Volume 21, Number 7 December 15, 2016
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of Computational Analysis and Applications

EUDOXUS PRESS,LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE

SCOPE OF THE JOURNAL

An international publication of Eudoxus Press, LLC
(fourteen times annually)

Editor in Chief: George Anastassiou
Department of Mathematical Sciences,
University of Memphis, Memphis, TN 38152-3240, U.S.A
ganastss@memphis.edu
http://www.msci.memphis.edu/~ganastss/jocaaa

The main purpose of "J.Computational Analysis and Applications" is to publish high quality research articles from all subareas of Computational Mathematical Analysis and its many potential applications and connections to other areas of Mathematical Sciences. Any paper whose approach and proofs are computational,using methods from Mathematical Analysis in the broadest sense is suitable and welcome for consideration in our journal, except from Applied Numerical Analysis articles. Also plain word articles without formulas and proofs are excluded. The list of possibly connected mathematical areas with this publication includes, but is not restricted to: Applied Analysis, Applied Functional Analysis, Approximation Theory, Asymptotic Analysis, Difference Equations, Differential Equations, Partial Differential Equations, Fourier Analysis, Fractals, Fuzzy Sets, Harmonic Analysis, Inequalities, Integral Equations, Measure Theory, Moment Theory, Neural Networks, Numerical Functional Analysis, Potential Theory, Probability Theory, Real and Complex Analysis, Signal Analysis, Special Functions, Splines, Stochastic Analysis, Stochastic Processes, Summability, Tomography, Wavelets, any combination of the above, e.t.c.

"J.Computational Analysis and Applications" is a peer-reviewed Journal. See the instructions for preparation and submission of articles to JoCAAA. Assistant to the Editor: Dr.Razvan Mezei,Lenoir-Rhyne University,Hickory,NC 28601, USA.

Journal of Computational Analysis and Applications(JoCAAA) is published by **EUDOXUS PRESS,LLC**,1424 Beaver Trail Drive,Cordova,TN38016,USA,anastassioug@yahoo.com
http://www.eudoxuspress.com. **Annual Subscription Prices**:For USA and Canada,Institutional:Print $700, Electronic OPEN ACCESS. Individual:Print $350. For any other part of the world add $130 more(handling and postages) to the above prices for Print. No credit card payments.

JoCAAA is reviewed and abstracted by AMS Mathematical Reviews,MATHSCI,and Zentralblaat MATH.

Editorial Board
Associate Editors of Journal of Computational Analysis and Applications

Dartmouth College
8000 Cummings Hall,
Hanover, NH 03755-8000
603-646-3843 (X 3546 Secr.)
e-mail:george.cybenko@dartmouth.edu
Approximation Theory and Neural Networks

Sever S. Dragomir
School of Computer Science and Mathematics, Victoria University,
PO Box 14428,
Melbourne City,
MC 8001, AUSTRALIA
Tel. +61 3 9688 4437
Fax +61 3 9688 4050
sever.dragomir@vu.edu.au
Inequalities, Functional Analysis, Numerical Analysis, Approximations, Information Theory, Stochastics.

Oktay Duman
TOBB University of Economics and Technology,
Department of Mathematics, TR-06530,
Ankara, Turkey,
oduman@etu.edu.tr
Classical Approximation Theory, Summability Theory, Statistical Convergence and its Applications

Saber N. Elaydi
Department Of Mathematics
Trinity University
715 Stadium Dr.
San Antonio, TX 78212-7200
210-736-8246
e-mail: selaydi@trinity.edu
Ordinary Differential Equations, Difference Equations

Christodoulos A. Floudas
Department of Chemical Engineering
Princeton University
Princeton,NJ 08544-5263
609-258-4595(x4619 assistant)
e-mail: floudas@titan.princeton.edu
Optimization Theory&Applications, Global Optimization

J .A. Goldstein
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152
901-678-3130
jgoldste@memphis.edu
Partial Differential Equations, Semigroups of Operators

H. H. Gonska
Department of Mathematics
University of Duisburg
Duisburg, D-47048
Germany
011-49-203-379-3542
e-mail: heiner.gonska@uni-due.de
Approximation Theory, Computer Aided Geometric Design

John R. Graef
Department of Mathematics
University of Tennessee at Chattanooga
Chattanooga, TN 37304 USA
John-Graef@utc.edu
Ordinary and functional differential equations, difference equations, impulsive systems, differential inclusions, dynamic equations on time scales, control theory and their applications

Weimin Han
Department of Mathematics
University of Iowa
Iowa City, IA 52242-1419
319-335-0770
e-mail: whan@math.uiowa.edu
Numerical analysis, Finite element method, Numerical PDE, Variational inequalities, Computational mechanics

Tian-Xiao He
Department of Mathematics and Computer Science
P.O. Box 2900, Illinois Wesleyan University
Bloomington, IL 61702-2900, USA
Tel (309)556-3089
Fax (309)556-3864
the@iwu.edu
Approximations, Wavelet, Integration Theory, Numerical Analysis, Analytic Combinatorics

Margareta Heilmann
Faculty of Mathematics and Natural Sciences, University of Wuppertal
Gaußstraße 20
D-42119 Wuppertal, Germany,
heilmann@math.uni-wuppertal.de

Instructions to Contributors
Journal of Computational Analysis and Applications

An international publication of Eudoxus Press, LLC, of TN.

Editor in Chief: George Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152-3240, U.S.A.

1. Manuscripts files in Latex and PDF and in English, should be submitted via email to the Editor-in-Chief:

Prof.George A. Anastassiou
Department of Mathematical Sciences
The University of Memphis
Memphis,TN 38152, USA.
Tel. 901.678.3144
e-mail: ganastss@memphis.edu

Authors may want to recommend an associate editor the most related to the submission to possibly handle it.

Also authors may want to submit a list of six possible referees, to be used in case we cannot find related referees by ourselves.

2. Manuscripts should be typed using any of TEX,LaTEX,AMS-TEX,or AMS-LaTEX and according to EUDOXUS PRESS, LLC. LATEX STYLE FILE. (Click HERE to save a copy of the style file.)They should be carefully prepared in all respects. Submitted articles should be brightly typed (not dot-matrix), double spaced, in ten point type size and in 8(1/2)x11 inch area per page. Manuscripts should have generous margins on all sides and should not exceed 24 pages.

3. Submission is a representation that the manuscript has not been published previously in this or any other similar form and is not currently under consideration for publication elsewhere. A statement transferring from the authors(or their employers,if they hold the copyright) to Eudoxus Press, LLC, will be required before the manuscript can be accepted for publication.The Editor-in-Chief will supply the necessary forms for this transfer.Such a written transfer of copyright,which previously was assumed to be implicit in the act of submitting a manuscript,is necessary under the U.S.Copyright Law in order for the publisher to carry through the dissemination of research results and reviews as widely and effective as possible.

4. The paper starts with the title of the article, author's name(s) (no titles or degrees), author's affiliation(s) and e-mail addresses. The affiliation should comprise the department, institution (usually university or company), city, state (and/or nation) and mail code.

The following items, 5 and 6, should be on page no. 1 of the paper.

5. An abstract is to be provided, preferably no longer than 150 words.

6. A list of 5 key words is to be provided directly below the abstract. Key words should express the precise content of the manuscript, as they are used for indexing purposes.

The main body of the paper should begin on page no. 1, if possible.

7. All sections should be numbered with Arabic numerals (such as: 1. INTRODUCTION) .
Subsections should be identified with section and subsection numbers (such as 6.1. Second-Value Subheading).
If applicable, an independent single-number system (one for each category) should be used to label all theorems, lemmas, propositions, corollaries, definitions, remarks, examples, etc. The label (such as Lemma 7) should be typed with paragraph indentation, followed by a period and the lemma itself.

8. Mathematical notation must be typeset. Equations should be numbered consecutively with Arabic numerals in parentheses placed flush right, and should be thusly referred to in the text [such as Eqs.(2) and (5)]. The running title must be placed at the top of even numbered pages and the first author's name, et al., must be placed at the top of the odd numbed pages.

9. Illustrations (photographs, drawings, diagrams, and charts) are to be numbered in one consecutive series of Arabic numerals. The captions for illustrations should be typed double space. All illustrations, charts, tables, etc., must be embedded in the body of the manuscript in proper, final, print position. In particular, manuscript, source, and PDF file version must be at camera ready stage for publication or they cannot be considered.

Tables are to be numbered (with Roman numerals) and referred to by number in the text. Center the title above the table, and type explanatory footnotes (indicated by superscript lowercase letters) below the table.

10. List references alphabetically at the end of the paper and number them consecutively. Each must be cited in the text by the appropriate Arabic numeral in square brackets on the baseline.
References should include (in the following order):
initials of first and middle name, last name of author(s)
title of article,

name of publication, volume number, inclusive pages, and year of publication.

Authors should follow these examples:

Journal Article

1. H.H.Gonska,Degree of simultaneous approximation of bivariate functions by Gordon operators, (journal name in italics) *J. Approx. Theory*, 62,170-191(1990).

Book

2. G.G.Lorentz, (title of book in italics) *Bernstein Polynomials* (2nd ed.), Chelsea,New York,1986.

Contribution to a Book

3. M.K.Khan, Approximation properties of beta operators,in(title of book in italics) *Progress in Approximation Theory* (P.Nevai and A.Pinkus,eds.), Academic Press, New York,1991,pp.483-495.

11. All acknowledgements (including those for a grant and financial support) should occur in one paragraph that directly precedes the References section.

12. Footnotes should be avoided. When their use is absolutely necessary, footnotes should be numbered consecutively using Arabic numerals and should be typed at the bottom of the page to which they refer. Place a line above the footnote, so that it is set off from the text. Use the appropriate superscript numeral for citation in the text.

13. After each revision is made please again submit via email Latex and PDF files of the revised manuscript, including the final one.

14. Effective 1 Nov. 2009 for current journal page charges, contact the Editor in Chief. Upon acceptance of the paper an invoice will be sent to the contact author. The fee payment will be due one month from the invoice date. The article will proceed to publication only after the fee is paid. The charges are to be sent, by money order or certified check, in US dollars, payable to Eudoxus Press, LLC, to the address shown on the Eudoxus homepage.

No galleys will be sent and the contact author will receive one (1) electronic copy of the journal issue in which the article appears.

15. This journal will consider for publication only papers that contain proofs for their listed results.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 21, NO.7, 2016, COPYRIGHT 2016 EUDOXUS PRESS, LLC

PRODUCT-TYPE OPERATORS FROM WEIGHTED BERGMAN-ORLICZ SPACES TO BLOCH-ORLICZ SPACES

HONG-BIN BAI AND ZHI-JIE JIANG

Abstract. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk, φ an analytic self-map of $\mathbb{D}$ and ψ an analytic function on $\mathbb{D}$. Let D be the differentiation operator and $W_{\varphi,\psi}$ the weighted composition operator. The boundedness and compactness of the product-type operators $DW_{\varphi,\psi}$ from the weighted Bergman-Orlicz spaces to the Bloch-Orlicz spaces on $\mathbb{D}$ are characterized.

1. Introduction

Let $\mathbb{C}$ be the complex plane, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ the open unit disk and $H(\mathbb{D})$ the class of all analytic functions on $\mathbb{D}$. Let φ be an analytic self-map of $\mathbb{D}$ and $\psi \in H(\mathbb{D})$. Weighted composition operator $W_{\varphi,\psi}$ on $H(\mathbb{D})$ is defined by

$$W_{\varphi,\psi}f(z) = \psi(z)f(\varphi(z)),\ z \in \mathbb{D}.$$

If $\psi \equiv 1$, the operator is reduced to, so called, composition operator and usually denoted by C_φ. If $\varphi(z) = z$, it is reduced to, so called, multiplication operator and usually denoted by M_ψ. A standard problem is to provide function theoretic characterizations when φ and ψ induce a bounded or compact weighted composition operator. Composition operators and weighted composition operators between various spaces of holomorphic functions on different domains have been studied in many papers, see, for example, [1,3,8,11–15,17,19,22,26,27,31,33–36,40,42,48,53,55,60] and the references therein.

Let D be the differentiation operator on $H(\mathbb{D})$, that is,

$$Df(z) = f'(z),\ z \in \mathbb{D}.$$

Operator DC_φ has been studied, for example, in [6, 16, 18, 24, 25, 28, 41, 45, 50]. In [32] Sharma studied the operators $DM_\psi C_\varphi$ and $DC_\varphi M_\psi$ from Bergman spaces to Bloch type spaces. These operators on weighted Bergman spaces were also studied in [58] and [59] by Stević, Sharma and Bhat. If we consider the product-type operator $DW_{\varphi,\psi}$, it is clear that $DM_\psi C_\varphi = DW_{\varphi,\psi}$ and $DC_\varphi M_\psi = DW_{\varphi,\psi\circ\varphi}$. Quite recently, the present author has considered this operator in [7,9]. For some other product-type operators, see, for example, [10,20,21,23,37–39,43,44,46,47,51, 52,54,56,61] and the references therein. This paper is devoted to characterizing the boundedness and compactness of the operators $DW_{\varphi,\psi}$ from the weighted Bergman-Orlicz spaces to the Bloch-Orlicz spaces.

Next we are ready to introduce the needed spaces and some facts in [30]. The function $\Phi \not\equiv 0$ is called a growth function, if it is a continuous and nondecreasing

2000 *Mathematics Subject Classification.* Primary 47B38; Secondary 47B33, 47B37.
Key words and phrases. Weighted Bergman-Orlicz spaces, product-type operators, Bloch-Orlicz spaces, boundedness, compactness.
E-mail address: hbbai@suse.edu.cn, matjzj@126.com.

function from the interval $[0, \infty)$ onto itself. It is clear that these conditions imply that $\Phi(0) = 0$. It is said that the function Φ is of positive upper type (respectively, negative upper type), if there are $q > 0$ (respectively, $q < 0$) and $C > 0$ such that $\Phi(st) \leq Ct^q\Phi(s)$ for every $s > 0$ and $t \geq 1$. By $\mathfrak{U}^q$ we denote the family of all growth functions Φ of positive upper type q ($q \geq 1$), such that the function $t \mapsto \Phi(t)/t$ is nondecreasing on $[0, \infty)$. It is said that function Φ is of positive lower type (respectively, negative upper type), if there are $r > 0$ (respectively, $r < 0$) and $C > 0$ such that $\Phi(st) \leq Ct^r\Phi(s)$ for every $s > 0$ and $0 < t \leq 1$. By $\mathfrak{L}_r$ we denote the family of all growth functions Φ of positive lower type r ($0 < r \leq 1$), such that the function $t \mapsto \Phi(t)/t$ is nonincreasing on $[0, \infty)$. If $f \in \mathfrak{U}^q$, we will also assume that it is convex.

Let $dA(z) = \frac{1}{\pi}dxdy$ be the normalized Lebesgue measure on $\mathbb{D}$. For $\alpha > -1$, let $dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z)$ be the weighted Lebesgue measure on $\mathbb{D}$. Let Φ be a growth function. The weighted Bergman-Orlicz space $A^\Phi_\alpha(\mathbb{D}) := A^\Phi_\alpha$ is the space of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{A^\Phi_\alpha} = \int_{\mathbb{D}} \Phi(|f(z)|)dA_\alpha(z) < \infty.$$

On A^Φ_α is defined the following quasi-norm

$$\|f\|^{lux}_{A^\Phi_\alpha} = \inf\Big\{\lambda > 0 : \int_{\mathbb{D}} \Phi\Big(\frac{|f(z)|}{\lambda}\Big)dA_\alpha(z) \leq 1\Big\}.$$

If $\Phi \in \mathfrak{U}^q$ or $\Phi \in \mathfrak{L}_r$, then the quasi-norm on A^Φ_α is finite and called the Luxembourg norm. The classical weighted Bergman space A^p_α, $p > 0$, corresponds to $\Phi(t) = t^p$, consisting of all $f \in H(\mathbb{D})$ such that

$$\|f\|^p_{A^p_\alpha} = \int_{\mathbb{D}} |f(z)|^p dA_\alpha(z) < \infty.$$

It is well known that for $p \geq 1$ it is a Banach space, while for $0 < p < 1$ it is a translation-invariant metric space with $d(f, g) = \|f - g\|^p_{A^p_\alpha}$. Moreover, if $\Phi \in \mathfrak{U}^s$, then $A^{\Phi_p}_\alpha$, where $\Phi_p(t) = \Phi(t^p)$, is a subspace of A^p_α ([30]).

Recently, the Bloch-Orlicz space was introduced in [29] by Ramos Fernández. More precisely, let Ψ be a strictly increasing convex function such that $\Psi(0) = 0$. From these conditions it follows that $\lim_{t\to\infty} \Psi(t) = \infty$. The Bloch-Orlicz space associated with the function Ψ, denoted by $\mathcal{B}^\Psi$, is the class of all $f \in H(\mathbb{D})$ such that

$$\sup_{z\in\mathbb{D}}(1-|z|^2)\Psi(\lambda|f'(z)|) < \infty$$

for some $\lambda > 0$ depending on f. On $\mathcal{B}^\Psi$ Minkowski's functional

$$\|f\|_\Psi = \inf\Big\{k > 0 : S_\Psi\Big(\frac{f'}{k}\Big) \leq 1\Big\}$$

defines a seminorm, where

$$S_\Psi(f) = \sup_{z\in\mathbb{D}}(1-|z|^2)\Psi(f(z)).$$

Moreover, $\mathcal{B}^\Psi$ is a Banach space with the norm

$$\|f\|_{\mathcal{B}^\Psi} = |f(0)| + \|f\|_\Psi.$$

In fact, Ramos Fernández in [29] proved that $\mathcal{B}^{\Psi}$ is isometrically equal to μ-Bloch space, where

$$\mu(z) = \frac{1}{\Psi^{-1}(\frac{1}{1-|z|^2})}, \ \ z \in \mathbb{D}.$$

Thus, for $f \in \mathcal{B}^{\Psi}$, we have

$$\|f\|_{\mathcal{B}^{\Psi}} = |f(0)| + \sup_{z\in\mathbb{D}} \mu(z)|f'(z)|.$$

We can study the operator $DW_{\varphi,\psi} : A^{\Phi_p}_{\alpha} \to \mathcal{B}^{\Psi}$ with the help of this equivalent norm. It is obviously seen that if $\Psi(t) = t^p$ with $p > 0$, then the space $\mathcal{B}^{\Psi}$ coincides with the weighted Bloch space $\mathcal{B}^{\alpha}$ (see [62]), where $\alpha = 1/p$. Also, if $\Psi(t) = t\log(1+t)$ then $\mathcal{B}^{\Psi}$ coincides with the Log-Bloch space (see [2]). For the generalization of Log-Bloch spaces, see, for example, [49, 57].

Let X and Y be topological vector spaces whose topologies are given by translation invariant metrics d_X and d_Y, respectively. It is said that a linear operator $L : X \to Y$ is metrically bounded if there exists a positive constant K such that

$$d_Y(Lf, 0) \le K d_X(f, 0)$$

for all $f \in X$. When X and Y are Banach spaces, the metrical boundedness coincides with the usual definition of bounded operators between Banach spaces. Operator $L : X \to Y$ is said to be metrically compact if it maps bounded sets into relatively compact sets. When X and Y are Banach spaces, the metrical compactness coincides with the usual definition of compact operators between Banach spaces. Let $X = A^{\Phi}_{\alpha}$ and Y a Banach space. The norm of operator $L : X \to Y$ is defined by

$$\|L\|_{A^{\Phi}_{\alpha}\to Y} = \sup_{\|f\|_{A^{\Phi}_{\alpha}}\le 1} \|Lf\|_Y$$

and often written by $\|L\|$.

Throughout this paper, an operator is bounded (respectively, compact), if it is metrically bounded (respectively, metrically compact). C will be a constant not necessary the same at each occurrence. The notation $a \lesssim b$ means that $a \le Cb$ for some positive constant C. When $a \lesssim b$ and $b \lesssim a$, we write $a \simeq b$.

2. Auxiliary results

In order to prove the compactness of the product-type operators, we need the following result which is similar to Proposition 3.11 in [4]. The details of the proof are omitted.

Lemma 2.1. *Let $p \ge 1$, $\alpha > -1$, and $\Phi \in \mathfrak{U}^s$ such that $\Phi_p \in \mathfrak{L}_r$. Then the bounded operator $DW_{\varphi,\psi} : A^{\Phi_p}_{\alpha} \to \mathcal{B}^{\Psi}$ is compact if and only if for every bounded sequence $\{f_n\}_{n\in\mathbb{N}}$ in $A^{\Phi_p}_{\alpha}$ such that $f_n \to 0$ uniformly on every compact subset of $\mathbb{D}$ as $n \to \infty$, it follows that*

$$\lim_{n\to\infty} \|DW_{\varphi,\psi} f_n\|_{\mathcal{B}^{\Psi}} = 0.$$

We formulate the following two useful point estimates. For the first, see Lemma 2.4 in [30], and for the second, see Lemma 2.3 in [9].

Lemma 2.2. *Let $p \ge 1$, $\alpha > -1$ and $\Phi \in \mathfrak{U}^s$. Then for every $f \in A_\alpha^{\Phi_p}$ and $z \in \mathbb{D}$ we have*

$$|f(z)| \le \Phi_p^{-1}\Big(\Big(\frac{4}{1-|z|^2}\Big)^{\alpha+2}\Big)\|f\|^{lux}_{A_\alpha^{\Phi_p}}.$$

Lemma 2.3. *Let $p \ge 1$, $\alpha > -1$, $\Phi \in \mathfrak{U}^s$ and $n \in \mathbb{N}$. Then there are two positive constants $C_n = C(\alpha, p, n)$ and $D_n = D(\alpha, p, n)$ independent of $f \in A_\alpha^{\Phi_p}$ and $z \in \mathbb{D}$ such that*

$$|f^{(n)}(z)| \le \frac{C_n}{(1-|z|^2)^n}\Phi_p^{-1}\Big(\frac{D_n}{(1-|z|^2)^{\alpha+2}}\Big)\|f\|^{lux}_{A_\alpha^{\Phi_p}}.$$

We also need the following lemma which provides a class of useful test functions in space $A_\alpha^{\Phi_p}$(see [9]).

Lemma 2.4. *Let $p > 0$, $\alpha > -1$ and $\Phi \in \mathfrak{U}^s$. Then for every $t \ge 0$ and $w \in \mathbb{D}$ the following function is in $A_\alpha^{\Phi_p}$*

$$f_{w,t}(z) = \Phi_p^{-1}\Big(\Big(\frac{C}{1-|w|^2}\Big)^{\alpha+2}\Big)\Big(\frac{1-|w|^2}{1-\overline{w}z}\Big)^{\frac{2(\alpha+2)}{p}+t},$$

where C is an arbitrary positive constant.

Moreover,

$$\sup_{w\in\mathbb{D}} \|f_{w,t}\|^{lux}_{A_\alpha^{\Phi_p}} \lesssim 1.$$

3. The operator $DW_{\varphi,\psi} : A_\alpha^{\Phi_p} \to \mathcal{B}^\Psi$

First we characterize the boundedness of operator $DW_{\varphi,\psi} : A_\alpha^{\Phi_p} \to \mathcal{B}^\Psi$. We assume that $\Phi \in \mathfrak{U}^s$ such that $\Phi_p \in \mathfrak{L}_r$. Under this assumption, $A_\alpha^{\Phi_p}$ is a complete metric space (see [30]).

Theorem 3.1. *Let $p \ge 1$, $\alpha > -1$, and $\Phi \in \mathfrak{U}^s$ such that $\Phi_p \in \mathfrak{L}_r$. Then the following conditions are equivalent:*

(i) The operator $DW_{\varphi,\psi} : A_\alpha^{\Phi_p} \to \mathcal{B}^\Psi$ is bounded.

(ii) Functions φ and ψ satisfy the following conditions:

$$M_1 := \sup_{z\in\mathbb{D}} \mu(z)\big|\psi''(z)\big|\Phi_p^{-1}\Big(\Big(\frac{4}{1-|\varphi(z)|^2}\Big)^{\alpha+2}\Big) < \infty,$$

$$M_2 := \sup_{z\in\mathbb{D}} \frac{\mu(z)}{1-|\varphi(z)|^2}\big|\psi(z)\varphi''(z) + 2\psi'(z)\varphi'(z)\big|\Phi_p^{-1}\Big(\Big(\frac{D_1}{1-|\varphi(z)|^2}\Big)^{\alpha+2}\Big) < \infty,$$

and

$$M_3 := \sup_{z\in\mathbb{D}} \frac{\mu(z)}{(1-|\varphi(z)|^2)^2}|\psi(z)||\varphi'(z)|^2\Phi_p^{-1}\Big(\Big(\frac{D_2}{1-|\varphi(z)|^2}\Big)^{\alpha+2}\Big) < \infty.$$

Moreover, if the operator $DW_{\varphi,\psi} : A_\alpha^{\Phi_p} \to \mathcal{B}^\Psi$ is nonzero and bounded, then

$$\big\|DW_{\varphi,\psi}\big\| \simeq 1 + M_1 + M_2 + M_3.$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 21, NO.7, 2016, COPYRIGHT 2016 EUDOXUS PRESS, LLC

Proof. $(i) \Rightarrow (ii)$. Suppose that $DW_{\varphi,\psi} : A_\alpha^{\Phi_p} \to \mathcal{B}^\Psi$ is bounded. For $w \in \mathbb{D}$, we choose the function

$$\begin{aligned} f_{1,\varphi(w)}(z) &= c_0\Big(\frac{1-|\varphi(w)|^2}{1-\overline{\varphi(w)}z}\Big)^{\frac{2(\alpha+2)}{p}} + c_1\Big(\frac{1-|\varphi(w)|^2}{1-\overline{\varphi(w)}z}\Big)^{\frac{2(\alpha+2)}{p}+1} \\ &+ c_2\Big(\frac{1-|\varphi(w)|^2}{1-\overline{\varphi(w)}z}\Big)^{\frac{2(\alpha+2)}{p}+2} - \Big(\frac{1-|\varphi(w)|^2}{1-\overline{\varphi(w)}z}\Big)^{\frac{2(\alpha+2)}{p}+3}, \end{aligned}$$

where

$$c_0 = \frac{2(\alpha+2)+3p}{2(\alpha+2)},\ \ c_1 = -\frac{6(\alpha+2)+9p}{2(\alpha+2)+p},\ \ c_2 = \frac{6(\alpha+2)+9p}{2(\alpha+2)+2p}.$$

By a direct calculation, we have

$$f'_{1,\varphi(w)}(\varphi(w)) = f''_{1,\varphi(w)}(\varphi(w)) = 0. \tag{1}$$

Using the function $f_{1,\varphi(w)}$, we define the function

$$f(z) = \Phi_p^{-1}\Big(\Big(\frac{4}{1-|\varphi(w)|^2}\Big)^{\alpha+2}\Big) f_{1,\varphi(w)}(z).$$

Applying (1) to f' and f'', we obtain

$$f'(\varphi(w)) = f''(\varphi(w)) = 0. \tag{2}$$

It is clear that

$$f(\varphi(w)) = C\Phi_p^{-1}\Big(\Big(\frac{4}{1-|\varphi(w)|^2}\Big)^{\alpha+2}\Big), \tag{3}$$

where

$$C = \frac{2(\alpha+2)+3p}{2(\alpha+2)} - \frac{6(\alpha+2)+9p}{2(\alpha+2)+p} + \frac{6(\alpha+2)+9p}{2(\alpha+2)+2p} - 1 \neq 0.$$

By Lemma 2.4, $f \in A_\alpha^{\Phi_p}$ and $\|f\|_{A_\alpha^{\Phi_p}} \le C$. By (2), (3) and the boundedness of $DW_{\varphi,\psi} : A_\alpha^{\Phi_p} \to \mathcal{B}^\Psi$,

$$\mu(w)\big|\psi''(w)\big|\Phi_p^{-1}\Big(\Big(\frac{4}{1-|\varphi(w)|^2}\Big)^{\alpha+2}\Big) \le C\|DW_{\varphi,\psi}\|, \tag{4}$$

which means that

$$M_1 = \sup_{z\in\mathbb{D}} \mu(z)|\psi''(z)|\Phi_p^{-1}\Big(\Big(\frac{4}{1-|\varphi(z)|^2}\Big)^{\alpha+2}\Big) \le C\|DW_{\varphi,\psi}\| < \infty. \tag{5}$$

Next we will prove $M_2 < \infty$. For this we consider the functions $f_1(z) = z$ and $f_2(z) \equiv 1$, respectively. Since the operator $DW_{\varphi,\psi} : A_\alpha^{\Phi_p} \to \mathcal{B}^\Psi$ is bounded, we have

$$\begin{aligned} \sup_{z\in\mathbb{D}} \mu(z)\big|\psi''(z)\varphi(z) + 2\psi'(z)\varphi'(z) + \psi(z)\varphi''(z)\big| \\ \le \|DW_{\varphi,\psi} f_1\|_{\mathcal{B}^\Psi} \le C\|DW_{\varphi,\psi}\| \end{aligned} \tag{6}$$

and

$$\sup_{z\in\mathbb{D}} \mu(z)|\psi''(z)| \le \|DW_{\varphi,\psi} f_2\|_{\mathcal{B}^\Psi} \le C\|DW_{\varphi,\psi}\|. \tag{7}$$

By (6), (7) and the boundedness of φ,

$$J_1 := \sup_{z\in\mathbb{D}} \mu(z)\big|\psi(z)\varphi''(z) + 2\psi'(z)\varphi'(z)\big| \le C\|DW_{\varphi,\psi}\|. \tag{8}$$

For $w \in \mathbb{D}$, choose the function

$$\begin{aligned} f_{2,\varphi(w)}(z) = & c_0\Big(\frac{1-|\varphi(w)|^2}{1-\overline{\varphi(w)}z}\Big)^{\frac{2(\alpha+2)}{p}} + c_1\Big(\frac{1-|\varphi(w)|^2}{1-\overline{\varphi(w)}z}\Big)^{\frac{2(\alpha+2)}{p}+1} \\ & + c_2\Big(\frac{1-|\varphi(w)|^2}{1-\overline{\varphi(w)}z}\Big)^{\frac{2(\alpha+2)}{p}+2} - \Big(\frac{1-|\varphi(w)|^2}{1-\overline{\varphi(w)}z}\Big)^{\frac{2(\alpha+2)}{p}+3}, \end{aligned}$$

where

$$c_1 = \frac{36p(\alpha+2)^2 + 78p^2(\alpha+2) + 36p^3}{[4(\alpha+2)+2p][2(\alpha+2)+2p][2(\alpha+2)+3p]},$$

$$c_2 = \frac{4(\alpha+2)^2 + 42p(\alpha+2) + 36p^2}{[2(\alpha+2)+2p][4(\alpha+2)+6p]},$$

and

$$c_0 = 1 - c_1 - c_2.$$

From a calculation, we obtain

$$f_{2,\varphi(w)}(\varphi(w)) = f''_{2,\varphi(w)}(\varphi(w)) = 0. \tag{9}$$

Define the function

$$g(z) = \Phi_p^{-1}\Big(\Big(\frac{D_1}{1-|\varphi(w)|^2}\Big)^{\alpha+2}\Big) f_{2,\varphi(w)}(z).$$

Then by (9),

$$g(\varphi(w)) = g''(\varphi(w)) = 0, \tag{10}$$

and by a direct calculation,

$$g'(\varphi(w)) = C\frac{\overline{\varphi(w)}}{1-|\varphi(w)|^2}\Phi_p^{-1}\Big(\Big(\frac{D_1}{1-|\varphi(w)|^2}\Big)^{\alpha+2}\Big), \tag{11}$$

where $C = c_1 + 2c_2 - 3$. Also by Lemma 2.4, $g \in A_\alpha^{\Phi_p}$ and $\|g\|_{A_\alpha^{\Phi_p}} \le C$. Since $DW_{\varphi,\psi} : A_\alpha^{\Phi_p} \to \mathcal{B}^\Psi$ is bounded, we have

$$\mu(z)|(DW_{\varphi,\psi}g)'(z)| \le C\|DW_{\varphi,\psi}\|, \tag{12}$$

for all $z \in \mathbb{D}$. By (10) and (11), letting $z = w$ in (12) gives

$$\begin{aligned} J(w) := & \frac{\mu(w)|\varphi(w)|}{1-|\varphi(w)|^2}\big|\psi(w)\varphi''(w) + 2\psi'(w)\varphi'(w)\big|\Phi_p^{-1}\Big(\Big(\frac{D_1}{1-|\varphi(w)|^2}\Big)^{\alpha+2}\Big) \\ \le & C\|DW_{\varphi,\psi}\|. \end{aligned} \tag{13}$$

Hence

$$\sup_{z\in\mathbb{D}} J(z) \le C\|DW_{\varphi,\psi}\|. \tag{14}$$

For the fixed $\delta \in (0,1)$, by (8)

$$\begin{aligned} & \sup_{\{z\in\mathbb{D}:|\varphi(z)|\le\delta\}} \frac{\mu(z)}{1-|\varphi(z)|^2}\big|\psi(z)\varphi''(z) + 2\psi'(z)\varphi'(z)\big|\Phi_p^{-1}\Big(\Big(\frac{D_1}{1-|\varphi(z)|^2}\Big)^{\alpha+2}\Big) \\ & \le \frac{J_1}{1-\delta^2}\Phi_p^{-1}\Big(\Big(\frac{D_1}{1-\delta^2}\Big)^{\alpha+2}\Big) \le C\|DW_{\varphi,\psi}\|, \end{aligned} \tag{15}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 21, NO.7, 2016, COPYRIGHT 2016 EUDOXUS PRESS, LLC

and by (14)

$$\sup_{\{z\in\mathbb{D}:|\varphi(z)|>\delta\}}\frac{\mu(z)}{1-|\varphi(z)|^2}\big|\psi(z)\varphi''(z)+2\psi'(z)\varphi'(z)\big|\Phi_p^{-1}\Big(\Big(\frac{D_1}{1-|\varphi(z)|^2}\Big)^{\alpha+2}\Big)$$
$$\leq\frac{1}{\delta}\sup_{z\in\mathbb{D}}J(z)\leq C\|DW_{\varphi,\psi}\|. \tag{16}$$

Consequently, it follows from (15) and (16) that

$$M_2\leq C\|DW_{\varphi,\psi}\|<\infty. \tag{17}$$

Now we prove that $M_3<\infty$. First taking the function $f(z)=z^2$, we have

$$\sup_{z\in\mathbb{D}}\mu(z)\big|\psi''(z)\varphi(z)^2+4\psi'(z)\varphi'(z)\varphi(z)+2\psi(z)\varphi''(z)\varphi(z)+2\psi(z)\varphi'(z)^2\big|$$
$$\leq\|DW_{\varphi,\psi}z^2\|_{\mathcal{B}^\Psi}\leq C\|DW_{\varphi,\psi}\| \tag{18}$$

By (7) and the boundedness of φ, we obtain

$$\sup_{z\in\mathbb{D}}\mu(z)|\psi''(z)||\varphi(z)|^2\leq C\|DW_{\varphi,\psi}\|. \tag{19}$$

From (8), (18), (19) and the boundedness of φ, it follows that

$$J_2:=\sup_{z\in\mathbb{D}}\mu(z)|\psi(z)||\varphi'(z)|^2\leq C\|DW_{\varphi,\psi}\|. \tag{20}$$

For $w\in\mathbb{D}$, consider the function

$$f_{3,\varphi(w)}(z)=c_0\Big(\frac{1-|\varphi(w)|^2}{1-\overline{\varphi(w)}z}\Big)^{\frac{2(\alpha+2)}{p}}+c_1\Big(\frac{1-|\varphi(w)|^2}{1-\overline{\varphi(w)}z}\Big)^{\frac{2(\alpha+2)}{p}+1}$$
$$+c_2\Big(\frac{1-|\varphi(w)|^2}{1-\overline{\varphi(w)}z}\Big)^{\frac{2(\alpha+2)}{p}+2}-\Big(\frac{1-|\varphi(w)|^2}{1-\overline{\varphi(w)}z}\Big)^{\frac{2(\alpha+2)}{p}+3},$$

where

$$c_0=\frac{2(\alpha+2)+p}{2(\alpha+2)+2p},\ \ c_1=-\frac{3(\alpha+2)+4p}{\alpha+2+p},\ \ c_2=\frac{6(\alpha+2)+7p}{2(\alpha+2)+2p}.$$

For the function $f_{3,\varphi(w)}$, we have

$$f_{3,\varphi(w)}(\varphi(w))=f'_{3,\varphi(w)}(\varphi(w))=0. \tag{21}$$

For the function

$$h(z)=\Phi_p^{-1}\Big(\Big(\frac{D_2}{1-|\varphi(w)|^2}\Big)^{\alpha+2}\Big)f_{3,\varphi(w)}(z),$$

it follows from (21) that

$$h(\varphi(w))=h'(\varphi(w))=0. \tag{22}$$

By (21) and (22), the boundedness of the operator $DW_{\varphi,\psi}:A_\alpha^{\Phi_p}\to\mathcal{B}^\Psi$ gives

$$K(w):=\frac{\mu(w)|\varphi(w)|^2}{(1-|\varphi(w)|^2)^2}|\psi(w)||\varphi'(w)|^2\Phi_p^{-1}\Big(\Big(\frac{D_2}{1-|\varphi(w)|^2}\Big)^{\alpha+2}\Big)\leq C\|DW_{\varphi,\psi}\|.$$

This yields

$$\sup_{z\in\mathbb{D}}K(z)\leq C\|DW_{\varphi,\psi}\|<\infty. \tag{23}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 21, NO.7, 2016, COPYRIGHT 2016 EUDOXUS PRESS, LLC

For the fixed $\delta \in (0,1)$, by (20) and (23) we respectively obtain

$$\sup_{\{z\in\mathbb{D}:|\varphi(z)|\le\delta\}} \frac{\mu(z)}{(1-|\varphi(z)|^2)^2}|\psi(z)||\varphi'(z)|^2\Phi_p^{-1}\Big(\Big(\frac{D_2}{1-|\varphi(z)|^2}\Big)^{\alpha+2}\Big)$$
$$\le \frac{J_2}{(1-\delta^2)^2}\Phi_p^{-1}\Big(\Big(\frac{D_2}{1-\delta^2}\Big)^{\alpha+2}\Big) \le C\|DW_{\varphi,\psi}\| \tag{24}$$

and

$$\sup_{\{z\in\mathbb{D}:|\varphi(z)|>\delta\}} \frac{\mu(z)}{(1-|\varphi(z)|^2)^2}|\psi(z)||\varphi'(z)|^2\Phi_p^{-1}\Big(\Big(\frac{D_2}{1-|\varphi(z)|^2}\Big)^{\alpha+2}\Big)$$
$$\le \frac{1}{\delta^2}\sup_{z\in\mathbb{D}} K(z) \le C\|DW_{\varphi,\psi}\|. \tag{25}$$

So, by (24) and (25) we have

$$M_3 \le C\|DW_{\varphi,\psi}\| < \infty. \tag{26}$$

$(ii) \Rightarrow (i)$. By Lemmas 2.2 and 2.3, for all $f \in A_\alpha^{\Phi_p}$ we have

$$\begin{aligned}\|DW_{\varphi,\psi}f\|_{\mathcal{B}^\Psi} &= \big|(\psi\cdot f\circ\varphi)'(0)\big| + \sup_{z\in\mathbb{D}}\mu(z)\big|(\psi\cdot f\circ\varphi)''(z)\big| \\ &\le \big|(\psi\cdot f\circ\varphi)'(0)\big| + +\sup_{z\in\mathbb{D}}\mu(z)\big|\psi''(z)\big|\big|f(\varphi(z))\big| \\ &\quad + \sup_{z\in\mathbb{D}}\mu(z)\big|\psi(z)\varphi''(z)+2\psi'(z)\varphi'(z)\big|\big|f'(\varphi(z))\big| \\ &\quad + \sup_{z\in\mathbb{D}}\mu(z)|\psi(z)||\varphi'(z)|^2\big|f''(\varphi(z))\big| \\ &\le C\big(1+M_1+M_2+M_3\big)\|f\|_{A_\alpha^{\Phi_p}}.\end{aligned} \tag{27}$$

From condition (ii) and (27), it follows that $DW_{\varphi,\psi}: A_\alpha^{\Phi_p} \to \mathcal{B}^\Psi$ is bounded.

Suppose that the operator $DW_{\varphi,\psi}: A_\alpha^{\Phi_p} \to \mathcal{B}^\Psi$ is nonzero and bounded. Then from the preceding inequalities (5), (17) and (26), we obtain

$$M_1 + M_2 + M_3 \lesssim \|DW_{\varphi,\psi}\|. \tag{28}$$

Since the operator $DW_{\varphi,\psi}: A_\alpha^{\Phi_p} \to \mathcal{B}^\Psi$ is nonzero, we have $\|DW_{\varphi,\psi}\| > 0$. From this, we can find a positive constant C such that $1 \le C\|DW_{\varphi,\psi}\|$. This means that

$$1 \lesssim \|DW_{\varphi,\psi}\|. \tag{29}$$

Hence, combing (28) and (29) gives

$$1 + M_1 + M_2 + M_3 \lesssim \|DW_{\varphi,\psi}\|. \tag{30}$$

From (27), it is clear that

$$\|DW_{\varphi,\psi}\| \lesssim 1 + M_1 + M_2 + M_3. \tag{31}$$

So, from (30) and (31), we obtain the asymptotic expression of $\|DW_{\varphi,\psi}\|$. The proof is finished. □

Remark 3.1. *If $DW_{\varphi,\psi}: A_\alpha^{\Phi_p} \to \mathcal{B}^\Psi$ is a zero operator, then is obviously $\|DW_{\varphi,\psi}\| = 0$. Hence, the case is usually excluded from such considerations.*

Now we characterize the compactness of operator $DW_{\varphi,\psi}: A_\alpha^{\Phi_p} \to \mathcal{B}^\Psi$.

Theorem 3.2. *Let $p \ge 1$, $\alpha > -1$, and $\Phi \in \mathfrak{U}^s$ such that $\Phi_p \in \mathfrak{L}_r$. Then the following conditions are equivalent:*

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 21, NO.7, 2016, COPYRIGHT 2016 EUDOXUS PRESS, LLC

(i) The operator $DW_{\varphi,\psi} : A^{\Phi_p}_{\alpha} \to \mathcal{B}^{\Psi}$ *is compact.*
(ii) Functions φ *and* ψ *are such that* $\psi' \in \mathcal{B}^{\Psi}$,

$$J_1 := \sup_{z\in\mathbb{D}} \mu(z)\big|\psi(z)\varphi''(z) + 2\psi'(z)\varphi'(z)\big| < \infty,$$

$$J_2 := \sup_{z\in\mathbb{D}} \mu(z)|\psi(z)||\varphi'(z)|^2 < \infty,$$

$$\lim_{|\varphi(z)|\to 1^-} \mu(z)\big|\psi''(z)\big|\Phi_p^{-1}\Big(\Big(\frac{4}{1-|\varphi(z)|^2}\Big)^{\alpha+2}\Big) = 0,$$

$$\lim_{|\varphi(z)|\to 1^-} \frac{\mu(z)}{1-|\varphi(z)|^2}\big|\psi(z)\varphi''(z) + 2\psi'(z)\varphi'(z)\big|\Phi_p^{-1}\Big(\Big(\frac{D_1}{1-|\varphi(z)|^2}\Big)^{\alpha+2}\Big) = 0,$$

and

$$\lim_{|\varphi(z)|\to 1^-} \frac{\mu(z)}{(1-|\varphi(z)|^2)^2}|\psi(z)||\varphi'(z)|^2\Phi_p^{-1}\Big(\Big(\frac{D_2}{1-|\varphi(z)|^2}\Big)^{\alpha+2}\Big) = 0.$$

Proof. $(i) \Rightarrow (ii)$. Suppose that (i) holds. Then the operator $DW_{\varphi,\psi} : A^{\Phi_p}_{\alpha} \to \mathcal{B}^{\Psi}$ is bounded. In the proof of Theorem 3.1, we have obtained that $\psi' \in \mathcal{B}^{\Psi}$ and J_1, $J_2 < \infty$.

Next consider a sequence $\{\varphi(z_n)\}_{n\in\mathbb{N}}$ in $\mathbb{D}$ such that $|\varphi(z_n)| \to 1^-$ as $n \to \infty$. If such sequence does not exist, then condition (ii) obviously holds. Using this sequence, we define the functions

$$f_n(z) = \Phi_p^{-1}\Big(\Big(\frac{4}{1-|\varphi(z_n)|^2}\Big)^{\alpha+2}\Big) f_{1,\varphi(z_n)}(z).$$

By Lemma 2.4, we know that the sequence $\{f_n\}_{n\in\mathbb{N}}$ is uniformly bounded in $A^{\Phi_p}_{\alpha}$. From the proof of Theorem 3.6 in [30], it follows that the sequence $\{f_n\}_{n\in\mathbb{N}}$ uniformly converges to zero on any compact subset of $\mathbb{D}$ as $n \to \infty$. Hence by Lemma 2.1,

$$\lim_{n\to\infty} \|DW_{\varphi,\psi} f_n\|_{\mathcal{B}^{\Psi}} = 0.$$

From this, (2) and (3), we have

$$\lim_{n\to\infty} \mu(z_n)\big|\psi''(z_n)\big|\Phi_p^{-1}\Big(\Big(\frac{4}{1-|\varphi(z_n)|^2}\Big)^{\alpha+2}\Big) = 0.$$

By using the sequence of functions

$$g_n(z) = \Phi_p^{-1}\Big(\Big(\frac{D_1}{1-|\varphi(z_n)|^2}\Big)^{\alpha+2}\Big) f_{2,\varphi(z_n)}(z),$$

similar to the above, we obtain

$$\lim_{n\to\infty} \frac{\mu(z_n)}{1-|\varphi(z_n)|^2}\Big|\psi(z_n)\varphi''(z_n) + 2\psi'(z_n)\varphi'(z_n)\Big|\Phi_p^{-1}\Big(\Big(\frac{D_1}{1-|\varphi(z_n)|^2}\Big)^{\alpha+2}\Big) = 0.$$

Also, by using sequence of functions

$$h_n(z) = \Phi_p^{-1}\Big(\Big(\frac{D_2}{1-|\varphi(z_n)|^2}\Big)^{\alpha+2}\Big) f_{3,\varphi(z_n)}(z),$$

we obtain

$$\lim_{n\to\infty} \frac{\mu(z_n)}{(1-|\varphi(z_n)|^2)^2}|\psi(z_n)||\varphi'(z_n)|^2\Phi_p^{-1}\Big(\Big(\frac{D_2}{1-|\varphi(z_n)|^2}\Big)^{\alpha+2}\Big) = 0.$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 21, NO.7, 2016, COPYRIGHT 2016 EUDOXUS PRESS, LLC

The proof of the implication is finished.

$(ii) \Rightarrow (i)$. We first check that $DW_{\varphi,\psi} : A^{\Phi_p}_\alpha \to \mathcal{B}^\Psi$ is bounded. For this we observe that condition (ii) implies that for every $\varepsilon > 0$, there is an $\eta \in (0,1)$ such that

$$L_1(z) := \mu(z)\big|\psi''(z)\big|\Phi_p^{-1}\Big(\Big(\frac{4}{1-|\varphi(z)|^2}\Big)^{\alpha+2}\Big) < \varepsilon, \tag{32}$$

$$L_2(z) := \frac{\mu(z)}{1-|\varphi(z)|^2}\big|\psi(z)\varphi''(z)+2\psi'(z)\varphi'(z)\big|\Phi_p^{-1}\Big(\Big(\frac{D_1}{1-|\varphi(z)|^2}\Big)^{\alpha+2}\Big) < \varepsilon, \tag{33}$$

and

$$L_3(z) := \frac{\mu(z)}{(1-|\varphi(z)|^2)^2}|\psi(z)||\varphi'(z)|^2\Phi_p^{-1}\Big(\Big(\frac{D_2}{1-|\varphi(z)|^2}\Big)^{\alpha+2}\Big) < \varepsilon, \tag{34}$$

for any $z \in K = \{z \in \mathbb{D} : |\varphi(z)| > \eta\}$. Then since $\psi' \in \mathcal{B}^\Psi$ and by (32), we have

$$M_1 = \sup_{z\in\mathbb{D}} L_1(z) \le \sup_{z\in\mathbb{D}\setminus K} L_1(z) + \sup_{z\in K} L_1(z) \le \|\psi'\|_{\mathcal{B}^\Psi}\Phi_p^{-1}\Big(\Big(\frac{4}{1-\eta^2}\Big)^{\alpha+2}\Big) + \varepsilon.$$

By (33) and $J_1 < \infty$, we obtain

$$M_2 = \sup_{z\in\mathbb{D}} L_2(z) \le \sup_{z\in\mathbb{D}\setminus K} L_2(z) + \sup_{z\in K} L_2(z) \le \frac{J_1}{1-\eta^2}\Phi_p^{-1}\Big(\Big(\frac{D_1}{1-\eta^2}\Big)^{\alpha+2}\Big) + \varepsilon.$$

By (34) and $J_2 < \infty$, it follows that $M_3 < \infty$. So by Theorem 3.1, $DW_{\varphi,\psi} : A^{\Phi_p}_\alpha \to \mathcal{B}^\Psi$ is bounded.

