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On structures of IVF approximation spaces *

Ningxin Xie[†]

March 2, 2015

Abstract: Rough set theory is a powerful mathematical tool for dealing with inexact, uncertain or vague information. An IVF rough set, which is the result of approximation of an IVF set with respect to an IVF approximation space, is an extension of fuzzy rough sets. In this paper, properties of IVF rough approximation operators and construction of IVF rough sets are investigated. Topological and lattice structures of IVF approximation spaces are given.

Keywords: IVF set; IVF relation; IVF approximate space; IVF rough set; Topology; Lattice.

1 Introduction

Rough set theory was proposed by Pawlak [14] as a mathematical tool to handle imprecision and uncertainty in data analysis. It has been successfully applied to machine learning, intelligent systems, inductive reasoning, pattern recognition, mereology, image processing, signal analysis, knowledge discovery, decision analysis, expert systems and many other fields [15, 16, 17, 18]. The foundation of its object classification is an equivalence relation. The upper and lower approximation operations are two core notions of this theory. They can also be seen as the closure operator and the interior operator of the topology induced by an equivalence relation on the universe, respectively. In the real world, the equivalence relation is, however, too restrictive for many practical applications. To address this issue, many interesting and meaningful extensions of Pawlak's rough sets have been presented in the literature. Equivalence relations can be replaced by tolerance relations [21], similarity relations [22], binary relations [10, 25].

By replacing crisp relations with fuzzy relations, various fuzzy generalizations of rough approximations have been proposed [1, 3, 9, 13, 19, 24, 28]. Dubois [3] first proposed the concept of rough fuzzy set and fuzzy rough set.

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Fuzzy rough set have been applied to solve a lot of practical problems. For example, medical time series, neural networks, case generation and descriptive dimensionality reduction.

As a generalization of Zadeh's fuzzy set, interval-valued fuzzy (IVF, for short) sets were introduced by Gorzalczany [5] and Turksen [23], and they were applied to the fields of approximate inference, signal transmission and controller, etc. Mondal et al. [12] defined IVF topologies and studied their properties.

By integrating Pawlak rough set theory with IVF set theory, Sun et al. [20] introduced IVF rough sets based on an IVF approximation space, defined IVF information systems and discussed their attribute reduction. Gong et al. [6] presented IVF rough sets based on approximation spaces, studied the knowledge discovery in IVF information systems. However, structures of IVF rough sets have not been deeply studied.

Topologies are widely used in the research field of machine learning and cybernetics. For example, Koretelainen [7, 8] used topologies to detect dependencies of attributes in information systems with respect to gradual rules. Choudhury et al. [2] applied topology to study the evolutionary impact of learning on social problems. Topological structures are the most powerful notions and are important bases in data and system analysis.

Lattices and ordered sets play an important role in many areas of computer science. These range from lattices as models for logics, which are fundamental to understanding computation, to the ordered sets as models for computation, to the role both lattices and ordered sets play in combinatorics, a fundamental aspect of computation. Some researchers investigated relationships between rough sets and lattices. For example, Yang et al. [26] studied lattice structures in generalized approximation spaces. Estaji et al. [4] considered rough set theory applied to lattice theory. Zheng al. [4] investigated topological structures in IVF approximation spaces where the universe may be infinite.

The purpose of this paper is to investigate construction of IVF rough sets and topological or lattice structures of IVF approximation spaces.

2 Preliminaries

Throughout this paper, "interval-valued fuzzy" denotes briefly by "IVF". U denotes a nonempty finite set called the universe of discourse. I denotes $[0, 1]$ and $[I]$ denotes $\{[a, b] : a, b \in I \text{ and } a \leq b\}$. $F^{(i)}(U)$ denotes the family of all IVF sets in U . \bar{a} denotes $[a, a]$ for each $a \in [0, 1]$.

2.1 IVF sets

For any $[a_j, b_j] \in [I]$ ($j = 1, 2$), we define

$$[a_1, b_1] = [a_2, b_2] \iff a_1 = a_2, b_1 = b_2;$$

$$[a_1, b_1] \leq [a_2, b_2] \iff a_1 \leq a_2, b_1 \leq b_2;$$

$$[a_1, b_1] < [a_2, b_2] \iff [a_1, b_1] \leq [a_2, b_2] \text{ and } [a_1, b_1] \neq [a_2, b_2];$$

$$\bar{1} - [a_1, b_1] \text{ or } [a_1, b_1]^c = [1 - b_1, 1 - a_1].$$

Obviously, $([a, b]^c)^c = [a, b]$ for each $[a, b] \in [I]$.

Definition 2.1 ([5, 23]). For each $\{[a_j, b_j] : j \in J\} \subseteq [I]$, we define

$$\bigvee_{j \in J} [a_j, b_j] = [\bigvee_{j \in J} a_j, \bigvee_{j \in J} b_j] \text{ and } \bigwedge_{j \in J} [a_j, b_j] = [\bigwedge_{j \in J} a_j, \bigwedge_{j \in J} b_j],$$

where $\bigvee_{j \in J} a_j = \sup \{a_j : j \in J\}$ and $\bigwedge_{j \in J} a_j = \inf \{a_j : j \in J\}$.

Definition 2.2 ([5, 23]). An IVF set A in U is defined by a mapping $A : U \rightarrow [I]$. Denote

$$A(x) = [A^-(x), A^+(x)] \quad (x \in U).$$

Then $A^-(x)$ (resp. $A^+(x)$) is called the lower (resp. upper) degree to which x belongs to A . A^- (resp. A^+) is called the lower (resp. upper) IVF set of A .

The set of all IVF sets in U is denoted by $F^{(i)}(U)$.

Let $a, b \in I$. $\widetilde{[a, b]}$ represents the IVF set which satisfies $\widetilde{[a, b]}(x) = [a, b]$ for each $x \in U$. We denoted $\widetilde{[a, a]}$ by \tilde{a} .

We recall some basic operations on $F^{(i)}(U)$ as follows ([5, 23]): for any $A, B \in F^{(i)}(U)$ and $[a, b] \in [I]$,

- (1) $A = B \iff A(x) = B(x)$ for each $x \in U$.
- (2) $A \subseteq B \iff A(x) \leq B(x)$ for each $x \in U$.
- (3) $A = B^c \iff A(x) = B(x)^c$ for each $x \in U$.
- (4) $(A \cap B)(x) = A(x) \wedge B(x)$ for each $x \in U$.
- (5) $(A \cup B)(x) = A(x) \vee B(x)$ for each $x \in U$.
- (6) $([a, b]A)(x) = [a, b] \wedge [A^-(x), A^+(x)]$ for each $x \in U$.

Obviously,

$$A = B \iff A^- = B^- \text{ and } A^+ = B^+ ; (\widetilde{[a, b]})^c = \widetilde{[a, b]^c} \quad ([a, b] \in [I]).$$

Definition 2.3 ([12]). $A \in F^{(i)}(U)$ is called an IVF point in U , if there exist $[a, b] \in [I] - \{\bar{0}\}$ and $x \in U$ such that

$$A(y) = \begin{cases} [a, b], & y = x, \\ \bar{0}, & y \neq x. \end{cases}$$

We denote A by $x_{[a, b]}$.

Remark 2.4. $A = \bigcup_{x \in U} (A(x)x_{\bar{1}})$ ($A \in F^{(i)}(U)$).

2.2 IVF topologies

Definition 2.5 ([12]). $\tau \subseteq F^{(i)}(U)$ is called an IVF topology on U , if

- (i) $\tilde{0}, \tilde{1} \in \tau$,
- (ii) $A, B \in \tau \implies A \cap B \in \tau$,
- (iii) $\{A_j : j \in J\} \subseteq \tau \implies \bigcup_{j \in J} A_j \in \tau$.

The pair (U, τ) is called an IVF topological space. Every member of τ is called an IVF open set in U . Its complement is called an IVF closed set in U .

An IVF topology τ is called Alexandrov, if (ii) in Definition 2.5 is replaced by

- (ii)' $\{A_j : j \in J\} \subseteq \tau \implies \bigcap_{j \in J} A_j \in \tau$.

We denote $\tau^c = \{A : A^c \in \tau\}$.

The interior and closure of $A \in F^{(i)}(U)$ denoted respectively by $int(A)$ and $cl(A)$, are defined as follows:

$$int(A) \text{ or } int_{\tau}(A) = \bigcup \{B \in \tau : B \subseteq A\}, \quad cl(A) \text{ or } cl_{\tau}(A) = \bigcap \{B \in \tau^c : B \supseteq A\}.$$

Proposition 2.6 ([12]). Let τ be an IVF topology on U . Then for any $A, B \in F^{(i)}(U)$,

- (1) $int(\tilde{1}) = \tilde{1}, cl(\tilde{0}) = \tilde{0}$.
- (2) $int(A) \subseteq A \subseteq cl(A)$.
- (3) $A \subseteq B \implies int(A) \subseteq int(B), cl(A) \subseteq cl(B)$.
- (4) $int(A^c) = (cl(A))^c, cl(A^c) = (int(A))^c$.
- (5) $int(A \cap B) = int(A) \cap int(B), cl(A \cup B) = cl(A) \cup cl(B)$.
- (6) $int(int(A)) = int(A), cl(cl(A)) = cl(A)$.

3 Construction of IVF rough sets

3.1 IVF rough sets and IVF rough approximation operators

Recall that R is called an IVF relation on U if $R \in F^{(i)}(U \times U)$.

Definition 3.1 ([20]). Let R be an IVF relation on U . Then R is called

- (1) serial, if $\bigvee_{y \in U} R(x, y) = \tilde{1}$ for each $x \in U$.
- (2) reflexive, if $R(x, x) = \tilde{1}$ for each $x \in U$.
- (3) symmetric, if $R(x, y) = R(y, x)$ for any $x, y \in U$.
- (4) transitive, if $R(x, z) \geq R(x, y) \wedge R(y, z)$ for any $x, y, z \in U$.
- (5) Euclidian, if $R(x, z) \geq R(y, x) \wedge R(y, z)$ for any $x, y, z \in U$.

Let R be an IVF relation on U . R^{-1} is called the inverse relation of R if $R^{-1}(x, y) = R(y, x)$ for each $(x, y) \in U \times U$. R is called preorder if R is reflexive and transitive (see [10]).

Definition 3.2 ([20]). Let R be an IVF relation on U . The pair (U, R) is called an IVF approximation space. For each $A \in F^{(i)}(U)$, the IVF lower and the IVF upper approximation of A with respect to (U, R) , denoted by $\underline{R}(A)$ and $\overline{R}(A)$, are two IVF sets and are respectively defined as follows:

$$\underline{R}(A)(x) = \bigwedge_{y \in U} (A(y) \vee (\bar{1} - R(x, y))), \quad \overline{R}(A)(x) = \bigvee_{y \in U} (A(y) \wedge R(x, y)) \quad (x \in U).$$

The pair $(\underline{R}(A), \overline{R}(A))$ is called the IVF rough set of A with respect to (U, R) .

$\underline{R} : F^{(i)}(U) \rightarrow F^{(i)}(U)$ and $\overline{R} : F^{(i)}(U) \rightarrow F^{(i)}(U)$ are called the IVF lower approximation operator and the IVF upper approximation operator, respectively.

Remark 3.3. Let (U, R) be an IVF approximation space. Then

- (1) For each $x, y \in U$,

$$\overline{R}(x_{\bar{1}})(y) = R(y, x) \quad \text{and} \quad \underline{R}((x_{\bar{1}})^c)(y) = \bar{1} - R(y, x).$$

- (2) For each $[a, b] \in [I]$, $\underline{R}(\widetilde{[a, b]}) \supseteq \widetilde{[a, b]} \supseteq \overline{R}(\widetilde{[a, b]})$.

Proposition 3.4 ([20]). Let (U, R) be an IVF approximation space. Then for each $A \in F^{(i)}(U)$,

$$\begin{aligned} (\underline{R}(A))^- &= \underline{R}^+(A^-), \quad (\underline{R}(A))^+ = \underline{R}^-(A^+), \\ (\overline{R}(A))^- &= \overline{R}^-(A^-) \quad \text{and} \quad (\overline{R}(A))^+ = \overline{R}^+(A^+). \end{aligned}$$

3.2 Properties of IVF rough approximation operators

Theorem 3.5 ([27]). Let (U, R) be an IVF approximation space. Then for any $A, B \in F^{(i)}(U)$, $\{A_j : j \in J\} \subseteq F^{(i)}(U)$ and $[a, b] \in [I]$,

- (1) $\underline{R}(\bar{1}) = \bar{1}$, $\overline{R}(\bar{0}) = \bar{0}$.
- (2) $A \subseteq B \implies \underline{R}(A) \subseteq \underline{R}(B)$, $\overline{R}(A) \subseteq \overline{R}(B)$.
- (3) $\underline{R}(A^c) = (\overline{R}(A))^c$, $\overline{R}(A^c) = (\underline{R}(A))^c$.
- (4) $\underline{R}(\bigcap_{j \in J} A_j) = \bigcap_{j \in J} \underline{R}(A_j)$, $\overline{R}(\bigcup_{j \in J} A_j) = \bigcup_{j \in J} \overline{R}(A_j)$.
- (5) $\underline{R}(\widetilde{[a, b]} \cup A) = \widetilde{[a, b]} \cup \underline{R}(A)$, $\overline{R}(\widetilde{[a, b]} \cup A) = \widetilde{[a, b]} \cup \overline{R}(A)$.

Theorem 3.6 ([27]). Let R be an IVF relation on U and let τ be an IVF topology on U . If one of the following conditions is satisfied, then R is preorder.

- (1) \underline{R} is the interior operator of τ .
- (2) \overline{R} is the closure operator of τ .

Theorem 3.7. *Let (U, R) be an IVF approximation space. Then*

- (1) R is serial $\iff (ILS^*) \forall [a, b] \in [I], \underline{R}(\widetilde{[a, b]}) = \widetilde{[a, b]}$.
 $\iff (IUS^*) \forall [a, b] \in [I], \overline{R}(\widetilde{[a, b]}) = \widetilde{[a, b]}$.
 $\iff (IUS^{**}) \overline{R}(\widetilde{1}) = \widetilde{1}$.
 $\iff (ILS^{**}) \underline{R}(\widetilde{1}) = \widetilde{1}$.
- (2) R is reflexive $\iff (ILR) \forall A \in F^{(i)}(U), \underline{R}(A) \subseteq A$.
 $\iff (IUR) \forall A \in F^{(i)}(U), A \subseteq \overline{R}(A)$.
- (3) R is symmetric $\iff (ILS) \forall (x, y) \in U \times U, \underline{R}((x_{\bar{1}})^c)(y) = \underline{R}((y_{\bar{1}})^c)(x)$.
 $\iff (IUS) \forall (x, y) \in U \times U, \overline{R}(x_{\bar{1}})(y) = \overline{R}(y_{\bar{1}})(x)$.
- (4) R is transitive $\iff (ILT) \forall A \in F^{(i)}(U), \underline{R}(A) \subseteq \underline{R}(\underline{R}(A))$.
 $\iff (IUT) \forall A \in F^{(i)}(U), \overline{R}(\overline{R}(A)) \subseteq \overline{R}(A)$.

Proof. (1) By Theorem 3.5(3), (ILS^*) and (IUS^*) are equivalent, (ILS^{**}) and (IUS^{**}) are equivalent. We only need to prove that the serialisation of R is equivalent to (IUS^*) or (IUS^{**}) .

For any $[a, b] \in [I]$ and $x \in U$, we have

$$\overline{R}(\widetilde{[a, b]})(x) = \bigvee_{y \in U} ([a, b] \wedge R(x, y)) = [a, b] \wedge (\bigvee_{y \in U} R(x, y)) \quad (\star).$$

Assume that R is serial. Then for each $x \in U$, $\bigvee_{y \in U} R(x, y) = \bar{1}$. By (\star) ,

$$\overline{R}(\widetilde{[a, b]})(x) = [a, b]. \text{ Thus } \overline{R}(\widetilde{[a, b]}) = \widetilde{[a, b]} \text{ and so } \overline{R}(\widetilde{1}) = \widetilde{1}.$$

Assume $\overline{R}(\widetilde{[a, b]}) = \widetilde{[a, b]}$ for each $[a, b] \in [I]$. For each $x \in U$, then $\underline{R}(\widetilde{[a, b]})(x) = [a, b]$. By (\star) , $\bigvee_{y \in U} R(x, y) \geq [a, b]$. Put $[a, b] = \bar{1}$, then $\bigvee_{y \in U} R(x, y) \geq \bar{1}$. Hence $\bigvee_{y \in U} R(x, y) = \bar{1}$. So R is serial.

Assume that $\overline{R}(\widetilde{1}) = \widetilde{1}$. For each $x \in U$, by (\star) , $\bigvee_{y \in U} R(x, y) = \bar{1}$. So R is serial.

(2), (3) and (4) hold by Theorem 13 in [27]. □

Corollary 3.8 ([27]). *Let (U, R) be an IVF approximation space. If R is pre-order, then*

$$\underline{R}(\underline{R}(A)) = \underline{R}(A) \text{ and } \overline{R}(\overline{R}(A)) = \overline{R}(A) \quad (A \in F^{(i)}(U)).$$

3.3 Lower and upper sets in IVF approximation spaces

Definition 3.9. *Let (U, R) be an IVF approximation space.*

(1) $A \in F^{(i)}(U)$ is called an upper set if $A(x) \wedge R(x, y) \leq A(y)$ for any $x, y \in U$.

(2) $A \in F^{(i)}(U)$ is called a lower set if $A(y) \wedge R(x, y) \leq A(x)$ for any $x, y \in U$.

Proposition 3.10. *Let (U, R) be an IVF approximation space. Then the following are equivalent.*

- (1) $\overline{R}(A) \subseteq A$;
- (2) A is a lower set in (U, R) ;
- (3) A is an upper set in (U, R^{-1}) .

Proof. (1) \implies (2). Suppose that $\overline{R}(A) \subseteq A$. Since for each $x \in U$,

$$\bigvee_{y \in U} (A(y) \wedge R(x, y)) = \overline{R}(A)(x) \leq A(x),$$

$$A(y) \wedge R(x, y) \leq A(x) \quad (x, y \in U).$$

Then A is a lower set in (U, R) .

(2) \implies (3). This is obvious.

(3) \implies (1). Suppose that A is an upper set in (U, R^{-1}) . Then for any $x, y \in U$, $A(x) \wedge R^{-1}(x, y) \leq A(y)$. So $A(x) \wedge R(y, x) \leq A(y)$. Thus

$$\overline{R}(A)(y) = \bigvee_{x \in U} (A(x) \wedge R(y, x)) \leq A(y) \quad (y \in U).$$

Hence $\overline{R}(A) \subseteq A$. □

Corollary 3.11. *Let (U, R) be an IVF approximation space. If R is reflexive, then the following are equivalent.*

- (1) $\overline{R}(A) = A$;
- (2) A is a lower set in (U, R) ;
- (3) A is an upper set in (U, R^{-1}) .

Proof. This holds by Theorem 3.7(2) and Proposition 3.10. □

Let R be an IVF relation on U . For each $z \in U$, we define IVF sets $[z]^R : U \rightarrow [I]$, $[z]^R(x) = R(z, x)$ and $[z]_R : U \rightarrow [I]$, $[z]_R(x) = R(x, z)$.

Theorem 3.12. *Let (U, R) be an IVF approximation space. Then*

- (1) R is reflexive $\iff (ILS') \forall x \in U, [x]_R(x) = \bar{1}$.
 $\iff (IUR') \forall x \in U, [x]^R(x) = \bar{1}$.
- (2) R is symmetric $\iff (ILS') \forall x \in U, [x]_R = [x]^R$.
 $\implies \forall A \in F^{(i)}(U), A$ is a lower set if and only if
 A is an upper set.
- (3) R is transitive $\iff (ILT') \forall x \in U, [x]_R$ is a lower set.
 $\iff (IUT') \forall x \in U, [x]^R$ is an upper set.
 $\iff (IUT'') \forall A \in F^{(i)}(U), \overline{R}(A)$ is a lower set.
- (4) R is Euclidian $\iff (ILE) \forall x \in U, [x]_R$ is an upper set.

Proof. (1) and (2) are obvious.

(3) (IUT'')

$$\iff \forall A \in F^{(i)}(U), \overline{R}(\overline{R}(A)) \subseteq \overline{R}(A). \quad (\text{Proposition 3.10})$$

$$\iff R \text{ is transitive.} \quad (\text{Theorem 3.7(4)})$$

$$\iff \forall x, y, z \in U, R(x, y) \wedge R(y, z) \leq R(x, z).$$

$$\iff \forall x, y, z \in U, [z]_R(y) \wedge R(x, y) \leq [z]_R(x).$$

$$\iff (ILT^l) \forall x \in U, [x]_R \text{ is a lower set.}$$

$$\iff \forall x, y, z \in U, [x]^R(y) \wedge R(y, z) \leq [x]^R(z).$$

$$\iff (IUT') \forall x \in U, [x]^R \text{ is an upper set.}$$

(4) The proof is similar to (3). □

3.4 IVF rough equal relations

Definition 3.13. Let (U, R) be an IVF approximation space. Then for any $A, B \in F^{(i)}(U)$,

(1) If $\underline{R}(A) = \underline{R}(B)$, then A and B are called IVF lower rough equal. We denote it by $A \approx B$.

(2) If $\overline{R}(A) = \overline{R}(B)$, then A and B are called IVF upper rough equal. We denote it by $A \simeq B$.

(3) If $\underline{R}(A) = \underline{R}(B)$ and $\overline{R}(A) = \overline{R}(B)$, then A and B are called IVF rough equal. We denote it by $A \sim B$.

Proposition 3.14. Let (U, R) be an IVF approximation space. Then for any $A, B, C, D \in F^{(i)}(U)$,

(1) $A \approx B \iff (A \cap B) \approx A, (A \cap B) \approx B.$

(2) $A \simeq B \iff (A \cup B) \simeq A, (A \cup B) \simeq B.$

(3) $A \approx B, C \approx D \implies (A \cap B) \approx (C \cap D), (A \cup B) \approx (C \cup D);$

$A \simeq B, C \simeq D \implies (A \cap B) \simeq (C \cap D), (A \cup B) \simeq (C \cup D).$

(4) $A \approx \tilde{0}$ or $B \approx \tilde{0} \implies (A \cap B) \approx \tilde{0};$

$A \simeq \tilde{1}$ or $B \simeq \tilde{1} \implies (A \cup B) \simeq \tilde{1}.$

(5) $A \subseteq B, B \approx \tilde{0} \implies A \approx \tilde{0};$

$A \subseteq B, A \simeq \tilde{1} \implies B \simeq \tilde{1}.$

(6) If R is reflexive, then

a) $A \approx \tilde{1} \iff A = \tilde{1};$ b) $A \simeq \tilde{0} \iff A = \tilde{0}.$

Proof. (1) Let $A \approx B$. Then $\underline{R}(A) = \underline{R}(B)$. By Theorem 3.5(4),

$$\underline{R}(A \cap B) = \underline{R}(A) \cap \underline{R}(B) = \underline{R}(A) = \underline{R}(B).$$

Hence $(A \cap B) \approx A, (A \cap B) \approx B$.

Let $(A \cap B) \approx A, (A \cap B) \approx B$. Then $\underline{R}(A) = \underline{R}(A \cap B) = \underline{R}(B)$. So $A \approx B$.

(2) Let $A \simeq B$. Then $\overline{R}(A) = \overline{R}(B)$. By Theorem 3.5(4),

$$\overline{R}(A \cup B) = \overline{R}(A) \cup \overline{R}(B) = \overline{R}(A) = \overline{R}(B).$$

Hence $(A \cup B) \simeq A$, $(A \cup B) \simeq B$.

Let $(A \cap B) \simeq A$, $(A \cap B) \simeq B$. Then $\underline{R}(A) = \underline{R}(A \cup B) = \underline{R}(B)$. So $A \simeq B$.

(3) This holds by Theorem 3.5(4).

(4) This holds by (1) and (2).

(5) This holds by Theorem 3.5(2).

(6) a) Obviously, $A = \bar{1}$ implies $A \approx \tilde{1}$.

Let $A \approx \tilde{1}$. Then $\underline{R}(A) = \underline{R}(\bar{1})$. By Theorem 3.5(1), $\underline{R}(\bar{1}) = \tilde{1}$. Note that R is reflexive. Then for each $x \in U$,

$$A(x) = A(x) \vee (\bar{1} - R(x, x)) \geq \bigwedge_{y \in U} (A(y) \vee (\bar{1} - R(x, y))) = \underline{R}(A)(x) = \tilde{1}(x) = \bar{1}.$$

Thus $A = \tilde{1}$.

b) The proof is similar to a). □

Theorem 3.15. *Let (U, R) be an IVF approximation space. If R is preorder, then for each $A \in F^{(i)}(U)$,*

(1) $\underline{R}(A) = \bigcap \{B \in F^{(i)}(U) : B \approx A\}$.

(2) $\overline{R}(A) = \bigcup \{B \in F^{(i)}(U) : B \simeq A\}$.

Proof. (1) By Theorem 3.7(2), $\underline{R}(A) \subseteq \bigcap \{B \in F^{(i)}(U) : B \approx A\}$. By Corollary 3.8, $\underline{R}(A) \supseteq \bigcap \{B \in F^{(i)}(U) : B \approx A\}$. Then $\underline{R}(A) = \bigcap \{B \in F^{(i)}(U) : B \approx A\}$.

(2) The proof is similar to (1). □

4 Topological structures of IVF approximation spaces

Let (U, R) be an IVF approximation space. We denote

$$\tau_R = \{A \in F^{(i)}(U) : \underline{R}(A) = A\}, \theta_R = \{\underline{R}(A) : A \in F^{(i)}(U)\}.$$

4.1 IVF topologies based on IVF relations

Theorem 4.1 ([27]). *Let R be an IVF relation on U . If R is reflexive, then τ_R is an IVF topology on U .*

Definition 4.2 ([27]). *Let R be an IVF relation on U . If R is reflexive, then τ_R is called the IVF topology induced by R on U .*

Theorem 4.3. *Let R be a reflexive IVF relation on U and let τ_R be the IVF topology induced by R on U . Then the following properties hold.*

(1) a) $\tau_R \subseteq \theta_R$.

b) For each $A \in F^{(i)}(U)$,

$$\text{int}_{\tau_R}(A) \subseteq \underline{R}(A) \subseteq A \subseteq \overline{R}(A) \subseteq \text{cl}_{\tau_R}(A).$$

c) For each $[a, b] \in [I]$, $[\widetilde{a}, \widetilde{b}] \in \tau_R \cap \tau_R^c$.

- (2) If R is transitive, then
- a) $\tau_R = \theta_R$.
 - b) \underline{R} is the interior operator of τ_R .
 - c) \overline{R} is the closure operator of τ_R .
 - d) $int_\tau(A) = \bigcap \{B \in F^{(i)}(U) : B \approx A\}$.
 - e) $cl_\tau(A) = \bigcup \{B \in F^{(i)}(U) : B \simeq A\}$.

Proof. (1) holds by Theorem 17 in [27].

(2) a) b) and c) holds by Theorem 18 in [27].

d) This holds by (2) b) Proposition 3.18(1).

e) This holds by (2) c) Proposition 3.18(2). □

Theorem 4.4. Let R be a preorder IVF relation on U and let τ_R be the IVF topology induced by R on U . Then for any $x, y \in U$

$$R(x, y) = \bigwedge_{A \in (y)_{\tau_R}} A(x) ,$$

where $(y)_{\tau_R} = \{A \in \tau_R^c : A(y) = \bar{1}\}$.

Proof. For any $x, y \in U$, by Remark 3.3(1) and Theorem 4.3(2),

$$\begin{aligned} R(x, y) &= \overline{R}(y_{\bar{1}})(x) = cl_{\tau_R}(y_{\bar{1}})(x) \\ &= \left(\bigcap \{A \in \tau_R^c : A \supseteq y_{\bar{1}}\} \right)(x) \\ &= \bigwedge \{A(x) : A \in \tau_R^c, A \supseteq y_{\bar{1}}\}. \end{aligned}$$

Note that $A \supseteq y_{\bar{1}}$ if and only if $A(y) = \bar{1}$. Thus

$$R(x, y) = \bigwedge \{A(x) : A^c \in \tau_R, A(y) = \bar{1}\} = \bigwedge_{A \in (y)_{\tau_R}} A(x).$$

□

Theorem 4.5. Let R_1 and R_2 be two preorder IVF relations on U . Let τ_{R_1} and τ_{R_2} be the IVF topologies induced by R_1 and R_2 on U , respectively. Then the following properties hold.

- (1) If $R_1 \subseteq R_2$, then $\tau_{R_2} \subseteq \tau_{R_1}$.
- (2) $\tau_{R_1} = \tau_{R_2} \iff R_1 = R_2$.

Proof. (1) Let $R_1 \subseteq R_2$. For each $A \in \tau_{R_2}$, $\underline{R_2}(A) = A$. For each $x \in U$, by the

transitivity of R_2 ,

$$\begin{aligned}
 \underline{R_1}(A)(x) &= \underline{R_1}(\underline{R_2}(A))(x) \\
 &= \bigwedge_{y \in U} (\underline{R_2}(A)(y) \vee (\bar{1} - R_1(x, y))) \\
 &= \bigwedge_{y \in U} ((\bigwedge_{z \in U} (A(z) \vee (\bar{1} - R_2(y, z)))) \vee (\bar{1} - R_1(x, y))) \\
 &= \bigwedge_{y \in U} (\bigwedge_{z \in U} ((A(z) \vee (\bar{1} - R_2(y, z))) \vee (\bar{1} - R_1(x, y)))) \\
 &= \bigwedge_{y \in U} (\bigwedge_{z \in U} (A(z) \vee ((\bar{1} - R_2(y, z)) \vee (\bar{1} - R_1(x, y)))) \\
 &\geq \bigwedge_{y \in U} (\bigwedge_{z \in U} (A(z) \vee ((\bar{1} - R_2(y, z)) \vee (\bar{1} - R_2(x, y)))) \\
 &= \bigwedge_{y \in U} (\bigwedge_{z \in U} (A(z) \vee (\bar{1} - R_2(x, y) \wedge R_2(y, z)))) \\
 &\geq \bigwedge_{y \in U} (\bigwedge_{z \in U} (A(z) \vee (\bar{1} - R_2(x, z)))) \\
 &= \bigwedge_{z \in U} (A(z) \vee (\bar{1} - R_2(x, z))) \\
 &= \underline{R_2}(A)(x) = A(x).
 \end{aligned}$$

Then $\underline{R_1}(A) \supseteq A$.

By Theorem 3.7(2), $\underline{R_1}(A) \subseteq A$.

Then $\underline{R_1}(A) = A$ and so $A \in \tau_{R_1}$. Thus $\tau_{R_2} \subseteq \tau_{R_1}$.

(2) Let $\tau_{R_1} = \tau_{R_2}$. By Remark 3.3(1) and Theorem 4.3(2),

$$R_1(x, y) = \overline{R_1}(y_{\bar{1}})(x) = cl_{R_{\tau_1}}(y_{\bar{1}})(x) = cl_{R_{\tau_2}}(y_{\bar{1}})(x) = R_2(x, y)$$

for any $x, y \in U$. Then $R_1 = R_2$.

Conversely, this is obvious. □

4.2 IVF relations based on IVF topologies

4.2.1 IVF relations induced by IVF topologies

Definition 4.6. Let τ be an IVF topology. Define an IVF relation R_τ on U by

$$R_\tau(x, y) = cl_\tau(y_{\bar{1}})(x)$$

for each $(x, y) \in U \times U$. Then R_τ is called the IVF relation induced by τ on U .

Theorem 4.7. Let τ be an IVF topology on U and let R_τ be the IVF relation induced by τ on U . Then the following properties hold.

(1) R_τ is reflexive.

(2) If $\{\widetilde{[a, b]} : [a, b] \in [I]\} \subseteq \tau$, then

$$\underline{R}_\tau(A) \subseteq \text{int}_\tau(A) \subseteq A \subseteq \text{cl}_\tau(A) \subseteq \overline{R}_\tau(A) \quad (A \in F^{(i)}(U)).$$

Proof. (1) holds by Theorem 21 in [27].

(2) Since $\{\widetilde{[a, b]} : [a, b] \in [I]\} \subseteq \tau$, we have $\{\widetilde{[a, b]} : [a, b] \in [I]\} \subseteq \tau^c$. For each $A \in F^{(i)}(U)$, by Remark 2.4, Proposition 2.6 and Theorem 3.5,

$$\begin{aligned} \text{cl}_\tau(A) &= \text{cl}_\tau\left(\bigcup_{y \in U} (A(y)y_{\bar{1}})\right) = \bigcup_{y \in U} \text{cl}_\tau(A(y)y_{\bar{1}}) = \bigcup_{y \in U} \text{cl}_\tau(\widetilde{A(y)} \cap y_{\bar{1}}) \\ &\subseteq \bigcup_{y \in U} (\text{cl}_\tau(\widetilde{A(y)}) \cap \text{cl}_\tau(y_{\bar{1}})) = \bigcup_{y \in U} (\widetilde{A(y)} \cap \text{cl}_\tau(y_{\bar{1}})). \end{aligned}$$

Then for each $x \in U$,

$$\text{cl}_\tau(A)(x) \leq \bigvee_{y \in U} (\widetilde{A(y)}(x) \wedge \text{cl}_\tau(y_{\bar{1}})(x)) = \bigvee_{y \in U} (A(y) \wedge R_\tau(x, y)) = \overline{R}_\tau(A)(x).$$

Hence $\text{cl}_\tau(A) \subseteq \overline{R}_\tau(A)$.

By Proposition 2.6(4) and Theorem 3.5(3),

$$\text{int}_\tau(A) = (\text{cl}_\tau(A^c))^c \supseteq (\overline{R}_\tau(A^c))^c = \underline{R}_\tau(A).$$

So $\underline{R}_\tau(A) \subseteq \text{int}_\tau(A) \subseteq A \subseteq \text{cl}_\tau(A) \subseteq \overline{R}_\tau(A)$. □

Theorem 4.8. Let R be a reflexive IVF relation on U , let τ_R be the IVF topology induced by τ on U and let R_{τ_R} be the IVF relation induced by τ on U . If R is transitive, then $R_{\tau_R} = R$.

Proof. For each $(x, y) \in U \times U$, by Remark 3.3(1) and Theorem 4.3(2),

$$R(x, y) = \overline{R}(y_{\bar{1}})(x) = \text{cl}_{\theta_R}(y_{\bar{1}})(x) = \text{cl}_{\tau_R}(y_{\bar{1}})(x)$$

Note that $R_{\tau_R}(x, y) = \text{cl}_{\tau_R}(y_{\bar{1}})(x)$. Then $R_{\tau_R}(x, y) = R(x, y)$.

Thus $R_{\tau_R} = R$. □

4.2.2 The (CC) axiom

An IVF topology τ on U is said to satisfy the follows:

The (CC) axiom: for any $[a, b] \in [I]$ and $A \in F^{(i)}(U)$,

$$\text{cl}_\tau([a, b]A) = [a, b]\text{cl}_\tau(A).$$

Proposition 4.9. Let τ be an IVF topology on U . If τ satisfies the (CC) axiom, then

- (1) \overline{R}_τ is the closure operator of τ .
- (2) \underline{R}_τ is the interior operator of τ .
- (3) For each $[a, b] \in [I]$, $\widetilde{[a, b]} \in \tau$.
- (4) τ is Alexandrov.

Proof. (1) For each $A \in F^{(i)}(U)$, by Remark 2.4 and Proposition 2.6(5),

$$cl_\tau(A) = cl_\tau\left(\bigcup_{y \in U} (A(y)y_{\bar{1}})\right) = \bigcup_{y \in U} cl_\tau(A(y)y_{\bar{1}}) = \bigcup_{y \in U} (A(y)cl_\tau(y_{\bar{1}})).$$

Then for each $x \in U$,

$$cl_\tau(A)(x) = \bigvee_{y \in U} (A(y)(x) \wedge cl_\tau(y_{\bar{1}})(x)) = \bigvee_{y \in U} (A(y) \wedge R_\tau(x, y)) = \overline{R_\tau}(A)(x).$$

Hence $\overline{R_\tau}(A) = cl_\tau(A)$. Thus $\overline{R_\tau}$ is the closure operator of τ .

(2) This holds by (1), Proposition 2.6(4) and Theorem 3.5(3).

(3) For each $[a, b] \in [I]$, by (2), Remark 3.3(2) and Proposition 2.6(2),

$$[\widetilde{a}, \widetilde{b}] \supseteq int_\tau([\widetilde{a}, \widetilde{b}]) = \underline{R}([\widetilde{a}, \widetilde{b}]) \supseteq [\widetilde{a}, \widetilde{b}].$$

Then $int_\tau([\widetilde{a}, \widetilde{b}]) = [\widetilde{a}, \widetilde{b}]$ and so $[\widetilde{a}, \widetilde{b}] \in \tau$.

(4) Let $\{A_j : j \in J\} \subseteq \tau$. By (2), then for each $j \in J$, $A_j = int_\tau(A_j) = \underline{R}(A_j)$. By Proposition 2.6 and Theorem 3.5,

$$\bigcap_{j \in J} A_j = \bigcap_{j \in J} \underline{R}(A_j) = \underline{R}\left(\bigcap_{j \in J} A_j\right) = int_\tau\left(\bigcap_{j \in J} A_j\right).$$

So $\bigcap_{j \in J} A_j \in \tau$. Hence τ is Alexandrov. □

Proposition 4.10. *Let R be a preorder IVF relation on U . Then τ_R satisfies the (CC) axiom.*

Proof. For any $[a, b] \in [I]$ and $A \in F^{(i)}(U)$, by Theorems 4.3(2) and 3.5(5),

$$cl_{\tau_R}([a, b]A) = \overline{R}([a, b]A) = [a, b]\overline{R}(A) = [a, b]cl_{\tau_R}(A).$$

Thus τ_R satisfies the (CC) axiom. □

Theorem 4.11. *Let τ be an IVF topology on U and $\{[\widetilde{a}, \widetilde{b}] : [a, b] \in [I]\} \subseteq \tau$. Let R_τ be the IVF relation induced by τ on U and let τ_{R_τ} be the IVF topology induced by R_τ on U . Then*

$$\tau_{R_\tau} = \tau \text{ if and only if } \tau \text{ satisfies the (CC) axiom.}$$

Proof. Necessity. Let $\tau_{R_\tau} = \tau$. By Theorems 4.7(1), R_τ is reflexive.

For each $A \in F^{(i)}(U)$, by Theorems 4.3(2) and 4.7(2),

$$int_\tau(A) = int_{\tau_{R_\tau}}(A) \subseteq \underline{R_\tau}(A) \subseteq int_\tau(A).$$

Then $int_\tau(A) = \underline{R_\tau}(A)$. So $\underline{R_\tau}$ is the interior operator of τ . By Theorem 3.6(1), R_τ is a preorder. By Proposition 4.10, τ satisfies the (CC) axiom.

Sufficiency. By Theorem 4.7(1), R_τ is reflexive. For any $x, y, z \in U$, put $cl(z_{\bar{1}})(y) = [a, b]$. By Remark 2.4 and Theorem 3.5(2),

$$\begin{aligned} [a, b]cl_\tau(y_{\bar{1}}) &= cl_\tau([a, b]y_{\bar{1}}) = cl_\tau(cl_\tau(z_{\bar{1}})(y)y_{\bar{1}}) \\ &\subseteq cl_\tau\left(\bigcup_{t \in U} (cl_\tau(z_{\bar{1}})(t)t_{\bar{1}})\right) = cl_\tau(cl_\tau(z_{\bar{1}})) = cl_\tau(z_{\bar{1}}). \end{aligned}$$

Then

$$\begin{aligned} R_\tau(x, y) \wedge R_\tau(y, z) &= cl_\tau(y_{\bar{1}})(x) \wedge cl_\tau(z_{\bar{1}})(y) = cl_\tau(y_{\bar{1}})(x) \wedge [a, b] \\ &= [a, b] \wedge cl_\tau(y_{\bar{1}})(x) = ([a, b]cl_\tau(y_{\bar{1}}))(x) \\ &\leq cl_\tau(z_{\bar{1}})(x) = R_\tau(x, z). \end{aligned}$$

So R is transitive.

