\gamma \in h} \{\gamma^\alpha\}$

(8) $h^\alpha = \cup_{\gamma \in h} \{1 - (1 - \gamma)^\alpha\}$

In [11], Xia and Xu defined the score function of $HFEs$ and gave the comparison laws.

Definition 2.2. *Let h be an HFE, $s(h) = \frac{1}{n(h)} \sum_{\gamma \in h} \gamma$ is called the score function of h, where $n(h)$ is the number of values of h. For two $HFEs$ h_1 and h_2, if $s(h_1) > s(h_2)$, then $h_1 > h_2$; if $s(h_1) = s(h_2)$, then $h_1 = h_2$.*

Xia and Xu [11, 12] further gave some hesitant fuzzy aggregation operators as follows:

Let $h_j(j = 1, 2, \cdots, n)$ be a collection of HFEs, $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ be the weight vector of $h_j(j = 1, 2, \cdots, n)$ with $\omega_j \in [0, 1]$ and $\sum\limits_{j=1}^{n} \omega_j = 1$, then

(1) The hesitant fuzzy weighted averaging ($HFWA$) operator

$$HFWA(h_1, h_2, \cdots, h_n) = \bigoplus_{j=1}^{n} (\omega_j h_j) = \bigcup_{\substack{\gamma_j \in h_j, \\ i=1,\cdots,n}} \left\{ 1 - \prod_{j=1}^{n} (1-\gamma_j)^{\omega_j} \right\}$$

Especially, if $\omega = (1/n, 1/n, \cdots, 1/n)^T$, then the $HFWA$ operator reduces to the hesitant fuzzy averaging (HFA) operator

$$HFA(h_1, h_2, \cdots, h_n) = \bigcup_{\substack{\gamma_j \in h_j, \\ i=1,\cdots,n}} \left\{ 1 - \prod_{j=1}^{n} (1-\gamma_j)^{1/n} \right\} \tag{1}$$

(2) The hesitant fuzzy weighted geometric ($HFWG$) operator

$$HFWG(h_1, h_2, \cdots, h_n) = \bigotimes_{j=1}^{n} h_j{}^{\omega_j} = \bigcup_{\substack{\gamma_j \in h_j, \\ i=1,\cdots,n}} \left\{ \prod_{j=1}^{n} \gamma_j{}^{\omega_j} \right\}$$

Especially, if $\omega = (1/n, 1/n, \cdots, 1/n)^T$, then the $HFWG$ operator becomes to the hesitant fuzzy geometric (HFG) operator

$$HFG(h_1, h_2, \cdots, h_n) = \bigcup_{\substack{\gamma_j \in h_j, \\ i=1,\cdots,n}} \left\{ \prod_{j=1}^{n} \gamma_j{}^{1/n} \right\} \tag{2}$$

2.2 Maclaurin symmetric mean

The MSM introduced by Maclaurin [18] is a useful technique characterized by the ability to capture the interrelationship among the multi-input arguments. The definition of MSM is given as follows.

Definition 2.3. *[18] Let $a_i(i = 1, 2, \cdots, n)$ be a collection of nonnegative real numbers and $r = 1, 2, \cdots, n$. If*

$$MSM^{(r)}(a_1, a_2, \cdots, a_n) = \left(\frac{\sum\limits_{1 \le i_1 < \cdots < i_r \le i_n} \prod\limits_{j=1}^{r} a_{i_j}}{C_n^r} \right)^{\frac{1}{r}}$$

then $MSM^{(r)}$ is called the Maclaurin symmetric mean, where $(i_1, i_2, \cdots, i_r)$ traversal all the r-tuple combination of $(1, 2, \cdots, n)$, C_n^r is the binomial coefficient.

It is clear that the $MSM^{(r)}$ have the following properties:

(1)$MSM^{(r)}(0, 0, \cdots, 0) = 0$;

(2)$MSM^{(r)}(a, a, \cdots, a) = a$;

(3)$MSM^{(r)}(a_1, a_2, \cdots, a_n) \le MSM^{(r)}(b_1, b_2, \cdots, b_n)$, *if $a_i \le b_i$ for all i*;

(4)$\min\limits_{i}\{a_i\} \le MSM^{(r)}(a_1, a_2, \cdots, a_n) \le \max\limits_{i}\{a_i\}$.

3 Hesitant fuzzy MSM operator

In this section, we shall extend MSM to aggregate hesitant fuzzy information and obtain a hesitant fuzzy Maclaurin symmetric mean operator. We also investigate a variety of desirable properties and some special cases.

Definition 3.1. *Let $h_i(i = 1, 2, \cdots, n)$ be a collection of $HFEs$ and $r = 1, 2, \cdots, n$. If*

$$HFMSM^{(r)}(h_1, h_2, \cdots, h_n) = \left(\frac{\bigoplus\limits_{\substack{1 \le i_1 < \cdots \\ < i_r \le i_n}} \bigotimes\limits_{j=1}^{r} h_{i_j}}{C_n^r} \right)^{\frac{1}{r}} \tag{3}$$

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then $HFMSM^{(r)}$ is called the hesitant fuzzy Maclaurin symmetric mean $(HFMSM)$, where $(i_1, i_2, \cdots, i_r)$ traversal all the r-tuple combination of $(1, 2, \cdots, n)$, C_n^r is the binomial coefficient.

Based on the operations of $HFEs$ described in Section 2, we can derive the following Theorem 3.2.

Theorem 3.2. Let $h_i(i = 1, 2, \cdots, n)$ be a collection of $HFEs$ and $r = 1, 2, \cdots, n$. Then the aggregated value by using the $HFMSM^{(r)}$ is also an HFE, and

$$HFMSM^{(r)}(h_1, h_2, \cdots, h_n) = \bigcup_{\substack{\gamma_i \in h_i, \\ i=1,\cdots,n}} \left\{ \left(1 - \left(\prod_{\substack{1 \le i_1 < \cdots \\ < i_r \le n}} \left(1 - \prod_{j=1}^r \gamma_{i_j} \right) \right)^{\frac{1}{C_n^r}} \right)^{\frac{1}{r}} \right\} \tag{4}$$

Proof. By the operational laws (5)-(8) described in Section 2, we have

$$\bigotimes_{j=1}^r h_{i_j} = \bigcup_{\substack{\gamma_i \in h_i, \\ i=1,\cdots,n}} \left\{ \prod_{j=1}^r \gamma_{i_j} \right\}$$

and

$$\bigoplus_{\substack{1 \le i_1 < \cdots \\ < i_r \le n}} \bigotimes_{j=1}^r h_{i_j} = \bigcup_{\substack{\gamma_i \in h_i, \\ i=1,\cdots,n}} \left\{ 1 - \prod_{\substack{1 \le i_1 < \cdots \\ < i_r \le n}} \left(1 - \prod_{j=1}^r \gamma_{i_j} \right) \right\}$$

then we obtain

$$\frac{1}{C_n^r} \left(\bigoplus_{\substack{1 \le i_1 < \cdots \\ < i_r \le n}} \bigotimes_{j=1}^r h_{i_j} \right) = \bigcup_{\substack{\gamma_i \in h_i, \\ i=1,\cdots,n}} \left\{ 1 - \left(\prod_{\substack{1 \le i_1 < \cdots \\ < i_r \le n}} \left(1 - \prod_{j=1}^r \gamma_{i_j} \right) \right)^{\frac{1}{C_n^r}} \right\}$$

Thus

$$HFMSM^{(r)}(h_1, h_2, \cdots, h_n) = \bigcup_{\substack{\gamma_i \in h_i, \\ i=1,\cdots,n}} \left\{ \left(1 - \left(\prod_{\substack{1 \le i_1 < \cdots \\ < i_r \le n}} \left(1 - \prod_{j=1}^r \gamma_{i_j} \right) \right)^{\frac{1}{C_n^r}} \right)^{\frac{1}{r}} \right\},$$

which completes the proof of Theorem 3.2. □

In the following, we shall study some desirable properties of $HFMSM$.

Theorem 3.3. Let $h_i(i = 1, 2, \cdots, n)$ be a collection of $HFEs$. If $h_i = h = \{\gamma\}$ for all $i \in \{1, 2, \cdots, n\}$, then

$$HFMSM^{(r)}(h_1, h_2, \cdots, h_n) = h$$

Proof. Let $h_i = \{\gamma_i\}$, then $\gamma_i = \gamma(i = 1, 2, \cdots, n)$. By Theorem 3.2, we have

$$HFMSM^{(r)}(h_1, h_2, \cdots, h_n)$$

$$= \bigcup_{\substack{\gamma_i \in h_i, \\ i=1,\cdots,n}} \left\{ \left(1 - \left(\prod_{\substack{1 \le i_1 < \cdots \\ < i_r \le n}} \left(1 - \prod_{j=1}^{r} \gamma_i \right) \right)^{\frac{1}{C_n^r}} \right)^{\frac{1}{r}} \right\}$$

$$= \bigcup_{\substack{\gamma_i \in h_i, \\ i=1,\cdots,n}} \left\{ \left(1 - \left(\prod_{\substack{1 \le i_1 < \cdots \\ < i_r \le n}} (1 - \gamma_i^r) \right)^{\frac{1}{C_n^r}} \right)^{\frac{1}{r}} \right\}$$

$$= \bigcup_{\substack{\gamma_i \in h_i, \\ i=1,\cdots,n}} \left\{ \left(1 - \left((1 - \gamma_i^r)^{C_n^r} \right)^{\frac{1}{C_n^r}} \right)^{\frac{1}{r}} \right\}$$

$$= \left\{ (1 - (1 - \gamma^r))^{\frac{1}{r}} \right\}$$

$$= \{\gamma\} = h.$$

□

Corollary 3.4. *Let $h_i(i = 1, 2, \cdots, n)$ be a collection of HFEs.*
(1) If $h_i = h = \{0\}$ for all i, then $HFMSM^{(r)}(h_1, h_2, \cdots, h_n) = \{0\}$;
(2) If $h_i = h = \{1\}$ for all i, then $HFMSM^{(r)}(h_1, h_2, \cdots, h_n) = \{1\}$.

Theorem 3.5. *Let $h_i(i = 1, 2, \cdots, n)$ be a collection of HFEs, and $h_i'(i = 1, 2, \cdots, n)$ be any permutation of $h_i(i = 1, 2, \cdots, n)$, then*

$$HFMSM^{(r)}(h_1, h_2, \cdots, h_n) = HFMSM^{(r)}(h_1', h_2', \cdots, h_n')$$

Proof. Since $h_i'(i = 1, 2, \cdots, n)$ is any permutation of $h_i(i = 1, 2, \cdots, n)$, by Definition 3.1, we have

$$HFMSM^{(r)}(h_1, h_2, \cdots, h_n) = \left(\frac{\bigoplus_{\substack{1 \le i_1 < \cdots \\ < i_r \le i_n}} \bigotimes_{j=1}^{r} h_{i_j}}{C_n^r} \right)^{\frac{1}{r}}$$

$$= \left(\frac{\bigoplus_{\substack{1 \le i_1 < \cdots \\ < i_r \le i_n}} \bigotimes_{j=1}^{r} h_{i_j}'}{C_n^r} \right)^{\frac{1}{r}} = HFMSM^{(r)}(h_1', h_2', \cdots, h_n').$$

□

Theorem 3.6. *Let $h_\alpha = \{h_{\alpha_1}, \cdots, h_{\alpha_n}\}$ and $h_\beta = \{h_{\beta_1}, \cdots, h_{\beta_n}\}$ be two collections of HFEs. If for any $\gamma_{\alpha_i} \in h_{\alpha_i}$ and $\gamma_{\beta_i} \in h_{\beta_i}$, we have $\gamma_{\alpha_i} \le \gamma_{\beta_i}$ for all $i(i = 1, \cdots, n)$, then*

$$HFMSM^{(r)}(h_{\alpha_1}, h_{\alpha_2}, \cdots, h_{\alpha_n}) \le HFMSM^{(r)}(h_{\beta_1}, h_{\beta_2}, \cdots, h_{\beta_n})$$

Proof. Since $\gamma_{\alpha_i} \leq \gamma_{\beta_i}$ for all i, $i = 1, \cdots, n$, we have

$$1 - \prod_{j=1}^{r} \gamma_{\alpha_i} \geq 1 - \prod_{j=1}^{r} \gamma_{\beta_i}$$

$$\Longrightarrow \left(\prod_{\substack{1 \leq i_1 < \cdots \\ < i_r \leq n}} \left(1 - \prod_{j=1}^{r} \gamma_{\alpha_i} \right) \right)^{\frac{1}{C_n^r}} \geq \left(\prod_{\substack{1 \leq i_1 < \cdots \\ < i_r \leq n}} \left(1 - \prod_{j=1}^{r} \gamma_{\beta_i} \right) \right)^{\frac{1}{C_n^r}}$$

$$\Longrightarrow \left(1 - \left(\prod_{\substack{1 \leq i_1 < \cdots \\ < i_r \leq n}} \left(1 - \prod_{j=1}^{r} \gamma_{\alpha_i} \right) \right)^{\frac{1}{C_n^r}} \right)^{\frac{1}{r}} \leq \left(1 - \left(\prod_{\substack{1 \leq i_1 < \cdots \\ < i_r \leq n}} \left(1 - \prod_{j=1}^{r} \gamma_{\beta_i} \right) \right)^{\frac{1}{C_n^r}} \right)^{\frac{1}{r}}$$

According to Definition 2.2 and Eq. (4), we can complete the proof of Theorem 3.6. $\qquad \square$

Theorem 3.7. *Let* $h_i(i = 1, 2, \cdots, n)$ *be a collection of HFEs,* $h_{min}^{-} = \min_i \{ h_i^{-} | h_i^{-} = \min \{ \gamma_i \in h_i \} \}$, *and* $h_{max}^{+} = \max_i \{ h_i^{+} | h_i^{+} = \max \{ \gamma_i \in h_i \} \}$. *Then*

$$h_{min}^{-} \leq HFMSM^{(r)}(h_1, h_2, \cdots, h_n) \leq h_{max}^{+}$$

Proof. Since $h_{min}^{-} \leq h_i^{-} \leq \gamma_i \leq h_i^{+} \leq h_{max}^{+}$ for any $\gamma_i \in h_i (i = 1, 2, \cdots, n)$, then we have

$$(h_{min}^{-})^r \leq \bigcup_{\substack{\gamma_i \in h_i, \\ i=1,\cdots,n}} \left\{ \prod_{j=1}^{r} \gamma_{i_j} \right\} \leq (h_{max}^{+})^r$$

$$\Longrightarrow 1 - (h_{min}^{-})^r \geq \bigcup_{\substack{\gamma_i \in h_i, \\ i=1,\cdots,n}} \left\{ \left(\prod_{\substack{1 \leq i_1 < \cdots \\ < i_r \leq n}} \left(1 - \prod_{j=1}^{r} \gamma_{i_j} \right) \right)^{\frac{1}{C_n^r}} \right\} \geq 1 - (h_{max}^{+})^r$$

$$\Longrightarrow h_{min}^{-} \leq \bigcup_{\substack{\gamma_i \in h_i, \\ i=1,\cdots,n}} \left\{ \left(1 - \left(\prod_{\substack{1 \leq i_1 < \cdots \\ < i_r \leq n}} \left(1 - \prod_{j=1}^{r} \gamma_{i_j} \right) \right)^{\frac{1}{C_n^r}} \right)^{\frac{1}{r}} \right\} \leq h_{max}^{+}.$$

Thus the proof is completed. $\qquad \square$

Next, we present some special cases of the $HFMSM^{(r)}$ operator by changing the parameter r.

Theorem 3.8. *If* $r = 1$, *then* $HFMSM^{(r)}$ *operator reduces to the hesitant fuzzy averaging* (*HFA*) *operator* (*i.e., Eq.* (1)).

Proof. By the definition of $HFMSM^{(r)}$, we have

$$HFMSM^{(1)}(h_1, h_2, \cdots, h_n)$$

$$= \bigcup_{\substack{\gamma_i \in h_i, \\ i=1,\cdots,n}} \left\{ \left(1 - \left(\prod_{1 \leq i_1 \leq n} \left(1 - \prod_{j=1}^{1} \gamma_{i_j} \right) \right)^{\frac{1}{C_n^1}} \right)^{\frac{1}{1}} \right\}$$

$$= \bigcup_{\substack{\gamma_i \in h_i, \\ i=1,\cdots,n}} \left\{ 1 - \left(\prod_{1 \leq i_1 \leq n} (1 - \gamma_{i_1}) \right)^{\frac{1}{n}} \right\} (let\ i_1 = i)$$

$$= \bigcup_{\substack{\gamma_i \in h_i, \\ i=1,\cdots,n}} \left\{ 1 - \prod_{i=1}^{n} (1 - \gamma_i)^{\frac{1}{n}} \right\}$$

$$= HFA(h_1, h_2, \cdots, h_n)$$

$\qquad \square$

Theorem 3.9. *If $r = 2$, then $HFMSM^{(r)}$ operator reduces to the hesitant fuzzy interrelated square Bonferroni mean ($HFBM^{1,1}$) which was introduced by Zhu et al. in [16].*

Proof. Let $\rho_{i,j,i\neq j} = h_i \otimes h_j = \bigcup\limits_{\gamma_i \in h_i, \gamma_j \in h_j, i\neq j} \{1 - \gamma_i\gamma_j\} = \bigcup\limits_{\delta_{i,j} \in \rho_{i,j}} \{1 - \delta_{i,j}\}$, then by the definition of $HFMSM^{(r)}$, we have

$$HFMSM^{(2)}(h_1, h_2, \cdots, h_n)$$

$$= \bigcup_{\substack{\gamma_i \in h_i, \\ i=1,\cdots,n}} \left\{ \left(1 - \left(\prod_{\substack{1\leq i_1 < \\ i_2 \leq n}} \left(1 - \prod_{j=1}^{2} \gamma_{i_j} \right) \right)^{\frac{1}{C_n^2}} \right)^{\frac{1}{2}} \right\}$$

$$= \bigcup_{\substack{\gamma_i \in h_i, \\ i=1,\cdots,n}} \left\{ \left(1 - \left(\prod_{\substack{1\leq i_1 < \\ i_2 \leq n}} (1 - \gamma_{i_1}\gamma_{i_2}) \right)^{\frac{2}{n(n-1)}} \right)^{\frac{1}{2}} \right\} (let\ i_1 = i, i_2 = j)$$

$$= \bigcup_{\substack{\gamma_i \in h_i, \gamma_j \in h_j, \\ i,j=1,\cdots,n, i<j}} \left\{ \left(1 - \prod_{\substack{i,j=1 \\ i<j}}^{n} (1 - \gamma_i\gamma_j)^{\frac{2}{n(n-1)}} \right)^{\frac{1}{2}} \right\}$$

$$= \bigcup_{\substack{\delta_{i,j} \in \rho_{i,j}, \\ i,j=1,\cdots,n, i\neq j}} \left\{ \left(1 - \prod_{\substack{i,j=1 \\ i\neq j}}^{n} (1 - \delta_{i,j})^{\frac{1}{n(n-1)}} \right)^{\frac{1}{2}} \right\}$$

$$= HFB^{1,1}(h_1, h_2, \cdots, h_n)$$

$\square$

Theorem 3.10. *If $r = n$, then $HFMSM$ operator reduces to the hesitant fuzzy geometric (IFG) operator (i.e., Eq. (2)).*

Proof. By the definition of $HFMSM$, we have

$$HFMSM^{(1)}(h_1, h_2, \cdots, h_n)$$

$$= \bigcup_{\substack{\gamma_i \in h_i, \\ i=1,\cdots,n}} \left\{ \left(1 - \left(\prod_{\substack{1=i_1 < \cdots \\ <i_n = n}} \left(1 - \prod_{j=1}^{n} \gamma_{i_j} \right) \right)^{\frac{1}{C_n^n}} \right)^{\frac{1}{n}} \right\}$$

$$= \bigcup_{\substack{\gamma_i \in h_i, \\ i=1,\cdots,n}} \left\{ \left(1 - \left(1 - \prod_{j=1}^{n} \gamma_{i_j} \right)^{\frac{1}{1}} \right)^{\frac{1}{n}} \right\} (let\ i_j = j)$$

$$= \bigcup_{\substack{\gamma_j \in h_j, \\ j=1,\cdots,n}} \left\{ \left(\prod_{j=1}^{n} \gamma_j \right)^{\frac{1}{n}} \right\}$$

$$= HFG(h_1, h_2, \cdots, h_n)$$

$\square$

Theorem 3.8-3.10 show that some exiting hesitant fuzzy aggregation operators are the special cases of the $HFMSM$ oprator.

4   The weighted hesitant fuzzy operator and its application in decision making

In many practical applications, the weights of attributes should be taken into account. Especially for multiple attribute decision making problems, the considered attributes usually are of different importance. To overcome the limitations of the $HFMSM$ operator defined in the previous section, in this section, we shall introduce the weighted hesitant fuzzy Maclaurin symmetric mean ($WHFMSM$) operator and apply it to solve multiple attribute decision making problems.

4.1 $WHFMSM$ operator

we first introduce the definition of $WHFMSM$ operator as follows.

Definition 4.1. *Let $h_i(i = 1, 2, \cdots, n)$ be a collection of $HFEs$, $r = 1, 2, \cdots, n$, $w = (w_1, w_2, \cdots, w_n)^T$ is the weight vector of $h_i(i = 1, 2, \cdots, n)$ with $w_i \in [0, 1]$ and $\sum_{i=1}^{n} = 1$. If*

$$WHFMSM_w^{(r)}(h_1, h_2, \cdots, h_n) = \left(\frac{\bigoplus\limits_{\substack{1 \le i_1 < \cdots \\ < i_r \le i_n}} \bigotimes\limits_{j=1}^{r} w_{i_j} h_{i_j}}{C_n^r} \right)^{\frac{1}{r}} \tag{5}$$

then $WHFMSM_w^{(r)}$ is called the weighted hesitant fuzzy Maclaurin symmetric mean, where $(i_1, i_2, \cdots, i_r)$ traversal all the r-tuple combination of $(1, 2, \cdots, n)$, C_n^r is the binomial coefficient.

According to the operations of $HFEs$ described in Section 2, we can derive the following Theorem 4.2.

Theorem 4.2. *Let $h_i(i = 1, 2, \cdots, n)$ be a collection of $HFEs$ and $r = 1, 2, \cdots, n$. Then the aggregated value, by using the $WHFMSM^{(r)}$, is also an HFE, and*

$$WHFMSM_w^{(r)}(h_1, h_2, \cdots, h_n) = \bigcup_{\substack{\gamma_i \in h_i, \\ i=1,\cdots,n}} \left\{ \left(1 - \left(\prod_{\substack{1 \le i_1 < \cdots \\ < i_r \le n}} \left(1 - \prod_{j=1}^{r} \left(1 - (1 - \gamma_{i_j})^{w_{i_j}} \right) \right) \right)^{\frac{1}{C_n^r}} \right)^{\frac{1}{r}} \right\}$$

Proof. The proof is similar to one of Theorem 3.2. □

4.2   An application to multiple attribute decision making

Based on $WHFMSM$ operator, below we develop an approach to multiple attribute decision making under hesitant fuzzy environment.

For a multiple attribute decision making problem, let $Y = \{Y_1, Y_2, \cdots, Y_m\}$ be a discrete set of alternatives, $A = \{A_1, A_2, \cdots, A_n\}$ be a collection of attributes, whose weight vector is $w = (w_1, w_2, \cdots, w_n)^T$, satisfying $w_i \in [0, 1]$ and $\sum_{i=1}^{n} = 1$, where w_i represents the importance degree of the attribute A_i. The decision makers provide several values for the alternative $Y_i(i = 1, 2, \cdots, m)$ under the attribute $A_j(j = 1, 2, \cdots, n)$ with anonymity, these values can be considered as an HFE $h_{ij} = \cup_{\gamma_{ij} \in h_{ij}} \{\gamma_{ij}\}$. All elements $h_{ij}(i = 1, 2, \cdots, m, j = 1, 2, \cdots, n)$ construct a hesitant fuzzy decision matrix the decision matrix $H = (h_{ij})_{m \times n}$.

Then, we use the $WHFMSM$ operator to develop an approach to multiple attribute decision making problems with hesitant fuzzy information, which can be described as follows:

Step1. According to the decision information provided by the decision makers, construct the hesitant fuzzy decision matrix $H = (h_{ij})_{m \times n}$. If there are some cost attributes in decision making problems, then we need to transform the decision matrix $H = (h_{ij})_{m \times n}$ into a normalization matrix $P = (p_{ij})_{m \times n}$, where

$$p_{ij} = \begin{cases} p_{ij}, & \text{for benefit attribute } A_{ij}, \\ p_{ij}^c, & \text{for for cost attribute } A_{ij}. \end{cases}$$

Here $p_{ij} = \cup_{\gamma_{ij} \in p_{ij}} \{\gamma_{ij}\}$, p_{ij}^c is the complement of p_{ij} and $p_{ij}^c = \cup_{\gamma_{ij} \in p_{ij}} \{1 - \gamma_{ij}\}$.

Step2. Utilize the $WHFMSM$ operator

$$p_i = WHFMSM_w^{(r)}(p_{i1}, p_{i2}, \cdots, p_{in})$$

to aggregate all the performance values p_{ij} $(j = 1, 2, \cdots, n)$ of the ith line and get the overall performance value p_i corresponding to the alternative Y_i $(i = 1, 2, \cdots, m)$.

Step3. Calculate the score values $s(p_i)$ of the overall preference value p_i $(i = 1, 2, \cdots, m)$.

Step4. Rank all the alternatives Y_i $(i = 1, 2, \cdots, m)$ according to $s(p_i)$ in descending order, and then select the best one.

4.3   Illustrative example

Let us consider a Management School in a Chinese university, which wants to introduce a teacher (adapted from [24]). There is a panel with five possible alternatives. A set of four factors are considered: $A = \{A_1, A_2, A_3, A_4\}$={morality, research capability, teaching skill, education background}, whose weight vector is $w = (0.3, 0.2, 0.1, 0.4)^T$. The experts evaluate four alternatives $Y_i(i = 1, 2, \cdots, 4)$ in relation to the factors $A = \{A_1, A_2, A_3, A_4\}$. The evaluation information on the four alternatives $Y_i(i = 1, 2, \cdots, 4)$ under the factors $A = \{A_1, A_2, A_3, A_4\}$ are represented by the $HFEs$.

Step1. Construct the hesitant fuzzy decision matrix $H = (h_{ij})_{5\times4}$, which is listed in Table 1. Considering that all the attributes $A_j(j = 1, 2, 3, 4)$ are the benefit type attributes, the performance values of the alternatives $Y_i(j = 1, 2, \cdots, 5)$ do not need normalization.

Step2. Utilize the $WHFMSM$ operator aggregate all the performance values $h_{ij}(j = 1, 2, 3, 4)$ of the ith line and obtain the overall preference value h_i corresponding to the alternative Y_i. Take alternative Y_1 for an example, and let $r = 2$, we have

$$h_1 = WHFDMSM_w^{(r)}(h_{11}, h_{12}, \cdots, h_{14})$$

$$= \bigcup_{\substack{\gamma_i \in h_{1i}, \\ i=1,\cdots,n}} \left\{ \left(1 - \left(\prod_{\substack{1 \le i_1 < \cdots \\ < i_r \le n}}^{r} \left(1 - \prod_{j=1}^{r} \gamma_{i_j}^{w_{i_j}} \right) \right)^{\frac{1}{C_n^r}} \right)^{\frac{1}{r}} \right\}$$

$$= \{0.814019, 0.816764, 0.819298, 0.818635, 0.821303, 0.823767, 0.823095, 0.82569, 0.828086,$$
$$0.820225, 0.822867, 0.825307, 0.824669, 0.827238, 0.829611, 0.828963, 0.831463, 0.833771,$$
$$0.826476, 0.829016, 0.831362, 0.830748, 0.833218, 0.8355, 0.834877, 0.837281, 0.839501\}.$$

As the parameter r changes we can get different results for each alternative, here we will not list them for vast amounts of data.

Step3. Compute the score values $s(h_i)(i = 1, 2, \cdots, 5)$ of $h_i(i = 1, 2, 3, 4)$ by Definition 2.2. The score values for the alternatives are listed in Table 2.

Step4. By ranking $s(h_i)(i = 1, 2, \cdots, 5)$, we can get the priorities of the alternatives $Y_i(i = 1, 2, \cdots, 5)$ as the parameter r changes, which are shown in Table 2.

From Table 2, it can be seen that the ranking results are slightly different when the parameter change, which indicates the parameter can reflect the decision maker's risk preferences. Furthermore, we can find that the score values obtained by the $WHFMSM$ operator become smaller when the parameter r increases for the same aggregation arguments. Therefore, the decision makers can choose a proper value of the parameter r according to their risk preferences in real practical decision making process.

Table 1: Hesitant fuzzy decision making matrix H

	A_1	A_2	A_3	A_4
Y_1	{{0.4, 0.5,0.6}	{0.7}	{0.2}	{0.4}
Y_2	{0.2, 0.5, 0.8}	{(0.5, 0.7}	{0.6, 0.8}	{0.6, 0.7}
Y_3	{0.8, 0.9}	{0.4, 0.6}	{0.3, 0.4}	{0.1, 0.3}
Y_4	{0.3, 0.4, 0.6, 0.8}	{0.4, 0.8}	{0.3, 0.5}	{0.5, 0.6}
Y_5	{0.3, 0.5, 0.7}	{0.2, 0.4}	{0.6, 0.7}	{0.4, 0.6}

Table 2: Score values obtained by the $WHFMSM$ and the rankings of alternatives

	Y_1	Y_2	Y_3	Y_4	Y_5	Rankings
$r=1$	0.156157	0.218865	0.194147	0.188703	0.158	$Y_2 \succ Y_3 \succ Y_4 \succ Y_5 \succ Y_1$
$r=2$	0.14665	0.205581	0.15069	0.175264	0.147543	$Y_2 \succ Y_4 \succ Y_3 \succ Y_5 \succ Y_1$
$r=3$	0.135688	0.198208	0.132396	0.163811	0.140404	$Y_2 \succ Y_4 \succ Y_5 \succ Y_1 \succ Y_3$
$r=4$	0.113003	0.185361	0.116269	0.14538	0.130822	$Y_2 \succ Y_4 \succ Y_5 \succ Y_3 \succ Y_1$

Remark 4.3. To demonstrate the advantages of our method, in the following, we compare our method with the existing methods, such as the $HFWA$ and $HFWG$ operators introduced by Xia and Xu [11], and the weighted hesitant fuzzy Bonferroni mean ($WHFB_w^{p;q}$) and weighted hesitant fuzzy geometric Bonferroni mean ($WHFGB_w^{p;q}$) proposed by [15,16]. The rankings obtained by different aggregation operators are listed in Table 3.

From Table 3, we can see that i) when $r = 1$, the $WHFMSM$ and $HFWA$ operators have the same rankings; ii) when $r = n$, the $WHFMSM$ and $HFWG$ operators have the same rankings; iii) when $r = 2$, the $WHFMSM$, $WHFB_w^{p;q}$ and $WHFGB_w^{p;q}$ operators have the same rankings. It verifies the proposed method is reasonable and validity.

(1) Compare with the $HFWA$ and $HFWG$ operators. Our method can deal with the multiple attribute decision making problems where the attributes are independent, whereas the $HFWA$ and $HFWG$ operators can not do them. In addition, the $WHFMSM$ has an alterable parameter, With the change of the parameter, the proposed operator can be evolved into lots of different aggregation operators, which make decision making more flexible and can meet the needs of different types of decision makers. But the $HFWA$ (or $HFWG$) operator has not alterable parameter, so they can only satisfy the demand of a type of decision makers.

(2) Compare with the $WHFB_w^{p;q}$ and $WHFGB_w^{p;q}$ operators. The main advantage of the proposed method is that it can capture the interrelationship among the multi-input arguments, while the $WHFB_w^{p;q}$ and $WHFGB_w^{p;q}$ operators can only capture the interrelationship between two arguments. That is to say, our method is more general. In addition, the $WHFB_w^{p;q}$ and $WHFGB_w^{p;q}$ operators consider two parameters, while our method only needs to take one parameter. Therefore, the computational complexity of the $WHFB_w^{p;q}$ and $WHFGB_w^{p;q}$ operators are much higher than our method. Moreover, the $WHFMSM$ has a desirable property that the score values are more smaller when the parameter r increases, which indicates the decision makers can select easily a proper value for the parameter r according to their risk preferences. But the $WHFB_w^{p;q}$ and $WHFGB_w^{p;q}$ operators do not have the property. It follows that they are difficult to determine the values of the parameters p and q to reflect the decision makers' risk preferences in real practical decision making process.

According to the comparisons and analysis above, it is clear that our method is more flexible and robust to aggregate hesitant fuzzy information. Therefore, It is more suitable than the exiting aggregation operators to solve hesitant fuzzy multiple attribute decision making problems in which the attributes are independent.

Table 3: Comparisons with the exiting aggregation operators

Aggregation operator	Rankings	Aggregation operator	Rankings
$WHFMSM_w^{(1)}$	$Y_2 \succ Y_3 \succ Y_4 \succ Y_5 \succ Y_1$	$WHFMSM_w^{(2)}$	$Y_2 \succ Y_4 \succ Y_3 \succ Y_5 \succ Y_1$
$HFWA$	$Y_2 \succ Y_3 \succ Y_4 \succ Y_5 \succ Y_1$	$WHFB_w^{1,1}$	$Y_2 \succ Y_4 \succ Y_3 \succ Y_5 \succ Y_1$
$WHFMSM_w^{(n)}$	$Y_2 \succ Y_4 \succ Y_5 \succ Y_3 \succ Y_1$	$WHFGB_w^{1,1}$	$Y_2 \succ Y_4 \succ Y_3 \succ Y_5 \succ Y_1$
$HFWG$	$Y_2 \succ Y_4 \succ Y_5 \succ Y_3 \succ Y_1$		

5   Conclusions

The MSM is a classical averaging mean operator, which has been widely used in information fusion. However, it can not deal with the hesitant fuzzy information. To fill this gap, in this paper, we have extended the MSM to hesitant fuzzy environment, and defined a hesitant fuzzy Maclaurin symmetric mean. Some desirable properties and special cases have been discussed in detail. Considering the weight vector of the arguments, we have further developed a weighted hesitant fuzzy Maclaurin symmetric mean which can consider the importance of each attribute and the interrelationship among multi-input arguments. We also have proposed a method to solve hesitant fuzzy multiple attribute decision making problems. The illustrative example has shown that the proposed method is not only reasonable and validity but also more suitable to deal with multiple attribute decision making problems in which the attributes are independent under hesitant fuzzy environment.

Acknowledgements

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A NOTE ON THE GENERALIZED q-CHANGHEE NUMBERS OF HIGHER ORDER

EUN-JUNG MOON[1] AND JIN-WOO PARK[2],[*]

ABSTRACT. Recently, Changhee numbers and polynomials are introduced by T. Kim et al in [3]. In this paper, we consider the generalized q-Changhee polynomials and numbers of higher order by using the fermionic p-adic q-integral and give some relations between the generalized q-Changhee numbers of higher order and special numbers.

1. INTRODUCTION

Let d be fixed odd positive integer and let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will respectively denote the ring of p-adic rational integers, the field of p-adic rational numbers and the completions of algebraic closure of $\mathbb{Q}_p$. The p-adic norm is defined $|p|_p = \frac{1}{p}$.

We set

$$X = X_d = \varprojlim_N \mathbb{Z} \big/ dp^N \mathbb{Z}, \ X^* = \bigcup_{\substack{0 < a < dp \\ (a,p)=1}} (a + dp\mathbb{Z}_p),$$

$$a + dp^N \mathbb{Z}_p = \left\{ x \in X | x \equiv a \pmod{dp^N} \right\},$$

where $a \in \mathbb{Z}$ and $0 \leq a < dp^n$.

When one talks of q-extension, q is various considered as an indeterminate, a complex $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume that $|q - 1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for each $x \in \mathbb{Z}_p$. Throughout this paper, we use the notation :

$$[x]_{-q} = \frac{1 - (-q)^x}{1 - (-q)} \text{ and } [x]_q = \frac{1 - q^x}{1 - q}.$$

Hence, $\lim_{q \to 1} [x]_q = x$ for each $x \in \mathbb{Z}_p$.

Let $C(\mathbb{Z}_p)$ be the space of continuous functions on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, the *fermionic p-adic q-integral on $\mathbb{Z}_p$* is defined by T. Kim as follows :

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N \to \infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N - 1} f(x)(-q)^x, \text{ (see [4, 5]).} \quad (1.1)$$

Then, by (1.1), we can get the following well-known integral identity

$$I_{-q}(f_1) + I_{-q}(f) = [2]_q f(0), \quad (1.2)$$

where $f_1(x) = f(x+1)$ (see [1, 4, 5, 6]).

Recently, q-Changhee numbers and polynomials are introduced by Kim et. al. in [9], and have been studied by many mathematicians, and possess many interesting properties (see [3, 7, 9, 10]). In this paper, we consider the generalized q-Changhee polynomials and numbers of higher order by using the fermionic p-adic q-integral and give some relations between the generalized q-Changhee numbers of higher order and special numbers.

2. The generalized q-Daehee numbers attached to χ

Let χ be the Dirichlet character with conductor $d \in \mathbb{N} = \{1, 2, \ldots\}$ with $d \equiv 1 \pmod 2$. Then the *generalized q-Changhee numbers $Ch_{n,\chi,q}$ attached to χ* are defined by the generating function to be

$$\frac{[2]_q}{1 + q^d(1+t)^d} \sum_{a=0}^{d-1} (-1)^a \chi(a) q^a (1+t)^a = \sum_{n=0}^{\infty} Ch_{n,\chi,q} \frac{t^n}{n!}, \qquad (2.1)$$

where $t \in \mathbb{C}_p$ and $|t|_p < p^{-\frac{1}{p-1}}$.

As is well known, the *generalized q-Euler numbers $E_{n,\chi,q}$ attached to χ* are defined by the generating function to be

$$\frac{[2]_q}{1 + q^d e^{dt}} \sum_{a=0}^{d-1} (-1)^a \chi(a) q^a e^{at} = \sum_{n=0}^{\infty} E_{n,\chi,q} \frac{t^n}{n!}, \ (\text{see } [12]).$$

The Stirling numbers of the first kind is given by

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l) x^l \ (x \geq 0),$$

and the Stirling numbers of the second kind is defined by the generating function to be

$$(e^t - 1)^n = n! \sum_{l=n}^{\infty} S_2(l,n) \frac{t^l}{l!}$$

(see [2, 11]).

By replacing t by $e^t - 1$ in (2.1), we can have

$$\sum_{n=0}^{\infty} Ch_{n,\chi,q} \frac{(e^t-1)^n}{n!} = \frac{[2]_q}{1+q^d e^{dt}} \sum_{a=0}^{d-1} (-1)^a \chi(a) q^a e^{at} = \sum_{m=0}^{\infty} E_{m,\chi,q} \frac{t^m}{m!}, \qquad (2.2)$$

and

$$\begin{aligned} \sum_{n=0}^{\infty} Ch_{n,\chi,q} \frac{(e^t-1)^n}{n!} &= \sum_{n=0}^{\infty} \frac{Ch_{n,\chi,q}}{n!} n! \sum_{m=n}^{\infty} S_2(m,n) \frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty} \left(\sum_{n=0}^{m} Ch_{n,\chi,q} S_2(m,n) \right) \frac{t^m}{m!}. \end{aligned} \qquad (2.3)$$

Therefore, by (2.2) and (2.3), we obtain the following theorem.

Theorem 2.1. *For $m \geq 0$, we have*

$$E_{m,\chi,q} = \sum_{n=0}^{m} Ch_{n,\chi,q} S_2(m,n).$$

Now, we define the *generalized q-Changhee polynomials $Ch_{n,\chi,q}(x)$* as follows:

$$\frac{[2]_q}{1+q^d(1+t)^d}\sum_{a=0}^{d-1}(-1)^a\chi(a)q^a(1+t)^{a+x}=\sum_{n=0}^{\infty}Ch_{n,\chi,q}(x)\frac{t^n}{n!}, \qquad (2.4)$$

where $t\in\mathbb{C}_p$ and $|t|_p<p^{-\frac{1}{p-1}}$.

Note that, in the special case, $x=0$, $Ch_{n,\chi,q}(0)=Ch_{n,\chi,q}$ are generalized q-Changhee numbers.

From (1.2), we can derive the following equation.:

$$q^nI_{-q}(f_n)+(-1)^{n-1}I_{-q}(f)=[2]_q\sum_{l=0}^{n-1}(-1)^{n-1-l}q^lf(l), \qquad (2.5)$$

where $f_n(x)=f(x+n)$ and $n\geq 0$.

If taking $f(x)=\chi(x)(1+t)^x$ in (2.5), we can have

$$q^d\int_X\chi(x)(1+t)^{x+d}d\mu_{-q}(x)+\int_X\chi(x)(1+t)^xd\mu_{-q}(x)$$
$$=[2]_q\sum_{a=0}^{d-1}(-1)^a\chi(a)q^a(1+t)^a. \qquad (2.6)$$

By (2.6), we can easily have

$$\int_X\chi(x)(1+t)^xd\mu_{-q}(x)=\frac{[2]_q}{1+q^d(1+t)^d}\sum_{a=0}^{d-1}(-1)^a\chi(a)q^a(1+t)^a$$
$$=\sum_{n=0}^{\infty}Ch_{n,\chi,q}\frac{t^n}{n!}, \qquad (2.7)$$

and

$$\int_X\chi(x)(1+t)^xd\mu_{-q}(x)=\sum_{n=0}^{\infty}\left(\int_X\chi(x)(x)_nd\mu_{-q}(x)\right)\frac{t^n}{n!}. \qquad (2.8)$$

Therefore, by (2.7) and (2.8), we obtain the following theorem.

Theorem 2.2. *For $n\geq 0$, we have*

$$\frac{Ch_{n,\chi,q}}{n!}=\int_X\chi(x)\binom{x}{n}d\mu_{-q}(x).$$

By (2.4), we note that

$$\sum_{n=0}^{\infty}Ch_{n,\chi,q}(x)\frac{t^n}{n!}=\frac{[2]_q}{1+q^d(1+t)^d}\sum_{a=0}^{d-1}(-1)^a\chi(a)q^a(1+t)^{a+x}$$
$$=\left(\sum_{m=0}^{\infty}Ch_{m,\chi,q}\frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty}\binom{x}{l}t^l\right) \qquad (2.9)$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{x}{n-m}\frac{Ch_{m,\chi,q}}{m!}t^n.$$

So, by (2.9), we can have

$$\frac{Ch_{n,\chi,q}(x)}{n!} = \sum_{m=0}^{n} \binom{x}{n-m} \frac{Ch_{m,\chi,q}}{m!}. \tag{2.10}$$

From Theorem 2.2 and (2.10), we can derive the equations

$$\frac{Ch_{n,\chi,q}(x)}{n!} = \sum_{m=0}^{n} \binom{x}{n-m} \frac{1}{m!} \int_X \chi(y) \binom{y}{m} d\mu_{-q}(y)$$
$$= \int_X \chi(y) \binom{x+y}{n} d\mu_{-q}(y). \tag{2.11}$$

Therefore, by (2.11), we obtain the following corollary.

Corollary 2.3. *For $n \geq 0$, we have*

$$\frac{Ch_{n,\chi,q}(x)}{n!} = \int_X \chi(y) \binom{x+y}{n} d\mu_{-q}(y).$$

For $r \in \mathbb{N}$, let us consider the *generalized q-Changhee numbers of order r attached to χ* as follows:

$$\left(\sum_{a=0}^{d-1} \frac{[2]_q}{1+q^d(1+t)^d} (-1)^a \chi(a) q^a (1+t)^a \right)^r$$
$$= \sum_{a_1,\dots,a_r=0}^{d-1} \left(\frac{[2]_q}{1+q^d(1+t)^d} \right)^r (-1)^{a_1+\cdots+a_r} \chi(a_1)\cdots\chi(a_r) q^{a_1+\cdots+a_r} (1+t)^{a_1+\cdots+a_r}$$
$$= \sum_{n=0}^{\infty} Ch_{n,\chi,q}^{(r)} \frac{t^n}{n!}. \tag{2.12}$$

By (2.7), we can see that

$$\int_X \cdots \int_X \chi(x_1)\cdots\chi(x_r)(1+t)^{x_1+\cdots+x_r} d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r)$$
$$= \sum_{a_1,\dots,a_r=0}^{d-1} \left(\frac{[2]_q}{1+q^d(1+t)^d} \right)^r (-1)^{a_1+\cdots+a_r} \chi(a_1)\cdots\chi(a_r) q^{a_1+\cdots+a_r} (1+t)^{a_1+\cdots+a_r}. \tag{2.13}$$

Thus, by (2.12) and (2.13), we get

$$Ch_{n,\chi,q}^{(r)} = \int_X \cdots \int_X \chi(x_1)\cdots\chi(x_r)(x_1+\cdots+x_r)_n d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r). \tag{2.14}$$

From (2.14) and Theorem 2.2, we can drive

$$\frac{Ch_{n,\chi,q}^{(r)}}{n!}$$

$$= \int_X \cdots \int_X \chi(x_1) \cdots \chi(x_r) \binom{x_1 + \cdots + x_r}{n} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r)$$

$$= \int_X \cdots \int_X \chi(x_1) \cdots \chi(x_r) \sum_{l_1=0}^{n} \binom{x_1}{l_1} \sum_{l_2=0}^{n-l_1} \binom{x_2}{l_2} \cdots \sum_{l_{r-1}=0}^{n-l_1 \cdots -l_{r-2}} \binom{x_{r-1}}{l_{r-1}} \quad (2.15)$$

$$\times \binom{x_r}{n - l_1 - \cdots - l_{r-1}} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r)$$

$$= \sum_{l_1=0}^{n} \sum_{l_2=0}^{n-l_1} \cdots \sum_{l_{r-1}=0}^{n-l_1 \cdots -l_{r-2}} \frac{Ch_{l_1,\chi,q} Ch_{l_2,\chi,q} \cdots Ch_{l_{r-1},\chi,q} Ch_{n-l_1-\cdots-l_{r-1},\chi,q}}{l_1! l_2! \cdots l_{r-1}! (n - l_1 - l_2 - \cdots - l_{r-1})!}.$$

Therefore, by (2.13), (2.14) and (2.15), we obtain the following theorem.

Theorem 2.4. *For $n \geq 0$, we have*

$$Ch_{n,\chi,q}^{(r)} = \sum_{l_1=0}^{n} \sum_{l_2=0}^{n-l_1} \cdots \sum_{l_{r-1}=0}^{n-l_1 \cdots -l_{r-2}} \binom{n}{l_1, l_2, \cdots, l_{r-1}, n - l_1 - \cdots - l_{r-1}}$$

$$\times Ch_{l_1,\chi,q} Ch_{l_2,\chi,q} \cdots Ch_{l_{r-1},\chi,q} Ch_{n-l_1-\cdots-l_{r-1},\chi,q}$$

where $\binom{n}{l_1, l_2, \cdots, l_r} = \frac{n!}{l_1! l_2! \cdots l_r!}$.

From (2.14), we note that

$$Ch_{n,\chi,q}^{(r)}$$

$$= \int_X \cdots \int_X \chi(x_1) \cdots \chi(x_r)(x_1 + \cdots + x_r)_n d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r)$$

$$= \sum_{l=0}^{n} S_1(n,l) \int_X \cdots \int_X \chi(x_1) \cdots \chi(x_r)(x_1 + \cdots + x_r)^l d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r)$$

$$= \sum_{l=0}^{n} S_1(n,l) E_{l,\chi,q}^{(r)},$$

$$(2.16)$$

where $E_{l,\chi,q}^{(r)}$ are the l-th generalized q-Euler numbers of order r attached to χ, which given by

$$\left(\frac{[2]_q}{1 + q^d e^{dt}} \sum_{a=0}^{d-1} (-1)^a \chi(a) q^a e^{at} \right)^r = \sum_{n=0}^{\infty} E_{n,\chi,q}^{(r)} \frac{t^n}{n!}, \text{ (see [8])}.$$

Therefore, by (2.16), we obtain the following theorem.

Theorem 2.5. *For $n \geq 0$, we have*

$$Ch_{n,\chi,q}^{(r)} = \sum_{l=0}^{n} S_1(n,l) E_{l,\chi,q}^{(r)}.$$

By replacing t by $e^t - 1$ in (2.12), we can get

$$
\sum_{n=0}^{\infty} Ch_{n,\chi,q}^{(r)} \frac{(e^t-1)^n}{n!}
$$

$$
= \sum_{a_1,\ldots,a_r=0}^{d-1} \left(\frac{[2]_q}{1+q^d e^{dt}}\right)^r (-1)^{a_1+\cdots+a_r} \chi(a_1)\cdots\chi(a_r) q^{a_1+\cdots+a_r} e^{(a_1+\cdots+a_r)t} \quad (2.17)
$$

$$
= \sum_{m=0}^{\infty} E_{m,\chi,q}^{(r)} \frac{t^m}{m!},
$$

and

$$
\sum_{n=0}^{\infty} Ch_{n,\chi,q}^{(r)} \frac{(e^t-1)^n}{n!} = \sum_{n=0}^{\infty} \frac{Ch_{n,\chi,q}^{(r)}}{n!} n! \sum_{m=n}^{\infty} S_2(m,n) \frac{t^m}{m!}
$$
$$
= \sum_{m=0}^{\infty} \left(\sum_{n=0}^{m} Ch_{n,\chi,q}^{(r)} S_2(m,n)\right) \frac{t^m}{m!}. \quad (2.18)
$$

Therefore, by (2.17) and (2.18), we obtain the following theorem.

Theorem 2.6. *For $n \geq 0$, we have*

$$
E_{m,\chi,q}^{(r)} = \sum_{n=0}^{m} Ch_{n,\chi,q}^{(r)} S_2(m,n).
$$

From (2.12), we can consider the *generalized q-Changhee polynomials of order r attached to χ* as follows:

$$
\left(\sum_{a=0}^{d-1} \frac{[2]_q}{1+q^d(1+t)^d} (-1)^a \chi(a) q^a (1+t)^a\right)^r (1+t)^x
$$

$$
= \sum_{a_1,\ldots,a_r=0}^{d-1} \left(\frac{[2]_q}{1+q^d(1+t)^d}\right)^r (-1)^{a_1+\cdots+a_r} \chi(a_1)\cdots\chi(a_r) q^{a_1+\cdots+a_r} (1+t)^{a_1+\cdots+a_r+x}
$$

$$
= \sum_{n=0}^{\infty} Ch_{n,\chi,q}^{(r)}(x) \frac{t^n}{n!}.
$$

$$(2.19)$$

and

$$
\int_X \cdots \int_X \chi(x_1)\cdots\chi(x_r)(1+t)^{x_1+\cdots+x_r+x} d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r)
$$

$$
= \sum_{a_1,\ldots,a_r=0}^{d-1} \left(\frac{[2]_q}{1+q^d(1+t)^d}\right)^r (-1)^{a_1+\cdots+a_r} \chi(a_1)\cdots\chi(a_r) q^{a_1+\cdots+a_r} (1+t)^{a_1+\cdots+a_r+x}.
$$

Thus, we get

$$
Ch_{n,\chi,q}^{(r)}(x) = \int_X \cdots \int_X \chi(x_1)\cdots\chi(x_r)(x_1+\cdots+x_r+x)_n d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r).
$$

$$(2.20)$$

From (2.20), we have

$$
\begin{aligned}
&Ch_{n,\chi,q}^{(r)}(x) \\
&= \int_X \cdots \int_X \chi(x_1) \cdots \chi(x_r)(x_1 + \cdots + x_r + x)_n d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r) \\
&= \sum_{l=0}^{n} S_1(n,l) \int_X \cdots \int_X \chi(x_1) \cdots \chi(x_r)(x_1 + \cdots + x_r + x)^l d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r) \\
&= \sum_{l=0}^{n} S_1(n,l) E_{l,\chi,q}^{(r)}(x).
\end{aligned}
$$

$$(2.21)$$

Therefore, by (2.21), we obtain the following theorem.

Theorem 2.7. *For $n \geq 0$, we have*

$$
Ch_{n,\chi,q}^{(r)}(x) = \sum_{l=0}^{n} S_1(n,l) E_{l,\chi,q}^{(r)}(x).
$$

In (2.19), by replacing t by $e^t - 1$, we can get

$$
\begin{aligned}
\sum_{n=0}^{\infty} Ch_{n,\chi,q}^{(r)}(x) \frac{(e^t - 1)^n}{n!} &= \left(\sum_{a=0}^{d-1} \frac{[2]_q}{1 + q^d e^{dt}} (-1)^a \chi(a) q^a e^{at} \right)^r e^{xt} \\
&= \sum_{m=0}^{\infty} E_{m,\chi,q}^{(r)}(x) \frac{t^m}{m!}.
\end{aligned}
$$

$$(2.22)$$

and

$$
\sum_{n=0}^{\infty} Ch_{n,\chi,q}^{(r)}(x) \frac{(e^t - 1)^n}{n!} = \sum_{m=0}^{\infty} \left(\sum_{n=0}^{m} Ch_{n,\chi,q}^{(r)}(x) S_2(m,n) \right) \frac{t^m}{m!}. \qquad (2.23)
$$

Therefore, by (2.22) and (2.23), we obtain the following theorem.

Theorem 2.8. *For $n \geq 0$, we have*

$$
E_{m,\chi,q}^{(r)}(x) = \sum_{n=0}^{m} Ch_{n,\chi,q}^{(r)}(x) S_2(m,n).
$$

As is well-known, the rising factorial is given by

$$
(x)^{(n)} = x(x+1) \cdots (x+n-1) = (-1)^n (-x)_n = \sum_{l=0}^{n} (-1)^{n-l} S_1(n,l) x^l, \quad (2.24)
$$

where $n \geq 0$ (see [2, 11]).