To prove that the operator $DW_{\varphi,\psi} : A^{\Phi_p}_\alpha \to \mathcal{B}^\Psi$ is compact, by Lemma 2.1 we just need to prove that, if $\{f_n\}_{n\in\mathbb{N}}$ is a sequence in $A^{\Phi_p}_\alpha$ such that $\|f_n\|_{A^{\Phi_p}_\alpha} \le M$ and $f_n \to 0$ uniformly on any compact subset of $\mathbb{D}$ as $n \to \infty$, then

$$\lim_{n\to\infty} \|DW_{\varphi,\psi} f_n\|_{\mathcal{B}^\Psi} = 0.$$

For any $\varepsilon > 0$ and the above η, we have, by using again the condition (ii), Lemma 2.2 and Lemma 2.3,

$$\begin{aligned}
&\sup_{z\in\mathbb{D}}\mu(z)\big|(DW_{\varphi,\psi}f_n)'(z)\big| = \sup_{z\in\mathbb{D}}\mu(z)\big|(\psi\cdot f_n\circ\varphi)''(z)\big| \le \sup_{z\in\mathbb{D}}\mu(z)\big|\psi''(z)\big|\big|f_n(\varphi(z))\big| \\
&\quad + \sup_{z\in\mathbb{D}}\mu(z)\big|\psi(z)\varphi''(z)+2\psi'(z)\varphi'(z)\big|\big|f_n'(\varphi(z))\big| + \sup_{z\in\mathbb{D}}\mu(z)|\psi(z)||\varphi'(z)|^2\big|f_n''(\varphi(z))\big| \\
&\le \sup_{z\in\mathbb{D}\setminus K}\mu(z)\big|\psi''(z)\big|\big|f_n(\varphi(z))\big| + \sup_{z\in K}\mu(z)\big|\psi''(z)\big|\big|f_n(\varphi(z))\big| \\
&\quad + \sup_{z\in\mathbb{D}\setminus K}\mu(z)\big|\psi(z)\varphi''(z)+2\psi'(z)\varphi'(z)\big|\big|f_n'(\varphi(z))\big| \\
&\quad + \sup_{z\in K}\mu(z)\big|\psi(z)\varphi''(z)+2\psi'(z)\varphi'(z)\big|\big|f_n'(\varphi(z))\big| \\
&\quad + \sup_{z\in\mathbb{D}\setminus K}\mu(z)|\psi(z)||\varphi'(z)|^2\big|f_n''(\varphi(z))\big| + \sup_{z\in K}\mu(z)|\psi(z)||\varphi'(z)|^2\big|f_n''(\varphi(z))\big| \\
&\le K_n + M\sup_{z\in K}L_1(z) + M\sup_{z\in K}L_2(z) + M\sup_{z\in K}L_3(z) \\
&\le K_n + 3M\varepsilon,
\end{aligned}$$

where

$$K_n = \|\psi'\|_{\mathcal{B}^\Psi}\sup_{\{z:|z|\le\eta\}}\big|f_n(z)\big| + \sum_{i=1}^{2}J_i\sup_{\{z:|z|\le\eta\}}\big|f_n^{(i)}(z)\big|.$$

Hence

$$\begin{aligned}\|DW_{\varphi,\psi}f_n\|_{\mathcal{B}^\Psi} &\leq K_n + 3M\varepsilon + \left|(\psi\cdot f_n\circ\varphi)'(0)\right| \\ &= K_n + 3M\varepsilon + \left|\psi'(0)f_n(\varphi(0)) + \psi(0)f_n'(\varphi(0))\varphi'(0)\right|. \qquad (35)\end{aligned}$$

It is easy to see that, when $\{f_n\}_{n\in\mathbb{N}}$ uniformly converges to zero on any compact subset of $\mathbb{D}$, $\{f_n'\}_{n\in\mathbb{N}}$ and $\{f_n''\}_{n\in\mathbb{N}}$ also do as $n\to\infty$. From this, we obtain $K_n\to 0$ as $n\to\infty$. Since $\{z:|z|\leq\eta\}$ and $\{\varphi(0)\}$ are compact subsets of $\mathbb{D}$, letting $n\to\infty$ in (35) gives

$$\lim_{n\to\infty}\|DW_{\varphi,\psi}f_n\|_{\mathcal{B}^\Psi} = 0.$$

From Lemma 2.1, it follows that the operator $DW_{\varphi,\psi}: A_\alpha^{\Phi_p}\to\mathcal{B}^\Psi$ is compact. The proof is finished. □

Acknowledgments. This work is supported by the National Natural Science Foundation of China (No.11201323), the Sichuan Province University Key Laboratory of Bridge Non-destruction Detecting and Engineering Computing (No.2013QZJ01, No.2013YY01) and the Fund Project of Sichuan Provincial Department of Education (No.12ZB288).

References

[1] R. F. Allen, F. Colonna, Weighted composition operators on the Bloch space of a bounded homogeneous domain, *Oper. Theory: Adv. Appl.* **202** (2010), 11-37.
[2] K. Attele, Toeplitz and Hankel operators on Bergman spaces, *Hokkaido Math. J.* **21** (1992), 279-293.
[3] F. Colonna, S. Li, Weighted composition operators from the minimal Möbius invariant space into the Bloch space, *Mediter. J. Math.* **10** (1) (2013), 395-409.
[4] C. C. Cowen, B. D. MacCluer, Composition operators on spaces of analytic functions, CRC Press, 1995.
[5] P. L. Duren, Theory of H^p spaces, Academic Press, New York, NY, USA, 1970.
[6] R. A. Hibschweiler, N. Portnoy, Composition followed by differentiation between Bergman and Hardy spaces, *Rocky Mountain J. Math.* **35** (3) (2005), 843-855.
[7] Z. J. Jiang, On a class of operators from weighted Bergman spaces to some spaces of analytic functions, *Taiwan. J. Math. Soc.* **15** (5) (2011), 2095-2121.
[8] Z. J. Jiang, Weighted composition operators from weighted Bergman spaces to some spaces of analytic functions on the upper half plane, *Util. Math.* **93** (2014), 205-212.
[9] Z. J. Jiang, On a product-type operator from weighted BergmanCOrlicz space to some weighted type spaces, *Appl. Math. Comput.* **256** (2015), 37-51.
[10] S. Krantz, S. Stević, On the iterated logarithmic Bloch space on the unit ball, *Nonlinear Anal. TMA* **71** (2009), 1772-1795.
[11] P. Kumar, S. D. Sharma, Weighted composition operators from weighted Bergman Nevanlinna spaces to Zygmund spaces, *Int. J. Modern Math. Sci.* **3** (1) (2012), 31-54.
[12] L. Luo, S. Ueki, Weighted composition operators between weighted Bergman and Hardy spaces on the unit ball of $\mathbb{C}^n$, *J. Math. Anal. Appl.* **326** (2007), 88-100.
[13] L. Luo, S. Ueki, Compact weighted composition operators and multiplication opeators between Hardy spaces, *Abstr. Appl. Anal.* **2008** (2008), Article ID 196498, 12 pages.
[14] S. Li, S. Stević, Weighted composition operators from Bergman-type spaces into Bloch spaces, *Proc. Indian Acad. Sci. Math. Sci.* **117** (3) (2007), 371-385.
[15] S. Li, S. Stević, Weighted composition operators from H^∞ to the Bloch space on the polydisc, *Abstr. Appl. Anal.* **2007** (2007), Article ID 48478, 12 pages.
[16] S. Li, S. Stević, Composition followed by differentiation between Bloch type spaces, *J. Comput. Anal. Appl.* **9** (2) (2007), 195-205.
[17] S. Li, S. Stević, Weighted composition operators from Zygmund spaces into Bloch spaces, *Appl. Math. Comput.* **206** (2) (2008), 825-831.

[18] S. Li, S. Stević, Composition followed by differentiation from mixed norm spaces to α-Bloch spaces, *Sb. Math.* **199** (12) (2008), 1847-1857.
[19] S. Li, S. Stević, Weighted composition operators between H^∞ and α-Bloch spaces in the unit ball, *Taiwan. J. Math. Soc.* **12** (2008), 1625-1639.
[20] S. Li, S. Stević, Generalized composition operators on Zygmund spaces and Bloch type spaces, *J. Math. Anal. Appl.* **338** (2008), 1282-1295.
[21] S. Li, S. Stević, Products of composition and integral type operators from H^∞ to the Bloch space, *Complex Var. Elliptic Equ.* **53** (5) (2008), 463-474.
[22] S. Li, S. Stević, Weighted composition operators from Zygmund spaces into Bloch spaces, *Appl. Math. Comput.* **206** (2) (2008), 825-831.
[23] S. Li, S. Stević, Products of Volterra type operator and composition operator from H^∞ and Bloch spaces to the Zygmund space, *J. Math. Anal. Appl.* **345** (2008), 40-52.
[24] S. Li, S. Stević, Composition followed by differentiation between H^∞ and α-Bloch spaces, *Houston J. Math.* **35** (1) (2009), 327-340.
[25] S. Li, S. Stević, Products of composition and differentiation operators from Zygmund spaces to Bloch spaces and Bers spaces, *Appl. Math. Comput.* **217** (2010), 3144-3154.
[26] P. J. Nieminen, Compact differences of composition operators on Bloch and Lipschitz spaces, *Comput. Methods Funct. Theory.* **7** (2) (2007), 325-344.
[27] S. Ohno, Weighted composition operators between H^∞ and the Bloch space, *Taiwan. J. Math. Soc.* **5**(3)(2001), 555-563.
[28] S. Ohno, Products of composition and differentiation on Bloch spaces, *Bull. Korean Math. Soc.* **46** (6) (2009), 1135-1140.
[29] J.C. Ramos Fernández, Composition operators on Bloch-Orlicz type spaces, *Appl. Math. Comput.* **217** (2010), 3392-3402.
[30] B. Sehba, S. Stević, On some product-type operators from Hardy-Orlicz and Bergman-Orlicz spaces to weighted-type spaces, *Appl. Math. Comput.* **233** (2014), 565-581.
[31] A. K. Sharma, Z. Abbas, Weighted composition operators between weighted Bergman-Nevanlinna and Bloch-type spaces, *Appl. Math. Sci.* **41** (4) (2010), 2039-2048.
[32] A. K. Sharma, Products of composition multiplication and differentiation between Bergman and Bloch type spaces, *Turkish. J. Math.* **35** (2011), 275-291.
[33] S. D. Sharma, A. K. Sharma, S. Ahmed, Composition operators between Hardy and Bloch-type spaces of the upper half-plane, *Bull. Korean Math. Soc.* **43** (3) (2007), 475-482.
[34] S. Stević, Essential norms of weighted composition operators from the α-Bloch space to a weighted-type space on the unit ball, *Abstr. Appl. Anal.* Vol. 2008, Article ID 279691, (2008), 11 pages.
[35] S. Stević, Norm of weighted composition operators from Bloch space to H^∞ on the unit ball, *Ars Combin.* **88** (2008), 125-127.
[36] S. Stević, Norms of some operators from Bergman spaces to weighted and Bloch-type space, *Util. Math.* **76** (2008), 59-64.
[37] S. Stević, On a new integral-type operator from the weighted Bergman space to the Bloch-type space on the unit ball, *Discrete Dyn. Nat. Soc.* Vol. 2008, Article ID 154263, (2008), 14 pages.
[38] S. Stević, On a new operator from H^∞ to the Bloch-type space on the unit ball, *Util. Math.* **77** (2008), 257-263.
[39] S. Stević, On a new operator from the logarithmic Bloch space to the Bloch-type space on the unit ball, *Appl. Math. Comput.* **206** (2008), 313-320.
[40] S. Stević, Essential norm of an operator from the weighted Hilbert-Bergman space to the Bloch-type space, *Ars Combin.* **91** (2009), 123-127.
[41] S. Stević, Norm and essential norm of composition followed by differentiation from α-Bloch spaces to H^∞, *Appl. Math. Comput.* **207** (2009), 225-229.
[42] S. Stević, Norm of weighted composition operators from α-Bloch spaces to weighted-type spaces, *Appl. Math. Comput.* **215** (2009), 818-820.
[43] S. Stević, On a new integral-type operator from the Bloch space to Bloch-type space on the unit ball, *J. Math. Anal. Appl.* **354** (2009), 426-434.
[44] S. Stević, On an integral-type operator from logarithmic Bloch-type and mixed-norm spaces to Bloch-type spaces, *Nonlinear Anal. TMA* **71** (2009), 6323-6342.
[45] S. Stević, Products of composition and differentiation operators on the weighted Bergman space, *Bull. Belg. Math. Soc.* **16** (2009), 623-635.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 21, NO.7, 2016, COPYRIGHT 2016 EUDOXUS PRESS, LLC

[46] S. Stević, Products of integral-type operators and composition operators from the mixed norm space to Bloch-type spaces, *Siberian Math. J.* **50** (4) (2009), 726-736.
[47] S. Stević, Weighted differentiation composition operators from mixed-norm spaces to weighted-type spaces, *Appl. Math. Comput.* **211** (2009), 222-233.
[48] S. Stević, Weighted composition operators from weighted Bergman spaces to weighted-type spaces on the unit ball, *Appl. Math. Comput.* **212** (2009), 499-504.
[49] S. Stević, On new Bloch-type spaces, *Appl. Math. Comput.* **215** (2009), 841-849.
[50] S. Stević, Composition followed by differentiation from H^∞ and the Bloch space to n-th weighted-type spaces on the unit disk, *Appl. Math. Comput.* **216** (2010), 3450-3458.
[51] S. Stević, On an integral operator between Bloch-type spaces on the unit ball, *Bull. Sci. Math.* **134** (2010), 329-339.
[52] S. Stević, On operator P_φ^g from the logarithmic Bloch-type space to the mixed-norm space on unit ball, *Appl. Math. Comput.* **215** (2010), 4248-4255.
[53] S. Stević, Weighted composition operators from Bergman-Privalov-type spaces to weighted-type spaces on the unit ball, *Appl. Math. Comput.* **217** (2010) 1939-1943.
[54] S. Stević, Weighted differentiation composition operators from H^∞ and Bloch spaces to n-th weighted-type spaces on the unit disk, *Appl. Math. Comput.* **216** (2010), 3634-3641.
[55] S. Stević, R. P. Agarwal, Weighted composition operators from logarithmic Bloch-type spaces to Bloch-type spaces, *J. Inequal. Appl.* Vol. 2009, Article ID 964814, (2009), 21 pages.
[56] S. Stević, S. I. Ueki, Integral-type operators acting between weighted-type spaces on the unit ball, *Appl. Math. Comput.* **215** (2009), 2464-2471.
[57] S. Stević, R. Chen, Z. Zhou, Weighted composition operators between Bloch type spaces in the polydisc, *Mat. Sb.* **201** (1-2) (2010), 289-319.
[58] S. Stević, A. K. Sharma, A.Bhat, Products of multiplication composition and differentiation operators on weighted Bergman spaces, *Appl. Math. Comput.* **217** (2011), 8115-8125.
[59] S. Stević, A. K. Sharma, A.Bhat, Essential norm of multiplication composition and differentiation operators on weighted Bergman spaces, *Appl. Math. Comput.* **218** (2011), 2386-2397.
[60] W. Yang, Weighted composition operators from Bloch-type spaces to weighted-type spaces, *Ars. Combin.* **93** (2009), 265-274.
[61] W. Yang, W. Yan, Generalized weighted composition operators from area Nevanlinna spaces to weighted-type spaces, *Bull. Korean Math. Soc.* **48** (6) (2011), 1195-1205.
[62] K. Zhu, Spaces of holomorphic functions in the unit ball, Springer, New York, 2005.

Hong-bin Bai, School of Science, Sichuan University of Science and Engineering, Zigong, Sichuan, 643000, P. R. China

Zhi-jie Jiang, Institute of Nonlinear Science and Engineering Computing, Sichuan University of Science and Engineering, Zigong, Sichuan, 643000, P. R. China

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 21, NO.7, 2016, COPYRIGHT 2016 EUDOXUS PRESS, LLC

Lyapunov inequalities of linear Hamiltonian systems on time scales

Jing Liu[1] Taixiang Sun[1] Xin Kong[1] Qiuli He[2,*]

[1]Guangxi Colleges and Universities Key Laboratory of Mathematics and Its Applications, Nanning, 530004, China

[2]College of Electrical Engineering, Guangxi University, Nanning, Guangxi 530004, China

Abstract In this paper, we establish several Lyapunov-type inequalities for the following linear Hamiltonian systems

$$x^{\Delta}(t) = -A(t)x(\sigma(t)) - B(t)y(t), \quad y^{\Delta}(t) = C(t)x(\sigma(t)) + A^T(t)y(t)$$

on the time scale interval $[a,b]_{\mathbb{T}} \equiv [a,b] \cap \mathbb{T}$ for some $a, b \in \mathbb{T}$, where B and C are real $n \times n$ symmetric matrix-valued functions on $[a,b]_{\mathbb{T}}$ with B being semi-positive definite, A is real $n \times n$ matrix-valued function on $[a,b]_{\mathbb{T}}$ with $I + \mu(t)A$ being invertible, and x, y are real vector-valued functions on $[a,b]_{\mathbb{T}}$.

AMS Subject Classification: 34K11, 34N05, 39A10.

Keywords: Lyapunov inequality; Hamiltonian system; Time scale

1. Introduction

In 1990, Hilger introduced in [9] the theory of time scales with one goal being the unified treatment of differential equations (the continuous case) and difference equations (the discrete case). A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$, which has the topology that it inherits from the standard topology on $\mathbb{R}$. The two most popular examples are $\mathbb{R}$ and the integers $\mathbb{Z}$. The study of dynamic equations on time scales reveals such discrepancies, and helps avoid proving results twice-once for differential equations and once again for difference equations. Not only can the theory of dynamic equations unify the theories of differential equations and difference equations, but also extends these classical cases to cases "in between", e.g., to the so-called q-difference equations when $\mathbf{T} = \{1, q, q^2, \cdots, q^n, \cdots\}$, which has important applications in quantum theory (see [11]). For the time scale calculus, and some related basic concepts, and the basic notions connected to time scales, we refer the readers to the books by Bohner and Peterson [2,3] for further details.

In this paper, we study Lyapunov-type inequalities for the following linear Hamiltonian

Project Supported by NNSF of China (11461003) and NSF of Guangxi (2012GXNSFDA276040).

$*$ Corresponding author: E-mail: hql19680914@163.com

systems

$$x^{\Delta}(t) = -A(t)x(\sigma(t)) - B(t)y(t), \quad y^{\Delta}(t) = C(t)x(\sigma(t)) + A^T(t)y(t), \tag{1.1}$$

on the time scale interval $[a, b]_{\mathbb{T}} \equiv [a, b] \cap \mathbb{T}$ for some $a, b \in \mathbb{T}$, where B and C are real $n \times n$ symmetric matrix-valued functions on $[a, b]_{\mathbb{T}}$ with B being semi-positive definite, A is real $n \times n$ matrix-valued function on $[a, b]_{\mathbb{T}}$ with $I + \mu(t)A$ being invertible, and x, y are real vector-valued functions on $[a, b]_{\mathbb{T}}$.

When $n = 1$, (1.1) reduces to

$$x^{\Delta}(t) = \alpha(t)x(\sigma(t)) + \beta(t)y(t), \quad y^{\Delta}(t) = -\gamma(t)x(\sigma(t)) - \alpha(t)y(t) \tag{1.2}$$

on an arbitrary time scale $\mathbb{T}$, where $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ are real-valued rd-continuous functions defined on $\mathbb{T}$ with $\beta(t) \geq 0$ for any $t \in \mathbb{T}$.

In [10], Jiang and Zhou obtained some interesting Lyapunov-type inequalities.

Theorem 1.1[10] Suppose that for any $t \in \mathbb{T}$,

$$1 - \mu(t)\alpha(t) > 0, \;\; \beta(t) > 0, \;\; \gamma(t) > 0,$$

and let a, $b \in \mathbb{T}^k$ with $\sigma(a) < b$. Assume that (1.2) has a real solution $(x(t), y(t))$ such that $x(a)x(\sigma(a)) < 0$, and $x(b)x(\sigma(b)) < 0$. Then the inequality

$$\int_a^b |\alpha(t)| \triangle (t) + \Big[\int_a^{\sigma(b)} \beta(t) \triangle (t) \int_a^b \gamma(t) \triangle (t)\Big]^{1/2} > 1 \tag{1.3}$$

holds.

Theorem 1.2[10] Suppose that for any $t \in \mathbb{T}$,

$$1 - \mu(t)\alpha(t) > 0, \;\; \beta(t) > 0,$$

and let a, $b \in \mathbb{T}^k$ with $\sigma(a) < b$. Assume that (1.2) has a real solution $(x(t), y(t))$ such that $x(a)x(\sigma(a)) < 0$, and $x(\sigma(b)) = 0$. Then the inequality

$$\int_{\sigma(a)}^b |\alpha(t)| \triangle (t) + \Big[\int_{\sigma(a)}^{\sigma(b)} \beta(t) \triangle (t) \int_a^b \gamma^+(t) \triangle (t)\Big]^{1/2} > 1 \tag{1.4}$$

holds, where $\gamma^+(t) = \max\{\gamma(t), 0\}$.

In [8], He et al. obtained the following Lyapunov-type inequality.

Theorem 1.3[8] Suppose for any $t \in \mathbb{T}$,

$$1 - \mu(t)\alpha(t) > 0,$$

and let a, $b \in \mathbb{T}^k$ with $\sigma(a) \leq b$. Assume that (1.2) has a real solution $(x(t), y(t))$ such that $x(t)$ has generalized zeros at end-points a and b and $x(t)$ is not identically zero on $[a, b]_{\mathbb{T}} \equiv \{t \in_{\mathbb{T}}: a \leq t \leq b\}$, i.e.,

$$x(a) = 0 \text{ or } x(a)x(\sigma(a)) < 0; \;\; x(b) = 0 \text{ or } x(b)x(\sigma(b)) < 0; \;\; \max_{t \in [a,b]_{\mathbb{T}}} |x(t)| > 0.$$

Then the inequality

$$\int_a^b |\alpha(t)| \triangle (t) + \Big[\int_a^{\sigma(b)} \beta(t) \triangle (t) \int_a^b \gamma^+(t) \triangle (t) \Big]^{1/2} \geq 2 \tag{1.5}$$

holds.

For some other related results on Lyapunov-type inequality, see, for example, [1,4-6,8,10,12-16].

2. Preliminaries and some lemmas

For any $x \in \mathbb{R}^n$ and any $A \in \mathbb{R}^{n\times n}$ (the space of real $n \times n$ matrices), denote by

$$|x| = \sqrt{x^T x} \quad \text{and} \quad |A| = \max_{x\in\mathbb{R}^n, |x|=1} |Ax|$$

the Euclidean norm of x and the matrix norm of A respectively, where C^T is the transpose of a $n \times m$ matrix C. It is easy to show

$$|Ax| \leq |A||x|$$

for any $x \in \mathbb{R}^n$ and any $A \in \mathbb{R}^{n\times n}$. Denote by $\mathbb{R}_s^{n\times n}$ the space of all symmetric real $n \times n$ matrices. For $A \in \mathbb{R}_s^{n\times n}$, we say that A is semi-positive definite (resp. positive definite), written as $A \geq 0$ (resp. $A > 0$), if $x^T Ax \geq 0$ (resp. $x^T Ax > 0$) for all $x \in \mathbb{R}^n$. If A is semi-positive definite (resp. positive definite), then there exists a unique semi-positive definite matrix (resp. positive definite matrix), written as $\sqrt{A}$, such that $[\sqrt{A}]^2 = A$.

In this paper, we study Lyapunov-type inequalities of (1.1) which admits some solution $(x(t), y(t))$ satisfying

$$x(a) = x(b) = 0 \text{ and } \max_{t\in[a,b]_{\mathbb{T}}} |x(t)| > 0, \tag{2.1}$$

where $a, b \in \mathbb{T}$ with $\sigma(a) < b$, A, B, $C \in C_{rd}(\mathbb{T}, \mathbb{R}^{n\times n})$ are $n \times n$-matrix-valued functions on $\mathbb{T}$ with $I + \mu(t)A$ being invertible, B,$C \in \mathbb{R}_s^{n\times n}$ and $B \geq 0$. we first introduce the following notions and lemmas.

A partition of $[a, b)_{\mathbb{T}}$ is any finite ordered subset $P = \{t_0, t_1, \cdots, t_n\} \subset [a, b]_{\mathbb{T}}$ with $a = t_0 < t_1 < \cdots < t_n = b$. For given $\delta > 0$, we denote by $\mathcal{P}_\delta([a, b)_{\mathbb{T}})$ the set of all partitions $P : a = t_0 < t_1 < \cdots < t_n = b$ that possess the property: for every $i \in \{1, 2, \cdots, n\}$, either $t_i - t_{i-1} \leq \delta$ or $t_i - t_{i-1} > \delta$ and $\sigma(t_i) = t_{i-1}$.

Definition 2.1[7] Let f be a bounded function on $[a, b)_{\mathbb{T}}$, and let $P : a = t_0 < t_1 < \cdots < t_n = b$ be a partition of $[a, b)_{\mathbb{T}}$. In each interval $[t_{i-1}, t_i)_{\mathbb{T}}$ $(1 \leq i \leq n)$, choose an arbitrary point ξ_i and form the sum

$$S(P, f) = \Sigma_{i=1}^n f(\xi_i)(t_i - t_{i-1}).$$

We say that f is Δ-integrable from a to b (or on $[a, b)_{\mathbb{T}}$) if there exists a constant number I with the following property: for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|S(P, f) - I| < \varepsilon$$

for every $P \in \mathcal{P}_\delta([a,b)_{\mathbb{T}})$ independent of the way in which we choose $\xi_i \in [t_{i-1}, t_i)_{\mathbb{T}}$ $(1 \le i \le n)$.

It is easily seen that such a constant number I is unique. The number I, written as $\int_a^b f(t)\Delta t$, is called the Δ-integral of f from a to b.

Remark 2.2 In [7], Guseinov showed that if there exists $F : \mathbb{T} \to \mathbb{R}$ such that $F^{\triangle}(t) = f(t)$ holds for all $t \in \mathbb{T}^k$, then

$$\int_a^b f(t) \triangle t = F(b) - F(a), \text{ for any } a, b \in \mathbb{T}.$$

Lemma 2.3 Let $a_i, b_i, c_i \in \mathbb{R}$ $(i \in \{1, 2, \cdots, n\})$ with $c_i \ge 0$. Then

$$\Big(\sum_{i=1}^n a_i c_i\Big)^2 + \Big(\sum_{i=1}^n b_i c_i\Big)^2 \le \Big[\sum_{i=1}^n \sqrt{a_i^2 + b_i^2}\ c_i\Big]^2. \tag{2.2}$$

Proof. Since $2a_ib_ia_jb_j \le b_i^2a_j^2 + b_j^2a_i^2$ for any $i, j \in \{1, 2, \cdots, n\}$, we have

$$a_ic_ia_jc_j + b_ic_ib_jc_j \le \sqrt{a_i^2 + b_i^2}c_i\sqrt{a_j^2 + b_j^2}c_j,$$

which implies

$$\sum_{i=1}^n\sum_{j=1}^n(a_ic_ia_jc_j + b_ic_ib_jc_j) \le \sum_{i=1}^n\sum_{j=1}^n \sqrt{a_i^2 + b_i^2}c_i\sqrt{a_j^2 + b_j^2}c_j.$$

That is

$$\Big(\sum_{i=1}^n a_i c_i\Big)^2 + \Big(\sum_{i=1}^n b_i c_i\Big)^2 \le \Big[\sum_{i=1}^n \sqrt{a_i^2 + b_i^2}\ c_i\Big]^2.$$

This completes the proof of Lemma 2.3

Lemma 2.4 Let $f, g, f^2 + g^2$ be Δ-integrable from a to b. Then

$$\Big[\int_a^b f(t)\Delta t\Big]^2 + \Big[\int_a^b g(t)\Delta t\Big]^2 \le \Big[\int_a^b \sqrt{f^2(t) + g^2(t)}\Delta t\Big]^2. \tag{2.3}$$

Proof. By Definition 2.1, for any $\varepsilon > 0$ there exists $\delta_i > 0$ $(i = 1, 2, 3)$ such that

$$|S(P_1, f) - \int_a^b f(t)\Delta t| < \varepsilon, \tag{2.4}$$

$$|S(P_2, g) - \int_a^b g(t)\Delta t| < \varepsilon \tag{2.5}$$

and

$$|S(P_3, \sqrt{f^2(t) + g^2(t)}) - \int_a^b \sqrt{f^2(t) + g^2(t)}\Delta t| < \varepsilon \tag{2.6}$$

for every $P_i \in \mathcal{P}_{\delta_i}([a,b)_{\mathbb{T}})$. Let $P = P_1 \cup P_2 \cup P_3$ $(\in \cap_{i=1}^3 \mathcal{P}_{\delta_i}([a,b)_{\mathbb{T}}))$: $a = t_0 < t_1 < \cdots < t_n = b$

and choose an arbitrary point $\xi_i \in [t_{i-1}, t_i)$. Then from (2.4)-(2.6) and Lemma 2.3 we have

$$\begin{aligned}
\Big[\int_a^b f(t)\Delta t\Big]^2 + \Big[\int_a^b g(t)\Delta t\Big]^2 &\leq [|S(P,f)| + \varepsilon]^2 + [|S(P,g)| + \varepsilon]^2 \\
&= [|\Sigma_{i=1}^n f(\xi_i)(t_i - t_{i-1})| + \varepsilon]^2 + [|\Sigma_{i=1}^n g(\xi_i)(t_i - t_{i-1})| + \varepsilon]^2 \\
&\leq [\Sigma_{i=1}^n f(\xi_i)(t_i - t_{i-1})]^2 + [\Sigma_{i=1}^n g(\xi_i)(t_i - t_{i-1})]^2 \\
&\quad +2\varepsilon\Big[|\int_a^b f(t)\Delta t| + |\int_a^b g(t)\Delta t| + 3\varepsilon\Big] \\
&\leq [\Sigma_{i=1}^n \sqrt{f^2(\xi_i) + g^2(\xi_i)}(t_i - t_{i-1})]^2 \\
&\quad +2\varepsilon\Big[|\int_a^b f(t)\Delta t| + |\int_a^b g(t)\Delta t| + 3\varepsilon\Big] \\
&\leq [\int_a^b \sqrt{f^2(t) + g^2(t)}\Delta t + \varepsilon]^2 \\
&\quad +2\varepsilon\Big[|\int_a^b f(t)\Delta t| + |\int_a^b g(t)\Delta t| + 3\varepsilon\Big].
\end{aligned}$$

Let $\varepsilon \longrightarrow 0$, we obtain (2.3). This completes the proof of Lemma 2.4.

Corollary 2.5 Let $a, b \in \mathbb{T}$ with $a < b$ and $f_1(t), f_2(t), \cdots, f_n(t)$ be Δ-integrable on $[a,b]_{\mathbb{T}}$. write $x(t) = (f_1(t), f_2(t), \cdots, f_n(t))$. Then

$$|\int_a^b x(t)\Delta t| = \Big\{\sum_{i=1}^n \Big(\int_a^b f_i(t)\Delta t\Big)^2\Big\}^{\frac{1}{2}} \leq \int_a^b \Big\{\sum_{i=1}^n f_i^2(t)\Big\}^{\frac{1}{2}}\Delta t = \int_a^b |x(t)|\Delta t. \tag{2.7}$$

Proof. By Lemma 2.4, we know that (2.7) holds when $n = 2$. Assume that (2.7) holds when $n = k \geq 2$, that is

$$\sum_{i=1}^k \Big(\int_a^b f_i(t)\Delta t\Big)^2 \leq \Big[\int_a^b \Big\{\sum_{i=1}^k f_i^2(t)\Big\}^{\frac{1}{2}}\Delta t\Big]^2.$$

Then

$$\begin{aligned}
\Big[\int_a^b \Big\{\sum_{i=1}^{k+1} f_i^2(t)\Big\}^{\frac{1}{2}}\Delta t\Big]^2 &= \Big\{\int_a^b \{f_{k+1}^2(t) + [(\sum_{i=1}^k f_i^2(t))^{\frac{1}{2}}]^2\}^{\frac{1}{2}}\Delta t\Big\}^2 \\
&\geq \Big(\int_a^b f_{k+1}(t)\Delta t\Big)^2 + \Big[\int_a^b \{\sum_{i=1}^k f_i^2(t)\}^{\frac{1}{2}}\Delta t\Big]^2 \\
&\geq \sum_{i=1}^{k+1}\Big(\int_a^b f_i(t)\Delta t\Big)^2.
\end{aligned}$$

This completes the proof of Corollary 2.5.

Lemma 2.6[2] (Cauchy-Schwarz inequality) Let $a, b \in \mathbb{T}$ and $f, g \in C_{rd}(\mathbb{T}, \mathbb{R})$. Then

$$\int_a^b |f(t)g(t)| \bigtriangleup (t) \leq \Big\{\int_a^b f^2(t) \bigtriangleup (t) \cdot \int_a^b g^2(t) \bigtriangleup (t)\Big\}^{\frac{1}{2}}. \tag{2.8}$$

Lemma 2.7[2] Suppose that $A \in C_{rd}(\mathbb{T}, \mathbb{R}^{n\times n})$ with $I + \mu(t)A$ being invertible and $f \in C_{rd}(\mathbb{T}, \mathbb{R}^n)$. Let $t_0 \in \mathbb{T}$ and $x_o \in \mathbb{R}^n$. Then the initial value problem

$$x^{\Delta}(t) = -A(t)x(\sigma(t)) + f(t),\ x(t_0) = x_0$$

has a unique solution $x : \mathbb{T} \to \mathbb{R}^n$. Moreover, this solution is given by

$$x(t) = e_{\ominus A}(t, t_0)x_0 + \int_{t_0}^{t} e_{\ominus A}(t, \tau)f(\tau)\Delta\tau. \tag{2.9}$$

Lemma 2.8 Let $C \in \mathbb{R}_s^{n\times n}$. Then for any $C_1 \in \mathbb{R}_s^{n\times n}$ with $C_1 \geq C$ (i.e.,$C_1 - C \geq 0$), we have

$$(x^\sigma)^T Cx^\sigma \leq |C_1||x^\sigma|^2, x \in \mathbb{R}^n. \tag{2.10}$$

Proof. For $C, C_1 \in \mathbb{R}_s^{n\times n}$ with $C_1 \geq C$, we have $C_1 - C \geq 0$. Then for all $x \in \mathbb{R}^n$, we obtain $(x^\sigma)^T(C_1 - C)x^\sigma \geq 0$. Thus

$$\begin{aligned} (x^\sigma)^T Cx^\sigma &\leq (x^\sigma)^T C_1 x^\sigma \leq |x^\sigma||C_1 x^\sigma| \\ &\leq |x^\sigma||C_1||x^\sigma| = |C_1||x^\sigma|^2. \end{aligned}$$

This completes the proof of Lemma 2.8.

3. Main results and proofs

Denote

$$\xi(\sigma(t)) = \int_a^{\sigma(t)} |B(s)||e_{\ominus A}(\sigma(t), s)|^2\Delta s \tag{3.1}$$

and

$$\eta(\sigma(t)) = \int_{\sigma(t)}^{b} |B(s)||e_{\ominus A}(\sigma(t), s)|^2\Delta s. \tag{3.2}$$

Theorem 3.1 Let $a, b \in \mathbb{T}$ with $\sigma(a) < b$. If (1.1) has a solution $(x(t), y(t))$ satisfying (2.1) on the interval $[a, b]_\mathbb{T}$, then for any $C_1 \in \mathbb{R}_s^{n\times n}$ with $C_1(t) \geq C(t)$, one has the following inequality

$$\int_a^b \frac{\xi(\sigma(t))\eta(\sigma(t))}{\xi(\sigma(t)) + \eta(\sigma(t))}|C_1(t)| \triangle t \geq 1. \tag{3.3}$$

Proof. At first let us notice that any solution $(x(t), y(t))$ of (1.1) satisfies the following equality

$$\begin{aligned} (y^T(t)x(t))^\Delta &= (y^T(t))^\Delta x^\sigma(t) + y^T(t)x^\Delta(t) \\ &= (x^\sigma(t))^T y^\Delta(t) + y^T(t)x^\Delta(t) \\ &= (x^\sigma(t))^T C(t)x^\sigma(t) - y^T(t)B(t)y(t). \end{aligned} \tag{3.4}$$

By integrating (3.4) from a to b and taking into account that $x(a) = x(b) = 0$, one has

$$\int_a^b y^T(t)B(t)y(t) \triangle t = \int_a^b (x^\sigma(t))^T C(t)x^\sigma(t) \triangle t.$$

Moreover, since $B(t)$ is semi-positive definite, we have

$$y^T(t)B(t)y(t) \geq 0,\ t \in [a, b]_\mathbb{T}.$$

If

$$y^T(t)B(t)y(t) \equiv 0,\ t \in [a, b]_\mathbb{T}$$

then

$$B(t)y(t) = 0.$$

Thus the first equation of (1.1) would read as

$$x^{\Delta}(t) = -A(t)x(\sigma(t)),\ x(a) = 0.$$

By Lemma 2.7, it follows

$$x(t) = e_{\ominus A}(t, a) \cdot 0 = 0,$$

a contradiction with (2.1). Hence we have that

$$\int_a^b y^T(t)B(t)y(t) \triangle t = \int_a^b (x^{\sigma})^T(t)C(t)x^{\sigma}(t) \triangle t > 0, \tag{3.5}$$

and for $t \in [a, b]_{\mathbb{T}}$, let $t_0 = a$ and $t_0 = b$, from Lemma 2.7, we obtain

$$x(t) = -\int_a^t e_{\ominus A}(t, \tau)B(\tau)y(\tau)\Delta\tau = -\int_b^t e_{\ominus A}(t, \tau)B(\tau)y(\tau)\Delta\tau. \tag{3.6}$$

Which follows that for $t \in [a, b)_{\mathbb{T}}$,

$$x^{\sigma}(t) = -\int_a^{\sigma(t)} e_{\ominus A}(\sigma(t), \tau)B(\tau)y(\tau)\Delta\tau = +\int_{\sigma(t)}^b e_{\ominus A}(\sigma(t), \tau)B(\tau)y(\tau)\Delta\tau. \tag{3.7}$$

Note that for $a \le \tau \le \sigma(t) \le b$,

$$\begin{aligned}
|e_{\ominus A}(\sigma(t), \tau)B(\tau)y(\tau)| &\le |e_{\ominus A}(\sigma(t), \tau)||B(\tau)y(\tau)| \\
&= |e_{\ominus A}(\sigma(t), \tau)|\{y^T(\tau)B^T(\tau)B(\tau)y(\tau)\}^{\frac{1}{2}} \\
&= |e_{\ominus A}(\sigma(t), \tau)|\{(\sqrt{B(\tau)}y(\tau))^T B(\tau)\sqrt{B(\tau)}y(\tau)\}^{\frac{1}{2}} \\
&\le |e_{\ominus A}(\sigma(t), \tau)|\{|\sqrt{B(\tau)}y(\tau)||B(\tau)||\sqrt{B(\tau)}y(\tau)|\}^{\frac{1}{2}} \\
&= |e_{\ominus A}(\sigma(t), \tau)||B(\tau)|^{\frac{1}{2}}(y^T(\tau)B(\tau)y(\tau))^{\frac{1}{2}}.
\end{aligned}$$

Then from Corollary 2.5 and Lemma 2.6 we obtain

$$\begin{aligned}
|x^{\sigma}(t)| &= |\int_a^{\sigma(t)} e_{\ominus A}(\sigma(t), \tau)B(\tau)y(\tau)\Delta\tau| \\
&\le \int_a^{\sigma(t)} |e_{\ominus A}(\sigma(t), \tau)B(\tau)y(\tau)|\Delta\tau \\
&\le \int_a^{\sigma(t)} |e_{\ominus A}(\sigma(t), \tau)||B(\tau)|^{\frac{1}{2}}(y^T(\tau)B(\tau)y(\tau))^{\frac{1}{2}}\Delta\tau \\
&\le \Big(\int_a^{\sigma(t)} |e_{\ominus A}(\sigma(t), \tau)|^2|B(\tau)|\Delta\tau\Big)^{\frac{1}{2}}\Big(\int_a^{\sigma(t)} y^T(\tau)B(\tau)y(\tau)\Delta\tau\Big)^{\frac{1}{2}},
\end{aligned}$$

that is

$$|x^{\sigma}(t)|^2 \le \xi(\sigma(t))\int_a^{\sigma(t)} y^T(\tau)B(\tau)y(\tau)\Delta\tau. \tag{3.8}$$

Similarly, by letting $\eta(\sigma(t))$ be as in (3.2), for $a \le \sigma(t) \le \tau \le b$, we have

$$|x^{\sigma}(t)|^2 \le \eta(\sigma(t))\int_{\sigma(t)}^b y^T(\tau)B(\tau)y(\tau)\Delta\tau. \tag{3.9}$$

It follows from (3.8) and (3.9) that

$$\eta(\sigma(t))\xi(\sigma(t))\int_a^{\sigma(t)} y^T(\tau)B(\tau)y(\tau)\Delta\tau \geq |x^\sigma(t)|^2\eta(\sigma(t))$$

and

$$\eta(\sigma(t))\xi(\sigma(t))\int_{\sigma(t)}^{b} y^T(\tau)B(\tau)y(\tau)\Delta\tau \geq |x^\sigma(t)|^2\xi(\sigma(t)).$$

Thus

$$|x^\sigma(t)|^2 \leq \frac{\xi(\sigma(t))\eta(\sigma(t))}{\xi(\sigma(t))+\eta(\sigma(t))}\int_a^b y^T(\tau)B(\tau)y(\tau)\Delta\tau.$$

By Lemma 2.8 we see

$$\begin{aligned}
\int_a^b |C_1(t)||x^\sigma(t)|^2\Delta t &\leq \int_a^b (|C_1(t)|\frac{\xi(\sigma(t))\eta(\sigma(t))}{\xi(\sigma(t))+\eta(\sigma(t))}\int_a^b y^T(\tau)B(\tau)y(\tau)\Delta\tau)\Delta t \\
&= \int_a^b |C_1(t)|\frac{\xi(\sigma(t))\eta(\sigma(t))}{\xi(\sigma(t))+\eta(\sigma(t))}\Delta t\int_a^b y^T(\tau)B(\tau)y(\tau)\Delta\tau \\
&= \int_a^b |C_1(t)|\frac{\xi(\sigma(t))\eta(\sigma(t))}{\xi(\sigma(t))+\eta(\sigma(t))}\Delta t\int_a^b (x^\sigma(t))^T C(t)x^\sigma(t)\bigtriangleup t \\
&\leq \int_a^b |C_1(t)|\frac{\xi(\sigma(t))\eta(\sigma(t))}{\xi(\sigma(t))+\eta(\sigma(t))}\Delta t\int_a^b |C_1(t)||x^\sigma(t)|^2\Delta t.
\end{aligned}$$

Since

$$\int_a^b |C_1(t)||x^\sigma(t)|^2\Delta t \geq \int_a^b (x^\sigma)^T(t)C(t)x^\sigma(t)\bigtriangleup t = \int_a^b y^T(t)B(t)y(t)\Delta t > 0,$$

we get

$$\int_a^b \frac{\xi(\sigma(t))\eta(\sigma(t))}{\xi(\sigma(t))+\eta(\sigma(t))}|C_1(t)|\bigtriangleup t \geq 1.$$

This completes the proof of Theorem 3.1.

Theorem 3.2 Let $a,b\in\mathbb{T}$ with $\sigma(a)<b$. If (1.1) has a solution $(x(t),y(t))$ satisfying (2.1) on the interval $[a,b]_{\mathbb{T}}$, then for any $C_1\in\mathbb{R}_s^{n\times n}$ with $C_1(t)\geq C(t)$, one has the following inequality

$$\int_a^b |C_1(t)|\Big\{\int_a^b |B(s)||e_{\ominus A}(\sigma(t),s)|^2\Delta s\Big\}\bigtriangleup t \geq 4. \tag{3.10}$$

Proof. Note

$$\frac{\xi(\sigma(t))\eta(\sigma(t))}{\xi(\sigma(t))+\eta(\sigma(t))} \leq \frac{\xi(\sigma(t))+\eta(\sigma(t))}{4}.$$

It follows from (3.3) that

$$\int_a^b \frac{\xi(\sigma(t))+\eta(\sigma(t))}{4}|C_1(t)|\bigtriangleup t \geq 1.$$

Combining (3.1) and (3.2), we obtain

$$\int_a^b \Big(\int_a^b |B(s)||e_{\ominus A}(\sigma(t),s)|^2\Delta s|C_1(t)|\Big)\bigtriangleup t \geq 4.$$

That is

$$\int_a^b |C_1(t)|\Big\{\int_a^b |B(s)||e_{\ominus A}(\sigma(t),s)|^2\Delta s\Big\}\triangle t\geq 4.$$

This completes the proof of Theorem 3.2.

Theorem 3.3 Let $a,b\in\mathbb{T}$ with $\sigma(a)<b$. If (1.1) has a solution $(x(t),y(t))$ satisfying (2.1) on the interval $[a,b]_{\mathbb{T}}$, then for any $C_1\in\mathbb{R}_s^{n\times n}$ with $C_1(t)\geq C(t)$, one has the following inequality

$$\int_a^b |A(t)|\triangle t+\Big(\int_a^b |\sqrt{B(t)}|^2\triangle t\Big)^{1/2}\Big(\int_a^b |C_1(t)|\triangle t\Big)^{1/2}\geq 2. \tag{3.11}$$

Proof. From the proof of Theorem 3.1, we have

$$\int_a^b y^T(t)B(t)y(t)\triangle t=\int_a^b (x^\sigma(t))^TC(t)x^\sigma(t)\triangle t.$$

It follows from the first equation of (1.1) that for all $a\leq t\leq b$, we get

$$x(t)=\int_a^t(-A(\tau)x^\sigma(\tau)-B(\tau)y(\tau))\triangle\tau$$

$$x(t)=\int_t^b(A(\tau)x^\sigma(\tau)+B(\tau)y(\tau))\triangle\tau.$$

Thus, from Corollary 2.5, Lemma 2.6 and Lemma 2.8 we obtain

$$\begin{aligned}
|x(t)| &= \frac{1}{2}\Big[\Big|\int_a^t(A(\tau)x^\sigma(\tau)+B(\tau)y(\tau))\triangle\tau\Big|+\Big|\int_t^b(A(\tau)x^\sigma(\tau)+B(\tau)y(\tau))\triangle\tau\Big|\Big]\\
&\leq \frac{1}{2}\Big[\int_a^t|A(\tau)x^\sigma(\tau)+B(\tau)y(\tau)|\triangle\tau+\int_t^b|A(\tau)x^\sigma(\tau)+B(\tau)y(\tau)|\triangle\tau\Big]\\
&\leq \frac{1}{2}\Big[\int_a^b(|A(\tau)x^\sigma(\tau)|+|B(\tau)y(\tau)|)\triangle\tau\Big]\\
&\leq \frac{1}{2}\Big[\int_a^b|A(\tau)||x^\sigma(\tau)|\triangle\tau+\int_a^b|\sqrt{B(\tau)}||\sqrt{B(\tau)}y(\tau)|\triangle\tau\Big]\\
&\leq \frac{1}{2}\Big[\int_a^b|A(t)||x^\sigma(t)|\triangle t+\Big(\int_a^b|\sqrt{B(t)}|^2\triangle t\Big)^{1/2}\Big(\int_a^b|\sqrt{B(t)}y(t)|^2\triangle t\Big)^{1/2}\Big]\\
&= \frac{1}{2}\Big[\int_a^b|A(t)||x^\sigma(t)|\triangle t+\Big(\int_a^b|\sqrt{B(t)}|^2\triangle t\Big)^{1/2}\Big(\int_a^b(\sqrt{B(t)}y(t))^T\sqrt{B(t)}y(t)\triangle t\Big)^{1/2}\Big]\\
&= \frac{1}{2}\Big[\int_a^b|A(t)||x^\sigma(t)|\triangle t+\Big(\int_a^b|\sqrt{B(t)}|^2\triangle t\Big)^{1/2}\Big(\int_a^b(x^\sigma)^T(t)C(t)(x^\sigma(t))\triangle t\Big)^{1/2}\Big]\\
&\leq \frac{1}{2}\Big[\int_a^b|A(t)||x^\sigma(t)|\triangle t+\Big(\int_a^b|\sqrt{B(t)}|^2\triangle t\Big)^{1/2}\Big(\int_a^b|C_1(t)||x^\sigma(t)|^2\triangle t\Big)^{1/2}\Big].
\end{aligned}$$

Denote $M=\max_{a\leq t\leq b}|x(t)|>0$, then

$$M\leq\frac{1}{2}\Big[\int_a^b|A(t)|M\triangle t+\Big(\int_a^b|\sqrt{B(t)}|^2\triangle t\Big)^{1/2}\Big(\int_a^b|C_1(t)|M^2\triangle t\Big)^{1/2}\Big]. \tag{3.12}$$

Thus inequality (3.11) follows from (3.12).This completes the proof of Theorem 3.3.

REFERENCES

[1] R.Agarwal, M.Bohner, P.Rehak, Half-linear dynamic equations, Nonlinear Anal.Appl.1(2003):1-56.

[2] M.Bohner, A.Peterson, Dynamic Equations on Time Scales: An Introduction with Applications, Birkhauser, Boston, 2001.

[3] M.Bohner, A.Peterson, Advances in Dynamic Equations on Time Scales, Birkhauser, Boston, 2003.