So R_τ is preorder. For each $A \in F^{(i)}(U)$, by Theorem 4.3(2),

$$cl_{\tau_{R_\tau}}(A) = cl_{\theta_{R_\tau}}(A) = \overline{R_\tau}(A).$$

By Proposition 4.9(1), $\overline{R_\tau}(A) = cl_\tau(A)$. So $cl_{\tau_{R_\tau}}(A) = cl_\tau(A)$.

Thus $\tau_{R_\tau} = \tau$. □

Theorem 4.12. *Let*

$$\Sigma = \{R : R \text{ is a preorder IVF relation on } U\}$$

and

$$\Gamma = \{\tau : \tau \text{ is an IVF topology on } U \text{ satisfying the (CC) axiom}\}.$$

Then there exists a one-to-one correspondence between Σ and Γ .

Proof. Two mappings $f : \Sigma \rightarrow \Gamma$ and $g : \Gamma \rightarrow \Sigma$ are defined as follows:

$$f(R) = \tau_R \ (R \in \Sigma), \quad g(\tau) = R_\tau \ (\tau \in \Gamma).$$

By Theorem 4.8, $g \circ f = i_\Sigma$, where $g \circ f$ is the composition of f and g , and i_Σ is the identity mapping on Σ .

By Proposition 4.9(3) and Theorem 4.11, $f \circ g = i_\Gamma$, where $f \circ g$ is the composition of g and f , and i_Γ is the identity mapping on Γ .

Hence f and g are two one-to-one correspondences. This prove that there exists a one-to-one correspondence between Σ and Γ . □

Theorem 4.13. *Let τ be an IVF topology on U . Then the following are equivalent.*

- (1) τ satisfies the (CC) axiom;
- (2) For any $[a, b] \in [I]$ and $A \in F^{(i)}(U)$,

$$int_\tau([\widetilde{a, b}] \cup A) = [\widetilde{a, b}] \cup int_\tau(A);$$

- (3) There exists a preorder IVF relation ρ on U such that $\bar{\rho}$ is the closure operator of τ ;
- (4) There exists a preorder IVF relation ρ on U such that $\underline{\rho}$ is the interior operator of τ ;
- (5) $\overline{R_\tau}$ is the closure operator of τ ;
- (6) $\underline{R_\tau}$ is the interior operator of τ .

Proof. (1) \iff (2) is obvious.

(1) \implies (3). Suppose that τ satisfies the (CC) axiom. Pick $\rho = R_\tau$. By Proposition 4.9(1), $\bar{\rho}$ is the closure operator of τ . By Theorem 3.6(2), ρ is preorder.

(3) \implies (4). Let $\bar{\rho}$ be the closure operator of τ for some preorder IVF relation ρ on U . For each $A \in F^{(i)}(U)$, by Proposition 2.6(4) and Theorem 3.5(3),

$$\underline{\rho}(A) = (\bar{\rho}(A^c))^c = (cl_\tau(A^c))^c = int_\tau(A).$$

Thus, $\underline{\rho}$ is the interior operator of τ .

(4) \implies (6). Let $\underline{\rho}$ be the interior operator of τ for some preorder IVF relation ρ on U . For each $(x, y) \in U \times U$, by Remark 3.3(1),

$$\rho(x, y) = \bar{1} - \underline{\rho}((y_1)^c)(x) = \bar{1} - int_\tau((y_1)^c)(x) = cl_\tau(y_1)(x) = R_\tau(x, y).$$

Then $\rho = R_\tau$. Note that $\underline{\rho}$ is the interior operator of τ . Then $\underline{R_\tau}$ is the interior operator of τ .

(6) \iff (5) holds by Proposition 2.6(4) and Theorem 3.5(3).

(5) \implies (1). For any $[a, b] \in [I]$ and $A \in F^{(i)}(U)$, by Theorem 3.5(5),

$$cl_\tau([a, b]A) = \overline{R_\tau}([a, b]A) = [a, b]\overline{R_\tau}(A) = [a, b]cl_\tau(A).$$

Thus τ satisfies the (CC) axiom. □

5 Lattice structures of IVF approximation spaces

Let (U, R) be an IVF approximation space. We denote

$$Fix(\underline{R}) = \{A \in F^{(i)}(U) : \underline{R}(A) = A\}, \quad Fix(\overline{R}) = \{A \in F^{(i)}(U) : \overline{R}(A) = A\};$$

$$Im(\underline{R}) = \{\underline{R}(A) : A \in F^{(i)}(U)\}, \quad Im(\overline{R}) = \{\overline{R}(A) : A \in F^{(i)}(U)\};$$

$$Def(\underline{R}) = \{A \in F^{(i)}(U) : \underline{R}(A) = \overline{R}(A)\};$$

$$Fix(\underline{R} \circ \underline{R}) = \{A \in F^{(i)}(U) : \underline{R}(\underline{R}(A)) = A\}, \quad Fix(\overline{R} \circ \overline{R}) = \{A \in F^{(i)}(U) : \overline{R}(\overline{R}(A)) = A\};$$

$$\mathcal{O}(\underline{R}) = \{A \in F^{(i)}(U) : \underline{R}(\underline{R}(A)) = \underline{R}(A)\}, \quad \mathcal{O}(\overline{R}) = \{A \in F^{(i)}(U) : \overline{R}(\overline{R}(A)) = \overline{R}(A)\}.$$

For $\mathcal{A} \subseteq F^{(i)}(U)$, denote $\mathcal{A}^c = \{A : A^c \in \mathcal{A}\}$.

Proposition 5.1. *Let (U, R) be an IVF approximation space.*

(1)

$$\begin{aligned} \text{Fix}(\underline{R}) &\subseteq \text{Im}(\underline{R}), \quad \text{Fix}(\overline{R}) \subseteq \text{Im}(\overline{R}); \\ \text{Fix}(\underline{R}) &\subseteq \text{Fix}(\underline{R} \circ \underline{R}), \quad \text{Fix}(\overline{R}) \subseteq \text{Fix}(\overline{R} \circ \overline{R}). \end{aligned}$$

(2) *If R is reflexive, then*

a) *For each $[a, b] \in [I]$,*

$$\begin{aligned} [a, b] &\in \text{Fix}(\underline{R}) \cap \text{Fix}(\overline{R}) \cap \text{Fix}(\underline{R} \circ \underline{R}) \cap \text{Fix}(\overline{R} \circ \overline{R}) \\ &\cap \text{Im}(\underline{R}) \cap \text{Im}(\overline{R}) \cap \text{Def}(R) \cap \mathcal{O}(\underline{R}) \cap \mathcal{O}(\overline{R}). \end{aligned}$$

b)

$$\begin{aligned} (\text{Fix}(\underline{R}))^c &= \text{Fix}(\overline{R}); \\ (\text{Fix}(\underline{R} \circ \underline{R}))^c &= \text{Fix}(\overline{R} \circ \overline{R}); \\ (\text{Im}(\underline{R}))^c &= \text{Im}(\overline{R}); \\ (\mathcal{O}(\underline{R}))^c &= \mathcal{O}(\overline{R}). \end{aligned}$$

c)

$$\begin{aligned} \text{Fix}(\underline{R}) = \text{Fix}(\underline{R} \circ \underline{R}) &\subseteq \mathcal{O}(\underline{R}), \quad \text{Fix}(\overline{R}) = \text{Fix}(\overline{R} \circ \overline{R}) \subseteq \mathcal{O}(\overline{R}); \\ \text{Fix}(\underline{R} \circ \underline{R}) &\subseteq \text{Im}(\underline{R}), \quad \text{Fix}(\overline{R} \circ \overline{R}) \subseteq \text{Im}(\overline{R}). \end{aligned}$$

d)

$$\text{Def}(R) = \text{Fix}(\underline{R}) \cap \text{Fix}(\overline{R}).$$

(3) *If R is proorder, then*

$$\begin{aligned} \text{Fix}(\underline{R}) &= \text{Im}(\underline{R}), \quad \text{Fix}(\overline{R}) = \text{Im}(\overline{R}); \\ \mathcal{O}(\underline{R}) &= F^{(i)}(U) = \mathcal{O}(\overline{R}). \end{aligned}$$

Proof. These hold by Theorem 3.5, Theorem 3.7 and Corollary 3.8. □

Theorem 5.2. *Let (U, R) be an IVF approximation space. If R is reflexive, then*

- (1) *$(\text{Fix}(\underline{R}), \cap, \cup)$ is a complete distributive lattice.*
- (2) *$(\text{Fix}(\overline{R}), \cap, \cup)$ is a complete distributive lattice.*

Proof. (1) By Proposition 5.1(1), $\text{Fix}(\underline{R}) \neq \emptyset$.

Let $\{A_j : j \in J\} \subseteq \text{Fix}(\underline{R})$. Then $\underline{R}(A_j) = A_j$ for each $j \in J$. By Theorem 3.5,

$$\underline{R}\left(\bigcap_{j \in J} A_j\right) = \bigcap_{j \in J} \underline{R}(A_j) = \bigcap_{j \in J} A_j, \quad \underline{R}\left(\bigcup_{j \in J} A_j\right) \supseteq \bigcup_{j \in J} \underline{R}(A_j) = \bigcup_{j \in J} A_j.$$

By Theorem 3.7(2), $\underline{R}\left(\bigcup_{j \in J} A_j\right) \subseteq \bigcup_{j \in J} A_j$. Then $\underline{R}\left(\bigcup_{j \in J} A_j\right) = \bigcup_{j \in J} A_j$. So $\bigcap_{j \in J} A_j, \bigcup_{j \in J} A_j \in \text{Fix}(\underline{R})$.

Thus $(\text{Fix}(\underline{R}), \cap, \cup)$ is a complete lattice. Note that $(\text{Fix}(\underline{R}), \cap, \cup)$ satisfies distributive law. Then $(\text{Fix}(\underline{R}), \cap, \cup)$ is a complete distributive lattice.

(2) The proof is similar to (1). □

Theorem 5.3. *Let (U, R) be an IVF approximation space. If R is preorder, then $(Im(\underline{R}), \cap, \cup)$ and $(Im(\overline{R}), \cap, \cup)$ are both complete distributive lattice.*

Proof. This hold by Proposition 5.1(4) and Theorem 5.2. □

Theorem 5.4. *Let (U, R) be an IVF approximation space. If R is reflexive, then $(Def(R), \cap, \cup)$ is a complete lattice.*

Proof. Let $\{A_j : j \in J\} \subseteq Def(R)$. Then $\underline{R}(A_j) = \overline{R}(A_j)$ for each $j \in J$. By Theorems 3.5 and 3.7(2),

$$\begin{aligned} \underline{R}\left(\bigcap_{j \in J} A_j\right) &= \bigcap_{j \in J} \underline{R}(A_j) = \bigcap_{j \in J} \overline{R}(A_j) \supseteq \overline{R}\left(\bigcap_{j \in J} A_j\right), \quad \underline{R}\left(\bigcap_{j \in J} A_j\right) \subseteq \overline{R}\left(\bigcap_{j \in J} A_j\right); \\ \underline{R}\left(\bigcup_{j \in J} A_j\right) &\supseteq \bigcup_{j \in J} \underline{R}(A_j) = \bigcup_{j \in J} \overline{R}(A_j) = \overline{R}\left(\bigcup_{j \in J} A_j\right), \quad \underline{R}\left(\bigcup_{j \in J} A_j\right) \subseteq \overline{R}\left(\bigcup_{j \in J} A_j\right). \end{aligned}$$

Then $\underline{R}\left(\bigcap_{j \in J} A_j\right) = \overline{R}\left(\bigcap_{j \in J} A_j\right)$, $\underline{R}\left(\bigcup_{j \in J} A_j\right) = \overline{R}\left(\bigcup_{j \in J} A_j\right)$. So $\bigcap_{j \in J} A_j, \bigcup_{j \in J} A_j \in Def(R)$.

Thus $(Def(R), \cap, \cup)$ is a complete lattice. □

Theorem 5.5. *Let (U, R) be an IVF approximation space. If R is reflexive, then*

- (1) $(Fix(\underline{R} \circ \underline{R}), \cap, \cup)$ is a complete distributive lattice.
- (2) $(Fix(\overline{R} \circ \overline{R}), \cap, \cup)$ is a complete distributive lattice.

Proof. (1) By Proposition 5.1(1), $Fix(\underline{R} \circ \underline{R}) \neq \emptyset$.

Let $\{A_j : j \in J\} \subseteq Fix(\underline{R} \circ \underline{R})$. Then $\underline{R}(\underline{R}(A_j)) = A_j$ for each $j \in J$. By Theorem 3.5,

$$\begin{aligned} \underline{R}\left(\underline{R}\left(\bigcap_{j \in J} A_j\right)\right) &= \underline{R}\left(\bigcap_{j \in J} \underline{R}(A_j)\right) = \bigcap_{j \in J} \underline{R}(\underline{R}(A_j)) = \bigcap_{j \in J} A_j, \\ \underline{R}\left(\underline{R}\left(\bigcup_{j \in J} A_j\right)\right) &\supseteq \underline{R}\left(\bigcup_{j \in J} \underline{R}(A_j)\right) \supseteq \bigcup_{j \in J} \underline{R}(\underline{R}(A_j)) = \bigcup_{j \in J} A_j. \end{aligned}$$

By Theorem 3.7(2), $\underline{R}(\underline{R}\left(\bigcup_{j \in J} A_j\right)) \subseteq \underline{R}\left(\bigcup_{j \in J} A_j\right) \subseteq \bigcup_{j \in J} A_j$. Then $\underline{R}(\underline{R}\left(\bigcup_{j \in J} A_j\right)) = \bigcup_{j \in J} A_j$. So $\bigcap_{j \in J} A_j, \bigcup_{j \in J} A_j \in Fix(\underline{R} \circ \underline{R})$. Thus $(Fix(\underline{R} \circ \underline{R}), \cap, \cup)$ is a complete lattice.

Note that $(Fix(\underline{R} \circ \underline{R}), \cap, \cup)$ satisfies distributive law. Then $(Fix(\underline{R} \circ \underline{R}), \cap, \cup)$ is a complete distributive lattice.

(2) The proof is similar to (1). □

Theorem 5.6. *Let (U, R) be an IVF approximation space. If R is reflexive, then*

- (1) $(\mathcal{O}(\underline{R}), \cap, \cup)$ is a complete distributive lattice.
- (2) $(\mathcal{O}(\overline{R}), \cap, \cup)$ is a complete distributive lattice.

Proof. (1) By Proposition 5.1(1), $\mathcal{O}(\underline{R}) \neq \emptyset$.

Let $\{A_j : j \in J\} \subseteq \mathcal{O}(\underline{R})$. Then $\underline{R}(\underline{R}(A_j)) = \underline{R}(A_j)$ for each $j \in J$. By Theorem 3.5,

$$\underline{R}(\underline{R}(\bigcap_{j \in J} A_j)) = \underline{R}(\bigcap_{j \in J} \underline{R}(A_j)) = \bigcap_{j \in J} \underline{R}(\underline{R}(A_j)) = \bigcap_{j \in J} \underline{R}(A_j) = \underline{R}(\bigcap_{j \in J} A_j),$$

$$\underline{R}(\underline{R}(\bigcup_{j \in J} A_j)) \supseteq \underline{R}(\bigcup_{j \in J} \underline{R}(A_j)) \supseteq \bigcup_{j \in J} \underline{R}(\underline{R}(A_j)) = \bigcup_{j \in J} \underline{R}(A_j).$$

By Theorem 3.7(2), $\underline{R}(\underline{R}(\bigcup_{j \in J} A_j)) \subseteq \underline{R}(\bigcup_{j \in J} A_j)$. Then $\underline{R}(\underline{R}(\bigcup_{j \in J} A_j)) = \bigcup_{j \in J} A_j$.

So $\bigcap_{j \in J} A_j, \bigcup_{j \in J} A_j \in \mathcal{O}(\underline{R})$.

Thus $(\mathcal{O}(\underline{R}), \cap, \cup)$ is a complete distributive lattice.

(3) The proof is similar to (2). □

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Stability of ternary m -derivations on ternary Banach algebras

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Abstract. In this paper, we use a fixed point method to prove the stability of ternary m -derivations on ternary Banach algebras.

1. Introduction and preliminaries

Consider the functional equation $\mathfrak{S}_1(f) = \mathfrak{S}_2(f)$ (\mathfrak{S}) in a certain general setting. A mapping g is an approximate solution of (\mathfrak{S}) if $\mathfrak{S}_1(g)$ and $\mathfrak{S}_2(g)$ are close in some sense. The Ulam stability problem asks whether or not there is a true solution of (\mathfrak{S}) near g . A functional equation is superstable if every approximate solution of the equation is an exact solution of it. For more details about various results concerning such problems the reader is referred to [3, 7, 9, 11, 14, 15, 18, 19, 20, 22, 28].

Ternary algebraic operations were considered in the 19th century by several mathematicians: Cayley [6] introduced the notion of cubic matrix which in turn was generalized by Kapranov, Gelfand and Zelevinskii in 1990 [13]. As an application in physics, the quark model inspired a particular brand of ternary algebraic systems. There are also some applications, although still hypothetical, in the fractional quantum Hall effect, the non-standard statistics (the anyons), supersymmetric theories, Yang-Baxter equation, etc, (cf. [1, 29]).

The comments on physical applications of ternary structures can be found in (see [10, 24, 26]).

The monomial $f(x) = ax^m$ ($x \in \mathbb{R}$, $m = 1, 2, 3, 4$) is a solution of the following functional equation

$$f(ax + y) + f(ax - y) = a^{m-2}[f(x + y) + f(x - y)] + 2(a^2 - 1)[a^{m-2}f(x) + \frac{(m-2)(1-(m-2)^2)}{6}f(y)]. \quad (1.1)$$

For $m = 1, 2, 3, 4$, the functional equation (1.1) is equivalent to the additive, quadratic, cubic and quartic functional equation, respectively. The general solution of the functional equation (1.1) for any fixed integer a with $a \neq 0, \pm 1$, was obtained by Eshaghi Gordji et al. [8].

Let A be an algebra. An additive mapping $f : A \rightarrow A$ is called a derivation if $f(xy) = xf(y) + f(x)y$ holds for all $x, y \in A$. If, in addition, $f(\lambda x) = \lambda f(x)$ for all $x \in A$ and all $\lambda \in \mathbb{F}$, then f is called a linear derivation, where \mathbb{F} denotes the scalar field of A . The stability result concerning derivations between operator algebras was first obtained by Šemrl [23]. In [2], Badora proved the stability of functional equation $f(xy) = xf(y) + f(x)y$, where f is a mapping on normed algebra A with unit. Recently, Miura et al. [17] examined the stability of derivations on Banach algebras.

Suppose that A is a Banach algebra. Let θ, r be nonnegative real numbers. If $r \neq 1$ and $f : A \rightarrow A$ is a mapping such that

$$\|f(x + y) - f(x) - f(y)\| \leq \theta(\|x\|^r + \|y\|^r),$$

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$$\|f(xy) - xf(y) - f(x)y\| \leq \theta(\|x\|^r \cdot \|y\|^r)$$

for all $x, y \in A$. Then there exists a unique derivation $D : A \rightarrow A$ satisfying

$$\|f(x) - D(x)\| \leq \frac{2\theta}{|2 - 2^r|} \|x\|^r,$$

for all $x, y \in A$. In particular, if A is a Banach algebra, then f is a derivation.

The various problems of the stability of derivations have been studied during last few years (see, for instance, [10, 25, 27]).

Definition 1.1. Let A be a ternary algebra.

(i) A mapping $f : A \rightarrow A$ is called a ternary additive derivation (briefly, ternary 1-derivation) if f is an additive mapping satisfying $f([a, b, c]) = [f(a), b, c] + [a, f(b), c] + [a, b, f(c)]$ for all $a, b, c \in A$;

(ii) A mapping $f : A \rightarrow A$ is called a ternary quadratic derivation (briefly, ternary 2-derivation) if f is a quadratic mapping satisfying $f([a, b, c]) = [f(a), b, [b, c, c]] + [a, a, [f(b), c, c]] + [a, a, [b, b, f(c)]]$ for all $a, b, c \in A$;

(iii) A mapping $f : A \rightarrow A$ is called a ternary cubic derivation (briefly, ternary 3-derivation) if f is a cubic mapping satisfying $f([a, b, c]) = [f(a), b, [b, b, [c, c, c]]] + [a, a, [a, f(b), [c, c, c]]] + [a, a, [a, b, [b, b, f(c)]]]$ for all $a, b, c \in A$;

(iv) A mapping $f : A \rightarrow A$ is called a ternary quartic derivation (briefly, ternary 4-derivation) if f is a quartic mapping satisfying $f([a, b, c]) = [f(a), b, [b, b, [b, c, [c, c, c]]]] + [a, a, [a, a, [f(b), c, [c, c, c]]]] + [a, a, [a, a, [b, b, [b, b, f(c)]]]]$ for all $a, b, c \in A$.

The main theorem of [16], which is called the alternative of fixed point, plays an important role in proving the stability problem. Recently, Cădariu and Radu [4] applied the fixed point method to the investigation of the Cauchy additive functional equation (see also [5, 12, 21]).

In this paper, we adopt the idea of Cădariu and Radu to establish the stability of m -derivations on ternary Banach algebras related to the functional equation (1.1). In addition, we study the superstability of the functional equation (1.1) by suitable control functions.

2. Stability of ternary m -derivations on ternary Banach algebras via fixed point method

Throughout this section, we suppose that A is a ternary Banach algebra, and m is a fixed positive integer less than 5. For convenience, we use the following abbreviation for a given mapping $f : A \rightarrow A$

$$\begin{aligned} \Delta_m f(x, y) &= f(wx + y) + f(wx - y) \\ &- w^{m-2}[f(x + y) + f(x - y)] - 2(w^2 - 1)[w^{m-2}f(x) + \frac{(m-2)(1 - (m-2)^2)}{6}f(y)] \end{aligned}$$

for all $x, y \in A$ and any fixed integers $w \neq 0, \pm 1$.

Let

$$\begin{aligned} F_1 f(a, b, c) &:= [f(a), b, c] + [a, f(b), c] + [a, b, f(c)], \\ F_2 f(a, b, c) &:= [f(a), b, [b, c, c]] + [a, a, [f(b), c, c]] + [a, a, [b, b, f(c)]], \\ F_3 f(a, b, c) &:= [f(a), b, [b, b, [c, c, c]]] + [a, a, [a, f(b), [c, c, c]]] + [a, a, [a, b, [b, b, f(c)]]], \\ F_4 f(a, b, c) &:= [f(a), b, [b, b, [b, c, [c, c, c]]]] + [a, a, [a, a, [f(b), c, [c, c, c]]]] + [a, a, [a, a, [b, b, [b, b, f(c)]]]] \end{aligned}$$

for all $a, b, c \in A$.

Theorem 2.1. Let $f : A \rightarrow A$ be a mapping for which there exists function $\varphi_m : A^5 \rightarrow [0, \infty)$ such that

$$\|\Delta_m f(x, y) + f([a, b, c]) - F_m f(a, b, c)\| \leq \varphi_m(x, y, a, b, c) \tag{2.1}$$

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for all $x, y, a, b, c \in A$. If there exists a constant $0 < L < 1$ such that

$$\varphi_m \left(\frac{x}{w}, \frac{y}{w}, \frac{a}{w}, \frac{b}{w}, \frac{c}{w} \right) \leq \frac{L}{|w|^m} \varphi_m(x, y, a, b, c), \tag{2.2}$$

for all $x, y, a, b, c \in A$, then there exists a unique ternary m -derivation $D_m : A \rightarrow A$ such that

$$\|f(x) - D_m(x)\| \leq \frac{L}{2|w|^m(1-L)} \varphi_m(x, 0, 0, 0, 0), \tag{2.3}$$

for all $x \in A$.

Proof. First of all, if we take $x = y = a = b = c = 0$ in (2.2), then we obtain that $\varphi_m(0, 0, 0, 0, 0) = 0$, since $0 < L < 1$ and $w \neq 0, \pm 1$. Letting $x = y = a = b = c = 0$ in (2.1), we obtain $f(0) = 0$.

It follows from (2.2) that

$$\lim_{n \rightarrow \infty} |w|^{mn} \varphi_m \left(\frac{x}{w^n}, \frac{y}{w^n}, \frac{a}{w^n}, \frac{b}{w^n}, \frac{c}{w^n} \right) = 0 \tag{2.4}$$

for all $x, y, a, b, c \in A$.

Let us define Ω to be the set of all mappings $g : A \rightarrow A$ and introduce a generalized metric on Ω as follows:

$$d(g, h) = d_{\varphi_m}(g, h) = \inf\{K \in (0, \infty) : \|g(x) - h(x)\| \leq K\varphi_m(x, 0, 0, 0, 0), x \in A\}$$

It is easy to show that (Ω, d) is a generalized complete metric space [4, 5].

Now we consider the mapping $T : \Omega \rightarrow \Omega$ defined by $Tg(x) = w^m g(\frac{x}{w})$ for all $x \in A$ and all $g \in \Omega$.

Note that for all $g, h \in \Omega$,

$$\begin{aligned} d(g, h) < K &\Rightarrow \|g(x) - h(x)\| \leq K\varphi_m(x, 0, 0, 0, 0) && \text{for all } x \in A, \\ &\Rightarrow \left\| w^m g\left(\frac{x}{w}\right) - w^m h\left(\frac{x}{w}\right) \right\| \leq |w|^m K \varphi_m\left(\frac{x}{w}, 0, 0, 0, 0\right) && \text{for all } x \in A, \\ &\Rightarrow \left\| w^m g\left(\frac{x}{w}\right) - w^m h\left(\frac{x}{w}\right) \right\| \leq L K \varphi_m(x, 0, 0, 0, 0) && \text{for all } x \in A, \\ &\Rightarrow d(Tg, Th) \leq L K. \end{aligned}$$

Hence we see that

$$d(Tg, Th) \leq L d(g, h)$$

for all $g, h \in \Omega$, that is, T is a strictly contractive mapping of Ω with the Lipschitz constant L .

Putting $y = a = b = c = 0$ in (2.1), we have

$$\|2f(wx) - 2w^m f(x)\| \leq \varphi_m(x, 0, 0, 0, 0) \tag{2.5}$$

for all $x \in A$. So

$$\left\| f(x) - w^m f\left(\frac{x}{w}\right) \right\| \leq \frac{1}{2} \varphi_m\left(\frac{x}{w}, 0, 0, 0, 0\right) \leq \frac{L}{2|w|^m} \varphi_m(x, 0, 0, 0, 0)$$

for all $x \in A$, that is, $d(f, Tf) \leq \frac{L}{2|w|^m} < \infty$.

Now, from the fixed point alternative, it follows that there exists a fixed point D_m of T in Ω such that

$$D_m(x) = \lim_{n \rightarrow \infty} w^{mn} f\left(\frac{x}{w^n}\right) \tag{2.6}$$

for all $x \in A$, since $\lim_{n \rightarrow \infty} d(T^n f, D_m) = 0$.

On the other hand, it follows from (2.1), (2.4) and (2.6) that

$$\|\Delta_m D_m(x, y)\| = \lim_{n \rightarrow \infty} |w|^{mn} \left\| \Delta_m f\left(\frac{x}{w^n}, \frac{y}{w^n}\right) \right\| \leq \lim_{n \rightarrow \infty} |w|^{mn} \varphi_m\left(\frac{x}{w^n}, \frac{y}{w^n}, 0, 0, 0\right) = 0$$

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for all $x, y \in A$. So $\Delta_m D_m(x, y) = 0$. By [8], D_m is an m -mapping. So it follows from the definition of D_m , (2.2) and (2.4) that

$$\begin{aligned} \|D_m([a, b, c]) - F_m D_m(a, b, c)\| &= \lim_{n \rightarrow \infty} |w|^{3mn} \left\| f\left(\left[\frac{a}{w^n}, \frac{b}{w^n}, \frac{c}{w^n}\right]\right) - F_m f\left(\frac{a}{w^n}, \frac{b}{w^n}, \frac{c}{w^n}\right) \right\| \\ &\leq \lim_{n \rightarrow \infty} |w|^{3mn} \varphi_m\left(0, 0, \frac{a}{w^n}, \frac{b}{w^n}, \frac{c}{w^n}\right) = 0 \end{aligned}$$

for all $a, b, c \in A$. So $D_m([a, b, c]) = F_m D_m(a, b, c)$ for all $a, b, c \in A$.

According to the fixed point alternative, since D_m is the unique fixed point of T in the set $\Lambda = \{g \in \Omega : d(f, g) < \infty\}$, D_m is the unique mapping such that

$$\|f(x) - D_m(x)\| \leq K \varphi_m(x, 0, 0, 0, 0)$$

for all $x \in A$ and $K > 0$. Again using the fixed point alternative, we obtain

$$d(f, D_m) \leq \frac{1}{1-L} d(f, Tf) \leq \frac{L}{2|w|^m(1-L)}$$

and so we conclude that

$$\|f(x) - D_m(x)\| \leq \frac{L}{2|w|^m(1-L)} \varphi_m(x, 0, 0, 0, 0)$$

for all $x \in A$. This completes the proof. □

Corollary 2.2. *Let θ, r, s be nonnegative real numbers with $s > m$ and $r > m$. Suppose that $f : A \rightarrow A$ is a mapping such that*

$$\|\Delta_m f(x, y) + f([a, b, c]) - F_m f(a, b, c)\| \leq \theta(\|x\|^r + \|y\|^r + \|a\|^s \cdot \|b\|^s \cdot \|c\|^s)$$

for all $x, y, a, b, c \in A$. Then there exists a unique ternary m -derivation $D_m : A \rightarrow A$ satisfying

$$\|f(x) - D_m(x)\| \leq \frac{\theta}{2(|w|^r - |w|^m)} \|x\|^r$$

for all $x \in A$.

Proof. The proof follows from Theorem 2.1 by taking

$$\varphi_m(x, y, a, b, c) := \theta(\|x\|^r + \|y\|^r + \|a\|^s \cdot \|b\|^s \cdot \|c\|^s)$$

for all $x, y, a, b, c \in A$. Then we can choose $L = |w|^{m-r}$ and we get the desired results. □

Remark 2.3. Let $f : A \rightarrow A$ be a mapping with $f(0) = 0$ for which there exist functions $\varphi_m : A^5 \rightarrow [0, \infty)$ satisfying (2.1) and (2.2). Let $0 < L < 1$ be a constant such that $\varphi_m(wx, wy, wa, wb, wc) \leq |w|^m L \varphi_m(x, y, a, b, c)$ for all $x, y, a, b, c \in A$. By a similar method to the proof of Theorem 2.1, we can show that there exists a unique ternary m -derivation $D_m : A \rightarrow X$ satisfying

$$\|f(x) - D_m(x)\| \leq \frac{1}{2|w|^m(1-L)} \varphi_m(x, 0, 0, 0, 0)$$

for all $x \in A$.

For the case $\varphi_m(x, y, a, b, c) := \delta + \theta(\|x\|^r + \|y\|^r + \|a\|^s \cdot \|b\|^s \cdot \|c\|^s)$ (where θ, δ are nonnegative real numbers and $0 < r, s < m$), there exists a unique ternary m -derivation $D_m : A \rightarrow A$ satisfying

$$\|f(x) - D_m(x)\| \leq \frac{\delta}{2(|w|^m - |w|^r)} + \frac{\theta}{2(|w|^m - |w|^r)} \|x\|^r$$

for all $x \in A$.

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On IF approximating spaces *

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Abstract: In this paper, a IF approximating space is introduced. It is a particular type of IF topological spaces which associate with IF relations. A characteristic condition for IF topological spaces to be IF approximating spaces is established.

Keywords: IF set; IF relation; IF approximate space; IF rough set; IF topology; IF approximating space.

1 Introduction

Rough set theory was proposed by Pawlak [16, 17] as a mathematical tool to handle imprecision and uncertainty in data analysis. Usefulness and versatility of this theory have amply been demonstrated by successful applications in a variety of problems [21, 22].

The basic structure of rough set theory is an approximation space. Based on it, rough approximations can be induced. Using them, knowledge hidden in information systems may be revealed and expressed in the form of decision rules [16].

Intuitionistic fuzzy (IF, for short) sets were originated by Atanassov [1, 2]. It is a straightforward extension of Zadeh's fuzzy sets [26]. IF sets have played an useful role in the research of uncertainty theories. Unlike a fuzzy set, which gives a degree of which element belongs to a set, an IF set gives both a membership degree and a nonmembership degree. Thus, an IF set is more objective than a fuzzy set to describe the vagueness of data or information.

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Recently, IF approximate spaces were introduced and then IF rough sets were presented [6, 7, 8, 20, 23, 27, 28, 29, 30]. For example, Zhou et al. [27, 28, 29, 30] studied structures of IF rough sets, Wu et al. [23] researched IF topologies based on preorder IF relations, Zhang et al. [31] investigated IF rough sets on two universes.

It is well known that topology is a branch of mathematics, whose concepts exist not only in almost all branches of mathematics but also in many real life applications. Topology and rough set theory have been widely used in research field of computer science. An interesting and natural research topic is to study the relationship between rough sets and topologies.

The purpose of this paper is to investigate IF approximating space where the given IF topology coincides with the IF topology induced by some reflexive IF relation.

2 Preliminaries

Throughout this paper, “ Intuitionistic fuzzy ” is briefly written “ IF ”, X denotes a infinite universe. I denotes $[0, 1]$, $J = \{\lambda \in I \times I : a + b \leq 1\}$, $F(X)$ denotes the family of all fuzzy sets in X and $IF(X)$ denotes the family of all IF sets in X .

In this section, we recall some basic notions and properties related to IF sets, IF topologies and fuzzy rough sets.

2.1 IF sets

Definition 2.1 ([11]). Let $(a, b), (c, d) \in I \times I$. Define

- (1) $(a, b) = (c, d) \iff a = c, b = d$.
- (2) $(a, b) \sqcup (c, d) = (a \vee c, b \wedge d)$, $(a, b) \sqcap (c, d) = (a \wedge c, b \vee d)$.
- (3) $(a, b)^c = (b, a)$.

Moreover, for $\{(a_\alpha, b_\alpha) : \alpha \in \Gamma\} \subseteq I \times I$,

$$\bigsqcup_{\alpha \in \Gamma} (a_\alpha, b_\alpha) = (\bigvee_{\alpha \in \Gamma} a_\alpha, \bigwedge_{\alpha \in \Gamma} b_\alpha) \text{ and } \bigsqcap_{\alpha \in \Gamma} (a_\alpha, b_\alpha) = (\bigwedge_{\alpha \in \Gamma} a_\alpha, \bigvee_{\alpha \in \Gamma} b_\alpha).$$

Definition 2.2 ([11]). Let $(a, b), (c, d) \in J$ and let $S \subseteq J \times J$. $(a, b)S(c, d)$, if $a \leq c$ and $b \geq d$. We denote S by \leq .

Obviously, $(a, b) = (c, d) \iff (a, b) \leq (c, d)$ and $(c, d) \leq (a, b)$.

Remark 2.3. (1) (J, \leq) be a poset with $0_J = (0, 1)$ and $1_J = (1, 0)$.

- (2) $(a, b)^{cc} = (a, b)$.
- (3) $((a, b) \sqcup (c, d)) \sqcup (e, f) = (a, b) \sqcup ((c, d) \sqcup (e, f))$,
 $((a, b) \sqcap (c, d)) \sqcap (e, f) = (a, b) \sqcap ((c, d) \sqcap (e, f))$.
- (4) $(a, b) \sqcup (c, d) = (c, d) \sqcup (a, b)$, $(a, b) \sqcap (c, d) = (c, d) \sqcap (a, b)$.
- (5) $((a, b) \sqcup (c, d)) \sqcap (e, f) = ((a, b) \sqcap (e, f)) \sqcup ((c, d) \sqcap (e, f))$.
 $((a, b) \sqcap (c, d)) \sqcup (e, f) = ((a, b) \sqcup (e, f)) \sqcap ((c, d) \sqcup (e, f))$.
- (6) $(\bigsqcup_{\alpha \in \Gamma} (a_\alpha, b_\alpha))^c = \bigsqcap_{\alpha \in \Gamma} (a_\alpha, b_\alpha)^c$, $(\bigsqcap_{\alpha \in \Gamma} (a_\alpha, b_\alpha))^c = \bigsqcup_{\alpha \in \Gamma} (a_\alpha, b_\alpha)^c$.

Proposition 2.4 ([11]). $(J, \leq, \sqcap, \sqcup)$ be a complete distributive lattice.

Definition 2.5 ([1]). An IF set A in X is an object having the form

$$A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle : x \in X \},$$

where $\mu_A, \nu_A \in F(X)$ satisfying $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$, and $\mu_A(x), \nu_A(x)$ are used to define the degree of membership and the degree of non-membership of the element x to A , respectively.

For the sake of simplicity, we give the following definition.

Definition 2.6. A is called an IF set in X , if $A = (A^*, A_*) \in F(X) \times F(X)$ and for each $x \in X$, $A(x) = (A^*(x), A_*(x)) \in J$, where $A^*(x), A_*(x)$ are used to define the degree of membership and the degree of non-membership of the element x to A , respectively.

For each $\mathcal{A} \subseteq IF(X)$, we denote

$$\mathcal{A}^c = \{A^c : A \in \mathcal{A}\},$$

$$\mathcal{A}^* = \{A^* : A \in \mathcal{A}\} \text{ and } \mathcal{A}_* = \{A_* : A \in \mathcal{A}\}.$$

Let $\lambda \in J$. $\widehat{\lambda}$ represents a constant IF set which satisfies $\widehat{\lambda}(x) = \lambda$ for each $x \in X$. Denote $1_\sim = \widehat{(1, 0)}$ and $0_\sim = \widehat{(0, 1)}$.

Some IF relations and IF operations are defined as follows ([1, 2]): for any $A, B \in IF(X)$ and $\{A_\alpha : \alpha \in \Gamma\} \subseteq IF(X)$,

$$(1) A = B \iff A(x) = B(x) \text{ for each } x \in X.$$

$$(2) A \subseteq B \iff A(x) \leq B(x) \text{ for each } x \in X.$$

$$(3) \left(\bigcup_{\alpha \in \Gamma} A_\alpha \right)(x) = \bigsqcup_{\alpha \in \Gamma} A_\alpha(x) \text{ for each } x \in X.$$

$$(4) \left(\bigcap_{\alpha \in \Gamma} A_\alpha \right)(x) = \bigsqcap_{\alpha \in \Gamma} A_\alpha(x) \text{ for each } x \in X.$$

$$(5) A^c(x) = A(x)^c \text{ for each } x \in X.$$

$$(6) (\lambda A)(x) = \lambda \sqcap (A^*(x), A_*(x)) \text{ for any } x \in X \text{ and } \lambda \in J.$$

Obviously, $A = B \iff A^* = B^*$ and $A_* = B_* \iff A \subseteq B$ and $B \subseteq A$.

We define two special IF sets $1_y = ((1_y)^*, (1_y)_*)$ and $0_y = ((0_y)^*, (0_y)_*)$ for some $y \in X$ as follows:

$$(1_y)^*(x) = \begin{cases} 1, & x = y, \\ 0, & x \neq y. \end{cases} \quad (1_y)_*(x) = \begin{cases} 0, & x = y, \\ 1, & x \neq y. \end{cases}$$

$$(0_y)^*(x) = \begin{cases} 0, & x = y, \\ 1, & x \neq y. \end{cases} \quad (0_y)_*(x) = \begin{cases} 1, & x = y, \\ 0, & x \neq y. \end{cases}$$

Remark 2.7. For each $A \in IF(X)$,

$$A = \bigcup_{y \in X} (A(y)1_y).$$

2.2 IF topologies

Definition 2.8 ([5]). Let $\tau \subseteq IF(X)$. Then τ is called an IF topology on X , if

- (i) $0_{\sim}, 1_{\sim} \in \tau$,
- (ii) $A, B \in \tau$ implies $A \cap B \in \tau$,
- (iii) $\{A_{\alpha} : \alpha \in \Gamma\} \subseteq \tau$ implies $\bigcup\{A_{\alpha} : \alpha \in \Gamma\} \in \tau$.

