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Journal of Computational Analysis and Applications

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A Recurrent Neural Fuzzy Network

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Abstract

Besides the feedforward neural networks, there are the recurrent networks, where the impulses can be transmitted in both directions due to some reaction connections in these networks. Recurrent neural networks are linear or nonlinear dynamic systems. The dynamic behavior presented by the recurrent neural networks can be described both in continuous time, by differential equations and at discrete times by the recurrence relations (difference equations). The distinction between recurrent (or dynamic) neural networks and static neural networks is due to recurrent connections both between the layers of neurons of these networks and within the same layer, too. The aim of this paper is to describe a Recurrent Fuzzy Neural Network (RFNN) model, whose learning algorithm is based on the Improved Particle Swarm Optimization (IPSO) method. Each particle (candidate solution), which is moving permanently includes the parameters of the membership function and the weights of the recurrent neural-fuzzy network; initially, their values are randomly generated. The RFNN presented in this paper is unlike the others variants of RFNN models, by the number of the evolution directions that they use: in this paper, we update the velocity and the position of all particles along three dimensions, while in [8] are used two dimensions.

Keywords: recurrent networks; Improved Particle Swarm Optimization method; fuzzy rules; Wavelet Neural Network; feedback weight; delayed operator.

1 Introduction

Neural network (NN) is one of the important components in Artificial Intelligence (AI). NN architectures used in modelling of the nervous systems can be classified into three categories, each with a different philosophy: feedforward, recurrent (feedback), self-organizing map. Neural networks (NNs) are used in many different application domains in order to solve various information processing problems. For several years now, neural network models have enjoyed wide popularity [4], being applied to problems of regression, classification, computational science, computer vision, data processing and time series analysis.

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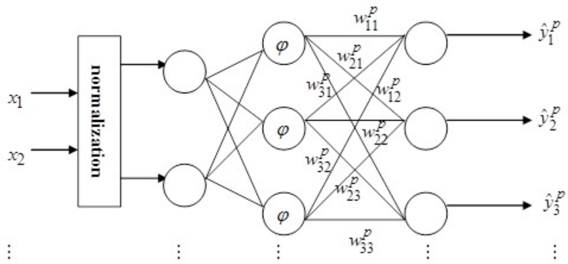

Figure 1: Schematic diagram of the WNN.

The main drawback of the feedforward neural networks is that the updating of the weights can fall [17] in a local minimum. An other major drawback of the feedforward neural networks consists in the fact that their application domain is limited to static problems due their inherent feedforward structure.

Since recurrent networks incorporate feedback, they have powerful representation capability and can [17] successfully overcome disadvantages of feedforward networks. This feedback implies that the network has [12] local memory characteristics that is able to store activity patterns and present those patterns to the network more than once, allowing the layer with feedback connections to use its own past activation in its preceding behavior.

The Recurrent Neural Network (RNN) has the feedforward and feedback connections contrasted which provides it with nonlinear mapping capacity and dynamical characteristics, so it can be used [22] to simulate dynamical system and solve dynamic problems. Different architectures can be created [12] by adding recurrent connections at different points in the basic feedforward architecture.

Recently some researchers have proposed several recurrent neuro- fuzzy networks. Kumar et al. 2004 compares the traditional feedforward approach of RNNs to forecast monthly river flows. Lin & Hsu, 2007 has proposed [10] a recurrent wavelet-based neuro- fuzzy system with the reinforcement hybrid evolutionary learning algorithm for solving various control problems. Carcano et al., 2008 has simulated [3] daily river flows for water resource purposes using the Jordan Recurrent Neural Network. Maraqua et al., 2012 has proposed [12] the use of a recurrent network architecture as a classification engine for automatic Arabic Sign Language recognition system. Šter, 2013 has introduced [18] an extended architecture of recurrent neural networks (called *Selective Recurrent Neural Network*) for dealing with long term dependencies.

1.1 Wavelet Neural Networks

Neural networks employing wavelet neurons are referred to as Wavelet Neural Networks(WNNs) [10]; they are characterized by weights and wavelet bases.

Lin & Chin, 2004 was proposed a Recurrent Neural Fuzzy Network (RNFN) where each fuzzy rule corresponding to a WNN (see Figure 1) consists (see [11], [8]) of single-scaling wavelets. The shape and position of the wavelet bases are shown [11] in Figure 2.

An ordinary wavelet neural network model is often used to normalize input vectors in the interval [0, 1]. The functions $\phi_{a.b}(x_i)$ are used to input vectors

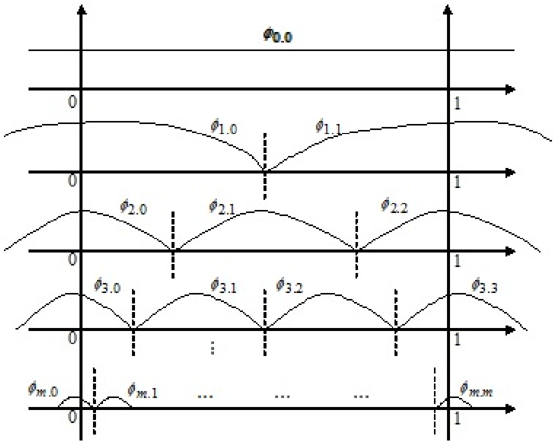

Figure 2: Wavelet bases are over-complete and compactly supported.

to fire up the wavelet interval; a such value is given in the following equation, which gives the shape of the M wavelet bases $\phi_{0.0},\ \phi_{1.0},\dots,\phi_{m.m}$:

$$\begin{cases} \phi(x_i) = \cos(x_i),\ -0.5 \leq x_i \leq 0.5 \\ 0 \text{ otherwise}, \qquad \phi_{a.b}(x_i) = \cos(ax_i - b), \end{cases} \tag{1}$$

$b = \overline{1,a},\ a = \overline{1,m}$, b being a shifting parameter and a meaning a scaling parameter corresponding to the maximum value of b.

A crisp value $\varphi_{a.b}$ can be obtained as follows:

$$\varphi_{a.b} = \frac{\sum_{j=1}^{n} \phi_{a.b}(x_i)}{|X|}, \tag{2}$$

where $|X|$ represents the number of input dimensions and n is the dimension of the input vector to the model.

1.2 Z- transform

The Z- transform is [20] the discrete- time counterpart of the Laplace transform. The Z- transform can be considered to be an extension of the discrete- time Fourier transform as the Laplace transform can be considered an extension of the Fourier transform.

The *bilateral* Z- transform of a discrete- time sequence $x(n)$ is:

$$Z\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}. \tag{3}$$

For causal sequences ($n \geq 0$) the Z- transform becomes:

$$Z\{x(n)\} = X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}. \tag{4}$$

The equation (4) is called the *unilateral* Z- transform; it exists only if the power series from its expression converges.

3

There are several methods for computing the inverse Z- transform, namely the sequence $x(n)$, given $X(z)$:

1. using the *inversion integral*:

$$x(n) = \frac{1}{2\pi j} \oint_{\Gamma} X(z) z^{n-1} dz, \tag{5}$$

   where $\oint_{\Gamma}$ means the integration along the closed contour Γ in the counterclockwise closed contour in the region of convergence of $X(z)$;

2. by a power series expansion: expressing $X(z)$ in a power series in z^{-1}, $x(n)$ can be achieved by identifying it with the coefficient of z^{-n} in the power series expansion;

3. by partial fraction expansion: for a rational functions, can be obtained a partial fraction expansion of $X(z)$ over its poles and the table of Z-transform helps to identify the sequences corresponding to the terms in that partial fraction expansion.

1.3 Application of Genetic Algorithms

The specialists think that the Genetic Algorithms are a computational intelligence application as well as the expert systems, fuzzy systems, neural networks, the intelligent agents, hybrid intelligent systems, electronic voice.

The genetic algorithms are some adaptive techniques of heuristic search, based on the genetic and selection natural principles, enunciated by Darwin (the best adapted will survive). The mechanism is similar to the evolutionary biological process. This process has a feature through that only the species which one adapt better to the environment are capable to survive and to develop into generations, while that those less adapted fail to survive and they disappear in time, as a result of the natural selection. The main notions that allow the analogy between the solution of the search problems and the natural evolution are:

1. *Population.* A population consists in some individuals (*chromosomes*) that have to live in an environment to which they must adapt.

2. *Fitness.* Each of the population individuals is adapted more or less to the environment. The fitness is a measure of the degree of adaptation to the environment.

3. *Chromosome.* It is a ordered set of elements, named *genes*, whose values establish the individual features.

4. *Generation.* A stage in a population evolution. If we see evolution as an iterative process in which a population turns to another population, then the generation is an iteration in this process.

5. *Selection.* The process of natural selection has the survival of individuals with a high environmental fitness (high fitness) as effect.

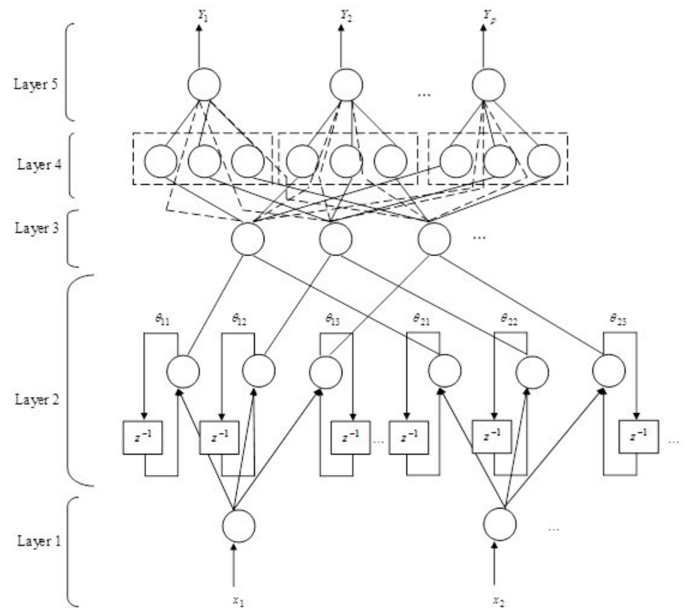

Figure 3: The RNFN architecture.

6. *Reproduction.* It is the process through which one passes from one generation to another. The individuals of the new generation inherit some features from their precursors (parents) but they can also get some new features as a result of some processes of mutation that have a random character. In the case when in the reproduction process at least two parents occur, the inherited features of the survivor (son) are obtained by combining (crossover) of the parent features.

The remainder of the paper is organized as follows. In Section 2 is discussed and analyzed the RNFN. We follow with the learning algorithm of the recurrent model in Section 3. We conclude in Section 4.

2 RNFN Architecture

The network construction is based on fuzzy rules, each corresponding to a Wavelet Neural Network (WNN).

The figure Figure 3 illustrates the RNFN model, whose training algorithm is based on Improved Particle Swarm Optimization (IPSO) method.

The nodes from the first layer constitute some input nodes; hence they only pass the input signal to the next layer, namely:

$$O_i^{(1)} = x_i^{(1)}. \tag{6}$$

The neurons in the second layer act as a membership function, meaning that they determine how an input value belongs to a fuzzy set. The following Gaussian function is chosen as the membership function:

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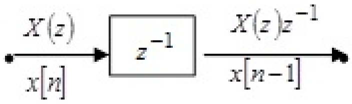

Figure 4: Delayed cell.

$$O_{ij}^{(2)} = \mathrm{e}^{-\frac{\left(I_{ij}^{(2)} - m_{ij}\right)^2}{\sigma_{ij}^2}}, \tag{7}$$

where:

- m_{ij} and σ_{ij} are the mean and standard deviation, respectively;
- $I_{ij}^{(2)}$ denotes the input of this layer for the discrete time scan:

$$I_{ij}^{(2)} = O_i^{(2)} + O_{ij}^{(f)}, \tag{8}$$

where

$$O_{ij}^{(f)} = O_{ij}^{(2)}(t-1)\theta_{ij}. \tag{9}$$

The inputs of this layer contain the terms of memory $O_{ij}^{(2)}(t-1)$, that store network information at a previous time; this information, which is an additional input of the network will be reintroduced at the entrance of the second layer.

The weight θ_{ij} constitutes the feedback weight of the network and z^{-1} signifies the delayed operator.

Figure 4 represents [14] a delayed cell, $X(z)$ being the Z- transform of the signal $x[n]$.

The neurons of the third layer achieve the product operation of their input signals:

$$O_j^3 = \prod_{i=1}^{n} O_{ij}^{(2)} = \prod_{i=1}^{n} \mathrm{e}^{-\frac{\left(I_{ij}^{(2)} - m_{ij}\right)^2}{\sigma_{ij}^2}}, \tag{10}$$

where n is the number of external dimensions.

The neurons of the fourth layer receive both the output of a WNN, denoted $\hat{y}_j$ and of a neuron from the third layer, namely O_j^3. The mathematical function of each node j is:

$$O_j^4 = \hat{y}_j^p \cdot O_j^3, \tag{11}$$

$\hat{y}_j^p$ being the local output of the WNN for the output y_p and the j-th rule:

$$\hat{y}_j^p = \sum_{k=1}^{M} w_{jk}^p \varphi_{a.b}, \tag{12}$$

with $\varphi_{a.b}$ from (2), where:

- $M = m+1$ denotes the number of wavelet bases, which equals the number of existing fuzzy rules in the considered model,

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- the link w_{jk}^p is the output action strength associated with in the p output, j-th rule and k-th $\varphi_{a.b}$.

The fifth layer acts as a defuzzifier namely it provides the nonfuzzy outputs y_p of the fuzzy recurrent neural network:

$$y_p = \frac{1}{1+\mathrm{e}^{-\lambda\cdot\frac{\sum_{j=1}^{M} O_j^4}{\sum_{j=1}^{M} O_j^3}}} = \frac{1}{1+\mathrm{e}^{-\lambda\cdot\frac{\sum_{j=1}^{M} \hat{y}_j^p\cdot O_j^3}{\sum_{j=1}^{M} O_j^3}}}, \tag{13}$$

namely:

$$y_p = \frac{1}{1+\mathrm{e}^{-\lambda\cdot\frac{\sum_{j=1}^{M}(w_{j1}^p\varphi_{1.1}+w_{j2}^p\varphi_{2.1}+\ldots+w_{jM}^p\varphi_{m.m})\cdot O_j^3}{\sum_{j=1}^{M} O_j^3}}},\ \lambda\in\Re. \tag{14}$$

3 Learning Algorithm of RNFN

The training algorithm of the network is based on the Improved optimization method Particle Swarm Optimization (IPSO). The new optimization algorithm called the IPSO enhances the traditional PSO (Particle Swarm Optimization) to enable it to obtain optimal solution capability.

We assume that each particle includes the mean, deviation and weight variables of the RNFN, being d- dimensional.

The following parameters will be determined by the learning procedure:

- the position vector $X_i = (x_{i1}, x_{i2}, \ldots, x_{id})$,
  and respectively
- the velocity vector $V_i = (v_{i1}, v_{i2}, \ldots, v_{id})$

of the i- th particle in the N-dimensional search space.

We denote by:

- $P_i = (P_{i1}, P_{i2}, \ldots, P_{id})$ the best position of each particle,
- $P_g = (P_{g1}, P_{g2}, \ldots, P_{gd})$ the fittest particle found so far,

according to an user-defined fitness function.

The steps of the learning procedure are:

Step 1 (*Individual initialization*). Set the initial values for every particle like being random values.

Step 2 (*Evaluate fitness*). Evaluate each particle in a swarm, by defining the fitness function:

$$f_i = \frac{1}{Y}, \tag{15}$$

where

$$Y = \sqrt{\frac{1}{N}\sum_{p=1}^{N}(y_p - \overline{y}_p)^2}, \tag{16}$$

7

- N represents the number of input data,
- $y_p,\ p = \overline{1, N}$ are the model outputs,
- $\overline{y}_p,\ p = \overline{1, N}$ constitute the desired outputs.

After a generation of learning, we achieve the following fifth best particles, ordered according to their fitness: *unimportant*, *rather unimportant*, *moderately important*, *rather important*, *very important* particles.

The input (preferred) particles are:

1. *unimportant* particle
$$C_u = (C_{u1}, C_{u2}, \dots, C_{ud}),$$
with the fitness F_u;
2. *rather unimportant* particle
$$C_r = (C_{r1}, C_{r2}, \dots, C_{rd}),$$
with the fitness F_r;
3. *moderately important* particle
$$C_m = (C_{m1}, C_{m2}, \dots, C_{md}),$$
with the fitness F_m;
4. *rather important* particle
$$C_R = (C_{R1}, C_{R2}, \dots, C_{Rd}),$$
with the fitness F_R;
5. *very important* particle
$$C_v = (C_{v1}, C_{v2}, \dots, C_{vd}),$$
with the fitness F_v.

The membership functions of the fuzzy terms *unimportant*, *rather unimportant*, *moderately important*, *rather important*, and respectively *very important* can be represented as fuzzy numbers in Figure 5,

being defined in the following relations:

$$\mu_{unimportant} = \begin{cases} 1 - \frac{x}{0.25},\ \text{if } 0 \le x \le 0.25, \\ 0,\ \text{otherwise.} \end{cases} \tag{17}$$

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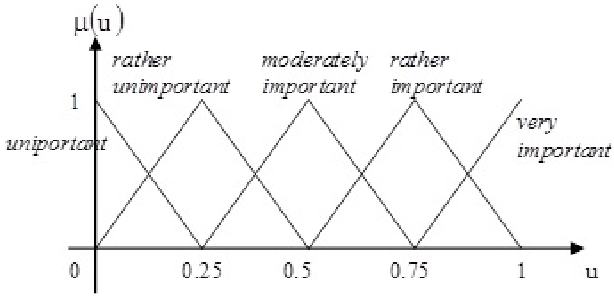

Figure 5: The membership functions of *importance*.

$$\mu_{rather\ unimportant} = \begin{cases} \frac{x}{0.25} \text{ if } 0 \leq x \leq 0.25, \\ 1 - \frac{x-0.25}{0.25} \text{ if } 0.25 < x \leq 0.5, \end{cases} \tag{18}$$

$$\mu_{moderately\ important} = \begin{cases} \frac{x-0.25}{0.25}, \text{ if } 0.25 \leq x \leq 0.5, \\ 1 - \frac{x-0.5}{0.25}, \text{ if } 0.5 < x \leq 0.75, \end{cases} \tag{19}$$

$$\mu_{rather\ important} = \begin{cases} \frac{x-0.5}{0.25}, \text{ if } 0.5 \leq x \leq 0.75, \\ 1 - \frac{x-0.75}{0.25}, \text{ if } 0.75 < x \leq 1, \end{cases} \tag{20}$$

$$\mu_{very\ important} = \begin{cases} \frac{x-0.75}{0.25}, \text{ if } 0.75 \leq x \leq 1, \\ 0, \text{ otherwise.} \end{cases} \tag{21}$$

The output (created) particle is *output particle*:

$$C_o = (C_{o1}, C_{o2}, \dots, C_{od}),$$

with the fitness F_o.

Step 3 (*Improve the capability of finding the global solution* (ICFGS)). Set: $D_1 = D_2 = D_3 = 1$ the magnitudes of the three evolution directions, $T_s = 1$ the initial index of the ICFGS, the number N_L of the ICFGS loop, the fifth particles with the best fitness values from the local best swarm to C_u, C_r, C_m, C_R, C_v.

Use a special equation to update the: *unimportant* particle, *rather unimportant* particle, *moderately important* particle and *rather important* particle to generate the migrant individuals, based on the best individual, $X_i = (x_{i1}, \dots, x_{id})$ in the aim of improving the fitness value [8]:

$$x_{id} = \begin{cases} x_{id} + \rho(x_{id}^L - x_{id}), \text{ if } r_1 < \frac{x_{id} - x_{id}^L}{x_{id}^L - x_{id}^U} \\ x_{id} + \rho(x_{id}^U - x_{id}) \text{ otherwise,} \end{cases} \tag{22}$$

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where ρ and r_1 are random numbers in the range of $[0,1]$ and L, U meaning "lower" and "upper".

Compute C_o:

$$C_{oj} = C_{uj} + D_1(C_{uj} - C_{rj}) + \\ + D_2(C_{uj} - C_{mj}) + D_3(C_{uj} - C_{Rj}) \quad (23)$$

Evaluate the new fitness F_o corresponding to the newly created output particle C_o.

Update the *unimportant* particle C_u, the *rather unimportant* particle C_r, *moderately important* particle C_m, *rather important* particle C_R and the *very important* particle C_v as follows:

(1) If $F_o > F_v$ then

$$\begin{cases} C_v = C_o \\ C_R = C_v \\ C_m = C_R \\ C_r = C_m \\ C_u = C_r. \end{cases}$$

(2) Else if $F_o > F_R$ and $F_o < F_v$ then

$$\begin{cases} C_R = C_o \\ C_m = C_R \\ C_r = C_m \\ C_u = C_r. \end{cases}$$

(3) Else if $F_o > F_m$ and $F_o < F_R$ then

$$\begin{cases} C_m = C_o \\ C_r = C_m \\ C_u = C_r. \end{cases}$$

(4) Else if $F_o > F_r$ and $F_o < F_m$ then

$$\begin{cases} C_r = C_o \\ C_u = C_r. \end{cases}$$

(5) Else if $F_o > F_u$ and $F_o < F_r$ then

$$C_u = C_o.$$

(6) Else if $F_o = F_u = F_r = F_m = F_R = F_v$ then

$$C_o = C_o + N_r \ (N_r \in [0,1]).$$

(7) Else if $F_o <= F_u$ then it will decrease the moving velocity:

$$\begin{cases} D_1 = -0.5D_1 \\ D_2 = -0.5D_2 \\ D_3 = -0.5D_3 \end{cases}$$

to obtain a good fitness.

The random number N_r is added at the statement (23) to prevent the learning algorithm from falling into a local optimum.

Test If *Step 3* isn't finished then $T_s = T_s + 1$; else update the global best: if the fitness value of the new particle is higher than that of the global best, then the global best will also be replaced with the particle.

Step 4 (*Update the velocity and the position*). Update the velocity and the position of all particles along each dimension using the equations:

$$v_{id}^{k+1} = \omega \cdot v_{id}^{k} + c_1 \cdot \mathrm{rand}(\cdot)(P_{id} - x_{id}^{k}) + c_2 \cdot \mathrm{rand}(\cdot)(P_{gd} - x_{id}^{k}) \tag{24}$$

$$x_{id}^{k+1} = x_{id}^{k} + v_{id}^{k+1}, \tag{25}$$

where: w is the coefficient of the inertia term; c_1 and c_2 are called the cognitive term and the society term, respectively; the function rand$(\cdot)$ yields uniformly distributed random numbers in $[0, 1]$.

The second term from (24) known as the cognitive component, represents the personal thinking of each particle, which encourages the particles to move toward their own best positions. The third term from (24) called the social component represents the collaborative effect of the particles, in finding the global optimal solution.

4 Conclusion

Dynamic or Recurrent Neural Networks (RNNs) are unlike from static neural networks since they include feedback or recurrent connections between the network layers and within the layer itself.

The learning algorithm of the Recurrent Neural Fuzzy Network (RNFN) model presented in this paper is based on the Improved Particle Swarm Optimization (IPSO) method, which is similar to evolutionary algorithms, but requires less computational bookkeeping and generally fewer lines of code. The new optimization algorithm called the IPSO enhances the traditional PSO (Particle Swarm Optimization) to enable it to obtain optimal solution capability.

The RFNN presented in this paper is unlike the others variants of RFNN models, by the number of the evolution directions that they use: in this paper, we update the velocity and the position of all particles along three dimensions.

The network construction is based on fuzzy rules, each corresponding to a WNN (Wavelet Neural Network).

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Qualitative Behavior of some Rational Difference Equations

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Abstract

We obtain in this paper the analytical forms of the solutions for the following difference equations

$$x_{n+1} = \frac{x_{n-1}x_{n-4}}{x_{n-2}(\pm 1 \pm x_{n-1}x_{n-4})}, \quad n = 0, 1, ...,$$

where the initial conditions are arbitrary real numbers. Also, we study the dynamics behavior of the solutions of the considered equations.

Keywords: difference equations, recursive sequences, stability, periodic solution.
Mathematics Subject Classification: 39A10

1 Introduction

The behavior of the solutions of the difference equations has been investigated by many authors, see for examples: Agarwal et al. [1] investigated the global stability, periodicity character and gave the solution of some special cases of the difference equations

$$x_{n+1} = ax_n + \frac{bx_nx_{n-3}}{cx_{n-2}+dx_{n-3}}.$$

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Cinar [2] investigated the solutions of the following difference equation

$$x_{n+1} = \frac{ax_{n-1}}{1+bx_n x_{n-1}}.$$

Elabbasy et al. [3] investigated the global attractivity of the equilibrium point and the asymptotic behavior of the solutions of the following difference equation and gave the solution of some special cases of the difference equation

$$x_{n+1} = \frac{ax_{n-l}x_{n-k}}{bx_{n-p}-cx_{n-q}}.$$

Elsayed [8] deal with some properties of the solutions of the difference equation

$$x_{n+1} = ax_n + \frac{bx_n}{cx_n - dx_{n-1}},$$

and obtained the form of the solution of special case of this difference equation

Karatas et al. [11] get the form of the solution of the difference equation

$$x_{n+1} = \frac{x_{n-5}}{1+x_{n-2}x_{n-5}}.$$

In [14] Wang et al. investigated the global attractivity of the equilibrium point, and the asymptotic behavior of the solutions of the following difference equation

$$x_{n+1} = \frac{\sum_{i=1}^{s} A_{k_i} x_{n-k_i}}{B_0 + \sum_{j=1}^{t} B_{l_j} x_{n-l_j}}.$$

In [15] Yalçınkaya studied the behavior of the following difference equation

$$x_{n+1} = \alpha + \frac{x_{n-m}}{x_n^k}.$$

Zayed et al. [17] studied a qualitative behavior of the rational recursive sequence

$$x_{n+1} = \frac{\alpha+\beta x_n+\gamma x_{n-1}}{A+Bx_n+Cx_{n-1}}.$$

Other related results on rational difference equations can be found in refs. [4-16].

In this paper, we study the existence of the analytical solutions for the following difference equations

$$x_{n+1} = \frac{x_{n-1}x_{n-4}}{x_{n-2}(\pm 1 \pm x_{n-1}x_{n-4})}, \quad n = 0, 1, ..., \tag{1}$$

where the initial conditions are arbitrary real numbers. Also, we study the global behavior of the solutions.

The following theorem will be useful in our current study.

Theorem A [13]: Assume that $p_i \in R,\ i = 1, 2, ..., k$ and $k \in \{0, 1, 2, ...\}$. Then

$$\sum_{i=1}^{k} |p_i| < 1,$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+k} + p_1 x_{n+k-1} + ... + p_k x_n = 0, \quad n = 0, 1, ...$$

In the following we investigate the behavior of the solutions for some different cases of Eq.(1).

2 On the Equation $x_{n+1} = \frac{x_{n-1}x_{n-4}}{x_{n-2}(1+x_{n-1}x_{n-4})}$

In this section we give a specific form of the solution of the equation

$$x_{n+1} = \frac{x_{n-1}x_{n-4}}{x_{n-2}(1+x_{n-1}x_{n-4})}, \quad n = 0, 1, ..., \tag{2}$$

where the initial values are arbitrary positive real numbers.

Theorem 1 *Let $\{x_n\}_{n=-4}^{\infty}$ be a solution of Eq.(2). Then for $n = 0, 1, ...$*

$$x_{6n-2} = \frac{x_{-2}x_0^n x_{-3}^n}{x_{-1}^n x_{-4}^n} \prod_{i=0}^{n-1} \left(\frac{(1+(3i+1)x_{-1}x_{-4})}{(1+(3i+2)x_0x_{-3})} \right), \ x_{6n-1} = \frac{x_{-1}^{n+1}x_{-4}^n}{x_0^n x_{-3}^n} \prod_{i=0}^{n-1} \left(\frac{(1+(3i+1)x_0x_{-3})}{(1+(3i+3)x_{-1}x_{-4})} \right),$$

$$x_{6n} = \frac{x_0^{n+1}x_{-3}^n}{x_{-1}^n x_{-4}^n} \prod_{i=0}^{n-1} \left(\frac{(1+(3i+2)x_{-1}x_{-4})}{(1+(3i+3)x_0x_{-3})} \right),$$

$$x_{6n+1} = \frac{x_{-1}^{n+1}x_{-4}^{n+1}}{x_{-2}x_0^n x_{-3}^n(1+x_{-1}x_{-4})} \prod_{i=0}^{n-1} \left(\frac{(1+(3i+2)x_0x_{-3})}{(1+(3i+4)x_{-1}x_{-4})} \right),$$

$$x_{6n+2} = \frac{x_0^{n+1}x_{-3}^{n+1}}{x_{-1}^{n+1}x_{-4}^n(1+x_0x_{-3})} \prod_{i=0}^{n-1} \left(\frac{(1+(3i+3)x_{-1}x_{-4})}{(1+(3i+4)x_0x_{-3})} \right),$$

$$x_{6n+3} = \frac{x_{-1}^{n+1}x_{-4}^{n+1}}{x_0^{n+1}x_{-3}^n(1+2x_{-1}x_{-4})} \prod_{i=0}^{n-1} \left(\frac{(1+(3i+3)x_0x_{-3})}{(1+(3i+5)x_{-1}x_{-4})} \right).$$

Proof: For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 1$. That is;

$$x_{6n-8} = \frac{x_{-2}x_0^{n-1}x_{-3}^{n-1}}{x_{-1}^{n-1}x_{-4}^{n-1}} \prod_{i=0}^{n-2} \frac{(1+(3i+1)x_{-1}x_{-4})}{(1+(3i+2)x_0x_{-3})}, \ x_{6n-7} = \frac{x_{-1}^n x_{-4}^{n-1}}{x_0^{n-1}x_{-3}^{n-1}} \prod_{i=0}^{n-2} \frac{(1+(3i+1)x_0x_{-3})}{(1+(3i+3)x_{-1}x_{-4})},$$

$$x_{6n-6} = \frac{x_0^n x_{-3}^{n-1}}{x_{-1}^{n-1}x_{-4}^{n-1}} \prod_{i=0}^{n-2} \frac{1+(3i+2)x_{-1}x_{-4}}{1+(3i+3)x_0x_{-3}}, \ x_{6n-5} = \frac{x_{-1}^n x_{-4}^n}{x_{-2}x_0^{n-1}x_{-3}^{n-1}(1+x_{-1}x_{-4})} \prod_{i=0}^{n-2} \frac{1+(3i+2)x_0x_{-3}}{1+(3i+4)x_{-1}x_{-4}},$$

$$x_{6n-4} = \frac{x_0^n x_{-3}^n}{x_{-1}^n x_{-4}^{n-1}(1+x_0x_{-3})} \prod_{i=0}^{n-2} \left(\frac{(1+(3i+3)x_{-1}x_{-4})}{(1+(3i+4)x_0x_{-3})} \right),$$

$$x_{6n-3} = \frac{x_{-1}^n x_{-4}^n}{x_0^n x_{-3}^{n-1}(1+2x_{-1}x_{-4})} \prod_{i=0}^{n-2} \left(\frac{(1+(3i+3)x_0x_{-3})}{(1+(3i+5)x_{-1}x_{-4})} \right).$$

Now, it follows from Eq.(2) that

$$x_{6n-2} = \frac{x_{6n-4}x_{6n-7}}{x_{6n-5}(1+x_{6n-4}x_{6n-7})}$$

$$= \frac{\frac{x_0^n x_{-3}^n}{x_{-1}^n x_{-4}^{n-1}(1+x_0x_{-3})} \prod_{i=0}^{n-2}\left(\frac{(1+(3i+3)x_{-1}x_{-4})}{(1+(3i+4)x_0x_{-3})}\right) \frac{x_{-1}^n x_{-4}^{n-1}}{x_0^{n-1}x_{-3}^{n-1}} \prod_{i=0}^{n-2}\left(\frac{(1+(3i+1)x_0x_{-3})}{(1+(3i+3)x_{-1}x_{-4})}\right)}{\left(\frac{x_{-1}^n x_{-4}^n}{x_{-2}x_0^{n-1}x_{-3}^{n-1}(1+x_{-1}x_{-4})} \prod_{i=0}^{n-2}\left(\frac{(1+(3i+2)x_0x_{-3})}{(1+(3i+4)x_{-1}x_{-4})}\right)\right) \left(1+\frac{x_0^n x_{-3}^n}{x_{-1}^n x_{-4}^{n-1}(1+x_0x_{-3})} \prod_{i=0}^{n-2}\left(\frac{(1+(3i+3)x_{-1}x_{-4})}{(1+(3i+4)x_0x_{-3})}\right) \frac{x_{-1}^n x_{-4}^{n-1}}{x_0^{n-1}x_{-3}^{n-1}} \prod_{i=0}^{n-2}\left(\frac{(1+(3i+1)x_0x_{-3})}{(1+(3i+3)x_{-1}x_{-4})}\right)\right)}$$

$$= \frac{\frac{x_0x_{-3}}{(1+x_0x_{-3})} \prod_{i=0}^{n-2}\left(\frac{(1+(3i+1)x_0x_{-3})}{(1+(3i+4)x_0x_{-3})}\right)}{\left(\frac{x_{-1}^n x_{-4}^n}{x_{-2}x_0^{n-1}x_{-3}^{n-1}(1+x_{-1}x_{-4})} \prod_{i=0}^{n-2}\left(\frac{(1+(3i+2)x_0x_{-3})}{(1+(3i+4)x_{-1}x_{-4})}\right)\right)\left(1+\frac{x_0x_{-3}}{(1+x_0x_{-3})} \prod_{i=0}^{n-2}\left(\frac{(1+(3i+1)x_0x_{-3})}{(1+(3i+4)x_0x_{-3})}\right)\right)}$$

$$= \left(\frac{x_{-2}x_0^{n-1}x_{-3}^{n-1}(1+x_{-1}x_{-4})}{x_{-1}^n x_{-4}^n}\right) \prod_{i=0}^{n-2}\left(\frac{(1+(3i+4)x_{-1}x_{-4})}{(1+(3i+2)x_0x_{-3})}\right) \frac{\frac{x_0x_{-3}}{(1+(3n-2)x_0x_{-3})}}{\left(1+\frac{x_0x_{-3}}{(1+(3n-2)x_0x_{-3})}\right)}$$

$$= \left(\frac{x_{-2}x_0^n x_{-3}^n(1+x_{-1}x_{-4})}{x_{-1}^n x_{-4}^n}\right) \prod_{i=0}^{n-2}\left(\frac{(1+(3i+4)x_{-1}x_{-4})}{(1+(3i+2)x_0x_{-3})}\right) \frac{1}{(1+(3n-1)x_0x_{-3})}.$$

Hence, we have

$$x_{6n-2} = \frac{x_{-2}x_0^n x_{-3}^n}{x_{-1}^n x_{-4}^n} \prod_{i=0}^{n-1}\left(\frac{(1+(3i+1)x_{-1}x_{-4})}{(1+(3i+2)x_0x_{-3})}\right).$$

Similarly

$$x_{6n-1} = \frac{x_{6n-3}x_{6n-6}}{x_{6n-4}(1+x_{6n-3}x_{6n-6})}$$

$$= \frac{\frac{x_{-1}^n x_{-4}^n}{x_0^n x_{-3}^{n-1}(1+2x_{-1}x_{-4})} \prod_{i=0}^{n-2}\left(\frac{(1+(3i+3)x_0x_{-3})}{(1+(3i+5)x_{-1}x_{-4})}\right) \frac{x_0^n x_{-3}^{n-1}}{x_{-1}^{n-1}x_{-4}^{n-1}} \prod_{i=0}^{n-2}\left(\frac{(1+(3i+2)x_{-1}x_{-4})}{(1+(3i+3)x_0x_{-3})}\right)}{\left(\frac{x_0^n x_{-3}^n}{x_{-1}^n x_{-4}^{n-1}(1+x_0x_{-3})} \prod_{i=0}^{n-2}\left(\frac{(1+(3i+3)x_{-1}x_{-4})}{(1+(3i+4)x_0x_{-3})}\right)\right)\left(1+\frac{x_{-1}^n x_{-4}^n}{x_0^n x_{-3}^{n-1}(1+2x_{-1}x_{-4})} \prod_{i=0}^{n-2}\left(\frac{(1+(3i+3)x_0x_{-3})}{(1+(3i+5)x_{-1}x_{-4})}\right) \frac{x_0^n x_{-3}^{n-1}}{x_{-1}^{n-1}x_{-4}^{n-1}} \prod_{i=0}^{n-2}\left(\frac{(1+(3i+2)x_{-1}x_{-4})}{(1+(3i+3)x_0x_{-3})}\right)\right)}$$

$$= \frac{\frac{x_{-1}x_{-4}}{(1+2x_{-1}x_{-4})} \prod_{i=0}^{n-2}\left(\frac{(1+(3i+2)x_{-1}x_{-4})}{(1+(3i+5)x_{-1}x_{-4})}\right)}{\left(\frac{x_0^n x_{-3}^n}{x_{-1}^n x_{-4}^{n-1}(1+x_0x_{-3})} \prod_{i=0}^{n-2}\left(\frac{(1+(3i+3)x_{-1}x_{-4})}{(1+(3i+4)x_0x_{-3})}\right)\right)\left(1+\frac{x_{-1}x_{-4}}{(1+2x_{-1}x_{-4})} \prod_{i=0}^{n-2}\left(\frac{(1+(3i+2)x_{-1}x_{-4})}{(1+(3i+5)x_{-1}x_{-4})}\right)\right)}$$

$$= \left(\frac{x_{-1}^n x_{-4}^{n-1}(1+x_0x_{-3})}{x_0^n x_{-3}^n}\right) \prod_{i=0}^{n-2}\left(\frac{(1+(3i+4)x_0x_{-3})}{(1+(3i+3)x_{-1}x_{-4})}\right) \frac{\frac{x_{-1}x_{-4}}{(1+(3n-1)x_{-1}x_{-4})}}{\left(1+\frac{x_{-1}x_{-4}}{(1+(3n-1)x_{-1}x_{-4})}\right)}$$

$$= \left(\frac{x_{-1}^n x_{-4}^{n-1}(1+x_0x_{-3})}{x_0^n x_{-3}^n}\right) \prod_{i=0}^{n-2}\left(\frac{(1+(3i+4)x_0x_{-3})}{(1+(3i+3)x_{-1}x_{-4})}\right) \frac{x_{-1}x_{-4}}{(1+(3n)x_{-1}x_{-4})}.$$

Hence, we have

$$x_{6n-1} = \frac{x_{-1}^{n+1}x_{-4}^n}{x_0^n x_{-3}^n} \prod_{i=0}^{n-1}\left(\frac{(1+(3i+1)x_0x_{-3})}{(1+(3i+3)x_{-1}x_{-4})}\right).$$

Similarly, we can easily obtain the other relations. Thus, the proof is completed.

Theorem 2 *Eq.(2) has $\overline{x}=0$ as a unique equilibrium point and it is unstable.*

Proof: For the equilibrium points of Eq.(2), set

$$\overline{x}=\frac{\overline{x}^2}{\overline{x}\left(1+\overline{x}^2\right)}.$$

Then

$$\overline{x}^2\left(1+\overline{x}^2\right)=\overline{x}^2, \quad \Rightarrow \quad \overline{x}^2\left(1+\overline{x}^2-1\right)=0, \;\Rightarrow\; \overline{x}^4=0.$$

Thus the equilibrium point of Eq.(2) is $\overline{x}=0$.
Let $f:(0,\infty)^3\longrightarrow(0,\infty)$ be a function defined by

$$f(t,u,v,w)=\frac{vw}{u(1+vw)}.$$

Thus the linearized equation of Eq.(2) about the equilibrium point $\overline{x}$ is given by

$$y_{n+1}=\sum_{i=0}^{4}\frac{\partial f(\overline{x},\overline{x},\overline{x},\overline{x})}{\partial x_{n-i}}.$$

The proof follows by Theorem A.

Numerical examples

For confirming the results of this section, we consider some numerical examples which represent different types of solutions to Eq.(2).

Example 1. Consider Eq.(2) with $x_{-4}=0.21,\ x_{-3}=2,\ x_{-2}=0.5,\ x_{-1}=7,\ x_0=0.3$. See Fig. 1.

Example 2. Consider Eq.(2) with $x_{-4}=9,\ x_{-3}=2,\ x_{-2}=6,\ x_{-1}=7,\ x_0=3$. See Fig. 2.

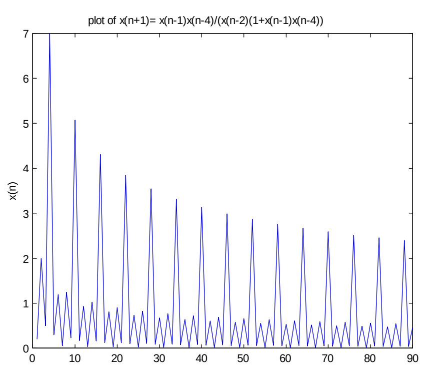

Figure 1.

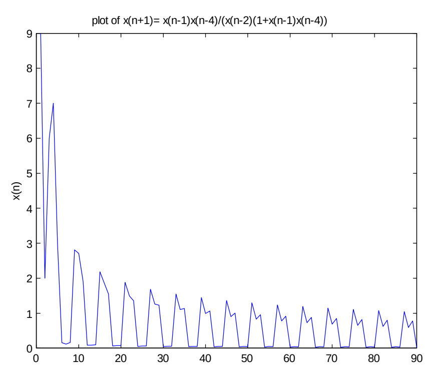

Figure 2.

5

3 On the Equation $x_{n+1}=\frac{x_{n-1}x_{n-4}}{x_{n-2}(-1+x_{n-1}x_{n-4})}$

In this section we obtain the solution of the difference equation

$$x_{n+1}=\frac{x_{n-1}x_{n-4}}{x_{n-2}(-1+x_{n-1}x_{n-4})},\quad n=0,1,..., \tag{3}$$

where the initial values are arbitrary non zero real numbers with $x_{-1}x_{-4}\neq 1,\ x_{-3}x_0\neq 1$.

Theorem 3 *Every solution $\{x_n\}_{n=-4}^{\infty}$ of Eq.(3) has the form*

$$x_{12n-4}=\frac{x_0^{2n}x_{-3}^{2n}}{x_{-1}^{2n}x_{-4}^{2n-1}}\frac{(-1+x_{-1}x_{-4})^n}{(-1+x_{-3}x_0)^n},\qquad x_{12n-3}=\frac{x_{-1}^{2n}x_{-4}^{2n}}{x_0^{2n}x_{-3}^{2n-1}}\frac{(-1+x_{-3}x_0)^n}{(-1+x_{-1}x_{-4})^n},$$

$$x_{12n-2}=\frac{x_{-2}x_0^{2n}x_{-3}^{2n}}{x_{-1}^{2n}x_{-4}^{2n}}\frac{(-1+x_{-1}x_{-4})^n}{(-1+x_{-3}x_0)^n},\qquad x_{12n-1}=\frac{x_{-1}^{2n+1}x_{-4}^{2n}}{x_0^{2n}x_{-3}^{2n}}\frac{(-1+x_{-3}x_0)^n}{(-1+x_{-1}x_{-4})^n},$$

$$x_{12n}=\frac{x_0^{2n+1}x_{-3}^{2n}}{x_{-1}^{2n}x_{-4}^{2n}}\frac{(-1+x_{-1}x_{-4})^n}{(-1+x_{-3}x_0)^n},\qquad x_{12n+1}=\frac{x_{-1}^{2n+1}x_{-4}^{2n+1}}{x_{-2}x_0^{2n}x_{-3}^{2n}}\frac{(-1+x_{-3}x_0)^n}{(-1+x_{-1}x_{-4})^{n+1}},$$

$$x_{12n+2}=\frac{x_0^{2n+1}x_{-3}^{2n+1}}{x_{-1}^{2n+1}x_{-4}^{2n}}\frac{(-1+x_{-1}x_{-4})^n}{(-1+x_{-3}x_0)^{n+1}},\qquad x_{12n+3}=\frac{x_{-1}^{2n+1}x_{-4}^{2n+1}}{x_0^{2n+1}x_{-3}^{2n}}\frac{(-1+x_{-3}x_0)^n}{(-1+x_{-1}x_{-4})^n},$$

$$x_{12n+4}=\frac{x_{-2}x_0^{2n+1}x_{-3}^{2n+1}}{x_{-1}^{2n+1}x_{-4}^{2n+1}}\frac{(-1+x_{-1}x_{-4})^{n+1}}{(-1+x_{-3}x_0)^n},\qquad x_{12n+5}=\frac{x_{-1}^{2n+2}x_{-4}^{2n+1}}{x_0^{2n+1}x_{-3}^{2n+1}}\frac{(-1+x_{-3}x_0)^{n+1}}{(-1+x_{-1}x_{-4})^{n+1}},$$

$$x_{12n+6}=\frac{x_0^{2n+2}x_{-3}^{2n+1}}{x_{-1}^{2n+1}x_{-4}^{2n+1}}\frac{(-1+x_{-1}x_{-4})^n}{(-1+x_{-3}x_0)^{n+1}},\qquad x_{12n+7}=\frac{x_{-1}^{2n+2}x_{-4}^{2n+2}}{x_{-2}x_0^{2n+1}x_{-3}^{2n+1}}\frac{(-1+x_{-3}x_0)^n}{(-1+x_{-1}x_{-4})^{n+1}}.$$

Proof: For $n=0$ the result holds. Now suppose that $n>0$ and that our assumption holds for $n-1$. That is;

$$x_{12n-16}=\frac{x_0^{2n-2}x_{-3}^{2n-2}}{x_{-1}^{2n-2}x_{-4}^{2n-3}}\frac{(-1+x_{-1}x_{-4})^{n-1}}{(-1+x_{-3}x_0)^{n-1}},\qquad x_{12n-15}=\frac{x_{-1}^{2n-2}x_{-4}^{2n-2}}{x_0^{2n-2}x_{-3}^{2n-3}}\frac{(-1+x_{-3}x_0)^{n-1}}{(-1+x_{-1}x_{-4})^{n-1}},$$

$$x_{12n-14}=\frac{x_{-2}x_0^{2n-2}x_{-3}^{2n-2}}{x_{-1}^{2n-2}x_{-4}^{2n-2}}\frac{(-1+x_{-1}x_{-4})^{n-1}}{(-1+x_{-3}x_0)^{n-1}},\qquad x_{12n-13}=\frac{x_{-1}^{2n-1}x_{-4}^{2n-2}}{x_0^{2n-2}x_{-3}^{2n-2}}\frac{(-1+x_{-3}x_0)^{n-1}}{(-1+x_{-1}x_{-4})^{n-1}},$$

$$x_{12n-12}=\frac{x_0^{2n-1}x_{-3}^{2n-2}}{x_{-1}^{2n-2}x_{-4}^{2n-2}}\frac{(-1+x_{-1}x_{-4})^{n-1}}{(-1+x_{-3}x_0)^{n-1}},\qquad x_{12n-11}=\frac{x_{-1}^{2n-1}x_{-4}^{2n-1}}{x_{-2}x_0^{2n-2}x_{-3}^{2n-2}}\frac{(-1+x_{-3}x_0)^{n-1}}{(-1+x_{-1}x_{-4})^n},$$

$$x_{12n-10}=\frac{x_0^{2n-1}x_{-3}^{2n-1}}{x_{-1}^{2n-1}x_{-4}^{2n-2}}\frac{(-1+x_{-1}x_{-4})^{n-1}}{(-1+x_{-3}x_0)^n},\qquad x_{12n-9}=\frac{x_{-1}^{2n-1}x_{-4}^{2n-1}}{x_0^{2n-1}x_{-3}^{2n-2}}\frac{(-1+x_{-3}x_0)^{n-1}}{(-1+x_{-1}x_{-4})^{n-1}},$$

$$x_{12n-8}=\frac{x_{-2}x_0^{2n-1}x_{-3}^{2n-1}}{x_{-1}^{2n-1}x_{-4}^{2n-1}}\frac{(-1+x_{-1}x_{-4})^n}{(-1+x_{-3}x_0)^{n-1}},\qquad x_{12n-7}=\frac{x_{-1}^{2n}x_{-4}^{2n-1}}{x_0^{2n-1}x_{-3}^{2n-1}}\frac{(-1+x_{-3}x_0)^n}{(-1+x_{-1}x_{-4})^n},$$

$$x_{12n-6}=\frac{x_0^{2n}x_{-3}^{2n-1}}{x_{-1}^{2n-1}x_{-4}^{2n-1}}\frac{(-1+x_{-1}x_{-4})^{n-1}}{(-1+x_{-3}x_0)^n},\qquad x_{12n-5}=\frac{x_{-1}^{2n}x_{-4}^{2n}}{x_{-2}x_0^{2n-1}x_{-3}^{2n-1}}\frac{(-1+x_{-3}x_0)^{n-1}}{(-1+x_{-1}x_{-4})^n}.$$

Now, it follows from Eq.(3) that

$$\begin{aligned}
x_{12n-4} &= \frac{x_{12n-6}x_{12n-9}}{x_{12n-7}(-1+x_{12n-6}x_{12n-9})}\\
&= \frac{\left(\frac{x_0^{2n}x_{-3}^{2n-1}}{x_{-1}^{2n-1}x_{-4}^{2n-1}}\frac{(-1+x_{-1}x_{-4})^{n-1}}{(-1+x_{-3}x_0)^n}\right)\left(\frac{x_{-1}^{2n-1}x_{-4}^{2n-1}}{x_0^{2n-1}x_{-3}^{2n-2}}\frac{(-1+x_{-3}x_0)^{n-1}}{(-1+x_{-1}x_{-4})^{n-1}}\right)}{\frac{x_{-1}^{2n}x_{-4}^{2n-1}}{x_0^{2n-1}x_{-3}^{2n-1}}\frac{(-1+x_{-3}x_0)^n}{(-1+x_{-1}x_{-4})^n}\left(-1+\frac{x_0^{2n}x_{-3}^{2n-1}}{x_{-1}^{2n-1}x_{-4}^{2n-1}}\frac{(-1+x_{-1}x_{-4})^{n-1}}{(-1+x_{-3}x_0)^n}\frac{x_{-1}^{2n-1}x_{-4}^{2n-1}}{x_0^{2n-1}x_{-3}^{2n-2}}\frac{(-1+x_{-3}x_0)^{n-1}}{(-1+x_{-1}x_{-4})^{n-1}}\right)}\\
&= \frac{\left(\frac{x_0x_{-3}}{(-1+x_{-3}x_0)}\right)}{\left(\frac{x_{-1}^{2n}x_{-4}^{2n-1}}{x_0^{2n-1}x_{-3}^{2n-1}}\frac{(-1+x_{-3}x_0)^n}{(-1+x_{-1}x_{-4})^n}\right)\left(-1+\left(\frac{x_0x_{-3}}{(-1+x_{-3}x_0)}\right)\right)} = \frac{x_0^{2n}x_{-3}^{2n}}{x_{-1}^{2n}x_{-4}^{2n-1}}\frac{(-1+x_{-1}x_{-4})^n}{(-1+x_{-3}x_0)^n},
\end{aligned}$$

$$\begin{aligned}
x_{12n-3} &= \frac{x_{12n-5}x_{12n-8}}{x_{12n-6}(-1+x_{12n-5}x_{12n-8})} \\
&= \frac{\frac{x_{-1}^{2n}x_{-4}^{2n}}{x_{-2}x_0^{2n-1}x_{-3}^{2n-1}}\frac{(-1+x_{-3}x_0)^{n-1}}{(-1+x_{-1}x_{-4})^{n}}\frac{x_{-2}x_0^{2n-1}x_{-3}^{2n-1}}{x_{-1}^{2n-1}x_{-4}^{2n-1}}\frac{(-1+x_{-1}x_{-4})^{n}}{(-1+x_{-3}x_0)^{n-1}}}{\frac{x_0^{2n}x_{-3}^{2n-1}(-1+x_{-1}x_{-4})^{n-1}}{x_{-1}^{2n-1}x_{-4}^{2n-1}(-1+x_{-3}x_0)^{n}}\left(-1+\frac{x_{-1}^{2n}x_{-4}^{2n}(-1+x_{-3}x_0)^{n-1}}{x_{-2}x_0^{2n-1}x_{-3}^{2n-1}(-1+x_{-1}x_{-4})^{n}}\frac{x_{-2}x_0^{2n-1}x_{-3}^{2n-1}(-1+x_{-1}x_{-4})^{n}}{x_{-1}^{2n-1}x_{-4}^{2n-1}(-1+x_{-3}x_0)^{n-1}}\right)} \\
&= \frac{x_{-1}^{2n-1}x_{-4}^{2n-1}}{x_0^{2n}x_{-3}^{2n-1}}\frac{(-1+x_{-3}x_0)^{n}}{(-1+x_{-1}x_{-4})^{n-1}}\frac{x_{-1}x_{-4}}{(-1+x_{-1}x_{-4})} = \frac{x_{-1}^{2n}x_{-4}^{2n}}{x_0^{2n}x_{-3}^{2n-1}}\frac{(-1+x_{-3}x_0)^{n}}{(-1+x_{-1}x_{-4})^{n}},
\end{aligned}$$

Similarly, we can easily obtain the other relations. Thus, the proof is completed.

Theorem 4 *Eq.(3) has three equilibrium points which are $\overline{x}=0$ and $\overline{x}=\pm\sqrt{2}$and all of them are unstable.*

Proof: The proof is similar to Theorem 2 and will be omitted.

Lemma 1. It is easy to see that every solution of Eq.(3) is unbounded except in the case $x_{-3}x_0 = x_{-1}x_{-4}$.

Theorem 5 *Eq.(3) has a periodic solution of period twelve iff $x_{-3}x_0 = x_{-1}x_{-4}$. Moreover the periodic solution has the following form*

$$\left\{\begin{array}{c} x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0, \frac{x_{-1}x_{-4}}{x_{-2}(-1+x_{-1}x_{-4})}, \frac{x_0x_{-3}}{x_{-1}(-1+x_{-3}x_0)}, x_{-3}, x_{-2}\left(-1+x_{-1}x_{-4}\right), \\ x_{-1}, \frac{x_0}{(-1+x_{-3}x_0)}, \frac{x_{-1}x_{-4}}{x_{-2}(-1+x_{-1}x_{-4})}, x_{-4}, x_{-3}, \ldots \end{array}\right\}.$$

Proof: First suppose that there exists a prime period twelve solution of Eq.(3) of the following form

$$\left\{\begin{array}{c} x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0, \frac{x_{-1}x_{-4}}{x_{-2}(-1+x_{-1}x_{-4})}, \frac{x_0x_{-3}}{x_{-1}(-1+x_{-3}x_0)}, x_{-3}, x_{-2}\left(-1+x_{-1}x_{-4}\right), \\ x_{-1}, \frac{x_0}{(-1+x_{-3}x_0)}, \frac{x_{-1}x_{-4}}{x_{-2}(-1+x_{-1}x_{-4})}, x_{-4}, x_{-3}, \ldots \end{array}\right\}.$$

Then we see from Theorem 3 that

$$\begin{aligned}
x_{12n-4} &= \frac{x_0^{2n}x_{-3}^{2n}}{x_{-1}^{2n}x_{-4}^{2n-1}}\frac{(-1+x_{-1}x_{-4})^{n}}{(-1+x_{-3}x_0)^{n}} = x_{-4}, \quad x_{12n-3} = \frac{x_{-1}^{2n}x_{-4}^{2n}}{x_0^{2n}x_{-3}^{2n-1}}\frac{(-1+x_{-3}x_0)^{n}}{(-1+x_{-1}x_{-4})^{n}} = x_{-3}, \\
x_{12n-2} &= \frac{x_{-2}x_0^{2n}x_{-3}^{2n}}{x_{-1}^{2n}x_{-4}^{2n}}\frac{(-1+x_{-1}x_{-4})^{n}}{(-1+x_{-3}x_0)^{n}} = x_{-2}, \quad x_{12n-1} = \frac{x_{-1}^{2n+1}x_{-4}^{2n}}{x_0^{2n}x_{-3}^{2n}}\frac{(-1+x_{-3}x_0)^{n}}{(-1+x_{-1}x_{-4})^{n}} = x_{-1}, \\
x_{12n} &= \frac{x_0^{2n+1}x_{-3}^{2n}(-1+x_{-1}x_{-4})^{n}}{x_{-1}^{2n}x_{-4}^{2n}(-1+x_{-3}x_0)^{n}} = x_0, x_{12n+1} = \frac{x_{-1}^{2n+1}x_{-4}^{2n+1}(-1+x_{-3}x_0)^{n}}{x_{-2}x_0^{2n}x_{-3}^{2n}(-1+x_{-1}x_{-4})^{n+1}} = \frac{x_{-1}x_{-4}}{x_{-2}(-1+x_{-1}x_{-4})}, \\
x_{12n+2} &= \frac{x_0^{2n+1}x_{-3}^{2n+1}}{x_{-1}^{2n+1}x_{-4}^{2n}}\frac{(-1+x_{-1}x_{-4})^{n}}{(-1+x_{-3}x_0)^{n+1}} = \frac{x_0x_{-3}}{x_{-1}(-1+x_{-3}x_0)}, \\
x_{12n+3} &= \frac{x_{-1}^{2n+1}x_{-4}^{2n+1}}{x_0^{2n+1}x_{-3}^{2n}}\frac{(-1+x_{-3}x_0)^{n}}{(-1+x_{-1}x_{-4})^{n}} = x_{-3}, \\
x_{12n+4} &= \frac{x_{-2}x_0^{2n+1}x_{-3}^{2n+1}}{x_{-1}^{2n+1}x_{-4}^{2n+1}}\frac{(-1+x_{-1}x_{-4})^{n+1}}{(-1+x_{-3}x_0)^{n}} = x_{-2}\left(-1+x_{-1}x_{-4}\right), \\
x_{12n+5} &= \frac{x_{-1}^{2n+2}x_{-4}^{2n+1}}{x_0^{2n+1}x_{-3}^{2n+1}}\frac{(-1+x_{-3}x_0)^{n+1}}{(-1+x_{-1}x_{-4})^{n+1}} = x_{-1}, \\
x_{12n+6} &= \frac{x_0^{2n+2}x_{-3}^{2n+1}}{x_{-1}^{2n+1}x_{-4}^{2n+1}}\frac{(-1+x_{-1}x_{-4})^{n}}{(-1+x_{-3}x_0)^{n+1}} = \frac{x_0}{(-1+x_{-3}x_0)}, \\
x_{12n+7} &= \frac{x_{-1}^{2n+2}x_{-4}^{2n+2}}{x_{-2}x_0^{2n+1}x_{-3}^{2n+1}}\frac{(-1+x_{-3}x_0)^{n}}{(-1+x_{-1}x_{-4})^{n+1}} = \frac{x_{-1}x_{-4}}{x_{-2}(-1+x_{-1}x_{-4})}.
\end{aligned}$$

Then we get $\left(-1+x_{-3}x_{0}\right)=\left(-1+x_{-1}x_{-4}\right)$.
Second assume that $\left(-1+x_{-3}x_{0}\right)=\left(-1+x_{-1}x_{-4}\right)$. Then we see from the form of the solution of Eq.(3) that

$$\begin{aligned}
x_{12n-4} &= x_{-4}, \quad x_{12n-3}=x_{-3}, \quad x_{12n-2}=x_{-2}, \quad x_{12n-1}=x_{-1}, \quad x_{12n}=x_{0},\\
x_{12n+1} &= \frac{x_{-1}x_{-4}}{x_{-2}(-1+x_{-1}x_{-4})}, \quad x_{12n+2}=\frac{x_{0}x_{-3}}{x_{-1}(-1+x_{-3}x_{0})}, \quad x_{12n+3}=\frac{x_{-1}x_{-4}}{x_{0}}=x_{-3},\\
x_{12n+4} &= x_{-2}\left(-1+x_{-1}x_{-4}\right), \; x_{12n+5}=x_{-1},\\
x_{12n+6} &= \frac{x_{0}}{-1+x_{-3}x_{0}}, \; x_{12n+7}=\frac{x_{-1}x_{-4}}{x_{-2}(-1+x_{-1}x_{-4})}.
\end{aligned}$$

Thus we have a periodic solution of period twelve and the proof is complete.

Theorem 6 *Eq.(3) has a periodic solution of period six iff* $x_{-1}x_{-4}=x_{-3}x_{0}=2$ *and has the form* $\left\{x_{-4},x_{-3},x_{-2},x_{-1},x_{0},\frac{2}{x_{-2}},x_{-4},x_{-3},x_{-2},...\right\}$.

Proof: The proof is consequently from the previous Theorems and will be omitted.
In the following we present some figures illustrate the behavior of the solutions of Eq.(3) under some different initial values.

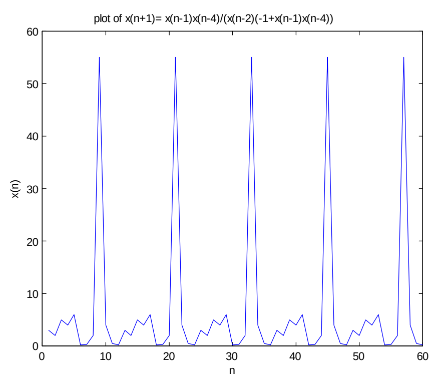

Figure 3.
$x_{-4}=3,x_{-3}=2,x_{-2}=5,$
$x_{-1}=4,\; x_{0}=6.$

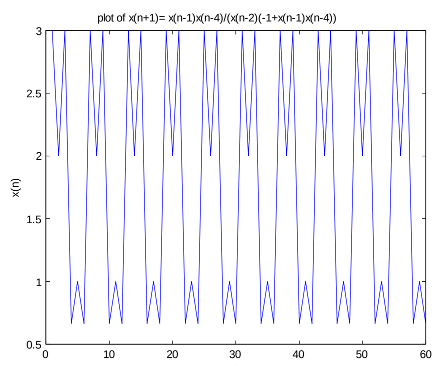

Figure 4.
$x_{-4}=3,x_{-3}=2,x_{-2}=3,$
$x_{-1}=2/3,\; x_{0}=1.$

The following cases can be treated similarly.

4 On the Equation $x_{n+1}=\frac{x_{n-1}x_{n-4}}{x_{n-2}(1-x_{n-1}x_{n-4})}$

In this section we get the solution of the third following equation

$$x_{n+1}=\frac{x_{n-1}x_{n-4}}{x_{n-2}(1-x_{n-1}x_{n-4})}, \quad n=0,1,..., \tag{4}$$

where the initial values are arbitrary positive real numbers.

Theorem 7 *Assume that $\{x_n\}_{n=-4}^{\infty}$ be a solution of Eq.(4). Then for $n = 0, 1, ...$*

$$x_{6n-2} = \frac{x_{-2}x_0^n x_{-3}^n}{x_{-1}^n x_{-4}^n}\prod_{i=0}^{n-1}\left(\frac{(1-(3i+1)x_{-1}x_{-4})}{(1-(3i+2)x_0x_{-3})}\right), \quad x_{6n-1} = \frac{x_{-1}^{n+1}x_{-4}^n}{x_0^n x_{-3}^n}\prod_{i=0}^{n-1}\left(\frac{(1-(3i+1)x_0x_{-3})}{(1-(3i+3)x_{-1}x_{-4})}\right),$$

$$x_{6n} = \frac{x_0^{n+1}x_{-3}^n}{x_{-1}^n x_{-4}^n}\prod_{i=0}^{n-1}\left(\frac{1-(3i+2)x_{-1}x_{-4}}{1-(3i+3)x_0x_{-3}}\right), \quad x_{6n+1} = \frac{x_{-1}^{n+1}x_{-4}^{n+1}}{x_{-2}x_0^n x_{-3}^n(1-x_{-1}x_{-4})}\prod_{i=0}^{n-1}\left(\frac{1-(3i+2)x_0x_{-3}}{1-(3i+4)x_{-1}x_{-4}}\right),$$

$$x_{6n+2} = \frac{x_0^{n+1}x_{-3}^{n+1}}{x_{-1}^{n+1}x_{-4}^n(1-x_0x_{-3})}\prod_{i=0}^{n-1}\left(\frac{(1-(3i+3)x_{-1}x_{-4})}{(1-(3i+4)x_0x_{-3})}\right),$$

$$x_{6n+3} = \frac{x_{-1}^{n+1}x_{-4}^{n+1}}{x_0^{n+1}x_{-3}^n(1-2x_{-1}x_{-4})}\prod_{i=0}^{n-1}\left(\frac{(1-(3i+3)x_0x_{-3})}{(1-(3i+5)x_{-1}x_{-4})}\right).$$

Theorem 8 *Eq.(4) has the unique equilibrium point $\overline{x} = 0$ and it is unstable.*

Example 3. Consider Eq.(4) with $x_{-4} = 3,\ x_{-3} = 5,\ x_{-2} = 2,\ x_{-1} = 2/3,\ x_0 = 0.4$. See Fig. 5.