[4] S.S.Cheng, A discrete analogue of the inequality of Lyapunov, Hokkaido Math.J.12(1983):10 5-112.

[5] S.Cheng, Lyapunov inequalities for differential and difference equations, Fasc.Math.23(19 91):25-41.

[6] G.Sh.Guseinov, B.Kaymakcalan, Lyapunov inequalities for discrete linear Hamiltonian systems, Comput.Math.Appl.45(2003):1399-1416.

[7] G.Sh.Guseinov, Integration on time scales, J.Math.Anal.Appl.285(2003):107-127.

[8] X.He, Q.Zhang, X.Tang, On inequalities of Lyapunov for linear Hamiltonian systems on time scales, J.Math.Anal.Appl.381(2011):695-705.

[9] S.Hilger, Analysis on measure chains − a unified approach to continuous and discrete calculus, Results Math., 18(1990):18-56.

[10] L.Jiang, Z.Zhou, Lyapunov inequality for linear Hamiltonian systems on time scales, J. Math.Anal.Appl. 310(2005):579-593.

[11] V.Kac, P.Chueng, Quantum Calculus, Universitext,2002.

[12] X.Liu, M.Tang, Lyapunov-type inequality for linear Hamiltonian systems on time scales, J.Math.Anal.Appl. 310(2005):579-593.

[13] W.Reid, A generalized Lyapunov inequality, J.Differential equations, 13(1973):182-196.

[14] X.Tang, M.Zhang, Lyapunov inequalities and stability for linear Hamiltonian systems, J.Differential Equations. 252(2012):358-381.

[15] F.Wong, S.Yu, C.Yeh, W.Lian, Lyapunov's inequality on time scales,Appl.Math.Lett.19(20 06):1293-1299.

[16] Q.Zhang, X.He, J.Jiang, On Lyapunov-type inequalities for nonlinear dynamic systems on time scales, Comput.Math.Appl. 62(2011):4028-4038.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 21, NO.7, 2016, COPYRIGHT 2016 EUDOXUS PRESS, LLC

Error analysis of distributed algorithm for large scale data classification *

Cheng Wang Feilong Cao †

Department of Applied Mathematics, China Jiliang University,
Hangzhou 310018, Zhejiang Province, P R China

Abstract

The distributed algorithm is an important and basic approach, and it is usually used for large scale data processing. This paper aims to error analysis of distributed algorithm for large scale data classification generated from Tikhonov regularization schemes associated with varying Gaussian kernels and convex loss functions. The main goal is to provide fast convergence rates for the excess misclassification error. The number of subsets randomly divided from a large scale datasets is determined to guarantee that the distributed algorithm have lower time complexity and memory complexity.

Keywords: Distributed algorithm; Classification; Large scale data; Generalization error
Mathematics Subject Classification: 68T05, 68P30.

1 Introduction

In [11], a binary classification problem, which is generated from Tikhonov regularization schemes with general convex loss functions and varying Gaussian kernels, was studied well. This paper addresses error analysis of distributed algorithm for the classification with large scale datasets. For ease of description, we first introduce some concepts and notations. Most of them are the same as that of [11].

We denote the input space by a compact subset X of $\mathbb{R}^p$. To represent the two classes, we write the output space $Y = \{-1, 1\}$. Clearly, classification algorithms produce binary classifiers $\mathcal{C} : X \to Y$, and the prediction power of such classifier $\mathcal{C}$ can be measured by using its misclassification error defined by

$$\mathcal{R}(\mathcal{C}) = \mathrm{Prob}(\mathcal{C}(x) \neq y) = \int_X P(y \neq \mathcal{C}(x)|x)\, \mathrm{d}\rho_X,$$

where ρ is a probability distribution on $Z := X \times Y$, ρ_X is the marginal distribution of ρ on X, and $P(y|x)$ is the conditional distribution at $x \in X$. So-called Bayes rule is the classifier minimizing $\mathcal{R}(\mathcal{C})$, and is given by

$$f_c(x) = \begin{cases} 1, & \text{if } P(y = 1|x) \geq P(y = -1|x), \\ -1, & \text{otherwise.} \end{cases}$$

So the excess misclassification error $\mathcal{R}(\mathcal{C}) - \mathcal{R}(f_c)$ of a classifier $\mathcal{C}$ can be used to measure the performance of the classifier $\mathcal{C}$.

In this paper we consider classifiers $\mathcal{C}_f$ induced by real-valued functions $f : X \to \mathbb{R}$, which is defined by

$$\mathcal{C}_f = \mathrm{sgn}(f)(x) = \begin{cases} 1, & \text{if } f(x) \geq 0, \\ -1, & \text{otherwise.} \end{cases}$$

The real-valued functions are generated from Tikhonov regularization schemes associated with general convex loss functions and varying Gaussian kernels.

Now we give a definition for loss function [11].

*This work was supported by the National Natural Science Foundation of China (Grant Nos. 10901137, 91330118, and 61272023) and Zhejiang Provincial Natural Science Foundation of China (No. LY14A010026)
†Corresponding author. E-mail: feilongcao@gmail.com

Definition 1.1. *(see [11]) We say $\varphi : \mathbb{R} \to \mathbb{R}_+$ is a classifying loss (function) if it is convex, differentiable at 0 with $\varphi'(0) < 0$, and the smallest zero of φ is 1.*

For details of such loss function, we refer reader to Cucker and Zhou [4].

The function on $X \times X$ given by $K^\sigma(x, x') = \exp\left\{-\frac{|x-x'|}{2\sigma^2}\right\}$ is called the Gaussian kernel with variance $\sigma > 0$. From [1], this function can be used to define a reproducing kernel Hilbert space (RKHS). We denote the RKHS by $\mathcal{H}_\sigma$.

From [10] and [5], the Tikhonov regularization scheme with the loss φ, Gaussian kernel K^σ, and a sample $\mathbf{z} = \{(x_i, y_i)\}_{i=1}^n \in Z^n$ can be defined as the solution $f_{\mathbf{z}}$ of the following minimization problem

$$f_{\mathbf{z}} = \operatorname*{argmin}_{f \in \mathcal{H}_\sigma} \left\{ \frac{1}{m} \sum_{i=1}^n \varphi(y_i f(x_i)) + \lambda \|f\|^2_{\mathcal{H}_\sigma} \right\}, \tag{1.1}$$

where $\lambda > 0$ is called the regularization parameter. The regularizing function in terms of the generalization error $\mathcal{E}^\varphi$ is defined as

$$\tilde{f}_{\sigma,\lambda} := \arg\min_{f \in \mathcal{H}_\sigma} \{\mathcal{E}^\varphi(f) + \lambda \|f\|^2_{\mathcal{H}_\sigma}\}, \quad \text{where } \mathcal{E}^\varphi(f) = \int_Z \varphi(yf(x))\, \mathrm{d}\rho.$$

This function was used in Zhang [13], De Vito et al. [6], and Yao [12]. Zhou and Xiang [11] constructed a function (denoted by $f_{\sigma,\lambda}$) which works better than $\tilde{f}_{\sigma,\lambda}$ due to the special approximation ability of varying Gaussian kernels. The construction of $f_{\sigma,\lambda}$ is done under a Sobolev smoothness condition of a measurable function f^φ_ρ minimizing $\mathcal{E}^\varphi$, i.e., for almost everywhere $x \in X$,

$$f^\varphi_\rho(x) = \operatorname*{argmin}_{t \in \mathbb{R}} \int_Y \varphi(yt)\, \mathrm{d}\rho(y|x) = \operatorname*{argmin}_{t \in \mathbb{R}} \{\varphi(t) P(y = 1|x) + \varphi(-t) P(y = -1|x)\}.$$

The constructed function $f_{\sigma,\lambda}$ was used to estimate the excess misclassification error in [11]. The following Lemma 2.2 is a key result in [11], which will be employed as a base of our proof.

We will use the concept of Sobolev space with index $s > 0$ and denote the space by $H^s(\mathbb{R}^p)$. In fact, the space is consisted by all functions in $L^2(\mathbb{R}^p)$ with the finite semi-norm

$$|f|_{H^s(\mathbb{R}^p)} = \{(2\pi)^{-n} \int_{\mathbb{R}^p} |\xi|^{2s} |\hat{f}(\xi)|^2\, \mathrm{d}\xi\}^{\frac{1}{2}},$$

where $\hat{f}$ is the Fourier transform of f defined for $f \in L^1(\mathbb{R}^p)$ as $\hat{f}(\xi) = \int_{\mathbb{R}^p} f(x) \mathrm{e}^{-\mathrm{i}x\xi}\, \mathrm{d}x$.

It was proved in Chen et al. [3] and Bartlett et al. [2] that

$$\mathcal{R}(\mathrm{sgn}(f)) - \mathcal{R}(f_c) \le c_\varphi \sqrt{\mathcal{E}^\varphi(f) - \mathcal{E}^\varphi(f^\varphi_\rho)} \tag{1.2}$$

holds for some $c_\varphi > 0$.

Although the statistical aspects of (1.1) are well investigated, the computation of (1.1) can be complicated for large data with size N. For example, in a standard implementation [9], it requires costs $\mathcal{O}(N^3)$ and $\mathcal{O}(N^2)$ in time and memory, respectively. Such scaling are prohibitive when the sample size is large.

In this work, we study a decomposition-based learning approach for large datasets, which is also called distributed algorithm for large datasets. Recently, the approach has attacked more attentions of researchers, and more results have been explored, such as McDonald et al. [8] for perceptron-based algorithms, Kleiner et al. [7] for bootstrap, and Zhang et al. [14] for parametric smooth convex optimization problems. The aim of this paper is to study the binary classification error of the distributed algorithm with varying λ and σ for general loss functions. For this purpose, we first describe the distributed algorithm [15].

We are given N samples $(x_1, y_1), \ldots, (x_N, y_N)$ drawn independent identically distributed (i.i.d.) according to the distribution ρ on $Z = X \times Y$. Rather than solving the problem (1.1) on all N samples, we execute the following three steps: (1) Divide the set of samples $\{(x_1, y_1), \ldots, (x_N, y_N)\}$ randomly and evenly into m disjoint subsets $S_1, \ldots, S_m \subset Z$, and each

S_i has $n = \frac{N}{m}$ samples; (2) For each $i = 1, 2, \ldots, m$, compute the local estimate

$$\hat{f}_i := \operatorname*{argmin}_{f \in \mathcal{H}_\sigma} \left\{ \frac{1}{n} \sum_{(x,y)\in S_i} \varphi(yf(x)) + \lambda \|f\|^2_{\mathcal{H}_\sigma} \right\};$$

(3) Average together the local estimates and output $\bar{f} = \frac{1}{m}\sum_{i=1}^m \hat{f}_i$.

Our aim is to estimate the error $\mathcal{R}(\operatorname{sgn}(\bar{f})) - \mathcal{R}(f_c)$. However, from (1.2), we only need to estimate $\mathcal{E}^\varphi(\bar{f}) - \mathcal{E}^\varphi(f^\varphi_\rho)$. The following section presents some results to bound $\mathcal{E}^\varphi(\bar{f}) - \mathcal{E}^\varphi(f^\varphi_\rho)$ and $\mathcal{R}(\operatorname{sgn}(\bar{f})) - \mathcal{R}(f_c)$. When solving each $\hat{f}_i$, similarly to [11], we take $\lambda = \lambda(n) = n^{-\gamma}$, $\sigma = \sigma(n) = \lambda^\zeta = n^{-\gamma\zeta}$, for some $\gamma, \zeta > 0$.

2 Main results

Lemma 2.1. *We have* $\mathcal{E}^\varphi(\bar{f}) - \mathcal{E}^\varphi(f^\varphi_\rho) \le \frac{1}{m}\sum_{i=1}^m \left(\mathcal{E}^\varphi(\hat{f}_i) - \mathcal{E}^\varphi(f^\varphi_\rho)\right)$.

Proof. Due to the convexity of φ, we have

$$\mathcal{E}^\varphi(\bar{f}) = \int_Z \varphi(y\bar{f}(x))\,\mathrm{d}\rho \le \int_Z \frac{1}{m}\sum_{i=1}^m \varphi(y\hat{f}_i(x))\,\mathrm{d}\rho = \frac{1}{m}\sum_{i=1}^m \int_Z \varphi(y\hat{f}_i(x))\,\mathrm{d}\rho = \frac{1}{m}\sum_{i=1}^m \mathcal{E}^\varphi(\hat{f}_i).$$

So $\mathcal{E}^\varphi(\bar{f}) - \mathcal{E}^\varphi(f^\varphi_\rho) \le \frac{1}{m}\sum_{i=1}^m \left(\mathcal{E}^\varphi(\hat{f}_i) - \mathcal{E}^\varphi(f^\varphi_\rho)\right)$. □

Now in order to bound $\mathcal{E}^\varphi(\bar{f}) - \mathcal{E}^\varphi(f^\varphi_\rho)$, we only need to estimate $\mathcal{E}^\varphi(\hat{f}_i) - \mathcal{E}^\varphi(f^\varphi_\rho)$ for each i. In fact, the results for each i are the same because $\hat{f}_i$ $(i = 1, 2, \ldots, m)$ are i.i.d., and share the same properties. We take Xiang and Zhou's approach [11] and make some modifications.

Lemma 2.2. *(see [11]) Assume that for some* $s > 0$,

$$f^\varphi_\rho = \tilde{f}^\varphi_\rho|_X \text{ for some } \tilde{f}^\varphi_\rho \in H^s(\mathbb{R}^p) \cap L^\infty(\mathbb{R}^p) \text{ and } \frac{\mathrm{d}\rho_X}{\mathrm{d}x} \in L^2(X). \tag{2.1}$$

Then we can find functions $\{f_{\sigma,\lambda} \in \mathcal{H}_\sigma : 0 < \sigma \le 1, \lambda > 0\}$ *such that*

$$\|f_{\sigma,\lambda}\|_{L^\infty(X)} \le \tilde{A}, \tag{2.2}$$

$$\mathcal{D}(\sigma,\lambda) := \mathcal{E}^\varphi(f_{\sigma,\lambda}) - \mathcal{E}^\varphi(f^\varphi_\rho) + \lambda\|f_{\sigma,\lambda}\|^2_{\mathcal{H}_\sigma} \le \tilde{A}(\sigma^s + \lambda\sigma^{-p})$$

for $0 < \sigma \le 1$, $\lambda > 0$, *where* $\tilde{A} \ge 1$ *is a constant independent of* σ *and* λ.

Using the method of error decomposition of [11], we easily obtain the following Lemma 2.3.

Lemma 2.3. *Let* φ *be a classifying loss function, we have*

$$\mathcal{E}^\varphi(f_i) - \mathcal{E}^\varphi(f^\varphi_\rho) \le \mathcal{D}(\sigma,\lambda) + \mathcal{S}_{\mathbf{z}}(f_{\sigma,\lambda}) - \mathcal{S}_{\mathbf{z}}(\hat{f}_i), \tag{2.3}$$

where $\mathcal{S}_{\mathbf{z}}(f)$ *is defined for any* f *by* $\mathcal{S}_{\mathbf{z}}(f) = [\mathcal{E}^\varphi_{\mathbf{z}}(f) - \mathcal{E}^\varphi_{\mathbf{z}}(f^\varphi_\rho)] - [\mathcal{E}^\varphi(f) - \mathcal{E}^\varphi(f^\varphi_\rho)]$, *and* $\mathcal{E}^\varphi_{\mathbf{z}}(f) = \frac{1}{n}\sum_{(x,y)\in S_i} \varphi(yf(x))$.

We also need the following Definition 2.1.

Definition 2.1. *(see [11]) A variancing power* $\tau = \tau_{\varphi,\rho}$ *of the pair* (φ, ρ) *is the maximal number* τ *in* $[0, 1]$ *such that for any* $\tilde{B} \ge 1$, *there exists* $C_1 = C_1(\tilde{B})$ *satisfying*

$$\mathbb{E}[\varphi(yf(x)) - \varphi(yf^\varphi_\rho(x))]^2 \le C_1[\mathcal{E}^\varphi(f) - \mathcal{E}^\varphi(f^\varphi_\rho)]^\tau \quad \forall f : X \to [-\tilde{B}, \tilde{B}], \tag{2.4}$$

where $\mathbb{E}\xi$ *denotes the expected value of* ξ.

The following Lemma 2.4 is to bound the second term of (2.3).

Lemma 2.4. *(see [11]) Suppose $\tilde{A}$ and $f_{\sigma,\lambda}$ are as in Lemma 2.2, $\tau = \tau_{\varphi,\rho}$ and $C_1 = C_1(\tilde{A})$ are as in Definition 2.1. Then for any $0 < \delta < 1$, with confidence $1 - \frac{\delta}{2}$, we have*

$$\mathcal{S}_{\mathbf{Z}}(f_{\sigma,\lambda}) \leq 2\left(\|\varphi\|_{C[-\tilde{A},\tilde{A}]} + C_1\right) \ln \frac{2}{\delta} n^{-\frac{1}{2-\tau}} + \left(\mathcal{E}^{\varphi}(f_{\sigma,\lambda}) - \mathcal{E}^{\varphi}(f_{\rho}^{\varphi})\right).$$

To bound the third term of (2.3), $-\mathcal{S}_{\mathbf{Z}}(\hat{f}_i)$, we need the following Lemma 2.5, Lemma 2.6, and Lemma 2.7.

Lemma 2.5. *For any $\lambda > 0$, we have $\|\hat{f}_i\|_{\mathcal{H}_\sigma} \leq \sqrt{\varphi(0)/\lambda}$.*

The proof is easy by taking $f = 0$ in the definition of $\hat{f}_i$, referring to De Vito et al. [6]. The next Lemma 2.6 is from Cucker and Zhou [4].

Lemma 2.6. *(see [4]) Let $0 \leq \tau \leq 1$, $c, B \geq 0$, and $\mathcal{G}$ be a set of functions on Z such that for every $g \in \mathcal{G}$, $\mathbb{E}(g) \geq 0$, $\|g - \mathbb{E}(g)\|_\infty \leq B$ and $\mathbb{E}(g^2) \leq c(\mathbb{E}(g))^\tau$. Then for all $\varepsilon > 0$,*

$$\operatorname*{Prob}_{\mathbf{Z}\in Z^n}\left\{\sup_{g\in\mathcal{G}} \frac{\mathbb{E}(g) - \frac{1}{n}\sum_{i=1}^n f(z_i)}{\sqrt{(\mathbb{E}(g))^\tau + \varepsilon^\tau}} > 4\varepsilon^{1-\frac{\tau}{2}}\right\} \leq \mathcal{N}(\mathcal{G},\varepsilon)\exp\left\{-\frac{n\varepsilon^{2-\tau}}{2(c + \frac{1}{3}B\varepsilon^{1-\tau})}\right\},$$

where $\mathcal{N}(\mathcal{G},\varepsilon)$ denotes the covering number to be the minimal $\ell \in \mathbb{N}$ such that there exist ℓ disks in $\mathcal{G}$ with radius ε covering $\mathcal{G}$.

Note that if $\|f\|_{\mathcal{H}_\sigma} \leq \sqrt{\varphi(0)/\lambda}$, then $\|f\|_\infty \leq C_\sigma\sqrt{\varphi(0)/\lambda}$. From the above Lemma 2.6, we obtain the following Lemma 2.7.

Lemma 2.7. *Let $\tau = \tau_{\varphi,\rho}$ with $\tilde{B} = C_\sigma\sqrt{\varphi(0)/\lambda}$ and $C_1 = C_1(\tilde{B})$ in Definition 2.1. For any $\varepsilon > 0$, there holds*

$$\operatorname*{Prob}_{\mathbf{Z}\in Z^n}\left\{\sup_{\|f\|_{\mathcal{H}_\sigma}\leq\sqrt{\varphi(0)/\lambda}} \frac{[\mathcal{E}^\varphi(f) - \mathcal{E}^\varphi(f_\rho^\varphi)] - [\mathcal{E}_{\mathbb{Z}}^\varphi(f) - \mathcal{E}_{\mathbb{Z}}^\varphi(f_\rho^\varphi)]}{\sqrt{(\mathcal{E}^\varphi(f) - \mathcal{E}^\varphi(f_\rho^\varphi))^\tau + \varepsilon^\tau}} \leq 4\varepsilon^{1-\frac{\tau}{2}}\right\} \geq$$
$$1 - \mathcal{N}\left(B_1, \frac{\varepsilon\sqrt{\lambda}}{D_1\sqrt{\varphi(0)}}\right)\exp\left\{-\frac{n\varepsilon^{2-\tau}}{2C_1 + \frac{4}{3}D_2\varepsilon^{1-\tau}}\right\},$$

where $D_1 = \max\{|\varphi'_+(-\tilde{B})|, |\varphi'_-(\tilde{B})|\}$, and $D_2 = \max\{\varphi(-1), \|\varphi\|_{C[-\tilde{B},\tilde{B}]}\}$.

Proof. We apply the above Lemma 2.6 to the function set

$$\mathcal{G} = \left\{\varphi(yf(x)) - \varphi(yf_\rho^\varphi(x)) : \|f\|_{\mathcal{H}_\sigma} \leq \sqrt{\varphi(0)/\lambda}\right\},$$

and see that each function $g \in \mathcal{G}$ satisfies $\mathbb{E}(g^2) \leq c(\mathbb{E}(g))^\tau$ for $c = C_1$. Obviously $\|g\|_\infty \leq D_2 := \max\{\varphi(-1), \|\varphi\|_{C[-\tilde{B},\tilde{B}]}\}$, so $\|g - \mathbb{E}(g)\|_\infty \leq B := 2D_2$. To draw our conclusion, we only need to bound the covering number $\mathcal{N}(\mathcal{G},\varepsilon)$. To do so, note that for f_1 and f_2 satisfying $\|f\|_{\mathcal{H}_\sigma} \leq \sqrt{\varphi(0)/\lambda}$ and $(x,y) \in Z$, we have

$$\begin{aligned}&|\{\varphi(yf_1(x)) - \varphi(yf_\rho^\varphi(x))\} - \{\varphi(yf_2(x)) - \varphi(yf_\rho^\varphi(x))\}|\\ &= |\varphi(yf_1(x)) - \varphi(yf_2(x))| \leq D_1\|f_1 - f_2\|_\infty.\end{aligned}$$

Therefore, $\mathcal{N}(\mathcal{G},\varepsilon) \leq \mathcal{N}\left(B_{\sqrt{\varphi(0)/\lambda}}, \frac{\varepsilon}{D_1}\right) = \mathcal{N}(B_1, \frac{\varepsilon\sqrt{\lambda}}{D_1\sqrt{\varphi(0)}})$, where $B_{\sqrt{\varphi(0)/\lambda}}$ denotes the ball with radius $\sqrt{\varphi(0)/\lambda}$ in $\mathcal{H}_\sigma$. The statement is proved. □

Let $\varepsilon^*(n,\lambda,\sigma,\delta)$ denote the smallest positive number ε satisfying

$$1 - \mathcal{N}\left(B_1, \frac{\varepsilon\sqrt{\lambda}}{D_1\sqrt{\varphi(0)}}\right)\exp\left\{-\frac{n\varepsilon^{2-\tau}}{2C_1 + \frac{4}{3}D_2\varepsilon^{1-\tau}}\right\} \geq 1 - \frac{\delta}{2}.$$

Then we have the following proposition.

Proposition 2.1. *Let $\sigma = \lambda^\zeta$ with $0 < \zeta < \frac{1}{p}$ (Noting p is the dimension of X), s be as in Lemma 2.2, and $f_{\sigma,\lambda} \in \mathcal{H}_\sigma$ satisfy (2.2). For any $0 < \delta < 1$, with confidence at least $1-\delta$, we have*

$$\mathcal{E}^\varphi(\hat{f}_i) - \mathcal{E}^\varphi(f_\rho^\varphi) \leq 8\tilde{A}\lambda^{\min\{s\zeta, 1-p\zeta\}} + 40\varepsilon^*(n,\lambda,\sigma,\delta) + 4(\|\varphi\|_{C[-\tilde{A},\tilde{A}]} + C_1)\ln\frac{2}{\delta}n^{-\frac{1}{2-\tau}}. \quad (2.5)$$

Proof. Xiang and Zhou [11] (see Proposition 2 in [11]) have proved that for any $0 < \delta < 1$, with confidence at least $1-\delta$,

$$\mathcal{E}^\varphi(\hat{f}_i) - \mathcal{E}^\varphi(f_\rho^\varphi) \leq 4\mathcal{D}(\sigma,\lambda) + 40\varepsilon^*(n,\lambda,\sigma,\delta) + 4(\|\varphi\|_{C[-\tilde{A},\tilde{A}]} + C_1)\ln\frac{2}{\delta}n^{-\frac{1}{2-\tau}}.$$

With Lemma 2.2 and $\sigma = \lambda^\zeta$, we have $\mathcal{D} \leq \tilde{A}(\lambda^{s\zeta} + \lambda^{1-p\zeta}) \leq 2\tilde{A}\lambda^{\min\{s\zeta,1-p\zeta\}}$. So Proposition 2.1 is proved. □

To get the more explicit bound, we need the following Lemma 2.8 to bound $\varepsilon^*(m,\lambda,\sigma,\delta)$. It can be proved via the same method as in [11].

Lemma 2.8. *Let $0 \leq \tau \leq 1$, $\lambda = n^{-\gamma}$ and $\sigma = \lambda^\zeta$ with $\gamma > 0$ and $0 < \zeta < \frac{1}{2\gamma(p+1)}$. Then we have*

$$\varepsilon^*(m,\lambda,\sigma,\delta) \leq C_2 n^{-\frac{1-2\gamma\zeta(p+1)}{2-\tau}\ln\frac{2}{\delta}}. \quad (2.6)$$

From Proposition 2.1 and Lemma 2.8, we have the following Proposition 2.2.

Proposition 2.2. *Let $\sigma = \lambda^\zeta$ and $\lambda = n^{-\gamma}$ for some $0 < \zeta < \frac{1}{p}$ and $0 < \gamma < \frac{1}{2\zeta(p+1)}$. If (2.1) is valid for some $s > 0$, then for any $0 < \delta < 1$, with confidence $1-\delta$ we have*

$$\mathcal{E}^\varphi(\hat{f}_i) - \mathcal{E}^\varphi(f_\rho^\varphi) \leq \tilde{C}n^{-\theta}\ln\frac{2}{\delta}, \quad (2.7)$$

where

$$\theta = \min\left\{s\zeta\gamma, \gamma(1-p\zeta), \frac{1-2\gamma\zeta(p+1)}{2-\tau}\right\}, \quad (2.8)$$

and $\tilde{C}$ is a constant independent of n and δ.

Proof. Applying the bound for ε^* from Lemma 2.8 on Proposition 2.1, with confidence at least $1-\delta$, we have

$$\mathcal{E}^\varphi(\hat{f}_i) - \mathcal{E}^\varphi(f_\rho^\varphi) \leq 8\tilde{A}\lambda^{\min\{s\zeta,1-p\zeta\}} + 40C_2 n^{-\frac{1-2\gamma\zeta(p+1)}{2-\tau}\ln\frac{2}{\delta}} + 4(\|\varphi\|_{C[-\tilde{A},\tilde{A}]} + C_1)\ln\frac{2}{\delta}n^{-\frac{1}{2-\tau}}.$$

Putting $\lambda = n^{-\gamma}$ into the above formula, we easily see that $\mathcal{E}^\varphi(\hat{f}_i) - \mathcal{E}^\varphi(f_\rho^\varphi) \leq \tilde{C}n^{-\theta}\ln\frac{2}{\delta}$. Here θ is given by (2.8) and $\tilde{C}$ is the constant independent of n and δ given by $\tilde{C} = 8\tilde{A} + 40C_2 + 4(\|\varphi\|_{C[-\tilde{A},\tilde{A}]} + C_1)$. □

Now we can obtain our main result to bound $\mathcal{E}^\varphi(\overline{f}) - \mathcal{E}^\varphi(f_\rho^\varphi)$.

Theorem 2.1. *Under the condition of Proposition 2.2, for any $0 < \delta < 1$, with confidence $1-\delta$ we have*

$$\mathcal{E}^\varphi(\overline{f}) - \mathcal{E}^\varphi(f_\rho^\varphi) \leq \tilde{C}n^{-\theta}\ln\frac{2m}{\delta}, \quad (2.9)$$

where θ and $\tilde{C}$ are as in Proposition 2.2.

Proof. From Proposition 2.2, for any $\delta > 0$, with confidence $1-\frac{\delta}{m}$, $\mathcal{E}^\varphi(\hat{f}_i)-\mathcal{E}^\varphi(f_\rho^\varphi) \leq \tilde{C}n^{-\theta}\ln\frac{2m}{\delta}$. From Lemma 2.1,

$$\text{Prob}\left\{\mathcal{E}^\varphi(\overline{f}) - \mathcal{E}^\varphi(f_\rho^\varphi) \leq \tilde{C}n^{-\theta}\ln\frac{2m}{\delta}\right\} \geq \text{Prob}\left\{\frac{1}{m}\sum_{i=1}^m\left(\mathcal{E}^\varphi(\hat{f}_i) - \mathcal{E}^\varphi(f_\rho^\varphi)\right) \leq \tilde{C}n^{-\theta}\ln\frac{2m}{\delta}\right\}$$
$$\geq \text{Prob}\left\{\bigcap_{i=1}^m\left\{\mathcal{E}^\varphi(\hat{f}_i) - \mathcal{E}^\varphi(f_\rho^\varphi) \leq \tilde{C}n^{-\theta}\ln\frac{2m}{\delta}\right\}\right\} \geq 1 - m \times \frac{\delta}{m} = 1-\delta.$$

□

Remark 2.1. *Given N, we take $n = m^a$, i.e. $m = N^{\frac{1}{a+1}}$ and $n = N^{\frac{a}{a+1}}$. We easily see that the above bound $\frac{1}{m^{a\theta}} \ln \frac{2m}{\delta} \to 0 (m \to \infty)$ for all $a > 0$.*

As mentioned in Introduction, the Tikhonov regularization scheme for all N samples have time complexity $\mathcal{O}(N^3)$ and memory complexity $\mathcal{O}(N^2)$. Now we can determine m (also n) to guarantee that the distributed algorithm have lower time complexity and memory complexity.

Corollary 2.1. *For any $k < 3$, the time complexity of the distributed algorithm is less than $\mathcal{O}(N^k)$ if and only if $m > N^{\frac{3-k}{2}}$.*

Proof. Let $n = m^a$, i.e. $m = N^{\frac{1}{a+1}}$. The time complexity is $m \cdot \mathcal{O}(n^3) = \mathcal{O}(m^{3a+1}) = \mathcal{O}(N^{\frac{3a+1}{a+1}})$. For $k < 3$, to ensure $\frac{3a+1}{a+1} < k$, it only needs $a < \frac{k-1}{3-k}$. So $m = N^{\frac{1}{a+1}} > N^{\frac{3-k}{2}}$. □

For memory complexity, we have a similar result as follows.

Corollary 2.2. *For any $k < 2$, the memory complexity of the distributed algorithm is less than $\mathcal{O}(N^k)$ if and only if $m > N^{2-k}$.*

Due to (1.2), we have

Theorem 2.2. $\mathcal{R}(\mathrm{sgn}(\bar{f})) - \mathcal{R}(f_c) \le c_\varphi \sqrt{\tilde{C} n^{-\theta} \ln \frac{2m}{\delta}}$.

References

[1] N. Aronszajn. Theory of reproducing kernels. Trans. Amer. Math. Soc., 68: 337-404, 1950.

[2] P. L. Bartlett, M. I. Jordan, J. D. McAuliffe. Convexity, classification, and risk bounds. J. Amer. Statis. Assoc., 101: 138-156, 2006.

[3] D. R. Chen, Q. Wu, Y. Ying, D. X. Zhou. Support vector machine soft margin classifiers: error analysis. J. Mach. Lear. Res., 5: 1143-1175, 2004.

[4] F. Cucker, D. X. Zhou. Learning Theory: An Approximation Theory Viewpoint. Cambridge University Press, 2007.

[5] N. Cristianini, J. Shawe-Taylor. An Introduction to Support Vector Machines. Cambridge University Press, 2000.

[6] E. De Vito, A. Caponnetto, L. Rosasco. Model selection for regularized least-squares algorithm in learning theory. Found. Comput. Math., 5: 59-85, 2006.

[7] A. Kleiner, A. Talwalkar, P. Sarkar, M. Jordan. Bootstrapping big data. in: Proc. 29th Inter. Conf. Mach. Lear., Edinburgh, Scotland, UK, 2012. http://www.cs.ucla.edu/ ameet/blb-icml2012-final.pdf

[8] R. McDonald, K. Hall, G. Mann. Distributed training strategies for the structured perceptron. in: Proc. 2010 Annual Conference North American Chapter of the Association for Computational Linguistics (NAACL), pp. 456-464, Association for Computational Linguistics Stroudsburg, PA, USA, 2010.

[9] C. Saunders, A. Gammerman, V. Vovk. Ridge regression learning algorithm in dual variables. in Proc. 15th Inter. Conf. Mach. Lear., pp. 515-521, Morgan Kaufmann, 1998.

[10] G. Wahba. Spline Models for Observatianal Data. SIAM, 1990.

[11] D. H. Xiang, D. X. Zhou. Classification with Gaussians and convex loss. J. Mach. Lear. Res., 10: 1147-1468, 2009.

[12] Y. Yao. On complexity issue of online learning algorithms. IEEE Trans. Inform. Theory, 56(12): 6470-6481, 2010.

[13] T. Zhang. Statistical behavior and consistency of classification methods based on convex risk minimization. Annals of Statis., 32: 56-85, 2004.

[14] Y. Zhang, J. C. Duchi, M.J. Wainwright. Communication-efficient algorithms for statistical optimization. J. Mach. Lear. Res., 14: 3321-3363, 2013.

[15] Y. Zhang, J. C. Duchi, M.J. Wainwright. Divide and Conquer Kernel Ridge Regression. JMLR: Workshop and Conference Proceedings, 30: 1-26, 2013.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 21, NO.7, 2016, COPYRIGHT 2016 EUDOXUS PRESS, LLC

Korovkin type statistical approximation theorem for a function of two variables

G. A. Anastassiou[1] and M. Arsalan Khan[2]

[1]Department of Mathematical Sciences, The University of Memphis, Memphis, TN 38152, USA

ganastss@gmail.com

[2]Department of Civil Engineering, Aligarh Muslim University, Aligarh 202002, India

mohd.arsalan.khan@hotmail.co.uk

Abstract. In this paper, we prove a Korovkin type approximation theorem for a function of two variables by using the notion of convergence in the Pringsheim's sense and statistical convergence of double sequences. We also display an example in support of our results.

Keywords and phrases: Double sequence; statistical convergence; positive linear operator; Korovkin type approximation theorem.

AMS subject classification (2000): 41A10, 41A25, 41A36, 40A30, 40G15.

1. Introduction and preliminaries

The concept of statistical convergence for sequences of real numbers was introduced by Fast [8] and further studied Fridy [9] and many others.

Let $K \subseteq \mathbb{N}$ and $K_n = \{k \leq n : k \in K\}$ .Then the *natural density* of K is defined by $\delta(K) = \lim_n n^{-1}|K_n|$ if the limit exists, where $|K_n|$ denotes the cardinality of K_n.

A sequence $x = (x_k)$ of real numbers is said to be *statistically convergent* to L provided that for every $\epsilon > 0$ the set $K_\epsilon := \{k \in \mathbb{N} : |x_k - L| \geq \epsilon\}$ has natural density zero, i.e. for each $\epsilon > 0$,

$$\lim_n \frac{1}{n}|\{j \leq n : |x_j - L| \geq \epsilon\}| = 0.$$

By the convergence of a double sequence we mean the convergence in the Pringsheim's sense [20]. A double sequence $x = (x_{jk})$ is said to be *Pringsheim's convergent* (or *P-convergent*) if for given $\epsilon > 0$ there exists an integer N such that $|x_{jk} - \ell| < \epsilon$ whenever $j, k > N$. In this case, ℓ is called the Pringsheim limit of $x = (x_{jk})$ and it is written as $P - \lim x = \ell$.

A double sequence $x = (x_{jk})$ is said to be *bounded* if there exists a positive number M such that $|x_{jk}| < M$ for all j, k.

Note that, in contrast to the case for single sequences, a convergent double sequence need not be bounded.

The idea of statistical convergence for double sequences was introduced and studied by Moricz [17] and Mursaleen and Edely [18], independently in the same year and further studied in [15].

Let $K \subseteq \mathbb{N} \times \mathbb{N}$ be a two-dimensional set of positive integers and let $K(m,n) = \{(j,k) : j \leq m, k \leq n\}$. Then the *double natural density* of the set K is defined as

$$P - \lim_{m,n} \frac{\mid K(m,n) \mid}{mn} = \delta_2(K)$$

provided that the sequence $(\mid K(m,n) \mid /mn)$ has a limit in Pringsheim's sense.

For example, let $K = \{(i^2, j^2) : i,j \in \mathbb{N}\}$. Then

$$\delta_2(K) = P - \lim_{m,n} \frac{\mid K(m,n) \mid}{mn} \leq P - \lim_{m,n} \frac{\sqrt{m}\sqrt{n}}{mn} = 0,$$

i.e. the set K has double natural density zero, while the set $\{(i, 2j) : i,j \in \mathbb{N}\}$ has double natural density $\frac{1}{2}$.

A real double sequence $x = (x_{jk})$ is said to be *statistically convergent* to the number L if for each $\epsilon > 0$, the set

$$\{(j,k), j \leq m \text{ and } k \leq n : \mid x_{jk} - L \mid \geq \epsilon\}$$

has double natural density zero. In this case we write $st_2\text{-}\lim\limits_{j,k\to\infty} x_{jk} = L$.

Remark 1.1. Note that if $x = (x_{jk})$ is P-convergent then it is statisically convergent but not conversely. See the following example.

Example 1.1. The double sequence $w = (w_{jk})$ defined by

$$w_{jk} = \begin{cases} 1 \ , & \text{if } j \text{ and } k \text{ are squares;} \\ 0 \ , & \text{otherwise} \quad . \end{cases} \tag{1.1.1}$$

Then w is statistically convergent to zero but not P-convergent.

Let $C[a,b]$ be the space of all functions f continuous on $[a,b]$ equipped with the norm

$$\|f\|_{C[a,b]} := \sup_{x\in[a,b]} |f(x)|, \ f \in C[a,b].$$

The classical Korovkin approximation theorem states as follows (cf. [10], [13]):

Let (T_n) be a sequence of positive linear operators from $C[a,b]$ into $C[a,b]$. Then $\lim_n \|T_n(f,x) - f(x)\|_{C[a,b]} = 0$, for all $f \in C[a,b]$ if and only if $\lim_n \|T_n(f_i,x) - f_i(x)\|_{C[a,b]} = 0$, for $i = 0,1,2$, where $f_0(x) = 1, f_1(x) = x$ and $f_2(x) = x^2$.

Korovkin type approximation theorems are also proved for different summability methods to replace the ordinary convergence, e.g. [4], [7], [11], [14], [16] etc..

Quite recently, such type of approximation theorems are proved in [1], [2], [3], [6] and [19] for functions of two variables by using almost convergence and statistical convergence of double sequences, respectively. For single sequences, Boyanov and Veselinov [2] have proved the Korovkin theorem on $C[0,\infty)$ by using the test functions $1, e^{-x}, e^{-2x}$. In this paper, we extend the result of Boyanov and Veselinov for functions of two variables by using the notion of Pringsheim's convergence and statistical convergence of double sequences.

2. Main result

Let $C(I^2)$ be the Banach space with the uniform norm $\| \, . \, \|$ of all real-valued two dimensional continuous functions on $I \times I$, where $I = [0,\infty)$; provided that $\lim_{(x,y)\to(\infty,\infty)} f(x,y)$ is finite. Suppose that $T_{m,n} : C(I^2) \to C(I^2)$. We write $T_{m,n}(f;x,y)$ for $T_{m,n}(f(s,t);x,y)$; and we say that T is a positive operator if $T(f;x,y) \geq 0$ for all $f(x,y) \geq 0$.

The following result is an extension of Boyanov and Veselinov theorem [5] for functions of two variables.

Theorem 2.1. Let $(T_{j,k})$ be a double sequence of positive linear operators from $C(I^2)$ into $C(I^2)$. Then for all $f \in C(I^2)$

$$P\text{-}\lim_{j,k\to\infty} \left\| T_{j,k}(f;x,y) - f(x,y) \right\| = 0. \tag{2.1.0}$$

if and only if

$$P\text{-}\lim_{j,k\to\infty} \left\| T_{j,k}(1;x,y) - 1 \right\| = 0, \tag{2.1.1}$$

$$P\text{-}\lim_{j,k\to\infty} \left\| T_{j,k}(e^{-s};x,y) - e^{-x} \right\| = 0, \tag{2.1.2}$$

$$P\text{-}\lim_{j,k\to\infty} \left\| T_{j,k}(e^{-t};x,y) - e^{-y} \right\| = 0, \tag{2.1.3}$$

$$P\text{-}\lim_{j,k\to\infty} \left\| T_{j,k}(e^{-2s}+e^{-2t};x,y) - (e^{-2x}+e^{-2y}) \right\| = 0. \tag{2.1.4}$$

Proof. Since each $1, e^{-x}, e^{-y}, e^{-2x}+e^{-2y}$ belongs to $C(I^2)$, conditions (2.1.1)-(2.1.4) follow immediately from (2.1.0). Let $f \in C(I^2)$. There exist aconstant $M > 0$ such that $|f(x,y)| \leq M$ for each $(x,y) \in I^2$. Therefore,

$$|f(s,t)-f(x,y)| \leq 2M, \quad -\infty < s,t,x,y < \infty. \tag{2.1.5}$$

It is easy to prove that for a given $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|f(s,t)-f(x,y)| < \varepsilon, \tag{2.1.6}$$

whenever $|e^{-s}-e^{-x}| < \delta$ and $|e^{-t}-e^{-y}| < \delta$ for all $(x,y) \in I^2$.
Using (2.1.5), (2.1.6), putting $\psi_1 = \psi_1(s,x) = (e^{-s}-e^{-x})^2$ and $\psi_2 = \psi_2(t,y) = (e^{-t}-e^{-y})^2$, we get

$$|f(s,t)-f(x,y)| < \varepsilon + \frac{2M}{\delta^2}(\psi_1+\psi_2), \quad \forall\ |s-x| < \delta \text{ and } |t-y| < \delta.$$

This is,

$$-\varepsilon - \frac{2M}{\delta^2}(\psi_1+\psi_2) < f(s,t)-f(x,y) < \varepsilon + \frac{2M}{\delta^2}(\psi_1+\psi_2).$$

Now, operating $T_{j,k}(1;x,y)$ to this inequality since $T_{j,k}(f;x,y)$ is monotone and linear. We obtain

$$\begin{aligned} T_{j,k}(1;x,y)\left(-\varepsilon - \frac{2M}{\delta^2}(\psi_1+\psi_2)\right) &< T_{j,k}(1;x,y)(f(s,t)-f(x,y)) \\ &< T_{j,k}(1;x,y)\left(\varepsilon + \frac{2M}{\delta^2}(\psi_1+\psi_2)\right). \end{aligned}$$

Note that x and y are fixed and so $f(x,y)$ is constant number. Therefore

$$\begin{aligned} -\varepsilon T_{j,k}(1;x,y) - \frac{2M}{\delta^2}T_{j,k}(\psi_1+\psi_2;x,y) &< T_{j,k}(f;x,y) - f(x,y)T_{j,k}(1;x,y) \\ &< \varepsilon T_{j,k}(1;x,y) + \frac{2M}{\delta^2}T_{j,k}(\psi_1+\psi_2;x,y). \end{aligned} \tag{2.1.7}$$

But

$$\begin{aligned} T_{j,k}(f;x,y) - f(x,y) &= T_{j,k}(f;x,y) - f(x,y)T_{j,k}(1;x,y) + f(x,y)T_{j,k}(1;x,y) - f(x,y) \\ &= [T_{j,k}(f;x,y) - f(x,y)T_{j,k}(1;x,y)] + f(x,y)[T_{j,k}(1;x,y)-1]. \end{aligned} \tag{2.1.8}$$

Using (2.1.7) and (2.1.8), we have

$$T_{j,k}(f;x,y) - f(x,y) < \varepsilon T_{j,k}(1;x,y) + \frac{2M}{\delta^2}T_{j,k}(\psi_1+\psi_2;x,y) + f(x,y)(T_{j,k}(1;x,y)-1). \tag{2.1.9}$$

Now

$$T_{j,k}(\psi_1+\psi_2;x,y) = T_{j,k}((e^{-s}-e^{-x})^2 + (e^{-t}-e^{-y})^2;x,y)$$

$$= T_{j,k}(e^{-2s} - 2e^{-s}e^{-x} + e^{-2x} + e^{-2t} - 2e^{-t}e^{-y} + e^{-2y}; x, y)$$
$$= T_{j,k}(e^{-2s} + e^{-2t}; x, y) - 2e^{-x}T_{j,k}(s; x, y) - 2e^{-y}T_{j,k}(t; x, y)$$
$$+(e^{-2x} + e^{-2y})T_{j,k}(1; x, y)$$
$$= [T_{j,k}(e^{-2s} + e^{-2t}; x, y) - (e^{-2x} + e^{-2y})] - 2e^{-x}[T_{j,k}(e^{-s}; x, y) - e^{-x}]$$
$$- 2e^{-y}[T_{j,k}(e^{-t}; x, y) - e^{-y}] + (e^{-2x} + e^{-2y})[T_{j,k}(1; x, y) - 1].$$

Using (2.1.9), we obtain

$$T_{j,k}(f; x, y) - f(x, y) < \varepsilon T_{j,k}(1; x, y) + \frac{2M}{\delta^2}\{[T_{j,k}((e^{-2s} + e^{-2t}); x, y) - (e^{-2x} + e^{-2y})]$$
$$- 2e^{-x}[T_{j,k}(e^{-s}; x, y) - e^{-x}] - 2e^{-y}[T_{j,k}(e^{-t}; x, y) - e^{-y}]$$
$$+ (e^{-2x} + e^{-2y})[T_{j,k}(1; x, y) - 1]\} + f(x, y)(T_{j,k}(1; x, y) - 1)$$
$$= \varepsilon[T_{j,k}(1; x, y) - 1] + \varepsilon + \frac{2M}{\delta^2}\{[T_{j,k}((e^{-2s} + e^{-2t}); x, y) - (e^{-2x} + e^{-2y})]$$
$$- 2e^{-x}[T_{j,k}(e^{-s}; x, y) - e^{-x}] - 2e^{-y}[T_{j,k}(e^{-t}; x, y) - e^{-y}]$$
$$+ (e^{-2x} + e^{-2y})[T_{j,k}(1; x, y) - 1]\} + f(x, y)(T_{j,k}(1; x, y) - 1).$$

Since ε is arbitrary, we can write

$$T_{j,k}(f; x, y) - f(x, y) \leq \varepsilon[T_{j,k}(1; x, y) - 1] + \frac{2M}{\delta^2}\{[T_{j,k}((e^{-2s} + e^{-2t}); x, y) - (e^{-2x} + e^{-2y})]$$
$$- 2e^{-x}[T_{j,k}(e^{-s}; x, y) - e^{-x}] - 2e^{-y}[T_{j,k}(e^{-t}; x, y) - e^{-y}]$$
$$+ (e^{-2x} + e^{-2y})[T_{j,k}(1; x, y) - 1]\} + f(x, y)(T_{j,k}(1; x, y) - 1).$$

Therefore

$$| T_{j,k}(f; x, y) - f(x, y) | \leq \varepsilon + (\varepsilon + M) | T_{j,k}(1; x, y) - 1 | + \frac{2M}{\delta^2} | e^{-2x} + e^{-2y} || T_{j,k}(1; x, y) - 1 |$$
$$+ \frac{2M}{\delta^2} | T_{j,k}(e^{-2s} + e^{-2t}; x, y) | - (e^{-2x} + e^{-2y}) |$$
$$+ \frac{4M}{\delta^2} | e^{-x} || T_{j,k}(e^{-s}; x, y) - e^{-x} | + \frac{4M}{\delta^2} | e^{-y} || T_{j,k}(e^{-t}; x, y) - e^{-y} |$$
$$\leq \varepsilon + (\varepsilon + M + \frac{4M}{\delta^2}) | T_{j,k}(1; x, y) - 1 | + \frac{2M}{\delta^2} | e^{-2x} + e^{-2y} || T_{j,k}(1; x, y) - 1 |$$
$$+ \frac{2M}{\delta^2} | T_{j,k}(e^{-2s} + e^{-2t}; x, y) - (e^{-2x} + e^{-2y}) |$$
$$+ \frac{4M}{\delta^2} | T_{j,k}(e^{-s}; x, y) - e^{-x} | + \frac{4M}{\delta^2} | T_{j,k}(e^{-t}; x, y) - e^{-y} | . \tag{2.1.10}$$

since $| e^{-x} |, | e^{-y} | \leq 1$ for all $x, y \in I$. Now, taking $\sup_{(x,y) \in I^2}$, we get

$$\left\| T_{j,k}(f; x, y) - f(x, y) \right\| \leq \varepsilon + K\Bigg(\left\| T_{j,k}(1; x, t) - 1 \right\|$$

$$+\left\|T_{j,k}(e^{-s};x,y)-e^{-x}\right\|+\left\|T_{j,k}(e^{-t};x,y)-e^{-y}\right\|$$

$$+\left\|T_{j,k}(e^{-2s}+e^{-2t};x,y)-(e^{-2x}+e^{-2y})\right\|\Big), \tag{2.1.11}$$

where where $K=\max\{\varepsilon+M+\frac{4M}{\delta^2},\frac{4M}{\delta^2},\frac{2M}{\delta^2}\}$. Taking P-lim as $j,k\to\infty$ and using (2.1.1), (2.1.2), (2.1.3), (2.1.4), we get

$$P-\lim_{p,q\to\infty}\left\|T_{j,k}(f;x,y)-f(x,y)\right\|=0, \text{ uniformly in } m,n.$$

This completes the proof of the theorem.