The pair (X, τ) is called an IF topological space and every member of τ is called an IF open set in X . Its complement is called an IF closed set in X .

We denote $\tau^c = \{A : A^c \in \tau\}$.

The interior and closure of $A \in IF(X)$ denoted respectively by $int(A)$ and $cl(A)$, are defined as follows:

$$int(A) \text{ or } int_{\tau}(A) = \bigcup\{B \in \tau : B \subseteq A\},$$

$$cl(A) \text{ or } cl_{\tau}(A) = \bigcap\{B \in \tau^c : B \supseteq A\}.$$

An IF topology τ is called Alexandrov, if (i) and (ii) in Definition 2.8 are replaced by

- (i)' For each $\lambda \in J, \hat{\lambda} \in \tau$.
- (ii)' $\{A_{\alpha} : \alpha \in \Gamma\} \subseteq \tau$ implies $\bigcap_{\alpha \in \Gamma} A_{\alpha} \in \tau$.

Proposition 2.9 ([5]). Let (X, τ) be an IF topological space. Then

- (1) τ^* is the fuzzy topology on X in Chang' sense.
- (2) $(\tau_*)^c = \{(A_*)^c : A \in \tau\}$ is the fuzzy topology on X in Chang' sense and τ_* is the family of all fuzzy closed sets in X .

Proposition 2.10 ([5]). Let (X, τ) be an IF topological space and $A \in IF(X)$. Then

- (1) If $A \in \tau$, then $A^* \in \tau^*$ and $A_* \in (\tau_*)^c$.
- (2) If $A \in \tau^c$, then $A^* \in (\tau_*)^c$ and $A_* \in \tau^*$.

2.3 Fuzzy rough sets

Recall that R is called a fuzzy relation on X if $R \in F(X \times X)$.

Definition 2.11 ([18]). Let R be a fuzzy relation on X . Then the pair (X, R) is called a fuzzy approximation space. Based on (X, R) , the fuzzy lower and the fuzzy upper approximation of $A \in F(X)$ with respect to (X, R) , denoted by $\underline{R}(A)$ and $\overline{R}(A)$ are respectively, defined as follows:

$$\underline{R}(A)(x) = \bigwedge_{y \in X} (A(y) \vee (1 - R(x, y))) \quad (x \in X)$$

and

$$\overline{R}(A)(x) = \bigvee_{y \in X} (A(y) \wedge R(x, y)) \quad (x \in X).$$

The pair $(\underline{R}(A), \overline{R}(A))$ is called the fuzzy rough set of A with respect to (X, R) .

$\underline{R} : F(X) \rightarrow F(X)$ and $\overline{R} : F(X) \rightarrow F(X)$ are called the fuzzy lower approximation operator and the fuzzy upper approximation operator, respectively. In general, we refer to \underline{R} and \overline{R} as the fuzzy rough approximation operators.

Proposition 2.12 ([18]). Let (X, R) be a fuzzy approximation space. Then for any $A, B \in F(X)$, $\{A_\alpha : \alpha \in \Gamma\} \subseteq F(X)$ and $\lambda \in I$,

- (1) $\underline{R}(\overline{1}) = \overline{1}$, $\overline{R}(\overline{0}) = \overline{0}$.
- (2) $A \subseteq B \implies \underline{R}(A) \subseteq \underline{R}(B)$, $\overline{R}(A) \subseteq \overline{R}(B)$.
- (3) $\underline{R}(A^c) = (\overline{R}(A))^c$, $\overline{R}(A^c) = (\underline{R}(A))^c$.
- (4) $\underline{R}(\bigcap_{\alpha \in \Gamma} A_\alpha) = \bigcap_{\alpha \in \Gamma} (\underline{R}(A_\alpha))$, $\overline{R}(\bigcup_{\alpha \in \Gamma} A_\alpha) = \bigcup_{\alpha \in \Gamma} (\overline{R}(A_\alpha))$.
- (5) $\underline{R}(\overline{\lambda} \cup A) = \overline{\lambda} \cup \underline{R}(A)$, $\overline{R}(\lambda A) = \lambda \overline{R}(A)$.

3 IF approximation spaces and IF rough sets

In this section, we investigate properties related to IF approximation spaces.

An IF relation R on X is an IF set in $X \times X$ ([3]), we write $R \in IF(X \times X)$, namely,

$$R = (R^*, R_*)$$

$$R(x, y) = (R^*(x, y), R_*(x, y)) \in J \text{ for any } x, y \in X$$

where

R^* and R_* are two fuzzy relations on X .

Let R be an IF relation on a finite set X . R may be represent by a matrix. That is, if $X = \{t_1, t_2, \dots, t_n\}$, then R may be represented by the following matrix

$$\begin{pmatrix} R(t_1, t_1) & R(t_1, t_2) & \dots & R(t_1, t_n) \\ R(t_2, t_1) & R(t_2, t_2) & \dots & R(t_2, t_n) \\ \vdots & \vdots & & \vdots \\ R(t_n, t_1) & R(t_n, t_2) & \dots & R(t_n, t_n) \end{pmatrix}.$$

Definition 3.1 ([3]). Let R be an IF relation on X . Then R is called

- (1) reflexive, if $R(x, x) = (1, 0)$ for each $x \in X$.
- (2) symmetric, if $R(x, y) = R(y, x)$ for any $x, y \in X$.
- (3) transitive, if $R(x, z) \geq R(x, y) \sqcap R(y, z)$ for any $x, y, z \in X$.

Remark 3.2. Let R be an IF relation on X . Then

- (1) If R is reflexive, then R^* and $(R_*)^c$ are reflexive.
- (2) If R is symmetric, then R^* and $(R_*)^c$ are symmetric.
- (3) If R is transitive, then R^* and $(R_*)^c$ are transitive.

Let R be an IF relation on X . R is called preorder if R is reflexive and transitive. R^c is called the dual of R if $R^c(x, y) = (R_*(x, y), R^*(x, y))$ for any $x, y \in X$.

Definition 3.3 ([29]). Let R be an IF relation on X . Then the pair (X, R) is called an IF approximation space. Based on (X, R) , the IF lower and the IF upper approximation of $A \in IF(X)$ with respect to (X, R) , denoted by $\underline{R}(A)$ and $\overline{R}(A)$, are two IF sets and are respectively defined as follows:

$$\begin{aligned} \underline{R}(A)(x) &= ((\underline{R}(A))^*(x), (\underline{R}(A))_*(x)) \quad (x \in X), \\ \overline{R}(A)(x) &= ((\overline{R}(A))^*(x), (\overline{R}(A))_*(x)) \quad (x \in X), \end{aligned}$$

where

$$\begin{aligned} (\underline{R}(A))^*(x) &= \bigwedge_{y \in X} (A^*(y) \vee R_*(x, y)), & (\underline{R}(A))_*(x) &= \bigvee_{y \in X} (A_*(y) \wedge R^*(x, y)), \\ (\overline{R}(A))^*(x) &= \bigvee_{y \in X} (A^*(y) \wedge R^*(x, y)), & (\overline{R}(A))_*(x) &= \bigwedge_{y \in X} (A_*(y) \vee R_*(x, y)). \end{aligned}$$

The pair $(\underline{R}(A), \overline{R}(A))$ is called the IF rough set of A with respect to (X, R) . $\underline{R} : IF(X) \rightarrow IF(X)$ and $\overline{R} : IF(X) \rightarrow IF(X)$ are called the IF lower approximation operator and the IF upper approximation operator, respectively. In general, we refer to \underline{R} and \overline{R} as the IF rough approximation operators.

Remark 3.4 ([29]). Let (X, R) be an IF approximation space. Then

$$\overline{R}(1_x)(y) = R(y, x) \quad \text{and} \quad \underline{R}(0_x)(y) = R^c(y, x) \quad (x, y \in X).$$

Proposition 3.5. Let (X, R) be an IF approximation space. Then for any $A \in IF(X)$ and $x \in X$,

$$\begin{aligned} (1) \quad & (\underline{R}(A))^* = \underline{(R_*)^c(A^*)}, \quad (\underline{R}(A))_* = \overline{R^*(A_*)}, \\ & (\overline{R}(A))^* = \overline{R^*(A^*)}, \quad (\overline{R}(A))_* = \underline{(R_*)^c(A_*)}. \\ (2) \quad & \underline{R}(A)(x) = \bigcap_{y \in X} (A(y) \sqcup R^c(x, y)), \quad \overline{R}(A)(x) = \bigcup_{y \in X} (A(y) \sqcap R(x, y)). \end{aligned}$$

Proof. (1) This is obvious.

(2) For any $A \in IF(X)$ and $x \in X$, since

$$\begin{aligned} \bigcap_{y \in X} (A(y) \sqcup R^c(x, y)) &= \bigcap_{y \in X} ((A^*(y), A_*(y)) \sqcup (R_*(x, y), R^*(x, y))) \\ &= \bigcap_{y \in X} (A^*(y) \vee R_*(x, y), A_*(y) \wedge R^*(x, y)) \\ &= (\bigwedge_{y \in X} (A^*(y) \vee R_*(x, y)), \bigvee_{y \in X} (A_*(y) \wedge R^*(x, y))) \\ &= ((\underline{R}(A))^*(x), (\underline{R}(A))_*(x)) \\ &= \underline{R}(A)(x), \end{aligned}$$

we have $\underline{R}(A)(x) = \bigcap_{y \in X} (A(y) \sqcup R^c(x, y))$.

Similarly, we can prove $\overline{R}(A)(x) = \bigcup_{y \in X} (A(y) \sqcap R(x, y))$. □

Proposition 3.6 ([29]). *Let (X, R) be an IF approximation space. Then for any $A, B \in IF(X)$, $\{A_\alpha : \alpha \in \Gamma\} \subseteq IF(X)$ and $\lambda \in J$,*

- (1) $\underline{R}(1_\sim) = 1_\sim, \overline{R}(0_\sim) = 0_\sim;$
- (2) $A \subseteq B \implies \underline{R}(A) \subseteq \underline{R}(B), \overline{R}(A) \subseteq \overline{R}(B);$
- (3) $\underline{R}(A^c) = (\overline{R}(A))^c, \overline{R}(A^c) = (\underline{R}(A))^c;$
- (4) $\underline{R}(\bigcap_{\alpha \in \Gamma} A_\alpha) = \bigcap_{\alpha \in \Gamma} (\underline{R}(A_\alpha)), \overline{R}(\bigcup_{\alpha \in \Gamma} A_\alpha) = \bigcup_{\alpha \in \Gamma} (\overline{R}(A_\alpha));$
- (5) $\underline{R}(\widehat{\lambda} \cup A) = \widehat{\lambda} \cup \underline{R}(A), \overline{R}(\lambda A) = \lambda \overline{R}(A).$

Theorem 3.7 ([29]). *Let (X, R) be an IF approximation space. Then*

- (1) *R is reflexive* $\iff (ILR) \forall A \in IF(X), \underline{R}(A) \subseteq A.$
 $\iff (IUR) \forall A \in IF(X), A \subseteq \overline{R}(A).$
- (2) *R is symmetric* $\iff (ILS) \forall x, y \in X, \underline{R}(0_x)(y) = \underline{R}(0_y)(x).$
 $\iff (IUS) \forall x, y \in X, \overline{R}(1_x)(y) = \overline{R}(1_y)(x).$
- (3) *R is transitive* $\iff (ILT) \forall A \in IF(X), \underline{R}(A) \subseteq \underline{R}(\underline{R}(A)).$
 $\iff (IUT) \forall A \in IF(X), \overline{R}(\overline{R}(A)) \subseteq \overline{R}(A).$

Proposition 3.8. *Let (X, R) be an IF approximation space.*

- (1) *For each $\lambda \in J$,*

$$\underline{R}(\widehat{\lambda}) \supseteq \widehat{\lambda} \supseteq \overline{R}(\widehat{\lambda}).$$

- (2) *If R is reflexive, then for each $\lambda \in J$,*

$$\underline{R}(\widehat{\lambda}) = \widehat{\lambda} = \overline{R}(\widehat{\lambda}).$$

Proof. (1) For any $\lambda \in J$ and $x \in X$, by Proposition 3.5(2),

$$\overline{R}(\widehat{\lambda})(x) = \bigsqcup_{y \in X} (\lambda \cap R(x, y)) = \lambda \cap (\bigsqcup_{y \in X} R(x, y)) \leq \lambda.$$

Hence $\widehat{\lambda} \supseteq \overline{R}(\widehat{\lambda})$. By Proposition 3.6(3),

$$\underline{R}(\widehat{\lambda}) = (\overline{R}(\widehat{\lambda}^c))^c = (\overline{R}(\widehat{\lambda}^c))^c \supseteq (\widehat{\lambda}^c)^c = \widehat{\lambda}.$$

- (2) This holds by (1) and Theorem 3.7(1). □

Theorem 3.9. *Let R be an IF relation on X and let τ be an IF topology on X . If one of the following conditions is satisfied, then R is preorder.*

- (1) \overline{R} is the closure operator of τ .
- (2) \underline{R} is the interior operator of τ .

Proof. (1) By Remark 3.4, $\overline{R}(1_x)(y) = R(y, x)$ for any $x, y \in X$. Note that \underline{R} is the interior operator of τ . Then for each $x \in X$,

$$R(x, x) = \overline{R}(1_x)(x) = cl_\tau(1_x)(x) \geq 1_x(x) = 1,$$

Thus R is reflexive.

For any $x, y, z \in X$, denote $cl_\tau(1_z)(y) = \lambda$, by Remark 2.7, Remark 3.4 and Proposition 3.6(5),

$$\begin{aligned} R(x, y) \sqcap R(y, z) &= \overline{R}(1_y)(x) \sqcap \overline{R}(1_z)(y) = \overline{R}(1_y)(x) \sqcap cl_\tau(1_z)(y) \\ &= \overline{R}(1_y)(x) \sqcap \lambda = \lambda \overline{R}(1_y)(x) = \overline{R}(\lambda 1_y)(x) \\ &= cl_\tau(\lambda 1_y)(x) = cl_\tau(cl_\tau(1_z)(y) 1_y)(x) \\ &\leq cl_\tau\left(\bigcup_{t \in X} (cl_\tau(1_z)(t) 1_t)\right)(x) \\ &= cl_\tau(cl_\tau(1_z))(x) = cl_\tau(1_z)(x) = R(x, z). \end{aligned}$$

Then R is transitive. Hence R is preorder.

(2) The proof is similar to (1). □

4 Relationships between IF relations and IF topologies

In this section we establish relationships between IF relations and IF topologies.

4.1 IF topologies induced by IF relations

For $R \in IF(X \times X)$, we denote

$$\tau_R = \{A \in IF(X) : A = \underline{R}(A)\}, \quad \theta_R = \{\underline{R}(A) : A \in IF(X)\}.$$

Proposition 4.1. *Let (X, R) be an IF approximation space. If R is preorder, then*

$$\tau_R = \theta_R.$$

Proof. Obviously, $\tau_R \subseteq \theta_R$. For each $\underline{R}(A) \in \theta_R$, by Theorem 3.7, $\underline{R}(\underline{R}(A)) = \underline{R}(A)$. So $\underline{R}(A) \in \tau_R$. Thus $\theta_R \subseteq \tau_R$. Hence $\tau_R = \theta_R$. □

Theorem 4.2 ([29]). *Let R be a preorder IF relation. Then*

- (1) θ_R is an IF topology on X .
- (2) \underline{R} is the interior operator of θ_R .
- (3) \overline{R} is the closure operator of θ_R .

Theorem 4.3. *Let R be a reflexive IF relation. Then*

- (1) τ_R is an Alexandrov IF topology on X .
- (2) For each $A \in IF(X)$,

$$int_{\tau_R}(A) \subseteq \underline{R}(A) \subseteq A \subseteq \overline{R}(A) \subseteq cl_{\tau_R}(A).$$

- (3) $A \in (\tau_R)^c \iff A = \overline{R}(A)$.
- (4) For each $\lambda \in J$, $\widehat{\lambda} \in (\tau_R)^c$.
- (5) $(\tau_R)^* = \tau_{(R^*)^c}$ where $\tau_{(R^*)^c} = \{V \in F(X) : (R^*)^c(V) = V\}$.
 $(\tau_R)_* = (\tau_{R^*})^c$ where $\tau_{R^*} = \{V \in F(X) : \underline{R}^*(V) = V\}$.

Proof. (1) (i) For each $\lambda \in J$, by Proposition 3.8(2), $\underline{R}(\widehat{\lambda}) = \widehat{\lambda}$. Then $\widehat{\lambda} \in \tau_R$.

(ii) Let $\{A_\alpha : \alpha \in \Gamma\} \subseteq \tau_R$. Then $\underline{R}(A_\alpha) = A_\alpha$ for each $\alpha \in \Gamma$. By Proposition 3.6(4),

$$\underline{R}\left(\bigcap_{\alpha \in \Gamma} A_\alpha\right) = \bigcap_{\alpha \in \Gamma} \underline{R}(A_\alpha) = \bigcap_{i \in J} A_\alpha.$$

Hence $\bigcap_{\alpha \in \Gamma} A_\alpha \in \tau_R$.

(iii) Let $\{A_\alpha : \alpha \in \Gamma\} \subseteq \tau_R$. Then $\underline{R}(A_\alpha) = A_\alpha$ for each $\alpha \in \Gamma$. By the reflexivity of R and Theorem 3.7(1), $\underline{R}\left(\bigcup_{\alpha \in \Gamma} A_\alpha\right) \subseteq \bigcup_{\alpha \in \Gamma} A_\alpha$. Note that

$$\underline{R}\left(\bigcup_{\alpha \in \Gamma} A_\alpha\right) \supseteq \bigcup_{\alpha \in \Gamma} \underline{R}(A_\alpha) = \bigcup_{\alpha \in \Gamma} A_\alpha.$$

Then $\underline{R}\left(\bigcup_{\alpha \in \Gamma} A_\alpha\right) = \bigcup_{\alpha \in \Gamma} A_\alpha$. Hence $\bigcup_{\alpha \in \Gamma} A_\alpha \in \tau_R$.

So τ_R is an Alexandrov IF topology on X .

(2) For each $A \in IF(X)$, by Proposition 3.6(2),

$$\begin{aligned} \text{int}_{\tau_R}(A) &= \bigcup\{B \in \tau_R : B \subseteq A\} \subseteq \bigcup\{B \in \tau_R : \underline{R}(B) \subseteq \underline{R}(A)\} \\ &= \bigcup\{B \in IF(X) : B = \underline{R}(B) \subseteq \underline{R}(A)\} \subseteq \underline{R}(A). \end{aligned}$$

By Proposition 3.6(3),

$$\text{cl}_{\tau_R}(A) = (\text{int}_{\tau_R}(A^c))^c \supseteq (\underline{R}(A^c))^c = \overline{R}(A).$$

By the reflexivity of R and Proposition 3.6(1),

$$\text{int}_{\tau_R}(A) \subseteq \underline{R}(A) \subseteq A \subseteq \overline{R}(A) \subseteq \text{cl}_{\tau_R}(A).$$

(3) This holds by Proposition 3.6(3).

(4) This holds by (3) and Proposition 3.8(2).

(5) Let $V \in (\tau_R)^*$. Then $A^* = V$ for some $A \in \tau_R$ and so $\underline{R}(A) = A$. By Proposition 3.5(1),

$$(\underline{R}_*)^c(V) = (\underline{R}_*)^c(A^*) = (\underline{R}(A))^* = A^* = V.$$

So $V \in \tau_{(\underline{R}_*)^c}$. Thus $(\tau_R)^* \subseteq \tau_{(\underline{R}_*)^c}$.

Let $V \in \tau_{(\underline{R}_*)^c}$. Put $A = (V, \bar{0})$. By Remark 3.2, $(\underline{R}_*)^c$ is reflexive. Then $(\underline{R}_*)^c(\bar{0}) = \bar{0}$. Thus $A^* = V$, $A_* = \bar{0}$. By Proposition 3.5(1), we have

$$(\underline{R}(A))^* = (\underline{R}_*)^c(A^*) = (\underline{R}_*)^c(V) = V = A^*$$

and

$$(\underline{R}(A))_* = (\underline{R}^*)^c(A_*) = (\underline{R}^*)^c(\bar{0}) = \bar{0} = A_*.$$

Then $\underline{R}(A) = A$ and so $A \in \tau_R$. This implies that $V = A^* \in (\tau_R)^*$. Thus $(\tau_R)^* \supseteq \tau_{(\underline{R}_*)^c}$.

Hence $(\tau_R)^* = \tau_{(\underline{R}_*)^c}$.

Similarly, we can prove that $(\tau_R)_* = (\tau_{R^*})^c$. □

Definition 4.4. Let R be a reflexive IF relation. Then τ_R is called the IF topology induced by R on X .

Example 4.5. Let $U = \{x, y, z, w\}$ and let R be an IF relation on X where

$$R = \begin{pmatrix} (0, 1) & (1, 0) & (1, 0) & (1, 0) \\ (1, 0) & (0, 1) & (1, 0) & (1, 0) \\ (1, 0) & (1, 0) & (0, 1) & (1, 0) \\ (1, 0) & (1, 0) & (1, 0) & (0, 1) \end{pmatrix}.$$

Then R is not reflexive.

For any $A \in IF(X)$ and $t \in X$,

$$\underline{R}(A)(t) = \bigcap_{s \in X} (A(s) \sqcup R^c(t, s)) = \bigcap_{s \in X - \{t\}} A(s).$$

Suppose that $A(x) \leq A(y) \leq A(z) \leq A(w)$. Since $\underline{R}(A) = A$, we have

$$A(x) \wedge A(y) \wedge A(z) = A(w).$$

Then $A(t) \geq A(w)$ for each $t \in \{x, y, z\}$. So $A(x) = A(y) = A(z) = A(w)$.

Thus $\tau_R = \{\hat{\lambda} : \lambda \in J\}$.

Obviously, τ_R is an Alexandrov IF topology on X .

4.2 IF relations induced by IF topologies

Definition 4.6. Let τ be an IF topology on X . Define an IF relation R_τ on X by

$$R_\tau(x, y) = cl_\tau(1_y)(x)$$

for each $x, y \in X$. Then R_τ is called the IF relation induced by τ on X and (X, R_τ) is called the IF approximation space induced by τ on X .

Theorem 4.7. Let τ be an IF topology on X and let R_τ be the IF relation induced by τ on X . Then the following properties hold.

- (1) R_τ is reflexive.
- (2) If $\{\hat{\lambda} : \lambda \in J\} \subseteq \tau^c$, then for each $A \in IF(X)$,

$$\underline{R}_\tau(A) \subseteq int_\tau(A) \subseteq A \subseteq cl_\tau(A) \subseteq \overline{R}_\tau(A).$$

Proof. (1) For each $x \in X$,

$$R_\tau(x, x) = cl_\tau(1_x)(x) \geq (1_x)(x) = (1, 0).$$

Then R_τ is reflexive.

- (2) For each $A \in IF(X)$, by Remark 2.7 and Proposition 3.6(2),

$$\begin{aligned} cl_\tau(A) &= cl_\tau\left(\bigcup_{y \in X} (A(y)1_y)\right) = \bigcup_{y \in X} cl_\tau(A(y)1_y) = \bigcup_{y \in X} cl_\tau(\widehat{A(y)} \cap 1_y) \\ &\subseteq \bigcup_{y \in X} (cl_\tau(\widehat{A(y)}) \cap cl_\tau(1_y)) = \bigcup_{y \in X} (\widehat{A(y)} \cap cl_\tau(1_y)). \end{aligned}$$

Then for each $x \in X$,

$$cl_\tau(A)(x) \leq \bigsqcup_{y \in X} (\widehat{A(y)}(x) \sqcap cl_\tau(1_y)(x)) = \bigsqcup_{y \in X} (A(y) \sqcap R_\tau(x, y)) = \overline{R_\tau}(A)(x).$$

Hence $cl_\tau(A) \subseteq \overline{R_\tau}(A)$.

By Proposition 3.6(3),

$$int_\tau(A) = (cl_\tau(A^c))^c \supseteq (\overline{R_\tau}(A^c))^c = \underline{R_\tau}(A).$$

So

$$\underline{R_\tau}(A) \subseteq int_\tau(A) \subseteq A \subseteq cl_\tau(A) \subseteq \overline{R_\tau}(A).$$

□

Theorem 4.8 ([23]). *Let R be a reflexive IF relation on X , let τ_R be the IF topology by R on X and let R_{τ_R} be the IF relation induced by τ_R on X . If R is transitive, then $R_{\tau_R} = R$.*

4.3 (C_1) and (C_2) axioms

The following conditions for an IF topology τ on X are respectively called (C_1) axiom and (C_2) axiom: for any $\lambda \in J$, $A \in IF(X)$ and $\{A_\alpha : \alpha \in \Gamma\} \subseteq IF(X)$,

$$(C_1) \text{ axiom : } cl_\tau(\lambda A) = \lambda cl_\tau(A); \quad (C_2) \text{ axiom : } cl_\tau\left(\bigcup_{\alpha \in \Gamma} A_\alpha\right) = \bigcup_{\alpha \in \Gamma} cl_\tau(A_\alpha).$$

Proposition 4.9. *Let τ be an IF topology on U . If τ satisfies (C_1) and (C_2) axioms, then*

- (1) $\overline{R_\tau}$ is the closure operator of τ .
- (2) $\underline{R_\tau}$ is the interior operator of τ .
- (3) For each $\lambda \in J$, $\widehat{\lambda} \in \tau$.
- (4) τ is Alexandrov.

Proof. (1) For each $A \in IF(X)$, by Remark 2.7, (C_1) axiom and (C_2) axiom,

$$cl_\tau(A) = cl_\tau\left(\bigcup_{y \in X} (A(y)1_y)\right) = \bigcup_{y \in X} cl_\tau(A(y)1_y) = \bigcup_{y \in X} (A(y)cl_\tau(1_y)).$$

Then for each $x \in X$,

$$cl_\tau(A)(x) = \bigsqcup_{y \in X} (\widehat{A(y)}(x) \sqcap cl_\tau(1_y)(x)) = \bigsqcup_{y \in X} (A(y) \sqcap R_\tau(x, y)) = \overline{R_\tau}(A)(x).$$

Thus $\overline{R_\tau}(A) = cl_\tau(A)$. So $\overline{R_\tau}$ is the closure operator of τ .

- (2) This holds by (1) and Proposition 3.6(3).
- (3) For each $\lambda \in J$, by (2) and Proposition 3.8(1),

$$\widehat{\lambda} \supseteq int_\tau(\widehat{\lambda}) = \underline{R}(\widehat{\lambda}) \supseteq \widehat{\lambda}.$$

Then $int_\tau(\widehat{\lambda}) = \widehat{\lambda}$ and so $\widehat{\lambda} \in \tau$.

(4) Suppose that τ satisfies (C_1) and (C_2) axioms. For each $\lambda \in J$, by (1) and Proposition 3.6(3),

$$int_\tau(\widehat{\lambda}) = (cl_\tau(\widehat{\lambda}^c))^c = (\overline{R}(\widehat{\lambda}^c))^c = \underline{R}(\widehat{\lambda}) \supseteq \widehat{\lambda}.$$

Note that $int_\tau(\widehat{\lambda}) \subseteq \widehat{\lambda}$. Then $int_\tau(\widehat{\lambda}) = \widehat{\lambda}$. Thus $\widehat{\lambda} \in \tau$.

For each $A \in IF(X)$, by (1), (C_1) axiom and (C_2) axiom, $\overline{R}(A^c) = cl_\tau(A^c)$. Then $\underline{R}(A) = int_\tau(A)$.

Let $\{A_\alpha : \alpha \in \Gamma\} \subseteq \tau$. Note that $\underline{R}(A_\alpha) = int_\tau(A_\alpha)$. Then $A_\alpha = \underline{R}(A_\alpha)$. By Proposition 3.6(4),

$$\bigcap_{\alpha \in \Gamma} A_\alpha = \bigcap_{\alpha \in \Gamma} \underline{R}(A_\alpha) = \underline{R}(\bigcap_{\alpha \in \Gamma} A_\alpha) = int_\tau(\bigcap_{\alpha \in \Gamma} A_\alpha).$$

So $\bigcap_{\alpha \in \Gamma} A_\alpha \in \tau$. Hence τ is Alexandrov. □

Proposition 4.10 ([29]). *Let R be a preorder IF relation on X . Then τ_R satisfies (C_1) and (C_2) axioms.*

Theorem 4.11. *Let τ be an IF topology on X , let R_τ be the IF relation induced by τ on X and let τ_{R_τ} be the IF topology induced by R_τ on X . Then*

$$\tau_{R_\tau} = \tau \quad \text{if and only if } \tau \text{ satisfies } (C_1) \text{ and } (C_2) \text{ axioms.}$$

Proof. Necessity. For each $A \in IF(X)$, by Theorem 4.7(2), $\underline{R}_\tau(A) \subseteq int_\tau(A)$. By Theorem 4.3(2),

$$int_\tau(A) = int_{\tau_{R_\tau}}(A) \subseteq \underline{R}_\tau(A).$$

Then $int_\tau(A) = \underline{R}_\tau(A)$. So \underline{R}_τ is the interior operator of τ . By Theorem 3.9(2), R_τ is a preorder IF relation on X . By Theorem 4.10, τ_{R_τ} satisfies (C_1) and (C_2) axioms.

Sufficiency. By Theorem 4.7(1), R_τ is reflexive. For any $x, y, z \in X$, put $cl_\tau(1_z)(y) = \lambda$. By Remark 2.7, Proposition 3.6(2),

$$\begin{aligned} \lambda cl_\tau(1_y) &= cl_\tau(\lambda 1_y) = cl_\tau(cl_\tau(1_z)(y) 1_y) \\ &\subseteq cl_\tau\left(\bigcup_{t \in X} (cl_\tau(1_z)(t) 1_t)\right) = cl_\tau(cl_\tau(1_z)) = cl_\tau(1_z). \end{aligned}$$

Then

$$\begin{aligned} R_\tau(x, y) \sqcap R_\tau(y, z) &= cl_\tau(1_y)(x) \sqcap cl_\tau(1_z)(y) = cl_\tau(1_y)(x) \sqcap \lambda \\ &= \lambda \sqcap cl_\tau(1_y)(x) = (\lambda cl_\tau(1_y))(x) \\ &\leq cl_\tau(1_z)(x) = R_\tau(x, z). \end{aligned}$$

Then R_τ is transitive.

So R_τ is preorder. For each $A \in IF(X)$, by Proposition 4.1 and Theorem 4.2(3),

$$cl_{\tau_{R_\tau}}(A) = cl_{\theta_{R_\tau}}(A) = \overline{R_\tau}(A).$$

By (C_1) axiom, (C_2) axiom and Proposition 4.9(1), $\overline{R_\tau}(A) = cl_\tau(A)$. So $cl_{\tau_{R_\tau}}(A) = cl_\tau(A)$. Thus $\tau_{R_\tau} = \tau$. \square

Theorem 4.12. *Let*

$$\Sigma = \{R : R \text{ is a preorder IF relation on } X\}$$

and

$$\Gamma = \{\tau : \tau \text{ is an IF topology on } X \text{ satisfying } (C_1) \text{ and } (C_2) \text{ axioms}\}.$$

Then there exists a one-to-one correspondence between Σ and Γ .

Proof. Two mappings $f : \Sigma \rightarrow \Gamma$ and $g : \Gamma \rightarrow \Sigma$ are defined as follows:

$$\begin{aligned} f(R) &= \tau_R \quad (R \in \Sigma), \\ g(\tau) &= R_\tau \quad (\tau \in \Gamma). \end{aligned}$$

By Theorem 4.8,

$$g \circ f = i_\Sigma,$$

where $g \circ f$ is the composition of f and g , and i_Σ is the identity mapping on Γ .

By Theorem 4.11,

$$f \circ g = i_\Gamma,$$

where $f \circ g$ is the composition of g and f , and i_Γ is the identity mapping on Σ .

Hence f and g are two one-to-one correspondences. This prove that there exists a one-to-one correspondence between Σ and Γ . \square

5 IF approximating spaces

As can be seen from Section 4, a reflexive IF relation yields an IF topology. In this section, we consider the reverse problem, that is, under which conditions can an IF topology be associated with an IF relation which produces the given IF topology?

Definition 5.1. *Let (X, τ) be an IF topological space. If there exists a reflexive IF relation R on X such that $\tau_R = \tau$, then (X, τ) is called an IF approximating space.*

Theorem 5.2. *Let τ be an IF topology on X . Then the following are equivalent.*

- (1) τ satisfies (C_1) and (C_2) axioms;
- (2) For any $\lambda \in J$, $A \in IF(X)$ and $\{A_\alpha : \alpha \in \Gamma\} \subseteq IF(X)$,

$$int_\tau(\widehat{\lambda} \cup A) = \widehat{\lambda} \cup int_\tau(A), \quad int_\tau\left(\bigcap_{\alpha \in \Gamma} A_\alpha\right) = \bigcap_{\alpha \in \Gamma} int_\tau(A_\alpha).$$

(3) There exists a preorder IF relation ρ on X such that $\bar{\rho}$ is the closure operator of τ ;

(4) There exists a preorder IF relation ρ on X such that $\underline{\rho}$ is the interior operator of τ ;

(5) \bar{R}_τ is the closure operator of τ ;

(6) \underline{R}_τ is the interior operator of τ .

Proof. (1) \iff (2). This is obvious.

(1) \implies (3). Suppose that τ satisfies (C_1) and (C_2) axioms. Pick $\rho = R_\tau$. By Theorem 4.9, $\bar{\rho}$ is the closure operator of τ . By Theorem 3.9(1), ρ is preorder.

(3) \implies (4). Let $\bar{\rho}$ be the closure operator of τ for some preorder IF relation ρ on X . For each $A \in IF(X)$, by Proposition 3.6(3),

$$\underline{\rho}(A) = (\bar{\rho}(A^c))^c = (cl_\tau(A^c))^c = int_\tau(A).$$

Thus, $\underline{\rho}$ is the interior operator of τ .

(4) \implies (6). Let $\underline{\rho}$ be the interior operator of τ for some preorder IF relation ρ on X .

For $x, y \in X$, by Remark 3.4,

$$\rho(x, y) = (\underline{\rho}((1_y)^c)(x))^c = (int_\tau((1_y)^c)(x))^c = cl_\tau(1_y)(x) = R_\tau(x, y).$$

Then $\rho = R_\tau$. Note that $\underline{\rho}$ is the interior operator of τ . Then \underline{R}_τ is the interior operator of τ .

(6) \implies (5) holds by Proposition 3.6(3).

(5) \implies (1). For any $\lambda \in J$ and $A \in IF(X)$, by Proposition 3.6,

$$cl_\tau(\lambda A) = \bar{R}_\tau(\lambda A) = \lambda \bar{R}_\tau(A) = \lambda cl_\tau(A),$$

and

$$cl_\tau\left(\bigcup_{\alpha \in \Gamma} A_\alpha\right) = \bar{R}_\tau\left(\bigcup_{\alpha \in \Gamma} A_\alpha\right) = \bigcup_{\alpha \in \Gamma} \bar{R}_\tau(A_\alpha) = \bigcup_{\alpha \in \Gamma} cl_\tau(A_\alpha)$$

Thus τ satisfies (C_1) and (C_2) axioms. □

Theorem 5.3. *Let (X, τ) be an IF topological space. If one of the following conditions is satisfied, then (X, τ) is an IF approximating space.*

(1) τ satisfies (C_1) and (C_2) axioms.

(2) For any $\lambda \in J$, $A \in IF(X)$ and $\{A_\alpha : \alpha \in \Gamma\} \subseteq IF(X)$,

$$int_\tau(\widehat{\lambda} \cup A) = \widehat{\lambda} \cup int_\tau(A), \quad int_\tau\left(\bigcap_{\alpha \in \Gamma} A_\alpha\right) = \bigcap_{\alpha \in \Gamma} int_\tau(A_\alpha).$$

(3) There exists a preorder IF relation R on X such that \bar{R} is the closure operator of τ .

(4) There exists a preorder IF relation R on X such that \underline{R} is the interior operator of τ .

(5) \bar{R}_τ is the closure operator of τ .

(6) \underline{R}_τ is the interior operator of τ .

Proof. These hold by Theorems 4.11 and 5.2. □

Example 5.4. $\{\widehat{\lambda} : \lambda \in J\}$ is an IF approximating space.

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On Cauchy problems with Caputo Hadamard fractional derivatives

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Abstract

The current work is motivated by the so-called Caputo-type modification of the Hadamard or Caputo Hadamard fractional derivative discussed in [4]. The main aim of this paper is to study Cauchy problems for a differential equation with a left Caputo Hadamard fractional derivative in spaces of continuously differentiable functions. The equivalence of this problem to a nonlinear Volterra type integral equation of the second kind is shown. On the basis of the obtained results, the existence and uniqueness of the solution to the considered Cauchy problem is proved by using Banach's fixed point theorem. Finally, two examples are provided to explain the applications of the results.

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Keywords: Caputo Hadamard fractional derivatives, Cauchy problem, Volterra integral equation, continuously differentiable function, fixed point theorem.

1 Introduction

Fractional calculus, that is, the theory of derivatives and integrals of fractional non-integer order, are used in many fields like: mathematics, physics, chemistry, engineering, and other sciences.

Few years ago, many scholars started making deeper researches on fractional differential equations. Intensive development of this latter and its applications led to that. (e.g.; [1, 2, 3, 10, 11, 12]). Many definitions were supplied for the Fractional order differential operators and many reports on the existence and uniqueness of solutions to differential equations in the frame of these operators appeared. (see for example [14] and the references therein).

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J. Hadamard [6] in 1892, introduced a new definition of fractional derivatives and integrals in which he claims:

$$\left(\mathcal{J}_{a+}^{\alpha} g\right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{\tau}\right)^{\alpha-1} g(\tau) \frac{d\tau}{\tau}, \quad (0 < a < t), \quad Re(\alpha) > 0, \quad (1)$$

for suitable functions g , where Γ represents gamma function. This is the generalization of the n^{th} integral

$$\left(\mathcal{J}_{a+}^n g\right)(t) = \int_a^t \frac{d\tau_1}{\tau_1} \int_a^{\tau_1} \frac{d\tau_2}{\tau_2} \dots \int_a^{\tau_{n-1}} g(\tau_n) \frac{d\tau_n}{\tau_n} \equiv \frac{1}{\Gamma(n)} \int_a^t \left(\ln \frac{t}{\tau}\right)^{n-1} g(\tau) \frac{d\tau}{\tau}, \quad (2)$$

where $n = [Re(\alpha)] + 1$ and $[Re(\alpha)]$ means the integer part of $Re(\alpha)$.

The corresponding left-sided Hadamard fractional derivative of order α is defined by

$$\left(\mathcal{D}_{a+}^{\alpha} g\right)(t) = \delta^n \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(\ln \frac{t}{\tau}\right)^{n-\alpha-1} g(\tau) \frac{d\tau}{\tau}, \quad \alpha \in [n-1, n), \quad (3)$$

where $\delta = t \frac{d}{dt}$. The main difference between Hadamard's definition and the previous ones is that the kernel integral contains logarithmic function of arbitrary exponent. The present paper follows the Caputo-type definition based on the modification of Hadamard fractional derivatives. This approach is given by the equality,

$$\left({}^c\mathcal{D}_{a+}^{\alpha} g\right)(t) = \left(\mathcal{D}_{a+}^{\alpha} g\right) \left[g(\tau) - \sum_{k=0}^{n-1} \frac{\delta^k g(a)}{k!} \left(\ln \frac{\tau}{a}\right)^k \right] (t), \quad (0 < a < t). \quad (4)$$

We can use the following equivalent representation, which follows from (3) and (4)

$$\left({}^c\mathcal{D}_{a+}^{\alpha} g\right)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(\log \frac{t}{\tau}\right)^{n-\alpha-1} \delta^n g(\tau) \frac{d\tau}{\tau}. \quad (5)$$

The Caputo Hadamard derivative is obtained from the Hadamard derivative by changing the order of its differential and integral parts. Despite the different requirements on the function itself, the main difference between the Caputo Hadamard fractional derivative and the Hadamard fractional derivative is that the Caputo Hadamard derivative of a constant is zero [4]. The most important advantage of Caputo Hadamard is that it brought a new definition through which the integer order initial conditions can be defined for fractional order differential equations in the frame of the Hadamard fractional derivative.