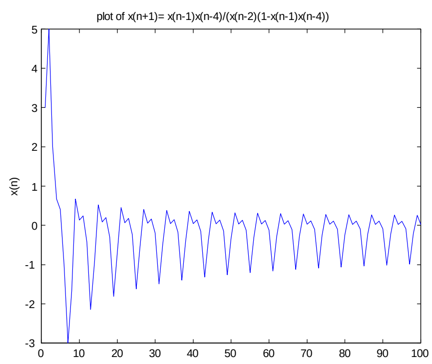

Figure 5.

5 On the Equation $x_{n+1} = \frac{x_{n-1}x_{n-4}}{x_{n-2}(-1-x_{n-1}x_{n-4})}$

Here we obtain the analytical form of the solutions of the equation

$$x_{n+1} = \frac{x_{n-1}x_{n-4}}{x_{n-2}(-1-x_{n-1}x_{n-4})}, \quad n = 0, 1, ..., \tag{5}$$

where the initial values are arbitrary non zero real numbers with $x_{-1}x_{-4} \neq -1,\ x_{-3}x_0 \neq -1$.

Theorem 9 *Let $\{x_n\}_{n=-4}^{\infty}$ be a solution of Eq.(5). Then for $n = 0, 1, 2, ...$ the solution of Eq.(5) is given by*

$$x_{12n-4} = \frac{x_0^{2n}x_{-3}^{2n}}{x_{-1}^{2n}x_{-4}^{2n-1}}\frac{(-1-x_{-1}x_{-4})^n}{(-1-x_{-3}x_0)^n}, \qquad x_{12n-3} = \frac{x_{-1}^{2n}x_{-4}^{2n}}{x_0^{2n}x_{-3}^{2n-1}}\frac{(-1-x_{-3}x_0)^n}{(-1-x_{-1}x_{-4})^n},$$

$$x_{12n-2} = \frac{x_{-2}x_0^{2n}x_{-3}^{2n}}{x_{-1}^{2n}x_{-4}^{2n}}\frac{(-1-x_{-1}x_{-4})^n}{(-1-x_{-3}x_0)^n}, \qquad x_{12n-1} = \frac{x_{-1}^{2n+1}x_{-4}^{2n}}{x_0^{2n}x_{-3}^{2n}}\frac{(-1-x_{-3}x_0)^n}{(-1-x_{-1}x_{-4})^n},$$

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$$x_{12n} = \frac{x_0^{2n+1}x_{-3}^{2n}}{x_{-1}^{2n}x_{-4}^{2n}}\frac{(-1-x_{-1}x_{-4})^n}{(-1-x_{-3}x_0)^n}, \quad x_{12n+1} = \frac{x_{-1}^{2n+1}x_{-4}^{2n+1}}{x_{-2}x_0^{2n}x_{-3}^{2n}}\frac{(-1-x_{-3}x_0)^n}{(-1-x_{-1}x_{-4})^{n+1}},$$

$$x_{12n+2} = \frac{x_0^{2n+1}x_{-3}^{2n+1}}{x_{-1}^{2n+1}x_{-4}^{2n}}\frac{(-1-x_{-1}x_{-4})^n}{(-1-x_{-3}x_0)^{n+1}}, \quad x_{12n+3} = \frac{x_{-1}^{2n+1}x_{-4}^{2n+1}}{x_0^{2n+1}x_{-3}^{2n}}\frac{(-1-x_{-3}x_0)^n}{(-1-x_{-1}x_{-4})^n},$$

$$x_{12n+4} = \frac{x_{-2}x_0^{2n+1}x_{-3}^{2n+1}}{x_{-1}^{2n+1}x_{-4}^{2n+1}}\frac{(-1-x_{-1}x_{-4})^{n+1}}{(-1-x_{-3}x_0)^n}, \quad x_{12n+5} = \frac{x_{-1}^{2n+2}x_{-4}^{2n+1}}{x_0^{2n+1}x_{-3}^{2n+1}}\frac{(-1-x_{-3}x_0)^{n+1}}{(-1-x_{-1}x_{-4})^{n+1}},$$

$$x_{12n+6} = \frac{x_0^{2n+2}x_{-3}^{2n+1}}{x_{-1}^{2n+1}x_{-4}^{2n+1}}\frac{(-1-x_{-1}x_{-4})^n}{(-1-x_{-3}x_0)^{n+1}}, \quad x_{12n+7} = \frac{x_{-1}^{2n+2}x_{-4}^{2n+2}}{x_{-2}x_0^{2n+1}x_{-3}^{2n+1}}\frac{(-1-x_{-3}x_0)^n}{(-1-x_{-1}x_{-4})^{n+1}}.$$

Theorem 10 *Eq.(5) has* $\overline{x} = 0$ *as a unique equilibrium point which is unstable.*

Lemma 2. It is easy to see that every solution of Eq.(5) is unbounded except in the case $x_{-3}x_0 = x_{-1}x_{-4}$.

Theorem 11 *Eq.(5) has a periodic solution of period twelve iff* $x_{-3}x_0 = x_{-1}x_{-4}$. *Moreover the periodic solution has the form* $\{x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0, \frac{x_{-1}x_{-4}}{x_{-2}(-1-x_{-1}x_{-4})},$

$\frac{x_0x_{-3}}{x_{-1}(-1-x_{-3}x_0)}, x_{-3}, x_{-2}\left(-1 - x_{-1}x_{-4}\right), x_{-1}, \frac{x_0}{-1-x_{-3}x_0}, \frac{x_{-1}x_{-4}}{x_{-2}(-1-x_{-1}x_{-4})}, x_{-4}, ...\}$

Theorem 12 *Eq.(5) has a periodic solution of period six iff* $x_{-3}x_0 = x_{-1}x_{-4} = -2$ *and will be taken the form* $\{x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0, \frac{-2}{x_{-2}}, x_{-4}, ...\}$.

Example 4. Fig. 6 below shows the behavior of the solution of Eq.(5) whenever $x_{-4} = 3,\ x_{-3} = 5,\ x_{-2} = -7,\ x_{-1} = 4,\ x_0 = 2.$

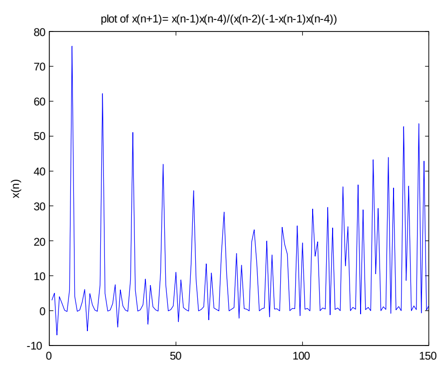

Figure 6.

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Worse-Case Conditional Value-at-Risk for Asymmetrically Distributed Asset Scenarios Returns

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Abstract: Many studies have reported empirical evidence of asymmetries in asset return distributions. Meanwhile, optimal solutions to the Conditional Value-at-Risk (CVaR) minimization are highly susceptible to estimation error of the risk measure because the estimate depends on only a small portion of sampled scenarios. In this paper, based on the robust optimization techniques Chen et al.(2007)[19], we propose a computationally tractable worst-case Conditional Value-at-Risk (CVaR). In the situation, the sampled scenario returns are generated by a factor model with some asymmetric affine uncertainty set. The remarkable characteristic of the new method is that the robust optimization model retains the complexity of original portfolio optimization problem, i.e., the robust counterpart problem is still a linear programming problem. Moreover, it takes into consideration asymmetries in the distributions of scenarios returns used for defining CVaR. We present some numerical experiments with simulated and real market data to illustrate the behavior of the robust optimization model.

Keywords: Portfolio optimization, Conditional value at risk(CVaR), Robust optimization, Linear programming(LP).

1. Introduction

Portfolio optimization problem is an attractive and important research topic since the pioneering Markowitz work on optimal portfolio selection [1]. It is now well known that while mean-variance optimization is appropriate for symmetrically distributed portfolio returns, it results in unsatisfactory asset allocations when returns are asymmetrically distributed, or when downside risk is more weighted than upside risk.

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Since the middle of 1990s, Value-at-Risk (VaR, [4]), a new measure of downside risk, has become popular in financial risk management. It has even been recommended as a standard on banking supervision by the Basel Committee. However, Critics have pointed out numerous shortcomings of VaR [5]. On the other hand, Conditional Value-at-Risk (CVaR), defined as the mean of the tail distribution exceeding VaR, has attracted much attention in recent years. As a measure of risk, CVaR exhibits some better properties than VaR. First, it can deal with the asymmetric distribution of asset return better than mean-variance analysis, especially for assets with returns that are heavy-tailed. Secondly, minimizing CVaR usually results in solving a convex programming problem, such as a linear programming problem, which allows the decision maker to deal with a large-scale portfolio problem efficiently [6, 7]. Finally, Artzner et al.[5] demonstrate that CVaR is a coherent measure of risk, which has been widely accepted as a benchmark to evaluate risk measures. All these stimulate the application of CVaR in practice, and CVaR is getting more and more popular in financial management.

In fact, it is noted that in the process of portfolio selection, the original data brought to the model are not always accurate, i.e., it may be subject to some errors. Thus the result may be influenced by perturbations in the parameters. As pointed out by Black and Litterman [8], in the classical mean-variance model, the portfolio decision is very sensitive to the mean and the covariance matrix, especially to the mean. Chopra and Ziemba [9] showed that small changes in the input parameters can result in large changes in the optimal portfolio allocation. Thus, the modeling risk arises due to the uncertainty of the underlying probability distribution.

Being aware of the importance of robustness in recent years, researchers from both finance and operations research have paid increasing attentions to the robust version of portfolio selection problems. Lobo and Boyd (2000)[10], Goldfarb and Iyengar (2003)[11] studied the robust portfolio problem under the mean-variance framework. Instead of assuming precise information on the mean and the covariance matrix of asset returns, they introduced some types of uncertainties, such as polyhedral uncertainty, box uncertainty and ellipsoidal uncertainty, in the parameters in determining the mean and the covariance matrix, and they then transformed the problem into semidefinite programs(SDP) or second-order cone programs(SOCP), which can be efficiently solved by interior-point algorithms developed in recent years. Halldórsson and Tütüncu (2004) [12] applied their interior-point method for saddle-point problems to the robust mean-variance portfolio selection under the box uncertainty of the elements in the mean vector and the covariance matrix. El Ghaoui, Oks and Oustry (2003)[13] investigated the robust portfolio optimization problem using worst-case VaR, where only the first- and second-moment information on the distribution is available. Several formulations corresponding to various structures of partial information have been extensively exploited to derive the resulting portfolio selection problems in a form of a semidefinite program(SDP). Natarajan, Pachamanova, and Sim, (2008) [14] proposed a computationally tractable approximation method for minimizing the VaR of a portfolio based on robust optimization techniques in Chen et al.(2007)[19]. The method results in the optimization of a modified VaR measure, Asymmetry-Robust VaR, that takes into consideration asymmetries in the distributions of returns and is coherent. Zhu and Fukushima (2009)[15] further investigated the worst-case CVaR risk measure with several structures of uncertainty in the underlying distribution. They focus on the uncertainty in the probability distribution used for defining CVaR. Such a modeling is called distributionally robust modeling. It is true that the probability estimation itself is under uncertainty and we cannot know the true one. However, it is not easy to imagine what form of uncertainty set is proper for the probability measure. In this sense, employing the uncertainty of probability distribution may not provide investors with a satisfactory solution.

On the other hand, since the estimate of CVaR is computed by using only an upper tail part of the loss distribution, a large number of samples are required for assuring the statistical reliability of the estimate. Especially when CVaR is employed as the objective of a portfolio optimization, a much larger number of samples are required for ensuring the accuracy of the optimal portfolio. In practice, however, the number of samples which is available for the estimation is limited, and the estimated CVaR and the resulting optimal portfolio may contain considerable estimation error.

Meanwhile, many studies have reported empirical evidence of asymmetries and large kurtosis in asset return distributions. Empirically, however, there is evidence that both short- and long-horizon stock returns can be skewed and highly leptokurtic (Fama 1976 [22], Duffee 2002 [23]). Furthermore, the returns of portfolios involving derivatives or credit risky assets can have extremely left-skewed distributions (Schönbucher 2000 [24]). More recently, Ang and Chen (2002)[25] find that the asymmetries in the data reject the null hypothesis of multivariate normal distributions. Conine and Tamarkin (1981) [26] also claim that though diversification can change skewness exposure, the remaining idiosyncratic skewness is relevant in asset pricing and thus portfolio optimization under asymmetric distribution is a significant topic for research.

In this paper, we further study the Worse-Case Conditional Value-at-Risk by supposing the sampled scenario returns are generated by a factor model with some asymmetric affine uncertainty set in order to Mitigate the fragility of CVaR-based portfolio optimization problem. Motivated by the works in Chen et al.(2007)[19], we provide a computationally tractable robust optimization method for minimizing the Worse-Case CVaR of a portfolio. Moreover, it takes into consideration asymmetries in the distributions of returns used for defining CVaR.

Notations: Throughout this paper, we use boldface letter such as $\boldsymbol{x}$ for vector to distinguish it from scalar x.

2. Conditional value-at-risk (CVaR)

The conditional value-at-risk (CVaR) has gained growing popularity in financial risk management due to the coherence property and tractability in its optimization.

Let $f(\boldsymbol{x}, \boldsymbol{y})$ be the loss associated with the decision vector $\boldsymbol{x}$, to be chosen from a certain subset X of R^n, and the random vector $\boldsymbol{y}$ in R^m. For convenience, the underling probability of $\boldsymbol{y}$ will be assumed to have a density function $p(.)$.

The probability of $f(\boldsymbol{x}, \boldsymbol{y})$ not exceeding a threshold α is then given by

$$\Psi(\boldsymbol{x}, \alpha) = \int_{f(\boldsymbol{x},\boldsymbol{y})\leq\alpha} p(\boldsymbol{y})d\boldsymbol{y}. \tag{2.1}$$

As a function of α for fixed $\boldsymbol{x}$, $\Psi(\boldsymbol{x}, \alpha)$ is the cumulative distribution function for the loss associated with $\boldsymbol{x}$.

For a confidence level β and a fixed $\boldsymbol{x} \in \mathrm{X}$ the value-at-risk, denoted by $\mathrm{VaR}_\beta(\boldsymbol{x})$ is defined as

$$\mathrm{VaR}_\beta(\boldsymbol{x}) = \min\{\alpha \in \mathrm{R} : \Psi(\boldsymbol{x}, \alpha) \geq \beta\}. \tag{2.2}$$

The conditional value-at-risk, denoted by $\mathrm{CVaR}_\beta(\boldsymbol{x})$, is defined as the expected value of the loss that exceeds $\mathrm{VaR}_\beta(\boldsymbol{x})$, that is,

$$\mathrm{CVaR}_\beta(\boldsymbol{x}) = (1-\beta)^{-1} \int_{f(\boldsymbol{x},\boldsymbol{y})\geq\mathrm{VaR}_\beta(\boldsymbol{x})} f(\boldsymbol{x}, \boldsymbol{y})p(\boldsymbol{y})d\boldsymbol{y}. \tag{2.3}$$

The CVaR is a coherent risk measure [5]. We note that the problem involved $\mathrm{CVaR}_\beta(\boldsymbol{x})$ is difficult to proceed due to its convoluted and implicit version. Rockafellar and Uryasev made a remarkable contribution in [6] by introducing a simpler auxiliary function F_β on $\mathrm{X} \times \mathrm{R}$, defined by

$$F_\beta(\boldsymbol{x}, \alpha) = \alpha + (1-\beta)^{-1} \int_{\boldsymbol{y}\in R^m} [f(\boldsymbol{x}, \boldsymbol{y}) - \alpha]^+ p(\boldsymbol{y})d\boldsymbol{y}, \tag{2.4}$$

In practice, the probability density function $p(\boldsymbol{y})$ is often not available, or is very difficult to estimate. Instead, we might have T different scenarios $\mathrm{Y} = (\boldsymbol{y}_{[1]}, \boldsymbol{y}_{[2]}, \ldots, \boldsymbol{y}_{[T]})$ that are sampled from the probability distribution or that have been obtained from computer simulations. Evaluating the auxiliary function $\tilde{F}_\beta(\boldsymbol{x}, \alpha)$ using the scenarios Y, we have

$$\tilde{F}_\beta(\boldsymbol{x}, \alpha) = \alpha + (1-\beta)^{-1} \sum_{t=1}^{T} \pi_t [f(\boldsymbol{x}, \boldsymbol{y}_{[t]}) - \alpha]^+, \tag{2.5}$$

where $\boldsymbol{y}_{[t]}$ denotes the tth sample (the subscript $[t]$ is used to distinguish a vector from a scalar) generated by simple random sampling with respect to $\boldsymbol{x}$ according to its density function $p(.)$, and T denotes the number of samples, where π_t are probabilities of scenarios $\boldsymbol{y}_{[t]}$. If π_t is equal to T^{-1} for all t, then (2.5) reduces to

$$\tilde{F}_\alpha(\boldsymbol{x}, \alpha) = \alpha + \frac{1}{T(1-\beta)} \sum_{t=1}^{T} [f(\boldsymbol{x}, \boldsymbol{y}_{[t]}) - \alpha]^+. \tag{2.6}$$

Obviously, $\tilde{F}_\alpha(\boldsymbol{x}, \alpha)$ is convex and piecewise linear with respect to α. Further, $\tilde{F}_\alpha(\boldsymbol{x}, \alpha)$ is convex for $(\boldsymbol{x}, \alpha)$ if $f(\boldsymbol{x}, \boldsymbol{y})$ is convex (see Theorem 2 in [6]). Replacing $[f(\boldsymbol{x}, \boldsymbol{y}_{[t]}) - \alpha]^+$ by the auxiliary variables d_t along with appropriate constraints, we obtain the equivalent optimization problem

$$\begin{aligned} \min_{(\boldsymbol{x},\boldsymbol{d},\alpha)\in\mathrm{R}^n\times\mathrm{R}^T\times\mathrm{R}} \quad & \alpha + \frac{1}{T(1-\beta)} \sum_{t=1}^{T} d_t, \\ s.t. \quad & \boldsymbol{x} \in \mathrm{X} \\ & d_t \geq f(\boldsymbol{x}, \boldsymbol{y}_{[t]}) - \alpha, \qquad t = 1, \ldots, T, \\ & \boldsymbol{d} \geq 0. \end{aligned} \tag{2.7}$$

Generally, the loss and return functions of portfolio allocation are chosen by:

$$f(\boldsymbol{x},\boldsymbol{y}) = -\boldsymbol{x}^T\boldsymbol{y}, \quad \mathrm{R}_p(x) = \mathrm{E}_p[\boldsymbol{x}^T\boldsymbol{y}] = \boldsymbol{x}^T\mathrm{E}_p[\boldsymbol{y}] = \boldsymbol{x}^T\boldsymbol{r}, \tag{2.8}$$

in which $\boldsymbol{y}$ is the vector of the assets' return, $\boldsymbol{r}$ is the vector of the expected assets' return, and $\boldsymbol{x}^T\boldsymbol{r}$ is the mean return of the portfolio. Hence, adding an auxiliary variable $\theta \in R$, the minimization model of CVaR (2.9) becomes the following linear programming (LP) problem with variables $(\boldsymbol{x},\boldsymbol{d},\alpha,\theta) \in \mathrm{R}^n \times \mathrm{R}^T \times \mathrm{R} \times \mathrm{R}$.

$$\begin{aligned} \min \quad & \theta \\ s.t. \quad & \boldsymbol{x} \in \mathrm{X} \\ & \alpha + \frac{1}{T(1-\beta)}\sum_{t=1}^{T} d_t \le \theta, \\ & d_t \ge -\boldsymbol{x}^T\boldsymbol{y}_{[t]} - \alpha, \qquad t = 1,\ldots,T, \\ & \boldsymbol{d} \ge 0. \end{aligned} \tag{2.9}$$

Portfolio optimization tries to find an optimal trade-off between the risk and the return according to the investor's preference. Thus, the portfolio selection problem using CVaR as a risk measure can be represented as

$$\min_{\boldsymbol{x}\in\mathrm{X}} \quad \mathrm{CVaR}_\beta(\boldsymbol{x})$$

where X denotes the constraint on the portfolio position, which usually includes the budget constraint and no short sales constraint

$$\boldsymbol{x}^T\mathbf{1} = 1, \quad \boldsymbol{x} \ge 0. \tag{2.10}$$

Let μ be the smallest expected return of the portfolio required by the investor. From (2.8), this return requirement can be represented as

$$\boldsymbol{x}^T\boldsymbol{r} \ge \mu. \tag{2.11}$$

Therefore, the feasible decision set of portfolios can be denoted as

$$\mathrm{X} = \{\boldsymbol{x} | \quad \boldsymbol{x}^T\mathbf{1} = 1, \quad \boldsymbol{x} \ge 0, \quad \boldsymbol{x}^T\boldsymbol{r} \ge \mu\}. \tag{2.12}$$

From (2.9) and 2.12, the mean-CVaR Portfolio optimization can be be written as the following linear program

$$\begin{aligned} \min \quad & \theta \\ s.t. \quad & \alpha + \frac{1}{T(1-\beta)}\sum_{t=1}^{T} d_t \le \theta, \\ & d_t \ge -\boldsymbol{x}^T\boldsymbol{y}_{[t]} - \alpha, \qquad t = 1,\ldots,T, \\ & \boldsymbol{d} \ge 0. \\ & \boldsymbol{x}^T\mathbf{1} = 1, \quad \boldsymbol{x} \ge 0, \quad \boldsymbol{x}^T\boldsymbol{r} \ge \mu. \end{aligned} \tag{2.13}$$

3. Worst-Case Conditional value-at-risk (CVaR)

However, optimal solutions to the CVaR minimization are highly susceptible to estimation error of the risk measure because the estimate depends on only a small portion of sampled scenarios, for example $\mathrm{Y} = (\boldsymbol{y}_{[1]}, \boldsymbol{y}_{[2]}, \ldots, \boldsymbol{y}_{[T]})$.

A practical way to alleviate the effect of such a perturbation is to employ a statistical model. For example, Konno, Waki and Yuuki (2002) replace the observed returns $\mathrm{Y} = (\boldsymbol{y}_{[1]}, \boldsymbol{y}_{[2]}, \ldots, \boldsymbol{y}_{[T]})$ in 2.6 with values estimated by a regression approach. Based on the robust optimization techniques in Chen et al.(2007)[19], we suppose that future asset returns $\tilde{r}$ are generated by the following factor model

$$\boldsymbol{r} = \boldsymbol{r}^0 + \Delta \boldsymbol{r}\, \boldsymbol{z}, \boldsymbol{z} \in C \tag{3.1}$$

in which $\boldsymbol{r}^0$ is a vector of expected returns, and $\Delta \boldsymbol{r}$ is a matrix of factor loadings. The factors $\boldsymbol{z}$ are stochastically independent with following support set

$$C = \Big\{ \boldsymbol{z} : \exists \boldsymbol{v},\ \boldsymbol{w} \in R_+^N, \boldsymbol{z} = \boldsymbol{v} - \boldsymbol{w}, \|\mathbf{P}^{-1}\mathbf{v} + \mathbf{Q}^{-1}\mathbf{w}\| \leq \Omega \Big\}, \tag{3.2}$$

and $\mathbf{P} = \mathrm{diag}(p_1, \ldots, p_N)$, $\mathbf{Q} = \mathrm{diag}(q_1, \ldots, q_N)$. The parameters $p_j > 0$ and $q_j > 0$ are the "forward"and the "backward" deviations of random variable $z_j, j = 1, \ldots, N$, respectively. The uncertainty set C is convex, and its size is controlled by Ω. Intuitively speaking, the uncertain factors $\boldsymbol{z}$ are decomposed into two random variables: $\boldsymbol{v} = \max\{\boldsymbol{z}, 0\}$ and $\boldsymbol{w} = \max\{-\boldsymbol{z}, 0\}$, so that $\boldsymbol{z} = \boldsymbol{v} - \boldsymbol{w}$. The multipliers $\frac{1}{p_j}$ and $\frac{1}{q_j}$ normalize the effective perturbation contributed by both $\boldsymbol{v}$ and $\boldsymbol{w}$ such that the norm of the aggregated values falls within the budget of uncertainty. Therefore, considered sampling error of the samples, we present the Sample-based Worst-Case CVaR, its mathematical definition is as follows:

$$\mathrm{WSCVaR}_\beta(\boldsymbol{x}) = \sup_{(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_T) \in S_\Omega} \mathrm{CVaR}_\beta(\boldsymbol{x}),) \tag{3.3}$$

where

$$S_\Omega = \Big\{ \boldsymbol{r}_t : \boldsymbol{r}_t = \boldsymbol{r}_t^0 + \Delta \boldsymbol{r}_t \boldsymbol{z}_t, \boldsymbol{z}_t \in C_t) \Big\}, \tag{3.4}$$

$$C_t = \Big\{ \boldsymbol{z}_t : \exists \boldsymbol{v},\ \boldsymbol{w} \in R_+^N, \boldsymbol{z}_t = \boldsymbol{v}_t - \boldsymbol{w}_t, \|\mathbf{P}_t^{-1}\mathbf{v}_t + \mathbf{Q}_t^{-1}\mathbf{w}_t\| \leq \Omega \Big\}. \tag{3.5}$$

Next, we prove the WSCVaR 3.3 is a coherent risk measure.

Theorem 3.1 *If* $(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_T) \in S_\Omega$, *then WSCVaR is a coherent risk measure.*

Proof. Letting $\rho(\boldsymbol{x}) = \mathrm{CVaR}_\beta(\boldsymbol{x}), \rho_w(\boldsymbol{x}) = \mathrm{WSCVaR}_\beta(\boldsymbol{x})$, we have

$$\rho_w(\boldsymbol{x}) = \sup_{(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_T) \in S_\Omega} \rho(\boldsymbol{x}).$$

As $\mathrm{CVaR}_\beta(\boldsymbol{x})$ is a coherent risk measure, so $\rho(\boldsymbol{x})$ satisfies four axioms of Coherent risk measure.

In what following, we prove $\rho_w(\boldsymbol{x})$ also satisfies four axioms of Coherent risk measure.

- Monotonicity: if $\boldsymbol{x} < \boldsymbol{y}$, then $\rho(\boldsymbol{x}) < \rho(\boldsymbol{y})$. Therefore

$$\rho_w(\boldsymbol{x}) = \sup_{(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_T)\in S_\Omega} \rho(\boldsymbol{x}) < \sup_{(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_T)\in S_\Omega} \rho(\boldsymbol{y}) = \rho_w(\boldsymbol{y});$$

- subadditivity: for all $\boldsymbol{x}, \boldsymbol{y}$, we have

$$\rho_w(\boldsymbol{x}+\boldsymbol{y}) = \sup_{(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_T)\in S_\Omega} \rho(\boldsymbol{x}+\boldsymbol{y}) \le \sup_{(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_T)\in S_\Omega} [\rho(\boldsymbol{x})+\rho(\boldsymbol{y})] = \rho_w(\boldsymbol{x})+\rho_w(\boldsymbol{y});$$

- positive homogeneity: for any $\lambda > 0$, we have

$$\rho_w(\lambda\boldsymbol{x}) = \sup_{(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_T)\in S_\Omega} \rho(\lambda\boldsymbol{x}) = \lambda \sup_{(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_T)\in S_\Omega} \rho(\boldsymbol{x}) = \lambda\rho_w(\boldsymbol{x});$$

- translation invariance: for any constant $a \in R$, we have

$$\rho_w(\boldsymbol{x}+a) = \sup_{(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_T)\in S_\Omega} \rho(\boldsymbol{x}+a) = \sup_{(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_T)\in S_\Omega} \rho(\boldsymbol{x})+a = \rho_w(\boldsymbol{x})+a.$$

Therefore, the theorem is true.

Chen, Sim and Sun [19] stated the uncertainty set S_Ω is convex, and its size is determined by Ω. Therefore, S_Ω is a compact convex set. Let $f(\boldsymbol{x},\boldsymbol{y}) = -\boldsymbol{x}^T\boldsymbol{r}$ be the loss associated with the decision vector $\boldsymbol{x}$, to be chosen from a certain subset X of R^n, and the random vector $\boldsymbol{r}$ in R^m. So, from 2.6, WSCVaR can be converted to the following form:

$$\mathrm{WSCVaR}_\beta(\boldsymbol{x}) = \max_{(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_T)\in S_\Omega} \min\Big\{\alpha + \frac{1}{T(1-\beta)}\sum_{t=1}^{T}\max\{-\boldsymbol{r}_t^T\boldsymbol{x}-\alpha,0\}\Big\}. \tag{3.6}$$

Next, we will show the WSCVaR enjoys an important nature, in the process the dual-norm $\|\boldsymbol{u}\|^*$, (see Bertsimas and Sim [18]) is required. It is defined as:

$$\|\boldsymbol{u}\|^* = \max_{\{\|\boldsymbol{x}\|\le 1\}} \boldsymbol{u}^T\boldsymbol{x}.$$

Theorem 3.2 *If* $(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_T) \in S_\Omega$, *we have*

$$WSCVaR_\beta(\boldsymbol{x}) = CVaR_\beta(\boldsymbol{x}) + \frac{\Omega}{T(1-\beta)}\sum_{t=1}^{T}\|\boldsymbol{u}_t\|^*. \tag{3.7}$$

Proof. From 3.6, we can obtain

$$\begin{aligned}
\mathrm{WSCVaR}_\beta(\boldsymbol{x}) &= \max_{(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_T)\in S_\Omega} \min\Big\{\alpha + \frac{1}{T(1-\beta)}\sum_{t=1}^{T}\max\{-\boldsymbol{r}_t^T\boldsymbol{x}-\alpha,0\}\Big\} \\
&= \max_{\boldsymbol{z}_t\in C_t} \min\Big\{\alpha + \frac{1}{T(1-\beta)}\sum_{t=1}^{T}\max\{-(\boldsymbol{r}_t^0)^T\boldsymbol{x}-(\Delta\boldsymbol{r}_t\boldsymbol{z}_t)^T\boldsymbol{x}-\alpha,0\}\Big\} \\
&= \mathrm{CVaR}_\beta(\boldsymbol{x}) + \max_{\boldsymbol{z}_t\in C_t} \max\Big\{\frac{1}{T(1-\beta)}\sum_{t=1}^{T}(\Delta\boldsymbol{r}_t\boldsymbol{z}_t)^T\boldsymbol{x}\Big\} \\
&= \mathrm{CVaR}_\beta(\boldsymbol{x}) + \frac{1}{T(1-\beta)}\sum_{t=1}^{T}\max_{\boldsymbol{z}_t\in C_t}\Big\{\boldsymbol{z}_t^T\boldsymbol{y}_t\Big\}, \boldsymbol{y}_t = \Delta\boldsymbol{r}_t^T\boldsymbol{x}.
\end{aligned}$$

Observe that

$$\begin{aligned}
& \max_{\{\boldsymbol{z}_t \in C_t\}} \boldsymbol{z}_t^T \boldsymbol{y}_t \\
= & \max_{\{\mathbf{v}_t, \mathbf{w}_t \in R_+^N : \|\mathbf{P}_t^{-1}\mathbf{v}_t + \mathbf{Q}_t^{-1}\mathbf{w}_t\| \le \Omega\}} (\boldsymbol{v}_t - \boldsymbol{w}_t)^T \boldsymbol{y}_t \\
= & \max_{\{\mathbf{v}_t, \mathbf{w}_t \in R_+^N : \|\boldsymbol{v}_t + \boldsymbol{w}_t\| \le \Omega\}} (\mathbf{P}_t \boldsymbol{y}_t)^T \boldsymbol{v}_t - (\mathbf{Q}_t \boldsymbol{y}_t)^T \boldsymbol{w}_t \\
= & \Omega \|\mathbf{u}_t\|^*
\end{aligned}$$

where $\mathbf{u}_t = \max\{\mathbf{P}_t \boldsymbol{y}_t, -\mathbf{Q}_t \boldsymbol{y}_t, 0\} = \max\{\mathbf{P}_t \boldsymbol{y}_t, -\mathbf{Q}_t \boldsymbol{y}_t\}$

Note: Theorem 3.7 indicates that the WSCVaR can be seen as the original CVaR plus a regular item. It is easy to know that $\text{CVaR}_\beta(\boldsymbol{x}) \le \text{WSCVaR}_\beta(\boldsymbol{x})$. Obvious, WSCVaR is more cautious than the original CVaR.

4. Computing WSCVaR and its application in portfolio management

By the Chen, Sim and Sun [19] Theorem 2 and Theorem 3.2, adding an auxiliary variable $h_t \in R, t = 1, 2, \ldots, T$, the WSCVaR (3.7) can be transformed into the following form

$$\begin{aligned}
\min \quad & \alpha + \frac{1}{T(1-\beta)} \sum_{t=1}^{T} d_t + + \frac{\Omega}{T(1-\beta)} \sum_{t=1}^{T} h_t, \\
s.t. \quad & \|\boldsymbol{u}_t\|^* \le h_t, t = 1, 2, \ldots T, \\
& \mathbf{u}_t \ge -\mathbf{P}_t \Delta \boldsymbol{r}_t^T \boldsymbol{x}, t = 1, 2, \ldots T, \\
& \mathbf{u}_t \ge \mathbf{Q}_t \Delta \boldsymbol{r}_t^T \boldsymbol{x}, t = 1, 2, \ldots T, \\
& d_t \ge (\boldsymbol{r}_t^0)^T \boldsymbol{x} - \alpha, \qquad t = 1, \ldots, T, \\
& \boldsymbol{d} \ge 0.
\end{aligned} \tag{4.1}$$

The complete formulation and complexity class of the robust counterpart depends on the representation of the dual norm constraint, $\|\boldsymbol{u}_t\|^* \le h_t, t = 1, 2, \ldots T$. Table 1 lists the common choices of norms, the representation of their dual norms which is come from reference [18](See page 14, Table 2).

Table 1: Representation of the dual norm for $u \ge 0$.

Norms	$\|t\|$	$\|u\|^* \le h$
l_2	$\|t\|_2$	$\|u\|_2 \le h$
l_1	$\|t\|_1$	$u_j \le h, \forall j = \{1, \ldots, N\}$
l_∞	$\|t\|_\infty$	$\sum_{j=1}^{N} u_j \le h$
$l_1 \bigcap l_\infty$	$\max\{\frac{1}{\Omega}\|t\|_1, \|t\|_\infty\}$	$\Omega\delta + \sum_{j=1}^{N} v_j \le h; v_j + \delta \ge u_j, \forall j \in N; \delta \in R_+, \boldsymbol{v} \in R_+^N$

In [18], Bertsimas and Sim discussed the nature and size of the proposed robust conic problem. In terms of keeping the model linear and simplicity in size, the l_1 norm also is an attractive choice. In this paper, we adopt l_1 norm. So under l_1 norm, the constraints $\|\boldsymbol{u}_t\|^* \leq h_t, t = 1, 2, \ldots T$ in (4.1) is equivalent to

$$u_t^j \leq h_t, \forall j = \{1, \ldots, N\}, t = 1, 2, \ldots T. \tag{4.2}$$

Hence, the resulting problem (4.2) is still a linear constraint.

For the constraint term $\mathbf{u}_t \geq -\mathbf{P}_t \Delta \boldsymbol{r}_t^T \boldsymbol{x}, t = 1, 2, \ldots T$ in (4.1), as discussed in [18], when all the data entries of the problem have independent random perturbation, we can further reduce the size of the robust model. In this article, we assume that the dimension of $\boldsymbol{x}$ and $\mathbf{u}$ is identical (n=N), that is, z_t^j in (3.4)is the independent random variable associated with the j-th data element, and $\Delta \boldsymbol{r}_j$ contains mostly zeros except at the entries corresponding to the data element, such as $\Delta \boldsymbol{r}_t^j = (0, \ldots, 0, \Delta r_t^j, 0, \ldots, 0)^T$. Then $u_t^j \geq -p_t^j (\Delta \boldsymbol{r}_t^j)^T \boldsymbol{x}$ will reduce to $u_t^j \geq -p_t^j \Delta r_t^j \cdot x^j$. Then, the constraint term $\mathbf{u}_t \geq -\mathbf{P}_t \Delta \boldsymbol{r}_t^T \boldsymbol{x}, t = 1, 2, \ldots T$ in (4.1) can be transformed into the following form

$$u_t^j \geq -p_t^j \Delta r_t^j \cdot x^j, j = 1, \ldots, n, t = 1, 2, \ldots T. \tag{4.3}$$

Based on investor preferences, portfolio optimization try to find the balance between risk and return. Therefore, the WSCVaR-based portfolio problem can be expressed as

$$\min_{\boldsymbol{x} \in \mathrm{X}} \quad \mathrm{WSCVaR}_\beta(\boldsymbol{x}),$$

where X denotes the constraint on the portfolio position, which usually includes the budget constraint, no short sales constraint, and the return requirement. Therefore, the feasible decision set of portfolios can be denoted as

$$\mathrm{X} = \{\boldsymbol{x} | \quad \boldsymbol{x}^T \mathbf{1} = 1, \quad \boldsymbol{x} \geq 0, \quad \boldsymbol{x}^T \boldsymbol{r} \geq \mu\}. \tag{4.4}$$

From (4.1) (4.2) and (4.3), adding an auxiliary variable $\theta \in R$, the AWCVaR-based robust portfolio selection problem can be written as the following linear programming problem with variables $(\boldsymbol{x}, \boldsymbol{d}, \boldsymbol{u}_t, h_t, \theta, \alpha)$

$$\begin{aligned}
\min \quad & \theta \\
s.t. \quad & \alpha + \frac{1}{T(1-\beta)} \sum_{t=1}^{T} d_t + + \frac{\Omega}{T(1-\beta)} \sum_{t=1}^{T} h_t \leq \theta, \\
& u_t^j \leq h_t, \forall j = \{1, \ldots, N\}, t = 1, 2, \ldots T, \\
& u_t^j \geq -p_t^j \Delta r_t^j \cdot x^j, j = 1, \ldots, n, t = 1, 2, \ldots T, \\
& u_t^j \geq q_t^j \Delta r_t^j \cdot x^j, j = 1, \ldots, n, t = 1, 2, \ldots T, \\
& d_t \geq (\boldsymbol{r}_t^0)^T \boldsymbol{x} - \alpha, \qquad t = 1, \ldots, T, \\
& \boldsymbol{d} \geq 0, \boldsymbol{u}_t \geq 0, \\
& \boldsymbol{x}^T \mathbf{1} = 1, \quad \boldsymbol{x} \geq 0, \quad \boldsymbol{x}^T \boldsymbol{r} \geq \mu.
\end{aligned} \tag{4.5}$$

5. Computational Experiments

We compare the performance of minimizing-portfolio WSCVaR under our approach with the initial CVaR method [6]. Firstly, we use simulated asset returns and show that our WSCVaR approach performs well for negatively-skewed returns. Secondly, we compare initial CVaR method and the robust portfolio optimization methods by employing a widely available data set of Hedge Funds returns, from http://www.hedgeindex.com.

In our numerical experiments, the methods have the following meanings:

- "CVaR" stands for the initial mean-CVaR Portfolio optimization model (2.13)[6];
- "WSCVaR" stands for the robust mean-WSCVaR Portfolio optimization model (4.5).

We utilize Matlab2012 to solve models CVaR and WSCVaR, which are linear programming problems.

5.1. Experiments with Simulated Data

Consider a portfolio of $n = 20$ assets with uncertain returns $\tilde{r}_i^t, i = 1, \ldots, n, t = 1, \ldots, T$. Each return $\tilde{r}_i^t$ is determined by a simple single factor model $\tilde{r}_i^t = \hat{r}_i^t + \tilde{z}(\omega_i^t)$, where $\hat{r}_i^t = 1$. The factors $\tilde{z}^t(\omega_i)$ are independent and distributed as follows:

$$\tilde{z}(\omega_i^t) = \begin{cases} \frac{\sqrt{\omega_i^t(1-\omega_i^t)}}{\omega_i^t}, & \text{with probability} \quad \omega_i^t, \\ -\frac{\sqrt{\omega_i^t(1-\omega_i^t)}}{1-\omega_i^t}, & \text{with probability} \quad 1-\omega_i^t. \end{cases} \tag{5.1}$$

Note that the mean and the standard deviation of $\tilde{z}(\omega_i^t)$ are the same for all $\omega_i^t \in (0, 1)$ - they are 0 and 1, respectively. However, the degree of symmetry of $\tilde{z}^t(\omega_i^t)$ can be different. Higher values for $\omega_i^t(e.g., \omega_i^t = 0.9)$ result in larger negative skew. We generate values for ω_i^t as follows:

$$\omega_i^t = \frac{1}{2}\Big(1 + \frac{i}{N+t}\Big), i = 1, \ldots, n, t = 1, \ldots, T. \tag{5.2}$$

Therefore, the return distributions for stocks with high index numbers in the portfolio are more negatively skewed than those for stocks with low index numbers.

We use exact values for the parameters in the CVaR and WSCVaR optimization problems. These parameters include the standard deviation and average returns for the CVaR approaches, and the backward and forward deviations for the WSCVaR approach are set to $p_j^t = 1.5, q_j^t = 2$. Δr_j^t is set to the vector of standard deviation of asset returns estimated by the T samples. We use a training set of 1,000 simulated returns from the above distributions that is $T = 1000$. The optimal portfolio allocations resulting from the five approximate CVaR optimization approaches for $\beta = 1\%$ are shown in Figure 1.

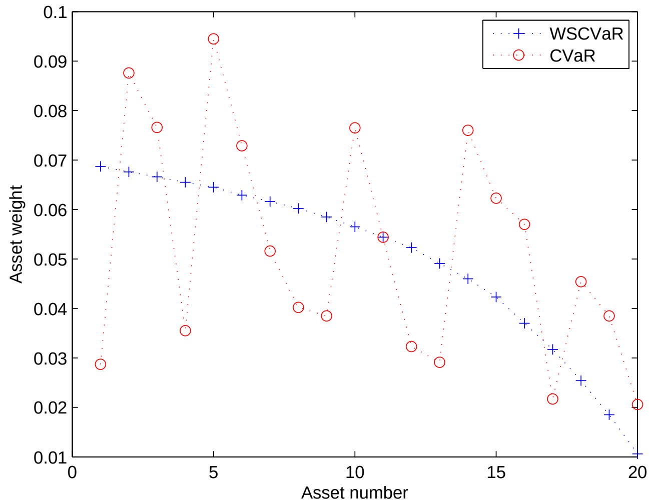

Figure 1-Optimal portfolio weights (as proportions) for assets numbered 1 through 20 resulting from different optimization formulations.

The behavior of the CVaR approach is erratic. In fact, the optimal weights for the portfolios found by the CVaR approach vary widely from sample to sample. WSCVaR is able to detect the asymmetry in the distributions, and allocates less in assets with more negatively skewed return distributions (those with high index numbers).

5.2. Experiments with Hedge Funds

We select 12 Credit Suisse/Tremont Hedge Fund Indices (listed in Table 2) as the candidates for constructing hedge fund portfolios. Monthly returns of these indices, from January 1994 to December 2012 (240 samples in total) are used as the data set, which can be freely downloaded from http://www.hedgeindex.com.

Table 2: Credit Suisse/Tremont Hedge Fund Indices

1	Convertible Arbitrage
2	Dedicated Short Bias
3	Emerging Markets
4	Equity Market Neutral
5	Event Driven
6	Distressed
7	Multi-Strategy
8	Risk Arbitrage
9	Fixed Income Arbitrage
10	Global Macro
11	Long/Short Equity
12	Managed Futures

To construct an optimal portfolio with an accuracy to certain degree, we need to generate adequate scenarios with the given 240 samples. A question we face first in scenario generation is which distribution the asset returns follow. Statistic test shows that most of the distributions of returns of these hedge fund indices are skewed and exhibit a high kurtosis. Thus, the returns should not be modeled by a normal distribution. Table 3 shows the means and standard deviations of these 12 asset returns within three different but overlapped time periods. Each of these three time periods covers 100 months. The beginning and the end dates for each time period are specified in Table 3. We find that, for most assets, there exist remarkable differences among three periods for both the mean and the standard deviation, especially for the mean. For example, the mean of asset 4 during the time period of 1/31/1994-4/30/2002 is 15 times of that during the time period of 6/30/2002-9/30/2010.

Table 3: Mean and standard deviation of asset returns within different time periods

Time	1/31/1994-4/30/2002		8/31/1997-11/30/2005		6/30/2002-9/30/2010	
Asset	Mean	Std	Mean	Std	Mean	Std
1	0.0084	0.0143	0.0066	0.0147	0.0046	0.0254
2	0.0005	0.0534	-0.0003	0.0534	-0.0036	0.0454
3	0.0061	0.0554	0.0048	0.0448	0.0091	0.0296
4	**0.0090**	0.0094	0.0076	0.0070	**0.0006**	0.0424
5	0.0094	0.0178	0.0080	0.0178	0.0072	0.0175
6	0.0110	0.0202	0.0092	0.0193	0.0070	0.0182
7	0.0086	0.0193	0.0073	0.0192	0.0074	0.0184
8	0.0078	0.0130	0.0056	0.0134	0.0041	0.0109
9	0.0057	0.0117	0.0039	0.0117	0.0029	0.0214
10	0.0117	0.0381	0.0088	0.0270	0.0087	0.0160
11	0.0107	0.0342	0.0096	0.0320	0.0062	0.0226
12	0.0038	0.0332	0.0062	0.0358	0.0075	0.0347

Since the distribution of asset returns is unknown, we adopt a distribution free method to generate scenarios given in Topaloglou et al. (2002)[20] and Zhu et al. (2013)[16].

We use back test method to check the performances of the robust approaches and the traditional approach in portfolio management, and the initial wealth is set at 1. Firstly, asset returns of the first N=162 (from 1/31/1994 to 7/31/2007) months are used to generate T=500 scenarios. Portfolio optimization models of the CVaR, and WSCVaR are then, respectively, solved to generate the traditional and the robust portfolio strategies. In month N+1, the two portfolios are constructed according to the derived strategies. At the beginning of month N+2, the scenarios are reproduced using the data from month 2 up to month N+1. The portfolio models are then re-solved, respectively, using the updated scenarios to generate new portfolio strategies for month N+1. The above procedure repeats until the end of the data set.

In this experiments, we also use exact values for the parameters in the CVaR, WSCVaR optimization problems. These parameters include the standard deviation and average returns for the CVaR, and the backward and forward deviations for the WSCVaR approach are set to

$p_t^j = 1.5, q_t^j = 2$. Δr_t^j is set to the vector of standard deviation of asset returns estimated by the $i-th$ T samples.

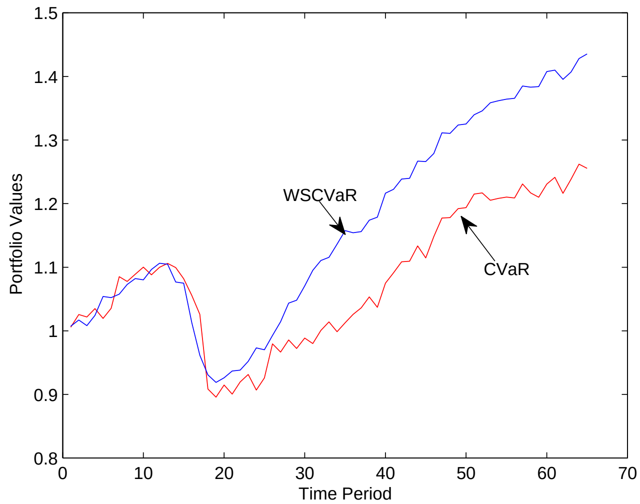

Figure 2-Portfolio Values for Out-of-Sample Observations When a Simple Buy-and-Hold Strategy is Employed

From Figure 2, we can see the optimal portfolio allocation based on the WSCVaR approach tends to result in stable returns, whereas, for example, the behavior of the optimal portfolio obtained with the CVaR approach is some erratic. In addition, the portfolio Values for generated by the WSCVaR model is better than the initial CVaR model at the end of investment period. But, during the gradually declining period from June to October, 2008, robust portfolio strategies perform better than the traditional ones in most cases.

6. Conclusion

With an asymmetric affine uncertainty set based on the factor model, which is often employed in practice for estimating the asset return distribution, we propose a computationally tractable robust optimization method for minimizing the Worse-Case CVaR of a portfolio. The remarkable characteristic of the new method is that the robust optimization model retains the complexity of original portfolio optimization problem, i.e., the robust counterpart problem is still a linear programming problem. Specially in the new method, we incorporate information about asymmetries in the distributions of uncertainties. We present some numerical experiments with simulated and real market data to illustrate the behavior of robust optimization model.

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A NOTE ON THE INTERVAL-VALUED SIMILARITY MEASURE AND THE INTERVAL-VALUED DISTANCE MEASURE INDUCED BY THE CHOQUET INTEGRAL WITH RESPECT TO AN INTERVAL-VALUED CAPACITY

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Abstract. In this paper, we introduce an interval-valued capacity which is motivated by the goal to represent reasonable capacity and to define the Choquet integral with respect to an interval-valued capacity. We also investigate some properties of the Choquet integral with respect to an interval-valued capacity on the space of fuzzy sets and discuss their applications, for examples, interval-valued similarity measure and interval-valued distance measure induced by the Choquet integral with respect to an interval-valued capacity.

1. Introduction

The theory of fuzzy sets defined by Zadeh (1965) has been researching many new approaches and theories, for examples, entropy, similarity measures, distance measures, Choquet integrals, fuzzy sets, and intuitionistic fuzzy sets which are applied to theories treating reasonability and uncertainty. Note that measuring the similarity between fuzzy sets is important in pattern recognition research and decision making.

Balopoulos-Hatzimichailidis-Papadopoulos [2], Fan-Ma-Xie [5], Hong-Lee [6], Li-Sheng [13], Liu [11], Turksen [22], Wang-Li [23], Wei-Chen [25], Xu-Xia [26], Zeng-Li [27], Zeng-Guo [28], and Zhang-Zhang-Mei [29] have studied some properties and applications of similarity measures, entropy, and distance measures on interval-valued fuzzy sets (or fuzzy set), and Choquet [3], Murofushi-Sugeno [15,16], and Narukawa-Murofushi-Sugeno [18,19] have studied the theory of fuzzy measures(or capacity) and Choquet integrals. Couso-Montes-Gil [4], Jang [12], Murofushi-Sugen0-Suzaki [17], Pedrycz-Yang-Ha [20], and Wang [24] have studied various convergence properties of the Choquet integral with respect to a capacity.

By using interval-valued functions, we have studied the Choquet integral with respect to a fuzzy measure of interval-valued functions which are able to better handle the representation of decision making and information theory (see [7-11]). Recently, we studied some convergence properties of the Choquet integral with respect to an interval-valued capacity functional (see

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[12]). Main purpose of this paper is to provide some applications of the Choquet integral with respect to an interval-valued capacity on the space of all fuzzy sets.

In section 2, we define an interval-valued similarity measure and an interval-valued distance measure, and discuss some basic properties of them. In section 3, we define an interval-valued capacity and the Choquet integral with respect to an interval-valued capacity of a fuzzy set, and discuss some properties of them. In section 4, we prove that an interval-valued mapping induced by the Choquet integral with respect to a continuous from below interval-valued capacity is an interval-valued similarity measure on the space of fuzzy sets, and discuss their applications, for examples, the interval-valued similarity measure and the interval-valued distance measure. In section 5, we discuss various convergence properties of the interval-valued distance measure induced by the Choquet integral with respect to an interval-valued capacity. In section 6, we give a brief summary results and some conclusions.

2. Choquet integrals and interval-valued similarity measures

In this section, we consider the Choquet integral with respect to a capacity and discuss their properties. Let $[0,1]$ be the unit interval in the set of real numbers and Ω be a σ-algebra on a set X.

Definition 2.1. ([14-17]) (1) A real-valued set function $\mu : \Omega \longrightarrow [0,1]$ is called a capacity if it satisfies the following properties:

(i) $\mu(\emptyset) = 0$ and $\mu(X) = 1$, and

(ii) $\mu(E_1) \leq \mu(E_2)$ whenever $E_1, E_2 \in \Omega$ and $E_1 \subset E_2$.

(2) A capacity μ is said to be continuous from below if for each increasing sequence $\{E_n\} \subset \Omega$, $\mu(\cup_{n=1}^{\infty} E_n) = \lim_{n\to\infty} \mu(E_n)$.

(3) A capacity μ is said to be continuous from above if for each decreasing sequence $\{E_n\} \subset \Omega$, $\mu(\cap_{n=1}^{\infty} E_n) = \lim_{n\to\infty} \mu(E_n)$.

(4) A capacity μ is said to be continuous if it is continuous from above and continuous from below.

(5) A capacity μ is said to be subadditive if $\mu(E_1 \cup E_2) \leq \mu(E_1) + \mu(E_2)$ whenever $E_1, E_2 \in \Omega$ and $E_1 \cap E_2 = \emptyset$.

We consider the Choquet integral with respect to a capacity which was introduced by Murofushi at el ([15-17]). Throughout this paper, we assume that the membership function of a fuzzy set A is a measurable function η_A from X to $[0,1]$.

Definition 2.2. ([14-17]) (1) The Choquet integral with respect to a capacity μ of a fuzzy set A is defined by

$$(C)\int A d\mu = \int_0^1 \mu_{\eta_A}(r) dr \tag{1}$$

where $\mu_{\eta_A}(r) = \mu(\{x \in X | \eta_A(x) > r\})$ for all $r \in [0,1]$ and the integral on the right-hand side is the Lebesgue integral of μ_{η_A}.

(2) A fuzzy set A is said to be μ-integrable if the Choquet integral of A on X exists.

We note that if A, B are fuzzy sets on X, then $A \leq B$ means $\eta_A(x) \leq \eta_B(x)$ for all $x \in X$ and that $\eta_{A\vee B}(x) = \eta_A(x) \vee \eta_B(x)$ and $\eta_{A\wedge B}(x) = \eta_A(x) \wedge \eta_B(x)$ for all $x \in X$.

Theorem 2.1. ([14-17]) *Let A and B be μ-integrable fuzzy sets.*

(1) *If $A \leq B$, then $(C)\int A d\mu \leq (C)\int B d\mu$.*

(2) *If $E_1, E_2 \in \Omega$ and $E_1 \subset E_2$, then $(C)\int_{E_1} A d\mu \leq (C)\int_{E_2} A d\mu$.*

(3) *If we define* $\eta_{A\vee B} = \eta_A(x) \vee \eta_B(x)$ *and* $\eta_{A\wedge B}(x) = \eta_A(x) \wedge \eta_B(x)$ *for all* $x \in X$, *then*

$$(C)\int A \vee B d\mu \geq (C)\int A d\mu \vee (C)\int B d\mu,$$

and

$$(C)\int A \wedge B d\mu \leq (C)\int A d\mu \wedge (C)\int B d\mu.$$

Let $[[0,1]]$ is the set of all closed intervals in $[0,1]$ as follows:

$$[[0,1]] = \{\bar{a} = [a^-, a^+] | a^-, a^+ \in [0,1] \text{ and } a^- \leq a^+\}.$$

For any $a \in [0,1]$, we define $a = [a,a]$. Obviously, $a \in [[0,1]]$(see [7-13, 21-223, 25, 27-29]).

Definition 2.3. Let I be an index set. If $\bar{a} = [a^-, a^+], \bar{b} = [b^-, b^+], \bar{a}_n = [a_n^-, a_n^+] \in [[0,1]]$ for all $n \in \mathbb{N}$ and $k \in [0,1]$, then we define arithmetic, minimum, maximum, order, and inclusion operations as follows:
(1) $k\bar{a} = [ka^-, ka^+]$,
(2) $\bar{a}\bar{b} = [a^-b^-, a^+b^+]$,
(3) $\bar{a} \wedge \bar{b} = [a^- \wedge b^-, a^+ \wedge b^+]$,
(4) $\bar{a} \vee \bar{b} = [a^- \vee b^-, a^+ \vee b^+]$,
(5) $\bar{a} \leq \bar{b}$ if and only if $a^- \leq b^-$ and $a^+ \leq b^+$,
(6) $\bar{a} < \bar{b}$ if and only if $\bar{a} \leq \bar{b}$ and $\bar{a} \neq \bar{b}$,
(7) $\bar{a} \subset \bar{b}$ if and only if $b^- \leq a^-$ and $a^+ \leq b^+$,
(8) $1 - \bar{a} = [1 - a^+, 1 - a^-]$,
(9) $\sup_{n\in I} \bar{a}_n = [\sup_{n\in I} a_n^-, \sup_{n\in I} a_n^+]$, and
(10) $\inf_{n\in I} \bar{a}_n = [\inf_{n\in I} a_n^-, \inf_{n\in I} a_n^+]$.

Theorem 2.2. *For* $\bar{a}, \bar{b}, \bar{c} \in [[0,1]]$, *we have*
(1) *idempotent law:* $\bar{a} \wedge \bar{a} = \bar{a}$ *and* $\bar{a} \vee \bar{a} = \bar{a}$,
(2) *commutative law:* $\bar{a} \wedge \bar{b} = \bar{b} \wedge \bar{a}$ *and* $\bar{a} \vee \bar{b} = \bar{b} \vee \bar{a}$,
(3) *associative law:* $(\bar{a} \wedge \bar{b}) \wedge \bar{c} = \bar{a} \wedge (\bar{b} \wedge \bar{c})$ *and* $(\bar{a} \vee \bar{b}) \vee \bar{c} = \bar{a} \vee (\bar{b} \vee \bar{c})$,
(4) *absorptive law:* $\bar{a} \wedge (\bar{a} \vee \bar{b}) = \bar{a} \vee (\bar{a} \wedge \bar{b}) = \bar{a}$, *and*
(5) *distributive law:* $\bar{a} \wedge (\bar{b} \vee \bar{c}) = (\bar{a} \wedge \bar{b}) \vee (\bar{a} \wedge \bar{c})$ *and* $\bar{a} \vee (\bar{b} \wedge \bar{c}) = (\bar{a} \vee \bar{b}) \wedge (\bar{a} \vee \bar{c})$.

Let $\mathfrak{F}(X)$ be the family of all fuzzy sets A of X with the membership measurable function $\eta_A : X \to [0,1]$. Recall that for $A, B \in \mathfrak{F}(X)$, $A \equiv B$ means $\mu(\{x \in X | \eta_A(x) \neq \eta_B(x)\}) = 0$, where μ is a capacity on X. We introduce the definitions of similarity measures and distance measures on $\mathfrak{F}(X)$, and some characterizations of them(see [2,5,6,14,26-29]).

Definition 2.4. (1) A real-valued function $s : \mathfrak{F}(X) \times \mathfrak{F}(X) \longrightarrow [0,1]$ is called a similarity measure if it satisfies the following properties:
(i) $s(A, A^c) = 0$ if A is a crisp set,
(ii) for $A, B \in \mathfrak{F}(X)$, $s(A,B) = 1$ if and only if $A \equiv B$,
(iii) for $A, B \in \mathfrak{F}(X)$, $s(A,B) = s(B,A)$, and
(iv) if $A, B, C \in \mathfrak{F}(X)$ and $A \leq B \leq C$, then $s(A,C) \leq s(A,B)$ and $s(A,C) \leq s(B,C)$.
(2) A real-valued function $d : \mathfrak{F}(X) \times \mathfrak{F}(X) \longrightarrow [0,1]$ is called a distance measure if it satisfies the following properties:
(i) $d(A, A^c) = 1$ if A is a crisp set,
(ii) for $A, B \in \mathfrak{F}(X)$, $d(A,B) = 0$ if and only if $A \equiv B$,
(iii) for $A, B \in \mathfrak{F}(X)$, $d(A,B) = d(B,A)$, and
(iv) if $A, B, C \in \mathfrak{F}(X)$ and $A \leq B \leq C$, then $d(A,C) \geq d(A,B)$ and $d(A,C) \geq d(B,C)$.

It is easy to see that if s is a similarity measure and we define $l_1 = 1 - s$, then l_1 is a distance measure and that if d is a distance measure and we define $l_2 = 1 - d$, then l_2 is a similarity measure.

Definition 2.5. (1) An interval-valued function $S = [s^-, s^+] : \mathfrak{F}(X) \times \mathfrak{F}(X) \longrightarrow [[0,1]]$ is called an interval-valued similarity measure if it satisfies the following properties:

(i) $S(A, A^c) = 0$ if A is a crisp set,

(ii) for $A, B \in \mathfrak{F}(X)$, $S(A, B) = 1$ if and only if $A \equiv B$,

(iii) for $A, B \in \mathfrak{F}(X)$, $S(A, B) = S(B, A)$, and

(iv) if $A, B, C \in \mathfrak{F}(X)$ and $A \leq B \leq C$, then $S(A, C) \leq S(A, B)$ and $S(A, C) \leq S(B, C)$.

(2) An interval-valued function $D = [d^-, d^+] : \mathfrak{F}(X) \times \mathfrak{F}(X) \longrightarrow [[0,1]]$ is called a distance measure if it satisfies the following properties:

(i) $D(A, A^c) = 1$ if A is a crisp set,

(ii) for $A, B \in \mathfrak{F}(X)$, $D(A, B) = 0$ if and only if $A \equiv B$,

(iii) for $A, B \in \mathfrak{F}(X)$, $D(A, B) = D(B, A)$, and

(iv) if $A, B, C \in \mathfrak{F}(X)$ and $A \leq B \leq C$, then $D(A, C) \geq D(A, B)$ and $D(A, C) \geq D(B, C)$.

By the definitions of an interval-valued similarity measure and an interval-valued distance measure, we can obtain the following theorem.

Theorem 2.3. (1) *An interval-valued function $S = [s^-, s^+]$ is an interval-valued similarity measure if and only if real-valued functions s^- and s^+ are real-valued similarity measures, and $0 \leq s^- \leq s^+ \leq 1$.*

(2) *An interval-valued function $D = [d^-, d^+]$ is an interval-valued distance measure if and only if real-valued functions d^- and d^+ are real-valued distance measures, and $0 \leq d^- \leq d^+ \leq 1$.*

(3) *If S is an interval-valued similarity measure and we define $H = 1 - S = [1 - s^+, 1 - s^-]$, then H is an interval-valued distance measure.*

(4) *If D is an interval-valued distance measure and we define $L = 1 - D = [1 - d^+, 1 - d^-]$, then L is an interval-valued similarity measure.*

Proof. (1) ($\Longrightarrow$) Suppose that S is an interval-valued similarity measure. If A is a crisp set, then

$$0 = S(A, A^c) = [s^-(A, A^c), s^+(A, A^c)].$$

Thus $s^-(A, A^c) = 0$ and $s^+(A, A^c) = 0$. Since $S(A, B) = S(B, A)$ for all $A, B \in \mathfrak{F}(X)$,

$$s^-(A, B) = s^-(B, A) \text{ and } s^+(A, B) = S^+(B, A).$$

Let $A, B, C \in \mathfrak{F}(X)$ and $A \leq B \leq C$. Then we have

$$S(A, C) \leq S(A, B) \text{ and } S(A, C) \leq S(B, C).$$

Thus, we have

$$s^-(A, C) \leq s^-(A, B) \text{ and } s^-(A, C) \leq s^-(B, C),$$

and

$$s^+(A, C) \leq s^+(A, B) \text{ and } s^+(A, C) \leq s^+(B, C),$$

Therefore, we obtain that s^- and s^+ are real-valued similarity measures and $0 \leq s^- \leq s^+ \leq 1$.