3. Statistical version

In the following theorem we use the notion of statistical convergence of double sequences to generalize the above theorem. We also display an interesting example to show its importance.

Theorem 3.1. Let $(T_{j,k})$ be a double sequence of positive linear operators from $C(I^2)$ into $C(I^2)$. Then for all $f\in C(I^2)$

$$st_2\text{-}\lim_{j,k\to\infty}\left\|T_{j,k}(f;x,y)-f(x,y)\right\|=0. \tag{3.1.0}$$

if and only if

$$st_2\text{-}\lim_{j,k\to\infty}\left\|T_{j,k}(1;x,y)-1\right\|=0, \tag{3.1.1}$$

$$st_2\text{-}\lim_{j,k\to\infty}\left\|T_{j,k}(e^{-s};x,y)-e^{-x}\right\|=0, \tag{3.1.2}$$

$$st_2\text{-}\lim_{j,k\to\infty}\left\|T_{j,k}(e^{-t};x,y)-e^{-y}\right\|=0, \tag{3.1.3}$$

$$st_2\text{-}\lim_{j,k\to\infty}\left\|T_{j,k}(e^{-2s}+e^{-2t};x,y)-(e^{-2x}+e^{-2y})\right\|=0. \tag{3.1.4}$$

Proof. For a given $r>0$ choose $\varepsilon>0$ such that $\varepsilon<r$. Define the following sets

$$D:=\{(j,k),j\le m \text{ and } k\le n:\left\|T_{j,k}(f;x,y)-f(x,y)\right\|\ge r\},$$

$$D_1:=\{(j,k),j\le m \text{ and } k\le n:\left\|T_{j,k}(1;x,y)-1\right\|\ge\frac{r-\varepsilon}{4K}\},$$

$$D_2:=\{(j,k),j\le m \text{ and } k\le n:\left\|T_{j,k}(e^{-s};x,y)-e^{-x}\right\|\ge\frac{r-\varepsilon}{4K}\},$$

$$D_3 := \{(j,k), j \le m \text{ and } k \le n : \left\| T_{j,k}(e^{-t}; x, y) - e^{-y} \right\| \ge \frac{r-\varepsilon}{4K}\}.$$

$$D_4 := \{(j,k), j \le m \text{ and } k \le n : \left\| T_{j,k}(e^{-2s} + e^{-2t}; x, y) - (e^{-2x} + e^{-2y}) \right\| \ge \frac{r-\varepsilon}{4K}\}.$$

Then from (2.1.11), we see that $D \subset D_1 \cup D_2 \cup D_3 \cup D_4$ and therefore $\delta_2(D) \le \delta_2(D_1) + \delta_2(D_2) + \delta_2(D_3) + \delta_2(D_4)$. Hence conditions (3.1.1)–(3.1.4) imply the condition (3.1.0).

This completes the proof of the theorem.

We show that the following double sequence of positive linear operators satisfies the conditions of Theorem 3.1 but does not satisfy the conditions of Theorem 2.1.

Example 3.2. Consider the sequence of classical Baskakov operators of two variables [12]

$$B_{m,n}(f; x, y) := \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} f\left(\frac{j}{m}, \frac{k}{n}\right) \binom{m-1+j}{j} \binom{n-1+k}{k} x^j (1+x)^{-m-j} y^k (1+y)^{-n-k};$$

where $0 \le x, y < \infty$. Let $L_{m,n} : C(I^2) \to C(I^2)$ be defined by

$$L_{m,n}(f; x, y) = (1 + w_{mn}) B_{m,n}(f; x, y),$$

where the sequence (w_{mn}) is defined by (1.1.1). Since

$$B_{m,n}(1; x, y) = 1,$$

$$B_{m,n}(e^{-s}; x, y) = (1 + x - xe^{-\frac{1}{m}})^{-m},$$

$$B_{m,n}(e^{-t}; x, y) = (1 + y - ye^{-\frac{1}{n}})^{-n},$$

$$B_{m,n}(e^{-2s} + e^{-2t}; x, y) = (1 + x^2 - x^2 e^{-\frac{1}{m}})^{-m} + (1 + y^2 - y^2 e^{-\frac{1}{n}})^{-n},$$

we have that the sequence $(L_{m,n})$ satisfies the conditions (3.1.1), (3.2.2), (3.1.3) and (3.1.4). Hence by Theorem 3.1, we have

$$st_2\text{-}\lim_{m,n\to\infty} \|L_{m,n}(f; x, y) - f(x, y)\| = 0.$$

On the other hand, we get $L_{m,n}(f; 0, 0) = (1+w_{mn}) f(0, 0)$, since $B_{m,n}(f; 0, 0) = f(0, 0)$, and hence

$$\|L_{m,n}(f; x, y) - f(x, y)\| \ge |L_{m,n}(f; 0, 0) - f(0, 0)| = w_{mn} |f(0, 0)|.$$

We see that $(L_{m,n})$ does not satisfy the conditions of Theorem 2.1, since $P\text{-}\lim_{m,n\to\infty} w_{mn}$ does not exist.

References

[1] A. Alotaibi, M. Mursaleen and S.A. Mohiuddine, Statistical approximation for periodic functions of two variables, Jour. Function Spaces Appl., Volume 2013, Article ID 491768, 5 pages http://dx.doi.org/10.1155/2013/491768.

[2] G. A. Anastassiou, M. Mursaleen and S. A. Mohiuddine, Some approximation theorems for functions of two variables through almost convergence of double sequences, Jour. Comput. Analy. Appl., 13(1)(2011) 37-40.

[3] C. Belen, M. Mursaleen and M. Yildirim, Statistical A-summability of double sequences and a Korovkin type approximation theorem, Bull. Korean Math. Soc., 49(4) (2012) 851-861.

[4] C. Belen and S. A. Mohiuddine, Generalized weighted statistical convergence and application," Appl. Math. Comput., 219 (2013) 9821-9826.

[5] B. D. Boyanov and V. M. Veselinov, A note on the approximation of functions in an infinite interval by linear positive operators, Bull. Math. Soc. Sci. Math. Roum., 14(62) (1970) 9-13.

[6] F. Dirik and K. Demirci, Korovkin type approximation theorem for functions of two variables in statistical sense, Turk. J. Math. 33(2009)1-11.

[7] O.H.H. Edely, S.A. Mohiuddine and A. K. Noman, Korovkin type approximation theorems obtained through generalized statistical convergence, Appl. Math. Lett., 23(2010)1382-1387.

[8] H. Fast, Sur la convergence statistique, Colloq. Math, 2 (1951), 241-244.

[9] J. A. Fridy, On statistical convergence, Analysis, 5 (1985), 301-313.

[10] A. D. Gadžiev, The convergence problems for a sequence of positive linear operators on unbounded sets, and theorems analogous to that of P.P.Korovkin, Soviet Math. Dokl., 15(1974) 1433-1436.

[11] A.D. Gadjiv and C. Orhan, Sme approximzation theorems via statistical convergence, Rocky Mountain J. Math. 32 (2002) 129-138.

[12] M. Gurdek, L. Rempulska and M. Skorupka, The Baskakov operators for functions of two variables, Collect. Math. 50, 3 (1999) 289–302.

[13] P. P. Korovkin, Linear operators and approximation theory, Hindustan Publ. Co., Delhi,1960.

[14] S.A. Mohiuddine, An application of almost convergence in approximation theorems, Appl. Math. Lett., 24(2011)1856-1860.

[15] S.A. Mohiuddine, A. Alotaibi and M. Mursaleen, Statistical convergence of double sequences in locally solid Riesz spaces, Abstract Appl. Anal., Volume 2012, Article ID 719729, 9 pages, doi:10.1155/2012/719729.

[16] S.A. Mohiuddine and A. Alotaibi, Statistical convergence and approximation theorems for functions of two variables, J. Comput. Anal. Appl., 15 (2) (2013) 218-223.

[17] F. Moricz, Statistical convergence of multiple sequences, Arch. Math. 81 (2003) 82-89.

[18] M. Mursaleen and Osama H. H. Edely, Statistical convergence of double sequences, J. Math. Anal. Appl., 288 (2003), 223-231.

[19] M. Mursaleen and A. Alotaib, Korovkin type approximation theorem for statistical A-summability of double sequences, Jour. Comput. Anal. Appl., 15(6) (2013) 1036-1045.

[20] A. Pringsheim, Zur theorie der zweifach unendlichen Zahlenfolgen, Math. Z., 53(1900) 289-321.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 21, NO.7, 2016, COPYRIGHT 2016 EUDOXUS PRESS, LLC

Advanced Fractional Taylor's formulae

George A. Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

Abstract

Here are presented five new advanced fractional Taylor's formulae under as weak as possible assumptions.

2010 AMS Subject Classification: 26A33.
Key Words and Phrases: fractional integral, fractional derivative, fractional Taylor's formula.

1 Introduction

In [3] we proved

Theorem 1 *Let $f, f', ..., f^{(n)}; g, g'$ be continuous functions from $[a,b]$ (or $[b,a]$) into $\mathbb{R}$, $n \in \mathbb{N}$. Assume that $\left(g^{-1}\right)^{(k)}$, $k = 0, 1, ..., n$, are continuous functions. Then it holds*

$$f(b) = f(a) + \sum_{k=1}^{n-1} \frac{\left(f \circ g^{-1}\right)^{(k)}(g(a))}{k!} (g(b) - g(a))^k + R_n(a,b), \tag{1}$$

where

$$R_n(a,b) := \frac{1}{(n-1)!} \int_a^b (g(b) - g(s))^{n-1} \left(f \circ g^{-1}\right)^{(n)} (g(s))\, g'(s)\, ds \tag{2}$$

$$= \frac{1}{(n-1)!} \int_{g(a)}^{g(b)} (g(b) - t)^{n-1} \left(f \circ g^{-1}\right)^{(n)} (t)\, dt.$$

Remark 2 *Let g be strictly increasing and $g \in AC([a,b])$ (absolutely continuous functions). Set $g([a,b]) = [c,d]$, where $c, d \in \mathbb{R}$, i.e. $g(a) = c$, $g(b) = d$, and call $l := f \circ g^{-1}$.*

1

Assume that $l \in AC^n([c,d])$ *(i.e.* $l^{(n-1)} \in AC([c,d])$*).*

[Obviously here it is implied that $f \in C([a,b])$ *.]*

Furthermore assume that $(f \circ g^{-1})^{(n)} \in L_\infty([c,d])$*. [By this very last assumption, the function* $(g(b)-t)^{n-1}(f \circ g^{-1})^{(n)}(t)$ *is integrable over* $[c,d]$*. Since* $g \in AC([a,b])$ *and it is increasing, by [9] the function* $(g(b)-g(s))^{n-1}(f \circ g^{-1})^{(n)}(g(s))\,g'(s)$ *is integrable on* $[a,b]$*, and again by [9], (2) is valid in this general setting.] Clearly (1) is now valid under these general assumptions.*

2 Results

We need

Lemma 3 *Let* g *be strictly increasing and* $g \in AC([a,b])$*. Assume that* $(f \circ g^{-1})^{(m)}$ *is Lebesgue measurable function over* $[c,d]$*. Then*

$$\left\| \left(f \circ g^{-1}\right)^{(m)} \right\|_{\infty,[c,d]} \leq \left\| \left(f \circ g^{-1}\right)^{(m)} \circ g \right\|_{\infty,[a,b]}, \tag{3}$$

where $(f \circ g^{-1})^{(m)} \circ g \in L_\infty([a,b])$*.*

Proof. We observe by definition of $\|\cdot\|_\infty$ that:

$$\left\| \left(f \circ g^{-1}\right)^{(m)} \circ g \right\|_{\infty,[a,b]} = \tag{4}$$

$$\inf\left\{M : m\left\{t \in [a,b] : \left| \left(\left(f \circ g^{-1}\right)^{(m)} \circ g\right)(t) \right| > M\right\} = 0\right\},$$

where m is the Lebesgue measure.

Because g is absolutely continuous and strictly increasing function on $[a,b]$, by [11], p. 108, exercise 14, we get that

$$m\left\{z \in [c,d] : \left| \left(f \circ g^{-1}\right)^{(m)}(z) \right| > M\right\} =$$

$$m\left\{g(t) \in [c,d] : \left| \left(f \circ g^{-1}\right)^{(m)}(g(t)) \right| > M\right\} =$$

$$m\left(g\left(\left\{t \in [a,b] : \left| \left(f \circ g^{-1}\right)^{(m)}(g(t)) \right| > M\right\}\right)\right) = 0,$$

given that

$$m\left\{t \in [a,b] : \left| \left(\left(f \circ g^{-1}\right)^{(m)} \circ g\right)(t) \right| > M\right\} = 0.$$

Therefore each M of (4) fulfills

$$M \in \left\{L : m\left\{z \in [c,d] : \left| \left(f \circ g^{-1}\right)^{(m)}(z) \right| > L\right\} = 0\right\}. \tag{5}$$

The last implies (3). ■

We give

2

Definition 4 *(see also [10, p. 99]) The left and right fractional integrals, respectively, of a function f with respect to given function g are defined as follows:*

Let $a, b \in \mathbb{R}$, $a < b$, $\alpha > 0$. Here $g \in AC([a,b])$ and is strictly increasing, $f \in L_\infty([a,b])$. We set

$$\left(I_{a+;g}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (g(x) - g(t))^{\alpha-1} g'(t) f(t) \, dt, \quad x \geq a, \tag{6}$$

where Γ is the gamma function, clearly $\left(I_{a+;g}^{\alpha} f\right)(a) = 0$, $I_{a+;g}^{0} f := f$ and

$$\left(I_{b-;g}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (g(t) - g(x))^{\alpha-1} g'(t) f(t) \, dt, \quad x \leq b, \tag{7}$$

clearly $\left(I_{b-;g}^{\alpha} f\right)(b) = 0$, $I_{b-;g}^{0} f := f$.

When g is the identity function id, we get that $I_{a+;id}^{\alpha} = I_{a+}^{\alpha}$, and $I_{b-;id}^{\alpha} = I_{b-}^{\alpha}$, the ordinary left and right Riemann-Liouville fractional integrals, where

$$\left(I_{a+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) \, dt, \quad x \geq a, \tag{8}$$

$\left(I_{a+}^{\alpha} f\right)(a) = 0$ and

$$\left(I_{b-}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) \, dt, \quad x \leq b, \tag{9}$$

$\left(I_{b-}^{\alpha} f\right)(b) = 0$.

In [5], we proved

Lemma 5 *Let $g \in AC([a,b])$ which is strictly increasing and f Borel measurable in $L_\infty([a,b])$. Then $f \circ g^{-1}$ is Lebesgue measurable, and*

$$\|f\|_{\infty,[a,b]} \geq \left\|f \circ g^{-1}\right\|_{\infty,[g(a),g(b)]}, \tag{10}$$

i.e. $\left(f \circ g^{-1}\right) \in L_\infty([g(a), g(b)])$.

If additionally $g^{-1} \in AC([g(a), g(b)])$, then

$$\|f\|_{\infty,[a,b]} = \left\|f \circ g^{-1}\right\|_{\infty,[g(a),g(b)]}. \tag{11}$$

Remark 6 *We proved ([5]) that*

$$\left(I_{a+;g}^{\alpha} f\right)(x) = \left(I_{g(a)+}^{\alpha}\left(f \circ g^{-1}\right)\right)(g(x)), \quad x \geq a \tag{12}$$

and

$$\left(I_{b-;g}^{\alpha} f\right)(x) = \left(I_{g(b)-}^{\alpha}\left(f \circ g^{-1}\right)\right)(g(x)), \quad x \leq b. \tag{13}$$

It is well known that, if f is a Lebesgue measurable function, then there exists f^* a Borel measurable function, such that $f = f^*$, a.e. Also it holds $\|f\|_\infty = \|f^*\|_\infty$, and $\int ...f...dx = \int ...f^*...dx$.

Of course a Borel measurable function is a Lebesgue measurable function.

Thus, by Lemma 5, we get

$$\|f\|_{\infty,[a,b]} = \|f^*\|_{\infty,[a,b]} \geq \left\|f^* \circ g^{-1}\right\|_{\infty,[g(a),g(b)]}. \tag{14}$$

We observe the following:

Let $\alpha, \beta > 0$, then

$$\begin{gathered}\left(I^\beta_{a+;g}\left(I^\alpha_{a+;g}f\right)\right)(x) = \left(I^\beta_{a+;g}\left(I^\alpha_{a+;g}f^*\right)\right)(x) = \\ I^\beta_{g(a)+}\left(\left(\left(I^\alpha_{a+;g}f^*\right)\circ g^{-1}\right)(g(x)) = I^\beta_{g(a)+}\left(I^\alpha_{g(a)+}\left(f^*\circ g^{-1}\right)\circ g\circ g^{-1}\right)(g(x)) = \\ \left(I^\beta_{g(a)+}I^\alpha_{g(a)+}\left(f^*\circ g^{-1}\right)\right)(g(x)) \overset{(\text{by [8], p. 14})}{=} \\ \left(I^{\beta+\alpha}_{g(a)+}f^*\circ g^{-1}\right)(g(x)) = \left(I^{\beta+\alpha}_{a+;g}f^*\right)(x) = \left(I^{\beta+\alpha}_{a+;g}f\right)(x) \ \text{a.e.}\end{gathered} \tag{15}$$

The last is true for all x, if $\alpha + \beta \geq 1$ or $f \in C([a,b])$.

We have proved the semigroup composition property

$$\left(I^\alpha_{a+;g}I^\beta_{a+;g}f\right)(x) = \left(I^{\alpha+\beta}_{a+;g}f\right)(x) = \left(I^\beta_{a+;g}I^\alpha_{a+;g}f\right)(x), \quad x \geq a, \tag{16}$$

a.e., which is true for all x, if $\alpha + \beta \geq 1$ or $f \in C([a,b])$.

Similarly we get

$$\begin{gathered}\left(I^\beta_{b-;g}\left(I^\alpha_{b-;g}f\right)\right)(x) = \left(I^\beta_{b-;g}\left(I^\alpha_{b-;g}f^*\right)\right)(x) = \\ I^\beta_{g(b)-}\left(\left(\left(I^\alpha_{b-;g}f^*\right)\circ g^{-1}\right)(g(x)) = I^\beta_{g(b)-}\left(I^\alpha_{g(b)-}\left(f^*\circ g^{-1}\right)\circ g\circ g^{-1}\right)(g(x)) = \\ I^\beta_{g(b)-}\left(I^\alpha_{g(b)-}\left(f^*\circ g^{-1}\right)\right)(g(x)) \overset{(\text{by [1]})}{=} \\ \left(I^{\beta+\alpha}_{g(b)-}\left(f^*\circ g^{-1}\right)\right)(g(x)) = \left(I^{\beta+\alpha}_{b-;g}f^*\right)(x) = \left(I^{\beta+\alpha}_{b-;g}f\right)(x) \ \text{a.e.,}\end{gathered} \tag{17}$$

true for all $x \in [a,b]$, if $\alpha + \beta \geq 1$ or $f \in C([a,b])$.

We have proved the semigroup property that

$$\left(I^\alpha_{b-;g}I^\beta_{b-;g}f\right)(x) = \left(I^{\alpha+\beta}_{b-;g}f\right)(x) = \left(I^\beta_{b-;g}I^\alpha_{b-;g}f\right)(x), \ \text{a.e.}, \ x \leq b, \tag{18}$$

which is true for all $x \in [a,b]$, if $\alpha + \beta \geq 1$ or $f \in C([a,b])$.

From now on without loss of generality, within integrals we may assume that $f = f^*$, and we mean that $f = f^*$, a.e.

We make

4

Definition 7 *Let $\alpha > 0$, $\lceil \alpha \rceil = n$, $\lceil \cdot \rceil$ the ceiling of the number. Again here $g \in AC([a,b])$ and strictly increasing. We assume that $\left(f \circ g^{-1}\right)^{(n)} \circ g \in L_{\infty}([a,b])$. We define the left generalized g-fractional derivative of f of order α as follows:*

$$\left(D_{a+;g}^{\alpha} f\right)(x) := \frac{1}{\Gamma(n-\alpha)} \int_a^x (g(x) - g(t))^{n-\alpha-1} g'(t) \left(f \circ g^{-1}\right)^{(n)} (g(t))\, dt, \tag{19}$$

$x \geq a$.

If $\alpha \notin \mathbb{N}$, by [6], we have that $D_{a+;g}^{\alpha} f \in C([a,b])$.

We see that

$$\left(I_{a+;g}^{n-\alpha}\left(\left(f \circ g^{-1}\right)^{(n)} \circ g\right)\right)(x) = \left(D_{a+;g}^{\alpha} f\right)(x), \quad x \geq a. \tag{20}$$

We set

$$D_{a+;g}^{n} f(x) := \left(\left(f \circ g^{-1}\right)^{(n)} \circ g\right)(x), \tag{21}$$

$$D_{a+;g}^{0} f(x) = f(x), \quad \forall\, x \in [a,b]. \tag{22}$$

When $g = id$, then

$$D_{a+;g}^{\alpha} f = D_{a+;id}^{\alpha} f = D_{*a}^{\alpha} f, \tag{23}$$

the usual left Caputo fractional derivative.

We make

Remark 8 *Under the assumption that $\left(f \circ g^{-1}\right)^{(n)} \circ g \in L_{\infty}([a,b])$, which could be considered as Borel measurable within integrals, we obtain*

$$\left(I_{a+;g}^{\alpha} D_{a+;g}^{\alpha} f\right)(x) = \left(I_{a+;g}^{\alpha}\left(I_{a+;g}^{n-\alpha}\left(\left(f \circ g^{-1}\right)^{(n)} \circ g\right)\right)\right)(x) =$$

$$\left(I_{a+;g}^{\alpha+n-\alpha}\left(\left(f \circ g^{-1}\right)^{(n)} \circ g\right)\right)(x) = I_{a+;g}^{n}\left(\left(f \circ g^{-1}\right)^{(n)} \circ g\right)(x) = \tag{24}$$

$$\frac{1}{(n-1)!} \int_a^x (g(x) - g(t))^{n-1} g'(t) \left(\left(f \circ g^{-1}\right)^{(n)} \circ g\right)(t)\, dt.$$

We have proved that

$$\left(I_{a+;g}^{\alpha} D_{a+;g}^{\alpha} f\right)(x) = \frac{1}{(n-1)!} \int_a^x (g(x) - g(t))^{n-1} g'(t) \left(f \circ g^{-1}\right)^{(n)} (g(t))\, dt$$

$$= R_n(a,x), \quad \forall\, x \geq a, \tag{25}$$

see (2).

But also it holds

$$R_n(a,x) = \left(I_{a+;g}^{\alpha} D_{a+;g}^{\alpha} f\right)(x) = \tag{26}$$

$$\frac{1}{\Gamma(\alpha)} \int_a^x (g(x) - g(t))^{\alpha-1} g'(t) \left(D_{a+;g}^{\alpha} f\right)(t)\, dt, \quad x \geq a.$$

5

We have proved the following g-left fractional generalized Taylor's formula:

Theorem 9 *Let g be strictly increasing function and $g \in AC([a,b])$. We assume that $(f \circ g^{-1}) \in AC^n([g(a), g(b)])$, where $\mathbb{N} \ni n = \lceil \alpha \rceil$, $\alpha > 0$. Also we assume that $(f \circ g^{-1})^{(n)} \circ g \in L_\infty([a,b])$. Then*

$$f(x) = f(a) + \sum_{k=1}^{n-1} \frac{(f \circ g^{-1})^{(k)}(g(a))}{k!} (g(x) - g(a))^k +$$

$$\frac{1}{\Gamma(\alpha)} \int_a^x (g(x) - g(t))^{\alpha-1} g'(t) \left(D_{a+;g}^{\alpha} f\right)(t)\, dt, \quad \forall\, x \in [a,b]. \tag{27}$$

Calling $R_n(a,x)$ the remainder of (27), we get that

$$R_n(a,x) = \frac{1}{\Gamma(\alpha)} \int_{g(a)}^{g(x)} (g(x) - z)^{\alpha-1} \left(\left(D_{a+;g}^{\alpha} f\right) \circ g^{-1}\right)(z)\, dz, \quad \forall\, x \in [a,b]. \tag{28}$$

Remark 10 *By [6], $R_n(a,x)$ is a continuous function in $x \in [a,b]$. Also, by [9], change of variable in Lebesgue integrals, (28) is valid.*

By [3] we have

Theorem 11 *Let $f, f', ..., f^{(n)}; g, g'$ be continuous from $[a,b]$ into $\mathbb{R}$, $n \in \mathbb{N}$. Assume that $(g^{-1})^{(k)}$, $k = 0, 1, ..., n$, are continuous. Then*

$$f(x) = f(b) + \sum_{k=1}^{n-1} \frac{(f \circ g^{-1})^{(k)}(g(b))}{k!} (g(x) - g(b))^k + R_n(b,x), \tag{29}$$

where

$$R_n(b,x) := \frac{1}{(n-1)!} \int_b^x (g(x) - g(s))^{n-1} (f \circ g^{-1})^{(n)}(g(s))\, g'(s)\, ds \tag{30}$$

$$= \frac{1}{(n-1)!} \int_{g(b)}^{g(x)} (g(x) - t)^{n-1} (f \circ g^{-1})^{(n)}(t)\, dt, \quad \forall\, x \in [a,b]. \tag{31}$$

Notice that (29), (30) and (31) are valid under more general weaker assumptions, as follows: g is strictly increasing and $g \in AC([a,b])$, $(f \circ g^{-1}) \in AC^n([g(a), g(b)])$, and $(f \circ g^{-1})^{(n)} \in L_\infty([g(a), g(b)])$.

We make

Definition 12 *Here we assume that $\left(f \circ g^{-1}\right)^{(n)} \circ g \in L_{\infty}([a,b])$, where $N \ni n = \lceil \alpha \rceil$, $\alpha > 0$. We define the right generalized g-fractional derivative of f of order α as follows:*

$$\left(D_{b-;g}^{\alpha} f\right)(x) := \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b (g(t) - g(x))^{n-\alpha-1} g'(t) \left(f \circ g^{-1}\right)^{(n)} (g(t))\, dt, \tag{32}$$

all $x \in [a,b]$.

If $\alpha \notin \mathbb{N}$, by [7], we get that $\left(D_{b-;g}^{\alpha} f\right) \in C([a,b])$.

We see that

$$I_{b-;g}^{n-\alpha} \left((-1)^n \left(f \circ g^{-1}\right)^{(n)} \circ g \right)(x) = \left(D_{b-;g}^{\alpha} f\right)(x), \quad a \le x \le b. \tag{33}$$

We set

$$D_{b-;g}^{n} f(x) = (-1)^n \left(\left(f \circ g^{-1}\right)^{(n)} \circ g \right)(x), \tag{34}$$

$$D_{b-;g}^{0} f(x) = f(x), \quad \forall\, x \in [a,b].$$

When $g = id$, then

$$D_{b-;g}^{\alpha} f(x) = D_{b-;id}^{\alpha} f(x) = D_{b-}^{\alpha} f, \tag{35}$$

the usual right Caputo fractional derivative.

We make

Remark 13 *Furthermore it holds*

$$\left(I_{b-;g}^{\alpha} D_{b-;g}^{\alpha} f\right)(x) = \left(I_{b-;g}^{\alpha} I_{b-;g}^{n-\alpha} \left((-1)^n \left(f \circ g^{-1}\right)^{(n)} \circ g \right) \right)(x) =$$

$$\left(I_{b-;g}^{n} \left((-1)^n \left(f \circ g^{-1}\right)^{(n)} \circ g \right) \right)(x) = (-1)^n \left(I_{b-;g}^{n} \left(\left(f \circ g^{-1}\right)^{(n)} \circ g \right) \right)(x) = \tag{36}$$

$$\frac{(-1)^n}{(n-1)!} \int_x^b (g(t) - g(x))^{n-1} g'(t) \left(\left(f \circ g^{-1}\right)^{(n)} \circ g \right)(t)\, dt =$$

$$\frac{(-1)^{2n}}{(n-1)!} \int_b^x (g(x) - g(t))^{n-1} g'(t) \left(\left(f \circ g^{-1}\right)^{(n)} \circ g \right)(t)\, dt =$$

$$\frac{1}{(n-1)!} \int_b^x (g(x) - g(t))^{n-1} g'(t) \left(\left(f \circ g^{-1}\right)^{(n)} \circ g \right)(t)\, dt = R_n(b,x), \tag{37}$$

as in (30).

That is

$$R_n(b,x) = \left(I_{b-;g}^{\alpha} D_{b-;g}^{\alpha} f\right)(x) =$$

$$\frac{1}{\Gamma(\alpha)} \int_x^b (g(t) - g(x))^{\alpha-1} g'(t) \left(D_{b-;g}^{\alpha} f\right)(t)\, dt, \quad \text{all } a \le x \le b. \tag{38}$$

We have proved the g-right generalized fractional Taylor's formula:

Theorem 14 *Let g be strictly increasing function and $g \in AC([a,b])$. We assume that $\left(f \circ g^{-1}\right) \in AC^n([g(a), g(b)])$, where $\mathbb{N} \ni n = \lceil \alpha \rceil$, $\alpha > 0$. Also we assume that $\left(f \circ g^{-1}\right)^{(n)} \circ g \in L_\infty([a,b])$. Then*

$$f(x) = f(b) + \sum_{k=1}^{n-1} \frac{\left(f \circ g^{-1}\right)^{(k)}(g(b))}{k!} (g(x) - g(b))^k +$$

$$\frac{1}{\Gamma(\alpha)} \int_x^b (g(t) - g(x))^{\alpha-1} g'(t) \left(D_{b-;g}^{\alpha} f\right)(t)\, dt, \quad \text{all } a \leq x \leq b. \tag{39}$$

Calling $R_n(b,x)$ the remainder in (39), we get that

$$R_n(b,x) = \frac{1}{\Gamma(\alpha)} \int_{g(x)}^{g(b)} (z - g(x))^{\alpha-1} \left(\left(D_{b-;g}^{\alpha} f\right) \circ g^{-1}\right)(z)\, dz, \quad \forall\, x \in [a,b]. \tag{40}$$

Remark 15 *By [7], $R_n(b,x)$ is a continuous function in $x \in [a,b]$. Also, by [9], change of variable in Lebesgue integrals, (40) is valid.*

Basics 16 *The right Riemann-Liouville fractional integral of order $\alpha > 0$, $f \in L_1([a,b])$, $a < b$, is defined as follows:*

$$I_{b-}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (z - x)^{\alpha-1} f(z)\, dz, \ \ \forall\, x \in [a,b]. \tag{41}$$

$$I_{b-}^{0} := I \ \text{(the identity operator)}.$$

Let $\alpha, \beta \geq 0$, $f \in L_1([a,b])$. Then, by [1], we have

$$I_{b-}^{\alpha} I_{b-}^{\beta} f = I_{b-}^{\alpha+\beta} f = I_{b-}^{\beta} I_{b-}^{\alpha} f, \tag{42}$$

valid a.e. on $[a,b]$. If $f \in C([a,b])$ or $\alpha + \beta \geq 1$, then the last identity is true on all of $[a,b]$.

The right Caputo fractional derivative of order $\alpha > 0$, $m = \lceil \alpha \rceil$, $f \in AC^m([a,b])$ is defined as follows:

$$D_{b-}^{\alpha} f(x) := (-1)^m I_{b-}^{m-\alpha} f^{(m)}(x), \tag{43}$$

that is

$$D_{b-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (z - x)^{m-\alpha-1} f^{(m)}(z)\, dz, \quad \forall\, x \in [a,b], \tag{44}$$

with $D_{b-}^{m} f(x) := (-1)^m f^{(m)}(x)$.

8

By [1], we have the following right fractional Taylor's formula:
Let $f \in AC^m([a,b])$, $x \in [a,b]$, $\alpha > 0$, $m = \lceil \alpha \rceil$, then

$$f(x) - \sum_{k=0}^{m-1} \frac{f^{(k)}(b)}{k!}(x-b)^k = \frac{1}{\Gamma(\alpha)} \int_x^b (z-x)^{\alpha-1} D_{b-}^{\alpha} f(z)\, dz = \tag{45}$$

$$\begin{gathered} \left(I_{b-}^{\alpha} D_{b-}^{\alpha} f\right)(x) = (-1)^m \left(I_{b-}^{\alpha} I_{b-}^{m-\alpha} f^{(m)}\right)(x) = (-1)^m \left(I_{b-}^{m} f^{(m)}\right)(x) = \\ (-1)^m \frac{1}{(m-1)!} \int_x^b (z-x)^{m-1} f^{(m)}(z)\, dz = \\ (-1)^m \frac{(-1)^m}{(m-1)!} \int_b^x (x-z)^{m-1} f^{(m)}(z)\, dz = \\ \frac{1}{(m-1)!} \int_b^x (x-z)^{m-1} f^{(m)}(z)\, dz. \end{gathered} \tag{46}$$

That is

$$\left(I_{b-}^{\alpha} D_{b-}^{\alpha} f\right)(x) = (-1)^m \left(I_{b-}^{m} f^{(m)}\right)(x) =$$

$$f(x) - \sum_{k=0}^{m-1} \frac{f^{(k)}(b)}{k!}(x-b)^k = \frac{1}{(m-1)!} \int_b^x (x-z)^{m-1} f^{(m)}(z)\, dz. \tag{47}$$

We make

Remark 17 *If $0 < \alpha \leq 1$, then $m = 1$, hence*

$$\left(I_{b-}^{\alpha} D_{b-}^{\alpha} f\right)(x) = f(x) - f(b) \tag{48}$$

$$= \frac{1}{\Gamma(\alpha)} \int_x^b (z-x)^{\alpha-1} D_{b-}^{\alpha} f(z)\, dz =: (\psi_1).$$

[Let $f' \in L_\infty([a,b])$, then by [4], we get that $D_{b-}^{\alpha} f \in C([a,b])$, $0 < \alpha < 1$, where

$$\left(D_{b-}^{\alpha} f\right)(x) = \frac{(-1)}{\Gamma(1-\alpha)} \int_x^b (z-x)^{-\alpha} f'(z)\, dz, \tag{49}$$

with $\left(D_{b-}^{1} f\right)(x) = -f'(x)$.

Also $(z-x)^{\alpha-1} > 0$, over (x,b), and

$$\int_x^b (z-x)^{\alpha-1} dz = \frac{(b-x)^{\alpha}}{\alpha} < \infty, \quad \textit{for any } 0 < \alpha \leq 1, \tag{50}$$

thus $(z-x)^{\alpha-1}$ is integrable over $[x,b]$.]

By the first mean value theorem for integration, when $0 < \alpha < 1$, we get that

$$(\psi_1) = \frac{\left(D_{b-}^{\alpha} f\right)(\xi_x)}{\Gamma(\alpha)} \int_x^b (z-x)^{\alpha-1} dz = \frac{\left(D_{b-}^{\alpha} f\right)(\xi_x)}{\Gamma(\alpha)} \frac{(b-x)^{\alpha}}{\alpha} \tag{51}$$

$$= \frac{\left(D_{b-}^{\alpha} f\right)(\xi_x)}{\Gamma(\alpha+1)} (b-x)^{\alpha}, \ \ \xi_x \in [x,b].$$

Thus, we obtain

$$f(x) - f(b) = \frac{\left(D_{b-}^{\alpha} f\right)(\xi_x)}{\Gamma(\alpha+1)} (b-x)^{\alpha}, \ \ \xi_x \in [x,b], \tag{52}$$

where $f \in AC([a,b])$.

We have proved

Theorem 18 *(Right generalized mean value theorem). Let $f \in AC([a,b])$, $f' \in L_{\infty}([a,b])$, $0 < \alpha < 1$. Then*

$$f(x) - f(b) = \frac{\left(D_{b-}^{\alpha} f\right)(\xi_x)}{\Gamma(\alpha+1)} (b-x)^{\alpha}, \tag{53}$$

with $x \leq \xi_x \leq b$, where $x \in [a,b]$.

If $f \in C([a,b])$ and there exists $f'(x)$, for any $x \in (a,b)$, then

$$f(x) - f(b) = (-1) f'(\xi_x)(b-x), \tag{54}$$

equivalently,

$$f(b) - f(x) = f'(\xi_x)(b-x), \tag{55}$$

the usual mean value theorem.

We make

Remark 19 *In general: we notice the following*

$$\left|D_{b-}^{\alpha} f(x)\right| \leq \frac{1}{\Gamma(m-\alpha)} \int_x^b (z-x)^{m-\alpha-1} \left|f^{(m)}(z)\right| dz$$

(assuming $f^{(m)} \in L_{\infty}([a,b])$)

$$\leq \frac{\left\|f^{(m)}\right\|_{\infty}}{\Gamma(m-\alpha)} \int_x^b (z-x)^{m-\alpha-1} dz = \frac{\left\|f^{(m)}\right\|_{\infty}}{\Gamma(m-\alpha)} \frac{(b-x)^{m-\alpha}}{m-\alpha} \tag{56}$$

$$= \frac{\left\|f^{(m)}\right\|_{\infty}}{\Gamma(m-\alpha+1)} (b-x)^{m-\alpha} \leq \frac{\left\|f^{(m)}\right\|_{\infty}}{\Gamma(m-\alpha+1)} (b-a)^{m-\alpha}.$$

So when $f^{(m)} \in L_\infty([a,b])$ *we get that*

$$D_{b-}^{\alpha} f(b) = 0, \quad \textit{where } \alpha \notin \mathbb{N}, \tag{57}$$

and

$$\left\| D_{b-}^{\alpha} f \right\|_\infty \leq \frac{\left\| f^{(m)} \right\|_\infty}{\Gamma(m-\alpha+1)} (b-a)^{m-\alpha}. \tag{58}$$

In particular when $f' \in L_\infty([a,b])$, $0 < \alpha < 1$, we have that

$$D_{b-}^{\alpha} f(b) = 0. \tag{59}$$

Notation 20 *Denote by*

$$D_{b-}^{n\alpha} := D_{b-}^{\alpha} D_{b-}^{\alpha} ... D_{b-}^{\alpha} \quad (n \textit{ times}),\ n \in \mathbb{N}. \tag{60}$$

Also denote by

$$I_{b-}^{n\alpha} := I_{b-}^{\alpha} I_{b-}^{\alpha} ... I_{b-}^{\alpha} \quad (n \textit{ times}),\ n \in \mathbb{N}. \tag{61}$$

We have

Theorem 21 *Suppose that* $D_{b-}^{n\alpha} f$, $D_{b-}^{(n+1)\alpha} f \in C([a,b])$, $0 < \alpha \leq 1$. *Then*

$$\left(I_{b-}^{n\alpha} D_{b-}^{n\alpha} f \right)(x) - \left(I_{b-}^{(n+1)\alpha} D_{b-}^{(n+1)\alpha} f \right)(x) = \frac{(b-x)^{n\alpha}}{\Gamma(n\alpha+1)} \left(D_{b-}^{n\alpha} f \right)(b). \tag{62}$$

Proof. By (42) we get that

$$\begin{gathered}
\left(I_{b-}^{n\alpha} D_{b-}^{n\alpha} f \right)(x) - \left(I_{b-}^{(n+1)\alpha} D_{b-}^{(n+1)\alpha} f \right)(x) = \\
I_{b-}^{n\alpha} \left(\left(D_{b-}^{n\alpha} f \right)(x) - \left(I_{b-}^{\alpha} D_{b-}^{(n+1)\alpha} f \right)(x) \right) = \\
I_{b-}^{n\alpha} \left(\left(D_{b-}^{n\alpha} f \right)(x) - \left(\left(I_{b-}^{\alpha} D_{b-}^{\alpha} \right) \left(D_{b-}^{n\alpha} f \right) \right)(x) \right) \overset{(48)}{=} \\
I_{b-}^{n\alpha} \left(\left(D_{b-}^{n\alpha} f \right)(x) - \left(D_{b-}^{n\alpha} f \right)(x) + \left(D_{b-}^{n\alpha} f \right)(b) \right) = \\
I_{b-}^{n\alpha} \left(\left(D_{b-}^{n\alpha} f \right)(b) \right) = \frac{(b-x)^{n\alpha}}{\Gamma(n\alpha+1)} \left(D_{b-}^{n\alpha} f \right)(b).
\end{gathered} \tag{63}$$

■

Remark 22 *Suppose that* $D_{b-}^{k\alpha} f \in C([a,b])$, *for* $k = 0, 1, ..., n+1; 0 < \alpha \leq 1$. *By (62) we get that*

$$\begin{gathered}
\sum_{i=0}^{n} \left(\left(I_{b-}^{i\alpha} D_{b-}^{i\alpha} f \right)(x) - \left(I_{b-}^{(i+1)\alpha} D_{b-}^{(i+1)\alpha} f \right)(x) \right) = \\
\sum_{i=0}^{n} \frac{(b-x)^{i\alpha}}{\Gamma(i\alpha+1)} \left(D_{b-}^{i\alpha} f \right)(b).
\end{gathered} \tag{64}$$

That is

$$f(x) - \left(I_{b-}^{(n+1)\alpha} D_{b-}^{(n+1)\alpha} f\right)(x) = \sum_{i=0}^{n} \frac{(b-x)^{i\alpha}}{\Gamma(i\alpha+1)} \left(D_{b-}^{i\alpha} f\right)(b). \tag{65}$$

Hence it holds

$$f(x) = \sum_{i=0}^{n} \frac{(b-x)^{i\alpha}}{\Gamma(i\alpha+1)} \left(D_{b-}^{i\alpha} f\right)(b) + \left(I_{b-}^{(n+1)\alpha} D_{b-}^{(n+1)\alpha} f\right)(x) = \tag{66}$$

$$\sum_{i=0}^{n} \frac{(b-x)^{i\alpha}}{\Gamma(i\alpha+1)} \left(D_{b-}^{i\alpha} f\right)(b) + R^*(x,b),$$

where

$$R^*(x,b) := \frac{1}{\Gamma((n+1)\alpha)} \int_x^b (z-x)^{(n+1)\alpha-1} \left(D_{b-}^{(n+1)\alpha} f\right)(z)\, dz. \tag{67}$$

We see that (there exists $\xi_x \in [x,b]$:)

$$R^*(x,b) = \frac{\left(D_{b-}^{(n+1)\alpha} f\right)(\xi_x)}{\Gamma((n+1)\alpha)} \int_x^b (z-x)^{(n+1)\alpha-1}\, dz =$$

$$\frac{\left(D_{b-}^{(n+1)\alpha} f\right)(\xi_x)}{\Gamma((n+1)\alpha)} \frac{(b-x)^{(n+1)\alpha}}{(n+1)\alpha} = \frac{\left(D_{b-}^{(n+1)\alpha} f\right)(\xi_x)}{\Gamma((n+1)\alpha+1)} (b-x)^{(n+1)\alpha}. \tag{68}$$

We have proved the following right generalized fractional Taylor's formula:

Theorem 23 *Suppose that $D_{b-}^{k\alpha} f \in C([a,b])$, for $k = 0, 1, ..., n+1$, where $0 < \alpha \leq 1$. Then*

$$f(x) = \sum_{i=0}^{n} \frac{(b-x)^{i\alpha}}{\Gamma(i\alpha+1)} \left(D_{b-}^{i\alpha} f\right)(b) + \tag{69}$$

$$\frac{1}{\Gamma((n+1)\alpha)} \int_x^b (z-x)^{(n+1)\alpha-1} \left(D_{b-}^{(n+1)\alpha} f\right)(z)\, dz =$$

$$\sum_{i=0}^{n} \frac{(b-x)^{i\alpha}}{\Gamma(i\alpha+1)} \left(D_{b-}^{i\alpha} f\right)(b) + \frac{\left(D_{b-}^{(n+1)\alpha} f\right)(\xi_x)}{\Gamma((n+1)\alpha+1)} (b-x)^{(n+1)\alpha}, \tag{70}$$

where $\xi_x \in [x,b]$, with $x \in [a,b]$.

We make

Remark 24 *Let $\alpha > 0$, $m = \lceil \alpha \rceil$, g is strictly increasing and $g \in AC([a,b])$. Call $l = f \circ g^{-1}$, $f : [a,b] \to \mathbb{R}$. Assume that $l \in AC^m([c,d])$ (i.e. $l^{(m-1)} \in AC([c,d])$) (where $g([a,b]) = [c,d]$, $c, d \in \mathbb{R} : g(a) = c$, $g(b) = d$; hence here f is continuous on $[a,b]$).*

Assume also that $\left(f \circ g^{-1}\right)^{(m)} \circ g \in L_\infty([a,b])$.

The right generalized g-fractional derivative of f of order α is defined as follows:

$$\left(D_{b-;g}^{\alpha}f\right)(x) := \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (g(t)-g(x))^{m-\alpha-1} g'(t) \left(f \circ g^{-1}\right)^{(m)} (g(t))\, dt, \tag{71}$$

$a \leq x \leq b$.

We saw that

$$I_{b-;g}^{m-\alpha}\left((-1)^m \left(f \circ g^{-1}\right)^{(m)} \circ g\right)(x) = \left(D_{b-;g}^{\alpha}f\right)(x), \quad a \leq x \leq b. \tag{72}$$

We proved earlier (37), (38), (39) that ($a \leq x \leq b$)

$$\begin{gathered}\left(I_{b-;g}^{\alpha}D_{b-;g}^{\alpha}f\right)(x) = \\ \frac{1}{(m-1)!}\int_b^x (g(x)-g(t))^{m-1} g'(t) \left(\left(f \circ g^{-1}\right)^{(m)} \circ g\right)(t)\, dt = \\ \frac{1}{\Gamma(\alpha)}\int_x^b (g(t)-g(x))^{\alpha-1} g'(t) \left(D_{b-;g}^{\alpha}f\right)(t)\, dt = \\ f(x) - f(b) - \sum_{k=1}^{m-1} \frac{\left(f \circ g^{-1}\right)^{(k)}(g(b))}{k!} (g(x)-g(b))^k .\end{gathered} \tag{73}$$

If $0 < \alpha \leq 1$, then $m = 1$, hence

$$\left(I_{b-;g}^{\alpha}D_{b-;g}^{\alpha}f\right)(x) = f(x) - f(b) \tag{74}$$

$$= \frac{1}{\Gamma(\alpha)}\int_x^b (g(t)-g(x))^{\alpha-1} g'(t) \left(D_{b-;g}^{\alpha}f\right)(t)\, dt$$

(when $\alpha \in (0,1)$, $D_{b-;g}^{\alpha}f$ is continuous on $[a,b]$ and)

$$= \frac{\left(D_{b-;g}^{\alpha}f\right)(\xi_x)}{\Gamma(\alpha)} \int_x^b (g(t)-g(x))^{\alpha-1} g'(t)\, dt = \frac{\left(D_{b-;g}^{\alpha}f\right)(\xi_x)}{\Gamma(\alpha+1)} (g(b)-g(x))^{\alpha}, \tag{75}$$

where $\xi_x \in [x,b]$.

We have proved

Theorem 25 *(right generalized g-mean value theorem). Let $0 < \alpha < 1$, and $f \circ g^{-1} \in AC([c,d])$, $\left(f \circ g^{-1}\right)' \circ g \in L_{\infty}([a,b])$, where g strictly increasing, $g \in AC([a,b])$, $f : [a,b] \to \mathbb{R}$. Then*

$$f(x) - f(b) = \frac{\left(D_{b-;g}^{\alpha}f\right)(\xi_x)}{\Gamma(\alpha+1)} (g(b)-g(x))^{\alpha}, \tag{76}$$

where $\xi_x \in [x,b]$, for $x \in [a,b]$.