In this article, we extend the approach of Kilbas et al. [10] to fractional Cauchy problems with a left Caputo Hadamard in spaces of continuously differentiable functions and prove the existence and uniqueness of solutions to these problems.

To get to our aim, the equivalence of the Cauchy type problems to a nonlinear Volterra type integral equation of the second kind is first proved. Once that is done, Banach's fixed point theorem is applied. By the end, some examples are given to illustrate the obtained results.

2 Preliminaries

Below, we recall some basic definitions, properties, theorems and lemmas needed in the rest of this paper.

Let $C^n([a, b], \mathbb{R})$ be the Banach space of all continuously differentiable functions from $[a, b]$ to \mathbb{R} . We will introduce the weighted space $C_{\gamma, \ln}[a, b]$, $C_{\delta, \gamma, \ln}^n[a, b]$ and $C_{\delta, \gamma, \ln}^{\alpha, r}[a, b]$ of the function g on the finite interval $[a, b]$.

Definition 2.1. If $\alpha \in (n - 1, n]$ and $\gamma \in (0, 1]$, then

(1) The space $C_{\gamma, \ln} [a, b]$ is defined by

$$C_{\gamma, \ln} [a, b] = \left\{ g : \left(\ln \frac{t}{a} \right)^\gamma g(t) \in C [a, b] \right\}, C_{0, \ln} [a, b] = C [a, b],$$

and on this space we define the norm $\|\cdot\|_{C_{\gamma, \ln}}$ by

$$\|g\|_{C_{\gamma, \ln}} = \left\| \left(\ln \frac{t}{a} \right)^\gamma g(t) \right\|_C = \max_{t \in [a, b]} \left| \left(\ln \frac{t}{a} \right)^\gamma g(t) \right|.$$

(2) The space $C_{\delta, \gamma, \ln}^n [a, b]$ is defined by

$$C_{\delta, \gamma, \ln}^n [a, b] = \left\{ g : \delta^k g \in C [a, b], k = 0, \dots, n - 1 \text{ and } \delta^n g \in C_{\gamma, \ln} [a, b] \right\},$$

and on this space we define the norm $\|\cdot\|_{C_{\delta, \gamma, \ln}^n}$ by

$$\|g\|_{C_{\delta, \gamma, \ln}^n} = \sum_{k=0}^{n-1} \|\delta^k g\|_C + \|\delta^n g\|_{C_{\gamma, \ln}}, \|g\|_{C_\delta^n} = \sum_{k=0}^n \max_{t \in [a, b]} |\delta^k g(t)|.$$

(3) We denote by $C_{\delta, \gamma, \ln}^{\alpha, r} [a, b]$ the space of functions g given on $[a, b]$ and such that

$$C_{\delta, \gamma, \ln}^{\alpha, r} [a, b] = \left\{ g \in C_\delta^r [a, b] : \left({}^c \mathcal{D}_{a+}^\alpha g \right) \in C_{\gamma, \ln} [a, b], r \in \mathbb{N} \right\},$$

$$C_{\delta, \gamma, \ln}^{r, r} [a, b] = C_{\delta, \gamma, \ln}^r [a, b].$$

Property 2.2 ([10]). *The fractional integral operators $\left(\mathcal{J}_{a+}^\alpha \right)$ satisfy the semigroup property*

$$\left(\mathcal{J}_{a+}^\alpha \mathcal{J}_{a+}^\beta g \right) (t) = \left(\mathcal{J}_{a+}^{\alpha+\beta} g \right) (t), \operatorname{Re}(\alpha) > 0, \operatorname{Re}(\beta) > 0.$$

The fractional derivative operators $\left(\mathcal{D}_{a+}^\alpha \right)$ fulfill the semigroup property

$$\left(\mathcal{D}_{a+}^\alpha \mathcal{J}_{a+}^\beta g \right) (t) = \left(\mathcal{J}_{a+}^{\beta-\alpha} g \right) (t).$$

Property 2.3 ([4]). *Let $\operatorname{Re}(\alpha) \geq 0, n = [\operatorname{Re}(\alpha)] + 1$ and $\operatorname{Re}(\beta) > 0$, then*

$$\left({}^c \mathcal{D}_{a+}^\alpha \left(\ln \frac{t}{a} \right)^{\beta-1} \right) = \frac{\Gamma(\beta)}{\Gamma(\beta - \alpha)} \left(\ln \frac{t}{a} \right)^{\beta-\alpha-1}, \operatorname{Re}(\beta) > n.$$

On the other hand, for $k = 0, 1, \dots, n - 1$,

$$\left({}^c \mathcal{D}_{a+}^\alpha \left(\ln \frac{t}{a} \right)^k \right) = 0.$$

Lemma 2.4 ([4]). *Let $\alpha \in \mathbb{C}, n = [\operatorname{Re}(\alpha)] + 1$, let $g(t) \in AC_\delta^n [a, b]$ or $C_\delta^n [a, b]$, then*

$$\left(\mathcal{J}_{a+}^\alpha \left({}^c \mathcal{D}_{a+}^\alpha g \right) \right) (t) = g(t) - \sum_{k=0}^{n-1} \frac{(\delta^k g)(a)}{k!} \left(\ln \frac{t}{a} \right)^k.$$

Lemma 2.5 ([10]). *Let $n \in \mathbb{N}$ and $0 \leq \gamma < 1$. The space $C_{\delta, \gamma, \ln}^n [a, b]$ consists of those and only those functions g which are represented in the form*

$$g(t) = \frac{1}{(n-1)!} \int_a^t \left(\ln \frac{t}{\tau}\right)^{n-1} \varphi(\tau) \frac{d\tau}{\tau} + \sum_{k=0}^{n-1} d_k \left(\ln \frac{t}{a}\right)^k,$$

where $\varphi \in C_{\gamma, \ln} [a, b]$ and d_k ($k = 0, 1, \dots, n-1$) are arbitrary constants, such that

$$\varphi(t) = \delta^n g(t), \quad d_k = \frac{\delta^k g(a)}{k!} \quad (k = 0, 1, \dots, n-1).$$

Lemma 2.6 ([10]). *Let $0 < a < b < +\infty$, $Re(\alpha) > 0$, and $0 \leq \gamma < 1$, then*

a. *If $\gamma > \alpha > 0$, then $(\mathcal{J}_{a+}^\alpha)$ is bounded from $C_{\gamma, \ln} [a, b]$ into $C_{\gamma-\alpha, \ln} [a, b]$:*

$$\left\| \mathcal{J}_{a+}^\alpha g \right\|_{C_{\gamma-\alpha, \ln}} \leq k \|g\|_{C_{\gamma, \ln}}, \quad k = \left(\ln \frac{b}{a}\right)^{Re(\alpha)} \frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)}.$$

In particular $(\mathcal{J}_{a+}^\alpha)$ is bounded in $C_{\gamma, \ln} [a, b]$.

b. *If $\gamma \leq \alpha$, then $(\mathcal{J}_{a+}^\alpha)$ is bounded from $C_{\gamma, \ln} [a, b]$ into $C [a, b]$:*

$$\left\| \mathcal{J}_{a+}^\alpha g \right\|_C \leq k \|g\|_{C_{\gamma, \ln}}, \quad k = \left(\ln \frac{b}{a}\right)^{Re(\alpha)-\gamma} \frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)}.$$

In particular $(\mathcal{J}_{a+}^\alpha)$ is bounded in $C_{\gamma, \ln} [a, b]$.

Lemma 2.7 ([10]). *The fractional operator $(\mathcal{J}_{a+}^\alpha)$ represents a mapping from $C [a, b]$ to $C [a, b]$ and*

$$\left\| \mathcal{J}_{a+}^\alpha g \right\|_C \leq \frac{1}{Re(\alpha) \Gamma(\alpha)} \left(\ln \frac{b}{a}\right)^{Re(\alpha)} \|g\|_C.$$

Theorem 2.8 (Banach fixed point Theorem, [10]). *Let (X, d) be a nonempty complete metric space, let $0 \leq w < 1$, and let $T : X \rightarrow X$ be a map such that for every $x, \tilde{x} \in X$, the relation*

$$d(Tx, T\tilde{x}) \leq wd(x, \tilde{x}),$$

holds. Then the operator T has a uniquely defined fixed point $x^ \in X$.*

Furthermore, if T^k ($k \in \mathbb{N}$) is the sequence defined by

$$T^1 = T, \quad T^k = TT^{k-1} \quad (k \in \mathbb{N} - \{1\}),$$

then, for any $x_0 \in X$ $\{T^k x_0\}_{k=1}^{k=\infty}$ converges to the above fixed point x^ .*

Definition 2.9 ([10]). *Let $l \in \mathbb{N}$, $G \subset \mathbb{R}^l$, $[a, b] \subset \mathbb{R}$, $g : [a, b] \times G \rightarrow \mathbb{R}$ be a function such that, for any $(x_1, \dots, x_l), (\tilde{x}_1, \dots, \tilde{x}_l) \in G$, g satisfies generalized Lipschitzian condition:*

$$|g[t, x_1, \dots, x_l] - g[t, \tilde{x}_1, \dots, \tilde{x}_l]| \leq A_1 |x_1 - \tilde{x}_1| + \dots + A_l |x_l - \tilde{x}_l|, \quad A_j \geq 0, \quad j = 1, \dots, l. \quad (6)$$

In particular, g satisfies the Lipschitzian condition with respect to the second variable if for all $t \in (a, b]$ and for any $x, \tilde{x} \in G$ one has

$$|g[t, x] - g[t, \tilde{x}]| \leq A |x - \tilde{x}|, \quad A > 0. \quad (7)$$

3 Nonlinear Cauchy problem

In this section, we present the existence and uniqueness results in the space $C_{\delta, \gamma, \ln}^{\alpha, r} [a, b]$ of the Cauchy problem for the nonlinear fractional differential equation in the frame of Caputo Hadamard fractional derivative. That is we consider the equation

$$\left({}^c \mathcal{D}_{a+}^{\alpha} x \right) (t) = h [t, x (t)], \operatorname{Re} (\alpha) > 0, t > a > 0, \tag{8}$$

subject to the initial conditions

$$(\delta^k x) (a_+) = d_k, d_k \in \mathbb{R}, k = 0, \dots, n - 1, n = [\operatorname{Re}(\alpha)] + 1. \tag{9}$$

The Volterra type integral equation corresponding to problem (8)-(9) is :

$$x(t) = \sum_{j=0}^{n-1} \frac{d_j}{j!} \left(\ln \frac{t}{a} \right)^j + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{\tau} \right)^{\alpha-1} h [\tau, x(\tau)] \frac{d\tau}{\tau}, a \leq t \leq b. \tag{10}$$

In particular, if $\alpha = n \in \mathbb{N}$ then the problem (8)-(9) is as follows:

$$(\delta^n x) (t) = h [t, x (t)], a \leq t \leq b, (\delta^k x) (a_+) = d_k \in \mathbb{R}, k = 0, 1, \dots, n - 1. \tag{11}$$

The corresponding integral equation to the problem (11) has the form:

$$x(t) = \sum_{j=0}^{n-1} \frac{d_j}{j!} \left(\ln \frac{t}{a} \right)^j + \left(\mathcal{J}_{a+}^n h \right) (t), a \leq t \leq b. \tag{12}$$

Firstly, we we have to prove the equivalence of the Cauchy problem to the Volterra type integral equation in the sense that, if $x \in C_{\delta}^r [a, b]$ satisfies one of them, then it also satisfies the other one.

Theorem 3.1. *Let $\operatorname{Re} (\alpha) > 0, n = [\operatorname{Re}(\alpha)] + 1, (0 < a < b < +\infty)$, and $0 \leq \gamma < 1$ be such that $\alpha \geq \gamma$. Let G be an open set in \mathbb{R} and let $h : [a, b] \times G \rightarrow \mathbb{R}$ be a function such that $h [t, x] \in C_{\gamma, \ln} [a, b]$ for any $x \in C_{\gamma, \ln} [a, b]$.*

- (i) *Let $r = n - 1$ for $\alpha \notin \mathbb{N}$, if $x \in C_{\delta}^{n-1} [a, b]$ then x satisfies the relations (8) and (9) iff x satisfies equation (10).*
- (ii) *Let $r = n$ for $\alpha \in \mathbb{N}$, if $x \in C_{\delta}^n [a, b]$ then x satisfies the relation (11) if and only if, x satisfies equation (12).*

Proof. (i) Let $\alpha \notin \mathbb{N}, n - 1 < \alpha < n$ and $x \in C_{\delta}^{n-1} [a, b]$.

(i.a) Here we prove the necessity. From definition of ${}^c \mathcal{D}_{a+}^{\alpha}$ and (3) we obtain

$${}^c \mathcal{D}_{a+}^{\alpha} x (t) = (\delta^n) \left(\mathcal{J}_{a+}^{n-\alpha} \left[x (\tau) - \sum_{j=0}^{n-1} \frac{\delta^j x (a)}{j!} \left(\ln \frac{\tau}{a} \right)^j \right] \right) (t).$$

By hypothesis, $h [t, x] \in C_{\gamma, \ln} [a, b]$ and it follows from (8) that ${}^c \mathcal{D}_{a+}^{\alpha} x (t) \in C_{\gamma, \ln} [a, b]$, and hence, by applying Lemma 2.5, we have

$$\left(\mathcal{J}_{a+}^{n-\alpha} \left[x (\tau) - \sum_{j=0}^{n-1} \frac{\delta^j x (a)}{j!} \left(\ln \frac{t}{\tau} \right)^j \right] \right) (t) \in C_{\delta, \gamma, \ln}^n [a, b].$$

By using Lemma 2.4, we obtain

$$\mathcal{J}_{a_+}^\alpha \left({}^C \mathcal{D}_{a_+}^\alpha x \right) (t) = x(t) - \sum_{j=0}^{n-1} \frac{\delta^j x(a)}{j!} \left(\ln \frac{t}{a} \right)^j. \tag{13}$$

In view of Lemma 2.6-(b), $\mathcal{J}_{a_+}^\alpha h[t, x]$ belongs to the $C[a, b]$ space, Applying $\left(\mathcal{J}_{a_+}^\alpha \right)$ to the both sides of (8) and utilizing (13), with respect to the initial conditions (9), we deduce that there exists a unique solution $x \in C_\delta^{n-1}[a, b]$ to equation (10).

(i.b) Let $x \in C_\delta^{n-1}[a, b]$ satisfies the equation (10).

– We want to show that x satisfies equation (8). Applying $\left(\mathcal{D}_{a_+}^\alpha \right)$ to both sides of (10), and taking into account (4), (9), Property 2.2 and Property 2.3, we get

$$\mathcal{D}_{a_+}^\alpha \left(x(t) - \sum_{j=0}^{n-1} \frac{\delta^j x(a)}{j!} \left(\ln \frac{t}{a} \right)^j \right) = \mathcal{D}_{a_+}^\alpha \left(\frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{\tau} \right)^{\alpha-1} h[\tau, x(\tau)] \frac{d\tau}{\tau} \right),$$

then

$$\left({}^C \mathcal{D}_{a_+}^\alpha x \right) (t) = \left(\mathcal{D}_{a_+}^\alpha \right) \left(\mathcal{J}_{a_+}^\alpha h \right) (t) \equiv h[t, x(t)].$$

– Now, we show that x satisfies the initial relations (9). We obtain by differentiation both sides of (10) that,

$$\delta^k x(t) = \sum_{j=k}^{n-1} \frac{d_j}{(j-k)!} \left(\ln \frac{t}{a} \right)^{j-k} + \frac{1}{\Gamma(\alpha-k)} \int_a^t \left(\ln \frac{t}{\tau} \right)^{\alpha-k-1} h[\tau, x(\tau)] d\tau.$$

Changing the variable $\tau = a \left(\frac{t}{a} \right)^s$, yields

$$\begin{aligned} \delta^k x(t) &= \sum_{j=k}^{n-1} \frac{d_j}{(j-k)!} \left(\ln \frac{t}{a} \right)^{j-k} + \frac{1}{\Gamma(\alpha-k)} \int_0^1 \left(\ln \frac{t}{a \left(\frac{t}{a} \right)^s} \right)^{\alpha-k-1} \\ &\quad \times h \left[a \left(\frac{t}{a} \right)^s, x \left(a \left(\frac{t}{a} \right)^s \right) \right] a \ln \left(\frac{t}{a} \right) \left(\frac{t}{a} \right)^s ds \\ &= \sum_{j=k}^{n-1} \frac{d_j}{(j-k)!} \left(\ln \frac{t}{a} \right)^{j-k} + \frac{\ln \left(\frac{t}{a} \right)^{\alpha-k}}{\Gamma(\alpha-k)} \int_0^1 (1-s)^{\alpha-k-1} h \left[a \left(\frac{t}{a} \right)^s, x \left(a \left(\frac{t}{a} \right)^s \right) \right] ds. \end{aligned}$$

for $k = 0, \dots, n-1$. Because $\alpha - k > n - 1 - k \geq 0$, using the continuity of h , Property 2.3 and Lemma 2.7 we get $\mathcal{J}_{a_+}^\alpha h[t, x] \in C[a, b]$, and taking a limit as $t \rightarrow a_+$, we obtain $\delta^k x(a_+) = d_k$.

(ii) For $\alpha \in \mathbb{N}$ and $x(t) \in C_\delta^n[a, b]$ be the solution to the Cauchy problem (11).

(ii.a) Firstly, we prove the necessity. Applying $\left(\mathcal{J}_{a_+}^n \right)$ to both sides of equation (11), using (4) and Lemma 2.4, we have

$$\mathcal{J}_{a_+}^n \delta^n x(t) = x(t) - \sum_{k=0}^{n-1} \frac{\delta^k x(a)}{k!} \left(\ln \frac{t}{a} \right)^k = \mathcal{J}_{a_+}^n h(t),$$

since $\delta^k x(a_+) = d_k$, we arrive at equation (12) and hence the necessity is proved.

(ii.b) If $x \in C_\delta^n [a, b]$ satisfies the equation (12), in addition, by term-by-term differentiation of (12) in the usual sense k times, we get

$$\delta^k x(t) = \sum_{j=k}^{n-1} \frac{d_j}{(j-k)!} \left(\ln \frac{t}{a}\right)^{j-k} + \frac{1}{(n-k-1)!} \int_a^t \left(\ln \frac{t}{\tau}\right)^{n-k-1} h[\tau, x(\tau)] \frac{d\tau}{\tau},$$

for $k = 0, \dots, n$. Using Property 2.3, taking the limit as $t \rightarrow a_+$, we obtain $\delta^k x(a_+) = d_k$, and $\delta^n x(t) = h[t, x(t)]$. Thus the Theorem 3.1 is proved for $\alpha \in \mathbb{N}$.

This completes the proof of the theorem. □

Corollary 3.2. Under the hypotheses of Theorem 3.1, with $0 < \operatorname{Re}(\alpha) < 1$, if $x \in C_\delta [a, b]$ then $x(t)$ satisfies the relation

$$\left({}^C \mathcal{D}_{a_+}^\alpha x\right)(t) = h[t, x(t)], \quad t > a > 0, \quad x(a) = d_0,$$

if and only if, x satisfies the equation

$$x(t) = d_0 + \left(\mathcal{J}_{a_+}^\alpha h\right)(t), \quad a \leq t \leq b.$$

The next step is to prove the existence of a unique solution to the Cauchy problem (8)-(9) in the space of functions $C_{\delta, \gamma, \ln}^{\alpha, r} [a, b]$ by using the Banach's fixed point theorem.

Theorem 3.3. Let $\alpha > 0$, and $n = [\Re(\alpha)] + 1$, $0 \leq \gamma < 1$ be such that $\alpha \geq \gamma$. Let G be an open set in \mathbb{R} and $h :]a, b[\times G \rightarrow \mathbb{C}$ be a function such that, for any $x \in G$, $h[t, x] \in C_{\gamma, \ln} [a, b]$, $x \in C_{\gamma, \ln} [a, b]$, and the Lipschitz condition (7) holds with respect to the second variable.

- (i) If $n-1 < \alpha < n$, then there exists a unique solution x to (8)-(9) in the space $C_{\delta, \gamma, \ln}^{\alpha, n-1} [a, b]$.
- (ii) If $\alpha = n$, then there exists a unique solution $x \in C_{\delta, \gamma, \ln}^n [a, b]$.

Since the problem (8)-(9) and the equation (10) are equivalent, it is enough to prove that there exists only one solution to (10).

Proof. Here we prove (i) only as (ii) can be proved similarly.

Step 1. First we show that there exists a unique solution $x \in C_\delta^{n-1} [a, b]$.

Divide the interval $[a, b]$ into M subdivisions $[a, t_1], [t_1, t_2], \dots, [t_{M-1}, b]$ such that $a < t_1 < t_2 < \dots < t_{M-1} < b$.

(a) Choose $t_1 \in]a, b[$ such that the inequality

$$w_1 = A \sum_{k=0}^{n-1} \frac{\Gamma(1-\gamma)}{\Gamma(\alpha-k-\gamma+1)} \left(\ln \frac{t_1}{a}\right)^{\operatorname{Re}(\alpha)-k} < 1, \quad A > 0, \tag{14}$$

holds. Now we prove that there exists a unique solution $x(t) \in C_\delta^{n-1} [a, t_1]$ to equation (10) in the interval $[a, t_1]$.

It is easy to see that $C_\delta^{n-1}[a, t_1]$ is a complete metric space equipped with the distance

$$d(x_1, x_2) = \|x_1 - x_2\|_{C_\delta^{n-1}[a, t_1]} = \sum_{k=0}^{n-1} \|(\delta^k x_1 - \delta^k x_2)\|_{C[a, t_1]}.$$

Now, for any $x \in C_\delta^{n-1}[a, t_1]$, define operator T as follows

$$(Tx)(t) \equiv Tx(t) = x_0(t) + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{\tau}\right)^{\alpha-1} h[\tau, x(\tau)] \frac{d\tau}{\tau}, \tag{15}$$

with

$$x_0(t) = \sum_{j=0}^{n-1} \frac{d_j}{j!} \left(\ln \frac{t}{a}\right)^j. \tag{16}$$

Transforming the problem (10) into a fixed point problem, $x(t) = Tx(t)$, where T is defined by (15). One can see that the fixed points of T are nothing but solutions to problem (8)-(9). Applying the Banach contraction mapping, we shall prove that T has a unique fixed point.

Firstly, we have to show that:

- (a.i) if $x(t) \in C_\delta^{n-1}[a, t_1]$, then $(Tx)(t) \in C_\delta^{n-1}[a, t_1]$.
- (a.ii) $\forall x_1, x_2 \in C_\delta^{n-1}[a, t_1]$ the following inequality holds:

$$\|Tx_1 - Tx_2\|_{C_\delta^{n-1}[a, t_1]} \leq w_1 \|x_1 - x_2\|_{C_\delta^{n-1}[a, t_1]}, \quad 0 < w_1 < 1.$$

(a.i) Let us prove that $Tx : C_\delta^{n-1}[a, t_1] \rightarrow C_\delta^{n-1}[a, t_1]$ is a continuous operator. Differentiating (15) k ($k = 0, \dots, n - 1$) times, we arrive at the equality

$$(\delta^k Tx)(t) = \delta^k x_0(t) + \frac{1}{\Gamma(\alpha - k)} \int_a^t \left(\ln \frac{t}{\tau}\right)^{\alpha-1-k} h[\tau, x(\tau)] \frac{d\tau}{\tau},$$

with

$$\delta^k x_0(t) = \sum_{j=k}^{n-1} \frac{d_j}{(j - k)!} \left(\ln \frac{t}{a}\right)^{j-k}.$$

It follows that $\delta^k x_0(t) \in C_\delta[a, t_1]$ because $x_0(t)$ might be further decomposed as a finite sum of functions in $C_\delta^{n-1}[a, t_1]$. When $x_0(t) \in C_\delta^{n-1}[a, t_1]$ then

$$\|x_0(t)\|_{C[a, t_1]} \leq \|x_0(t)\|_{C_\delta^{n-1}[a, t_1]} = \sum_{k=1}^{n-1} \|(\delta^k x_0(t))\|_{C[a, t_1]} + \|x_0(t)\|_{C[a, t_1]}.$$

On the other hand, we can apply Lemma 2.6-(b) with $\alpha \geq \gamma$, and α being replaced by $(\alpha - k)$, we have

$$\mathcal{J}_{a+}^{\alpha-k} h[\tau, x(\tau)](t) \in C_\delta[a, t_1].$$

In view of Lemma 2.6 and (7), for all $k = 0, \dots, n - 1$, we have

$$\begin{aligned} \left\| \mathcal{J}_{a+}^{\alpha-k} h[\tau, x(\tau)] \right\|_{C[a, t_1]} &\leq \frac{\Gamma(1 - \gamma)}{\Gamma(1 + \alpha - k - \gamma)} \left(\ln \frac{t_1}{a}\right)^{Re(\alpha) - k - \gamma} \|h[t, x(t)]\|_{C_{\gamma, \ln}[a, t_1]} \\ &\leq A \frac{\Gamma(1 - \gamma)}{\Gamma(1 + \alpha - k - \gamma)} \left(\ln \frac{t_1}{a}\right)^{Re(\alpha) - k - \gamma} \|x(t)\|_{C_{\gamma, \ln}[a, t_1]} \\ &\leq A \frac{\Gamma(1 - \gamma)}{\Gamma(1 + \alpha - k - \gamma)} \left(\ln \frac{t_1}{a}\right)^{Re(\alpha) - k} \|x(t)\|_{C[a, t_1]}. \end{aligned}$$

As fractional integrals are bounded in the space of functions continuous in interval $[a, t_1]$. The above implies that $Tx(t)$ belongs to the $C_\delta^{n-1}[a, t_1]$ space.

(a.ii) Next, we let $x_1, x_2 \in C_\delta^{n-1}[a, t_1]$ the following estimate holds:

$$\begin{aligned} \|Tx_1 - Tx_2\|_{C_\delta^{n-1}[a, t_1]} &= \left\| \mathcal{J}_{a+}^\alpha (h[\tau, x_1(\tau)] - h[\tau, x_2(\tau)])(t) \right\|_{C_\delta^{n-1}[a, t_1]} \\ &= \sum_{k=0}^{n-1} \left\| \mathcal{J}_{a+}^{\alpha-k} (h[\tau, x_1(\tau)] - h[\tau, x_2(\tau)])(t) \right\|_{C[a, t_1]} \\ &\leq \sum_{k=0}^{n-1} \frac{\Gamma(1-\gamma)}{\Gamma(\alpha-k-\gamma+1)} \left(\ln \frac{t_1}{a}\right)^{Re(\alpha)-k-\gamma} \|h[\tau, x_1(\tau)] - h[\tau, x_2(\tau)]\|_{C_{\gamma, \ln}[a, t_1]} \\ &\leq A \sum_{k=0}^{n-1} \frac{\Gamma(1-\gamma)}{\Gamma(\alpha-k-\gamma+1)} \left(\ln \frac{t_1}{a}\right)^{Re(\alpha)-k-\gamma} \|x_1(t) - x_2(t)\|_{C_{\gamma, \ln}[a, t_1]} \\ &\leq A \sum_{k=0}^{n-1} \frac{\Gamma(1-\gamma)}{\Gamma(\alpha-k-\gamma+1)} \left(\ln \frac{t_1}{a}\right)^{Re(\alpha)-k} \|x_1(t) - x_2(t)\|_{C[a, t_1]} \\ &\leq A \sum_{k=0}^{n-1} \frac{\Gamma(1-\gamma)}{\Gamma(\alpha-k-\gamma+1)} \left(\ln \frac{t_1}{a}\right)^{Re(\alpha)-k} \|x_1(t) - x_2(t)\|_{C_\delta^{n-1}[a, t_1]}. \end{aligned}$$

Thus

$$\|Tx_1 - Tx_2\|_{C_\delta^{n-1}[a, t_1]} \leq A \sum_{k=0}^{n-1} \frac{\Gamma(1-\gamma)}{\Gamma(\alpha-k-\gamma+1)} \left(\ln \frac{t_1}{a}\right)^{Re(\alpha)-k} \|x_1(t) - x_2(t)\|_{C_\delta^{n-1}[a, t_1]}.$$

The last estimate shows that the operator T is a contraction mapping from $C_\delta^{n-1}[a, t_1]$. Thus, the Banach fixed point theorem implies that there exists a unique function (solution) $x_0^* \in C_\delta^{n-1}[a, t_1]$ and this given as:

$$x_0^* = \lim_{m \rightarrow +\infty} T^m x_{00}^*, \quad (m \in \mathbb{N}^*),$$

where

$$(T^m x_{00}^*)(t) = x_0(t) + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{\tau}\right)^{\alpha-1} h[\tau, (T^{m-1} x_{00}^*)(\tau)] \frac{d\tau}{\tau},$$

with $x_{00}^* \in C_\delta^{n-1}[a, t_1]$ is an arbitrary starting function.

Let us take $x_{00}^*(t) = x_0(t)$ when $d_k \neq 0$ with $x_0(t)$ defined by (16), if we denote by

$$x_m(t) = (T^m x_{00}^*)(t), \quad (m \in \mathbb{N}^*),$$

then

$$\lim_{m \rightarrow +\infty} \|x_m(t) - x_0^*(t)\|_{C_\delta^{n-1}[a, t_1]} = 0.$$

Now we show that this solution $x_0^*(t)$ is unique. Suppose that there exist two solutions $x_0^*(t), \tilde{x}_0^*(t)$ of equation (10) on $[a, t_1]$. Using Lemma 2.6 and substituting them into (10), we get

$$\|x_0^*(t) - \tilde{x}_0^*(t)\|_{C_\delta^{n-1}[a, t_1]} \leq A \sum_{k=0}^{n-1} \frac{\Gamma(1-\gamma)}{\Gamma(\alpha-k-\gamma+1)} \left(\ln \frac{t_1}{a}\right)^{Re(\alpha)-k} \|x_0^*(t) - \tilde{x}_0^*(t)\|_{C_\delta^{n-1}[a, t_1]}.$$

This relation yields

$$A \sum_{k=0}^{n-1} \frac{\Gamma(1-\gamma)}{\Gamma(\alpha-k-\gamma+1)} \left(\ln \frac{t_1}{a}\right)^{Re(\alpha)-k} \geq 1,$$

which contradicts the assumption (14). Thus there is a unique solution $x_0^*(t) \in C_\delta^{n-1}[a, t_1]$.

(b) We prove the existence of an unique solution $x(t) \in C_\delta^{n-1}[t_1, b]$. analogously

Further, if we consider the closed interval $[t_1, b]$, we can rewrite equation (10) in the form $x(t) = (Tx)(t)$ where

$$(Tx)(t) = x_{01}(t) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t \left(\ln \frac{t}{\tau}\right)^{\alpha-1} \frac{h[\tau, x(\tau)]}{\tau} d\tau, \tag{17}$$

where $x_{01}(t)$ defined by

$$x_{01}(t) = x_0(t) + \frac{1}{\Gamma(\alpha)} \int_a^{t_1} \left(\ln \frac{t}{\tau}\right)^{\alpha-1} \frac{h[\tau, x(\tau)]}{\tau} d\tau,$$

is a known function.

We note that $x_{01}(t) \in C_\delta^{n-1}[t_1, b]$. Differentiating (17) k ($k = 0, \dots, n-1$) times, we arrive at the equality

$$(\delta^k Tx)(t) = \delta^k x_{01}(t) + \frac{1}{\Gamma(\alpha - k)} \int_a^t \left(\ln \frac{t}{\tau}\right)^{\alpha-k-1} h[\tau, x(\tau)] \frac{d\tau}{\tau}.$$

It follows that $\delta^k x_{01}(t) \in C_\delta[t_1, b]$ and $\mathcal{J}_{a+}^{\alpha-k} h[\tau, x(\tau)] \in C_\delta[t_1, b]$ thus $(Tx)(t) \in C_\delta^{n-1}[t_1, b]$.

(b.i) Choose $t_2 \in]t_1, b]$ such that the inequality

$$w_2 = A \sum_{k=1}^{n-1} \frac{\Gamma(1-\gamma)}{\Gamma(\alpha - k - \gamma + 1)} \left(\ln \frac{t_2}{t_1}\right)^{Re(\alpha)-k} < 1,$$

hold. Let $x_1, x_2 \in C_\delta^{n-1}[t_1, t_2]$ the following estimate holds:

$$\begin{aligned} \|Tx_1 - Tx_2\|_{C_\delta^{n-1}[t_1, t_2]} &\leq \sum_{k=0}^{n-1} \left\| \mathcal{J}_{a+}^{\alpha-k} (h[\tau, x_1(\tau)] - h[\tau, x_2(\tau)])(t) \right\|_{C[t_1, t_2]} \\ &\leq A \sum_{k=0}^n \frac{\Gamma(1-\gamma)}{\Gamma(\alpha - k + 1)} \left(\ln \frac{t_2}{t_1}\right)^{Re(\alpha)-k} \|x_1(t) - x_2(t)\|_{C_\delta^{n-1}[t_1, t_2]}. \end{aligned}$$

Hence Tx is a contraction in $C_\delta^{n-1}[t_1, t_2]$.

By Lemma 2.6-(b) and α being replaced by $\alpha - k$, we obtain that $\mathcal{J}_{t_1+}^{\alpha-k} (h[\tau, x_1(\tau)] - h[\tau, x_2(\tau)])$ is continuous in $[t_1, t_2]$. Then, the Banach fixed point theorem implies that there exists a unique solution $x_1^* \in C_\delta^{n-1}[t_1, t_2]$ to the equation (10) on the interval $[t_1, t_2]$.

Notice that $x_1^*(t_1) = x_0^*(t_1) = x_{01}(t_1)$. Further, Theorem 2.8 guarantees that this solution $x_1^*(t)$ is the limit of the convergent sequence $T^m x_{01}^*$. Thus, we have

$$\lim_{m \rightarrow +\infty} \|T^m x_{01}^* - x_1^*\|_{C_\delta^{n-1}[t_1, t_2]} = 0,$$

with

$$(T^m x_{01}^*)(t) = x_{01}(t) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t \left(\ln \frac{t}{\tau}\right)^{\alpha-1} h[\tau, (T^{m-1} x_{01}^*)(\tau)] \frac{d\tau}{\tau}, (m \in \mathbb{N}^*).$$

If $x_0(t) \neq 0$ then we can take $x_{01}^*(t) = x_0(t)$, therefore,

$$\lim_{m \rightarrow +\infty} \|x_m(t) - x_1^*(t)\|_{C_\delta^{n-1}[t_1, t_2]} = 0, \quad x_m(t) = (T^m x_{01}^*)(t).$$

Now let

$$x^*(t) = \begin{cases} x_0^*(t) & t \in [t_1, t_2], \\ x_1^*(t) & t \in [a, t_1]. \end{cases}$$

Moreover, since $x^* \in C_\delta^{n-1}[a, t_1]$ and $x^* \in C_\delta^{n-1}[t_1, t_2]$, we have $x^* \in C_\delta^{n-1}[a, t_2]$, and hence there is a unique solution $x^* \in C_\delta^{n-1}[a, t_2]$ to the equation (10) on the interval $[a, t_2]$.

(b.ii) Finally, we prove that a unique solution $x(t) \in C_\delta^{n-1}[t_2, b]$ exists.

If $t_2 \neq b$, we choose $t_{i+1} \in]t_i, b]$ such that the relation

$$w_{i+1} = A \sum_{k=0}^{n-1} \frac{\Gamma(1-\gamma)}{\Gamma(\alpha-k-\gamma+1)} \left(\ln \frac{t_{i+1}}{t_i}\right)^{Re(\alpha)-k} < 1, \quad i = 2, 3, \dots, M, \quad b = t_M.$$

Repeating the above process i times, we also deduce that there exists a unique solution $x_i^* \in C_\delta^{n-1}[t_i, t_{i+1}]$ given as a limit of a convergent sequence $T^m x_{0i}^*$ i.e.,

$$\lim_{m \rightarrow +\infty} \|T^m x_{0i}^* - x_i^*\|_{C_\delta^{n-1}[t_i, t_{i+1}]} = 0, \quad i = 2, 3, \dots, M.$$

Consequently, the previous relation can be rewritten as

$$\lim_{m \rightarrow +\infty} \|x_m(t) - x^*(t)\|_{C_\delta^{n-1}[a, b]} = 0, \tag{18}$$

with

$$x_m(t) = T^m x_{0i}^*, \quad x_{0i}^*(t) = x_0(t), \quad x^*(t) = x_i^*(t), \quad i = 0, 1, \dots, M,$$

and

$$x_i^*(t_{i+1}) = x_{i+1}^*(t_{i+1}), \quad [a, b] = \cup [t_i, t_{i+1}], \quad a = t_0 < \dots < t_M = b.$$

Step 2. Now we show that $({}^c\mathcal{D}_{a+}^\alpha x^*)(t) \in C_{\gamma, \ln}[a, b]$.

By (8), (18) and the Lipschitzian condition (7), we have that

$$\begin{aligned} \lim_{m \rightarrow +\infty} \left\| \left({}^c\mathcal{D}_{a+}^\alpha x_m \right) (t) - \left({}^c\mathcal{D}_{a+}^\alpha x^* \right) (t) \right\|_{C_{\gamma, \ln}[a, b]} &= \lim_{m \rightarrow +\infty} \|h[t, x_m(t)] - h[t, x^*(t)]\|_{C_{\gamma, \ln}[a, b]} \\ &\leq A \lim_{m \rightarrow +\infty} \|x_m(t) - x^*(t)\|_{C_{\gamma, \ln}[a, b]} \\ &\leq A \left(\ln \frac{b}{a}\right)^\gamma \lim_{m \rightarrow +\infty} \|x_m(t) - x^*(t)\|_{C[a, b]} \\ &\leq A \left(\ln \frac{b}{a}\right)^\gamma \lim_{m \rightarrow +\infty} \|x_m(t) - x^*(t)\|_{C_\delta^{n-1}[a, b]}. \end{aligned}$$

It is obvious that the right hand side of the above inequality approaches to zero independently, thus

$$\lim_{m \rightarrow +\infty} \left\| \left({}^c\mathcal{D}_{a+}^\alpha x_m \right) (t) - \left({}^c\mathcal{D}_{a+}^\alpha x^* \right) (t) \right\|_{C_{\gamma, \ln}[a, b]} = 0.$$

By hypothesis, $({}^c\mathcal{D}_{a+}^\alpha x_m)(t) = h[t, x_m(t)]$ and $h[t, x(t)] \in C_{\gamma, \ln}[a, b]$ for $x \in C_\delta^{n-1}[a, b]$, we have $({}^c\mathcal{D}_{a+}^\alpha x^*)(t) \in C_{\gamma, \ln}[a, b]$.