Next, we consider the *generalized q-Changhee numbers of order r attached to χ of the second kind* as follows:

$$\widehat{Ch}_{n,\chi,q}^{(r)}$$

$$= \int_X \cdots \int_X \chi(x_1)\cdots\chi(x_r)(-x_1-\cdots-x_r)_n d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r)$$

$$= \sum_{l=0}^{n}(-1)^l S_1(n,l) \int_X \cdots \int_X \chi(x_1)\cdots\chi(x_r)(x_1+\cdots+x_r)^l d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r)$$

$$= \sum_{l=0}^{n}(-1)^l S_1(n,l) E_{l,\chi,q}^{(r)}.$$

The generating function of $\widehat{Ch}_{n,\chi,q}^{(r)}$ is given by

$$\sum_{n=0}^{\infty} \widehat{Ch}_{n,\chi,q}^{(r)} \frac{t^n}{n!}$$

$$= \int_X \cdots \int_X \chi(x_1)\cdots\chi(x_r)(1+t)_n^{-x_1-\cdots-x_r} d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r) \qquad (2.25)$$

$$= \left(\sum_{a=0}^{d-1} \frac{[2]_q}{1+q^d(1+t)^d}(-1)^a \chi(a) q^a (1+t)^a \right)^r (1+t)^r.$$

Now, we can observe that

$$\left(\sum_{a=0}^{d-1} \frac{[2]_q}{1+q^d(1+t)^d}(-1)^a \chi(a) q^a (1+t)^a \right)^r (1+t)^r$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{r}{m} Ch_{n-m,\chi,q}^{(r)} \frac{n!}{(n-m)!} \right) \frac{t^n}{n!}. \qquad (2.26)$$

Thus, by (2.25) and (2.26), we get

$$\widehat{Ch}_{n,\chi,q}^{(r)} = \sum_{m=0}^{n} m! \binom{r}{m}\binom{n}{m} Ch_{n-m,\chi,q}^{(r)}. \qquad (2.27)$$

Therefore, by (2.27), we obtain the following theorem.

Theorem 2.9. *For $n \geq 0$, we have*

$$\widehat{Ch}_{n,\chi,q}^{(r)} = \sum_{m=0}^{n} m! \binom{r}{m}\binom{n}{m} Ch_{n-m,\chi,q}^{(r)}.$$

In (2.25), by replacing t by $e^t - 1$, we can get

$$\sum_{n=0}^{\infty} \widehat{Ch}_{n,\chi,q}^{(r)} \frac{(e^t-1)^n}{n!} = \left(\sum_{a=0}^{d-1} \frac{[2]_q}{1+q^d e^{dt}}(-1)^a \chi(a) q^a e^{at} \right)^r e^{rt}$$

$$= \sum_{m=0}^{\infty} E_{m,\chi,q}^{(r)}(r) \frac{t^m}{m!}. \qquad (2.28)$$

and

$$\sum_{n=0}^{\infty} \widehat{Ch}_{n,\chi,q}^{(r)} \frac{(e^t-1)^n}{n!} = \sum_{m=0}^{\infty} \left(\sum_{n=0}^{m} \widehat{Ch}_{n,\chi,q}^{(r)} S_2(m,n) \right) \frac{t^m}{m!}. \qquad (2.29)$$

Therefore, by (2.28) and (2.29), we obtain the following theorem.

Theorem 2.10. *For $n \geq 0$, we have*

$$E_{m,\chi,q}^{(r)}(r) = \sum_{n=0}^{m} \widehat{Ch}_{n,\chi,q}^{(r)} S_2(m,n).$$

Now, we define the *generalized q-Changhee polynomials of order r attached to χ* of the second kind as follows:

$$\widehat{Ch}_{n,\chi,q}^{(r)}(x) = \int_X \cdots \int_X \chi(x_1) \cdots \chi(x_r)(-x_1 - \cdots - x_r + x)_n d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r). \tag{2.30}$$

Thus, by (2.30), we get

$$\sum_{n=0}^{\infty} \widehat{Ch}_{n,\chi,q}^{(r)}(x) \frac{t^n}{n!}$$

$$= \int_X \cdots \int_X \chi(x_1) \cdots \chi(x_r)(1+t)_n^{-x_1 - \cdots - x_r + x} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r) \tag{2.31}$$

$$= \left(\sum_{a=0}^{d-1} \frac{[2]_q}{1 + q^d(1+t)^d} (-1)^a \chi(a) q^a (1+t)^a \right)^r (1+t)^{x+r}.$$

It is easy to show that

$$\left(\sum_{a=0}^{d-1} \frac{[2]_q}{1 + q^d(1+t)^d} (-1)^a \chi(a) q^a (1+t)^a \right)^r (1+t)^{x+r}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} m! \binom{x}{m} \binom{n}{m} Ch_{n-m,\chi,q}^{(r)} \right) \frac{t^n}{n!}. \tag{2.32}$$

Therefore, by (2.31) and (2.32), we obtain the following theorem.

Theorem 2.11. *For $n \geq 0$, we have*

$$\widehat{Ch}_{n,\chi,q}^{(r)}(x) = \sum_{m=0}^{n} m! \binom{x}{m} \binom{n}{m} Ch_{n-m,\chi,q}^{(r)}.$$

By (2.30), we get

$$\widehat{Ch}_{n,\chi,q}^{(r)}(x)$$

$$= \sum_{l=0}^{n} (-1)^l S_1(n,l) \int_X \cdots \int_X \chi(x_1) \cdots \chi(x_r)(x_1 + \cdots + x_r - x)^l d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r)$$

$$= \sum_{l=0}^{n} (-1)^l S_1(n,l) E_{l,\chi,q}^{(r)}(-x).$$

In (2.32), by replacing t by $e^t - 1$, we can get

$$\sum_{n=0}^{\infty} \widehat{Ch}_{n,\chi,q}^{(r)}(x) \frac{(e^t-1)^n}{n!} = \left(\sum_{a=0}^{d-1} \frac{[2]_q}{1 + q^d e^{dt}} (-1)^a \chi(a) q^a e^{at} \right)^r e^{(x+r)t}$$

$$= \sum_{m=0}^{\infty} E_{m,\chi,q}^{(r)}(x+r) \frac{t^m}{m!}. \tag{2.33}$$

and

$$\sum_{n=0}^{\infty} \widehat{Ch}_{n,\chi,q}^{(r)}(x) \frac{(e^t-1)^n}{n!} = \sum_{m=0}^{\infty} \left(\sum_{n=0}^{m} \widehat{Ch}_{n,\chi,q}^{(r)}(x) S_2(m,n) \right) \frac{t^m}{m!}. \qquad (2.34)$$

Therefore, by (2.33) and (2.34), we obtain the following theorem.

Theorem 2.12. *For $n \geq 0$, we have*

$$E_{m,\chi,q}^{(r)}(x+r) = \sum_{n=0}^{m} \widehat{Ch}_{n,\chi,q}^{(r)}(x) S_2(m,n).$$

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[1] Department of Mathematics, Kyungpook National University, Taegu 702-701, Republic of Korea.
   E-mail address: mej0917@naver.com

[2] Department of Mathematics Education, Daegu University, Jillyang, Gyeongsan, Gyeongbuk 712-714, Republic of Korea.
   E-mail address: a0417001@knu.ac.kr

An Investigation of the Certain Class of Multivalent Harmonic Mappings

H. Esra Özkan Uçar, Yaşar Polatoğlu and Melike Aydoğan

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The main purpose of the present paper is to investigate some properties of the certain class of sense-preserving p-valent harmonic mappings in the open unit disc $\mathbb{D} = \{z \in \mathbb{C} \,||z| < 1\}$.

1 Introduction

Let Ω_1 be the family of functions $\varphi(z)$ which are analytic in the open unit disc $\mathbb{D}$, and satisfying the condition $|\varphi(z)| < 1$ for all $z \in \mathbb{D}$, and let Ω_2 be the family of functions $\phi(z)$ which are regular in $\mathbb{D}$ and satisfying the conditions $\phi(0) = 0$ and $|\phi(z)| < 1$ for every $z \in \mathbb{D}$. Denote by $\mathcal{P}(p, n)$, $p \geq 1$, $n \geq 1$ the family of functions $p(z) = p + p_1 z + \cdots$ which are regular in $\mathbb{D}$ and satisfying the condition $\mathrm{Re}\, p(z) > 0$. Let $s_1(z) = z + d_2 z^2 + \cdots$ and $s_2(z) = z + e_2 z^2 + \cdots$ be analytic functions in $\mathbb{D}$. If there exists $\phi(z) \in \Omega_2$ such that $s_1(z) = s_2(\phi(z))$ for every $z \in \mathbb{D}$, then we say that $s_1(z)$ is subordinate to $s_2(z)$ and we write $s_1 \prec s_2$. Specially, if $s_2(z)$ is univalent in $\mathbb{D}$, then $s_1 \prec s_2$ if and only if $s_1(\mathbb{D}) \subset s_2(\mathbb{D})$, and $s_1(0) = s_2(0)$ implies $s_1(\mathbb{D}_r) \subset s_2(\mathbb{D}_r)$, where $\mathbb{D}_r = \{z \,||z| < r, 0 < r < 1\}$ (see [1], [4]).

We denote by $\mathcal{S}(p, n)$ ($p \geq 1$ and $n \geq 1$, integers) the class of all regular and p-valent functions in $\mathbb{D}$, having the series expansion of the form

$$s(z) = z^p + c_{np+1} z^{np+1} + c_{np+2} z^{np+2} + c_{np+3} z^{np+3} + \cdots + c_{np+m} z^{np+m} + \cdots \quad (1)$$

for all $z \in \mathbb{D}$. It is clear that $\mathcal{S}(p, 1) \supset \mathcal{S}(p, 2) \supset \mathcal{S}(p, 3) \supset \cdots \mathcal{S}(p, m) \supset \cdots$. Let $\mathcal{S}^*(p, n)$ ($p \geq 1$ and $n \geq 1$ integers) denote the class of functions of the form (1) which are regular in $\mathbb{D}$ and satisfying

$$\mathrm{Re}\left(z \frac{s'(z)}{s(z)}\right) > 0 \quad (2)$$

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and

$$\int_0^{2\pi} \text{Re}\left(z\frac{s'(z)}{s(z)} \right) d\theta = 2pn\pi \tag{3}$$

for every $z \in \mathbb{D}$. A member of $\mathcal{S}^*(p,n)$ is called p-valent starlike function in the unit disc $\mathbb{D}$.

Finally, a planar harmonic mapping in the open unit disc $\mathbb{D}$ is a complex-valued harmonic function f, which maps $\mathbb{D}$ onto the some planar domain $f(\mathbb{D})$. Since $\mathbb{D}$ is a simply connected domain, the mapping f has a canonical decomposition $f = h + \overline{g}$, where $h(z)$ and $g(z)$ are analytic in $\mathbb{D}$ and have the following power series expansion

$$h(z) = z^p + a_{np+1}z^{np+1} + a_{np+2}z^{np+2} + \cdots + a_{np+m}z^{np+m} + \cdots$$

and

$$g(z) = b_{np}z^{np} + b_{np+1}z^{np+1} + b_{np+2}z^{np+2} + \cdots + b_{np+m}z^{np+m} + \cdots$$

where $|b_{np}| < 1$, $p \geq 1$ and $n \geq 1$ integers, $a_{np+m}, b_{np+m} \in \mathbb{C}$ and every $z \in \mathbb{D}$. As usual, we call $h(z)$ the analytic part and $g(z)$ the co-analytic part of f, respectively, and let the class of such harmonic mappings is denoted by $\mathcal{SH}(p,n)$. Lewy (see [2]) proved in 1936 that the harmonic mapping f is locally univalent in $\mathbb{D}$ if and only if its Jacobian $J_f = |h'(z)|^2 - |g'(z)|^2$ is strictly positive in $\mathbb{D}$. In view of this result, locally univalent harmonic mappings in the open unit disc are either sense-reversing if $|g'(z)| > |h'(z)|$ or sense-preserving if $|g'(z)| < |h'(z)|$ in $\mathbb{D}$. Throughout this paper, we restrict ourselves to the study of sense-preserving harmonic mappings. We also note that an elegant and complete treatment theory of the harmonic mapping is given Duren's monograph (see [2]).

The main aim of this paper is to investigate the some properties of the following class

$$\mathcal{S}^*\mathcal{H}(p,n) = \left\{ f = h + \overline{g} \in \mathcal{SH}(p,n) | w(z) = \frac{g'(z)}{h'(z)} \prec b_{np}\frac{1+\phi(z)}{1-\phi(z)}, \right.$$

$$\left. \phi(z) = z^n\psi(z), \ \psi(z) \in \Omega_1, \ h(z) \in \mathcal{S}^*(p,n), \ z \in \mathbb{D} \right\}$$

and for this aim we need the following lemma

Lemma 1.1 ([3]) Let $w(z) = a_n z^n + a_{n+1}z^{n+1} + a_{n+1}z^{n+2} + \cdots$ $(a_n \neq 0, \ n \geq 1)$ be analytic in $\mathbb{D}$. If the maximum value of $|w(z)|$ on the circle $|z| = r < 1$ is attained at $z = z_0$, then we have $z_0 w'(z_0) = pw(z_0)$ where $p \geq n$ and every $z \in \mathbb{D}$.

2 Main Results

Lemma 2.1 If $p(z) \in \mathcal{P}(p,n)$ then

$$p(z) = p\frac{1+z^n\psi(z)}{1-z^n\psi(z)}, \ z \in \mathbb{D} \tag{4}$$

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where $\psi(z) \in \Omega_1$.

Proof. Consider the function $H(z)$ such that

$$H(z) = \frac{p(z)}{p}, \ z \in \mathbb{D} \tag{5}$$

where $p(z) \in \mathcal{P}(p, n)$. So, that $H(z)$ is regular and satisfies the conditions $\text{Re}H(z) > 0$ and $H(0) = 1$ in $\mathbb{D}$. Let $\varphi(z) = (1 + H(z))/(1 - H(z))$, then $\varphi(z)$ is regular and $|\varphi(z)| < 1$ in the unit disc $\mathbb{D}$, and also $\varphi(z)$ has n^{th} order zero at the origin. Hence, $\varphi(z) = z^n \psi(z)$ where $\psi(z)$ in Ω_1 for all $z \in \mathbb{D}$. Expressing $H(z)$ in terms of $\varphi(z)$ we have

$$H(z) = \frac{1 + \varphi(z)}{1 - \varphi(z)}, \ z \in \mathbb{D}. \tag{6}$$

Thus,

$$H(z) = \frac{p(z)}{p} = \frac{1 + \varphi(z)}{1 - \varphi(z)} = \frac{1 + z^n \psi(z)}{1 - z^n \psi(z)}$$

or

$$p(z) = p\frac{1 + z^n \psi(z)}{1 - z^n \psi(z)}$$

for all $z \in \mathbb{D}$.

Lemma 2.2 *Let $f = h + \overline{g}$ be an element of $\mathcal{S}^*\mathcal{H}(p, n)$, then*

$$\left| w(z) - \frac{b_{np}(1 - r^{2m})}{1 - |b_{np}|^2 r^{2m}} \right| \leq \frac{(1 - |b_{np}|^2)r^m}{1 - |b_{np}|^2 r^{2m}}, \ |z| = r < 1 \tag{7}$$

where $m = np - p + 1$.

Proof. Since $f = h + \overline{g} \in \mathcal{S}^*\mathcal{H}(p, n)$, then

$$w(z) = \frac{g'(z)}{h'(z)} = \frac{(b_{np}z^p + b_{np+1}z^{np+1} + b_{np+2}z^{np+2} + \cdots)'}{(z^p + a_{np+1}z^{np+1} + a_{np+2}z^{np+2} + \cdots)'}$$

$$= \frac{b_{np} + \frac{(np+1)b_{np+1}}{p}z^{np+1-p} + \cdots}{1 + \frac{(np+1)a_{np+1}}{p}z^{np+1-p} + \cdots}$$

so that $w(0) = b_{np}$. On the other hand, because of the sense-preserving property we have that $|w(z)| < 1$ for every $z \in \mathbb{D}$. Thus, the function defined by

$$\phi(z) = \frac{w(z) - w(0)}{1 - \overline{w(0)}w(z)}, \ z \in \mathbb{D}$$

satisfies the conditions of Schwarz Lemma (see [1]). Therefore, we have the following subordination relation

$$w(z) = \frac{b_{np} + \phi(z)}{1 + \overline{b}_{np}\phi(z)} \text{ if and only if } w(z) \prec \frac{b_{np} + z^m}{1 + \overline{b}_{np}z^m} \ z \in \mathbb{D}. \tag{8}$$

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It is easy to see that the linear transformation $\frac{b_{np}+z^m}{1+\overline{b}_{np}z^m}$ maps $|z| = r$ onto the circle with the center $C(r) = \left(\frac{\alpha_1(1-r^{2m})}{1-|b_{np}|^2r^{2m}}, \frac{\alpha_2(1-r^{2m})}{1-|b_{np}|^2r^{2m}} \right)$ and having the radius $\rho(r) = \frac{(1-|b_{np}|^2)r^m}{1-|b_{np}|^2r^{2m}}$, where $\alpha_1 =$Reb_{np} and $\alpha_2 =$Imb_{np}, then we can write

$$\left| w(z) - \frac{b_{np}(1-r^{2m})}{1-|b_{np}|^2r^{2m}} \right| \leq \frac{(1-|b_{np}|^2)r^m}{1-|b_{np}|^2r^{2m}}$$

for all $|z| = r < 1$. As a simple consequence of Lemma 2.2, we give the following corollary.

Corollary 2.3 *If $f = h(z) + \overline{g(z)} \in \mathcal{S}^*\mathcal{H}(p,n)$, then*

$$\frac{|b_{np}| - r^n}{1 - |b_{np}|r^n} \leq |w(z)| \leq \frac{|b_{np}| + r^n}{1 + |b_{np}|r^n},$$

$$\frac{(1-r^n)(1-|b_{np}|)}{1+|b_{np}|r^n} \leq 1 - |w(z)| \leq \frac{(1+r^n)(1-|b_{np}|)}{1+|b_{np}|r^n}$$

$$\frac{(1-r^n)(1+|b_{np}|)}{1-|b_{np}|r^n} \leq 1 + |w(z)| \leq \frac{(1+r^n)(1+|b_{np}|)}{1+|b_{np}|r^n}$$

and

$$\frac{(1-|b_{np}|^2)(1-r^{2n})}{(1+|b_{np}|r^n)^2} \leq 1 - |w(z)|^2 \leq \frac{(1-|b_{np}|^2)(1-r^{2n})}{(1-|b_{np}|r^n)^2},$$

for all $|z| = r < 1$.

Theorem 2.4 *Let $s(z)$ be an element of $\mathcal{S}^*(p,n)$, then the inequalities*

$$\frac{r^p}{(1+r^n)^{2p/n}} \leq |s(z)| \leq \frac{r^p}{(1-r^n)^{2p/n}} \tag{9}$$

and

$$\frac{pr^{p-1}(1-r^n)}{(1+r^n)^{(2p/n)+1}} \leq |s'(z)| \leq \frac{pr^{p-1}(1+r^n)}{(1-r^n)^{(2p/n)+1}} \tag{10}$$

hold for every $|z| = r < 1$.

Proof. Since $f = h(z) + \overline{g(z)} \in \mathcal{S}^*\mathcal{H}(p,n)$ then we have $z\frac{s'(z)}{s(z)} \prec p\frac{1+z^n}{1-z^n}$ for all z in $\mathbb{D}$. Therefore, the inequality $\left| z\frac{s'(z)}{s(z)} \frac{p(1+r^{2n})}{1-r^{2n}} \right| \leq \frac{2pr^n}{1-r^{2n}}$ holds for every $|z| = r < 1$. Thus, we have

$$\frac{p(1-r^n)}{1+r^n} \leq \left| z\frac{s'(z)}{s(z)} \right| \leq \frac{p(1+r^n)}{1-r^n} \tag{11}$$

or

$$\frac{p(1-r^n)}{1+r^n} \leq \text{Re}\left(z\frac{s'(z)}{s(z)} \right) \leq \frac{p(1+r^n)}{1-r^n} \tag{12}$$

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for all $|z| = r < 1$. It is fact that

$$\text{Re}\left(z\frac{s'(z)}{s(z)}\right) = r\frac{\partial}{\partial r}\log|s(z)| \tag{13}$$

true for every $|z| = r < 1$. Considering (12) and (13) together we obtain

$$\frac{p(1-r^n)}{r(1+r^n)} \le \frac{\partial}{\partial r}\log|s(z)| \le \frac{p(1+r^n)}{r(1-r^n)}, \ |z| = r < 1. \tag{14}$$

Integrating (14), we get (9). On the other hand the inequality (11) can be written in the form

$$\frac{p(1-r^n)}{r(1+r^n)}|s(z)| \le |s'(z)| \le \frac{p(1+r^n)}{r(1-r^n)}|s(z)|, |z| = r < 1. \tag{15}$$

Using (9) in (15) we get (10).

Theorem 2.5 *Let $f = h(z) + \overline{g(z)}$ be an element of $\mathcal{S}^*\mathcal{H}(p,n)$, then*

$$\frac{g(z)}{h(z)} = b_{np}\frac{1+\phi(z)}{1-\phi(z)}$$

where $|b_{np}| < 1$, $\phi(z) = z^n\psi(z)$ and $\psi(z) \in \Omega_1$ for every $z \in \mathbb{D}$.

 Proof. Since $f = h(z) + \overline{g(z)} \in \mathcal{S}^*\mathcal{H}(p,n)$, we can write

$$w(\mathbb{D}_r) = \left\{z \in \mathbb{C} : \left|\frac{g'(z)}{h'(z)} - b_{np}\frac{1+r^{2n}}{1-r^{2n}}\right| \le \frac{2|b_{np}|r^n}{1-r^{2n}}, \ |z| = r < 1\right\}. \tag{16}$$

On the other hand, since $h(z)$ is an element of $\mathcal{S}^*(p,n)$, the value of $h(z)/(zh'(z))$ at a point z_1 on the circle $|z| = r$ is

$$\frac{h(z_1)}{z_1 h'(z_1)} = \frac{1}{p}\frac{1-r^n}{1+r^n}. \tag{17}$$

Now, we define the function

$$\frac{g(z)}{h(z)} = \frac{1+\phi(z)}{1-\phi(z)}, \tag{18}$$

where $\phi(z) = z^n\psi(z)$, $\psi(z) \in \Omega_1$ and $z \in \mathbb{D}$, then $\phi(z)$ analytic in $\mathbb{D}$ and $\phi(0) = 0$. We need to show that $|\phi(z)| < 1$ for all $z \in \mathbb{D}$. Assume to the contrary, that there exists a $z_1 \in \mathbb{D}$ such that $|\phi(z_1)| = 1$. If we take the derivative of (18) and after simple calculations we get

$$w(z) = \frac{g'(z)}{h'(z)} = b_{np}\left(\frac{1+\phi(z)}{1-\phi(z)} + \frac{2z\phi'(z)}{(1-\phi(z))^2}\frac{h(z)}{zh'(z)}\right), z \in \mathbb{D}.$$

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Considering (12), (13), (15) and Lemma 1.1 together we obtain that

$$w(z_1) = \frac{g'(z_1)}{h'(z_1)} = b_{np} \left(\frac{1 + \phi(z_1)}{1 - \phi(z_1)} + \frac{2p\phi'(z_1)}{(1 - \phi(z_1))^2} \frac{1}{p} \frac{1 - r^n}{1 + r^n} \right) \notin w(\mathbb{D}_r), |z| = r.$$

But this is a contradiction, therefore, $|\phi(z)| < 1$ for all $z \in \mathbb{D}$. Thus, for a function $f = h(z) + \overline{g(z)}$ in $\mathcal{S}^*\mathcal{H}(p,n)$ we have

$$\frac{g(z)}{h(z)} = b_{np} \frac{1 + \phi(z)}{1 - \phi(z)}, z \in \mathbb{D}.$$

Corollary 2.6 *Let $f = h(z) + \overline{g(z)}$ be an element of $\mathcal{S}^*\mathcal{H}(p,n)$, then*

$$\frac{p|b_{np}|r^{p-1}(1 - r^n)^2}{(1 + r^n)^{\frac{2p}{n}+2}} \leq |g'(z)| \leq \frac{p|b_{np}|r^{p-1}(1 + r^n)^2}{(1 - r^n)^{\frac{2p}{n}+2}}, \tag{19}$$

and

$$\frac{|b_{np}|r^p(1 - r^n)}{(1 + r^n)^{\frac{2p}{n}+1}} \leq |g(z)| \leq \frac{|b_{np}|r^p(1 + r^n)}{(1 - r^n)^{\frac{2p}{n}+1}}, \tag{20}$$

for every $|z| = r < 1$.

Proof. Using the definition of the class $\mathcal{S}^*\mathcal{H}(p,n)$ and Theorem 2.5, we obtain

$$\frac{|b_{np}|(1 - r^n)}{1 + r^n}|h'(z)| \leq |g'(z)| \leq \frac{|b_{np}|(1 + r^n)}{1 - r^n}|h'(z)|$$

and

$$\frac{|b_{np}|(1 - r^n)}{1 + r^n}|h(z)| \leq |g(z)| \leq \frac{|b_{np}|(1 + r^n)}{1 - r^n}|h(z)|$$

for all z in $\mathbb{D}$. If we use Theorem 2.4 in the last inequalities we obtain (19) and (20).

Corollary 2.7 *If $f = h(z) + \overline{g(z)} \in \mathcal{S}^*\mathcal{H}(p,n)$, then*

$$\frac{p^2 r^{2(p-1)}(1 - r^n)^3(1 + |b_{np}|^2)}{(1 + r^n)^{\frac{4p}{n}+1}(1 + |b_{np}|r^n)^2} \leq J_f \leq \frac{p^2 r^{2(p-1)}(1 + r^n)^3(1 - |b_{np}|^2)}{(1 - r^n)^{\frac{4p}{n}+1}(1 - |b_{np}|r^n)^2}, |z| = r < 1.$$

This corollary is a simple consequence of Corollary 2.3, Theorem 2.4 and the following equalities

$$J_f = |h'(z)|^2 - |g'(z)|^2 = |h'(z)|^2(1 - |w(z)|^2), z \in \mathbb{D}.$$

Corollary 2.8 *Let $f = h(z) + \overline{g(z)}$ be an element of $\mathcal{S}^*\mathcal{H}(p,n)$, then*

$$p(1 - |b_{np}|) \int \frac{r^{p-1}(1 - r^n)^2}{(1 + r^n)^{\frac{2p}{n}+1}(1 + |b_{np}|r^n)} dr \leq |f|$$

$$\leq p(1 + |b_{np}|) \int \frac{r^{p-1}(1 + r^n)^2}{(1 - r^n)^{\frac{2p}{n}+1}(1 + |b_{np}|r^n)} dr$$

This corollary is a simple consequence of Corollary 2.3, Theorem 2.4 and the following inequalities

$$|h'(z)|(1 - |w(z)|)|dz| \leq |df| \leq |h'(z)|(1 + |w(z)|)|dz|, z \in \mathbb{D}.$$

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Robust Stabilization Based on Periodic Observers for LDP Systems [*]

Ling-Ling Lv [†], Lei Zhang [‡]

Abstract

In this paper, the problem of robust stabilization based on observers for linear discrete-time periodic (LDP) systems is studied. It is proofed that principle of separating exists in this type of systems. Based on this, periodic controllers and periodic state observers can be builded independently. Utilizing parametric poles assignment algorithm and robust performance index, an algorithm of robust stabilization based on periodic observers is proposed. A numerical example is employed to verify the effectiveness of the presented approaches.

Keywords: Robust stabilization; Periodic observers; Principle of separating; LDP systems.

1   Introduction

The analysis and control of linear discrete periodic (LDP) systems have long been interesting problems in the control fields, because LDP systems, such as cyclostationary process, and multirate digital control which occur in control systems, arise often in nature and in engineering ([1]). Thus, this type of systems have been widely researched (see [2]-[8] and references therein). The lifting technique and the cyclic technique are used to carry out such analysis studies, since they can preserve the system's algebraic structure and norms. Based on their lifted LTI reformulation, structural properties such as observability, reachability, detectability, and stabilizability are analyzed [9].

Periodic linear systems have received renewed interested in recent years. For example, semi-global stabilization of discrete-time periodic systems subject to actuator saturation is investigated in [10] by solutions to a parametric periodic Lyapunov equation, stability and stabilization of discrete-time periodic linear systems with actuator saturation is studied in [11] via periodic invariant set, stabilization of continuous-time periodic linear systems is solved in [12] via a periodic Lyapunov equation based approach, L_∞ and L_2 semi-global stabilization of continuous-time periodic linear systems with bounded controls is studied in [13], and stabilization of periodic systems with input and output delays is investigated in [14]. For more related recent work on the control of periodic systems, interested readers may refer to the references cited in [10, 11, 12] and [13].

In engineering, it is usually required to stabilize an unstable periodic motion or a critically stable periodic motion by using proper control. The stabilization problem has a fundamental importance in engineering, and hence the stabilization of periodic motions of dynamic systems has drawn much attention over the past years (see [11]-[15] and references therein). Observers can extract real-time information of a plant's internal state from its input-output data. Therefore, Observer-based control has been widely investigated (e.g., [16]-[21]).

In this paper, we investigate the problem of robust stabilization for uncertain LDP systems. On the problem of observer based control without robustness considerations, a trivial result has been present at [22]. According to the principle of separating, the problem of stabilization based on observer is converted into problems of stabilizing an argumented system and designing a periodic observer respectively. By adopting parametric poles assignment approach combined with a sensitivity index, robust stabilization problem is solved. Two

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detailed algorithms are presented and an example is utilized to illustrate the design procedures proposed in this paper.

Notation 1 *The superscripts "T" and "-1" stand for matrix transposition and matrix inverse, respectively; $\mathbb{R}^n$ denotes the n-dimensional Euclidean space; $\overline{i,j}$ represents the integer set $\{i, i+1, \ldots, j-1, j\}$; For a square time-varying matrix $A(t), t = 0, 1, \cdots$, we denote $\Phi_A(j, i) = A(j-1)A(j-2)\cdots A(i)$ for $j > i$ and $\Phi_A(i, i) = I$; The notation $\|\cdot\|_F$ is Frobenius norm.*

2 Preliminaries

Consider LDP systems with the following state space representation

$$\begin{cases} x(t+1) = A(t)x(t) + B(t)u(t) \\ y(t) = C(t)x(t) \end{cases} \tag{1}$$

where $t \in \mathbb{Z}$, the set of integers, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^r$ and $y(t) \in \mathbb{R}^m$ are respectively the state vector, the input vector and the output vector, $A(t)$, $B(t)$, $C(t)$ are matrices of compatible dimensions satisfying

$$A(t+T) = A(t), \quad B(t+T) = B(t), C(t+T) = C(t).$$

In case that the state of system (1) can be measured, by periodic feedback control law

$$u(t) = K(t)x(t) + v(t), \quad K(t+T) = K(t), \quad K(t) \in \mathbb{R}^{r\times n} \tag{2}$$

where $v(t)$ is the reference input, we can obtain the following combined system with period T

$$\begin{cases} x(t+1) = (A(t) + B(t)K(t))x(t) + B(t)v(t) \\ y(t) = C(t)x(t) \end{cases} \tag{3}$$

When there exists some restrictions in practice, the state of system (1) can not be gotten by hardware, but the input $u(t)$ and the output $y(t)$ can be measured. In this case, we need build another periodic system giving an asymptotic estimation of system states. The system with the following form can be adopted:

$$\begin{cases} \hat{x}(t+1) = A(t)\hat{x}(t) + B(t)u(t) + L(t)(C(t)\hat{x} - y(t)) \\ \hat{x}(0) = \hat{x}_0 \end{cases} \tag{4}$$

where $\hat{x} \in \mathbb{R}^n$ and $L(t) \in \mathbb{R}^{n\times m}$, $t \in \mathbb{Z}$ are real matrices of period T.

Utilizing observer (4), we can build a periodic control law based on the observed states as

$$u(t) = K(t)\hat{x}(t) + v(t) \tag{5}$$

such that the combined system meets some control aims, e.g., stability.

Similar to its LTI counterpart, for LDP systems, we present a simple existence condition of observers and omit its proof.

Proposition 1 *There exist matrices $L(t)$, $t \in \overline{0, T-1}$ such that system (4) becomes a full order state observer of system (1) if and only if periodic matrix pairs $(A(t), C(t))$ are detectable. In this case, we only need to choose $L(t)$, $t \in \overline{0, T-1}$ such that matrix*

$$\Phi_{A+LC}(T, 0) = (A(T-1) + L(T-1)C(T-1))(A(T-2) + L(T-2)C(T-2))\cdots(A(0) + L(0)C(0))$$

is stable.

Plugging (5) into (4) gives

$$\begin{cases} \hat{x}(t+1) = (A(t) + L(t)C(t))\hat{x}(t) - L(t)y(t) + B(t)u(t) \\ u(t) = K(t)\hat{x}(t) + v(t) \end{cases} \tag{6}$$

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Integrating control law (6) into system (1), we can get

$$
\begin{cases}
\begin{bmatrix} x(t+1) \\ \hat{x}(t+1) \end{bmatrix} = \begin{bmatrix} A(t) & B(t)K(t) \\ -L(t)C(t) & F(t)+B(t)K(t) \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} + \begin{bmatrix} B(t) \\ B(t) \end{bmatrix} v(t) \\
y(t) = \begin{bmatrix} C(t) & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}
\end{cases}
\tag{7}
$$

where $F(t) = A(t) + L(t)C(t)$.

With the preparation, the stabilization problem of system (1) based on observers can be formed as the following:

Problem 1 *Given a completely observable and reachable LDP system (1), find matrices $L(t) \in \mathbf{R}^{n \times m}, t \in \overline{0, T-1}$ and $K(t) \in \mathbf{R}^{r \times n}, t \in \overline{0, T-1}$, such that the augmented system (7) is stable.*

Because of the inaccuracy of modelling and the influence of their internal perturbation and external disturbance from environment, unavoidably, system model has uncertainties, leading to the necessity of the study of robustness for LDP systems. Robust stabilization of system (1) based on observers can be formed as follows:

Problem 2 *Given a completely observable and reachable LDP system (1), find matrices $L(t) \in \mathbf{R}^{n \times m}, t \in \overline{0, T-1}$ and $K(t) \in \mathbf{R}^{r \times n}, t \in \overline{0, T-1}$, such that the augmented system (7) is stable and as insensitive as possible to small changes of system data.*

3 Main result

Consider the following LDP system

$$
\begin{cases}
x(t+1) = \tilde{A}(t)x(t) + \tilde{B}(t)u(t) \\
y(t) = \tilde{C}(t)x(t)
\end{cases}
\tag{8}
$$

where the system data possess the same dimensions with that of system (1).

Lemma 1 *Given two LDP systems (1) and (8). If there exists a nonsingular matrix P satisfying*

$$
\tilde{A}(t) = PA(t)P^{-1}, \quad \tilde{B}(t) = PB(t), \quad \tilde{C}(t) = C(t)P^{-1},
\tag{9}
$$

then the lifted systems of this two systems are equivalent.

Proof. Lifting system (1) gives the following LTI system

$$
\begin{cases}
x^{\mathrm{L}}(t+1) = A^{\mathrm{L}}x^{\mathrm{L}}(t) + B^{\mathrm{L}}u^{\mathrm{L}}(t) \\
y^{\mathrm{L}}(t) = C^{\mathrm{L}}x^{\mathrm{L}}(t)
\end{cases},
\tag{10}
$$

where

$$
A^{\mathrm{L}} = A(T-1)A(T-2)\cdots A(0)
$$

$$
B^{\mathrm{L}} = \begin{bmatrix} A(T-1)A(T-2)\cdots A(1)B(0) & \cdots & A(T-1)B(T-2) & B(T-1) \end{bmatrix}
$$

$$
C^{\mathrm{L}} = \begin{bmatrix} C(0) \\ C(1)A(0) \\ \vdots \\ C(T-1)A(T-2)\cdots A(0) \end{bmatrix}
$$

3

Lifting system (8) gives the following LTI system

$$\begin{cases} x^{\mathrm{L}}(t+1) = \tilde{A}^{\mathrm{L}} x^{\mathrm{L}}(t) + \tilde{B}^{\mathrm{L}} u^{\mathrm{L}}(t) \\ y^{\mathrm{L}}(t) = \tilde{C}^{\mathrm{L}} x^{\mathrm{L}}(t) \end{cases} \tag{11}$$

where

$$\tilde{A}^{\mathrm{L}} = \tilde{A}(T-1)\tilde{A}(T-2)\cdots\tilde{A}(0),$$

$$\tilde{B}^{\mathrm{L}} = \begin{bmatrix} \tilde{A}(T-1)\tilde{A}(T-2)\cdots\tilde{A}(1)\tilde{B}(0) & \cdots & \tilde{A}(T-1)\tilde{B}(T-2) & \tilde{B}_{T-1} \end{bmatrix},$$

$$\tilde{C}^{\mathrm{L}} = \begin{bmatrix} \tilde{C}(0) \\ \tilde{C}(1)\tilde{A}(0) \\ \vdots \\ \tilde{C}(T-1)\tilde{A}(T-2)\cdots\tilde{A}(0) \end{bmatrix}.$$

According to (9), we get

$$\begin{aligned} \tilde{A}^{\mathrm{L}} &= \tilde{A}(T-1)\tilde{A}(T-2)\cdots\tilde{A}(0) \\ &= PA(T-1)P^{-1}PA(T-2)P^{-1}\cdots PA(0)P^{-1} \\ &= PA^{\mathrm{L}}P^{-1}, \end{aligned}$$

$$\begin{aligned} \tilde{B}^{\mathrm{L}} &= \begin{bmatrix} \tilde{A}(T-1)\tilde{A}(T-2)\cdots\tilde{A}(1)\tilde{B}(0) & \cdots \\ & \tilde{A}(T-1)\tilde{B}(T-2) & \tilde{B}(T-1) \end{bmatrix} \\ &= \begin{bmatrix} PA(T-1)A(T-2)\cdots A(1)B(0) & \cdots \\ & PA(T-1)B(T-2) & PB(T-1) \end{bmatrix} \\ &= PB^{\mathrm{L}}, \end{aligned}$$

$$\begin{aligned} \tilde{C}^{\mathrm{L}} &= \begin{bmatrix} \tilde{C}(0) \\ \tilde{C}_1\tilde{A}(0) \\ \vdots \\ \tilde{C}(T-1)\tilde{A}(T-2)\cdots\tilde{A}(0) \end{bmatrix} \\ &= \begin{bmatrix} C(0)P^{-1} \\ C_1 A(0)P^{-1} \\ \vdots \\ C(T-1)A(T-2)\cdots A(0)P^{-1} \end{bmatrix} \\ &= C^{\mathrm{L}}P^{-1}. \end{aligned}$$

Thus, we can see the lifted systems (10) and (11) are algebraically equivalent, which means the equivalence between system (1) and system (8). ∎

By virtue of this conclusion, we can form the following Theorem.

Theorem 1 *Consider systems (3) and (7). The eigenvalue set of system (7) are composed by sets $\sigma(\Phi_{\mathrm{A+BK}}(T,0))$ and $\sigma(\Phi_{\mathrm{F}}(T,0))$ corresponding to systems (3) and (4), respectively.*

Proof. Let

$$P = \begin{bmatrix} I & 0 \\ -I & I \end{bmatrix}.$$

It is easily computed that

$$P^{-1} = \begin{bmatrix} I & 0 \\ I & I \end{bmatrix}.$$

4

Therefore,

$$
P \begin{bmatrix} A(t) & B(t)K(t) \\ -L(t)C(t) & F(t)+B(t)K(t) \end{bmatrix} P^{-1} = \begin{bmatrix} A(t)+B(t)K(t) & B(t)K(t) \\ 0 & F(t) \end{bmatrix},
$$

$$
P \begin{bmatrix} B(t) \\ B(t) \end{bmatrix} = \begin{bmatrix} B(t) \\ 0 \end{bmatrix},
$$

$$
\begin{bmatrix} C(t) & 0 \end{bmatrix} P^{-1} = \begin{bmatrix} C(t) & 0 \end{bmatrix}.
$$

By lemma 1, system (7) and the following system have equivalent lifted systems

$$
\left(\begin{bmatrix} A(t)+B(t)K(t) & B(t)K(t) \\ 0 & F(t) \end{bmatrix}, \begin{bmatrix} B(t) \\ 0 \end{bmatrix}, \begin{bmatrix} C(t) & 0 \end{bmatrix} \right). \tag{12}
$$

Thus, all the eigenvalues of the two lifted systems are the same. Since eigenvalues of LDP systems are defined to be eigenvalues of their lifted system, the proof is completed. ■

We call the above result as principle of separating for LDP systems. It is shown that the introduction of full order state observers has no influence on the stability of the close-loop system by state feedback law (2). At the same time, the introduction of state feedback has no influence on the designed observers. By this theorem, when discussing the problem of stabilizing LDP systems based on observers, periodic control laws and periodic observers can be designed independently. The work remaining is to find matrices $K(t)$ and $L(t)$ such that matrix $\Phi_{A+BK}(T,0)$ and matrix $\Phi_F(T,0)$ are stable respectively. Here, we adopt poles assignment approach.

Let A^L and B^L denote the lifted system matrices corresponding to periodic matrix pair $(A(\cdot), C(\cdot))$, A^{LT} and C^{LT} denote the lifted system matrices corresponding to periodic matrix pair $(A^T(\cdot), C^T(\cdot))$, and matrices F and G are real matrices possessing the desired pole set of matrices $\Phi_{A+BK}(T,0)$ and matrix $\Phi_F(T,0)$ respectively. Introducing the following polynomial matrix factorizations:

$$
(zI - A^L)^{-1} B^L = N(z) D^{-1}(z) \tag{13}
$$

$$
(zI - A^{LT})^{-1} C^{LT} = H(z) L^{-1}(z) \tag{14}
$$

where $N(z) \in \mathbb{R}^{n \times Tr}$, $D(z) \in \mathbb{R}^{Tr \times Tr}$ are right coprime matrix polynomials in z, and $H(z) \in \mathbb{R}^{n \times Tm}$, $L(z) \in \mathbb{R}^{Tm \times Tm}$ are the same. If we denote

$$
D(z) = [d_{ij}(z)]_{Tr \times Tr}, N(z) = [n_{ij}(z)]_{n \times Tr}
$$

$$
H(z) = [h_{ij}(z)]_{Tm \times Tm}, L(z) = [l_{ij}(z)]_{n \times Tm}
$$

and $\alpha = \max\{\alpha_1, \alpha_2\}, \beta = \max\{\beta_1, \beta_2\}$, where

$$
\alpha_1 = \max_{i,j \in \overline{1,Tr}} \{\deg(d_{ij}(z))\}
$$

$$
\alpha_2 = \max_{i \in \overline{1,n}, j=\overline{1,Tr}} \{\deg(n_{ij}(z))\}
$$

$$
\beta_1 = \max_{i,j \in \overline{1,Tm}} \{\deg(h_{ij}(z))\}
$$

$$
\beta_2 = \max_{i \in \overline{1,n}, j=\overline{1,Tm}} \{\deg(l_{ij}(z))\}
$$

then $N(z)$ and $D(z)$ can be rewritten as

$$
\begin{cases} N(z) = \sum\limits_{i=0}^{\alpha} N_i z^i, N_i \in \mathbb{C}^{n \times Tr} \\ D(z) = \sum\limits_{i=0}^{\alpha} D_i z^i, D_i \in \mathbb{C}^{Tr \times Tr} \end{cases} \tag{15}
$$

5

$H(z)$ and $L(z)$ can be rewritten as

$$
\begin{cases}
H(z) = \sum\limits_{i=0}^{\beta} H_i z^i, H_i \in \mathbb{C}^{n \times Tm} \\
L(z) = \sum\limits_{i=0}^{\beta} L_i z^i, L_i \in \mathbb{C}^{Tm \times Tm}
\end{cases}
\tag{16}
$$

Denote

$$
\begin{cases}
V_{\mathrm{K}}(Z_1) = N_0 Z_1 + N_1 Z_1 F + \cdots + N_\alpha Z_1 F^\alpha \\
W_{\mathrm{K}}(Z_1) = D_0 Z_1 + D_1 Z_1 F + \cdots + D_\alpha Z_1 F^\alpha
\end{cases}
\tag{17}
$$

$$
\begin{cases}
V_{\mathrm{L}}(Z_2) = H_0 Z_2 + H_1 Z_2 G + \cdots + H_\beta Z_2 G^\beta \\
W_{\mathrm{L}}(Z_2) = L_0 Z + L_1 Z_2 G + \cdots + L_\beta Z_2 G^\beta
\end{cases}
\tag{18}
$$

and

$$
\mathcal{Z}_1 = \left\{ Z_1 \,\middle|\, \det\left(\sum_{i=0}^{\alpha} N_i Z_1 F^i \right) \neq 0 \right\}
\tag{19}
$$

$$
\mathcal{Z}_2 = \left\{ Z_2 \,\middle|\, \det\left(\sum_{i=0}^{\beta} H_i Z_2 G^i \right) \neq 0 \right\}
\tag{20}
$$

where Z_1 and Z_2 are arbitrary parameter matrices with compatible dimension.

Let

$$
X(Z_1) = W_{\mathrm{K}}(Z_1) V_{\mathrm{K}}^{-1}(Z_1) \triangleq \begin{bmatrix} X_0^{\mathrm{T}} & X_1^{\mathrm{T}} & \cdots & X_{T-1}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}
\tag{21}
$$

$$
Y(Z_2) = W_{\mathrm{L}}(Z_2) V_{\mathrm{L}}^{-1}(Z_2) \triangleq \begin{bmatrix} Y_0^{\mathrm{T}} & Y_1^{\mathrm{T}} & \cdots & Y_{T-1}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}
\tag{22}
$$

where $Z_1 \in \mathcal{Z}_1$ and $Z_2 \in \mathcal{Z}_2$.

According to theorem 1 in this paper and the theorem 1 of literature [23], we have the following conclusion.

Theorem 2 *For given LDP system (1) and stable real constant matrices F, G with compatible dimensions and the desired poles, if $V_{\mathrm{K}}(Z_1)$ and $W_{\mathrm{K}}(Z_1)$ are given by (17), $V_{\mathrm{L}}(Z_2)$ and $W_{\mathrm{L}}(Z_2)$ are given by (18), X_i, $i \in \overline{0, T-1}$ and Y_i, $i \in \overline{0, T-1}$ are given by (21) and (22) respectively, then the whole set of solutions to Problem 1 can be given by (23) and (24).*

$$
\mathcal{K} = \left\{ \begin{pmatrix} K(0) \\ K(1) \\ \vdots \\ K(T-1) \end{pmatrix} \,\middle|\, \begin{cases} X(Z) = W_{\mathrm{K}}(Z_1) V_{\mathrm{K}}^{-1}(Z_1), Z_1 \in \mathcal{Z}_1 \\ K(0) = [X_1]^{\mathrm{T}}, \\ K(t) = \left[X_{t+1} \prod\limits_{j=0}^{i-1} (A(j) + B(j)K(j))^{-1} \right]^{\mathrm{T}}, \ t \in \overline{1, T-1} \end{cases} \right\}
\tag{23}
$$

$$
\mathcal{L} = \left\{ \begin{pmatrix} L(0) \\ L(1) \\ \vdots \\ L(T-1) \end{pmatrix} \,\middle|\, \begin{cases} Y(Z) = W_{\mathrm{L}}(Z_2) V_{\mathrm{L}}^{-1}(Z_2), Z_2 \in \mathcal{Z}_2 \\ L(0) = [Y_1]^{\mathrm{T}}, \\ L(t) = \left[Y_{t+1} \prod\limits_{j=0}^{i-1} \left(A^{\mathrm{T}}(j) + C^{\mathrm{T}}(j)L^{\mathrm{T}}(j) \right)^{-1} \right]^{\mathrm{T}}, \ t \in \overline{1, T-1} \end{cases} \right\}
\tag{24}
$$

Based upon theorem 2, an algorithm for solving problem 1 follows.

Algorithm 1 *(Stabilization of LDP systems)*

1. *Select constant matrices F and G such that all of their poles lie in the unit circle.*

2. *Solve the right coprime polynomial matrices $N(z), D(z)$ satisfying factorization (13) and the right coprime polynomial matrices $H(z), L(z)$ satisfying factorization (14).*

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3. *According to formulae (15), compute matrices $N_i, D_i, i \in \overline{0, \alpha}$; According to formulae (16), compute matrices $H_i, L_i, i \in \overline{0, \beta}$.*

4. *Compute $V_{\mathrm{K}}(Z_1)$ and $W_{\mathrm{K}}(Z_1)$ by formulae (17); Compute $V_{\mathrm{L}}(Z_2)$ and $W_{\mathrm{L}}(Z_2)$ by formulae (18).*

5. *According to formulaes (21) and (23), compute periodic state feedback matrices $K(t), t \in \overline{0, T-1}$; According to formulaes (22) and (24), compute periodic observer gains $L(t), t \in \overline{0, T-1}$.*

Because of the arbitrariness of the choose of parameter matrices Z_1 and Z_2 in the design process, the above parametric design algorithm can provide numerous solutions to problem 1. This makes multi-object design possible for LDP systems. Here, we only consider robustness. According to literature [23], the following robustness performance index can be adopted:

$$J_1(Z_1) \triangleq \kappa_{\mathrm{F}}(V_{\mathrm{K}}) \sum_{t=0}^{T-1} \|A(t) + B(t)K(t)\|_{\mathrm{F}}^{T-1}, \tag{25}$$

$$J_2(Z_2) \triangleq \kappa_{\mathrm{F}}(V_{\mathrm{L}}) \sum_{t=0}^{T-1} \|A(t) + L(t)C(t)\|_{\mathrm{F}}^{T-1}, \tag{26}$$

where $\kappa_{\mathrm{F}}(V_{\mathrm{K}}) \triangleq \|V_{\mathrm{K}}^{-1}\|_{\mathrm{F}} \|V_{\mathrm{K}}\|_{\mathrm{F}}$ and $\kappa_{\mathrm{F}}(V_{\mathrm{L}}) \triangleq \|V_{\mathrm{L}}^{-1}\|_{\mathrm{F}} \|V_{\mathrm{L}}\|_{\mathrm{F}}$ are the Frobenius-norm conditional numbers of matrix V_{K} and matrix V_{L} respectively. Thus, we can summarize the robust stabilization algorithm based on observers as follows.

Algorithm 2 *(Robust stabilization algorithm of LDP systems)*

1. *Select constant matrices F and G such that all of their poles lie in the unit circle.*

2. *Solve the right coprime polynomial matrices $N(z), D(z)$ satisfying factorization (13) and the right coprime polynomial matrices $H(z), L(z)$ satisfying factorization (14).*

3. *According to formulae (15), compute matrices $N_i, D_i, i \in \overline{0, \alpha}$; According to formulae (16), compute matrices $H_i, L_i, i \in \overline{0, \beta}$.*

4. *Construct general expressions for matrices V_{K} and $K(t), t \in \overline{0, T-1}$ according to formulaes (17), (21) and (23), construct general expressions for matrices V_{L} and $L(t), t \in \overline{0, T-1}$ according to formulaes (18), (22) and (24).*

5. *Solving optimization problems*

$$\text{Minimize} J_1(Z_1)$$

and

$$\text{Minimize} J_2(Z_2)$$

by using gradient based searching method. The optimal decision matrix is denoted by Z_1^{opt} and Z_2^{opt} respectively.

6. *Compute matrices $K^{\mathrm{opt}}(t), t \in \overline{0, T-1}$ according to (17), (21) and (23) by using optimal decision matrix Z_1^{opt}; Compute matrices $L^{\mathrm{opt}}(t), t \in \overline{0, T-1}$ according to (18), (22) and (24) by using optimal decision matrix Z_2^{opt}.*

4   Numerical example

Consider LDP system (1) with parameters as follows:

$$A(0) = \begin{bmatrix} -4.5 & -1 \\ 2.5 & 0.5 \end{bmatrix}, \quad A(1) = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix},$$

$$A(2) = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}, \; B(0) = B(1) = B(2) = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$C(0) = \begin{bmatrix} 2 & 1 \end{bmatrix}, \quad C(1) = \begin{bmatrix} -1 & 1 \end{bmatrix},$$

$$C(2) = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

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It is an oscillation system possessing performances of complete reachability and complete observability. In the following, we will design a robust stabilization law for this system.

For convenience, we can choose matrices F and G as

$$F = \begin{bmatrix} -0.3 & 0 \\ 0 & 0.3 \end{bmatrix}, G = \begin{bmatrix} -0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$$

According to algorithm 1, by randomly choosing parameter matrices Z_1 and Z_2, we obtain a group of solutions as follows:

$$K_{\mathrm{rand}}(0) = \begin{bmatrix} 0.7900 & 0.3400 \end{bmatrix},$$
$$K_{\mathrm{rand}}(1) = \begin{bmatrix} 2.0000 & 2.2857 \end{bmatrix},$$
$$K_{\mathrm{rand}}(2) = \begin{bmatrix} -0.6667 & -1.2593 \end{bmatrix},$$

$$L_{\mathrm{rand}}(0) = \begin{bmatrix} 1.6377 \\ -0.8841 \end{bmatrix},$$
$$L_{\mathrm{rand}}(1) = \begin{bmatrix} 0.3871 \\ -0.0645 \end{bmatrix},$$
$$L_{\mathrm{rand}}(2) = \begin{bmatrix} -33.0000 \\ -67.3333 \end{bmatrix}.$$

Applying algorithm 2 gives solutions to problem 2 with the following gains:

$$K_{\mathrm{robu}}(0) = \begin{bmatrix} 1.0448 & 0.0428 \end{bmatrix},$$
$$K_{\mathrm{robu}}(1) = \begin{bmatrix} -0.6217 & -1.3782 \end{bmatrix},$$
$$K_{\mathrm{robu}}(2) = \begin{bmatrix} -0.6217 & -1.6218 \end{bmatrix},$$

$$L_{\mathrm{robu}}(0) = \begin{bmatrix} 2.0876 \\ -0.8805 \end{bmatrix},$$
$$L_{\mathrm{robu}}(1) = \begin{bmatrix} -0.7062 \\ -0.3082 \end{bmatrix},$$
$$L_{\mathrm{robu}}(2) = \begin{bmatrix} -2.4536 \\ -1.3617 \end{bmatrix}.$$

Denote

$$K_{\mathrm{rand}} = (K_{\mathrm{rand}}(0), \ K_{\mathrm{rand}}(1), \ K_{\mathrm{rand}}(2))$$
$$L_{\mathrm{rand}} = (L_{\mathrm{rand}}(0), \ L_{\mathrm{rand}}(1), \ L_{\mathrm{rand}}(2))$$
$$K_{\mathrm{robu}} = (K_{\mathrm{robu}}(0), \ K_{\mathrm{robu}}(1), \ K_{\mathrm{robu}}(2))$$
$$L_{\mathrm{robu}} = (L_{\mathrm{robu}}(0), \ L_{\mathrm{robu}}(1), \ L_{\mathrm{robu}}(2))$$

Choose the sine signal $v(t) = 0.1 * \sin(t + \pi/2)$ as reference input and $x_0 = \begin{bmatrix} -1 & 1 \end{bmatrix}^{\mathrm{T}}, \hat{x}_0 = \begin{bmatrix} 0 & 0 \end{bmatrix}^{\mathrm{T}}$ as the initial states of systems (1) and (4), respectively. We give the state histories of the system (1) in Figure. 1. With $(K_{\mathrm{rand}}, L_{\mathrm{rand}})$, Figure. 2 shows the state $x(t)$ of system (7). From this figure, we can see the good control effectiveness of Algorithm 1 when there is no uncertainty in system data.