($\Longleftarrow$) The proof is similar to the proof of ($\Longrightarrow$).

(2) The proof is similar to the proof of (1).

(3) Let S be an interval-valued similarity measure and we define $H = 1 - S = [1 - s^+, 1 - s^-]$. If A is a crisp set, then $S(A, A^c) = 0$. Thus, $H(A, A^c) = 1 - S(A, A^c) = 1 - 1 = 0$. Let $A, B \in \mathfrak{F}(X)$. Then, $A \equiv B$ if and only if $S(A, B) = 1$, that is, $H(A, B) = 1 - S(A, B) = 0$. If $A, B \in \mathfrak{F}(X)$, then $S(A, B) = S(B, A)$. Then,

$$H(A, B) = 1 - S(A, B) = 1 - S(B, A) = H(B, A).$$

If $A, B, C \in \mathfrak{F}(X)$ and $A \leq B \leq C$, then

$$S(A, C) \leq S(A, B) \text{ and } S(A, C) \leq S(B, C).$$

Thus, we have

$$H(A, C) = 1 - S(A, C) \geq 1 - S(A, B) = H(A, B)$$

and

$$H(A, C) = 1 - S(A, C) \geq 1 - S(B, C) = H(B, C).$$

Therefore, H is an interval-valued distance measure.

(4) The proof is similar to the proof of (3).

3. The Choquet integral with respect to an interval-valued capacity

In this section, we define an interval-valued capacity and the Choquet integral with respect to an interval-valued capacity of a fuzzy set. Note that a mapping $d_H : [[0,1]] \times [[0,1]] \longrightarrow [0, \infty)$ is the Hausdorff metric defined by

$$d_H(A, B) = \max \left\{ \sup_{x \in A} \inf_{y \in B} |x - y|, \sup_{y \in B} \inf_{x \in A} |x - y| \right\} \quad (2)$$

for all $A, B \in [[0,1]]$, and $([[0,1]], d_H)$ is a metric space. By the definition of the Hausdorff metric, it is easy to see that for any $\bar{a} = [a^-, a^+], \bar{b} = [b^-, b^+] \in [[0,1]]$, we have

$$d_H(\bar{a}, \bar{b}) = \max \left\{ |a^- - b^-|, |a^+ - b^+| \right\}. \quad (3)$$

We recall that for any $\{\bar{a}_n\} \subset [[0,1]]$ and $\bar{a} \in [[0,1]]$,

$$d_H - \lim_{n \to \infty} \bar{a}_n = \bar{a} \text{ means } \lim_{n \to \infty} d_H(\bar{a}_n, \bar{a}) = 0. \quad (4)$$

We define an interval-valued capacity $\bar{\mu} = [\mu^-, \mu^+] : \Omega \longrightarrow [[0,1]]$ on a measurable space (X, Ω) as follows:

Definition 3.1. (1) An interval-valued set function $\bar{\mu} : \Omega \longrightarrow [[0,1]]$ is called an interval-valued capacity if it satisfies the following properties:

(i) $\bar{\mu}(\emptyset) = 0$ and $\bar{\mu}(X) = 1$, and

(ii) $\bar{\mu}(E_1) \leq \bar{\mu}(E_2)$ whenever $E_1, E_2 \in \Omega$ and $E_1 \subset E_2$.

(2) An interval-valued capacity $\bar{\mu}$ is said to be continuous from above if for each increasing sequence $\{E_n\} \subset \Omega$, $\bar{\mu}(\cup_{n=1}^{\infty} E_n) = d_H - \lim_{n \to \infty} \bar{\mu}(E_n)$.

(3) An interval-valued capacity $\bar{\mu}$ is said to be continuous from below if for each decreasing sequence $\{E_n\} \subset \Omega$, $\bar{\mu}(\cap_{n=1}^{\infty} E_n) = d_H - \lim_{n \to \infty} \bar{\mu}(E_n)$.

(4) An interval-valued capacity $\bar{\mu}$ is said to be continuous if it is continuous from above and continuous from below.

(5) An interval-valued capacity $\bar{\mu}$ is said to be subadditive if $\bar{\mu}(E_1 \cup E_2) \leq \bar{\mu}(E_1) + \bar{\mu}(E_2)$, whenever $E_1, E_2 \in \Omega$ and $E_1 \cap E_2 = \emptyset$.

It is easy to see that for each increasing sequence $\{E_n\} \subset \Omega$ with $E = \cup_{n=1}^{\infty} E_n$,

$$\lim_{n \to \infty} d_H(\bar{\mu}(E_n), \bar{\mu}(E)) = 0 \text{ if and only if } \lim_{n \to \infty} \mu^-(E_n) = \mu^-(E) \text{ and } \lim_{n \to \infty} \mu^+(E_n) = \mu^+(E), \quad (5)$$

and for each decreasing sequence $\{E_n\} \subset \Omega$ with $F = \cap_{n=1}^{\infty} E_n$,

$$\lim_{n \to \infty} d_H(\bar{\mu}(E_n), \bar{\mu}(F)) = 0 \text{ if and only if } \lim_{n \to \infty} \mu^-(E_n) = \mu^-(F) \text{ and } \lim_{n \to \infty} \mu^+(E_n) = \mu^+(F). \quad (6)$$

By (5) and (6), we can directly derive the following theorem.

Theorem 3.1. (1) *An interval-valued set function* $\bar{\mu} = [\mu^-, \mu^+] : \Omega \longrightarrow [[0,1]]$ *is an interval-valued capacity if and only if* μ^- *and* μ^+ *are capacities and* $\mu^- \leq \mu^+$.

(2) *An interval-valued capacity* $\bar{\mu} = [\mu^-, \mu^+]$ *is continuous from below if and only if* μ^- *and* μ^+ *are continuous from below and* $\mu^- \leq \mu^+$.

(3) *An interval-valued capacity* $\bar{\mu} = [\mu^-, \mu^+]$ *is continuous from above if and only if* μ^- *and* μ^+ *are continuous from above and* $\mu^- \leq \mu^+$.

(4) *An interval-valued capacity* $\bar{\mu} = [\mu^-, \mu^+]$ *is continuous if and only if* μ^- *and* μ^+ *are continuous and* $\mu^- \leq \mu^+$.

(5) *An interval-valued capacity* $\bar{\mu} = [\mu^-, \mu^+]$ *is subadditive if and only if* μ^- *and* μ^+ *are subadditive and* $\mu^- \leq \mu^+$.

Recall that if $([0,1], \mathfrak{M}, m)$ is the Lebesgue measure space and $C([0,1])$ is the family of all closed subsets of I, then the Aumann integral of a closed set-valued function $G : [0,1] \longrightarrow C([0,1])$ is defined by

$$(A)\int Gdm = \left\{ \int gdm |\ g \in S(G) \right\}, \tag{7}$$

where $S(G)$ is the set of all integrable selections of G, that is,

$$S(G) = \{g : [0,1] \longrightarrow [0,1] |\ \int gdm < \infty \text{ and } g(r) \in G(r)\ m - a.e.\}. \tag{8}$$

We note that $m - a.e.$ means almost everywhere in the Lebesgue measure m (see[1,16]). Then, we introduce the following theorems which are used to define the Choquet integral with respect to an interval-valued capacity of a fuzzy set.

Theorem 3.2. *(*[13, Lemma 2.1]*) If a closed set-valued function* $G : [0,1] \longrightarrow C([0,1])$ *is* $\mathfrak{M}$*-measurable, then* $(A)\int Gdm$ *is convex in* $[0,1]$.

Theorem 3.3. *(*[13, Lemma 2.2]*) If a closed set-valued function* $G : [0,1] \longrightarrow C([0,1])$ *is* $\mathfrak{M}$*-measurable and integrably bounded, that is, there exists a integrable function* $\varphi : [0,1] \longrightarrow [0,1]$ *such that*

$$\sup_{x \in G(r)} x \leq \varphi(r) \qquad \text{for } r \in [0,1], \tag{9}$$

then $(A)\int Gdm$ *is nonempty compact convex in* $[0,1]$.

From Theorem 3.3, we can see that $(A)\int Gdm$ is a nonempty bounded and closed subset in $[0,1]$ under the same assumption of G. Thus, we obtain the following corollary (see [12,13,21]).

Corollary 3.4. *If an interval-valued function* $G = [g^-, g^+] : I \longrightarrow [[0,1]]$ *is* $\mathfrak{M}$*-measurable and integrably bounded, then* $g^-, g^+ \in S(F)$ *and*

$$(A)\int Gdm = \left[\int g^- dm, \int g^+ dm \right], \tag{10}$$

where the integrals on the right-hand side are the Lebesgue integral with respect to m.

We write $\int gdm = \int_0^1 g(r)dm(r)$ for all measurable functions g. By using an interval-valued capacity, we define the Choquet integral with respect to an interval-valued capacity of a fuzzy set A.

Definition 3.2. (1) The Choquet integral with respect to an interval-valued capacity $\bar{\mu}$ of a fuzzy set $A \in \mathfrak{F}$ is defined by

$$(C)\int Ad\bar{\mu} = (A)\int_0^1 \bar{\mu}_A(r)dr, \tag{11}$$

where η_A is the membership measurable function of A, $\bar{\mu}_A(r) = \bar{\mu}(\{x \in X | \eta_A(x) > r\})$ for all $r \in [0,1]$, and the integral on the right-hand side is the Aumann integral in (7).

(2) A fuzzy set $A \in \mathfrak{F}$ is said to be $\bar{\mu}$-integrable if $(C)\int Ad\bar{\mu} \in [[0,1]]$.

Note that if an interval-valued capacity $\bar{\mu}$ is continuous from below and $A \in \mathfrak{F}(X)$, then $\bar{\mu}_A : I \longrightarrow [[0,1]]$ is continuous from below on $[0,1]$. Thus, we obtain that $\bar{\mu}_A$ is $\mathfrak{M}$-measurable and integrably bounded on $[0,1]$. Thus, by Definition 3.2 and Corollary 3.4, we can easily obtain the following theorem.

Theorem 3.5. *If an interval-valued capacity $\bar{\mu}$ is continuous from below and $A \in \mathfrak{F}$, then we have*

$$(C)\int Ad\bar{\mu} = \left[(C)\int Ad\mu^-, (C)\int Ad\mu^+\right], \tag{12}$$

where the integrals on the right-hand side are Choquet integrals.

Proof. By Definition 3.2 and Corollary 3.4, we can derive

$$\begin{aligned}
(C)\int Ad\bar{\mu} &= (A)\int_0^1 \bar{\mu}_A(r)dr \\
&= (A)\int_0^1 [\mu_A^-(r), \mu_A^+(r)]dr \\
&= \left[\int_0^1 \mu_A^-(r)dr, \int_0^1 \mu_A^+(r)dr\right] \\
&= \left[(C)\int Ad\mu^-, (C)\int Ad\mu^+\right].
\end{aligned}$$

By Theorem 3.5, we can easily obtain the following basic properties of the Choquet integrals with respect to a continuous from below interval-valued capacity of a fuzzy set.

Theorem 3.6. *Let (X, Ω) be a measurable space. Assume that an interval-valued $\bar{\mu}$ is continuous from below.*

(1) *If $A, B \in \mathfrak{F}(X)$ and $A \leq B$, then*

$$(C)\int Ad\bar{\mu} \leq (C)\int Bd\bar{\mu}.$$

(2) *If $A, B \in \mathfrak{F}(X)$ and we define $\eta_{(A\vee B)}(x) = \eta_A(x) \vee \eta_B(x)$ for all $x \in X$, then*

$$(C)\int A \vee Bd\bar{\mu} \geq (C)\int Ad\bar{\mu} \vee (C)\int Ad\bar{\mu}.$$

(3) *If $A, B \in \mathfrak{F}(X)$ and we define $\eta_{(A\wedge B)}(x) = \eta_A(x) \wedge \eta_B(x)$ for all $x \in X$, then*

$$(C)\int A \wedge Bd\bar{\mu} \leq (C)\int Ad\bar{\mu} \wedge (C)\int Ad\bar{\mu}.$$

4. Interval-valued similarity measures induced by the Choquet integral

In this section, we discuss some applications of the Choquet integral with respect to a continuous from below interval-valued capacity of a fuzzy set.

Theorem 4.1. *Assume that an interval-valued $\bar{\mu}$ is continuous from below and $\bar{\mu}(X) = \bar{\{\mu\}}(X) = 1$. If we define an interval-valued function $S_{\bar{\mu}} : \mathfrak{F} \times \mathfrak{F} \longrightarrow [[0,1]]$ as following*

$$S_{\bar{\mu}}(A,B) = 1 - (C)\int |\eta_A - \eta_B| d\bar{\mu} \tag{13}$$

for all $A, B \in \mathfrak{F}(X)$, then $S_{\bar{\mu}}$ is an interval-valued similarity measure.

Proof. (i) If A is a crisp measurable set, then the membership measurable function η_A of a fuzzy set A is defined by

$$\eta_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \in A^c = I \setminus A. \end{cases}$$

We note that if the membership measurable function η_{A^c} of the complement of a fuzzy set A, then

$$\eta_{A^c}(x) = \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \in A^c = I \setminus A. \end{cases}$$

Thus, we have $|\eta_A(x) - \eta_{A^c}(x)| = 1$ for all $x \in X$. Therefore, we have

$$\begin{aligned} S_{\bar{\mu}}(A, A^c) &= 1 - (C)\int |\eta_A - \eta_{A^c}| d\bar{\mu} \\ &= 1 - \int_0^1 \bar{\mu}(\{x \in X|\ |\eta_A(x) - \eta_{A^c}(x)| > r\})dr \\ &= 1 - \int_0^1 \bar{\mu}(X)dr = 0. \end{aligned}$$

(ii) If $A \equiv B$, then $\eta_A = \eta_B$ $\bar{\mu} - a.e.$ on X. Thus, we have

$$\begin{aligned} S_{\bar{\mu}}(A, B) &= 1 - (C)\int |\eta_A - \eta_B| d\bar{\mu} \\ &= 1 - \int_0^1 \bar{\mu}(\{x \in X|\ |\eta_A(x) - \eta_B(x)| > r\})dr \\ &= 1 - \int_0^1 \bar{\mu}(\emptyset)dr = 1. \end{aligned}$$

If $S_{\bar{\mu}}(A, B) = 1$, then

$$\int_0^1 \bar{\mu}(\{x \in X|\ |\eta_A(x) - \eta_B(x)| > r\})dr = 0.$$

Then, it is easy to see that

$$\bar{\mu}(\{x \in X|\ |\eta_A(x) - \eta_B(x)| > r\}) = 0 \ m - a.e. \text{ on } I. \tag{14}$$

From (14), we have

$$\bar{\mu}(\{x \in X|\ |\eta_A(x) - \eta_B(x)| \neq 0\}) = 0,$$

that is, $\eta_A = \eta_B$ $\bar{\mu} - a.e.$ on X and hence $A \equiv B$.

(iii) If $A, B \in \mathfrak{F}(X)$, then we have

$$\begin{aligned} S_{\bar{\mu}}(A, B) &= 1 - (C)\int |\eta_A - \eta_B| d\bar{\mu} \\ &= 1 - (C)\int |\eta_B - \eta_A| d\bar{\mu} = S_{\bar{\mu}}(B, A). \end{aligned}$$

(iv) If $A, B, C \in \mathfrak{F}(X)$ and $A \leq B \leq C$, then $\eta_A \leq \eta_B \leq \eta_C$. Thus, we have

$$|\eta_A(x) - \eta_B(x)| \leq |\eta_A(x) - \eta_C(x)| \text{ and } |\eta_B(x) - \eta_C(x)| \leq |\eta_A(x) - \eta_C(x)|, \tag{15}$$

for all $x \in X$. By (15) and Theorem 2.2 (1), we have

$$\begin{aligned} S_{\bar{\mu}}(A, C) &= 1 - (C)\int |\eta_A - \eta_C| d\bar{\mu} \\ &\leq 1 - (C)\int |\eta_A - \eta_B| d\bar{\mu} = S_{\bar{\mu}}(A, B), \end{aligned}$$

and

$$\begin{aligned} S_{\bar{\mu}}(A,C) &= 1-(C)\int |\eta_A-\eta_C|d\bar{\mu} \\ &\leq 1-(C)\int |\eta_B-\eta_C|d\bar{\mu} = S_{\bar{\mu}}(B,C), \end{aligned}$$

By (i),(ii),(iii), and (iv), we see that $S_{\bar{\mu}}$ is an interval-valued similarity measure.

By Theorem 4.1 and Theorem 2.3(3), we can easily obtain the following corollary.

Corollary 4.2. *Assume that an interval-valued* $\bar{\mu}$ *is continuous from below and* $\bar{\mu}(X) = \bar{\{}\mu\}(X) = 1$*. If we define an interval-valued function* $D_{\bar{\mu}} = 1 - S_{\bar{\mu}} = (C)\int |\eta_A - \eta_B| d\bar{\mu}$ *for all* $A, B \in \mathfrak{F}(X)$*, then* $D_{\bar{\mu}}$ *is an interval-valued distance measure.*

In order to illustrate the proposed similarity measure are reasonable, we give the following example.

Example 4.1. Let $X = \{x_1, x_2, x_3\}$ and $\Omega = \wp(X)$ be the power set of X. Suppose that $\bar{\mu} : \Omega \longrightarrow [[0,1]]$ is defined by

$$\bar{\mu}(E) = [\mu^-(E), \mu^+(E)], \tag{16}$$

where $m(E)$ is the cardinality of $E \in \Omega$, $\mu^-(E) = \left(\frac{m(E)}{m(X)}\right)^2$, and $\mu^+(E) = \frac{m(E)}{m(X)}$. Since X is a finite set, clearly, we see that $\bar{\mu}$ is a continuous from below interval-valued capacity on a measurable space (X, Ω) and $\bar{\mu}(X) = \bar{\{}\mu\}(X) = 1$. . The three patterns are denoted as follows:

$$A_1 = \{(x_1, 0.3), (x_2, 0.2), (x_3, 0.1)\},$$
$$A_2 = \{(x_1, 0.2), (x_2, 0.2), (x_3, 0.2)\}, \text{ and}$$
$$A_3 = \{(x_1, 0.4), (x_2, 0.4), (x_3, 0.4)\}.$$

Assume that a sample $B = \{(x_1, 0.3), (x_2, 0.2), (x_3, 0.1)\}$ is given. In order to interpret the measure of similarity of B with these patterns, we calculate the proposed interval-valued similarity measure $S_{\bar{\mu}}$ as follows:

$$S_{\bar{\mu}}(A_1, B) = 1 - \sum_{i=1}^{3} (|\eta_{A_1}(x_{(i)}) - \eta_B(x_{(i)})|)(\bar{\mu}(A_{(i)}) = 1, \tag{17}$$

$$S_{\bar{\mu}}(A_2, B) = 1 - \sum_{i=1}^{3} (|\eta_{A_2}(x_{(i)}) - \eta_B(x_{(i)})|)(\bar{\mu}(A_{(i)}) = \left[\frac{14}{15}, \frac{43}{45}\right], \text{ and} \tag{18}$$

$$S_{\bar{\mu}}(A_3, B) = 1 - \sum_{i=1}^{3} (|\eta_{A_3}(x_{(i)}) - \eta_B(x_{(i)})|)(\bar{\mu}(A_{(i)}) = \left[\frac{4}{5}, \frac{38}{45}\right]. \tag{19}$$

By (17), (18), and (19), we interpret that B is equal(or, absolutely similar) to A_1 and B is more similar to A_2 than similar to A_3.

Example 4.2. Let $X = \{x_1, x_2, x_3\}$ and $\Omega = \wp(X)$ be the power set of X. Suppose that $\bar{\nu} : \Omega \longrightarrow [I]$ is defined by

$$\bar{\nu}(E) = [\nu^-(E), \nu^+(E)], \tag{20}$$

where $m(E)$ is the cardinality of $E \in \Omega$, $\nu^-(E) = \left(\frac{m(E)}{m(X)}\right)^3$, and $\nu^+(E) = \left(\frac{m(E)}{m(X)}\right)^2$.

The three patterns are denoted as follows:

$$A_1 = \{(x_1, 0.3), (x_2, 0.2), (x_3, 0.1)\},$$
$$A_2 = \{(x_1, 0.2), (x_2, 0.2), (x_3, 0.2)\}, \text{ and}$$
$$A_3 = \{(x_1, 0.4), (x_2, 0.4), (x_3, 0.4)\}.$$

Assume that a sample $B = \{(x_1, 0.3), (x_2, 0.2), (x_3, 0.1)\}$ is given. In order to interpret the measure of similarity of B with these patterns, we calculate the proposed interval-valued similarity measure $S_{\bar{\nu}}$ as follows:

$$S_{\bar{\nu}}(A_1, B) = 1,\ S_{\bar{\nu}}(A_2, B) = \left[\frac{43}{45}, \frac{131}{135}\right],\ \text{and}\ S_{\bar{\nu}}(A_3, B) = \left[\frac{38}{45}, \frac{13}{15}\right]. \tag{21}$$

Thus, we can see that there is an interpretation of the notions of these patterns under two different interval-valued capacity $\bar{\mu}$ and $\bar{\nu}$ as follows:

$$S_{\bar{\mu}}(A_1, B) = 1 = S_{\bar{\nu}}(A_1, B),$$

$$S_{\bar{\mu}}(A_2, B) = \left[\frac{14}{15}, \frac{43}{45}\right] < \left[\frac{43}{45}, \frac{131}{135}\right] = S_{\bar{\nu}}(A_2, B),\ \text{and}$$

$$S_{\bar{\mu}}(A_3, B) = \left[\frac{4}{5}, \frac{38}{45}\right] < \left[\frac{38}{45}, \frac{13}{15}\right] = S_{\bar{\nu}}(A_3, B).$$

Therefore, this means that $\bar{\nu}$ has more positive sense than $\bar{\mu}$.

5. Convergence in the interval-valued distance measure

Throughout this section, we assume that $\bar{\mu} = [\mu^-, \mu^+]$ is continuous from below. At first, we introduce uniformly μ-integrability and convergence in the interval-valued distance measure on $\mathfrak{F}(X)$.

Definition 5.1. ([26]) Let μ be a capacity on a measurable space (X, Ω), $\{A_n\}$ be a sequence of fuzzy sets and A be a fuzzy set.

(1) A sequence $\{A_n\}$ converges to A almost everywhere on X if there exist a null set $N \in \Omega$ with $\mu(N) = 0$ such that

$$\eta_A(x) = \lim_{n\to\infty} \eta_{A_n}(x), \quad \text{for all}\ \ x \in N^c. \tag{22}$$

(2) A sequence $\{A_n\}$ converges in the distance measure d_μ to A if

$$\lim_{n\to\infty} d_\mu(\eta_{A_n}, \eta_A) = 0, \tag{23}$$

where $d_\mu(\eta_{A_n}, \eta_A) = (C)\int |\eta_{A_n}(x) - \eta_A(x)| d\mu$ for all $n \in \mathbb{N}$.

Remark that convergence in the distance measure $d\mu$ is equal to convergence in μ-mean(see [4])

Definition 5.2. ([4]) Let μ be a capacity on a measurable space (X, Ω) and $I \subset \mathbb{N}$ be an index set. A class $\{A_n\}_{n\in I}$ of fuzzy sets is said to be uniform μ-integrable if

(i) $\sup_{n\in I} d_\mu(A_n, 0) < \infty$, (24)

(ii) $\forall\varepsilon > 0,\ \exists\delta(\varepsilon) > 0$ such that $d_{E,\mu}(A_n, 0) < \varepsilon$ if $E \in \Omega$ and $\mu(E) < \delta(\varepsilon)$, (25)

where $d_{E,\mu}(A_n, 0) = (C)\int_E |\eta_{A_n}| d\mu$ for all $n \in \mathbb{N}$.

We also introduce various convergence properties of the Choquet integral on $\mathfrak{F}(X)$ as follows:

Theorem 5.1. ([4]) *Let a capacity μ be subadditive and $\{A_n\}$ a sequence of fuzzy sets in $\mathfrak{F}(X)$. Then $\{A_n\}$ is an uniformly μ-integrable if and only if*

$$\lim_{a\to\infty} \sup_{n\in\mathbb{N}} d_{[|\eta_{A_n}|>a],\mu}(A_n, 0) = 0. \tag{26}$$

Theorem 5.2. ([4]) *Let a capacity μ be subadditive and a sequence $\{A_n\}$ of fuzzy sets in $\mathfrak{F}(X)$ converges to a fuzzy set A in $\mathfrak{F}(X)$ μ-almost everywhere on X and $A_n \leq B$ for some μ-integrable fuzzy set B, then we have*

(1) *A_n and A are μ-integrable for all $n \in \mathbb{N}$, and*

(2) *$\{A_n\}$ converges to A in the distance measure d_μ, that is,*

$$\lim_{n\to\infty} d_\mu(A_n, 0) = 0. \tag{27}$$

We assume that an interval-valued capacity $\bar{\mu}(X) = [\mu^-, \mu^+]$ is continuous from below. Then we define convergence in the interval-valued distance measure $D_{\bar{\mu}}$ and uniform $\bar{\mu}$-integrability on $\mathfrak{f}(X)$. It is easy to see that

$$D_{\bar{\mu}}(A, B) = [d_{\mu^-}(A, B), d_{\mu^+}(A, B)], \text{ for all } A, B \in \mathfrak{F}(X). \tag{28}$$

Definition 5.3. Let $I \subset \mathbb{N}$ be an index set.

(1) A sequence $\{A_n\}$ converges in the interval-valued distance measure $D_{\bar{\mu}}$ to A if

$$d_H - \lim_{n\to\infty} D_{\bar{\mu}}(A_n, A) = 0, \tag{29}$$

where

$$d_H - \lim_{n\to\infty} D_{\bar{\mu}}(A_n, A) = \lim_{n\to\infty} d_H\{D_{\bar{\mu}}(A_n, A), 0\}$$

and

$$D_{\bar{\mu}}(A_n, A) = [d_{\mu^-}(A_n, A), d_{\mu^-}(A_n, A)]$$

for all $n \in \mathbb{N}$.

(2) A class $\{A_n\}_{n\in I}$ of fuzzy sets in $\mathfrak{F}(X)$ is said to be $\bar{\mu}$-integrable if

$$\text{(i) } \sup_{n\in I} D_{\bar{\mu}}(A_n, 0) < \infty, \tag{30}$$

$$\text{(ii) } \forall\varepsilon > 0,\ \exists\delta(\varepsilon) > 0 \text{ such that } D_{E,\bar{\mu}}(A_n, 0) < \varepsilon \text{ if } E \in \Omega \text{ and } \bar{\mu}(E) < \delta(\varepsilon), \tag{31}$$

where $D_{E,\bar{\mu}}(A_n, 0) = (C)\int_E |\eta_{A_n}| d\bar{\mu}$ for all $n \in \mathbb{N}$.

By (3), it is easy to see that (29) holds if and only if

$$\lim_{n\to\infty} \max\{d_{\mu^-}(A_n, A), d_{\mu^+}(A_n, A)\}) = 0, \tag{32}$$

By Definition 5.1 and Definition 5.3, we obtain various convergence properties of the interval-valued distance measure $D_{\bar{\mu}}$ as follows:

Theorem 5.3. *Let $I \subset \mathbb{N}$ be an index set.*

(1) *A class $\{A_n\}_{n\in I}$ is uniformly $\bar{\mu}$-integrable if and only if it is uniformly μ^--integrable and uniformly μ^+-integrable, and $\mu^- \leq \mu^+$.*

(2) *A sequence $\{A_n\}$ of fuzzy sets in $\mathfrak{F}(X)$ converges to a fuzzy set $A \in \mathfrak{F}(X)$ in the interval-valued distance measure $D_{\bar{\mu}}$ if and only if $\{A_n\}$ converges to A in the distance measures d_{μ^-} and d_{μ^+}, and $d_{\mu^-} \leq d_{\mu^+}$.*

Proof. (1) Let $\{A_n\}$ be a sequence of fuzzy sets in $\mathfrak{F}(X)$. If $\{A_n\}$ converges to A in the interval-valued distance measure $D_{\bar{\mu}}$, then, by (12) and (29),

$$\begin{aligned}\lim_{n\to\infty} d_{\mu^-}(A_n, A) &\leq \lim_{n\to\infty}(\max\{d_{\mu^-}(A_n, A), d_{\mu^+}(A_n, A)\}) \\ &= \lim_{n\to\infty} d_H(D_{\bar{\mu}}(A_n, A), 0) = 0.\end{aligned} \tag{33}$$

As in the same method with (33), we obtain

$$\lim_{n\to\infty} d_{\mu^+}(A_n, A) = 0. \tag{34}$$

Thus, by (33) and (34), $\{A_n\}$ converges to A in the distance measure d_{μ^-} and d_{μ^+}.

Conversely, if we take an interval-valued distance measure $\bar{\mu} = [\mu^-, \mu^+]$, then, similarly, we can obtain the converse result.

(2) Suppose that $\{A_n\}_{n\in I}$ is uniformly $\bar{\mu}$-integrable and $\bar{\mu}$ is continuous from below. By (12) and Definition2.3 (9), we have

$$\begin{aligned}\sup_{n\in I} D_{\bar{\mu}}(A_n, 0) &= \sup_{n\in I}[d_{\mu^-}(A_n, 0), d_{\mu^+}(A_n, 0)] \\ &= [\sup_{n\in I} d_{\mu^-}(A_n, 0), \sup_{n\in I} d_{\mu^+}(A_n, 0)] < \infty, \qquad (35)\end{aligned}$$

and for arbitrary $\varepsilon > 0$ and $E \in \Omega$, there exists $\delta(\varepsilon) > 0$ such that

$$\begin{aligned}\sup_{n\in I} D_{E,\bar{\mu}}(A_n, 0) &= \sup_{n\in I}[d_{E,\mu^-}(A_n, 0), d_{E,\mu^+}(A_n, 0)] \\ &= [\sup_{n\in I} d_{E,\mu^-}(A_n, 0), \sup_{n\in I} d_{E,\mu^+}(A_n, 0)] < \varepsilon, \qquad (36)\end{aligned}$$

if $\bar{\mu} < \delta(\varepsilon)$. By (35) and (36), $\{A_n\}$ converges to A in the distance measures d_{μ^-} and d_{μ^+}, nd $d_{\mu^-} \leq d_{\mu^+}$.

Conversely, if we take an interval-valued distance measure $\bar{\mu} = [\mu^-, \mu^+]$, then, similarly, we can obtain the converse result.

Theorem 5.4. *Let an interval-valued capacity $\bar{\mu}$ be subadditive and $\{A_n\}$ a sequence of fuzzy sets in $\mathfrak{F}(X)$. Then, $\{A_n\}$ is an uniformly $\bar{\mu}$-integrable if and only if*

$$\lim_{a\to\infty} \sup_{n\in\mathbb{N}} D_{[|\eta_{A_n}|>a],\bar{\mu}}(A_n, 0) = 0. \qquad (37)$$

Proof. Since an interval-valued capacity $\bar{\mu} = [\mu^-, \mu^+]$ is subadditive, by Theorem 3.1(5), μ^- and μ^+ are subadditive. From Theorem 5.3 (1), $\{A_n\}$ is an uniformly $\bar{\mu}$-integrable if and only if $\{A_n\}$ is an uniformly μ^--integrable and an uniformly μ^+-integrable. Thus, by Theorem 5.1, $\{A_n\}$ is an uniformly μ^--integrable if and only if

$$\lim_{a\to\infty} \sup_{n\in\mathbb{N}} d_{[|\eta_{A_n}|>a],\mu^-}(A_n, 0) = 0 \qquad (38)$$

and $\{A_n\}$ is an uniformly μ^+-integrable if and only if

$$\lim_{a\to\infty} \sup_{n\in\mathbb{N}} d_{[|\eta_{A_n}|>a],\mu^+}(A_n, 0) = 0 \qquad (39)$$

By (38) and (39), and (12), we have

$$\begin{aligned}&\lim_{a\to\infty} \sup_{n\in\mathbb{N}} d_H(D_{|\eta_{A_n}|>a],\bar{\mu}}(A_n, 0), 0) \\ = &\lim_{a\to\infty} \sup_{n\in\mathbb{N}} \max\{d_{[|\eta_{A_n}|>a],\mu^-}(A_n, 0), d_{[|\eta_{A_n}|>a],\mu^+}(A_n, 0)\} = 0. \qquad (40)\end{aligned}$$

Conversely, by the similar method of the above proof, we can obtain the converse result.

Lemma 5.5. *Assume that an interval-valued capacity $\bar{\mu} = [\mu^-, \mu^+]$ is continuous from below. Then $\{A_n\}$ is $\bar{\mu}$-integrable if and only if $\{A_n\}$ is μ^--integrable and μ^+-integrable*

Proof. The proof is trivial.

Theorem 5.6. *Let an interval-valued capacity μ be subadditive. If a sequence $\{A_n\}$ of fuzzy sets in $\mathfrak{F}(X)$ converges to a fuzzy set A in $\mathfrak{F}(X)$ μ-almost everywhere on X and $A_n \leq B$ for some $\bar{\mu}$-integrable fuzzy set B, then we have*

(1) *A_n and A are $\bar{\mu}$-integrable for all $n \in \mathbb{N}$, and*

(2) *$\{A_n\}$ converges to A in the interval-valued distance measure $D_{\bar{\mu}}$, that is,*

$$d_H - \lim_{n\to\infty} D_{\bar{\mu}}(A_n, 0) = 0. \qquad (41)$$

Proof. Since B is $\bar{\mu}$-integrable fuzzy set and $A_n \leq B$, by Theorem 5.3 (1), we have
(i)A_n and A are μ^--integrable and μ^--integrable for all $n \in \mathbb{N}$, and
(ii) $\{A_n\}$ converges to A in the distance measure d_{μ^-} and in the distance measure d_{μ^+}.
Thus, by Lemma 5.5 and Theorem 5.3 (1) and (12), we obtain
(1) A_n and A are $\bar{\mu}$-integrable for all $n \in \mathbb{N}$, and
(2) $\{A_n\}$ converges to A in the interval-valued distance measure $D_{\bar{\mu}}$, that is,

$$d_H - \lim_{n\to\infty} D_{\bar{\mu}}(A_n, 0) = 0. \tag{42}$$

6. Conclusions

In this paper, we define the concept of interval-valued capacity which means reasonable capacity. By using Aumann integral of integrably bounded interval-valued functions in Corollary 3.4, we consider the Choquet integral with respect to a continuous interval-valued capacity of a fuzzy set.

From Definitions 2.3, 3.1, 3.2 and Theorems 3.5, 3.6, we discuss interval-valued similarity measures induced by the Choquet integral with respect to a continuous interval-valued capacity on $\mathfrak{F}(X)$. By Examples 4.1 and 4.2, it is possible that we interpret the interval-valued measure of similarity of a sample with the three patterns. From Definitions 5.1, 5.2, 5.3, and Theorems 5.3, 5.4, and 5.6, we can provide the concept of convergence in the interval-valued distance measure and discuss various convergence properties of the interval-valued distance3 measure on the space of fuzzy sets for the Choquet integral.

In the future, by using these results of this paper, we can develop various problems and models for representing uncertain similarity measures and uncertain distance measures in pattern recognition research, information theory, decision making, and fuzzy risk analysis, etc.

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n-JORDAN $*$-DERIVATIONS ON INDUCED FUZZY C^*-ALGEBRAS

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Abstract. Using the fixed point alternative theorem, we investigate the Hyers-Ulam stability of of n-Jordan $*$-derivations on induced fuzzy C^*-algebras associated with the following functional equation $f(y-x)+f(x-z)+f(3x-y+z)=f(3x)$.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [43] concerning the stability of group homomorphisms. Hyers [22] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [38] for linear mappings by considering an unbounded Cauchy difference. Those results have been recently complemented in [9]. A generalization of the Aoki and Rassias theorem was obtained by Găvruta [21], who used a more general function controlling the possibly unbounded Cauchy difference in the spirit of Rassias' approach. The stability problems for several functional equations or inequalities have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [8, 15], [23]–[31], [39]–[41]).

We recall a fundamental result in fixed point theory.

Let X be a set. A function $d : X \times X \to [0,\infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x,y)=0$ if and only if $x=y$;
(2) $d(x,y)=d(y,x)$ for all $x,y\in X$;
(3) $d(x,z)\le d(x,y)+d(y,z)$ for all $x,y,z\in X$.

Theorem 1.1 (see [14, 18]). *Let (X,d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L<1$. Then for each given element $x\in X$, either*

$$d(J^n x, J^{n+1}x)=\infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) *$d(J^n x, J^{n+1}x)<\infty$, for all $n\ge n_0$;*
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y=\{y\in X|d(J^{n_0}x,y)<\infty\}$;*
(4) *$d(y,y^*)\le \frac{1}{1-L}d(y,Jy)$ for all $y\in Y$.*

By using the fixed point method, the stability problems of several functional equations have been extensively investigated by a number of authors (see [10, 13, 14, 17, 19, 28, 33, 34, 37, 46]).

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In 1984, Katsaras [27] defined a fuzzy norm on a linear space and at the same year Wu and Fang [44] also introduced a notion of fuzzy normed space and gave the generalization of the Kolmogoroff normalized theorem for fuzzy topological linear space. In [7], Biswas defined and studied fuzzy inner product spaces in linear space. Since then some mathematicians have defined fuzzy metrics and norms on a linear space from various points of view [6, 20, 30, 42, 45]. In 1994, Cheng and Mordeson introduced a definition of fuzzy norm on a linear space in such a manner that the corresponding induced fuzzy metric is of Kramosil and Michalek type [29]. In 2003, Bag and Samanta [6] modified the definition of Cheng and Mordeson [16] by removing a regular condition. They also established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy norms (see [3]). Following [2], we give the employing notion of a fuzzy norm.

Let X be a real linear space. A function $N : X \times \mathbb{R} \to [0,1]$(the so-called fuzzy subset) is said to be a fuzzy norm on X if for all $x, y \in X$ and all $a, b \in \mathbb{R}$:
(N_1) $N(x,a) = 0$ for $a \le 0$;
(N_2) $x = 0$ if and only if $N(x,a) = 1$ for all $a > 0$;
(N_3) $N(ax,b) = N(x, \frac{b}{|a|})$ if $a \neq 0$;
(N_4) $N(x+y, a+b) \ge min\{N(x,a), N(y,b)\}$;
(N_5) $N(x,.)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{a\to\infty} N(x,a) = 1$;
(N_6) For $x \neq 0$, $N(x,.)$ is (upper semi) continuous on $\mathbb{R}$.
The pair (X,N) is called a fuzzy normed linear space. One may regard $N(x,a)$ as the truth value of the statement the norm of x is less than or equal to the real number a'.

Definition 1.2. Let (X,N) be a fuzzy normed linear space. Let x_n be a sequence in X. Then x_n is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, a) = 1$ for all $a > 0$. In that case, x is called the limit of the sequence x_n and we denote it by $N\text{-}lim_{n\to\infty}\, x_n = x$.

Definition 1.3. A sequence x_n in X is called *Cauchy* if for each $\epsilon > 0$ and each $a > 0$ there exists n_0 such that for all $n \ge n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, a) > 1 - \epsilon$.

It is known that every convergent sequence in fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a *fuzzy Banach space*.

We say that a mapping $f : X \to Y$ between fuzzy normed vector space X, Y is continuous at point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to x_0 in X, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be *continuous* on X(see [2])

Definition 1.4. [36] Let X be a $*$-algebra and (X,N) a fuzzy normed space.

(1) The fuzzy normed space (X,N) is called a fuzzy normed $*$-algebra if

$$N(xy, st) \ge N(x,s) \cdot N(y,t) \quad \text{and} \quad N(x^*, t) = N(x,t).$$

(2) A complete fuzzy normed $*$-algebra is called a *fuzzy Banach $*$-algebra.*

Example 1.5. *Let $(X, \|.\|)$ be a normed $*$-algebras. Let*

$$N(x,a) = \begin{cases} \frac{a}{a+\|x\|}, & a > 0\ , x \in X, \\ 0, & a \le 0, x \in X \end{cases}$$

Then $N(x,t)$ is a fuzzy norm on X and $(X, N(x,t))$ is a fuzzy normed $*$-algebra.

Definition 1.6. Let $(X, \|\cdot\|)$ be a C^*-algebra and N a fuzzy norm on X.

(1) The fuzzy normed $*$-algebra (X, N) is called an induced fuzzy normed $*$-algebra.
(2) The fuzzy Banach $*$-algebra (X, N) is called an induced fuzzy C^*-algebra.

Definition 1.7. Let $(X, \|\cdot\|)$ be an induced fuzzy normed $*$-algebra. Then a $\mathbb{C}$-linear mapping $D : (X, N) \to (X, N)$ is called a *fuzzy n-Jordan $*$-derivation* if

$$\begin{aligned} D(x^n) &= D(x)x^{n-1} + xD(x)x^{n-2} + \cdots + x^{n-2}D(x)x + x^{n-1}D(x), \\ D(x^*) &= D(x)^* \end{aligned}$$

for all $x \in X$.

Throughout this paper, assume that (X, N) is an induced fuzzy C^*-algebra.

2. Main results

Lemma 2.1. *Let (Z, N) be a fuzzy normed vector space and $f : X \to Z$ be a mapping such that*

$$N\left(f\left(y-x\right)+f\left(x-z\right)+f\left(3x-y+z\right),t\right) \geq N\left(f\left(3x\right),\frac{t}{2}\right) \tag{2.1}$$

for all $x, y, z \in X$ and all $t > 0$. Then f is additive.

Proof. Letting $x = y = z = 0$ in (2.1), we get

$$N(3f(0), t) = N\left(f(0), \frac{t}{3}\right) \geq N\left(f(0), \frac{t}{2}\right)$$

for all $t > 0$. By (N_5) and (N_6), $N(f(0), t) = 1$ for all $t > 0$. It follows from (N_2) that $f(0) = 0$.

Letting $x = z = 0$ in (2.1), we get

$$N(f(y) + f(0) + f(-y), t) \geq N\left(f(0), \frac{t}{2}\right) = 1$$

for all $t > 0$. It follows from (N_2) that $f(-y) + f(y) = 0$ for all $y \in X$. Thus

$$f(-y) = -f(y)$$

for all $y \in X$.

Letting $x = 0$ and replacing z by $-z$ in (2.1), we get

$$N(f(y) + f(z) + f(-y-z), t) \geq N\left(f(0), \frac{t}{2}\right) = 1$$

for all $t > 0$. It follows from (N_2) that

$$f(y) + f(z) + f(-y-z) = 0$$

for all $y, z \in X$. Thus

$$f(y+z) = f(y) + f(z)$$

for all $y, z \in X$, as desired. □

Theorem 2.2. *Let $\phi : X^3 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\phi\left(\frac{x}{3}, \frac{y}{3}, \frac{z}{3}\right) \leq \frac{L}{3}\phi(x, y, z) \tag{2.2}$$

for all $x, y, z \in X$. Let $f : X \to X$ be a mapping such that

$$\begin{aligned} &N\left(f\left(\mu(y-x)\right) + f\left(\mu(x-z)\right) + f\left(\mu(3x-y+z)\right) - \mu f\left(3x\right), t\right) \\ &\geq \frac{t}{t + \phi(x, y, z)}, \end{aligned} \tag{2.3}$$

$$\begin{aligned} &N\left(f(w^n) - f(w)w^{n-1} - wf(w)w^{n-2} - \cdots - w^{n-2}f(w)w - w^{n-1}f(w)\right. \\ &\left. + f(v^*) - f(v)^*, t\right) \geq \frac{t}{t + \phi(w, v, 0)} \end{aligned} \tag{2.4}$$

for all $x, y, z, w, v \in X$, all $t > 0$ and all $\mu \in \mathbb{T}^1 := \{c \in \mathbb{C} : |c| = 1\}$. Then the limit $A(x) = N - \lim_{n\to\infty} 3^n f\left(\frac{x}{3^n}\right)$ exists for each $x \in X$ and the mapping $A : X \to X$ is a fuzzy n-Jordan $*$-derivation satisfying

$$N(f(x) - A(x), t) \geq \frac{3(1-L)t}{3(1-L)t + L\phi\left(x, 2x, 0\right)} \tag{2.5}$$

for all $x \in X$ and all $t > 0$.

Proof. Letting $\mu = 1, y = 2x$, $z = 0$ in (2.3), we have

$$N\left(3f\left(x\right) - f(3x), t\right) \geq \frac{t}{t + \phi\left(x, 2x, 0\right)} \tag{2.6}$$

and so

$$N\left(3f\left(\frac{x}{3}\right) - f(x), t\right) \geq \frac{t}{t + \phi\left(\frac{x}{3}, \frac{2x}{3}, 0\right)} = \frac{t}{t + \frac{L}{3}\phi\left(x, 2x, 0\right)}$$

for all $x \in X$. Thus

$$N\left(3f\left(\frac{x}{3}\right) - f(x), \frac{L}{3}t\right) \geq \frac{\frac{L}{3}t}{\frac{L}{3}t + \frac{L}{3}\phi\left(x, 2x, 0\right)} = \frac{t}{t + \phi\left(x, 2x, 0\right)} \tag{2.7}$$

for all $x \in X$.

Consider the set

$$G := \{g : X \to X\}$$

and introduce the generalized metric on G:

$$d(g, h) := \inf\{a \in \mathbb{R}^+ : N(g(x) - h(x), at) \geq \frac{t}{t + \phi\left(x, 2x, 0\right)}\}$$

for all $x \in X$ and all $t > 0$, where $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see the proof of [32, Lemma 2.1]

Now, we consider the linear mapping $Q : G \to G$ such that

$$Qg(x) := 3g\left(\frac{x}{3}\right)$$

for all $x \in X$.

Let $g, h \in G$ be given such that $d(g,h) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \geq \frac{t}{t + \phi(x, 2x, 0)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Qg(x) - Qh(x), L\varepsilon t) &= N\left(3g\left(\frac{x}{3}\right) - 3h\left(\frac{x}{3}\right), L\varepsilon t\right) = N\left(g\left(\frac{x}{3}\right) - h\left(\frac{x}{3}\right), \frac{L}{3}\varepsilon t\right) \\ &\geq \frac{\frac{Lt}{3}}{\frac{Lt}{3} + \phi\left(\frac{x}{3}, \frac{2x}{3}, 0\right)} \geq \frac{\frac{Lt}{3}}{\frac{Lt}{3} + \frac{L}{3}\phi(x, 2x, 0)} \\ &= \frac{t}{t + \phi(x, 2x, 0)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. Thus $d(g,h) = \varepsilon$ implies that $d(Qg, Qh) \leq L\varepsilon$. This means that

$$d(Qg, Qh) \leq L d(g,h)$$

for all $g, h \in G$.

It follows from (2.7) that $d(f, Qf) \leq \frac{L}{3}$.

By Theorem 1.1, there exists a mapping $A : X \to X$ satisfying the following:

(1) A is a fixed point of Q, *i.e.*,

$$A\left(\frac{x}{3}\right) = \frac{1}{3}A(x) \tag{2.8}$$

for all $x \in X$. The mapping A is a unique fixed point of Q in the set

$$M = \{g \in G : d(f, g) < \infty\}.$$

This implies that A is a unique mapping satisfying (2.8) such that there exists an $a \in (0, \infty)$ satisfying

$$N(f(x) - A(x), at) \geq \frac{t}{t + \phi(x, 2x, 0)}$$

for all $x \in X$.

(2) $d(Q^k f, A) \to 0$ as $k \to \infty$. This implies the equality

$$N - \lim_{k \to \infty} 3^k f\left(\frac{x}{3^k}\right) = A(x)$$

for all $x \in X$;

(3) $d(f, A) \leq \frac{1}{1-L} d(f, Qf)$, which implies the inequality

$$d(f, A) \leq \frac{L}{3(1-L)}.$$

This implies that the inequality (2.5) holds.

Next we show that A is additive. It follows from (2.2) that

$$\begin{aligned}\sum_{k=0}^{\infty} 3^k\phi\left(\frac{x}{3^k},\frac{y}{3^k},\frac{z}{3^k}\right) &= \phi(x,y,z) + 3\phi\left(\frac{x}{3},\frac{y}{3},\frac{z}{3}\right) + 3^2\phi\left(\frac{x}{3^2},\frac{y}{3^2},\frac{z}{3^2}\right) + \cdots \\ &\leq \phi(x,y,z) + L\phi(x,y,z) + L^2\phi(x,y,z) + \cdots \\ &= \frac{1}{1-L}\phi(x,y,z) < \infty\end{aligned}$$

for all $x,y,z \in X$.

By (2.3),

$$\begin{aligned}&N\left(3^k f\left(\mu\frac{y-x}{3^k}\right) + 3^k f\left(\mu\frac{x-z}{3^k}\right) + f\left(\mu\frac{3x-y+z}{3^k}\right) - 3^k\mu f\left(\frac{3}{3^k}x\right), 3^k t\right) \\ &\geq \frac{t}{t+\phi\left(\frac{x}{3^k},\frac{y}{3^k},\frac{z}{3^k}\right)}\end{aligned}$$

and so

$$\begin{aligned}&N\left(3^k f\left(\mu\frac{y-x}{3^k}\right) + 3^k f\left(\mu\frac{x-z}{3^k}\right) + 3^k f\left(\mu\frac{3x-y+z}{3^k}\right) - 3^k\mu f\left(\frac{3}{3^k}x\right), t\right) \\ &\geq \frac{\frac{t}{3^k}}{\frac{t}{3^k}+\phi\left(\frac{x}{3^k},\frac{y}{3^k},\frac{z}{3^k}\right)} = \frac{t}{t+3^k\phi\left(\frac{x}{3^k}\cdot\frac{y}{3^k},\frac{z}{3^k}\right)}\end{aligned}$$

for all $x,y,z \in X$, all $t > 0$ and all $\mu \in \mathbb{T}^1$. Since $\lim_{k\to\infty} \frac{t}{t+3^k\phi\left(\frac{x}{3^k},\frac{y}{3^k},\frac{z}{3^k}\right)} = 1$ for all $x,y,z \in X$ and all $t > 0$,

$$N\left(A\left(\mu(y-x)\right) + A\left(\mu(x-z)\right) + A\left(\mu(3x-y+z)\right) - \mu A\left(3x\right), t\right) = 1$$

for all $x,y,z \in X$, all $t > 0$ and all $\mu \in \mathbb{T}^1$. So

$$A\left(\mu(y-x)\right) + A\left(\mu(x-z)\right) + A\left(\mu(3x-y+z)\right) = \mu A\left(3x\right) \tag{2.9}$$

for all $x,y,z \in X$, all $t > 0$ and all $\mu \in \mathbb{T}^1$. Letting $x = y = z = 0$ in (2.9), we have $A(0) = 0$. Let $\mu = 1$, $x = 0$ and replace z by $-z$ in (2.9). By the same reasoning as in the proof of Lemma 2.1, one can easily show that A is additive. Letting $y = 2x$, $z = 0$ in (2.9), we get

$$\mu A(x) = 3A\left(\mu\frac{x}{3}\right) = A(\mu x)$$

for all $x \in X$ and $\mu \in \mathbb{T}^1$. The mapping $A : X \to X$ is $\mathbb{C}$-linear by [35, Theorem 2.1].

By (2.4) and letting $v = 0$ in (2.4), we get

$$\begin{aligned}&N\left(3^{nk} f\left(\frac{w^n}{3^{nk}}\right) - 3^{nk} f\left(\frac{w}{3^k}\right)\left(\frac{w}{3^k}\right)^{n-1} - 3^{nk}\frac{w}{3^k} f\left(\frac{w}{3^k}\right)\left(\frac{w}{3^k}\right)^{n-2} - \cdots\right. \\ &\left.-3^{nk}\left(\frac{w}{3^k}\right)^{n-2} f\left(\frac{w}{3^k}\right) w - 3^{nk}\left(\frac{w}{3^k}\right)^{n-1} f\left(\frac{w}{3^k}\right), 3^{nk}t\right) \geq \frac{t}{t+\phi(\frac{w}{3^k},0,0)}\end{aligned}$$

for all $w \in X$ and all $t > 0$. Thus

$$N\left(3^{nk} f\left(\frac{w^n}{3^{nk}}\right) - 3^{nk} f\left(\frac{w}{3^k}\right)\left(\frac{w}{3^k}\right)^{n-1} - 3^{nk}\frac{w}{3^k} f\left(\frac{w}{3^k}\right)\left(\frac{w}{3^k}\right)^{n-2} - \cdots\right.$$
$$\left. -3^{nk}\left(\frac{w}{3^k}\right)^{n-2} f\left(\frac{w}{3^k}\right) w - 3^{nk}\left(\frac{w}{3^k}\right)^{n-1} f\left(\frac{w}{3^k}\right), t\right) \geq \frac{\frac{t}{3^{nk}}}{\frac{t}{3^{nk}} + \phi(\frac{w}{3^k}, 0, 0)}$$
$$\geq \frac{t}{t + (3^{n-1}L)^k \phi(w, 0, 0)}$$

for all $w \in X$ and all $t > 0$. Since $\lim_{k\to\infty} \frac{t}{t+(3^{n-1}L)^k\phi(w,0,0)} = 1$ for all $w \in X$ and all $t > 0$, we get

$$N(D(w^n) - D(w)w^{n-1} - wD(w)w^{n-2} - \cdots - w^{n-2}D(w)w - w^{n-1}D(w), t) = 1$$

for all $x \in X$ and all $t > 0$. So

$$D(w^n) - D(w)w^{n-1} - wD(w)w^{n-2} - \cdots - w^{n-2}D(w)w - w^{n-1}D(w) = 0$$

for all $w \in X$.

Letting $w = 0$ in (2.4), similarly, we get $D(v^*) - D(v)^* = 0$ for all $v \in X$.

Therefore, the mapping $D : X \to X$ is a fuzzy n-Jordan $*$-derivation. □

Corollary 2.3. *Let p be a real number with $p > 1$, $\theta \geq 0$, and X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to X$ be a mapping satisfying*

$$N\left(f\left(\mu(y-x)\right) + f\left(\mu(x-z)\right) + f\left(\mu(3x-y+z)\right) - \mu f\left(3x\right), t\right) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p + \|z\|^p)}, \tag{2.10}$$

$$N\left(f(w^n) - f(w)w^{n-1} - wf(w)w^{n-2} - \cdots - w^{n-2}f(w)w - w^{n-1}f(w) + f(v^*) - f(v)^*, t\right) \geq \frac{t}{t + \theta(\|w\|^p + \|v\|^p)} \tag{2.11}$$

for all $x, y, w, v \in X$, all $t > 0$ and all $\mu \in \mathbb{T}^1$. Then the limit $A(x) = N-\lim_{n\to\infty} 3^n f\left(\frac{x}{3^n}\right)$ exists for each $x \in X$ and the mapping $A : X \to X$ is a fuzzy n-Jordan $*$-derivation satisfying

$$N(f(x) - A(x), t) \geq \frac{(3^p - 3)t}{(3^p - 3)t + \theta(1 + 2^p)\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.2 by taking

$$\phi(x, y, z) = \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

and $L = 3^{1-p}$. □

Theorem 2.4. *Let $\phi : X^3 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$3L\phi\left(\frac{x}{3}, \frac{y}{3}, \frac{z}{3}\right) \leq \phi(x, y, z) \tag{2.12}$$

for all $x, y, z \in X$. Let $f : X \to X$ be a mapping satisfying (2.3) and (2.4). Then the limit $A(x) = N - \lim_{n\to\infty} \frac{1}{3^n} f(3^n x)$ exists for each $x \in X$ and the mapping $A : X \to X$ is a fuzzy n-Jordan ∗-derivation satisfying

$$N(f(x) - A(x), t) \geq \frac{3(1-L)t}{3(1-L)t + \phi(x, 2x, 0)} \tag{2.13}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (G, d) be generalized metric space defined in the proof of Theorem 2.2. Consider the linear mapping $Q : G \to G$ such that

$$Qg(x) := \frac{1}{3} g(3x)$$

for all $x \in X$.

It follow from (2.6) that

$$N\left(f(x) - \frac{1}{3} f(3x), \frac{1}{3} t\right) \geq \frac{t}{t + \phi(x, 2x, 0)}$$

for all $x \in X$ and all $t > 0$. Thus $d(f, Qf) \leq \frac{1}{3}$. Hence

$$d(f, A) \leq \frac{1}{3(1-L)},$$

which implies that the inequality (2.13) holds.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let $\theta \geq 0$ and let p be a positive real number with $p < 1$. Let X be a normed vector space with normed $\|\cdot\|$. Let $f : X \to X$ be a mapping satisfying (2.10) and (2.11). Then $A(x) = N - \lim_{n\to\infty} \frac{1}{3^n} f(3^n x)$ exists for each $x \in X$ and defines a fuzzy n-Jordan ∗-derivation $A : X \to X$ such that*

$$N(f(x) - A(x), t) \geq \frac{(3 - 3^p)t}{(3 - 3^p)t + \theta(1 + 2^p)\|x\|^p}$$

for every $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.4 by taking

$$\phi(x, y, z) = \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

and $L = 3^{p-1}$. □

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Global stability analysis of a delayed viral infection model with antibodies and general nonlinear incidence rate

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Abstract

In this paper, we study the global properties of a viral infection model with antibody immune response. The incidence rate is given by a general function of the populations of the uninfected target cells, infected cells and free viruses. The model contains two types of intracellular discrete time delays to describe the time required for viral contacting an uninfected target cell and viral emission. We have established a set of conditions on the general incidence rate function and determined two threshold parameters R_0 (the basic infection reproduction number) and R_1 (the antibody immune response activation number) which are sufficient to determine the global behavior of the model. The global asymptotic stability of the equilibria of the model has been proven by using direct Lyapunov method and applying LaSalle's invariance principle.

Keywords: Virus dynamics; Intracellular delay; global stability; antibody immune response; Lyapunov functional.

1 Introduction

In recent years, several works have been devoted to study and develop mathematical models of the virus dynamics such as human immunodeficiency virus (HIV) (see e.g. [1]-[14]), hepatitis B virus (HBV) [15]-[18], hepatitis C virus (HCV) [19]-[21] and human T cell leukemia HTLV [22], etc. Mathematical models of viral infection can help for understanding the viral dynamics and developing antiviral drug therapies. In reality, the immune response needs an indispensable components to do its job such as antibodies, cytokines, natural killer cells, and T cells. The antibody immune response is a part of the adaptive system in which the body responds to pathogens by primarily using antibodies that produced from the B cells. While the other part is the Cytotoxic T Lymphocytes (CTL) immune response where the CTL attacks and kills the infected cells [4]. In some infections such as in malaria, the CTL immune response is less effective than the antibody immune response [23]. Mathematical models of viral infection with antibody immune response have been proposed and analyzed in ([24]-[29]). The basic model of viral infection with antibody immune response has introduced by Murase et. al. [24] and Shifi Wang [29] as:

$$\dot{x}(t) = s - dx(t) - \beta v(t)x(t), \tag{1}$$

$$\dot{y}(t) = \beta v(t)x(t) - ay(t), \tag{2}$$

$$\dot{v}(t) = ky(t) - bz(t)v(t) - cv(t), \tag{3}$$

$$\dot{z}(t) = rz(t)v(t) - \mu z(t), \tag{4}$$

where $x(t)$, $y(t)$, $v(t)$ and $z(t)$ denote the populations of uninfected target cells, infected cells, free virus particles and antibody immune cells at time t, respectively. Parameters s, k and r represent, respectively, the rate at which new healthy cells are generated from the source within the body, the generation rate constant of free viruses produced from the infected cells and the proliferation rate constant of antibody immune cells. Parameters d, a, c and μ are the natural death rate constants of the uninfected cells, infected cells, free virus particles and antibody immune cells, respectively. Parameter β is the infection rate constant and b is the removal rate constant of the virus due to the antibodies. All the parameters given in model (1)-(4) are positive.

The intracellular time delay between the time of the virus contacting the target cells and the time of generating new infectious viruses has been neglected in system (1)-(4). In fact, the intracellular delay in the infection process is actually exists (see e.g. [8]-[12]). Note that, the infection rate in model (1)-(4) is presented to be bilinear in x and v, which can not be completely describe the interaction between the uninfected target cells and viruses. Nevertheless, there are many types of improved incidence rates which are more commonly used due to their benefit for helping us gain the unification theory through passing over the unessential details (see e.g. [30] and [31]). Variety of viral infection models with antibody immune response have been considered with different forms of the incidence rate such as saturated incidence rate, $\frac{\beta xv}{1+\alpha v}$ where $\alpha \geq 0$, [27], Beddington-DeAngelis functional response, $\frac{\beta xv}{1+\gamma x+\alpha v}$, $\alpha, \gamma \geq 0$ [26], and general form, $\psi(x,v)v$ [28]. In [28], a discrete time delay has been incorporated within the model. However, the infection rate does not depend on the infected cells y. In some viral infections such as HBV, the infection rate depends on x, y and v [17], [16]. In [32], the infection rate is given by $\psi(x,y,v)v$, however the antibody immune response has been neglected. Our aim in this paper is to investigate the global stability analysis of a viral infection model with general incidence rate function and antibody immune response taking into consideration two types of discrete time delays.

The rest of the paper is designed as follows. In the next section, we introduce the model and discuss the non-negativity and boundedness of the solutions. In Section 3, we define two threshold parameters and discuss the existence of the model's equilibria. In Section 4, we study the global asymptotic stability of the equilibria using suitable Lyapunov functional and applying LaSalle's invariance principle. Finally, conclusion is given in Section 5.

2 The mathematical model

In this section, we consider the following viral infection model with general incidence rate taking into consideration the antibody immune response.

$$\dot{x}(t) = s - dx(t) - \psi(x(t), y(t), v(t))v(t), \tag{5}$$

$$\dot{y}(t) = e^{-\mu_1\tau_1}\psi(x(t-\tau_1), y(t-\tau_1), v(t-\tau_1))v(t-\tau_1) - ay(t), \tag{6}$$

$$\dot{v}(t) = ke^{-\mu_2\tau_2}y(t-\tau_2) - bz(t)v(t) - cv(t), \tag{7}$$

$$\dot{z}(t) = rz(t)v(t) - \mu z(t), \tag{8}$$

where τ_1 and τ_2 are the delay parameters. We assume that, the virus contacts an uninfected target cell at time $t-\tau_1$, the cell becomes infected at time t. The term $e^{-\mu_1\tau_1}$ represents the probability of surviving the contacted cell during the time delay interval, where μ_1 is the death rate constant of the contacted cells. In addition, we assume that a cell infected at time $t-\tau_2$ starts to generate new infectious viruses at time t. The term $e^{-\mu_2\tau_2}$ denotes the probability of surviving the infected cell during the time delay interval, where μ_2 is a constant. The definitions of all variables and parameters are identical to those given in Section 1. The incidence rate of infection is presented by a general function in the form $\psi(x, y, v)v$, where ψ is continuously differentiable and satisfies the following assumptions [28] and [32]:

Assumption A1. $\psi(0, y, v) = 0$ for all $y, v \geq 0$ and $\psi(x, y, v) > 0$ for all $x > 0$, $y \geq 0$, $v \geq 0$.

Assumption A2. $\dfrac{\partial\psi(x, y, v)}{\partial x} > 0$ for all $x > 0$, $y \geq 0$ and $v \geq 0$.