Consequently, $x^* \in C_{\delta, \gamma, \ln}^{\alpha, n-1}[a, b]$ is the unique solution to the problem (8)-(9). \square

Corollary 3.4. *Under the hypotheses of Theorem 3.3, with $\gamma = 0$, there exists a unique solution x to the problem (8)-(9) in the space $C_\delta^{\alpha, n-1} [a, b]$ and to the problem (11) in the space $C_\delta^n [a, b]$.*

Proof. The above Corollary can be demonstrated in a similar way to that of Theorem 3.3, using the following inequality

$$w_{i+1} = A \sum_{k=0}^{n-1} \frac{1}{\operatorname{Re}(\alpha - k) \Gamma(\alpha - k + 1)} \left(\ln \frac{t_{i+1}}{t_i} \right)^{\operatorname{Re}(\alpha) - k} < 1, \quad i = 0, \dots, M, \quad a = t_0, \quad b = t_M,$$

where $t_i \in [a, b]$ and we observe that T is a contractive mapping when the following inequality holds, indeed, for any $x_1, x_2 \in C_\delta^{n-1} [t_i, t_{i+1}]$

$$\begin{aligned} \|Tx_1 - Tx_2\|_{C_\delta^{n-1} [t_i, t_{i+1}]} &= \sum_{k=0}^{n-1} \left\| \mathcal{J}_{t_i+}^{\alpha-k} (h[\tau, x_1(\tau)] - h[\tau, x_2(\tau)])(t) \right\|_{C[t_i, t_{i+1}]} \\ &\leq \sum_{k=0}^{n-1} \frac{\left(\ln \frac{t_{i+1}}{t_i} \right)^{\operatorname{Re}(\alpha) - k}}{\operatorname{Re}(\alpha - k) \Gamma(\alpha - k + 1)} \|h[t, x_1(t)] - h[t, x_2(t)]\|_{C[t_i, t_{i+1}]} \\ &\leq A \sum_{k=0}^{n-1} \frac{\left(\ln \frac{t_{i+1}}{t_i} \right)^{\operatorname{Re}(\alpha) - k}}{\operatorname{Re}(\alpha - k) \Gamma(\alpha - k + 1)} \|x_1(t) - x_2(t)\|_{C[t_i, t_{i+1}]} \\ &\leq A \sum_{k=0}^{n-1} \frac{\left(\ln \frac{t_{i+1}}{t_i} \right)^{\operatorname{Re}(\alpha) - k}}{\operatorname{Re}(\alpha - k) \Gamma(\alpha - k + 1)} \|x_1(t) - x_2(t)\|_{C_\delta^{n-1} [t_i, t_{i+1}]} . \end{aligned}$$

□

4 The Generalized Cauchy type problem

The results in the previous section can be extended to the following equation, which is more general than (8) :

$$\left({}^c \mathcal{D}_{a+}^\alpha x \right) (t) = h \left[t, x(t), \left({}^c \mathcal{D}_{a+}^{\alpha_1} x \right) (t), \dots, \left({}^c \mathcal{D}_{a+}^{\alpha_l} x \right) (t) \right], \quad (19)$$

with $\alpha_j \in (j - 1, j]$, $j = 1, 2, \dots, l$, $\alpha_0 = 0$, and $({}^c \mathcal{D}_{a+}^{\alpha_j})$ denotes the Caputo Hadamard operator of order α_j .

The initial conditions for (19) are

$$(\delta^k x)(a_+) = d_k, \quad d_k \in \mathbb{R} \quad (k = 0, \dots, n - 1). \quad (20)$$

For simplicity, we denote by $h[t, \varphi(t, x)]$ instead of $h \left[t, x(t), \left({}^c \mathcal{D}_{a+}^{\alpha_1} x \right) (t), \dots, \left({}^c \mathcal{D}_{a+}^{\alpha_l} x \right) (t) \right]$.

Similar to the things discussed in the previous, our investigations are based on reducing the problem (19)-(20) to the Volterra equation

$$x(t) = \sum_{j=0}^{n-1} \frac{d_j}{j!} \left(\ln \frac{t}{a} \right)^j + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{\tau} \right)^{\alpha-1} h[\tau, \varphi(\tau, x)] \frac{d\tau}{\tau}, \quad (t > a). \quad (21)$$

Theorem 4.1. *Let $\alpha > 0$, $n = [\operatorname{Re}(\alpha)] + 1$ and $\alpha_j \in \mathbb{C}$ ($j = 0, \dots, l$) be such that*

$$0 = \operatorname{Re}(\alpha_0) < \operatorname{Re}(\alpha_1) < \dots < \operatorname{Re}(\alpha_l) < n - 1. \quad (22)$$

Let $G \in \mathbb{R}^{l+1}$ be open subsets and let $h : (a, b] \times G \rightarrow \mathbb{R}$ be a function such that $h[t, x, x_1, \dots, x_l] \in C_{\gamma, \ln} [a, b]$ for arbitrary $x, x_1, \dots, x_l \in C_{\gamma, \ln} [a, b]$ and the Lipschitz condition (6) is fulfilled.

(i) If $x \in C_{\delta, \gamma, \ln}^{\alpha, n-1} [a, b]$, then x holds the relations (19)-(20) if and only if x holds the equation (21).

(ii) If $0 < \alpha < 1$, then $x \in C_{\delta, \gamma, \ln}^{\alpha} [a, b]$ satisfies the relations

$$\left({}^c \mathcal{D}_{a+}^{\alpha} x \right) (t) = h [t, \varphi (t, x)], \quad x (a_+) = d_0, \quad d_0 \in \mathbb{R}, \tag{23}$$

iff x satisfies the equation

$$x(t) = d_0 + \left(\mathcal{J}_{a+}^{\alpha} \right) h [\tau, \varphi (\tau, x)] (t), \quad (t > a). \tag{24}$$

Proof. Let $\alpha \in (n - 1, n]$ and $x \in C_{\delta}^{n-1} [a, b]$ satisfies the relations (19)-(20).

(i.a) According to (4) and (19),

$$\left({}^c \mathcal{D}_{a+}^{\alpha} x \right) (t) = \left(\mathcal{D}_{a+}^{\alpha} \right) \left[x (\tau) - \sum_{k=0}^{n-1} \frac{\delta^k x (a)}{k!} \left(\ln \frac{\tau}{a} \right)^k \right] (t).$$

We have $\left({}^c \mathcal{D}_{a+}^{\alpha} x \right) (t) \in C_{\gamma, \ln} [a, b]$ and hence

$$\delta^n \mathcal{J}_{a+}^{n-\alpha} \left(x (\tau) - \sum_{j=0}^{n-1} \frac{\delta^j x (a)}{j!} \left(\ln \frac{\tau}{a} \right)^j \right) \in C_{\gamma, \ln} [a, b].$$

Thus,

$$\mathcal{J}_{a+}^{n-\alpha} \left(x (\tau) - \sum_{j=0}^{n-1} \frac{\delta^j x (a)}{j!} \left(\ln \frac{\tau}{a} \right)^j \right) \in C_{\delta, \gamma, \ln}^n [a, b],$$

and by Lemma 2.4

$$\left(\mathcal{J}_{a+}^{\alpha} \right) \left({}^c \mathcal{D}_{a+}^{\alpha} \right) x (t) = x (t) - \sum_{j=1}^{n-1} \frac{\delta^j x (a)}{(j-1)!} \left(\ln \frac{t}{a} \right)^{j-1},$$

Then, from (19), (20) and the last relation, we obtain

$$x(t) = \sum_{j=0}^{n-1} \frac{d_j}{j!} \left(\ln \frac{t}{a} \right)^j + \left(\mathcal{J}_{a+}^{\alpha} \right) h [\tau, \varphi (\tau, x)] (t), \quad (t > a).$$

That is $x \in C_{\delta}^{n-1} [a, b]$ satisfy the equation (21).

(i.b) Now we prove the sufficiency. Let $x \in C_{\delta}^{n-1} [a, b]$ satisfies equation (21).

– From (21) we have

$$x(t) - \sum_{j=0}^{n-1} \frac{d_j}{j!} \left(\ln \frac{t}{a} \right)^j = \left(\mathcal{J}_{a+}^{\alpha} \right) h \left[\tau, x (\tau), \left({}^c \mathcal{D}_{a+}^{\alpha_1} x \right) (\tau), \dots, \left({}^c \mathcal{D}_{a+}^{\alpha_l} x \right) (\tau) \right] (t).$$

Applying $(\mathcal{D}_{a_+}^\alpha)$ on both sides of this relation, taking into account the conditions for h and using Property 2.2, we get

$$\begin{aligned} (\mathcal{D}_{a_+}^\alpha) \left(x(t) - \sum_{j=0}^{n-1} \frac{d_j}{j!} \left(\ln \frac{t}{a} \right)^j \right) &= (\mathcal{D}_{a_+}^\alpha) (\mathcal{J}_{a_+}^\alpha) h[\tau, \varphi(\tau, x)](t) \\ &= h[t, \varphi(t, x)]. \end{aligned}$$

By (4), the left hand of the above expression is $({}^c\mathcal{D}_{a_+}^\alpha)$ and thus

$$({}^c\mathcal{D}_{a_+}^\alpha) x(t) = h \left[t, x(t), ({}^c\mathcal{D}_{a_+}^{\alpha_1} x)(t), \dots, ({}^c\mathcal{D}_{a_+}^{\alpha_l} x)(t) \right].$$

Hence $x \in C_\delta^{n-1}[a, b]$ satisfies (19).

– Applying δ^k ($k = 0, \dots, n - 1$) to both sides of (21), we have

$$\delta^k x(t) = \sum_{j=k}^{n-1} \frac{d_j}{(j-k)!} \left(\ln \frac{t}{a} \right)^{j-k} + (\delta^k) (\mathcal{J}_{a_+}^\alpha) h[\tau, \varphi(\tau, x)](t), \quad (t > a), \quad (25)$$

Since $x \in C_\delta^{n-1}[a, b]$ for any $(({}^c\mathcal{D}_{a_+}^{\alpha_1} x), \dots, ({}^c\mathcal{D}_{a_+}^{\alpha_l} x)) \in \mathbb{R}^{n-1}$ and $\alpha - k > \gamma - (n - 1) > 0$, we have

$$(\mathcal{J}_{a_+}^{\alpha-k}) h \left[\tau, x(\tau), ({}^c\mathcal{D}_{a_+}^{\alpha_1} x)(\tau), \dots, ({}^c\mathcal{D}_{a_+}^{\alpha_l} x)(\tau) \right] \in C[a, b]. \quad (26)$$

On the other hand, by Lemma 2.3, we let $\tau \rightarrow a_+$ on the both sides of (25), then we obtain

$$\delta^k x(\tau)|_{\tau=a_+} = d_k, \quad k = 0, \dots, n - 1.$$

Hence, x satisfying (21) satisfies the initial condition (20). That is $x \in C_\delta^{n-1}[a, b]$ satisfies the Cauchy problem (19)-(20).

Similarly, we prove the second part of the Theorem. □

Theorem 4.2. *Let $\alpha \in \mathbb{C}$, $n = [\text{Re}(\alpha)] + 1$, $0 \leq \gamma < 1$ be such that $\gamma \leq \alpha$. Let $\alpha_j > 0$ ($j = 1, \dots, l$) be such that conditions in (22) are satisfied. Let G be an open set in \mathbb{R}^{l+1} and let $h : (a, b] \times G \rightarrow \mathbb{R}$ be a function such that $h[t, x, x_1, \dots, x_l] \in C_{\gamma, \ln}[a, b]$ for any $x, x_1, \dots, x_l \in C_{\gamma, \ln}[a, b]$ and the Lipschitz condition (6) is fulfilled.*

(i) *If $n - 1 < \alpha < n$, then there is a unique solution x to the problem (19)-(20) in the space $C_{\delta, \gamma, \ln}^{\alpha, n-1}[a, b]$.*

(ii) *If $0 < \alpha < 1$, then there is a unique solution $x \in C_{\delta, \gamma, \ln}^\alpha[a, b]$ to (19) with the condition*

$$x(a_+) = d_0 \in \mathbb{R}.$$

Proof. By Theorem 4.1 it is sufficient to establish the existence of a unique solution $x \in C_{\delta, \gamma, \ln}^{\alpha, n-1}[a, b]$ to the integral equation (21).

Step 1. First we show that there exists a unique solution $x \in C_\delta^{n-1}[a, b]$.

(a) We choose $t_1 \in]a, b]$, we prove the existence of a unique solution $x \in C_\delta^{n-1}[a, t_1]$, so that the conditions

$$w_1 = \sum_{k=0}^{n-1} \sum_{j=0}^l A_j (\ln \frac{t_1}{a})^{Re(\alpha-\alpha_j)-k} \frac{\Gamma(1-\gamma)}{\Gamma(1-\gamma+\alpha-\alpha_j-k)} < 1 \quad \text{if } \gamma \leq \alpha,$$

holds, and apply the Banach fixed point theorem to prove the existence of a unique solution $x \in C_\delta^{n-1}[a, t_1]$ of the integral equation (21).

We rewrite the equation (21) in the form $x(t) = (Tx)(t)$, where

$$(Tx)(t) = x_0(t) + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{\tau}\right)^{\alpha-1} h[\tau, \varphi(\tau, x)] \frac{d\tau}{\tau},$$

with

$$x_0(t) = \sum_{j=0}^{n-1} \frac{d_j}{j!} \left(\ln \frac{t}{a}\right)^j.$$

It follows that $x_0(t) \in C_\delta^{n-1}[a, t_1]$ because $x_0(t)$ may be further decomposed as a finite sum of functions in $C_\delta^{n-1}[a, t_1]$,

$$h[\tau, \varphi(\tau, x)] \in C_{\gamma, \ln}[a, b] \implies h[\tau, \varphi(\tau, x)] \in C_{\gamma, \ln}[a, t_1],$$

and, from Lemma 2.6-(b), we have, using the fact that $\alpha > 0$ and $0 \leq \gamma < 1$,

$$\mathcal{J}_{a+}^\alpha h[\tau, \varphi(\tau, x)] \in C[a, t_1] \quad \text{if } \gamma \leq \alpha.$$

Let $x \in C_\delta^{n-1}[a, t_1]$, by Lemma 2.7, the integral in the right-hand side of (21) also belongs to $C_\delta^{n-1}[a, t_1]$ i.e., $\mathcal{J}_{a+}^\alpha h[\tau, \varphi(\tau, x)] \in C_\delta^{n-1}[a, t_1]$, and hence $Tx \in C_\delta^{n-1}[a, t_1]$, this proves T is continuous on $C_\delta^{n-1}[a, t_1]$.

To show that T is a contraction we have to prove that, for any $x_1, x_2 \in C_\delta^{n-1}[a, t_1]$ there exists $w_1 > 0$ such that

$$\|Tx_1 - Tx_2\|_{C_\delta^{n-1}[a, t_1]} \leq w_1 \|x_1 - x_2\|_{C_\delta^{n-1}[a, t_1]}.$$

By Lipschitzian condition (6), Property 2.2 and Lemma 2.4, thus

$$\begin{aligned} & \left\| \left(\mathcal{J}_{a+}^\alpha \left(h \left[\tau, x_1, {}^c \mathcal{D}_{a+}^{\alpha_1} x_1, \dots, {}^c \mathcal{D}_{a+}^{\alpha_l} x_1 \right] - h \left[\tau, x_2, {}^c \mathcal{D}_{a+}^{\alpha_1} x_2, \dots, {}^c \mathcal{D}_{a+}^{\alpha_l} x_2 \right] \right) \right) (t) \right\| \\ & \leq \mathcal{J}_{a+}^\alpha \left(\left\| h \left[\tau, x_1, {}^c \mathcal{D}_{a+}^{\alpha_1} x_1, \dots, {}^c \mathcal{D}_{a+}^{\alpha_l} x_1 \right] - h \left[\tau, x_2, {}^c \mathcal{D}_{a+}^{\alpha_1} x_2, \dots, {}^c \mathcal{D}_{a+}^{\alpha_l} x_2 \right] \right\| \right) (t) \\ & \leq \sum_{j=0}^l A_j \left\| \left(\mathcal{J}_{a+}^{\alpha-\alpha_j} \right) \mathcal{J}_{a+}^{\alpha_j} \left({}^c \mathcal{D}_{a+}^{\alpha_j} \right) (x_1 - x_2) \right\| (t) \\ & = \left(\sum_{j=0}^l A_j \mathcal{J}_{a+}^{\alpha-\alpha_j} \left\| \mathcal{J}_{a+}^{\alpha_j} \left({}^c \mathcal{D}_{a+}^{\alpha_j} \right) (x_1 - x_2) \right\| \right) (t) \\ & = \left[\left(\sum_{j=0}^l A_j \mathcal{J}_{a+}^{\alpha-\alpha_j} \|x_1 - x_2\| \right) (\tau) - \sum_{k_j=0}^{n_j-1} \frac{\delta^{k_j} (x_1 - x_2)(a_+)}{k_j!} \left(\ln \frac{t}{a}\right)^{k_j} \right]. \end{aligned}$$

By the hypothesis and Lemma 2.4, $\delta^{k_j} x_1(a_+) = \delta^{k_j} (x_2)(a_+)$, $k_j = 0, \dots, n_j - 1$, $n_j = Re(\alpha_j) + 1$, thus

$$\begin{aligned} \left\| \mathcal{J}_{a+}^{\alpha_j} \left({}^c \mathcal{D}_{a+}^{\alpha_j} \right) (x_1 - x_2) (t) \right\| &= \left\| (x_1 - x_2) (t) - \sum_{k_j=0}^{n_j-1} \frac{\delta^{k_j} (x_1 - x_2)(a_+)}{k_j!} \left(\ln \frac{t}{a}\right)^{k_j} \right\| \\ &= \|(x_1 - x_2) (t)\| \end{aligned}$$

for arbitrary $t \in [a, t_1]$. Thus we may continue our estimation above according to

$$\left\| \left(\mathcal{J}_{a_+}^\alpha \{h[\tau, \varphi(\tau, x_1)] - h[\tau, \varphi(\tau, x_2)]\} \right) (t) \right\| \leq \sum_{j=0}^l A_j \left(\mathcal{J}_{a_+}^{\alpha-\alpha_j} (\|x_1 - x_2\|) \right) (t). \quad (27)$$

Moreover by Lemma 2.6-(b), (27) and by (a.ii) in Theorem 3.3 the following holds, indeed, for $x_1, x_2 \in C_\delta^{n-1}[a, t_1]$

$$\begin{aligned} \left\| \mathcal{J}_{a_+}^\alpha (h[\tau, \varphi(\tau, x_1)] - h[\tau, \varphi(\tau, x_2)]) (t) \right\|_{C_\delta^{n-1}[a, t_1]} &\leq \left\| \sum_{k=0}^{n-1} \mathcal{J}_{a_+}^{\alpha-k} (h[\tau, \varphi(\tau, x_1)] - h[\tau, \varphi(\tau, x_2)]) (t) \right\|_{C_\delta[a, t_1]} \\ &\leq \sum_{k=0}^{n-1} \sum_{j=0}^l A_j \left(\ln \frac{t_1}{a} \right)^{Re(\alpha-\alpha_j)-k} \frac{\Gamma(1-\gamma)}{\Gamma(1-\gamma+\alpha-\alpha_j-k)} \|x_1(t) - x_2(t)\|_{C_\delta^{n-1}[a, t_1]}. \end{aligned}$$

We conclude that mapping T satisfies

$$\|Tx_1 - Tx_2\|_{C_\delta^{n-1}[a, t_1]} \leq w_1 \|x_1 - x_2\|'_{C_\delta^{n-1}[a, t_1]}$$

for any functions $x_1, x_2 \in C_\delta^{n-1}[a, t_1]$. Hence, a unique fixed point in space $C_\delta^{n-1}[a, t_1]$ exists and it is explicitly given as a limit of iterations of the mapping T i.e., $\exists x_0^* \in C_\delta^{n-1}[a, t_1]$ such that

$$\lim_{m \rightarrow +\infty} \|x_m(t) - x_0^*(t)\|_{C_\delta^{n-1}[a, t_1]} = 0,$$

Thus we deduce that a unique solution $x^*(t) \in C_\delta^{n-1}[a, b]$ exists such that

$$\lim_{m \rightarrow +\infty} \|x_m(t) - x^*(t)\|_{C_\delta^{n-1}[a, b]} = 0,$$

where

$$x_m(t) = T^m x_{0i}^*, \quad x_{0i}^*(t) = x_0(t), \quad x^*(t) = x_i^*(t), \quad i = 0, 1, \dots, M,$$

and

$$x_i^*(t_{i+1}) = x_{i+1}^*(t_{i+1}), \quad [a, b] = \cup [t_i, t_{i+1}], \quad a = t_0 < \dots < t_M = b.$$

Step 2. To complete the proof of Theorem 4.2, we show that this unique solution $x(t) = x^*(t) \in C_\delta^{n-1}[a, b]$ belongs to the space $C_{\delta, \gamma, \ln}^{\alpha, n-1}[a, b]$. It is sufficient to prove that $({}^c\mathcal{D}_{a_+}^\alpha x)(t) \in C_{\delta, \gamma, \ln}^\alpha[a, b]$. Using the estimate (27), we have

$$\begin{aligned} \left\| ({}^c\mathcal{D}_{a_+}^\alpha x_m)(t) - ({}^c\mathcal{D}_{a_+}^\alpha x^*)(t) \right\|_{C_{\gamma, \ln}[a, b]} &= \|h[t, \varphi(t, x_m)] - h[t, \varphi(t, x^*)]\|_{C_{\gamma, \ln}[a, b]} \\ &\leq \sum_{j=0}^l A_j \|{}^c\mathcal{D}_{a_+}^{\alpha_j} (x_m(t) - x^*(t))\|_{C_{\gamma, \ln}[a, b]} \\ &\leq \sum_{j=0}^l A_j \left\| \mathcal{J}_{a_+}^{n-1-\alpha_j} \delta^{n-1} (x_m(t) - x^*(t)) \right\|_{C_{\gamma, \ln}[a, b]} \\ &\leq \sum_{j=0}^l A_j \left(\ln \frac{b}{a} \right)^\gamma \left\| \mathcal{J}_{a_+}^{n-1-\alpha_j} \delta^{n-1} (x_m(t) - x^*(t)) \right\|_{C[a, b]} \\ &\leq \sum_{j=0}^l A_j \frac{\left(\ln \frac{b}{a} \right)^\gamma}{Re(n-1-\alpha_j)\Gamma(n-1-\alpha_j)} \|\delta^{n-1} (x_m(t) - x^*(t))\|_{C[a, b]} \\ &\leq \sum_{j=0}^l A_j \frac{\left(\ln \frac{b}{a} \right)^\gamma}{Re(n-1-\alpha_j)\Gamma(n-1-\alpha_j)} \|x_m(t) - x^*(t)\|_{C^{n-1}[a, b]}, \end{aligned}$$

It is clear that the right hand side of the above inequality approaches to zero independently. Hence,

$$\lim_{m \rightarrow +\infty} \left\| \left({}^c \mathcal{D}_{a+}^\alpha x_m \right) (t) - \left({}^c \mathcal{D}_{a+}^\alpha x^* \right) (t) \right\|_{C_{\gamma, \ln}[a, b]} = 0.$$

Consequently, a unique solution $x^* \in C_{\delta, \gamma, \ln}^{\alpha, n-1}[a, b]$ of equation (21) exists. The second part of the theorem can be proved analogously. \square

Corollary 4.3. *Under the hypotheses of Theorem 4.2 with $\gamma = 0$. Then there exists a unique solution $x^*(t) \in C_\delta^{m-1}[a, b]$ to the Cauchy problem (19)-(20).*

Proof. The above Corollary can be demonstrated in a similar way to that of Theorem 4.2, using the following inequality

$$\begin{aligned} & \left\| \mathcal{J}_{a+}^\alpha (h[\tau, \varphi(\tau, x_1)] - h[\tau, \varphi(\tau, x_2)])(t) \right\|_{C[t_i, t_{i+1}]} \\ & \leq \sum_{k=0}^{n-1} \sum_{j=0}^l A_j \frac{\left(\ln \frac{t_i}{t_{i+1}} \right)^{\operatorname{Re}(\alpha - \alpha_j) - k}}{\Re(\alpha - \alpha_j - k) \Gamma(\alpha - \alpha_j - k)} \|x_1(t) - x_2(t)\|_{C[t_i, t_{i+1}]}, \end{aligned}$$

for $i = 0, 1, \dots, M$, $a = t_0$, $b = t_M$, and

$$\begin{aligned} & \left\| \left({}^c \mathcal{D}_{a+}^\alpha x_m \right) (t) - \left({}^c \mathcal{D}_{a+}^\alpha x^* \right) (t) \right\|_{C_{\gamma, \ln}[a, b]} \leq \\ & \sum_{j=0}^l A_j \frac{\left(\ln \frac{b}{a} \right)^\gamma}{\operatorname{Re}(n-1-\alpha_j) \Gamma(n-1-\alpha_j)} \|x_m(t) - x^*(t)\|_{C^{n-1}[a, b]}. \end{aligned}$$

\square

We can derive the corresponding results for the Cauchy problems for linear fractional equations.

Corollary 4.4. *Let $\alpha > 0$, $n = [\operatorname{Re}(\alpha)] + 1$ and $0 \leq \gamma < 1$ be such that $\alpha \geq \gamma$. Let $l \in \mathbb{N}$, $\alpha_j > 0$ ($j = 1, \dots, l$) be such that conditions in (22) are satisfied and let $d_j(t) \in C[a, b]$ ($j = 1, \dots, l$) and $f(t) \in C_{\gamma, \ln}[a, b]$.*

Then the Cauchy problem for the following linear differential equation of order α

$$\left({}^c \mathcal{D}_{a+}^\alpha x \right) (t) + \sum_{j=1}^l d_j(t) \left({}^c \mathcal{D}_{a+}^{\alpha_j} x \right) (t) + d_0(t) x(t) = f(t) \quad (t > a),$$

with the initial conditions (9) has a unique solution $x(t)$ in the space $C_{\delta, \gamma, \ln}^{\alpha, n-1}[a, b]$.

In particular, there exists a unique solution $x(t)$ in the space $C_{\delta, \gamma, \ln}^{\alpha, n-1}[a, b]$ to the Cauchy problem for the equation with $\lambda_j \in \mathbb{R}$ and $\beta_j \geq 0$ ($j = 1, \dots, l$):

$$\left({}^c \mathcal{D}_{a+}^\alpha x \right) (t) + \sum_{j=1}^l \lambda_j \left(\ln \frac{t}{a} \right)^{\beta_j} \left({}^c \mathcal{D}_{a+}^{\alpha_j} x \right) (t) + \lambda_0 \left(\ln \frac{t}{a} \right)^{\beta_0} x(t) = f(t) \quad (t > a).$$

Proof. The proof is a direct consequence of Theorem 4.2. \square

5 Illustrative Examples

We give here some applications of the above results to Cauchy problems with the Caputo Hadamard derivative.

Example 5.1. We consider the fractional differential equation of the form

$$\left({}^c\mathcal{D}_{a+}^{\alpha} x\right)(t) = \lambda \left(\ln \frac{t}{a}\right)^{\beta} [x(t)]^m; \quad t > a > 0; \quad \operatorname{Re}(\alpha) > 0, \quad m > 0; \quad m \neq 1, \quad (28)$$

with $\lambda, \beta \in \mathbb{R}$ ($\lambda \neq 0$), with the initial conditions

$$\left(\delta^k x\right)\left(a_{+}\right) = 0, \quad k = 0, \dots, n - 1. \quad (29)$$

(a) Suppose that the solution has the following form:

$$x(t) = c \left(\ln \frac{t}{a}\right)^{\nu},$$

then, this equation has the explicit solution

$$x(t) = \left[\frac{\Gamma(\gamma - \alpha + 1)}{\lambda \Gamma(\gamma + 1)}\right]^{\frac{1}{(m-1)}} \left(\ln \frac{t}{a}\right)^{\alpha - \gamma}, \quad \gamma = \frac{(\beta + m\alpha)}{(m - 1)}. \quad (30)$$

Moreover, the condition (29) is satisfied.

Hence $x(t)$ is an eigenfunction if both of $\gamma + 1$ and $\gamma - \alpha + 1$ are not equal to 0 or negative integer. also using Property 2.3 it is easily verified that if the condition

$$\frac{(\beta + \alpha)}{(m - 1)} \geq -1, \quad (31)$$

holds, this solution $x(t)$ belongs to $C_{\gamma}[a, b]$ and to $C[a, b]$ in the respective cases $0 \leq \alpha$ and $\gamma - \alpha \leq 0$.

$$x(t) \in C_{\gamma}[a, b] \quad \text{if} \quad 0 \leq \gamma < 1 \quad \text{and} \quad 0 \leq \alpha, \quad (32)$$

$$x(t) \in C[a, b] \quad \text{if} \quad \gamma - \alpha \leq 0.$$

The right-hand side of the equation (28) takes the form

$$h[t, x(t)] = \left[\frac{\Gamma(\gamma - \alpha + 1)}{\lambda \Gamma(\gamma + 1)}\right]^{\frac{m}{(m-1)}} \left(\ln \frac{t}{a}\right)^{-\gamma}. \quad (33)$$

The function $h[t, x(t)] \in C_{\gamma}[a, b]$ when $0 \leq \gamma < 1$ and $h[t, x(t)] \in C[a, b]$ when $\gamma \leq 0$

$$h[t, x(t)] \in C_{\gamma}[a, b] \quad \text{if} \quad 0 \leq \gamma < 1, \quad (34)$$

$$h[t, x(t)] \in C[a, b] \quad \text{if} \quad \gamma \leq 0.$$

In accordance with (31), the following case is possible for the space of the right-hand side (33) and of the solution (30) :

1. When $\alpha > 0$ and

$$\begin{aligned} m > 1, \quad -m\alpha \leq \beta < m - 1 - m\alpha, \quad \beta \leq -\alpha, \\ \text{or} \\ 0 < m < 1, \quad m - 1 - m\alpha < \beta \leq -m\alpha, \quad \beta \geq -\alpha. \end{aligned}$$

2. If $0 < \alpha < 1$ these conditions take the following forms

$$m > 1, \quad -m\alpha \leq \beta \leq -\alpha \text{ or } 0 < m < 1, \quad -\alpha \leq \beta \leq -m\alpha. \tag{35}$$

3. If $\alpha \geq 1$ then

$$m > 1, \quad -m\alpha \leq \beta < m - 1 - m\alpha \text{ or } 0 < m < 1, \quad m - 1 - m\alpha < \beta \leq -m\alpha. \tag{36}$$

(b) Now we establish the conditions for the uniqueness of the solution (30) to the above problem (28)-(29). For this we have to choose the domain G and check when the Lipschitz condition (7) with the right-hand side of (28) is valid.

We choose the following domain:

$$G = \left\{ (t, x) \in \mathbb{R}^2 : 0 < a < t \leq b, \quad 0 < x < p \left(\ln \frac{t}{a} \right)^q, \quad q \in \mathbb{R}, \quad p > 0 \right\}. \tag{37}$$

To prove the Lipschitz condition (7) with

$$h[t, x(t)] = \lambda \left(\ln \frac{t}{a} \right)^\beta (x(t))^m, \tag{38}$$

we have, for any $(t, x_1), (t, x_2) \in G$:

$$|h[t, x_1] - h[t, x_2]| \leq |\lambda| \left(\ln \frac{t}{a} \right)^\beta |x_1^m - x_2^m|. \tag{39}$$

By definition (37), we have

$$|x_1^m - x_2^m| < mK \left(\ln \frac{t}{a} \right)^q |x_1 - x_2|, \quad m > 0.$$

Substituting this estimate into (39), we obtain

$$|h[t, x_1] - h[t, x_2]| \leq |\lambda| mK \left(\ln \frac{t}{a} \right)^{\beta+(m-1)q} |x_1 - x_2|.$$

Then the functions $h[t, x(t)]$ fulfil the Lipschitzian condition provided that $\beta+(m-1)q \geq 0$.

Proposition 5.2. *Let $\lambda, \beta \in \mathbb{R} (\lambda \neq 0)$ and $m > 0 (m \neq 1), \gamma = (\beta + m\alpha) \setminus (m - 1)$. Let G be the domain (37), where $q \in \mathbb{R}$ is such that $\beta + (m - 1)q \geq 0$.*

(i) *Let $0 < \alpha < 1$, if either of the conditions (35) holds, then the Cauchy problem*

$$\left({}^c \mathcal{D}_{a+}^\alpha x \right) (t) = \lambda \left(\ln \frac{t}{a} \right)^\beta [x(t)]^m \text{ and } x(a_+) = 0, \tag{40}$$

has a unique solution $x(t) \in C_{\delta, \gamma, \ln}^\alpha [a, b]$ and this solution is given by (30).

(ii) Let $n-1 < \alpha < n$ ($n \in \mathbb{N} \setminus \{1\}$), if either of the conditions (36) holds, then the problem

$$\left({}^c\mathcal{D}_{a_+}^\alpha x\right)(t) = \lambda \left(\ln \frac{t}{a}\right)^\beta [x(t)]^m \text{ and } (\delta^k x)(a_+) = 0, \quad k = 0, \dots, n-1, \quad (41)$$

has a unique solution $x(t) \in C_{\delta, \gamma, \ln}^{\alpha, n-1}[a, b]$ and this solution is given by (30).

Remark 5.3. If $\beta = 0$, $0 < \text{Re}(\alpha) < 1$ then the Lipschitz condition is violated in the domain (37). The Cauchy problem (41) admits of two continuous solutions $x = 0$ and

$$x(t) = \left[\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)}\right]^{\frac{1}{(m-1)}} \left(\ln \frac{t}{a}\right)^\gamma, \quad \gamma = \frac{\alpha}{(1-m)}.$$

Example 5.4. Let us consider the following problem of order α ($\text{Re}(\alpha) > 0$)

$$\left({}^c\mathcal{D}_{a_+}^\alpha x\right)(t) = \lambda \left(\ln \frac{t}{a}\right)^\beta [x(t)]^m + c \left(\ln \frac{t}{a}\right)^\nu, \quad \lambda, c \in \mathbb{R} \ (\lambda \neq 0) \text{ and } \nu, \beta \in \mathbb{R}. \quad (42)$$

Then it is verified that the equation (42) has the solution of the form

$$x(t) = \mu \left(\ln \frac{t}{a}\right)^\gamma, \quad \gamma = (\beta + \alpha) \setminus (1-m). \quad (43)$$

In this case the right-hand side of 42 takes the form

$$h[t, x(t)] = (\lambda + c) \left(\ln \frac{t}{a}\right)^{(\beta + \alpha m) \setminus (1-m)}. \quad (44)$$

Using the same arguments as in the proof of Proposition 5.2 we derive the uniqueness result for the Cauchy problem 42.

Proposition 5.5. Let $\lambda, \beta \in \mathbb{R} \ (\lambda \neq 0)$ and $m > 0 \ (m \neq 1)$, $\gamma = (\beta + m\alpha) \setminus (m-1)$. Let G be the domain (37), where $q \in \mathbb{R}$ is such that $\beta + (m-1)q \geq 0$. Let $\nu = -\gamma$ and let the transcendental equation

$$\Gamma\left(\frac{\alpha + \beta}{1-m} + 1 - \alpha\right) [\lambda y^m + c] - \Gamma\left(\frac{\alpha + \beta}{1-m} + 1\right) y = 0,$$

have a unique solution $y = \mu$.

(i) Let $0 < \alpha < 1$, if either of the conditions (35) holds, then the Cauchy problem

$$\left({}^c\mathcal{D}_{a_+}^\alpha x\right)(t) = \lambda \left(\ln \frac{t}{a}\right)^\beta [x(t)]^m + c \left(\ln \frac{t}{a}\right)^\nu, \quad x(a_+) = 0,$$

has a unique solution $x(t) \in C_{\delta, \gamma, \ln}^{\alpha, n-1}[a, b]$ and this solution is given by (43).

(ii) Let $n-1 < \alpha < n$, if either of the conditions (36) holds, then the problem (42)-(29) has a unique solution $x(t) \in C_{\delta, \gamma, \ln}^{\alpha, n-1}[a, b]$ and this solution is given by (43).

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Multivalued Generalized Contractive Maps and Fixed Point Results

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Abstract

In this paper, we prove some fixed point results for generalized contractive multimaps with respect to generalized distance. Consequently, several known fixed point results either generalized or improved including the corresponding recent fixed point results of Ciric, BinDehaish-Latif, Latif-Albar, Klim-Wardowski, Feng-Liu.

1 Introduction and Preliminaries

Let (X, d) be a metric space, 2^X a collection of nonempty subsets of X , and $CB(X)$ a collection of nonempty closed bounded subsets of X , $Cl(X)$ a collection of nonempty closed subsets of X , $K(X)$ a collection of nonempty compact subsets of X and H the Hausdorff metric induced by d . Then for any $A, B \in CB(X)$,

$$H(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\},$$

where $d(x, B) = \inf_{y \in B} d(x, y)$.

An element $x \in X$ is called a *fixed point* of a multivalued map $T : X \rightarrow 2^X$ if $x \in T(x)$. We denote $Fix(T) = \{x \in X : x \in T(x)\}$. A sequence $\{x_n\}$ in X is called an *orbit* of T at $x_0 \in X$ if $x_n \in T(x_{n-1})$ for all $n \geq 1$. A map $f : X \rightarrow \mathbb{R}$

0

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is called *T-orbitally lower semicontinuous* if for any orbit $\{x_n\}$ of T and $x \in X$, $x_n \rightarrow x$ imply that $f(x) \leq \liminf_{n \rightarrow \infty} f(x_n)$.

Using the concept of Hausdorff metric, Nadler [13] introduced a notion of multivalued contraction maps and proved a multivalued version of the well-known Banach contraction principle, which states that each closed bounded valued contraction map on a complete metric space has a fixed point. Since then various fixed point results concerning multivalued contractions have appeared. Feng and Liu [4] extended Nadler's fixed point theorem without using the concept of Hausdorff metric. While in [7] Klim and Wardowski generalized their result. Ciric [3] obtained some interesting fixed point results which extend and generalize these cited results.

In [6], Kada et al. introduced the concept of w -distance on a metric space and studied the properties, examples and some classical results with respect to w -distance. Using this generalized distance, Suzuki and Takahashi [14] have introduced notions of single-valued and multivalued weakly contractive maps and proved fixed point results for such maps. Consequently, they generalized the Banach contraction principle and Nadler's fixed point result. Some other fixed point results concerning w -distance can be found in [8, 9, 10, 16, 18].

In [15], Susuki generalized the concept of w -distance by introducing the notion of τ -distance on metric space (X, d) . In [15], Suzuki improved several classical results including the Caristi's fixed point theorem for single-valued maps with respect to τ -distance.

In the literature, several other kinds of distances and various versions of known results are appeared. Most recently, Ume [17] generalized the notion of τ -distance by introducing a concept of u -distance as follows:

A function $p : X \times X \rightarrow \mathbb{R}_+$ is called u -distance on X if there exists a function $\theta : X \times X \times \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that the following hold for each $x, y, z \in X$:

- (u₁) $p(x, z) \leq p(x, y) + p(y, z)$.
- (u₂) $\theta(x, y, 0, 0) = 0$ and $\theta(x, y, s, t) \geq \min\{s, t\}$ for each $s, t \in \mathbb{R}_+$, and for every $\epsilon > 0$, there exists $\delta > 0$ such that $|s - s_0| < \delta$, $|t - t_0| < \delta$, $s, s_0, t, t_0 \in \mathbb{R}_+$ and $y \in X$ imply

$$|\theta(x, y, s, t) - \theta(x, y, s_0, t_0)| < \epsilon.$$

$$(u_3) \quad \lim_{n \rightarrow \infty} x_n = x$$

$$\lim_{n \rightarrow \infty} \sup \{ \theta(w_n, z_n, p(w_n, x_m), p(z_n, x_m)) : m \geq n \} = 0$$

imply

$$p(y, x) \leq \liminf_{n \rightarrow \infty} p(y, x_n)$$

$$(u_4)$$

$$\lim_{n \rightarrow \infty} \sup \{ p(x_n, w_m) : m \geq n \} = 0,$$

$$\lim_{n \rightarrow \infty} \sup \{ p(y_n, z_m) : m \geq n \} = 0,$$

$$\lim_{n \rightarrow \infty} \theta(x_n, w_n, s_n, t_n) = 0,$$

$$\lim_{n \rightarrow \infty} \theta(y_n, z_n, s_n, t_n) = 0$$

imply

$$\lim_{n \rightarrow \infty} \theta(w_n, z_n, s_n, t_n) = 0$$

or

$$\lim_{n \rightarrow \infty} \sup \{ p(w_n, x_m) : m \geq n \} = 0,$$

$$\lim_{n \rightarrow \infty} \sup \{ p(z_m, y_n) : m \geq n \} = 0,$$

$$\lim_{n \rightarrow \infty} \theta(x_n, w_n, s_n, t_n) = 0,$$

$$\lim_{n \rightarrow \infty} \theta(y_n, z_n, s_n, t_n) = 0$$

imply

$$\lim_{n \rightarrow \infty} \theta(w_n, z_n, s_n, t_n) = 0;$$

$$(u_5)$$

$$\lim_{n \rightarrow \infty} \theta(w_n, z_n, p(w_n, x_n), p(z_n, x_n)) = 0,$$

$$\lim_{n \rightarrow \infty} \theta(w_n, z_n, p(w_n, y_n), p(z_n, y_n)) = 0$$

imply

$$\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$$

or

$$\begin{aligned} \lim_{n \rightarrow \infty} \theta(a_n, b_n, p(x_n, a_n), p(x_n, b_n)) &= 0, \\ \lim_{n \rightarrow \infty} \theta(a_n, b_n, p(y_n, a_n), p(y_n, b_n)) &= 0 \end{aligned}$$

imply

$$\lim_{n \rightarrow \infty} d(x_n, y_n) = 0.$$

Remark 1.1 [17] (a) Suppose that θ from $X \times X \times \mathbb{R}_+ \times \mathbb{R}_+$ into \mathbb{R}_+ is a mapping satisfying (u2) to (u5). Then there exists a mapping η from $X \times X \times \mathbb{R}_+ \times \mathbb{R}_+$ into \mathbb{R}_+ such that η is nondecreasing in its third and fourth variable, respectively, satisfying (u2) η to (u5) η , where (u2) η to (u5) η stand for substituting η for θ in (u2) to (u5), respectively.