To verify the effectiveness of the robust controller, let the system matrices be perturbed as follows:

$$A(t) \mapsto A(t) + \mu\Delta_{at}, t \in \overline{0,2}$$
$$B(t) \mapsto B(t) + \mu\Delta_{bt}, t \in \overline{0,2}$$
$$C(t) \mapsto C(t) + \mu\Delta_{ct}, \ t \in \overline{0,2}$$
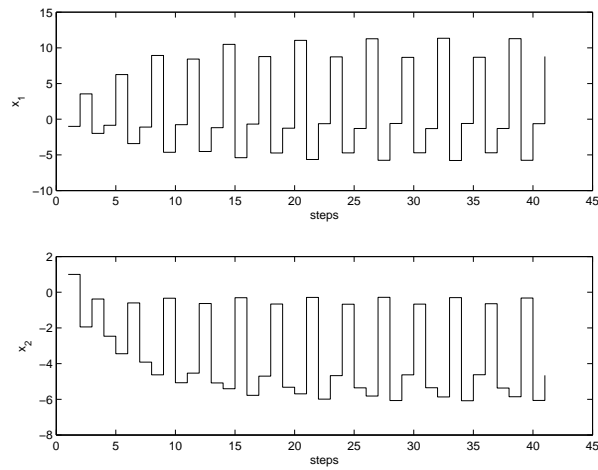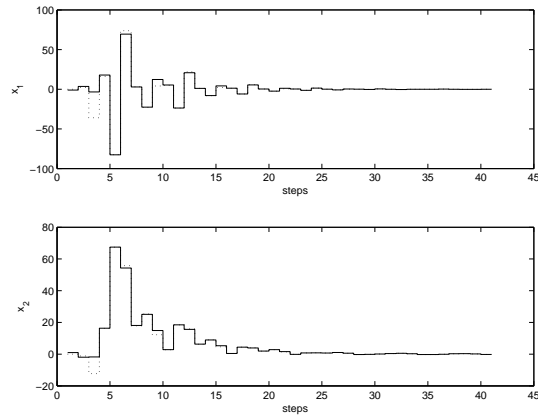
8

Figure 1: State $x(t)$ of the original system

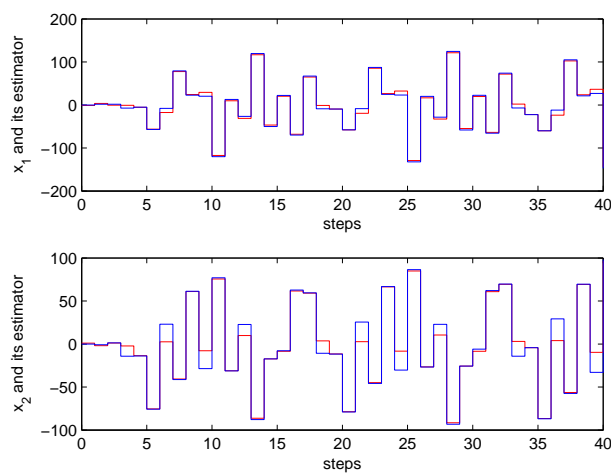Figure 2: $x(t)$ and $\hat{x}(t)$ with $(K_{\mathrm{rand}}, L_{\mathrm{rand}})$ when $\mu = 0$

Figure 3: $x(t)$ with $(K_{\mathrm{rand}}, L_{\mathrm{rand}})$ when $\mu = 0.015$
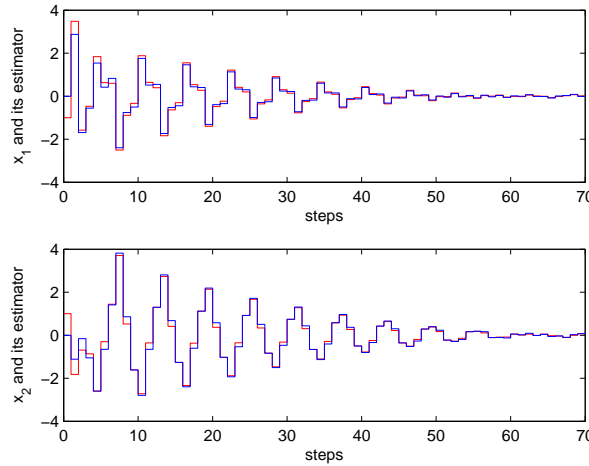
9

Figure 4: $x(t)$ and $\hat{x}(t)$ with $(K_{\mathrm{robu}}, L_{\mathrm{robu}})$ when $\mu = 0.35$

where $\Delta_{at} \in \mathbb{R}^{2\times2}$, $\Delta_{bt} \in \mathbb{R}^{2\times1}$, $\Delta_{ct} \in \mathbb{R}^{1\times2}, t \in \overline{0,2}$ are random perturbations normalized such that $\|\Delta_{at}\|_{\mathrm{F}} = 1, \|\Delta_{bt}\|_{\mathrm{F}} = 1, \|\Delta_{ct}\|_{\mathrm{F}} = 1, t \in \overline{0,2}$ and $\mu > 0$ is a parameter controlling the level of perturbations. Let $\mu = 0.015$, we depict the response histories of $x(t)$ and $\hat{x}(t)$ with gains $(K_{\mathrm{rand}}, L_{\mathrm{rand}})$ in figure. 3, where the solid line denotes $x(t)$ and the dotted line denotes $\hat{x}(t)$. It is obvious that system (7) with gains $(K_{\mathrm{rand}}, L_{\mathrm{rand}})$ is not stable even the perturbation level is reduced to $\mu = 0.015$. To measure robustness of the designed robust controller based on periodic observers, we continuously increase the perturbation controlling level until $\mu = 0.35$ and depict the results in figure. 4. From simulation results, we can see the designed robust controller has strong anti-interference ability. In addition, we notice that $(K_{\mathrm{robu}}, L_{\mathrm{robu}})$ has a very small norm compared with $(K_{\mathrm{rand}}, L_{\mathrm{rand}})$. This means that the robust controllers and observers can possess less energy consumption, since small gains lead to small control signals.

From the simulation results, we can see the approaches proposed in this paper are very effective.

5 Conclusion

In this paper, the observer-based robust stabilization problem for LDP systems is considered. It is proofed that the principle of separating exists in this type of systems. Thus, periodic controllers and periodic observers can be designed separately. By using poles assignment technique, numerous periodic controllers and observers are obtained in the form of iteration and parametrization. Combined with our recent result about robustness, robust stabilization problem based on observers is solved. Two detailed algorithms are presented. The proposed approaches are checked by a numerical example and the simulation results are of great satisfaction. A possible future study is to combine the developed approach with the truncated predictor feedback [14, 24, 25] and constrained control theory [26, 27] to investigate the observer-based robust stabilization problem for LDP systems with time delays and input saturation.

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Embedding relations of Besov classes under GBV

W. T. CHENG, X. W. XU and X. M. ZENG

School of Mathematical Sciences, Xiamen University

Xiamen 361005 P. R. China

e-mails:chengwentao_231@sina.com, lampminket@263.net, xmzeng@xmu.edu.cn

Abstract. In this paper, we strengthen some of Leindler's results from [L. Leindler. Embedding relations of Besov classes. Acta Sci. Math. (Szeged), 73(2007)133-149.] under GBV condition. First, we discuss embedding relations between two Besov classes. Next, we give an equivalent estimate for the k-order modulus of continuity of $f(x)$ in L^p norm under GBV condition. Finally, we give the condition to ensure a function $f \in L^p$ have Fourier coefficients of GBV belongs to the Besov class.

Keywords. GBV, Besov classes, embedding relations, Fourier coefficients.

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1 Introduction

Many classical results in Fourier analysis have been generalized by weakening the condition imposed on the coefficients of trigonometric series from MS to RBVS, GBVS and, Finally, to MVBVS(see [26] for more details). In [15], Leindler defined the class of sequences of rest bounded variation, in symbol: RBVS, and showed that it is not comparable to the classical quasi monotone sequences, in symbol: CQMS. In [6], Le and Zhou defined the class GBVS containing both RBVS and CQMS. In [10], Leindler introduced a new class of sequences, the class γRBVS.

Definition 1.1. *Let $\gamma := \{\gamma_n\}$ be a positive sequence. A null-sequence $A := \{a_n\}(a_n \to 0)$ of real number satisfying the inequalities*

$$(1.1) \qquad \sum_{i=n}^{\infty} |\Delta a_i| \leq K(A)\gamma_n \quad (\Delta a_i := a_i - a_{i+1}), \quad n = 1, 2, \cdots$$

with a positive constant $K(A)$ is said to be a sequence of γ rest bounded variation, in symbol: $A \in \gamma RBVS$.

If $\gamma \equiv A$ and $a_n > 0$, then γRBVS $\equiv$ RBVS. It is easy to see that if $A \in$ RBVS, then it is also almost monotone, in symbol: $A \in$ AMS, that is for all $n \geq m$, we have

$$a_n \leq K(A)a_m.$$

In [11] and [10], Leindler introduced the class of mean rest bounded variation sequences, where γ is defined by a certain arithmetical mean of the coefficients, e.g.,

$$\gamma_n^* := \frac{1}{n}\sum_{i \geq n/2}^{n} a_i \quad or \quad \overline{\gamma}_n := \frac{1}{n}\sum_{i=n}^{2n-1} a_i.$$

It is easy to see that the class γ^*MRBVS includes the class RBVS, consequently the almost monotone and monotone sequences, too; but $\overline{\gamma}$MRBVS does not, in general. In [21], B. Szal proved that RBVS $\neq \gamma^*$MRBVS. Namely, he showed that the sequence

$$d_n := \begin{cases} 1, & \text{if} \quad n = 1, \\ \frac{1+m+(-1)^m}{(2^{\mu_m})^2 m}, & \text{if} \quad \mu_m \leq n < \mu_{m+1} \end{cases}$$

where $\mu_m = 2^m$ for $m = 1, 2, 3 \cdots$, belongs to the class γ^*MRBVS but it does not belong to the class RBVS. In [23], B. Szal showed that $\overline{\gamma}$MRBVS $\subset \gamma^*$MRBVS and $\overline{\gamma}$MRBVS $\neq \gamma^*$MRBVS. Namely, he showed that the above sequence d_n belongs to the class γ^*MRBVS but it does not belong to the class $\overline{\gamma}$MRBVS. In [22], B. Szal introduced the class of infinity mean rest bounded variation, briefly A $\in$ IMRBVS, if $\sum_{n=1}^{\infty} \frac{a_n}{n} < \infty$ and $\gamma_n = \sum_{i=n}^{\infty} \frac{a_i}{i}$. Moreover, he showed that $\overline{\gamma}$MRBVS $\neq$ IMRBVS and γ^*MRBVS $\neq$ IMRBVS.

In [6], Le and Zhou first defined the class GBVS as follow:

Definition 1.2. *A positive sequence* A $:= \{a_n\}_{n=1}^{\infty}$ *satisfying the inequalities*

$$\sum_{i=n}^{2n-1} |\Delta a_i| \leq K(A) a_n, \ n = 1, 2, \cdots$$

with a positive constant $K(A)$ is said to be a sequence of group bounded variation, in symbol: A $\in$ GBVS.

Moreover, they proved that RBVS $\subseteq$ GBVS. If A$\in$ GBVS, then for all $m \leq n \leq 2m$, we have $a_n \leq K(A) a_m$. Thus, GBVS also named general monotone sequences in [16] and [24] (in symbol: GMS). In [11], Leindler proved that MRBVS $\not\sim$ GBVS.

Many classical theorems were generalized under RBV condition or GBV condition in [9], [5], [8], [7] and so on .The properties of the Besov classes have been studied by many authors (see [22], [10], [14], [18], [19]). Their major work studied three theorems in connection with Besov classes of functions $f \in L_{[-\pi,\pi]}^p$ under coefficient sequence satisfying restricted condition. In [22], [23], [10], [14], [18] and [19] studied them under IMRBV condition, γ^*MRBV condition, $\overline{\gamma}$MRBV condition, RBV condition, M condition, respectively. In view of the relation between GBVS and other RBVS, we make further efforts to generalize the three theorems under GBV.

The rest of the paper is organized as follows. In Section 2 we give notions and notations used in the paper. In Section 3 we give our main results. In Section 4 we introduce some lemmas to prove our results. In Section 5 we prove the main results.

2 Notions and notations

Let $L_{[-\pi,\pi]}^p (1 \leq p \leq \infty)$ be the space of all p-power integrable real functions of period 2π with the norms

$$\|f\|_p := \begin{cases} \left(\int_{-\pi}^{\pi} |f(t)|^p \mathrm{d}t \right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \operatorname*{ess\,sup}_{x \in [-\pi,\pi]} |f(t)|, & p = \infty. \end{cases}$$

The best trigonometric approximation $E_n(f)_p$ and the modulus of smoothness $\omega_k(f; \delta)_p$ are defined as follows:

$$E_n(f)_p = \min \left(\|f - T\|_p : T \in \mathbf{T}_n \right), \ \mathbf{T}_n = \operatorname{span}\{\cos mx, \sin mx : |m| \leq n\}$$

and

$$\omega_k(f;\delta)_p = \sup_{|h|<\delta} \left\| \Delta_h^k f(x) \right\|_p$$

$$\Delta_h^k f(x) = \Delta_h^{k-1}(\Delta_h f(x)) \; \Delta_h f(x) = f(x+h) - f(x),$$

respectively.

A function $\alpha(t)$ is called σ-type if it is measurable on $[0,1]$, integrable on $[\delta,1]$ for every $\delta \in (0,1)$, and there exist positive constants C_1 and C_2 such that

(i) $\alpha(t) \geq C_1$ for all $t \in [0,1]$,

(ii) $\int_0^\delta \alpha(t)t^\sigma \mathrm{d}t \leq C_2\delta^\sigma \int_\delta^{2\delta} \alpha(t)\mathrm{d}t$ for all $\delta \in (0,\delta_0)$, where $0 < \delta_0 \leq \frac{1}{2}$ is given.

A positive function $\alpha(t)$ is said to satisfy the λ-condition, $\lambda > 0$, if there exists a positive constant C_3 such that

$$\int_{2\delta}^1 \alpha(t)t^\lambda \mathrm{d}t \leq C_3\delta^\lambda \int_\delta^{2\delta} \alpha(t)\mathrm{d}t, \text{ for all } \delta \in (0,\delta_0).$$

We say that $f \in B(p,\gamma,\alpha)$ if

(i) $f \in L_{[-\pi,\pi]}^p$,

(ii) $0 < \gamma < \infty$,

(iii) $\alpha(t)$ is σ-type,

(iv) $\int_0^1 \omega_k^\gamma(f;t)_p \, \mathrm{d}t < \infty, k \geq \frac{\sigma}{\gamma}$.

We use the notation $L \ll R$ at inequalities if there exists a positive constant K such that $L \leq KR$; and if $L \ll R$ and $R \ll L$ hold simultaneously, then we shall write $L \asymp R$.

3   Main results

We formulate our results as follow:

Theorem 3.1. *If $1 < p < q \leq \infty$, the function $\alpha(t)$ satisfies λ-condition with*

$$\lambda = \left(\frac{1}{p} - \frac{1}{q}\right)\gamma, \; 0 < \gamma < \infty, \; \alpha^*(t) := \alpha(t)t^\lambda,$$

$A := \{a_n\}_{n=1}^\infty \in GBVS$, and f has the Fourier expansion

(3.1)
$$f(x) \sim \sum_{n=1}^\infty a_n \cos nx,$$

then the Besov classes $B(p,\gamma,\alpha)$ and $B(p,\gamma,\alpha^*)$ coincide. Furthermore, for any

$$k_1 \geq \frac{\sigma}{\gamma}, k_2 \geq \frac{\sigma^*}{\gamma}, k_3 \geq \frac{\sigma^*}{\gamma}, \sigma^* = \sigma - \lambda,$$

we have

(3.2)
$$\int_0^1 \alpha^*(t)\omega_{k_2}^\gamma(f;t)_q \mathrm{d}t \ll \int_0^1 \alpha(t)\omega_{k_1}^\gamma(f;t)_q \mathrm{d}t \ll \int_0^1 \alpha^*(t)\omega_{k_3}^\gamma(f;t)_q \mathrm{d}t.$$

Theorem 3.2. *If $f \in L_{[-\pi,\pi]}^p$, $1 < p < \infty$, f has the Fourier expansion (3.1) with $A := \{a_n\} \in GBVS$, then*

(3.3)
$$S(A,p,k,n) \ll \omega_k\left(f;\frac{1}{n}\right)_p \ll S(A,p,k,n),$$

where

$$S(A, q, k, n) := \begin{cases} a_n, & \text{if } q = 1, \\ n^{-k}\left(\sum_{i=1}^{n} a_i^q i^{(k+1)q-2}\right)^{1/q} + \left(\sum_{i=n+1}^{\infty} a_i^q i^{q-2}\right)^{1/q}, & \text{if } 1 < q < \infty, \\ n^{-k}\sum_{i=1}^{n} a_i i^k + \sum_{i=n+1}^{\infty} a_i, & \text{if } q = \infty. \end{cases}$$

Theorem 3.3. *If* $f \in L^p_{[-\pi,\pi]}$, $1 < p < \infty$, *f has the Fourier expansion* (3.1) *with* $A := \{a_n\} \in GBVS$, $\alpha(t) = t^{-r\gamma-1}$ *and* $k > r$. *If* $\gamma \geq 1$, *then* $f \in B(p, \gamma, \alpha)$ *if and only if*

$$(3.4) \qquad\qquad J_1 := \sum_{n=1}^{\infty} a_n^\gamma n^{r\gamma+\gamma-\frac{\gamma}{p}-1} < \infty.$$

If $0 < \gamma \leq 1$, *then a sufficient condition for* $f \in B(p, \gamma, \alpha)$ *is*

$$(3.5) \qquad\qquad J_2 := \sum_{n=1}^{\infty} a_n^\gamma n^{r\gamma-\gamma/p} < \infty$$

and a necessary condition is

$$(3.6) \qquad\qquad J_1 := \sum_{n=1}^{\infty} a_n^\gamma n^{r\gamma+\gamma-\frac{\gamma}{p}-1} < \infty.$$

4 Auxiliary lemmas

In order to verify our theorems we need several lemmas: most of them are the analogues of the lemmas used in the proofs of the theorems with monotone coefficients or other conditions.

Lemma 4.1. ([*13*], Corollary 1) *If* $\lambda_n > 0$ *and* $a_n \geq 0$, *then*

$$(4.1) \qquad \sum_{n=1}^{\infty} \lambda_n \left(\sum_{k=1}^{n} a_k\right)^p \leq p^p \sum_{n=1}^{\infty} \lambda_n^{1-p} a_n^p \left(\sum_{k=n}^{\infty} \lambda_k\right)^p$$

$$(4.2) \qquad \sum_{n=1}^{\infty} \lambda_n \left(\sum_{k=n}^{\infty} a_k\right)^p \leq p^p \sum_{n=1}^{\infty} \lambda_n^{1-p} a_n^p \left(\sum_{k=1}^{n} \lambda_k\right)^p$$

hold for any $p \geq 1$; *while if* $0 < p < 1$, *then the inequality in* (4.1) *and* (4.2) *hold with opposite direction.*

Lemma 4.2. ([*2*], Theorem 19) *If* $a_n \geq 0$ *and* $0 < p_1 < p_2 < \infty$, *then*

$$(4.3) \qquad \left(\sum_{n=1}^{\infty} a_n^{p_2}\right)^{\frac{1}{p_2}} \leq \left(\sum_{n=1}^{\infty} a_n^{p_1}\right)^{\frac{1}{p_1}}.$$

Lemma 4.3. ([*1*], p. 293) *If* $f \in L^\infty_{[-\pi,\pi]} \equiv C_{[-\pi,\pi]}$ *and* $a_n \geq 0$,

$$f(x) = \sum_{n=1}^{\infty} a_n \cos nx, \ x \in [-\pi, \pi],$$

then

$$\sum_{k=2n}^{\infty} a_k \leq 4E_n(f)_C.$$

Lemma 4.4. [25] *If $f \in L^p_{[-\pi,\pi]}$, $1 < p \le 2$, then*

$$\omega_k\left(f; \frac{1}{n}\right)_p \ll n^{-k}\left(\sum_{i=1}^n i^{kp-1} E_i^p(f)_p\right)^{\frac{1}{p}};$$

while if $p > 2$, then the reverse inequality holds.

Lemma 4.5. ([19], pp. $847 - 848$) *If $f \in L^p_{[-\pi,\pi]}$, $1 \le p \le \infty$, $0 < \gamma < \infty$, α is a σ-type function and $k \ge \frac{\sigma}{\gamma}$, then*

$$E_0^r(f)_p + E_1^r(f)_p + \sum_{i=1}^\infty \mu(i)E_{2^i}^\gamma(f)_p \asymp \int_0^1 \alpha(t)\omega_k^\gamma(f;t)_p \mathrm{d}t,$$

where

$$\mu(n) := \int_{2^{-n}}^{2^{-n+1}} \alpha(t)\mathrm{d}t, n \ge 1 \text{ and } \mu(0) = 1.$$

Lemma 4.6. ([23], Lemma 6) *If α is a σ-type function, then*

(4.4) $$\mu(n + 1) \ll \mu(n)$$

hold for all n.

Lemma 4.7. ([20], Theorem 1) *If $f \in L^p_{[-\pi,\pi]}$, $1 \le p \le \infty$, f has the Fourier expansion (3.1), and $P_1 := \min\{2, p\}$, $P_2 := \max\{2, p\}$, then*

$$S(A, P_1, k, n) \ll \omega_k\left(f; \frac{1}{n}\right)_p \ll S(A, P_2, k, n).$$

Lemma 4.8. ([18], Theorem 1) *If $f \in B(p, \gamma, \alpha)$, $1 < p < q \le \infty$ and α satisfies λ-condition with $\lambda = \left(\frac{1}{p} - \frac{1}{q}\right)\gamma$, then $f \in B(q, \gamma, \alpha^*)$, where*

$$\alpha^*(t) := \alpha(t)t^\lambda, \text{ that is, } B(p, \gamma, \alpha) \subset B(q, \gamma, \alpha^*);$$

furthermore,

$$\int_0^1 \alpha^*(t)\omega_{k_2}^\gamma(f;t)_q \mathrm{d}t \ll \int_0^1 \alpha(t)\omega_{k_1}^\gamma(f;t)_p \mathrm{d}t$$

for any

$$k_1 \ge \frac{\sigma}{\gamma}, k_2 \ge \frac{\sigma^*}{\gamma} \text{ and } \sigma^* := \sigma - \left(\frac{1}{p} - \frac{1}{q}\right) + \varepsilon, \varepsilon > 0.$$

Lemma 4.9. [6] *Let $\{a_n\} \in GBVS$, then for all $n \ge 1$, the following inequalities hold*

(4.5) $$\sum_{i=1}^\infty a_{2^i n} \ll \sum_{i=n}^\infty \frac{a_i}{i}.$$

(4.6) $$a_{n+1} \ll \sum_{i=[n/2]+1}^{2n} \frac{a_i}{i}.$$

Lemma 4.10. [7] *If $1 < p < \infty$, and f has the Fourier expansion (3.1) with $\{a_n\} \in GBVS$, then $f \in L^p_{[-\pi,\pi]}$ if and only if*

(4.7) $$\sum_{n=1}^\infty n^{p-2}a_n^p < \infty$$

or, more precisely

(4.8) $$\|f\|_p^p \asymp \sum_{n=1}^\infty n^{p-2}a_n^p.$$

Lemma 4.11. [3] *Assume that f has the Fourier expansion (3.1) with $\{a_n\} \in GBVS$. If $1 < p < \infty$ and (4.7) holds, then*

$$(4.9) \qquad E_n(f)_p \ll a_{n+1}(n+1)^{1-\frac{1}{p}} + \left(\sum_{i=n+1}^{\infty} i^{p-2} a_i^p \right)^{\frac{1}{p}}.$$

Lemma 4.12. ([4], Theorem 5) *If $f \in L^p_{[-\pi,\pi]}$, $1 < p < \infty$, and f has the Fourier expansion (3.1) with $a_n \geq 0$, then for $\eta > \frac{1}{p}$*

$$\sum_{i=n}^{\infty} \frac{a_i}{i^{\eta}} \leq n^{-\eta+\frac{1}{p}} E_n(f)_p.$$

Lemma 4.13. *If $f \in L^p_{[-\pi,\pi]}$, $1 < p < \infty$, f has the Fourier expansion (3.1) with $\{a_n\} \in GBVS$, then*

$$E_n^p(f)_p \gg \sum_{i=2n}^{\infty} a_i^p i^{p-2}.$$

Proof. We want to apply Lemma 4.10 to the following function:

$$f_0(x) := f(x) - \sum_{i=1}^{2n-1} a_i \cos ix + a_{2n} \sum_{i=1}^{2n-1} \cos ix.$$

First, we show that the $A^0 := \{a_n^0\}$ of coefficients of f_0 belongs to GBVS, that is, that

$$(4.10) \qquad \sum_{i=m}^{2m-1} \left| \Delta a_i^0 \right| \ll a_m^0, m = 1, 2, \cdots .$$

We consider three cases:

(i) If $m \geq 2n$, then $a_i^0 = a_i$ for all $i \geq m$, we easily know

$$\sum_{i=m}^{2m-1} \left| \Delta a_i^0 \right| = \sum_{i=m}^{2m-1} |\Delta a_i| \ll a_m = a_m^0.$$

(ii) If $m \leq n$, then $a_i^0 = a_{2n}$ for all $1 \leq i \leq 2m$, we easily know

$$\sum_{i=m}^{2m-1} \left| \Delta a_i^0 \right| = 0 < a_m^0.$$

(iii) If $n < m < 2n$, then $a_i^0 = a_{2n}$ for all $m \leq i \leq 2n$ and $a_i^0 = a_k$ for all $i \geq 2n$, we easily know

$$\sum_{i=m}^{2m-1} \left| \Delta a_i^0 \right| = \sum_{i=m}^{2n-1} \left| \Delta a_i^0 \right| + \sum_{i=2n}^{2m-1} \left| \Delta a_i^0 \right| < 0 + \sum_{i=2n}^{4n-1} |\Delta a_i| \ll a_{2n} = a_m^0.$$

That means $A^0 \in GBVS$, we can apply Lemma 4.10 to f_0, thus we obtain

$$\|f - S_{2n-1}(f)\|_p^p + a_{2n}^p \left\| \sum_{i=1}^{2n-1} \cos ix \right\|_p^p \gg \|f_0\|_p^p \gg \sum_{i=2n}^{\infty} a_i^p i^{p-2}.$$

Since

$$\left\| \sum_{i=1}^{2n-1} \cos ix \right\|_p^p = 2 \int_0^\pi \left| \sum_{i=1}^{2n-1} \cos ix \right|^p dx$$

$$= 2 \left(\int_0^{\frac{\pi}{n}} + \int_{\frac{\pi}{n}}^\pi \right) \left| \frac{\cos nx \sin \frac{2n-1}{2} x}{\sin \frac{x}{2}} \right|^p dx$$

$$\ll n^p \int_0^{\frac{\pi}{n}} dx + \int_{\frac{\pi}{n}}^\pi \frac{1}{x^p} dx \ll n^{p-1},$$

by a theorem of M. Riesz ([17], Theorem 3, p. 221), we obtain

$$(4.11) \qquad \sum_{i=2n}^\infty a_i^p i^{p-2} \ll E_{2n-1}^p(f)_p + a_{2n}^p n^{p-1} < E_n^p(f)_p + a_{2n}^p n^{p-1}.$$

Applying Lemma 4.12 with $\eta = 1$ and (4.5), we obtain

$$(4.12) \qquad a_{2n}^p n^{p-1} \le n^{p-1} \left(\sum_{i=1}^\infty a_{2^i n} \right)^p \ll n^{p-1} \left(\sum_{i=n}^\infty \frac{a_i}{i} \right)^p \ll E_n^p(f)_p.$$

The inequalities (4.11) and (4.12) imply the assertion. □

Lemma 4.14. *If $f \in L_{[-\pi,\pi]}^p$, $1 < p < q \le \infty$, and f has the Fourier expansion (3.1) with $A := \{a_n\}_{n=1}^\infty \in GBVS$. If $q < \infty$, then*

$$S_1 := \sum_{i=8n}^\infty i^{\frac{q}{p}-2} E_i^q(f)_p \ll E_n^q(f)_q;$$

while if $q = \infty$, then

$$S_2 := \sum_{i=8n}^{16n} i^{\frac{1}{p}-2} E_i(f)_p \ll E_n(f)_q.$$

Proof. By Lemma 4.11, we have

$$S_1 \ll \sum_{i=8n}^\infty i^{\frac{q}{p}-2} a_{i+1}^q (i+1)^{q\left(1-\frac{1}{p}\right)} + \sum_{i=8n}^\infty i^{\frac{q}{p}-2} \left(\sum_{l=i+1}^\infty l^{p-2} a_l^p \right)^{\frac{q}{p}}$$

Using the inequalities of Lemma 4.1 and Lemma 4.13, we obtain

$$S_1 \ll \sum_{i=8n}^\infty a_{i+1}^q (i+1)^{q-2} + \sum_{i=8n}^\infty a_{i+1}^q (i+1)^{q-2+\frac{q}{p}-\left(\frac{q}{p}\right)^2} \left(\sum_{l=1}^{i+1} l^{\frac{q}{p}-2} \right)^{\frac{q}{p}}$$

$$\le \sum_{i=8n}^\infty a_i^q i^{q-2} \ll E_n^q(f)_q.$$

To estimate S_2, we apply Lemma 4.11 again. Thus

$$S_2 \ll \sum_{i=8n}^\infty i^{\frac{1}{p}-1} a_{i+1} (i+1)^{1-\frac{1}{p}} + \sum_{i=8n}^\infty i^{\frac{1}{p}-1} \left(\sum_{l=i+1}^\infty a_l^p l^{p-2} \right)^{\frac{1}{p}}$$

$$:= S_{21} + S_{22}.$$

First, we you

$$S_{21} \ll \sum_{i=2n}^{\infty} a_{i+1} < \sum_{i=2n}^{\infty} a_i \ll E_n(f)_q$$

and since $A \in GBVS$, for all $m \le i \le 2m$, we have $a_i \ll a_m$, if $2^j n \le i < 2^{j+1} n$,

$$a_i \ll a_{2^j n} \ll \sum_{i=2^j n}^{\infty} |\Delta a_i| \ll \sum_{v=j}^{\infty} a_{2^v n} \ll \sum_{v=2^{j-1}n}^{\infty} \frac{a_v}{v} \ll \sum_{v=[i/4]}^{\infty} \frac{a_v}{v+1}$$

we obtain

$$
\begin{aligned}
S_{22} &\le \sum_{i=8n}^{16n} i^{\frac{1}{p}-1} \left(\sum_{l=i}^{\infty} a_l^p l^{p-2} \right)^{\frac{1}{p}} \ll \sum_{i=8n}^{16n} i^{\frac{1}{p}-1} \left(\sum_{l=i}^{\infty} l^{p-2} \left(\sum_{v=[l/4]}^{\infty} \frac{a_v}{v+1} \right)^p \right)^{\frac{1}{p}} \\
&\ll \sum_{i=8n}^{16n} i^{\frac{1}{p}-1} \left(\sum_{l=i}^{\infty} l^{p-2} \left(\sum_{v=[l/4]}^{\infty} \frac{a_v}{v+1} \right)^p \right)^{\frac{1}{p}} \ll \sum_{i=8n}^{16n} i^{\frac{1}{p}-1} \left(\sum_{l=i}^{\infty} l^{-2} \left(\sum_{v=[l/4]}^{\infty} a_v \right)^p \right)^{\frac{1}{p}} \\
&\ll \sum_{i=2n}^{\infty} a_i \sum_{i=8n}^{16n} i^{\frac{1}{p}-1} \left(\sum_{l=i}^{\infty} l^{-2} \right)^{\frac{1}{p}} \ll \sum_{i=2n}^{\infty} a_i \sum_{i=8n}^{16n} i^{-1} \ll \sum_{i=2n}^{\infty} a_i.
\end{aligned}
$$

Collecting our estimates, by Lemma 4.3, we obtain that $S_2 \ll E_n(f)_\infty$, herewith the proof of lemma is complete.   □

5   Proofs of the theorems

5.1   Proof of Theorem 3.1

By Lemma 4.8 the first inequality in (3.2) is proved, whence

(5.1) $B(p, \gamma, \alpha) \subset B(q, \gamma, \alpha^*)$

also holds. To prove the second inequality of (3.2), we use Lemma 4.5, Assume $f \in B(q, \gamma, \alpha^*)$, then

$$I_q := E_0^{\gamma}(f)_q + E_1^{\gamma}(f)_q + \sum_{n=1}^{\infty} \mu^*(n) E_{2^n}^{\gamma}(f)_q \ll \int_0^1 \alpha^*(t) \omega_{k_3}^{\gamma}(f; t)_q \mathrm{d}t < \infty,$$

where $k_3 \ge \frac{\sigma^*}{\gamma}$ and

$$\mu^*(n) := \int_{2^{-n}}^{2^{1-n}} \alpha^*(t) \mathrm{d}t, n > 1 \text{ and } \mu^*(0) = 1.$$

Since $1 < p < q$, by Lemma 4.5 and Lemma 4.6, we have

$$\mu(n) \ll \mu^*(n) 2^{n(1/p-1/q)\gamma}, \ \mu(4) \ll \mu(3) \ll \mu(2) \ll \mu(1) \ll 1 \text{ and } \mu(n+4) \ll \mu(n).$$

It is clear that

$$I_p := E_0^\gamma(f)_q + E_1^\gamma(f)_q + \sum_{n=1}^{\infty} \mu(n) E_{2^n}^\gamma(f)_q$$

$$\ll E_0^\gamma(f)_q + E_1^\gamma(f)_q + \sum_{n=1}^{4} \mu(n) E_{2^n}^\gamma(f)_q + \sum_{n=1}^{\infty} \mu(n+4) E_{2^{n+4}}^\gamma(f)_q$$

$$\ll E_0^\gamma(f)_q + E_1^\gamma(f)_q + \sum_{n=1}^{\infty} \mu(n) E_{2^{n+4}}^\gamma(f)_q$$

$$\ll E_0^\gamma(f)_q + E_1^\gamma(f)_q + \sum_{n=1}^{\infty} \mu^*(n) 2^{n(1/p-1/q)\gamma} E_{2^{n+4}}^\gamma(f)_q$$

$$\ll E_0^\gamma(f)_q + E_1^\gamma(f)_q + \sum_{n=1}^{\infty} \mu^*(n) \left(2^{n(1/p-1/q)} E_{2^{n+4}}(f)_q \right)^\gamma$$

$$\ll E_0^\gamma(f)_q + E_1^\gamma(f)_q + \sum_{n=1}^{\infty} \mu^*(n) \left(\sum_{i=2^{n+3}}^{2^{n+4}} i^{(1/p-1/q)-1} E_i(f)_q \right)^\gamma .$$

Hence, if $q = \infty$, by Lemma 4.14, we obtain

$$I_p \ll E_0^\gamma(f)_q + E_1^\gamma(f)_q + \sum_{n=1}^{\infty} \mu^*(n) E_{2^n}^\gamma(f)_q$$

and immediately $I_p \ll I_q$. If $1 < q < \infty$, then applying Hölder's inequality and Lemma 4.14, we have

$$\sum_{i=2^{n+3}}^{2^{n+4}} i^{(1/p-1/q)-1} E_i(f)_q = \sum_{i=2^{n+3}}^{2^{n+4}} i^{1/p-2/q} E_i(f)_q i^{1/q-1}$$

$$\leq \left(\sum_{i=2^{n+3}}^{2^{n+4}} i^{q/p-2} E_i^q(f)_q \right)^{1/q} \left(\sum_{i=2^{n+3}}^{2^{n+4}} \left(i^{1/q-1} \right)^{q/(q-1)} \right)^{1-1/q}$$

$$\ll \left(\sum_{i=2^{n+3}}^{2^{n+4}} i^{q/p-2} E_i^q(f)_q \right)^{1/q} .$$

From this and Lemma 4.5, we can obtain

$$I_p \ll E_0^\gamma(f)_q + E_1^\gamma(f)_q + \left(\sum_{i=2^{n+3}}^{2^{n+4}} i^{q/p-2} E_i^q(f)_q \right)^{\gamma/q}$$

$$\ll E_0^\gamma(f)_q + E_1^\gamma(f)_q + \sum_{n=1}^{\infty} \mu^*(n) E_{2^n}^\gamma(f)_q$$

then by Lemma 4.14, $I_p \ll I_q$ is visible.

Finally, by Lemma 4.5, we obtain that

$$\int_0^1 \alpha(t) \omega_{k_1}^\gamma(f;t)_p dt \ll I_p \ll I_q \ll \int_0^1 \alpha^*(t) \omega_{k_3}^\gamma(f;t)_p dt < \infty$$

follows with $k_1 \geq \frac{\sigma}{\gamma}$.

This proves the second inequality of (3.2), consequently

(5.2) $$B(q, \gamma, \alpha^*) \subset B(p, \gamma, \alpha).$$

Thus, (5.1) and (5.2) completes the proof of Theorem 3.1 with $\{a_n\} \in$ GBVS. ∎

5.2 Proof of Theorem 3.2

First, we prove $S(A, p, k, n) \ll \omega_k\left(f; \frac{1}{n}\right)_p$. We separate two cases:

(i) If $1 < p \leq 2$, by Lemma 4.7, we easily know $S(A, p, k, n) \ll \omega_k\left(f; \frac{1}{n}\right)_p$ holds.

(ii) If $p \geq 2$, then by Lemma 4.13, Jackson's theorem and the properties of $\omega_k(f; \delta)_p$, we obtain

$$(5.3) \qquad \left(\sum_{i=n+1}^{\infty} a_i^p i^{p-2}\right)^{1/p} \ll E_{n^*}(f)_p \ll \omega_k\left(f; \frac{1}{n}\right)_p,$$

where

$$n^* = \begin{cases} m, & \text{if } n = 2m, \\ m, & \text{if } n = 2m - 1. \end{cases}$$

By (4.6) and Lemma 4.13, we easily obtain

$$\begin{aligned}
a_i^p i^{p-1} &\ll \left(\sum_{j=[(i-1)/2]+1}^{2(i-1)} \frac{a_j}{j}\right)^p i^{p-1} \ll \left(\sum_{j=[i/4]+1}^{2i} \frac{a_j}{j}\right)^p i^{p-1} \ll \left(\sum_{j=[i/4]+1}^{2i} a_j\right)^p i^{-1} \ll \left(\sum_{j=[i/4]+1}^{2i} j^{-1/p} a_j\right)^p \\
&\ll \sum_{j=[i/4]+1}^{2i} a_j^p \left(\sum_{j=[i/4]+1}^{2i} j^{-1/(p-1)}\right)^{p-1} \ll i^{p-2} \sum_{j=[i/4]+1}^{2i} a_j^p \ll \sum_{j=[i/4]+1}^{2i} j^{p-2} a_j^p \ll E_{[i/8]}^p(f)_p.
\end{aligned}$$

Putting this into the following sum and applying Lemma 4.4, we find the following estimates:

$$\begin{aligned}
(5.4) \qquad n^{-k}\left(\sum_{i=1}^{n} a_i^p i^{(k+1)p-2}\right)^{1/p} &\ll n^{-k}\left(\sum_{i=1}^{n} E_{[i/8]}^p(f)_p i^{kp-1}\right)^{1/p} \\
&\ll n^{-k}\left(\sum_{i=1}^{n} i^{kp-1} E_i^p(f)_p\right)^{1/p} \ll \omega_k\left(f; \frac{1}{n}\right)_p.
\end{aligned}$$

The inequalities (5.3) and (5.4) verify $S(A, p, k, n) \ll \omega_k\left(f; \frac{1}{n}\right)_p$ for $2 \leq p < \infty$, thus it is proved for any $1 < p < \infty$.

Next, we prove that $\omega_k\left(f; \frac{1}{n}\right)_p \ll S(A, p, k, n)$. We consider two cases:

(i) If $2 \leq p < \infty$, by Lemma 4.7, we easily know $\omega_k\left(f; \frac{1}{n}\right)_p \ll S(A, p, k, n)$ holds.

(ii) If $1 < p \leq 2$, then we use Lemma 4.4 and Lemma 4.2, thus an elementary calculation, we obtain that

$$(5.5)$$
$$\begin{aligned}
\omega_k\left(f; \frac{1}{n}\right)_p &\ll n^{-k}\left(\sum_{i=1}^{n} i^{kp-1} E_n^p(f)_p\right)^{1/p} \\
&\ll n^{-k}\left(\sum_{i=1}^{n} i^{kp-1} a_{i+1}^p (i+1)^{p-1} + \sum_{i=1}^{n} i^{kp-1} \sum_{j=i+1}^{\infty} j^{p-2} a_j^p\right)^{1/p} \\
&\ll n^{-k}\left(\sum_{i=1}^{n} i^{(k+1)p-2} a_{i+1}^p + \sum_{i=1}^{n} i^{kp-1} \sum_{j=i+1}^{n} j^{p-2} a_j^p + \sum_{i=1}^{n} i^{kp-1} \sum_{j=n+1}^{\infty} j^{p-2} a_j^p\right)^{1/p} \\
&\ll n^{-k}\left(\sum_{i=1}^{n} i^{(k+1)p-2} a_i^p + \sum_{j=2}^{n} j^{p-2} a_j^p \sum_{i=1}^{j} i^{kp-1} + (n+1)^{(k+1)p-2} a_{n+1}^p + n^{kp} \sum_{j=n+1}^{\infty} j^{p-2} a_j^p\right)^{1/p} \\
&\ll n^{-k}\left(\sum_{i=1}^{n} i^{(k+1)p-2} a_i^p + n^{kp} \sum_{j=n+1}^{\infty} j^{p-2} a_j^p\right)^{1/p} \\
&\ll n^{-k}\left(\sum_{i=1}^{n} i^{(k+1)p-2} a_i^p\right)^{1/p} + \left(\sum_{j=n+1}^{\infty} j^{p-2} a_j^p\right)^{1/p} = S(A, p, k, n).
\end{aligned}$$

This proves $\omega_k\left(f;\dfrac{1}{n}\right)_p \ll S(A,p,k,n)$ for $1 < p \le 2$, and consequently for any $1 < p < \infty$.

Herewith the proof of Theorem 3.2 is complete. ∎

5.3   Proof of Theorem 3.3

By the following inequality

$$(5.6) \qquad J := \int_0^1 t^{-r\gamma-1}\omega_k^\gamma(f;t)_p\,\mathrm{d}t \asymp \sum_{n=1}^\infty n^{r\gamma-1}\omega_k^\gamma\left(f;\frac{1}{n}\right)_p$$

and Theorem 3.2, we can obtain

$$
\begin{aligned}
(5.7) \qquad J &\ll \sum_{n=1}^\infty n^{r\gamma-1}\omega_k^\gamma\left(f;\frac{1}{n}\right)_p \\
&\ll \sum_{n=1}^\infty n^{r\gamma-1}\left(n^{-k}\left(\sum_{i=1}^n i^{(k+1)p-2}a_i^p\right)^{1/p} + \left(\sum_{j=n+1}^\infty j^{p-2}a_j^p\right)^{1/p}\right)^\gamma \\
&\ll \sum_{n=1}^\infty n^{(r-k)\gamma-1}\left(\sum_{i=1}^n i^{(k+1)p-2}a_i^p\right)^{\gamma/p} + \sum_{n=1}^\infty n^{r\gamma-1}\left(\sum_{j=n+1}^\infty j^{p-2}a_j^p\right)^{\gamma/p}.
\end{aligned}
$$

Case (i): $\gamma \ge 1$

Sufficiency. We distinguish two cases listed under (A) and (B):

Case (A): $\gamma/p \ge 1$, by Lemma 4.1, we can obtain

$$
\begin{aligned}
(5.8) \qquad J &\ll \sum_{n=1}^\infty n^{r\gamma+\gamma-\frac{\gamma}{p}-1-\frac{(r-k)\gamma 2}{p}}a_n^\gamma\left(\sum_{i=n}^\infty i^{(r-k)\gamma-1}\right)^{\gamma/p} + \sum_{n=1}^\infty n^{r\gamma+\gamma-\frac{\gamma}{p}-1-\frac{r\gamma 2}{p}}a_n^\gamma\left(\sum_{i=1}^n i^{r\gamma-1}\right)^{\gamma/p} \\
&\ll \sum_{n=1}^\infty n^{r\gamma+\gamma-\gamma/p-1}a_n^\gamma.
\end{aligned}
$$

From the above estimate we get that $J \ll J_1$. under $\gamma/p \ge 1$.

Case (B): $\gamma/p < 1$, by (5.6), we can yields that

$$(5.9) \qquad J \asymp \sum_{n=0}^\infty 2^{nr\gamma}\omega_k^\gamma\left(f;\frac{1}{2^n}\right)_p$$

then using again Theorem 3.2, we obtain that

$$
\begin{aligned}
(5.10) \qquad J &\ll \sum_{n=0}^\infty 2^{n(r-k)\gamma}\left(\sum_{i=1}^{2^n} i^{(k+1)p-2}a_i^p\right)^{\gamma/p} + \sum_{n=0}^\infty 2^{nr\gamma}\left(\sum_{i=2^n+1}^\infty i^{p-2}a_i^p\right)^{\gamma/p} \\
&:\ = \ J_{11} + J_{12}.
\end{aligned}
$$

Applying Lemma 4.1, Lemma 4.2, Lemma 4.9 and Hölder's inequality, we can obtain that

(5.11)

$$
\begin{aligned}
J_{11} &= \sum_{n=0}^\infty 2^{n(r-k)\gamma}\left(\sum_{i=1}^{2^n} i^{(k+1)p-2}a_i^p\right)^{\gamma/p} \ll \sum_{n=0}^\infty 2^{n(r-k)\gamma}\left(\sum_{j=0}^n \sum_{i=2^j}^{2^{j+1}} i^{(k+1)p-2}\left(\sum_{t=[i/4]+1}^{2i}\frac{a_t}{t}\right)^p\right)^{\gamma/p} \\
&\ll \sum_{n=0}^\infty 2^{n(r-k)\gamma}\left(\sum_{j=0}^n 2^{j((k+1)p-1)}\left(\frac{1}{2^j}\sum_{t=[2^{j-2}]+1}^{2^{j+1}}a_t\right)^p\right)^{\gamma/p} \ll \sum_{n=0}^\infty 2^{n(r-k)\gamma}\sum_{j=0}^n 2^{j((k+1)\gamma-\gamma/p)}\left(\frac{1}{2^j}\sum_{t=[2^{j-2}]+1}^{2^{j+1}}a_t\right)^\gamma \\
&\ll \sum_{n=0}^\infty 2^{n(r-k)\gamma}\sum_{j=0}^n 2^{j((k+1)\gamma-\gamma/p-1)}\sum_{t=[2^{j-2}]+1}^{2^{j+1}}a_t^\gamma \ll \sum_{j=0}^\infty 2^{j((k+1)\gamma-\gamma/p-1)}\sum_{t=[2^{j-2}]+1}^{2^{j+1}}a_t^\gamma \sum_{n=j}^\infty 2^{n(r-k)\gamma} \\
&\ll \sum_{j=0}^\infty 2^{j((r+1)\gamma-\gamma/p-1)}\sum_{t=[2^{j-2}]+1}^{2^{j+1}}a_t^\gamma \ll \sum_{j=0}^\infty \sum_{t=[2^{j-2}]+1}^{2^{j+1}} t^{(r+1)\gamma-\gamma/p-1}a_t^\gamma \ll \sum_{n=1}^\infty n^{r\gamma+\gamma-\gamma/p-1}a_n^\gamma
\end{aligned}
$$

and

$$
\begin{aligned}
J_{12} &= \sum_{n=0}^{\infty} 2^{nr\gamma} \left(\sum_{i=2^n+1}^{\infty} i^{p-2} a_i^p \right)^{\gamma/p} \ll \sum_{n=0}^{\infty} 2^{nr\gamma} \left(\sum_{i=2^n}^{\infty} i^{p-2} a_{i+1}^p \right)^{\gamma/p} \\
&\ll \sum_{n=0}^{\infty} 2^{nr\gamma} \left(\sum_{j=n}^{\infty} \sum_{i=2^j}^{2^{j+1}} i^{p-2} \left(\sum_{t=[i/2]+1}^{2i} \frac{a_t}{t} \right)^p \right)^{\gamma/p} \ll \sum_{n=0}^{\infty} 2^{nr\gamma} \left(\sum_{j=n}^{\infty} 2^{jp-1} \left(\sum_{t=[2^{j-1}]+1}^{2^{j+1}} \frac{a_t}{t} \right)^p \right)^{\gamma/p} \\
&\ll \sum_{n=0}^{\infty} 2^{nr\gamma} \sum_{j=n}^{\infty} 2^{j\gamma - \gamma/p} \left(\sum_{t=[2^{j-1}]+1}^{2^{j+1}} \frac{a_t}{t} \right)^{\gamma} \ll \sum_{j=0}^{\infty} 2^{j\gamma - \gamma/p} \left(\sum_{t=[2^{j-1}]+1}^{2^{j+1}} \frac{a_t}{t} \right)^{\gamma} \sum_{n=0}^{j} 2^{nr\gamma} \\
&\ll \sum_{j=0}^{\infty} 2^{jr\gamma + \gamma - \gamma/p - 1} \sum_{t=[2^{j-1}]+1}^{2^{j+1}} a_t^{\gamma} \ll \sum_{n=1}^{\infty} n^{r\gamma + \gamma - \gamma/p - 1} a_n^{\gamma}.
\end{aligned}
\tag{5.12}
$$

The inequalities (5.11) and (5.12) verify $J \ll J_1$ for $\gamma/p \le 1$, and consequently we complete the proof of sufficiency under $\gamma \ge 1$.

Necessity. Now, we prove that $J \gg J_1$, we start again with (5.9) and use Theorem 3.2, thus we get that

$$
\begin{aligned}
J &\gg \sum_{n=0}^{\infty} 2^{n(r-k)\gamma} \left(\sum_{i=1}^{2^n} i^{(k+1)p-2} a_i^p \right)^{\gamma/p} + \sum_{n=0}^{\infty} 2^{nr\gamma} \left(\sum_{i=2^n+1}^{\infty} i^{p-2} a_i^p \right)^{\gamma/p} \\
&: = J_{21} + J_{22}
\end{aligned}
\tag{5.13}
$$

Similarly, we distinguish two cases listed under (C) and (D):

Case (C): If $\gamma/p \ge 1$, since $A \in$ GBVS, we know that $a_n \le a_m$ when $m \le n \le 2m$. From this the property, we can deduce that

$$
2^j a_{2^j+1}^{\gamma} \gg \sum_{i=2^j+1}^{2^{j+1}} a_i^{\gamma}.
$$

Combining Lemma 4.1, Lemma 4.2, Lemma 4.9 and Hölder's inequality, we can obtain that

$$
\begin{aligned}
J_{21} &\gg \sum_{n=0}^{\infty} 2^{n(r-k)\gamma} \left(\sum_{i=1}^{2^n} i^{(k+1)p-2} a_i^p \right)^{\gamma/p} \gg \sum_{n=0}^{\infty} 2^{n(r-k)\gamma} \left(\sum_{j=0}^{n-1} \sum_{i=2^j}^{2^{j+1}} i^{(k+1)p-2} a_i^p \right)^{\gamma/p} \\
&\gg \sum_{n=0}^{\infty} 2^{n(r-k)\gamma} \left(\sum_{j=0}^{n-1} \sum_{i=[2^{j-1}]+1}^{2^{j+1}} i^{(k+1)p-2} a_i^p \right)^{\gamma/p} \gg \sum_{n=0}^{\infty} 2^{n(r-k)\gamma} \left(\sum_{j=0}^{n-1} 2^{j((k+1)p-1)} \frac{1}{2^j} \sum_{i=[2^{j-1}]+1}^{2^{j+1}} a_i^p \right)^{\gamma/p} \\
&\gg \sum_{n=0}^{\infty} 2^{n(r-k)\gamma} \sum_{j=0}^{n-1} 2^{j((k+1)r-\gamma/p)} \left(\frac{1}{2^j} \sum_{i=[2^{j-1}]+1}^{2^{j+1}} a_i^p \right)^{\gamma/p} \gg \sum_{n=0}^{\infty} 2^{n(r-k)\gamma} \sum_{j=0}^{n-1} 2^{j((k+1)r-\gamma/p)} \left(\frac{1}{2^j} \sum_{i=[2^{j-1}]+1}^{2^{j+1}} a_i \right)^{\gamma} \\
&\gg \sum_{n=0}^{\infty} 2^{n(r-k)\gamma} \sum_{j=0}^{n-1} 2^{j((k+1)r-\gamma/p)} a_{2^j+1}^{\gamma} \gg \sum_{j=0}^{\infty} 2^{j((k+1)r-\gamma/p)} a_{2^j+1}^{\gamma} \sum_{n=j}^{\infty} 2^{n(r-k)\gamma} \gg \sum_{j=0}^{\infty} 2^{j(r\gamma + r - \gamma/p - 1)} 2^j a_{2^j+1}^{\gamma} \\
&\gg \sum_{j=0}^{\infty} 2^{j(r\gamma + r - \gamma/p - 1)} \sum_{i=2^j+1}^{2^{j+1}} a_i^{\gamma} \gg \sum_{j=0}^{\infty} \sum_{i=2^j+1}^{2^{j+1}} i^{(r\gamma + r - \gamma/p - 1)} a_i^{\gamma} \gg \sum_{n=1}^{\infty} n^{r\gamma + \gamma - \gamma/p - 1} a_n^{\gamma}.
\end{aligned}
$$

Similarly, we can get that

$$
J_{22} \gg \sum_{n=1}^{\infty} n^{r\gamma + \gamma - \gamma/p - 1} a_n^{\gamma}.
\tag{5.14}
$$

From the above two estimates we get that $J \gg J_1$ under $\gamma/p \ge 1$.