Assumption A3. $\dfrac{\partial\psi(x, y, v)}{\partial y} < 0$, $\dfrac{\partial\psi(x, y, v)}{\partial v} < 0$ for all $x, y, v > 0$.

Assumption A4. $\dfrac{\partial\left(\psi(x, y, v)v\right)}{\partial v} > 0$ for all $x, y, v > 0$.

Let the initial states of system (5)-(8) be given as:

$$\begin{aligned} &x(\eta) = \zeta_1(\eta),\ y(\eta) = \zeta_2(\eta), v(\eta) = \zeta_3(\eta),\ z(\eta) = \zeta_4(\eta), \\ &\zeta_j(\eta) \geq 0,\ \eta \in [-\tau, 0),\quad j = 1, ..., 4, \\ &\zeta_j(0) > 0,\ j = 1, ..., 4, \end{aligned} \tag{9}$$

where $\tau = \max\{\tau_1, \tau_2\}$, $(\zeta_1(\eta), \zeta_2(\eta), \zeta_3(\eta), \zeta_4(\eta)) \in C([-\tau, 0], \mathbb{R}^4_{\geq 0})$. We denote by $C = C([-\tau, 0], \mathbb{R}^4_{\geq 0})$ the

Banach space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^4_{\geq 0}$; with norm $\|\zeta\| = \sup_{-\tau \leq \eta \leq 0} |\zeta(\eta)|$ for $\zeta \in C$. We note that the system (5)-(8) with initial states (9) has a unique solution [33].

2.1 Non-negativity and boundedness of solutions

In this section, we show that the solutions of model (5)-(8) with initial states (9) are non-negative and ultimately bounded.

Proposition 1. Assume that Assumption A1 is satisfied. Then the solutions of (5)-(8) with the initial states (9) are non-negative and ultimately bounded.

Proof. At the beginning, we show that $x(t)$ is positive for all $t \geq 0$. Let us assume in contrary that $x(t) \leq 0$ on the time interval $[0, \gamma]$ where γ is a constant, and let where $\bar{t} \in [0, \gamma]$ be such that $x(\bar{t}) = 0$, . Then from Eq. (5) we get $\dot{x}(\bar{t}) = s > 0$. Thus, for sufficiently small $\varepsilon > 0$, we have $x(t) > 0$ for some $t \in (\bar{t}, \bar{t} + \varepsilon)$. This contradicts our assumption and then $x(t) > 0$, $\forall\ t \geq 0$. Now from Eqs. (6)-(8) we get

$$y(t) = y(0)e^{-at} + e^{-\mu_1\tau_1} \int_0^t e^{-a(t-\eta)} \psi(x(\eta - \tau_1), y(\eta - \tau_1), v(\eta - \tau_1))v(\eta - \tau_1)d\eta,$$

$$v(t) = v(0)e^{-\int_0^t (c+bz(\xi))d\xi} + ke^{-\mu_2\tau_2} \int_0^t e^{-\int_\eta^t (c+bz(\xi))d\xi} y(\eta - \tau_2)d\eta,$$

$$z(t) = z(0)e^{-\int_0^t (\mu - rv(\xi))d\xi},$$

which yield $y(t)$, $v(t)$, $z(t) \geq 0$ for all $t \in [0, \tau]$. By a recursive argument, we get that $y(t), v(t), z(t) \geq 0$ for all $t \geq 0$.

Next we prove the ultimate bound of the solutions of system (5)-(8). From Eq. (5) we get $\dot{x}(t) \leq s - dx(t)$ and thus $\limsup_{t\to\infty} x(t) \leq \frac{s}{d}$. Let $T_1(t) = e^{-\mu_1\tau_1}x(t - \tau_1) + y(t)$, then

$$\begin{aligned}
\dot{T_1}(t) &= e^{-\mu_1\tau_1} \left(s - dx(t - \tau_1) - \psi(x(t - \tau_1), y(t - \tau_1), v(t - \tau_1))v(t - \tau_1)\right) \\
&\quad + e^{-\mu_1\tau_1}\psi(x(t - \tau_1), y(t - \tau_1), v(t - \tau_1))v(t - \tau_1) - ay(t), \\
&= se^{-\mu_1\tau_1} - de^{-\mu_1\tau_1}x(t - \tau_1) - ay(t) \leq se^{-\mu_1\tau_1} - \sigma_1 \left(e^{-\mu_1\tau_1}x(t - \tau_1) + y(t)\right) \\
&= se^{-\mu_1\tau_1} - \sigma_1 T_1(t) \leq s - \sigma_1 T_1(t),
\end{aligned}$$

where $\sigma_1 = \min\{d, a\}$. Hence $\limsup_{t\to\infty} T_1(t) \le L_1$, where $L_1 = \dfrac{s}{\sigma_1}$. Since $x(t)$ and $y(t)$ are non-negative, then $\limsup_{t\to\infty} y(t) \le L_1$. Moreover, let $T_2(t) = v(t) + \frac{b}{r}z(t)$, then

$$\begin{aligned}\dot{T}_2(t) &= ke^{-\mu_2\tau_2}y(t-\tau_2) - cv(t) - \frac{b\mu}{r}z(t) \le ke^{-\mu_2\tau_2}L_1 - \sigma_2(v(t) + \frac{b}{r}z(t))\\ &= ke^{-\mu_2\tau_2}L_1 - \sigma_2 T_2(t) \le kL_1 - \sigma_2 T_2(t),\end{aligned}$$

where $\sigma_2 = \min\{c, \mu\}$. It follows that, $\limsup_{t\to\infty} T_2(t) \le L_2$, where $L_2 = \dfrac{kL_1}{\sigma_2}$. Since $v(t)$ and $z(t)$ are non-negative, then $\limsup_{t\to\infty} v(t) \le L_2$ and $\limsup_{t\to\infty} z(t) \le L_3$, where $L_3 = \frac{r}{b}L_2$. Therefore, all the state variables of the model are ultimately bounded.

2.2 The equilibria and threshold parameters

At any equilibrium we have

$$s - dx - \psi(x,y,v)v = 0, \tag{10}$$

$$e^{-\mu_1\tau_1}\psi(x,y,v)v - ay = 0, \tag{11}$$

$$ke^{-\mu_2\tau_2}y - bvz - cv = 0, \tag{12}$$

$$(rv - \mu)z = 0. \tag{13}$$

From Eq. (13), either $z = 0$ or $z \ne 0$. If $z = 0$, then from Eqs. (10)-(12) we get

$$y = \frac{s-dx}{ae^{\mu_1\tau_1}} = \frac{c}{ke^{-\mu_2\tau_2}}v, \qquad v = \frac{k(s-dx)}{ace^{\mu_1\tau_1+\mu_2\tau_2}}. \tag{14}$$

Substituting from Eq. (14) into Eq. (11) we get:

$$\left[\psi\left(x, \frac{s-dx}{ae^{\mu_1\tau_1}}, \frac{k(s-dx)}{ace^{\mu_1\tau_1+\mu_2\tau_2}}\right) - \frac{ac}{k}e^{\mu_1\tau_1+\mu_2\tau_2}\right]v = 0. \tag{15}$$

Eq. (15) has two possible solutions $v = 0$ or $v \ne 0$. If $v = 0$, then from Eqs. (10) and (11), we get $x = s/d$ and $y = 0$ which leads to the infection-free equilibrium $E_0(x_0, 0, 0, 0)$ where $x_0 = s/d$. If $v \ne 0$, then we have

$$\psi\left(x, \frac{s-dx}{ae^{\mu_1\tau_1}}, \frac{k(s-dx)}{ace^{\mu_1\tau_1+\mu_2\tau_2}}\right) - \frac{ac}{k}e^{\mu_1\tau_1+\mu_2\tau_2} = 0.$$

Let

$$\Phi_1(x) = \psi\left(x, \frac{s-dx}{ae^{\mu_1\tau_1}}, \frac{k(s-dx)}{ace^{\mu_1\tau_1+\mu_2\tau_2}}\right) - \frac{ac}{k}e^{\mu_1\tau_1+\mu_2\tau_2} = 0,$$

then, we have

$$\Phi_1'(x) = \frac{\partial \psi}{\partial x} - \frac{d}{ae^{\mu_1\tau_1}}\frac{\partial \psi}{\partial y} - \frac{kd}{ace^{\mu_1\tau_1+\mu_2\tau_2}}\frac{\partial \psi}{\partial v}.$$

Because of Assumptions A2 and A3, we have $\Phi_1'(x) > 0$ which implies that function $\Phi_1(x)$ is strictly increasing w.r.t. x. Moreover,

$$\Phi_1(0) = \psi\left(0, \frac{s}{ae^{\mu_1\tau_1}}, \frac{ks}{ace^{\mu_1\tau_1+\mu_2\tau_2}}\right) - \frac{ac}{k}e^{\mu_1\tau_1+\mu_2\tau_2} = -\frac{ac}{k}e^{\mu_1\tau_1+\mu_2\tau_2} < 0,$$

$$\Phi_1(x_0) = \psi(x_0,0,0) - \frac{ac}{k}e^{\mu_1\tau_1+\mu_2\tau_2} = \frac{ac}{k}e^{\mu_1\tau_1+\mu_2\tau_2}\left(\frac{k\psi(x_0,0,0)}{ac}e^{-\mu_1\tau_1-\mu_2\tau_2} - 1\right).$$

Therefore, if $\frac{k\psi(x_0,0,0)}{ac}e^{-\mu_1\tau_1-\mu_2\tau_2} > 1$, then there exist a unique $x_1 \in (0, x_0)$ such that $\Phi_1(x_1) = 0$. It follows from (12) and (14) that $y_1 = \frac{d(x_0 - x_1)}{ae^{\mu_1\tau_1}} > 0$ and $v_1 = \frac{kd(x_0 - x_1)}{ace^{\mu_1\tau_1+\mu_2\tau_2}} > 0$. It means that, a chronic-infection equilibrium without antibody immune response $E_1(x_1, y_1, v_1, 0)$ exists when $\frac{k\psi(x_0,0,0)}{ac}e^{-\mu_1\tau_1-\mu_2\tau_2} > 1$. Let us define the basic infection reproduction number as:

$$R_0 = \frac{k\psi(x_0,0,0)}{ac}e^{-\mu_1\tau_1-\mu_2\tau_2}.$$

The parameter R_0 determines whether a chronic-infection can be established. The other possibility of Eq. (13) is $z \neq 0$ which leads to $v_2 = \frac{\mu}{r}$. From Eq. (10) we let

$$\Phi_2(x) = s - dx - \psi(x, \frac{s - dx}{ae^{\mu_1\tau_1}}, v_2)v_2 = 0.$$

According to Assumptions A2 and A3, we know that Φ_2 is a decreasing function of x. Clearly, $\Phi_2(0) = s > 0$ and $\Phi_2(x_0) = -\psi(x_0, 0, v_2)v_2 < 0$. Thus, there exists a unique $x_2 \in (0, x_0)$ such that $\Phi_2(x_2) = 0$. It follows from Eq. (14) that, $y_2 = \frac{d(x_0 - x_2)}{ae^{\mu_1\tau_1}} > 0$ and $z_2 = \frac{k\psi(x_2,y_2,v_2)}{abe^{\mu_1\tau_1+\mu_2\tau_2}} - \frac{c}{b} = \frac{c}{b}\left(\frac{k\psi(x_2,y_2,v_2)}{ace^{\mu_1\tau_1+\mu_2\tau_2}} - 1\right)$. Then, if $\frac{k\psi(x_2,y_2,v_2)}{ace^{\mu_1\tau_1+\mu_2\tau_2}} > 1$ then $z_2 > 0$. Now we define the antibody immune response activation number as

$$R_1 = \frac{k\psi(x_2,y_2,v_2)}{ace^{\mu_1\tau_1+\mu_2\tau_2}},$$

which determines whether a persistent antibody immune response can be established. Hence, z_2 can be rewritten as $z_2 = \frac{c}{b}(R_1 - 1)$. It follows that, there is a chronic-infection equilibrium with antibody immune response $E_2(x_2, y_2, v_2, z_2)$ when $R_1 > 1$.

Clearly from Assumptions A2 and A3, we have

$$R_1 = \frac{k\psi(x_2,y_2,v_2)}{ace^{\mu_1\tau_1+\mu_2\tau_2}} < \frac{k\psi(x_0,y_2,v_2)}{ace^{\mu_1\tau_1+\mu_2\tau_2}} < \frac{k\psi(x_0,0,0)}{ace^{\mu_1\tau_1+\mu_2\tau_2}} = R_0.$$

2.3 Global stability analysis

In this section, the global asymptotic stability of the three equilibria of model (5)-(8) will be established by using direct Lyapunov method and applying LaSalle's invariance principle. In the remaining parts of the paper we shall use the following function: $H:(0,\infty)\to[0,\infty)$,

$$H(u)=u-1-\ln u.$$

Theorem 1. Let Assumptions A1-A3 be hold true and $R_0\le 1$, then the infection-free equilibrium E_0 is globally asymptotically stable (GAS).

Proof. We construct a Lyapunov functional as:

$$\begin{aligned}U_0 &= x-x_0-\int_{x_0}^{x}\frac{\psi(x_0,0,0)}{\psi(\eta,0,0)}d\eta+e^{\mu_1\tau_1}y+\frac{a}{k}e^{\mu_1\tau_1+\mu_2\tau_2}v+\frac{ab}{rk}e^{\mu_1\tau_1+\mu_2\tau_2}z\\ &+\int\limits_{t-\tau_1}^{t}\psi(x(\eta),y(\eta),v(\eta))v(\eta)d\eta+ae^{\mu_1\tau_1}\int\limits_{t-\tau_2}^{t}y(\eta)d\eta. \qquad (16)\end{aligned}$$

We calculate $\frac{dU_0}{dt}$ along the solutions of model (5)-(8) as:

$$\begin{aligned}\frac{dU_0}{dt} &= \left(1-\frac{\psi(x_0,0,0)}{\psi(x,0,0)}\right)(s-dx-\psi(x,y,v)\,v)+\psi(x(t-\tau_1),y(t-\tau_1),v(t-\tau_1))v(t-\tau_1)-ae^{\mu_1\tau_1}y\\ &+ae^{\mu_1\tau_1}y(t-\tau_2)-\frac{ac}{k}e^{\mu_1\tau_1+\mu_2\tau_2}v-\frac{ab}{k}e^{\mu_1\tau_1+\mu_2\tau_2}zv+\frac{ab}{k}e^{\mu_1\tau_1+\mu_2\tau_2}zv-\frac{ab\mu}{rk}e^{\mu_1\tau_1+\mu_2\tau_2}z\\ &+\psi(x,y,v)v-\psi(x(t-\tau_1),y(t-\tau_1),v(t-\tau_1))v(t-\tau_1)+ae^{\mu_1\tau_1}(y-y(t-\tau_2))\\ &= s\left(1-\frac{\psi(x_0,0,0)}{\psi(x,0,0)}\right)\left(1-\frac{x}{x_0}\right)+\left(\psi(x,y,v)\frac{\psi(x_0,0,0)}{\psi(x,0,0)}-\frac{ac}{k}e^{\mu_1\tau_1+\mu_2\tau_2}\right)v-\frac{ab\mu}{rk}e^{\mu_1\tau_1+\mu_2\tau_2}z\\ &= s\left(1-\frac{\psi(x_0,0,0)}{\psi(x,0,0)}\right)\left(1-\frac{x}{x_0}\right)+\frac{ac}{k}e^{\mu_1\tau_1+\mu_2\tau_2}\left(\frac{\psi(x,y,v)}{\psi(x,0,0)}R_0-1\right)v-\frac{ab\mu}{rk}e^{\mu_1\tau_1+\mu_2\tau_2}z. \qquad (17)\end{aligned}$$

From Assumptions A2-A3 we know that $\psi(x,y,v)$ is an increasing function of x and decreasing function of y and v. Then, the first term of Eq. (17) is less than or equal zero and

$$\psi(x,y,v)<\psi(x,0,0),\qquad x,y,v>0.$$

It follows that

$$\frac{dU_0}{dt}\le s\left(1-\frac{\psi(x_0,0,0)}{\psi(x,0,0)}\right)\left(1-\frac{x}{x_0}\right)+\frac{ac}{k}e^{\mu_1\tau_1+\mu_2\tau_2}\left(R_0-1\right)v-\frac{ab\mu}{rk}e^{\mu_1\tau_1+\mu_2\tau_2}z. \qquad (18)$$

Therefore, if $R_0 \leq 1$, then $\frac{dU_0}{dt} \leq 0$ for all $x, y, v, z > 0$. We note that the solutions of system (5)-(8) converge to Ω, the largest invariant subset of $\left\{\frac{dU_0}{dt} = 0\right\}$[33]. From (18), we have $\frac{dU_0}{dt} = 0$ iff $x = x_0$, $v = 0$ and $z = 0$. The set Ω is invariant and for any element belongs to Ω satisfies $v = 0$ and $z = 0$. We can see from Eq. (7) that

$$\dot{v} = 0 = ke^{-\mu_2\tau_2}y(t-\tau_2).$$

It follows that, $y = 0$. Hence $\frac{dU_0}{dt} = 0$ iff $x = x_0$ and $y = v = z = 0$. Using LaSalle's invariance principle, we derive that E_0 is GAS.

Assumption A5.

$$\left(1 - \frac{\psi(x,y,v)}{\psi(x,y_i,v_i)}\right)\left(\frac{\psi(x,y_i,v_i)}{\psi(x,y,v)} - \frac{v}{v_i}\right) \leq 0,\ \ i = 1,2 \text{ for all } x,y,v > 0.$$

Theorem 2. Let Assumptions A1-A5 be hold true and $R_1 \leq 1 < R_0$, then the chronic-infection equilibrium without antibody immune response E_1 is GAS.

Proof. Define:

$$\begin{aligned}
U_1 &= x - x_1 - \int_{x_1}^{x} \frac{\psi(x_1,y_1,v_1)}{\psi(\eta,y_1,v_1)} d\eta + e^{\mu_1\tau_1} y_1 H\left(\frac{y}{y_1}\right) \\
&+ \frac{a}{k} e^{\mu_1\tau_1+\mu_2\tau_2} v_1 H\left(\frac{v}{v_1}\right) + \frac{ab}{rk} e^{\mu_1\tau_1+\mu_2\tau_2} z \\
&+ \psi(x_1,y_1,v_1)v_1 \int\limits_{t-\tau_1}^{t} H\left(\frac{\psi(x(\eta),y(\eta),v(\eta))v(\eta)}{\psi(x_1,y_1,v_1)v_1}\right) d\eta + ae^{\mu_1\tau_1} y_1 \int\limits_{t-\tau_2}^{t} H\left(\frac{y(\eta)}{y_1}\right) d\eta. \qquad (19)
\end{aligned}$$

Calculating the time derivative of U_1 along the trajectories of system (5)-(8), we obtain

$$\begin{aligned}
\frac{dU_1}{dt} &= \left(1 - \frac{\psi(x_1,y_1,v_1)}{\psi(x,y_1,v_1)}\right)(s - dx - \psi(x,y,v)\,v) \\
&+ e^{\mu_1\tau_1}\left(1 - \frac{y_1}{y}\right)\left(e^{-\mu_1\tau_1}\psi(x(t-\tau_1),y(t-\tau_1),v(t-\tau_1))v(t-\tau_1) - ay\right) \\
&+ \frac{a}{k}e^{\mu_1\tau_1+\mu_2\tau_2}\left(1 - \frac{v_1}{v}\right)\left(ke^{-\mu_2\tau_2}y(t-\tau_2) - cv - bvz\right) + \frac{ab}{rk}e^{\mu_1\tau_1+\mu_2\tau_2}(rvz - \mu z) \\
&+ \psi(x,y,v)v - \psi(x(t-\tau_1),y(t-\tau_1),v(t-\tau_1))v(t-\tau_1) \\
&+ \psi(x_1,y_1,v_1)v_1 \ln\left(\frac{\psi(x(t-\tau_1),y(t-\tau_1),v(t-\tau_1))v(t-\tau_1)}{\psi(x,y,v)v}\right) \\
&+ ae^{\mu_1\tau_1}\left(y - y(t-\tau_2) + y_1 \ln\left(\frac{y(t-\tau_2)}{y}\right)\right) \\
&= \left(1 - \frac{\psi(x_1,y_1,v_1)}{\psi(x,y_1,v_1)}\right)(s - dx) + \psi(x_1,y_1,v_1)\frac{\psi(x,y,v)v}{\psi(x,y_1,v_1)} \\
&- \frac{y_1}{y}\psi(x(t-\tau_1),y(t-\tau_1),v(t-\tau_1))v(t-\tau_1) + ay_1e^{\mu_1\tau_1} \\
&- \frac{ac}{k}e^{\mu_1\tau_1+\mu_2\tau_2}v - ay(t-\tau_2)\frac{v_1}{v}e^{\mu_1\tau_1} + \frac{ac}{k}e^{\mu_1\tau_1+\mu_2\tau_2}v_1 \\
&+ \frac{ab}{k}e^{\mu_1\tau_1+\mu_2\tau_2}v_1 z - \frac{ab\mu}{rk}e^{\mu_1\tau_1+\mu_2\tau_2}z \\
&+ \psi(x_1,y_1,v_1)v_1 \ln\left(\frac{\psi(x(t-\tau_1),y(t-\tau_1),v(t-\tau_1))v(t-\tau_1)}{\psi(x,y,v)v}\right) \\
&+ ae^{\mu_1\tau_1}y_1 \ln\left(\frac{y(t-\tau_2)}{y}\right).
\end{aligned} \tag{20}$$

Using the equilibrium conditions for E_1:

$$s = dx_1 + ae^{\mu_1\tau_1}y_1, \quad \psi(x_1,y_1,v_1)v_1 = ae^{\mu_1\tau_1}y_1 = \frac{ac}{k}e^{\mu_1\tau_1+\mu_2\tau_2}v_1,$$

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we obtain

$$\begin{aligned}\frac{dU_1}{dt} &= dx_1\left(1-\frac{\psi(x_1,y_1,v_1)}{\psi(x,y_1,v_1)}\right)\left(1-\frac{x}{x_1}\right)+3ae^{\mu_1\tau_1}y_1\\ &- ae^{\mu_1\tau_1}y_1\frac{\psi(x_1,y_1,v_1)}{\psi(x,y_1,v_1)}+ae^{\mu_1\tau_1}y_1\frac{\psi(x,y,v)v}{\psi(x,y_1,v_1)v_1}\\ &- ae^{\mu_1\tau_1}y_1\frac{y_1\psi(x(t-\tau_1),y(t-\tau_1),v(t-\tau_1))v(t-\tau_1)}{y\psi(x_1,y_1,v_1)v_1}\\ &- ae^{\mu_1\tau_1}y_1\frac{v}{v_1}-ae^{\mu_1\tau_1}y_1\frac{v_1y(t-\tau_2)}{vy_1}\\ &+ ae^{\mu_1\tau_1}y_1\ln\left(\frac{\psi(x(t-\tau_1),y(t-\tau_1),v(t-\tau_1))v(t-\tau_1)}{\psi(x,y,v)v}\right)\\ &+ ae^{\mu_1\tau_1}y_1\ln\left(\frac{y(t-\tau_2)}{y}\right)+\frac{ab}{k}e^{\mu_1\tau_1+\mu_2\tau_2}\left(v_1-\frac{\mu}{r}\right)z. \end{aligned} \tag{21}$$

Using the following equalities:

$$\begin{aligned}\ln\left(\frac{\psi(x(t-\tau_1),y(t-\tau_1),v(t-\tau_1))v(t-\tau_1)}{\psi(x,y,v)v}\right) &= \ln\left(\frac{y_1\psi(x(t-\tau_1),y(t-\tau_1),v(t-\tau_1))v(t-\tau_1)}{y\psi(x_1,y_1,v_1)v_1}\right)\\ &+\ln\left(\frac{\psi(x_1,y_1,v_1)}{\psi(x,y_1,v_1)}\right)+\ln\left(\frac{\psi(x,y_1,v_1)}{\psi(x,y,v)}\right)+\ln\left(\frac{v_1y}{vy_1}\right),\\ \ln\left(\frac{y(t-\tau_2)}{y}\right) &= \ln\left(\frac{vy_1}{v_1y}\right)+\ln\left(\frac{v_1y(t-\tau_2)}{vy_1}\right),\end{aligned}$$

we get

$$\begin{aligned}\frac{dU_1}{dt} &= dx_1\left(1-\frac{\psi(x_1,y_1,v_1)}{\psi(x,y_1,v_1)}\right)\left(1-\frac{x}{x_1}\right)+ae^{\mu_1\tau_1}y_1\left(\frac{\psi(x,y,v)v}{\psi(x,y_1,v_1)v_1}-\frac{v}{v_1}-1+\frac{\psi(x,y_1,v_1)}{\psi(x,y,v)}\right)\\ &- ae^{\mu_1\tau_1}y_1\left[\left(\frac{\psi(x_1,y_1,v_1)}{\psi(x,y_1,v_1)}-1-\ln\left(\frac{\psi(x_1,y_1,v_1)}{\psi(x,y_1,v_1)}\right)\right)\right.\\ &+\left(\frac{y_1\psi(x(t-\tau_1),y(t-\tau_1),v(t-\tau_1))v(t-\tau_1)}{y\psi(x_1,y_1,v_1)v_1}-1-\ln\left(\frac{y_1\psi(x(t-\tau_1),y(t-\tau_1),v(t-\tau_1))v(t-\tau_1)}{y\psi(x_1,y_1,v_1)v_1}\right)\right)\\ &+\left(\frac{v_1y(t-\tau_2)}{vy_1}-1-\ln\left(\frac{v_1y(t-\tau_2)}{vy_1}\right)\right)\\ &+\left.\left(\frac{\psi(x,y_1,v_1)}{\psi(x,y,v)}-1-\ln\left(\frac{\psi(x,y_1,v_1)}{\psi(x,y,v)}\right)\right)\right]+\frac{ab}{k}e^{\mu_1\tau_1+\mu_2\tau_2}\left(v_1-\frac{\mu}{r}\right)z. \end{aligned} \tag{22}$$

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Eq. (22) can be simplified as:

$$\begin{aligned}
\frac{dU_1}{dt} &= dx_1\left(1-\frac{\psi(x_1,y_1,v_1)}{\psi(x,y_1,v_1)}\right)\left(1-\frac{x}{x_1}\right)\\
&+ae^{\mu_1\tau_1}y_1\left(1-\frac{\psi(x,y,v)}{\psi(x,y_1,v_1)}\right)\left(\frac{\psi(x,y_1,v_1)}{\psi(x,y,v)}-\frac{v}{v_1}\right)\\
&-ae^{\mu_1\tau_1}y_1\left[H\left(\frac{\psi(x_1,y_1,v_1)}{\psi(x,y_1,v_1)}\right)+H\left(\frac{v_1y(t-\tau_2)}{vy_1}\right)\right.\\
&\left.+H\left(\frac{y_1\psi(x(t-\tau_1),y(t-\tau_1),v(t-\tau_1))v(t-\tau_1)}{y\psi(x_1,y_1,v_1)v_1}\right)+H\left(\frac{\psi(x,y_1,v_1)}{\psi(x,y,v)}\right)\right]\\
&+\frac{ab}{k}e^{\mu_1\tau_1+\mu_2\tau_2}\left(v_1-\frac{\mu}{r}\right)z. \qquad (23)
\end{aligned}$$

From Assumptions A1 and A5, we get that the first and second terms of Eq. (23) are less than or equal zero. Now we show that if $R_1 \leq 1$ then $v_1 \leq \frac{\mu}{r} = v_2$. Let $R_0 > 1$, then we want to show that

$$sgn(x_2-x_1) = sgn(v_1-v_2) = sgn(y_1-y_2) = sgn(R_1-1).$$

From Assumptions A2-A4, for $x_1, x_2, y_1, y_2, v_1, v_2 > 0$, we have

$$(\psi(x_2,y_2,v_2)-\psi(x_1,y_2,v_2))(x_2-x_1) > 0, \qquad (24)$$

$$(\psi(x_1,y_1,v_1)-\psi(x_1,y_2,v_1))(y_2-y_1) > 0, \qquad (25)$$

$$(\psi(x_1,y_1,v_1)-\psi(x_1,y_1,v_2))(v_2-v_1) > 0, \qquad (26)$$

$$(\psi(x_2,y_2,v_2)v_2-\psi(x_2,y_2,v_1)v_1)(v_2-v_1) > 0. \qquad (27)$$

First, we claim $sgn(x_2-x_1) = sgn(v_1-v_2)$. Suppose this is not true, i.e., $sgn(x_2-x_1) = sgn(v_2-v_1)$. Using the conditions of the equilibria E_1 and E_2 we have

$$\begin{aligned}
(s-dx_2)-(s-dx_1) &= \psi(x_2,y_2,v_2)v_2-\psi(x_1,y_1,v_1)v_1\\
&= ae^{\mu_1\tau_1}(y_2-y_1). \qquad (28)
\end{aligned}$$

Then,

$$sgn(x_2-x_1) = sgn(y_1-y_2) \qquad (29)$$

Moreover,

$$(s-dx_2)-(s-dx_1)=\psi(x_2,y_2,v_2)v_2-\psi(x_1,y_1,v_1)v_1$$
$$=(\psi(x_2,y_2,v_2)v_2-\psi(x_2,y_2,v_1)v_1)+(\psi(x_2,y_2,v_1)v_1-\psi(x_1,y_2,v_1)v_1)$$
$$+(\psi(x_1,y_2,v_1)v_1-\psi(x_1,y_1,v_1)v_1).$$

Therefore, from inequalities (24) and (29) we get:

$$sgn\,(x_1-x_2)=sgn\,(x_2-x_1)\,,$$

which leads to contradiction. Thus, $sgn\,(x_2-x_1)=sgn\,(v_1-v_2)$. Using the equilibrium conditions for E_1 we have $\frac{k\psi(x_1,y_1,v_1)}{ace^{\mu_1\tau_1+\mu_2\tau_2}}=1$, then

$$\begin{aligned}R_1-1&=\frac{k\psi(x_2,y_2,v_2)}{ace^{\mu_1\tau_1+\mu_2\tau_2}}-\frac{k\psi(x_1,y_1,v_1)}{ace^{\mu_1\tau_1+\mu_2\tau_2}}\\&=\frac{k}{ac}e^{-\mu_1\tau_1-\mu_2\tau_2}[\psi(x_2,y_2,v_2)-\psi(x_2,y_2,v_1)+\psi(x_2,y_2,v_1)\\&-\psi(x_1,y_2,v_1)+\psi(x_1,y_2,v_1)-\psi(x_1,y_1,v_1)]\,.\end{aligned}$$

We get $sgn(R_1-1)=sgn(v_1-v_2)$. Hence, if $R_0>1$, then $x_1,y_1,v_1>0$, and if $R_1\leq 1$, then $v_1\leq v_2=\frac{\mu}{r}$. It follows from the above discussion that $\frac{dU_1}{dt}\leq 0$ for all $x,y,v,z>0$. The solutions of system (5)-(8) converge to Ω, the largest invariant subset of $\left\{(x,y,v,z):\frac{dU_1}{dt}=0\right\}$ [33]. We have $\frac{dU_1}{dt}=0$ iff $x=x_1, v=v_1,\ z=0$ and $H=0$ i.e.

$$\frac{y_1\psi(x(t-\tau_1),y(t-\tau_1),v(t-\tau_1))v(t-\tau_1)}{y\psi(x_1,y_1,v_1)v_1}=\frac{v_1y(t-\tau_2)}{vy_1}=1 \text{ for almost all } \tau_i\in[0,\tau], i=1,2. \qquad (30)$$

From Eq. (30), if $v=v_1$ then $y=y_1$ and hence $\frac{dU_1}{dt}=0$ iff $x=x_1,y=y_1,v=v_1$ and $z=0$. So Ω contains a unique point, that is E_1. Thus, the global asymptotic stability of the chronic-infection equilibrium without antibody immune response E_1 follows from LaSalle's invariance principle.

Theorem 3. Let Assumptions A1-A5 be hold true and $R_1>1$, then the chronic-infection equilibrium with antibody immune response E_2 is GAS.

Proof. We construct a Lyapunov functional as follows:

$$
\begin{aligned}
U_2 &= x - x_2 - \int_{x_2}^{x} \frac{\psi(x_2,y_2,v_2)}{\psi(\eta,y_2,v_2)} d\eta + e^{\mu_1\tau_1} y_2 H\left(\frac{y}{y_2}\right) \\
&+ \frac{a}{k} e^{\mu_1\tau_1+\mu_2\tau_2} v_2 H\left(\frac{v}{v_2}\right) + \frac{ab}{rk} e^{\mu_1\tau_1+\mu_2\tau_2} z_2 H\left(\frac{z}{z_2}\right) \\
&+ \psi(x_2,y_2,v_2)v_2 \int_{t-\tau_1}^{t} H\left(\frac{\psi(x(\eta),y(\eta),v(\eta))v(\eta)}{\psi(x_2,y_2,v_2)v_2}\right) d\eta + ae^{\mu_1\tau_1} y_2 \int_{t-\tau_2}^{t} H\left(\frac{y(\eta)}{y_2}\right) d\eta. \quad (31)
\end{aligned}
$$

Function U_2 satisfies:

$$
\begin{aligned}
\frac{dU_2}{dt} &= \left(1 - \frac{\psi(x_2,y_2,v_2)}{\psi(x,y_2,v_2)}\right)(s - dx - \psi(x,y,v)v) \\
&+ e^{\mu_1\tau_1}\left(1 - \frac{y_2}{y}\right)\left(e^{-\mu_1\tau_1}\psi(x(t-\tau_1),y(t-\tau_1),v(t-\tau_1))v(t-\tau_1) - ay\right) \\
&+ \frac{a}{k} e^{\mu_1\tau_1+\mu_2\tau_2}\left(1 - \frac{v_2}{v}\right)\left(ke^{-\mu_2\tau_2} y(t-\tau_2) - cv - bvz\right) + \frac{ab}{rk} e^{\mu_1\tau_1+\mu_2\tau_2}\left(1 - \frac{z_2}{z}\right)(rvz - \mu z) \\
&+ \psi(x,y,v)v - \psi(x(t-\tau_1),y(t-\tau_1),v(t-\tau_1))v(t-\tau_1) \\
&+ \psi(x_2,y_2,v_2)v_2 \ln\left(\frac{\psi(x(t-\tau_1),y(t-\tau_1),v(t-\tau_1))v(t-\tau_1)}{\psi(x,y,v)v}\right) \\
&+ ae^{\mu_1\tau_1}\left(y - y(t-\tau_2) + y_2 \ln\left(\frac{y(t-\tau_2)}{y}\right)\right). \quad (32)
\end{aligned}
$$

Applying $s = dx_2 + ae^{\mu_1\tau_1} y_2$, we get

$$
\begin{aligned}
\frac{dU_2}{dt} &= d\left(1 - \frac{\psi(x_2,y_2,v_2)}{\psi(x,y_2,v_2)}\right)(x_2 - x) + ae^{\mu_1\tau_1} y_2 - ae^{\mu_1\tau_1} y_2 \frac{\psi(x_2,y_2,v_2)}{\psi(x,y_2,v_2)} \\
&+ \psi(x,y,v)v \frac{\psi(x_2,y_2,v_2)}{\psi(x,y_2,v_2)} - \psi(x_2,y_2,v_2)v_2 \frac{y_2\psi(x(t-\tau_1),y(t-\tau_1),v(t-\tau_1))v(t-\tau_1)}{y\psi(x_2,y_2,v_2)v_2} \\
&+ ae^{\mu_1\tau_1} y_2 - \frac{ac}{k} e^{\mu_1\tau_1+\mu_2\tau_2} v - ae^{\mu_1\tau_1} y(t-\tau_2)\frac{v_2}{v} + \frac{ac}{k} e^{\mu_1\tau_1+\mu_2\tau_2} v_2 \\
&+ \frac{ab}{k} e^{\mu_1\tau_1+\mu_2\tau_2} v_2 z - \frac{ab\mu}{rk} e^{\mu_1\tau_1+\mu_2\tau_2} z - \frac{ab}{k} e^{\mu_1\tau_1+\mu_2\tau_2} z_2 v + \frac{ab\mu}{rk} e^{\mu_1\tau_1+\mu_2\tau_2} z_2 \\
&+ \psi(x_2,y_2,v_2)v_2 \ln\left(\frac{\psi(x(t-\tau_1),y(t-\tau_1),v(t-\tau_1))v(t-\tau_1)}{\psi(x,y,v)v}\right) + ae^{\mu_1\tau_1} y_2 \ln\left(\frac{y(t-\tau_2)}{y}\right) \quad (33)
\end{aligned}
$$

By using the equilibrium conditions of E_2

$$
\psi(x_2,y_2,v_2)v_2 = ae^{\mu_1\tau_1} y_2, \quad cv_2 = ke^{-\mu_2\tau_2} y_2 - bv_2 z_2, \quad \mu = rv_2,
$$

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and the following equalities

$$
\begin{aligned}
cv &= cv_2\frac{v}{v_2} = \left(ke^{-\mu_2\tau_2}y_2 - bv_2z_2\right)\frac{v}{v_2},\\
\ln\left(\frac{\psi(x(t-\tau_1),y(t-\tau_1),v(t-\tau_1))v(t-\tau_1)}{\psi(x,y,v)v}\right) &= \ln\left(\frac{y_2\psi(x(t-\tau_1),y(t-\tau_1),v(t-\tau_1))v(t-\tau_1)}{y\psi(x_2,y_2,v_2)v_2}\right)\\
&\quad+\ln\left(\frac{\psi(x_2,y_2,v_2)}{\psi(x,y_2,v_2)}\right)+\ln\left(\frac{\psi(x,y_2,v_2)}{\psi(x,y,v)}\right)+\ln\left(\frac{v_2y}{vy_2}\right),\\
\ln\left(\frac{y(t-\tau_2)}{y}\right) &= \ln\left(\frac{vy_2}{v_2y}\right)+\ln\left(\frac{v_2y(t-\tau_2)}{vy_2}\right),
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\frac{dU_2}{dt} &= d\left(1-\frac{\psi(x_2,y_2,v_2)}{\psi(x,y_2,v_2)}\right)(x_2-x) + ae^{\mu_1\tau_1}y_2\left(\frac{\psi(x,y,v)v}{\psi(x,y_2,v_2)v_2}-\frac{v}{v_2}-1+\frac{\psi(x,y_2,v_2)}{\psi(x,y,v)}\right)\\
&\quad - ae^{\mu_1\tau_1}y_2\left[\left(\frac{\psi(x_2,y_2,v_2)}{\psi(x,y_2,v_2)}-1-\ln\left(\frac{\psi(x_2,y_2,v_2)}{\psi(x,y_2,v_2)}\right)\right)\right.\\
&\quad+\left(\frac{y_2\psi(x(t-\tau_1),y(t-\tau_1),v(t-\tau_1))v(t-\tau_1)}{y\psi(x_2,y_2,v_2)v_2}-1-\ln\left(\frac{y_2\psi(x(t-\tau_1),y(t-\tau_1),v(t-\tau_1))v(t-\tau_1)}{y\psi(x_2,y_2,v_2)v_2}\right)\right)\\
&\quad\left.+\left(\frac{v_2y(t-\tau_2)}{vy_2}-1-\ln\left(\frac{v_2y(t-\tau_2)}{vy_2}\right)\right)+\left(\frac{\psi(x,y_2,v_2)}{\psi(x,y,v)}-1-\ln\left(\frac{\psi(x,y_2,v_2)}{\psi(x,y,v)}\right)\right)\right]. \qquad (34)
\end{aligned}
$$

We can rewrite (34) as

$$
\begin{aligned}
\frac{dU_2}{dt} &= dx_2\left(1-\frac{\psi(x_2,y_2,v_2)}{\psi(x,y_2,v_2)}\right)\left(1-\frac{x}{x_2}\right)+ae^{\mu_1\tau_1}y_2\left(1-\frac{\psi(x,y,v)}{\psi(x,y_2,v_2)}\right)\left(\frac{\psi(x,y_2,v_2)}{\psi(x,y,v)}-\frac{v}{v_2}\right)\\
&\quad - ae^{\mu_1\tau_1}y_2\left[H\left(\frac{\psi(x_2,y_2,v_2)}{\psi(x,y_2,v_2)}\right)+H\left(\frac{y_2\psi(x(t-\tau_1),y(t-\tau_1),v(t-\tau_1))v(t-\tau_1)}{y\psi(x_2,y_2,v_2)v_2}\right)\right.\\
&\quad\left.+H\left(\frac{v_2y(t-\tau_2)}{vy_2}\right)+H\left(\frac{\psi(x,y_2,v_2)}{\psi(x,y,v)}\right)\right]. \qquad (35)
\end{aligned}
$$

We note that from Assumptions A2 and A5, the first and second terms of Eq. (35) are less than or equal zero. Noting that $x, y, v, z > 0$, we have that $\frac{dU_2}{dt} \le 0$. The solutions of model (5)-(8) converge to Ω, the largest invariant subset of $\left\{(x,y,v,z) : \frac{dU_2}{dt} = 0\right\}$[33]. We have $\frac{dU_2}{dt} = 0$ iff $x = x_2, v = v_2$ and $H = 0$ i.e.,

$$
\frac{y_2\psi(x(t-\tau_1),y(t-\tau_1),v(t-\tau_1))v(t-\tau_1)}{y\psi(x_2,y_2,v_2)v_2} = \frac{v_2y(t-\tau_2)}{vy_2} = 1 \text{ for almost all } \tau_i \in [0,\tau], i = 1,2. \qquad (36)
$$

If $v = v_2$, then from Eq. (36) we get $y = y_2$. The set Ω is invariant and for any element belongs to Ω satisfies $v = v_2 = \frac{\mu}{r}$. From Eq. (7) we get $z = z_2$. Therefore, $\frac{dU_2}{dt} = 0$ iff $x = x_2, y = y_2, v = v_2$ and $z = z_2$. The global asymptotic stability of the chronic-infection equilibrium with antibody immune response E_2 follows from LaSalle's invariance principle.

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3 Conclusion

In this paper, we have proposed a delayed viral infection model with general incidence rate function and antibody immune response. The model has been incorporated with two kinds of discrete time delays representing the time needed for infecting an uninfected target cell and viral production. We have derived a set of conditions on the general functional response and have determined two threshold parameters R_0 and R_1 to prove the existence and the global stability of the model's equilibria. The global asymptotic stability of the three equilibria, infection-free, chronic-infection without antibody immune response and chronic-infection with antibody immune response has been proven by using direct Lyapunov method and LaSalle's invariance principle.

4 Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this article.

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STABILITY OF GENERALIZED CUBIC SET-VALUED FUNCTIONAL EQUATIONS

DONGSEUNG KANG

Abstract. We will show the general solution of the functional equation

$$\begin{aligned} & f(ax+by)+f(bx-ay)+(a+b)^2(a-b)f(y) \\ = \ & a^2bf(x+y)+ab^2f(x-y)+(a+b)(a-b)^2f(x) \end{aligned}$$

and investigate the Hyers-Ulam stability of cubic set-valued functional equation when $b=1$.

1. Introduction

The theory of set-valued functions in Banach spaces is connected to the control theory and the mathematical economics. Aumann [4] and Debreu [8] wrote papers that were motivated from the topic. We refer the reader to the papers by [1], [18], [10], [3], [17], [7] and [9].

The stability problem of functional equations originated from a question of Ulam [25] concerning the stability of group homomorphisms. Hyers [11] gave a first affirmative partial answer to the question of Ulam. Afterwards, the result of Hyers was generalized by Aoki [2] for additive mapping and by Rassias [23] for linear mappings by considering a unbounded Cauchy difference. Later, the result of Rassias has provided a lot of influence in the development of what we call Hyers-Ulam stability or Hyers-Ulam-Rassias stability of functional equations. For further information about the topic, we also refer the reader to [13], [12], [5] and [6].

Jun and Kim [15] introduced the following cubic functional equation:

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x)$$

and established a general solution. Najati [20] investigated the following generalized cubic functional equation:

$$f(ax+y)+f(ax-y)=af(x+y)+af(x-y)+2(a^3-a)f(x)\,. \tag{1.1}$$

In this paper, we deal with the following functional equation:

$$\begin{aligned} & f(ax+by)+f(bx-ay)+(a+b)^2(a-b)f(y) \\ = \ & a^2bf(x+y)+ab^2f(x-y)+(a+b)(a-b)^2f(x) \end{aligned} \tag{1.2}$$

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for all $x, y \in X$ and integers $a, b\,(a > b \geq 1)$. We will show the general solution of the functional equation (1.2) and investigate the Hyers-Ulam stability of cubic set-valued functional equation when $b = 1$.

2. A generalized cubic functional equation

In this section let X and Y be vector spaces and we investigate the general solution of the functional equation (1.2).

Theorem 2.1. *A function $f : X \to Y$ satisfies the functional equation (1.1) if and only if it satisfies the functional equation*

$$\begin{aligned} f(ax+y) + f(x-ay) - a^2 f(x+y) - af(x-y) \\ = (a-1)(a^2-1)f(x) - (a+1)(a^2-1)f(y) \end{aligned} \tag{2.1}$$

Proof. See [16, Theorem 2.1]. □

Theorem 2.2. *A function $f : X \to Y$ satisfies the functional equation (1.1) if and only if it satisfies the functional equation (1.2).*

Proof. Suppose that f satisfies the equation (1.1). Since f satisfies the equation (1.1), it is easy to show $f(0) = 0$, $f(x) = -f(-x)$ and $f(ax) = a^3 f(x)$ for all $x \in X$ and integer a ($a \neq 0, \pm 1$). Replacing x and y in the equation (1.1), we obtain

$$f(x+ay) - f(x-ay) = a[f(x+y) - f(x-y)] + 2a(a^2-1)f(y) \tag{2.2}$$

for all $x, y \in X$ and an integer a ($a \neq 0, \pm 1$). By letting $x = ax$ in the equation (2.2), we have

$$f(ax+y) - f(ax-y) = a^2[f(x+y) - f(x-y)] + 2(1-a^2)f(y) \tag{2.3}$$

for all $x, y \in X$ and an integer a ($a \neq 0, \pm 1$). By replacing x and y in the equation (2.3), we get

$$f(x+ay) + f(x-ay) = a^2[f(x+y) + f(x-y)] + 2(1-a^2)f(x) \tag{2.4}$$

for all $x, y \in X$ and an integer a ($a \neq 0, \pm 1$). Replacing a by b in the equation (1.1), we have

$$f(bx+y) + f(bx-y) = bf(x+y) + bf(x-y) + 2(b^3-b)f(x) \tag{2.5}$$

Letting $y = by$ in the equation (1.1),

$$f(ax+by) + f(ax-by) = af(x+by) + af(x-by) + 2(a^3-a)f(x) \tag{2.6}$$

Letting $y = ay$ in equation (2.5),

$$f(bx+ay) + f(bx-ay) = bf(x+ay) + bf(x-ay) + 2(b^3-b)f(x) \tag{2.7}$$

Replacing x and y in the equation (2.7),

$$f(ax+by) - f(ax-by) = bf(ax+y) - bf(ax-by) + 2(b^3-b)f(y) \tag{2.8}$$

Replacing x and y in equation (2.6),

$$f(bx+ay) - f(bx-ay) = af(bx+y) - af(bx-y) + 2(a^3-a)f(y) \tag{2.9}$$

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Adding two equations (2.6) and (2.8), we obtain

$$2f(ax+by) = af(x+by) + af(x-by) + 2(a^3-a)f(x) + bf(ax+y) - bf(ax-y) + 2(b^3-b)f(y) \tag{2.10}$$

Subtracting (2.9) from (2.7), we have

$$2f(bx-ay) = bf(x+ay) + bf(x-ay) + 2(b^3-b)f(x) - af(bx+y) + af(bx-y) - 2(a^3-a)f(y) \tag{2.11}$$

Now, adding two equations (2.10) and (2.11), we get

$$\begin{aligned} 2[f(ax+by) + f(bx-ay)] &= a[f(x+by) + f(x-by)] \\ &+ b[f(ax+y) - f(ax-y)] + 2(a^3-a)f(x) + 2(b^3-b)f(y) \\ &+ b[f(x+ay) + f(x-ay)] - a[f(bx+y) - f(bx-y)] \\ &+ 2(b^3-b)f(x) - 2(a^3-a)f(y) \end{aligned} \tag{2.12}$$

The desired result is obtained from the equation (2.12) by using the equations (2.3) and (2.4). Conversely, suppose that f satisfies the equation (1.2). Letting $b=1$ in the equation (1.2), we have the equation (2.1). The remains follow from Theorem 2.1. □

If f satisfies the equation (1.2), we call f a *generalized cubic mapping.*

3. Stability of the generalized cubic set-valued functional equation

In this section, we first introduce some definitions and notations which are needed to prove the main theorems. Let Y be a Banach space. The family of all closed subsets, containing 0, of Y will be denoted by $C_z(Y)$. Let A, B be nonempty subsets of a real vector space X and λ a real number. We define

$$\begin{aligned} A+B &= \{a+b \in X \mid a \in A, b \in B\} \\ \lambda A &= \{\lambda a \in X \mid a \in A\}. \end{aligned}$$

Lemma 3.1 ([21]). *Let λ and μ be real numbers. If A and B are nonempty subset of a real vector space, then*

$$\begin{aligned} \lambda(A+B) &= \lambda A + \lambda B \\ (\lambda+\mu)A &\subseteq \lambda A + \mu A. \end{aligned}$$

Moreover, if A is a convex set and $\lambda\mu \geq 0$, then we have

$$(\lambda+\mu)A = \lambda A + \mu A.$$

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A subset $A \subseteq X$ is said to be a *cone* if $A + A \subseteq A$ and $\lambda A \subseteq A$ for all $\lambda > 0$. If the zero vector in X belongs to A, then we say that A is a *cone with zero*.

Let $C_b(Y)$ be the set of all closed bounded subsets of Y, $C_c(Y)$ the set of all closed convex subsets of Y and $C_{cb}(Y)$ the set of all closed bounded convex subsets of Y. For elements A, B of $C_c(Y)$ and positive real values λ, μ, we denote

$$A \oplus B = \overline{A + B}.$$

For a subset A of Y, the distance function $d(\cdot, A)$ and the support function $s(\cdot, A)$ are defined by

$$\begin{aligned} d(x, A) &:= \inf \{||x - y|| \,|\, y \in A\} \text{ for all } x \in Y \\ s(x^*, A) &:= \sup \{\langle x^*, x\rangle|| \,|\, x \in A\} \text{ for all } x^* \in Y^*. \end{aligned}$$

For $A, A' \in C_b(Y)$, the *Hausdorff distance* $h(A, A')$ between A and A' is defined by

$$h(A, A') := \inf \{\alpha \geq 0 \,|\, A \subseteq A' + \alpha B_Y, A' \subseteq A + \alpha B_Y\},$$

where B_Y is the closed unit ball in Y. Castaing and Valadier [7] proved that $(C_{cb}(Y), \oplus, h)$ is a complete metric semigroup. Rådström [22] showed that $(C_{cb}(Y), \oplus, h)$ is isometrically embedded in a Banach space. The following remark is directly obtained from the notion of the Hausdorff distance.

Remark 3.2. Let $A, A', B, B', C \in C_{cb}(Y)$ and $\alpha > 0$. Then the following properties hold:

(1) $h(A \oplus A', B \oplus B') \leq h(A, B) + h(A', B')$

(2) $h(\alpha A, \alpha B) = \alpha h(A, B)$

(3) $h(A, B) = h(A \oplus C, B \oplus C)$.

First, let X be a real vector space, $A \subset X$ a cone with zero and Y a Banach space.

Theorem 3.3. *If $f : A + (-1)A \to C_z(Y)$ is a set-valued mapping with $f(0) = \{0\}$ satisfying*

$$\begin{aligned} f(ax + y) + f(x - ay) + (a^2 - 1)(a + 1)f(y) \\ \subseteq a^2 f(x + y) + af(x - y) + (a^2 - 1)(a - 1)f(x) \end{aligned} \tag{3.1}$$

and

$$\sup\{\operatorname{diam}(f(x)) \,|\, x \in A\} < \infty$$

for all $x, y \in A$ and an integer $a\,(a \geq 2)$, then there exists a unique generalized cubic mapping $C : A + (-1)A \to Y$ such that $C(x) \in f(x)$ for all $x \in A$.

Proof. Letting $y = 0$ in (3.1), we have

$$f(ax) \subseteq a^3 f(x) \tag{3.2}$$

for all $x \in A$ and an integer $a\,(a \geq 2)$. Replacing x by $a^n x$, $n \in \mathbb{N}$ in (3.2), we get

$$f(a^{n+1}x) \subseteq a^3 f(a^n x)$$

and

$$\frac{1}{a^{3(n+1)}} f(a^{n+1}x) \subseteq \frac{1}{a^{3n}} f(a^n x)$$

for all $x \in A$ and an integer $a\,(a \geq 2)$. Let $f_n(x) = \frac{1}{a^{3n}} f(a^n x)$ for each $x \in A$, $n \in \mathbb{N}$. Then $\{f_n(x)\}_{n \geq 0}$ is a decreasing sequence of closed subsets of the Banach space Y. Also, we obtain

$$\mathrm{diam}(f_n(x)) = \frac{1}{a^{3n}} \mathrm{diam}(f(a^n x)).$$

Since $\sup\{\mathrm{diam}(f(x)) \,|\, x \in A\} < \infty$, we have

$$\lim_{n \to \infty} \mathrm{diam}(f_n(x)) = 0.$$

Using the Cantor theorem for the sequence $\{f_n(x)\}_{n \geq 0}$, we get that $\cap_{n \geq 0} f_n(x)$ is a singleton set and we denote this intersection by $C(x)$ for all $x \in A$. Hence we obtain a map $C : A + (-1)A \to Y$ and

$$C(x) \in f_0(x) = f(x)$$

for all $x \in A$. We claim that C is generalized cubic. We note that

$$\begin{aligned}
& f_n(ax+y) + f_n(x-ay) + (a^2-1)(a+1) f_n(y) \\
&= \frac{f(a^n(ax+y))}{a^{3n}} + \frac{f(a^n(x-ay))}{a^{3n}} + \frac{(a^2-1)(a+1) f(a^n y)}{a^{3n}} \\
&\subseteq \frac{a^2 f(a^n(x+y))}{a^{3n}} + \frac{a f(a^n(x-y))}{a^{3n}} + \frac{(a^2-1)(a-1) f(a^n x)}{a^{3n}} \\
&= a^2 f_n(x+y) + a f_n(x-y) + (a^2-1)(a-1) f_n(x)
\end{aligned}$$

for all $x \in A$ and an integer $a\,(a \geq 2)$. By the definition of C, we obtain

$$\begin{aligned}
& C(ax+y) + C(x-ay) + (a^2-1)(a+1) C(y) \\
&= \cap_{n=0}^{\infty} \Big(f_n(ax+y) + f_n(x-ay) + (a^2-1)(a+1) f_n(y) \Big) \\
&\subseteq \cap_{n=0}^{\infty} \Big(a^2 f_n(x+y) + a f_n(x-y) + (a^2-1)(a-1) f_n(x) \Big)
\end{aligned}$$

for all $x \in A$ and an integer $a\,(a \geq 2)$. Hence we have

$$\begin{aligned}
& ||C(ax+y) + C(x-ay) + (a^2-1)(a+1) C(y) \\
& \qquad - a^2 C(x+y) - a C(x-y) - (a^2-1)(a-1) C(x)|| \\
& \leq a^2 \mathrm{diam}(f_n(x+y)) + a\,\mathrm{diam}(f_n(x-y)) + (a^2-1)(a-1) \mathrm{diam}(f_n(x)),
\end{aligned}$$

which tends to zero as $n \to \infty$. Thus C satisfies the equality (1.2). Hence C is a generalized cubic, as claimed. Next, let us prove the uniqueness of C. Assume f has two generalized cubic functional equations C_1 and C_2 from $A + (-1)A$ into Y. Then we have

$$(an)^3 C_i(x) = C_i(anx) \in f(anx)$$

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for all $x \in X$, $n \in \mathbb{N}$ and $i \in \{1,2\}$. Then we have

$$\begin{aligned}(an)^3||C_1(x)-C_2(x)|| &= ||(an)^3C_1(x)-(an)^3C_2(x)||\\ &= ||(C_1(anx)-C_2(anx)||\\ &\le \operatorname{diam}(f(anx))\end{aligned}$$

for all $x \in X$, $n \in \mathbb{N}$. Since $\sup\{\operatorname{diam}(f(x))\,|\, x \in A\} < \infty$, $C_1(x) = C_2(x)$, for all $x \in X$. □

Definition 3.4. Let $f : X \to C_{cb}(Y)$. The generalized cubic set-valued functional equation is defined by

$$\begin{aligned}(3.3)\qquad & f(ax+y)\oplus f(x-ay)\oplus(a^2-1)(a+1)f(y)\\ &= a^2f(x+y)\oplus af(x-y)\oplus(a^2-1)(a-1)f(x)\end{aligned}$$

for all $x \in A$ and an integer $a\,(a \ge 2)$. Every solution of the generalized cubic set-valued functional equation is called a *generalized cubic set-valued mapping.*

Theorem 3.5. *Let $\phi : X \times X \to [0,\, \infty)$ be a function such that*

$$(3.4)\qquad \widetilde{\phi}(x,y) := \sum_{j=0}^{\infty}\frac{1}{a^{3j}}\phi(a^jx,\, a^jy) < \infty$$

for all $x,\, y \in X$ and an integer $a\,(a \ge 2)$. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping with $f(0) = \{0\}$ satisfying

$$\begin{aligned}(3.5)\qquad & h\Big(f(ax+y)\oplus f(x-ay)\oplus(a^2-1)(a+1)f(y),\\ & a^2f(x+y)\oplus af(x-y)\oplus(a^2-1)(a-1)f(x)\Big) \le \phi(x,\, y)\end{aligned}$$

for all $x,\, y \in X$ and an integer $a\,(a \ge 2)$. Then there exists a unique generalized cubic set-valued mapping $C : X \to (C_{cb}(Y),\, h)$ such that

$$(3.6)\qquad h(f(x),\, C(x)) \le \frac{1}{a^3}\widetilde{\phi}(x,0)$$

for all $x,\, y \in X$ and an integer $a\,(a \ge 2)$.

Proof. Let $y = 0$ in the inequality (3.5). Since $f(x)$ is convex, we have

$$h\Big(f(ax)\oplus f(x),\, a^2f(x)\oplus af(x)\oplus(a^2-1)(a-1)f(x)\Big) \le \phi(x,\, 0)\,,$$

that is,

$$(3.7)\qquad h\Big(f(x),\, \frac{1}{a^3}f(ax)\Big) \le \frac{1}{a^3}\phi(x,0)$$

for all $x \in X$. Replacing x by a^kx , $k \in \mathbb{N}$, we get

$$h\Big(f(a^kx), \frac{1}{a^3}f(a^{k+1}x)\Big) \le \frac{1}{a^3}\phi(a^kx,0)$$

and

$$h\Big(\frac{1}{a^{3k}}f(a^kx), \frac{1}{a^{3(k+1)}}f(a^{k+1}x)\Big) \le \frac{1}{a^{3(k+1)}}\phi(a^kx,0)$$

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for all $x \in X$. Using the induction on k , we obtain

$$h\Big(f(x), \frac{1}{a^{3n}} f(a^n x)\Big) \le \frac{1}{a^3} \sum_{k=0}^{n-1} \frac{1}{a^{3k}} \phi(a^k x, 0) \tag{3.8}$$

for all $x \in X$ and $n \in \mathbb{N}$. Dividing the inequality (3.8) by a^{3m} and replacing x by $a^m x$, we have

$$h\Big(\frac{1}{a^{3m}} f(a^m x), \frac{1}{a^{3(n+m)}} f(a^{n+m} x)\Big) \le \frac{1}{a^3} \frac{1}{a^m} \sum_{k=0}^{n-1} \frac{1}{a^{3k}} \phi(a^{m+k} x, 0) \tag{3.9}$$

for all $x \in X$ and $n, m \in \mathbb{N}$. Since the right-hand side of the inequality (3.9) tends to zero as $m \to \infty$, the sequence $\{\frac{1}{a^{3n}} f(a^n x)\}$ is a Cauchy sequence in $(C_{cb}(Y), h)$. By the completeness of $C_{cb}(Y)$, we can define

$$C(x) := \lim_{n\to\infty} \frac{1}{a^{3n}} f(a^n x)$$

for all $x \in X$ and an integer $a\,(a \ge 2)$. We note that

$$\begin{aligned} & h\Big(\frac{f(a^n(ax+y))}{a^{3n}} \oplus \frac{f(a^n(x-ay))}{a^{3n}} \oplus \frac{(a^2-1)(a+1)f(a^n y)}{a^{3n}}, \\ & \frac{a^2 f(a^n(x+y))}{a^{3n}} \oplus \frac{af(a^n(x-y))}{a^{3n}} \oplus \frac{(a^2-1)(a-1)f(a^n x)}{a^{3n}}\Big) \\ & \le \frac{1}{a^{3n}} \phi(a^n x, a^n y) \end{aligned}$$

for all $x, y \in X$ and an integer $a\,(a \ge 2)$. By the definition of C , we have

$$\begin{aligned} & h\Big(C(ax+y) \oplus C(x-ay) \oplus (a^2-1)(a+1)C(y), \\ & \qquad a^2 C(x+y) \oplus aC(x-y) \oplus (a^2-1)(a-1)C(x)\Big) \\ & \le \lim_{n\to\infty} \frac{1}{a^{3n}} \phi(a^n x, a^n y) = 0\,. \end{aligned}$$

Hence C is a generalized cubic set-valued mapping. Now, by taking $n \to \infty$ in the inequality (3.8), we have the inequality (3.6). It remains to show the uniqueness of C . Assume $C' : X \to (C_{cb}(Y), h)$ is another generalized cubic set-valued mapping satisfying the inequality (3.6). Then

$$\begin{aligned} h\Big(C(x), C'(x)\Big) &= \frac{1}{a^{3n}} h\Big(C(a^n x), C'(a^n x)\Big) \\ &\le \frac{1}{a^{3n}} h\Big(C(a^n x), f(a^n x)\Big) + \frac{1}{a^{3n}} h\Big(f(a^n x), C'(a^n x)\Big) \\ &\le \frac{2}{a^{3(n+1)}} \widetilde{\phi}(a^n x, 0) \end{aligned}$$

for all $x \in X$. Since $\frac{2}{a^{3(n+1)}} \widetilde{\phi}(a^n x, 0) \to 0$ as $n \to \infty$, we may conclude that the generalized cubic set-valued mapping C is unique. □

Corollary 3.6. *Let $0 < p < 3$, $\theta \geq 0$ be real numbers and let X be a real normed space. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping with $f(0) = \{0\}$ satisfying*

$$h\Big(f(ax+y) \oplus f(x-ay) \oplus (a^2-1)(a+1)f(y),$$
$$a^2 f(x+y) \oplus af(x-y) \oplus (a^2-1)(a-1)f(x)\Big) \leq \theta(||x||^p + ||y||^p)$$

for all $x, y \in X$ and an integer $a\,(a \geq 2)$. Then there exists a unique generalized cubic set-valued mapping $C : X \to (C_{cb}(Y), h)$ satisfying

$$h(f(x), C(c)) \leq \frac{\theta}{a^3 - a^p}||x||^p$$

for all $x, y \in X$ and an integer $a\,(a \geq 2)$.

Proof. It follows from Theorem 3.5 by letting $\phi(x, y) = \theta(||x||^p + ||y||^p)$ for all $x, y \in X$. □

4. Stability of set-valued functional equation by the fixed point method

Now, we will investigate the stability of the given functional equation (3.3) using the alternative fixed point method. Before proceeding the proof, we will state the theorem, the alternative of fixed point; see [19] and [24].