(b) In the light of (a), we may assume that θ is nondecreasing in its third and fourth variables, respectively, for a function θ from $X \times X \times \mathbb{R}_+ \times \mathbb{R}_+$ into \mathbb{R}_+ satisfying (u2) to (u5).

(c) Each τ -distance p on a metric space (X, d) is also a u -distance on X .

Here we present some examples of u -distance which are not τ -distance. (For the detail, see [17]).

Example 1.2. Let $X = \mathbb{R}_+$ with the usual metric. Define $p : X \times X \rightarrow \mathbb{R}_+$ by $p(x, y) = (\frac{1}{4})x^2$. Then p is a u -distance on X but not a τ distance on X .

Example 1.3. Let X be a normed space with norm $\|\cdot\|$. Then a function $p : X \times X \rightarrow \mathbb{R}_+$ defined by $p(x, y) = \|x\|$ for every $x, y \in X$ is a u -distance on X but not a τ -distance.

It follows from the above examples and Remark 1.1(c) that u -distance is a proper extension of τ -distance. Other useful examples on u -distance are also given in [17].

Let (X, d) be a metric space and let p be a u -distance on X . A sequence $\{x_n\}$ in X is called p -Cauchy [17] if there exists a function θ from $X \times X \times \mathbb{R}_+ \times \mathbb{R}_+$ into \mathbb{R}_+ satisfying (u2)~(u5) and a sequence $\{z_n\}$ of X such that

$$\lim_{n \rightarrow \infty} \sup\{\theta(z_n, z_n, p(z_n, x_m), p(z_n, x_m)) : m \geq n\} = 0,$$

or

$$\lim_{n \rightarrow \infty} \sup\{\theta(z_n, z_n, p(x_m, z_n), p(x_m, z_n)) : m \geq n\} = 0.$$

The following lemmas concerning u -distance are crucial for the proofs of our results.

Lemma 1.4 [17] *Let (X, d) be a metric space and let p be a u -distance on X . If $\{x_n\}$ is a p -Cauchy sequence in X , then $\{x_n\}$ is a Cauchy sequence.*

Lemma 1.5 [5] *Let (X, d) be a metric space and let p be a u -distance on X . If $\{x_n\}$ is a p -Cauchy sequence and $\{y_n\}$ is a sequence satisfying*

$$\lim_{n \rightarrow \infty} \sup\{p(x_n, y_m) : m \geq n\} = 0,$$

then $\{y_n\}$ is also a p -Cauchy sequence and $\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$.

Lemma 1.6 [17] *Let (X, d) be a metric space and let p be a u -distance on X . Suppose that a sequence $\{x_n\}$ of X satisfies*

$$\lim_{n \rightarrow \infty} \sup\{p(x_n, x_m) : m > n\} = 0,$$

or

$$\lim_{n \rightarrow \infty} \sup\{p(x_m, x_n) : m > n\} = 0.$$

Then $\{x_n\}$ is a p -Cauchy sequence.

The aim of this paper is to present some more general fixed point results with respect to u -distance for multivalued maps satisfying certain conditions. Our results unify and generalize the corresponding results of Mizoguchi and Takahashi [12], Klim and Wardowski [7], Latif and Abdou [10], BinDehaish and Latif [2], Ciric [3], Feng and Liu [4], and several others.

2 The Results

Using the u -distance, we prove a general result on the existence of fixed points for multivalued maps.

Theorem 2.1 *Let (X, d) be a complete metric space. Let $T : X \rightarrow Cl(X)$ be a multivalued map and let $\varphi : [0, \infty) \rightarrow [0, 1)$ be such that $\limsup_{r \rightarrow t^+} \varphi(r) < 1$*

for each $t \in [0, \infty)$. Let p be a u -distance on X and assume that the following conditions hold:

(I) for any $x \in X$, there exists $y \in T(x)$ satisfying

$$p(x, y) \leq (2 - \varphi(p(x, y)))p(x, T(x)),$$

and

$$p(y, T(y)) \leq \varphi(p(x, y))p(x, y)$$

(II) the map $f : X \rightarrow \mathbb{R}$, defined by $f(x) = p(x, T(x))$ is T -orbitally lower semicontinuous.

Then there exists $v_0 \in X$ such that $f(v_0) = 0$. Further if $p(v_0, v_0) = 0$, then $v_0 \in T(v_0)$.

Proof. let $x_0 \in X$ be an arbitrary but fixed element in X . Then there exists $x_1 \in T(x_0)$ such that

$$p(x_0, x_1) \leq (2 - \varphi(p(x_0, x_1)))p(x_0, T(x_0)), \tag{1}$$

and

$$p(x_1, T(x_1)) \leq \varphi(p(x_0, x_1))p(x_0, x_1). \tag{2}$$

From (1) and (2), we get

$$p(x_1, T(x_1)) \leq \varphi(p(x_0, x_1))(2 - \varphi(p(x_0, x_1)))p(x_0, T(x_0)). \tag{3}$$

Define a function $\psi : [0, \infty) \rightarrow [0, \infty)$ by

$$\psi(t) = \varphi(t)(2 - \varphi(t)) = 1 - (1 - \varphi(t))^2. \tag{4}$$

Using the facts that for each $t \in [0, \infty)$, $\varphi(t) < 1$ and $\lim_{r \rightarrow t^+} \sup \varphi(r) < 1$, we have

$$\psi(t) < 1 \tag{5}$$

and

$$\limsup_{r \rightarrow t^+} \psi(r) < 1, \text{ for all } t \in [0, \infty) \tag{6}$$

From (3) and (4), we have

$$p(x_1, T(x_1)) \leq \psi(p(x_0, x_1))p(x_0, T(x_0)). \tag{7}$$

Similarly, for $x_1 \in X$, there exists $x_2 \in T(x_1)$ such that

$$p(x_1, x_2) \leq (2 - \varphi(p(x_1, x_2)))p(x_1, T(x_1)),$$

and

$$p(x_2, T(x_2)) \leq \varphi(p(x_1, x_2))p(x_1, x_2).$$

Thus

$$p(x_2, T(x_2)) \leq \psi(p(x_1, x_2))p(x_1, T(x_1)).$$

Continuing this process we can get an orbit $\{x_n\}$ of T in X satisfying the following

$$p(x_n, x_{n+1}) \leq (2 - \varphi(p(x_n, x_{n+1})))p(x_n, T(x_n)) \tag{8}$$

and

$$p(x_{n+1}, T(x_{n+1})) \leq \psi(p(x_n, x_{n+1}))p(x_n, T(x_n)), \tag{9}$$

for each integer $n \geq 0$. Since $\psi(t) < 1$ for each $t \in [0, \infty)$ and from (9), we have for all $n \geq 0$

$$p(x_{n+1}, T(x_{n+1})) < p(x_n, T(x_n)). \tag{10}$$

Thus the sequence of non-negative real numbers $\{p(x_n, T(x_n))\}$ is decreasing and bounded below, thus convergent. Therefore, there is some $\delta \geq 0$ such that

$$\lim_{n \rightarrow \infty} p(x_n, T(x_n)) = \delta. \tag{11}$$

From (8), as $\varphi(t) < 1$ for all $t \geq 0$, we get

$$p(x_n, T(x_n)) \leq p(x_n, x_{n+1}) < 2p(x_n, T(x_n)), \tag{12}$$

Thus, we conclude that the sequence of non-negative reals $\{\omega(x_n, x_{n+1})\}$ is bounded. Therefore, there is some $\theta \geq 0$ such that

$$\liminf_{n \rightarrow \infty} p(x_n, x_{n+1}) = \theta. \tag{13}$$

Note that $p(x_n, x_{n+1}) \geq p(x_n, T(x_n))$ for each $n \geq 0$, so we have $\theta \geq \delta$. Now we shall show that $\theta = \delta$. If $\delta = 0$. Then we get

$$\lim_{n \rightarrow \infty} p(x_n, x_{n+1}) = 0.$$

Now consider $\delta > 0$. Suppose to the contrary, that $\theta > \delta$. Then $\theta - \delta > 0$ and so from (11) and (13) there is a positive integer n_0 such that

$$p(x_n, T(x_n)) < \delta + \frac{\theta - \delta}{4} \quad \text{for all } n \geq n_0 \tag{14}$$

and

$$\theta - \frac{\theta - \delta}{4} < p(x_n, x_{n+1}) \quad \text{for all } n \geq n_0 \tag{15}$$

Then from (15), (8) and (14), we get

$$\begin{aligned} \theta - \frac{\theta - \delta}{4} &< p(x_n, x_{n+1}) \\ &\leq (2 - \varphi(p(x_n, x_{n+1})))p(x_n, T(x_n)) \\ &< (2 - \varphi(p(x_n, x_{n+1}))) \left[\delta + \frac{\theta - \delta}{4} \right]. \end{aligned}$$

Thus for all $n \geq n_0$,

$$(2 - \varphi(p(x_n, x_{n+1}))) > \frac{3\theta + \delta}{3\delta + \theta},$$

that is;

$$1 + (1 - \varphi(p(x_n, x_{n+1}))) > 1 + \frac{2(\theta - \delta)}{3\delta + \theta},$$

and we get

$$-(1 - \varphi(p(x_n, x_{n+1})))^2 < - \left[\frac{2(\theta - \delta)}{3\delta + \theta} \right]^2.$$

Thus for all $n \geq n_0$,

$$\begin{aligned} \psi(p(x_n, x_{n+1})) &= 1 - (1 - \varphi(p(x_n, x_{n+1})))^2 \\ &< 1 - \left[\frac{2(\theta - \delta)}{3\delta + \theta} \right]^2. \end{aligned} \tag{16}$$

Thus, from (9) and (16) , we get

$$p(x_{n+1}, T(x_{n+1})) \leq hp(x_n, T(x_n)) \quad \text{for all } n \geq n_0, \tag{17}$$

where $h = 1 - \left[\frac{2(\theta - \delta)}{3\delta + \theta} \right]^2$. Clearly $h < 1$ as $\theta > \delta$. From (14) and (17), we have for any $k \geq 1$

$$p(x_{n_0+k}, T(x_{n_0+k})) \leq h^k p(x_{n_0}, T(x_{n_0})). \tag{18}$$

Since $\delta > 0$ and $h < 1$, there is a positive integer k_0 such that $h^{k_0}p(x_{n_0}, T(x_{n_0})) < \delta$. Now, since $\delta \leq p(x_n, T(x_n))$ for each $n \geq 0$, by (18) we have

$$\delta \leq p(x_{n_0+k_0}, T(x_{n_0+k_0})) \leq h^{k_0}p(x_{n_0}, T(x_{n_0})) < \delta.$$

a contradiction. Hence, our assumption $\theta > \delta$ is wrong. Thus $\delta = \theta$. Now we show that $\theta = 0$. Since $\theta = \delta \leq p(x_n, T(x_n)) \leq p(x_n, x_{n+1})$, then from (13) we can read as

$$\liminf_{n \rightarrow \infty} p(x_n, x_{n+1}) = \theta +,$$

so, there exists a subsequence $\{p(x_{n_k}, x_{n_k+1})\}$ of $\{p(x_n, x_{n+1})\}$ such that

$$\lim_{k \rightarrow \infty} p(x_{n_k}, x_{n_k+1}) = \theta + .$$

Now from (6) we have

$$\limsup_{p(x_{n_k}, x_{n_k+1}) \rightarrow \theta+} \psi(p(x_{n_k}, x_{n_k+1})) < 1, \tag{19}$$

and from (9), we have

$$p(x_{n_k}, T(x_{n_k+1})) \leq \psi(p(x_{n_k}, x_{n_k+1}))p(x_{n_k}, T(x_{n_k})).$$

Taking the limit as $k \rightarrow \infty$ and using (11), we get

$$\begin{aligned} \delta &= \limsup_{k \rightarrow \infty} p(x_{n_{k+1}}, T(x_{n_{k+1}})) \\ &\leq (\limsup_{k \rightarrow \infty} \psi(p(x_{n_k}, x_{n_k+1}))) (\limsup_{k \rightarrow \infty} p(x_{n_k}, T(x_{n_k}))) \\ &= (\limsup_{p(x_{n_k}, x_{n_k+1}) \rightarrow \theta+} \psi(p(x_{n_k}, x_{n_k+1}))) \delta. \end{aligned}$$

If we suppose that $\delta > 0$, then from last inequality, we have

$$\limsup_{p(x_{n_k}, x_{n_k+1}) \rightarrow \theta+} \psi(p(x_{n_k}, x_{n_k+1})) \geq 1,$$

which contradicts with (19). Thus $\delta = 0$. Then from (11) and (12), we have

$$\lim_{n \rightarrow \infty} p(x_n, T(x_n)) = 0+, \tag{20}$$

and thus

$$\lim_{n \rightarrow \infty} p(x_n, x_{n+1}) = 0 + . \tag{21}$$

Now, let

$$\alpha = \lim_{p(x_{n_k}, x_{n_k+1}) \rightarrow 0^+} \sup \psi(p(x_{n_k}, x_{n_k+1})).$$

Then by (6), $\alpha < 1$. Let q be such that $\alpha < q < 1$. Then there is some $n_0 \in \mathbb{N}$ such that

$$\psi(p(x_n, x_{n+1})) < q, \quad \text{for all } n \geq n_0.$$

Thus it follows from (9)

$$p(x_{n+1}, T(x_{n+1})) \leq qp(x_n, T(x_n)) \quad \text{for all } n \geq n_0.$$

By induction we get

$$p(x_{n+1}, T(x_{n+1})) \leq q^{n+1-n_0} p(x_{n_0}, T(x_{n_0})) \quad \text{for all } n \geq n_0. \quad (22)$$

Now, using (12) and (22), we have

$$p(x_n, x_{n+1}) \leq 2q^{n-n_0} p(x_{n_0}, T(x_{n_0})) \quad \text{for all } n \geq n_0. \quad (23)$$

Now, we show that $\{x_n\}$ is a Cauchy sequence, for all $m \geq n \geq n_0$, we get

$$\begin{aligned} p(x_n, x_m) &\leq \sum_{k=n}^{m-1} p(x_k, x_{k+1}) \\ &\leq 2 \sum_{k=n}^{m-1} q^{k-n_0} p(x_{n_0}, T(x_{n_0})) \\ &\leq 2 \left(\frac{q^{m-n_0}}{1-q} \right) p(x_{n_0}, T(x_{n_0})). \end{aligned} \quad (24)$$

and hence

$$\lim_{n \rightarrow \infty} \sup \{p(x_n, x_m) : m \geq n\} = 0.$$

Thus, by Lemma 1.6, $\{x_n\}$ is a p -Cauchy sequence and hence by Lemma 1.4, $\{x_n\}$ is a Cauchy sequence. Due to the completeness of X , there exists some $v_0 \in X$ such that $\lim_{n \rightarrow \infty} x_n = v_0$. Since f is T-orbitally lower semicontinuous and from (20), we have

$$0 \leq f(v_0) \leq \liminf_{n \rightarrow \infty} f(x_n) = \liminf_{n \rightarrow \infty} p(x_n, T(x_n)) = 0,$$

and thus, $f(v_0) = p(v_0, T(v_0)) = 0$. Thus there exists a sequence $\{v_n\} \subset T(v_0)$ such that $\lim_{n \rightarrow \infty} p(v_0, v_n) = 0$. It follows that

$$0 \leq \lim_{n \rightarrow \infty} \sup \{p(x_n, v_m) : m \geq n\} \leq \lim_{n \rightarrow \infty} \sup \{p(x_n, v_0) + p(v_0, v_m) : m \geq n\} = 0. \quad (25)$$

Since $\{x_n\}$ is a p -Cauchy sequence, thus it follows from (25) and Lemma 1.5 that $\{v_n\}$ is also a p -Cauchy sequence and $\lim_{n \rightarrow \infty} d(x_n, v_n) = 0$. Thus, by Lemma 1.4, $\{v_n\}$ is a Cauchy sequence in the complete space. Due to closedness of $T(v_0)$, there exists $z_0 \in X$ such that $\lim_{n \rightarrow \infty} v_n = z_0 \in T(v_0)$. Consequently, using (u_3) we get

$$p(v_0, z_0) \leq \liminf_{n \rightarrow \infty} p(v_0, v_n) = 0,$$

and thus $p(v_0, z_0) = 0$. But, since $\lim_{n \rightarrow \infty} x_n = v_0$, $\lim_{n \rightarrow \infty} v_n = z_0$ and $\lim_{n \rightarrow \infty} d(x_n, v_n) = 0$, we have $v_0 = z_0$. Hence $v_0 \in \text{Fix}(T)$ and $p(v_0, v_0) = 0$.

Remarks 2.2 Theorem 2.1 generalizes fixed point theorems of Latif and Abdou [10, Theorem 2.1], Ćirić [3, Theorem 5], Bin Dehaish and Latif [2, Theorem 2.2], Latif and Abdou [8, Theorem 2.2], Suzuki [15, Theorem 2], Bin Dehaish and Latif [1, Theorem 2.2], Suzuki and Takahashi [14, Theorem 1], Klim and Wardowski [7, Theorem 2.1] and Feng and Liu [4, Theorem 3.1] which contains Nadler' fixed point theorem.

We also have the following interesting result by replacing the hypothesis (II) of Theorem 2.1 with another suitable condition.

Theorem 2.3 *Suppose that all the hypotheses of Theorem 2.1 except (II) hold. Assume that*

$$\inf\{p(x, v) + p(x, T(x)) : x \in X\} > 0,$$

for every $v \in X$ with $v \notin T(v)$. Then $\text{Fix}(T) \neq \emptyset$.

Proof. Following the proof of Theorem 2.1, there exists there exists an orbit $\{x_n\}$ of T , which is Cauchy sequence in a complete metric space X . Thus, there exists $v_0 \in X$ such that $\lim_{n \rightarrow \infty} x_n = v_0$. Thus, using (u_3) and (24) we have for all $n \geq n_0$

$$p(x_n, v_0) \leq \liminf_{m \rightarrow \infty} p(x_n, x_m) \leq \left(\frac{2q^{n-n_0}}{1-q}\right)p(x_{n_0}, T(x_{n_0})),$$

and

$$p(x_n, T(x_n)) \leq p(x_n, x_{n+1}) \leq 2q^{n-n_0}p(x_{n_0}, T(x_{n_0})).$$

Assume that $v_0 \notin T(v_0)$. Then, we have

$$0 < \inf\{p(x, v_0) + p(x, T(x)) : x \in X\}$$

$$\begin{aligned}
&\leq \inf\{p(x_n, v_0) + p(x_n, T(x_n)) : n \geq n_0\} \\
&\leq \inf\left\{\left(\frac{2q^{n-n_0}}{1-q}\right)p(x_{n_0}, T(x_{n_0})) + 2q^{n-n_0}p(x_{n_0}, T(x_{n_0})) : n \geq n_0\right\} \\
&= \frac{2(2-q)}{(1-q)q^{n_0}} p(x_{n_0}, T(x_{n_0})) \inf\{q^n : n \geq n_0\} = 0,
\end{aligned}$$

which is impossible and hence $v_0 \in \text{Fix}(T)$.

Remarks 2.4 Theorem 2.3 generalizes [8, Theorem 2.4], [10, Theorem 3.3] and [2, Theorem 2.5].

Competing interests

The author declares that he has no competing interests.

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ESSENTIAL COMMUTATIVITY AND ISOMETRY OF COMPOSITION OPERATOR AND DIFFERENTIATION OPERATOR

GENG-LEI LI

ABSTRACT. In this paper, we characterize the essential commutativity and isometry of composition operator and differentiation operator from the Bloch type space to space of all weighted bounded analytic functions in the disk.

1. INTRODUCTION

Let \mathbb{D} be the unit disk of the complex plane, and $S(\mathbb{D})$ be the set of analytic self-maps of \mathbb{D} . The algebra of all holomorphic functions with domain \mathbb{D} will be denoted by $H(\mathbb{D})$.

A positive continuous function v on $[0, 1)$ is called normal (see, e.g., [17]), if there exist three constants $0 \leq \delta < 1$, and $0 < a < b < \infty$, such that for $r \in [\delta, 1)$

$$\frac{v(r)}{(1-r)^a} \downarrow 0, \quad \frac{v(r)}{(1-r)^b} \uparrow \infty$$

as $r \rightarrow 1$.

Assume v is normal, the weighted-type space H_v^∞ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{H_v^\infty} = \sup_{z \in \mathbb{D}} v(z)|f(z)| < \infty.$$

When $v(z) = 1$, we know that $H_v^\infty = H^\infty$, that is

$$H^\infty = \{f \in H(\mathbb{D}), \sup_{z \in \mathbb{D}} |f(z)| < \infty\}.$$

We recall that the Bloch type space \mathcal{B}^α ($\alpha > 0$) consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}^\alpha} = \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| < \infty,$$

then $\|\cdot\|_{\mathcal{B}^\alpha}$ is a complete semi-norm on \mathcal{B}^α , which is Möbius invariant.

It is well known that \mathcal{B}^α is a Banach space under the norm

$$\|f\| = |f(0)| + \|f\|_{\mathcal{B}^\alpha}.$$

Let φ be an analytic self-map of \mathbb{D} , the composition operator C_φ induced by φ is defined by

$$(C_\varphi f)(z) = f(\varphi(z))$$

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for $z \in \mathbb{D}$ and $f \in H(\mathbb{D})$.

Let D be the differentiation operator on $H(\mathbb{D})$, that is $Df(z) = f'(z)$. For $f \in H(\mathbb{D})$, the products of composition and differentiation operators DC_φ and $C_\varphi D$ are defined by

$$\begin{aligned} C_\varphi D(f) &= f'(\varphi) \\ DC_\varphi(f) &= (f \circ \varphi)' = f'(\varphi)\varphi' \end{aligned}$$

The boundedness and compactness of DC_φ on the Hardy space were discussed by Hirschweiler and Portnoy in [7] and by Ohno in [14].

We write T_φ for the operators $DC_\varphi - C_\varphi D$, which is from the Bloch type space \mathcal{B}^α to H_v^∞ . Generally speaking, it is clear that $DC_\varphi \neq C_\varphi D$, but it is interesting to study when

$$DC_\varphi(\mathcal{B}^\alpha \rightarrow H_v^\infty) \equiv C_\varphi D(\mathcal{B}^\alpha \rightarrow H_v^\infty), \text{ mod } K$$

where K denotes the collection of all compact operators from Bloch type space \mathcal{B}^α to H_v^∞ . If the upper properties is satisfied, we say they are essential commutative.

In the past few decades, boundedness, compactness, isometries and essential norms of composition and closely related operators between various spaces of holomorphic functions have been studied by many authors, see, e.g., [1, 3, 5, 9, 12, 15, 16, 21, 22]; the results about difference and other properties can be seen [?, 4, 6, 10, 11, 13, 18, 20] and the related references therein. Recently, many interests focused on studying the essential commutativity of various different composition operators.

In [23], Zhou and Zhang studied the essential commutativity of the integral operators and composition operators from a mixed-norm space to Bloch type space. In [19, ?], Tong and Zhou characterized the intertwining relations for Volterra operators on the Bergman space, and compact intertwining relations for composition operators between the weighted Bergman spaces and the weighted Bloch spaces, respectively.

The paper continues this line of research, and discusses the essential commutativity of composition operator and differentiation operator from the Bloch type space to the space of all weighted bounded analytic functions in the disk.

2. NOTATION AND LEMMAS

To begin the discussion, let us introduce some notations and state a couple of lemmas.

For $a \in \mathbb{D}$, the involution φ_a which interchanges the origin and point a , is defined by

$$\varphi_a(z) = \frac{a - z}{1 - \bar{a}z}.$$

For z, w in \mathbb{D} , the pseudo-hyperbolic distance between z and w is given by

$$\rho(z, w) = |\varphi_z(w)| = \left| \frac{z - w}{1 - \bar{z}w} \right|,$$

and the hyperbolic metric is given by

$$\beta(z, w) = \inf_{\gamma} \int_{\gamma} \frac{|d\xi|}{1 - |\xi|^2} = \frac{1}{2} \log \frac{1 + \rho(z, w)}{1 - \rho(z, w)},$$

where γ is any piecewise smooth curve in \mathbb{D} from z to w .

The following lemma is well known [24].

Lemma 1. *For all $z, w \in \mathbb{D}$, we have*

$$1 - \rho^2(z, w) = \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \bar{z}w|^2}.$$

For $\varphi \in S(D)$, the Schwarz-Pick lemma shows that $\rho(\varphi(z), \varphi(w)) \leq \rho(z, w)$, and if equality holds for some $z \neq w$, then φ is an automorphism of the disk. It is also well known that for $\varphi \in S(\mathbb{D})$, C_φ is always bounded on \mathcal{B} .

Lemma 2. [8, Lemma 3] *Assume that $f \in H^\infty(\mathbb{D})$, then for each $n \in \mathbb{N}$, there is a positive constant C independent of f such that*

$$\sup_{z \in D} (1 - |z|^2)^n |f^{(n)}(z)| < C \|f\|_\infty.$$

A little modification of Lemma 1 in [?] shows the following lemma.

Lemma 3. *There exists a constant $C > 0$ such that*

$$\left| \left(1 - |z|^2\right)^\alpha f'(z) - \left(1 - |w|^2\right)^\alpha f'(w) \right| \leq C \|f\|_{\mathcal{B}^\alpha} \cdot \rho(z, w)$$

for all $z, w \in \mathbb{D}$ and $f \in \mathcal{B}^\alpha$.

The following lemma is an easy modification of Proposition 3.11 in [2].

Lemma 4. *Let $0 < \alpha < \infty$, $g \in H(\mathbb{B})$ and φ be a holomorphic self-map of \mathbb{B} . Then $P_\varphi^g : H^\infty \rightarrow \mathcal{B}^\alpha$ is compact if and only if $P_\varphi^g : H^\infty \rightarrow \mathcal{B}^\alpha$ is bounded and for any bounded sequence $(f_k)_{k \in \mathbb{N}}$ in H^∞ which converges to zero uniformly on \mathbb{B} as $k \rightarrow \infty$, we have $\|(P_{\varphi_1}^{g_1} - P_{\varphi_2}^{g_2})f_k\|_{\mathcal{B}^\alpha} \rightarrow 0$ as $k \rightarrow \infty$.*

Throughout the remainder of this paper, C will denote a positive constant, the exact value of which will vary from one appearance to the next.

3. MAIN THEOREMS

Theorem 1. *Let $0 < \alpha < \infty$ and φ be a analytic self map of the unit disk. Then $T_\varphi = DC_\varphi - C_\varphi D$ is a bounded operator from \mathcal{B}^α to H_v^∞ if and only if*

$$\sup_{z \in D} \frac{v(z) |\varphi'(z) - 1|}{(1 - |\varphi(z)|^2)^\alpha} < \infty. \tag{1}$$

Proof. We prove the sufficiency first.

Assume that (1) is true, for every $f \in \mathcal{B}^\alpha$, we have

$$\begin{aligned} \|T_\varphi f\|_{H_v^\infty} &= \sup_{z \in D} v(z) |f'(\varphi(z))\varphi'(z) - f'(\varphi(z))| \\ &= \sup_{z \in \mathbb{D}} \frac{v(z) |\varphi'(z) - 1|}{(1 - |\varphi(z)|^2)^\alpha} (1 - |\varphi(z)|^2)^\alpha |f'(\varphi(z))| \\ &\leq C \|f\|_{\mathcal{B}^\alpha}. \end{aligned}$$

This means that $T_\varphi = DC_\varphi - C_\varphi D$ is a bounded operator from \mathcal{B}^α to H_v^∞ .

Now we turn to the necessity.

Suppose that $T_\varphi : \mathcal{B}^\alpha \rightarrow H_v^\infty$ is a bounded operator, that is, there exists a constant C such that $\|T_\varphi f\|_{H_v^\infty} \leq C \|f\|_{\mathcal{B}^\alpha}$, for any $f \in \mathcal{B}^\alpha$.

For any $a \in \mathbb{D}$, we begin by taking test function

$$f_a(z) = \int_0^z \frac{(1 - |a|^2)^\alpha}{(1 - \bar{a}t)^{2\alpha}} dt.$$

It is clear that $f'_a(z) = \frac{(1 - |a|^2)^\alpha}{(1 - \bar{a}z)^{2\alpha}}$. Using Lemma 1, we have

$$(1 - |z|^2)^\alpha |f'_a(z)| = \frac{(1 - |z|^2)^\alpha (1 - |a|^2)^\alpha}{|1 - \bar{a}z|^{2\alpha}} = (1 - \rho^2(a, z))^\alpha.$$

So

$$\|f_a\|_{\mathcal{B}^\alpha} = \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f'_a(z)| \leq 1.$$

that is $f_a(z) \in \mathcal{B}^\alpha$.

Therefore

$$\begin{aligned} \infty &> C \|T_\varphi\|_{\mathcal{B}^\alpha \rightarrow H_v^\infty} > \|T_\varphi f_{\varphi(a)}\|_{H_v^\infty} \\ &= \sup_{z \in \mathbb{D}} \frac{v(z) |\varphi'(z) - 1|}{(1 - |\varphi(z)|^2)^\alpha} (1 - |\varphi(z)|^2)^\alpha |f'_{\varphi(a)}(\varphi(z))| \\ &\geq \frac{v(z) |\varphi'(a) - 1|}{(1 - |\varphi(a)|^2)^\alpha}. \end{aligned}$$

So (1) follows by noticing a is arbitrary.

This completes the proof of this theorem. □

Theorem 2. Let $0 < \alpha < \infty$ and φ be a analytic self map of the unit disk. Then $T_\varphi = DC_\varphi - C_\varphi D$ is operator from \mathcal{B}^α to H_v^∞ . Then C_φ and D are essential commutative if and only if T_φ is bounded and

$$\lim_{|\varphi(z)| \rightarrow 1} \frac{v(z) |\varphi'(z) - 1|}{(1 - |\varphi(z)|^2)^\alpha} = 0. \tag{2}$$

Proof. We prove the sufficiency first.

Assume that T_φ is bounded and condition (2) holds. By the Theorem 1, we have

$$\sup_{z \in \mathbb{D}} v(z) |\varphi'(z) - 1| < \infty \tag{3}$$

for any $z \in \mathbb{D}$.

Let $\{f_k\}_{k \in \mathbb{N}}$ be a arbitrary sequence in \mathcal{B}^α which converges to zero uniformly on compact subset of \mathbb{D} as $k \rightarrow \infty$, and its norm $\|f_k\|_{\mathcal{B}^\alpha} \leq C$.

Then, it follows from (2) that for any $\varepsilon > 0$, there is a $\delta > 0$, with $\delta < |\varphi(z)| < 1$, such that

$$\sup_{z \in \mathbb{D}} \frac{v(z) |\varphi'(z) - 1|}{(1 - |\varphi(z)|^2)^\alpha} < \frac{\varepsilon}{C}. \tag{4}$$

Let $A = \{z \in \mathbb{D} : |\varphi(z)| \leq \delta\}$ and $B = \{w : |w| \leq \delta\}$, then B is a compact subset of \mathbb{D} .

The boundedness of T_φ implies (1) is true by the Theorem 1. Combining (3) and(4), it follows from Lemma 2 that

$$\begin{aligned} \|T_\varphi f_k\|_{H_v^\infty} &= \sup_{z \in \mathbb{D}} v(z) |f'_k(\varphi(z))\varphi'(z) - f'_k(\varphi(z))| \\ &= \sup_{z \in \mathbb{D}} \frac{v(z) |\varphi'(z) - 1|}{(1 - |\varphi(z)|^2)^\alpha} (1 - |\varphi(z)|^2)^\alpha |f'_k(\varphi(z))| \\ &\leq \sup_{z \in A} v(z) |\varphi'(z) - 1| |f'_k(\varphi(z))| \\ &+ \sup_{z \in \mathbb{D} \setminus A} \frac{v(z) |\varphi'(z) - 1|}{(1 - |\varphi(z)|^2)^\alpha} (1 - |\varphi(z)|^2)^\alpha |f'_k(\varphi(z))| \\ &\leq C \sup_{w \in B} |f_k(w)| + \varepsilon. \end{aligned}$$

As we are assume that $f_k \rightarrow 0$ on compact subset of \mathbb{D} as $k \rightarrow \infty$, and ε is an arbitrary positive number. Letting $k \rightarrow \infty$, we have $\|T_\varphi f_k\|_{H_v^\infty} \rightarrow 0$. Therefore, the operator T_φ is a compact operator by Lemma 3, so the operators C_φ and D are essentially commutative.

Now we turn to the necessity.

Assume that C_φ and D are essentially commutative. Then $T_\varphi = DC_\varphi - C_\varphi D$ is obvious bounded since it is a compact operator.

Nest, let $\{z_k\}_{k \in \mathbb{N}}$ is a arbitrary sequence in \mathbb{D} such that $|\varphi(z_k)| \rightarrow 1$ as $k \rightarrow \infty$. we will show (2) holds.

For any z_k , we begin by taking test function

$$f_k(z) = \int_0^z \frac{(1 - |\varphi(z_k)|^2)^\alpha}{(1 - \varphi(z_k)t)^{2\alpha}} dt.$$

It is clear that $f'_k(z) = \frac{(1 - |\varphi(z_k)|^2)^\alpha}{(1 - \varphi(z_k)z)^{2\alpha}}$. Using Lemma 1, we have

$$(1 - |z|^2)^\alpha |f'_k(z)| = \frac{(1 - |z|^2)^\alpha (1 - |\varphi(z_k)|^2)^\alpha}{|1 - \varphi(z_k)z|^{2\alpha}} = (1 - \rho^2(\varphi(z_k), z))^\alpha.$$

So

$$\|f_k\|_{\mathcal{B}^\alpha} = \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f'_k(z)| \leq 1.$$

that is $f_k(z) \in \mathcal{B}^\alpha$, and the sequence $\{f_k\}$ converges to 0 uniformly on compact subset of \mathbb{D} as $k \rightarrow \infty$. As the operator $T_\varphi = DC_\varphi - C_\varphi D$ is a compact operator, it follows from Lemma 3 that

$$\lim_{k \rightarrow \infty} \|T_\varphi f_k\|_{H_v^\infty} = 0. \tag{5}$$

So, we have

$$\begin{aligned} \|T_\varphi f_k\|_{H_v^\infty} &= \sup_{z \in \mathbb{D}} (v(z) |f'_k(\varphi(z))\varphi'(z) - f'_k(\varphi(z))|) \\ &\geq v(z_k) |f'_k(\varphi(z_k))\varphi'(z_k) - f'_k(\varphi(z_k))| \\ &= v(z_k) |\varphi'(z_k) - 1| |f'_k(\varphi(z_k))| \\ &= v(z_k) |\varphi'(z_k) - 1| \frac{1}{(1 - |\varphi(z_k)|^2)^\alpha} \end{aligned}$$

So, the condition (2) is followed by combining (5) and the above result.

This completes the proof of this theorem. □

Remark If $\alpha = 1$, $v(z) = 1$ then the space \mathcal{B}^α and H_v^∞ will be Bloch space \mathcal{B} and H^∞ . The similar results from Bloch space \mathcal{B} to the H^∞ corresponding to Theorems 1 and 2 also hold.

In the next, we study the isometry of the operator $T_\varphi = DC_\varphi - C_\varphi$, which is from \mathcal{B}^α to space H_β^∞ , and give the following theorem.

Theorem 3. *Let $0 < \alpha < \infty$ and φ be a analytic self maps of the unit disk . Then the operator $T_\varphi = DC_\varphi - C_\varphi D : \mathcal{B}^\alpha \rightarrow H_v^\infty$ is an isometry in the semi-norm if and only if the following conditions hold:*

- (a) $\sup_{z \in D} \frac{v(z)|\varphi'(z)-1|}{(1-|\varphi(z)|^2)^\alpha} \leq 1$;
- (b) For every $a \in \mathbb{D}$, there at least exists a sequence $\{z_n\}$ in \mathbb{D} , such that $\lim_{n \rightarrow \infty} \rho(\varphi(z_n), a) = 0$ and $\lim_{n \rightarrow \infty} \frac{(1-|z_n|^2)^\beta |\varphi'(z_n)-1|}{(1-|\varphi(z_n)|^2)^\alpha} = 1$.

Proof. We prove the sufficiency first.

As condition (a), for every $f \in \mathcal{B}^\alpha$, we have

$$\begin{aligned} \|T_\varphi f\|_{H_v^\infty} &= \sup_{z \in D} v(z) |f'(\varphi(z))\varphi'(z) - f'(\varphi(z))| \\ &= \sup_{z \in D} \frac{v(z) |\varphi'(z) - 1|}{(1 - |\varphi(z)|^2)^\alpha} (1 - |\varphi(z)|^2)^\alpha |f'(\varphi(z))| \\ &\leq \|f\|_{\mathcal{B}^\alpha}. \end{aligned}$$

Next we show that the property (b) implies $\|T_\varphi f\|_{H_v^\infty} \geq \|f\|_{\mathcal{B}^\alpha}$.

Given any $f \in \mathcal{B}^\alpha$, then $\|f\|_{\mathcal{B}^\alpha} = \lim_{m \rightarrow \infty} (1 - |a_m|^2)^\alpha |f'(a_m)|$ for some sequence $\{a_m\} \subset D$. For any fixed m , by property (b), there is a sequence $\{z_k^m\} \subset \mathbb{D}$ such that

$$\rho(\varphi(z_k^m), a_m) \rightarrow 0 \text{ and } \frac{v(z_k^m) |\varphi'(z_k^m) - 1|}{(1 - |\varphi(z_k^m)|^2)^\alpha} \rightarrow 1$$

as $k \rightarrow \infty$. By Lemma 3, for all m and k ,

$$|(1 - |\varphi(z_k^m)|^2)^\alpha f'(\varphi(z_k^m)) - (1 - |a_m|^2)^\alpha f'(a_m)| \leq C \|f\|_{\mathcal{B}^\alpha} \cdot \rho(\varphi(z_k^m), a_m).$$

Hence

$$(1 - |\varphi(z_k^m)|^2)^\alpha |f'(\varphi(z_k^m))| \geq (1 - |a_m|^2)^\alpha |f'(a_m)| - C \|f\|_{\mathcal{B}^\alpha} \cdot \rho(\varphi(z_k^m), a_m).$$

Therefore,

$$\begin{aligned} \|T_\varphi f\|_{H_v^\infty} &= \sup_{z \in D} v(z) |f'(\varphi(z))\varphi'(z) - f'(\varphi(z))| \\ &= \sup_{z \in \mathbb{D}} \frac{v(z) |\varphi'(z) - 1|}{(1 - |\varphi(z)|^2)^\alpha} (1 - |\varphi(z)|^2)^\alpha |f'(\varphi(z))| \\ &\geq \limsup_{k \rightarrow \infty} \frac{v(z_k^m) |\varphi'(z_k^m) - 1|}{(1 - |\varphi(z_k^m)|^2)^\alpha} (1 - |\varphi(z_k^m)|^2)^\alpha |f'(\varphi(z_k^m))| \\ &= (1 - |a_m|^2)^\alpha |f'(a_m)|. \end{aligned}$$

The inequality $\|T_\varphi f\|_{H_v^\infty} \geq \|f\|_{\mathcal{B}^\alpha}$ follows by letting $m \rightarrow \infty$.