Case (D): If $\gamma/p < 1$, using (5.6), Theorem 3.2 and Lemma 4.1, we can obtain that

$$
\begin{aligned}
J \quad &\gg \quad \sum_{n=1}^{\infty} n^{(r-k)\gamma-1}\left(\sum_{i=1}^{n} i^{(k+1)p-2}a_i^p\right)^{\gamma/p} + \sum_{n=1}^{\infty} n^{r\gamma-1}\left(\sum_{j=n+1}^{\infty} j^{p-2}a_j^p\right)^{\gamma/p} \\
&\gg \quad \sum_{n=1}^{\infty} n^{r\gamma+\gamma-\frac{\gamma}{p}-1-\frac{(r-k)\gamma^2}{p}}a_n^\gamma\left(\sum_{i=n}^{\infty} i^{(r-k)\gamma-1}\right)^{\gamma/p} + \sum_{n=1}^{\infty} n^{r\gamma+\gamma-\frac{\gamma}{p}-1-\frac{r\gamma^2}{p}}a_n^\gamma\left(\sum_{i=1}^{n} i^{r\gamma-1}\right)^{\gamma/p} \\
&\gg \quad \sum_{n=1}^{\infty} n^{r\gamma+\gamma-\gamma/p-1}a_n^\gamma = J_1.
\end{aligned}
$$

(5.15)

The inequality (5.15) verify $J \gg J_1$ for $\gamma/p < 1$, and consequently we complete the proof of necessity under $\gamma \geq 1$.

Case (ii): $0 < \gamma < 1$, in this case, we easily know that $\gamma/p < 1$.

Necessity. Necessity can be proved by (5.15).

Sufficiency. Applying (5.10), Lemma 4.1, Lemma 4.2 and Lemma 4.9, we can obtain that

$$
\begin{aligned}
J &\ll \sum_{n=0}^{\infty} 2^{n(r-k)\gamma}\left(\sum_{i=1}^{2^n} i^{(k+1)p-2}a_i^p\right)^{\gamma/p} + \sum_{n=0}^{\infty} 2^{nr\gamma}\left(\sum_{i=2^n+1}^{\infty} i^{p-2}a_i^p\right)^{\gamma/p} \\
&\ll \sum_{n=0}^{\infty} 2^{n(r-k)\gamma}\left(\sum_{j=0}^{n}\sum_{i=2^j}^{2^{j+1}} i^{(k+1)p-2}a_i^p\right)^{\gamma/p} + \sum_{n=0}^{\infty} 2^{nr\gamma}\left(\sum_{j=n}^{\infty}\sum_{i=2^j}^{2^{j+1}} i^{p-2}a_i^p\right)^{\gamma/p} \\
&\ll \sum_{n=0}^{\infty} 2^{n(r-k)\gamma}\left(\sum_{j=0}^{n}\sum_{i=2^j}^{2^{j+1}} i^{(k+1)p-2}\left(\sum_{l=[i/4]+1}^{2i}\frac{a_l}{l}\right)^p\right)^{\gamma/p} + \sum_{n=0}^{\infty} 2^{nr\gamma}\left(\sum_{j=n}^{\infty}\sum_{i=2^j}^{2^{j+1}} i^{p-2}\left(\sum_{l=[i/4]+1}^{2i}\frac{a_l}{l}\right)^p\right)^{\gamma/p} \\
&\ll \sum_{n=0}^{\infty} 2^{n(r-k)\gamma}\left(\sum_{j=0}^{n}\sum_{i=2^j}^{2^{j+1}} i^{kp-2}\left(\sum_{l=[2^{j-2}]+1}^{2^{j+2}} a_l\right)^p\right)^{\gamma/p} + \sum_{n=0}^{\infty} 2^{nr\gamma}\left(\sum_{j=n}^{\infty}\sum_{i=2^j}^{2^{j+1}} i^{-2}\left(\sum_{l=[2^{j-2}]+1}^{2^{j+2}} a_l\right)^p\right)^{\gamma/p} \\
&\ll \sum_{n=0}^{\infty} 2^{n(r-k)\gamma}\left(\sum_{j=0}^{n} 2^{j(kp-1)}\left(\sum_{l=[2^{j-2}]+1}^{2^{j+2}} a_l\right)^p\right)^{\gamma/p} + \sum_{n=0}^{\infty} 2^{nr\gamma}\left(\sum_{j=n}^{\infty} 2^{-j}\left(\sum_{l=[2^{j-2}]+1}^{2^{j+2}} a_l\right)^p\right)^{\gamma/p} \\
&\ll \sum_{n=0}^{\infty} 2^{n(r-k)\gamma}\sum_{j=0}^{n} 2^{j(k\gamma-\gamma/p)}\left(\sum_{l=[2^{j-2}]+1}^{2^{j+2}} a_l\right)^\gamma + \sum_{n=0}^{\infty} 2^{nr\gamma}\sum_{j=n}^{\infty} 2^{-j\gamma/p}\left(\sum_{l=[2^{j-2}]+1}^{2^{j+2}} a_l\right)^\gamma \\
&\ll \sum_{n=0}^{\infty} 2^{n(r-k)\gamma}\sum_{j=0}^{n} 2^{j(k\gamma-\gamma/p)}\sum_{l=[2^{j-2}]+1}^{2^{j+2}} a_l^\gamma + \sum_{n=0}^{\infty} 2^{nr\gamma}\sum_{j=n}^{\infty} 2^{-j\gamma/p}\sum_{l=[2^{j-2}]+1}^{2^{j+2}} a_l^\gamma \\
&\ll \sum_{j=0}^{\infty} 2^{j(k\gamma-\gamma/p)}\sum_{l=[2^{j-2}]+1}^{2^{j+2}} a_l^\gamma\sum_{n=j}^{\infty} 2^{n(r-k)\gamma} + \sum_{j=0}^{\infty} 2^{-j\gamma/p}\sum_{l=[2^{j-2}]+1}^{2^{j+2}} a_l^\gamma\sum_{n=0}^{j} 2^{nr\gamma} \\
&\ll \sum_{j=0}^{\infty} 2^{j(r\gamma-\gamma/p)}\sum_{l=[2^{j-2}]+1}^{2^{j+2}} a_l^\gamma \ll \sum_{n=0}^{\infty}\sum_{i=[2^{n-2}]+1}^{2^{n+2}} i^{(r\gamma-\gamma/p)}a_i^\gamma \\
&\ll \sum_{n=1}^{\infty} n^{r\gamma-\gamma/p}a_n^\gamma \ll J_2.
\end{aligned}
$$

This ends our proof of Theorem 3.3. ∎

6   Acknowledgement

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Existence and uniqueness results for a nonlocal q-fractional integral boundary value problem of sequential orders

Bashir Ahmad[1], Yong Zhou[2], Ahmed Alsaedi[1] and Hana Al-Hutami[1]

[1]Department of Mathematics, Faculty of Science, King Abdulaziz University,
P.O. Box 80203, Jeddah 21589, Saudi Arabia
e-mail: bashirahmad_qau@yahoo.com (B. Ahmad), aalsaedi@hotmail.com (A. Alsaedi),
hanno.1407@hotmail.com (H. Al-Hutami)

[2]Faculty of Mathematics and Computational Science, Xiangtan University, P.R. China
e-mail: yzhou@xtu.edu.cn

Abstract

In this paper, we discuss the existence of solutions for a new boundary value problem of nonlinear q-fractional integral equations involving fractional orders $0 < \beta \le 1$, $1 < \gamma \le 2$ and nonlocal q-integral boundary conditions. Our results rely on classical tools of fixed point theory. We demonstrate the application of our work with the aid of an example.

Key words and phrases: Sequential; fractional integro-differential equations; boundary conditions; existence; fixed point.
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1   Introduction

Fractional calculus has developed into a useful mathematical tool for modelling of several real world phenomena occurring in applied and technical sciences ([1]-[3]). As a matter of fact, fractional-order models are replacing their integer-order counterparts due to the ability of fractional-order operators to describe the hereditary properties of processes and phenomena involved in the models under consideration. For examples and details, we refer to a series of papers [4]-[10]) and the references cited therein.

Motivated by the popularity of fractional differential equations, q-difference equations of fractional-order are also attracting a considerable attention. Fractional q-difference equations may be regarded as fractional analogue of q-difference equations. For earlier work on the topic, we refer to ([11]-[12]), while some recent development of fractional q-difference equations, for instance, can be found in ([13]-[21]). The basic concepts of q-fractional calculus can be found in a recent text [22].

In this paper, we consider a nonlocal fractional q-difference integral boundary value problem of sequential orders given by

$$^cD_q^\beta(^cD_q^\gamma + \lambda)x(t) = pf(t, x(t)) + kI_q^\xi g(t, x(t)), \quad 0 \le t \le 1, \ 0 < q < 1, \tag{1}$$

$$\begin{cases} x(0) = aI_q^{\alpha-1}x(\eta) = a\int_0^\eta \frac{(\eta - qs)^{(\alpha-2)}}{\Gamma_q(\alpha - 1)}x(s)d_qs, \\ x(1) = bI_q^{\alpha-1}x(\sigma) = b\int_0^\sigma \frac{(\sigma - qs)^{(\alpha-2)}}{\Gamma_q(\alpha - 1)}x(s)d_qs, \ \alpha > 2, \ 0 < \eta, \sigma < 1, \\ D_qx(1) = 0, \end{cases} \tag{2}$$

where $^cD_q^\beta$ and $^cD_q^\gamma$ denote the fractional q-derivative of the Caputo type, $0 < \beta \le 1$, $1 < \gamma \le 2$, $I_{q,0}^\xi(.) = I_q^\xi(.)$ denotes Riemann-Liouville integral with $0 < \xi < 1$, f, g are given continuous functions, and $\lambda, \ p, \ k$ are real constants.

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The paper is organized as follows. Section 2 contains some necessary background material on the topic, while the main results are presented in Section 3. We make use of Banach's contraction principle, Krasnoselskii's fixed point theorem and Leray-Schauder nonlinear alternative to establish the existence results for the problem at hand. Although these tools are standard, yet their exposition in the framework of the present problem is new.

2 Preliminaries on fractional q-calculus

This section is devoted to the notations of and basic concepts of q-fractional calculus [23]-[24].

A q-real number for a real parameter $q \in \mathbb{R}^+ \setminus \{1\}$, denoted by $[u]_q$, is defined by

$$[u]_q = \frac{1 - q^u}{1 - q}, \quad u \in \mathbb{R}.$$

The q-analogue of the Pochhammer symbol (q-shifted factorial) is defined as

$$(u; q)_0 = 1, \quad (u; q)_k = \prod_{i=0}^{k-1} (1 - uq^i), \ k \in \mathbb{N} \cup \{\infty\}.$$

The q-analogue of the exponent $(u - v)^k$ is

$$(u - v)^{(0)} = 1, \quad (u - v)^{(k)} = \prod_{j=0}^{k-1} (u - vq^j), \ k \in \mathbb{N}, \ u, v \in \mathbb{R}.$$

The q-gamma function $\Gamma_q(u)$ is defined as

$$\Gamma_q(u) = \frac{(1 - q)^{(u-1)}}{(1 - q)^{u-1}},$$

where $u \in \mathbb{R} \setminus \{0, -1, -2, \ldots\}$. Observe that $\Gamma_q(v + 1) = [v]_q \Gamma_q(v)$.

Definition 2.1 *([23]) Let f be a function defined on $[0, 1]$. The fractional q-integral of the Riemann-Liouville type of order $\beta \geq 0$ is $(I_q^0 f)(t) = f(t)$ and*

$$I_q^\beta f(t) := \int_0^t \frac{(t - qs)^{(\beta-1)}}{\Gamma_q(\beta)} f(s) d_q s = t^\beta (1 - q)^\beta \sum_{k=0}^\infty q^k \frac{(q^\beta; q)_n}{(q; q)_n} f(tq^k), \quad \beta > 0, \ t \in [0, 1].$$

Observe that $\beta = 1$ in the Definition 2.1 yields q-integral

$$I_q f(t) := \int_0^t f(s) d_q s = t(1 - q) \sum_{k=0}^\infty q^k f(tq^k).$$

For more details on q-integral and fractional q-integral, see Section 1.3 and Section 4.2 respectively in [22].

Remark 2.2 *The q-fractional integration possesses the semigroup property (Proposition 4.3 [22]):*

$$I_q^\gamma I_q^\beta f(t) = I_q^{\beta+\gamma} f(t); \ \gamma, \beta \in \mathbb{R}^+.$$

Further, it has been shown in Lemma 6 of [24] that

$$I_q^\beta(x)^{(\sigma)} = \frac{\Gamma_q(\sigma + 1)}{\Gamma_q(\beta + \sigma + 1)} (x)^{(\beta+\sigma)}, \ 0 < x < a, \beta \in \mathbb{R}^+, \sigma \in (-1, \infty).$$

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Before giving the definition of fractional q-derivative, we recall the concept of q-derivative. We know that the q-derivative of a function $f(t)$ is defined as

$$(D_q f)(t) = \frac{f(t) - f(qt)}{t - qt}, \ t \neq 0, \ (D_q f)(0) = \lim_{t \to 0}(D_q f)(t).$$

Furthermore,

$$D_q^0 f = f, \ D_q^n f = D_q(D_q^{n-1} f), \ n = 1, 2, 3, \ldots. \tag{3}$$

Definition 2.3 *([22]) The Caputo fractional q-derivative of order $\beta > 0$ is defined by*

$$^c D_q^\beta f(t) = I_q^{\lceil \beta \rceil - \beta} D_q^{\lceil \beta \rceil} f(t),$$

where $\lceil \beta \rceil$ is the smallest integer greater than or equal to β.

Next we recall some properties involving Riemann-Liouville q-fractional integral and Caputo fractional q-derivative (Theorem 5.2 [22]):

$$I_q^\beta \, ^c D_q^\beta f(t) = f(t) - \sum_{k=0}^{\lceil \beta \rceil - 1} \frac{t^k}{\Gamma_q(k+1)} (D_q^k f)(0^+), \ \forall \ t \in (0, a], \ \beta > 0; \tag{4}$$

$$^c D_q^\beta I_q^\beta f(t) = f(t), \quad \forall \ t \in (0, a], \ \beta > 0. \tag{5}$$

In order to define the solution for the problem (1)-(2), we need the following lemma.

Lemma 2.4 *For a given $h \in C([0,1], \mathbb{R})$, the unique solution of the linear boundary value problem:*

$$^c D_q^\beta (^c D_q^\gamma + \lambda) x(t) = h(t), \quad 0 \leq t \leq 1, \ 0 < q < 1, \tag{6}$$

$$\begin{cases} x(0) = a I_q^{\alpha-1} x(\eta) = a \int_0^\eta \frac{(\eta - qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} x(s) d_q s, \\ x(1) = b I_q^{\alpha-1} x(\sigma) = b \int_0^\sigma \frac{(\sigma - qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} x(s) d_q s, \ \alpha > 2, \ 0 < \eta, \sigma < 1, \\ D_q x(1) = 0, \end{cases} \tag{7}$$

is given by

$$\begin{aligned} x(t) &= \int_0^t \frac{(t - qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(I_q^\beta h(u) - \lambda x(u) \Big) d_q u \\ &+ a A(t) \int_0^\eta \frac{(\eta - qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \Big(\int_0^s \frac{(s - qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(I_q^\beta h(u) - \lambda x(u) \Big) d_q u \Big) d_q s \\ &- b B(t) \int_0^\sigma \frac{(\sigma - qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \Big(\int_0^s \frac{(s - qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(I_q^\beta h(u) - \lambda x(u) \Big) d_q u \Big) d_q s \\ &+ B(t) \int_0^1 \frac{(1 - qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(I_q^\beta h(u) - \lambda x(u) \Big) d_q u \\ &- C(t) \int_0^1 \frac{(1 - qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)} \Big(I_q^\beta h(u) - \lambda x(u) \Big) d_q u \Big) d_q s, \end{aligned} \tag{8}$$

where

$$A(t) = \frac{1}{\Delta} \Big[\Big(\mu_5 [\gamma-2]_q - \mu_6 [\gamma-1]_q \Big) t^\gamma - \Big(\mu_4 [\gamma-2]_q - \mu_6 [\gamma]_q \Big) t^{\gamma-1} + \Big(\mu_4 [\gamma-1]_q - \mu_5 [\gamma]_q \Big) t^{\gamma-2} \Big], \tag{9}$$

$$B(t) = \frac{1}{\Delta} \Big[\Big(\mu_2 [\gamma-2]_q - \mu_3 [\gamma-1]_q \Big) t^\gamma - \Big(\mu_1 [\gamma-2]_q - \mu_3 [\gamma]_q \Big) t^{\gamma-1} + \Big(\mu_1 [\gamma-1]_q - \mu_2 [\gamma]_q \Big) t^{(\gamma-2)} \Big], \tag{10}$$

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$$C(t) = \frac{1}{\Delta}\left[\left(\mu_3\mu_5 - \mu_2\mu_6\right)t^\gamma - \left(\mu_3\mu_4 - \mu_1\mu_6\right)t^{\gamma-1} + \left(\mu_2\mu_4 - \mu_1\mu_5\right)t^{\gamma-2}\right], \tag{11}$$

$$\mu_1 = \left(\frac{a\eta^{(\gamma+\alpha-1)}\Gamma_q(\gamma+1)}{\Gamma_q(\gamma+\alpha)}\right), \quad \mu_2 = \left(\frac{a\eta^{(\gamma+\alpha-2)}\Gamma_q(\gamma)}{\Gamma_q(\gamma+\alpha-1)}\right),$$

$$\mu_3 = \left(\frac{a\eta^{(\gamma+\alpha-3)}\Gamma_q(\gamma-1)}{\Gamma_q(\gamma+\alpha-2)}\right), \quad \mu_4 = \left(\frac{b\sigma^{(\gamma+\alpha-1)}\Gamma_q(\gamma+1)}{\Gamma_q(\gamma+\alpha)} - 1\right),$$

$$\mu_5 = \left(\frac{b\sigma^{(\gamma+\alpha-2)}\Gamma_q(\gamma)}{\Gamma_q(\gamma+\alpha-1)} - 1\right), \quad \mu_6 = \left(\frac{b\sigma^{(\gamma+\alpha-3)}\Gamma_q(\gamma-1)}{\Gamma_q(\gamma+\alpha-2)} - 1\right),$$

$$\Delta = (\mu_1\mu_5 - \mu_2\mu_4)[\gamma-2]_q + (\mu_3\mu_4 - \mu_1\mu_6)[\gamma-1]_q + (\mu_2\mu_6 - \mu_3\mu_5)[\gamma]_q \neq 0.$$

Proof. Using (4), the solution $x(t)$ of (6) can be written as

$$x(t) = \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\left(I_q^\beta h(u) - \lambda x(u)\right)d_q u - \frac{t^\gamma}{\Gamma_q(\gamma+1)}c_0 - t^{\gamma-1}c_1 - t^{\gamma-2}c_2. \tag{12}$$

q-differentiating both sides of (12), we obtain

$$D_q x(t) = \int_0^t \frac{(t-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)}\left(I_q^\beta h(u) - \lambda x(u)\right)d_q u$$
$$-\frac{[\gamma]_q t^{\gamma-1}}{\Gamma_q(\gamma+1)}c_0 - [\gamma-1]_q t^{\gamma-2}c_1 - [\gamma-2]_q t^{\gamma-3}c_2, \quad t \in [0,1]. \tag{13}$$

Using the boundary conditions (7) in (12), we have

$$\frac{1}{\Gamma_q(\gamma+1)}\left(\frac{a\eta^{(\gamma+\alpha-1)}\Gamma_q(\gamma+1)}{\Gamma_q(\gamma+\alpha)}\right)c_0 + \left(\frac{a\eta^{(\gamma+\alpha-2)}\Gamma_q(\gamma)}{\Gamma_q(\gamma+\alpha-1)}\right)c_1 + \left(\frac{a\eta^{(\gamma+\alpha-3)}\Gamma_q(\gamma-1)}{\Gamma_q(\gamma+\alpha-2)}\right)c_2$$
$$= a\int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\left(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\left(I_q^\beta h(u) - \lambda x(u)\right)d_q u\right)d_q s,$$

$$\frac{1}{\Gamma_q(\gamma+1)}\left(\frac{b\sigma^{(\gamma+\alpha-1)}\Gamma_q(\gamma+1)}{\Gamma_q(\gamma+\alpha)} - 1\right)c_0 + \left(\frac{b\sigma^{(\gamma+\alpha-2)}\Gamma_q(\gamma)}{\Gamma_q(\gamma+\alpha-1)} - 1\right)c_1$$
$$+\left(\frac{b\sigma^{(\gamma+\alpha-3)}\Gamma_q(\gamma-1)}{\Gamma_q(\gamma+\alpha-2)} - 1\right)c_2$$
$$= b\int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\left(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\left(I_q^\beta h(u) - \lambda x(u)\right)d_q u\right)d_q s$$
$$-\int_0^1 \frac{(1-qu)^{\gamma-1}}{\Gamma_q(\gamma)}\left(I_q^\beta h(u) - \lambda x(u)\right)d_q u,$$

$$\frac{1}{\Gamma_q(\gamma+1)}[\gamma]_q c_0 + [\gamma-1]_q c_1 + [\gamma-2]_q c_2 = \int_0^1 \frac{(1-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)}\left(I_q^\beta h(u) - \lambda x(u)\right)d_q u.$$

Solving the above system of equations for c_0, c_1, c_2, we get

$$c_0 = \frac{\Gamma_q(\gamma+1)}{\Delta}\left[\left(\mu_5[\gamma-2]_q - \mu_6[\gamma-1]_q\right)a\int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\left(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\right.\right.$$
$$\times\left(I_q^\beta h(u) - \lambda x(u)\right)d_q u\bigg)d_q s$$
$$-\left(\mu_2[\gamma-2]_q - \mu_3[\gamma-1]_q\right)b\int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\left(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\right.$$
$$\times\left(I_q^\beta h(u) - \lambda x(u)\right)d_q u\bigg)d_q s$$

$$+\Big(\mu_2[\gamma-2]_q-\mu_3[\gamma-1]_q\Big)\int_0^1\frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(I_q^\beta h(u)-\lambda x(u)\Big)d_qu$$

$$-\Big(\mu_3\mu_5-\mu_2\mu_6\Big)\int_0^1\frac{(1-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)}\Big(I_q^\beta h(u)-\lambda x(u)\Big)d_qu\Big],$$

$$c_1=\frac{-1}{\Delta}\Big[\Big(\mu_4[\gamma-2]_q-\mu_6[\gamma]_q\Big)a\int_0^\eta\frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s\frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}$$

$$\times\Big(I_q^\beta h(u)-\lambda x(u)\Big)d_qu\Big)d_qs$$

$$-\Big(\mu_1[\gamma-2]_q-\mu_3[\gamma]_q\Big)b\int_0^\sigma\frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s\frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}$$

$$\times\Big(I_q^\beta h(u)-\lambda x(u)\Big)d_qu\Big)d_qs$$

$$+\Big(\mu_1[\gamma-2]_q-\mu_3[\gamma]_q\Big)\int_0^1\frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(I_q^\beta h(u)-\lambda x(u)\Big)d_qu$$

$$-\Big(\mu_3\mu_4-\mu_1\mu_6\Big)\int_0^1\frac{(1-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)}\Big(I_q^\beta h(u)-\lambda x(u)\Big)d_qu\Big],$$

$$c_2=\frac{1}{\Delta}\Big[\Big(\mu_4[\gamma-1]_q-\mu_5[\gamma]_q\Big)a\int_0^\eta\frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s\frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}$$

$$\times\Big(I_q^\beta h(u)-\lambda x(u)\Big)d_qu\Big)d_qs$$

$$-\Big(\mu_1[\gamma-1]_q-\mu_2[\gamma]_q\Big)b\int_0^\sigma\frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s\frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}$$

$$\times\Big(I_q^\beta h(u)-\lambda x(u)\Big)d_qu\Big)d_qs$$

$$+\Big(\mu_1[\gamma-1]_q-\mu_2[\gamma]_q\Big)\int_0^1\frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(I_q^\beta h(u)-\lambda x(u)\Big)d_qu$$

$$-\Big(\mu_2\mu_4-\mu_1\mu_5\Big)\int_0^1\frac{(1-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)}\Big(I_q^\beta h(u)-\lambda x(u)\Big)d_qu\Big].$$

Substituting the values of c_0, c_1 and c_2 in (12) yields the solution (8). This completes the proof. $\square$

3 Main results

Let $\mathcal{C}=C([0,1],\mathbb{R})$ denote the Banach space of all continuous functions from $[0,1]$ into $\mathbb{R}$ endowed with the usual norm defined by $\|x\|=\sup\{|x(t)|, t\in[0,1]\}$.

In the sequel, we need the following assumptions:

(A_1) $f,g:[0,1]\times\mathbb{R}\to\mathbb{R}$ are continuous functions such that $|f(t,x)-f(t,y)|\le L_1|x-y|$ and $|g(t,x)-g(t,y)|\le L_2|x-y|$, $\forall t\in[0,1]$, L_1, $L_2>0$, $x,y\in\mathbb{R}$;

(A_2) there exist $\delta_1,\delta_2\in C([0,1],\mathbb{R}^+)$ with $|f(t,x)|\le\delta_1(t)$, $|g(t,x)|\le\delta_2(t)$, $\forall(t,x)\in[0,1]\times\mathbb{R}$, where $\sup_{t\in[0,1]}|\delta_i(t)|=\|\delta_i\|$, $i=1,2$.

For the sake of computational convenience, let us set the following notations:

$$\omega_1=\frac{(1+B_1)}{\Gamma_q(\beta+\gamma+1)}+\frac{1}{\Gamma_q(\beta+\gamma+\alpha)}\Big(|a|A_1\eta^{(\beta+\gamma+\alpha-1)}+|b|B_1\sigma^{(\beta+\gamma+\alpha-1)}\Big)+\frac{C_1}{\Gamma_q(\beta+\gamma)}, \tag{14}$$

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$$\omega_2 = \frac{(1+B_1)}{\Gamma_q(\beta+\xi+\gamma+1)} + \frac{1}{\Gamma_q(\beta+\xi+\gamma+\alpha)}\Big(|a|A_1\eta^{(\beta+\xi+\gamma+\alpha-1)} + |b|B_1\sigma^{(\beta+\xi+\gamma+\alpha-1)}\Big)$$
$$+\frac{C_1}{\Gamma_q(\beta+\xi+\gamma)}, \tag{15}$$

$$\omega_3 = \frac{(1+B_1)}{\Gamma_q(\gamma+1)} + \frac{1}{\Gamma_q(\gamma+\alpha)}\Big(|a|A_1\eta^{(\gamma+\alpha-1)} + |b|B_1\eta^{(\gamma+\alpha-1)}\Big) + \frac{C_1}{\Gamma_q(\gamma)}, \tag{16}$$

$$\Omega = L\Big[|p|\Big(\frac{1}{\Gamma_q(\beta+\gamma+\alpha)}\Big(|a|A_1\eta^{(\beta+\gamma+\alpha-1)} + |b|B_1\sigma^{(\beta+\gamma+\alpha-1)}\Big)$$
$$+\frac{B_1}{\Gamma_q(\beta+\gamma+1)} + \frac{C_1}{\Gamma_q(\beta+\gamma)}\Big)$$
$$+|k|\Big(\frac{1}{\Gamma_q(\beta+\xi+\gamma+\alpha)}\Big(|a|A_1\eta^{(\beta+\xi+\gamma+\alpha-1)} + |b|B_1\sigma^{(\beta+\xi+\gamma+\alpha-1)}\Big)$$
$$+\frac{B_1}{\Gamma_q(\beta+\xi+\gamma+1)} + \frac{C_1}{\Gamma_q(\beta+\xi+\gamma)}\Big)\Big] \tag{17}$$
$$+|\lambda|\Big[\frac{1}{\Gamma_q(\gamma+\alpha)}\Big(|a|A_1\eta^{(\gamma+\alpha-1)} + |b|B_1\sigma^{(\gamma+\alpha-1)}\Big) + \frac{B_1}{\Gamma_q(\gamma+1)} + \frac{C_1}{\Gamma_q(\gamma)}\Big],$$

where $A_1 = \max_{t\in[0,1]}|A(t)|$, $B_1 = \max_{t\in[0,1]}|B(t)|$, $C_1 = \max_{t\in[0,1]}|C(t)|$, $L = \max\{L_1, L_2\}$ and $A(t), B(t), C(t)$ are respectively given by (9), (10) and (11).

In view of Lemma 2.4, we define an operator $\mathcal{F}: \mathcal{C} \to \mathcal{C}$ as

$$(\mathcal{F}x)(t)$$
$$= \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(p\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}f(m, x(m))d_qm$$
$$+k\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}g(m, x(m))d_qm - \lambda x(u)\Big)d_qu$$
$$+aA(t)\int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(p\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}f(m, x(m))d_qm$$
$$+k\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}g(m, x(m))d_qm - \lambda x(u)\Big)d_qu\Big)d_qs$$
$$-bB(t)\int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(p\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}f(m, x(m))d_qm$$
$$+k\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}g(m, x(m))d_qm - \lambda x(u)\Big)d_qu\Big)d_qs \tag{18}$$
$$+B(t)\int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(p\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}f(m, x(m))d_qm$$
$$+k\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}g(m, x(m))d_qm - \lambda x(u)\Big)d_qu$$
$$-C(t)\int_0^1 \frac{(1-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)}\Big(p\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}f(m, x(m))d_qm$$
$$+k\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}g(m, x(m))d_qm - \lambda x(u)\Big)d_qu.$$

Observe that problem (1)-(2) has solutions only if the operator equation $x = \mathcal{F}x$ has fixed points.

Our first existence result is based on Krasnoselskii's fixed point theorem.

Lemma 3.1 *(Krasnoselskii) [25]. Let Y be a closed, convex, bounded and nonempty subset of a Banach space X. Let Q_1, Q_2 be the operators such that (a) $Q_1x + Q_2y \in Y$ whenever $x, y \in Y$; (b) Q_1 is compact and continuous and (c) Q_2 is a contraction mapping. Then there exists $z \in Y$ such that $z = Q_1z + Q_2z$.*

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Theorem 3.2 *Let $f, g : [0,1] \times \mathbb{R} \to \mathbb{R}$ be continuous functions satisfying the assumption $(A_1) - (A_2)$. Furthermore $\Omega < 1$, where Ω is given by (17) Then the problem (1)-(2) has at least one solution on $[0,1]$.*

Proof. Let us fix

$$\varepsilon \geq \frac{|p|\|\delta_1\|\omega_1 + |k|\|\delta_2\|\omega_2}{1 - |\lambda|\omega_3},$$

where ω_1, ω_2, ω_3 are respectively given by (14), (15), (16), and consider $B_\varepsilon = \{x \in \mathcal{C} : \|x\| \leq \varepsilon\}$. We define operators $\mathcal{S}_1$ and $\mathcal{S}_2$ on B_ε as

$$
\begin{aligned}
&(\mathcal{S}_1 x)(t)\\
&= \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(p\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}f(m,x(m))d_q m\\
&\quad + k\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}g(m,x(m))d_q m - \lambda x(u)\Big)d_q u, \quad t \in [0,1],
\end{aligned}
$$

$$
\begin{aligned}
&(\mathcal{S}_2 x)(t)\\
&= aA(t)\int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(p\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}f(m,x(m))d_q m\\
&\quad + k\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}g(m,x(m))d_q m - \lambda x(u)\Big)d_q u\Big)d_q s\\
&\quad - bB(t)\int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(p\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}f(m,x(m))d_q m\\
&\quad + k\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}g(m,x(m))d_q m - \lambda x(u)\Big)d_q u\Big)d_q s\\
&\quad + B(t)\int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(p\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}f(m,x(m))d_q m\\
&\quad + k\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}g(m,x(m))d_q m - \lambda x(u)\Big)d_q u\\
&\quad - C(t)\int_0^1 \frac{(1-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)}\Big(p\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}f(m,x(m))d_q m\\
&\quad + k\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}g(m,x(m))d_q m - \lambda x(u)\Big)d_q u, \quad t \in [0,1].
\end{aligned}
$$

For $x, y \in B_\varepsilon$, we find that

$$\|\mathcal{S}_1 x + \mathcal{S}_2 y\| \leq |p|\|\delta_1\|\omega_1 + |k|\|\delta_2\|\omega_2 + |\lambda|\varepsilon\omega_3 \leq \varepsilon.$$

Thus, $\mathcal{S}_1 x + \mathcal{S}_2 y \in B_\varepsilon$. Continuity of f and g imply that the operator $\mathcal{S}_1$ is continuous. Also, $\mathcal{S}_1$ is uniformly bounded on B_ε as

$$\|\mathcal{S}_1 x\| \leq \frac{|p|\|\delta_1\|}{\Gamma_q(\beta+\gamma+1)} + \frac{|k|\|\delta_2\|}{\Gamma_q(\beta+\xi+\gamma+1)} + \frac{|\lambda|\varepsilon}{\Gamma_q(\gamma+1)}.$$

Now we prove the compactness of the operator $\mathcal{S}_1$. In view of (A_1), we define

$$\sup_{(t,x)\in[0,1]\times B_\varepsilon} |f(t,x)| = \overline{f}, \qquad \sup_{(t,x)\in[0,1]\times B_\varepsilon} |g(t,x)| = \overline{g}.$$

Consequently we have

$$
\begin{aligned}
&\|(\mathcal{S}_1 x)(t_2) - (\mathcal{S}_1 x)(t_1)\|\\
&\leq \int_0^{t_1} \frac{(t_2-qu)^{(\gamma-1)} - (t_1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big[|p|\overline{f}\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}d_q m\\
&\quad + |k|\overline{g}\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}d_q m + |\lambda|\varepsilon\Big]d_q u
\end{aligned}
$$

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$$+ \int_{t_1}^{t_2} \frac{(t_2 - qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big[|p| \overline{f} \int_0^u \frac{(u - qm)^{(\beta-1)}}{\Gamma_q(\beta)} d_q m + |k| \overline{g} \int_0^u \frac{(u - qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)} d_q m + |\lambda| \varepsilon \Big] d_q u,$$

which is independent of x and tends to zero as $t_2 \to t_1$. Thus, $\mathcal{S}_1$ is relatively compact on B_ε. Hence, by the Arzelá-Ascoli Theorem, $\mathcal{S}_1$ is compact on B_ε. Now, we shall show that $\mathcal{S}_2$ is a contraction. From (A_1) and for $x, y \in B_\varepsilon$, we have

$$\|\mathcal{S}_2 x - \mathcal{S}_2 y\|$$

$$\leq \sup_{t \in [0,1]} \Bigg\{ |a||A(t)| \int_0^\eta \frac{(\eta - qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \Big(\int_0^s \frac{(s - qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(|p| \int_0^u \frac{(u - qm)^{(\beta-1)}}{\Gamma_q(\beta)}$$

$$\times \Big| f(m, x(m)) - f(m, y(m)) \Big| d_q m$$

$$+ |k| \int_0^u \frac{(u - qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)} \Big| g(m, x(m)) - g(m, y(m)) \Big| d_q m + |\lambda| |x(u) - y(u)| \Big) d_q u \Big) d_q s$$

$$+ |b||B(t)| \int_0^\sigma \frac{(\sigma - qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \Big(\int_0^s \frac{(s - qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(|p| \int_0^u \frac{(u - qm)^{(\beta-1)}}{\Gamma_q(\beta)}$$

$$\times \Big| f(m, x(m)) - f(m, y(m)) \Big| d_q m$$

$$+ |k| \int_0^u \frac{(u - qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)} \Big| g(m, x(m)) - g(m, y(m)) \Big| d_q m + |\lambda| |x(u) - y(u)| \Big) d_q u \Big) d_q s$$

$$+ |B(t)| \int_0^1 \frac{(1 - qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(|p| \int_0^u \frac{(u - qm)^{(\beta-1)}}{\Gamma_q(\beta)} \Big| f(m, x(m)) - f(m, y(m)) \Big| d_q m$$

$$+ |k| \int_0^u \frac{(u - qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)} \Big| g(m, x(m)) - g(m, y(m)) \Big| d_q m + |\lambda| |x(u) - y(u)| \Big) d_q u$$

$$+ |C(t)| \int_0^1 \frac{(1 - qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)} \Big(|p| \int_0^u \frac{(u - qm)^{(\beta-1)}}{\Gamma_q(\beta)} \Big| f(m, x(m)) - f(m, y(m)) \Big| d_q m$$

$$+ |k| \int_0^u \frac{(u - qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)} \Big| g(m, x(m)) - g(m, y(m)) \Big| d_q m + |\lambda| |x(u) - y(u)| \Big) d_q u \Bigg\}$$

$$\leq \sup_{t \in [0,1]} \Bigg\{ |a||A(t)| \int_0^\eta \frac{(\eta - qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \Big(\int_0^s \frac{(s - qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(|p| \int_0^u \frac{(u - qm)^{(\beta-1)}}{\Gamma_q(\beta)} L_1 \Big| x(m) - y(m) \Big| d_q m$$

$$+ |k| \int_0^u \frac{(u - qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)} L_2 \Big| x(m) - y(m) \Big| d_q m + |\lambda| |x(u) - y(u)| \Big) d_q u \Big) d_q s$$

$$+ |b||B(t)| \int_0^\sigma \frac{(\sigma - qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \Big(\int_0^s \frac{(s - qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(|p| \int_0^u \frac{(u - qm)^{(\beta-1)}}{\Gamma_q(\beta)} L_1 \Big| x(m) - y(m) \Big| d_q m$$

$$+ |k| \int_0^u \frac{(u - qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)} L_2 \Big| x(m) - y(m) \Big| d_q m + |\lambda| |x(u) - y(u)| \Big) d_q u \Big) d_q s$$

$$+ |B(t)| \int_0^1 \frac{(1 - qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(|p| \int_0^u \frac{(u - qm)^{(\beta-1)}}{\Gamma_q(\beta)} L_1 \Big| x(m) - y(m) \Big| d_q m$$

$$+ |k| \int_0^u \frac{(u - qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)} L_2 \Big| x(m) - y(m) \Big| d_q m + |\lambda| |x(u) - y(u)| \Big) d_q u$$

$$+ |C(t)| \int_0^1 \frac{(1 - qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)} \Big(|p| \int_0^u \frac{(u - qm)^{(\beta-1)}}{\Gamma_q(\beta)} L_1 \Big| x(m) - y(m) \Big| d_q m$$

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$$+|k|\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}L_2\Big|x(m)-y(m)\Big|d_qm+|\lambda||x(u)-y(u)|\Big)d_qu\Big\}$$

$$\leq \Big[L\Big[|p|\Big(\frac{1}{\Gamma_q(\beta+\gamma+\alpha)}\Big(|a|A_1\eta^{(\beta+\gamma+\alpha-1)}+|b|B_1\sigma^{(\beta+\gamma+\alpha-1)}\Big)+\frac{B_1}{\Gamma_q(\beta+\gamma+1)}+\frac{C_1}{\Gamma_q(\beta+\gamma)}\Big)$$

$$+|k|\Big(\frac{1}{\Gamma_q(\beta+\xi+\gamma+\alpha)}\Big(|a|A_1\eta^{(\beta+\xi+\gamma+\alpha-1)}+|b|B_1\sigma^{(\beta+\xi+\gamma+\alpha-1)}\Big)$$

$$+\frac{B_1}{\Gamma_q(\beta+\xi+\gamma+1)}+\frac{C_1}{\Gamma_q(\beta+\xi+\gamma)}\Big)\Big]$$

$$+|\lambda|\Big[\frac{1}{\Gamma_q(\gamma+\alpha)}\Big(|a|A_1\eta^{(\gamma+\alpha-1)}+|b|B_1\sigma^{(\gamma+\alpha-1)}\Big)+\frac{B_1}{\Gamma_q(\gamma+1)}+\frac{C_1}{\Gamma_q(\gamma)}\Big]\Big]\|x-y\|=\Omega\|x-y\|,$$

where we have used (17). Since $\Omega < 1$ by our assumption, therefore $\mathcal{S}_2$ is a contraction. Thus all the assumptions of Lemma 3.1 are satisfied. So, by the conclusion of Lemma 3.1, the problem $(1)-(2)$ has at least one solution on $[0,1]$. $\qquad\square$

In the next result, we make use of Leray-Schauder Alternative.

Lemma 3.3 *(Nonlinear alternative for single valued maps)[26]. Let E be a Banach space, C a closed, convex subset of E, W an open subset of C and $0 \in W$. Suppose that $\mathcal{F} : \overline{W} \to C$ is a continuous, compact (that is, $\mathcal{F}(\overline{W})$ is a relatively compact subset of C) map. Then either*

(i) *$\mathcal{F}$ has a fixed point in $\overline{W}$, or*

(ii) *there is a $x \in \partial W$ (the boundary of W in C) and $\tau \in (0,1)$ with $x = \tau\mathcal{F}(x)$.*

Theorem 3.4 *Let $f, g : [0,1] \times \mathbb{R} \to \mathbb{R}$ be continuous functions and the following assumptions hold:*

(A_3) *there exist functions $\phi_1, \phi_2 \in C([0,1], \mathbb{R}^+)$, and nondecreasing functions $\Psi_1, \Psi_2 : \mathbb{R}^+ \to \mathbb{R}^+$ such that*
$$|f(t,x)| \leq \phi_1(t)\Psi_1(\|x\|), \quad |g(t,x)| \leq \phi_2(t)\Psi_2(\|x\|), \quad \forall(t,x) \in [0,1] \times \mathbb{R}.$$

(A_4) *There exists a constant $H > 0$ such that*

$$H > \frac{|p|\|\phi_1\|\Psi_1(H)\omega_1+|k|\|\phi_2\|\Psi_2(H)\omega_2}{1-|\lambda|\omega_3},$$

where $|\lambda| < \frac{1}{\omega_3}$.

Then the boundary value problem $(1)-(2)$ has at least one solution on $[0,1]$.

Proof. Consider the operator $\mathcal{F} : \mathcal{C} \to \mathcal{C}$ defined by (18). The proof consists of several steps.

(i) *$\mathcal{F}$ is continuous.*
It is easy to show that $\mathcal{F}$ is continuous.

(ii) *$\mathcal{F}$ maps bounded sets into bounded sets in $C([0,1] \times \mathbb{R})$.*
For a positive number $\overline{r}$, let $B_{\overline{r}} = \{x \in \mathcal{C} : \|x\| \leq \overline{r}\}$ be a bounded set in $C([0,1] \times \mathbb{R})$ and $x \in B_{\overline{r}}$. Then, we have

$$\|(\mathcal{F}x)\|$$
$$\leq \sup_{t\in[0,1]}\Big\{\int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(|p|\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}|f(m,x(m))|d_qm$$

$$+ \quad |k|\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}|g(m,x(m))|d_qm+|\lambda||x(u)|\Big)d_qu$$

$$+ \quad |a||A(t)|\int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(|p|\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}|f(m,x(m))|d_qm$$

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$$
+ \quad |k| \int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)} |g(m,x(m))| d_q m + |\lambda||x(u)| \Big) d_q u \Big) d_q s
$$

$$
+ \quad |b||B(t)| \int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(|p| \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} |f(m,x(m))| d_q m
$$

$$
+ \quad |k| \int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)} |g(m,x(m))| d_q m + |\lambda||x(u)| \Big) d_q u \Big) d_q s
$$

$$
+ \quad |B(t)| \int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(|p| \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} |f(m,x(m))| d_q m
$$

$$
+ \quad |k| \int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)} |g(m,x(m))| d_q m + |\lambda||x(u)| \Big) d_q u
$$

$$
+ \quad |C(t)| \int_0^1 \frac{(1-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)} \Big(|p| \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} |f(m,x(m))| d_q m
$$

$$
+ \quad |k| \int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)} |g(m,x(m))| d_q m + |\lambda||x(u)| \Big) d_q u \Big\}
$$

$$
\leq \quad \sup_{t\in[0,1]} \Big\{ \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(|p| \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} \phi_1(m)\Psi_1(\|x\|) d_q m
$$

$$
+ \quad |k| \int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)} \phi_2(m)\Psi_2(\|x\|) d_q m + |\lambda||x(u)| \Big) d_q u
$$

$$
+ \quad |a||A(t)| \int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(|p| \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} \phi_1(m)\Psi_1(\|x\|) d_q m
$$

$$
+ \quad |k| \int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)} \phi_2(m)\Psi_2(\|x\|) d_q m + |\lambda||x(u)| \Big) d_q u \Big) d_q s
$$

$$
+ \quad |b||B(t)| \int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(|p| \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} \phi_1(m)\Psi_1(\|x\|) d_q m
$$

$$
+ \quad |k| \int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)} \phi_2(m)\Psi_2(\|x\|) d_q m + |\lambda||x(u)| \Big) d_q u \Big) d_q s
$$

$$
+ \quad |B(t)| \int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(|p| \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} \phi_1(m)\Psi_1(\|x\|) d_q m
$$

$$
+ \quad |k| \int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)} \phi_2(m)\Psi_2(\|x\|) d_q m + |\lambda||x(u)| \Big) d_q u
$$

$$
+ \quad |C(t)| \int_0^1 \frac{(1-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)} \Big(|p| \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} \phi_1(m)\Psi_1(\|x\|) d_q m
$$

$$
+ \quad |k| \int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)} \phi_2(m)\Psi_2(\|x\|) d_q m + |\lambda||x(u)| \Big) d_q u \Big\}
$$

$$
\leq \quad |p|\|\phi_1\|\Psi_1(\|x\|) \sup_{t\in[0,1]} \Big\{ \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big[\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} d_q m \Big] d_q u
$$

$$
+ \quad |a||A(t)| \int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \Big[\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} d_q m \Big) d_q u \Big] d_q s
$$

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$$+ \quad |b||B(t)| \int_0^\sigma \frac{(\sigma - qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \Big[\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} d_q m \Big) d_q u \Big] d_q s$$

$$+ \quad |B(t)| \int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big[\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} d_q m \Big] d_q u$$

$$+ \quad |C(t)| \int_0^t \frac{(t-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)} \Big[\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} d_q m \Big] d_q u \Big\}$$

$$+ \quad |k|\|\phi_2\|\Psi_2(\|x\|) \sup_{t\in[0,1]} \Big\{ \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big[\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)} d_q m \Big] d_q u$$

$$+ \quad |a||A(t)| \int_0^\eta \frac{(\eta - qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \Big[\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)} d_q m \Big) d_q u \Big] d_q s$$

$$+ \quad |b||B(t)| \int_0^\sigma \frac{(\sigma - qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \Big[\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)} d_q m \Big) d_q u \Big] d_q s$$

$$+ \quad |B(t)| \int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big[\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)} d_q m \Big] d_q u$$

$$+ \quad |C(t)| \int_0^t \frac{(t-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)} \Big[\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)} d_q m \Big] d_q u \Big\}$$

$$+ \quad |\lambda|\|x\| \sup_{t\in[0,1]} \Big\{ \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} d_q u + |a||A(t)| \int_0^\eta \frac{(\eta - qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \Big[\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} d_q u \Big] d_q s$$

$$+ \quad |b||B(t)| \int_0^\sigma \frac{(\sigma - qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \Big[\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} d_q u \Big] d_q s + |B(t)| \int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} d_q u$$

$$+ \quad |C(t)| \int_0^t \frac{(t-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)} d_q u \Big\}$$

$$\leq \quad |p|\|\phi_1\|\Psi_1(\|x\|)\omega_1 + |k|\|\phi_2\|\Psi_2(\|x\|)\omega_2 + |\lambda|\|x\|\omega_3.$$

(iii) $\mathcal{F}$ maps bounded sets into equicontinuous sets of $C([0,1] \times \mathbb{R})$.

Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and $x \in B_{\overline{r}}$, where $B_{\overline{r}}$ is a bounded set of $C([0,1], \mathbb{R})$. Then, we obtain

$$\|(\mathcal{F}x)(t_2) - (\mathcal{F}x)(t_1)\|$$

$$\leq \quad \Big| \int_0^{t_1} \frac{(t_2 - qu)^{(\gamma-1)} - (t_1 - qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big[|p| \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} \phi_1(m)\Psi_1(\overline{r}) d_q m$$

$$+ \quad |k| \int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)} \phi_2(m)\Psi_2(\overline{r}) d_q m + |\lambda|\overline{r} \Big] d_q u$$

$$+ \quad \int_{t_1}^{t_2} \frac{(t_2 - qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big[|p| \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} \phi_1(m)\Psi_1(\overline{r}) d_q m$$

$$+ \quad |k| \int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)} \phi_2(m)\Psi_2(\overline{r}) d_q m + |\lambda|\overline{r} \Big] d_q u \Big|$$

$$+ \quad \frac{|a|}{|\Delta|} \Big[\Big| \mu_5[\gamma-2]_q - \mu_6[\gamma-1]_q \Big| \Big| t_2^\gamma - t_1^\gamma \Big| + \Big| \mu_4[\gamma-2]_q - \mu_6[\gamma-1]_q \Big| \Big| t_2^{\gamma-1} - t_1^{\gamma-1} \Big|$$

$$+ \quad \Big| \mu_4[\gamma-1]_q - \mu_5[\gamma]_q \Big| \Big| t_2^{\gamma-2} - t_1^{\gamma-2} \Big| \Big] \int_0^\eta \frac{(\eta - qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}$$

$$\times \quad \Big(|p| \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} \phi_1(m)\Psi_1(\overline{r}) d_q m$$

$$+ \quad |k| \int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)} \phi_2(m)\Psi_2(\overline{r}) d_q m + |\lambda|\overline{r} \Big) d_q u \Big) d_q s$$

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$$+ \quad \frac{|b|}{|\Delta|}\left[\left|\mu_2[\gamma-2]_q - \mu_3[\gamma-1]_q\right|\left|t_2^\gamma - t_1^\gamma\right| + \left|\mu_1[\gamma-2]_q - \mu_3[\gamma-1]_q\right|\left|t_2^{\gamma-1} - t_1^{\gamma-1}\right|\right.$$

$$+ \quad \left|\mu_1[\gamma-1]_q - \mu_2[\gamma]_q\right|\left|t_2^{\gamma-2} - t_1^{\gamma-2}\right|\right]\int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\left(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\right.$$

$$\times \quad \left(|p|\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}\phi_1(m)\Psi_1(\bar{r})d_qm\right.$$

$$+ \quad |k|\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}\phi_2(m)\Psi_2(\bar{r})d_qm + |\lambda|\bar{r}\right)d_qu\Big)d_qs$$

$$+ \quad \frac{1}{|\Delta|}\left[\left|\mu_2[\gamma-2]_q - \mu_3[\gamma-1]_q\right|\left|t_2^\gamma - t_1^\gamma\right| + \left|\mu_1[\gamma-2]_q - \mu_3[\gamma-1]_q\right|\left|t_2^{\gamma-1} - t_1^{\gamma-1}\right|\right.$$

$$+ \quad \left|\mu_1[\gamma-1]_q - \mu_2[\gamma]_q\right|\left|t_2^{\gamma-2} - t_1^{\gamma-2}\right|\right]\int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\left(|p|\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}\right.$$

$$\times \quad \phi_1(m)\Psi_1(\bar{r})d_qm + |k|\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}\phi_2(m)\Psi_2(\bar{r})d_qm + |\lambda||x(u)|\right)d_qu$$

$$+ \quad \frac{1}{|\Delta|}\left[\left|\mu_3\mu_5 - \mu_2\mu_6\right|\left|t_2^\gamma - t_1^\gamma\right| + \left|\mu_3\mu_4 - \mu_1\mu_6\right|\left|t_2^{\gamma-1} - t_1^{\gamma-1}\right| + \left|\mu_2\mu_4 - \mu_1\mu_5\right|\left|t_2^{\gamma-2} - t_1^{\gamma-2}\right|\right]$$

$$\times \quad \int_0^1 \frac{(1-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)}\left(|p|\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}\phi_1(m)\Psi_1(\bar{r})d_qm\right.$$

$$+ \quad |k|\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}\phi_2(m)\Psi_2(\bar{r})d_qm + |\lambda|\bar{r}\right)d_qu.$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in B_{\bar{r}}$ as $t_2 - t_1 \to 0$. As $\mathcal{F}$ satisfies the above assumptions, therefore it follows by the Arzelá-Ascoli theorem that $\mathcal{F} : \mathcal{C} \to \mathcal{C}$ is completely continuous.

(iv) Let x be a solution and $x = \tau\mathcal{F}x$ for $\tau \in (0, 1)$. Then, for $t \in [0, 1]$, and using the computations in proving that $\mathcal{F}$ is bounded, we have

$$|x(t)| = |\tau(\mathcal{F}x)(t)| \leq |p|\|\phi_1\|\Psi_1(\|x\|)\omega_1 + |k|\|\phi_2\|\Psi_2(\|x\|)\omega_2 + |\lambda|\|x\|\omega_3,$$

which implies that

$$\|x\| \leq \frac{|p|\|\phi_1\|\Psi_1(\|x\|)\omega_1 + |k|\|\phi_2\|\Psi_2(\|x\|)\omega_2}{1 - |\lambda|\omega_3}.$$

In view of (A_4), there exists H such that $\|x\| \neq H$. Let us set

$$W = \{x \in \mathcal{C} : \|x\| < H\}.$$

Note that the operator $\mathcal{F} : \overline{W} \to C([0, 1], \mathbb{R})$ is continuous and completely continuous. From the choice of W, there is no $x \in \partial W$ such that $x = \tau\mathcal{F}(x)$ for some $\tau \in (0, 1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 3.3), we deduce that $\mathcal{F}$ has a fixed point $x \in \overline{W}$ which is a solution of the problem $(1) - (2)$. This completes the proof. $\quad\square$

The third existence result is based on Banach's contraction principle (Banach fixed point theorem).