Definition 4.1. Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 4.2. *[The alternative of fixed point* [19], [24] *] Suppose that we are given a complete generalized metric space (Ω, d) and a strictly contractive mapping $T : \Omega \to \Omega$ with Lipschitz constant L. Then for each given $x \in \Omega$, either*

$$d(T^n x, T^{n+1}x) = \infty \text{ for all } n \geq 0,$$

or there exists a natural number n_0 such that

(1) *$d(T^n x, T^{n+1}x) < \infty$ for all $n \geq n_0$;*
(2) *The sequence $(T^n x)$ is convergent to a fixed point y^* of T;*
(3) *y^* is the unique fixed point of T in the set*

$$\triangle = \{y \in \Omega | d(T^{n_0}x, y) < \infty\};$$

(4) *$d(y, y^*) \leq \frac{1}{1-L}\, d(y, Ty)$ for all $y \in \triangle$.*

Theorem 4.3. *Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping with $f(0) = \{0\}$ satisfying*

$$h\Big(f(ax+y) \oplus f(x-ay) \oplus (a^2-1)(a+1)f(y), \tag{4.1}$$

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$$a^2 f(x+y) \oplus af(x-y) \oplus (a^2-1)(a-1)f(x)\Big) \le \phi(x,\, y)$$

for all $x,\, y \in X$ *and an integer* $a\,(a \ge 2)$ *and there exists a constant* L *with* $0 < L < 1$ *for which the function* $\phi : X^2 \to \mathbb{R}^+$ *satisfies*

$$\phi(ax, 0) \le a^3 L\phi(x, 0) \tag{4.2}$$

for all $x \in X$ *. Then there exists a unique generalized cubic set-valued mapping* $C : X \to (C_{cb}(Y),\, h)$ *given by* $C(x) = \lim_{n\to\infty} \frac{f(a^n x)}{a^{3n}}$ *such that*

$$h(f(x),\, C(x)) \le \frac{1}{a^3(1-L)}\widetilde{\phi}(x, 0) \tag{4.3}$$

for all $x,\, y \in X$ *and an integer* $a\,(a \ge 2)$ *.*

Proof. Consider the set

$$\Omega = \{g \,|\, g : X \to C_{cb}(Y)\,, g(0) = \{0\}\}$$

and introduce the generalized metric on Ω defined by

$$d(g_1,\, g_2) = \inf\{\mu \in (0, \infty) \,|\, h(g_1(x)\,, g_2(x)) \le \mu\phi(x, 0)\,, \text{for all } x \in X\}\,.$$

We note that $\inf \emptyset := \infty$. It is easy to show that (Ω, d) is complete; see [14]. Now we define a function $T : \Omega \to \Omega$ by

$$T(g)(x) = \frac{1}{a^3} g(ax) \tag{4.4}$$

for all $x \in X$. Note that for all $g_1,\, g_2 \in \Omega$, let $\mu \in (0,\, \infty)$ be an arbitrary constant with $d(g_1,\, g_2) = \mu$. Then

$$h(\frac{1}{a^3} g_1(ax)\,, \frac{1}{a^3} g_2(ax)) \le \frac{\mu}{a^3}\phi(ax, 0) \tag{4.5}$$

for all $x \in X$. By using (4.2), we have

$$h(\frac{1}{a^3} g_1(ax)\,, \frac{1}{a^3} g_2(ax)) \le \mu\, L\phi(x, 0) \tag{4.6}$$

for all $x \in X$. Hence we obtain

$$d(Tg_1,\, Tg_2) \le Ld(g_1,\, g_2)$$

for all $g_1\,, g_2 \in \Omega$, that is, T is a strictly self-mapping of Ω with the Lipschitz constant L . Letting $y = 0$ in the inequality (4.1), we get

$$h(\frac{1}{a^3} f(ax)\,, f(x)) \le \frac{1}{a^3}\phi(x, 0)$$

for all $x \in X$. This means that

$$d(Tf, f) \le \frac{1}{a^3}\,.$$

By Theroem 4.2, there exists a fixed point $C : X \to (C_{cb}(Y),\, h)$ of T in $\{g \in \Omega \,|\, d(g_1\,, g_2) < \infty\}$ such that $\{T^k f\} \to 0$ ad $k \to \infty$. Hence we have

$$C(x) = \lim_{n\to\infty} \frac{f(a^n x)}{a^{3n}}\,, \tag{4.7}$$

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for all $x \in X$. Also, we have

$$d(f, C) \leq \frac{1}{1-L} d(Tf, f) \leq \frac{1}{a^3} \frac{1}{1-L} .$$

This implies that the inequality (4.3) holds for all $x \in X$. By the inequalities (4.1) and (4.2), we have

$$\begin{aligned} & h\Big(C(ax+y) \oplus C(x-ay) \oplus (a^2-1)(a+1)C(y), \\ & \qquad a^2 C(x+y) \oplus aC(x-y) \oplus (a^2-1)(a-1)C(x) \Big) \\ & \leq \lim_{n\to\infty} L^n \phi(a^n x,\, a^n y) = 0 \end{aligned}$$

for all $x, y \in X$ and an integer $a\,(a \geq 2)$. Thus C is a unique generalized cubic set-valued mapping. □

Corollary 4.4. *Let $0 < p < 3$ and $\theta \geq 0$ be real numbers and let X be a real normed space. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping with $f(0) = \{0\}$ satisfying*

$$\begin{aligned} & h\Big(f(ax+y) \oplus f(x-ay) \oplus (a^2-1)(a+1)f(y), \\ & \qquad a^2 f(x+y) \oplus af(x-y) \oplus (a^2-1)(a-1)f(x) \Big) \leq \theta(||x||^p + ||y||^p) \end{aligned} \tag{4.8}$$

for all $x, y \in X$ and an integer $a\,(a \geq 2)$. Then there exists a unique generalized cubic set-valued mapping $C : X \to (C_{cb}(Y), h)$ such that

$$h(f(x), C(x)) \leq \frac{\theta}{a^3 - a^p} ||x||^p \tag{4.9}$$

for all $x \in X$ and an integer $a\,(a \geq 2)$.

Proof. It follows from Theorem 4.3 by letting $\phi(x, y) = \theta(||x||^p + ||y||^p)$ for all $x, y \in X$. Then we can choose $L = a^{p-3}$ and hence we have the desired result. □

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A new regularity (p-regularity) of stratified L-generalized convergence spaces

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Abstract: In the classical theory of convergence spaces, both regularity (p-regularity) and topologicallness (p-topologicallness) are important notions. It is well known that topologicallness (p-topologicallness) can be described by a sophisticated Fischer-type diagonal condition, and regularity (p-regularity) can be described by dualizing that diagonal condition. Additionally, regularity (p-regularity) can also be characterized by the notion of closures of filters. In this paper, for stratified L-generalized convergence spaces, a new regularity (p-regularity) is defined by duzlizing a Fischer-type diagonal condition, which is used to describe the L-topologicallness of stratified L-convergence spaces (a subcategory of stratified L-generalized convergence spaces). Additionally, a characterization on this new regularity (p-regularity) by a notion of closures of stratified L-filters, is also presented.

Keywords: Topology; Lattice-valued topology; Lattice-valued convergence space; regularity; Diagonal condition

1 Introduction

p-topologicallness [17] and p-regularity [11] are dual notions in the classical theory of convergence spaces [16]. For a set X, let $\mathbb{F}(X)$ denote the set of all filters on X. Let q and p be convergence structures on a set X. Then the space (X, q) is called p-topological if it satisfies either of the two conditions below.

(1) $\mathbb{U}_p(\mathbb{F}) \xrightarrow{q} x$ whenever $\mathbb{F} \xrightarrow{q} x$, where $\mathbb{U}_p(\mathbb{F})$ is the neighborhood of $\mathbb{F}$ w.r.t p.

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(2) (Fischer-type diagonal condition) Let J be any set, $\psi : J \longrightarrow X$, and let $\sigma : J \longrightarrow \mathbb{F}(X)$ have the condition that $\sigma(j) \xrightarrow{p} \psi(j)$, for all $j \in J$. If $\mathbb{F} \in \mathbb{F}(J)$ is such that $\psi(\mathcal{F}) \xrightarrow{q} x$, then $k\sigma\mathbb{F} \xrightarrow{q} x$. Here, $k\sigma\mathbb{F} = \bigcup\limits_{F\in\mathbb{F}} \bigcap\limits_{j\in F} \sigma(j) \in \mathbb{F}(X)$ is called the compression of $\mathbb{F}$ relative to σ.

The space (X, q) is called p-regular if it satisfies either of the two conditions below.

(1) $\overline{\mathbb{F}}_p \xrightarrow{q} x$ whenever $\mathbb{F} \xrightarrow{q} x$, where $\overline{\mathbb{F}}_p$ is the closure of $\mathbb{F}$ w.r.t p.

(2) (Dual Fischer-type diagonal condition) Let J be any set, $\psi : J \longrightarrow X$, and let $\sigma : J \longrightarrow \mathbb{F}(X)$ have the condition that $\sigma(j) \xrightarrow{p} \psi(j)$, for all $j \in J$. If $\mathbb{F} \in \mathbb{F}(J)$ is such that $k\sigma\mathbb{F} \xrightarrow{q} x$, then $\psi(\mathcal{F}) \xrightarrow{q} x$.

When $p = q$, p-topologicallness and p-regularity are refereed to topologicallness and regularity [1, 3, 12], respectively.

Stratified L-generalized convergence spaces defined by Jäger [7] are lattice- valued extensions of convergence spaces. In [9], Jäger studied a regularity of stratified L-generalized convergence spaces both by a dual Fischer-type diagonal condition and a notion of α-lever closures of stratified L-filters. Later, Li and Jin [14] generalized Jäger's regularity to p-regularity. Quite recently, by modifying Jäger's Fischer-type diagonal condition, the first author and his co-author [15] introduced a new Fischer-type diagonal condition, and proved that this condition happens to characterize the topologicallness of stratified L-convergence spaces [4, 13] (a subcategory of stratified L-generalized convergence spaces). In this paper, by dualizing that diagonal condition, a new regularity (p-regularity) of stratified L-generalized convergence spaces is defined, and a characterization on this new regularity (p-regularity) by the notion of closures of stratified L-filters, is also presented.

The contents are arranged as follows. Section 2 fixes some notions and notations used in this note. Section 3 recalls the Fischer-type diagonal notion such that stratified L-convergence spaces are L-topological. Section 4 presents the main results. That is, by dualizing a Fischer-type diagonal condition in Section 3, we define a new regularity (p-regularity) of stratified L-generalized convergence spaces and then present a characterization on that regularity (p-regularity) by a notion of closures of stratified L-filters.

In this paper, if not otherwise specified, $L = (L, \leq)$ is always a complete lattice with a top element 1 and a bottom element 0, which satisfies the distributive law $\alpha \wedge (\bigvee_{i\in I} \beta_i) = \bigvee_{i\in I}(\alpha \wedge \beta_i)$. A lattice with these conditions is called a complete Heyting algebra or a frame. The operation $\rightarrow: L \times L \longrightarrow L$ given by $\alpha \rightarrow \beta = \bigvee\{\gamma \in L : \alpha \wedge \gamma \leq \beta\}$, is called the residuation with respect to $\wedge$. For the properties of $\wedge$ and $\rightarrow$, please refer to the literatures [6, 7, 14].

For a set X, the set L^X of functions from X to L with the pointwise order becomes a complete lattice. Each element of L^X is called an L-set (or a fuzzy subset) of X. And we make no difference between a constant function and its value since no confusion will arise. Let $f : X \longrightarrow Y$ be a function. We define$f^{\leftarrow} : L^Y \longrightarrow L^X$ [6] by $f^{\leftarrow}(\mu) = \mu \circ f$ for $\mu \in L^Y$.

Let X be a set. A fuzzy partial order (or, an L-partial order) on X [2] is a reflexive, transitive and antisymmetric fuzzy relation on X. The pair (X, R) is called an L-partially ordered set. Let $[L^X] : L^X \times L^X \longrightarrow L$ be a function defined by $[L^X](\lambda, \mu) = \bigwedge_{x \in X}(\lambda(x) \to \mu(x))$. Then $[L^X]$ is an L-partial order on L^X [2, 19]. The value $[L^X](\lambda, \mu) \in L$ is interpreted as the degree that λ is contained in μ. In the sequel, we use the symbol $[\lambda, \mu]$ to denote $[L^X](\lambda, \mu)$ for simplicity. The following lemma is useful to the subsequent section.

Lemma 1.1. *[14] Let $f : X \longrightarrow Y$ be an function. For any $\lambda, \mu, \nu \in L^X$ and any $\{\lambda_i\}_{i \in I}, \{\mu_i\}_{i \in I} \subseteq L^X$, we have* (1) *$\lambda \leq \mu$ implies $[\lambda, \nu] \geq [\mu, \nu]$;* (2) *$[\lambda, \wedge_{i \in I}\mu_i] = \wedge_{i \in I}[\lambda, \mu_i]$;* (3) *$\lambda \wedge [\lambda, \mu] \leq \mu$;* (4) *$[\vee_{i \in I}\lambda_i, \mu] = \wedge_{i \in I}[\lambda_i, \mu]$;* (5) *$[\lambda, \mu] \leq [f^{\rightarrow}(\lambda), f^{\rightarrow}(\mu)]$.*

A stratified L-filter [6] on a set X is a function $\mathcal{F} : L^X \longrightarrow L$ such that: (F0) $\mathcal{F}(0) = 0$, (F1) $\mathcal{F}(1) = 1$, (F2) $\forall \lambda, \mu \in L^X$, $\mathcal{F}(\lambda) \wedge \mathcal{F}(\mu) = \mathcal{F}(\lambda \wedge \mu)$, (Fs) $\forall \alpha \in L$, $\mathcal{F}(\alpha) \geq \alpha$. The set $\mathcal{F}^s_L(X)$ of all stratified L-filters on X is ordered by $\mathcal{F} \leq \mathcal{G} \Leftrightarrow \forall \lambda \in L^X, \mathcal{F}(\lambda) \leq \mathcal{G}(\lambda)$. There is a natural fuzzy partial order on $\mathcal{F}^s_L(X)$ inherited from $L^{(L^X)}$. Precisely, for all $\mathcal{F}, \mathcal{G} \in \mathcal{F}^s_L(X)$, if we let $[\mathcal{F}^s_L(X)](\mathcal{F}, \mathcal{G}) = [L^{L^X}](\mathcal{F}, \mathcal{G}) = \bigwedge_{\lambda \in L^X}(\mathcal{F}(\lambda) \to \mathcal{G}(\lambda))$, then $[\mathcal{F}^s_L(X)]$ is an L-partially order.

Example 1.2. (1) For each point x in a set X, the function $[x] : L^X \longrightarrow L, [x](\lambda) = \lambda(x)$ is a stratified L-filter on X. (2) If $\{\mathcal{F}_j | j \in J\} \subseteq \mathcal{F}^s_L(X)$, then $\bigwedge_{j \in J} \mathcal{F}_j \in \mathcal{F}^s_L(X)$.

(3) Let $f : X \longrightarrow Y$ be a function. If $\mathcal{F} \in \mathcal{F}^s_L(X)$, then the function $f^{\Rightarrow}(\mathcal{F}) \in \mathcal{F}^s_L(Y)$, where $f^{\Rightarrow}(\mathcal{F}) : L^Y \longrightarrow L$ is defined by $\lambda \mapsto \mathcal{F}(\lambda \circ f) = \mathcal{F}(f^{\leftarrow}(\lambda))$.

2 Fischer-type diagonal condition of stratified L-convergence spaces

In this section, we shall recall the Fischer-type diagonal condition such that a stratified L-convergence space is L-topological.

Definition 2.1. A stratified L-generalized convergence structure [7, 18] on a set X is a function $\lim^q : \mathcal{F}^s_L(X) \longrightarrow L^X$ satisfying **(LC1)** $\forall x \in X, \lim^q[x](x) = 1$; and **(LC2)** $\forall \mathcal{F}, \mathcal{G} \in \mathcal{F}^s_L(X)$, $\mathcal{F} \leq \mathcal{G} \Longrightarrow \lim^q\mathcal{F} \leq \lim^q\mathcal{G}$. The pair $(X, \lim^q)$ is called a stratified L-generalized convergence space. The pair $(X, \lim^q)$ is called a stratified L-convergence space [13] (or, a stratified L-ordered convergence space in [4]) if $\lim : \mathcal{F}^s_L(X) \longrightarrow L^X$ is a function satisfying **(LC1)** and **(LC2′)** $\forall \mathcal{F}, \mathcal{G} \in \mathcal{F}^s_L(X), [\mathcal{F}^s_L(X)](\mathcal{F}, \mathcal{G}) \leq [L^X](\lim^q\mathcal{F}, \lim^q\mathcal{G})$. Because **(LC2)′⇒(LC2)**, a stratified L-convergence space is a

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stratified L-generalized convergence space. A function $f : X \longrightarrow X'$ between two stratified L-generalized convergence spaces $(X, \lim^q)$, $(X', \lim^{q'})$ is called continuous if for all $\mathcal{F} \in \mathcal{F}_L^s(X)$ and all $x \in X$ we have $\lim^q \mathcal{F}(x) \leq \lim^{q'} f^{\Rightarrow}(\mathcal{F})(f(x))$.

For a given source $(X \xrightarrow{f_i} (X_i, \lim^{q_i}))_{i\in I}$, the *initial structure*, $\lim^q$ on X is defined by $\forall \mathcal{F} \in \mathcal{F}_L^s(X), \forall x \in X$, $\lim^q \mathcal{F}(x) = \bigwedge_{i\in I} \lim^{q_i} f_i^{\Rightarrow}(\mathcal{F})(f_i(x))$.

For a given sink $((X_i, \lim^{q_i}) \xrightarrow{f_i} X)_{i\in I}$, the *final structure*, $\lim^q$ on X is defined by

$$\lim^q \mathcal{F}(x) = \begin{cases} 1, & \mathcal{F} \geq [x]; \\ \bigvee_{i\in I, x_i \in X_i, \mathcal{G}_i \in \mathcal{F}_L^s(X_i), f_i(x_i)=x, f_i^{\Rightarrow}(\mathcal{G}_i) \leq \mathcal{F}} \lim^{q_i} \mathcal{G}_i(x_i), & \mathcal{F} \not\geq [x]. \end{cases}$$

When $X = \cup_{i\in I} f_i(X_i)$, the final structure $\lim^q$ can be simplified as [14]

$$\lim^q \mathcal{F}(x) = \bigvee_{i\in I, x_i \in X_i, \mathcal{G}_i \in \mathcal{F}_L^s(X_i), f_i(x_i)=x, f_i^{\Rightarrow}(\mathcal{G}_i) \leq \mathcal{F}} \lim^{q_i} \mathcal{G}_i(x_i).$$

In the theory of convergence spaces, Fischer-type diagonal condition is formulated by the aid of the notion of compression. The situation with lattice-valued convergence is similar. In [8], Jäger introduced an lattice-valued version of compression, which first appeared in [5] with a slightly different formalization.

Let $\sigma : J \longrightarrow \mathcal{F}_L^s(X)$ be a function and $\mathcal{F} \in \mathcal{F}_L^s(X)$. Then the function $k_L\sigma\mathcal{F} : L^X \longrightarrow L$ defined by

$$\forall \lambda \in L^X, k_L\sigma\mathcal{F}(\lambda) := \mathcal{F}(\widehat{\sigma}(\lambda)), \text{ where } \widehat{\sigma}(\lambda) = \sigma(-)(\lambda) \in L^J$$

forms a stratified L-filter on X; and it is called the compression of $\mathcal{F}$ w.r.t σ.

In [15], the first author and his co-author modified Jäger's compression and introduced a Fischer-type diagonal condition. It was proved that a stratified L-convergence space with this diagonal condition is L-topological.

Note that when a function $\sigma : J \longrightarrow \mathcal{F}_L^s(X)$ being given, that means an L-filter $\sigma(j)$ is selected for each $j \in J$. In this sense, we call $\sigma : J \longrightarrow \mathcal{F}_L^s(X)$ an L-filter select function. The definition below generalizes that notion.

Definition 2.2. [15] A function $\sigma = (\sigma_1, \sigma_2) : J \longrightarrow \mathcal{F}_L^s(X) \times L_0$, where $L_0 = L - \{0\}$, is said to be an L-filter select degree function. For any $j \in J$, the value $\sigma_2(j) \in L$ is interpreted as the degree to which the stratified L-filter $\sigma_1(j)$ is selected. Obviously, an L-filter select function can be regarded as an L-filter select degree function with $\sigma_2 \equiv 1$.

Definition 2.3. [15] Let $\sigma : J \longrightarrow \mathcal{F}_L^s(X) \times L_0$ be an L-filter select degree function and $\mathcal{F} \in \mathcal{F}_L^s(J)$. If the function $k_L\sigma\mathcal{F} : L^X \longrightarrow L$ defined by

$$\forall \lambda \in L^X, k_L\sigma\mathcal{F}(\lambda) := \mathcal{F}(\widehat{\sigma}(\lambda)), \text{ where } \widehat{\sigma}(\lambda) = \sigma_2(-) \to \sigma_1(-)(\lambda) \in L^J$$

forms a stratified L-filter on X, then we call such $\mathcal{F}$ compressible w.r.t σ and call $k_L\sigma\mathcal{F}$ as the compression of $\mathcal{F}$ w.r.t σ. It is easily seen that $k_L\sigma\mathcal{F}$ satisfies (F1), (F2) and (Fs) for any $\mathcal{F} \in \mathcal{F}_L^s(J)$.

If $\sigma : J \longrightarrow \mathcal{F}_L^s(X) \times L_0$ is an L-filter select function, then $k_L\sigma\mathcal{F} \in \mathcal{F}_L^s(X)$ for any $\mathcal{F} \in \mathcal{F}_L^s(J)$. In this case, $k_L\sigma\mathcal{F}$ coincides with Jäger's compression. Thus, Definition 2.3 generalizes Jäger's compression.

Theorem 2.4. *[15] Let $(X, \lim^q)$ be a stratified L-convergence spaces. Then $(X, \lim^q)$ is L-topological if and only if it satisfies the following condition* **(Lf)**.

(Lf) *Let J be any set, $\psi : J \longrightarrow X$, and let $\sigma : J \longrightarrow \mathcal{F}_L^s(X) \times L_0$. If $\mathcal{F} \in \mathcal{F}_L^s(J)$ is compressible w.r.t σ, then for each $x \in X$,*

$$\lim{}^q\psi^{\Rightarrow}(\mathcal{F})(x) * \bigwedge_{j\in J} \lim{}^q\sigma(j)(\psi(j)) \leq \lim{}^q k_L\sigma\mathcal{F}(x),$$

where $\lim^q \sigma(j)(\psi(j)) := \sigma_2(j) \to \lim^q \sigma_1(j)(\psi(j))$.

Obviously, the condition **(Lf)** implies the following condition **(Lfw)**.

(Lfw): Let J be any set, $\psi : J \longrightarrow X$, and let $\sigma : J \longrightarrow \mathcal{F}_L^s(X) \times L_0$ have the condition $\forall j \in J$, $\sigma_2(j) = \lim^q \sigma_1(j)(\psi(j))$ (which means that $\lim^q \sigma(j)(\psi(j)) \equiv 1$). If $\mathcal{F} \in \mathcal{F}_L^s(J)$ is compressible w.r.t σ, then $\lim^q \psi^{\Rightarrow}(\mathcal{F})(x) \leq \lim^q k_L\sigma\mathcal{F}(x)$ for each $x \in X$.

Note that in the proof of the sufficiency of Theorem 2.4, the selected σ, ψ satisfies the condition $\sigma_2(j) = \lim^q \sigma_1(j)(\psi(j))$ (see Theorem 4.9 in [15]). It follows immediately that **(Lf)**$\Leftrightarrow$**(Lfw)**. In addition, the characterization on L-topologicallness of stratified L-convergence spaces by the notion of neighborhoods of stratified L-filters, was presented in [10].

3 regularity and p-regularity of stratified L-generalized convergence spaces

In this section, by dualizing the condition **(Lfw)** we define a new regularity (p-regularity) of stratified L-generalized convergence spaces. Then we also present a characterization on that regularity (p-regularity) by a notion of closures of stratified L-filters.

Let $(X, \lim^p, \lim^q)$ be a pair of stratified L-generalized convergence spaces.

p-**(DLfw)**: Let J be any set, $\psi : J \longrightarrow X$, and let $\sigma : J \longrightarrow \mathcal{F}_L^s(X) \times L_0$ have the condition $\forall j \in J$, $\sigma_2(j) = \lim^p \sigma_1(j)(\psi(j))$. If $\mathcal{F} \in \mathcal{F}_L^s(J)$ is compressible w.r.t σ, then $\lim^q k_L\sigma\mathcal{F}(x) \leq \lim^q \psi^{\Rightarrow}(\mathcal{F})(x)$ for each $x \in X$.

When $\lim^p = \lim^q$, the condition p-**(DLfw)** is denoted as **(DLfw)**. Obviously, the condition **(DLfw)** is obtained by dualizing the condition **(Lfw)**.

It is easily seen that when $L = \{0, 1\}$, the condition p-**(DLfw)** is equivalent to the crisp dual Fischer-type diagonal condition.

Definition 3.1. Let $(X, \lim^p, \lim^q)$ be a pair of stratified L-generalized convergence spaces. Then $(X, \lim^q)$ is called p-regular if it satisfies the dual Fischer-type diagonal condition p-**(DLfw)**. When $\lim^p = \lim^q$, then $(X, \lim^q)$ is called regular if it is p-regular.

In the following, we shall give a characterization on regularity (p-regularity) by the notion of closures of stratified L-filters.

Definition 3.2. Let $(X, \lim^p)$ be a stratified L-generalized convergence space, and let $\lambda \in L^X$. Then the L-set $\overline{\lambda}_p \in L^X$ defined by

$$\forall x \in X, \overline{\lambda}_p(x) = \bigvee_{\mathcal{F} \in \mathcal{F}_L^s(X):\ \lim^p \mathcal{F}(x) \neq 0} (\lim^p \mathcal{F}(x) \to \mathcal{F}(\lambda))$$

is called the closure of λ w.r.t $(X, \lim^p)$.

Remark 3.3. When $L = \{0, 1\}$, a stratified L-generalized convergence space reduces to a convergence space. Then it is easily seen that $x \in \overline{\lambda}_p \Leftrightarrow \exists \mathbb{F} \xrightarrow{p} x$ s.t. $\lambda \in \mathbb{F}$. This shows that closure is precisely the crisp closure in [16] when $L = \{0, 1\}$.

Lemma 3.4. *Let $(X, \lim^p)$ be a stratified L-generalized convergence space. Then for all $\lambda, \mu \in L^X$ and all $\alpha \in L$ we get* (1) *$\lambda \leq \overline{\lambda}_p$;* (2) *$\lambda \leq \mu$ implies $\overline{\lambda}_p \leq \overline{\mu}_p$;* (3) *$\overline{\alpha}_p \geq \alpha$.*

Proof. (1) For each $x \in X$, by $\lim^p[x](x) = 1$ we get $\overline{\lambda}_p(x) \geq [x](\lambda) = \lambda(x)$. So, $\lambda \leq \overline{\lambda}_p$. Take $\lambda = 1$ in (1), we obtain $\overline{1}_p = 1$.

(2) It follows from the property (F2) of stratified L-filters.

(3) For each $x \in X$ we have

$$\overline{\alpha}_p(x) = \bigvee_{\lim \mathcal{F}(x) \neq 0} (\lim^p \mathcal{F}(x) \to \mathcal{F}(\alpha)) \overset{\text{(Fs)}}{\geq} \bigvee_{\lim \mathcal{F}(x) \neq 0} (\lim^p \mathcal{F}(x) \to \alpha) \geq \alpha. \quad \square$$

Theorem 3.5. *Let $(X, \lim^p)$ be a stratified L-generalized convergence space. For each $\mathcal{F} \in \mathcal{F}_L^s(X)$, the function $\overline{\mathcal{F}}_p : L^X \longrightarrow L$, defined by $\forall \lambda \in L^X, \overline{\mathcal{F}}_p(\lambda) = \bigvee_{\mu \in L^X} (\mathcal{F}(\mu) \wedge [\overline{\mu}_p, \lambda])$, is a stratified L-filter, called the closure of $\mathcal{F}$ w.r.t $(X, \lim^p)$.*

Proof. (F1) That $\overline{\mathcal{F}}_p(1) = 1$ is obvious. That $\overline{\mathcal{F}}_p(0) = 0$ follows by

$$\overline{\mathcal{F}}_p(\lambda) = \bigvee_{\mu \in L^X} (\mathcal{F}(\mu) \wedge [\overline{\mu}_p, \lambda]) \leq \bigvee_{\mu \in L^X} (\mathcal{F}(\mu) \wedge [\mu, \lambda]) \leq \mathcal{F}(\lambda).$$

(F2) Obviously, $\overline{\mathcal{F}}_p(\lambda\wedge\mu)\leq\overline{\mathcal{F}}_p(\lambda)\wedge\overline{\mathcal{F}}_p(\mu)$. Conversely,

$$\begin{aligned}\overline{\mathcal{F}}_p(\lambda)\wedge\overline{\mathcal{F}}_p(\mu) &= \bigvee_{a\in L^X}(\mathcal{F}(a)\wedge[\overline{a}_p,\lambda])\wedge\bigvee_{b\in L^X}(\mathcal{F}(b)\wedge[\overline{b}_p,\mu])\\ &= \bigvee_{a,b\in L^X}(\mathcal{F}(a)\wedge\mathcal{F}(b)\wedge[\overline{a}_p,\lambda]\wedge[\overline{b}_p,\mu])\\ &\leq \bigvee_{a,b\in L^X}(\mathcal{F}(a\wedge b)\wedge[\overline{(a\wedge b)}_p,\lambda\wedge\mu])\\ &\leq \bigvee_{c\in L^X}(\mathcal{F}(c)\wedge[\overline{c}_p,\lambda\wedge\mu])=\overline{\mathcal{F}}_p(\lambda\wedge\mu).\end{aligned}$$

(Fs) By $\overline{1}_p=1$, it follows that $\overline{\mathcal{F}}_p(\alpha)=\bigvee_{\mu\in L^X}(\mathcal{F}(\mu)\wedge[\overline{\mu}_p,\alpha])\geq\mathcal{F}(1)\wedge\alpha=\alpha$. □

Remark 3.6. When $L=\{0,1\}$, a stratified L-generalized convergence space reduces to a convergence space. It is easily seen that $\overline{\mathcal{F}}_p$ is precisely the filter generated by $\{\overline{A}:A\in\mathbb{F}\}$ as a filterbasis [16].

Lemma 3.7. *Let J,X,σ,ψ satisfy the condition in p-***(DLfw)***. Then for any $\lambda,\mu\in L^X$ we have $[\overline{\mu}_p,\lambda]\leq[\hat{\phi}(\mu),\psi^{\leftarrow}(\lambda)]$.*

Proof. Note that $\forall j\in J$, $\sigma_2(j)=\lim^p\sigma_1(j)(\psi(j))\neq 0$. Then

$$\begin{aligned}[\overline{\mu}_p,\lambda] &= \bigwedge_{x\in X}\Big(\bigvee_{\mathcal{G}\in\mathcal{F}^s_L(X):\lim^p\mathcal{G}(x)\neq 0}(\lim^p\mathcal{G}(x)\to\mathcal{G}(\mu))\to\lambda(x)\Big)\\ &= \bigwedge_{x\in X}\bigwedge_{\mathcal{G}\in\mathcal{F}^s_L(X):\lim^p\mathcal{G}(x)\neq 0}((\lim^p\mathcal{G}(x)\to\mathcal{G}(\mu))\to\lambda(x))\\ &\leq \bigwedge_{j\in J}(\lim^p\sigma_1(j)(\psi(j))\to\sigma_1(j)(\mu))\to\lambda(\psi(j)))\\ &\leq \bigwedge_{j\in J}(\sigma_2(j)\to\sigma_1(j)(\mu))\to\psi^{\leftarrow}(\lambda)(j))\\ &= \bigwedge_{j\in J}(\hat{\sigma}(\mu)(j)\to\psi^{\leftarrow}(\lambda)(j))=[\hat{\sigma}(\mu),\psi^{\leftarrow}(\lambda)].\end{aligned}$$ □

Lemma 3.8. *Let J,X,σ,ψ satisfy the condition in p-***(DLfw)***, and let $\mathcal{F}\in\mathcal{F}^s_L(X)$. Then the function $\mathcal{F}^\sigma:L^J\longrightarrow L$, defined by $\mathcal{F}^\sigma(\lambda)=\bigvee_{\mu\in L^X}(\mathcal{F}(\mu)\wedge[\hat{\sigma}(\mu),\lambda])$, satisfies (F1), (F2), (Fs) and $k_L\sigma\mathcal{F}^\sigma\geq\mathcal{F}$.*

Proof. (F1): It is obvious. (F2): Obviously, $\mathcal{F}^\sigma(\lambda\wedge\mu)\leq\mathcal{F}^\sigma(\lambda)\wedge\mathcal{F}^\sigma(\mu)$. Conversely,

$$\begin{aligned}\mathcal{F}^\sigma(\lambda)\wedge\mathcal{F}^\sigma(\mu) &= \bigvee_{a\in L^X}(\mathcal{F}(a)\wedge[\widehat{\sigma}(a),\lambda])\wedge\bigvee_{b\in L^X}(\mathcal{F}(b)\wedge[\widehat{\sigma}(b),\mu])\\ &= \bigvee_{a,b\in L^X}(\mathcal{F}(a)\wedge\mathcal{F}(b)\wedge[\widehat{\sigma}(a),\lambda]\wedge[\widehat{\sigma}(b),\mu])\\ &\leq \bigvee_{a,b\in L^X}(\mathcal{F}(a\wedge b)\wedge[\widehat{\sigma}(a\wedge b),\lambda\wedge\mu])\\ &\leq \bigvee_{c\in L^X}(\mathcal{F}(c)\wedge[\widehat{\sigma}(c),\lambda\wedge\mu])=\mathcal{F}^\sigma(\lambda\wedge\mu).\end{aligned}$$

(Fs): For any $\beta \in L$, we have

$$\mathcal{F}^{\sigma}(\beta) = \bigvee_{\mu \in L^X} (\mathcal{F}(\mu) \wedge [\widehat{\sigma}(\mu), \beta]) \geq \mathcal{F}(1) \wedge [\widehat{\sigma}(1), \beta] = 1 \wedge \beta = \beta.$$

It follows by the following inequality that $k_L \sigma \mathcal{F}^{\sigma} \geq \mathcal{F}$. For any $\lambda \in L^X$,

$$k_L \sigma \mathcal{F}^{\sigma}(\lambda) = \mathcal{F}^{\sigma}(\hat{\sigma}(\lambda)) = \bigvee_{\mu \in L^X} (\mathcal{F}(\mu) \wedge [\hat{\sigma}(\mu), \hat{\sigma}(\lambda)]) \geq \mathcal{F}(\lambda). \quad \square$$

Theorem 3.9. *Let $(X, \lim^p, \lim^q)$ be a pair of stratified L-generalized convergence spaces. Then $(X, \lim^q)$ is p-regular if and only if $\lim^q \mathcal{F} \leq \lim^q \overline{\mathcal{F}}_p$ for any $\mathcal{F} \in \mathcal{F}_L^s(X)$.*

Proof. Necessity: Let

$$J = \{(\mathcal{G}, y) \in \mathcal{F}_L^s(X) \times X | \lim^p \mathcal{G}(y) \neq 0\}, \ \psi : J \longrightarrow X, (\mathcal{G}, y) \mapsto y,$$

$$\sigma : J \longrightarrow \mathcal{F}_L^s(X) \times L_0, (\mathcal{G}, y) \mapsto (\mathcal{G}, \lim^p \mathcal{G}(y)).$$

Then (1) $\sigma_2(j) = \lim^p \sigma_1(j)(\psi(j)) \neq 0$. (2) For any $\mathcal{F} \in \mathcal{F}_L^s(X)$ we have $\mathcal{F}^{\sigma} \in \mathcal{F}_L^s(J)$.

Indeed, by Lemma 3.8, we need only to check that $\mathcal{F}^{\sigma}(0) = 0$.

$$\begin{aligned}
\mathcal{F}^{\sigma}(0) &= \bigvee_{\mu \in L^X} (\mathcal{F}(\mu) \wedge [\hat{\sigma}(\mu), 0]) = \bigvee_{\mu \in L^X} (\mathcal{F}(\mu) \wedge \bigwedge_{j \in J} (\hat{\sigma}(\mu)(j) \to 0)) \\
&\leq \bigvee_{\mu \in L^X} (\mathcal{F}(\mu) \wedge (\bigwedge_{y \in X} (\hat{\sigma}(\mu)([y], y) \to 0)) \\
&= \bigvee_{\mu \in L^X} (\mathcal{F}(\mu) \wedge (\bigwedge_{y \in X} ((\lim^p [y](y) \to [y](\mu)) \to 0)) \\
&= \bigvee_{\mu \in L^X} (\mathcal{F}(\mu) \wedge (\bigwedge_{y \in X} (\mu(y) \to 0)) = \bigvee_{\mu \in L^X} (\mathcal{F}(\mu) \wedge [\mu, 0]) \\
&\leq \bigvee_{\mu \in L^X} \mathcal{F}(\mu \wedge [\mu, 0]) \leq \mathcal{F}(0) = 0.
\end{aligned}$$

(3) $\psi^{\Rightarrow}(\mathcal{F}^{\sigma}) = \overline{\mathcal{F}}_p$. For any $\lambda, \mu \in L^X$,

$$\begin{aligned}
[\overline{\mu}_p, \lambda] &= \bigwedge_{x \in X} (\bigvee_{\mathcal{G} \in \mathcal{F}_L^s(X) : \lim^p \mathcal{G}(x) \neq 0} (\lim^p \mathcal{G}(x) \to \mathcal{G}(\mu)) \to \lambda(x)) \\
&= \bigwedge_{x \in X} \bigwedge_{\mathcal{G} \in \mathcal{F}_L^s(X) : \lim^p \mathcal{G}(x) \neq 0} ((\lim^p \mathcal{G}(x) \to \mathcal{G}(\mu)) \to \lambda(x)) \\
&= \bigwedge_{j \in J} (\lim^p \sigma_1(j)(\psi(j)) \to \sigma_1(j)(\mu)) \to \lambda(\psi(j))) \\
&= \bigwedge_{j \in J} (\sigma_2(j) \to \sigma_1(j)(\mu)) \to \psi^{\leftarrow}(\lambda)(j)) \\
&= \bigwedge_{j \in J} (\hat{\sigma}(\mu)(j) \to \psi^{\leftarrow}(\lambda)(j)) = [\hat{\sigma}(\mu), \psi^{\leftarrow}(\lambda)].
\end{aligned}$$

It follows that

$$\psi^{\Rightarrow}(\mathcal{F}^{\sigma})(\lambda)=\mathcal{F}^{\sigma}(\psi^{\leftarrow}(\lambda))=\bigvee_{\mu\in L^X}(\mathcal{F}(\mu)\wedge[\hat{\sigma}(\mu),\psi^{\leftarrow}(\lambda)])=\bigvee_{\mu\in L^X}(\mathcal{F}(\mu)\wedge[\overline{\mu}_p,\lambda])=\overline{\mathcal{F}}_p(\lambda).$$

(4) $\mathcal{F}^{\sigma}$ is compressible w.r.t. σ. For any $\lambda,\mu\in L^X$,

$$\begin{aligned}[\widehat{\sigma}(\lambda),\widehat{\sigma}(\mu)] &= \bigwedge_{j\in J}(\widehat{\sigma}(\lambda)(j)\to\widehat{\sigma}(\mu)(j))\\ &= \bigwedge_{(\mathcal{G},y):\lim^p\mathcal{G}(y)\neq 0}((\sigma_2(j)\to\sigma_1(j)(\lambda))\to(\sigma_2(j)\to\sigma_1(j)(\mu))\\ &\leq \bigwedge_{([y],y):y\in X}((\lim^p[y](y)\to[y](\lambda))\to(\lim^p[y](y)\to[y](\mu)))\\ &= \bigwedge_{y\in X}(\lambda(y)\to\mu(y))=[\lambda,\mu].\end{aligned}$$

Therefore, for any $\lambda\in L^X$,

$$k_L\sigma\mathcal{F}^{\sigma}(\lambda)=\mathcal{F}^{\sigma}(\hat{\sigma}(\lambda))=\bigvee_{\mu\in L^X}(\mathcal{F}(\mu)\wedge[\hat{\sigma}(\mu),\hat{\sigma}(\lambda)])\leq\bigvee_{\mu\in L^X}(\mathcal{F}(\mu)\wedge[\mu,\lambda])\leq\mathcal{F}(\lambda).$$

By Lemma 3.8, we have $k_L\sigma\mathcal{F}^{\sigma}=\mathcal{F}\in\mathcal{F}_L^s(X)$. Thus $k_L\sigma\mathcal{F}^{\sigma}$ is compressible w.r.t. σ.

Applying (1)-(4) in p-**(DLfw)** we have $\lim^q\mathcal{F}\leq\lim^q\overline{\mathcal{F}}_p$.

Sufficiency: Let J,X,σ,ψ satisfy the condition in **(DLfw)**. Then for any $\mathcal{F}\in\mathcal{F}_L^s(J)$, by $(X,\lim^q)$ is p-regular we have that $\lim^q k_L\sigma\mathcal{F}\leq\lim^q\overline{k_L\sigma\mathcal{F}}_p(x)$. For any $\lambda\in L^X$, by Lemma 3.7 we have

$$\begin{aligned}\overline{k_L\sigma\mathcal{F}}_p(\lambda) &= \bigvee_{\mu\in L^X}(k_L\sigma\mathcal{F}(\mu)\wedge[\overline{\mu}_p,\lambda])=\bigvee_{\mu\in L^X}(\mathcal{F}(\hat{\sigma}(\mu))\wedge[\overline{\mu}_p,\lambda])\\ &\leq \bigvee_{\mu\in L^X}(\mathcal{F}(\hat{\sigma}(\mu))\wedge[\hat{\sigma}(\mu),\psi^{\leftarrow}(\lambda)])\leq\mathcal{F}(\psi^{\leftarrow}(\lambda))=\psi^{\Rightarrow}(\mathcal{F})(\lambda).\end{aligned}$$

So, $\overline{k_L\sigma\mathcal{F}}_p\leq\psi^{\Rightarrow}(\mathcal{F})$, and hence $\lim^q\psi^{\Rightarrow}(\mathcal{F})\geq\lim^q\overline{k_L\sigma\mathcal{F}}_p\geq\lim^q k_L\sigma\mathcal{F}$, i.e., the condition p-**(DLfw)** holds. □

The next two theorems show that p-regularity behave reasonably well relative to initial and final structures.

Definition 3.10. Let $f:(X,\lim^q)\longrightarrow(Y,\lim^p)$ be a function between stratified L-generalized convergence spaces. Then f is said to be a closure function if $f^{\rightarrow}(\overline{\lambda}_q)\geq\overline{f^{\rightarrow}(\lambda)}_p$ for all $\lambda\in L^X$.

Lemma 3.11. *Let $f:(X,\lim^q)\longrightarrow(Y,\lim^p)$ be a function between stratified L-generalized convergence spaces, and let $\mathcal{F}\in\mathcal{F}_L^s(X)$. (1) If f is continuous, then $f^{\Rightarrow}(\overline{\mathcal{F}}_q)\geq\overline{f^{\Rightarrow}(\mathcal{F})}_p$. (2) If f is a closure function, then $f^{\Rightarrow}(\overline{\mathcal{F}}_q)\leq\overline{f^{\Rightarrow}(\mathcal{F})}_p$.*

Proof. (1) Let f be a continuous function. Then for each $\lambda \in L^Y$ we check below that $\overline{(f^{\leftarrow}(\lambda))}_q \leq f^{\leftarrow}(\overline{\lambda}_p)$. Indeed, for each $x \in X$,

$$\begin{aligned}\overline{(f^{\leftarrow}(\lambda))}_q(x) &= \bigvee_{\mathcal{G}\in\mathcal{F}_L^s(X):\lim^q\mathcal{G}(x)\neq 0}\left(\lim{}^q\mathcal{G}(x)\to\mathcal{G}(f^{\leftarrow}(\lambda))\right)\\ &\leq \bigvee_{\mathcal{G}\in\mathcal{F}_L^s(X):\lim^q\mathcal{G}(x)\neq 0}\left(\lim{}^p f^{\Rightarrow}(\mathcal{G})(f(x))\to f^{\Rightarrow}(\mathcal{G})(\lambda)\right)\\ &\leq \bigvee_{\mathcal{H}\in\mathcal{F}_L^s(Y):\lim^p\mathcal{H}(x)\neq 0}\left(\lim{}^p\mathcal{H}(f(x))\to\mathcal{H}(\lambda)\right)=f^{\leftarrow}(\overline{\lambda}_p)(x),\end{aligned}$$

where the first inequality holds for the continuity of f. Then for each $\mathcal{F}\in\mathcal{F}_L^s(X)$ and each $\lambda\in L^Y$

$$\begin{aligned}f^{\Rightarrow}(\overline{\mathcal{F}}_q)(\lambda) &= \overline{\mathcal{F}}_q(f^{\leftarrow}(\lambda))=\bigvee_{\mu\in L^X}([\overline{\mu}_q,f^{\leftarrow}(\lambda)]\wedge\mathcal{F}(\mu))\\ &\geq \bigvee_{\nu\in L^Y}([\overline{(f^{\leftarrow}(\nu))}_q,f^{\leftarrow}(\lambda)]\wedge\mathcal{F}(f^{\leftarrow}(\nu)))\geq\bigvee_{\nu\in L^Y}([f^{\leftarrow}(\overline{\nu}_p),f^{\leftarrow}(\lambda)]\wedge\mathcal{F}(f^{\leftarrow}(\nu)))\\ &\geq \bigvee_{\nu\in L^Y}([\overline{\nu}_p,\lambda]\wedge f^{\Rightarrow}(\mathcal{F})(\nu))=\overline{f^{\Rightarrow}(\mathcal{F})}_p(\lambda).\end{aligned}$$

Thus $f^{\Rightarrow}(\overline{\mathcal{F}}_q)\geq\overline{f^{\Rightarrow}(\mathcal{F})}_p$. (2) Let f be a closure function. Then for each $\lambda\in L^Y$,

$$\begin{aligned}\overline{f^{\Rightarrow}(\mathcal{F})}_p(\lambda) &= \bigvee_{\mu\in L^Y}(f^{\Rightarrow}(\mathcal{F})(\mu)\wedge[\overline{\mu}_p,\lambda])=\bigvee_{\mu\in L^Y}(\mathcal{F}(f^{\leftarrow}(\mu))\wedge[\overline{\mu}_p,\lambda])\\ &\geq \bigvee_{\nu\in L^X}(\mathcal{F}(f^{\leftarrow}f^{\rightarrow}(\nu))\wedge[\overline{f^{\rightarrow}(\nu)}_p,\lambda])\geq\bigvee_{\nu\in L^X}(\mathcal{F}(\nu)\wedge[\overline{f^{\rightarrow}(\nu)}_p,\lambda])\\ &\geq \bigvee_{\nu\in L^X}(\mathcal{F}(\nu)\wedge[f^{\rightarrow}(\overline{\nu}_q),\lambda])=\bigvee_{\nu\in L^X}(\mathcal{F}(\nu)\wedge[\overline{\nu}_q,f^{\leftarrow}(\lambda)])\\ &= \overline{\mathcal{F}}_q(f^{\leftarrow}(\lambda))=f^{\Rightarrow}(\overline{\mathcal{F}}_q)(\lambda),\end{aligned}$$

where the third inequality holds for f being a closure function, and the third equality follows from Lemma 1.1 (7). By the arbitrariness of λ, we get $f^{\Rightarrow}(\overline{\mathcal{F}}_q)\leq\overline{f^{\Rightarrow}(\mathcal{F})}_p$. □

Theorem 3.12. *Let $\{(X_i,\lim^{q_i},\lim^{p_i})\}_{i\in I}$ be pairs of stratified L-generalized convergence spaces and let $\lim^q$ (resp., $\lim^p$) be the initial structure on X relative to the source $(X\xrightarrow{f_i}(X_i,\lim^{q_i}))_{i\in I}$ (resp., $(X\xrightarrow{f_i}(X_i,\lim^{p_i}))_{i\in I}$). If each $\lim^{q_i}$ is p_i-regular, then $(X,\lim^q)$ is p-regular.*

Proof. Let $\mathcal{F}\in\mathcal{F}_L^s(X)$ and $x\in X$. Then by Lemma 3.11 (1) we have $f_i^{\Rightarrow}(\overline{\mathcal{F}}_p)\geq\overline{f_i^{\Rightarrow}(\mathcal{F})}_{p_i}$ for all $i\in I$. It follows by each $(X_i,\lim^{q_i})$ being p_i-regular that

$$\begin{aligned}\lim{}^q\overline{\mathcal{F}}_p(x) &= \bigwedge_{i\in I}\lim{}^{q_i}f_i^{\Rightarrow}(\overline{\mathcal{F}}_p)(f_i(x))\geq\bigwedge_{i\in I}\lim{}^{q_i}\overline{f_i^{\Rightarrow}(\mathcal{F})}_{p_i}(f_i(x))\\ &\geq \bigwedge_{i\in I}\lim{}^{q_i}f_i^{\Rightarrow}(\mathcal{F})(f_i(x))=\lim{}^q\mathcal{F}(x).\end{aligned}$$

Thus $(X,\lim^q)$ is p-regular. □

Theorem 3.13. *Let $\{(X_i, \lim^{q_i}, \lim^{p_i})\}_{i\in I}$ be pairs of stratified L-generalized convergence spaces, and let $\lim^q$ be the final structure on X w.r.t. the sink $((X_i, \lim^{q_i}) \xrightarrow{f_i} X)_{i\in I}$ with $X = \cup_{i\in I} f_i(X_i)$. If each $\lim^{q_i}$ is p_i-regular and $\lim^p$ is a stratified L-generalized convergence structure on X such that each $f_i : (X_i, \lim^{p_i}) \longrightarrow (X, \lim^p)$ is a closure function, then $(X, \lim^q)$ is p-regular.*

Proof. Let $\mathcal{F} \in \mathcal{F}_L^s(X)$ and $x \in X$. Then

$$\begin{aligned}
\lim{}^q\mathcal{F}(x) &= \bigvee_{i\in I, x_i\in X_i, \mathcal{G}_i\in\mathcal{F}_L^s(X_i), f_i(x_i)=x, f_i^{\Rightarrow}(\mathcal{G}_i)\leq\mathcal{F}} \lim{}^{q_i}\mathcal{G}_i(x_i)\\
&\leq \bigvee_{i\in I, x_i\in X_i, \mathcal{G}_i\in\mathcal{F}_L^s(X_i), f_i(x_i)=x, f_i^{\Rightarrow}(\mathcal{G}_i)\leq\mathcal{F}} \lim{}^{q_i}\overline{\mathcal{G}_i}_{p_i}(x_i)\\
&\leq \bigvee_{i\in I, x_i\in X_i, \mathcal{G}_i\in\mathcal{F}_L^s(X_i), f_i(x_i)=x, \overline{f_i^{\Rightarrow}(\mathcal{G}_i)}_p\leq\overline{\mathcal{F}}_p} \lim{}^{q_i}\overline{\mathcal{G}_i}_{p_i}(x_i)\\
&= \bigvee_{i\in I, x_i\in X_i, \mathcal{G}_i\in\mathcal{F}_L^s(X_i), f_i(x_i)=x, f_i^{\Rightarrow}(\overline{\mathcal{G}_i}_{p_i})\leq\overline{f_i^{\Rightarrow}(\mathcal{G}_i)}_p\leq\overline{\mathcal{F}}_p} \lim{}^{q_i}\overline{\mathcal{G}_i}_{p_i}(x_i)\\
&\leq \bigvee_{i\in I, x_i\in X_i, \mathcal{H}_i\in\mathcal{F}_L^s(X_i), f_i(x_i)=x, f_i^{\Rightarrow}(\mathcal{H}_i)\leq\overline{\mathcal{F}}_p} \lim{}^{q_i}\mathcal{H}_i(x_i) = \lim{}^q\overline{\mathcal{F}}_p(x),
\end{aligned}$$

where the first inequality holds for p_i-regularity of $\lim^{q_i}$, the second equality follows from Lemma 3.11 (2). Then $\lim^q$ is p-regular by $\lim^q \mathcal{F} \leq \lim^q \overline{\mathcal{F}}_p$. □

The regularity has similar characterization and properties, we omit them here.

4 Conclusion

In this paper, by dualizing the Fischer-type diagonal condition **(Lfw)**, which is used to describe the L-topologicallness of stratified L-convergence spaces, we define a new regularity (p-regularity) of stratified L-generalized convergence spaces. Then we also present a characterization on that regularity (p-regularity) by the notion of closures of stratified L-filters. The regularity (p-regularity) is proved to behave reasonably well relative to initial and final structures.

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Uni-soft filters and uni-soft G-filters in residuated lattices

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Abstract

The notions of uni-soft filters and uni-soft G-filters in residuated lattices are introduced, and their relations, properties and characterizations are investigated. Conditions for a uni-soft filter to be a uni-soft G-filter are provided.

Keywords: Residuated lattice, Uni-soft filter, Uni-soft G-filter.

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1 Introduction

Non-classical logic has become a formal and useful tool in dealing with fuzzy and uncertain informations. Various logical algebras have been proposed as the semantical systems of non-classical logic systems. Residuated lattices are important algebraic structures which are basic of BL-algebras, MV-algebras, MTL-algebras, Gödel algebras, R_0-algebras, lattice implication algebras, and so forth. The (fuzzy) filter theory in the logical algebras has an important role in studying these algebras and completeness of the corresponding non-classical logics, and it is studied in the papers [1], [2], [3], [9], [12], [13] and [14]. Filter theory, which is an important notion, in residuated lattices is studied by Shen and Zhang [11] and Zhu and Xu [16]. Uncertainty is an attribute of information. As a new mathematical tool for dealing with uncertainties, Molodtsov [10] introduced the concept of soft sets. Since then several authors studied (fuzzy) algebraic structures based on soft

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set theory in several algebraic structures. In order to lay a foundation for providing a soft algebraic tool in considering many problems that contain uncertainties, Jun [7] discussed the union soft sets with applications in BCK/BCI-algebras. Also, Jun et al. [8] discussed uni-soft sets applied to commutative BCI-ideals.

In this paper, we introduce uni-soft filters and uni-soft G-filters in residuated lattices, and investigate their properties. We consider characterizations of uni-soft filters and uni-soft G-filters. We provide conditions for a uni-soft filter to be a uni-soft G-filter.

2 Preliminaries

Definition 2.1 ([1, 5, 6])**.** A residuated lattice is an algebra $\mathcal{L} := (L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ such that

(1) $(L, \vee, \wedge, 0, 1)$ is a bounded lattice.

(2) $(L, \odot, 1)$ is a commutative monoid.

(3) $\odot$ and $\rightarrow$ form an adjoint pair, that is,

$$(\forall x, y, z \in L)\,(x \leq y \rightarrow z \;\Leftrightarrow\; x \odot y \leq z)\,.$$

In a residuated lattice $\mathcal{L}$, the ordering $\leq$ is defined as follows:

$$(\forall x, y \in L)\,(x \leq y \;\Leftrightarrow\; x \wedge y = x \;\Leftrightarrow\; x \vee y = y \;\Leftrightarrow\; x \rightarrow y = 1)$$

and x' will be reserved for $x \rightarrow 0$, and $x'' = (x')'$, etc. for all $x \in L$.

Proposition 2.2 ([1, 5, 6, 12, 13])**.** *In a residuated lattice L, the following properties are valid.*

$$1 \rightarrow x = x,\; x \rightarrow 1 = 1,\; x \rightarrow x = 1,\; 0 \rightarrow x = 1,\; x \rightarrow (y \rightarrow x) = 1. \tag{2.1}$$

$$x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z = y \rightarrow (x \rightarrow z). \tag{2.2}$$

$$x \leq y \;\Rightarrow\; z \rightarrow x \leq z \rightarrow y,\; y \rightarrow z \leq x \rightarrow z. \tag{2.3}$$

$$z \rightarrow y \leq (x \rightarrow z) \rightarrow (x \rightarrow y),\; z \rightarrow y \leq (y \rightarrow x) \rightarrow (z \rightarrow x). \tag{2.4}$$

Definition 2.3 ([11])**.** A nonempty subset F of a residuated lattice $\mathcal{L}$ is called a filter of $\mathcal{L}$ if it satisfies the conditions:

$$(\forall x, y \in L)\,(x, y \in F \;\Rightarrow\; x \odot y \in F)\,. \tag{2.5}$$

$$(\forall x, y \in L)\,(x \in F,\; x \leq y \;\Rightarrow\; y \in F)\,. \tag{2.6}$$

Proposition 2.4 ([11]). *A nonempty subset F of a residuated lattice $\mathcal{L}$ is a filter of $\mathcal{L}$ if and only if it satisfies:*

$$1 \in F. \tag{2.7}$$

$$(\forall x \in F)\,(\forall y \in L)\,(x \to y \in F \;\Rightarrow\; y \in F)\,. \tag{2.8}$$

Definition 2.5 ([15]). A nonempty subset F of $\mathcal{L}$ is called a G-filter of $\mathcal{L}$ if it is a filter of $\mathcal{L}$ that satisfies the following condition:

$$(\forall x, y \in L)\,((x \odot x) \to y \in F \;\Rightarrow\; x \to y \in F)\,. \tag{2.9}$$

A soft set theory is introduced by Molodtsov [10], and Çağman et al. [4] provided new definitions and various results on soft set theory.

In what follows, let U be an initial universe set and E be a set of parameters. Let $\mathscr{P}(U)$ denotes the power set of U and $A, B, C, \cdots \subseteq E$.

Definition 2.6 ([4, 10]). A soft set $(\tilde{f}, A)$ over U is defined to be the set of ordered pairs

$$(\tilde{f}, A) := \left\{ (x, \tilde{f}_A(x)) : x \in E,\ \tilde{f}_A(x) \in \mathscr{P}(U) \right\},$$

where $\tilde{f}_A : E \to \mathscr{P}(U)$ such that $\tilde{f}(x) = \emptyset$ if $x \notin A$. The soft set $(\tilde{f}, A)$ is simply denoted by $\tilde{f}_A$.

For a soft set $\tilde{f}_A$ over U and a subset τ of U, the τ-exclusive set of $\tilde{f}_A$, denoted by $e(\tilde{f}_A; \tau)$, is defined to be the set

$$e(\tilde{f}_A; \tau) := \left\{ x \in A \mid \tilde{f}_A(x) \subseteq \tau \right\}.$$

3 Uni-soft filters

In what follows, we take a residuated lattice $\mathcal{L}$ as a set of parameters.

Definition 3.1. A soft set $\tilde{f}_{\mathcal{L}}$ over U is called a uni-soft filter of $\mathcal{L}$ if it satisfies:

$$(\forall x, y \in L)\left(x \le y \;\Rightarrow\; \tilde{f}_{\mathcal{L}}(x) \supseteq \tilde{f}_{\mathcal{L}}(y)\right), \tag{3.1}$$

$$(\forall x, y \in L)\left(\tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}(y) \supseteq \tilde{f}_{\mathcal{L}}(x \odot y)\right). \tag{3.2}$$

Proposition 3.2. *Every uni-soft filter $\tilde{f}_{\mathcal{L}}$ of $\mathcal{L}$ satisfies:*

$$(\forall x \in L) \left(\tilde{f}_{\mathcal{L}}(x) \supseteq \tilde{f}_{\mathcal{L}}(1) \right). \tag{3.3}$$

$$(\forall x, y \in L) \left(\tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}(x \to y) \supseteq \tilde{f}_{\mathcal{L}}(y) \right). \tag{3.4}$$

Proof. Let $x, y \in L$. Since $x \leq 1$, we have $\tilde{f}_{\mathcal{L}}(x) \supseteq \tilde{f}_{\mathcal{L}}(1)$ by (3.1). Since $x \odot (x \to y) \leq y$, it follows from (3.2) and (3.1) that

$$\tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}(x \to y) \supseteq \tilde{f}_{\mathcal{L}}(x \odot (x \to y)) \supseteq \tilde{f}_{\mathcal{L}}(y).$$

This completes the proof. □

Lemma 3.3. *If a soft set $\tilde{f}_{\mathcal{L}}$ over U satisfies two conditions (3.3) and (3.4), then*

$$(\forall x, y, z \in L) \left(x \leq y \to z \;\Rightarrow\; \tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}(y) \supseteq \tilde{f}_{\mathcal{L}}(z) \right), \tag{3.5}$$

$$(\forall x, y, z \in L) \left(x \odot y \leq z \;\Rightarrow\; \tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}(y) \supseteq \tilde{f}_{\mathcal{L}}(z) \right). \tag{3.6}$$

Proof. Assume that $x \leq y \to z$ for all $x, y, z \in L$. Then $x \to (y \to z) = 1$, and so

$$\begin{aligned}
\tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}(y) &= \left(\tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}(1) \right) \cup \tilde{f}_{\mathcal{L}}(y) \\
&= \left(\tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}(x \to (y \to z)) \right) \cup \tilde{f}_{\mathcal{L}}(y) \\
&\supseteq \tilde{f}_{\mathcal{L}}(y) \cup \tilde{f}_{\mathcal{L}}(y \to z) \supseteq \tilde{f}_{\mathcal{L}}(z).
\end{aligned}$$

Since $x \leq y \to z \;\Leftrightarrow\; x \odot y \leq z$, we know that (3.5) induces (3.6). □

We consider characterizations of uni-soft filters.

Theorem 3.4. *A soft set $\tilde{f}_{\mathcal{L}}$ over U is a uni-soft filter of $\mathcal{L}$ if and only if it satisfies two conditions (3.3) and (3.4).*

Proof. The necessity is from Proposition 3.2.

Conversely, let $\tilde{f}_{\mathcal{L}}$ be a soft set over U that satisfies (3.3) and (3.4). Let $x, y \in L$ be such that $x \leq y$. Then $x \to y = 1$ and so

$$\tilde{f}_{\mathcal{L}}(x) = \tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}(1) = \tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}(x \to y) \supseteq \tilde{f}_{\mathcal{L}}(y).$$

Since $x \odot y \leq x \odot y$ for all $x, y \in L$, it follows from (3.6) that $\tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}(y) \supseteq \tilde{f}_{\mathcal{L}}(x \odot y)$ for all $x, y \in L$. Therefore $\tilde{f}_{\mathcal{L}}$ is a uni-soft filter of $\mathcal{L}$. □

Theorem 3.5. *A soft set $\tilde{f}_{\mathcal{L}}$ over U is a uni-soft filter of $\mathcal{L}$ if and only if it satisfies the condition* (3.5).

Proof. The necessity is from Lemma 3.3 and Theorem 3.4.