From the above discussions, we have $\|T_\varphi f\|_{H_v^\infty} = \|f\|_{\mathcal{B}^\alpha}$, which means that T_φ is an isometry operator in the semi-norm from \mathcal{B}^α to H_v^∞ .

Now we turn to the necessity.

For any $a \in \mathbb{D}$, we begin by taking test function

$$f_a(z) = \int_0^z \frac{(1 - |a|^2)^\alpha}{(1 - \bar{a}t)^{2\alpha}} dt. \tag{6}$$

It is clear that $f'_a(z) = \frac{(1 - |a|^2)^\alpha}{(1 - \bar{a}z)^{2\alpha}}$. Using Lemma 1, we have

$$(1 - |z|^2)^\alpha |f'_a(z)| = \frac{(1 - |z|^2)^\alpha (1 - |a|^2)^\alpha}{|1 - \bar{a}z|^{2\alpha}} = (1 - \rho^2(a, z))^\alpha. \tag{7}$$

So

$$\|f_a\|_{\mathcal{B}^\alpha} = \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f'_a(z)| \leq 1.$$

On the other hand, since $(1 - |a|^2)^\alpha |f'_a(a)| = \frac{(1 - |a|^2)^{2\alpha}}{(1 - |a|^2)^{2\alpha}} = 1$, we have $\|f_a\|_{\mathcal{B}^\alpha} = 1$. By isometry assumption, for any $a \in \mathbb{D}$, we have

$$\begin{aligned} 1 &= \|f_{\varphi(a)}\|_{\mathcal{B}^\alpha} = \|T_\varphi f_{\varphi(a)}\|_{H_v^\infty} \\ &= \sup_{z \in D} v(z) \left| f'_{\varphi(a)}(\varphi(z))\varphi'(z) - f'_{\varphi(a)}(\varphi(z)) \right| \\ &= \sup_{z \in D} \frac{v(z) |\varphi'(z) - 1|}{(1 - |\varphi(z)|^2)^\alpha} (1 - |\varphi(z)|^2)^\alpha \left| f'_{\varphi(a)}(\varphi(z)) \right| \\ &\geq \frac{v(a) |\varphi'(a) - 1|}{(1 - |\varphi(a)|^2)^\alpha}. \end{aligned}$$

So (a) follows by noticing a is arbitrary.

Since $\|T_\varphi f_a\|_{H_v^\infty} = \|f_a\|_{\mathcal{B}^\alpha} = 1$, there exists a sequence $\{z_m\} \subset \mathbb{D}$ such that

$$v(z_m) |(T_\varphi f_a)(z_m)| = v(z_m) |f'_a(\varphi(z_m))| |\varphi'(z_m) - 1| \rightarrow 1 \tag{8}$$

as $m \rightarrow \infty$.

It follows from (a) that

$$\begin{aligned} &v(z_m) |f'_a(\varphi(z_m))| |\varphi'(z_m) - 1| \\ &= \frac{v(z_m) |\varphi'(z_m) - 1|}{(1 - |\varphi(z_m)|^2)^\alpha} (1 - |\varphi(z_m)|^2)^\alpha |f'_a(\varphi(z_m))| \tag{9} \end{aligned}$$

$$\leq (1 - |\varphi(z_m)|^2)^\alpha |f'_a(\varphi(z_m))|. \tag{10}$$

Combining (8) and (10), it follows that

$$\begin{aligned} 1 &\leq \liminf_{m \rightarrow \infty} (1 - |\varphi(z_m)|^2)^\alpha |f'_a(\varphi(z_m))| \\ &\leq \limsup_{m \rightarrow \infty} (1 - |\varphi(z_m)|^2)^\alpha |f'_a(\varphi(z_m))| \leq 1. \end{aligned}$$

The last inequality follows by (7) since $\varphi(z_m) \in \mathbb{D}$.

Consequently,

$$\lim_{m \rightarrow \infty} (1 - |\varphi(z_m)|^2)^\alpha |f'_a(\varphi(z_m))| = \lim_{m \rightarrow \infty} (1 - \rho^2(\varphi(z_m), a))^\alpha = 1. \tag{11}$$

That is, $\lim_{m \rightarrow \infty} \rho(\varphi(z_m), a) = 0$.

Combining (8), (9) and (11), we know

$$\lim_{m \rightarrow \infty} \frac{(1 - |z_m|^2)^\beta |\varphi'(z_m) - 1|}{(1 - |\varphi(z_m)|^2)^\alpha} = 1.$$

This completes the proof of this theorem. □

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APPROXIMATION OF JENSEN TYPE QUADRATIC-ADDITIVE MAPPINGS VIA THE FIXED POINT THEORY *

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ABSTRACT. In this article, we investigate the stability results of a Jensen type quadratic-additive functional equation

$$f(x + y) + f(x - y) + 2f(z) = 2f(x) + f(z + y) + f(z - y)$$

via the fixed point theory. And then, we present two counter-examples which do not satisfy the stability results.

1. INTRODUCTION

A classical question in the theory of functional equations is “when is it true that a mapping, which satisfies approximately a functional equation, must be somehow close to an exact solution of the equation?” Such a problem, called a stability problem of functional equations, was formulated by S. M. Ulam [31] in 1940 as follows: Let G_1 be a group and G_2 a metric group with metric $\rho(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G_1 \rightarrow G_2$ satisfies $\rho(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then a homomorphism $h : G_1 \rightarrow G_2$ exists with $\rho(f(x), h(x)) < \epsilon$ for all $x \in G_1$? When this problem has a solution, we say that the homomorphisms from G_1 to G_2 are *stable*. In 1941, D. H. Hyers [16] considered the case of approximately additive mappings between Banach spaces and proved the following result. Suppose that E_1 and E_2 are Banach spaces and $f : E_1 \rightarrow E_2$ satisfies the following condition: there is a constant $\epsilon \geq 0$ such that

$$\|f(x + y) - f(x) - f(y)\| \leq \epsilon$$

for all $x, y \in E_1$. Then the limit $h(x) = \lim_{n \rightarrow \infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in E_1$ and it is a unique additive mapping $h : E_1 \rightarrow E_2$ such that $\|f(x) - h(x)\| \leq \epsilon$.

The method which was provided by Hyers, and which produces the additive mapping h , was called a direct method. This method is the most important and most powerful tool for studying the stability of various functional equations. Hyers’ Theorem was generalized by T. Aoki [1] and D.G. Bourgin [3] for additive mappings by considering an unbounded Cauchy difference. In 1978, Th.M. Rassias [26] also provided a generalization of Hyers’ Theorem for linear mappings which allows the Cauchy difference to be unbounded like this $\|x\|^p + \|y\|^p$. A generalized result of

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Th.M. Rassias' theorem was obtained by P. Găvruta in [10] and S. Jung in [18]. In 1990, Th.M. Rassias [27] during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. In 1991, Z. Gajda [9] following the same approach as in [26], gave an affirmative solution to this question for $p > 1$. It was shown by Z. Gajda [9], as well as by Th.M. Rassias and P. Šemrl [28], that one cannot prove a Th.M. Rassias' type theorem when $p = 1$. In 2003-2007 J.M. Rassias and M.J. Rassias [23, 24] and J.M. Rassias [25] solved the above Ulam problem for Jensen and Jensen type mappings.

During the last decades, the stability problems of functional equations have been extensively investigated by a number of mathematicians, see [6, 14, 17, 29, 22, 15]. Almost all subsequent proofs, in this very active area, have used Hyers' direct method, namely, the mapping F , which is a solution of the functional equation, is explicitly constructed by the limit function of a Cauchy sequence starting from the given approximate mapping f .

The first result of the generalized Hyers-Ulam stability for Jensen equation was given in the paper [19] by the direct method. In 2003, L. Cădariu and V. Radu [4] observed that the existence of the solution F for a Cauchy functional equation and the estimation of the mapping F with the approximate mapping f of the equation can be obtained from the alternative fixed point theorem. This method is called a *fixed point method*. In addition, they applied this method to prove stability theorems of the *Jensen's functional equation*:

$$(1.1) \quad 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) = 0 \Leftrightarrow 2f(x) - f(x+y) - f(x-y) = 0.$$

On the other hand, some properties of generalized Hyers-Ulam stability for a functional equation of Jensen type were obtained in [7] by the fixed point method. Further, the authors [5] obtained the stability of the *quadratic functional equation*:

$$(1.2) \quad f(x+y) + f(x-y) - 2f(x) - 2f(y) = 0$$

by using the fixed point method. Notice that if we consider the functions $f_1, f_2 : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f_1(x) = ax + b$ and $f_2(x) = cx^2$, respectively, where a, b and c are real constants, then f_1 satisfies the equation (1.1) and f_2 is a solution of the equation (1.2), respectively.

Associating the equation (1.1) with the equation (1.2), we see the following well known Drygas functional equation:

$$(1.3) \quad f(x+y) + f(x-y) = 2f(x) + f(y) + f(-y),$$

which has quadratic solutions Q of equation (1.2) in the class of even functions, and has additive solutions A of equation (1.1) in the class of odd functions. Hence the general solution f of (1.3) is given by $f(x) = Q(x) + A(x)$ [30].

Now, adding the equation (1.3) and the following Drygas functional equation

$$(1.4) \quad 2f(z) + f(y) + f(-y) = f(z+y) + f(z-y),$$

we get the Jensen type quadratic-additive functional equation:

$$(1.5) \quad f(x+y) + f(x-y) + 2f(z) = f(z+y) + f(z-y) + 2f(x),$$

of which the general solution function $f(x) - f(0)$ has of the form $f(x) - f(0) = Q(x) + A(x)$, where $Q(x) := \frac{f(x)+f(-x)}{2} - f(0)$ is a quadratic mapping satisfying the equation (1.2) and $A(x) := \frac{f(x)-f(-x)}{2}$ is a Jensen mapping satisfying the equation (1.1). In the paper, without splitting the given approximate mapping $f : X \rightarrow Y$ of the equation (1.5) into two approximate even and odd parts, we are going to derive the desired approximate solution F near f at once. Precisely, we introduce a strictly contractive mapping with Lipschitz constant $0 < L < 1$, and then, we show that the contractive mapping has the fixed point F in a generalized metric function space by using the fixed point method in the sense of L. Cădariu and V. Radu, where, the fixed point F yields the precise solution of the equation (1.5) near f . In Section 2, we prove several stability results of the functional equation (1.5) using the fixed point method under suitable conditions. In Section 3, we use the results in the previous section to get stability results of the Jensen’s functional equation (1.1) and to get that of the quadratic functional equation (1.2), respectively.

2. GENERALIZED HYERS–ULAM STABILITY OF (1.5)

In this section, we prove the generalized Hyers–Ulam stability of the Jensen type quadratic-additive functional equation (1.5). We recall the following fundamental result of the fixed point theory by Margolis and Diaz [20].

Theorem 2.1. *Suppose that a complete generalized metric space (X, d) , which means that the metric d may assume infinite values, and a strictly contractive mapping $\Lambda : X \rightarrow X$ with the Lipschitz constant $0 < L < 1$ are given. Then, for each given element $x \in X$, either*

$$d(\Lambda^n x, \Lambda^{n+1} x) = +\infty, \forall n \in \mathbf{N} \cup \{0\},$$

or there exists a nonnegative integer k such that

- $d(\Lambda^n x, \Lambda^{n+1} x) < +\infty$ for all $n \geq k$;
- the sequence $\{\Lambda^n x\}$ is convergent to a fixed point y^* of Λ ;
- y^* is the unique fixed point of Λ in $X_1 := \{y \in X, d(\Lambda^k x, y) < +\infty\}$;
- $d(y, y^*) \leq \frac{1}{1-L} d(y, \Lambda y)$ for all $y \in X_1$.

Throughout this paper, let V be a (real or complex) linear space and Y a Banach space. For a given mapping $f : V \rightarrow Y$, we use the following abbreviation

$$Df(x, y, z) := f(x + y) + f(x - y) + 2f(z) - f(z + y) - f(z - y) - 2f(x)$$

for all $x, y, z \in V$.

In the following theorem, we prove the stability of the Jensen type quadratic-additive functional equation (1.5) using the fixed point method.

Theorem 2.2. *Let $f : V \rightarrow Y$ be a mapping for which there exists a mapping $\varphi : V^3 \rightarrow \mathbb{R}^+ := [0, \infty)$ such that*

$$(2.1) \quad \|Df(x, y, z)\| \leq \varphi(x, y, z)$$

for all $x, y, z \in V$. If $\varphi(x, y, z) = \varphi(-x, -y, -z)$ for all $x, y, z \in V$ and there exists a constant $0 < L < 1$ such that

$$(2.2) \quad \varphi(2x, 2y, 2z) \leq 2L\varphi(x, y, z),$$

for all $x, y, z \in V$, then there exists a unique Jensen type quadratic-additive mapping $F : V \rightarrow Y$ such that $DF(x, y, z) = 0$ for all $x, y, z \in V$ and

$$(2.3) \quad \|f(x) - F(x)\| \leq \frac{\varphi(x, x, 0)}{2(1 - L)}$$

for all $x \in V$. In particular, F is represented by

$$(2.4) \quad F(x) = f(0) + \lim_{n \rightarrow \infty} \left[\frac{f(2^n x) + f(-2^n x)}{2 \cdot 4^n} + \frac{f(2^n x) - f(-2^n x)}{2^{n+1}} \right]$$

for all $x \in V$.

Proof. If we consider the mapping $\tilde{f} := f - f(0)$, then $\tilde{f} : V \rightarrow Y$ satisfies $\tilde{f}(0) = 0$ and

$$(2.5) \quad \|D\tilde{f}(x, y, z)\| = \|Df(x, y, z)\| \leq \varphi(x, y, z)$$

for all $x, y, z \in V$. Let S be the set of all mappings $g : V \rightarrow Y$ with $g(0) = 0$, and then we introduce a generalized metric d on S by

$$(2.6) \quad d(g, h) := \inf \{ K \in \mathbb{R}^+ : \|g(x) - h(x)\| \leq K\varphi(x, x, 0) \quad \forall x \in V \}.$$

It is easy to show that (S, d) is a generalized complete metric space. Now, we consider an operator $\Lambda : S \rightarrow S$ defined by

$$(2.7) \quad \Lambda g(x) := \frac{g(2x) - g(-2x)}{4} + \frac{g(2x) + g(-2x)}{8}$$

for all $g \in S$ and all $x \in V$. Then we notice that

$$(2.8) \quad \Lambda^n g(x) = \frac{g(2^n x) - g(-2^n x)}{2^{n+1}} + \frac{g(2^n x) + g(-2^n x)}{2 \cdot 4^n}$$

for all $n \in \mathbb{N}$ and $x \in V$.

First, we show that Λ is a strictly contractive self-mapping of S with the Lipschitz constant L . Let $g, h \in S$ and let $K \in [0, \infty]$ be an arbitrary constant with $d(g, h) \leq K$. From the definition of d , we have

$$(2.9) \quad \begin{aligned} \|\Lambda g(x) - \Lambda h(x)\| &= \frac{3}{8}\|g(2x) - h(2x)\| + \frac{1}{8}\|g(-2x) - h(-2x)\| \\ &\leq \frac{1}{2}K\varphi(2x, 2x, 0) \leq LK\varphi(x, x, 0) \end{aligned}$$

for all $x \in V$, which implies that

$$(2.10) \quad d(\Lambda g, \Lambda h) \leq Ld(g, h)$$

for any $g, h \in S$. That is, Λ is a strictly contractive self-mapping of S with the Lipschitz constant L . Moreover, by (2.5) we see that

$$(2.11) \quad \begin{aligned} \|\tilde{f}(x) - \Lambda\tilde{f}(x)\| &= \frac{1}{8} \|-3D\tilde{f}(x, x, 0) + D\tilde{f}(-x, -x, 0)\| \\ &\leq \frac{1}{2} \varphi(x, x, 0) \end{aligned}$$

for all $x \in V$. It means that $d(\tilde{f}, \Lambda\tilde{f}) \leq \frac{1}{2} < \infty$ by the definition of d . Therefore, according to Theorem 2.1, the sequence $\{\Lambda^n \tilde{f}\}$ converges to the unique fixed point $\tilde{F} : V \rightarrow Y$ of Λ in the set $S_1 = \{g \in S | d(\tilde{f}, g) < \infty\}$, which is represented by

$$(2.12) \quad \tilde{F}(x) = \lim_{n \rightarrow \infty} \left[\frac{\tilde{f}(2^n x) - \tilde{f}(-2^n x)}{2^{n+1}} + \frac{\tilde{f}(2^n x) + \tilde{f}(-2^n x)}{2 \cdot 4^n} \right]$$

for all $x \in V$. Putting $F := \tilde{F} + f(0)$, then we have the equality $\|f(x) - F(x)\| = \|\tilde{f}(x) - \tilde{F}(x)\|$ for all $x \in V$. Thus one notes that

$$(2.13) \quad d(\tilde{f}, \tilde{F}) \leq \frac{1}{1-L} d(\tilde{f}, \Lambda\tilde{f}) \leq \frac{1}{2(1-L)},$$

which implies (2.3) and (2.4).

By the definitions of F and \tilde{F} , together with (2.5) and (2.2), we have that

$$\begin{aligned} \|DF(x, y, z)\| &= \|D\tilde{F}(x, y, z)\| \\ &= \lim_{n \rightarrow \infty} \left\| \frac{Df(2^n x, 2^n y, 2^n z) - f(-2^n x, -2^n y, -2^n z)}{2^{n+1}} \right. \\ &\quad \left. + \frac{Df(2^n x, 2^n y, 2^n z) + Df(-2^n x, -2^n y, -2^n z)}{2 \cdot 4^n} \right\| \\ &\leq \lim_{n \rightarrow \infty} \frac{2^n + 1}{2 \cdot 4^n} \left(\varphi(2^n x, 2^n y, 2^n z) + \varphi(-2^n x, -2^n y, -2^n z) \right) \\ &= 0 \end{aligned}$$

for all $x, y, z \in V$. Thus, the mapping F satisfies the Jensen type quadratic-additive functional equation (2.3). This completes the proof of this theorem. \square

Theorem 2.3. *Let $f : V \rightarrow Y$ be a mapping for which there exists a mapping $\varphi : V^3 \rightarrow [0, \infty)$ such that*

$$(2.14) \quad \|Df(x, y, z)\| \leq \varphi(x, y, z)$$

for all $x, y, z \in V$. If $\varphi(x, y, z) = \varphi(-x, -y, -z)$ for all $x, y, z \in V$ and there exists a constant $0 < L < 1$ such that the mapping φ has the property

$$(2.15) \quad \varphi(x, y, z) \leq \frac{L}{4} \varphi(2x, 2y, 2z)$$

for all $x, y, z \in V$, then there exists a unique Jensen type quadratic-additive mapping $F : V \rightarrow Y$ such that $DF(x, y, z) = 0$ for all $x, y, z \in V$ and

$$(2.16) \quad \|f(x) - F(x)\| \leq \frac{L\varphi(x, x, 0)}{4(1-L)}$$

for all $x \in V$. In particular, F is represented by

$$(2.17) \quad F(x) := f(0) + \lim_{n \rightarrow \infty} \left[\frac{4^n}{2} \left(f\left(\frac{x}{2^n}\right) + f\left(\frac{-x}{2^n}\right) - 2f(0) \right) + 2^{n-1} \left(f\left(\frac{x}{2^n}\right) - f\left(\frac{-x}{2^n}\right) \right) \right]$$

for all $x \in V$.

Proof. The proof is similarly verified by the same argument as that of Theorem 2.2. □

3. APPLICATIONS

For a given mapping $f : V \rightarrow Y$, we use the following abbreviations

$$\begin{aligned} Jf(x, y) &:= 2f\left(\frac{x+y}{2}\right) - f(x) - f(y), \\ Qf(x, y) &:= f(x+y) + f(x-y) - 2f(x) - 2f(y) \end{aligned}$$

for all $x, y \in V$. Using Theorem 2.2 and Theorem 2.3,

we will obtain the stability results of the quadratic functional equation $Qf \equiv 0$ and the Jensen's functional equation $Jf \equiv 0$ in the following corollaries.

Corollary 3.1. *Suppose that each $f_i : V \rightarrow Y, i = 1, 2$, and $\phi_i : V^2 \rightarrow [0, \infty), i = 1, 2$, are given functions satisfying*

$$\|Qf_i(x, y)\| \leq \phi_i(x, y)$$

and $\phi_i(x, y) = \phi_i(-x, -y)$ for all $x, y \in V$, respectively. If there exists $0 < L < 1$ such that

$$(3.1) \quad \phi_1(2x, 2y) \leq 2L\phi_1(x, y),$$

$$(3.2) \quad \phi_2(x, y) \leq \frac{L}{4}\phi_2(2x, 2y)$$

for all $x, y \in V$, then we have unique quadratic mappings $F_1, F_2 : V \rightarrow Y$ such that

$$(3.3) \quad \|f_1(x) - f_1(0) - F_1(x)\| \leq \frac{\phi_1(0, x) + \phi_1(x, x)}{2(1-L)},$$

$$(3.4) \quad \|f_2(x) - F_2(x)\| \leq \frac{L[\phi_2(0, x) + \phi_2(x, x)]}{4(1-L)}$$

for all $x \in V$. In particular, F_1 and F_2 are represented by

$$(3.5) \quad F_1(x) = \lim_{n \rightarrow \infty} \frac{f_1(2^n x)}{4^n},$$

$$(3.6) \quad F_2(x) = \lim_{n \rightarrow \infty} 4^n f_2\left(\frac{x}{2^n}\right)$$

for all $x \in V$. Moreover, if $0 < L < \frac{1}{2}$ and ϕ_1 is continuous, then $f_1 - f_1(0)$ is itself a Jensen type quadratic-additive mapping.

Proof. Notice that

$$\|Df_i(x, y, z)\| = \|Qf_i(x, y) - Qf_i(z, y)\| \leq \phi_i(x, y) + \phi_i(z, y)$$

for all $x, y, z \in V$ and $i = 1, 2$. Put

$$\varphi_i(x, y, z) := \phi_i(x, y) + \phi_i(z, y)$$

for all $x, y, z \in V$. Then φ_1 satisfies (2.2) and φ_2 satisfies (2.15). According to Theorem 2.2, there exists a unique mapping $F_1 : V \rightarrow Y$ satisfying (3.3) which is represented by

$$F_1(x) = \lim_{n \rightarrow \infty} \left(\frac{f_1(2^n x) + f_1(-2^n x)}{2 \cdot 4^n} + \frac{f_1(2^n x) - f_1(-2^n x)}{2^{n+1}} \right).$$

Observe that

$$\begin{aligned} \lim_{n \rightarrow \infty} \left\| \frac{f_1(2^n x) - f_1(-2^n x)}{2^{n+1}} \right\| &= \lim_{n \rightarrow \infty} \frac{1}{2^{n+1}} \|Qf_1(0, 2^n x)\| \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{2^{n+1}} \phi_1(0, 2^n x) \\ &\leq \lim_{n \rightarrow \infty} \frac{L^n}{2} \phi_1(0, x) = 0 \end{aligned}$$

as well as

$$\lim_{n \rightarrow \infty} \left\| \frac{f_1(2^n x) + f_1(-2^n x)}{2 \cdot 4^n} \right\| \leq \lim_{n \rightarrow \infty} \frac{L^n}{2^{n+1}} \phi_1(0, x) = 0$$

for all $x \in V$. From these two properties, we lead to the mapping (3.5) for all $x \in V$. Moreover, we have

$$\left\| \frac{Qf_1(2^n x, 2^n y)}{4^n} \right\| \leq \frac{\phi_1(2^n x, 2^n y)}{4^n} \leq \frac{L^n}{2^n} \phi_1(x, y)$$

for all $x, y \in V$. Taking the limit as $n \rightarrow \infty$ in the above inequality, we get

$$QF_1(x, y) = 0$$

for all $x, y \in V$ and so $F_1 : V \rightarrow Y$ is a quadratic mapping.

On the other hand, since $L\phi_2(0, 0) \geq 4\phi_2(0, 0)$ and

$$\|2f_2(0)\| = \|Qf_2(0, 0)\| \leq \phi_2(0, 0)$$

we can show that $\phi_2(0, 0) = 0$ and $f_2(0) = 0$. According to Theorem 2.3, there exists a unique mapping $F_2 : V \rightarrow Y$ satisfying (3.4), which is represented by (2.17). We have

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{4^n}{2} \left\| -f_2\left(\frac{x}{2^n}\right) + f_2\left(-\frac{x}{2^n}\right) \right\| &= \lim_{n \rightarrow \infty} \frac{4^n}{2} \left\| Qf_2\left(0, \frac{x}{2^n}\right) \right\| \\ &\leq \lim_{n \rightarrow \infty} \frac{4^n}{2} \phi_2\left(0, \frac{x}{2^n}\right) \\ &\leq \lim_{n \rightarrow \infty} \frac{L^n}{2} \phi_2(0, x) = 0 \end{aligned}$$

as well as

$$\lim_{n \rightarrow \infty} 2^{n-1} \left\| f_2\left(\frac{x}{2^n}\right) - f_2\left(-\frac{x}{2^n}\right) \right\| = 0$$

for all $x \in V$. From these and (2.8), we get (3.6). Notice that

$$\left\| 4^n Q f_2 \left(\frac{x}{2^n}, \frac{y}{2^n} \right) \right\| \leq 4^n \phi_2 \left(\frac{x}{2^n}, \frac{y}{2^n} \right) \leq L^n \phi_2(x, y)$$

for all $x, y \in V$. Taking the limit as $n \rightarrow \infty$, then we have shown that

$$QF_2(x, y) = 0$$

for all $x, y \in V$ and so $F_2 : V \rightarrow Y$ is a quadratic mapping. This completes the corollary. \square

Now, we obtain a stability result of Jensen functional equations.

Corollary 3.2. *Let $f_i : V \rightarrow Y$, $i = 1, 2$, be mappings for which there exist functions $\phi_i : V^2 \rightarrow [0, \infty)$, $i = 1, 2$, such that*

$$(3.7) \quad \|Jf_i(x, y)\| \leq \phi_i(x, y)$$

and $\phi_i(x, y) = \phi_i(-x, -y)$ for all $x, y \in V$, respectively. If there exists $0 < L < 1$ such that the mapping ϕ_1 has the property (3.1) and ϕ_2 holds (3.2) for all $x, y \in V$, then there exist unique Jensen mappings $F_i : V \rightarrow Y$, $i = 1, 2$, such that

$$(3.8) \quad \|f_1(x) - F_1(x)\| \leq \frac{\phi_1(0, 2x) + \phi_1(x, -x)}{2(1 - L)},$$

$$(3.9) \quad \|f_2(x) - F_2(x)\| \leq \frac{L(\phi_2(0, 2x) + \phi_2(x, -x))}{4(1 - L)}$$

for all $x \in V$. In particular, the mappings F_1, F_2 are represented by

$$(3.10) \quad F_1(x) = \lim_{n \rightarrow \infty} \frac{f_1(2^n x)}{2^n} + f_1(0),$$

$$(3.11) \quad F_2(x) = \lim_{n \rightarrow \infty} 2^n \left(f_2 \left(\frac{x}{2^n} \right) - f_2(0) \right) + f_2(0)$$

for all $x \in V$.

Proof. The proof is similar to that of Theorem 3.1. \square

Now, we obtain generalized Hyers-Ulam stability results in the framework of normed spaces using Theorem 2.2 and Theorem 2.3.

Corollary 3.3. *Let X be a normed space, $\theta \geq 0$, and $p \in (0, 1) \cup (2, \infty)$. Suppose that a mapping $f : X \rightarrow Y$ satisfies the inequality*

$$\|Df(x, y, z)\| \leq \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

for all $x, y, z \in X$. Then there exists a unique Jensen type quadratic-additive mapping $F : X \rightarrow Y$ such that

$$\|f(x) - F(x)\| \leq \begin{cases} \frac{2\theta}{2-2^p} \|x\|^p, & \text{if } 0 < p < 1; \\ \frac{2\theta}{2^p-4} \|x\|^p, & \text{if } p > 2, \end{cases}$$

for all $x \in X$.

Proof. It follows from Theorem 2.2 and Theorem 2.3, by putting $L := 2^{p-1} < 1$ if $0 < p < 1$, and $L := 2^{2-p} < 1$ if $p > 2$. \square

In the following, we present counter-examples for the singular cases $p = 1$ and $p = 2$ in Corollary 3.3.

Example 3.1. We remark that if we consider an odd function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f(x) = \sum_{i=1}^{\infty} \frac{\phi(2^i x)}{2^i}, \quad \phi(x) = \begin{cases} \mu x, & \text{if } -1 < x < 1; \\ \mu, & \text{if } x \geq 1; \\ -\mu, & \text{if } x \leq -1, \quad (\mu > 0), \end{cases}$$

which is the same type as that in the paper [9], then it follows from [28] that

$$\begin{aligned} |f(x+y) + f(x-y) - 2f(x)| &\leq \theta(|x| + |y|), \\ |f(z+y) + f(z-y) - 2f(z)| &\leq \theta(|z| + |y|), \end{aligned}$$

and so

$$|Df(x, y, z)| \leq 2\theta(|x| + |y| + |z|),$$

for all x, y, z and for some constant $\theta > 0$. However, there doesn't exist Jensen type quadratic-additive function $F : \mathbb{R} \rightarrow \mathbb{R}$ such that $|f(x) - F(x)| \leq K(\theta)|x|$ for all x and for some constant $K(\theta)$. Hence, there exists a counter-example for the case $p = 1$ in Corollary 3.3.

Also, we remark that if we consider an even function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f(x) = \sum_{i=1}^{\infty} \frac{\phi(2^i x)}{4^i}, \quad \phi(x) = \begin{cases} \mu x^2, & \text{if } |x| < 1; \\ \mu, & \text{if } |x| \geq 1, \quad (\mu > 0), \end{cases}$$

which is the same type as that in the paper [8], then it is well known that

$$\begin{aligned} |f(x+y) + f(x-y) - 2f(x) - 2f(y)| &\leq \theta(|x|^2 + |y|^2), \\ |f(z+y) + f(z-y) - 2f(z) - 2f(y)| &\leq \theta(|z|^2 + |y|^2), \end{aligned}$$

and so

$$|Df(x, y, z)| \leq 2\theta(|x|^2 + |y|^2 + |z|^2),$$

for all x, y, z and for some constant $\theta > 0$. However, there doesn't exist Jensen type quadratic-additive function $F : \mathbb{R} \rightarrow \mathbb{R}$ such that $|f(x) - F(x)| \leq K(\theta)|x|^2$ for all x and for some constant $K(\theta)$. Hence, there exists a counter-example for the case $p = 2$ in Corollary 3.3.

Corollary 3.4. Let X be a normed space, $\theta \geq 0$ and $p, q, r > 0$ be reals with $p+q+r \in (-\infty, 1) \cup (2, \infty)$. If a mapping $f : X \rightarrow Y$ satisfies

$$\|Df(x, y, z)\| \leq \theta \|x\|^p \|y\|^q \|z\|^r$$

for all $x, y, z \in X$, then f is itself a Jensen type quadratic-additive mapping.

Proof. It follows from Theorem 2.2 and Theorem 2.3, by putting $L := 2^{p+q+r-1} < 1$ if $p+q+r < 1$, and $L := 2^{2-p-q-r} < 1$ if $p+q+r > 2$. \square

Corollary 3.5. *Let X be a normed space, $\theta_i \geq 0, (i = 1, 2, 3)$ and $p, q, r > 0$ be reals such that either $\max\{p + q, q + r, p + r\} < 1$ or $\min\{p + q, q + r, p + r\} > 2$. If a mapping $f : X \rightarrow Y$ satisfies*

$$\|Df(x, y, z)\| \leq \theta_1 \|x\|^p \|y\|^q + \theta_2 \|y\|^q \|z\|^r + \theta_3 \|x\|^p \|z\|^r$$

for all $x, y, z \in X$, Then there exists a unique Jensen type quadratic-additive mapping $F : X \rightarrow Y$ such that

$$\|f(x) - F(x)\| \leq \begin{cases} \frac{\theta_1}{2 - 2^{\max\{p+q, q+r, p+r\}}} \|x\|^{p+q}, & \text{if } \max\{p + q, q + r, p + r\} < 1; \\ \frac{\theta_1}{2^{\min\{p+q, q+r, p+r\} - 4}} \|x\|^{p+q}, & \text{if } \min\{p + q, q + r, p + r\} > 2, \end{cases}$$

for all $x \in X$.

Proof. It follows from Theorem 2.2 and Theorem 2.3, by putting

$$L := 2^{\max\{p+q, q+r, p+r\} - 1} < 1, \text{ if } \max\{p + q, q + r, p + r\} < 1,$$

and

$$L := 2^{2 - \min\{p+q, q+r, p+r\}} < 1, \text{ if } \min\{p + q, q + r, p + r\} > 2.$$

□

In the following, we present counter-examples for the singular cases $\max\{p + q, q + r, p + r\} = 1$ and $\min\{p + q, q + r, p + r\} = 2$ in Corollary 3.5.

Example 3.2. We remark that if we consider an odd function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} x \ln|x|, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0, \end{cases}$$

then for any p with $0 < p < 1$ it follows from [11, 12] that there exists a constant $c > 0$ such that

$$|f(x + y) - f(x) - f(y)| \leq c|x|^p|y|^{1-p},$$

and so

$$\begin{aligned} |f(x - y) - f(x) + f(y)| &\leq c|x|^p|y|^{1-p}, \\ |f(x + y) + f(x - y) - 2f(x)| &\leq 2c|x|^p|y|^{1-p}, \\ |f(z + y) + f(z - y) - 2f(z)| &\leq 2c|z|^p|y|^{1-p}, \end{aligned}$$

which yield

$$|Df(x, y, z)| \leq 2c(|x|^p|y|^{1-p} + |y|^{1-p}|z|^p),$$

for all x, y, z . However, there doesn't exist Jensen type quadratic-additive function $F : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$|f(x) - F(x)| \leq K(c, p)|x|$$

for all x and for some constant $K(c, p)$. Hence, there exists a counter-example for the case $\max\{p + q, q + r, p + r\} = 1$ in Corollary 3.5.

Also, we remark that if we consider an even function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2 \ln|x|, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0, \end{cases}$$

then for any p with $0 < p < 2$ it follows from [13] that there exists a constant $k > 0$ such that

$$\begin{aligned} |f(x+y) + f(x-y) - 2f(x) - 2f(y)| &\leq k|x|^p|y|^{2-p}, \\ |f(z+y) + f(z-y) - 2f(z) - 2f(y)| &\leq k|z|^p|y|^{2-p}, \end{aligned}$$

which yield

$$|Df(x, y, z)| \leq k(|x|^p|y|^{2-p} + |y|^{2-p}|z|^p),$$

for all x, y, z . However, there doesn't exist Jensen type quadratic-additive function $F : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$|f(x) - F(x)| \leq K(k, p)|x|^2$$

for all x and for some constant $K(k, p)$. Hence, there exists a counter-example for the case $\min\{p+q, q+r, p+r\} = 2$ in Corollary 3.5.

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On certain subclasses of p -valent analytic functions involving Saitoh operator

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Abstract: The object of the present investigation is to solve Fekete-Szegö problem for a new class $\mathcal{V}_p^\lambda(a, c, A, B)$ of p -valent analytic functions involving the Saitoh operator in the unit disk. We also obtain subordination results and some interesting corollaries for functions in \mathcal{A}_p involving this operator. Relevant connections of the results obtained here with those given by earlier workers on the subject are also mentioned.

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1. INTRODUCTION AND PRELIMINARIES

Let \mathcal{A}_p be the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \tag{1.1}$$

analytic in the unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ with $p \in \mathbb{N} = \{1, 2, 3, \dots\}$. Let \mathcal{S} be the subclass of $\mathcal{A}_1 = \mathcal{A}$ consisting of univalent functions.

A function $f \in \mathcal{A}_p$ is said to be p -valent starlike of order α , denoted by $\mathcal{S}_p^*(\alpha)$, if and only if

$$\operatorname{Re} \left\{ \frac{z f'(z)}{f(z)} \right\} > \alpha \quad (0 \leq \alpha < p; z \in \mathcal{U}). \tag{1.2}$$

Similarly, a function $f \in \mathcal{A}_p$ is said to be p -valent convex of order α , denoted by $\mathcal{C}_p(\alpha)$, if and only if

$$\operatorname{Re} \left\{ 1 + \frac{z f''(z)}{f'(z)} \right\} > \alpha \quad (0 \leq \alpha < p; z \in \mathcal{U}). \tag{1.3}$$

From (1.2) and (1.3), it follows that

$$f(z) \in \mathcal{C}_p(\alpha) \iff \frac{z f'(z)}{p} \in \mathcal{S}_p^*(\alpha).$$

Furthermore, we say that a function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{R}_p(\alpha)$, if and only if

$$\operatorname{Re} \left\{ \frac{f'(z)}{z^{p-1}} \right\} > \alpha \quad (0 \leq \alpha < p; z \in \mathcal{U}). \tag{1.4}$$

For functions f and g , analytic in the unit disk \mathcal{U} , we say the f is said to be subordinate to g , written as $f \prec g$ or $f(z) \prec g(z)$ ($z \in \mathcal{U}$), if there exists an analytic function ω in \mathcal{U} with $\omega(0) = 0, |\omega(z)| \leq |z|$ ($z \in \mathcal{U}$) and $f(z) = g(\omega(z))$ for all $z \in \mathcal{U}$. In particular, if g is univalent in \mathcal{U} , then we have the following equivalence (cf., e.g., [14]):

$$f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(\mathcal{U}) \subset g(\mathcal{U}).$$

For the functions f and g given by the power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad g(z) = \sum_{n=0}^{\infty} b_n z^n, \quad (z \in \mathcal{U})$$

their Hadamard product (or convolution), denoted by $f \star g$ is defined as

$$(f \star g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n = (g \star f)(z) \quad (z \in \mathcal{U}).$$

Note that $f \star g$ is analytic in \mathcal{U} .

By making use of the Hadamard product, Saitoh [18] defined a linear operator $\mathcal{L}_p(a, c) : \mathcal{A}_p \rightarrow \mathcal{A}_p$ in terms of the function φ_p as

$$\mathcal{L}_p(a, c)f(z) = \varphi_p(a, c; z) \star f(z) \quad (f \in \mathcal{A}_p; z \in \mathcal{U}), \tag{1.5}$$

where

$$\varphi_p(a, c; z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{p+k} \quad (a \in \mathbb{C}, c \in \mathbb{C} \setminus \{\dots, -2, -1, 0\}; z \in \mathcal{U}). \tag{1.6}$$

and $(x)_k$ is the Pochhammer symbol (or shifted factorial) given by

$$(x)_k = \begin{cases} 1, & n = 0, \\ x(x+1) \cdots (x+k-1), & k \in \mathbb{N}. \end{cases}$$

For $p = 1$, the operator $\mathcal{L}_p(a, c)$ reduces to the Carlson-Shaffer operator $\mathcal{L}(a, c)$ [1]. If $f \in \mathcal{A}_p$ is given by (1.1), then it follows from (1.5) and (1.6) that

$$\mathcal{L}_p(a, c)f(z) = z^p + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} a_{p+k} z^{p+k} \quad (z \in \mathcal{U}) \tag{1.7}$$

and

$$z (\mathcal{L}_p(a, c)f)'(z) = a\mathcal{L}_p(a+1, c)f(z) - (a-p)\mathcal{L}_p(a, c)f(z) \quad (z \in \mathcal{U}). \tag{1.8}$$

It is easily seen that for $f \in \mathcal{A}_p$

(i) $\mathcal{L}_p(a, a)f(z) = f(z),$

(ii) $\mathcal{L}_p(p+1, p)f(z) = \frac{zf'(z)}{p},$

(iii) $\mathcal{L}_p(n+p, p)f(z) = \mathcal{D}^{n+p-1}f(z)$ ($n \in \mathbb{Z}; n > -p$), the operator studied by Goel and Sohi [5]. For the case $p = 1$, \mathcal{D}^n is the Ruschewyh derivative operator [17].