Theorem 3.5 *Suppose that the assumption (A_1) holds and that*

$$\bar{\Omega} = (L\Omega_1 + |\lambda|\omega_3) < 1, \quad \Omega_1 = |p|\omega_1 + |k|\omega_2, \tag{19}$$

where ω_1, ω_2, ω_3 are respectively given by (14), (15), (16), and $L = \max\{L_1, L_2\}$. Then the problem (1)-(2) has a unique solution on $[0, 1]$.

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Proof. Let us define $M = \max\{M_1, M_2\}$, where M_1, M_2 are finite numbers given by $\sup_{t\in[0,1]}|f(t, 0)| = M_1$, $\sup_{t\in[0,1]}|g(t,0)| = M_2$. Selecting $\varepsilon \geq \dfrac{M\Omega_1}{1-\bar{\Omega}}$, we show that $\mathcal{F}B_\varepsilon \subset B_\varepsilon$, where $B_\varepsilon = \{x \in \mathcal{C} : \|x\| \leq \varepsilon\}$ For $x \in B_\varepsilon$, we have

$$
\begin{aligned}
&\|(\mathcal{F}x)\| \\
&\leq \sup_{t\in[0,1]}\Bigg\{ \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(|p|\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}|f(m,x(m))|d_qm \\
&+\ |k|\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}|g(m,x(m))|d_qm + |\lambda||x(u)|\Big)d_qu \\
&+\ |a||A(t)|\int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(|p|\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}|f(m,x(m))|d_qm \\
&+\ |k|\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}|g(m,x(m))|d_qm + |\lambda||x(u)|\Big)d_qu\Big)d_qs \\
&+\ |b||B(t)|\int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(|p|\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}|f(m,x(m))|d_qm \\
&+\ |k|\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}|g(m,x(m))|d_qm + |\lambda||x(u)|\Big)d_qu\Big)d_qs \\
&+\ |B(t)|\int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(|p|\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}|f(m,x(m))|d_qm \\
&+\ |k|\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}|g(m,x(m))|d_qm + |\lambda||x(u)|\Big)d_qu \\
&+\ |C(t)|\int_0^1 \frac{(1-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)}\Big(|p|\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}|f(m,x(m))|d_qm \\
&+\ |k|\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}|g(m,x(m))|d_qm + |\lambda||x(u)|\Big)d_qu\Bigg\} \\
&\leq \sup_{t\in[0,1]}\Bigg\{ \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(|p|\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}\Big(|f(m,x(m))-f(m,0)|+|f(m,0)|\Big)d_qm \\
&+\ |k|\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}\Big(|g(m,x(m))-g(m,0)|+|g(m,0)|\Big)d_qm + |\lambda||x(u)|\Big)d_qu \\
&+\ |a||A(t)|\int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(|p|\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}\Big(|f(m,x(m))-f(m,0)| \\
&+\ |f(m,0)|\Big)d_qm + |k|\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}\Big(|g(m,x(m))-g(m,0)|+|g(m,0)|\Big)d_qm \\
&+\ |\lambda||x(u)|\Big)d_qu\Big)d_qs \\
&+\ |b||B(t)|\int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(|p|\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}\Big(|f(m,x(m))-f(m,0)| \\
&+\ |f(m,0)|\Big)d_qm + |k|\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}\Big(|g(m,x(m))-g(m,0)|+|g(m,0)|\Big)d_qm
\end{aligned}
$$

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$$+ \quad |\lambda||x(u)|\Big)d_q u\Big)d_q s$$

$$+ \quad |B(t)|\int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(|p|\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}\Big(|f(m,x(m))-f(m,0)|+|f(m,0)|\Big)d_q m$$

$$+ \quad |k|\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}\Big(|g(m,x(m))-g(m,0)|+|g(m,0)|\Big)d_q m + |\lambda||x(u)|\Big)d_q u$$

$$+ \quad |C(t)|\int_0^1 \frac{(1-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)}\Big(|p|\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}\Big(|f(m,x(m))-f(m,0)|+|f(m,0)|\Big)d_q m$$

$$+ \quad |k|\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}\Big(|g(m,x(m))-g(m,0)|+|g(m,0)|\Big)d_q m + |\lambda||x(u)|\Big)d_q u\Big\}$$

$$\leq \quad |p|(L_1\varepsilon+M_1)\sup_{t\in[0,1]}\Big\{\int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big[\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}d_q m\Big]d_q u$$

$$+ \quad |a||A(t)|\int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big[\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}d_q m\Big)d_q u\Big]d_q s$$

$$+ \quad |b||B(t)|\int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big[\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}d_q m\Big)d_q u\Big]d_q s$$

$$+ \quad |B(t)|\int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big[\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}d_q m\Big)d_q u$$

$$+ \quad |C(t)|\int_0^t \frac{(t-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)}\Big[\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}d_q m\Big]d_q u\Big\}$$

$$+ \quad |k|(L_2\varepsilon+M_2)\sup_{t\in[0,1]}\Big\{\int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big[\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}d_q m\Big]d_q u$$

$$+ \quad |a||A(t)|\int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big[\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}d_q m\Big)d_q u\Big]d_q s$$

$$+ \quad |b||B(t)|\int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big[\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}d_q m\Big)d_q u\Big]d_q s$$

$$+ \quad |B(t)|\int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big[\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}d_q m\Big)d_q u$$

$$+ \quad |C(t)|\int_0^t \frac{(t-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)}\Big[\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}d_q m\Big]d_q u\Big\}$$

$$+ \quad |\lambda|\varepsilon\sup_{t\in[0,1]}\Big\{\int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}d_q u+|a||A(t)|\int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big[\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}d_q u\Big]d_q s$$

$$+ \quad |b||B(t)|\int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big[\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}d_q u\Big]d_q s+|B(t)|\int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}d_q u$$

$$+ \quad |C(t)|\int_0^t \frac{(t-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)}d_q u\Big\}$$

$$\leq \quad (L\varepsilon+M)\Omega_1+|\lambda|\varepsilon\omega_3 \leq \varepsilon.$$

This shows that $\mathcal{F}B_\varepsilon \subset B_\varepsilon$. Now, for $x,y\in\mathcal{C}$, we obtain

$$\|\mathcal{F}x-\mathcal{F}y\|$$

$$\leq \quad \sup_{t\in[0,1]}\Big\{\int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(|p|\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}|f(m,x(m))-f(m,y(m))|d_q m$$

$$+ \quad |k|\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}|g(m,x(m))-g(m,y(m))|d_q m + |\lambda||x(u)-y(u)|\Big)d_q u$$

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$$+ \quad |a||A(t)| \int_0^\eta \frac{(\eta - qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \Big(\int_0^s \frac{(s - qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(|p| \int_0^u \frac{(u - qm)^{(\beta-1)}}{\Gamma_q(\beta)} |f(m, x(m)) - f(m, y(m))| d_q m$$

$$+ \quad |k| \int_0^u \frac{(u - qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)} |g(m, x(m)) - g(m, y(m))| d_q m + |\lambda||x(u) - y(u)| \Big) d_q u \Big) d_q s$$

$$+ \quad |b||B(t)| \int_0^\sigma \frac{(\sigma - qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \Big(\int_0^s \frac{(s - qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(|p| \int_0^u \frac{(u - qm)^{(\beta-1)}}{\Gamma_q(\beta)} |f(m, x(m)) - f(m, y(m))| d_q m$$

$$+ \quad |k| \int_0^u \frac{(u - qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)} |g(m, x(m)) - g(m, y(m))| d_q m + |\lambda||x(u) - y(u)| \Big) d_q u \Big) d_q s$$

$$+ \quad |B(t)| \int_0^1 \frac{(1 - qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(|p| \int_0^u \frac{(u - qm)^{(\beta-1)}}{\Gamma_q(\beta)} |f(m, x(m)) - f(m, y(m))| d_q m$$

$$+ \quad |k| \int_0^u \frac{(u - qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)} |g(m, x(m)) - g(m, y(m))| d_q m + |\lambda||x(u) - y(u)| \Big) d_q u$$

$$+ \quad |C(t)| \int_0^1 \frac{(1 - qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)} \Big(|p| \int_0^u \frac{(u - qm)^{(\beta-1)}}{\Gamma_q(\beta)} |f(m, x(m)) - f(m, y(m))| d_q m$$

$$+ \quad |k| \int_0^u \frac{(u - qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)} |g(m, x(m)) - g(m, y(m))| d_q m + |\lambda||x(u) - y(u)| \Big) d_q u \Big\}$$

$$\leq \quad \bar{\Omega}\|x - y\|,$$

which shows that $\mathcal{F}$ is a contraction as $\bar{\Omega} < 1$ by the given assumption. Therefore, it follows by Banach's contraction principle that the problem (1)-(2) has a unique solution. □

Example. Consider a boundary value problem of integro-differential equations of fractional order given by

$$\begin{cases} {}^cD_q^{1/2}({}^cD_q^{1/2} + \frac{1}{5})x(t) = \frac{1}{6}f(t, x(t)) + \frac{1}{9}I_q^{1/2}g(t, x(t)), & 0 \leq t \leq 1, \ 0 < q < 1, \\ x(0) = I_q^{\alpha-1}x(1/3), & x(1) = 1/2 I_q^{\alpha-1}x(2/3), \ D_q x(1) = 0, \end{cases} \tag{20}$$

Here $f(t, x) = \frac{1}{(4+t^2)^2}\Big(\sin t + \frac{|x|}{1+|x|} + |x| \Big)$, $g(t, x) = \frac{1}{2}\tan^{-1}x + t^3$. Clearly

$$|f(t, x) - f(t, y)| \leq \frac{1}{8}|x - y|, \quad |g(t, x) - g(t, y)| \leq \frac{1}{2}|x - y|.$$

With $\beta = \xi = 1/2, \gamma = 3/2, \ \lambda = 1/5, p = 1/6, k = 1/9, q = 1/2, L_1 = 1/8, \ L_2 = 1/2$, we find that

$$\bar{\Omega} = L(|p|\omega_1 + |k|\omega_2) + |\lambda|\omega_3 \simeq 0.49725 < 1.$$

Clearly $L = \max\{L_1, L_2\} = 1/2$. Thus all the assumptions of Theorem 3.5 are satisfied. Hence, by the conclusion of Theorem 3.5, the problem (20) has a unique solution.

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Reconstruction of bivariate functions by sparse sine coefficients [*]

Zhihua Zhang[1,2]

1. College of Global Change and Earth System Science, Beijing Normal University, Beijing, China, 100875.

2. Joint Center for Global Change Studies, Beijing 100875, China

E-mail: zhangzh@bnu.edu.cn

Abstract. In application, one often expands the functions f on $[0,1]^2$ into Fourier sine series and uses few Fourier sine coefficients to reconstruct f. In this paper, we give a decomposition formula of Fourier sine coefficients. Based on it, we discuss hyperbolic cross approximations of Fourier sine series and Fourier sine expansion with simple polynomial factors. In the end of this paper, we consider the three-dimensional case.

1. Introduction

In application, one often expands the functions f on $[0,1]^2$ into Fourier sine series and uses few Fourier sine coefficients to reconstruct f. But the precise representation of Fourier sine coefficients does not available. In Section 2, we will give the following decomposition of Fourier sine coefficients.

Suppose that f is a bivariate function with $\frac{\partial^4 f}{\partial x^2 \partial y^2}(x,y) \in C([0,1]^2)$. For its Fourier sine coefficients, we have

$$c_{n_1,n_2}(f)$$

$$= 4 \int_{[0,1]^2} f(x,y) \sin(\pi n_1 x) \sin(\pi n_2 y) \, \mathrm{d}x\mathrm{d}y$$

$$= \frac{4}{\pi^2 n_1 n_2} \left(J_{n_1,n_2} - \frac{1}{\pi n_1}(c_{n_1}(g_1) - (-1)^{n_2+1}c_{n_2}(g_2)) - \frac{1}{\pi n_2}(c_{n_2}(g_3) + (-1)^{n_1+1}c_{n_2}(g_4)) + \frac{1}{\pi^2 n_1 n_2}c_{n_1,n_2}(h) \right),$$

where

$$J_{n_1,n_2} = f(0,0) + (-1)^{n_1+1}f(1,0) + (-1)^{n_2+1}f(0,1) + (-1)^{n_1+n_2}f(1,1)$$

is an algebraic sum of values of f at vertexes of the square $[0,1]^d$ and

$$g_1(t) = \frac{\partial^2 f}{\partial t^2}(t,0), \qquad g_2(t) = \frac{\partial^2 f}{\partial t^2}(t,1),$$

$$g_3(t) = \frac{\partial^2 f}{\partial t^2}(0,t), \qquad g_4(t) = \frac{\partial^2 f}{\partial t^2}(1,t)$$

are the second-order derivatives of f on boundary of $[0,1]^2$ and

$$c_n(g_i) = 2 \int_0^1 g_i(t) \sin(\pi n t) \, \mathrm{d}t$$

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is Fourier sine coefficients of univariate functions g_i, and $h = \frac{\partial^4 f}{\partial x^2 \partial y^2}$ and

$$c_{n_1,n_2}(h) = 4 \int_{[0,1]^2} h(x,y) \sin(\pi n_1 x) \sin(\pi n_2 y) \, dx dy$$

is the Fourier sine coefficient of bivariate function $h(x,y)$.

It is well known that in order to reconstruct f by using fewer Fourier sine coefficients, we should replace full grid approximation by sparse grid approximation [1,3,4]. In Section 3, based on this decomposition, we prove that for the hyperbolic cross truncations

$$S_N^{(h)}(f; x, y) = \sum_{\substack{1 \le n_1, n_2 \le N-1 \\ 1 \le n_1 n_2 \le N-1}} c_{n_1,n_2}(f) \sin(\pi n_1 x) \sin(\pi n_2 y)$$

of Fourier sine series of f, the approximation errors satisfy

$$\| f - S_N^{(h)}(f) \|_2^2$$

$$= \tfrac{4}{\pi^4} (f^2(0,0) + f^2(0,1) + f^2(1,0) + f^2(1,1)) \tfrac{\log N}{N} + O\left(\tfrac{1}{N}\right).$$

Since the number of coefficients in $S_N^{(h)}(f)$ is $N_c \sim N \log N$. When we use the hyperbolic cross truncations to reconstruct f, we need fewer Fourier sine coefficients than that by partial sums of Fourier sine series.

To obtain these results, we need to use a decomposition of bivariate functions in [8].

Suppose that f is a second-order continuously differentiable on $[0,1]^2$, denote by $f \in W^{(2,2)}([0,1]^2)$. Let

$$P(x,y) = f(0,0)(1-x)(1-y) + f(0,1)(1-x)y + f(1,0)x(1-y) + f(1,1)xy \qquad (1.1)$$

which is a bivariate polynomial determined by the values of f at vertexes of $[0,1]^2$, and let

$$Q(x,y) = f_1(0,y)(1-x) + f_1(1,y)x + f_1(x,0)(1-y) + f_1(x,1)y \qquad (f_1 = f - P). \qquad (1.2)$$

The bivariate function $Q(x,y)$ is a sum of products of separated variable types. Denote the residual

$$R = f - P - Q. \qquad (1.3)$$

It is easy to check that

$$R(x,y) = 0 \qquad ((x,y) \in \partial([0,1]^2)),$$

$$\frac{\partial^2 R}{\partial x^2}(x,y) = \frac{\partial^2 R}{\partial x^2}(x,y) - \frac{\partial^2 f_1}{\partial x^2}(x,0)(1-y) - \frac{\partial^2 f_1}{\partial x^2}(x,1)y.$$

So it follows that

$$R(x,0) = R(x,1) = 0 \qquad (0 \le x \le 1),$$

$$R(0,y) = R(1,y) = 0 \qquad (0 \le y \le 1), \qquad (1.4)$$

$$\frac{\partial^2 R}{\partial x^2}(x,1) = \frac{\partial^2 R}{\partial x^2}(x,0) = 0,$$

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and we have a decomposition formula:

$$f(x, y) = P(x, y) + Q(x, y) + R(x, y), \tag{1.5}$$

where $P, Q,$ and R are stated in (1.1)-(1.3).

In Section 4, by using the decomposition (1.5), we expand f into Fourier sine series with simple polynomial factors whose hyperbolic cross truncation can reconstruct f by using fewest Fourier sine coefficients. In order to extend the above results to stochastic processes in Section 5, we need some concepts in Calculus of stochastic processes [2,7].

If $\{\xi_n\}_1^\infty$ is a sequence of stochastic variables and ξ is a stochastic variable, if the expectation

$$E[|\xi_n - \xi|^2] \to 0 \qquad (n \to \infty),$$

we say ξ is the limit of the sequence $\{\xi_n\}_1^\infty$. Based on this concept, one defines concepts of continuity, derivatives, and integrals. If $f(\mathbf{t})$ is a stochastic variable for each $\mathbf{t} \in [0, 1]^d$, we say $f(\mathbf{t})$ is a stochastic process on $[0, 1]^d$. If $f(\mathbf{t})$ is a stochastic process on $[0, 1]^d$ and $E\left[\int_{[0,1]^d} f^2(\mathbf{t}) \, d\mathbf{t}\right] < \infty$, then f can be expanded a Fourier sine series:

$$f(\mathbf{t}) = \sum_{\mathbf{n} \in \mathbb{Z}_+^d} c_{\mathbf{n}}(f) \left(\prod_{k=1}^d \sin(\pi n_k t_k) \right),$$

where the coefficients:

$$c_{\mathbf{n}}(f) = 2^d \int_{[0,1]^d} f(t_1, ..., t_d) \left(\prod_{k=1}^d \sin(\pi n_k t_k) \right) dt_1 \cdots dt_d.$$

For convenience, the notation $f \in W^{(l_1,...,l_d)}([0, 1]^d)$ means $\frac{\partial^{l_1 + \cdots + l_d}}{\partial t_1^{l_1} \partial t_2^{l_2} \cdots \partial t_d^{l_d}} f \in C([0, 1]^d)$, and the notation $\alpha_{n_1,...,n_d} = o(1)$ means that $\alpha_{n_1,...,n_d} \to 0$ as $n_1^2 + \cdots + n_d^2 \to \infty$.

At the end of this paper (i.e. Section 6), we consider the three-dimensional case.

2. Fourier sine coefficient decomposition

From this decomposition formula (1.5), it follows that the Fourier sine coefficients of f satisfy

$$c_{n_1,n_2}(f) = c_{n_1,n_2}(P) + c_{n_1,n_2}(Q) + c_{n_1,n_2}(R).$$

Suppose that $f \in W^{(2,2)}([0, 1]^2)$. Then

(i)

$$c_{n_1,n_2}(P) = \frac{4}{\pi^2 n_1 n_2} J_{\mathbf{n}},$$

where

$$J_{n_1,n_2} = f(0,0) + (-1)^{n_1+1} f(1,0) + (-1)^{n_2+1} f(0,1) + (-1)^{n_1+n_2} f(1,1). \tag{2.1}$$

(ii)

$$c_{n_1,n_2}(Q) = \frac{4}{\pi n_1} \int_0^1 F_1(y) \sin(\pi n_2 y) \, dy + \frac{4}{\pi n_2} \int_0^1 F_2(x) \sin(\pi n_1 x) \, dx,$$

where

$$F_1(y) = f_1(0, y) + f_1(1, y)(-1)^{n_1+1},$$

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$$F_2(x) = f_1(x,0) + f_1(x,1)(-1)^{n_2+1}.$$

By $f_1 = f - P$, we have

$$F_1(0) = F_1(1) = F_2(0) = F_2(1) = 0.$$

Since $\frac{\partial^2 P}{\partial x^2} = \frac{\partial^2 P}{\partial y^2} = 0$, we have

$$F_1''(y) = \frac{\partial^2 f}{\partial y^2}(0,y) + \frac{\partial^2 f}{\partial y^2}(1,y)(-1)^{n_1+1},$$

$$F_2''(x) = \frac{\partial^2 f}{\partial x^2}(x,0) + \frac{\partial^2 f}{\partial x^2}(x,1)(-1)^{n_2+1}.$$

Let

$$g_1(t) = \frac{\partial^2 f}{\partial t^2}(t,0), \qquad g_2(t) = \frac{\partial^2 f}{\partial t^2}(t,1),$$

$$g_3(t) = \frac{\partial^2 f}{\partial t^2}(0,t), \qquad g_4(t) = \frac{\partial^2 f}{\partial t^2}(1,t). \tag{2.2}$$

Then

$$F_1''(y) = g_3(y) + (-1)^{n_1+1}g_4(y),$$

$$F_2''(x) = g_1(x) + (-1)^{n_2+1}g_2(x).$$

From this, we deduce that

$$
\begin{aligned}
2\int_0^1 F_1(y)\,\sin(\pi n_1 y)\,dy &= -\frac{2}{(\pi n_1)^2}\int_0^1 F_1''(y)\,\sin(\pi n_1 y)\,dy \\
&= -\frac{c_{n_1}(g_3)+(-1)^{n_1+1}c_{n_1}(g_4)}{(\pi n_1)^2}, \\
2\int_0^1 F_2(x)\,\sin(\pi n_2 x)\,dx &= -\frac{2}{(\pi n_2)^2}\int_0^1 F_2''(x)\,\sin(\pi n_2 x)\,dx \\
&= -\frac{c_{n_2}(g_1)+(-1)^{n_2+1}c_{n_2}(g_2)}{(\pi n_2)^2},
\end{aligned}
$$

where $c_n(g_i) = 2\int_0^1 g_i(x)\,\sin(\pi n x)\,dx$, and so

$$
\begin{aligned}
c_{n_1,n_2}(Q) &= -\frac{1}{\pi^3 n_1 n_2}\left(\frac{c_{n_1}(g_3)+(-1)^{n_1+1}c_{n_1}(g_4)}{n_1} + \frac{c_{n_2}(g_1)+(-1)^{n_2+1}c_{n_2}(g_2)}{n_2}\right) \\
&= \frac{1}{n_1 n_2}\left(o\left(\frac{1}{n_1}\right) + o\left(\frac{1}{n_2}\right)\right).
\end{aligned}
$$

(iii)

$$\frac{1}{4}c_{n_1,n_2}(R) = \int_0^1 \int_0^1 R(x,y)\,\sin(\pi n_1 x)\,\sin(\pi n_2 y)\,dxdy.$$

4

Since $R \in W^{(2,,2)}([0,1]^2)$, using integration by parts, it follows by (1.4) that

$$\int_0^1 R(x,y) \sin(\pi n_1 x) \, dx$$

$$= -\frac{R(x,y)}{\pi n_1} \cos(\pi n_1 x)\Big|_0^1 + \frac{1}{\pi n_1} \int_0^1 \frac{\partial R}{\partial x}(x,y) \cos(\pi n_1 x) \, dx$$

$$= \frac{1}{\pi n_1} \left(\frac{1}{\pi n_1} \frac{\partial R}{\partial x}(x,y) \sin(\pi n_1 x)\Big|_0^1 - \frac{1}{\pi n_1} \int_0^1 \frac{\partial^2 R}{\partial x^2}(x,y) \sin(\pi n_1 x) \, dx \right)$$

$$= -\frac{1}{(\pi n_1)^2} \int_0^1 \frac{\partial^2 R}{\partial x^2}(x,y) \sin(\pi n_1 x) \, dx.$$

So

$$\frac{1}{4} c_{n_1,n_2}(R) = \frac{1}{(\pi n_1)^2} \int_0^1 \sin(\pi n_1 x) \left(\int_0^1 \frac{\partial^2 R}{\partial x^2}(x,y) \sin(\pi n_2 y) \, dy \right) dx.$$

By (1.4), we get

$$\int_0^1 \frac{\partial^2 R}{\partial x^2}(x,y) \sin(\pi n_2 y) \, dy = -\frac{1}{(\pi n_2)^2} \int_0^1 \frac{\partial^4 R}{\partial x^2 \partial y^2}(x,y) \sin(\pi n_2 y) \, dy.$$

From this, we get

$$
\begin{aligned}
c_{n_1,n_2}(R) &= \frac{4}{\pi^4 n_1^2 n_2^2} \int_0^1 \int_0^1 \frac{\partial^4 f}{\partial x^2 \partial y^2}(x,y) \sin(\pi n_1 x) \sin(\pi n_2 y) \, dx dy \\
&= \frac{c_{n_1,n_2}\left(\frac{\partial^4 f}{\partial x^2 \partial y^2}\right)}{\pi^4 n_1^2 n_2^2} = o\left(\frac{1}{n_1^2 n_2^2}\right).
\end{aligned}
\tag{2.3}
$$

Summarizing up all results, we get the following theorem.

Theorem 2.1. Let $f \in W^{(2,2)}([0,1]^2)$. Then its Fourier sine coefficients have the decomposition formula:

$$c_{n_1,n_2}(f)$$

$$= \frac{4}{\pi^2 n_1 n_2} \left(J_{n_1,n_2} - \frac{c_{n_1}(g_1) + (-1)^{n_2+1} c_{n_1}(g_2)}{\pi n_1} - \frac{c_{n_2}(g_3) + (-1)^{n_1+1}(g_4)}{\pi n_2} + \frac{c_{n_1,n_2}\left(\frac{\partial^4 f}{\partial x^2 \partial y^2}\right)}{\pi^2 n_1 n_2} \right), \tag{2.4}$$

where J_{n_1,n_2} is stated in (2.1) and g_i $(i = 1, 2, 3, 4)$ are stated in (2.2), and

$$c_n(g_i) = 2 \int_0^1 g_i(t) \sin(\pi n t) \, dt, \qquad n \in \mathbb{Z}_+ \quad (i = 1, 2, 3, 4).$$

By the Riemann-Lebesgue lemma [5],

$$c_{n_1,n_2}(f) = \frac{4}{\pi^2 n_1 n_2} \left(J_{n_1,n_2} + o\left(\frac{1}{n_1}\right) + o\left(\frac{1}{n_2}\right) \right). \tag{2.5}$$

In detail, we have the following asymptotic formulas:

$$c_{2n_1,2n_2}(f) = \frac{1}{\pi^2 n_1 n_2}\left(f(0,0) - f(0,1) - f(1,0) + f(1,1) + o\left(\frac{1}{n_1}\right) + o\left(\frac{1}{n_2}\right)\right),$$

$$c_{2n_1+1,2n_2+1}(f) = \frac{1}{\pi^2 n_1 n_2}\left(f(0,0) + f(0,1) + f(1,0) + f(1,1) + o\left(\frac{1}{n_1}\right) + o\left(\frac{1}{n_2}\right)\right),$$

$$c_{2n_1+1,2n_2}(f) = \frac{1}{\pi^2 n_1 n_2}\left(f(0,0) - f(0,1) + f(1,0) - f(1,1) + o\left(\frac{1}{n_1}\right) + o\left(\frac{1}{n_2}\right)\right),$$

$$c_{2n_1,2n_2+1}(f) = \frac{1}{\pi^2 n_1 n_2}\left(f(0,0) + f(0,1) - f(1,0) - f(1,1) + o\left(\frac{1}{n_1}\right) + o\left(\frac{1}{n_2}\right)\right).$$

Consider the sum of their squares:

$$\sum_{i,j=0}^{1} c_{2n_1+i,2n_2+j}^2(f) = \frac{4}{\pi^4 n_1^2 n_2^2}\left(f^2(0,0) + f^2(0,1) + f^2(1,0) + f^2(1,1) + o\left(\frac{1}{n_1}\right) + o\left(\frac{1}{n_2}\right)\right). \tag{2.6}$$

This implies that the equality:

$$\sum_{i,j=0}^{1} c_{2n_1+i,2n_2+j}^2(f) = o\left(\frac{1}{n_1}\right) + o\left(\frac{1}{n_2}\right)$$

holds if and only if

$$f(0,0) = f(0,1) = f(1,0) = f(1,1) = 0. \tag{2.7}$$

This is equivalent to that the equality:

$$c_{n_1,n_2}(f) = o\left(\frac{1}{n_1}\right) + o\left(\frac{1}{n_2}\right)$$

holds if and only if (2.7) holds. However, similar to an argument of (2.3), we can derive that $c_{n_1,n_2} = o\left(\frac{1}{n_1 n_2}\right)$ if and only if $f(x,y) = 0\,((x,y) \in \partial([0,1]^2))$.

If $f \in W^{(1,2)}([0,1]^2)$, $f \in W^{(2,1)}([0,1]^2)$, and $f \in W^{(1,1)}([0,1]^2)$, then we have the corresponding results.

Theorem 2.2. Let $f \in W^{(l_1,l_2)}([0,1]^2)$. Then Fourier sine coefficients of f satisfy asymptotic formulas:

(i) $c_{n_1,n_2}(f) = \frac{4}{\pi^2 n_1 n_2}\left(J_{n_1,n_2} + o\left(\frac{1}{n_1}\right) + o\left(\frac{1}{n_2}\right)\right)$, where $l_1 = l_2 = 2$;

(ii) $c_{n_1,n_2}(f) = \frac{4}{\pi^2 n_1 n_2}(J_{n_1,n_2} + \eta_1 + \eta_2)$, where $l_1 = l_2 = 1$;

(iii) $c_{n_1,n_2}(f) = \frac{4}{\pi^2 n_1 n_2}\left(J_{n_1,n_2} + \eta_1 + o\left(\frac{1}{n_2}\right)\right)$, where $l_1 = 1, l_2 = 2$;

(iv) $c_{n_1,n_2}(f) = \frac{4}{\pi^2 n_1 n_2}\left(J_{n_1,n_2} + o\left(\frac{1}{n_1}\right) + \eta_2\right)$, where $l_1 = l_2 = 1$.

Here J_{n_1,n_2} is stated in (2.1) and $\eta_i \to 0$ as $n_i \to \infty$.

3. Approximation of hyperbolic cross truncations

Suppose that $f \in W^{(2,2)}([0,1]^2)$. We expand it into a Fourier sine series:

$$f(x,y) = \sum_{n \in \mathbb{Z}^2} c_{n_1,n_2}(f) \sin(\pi n_1 x)\sin(\pi n_2 y) \qquad (L^2).$$

6

Consider its hyperbolic cross truncations:

$$S_N^{(h)}(f; x, y) = \sum_{\substack{1 \le n_1, n_2 \le N-1 \\ 1 \le n_1 n_2 \le N-1}} c_{n_1,n_2}(f) \sin(\pi n_1 x) \sin(\pi n_2 y).$$

Then

$$f(x, y) - S_N^{(h)}(f; x, y) = \left(\sum_{n_1=1}^{N-1} \sum_{n_2=\left[\frac{N-1}{n_1}\right]+1}^{\infty} + \sum_{n_1=N}^{\infty} \sum_{n_2=1}^{\infty} \right) c_{n_1,n_2}(f) \sin(\pi n_1 x) \sin(\pi n_2 y).$$

By the Parseval identity,

$$4 \parallel f - S_N^{(h)}(f) \parallel_2^2 = \sum_{n_1=1}^{N} \sum_{n_2=\left[\frac{N}{n_1}\right]}^{\infty} |c_{n_1,n_2}(f)|^2 + \sum_{n_1=N}^{\infty} \sum_{n_2=1}^{\infty} |c_{n_1,n_2}(f)|^2 = I_N^{(1)} + I_N^{(2)}. \tag{3.1}$$

By (2.5),

$$|c_{n_1,n_2}(f)|^2 = \frac{16}{\pi^4 n_1^2 n_2^2} \left(J_{n_1,n_2}^2 + o\left(\frac{1}{n_1}\right) + o\left(\frac{1}{n_2}\right) \right).$$

So

$$I_N^{(2)} = O\left(\frac{1}{N}\right)$$

and

$$
\begin{aligned}
I_N^{(1)} &= \frac{16}{\pi^4} \sum_{n_1=1}^{N} \sum_{n_2=\left[\frac{N-1}{n_1}\right]+1}^{\infty} \frac{J_{n_1,n_2}^2}{n_1^2 n_2^2} \\
&\quad + o(1) \sum_{n_1=1}^{N} \sum_{n_2=\left[\frac{N-1}{n_2}\right]+1}^{\infty} \frac{1}{n_1^3 n_2^2} + o(1) \sum_{n_1=1}^{N} \sum_{n_2=\left[\frac{N-1}{n_1}\right]+1}^{\infty} \frac{1}{n_1^2 n_2^3} \\
&= \frac{16}{\pi^4} \sum_{n_1=1}^{N-1} \sum_{n_2=\left[\frac{N-1}{n_1}\right]+1}^{\infty} \frac{J_{n_1,n_2}^2}{n_1^2 n_2^2} + o\left(\frac{1}{N}\right).
\end{aligned}
\tag{3.2}
$$

By (3.1), we get

$$4 \parallel f - S_N^{(h)}(f) \parallel_2^2 = K_N + o\left(\frac{1}{N}\right), \tag{3.3}$$

where

$$K_N = \frac{16}{\pi^4} \sum_{n_1=1}^{N-1} \sum_{n_2=\left[\frac{N}{n_1}\right]}^{\infty} \frac{J_{n_1,n_2}^2}{n_1^2 n_2^2}.$$

A direct computation shows that

$$
\begin{aligned}
K_N &= \frac{1}{\pi^4} \sum_{n_1=1}^{\left[\frac{N-1}{2}\right]} \sum_{n_2=\left[\frac{N}{4n_1}\right]}^{\infty} \frac{1}{n_1^2 n_2^2} \left(J_{2n_1,2n_2}^2 + J_{2n_1-1,2n_2}^2 + J_{2n_1,2n_2-1}^2 + J_{2n_1-1,2n_2-1}^2 \right) + O\left(\frac{1}{N}\right) \\
&= \frac{4M}{\pi^4} \sum_{n_1=1}^{\left[\frac{N-1}{2}\right]} \sum_{n_2=\left[\frac{N}{4n_1}\right]}^{\infty} \frac{1}{n_1^2 n_2^2} + O\left(\frac{1}{N}\right),
\end{aligned}
$$

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where $M = f^2(0,0) + f^2(0,1) + f^2(1,0) + f^2(1,1)$. Notice that

$$\sum_{n_2 = \left[\frac{N}{4n_1}\right]} \frac{1}{n_1^2 n_2^2} = \frac{1}{n_1^2}\left(\int_{\frac{N}{4n_1}}^{\infty} \frac{\mathrm{d}t}{t^2} + O\left(\frac{n_1^2}{N^2}\right)\right).$$

Then

$$K_N = \frac{16M}{\pi^4 N}\sum_{n_1=1}^{\left[\frac{N-1}{2}\right]} \frac{1}{n_1} + O\left(\frac{1}{N}\right) = \frac{16M\log N}{\pi^4 N} + O\left(\frac{1}{N}\right).$$

From this and (3.1)-(3.3), it follows that

$$\| f - S_N^{(h)}(f) \|_2^2 = \frac{4M}{\pi^4}\frac{\log N}{N} + O\left(\frac{1}{N}\right). \tag{3.4}$$

The number of Fourier sine coefficients in the Nth hyperbolic cross truncation $S_N^{(h)}(f)$ is

$$N_c = \sum_{n_1=1}^{N-1}\sum_{n_2=1}^{\left[\frac{N-1}{n_1}\right]} 1 = \sum_{n_1=1}^{N-1}\left[\frac{N-1}{n_1}\right] = \int_1^N \frac{N}{t}\,\mathrm{d}t + O(N) = N\log N + O(N).$$

Then (3.4) can be written into

$$\| f - S_N^{(h)}(f) \|_2^2 = \frac{4M}{\pi^4}\frac{\log^2 N_c}{N_c} + O\left(\frac{\log N_c}{N_c}\right).$$

Theorem 3.1. Let $f \in W^{(l_1, l_2)}([0,1]^2)$. Then the hyperbolic cross truncations of Fourier sine series of f satisfy the asymptotic formulas:
  (i) $\| f - S_N^{(h)}(f) \|_2^2 = \frac{4M}{\pi^4}\frac{\log N}{N} + O\left(\frac{1}{N}\right)$ $(l_1 = l_2 = 2)$;
  (ii) $\| f - S_N^{(h)}(f) \|_2^2 = \frac{4M}{\pi^4}\frac{\log N}{N} + o\left(\frac{\log N}{N}\right)$ $(l_1 = l_2$ or $l_1 = 2, l_2 = 1$ or $l_1 = 1, l_2 = 2$,
where the constant $M = f^2(0,0) + f^2(0,1) + f^2(1,0) + f^2(1,1)$.

4. Fourier sine expansion with polynomial factors

Suppose that $f \in W^{(2,2)}([0,1]^2)$. Then, by decomposition formula:

$$f(x,y) = P(x,y) + f_1(0,y)(1-x) + f_1(1,y)x + f_1(x,0)(1-y) + f_1(x,1)y + R(x,y),$$

denote

$$\alpha_1(y) = f_1(0,y), \qquad \alpha_2(y) = f_1(1,y),$$

$$\alpha_3(x) = f_1(x,0), \qquad \alpha_4(x) = f_1(x,1),$$

then $\alpha_i(0) = \alpha_i(1) = 0$ and $\alpha_i \in W([0,1])$ $(i = 1,2,3,4)$.

Expanding each α_i into a univariate Fourier sine series and $R(x,y)$ into a bivariate Fourier sine series, we

get a Fourier sine expansion of f with polynomial factors:

$$
\begin{aligned}
f(x,y) \ = \ & P(x,y) \\
\ + \ & (1-x)\sum_{1}^{\infty} c_m(\alpha_1)\sin(\pi m y) + x\sum_{1}^{\infty} c_m(\alpha_2)\sin(\pi m y) \\
\ + \ & (1-y)\sum_{1}^{\infty} c_m(\alpha_3)\sin(\pi m x) + y\sum_{1}^{\infty} c_m(\alpha_4)\sin(\pi m x) \\
\ + \ & \sum_{n_1,n_2=1}^{\infty} c_{n_1,n_2}(R)\sin(\pi n_1 x)\sin(\pi n_2 y),
\end{aligned}
\tag{4.1}
$$

where

$$
\begin{aligned}
c_m(\alpha_i) \ = \ & 2\int_0^1 \alpha_i(t)\,\sin(\pi m t)\,\mathrm{d}t \\
\ = \ & -\tfrac{2}{(\pi m)^2}\int_0^1 \alpha_i''(t)\,\sin(\pi m t)\,\mathrm{d}t \\
\ = \ & -\tfrac{1}{(\pi m)^2} c_m(\alpha_i'') \quad (i=1,2,3,4),
\end{aligned}
\tag{4.2}
$$

$$
c_{n_1,n_2}(R) = \frac{c_{n_1,n_2}\left(\frac{\partial^4 f}{\partial x^2 \partial y^2}\right)}{\pi^4 n_1^2 n_2^2}.
$$

By the definition of f_1, we have $\alpha_i''(t) = h_i''(t)$ $(i=1,2,3,4)$, where

$$
h_1(t) = f(0,t), \qquad h_2(t) = f(1,t),
$$

$$
h_3(t) = f(t,0), \qquad h_4(t) = f(t,1).
$$

Consider the hyperbolic cross truncations of the series (4.1):

$$
\begin{aligned}
\widetilde{S}_N^{(h)}(x,y) \ = \ & P(x,y) \\
\ + \ & (1-x)\sum_{1}^{N-1} c_n(\alpha_1)\sin(\pi m y) + x\sum_{1}^{N-1} c_n(\alpha_2)\sin(\pi m y) \\
\ + \ & (1-y)\sum_{1}^{N-1} c_n(\alpha_3)\sin(\pi m x) + y\sum_{1}^{N-1} c_n(\alpha_4)\sin(\pi m x) \\
\ + \ & \sum_{\substack{1\le n_1,n_2\le N-1 \\ 1\le n_1 n_2 \le N-1}} c_{n_1,n_2}(R)\sin(\pi n_1 x)\sin(\pi n_2 y).
\end{aligned}
$$

By using Parseval identity, it follows from (4.1) and (4.3) that

$$
\| f - \widetilde{S}_N^{(h)}(f) \|_2^2 = O(1)\left(\sum_{i=1}^{4}\sum_{m=N}^{\infty} |c_n(\alpha_i)|^2 + \left(\sum_{n_1=1}^{N-1}\sum_{n_2=\left[\frac{N}{n_1}\right]}^{\infty} + \sum_{n_1=N}^{\infty}\sum_{n_2=1}^{\infty} \right) |c_{n_1,n_2}(R)|^2 \right).
$$

Finally, by the estimates (4.2) and (2.3), we get

$$
\| f - \widetilde{S}_N^{(h)}(f) \|_2^2 = O\left(\frac{\log N}{N^3}\right).
$$

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The number of Fourier sine coefficients in the series (4.3) satisfies $N_c \sim N \log N$. Therefore,

$$\| f - \widetilde{S}_N^{(h)}(f) \|_2^2 = O\left(\frac{\log^4 N_c}{N_c^3}\right).$$

Theorem 4.1. Let $f \in W^{(l_1, l_2)}([0,1]^2)$ ($l_1 = 1 \text{ or } 2$, $l_2 = 1 \text{ or } 2$). Then the hyperbolic cross truncations of the series (4.1) satisfy

$$\| f - \widetilde{S}_N^{(h)}(f) \|_2^2 = O\left(\frac{\log N}{N^3}\right).$$

5. Uncertainty analysis

Suppose that f is a stochastic process and $f \in W^{(2,2)}([0,1]^2)$. Then the coefficient decomposition formula still holds:

$$c_{n_1,n_2}(f)$$
$$= \frac{4}{\pi^2 n_1 n_2} J_{n_1,n_2} - \frac{2}{\pi^3 n_1 n_2}\left(\frac{c_{n_1}(g_3) + (-1)^{n_1+1} c_{n_1}(g_4)}{n_1} + \frac{c_{n_2}(g_1) + (-1)^{n_2+1} c_{n_2}(g_2)}{n_2}\right) + r_{\mathbf{n}}, \tag{5.1}$$

where the error r_{n_1,n_2} is equal to

$$r_{n_1,n_2} = \frac{c_{n_1,n_2}\left(\frac{\partial^4 f}{\partial x^2 \partial y^2}\right)}{\pi^4 n_1^2 n_2^2} = \frac{4}{\pi^4 n_1^2 n_2^2} \int_{[0,1]^2} \frac{\partial^4 f}{\partial x^2 \partial y^2}(x,y) \sin(\pi n_1 x) \sin(\pi n_2 y)\, dx dy. \tag{5.2}$$

Consider the expectation of r_{n_1,n_2}. The expectation and integral can be exchanged, so

$$E[r_{n_1,n_2}] = \frac{4}{\pi^4 n_1^2 n_2^2} \int_{[0,1]^2} E\left[\frac{\partial^4 f}{\partial x^2 \partial y^2}(x,y)\right] \sin(\pi n_1 x) \sin(\pi n_2 y)\, dx dy. \tag{5.3}$$

The expectation and limit can be exchanged, so it follows from $\frac{\partial^4 f}{\partial x^2 \partial y^2} \in C([0,1]^2)$ that $E\left[\frac{\partial^4 f}{\partial x^2 \partial y^2}\right] \in C([0,1]^2)$. By the Riemann-Lebesgue lemma,

$$E[r_{n_1,n_2}] = \frac{c_{n_1,n_2}\left(E\left[\frac{\partial^4 f}{\partial x^2 \partial y^2}\right]\right)}{\pi^4 n_1^2 n_2^2} = o\left(\frac{1}{n_1^2 n_2^2}\right). \tag{5.4}$$

Consider the variance of $r_{\mathbf{n}}$. By (5.2), we have

$$r_{n_1,n_2}^2$$
$$= \frac{16}{\pi^8 n_1^4 n_2^4} \int_{[0,1]^4} \frac{\partial^4 f}{\partial x^2 \partial y^2}(x,y) \frac{\partial^4 f}{\partial t^2 \partial s^2}(t,s) \sin(\pi n_1 x) \sin(\pi n_2 y) \sin(\pi n_1 t) \sin(\pi n_2 s)\, dx dy dt ds.$$

From this and (5.4),

$$E[r_{n_1,n_2}^2]$$
$$= \frac{16}{\pi^8 n_1^4 n_2^4} \int_{[0,1]^4} E\left[\frac{\partial^4 f}{\partial x^2 \partial y^2}(x,y) \frac{\partial^4 f}{\partial t^2 \partial s^2}(t,s)\right] \sin(\pi n_1 x) \sin(\pi n_1 t) \sin(\pi n_2 x) \sin(\pi n_2 s)\, dx dy dt ds \tag{5.5}$$
$$= \frac{c_{n_1,n_2,n_1,n_2}\left(E\left[\frac{\partial^4 f}{\partial x^2 \partial y^2}(x,y) \frac{\partial^4 f}{\partial t^2 \partial s^2}(t,s)\right]\right)}{\pi^8 n_1^4 n_2^4} = o\left(\frac{1}{n_1^4 n_2^4}\right).$$

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By (5.3),

$$(E[r_{n_1,n_2}])^2$$

$$= \frac{16}{\pi^8 n_1^4 n_2^4} \int_{[0,1]^4} E\left[\frac{\partial^4 f}{\partial x^2 \partial y^2}(x,y)\right] E\left[\frac{\partial^4 f}{\partial t^2 \partial s^2}(t,s)\right] \sin(\pi n_1 x)\sin(\pi n_2 y)\sin(\pi n_1 t)\sin(\pi n_2 s)\,\mathrm{d}x\mathrm{d}y\mathrm{d}t\mathrm{d}s.$$

Notice that

$$\mathrm{Var}(r_{n_1,n_2}) = E[r_{n_1,n_2}^2] - (E[r_{n_1,n_2}])^2,$$

$$\mathrm{Cov}\left(\frac{\partial^4 f}{\partial x^2 \partial y^2}, \frac{\partial^4 f}{\partial t^2 \partial s^2}\right) = E\left[\frac{\partial^4 f}{\partial x^2 \partial y^2}\frac{\partial^4 f}{\partial t^2 \partial s^2}\right] - E\left[\frac{\partial^4 f}{\partial x^2 \partial y^2}\right] E\left[\frac{\partial^4 f}{\partial t^2 \partial s^2}\right].$$

Then, by (5.4) and (5.5), the variance of $r_{\mathbf{n}}$:

$$\mathrm{Var}(r_{n_1,n_2})$$

$$= \frac{16}{\pi^8 n_1^4 n_2^4} \int_{[0,1]^4} \mathrm{Cov}\left(\frac{\partial^4 f}{\partial x^2 \partial y^2}(x,y), \frac{\partial^4 f}{\partial t^2 \partial s^2}(t,s)\right) \sin(\pi n_1 x)\sin(\pi n_2 y)\sin(\pi n_1 t)\sin(\pi n_2 s)\,\mathrm{d}x\mathrm{d}y\mathrm{d}t\mathrm{d}s$$

$$= \frac{c_{n_1,n_2,n_1,n_2}\left(\mathrm{Cov}(\frac{\partial^4 f}{\partial x^2 \partial y^2}, \frac{\partial^4 f}{\partial t^2 \partial s^2})\right)}{\pi^8 n_1^4 n_2^4} = o\left(\frac{1}{n_1^4 n_2^4}\right).$$

Similarly, for $i = 1, 2, 3$, as $n_i \to \infty$, we have

$$E[c_{n_i}(g_i)] = c_{n_i}(E[g_i]) = o(1);$$

$$E[c_{n_i}^2(g_i)] = c_{n_i,n_i}(E[g_i(x)g_i(y)]) = o(1);$$

$$\mathrm{Var}(c_{n_i}(g_i)) = c_{n_i,n_i}(\mathrm{Cov}(g_i(x), g_i(y))) = o(1).$$

By (5.1), we get

$$E[c_{n_1,n_2}(f)] = \frac{4}{\pi^2 n_1 n_2}\left(E[J_{n_1,n_2}] + o\left(\frac{1}{n_1}\right) + o\left(\frac{1}{n_2}\right)\right).$$

For convenience, denote

$$\tau_{n_1,n_2} = -\frac{2}{\pi^3 n_1 n_2}\left(\frac{c_{n_2}(g_1) + (-1)^{n_2+1}(g_2)}{n_1} + \frac{c_{n_1}(g_3) + (-1)^{n_1+1}(g_4)}{n_2}\right)$$

$$\mu_{n_1,n_2} = \frac{4}{\pi^2 n_1 n_2} J_{n_1,n_2}.$$

So $c_{n_1,n_2}(f) = \mu_{n_1,n_2} + \tau_{n_1,n_2} + r_{n_1,n_2}$, and so

$$E[c_{n_1,n_2}^2(f)] = E[\mu_{n_1,n_2}^2] + A_{n_1,n_2},$$

where

$$A_{n_1,n_2}$$

$$= E[\tau_{n_1,n_2}^2] + E[r_{n_1,n_2}^2] + 2E[\mu_{n_1,n-2}\tau_{n_1,n_2}] + 2E[\mu_{n_1,n_2}r_{n_1,n_2}] + 2E[\tau_{n_1,n_2}r_{n_1,n_2}]$$

$$= \frac{1}{n_1^2 n_2^2}\left(o\left(\frac{1}{n_1}\right) + o\left(\frac{1}{n_2}\right)\right).$$

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Therefore,

$$E[c_{n_1,n_2}^2(f)] = \frac{16}{\pi^4 n_1^2 n_2^2}\left(E[J_{n_1,n_2}^2] + o\left(\frac{1}{n_1}\right) + o\left(\frac{1}{n_2}\right)\right). \tag{5.6}$$

Denote

$$\widetilde{M} = E[f^2(0,0)] + E[f^2(0,1)] + E[f^2(1,0)] + E[f^2(1,1)].$$

Similar to the argument from (3.1) to (3.4), we can deduce from (5.6) that

$$E\left[\parallel f - S_N^{(h)}(f)\parallel_2^2\right] = \frac{4\widetilde{M}}{\pi^4}\frac{\log N}{N} + O\left(\frac{1}{N}\right).$$

Theorem 5.1. Let f be a stochastic process and $f \in W^{(2,2)}([0,1]^2)$. Then
(i) Fourier sine coefficients of f satisfy

$$c_{n_1,n_2}(f)$$

$$= \frac{4}{\pi^2 n_1 n_2}J_{n_1,n_2} - \frac{2}{\pi^3 n_1 n_2}\left(\frac{c_{n_2}(g_1)+(-1)^{n_2+1}c_{n_2}(g_2)}{n_1} + \frac{c_{n_1}(g_3)+(-1)^{n_1+1}(g_4)}{n_2}\right) + r_{n_1,n_2},$$

where

$$E[r_{n_1,n_2}] = \frac{c_{n_1,n_2}\left(E[\frac{\partial^4 f}{\partial x^2 \partial y^2}]\right)}{\pi^4 n_1^2 n_2^2},$$

$$\mathrm{Var}(r_{n_1,n_2}) = \frac{c_{n_1,n_2,n_1,n_2}\left(\mathrm{Cov}\left(\frac{\partial^4 f}{\partial x^2 \partial y^2}, \frac{\partial^4 f}{\partial t^2 \partial s^2}\right)\right)}{\pi^8 n_1^4 n_2^4},$$

where $c_{n_1,n_2,n_1,n_2}(\mathrm{Cov}\left(\frac{\partial^4 f}{\partial x^2 \partial y^2}, \frac{\partial^4 f}{\partial t^2 \partial s^2}\right))$ is the four-variate Fourier sine coefficient of the covariance of $\frac{\partial^4 f}{\partial x^2 \partial y^2}$ and $\frac{\partial^4 f}{\partial t^2 \partial s^2}$ at $\mathbf{n} = (n_1, n_2, n_1, n_2)$.
(ii) the hyperbolic cross truncations of Fourier sine series of f satisfy

$$E\left[\parallel f - S_N^{(h)}(f)\parallel_2^2\right] = \frac{4\widetilde{M}}{\pi^4}\frac{\log N}{N} + O\left(\frac{1}{N}\right),$$

where $\widetilde{M} = E[f^2(0,0)] + E[f^2(0,1)] + E[f^2(1,0)] + E[f^2(1,1)]$.

6. The three-dimensional case

For a three-dimensional function f on $[0,1]^3$, we can decompose f as follows:

$$f(x,y,z) = P(x,y,z) + Q(x,y,z) + R(x,y,z) + T(x,y,z), \tag{6.1}$$

where

$$\begin{aligned} P(x,y,z) &= f(0,0,0)(1-x)(1-y)(1-z) + f(0,1,0)(1-x)y(1-z)\\ &+ f(0,1,1)(1-x)yz + f(0,0,1)(1-x)(1-y)z\\ &+ f(1,0,0)x(1-y)(1-z) + f(1,1,0)xy(1-z)\\ &+ f(1,1,1)xyz + f(1,0,1)x(1-y)z \end{aligned} \tag{6.2}$$

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is a three-variate polynomial;

$$
\begin{aligned}
Q(x,y,z) \ = \ & f_1(x,0,0)(1-y)(1-z) + f_1(x,0,1)(1-y)z \\
+ \ & f_1(x,1,0)y(1-z) + f_1(x,1,1)yz \\
+ \ & f_1(1,y,0)(1-x)(1-z) + f_1(0,y,1)(1-x)z \\
+ \ & f_1(1,y,0)x(1-z) + f_1(1,y,1)xz \\
+ \ & f_1(0,0,z)(1-x)(1-y) + f_1(0,1,z)(1-x)y \\
+ \ & f_1(1,0,z)x(1-y) + f(1,1,z)xy \qquad (f_1 = f - P)
\end{aligned}
\tag{6.3}
$$

is a sum of products of separated variable types, where one factor is the restriction of f_1 is each edge, the other factor is a bivariate polynomial;

$$
\begin{aligned}
R(x,y,z) \ = \ & f_2(x,y,0)(1-z) + f_2(x,0,z)(1-y) \\
+ \ & f_2(0,y,z)(1-x) + f_2(x,y,1)z \\
+ \ & f_2(x,1,z)y + f_2(1,y,z)x \qquad (f_2 = f - P - Q)
\end{aligned}
\tag{6.4}
$$

is a sum of products of separated variable types, where one factor is the restriction of f_2, the other factor is a univariate polynomial and

$$
T(x,y,z) = f(x,y,z) - P(x,y,z) - Q(x,y,z) - R(x,y,z).
$$

It is easy to check the following proposition.

Proposition 6.1. $f_1(x,y,z) = 0$ for each vertex of $[0,1]^3$ and $f_2(x,y,z) = 0$ for each edge of $[0,1]^3$, and $T(x,y,z) = 0$ for each face of $[0,1]^3$.