Conversely let $\tilde{f}_{\mathcal{L}}$ be a soft set over U satisfying (3.5). Since

$$x \leq x \to 1 \text{ and } x \to y \leq x \to y$$

for all $x, y \in L$, it follows from (3.5) that

$$\tilde{f}_{\mathcal{L}}(x) = \tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}(x) \supseteq \tilde{f}_{\mathcal{L}}(1) \text{ and } \tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}(x \to y) \supseteq \tilde{f}_{\mathcal{L}}(y)$$

for all $x, y \in L$. Hence $\tilde{f}_{\mathcal{L}}$ is a uni-soft filter of $\mathcal{L}$ by Theorem 3.4. □

Proposition 3.6. *Every uni-soft filter $\tilde{f}_{\mathcal{L}}$ of $\mathcal{L}$ satisfies the following condition:*

$$(\forall x, y, z \in L)\left(\tilde{f}_{\mathcal{L}}(x \to (y \to z)) \cup \tilde{f}_{\mathcal{L}}(x \to y) \supseteq \tilde{f}_{\mathcal{L}}(x \to (x \to z))\right). \tag{3.7}$$

Proof. Let $x, y, z \in L$. Using (2.2) and (2.4), we have

$$x \to (y \to z) = y \to (x \to z) \leq (x \to y) \to (x \to (x \to z)).$$

It follows from Theorem 3.5 that

$$\tilde{f}_{\mathcal{L}}(x \to (y \to z)) \cup \tilde{f}_{\mathcal{L}}(x \to y) \supseteq \tilde{f}_{\mathcal{L}}(x \to (x \to z)).$$

This completes the proof. □

Theorem 3.7. *A soft set $\tilde{f}_{\mathcal{L}}$ over U is a uni-soft filter of $\mathcal{L}$ if and only if $\tilde{f}_{\mathcal{L}}$ satisfies the condition* (3.3) *and*

$$(\forall x, y, z \in L)\left(\tilde{f}_{\mathcal{L}}(x \to (y \to z)) \cup \tilde{f}_{\mathcal{L}}(y) \supseteq \tilde{f}_{\mathcal{L}}(x \to z)\right). \tag{3.8}$$

Proof. Assume that $\tilde{f}_{\mathcal{L}}$ is a uni-soft filter of $\mathcal{L}$. Then the condition (3.3) is valid. Using (3.4) and (2.2), we have

$$\begin{aligned} \tilde{f}_{\mathcal{L}}(x \to z) &\subseteq \tilde{f}_{\mathcal{L}}(y) \cup \tilde{f}_{\mathcal{L}}(y \to (x \to z)) \\ &= \tilde{f}_{\mathcal{L}}(y) \cup \tilde{f}_{\mathcal{L}}(x \to (y \to z)) \end{aligned}$$

for all $x, y, z \in L$.

Conversely, let $\tilde{f}_{\mathcal{L}}$ be a soft set over U satisfying (3.3) and (3.8). Taking $x := 1$ in (3.8) and using (2.1), we have

$$\begin{aligned} \tilde{f}_{\mathcal{L}}(z) = \tilde{f}_{\mathcal{L}}(1 \to z) &\subseteq \tilde{f}_{\mathcal{L}}(1 \to (y \to z)) \cup \tilde{f}_{\mathcal{L}}(y) \\ &= \tilde{f}_{\mathcal{L}}(y \to z) \cup \tilde{f}_{\mathcal{L}}(y) \end{aligned}$$

for all $y, z \in L$. Thus $\tilde{f}_{\mathcal{L}}$ is a uni-soft filter of $\mathcal{L}$ by Theorem 3.4. □

Proposition 3.8. *Every uni-soft filter $\tilde{f}_{\mathcal{L}}$ of $\mathcal{L}$ satisfies the following condition:*

$$(\forall a, x \in L)\left(\tilde{f}_{\mathcal{L}}(a) \supseteq \tilde{f}_{\mathcal{L}}((a \to x) \to x)\right). \tag{3.9}$$

Proof. If we take $y := (a \to x) \to x$ and $x := a$ in (3.4), then

$$\begin{aligned}\tilde{f}_{\mathcal{L}}((a \to x) \to x) &\subseteq \tilde{f}_{\mathcal{L}}(a) \cup \tilde{f}_{\mathcal{L}}(a \to ((a \to x) \to x)) \\ &= \tilde{f}_{\mathcal{L}}(a) \cup \tilde{f}_{\mathcal{L}}((a \to x) \to (a \to x)) \\ &= \tilde{f}_{\mathcal{L}}(a) \cup \tilde{f}_{\mathcal{L}}(1) = \tilde{f}_{\mathcal{L}}(a).\end{aligned}$$

This completes the proof. □

Theorem 3.9. *A soft set $\tilde{f}_{\mathcal{L}}$ over U is a uni-soft filter of $\mathcal{L}$ if and only if it satisfies the following conditions:*

$$(\forall x, y \in L)\left(\tilde{f}_{\mathcal{L}}(x) \supseteq \tilde{f}_{\mathcal{L}}(y \to x)\right), \tag{3.10}$$

$$(\forall x, a, b \in L)\left(\tilde{f}_{\mathcal{L}}(a) \cup \tilde{f}_{\mathcal{L}}(b) \supseteq \tilde{f}_{\mathcal{L}}((a \to (b \to x)) \to x)\right). \tag{3.11}$$

Proof. Assume that $\tilde{f}_{\mathcal{L}}$ is a uni-soft filter of $\mathcal{L}$. Using (3.4), (2.1) and (3.3), we have

$$\tilde{f}_{\mathcal{L}}(y \to x) \subseteq \tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}(x \to (y \to x)) = \tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}(1) = \tilde{f}_{\mathcal{L}}(x)$$

for all $x, y \in L$. Using (3.8) and (3.9), we get

$$\tilde{f}_{\mathcal{L}}((a \to (b \to x)) \to x) \subseteq \tilde{f}_{\mathcal{L}}((a \to (b \to x)) \to (b \to x)) \cup \tilde{f}_{\mathcal{L}}(b) \subseteq \tilde{f}_{\mathcal{L}}(a) \cup \tilde{f}_{\mathcal{L}}(b)$$

for all $a, b, x \in L$.

Conversely, let $\tilde{f}_{\mathcal{L}}$ be a soft set over U satisfying two conditions (3.10) and (3.11). If we take $y := x$ in (3.10), then $\tilde{f}_{\mathcal{L}}(x) \supseteq \tilde{f}_{\mathcal{L}}(x \to x) = \tilde{f}_{\mathcal{L}}(1)$ for all $x \in L$. Using (3.11) induces

$$\tilde{f}_{\mathcal{L}}(y) = \tilde{f}_{\mathcal{L}}(1 \to y) = \tilde{f}_{\mathcal{L}}(((x \to y) \to (x \to y)) \to y) \subseteq \tilde{f}_{\mathcal{L}}(x \to y) \cup \tilde{f}_{\mathcal{L}}(x)$$

for all $x, y \in L$. Therefore $\tilde{f}_{\mathcal{L}}$ is a uni-soft filter of $\mathcal{L}$ by Theorem 3.4. □

Theorem 3.10. *A soft set $\tilde{f}_{\mathcal{L}}$ over U is a uni-soft filter of $\mathcal{L}$ if and only if the nonempty τ-exclusive set of $\tilde{f}_{\mathcal{L}}$ is a filter of $\mathcal{L}$ for all $\tau \in \mathscr{P}(U)$.*

Proof. Assume that $\tilde{f}_{\mathcal{L}}$ is a uni-soft filter of $\mathcal{L}$ and let $\tau \in \mathscr{P}(U)$ be such that $e(\tilde{f}_{\mathcal{L}};\tau) \neq \emptyset$. Let $x, y \in L$ be such that $x \in e(\tilde{f}_{\mathcal{L}};\tau)$ and $x \to y \in e(\tilde{f}_{\mathcal{L}};\tau)$. Then $\tau \supseteq \tilde{f}_{\mathcal{L}}(x)$ and $\tau \supseteq \tilde{f}_{\mathcal{L}}(x \to y)$. It follows from (3.3) and (3.4) that $\tilde{f}_{\mathcal{L}}(1) \subseteq \tilde{f}_{\mathcal{L}}(x) \subseteq \tau$ and $\tilde{f}_{\mathcal{L}}(y) \subseteq \tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}(x \to y) \subseteq \tau$. Hence $1 \in e(\tilde{f}_{\mathcal{L}};\tau)$ and $y \in e(\tilde{f}_{\mathcal{L}};\tau)$, and therefore $e(\tilde{f}_{\mathcal{L}};\tau)$ is a filter of $\mathcal{L}$ by Proposition 2.4.

Conversely, suppose that $e(\tilde{f}_{\mathcal{L}};\tau)$ is a filter of $\mathcal{L}$ for all $\tau \in \mathscr{P}(U)$ with $e(\tilde{f}_{\mathcal{L}};\tau) \neq \emptyset$. For any $x \in L$, let $\tilde{f}_{\mathcal{L}}(x) = \delta$. Then $x \in e(\tilde{f}_{\mathcal{L}};\delta)$ and $e(\tilde{f}_{\mathcal{L}};\delta)$ is a filter of $\mathcal{L}$. Hence $1 \in e(\tilde{f}_{\mathcal{L}};\delta)$ and so $\tilde{f}_{\mathcal{L}}(x) = \delta \supseteq \tilde{f}_{\mathcal{L}}(1)$. For any $x, y \in L$, let $\tilde{f}_{\mathcal{L}}(x) = \delta_x$ and $\tilde{f}_{\mathcal{L}}(x \to y) = \delta_{x \to y}$. If we take $\delta = \delta_x \cup \delta_{x \to y}$, then $x \in e(\tilde{f}_{\mathcal{L}};\delta)$ and $x \to y \in e(\tilde{f}_{\mathcal{L}};\delta)$ which imply that $y \in e(\tilde{f}_{\mathcal{L}};\delta)$. Thus

$$\tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}(x \to y) = \delta_x \cup \delta_{x \to y} = \delta \supseteq \tilde{f}_{\mathcal{L}}(y).$$

Therefore $\tilde{f}_{\mathcal{L}}$ is a uni-soft filter of $\mathcal{L}$ by Theorem 3.4. □

Theorem 3.11. *For a soft set $\tilde{f}_{\mathcal{L}}$ over U, let $\tilde{f}^*_{\mathcal{L}}$ be a soft set over U which is given as follows:*

$$\tilde{f}^*_{\mathcal{L}} : L \to \mathscr{P}(U),\ x \mapsto \begin{cases} \tilde{f}_{\mathcal{L}}(x) & \text{if } x \in e(\tilde{f}_{\mathcal{L}};\tau), \\ U & \text{otherwise,} \end{cases}$$

where $\tau \in \mathscr{P}(U)$ with $\tau \neq U$. If $\tilde{f}_{\mathcal{L}}$ is a uni-soft filter of $\mathcal{L}$, then so is $\tilde{f}^*_{\mathcal{L}}$.

Proof. Suppose that $\tilde{f}_{\mathcal{L}}$ is a uni-soft filter of $\mathcal{L}$. Then $e(\tilde{f}_{\mathcal{L}};\tau)$ is a filter of $\mathcal{L}$ for all $\tau \in \mathscr{P}(U)$ with $e(\tilde{f}_{\mathcal{L}};\tau) \neq \emptyset$ by Theorem 3.10. Thus $1 \in e(\tilde{f}_{\mathcal{L}};\tau)$, and so $\tilde{f}^*_{\mathcal{L}}(1) = \tilde{f}_{\mathcal{L}}(1) \subseteq \tilde{f}_{\mathcal{L}}(x) \subseteq \tilde{f}^*_{\mathcal{L}}(x)$ for all $x \in L$. Let $x, y \in L$. If $x \in e(\tilde{f}_{\mathcal{L}};\tau)$ and $x \to y \in e(\tilde{f}_{\mathcal{L}};\tau)$, then $y \in e(\tilde{f}_{\mathcal{L}};\tau)$. Hence

$$\tilde{f}^*_{\mathcal{L}}(x) \cup \tilde{f}^*_{\mathcal{L}}(x \to y) = \tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}(x \to y) \supseteq \tilde{f}_{\mathcal{L}}(y) = \tilde{f}^*_{\mathcal{L}}(y).$$

If $x \notin e(\tilde{f}_{\mathcal{L}};\tau)$ or $x \to y \notin e(\tilde{f}_{\mathcal{L}};\tau)$, then $\tilde{f}^*_{\mathcal{L}}(x) = U$ or $\tilde{f}^*_{\mathcal{L}}(x \to y) = U$. Thus

$$\tilde{f}^*_{\mathcal{L}}(x) \cup \tilde{f}^*_{\mathcal{L}}(x \to y) = U \supseteq \tilde{f}^*_{\mathcal{L}}(y).$$

Therefore $\tilde{f}^*_{\mathcal{L}}$ is a uni-soft filter of $\mathcal{L}$. □

Theorem 3.12. *If $\tilde{f}_{\mathcal{L}}$ is a uni-soft filter of L, then the set*

$$\mathcal{L}_a := \{x \in L \mid \tilde{f}_{\mathcal{L}}(a) \supseteq \tilde{f}_{\mathcal{L}}(x)\}$$

is a filter of $\mathcal{L}$ for every $a \in L$.

Proof. Since $\tilde{f}_{\mathcal{L}}(1) \subseteq \tilde{f}_{\mathcal{L}}(a)$ for all $a \in L$, we have $1 \in \mathcal{L}_a$. Let $x, y \in L$ be such that $x \in \mathcal{L}_a$ and $x \to y \in \mathcal{L}_a$. Then $\tilde{f}_{\mathcal{L}}(x) \subseteq \tilde{f}_{\mathcal{L}}(a)$ and $\tilde{f}_{\mathcal{L}}(x \to y) \subseteq \tilde{f}_{\mathcal{L}}(a)$. Since $\tilde{f}_{\mathcal{L}}$ is a uni-soft filter of L, it follows from (3.4) that

$$\tilde{f}_{\mathcal{L}}(a) \supseteq \tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}(x \to y) \supseteq \tilde{f}_{\mathcal{L}}(y)$$

so that $y \in \mathcal{L}_a$. Hence $\mathcal{L}_a$ is a filter of $\mathcal{L}$ by Proposition 2.4. □

Theorem 3.13. *Let $a \in L$ and let $\tilde{f}_{\mathcal{L}}$ be a soft set over U. Then*

(1) *If $\mathcal{L}_a$ is a filter of L, then $\tilde{f}_{\mathcal{L}}$ satisfies the following condition:*

$$(\forall x, y \in L)\,(\tilde{f}_{\mathcal{L}}(a) \supseteq \tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}(x \to y) \;\Rightarrow\; \tilde{f}_{\mathcal{L}}(a) \supseteq \tilde{f}_{\mathcal{L}}(y)). \tag{3.12}$$

(2) *If $\tilde{f}_{\mathcal{L}}$ satisfies (3.3) and (3.12), then $\mathcal{L}_a$ is a filter of L.*

Proof. (1) Assume that $\mathcal{L}_a$ is a filter of L. Let $x, y \in L$ be such that

$$\tilde{f}_{\mathcal{L}}(a) \supseteq \tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}(x \to y).$$

Then $x \to y \in \mathcal{L}_a$ and $x \in \mathcal{L}_a$. Using (2.8), we have $y \in \mathcal{L}_a$ and so $\tilde{f}_{\mathcal{L}}(a) \supseteq \tilde{f}_{\mathcal{L}}(y)$.

(2) Suppose that $\tilde{f}_{\mathcal{L}}$ satisfies (3.3) and (3.12). Then $1 \in \mathcal{L}_a$ by (3.3). Let $x, y \in L$ be such that $x \in \mathcal{L}_a$ and $x \to y \in \mathcal{L}_a$. Then $\tilde{f}_{\mathcal{L}}(a) \supseteq \tilde{f}_{\mathcal{L}}(x)$ and $\tilde{f}_{\mathcal{L}}(a) \supseteq \tilde{f}_{\mathcal{L}}(x \to y)$, which imply that $\tilde{f}_{\mathcal{L}}(a) \supseteq \tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}(x \to y)$. Thus $\tilde{f}_{\mathcal{L}}(a) \supseteq \tilde{f}_{\mathcal{L}}(y)$ by (3.12), and so $y \in \mathcal{L}_a$. Therefore $\mathcal{L}_a$ is a filter of $\mathcal{L}$ by Proposition 2.4. □

4 Uni-soft G-filters

Definition 4.1. A soft set $\tilde{f}_{\mathcal{L}}$ over U is called a uni-soft G-filter of $\mathcal{L}$ if it is a uni-soft filter of $\mathcal{L}$ that satisfies:

$$(\forall x, y \in L)\left(\tilde{f}_{\mathcal{L}}((x \odot x) \to y) \supseteq \tilde{f}_{\mathcal{L}}(x \to y)\right). \tag{4.1}$$

Note that the condition (4.1) is equivalent to the following condition:

$$(\forall x, y \in L)\left(\tilde{f}_{\mathcal{L}}(x \to (x \to y)) \supseteq \tilde{f}_{\mathcal{L}}(x \to y)\right). \tag{4.2}$$

Example 4.2. Let $L := [0, 1]$ (unit interval). For any $a, b \in L$, define

$$a \vee b = \max\{a, b\},\ \ a \wedge b = \min\{a, b\},$$
$$a \to b = \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{otherwise,} \end{cases} \quad \text{and} \quad a \odot b = \min\{a, b\}.$$

Then $\mathcal{L} := (L, \vee, \wedge, \odot, \to, 0, 1)$ is a residuated lattice. Let $\tilde{f}_{\mathcal{L}}$ be a soft set over U defined by

$$\tilde{f}_{\mathcal{L}} : L \to \mathscr{P}(U),\ \ x \mapsto \begin{cases} \tau & \text{if } x \in [\frac{1}{2}, 1], \\ U & \text{otherwise,} \end{cases}$$

where $\tau \in \mathscr{P}(U)$ with $\tau \neq U$. Then $\tilde{f}_{\mathcal{L}}$ is a uni-soft G-filter of $\mathcal{L}$.

Theorem 4.3. *Let $\tilde{f}_{\mathcal{L}}$ be a soft set over U. Then $\tilde{f}_{\mathcal{L}}$ is a uni-soft G-filter of $\mathcal{L}$ if and only if it is a uni-soft filter of $\mathcal{L}$ that satisfies the following condition:*

$$(\forall x, y, z \in L) \left(\tilde{f}_{\mathcal{L}}(x \to (y \to z)) \cup \tilde{f}_{\mathcal{L}}(x \to y) \supseteq \tilde{f}_{\mathcal{L}}(x \to z)\right). \tag{4.3}$$

Proof. Assume that $\tilde{f}_{\mathcal{L}}$ is a uni-soft G-filter of $\mathcal{L}$. Then $\tilde{f}_{\mathcal{L}}$ is a uni-soft filter of $\mathcal{L}$. Note that $x \leq 1 = (x \to y) \to (x \to y)$, and thus $x \to y \leq x \to (x \to y)$ for all $x, y \in L$. It follows from (3.1) that $\tilde{f}_{\mathcal{L}}(x \to y) \supseteq \tilde{f}_{\mathcal{L}}(x \to (x \to y))$. Combining this and (4.2), we have

$$\tilde{f}_{\mathcal{L}}(x \to y) = \tilde{f}_{\mathcal{L}}(x \to (x \to y)) \tag{4.4}$$

for all $x, y \in L$. Using (3.7) and (4.4), we have

$$\tilde{f}_{\mathcal{L}}(x \to (y \to z)) \cup \tilde{f}_{\mathcal{L}}(x \to y) \supseteq \tilde{f}_{\mathcal{L}}(x \to z)$$

for all $x, y, z \in L$.

Conversely, let $\tilde{f}_{\mathcal{L}}$ be a uni-soft filter of $\mathcal{L}$ that satisfies the condition (4.3). If we put $y = x$ and $z = y$ in (4.3) and use (2.1) and (3.3), then

$$\begin{aligned} \tilde{f}_{\mathcal{L}}(x \to y) &\subseteq \tilde{f}_{\mathcal{L}}(x \to (x \to y)) \cup \tilde{f}_{\mathcal{L}}(x \to x) \\ &= \tilde{f}_{\mathcal{L}}(x \to (x \to y)) \cup \tilde{f}_{\mathcal{L}}(1) \\ &= \tilde{f}_{\mathcal{L}}(x \to (x \to y)) \end{aligned}$$

for all $x, y \in L$. Therefore $\tilde{f}_{\mathcal{L}}$ is a uni-soft G-filter of $\mathcal{L}$. □

Theorem 4.4. *Let $\tilde{f}_{\mathcal{L}}$ be a soft set over U that satisfies the condition (3.3) and*

$$(\forall x, y, z \in L) \left(\tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}((y \to z) \to (x \to y)) \supseteq \tilde{f}_{\mathcal{L}}(y) \right). \tag{4.5}$$

Then $\tilde{f}_{\mathcal{L}}$ is a uni-soft G-filter of $\mathcal{L}$.

Proof. If we take $z := 1$ in (4.5) and use (2.1), then

$$\begin{aligned} \tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}(x \to y) &= \tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}(1 \to (x \to y)) \\ &= \tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}((y \to 1) \to (x \to y)) \\ &\supseteq \tilde{f}_{\mathcal{L}}(y). \end{aligned}$$

Hence $\tilde{f}_{\mathcal{L}}$ is a uni-soft filter of $\mathcal{L}$ by Theorem 3.4. Let $x, y, z \in L$. Since

$$x \to (y \to z) \leq (x \to y) \to (x \to (x \to z))$$

by (2.2) and (2.4), we have $\tilde{f}_{\mathcal{L}}(x \to (y \to z)) \supseteq \tilde{f}_{\mathcal{L}}((x \to y) \to (x \to (x \to z)))$ by (3.1). It follows from (3.1), (3.3), (3.4), (2.4) and (4.5) that

$$\begin{aligned} \tilde{f}_{\mathcal{L}}(x \to y) \cup \tilde{f}_{\mathcal{L}}(x \to (y \to z)) &\supseteq \tilde{f}_{\mathcal{L}}(x \to y) \cup \tilde{f}_{\mathcal{L}}((x \to y) \to (x \to (x \to z))) \\ &\supseteq \tilde{f}_{\mathcal{L}}(x \to (x \to z)) \\ &\supseteq \tilde{f}_{\mathcal{L}}(((x \to z) \to z) \to (x \to z)) \\ &= \tilde{f}_{\mathcal{L}}(((x \to z) \to z) \to (1 \to (x \to z))) \\ &\supseteq \tilde{f}_{\mathcal{L}}(x \to z). \end{aligned}$$

Therefore $\tilde{f}_{\mathcal{L}}$ is a uni-soft G-filter of $\mathcal{L}$ by Theorem 4.3. □

The following example shows that any uni-soft G-filter may not satisfy the condition (4.5).

Example 4.5. The uni-soft G-filter $\tilde{f}_{\mathcal{L}}$ of $\mathcal{L}$ in Example 4.2 does not satisfy the condition (4.5) since

$$\tilde{f}_{\mathcal{L}}(\tfrac{2}{3}) \cup \tilde{f}_{\mathcal{L}}((\tfrac{1}{3} \to \tfrac{1}{4}) \to (\tfrac{2}{3} \to \tfrac{1}{3})) = \tilde{f}_{\mathcal{L}}(\tfrac{2}{3}) \cup \tilde{f}_{\mathcal{L}}(1) = \tau \not\supseteq U = \tilde{f}_{\mathcal{L}}(\tfrac{1}{3}).$$

Proposition 4.6. *For a uni-soft filter $\tilde{f}_{\mathcal{L}}$ of $\mathcal{L}$, the condition (4.5) is equivalent to the following condition.*

$$(\forall x, y \in L) \left(\tilde{f}_{\mathcal{L}}((x \to y) \to x) \supseteq \tilde{f}_{\mathcal{L}}(x) \right). \tag{4.6}$$

Proof. Assume that the condition (4.5) is valid. It follows from (3.3) and (2.1) that

$$\begin{aligned}\tilde{f}_{\mathcal{L}}((x \to y) \to x) &= \tilde{f}_{\mathcal{L}}(1) \cup \tilde{f}_{\mathcal{L}}((x \to y) \to x)\\ &= \tilde{f}_{\mathcal{L}}(1) \cup \tilde{f}_{\mathcal{L}}((x \to y) \to (1 \to x))\\ &\supseteq \tilde{f}_{\mathcal{L}}(x)\end{aligned}$$

for all $x, y \in L$.

Conversely, suppose that the condition (4.6) is valid. It follows from (2.2) and (3.4) that

$$\begin{aligned}\tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}((y \to z) \to (x \to y)) &= \tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}(x \to ((y \to z) \to y))\\ &\supseteq \tilde{f}_{\mathcal{L}}((y \to z) \to y) \supseteq \tilde{f}_{\mathcal{L}}(y)\end{aligned}$$

for all $x, y \in L$. □

Combining Theorem 4.4 and Proposition 4.6, we have the following result.

Theorem 4.7. *Every uni-soft filter satisfying the condition (4.6) is a uni-soft G-filter.*

Proposition 4.8. *Every uni-soft filter $\tilde{f}_{\mathcal{L}}$ of $\mathcal{L}$ with the condition (4.5) satisfies the following condition.*

$$(\forall x, y \in L)\left(\tilde{f}_{\mathcal{L}}((x \to y) \to y) \supseteq \tilde{f}_{\mathcal{L}}((y \to x) \to x)\right). \tag{4.7}$$

Proof. Let $\tilde{f}_{\mathcal{L}}$ be a uni-soft filter of $\mathcal{L}$ that satisfies the condition (4.5) and let $x, y \in L$. Since $x \to ((y \to x) \to x) = (y \to x) \to (x \to x) = (y \to x) \to 1 = 1$, that is, $x \leq (y \to x) \to x$, we have $((y \to x) \to x) \to y \leq x \to y$ by (2.3). It follows from (2.4), (2.2) and (2.3) that

$$\begin{aligned}(x \to y) \to y &\leq (y \to x) \to ((x \to y) \to x)\\ &= (x \to y) \to ((y \to x) \to x)\\ &\leq (((y \to x) \to x) \to y) \to ((y \to x) \to x).\end{aligned}$$

Using (3.1), (3.3), (2.1), (2.2) and (4.5), we have

$$\begin{aligned}\tilde{f}_{\mathcal{L}}((x \to y) \to y) &\supseteq \tilde{f}_{\mathcal{L}}((((y \to x) \to x) \to y) \to ((y \to x) \to x))\\ &= \tilde{f}_{\mathcal{L}}(1) \cup \tilde{f}_{\mathcal{L}}(1 \to ((((y \to x) \to x) \to y) \to ((y \to x) \to x)))\\ &= \tilde{f}_{\mathcal{L}}(1) \cup \tilde{f}_{\mathcal{L}}((((y \to x) \to x) \to y) \to (1 \to ((y \to x) \to x)))\\ &\supseteq \tilde{f}_{\mathcal{L}}((y \to x) \to x).\end{aligned}$$

Hence the condition (4.7) is valid. □

Corollary 4.9. *Every uni-soft filter $\tilde{f}_{\mathcal{L}}$ of $\mathcal{L}$ with the condition (4.6) satisfies the condition (4.7).*

Proposition 4.10. *Every uni-soft G-filter $\tilde{f}_{\mathcal{L}}$ of $\mathcal{L}$ with the condition (4.7) satisfies the condition (4.5).*

Proof. Let $\tilde{f}_{\mathcal{L}}$ be a uni-soft G-filter of $\mathcal{L}$ that satisfies the condition (4.7). For any $x, y, z \in L$, we have

$$\begin{aligned}\tilde{f}_{\mathcal{L}}(z) \cup \tilde{f}_{\mathcal{L}}((x \to y) \to (z \to x)) &= \tilde{f}_{\mathcal{L}}(z) \cup \tilde{f}_{\mathcal{L}}(z \to ((x \to y) \to x)) \\ &\supseteq \tilde{f}_{\mathcal{L}}((x \to y) \to x) \\ &\supseteq \tilde{f}_{\mathcal{L}}((x \to y) \to ((x \to y) \to y)) \\ &\supseteq \tilde{f}_{\mathcal{L}}((x \to y) \to y) \\ &\supseteq \tilde{f}_{\mathcal{L}}((y \to x) \to x)\end{aligned}$$

by (2.2), (3.4), (3.1), (2.4), (4.2) and (4.7). Since $(x \to y) \to x \leq y \to x \leq z \to (y \to x)$, it follows from (3.1) that $\tilde{f}_{\mathcal{L}}((x \to y) \to x) \supseteq \tilde{f}_{\mathcal{L}}(z \to (y \to x))$ and so from (3.4) that

$$\begin{aligned}\tilde{f}_{\mathcal{L}}(z) \cup \tilde{f}_{\mathcal{L}}((x \to y) \to (z \to x)) &\supseteq \tilde{f}_{\mathcal{L}}(z) \cup \tilde{f}_{\mathcal{L}}((x \to y) \to x) \\ &\supseteq \tilde{f}_{\mathcal{L}}(z) \cup \tilde{f}_{\mathcal{L}}(z \to (y \to x)) \\ &\supseteq \tilde{f}_{\mathcal{L}}(y \to x).\end{aligned}$$

Therefore

$$\tilde{f}_{\mathcal{L}}(z) \cup \tilde{f}_{\mathcal{L}}((x \to y) \to (z \to x)) \supseteq \tilde{f}_{\mathcal{L}}(y \to x) \cup \tilde{f}_{\mathcal{L}}((y \to x) \to x) \supseteq \tilde{f}_{\mathcal{L}}(x).$$

Hence the condition (4.5) is valid. □

Theorem 4.11. *Let $\tilde{f}_{\mathcal{L}}$ be a uni-soft filter of $\mathcal{L}$. Then $\tilde{f}_{\mathcal{L}}$ is a uni-soft G-filter of $\mathcal{L}$ if and only if the following condition holds:*

$$(\forall x \in L)\left(\tilde{f}_{\mathcal{L}}(x \to (x \odot x)) = \tilde{f}_{\mathcal{L}}(1)\right). \tag{4.8}$$

Proof. Suppose that $\tilde{f}_{\mathcal{L}}$ is a uni-soft G-filter of L. Since $x \to (x \to (x \odot x)) = 1$ for all $x \in L$, we have $\tilde{f}_{\mathcal{L}}(x \to (x \to (x \odot x))) = \tilde{f}_{\mathcal{L}}(1)$. It follows from (4.3) and (2.1) that

$$\tilde{f}_{\mathcal{L}}(x \to (x \odot x)) \subseteq \tilde{f}_{\mathcal{L}}(x \to (x \to (x \odot x))) \cup \tilde{f}_{\mathcal{L}}(x \to x) = \tilde{f}_{\mathcal{L}}(1)$$

and so from (3.3) that $\tilde{f}_{\mathcal{L}}(x \to (x \odot x)) = \tilde{f}_{\mathcal{L}}(1)$ for all $x \in L$.

Conversely, let $\tilde{f}_{\mathcal{L}}$ be a uni-soft filter of $\mathcal{L}$ which satisfies the condition (4.8) and let $x, y \in L$. Since

$$x \to (x \to y) = (x \odot x) \to y \leq (x \to (x \odot x)) \to (x \to y)$$

by (2.2) and (2.4), it follows from (3.1) that

$$\tilde{f}_{\mathcal{L}}(x \to (x \to y)) \supseteq \tilde{f}_{\mathcal{L}}((x \to (x \odot x)) \to (x \to y)).$$

Hence, we have

$$\begin{aligned} \tilde{f}_{\mathcal{L}}(x \to y) &\subseteq \tilde{f}_{\mathcal{L}}((x \to (x \odot x)) \to (x \to y)) \cup \tilde{f}_{\mathcal{L}}(x \to (x \odot x)) \\ &\subseteq \tilde{f}_{\mathcal{L}}(x \to (x \to y)) \cup \tilde{f}_{\mathcal{L}}(x \to (x \odot x)) \\ &= \tilde{f}_{\mathcal{L}}(x \to (x \to y)) \cup \tilde{f}_{\mathcal{L}}(1) \\ &= \tilde{f}_{\mathcal{L}}(x \to (x \to y)) \end{aligned}$$

by using (3.4), (4.8) and (3.3). Hence $\tilde{f}_{\mathcal{L}}$ is a uni-soft G-filter of $\mathcal{L}$. □

Theorem 4.12. *A soft set $\tilde{f}_{\mathcal{L}}$ over U is a uni-soft G-filter of $\mathcal{L}$ if and only if it is a uni-soft filter of $\mathcal{L}$ with an additional condition:*

$$(\forall x, y \in L) \left(\tilde{f}_{\mathcal{L}}(x \to y) = \tilde{f}_{\mathcal{L}}(x \to (x \to y)) \right). \tag{4.9}$$

Proof. Suppose that $\tilde{f}_{\mathcal{L}}$ is a uni-soft G-filter of $\mathcal{L}$. Then $\tilde{f}_{\mathcal{L}}$ is a uni-soft filter of $\mathcal{L}$. Let $x, y \in L$. Since $x \to y \leq x \to (x \to y)$, we have $\tilde{f}_{\mathcal{L}}(x \to y) \supseteq \tilde{f}_{\mathcal{L}}(x \to (x \to y))$ by (3.1). Hence $\tilde{f}_{\mathcal{L}}(x \to y) = \tilde{f}_{\mathcal{L}}(x \to (x \to y))$ by using (4.2).

Conversely, let $\tilde{f}_{\mathcal{L}}$ be a uni-soft filter of $\mathcal{L}$ with the condition (4.9). It follows from Proposition 3.6 that

$$\tilde{f}_{\mathcal{L}}(x \to (y \to z)) \cup \tilde{f}_{\mathcal{L}}(x \to y) \supseteq \tilde{f}_{\mathcal{L}}(x \to (x \to z)) = \tilde{f}_{\mathcal{L}}(x \to z)$$

for all $x, y, z \in L$. Therefore $\tilde{f}_{\mathcal{L}}$ is a uni-soft G-filter of $\mathcal{L}$ by Theorem 4.3. □

Proposition 4.13. *Every uni-soft G-filter $\tilde{f}_{\mathcal{L}}$ of $\mathcal{L}$ satisfies the following conditions:*

$$(\forall x, y, z \in L) \left(\tilde{f}_{\mathcal{L}}(x \to (y \to z)) \supseteq \tilde{f}_{\mathcal{L}}((x \to y) \to (x \to z)) \right). \tag{4.10}$$

$$(\forall x, y, z \in L) \left(\tilde{f}_{\mathcal{L}}(x \to (y \to z)) = \tilde{f}_{\mathcal{L}}((x \to y) \to (x \to z)) \right). \tag{4.11}$$

Proof. Let $\tilde{f}_{\mathcal{L}}$ be a uni-soft G-filter of $\mathcal{L}$. Using (2.2), (4.3), (2.4) and (3.3), we have

$$\begin{aligned}\tilde{f}_{\mathcal{L}}((x \to y) \to (x \to z)) &= \tilde{f}_{\mathcal{L}}(x \to ((x \to y) \to z)) \\ &\subseteq \tilde{f}_{\mathcal{L}}(x \to (y \to z)) \cup \tilde{f}_{\mathcal{L}}(x \to ((y \to z) \to ((x \to y) \to z))) \\ &= \tilde{f}_{\mathcal{L}}(x \to (y \to z)) \cup \tilde{f}_{\mathcal{L}}((y \to z) \to ((x \to y) \to (x \to z))) \\ &= \tilde{f}_{\mathcal{L}}(x \to (y \to z)) \cup \tilde{f}_{\mathcal{L}}(1) \\ &= \tilde{f}_{\mathcal{L}}(x \to (y \to z))\end{aligned}$$

for all $x, y, z \in L$. Thus (4.10) holds. Since $(x \to y) \to (x \to z) \leq x \to (y \to z)$ for all $x, y, z \in L$, it follows from (3.1) that $\tilde{f}_{\mathcal{L}}((x \to y) \to (x \to z)) \supseteq \tilde{f}_{\mathcal{L}}(x \to (y \to z))$ and so that

$$\tilde{f}_{\mathcal{L}}(x \to (y \to z)) = \tilde{f}_{\mathcal{L}}((x \to y) \to (x \to z))$$

for all $x, y, z \in L$ by using (4.10). □

Proposition 4.14. *Assume that $\mathcal{L}$ satisfies the divisibility, that is, $x \wedge y = x \odot (x \to y)$ for all $x, y \in L$. If $\tilde{f}_{\mathcal{L}}$ is a uni-soft G-filter of $\mathcal{L}$ satisfying (4.11), then the following equality is true.*

$$(\forall x, y, z \in L) \left(\tilde{f}_{\mathcal{L}}((x \odot y) \to z) = \tilde{f}_{\mathcal{L}}((x \wedge y) \to z) \right). \tag{4.12}$$

Proof. Using the divisibility and (2.2), we have

$$(x \wedge y) \to z = (x \odot (x \to y)) \to z = (x \to y) \to (x \to z)$$

for all $x, y, z \in L$. It follows from (2.2) and (4.11) that

$$\begin{aligned}\tilde{f}_{\mathcal{L}}((x \odot y) \to z) &= \tilde{f}_{\mathcal{L}}(x \to (y \to z)) \\ &= \tilde{f}_{\mathcal{L}}((x \to y) \to (x \to z)) \\ &= \tilde{f}_{\mathcal{L}}((x \wedge y) \to z)\end{aligned}$$

for all $x, y, z \in L$. □

Theorem 4.15. *Let $\mathcal{L}$ satisfy the divisibility, that is, $x \wedge y = x \odot (x \to y)$ for all $x, y \in L$. Then every uni-soft filter $\tilde{f}_{\mathcal{L}}$ of $\mathcal{L}$ satisfying the condition (4.12) is a uni-soft G-filter of $\mathcal{L}$.*

Proof. Using Proposition 3.6, (2.2) and (4.12), we have

$$\begin{aligned}&\tilde{f}_{\mathcal{L}}(x \to (y \to z)) \cup \tilde{f}_{\mathcal{L}}(x \to y) \supseteq \tilde{f}_{\mathcal{L}}(x \to (x \to z)) \\ &= \tilde{f}_{\mathcal{L}}((x \odot x) \to z) = \tilde{f}_{\mathcal{L}}((x \wedge x) \to z) = \tilde{f}_{\mathcal{L}}(x \to z)\end{aligned}$$

for all $x, y, z \in L$. Therefore $\tilde{f}_{\mathcal{L}}$ is a uni-soft G-filter of $\mathcal{L}$ by Theorem 4.3. □

Theorem 4.16. *Let $\tilde{f}_{\mathcal{L}}$ and $\tilde{g}_{\mathcal{L}}$ be uni-soft filters of $\mathcal{L}$ such that $\tilde{f}_{\mathcal{L}}(1) = \tilde{g}(1)$ and $\tilde{f}_{\mathcal{L}} \supseteq \tilde{g}_{\mathcal{L}}$, i.e., $\tilde{f}_{\mathcal{L}}(x) \supseteq \tilde{g}_{\mathcal{L}}(x)$ for all $x \in L$. If $\tilde{f}_{\mathcal{L}}$ is a uni-soft G-filter of $\mathcal{L}$, then so is $\tilde{g}_{\mathcal{L}}$.*

Proof. Assume that $\tilde{f}_{\mathcal{L}}$ is a uni-soft G-filter of $\mathcal{L}$. Using (2.2) and (2.1), we have

$$x \to (x \to ((x \to (x \to y)) \to y)) = (x \to (x \to y)) \to (x \to (x \to y)) = 1$$

for all $x, y \in L$. Thus

$$\begin{aligned}\tilde{g}(x \to ((x \to (x \to y)) \to y)) &\subseteq \tilde{f}_{\mathcal{L}}(x \to ((x \to (x \to y)) \to y)) \\ &= \tilde{f}_{\mathcal{L}}(x \to (x \to ((x \to (x \to y)) \to y))) \\ &= \tilde{f}_{\mathcal{L}}(1) = \tilde{g}(1)\end{aligned}$$

by hypotheses and (4.4), and so

$$\tilde{g}(x \to ((x \to (x \to y)) \to y)) = \tilde{g}(1)$$

for all $x, y \in L$ by (3.3). Since $\tilde{g}_{\mathcal{L}}$ is a uni-soft filter of $\mathcal{L}$, it follows from (3.4), (2.2) and (3.3) that

$$\begin{aligned}\tilde{g}(x \to y) &\subseteq \tilde{g}(x \to (x \to y)) \cup \tilde{g}((x \to (x \to y)) \to (x \to y)) \\ &= \tilde{g}(x \to (x \to y)) \cup \tilde{g}(x \to ((x \to (x \to y)) \to y)) \\ &= \tilde{g}(x \to (x \to y)) \cup \tilde{g}(1) \\ &= \tilde{g}(x \to (x \to y))\end{aligned}$$

for all $x, y \in L$. Therefore $\tilde{g}_{\mathcal{L}}$ is a uni-soft G-filter of $\mathcal{L}$. □

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Mathematical analysis of a general viral infection model with immune response

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Abstract

In this paper, we study the global dynamics of a viral infection model with antibody immune response. The incidence rate is given by a general function of the population of the uninfected target cells, infected cells and free viruses. We have established a set of conditions on the general incidence rate function and determined two threshold parameters R_0 (the basic infection reproduction number) and R_1 (the antibody immune response activation number) which are sufficient to determine the global behavior of the model. The global asymptotic stability of the equilibria of the model has been proven by using direct Lyapunov method and applying LaSalle's invariance principle.

Keywords: Virus dynamics; global stability; antibody immune response; Lyapunov functional.

Mathematics Subject Classification: 34D20; 34D23; 37N25; 92D30

1 Introduction

Several works have been devoted to propose mathematical models of viral infectious dynamics such as human immunodeficiency virus (HIV) (see, for example, [1]-[22]), hepatitis B virus (HBV) [23]-[26], hepatitis C virus (HCV) [27]-[29] and human T cell leukemia HTLV [30], etc. Mathematical models of viral infection can help for understanding the viral dynamics and developing antiviral drug therapies. In reality, the immune response needs an indispensable components to do its job such as antibodies, cytokines, natural killer cells, and T cells. The antibody immune response is a part of the adaptive system in which the body responds to pathogens by primarily using antibodies that produced from the B cells. While the other part is the Cytotoxic T Lymphocytes (CTL) immune response where the CTL attacks and kills the infected cells [7]. In some infections such as malaria, the CTL immune response is less effective than the antibody immune response [31]. Mathematical models of viral infection with antibody immune response have been proposed and analyzed in ([32]-[39]). The basic model of viral infection with antibody immune response has introduced by Murase et. al. [32] and Shifi Wang [39] as:

$$\dot{x} = s - dx - \beta vx, \tag{1}$$

$$\dot{y} = \beta vx - ay, \tag{2}$$

$$\dot{v} = ky - bzv - cv, \tag{3}$$

$$\dot{z} = rzv - \mu z, \tag{4}$$

where x, y, v and z denote the populations of uninfected target cells, infected cells, free virus particles and antibody immune cells at time t, respectively. Parameters s, k and r represent, respectively, the rate at which new healthy cells are generated from the source within the body, the generation rate constant of free viruses produced from the infected cells and the proliferation rate constant of antibody immune cells. Parameters d, a, c and μ are the natural death rate constants of the uninfected target cells, infected cells, free virus particles and antibody immune cells, respectively. Parameter β is the infection rate constant and b is the removal rate constant of the viruses due to the antibodies. All the parameters given in model (1)-(4) are positive.

Note that, the infection rate in model (1)-(4) is presented to be bilinear in x and v, which can not be completely describe the interaction between the uninfected target cells and viruses. Nevertheless, there are many types of an improved incidence rate which are more commonly used due to their benefit for helping us gain the unification theory through passing over the unessential details (see e.g. [40] and [41]). Variety of viral infection models with antibody immune response have been considered different forms of the incidence rate such as saturated incidence rate, $\frac{\beta xv}{1+\alpha v}$ where $\alpha \geq 0$ [42], [37], [35], Beddington-DeAngelis functional response, $\frac{\beta xv}{1+\gamma x+\alpha v}$, $\alpha, \gamma \geq 0$ [36], and general form, $\psi(x, v)v$ [38].

However the infection rate does not depend on the infected cells y. In some viral infections such as HBV, the infection rate depends on x, y and v [25], [24]. In [43], the infection rate is given by $\psi(x, y, v)v$, however the antibody immune response has been neglected. Our aim in this paper is to investigate the global stability analysis of the viral infection model with general incidence rate function and antibody immune response.

The rest of the paper is designed as follows. In the next section, we introduce the model and discuss the non-negativity and boundedness of the solutions. In Section 3, we define two threshold parameters and discuss the existence of the model's equilibria. In Section 4, we study the global asymptotic stability of the equilibria using suitable Lyapunov functional and applying LaSalle's invariance principle. Finally, conclusion is given in Section 5.

2 The mathematical model

In this section, we consider the following viral infection model with general incidence rate taking into consideration the antibody immune response.

$$\dot{x} = s - dx - \psi(x, y, v)v, \tag{5}$$

$$\dot{y} = \psi(x, y, v)v - ay, \tag{6}$$

$$\dot{v} = ky - bzv - cv, \tag{7}$$

$$\dot{z} = rzv - \mu z. \tag{8}$$

The definitions of all variables and parameters are identical to those given in Section 1. The incidence rate of infection is presented by a general function in the form $\psi(x, y, v)v$, where ψ is continuously differentiable and satisfies the following assumptions (see [38] and [43]):

Assumption A1. $\psi(x, y, v) > 0$ for all $x > 0$, $y \geq 0$, $v \geq 0$, and $\psi(0, y, v) = 0$ for all $y \geq 0$, $v \geq 0$.

Assumption A2. $\dfrac{\partial\psi(x, y, v)}{\partial x} > 0$ for all $x > 0$, $y \geq 0$ and $v \geq 0$.

Assumption A3. $\dfrac{\partial\psi(x, y, v)}{\partial y} < 0$, $\dfrac{\partial\psi(x, y, v)}{\partial v} < 0$ for all $x > 0$, $y > 0$ and $v > 0$.

Assumption A4. $\dfrac{\partial\left(\psi(x, y, v)v\right)}{\partial v} > 0$ for all $x > 0$, $y > 0$ and $v > 0$.

2.1 Positive invariance

In the following proposition, we show that the non-negative orthant $\mathbb{R}^4_{\geq 0}$ is the positively invariant and there exists a compact set which is positively invariant for model (5)-(8).

Proposition 1. Assume that Assumption A1 is satisfied. Then there exist positive numbers L_i, $i = 1, 2, 3$, such that the compact set

$$\Gamma = (x, y, v, z) \in \mathbb{R}^4_{\geq 0} : 0 \leq x, y \leq L_1, 0 \leq v \leq L_2, 0 \leq z \leq L_3$$

is positively invariant.

Proof. First, we prove that the orthant $\mathbb{R}^4_{\geq 0}$ is positively invariance for system (5)-(8). We have

$$\dot{x} \mid_{x=0} = s > 0,$$

$$\dot{y} \mid_{y=0} = \psi(x, 0, v)v \geq 0 \text{ for all } x > 0, v \geq 0,$$

$$\dot{v} \mid_{v=0} = ky \geq 0 \text{ for all } y \geq 0,$$

$$\dot{z} \mid_{z=0} = 0.$$

Hence, all the solutions are nonnegative.

Next we show that the solutions of system are bounded. Let $T_1(t) = x(t) + y(t)$, then

$$\dot{T}_1(t) = (s - dx - \psi(x, y, v)v) + \psi(x, y, v)v - ay,$$
$$= s - dx - ay \leq s - \sigma_1(x + y) = s - \sigma_1 T_1(t),$$

where $\sigma_1 = \min\{d, a\}$. Hence $0 \leq T_1(t) \leq \frac{s}{\sigma_1}$ for all $t \geq 0$ if $T_1(0) \leq \frac{s}{\sigma_1}$. It follows that, $0 \leq x(t), y(t) \leq L_1$ for all $t \geq 0$ if $x(0) + y(0) \leq L_1$, where $L_1 = \frac{s}{\sigma_1}$. Moreover, let $T_2(t) = v(t) + \frac{b}{r}z(t)$, then

$$\dot{T}_2(t) = ky - cv - \frac{b\mu}{r}z \leq kL_1 - \sigma_2(v + \frac{b}{r}z) = kL_1 - \sigma_2 T_2(t),$$

where $\sigma_2 = \min\{c, \mu\}$. Hence $0 \leq T_2(t) \leq L_2$ for all $t \geq 0$ when $T_2(0) \leq L_2$. It follows that $0 \leq v(t) \leq L_2$ and $0 \leq z(t) \leq L_3$ for all $t \geq 0$ if $v(0) + \frac{b}{r}z(0) \leq L_2$, where $L_2 = \dfrac{kL_1}{\sigma_2}$ and $L_3 = \frac{r}{b}L_2$. Therefore, $x(t), y(t), v(t)$ and $z(t)$ are all bounded.

2.2 The equilibria and threshold parameters

At any equilibrium we have

$$s - dx - \psi(x, y, v)v = 0, \tag{9}$$

$$\psi(x, y, v)v - ay = 0, \tag{10}$$

$$ky - bzv - cv = 0, \tag{11}$$

$$rzv - \mu z = 0. \tag{12}$$

From Eq. (12), either $z = 0$ or $z \neq 0$. If $z = 0$, then from Eqs. (9)-(11) we get

$$y = \frac{s - dx}{a} = \frac{c}{k}v, \qquad v = \frac{k(s - dx)}{ac}. \tag{13}$$

Substituting from Eq. (13) into Eq. (10) we get:

$$\left[\psi\left(x, \frac{s - dx}{a}, \frac{k(s - dx)}{ac}\right) - \frac{ac}{k}\right] v = 0. \tag{14}$$

Eq. (14) has two possible solutions $v = 0$ or $v \neq 0$. If $v = 0$, then from Eqs. (9) and (10), we get $x = s/d$ and $y = 0$ which leads to the infection-free equilibrium $E_0(x_0, 0, 0, 0)$ where $x_0 = s/d$. If $v \neq 0$, then we have

$$\psi\left(x, \frac{s - dx}{a}, \frac{k(s - dx)}{ac}\right) - \frac{ac}{k} = 0.$$

Let

$$\Phi_1(x) = \psi\left(x, \frac{s - dx}{a}, \frac{k(s - dx)}{ac}\right) - \frac{ac}{k} = 0.$$

Then, we have

$$\Phi_1'(x) = \frac{\partial \psi}{\partial x} - \frac{d}{a}\frac{\partial \psi}{\partial y} - \frac{kd}{ac}\frac{\partial \psi}{\partial v}.$$

Because of Assumptions A2 and A3, we have $\Phi_1'(x) > 0$ which implies that function $\Phi_1(x)$ is strictly increasing w.r.t. x. Moreover,

$$\Phi_1(0) = \psi\left(0, \frac{s}{a}, \frac{ks}{ac}\right) - \frac{ac}{k} = -\frac{ac}{k} < 0,$$

$$\Phi_1(x_0) = \psi(x_0, 0, 0) - \frac{ac}{k} = \frac{ac}{k}\left(\frac{k\psi(x_0, 0, 0)}{ac} - 1\right).$$

Therefore, if $\dfrac{k\psi(x_0,0,0)}{ac} > 1$, then there exists a unique $x_1 \in (0, x_0)$ such that $\Phi_1(x_1) = 0$. Therefore from Eq. (13) we obtain $y_1 = \dfrac{d(x_0 - x_1)}{a} > 0$ and $v_1 = \dfrac{kd(x_0 - x_1)}{ac} > 0$. It follows that, if $\dfrac{k\psi(x_0,0,0)}{ac} > 1$, then there exists a chronic-infection equilibrium without antibody immune response $E_1(x_1, y_1, v_1, 0)$.

Let us define the basic reproduction number as:

$$R_0 = \frac{k\psi(x_0, 0, 0)}{ac}.$$

The parameter R_0 determines whether a chronic-infection can be established. The other possibility of Eq. (12) is $z \neq 0$ which leads to $v_2 = \dfrac{\mu}{r}$. From Eq. (9) we let

$$\Phi_2(x) = s - dx - \psi\left(x, \frac{s - dx}{a}, v_2\right) v_2 = 0.$$

Assumptions A2 and A3 provide that Φ_2 is a decreasing function of x. Clearly, $\Phi_2(0) = s > 0$ and $\Phi_2(x_0) = -\psi(x_0, 0, v_2)v_2 < 0$. Thus, there exists a unique $x_2 \in (0, x_0)$ such that $\Phi_2(x_2) = 0$. It follows from Eqs. (11) and (13) that, $y_2 = \dfrac{d(x_0 - x_2)}{a} > 0$ and $z_2 = \dfrac{k\psi(x_2, y_2, v_2)}{ab} - \dfrac{c}{b} = \dfrac{c}{b}\left(\dfrac{k\psi(x_2, y_2, v_2)}{ac} - 1\right)$. Then if $\dfrac{k\psi(x_2, y_2, v_2)}{ac} > 1$ then $z_2 > 0$. Now we Define the antibody immune response activation number as:

$$R_1 = \frac{k\psi(x_2, y_2, v_2)}{ac},$$

which determines whether a persistent antibody immune response can be established. Hence, z_2 can be rewritten as $z_2 = \dfrac{c}{b}(R_1 - 1)$. It follows that, there is a chronic-infection equilibrium with antibody immune response $E_2(x_2, y_2, v_2, z_2)$ iff $R_1 > 1$.

Clearly from Assumptions A2 and A3, we have

$$R_1 = \frac{k\psi(x_2, y_2, v_2)}{ac} < \frac{k\psi(x_0, y_2, v_2)}{ac} < \frac{k\psi(x_0, 0, 0)}{ac} = R_0.$$

2.3 Global stability analysis

In this section, the global asymptotic stability of the three equilibria of model (5)-(8) will be established by using direct Lyapunov method and applying LaSalle's invariance principle. Let us define the function $H : (0, \infty) \to [0, \infty)$ as

$$H(w) = w - 1 - \ln w.$$

Theorem 1. Let Assumptions A1-A3 be hold true and $R_0 \leq 1$, then the infection-free equilibrium E_0 is globally asymptotically stable (GAS).

Proof. We construct a Lyapunov functional as:

$$U_0 = x - x_0 - \int_{x_0}^{x} \frac{\psi(x_0, 0, 0)}{\psi(\eta, 0, 0)} d\eta + y + \frac{a}{k} v + \frac{ab}{rk} z. \tag{15}$$

We calculate $\frac{dU_0}{dt}$ along the solutions of model (5)-(8) as:

$$\begin{aligned} \frac{dU_0}{dt} &= d\left(1 - \frac{\psi(x_0, 0, 0)}{\psi(x, 0, 0)}\right)(x_0 - x) + \left(\psi(x, y, v)\frac{\psi(x_0, 0, 0)}{\psi(x, 0, 0)} - \frac{ac}{k}\right) v - \frac{ab\mu}{rk} z \\ &= s\left(1 - \frac{\psi(x_0, 0, 0)}{\psi(x, 0, 0)}\right)\left(1 - \frac{x}{x_0}\right) + \frac{ac}{k}\left(\frac{\psi(x, y, v)}{\psi(x, 0, 0)} R_0 - 1\right) v - \frac{ab\mu}{rk} z. \end{aligned} \tag{16}$$

From Assumptions A2 and A3 we know that $\psi(x, y, v)$ is an increasing function of x and decreasing function of y and v. Then the first term of Eq. (16) is less than or equal zero and

$$\psi(x, y, v) < \psi(x, 0, 0), \quad x, y, v > 0.$$

It follows that

$$\frac{dU_0}{dt} \leq s\left(1-\frac{\psi(x_0,0,0)}{\psi(x,0,0)}\right)\left(1-\frac{x}{x_0}\right)+\frac{ac}{k}\left(R_0-1\right)v-\frac{ab\mu}{rk}z. \tag{17}$$

Therefore, if $R_0 \leq 1$, then $\frac{dU_0}{dt} \leq 0$ for all $x, y, v, z > 0$. We note that the solutions of system (5)-(8) converge to Ω, the largest invariant subset of $\left\{\frac{dU_0}{dt}=0\right\}$ [44]. From (17), we have $\frac{dU_0}{dt}=0$ iff $x = x_0$, $v = 0$ and $z = 0$. The set Ω is invariant and for any element belong to Ω satisfies $v = 0$ and $z = 0$. We can see from Eq. (7) that

$$\dot{v} = 0 = ky.$$

It follows that, $y = 0$. Hence $\frac{dU_0}{dt}=0$ iff $x = x_0$ and $y = v = z = 0$. Using LaSalle's invariance principle, we derive that E_0 is GAS.

Assumption A5

$$\left(1-\frac{\psi(x,y,v)}{\psi(x,y_i,v_i)}\right)\left(\frac{\psi(x,y_i,v_i)}{\psi(x,y,v)}-\frac{v}{v_i}\right) \leq 0,\ i=1,2 \text{ for all } x,y,v>0.$$

Theorem 2. Assume that Assumptions A1-A5 are satisfied and $R_1 \leq 1 < R_0$, then the chronic-infection equilibrium without antibody immune response E_1 is GAS.

Proof. Define a Lyapunov functional as:

$$U_1 = x - x_1 - \int_{x_1}^{x}\frac{\psi(x_1,y_1,v_1)}{\psi(\eta,y_1,v_1)}d\eta + y_1 H\left(\frac{y}{y_1}\right)+\frac{a}{k}v_1 H\left(\frac{v}{v_1}\right)+\frac{ab}{rk}z.$$

Calculating the time derivative of U_1 along the trajectories of system (5)-(8), we obtain

$$\begin{aligned}\frac{dU_1}{dt} &= \left(1-\frac{\psi(x_1,y_1,v_1)}{\psi(x,y_1,v_1)}\right)(s-dx-\psi(x,y,v)\,v)+\left(1-\frac{y_1}{y}\right)(\psi(x,y,v)v-ay)\\ &+\frac{a}{k}\left(1-\frac{v_1}{v}\right)(ky-bzv-cv)+\frac{ab}{rk}(rzv-\mu z)\\ &= \left(1-\frac{\psi(x_1,y_1,v_1)}{\psi(x,y_1,v_1)}\right)(s-dx)+\psi(x_1,y_1,v_1)\frac{\psi(x,y,v)v}{\psi(x,y_1,v_1)}\\ &-\frac{y_1}{y}\psi(x,y,v)v+ay_1-\frac{ac}{k}v-ay\frac{v_1}{v}+\frac{ac}{k}v_1+\frac{ab}{k}v_1 z-\frac{ab\mu}{rk}z.\end{aligned} \tag{18}$$

Using the equilibrium conditions for E_1:

$$s = dx_1 + ay_1, \quad \psi(x_1, y_1, v_1)v_1 = ay_1 = \frac{ac}{k}v_1,$$

we obtain

$$\frac{dU_1}{dt} = d\left(1 - \frac{\psi(x_1,y_1,v_1)}{\psi(x,y_1,v_1)}\right)(x_1 - x) + 3ay_1 - ay_1\frac{\psi(x_1,y_1,v_1)}{\psi(x,y_1,v_1)} + ay_1\frac{\psi(x,y,v)v}{\psi(x,y_1,v_1)v_1}$$
$$- ay_1\frac{y_1\psi(x,y,v)v}{y\psi(x_1,y_1,v_1)v_1} - ay_1\frac{v}{v_1} - ay_1\frac{v_1 y}{v y_1} + \frac{ab}{k}\left(v_1 - \frac{\mu}{r}\right)z. \tag{19}$$

Collecting terms of Eq. (19) we get

$$\begin{aligned}\frac{dU_1}{dt} &= dx_1\left(1 - \frac{\psi(x_1,y_1,v_1)}{\psi(x,y_1,v_1)}\right)\left(1 - \frac{x}{x_1}\right)\\ &+ ay_1\left(\frac{\psi(x,y,v)v}{\psi(x,y_1,v_1)v_1} - \frac{v}{v_1} - 1 + \frac{\psi(x,y_1,v_1)}{\psi(x,y,v)}\right)\\ &+ ay_1\left[4 - \frac{\psi(x_1,y_1,v_1)}{\psi(x,y_1,v_1)} - \frac{y_1\psi(x,y,v)v}{y\psi(x_1,y_1,v_1)v_1} - \frac{v_1 y}{v y_1} - \frac{\psi(x,y_1,v_1)}{\psi(x,y,v)}\right]\\ &+ \frac{ab}{k}\left(v_1 - \frac{\mu}{r}\right)z. \end{aligned} \tag{20}$$

Eq. (20) can be simplified as:

$$\begin{aligned}\frac{dU_1}{dt} &= dx_1\left(1 - \frac{\psi(x_1,y_1,v_1)}{\psi(x,y_1,v_1)}\right)\left(1 - \frac{x}{x_1}\right)\\ &+ ay_1\left(1 - \frac{\psi(x,y,v)}{\psi(x,y_1,v_1)}\right)\left(\frac{\psi(x,y_1,v_1)}{\psi(x,y,v)} - \frac{v}{v_1}\right)\\ &+ ay_1\left[4 - \frac{\psi(x_1,y_1,v_1)}{\psi(x,y_1,v_1)} - \frac{y_1\psi(x,y,v)v}{y\psi(x_1,y_1,v_1)v_1} - \frac{v_1 y}{v y_1} - \frac{\psi(x,y_1,v_1)}{\psi(x,y,v)}\right]\\ &+ \frac{ab}{k}\left(v_1 - \frac{\mu}{r}\right)z. \end{aligned} \tag{21}$$

From Assumptions A1 and A5, we get that the first and second terms of Eq. (21) is less than or equal zero. Since the geometrical mean is less than or equal to the arithmetical mean, then the third term of Eq. (21) is also less than or equal zero.

Now we show that if $R_1 \leq 1$ then $v_1 \leq \frac{\mu}{r} = v_2$. Let $R_0 > 1$, then we want to show that

$$sgn(x_2 - x_1) = sgn(v_1 - v_2) = sgn(y_1 - y_2) = sgn(R_1 - 1).$$

From Assumptions A2-A4, for $x_1, x_2, y_1, y_2, v_1, v_2 > 0$, we have

$$(\psi(x_2, y_2, v_2) - \psi(x_1, y_2, v_2))(x_2 - x_1) > 0, \tag{22}$$

$$(\psi(x_1, y_1, v_1) - \psi(x_1, y_2, v_1))(y_2 - y_1) > 0 \tag{23}$$

$$(\psi(x_1, y_1, v_1) - \psi(x_1, y_1, v_2))(v_2 - v_1) > 0, \tag{24}$$

$$(\psi(x_2, y_2, v_2)v_2 - \psi(x_2, y_2, v_1)v_1)(v_2 - v_1) > 0. \tag{25}$$

First, we claim $sgn(x_2 - x_1) = sgn(v_1 - v_2)$. Suppose this is not true, i.e., $sgn(x_2 - x_1) = sgn(v_2 - v_1)$. Using the conditions of the equilibria E_1 and E_2 we have

$$\begin{aligned}(s - dx_2) - (s - dx_1) &= \psi(x_2, y_2, v_2)v_2 - \psi(x_1, y_1, v_1)v_1 \\ &= a(y_2 - y_1),\end{aligned} \tag{26}$$

then $sgn(x_1 - x_2) = sgn(y_2 - y_1)$. Moreover

$$\begin{aligned}(s - dx_2) - (s - dx_1) &= \psi(x_2, y_2, v_2)v_2 - \psi(x_1, y_1, v_1)v_1 \\ &= (\psi(x_2, y_2, v_2)v_2 - \psi(x_2, y_2, v_1)v_1) + (\psi(x_2, y_2, v_1)v_1 - \psi(x_1, y_2, v_1)v_1) \\ &+ (\psi(x_1, y_2, v_1)v_1 - \psi(x_1, y_1, v_1)v_1).\end{aligned}$$

Therefore, from inequalities (22)-(26) we get:

$$sgn\,(x_1 - x_2) = sgn\,(x_2 - x_1)\,,$$

which leads to contradiction. Thus, $sgn\,(x_2 - x_1) = sgn\,(v_1 - v_2)\,$. Using the equilibrium conditions for E_1 we have $\frac{k\psi(x_1, y_1, v_1)}{ac} = 1$, then

$$\begin{aligned}R_1 - 1 &= \frac{k\psi(x_2, y_2, v_2)}{ac} - \frac{k\psi(x_1, y_1, v_1)}{ac} \\ &= \frac{k}{ac}(\psi(x_2, y_2, v_2) - \psi(x_2, y_2, v_1) + \psi(x_2, y_2, v_1) \\ &- \psi(x_1, y_2, v_1) + \psi(x_1, y_2, v_1) - \psi(x_1, y_1, v_1)).\end{aligned}$$

We get $sgn(R_1 - 1) = sgn(v_1 - v_2)$. Hence, if $R_0 > 1$, then $x_1, y_1, v_1 > 0$, and if $R_1 \leq 1$, then $v_1 \leq v_2 = \frac{\mu}{r}$. It follows from the above discussion that $\frac{dU_1}{dt} \leq 0$ for all $x, y, v, z > 0$ and $\frac{dU_1}{dt} = 0$ iff $x = x_1, y = y_1, v = v_1$ and $z = 0$. So Ω contains a unique point, the equilibrium E_1. Thus, we prove the global asymptotic stability of the chronic-infection equilibrium without antibody immune response E_1 by using LaSalle's invariance principle.