13

Denote by

$$D_{b-;g}^{n\alpha} := D_{b-;g}^{\alpha} D_{b-;g}^{\alpha} ... D_{b-;g}^{\alpha} \quad (n \text{ times}),\ n \in \mathbb{N}. \tag{77}$$

Also denote by

$$I_{b-;g}^{n\alpha} := I_{b-;g}^{\alpha} I_{b-;g}^{\alpha} ... I_{b-;g}^{\alpha} \quad (n \text{ times}). \tag{78}$$

Here to remind

$$\left(I_{b-;g}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (g(t) - g(x))^{\alpha-1} g'(t) f(t)\, dt, \quad x \leq b. \tag{79}$$

We need

Theorem 26 *Suppose that $F_k := D_{b-;g}^{k\alpha} f$, $k = n, n+1$, fulfill $F_k \circ g^{-1} \in AC([c,d])$, and $\left(F_k \circ g^{-1}\right)' \circ g \in L_\infty([a,b])$, $0 < \alpha \leq 1$, $n \in \mathbb{N}$. Then*

$$\left(I_{b-;g}^{n\alpha} D_{b-;g}^{n\alpha} f\right)(x) - \left(I_{b-;g}^{(n+1)\alpha} D_{b-;g}^{(n+1)\alpha} f\right)(x) = \frac{(g(b) - g(x))^{n\alpha}}{\Gamma(n\alpha+1)} \left(D_{b-;g}^{n\alpha} f\right)(b). \tag{80}$$

Proof. By semigroup property of $I_{b-;g}^{\alpha}$, we get

$$\left(I_{b-;g}^{n\alpha} D_{b-;g}^{n\alpha} f\right)(x) - \left(I_{b-;g}^{(n+1)\alpha} D_{b-;g}^{(n+1)\alpha} f\right)(x) =$$

$$\left(I_{b-;g}^{n\alpha} \left(D_{b-;g}^{n\alpha} f - I_{b-;g}^{\alpha} D_{b-;g}^{(n+1)\alpha} f\right)\right)(x) = \tag{81}$$

$$\left(I_{b-;g}^{n\alpha} \left(D_{b-;g}^{n\alpha} f - \left(I_{b-;g}^{\alpha} D_{b-;g}^{\alpha}\right)\left(D_{b-;g}^{n\alpha} f\right)\right)\right)(x) \overset{(74)}{=}$$

$$\left(I_{b-;g}^{n\alpha} \left(D_{b-;g}^{n\alpha} f - D_{b-;g}^{n\alpha} f + D_{b-;g}^{n\alpha} f(b)\right)\right)(x) =$$

$$\left(I_{b-;g}^{n\alpha} \left(D_{b-;g}^{n\alpha} f(b)\right)\right)(x) = \left(D_{b-;g}^{n\alpha} f(b)\right)\left(I_{b-;g}^{n\alpha}(1)\right)(x) = \tag{82}$$

[Notice that

$$\left(I_{b-;g}^{\alpha} 1\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (g(t) - g(x))^{\alpha-1} g'(t)\, dt = \tag{83}$$

$$\frac{1}{\Gamma(\alpha)} \frac{(g(b) - g(x))^{\alpha}}{\alpha} = \frac{1}{\Gamma(\alpha+1)} (g(b) - g(x))^{\alpha}.$$

Thus we have

$$\left(I_{b-;g}^{\alpha} 1\right)(x) = \frac{(g(b) - g(x))^{\alpha}}{\Gamma(\alpha+1)}. \tag{84}$$

Hence it holds

$$\left(I_{b-;g}^{2\alpha} 1\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (g(t) - g(x))^{\alpha-1} g'(t) \frac{(g(b) - g(t))^{\alpha}}{\Gamma(\alpha+1)} dt =$$

$$\frac{1}{\Gamma(\alpha)\Gamma(\alpha+1)}\int_x^b (g(b)-g(t))^{\alpha}(g(t)-g(x))^{\alpha-1}g'(t)\,dt =$$

$$\frac{1}{\Gamma(\alpha)\Gamma(\alpha+1)}\int_{g(x)}^{g(b)} (g(b)-z)^{(\alpha+1)-1}(z-g(x))^{\alpha-1}\,dz =$$

$$\frac{1}{\Gamma(\alpha)\Gamma(\alpha+1)}\frac{\Gamma(\alpha+1)\Gamma(\alpha)}{\Gamma(2\alpha+1)}(g(b)-g(x))^{2\alpha} = \frac{1}{\Gamma(2\alpha+1)}(g(b)-g(x))^{2\alpha}, \tag{85}$$

etc.]

$$= \left(D_{b-;g}^{n\alpha}f\right)(b)\frac{(g(b)-g(x))^{n\alpha}}{\Gamma(n\alpha+1)}, \tag{86}$$

proving the claim. ■

We make

Remark 27 *Suppose that* $F_k = D_{b-;g}^{k\alpha}f$, *for* $k = 0, 1, ..., n+1$; *are as in last Theorem 26,* $0 < \alpha \leq 1$. *By (80) we get*

$$\sum_{i=0}^{n}\left(\left(I_{b-;g}^{i\alpha}D_{b-;g}^{i\alpha}f\right)(x) - I_{b-;g}^{(i+1)\alpha}D_{b-;g}^{(i+1)\alpha}f(x)\right) = \tag{87}$$

$$\sum_{i=0}^{n}\frac{(g(b)-g(x))^{i\alpha}}{\Gamma(i\alpha+1)}\left(D_{b-;g}^{i\alpha}f\right)(b).$$

That is

(notice that $I_{b-;g}^{0}f = D_{b-;g}^{0}f = f$*)*

$$f(x) - \left(I_{b-;g}^{(n+1)\alpha}D_{b-;g}^{(n+1)\alpha}f\right)(x) = \sum_{i=0}^{n}\frac{(g(b)-g(x))^{i\alpha}}{\Gamma(i\alpha+1)}\left(D_{b-;g}^{i\alpha}f\right)(b). \tag{88}$$

Hence

$$f(x) = \sum_{i=0}^{n}\frac{(g(b)-g(x))^{i\alpha}}{\Gamma(i\alpha+1)}\left(D_{b-;g}^{i\alpha}f\right)(b) + \left(I_{b-;g}^{(n+1)\alpha}D_{b-;g}^{(n+1)\alpha}f\right)(x) = \tag{89}$$

$$\sum_{i=0}^{n}\frac{(g(b)-g(x))^{i\alpha}}{\Gamma(i\alpha+1)}\left(D_{b-;g}^{i\alpha}f\right)(b) + R_g(x,b), \tag{90}$$

where

$$R_g(x,b) := \frac{1}{\Gamma((n+1)\alpha)}\int_x^b (g(t)-g(x))^{(n+1)\alpha-1}g'(t)\left(D_{b-;g}^{(n+1)\alpha}f\right)(t)\,dt. \tag{91}$$

(here $D_{b-;g}^{(n+1)\alpha}f$ *is continuous over* $[a,b]$*)*

15

Hence it holds

$$R_g(x,b) = \frac{\left(D_{b-;g}^{(n+1)\alpha} f\right)(\psi_x)}{\Gamma((n+1)\alpha)} \int_x^b (g(t) - g(x))^{(n+1)\alpha - 1} g'(t)\, dt =$$

$$\frac{\left(D_{b-;g}^{(n+1)\alpha} f\right)(\psi_x)}{\Gamma((n+1)\alpha)} \frac{(g(b) - g(x))^{(n+1)\alpha}}{(n+1)\alpha} = \frac{\left(D_{b-;g}^{(n+1)\alpha} f\right)(\psi_x)}{\Gamma((n+1)\alpha + 1)} (g(b) - g(x))^{(n+1)\alpha}, \tag{92}$$

where $\psi_x \in [x,b]$.

We have proved the following g-right generalized modified Taylor's formula:

Theorem 28 *Suppose that* $F_k := D_{b-;g}^{k\alpha} f$, *for* $k = 0, 1, ..., n+1$, *fulfill:* $F_k \circ g^{-1} \in AC([c,d])$ *and* $\left(F_k \circ g^{-1}\right)' \circ g \in L_\infty([a,b])$, *where* $0 < \alpha \leq 1$. *Then*

$$f(x) = \sum_{i=0}^{n} \frac{(g(b) - g(x))^{i\alpha}}{\Gamma(i\alpha + 1)} \left(D_{b-;g}^{i\alpha} f\right)(b) +$$

$$\frac{1}{\Gamma((n+1)\alpha)} \int_x^b (g(t) - g(x))^{(n+1)\alpha - 1} g'(t) \left(D_{b-;g}^{(n+1)\alpha} f\right)(t)\, dt = \tag{93}$$

$$\sum_{i=0}^{n} \frac{(g(b) - g(x))^{i\alpha}}{\Gamma(i\alpha + 1)} \left(D_{b-;g}^{i\alpha} f\right)(b) + \frac{\left(D_{b-;g}^{(n+1)\alpha} f\right)(\psi_x)}{\Gamma((n+1)\alpha + 1)} (g(b) - g(x))^{(n+1)\alpha}, \tag{94}$$

where $\psi_x \in [x,b]$, *any* $x \in [a,b]$.

We make

Remark 29 *Let* $\alpha > 0$, $m = \lceil \alpha \rceil$, *g is strictly increasing and* $g \in AC([a,b])$. *Call* $l = f \circ g^{-1}$, $f : [a,b] \to \mathbb{R}$. *Assume* $l \in AC^m([c,d])$ *(i.e.* $l^{(m-1)} \in AC([c,d])$*) (where* $g([a,b]) = [c,d]$, $c, d \in \mathbb{R} : g(a) = c$, $g(b) = d$, *hence here f is continuous on* $[a,b]$*).*

Assume also that $\left(f \circ g^{-1}\right)^{(m)} \circ g \in L_\infty([a,b])$.

The left generalized g-fractional derivative of f of order α is defined as follows:

$$\left(D_{a+;g}^{\alpha} f\right)(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x (g(x) - g(t))^{m-\alpha-1} g'(t) \left(f \circ g^{-1}\right)^{(m)}(g(t))\, dt, \tag{95}$$

$x \geq a$.

If $\alpha \notin \mathbb{N}$, then $\left(D_{a+;g}^{\alpha} f\right) \in C([a,b])$.

We see that

$$\left(I_{a+;g}^{m-\alpha} \left(\left(f \circ g^{-1}\right)^{(m)} \circ g\right)\right)(x) = \left(D_{a+;g}^{\alpha} f\right)(x), \quad x \geq a. \tag{96}$$

We proved earlier (24), (25), (26), (27), that $(a \le x \le b)$

$$\left(I_{a+;g}^{\alpha} D_{a+;g}^{\alpha} f\right)(x) =$$

$$\frac{1}{(m-1)!} \int_a^x (g(x) - g(t))^{m-1} g'(t) \left(\left(f \circ g^{-1}\right)^{(m)} \circ g\right)(t)\, dt = \tag{97}$$

$$\frac{1}{\Gamma(\alpha)} \int_a^x (g(x) - g(t))^{\alpha-1} g'(t) \left(D_{a+;g}^{\alpha} f\right)(t)\, dt =$$

$$f(x) - f(a) - \sum_{k=1}^{m-1} \frac{\left(f \circ g^{-1}\right)^{(k)}(g(a))}{k!} (g(x) - g(a))^k . \tag{98}$$

If $0 < \alpha \le 1$, then $m = 1$, and then

$$\left(I_{a+;g}^{\alpha} D_{a+;g}^{\alpha} f\right)(x) = f(x) - f(a) \tag{99}$$

$$= \frac{1}{\Gamma(\alpha)} \int_a^x (g(x) - g(t))^{\alpha-1} g'(t) \left(D_{a+;g}^{\alpha} f\right)(t)\, dt$$

$$\overset{(\alpha \in (0,1) \text{ case})}{=} \frac{\left(D_{a+;g}^{\alpha} f\right)(\xi_x)}{\Gamma(\alpha+1)} (g(x) - g(a))^{\alpha} , \tag{100}$$

where $\xi_x \in [a, x]$, any $x \in [a, b]$.

We have proved

Theorem 30 *(left generalized g-mean value theorem). Let $0 < \alpha < 1$ and $f \circ g^{-1} \in AC([c, d])$ and $\left(f \circ g^{-1}\right)' \circ g \in L_{\infty}([a, b])$, where g strictly increasing, $g \in AC([a, b])$, $f : [a, b] \to \mathbb{R}$. Then*

$$f(x) - f(a) = \frac{\left(D_{a+;g}^{\alpha} f\right)(\xi_x)}{\Gamma(\alpha+1)} (g(x) - g(a))^{\alpha} , \tag{101}$$

where $\xi_x \in [a, x]$, any $x \in [a, b]$.

Denote by

$$D_{a+;g}^{n\alpha} := D_{a+;g}^{\alpha} D_{a+;g}^{\alpha} ... D_{a+;g}^{\alpha} \quad (n \text{ times}),\ n \in \mathbb{N}. \tag{102}$$

Also denote by

$$I_{a+;g}^{n\alpha} := I_{a+;g}^{\alpha} I_{a+;g}^{\alpha} ... I_{a+;g}^{\alpha} \quad (n \text{ times}). \tag{103}$$

Here to remind

$$\left(I_{a+;g}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (g(x) - g(t))^{\alpha-1} g'(t) f(t)\, dt, \quad x \ge a. \tag{104}$$

By convention $I_{a+;g}^{0} = D_{a+;g}^{0} = I$ (identity operator).

We give

Theorem 31 *Suppose that* $F_k := D^{k\alpha}_{a+;g} f$, $k = n, n+1$, *fulfill* $F_k \circ g^{-1} \in AC([c,d])$, *and* $\left(F_k \circ g^{-1}\right)' \circ g \in L_\infty([a,b])$, $0 < \alpha \leq 1$, $n \in \mathbb{N}$. *Then*

$$\left(I^{n\alpha}_{a+;g} D^{n\alpha}_{a+;g} f\right)(x) - \left(I^{(n+1)\alpha}_{a+;g} D^{(n+1)\alpha}_{a+;g} f\right)(x) = \frac{(g(x) - g(a))^{n\alpha}}{\Gamma(n\alpha + 1)} \left(D^{n\alpha}_{a+;g} f\right)(a). \tag{105}$$

Proof. By semigroup property of $I^{\alpha}_{a+;g}$, we get

$$\left(I^{n\alpha}_{a+;g} D^{n\alpha}_{a+;g} f\right)(x) - \left(I^{(n+1)\alpha}_{a+;g} D^{(n+1)\alpha}_{a+;g} f\right)(x) =$$

$$\left(I^{n\alpha}_{a+;g}\left(D^{n\alpha}_{a+;g} f - I^{\alpha}_{a+;g} D^{(n+1)\alpha}_{a+;g} f\right)\right)(x) = \tag{106}$$

$$\left(I^{n\alpha}_{a+;g}\left(D^{n\alpha}_{a+;g} f - \left(I^{\alpha}_{a+;g} D^{\alpha}_{a+;g}\right)\left(D^{n\alpha}_{a+;g} f\right)\right)\right)(x) \overset{(99)}{=}$$

$$\left(I^{n\alpha}_{a+;g}\left(D^{n\alpha}_{a+;g} f - D^{n\alpha}_{a+;g} f + D^{n\alpha}_{a+;g} f(a)\right)\right)(x) =$$

$$\left(I^{n\alpha}_{a+;g}\left(D^{n\alpha}_{a+;g} f(a)\right)\right)(x) = \left(D^{n\alpha}_{a+;g} f(a)\right)\left(I^{n\alpha}_{a+;g}(1)\right)(x) = \tag{107}$$

[notice that

$$\left(I^{\alpha}_{a+;g} 1\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (g(x) - g(t))^{\alpha - 1} g'(t)\, dt$$

$$= \frac{(g(x) - g(a))^{\alpha}}{\Gamma(\alpha + 1)}. \tag{108}$$

Hence

$$\left(I^{2\alpha}_{a+;g} 1\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (g(x) - g(t))^{\alpha - 1} g'(t) \frac{(g(t) - g(a))^{\alpha}}{\Gamma(\alpha + 1)} dt = \tag{109}$$

$$\frac{1}{\Gamma(\alpha)\Gamma(\alpha + 1)} \int_a^x (g(x) - g(t))^{\alpha - 1} g'(t) (g(t) - g(a))^{\alpha}\, dt =$$

$$\frac{1}{\Gamma(\alpha)\Gamma(\alpha + 1)} \int_{g(a)}^{g(x)} (g(x) - z)^{\alpha - 1} (z - g(a))^{(\alpha + 1) - 1}\, dt =$$

$$\frac{1}{\Gamma(\alpha)\Gamma(\alpha + 1)} \frac{\Gamma(\alpha)\Gamma(\alpha + 1)}{\Gamma(2\alpha + 1)} (g(x) - g(a))^{2\alpha}.$$

That is

$$\left(I^{2\alpha}_{a+;g} 1\right)(x) = \frac{(g(x) - g(a))^{2\alpha}}{\Gamma(2\alpha + 1)}, \tag{110}$$

etc.]

$$= \left(D^{n\alpha}_{a+;g} f(a)\right) \frac{(g(x) - g(a))^{n\alpha}}{\Gamma(n\alpha + 1)}, \tag{111}$$

proving the claim. ■

Remark 32 *Suppose that $F_k = D^{k\alpha}_{a+;g} f$, for $k = 0, 1, ..., n+1$; are as in Theorem 31, $0 < \alpha \leq 1$. By (105) we get*

$$\sum_{i=0}^{n} \left(\left(I^{i\alpha}_{a+;g} D^{i\alpha}_{a+;g} f \right)(x) - I^{(i+1)\alpha}_{a+;g} D^{(i+1)\alpha}_{a+;g} f(x) \right) = \tag{112}$$

$$\sum_{i=0}^{n} \frac{(g(x) - g(a))^{i\alpha}}{\Gamma(i\alpha + 1)} \left(D^{i\alpha}_{a+;g} f \right)(a).$$

That is

$$f(x) - \left(I^{(n+1)\alpha}_{a+;g} D^{(n+1)\alpha}_{a+;g} f \right)(x) = \sum_{i=0}^{n} \frac{(g(x) - g(a))^{i\alpha}}{\Gamma(i\alpha + 1)} \left(D^{i\alpha}_{a+;g} f \right)(a).$$

Hence

$$f(x) = \sum_{i=0}^{n} \frac{(g(x) - g(a))^{i\alpha}}{\Gamma(i\alpha + 1)} \left(D^{i\alpha}_{a+;g} f \right)(a) + \left(I^{(n+1)\alpha}_{a+;g} D^{(n+1)\alpha}_{a+;g} f \right)(x) = \tag{113}$$

$$\sum_{i=0}^{n} \frac{(g(x) - g(a))^{i\alpha}}{\Gamma(i\alpha + 1)} \left(D^{i\alpha}_{a+;g} f \right)(a) + R_g(a, x), \tag{114}$$

where

$$R_g(a, x) := \frac{1}{\Gamma((n+1)\alpha)} \int_a^x (g(x) - g(t))^{(n+1)\alpha - 1} g'(t) \left(D^{(n+1)\alpha}_{a+;g} f \right)(t)\, dt. \tag{115}$$

(there $D^{(n+1)\alpha}_{a+;g} f$ is continuous over $[a, b]$.)

Hence it holds

$$R_g(a, x) = \frac{\left(D^{(n+1)\alpha}_{a+;g} f \right)(\psi_x)}{\Gamma((n+1)\alpha)} \left(\int_a^x (g(x) - g(t))^{(n+1)\alpha - 1} g'(t)\, dt \right) =$$

$$\frac{\left(D^{(n+1)\alpha}_{a+;g} f \right)(\psi_x)}{\Gamma((n+1)\alpha + 1)} (g(x) - g(a))^{(n+1)\alpha}, \tag{116}$$

where $\psi_x \in [a, x]$.

We have proved the following g-left generalized modified Taylor's formula:

Theorem 33 *Suppose that $F_k := D^{k\alpha}_{a+;g} f$, for $k = 0, 1, ..., n+1$, fulfill: $F_k \circ g^{-1} \in AC([c, d])$ and $\left(F_k \circ g^{-1} \right)' \circ g \in L_\infty([a, b])$, where $0 < \alpha \leq 1$. Then*

$$f(x) = \sum_{i=0}^{n} \frac{(g(x) - g(a))^{i\alpha}}{\Gamma(i\alpha + 1)} \left(D^{i\alpha}_{a+;g} f \right)(a) + \tag{117}$$

$$\frac{1}{\Gamma\left(\left(n+1\right)\alpha\right)}\int_{a}^{x}\left(g\left(x\right)-g\left(t\right)\right)^{\left(n+1\right)\alpha-1}g'\left(t\right)\left(D_{a+;g}^{\left(n+1\right)\alpha}f\right)\left(t\right)dt=$$

$$\sum_{i=0}^{n}\frac{\left(g\left(x\right)-g\left(a\right)\right)^{i\alpha}}{\Gamma\left(i\alpha+1\right)}\left(D_{a+;g}^{i\alpha}f\right)\left(a\right)+\frac{\left(D_{a+;g}^{\left(n+1\right)\alpha}f\right)\left(\psi_{x}\right)}{\Gamma\left(\left(n+1\right)\alpha+1\right)}\left(g\left(x\right)-g\left(a\right)\right)^{\left(n+1\right)\alpha}, \tag{118}$$

where $\psi_x \in [a,x]$, *any* $x \in [a,b]$.

References

[1] G. Anastassiou, *On right fractional calculus*, Chaos, Solitons & Fractals, 42 (2009), 365-376.

[2] G. Anastassiou, *Fractional Differentiation Inequalities*, Springer, New York, 2009.

[3] G. Anastassiou, *Basic Inequalities, Revisited*, Mathematica Balkanica, New Series, Vol. 24, 2010, Fasc. 1-2, pp. 59-84.

[4] G. Anastassiou, *Fractional Representation Formulae and Right Fractional Inequalities*, Mathematical and Computer Modelling, 54, (11-12), (2011), 3098-3115.

[5] G. Anastassiou, *The Reduction Method in Fractional Calculus and Fractional Ostrowski type Inequalities*, Indian Journal of Mathematics, Vol. 56, No. 3, 2014, pp. 333-357.

[6] G. Anastassiou, *Univariate left General High order Fractional Monotone Approximation*, submitted, 2015.

[7] G. Anastassiou, *Univariate Right General High order Fractional Monotone Approximation Theory*, submitted, 2015.

[8] K. Diethelm, *The Analysis of Fractional Differential Equations*, Lecture Notes in Mathematics, Vol. 2004, 1st edition, New York, Heidelberg, 2010.

[9] Rong-Qing Jia, *Chapter 3. Absolutely Continuous Functions*, https://www.ualberta.ca/~rjia/Math418/Notes/Chap.3.pdf.

[10] A.A. Kilbas, H.M. Srivastava and J.J. Tujillo, *Theory and Applications of Fractional Differential Equations*, Vol. 204 of North-Holland Mathematics Studies, Elsevier, New York, USA, 2006.

[11] H.L. Royden, *Real Analysis*, Second edition, Macmillan Publishing Co., Inc., New York, 1968.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 21, NO.7, 2016, COPYRIGHT 2016 EUDOXUS PRESS, LLC

Generalized Canavati type Fractional Taylor's formulae

George A. Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

Abstract

We present here four new generalized Canavati type fractional Taylor's formulae.

2010 AMS Subject Classification: 26A33.

Key Words and Phrases: fractional integral, fractional derivative of Canavati type, fractional Taylor's formula.

1 Results

Let $g : [a, b] \to \mathbb{R}$ be a strictly increasing function. Let $f \in C^n([a, b])$, $n \in \mathbb{N}$. Assume that $g \in C^1([a, b])$, and $g^{-1} \in C^n([a, b])$. Call $l := f \circ g^{-1} : [g(a), g(b)] \to \mathbb{R}$. It is clear that $l, l', ..., l^{(n)}$ are continuous functions from $[g(a), g(b)]$ into $f([a, b]) \subseteq \mathbb{R}$.

Let $\nu \geq 1$ such that $[\nu] = n$, $n \in \mathbb{N}$ as above, where $[\cdot]$ is the integral part of the number.

Clearly when $0 < \nu < 1$, $[\nu] = 0$. Next we follow [1], pp. 7-9.

I) Let $h \in C([g(a), g(b)])$, we define the left Riemann-Liouville fractional integral as

$$\left(J_{\nu}^{z_0} h\right)(z) := \frac{1}{\Gamma(\nu)} \int_{z_0}^{z} (z - t)^{\nu - 1} h(t)\, dt, \tag{1}$$

for $g(a) \leq z_0 \leq z \leq g(b)$, where Γ is the gamma function; $\Gamma(\nu) = \int_0^\infty e^{-t} t^{\nu-1} dt$.

We set $J_0^{z_0} h = h$.

Let $\alpha := \nu - [\nu]$ $(0 < \alpha < 1)$. We define the subspace $C_{g(x_0)}^{\nu}([g(a), g(b)])$ of $C^{[\nu]}([g(a), g(b)])$, where $x_0 \in [a, b]$:

$$C_{g(x_0)}^{\nu}([g(a), g(b)]) := \left\{ h \in C^{[\nu]}([g(a), g(b)]) : J_{1-\alpha}^{g(x_0)} h^{([\nu])} \in C^1([g(x_0), g(b)]) \right\}. \tag{2}$$

So let $h \in C^{\nu}_{g(x_0)}([g(a), g(b)])$; we define the left g-generalized fractional derivative of h of order ν, of Canavati type, over $[g(x_0), g(b)]$ as

$$D^{\nu}_{g(x_0)} h := \left(J^{g(x_0)}_{1-\alpha} h^{([\nu])} \right)'. \tag{3}$$

Clearly, for $h \in C^{\nu}_{g(x_0)}([g(a), g(b)])$, there exists

$$\left(D^{\nu}_{g(x_0)} h \right)(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_{g(x_0)}^{z} (z-t)^{-\alpha} h^{([\nu])}(t)\, dt, \tag{4}$$

for all $g(x_0) \leq z \leq g(b)$.

In particular, when $f \circ g^{-1} \in C^{\nu}_{g(x_0)}([g(a), g(b)])$ we have that

$$\left(D^{\nu}_{g(x_0)} \left(f \circ g^{-1} \right) \right)(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_{g(x_0)}^{z} (z-t)^{-\alpha} \left(f \circ g^{-1} \right)^{([\nu])}(t)\, dt, \tag{5}$$

for all $g(x_0) \leq z \leq g(b)$. We have $D^{n}_{g(x_0)} \left(f \circ g^{-1} \right) = \left(f \circ g^{-1} \right)^{(n)}$ and $D^{0}_{g(x_0)} \left(f \circ g^{-1} \right) = f \circ g^{-1}$.

By Theorem 2.1, p. 8 of [1], we have for $f \circ g^{-1} \in C^{\nu}_{g(x_0)}([g(a), g(b)])$, where $x_0 \in [a, b]$ is fixed, that

(i) if $\nu \geq 1$, then

$$\left(f \circ g^{-1} \right)(z) = \sum_{k=0}^{[\nu]-1} \frac{\left(f \circ g^{-1} \right)^{(k)}(g(x_0))}{k!} (z - g(x_0))^k +$$

$$\frac{1}{\Gamma(\nu)} \int_{g(x_0)}^{z} (z-t)^{\nu-1} \left(D^{\nu}_{g(x_0)} \left(f \circ g^{-1} \right) \right)(t)\, dt, \tag{6}$$

all $z \in [g(a), g(b)] : z \geq g(x_0)$,

(ii) if $0 < \nu < 1$, we get

$$\left(f \circ g^{-1} \right)(z) = \frac{1}{\Gamma(\nu)} \int_{g(x_0)}^{z} (z-t)^{\nu-1} \left(D^{\nu}_{g(x_0)} \left(f \circ g^{-1} \right) \right)(t)\, dt, \tag{7}$$

all $z \in [g(a), g(b)] : z \geq g(x_0)$.

We have proved the following left generalized g-fractional, of Canavati type, Taylor's formula:

Theorem 1 *Let $f \circ g^{-1} \in C^{\nu}_{g(x_0)}([g(a), g(b)])$, where $x_0 \in [a, b]$ is fixed.*

(i) if $\nu \geq 1$, then

$$f(x) - f(x_0) = \sum_{k=1}^{[\nu]-1} \frac{\left(f \circ g^{-1} \right)^{(k)}(g(x_0))}{k!} (g(x) - g(x_0))^k +$$

$$\frac{1}{\Gamma(\nu)} \int_{g(x_0)}^{g(x)} (g(x)-t)^{\nu-1} \left(D^{\nu}_{g(x_0)} \left(f \circ g^{-1} \right) \right)(t)\, dt, \quad \text{all } x \in [a, b] : x \geq x_0, \tag{8}$$

2

(ii) if $0<\nu<1$*, we get*

$$f(x)=\frac{1}{\Gamma(\nu)}\int_{g(x_0)}^{g(x)}(g(x)-t)^{\nu-1}\left(D_{g(x_0)}^{\nu}\left(f\circ g^{-1}\right)\right)(t)\,dt,\quad \text{all } x\in[a,b]:x\geq x_0. \tag{9}$$

By the change of variable method, see [3], we may rewrite the remainder of (8), (9), as

$$\frac{1}{\Gamma(\nu)}\int_{g(x_0)}^{g(x)}(g(x)-t)^{\nu-1}\left(D_{g(x_0)}^{\nu}\left(f\circ g^{-1}\right)\right)(t)\,dt= \tag{10}$$

$$\frac{1}{\Gamma(\nu)}\int_{x_0}^{x}(g(x)-g(s))^{\nu-1}\left(D_{g(x_0)}^{\nu}\left(f\circ g^{-1}\right)\right)(g(s))\,g'(s)\,ds,$$

all $x\in[a,b]:x\geq x_0$.

We may rewrite (9) as follows:

if $0<\nu<1$, we have

$$f(x)=\left(J_{\nu}^{g(x_0)}\left(D_{g(x_0)}^{\nu}\left(f\circ g^{-1}\right)\right)\right)(g(x)), \tag{11}$$

all $x\in[a,b]:x\geq x_0$.

II) Next we follow [2], pp. 345-348.

Let $h\in C([g(a),g(b)])$, we define the right Riemann-Liouville fractional integral as

$$\left(J_{z_0-}^{\nu}h\right)(z):=\frac{1}{\Gamma(\nu)}\int_{z}^{z_0}(t-z)^{\nu-1}h(t)\,dt, \tag{12}$$

for $g(a)\leq z\leq z_0\leq g(b)$. We set $J_{z_0-}^{0}h=h$.

Let $\alpha:=\nu-[\nu]$ $(0<\alpha<1)$. We define the subspace $C_{g(x_0)-}^{\nu}([g(a),g(b)])$ of $C^{[\nu]}([g(a),g(b)])$, where $x_0\in[a,b]$:

$$C_{g(x_0)-}^{\nu}([g(a),g(b)]):=$$

$$\left\{h\in C^{[\nu]}([g(a),g(b)]):J_{g(x_0)-}^{1-\alpha}h^{([\nu])}\in C^{1}([g(x_0),g(b)])\right\}. \tag{13}$$

So let $h\in C_{g(x_0)-}^{\nu}([g(a),g(b)])$; we define the right g-generalized fractional derivative of h of order ν, of Canavati type, over $[g(a),g(x_0)]$ as

$$D_{g(x_0)-}^{\nu}h:=(-1)^{n-1}\left(J_{g(x_0)-}^{1-\alpha}h^{([\nu])}\right)'. \tag{14}$$

Clearly, for $h\in C_{g(x_0)-}^{\nu}([g(a),g(b)])$, there exists

$$\left(D_{g(x_0)-}^{\nu}h\right)(z)=\frac{(-1)^{n-1}}{\Gamma(1-\alpha)}\frac{d}{dz}\int_{z}^{g(x_0)}(t-z)^{-\alpha}h^{([\nu])}(t)\,dt, \tag{15}$$

3

for all $g(a) \leq z \leq g(x_0) \leq g(b)$.

In particular, when $f \circ g^{-1} \in C^{\nu}_{g(x_0)-}([g(a), g(b)])$ we have that

$$\left(D^{\nu}_{g(x_0)-}\left(f \circ g^{-1}\right)\right)(z) = \frac{(-1)^{n-1}}{\Gamma(1-\alpha)} \frac{d}{dz} \int_z^{g(x_0)} (t-z)^{-\alpha} \left(f \circ g^{-1}\right)^{([\nu])}(t)\, dt, \tag{16}$$

for all $g(a) \leq z \leq g(x_0) \leq g(b)$.

We get that

$$\left(D^{n}_{g(x_0)-}\left(f \circ g^{-1}\right)\right)(z) = (-1)^n \left(f \circ g^{-1}\right)^{(n)}(z) \tag{17}$$

and $\left(D^{0}_{g(x_0)-}\left(f \circ g^{-1}\right)\right)(z) = \left(f \circ g^{-1}\right)(z)$, all $z \in [g(a), g(x_0)]$.

By Theorem 23.19, p. 348 of [2], we have for $f \circ g^{-1} \in C^{\nu}_{g(x_0)-}([g(a), g(b)])$, where $x_0 \in [a, b]$ is fixed, that

(i) if $\nu \geq 1$, then

$$\left(f \circ g^{-1}\right)(z) = \sum_{k=0}^{[\nu]-1} \frac{\left(f \circ g^{-1}\right)^{(k)}(g(x_0))}{k!} (z - g(x_0))^k + \tag{18}$$

$$\frac{1}{\Gamma(\alpha)} \int_z^{g(x_0)} (t-z)^{\nu-1} \left(D^{\nu}_{g(x_0)-}\left(f \circ g^{-1}\right)\right)(t)\, dt,$$

all $z \in [g(a), g(b)] : z \leq g(x_0)$,

(ii) if $0 < \nu < 1$, we get

$$\left(f \circ g^{-1}\right)(z) = \frac{1}{\Gamma(\nu)} \int_z^{g(x_0)} (t-z)^{\nu-1} \left(D^{\nu}_{g(x_0)-}\left(f \circ g^{-1}\right)\right)(t)\, dt, \tag{19}$$

all $z \in [g(a), g(b)] : z \leq g(x_0)$.

We have proved the following right generalized g-fractional, of Canavati type, Taylor's formula:

Theorem 2 *Let $f \circ g^{-1} \in C^{\nu}_{g(x_0)-}([g(a), g(b)])$, where $x_0 \in [a, b]$ is fixed.*

(i) if $\nu \geq 1$, then

$$f(x) - f(x_0) = \sum_{k=1}^{[\nu]-1} \frac{\left(f \circ g^{-1}\right)^{(k)}(g(x_0))}{k!} (g(x) - g(x_0))^k +$$

$$\frac{1}{\Gamma(\nu)} \int_{g(x)}^{g(x_0)} (t - g(x))^{\nu-1} \left(D^{\nu}_{g(x_0)-}\left(f \circ g^{-1}\right)\right)(t)\, dt, \quad \text{all } a \leq x \leq x_0, \tag{20}$$

(ii) if $0 < \nu < 1$, we get

$$f(x) = \frac{1}{\Gamma(\nu)} \int_{g(x)}^{g(x_0)} (t - g(x))^{\nu-1} \left(D^{\nu}_{g(x_0)-}\left(f \circ g^{-1}\right)\right)(t)\, dt, \quad \text{all } a \leq x \leq x_0. \tag{21}$$

4

By change of variable, see [3], we may rewrite the remainder of (20), (21), as

$$\frac{1}{\Gamma(\nu)} \int_{g(x)}^{g(x_0)} (t - g(x))^{\nu-1} \left(D_{g(x_0)-}^{\nu} \left(f \circ g^{-1} \right) \right)(t)\, dt = \tag{22}$$

$$\frac{1}{\Gamma(\nu)} \int_{x}^{x_0} (g(s) - g(x))^{\nu-1} \left(D_{g(x_0)-}^{\nu} \left(f \circ g^{-1} \right) \right)(g(s))\, g'(s)\, ds,$$

all $a \leq x \leq x_0$.

We may rewrite (21) as follows:

if $0 < \nu < 1$, we have

$$f(x) = \left(J_{g(x_0)-}^{\nu} \left(D_{g(x_0)-}^{\nu} \left(f \circ g^{-1} \right) \right) \right)(g(x)), \tag{23}$$

all $a \leq x \leq x_0 \leq b$.

III) Denote by

$$D_{g(x_0)}^{m\nu} = D_{g(x_0)}^{\nu} D_{g(x_0)}^{\nu} \ldots D_{g(x_0)}^{\nu} \quad (m\text{-times}),\ m \in \mathbb{N}. \tag{24}$$

Also denote by

$$J_{m\nu}^{g(x_0)} = J_{\nu}^{g(x_0)} J_{\nu}^{g(x_0)} \ldots J_{\nu}^{g(x_0)} \quad (m\text{-times}),\ m \in \mathbb{N}. \tag{25}$$

We need

Theorem 3 *Here $0 < \nu < 1$. Assume that $\left(D_{g(x_0)}^{m\nu} \left(f \circ g^{-1} \right) \right) \in C_{g(x_0)}^{\nu}([g(a), g(b)])$, where $x_0 \in [a, b]$ is fixed. Assume also that $\left(D_{g(x_0)}^{(m+1)\nu} \left(f \circ g^{-1} \right) \right) \in C([g(x_0), g(b)])$. Then*

$$\left(J_{m\nu}^{g(x_0)} D_{g(x_0)}^{m\nu} \left(f \circ g^{-1} \right) \right)(g(x)) - \left(J_{(m+1)\nu}^{g(x_0)} D_{g(x_0)}^{(m+1)\nu} \left(f \circ g^{-1} \right) \right)(g(x)) = 0, \tag{26}$$

for all $x_0 \leq x \leq b$.

Proof. We observe that $(l := f \circ g^{-1})$

$$\left(J_{m\nu}^{g(x_0)} D_{g(x_0)}^{m\nu}(l) \right)(g(x)) - \left(J_{(m+1)\nu}^{g(x_0)} D_{g(x_0)}^{(m+1)\nu}(l) \right)(g(x)) =$$

$$\left(J_{m\nu}^{g(x_0)} \left(D_{g(x_0)}^{m\nu}(l) - J_{\nu}^{g(x_0)} D_{g(x_0)}^{(m+1)\nu}(l) \right) \right)(g(x)) = \tag{27}$$

$$\left(J_{m\nu}^{g(x_0)} \left(D_{g(x_0)}^{m\nu}(l) - \left(J_{\nu}^{g(x_0)} D_{g(x_0)}^{\nu} \right) \left(\left(D_{g(x_0)}^{m\nu}(l) \right) \circ g \circ g^{-1} \right) \right) \right)(g(x)) =$$

$$\left(J_{m\nu}^{g(x_0)} \left(D_{g(x_0)}^{m\nu}(l) - \left(D_{g(x_0)}^{m\nu}(l) \right) \right) \right)(g(x)) = \left(J_{m\nu}^{g(x_0)}(0) \right)(g(x)) = 0.$$

■

We make

5

Remark 4 *Let $0 < \nu < 1$. Assume that $\left(D^{i\nu}_{g(x_0)}\left(f \circ g^{-1}\right)\right) \in C^{\nu}_{g(x_0)}\left(\left[g(a), g(b)\right]\right)$, $x_0 \in [a,b]$, for all $i = 0, 1, ..., m$. Assume also that $\left(D^{(m+1)\nu}_{g(x_0)}\left(f \circ g^{-1}\right)\right) \in C\left(\left[g(x_0), g(b)\right]\right)$. We have that*

$$\sum_{i=0}^{m}\left[\left(J^{g(x_0)}_{i\nu} D^{i\nu}_{g(x_0)}\left(f \circ g^{-1}\right)\right)(g(x)) - \left(J^{g(x_0)}_{(i+1)\nu} D^{(i+1)\nu}_{g(x_0)}\left(f \circ g^{-1}\right)\right)(g(x))\right] = 0. \tag{28}$$

Hence it holds

$$f(x) - \left(J^{g(x_0)}_{(m+1)\nu} D^{(m+1)\nu}_{g(x_0)}\left(f \circ g^{-1}\right)\right)(g(x)) = 0, \tag{29}$$

for all $x_0 \leq x \leq b$.

That is

$$f(x) = \left(J^{g(x_0)}_{(m+1)\nu} D^{(m+1)\nu}_{g(x_0)}\left(f \circ g^{-1}\right)\right)(g(x)), \tag{30}$$

for all $x_0 \leq x \leq b$.

We have proved the following modified and generalized left fractional Taylor's formula of Canavati type:

Theorem 5 *Let $0 < \nu < 1$. Assume that $\left(D^{i\nu}_{g(x_0)}\left(f \circ g^{-1}\right)\right) \in C^{\nu}_{g(x_0)}\left(\left[g(a), g(b)\right]\right)$, $x_0 \in [a,b]$, for $i = 0, 1, ..., m$. Assume also that $\left(D^{(m+1)\nu}_{g(x_0)}\left(f \circ g^{-1}\right)\right) \in C\left(\left[g(x_0), g(b)\right]\right)$. Then*

$$f(x) = \frac{1}{\Gamma((m+1)\nu)} \int_{g(x_0)}^{g(x)} (g(x) - z)^{(m+1)\nu - 1}\left(D^{(m+1)\nu}_{g(x_0)}\left(f \circ g^{-1}\right)\right)(z)\,dz \tag{31}$$

$$= \frac{1}{\Gamma((m+1)\nu)} \int_{x_0}^{x} (g(x) - g(s))^{(m+1)\nu - 1}\left(D^{(m+1)\nu}_{g(x_0)}\left(f \circ g^{-1}\right)\right)(g(s))\, g'(s)\,ds,$$

all $x_0 \leq x \leq b$.

IV) Denote by

$$D^{m\nu}_{g(x_0)-} = D^{\nu}_{g(x_0)-} D^{\nu}_{g(x_0)-} ... D^{\nu}_{g(x_0)-} \quad (m\text{-times}),\ m \in \mathbb{N}. \tag{32}$$

Also denote by

$$J^{m\nu}_{g(x_0)-} = J^{\nu}_{g(x_0)-} J^{\nu}_{g(x_0)-} ... J^{\nu}_{g(x_0)-} \quad (m\text{-times}),\ m \in \mathbb{N}. \tag{33}$$

We need

Theorem 6 *Here $0 < \nu < 1$. Assume that $\left(D^{m\nu}_{g(x_0)-}\left(f \circ g^{-1}\right)\right) \in C^{\nu}_{g(x_0)-}\left(\left[g(a), g(b)\right]\right)$, where $x_0 \in [a,b]$ is fixed. Assume also that $\left(D^{(m+1)\nu}_{g(x_0)-}\left(f \circ g^{-1}\right)\right) \in C\left(\left[g(a), g(x_0)\right]\right)$. Then*

$$\left(J^{m\nu}_{g(x_0)-} D^{m\nu}_{g(x_0)-}\left(f \circ g^{-1}\right)\right)(g(x)) - \left(J^{(m+1)\nu}_{g(x_0)-} D^{(m+1)\nu}_{g(x_0)-}\left(f \circ g^{-1}\right)\right)(g(x)) = 0, \tag{34}$$

for all $a \leq x \leq x_0$.

6

Proof. We observe that $(l := f \circ g^{-1})$

$$\left(J_{g(x_0)-}^{m\nu} D_{g(x_0)-}^{m\nu}(l)\right)(g(x)) - \left(J_{g(x_0)-}^{(m+1)\nu} D_{g(x_0)-}^{(m+1)\nu}(l)\right)(g(x)) =$$

$$\left(J_{g(x_0)-}^{m\nu}\left(D_{g(x_0)-}^{m\nu}(l) - J_{g(x_0)-}^{\nu} D_{g(x_0)-}^{(m+1)\nu}(l)\right)\right)(g(x)) =$$

$$\left(J_{g(x_0)-}^{m\nu}\left(D_{g(x_0)-}^{m\nu}(l) - \left(J_{g(x_0)-}^{\nu} D_{g(x_0)-}^{\nu}\right)\left(\left(D_{g(x_0)-}^{m\nu}(l)\right)\circ g \circ g^{-1}\right)\right)\right)(g(x)) =$$

$$\left(J_{g(x_0)-}^{m\nu}\left(D_{g(x_0)-}^{m\nu}(l) - D_{g(x_0)-}^{m\nu}(l)\right)\right)(g(x)) = J_{g(x_0)-}^{m\nu}(0)(g(x)) = 0. \tag{35}$$

■

We make

Remark 7 *Let $0 < \nu < 1$. Assume that $\left(D_{g(x_0)-}^{i\nu}\left(f \circ g^{-1}\right)\right) \in C_{g(x_0)-}^{\nu}([g(a), g(b)])$, $x_0 \in [a,b]$, for all $i = 0, 1, ..., m$. Assume also that $\left(D_{g(x_0)-}^{(m+1)\nu}\left(f \circ g^{-1}\right)\right) \in C([g(a), g(x_0)])$. We have that (by (34))*

$$\sum_{i=0}^{m}\left[\left(J_{g(x_0)-}^{i\nu} D_{g(x_0)-}^{i\nu}\left(f \circ g^{-1}\right)\right)(g(x)) - \left(J_{g(x_0)-}^{(i+1)\nu} D_{g(x_0)-}^{(i+1)\nu}\left(f \circ g^{-1}\right)\right)(g(x))\right] = 0. \tag{36}$$

Hence it holds

$$f(x) - \left(J_{g(x_0)-}^{(m+1)\nu} D_{g(x_0)-}^{(m+1)\nu}\left(f \circ g^{-1}\right)\right)(g(x)) = 0, \tag{37}$$

for all $a \leq x \leq x_0 \leq b$.

That is

$$f(x) = \left(J_{g(x_0)-}^{(m+1)\nu} D_{g(x_0)-}^{(m+1)\nu}\left(f \circ g^{-1}\right)\right)(g(x)), \tag{38}$$

for all $a \leq x \leq x_0 \leq b$.

We have proved the following modified and generalized right fractional Taylor's formula of Canavati type:

Theorem 8 *Let $0 < \nu < 1$. Assume that $\left(D_{g(x_0)-}^{i\nu}\left(f \circ g^{-1}\right)\right) \in C_{g(x_0)-}^{\nu}([g(a), g(b)])$, $x_0 \in [a,b]$, for all $i = 0, 1, ..., m$. Assume also that $\left(D_{g(x_0)-}^{(m+1)\nu}\left(f \circ g^{-1}\right)\right) \in C([g(a), g(x_0)])$. Then*

$$f(x) = \frac{1}{\Gamma((m+1)\nu)} \int_{g(x)}^{g(x_0)} (z - g(x))^{(m+1)\nu - 1}\left(D_{g(x_0)-}^{(m+1)\nu}\left(f \circ g^{-1}\right)\right)(z)\, dz \tag{39}$$

$$= \frac{1}{\Gamma((m+1)\nu)} \int_{x}^{x_0} (g(s) - g(x))^{(m+1)\nu - 1}\left(D_{g(x_0)-}^{(m+1)\nu}\left(f \circ g^{-1}\right)\right)(g(s))\, g'(s)\, ds,$$

all $a \leq x \leq x_0 \leq b$.

References

[1] G. Anastassiou, *Fractional Differentiation Inequalities*, Springer, New York, 2009.

[2] G. Anastassiou, *Inteligent Mathematics: Computational Analysis*, Springer, Heidelberg, 2011.

[3] Rong-Qing Jia, *Chapter 3. Absolutely Continuous Functions*, https://www.ualberta.ca/~rjia/Math418/Notes/Chap.3.pdf.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 21, NO.7, 2016, COPYRIGHT 2016 EUDOXUS PRESS, LLC

Properties on a subclass of univalent functions defined by using Sălăgean operator and Ruscheweyh derivative

Alina Alb Lupaş
Department of Mathematics and Computer Science
University of Oradea
str. Universitatii nr. 1, 410087 Oradea, Romania
dalb@uoradea.ro

Abstract

In this paper we have introduced and studied the subclass $\mathcal{L}(d,\alpha,\beta)$ of univalent functions defined by the linear operator $L_\gamma^n f(z)$ defined by using the Ruscheweyh derivative $R^n f(z)$ and the Sălăgean operator $S^n f(z)$, as $L_\gamma^n : \mathcal{A} \to \mathcal{A}$, $L_\gamma^n f(z) = (1-\gamma) R^n f(z) + \gamma S^n f(z)$, $z \in U$, where $\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1} z^{n+1} + \ldots,\ z \in U\}$ is the class of normalized analytic functions with $\mathcal{A}_1 = \mathcal{A}$. The main object is to investigate several properties such as coefficient estimates, distortion theorems, closure theorems, neighborhoods and the radii of starlikeness, convexity and close-to-convexity of functions belonging to the class $\mathcal{L}(d,\alpha,\beta)$.

Keywords: univalent function, Starlike functions, Convex functions, Distortion theorem.
2000 Mathematical Subject Classification: 30C45, 30A20, 34A40.

1 Introduction

Denote by U the unit disc of the complex plane, $U = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}(U)$ the space of holomorphic functions in U.

Let $\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1} z^{n+1} + \ldots,\ z \in U\}$ with $\mathcal{A}_1 = \mathcal{A}$.

Definition 1.1 *(Sălăgean [8]) For $f \in \mathcal{A}$, $n \in \mathbb{N}$, the operator S^n is defined by $S^n : \mathcal{A} \to \mathcal{A}$,*

$$\begin{aligned} S^0 f(z) &= f(z), \quad S^1 f(z) = z f'(z), \ \ldots \\ S^{n+1} f(z) &= z\left(S^n f(z)\right)', \quad z \in U. \end{aligned}$$

Remark 1.1 *If $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $S^n f(z) = z + \sum_{j=2}^{\infty} j^n a_j z^j$, for $z \in U$.*

Definition 1.2 *(Ruscheweyh [7]) For $f \in \mathcal{A}$, $n \in \mathbb{N}$, the operator R^n is defined by $R^n : \mathcal{A} \to \mathcal{A}$,*

$$\begin{aligned} R^0 f(z) &= f(z), \quad R^1 f(z) = z f'(z), \ \ldots \\ (n+1) R^{n+1} f(z) &= z\left(R^n f(z)\right)' + n R^n f(z), \quad z \in U. \end{aligned}$$

Remark 1.2 *If $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $R^n f(z) = z + \sum_{j=2}^{\infty} \frac{(n+j-1)!}{n!(j-1)!} a_j z^j$, $z \in U$.*

Definition 1.3 *[1] Let $\gamma \geq 0$, $n \in \mathbb{N}$. Denote by L_γ^n the operator given by $L_\gamma^n : \mathcal{A} \to \mathcal{A}$, $L_\gamma^n f(z) = (1-\gamma) R^n f(z) + \gamma S^n f(z)$, $z \in U$.*

Remark 1.3 *If $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $L_\gamma^n f(z) = z + \sum_{j=2}^{\infty} \left(\gamma j^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!}\right) a_j z^j$, $z \in U$.*

This operator was studied also in [2], [3], [4], [5].