Consider the Fourier sine coefficients $c_{n_1,n_2,n_3}(f)$. From the decomposition formula, it follows that

$$
c_{n_1,n_2,n_3}(f) = c_{n_1,n_2,n_3}(P) + c_{n_1,n_2,n_3}(Q) + c_{n_1,n_2,n_3}(R) + c_{n_1,n_2,n_3}(T).
$$

Since the Fourier sine coefficients:

$$
c_{n_1,n_2,n_3}(f) = 8\int_{[0,1]^3} f(x,y,z)\,\sin(\pi n_1 x)\sin(\pi n_2 y)\sin(\pi n_3 z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z,
$$

we obtain that

(i)

$$
c_{n_1,n_2,n_3}(P) = \frac{8U_{n_1,n_2,n_3}}{\pi^3 n_1 n_2 n_3},
$$

where

$$
\begin{aligned}
U_{n_1,n_2,n_3} \ = \ & f(0,0,0) + (-1)^{n_2+1}f(0,1,0) + (-1)^{n_2+n_3}f(0,1,1) \\
+ \ & (-1)^{n_3+1}f(0,0,1) + (-1)^{n_1+1}f(1,0,0) + (-1)^{n_1+n_2}f(1,1,0) \\
+ \ & (-1)^{n_1+n_2+n_3+1}f(1,1,1) + (-1)^{n_1+n_3}f(1,0,1);
\end{aligned}
\tag{6.5}
$$

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(ii)

$$c_{n_1,n_2,n_3}(Q) = -\frac{V^{(1)}_{n_1,n_2,n_3}}{\pi^4 n_1^2 n_2 n_3} - \frac{V^{(2)}_{n_1,n_2,n_3}}{\pi^4 n_1 n_2^2 n_3} - \frac{V^{(3)}_{n_1,n_2,n-3}}{\pi^4 n_1 n_2 n_3^2},$$

where

$$
\begin{aligned}
V^{(1)}_{n_1,n_2,n_3} &= c_{n_1}\left(\frac{\partial^2 f}{\partial x^2}(\cdot,0,0)\right) + (-1)^{n_3+1} c_{n_1}\left(\frac{\partial^2 f}{\partial x^2}(\cdot,0,1)\right) \\
&\quad + (-1)^{n_2+1} c_{n_1}\left(\frac{\partial^2 f}{\partial x^2}(\cdot,1,0)\right) + (-1)^{n_2+n_3} c_{n_1}\left(\frac{\partial^2 f}{\partial x^2}(\cdot,1,1)\right), \\
V^{(2)}_{n_1,n_2,n_3} &= c_{n_2}\left(\frac{\partial^2 f}{\partial y^2}(0,\cdot,0)\right) + (-1)^{n_3+1} c_{n_2}\left(\frac{\partial^2 f}{\partial y^2}(0,\cdot,1)\right) \\
&\quad + (-1)^{n_1+1} c_{n_2}\left(\frac{\partial^2 f}{\partial y^2}(1,\cdot,0)\right) + (-1)^{n_1+n_3} c_{n_2}\left(\frac{\partial^2 f}{\partial y^2}(1,\cdot,1)\right), \\
V^{(3)}_{n_1,n_2,n_3} &= c_{n_3}\left(\frac{\partial^2 f}{\partial z^2}(0,0,\cdot)\right) + (-1)^{n_2+1} c_{n_3}\left(\frac{\partial^2 f}{\partial z^2}(0,1,\cdot)\right) \\
&\quad + (-1)^{n_1+1} c_{n_3}\left(\frac{\partial^2 f}{\partial z^2}(1,0,\cdot)\right) + (-1)^{n_1+n_2} c_{n_3}\left(\frac{\partial^2 f}{\partial z^2}(1,1,\cdot)\right);
\end{aligned}
$$

(iii)

$$c_{n_1,n_2,n_3}(R) = \frac{M^{(1)}_{n_1,n_2,n_3}}{\pi^5 n_1^2 n_2^2 n_3} + \frac{M^{(2)}_{n_1,n_2,n_3}}{\pi^5 n_1^2 n_2 n_3^2} + \frac{M^{(3)}_{n_1,n_2,n_3}}{\pi^5 n_1 n_2^2 n_3^2},$$

where

$$
\begin{aligned}
M^{(1)}_{n_1,n_2,n_3} &= c_{n_1,n_2}\left(\frac{\partial^4 f}{\partial x^2 \partial y^2}(\cdot,\cdot,0)\right) + (-1)^{n_3+1} c_{n_1,n_2}\left(\frac{\partial^4 f}{\partial x^2 \partial y^2}(\cdot,\cdot,1)\right), \\
M^{(2)}_{n_1,n_2,n-3} &= c_{n_1,n_3}\left(\frac{\partial^4 f}{\partial x^2 \partial z^2}(\cdot,0,\cdot)\right) + (-1)^{n_2+1} c_{n_1,n_3}\left(\frac{\partial^4 f}{\partial x^2 \partial z^2}(\cdot,1,\cdot)\right), \\
M^{(3)}_{n_1,n_2,n_3} &= c_{n_2,n_3}\left(\frac{\partial^4 f}{\partial y^2 \partial z^2}(0,\cdot,\cdot)\right) + (-1)^{n_1+1} c_{n_2,n_3}\left(\frac{\partial^4 f}{\partial y^2 \partial z^2}(1,\cdot,\cdot)\right);
\end{aligned}
$$

(iv)

$$c_{n_1,n_2,n_3}(T) = \frac{c_{n_1,n_2,n_3}\left(\frac{\partial^6 f}{\partial x^2 \partial y^2 z^2}\right)}{\pi^6 n_1^2 n_2^2 n_3^2}.$$

From this and Proposition 6.1, we get the following theorem.

Theorem 6.2. Suppose that $f \in W^{(2,2,2)}([0,1]^3)$, i.e., $\frac{\partial^6 f}{\partial x^2 \partial y^2 \partial z^2}(x,y,z) \in C([0,1]^3)$. Then

$$c_{n_1,n_2,n_3}(f) = \frac{8}{\pi^3 n_1 n_2 n_3}\left(U_{n_1,n_2,n_3} + o\left(\frac{1}{n_1}\right) + o\left(\frac{1}{n_2}\right) + o\left(\frac{1}{n_3}\right)\right),$$

where U_{n_1,n_2,n_3} is stated in (6.5).

Let $n_i = 2p_i + q_i$ $(i = 1,2,3)$. Then

$$c_{2p_1+q_1,2p_2+q_2,2p_3+q_3}(f) = \frac{1}{\pi^3 p_1 p_2 p_3}\left(U_{2p_1+q_1,2p_2+q_2,2p_3+q_3} + o\left(\frac{1}{p_1}\right) + o\left(\frac{1}{p_2}\right) + o\left(\frac{1}{p_3}\right)\right),$$

where U_{n_1,n_2,n_3} is stated in (6.5). It is clear from (6.5) that $U_{2p_1+q_1,2p_2+q_2,2p_3+q_3} = U_{q_1,q_2,q_3}$. So

$$c^2_{2p_1+q_1,2p_2+q_2,2p_3+q_3}(f) = \frac{1}{\pi^2 p_1^2 p_2^2 p_3^2}\left(U_{q_1,q_2,q_3} + o\left(\frac{1}{p_1}\right) + o\left(\frac{1}{p_2}\right) + o\left(\frac{1}{p_3}\right)\right)$$

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and

$$\sum_{(q_1,q_2,q_3)\in\{0,1\}^3} c^2_{2p_1+q_1,2p_2+q_2,2p_3+q_3}(f) = \frac{1}{\pi^2 p_1^2 p_2^2 p_3^2}\left(\sum_{(q_1,q_2,q_3)\in\{0,1\}^3} U^2_{q_1,q_2,q_3} + o\left(\frac{1}{p_1}\right) + o\left(\frac{1}{p_2}\right) + o\left(\frac{1}{p_3}\right)\right).$$

By (6.5),

$$\begin{aligned}
U_{q_1,q_2,q_3} &= f(0,0,0) + (-1)^{q_2+1} f(0,1,0) + (-1)^{q_2+q_3} f(0,1,1) \\
&+ (-1)^{q_3+1} f(0,0,1) + (-1)^{q_1+1} f(1,0,0) + (-1)^{q_1+q_2} f(1,1,0) \\
&+ (-1)^{q_1+q_2+q_3+1} f(1,1,1) + (-1)^{q_1+q_3} f(1,0,1).
\end{aligned}$$

A direct computation shows that

$$\begin{aligned}
\sum_{(q_1,q_2,q_3)\in\{0,1\}^3} U_{q_1,q_2,q_3} &= 8(f^2(0,0,0) + f^2(0,1,0) + f^2(0,1,1) + f^2(0,0,1) \\
&+ f^2(1,0,0) + f^2(1,1,0) + f^2(1,1,1) + f^2(1,0,1)) \\
&= 8\sum_{\lambda\in\{0,1\}^3} f^2(\lambda).
\end{aligned}$$

Therefore,

$$\sum_{(q_1,q_2,q_3)\in\{0,1\}^3} c^2_{2p_1+q_1,2p_2+q_2,2p_3+q_3}(f) = \frac{1}{\pi^6 p_1^2 p_2^2 p_3^2}\left(8\sum_{\lambda\in\{0,1\}^3} f^2(\lambda) + o\left(\frac{1}{p_1}\right) + o\left(\frac{1}{p_2}\right) + o\left(\frac{1}{p_3}\right)\right).$$

From this, we deduce the following proposition.

Proposition 6.3. Let $f \in W^{(2,2,2)}([0,1]^3)$. Then its Fourier sine coefficients

$$c_{n_1,n_2,n_3}(f) = o\left(\frac{1}{n_1 n_2 n_3}\right)\left(o\left(\frac{1}{n_1}\right) + o\left(\frac{1}{n_2}\right) + o\left(\frac{1}{n_3}\right)\right)$$

if and only if $f(\lambda) = 0$ for all $\lambda \in \{0,1\}^3$.

Suppose that $f \in W^{(2,2,2)}([0,1]^3)$. Then the hyperbolic cross truncation of its Fourier sine series:

$$S_N^{(h)}(f; x, y, z)$$

$$= \sum_{\substack{1\le n_1,n_2,n_3\le N-1 \\ 1\le n_1 n_2 n_3 \le N-1}} c_{n_1,n_2,n_3}(f)\, \sin(\pi n_1 x)\sin(\pi n_2 y)\sin(\pi n_3 z)$$

$$= \sum_{\substack{1\le p_1,p_2,p_3\le\left[\frac{N-1}{2}\right] \\ 1\le p_1 p_2 p_3 \le\left[\frac{N-1}{8}\right]}} \sum_{(q_1,q_2,q_3)\in\{0,1\}^3} c_{2p_1+q_1,2p_2+q_2,2p_3+q_3}(f)\sin\pi(2p_1+q_1)x\,\sin\pi(2p_2+q_2)y\,\sin\pi(2p_3+q_3)z.$$

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By the Parseval identity and (6.3), it follows that

$$8 \parallel f - S_N^{(h)}(f) \parallel_2^2$$

$$= \left(\sum_{p_1,p_2,p_3=1}^{\infty} - \sum_{\substack{1 \le p_1,p_2,p_3 \le \left[\frac{N-1}{2}\right] \\ 1 \le p_1 p_2 p_3 \le \left[\frac{N-1}{8}\right]}} \right) \sum_{(q_1,q_2,q_3) \in \{0,1\}^3} c_{2p_1+q_1, 2p_2+q_2, 2p_3+q_3}^2(f)$$

$$= \frac{1}{\pi^6} \left(\sum_{p_1,p_2,p_3=1}^{\infty} - \sum_{\substack{1 \le p_1,p_2,p_3 \le \left[\frac{N-1}{2}\right] \\ 1 \le p_1 p_2 p_3 \le \left[\frac{N-1}{8}\right]}} \right) \frac{8}{p_1^2 p_2^2 p_3^2} \left(\sum_{\lambda \in \{0,1\}^3} f^2(\lambda) + o(1) \right)$$

$$= \frac{1}{\pi^6} \sum_{p_1=1}^{\left[\frac{N-1}{2}\right]} \sum_{p_2=1}^{\left[\frac{N}{4p_1}\right]} \sum_{p_3=\left[\frac{N}{2p_1p_2}\right]}^{\infty} \left(\sum_{\lambda \in \{0,1\}^3} f^2(\lambda) + o(1) \right).$$

Notice that

$$\sum_{p_3=\left[\frac{N}{8p_1p_2}\right]}^{\infty} \frac{1}{p_1^2 p_2^2 p_3^2}$$

$$= \frac{1}{p_1^2 p_2^2} \int_{\frac{N}{8p_1p_2}}^{\infty} \frac{\mathrm{d}t}{t^2} + O\left(\frac{1}{N^2}\right) = \frac{8}{p_1 p_2 N} + O\left(\frac{1}{N^2}\right)$$

and

$$\sum_{p_2=1}^{\left[\frac{N}{4p_1}\right]} \sum_{p_3=\left[\frac{N}{8p_1p_2}\right]}^{\infty} \frac{1}{p_1^2 p_2^2 p_3^2}$$

$$= \sum_{p_2=1}^{\left[\frac{N}{4p_1}\right]} \frac{8}{p_1 p_2 N} + O\left(\frac{1}{Np_1}\right) = \frac{8}{p_1 N} \int_1^{\frac{N}{4p_1}} \frac{\mathrm{d}t}{t} + O\left(\frac{1}{Np_1}\right)$$

$$= \frac{8}{Np_1} (\log N - \log p_1) + O\left(\frac{1}{Np_1}\right).$$

Since

$$\sum_{p_1=1}^{\left[\frac{N-1}{2}\right]} \frac{1}{p_1} = \log N + O(1)$$

and

$$\sum_{p_1=1}^{\left[\frac{N-1}{2}\right]} \frac{\log p_1}{p_1} = \int_1^{\frac{N}{2}} \frac{\log t}{t} \, \mathrm{d}t + O(1) = \frac{1}{2} \int_1^{\frac{N}{2}} \mathrm{d}\log^2 t + O(1) = \frac{1}{2} \log^2 N + O(1),$$

we have

$$\sum_{p_1=1}^{\left[\frac{N-1}{2}\right]} \sum_{p_2=1}^{\left[\frac{N}{4p_1}\right]} \sum_{p_3=\left[\frac{N}{8p_1p_2}\right]}^{\infty} \frac{1}{p_1^2 p_2^2 p_3^2}$$

$$= \frac{8\log N}{N} \sum_{p_1=1}^{\left[\frac{N-1}{2}\right]} \frac{1}{p_1} - \frac{8}{N} \sum_{p_1=1}^{\left[\frac{N-1}{2}\right]} \frac{\log p_1}{p_1} + O\left(\frac{\log N}{N}\right)$$

$$= \frac{8\log^2 N}{N} - \frac{4\log^2 N}{N} + O\left(\frac{\log N}{N}\right) = \frac{4\log^2 N}{N} + O\left(\frac{\log N}{N}\right).$$

Finally, we have

$$\| f - S_N^{(h)}(f) \|_2^2 = \frac{4}{\pi^6} \left(\sum_{\lambda \in \{0,1\}^3} f^2(\lambda) \right) \frac{\log^2 N}{N}(1 + o(1)) \qquad (N \to \infty).$$

For stochastic processes, we have the corresponding result.

Theorem 6.4. Suppose that f is a stochastic process and $f \in W^{(2,2,2)}([0,1]^3)$. Then

$$E\left[\| f - S_N^{(h)}(f) \|_2^2 \right] = \frac{4}{\pi^6} \left(\sum_{\lambda \in \{0,1\}^3} E[f^2(\lambda)] \right) \frac{\log^2 N}{N}(1 + o(1)) \qquad (N \to \infty).$$

For a three-variate function f on $[0,1]^3$, in its decomposition formula (6.1)-(6.4), we expand univariate functions $f_1(x,0,0), ..., f_1(1,y,1)$, bivariate functions $f_2(x,y,0), ..., f_2(x,y,1)$, and three-variate function $T(x,y,z)$ into Fourier sine series, we get the Fourier sine series with polynomial factors. We again define the corresponding hyperbolic cross truncations as follows:

$$(\widetilde{S}_N^{(h)} f)(x,y,z) = P(x,y,z) + Q_N(x,y,z) + R_N(x,y,z) + T_N(x,y,z), \qquad (6.6)$$

where $P(x,y,z)$ is stated in (6.2), $Q_N(x,y,z)$ is obtained by replacing eight univariate functions by their Nth partial sums in (6.3), $R_N(x,y,z)$ is obtained by replacing four bivariate functions by their Nth hyperbolic cross truncations in (6.4), and $T_N^{(h)}$ is the Nth hyperbolic cross truncation of $T(x,y,z)$.

Theorem 6.5. Let $f \in W^{(2,2,2)}([0,1]^3)$. Then hyperbolic cross truncations of the Fourier sine series of f with polynomial factors satisfy

$$\| f - \widetilde{S}_N^{(h)}(f) \|_2^2 = O\left(\frac{\log^2 N}{N^3}\right),$$

where $\widetilde{S}_N^{(h)}(f)$ is defined in (6.6).

The number of Fourier sine coefficients in $\widetilde{S}_N^{(h)}(f)$ satisfy $N_c \sim N \log^2 N$. From this and (6.7), we have

$$\| f - S_N^{(h)}(f) \|_2^2 = O\left(\frac{\log^8 N_c}{N_c^3}\right).$$

Therefore, we can use fewest Fourier sine coefficients to reconstruct f. For stochastic processes, the corresponding result is

$$E\left[\| f - S_N^{(h)}(f) \|_2^2 \right] = O\left(\frac{\log^8 N_c}{N_c^3}\right).$$

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A new relaxation method for mathematical programs with nonlinear complementarity constraints [*]

Jianling Li[1], Xiaojin Huang[2], Jinbao Jian[3,†]

[1] College of Mathematics and Informatics Science, Guangxi University,
530004, Nanning, P.R. China

[2] Department of Mathematics, Shanghai Normal University,
200234 , Shanghai, P.R. China

[3] School of Mathematics and Informatics Science, Yulin Normal University,
537000, Yulin, P.R. China

Abstract. In this paper, mathematical programs with nonlinear complementarity constraints (MPCC) are investigated and a new relaxed method is proposed. Firstly, based on Mangasarian complementarity function, MPCC is relaxed. The relaxed problem is a parametrized nonlinear programming. Secondly, it is proved that the sequence of stationary points of the relaxed problems converges to M-stationary point of MPCC under some mild assumptions; further, it is shown that the stationary point is strong for MPCC if some additional conditions are satisfied. Thirdly, we analyze the existence of the Lagrange multipliers for the relaxed problem. We show that Guignard constraint qualification holds for the relaxed problem under MPCC-linear independence constraint qualifications, and then obtain the existence theorem of the Lagrange multipliers.

Key words. Nonlinear complementarity constraints; Mathematical programs; Relaxed method; Constraint qualifications; Stationary points; Global convergence

AMS subject classification 90C, 49M.

1. Introduction

In this paper, we consider the following MPCC :

$$
\begin{aligned}
\min \quad & f(z) \\
\text{s.t.} \quad & g_i(z) \leq 0, \quad i \in I = \{1, \cdots, m\}, \\
& h_i(z) = 0, \quad i \in I_e = \{1, \cdots, m_e\}, \\
& G_i(z) \geq 0, \quad i \in I_c = \{1, \cdots, m_c\}, \\
& H_i(z) \geq 0, \quad i \in I_c, \\
& G_i(z)H_i(z) \leq 0, \quad i \in I_c,
\end{aligned}
\tag{1.1}
$$

where f, g_i, h_i, G_i, $H_i : \mathbb{R}^n \to \mathbb{R}$ are all continuously differentiable. The MPCC (1.1) has many applications in game theory, traffic transportation, engineering design and so on. The interested reader is referred to the monograph [1] for more details.

As we know, the MPCC (1.1) is a highly difficult nonlinear program since the standard Mangasarian-Fromovitz constraint qualification (MFCQ) is violated at any feasible point (see [2]). This implies that the well-developed approaches for the standard nonlinear programs typically have severe difficulties if they are directly used to solve the MPCC (1.1). So MPCC-tailed algorithms have to be studied.

During last decade, several kinds of efficient methods for the MPCC (1.1) have been developed, such as relaxation (or regularization) ([4–8]), smoothing ([1, 9–17]), interior point method ([1, 18–20]) and penalization ([21]). In this paper, our focus is on relaxation method. The basic idea of the relaxation method is to relax the complicated complementarity constraints

$$G_i(z) \geq 0, \ H_i(z) \geq 0, \ G_i(z)H_i(z) \leq 0, \ \ i \in I_c$$

in a suitable way. The interested reader is referred to the recent review paper on relaxation method [5] for more knowledge.

Kadrani et al. proposed a relaxation scheme in [8] as follows:

$$
\begin{aligned}
\min \quad & f(z) \\
\text{s.t.} \quad & g_i(z) \leq 0, \quad i \in I, \\
& h_i(z) = 0, \quad i \in I_e, \\
& G_i(z) \geq -t, \quad i \in I_c, \\
& H_i(z) \geq -t, \quad i \in I_c, \\
& (G_i(z) - t)(H_i(z) - t) \leq 0, \quad i \in I_c,
\end{aligned}
\tag{1.2}
$$

where t is a nonnegative parameter. It is shown that any accumulation point of the stationary point sequence of (1.2) converges to an M-stationary point of MPCC (1.1) when $t \to 0$ under the MPCC-linear independence constraint qualification (MPCC-LICQ) condition and some mild conditions. They also showed that existence of KKT multipliers for the relaxed problem (1.2) under the MPCC-LICQ assumption. Figure 1, however, shows that there exist two disadvantages: (1) the feasible region of the relaxed problem (1.2) is almost disconnected. Therefore, one has to meet severe difficulties when solving (1.2) by means of a standard NLP algorithm; (2) the feasible region of the MPCC (1.1) is not included in that of the relaxed problem (1.2), regardless of the choice of $t > 0$.

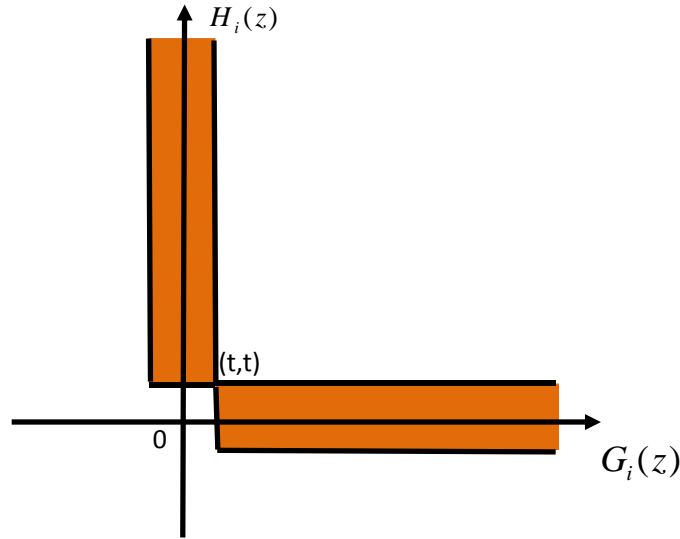In order to overcome the above drawbacks, Kanzow et al. recently proposed a new relaxation

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Figure 1: Geometric interpretation of relaxation in [8]

scheme in [5] as follows:

$$
\begin{aligned}
\min \quad & f(z) \\
\text{s.t.} \quad & g_i(z) \leq 0, \ i \in I, \\
& h_i(z) = 0, \ i \in I_e, \\
& G_i(z) \geq -t, \ i \in I_c, \\
& H_i(z) \geq -t, \ i \in I_c, \\
& \Psi(z;t) = (\Psi_i(z;t), i \in I_c) \leq 0,
\end{aligned}
\tag{1.3}
$$

where $t \geq 0$ is a parameter, $\Psi_i(z;t) = \varphi(G_i(z) - t, H_i(z) - t)$, the complementarity function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$
\varphi(x,y) = \begin{cases} xy, & x + y \geq 0, \\ -\frac{1}{2}(x^2 + y^2), & x + y < 0. \end{cases}
$$

The geometric interpretation of the relaxation scheme (1.3) is given in Figure 2. It is shown that any accumulation point of the stationary point sequence of (1.3) converges to an M-stationary point of MPCC (1.1) when $t \to 0$ under much weaker MPCC-constant positive linear dependence (MPCC-CPLD) condition and some mild conditions. And they also showed the existence of the Lagrange multipliers for the relaxed problem (1.3) under the MPCC-LICQ assumption.

It is worth noting that the feasible region of the original problem (1.1) is part of the boundary of that of the relaxed problem (1.3). Consequently, some additional stricter conditions is required for the search directions when solving the relaxed problem (1.3) by a standard NLP algorithm.

In this paper, motivated from the ideas in [5, 8] and based on the Mangasarian complementarity
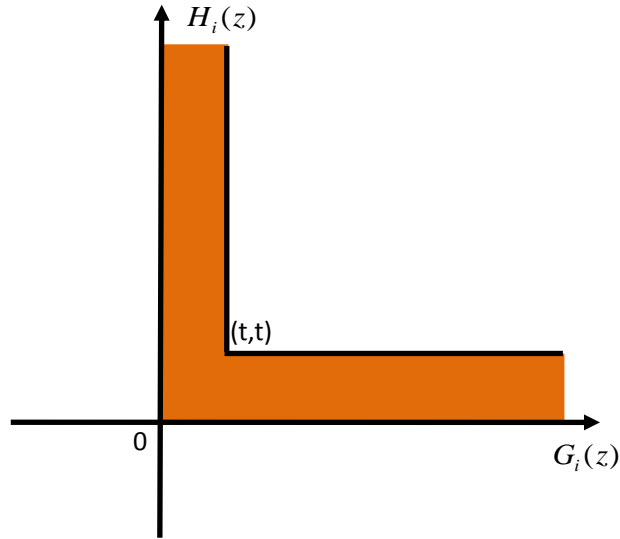
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Figure 2: Geometric interpretation of relaxation in [5]

function ([25]) defined by

$$\phi(a,b) = \rho(a) + \rho(b) - \rho(|a-b|) \tag{1.4}$$

with $\rho : \mathbb{R} \to \mathbb{R}$ being given by

$$\rho(\tau) = \begin{cases} \tau^2, & \text{if } \tau \geq 0 \\ -\tau^2, & \text{if } \tau < 0, \end{cases}$$

we propose a new relaxation scheme:

$$\begin{aligned} \min \quad & f(z) \\ \text{s.t.} \quad & g_i(z) \leq 0, \ i \in I, \\ & h_i(z) = 0, \ i \in I_e, \\ & G_i(z) \geq -t, \ i \in I_c, \\ & H_i(z) \geq -t, \ i \in I_c, \\ & \Phi_i(z;t) \leq 0, \ i \in I_c, \end{aligned} \tag{1.5}$$

where t is a nonnegative parameter and

$$\Phi_i(z;t) = \phi(G_i(z) - t, H_i(z) - t). \tag{1.6}$$

The geometric interpretation of the relaxation scheme (1.5) is given in Figure 3.

We show that any accumulation point of the stationary point sequence of (1.5) converges to an M-stationary point of MPCC (1.1) when $t \to 0$ under much weaker MPCC-CPLD condition and some mild conditions, and converges to a strongly stationary point of MPCC (1.1) under additional conditions. We also show that the standard Guignard constraint qualification (GCQ) holds at every
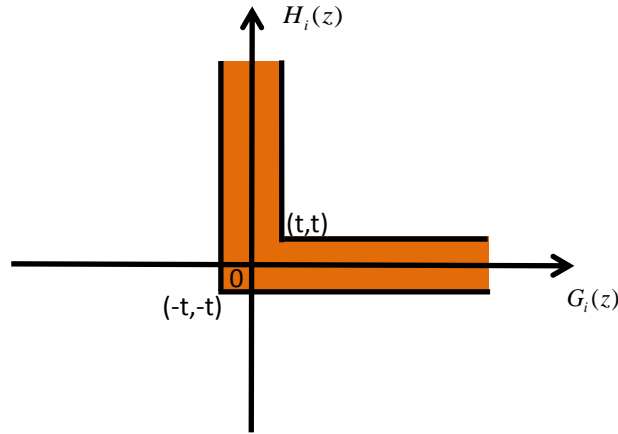
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Figure 3: Geometric interpretation of the proposed relaxation scheme

feasible point of the relaxed problem (1.5) and the existence of Lagrange multipliers of the relaxed problem (1.5) is verified under some mild conditions.

The rest of the paper is organized as follows. Some definitions of different stationary points and constraint qualifications and preliminary results about MPCC are restated in Section 2. Section 3 contains the analysis and proof of the convergent results. The existence of Lagrange multipliers for the relaxed problem is analyzed and verified in Section 4 and some concluding remarks are given in the final section.

2. Preliminaries

As we know, except for Guignard CQ, all standard constraint qualifications are far too restrictive for MPCCs ([22]). Some MPCC-tailed CQs are introduced in the past. Furthermore, due to the fact that most standard CQs are likely to be violated at a local minimum of an MPCC, the KKT conditions can not be considered as the optimality conditions. Hence, several weaker stationarity notions have been proposed. For convenience and completeness, in this section we briefly restate some concepts and results about the MPCC (1.1) which are needed in the sequel analysis. The reader is also referred to [5, 22–24].

Let S be the feasible set of the MPCC (1.1). For any $z^* \in S$, define different index sets for the MPCC (1.1) as follows:

$$
\begin{aligned}
I_{0+}(z^*) &= \{i \in I_c \mid G_i(z^*) = 0, \ H_i(z^*) > 0\}, \\
I_{00}(z^*) &= \{i \in I_c \mid G_i(z^*) = 0, \ H_i(z^*) = 0\}, \\
I_{+0}(z^*) &= \{i \in I_c \mid G_i(z^*) > 0, \ H_i(z^*) = 0\}.
\end{aligned}
\tag{2.1}
$$

Definition 2.1 [5, 23] Let z^* be a feasible point of the MPCC (1.1). Then z^* is said to be

(1) weakly stationary for the MPCC (1.1), if there exist multipliers $(\alpha^*, \beta^*, \gamma^*, \delta^*) \in \mathbb{R}^m \times \mathbb{R}^{m_e} \times$

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$\mathbb{R}^{m_c} \times \mathbb{R}^{m_c}$ such that

$$\nabla f(z^*) + \sum_{i \in I} \alpha_i^* \nabla g_i(z^*) + \sum_{i \in I_e} \beta_i^* \nabla h_i(z^*) - \sum_{i \in I_c} \gamma_i^* \nabla G_i(z^*) - \sum_{i \in I_c} \delta_i^* \nabla H_i(z^*) = 0,$$

$$\alpha_i^* \geq 0, \quad \alpha_i^* g_i(z^*) = 0 \ (i \in I), \qquad \gamma_i^* = 0 \ (i \in I_{+0}(z^*)), \quad \delta_i^* = 0 \ (i \in I_{0+}(z^*)); \tag{2.2}$$

(2) C-stationarity, if it is weakly stationarity and $\gamma_i^* \delta_i^* \geq 0, \ \forall \ i \in I_{00}(z^*)$;

(3) M-stationary, if it is weakly stationarity and $\gamma_i^* > 0, \ \delta_i^* > 0 \ $ or $\ \gamma_i^* \delta_i^* = 0, \ \forall \ i \in I_{00}(z^*)$;

(4) strongly stationary, if it is weakly stationarity and $\gamma_i^*, \ \delta_i^* \geq 0, \ \forall \ i \in I_{00}(z^*)$.

Obviously, we know that strong stationarity implies M-stationarity, M-stationarity implies C-stationarity and C-stationarity implies weak stationarity. Moreover, it is shown in [22] that strong stationarity is equivalent to the standard KKT conditions of an MPCC. However, a counterexample given in [23] indicates that strong stationarity may not hold at a global minimum, even for very simple MPCCs.

Definition 2.2 [5, 23] Let z^* be a feasible point of the MPCC (1.1). Then

(1) MPCC-LICQ is said to hold at z^* if the gradients

$$\{\nabla g_i(z^*) \mid i \in I_g(z^*)\} \cup \{\nabla h_i(z^*) \mid i \in I_e\} \cup \{\nabla G_i(z^*) \mid i \in I_{00}(z^*) \cup I_{0+}(z^*)\}$$
$$\cup \{\nabla H_i(z^*) \mid i \in I_{00}(z^*) \cup I_{+0}(z^*)\}$$

are linearly independent.

(2) MPCC-CPLD is said to hold at z^* if, for any subsets $I_1 \subseteq I_g(z^*)$, $I_2 \subseteq I_e$, $I_3 \subseteq I_{00}(z^*) \cup I_{0+}(z^*)$, $I_4 \subseteq I_{00}(z^*) \cup I_{+0}(z^*)$ such that the gradients

$$\{\nabla g_i(z^*) \mid i \in I_1\} \cup \{\{\nabla h_i(z^*) \mid i \in I_2\} \cup \{\nabla G_i(z^*) \mid i \in I_3\} \cup \{\nabla H_i(z^*) \mid i \in I_4\}\}$$

are positive-linearly dependent, there exists a neighborhood $N(z^*)$ of z^* such that the gradients

$$\{\nabla g_i(z) \mid i \in I_1\} \cup \{\nabla h_i(z) \mid i \in I_2\} \cup \{\nabla G_i(z) \mid i \in I_3\} \cup \{\nabla H_i(z) \mid i \in I_4\}$$

are linearly dependent for all $z \in N(z^*)$.

It follows from [24] that MPCC-LICQ implies MPCC-CPLD.

3. Convergence results

In this section, we analyze the convergence behavior of the relaxed problem (1.5) as $t \to 0$. For convenience, denote by $R_{MPCC}(t)$ (1.5) the relaxed problem (1.5), and define the following index sets for $R_{MPCC}(t)$ (1.5):

$$I_g(z) = \{i \in I \mid g_i(z) = 0\}, \quad I_G(z; t) = \{i \in I_c \mid G_i(z) = -t\},$$
$$I_H(z; t) = \{i \in I_c \mid H_i(z) = -t\}, \quad I_\Phi(z; t) = \{i \in I_c \mid \Phi_i(z; t) = 0\},$$
$$I_\Phi^{0+}(z; t) = \{i \in I_\Phi(z; t) \mid G_i(z) - t = 0, \ H_i(z) - t > 0\},$$
$$I_\Phi^{00}(z; t) = \{i \in I_\Phi(z; t) \mid G_i(z) - t = 0, \ H_i(z) - t = 0\},$$
$$I_\Phi^{+0}(z; t) = \{i \in I_\Phi(z; t) \mid G_i(z) - t > 0, \ H_i(z) - t = 0\},$$
$$supp(c) = \{i \mid c_i \neq 0, \ i = 1, \ldots, l. \ c = (c_i) \in \mathbb{R}^l\}.$$

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Obviously, we have $I_\Phi^{0+}(z;t) \cap I_\Phi^{00}(z;t) \cap I_\Phi^{+0}(z;t) = \emptyset$, $I_\Phi^{0+}(z;t) \cup I_\Phi^{00}(z;t) \cup I_\Phi^{+0}(z;t) = I_\Phi(z;t)$.

By elementary computation and analysis, we can obtain the important properties of the complementarity function ϕ given in (1.4), which play a key role in the subsequently analysis.

Lemma 3.1 [25] (1) $\phi(a,b) = 0$ if and only if $a \geq 0$, $b \geq 0$, $ab = 0$.

(2) ϕ is continuously differentiable, and its gradient is

$$
\nabla\phi(a,b) = \begin{cases} \begin{pmatrix} -4a + 2b \\ -4b + 2a \end{pmatrix}, & \text{if } a < 0 \text{ and } b < 0, \\[2mm] \begin{pmatrix} -4a + 2b \\ 2a \end{pmatrix}, & \text{if } a < 0 \text{ and } b \geq 0, \\[2mm] \begin{pmatrix} 2b \\ -4b + 2a \end{pmatrix}, & \text{if } a \geq 0 \text{ and } b < 0, \\[2mm] \begin{pmatrix} 2b \\ 2a \end{pmatrix}, & \text{if } a \geq 0 \text{ and } b \geq 0. \end{cases}
$$

(3) The following inequality holds:

$$
\phi(a,b) \begin{cases} > 0, & \text{if } a > 0 \text{ and } b > 0, \\ < 0, & \text{if } a < 0 \text{ or } b < 0. \end{cases} \tag{3.1}
$$

According to the definition (1.6) of Φ_i, one obtains the expressions of $\Phi_i(z;t)$ and the gradient of $\Phi_i(z;t)$, respectively :

$$
\begin{aligned}
\Phi_i(z;t) &= \phi(G_i(z) - t, H_i(z) - t) \\
&= \begin{cases} -2(G_i(z) - t)^2 - 2(H_i(z) - t)^2 + 2(G_i(z) - t)(H_i(z) - t), & G_i(z) - t < 0 \text{ and } H_i(z) - t < 0, \\ -2(G_i(z) - t)^2 + 2(G_i(z) - t)(H_i(z) - t), & G_i(z) - t < 0 \text{ and } H_i(z) - t \geq 0, \\ -2(H_i(z) - t)^2 + 2(G_i(z) - t)(H_i(z) - t), & G_i(z) - t \geq 0 \text{ and } H_i(z) - t < 0, \\ 2(G_i(z) - t)(H_i(z) - t), & G_i(z) - t \geq 0 \text{ and } H_i(z) - t \geq 0, \end{cases}
\end{aligned} \tag{3.2}
$$

$$
\begin{aligned}
&\nabla\Phi_i(z;t) \\
&= \begin{cases} (2H_i(z) - 4G_i(z) + 2t)\nabla G_i(z) + (2G_i(z) - 4H_i(z) + 2t)\nabla H_i(z), & G_i(z) - t < 0 \text{ and } H_i(z) - t < 0, \\ (2H_i(z) - 4G_i(z) + 2t)\nabla G_i(z) + 2(G_i(z) - t)\nabla H_i(z), & G_i(z) - t < 0 \text{ and } H_i(z) - t \geq 0, \\ 2(H_i(z) - t)\nabla G_i(z) + (2G_i(z) - 4H_i(z) + 2t)\nabla H_i(z), & G_i(z) - t \geq 0 \text{ and } H_i(z) - t < 0, \\ 2(H_i(z) - t)\nabla G_i(z) + 2(G_i(z) - t)\nabla H_i(z), & G_i(z) - t \geq 0 \text{ and } H_i(z) - t \geq 0, \end{cases}
\end{aligned} \tag{3.3}
$$

where the parameter $t \geq 0$.

Let $S(t)$ be the feasible set of $R_{MPCC}(t)$ (1.5). Then the following result is true.

Lemma 3.2 (1) $S(0) = S$; (2) $S(t_1) \subseteq S(t_2)$, $0 \leq t_1 \leq t_2$; (3) $S = \bigcap_{t>0} S(t)$.

Next, we establish the convergence theorem of the proposed relaxation method.

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Theorem 3.1 Suppose that $\{t_k\} \downarrow 0$, $(z^k, \alpha^k, \beta^k, \gamma^k, \delta^k, \nu^k)$ is a KKT pair of $R_{MPCC}(t_k)$ (1.5) for all k, z^* is an accumulation point of the sequence $\{z^k\}$, and MPCC-CPLD holds at z^*. Then the following statements hold:

(1) z^* is M-stationary for the MPCC (1.1);

(2) If $\{z^k\}$ additionally satisfies $I_\Phi^{0+}(z^k; t_k) = I_\Phi^{+0}(z^k; t_k) = \emptyset$, then z^* is strongly stationary for the MPCC (1.1).

Proof. (1) Note that $z^k \to z^*$, $t_k \to 0$ and the continuity of g_i, h_i, G_i, H_i, we obtain that z^* is feasible for MPCC (1.1) and the following inclusion relations are true:

$$
\begin{aligned}
I_g(z^k) &\subseteq I_g(z^*), \\
I_G(z^k; t_k) \cup I_\Phi^{00}(z^k; t_k) \cup I_\Phi^{0+}(z^k; t_k) &\subseteq I_{00}(z^*) \cup I_{0+}(z^*), \\
I_H(z^k; t_k) \cup I_\Phi^{00}(z^k; t_k) \cup I_\Phi^{+0}(z^k; t_k) &\subseteq I_{00}(z^*) \cup I_{+0}(z^*).
\end{aligned}
\tag{3.4}
$$

Since $(z^k, \alpha^k, \beta^k, \gamma^k, \delta^k, \nu^k)$ is a KKT pair of $R_{MPCC}(t_k)$ (1.5), we have

$$
\begin{aligned}
0 = \ &\nabla f(z^k) + \sum_{i \in I} \alpha_i^k \nabla g_i(z^k) + \sum_{i \in I_e} \beta_i^k \nabla h_i(z^k) - \sum_{i \in I_c} \gamma_i^k \nabla G_i(z^k) - \sum_{i \in I_c} \delta_i^k \nabla H_i(z^k) \\
&+ \sum_{i \in I_c} \nu_i^k \nabla \Phi_i(z^k; t_k),
\end{aligned}
\tag{3.5}
$$

$$
\alpha_i^k \begin{cases} \geq 0, & i \in I_g(z^k), \\ = 0, & i \notin I_g(z^k); \end{cases}
\qquad
\gamma_i^k \begin{cases} \geq 0, & i \in I_G(z^k; t_k), \\ = 0, & i \notin I_G(z^k; t_k); \end{cases}
$$
$$
\delta_i^k \begin{cases} \geq 0, & i \in I_H(z^k; t_k), \\ = 0, & i \notin I_H(z^k; t_k); \end{cases}
\qquad
\nu_i^k \begin{cases} \geq 0, & i \in I_\Phi(z^k; t_k), \\ = 0, & i \notin I_\Phi(z^k; t_k). \end{cases}
\tag{3.6}
$$

From (3.3), one has

$$
\nabla \Phi_i(z^k; t_k) = \begin{cases} 2(H_i(z^k) - t_k)\nabla G_i(z^k), & i \in I_\Phi^{0+}(z^k; t_k), \\ 2(G_i(z^k) - t_k)\nabla H_i(z^k), & i \in I_\Phi^{+0}(z^k; t_k), \\ 0, & i \in I_\Phi^{00}(z^k; t_k). \end{cases}
$$

Define $\nu^{G,k} = (\nu_i^{G,k}, i \in I_c)$ and $\nu^{H,k} = (\nu_i^{H,k}, i \in I_c)$ with

$$
\nu_i^{G,k} = \begin{cases} 2\nu_i^k(H_i(z^k) - t_k), & \text{if } i \in I_\Phi^{0+}(z^k; t_k), \\ 0, & \text{otherwise;} \end{cases}
\qquad
\nu_i^{H,k} = \begin{cases} 2\nu_i^k(G_i(z^k) - t_k), & \text{if } i \in I_\Phi^{+0}(z^k; t_k), \\ 0, & \text{otherwise.} \end{cases}
$$

Note that $I_\Phi(z^k; t_k) = I_\Phi^{0+}(z^k; t_k) \cup I_\Phi^{00}(z^k; t_k) \cup I_\Phi^{+0}(z^k; t_k)$, (3.5) can be rewritten as follows:

$$
\begin{aligned}
0 = \ &\nabla f(z^k) + \sum_{i \in I} \alpha_i^k \nabla g_i(z^k) + \sum_{i \in I_e} \beta_i^k \nabla h_i(z^k) - \sum_{i \in I_c} \gamma_i^k \nabla G_i(z^k) - \sum_{i \in I_c} \delta_i^k \nabla H_i(z^k) \\
&+ \sum_{i \in I_c} \nu_i^{G,k} \nabla G_i(z^k) + \sum_{i \in I_c} \nu_i^{H,k} \nabla H_i(z^k).
\end{aligned}
\tag{3.7}
$$

Note that the multipliers $v_i^{G,k}$ and $\delta_i^{H,k}$ are nonnegative, too, according to [4, Lemma A.1], we suppose, without loss of generality, the gradients corresponding to nonzero multipliers, that is,

$$
\begin{aligned}
&\{\nabla g_i(z^k) \mid i \in supp(\alpha^k)\} \cup \{\nabla h_i(z^k) \mid i \in supp(\beta^k)\} \cup \{\nabla G_i(z^k) \mid i \in supp(\gamma^k) \cup supp(\nu^{G,k})\} \\
&\cup \{\nabla H_i(z^k) \mid i \in supp(\delta^k) \cup supp(\nu^{H,k})\},
\end{aligned}
\tag{3.8}
$$

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are linearly independent.

In what follows, we show that the sequence $\{(\alpha^k, \beta^k, \gamma^k, \delta^k, \nu^{G,k}, \nu^{H,k})\}$ is bounded. Suppose, by contradiction, that the conclusion is not true. Then there is a vector $(\alpha, \beta, \gamma, \delta, \nu^G, \nu^H)$ and a subset $K \subseteq \{1, 2, \ldots\}$ such that

$$\frac{(\alpha^k, \beta^k, \gamma^k, \delta^k, \nu^{G,k}, \nu^{H,k})}{\|(\alpha^k, \beta^k, \gamma^k, \delta^k, \nu^{G,k}, \nu^{H,k})\|} \xrightarrow{K} (\alpha, \beta, \gamma, \delta, \nu^G, \nu^H) \neq 0.$$

Dividing by $\|(\alpha^k, \beta^k, \gamma^k, \delta^k, \nu^{G,k}, \nu^{H,k})\|$ in (3.7) and passing to the limit, we obtain

$$
\begin{aligned}
0 = {} & \sum_{i \in I} \alpha_i \nabla g_i(z^*) + \sum_{i \in I_e} \beta_i \nabla h_i(z^*) - \sum_{i \in I_c} \gamma_i \nabla G_i(z^*) - \sum_{i \in I_c} \delta_i \nabla H_i(z^*) \\
& + \sum_{i \in I_c} \nu_i^G \nabla G_i(z^*) + \sum_{i \in I_c} \nu_i^H \nabla H_i(z^*),
\end{aligned}
$$

which implies the gradients

$$\{\nabla g_i(z^*) \mid i \in supp(\alpha)\} \cup \{\nabla h_i(z^*) \mid i \in supp(\beta)\} \cup \{\nabla G_i(z^*) \mid i \in supp(\gamma) \cup supp(\nu^G)\}$$
$$\cup \{\nabla H_i(z^*) \mid i \in supp(\delta) \cup supp(\nu^H)\}$$

are positive-linearly dependent.

Since MPCC-CPLD holds at z^*, there exists a neighbourbood $U(z^*)$ of z^* such that $\forall z \in U(z^*)$, the gradients

$$\{\nabla g_i(z) \mid i \in supp(\alpha)\} \cup \{\nabla h_i(z) \mid i \in supp(\beta)\} \cup \{\nabla G_i(z) \mid i \in supp(\gamma) \cup supp(\nu^G)\}$$
$$\cup \{\nabla H_i(z) \mid i \in supp(\delta) \cup supp(\nu^H)\}$$

are linearly dependent. This contradicts the linear independence in (3.8) since $supp(\alpha, \beta, \gamma, \delta, \nu^G, \nu^H) \subseteq supp(\alpha^k, \beta^k, \gamma^k, \delta^k, \nu^{G,k}, \nu^{H,k})$ for k sufficiently large. Therefore, $\{(\alpha^k, \beta^k, \gamma^k, \delta^k, \nu^{G,k}, \nu^{H,k})\}$ is bounded.

We suppose, without loss of generality, that $\{(\alpha^k, \beta^k, \gamma^k, \delta^k, \nu^{G,k}, \nu^{H,k})\}$ converges to $(\alpha^*, \beta^*, \gamma^*, \delta^*, \nu^{G,*}, \nu^{H,*})$. Since $I_G(z^k; t_k) \cap I_\Phi^{0+}(z^k; t_k) = \emptyset$, $I_H(z^k; t_k) \cap I_\Phi^{+0}(z^k; t_k) = \emptyset$, we define

$$
\widetilde{\gamma}_i = \begin{cases} \gamma_i^*, & \text{if } i \in supp(\gamma^*), \\ -\nu_i^{G,*}, & \text{if } i \in supp(\nu^{G,*}), \\ 0, & \text{otherwise.} \end{cases}
\quad \text{and} \quad
\widetilde{\delta}_i = \begin{cases} \delta_i^*, & \text{if } i \in supp(\delta^*), \\ -\nu_i^{H,*}, & \text{if } i \in supp(\nu^{H,*}), \\ 0, & \text{otherwise.} \end{cases}
\tag{3.9}
$$

By passing to the limit in (3.7), we have

$$0 = \nabla f(z^*) + \sum_{i \in I} \alpha_i^* \nabla g_i(z^*) + \sum_{i \in I_e} \beta_i^* \nabla h_i(z^*) - \sum_{i \in I_c} \widetilde{\gamma}_i \nabla G_i(z^*) - \sum_{i \in I_c} \widetilde{\delta}_i \nabla H_i(z^*), \tag{3.10}$$

where $\alpha_i^* \geq 0$, $\alpha_i^* g_i(z^*) = 0$, $i \in I$. And it follows for k sufficiently large that

$$supp(\alpha^*) \subseteq I_g(z^k) \subseteq I_g(z^*),$$
$$supp(\widetilde{\gamma}) = supp(\gamma^*) \cup supp(\nu^{G,*}) \subseteq I_G(z^k; t_k) \cup I_\Phi^{0+}(z^k; t_k) \subseteq I_{00}(z^*) \cup I_{0+}(z^*), \tag{3.11}$$
$$supp(\widetilde{\delta}) = supp(\delta^*) \cup supp(\nu^{H,*}) \subseteq I_H(z^k; t_k) \cup I_\Phi^{+0}(z^k; t_k) \subseteq I_{00}(z^*) \cup I_{+0}(z^*).$$

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From (3.11), one has $\widetilde{\gamma}_i = 0$, $i \in I_{+0}(z^*)$; $\widetilde{\delta}_i = 0$, $i \in I_{0+}(z^*)$, together with (3.10), we can conclude that z^* is weakly stationary for MPCC (1.1).

In what follows, we prove z^* is M-stationary, i.e., either $\widetilde{\gamma}_i > 0$, $\widetilde{\delta}_i > 0$ or $\widetilde{\gamma}_i \widetilde{\delta}_i = 0$, for all $i \in I_{00}(z^*)$. Suppose, by contradiction, that there is an $i \in I_{00}(z^*)$ with $\widetilde{\gamma}_i < 0$ and $\widetilde{\delta}_i \neq 0$ (the case $\widetilde{\gamma}_i \neq 0$ and $\widetilde{\delta}_i < 0$ can be proven in a similar way). According to (3.9) and (3.11), one has $i \in supp(\nu^{G,*}) \subseteq I_{\Phi}^{0+}(z^k; t_k)$ for k sufficiently large. Note that $I_{\Phi}^{0+}(z^k; t_k) \cap (I_H(z^k; t_k) \cup I_{\Phi}^{+0}(z^k; t_k)) = \emptyset$, it follows from (3.9) that $\widetilde{\delta}_i = 0$, which yields a contradiction. Hence, z^* is an M-stationary point.

(2) In order to show z^* is a strongly stationary point of the MPCC (1.1), based on the result (1), it is sufficient to show that $\widetilde{\gamma}_i \geq 0$, $\forall i \in I_{00}(z^*)$; $\widetilde{\delta}_i \geq 0$, $\forall i \in I_{00}(z^*)$.

By (3.11), the equality (3.10) can be rewirtten as

$$0 = \nabla f(z^*) + \sum_{i \in supp(\alpha^*)} \alpha_i^* \nabla g_i(z^*) + \sum_{i \in I_e} \beta_i^* \nabla h_i(z^*) - \sum_{i \in supp(\widetilde{\gamma})} \widetilde{\gamma}_i \nabla G_i(z^*) - \sum_{i \in supp(\widetilde{\delta})} \widetilde{\delta}_i \nabla H_i(z^*). \quad (3.12)$$

In view of $I_{\Phi}^{0+}(z^k; t_k) = \emptyset$, one gets from (3.9)

$$\widetilde{\gamma}_i = \begin{cases} \gamma_i^*, & i \in supp(\gamma^*), \\ 0, & \text{else.} \end{cases} \quad (3.13)$$

For $\forall\ i \in I_{00}(z^*)$, if $i \in supp(\gamma^*)$, then one obtains from (3.11) that $\widetilde{\gamma} = \gamma_i^* > 0$; otherwise, $\widetilde{\gamma} = \gamma_i^* = 0$. This indicates $\widetilde{\gamma}_i \geq 0$ for all $i \in I_{00}(z^*)$.

Similarly, one can show $\widetilde{\delta}_i \geq 0$ for all $i \in I_{00}(z^*)$.

Thus, z^* is a strongly stationary point of the MPCC (1.1). □

4. Existence of multipliers

In the convergent theorem, i.e., Theorem 3.1, we assume that there exists a KKT point for $R_{MPCC}(t_k)$ (1.5). Whether does a KKT point for $R_{MPCC}(t_k)$ (1.5) exist or not, or what conditions can ensure the existence of KKT point ? In order to answer these questions, we will further discuss the existence of Lagrange multipliers of $R_{MPCC}(t)$ (1.5) in this section.

Let $\widetilde{z}$ be feasible for $R_{MPCC}(t)$ (1.5) and J be an arbitrary subset of $I_{\Phi}^{00}(\widetilde{z}; t)$, define an auxiliary program $(AP(t, J)$ for short) as follows:

$$\begin{aligned} \min\quad & f(z) \\ \text{s.t.}\quad & g_i(z) \leq 0, \quad i \in I, \\ & h_i(z) = 0, \quad i \in I_e, \\ & G_i(z) \geq -t,\ H_i(z) \geq -t, G_i(z) \leq t,\ i \in J, \\ & G_i(z) \geq -t,\ H_i(z) \geq -t, H_i(z) \leq t,\ i \in \overline{J}, \\ & G_i(z) \geq -t,\ H_i(z) \geq -t, \Phi_i(z; t) \leq 0,\ i \notin I_{\Phi}^{00}(\widetilde{z}; t), \end{aligned} \quad (4.1)$$

where $\overline{J}$ means the complement of J in $I_{\Phi}^{00}(\widetilde{z}; t)$.

It is obvious that $\widetilde{z}$ is feasible for $AP(t, J)$. Denote by $S(t, J)$ the feasible set of $AP(t, J)$ (4.1). It is not difficult to obtain the relation of feasible sets between $AP(t, J)$ (4.1) and $R_{MPCC}(t)$ (1.5).

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Lemma 4.1 Let J be an arbitrary subset of $I_\Phi^{00}(\widetilde{z};t)$ and $t \geq 0$. Then $S(t,J) \subseteq S(t)$.