Theorem 3. Let Assumptions A1-A5 be hold true and $R_1 > 1$, then the chronic-infection equilibrium with antibody immune response E_2 is GAS.

Proof. We construct a Lyapunov functional as follows:

$$U_2 = x - x_2 - \int_{x_2}^{x} \frac{\psi(x_2, y_2, v_2)}{\psi(\eta, y_2, v_2)} d\eta + y_2 H\left(\frac{y}{y_2}\right) + \frac{a}{k} v_2 H\left(\frac{v}{v_2}\right) + \frac{ab}{rk} z_2 H\left(\frac{z}{z_2}\right). \tag{27}$$

Function U_2 satisfies:

$$\begin{aligned}\frac{dU_2}{dt} &= \left(1 - \frac{\psi(x_2, y_2, v_2)}{\psi(x, y_2, v_2)}\right)(s - dx - \psi(x, y, v)v) + \left(1 - \frac{y_2}{y}\right)(\psi(x, y, v)v - ay) \\ &\quad + \frac{a}{k}\left(1 - \frac{v_2}{v}\right)(ky - bzv - cv) + \frac{ab}{rk}\left(1 - \frac{z_2}{z}\right)(rzv - \mu z). \end{aligned} \tag{28}$$

Applying $s = dx_2 + ay_2$, we get

$$\begin{aligned}\frac{dU_2}{dt} &= d\left(1 - \frac{\psi(x_2, y_2, v_2)}{\psi(x, y_2, v_2)}\right)(x_2 - x) + ay_2 \\ &\quad - ay_2 \frac{\psi(x_2, y_2, v_2)}{\psi(x, y_2, v_2)} + \psi(x, y, v)v \frac{\psi(x_2, y_2, v_2)}{\psi(x, y_2, v_2)} \\ &\quad - \psi(x_2, y_2, v_2)v_2 \frac{y_2 \psi(x, y, v)v}{y \psi(x_2, y_2, v_2)v_2} + ay_2 - \frac{ac}{k} v - ay \frac{v_2}{v} \\ &\quad + \frac{ac}{k} v_2 + \frac{ab}{k} v_2 z - \frac{ab\mu}{rk} z - \frac{ab}{k} z_2 v + \frac{ab\mu}{rk} z_2. \end{aligned} \tag{29}$$

By using the equilibrium conditions of E_2

$$\psi(x_2, y_2, v_2)v_2 = ay_2, \quad cv_2 = ky_2 - bv_2 z_2, \quad \mu = rv_2,$$

and the following equality

$$cv = cv_2 \frac{v}{v_2} = \frac{v}{v_2}(ky_2 - bv_2 z_2),$$

we obtain

$$\frac{dU_2}{dt} = d\left(1 - \frac{\psi(x_2, y_2, v_2)}{\psi(x, y_2, v_2)}\right)(x_2 - x) + ay_2\left(\frac{\psi(x, y, v)v}{\psi(x, y_2, v_2)v_2} - \frac{v}{v_2} - 1 + \frac{\psi(x, y_2, v_2)}{\psi(x, y, v)}\right)$$
$$+ ay_2\left[4 - \frac{\psi(x_2, y_2, v_2)}{\psi(x, y_2, v_2)} - \frac{y_2\psi(x, y, v)v}{y\psi(x_2, y_2, v_2)v_2} - \frac{v_2 y}{v y_2} - \frac{\psi(x, y_2, v_2)}{\psi(x, y, v)}\right]. \tag{30}$$

We can simplify (30) as:

$$\frac{dU_2}{dt} = dx_2\left(1 - \frac{\psi(x_2, y_2, v_2)}{\psi(x, y_2, v_2)}\right)\left(1 - \frac{x}{x_2}\right) + ay_2\left(1 - \frac{\psi(x, y, v)}{\psi(x, y_2, v_2)}\right)\left(\frac{\psi(x, y_2, v_2)}{\psi(x, y, v)} - \frac{v}{v_2}\right)$$
$$+ ay_2\left[4 - \frac{\psi(x_2, y_2, v_2)}{\psi(x, y_2, v_2)} - \frac{y_2\psi(x, y, v)v}{y\psi(x_2, y_2, v_2)v_2} - \frac{v_2 y}{v y_2} - \frac{\psi(x, y_2, v_2)}{\psi(x, y, v)}\right]. \tag{31}$$

We note that from assumptions A2, A5 and the relationship between the arithmetical and geometrical means, we have $\frac{dU_2}{dt} \leq 0$. One can easily see that $\frac{dU_2}{dt} = 0$ at E_2. The global asymptotic stability of the chronic-infection equilibrium with antibody immune response E_2 follows from LaSalle's invariance principle.

3 Conclusion

In this paper, we have proposed a viral infection model with general incidence rate function and antibody immune response. We have derived a set of conditions on the general functional response and have determined two thresholds parameters R_0 and R_1 to prove the existence and global stability of the model's equilibria. The global asymptotic stability of the three equilibria, infection-free, chronic-infection without antibody immune response and chronic-infection with antibody immune response has been proven by using direct Lyapunov method and LaSalle's invariance principle.

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NEWTON'S METHOD FOR COMPUTING THE FIFTH ROOTS OF p-ADIC NUMBERS

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Abstract We consider Newton's method to compute the fifth root of a p-adic number in $\mathbb{Q}_p$. We have the sufficient conditions for the convergence of Newton's method and the speed of its convergence. We also calculate the number of iterations to obtain a number of corrected digits in the approximation.

1. Introduction

Let p be a prime and $\mathbb{Q}_p$ be the field of p-adic numbers. The theory of the field of p-adic numbers introduced by Hensel has been related to several areas of mathematics including number theory, analysis and other modern mathematics, and recently to physics. The study of this field has been an important area of research in mathematics([9]).

The application of classical methods in numerical analysis to p-adic numbers and polynomials and the analysis of their convergence in $\mathbb{Q}_p$ have been a recent development([2-3], [5], [7], [10-11]). Newton's method is the most often used method to find zeros of polynomials. In [7], the authors applied Newton's method to compute the cubic root of a p-adic number. In [2-3], the authors also used Newton-Raphson method to compute square and cube roots of p-adic numbers in $\mathbb{Q}_p$. Computing the q-th root of a p-adic number is useful in the field of computer science and cryptography, specially when q is a prime. In [6], Kim-Choi give the conditions for the existence of the q-th roots of p-adic numbers in $\mathbb{Q}_p$ when $(p, q) = 1$, and also have the condition for the existence the fifth roots including $p = q$.

In this paper, we use Newton's method to compute the fifth root of a p-adic number in $\mathbb{Q}_p$. We have the sufficient conditions for the convergence of Newton's method and the speed of its convergence. We also calculate the number of iterations to obtain a number of corrected digits in the approximation.

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2. Preliminaries

The following definitions and results are needed for our discussion. See [4] and [8] for details.

Definition 1. *Let $p \in \mathbb{N}$ be a prime number and $x \in \mathbb{Q}$ $(x \neq 0)$. The p-adic order of x, $\mathrm{ord}_p x$, is defined by*

$$\mathrm{ord}_p x = \begin{cases} \text{the highest power of } p \text{ which divides } x, & \text{if } x \in \mathbb{Z}, \\ \mathrm{ord}_p a - \mathrm{ord}_p b, & \text{if } x = \frac{a}{b},\ a, b \in \mathbb{Z},\ b \neq 0. \end{cases}$$

Consider a map $|\cdot|_p : \mathbb{Q} \to \mathbb{R}^+$ as follows.

Definition 2. *Let $p \in \mathbb{N}$ be a prime number and $x \in \mathbb{Q}$. The p-adic norm $|\cdot|_p$ of x is defined by*

$$|x|_p = \begin{cases} p^{-\mathrm{ord}_p x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

The field of p-adic numbers $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to the p-adic norm $|\cdot|_p$ of Definition 2. The elements of $\mathbb{Q}_p$ are equivalence classes of Cauchy sequences in $\mathbb{Q}$ with respect to the extension of the p-adic norm defined by

$$|a|_p = \lim_{n \to \infty} |a_n|_p,$$

where $\{a_n\}$ is a Cauchy sequence in $\mathbb{Q}$ representing $a \in \mathbb{Q}_p$.

Theorem 1. *Every equivalence class a in $\mathbb{Q}_p$ satisfying $|a|_p \leq 1$ has exactly one representative Cauchy sequence $\{a_i\}$ such that*
(1) $a_i \in \mathbb{Z}$, $0 \leq a_i < p^i$ for $i = 1, 2, \ldots$,
(2) $a_i \equiv a_{i+1}$ (mod p^i) for $i = 1, 2, \ldots$.

From this, every p-adic number $a \in \mathbb{Q}_p$ has a unique representation

$$a = \sum_{n=-m}^{\infty} a_n p^n,$$

where $a_{-m} \neq 0$ and $a_n \in \{0, 1, 2, \ldots, p-1\}$ for $n \geq -m$. We represent the given p-adic number a as a fraction in the base p as follows:

$$a = \ldots a_n \ldots a_2 a_1 a_0 . a_{-1} \ldots a_{-m}.$$

This representation is called the canonical p-adic expansion of a.

Definition 3. *Let $\mathbb{Z}_p = \{a \in \mathbb{Q}_p |\ a = \sum_{i=0}^{\infty} a_i p^i\}$ be the set of p-adic integers and $\mathbb{Z}_p^{\times} = \{a \in \mathbb{Q}_p |\ a = \sum_{i=0}^{\infty} a_i p^i,\ a_0 \neq 0\}$ be the set of p-adic units.*

From Definition 3, it is easy to see that $\mathbb{Z}_p = \{a \in \mathbb{Q}_p | \ |a|_p \le 1\}$ and $\mathbb{Z}_p^\times = \{a \in \mathbb{Q}_p | \ |a|_p = 1\}$. Hence the following theorem follows.

Theorem 2. *Let a be a p-adic number of norm p^{-n}. Then $a = p^n u$ for some $u \in \mathbb{Z}_p^\times$.*

From now, we discuss the conditions for the existence of p-adic roots.

Definition 4. *A p-adic number $x \in \mathbb{Q}_p$ is said to be a q-th root of $a \in \mathbb{Q}_p$ of order $k \in \mathbb{N}$ if and only if $x^q \equiv a \pmod{p^k}$.*
When $q = 5$, the q-th root of $a \in \mathbb{Q}_p$ is called the fifth root of a.

The following lemmata are essential for our discussions([4]).

Lemma 3. *Let $a, b \in \mathbb{Q}_p$. Then a and b are congruent modulo p^k and write $a \equiv b \pmod{p^k}$ if and only if $|a-b|_p \le 1/p^k$.*

Lemma 4. *Let $a, b \in \mathbb{Q}_p$. If $|a-b|_p < |b|_p$, then $|a|_p = |b|_p$.*

The next theorem is the basis for the existence of p-adic roots([8]).

Theorem 5. *(Hensel's lemma) Let $F(x) = c_0 + c_1 x + \cdots + c_n x^n$ be a polynomial whose coefficients are p-adic integers. Let $F'(x) = c_1 + c_2 x + 3c_3 x^2 + \cdots + nc_n x^n$ be the derivative of $F(x)$. Let a_0 be a p-adic integer such that $F(a_0) \equiv 0 \pmod{p}$ and $F'(a_0) \not\equiv 0 \pmod{p}$. Then there exists a unique p-adic integer a such that*

$$F(a) = 0 \quad \text{and} \quad a \equiv a_0 \pmod{p}.$$

The following theorem follows from Theorem 5, and provides the condition between p-adic numbers and congruence([4]).

Theorem 6. *A polynomial with integer coefficients has a root in $\mathbb{Z}_p$ if and only if it has an integer root modulo p^k for any $k \ge 1$.*

Some results of the existence of square roots of p-adic numbers are obtained from Theorem 6([4]). In [6], we have the conditions for the existence of the fifth roots of p-adic numbers in $\mathbb{Q}_p$ as followings.

Theorem 7. *A rational integer a not divisible by p has a fifth root in $\mathbb{Z}_p$ $(p \ne 5)$ if and only if a is a fifth residue modulo p.*

From Theorem 7, we have the following theorem([6]).

Theorem 8. *Let p be a prime number. Then we have:*
(1) If $p \ne 5$, then $a = p^{\mathrm{ord}_p a} u \in \mathbb{Q}_p$ for some $u \in \mathbb{Z}_p^\times$ has a fifth root in $\mathbb{Q}_p$ if and only if $\mathrm{ord}_p a = 5m$ for $m \in \mathbb{Z}$ and $u = v^5$ for some unit $v \in \mathbb{Z}_p^\times$.
(2) If $p = 5$, then $a = 5^{\mathrm{ord}_5 a} u \in \mathbb{Q}_5$ for some $u \in \mathbb{Z}_5^\times$ has a fifth root in $\mathbb{Q}_5$ if and only if $\mathrm{ord}_5 a = 5m$ for $m \in \mathbb{Z}$ and $u \equiv 1 \pmod{25}$ or $u \equiv k \pmod{5}$ for some k $(2 \le k \le 4)$.

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3. Newton's Method

Newton's method is a well known numerical method to find zeros of a polynomial $f(x)$ in $\mathbb{R}$([1]). The iterative formula for this method is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots. \tag{3.1}$$

To seek the fifth root of a is to find the zero of $f(x) = x^5 - a$. The iteration (3.1) for Newton's method becomes the recurrence relation

$$x_{n+1} = \frac{4x_n^5 + a}{5x_n^4}, \quad n = 0, 1, 2, \ldots. \tag{3.2}$$

Like for real numbers, we can show that Newton's method also converges quadratically for convergence.

Let $a(\neq 0) \in \mathbb{Q}_p$ be a p-adic number such that

$$|a|_p = p^{-\mathrm{ord}_p a} = p^{-5m}, \quad m \in \mathbb{Z}.$$

The following theorem is the result when $p \neq 5$.

Theorem 9. *Let $p \neq 5$ and $\{x_n\}$ be the sequence of p-adic numbers obtained from the Newton's iteration (3.2). If x_0 is a fifth root of a of order r with $|x_0|_p = p^{-m}$ and $r > 5m$, then*

(1) $|x_n|_p = p^{-m}$, $n = 1, 2, \ldots,$
(2) $x_n^5 \equiv a \ (mod\ p^{2^n r - 5m(2^n - 1)})$,
(3) $\{x_n\}$ converges to the fifth root of a.

Proof. We will prove (1) and (2) by induction.
(i) First, we prove it when $p > 5$. Let $n = 1$. By assumption, we have

$$x_0^5 = a + bp^r \quad (0 < b < p). \tag{3.3}$$

From (3.2), (3.3) and Lemma 4, we have

$$|x_1|_p = \frac{|4x_0^5 + a|_p}{|5x_0^4|_p} = \frac{|5a + 4bp^r|_p}{|5x_0^4|_p} = \frac{\max\{|5a|_p, |4bp^r|_p\}}{|5x_0^4|_p} = p^{-m}. \tag{3.4}$$

Also by (3.2), we have

$$x_1^5 - a = \frac{(x_0^5 - a)^2}{3125x_0^{20}}(1024x_0^{15} + 203ax_0^{10} + 22a^2x_0^5 + a^3). \tag{3.5}$$

To calculate the p-adic norm of $x_1^5 - a$, we let

$$h(x) = 1024x^{15} + 203ax^{10} + 22a^2x^5 + a^3. \tag{3.6}$$

From (3.3), we have

$$h(x_0) = 1250a^3 + 3500a^2bp^r + 3275ab^2p^{2r} + 1024b^3p^{3r}. \tag{3.7}$$

Using the strong triangle inequality, we have from (3.7) that

$$\begin{aligned}&|h(x_0)|_p\\&\leq \max\left\{|2\cdot 5^4a^3|_p, |2^25^37a^2bp^r|_p, |5^2131ab^2p^{2r}|_p, |2^{10}b^3p^{3r}|_p\right\}\\&= \max\left\{p^{-15m}, p^{-10m-r}, p^{-5m-2r}, p^{-3r}\right\}\\&= p^{-15m}.\end{aligned} \tag{3.8}$$

Also the p-adic norm of the denominator of the right hand of (3.5) is

$$|3125x_0^{20}|_p = |5^5x_0^{20}|_p = p^{-20m}. \tag{3.9}$$

Since x_0 is a fifth root of a of order r, we have

$$|(x_0^5-a)^2|_p = p^{-2r}. \tag{3.10}$$

By (3.5), (3.8), (3.9) and (3.10), we have

$$|x_1^5-a|_p \leq p^{5m-2r}.$$

By Lemma 3, $x_1^5 - a \equiv 0 \pmod{p^{2r-5m}}$. Hence (1) and (2) is true when $n = 1$.

Now assume that

$$|x_{n-1}|_p = p^{-m}, \tag{3.11}$$

$$x_{n-1}^5 = a \pmod{p^{2^{n-1}r-5m(2^{n-1}-1)}}, \tag{3.12}$$

and so

$$x_{n-1}^5 = a + bp^{2^{n-1}r-5m(2^{n-1}-1)} \quad (0 < b < p). \tag{3.13}$$

From (3.2), (3.11) and (3.13), we have

$$\begin{aligned}|x_n|_p &= \frac{|4x_{n-1}^5+a|_p}{|5x_{n-1}^4|_p} = \frac{|5a+4bp^{2^{n-1}r-5m(2^{n-1}-1)}|_p}{|5x_{n-1}^4|_p}\\&= \frac{\max\{|5a|_p, |4bp^{2^{n-1}r-5m(2^{n-1}-1)}|_p\}}{|5x_{n-1}^4|_p} = p^{-m}.\end{aligned} \tag{3.14}$$

Thus (1) is proved by (3.4), (3.11) and (3.14). Also from (3.2), it follows that

$$x_n^5 - a = \frac{(x_{n-1}^5-a)^2}{5^5x_{n-1}^{20}}h(x_{n-1}). \tag{3.15}$$

Let $Q = p^{2^{n-1}r-5m(2^{n-1}-1)}$ for simplicity. From (3.13),

$$h(x_{n-1}) = 2\cdot 5^4a^3 + 2^2\cdot 5^3\cdot 7a^2bQ + 5^2\cdot 131ab^2Q^2 + 2^{10}b^3Q^3. \tag{3.16}$$

Since $r > 5m$, the p-adic norm of $h(x_{n-1})$ in (3.16) is

$$\begin{aligned}|h(x_{n-1})|_p &\leq \max\{p^{-15m}, p^{-15m-2^{n-1}(r-5m)},\\&\qquad p^{-15m-2^n(r-5m)}, p^{-15m-3\cdot 2^{n-1}(r-5m)}\}\\&= p^{-15m}.\end{aligned} \tag{3.17}$$

Since x_{n-1} is a fifth root of a of order $2^{n-1}r - 5m(2^{n-1}-1)$, we have from (3.12), (3.15) and (3.17) that

$$|x_n^5 - a|_p \le p^{-2^n r + 5m(2^n - 1)}.$$

By Lemma 3, we have $x_n^5 - a \equiv 0 \pmod{p^{2^n r - 5m(2^n-1)}}$. Thus (2) is true for all $n \in \mathbb{N}$.

(ii) When $p < 5$, there are two cases, $p = 3$ and $p = 2$.

The proof is the same with (i) when the first case $p = 3$, because 3 is no factor of any coefficients of terms of $h(x_0)$ in (3.7). It means that $|h(x_0)|_p \le p^{-15m}$, and so $x_1^5 \equiv a \pmod{p^{2r-5m}}$. By assuming $x_{n-1}^5 \equiv a$ $\pmod{p^{2^{n-1}r-5m(2^{n-1}-1)}}$, we have $x_n^5 \equiv a \pmod{p^{2^n r-5m(2^n-1)}}$ using the same process of (i). Moreover we can check easily $|x_n|_3 = 3^{-m}$ by induction.

The other case is $p = 2$. Let $n = 1$, $|x_1|_p = p^{-m}$ is obtained easily from (3.4). And we have

$$x_1^5 - a = \frac{(x_0^5 - a)^2}{3125 x_0^{20}} h(x_0), \tag{3.18}$$

where $h(x)$ is the polynomial in (3.6). Since $r > 5m$, we have

$$|h(x_0)|_p \le \max\{p^{-15m-1},\ p^{-10m-r-2},\ p^{-5m-2r},\ p^{-3r-10}\} \le p^{-15m}. \tag{3.19}$$

In (3.18), we have

$$|3125 x_0^{20}|_p = p^{-20m}, \tag{3.20}$$

and, by assumption,

$$|(x_0^5 - a)^2|_p = p^{-2r}. \tag{3.21}$$

Also (3.19), (3.20) and (3.21) imply $|x_1^5 - a|_p \le p^{-2r+5m}$, and so $x_1^5 \equiv a \pmod{p^{2r-5m}}$. Thus (1) and (2) are true when $n = 1$ if $p = 2$.

Assume that $|x_{n-1}|_p = p^{-m}$ and $x_{n-1}^5 \equiv a \pmod{p^{2^{n-1}r-5m(2^{n-1}-1)}}$. That is,

$$x_{n-1}^5 = a + bp^{2^{n-1}r - 5m(2^{n-1}-1)} \ (0 < b < p). \tag{3.22}$$

It follows (3.15) and (3.16), and so we have

$$\begin{aligned} |h(x_{n-1})|_p \le \max\{&p^{-15m-1},\ p^{-15m-2-2^{n-1}(r-5m)}, \\ &p^{-15m-2^n(r-5m)},\ p^{-15m-10-3\cdot 2^{n-1}(r-5m)}\} \\ \le p^{-15m}&. \end{aligned} \tag{3.23}$$

By (3.15), (3.17), (3.20) and (3.23), we have

$$|x_n^5 - a|_p \le p^{-2^n r + 5m(2^n - 1)}.$$

Hence we have that for all $n \in \mathbb{N}$, $x_n^5 \equiv a \pmod{p^{2^n r - 5m(2^n-1)}}$. We note that $|x_n|_2 = 2^{-m}$ is obtained easily from (3.14). So we complete the proof of (1) and (2).

From (2), we have

$$|x_n^5 - a|_p \le p^{-2^n r + 5m(2^n-1)} \tag{3.24}$$

for each prime $p(\neq 5)$. (3) follows immediately from the inequality (3.24) as $n \to \infty$. □

When $p = 5$, we have the following theorem.

Theorem 10. *Let $p = 5$ and $\{x_n\}$ be the sequence of p-adic numbers obtained from the Newton's iteration (3.2). If x_0 is a fifth root of a of order r with $|x_0|_p = p^{-m}$ and $r > 5m+1$, then*

(1) $|x_n|_p = p^{-m}$, $n = 1, 2, \ldots,$
(2) $x_n^5 \equiv a \pmod{p^{2^n r - (5m+1)(2^n-1)}}$,
(3) $\{x_n\}$ converges to the fifth root of a.

Proof. (1) and (2) will be proved by induction. Let $n = 1$. By assumption $x_0^5 \equiv a \pmod{p^r}$, and from (3.2) and Lemma 4, we have

$$|x_1|_p = \frac{|5a + 4bp^r|_p}{|5x_0^4|_p} = \frac{\max\{|5a|_p, |4bp^r|_p\}}{|5x_0^4|_p} = \frac{p^{-5m-1}}{p^{-4m-1}} = p^{-m}.$$

By calculating the p-adic norms of $h(x_0)$ in (3.7), we have

$$|h(x_0)|_p \le \max\{p^{-15m-4}, p^{-10m-r-3}, p^{-5m-2r-2}, p^{-3r}\} = p^{-15m-4},$$

since $r > 5m+1$. Also we have $|3125x_0^{20}|_p = p^{-20m-5}$. Thus

$$|x_1^5 - a|_p \le p^{-2r+5m+1},$$

and so $x_1^5 \equiv a \pmod{p^{2r-(5m+1)}}$ by Lemma 3. Hence it is true when $n = 1$. Now we assume that

$$|x_{n-1}|_p = p^{-m}$$

and

$$x_{n-1}^5 \equiv a \pmod{p^{2^{n-1} r - (5m+1)(2^{n-1}-1)}}.$$

In the similar manner as (3.14), (3.16) and (3.17), we have

$$\begin{aligned} |x_n|_p &= \frac{|4x_{n-1}^5 + a|_p}{|5x_{n-1}^4|_p} = \frac{|5a + 4bp^{2^{n-1}r-(5m+1)(2^{n-1}-1)}|_p}{|5x_{n-1}^4|_p} \\ &= \frac{\max\{|5a|_p, |4bp^{2^{n-1}r-(5m+1)(2^{n-1}-1)}|_p\}}{|5x_{n-1}^4|_p} = \frac{p^{-5m-1}}{p^{-4m-1}} = p^{-m} \end{aligned}$$

and

$$\begin{aligned}|h(x_{n-1})|_p &\le \max\{p^{-15m-4},\ p^{-15m-4-2^{n-1}[r-(5m+1)]}\\ &\qquad p^{-15m-4-2^n[r-(5m+1)]},\ p^{-15m-3-3\cdot 2^{n-1}[r-(5m+1)]}\}\\ &= p^{-15m-4}.\end{aligned}$$

And so we have

$$|x_n^5 - a|_p \le p^{-2^n r + (5m+1)(2^n-1)}. \tag{3.25}$$

It follows that (1) and (2) are true for all $n \in \mathbb{N}$.
(3) follows from the inequality (3.25) as $n \to \infty$. □

To determine the rate of convergence of the sequence $\{x_n\}$ given by (3.2), we consider the sequence $\{e_n\}$ defined by

$$e_n = x_{n+1} - x_n, \quad \forall n \in \mathbb{N}. \tag{3.26}$$

From Theorem 9 and Theorem 10, we obtain the following theorem.

Theorem 11. *If x_0 is the fifth root of a of order r, then the sequence $\{e_n\}$ in (3.26) is $e_n \equiv 0$ (mod p^{α_n}), where*

$$\alpha_n = \begin{cases} 2^n r - 5m\cdot 2^n + m, & \text{if } p \ne 5,\\ 2^n r - (5m+1)\cdot 2^n + m, & \text{if } p = 5.\end{cases}$$

Proof. (i) First, let $p \ne 5$. Then, from the Newton's iteration formula (3.2), we have

$$e_n = x_{n+1} - x_n = \frac{1}{5x_n^4}(a - x_n^5), \quad \forall n \in \mathbb{N}. \tag{3.27}$$

By computing the p-adic norms of each side of the equation (3.27), we have from Theorem 8 that

$$|e_n|_p = |x_{n+1} - x_n|_p = \left|\frac{1}{5x_n^4}\right|_p \cdot |a - x_n|_p \le p^{-2^n r + 5m\cdot 2^n - m}.$$

Hence $e_n \equiv 0$ (mod p^{α_n}) by Lemma 3.
(ii) Let $p = 5$. By a similar way as (i), we have from Theorem 9 that

$$|e_n|_p = \left|\frac{1}{5x_n^4}\right|_p \cdot |a - x_n|_p \le p^{-2^n r + (5m+1)\cdot 2^n - m}.$$

Hence $e_n \equiv 0$ (mod p^{α_n}) by Lemma 3. This completes the proof. □

From Theorem 11, we have that the rate of convergence of the sequence $\{x_n\}$ is of order α_n. Thus the number of correct digits in the approximation increases by α_n for every iteration.

We can compute the number of iterations to obtain certain finite digits. From Theorem 9 and Theorem 10, we have the following corollary.

Corollary 12. *(1) For $p \neq 5$, let $\{x_n\}$ be the sequence of approximation in Theorem 9. Then the number of iterations to obtain at least M correct digits is*

$$n = \left\lceil \frac{\ln\left(\frac{M-4m}{r-5m}\right)}{\ln 2} \right\rceil. \tag{3.28}$$

(2) Let $p = 5$ and $\{x_n\}$ be the sequence of approximation in Theorem 10. Then the number of iterations to obtain at least M correct digits is

$$n = \left\lceil \frac{\ln\left(\frac{M-(4m+1)}{r-(5m+1)}\right)}{\ln 2} \right\rceil. \tag{3.29}$$

Proof. (1) Since we need M correct digits in the approximation, we must set the order to $M + m$ to find the number of iterations with M correct digits. That is,

$$2^n r - 5m(2^n - 1) = M + m. \tag{3.30}$$

From (3.30), we have

$$2^n = \frac{M - 4m}{r - 5m}.$$

Since $\{x_n\}$ converges to the fifth root of a by Theorem 8 (3) and $r > 5m$, we have the equation (3.28).

(2) As in the proof of (1), we set

$$2^n r - (5m + 1)(2^n - 1) = M + m. \tag{3.31}$$

From (3.31), we have

$$2^n = \frac{M - 4m - 1}{r - (5m + 1)}. \tag{3.32}$$

Since $r > 5m + 1$, the result follows from (3.32). □

The numbers in (3.28) and (3.29) are sufficient numbers of iterations to provide at least M correct digits in the approximation.

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Solution of the Ulam stability problem for Euler-Lagrange $(\alpha, \beta; k)$-quadratic mappings

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Abstract. In 1940 S. M. Ulam proposed at the University of Wisconsin the problem: "Give conditions in order for a linear mapping near an approximately linear mapping to exist". In 1982-2013, the second author solved the above Ulam problem for a variety of quadratic mappings. Interesting stability results have been achieved by S. A. Mohiuddine et al., since 2009. In this paper, we solve the Ulam stability problem for Euler-Lagrange $(\alpha, \beta; k)$ quadratic mapping. The other authors of this research area have established important results also on functional inequalities.

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1. Introduction

In 1940 S. M. Ulam [36] proposed the famous "Ulam stability problem", which was solved by D. H. Hyers [4], in 1941, for additive mappings. In 1950 T. Aoki [3] solved this Ulam problem for weaker additive mappings. In 1978 Th. M. Rassias [33] generalized the theorem of Hyers for linear mappings. In 1982-1999, J. M. Rassias ([23–30]) generalized this problem. For more detail of Ulam stability problem, we refer to [5, 6, 8–11, 19, 20, 32, 34] and references therein.

In 1992, the second author [23, 24] introduced the term "Euler-Lagrange functional equation" and "Euler-Lagrange quadratic mappings", of satisfying

$$Q(x+y) + Q(x-y) = 2\big[Q(x) + Q(y)\big] \tag{1.1}$$

and then solved the Ulam stability problem of the Euler-Lagrange quadratic functional equation (1.1). In 1996, J. M. Rassias [30] established the Ulam stablity of the general Euler-Lagrange quadratic functional equation

$$Q(\alpha x + \beta y) + Q(\beta x - \alpha y) = (\alpha^2 + \beta^2)\big[Q(x) + Q(y)\big]. \tag{1.2}$$

In 2009-2014, S. A. Mohiuddine et al. ([1,2,12–18]) solved this problem in several normed spaces. In 2008-2012 J. M. Rassias et al. ([21, 22, 31, 37]) solved the generalized Ulam problem via various methods. In 2010, M. E. Gordji et al [7] established Ulam stabilities on Banach algebras. Also J. Rätz [35] results are interesting on orthogonal mappings.

In this paper, we solve the Ulam stability problem for the Euler-Lagrange $(\alpha, \beta; k)$ quadratic mapping satisfying

$$kQ(\alpha x + \beta y) + Q(k\beta x - \alpha y) = (\alpha^2 + k\beta^2)\big[kQ(x) + Q(y)\big]. \tag{1.3}$$

Let us note that $Q(x) = |x|^2$ satisfies equation (1.3) because the following Euler-Lagrange quadratic identity

$$k|\alpha x+\beta y|^2+|k\beta x-\alpha y|^2=(\alpha^2+k\beta^2)\big[k|x|^2+|y|^2\big] \tag{1.4}$$

holds with any fixed reals α, β and k.

Definition 1.1. Let X be a normed linear space and let Y be a real complete normed linear space. Then a non-linear mapping $Q : X \to Y$ is called Euler-Lagrange quadratic if equation (1.3) holds for all 2-dimensional vectors $(x, y) \in X^2$, and any fixed reals α, β and k. We note that Q may be called quadratic because the above Euler-Lagrange identity (1.4) holds and because the functional equation

$$Q(m^n x)=(m^n)^2Q(x) \tag{1.5}$$

holds for all $x \in X$, all $n \in \mathbb{N}$:

$$m=\alpha^2+k\beta^2. \tag{1.6}$$

Assume $m \in \mathbb{R}-\{0,1\}$ and $k \in \mathbb{R}-\{-1,0\}$.

In fact, substitution of $x = y = 0$ in equation (1.3) yields

$$(k+1)(1-m)Q(0)=0,$$

or

$$Q(0)=0, \quad m\neq 1 \quad (\text{and } \ k\neq -1). \tag{1.7}$$

Substituting $x = x, y = 0$ in (1.3), one gets that

$$kQ(\alpha x)+Q(k\beta x)=kmQ(x)+mQ(0), \tag{1.8}$$

or

$$Q(\alpha x)+\frac{1}{k}Q(k\beta x)=mQ(x)+\frac{m}{k}Q(0), \tag{1.9}$$

holds for all $x \in X$, and any fixed real $k \neq 0$. Employing (1.7), we obtain from (1.8) that

$$Q(\alpha x)+Q(k\beta x)=kmQ(x). \tag{1.10}$$

Moreover, substitution $x \to \alpha x$, $y = k\beta x$ in (1.3), we find that

$$kQ(mx)+Q(0)=m\big[kQ(\alpha x)+Q(k\beta x)\big],$$

or

$$kQ(\alpha x)+Q(k\beta x)=km^{-1}Q(mx)+\frac{1}{m}Q(0), \tag{1.11}$$

or

$$Q(\alpha x)+\frac{1}{k}Q(k\beta x)=m^{-1}Q(mx)+\frac{1}{km}Q(0) \tag{1.12}$$

holds for all $x \in X$, and any fixed reals $k \neq 0$, $m \neq 0$. Functional Equations (1.8) and (1.11), or (1.9) and (1.12) yield

$$km^{-1}Q(mx)+\frac{1}{m}Q(0)=kmQ(x)+mQ(0),$$

or

$$km\big[Q(x)-m^{-2}Q(mx)\big]=\left(\frac{1}{m}-m\right)Q(0),$$

or

$$km\big[Q(x) - m^2 Q(mx)\big] = \left(\frac{1-m^2}{m}\right) Q(0),$$

or

$$Q(x) - m^{-2} Q(mx) = \left(\frac{1}{k}\ \frac{1-m^2}{m^2}\right) Q(0). \tag{1.13}$$

Employing (1.7), one gets

$$Q(x) = m^{-2} Q(mx), \tag{1.14}$$

or

$$Q(mx) = m^2 Q(x) \tag{1.15}$$

Replaying $x \to mx$ in (1.15), we find

$$Q(m^2 x) = m^2 Q(mx),$$

or

$$Q(m^2 x) = m^4 Q(x) \tag{1.16}$$

Then by induction on $n \in \mathbb{N}$ with $x \to m^{n-1} x$ yields equation (1.5).

Definition 1.2. Let X be a normed linear space and let Y be a real complete normed linear space. Then we call the non-linear mapping $\bar{Q} : X \to Y$, a 2-dimensional quadratic weighted mean if

$$\bar{Q}(x) = \frac{kQ(\alpha x) + Q(k\beta x)}{km} \tag{1.17}$$

holds for all $x \in X$ and any fixed reals $k, m \neq 0$.

Let us note that from (1.8) and (1.17), one get

$$\bar{Q}(x) = \frac{kmQ(x) + mQ(o)}{km},$$

or

$$\bar{Q}(x) = Q(x) + \frac{1}{k} Q(o), \tag{1.18}$$

for all $x \in X$, and any fixed real $k \neq 0$. From (1.7) and (1.18), we obtain

$$\bar{Q}(x) = Q(x), \tag{1.19}$$

for all $x \in X$.

2. Stability for Euler-Lagrange quadratic mappings

Let us introduce the Euler-Lagrange $(\alpha, \beta; k)$ quadratic functional inequality

$$\big\| kf(\alpha x + \beta y) + f(k\beta x - \alpha y) - (\alpha^2 + k\beta^2)\big[kf(x) + f(y)\big] \big\| \leq c, \tag{2.1}$$

for all 2-dimensional vectors $(x, y) \in X^2$ and any fixed reals α, β and k as well as $m = \alpha^2 + k\beta^2$, with $m \in \mathbb{R} - \{0, 1\}$ $\quad (k \in \mathbb{R} - \{-1, 0\})$, and $c(:=$ constant inde of $x, y) \geq 0$.

Then we prove the following theorem.

Theorem 2.1. Let X be a normed linear space and let Y be a real complete normed linear space. Let us denote,

$$\bar{f}(x) = \frac{kf(\alpha x) + f(k\beta x)}{km} \tag{2.2}$$

holds for all $x \in X$ and any fixed reals $k, m \neq 0$. Also let us assume $m : |m| > 1$. Then the limit

$$Q(x) = \lim_{n\to\infty} m^{-2n} f(m^n x), \tag{2.3}$$

exists for all $x \in X$, all $n \in \mathbb{N}$, and any fixed real $m : |m| > 1$ and $Q : X \to Y$ is the unique quadratic mapping satisfying functional equation (1.3) such that

$$\|f(x) - Q(x)\| \leq c_3 = \frac{c_2}{m^2 - 1}, \ |m| > 1, \tag{2.4}$$

where

$$c_2 = m^2 c_1 = \frac{|k+1|\big(1+|m|\big) + |1+m|}{|k|\ |k+1|} c,$$
$$c_3 = \frac{c_2}{m^2 - 1}.$$

Moreover , identity

$$Q(x) = m^{-2n} Q(m^n x) \tag{2.5}$$

holds for all $x \in X$ all $n \in \mathbb{N}$, and any fixed reals: α, β; k, $m : |m| > 1$ with $m \in \mathbb{R} - \{0, 1\}$, $(k \in \mathbb{R} - \{-1, 0\})$.

Proof of Existence in Theorem 2.1.

In fact, substitution of $x = y = 0$ in equality (2.1) yields

$$|k+1|\ |1-m|\ \|f(0)\| \leq c,$$

or

$$\|f(0)\| \leq \frac{c}{|k+1|\ |1-m|}, k \neq -1, m \neq 1. \tag{2.6}$$

Substituting $x = x$, $y = 0$ in (2.1), one gets that

$$\|kmf(x) - [kf(\alpha x) + f(k\beta x)] + mf(0)\| \leq c,$$

or

$$\|f(x) - \bar{f}(x) + \frac{1}{k} f(0)\| \leq \frac{c}{|k||m|}, k \neq 0, m \neq 0, |m| > 1 \tag{2.7}$$

from (2.2). Moreover substitution $x \to \alpha x, y = k\beta x$ in (2.1), we find that

$$\|kf(mx) + f(0) - m[kf(\alpha x) + f(k\beta x)]\| \leq c,$$

or

$$\|kf(\alpha x) + f(k\beta x) - km^{-1} f(mx) - \frac{1}{m} f(0)\| \leq \frac{c}{|m|},$$

or

$$\|\bar{f}(x) - m^{-2} f(mx) - \frac{1}{km^2} f(0)\| \leq \frac{c}{|k|\ m^2}, \tag{2.8}$$

Functional inequalities (2.6),(2.7),(2.8) and triangle inequality yields

$$\begin{aligned}
\|f(x) - m^{-2} f(mx)\| &\leq \|f(x) - \bar{f}(x) + \frac{1}{k} f(0)\| + \|\bar{f}(x) - m^{-2} f(mx) - \frac{1}{km^2} f(0)\| \\
&\quad + \Big\|\frac{1}{km^2} f(0) - \frac{1}{k} f(0)\Big\| \\
&\leq \frac{c}{|k|\ |m|} + \frac{c}{|k|\ m^2} + \frac{|1-m^2|}{|k|m^2} \|f(0)\| \\
&= \frac{1+|m|}{|k|\ m^2} c + \frac{|1-m^2|}{|k|m^2} \|f(0)\|
\end{aligned}$$

$$\begin{aligned} &\le \left(\frac{1+|m|}{|k|\,m^2}+\frac{|1-m^2|}{|k|m^2}\,\frac{1}{|k+1||1-m|}\right)c \\ &= \left(\frac{1+|m|}{|k|\,m^2}+\frac{|1+m|}{|k||k+1|m^2}\right)c \\ &= c_1=\frac{|k+1|\big(1+|m|\big)+|1+m|}{|k||k+1|m^2}c, \end{aligned}$$

or

$$\big\|f(x)-m^{-2}f(mx)\big\|\le c_1=\frac{c_2}{m^2}, \tag{2.9}$$

where

$$c_2=m^2c_1=\frac{|k+1|\big(1+|m|\big)+|1+m|}{|k||k+1|}c, \tag{2.10}$$

holds for fixed k, $m\neq 0$, $m\neq 1$, $m>1$. Replacing $x\to mx$ in (2.9) and then multiplying by m^{-2}, we find

$$\big\|m^{-2}f(mx)-m^{-4}f(m^2x)\big\|\le m^{-2}c_1, m\neq 0 \tag{2.11}$$

From (2.9) and (2.11), one gets

$$\big\|f(x)-m^{-4}f(m^2x)\big\|\le\big\|f(x)-m^{-2}f(mx)\big\|+\big\|m^{-2}f(mx)-m^{-4}f(m^2x)\big\|\le 1+m^{-2}c_1,$$

or

$$\big\|f(x)-m^{-4}f(m^2x)\big\|\le(1+m^{-2})c_1, m\neq 0. \tag{2.12}$$

Employing (2.9) and (2.12) without induction, we obtain

$$\begin{aligned} \big\|f(x)-m^{-2n}f(m^nx)\big\| &\le \big\|f(x)-m^{-2}f(mx)\big\|+\big\|m^{-2}f(mx)-m^{-4}f(m^2x)\big\|+\cdots \\ &\quad +\|m^{-2(n-1)}f(m^{n-1}x)-m^{-2n}f(m^nx)\big\| \\ &\le \big(1+m^{-2}+\cdots+m^{-2(n-1)}\big)c_1, \end{aligned}$$

or

$$\big\|f(x)-m^{-2n}f(m^nx)\big\|\le\frac{1-m^{-2n}}{1-m^{-2}}c_1=\frac{m^2}{m^2-1}\big(1-m^{-2n}\big)c_1, \tag{2.13}$$

or the general inequality:

$$\big\|f(x)-m^{-2n}f(m^nx)\big\|\le\frac{1}{m^2-1}\big(1-m^{-2n}\big)c_2, \tag{2.14}$$

where $|m|>1$, $c_2=m^2c_1$.

Claim now that the sequence

$$\big\{f_n(x)\big\}, f_n(x)=\big\{m^{-2n}f(m^nx)\big\} \tag{2.15}$$

converges. Note that from the general inequality (2.14) and the completeness of Y, one proves that the above sequence (2.15) is a Cauchy sequence. In fact, if $i>j>0$, then

$$\begin{aligned} \big\|f_i(x)-f_j(x)\big\| &= \big\|m^{-2i}f(m^ix)-m^{-2j}f(m^jx)\big\| \\ &= m^{-2j}\big\|m^{-2(i-j)}f(m^ix)-f(m^jx)\big\| \\ &= m^{-2j}\big\|f(m^jx)-m^{-2(i-j)}f(m^{i-j}\cdot m^jx)\big\| \\ &\le m^{-2j}\cdot\frac{1}{m^2-1}(1-m^{-2(i-j)})c_2, \end{aligned}$$

or

$$\big\|f_i(x)-f_j(x)\big\|\le\frac{1}{m^2-1}(m^{-2j}-m^{-2i})c_2,\ |m|>1, \tag{2.16}$$

or

$$0 \leq \lim_{i>j\to\infty} \big\| f_i(x) - f_j(x) \big\| \leq 0,$$

or

$$\lim_{i>j\to\infty} \big\| f_i(x) - f_j(x) \big\| = 0, \tag{2.17}$$

completing the proof that the sequence $\{f_n(x)\}$ converges. Hence $Q = Q(x)$ is well-defined via the formula (2.3). This means that the limit (2.3) exists for all $x \in X$.

In addition claim that mapping Q satisfies the functional equation (1.3) for all vectors $(x, y) \in X^2$. In fact, it is clear from functional inequality (2.1) and the limit (2.3) that inequality

$$\begin{aligned} \Big\| k \lim_{n\to\infty} m^{-2n} f\big[m^n(\alpha x + \beta y)\big] \quad &+ \quad \lim_{n\to\infty} m^{-2n} f\big[m^n(k\beta x - \alpha y)\big] \\ &-(\alpha^2 + k\beta^2)\big[k \lim_{n\to\infty} m^{-2n} f(m^n x) + \lim_{n\to\infty} m^{-2n} f(m^n y)\big] \Big\| \\ &\leq \quad c(\lim_{n\to 0} m^{-2n}) = 0, \qquad |m| > 1, \end{aligned} \tag{2.18}$$

or

$$\big\| kQ(\alpha x + \beta y) + Q(k\beta x - \alpha y) - (\alpha^2 + k\beta^2)[kQ(x) + Q(y)\big\| = 0,$$

or mapping Q satisfies the functional equation (1.3) for all $x, y \in X$, and $|m| > 1$. Thus Q is a 2-dimensional quadratic mapping. It is now clear from general inequality (2.14), $n \to \infty$, and the formula (2.3) that inequality (2.4) holds in X, completing the existence proof of this Theorem 2.1.

Proof of Uniqueness in Theorem 2.1.

Let $Q' : X \to Y$ be another 2-dimensional quadratic mapping satisfying equation (1.3), such that

$$\big\| f(x) - Q'(x) \big\| \leq c_3 \Big(= \frac{c_2}{m^2 - 1} \Big), \tag{2.4$'$}$$

for all $x \in X$, and any fixed real $m : |m| > 1$.

To prove the above-mentioned uniqueness employ (2.5) for Q and Q', as well, so that

$$Q'(x) = m^{-2n} Q'(m^n x) \tag{2.5$'$}$$

holds for all $x \in X$, all $n \in \mathbb{N}$, and any fixed real $m : |m| > 1$.

Moreover, the triangle inequality and functional inequalities (2.4)-(2.4)$'$ yield

$$\big\| Q(m^n x) - Q'(m^n x) \big\| \leq \big\| Q(m^n x) - f(m^n x) \big\| + \big\| f(m^n x) - Q'(m^n x) \big\|,$$

or

$$\big\| Q(m^n x) - Q'(m^n x) \big\| \leq 2c_3, \tag{2.19}$$

for all $x \in X$, all $n \in \mathbb{N}$, and any fixed real $m : |m| > 1$. Then from (2.5)-(2.5)$'$, and (2.19), one proves that

$$\big\| Q(x) - Q'(x) \big\| = \big\| m^{-2n} Q(m^n x) - m^{-2n} Q'(m^n x) \big\|,$$

or

$$\big\| Q(x) - Q'(x) \big\| \leq 2m^{-2n} c_3, \tag{2.20}$$

holds for all $x \in X$, all $n \in \mathbb{N}$, and any fixed real $m : |m| > 1$. Therefore from (2.20), and $n \to \infty$, one establishes

$$0 \leq \lim_{n\to\infty} \big\| Q(x) - Q'(x) \big\| \leq 2\Big(\lim_{n\to\infty} m^{-2n} \Big) c_3 = 0, \quad |m| > 1,$$

or

$$\left\|Q(x)-Q'(x)\right\|=0,$$

or

$$Q(x)=Q'(x), \quad |m|>1, \tag{2.21}$$

for all $x \in X$, completing the proof of uniqueness and thus the stability of Theorem 2.1.

Theorem 2.2. Let X be a normed linear space and let Y be a real complete normed linear space. Let us denote

$$\bar{\bar{f}}(x)=m^2\bar{f}(m^{-1}x)=\frac{m}{k}\left[kf\left(\frac{1}{m}\alpha x\right)+f\left(\frac{k}{m}\beta x\right)\right] \tag{2.2$'$}$$

holds for all $x \in X$ and any fixed reals $k, m \neq 0$. Also let us assume $|m|<1$. Then the limit

$$Q(x)=\lim_{n\to\infty} m^{2n} f(m^{-n}x), \tag{2.3$'$}$$

exists for all $x \in X$, all $n \in \mathbb{N}$, and any fixed real $m: |m|<1$, and $Q: X \to Y$ is the unique quadratic mapping satisfying functional equation $(2.3)'$, such that

$$\left\|f(x)-Q(x)\right\| \leq c_4=\frac{c_1}{1-m^2}.$$

Moreover, identity

$$Q(x)=m^{2n}Q(m^{-n}x) \tag{2.5$'$}$$

holds for all $x \in X$, $n \in \mathbb{N}$ and $|m|<1, m \neq 0$. From (2.7) with $x \to m^{-1}x (m \neq 0, |m|<1)$ and multiplying by m^2, one find

$$\left\|m^2 f(m^{-1}x)-\bar{\bar{f}}(x)+\frac{m^2}{k}f(0)\right\| \leq \frac{|m|}{|k|}c, \tag{2.22}$$

where

$$\bar{\bar{f}}(x)=m^2\bar{f}(m^{-1}x)=\frac{m}{k}\left[kf\big(m^{-1}\alpha x\big)+f\left(\frac{k}{m}\beta x\right)\right], \quad m \neq 0, |m|<1. \tag{2.23}$$

From (2.8) with $x \to m^{-1}x \quad (m \neq 0, |m|<1)$, one obtains

$$\left\|\bar{f}(m^{-1}x)-m^{-2}f(x)-\frac{1}{km^2}f(0)\right\| \leq \frac{c}{|k|\, m^2}.$$

Multiplying by m^2, we get

$$\left\|\bar{\bar{f}}(x)-f(x)-\frac{1}{k}f(0)\right\| \leq \frac{c}{|k|}. \tag{2.24}$$

Functional inequalities (2.6),(2.23),(2.24) and triangle inequality yield

$$\begin{aligned}
\left\|f(x)-m^2 f(m^{-1}x)\right\| &\leq \left\|f(x)-\bar{\bar{f}}(x)+\frac{1}{k}f(0)\right\|+\left\|\bar{\bar{f}}(x)-m^2 f(m^{-1}x)-\frac{m^2}{k}f(0)\right\| \\
&\quad +\left\|\frac{m^2}{k}f(0)-\frac{1}{k}f(0)\right\| \\
&\leq \frac{c}{|k|}+\frac{|m|}{|k|}c+\frac{|m^2-1|}{|k|}\left\|f(0)\right\| \\
&= \frac{1+|m|}{|k|}c+\frac{|1-m^2|}{|k|}\|f(0)\| \\
&\leq \left(\frac{1+|m|}{|k|}+\frac{|1+m|}{|k||k+1|}\right)c \\
&= \frac{|k+1|\big(1+|m|\big)+|1+m|}{|k||k+1|}c=c_2,
\end{aligned}$$

or

$$\left\|f(x)-m^2f(m^{-1}x)\right\|\le c_2, \tag{2.25}$$

where

$$c_2=\frac{|k+1|\big(1+|m|\big)+|1+m|}{|k||k+1|}c,\quad |m|<1, k\neq 0, k\neq -1, m\neq 0.$$

Replacing $x\to m^{-1}x$ in (2.25) and multiplying by m^2, we get

$$m^2f(m^{-1}x)-m^4f(m^{-2}x)\|\le m^2c_2, \tag{2.26}$$

From (2.25)-(2.26), one finds

$$\left\|f(x)-m^4f(m^{-2}x)\right\|\le\left\|f(x)-m^2f(m^{-1}x)\right\|+\left\|m^2f(m^{-1}x)-m^4f(m^{-2}x)\right\|\le(1+m^2)c_2,$$

or

$$\left\|f(x)-m^4f(m^{-2}x)\right\|\le\big(1+m^2\big)c_2,\ m\neq 0. \tag{2.27}$$

Employing (2.25) and (2.27), without induction, we get

$$\begin{aligned}\left\|f(x)-m^{2n}f(m^{-n}x)\right\| &\le \left\|f(x)-m^2f(m^{-1}x)\right\|+\left\|m^2f(m^{-1}x)-m^4f(m^{-2}x)\right\|+\cdots\\ &\quad+\left\|m^{2(n-1)}f(m^{-(n-1)}x)-m^{2n}f(m^{-n}x)\right\|\\ &\le \big(1+m^2+\cdots+m^{2(n-1)}\big)c_2\end{aligned}$$

or

$$\left\|f(x)-m^{2n}f(m^{-n}x)\right\|\le\frac{1-m^{2(n-1)}}{1-m^2}c_2=\frac{c_2}{1-m^2}(1-m^{2(n-1)}), \tag{2.28}$$

or the general inequality:

$$\left\|f(x)-m^{2n}f(m^{-n}x)\right\|\le\frac{c_2}{1-m^2}, \tag{2.29}$$

where $|m|<1, m\neq 0$.

Rest of the proof is similar to the proof of Theorem 2.1.

Assume the following condition on f:

$$f(0)=0. \tag{2.30}$$

From (2.30) and (2.7)-(2.8), we get

$$\left\|f(x)-\bar f(x)\right\|\le\frac{c}{|k|\ |m|}, \tag{2.31}$$

and

$$\left\|\bar f(x)-m^{-2}f(mx)\right\|\le\frac{c}{|k|\ m^2},\quad k\neq 0, m\neq 0, |m|>1. \tag{2.32}$$

From (2.31)-(2.32), one obtains

$$\left\|f(x)-m^{-2}f(mx)\right\|\le\left\|f(x)-\bar f(x)\right\|+\left\|\bar f(x)-m^{-2}f(mx)\right\|,$$

or

$$\left\|f(x)-m^{-2}f(mx)\right\|\le c_1'=\frac{|m|+1}{|k|\ m^2}c,\quad k\neq 0, m\neq 0, |m|>1. \tag{2.33}$$

Thus

$$\begin{aligned}\left\|f(x)-m^{-2n}f(m^nx)\right\| &\le \left\|f(x)-m^{-2}f(mx)\right\|+\left\|m^{-2}f(mx)-m^{-4}f(m^2x)\right\|\\ &\quad+\cdots+\left\|m^{-2(n-1)}f(m^{n-1}x)-m^{-2n}f(m^nx)\right\|\\ &\le \big(1+m^{-2}+\cdots+m^{-2(n-1)}\big)c_1',\end{aligned}$$

or

$$\left\|f(x)-m^{-2n}f(m^nx)\right\| \le \frac{1-m^{-2n}}{1-m^{-2}}c_1' = \frac{m^2}{m^2-1}\left(1-m^{-2n}\right)c_1',$$

or

$$\left\|f(x)-m^{-2n}f(m^nx)\right\| \le \frac{1}{m^2-1}\left(1-m^{-2n}\right)c_2', \tag{2.34}$$

where

$$|m|>1, \text{ with } c_2' = m^2c_1' = \frac{|m|+1}{|k|}c.$$

Therefore the following Theorem 2.1a holds.

Theorem 2.1a. Let X be a normed linear space and let Y be a real complete normed linear space. Then the limit (2.3) exists for all $x \in X$, all $n \in \mathbb{N}$, $|m|>1$ and $Q: X \to Y$ is the unique quadratic mapping satisfying equation (1.3), such that

$$\left\|f(x)-Q(x)\right\| \le \frac{c_2'}{m^2-1} = \frac{|m|+1}{m^2-1}\,\frac{1}{|k|}c, \quad k \neq 0, |m|>1. \tag{2.35}$$

The proof of this Theorem 2.1a is similar to the proof of the previous Theorem 2.1.

Alternatively: $|m|<1$, $f(0)=0$:

From (2.30) and (2.22) , (2.24), we get

$$\left\|f(x)-\bar{\bar{f}}(x)\right\| \le \frac{c}{|k|}, \tag{2.36}$$

and

$$\left\|\bar{\bar{f}}-m^2f(m^{-1}x)\right\| \le \frac{|m|}{|k|}c, \tag{2.37}$$

$k\neq 0, m\neq 0, |m|<1$. From (2.36)-(2.37), one obtains

$$\left\|f(x)-m^2f(m^{-1}x)\right\| \le \left\|f(x)-\bar{\bar{f}}(x)\right\| + \left\|\bar{\bar{f}}(x)-m^2f(m^{-1}x)\right\|$$

or

$$\left\|f(x)-m^2f(m^{-1}x)\right\| \le c_2' = \frac{|m|+1}{|k|}c \tag{2.38}$$

$k\neq 0, m\neq 0, |m|<1$. Thus

$$\begin{aligned}\left\|f(x)-m^{2n}f(m^{-n}x)\right\| &\le \left\|f(x)-m^{-2}f(mx)\right\| \\ &\quad + \cdots + \left\|m^{2(n-1)}f(m^{-(n-1)}x)-m^{2n}f(m^{-n}x)\right\| \\ &\le \left(1+m^2+\cdots+m^{2(n-1)}\right)c_2',\end{aligned}$$

or

$$\left\|f(x)-m^{2n}f(m^{-n}x)\right\| \le \frac{1}{1-m^2}\left(1-m^{2n}\right)c_2', \tag{2.39}$$

where $|m|<1, m\neq 0$.

Therefore the following Theorem 2.2a (analogous to Theorem 2.1a) holds for $|m|<1$, $m\neq 0$.

Theorem 2.2a. Let X be a normed linear space, and Y a real complete normed linear space. Then the limit (2.3)′ exists for all $x\in X$, $n\in\mathbb{N}$, $|m|<1$; $m\neq 0$, and $Q: X\to Y$ is the unique quadratic mapping satisfying equation (1.3), such that

$$\left\|f(x)-Q(x)\right\| \le \frac{c_2'}{1-m^2} = \frac{1+|m|}{1-m^2}\,\frac{1}{|k|}c, \tag{2.40}$$

$k \neq 0$, $|m| < 1$; $m \neq 0$.

Special case: Replacing $\alpha = \beta = 1$ in equation (1.3) and (2.1), one gets

$$kf(x+y) + f(kx-y) = (k+1)\big[kf(x) + f(y)\big], \quad k \in \mathbb{R} - \{-1, 0\}. \tag{2.41}$$

Thus

$$m = k + 1 \in \mathbb{R} - \{0, 1\}.$$

Also

$$\big\|kf(x+y) + f(kx-y) - (k+1)\big[kf(x) + f(y)\big]\big\| \le c, \quad k \in \mathbb{R} - \{-1, 0\}. \tag{2.42}$$

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Some integral inequalities via $(h-(\alpha,m))-$logarithmically convexity

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Abstract. In this paper, we introduce the concept of $(h-(\alpha,m))-$logarithmically convex functions and establish some new integral inequalities of these classes of functions.

Keywords: Hermite's inequalities; $m-$logarithmically convex; $(\alpha,m)-$logarithmically convex; $(h-(\alpha,m))-$logarithmically convex;

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1 Introduction and preliminaries

The mathematical inequalities play an important role in the mathematical branches and their enormous application can not be underestimated. Afterwards, many researchers[1-13] studied the properties of convexity and achieve some different integral inequalities. The purpose of this paper is to introduce the definition of $(h-(\alpha,m))-$logarithmically convex functions and establish some new integral inequalities of these classes of functions. Before stating our results, we need recall some notions.

Throughout this paper, by $\Re$, we denote the set of all real numbers.

Definition 1.1 *Let $f : I \subset \Re \to \Re$ be a function define on interval I of real numbers. Then f is called convex (see[4]) if*

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)$$

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for all $x, y \in I$ *and* $t \in [0,1]$.

In [2] , Toader gave the definition of $m-$convexity as follows.

Definition 1.2 *The function* $f : [a,b] \to \Re$, $0 \le a < b$ *is said to be* $m-$*convex , where* $m \in [0,1]$, *if*

$$f(tx + m(1-t)y) \le tf(x) + m(1-t)f(y)$$

holds for all $x, y \in [0,1]$ *and* $t \in [0,1]$. *We say that* f *is* $m-$*concave if* $-f$ *is* $m-$*convex.*

In [3] , Mihesan gave the definition of $(\alpha, m)-$convexity as follows.

Definition 1.3 *The function* $f : [a,b] \to \Re$, $0 \le a < b$ *is said to be* $(\alpha, m)-$*convex , where* $(\alpha, m) \in [0,1]^2$, *if*

$$f(tx + m(1-t)y) \le t^\alpha f(x) + m(1-t^\alpha)f(y)$$

holds for all $x, y \in [0,1]$ *and* $t \in [0,1]$.

In [1], Özedemir et al. gave the definition of $(h-(\alpha, m))-$convexity as follows.

Definition 1.4 *Let* $h : K \subset \Re \to \Re$ *be a nonnegative function,* $h \neq 0$. *The function* $f : L \subset \Re \to \Re$ *is said to be* $(h-(\alpha, m))-$*convex function if* f *is non-negative and for all* $x, y \in [0,1]$ *and* $t \in (0,1)$ *for* $(\alpha, m) \in [0,1]^2$, *we have*

$$f(tx + m(1-t)y) \le h^\alpha(t)f(x) + m(1-h^\alpha(t))f(y).$$

In [5], Bai gave the definition of $m-$ and $(\alpha, m)-$logarithmically convex functions as follows.

Definition 1.5 *The function* $f : [a,b] \to (0,\infty)$,$0 \le a < b$ *is said to be* $m-$*logarithmically convex, where* $m \in (0,1]$, *if*

$$f(tx + m(1-t)y) \le [f(x)]^t [f(y)]^{m(1-t)}$$

holds for all $x, y \in [0,1]$ *and* $t \in [0,1]$.

Definition 1.6 *The function* $f : [a,b] \to (0,\infty)$,$0 \le a < b$ *is said to be* $(\alpha, m)-$*logarithmically convex, where* $(\alpha, m) \in (0,1]^2$, *if*

$$f(tx + m(1-t)x) \le [f(x)]^{t^\alpha} [f(x)]^{m(1-t^\alpha)}$$

holds for all $x, y \in [0,1]$ *and* $t \in [0,1]$.

2 Main results

In this section, we will introduce the concept of $(h-(\alpha,m))-$logarithmically convex functions. We give some new integral inequalities of these classes of functions. First, we present the definition of $(h-(\alpha,m))-$logarithmically convex functions as follow.

Definition 2.1 *Let $h : K \subset \Re \to \Re$ be a nonnegative function, $h \neq 0$. The function $f : L \subset \Re \to \Re$ is said to be $(h-(\alpha,m))-$logarithmically convex function if f is nonnegative and for all $x, y \in L$ and $t \in (0,1)$ for $(\alpha,m) \in (0,1]^2$, we have*

$$f(tx + m(1-t)y) \leq [f(x)]^{h^\alpha(t)}[f(y)]^{m\left(1-h^\alpha(t)\right)}.$$

Obviously, if $h(t) = t$, then $(h-(\alpha,m))-$logarithmically convex function is a $(\alpha,m)-$logarithmically convex function; if $h(t) = t$,$\alpha = 1$, then $(h-(\alpha,m))-$logarithmically convex function is a $m-$logarithmically convex function.

Before giving our results, we need the following lemma which is proved by Özdemir et al. [13].

Lemma 2.1 *Let $f : [a,b] \to \Re$, $0 \leq a < b$ be continuous on $[a,b]$ such that $f \in L([a,b])$. Then the equality*

$$\int_a^b (x-a)^p(x-b)^q f(x)dx = (b-a)^{p+q+1}\int_0^1 (1-t)^p t^q f(tx+(1-t)y)dt$$

holds for some fixed $p, q > 0$.