We follow the works of A.R. Juma and H. Ziraz .

Definition 1.4 *Let the function $f \in \mathcal{A}$. Then $f(z)$ is said to be in the class $\mathcal{L}(d,\alpha,\beta)$ if it satisfies the following criterion:*

$$\left| \frac{1}{d} \left(\frac{z(L_\gamma^n f(z))' + \alpha z^2 (L_\gamma^n f(z))''}{(1-\alpha) L_\gamma^n f(z) + \alpha z (L_\gamma^n f(z))'} - 1 \right) \right| < \beta, \tag{1.1}$$

where $d \in \mathbb{C} - \{0\}$, $0 \leq \alpha \leq 1$, $0 < \beta \leq 1$, $z \in U$.

In this paper we shall first deduce a necessary and sufficient condition for a function $f(z)$ to be in the class $\mathcal{L}(d,\alpha,\beta)$. Then obtain the distortion and growth theorems, closure theorems, neighborhood and radii of univalent starlikeness, convexity and close-to-convexity of order δ, $0\le\delta<1$, for these functions.

2 Coefficient Inequality

Theorem 2.1 *Let the function $f\in\mathcal{A}$. Then $f(z)$ is said to be in the class $\mathcal{L}(d,\alpha,\beta)$ if and only if*

$$\sum_{j=2}^{\infty}(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma j^n+(1-\gamma)\frac{(n+j-1)!}{n!\,(j-1)!}\right\}a_j\le\beta|d|, \tag{2.1}$$

where $d\in\mathbb{C}-\{0\}$, $0\le\alpha\le1$, $0<\beta\le1$, $z\in U$.

Proof. Let $f(z)\in\mathcal{L}(d,\alpha,\beta)$. Assume that inequality (2.1) holds true. Then we find that $\left|\frac{z(L_\gamma^n f(z))'+\alpha z^2(L_\gamma^n f(z))''}{(1-\alpha)L_\gamma^n f(z)+\alpha z(L_\gamma^n f(z))'}-1\right|=\left|\frac{\sum_{j=2}^{\infty}(1+\alpha(j-1))(j-1)\left[\gamma j^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right]a_jz^j}{z+\sum_{j=2}^{\infty}(1+\alpha(j-1))\left[\gamma j^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right]a_jz^j}\right|\le$ $\frac{\sum_{j=2}^{\infty}(1+\alpha(j-1))(j-1)\left[\gamma j^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right]a_j|z|^{j-1}}{1-\sum_{j=2}^{\infty}(1+\alpha(j-1))\left[\gamma j^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right]a_j|z|^{j-1}}<\beta|d|$.

Choosing values of z on real axis and letting $z\to1^-$, we have $\sum_{j=2}^{\infty}(1+\alpha(j-1))(j-1+\beta|d|)\left[\gamma j^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right]a_j\le\beta|d|$. Conversely, assume that $f(z)\in\mathcal{L}(d,\alpha,\beta)$, then we get the following inequality $Re\left\{\left|\frac{z(L_\gamma^n f(z))'+\alpha z^2(L_\gamma^n f(z))''}{(1-\alpha)L_\gamma^n f(z)+\alpha z(L_\gamma^n f(z))'}-1\right|\right\}>-\beta|d|$,
$Re\left\{\frac{z+\sum_{j=2}^{\infty}j(1+\alpha(j-1))\left[\gamma j^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right]a_jz^j}{z+\sum_{j=2}^{\infty}(1+\alpha(j-1))\left[\gamma j^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right]a_jz^j}-1+\beta|d|\right\}>0$
$Re\frac{\beta|d|z+\sum_{j=2}^{\infty}(1+\alpha(j-1))(j-1+\beta|d|)\left[\gamma j^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right]a_jz^j}{z+\sum_{j=2}^{\infty}(1+\alpha(j-1))\left[\gamma j^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right]a_jz^j}>0$. Since $Re(-e^{i\theta})\ge-|e^{i\theta}|=-1$, the above inequality reduces to $\frac{\beta|d|r-\sum_{j=2}^{\infty}(1+\alpha(j-1))(j-1+\beta|d|)\left[\gamma j^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right]a_jr^j}{r-\sum_{j=2}^{\infty}(1+\alpha(j-1))\left[\gamma j^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right]a_jr^j}>0$. Letting $r\to1^-$ and by the mean value theorem we have desired inequality (2.1). This completes the proof of Theorem 2.1 ■

Corollary 2.2 *Let the function $f\in\mathcal{A}$ be in the class $\mathcal{L}(d,\alpha,\beta)$. Then $a_j\le\frac{\beta|d|}{(1+\alpha(j-1))(j-1+\beta|d|)\left[\gamma j^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right]}$, $j\ge2$.*

3 Distortion Theorems

Theorem 3.1 *Let the function $f\in\mathcal{A}$ be in the class $\mathcal{L}(d,\alpha,\beta)$. Then for $|z|=r<1$, we have* $r-\frac{\beta|d|}{(1+\alpha)(1+\beta|d|)[2^n\gamma+(1-\gamma)(n+1)]}r^2\le|f(z)|\le r+\frac{\beta|d|}{(1+\alpha)(1+\beta|d|)[2^n\gamma+(1-\gamma)(n+1)]}r^2$.

The result is sharp for the function $f(z)$ given by $f(z)=z+\frac{\beta|d|}{(1+\alpha)(1+\beta|d|)[2^n\gamma+(1-\gamma)(n+1)]}z^2$.

Proof. Given that $f(z)\in\mathcal{L}(d,\alpha,\beta)$, from the equation (2.1) and since $(1+\alpha)(1+\beta|d|)\left[2^n\gamma+(1-\gamma)(n+1)\right]$ is non decreasing and positive for $j\ge2$, then we have $(1+\alpha)(1+\beta|d|)\left[2^n\gamma+(1-\gamma)(n+1)\right]\sum_{j=2}^{\infty}a_j\le$ $\sum_{j=2}^{\infty}(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma j^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_j\le\beta|d|$, which is equivalent to,

$$\sum_{j=2}^{\infty}a_j\le\frac{\beta|d|}{(1+\alpha)(1+\beta|d|)\left[2^n\gamma+(1-\gamma)(n+1)\right]}. \tag{3.1}$$

Using (3.1), we obtain for $f(z)=z+\sum_{j=2}^{\infty}a_jz^j$ that $|f(z)|\le|z|+\sum_{j=2}^{\infty}a_j|z|^j\le r+\sum_{j=2}^{\infty}a_jr^j\le r+r^2\sum_{j=2}^{\infty}a_j$ $\le r+\frac{\beta|d|}{(1+\alpha)(1+\beta|d|)[2^n\gamma+(1-\gamma)(n+1)]}r^2$. Similarly, $|f(z)|\ge r^2-\frac{\beta|d|}{(1+\alpha)(1+\beta|d|)[2^n\gamma+(1-\gamma)(n+1)]}r^2$. This completes the proof of Theorem 3.1. ■

Theorem 3.2 *Let the function $f\in\mathcal{A}$ be in the class $\mathcal{L}(d,\alpha,\beta)$. Then for $|z|=r<1$, we have* $-\frac{2\beta|d|}{(1+\alpha)(1+\beta|d|)[2^n\gamma+(1-\gamma)(n+1)]}r\le|f'(z)|\le\frac{2\beta|d|}{(1+\alpha)(1+\beta|d|)[2^n\gamma+(1-\gamma)(n+1)]}r$.

The result is sharp for the function $f(z)$ given by $f(z)=z+\frac{\beta|d|}{(1+\alpha)(1+\beta|d|)[2^n\gamma+(1-\gamma)(n+1)]}z^2$.

Proof. From (3.1) we obtain $f'(z) = 1+\sum_{j=2}^{\infty} ja_j z^{j-1}$ and $|f'(z)| \le 1-\sum_{j=2}^{\infty} ja_j|z|^{j-1} \le 1+\sum_{j=2}^{\infty} ja_j r^{j-1} \le 1+\frac{2\beta|d|}{(1+\alpha)(1+\beta|d|)[2^n\gamma+(1-\gamma)(n+1)]}r$. Similarly, $|f'(z)| \ge 1-\frac{2\beta|d|}{(1+\alpha)(1+\beta|d|)[2^n\gamma+(1-\gamma)(n+1)]}r$. This completes the proof of Theorem 3.2. ■

4 Closure Theorems

Theorem 4.1 *Let the functions f_k, $k = 1, 2, ..., m$, defined by*

$$f_k(z) = z + \sum_{j=2}^{\infty} a_{j,k} z^j, \quad a_{j,k} \ge 0, \tag{4.1}$$

be in the class $\mathcal{L}(d, \alpha, \beta)$. Then the function $h(z)$ defined by $h(z) = \sum_{k=1}^{m} \mu_k f_k(z)$, $\mu_k \ge 0$, is also in the class $\mathcal{L}(d, \alpha, \beta)$, where $\sum_{k=1}^{m} \mu_k = 1$.

Proof. We can write $h(z) = \sum_{k=1}^{m} \mu_m z + \sum_{k=1}^{m}\sum_{j=2}^{\infty} \mu_k a_{j,k} z^j = z + \sum_{j=2}^{\infty}\sum_{k=1}^{m} \mu_k a_{j,k} z^j$. Furthermore, since the functions $f_k(z)$, $k = 1, 2, ..., m$, are in the class $\mathcal{L}(d, \alpha, \beta)$, then from Theorem 2.1 we have $\sum_{j=2}^{\infty}(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma j^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\} a_{j,k} \le \beta|d|$. Thus it is enough to prove that $\sum_{j=2}^{\infty}(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma j^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}\left(\sum_{k=1}^{m} \mu_k a_{j,k}\right) = \sum_{k=1}^{m} \mu_k \sum_{j=2}^{\infty}(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma j^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\} a_{j,k} \le \sum_{k=1}^{m} \mu_k \beta|d| = \beta|d|$. Hence the proof is complete. ■

Corollary 4.2 *Let the functions f_k, $k = 1, 2$, defined by (4.1) be in the class $\mathcal{L}(d, \alpha, \beta)$. Then the function $h(z)$ defined by $h(z) = (1-\zeta)f_1(z) + \zeta f_2(z)$, $0 \le \zeta \le 1$, is also in the class $\mathcal{L}(d, \alpha, \beta)$.*

Theorem 4.3 *Let $f_1(z) = z$, and $f_j(z) = z+\frac{\beta|d|}{(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma j^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}} z^j$, $j \ge 2$. Then the function $f(z)$ is in the class $\mathcal{L}(d, \alpha, \beta)$ if and only if it can be expressed in the form $f(z) = \mu_1 f_1(z) + \sum_{j=2}^{\infty} \mu_j f_j(z)$, where $\mu_1 \ge 0$, $\mu_j \ge 0$, $j \ge 2$ and $\mu_1 + \sum_{j=2}^{\infty} \mu_j = 1$.*

Proof. Assume that $f(z)$ can be expressed in the form $f(z) = \mu_1 f_1(z) + \sum_{j=2}^{\infty} \mu_j f_j(z) = z + \sum_{j=2}^{\infty} \frac{\beta|d|}{(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma j^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}} \mu_j z^j$.

Thus $\sum_{j=2}^{\infty} \frac{(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma j^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}{\beta|d|} \frac{\beta|d|}{(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma j^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}\mu_j = \sum_{j=2}^{\infty} \mu_j = 1-\mu_1 \le 1$. Hence $f(z) \in \mathcal{L}(d, \alpha, \beta)$.

Conversely, assume that $f(z) \in \mathcal{L}(d, \alpha, \beta)$.

Setting $\mu_j = \frac{(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma j^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}{\beta|d|} a_j$, since $\mu_1 = 1 - \sum_{j=2}^{\infty} \mu_j$. Thus $f(z) = \mu_1 f_1(z) + \sum_{j=2}^{\infty} \mu_j f_j(z)$. Hence the proof is complete. ■

Corollary 4.4 *The extreme points of the class $\mathcal{L}(d, \alpha, \beta)$ are the functions $f_1(z) = z$, and $f_j(z) = z + \frac{\beta|d|}{(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma j^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}} z^j$, $j \ge 2$.*

5 Inclusion and Neighborhood Results

We define the δ- neighborhood of a function $f(z) \in \mathcal{A}$ by

$$N_\delta(f) = \{g \in \mathcal{A} : g(z) = z + \sum_{j=2}^{\infty} b_j z^j \text{ and } \sum_{j=2}^{\infty} j|a_j - b_j| \le \delta\}. \tag{5.1}$$

In particular, for $e(z) = z$

$$N_\delta(e) = \{g \in \mathcal{A} : g(z) = z + \sum_{j=2}^{\infty} b_j z^j \text{ and } \sum_{j=2}^{\infty} j|b_j| \le \delta\}. \tag{5.2}$$

Furthermore, a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{L}^{\xi}(d, \alpha, \beta)$ if there exists a function $h(z) \in \mathcal{L}(d, \alpha, \beta)$ such that

$$\left|\frac{f(z)}{h(z)} - 1\right| < 1-\xi, \quad z \in U, \quad 0 \le \xi < 1. \tag{5.3}$$

Theorem 5.1 *If* $\left\{\gamma j^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\} \geq [2^n\gamma + (1-\gamma)(n+1)]$, $j \geq 2$, *and* $\delta = \frac{2\beta|d|}{(1+\alpha)(1+\beta|d|)[2^n\gamma+(1-\gamma)(n+1)]}$, *then* $\mathcal{L}(d,\alpha,\beta) \subset N_\delta(e)$.

Proof. Let $f \in \mathcal{L}(d,\alpha,\beta)$. Then in view of assertion (2.1) of Theorem 2.1 and the condition $\left\{\gamma j^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\} \geq [2^n\gamma + (1-\gamma)(n+1)]$ for $j \geq 2$, we get $(1+\alpha)(1+\beta|d|)[2^n\gamma + (1-\gamma)(n+1)]\sum_{j=2}^{\infty} a_j \leq \sum_{j=2}^{\infty}(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma j^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\} a_j \leq \beta|d|$, which implise

$$\sum_{j=2}^{\infty} a_j \leq \frac{\beta|d|}{(1+\alpha)(1+\beta|d|)[2^n\gamma + (1-\gamma)(n+1)]}. \tag{5.4}$$

Applying assertion (2.1) of Theorem 2.1 in conjunction with (5.4), we obtain $(1+\alpha)(1+\beta|d|)[2^n\gamma + (1-\gamma)(n+1)]\sum_{j=2}^{\infty} a_j \leq \beta|d|$, $2(1+\alpha)(1+\beta|d|)[2^n\gamma + (1-\gamma)(n+1)]\sum_{j=2}^{\infty} a_j \leq 2\beta|d|$ and $\sum_{j=2}^{\infty} ja_j \leq \frac{2\beta|d|}{(1+\alpha)(1+\beta|d|)[2^n\gamma+(1-\gamma)(n+1)]} = \delta$, by virtue of (5.1), we have $f \in N_\delta(e)$.

This completes the proof of the Theorem 5.1. ∎

Theorem 5.2 *If* $h \in \mathcal{L}(d,\alpha,\beta)$ *and*

$$\xi = 1 - \frac{\delta}{2}\frac{(1+\alpha)(1+\beta|d|)[2^n\gamma + (1-\gamma)(n+1)]}{(1+\alpha)(1+\beta|d|)[2^n\gamma + (1-\gamma)(n+1)] - \beta|d|}, \tag{5.5}$$

then $N_\delta(h) \subset \mathcal{L}^\xi(d,\alpha,\beta)$.

Proof. Suppose that $f \in N_\delta(h)$, we then find from (5.1) that $\sum_{j=2}^{\infty} j|a_j - b_j| \leq \delta$, which readily implies the following coefficient inequality

$$\sum_{j=2}^{\infty}|a_j - b_j| \leq \frac{\delta}{2}. \tag{5.6}$$

Next, since $h \in \mathcal{L}(d,\alpha,\beta)$ in the view of (5.4), we have

$$\sum_{j=2}^{\infty} b_j \leq \frac{\beta|d|}{(1+\alpha)(1+\beta|d|)[2^n\gamma + (1-\gamma)(n+1)]}. \tag{5.7}$$

Using (5.6) and (5.7), we get $\left|\frac{f(z)}{h(z)} - 1\right| \leq \frac{\sum_{j=2}^{\infty}|a_j-b_j|}{1-\sum_{j=2}^{\infty} b_j} \leq \frac{\delta}{2\left(1-\frac{\beta|d|}{(1+\alpha)(1+\beta|d|)[2^n\gamma+(1-\gamma)(n+1)]}\right)} \leq \frac{\delta}{2}\frac{(1+\alpha)(1+\beta|d|)[2^n\gamma+(1-\gamma)(n+1)]}{(1+\alpha)(1+\beta|d|)[2^n\gamma+(1-\gamma)(n+1)]-\beta|d|} = 1-\xi$, provided that ξ is given by (5.5), thus by condition (5.3), $f \in \mathcal{L}^\xi(d,\alpha,\beta)$, where ξ is given by (5.5). ∎

6 Radii of Starlikeness, Convexity and Close-to-Convexity

Theorem 6.1 *Let the function* $f \in \mathcal{A}$ *be in the class* $\mathcal{L}(d,\alpha,\beta)$. *Then* f *is univalent starlike of order* δ, $0 \leq \delta < 1$, *in* $|z| < r_1$, *where* $r_1 = \inf_j \left\{\frac{(1-\delta)(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma j^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}{\beta|d|(1-\delta)}\right\}^{\frac{1}{j-1}}$. *The result is sharp for the function* $f(z)$ *given by* $f_j(z) = z + \frac{\beta|d|}{(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma j^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}} z^j$, $j \geq 2$.

Proof. It suffices to show that $\left|\frac{zf'(z)}{f(z)} - 1\right| \leq 1-\delta$, $|z| < r_1$. Since $\left|\frac{zf'(z)}{f(z)} - 1\right| = \left|\frac{\sum_{j=2}^{\infty}(j-1)a_j z^{j-1}}{1+\sum_{j=2}^{\infty} a_j z^{k-1}}|\right| \leq \frac{\sum_{j=2}^{\infty}(j-1)a_j|z|^{j-1}}{1-\sum_{j=2}^{\infty} a_j|z|^{j-1}}$. To prove the theorem, we must show that $\frac{\sum_{j=2}^{\infty}(j-1)a_j|z|^{j-1}}{1-\sum_{j=2}^{\infty} a_j|z|^{j-1}} \leq 1-\delta$. It is equivalent to $\sum_{j=2}^{\infty}(j-\delta)a_j|z|^{j-1} \leq 1-\delta$, using Theorem 2.1, we obtain $|z| \leq \left\{\frac{(1-\delta)(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma j^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}{\beta|d|(1-\delta)}\right\}^{\frac{1}{j-1}}$. Hence the proof is complete. ∎

Theorem 6.2 *Let the function $f \in \mathcal{A}$ be in the class $\mathcal{L}(d,\alpha,\beta)$. Then f is univalent convex of order δ, $0 \le \delta \le 1$, in $|z| < r_2$, where $r_2 = \inf_j \left\{ \frac{(1-\delta)(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma j^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}{2(j-\delta)\beta|d|} \right\}^{\frac{1}{k-p}}$. The result is sharp for the function $f(z)$ given by*

$$f_j(z) = z + \frac{\beta|d|}{(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma j^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}} z^j, \quad j \ge 2. \tag{6.1}$$

Proof. It suffices to show that $\left|\frac{zf''(z)}{f'(z))}\right| \le 1-\delta$, $|z| < r_2$. Since $\left|\frac{zf''(z)}{f'(z)}\right| = \left|\frac{\sum_{j=2}^{\infty} j(j-1)a_j z^{j-1}}{1+\sum_{j=2}^{\infty} ja_j z^{j-1}}\right| \le \frac{\sum_{j=2}^{\infty} j(j-1)a_j |z|^{j-1}}{1-\sum_{j=2}^{\infty} ja_j |z|^{j-1}}$. To prove the theorem, we must show that $\frac{\sum_{j=2}^{\infty} j(j-1)a_j |z|^{j-1}}{1-\sum_{j=2}^{\infty} ja_j |z|^{j-1}} \le 1-\delta$, and $\sum_{j=2}^{\infty} j(j-\delta)a_j|z|^{j-1} \le 1-\delta$, using Theorem 2.1, we obtain $|z|^{j-1} \le \frac{(1-\delta)(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma j^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}{2(j-\delta)\beta|d|}$, or $|z| \le \left\{ \frac{(1-\delta)(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma j^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}{2(j-\delta)\beta|d|} \right\}^{\frac{1}{j-1}}$. Hence the proof is complete. ■

Theorem 6.3 *Let the function $f \in \mathcal{A}$ be in the class $\mathcal{L}(d,\alpha,\beta)$. Then f is univalent close-to-convex of order δ, $0 \le \delta < 1$, in $|z| < r_3$, where $r_3 = \inf_j \left\{ \frac{(1-\delta)(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma j^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}{j\beta|d|} \right\}^{\frac{1}{j-1}}$. The result is sharp for the function $f(z)$ given by (6.1).*

Proof. It suffices to show that $|f'(z)-1| \le 1-\delta$, $|z| < r_3$. Then $|f'(z)-1| = \left|\sum_{j=2}^{\infty} ja_j z^{j-1}\right| \le \sum_{j=2}^{\infty} ja_j|z|^{j-1}$. Thus $|f'(z)-1| \le 1-\delta$ if $\sum_{j=2}^{\infty} \frac{ja_j}{1-\delta}|z|^{j-1} \le 1$. Using Theorem 2.1, the above inequality holds true if $|z|^{j-1} \le \frac{(1-\delta)(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma j^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}{j\beta|d|}$ or $|z| \le \left\{ \frac{(1-\delta)(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma j^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}{j\beta|d|} \right\}^{\frac{1}{j-1}}$. Hence the proof is complete. ■

References

[1] A. Alb Lupaş, *On special differential subordinations using Sălăgean and Ruscheweyh operators*, Mathematical Inequalities and Applications, Volume 12, Issue 4, 2009, 781-790.

[2] A. Alb Lupaş, *On a certain subclass of analytic functions defined by Salagean and Ruscheweyh operators*, Journal of Mathematics and Applications, No. 31, 2009, 67-76.

[3] A. Alb Lupaş, D. Breaz, *On special differential superordinations using Sălăgean and Ruscheweyh operators*, Geometric Function Theory and Applications' 2010 (Proc. of International Symposium, Sofia, 27-31 August 2010), 98-103.

[4] A. Alb Lupaş, *Some differential subordinations using Ruscheweyh derivative and Sălăgean operator*, Advances in Difference Equations.2013, 2013:150., DOI: 10.1186/1687-1847-2013-150.

[5] A. Alb Lupaş, *Differential subordinations using Ruscheweyh derivative and Sălăgean operator*, Proceedings of GFTA 2013, TWMS J. App. Eng. Math. V.4, No.1, 2014, 33-38.

[6] A.R. Juma, H. Zirar, *Properties on a subclass of p-valent functions defined by new operator V_p^λ*, Analele Univ. Oradea, Fasc. Math, Tom XXI (2014), Issue No. 1, 73–82.

[7] St. Ruscheweyh, *New criteria for univalent functions*, Proc. Amet. Math. Soc., 49(1975), 109-115.

[8] G. St. Sălăgean, *Subclasses of univalent functions*, Lecture Notes in Math., Springer Verlag, Berlin, 1013(1983), 362-372.

About some differential sandwich theorems using a multiplier transformation and Ruscheweyh derivative

Alb Lupaş Alina
Department of Mathematics and Computer Science
University of Oradea
1 Universitatii street, 410087 Oradea, Romania
alblupas@gmail.com

Abstract

In this work we study a new operator $IR_{\lambda,l}^{m,n}$ defined as the Hadamard product of the multiplier transformation $I(m,\lambda,l)$ and Ruscheweyh derivative R^n, given by $IR_{\lambda,l}^{m,n}:\mathcal{A}\to\mathcal{A}$, $IR_{\lambda,l}^{m,n}f(z)=(I(m,\lambda,l)*R^n)f(z)$ and $\mathcal{A}_n=\{f\in\mathcal{H}(U):f(z)=z+a_{n+1}z^{n+1}+...,\ z\in U\}$ is the class of normalized analytic functions with $\mathcal{A}_1=\mathcal{A}$. The purpose of this paper is to derive certain subordination and superordination results involving the operator $IR_{\lambda,l}^{m,n}$ and we establish differential sandwich-type theorems.

Keywords: analytic functions, differential operator, differential subordination, differential superordination.
2010 Mathematical Subject Classification: 30C45.

1 Introduction

Let $\mathcal{H}(U)$ be the class of analytic function in the open unit disc of the complex plane $U=\{z\in\mathbb{C}:|z|<1\}$. Let $\mathcal{H}(a,n)$ be the subclass of $\mathcal{H}(U)$ consisting of functions of the form $f(z)=a+a_nz^n+a_{n+1}z^{n+1}+\ldots$.

Let $\mathcal{A}_n=\{f\in\mathcal{H}(U):f(z)=z+a_{n+1}z^{n+1}+\ldots,\ z\in U\}$ and $\mathcal{A}=\mathcal{A}_1$.

Let the functions f and g be analytic in U. We say that the function f is subordinate to g, written $f\prec g$, if there exists a Schwarz function w, analytic in U, with $w(0)=0$ and $|w(z)|<1$, for all $z\in U$, such that $f(z)=g(w(z))$, for all $z\in U$. In particular, if the function g is univalent in U, the above subordination is equivalent to $f(0)=g(0)$ and $f(U)\subset g(U)$.

Let $\psi:\mathbb{C}^3\times U\to\mathbb{C}$ and h be an univalent function in U. If p is analytic in U and satisfies the second order differential subordination

$$\psi(p(z),zp'(z),z^2p''(z);z)\prec h(z),\quad \text{for } z\in U, \tag{1.1}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant, if $p\prec q$ for all p satisfying (1.1). A dominant $\widetilde{q}$ that satisfies $\widetilde{q}\prec q$ for all dominants q of (1.1) is said to be the best dominant of (1.1). The best dominant is unique up to a rotation of U.

Let $\psi:\mathbb{C}^2\times U\to\mathbb{C}$ and h analytic in U. If p and $\psi\left(p(z),zp'(z),z^2p''(z);z\right)$ are univalent and if p satisfies the second order differential superordination

$$h(z)\prec\psi(p(z),zp'(z),z^2p''(z);z),\qquad z\in U, \tag{1.2}$$

then p is a solution of the differential superordination (1.2) (if f is subordinate to F, then F is called to be superordinate to f). An analytic function q is called a subordinant if $q\prec p$ for all p satisfying (1.2). An univalent subordinant $\widetilde{q}$ that satisfies $q\prec\widetilde{q}$ for all subordinants q of (1.2) is said to be the best subordinant.

Miller and Mocanu [4] obtained conditions h, q and ψ for which the following implication holds $h(z)\prec\psi(p(z),zp'(z),z^2p''(z);z)\Rightarrow q(z)\prec p(z)$.

For two functions $f(z)=z+\sum_{j=2}^{\infty}a_jz^j$ and $g(z)=z+\sum_{j=2}^{\infty}b_jz^j$ analytic in the open unit disc U, the Hadamard product (or convolution) of $f(z)$ and $g(z)$, written as $(f*g)(z)$ is defined by $f(z)*g(z)=(f*g)(z)=z+\sum_{j=2}^{\infty}a_jb_jz^j$.

Definition 1.1 *([1]) Let $\lambda, l \geq 0$ and $n, m \in \mathbb{N}$. Denote by $IR_{\lambda,l}^{m,n} : \mathcal{A} \to \mathcal{A}$ the operator given by the Hadamard product of the multiplier transformation $I(m,\lambda,l)$ and the Ruscheweyh derivative R^n, $IR_{\lambda,l}^{m,n} f(z) = (I(m,\lambda,l) * R^n) f(z)$, for any $z \in U$ and each nonnegative integers m, n.*

Remark 1.1 *If $f \in \mathcal{A}$ and $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $IR_{\lambda,l}^{m,n} f(z) = z + \sum_{j=2}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m \frac{(n+j-1)!}{n!(j-1)!} a_j^2 z^j$, $z \in U$.*

Using simple computation one obtains the next result.

Proposition 1.1 *[2]For $m, n \in \mathbb{N}$ and $\lambda, l \geq 0$ we have*

$$(n+1) IR_{\lambda,l}^{m,n+1} f(z) - n IR_{\lambda,l}^{m,n} f(z) = z \left(IR_{\lambda,l}^{m,n} f(z)\right)'. \tag{1.3}$$

The purpose of this paper is to derive the several subordination and superordination results involving a differential operator. Furthermore, we studied the results of Selvaraj and Karthikeyan [6], Shanmugam, Ramachandran, Darus and Sivasubramanian [7] and Srivastava and Lashin [8].

In order to prove our subordination and superordination results, we make use of the following known results.

Definition 1.2 *[5] Denote by Q the set of all functions f that are analytic and injective on $\overline{U} \backslash E(f)$, where $E(f) = \{\zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty\}$, and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \backslash E(f)$.*

Lemma 1.1 *[5] Let the function q be univalent in the unit disc U and θ and ϕ be analytic in a domain D containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = zq'(z)\phi(q(z))$ and $h(z) = \theta(q(z)) + Q(z)$. Suppose that Q is starlike univalent in U and $Re\left(\frac{zh'(z)}{Q(z)}\right) > 0$ for $z \in U$. If p is analytic with $p(0) = q(0)$, $p(U) \subseteq D$ and $\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z))$, then $p(z) \prec q(z)$ and q is the best dominant.*

Lemma 1.2 *[3] Let the function q be convex univalent in the open unit disc U and ν and ϕ be analytic in a domain D containing $q(U)$. Suppose that $Re\left(\frac{\nu'(q(z))}{\phi(q(z))}\right) > 0$ for $z \in U$ and $\psi(z) = zq'(z)\phi(q(z))$ is starlike univalent in U. If $p(z) \in \mathcal{H}[q(0), 1] \cap Q$, with $p(U) \subseteq D$ and $\nu(p(z)) + zp'(z)\phi(p(z))$ is univalent in U and $\nu(q(z)) + zq'(z)\phi(q(z)) \prec \nu(p(z)) + zp'(z)\phi(p(z))$, then $q(z) \prec p(z)$ and q is the best subordinant.*

2 Main results

We begin with the following

Theorem 2.1 *Let $\frac{IR_{\lambda,l}^{m,n+1} f(z)}{IR_{\lambda,l}^{m,n} f(z)} \in \mathcal{H}(U)$ and let the function $q(z)$ be analytic and univalent in U such that $q(z) \neq 0$, for all $z \in U$. Suppose that $\frac{zq'(z)}{q(z)}$ is starlike univalent in U. Let*

$$Re\left(\frac{\alpha+\mu}{\mu} + \frac{2\beta}{\mu} q(z) + \frac{zq''(z)}{q'(z)}\right) > 0, \tag{2.1}$$

for $\alpha, \beta, \mu \in \mathbb{C}$, $\mu \neq 0$, $z \in U$ and

$$\psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z) := \mu(n+2) \frac{IR_{\lambda,l}^{m,n+2} f(z)}{IR_{\lambda,l}^{m,n} f(z)} + (\alpha-\mu) \frac{IR_{\lambda,l}^{m,n+1} f(z)}{IR_{\lambda,l}^{m,n} f(z)} + [\beta - \mu(n+1)] \left(\frac{IR_{\lambda,l}^{m,n+1} f(z)}{IR_{\lambda,l}^{m,n} f(z)}\right)^2. \tag{2.2}$$

If q satisfies the following subordination

$$\psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z) \prec \alpha q(z) + \beta(q(z))^2 + \mu z q'(z), \tag{2.3}$$

for $\alpha, \beta, \mu \in \mathbb{C}$, $\mu \neq 0$, then $\frac{IR_{\lambda,l}^{m,n+1} f(z)}{IR_{\lambda,l}^{m,n} f(z)} \prec q(z)$, and q is the best dominant.

Proof. Let the function p be defined by $p(z) := \frac{IR^{m,n+1}_{\lambda,l} f(z)}{IR^{m,n}_{\lambda,l} f(z)}$, $z \in U$, $z \neq 0$, $f \in \mathcal{A}$. We have $p'(z) = \frac{\left(IR^{m,n+1}_{\lambda,l} f(z)\right)' IR^{m,n}_{\lambda,l} f(z) - IR^{m,n+1}_{\lambda,l} f(z) \left(IR^{m,n}_{\lambda,l} f(z)\right)'}{\left(IR^{m,n}_{\lambda,l} f(z)\right)^2} = \frac{\left(IR^{m,n+1}_{\lambda,l} f(z)\right)'}{IR^{m,n}_{\lambda,l} f(z)} - \frac{IR^{m,n+1}_{\lambda,l} f(z)}{IR^{m,n}_{\lambda,l} f(z)} \cdot \frac{\left(IR^{m,n}_{\lambda,l} f(z)\right)'}{IR^{m,n}_{\lambda,l} f(z)}$. Then $zp'(z) = \frac{z\left(IR^{m,n+1}_{\lambda,l} f(z)\right)'}{IR^{m,n}_{\lambda,l} f(z)} - \frac{IR^{m,n+1}_{\lambda,l} f(z)}{IR^{m,n}_{\lambda,l} f(z)} \cdot \frac{z\left(IR^{m,n}_{\lambda,l} f(z)\right)'}{IR^{m,n}_{\lambda,l} f(z)}$.

By using the identity (1.3), we obtain

$$zp'(z) = (n+2) \frac{IR^{m,n+2}_{\lambda,l} f(z)}{IR^{m,n}_{\lambda,l} f(z)} - (n+1) \left(\frac{IR^{m,n+1}_{\lambda,l} f(z)}{IR^{m,n}_{\lambda,l} f(z)} \right)^2 - \frac{IR^{m,n+1}_{\lambda,l} f(z)}{IR^{m,n}_{\lambda,l} f(z)}. \tag{2.4}$$

By setting $\theta(w) := \alpha w + \beta w^2$ and $\phi(w) := \mu$, it can be easily verified that θ is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C}\backslash\{0\}$.

Also, by letting $Q(z) = zq'(z) \phi(q(z)) = \mu z q'(z)$ and $h(z) = \theta(q(z)) + Q(z) = \alpha q(z) + \beta (q(z))^2 + \mu z q'(z)$, we find that $Q(z)$ is starlike univalent in U.

We have $h'(z) = (\alpha + \mu) q'(z) + 2\beta q(z) q'(z) + \mu z q''(z)$ and $\frac{zh'(z)}{Q(z)} = \frac{zh'(z)}{\mu z q'(z)} = \frac{\alpha+\mu}{\mu} + \frac{2\beta}{\mu} q(z) + \frac{zq''(z)}{q'(z)}$.

We deduce that $Re\left(\frac{zh'(z)}{Q(z)}\right) = Re\left(\frac{\alpha+\mu}{\mu} + \frac{2\beta}{\mu} q(z) + \frac{zq''(z)}{q'(z)}\right) > 0$.

By using (2.4), we obtain

$$\alpha p(z) + \beta (p(z))^2 + \mu z p'(z) = \mu (n+2) \frac{IR^{m,n+2}_{\lambda,l} f(z)}{IR^{m,n}_{\lambda,l} f(z)} + (\alpha - \mu) \frac{IR^{m,n+1}_{\lambda,l} f(z)}{IR^{m,n}_{\lambda,l} f(z)} + [\beta - \mu(n+1)] \left(\frac{IR^{m,n+1}_{\lambda,l} f(z)}{IR^{m,n}_{\lambda,l} f(z)} \right)^2.$$

By using (2.3), we have $\alpha p(z) + \beta (p(z))^2 + \mu z p'(z) \prec \alpha q(z) + \beta (q(z))^2 + \mu z q'(z)$.

By an application of Lemma 1.1, we have $p(z) \prec q(z)$, $z \in U$, i.e. $\frac{IR^{m,n+1}_{\lambda,l} f(z)}{IR^{m,n}_{\lambda,l} f(z)} \prec q(z)$, $z \in U$ and q is the best dominant. ■

Corollary 2.2 *Let $m, n \in \mathbb{N}$, $\lambda, l \geq 0$. Assume that (2.1) holds. If $f \in \mathcal{A}$ and $\psi^{m,n}_{\lambda,l}(\alpha, \beta, \mu; z) \prec \alpha \frac{1+Az}{1+Bz} + \beta \left(\frac{1+Az}{1+Bz}\right)^2 + \mu \frac{(A-B)z}{(1+Bz)^2}$, for $\alpha, \beta, \mu \in \mathbb{C}$, $\mu \neq 0$, $-1 \leq B < A \leq 1$, where $\psi^{m,n}_{\lambda,l}$ is defined in (2.2), then $\frac{IR^{m,n+1}_{\lambda,l} f(z)}{IR^{m,n}_{\lambda,l} f(z)} \prec \frac{1+Az}{1+Bz}$, and $\frac{1+Az}{1+Bz}$ is the best dominant.*

Proof. For $q(z) = \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$ in Theorem 2.1 we get the corollary. ■

Corollary 2.3 *Let $m, n \in \mathbb{N}$, $\lambda, l \geq 0$. Assume that (2.1) holds. If $f \in \mathcal{A}$ and $\psi^{m,n}_{\lambda,l}(\alpha, \beta, \mu; z) \prec \alpha \left(\frac{1+z}{1-z}\right)^\gamma + \beta \left(\frac{1+z}{1-z}\right)^{2\gamma} + \frac{2\mu\gamma z}{(1-z)^2} \left(\frac{1+z}{1-z}\right)^{\gamma-1}$, for $\alpha, \beta, \mu \in \mathbb{C}$, $0 < \gamma \leq 1$, $\mu \neq 0$, where $\psi^{m,n}_{\lambda,l}$ is defined in (2.2), then $\frac{IR^{m,n+1}_{\lambda,l} f(z)}{IR^{m,n}_{\lambda,l} f(z)} \prec \left(\frac{1+z}{1-z}\right)^\gamma$, and $\left(\frac{1+z}{1-z}\right)^\gamma$ is the best dominant.*

Proof. Corollary follows by using Theorem 2.1 for $q(z) = \left(\frac{1+z}{1-z}\right)^\gamma$, $0 < \gamma \leq 1$. ■

Theorem 2.4 *Let q be analytic and univalent in U such that $q(z) \neq 0$ and $\frac{zq'(z)}{q(z)}$ be starlike univalent in U. Assume that*

$$Re\left(\frac{\alpha}{\mu} q'(z) + \frac{2\beta}{\mu} q(z) q'(z)\right) > 0, \ \text{for } \alpha, \beta, \mu \in \mathbb{C}, \ \mu \neq 0. \tag{2.5}$$

If $f \in \mathcal{A}$, $\frac{IR^{m,n+1}_{\lambda,l} f(z)}{IR^{m,n}_{\lambda,l} f(z)} \in \mathcal{H}[q(0), 1] \cap Q$ and $\psi^{m,n}_{\lambda,l}(\alpha, \beta, \mu; z)$ is univalent in U, where $\psi^{m,n}_{\lambda,l}(\alpha, \beta, \mu; z)$ is as defined in (2.2), then

$$\alpha q(z) + \beta (q(z))^2 + \mu z q'(z) \prec \psi^{m,n}_{\lambda,l}(\alpha, \beta, \mu; z) \tag{2.6}$$

implies $q(z) \prec \frac{IR^{m,n+1}_{\lambda,l} f(z)}{IR^{m,n}_{\lambda,l} f(z)}$, $z \in U$, and q is the best subordinant.

Proof. Let the function p be defined by $p(z) := \frac{IR^{m,n+1}_{\lambda,l} f(z)}{IR^{m,n}_{\lambda,l} f(z)}$, $z \in U$, $z \neq 0$, $f \in \mathcal{A}$.

By setting $\nu(w) := \alpha w + \beta w^2$ and $\phi(w) := \mu$ it can be easily verified that ν is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C}\backslash\{0\}$.

Since $\frac{\nu'(q(z))}{\phi(q(z))} = \frac{q'(z)[\alpha+2\beta q(z)]}{\mu}$, it follows that $Re\left(\frac{\nu'(q(z))}{\phi(q(z))}\right) = Re\left(\frac{\alpha}{\mu}q'(z) + \frac{2\beta}{\mu}q(z)q'(z)\right) > 0$, for $\alpha, \beta, \mu \in \mathbb{C}$, $\mu \neq 0$.

By using (2.4) and (2.6) we obtain $\alpha q(z) + \mu (q(z))^2 + \mu z q'(z) \prec \alpha p(z) + \beta (p(z))^2 + \mu z p'(z)$.

Using Lemma 1.2, we have $q(z) \prec p(z) = \frac{IR_{\lambda,l}^{m,n+1} f(z)}{IR_{\lambda,l}^{m,n} f(z)}$, $z \in U$, and q is the best subordinant. ■

Corollary 2.5 *Let $m, n \in \mathbb{N}$, $\lambda, l \geq 0$. Assume that (2.5) holds. If $f \in \mathcal{A}$, $\frac{IR_{\lambda,l}^{m,n+1} f(z)}{IR_{\lambda,l}^{m,n} f(z)} \in \mathcal{H}[q(0),1] \cap Q$ and $\alpha\frac{1+Az}{1+Bz} + \beta\left(\frac{1+Az}{1+Bz}\right)^2 + \mu\frac{(A-B)z}{(1+Bz)^2} \prec \psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z)$, for $\alpha, \beta, \mu \in \mathbb{C}$, $\mu \neq 0$, $-1 \leq B < A \leq 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.2), then $\frac{1+Az}{1+Bz} \prec \frac{IR_{\lambda,l}^{m,n+1} f(z)}{IR_{\lambda,l}^{m,n} f(z)}$, and $\frac{1+Az}{1+Bz}$ is the best subordinant.*

Proof. For $q(z) = \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$ in Theorem 2.4 we get the corollary. ■

Corollary 2.6 *Let $m, n \in \mathbb{N}$, $\lambda, l \geq 0$. Assume that (2.5) holds. If $f \in \mathcal{A}$, $\frac{IR_{\lambda,l}^{m,n+1} f(z)}{IR_{\lambda,l}^{m,n} f(z)} \in \mathcal{H}[q(0),1] \cap Q$ and $\alpha\left(\frac{1+z}{1-z}\right)^\gamma + \beta\left(\frac{1+z}{1-z}\right)^{2\gamma} + \frac{2\mu\gamma z}{(1-z)^2}\left(\frac{1+z}{1-z}\right)^{\gamma-1} \prec \psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z)$, for $\alpha, \beta, \mu \in \mathbb{C}$, $\mu \neq 0$, $0 < \gamma \leq 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.2), then $\left(\frac{1+z}{1-z}\right)^\gamma \prec \frac{IR_{\lambda,l}^{m,n+1} f(z)}{IR_{\lambda,l}^{m,n} f(z)}$, and $\left(\frac{1+z}{1-z}\right)^\gamma$ is the best subordinant.*

Proof. For $q(z) = \left(\frac{1+z}{1-z}\right)^\gamma$, $0 < \gamma \leq 1$ in Theorem 2.4 we get the corollary. ■

Combining Theorem 2.1 and Theorem 2.4, we state the following sandwich theorem.

Theorem 2.7 *Let q_1 and q_2 be analytic and univalent in U such that $q_1(z) \neq 0$ and $q_2(z) \neq 0$, for all $z \in U$, with $\frac{zq_1'(z)}{q_1(z)}$ and $\frac{zq_2'(z)}{q_2(z)}$ being starlike univalent. Suppose that q_1 satisfies (2.1) and q_2 satisfies (2.5). If $f \in \mathcal{A}$, $\frac{IR_{\lambda,l}^{m,n+1} f(z)}{IR_{\lambda,l}^{m,n} f(z)} \in \mathcal{H}[q(0),1] \cap Q$ and $\psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z)$ is as defined in (2.2) univalent in U, then $\alpha q_1(z) + \beta(q_1(z))^2 + \mu z q_1'(z) \prec \psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z) \prec \alpha q_2(z) + \beta(q_2(z))^2 + \mu z q_2'(z)$, for $\alpha, \beta, \mu \in \mathbb{C}$, $\mu \neq 0$, implies $q_1(z) \prec \frac{IR_{\lambda,l}^{m,n+1} f(z)}{IR_{\lambda,l}^{m,n} f(z)} \prec q_2(z)$, and q_1 and q_2 are respectively the best subordinant and the best dominant.*

For $q_1(z) = \frac{1+A_1 z}{1+B_1 z}$, $q_2(z) = \frac{1+A_2 z}{1+B_2 z}$, where $-1 \leq B_2 < B_1 < A_1 < A_2 \leq 1$, we have the following corollary.

Corollary 2.8 *Let $m, n \in \mathbb{N}$, $\lambda, l \geq 0$. Assume that (2.1) and (2.5) hold. If $f \in \mathcal{A}$, $\frac{IR_{\lambda,l}^{m,n+1} f(z)}{IR_{\lambda,l}^{m,n} f(z)} \in \mathcal{H}[q(0),1] \cap Q$ and $\alpha\frac{1+A_1 z}{1+B_1 z} + \beta\left(\frac{1+A_1 z}{1+B_1 z}\right)^2 + \mu\frac{(A_1-B_1)z}{(1+B_1 z)^2} \prec \psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z) \prec \alpha\frac{1+A_2 z}{1+B_2 z} + \beta\left(\frac{1+A_2 z}{1+B_2 z}\right)^2 + \mu\frac{(A_2-B_2)z}{(1+B_2 z)^2}$, for $\alpha, \beta, \mu \in \mathbb{C}$, $\mu \neq 0$, $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.2), then $\frac{1+A_1 z}{1+B_1 z} \prec \frac{IR_{\lambda,l}^{m,n+1} f(z)}{IR_{\lambda,l}^{m,n} f(z)} \prec \frac{1+A_2 z}{1+B_2 z}$, hence $\frac{1+A_1 z}{1+B_1 z}$ and $\frac{1+A_2 z}{1+B_2 z}$ are the best subordinant and the best dominant, respectively.*

For $q_1(z) = \left(\frac{1+z}{1-z}\right)^{\gamma_1}$, $q_2(z) = \left(\frac{1+z}{1-z}\right)^{\gamma_2}$, where $0 < \gamma_1 < \gamma_2 \leq 1$, we have the following corollary.

Corollary 2.9 *Let $m, n \in \mathbb{N}$, $\lambda, l \geq 0$. Assume that (2.1) and (2.5) hold. If $f \in \mathcal{A}$, $\frac{IR_{\lambda,l}^{m,n+1} f(z)}{IR_{\lambda,l}^{m,n} f(z)} \in \mathcal{H}[q(0),1] \cap Q$ and $\alpha\left(\frac{1+z}{1-z}\right)^{\gamma_1} + \beta\left(\frac{1+z}{1-z}\right)^{2\gamma_1} + \frac{2\mu\gamma_1 z}{(1-z)^2}\left(\frac{1+z}{1-z}\right)^{\gamma_1-1} \prec \psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z) \prec \alpha\left(\frac{1+z}{1-z}\right)^{\gamma_2} + \beta\left(\frac{1+z}{1-z}\right)^{2\gamma_2} + \frac{2\mu\gamma_2 z}{(1-z)^2}\left(\frac{1+z}{1-z}\right)^{\gamma_2-1}$, for $\alpha, \beta, \mu \in \mathbb{C}$, $\mu \neq 0$, $0 < \gamma_1 < \gamma_2 \leq 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.2), then $\left(\frac{1+z}{1-z}\right)^{\gamma_1} \prec \frac{IR_{\lambda,l}^{m,n+1} f(z)}{IR_{\lambda,l}^{m,n} f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\gamma_2}$, hence $\left(\frac{1+z}{1-z}\right)^{\gamma_1}$ and $\left(\frac{1+z}{1-z}\right)^{\gamma_2}$ are the best subordinant and the best dominant, respectively.*

We have also

Theorem 2.10 *Let $\left(\frac{IR_{\lambda,l}^{m,n+1} f(z)}{IR_{\lambda,l}^{m,n} f(z)}\right)^\delta \in \mathcal{H}(U)$, $f \in \mathcal{A}$, $z \in U$, $\delta \in \mathbb{C}, \delta \neq 0$, $m, n \in \mathbb{N}$, $\lambda, l \geq 0$ and let the function $q(z)$ be convex and univalent in U such that $q(0) = 1$, $z \in U$. Assume that*

$$Re\left(1 + \frac{\xi}{\beta}q(z) + \frac{2\mu}{\beta}q^2(z) - z\frac{q'(z)}{q(z)} + z\frac{q''(z)}{q'(z)}\right) > 0, \tag{2.7}$$

for $\alpha,\xi,\mu,\beta\in\mathbb{C}$, $\beta\neq 0$, $z\in U$, *and*

$$\psi_{\lambda,l}^{m,n}\left(\delta,\alpha,\xi,\mu,\beta;z\right):=\alpha+\xi\left(\frac{IR_{\lambda,l}^{m,n+1}f\left(z\right)}{IR_{\lambda,l}^{m,n}f\left(z\right)}\right)^{\delta}+\mu\left(\frac{IR_{\lambda,l}^{m,n+1}f\left(z\right)}{IR_{\lambda,l}^{m,n}f\left(z\right)}\right)^{2\delta}+ \tag{2.8}$$

$$\beta\delta\left(n+2\right)\frac{IR_{\lambda,l}^{m,n+2}f\left(z\right)}{IR_{\lambda,l}^{m,n+1}f\left(z\right)}-\beta\delta\left(n+1\right)\frac{IR_{\lambda,l}^{m,n+1}f\left(z\right)}{IR_{\lambda,l}^{m,n}f\left(z\right)}-\beta\delta.$$

If q *satisfies the following subordination*

$$\psi_{\lambda,l}^{m,n}\left(\delta,\alpha,\xi,\mu,\beta;z\right)\prec\alpha+\xi q\left(z\right)+\mu q^{2}\left(z\right)+\frac{\beta zq'\left(z\right)}{q\left(z\right)}, \tag{2.9}$$

for $\alpha,\xi,\mu,\beta\in\mathbb{C}$, $\beta\neq 0$, $z\in U$, *then* $\left(\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^{\delta}\prec q\left(z\right)$, $z\in U$, $\delta\in\mathbb{C}$, $\delta\neq 0$, *and* q *is the best dominant.*

Proof. Let the function p be defined by $p\left(z\right):=\left(\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^{\delta}$, $z\in U$, $z\neq 0$, $f\in\mathcal{A}$. The function p is analytic in U and $p\left(0\right)=1$

We have $zp'\left(z\right)=\delta\left(\frac{IR_{\lambda,l}^{m,n+1}f(z)}{z}\right)^{\delta}\left[\frac{z\left(IR_{\lambda,l}^{m,n+1}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}-\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\cdot\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}\right]$.