Lemma 4.2 For any $t \geq 0$ and any feasible point $\widetilde{z}$ of $R_{MPCC}(t)$ (1.5), the following equality holds true:

$$\mathcal{T}_{S(t)}(\widetilde{z}) = \bigcup_{J \subseteq I_\Phi^{00}(\widetilde{z};t)} \mathcal{T}_{S(t,J)}(\widetilde{z}),$$

where $\mathcal{T}_{S(t)}$ and $\mathcal{T}_{S(t,J)}(\widetilde{z})$ are the tangent cones of $R_{MPCC}(t)$ (1.5) and $AP(t,J)$ (4.1) at $\widetilde{z}$, respectively.

Proof. For any $d \in \mathcal{T}_{S(t)}(\widetilde{z})$, the definition of tangent cone tells us that there exists a sequence $\{z^k\} \subseteq S(t)$, $z^k \to \widetilde{z}$, and a sequence $\{\tau_k\} \downarrow 0$ such that $d = \lim\limits_{k\to\infty} \frac{z^k - \widetilde{z}}{\tau_k}$.

In the following, we show that $d \in \bigcup\limits_{J \subseteq I_\Phi^{00}(\widetilde{z};t)} \mathcal{T}_{S(t,J)}(\widetilde{z})$. Note that $z^k \in S(t)$, one has

$$g_i(z^k) \leq 0, \ i \in I, \quad h_i(z^k) = 0, \ i \in I_e,$$
$$G_i(z^k) \geq -t, \ H_i(z^k) \geq -t, \ \Phi_i(z^k;t) \leq 0, \ i \in I_c.$$

Hence, one has $\Phi_i(z^k;t) \leq 0$ for any $i \in I_c$.

If $i \in I_\Phi^{00}(\widetilde{z};t)$, there are six cases for $\Phi_i(z^k;t) \leq 0$ as follows:

$$G_i(z^k) - t < 0, \ H_i(z^k) - t < 0;$$
$$G_i(z^k) - t < 0, \ H_i(z^k) - t \geq 0;$$
$$G_i(z^k) - t \geq 0, \ H_i(z^k) - t < 0;$$
$$G_i(z^k) - t > 0, \ H_i(z^k) - t = 0;$$
$$G_i(z^k) - t = 0, \ H_i(z^k) - t = 0;$$
$$G_i(z^k) - t = 0, \ H_i(z^k) - t > 0.$$

Thus, there exists an infinity subset $K \subseteq \{1,2,\ldots\}$ such that $G_i(z^k) - t \leq 0, \ \forall \ k \in K$. Let $J = \{i \in I_\Phi^{00}(\widetilde{z};t) \mid G_i(z^k) - t \leq 0, \ \forall \ k \in K\}$, $\overline{J} = I_\Phi^{00}(\widetilde{z};t) \backslash J$, then one gets $\{z^k\} \subseteq S(t,J)$. This implies $d \in \bigcup\limits_{J \subseteq I_\Phi^{00}(\widetilde{z};t)} \mathcal{T}_{S(t,J)}(\widetilde{z})$. Therefore, we have $\mathcal{T}_{S(t)}(\widetilde{z}) \subseteq \bigcup\limits_{J \subseteq I_\Phi^{00}(\widetilde{z};t)} \mathcal{T}_{S(t,J)}(\widetilde{z})$.

Conversely, for any $d \in \bigcup\limits_{J \subseteq I_\Phi^{00}(\widetilde{z};t)} \mathcal{T}_{S(t,J)}(\widetilde{z})$, there exists a subset $J \subseteq I_\Phi^{00}(\widetilde{z};t)$ such that $d \in \mathcal{T}_{S(t,J)}(\widetilde{z})$. Accordingly, there exists a sequence $\{z^k\} \subseteq S(t,J)$, $z^k \to \widetilde{z}$ and a sequence $\{\tau_k\} \downarrow 0$ such that $d = \lim\limits_{k\to\infty} \frac{z^k - \widetilde{z}}{\tau_k}$.

By Lemma 4.1, one has $\{z^k\} \subseteq S(t)$, so $d \in \mathcal{T}_{S(t)}(\widetilde{z})$. Thus one obtains

$$\bigcup_{J \subseteq I_\Phi^{00}(\widetilde{z};t)} \mathcal{T}_{S(t,J)}(\widetilde{z}) \subseteq \mathcal{T}_{S(t)}(\widetilde{z}).$$

Hence, the result is true. The proof is finished. $\qquad\square$

For the sake of convenience, we now give a conclusion in [26], which is used in the proof of our Theorem 4.1.

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Lemma 4.3 Suppose that C_1, $C_2 \subseteq \mathbb{R}^n$ are cones defined by

$$C_1 = \{p \in \mathbb{R}^n \mid x_i^T p \leq 0, \ i \in I; \ y_i^T p = 0, \ i \in I_e\},$$
$$C_2 = \{q \in \mathbb{R}^n \mid q = \sum_{i \in I} \lambda_i x_i + \sum_{i \in I_e} \mu_i y_i, \ \lambda_i \geq 0, \ \forall \, i \in I\}.$$

Then $C_1 = C_2^\circ$ and $C_2 = C_1^\circ$, where C_1° and C_2° are the polar cones of C_1 and C_2, respectively.

The following theorem shows that standard Guignard CQ holds for $R_{MPCC}(t)$ (1.5) only under MPCC-LICQ assumption.

Theorem 4.1 Let z^* be feasible for MPCC (1.1) such that MPCC-LICQ holds at z^*. Then there exists a $\bar{t} > 0$ and a neighborhood $U(z^*)$ of z^* such that standard GCQ holds for $R_{MPCC}(t)$ (1.5) for $\forall \, t \in (0, \bar{t}]$ and $\forall \, z \in U(z^*) \cap S(t)$.

Proof. Since MPCC-LICQ holds at z^*, the gradients

$$
\begin{aligned}
&\{\nabla g_i(z) \mid i \in I_g(z^*)\} \cup \{\nabla h_i(z) \mid i \in I_e\} \cup \{\nabla G_i(z) \mid i \in I_{0+}(z^*) \cup I_{00}(z^*)\} \\
&\cup \{\nabla H_i(z) \mid i \in I_{+0}(z^*) \cup I_{00}(z^*)\}
\end{aligned}
\tag{4.2}
$$

are linearly independent at z^*. Since g_i, h_i, G_i and H_i are continuously differentiable, the gradients (4.2) remain linearly independent in some neighborhood of z^*. Hence, there exists a $\bar{t} > 0$ and sufficiently small neighborhood $U(z^*)$ of z^* such that for all $t \in (0, \bar{t}]$ and all $z \in U(z^*) \cap S(t)$, the gradients (4.2) are linearly independent at z, and the following inclusions hold from (3.4)

$$
\begin{aligned}
&I_g(z) \subseteq I_g(z^*), \quad I_G(z;t) \subseteq I_{00}(z^*) \cup I_{0+}(z^*), \quad I_H(z;t) \subseteq I_{00}(z^*) \cup I_{+0}(z^*), \\
&I_\Phi^{00}(z;t) \cup I_\Phi^{0+}(z;t) \subseteq I_{00}(z^*) \cup I_{0+}(z^*), \quad I_\Phi^{00}(z;t) \cup I_\Phi^{+0}(z;t) \subseteq I_{00}(z^*) \cup I_{+0}(z^*).
\end{aligned}
\tag{4.3}
$$

For any $t \in (0, \bar{t}]$ and $\widetilde{z} \in U(z^*) \cap S(t)$, we have $\widetilde{z} \in S(t, J)$ for any $J \subseteq I_\Phi^{00}(\widetilde{z};t)$, and the active gradients of $AP(t, J)$ (4.1) are

$$
\begin{aligned}
&\{\nabla g_i(\widetilde{z}) \mid i \in I_g(\widetilde{z})\} \cup \{\nabla h_i(\widetilde{z}) \mid i \in I_e\} \cup \{\nabla G_i(\widetilde{z}) \mid i \in I_G(\widetilde{z};t) \cup I_\Phi^{0+}(\widetilde{z};t) \cup J\} \\
&\cup \{\nabla H_i(\widetilde{z}) \mid i \in I_H(\widetilde{z};t) \cup I_\Phi^{+0}(\widetilde{z};t) \cup \overline{J}\}.
\end{aligned}
$$

Thus, the standard LICQ for $AP(t, J)$ (4.1) holds at $\widetilde{z}$. Since LICQ implies ACQ, we have

$$\mathcal{T}_{S(t,J)}(\widetilde{z}) = \mathcal{L}_{S(t,J)}(\widetilde{z}), \quad \forall \, J \subseteq I_\Phi^{00}(\widetilde{z};t),$$

which together with Lemma 4.1 yields

$$\mathcal{T}_{S(t)}(\widetilde{z}) = \bigcup_{J \subseteq I_\Phi^{00}(\widetilde{z};t)} \mathcal{T}_{S(t,J)}(\widetilde{z}) = \bigcup_{J \subseteq I_\Phi^{00}(\widetilde{z};t)} \mathcal{L}_{S(t,J)}(\widetilde{z}).$$

From Theorem 3.1.9 in [26], we obtain

$$\mathcal{T}_{S(t)}(\widetilde{z})^\circ = \bigcap_{J \subseteq I_\Phi^{00}(\widetilde{z};t)} \mathcal{L}_{S(t,J)}(\widetilde{z})^\circ.
\tag{4.4}$$

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To prove that GCQ for $R_{MPCC}(t)$ (1.5) holds at $\widetilde{z}$, we need to show that $\mathcal{L}_{S(t)}(\widetilde{z})^\circ = \mathcal{T}_{S(t)}(\widetilde{z})^\circ$. Note that $\mathcal{L}_{S(t)}(\widetilde{z})^\circ \subseteq \mathcal{T}_{S(t)}(\widetilde{z})^\circ$ since $\mathcal{T}_{S(t)}(\widetilde{z}) \subseteq \mathcal{L}_{S(t)}(\widetilde{z})$, we only prove the inclusion

$$\mathcal{T}_{S(t)}(\widetilde{z})^\circ \subseteq \mathcal{L}_{S(t)}(\widetilde{z})^\circ.$$

The linearized tangent cone of $AP(t, J)$ (4.1) at $\widetilde{z}$ is given by

$$
\begin{aligned}
\mathcal{L}_{S(t,J)}(\widetilde{z}) = \ & \{p \in \mathbb{R}^n \mid \nabla g_i(\widetilde{z})^T p \leq 0, \ i \in I_g(\widetilde{z}), \\
& \nabla h_i(\widetilde{z})^T p = 0, \ i \in I_e, \\
& \nabla G_i(\widetilde{z})^T p \geq 0, \ i \in I_G(\widetilde{z}; t), \\
& \nabla H_i(\widetilde{z})^T p \geq 0, \ i \in I_H(\widetilde{z}; t), \\
& \nabla G_i(\widetilde{z})^T p \leq 0, \ i \in I_\Phi^{0+}(\widetilde{z}; t) \cup J, \\
& \nabla H_i(\widetilde{z})^T p \leq 0, \ i \in I_\Phi^{+0}(\widetilde{z}; t) \cup \overline{J}\},
\end{aligned}
$$

so it follows from Lemma 4.2 that

$$
\begin{aligned}
\mathcal{L}_{S(t,J)}(\widetilde{z})^\circ = \Big\{ q \in \mathbb{R}^n \mid q = & \sum_{i \in I_g(\widetilde{z})} \alpha_i \nabla g_i(\widetilde{z}) + \sum_{i \in I_e} \beta_i \nabla h_i(\widetilde{z}) - \sum_{i \in I_G(\widetilde{z}; t)} \gamma_i \nabla G_i(\widetilde{z}) \\
& - \sum_{i \in I_H(\widetilde{z}; t)} \delta_i \nabla H_i(\widetilde{z}) + \sum_{i \in I_\Phi^{0+}(\widetilde{z}; t) \cup J} \nu_i \nabla G_i(\widetilde{z}) \\
& + \sum_{i \in I_\Phi^{+0}(\widetilde{z}; t) \cup \overline{J}} \sigma_i \nabla H_i(\widetilde{z}), \ \alpha, \ \gamma, \ \delta, \ \nu, \ \sigma \geq 0 \Big\}.
\end{aligned}
\tag{4.5}
$$

Now for $q \in \mathcal{T}_{S(t)}(\widetilde{z})^\circ$, one obtains from (4.4) that $q \in \mathcal{L}_{S(t,J)}(\widetilde{z})^\circ$ for any $J \subseteq I_\Phi^{00}(\widetilde{z}; t)$. So it follows from (4.5) that

$$
\begin{aligned}
q = \ & \sum_{i \in I_g(\widetilde{z})} \alpha_i \nabla g_i(\widetilde{z}) + \sum_{i \in I_e} \beta_i \nabla h_i(\widetilde{z}) - \sum_{i \in I_G(\widetilde{z}; t)} \gamma_i \nabla G_i(\widetilde{z}) - \sum_{i \in I_H(\widetilde{z}; t)} \delta_i \nabla H_i(\widetilde{z}) \\
& + \sum_{i \in I_\Phi^{0+}(\widetilde{z}; t) \cup J} \nu_i \nabla G_i(\widetilde{z}) + \sum_{i \in I_\Phi^{+0}(\widetilde{z}; t) \cup \overline{J}} \sigma_i \nabla H_i(\widetilde{z}),
\end{aligned}
\tag{4.6}
$$

where $\alpha_i, \ \gamma_i, \ \delta_i, \ \nu_i, \ \sigma_i \geq 0$.

On the other hand, in view of $\overline{J} \subseteq I_\Phi^{00}(\widetilde{z}; t)$, we have from (4.4) that $q \in \mathcal{L}_{S(t,\overline{J})}(\widetilde{z})^\circ$, thus it follows that

$$
\begin{aligned}
q = \ & \sum_{i \in I_g(\widetilde{z})} \overline{\alpha}_i \nabla g_i(\widetilde{z}) + \sum_{i \in I_e} \overline{\beta}_i \nabla h_i(\widetilde{z}) - \sum_{i \in I_G(\widetilde{z}; t)} \overline{\gamma}_i \nabla G_i(\widetilde{z}) - \sum_{i \in I_H(\widetilde{z}; t)} \overline{\delta}_i \nabla H_i(\widetilde{z}) \\
& + \sum_{i \in I_\Phi^{0+}(\widetilde{z}; t) \cup \overline{J}} \overline{\nu}_i \nabla G_i(\widetilde{z}) + \sum_{i \in I_\Phi^{+0}(\widetilde{z}; t) \cup J} \overline{\sigma}_i \nabla H_i(\widetilde{z}),
\end{aligned}
\tag{4.7}
$$

where $\overline{\alpha}_i, \ \overline{\gamma}_i, \ \overline{\delta}_i, \ \overline{\nu}_i, \ \overline{\sigma}_i \geq 0$.

Note that the gradients

$$
\begin{aligned}
& \{\nabla g_i(\widetilde{z}) \mid i \in I_g(\widetilde{z})\} \cup \{\nabla h_i(\widetilde{z}) \mid i \in I_e\} \cup \{\nabla G_i(\widetilde{z}) \mid i \in I_G(\widetilde{z}; t) \cup I_\Phi^{0+}(\widetilde{z}; t) \cup I_\Phi^{00}(\widetilde{z}; t)\} \\
& \cup \{\nabla H_i(\widetilde{z}) \mid i \in I_H(\widetilde{z}; t) \cup I_\Phi^{+0}(\widetilde{z}; t) \cup I_\Phi^{00}(\widetilde{z}; t)\}
\end{aligned}
$$

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are linearly independent, hence, the corresponding coefficients in (4.6) and (4.7) must be equal. In particular, we obtain

$$\nu_i = 0, \ i \in J; \qquad \sigma_i = 0, \ i \in \overline{J}.$$

Further, we obtain

$$
\begin{aligned}
q = & \sum_{i \in I_g(\widetilde{z})} \alpha_i \nabla g_i(\widetilde{z}) + \sum_{i \in I_e} \beta_i \nabla h_i(\widetilde{z}) - \sum_{i \in I_G(\widetilde{z};t)} \gamma_i \nabla G_i(\widetilde{z}) - \sum_{i \in I_H(\widetilde{z};t)} \delta_i \nabla H_i(\widetilde{z}) \\
& + \sum_{i \in I_\Phi^{0+}(\widetilde{z};t)} \nu_i \nabla G_i(\widetilde{z}) + \sum_{i \in I_\Phi^{+0}(\widetilde{z};t)} \sigma_i \nabla H_i(\widetilde{z}).
\end{aligned}
\tag{4.8}
$$

Note that the linearized cone of $R_{MPCC}(t)$ (1.5) is given as follows:

$$
\begin{aligned}
\mathcal{L}_{S(t)}(\widetilde{z}) = \ \big\{ p \in \mathbb{R}^n \mid & \nabla g_i(\widetilde{z})^T p \le 0, \ i \in I_g(\widetilde{z}), \\
& \nabla h_i(\widetilde{z})^T p = 0, \ i \in I_e, \\
& \nabla G_i(\widetilde{z})^T p \ge 0, \ i \in I_G(\widetilde{z};t), \\
& \nabla H_i(\widetilde{z})^T p \ge 0, \ i \in I_H(\widetilde{z};t), \\
& \nabla \Phi_i(\widetilde{z};t)^T p \le 0, \ i \in I_\Phi(\widetilde{z};t) \big\}.
\end{aligned}
$$

In view of $\nabla \Phi_i(\widetilde{z};t) = 0$, $i \in I_\Phi^{00}(\widetilde{z};t)$ and $I_\Phi(\widetilde{z};t) = I_\Phi^{0+}(\widetilde{z};t) \cup I_\Phi^{00}(\widetilde{z};t) \cup I_\Phi^{+0}(\widetilde{z};t)$, $I_\Phi^{0+}(\widetilde{z};t) \cap I_\Phi^{00}(\widetilde{z};t) \cap I_\Phi^{+0}(\widetilde{z};t) = \emptyset$, the representation above can be rewritten as follows:

$$
\begin{aligned}
\mathcal{L}_{S(t)}(\widetilde{z}) = \ \big\{ p \in \mathbb{R}^n \mid & \nabla g_i(\widetilde{z})^T p \le 0, \ i \in I_g(\widetilde{z}), \\
& \nabla h_i(\widetilde{z})^T p = 0, \ i \in I_e, \\
& \nabla G_i(\widetilde{z})^T p \ge 0, \ i \in I_G(\widetilde{z};t), \\
& \nabla H_i(\widetilde{z})^T p \ge 0, \ i \in I_H(\widetilde{z};t), \\
& \nabla G_i(\widetilde{z})^T p \le 0, \ i \in I_\Phi^{0+}(\widetilde{z};t), \\
& \nabla H_i(\widetilde{z})^T p \le 0, \ i \in I_\Phi^{+0}(\widetilde{z};t) \big\}.
\end{aligned}
$$

By Lemma 4.3 and (4.8), one obtains $q \in \mathcal{L}_{S(t)}(\widetilde{z})^\circ$. The arbitrariness of q implies $\mathcal{T}_{S(t)}(\widetilde{z})^\circ \subseteq \mathcal{L}_{S(t)}(\widetilde{z})^\circ$. The proof is finished. $\qquad \square$

The following result shows that stronger CQ for $R_{MPCC}(t)$ (1.5) holds at all points where $I_\Phi^{00}(z;t) = \emptyset$ holds.

Theorem 4.2 Let z^* be feasible for the MPCC (1.1) such that MPCC-CPLD (MPCC-LICQ) holds at z^*. Then there exists a $\bar{t} > 0$ and a neighborhood $U(z^*)$ of z^* such that the following statement holds for all $t \in (0, \bar{t}]$: If $z \in U(z^*) \cap S(t)$ with $I_\Phi^{00}(z;t) = \emptyset$, then standard CPLD (LICQ) for $R_{MPCC}(t)$ (1.5) holds at z.

Proof. We first prove the conclusion for MPCC-CPLD. Suppose, by contradiction, that there were a sequence $\{t^k\} \downarrow 0$ and $z^k \to z^*$ with z^k feasible for $R_{MPCC}(t^k)$ (1.5), and $I_\Phi^{00}(z^k;t_k) = \emptyset$ for all $k \in \{1, 2, \dots\}$ such that standard CPLD is not satisfied in z^k for all $k \in \{1, 2, \dots\}$. Thus there exist index subsets

$$I_1^k \subseteq I_g(z^k), \ I_2^k \subseteq I_e, \ I_3^k \subseteq I_G(z^k;t_k), \ I_4^k \subseteq I_H(z^k;t_k), \ I_5^k \subseteq I_\Phi^{0+}(z^k;t_k), \ I_6^k \subseteq I_\Phi^{+0}(z^k;t_k)$$

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such that the gradients

$$\left\{ \{\nabla g_i(z) \mid i \in I_1^k\} \cup \{-\nabla G_i(z) \mid i \in I_3^k\} \cup \{-\nabla H_i(z) \mid i \in I_4^k\} \cup \{(H_i(z) - t_k)\nabla G_i(z) \mid i \in I_5^k\} \right.$$
$$\left. \cup \{(G_i(z) - t_k)\nabla H_i(z) \mid i \in I_6^k\} \right\} \cup \{\nabla h_i(z) \mid i \in I_2^k\}$$

are positive-linearly dependent at z^k, but linearly independent at points arbitrary close to z^k. Since $I_g(z^k)$, I_e, $I_G(z^k; t_k)$, $I_H(z^k; t_k)$, $I_\Phi^{0+}(z^k; t_k)$, $I_\Phi^{+0}(z^k; t_k)$ are all finite sets, without loss of generality, we may assume $I_i^k \equiv I_i$ $(i = 1, 2, \dots 6)$. Note that $I_g(z^k) \subseteq I_g(z^*)$ for all k sufficiently large, thus $I_1 \subseteq I_g(z^*)$. Similarly, we obtain $I_3 \cup I_5 \subseteq I_{00}(z^*) \cup I_{0+}(z^*)$, $I_4 \cup I_6 \subseteq I_{00}(z^*) \cup I_{+0}(z^*)$. Positive-linearly dependence at z^k implies positive-linearly dependence of the gradients

$$\{\nabla g_i(z^k) \mid i \in I_1\} \cup \left\{ \{\nabla h_i(z^k) \mid i \in I_2\} \cup \{\nabla G_i(z^k) \mid i \in I_3 \cup I_5\} \cup \{\nabla H_i(z^k) \mid i \in I_4 \cup I_6\} \right\}. \tag{4.9}$$

Because the standard CPLD is not satisfied, there exists a sequence $\{y^k\} \to z^*$ such that the gradients (4.9) are linearly independent at y^k. If the gradients (4.9) were positive-linearly independent at z^*, then from Theorem 2.2 in [27] we know that these gradients are also positive-linearly independent at any point close to z^*. This is a contradiction. If the gradients (4.9) were positive-linearly dependent at z^*, MPCC-CPLD would imply that they remain linearly dependent in some neighborhood of z^*, which contradicts the statement the gradients (4.9) are linearly independent at y^k. Hence, CPLD holds at z.

Next we prove the conclusion for MPCC-LICQ. For all $z \in U(z^*) \cap S(t)$ and $t \in (0, \bar{t})$ sufficiently small, we have the following relations:

$$
\begin{aligned}
&I_g(z) \subseteq I_g(z^*), \\
&I_G(z; t) \cup I_\Phi^{00}(z; t) \cup I_\Phi^{0+}(z; t) \subseteq I_{00}(z^*) \cup I_{0+}(z^*), \\
&I_H(z; t) \cup I_\Phi^{00}(z; t) \cup I_\Phi^{+0}(z; t) \subseteq I_{00}(z^*) \cup I_{+0}(z^*), \\
&I_G(z; t) \cap \left(I_\Phi^{00}(z; t) \cup I_\Phi^{0+}(z; t) \right) = \emptyset, \\
&I_H(z; t) \cap \left(I_\Phi^{00}(z; t) \cup I_\Phi^{+0}(z; t) \right) = \emptyset.
\end{aligned}
\tag{4.10}
$$

In view of MPCC-LICQ assumption and (4.10), for any $z \in U(z^*)$, the gradients

$$\{\nabla g_i(z) \mid i \in I_g(z)\} \cup \{\nabla h_i(z) \mid i \in I_e\} \cup \{\nabla G_i(z) \mid i \in I_G(z; t) \cup I_\Phi^{0+}(z; t)\}$$
$$\cup \{\nabla H_i(z) \mid i \in I_H(z; t) \cup I_\Phi^{+0}(z; t)\} \tag{4.11}$$

are linearly independent.

The active gradients of $R_{MPCC}(t)$ (1.5) at feasible point $z \in U(z^*)$ are

$$
\begin{aligned}
&\nabla g_i(z), \ i \in I_g(z), \\
&\nabla h_i(z), \ i \in I_e, \\
&\nabla G_i(z), \ i \in I_G(z; t), \\
&\nabla H_i(z), \ i \in I_H(z; t), \\
&\nabla \Phi_i(z; t) = \begin{cases} 2(H_i(z) - t)\nabla G_i(z), & i \in I_\Phi^{0+}(z; t), \\ 2(G_i(z) - t)\nabla H_i(z), & i \in I_\Phi^{+0}(z; t). \end{cases}
\end{aligned}
\tag{4.12}
$$

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From (4.11), we know that the following equality

$$
\begin{aligned}
& \sum_{i\in I_g(z)} \alpha_i \nabla g_i(z) + \sum_{i\in I_e} \beta_i \nabla h_i(z) - \sum_{i\in I_G(z;t)} \gamma_i \nabla G_i(z) - \sum_{i\in I_H(z;t)} \delta_i \nabla H_i(z) + \sum_{i\in I_\Phi(z;t)} \nu_i \nabla \Phi_i(z;t) \\
= \ & \sum_{i\in I_g(z)} \alpha_i \nabla g_i(z) + \sum_{i\in I_e} \beta_i \nabla h_i(z) - \sum_{i\in I_G(z;t)} \gamma_i \nabla G_i(z) - \sum_{i\in I_H(z;t)} \delta_i \nabla H_i(z) \\
& + \sum_{i\in I_\Phi^{0+}(z;t)} \nu_i [2(H_i(z)-t)]\nabla G_i(z;t) + \sum_{i\in I_\Phi^{+0}(z;t)} \nu_i [2(G_i(z)-t)]\nabla H_i(z;t) \\
= \ & 0
\end{aligned}
$$

(4.13)

implies that

$$
\alpha_i = 0,\ i \in I_g(z);\ \ \beta_i = 0,\ i \in I_e;\ \ \gamma_i = 0,\ i \in I_G(z;t);\ \ \delta_i = 0,\ i \in I_H(z;t);
$$
$$
\nu_i[2(H_i(z)-t)] = 0,\ i \in I_\Phi^{0+}(z;t);\ \ \ \nu_i[2(G_i(z)-t)] = 0,\ i \in i \in I_\Phi^{+0}(z;t).
$$

Note that $H_i(z) - t > 0$ for $i \in I_\Phi^{0+}(z;t)$, so $\nu_i = 0$ for $i \in I_\Phi^{0+}(z;t)$.

Similarly, we have $\nu_i = 0$ for $i \in I_\Phi^{+0}(z;t)$.

Summing up the above discussion, we can conclude that standard LICQ holds at $z \in U(z^*)\cap S(t)$ for $R_{MPCC}(t)$ (1.5). $\qquad \square$

The following result shows that the existence of Lagrange multipliers in a local minimum of $R_{MPCC}(t)$ (1.5) can be guaranteed, which is a direct consequence of Theorem 4.1.

Theorem 4.3 Let z^* be feasible for MPCC (1.1) such that MPCC-LICQ holds at z^*. Then there is a $\bar{t} > 0$ and a neighborhood $U(z^*)$ of z^* such that for all $t \in (0,\bar{t}]$: If $z \in U(z^*)$ is a local minimizer of feasible point for $R_{MPCC}(t)$ (1.5), then there exists Lagrange multipliers such that z together with Lagrange multiplier vector w is a KKT point of $R_{MPCC}(t)$ (1.5).

5. Concluding remarks

In this paper, based on Mangasarian complementarity function, a new relaxed method for mathematical program with complementarity constraints is proposed. Under MPCC-CPLD, any limit point of a sequence of stationary points of a sequence of relaxed problems is M-stationary for MPCC (1.1), and it is strongly stationary under additional conditions which is easily to be checked. Moreover, we further analyze the existence of the Lagrange multipliers for relaxed problems. The existence of the Lagrange multipliers can be guaranteed under MPCC-LICQ.

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Some fixed point results of generalized Lipschitz mappings on cone b-metric spaces over Banach algebras

Huaping Huang[1]*, Stojan Radenović[2]

1. School of Mathematics and Statistics, Hubei Normal University, Huangshi, 435002, China

2. Faculty of Mathematics and Information Technology, Dong Thap University, Dong Thap, Viêt Nam

Abstract: In this paper, we introduce the concept of cone b-metric space over Banach algebra and obtain some fixed point theorems for generalized Lipschitz mappings in such settings without the assumption of normality. Moveover, we obtain some periodic properties of the fixed points. In addition, we give two examples to illustrate our assertions and show our results are never equivalent with the counterparts in b-metric versions.

MSC: 47H10; 54H25

Keywords: cone b-metric space over Banach algebra, generalized Lipschitz condition, P property, c-sequence

1 Introduction

In 2007, Huang and Zhang [1] defined cone metric spaces with a different view in respect to previous works. They substituted a normed space instead of the real line, but went further, defining convergent and Cauchy sequences in terms of interior points of the underlying cone. Moreover, they obtained some fixed point theorems in cone metric spaces. Afterwards, some scholars focused on the investigation of fixed point theorems in such spaces. According to incomplete statistics, more than six hundred papers dealing with cone metric spaces have been published so far (see [9]). But now it is not popular since

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*Corresponding author: Huaping Huang. E-mail: mathhhp@163.com

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some researchers constructed several mappings from cone metric spaces to metric spaces, and found some fixed point results in cone metric spaces could been directly obtained from the corresponding metric versions (see [3-10]). This makes it become meaningless to study fixed point theorems in cone metric spaces. However, the current situation changed, since, very recently, Liu and Xu [18] introduced cone metric space over Banach algebra and defined generalized Lipschitz mapping where the contractive coefficient is vector instead of usual real constant. They proved the existence of fixed points in such settings under the conditions that the underlying cones are normal cones. Furthermore, they gave an example to explain that the fixed point theorems in cone metric spaces over Banach algebras are not equivalent to those in metric spaces. Subsequently, Xu and Radenović [20] omitted the normality of cones by using c-sequences. Starting with the similar approach of [18], several papers have appeared (see [20-25]).

The main purpose of this article is to introduce a concept called cone b-metric space over Banach algebra, which is a great improvement since it expands the concept of cone metric space over Banach algebra. We present some fixed point theorems in such frameworks without the assumption of normal cones. Moreover, we obtain the P properties of the mappings. Further, by two examples, we support our results and establish the non-equivalence of fixed point results between cone b-metric spaces over Banach algebras and b-metric spaces.

We need the following definitions and results, consistent with [18], in the sequel.

Let $\mathbb{A}$ be a real Banach algebra, $\|\cdot\|$ be its norm and θ be its zero element. A nonempty closed subset K of $\mathbb{A}$ is called a cone if $K + K \subset K$, $K^2 = K \cap K \subset K$, $K \cap (-K) = \{\theta\}$ and $\lambda K \subset K$ for all $\lambda \geq 0$. We denote $\mathrm{int}K$ as the interior of K. If $\mathrm{int}K \neq \emptyset$, then K is said to be a solid cone. Define a partial ordering $\preceq$ with respect to K by $u \preceq v$ iff $v - u \in K$. Write $u \prec v$ iff $v - u \in K$ and $u \neq v$. Define $u \ll v$ iff $v - u \in \mathrm{int}K$. The cone K is called normal if there is a real number $M > 0$ such that for all $u, v \in \mathbb{A}$, $\theta \preceq u \preceq v$ implies $\|u\| \leq M\|v\|$. The least positive number satisfying above is called the normal constant of K.

In the sequel, unless otherwise specified, we always suppose that $\mathbb{A}$ is a real Banach algebra with a unit e, K is a solid cone in $\mathbb{A}$, and $\preceq$, $\prec$ and $\ll$ are partial orderings with respect to K.

Definition 1.1([18]) Let X be a nonempty set and $\mathbb{A}$ be a Banach algebra. Suppose

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that a mapping $d : X \times X \to \mathbb{A}$ satisfies for all $x, y, z \in X$,

(c1) $\theta \preceq d(x, y)$ and $d(x, y) = \theta$ iff $x = y$;

(c2) $d(x, y) = d(y, x)$;

(c3) $d(x, z) \preceq d(x, y) + d(y, z)$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space over Banach algebra.

Inspired by Definition 1.1 and [12, Definition 2.1], we introduce the notion of cone b-metric space over Banach algebra.

Definition 1.2 Let X be a nonempty set, $s \geq 1$ be a constant and $\mathbb{A}$ be a Banach algebra. Suppose that a mapping $d : X \times X \to \mathbb{A}$ satisfies for all $x, y, z \in X$,

(d1) $\theta \preceq d(x, y)$ and $d(x, y) = \theta$ iff $x = y$;

(d2) $d(x, y) = d(y, x)$;

(d3) $d(x, z) \preceq s[d(x, y) + d(y, z)]$.

Then d is called a cone b-metric on X, and (X, d) is called a cone b-metric space over Banach algebra.

Remark 1.3 A cone metric space over Banach algebra must be a cone b-metric space over Banach algebra. Conversely, it is not true. As a result, the concept of cone b-metric space over Banach algebra greatly generalizes the concept of cone metric space over Banach algebra.

We shall give some examples in an attempt to illustrate that it is an interesting increase from cone b-metric space over Banach algebra to cone metric space over Banach algebra, since there exist a lot of cone b-metric spaces over Banach algebras which are not cone metric spaces over Banach algebras.

Example 1.4 Let $\mathbb{A} = C[0, 1]$ be the usual Banach space with the supremum norm. Define multiplication in the usual way: $(xy)(t) = x(t)y(t)$. Then $\mathbb{A}$ is a Banach algebra with a unit 1. Put $K = \{x \in \mathbb{A} : x(t) \geq 0, t \in [0, 1]\}$ and $X = \mathbb{R}$. Define a mapping $d : X \times X \to \mathbb{A}$ by $d(x, y)(t) = |x - y|^p e^t$ for all $x, y \in X$, where $p > 1$ is a constant. This makes (X, d) into a cone b-metric space over Banach algebra with the coefficient $s = 2^{p-1}$, but it is not a cone metric space over Banach algebra.

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Example 1.5 Let $X = l^p = \{x = (x_n)_{n \geq 1} : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$ $(0 < p < 1)$. Let $d : X \times X \to \mathbb{R}^+$,

$$d(x, y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{\frac{1}{p}},$$

where $x = (x_n)_{n \geq 1}$, $y = (y_n)_{n \geq 1} \in l^p$. Then (X, d) is a b-metric space (see [11]). Put $\mathbb{A} = l^1 = \{a = (a_n)_{n \geq 1} : \sum_{n=1}^{\infty} |a_n| < \infty\}$ with convolution as multiplication:

$$ab = (a_n)_{n \geq 1}(b_n)_{n \geq 1} = \left(\sum_{i+j=n} a_i b_j \right)_{n \geq 1}.$$

Then $\mathbb{A}$ is a Banach algebra with a unit $e = (1, 0, 0, \ldots)$. Let $K = \{a = (a_n)_{n \geq 1} \in \mathbb{A} : a_n \geq 0,$ for all $n \geq 1\}$, which is a normal cone in $\mathbb{A}$. Defining a mapping $\tilde{d} : X \times X \to \mathbb{A}$ by $\tilde{d}(x, y) = (\frac{d(x,y)}{2^n})_{n \geq 1}$, we conclude that $(X, \tilde{d})$ is a cone b-metric space over Banach algebra with the coefficient $s = 2^{\frac{1}{p} - 1} > 1$, but it is not a cone metric space over Banach algebra.

Definition 1.6 Let (X, d) be a cone b-metric space over Banach algebra $\mathbb{A}$ and $\{x_n\}$ a sequence in X. We say that

   (i) $\{x_n\}$ is a convergent sequence if, for every $c \in \mathbb{A}$ with $\theta \ll c$, there is an N such that $d(x_n, x) \ll c$ for all $n \geq N$. Ones write it by $x_n \to x$ $(n \to \infty)$;

   (ii) $\{x_n\}$ is a Cauchy sequence if, for every $c \in \mathbb{A}$ with $\theta \ll c$, there is an N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$;

   (iii) (X, d) is a complete cone b-metric space if every Cauchy sequence in X is convergent.

Definition 1.7([17]) Let K be a solid cone in a Banach space $\mathbb{A}$. A sequence $\{u_n\} \subset K$ is said to be a c-sequence if for each $c \gg \theta$ there exists a natural number N such that $u_n \ll c$ for all $n > N$.

Lemma 1.8([20]) Let K be a solid cone in a Banach algebra $\mathbb{A}$, $\{u_n\}$ and $\{v_n\}$ be two c-sequences in K. If $\alpha, \beta \in K$ are two arbitrarily given vectors, then $\{\alpha u_n + \beta v_n\}$ is a c-sequence.

Proof It is evident that $\{u_n + v_n\}$ is a c-sequence (see [17]). We only show that $\{\alpha u_n\}$ is a c-sequence. Indeed, without loss of generality, put $\theta \prec \alpha$. For any $c \gg \theta$, there is a

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$\delta > 0$ such that

$$U(c, \delta) = \{u \in \mathbb{A} : \|u - c\| < \delta\} \subset K.$$

Set $c_0 \gg \theta$ and $\|c_0\| < \frac{\delta}{\|\alpha\|}$. Notice that

$$\|(c - \alpha c_0) - c\| = \|\alpha c_0\| \leq \|\alpha\| \|c_0\| < \delta \Rightarrow c - \alpha c_0 \in U(c, \delta) \subset K,$$

which implies that $c - \alpha c_0 \in \mathrm{int} K$, i.e., $\alpha c_0 \ll c$. Since $\{u_n\}$ is a c-sequence, then there exists N such that $u_n \ll c_0$ for all $n > N$, thus $\alpha u_n \ll c \ (n > N)$.

Lemma 1.9([19]) Let $\mathbb{A}$ be a Banach algebra with a unit e, then the spectral radius $\rho(u)$ of $u \in \mathbb{A}$ holds

$$\rho(u) = \lim_{n \to \infty} \|u^n\|^{\frac{1}{n}} = \inf \|u^n\|^{\frac{1}{n}}.$$

Further, $e - u$ is invertible and $(e - u)^{-1} = \sum_{i=0}^{\infty} u^i$ provided that $\rho(u) < 1$.

Lemma 1.10([19]) Let $\mathbb{A}$ be a Banach algebra with a unit e and $u, v \in \mathbb{A}$. If u commutes with v, then

$$\rho(u + v) \leq \rho(u) + \rho(v), \qquad \rho(uv) \leq \rho(u)\rho(v).$$

Lemma 1.11([20]) Let $\mathbb{A}$ be a Banach algebra with a unit e and let k be a vector in $\mathbb{A}$. If $\rho(k) < 1$, then

$$\rho((e - k)^{-1}) < \frac{1}{1 - \rho(k)}.$$

The following properties are often used (in particular when dealing with cone b-metric spaces over Banach algebras in which the cones need not be normal)(see [2], [20]).

(p1) If $\theta \preceq u \ll c$ for each $c \in \mathrm{int} K$, then $u = \theta$.

(p2) If $u \preceq v$ and $v \ll w$, then $u \ll w$.

(p3) If $u \in K$ and $\rho(u) < 1$, then $\|u^n\| \to 0 \ (n \to \infty)$.

(p4) If $u \preceq ku$, where $u, k \in K$ and $\rho(k) < 1$, then $u = \theta$.

(p5) If $c \in \mathrm{int} K$ and $u_n \to \theta \ (n \to \infty)$, then there exists N such that, for all $n > N$, one has $u_n \ll c$.

2   Main results

In this section, by deleting the assumption of normality of cones, we shall prove some fixed point theorems of generalized Lipschitz mappings in the setting of cone b-metric s-

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paces over Banach algebras. We also obtain the P properties of the mappings. Otherwise, we present two examples to verify our results. Our examples indicate that cone b-metric spaces over Banach algebras are never equivalent to b-metric spaces in terms of the existence of the fixed points of the mappings involved.

Theorem 2.1 Let (X, d) be a complete cone b-metric space over Banach algebra $\mathbb{A}$ with the coefficient $s \geq 1$. Let K be a solid not necessarily normal cone of $\mathbb{A}$. Suppose $T : X \to X$ is a mapping and suppose that there exists $k \in K$ such that, for all $x, y \in X$, at least one of the following generalized Lipschitz conditions holds:

(i) $d(Tx, Ty) \preceq kd(x, y)$ and $\rho(k) < \frac{1}{s}$;

(ii) $d(Tx, Ty) \preceq k(d(Tx, x) + d(Ty, y))$ and $\rho(k) < \frac{1}{1+s}$;

(iii) $d(Tx, Ty) \preceq k(d(Tx, y) + d(Ty, x))$ and $\rho(k) < \frac{1}{s+s^2}$.

Then T has a unique fixed point in X.

Proof Fix $x_0 \in X$ and set $x_1 = Tx_0$ and $x_{n+1} = Tx_n = T^{n+1}x_0$. Then for all three cases (i)-(iii), we shall prove that

$$d(x_{n+1}, x_n) \preceq \lambda d(x_n, x_{n-1}), \tag{2.1}$$

where $\lambda \in K$ and $\rho(\lambda) < \frac{1}{s}$.

For the case (i), it ensures us that

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \preceq kd(x_n, x_{n-1}).$$

Let $\lambda = k$, (2.1) is clear.

For the case (ii), it is easy to see that

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \preceq k(d(Tx_n, x_n) + d(Tx_{n-1}, x_{n-1}))$$
$$= k(d(x_{n+1}, x_n) + d(x_n, x_{n-1})). \tag{2.2}$$

As a consequence of $\rho(k) < \frac{1}{1+s} < 1$, it follows from Lemma 1.9 that $e - k$ is invertible. Hence by (2.2), we deduce that

$$d(x_{n+1}, x_n) \preceq (e - k)^{-1} kd(x_n, x_{n-1}).$$

By Lemma 1.10 and Lemma 1.11, we speculate that

$$\rho((e - k)^{-1}k) \leq \frac{\rho(k)}{1 - \rho(k)} < \frac{\frac{1}{1+s}}{1 - \frac{1}{1+s}} = \frac{1}{s}. \tag{2.3}$$

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Put $\lambda = (e - k)^{-1}k$, (2.1) is obvious.

For the case (iii), we have

$$
\begin{aligned}
d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) &\preceq k(d(Tx_n, x_{n-1}) + d(x_n, Tx_{n-1})) \\
&= k(d(x_{n+1}, x_{n-1}) + d(x_n, x_n)) = kd(x_{n+1}, x_{n-1}) \\
&\preceq sk(d(x_{n+1}, x_n) + d(x_n, x_{n-1})).
\end{aligned}
\tag{2.4}
$$

On account of $\rho(k) < \frac{1}{s}$, it follows from Lemma 1.9 that $e - sk$ is invertible, then by (2.4), one has

$$
d(x_{n+1}, x_n) \preceq (e - sk)^{-1}skd(x_n, x_{n-1}).
$$

Take advantage of Lemma 1.10 and Lemma 1.11, it establishes that

$$
\begin{aligned}
\rho((e - sk)^{-1}sk) &\leq \rho((e - sk)^{-1})\rho(sk) \\
&\leq \frac{\rho(sk)}{1 - \rho(sk)} = \frac{s\rho(k)}{1 - s\rho(k)} < \frac{\frac{s}{s+s^2}}{1 - \frac{s}{s+s^2}} = \frac{1}{s}.
\end{aligned}
\tag{2.5}
$$

Choose $\lambda = (e - sk)^{-1}sk$, (2.1) is valid.

Making full use of (2.1), we get

$$
d(x_{n+1}, x_n) \preceq \lambda d(x_n, x_{n-1}) \preceq \lambda^2 d(x_{n-1}, x_{n-2}) \preceq \cdots \preceq \lambda^n d(x_1, x_0).
$$

Note that $\rho(\lambda) < \frac{1}{s}$ implies $e - s\lambda$ is invertible and

$$
(e - s\lambda)^{-1} = \sum_{i=0}^{\infty}(s\lambda)^i.
$$

Hence, for any $m \geq 1$, $p \geq 1$ and $\lambda \in K$ with $\rho(\lambda) < \frac{1}{s}$, we have that

$$
\begin{aligned}
d(x_m, x_{m+p}) &\preceq s[d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+p})] \\
&\preceq sd(x_m, x_{m+1}) + s^2[d(x_{m+1}, x_{m+2}) + d(x_{m+2}, x_{m+p})] \\
&\preceq sd(x_m, x_{m+1}) + s^2 d(x_{m+1}, x_{m+2}) + s^3 d(x_{m+2}, x_{m+3}) \\
&\quad + \cdots + s^{p-1}d(x_{m+p-2}, x_{m+p-1}) + s^{p-1}d(x_{m+p-1}, x_{m+p}) \\
&\preceq s\lambda^m d(x_1, x_0) + s^2\lambda^{m+1}d(x_1, x_0) + s^3\lambda^{m+2}d(x_1, x_0) \\
&\quad + \cdots + s^{p-1}\lambda^{m+p-2}d(x_1, x_0) + s^p\lambda^{m+p-1}d(x_1, x_0) \\
&= s\lambda^m[e + s\lambda + s^2\lambda^2 + \cdots + (s\lambda)^{p-1}]d(x_1, x_0) \\
&\preceq s\lambda^m(e - s\lambda)^{-1}d(x_1, x_0).
\end{aligned}
$$

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Since $\rho(\lambda) < \frac{1}{s} \leq 1$ implies that $\|\lambda^m\| \to 0$ $(m \to \infty)$, further, $\{\lambda^m\}$ is a c-sequence. Thus we derive from Lemma l.8 that $\{s\lambda^m(e - s\lambda)^{-1}d(x_1, x_0)\}$ is a c-sequence. This means $\{x_n\}$ is a Cauchy sequence in X. Since (X, d) is complete, there exists $x^* \in X$ such that $x_n \to x^*$ $(n \to \infty)$. Next, we shall show x^* is the fixed point. In order to complete it, we consider three cases as follows.

For the case (i), one has

$$d(x^*, Tx^*) \preceq s[d(x_{n+1}, x^*) + d(Tx_n, Tx^*)] \preceq s[d(x_{n+1}, x^*) + kd(x_n, x^*)].$$

In view of $x_n \to x^*$ $(n \to \infty)$, it follows that $\{d(x_{n+1}, x^*)\}$ and $\{d(x_n, x^*)\}$ are c-sequences. So by Lemma 1.8 that $\{s[d(x_{n+1}, x^*) + kd(x_n, x^*)]\}$ is also a c-sequence. We obtain $Tx^* = x^*$.

For the case (ii), it is not hard to verify that

$$d(x^*, Tx^*) \preceq s[d(x_{n+1}, x^*) + d(Tx_n, Tx^*)]$$
$$\preceq sd(x_{n+1}, x^*) + sk[d(x_n, x_{n+1}) + d(x^*, Tx^*)]. \tag{2.6}$$

Note that $e - sk$ is invertible, then (2.6) implies that

$$d(x^*, Tx^*) \preceq s(e - sk)^{-1}[d(x_{n+1}, x^*) + kd(x_n, x_{n+1})].$$

Because $\{x_n\}$ is a Cauchy and convergent sequence, it means $\{d(x_{n+1}, x^*)\}$ and $\{d(x_n, x_{n+1})\}$ are c-sequences. Hence by Lemma 1.8 that $\{s(e - sk)^{-1}[d(x_{n+1}, x^*) + kd(x_n, x_{n+1})]\}$ is also a c-sequence. We have $Tx^* = x^*$.

For the case (iii), it is evident that

$$d(x^*, Tx^*) \preceq s[d(x_{n+1}, x^*) + d(Tx_n, Tx^*)]$$
$$\preceq sd(x_{n+1}, x^*) + sk[d(x_n, Tx^*) + d(x^*, x_{n+1})]$$
$$\preceq sd(x_{n+1}, x^*) + s^2k[d(x_n, x^*) + d(x^*, Tx^*)] + skd(x^*, x_{n+1}). \tag{2.7}$$

Now that $\rho(k) < \frac{1}{s+s^2} < \frac{1}{s^2}$ determines that $e - s^2k$ is invertible, then (2.7) leads to

$$d(x^*, Tx^*) \preceq s(e - s^2k)^{-1}[(e + k)d(x_{n+1}, x^*) + skd(x_n, x^*)].$$

Since $\{d(x_n, x^*)\}$ is a c-sequence, then by Lemma 1.8, $\{s(e - s^2k)^{-1}[(e + k)d(x_{n+1}, x^*) + skd(x_n, x^*)]\}$ is also a c-sequence. Accordingly, $Tx^* = x^*$.

Finally, we shall prove the fixed point is unique. To this end, we suppose for absurd that there exists another fixed point y^*. We need to show it for three cases.

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For the case (i), it may be verified that

$$d(x^*, y^*) = d(Tx^*, Ty^*) \preceq kd(x^*, y^*).$$

Consequently, $y^* = x^*$.

For the case (ii), it is valid that

$$d(x^*, y^*) = d(Tx^*, Ty^*) \preceq k[d(x^*, Tx^*) + d(y^*, Ty^*)] = \theta.$$

That is, $y^* = x^*$.

For the case (iii), we arrive at

$$d(x^*, y^*) = d(Tx^*, Ty^*) \preceq k[d(x^*, Ty^*) + d(y^*, Tx^*)] = 2kd(x^*, y^*).$$

Because $\rho(k) < \frac{1}{s+s^2} \leq \frac{1}{2}$ leads to $\rho(2k) < 1$, we get $y^* = x^*$. Finally, the claim holds.

Remark 2.2 Theorem 2.1 generalizes [20, Theorem 3.1-3.3]. Indeed, take $s = 1$ in Theorem 2.1. Otherwise, Theorem 2.1 also generalizes [27, Corollary 3.8] from b-metric (or metric type) space to cone b-metric (or cone metric type) space over Banach algebra.

It is well-known that if x^* is a fixed point of a mapping T, then x^* is also a fixed point of T^n for each $n \in \mathbb{N}$. But the converse is not true. If a mapping T holds $F(T) = F(T^n)$ for each $n \in \mathbb{N}$, where $F(T)$ denotes the set of all fixed points of T, then ones call T has a P property (see [28-30]). The following results are generalizations of the corresponding results in metric and cone metric spaces (see [28-30]). It will be obtained also without using normality of cones.

Theorem 2.3 Let (X, d) be a cone b-metric space over Banach algebra $\mathbb{A}$ with the coefficient $s \geq 1$. Let K be a solid not necessarily normal cone of $\mathbb{A}$. Suppose $T : X \to X$ is a mapping such that $F(T) \neq \emptyset$ and that

$$d(Tx, T^2x) \preceq \mu d(x, Tx) \tag{2.8}$$

for all $x \in X$, where $\mu \in K$ is a generalized Lipschitz constant with $\rho(\mu) < 1$. Then T has a P property.

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Proof We shall always assume that $n > 1$, since the statement for $n = 1$ is trivial. Let $z \in F(T^n)$. By the assumption, it is clear that

$$d(z, Tz) = d(TT^{n-1}z, T^2T^{n-1}z) \preceq \mu d(T^{n-1}z, T^n z) = \mu d(TT^{n-2}z, T^2T^{n-2}z)$$

$$\preceq \mu^2 d(T^{n-2}z, T^{n-1}z) \preceq \cdots \preceq \mu^n d(z, Tz).$$

By virtue of $\rho(\mu) < 1$, it follows that $\|\mu^n\| \to 0$ $(n \to \infty)$. Accordingly, $\{\mu^n d(z, Tz)\}$ is a c-sequence. Then $d(z, Tz) = \theta$, i.e., $Tz = z$.

Theorem 2.4 Let (X, d) be a complete cone b-metric space over Banach algebra $\mathbb{A}$ with the coefficient $s \geq 1$. Let K be a solid not necessarily normal cone of $\mathbb{A}$. Suppose $T : X \to X$ is a mapping and suppose that there exists $k \in K$ such that, for all $x, y \in X$, at least one of the following generalized Lipschitz conditions holds:

(i) $d(Tx, Ty) \preceq kd(x, y)$ and $\rho(k) < \frac{1}{s}$;

(ii) $d(Tx, Ty) \preceq k(d(Tx, x) + d(Ty, y))$ and $\rho(k) < \frac{1}{1+s}$;

(iii) $d(Tx, Ty) \preceq k(d(Tx, y) + d(Ty, x))$ and $\rho(k) < \frac{1}{s+s^2}$.

Then T has a P property.

Proof Making full use of Theorem 2.1, we claim T has a unique fixed point. In order to utilize Theorem 2.3, we have to show (2.8). To this end, we divide it into three cases.

For the case (i), it follows that

$$d(Tx, T^2x) = d(Tx, TTx) \preceq kd(x, Tx).$$

Let $\mu = k$, (2.8) is valid.

For the case (ii), we have

$$d(Tx, T^2x) = d(Tx, TTx) \preceq k(d(x, Tx) + d(Tx, T^2x)),$$

which establishes that

$$d(Tx, T^2x) \preceq (e - k)^{-1}kd(x, Tx).$$

Owing to (2.3), $\rho((e - k)^{-1}k) < \frac{1}{s} \leq 1$, then let $\mu = (e - k)^{-1}k$, (2.8) is trivial.

For the case (iii), we have

$$d(Tx, T^2x) = d(Tx, TTx) \preceq k(d(x, T^2x) + d(Tx, Tx))$$

$$= kd(x, T^2x) \preceq sk(d(x, Tx) + d(Tx, T^2x)),$$

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which means that

$$d(Tx, T^2x) \preceq (e - sk)^{-1}skd(x, Tx).$$

In view of (2.5), $\rho((e - sk)^{-1}sk) < \frac{1}{s} \leq 1$, then let $\mu = (e - sk)^{-1}sk$, (2.8) is trivial.

Theorem 2.5 Let (X, d) be a complete cone b-metric space over Banach algebra $\mathbb{A}$ with the coefficient $s \geq 1$. Let K be a solid not necessarily normal cone of $\mathbb{A}$. Suppose $T : X \to X$ is a mapping and there exists $k \in K$ and $\rho(k) < \frac{1}{s}$ such that, for all $x, y \in X$, the following generalized Lipschitz condition holds:

$$d(Tx, Ty) \preceq k \cdot u(x, y),$$

where

$$u(x, y) \in \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty)}{2s}, \frac{d(y, Tx)}{2s} \right\}.$$

Then T has a unique fixed point in X. Moreover, T has a P property.