Theorem 2.1 *Let $f : [a,b] \to \Re$, $0 \leq a < b$ be continuous on $[a,b]$ such that $f \in L([a,b])$. If the mapping f is $(h-(\alpha,m))-$logarithmically convex on $[a,b]$ for all $t \in (0,1)$ and $(\alpha,m) \in (0,1]^2$, then*

$$\begin{aligned}\int_a^b (x-a)^p(x-b)^q f(x)dx \leq (b-a)^{p+q+1}[\beta(\frac{q}{1-m}+1, \frac{p}{1-m}+1)]^{1-m} \\ \times\left\{\int_0^1 [f(a)]^{\frac{h^\alpha(t)}{m}} f(\frac{b}{m})^{1-h^\alpha(t)}dt\right\}^m\end{aligned} \tag{2.1}$$

where $\beta(x,y) = \int_0^1 (t)^{x-1}(1-t)^{y-1}dt$.

Proof. Using Lemma 2.1 , we have

$$\int_a^b (x-a)^p(x-b)^q f(x)dx = (b-a)^{p+q+1}\int_0^1 (1-t)^p t^q f(ta+(1-t)b)dt. \tag{2.2}$$

Since f is $(h-(\alpha,m))-$logarithmically convex on $[a,b]$, we know that for every $t \in (0,1)$

$$f(ta+(1-t)b) = f(ta+m(1-t)(\frac{b}{m})) \leq [f(a)]^{h^\alpha(t)}[f(\frac{b}{m})]^{m\left(1-h^\alpha(t)\right)}. \tag{2.3}$$

From (2.1), (2.2), (2.3) and Hölder inequality, we can conclude that

$$\begin{aligned}
&\int_a^b (x-a)^p(x-b)^q f(x)dx \\
&= (b-a)^{p+q+1}\int_0^1 (1-t)^p t^q f(ta+(1-t)b)dt \\
&= (b-a)^{p+q+1}\int_0^1 (1-t)^p t^q f(ta+m(1-t)\frac{b}{m})dt \\
&\leq (b-a)^{p+q+1}\int_0^1 (1-t)^p t^q [f(a)]^{h^\alpha(t)}[f(\frac{b}{m})]^{m\left(1-h^\alpha(t)\right)}dt \\
&\leq (b-a)^{p+q+1}\left\{\int_0^1 [(1-t)^p t^q]^{\frac{1}{1-m}}dt\right\}^{1-m}\left\{\int_0^1 \{[f(a)]^{h^\alpha(t)}[f(\frac{b}{m})^{m(1-h^\alpha(t))}]\}^{\frac{1}{m}}dt\right\}^m \\
&\leq (b-a)^{p+q+1}\left\{\int_0^1 [(1-t)^{\frac{p}{1-m}}t^{\frac{q}{1-m}}dt\right\}^{1-m}\left\{\int_0^1 \{[f(a)]^{\frac{h^\alpha(t)}{m}}[f(\frac{b}{m})^{1-h^\alpha(t)}]\}dt\right\}^m \\
&\leq (b-a)^{p+q+1}[\beta(\frac{q}{1-m}+1,\frac{p}{1-m}+1)]^{1-m}\left\{\int_0^1 \{[f(a)]^{\frac{h^\alpha(t)}{m}}[f(\frac{b}{m})^{1-h^\alpha(t)}]\}dt\right\}^m.
\end{aligned}$$

Hence, the proof of theorem 2.1 is completed.

Remark 2.1 *If $\alpha = 1$, then we can conclude the following inequality:*

$$\begin{aligned}
\int_a^b (x-a)^p(x-b)^q f(x)dx \leq (b-a)^{p+q+1}[\beta(\frac{q}{1-m}+1,\frac{p}{1-m}+1)]^{1-m} \\
\times\left\{\int_0^1 \{[f(a)]^{\frac{h(t)}{m}}[f(\frac{b}{m})^{1-h(t)}]\}dt\right\}^m.
\end{aligned}$$

Theorem 2.2 *Let $f : [a,b] \to \Re$, $0 \leq a < b$ be continuous on $[a,b]$ such that $f \in L([a,b])$. If the mapping $|f|^{\frac{k}{k-1}}$ $(k > 1)$ is $(h-(\alpha,m))-$logarithmically convex on $[a,b]$ for all $t \in (0,1)$ and $(\alpha,m) \in (0,1]^2$, then*

$$\begin{aligned}
\int_a^b (x-a)^p(x-b)^q f(x)dx \leq (b-a)^{p+q+1}[\beta(kq+1,kp+1)]^{\frac{1}{k}}\left[\int_0^1 |f(a)|^{\frac{k^2h^\alpha(t)}{k-1}}dt\right]^{\frac{k-1}{k^2}} \\
\times\left[\int_0^1 |f(\frac{b}{m})|^{\frac{k^2m}{(k-1)^2}(1-h^\alpha(t))}\right]^{\frac{(k-1)^2}{k^2}}
\end{aligned} \tag{2.4}$$

where $\beta(x,y) = \int_0^1 (t)^{x-1}(1-t)^{y-1}dt$.

Proof. Using Lemma 2.1 , we have

$$\int_a^b (x-a)^p(x-b)^q f(x)dx = (b-a)^{p+q+1}\int_0^1 (1-t)^p t^q f(ta+(1-t)b)dt. \tag{2.5}$$

Taking into account that $|f|^{\frac{k}{k-1}}$ is $(h-(\alpha,m))-$logarithmically convex on $[a,b]$, we deduce that

$$|f(ta+(1-t)b)|^{\frac{k}{k-1}} = |f(ta+m(1-t)(\frac{b}{m}))|^{\frac{k}{k-1}} \leq |f(a)|^{\frac{k}{k-1}h^\alpha(t)}|f(\frac{b}{m})|^{\frac{k}{k-1}m(1-h^\alpha(t))}. \tag{2.6}$$

Hence, from (2.4), (2.5), (2.6) and Hölder inequality, we can achieve the following inequality:

$$\begin{aligned}
&\int_a^b (x-a)^p (x-b)^q f(x)dx \\
&= (b-a)^{p+q+1}\int_0^1 (1-t)^p t^q f(ta+(1-t)b)dt \\
&\le (b-a)^{p+q+1}\left[\int_0^1 (1-t)^{kp} t^{kq} dt\right]^{\frac{1}{k}} \left\{\int_0^1 \left|f(ta+m(1-t)\frac{b}{m})\right|^{\frac{k}{k-1}} dt\right\}^{\frac{k-1}{k}} \\
&= (b-a)^{p+q+1}[\beta(kq+1,kp+1)]^{\frac{1}{k}} \left\{\int_0^1 \left|f(ta+m(1-t)\frac{b}{m})\right|^{\frac{k}{k-1}} dt\right\}^{\frac{k-1}{k}} \\
&\le (b-a)^{p+q+1}[\beta(kq+1,kp+1)]^{\frac{1}{k}} \left[\int_0^1 |f(a)|^{\frac{k}{k-1}h^\alpha(t)} |f(\frac{b}{m})|^{\frac{k}{k-1}m(1-h^\alpha(t))} dt\right]^{\frac{k-1}{k}}.
\end{aligned} \tag{2.7}$$

Using Hölder inequality again, we have

$$\begin{aligned}
&\left[\int_0^1 |f(a)|^{\frac{k}{k-1}h^\alpha(t)} |f(\frac{b}{m})|^{\frac{k}{k-1}m(1-h^\alpha(t))} dt\right]^{\frac{k-1}{k}} \\
&\le \left\{\left[\int_0^1 |f(a)|^{\frac{k^2}{k-1}h^\alpha(t)} dt\right]^{\frac{1}{k}} \left[\int_0^1 [|f(\frac{b}{m})|^{\frac{k}{k-1}m(1-h^\alpha(t))}]^{\frac{k}{k-1}} dt\right]^{\frac{k-1}{k}}\right\}^{\frac{k-1}{k}} \\
&\le \left\{\left[\int_0^1 |f(a)|^{\frac{k^2}{k-1}h^\alpha(t)} dt\right]^{\frac{1}{k}} \left[\int_0^1 [|f(\frac{b}{m})|^{(\frac{k}{k-1})^2 m(1-h^\alpha(t))}] dt\right]^{\frac{k-1}{k}}\right\}^{\frac{k-1}{k}}
\end{aligned} \tag{2.8}$$

Combining with (2.7) and (2.8), we can conclude that (2.4) holds. Hence, the proof of theorem 2.2 is completed.

Remark 2.2 *If $\alpha = 1$, then we can conclude the following inequality:*

$$\begin{aligned}
\int_a^b (x-a)^p (x-b)^q f(x)dx \le (b-a)^{p+q+1}[\beta(kq+1,kp+1)]^{\frac{1}{k}} \left[\int_0^1 |f(a)|^{\frac{k^2 h(t)}{k-1}} dt\right]^{\frac{k-1}{k^2}} \\
\times\left[\int_0^1 |f(\frac{b}{m})|^{\frac{k^2 m}{(k-1)^2}(1-h(t))}\right]^{\frac{(k-1)^2}{k^2}}.
\end{aligned}$$

Theorem 2.3 *Let $f:[a,b]\to\Re$, $0\le a<b$ be continuous on $[a,b]$ such that $f\in L([a,b])$. If the mapping $|f|^l$ $(l\ge 1)$ is $(h-(\alpha,m))-$logarithmically convex on $[a,b]$ for all $t\in(0,1)$ and $(\alpha,m)\in(0,1]^2$, then*

$$\begin{aligned}
\int_a^b (x-a)^p (x-b)^q f(x)dx \le (b-a)^{p+q+1}[\beta(q+1,p+1)]^{\frac{l-1}{l}} \left[\beta(ql+1,pl+1)\right]^{\frac{1}{l^2}} \left[\int_0^1 |f(a)|^{\frac{l^3 h^\alpha(t)}{l-1}} dt\right]^{\frac{l-1}{l^3}} \\
\times\left[\int_0^1 |f(\frac{b}{m})|^{\frac{l^2 m(1-h^\alpha(t))}{(l-1)^2}} dt\right]^{\frac{(l-1)^2}{l^3}}
\end{aligned} \tag{2.9}$$

where $\beta(x,y)=\int_0^1 (t)^{x-1}(1-t)^{y-1}dt$.

Proof. Using Lemma 2.1 , we have

$$\int_a^b (x-a)^p (x-b)^q f(x)dx = (b-a)^{p+q+1}\int_0^1 (1-t)^p t^q f(ta+(1-t)b)dt. \tag{2.10}$$

Since, $|f|^l$ is $(h-(\alpha,m))-$logarithmically convex on $[a,b]$, we have

$$|f(ta+(1-t)b)|^l = |f(ta+m(1-t)(\frac{b}{m}))|^l \leq |f(a)|^{lh^\alpha(t)}|f(\frac{b}{m})|^{lm(1-h^\alpha(t))}. \tag{2.11}$$

From (2.9), (2.10), (2.11) and Hölder inequality, we can achieve the following inequality:

$$\begin{aligned}
&\int_a^b (x-a)^p(x-b)^q f(x)dx \\
&= (b-a)^{p+q+1}\int_0^1 (1-t)^p t^q f(ta+m(1-t)(\frac{b}{m}))dt \\
&\leq (b-a)^{p+q+1}\int_0^1 [(1-t)^p(t)^q]^{\frac{l-1}{l}}[(1-t)^p(t)^q]^{\frac{1}{l}} f(ta+m(1-t)(\frac{b}{m}))dt \\
&\leq (b-a)^{p+q+1}\left[\int_0^1 (1-t)^p(t)^q dt\right]^{\frac{l-1}{l}}\left\{\int_0^1 [(1-t)^p(t)^q]|f(ta+m(1-t)(\frac{b}{m}))|^l dt\right\}^{\frac{1}{l}} \\
&= (b-a)^{p+q+1}[\beta(q+1,p+1)]^{\frac{l-1}{l}}\left\{\int_0^1 [(1-t)^p(t)^q]|f(ta+m(1-t)(\frac{b}{m}))|^l dt\right\}^{\frac{1}{l}}.
\end{aligned} \tag{2.12}$$

Using Hölder inequality again, we have

$$\begin{aligned}
&\left\{\int_0^1 [(1-t)^p(t)^q]|f(ta+m(1-t)(\frac{b}{m}))|^l dt\right\}^{\frac{1}{l}} \\
&\leq \left\{\int_0^1 [(1-t)^p(t)^q]|f(a)|^{lh^\alpha(t)}|f(\frac{b}{m})|^{lm(1-h^\alpha(t))}dt\right\}^{\frac{1}{l}} \\
&\leq \left\{\{\int_0^1 [(1-t)^p(t)^q]^l dt\}^{\frac{1}{l}}\{\int_0^1 \left[|f(a)|^{lh^\alpha(t)}|f(\frac{b}{m})|^{lm(1-h^\alpha(t))}\right]^{\frac{l}{l-1}}dt\}^{\frac{l-1}{l}}\right\}^{\frac{1}{l}} \\
&\leq \left[\beta(ql+1,pl+1)\right]^{\frac{1}{l^2}}\left[\int_0^1 |f(a)|^{\frac{l^2h^\alpha(t)}{l-1}}|f(\frac{b}{m})|^{\frac{l^2m(1-h^\alpha(t))}{l-1}}dt\right]^{\frac{l-1}{l^2}} \\
&\leq \left[\beta(ql+1,pl+1)\right]^{\frac{1}{l^2}}\left[\int_0^1 |f(a)|^{\frac{l^3h^\alpha(t)}{l-1}}dt\right]^{\frac{l-1}{l^3}}\left[\int_0^1 |f(\frac{b}{m})|^{\frac{l^3m(1-h^\alpha(t))}{(l-1)^2}}dt\right]^{\frac{(l-1)^2}{l^3}}.
\end{aligned} \tag{2.13}$$

By (2.12) and (2.13), we can achieve that (2.9) holds. Hence, the proof of theorem 2.3 is completed.

Remark 2.3 *If $\alpha=1$, then we can conclude the following inequality:*

$$\begin{aligned}
\int_a^b (x-a)^p(x-b)^q f(x)dx &\leq (b-a)^{p+q+1}[\beta(q+1,p+1)]^{\frac{l-1}{l}}\left[\beta(ql+1,pl+1)\right]^{\frac{1}{l^2}}\left[\int_0^1 |f(a)|^{\frac{l^3h(t)}{l-1}}dt\right]^{\frac{l-1}{l^3}} \\
&\times\left[\int_0^1 |f(\frac{b}{m})|^{\frac{l^2m(1-h(t))}{(l-1)^2}}dt\right]^{\frac{(l-1)^2}{l^3}}.
\end{aligned}$$

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On Gosper's q-Trigonometric Function

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Abstract. In this paper, we study about periodicity of q-trigonometric function which was introduced by Gosper and also we rewrite the q-analogue of Legendres duplication formula with the same bases. Furthermore, we modify some identities involving q-shifted factorial.

Keywords. Gosper's q-trigonometric function, q-Gamma function, Legendres duplication formula.

Mathematics Subject Classification. 11B65, 33D05.

1 Introduction

The q-shifted factorial [1, 3] is defined by

$$(a;q)_n = \begin{cases} 1 & n=0, \\ \prod_{m=0}^{n-1}(1-aq^m) & n=1,2,.... \end{cases} \tag{1}$$

and it is assumed that $a \neq q^{-m},\ m = 0, 1, ...$ The q-shifted factorial [1, 3] is also defined for any complex number α,

$$(a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty}, \tag{2}$$

where $(a;q)_\infty := \lim_{n\to\infty}\prod_{m=0}^{n}(1-aq^m)$ and the principal value of q^α is taken and it is assumed that $0 < q < 1$.

The q-Gamma function was introduced by Thomae [6] and Jackson [5], (see [3], page 20)

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \quad 0 < q < 1. \tag{3}$$

A q-analogue of Legendre's duplication formula [5, 7] has the form

$$\Gamma_q(2x)\Gamma_{q^2}(\frac{1}{2}) = (1+q)^{2x-1}\Gamma_{q^2}(x)\Gamma_{q^2}(x+\frac{1}{2}). \tag{4}$$

Gosper [4] defined q-trigonometric functions as follows:

$$sin_q(\pi z) := q^{(z-1/2)^2}\frac{(q^{2z};q^2)_\infty(q^{2-2z};q^2)_\infty}{(q;q^2)_\infty^2}, \quad 0 < q < 1, \tag{5}$$

$$cos_q(\pi z) := q^{z^2}\frac{(q^{1+2z};q^2)_\infty(q^{1-2z};q^2)_\infty}{(q;q^2)_\infty^2}, \quad 0 < q < 1. \tag{6}$$

1

It can be seen [4] that

$$cos_q(z) = sin_q(\frac{\pi}{2} - z). \tag{7}$$

By using (5), (6) and (7), one can see that, for the cases $x = 0$ and $x = \frac{\pi}{2}$, $sin_q(x)$ and $cos_q(x)$ are;

$$\begin{array}{ll} sin_q(0) = 0, & sin_q(\frac{\pi}{2}) = 1, \\ cos_q(0) = 1, & cos_q(\frac{\pi}{2}) = 0. \end{array} \tag{8}$$

There are many identities involving q-shifted factorial [1, 3], but in this paper we are using the following identities;

For all $a \in \mathbb{C}$ and $n \in \mathbb{N}$, following equations hold

$$(q^{2a}; q^2)_n = (q^a; q)_n(-q^a; q)_n, \tag{9}$$

$$(a; q)_{2n} = (a; q^2)_n(aq; q^2)_n, \tag{10}$$

$$(q^{1-a-n}; q)_n = (q^a; q)_n(-1)^n \, q^{-\binom{n}{2}-an}. \tag{11}$$

2 Main result

In the next lemma we show that the equations (9) and (10) are also valid for any complex number α,

Lemma 1. *For all a, $\alpha \in \mathbb{C}$, the following equations hold*

$$(q^{2a}; q^2)_\alpha = (q^a; q)_\alpha(-q^a; q)_\alpha, \tag{12}$$

$$(a; q)_{2\alpha} = (a; q^2)_\alpha(aq; q^2)_\alpha. \tag{13}$$

Proof. To prove (12) we use (2), then we have

$$(q^{2a}; q^2)_\alpha = \frac{(q^{2a}; q^2)_\infty}{(q^{2a+2\alpha}; q^2)_\infty}.$$

By using the definition of q-shifted factorial (1), we obtain

$$\begin{aligned} (q^{2a}; q^2)_\alpha &= \frac{(q^{2a}; q^2)_\infty}{(q^{2a+2\alpha}; q^2)_\infty} \\ &= \frac{\prod_{i=0}^{\infty}(1 - q^{2a+2i})}{\prod_{i=0}^{\infty}(1 - q^{2a+2\alpha+2i})} \\ &= \frac{\prod_{i=0}^{\infty}(1 - q^{a+i})(1 + q^{a+i})}{\prod_{i=0}^{\infty}(1 - q^{a+\alpha+i})(1 + q^{a+\alpha+i})} \\ &= \frac{\prod_{i=0}^{\infty}(1 - q^{a+i})}{\prod_{i=0}^{\infty}(1 - q^{a+\alpha+i})} \frac{\prod_{i=0}^{\infty}(1 + q^{a+i})}{\prod_{i=0}^{\infty}(1 + q^{a+\alpha+i})} \\ &= \frac{(q^a; q)_\infty}{(q^{a+\alpha}; q)_\infty} \frac{(-q^a; q)_\infty}{(-q^{a+\alpha}; q)_\infty} \\ &= (q^a; q)_\alpha(-q^a; q)_\alpha. \end{aligned}$$

2

The proof of (12) is complete. To Prove the next equation, we use (1) and (2), then we have

$$\begin{aligned}\frac{(a;q^2)_\alpha(aq;q^2)_\alpha}{(a;q)_{2\alpha}} &= \frac{(a;q^2)_\infty(aq;q^2)_\infty(aq^{2\alpha};q)_\infty}{(aq^{2\alpha};q^2)_\infty(aq^{2\alpha+1};q^2)_\infty(a;q)_\infty}\\ &= \frac{(a;q^2)_\infty(aq;q^2)_\infty}{(a;q)_\infty}\frac{(aq^{2\alpha};q)_\infty}{(aq^{2\alpha};q^2)_\infty(aq^{2\alpha+1};q^2)_\infty},\end{aligned}$$

each fraction in the last line is equal 1, since $(c;q)_\infty(cq^{\frac{1}{2}};q)_\infty = (c;q^{\frac{1}{2}})_\infty$ (see [8] , page 13). The proof of (13) is complete.

□

In the next lemma, we want to modify the equation (11).

Lemma 2. *For all α and $\beta \in \mathbb{C}$, the following equation holds*

$$(q^{1-\alpha-\beta};q)_\alpha = (q^\beta;q)_\alpha \frac{sin_{\sqrt{q}}\pi(\alpha+\beta)}{sin_{\sqrt{q}}\pi(\beta)}\, q^{-\binom{\alpha}{2}-\alpha\beta}, \tag{14}$$

where sin_q is defined as in (5).

Proof. After applying the equation (2) for both numerator and denominator of the left hand side of the following equation, we obtain that

$$\frac{(q^{1-\alpha-\beta};q)_\alpha}{(q^\beta;q)_\alpha} = \frac{(q^{1-\alpha-\beta};q)_\infty(q^{\alpha+\beta};q)_\infty}{(q^{1-\beta};q)_\infty(q^\beta;q)_\infty}$$

and by using the definition of sin_q which is written in (5), we have

$$\frac{(q^{1-\alpha-\beta};q)_\infty(q^{\alpha+\beta};q)_\infty}{(q^{1-\beta};q)_\infty(q^\beta;q)_\infty} = \frac{sin_{\sqrt{q}}\pi(\alpha+\beta)}{sin_{\sqrt{q}}\pi(\beta)}\, q^{-\binom{\alpha}{2}-\alpha\beta}.$$

Therefore proof is complete. □

Theorem 1. *For all $n \in \mathbb{N}$ and $x \in \mathbb{C}$, the following equations hold*

$$sin_q(x+n\pi) = (-1)^n sin_q(x), \tag{15}$$
$$cos_q(x+n\pi) = (-1)^n cos_q(x), \tag{16}$$
$$tan_q(x+n\pi) = tan_q(x), \tag{17}$$
$$cot_q(x+n\pi) = cot_q(x). \tag{18}$$

Proof. We use lemma 2 for prove the equation (15). Taking any arbitrary n $\in \mathbb{N}$ and a $\in \mathbb{C}$, then we have

$$(q^{1-n-a};q)_n = (q^a;q)_n \frac{sin_{\sqrt{q}}\pi(a+n)}{sin_{\sqrt{q}}\pi(a)}\, q^{-\binom{n}{2}-na}. \tag{19}$$

By comparing the equations (11) and (19), we obtain

$$\frac{sin_{\sqrt{q}}\pi(a+n)}{sin_{\sqrt{q}}\pi(a)} = (-1)^n.$$

Substituting q with $\sqrt{q}$ and x with $a\pi$ completes the proof of equation (15).

By using (7) and (15), one can shows that (16) is valid for all n $\in \mathbb{N}$ and $x \in \mathbb{C}$, and the last two equations (17) and (18) comes from $\frac{sin_q(x)}{cos_q(x)}$ and $\frac{cos_q(x)}{sin_q(x)}$, respectively.

□

Remark 1. *The $cos_q(x)$ is an even function, its come from the definition (6) directly. And the $sin_q(x)$ is an odd function, since by using (7), we can write $sin_q(x) = cos_q(\frac{\pi}{2}-x)$ and also we know that $cos_q(x)$ is an even function then we have $sin_q(x) = cos_q(x-\frac{\pi}{2})$, again apply (7), we obtain $cos_q(x-\frac{\pi}{2}) = sin_q(\pi-x)$. Now by using the Theorem 1, we obtain $sin_q(\pi - x) = -sin_q(-x)$. Therefore $sin_q(x) = -sin_q(-x)$.*

Lemma 3. *For all $k \in \mathbb{Z}$, zeroes of q-sine and q-cosine functions are $k\pi$ and $\frac{(2k+1)\pi}{2}$, respectively.*

Proof. Since sin_q is an odd function, therefore its enough to prove the lemma for positive value of k. We prove the lemma for positive value of k by induction. For $k = 1$ and using (15), we have

$$sin_q(\pi) = sin_q(0 + \pi) = sin_q(0) = 0,$$

since $sin_q(0) = 0$ comes from definition of sin_q. Then lemma is valid for $k = 1$. Assume that $sin_q(n\pi) = 0$ is true. We need to show that $sin_q((n+1)\pi) = 0$ is also true. By using (15), we have

$$sin_q((n+1)\pi) = sin_q(n\pi + \pi) = (-1)sin_q(n\pi) = 0.$$

Therefore zeroes of $sin_q(x)$ are $k\pi$, for all $k \in \mathbb{Z}$. About the zeroes of $cos_q(x)$, take any arbitrary $k \in \mathbb{Z}$, and by using the (7), we have

$$cos_q(\frac{(2k+1)\pi}{2}) = cos_q(k\pi + \frac{\pi}{2}) = sin_q(-k\pi) = 0.$$

Therefore zeroes of $cos_q(x)$ are $\frac{(2k+1)\pi}{2}$, for all $k \in \mathbb{Z}$.

□

Lemma 4. *For all $z \in \mathbb{C}$, the following equation holds*

$$(q^{z+1};q)_z = (-q^{\frac{1}{2}};q^{\frac{1}{2}})_{2z}(q^{\frac{1}{2}};q)_z.$$

4

Proof. Taking $a = 1$, $\alpha = 2z$ and substituting q with $q^{\frac{1}{2}}$ in (12), and applying to $(-q^{\frac{1}{2}}; q^{\frac{1}{2}})_{2z}$, we obtain

$$(-q^{\frac{1}{2}}; q^{\frac{1}{2}})_{2z} \frac{(q^{\frac{1}{2}}; q)_z}{(q^{z+1}; q)_z} = \frac{(q; q)_{2z}}{(q^{\frac{1}{2}}; q^{\frac{1}{2}})_{2z}} \frac{(q^{\frac{1}{2}}; q)_z}{(q^{z+1}; q)_z}. \tag{20}$$

By using (2), the right hand side of (20) can be written as

$$\begin{aligned} \frac{(q; q)_{2z}}{(q^{z+1}; q)_z} \frac{(q^{\frac{1}{2}}; q)_z}{(q^{\frac{1}{2}}; q^{\frac{1}{2}})_{2z}} &= \frac{\frac{(q;q)_\infty}{(q^{2z+1};q)_\infty}}{\frac{(q^{z+1};q)_\infty}{(q^{2z+1};q)_\infty}} \frac{(q^{\frac{1}{2}}; q)_z}{(q^{\frac{1}{2}}; q^{\frac{1}{2}})_{2z}}, \\ &= \frac{(q; q)_\infty}{(q^{z+1}; q)_\infty} \frac{(q^{\frac{1}{2}}; q)_z}{(q^{\frac{1}{2}}; q^{\frac{1}{2}})_{2z}}, \\ &= (q; q)_z \frac{(q^{\frac{1}{2}}; q)_z}{(q^{\frac{1}{2}}; q^{\frac{1}{2}})_{2z}}. \end{aligned} \tag{21}$$

By substituting q with $q^{\frac{1}{2}}$ and then taking $a = q^{\frac{1}{2}}$ in equation (13), one can see that the right hand side of (21) is equal 1 and this completes the proof. □

Lemma 5. *For all $z \in \mathbb{C}$, the following equation holds*

$$(q^{\frac{1}{2}-z}; q)_z = (q^{z+1}; q)_z \frac{q^{-\frac{z^2}{2}}}{(-q^{\frac{1}{2}}; q^{\frac{1}{2}})_{2z}} cos_{\sqrt{q}}(\pi z). \tag{22}$$

Proof. By using the Lemma 4 the equation (22) can be written as

$$(q^{\frac{1}{2}-z}; q)_z = (q^{\frac{1}{2}}; q)_z q^{-\frac{z^2}{2}} cos_{\sqrt{q}}(\pi z). \tag{23}$$

The equation (23) is a special case of lemma 2 when $\beta = \frac{1}{2}$, since $cos_q(z) = sin_q(\frac{\pi}{2} - z)$ and also cos_q is an even function. □

Corollary 1. *For the positive integers value of n, Lemma 5 deduce to*

$$(q^{\frac{1}{2}-n}; q)_n = (-1)^n (q^{n+1}; q)_n \frac{q^{-\frac{n^2}{2}}}{(-q^{\frac{1}{2}}; q^{\frac{1}{2}})_{2n}}.$$

Proof. The result is obtained by using Theorem 1. □

Euler (see [2], page 271 or [3], page 222) found the following formula in connection with partitions,

$$(-q; q)_\infty (q; q^2)_\infty = 1.$$

In the next lemma, we want to generalize this Euler's formula.

Theorem 2. *For all $z \in \mathbb{C}$, the following equation holds*

$$(q^{z+1};q)_z = (-q;q)_z(q;q^2)_z.$$

Proof. By substituting q with $q^{\frac{1}{2}}$ and then taking $a = -q^{\frac{1}{2}}$ in equation (13), we have

$$(-q^{\frac{1}{2}};q^{\frac{1}{2}})_{2z} = (-q^{\frac{1}{2}};q)_z(-q;q)_z,$$

now, we apply the result to Lemma 5 and obtain

$$(q^{z+1};q)_z = (-q^{\frac{1}{2}};q)_z(-q;q)_z(q^{\frac{1}{2}};q)_z. \tag{24}$$

Taking $a = \frac{1}{2}$ in the equation (12) and then applying to the right hand side of the equation (24) completes the proof.

□

Theorem 3. *For all $x \in \mathbb{C}$, the following equation holds,*

$$\Gamma_q(2x)\Gamma_q(\frac{1}{2}) = \Gamma_q(x)\Gamma_q(x+\frac{1}{2})(-q^{\frac{1}{2}};q^{\frac{1}{2}})_{2x-1}.$$

Proof. By using the definition of q-Gamma function (3) and then applying the equation (2), we can write

$$\frac{\Gamma_q(2x)\Gamma_q(\frac{1}{2})}{\Gamma_q(x)\Gamma_q(x+\frac{1}{2})} = \frac{(q^x;q)_\infty(q^{x+\frac{1}{2}};q)_\infty}{(q^{2x};q)_\infty(q^{\frac{1}{2}};q)_\infty} = \frac{(q^x;q)_x}{(q^{\frac{1}{2}};q)_x}, \tag{25}$$

the last equation holds since $(q^x;q)_x = \frac{(q^x;q)_\infty}{(q^{2x};q)_\infty}$ and $(q^{\frac{1}{2}};q)_x = \frac{(q^{\frac{1}{2}};q)_\infty}{(q^{\frac{1}{2}+x};q)_\infty}$. Taking $\beta = \frac{1}{2}$ in Lemma 2 and applying for the denominator of the last fraction in (25), we get

$$\begin{aligned}\frac{(q^x;q)_x}{(q^{\frac{1}{2}};q)_x} &= \frac{(q^x;q)_x}{(q^{\frac{1}{2}-x};q)_x}\frac{sin_{\sqrt{q}}\pi(\frac{1}{2}+x)}{sin_{\sqrt{q}}\pi(\frac{1}{2})}\,q^{-\frac{x^2}{2}},\\ &= \frac{(q^x;q)_x}{(q^{\frac{1}{2}-x};q)_x}cos_{\sqrt{q}}(\pi x)\,q^{-\frac{x^2}{2}}.\end{aligned}$$

Now, by using Lemma 4, we have

$$\frac{(q^x;q)_x}{(q^{\frac{1}{2}-x};q)_x}cos_{\sqrt{q}}(\pi x)\,q^{-\frac{x^2}{2}} = \frac{(q^x;q)_x}{(q^{x+1};q)_x}(-q^{\frac{1}{2}};q^{\frac{1}{2}})_{2x}.$$

Making use of (2), we have

$$\frac{(q^x;q)_x}{(q^{x+1};q)_x}(-q^{\frac{1}{2}};q^{\frac{1}{2}})_{2x} = \frac{(q^x;q)_\infty}{(q^{2x};q)_\infty}\frac{(q^{2x+1};q)_\infty}{(q^{x+1};q)_\infty}(-q^{\frac{1}{2}};q^{\frac{1}{2}})_{2x}.$$

After expanding the first and second fractions and then a simplification, yields

$$\begin{aligned}\frac{(q^x;q)_\infty}{(q^{2x};q)_\infty}\frac{(q^{2x+1};q)_\infty}{(q^{x+1};q)_\infty}(-q^{\frac{1}{2}};q^{\frac{1}{2}})_{2x} &= \frac{1-q^x}{1-q^{2x}}(-q^{\frac{1}{2}};q^{\frac{1}{2}})_{2x},\\ &= \frac{1}{1+q^x}(-q^{\frac{1}{2}};q^{\frac{1}{2}})_{2x},\\ &= (-q^{\frac{1}{2}};q^{\frac{1}{2}})_{2x-1}.\end{aligned}$$

□

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APPROXIMATE QUADRATIC FORMS ON RESTRICTED DOMAINS

WON-GIL PARK AND JAE-HYEONG BAE*

Abstract. Let r, s be nonzero real numbers with $r + s = 1$. In [9], Najati and Jung investigated a quadratic functional equation $g(rx + sy) + rs\,g(x - y) = rg(x) + sg(y)$. We introduce a functional equation $f(rx + sy, rz + sw) + rsf(x - y, z - w) = rf(x, z) + sf(y, w)$ and investigate the relation between the above two functional equations. And we find out the general solution and the Hyers-Ulam stability of the latter on restricted domains.

1. Introduction

In 1940 and in 1968, Ulam [12] proposed the general Ulam stability problem:

"When is it true that by slightly changing the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true?"

In 1941, Hyers [7] solved this problem for linear mappings. In 1950, Aoki [2] provided a generalization of the Hyers' theorem for additive mappings. This stability concept is also applied to the case of other functional equations. For more results on the stability of functional equations (see [5, 6, 11]). In 1998, S.-M. Jung [8] investigated the Hyers-Ulam stability for additive and quadratic mappings on restricted domains.

Let X and Y be real vector spaces. For a mapping $g : X \to Y$, consider the quadratic functional equation:

$$g(x + y) + g(x - y) = 2g(x) + 2g(y). \tag{1.1}$$

In 1989, J. Aczel [1] solved the solution of the equation (1.1). Later, many different quadratic functional equations were solved by numerous authors [3, 8, 10]. In recent, A. Najati and S.-M. Jung [9] introduced a generalized quadratic functional equation

$$g(rx + sy) + rs\,g(x - y) = rg(x) + sg(y), \tag{1.2}$$

where r, s are nonzero real numbers with $r + s = 1$. In 2007, the authors [4] solved the solution of the 2-variable quadratic functional equation

$$f(x + y, z + w) + f(x - y, z - w) = 2f(x, z) + 2f(y, w). \tag{1.3}$$

Consider a generalized 2-variable quadratic functional equation

$$f(rx + sy, rz + sw) + rsf(x - y, z - w) = rf(x, z) + sf(y, w), \tag{1.4}$$

where r, s are nonzero real numbers with $r + s = 1$.

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In this paper, we investigate the relation between (1.2) and (1.4) by the same method as the proofs of Theorem 1 and Theorem 2 in [4]. And we find out the general solution and the Hyers-Ulam stability of (1.4) in the spirit of Najati and Jung [9].

2. **Relation between (1.2) and (1.4)**

The functional equation (1.4) induces the quadratic functional equation (1.2) as follows.

THEOREM 2.1. *Let $f : X \times X \to Y$ be a mapping satisfying* (1.4) *and let $g : X \to Y$ be the mapping given by*

$$g(x) := f(x, x) \tag{2.1}$$

for all $x \in X$, then g satisfies (1.2).

Proof. By (1.4) and (2.1), we obtain

$$\begin{aligned} g(rx + sy) + rsg(x - y) &= f(rx + sy, rx + sy) + rsf(x - y, x - y) \\ &= rf(x, x) + sf(y, y) \\ &= rg(x) + sg(y) \end{aligned}$$

for all $x, y \in X$. □

EXAMPLE 1. *Let X be a real algebra and $D : X \to X$ a derivation on X. Define a mapping $f : X \times X \to X$ by*

$$f(x, y) := D(xy) = xD(y) + D(x)y$$

for all $x, y \in X$. Then we see that

$$\begin{aligned} &f(rx + sy, rz + sw) + rsf(x - y, z - w) = D[(rx + sy)(rz + sw)] + rsD[(x - y)(z - w)] \\ &= (rx + sy)D(rz + sw) + D(rx + sy)(rz + sw) + rs[(x - y)D(z - w) + D(x - y)(z - w)] \\ &= (rx + sy)[rD(z) + sD(w)] + [rD(x) + sD(y)](rz + sw) \\ &\quad + rs\big((x - y)[D(z) - D(w)] + [D(x) - D(y)](z - w)\big) \\ &= r^2xD(z) + s^2yD(w) + r^2D(x)z + s^2D(y)w + rsxD(z) + rsyD(w) + rsD(x)z + rsD(y)w \\ &= r[xD(z) + D(x)z] + s[yD(w) + D(y)w] = rD(xz) + sD(yw) = rf(x, z) + sf(y, w) \end{aligned}$$

for all $x, y, z, w \in X$. Thus f satisfies (1.4)*. Define a mapping $g : X \to X$ by*

$$g(x) := D(x^2) = xD(x) + D(x)x$$

for all $x \in X$. Then g satisfies (2.1)*. By Theorem* 2.1*, g satisfies* (1.2)*.*

The quadratic functional equation (1.2) induces the functional equation (1.4) with an additional condition.

THEOREM 2.2. *Let $a, b, c \in \mathbb{R}$ and $g : X \to Y$ be a mapping satisfying* (1.2)*. If $f : X \times X \to Y$ is the mapping given by*

$$f(x, y) := ag(x) + \frac{b}{4}[g(x + y) - g(x - y)] + cg(y) \tag{2.2}$$

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for all $x, y \in X$, then f satisfies (1.4). *Furthermore,* (2.1) *holds if r is a rational number and* $a+b+c=1$.

Proof. By (1.2) and (2.2), we see that

$$\begin{aligned}
&f(rx+sy, rz+sw) + rsf(x-y, z-w)\\
&= ag(rx+sy) + \frac{b}{4}\big[g\big(r(x+z)+s(y+w)\big) - g\big(r(x-z)+s(y-w)\big)\big] + cg(rz+sw)\\
&\quad + rs\Big(ag(x-y) + \frac{b}{4}\big[g(x-y+z-w) - g(x-y-z+w)\big] + cg(z-w)\Big)\\
&= ag(rx+sy) + rsag(x-y) + \frac{b}{4}\big[g\big(r(x+z)+s(y+w)\big) + rsg\big((x+z)-(y+w)\big)\big]\\
&\quad - \frac{b}{4}\big[g\big(r(x-z)+s(y-w)\big) + rsg\big((x-z)-(y-w)\big)\big] + cg(rz+sw) + rscg(z-w)\\
&= a\big[g(rx+sy) + rsg(x-y)\big] + \frac{b}{4}\big[rg(x+z) + sg(y+w)\big]\\
&\quad - \frac{b}{4}\big[rg(x-z) + sg(y-w)\big] + c\big[g(rz+sw) + rsg(z-w)\big]\\
&= a[rg(x) + sg(y)] + \frac{b}{4}\big(r\big[g(x+z) - g(x-z)\big] + s\big[g(y+w) - g(y-w)\big]\big) + c[rg(z) + sg(w)]\\
&= rf(x,z) + sf(y,w)
\end{aligned}$$

for all $x, y, z, w \in X$.

Let r be a rational number. Since g satisfies (1.2), it also satisfies (1.1) (see Theorem 2.3. in [9]). Letting $x = y = 0$ and $y = x$ in (1.1), respectively,

$$g(0) = 0 \quad \text{and} \quad g(2x) = 4g(x)$$

for all $x \in X$. By (2.2) and the above two equalities,

$$\begin{aligned}
f(x,x) &= ag(x) + \frac{b}{4}[g(2x) - g(0)] + cg(x)\\
&= (a+b+c)g(x)\\
&= g(x)
\end{aligned}$$

for all $x \in X$. □

Example 2. *Consider the function $g : \mathbb{R}^2 \to \mathbb{R}$ given by $g(\mathbf{x}) := \mathbf{x}^T A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^2$, where A is a 2×2 real matrix. Then we see that*

$$\begin{aligned}
g(r\mathbf{x}+s\mathbf{y}) + rs\,g(\mathbf{x}-\mathbf{y}) &= (r\mathbf{x}+s\mathbf{y})^T A(r\mathbf{x}+s\mathbf{y}) + rs(\mathbf{x}-\mathbf{y})^T A(\mathbf{x}-\mathbf{y})\\
&= (r\mathbf{x}^T + s\mathbf{y}^T)A(r\mathbf{x}+s\mathbf{y}) + rs(\mathbf{x}^T - \mathbf{y}^T)A(\mathbf{x}-\mathbf{y})\\
&= r^2\mathbf{x}^T A\mathbf{x} + rs(\mathbf{x}^T A\mathbf{y} + \mathbf{y}^T A\mathbf{x}) + s^2\mathbf{y}^T A\mathbf{y} + rs(\mathbf{x}^T A\mathbf{x} - \mathbf{x}^T A\mathbf{y} - \mathbf{y}^T A\mathbf{x} + \mathbf{y}^T A\mathbf{y})\\
&= r(r+s)\mathbf{x}^T A\mathbf{x} + s(r+s)\mathbf{y}^T A\mathbf{y} = rg(\mathbf{x}) + sg(\mathbf{y})
\end{aligned}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$*, where* r, s *are nonzero real numbers with* $r + s = 1$*. Thus* g *satisfies* (1.2)*. Let* $a, b, c \in \mathbb{R}$ *and define* $f(\mathbf{x}, \mathbf{y}) := ag(\mathbf{x}) + \frac{b}{4}[g(\mathbf{x}+\mathbf{y}) - g(\mathbf{x}-\mathbf{y})] + cg(\mathbf{y})$ *for all* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$*. By Theorem* 2.2*, the function* f *satisfies* (1.4)*. In fact,*

$$f(\mathbf{x}, \mathbf{y}) = \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}^T \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix} \begin{pmatrix} A\mathbf{x} \\ A\mathbf{y} \end{pmatrix}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$*.*

Example 3. *Let* M_n *be the algebra of* $n \times n$ *real matrices. Consider the mapping* $g : M_n \to M_n$ *given by* $g(A) := A^2$ *for all* $A \in M_n$*. Then we see that*

$$\begin{aligned} g(rA + sB) + rs\, g(A - B) &= (rA + sB)^2 + rs(A - B)^2 \\ &= r^2A^2 + rs(AB + BA) + s^2B^2 + rs(A^2 - AB - BA + B^2) \\ &= r^2A^2 + rsAB + rsBA + s^2B^2 + rsA^2 - rsAB - rsBA + rsB^2 \\ &= r(r + s)A^2 + s(r + s)B^2 = rg(A) + sg(B) \end{aligned}$$

for all $A, B \in \mathbb{R}^2$*, where* r, s *are nonzero real numbers with* $r + s = 1$*. Thus* g *satisfies* (1.2)*. Let* $a, b, c \in \mathbb{R}$ *and define*

$$f(A, B) := aA^2 + bA \circ B + cB^2,$$

where $A \circ B$ *the Jordan product* $\frac{1}{2}(AB + BA)$ *of* A *and* B *for all* $A, B \in M_n$*. Then the mapping* $f : M_n \times M_n \to M_n$ *satisfies* (2.2)*. By Theorem* 2.2*, the mapping* f *satisfies* (1.4)*.*

3. **Solution of the equation (1.4)**

We recall that r, s are nonzero real numbers with $r + s = 1$. In the following theorem, we find out the general solution of the functional equation (1.4).

Theorem 3.1. *Let* $f : X \times X \to Y$ *be a mapping such that* $f(x, y) = f(-x, -y)$ *for all* $x, y \in X$*. Then* f *satisfies* (1.3) *if it satisfies* (1.4)*. If* r *and* s *are rational numbers and* f *satisfies* (1.3)*, then it also satisfies* (1.4)*.*

Proof. Letting $x = y = z = w = 0$ in (1.4), we gain $f(0, 0) = 0$. Putting $y = w = 0$ in (1.4), we get $f(rx, rz) = r^2f(x, z)$ for all $x, z \in X$. Replacing x by $x + y$ and z by $z + w$ in (1.4), we have

$$f(rx + y, rz + w) = rf(x + y, z + w) + sf(y, w) - rsf(x, z) \tag{3.1}$$

for all $x, y, z, w \in X$. Replacing y by $-y$ and w by $-w$ in (3.1), we obtain

$$f(rx - y, rz - w) = rf(x - y, z - w) + sf(y, w) - rsf(x, z)$$

for all $x, y, z, w \in X$. Adding (3.1) to the above equation, we see that

$$f(rx+y, rz+w)+f(rx-y, rz-w) = r[f(x+y, z+w)+f(x-y, z-w)]+2sf(y, w)-2rsf(x, z) \tag{3.2}$$

for all $x, y, z, w \in X$. Replacing y by $x + ry$ and w by $z + rw$ in (3.1), we obtain

$$f(r(x + y) + x, r(z + w) + z) = rf(2x + ry, 2z + rw) + sf(x + ry, z + rw) - rsf(x, z) \tag{3.3}$$

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for all $x, y, z, w \in X$. Replacing x, y, z, w by $2x, ry, 2z, rw$ in (3.1), respectively, we obtain

$$rf(2x+ry, 2z+rw) = r^2 f(2x+y, 2z+w) - r^2 sf(y,w) + rsf(2x, 2z) \tag{3.4}$$

for all $x, y, z, w \in X$. Replacing y by ry and w by rw in (3.1), we obtain

$$sf(x+ry, z+rw) = rsf(x+y, z+w) - rs^2 f(y,w) + s^2 f(x,z) \tag{3.5}$$

for all $x, y, z, w \in X$. Replacing x, y, z, w by $x+y, x, z+w, z$ in (3.1), respectively, we obtain

$$f(r(x+y)+x, r(z+w)+z) = rf(2x+y, 2z+w) + sf(x,z) - rsf(x+y, z+w) \tag{3.6}$$

for all $x, y, z, w \in X$. By (3.3), (3.4), (3.5) and (3.6), we see that

$$f(2x+y, 2z+w) + 2f(x,z) + f(y,w) = 2f(x+y, z+w) + f(2x, 2z) \tag{3.7}$$

for all $x, y, z, w \in X$. Putting $y = -x$ and $w = -z$ in (3.7), we get $f(2x, 2z) = 4f(x,z)$ for all $x, z \in X$. Therefore, it follows from (3.7) that

$$f(2x+y, 2z+w) + f(y,w) = 2f(x+y, z+w) + 2f(x,z)$$

for all $x, y, z, w \in X$. Replacing y by $y-x$ and w by $w-z$ in the above equation, we have

$$f(x+y, z+w) + f(y-x, w-z) = 2f(x,z) + 2f(y,w)$$

for all $x, y, z, w \in X$. Hence f satisfies (1.3).

Conversely, let r and s be rational numbers and let f satisfy (1.3). Then there exist two symmetric bi-additive mappings $S_1, S_2 : X \times X \to Y$ and a bi-additive mapping $B : X \times X \to Y$ such that $f(x,y) = S_1(x,x) + B(x,y) + S_2(y,y)$ for all $x, y \in X$ (see [4]). Since r and s are rational numbers,

$$\begin{aligned}
&rf(x,z) + sf(y,w) - rsf(x-y, z-w) \\
&\quad = r^2 S_1(x,x) + 2rsS_1(x,y) + s^2 S_1(y,y) + r^2 B(x,z) + rsB(x,w) + rsB(y,z) + s^2 B(y,w) \\
&\qquad + r^2 S_2(z,z) + 2rsS_2(z,w) + s^2 S_2(w,w) \\
&\quad = S_1(rx,rx) + 2S_1(rx,sy) + S_1(sy,sy) + B(rx,rz) + B(rx,sw) + B(sy,rz) + B(sy,sw) \\
&\qquad + S_2(rz,rz) + 2S_2(rz,sw) + S_2(sw,sw) \\
&\quad = S_1(rx+sy, rx+sy) + B(rx+sy, rz+sw) + S_2(rz+sw, rz+sw) \\
&\quad = f(rx+sy, rz+sw)
\end{aligned}$$

for all $x, y, z, w \in X$. Therefore f satisfies (1.4). □

4. **Stability of the equation (1.4)**

From now on, let X be a real normed space and Y a Banach space.

The authors proved a generalized Hyers-Ulam stability theorem on a functional equation (1.3). The following theorem is a particular case of Theorem 4 in [4].

THEOREM 4.1 *Let $\delta \geq 0$ be fixed. If a mapping $f : X \times X \to Y$ satisfies the inequality*

$$\|f(x+y, z+w) + f(x-y, z-w) - 2f(x,z) - 2f(y,w)\| \leq \delta \tag{4.1}$$

for all $x, y, z, w \in X$, then there exists a unique 2-variable quadratic mapping $F : X \times X \to Y$ such that $\|f(x,y) - F(x,y)\| \leq \frac{1}{3}\delta$ for all $x, y \in X$.

Using a similar method used in the paper [8], we obtain the following theorem.

THEOREM 4.2 *Let $d > 0$ and $\delta \geq 0$ be fixed and let $X \neq \{0\}$. If a mapping $f : X \times X \to Y$ satisfies the inequality (4.1) for all $x, y, z, w \in X$ with $\|x+z\|+\|y+w\| \geq d$, then there exists a unique 2-variable quadratic mapping $F : X \times X \to Y$ such that*

$$\|f(x,y) - F(x,y)\| \leq \frac{5}{3}\delta \tag{4.2}$$

for all $x, y \in X$.

Proof. Assume that $\|x+z\| + \|y+w\| < d$. Let

$$t = \frac{1}{2}\Big(1 + \frac{d}{\|x+z\|}\Big)(x+z) \qquad \text{if} \qquad \|x+z\| \geq \|y+w\|;$$
$$t = \frac{1}{2}\Big(1 + \frac{d}{\|y+w\|}\Big)(y+w) \qquad \text{if} \qquad \|x+z\| < \|y+w\|.$$

If $x+z = y+w = 0$, then one can choose a $t \in X$ with $\|t\| = \frac{d}{2}$. Note that

$$2\|t\| = \|x+z\| + d \geq d \qquad \text{if} \qquad \|x+z\| \geq \|y+w\|;$$
$$2\|t\| = \|y+w\| + d > d \qquad \text{if} \qquad \|x+z\| < \|y+w\|.$$

Clearly, we see that

$$\begin{aligned}
&\|x+z-2t\| + \|y+w+2t\| \geq 4\|t\| - (\|x+z\| + \|y+w\|) \geq 2d - (\|x+z\| + \|y+w\|) \\
&\qquad\qquad\qquad\qquad\qquad\quad \geq 2d > d, \\
&\|x+z-y-w\| + 4\|t\| \geq \|x+z-y-w\| + 2d \geq 2d > d, \\
&\|x+z+2t\| + \|-y-w+2t\| \geq \max\{\|x+z+2t\|, \|-y-w+2t\|\} \geq d, \\
&\|x+z\| + 2\|t\| \geq 2\|t\| \geq d, \qquad 2\|t\| + \|y+w\| \geq 2\|t\| \geq d, \qquad 4\|t\| \geq 2d > d.
\end{aligned} \tag{4.3}$$

These inequalities (4.3) come from the corresponding substitutions attached between the right-hand sided parentheses of the following functional identity.

Besides from (4.1) with $x = y = z = w = 0$ we get $\|f(0,0)\| \leq \frac{\delta}{2}$. Therefore from (4.1), (4.3) and the new functional identity

$$\begin{aligned}
&2[f(x+y, z+w) + f(x-y, z-w) - 2f(x,z) - 2f(y,w) - f(0,0)] \\
&\quad = [f(x+y, z+w) + f(x-y-2t, z-w-2t) - 2f(x-t, z-t) - 2f(y+t, w+t)] \\
&\qquad - [f(x-y-2t, z-w-2t) + f(x-y+2t, z-w+2t) - 2f(x-y, z-w) - 2f(2t, 2t)] \\
&\qquad + [f(x-y+2t, z-w+2t) + f(x+y, z+w) - 2f(x+t, z+t) - 2f(-y+t, -w+t)] \\
&\qquad + 2[f(x+t, z+t) + f(x-t, z-t) - 2f(x,z) - 2f(t,t)] \\
&\qquad + 2[f(t+y, t+w) + f(t-y, t-w) - 2f(t,t) - 2f(y,w)] \\
&\qquad - 2[f(2t, 2t) + f(0,0) - 4f(t,t)],
\end{aligned}$$

we get

$$2\|f(x+y,z+w)+f(x-y,z-w)-2f(x,z)-2f(y,w)-f(0,0)\| \\ \leq \delta+\delta+\delta+2\delta+2\delta+2\delta=9\delta,$$

or

$$\|f(x+y,z+w)+f(x-y,z-w)-2f(x,z)-2f(y,w)\| \leq \frac{9}{2}\delta+\|f(0,0)\| \leq 5\delta. \tag{4.4}$$

Applying now Theorem 4.1 and the above inequality, there exists a unique 2-variable quadratic mapping $F : X \times X \to Y$ satisfying (4.2) such that $F(x,y) = \lim_{n\to\infty} 2^{-2n} f(2^n x, 2^n y)$, completing the proof. □

We recall that r, s are nonzero real numbers with $r + s = 1$.

THEOREM 4.3. *Let $d > 0$ and $\delta \geq 0$ be given. Assume that a mapping $f : X \times X \to Y$ such that $f(x,y) = f(-x,-y)$ and*

$$\|f(rx+sy,rz+sw)+rsf(x-y,z-w)-rf(x,z)-sf(y,w)\| \leq \delta \tag{4.5}$$

for all $x,y,z,w \in X$ with $\|x+z\|+\|y+w\| \geq d$. Then there exists $K > 0$ such that f satisfies

$$\|f(x+y,z+w)+f(x-y,z-w)-2f(x,z)-2f(y,w)\| \leq \frac{4(2+|r|+|s|)}{|rs|}\delta \tag{4.6}$$

for all $x,y,z,w \in X$ with $\|x+z\|+\|y+w\| \geq K$.

Proof. Let $x,y,z,w \in X$ with $\|x+z\|+\|y+w\| \geq 2d$. Since $2\|y+w\| = \|x+y+z+w+y+w-x-z\| \leq \|x+y+z+w\|+\|y+w\|+\|x+z\|$, we get

$$2\|y+w\|-\|x+z\| \leq \|x+y+z+w\|+\|y+w\|.$$

Since $\|x+z\| = \|x+y+z+w-y-w\| \leq \|x+y+z+w\|+\|y+w\|$, we have

$$\max\{\|x+z\|, 2\|y+w\|-\|x+z\|\} \leq \|x+y+z+w\|+\|y+w\|. \tag{4.7}$$

If $\|x+z\| < d$, then, since $\|x+z\|+\|y+w\| \geq 2d$, we get $2\|y+w\| > 2d = d+d > d+\|x+z\|$ and $2\|y+w\|-\|x+z\| > d$. So we have

$$\max\{\|x+z\|, 2\|y+w\|-\|x+z\|\} \geq d. \tag{4.8}$$

By (4.7) and (4.8), we have $\|x+y+z+w\|+\|y+w\| \geq d$. So it follows from (4.5) that

$$\|f(rx+y,rz+w)+rsf(x,z)-rf(x+y,z+w)-sf(y,w)\| \leq \delta \tag{4.9}$$

for all $x,y,z,w \in X$ with $\|x+z\|+\|y+w\| \geq 2d$. So

$$\|f(ry+x,rw+z)+rsf(y,w)-rf(x+y,z+w)-sf(x,z)\| \leq \delta \tag{4.10}$$

for all $x,y,z,w \in X$ with $\|x+z\|+\|y+w\| \geq 2d$.

Let $x,y,z,w \in X$ with $\|x+z\|+\|y+w\| \geq 4d\big(1/|r|+\big|1-1/|r|\big|\big)$. If $\|y+w\| > 2d/|r|$, then

$$\|x+z\|+\|x+ry+z+rw\| \geq |r|(\|y+w\|) \geq 2d. \tag{4.11}$$

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If $\|y+w\| \le 2d/|r|$, then $\|x+z\| \ge 2d(1/|r|+2|1-1/|r||)$ and

$$\|x+z\|+\|x+ry+z+rw\| \ge 2\|x+z\|-|r|\cdot\|y+w\| \ge \left(\frac{2}{|r|}+4\left|1-\frac{1}{|r|}\right|-1\right) \ge 2d. \tag{4.12}$$

Therefore we get that $\|x+z\|+\|x+ry+z+rw\| \ge 2d$ from (4.11) and (4.12). Hence by (4.9) we have

$$\|f(r(x+y)+x, r(z+w)+z)+rsf(x,z)-rf(2x+ry,2z+rw)-sf(x+ry,z+rw)\| \le \delta \tag{4.13}$$

for all $x,y,z,w \in X$ with $\|x+z\|+\|y+w\| \ge 4d\big(1/|r|+\big|1-1/|r|\big|\big)$. Set $M := 4d\big(1/|r|+\big|1-1/|r|\big|\big)$. Then

$$\|x+y+z+w\|+\|x+z\| \ge \frac{M}{2} \ge 2d, \quad \|2x+2z\|+\|y+w\| \ge M \ge 4d \tag{4.14}$$

for all $x,y,z,w \in X$ with $\|x+z\|+\|y+w\| \ge M$. From (4.9) and (4.10), we get the following inequalities:

$$\|f(r(x+y)+x, r(z+w)+z)+rsf(x+y,z+w)-rf(2x+y,2z+w)-sf(x,z)\| \le \delta,$$
$$\|rf(ry+2x, rw+2z)+r^2sf(y,w)-r^2f(2x+y,2z+w)-rsf(2x,2z)\| \le \delta|r|,$$
$$\|sf(ry+x, rw+z)+rs^2f(y,w)-rsf(x+y,z+w)-s^2f(x,z)\| \le \delta|s|.$$

Using (4.13) and the above three inequalities, we get

$$\|f(2x+y,2z+w)+2f(x,z)+f(y,w)-2f(x+y,z+w)-f(2x,2z)\| \le \frac{2+|r|+|s|}{|rs|}\delta \tag{4.15}$$

for all $x,y,z,w \in X$ with $\|x+z\|+\|y+w\| \ge M$. If $x,y,z,w \in X$ with $\|x+z\|+\|y+w\| \ge 2M$, then $\|x+z\|+\|y+w-x-z\| \ge M$. So it follows from (4.15) that

$$\|f(x+y,z+w)+2f(x,z)+f(y-x,w-z)-2f(y,w)-f(2x,2z)\| \le \frac{2+|r|+|s|}{|rs|}\delta \tag{4.16}$$

for all $x,y,z,w \in X$ with $\|x+z\|+\|y+w\| \ge 2M$.

Letting $y=0$ and $w=0$ in (4.16), we get

$$\|4f(x,z)-f(2x,2z)-2f(0,0)\| \le \frac{2+|r|+|s|}{|rs|}\delta \tag{4.17}$$

for all $x,z \in X$ with $\|x+z\| \ge 2M$. Letting $x=0$ and $z=0$ (and $y,w \in X$ with $\|y\| \ge 2M$, $\|w\| \ge 2M$) in (4.16), we get

$$\|f(0,0)\| \le \big((2+|r|+|s|)/|rs|\big)\delta. \tag{4.18}$$

Therefore it follows from (4.16), (4.17) and (4.18) that

$$\begin{aligned}\|f(x+y,&z+w)+f(y-x,w-z)-2f(x,z)-2f(y,w)\| \\ &\le \|f(x+y,z+w)+2f(x,z)+f(y-x,w-z)-2f(y,w)-f(2x,2z)\| \\ &\qquad + \|4f(x,z)-f(2x,2z)-2f(0,0)\|+2\|f(0,0)\| \\ &\le \frac{4(2+|r|+|s|)}{|rs|}\delta\end{aligned}$$

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for all $x, y, z, w \in X$ with $\|x + z\| \geq 2M$. Since $f(x, y) = f(-x, -y)$ for all $x, y \in X$, the above inequality holds for all $x, y, z, w \in X$ with $\|y + w\| \geq 2M$. Therefore

$$\|f(x+y, z+w) + f(y-x, w-z) - 2f(x, z) - 2f(y, w)\| \leq \frac{4(2+|r|+|s|)}{|rs|}\delta$$

for all $x, y, z, w \in X$ with $\|x+z\| + \|y+w\| \geq 4M$. This completes the proof by letting $K := 4M$. □

Theorem 4.4 *Let $d > 0$ and $\delta \geq 0$ be given. Assume that a mapping $f : X \times X \to Y$ such that $f(x, y) = f(-x, -y)$ and (4.5) for all $x, y, z, w \in X$ with $\|x + z\| + \|y + w\| \geq d$. Then there exists $K > 0$ such that f satisfies*

$$\|f(x+y, z+w) + f(x-y, z-w) - 2f(x, z) - 2f(y, w)\| \leq \frac{19(2+|r|+|s|)}{|rs|}\delta$$

for all $x, y, z, w \in X$.

Proof. By Theorem 4.3, there exists $K > 0$ such that f satisfies (4.6) for all $x, y, z, w \in X$ with $\|x + z\| + \|y + w\| > K$. By (4.4) and (4.18), we get that

$$\begin{aligned}\|f(x+y, z+w) + f(x-y, z-w) - 2f(x, z) - 2f(y, w)\| &\leq \frac{18(2+|r|+|s|)}{|rs|}\delta + \|f(0,0)\| \\ &\leq \frac{19(2+|r|+|s|)}{|rs|}\delta\end{aligned}$$

for all $x, y, z, w \in X$. □

Theorem 4.5 *Let $d > 0$ and $\delta \geq 0$ be given. Assume that a mapping $f : X \times X \to Y$ such that (4.5) and $f(x, y) = f(-x, -y)$ for all $x, y, z, w \in X$ with $\|x + z\| + \|y + w\| \geq d$. Then there exists a unique quadratic mapping $F : X \times X \to Y$ such that $F(x, y) = \lim_{n\to\infty} 4^{-n} f(2^n x, 2^n y)$ and*

$$\|f(x, y) - Q(x, y)\| \leq \frac{19(2+|r|+|s|)}{3|rs|}\delta$$

for all $x, y \in X$.

Proof. The result follows from Theorem 4.1 and Theorem 4.4. □

Corollary 4.6. *Let r and s be rational numbers and a mapping $f : X \times X \to Y$ satisfy $f(x, y) = f(-x, -y)$ for all $x, y \in X$. Then f is quadratic if and only if the asymptotic condition*

(4.19)

$$\|f(rx+sy, rz+sw) + rsf(x-y, z-w) - rf(x, z) - sf(y, w)\| \to 0 \quad \text{as} \quad \|x+z\| + \|y+w\| \to \infty$$

holds true.

Proof. The asymptotic condition (4.19) is equivalent to the condition that there exists a sequence $\{\delta_n\}$ monotonically decreasing to 0 such that

(4.20) $$\|f(rx+sy, rz+sw) + rsf(x-y, z-w) - rf(x, z) - sf(y, w)\| \leq \delta_n$$

for all $x, y, z, w \in X$ with $\|x + z\| + \|y + w\| \geq n$.

It follows from (4.20) and Theorem 4.4 that there exists a unique quadratic mapping $Q_n : X \times X \to Y$ such that

$$\|f(x,y) - Q_n(x,y)\| \leq \frac{19(2+|r|+|s|)}{|rs|}\delta_n \tag{4.21}$$

for all $x, y \in X$. Since $\{\delta_n\}$ is monotonically decreasing, the quadratic mapping Q_m satisfies (4.21) for all $m \geq n$. The uniqueness of Q_n implies $Q_m = Q_n$ for all $m \geq n$. By letting $n \to \infty$ in (4.21), we conclude that f is quadratic.

The converse is trivial. □

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