By using the identity (1.3), we obtain

$$\frac{zp'\left(z\right)}{p\left(z\right)}=\delta\left(n+2\right)\frac{IR_{\lambda,l}^{m,n+2}f\left(z\right)}{IR_{\lambda,l}^{m,n+1}f\left(z\right)}-\delta\left(n+1\right)\frac{IR_{\lambda,l}^{m,n+1}f\left(z\right)}{IR_{\lambda,l}^{m,n}f\left(z\right)}. \tag{2.10}$$

By setting $\theta\left(w\right):=\alpha+\xi w+\mu w^{2}$ and $\phi\left(w\right):=\frac{\beta}{w}$, it can be easily verified that θ is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi\left(w\right)\neq 0$, $w\in\mathbb{C}\backslash\{0\}$.

Also, by letting $Q\left(z\right)=zq'\left(z\right)\phi\left(q\left(z\right)\right)=\frac{\beta zq'(z)}{q(z)}$, we find that $Q\left(z\right)$ is starlike univalent in U.

Let $h\left(z\right)=\theta\left(q\left(z\right)\right)+Q\left(z\right)=\alpha+\xi q\left(z\right)+\mu q^{2}\left(z\right)+\frac{\beta zq'(z)}{q(z)}$.

We have $Re\left(\frac{zh'(z)}{Q(z)}\right)=Re\left(1+\frac{\xi}{\beta}q\left(z\right)+\frac{2\mu}{\beta}q^{2}\left(z\right)-z\frac{q'(z)}{q(z)}+z\frac{q''(z)}{q'(z)}\right)>0$.

By using (2.10), we obtain $\alpha+\xi p\left(z\right)+\mu\left(p\left(z\right)\right)^{2}+\beta\frac{zp'(z)}{p(z)}=\alpha+\xi\left(\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^{\delta}+\mu\left(\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^{2\delta}$ $+\beta\delta\left(n+2\right)\frac{IR_{\lambda,l}^{m,n+2}f(z)}{IR_{\lambda,l}^{m,n+1}f(z)}-\beta\delta\left(n+1\right)\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}-\beta\delta$.

By using (2.9), we have $\alpha+\xi p\left(z\right)+\mu\left(p\left(z\right)\right)^{2}+\beta\frac{zp'(z)}{p(z)}\prec\alpha+\xi q\left(z\right)+\mu q^{2}\left(z\right)+\frac{\beta zq'(z)}{q(z)}$.

From Lemma 1.1, we have $p\left(z\right)\prec q\left(z\right)$, $z\in U$, i.e. $\left(\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^{\delta}\prec q\left(z\right)$, $z\in U$, $\delta\in\mathbb{C},\delta\neq 0$ and q is the best dominant. ∎

Corollary 2.11 *Let* $q\left(z\right)=\frac{1+Az}{1+Bz}$, $z\in U$, $-1\leq B<A\leq 1$, $m,n\in\mathbb{N}$, $\lambda,l\geq 0$. *Assume that (2.7) holds. If* $f\in\mathcal{A}$ *and* $\psi_{\lambda,l}^{m,n}\left(\delta,\alpha,\xi,\mu,\beta;z\right)\prec\alpha+\xi\frac{1+Az}{1+Bz}+\mu\left(\frac{1+Az}{1+Bz}\right)^{2}+\beta\frac{(A-B)z}{(1+Az)(1+Bz)}$, *for* $\alpha,\xi,\mu,\beta,\delta\in\mathbb{C}$, $\beta,\delta\neq 0$, $-1\leq B<A\leq 1$, *where* $\psi_{\lambda,l}^{m,n}$ *is defined in (2.8), then* $\left(\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^{\delta}\prec\frac{1+Az}{1+Bz}$, *and* $\frac{1+Az}{1+Bz}$ *is the best dominant.*

Proof. For $q\left(z\right)=\frac{1+Az}{1+Bz}$, $-1\leq B<A\leq 1$, in Theorem 2.10 we get the corollary. ∎

Corollary 2.12 *Let* $q\left(z\right)=\left(\frac{1+z}{1-z}\right)^{\gamma}$, $m,n\in\mathbb{N}$, $\lambda,l\geq 0$. *Assume that (2.7) holds. If* $f\in\mathcal{A}$ *and* $\psi_{\lambda,l}^{m,n}\left(\delta,\alpha,\xi,\mu,\beta;z\right)\prec$ $\alpha+\xi\left(\frac{1+z}{1-z}\right)^{\gamma}+\mu\left(\frac{1+z}{1-z}\right)^{2\gamma}+\frac{2\beta\gamma z}{1-z^{2}}$, *for* $\alpha,\xi,\mu,\beta,\delta\in\mathbb{C}$, $0<\gamma\leq 1$, $\beta,\delta\neq 0$, *where* $\psi_{\lambda,l}^{m,n}$ *is defined in (2.8), then* $\left(\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^{\delta}\prec\left(\frac{1+z}{1-z}\right)^{\gamma}$, *and* $\left(\frac{1+z}{1-z}\right)^{\gamma}$ *is the best dominant.*

Proof. Corollary follows by using Theorem 2.10 for $q(z)=\left(\frac{1+z}{1-z}\right)^{\gamma}$, $0<\gamma\leq 1$. ■

Theorem 2.13 *Let q be convex and univalent in U such that $q(0)=1$. Assume that*

$$Re\left(\frac{\xi}{\beta}q(z)q'(z)+\frac{2\mu}{\beta}q^{2}(z)q'(z)\right)>0,\ for\ \alpha,\xi,\mu,\beta\in\mathbb{C},\ \beta\neq 0. \tag{2.11}$$

If $f\in\mathcal{A}$, $\left(\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^{\delta}\in\mathcal{H}[q(0),1]\cap Q$ and $\psi_{\lambda,l}^{m,n}(\delta,\alpha,\xi,\mu,\beta;z)$ is univalent in U, where $\psi_{\lambda,l}^{m,n}(\delta,\alpha,\xi,\mu,\beta;z)$ is as defined in (2.8), then

$$\alpha+\xi q(z)+\mu q^{2}(z)+\frac{\beta zq'(z)}{q(z)}\prec\psi_{\lambda,l}^{m,n}(\delta,\alpha,\xi,\mu,\beta;z) \tag{2.12}$$

implies $q(z)\prec\left(\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^{\delta}$, $\delta\in\mathbb{C}$, $\delta\neq 0$, $z\in U$, and q is the best subordinant.

Proof. Let the function p be defined by $p(z):=\left(\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^{\delta}$, $z\in U$, $z\neq 0$, $\delta\in\mathbb{C},\delta\neq 0$, $f\in\mathcal{A}$. The function p is analytic in U and $p(0)=1$.

By setting $\nu(w):=\alpha+\xi w+\mu w^{2}$ and $\phi(w):=\frac{\beta}{w}$ it can be easily verified that ν is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w)\neq 0$, $w\in\mathbb{C}\backslash\{0\}$.

Since $\frac{\nu'(q(z))}{\phi(q(z))}=\frac{\xi}{\beta}q(z)q'(z)+\frac{2\mu}{\beta}q^{2}(z)q'(z)$, it follows that $Re\left(\frac{\nu'(q(z))}{\phi(q(z))}\right)=Re\left(\frac{\xi}{\beta}q(z)q'(z)+\frac{2\mu}{\beta}q^{2}(z)q'(z)\right)>0$, for $\alpha,\xi,\mu,\beta\in\mathbb{C}$, $\beta\neq 0$.

Now, by using (2.12) we obtain $\alpha+\xi q(z)+\mu q^{2}(z)+\frac{\beta zq'(z)}{q(z)}\prec\alpha+\xi p(z)+\mu p^{2}(z)+\frac{\beta zp'(z)}{p(z)}$, $z\in U$. From Lemma 1.2, we have $q(z)\prec p(z)=\left(\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^{\delta}$, $z\in U$, $\delta\in\mathbb{C}$, $\delta\neq 0$, and q is the best subordinant. ■

Corollary 2.14 *Let $q(z)=\frac{1+Az}{1+Bz}$, $-1\leq B<A\leq 1$, $z\in U$, $m,n\in\mathbb{N}$, $\lambda,l\geq 0$. Assume that (2.11) holds. If $f\in\mathcal{A}$, $\left(\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^{\delta}\in\mathcal{H}[q(0),1]\cap Q$, $\delta\in\mathbb{C}$, $\delta\neq 0$ and $\alpha+\xi\frac{1+Az}{1+Bz}+\mu\left(\frac{1+Az}{1+Bz}\right)^{2}+\beta\frac{(A-B)z}{(1+Az)(1+Bz)}\prec\psi_{\lambda,l}^{m,n}(\delta,\alpha,\xi,\mu,\beta;z)$, for $\alpha,\xi,\mu,\beta\in\mathbb{C}$, $\beta\neq 0$, $-1\leq B<A\leq 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.8), then $\frac{1+Az}{1+Bz}\prec\left(\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^{\delta}$, $\delta\in\mathbb{C}$, $\delta\neq 0$, and $\frac{1+Az}{1+Bz}$ is the best subordinant.*

Proof. For $q(z)=\frac{1+Az}{1+Bz}$, $-1\leq B<A\leq 1$, in Theorem 2.13 we get the corollary. ■

Corollary 2.15 *Let $q(z)=\left(\frac{1+z}{1-z}\right)^{\gamma}$, $m,n\in\mathbb{N}$, $\lambda,l\geq 0$. Assume that (2.11) holds. If $f\in\mathcal{A}$, $\left(\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^{\delta}$ $\in\mathcal{H}[q(0),1]\cap Q$ and $\alpha+\xi\left(\frac{1+z}{1-z}\right)^{\gamma}+\mu\left(\frac{1+z}{1-z}\right)^{2\gamma}+\frac{2\beta\gamma z}{1-z^{2}}\prec\psi_{\lambda,l}^{m,n}(\delta,\alpha,\xi,\mu,\beta;z)$, for $\alpha,\xi,\mu,\beta,\delta\in\mathbb{C}$, $0<\gamma\leq 1$, β ,$\delta\neq 0$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.8), then $\left(\frac{1+z}{1-z}\right)^{\gamma}\prec\left(\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^{\delta}$, and $\left(\frac{1+z}{1-z}\right)^{\gamma}$ is the best subordinant.*

Proof. Corollary follows by using Theorem 2.13 for $q(z)=\left(\frac{1+z}{1-z}\right)^{\gamma}$, $0<\gamma\leq 1$. ■

Combining Theorem 2.10 and Theorem 2.13, we state the following sandwich theorem.

Theorem 2.16 *Let q_1 and q_2 be convex and univalent in U such that $q_1(z)\neq 0$ and $q_2(z)\neq 0$, for all $z\in U$. Suppose that q_1 satisfies (2.7) and q_2 satisfies (2.11). If $f\in\mathcal{A}$, $\left(\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^{\delta}\in\mathcal{H}[q(0),1]\cap Q$, $\delta\in\mathbb{C},\delta\neq 0$ and $\psi_{\lambda,l}^{m,n}(\delta,\alpha,\xi,\mu,\beta;z)$ is as defined in (2.8) univalent in U, then $\alpha+\xi q_1(z)+\mu q_1^{2}(z)+\frac{\beta zq_1'(z)}{q_1(z)}\prec\psi_{\lambda,l}^{m,n}(\delta,\alpha,\xi,\mu,\beta;z)\prec\alpha+\xi q_2(z)+\mu q_2^{2}(z)+\frac{\beta zq_2'(z)}{q_2(z)}$, for $\alpha,\xi,\mu,\beta\in\mathbb{C}$, $\beta\neq 0$, implies $q_1(z)\prec\left(\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^{\delta}\prec q_2(z)$, $z\in U$, $\delta\in\mathbb{C}$, $\delta\neq 0$, and q_1 and q_2 are respectively the best subordinant and the best dominant.*

For $q_1(z)=\frac{1+A_1z}{1+B_1z}$, $q_2(z)=\frac{1+A_2z}{1+B_2z}$, where $-1\le B_2<B_1<A_1<A_2\le 1$, we have the following corollary.

Corollary 2.17 *Let $m,n\in\mathbb{N}$, $\lambda,l\ge 0$. Assume that (2.7) and (2.11) hold for $q_1(z)=\frac{1+A_1z}{1+B_1z}$ and $q_2(z)=\frac{1+A_2z}{1+B_2z}$, respectively. If $f\in\mathcal{A}$, $\left(\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^{\delta}\in\mathcal{H}[q(0),1]\cap Q$ and $\alpha+\xi\frac{1+A_1z}{1+B_1z}+\mu\left(\frac{1+A_1z}{1+B_1z}\right)^2+\beta\frac{(A_1-B_1)z}{(1+A_1z)(1+B_1z)}\prec\psi_{\lambda,l}^{m,n}(\delta,\alpha,\xi,\mu,\beta;z)\prec\alpha+\xi\frac{1+A_2z}{1+B_2z}+\mu\left(\frac{1+A_2z}{1+B_2z}\right)^2+\beta\frac{(A_2-B_2)z}{(1+A_2z)(1+B_2z)}$, $z\in U$, for $\alpha,\xi,\mu,\beta\in\mathbb{C}$, $\beta\ne 0$, $-1\le B_2\le B_1<A_1\le A_2\le 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.2), then $\frac{1+A_1z}{1+B_1z}\prec\left(\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^{\delta}\prec\frac{1+A_2z}{1+B_2z}$, $z\in U$, $\delta\in\mathbb{C}$, $\delta\ne 0$, hence $\frac{1+A_1z}{1+B_1z}$ and $\frac{1+A_2z}{1+B_2z}$ are the best subordinant and the best dominant, respectively.*

For $q_1(z)=\left(\frac{1+z}{1-z}\right)^{\gamma_1}$, $q_2(z)=\left(\frac{1+z}{1-z}\right)^{\gamma_2}$, where $0<\gamma_1<\gamma_2\le 1$, we have the following corollary.

Corollary 2.18 *Let $m,n\in\mathbb{N}$, $\lambda,l\ge 0$. Assume that (2.7) and (2.11) hold for $q_1(z)=\left(\frac{1+z}{1-z}\right)^{\gamma_1}$ and $q_2(z)=\left(\frac{1+z}{1-z}\right)^{\gamma_2}$, respectively. If $f\in\mathcal{A}$, $\left(\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^{\delta}\in\mathcal{H}[q(0),1]\cap Q$ and $\alpha+\xi\left(\frac{1+z}{1-z}\right)^{\gamma_1}+\mu\left(\frac{1+z}{1-z}\right)^{2\gamma_1}+\frac{2\beta\gamma_1z}{1-z^2}\prec\psi_{\lambda,l}^{m,n}(\delta,\alpha,\xi,\mu,\beta;z)\prec\alpha+\xi\left(\frac{1+z}{1-z}\right)^{\gamma_2}+\mu\left(\frac{1+z}{1-z}\right)^{2\gamma_2}+\frac{2\beta\gamma_2z}{1-z^2}$, $z\in U$, for $\alpha,\xi,\mu,\beta\in\mathbb{C}$, $\beta\ne 0$, $0<\gamma_1<\gamma_2\le 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.2), then $\left(\frac{1+z}{1-z}\right)^{\gamma_1}\prec\left(\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^{\delta}\prec\left(\frac{1+z}{1-z}\right)^{\gamma_2}$, $z\in U$, $\delta\in\mathbb{C}$, $\delta\ne 0$, hence $\left(\frac{1+z}{1-z}\right)^{\gamma_1}$ and $\left(\frac{1+z}{1-z}\right)^{\gamma_2}$ are the best subordinant and the best dominant, respectively.*

References

[1] A. Alb Lupas, *Differential Sandwich Theorems using a multiplier transformation and Ruscheweyh derivative*, submitted GFTA 2015.

[2] A. Alb Lupas, *Some differential sandwich theorems using a multiplier transformation and Ruscheweyh derivative*, submitted 2015.

[3] T. Bulboacă, *Classes of first order differential superordinations*, Demonstratio Math., Vol. 35, No. 2, 287-292.

[4] S.S. Miller, P.T. Mocanu, *Subordinants of Differential Superordinations*, Complex Variables, vol. 48, no. 10, 815-826, October, 2003.

[5] S.S. Miller, P.T. Mocanu, *Differential Subordinations: Theory and Applications*, Marcel Dekker Inc., New York, 2000.

[6] C. Selvaraj, K.T. Karthikeyan, *Differential Subordination and Superordination for Analytic Functions Defined Using a Family of Generalized Differential Operators*, An. St. Univ. Ovidius Constanta, Vol. 17 (1) 2009, 201-210.

[7] T.N. Shanmugan, C. Ramachandran, M. Darus, S. Sivasubramanian, *Differential sandwich theorems for some subclasses of analytic functions involving a linear operator*, Acta Math. Univ. Comenianae, 16 (2007), no. 2, 287-294.

[8] H.M. Srivastava, A.Y. Lashin, *Some applications of the Briot-Bouquet differential subordination*, JIPAM. J. Inequal. Pure Appl. Math., 6 (2005), no. 2, Article 41, 7 pp. (electronic).

Approximating fixed points with applications in fractional calculus

George A. Anastassiou[1] and Ioannis K. Argyros[2]
[1]Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu
[2]Department of Mathematical Sciences
Cameron University
Lawton, Ok 73505, USA
iargyros@cameron.edu

Abstract

We approximate fixed points of some iterative methods on a generalized Banach space setting. Earlier studies such as [5, 6, 7, 12] require that the operator involved is Fréchet-differentiable. In the present study we assume that the operator is only continuous. This way we extend the applicability of these methods to include generalized fractional calculus and problems from other areas. Some applications include generalized fractional calculus involving the Riemann-Liouville fractional integral and the Caputo fractional derivative. Fractional calculus is very important for its applications in many applied sciences.

2010 AMS Subject Classification: 26A33, 65G99, 47J25.
Key Words and phrases: Generalized Banach space, Fixed point, semilocal convergence, Riemann-Liouville fractional integral, Caputo fractional derivative.

1 Introduction

Many problems in Computational sciences can be formulated as an operator equation using Mathematical Modelling [7, 10, 13, 14, 15]. The fixed points of these operators can rarely be found in closed form. That is why most solution methods are usually iterative.

1

The semilocal convergence is, based on the information around an initial point, to give conditions ensuring the convergence of the method.

We present a semilocal convergence analysis for some iterative methods on a generalized Banach space setting to approximate fixed point or a zero of an operator. A generalized norm is defined to be an operator from a linear space into a partially order Banach space (to be precised in section 2). Earlier studies such as [5, 6, 7, 12] for Newton's method have shown that a more precise convergence analysis is obtained when compared to the real norm theory. However, the main assumption is that the operator involved is Fréchet-differentiable. This hypothesis limits the applicability of Newton's method. In the present study we only assume the continuity of the operator. This may be expand the applicability of these methods.

The rest of the paper is organized as follows: section 2 contains the basic concepts on generalized Banach spaces and auxiliary results on inequalities and fixed points. In section 3 we present the semilocal convergence analysis of these methods. Finally, in the concluding sections 4-5, we present special cases and applications in generalized fractional calculus.

2 Generalized Banach spaces

We present some standard concepts that are needed in what follows to make the paper as self contained as possible. More details on generalized Banach spaces can be found in [5, 6, 7, 12], and the references there in.

Definition 2.1 *A generalized Banach space is a triplet $(x, E, /\cdot/)$ such that*

(i) X is a linear space over $\mathbb{R}(\mathbb{C})$.

(ii) $E = (E, K, \|\cdot\|)$ is a partially ordered Banach space, i.e.

(ii_1) $(E, \|\cdot\|)$ is a real Banach space,

(ii_2) E is partially ordered by a closed convex cone K,

(iii_3) The norm $\|\cdot\|$ is monotone on K.

(iii) The operator $/\cdot/ : X \to K$ satisfies

$/x/ = 0 \Leftrightarrow x = 0$, $/\theta x/ = |\theta| /x/$,

$/x + y/ \leq /x/ + /y/$ for each $x, y \in X$, $\theta \in \mathbb{R}(\mathbb{C})$.

(iv) X is a Banach space with respect to the induced norm $\|\cdot\|_i := \|\cdot\| \cdot /\cdot/$.

Remark 2.2 *The operator $/\cdot/$ is called a generalized norm. In view of (iii) and (ii_3) $\|\cdot\|_i$, is a real norm. In the rest of this paper all topological concepts will be understood with respect to this norm.*

Let $L\left(X^j, Y\right)$ stand for the space of j-linear symmetric and bounded operators from X^j to Y, where X and Y are Banach spaces. For X, Y partially

2

ordered $L_+\left(X^j,Y\right)$ stands for the subset of monotone operators P such that

$$0 \leq a_i \leq b_i \Rightarrow P\left(a_1,...,a_j\right) \leq P\left(b_1,...,b_j\right). \tag{2.1}$$

Definition 2.3 *The set of bounds for an operator $Q \in L(X,X)$ on a generalized Banach space $(X,E,/\cdot/)$ the set of bounds is defined to be:*

$$B(Q) := \{P \in L_+(E,E),\ /Qx/ \leq P/x/ \text{ for each } x \in X\}. \tag{2.2}$$

Let $D \subset X$ and $T: D \to D$ be an operator. If $x_0 \in D$ the sequence $\{x_n\}$ given by

$$x_{n+1} := T(x_n) = T^{n+1}(x_0) \tag{2.3}$$

is well defined. We write in case of convergence

$$T^{\infty}(x_0) := \lim\left(T^n(x_0)\right) = \lim_{n\to\infty} x_n. \tag{2.4}$$

We need some auxiliary results on inequations.

Lemma 2.4 *Let $(E,K,\|\cdot\|)$ be a partially ordered Banach space, $\xi \in K$ and $M,N \in L_+(E,E)$.*

(i) Suppose there exists $r \in K$ such that

$$R(r) := (M+N)r + \xi \leq r \tag{2.5}$$

and

$$(M+N)^k r \to 0 \quad \text{as} \quad k \to \infty. \tag{2.6}$$

Then, $b := R^{\infty}(0)$ is well defined satisfies the equation $t = R(t)$ and is the smaller than any solution of the inequality $R(s) \leq s$.

(ii) Suppose there exists $q \in K$ and $\theta \in (0,1)$ such that $R(q) \leq \theta q$, then there exists $r \leq q$ satisfying (i).

Proof. (i) Define sequence $\{b_n\}$ by $b_n = R^n(0)$. Then, we have by (2.5) that $b_1 = R(0) = \xi \leq r \Rightarrow b_1 \leq r$. Suppose that $b_k \leq r$ for each $k = 1,2,...,n$. Then, we have by (2.5) and the inductive hypothesis that $b_{n+1} = R^{n+1}(0) = R(R^n(0)) = R(b_n) = (M+N)b_n + \xi \leq (M+N)r + \xi \leq r \Rightarrow b_{n+1} \leq r$. Hence, sequence $\{b_n\}$ is bounded above by r. Set $P_n = b_{n+1} - b_n$. We shall show that

$$P_n \leq (M+N)^n r \quad \text{for each } n = 1,2,... \tag{2.7}$$

We have by the definition of P_n and (2.6) that

$$P_1 = R^2(0) - R(0) = R(R(0)) - R(0)$$

$$= R(\xi) - R(0) = \int_0^1 R'(t\xi)\,\xi dt \leq \int_0^1 R'(\xi)\,\xi dt$$

3

$$\leq \int_0^1 R'(r)\, r dt \leq (M+N)\, r,$$

which shows (2.7) for $n=1$. Suppose that (2.7) is true for $k=1,2,...,n$. Then, we have in turn by (2.6) and the inductive hypothesis that

$$P_{k+1} = R^{k+2}(0) - R^{k+1}(0) = R^{k+1}(R(0)) - R^{k+1}(0) =$$
$$R^{k+1}(\xi) - R^{k+1}(0) = R\left(R^k(\xi)\right) - R\left(R^k(0)\right) =$$
$$\int_0^1 R'\left(R^k(0) + t\left(R^k(\xi) - R^k(0)\right)\right)\left(R^k(\xi) - R^k(0)\right) dt \leq$$
$$R'\left(R^k(\xi)\right)\left(R^k(\xi) - R^k(0)\right) = R'\left(R^k(\xi)\right)\left(R^{k+1}(0) - R^k(0)\right) \leq$$
$$R'(r)\left(R^{k+1}(0) - R^k(0)\right) \leq (M+N)(M+N)^k r = (M+N)^{k+1} r,$$

which completes the induction for (2.7). It follows that $\{b_n\}$ is a complete sequence in a Banach space and as such it converges to some b. Notice that $R(b) = R\left(\lim_{n\to\infty} R^n(0)\right) = \lim_{n\to\infty} R^{n+1}(0) = b \Rightarrow b$ solves the equation $R(t) = t$. We have that $b_n \leq r \Rightarrow b \leq r$, where r a solution of $R(r) \leq r$. Hence, b is smaller than any solution of $R(s) \leq s$.

(ii) Define sequences $\{v_n\}$, $\{w_n\}$ by $v_0 = 0$, $v_{n+1} = R(v_n)$, $w_0 = q$, $w_{n+1} = R(w_n)$. Then, we have that

$$\begin{gathered} 0 \leq v_n \leq v_{n+1} \leq w_{n+1} \leq w_n \leq q, \\ w_n - v_n \leq \theta^n (q - v_n) \end{gathered} \tag{2.8}$$

and sequence $\{v_n\}$ is bounded above by q. Hence, it converges to some r with $r \leq q$. We also get by (2.8) that $w_n - v_n \to 0$ as $n \to \infty \Rightarrow w_n \to r$ as $n \to \infty$. ■

We also need the auxiliary result for computing solutions of fixed point problems.

Lemma 2.5 *Let $(X, (E, K, \|\cdot\|), /\cdot/)$ be a generalized Banach space, and $P \in B(Q)$ be a bound for $Q \in L(X, X)$. Suppose there exists $y \in X$ and $q \in K$ such that*

$$Pq + /y/ \leq q \text{ and } P^k q \to 0 \text{ as } k \to \infty. \tag{2.9}$$

Then, $z = T^\infty(0)$, $T(x) := Qx + y$ is well defined and satisfies: $z = Qz + y$ and $/z/ \leq P/z/ + /y/ \leq q$. Moreover, z is the unique solution in the subspace $\{x \in X | \exists\, \theta \in \mathbb{R} : \{x\} \leq \theta q\}$.

The proof can be found in [12, Lemma 3.2].

4

3 Semilocal convergence

Let $(X, (E, K, \|\cdot\|), /\cdot/)$ and Y be generalized Banach spaces, $D \subset X$ an open subset, $G : D \to Y$ a continuous operator and $A(\cdot) : D \to L(X, Y)$. A zero of operator G is to be determined by a method starting at a point $x_0 \in D$. The results are presented for an operator $F = JG$, where $J \in L(Y, X)$. The iterates are determined through a fixed point problem:

$$x_{n+1} = x_n + y_n, \quad A(x_n) y_n + F(x_n) = 0 \tag{3.1}$$
$$\Leftrightarrow y_n = T(y_n) := (I - A(x_n)) y_n - F(x_n).$$

Let $U(x_0, r)$ stand for the ball defined by

$$U(x_0, r) := \{x \in X : /x - x_0/ \le r\}$$

for some $r \in K$.

Next, we present the semilocal convergence analysis of method (3.1) using the preceding notation.

Theorem 3.1 *Let $F : D \subset X$, $A(\cdot) : D \to L(X, Y)$ and $x_0 \in D$ be as defined previously. Suppose:*

(H_1) There exists an operator $M \in B(I - A(x))$ for each $x \in D$.

(H_2) There exists an operator $N \in L_+(E, E)$ satisfying for each $x, y \in D$

$$/F(y) - F(x) - A(x)(y - x)/ \le N /y - x/.$$

(H_3) There exists a solution $r \in K$ of

$$R_0(t) := (M + N)t + /F(x_0)/ \le t.$$

(H_4) $U(x_0, r) \subseteq D$.

(H_5) $(M + N)^k r \to 0$ as $k \to \infty$.

Then, the following hold:

(C_1) The sequence $\{x_n\}$ defined by

$$x_{n+1} = x_n + T_n^\infty(0), \quad T_n(y) := (I - A(x_n)) y - F(x_n) \tag{3.2}$$

is well defined, remains in $U(x_0, r)$ for each $n = 0, 1, 2, ...$ and converges to the unique zero of operator F in $U(x_0, r)$.

(C_2) An apriori bound is given by the null-sequence $\{r_n\}$ defined by $r_0 := r$ and for each $n = 1, 2, ...$

$$r_n = P_n^\infty(0), \quad P_n(t) = Mt + Nr_{n-1}.$$

(C_3) An aposteriori bound is given by the sequence $\{s_n\}$ defined by

$$s_n := R_n^\infty(0), \quad R_n(t) = (M + N)t + Na_{n-1},$$

$$b_n := /x_n - x_0/ \leq r - r_n \leq r,$$

where

$$a_{n-1} := /x_n - x_{n-1}/ \text{ for each } n = 1, 2, ...$$

Proof. Let us define for each $n \in \mathbb{N}$ the statement:
(I_n) $x_n \in X$ and $r_n \in K$ are well defined and satisfy

$$r_n + a_{n-1} \leq r_{n-1}.$$

We use induction to show (I_n). The statement (I_1) is true: By Lemma 2.4 and (H_3), (H_5) there exists $q \leq r$ such that:

$$Mq + /F(x_0)/ = q \text{ and } M^k q \leq M^k r \to 0 \text{ as } k \to \infty.$$

Hence, by Lemma 2.5 x_1 is well defined and we have $a_0 \leq q$. Then, we get the estimate

$$\begin{aligned} P_1(r-q) &= M(r-q) + Nr_0 \\ &\leq Mr - Mq + Nr = R_0(r) - q \\ &\leq R_0(r) - q = r - q. \end{aligned}$$

It follows with Lemma 2.4 that r_1 is well defined and

$$r_1 + a_0 \leq r - q + q = r = r_0.$$

Suppose that (I_j) is true for each $j = 1, 2, ..., n$. We need to show the existence of x_{n+1} and to obtain a bound q for a_n. To achieve this notice that:

$$Mr_n + N(r_{n-1} - r_n) = Mr_n + Nr_{n-1} - Nr_n = P_n(r_n) - Nr_n \leq r_n.$$

Then, it follows from Lemma 2.4 that there exists $q \leq r_n$ such that

$$q = Mq + N(r_{n-1} - r_n) \text{ and } (M+N)^k q \to 0, \text{ as } k \to \infty. \tag{3.3}$$

By (I_j) it follows that

$$b_n = /x_n - x_0/ \leq \sum_{j=0}^{n-1} a_j \leq \sum_{j=0}^{n-1} (r_j - r_{j+1}) = r - r_n \leq r.$$

Hence, $x_n \in U(x_0, r) \subset D$ and by (H_1) M is a bound for $I - A(x_n)$.

We can write by (H_2) that

$$\begin{aligned} /F(x_n)/ &= /F(x_n) - F(x_{n-1}) - A(x_{n-1})(x_n - x_{n-1})/ \\ &\leq Na_{n-1} \leq N(r_{n-1} - r_n). \end{aligned} \tag{3.4}$$

It follows from (3.3) and (3.4) that

$$Mq + /F(x_n)/ \leq q.$$

By Lemma 2.5, x_{n+1} is well defined and $a_n \leq q \leq r_n$. In view of the definition of r_{n+1} we have that

$$P_{n+1}(r_n - q) = P_n(r_n) - q = r_n - q,$$

so that by Lemma 2.4, r_{n+1} is well defined and

$$r_{n+1} + a_n \leq r_n - q + q = r_n,$$

which proves (I_{n+1}). The induction for (I_n) is complete. Let $m \geq n$, then we obtain in turn that

$$/x_{m+1} - x_n/ \leq \sum_{j=n}^{m} a_j \leq \sum_{j=n}^{m} (r_j - r_{j+1}) = r_n - r_{m+1} \leq r_n. \tag{3.5}$$

Moreover, we get inductively the estimate

$$r_{n+1} = P_{n+1}(r_{n+1}) \leq P_{n+1}(r_n) \leq (M+N) r_n \leq ... \leq (M+N)^{n+1} r.$$

It follows from (H_5) that $\{r_n\}$ is a null-sequence. Hence, $\{x_n\}$ is a complete sequence in a Banach space X by (3.5) and as such it converges to some $x^* \in X$. By letting $m \to \infty$ in (3.5) we deduce that $x^* \in U(x_n, r_n)$. Furthermore, (3.4) shows that x^* is a zero of F. Hence, (C_1) and (C_2) are proved.

In view of the estimate

$$R_n(r_n) \leq P_n(r_n) \leq r_n$$

the apriori, bound of (C_3) is well defined by Lemma 2.4. That is s_n is smaller in general than r_n. The conditions of Theorem 3.1 are satisfied for x_n replacing x_0. A solution of the inequality of (C_2) is given by s_n (see (3.4)). It follows from (3.5) that the conditions of Theorem 3.1 are easily verified. Then, it follows from (C_1) that $x^* \in U(x_n, s_n)$ which proves (C_3). ■

In general the aposterior, estimate is of interest. Then, condition (H_5) can be avoided as follows:

Proposition 3.2 *Suppose: condition (H_1) of Theorem 3.1 is true.*

(H_3') There exists $s \in K$, $\theta \in (0,1)$ such that

$$R_0(s) = (M+N)s + /F(x_0)/ \leq \theta s.$$

(H_4') $U(x_0, s) \subset D$.

Then, there exists $r \leq s$ satisfying the conditions of Theorem 3.1. Moreover, the zero x^* of F is unique in $U(x_0, s)$.

7

Remark 3.3 *(i) Notice that by Lemma 2.4* $R_n^{\infty}(0)$ *is the smallest solution of* $R_n(s) \leq s$. *Hence any solution of this inequality yields on upper estimate for* $R_n^{\infty}(0)$. *Similar inequalities appear in* (H_2) *and* (H_2').

(ii) The weak assumptions of Theorem 3.1 do not imply the existence of $A(x_n)^{-1}$. *In practice the computation of* $T_n^{\infty}(0)$ *as a solution of a linear equation is no problem and the computation of the expensive or impossible to compute in general* $A(x_n)^{-1}$ *is not needed.*

(iii) We can used the following result for the computation of the aposteriori estimates. The proof can be found in [12, Lemma 4.2] by simply exchanging the definitions of R.

Lemma 3.4 *Suppose that the conditions of Theorem 3.1 are satisfied. If* $s \in K$ *is a solution of* $R_n(s) \leq s$, *then* $q := s - a_n \in K$ *and solves* $R_{n+1}(q) \leq q$. *This solution might be improved by* $R_{n+1}^k(q) \leq q$ *for each* $k = 1, 2, \ldots$.

4 Special cases and applications

Application 4.1 *The results obtained in earlier studies such as [5, 6, 7, 12] require that operator* F *(i.e.* G*) is Fréchet-differentiable. This assumption limits the applicability of the earlier results. In the present study we only require that* F *is a continuous operator. Hence, we have extended the applicability of these methods to include classes of operators that are only continuous.*

Example 4.2 *The* j*-dimensional space* $\mathbb{R}^j$ *is a classical example of a generalized Banach space. The generalized norm is defined by componentwise absolute values. Then, as ordered Banach space we set* $E = \mathbb{R}^j$ *with componentwise ordering with e.g. the maximum norm. A bound for a linear operator (a matrix) is given by the corresponding matrix with absolute values. Similarly, we can define the "N" operators. Let* $E = \mathbb{R}$. *That is we consider the case of a real normed space with norm denoted by* $\|\cdot\|$. *Let us see how the conditions of Theorem 3.1 look like.*

Theorem 4.3 *(H_1)* $\|I - A(x)\| \leq M$ *for some* $M \geq 0$.

(H_2) $\|F(y) - F(x) - A(x)(y - x)\| \leq N \|y - x\|$ *for some* $N \geq 0$.

(H_3) $M + N < 1$,

$$r = \frac{\|F(x_0)\|}{1 - (M + N)}. \tag{4.1}$$

(H_4) $U(x_0, r) \subseteq D$.

(H_5) $(M + N)^k r \to 0$ *as* $k \to \infty$, *where* r *is given by (4.1).*

Then, the conclusions of Theorem 3.1 hold.

5 Applications to k-Fractional Calculus

Background

We apply Theorem 4.3 in this section.

Let $f \in L_{\infty}([a,b])$, the k-left Riemann-Liouville fractional integral ([15]) of order $\alpha > 0$ is defined as follows:

$$ {}_kJ_{a+}^{\alpha} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)\, dt, \tag{5.1} $$

all $x \in [a,b]$, where $k > 0$, and $\Gamma_k(a)$ is the k-gamma function given by $\Gamma_k(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-\frac{t^k}{k}} dt$.

It holds ([4]) $\Gamma_k(\alpha+k) = \alpha\Gamma_k(\alpha)$, $\Gamma(\alpha) = \lim_{k\to 1} \Gamma_k(\alpha)$, and we set ${}_kJ_{a+}^{\alpha} f(x) = f(x)$.

Similarly, we define the k-right Riemann-Liouville fractional integral as

$$ {}_kJ_{b-}^{\alpha} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^b (t-x)^{\frac{\alpha}{k}-1} f(t)\, dt, \tag{5.2} $$

for all $x \in [a,b]$, and we set ${}_kJ_{b-}^{\alpha} f(x) = f(x)$.

Results

I) Here we work with ${}_kJ_{a+}^{\alpha} f(x)$. We observe that

$$ \begin{aligned} \left|{}_kJ_{a+}^{\alpha} f(x)\right| &\leq \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} |f(t)|\, dt \\ \leq \frac{\|f\|_{\infty}}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} dt &= \frac{\|f\|_{\infty}}{k\Gamma_k(\alpha)} \frac{(x-a)^{\frac{\alpha}{k}}}{\frac{\alpha}{k}} \\ = \frac{\|f\|_{\infty}}{\Gamma_k(\alpha+k)} (x-a)^{\frac{\alpha}{k}} &\leq \frac{\|f\|_{\infty}}{\Gamma_k(\alpha+k)} (b-a)^{\frac{\alpha}{k}}. \end{aligned} \tag{5.3} $$

We have proved that

$$ {}_kJ_{a+}^{\alpha} f(a) = 0, \tag{5.4} $$

and

$$ \left\|{}_kJ_{a+}^{\alpha} f\right\|_{\infty} \leq \frac{(b-a)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} \|f\|_{\infty}, \tag{5.5} $$

proving that ${}_kJ_{a+}^{\alpha}$ is a bounded linear operator.

By [3], p. 388, we get that $\left({}_kJ_{a+}^{\alpha} f\right)$ is a continuous function over $[a,b]$ and in particular continuous over $[a^*, b]$, where $a < a^* < b$.

Thus, there exist $x_1, x_2 \in [a^*, b]$ such that

$$ \left({}_kJ_{a+}^{\alpha} f\right)(x_1) = \min \left({}_kJ_{a+}^{\alpha} f\right)(x), \tag{5.6} $$

$$ \left({}_kJ_{a+}^{\alpha} f\right)(x_2) = \max \left({}_kJ_{a+}^{\alpha} f\right)(x), \;\; x \in [a^*, b]. \tag{5.7} $$

We assume that

$$\left({}_kJ^{\alpha}_{a+}f\right)(x_1) > 0. \tag{5.8}$$

Hence

$$\left\|{}_kJ^{\alpha}_{a+}f\right\|_{\infty,[a^*,b]} = \left({}_kJ^{\alpha}_{a+}f\right)(x_2) > 0. \tag{5.9}$$

Here it is

$$J(x) = mx, \;\; m \neq 0. \tag{5.10}$$

Therefore the equation

$$Jf(x) = 0, \quad x \in [a^*, b], \tag{5.11}$$

has the same solutions as the equation

$$F(x) := \frac{Jf(x)}{2\left({}_kJ^{\alpha}_{a+}f\right)(x_2)} = 0, \quad x \in [a^*, b]. \tag{5.12}$$

Notice that

$${}_kJ^{\alpha}_{a+}\left(\frac{f}{2\left({}_kJ^{\alpha}_{a+}f\right)(x_2)}\right)(x) = \frac{\left({}_kJ^{\alpha}_{a+}f\right)(x)}{2\left({}_kJ^{\alpha}_{a+}f\right)(x_2)} \leq \frac{1}{2} < 1, \quad x \in [a^*, b]. \tag{5.13}$$

Call

$$A(x) := \frac{\left({}_kJ^{\alpha}_{a+}f\right)(x)}{2\left({}_kJ^{\alpha}_{a+}f\right)(x_2)}, \quad \forall\, x \in [a^*, b]. \tag{5.14}$$

We notice that

$$0 < \frac{\left({}_kJ^{\alpha}_{a+}f\right)(x_1)}{2\left({}_kJ^{\alpha}_{a+}f\right)(x_2)} \leq A(x) \leq \frac{1}{2}, \;\; \forall\, x \in [a^*, b]. \tag{5.15}$$

Hence it holds

$$|1 - A(x)| = 1 - A(x) \leq 1 - \frac{\left({}_kJ^{\alpha}_{a+}f\right)(x_1)}{2\left({}_kJ^{\alpha}_{a+}f\right)(x_2)} =: \gamma_0, \quad \forall\, x \in [a^*, b]. \tag{5.16}$$

Clearly $\gamma_0 \in (0, 1)$.

We have proved that

$$|1 - A(x)| \leq \gamma_0, \quad \forall\, x \in [a^*, b]. \tag{5.17}$$

Next we assume that $F(x)$ is a contraction, i.e.

$$|F(x) - F(y)| \leq \lambda\,|x - y|\,; \;\; \forall\, x, y \in [a^*, b], \tag{5.18}$$

and $0 < \lambda < \frac{1}{2}$.

Equivalently we have

$$|Jf(x) - Jf(y)| \leq 2\lambda\left({}_kJ^{\alpha}_{a+}f\right)(x_2)\,|x - y|, \;\; \text{all } x, y \in [a^*, b]. \tag{5.19}$$

We observe that

$$|F(y) - F(x) - A(x)(y-x)| \leq |F(y) - F(x)| + |A(x)||y-x| \leq$$
$$\lambda |y-x| + |A(x)||y-x| = (\lambda + |A(x)|)|y-x| =: (\psi_1), \ \ \forall \ x, y \in [a^*, b]. \tag{5.20}$$

We have that

$$\left|\left({}_kJ^{\alpha}_{a+}f\right)(x)\right| \leq \frac{(b-a)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\|f\|_{\infty} < \infty, \ \ \forall \ x \in [a^*, b]. \tag{5.21}$$

Hence

$$|A(x)| = \frac{\left|\left({}_kJ^{\alpha}_{a+}f\right)(x)\right|}{2\left({}_kJ^{\alpha}_{a+}f\right)(x_2)} \leq \frac{(b-a)^{\frac{\alpha}{k}}\|f\|_{\infty}}{2\Gamma_k(\alpha+k)\left({}_kJ^{\alpha}_{a+}f\right)(x_2)} < \infty, \ \ \forall \ x \in [a^*, b]. \tag{5.22}$$

Therefore we get

$$(\psi_1) \leq \left(\lambda + \frac{(b-a)^{\frac{\alpha}{k}}\|f\|_{\infty}}{2\Gamma_k(\alpha+k)\left({}_kJ^{\alpha}_{a+}f\right)(x_2)}\right)|y-x|, \ \ \ \forall \ x, y \in [a^*, b]. \tag{5.23}$$

Call

$$0 < \gamma_1 := \lambda + \frac{(b-a)^{\frac{\alpha}{k}}\|f\|_{\infty}}{2\Gamma_k(\alpha+k)\left({}_kJ^{\alpha}_{a+}f\right)(x_2)}, \tag{5.24}$$

choosing $(b-a)$ small enough we can make $\gamma_1 \in (0,1)$.

We have proved that

$$|F(y) - F(x) - A(x)(y-x)| \leq \gamma_1 |y-x|, \ \ \forall \ x, y \in [a^*, b], \ \gamma_1 \in (0,1). \tag{5.25}$$

Next we call and we need that

$$0 < \gamma := \gamma_0 + \gamma_1 = 1 - \frac{\left({}_kJ^{\alpha}_{a+}f\right)(x_1)}{2\left({}_kJ^{\alpha}_{a+}f\right)(x_2)} + \lambda + \frac{(b-a)^{\frac{a}{k}}\|f\|_{\infty}}{2\Gamma_k(\alpha+k)\left({}_kJ^{\alpha}_{a+}f\right)(x_2)} < 1, \tag{5.26}$$

equivalently,

$$\lambda + \frac{(b-a)^{\frac{a}{k}}\|f\|_{\infty}}{2\Gamma_k(\alpha+k)\left({}_kJ^{\alpha}_{a+}f\right)(x_2)} < \frac{\left({}_kJ^{\alpha}_{a+}f\right)(x_1)}{2\left({}_kJ^{\alpha}_{a+}f\right)(x_2)}, \tag{5.27}$$

equivalently,

$$2\lambda\left({}_kJ^{\alpha}_{a+}f\right)(x_2) + \frac{(b-a)^{\frac{\alpha}{k}}\|f\|_{\infty}}{\Gamma_k(\alpha+k)} < \left({}_kJ^{\alpha}_{a+}f\right)(x_1), \tag{5.28}$$

which is possible for small λ, $(b-a)$. That is $\gamma \in (0,1)$. So our numerical method converges and solves (5.11).

II) Here we act on ${}_kJ_{b-}^{\alpha}f(x)$, see (5.2).

Let $f \in L_\infty([a,b])$. We have that

$$\left|{}_kJ_{b-}^{\alpha}f(x)\right| \leq \frac{1}{k\Gamma_k(\alpha)} \int_x^b (t-x)^{\frac{\alpha}{k}-1} |f(t)|\, dt$$

$$\leq \frac{\|f\|_\infty}{k\Gamma_k(\alpha)} \int_x^b (t-x)^{\frac{\alpha}{k}-1} dt = \frac{\|f\|_\infty}{k\Gamma_k(\alpha)} \frac{(b-x)^{\frac{\alpha}{k}}}{\frac{\alpha}{k}}$$

$$= \frac{\|f\|_\infty}{\Gamma_k(\alpha+k)} (b-x)^{\frac{\alpha}{k}} \leq \frac{\|f\|_\infty}{\Gamma_k(\alpha+k)} (b-a)^{\frac{\alpha}{k}}. \tag{5.29}$$

We observe that

$${}_kJ_{b-}^{\alpha}f(b) = 0, \tag{5.30}$$

and

$$\left\|{}_kJ_{b-}^{\alpha}f\right\|_\infty \leq \frac{(b-a)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} \|f\|_\infty. \tag{5.31}$$

That is ${}_kJ_{b-}^{\alpha}$ is a bounded linear operator.

Let here $a < b^* < b$.

By [4] we get that ${}_kJ_{b-}^{\alpha}f$ is continuous over $[a,b]$, and in particular it is continuous over $[a,b^*]$.

Thus, there exist $x_1, x_2 \in [a,b^*]$ such that

$$\left({}_kJ_{b-}^{\alpha}f\right)(x_1) = \min \left({}_kJ_{b-}^{\alpha}f\right)(x), \tag{5.32}$$

$$\left({}_kJ_{b-}^{\alpha}f\right)(x_2) = \max \left({}_kJ_{b-}^{\alpha}f\right)(x), \quad x \in [a,b^*].$$

We assume that

$$\left({}_kJ_{b-}^{\alpha}f\right)(x_1) > 0. \tag{5.33}$$

Hence

$$\left\|{}_kJ_{b-}^{\alpha}f\right\|_{\infty,[a^*,b]} = \left({}_kJ_{b-}^{\alpha}f\right)(x_2) > 0. \tag{5.34}$$

Here it is

$$J(x) = mx, \quad m \neq 0. \tag{5.35}$$

Therefore the equation

$$Jf(x) = 0, \quad x \in [a,b^*], \tag{5.36}$$

has the same solutions as the equation

$$F(x) := \frac{Jf(x)}{2\left({}_kJ_{b-}^{\alpha}f\right)(x_2)} = 0, \quad x \in [a,b^*]. \tag{5.37}$$

Notice that

$${}_kJ_{b-}^{\alpha}\left(\frac{f}{2\left({}_kJ_{b-}^{\alpha}f\right)(x_2)}\right)(x) = \frac{\left({}_kJ_{b-}^{\alpha}f\right)(x)}{2\left({}_kJ_{b-}^{\alpha}f\right)(x_2)} \leq \frac{1}{2} < 1, \quad x \in [a,b^*]. \tag{5.38}$$

Call

$$A(x) := \frac{\left({}_kJ_{b-}^{\alpha} f\right)(x)}{2\left({}_kJ_{b-}^{\alpha} f\right)(x_2)}, \quad \forall\, x \in [a, b^*]. \tag{5.39}$$

We notice that

$$0 < \frac{\left({}_kJ_{b-}^{\alpha} f\right)(x_1)}{2\left({}_kJ_{b-}^{\alpha} f\right)(x_2)} \leq A(x) \leq \frac{1}{2}, \ \ \forall\, x \in [a, b^*]. \tag{5.40}$$

Hence we have

$$|1 - A(x)| = 1 - A(x) \leq 1 - \frac{\left({}_kJ_{b-}^{\alpha} f\right)(x_1)}{2\left({}_kJ_{b-}^{\alpha} f\right)(x_2)} =: \gamma_0, \quad \forall\, x \in [a, b^*]. \tag{5.41}$$

Clearly $\gamma_0 \in (0, 1)$.