Proof If $u = d(x, y)$, then by Theorem 2.1(i) and Theorem 2.4(i), the proof is valid. We shall consider the other cases.

Fix $x_0 \in X$ and set $x_1 = Tx_0$ and $x_{n+1} = Tx_n = T^{n+1}x_0$. Then we have that

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \preceq k \cdot u(x_n, x_{n-1}),$$

where

$$u(x_n, x_{n-1}) \in \left\{ d(x_n, Tx_n), d(x_{n-1}, Tx_{n-1}), \frac{d(x_n, Tx_{n-1})}{2s}, \frac{d(x_{n-1}, Tx_n)}{2s} \right\}$$
$$= \left\{ d(x_n, x_{n+1}), d(x_{n-1}, x_n), \theta, \frac{d(x_{n-1}, x_{n+1})}{2s} \right\}.$$

If $d(x_{n+1}, x_n) \preceq kd(x_n, x_{n+1})$ or $d(x_{n+1}, x_n) \preceq \theta$, then for all $n \in \mathbb{N}$, $d(x_{n+1}, x_n) = \theta$. That is, $Tx_n = x_{n+1} = x_n$ for all $n \in \mathbb{N}$, thus x_n is the fixed point. If $d(x_{n+1}, x_n) \preceq kd(x_n, x_{n-1})$, i.e., (2.1) holds if $\lambda = k$. If $d(x_{n+1}, x_n) \preceq k \cdot \frac{d(x_{n-1}, x_{n+1})}{2s}$, then

$$d(x_{n+1}, x_n) \preceq k \cdot \frac{d(x_{n-1}, x_{n+1})}{2s} \preceq k \cdot \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2}. \tag{2.9}$$

Since $\rho(k) < \frac{1}{s}$ implies that $2e - k$ is invertible, then (2.9) leads to

$$d(x_{n+1}, x_n) \preceq (2e - k)^{-1}kd(x_n, x_{n-1}).$$

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Note that

$$\rho((2e-k)^{-1}k) = \frac{1}{2}\rho\left((e-\frac{k}{2})^{-1}k\right) \le \frac{1}{2} \cdot \frac{\rho(k)}{1 - \rho(\frac{k}{2})}$$

$$= \frac{\rho(k)}{2 - \rho(k)} < \frac{\frac{1}{s}}{2 - \frac{1}{s}} = \frac{1}{2s - 1} \le \frac{1}{s}.$$

Take $\lambda = (2e - k)^{-1}k$, hence (2.1) holds. Therefore, following an argument similar to that given in Theorem 2.1, we obtain that there exists $x^* \in X$ such that $x_n \to x^*$ $(n \to \infty)$.

In the following, we shall divide two cases to prove that x^* is the fixed point.

For the case that $u(x, y) = d(y, Ty)$, we have

$$d(x^*, Tx^*) \preceq s[d(x^*, Tx_n) + d(Tx_n, Tx^*)]$$
$$\preceq s[d(x^*, x_{n+1}) + kd(x^*, Tx^*)],$$

which follows that

$$d(x^*, Tx^*) \preceq s(e - sk)^{-1}d(x^*, x_{n+1}).$$

Because $\{d(x_{n+1}, x^*)\}$ is a c-sequence, then $x^* = Tx^*$.

For the case that $u(x, y) = \frac{d(y, Tx)}{2s}$, we arrive at

$$d(x^*, Tx^*) \preceq s[d(x^*, Tx_n) + d(Tx_n, Tx^*)]$$
$$\preceq s\left[d(x^*, x_{n+1}) + k \cdot \frac{d(x^*, x_{n+1})}{2s}\right]$$
$$= \left(se + \frac{1}{2}k\right)d(x^*, x_{n+1}).$$

Now that $\{d(x_{n+1}, x^*)\}$ is a c-sequence, then $x^* = Tx^*$.

Next, we shall prove that the fixed point is unique. Assume there exists another fixed point y^*, then

$$d(x^*, y^*) = d(Tx^*, Ty^*) \preceq k \cdot u(x^*, y^*),$$

where

$$u(x^*, y^*) \in \left\{d(x^*, Tx^*), d(y^*, Ty^*), \frac{d(x^*, Ty^*)}{2s}, \frac{d(y^*, Tx^*)}{2s}\right\} = \left\{\theta, \frac{d(x^*, y^*)}{2s}\right\}.$$

It is not hard to verify that $x^* = y^*$.

Finally, we shall prove T has a P property. In order to end this, we have to show (2.8). We divide it into four cases.

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For the case that $u(x, y) = d(x, Tx)$, from

$$d(Tx, T^2x) = d(Tx, TTx) \preceq kd(x, Tx),$$

we have (2.8).

For the case that $u(x, y) = d(y, Ty)$, we get

$$d(Tx, T^2x) = d(Tx, TTx) \preceq kd(Tx, T^2x),$$

which means that $d(Tx, T^2x) = \theta$. Hence (2.8) is clear.

For the case that $u(x, y) = \frac{d(x, Ty)}{2s}$, we obtain

$$d(Tx, T^2x) = d(Tx, TTx) \preceq k \cdot \frac{d(x, T^2x)}{2s} \preceq \frac{k}{2}[d(x, Tx) + d(Tx, T^2x)],$$

which implies that $d(Tx, T^2x) \preceq (2e - k)^{-1}kd(x, Tx)$. So (2.8) is obvious.

For the case that $u(x, y) = \frac{d(y, Tx)}{2s}$, we obtain

$$d(Tx, T^2x) = d(Tx, TTx) \preceq k \cdot \frac{d(Tx, Tx)}{2s} = \theta,$$

which establishes that $d(Tx, T^2x) = \theta$. Thus (2.8) is obvious.

Therefore, by using Theorem 2.3, T has a P property.

In the following, we shall furnish two nontrivial examples to support our main results.

Example 2.6(the case of a non-normal cone)　Let $\mathbb{A} = C^1_{\mathbb{R}}[0, 1]$ and $\|u\| = \|u\|_\infty + \|u'\|_\infty$ be its norm. Define usual pointwise multiplication as its multiplication. Clearly, $\mathbb{A}$ is a Banach algebra with a unit $e = 1$. Put $K = \{u \in \mathbb{A} : u = u(t) \geq 0, t \in [0, 1]\}$. Then K is a non-normal cone (see [2]). Set $X = \{a, b, c\}$ and define a mapping $d : X \times X \to \mathbb{A}$ by $d(a, b)(t) = d(b, a)(t) = e^t$, $d(b, c)(t) = d(c, b)(t) = 2e^t$, $d(a, c)(t) = d(c, a)(t) = 4e^t$, $d(a, a)(t) = d(b, b)(t) = d(c, c)(t) = 0$. One claims that (X, d) is a cone b-metric space over Banach algebra $\mathbb{A}$ with the coefficient $s = \frac{4}{3}$. But it is not a cone metric space over Banach algebra since it does not satisfy the triangle inequality. Now let a mapping $T : X \to X$ be $Ta = Tb = b$, $Tc = a$ and let $k \in K$ with $k(t) = \frac{1}{6}t + \frac{1}{2}$. It is not hard to verify that all conditions of Theorem 2.1 in the case of (i) hold. Therefore, $x^* = b$ is the unique fixed point. Clearly, T has a P property.

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Example 2.7(the case of a normal cone)   Let $\mathbb{A} = \mathbb{R}^2$ with the norm $\|(u_1, u_2)\| = |u_1| + |u_2|$ and the multiplication by

$$uv = (u_1, u_2)(v_1, v_2) = (u_1 v_1, u_1 v_2 + u_2 v_1).$$

Put $K = \{u = (u_1, u_2) \in \mathbb{A} : u_1, u_2 \geq 0\}$. It is easy to see that K is a normal cone and $\mathbb{A}$ is a Banach algebra with a unit $e = (1, 0)$. Let $X = [0, 0.55] \times [-2, 2]$ and for all $x = (x_1, x_2), y = (y_1, y_2) \in X$, $d(x, y) = (|x_1 - y_1|^2, |x_2 - y_2|^2)$. We demonstrate that (X, d) is a complete cone b-metric space over Banach algebra $\mathbb{A}$ with the coefficient $s = 2$.

Define a mapping $T : X \to X$ as

$$Tx = T(x_1, x_2) = \left(\frac{1}{2} \left(\cos \frac{x_1}{2} - |x_1 - \frac{1}{2}| \right), \arctan(2 + |x_2|) + \ln(x_1 + 1) \right).$$

By using mean value theorem of differentials, we have

$$
\begin{aligned}
d(Tx, Ty) &= d(T(x_1, x_2), T(y_1, y_2)) \\
&= \left(\left| \frac{1}{2} \left(\cos \frac{x_1}{2} - \cos \frac{y_1}{2} - |x_1 - \frac{1}{2}| + |y_1 - \frac{1}{2}| \right) \right|^2, \right. \\
&\quad \left. |\arctan(2 + |x_2|) - \arctan(2 + |y_2|) + \ln(x_1 + 1) - \ln(y_1 + 1)|^2 \right) \\
&\preceq \left(\left(\left| \frac{x_1 + y_1}{4} \right| \left| \frac{x_1 - y_1}{4} \right| + \frac{1}{2}|x_1 - y_1| \right)^2, \left(\frac{1}{5}|x_2 - y_2| + |x_1 - y_1| \right)^2 \right) \\
&\preceq \left(\left(\frac{|x_1 + y_1|}{16} + \frac{1}{2} \right)^2 |x_1 - y_1|^2, 2 \left(\frac{1}{25}|x_2 - y_2|^2 + |x_1 - y_1|^2 \right) \right) \\
&\preceq \left(\frac{1}{3}|x_1 - y_1|^2, \frac{2}{25}|x_2 - y_2|^2 + 2|x_1 - y_1|^2 \right) \\
&\preceq \left(\frac{1}{3}, 2 \right) \left(|x_1 - y_1|^2, |x_2 - y_2|^2 \right) \\
&= \left(\frac{1}{3}, 2 \right) d(x, y)
\end{aligned}
$$

for all $x, y \in X$. Denote $k = (\frac{1}{3}, 2)$. Careful calculations show that all conditions of Theorem 2.1 in the case of (i) hold. Thus T has a unique fixed point in X.

It is well-known that some results concerning fixed points and other results, in case of cone spaces with non-normal cones, cannot be provided by reducing to metric spaces (see [2]). In other words, if the underlying cones are non-normal, then some fixed point results in cone spaces are not equivalent to those of metric spaces. Otherwise, [3-10] appeal to the equivalence if the cones are normal cones. However, next, we shall claim our fixed point results in cone b-cone metric spaces over Banach algebras are never equivalent to the

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counterparts in b-metric spaces even if the cones are normal cones. For this, we consider Example 2.7. Put

$$d_\xi(x,y) = \xi_e \circ d(x,y) = \inf\{r \in \mathbb{R} : d(x,y) \preceq re\}, \quad x,y \in X,$$

where $e = (e_1, e_2) \in \text{int}K$, $\xi_e(y) = \inf\{r \in \mathbb{R} : y \in re - K\}$ $(y \in \mathbb{A})$. Then by Theorem 2.1 of [8], d_ξ is a b-metric. We shall prove our conclusions are not equivalent to the well-known Banach contraction principle in b-metric space. Indeed, taking $x' = (\frac{1}{2}, 0)$, $y' = (0,0)$ and $e = (1, \frac{1}{2})$, we have

$$
\begin{aligned}
d_\xi(Tx', Ty') &= \inf\left\{ r \in \mathbb{R} : \left(\frac{1}{4}\left(\cos\frac{1}{4} - \frac{1}{2} \right)^2, \left(\ln\frac{3}{2} \right)^2 \right) \preceq r\left(1, \frac{1}{2}\right) \right\} \\
&= \max\left\{ \left(\frac{1}{4}\left(\cos\frac{1}{4} - \frac{1}{2} \right)^2, 2\left(\ln\frac{3}{2} \right)^2 \right) \right\} = 2\left(\ln\frac{3}{2} \right)^2 \\
&\geq \frac{1}{4} = d_\xi(x', y'),
\end{aligned}
$$

which implies that there does not exist $\lambda \in [0,1)$ such that

$$d_\xi(Tx, Ty) \leq \lambda d_\xi(x,y)$$

for all $x, y \in X$. Thus it does not satisfy the contractive condition of Banach contraction principle in b-metric space. That is to say, the proof of [8, Theorem 2.6] will be unreasonable if under the setting of cone b-cone metric space over Banach algebra.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this article.

Authors' contributions

Both authors contribute equally and significantly in writing this paper. Both authors read and approve the final manuscript.

Authors details

[1]School of Mathematics and Statistics, Hubei Normal University, Huangshi, 435002, China. [2]Faculty of Mathematics and Information Technology, Dong Thap University, Dong Thap, Viêt Nam.

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SOME IDENTITIES OF BELL POLYNOMIALS

LEE-CHAE JANG AND TAEKYUN KIM

General Education Institute, Konkuk University, Chungju 138-701, Korea
E-mail : leechae.jang@kku.ac.kr

Department of Mathematics, Kwangwoon University, Seoul 139-701, S. Korea
E-mail : tkkim@kw.ac.kr

ABSTRACT. In this paper, we study some properties of Bell polynomials which are represented by the the linear combination of special polynomials. By using those properties, we give some new identities of Bell polynomials associated with special numbers and polynomials.

1. INTRODUCTION

The stirling number of the first kind is defined as

$$(x)_n = x(x-1)\cdots(x-n-1) = \sum_{l=0}^{n} s_1(n,l)x^l, \ (n \geq 0) \tag{1}$$

and the stirling number of the second kind is given by

$$x^n = \sum_{l=0}^{n} s_2(n,l)x^l, \ (n \geq 0) \qquad (\text{see } [10,13,17]). \tag{2}$$

It is known that the Bell polynomials are defined by the generating function to be

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty} Bel_n(x)\frac{t^n}{n!} \qquad (\text{see } [4,6,16,17,18]). \tag{3}$$

When $x = 1, Bel_n = Bel_n(1)$ are the Bell numbers. Note that $Bel_n(0) = \delta_{0,n}, \ (n \geq 0)$.
As is well known, the Bernoulli polynomials are defined by the generating function to be

$$\frac{t}{e^t - 1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \qquad (\text{see } [1,7]), \tag{4}$$

and the Euler polynomials are given by the generating function to be

$$\frac{2}{e^t + 1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} \qquad (\text{see } [1-18]). \tag{5}$$

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The Cauchy polynomials are given by

$$\frac{t}{log(t+1)}(1+t)^x = \sum_{n=0}^{\infty} C_n(x)\frac{t^n}{n!} \qquad \text{(see [6, 11])}, \tag{6}$$

and the Daehee polynomials are defined by the generating function to be

$$\frac{log(t+1)}{t}(1+t)^x = \sum_{n=0}^{\infty} D_n(x)\frac{t^n}{n!} \qquad \text{(see [9])}. \tag{7}$$

Finally, we introduce the Changhee polynomials which are given by the generating function to be

$$\frac{2}{t+2}(1+t)^x = \sum_{n=0}^{\infty} Ch_n(x)\frac{t^n}{n!} \qquad \text{(see [10])}. \tag{8}$$

Recently, several authors have studied these several special polynomials(see [1-18]).

In this paper, we study some properties of Bell polynomials which are represented by the the linear combination of special polynomials. By using those properties, we give some new identities of Bell polynomials associated with special numbers and polynomials.

2. Some identities of Bell polynomials

From (3), we easily derive the following equation:

$$Bel_n(x) = e^{-x}\sum_{l=0}^{\infty}\frac{x^l}{l!}l^n. \tag{9}$$

By replacing t by e^{e^t-1} in (4), we get

$$
\begin{aligned}
\frac{e^t-1}{e^{e^t-1}-1}e^{x(e^t-1)} &= \sum_{m=0}^{\infty}B_m(x)\frac{1}{m!}(e^t-1)^m \\
&= \sum_{m=0}^{\infty}B_m(x)\frac{m!}{m!}\sum_{n=m}^{\infty}s_2(n,m)\frac{t^m}{m!} \\
&= \sum_{m=0}^{\infty}\left\{\sum_{m=0}^{n}B_m(x)s_2(n,m)\right\}\frac{t^m}{m!},
\end{aligned}
\tag{10}
$$

and

$$
\begin{aligned}
\frac{e^t-1}{e^{e^t-1}-1}e^{x(e^t-1)} &= \left(\sum_{m=0}^{\infty}B_m\frac{1}{m!}(e^t-1)^m\right)\left(\sum_{l=0}^{\infty}Bel_l(x)\frac{t^l}{l!}\right) \\
&= \left(\sum_{k=0}^{\infty}\left(\sum_{m=0}^{k}B_m s_2(k,m)\right)\frac{t^k}{k!}\right)\left(\sum_{l=0}^{\infty}Bel_l(x)\frac{t^l}{l!}\right) \\
&= \sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\binom{n}{k}\sum_{m=0}^{k}B_m s_2(k,m)Bel_{n-k}(x)\right\}\frac{t^n}{n!}.
\end{aligned}
\tag{11}
$$

Therefore, by (10) and (11), we obtain the following theorem.

Theorem 2.1. *For $n \geq 0$, we have*

$$\sum_{m=0}^{n} B_m(x)s_2(n,m) = \sum_{k=0}^{n} \binom{n}{k} \sum_{m=0}^{k} B_m s_2(k,m)Bel_{n-k}(x). \tag{12}$$

Let us take $e^t - 1$ instead of t in (5). Then we have

$$
\begin{aligned}
\frac{2}{e^{e^t-1}+1}e^{x(e^t-1)} &= \sum_{m=0}^{\infty} E_m(x)\frac{1}{m!}(e^t-1)^m \\
&= \sum_{m=0}^{\infty} E_m(x)\sum_{n=m}^{\infty} s_2(n,m)\frac{t^n}{n!} \\
&= \sum_{n=0}^{i} nfty\left(\sum_{m=0}^{n} s_2(n,m)E_m(x)\right)\frac{t^n}{n!},
\end{aligned} \tag{13}
$$

and

$$
\begin{aligned}
\frac{e^{e^t-1}-1}{e^t-1}e^{x(e^t-1)} &= \left(\sum_{m=0}^{\infty} E_m\frac{(e^t-1)^m}{m!}\right)\left(\sum_{l=0}^{\infty} Bel_l(x)\frac{t^l}{l!}\right) \\
&= \left(\sum_{m=0}^{\infty} E_m \sum_{k=m}^{\infty} s_2(k,m)\frac{t^k}{k!}\right)\left(\sum_{l=0}^{\infty} Bel_l(x)\frac{t^l}{l!}\right) \\
&= \left(\sum_{k=0}^{\infty}\sum_{m=0}^{k} E_m s_2(k,m)\frac{t^k}{k!}\right)\left(\sum_{l=0}^{\infty} Bel_l(x)\frac{t^l}{l!}\right) \\
&= \sum_{n=0}^{\infty}\left\{\sum_{k=0}^{m}\binom{n}{k}\sum_{m=0}^{k} E_m s_2(k,m)Bel_{n-k}(x)\right\}\frac{t^n}{n!}.
\end{aligned} \tag{14}
$$

Therefore, by (13) and (14), we obtain the following theorem.

Theorem 2.2. *For $n \geq 0$, we have*

$$\sum_{m=0}^{n} s_2(n,m)E_m(x) = \sum_{k=0}^{m}\binom{n}{k}\sum_{m=0}^{k} E_m s_2(k,m)Bel_{n-k}(x). \tag{15}$$

From (6), by replacing t by $e^{e^t-1}-1$, we get

$$
\begin{aligned}
\frac{e^{e^t-1}-1}{e^t-1}e^{x(e^t-1)} &= \sum_{n=0}^{\infty} C_n(x)\frac{1}{n!}\left(e^{e^t-1}-1)^n\right) \\
&= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m} C_n(x)s_2(m,n)\right)\frac{t^m}{m!}
\end{aligned} \tag{16}
$$

and

$$
\begin{aligned}
\frac{e^{e^t-1}-1}{e^t-1}e^{x(e^t-1)} &= \frac{e^{(x+1)(e^t-1)}-e^{x(e^t-1)}}{e^t-1} \\
&= \frac{1}{t}\left(\frac{t}{e^t-1}\right)\left(\sum_{m=1}^{\infty}\{Bel_m(x+1)-Bel_m(x)\}\frac{t^m}{m!}\right) \\
&= \left(\sum_{k=0}^{\infty} B_k\frac{t^k}{k!}\right)\left(\sum_{m=0}^{\infty}\left\{\frac{Bel_{m+1}(x+1)-Bel_{m+1}(x)}{m+1}\right\}\frac{t^m}{m!}\right)
\end{aligned}
$$

$$= \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{n} \left(\frac{Bel_{m+1}(x+1) - Bel_{m+1}(x)}{m+1} \right) B_{n-m} \binom{n}{m} \right\} \frac{t^m}{m!}. \quad (17)$$

Therefore, by (16) and (17), we obtain the following theorem.

Theorem 2.3. *For $n \geq 0$, we have*

$$\sum_{m=0}^{n} C_m(x)s_2(n,m) = \sum_{m=0}^{n} \binom{n}{m} B_{n-m} \left(\frac{Bel_{m+1}(x+1) - Bel_{m+1}(x)}{m+1} \right). \quad (18)$$

Let us take $e^{e^t-1} - 1$ instead of t in (7). Then we have

$$
\begin{aligned}
\frac{e^t - 1}{e^{e^t-1} - 1} e^{x(e^t-1)} &= \sum_{n=0}^{\infty} D_n(x) \frac{1}{n!} (e^{e^t-1} - 1)^n \\
&= \sum_{n=0}^{\infty} D_n(x) \frac{1}{n} n! \sum_{m=n}^{\infty} s_2(m,n) \frac{(e^t-1)^m}{m!} \\
&= \sum_{k=0}^{\infty} \left\{ \sum_{m=0}^{k} \sum_{n=0}^{m} D_n(x) s_2(m,n) s_2(k,m) \right\} \frac{k!}{k!} \quad (19)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{e^t - 1}{e^{e^t-1} - 1} e^{x(e^t-1)} &= \left(\sum_{m=0}^{\infty} B_m \frac{1}{m!} (e^t-1)^m \right) \left(\sum_{l=0}^{\infty} Bel_l(x) \frac{t^l}{l!} \right) \\
&= \left(\sum_{m=0}^{\infty} B_m(x) \sum_{k=m}^{\infty} s_2(k,m) \frac{t^k}{k!} \right) \left(\sum_{l=0}^{\infty} Bel_l(x) \frac{t^l}{l!} \right) \\
&= \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{n} \binom{n}{k} \sum_{m=0}^{k} B_m s_2(k,m) Bel_{n-k}(x) \right\} \frac{t^n}{n!}. \quad (20)
\end{aligned}
$$

Therefore, by (19) and (20), we obtain the following theorem.

Theorem 2.4. *For $n \geq 0$, we have*

$$\sum_{k=0}^{n} \sum_{m=0}^{k} D_m(x)s_2(k,m)s_2(n,k) = \sum_{k=0}^{n} \binom{n}{k} \sum_{m=0}^{k} B_m s_2(k,m) Bel_{n-k}(x). \quad (21)$$

By replacing t by $e^{e^t-1} - 1$, we get

$$
\begin{aligned}
\frac{2}{e^{e^t-1} + 1} e^{x(e^t-1)} &= \sum_{n=0}^{\infty} Ch_n(x) \frac{(e^{e^t-1} - 1)^n}{n!} \\
&= \sum_{n=0}^{\infty} Ch_n(x) \sum_{m=n}^{\infty} s_2(m,n) \frac{(e^t-1)^m}{m!} \\
&= \sum_{m=0}^{\infty} \left\{ \sum_{n=0}^{m} Ch_n(x) s_2(m,n) \right\} \frac{1}{m!} (e^t-1)^m \\
&= \sum_{m=0}^{\infty} \left\{ \sum_{n=0}^{m} Ch_n(x) s_2(m,n) \right\} \sum_{k=m}^{\infty} s_2(k,m) \frac{t^k}{k!}
\end{aligned}
$$

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$$= \sum_{k=0}^{\infty} \left\{ \sum_{m=0}^{k} \sum_{n=0}^{m} Ch_n(x)s_2(m,n)s_2(k,m) \right) \frac{t^k}{k!}$$

(22)

and

$$\frac{2}{e^{e^t-1}+1}e^{x(e^t-1)} = \left(\sum_{n=0}^{\infty} E_n \frac{(e^t-1)^n}{n!} \right) \left(\sum_{l=0}^{\infty} Bel_l(x)\frac{t^l}{l!} \right)$$

$$= \left(\sum_{n=0}^{\infty} E_n(x)\frac{1}{n!}n! \sum_{k=n}^{\infty} s_2(k,n)\frac{t^k}{k!} \right) \left(\sum_{l=0}^{\infty} Bel_l(x)\frac{t^l}{l!} \right)$$

$$= \left(\sum_{m=0}^{\infty} \left(\sum_{n=0}^{m} E_n s_2(m,n) \right) \frac{t^m}{m!} \right) \left(\sum_{l=0}^{\infty} Bel_l(x)\frac{t^l}{l!} \right)$$

$$= \sum_{k=0}^{\infty} \left(\sum_{m=0}^{k} \binom{k}{m} \sum_{n=0}^{m} E_n s_2(m,n)Bel_{k-m}(x) \right) \frac{t^k}{k!}. \qquad (23)$$

Therefore, by (22) and (23), we obtain the following theorem.

Theorem 2.5. *For $k \geq 0$, we have*

$$\sum_{m=0}^{k} \binom{k}{m} \sum_{n=0}^{m} E_n s_2(m,n)Bel_{k-m}(x) = \sum_{m=0}^{k} \sum_{n=0}^{m} Ch_n(x)s_2(m,n)s_2(k,m). \qquad (24)$$

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On a type of rough intuitionistic fuzzy sets and its topological structure

Yan-lan Zhang^{a,} * Yin-bin Lei^{b},   Chang-qing Li^{c}

^{a}College of Computer,

Minnan Normal University, Zhangzhou, Fujian 363000, China

^{b}School of Applied Mathematics,

University of Electronic Science and Technology of China,Chengdu 610054,China

^{c}School of Mathematics and Statistics,

Minnan Normal University, Zhangzhou, Fujian 363000, China

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The rough intuitionistic fuzzy set theory is an extension of the theory of rough fuzzy sets. For further studying the theories and applications of rough intuitionistic fuzzy sets, in this paper, we propose a type of rough intuitionistic fuzzy sets and investigate its topological structure. It is proved that an intuitionistic fuzzy topology is induced by a binary relation in a crisp approximation space, and a preorder is generated by a family of intuitionistic fuzzy sets. Moreover, there exists a one-to-one correspondence between the set of all intuitionistic fuzzy topologies having property (∗) and the set of all preorders. That is to say, there exists a one-to-one correspondence between the set of all intuitionistic fuzzy topological spaces having property (∗) and the set of all crisp approximation spaces whose relations are preorders.

Keywords: approximation operator; preorder; rough intuitionistic fuzzy set; intuitionistic fuzzy topology.

1   Introduction

Rough set theory was proposed by Pawlak [14] to deal with imprecision, vagueness, and uncertainty in data analysis. In classical Pawlak rough set theory, the lower and upper approximation operators are two important basic concepts. The equivalence (indiscernibility) relations or partitions are the simplest formulation of the lower and upper approximation operators. However, the requirement of the equivalence relation in Pawlak rough set model seems to be a very restrictive condition that may limit the application domain of the rough set model. To solve this problem, many authors have generalized the notion of approximation operators by using more general binary relations [4, 20, 21, 26, 27] or by employing coverings [2, 3, 28, 33]. Moreover,

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rough sets can also be generalized into the fuzzy environment and the results are called rough fuzzy sets and fuzzy rough sets [8, 9, 10, 12, 13, 15, 16, 19, 22, 23, 24, 30].

As an extension of the theory of fuzzy sets, the theory of intuitionistic fuzzy (IF, for short) sets is originated by Atanassov [1]. A fuzzy set gives a degree of which element belongs to a set, but an IF set gives both a membership degree and a nonmembership degree. Obviously, an IF set is more objective than a fuzzy set to describe the vagueness of data or information. Many authors generalized the concepts and operations in fuzzy set theory into IF set theory, to enrich the theory of IF sets and enlarge the application of IF sets. Therefore, the combination of IF set theory and rough set theory is an interesting research issue over the years [5, 7, 17, 18, 29, 31]. The rough IF sets are indeed natural generalizations of rough fuzzy sets and will be applied in decision analysis.

Topology is a mathematical tool to study information systems and rough sets. It is important to discuss topological structures of rough sets. Many authors investigated topological structures of rough sets in the fuzzy environment [11, 32, 25]. Zhou et al. presented a one-to-one correspondence between the set of all IF reflexive and transitive approximation spaces and the set of all IF rough topological spaces [32]. Lin and Wang discussed the topological properties of IF rough sets [11]. Xu and Wu investigated topological structures of a type rough IF sets [25].

This paper is devoted to the discussion of a type of rough IF sets and its topological structure. Firstly, in a crisp approximation space, an IF topology is generated by the relation, whose interior and closure operators are IF lower and upper approximation operators respectively. Then, a preorder is induced by a family of IF sets. Moreover, there exists a one-to-one correspondence between the set of all intuitionistic fuzzy topological spaces having property $(*)$ and the set of all crisp approximation spaces whose relations are preorders.

2   Basic Concepts and properties

In this section, we introduce the basic concepts about binary relation, intuitionistic fuzzy set and intuitionistic fuzzy topological space.

Throughout this paper, U will be a nonempty set called the universe of discourse. The class of all subsets (intuitionistic fuzzy subsets, respectively) of U will be denoted by $\mathcal{P}(U)$ (by $\mathcal{IF}(U)$, respectively).

Definition 1. Let U be a set, $U \times U$ the product set of U and U. Any subset R of $U \times U$ is called a binary relation on U. For any $(x, y) \in U \times U$, if $(x, y) \in R$, we say x has relation R with y, and denote this relationship as xRy. For any $x \in U$, we call the set $\{y \in U | xRy\}$ the successor neighborhood of x in R and denote it as $R_s(x)$, and the set $\{y \in U | yRx\}$ the predecessor neighborhood of x in R and denote it as $R_p(x)$. Let R be a relation on U.

(Reflexive relation) If for any $x \in U$, xRx, we say R is reflexive. In another word, if for any $x \in U$, $x \in R_s(x)$, R is reflexive.

(Transitive relation) If for any $x, y, z \in U$, xRy and $yRz \Rightarrow xRz$, we say R is transitive. In another word, if for any $x, y \in U$, $y \in R_s(x) \Rightarrow R_s(y) \subseteq R_s(x)$, R is transitive.

(Preorder) A binary relation R is referred to as a preorder if R is reflexive and transitive.

Definition 2 [1]. Let U be a non-empty set. An intuitionistic fuzzy set A in U is an object

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having the form

$$A = \{< x, \mu_A(x), \gamma_A(x) > | x \in U\},$$

where $\mu_A : U \to [0,1]$ and $\gamma_A : U \to [0,1]$ satisfy $0 \le \mu_A(x) + \gamma_A(x) \le 1$ for all $x \in U$, and $\mu_A(x)$ and $\gamma_A(x)$ are, respectively, called the degree of membership and the degree of nonmembership of the element $x \in U$ to A.

Obviously, a fuzzy set $A = \{< x, \mu_A(x) > | x \in U\}$, can be identified with the IF set of the form $A = \{< x, \mu_A(x), 1 - \mu_A(x) > | x \in U\}$. Thus, an IF set is indeed an extension of a fuzzy set. We introduce some basic operations on $\mathcal{IF}(U)$ in the following definition.

Definition 3 [1]. Let $A, B \in \mathcal{IF}(U)$ and $\{A_j | j \in J\} \subseteq \mathcal{IF}(U)$, where J is an index set. Define the operations as follows:

$A \subseteq B \Leftrightarrow \mu_A(x) \le \mu_B(x)$ and $\gamma_A(x) \ge \gamma_B(x)$ for all $x \in U$,

$A \supseteq B \Leftrightarrow B \supseteq A$,

$A = B \Leftrightarrow A \subseteq B$ and $B \supseteq A$,

$A \cap B = \{< x, \mu_A(x) \wedge \mu_B(x), \gamma_A(x) \vee \gamma_B(x) > | x \in U\}$,

$A \cup B = \{< x, \mu_A(x) \vee \mu_B(x), \gamma_A(x) \wedge \gamma_B(x) > | x \in U\}$,

$A^c = \{< x, \gamma_A(x), \mu_A(x) > | x \in U\}$,

$\cap A_j = \{< x, \wedge \mu_{A_j}(x), \vee \gamma_{A_j}(x) > | x \in U\}$,

$\cup A_j = \{< x, \vee \mu_{A_j}(x), \wedge \gamma_{A_j}(x) > | x \in U\}$,

$0_\sim = \{< x, 0, 1 > | x \in U\}, 1_\sim = \{< x, 1, 0 > | x \in U\}$.

Definition 4 [6]. An IF topology τ on a nonempty set U is a family of IF sets in U satisfying the following axioms:

(T$_1$) $0_\sim, 1_\sim \in \tau$,

(T$_2$) $G_1 \cap G_2 \in \tau$ for all $G_1, G_2 \in \tau$,

(T$_3$) $\cup_{j \in J} G_j \in \tau$ for an arbitrary family $\{G_j | j \in J\} \subseteq \tau$.

In this case the pair (U, τ) is called an IF topological space and each IF set $G \in \tau$ is known as an IF open set in U, and the complement G^c of an IF open set G in (U, τ) is called an IF closed set in U. For any $A \in \mathcal{IF}(U)$, the IF interior and IF closure of A are, respectively, defined as follows:

$int(A) = \cup\{G | G \in \tau, \ G \subseteq A\}$,

$cl(A) = \cap\{K | K^c \in \tau, \ A \subseteq K\}$,

where int and cl are, respectively, called the IF interior operator and the IF closure operator of τ.

It can be shown that $cl(A)$ is an IF closed set and $int(A)$ is an IF open set in U, A is an IF open set in U if and only if $int(A) = A$, and A is an IF closed set in U if and only if $cl(A) = A$. Some properties of IF interior operator and IF closure operator are presented as

Proposition 1. *Let (U, τ) be an IF topological space and $A, B \in \mathcal{IF}(U)$. Then the following*

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properties hold:

(1) $cl(A^c) = (int(A))^c$, $int(A^c) = (cl(A))^c$,
(2) $int(A) \subseteq A \subseteq cl(A)$,
(3) $int(A \cap B) = int(A) \cap int(B)$, $cl(A \cup B) = cl(A) \cup cl(B)$,
(4) $int(int(A)) = int(A)$, $cl(cl(A)) = cl(A)$,
(5) $int(1_\sim) = 1_\sim$, $cl(0_\sim) = 0_\sim$.

Conversely, it is easy to verify that if an IF operator $i : \mathcal{IF}(U) \to \mathcal{IF}(U)$ ($c : \mathcal{IF}(U) \to \mathcal{IF}(U)$, respectively) satisfies the following properties: for any $A, B \in \mathcal{IF}(U)$,

(1) $i(1_\sim) = 1_\sim$, $(c(0_\sim) = 0_\sim,)$
(2) $i(A) \subseteq A$, $(A \subseteq c(A),)$
(3) $i(A \cap B) = i(A) \cap i(B)$, $(c(A \cup B) = c(A) \cup c(B),)$
(4) $i(i(A)) = i(A)$, $(c(c(A)) = c(A),)$

then $\{A|i(A) = A, \ A \in \mathcal{IF}(U)\}$ ($\{A^c|c(A) = A, \ A \in \mathcal{IF}(U)\}$, respectively) is an IF topology on U and denoted by $\tau(i)$ ($\tau(c)$, respectively).

3 The one-to-one correspondence between IF approximation operators and IF topological spaces

Firstly, we introduce the definition of IF approximation operators.

Definition 5. Let R be a binary relation on U. Then (U, R) is called a crisp approximation space. Define a family of IF sets as follows:

$$\mathcal{A}(R) = \{A \in \mathcal{IF}(U)|\forall(x,y) \in R, \mu_A(x) \leq \mu_A(y), \gamma_A(x) \geq \gamma_A(y)\}.$$

Then a pair of rough IF approximation operators are defined by

$$\underline{R}(X) = \cup\{A|A \subseteq X, A \in \mathcal{A}(R)\},$$
$$\overline{R}(X) = \cap\{A|X \subseteq A, A \in \mathcal{A}(R)\}.$$

Since Dubois and Prade proposed rough fuzzy set [8], much authors have discussed properties of rough fuzzy set [9, 23, 24]. At the same time, the definitions of rough fuzzy set in [9, 23, 24] were extended to rough IF set [17, 18, 25, 31]. It is easy to verify that the definition of the rough IF in Definition 5 is different from that in [17, 18, 25, 31].

It is easy to get properties of rough IF approximation operators: $\forall A, B \in \mathcal{IF}(U)$,

(1) $\underline{R}(1_\sim) = 1_\sim$, $\overline{R}(0_\sim) = 0_\sim$;
(2) $\underline{R}(A) = (\overline{R}(A^c))^c$, $\overline{R}(A) = (\underline{R}(A^c))^c$;
(3) $\underline{R}(A) \subseteq A \subseteq \overline{R}(A)$;
(4) $A \subseteq B \Rightarrow \underline{R}(A) \subseteq \underline{R}(B), \overline{R}(A) \subseteq \overline{R}(B)$.

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3.1 From a crisp approximation space to an intuitionistic fuzzy topological space

In this subsection, we will present more properties of $\mathcal{A}(R)$, $\underline{R}$ and $\overline{R}$.

Proposition 2. *Let R be a binary relation on U. Then, for any $\mathcal{B} \subseteq \mathcal{A}(R)$, $\cup\mathcal{B}, \cap\mathcal{B} \in \mathcal{A}(R)$.*

Proof. For any $(x, y) \in R$ and $B \in \mathcal{B}$, we get $\mu_B(x) \leq \mu_B(y)$ and $\gamma_B(x) \geq \gamma_B(y)$. Thus $\mu_{\cup\mathcal{B}}(x) \leq \mu_{\cup\mathcal{B}}(y)$ and $\gamma_{\cup\mathcal{B}}(x) \geq \gamma_{\cup\mathcal{B}}(y)$, $\mu_{\cap\mathcal{B}}(x) \leq \mu_{\cap\mathcal{B}}(y)$ and $\gamma_{\cap\mathcal{B}}(x) \geq \gamma_{\cap\mathcal{B}}(y)$. So $\cup\mathcal{B}, \cap\mathcal{B} \in \mathcal{A}(R)$.

Corollary 1. *Let R be a binary relation on U. Then*

(1) $\mathcal{A}(R)$ is an IF topology,
(2) $\underline{R}$ and $\overline{R}$ are, respectively, the IF interior operator and the IF closure operator of $\mathcal{A}(R)$.

Proof. (1) It is clear that $0_\sim, 1_\sim \in \mathcal{A}(R)$. Thus, according to Definition 4 and Proposition 2, $\mathcal{A}(R)$ is an IF topology.
(2) By (1) and Definition 4, we can get this proposition.

From Corollary 1, we know that $\mathcal{A}(R)$ is a IF topology if R is an arbitrarily relation, and $\underline{R}$ and $\overline{R}$ are, respectively, the IF interior and closure operators of $\mathcal{A}(R)$. Hence $\underline{R}(A) \cap \underline{R}(B) = \underline{R}(A \cap B)$ and $\overline{R}(A) \cup \overline{R}(B) = \overline{R}(A \cup B)$ for all $A, B \in \mathcal{IF}(U)$. To get more properties of $\mathcal{A}(R)$, we suppose R is a preorder in the following.

Proposition 3. *Let R be a preorder on U. Then, for any $x, y \in U$, xRy if and only if $\mu_A(x) \leq \mu_A(y)$ and $\gamma_A(x) \geq \gamma_A(y)$ for all $A \in \mathcal{A}(R)$.*

Proof. "$\Rightarrow$". If xRy, by the definition of $\mathcal{A}(R)$, it is easy to obtain that $\mu_A(x) \leq \mu_A(y)$ and $\gamma_A(x) \geq \gamma_A(y)$ for all $A \in \mathcal{A}(R)$.
"$\Leftarrow$". Suppose $(x, y) \notin R$, then define an IF set B as follows: for any $u \in U$,

$$\mu_B(u) = \begin{cases} 0, & u \in R_p(y); \\ 1, & u \notin R_p(y), \end{cases} \quad \gamma_B(u) = 1 - \mu_B(u).$$

For any $(u_1, u_2) \in R$, if $u_2 \notin R_p(y)$, then $\mu_B(u_2) = 1$. So $\mu_B(u_1) \leq \mu_B(u_2)$. If $u_2 \in R_p(y)$, hence $(u_2, y) \in R$. Since R is transitive and $(u_1, u_2) \in R$, we have $(u_1, y) \in R$. Thus $u_1 \in R_p(y)$, $\mu_B(u_1) = 0$, which implies $\mu_B(u_1) \leq \mu_B(u_2)$. So we can conclude that $\mu_B(u_1) \leq \mu_B(u_2)$. Then $\gamma_B(u_1) = 1 - \mu_B(u_1) \geq 1 - \mu_B(u_2) = \gamma_B(u_2)$. Consequently, $B \in \mathcal{A}(R)$.
Since R is reflexive, we obtain $y \in R_p(y)$, so $\mu_B(y) = 0$. By $(x, y) \notin R$, $x \notin R_p(y)$, thus $\mu_B(x) = 1$.
In conclusion, there exists $B \in \mathcal{A}(R)$ such that $\mu_B(x) > \mu_B(y)$, which contradicts the assumption of this theorem.

Proposition 4. *Let R be a preorder on U. Then, for any $x \in U$ and $a, b \in [0, 1]$ with $a + b \leq 1$, there exists an $A \in \mathcal{A}(R)$ such that for any $z \in U$,*

$$\mu_A(z) = \begin{cases} a, & z \in R_s(x); \\ 0, & z \notin R_s(x), \end{cases} \quad \gamma_A(z) = \begin{cases} b, & z \in R_s(x); \\ 1, & z \notin R_s(x). \end{cases}$$

5

Proof. We only prove $A \in \mathcal{A}(R)$. In fact, for any $(u, v) \in R$, if $u \in R_s(x)$, we have $v \in R_s(x)$ since R is transitive. R is reflexive, so $\mu_A(x) = \mu_A(u) = \mu_A(v) = a$ and $\gamma_A(x) = \gamma_A(u) = \gamma_A(v) = b$. If $u \notin R_s(x)$, then $\mu_A(u) = 0$ and $\gamma_A(u) = 1$. Hence $\mu_A(u) \leq \mu_A(v)$ and $\gamma_A(u) \geq \gamma_A(v)$. In conclusion, $\mu_A(u) \leq \mu_A(v)$ and $\gamma_A(u) \geq \gamma_A(v)$. Therefore, $A \in \mathcal{A}(R)$.

3.2 From an intuitionistic fuzzy topological space to a crisp approximation space

Definition 6. Let $\mathcal{A} \subseteq IF(U)$, then define a binary relation from $\mathcal{A}$ as follows:

$$R(\mathcal{A}) = \{(x, y) \in U \times U | \forall A \in \mathcal{A}, \mu_A(x) \leq \mu_A(y), \gamma_A(x) \geq \gamma_A(y)\}.$$

In Definition 6, a binary relation is induced from a family of intuitionistic fuzzy sets. Proposition 5 below gives the properties of $R(\mathcal{A})$.

Proposition 5. Let $\mathcal{A} \subseteq IF(U)$, then $R(\mathcal{A})$ is a preorder.

Proof. For any $x \in U$ and $A \in \mathcal{A}$, $\mu_A(x) = \mu_A(x)$ and $\gamma_A(x) = \gamma_A(x)$, then $(x, x) \in R(\mathcal{A})$. We obtain that $R(\mathcal{A})$ is reflexive.

For any $x, y, z \in U$, if $(x, y) \in R(\mathcal{A})$ and $(y, z) \in R(\mathcal{A})$, then for any $A \in \mathcal{A}$, $\mu_A(x) \leq \mu_A(y)$ and $\gamma_A(x) \geq \gamma_A(y)$, $\mu_A(y) \leq \mu_A(z)$ and $\gamma_A(y) \geq \gamma_A(z)$. So $\mu_A(x) \leq \mu_A(z)$ and $\gamma_A(x) \geq \gamma_A(z)$. Hence $(x, z) \in R(\mathcal{A})$. It follows that $R(\mathcal{A})$ is transitive.

By Proposition 5, we can induce a preorder from a family of IF sets. We first convert a preorder R into a family of IF sets $\mathcal{A}(R)$, then convert the family of IF sets $\mathcal{A}(R)$ into a preorder $R(\mathcal{A}(R))$, and consider the relationship between R and $R(\mathcal{A}(R))$.

Theorem 1. Let R be a preorder on U. Then $R = R(\mathcal{A}(R))$.

Proof. For any $(x, y) \in R$, by the definition of $\mathcal{A}(R)$, $\mu_A(x) \leq \mu_A(y)$ and $\gamma_A(x) \geq \gamma_A(y)$ for all $A \in \mathcal{A}(R)$. According to Definition 6, $(x, y) \in R(\mathcal{A}(R))$, so $R \subseteq R(\mathcal{A}(R))$. Conversely, for any $(x, y) \in R(\mathcal{A}(R))$, $\mu_A(x) \leq \mu_A(y)$ and $\gamma_A(x) \geq \gamma_A(y)$ for all $A \in \mathcal{A}(R)$. From Proposition 3, $(x, y) \in R$, so $R(\mathcal{A}(R)) \subseteq R$. Therefore, we obtain $R = R(\mathcal{A}(R))$.

If we first convert a family of IF sets $\mathcal{A}$ into a preorder $R(\mathcal{A})$, then change the preorder $R(\mathcal{A})$ into the family of IF sets $\mathcal{A}(R(\mathcal{A}))$, $\mathcal{A} = \mathcal{A}(R(\mathcal{A}))$?

Proposition 6. Let $\mathcal{A} \subseteq \mathcal{IF}(U)$, then $\mathcal{A} \subseteq \mathcal{A}(R(\mathcal{A}))$.

Proof. For any $A \in \mathcal{A}$, by the definition of $R(\mathcal{A})$, $\mu_A(x) \leq \mu_A(y)$ and $\gamma_A(x) \geq \gamma_A(y)$ for all $(x, y) \in R(\mathcal{A})$. According to Definition 5, we have $A \in \mathcal{A}(R(\mathcal{A}))$. So $\mathcal{A} \subseteq \mathcal{A}(R(\mathcal{A}))$.

Generally, $\mathcal{A}(R(\mathcal{A}))$ is not equal to $\mathcal{A}$.

Example 1. Let $U = \{a, b\}$ and $\mathcal{A} = \{A\}$, where $A = \{< a, \frac{1}{3}, \frac{1}{2} >, < b, \frac{1}{2}, \frac{1}{3} >\}$. Then $R(\mathcal{A}) = \{(a, a), (a, b), (b, b)\}$. Thus, we have $B = \{< a, \frac{1}{2}, \frac{1}{2} >, < b, \frac{1}{2}, \frac{1}{2} >\} \in \mathcal{A}(R(\mathcal{A}))$ and $B \notin \mathcal{A}$. So $\mathcal{A}(R(\mathcal{A})) \neq \mathcal{A}$.

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In order to give a sufficient and necessary condition for $\mathcal{A}(R(\mathcal{A})) = \mathcal{A}$, we propose two properties for a family of IF sets $\mathcal{A}$.

Property $(*)$: for any $x \in U$ and $a, b \in [0, 1]$ with $a + b \leq 1$, there exists an $A \in \mathcal{A}$ such that for any $z \in U$,

$$\mu_A(z) = \begin{cases} a, & z \in R(\mathcal{A})_s(x); \\ 0, & z \notin R(\mathcal{A})_s(x), \end{cases} \quad \gamma_A(z) = \begin{cases} b, & z \in R(\mathcal{A})_s(x); \\ 1, & z \notin R(\mathcal{A})_s(x). \end{cases}$$

Property $(**)$: for any $\mathcal{B} \subseteq \mathcal{A}$, $\cup \mathcal{B} \in \mathcal{A}$.

Theorem 2. *Let $\mathcal{A} \subseteq \mathcal{IF}(U)$, then $\mathcal{A} = \mathcal{A}(R(\mathcal{A}))$ if and only if $\mathcal{A}$ has properties $(*)$ and $(**)$.*

Proof. "$\Rightarrow$". From Proposition 5, $R(\mathcal{A})$ is a preorder. By Proposition 4, for any $x \in U$ and $a, b \in [0, 1]$ with $a + b \leq 1$, there exists an $A \in \mathcal{A}(R(\mathcal{A}))$ such that for any $z \in U$,

$$\mu_A(z) = \begin{cases} a, & z \in R(\mathcal{A})_s(x); \\ 0, & z \notin R(\mathcal{A})_s(x), \end{cases} \quad \gamma_A(z) = \begin{cases} b, & z \in R(\mathcal{A})_s(x); \\ 1, & z \notin R(\mathcal{A})_s(x). \end{cases}$$

Since $\mathcal{A} = \mathcal{A}(R(\mathcal{A}))$, $\mathcal{A}$ satisfies property $(*)$ by Proposition 2. According to Proposition 4, $\mathcal{A}(R(\mathcal{A}))$ satisfies property $(**)$, which implies that $\mathcal{A}$ has property $(**)$.

"$\Leftarrow$". By Proposition 6, $\mathcal{A} \subseteq \mathcal{A}(R(\mathcal{A}))$. Now we prove $\mathcal{A}(R(\mathcal{A})) \subseteq \mathcal{A}$. Let $A \in \mathcal{A}(R(\mathcal{A}))$, then for any $(u, v) \in R(\mathcal{A})$, $\mu_A(u) \leq \mu_A(v)$ and $\gamma_A(u) \geq \gamma_A(v)$. Since $\mathcal{A}$ satisfies property $(*)$, for any $x \in U$, there is $B_x \in \mathcal{A}$ such that for any $z \in U$,

$$\mu_{B_x}(z) = \begin{cases} \mu_A(x), & z \in R(\mathcal{A})_s(x); \\ 0, & z \notin R(\mathcal{A})_s(x), \end{cases} \quad \gamma_{B_x}(z) = \begin{cases} \gamma_A(x), & z \in R(\mathcal{A})_s(x); \\ 1, & z \notin R(\mathcal{A})_s(x). \end{cases}$$

Then $A = \bigcup_{x \in U} B_x$. In fact, for any $y \in U$, since $R(\mathcal{A})$ is reflexive, We have $y \in R(\mathcal{A})_s(y)$. So $\mu_{B_y}(y) = \mu_A(y)$ and $\gamma_{B_y}(y) = \gamma_A(y)$. Hence

$$\mu_A(y) \leq \vee_{x \in U} \mu_{B_x}(y) = \mu_{\left(\bigcup_{x \in U} B_x\right)}(y), \quad \gamma_A(y) \geq \wedge_{x \in U} \gamma_{B_x}(y) = \gamma_{\left(\bigcup_{x \in U} B_x\right)}(y).$$

Conversely, for any $x \in U$, if $y \notin R(\mathcal{A})_s(x)$, then $\mu_{B_x}(y) = 0$ and $\gamma_{B_x}(y) = 1$. If $y \in R(\mathcal{A})_s(x)$, then $\mu_{B_x}(y) = \mu_A(x) \leq \mu_A(y)$ and $\gamma_{B_x}(y) = \gamma_A(x) \geq \gamma_A(y)$. So

$$\mu_{\bigcup_{x \in U} B_x}(y) = \vee_{x \in U} \mu_{B_x}(y) \leq \mu_A(y), \quad \gamma_{\left(\bigcup_{x \in U} B_x\right)}(y) = \wedge_{x \in U} \gamma_{B_x}(y) \geq \gamma_A(y).$$

Therefore, by property $(**)$, $A = \bigcup_{x \in U} B_x \in \mathcal{A}$, which implies $\mathcal{A}(R(\mathcal{A})) \subseteq \mathcal{A}$.

From Theorem 2, $\mathcal{A}$ having properties $(*)$ and $(**)$ is a sufficiency and necessary condition for $\mathcal{A} = \mathcal{A}(R(\mathcal{A}))$. By Corollary 1 and Theorem 2, it is easy to obtain

Corollary 2. *Let $\mathcal{A} \subseteq \mathcal{IF}(U)$, then $\mathcal{A} = \mathcal{A}(R(\mathcal{A}))$ if and only if $\mathcal{A}$ is an IF topology satisfying property $(*)$.*

Denote the set of all preorders on U as $\widetilde{R}$, and denote the family of all IF topologies on U having property $(*)$ as $\widetilde{A}$. Combining Theorem 1, Corollaries 1 and 2, we have

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Theorem 2. *Let U be a non-empty set. Then there exists a one-to-one correspondence between $\widetilde{R}$ and $\widetilde{A}$.*

Proof. Define a mapping $f : \widetilde{R} \to \widetilde{A}$ by $f(R) = \mathcal{A}(R)$. And define a mapping $g : \widetilde{A} \to \widetilde{R}$ by $g(\mathcal{A}) = R(\mathcal{A})$. For any $R \in \widetilde{R}$, by Theorem 1 and Corollary 1, we get $g \circ f(R) = g(\mathcal{A}(R)) = R(\mathcal{A}(R)) = R$. For any $\mathcal{A} \in \widetilde{A}$, according to Proposition 5 and Corollary 2, $f \circ g(\mathcal{A}) = f(R(\mathcal{A})) = \mathcal{A}(R(\mathcal{A})) = \mathcal{A}$. Then there exists a one-to-one correspondence between $\widetilde{R}$ and $\widetilde{A}$.

By Theorem 2, there exists a one-to-one correspondence between crisp approximation spaces whose relations are preorders and IF topological spaces having property $(\ast)$.

4 Conclusion

In this paper, an IF topology has been induced in a crisp approximation space, whose IF interior and closure operators are IF lower and upper approximation operators respectively. Conversely, a preorder has been generated by a family of IF sets. The important contribution of this paper is that we establish a one-to-one correspondence between the set of all intuitionistic fuzzy topological spaces having property $(\ast)$ and the set of all crisp approximation spaces whose relations are preorders. In our future work, we will discuss relationships between this type of rough IF sets and other types of rough IF sets, and explore connections between rough IF sets and covering-based rough sets.

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