We have proved that

$$|1 - A(x)| \leq \gamma_0, \quad \forall\, x \in [a, b^*], \, \gamma_0 \in (0, 1). \tag{5.42}$$

Next we assume that $F(x)$ is a contraction, i.e.

$$|F(x) - F(y)| \leq \lambda |x - y|; \quad \forall\, x, y \in [a, b^*], \tag{5.43}$$

and $0 < \lambda < \frac{1}{2}$.

Equivalently we have

$$|Jf(x) - Jf(y)| \leq 2\lambda \left({}_kJ_{b-}^{\alpha} f\right)(x_2) |x - y|, \quad \text{all } x, y \in [a, b^*]. \tag{5.44}$$

We observe that

$$|F(y) - F(x) - A(x)(y - x)| \leq |F(y) - F(x)| + |A(x)| |y - x| \leq$$

$$\lambda |y - x| + |A(x)| |y - x| = (\lambda + |A(x)|) |y - x| =: (\psi_1), \ \ \forall\, x, y \in [a, b^*]. \tag{5.45}$$

We have that

$$\left|\left({}_kJ_{b-}^{\alpha} f\right)(x)\right| \leq \frac{(b - a)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha + k)} \|f\|_\infty < \infty, \ \ \forall\, x \in [a, b^*]. \tag{5.46}$$

Hence

$$|A(x)| = \frac{\left|\left({}_kJ_{b-}^{\alpha} f\right)(x)\right|}{2\left({}_kJ_{b-}^{\alpha} f\right)(x_2)} \leq \frac{(b - a)^{\frac{\alpha}{k}} \|f\|_\infty}{2\Gamma_k(\alpha + k)\left({}_kJ_{b-}^{\alpha} f\right)(x_2)} < \infty, \ \ \forall\, x \in [a, b^*]. \tag{5.47}$$

Therefore we get

$$(\psi_1) \leq \left(\lambda + \frac{(b - a)^{\frac{\alpha}{k}} \|f\|_\infty}{2\Gamma_k(\alpha + k)\left({}_kJ_{b-}^{\alpha} f\right)(x_2)}\right) |y - x|, \quad \forall\, x, y \in [a, b^*]. \tag{5.48}$$

Call

$$0<\gamma_1:=\lambda+\frac{(b-a)^{\frac{\alpha}{k}}\|f\|_\infty}{2\Gamma_k(\alpha+k)\left({}_kJ_{b-}^{\alpha}f\right)(x_2)}, \tag{5.49}$$

choosing $(b-a)$ small enough we can make $\gamma_1\in(0,1)$.

We have proved that

$$|F(y)-F(x)-A(x)(y-x)|\leq\gamma_1|y-x|,\ \ \forall\,x,y\in[a,b^*],\,\gamma_1\in(0,1). \tag{5.50}$$

Next we call and we need that

$$0<\gamma:=\gamma_0+\gamma_1=1-\frac{\left({}_kJ_{b-}^{\alpha}f\right)(x_1)}{2\left({}_kJ_{b-}^{\alpha}f\right)(x_2)}+\lambda+\frac{(b-a)^{\frac{a}{k}}\|f\|_\infty}{2\Gamma_k(\alpha+k)\left({}_kJ_{b-}^{\alpha}f\right)(x_2)}<1, \tag{5.51}$$

equivalently,

$$\lambda+\frac{(b-a)^{\frac{a}{k}}\|f\|_\infty}{2\Gamma_k(\alpha+k)\left({}_kJ_{b-}^{\alpha}f\right)(x_2)}<\frac{\left({}_kJ_{b-}^{\alpha}f\right)(x_1)}{2\left({}_kJ_{b-}^{\alpha}f\right)(x_2)}, \tag{5.52}$$

equivalently,

$$2\lambda\left({}_kJ_{b-}^{\alpha}f\right)(x_2)+\frac{(b-a)^{\frac{\alpha}{k}}\|f\|_\infty}{\Gamma_k(\alpha+k)}<\left({}_kJ_{b-}^{\alpha}f\right)(x_1), \tag{5.53}$$

which is possible for small λ, $(b-a)$. That is $\gamma\in(0,1)$. So our numerical method converges and solves (5.36).

III) Here we deal with the fractional M. Caputo-Fabrizio derivative defined as follows (see [9]):

let $0<\alpha<1$, $f\in C^1([0,b])$,

$${}^{CF}D_*^{\alpha}f(t)=\frac{1}{1-\alpha}\int_0^t\exp\left(-\frac{\alpha}{1-\alpha}(t-s)\right)f'(s)\,ds, \tag{5.54}$$

for all $0\leq t\leq b$.

Call

$$\gamma:=\frac{\alpha}{1-\alpha}>0. \tag{5.55}$$

I.e.

$${}^{CF}D_*^{\alpha}f(t)=\frac{1}{1-\alpha}\int_0^t e^{-\gamma(t-s)}f'(s)\,ds,\quad 0\leq t\leq b. \tag{5.56}$$

We notice that

$$\left|{}^{CF}D_*^{\alpha}f(t)\right|\leq\frac{1}{1-\alpha}\left(\int_0^t e^{-\gamma(t-s)}ds\right)\|f'\|_\infty$$

$$=\frac{e^{-\gamma t}}{\alpha}\left(e^{\gamma t}-1\right)\|f'\|_\infty=\frac{1}{\alpha}\left(1-e^{-\gamma t}\right)\|f'\|_\infty\leq\left(\frac{1-e^{-\gamma b}}{\alpha}\right)\|f'\|_\infty. \tag{5.57}$$

That is

$$\left({}^{CF}D_*^\alpha f\right)(0) = 0, \tag{5.58}$$

and

$$\left|{}^{CF}D_*^\alpha f(t)\right| \leq \left(\frac{1-e^{-\gamma b}}{\alpha}\right) \|f'\|_\infty, \ \forall\, t \in [0,b]. \tag{5.59}$$

Notice here that $1 - e^{-\gamma t}$, $t \geq 0$ is an increasing function.

Thus the smaller the t, the smaller it is $1 - e^{-\gamma t}$. We rewrite

$${}^{CF}D_*^\alpha f(t) = \frac{e^{-\gamma t}}{1-\alpha} \int_0^t e^{\gamma s} f'(s)\, ds, \tag{5.60}$$

proving that $\left({}^{CF}D_*^\alpha f\right)$ is a continuous function over $[0,b]$, in particular it is continuous over $[a,b]$, where $0 < a < b$.

Therefore there exist $x_1, x_2 \in [a,b]$ such that

$${}^{CF}D_*^\alpha f(x_1) = \min\ {}^{CF}D_*^\alpha f(x), \tag{5.61}$$

and

$${}^{CF}D_*^\alpha f(x_2) = \max\ {}^{CF}D_*^\alpha f(x), \ \text{ for } x \in [a,b].$$

We assume that

$${}^{CF}D_*^\alpha f(x_1) > 0. \tag{5.62}$$

(i.e. ${}^{CF}D_*^\alpha f(x) > 0$, $\forall\, x \in [a,b]$).

Furthermore

$$\left\|{}^{CF}D_*^\alpha fG\right\|_{\infty,[a,b]} =^{CF} D_*^\alpha f(x_2). \tag{5.63}$$

Here it is

$$J(x) = mx,\ m \neq 0. \tag{5.64}$$

The equation

$$Jf(x) = 0, \quad x \in [a,b], \tag{5.65}$$

has the same set of solutions as the equation

$$F(x) := \frac{Jf(x)}{{}^{CF}D_*^\alpha f(x_2)} = 0, \quad x \in [a,b]. \tag{5.66}$$

Notice that

$${}^{CF}D_*^\alpha \left(\frac{f(x)}{2\,{}^{CF}D_*^\alpha f(x_2)}\right) = \frac{{}^{CF}D_*^\alpha f(x)}{2\,{}^{CF}D_*^\alpha f(x_2)} \leq \frac{1}{2} < 1, \quad \forall\, x \in [a,b]. \tag{5.67}$$

We call

$$A(x) := \frac{{}^{CF}D_*^\alpha f(x)}{2\,{}^{CF}D_*^\alpha f(x_2)}, \quad \forall\, x \in [a,b]. \tag{5.68}$$

We notice that

$$0 < \frac{{}^{CF}D_*^\alpha f(x_1)}{2\,{}^{CF}D_*^\alpha f(x_2)} \leq A(x) \leq \frac{1}{2}. \tag{5.69}$$

15

Furthermore it holds

$$|1-A(x)| = 1-A(x) \leq 1 - \frac{{}^{CF}D_*^{\alpha} f(x_1)}{2^{CF}D_*^{\alpha} f(x_2)} =: \gamma_0, \quad \forall\, x \in [a,b]. \tag{5.70}$$

Clearly $\gamma_0 \in (0,1)$.

We have proved that

$$|1-A(x)| \leq \gamma_0 \in (0,1), \quad \forall\, x \in [a,b]. \tag{5.71}$$

Next we assume that $F(x)$ is a contraction over $[a,b]$, i.e.

$$|F(x) - F(y)| \leq \lambda |x-y|; \quad \forall\, x,y \in [a,b], \tag{5.72}$$

and $0 < \lambda < \frac{1}{2}$.

Equivalently we have

$$|Jf(x) - Jf(y)| \leq 2\lambda \left({}^{CF}D_*^{\alpha} f(x_2)\right) |x-y|, \quad \forall\, x,y \in [a,b]. \tag{5.73}$$

We observe that

$$|F(y) - F(x) - A(x)(y-x)| \leq |F(y) - F(x)| + |A(x)|\,|y-x| \leq$$

$$\lambda|y-x| + |A(x)|\,|y-x| = (\lambda + |A(x)|)\,|y-x| =: (\xi_2), \ \forall\, x,y \in [a,b]. \tag{5.74}$$

Here we have

$$\left|\left({}^{CF}D_*^{\alpha} f\right)(x)\right| \leq \left(\frac{1-e^{-\gamma b}}{\alpha}\right) \|f'\|_{\infty}, \ \forall\, t \in [a,b]. \tag{5.75}$$

Hence, $\forall\, x \in [a,b]$ we get that

$$|A(x)| = \frac{\left|{}^{CF}D_*^{\alpha} f(x)\right|}{2\left({}^{CF}D_*^{\alpha} f\right)(x_2)} \leq \frac{\left(1-e^{-\gamma b}\right)\|f'\|_{\infty}}{2\alpha\left({}^{CF}D_*^{\alpha} f\right)(x_2)} < \infty. \tag{5.76}$$

Consequently we observe

$$(\xi_2) \leq \left(\lambda + \frac{\left(1-e^{-\gamma b}\right)\|f'\|_{\infty}}{2\alpha\left({}^{CF}D_*^{\alpha} f\right)(x_2)}\right)|y-x|, \quad \forall\, x,y \in [a,b]. \tag{5.77}$$

Call

$$0 < \gamma_1 := \lambda + \frac{\left(1-e^{-\gamma b}\right)\|f'\|_{\infty}}{2\alpha\left({}^{CF}D_*^{\alpha} f\right)(x_2)}, \tag{5.78}$$

choosing b small enough, we can make $\gamma_1 \in (0,1)$.

We have proved

$$|F(y) - F(x) - A(x)(y-x)| \leq \gamma_1 |y-x|, \quad \gamma_1 \in (0,1), \ \forall\, x,y \in [a,b]. \tag{5.79}$$

16

Next we call and need

$$0 < \gamma := \gamma_0 + \gamma_1 = 1 - \frac{{}^{CF}D_*^{\alpha} f(x_1)}{2^{CF}D_*^{\alpha} f(x_2)} + \lambda + \frac{\left(1 - e^{-\gamma b}\right) \|f'\|_\infty}{2\alpha \left({}^{CF}D_*^{\alpha} f\right)(x_2)} < 1, \quad (5.80)$$

equivalently,

$$\lambda + \frac{\left(1 - e^{-\gamma b}\right) \|f'\|_\infty}{2\alpha \left({}^{CF}D_*^{\alpha} f\right)(x_2)} < \frac{{}^{CF}D_*^{\alpha} f(x_1)}{2^{CF}D_*^{\alpha} f(x_2)}, \quad (5.81)$$

equivalently,

$$2\lambda^{CF}D_*^{\alpha} f(x_2) + \frac{\left(1 - e^{-\gamma b}\right)}{\alpha} \|f'\|_\infty <^{CF} D_*^{\alpha} f(x_1), \quad (5.82)$$

which is possible for small λ, b.

We have proved that

$$\gamma = \gamma_0 + \gamma_1 \in (0,1). \quad (5.83)$$

Hence equation (5.65) can be solved with our presented numerical methods.

Conclusion:

In all three applications we have proved that

$$|1 - A(x)| \leq \gamma_0 \in (0,1), \quad (5.84)$$

and

$$|F(y) - F(x) - A(x)(y - x)| \leq \gamma_1 |y - x|, \quad (5.85)$$

where $\gamma_1 \in (0,1)$, and

$$\gamma = \gamma_0 + \gamma_1 \in (0,1), \quad (5.86)$$

for all $x, y \in [a^*, b]$, $[a, b^*]$, $[a, b]$, respectively.

Consequently, our presented Numerical methods here, Theorem 4.3, apply to solve

$$f(x) = 0. \quad (5.87)$$

References

[1] S. Amat, S. Busquier, *Third-order iterative methods under Kantorovich conditions*, J. Math. Anal. Applic. 336, (2007), 243-261.

[2] S. Amat, S. Busquier, S. Plaza, *Chaotic dynamics of a third-order Newton-type method*, J. Math. Anal. Applic. 366, 1, (2010), 164-174.

[3] G. Anastassiou, *Fractional Differentiation Inequalities*, Springer, New York, 2009.

[4] G. Anastassiou, *Fractional representation formulae and right fractional inequalities*, Mathematical and Computer Modelling, 54, (11-12), (2011), 3098-3115.

[5] I.K. Argyros, *Newton-like methods in partially ordered linear spaces*, J. Approx. Th. Applic. 9, 1, (1993), 1-10.

[6] I.K. Argyros, *Results on controlling the residuals of perturbed Newton-like methods on Banach spaces with a convergence structure*, Southwest J. Pure Appl. Math. 1, (1995), 32-38.

[7] I.K. Argyros, *Convergence and Applications of Newton-type iterations*, Springer-Verlag Publ., New York, 2008.

[8] J.A. Ezquerro, J.M. Gutierrez, M.A. Hernandez, N. Romero, M.J. Rubio, *The Newton method: From Newton to Kantorovich* (spanish), Gac. R. Soc. Mat. Esp. 13, (2010), 53-76.

[9] J. Losada, J.J. Nieto, *Properties of a new fractional derivative without singular kernel*, Progr. Fract. Differ. Appl., 1, 2, (2015), 87-92.

[10] A.A. Magrenan, *Different anomalies in a Surrutt family of iterative root finding methods*, Appl. Math. Comput. 233, (2014), 29-38.

[11] A.A. Magrenan, *A new tool to study real dynamics: The convergence plane*, Appl. Math. Comput. 248, (2014), 215-224.

[12] P.W. Meyer, *Newton's method in generalized Banach spaces*, Numer. Func. Anal. Optimiz. 9, 3 and 4, (1987), 244-259.

[13] S. Mukeen, G.M. Habibullah, *k-Fractional integrals and Application*, Int. J. Contemp. Math. Sciences, 7, 2, (2012), 89-94.

[14] F.A. Potra, V. Ptak, *Nondiscrete induction and iterative processes*, Pitman Publ., London, 1984.

[15] M. Zekisarikaya, A. Karaca, *On the Riemann-Liouville fractional integral and applications*, Intern. J. Stat. and Math., 1, 3, (2014), 33-43.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 21, NO.7, 2016, COPYRIGHT 2016 EUDOXUS PRESS, LLC

Some Sets of Sufficient Conditions for Carathéodory Functions

Young Jae Sim[1], Oh Sang Kwon[1], Nak Eun Cho[2,*] and H. M. Srivastava[3]

[1]Department of Mathematics, Kyungsung University,
Busan 608-736, Republic of Korea

E-Mail: yjsim@ks.ac.kr oskwon@ks.ac.kr

[2]Department of Applied Mathematics, Pukyung National University,
Busan 608-737, Republic of Korea

E-Mail: necho@pknu.ac.kr

***Corresponding Author**

[3]Department of Mathematics and Statistics, University of Victoria,
Victoria, British Columbia V8W 3R4, Canada

E-Mail: harimsri@math.uvic.ca

Abstract

In this paper, we first investigate and present several sets of sufficient conditions for Carathéodory functions in the open unit disk $\mathbb{U}$. We then apply the main results proven here in order to derive some conditions for starlike functions in $\mathbb{U}$. Relevant connections with various known results are also considered.

2010 Mathematics Subject Classification. Primary 30C45; Secondary 30C55.

Key Words and Phrases. Analytic functions; Starlike functions; Convex functions; Univalent functions; Carathéodory function; Principle of differential subordination.

1. Introduction, Definitions and Preliminaries

Let $\mathcal{P}$ denote the class of functions p of the form:

$$p(z) = \sum_{n=0}^{\infty} p_n z^n,$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

The function $p \in \mathcal{P}$ is called a *Carathéodory function* if it satisfies the following condition:

$$\Re\{p(z)\} > 0 \qquad (z \in \mathbb{U}).$$

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in $\mathbb{U}$. A function $f \in \mathcal{A}$ is in the class $\mathcal{S}^*$ of starlike functions in $\mathbb{U}$, if it satisfies the following condition:

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \qquad (z \in \mathbb{U}).$$

In recent years, many authors (see, for example, [1, 2, 3, 4, 6, 8, 9, 10, 12, 16, 18]) have investigated and derived sufficient conditions for Carathéodory functions and some of their results have been applied to find some sufficient conditions for starlikeness or convexity of analytic functions (see, for example, [5, 11, 13, 14, 15, 17]).

Following the principle of differential subordination, we say that a function f is subordinate to F in $\mathbb{U}$, written as $f \prec F$, if and only if

$$f(z) = F\big(w(z)\big) \qquad (z \in \mathbb{U})$$

for some Schwarz function $w(z)$, with

$$w(0) = 0 \qquad \text{and} \qquad |w(z)| < 1 \qquad (z \in \mathbb{U}).$$

If $F(z)$ is univalent in $\mathbb{U}$, then the subordination $f \prec F$ is equivalent to

$$f(0) = F(0) \qquad \text{and} \qquad f(\mathbb{U}) \subset F(\mathbb{U}).$$

We denote by $\mathcal{Q}$ the class of functions q that are analytic and injective on $\overline{\mathbb{U}} \setminus E(q)$, where

$$E(q) = \left\{\zeta : \zeta \in \partial\mathbb{U} \quad \text{and} \quad \lim_{z\to\zeta} q(z) = \infty\right\},$$

and are such that

$$q'(\zeta) \neq 0 \qquad \big(\zeta \in \partial\mathbb{U} \setminus E(q)\big).$$

Furthermore, let the subclass of $\mathcal{Q}$ for which $q(0) = a$ be denoted by $\mathcal{Q}(a)$.

The main object of this paper is to investigate and present several sets of sufficient conditions for Carathéodory functions in the open unit disk $\mathbb{U}$. The main results proven here are shown to lead to some conditions for starlike functions in $\mathbb{U}$. We also consider the relevant connections of our results with various known results.

2. A Set of Main Results

In order to prove our main results, we need the following lemma due to Miller and Mocanu [7, p. 24].

Lemma 1. *Let $q \in \mathcal{Q}(a)$ and let the function $p(z)$ given by*

$$p(z) = a + a_n z^n + \cdots \qquad (n \geqq 1)$$

be analytic in $\mathbb{U}$ with $p(0) = a$. If p is not subordinate to q, then there exist points $z_0 \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U} \setminus E(q)$ for which

(i) $p(z_0) = q(\zeta_0)$ *and*

(ii) $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0) \qquad (m \geqq n \geqq 1)$.

Applying Lemma 1, we can obtain the following results.

Theorem 1. *Let $P : \mathbb{U} \to \mathbb{C}$ with*

$$\Re\{P(z)\} \geqq \Im\{P(z)\}\tan\alpha \geqq 0 \qquad \left(0 \leqq \alpha < \frac{\pi}{2}\right).$$

If the function p is an analytic in $\mathbb{U}$ with $p(0) = 1$ and

$$\Re\left\{[p(z)]^2 + P(z)zp'(z)\right\} > \frac{B^2\sin^2\alpha}{4A\cos^2\alpha} - \frac{B}{2\cos\alpha}, \tag{1}$$

where

$$A = \cos 2\alpha + \frac{B}{2\cos\alpha} \tag{2}$$

and

$$B = \Re\{P(z)\}\cos\alpha - \Im\{P(z)\}\sin\alpha, \tag{3}$$

then

$$|\arg\{p(z)\}| < \frac{\pi}{2} - \alpha \qquad \left(0 \leqq \alpha < \frac{\pi}{2};\ z \in \mathbb{U}\right).$$

Proof. Let us define two functions $q(z)$ and $h_1(z)$ by

$$q(z) = e^{i\alpha}p(z) \qquad \left(q(z) \not\equiv e^{i\alpha};\ 0 \leqq \alpha < \frac{\pi}{2};\ z \in \mathbb{U}\right) \tag{4}$$

and

$$h_1(z) = \frac{e^{i\alpha} + \overline{e^{i\alpha}}z}{1-z} \qquad \left(0 \leqq \alpha < \frac{\pi}{2};\ z \in \mathbb{U}\right), \tag{5}$$

respectively. Then the functions $q(z)$ and $h_1(z)$ are analytic in $\mathbb{U}$ with

$$q(0) = h_1(0) = e^{i\alpha} \in \mathbb{C} \qquad \text{and} \qquad h_1(\mathbb{U}) = \{w : w \in \mathbb{C} \quad \text{and} \quad \Re\{w\} > 0\}.$$

We now suppose that the function q is not subordinate to h_1. Then, by Lemma 1, there exist points $z_1 \in \mathbb{U}$ and $\zeta_1 \in \partial\mathbb{U} \setminus \{1\}$ such that

$$q(z_1) = h_1(\zeta_1) = i\rho \quad (\rho \in \mathbb{R}) \quad \text{and} \quad z_1 q'(z_1) = m\zeta_1 h_1'(\zeta_1) = m\sigma_1 \quad (m \geqq 1), \tag{6}$$

where

$$\sigma_1 = -\frac{\rho^2 - 2\rho\sin\alpha + 1}{2\cos\alpha}. \tag{7}$$

Using the equations (4), (5), (6) and (7), we obtain

$$\begin{aligned}
&\Re\left\{[p(z_1)]^2 + P(z_1)z_1p'(z_1)\right\} \\
&= \Re\left\{\left[e^{-i\alpha}q(z_1)\right]^2 + P(z_1)e^{-i\alpha}z_1q'(z_1)\right\} \\
&= \Re\left\{e^{-2i\alpha}[h_1(\zeta_1)]^2 + P(z_1)e^{-i\alpha}m\zeta_1h_1'(\zeta_1)\right\} \\
&= \Re\left\{e^{-2i\alpha}(i\rho)^2 + P(z_1)e^{-i\alpha}m\sigma_1\right\} \\
&= -\rho^2\cos 2\alpha + m\sigma_1B_1 \\
&\leqq -\left(\cos 2\alpha + \frac{B_1}{2\cos\alpha}\right)\rho^2 + \left(\frac{B_1\sin\alpha}{\cos\alpha}\right)\rho - \frac{B_1}{2\cos\alpha} \\
&= -A_1\rho^2 + \left(\frac{B_1\sin\alpha}{\cos\alpha}\right)\rho - \frac{B_1}{2\cos\alpha} \\
&=: g(\rho),
\end{aligned} \tag{8}$$

where B_1 and A_1 are given by

$$B_1 = \Re\{P(z_1)\}\cos\alpha - \Im\{P(z_1)\}\sin\alpha$$

and

$$A_1 = \cos 2\alpha + \frac{B_1}{2\cos\alpha},$$

respectively. By a simple calculation, we see that the function $g_1(\rho)$ in (8) takes on the maximum value at ρ^* given by

$$\rho^* = \frac{B_1 \sin\alpha}{2A_1\cos\alpha}.$$

Hence we have

$$\begin{aligned}
&\Re\left\{[p(z_1)]^2 + P(z_1)z_1p'(z_1)\right\}\\
&\leqq g_1(\rho^*)\\
&= \frac{B_1^2\sin^2\alpha}{4A_1\cos^2\alpha} - \frac{B_1}{2\cos\alpha}\\
&\leqq \frac{B^2\sin^2\alpha}{4A\cos^2\alpha} - \frac{B}{2\cos\alpha},
\end{aligned}$$

where A and B are given by (2) and (3), respectively. Moreover, this inequality is a contradiction to (1). Therefore, we obtain

$$\Re\left\{e^{i\alpha}p(z)\right\} > 0 \qquad \left(0 \leqq \alpha < \frac{\pi}{2};\ z \in \mathbb{U}\right). \tag{9}$$

Next, let us define two analytic functions by

$$r(z) = e^{-i\alpha}p(z) \qquad \left(0 \leqq \alpha < \frac{\pi}{2};\ z \in \mathbb{U}\right) \tag{10}$$

and

$$h_2(z) = \frac{e^{-i\alpha} + \overline{e^{-i\alpha}}z}{1-z} \qquad \left(0 \leqq \alpha < \frac{\pi}{2};\ z \in \mathbb{U}\right). \tag{11}$$

Then the functions r and h_2 are analytic in $\mathbb{U}$ with

$$r(0) = h_2(0) = e^{-i\alpha} \in \mathbb{C} \quad \text{and} \quad h_2(\mathbb{U}) = \{w : w \in \mathbb{C} \quad \text{and} \quad \Re\{w\} > 0\} = h_1(\mathbb{U}).$$

Suppose that r is not subordinate to h_2. Then, by Lemma 1, there exist points $z_2 \in \mathbb{U}$ and $\zeta_2 \in \partial\mathbb{U} \setminus \{1\}$ such that

$$r(z_2) = h_2(\zeta_2) = i\rho \quad (\rho \in \mathbb{R}) \qquad \text{and} \qquad z_2r'(z_2) = m\zeta_2h_2'(\zeta_2) = m\sigma_2 \ (m \geqq 1), \tag{12}$$

where

$$\sigma_2 = -\frac{\rho^2 + 2\rho\sin\alpha + 1}{2\cos\alpha}. \tag{13}$$

From the equations (10), (11), (12) and (13), we get

$$\begin{aligned}
&\Re\left\{[p(z_2)]^2 + P(z_2)z_2p'(z_2)\right\}\\
&\quad= \Re\left\{e^{2i\alpha}[h_2(\zeta_2)]^2 + P(z_2)e^{i\alpha}m\zeta_2h_2'(\zeta_2)\right\}\\
&\quad= \Re\left\{e^{2i\alpha}(i\rho)^2 + P(z_2)e^{i\alpha}m\sigma_2\right\}\\
&\quad= -\rho^2\cos 2\alpha + m\sigma_2 B\\
&\quad\leqq -\rho^2\cos 2\alpha + \sigma_2 B\\
&\quad= -A\rho^2 - \left(\frac{B\sin\alpha}{\cos\alpha}\right)\rho - \frac{B}{2\cos\alpha}\\
&\quad= g_2(\rho)\\
&\quad\leqq g_2\left(-\frac{B\sin\alpha}{2A\cos\alpha}\right)\\
&\quad= \frac{B^2\sin^2\alpha}{4A\cos^2\alpha} - \frac{B}{2\cos\alpha},
\end{aligned}$$

which is a contradiction to (1). Therefore, we have

$$\Re\left\{e^{-i\alpha}p(z)\right\} > 0 \qquad \left(0 \leqq \alpha < \frac{\pi}{2};\ z \in \mathbb{U}\right). \tag{14}$$

Hence, by applying the inequalities (9) and (14), we find that

$$|\arg\{p(z)\}| < \frac{\pi}{2} - \alpha \qquad \left(0 \leqq \alpha < \frac{\pi}{2};\ z \in \mathbb{U}\right).$$

This evidently complete the proof of Theorem 1. □

If we take $P(z) \equiv \beta$ $(\beta > 0)$ in Theorem 1, then we have the following corollary.

Corollary 1. *Let the function p be analytic in $\mathbb{U}$ with $p(0) = 1$. If*

$$\Re\left\{[p(z)]^2 + \beta z p'(z)\right\} > \frac{1}{2\beta + 4\cos 2\alpha}\left\{(\beta^2 + 4\beta)\sin^2\alpha - \beta^2 - 2\beta\right\}$$

$$\left(\beta > 0;\ 0 \leqq \alpha < \frac{\pi}{2};\ z \in \mathbb{U}\right),$$

then

$$|\arg\{p(z)\}| < \frac{\pi}{2} - \alpha \qquad \left(0 \leqq \alpha < \frac{\pi}{2};\ z \in \mathbb{U}\right).$$

More specially, if we take $P(z) \equiv 1$ in Theorem 1 or set $\beta = 1$ in Corollary 1, we obtain the following corollary.

Corollary 2. *Let the function p be analytic in $\mathbb{U}$ with $p(0) = 1$. If*

$$\Re\left\{[p(z)]^2 + z p'(z)\right\} > \frac{5\sin^2\alpha - 3}{6 - 8\sin^2\alpha} \qquad \left(0 \leqq \alpha < \frac{\pi}{2};\ z \in \mathbb{U}\right),$$

then

$$|\arg\{p(z)\}| < \frac{\pi}{2} - \alpha \qquad \left(0 \leqq \alpha < \frac{\pi}{2};\ z \in \mathbb{U}\right).$$

Taking $\alpha = 0$ in Corollary 2, we have the following corollary.

Corollary 3. *Let the function p be analytic in $\mathbb{U}$ with $p(0) = 1$. If*

$$\Re\left\{[p(z)]^2 + zp'(z)\right\} > -\frac{1}{2} \qquad (z \in \mathbb{U}),$$

then

$$\Re\{p(z)\} > 0 \qquad (z \in \mathbb{U}).$$

The following corollary presents a sufficient condition for starlikeness of analytic functions in $\mathbb{U}$. It follows easily by taking

$$p(z) = \frac{zf'(z)}{f(z)} \qquad (f \in \mathcal{A})$$

in Corollary 3.

Corollary 4. *Let $f \in \mathcal{A}$. Then*

$$\Re\left\{\frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > -\frac{1}{2} \qquad (z \in \mathbb{U})$$

implies that $f \in \mathcal{S}^*$.

3. Further Sufficient Conditions

We now find another another set of sufficient conditions for Carathéodory functions.

Theorem 2. *Let $p(z)$ be a nonzero analytic function in $\mathbb{U}$ with $p(0) = 1$ and*

$$\left|\frac{zp'(z)}{[p(z)]^2}\right| < \frac{1}{2}\cos\alpha \qquad \left(0 \leqq \alpha < \frac{\pi}{2};\ z \in \mathbb{U}\right). \tag{15}$$

Then

$$|\arg\{p(z)\}| < \frac{\pi}{2} - \alpha \qquad \left(0 \leqq \alpha < \frac{\pi}{2};\ z \in \mathbb{U}\right).$$

Proof. As before, we define the functions $q(z)$ and $h_1(z)$ by (4) and (5), respectively. We also suppose that q is not subordinate to h_1. Then, by Lemma 1, there exist points $z_1 \in \mathbb{U}$ and $\zeta_1 \in \partial\mathbb{U} \setminus \{1\}$ satisfying (6). We note that $\rho \neq 0$ in (6), since the function $p(z)$ cannot vanish in $\mathbb{U}$. Thus, from the equations (4), (5), (6) and (7), we obtain

$$\left|\frac{z_1p'(z_1)}{[p(z_1)]^2}\right| = \left|\frac{z_1q'(z_1)}{[q(z_1)]^2}\right| = \left|\frac{m\zeta_1h_1'(\zeta_1)}{[h_1(\zeta_1)]^2}\right| = \left|\frac{m\sigma_1}{(i\rho)^2}\right|.$$

We also have

$$\left|\frac{m\sigma_1}{(i\rho)^2}\right| = m\frac{|\sigma_1|}{\rho^2} \geqq -\frac{\sigma_1}{\rho^2} = \frac{1}{2\cos\alpha}g_1(\rho),$$

where

$$g_1(\rho) = \frac{\rho^2 - 2\rho\sin\alpha + 1}{\rho^2}.$$

6

For the case when $\alpha \neq 0$, since g_1 has its minimum at

$$\rho^* = \frac{1}{\sin\alpha},$$

we have

$$\left|\frac{z_1 p'(z_1)}{[p(z_1)]^2}\right| \geqq \frac{1}{2\cos\alpha} g_1(\rho^*) = \frac{1}{2}\cos\alpha,$$

which is a contradiction to (15). We thus have

$$q(z) \prec h_1(z) \qquad (z \in \mathbb{U})$$

or, equivalently,

$$\Re\left\{e^{i\alpha} p(z)\right\} > 0 \qquad (z \in \mathbb{U}). \tag{16}$$

We next define the functions r and h_2 by (10) and (11), respectively. By using a similar method as the above, we obtain

$$\Re\left\{e^{-i\alpha} p(z)\right\} > 0 \qquad (z \in \mathbb{U}) \tag{17}$$

for the case when $\alpha \neq 0$. Thus, from (16) and (17), we have

$$|\arg\{p(z)\}| < \frac{\pi}{2} - \alpha \qquad \left(0 < \alpha < \frac{\pi}{2};\ z \in \mathbb{U}\right).$$

For the case when $\alpha = 0$, we have

$$g_1(\rho) = 1 + \rho^{-2} \geqq 1 \qquad (\rho \in \mathbb{R} \setminus \{0\}).$$

We thus have

$$\left|\frac{z_1 p'(z_1)}{[p(z_1)]^2}\right| \geqq \frac{1}{2} g_1(\rho) \geqq \frac{1}{2},$$

which is also a contradiction to (15). Finally, we have

$$q(z) \prec h_1(z) \quad (z \in \mathbb{U})$$

or, equivalently,

$$|\arg\{p(z)\}| < \frac{\pi}{2}.$$

We thus find that

$$|\arg\{p(z)\}| < \frac{\pi}{2} - \alpha \qquad \left(0 \leqq \alpha < \frac{\pi}{2};\ z \in \mathbb{U}\right).$$

□

By setting

$$p(z) = \frac{zf'(z)}{f(z)} \quad (f \in \mathcal{A}) \qquad \text{and} \qquad \alpha = 0$$

in Theorem 2, we can deduce the following corollary.

Corollary 5. *Let $f \in \mathcal{A}$. Then*

$$\left|\frac{1 + \frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}} - 1\right| < \frac{1}{2} \qquad \left(0 \leqq \alpha < \frac{\pi}{2};\ z \in \mathbb{U}\right)$$

implies that $f \in \mathcal{S}^*$.

7

Theorem 3. *Let $\beta \in \mathbb{C}$ with $u := \Re\{\beta\} > 0$. Let p be a nonzero analytic function with $p(0) = 1$ and*

$$\delta_1(\alpha) < \Im\left\{p(z) + \beta\frac{zp'(z)}{p(z)}\right\} < \delta_2(\alpha) \qquad \left(0 \leqq \alpha < \frac{\pi}{2};\ z \in \mathbb{U}\right), \tag{18}$$

where

$$\delta_1(\alpha) = -\frac{\sqrt{(2\cos^2\alpha + u)u} + u\sin\alpha}{\cos\alpha}$$

and

$$\delta_2(\alpha) = \frac{\sqrt{(2\cos^2\alpha + u)u} - u\sin\alpha}{\cos\alpha}.$$

Then

$$|\arg\{p(z)\}| < \frac{\pi}{2} - \alpha \qquad \left(0 \leqq \alpha < \frac{\pi}{2};\ z \in \mathbb{U}\right).$$

Proof. We define the functions q and h_1 by (4) and (5), respectively. We also suppose that q is not subordinate to h_1. Then, by Lemma 1, there exist points $z_1 \in \mathbb{U}$ and $\zeta_1 \in \partial\mathbb{U}\setminus\{1\}$ satisfying (6). We also have $\rho \neq 0$ in (6). Thus, from the equations (4), (5), (6) and (7), we have

$$\begin{aligned}
\Im\left\{p(z_1) + \beta\frac{z_1p'(z_1)}{p(z_1)}\right\} &= \Im\left\{e^{-i\alpha}q(z_1) + \beta\frac{z_1q'(z_1)}{q(z_1)}\right\} \\
&= \Im\left\{e^{-i\alpha}h(\zeta_1) + \beta\frac{m\zeta_1h'(\zeta_1)}{h(\zeta_1)}\right\} \\
&= \Im\left\{e^{-i\alpha}(i\rho) + \beta\frac{m\sigma_1}{i\rho}\right\} \\
&= \rho\cos\alpha - \frac{m\sigma_1u}{\rho},
\end{aligned}$$

where $u = \Re\{\beta\}$ and σ_1 is given by (7). For the case when $\rho > 0$, we have

$$\begin{aligned}
&\rho\cos\alpha - \frac{m\sigma_1u}{\rho} \\
&\quad \geqq \rho\cos\alpha - \frac{\sigma_1u}{\rho} \\
&\quad = \rho\cos\alpha + \frac{u(\rho^2 - 2\rho\sin\alpha + 1)}{2\rho\cos\alpha} \\
&\quad = \frac{1}{2\cos\alpha}\left\{(2\cos^2\alpha + u)\rho + \frac{u}{\rho} - 2u\sin\alpha\right\} \\
&\quad \geqq \frac{1}{2\cos\alpha}\left\{2\sqrt{(2\cos^2\alpha + u)u} - 2u\sin\alpha\right\} \\
&\quad = \delta_2(\alpha).
\end{aligned}$$

Therefore, we have

$$\Im\left\{p(z_1) + \beta\frac{z_1p'(z_1)}{p(z_1)}\right\} \geqq \delta_2(\alpha),$$

which is a contradiction to (18). For the case when $\rho < 0$, we put

$$\tilde{\rho} = -\rho > 0.$$

Then, by using the same method as the above, we obtain

$$\begin{aligned}
&\rho\cos\alpha - \frac{m\sigma_1 u}{\rho} \\
&\quad \leqq -\tilde{\rho}\cos\alpha + \frac{\sigma_1 u}{\tilde{\rho}} \\
&\quad = -\frac{1}{2\cos\alpha}\left\{(2\cos^2\alpha + u)\tilde{\rho} + \frac{u}{\tilde{\rho}} + 2u\sin\alpha\right\} \\
&\quad \leqq -\frac{1}{2\cos\alpha}\left\{2\sqrt{(2\cos^2\alpha + u)u} + 2u\sin\alpha\right\} \\
&\quad = \delta_1(\alpha).
\end{aligned}$$

Moreover, this last inequality yields

$$\Im\left\{p(z_1) + \beta\frac{z_1 p'(z_1)}{p(z_1)}\right\} \leqq \delta_1(\alpha),$$

which is a contradiction to (18). Hence we have

$$\Re\left\{e^{i\alpha}p(z)\right\} > 0 \qquad \left(0 \leqq \alpha < \frac{\pi}{2};\ z \in \mathbb{U}\right). \tag{19}$$

We next define the functions r and h_2 by (10) and (11), respectively. Then, by using a similar method as the above, we obtain

$$\Re\left\{e^{-i\alpha}p(z)\right\} > 0 \qquad \left(0 \leqq \alpha < \frac{\pi}{2};\ z \in \mathbb{U}\right). \tag{20}$$

Thus, from (19) and (20), we have

$$|\arg\{p(z)\}| < \frac{\pi}{2} - \alpha \qquad \left(0 \leqq \alpha < \frac{\pi}{2};\ z \in \mathbb{U}\right).$$

The proof of Theorem 3 is thus completed. □

Remark 1. If we put $\beta = 1$ in Theorem 3, then we can obtain the result given earlier by Kim and Cho [4, Theorem 2].

By setting

$$p(z) = \frac{zf'(z)}{f(z)} \quad (f \in \mathcal{A}) \qquad \text{and} \qquad \alpha = 0$$

in Theorem 3, we can deduce the following corollary.

Corollary 6. *Let $\beta \in \mathbb{C}$ with $u := \Re\{\beta\} > 0$ and let $f \in \mathcal{A}$. Then*

$$\left|\Im\left\{(1-\beta)\frac{zf'(z)}{f(z)} + \beta\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\}\right| < \sqrt{u^2 + 2u} \qquad (z \in \mathbb{U})$$

implies that $f \in \mathcal{S}^*$.

9

Theorem 4. *Let $\gamma \in \mathbb{R}$ with $\gamma > 0$. Let p be a nonzero analytic function with $p(0) = 1$ and*

$$\left| p(z) + \gamma \frac{zp'(z)}{p(z)} - 1 \right| < \left(\frac{\gamma}{2} + 1 \right) |p(z)| \cos \alpha \qquad \left(0 \leqq \alpha < \frac{\pi}{2};\ z \in \mathbb{U} \right). \tag{21}$$

Then

$$|\arg\{p(z)\}| < \frac{\pi}{2} - \alpha \qquad \left(0 \leqq \alpha < \frac{\pi}{2};\ z \in \mathbb{U} \right).$$

Proof. Let

$$q(z) = \frac{\mathrm{e}^{i\alpha}}{p(z)} \qquad \left(0 \leqq \alpha < \frac{\pi}{2};\ z \in \mathbb{U} \right).$$

Also let the function h_1 be defined by (5). If the function q is not subordinate to h_1, then there exist points $z_1 \in \mathbb{U}$ and $\zeta_1 \in \partial\mathbb{U} \setminus \{1\}$ satisfying (4). By using the equations (4), (5), (6) and (7), we have

$$\begin{aligned}
&\frac{|p(z_1) + \gamma \frac{z_1 p'(z_1)}{p(z_1)} - 1|}{|p(z_1)|} \\
&\quad = |\mathrm{e}^{-i\alpha} q(z_1) + \mathrm{e}^{-i\alpha} \gamma z_1 q'(z_1) - 1| \\
&\quad = |h(\zeta_1) + m\gamma\zeta_1 h_1'(\zeta_1) - \mathrm{e}^{i\alpha}| \\
&\quad = |i\rho + m\gamma\sigma_1 - \mathrm{e}^{i\alpha}| \\
&\quad = \sqrt{(m\gamma\sigma_1 - \cos\alpha)^2 + (\rho - \sin\alpha)^2} \\
&\quad \geqq \sqrt{(|\sigma_1|\gamma + \cos\alpha)^2 + (\rho - \sin\alpha)^2} \\
&\quad = \sqrt{\left(\frac{\gamma}{2\cos\alpha}(\rho - \sin\alpha)^2 + \frac{1}{2}\gamma\cos\alpha + \cos\alpha \right)^2 + (\rho - \sin\alpha)^2} \\
&\quad \geqq \left(\frac{\gamma}{2} + 1 \right) \cos\alpha.
\end{aligned}$$

We thus find that

$$\left| p(z_1) + \gamma \frac{z_1 p'(z_1)}{p(z_1)} - 1 \right| \geqq \left(\frac{\gamma}{2} + 1 \right) |p(z_1)| \cos\alpha,$$

which is a contradiction to (21). Therefore, we have

$$q(z) \prec h_1(z) \quad (z \in \mathbb{U}),$$

that is,

$$\Re\left\{ \frac{\mathrm{e}^{i\alpha}}{p(z)} \right\} > 0 \qquad \left(0 \leqq \alpha < \frac{\pi}{2};\ z \in \mathbb{U} \right). \tag{22}$$

We next consider the function $r(z)$ defined by

$$r(z) = \frac{\mathrm{e}^{-i\alpha}}{p(z)} \qquad \left(0 \leqq \alpha < \frac{\pi}{2};\ z \in \mathbb{U} \right)$$

and the function h_2 defined by (11). Using a similar method as the above, we obtain

$$\Re\left\{ \frac{\mathrm{e}^{-i\alpha}}{p(z)} \right\} > 0 \qquad \left(0 \leqq \alpha < \frac{\pi}{2};\ z \in \mathbb{U} \right). \tag{23}$$

Therefore, by (22) and (23), we have the assertion of Theorem 4. □

Remark 2. If we put $\gamma = 1$ in Theorem 4, then we can obtain the result proven earlier by Kim and Cho [4, Theorem 3].

If we take

$$p(z) = \frac{zf'(z)}{f(z)} \quad (f \in \mathcal{A}) \qquad \text{and} \qquad \alpha = 0$$

in Theorem 4, we obtain the following result.

Corollary 7. *Let $\gamma \in \mathbb{R}$ with $\gamma > 0$ and let $f \in \mathcal{A}$ with*

$$\frac{f(z)}{z} \neq 0 \qquad (z \in \mathbb{U}).$$

Then the following inequality:

$$\left|(1-\gamma)\frac{zf'(z)}{f(z)} + \gamma\left(1 + \frac{zf''(z)}{f'(z)}\right) - 1\right| < \left(\frac{\gamma}{2} + 1\right)\left|\frac{zf'(z)}{f(z)}\right|$$

implies that $f \in \mathcal{S}^*$.

Acknowledgements

This work was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology of the Republic of Korea (No. 2011-0007037).

References

[1] A. A. Attiya and M. A. Nasr, On sufficient conditions for Carathéodory functions with applications, *J. Inequal. Appl.* **191** (2003), 1–10.

[2] N. E. Cho and I. H. Kim, Conditions for Carathéodory functions, *J. Inequal. Appl.* **2009** (2009), Article ID 601597, 1–6.

[3] P. Delsarte and Y. Genin, A simple proof of Livingston's inequality for Carathéodory functions, *Proc. Amer. Math. Soc.* **107** (1989), 1017–1020.

[4] I. H. Kim and N. E. Cho, Sufficient conditions for Carathéodory functions, *Comput. Math. Appl.* **59** (2010), 2067–2073.

[5] J.-L. Li and S. Owa, Sufficient conditions for starlikeness, *Indian J. Pure Appl. Math.* **33** (2002), 313–318.

[6] S. Miller, Differential inequalities and Carathéodory functions, *Bull. Amer. Math. Soc.* **81** (1975), 79–81.

[7] S. S. Miller and P. T. Mocanu, *Differential Subordination: Theory and Applications*, Series on Monographs and Textbooks in Pure and Applied Mathematics, No. **225**, Marcel Dekker Incorporated, New York and Basel, 2000.

[8] M. Nunokawa, Differential inequalities and Carathéodory functions, *Proc. Japan Acad. Ser. A Math. Sci.* **65** (1989), 326–328.

[9] M. Nunokawa, On properties of non-Carathéodory functions, *Proc. Japan Acad. Ser. A Math. Sci.* **68** (1992), 152–153.

[10] M. Nunokawa, A. Ikeda, N. Koike, Y. Ota and H. Saitoh, Differential inequalities and Carathéodory functions, *J. Math. Anal. Appl.* **212** (1997), 324–332.

[11] M. Nunokawa, S. Owa, J. Nishiwaki and H. Saitoh, Sufficient conditions for starlikeness and convexity of analytic functions with real coefficients, *Southeast Asian Bull. Math.* **33** (2008), 1149–1155.

[12] M. Nunokawa, S. Owa, N. Takahashi and H. Saitoh, Sufficient conditions for Carathéodory functions, *Indian J. Pure Appl. Math.* **33** (2002), 1385–1390.

[13] M. Obradović and S. Owa, On certain properties for some classes of starlike functions, *J. Math. Anal. Appl.* **145** (1990), 357–364.

[14] K. S. Padmanabhan, On sufficient conditions for starlikeness, *Indian J. Pure Appl. Math.* **32** (1990), 543–550.

[15] V. Ravichandran, M. Darus and N. Seenivasagan, On a criteria for strong starlikeness, *Austral. J. Math. Anal. Appl.* **2** (1) (2005), Article ID 6, 1–12.

[16] H. Shiraishi, S. Owa and H. M. Srivastava, Sufficient conditions for strongly Carathéodory functions, *Comput. Math. Appl.* **62** (2011), 2978–2987.

[17] N. Tuneski, On certain sufficient conditions for starlikeness, *Internat. J. Math. Math. Sci.* **23** (2000), 521–527.

[18] D. Yang, S. Owa and K. Ochiai, Sufficient conditions for Carathéodory functions, *Comput. Math. Appl.* **51** (2006), 467–474.

TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 21, NO. 7, 2016