(iv) $\mathcal{L}_p(p+1, n+p)f(z) = \mathcal{J}_{n,p}f(z)$ ($n \in \mathbb{Z}; n > -p$), the extended Noor integral operator considered by Liu and Noor [10].

(v) $\mathcal{L}_p(p+1, p+1-\mu)f(z) = \Omega_z^{(\mu,p)}f(z)$ ($-\infty < \mu < p+1$), the extended fractional differintegral operator studied by Patel and Mishra [16]. Note that

$$\Omega_z^{(0,p)}f(z) = f(z), \Omega_z^{(1,p)}f(z) = \frac{zf'(z)}{p} \text{ and } \Omega_z^{(2,p)}f(z) = \frac{z^2f''(z)}{p(p-1)} \quad (p \geq 2).$$

As a special case, we get the operator $\Omega_z^\mu f(z)$ ($0 \leq \mu < 1$) for $p = 1$ introduced and studied by Owa-Srivastava [15].

With the aid of the operator $\mathcal{L}_p(a, c)$, we introduce a subclass of \mathcal{A}_p as follows.

Definition 1.1. For the fixed parameters A, B ($-1 \leq B < A \leq 1$), $a > 0$ and $c > 0$, we say that a function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{V}_p^\lambda(a, c, A, B)$, if it satisfies the following subordination relation

$$(1 - \lambda) \frac{\mathcal{L}_p(a, c)f(z)}{z^p} + \frac{\lambda z (\mathcal{L}_p(a, c)f)'(z)}{p \mathcal{L}_p(a, c)f(z)} \prec \frac{1 + Az}{1 + Bz} \quad (0 \leq \lambda \leq 1; z \in \mathcal{U}). \tag{1.9}$$

We note that the class $\mathcal{V}_p^\lambda(a, c, A, B)$ includes many known subclasses of \mathcal{A}_p as mentioned below.

$$\begin{aligned} \text{(i) } \mathcal{V}_p^1\left(a, c, 1 - \frac{2\alpha}{p}, -1\right) &= \mathcal{S}_p(a, c; \alpha) \quad (0 \leq \alpha < p) \\ &= \left\{ f \in \mathcal{A}_p : \operatorname{Re} \left(\frac{z (\mathcal{L}_p(a, c)f)'(z)}{\mathcal{L}_p(a, c)f(z)} \right) > \alpha, z \in \mathcal{U} \right\}. \end{aligned}$$

Note that $\mathcal{S}_p(a, a; \alpha) = \mathcal{S}_p^*(\alpha)$, the class of p -valent starlike functions of order α and $\mathcal{S}_p(p+1, p; \alpha) = \mathcal{C}_p(\alpha)$, the class of p -valent convex functions of order α .

$$\begin{aligned} \text{(ii) } \mathcal{V}_p^0\left(a, c, 1 - \frac{2\alpha}{p}, -1\right) &= \mathcal{R}_p(a, c; \alpha) \quad (0 \leq \alpha < p) \\ &= \left\{ f \in \mathcal{A}_p : \operatorname{Re} \left(\frac{\mathcal{L}_p(a, c)f(z)}{z^p} \right) > \frac{\alpha}{p}, z \in \mathcal{U} \right\} \end{aligned}$$

which, in turn yields the class $\mathcal{R}_p(\alpha)$ for $a = p + 1$ and $c = p$.

For $0 \leq \alpha < 1$, the functions in the class $\mathcal{R}_1(\alpha) = \mathcal{R}(\alpha)$ are called functions of bounded turning. By the Nashiro-Warschowski Theorem [3], the functions in $\mathcal{R}(\alpha)$ are univalent and also close-to-convex in \mathcal{U} . It is well-known that $\mathcal{R}(\alpha) \subsetneq \mathcal{S}_1^*(0) = \mathcal{S}^*$ and $\mathcal{S}^* \subsetneq \mathcal{R}(\alpha)$. For more information on the class $\mathcal{R}(0) = \mathcal{R}$ (cf., e.g., [12]).

Fekete and Szegö [4] proved a remarkable result that the estimate

$$|a_3 - \gamma a_2^2| \leq 1 + 2 \exp\left(-\frac{2\gamma}{1-\gamma}\right)$$

is sharp and holds for each $\gamma \in [0, 1]$ over the class \mathcal{S} consisting of functions $f \in \mathcal{A}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathcal{U}). \tag{1.10}$$

The coefficient functional

$$\Phi_\gamma(f) = a_3 - \gamma a_2^2 = \frac{1}{6} \left\{ f'''(0) - \frac{3\gamma}{2} (f''(0))^2 \right\}$$

on the functions in \mathcal{A} represents various geometrical properties, for example, when $\gamma = 1$, $\Phi_\gamma(f) = a_3 - a_2^2$ becomes $S_f(0)/6$, where S_f denote the Schwarzian derivative

$$S_f(z) = \left\{ \left(\frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left(\frac{f''(z)}{f'(z)} \right)^2 \right\} = \frac{f'''(z)}{f'(z)} - \frac{3}{2} \left(\frac{f''(z)}{f'(z)} \right)^2$$

of locally univalent functions in \mathcal{U} . For a family \mathcal{F} of functions in \mathcal{A} of the form (1.10), the more general problem of maximizing the absolute value for the functional $\Phi_\gamma(f)$ for some γ (real as well as complex) is popularly known as Fekete-Szegő problem for the class \mathcal{F} . In literature, there exists a large number of results about the inequalities for $|\Phi_\gamma(f)|$ corresponding to various subclasses of S (see, e.g., [4, 7, 8, 9]).

The object of the present study is to solve Fekete-Szegő problem for a new class $\mathcal{V}_p^\lambda(a, c, A, B)$ of p -valent analytic functions in \mathcal{U} involving the Saitoh operator. We also obtain some subordination results along with some interesting corollaries for functions in \mathcal{A}_p involving this operator. Relevant connections of the results obtained here with those given by earlier workers on the subject are pointed out.

2. PRELIMINARIES

Let \mathcal{P} denote the family of all functions of the form

$$\varphi(z) = 1 + q_1z + q_2z^2 + \dots \tag{2.1}$$

analytic in \mathcal{U} and satisfying the condition $\text{Re}\{\varphi(z)\} > 0$ in \mathcal{U} .

To establish our main results, we need the following lemmas.

Lemma 2.1. *If the function φ , given by (2.1) belongs to the class \mathcal{P} , then for any complex number γ ,*

$$|q_k| \leq 2 \tag{2.2}$$

and

$$|q_2 - \gamma q_1^2| \leq 2 \max\{1, |2\gamma - 1|\}. \tag{2.3}$$

The result in (2.2) is sharp for the function $\varphi_1(z) = (1+z)/(1-z)$ ($z \in \mathcal{U}$), where as, the result in (2.3) is sharp for the functions $\varphi_2(z) = (1+z^2)/(1-z^2)$ ($z \in \mathcal{U}$) and $\varphi_1(z)$.

We note that the estimate (2.2) is contained in [3], the estimate (2.3) is due to Ma and Minda [11].

The following lemma is due to Miller and Mocanu [14].

Lemma 2.2. *Let q be univalent in \mathcal{U} and let θ and ϕ be analytic in a domain Ω containing $q(\mathcal{U})$ with $\phi(w) \neq 0$, when $w \in q(\mathcal{U})$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$ and suppose that either*

- (i) h is convex, or
- (ii) Q is starlike.

In addition, assume that

$$(iii) \operatorname{Re} \left\{ \frac{zh'(z)}{Q(z)} \right\} > 0 \quad (z \in \mathcal{U}).$$

If ψ is analytic in \mathcal{U} with $\psi(0) = q(0), \psi(\mathcal{U}) \subset \Omega$ and

$$\theta(\psi(z)) + z\psi'(z)\phi(\psi(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)) \quad (z \in \mathcal{U}),$$

then $\psi(z) \prec q(z)$ ($z \in \mathcal{U}$) and the function q is the best dominant.

3. MAIN RESULTS

Unless otherwise mentioned, we assume throughout the sequel that $a > 0, c > 0, 0 \leq \lambda \leq 1$ and $-1 \leq B < A \leq 1$.

We first solve the Fekete-Szegő problem for the class $\mathcal{V}_p^\lambda(a, c, A, B)$.

Theorem 3.1. *If $\gamma \in \mathbb{R}$ and the function f , given by (1.1) belongs to the class $\mathcal{V}_p^\lambda(a, c, A, B)$, then*

$$\left| a_{p+2} - \gamma a_{p+1}^2 \right| \leq \begin{cases} \frac{-p(A-B)cQ}{(p+\lambda(1-p))^2(p+\lambda(2-p))a(a)_2}, & \gamma < \rho_1 \\ \frac{p(A-B)(c)_2}{\{p+(2-p)\lambda\}(a)_2}, & \rho_1 \leq \gamma \leq \rho_2 \\ \frac{p(A-B)cQ}{(p+\lambda(1-p))^2(p+\lambda(2-p))a(a)_2}, & \gamma > \rho_2, \end{cases} \quad (3.1)$$

where

$$Q = [\gamma p(p+\lambda(2-p))(A-B)(a+1)c + \{B(p+\lambda(1-p))^2 - \lambda p(A-B)\}a(c+1)],$$

$$\rho_1 = \frac{[\lambda p(A-B) - (1+B)\{p+\lambda(1-p)\}^2] a(c+1)}{p\{p+\lambda(2-p)\}(A-B)(a+1)c},$$

and

$$\rho_2 = \frac{[\lambda p(A-B) + (1-B)\{p+\lambda(1-p)\}^2] a(c+1)}{p\{p+\lambda(2-p)\}(A-B)(a+1)c}.$$

All these results are sharp.

Proof. From (1.9), it follows that

$$(1-\lambda) \frac{\mathcal{L}_p(a, c)f(z)}{z^p} + \frac{\lambda z (\mathcal{L}_p(a, c)f)'(z)}{p \mathcal{L}_p(a, c)f(z)} = \frac{1-A+(1+A)\varphi(z)}{1-B+(1+B)\varphi(z)} \quad (z \in \mathcal{U}),$$

where the function φ defined by (2.1) belongs to the class \mathcal{P} . Substituting the power series expansion of $\mathcal{L}_p(a, c)f$ and φ in the above expression, we deduce that

$$a_{p+1} = \frac{p(A-B)c}{2(p+\lambda(1-p))a} q_1, \quad (3.2)$$

and

$$\begin{aligned} a_{p+2} &= \frac{p(A-B)(c)_2}{2(p+\lambda(2-p))(a)_2} \left\{ q_2 - \left(\frac{1+B}{2} \right) q_1^2 + \frac{\lambda p(A-B)}{2(p+\lambda(1-p))^2} q_1^2 \right\} \\ &= q_2 + \frac{\lambda p(A-B) - (1+B)(p+\lambda(1-p))^2}{2(p+\lambda(1-p))^2} q_1^2. \end{aligned} \quad (3.3)$$

With the aid of (3.2) and (3.3), we get

$$\left| a_{p+2} - \gamma a_{p+1}^2 \right| = \frac{p(A - B)(c)_2}{2(p + \lambda(2 - p))(a)_2} \times \left| q_2 - \frac{\gamma p(p + \lambda(2 - p))(A - B)(a + 1)c + \{(1 + B)(p + \lambda(1 - p))^2 - \lambda p(A - B)\}a(c + 1)}{2(p + \lambda(1 - p))^2 a(c + 1)} q_1^2 \right|,$$

which in view of Lemma 2.1 yields

$$\left| a_{p+2} - \gamma a_{p+1}^2 \right| = \frac{p(A - B)(c)_2}{(p + \lambda(2 - p))(a)_2} \times \max \left\{ 1, \frac{\left| \gamma p(p + \lambda(2 - p))(A - B)(a + 1)c + \{B(p + \lambda(1 - p))^2 - \lambda p(A - B)\}a(c + 1) \right|}{(p + \lambda(1 - p))^2 a(c + 1)} \right\}. \tag{3.4}$$

Now, we consider the following cases.

(i) If

$$\frac{\left| \gamma p(p + \lambda(2 - p))(A - B)(a + 1)c + \{B(p + \lambda(1 - p))^2 - \lambda p(A - B)\}a(c + 1) \right|}{(p + \lambda(1 - p))^2 a(c + 1)} \leq 1,$$

then it is easily seen that $\rho_1 \leq \gamma \leq \rho_2$ and (3.4) gives the second estimate in (3.1).

(ii) For

$$\frac{\left| \gamma p(p + \lambda(2 - p))(A - B)(a + 1)c + \{B(p + \lambda(1 - p))^2 - \lambda p(A - B)\}a(c + 1) \right|}{(p + \lambda(1 - p))^2 a(c + 1)} > 1,$$

we have either

$$\frac{\gamma p(p + \lambda(2 - p))(A - B)(a + 1)c + \{B(p + \lambda(1 - p))^2 - \lambda p(A - B)\}a(c + 1)}{(p + \lambda(1 - p))^2 a(c + 1)} < -1$$

or

$$\frac{\gamma p(p + \lambda(2 - p))(A - B)(a + 1)c + \{B(p + \lambda(1 - p))^2 - \lambda p(A - B)\}a(c + 1)}{(p + \lambda(1 - p))^2 a(c + 1)} > 1.$$

The above inequalities implies that either $\gamma < \rho_1$ or $\gamma > \rho_2$. Thus, again by use of (3.4), we get the first and the third estimate in (3.1).

We note that the results are sharp for the function f defined in \mathcal{U} by

$$(1 - \lambda) \frac{\mathcal{L}_p(a, c)f(z)}{z^p} + \frac{\lambda z (\mathcal{L}_p(a, c)f)'(z)}{p \mathcal{L}_p(a, c)f(z)} = \begin{cases} \frac{1 + Az}{1 + Bz}, & \text{if } \gamma < \rho_1 \text{ or } \gamma > \rho_2 \\ \frac{1 + Az^2}{1 + Bz^2}, & \text{if } \rho_1 \leq \gamma \leq \rho_2, \end{cases}$$

where $0 \leq \lambda \leq 1, a > 0, c > 0$ and $-1 \leq B < A \leq 1$. This completes the proof of Theorem 3.1. □

Taking $\lambda = 1, A = 1 - (2\alpha/p)$ ($0 \leq \alpha < p$) and $B = -1$ in Theorem 3.1, we obtain

Corollary 3.1. *If $\gamma \in \mathbb{R}$ and the function f , given by (1.1) belongs to the class $\mathcal{S}_p(a, c; \alpha)$, then*

$$|a_{p+2} - \gamma a_{p+1}^2| \leq \begin{cases} \frac{(p-\alpha)\{2(p-\alpha)+1\}a(c)_2 - 4\gamma(p-\alpha)(a+1)c^2}{a(a)_2}, & \gamma < \frac{a(c+1)}{2(a+1)c} \\ \frac{(p-\alpha)(c)_2}{(a)_2}, & \frac{a(c+1)}{2(a+1)c} \leq \gamma \leq \frac{(p+1-\alpha)a(c+1)}{2(p-\alpha)(a+1)c} \\ \frac{(p-\alpha)\{4\gamma(p-\alpha)(a+1)c^2 - (2(p-\alpha)+1)a(c)_2\}}{a(a)_2}, & \gamma > \frac{(p+1-\alpha)a(c+1)}{2(p-\alpha)(a+1)c}. \end{cases}$$

These results are sharp for the function $f \in \mathcal{A}_p$ defined in \mathcal{U} by

$$f(z) = \begin{cases} \varphi_p(c, a; z) * \frac{z^p}{(1-z)^{2(p-\alpha)}}, & \gamma < \frac{a(c+1)}{2(a+1)c} \quad \text{or} \quad \gamma > \frac{(p+1-\alpha)a(c+1)}{2(p-\alpha)(a+1)c}, \\ \varphi_p(c, a; z) * \frac{z^p}{(1-z^2)^{p-\alpha}}, & \frac{a(c+1)}{2(a+1)c} \leq \gamma \leq \frac{(p+1-\alpha)a(c+1)}{2(p-\alpha)(a+1)c}. \end{cases}$$

Remark 3.1. (i) *Setting $c = a$ ($a = p + 1$ and $c = p$, respectively) in Corollary 3.1, we get the corresponding results obtained by Hayami and Owa [6, Theorem 3 and Theorem 4].*

(ii) *Using the fact that $|q_1| \leq 2$ in (3.2) and Lemma 2.1 in (3.3), we get the following coefficient estimates for a function f , given by (1.1) in the class $\mathcal{V}_p^\lambda(a, c, A, B)$,*

$$|a_{p+1}| \leq \frac{p(A-B)c}{\{p + \lambda(1-p)\}a}$$

and

$$|a_{p+2}| \leq \frac{p(A-B)(c)_2}{\{p + \lambda(2-p)\}(a)_2} \max \left\{ 1, \frac{|B\{p + \lambda(1-p)\}^2 - \lambda p(A-B)|}{\{p + \lambda(1-p)\}^2} \right\}.$$

Both the estimates are sharp.

For the case $\lambda = 0, A = 1 - (2\alpha/p)$ ($0 \leq \alpha < p$) and $B = -1$, Theorem 3.1 yields the following result.

Corollary 3.2. *If $\gamma \in \mathbb{R}$ and the function f , given by (1.1) belongs to the class $\mathcal{R}_{a,c}(\alpha)$, then*

$$|a_{p+2} - \gamma a_{p+1}^2| \leq \begin{cases} -\frac{2\left(1 - \frac{\alpha}{p}\right)\left\{2\gamma\left(1 - \frac{\alpha}{p}\right)(a+1)c - a(c+1)\right\}}{a(a)_2}, & \gamma < 0 \\ \frac{2\left(1 - \frac{\alpha}{p}\right)(c)_2}{(a)_2}, & 0 \leq \gamma \leq \frac{\left(1 - \frac{\alpha}{p}\right)^{-1} a(c+1)}{(a+1)c} \\ \frac{2\left(1 - \frac{\alpha}{p}\right)\left\{2\gamma\left(1 - \frac{\alpha}{p}\right)(a+1)c - a(c+1)\right\}}{a(a)_2}, & \gamma > \frac{\left(1 - \frac{\alpha}{p}\right)^{-1} a(c+1)}{(a+1)c}. \end{cases}$$

The results are sharp for the function $f \in \mathcal{A}_p$ defined in \mathcal{U} by

$$f(z) = \begin{cases} \varphi_p(c, a; z) * \frac{z^p \left\{ 1 + \left(1 - \frac{2\alpha}{p} \right) z \right\}}{1 - z}, & \gamma < 0 \text{ or } \gamma > \frac{\left(1 - \frac{\alpha}{p} \right)^{-1} a(c+1)}{(a+1)c} \\ \varphi_p(c, a; z) * \frac{z^p \left\{ 1 + \left(1 - \frac{2\alpha}{p} \right) z^2 \right\}}{1 - z^2}, & 0 \leq \gamma \leq \frac{\left(1 - \frac{\alpha}{p} \right)^{-1} a(c+1)}{(a+1)c}. \end{cases}$$

Letting $c = a$ in Corollary 3.2, we get

Corollary 3.3. *If $\gamma \in \mathbb{R}$ and the function $f \in \mathcal{A}$, given by (1.1) satisfies the condition*

$$\operatorname{Re} \left\{ \frac{f(z)}{z^p} \right\} > \frac{\alpha}{p} \quad (0 \leq \alpha < p; z \in \mathcal{U}),$$

then

$$\left| a_{p+2} - \gamma a_{p+1}^2 \right| \leq \begin{cases} -2 \left(1 - \frac{\alpha}{p} \right) \left\{ 2\gamma \left(1 - \frac{\alpha}{p} \right) - 1 \right\}, & \gamma < 0 \\ 2 \left(1 - \frac{\alpha}{p} \right), & 0 \leq \gamma \leq \left(1 - \frac{\alpha}{p} \right)^{-1} \\ 2 \left(1 - \frac{\alpha}{p} \right) \left\{ 2\gamma \left(1 - \frac{\alpha}{p} \right) - 1 \right\}, & \gamma > \left(1 - \frac{\alpha}{p} \right)^{-1}. \end{cases}$$

These results are sharp for the function $f \in \mathcal{A}$ defined in \mathcal{U} by

$$f(z) = \begin{cases} \frac{z^p \left\{ 1 + \left(1 - \frac{2\alpha}{p} \right) z \right\}}{1 - z}, & \gamma < 0 \text{ or } \gamma > \left(1 - \frac{\alpha}{p} \right)^{-1} \\ \frac{z^p \left\{ 1 + \left(1 - \frac{2\alpha}{p} \right) z^2 \right\}}{1 - z^2}, & 0 \leq \gamma \leq \left(1 - \frac{\alpha}{p} \right)^{-1}. \end{cases}$$

For the choice $a = p + 1$ and $c = p$ in Corollary 3.2, we obtain

Corollary 3.4. *If $\gamma \in \mathbb{R}$ and the function $f \in \mathcal{A}$, given by (1.1) satisfies the condition*

$$\operatorname{Re} \left\{ \frac{f'(z)}{z^{p-1}} \right\} > \alpha \quad (0 \leq \alpha < p; z \in \mathcal{U}),$$

then

$$\left| a_{p+2} - \gamma a_{p+1}^2 \right| \leq \begin{cases} \frac{2(p-\alpha) \{ 2\gamma(p+2)(p-\alpha) - (p+1)^2 \}}{(p+1)^2(p+2)}, & \gamma < 0 \\ \frac{2(p-\alpha)}{p+2}, & 0 \leq \gamma \leq \frac{(p+1)^2}{(p+2)(p-\alpha)} \\ \frac{2(p-\alpha) \{ 2\gamma(p+2)(p-\alpha) - (p+1)^2 \}}{(p+1)^2(p+2)}, & \gamma > \frac{(p+1)^2}{(p+2)(p-\alpha)}. \end{cases}$$

These results are sharp for the function $f \in \mathcal{A}$ defined in \mathcal{U} by

$$f(z) = \begin{cases} \varphi_p(p, p+1; z) * \frac{z^p \left\{ 1 + \left(1 - \frac{2\alpha}{p} \right) z \right\}}{1 - z}, & \gamma < 0 \text{ or } \gamma > \left(1 - \frac{\alpha}{p} \right)^{-1} \\ \varphi_p(p, p+1; z) * \frac{z^p \left\{ 1 + \left(1 - \frac{2\alpha}{p} \right) z^2 \right\}}{1 - z^2}, & 0 \leq \gamma \leq \left(1 - \frac{\alpha}{p} \right)^{-1}. \end{cases}$$

Next, we prove the following subordination result.

Theorem 3.2. *If a function $f \in \mathcal{A}_p$ satisfies the subordination relation*

$$(1 - \lambda) \frac{\mathcal{L}_p(a, c)f(z)}{z^p} + \frac{\lambda z (\mathcal{L}_p(a, c)f)'(z)}{p \mathcal{L}_p(a, c)f(z)} \prec 1 + (1 - \lambda) \frac{(A - B)z}{1 + Bz} + \frac{\lambda(A - B)z}{p(1 + Az)(1 + Bz)} \quad (0 < \lambda \leq 1, -1 \leq B < A \leq 1; z \in \mathcal{U}), \quad (3.5)$$

then

$$\frac{\mathcal{L}_p(a, c)f(z)}{z^p} \prec \frac{1 + Az}{1 + Bz} = \tilde{q}(z) \text{ (say)} \quad (z \in \mathcal{U}) \quad (3.6)$$

and the function \tilde{q} is the best dominant of (3.6).

Proof. Setting

$$q(z) = \frac{1 + Az}{1 + Bz} \quad (z \in \mathcal{U}), \theta(w) = \lambda + (1 - \lambda)w \quad (w \in \mathbb{C}) \quad \text{and} \quad \phi(w) = \frac{\lambda}{pw} \quad (0 \neq w \in \mathbb{C}),$$

we see that

$$Q(z) = \frac{\lambda z q'(z)}{p q(z)} = \frac{\lambda(A - B)z}{p(1 + Az)(1 + Bz)}$$

and

$$\operatorname{Re} \left\{ \frac{zQ'(z)}{Q(z)} \right\} = \operatorname{Re} \left\{ \frac{1}{1 + Az} - \frac{Bz}{1 + Bz} \right\} > 0,$$

so that Q is starlike in \mathcal{U} . Further, letting $h(z) = \theta(q(z)) + Q(z)$, we get

$$\operatorname{Re} \left\{ \frac{zh'(z)}{Q(z)} \right\} = \frac{(1 - \lambda)p}{\lambda} \operatorname{Re}\{q(z)\} + \operatorname{Re} \left\{ \frac{zQ'(z)}{Q(z)} \right\} > 0 \quad (z \in \mathcal{U}).$$

Suppose that

$$\psi(z) = \frac{\mathcal{L}_p(a, c)f(z)}{z^p} \quad (z \in \mathcal{U}).$$

Then the hypothesis (3.5) implies that

$$\theta(\psi(z)) + z\psi'(z)\phi(\psi(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)) \quad (z \in \mathcal{U}),$$

which in view of Lemma 2.2 gives the required assertion (3.6) and the function \tilde{q} is the best dominant. The proof of Theorem 3.2 is thus completed. \square

Taking $\lambda = 1, A = -\alpha/p$ and $B = -1$ in Theorem 3.2, we get

Corollary 3.5. *If a function $f \in \mathcal{A}_p$ satisfies the subordination relation*

$$\frac{z (\mathcal{L}_p(a, c)f)'(z)}{\mathcal{L}_p(a, c)f(z)} \prec p + \frac{(p - \alpha)z}{(p - \alpha z)(1 - z)} \quad (0 \leq \alpha < p; z \in \mathcal{U}),$$

then

$$\operatorname{Re} \left\{ \frac{\mathcal{L}_p(a, c)f(z)}{z^p} \right\} > \frac{p + \alpha}{2p} \quad (z \in \mathcal{U})$$

and the result is the best possible.

Putting $A = 1 - (2\alpha/p)$ ($0 \leq \alpha < p$) and $B = -1$ in Theorem 3.2, we obtain

Corollary 3.6. *If a function $f \in \mathcal{A}_p$ satisfies the subordination relation*

$$(1 - \lambda) \frac{\mathcal{L}_p(a, c)f(z)}{z^p} + \frac{\lambda z (\mathcal{L}_p(a, c)f)'(z)}{p \mathcal{L}_p(a, c)f(z)} < 1 + \frac{2(1 - \lambda)(p - \alpha)}{p} \frac{z}{1 - z} + \frac{2\lambda(p - \alpha)z}{p\{p + (p - 2\alpha)z\}(1 - z)} \quad (0 < \lambda \leq 1; z \in \mathcal{U}),$$

then

$$Re \left\{ \frac{\mathcal{L}_p(a, c)f(z)}{z^p} \right\} > \frac{\alpha}{p} \quad (z \in \mathcal{U})$$

and the result is the best possible.

For the choice $c = a$ ($a = p + 1$ and $c = p$, respectively), Corollary 3.5 yields the following result.

Corollary 3.7. *For $0 < \lambda \leq 1$ and $0 \leq \alpha < p$, let*

$$\Phi_p(\lambda, \alpha; z) = 1 + \frac{2(1 - \lambda)(p - \alpha)}{p} \frac{z}{1 - z} + \frac{2\lambda(p - \alpha)z}{p\{p + (p - 2\alpha)z\}(1 - z)} \quad (z \in \mathcal{U}).$$

(i) *If a function $f \in \mathcal{A}_p$ satisfies*

$$(1 - \lambda) \frac{f(z)}{z^p} + \frac{\lambda z f'(z)}{p f(z)} < \Phi_p(\lambda, \alpha; z) \quad (z \in \mathcal{U}),$$

then

$$Re \left\{ \frac{f(z)}{z^p} \right\} > \frac{\alpha}{p} \quad (z \in \mathcal{U}).$$

(ii) *If a function $f \in \mathcal{A}_p$ satisfies*

$$(1 - \lambda) \frac{f'(z)}{z^{p-1}} + \lambda \left\{ 1 + \frac{z f''(z)}{f'(z)} \right\} < p \Phi_p(\lambda, \alpha; z) \quad (z \in \mathcal{U}),$$

then

$$Re \left\{ \frac{f'(z)}{z^{p-1}} \right\} > \alpha \quad (z \in \mathcal{U}).$$

The results in (i) and (ii) are the best possible.

Remark 3.2. 1. *Letting $a = p + 1, c = p$ in Corollary 3.5 and noting that*

$$p + Re \left\{ \frac{(p - \alpha)z}{(p - \alpha z)(1 - z)} \right\} > \frac{(2p - 1)(p + \alpha) + 2\alpha}{2(p + \alpha)} \quad (0 \leq \alpha < p; z \in \mathcal{U}),$$

we get the corresponding result obtained by Deniz [2, Theorem 2.1].

2. *Setting $p = 1$ and $\alpha = 0$ ($p = 1$ and $\alpha = 1/2$, respectively) in Corollary 3.6, we get the following the following results due to Singh et al. [19, Theorem 1 and Theorem 2].*

(i) *If $f \in \mathcal{A}$ satisfies*

$$Re \left\{ (1 - \lambda)f'(z) + \lambda \left(1 + \frac{z f''(z)}{f'(z)} \right) \right\} > \lambda \quad (0 < \lambda < 1; z \in \mathcal{U}),$$

then

$$Re\{f'(z)\} > 0 \quad (z \in \mathcal{U})$$

and the result is sharp for the function $f(z) = -z - 2 \log(1 - z), z \in \mathcal{U}$.

(ii) If $f \in \mathcal{A}$ satisfies

$$\operatorname{Re} \left\{ (1 - \lambda)f'(z) + \lambda \left(1 + \frac{zf''(z)}{f'(z)} \right) \right\} > \frac{1}{2} \quad (0 < \lambda \leq 1; z \in \mathcal{U}),$$

then

$$\operatorname{Re}\{f'(z)\} > \frac{1}{2} \quad (z \in \mathcal{U})$$

and the result is sharp for the function $f(z) = -\log(1 - z)$, $z \in \mathcal{U}$.

Theorem 3.3. If $0 < \lambda < 1$ and a function $f \in \mathcal{A}_p$ satisfies

$$\operatorname{Re} \left\{ \frac{\mathcal{L}_p(a, c)f(z)}{z^p} \right\} > \frac{\alpha}{p} \quad (0 \leq \alpha < p; z \in \mathcal{U}), \tag{3.7}$$

then

$$(1 - \lambda) \frac{\mathcal{L}_p(a, c)f(z)}{z^p} + \frac{\lambda z (\mathcal{L}_p(a, c)f'(z))'}{p \mathcal{L}_p(a, c)f(z)} > (1 - \lambda) \frac{\alpha}{p} + \lambda \quad (|z| < R_p(\lambda, \alpha)), \tag{3.8}$$

where

$$R_p(\lambda, \alpha) = \begin{cases} \frac{\{\lambda + (1 - \lambda)(p - \alpha)\} - \sqrt{\lambda^2 + 2\lambda(1 - \lambda)(p - \alpha)}}{(1 - \lambda)(p - 2\alpha)}, & \alpha \neq \frac{p}{2} \\ \frac{(1 - \lambda)p}{2\lambda + (1 - \lambda)p}, & \alpha = \frac{p}{2}. \end{cases}$$

The result is the best possible.

Proof. From (3.7), it follows that

$$\frac{\mathcal{L}_p(a, c)f(z)}{z^p} = \frac{\alpha}{p} + \left(1 - \frac{\alpha}{p}\right) \varphi(z) \quad (z \in \mathcal{U}),$$

where $\varphi \in \mathcal{P}$. Differentiating the above expression logarithmically followed by a simple calculations, we deduce that

$$\begin{aligned} & (1 - \lambda) \frac{\mathcal{L}_p(a, c)f(z)}{z^p} + \frac{\lambda z (\mathcal{L}_p(a, c)f'(z))'}{p \mathcal{L}_p(a, c)f(z)} - (1 - \lambda) \frac{\alpha}{p} - \lambda \\ &= (1 - \lambda) \left(1 - \frac{\alpha}{p}\right) \left\{ \varphi(z) + \frac{\lambda z \varphi'(z)}{(1 - \lambda)\{\alpha + (p - \alpha)\varphi(z)\}} \right\} \end{aligned}$$

so that

$$\begin{aligned} & \operatorname{Re} \left\{ (1 - \lambda) \frac{\mathcal{L}_p(a, c)f(z)}{z^p} + \frac{\lambda z (\mathcal{L}_p(a, c)f'(z))'}{p \mathcal{L}_p(a, c)f(z)} \right\} - (1 - \lambda) \frac{\alpha}{p} - \lambda \\ & \geq (1 - \lambda) \left(1 - \frac{\alpha}{p}\right) \left[\operatorname{Re}\{\varphi(z)\} - \frac{\lambda |z \varphi'(z)|}{(1 - \lambda)\{|\alpha + (p - \alpha)\varphi(z)|\}} \right]. \end{aligned} \tag{3.9}$$

Using the estimates [13]

$$\frac{|z \varphi'(z)|}{\operatorname{Re}\{\varphi(z)\}} \leq \frac{2r}{1 - r^2} \quad \text{and} \quad \operatorname{Re}\{\varphi(z)\} \geq \frac{1 - r}{1 + r} \quad (|z| = r)$$

in (3.9), we get

$$\begin{aligned} & \operatorname{Re} \left\{ (1-\lambda) \frac{\mathcal{L}_p(a, c)f(z)}{z^p} + \frac{\lambda z (\mathcal{L}_p(a, c)f)'(z)}{p \mathcal{L}_p(a, c)f(z)} \right\} - (1-\lambda) \frac{\alpha}{p} - \lambda \\ & \geq (1-\lambda) \left(1 - \frac{\alpha}{p} \right) \operatorname{Re}\{\varphi(z)\} \left[1 - \frac{2\lambda r}{(1-\lambda) \{ \alpha(1-r^2) + (p-\alpha)(1-r)^2 \}} \right]. \end{aligned} \quad (3.10)$$

We note that the right hand side of (3.10) is positive, provided $r < R_p(\lambda, \alpha)$, where $R_p(\lambda, \alpha)$ is defined as in the theorem.

To show that the bound $R_p(\lambda, \alpha)$ is the best possible, we consider the function $f \in \mathcal{A}_p$ defined by

$$f(z) = \varphi_p(c, a; z) \star \frac{z^p \left\{ 1 + \left(1 - \frac{2\alpha}{p} \right) z \right\}}{1-z} \quad (0 \leq \alpha < p; z \in \mathcal{U}).$$

It follows that

$$\frac{\mathcal{L}_p(a, c)f(z)}{z^p} = \frac{\left\{ 1 + \left(1 - \frac{2\alpha}{p} \right) z \right\}}{1-z} \quad (0 \leq \alpha < p; z \in \mathcal{U}),$$

which on differentiating logarithmically followed by a routine calculation yields

$$\begin{aligned} & (1-\lambda) \frac{\mathcal{L}_p(a, c)f(z)}{z^p} + \frac{\lambda z (\mathcal{L}_p(a, c)f)'(z)}{p \mathcal{L}_p(a, c)f(z)} - (1-\lambda) \frac{\alpha}{p} - \lambda \\ & = (1-\lambda) \left(1 - \frac{\alpha}{p} \right) \frac{1+z}{1-z} \left[1 + \frac{2\lambda z}{\alpha(1-z^2) + (p-\alpha)(1-z)^2} \right] \\ & = 0 \quad \text{as } z \rightarrow -R_p(\lambda, \alpha). \end{aligned}$$

This completes the proof of Theorem 3.3. □

For the choice $c = a, p = 1$ and $\alpha = 0$ ($a = 2, c = p = 1$ and $\alpha = 0$, respectively), Theorem 3.3 yields the following result.

Corollary 3.8. *Let $0 < \lambda < 1$. If a function $f \in \mathcal{A}$ satisfies*

$$\operatorname{Re} \left\{ \frac{f(z)}{z} \right\} > 0 \quad (z \in \mathcal{U}),$$

then

$$\operatorname{Re} \left\{ (1-\lambda) \frac{f(z)}{z} + \lambda \frac{zf'(z)}{f(z)} \right\} > \lambda \quad (|z| < \tilde{R}(\lambda)),$$

and if it satisfies

$$\operatorname{Re}\{f'(z)\} > 0 \quad (z \in \mathcal{U}),$$

then

$$\operatorname{Re} \left\{ (1-\lambda)f'(z) + \lambda \left(1 + \frac{zf''(z)}{f'(z)} \right) \right\} > \lambda \quad (|z| < \tilde{R}(\lambda)),$$

where

$$\tilde{R}(\lambda) = \frac{1 - \sqrt{\lambda(2-\lambda)}}{1-\lambda}.$$

The results are the best possible.

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**GENERALIZED φ -WEAK CONTRACTIVE FUZZY MAPPINGS AND RELATED
FIXED POINT RESULTS ON COMPLETE METRIC SPACE**

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ABSTRACT. In this paper, we discuss the existence and uniqueness of a (common) fixed point of generalized φ -weak contractive fuzzy mappings on complete metric spaces. We present some examples to illustrate the obtained results.

1. INTRODUCTION AND PRELIMINARIES

1.1. Fuzzy fixed points of fuzzy mappings. In fixed point theory, the importance of various contractive inequalities cannot be overemphasized. Existence theorems of fixed points have been established for mappings defined on various types of spaces and satisfying different types of contractive inequalities. The notion of fuzzy sets was introduced by Zadeh [27] in 1965. Following this initial result, Weiss [24] and Butnariu [9] studied on the characterization of several notion in the sense of fuzzy numbers. Heilpern [14] introduced the fuzzy mapping and further he established fuzzy Banach contraction principle on a complete metric space. Subsequently several other researchers studied the existence of fixed points and common fixed points of fuzzy mappings satisfying a contractive type condition on a metric space (see [1, 3, 4, 7, 8, 10, 16, 19, 20, 22, 25]).

The following are some definitions and concepts required for our discussion in the paper. In fact most of these are discussed in [13, 14, 17] in metric linear spaces. We discuss them in metric spaces.

Suppose that (X, d) is a metric space. A fuzzy set A over X is defined by a function μ_A ,

$$\mu_A : X \rightarrow [0, 1],$$

where μ_A is called a membership function of A , and the value $\mu_A(x)$ is called the grade of membership of x in X . The value represents the degree of x belonging to the fuzzy set X . The α -level set of A is denoted by $[A]_\alpha$, and is defined as follows:

$$\begin{aligned} [A]_\alpha &= \{x : A(x) \geq \alpha\} \text{ if } \alpha \in (0, 1], \\ [A]_0 &= \overline{\{x : A(x) > 0\}}, \end{aligned}$$

where \overline{B} denotes the closure of the set B .

Let $\mathfrak{F}(X)$ be the collection of all fuzzy sets in a metric space X . For $A, B \in \mathfrak{F}(X)$, $A \subset B$ means $A(x) \leq B(x)$ for each $x \in X$. A fuzzy set A in a metric linear space V is said to be an approximate quantity if and only if $[A]_\alpha$ is compact and convex in V for each $\alpha \in [0, 1]$ and $\sup_{x \in V} A(x) = 1$. We

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denote the collection of all approximate quantities in a metric linear space V by $W(V)$. Clearly when X is a metric linear space $W(X) \subset \mathfrak{F}(X)$.

Let X be an arbitrary set and (Y, d) be a metric space. A mapping G is called fuzzy mapping if G is a mapping from X into $\mathfrak{F}(Y)$. A fuzzy mapping G is a fuzzy subset on $X \times Y$ with membership function $G(x)(y)$. The function $G(x)(y)$ is the grade of membership of y in $G(x)$. For convenience, we denote α -level set of $G(x)$ by $[Gx]_\alpha$ instead of $[G(x)]_\alpha$.

Definition 1. Let G, H be fuzzy mappings from X into $\mathfrak{F}(X)$. A point z in X is called an α -fuzzy fixed point of H if $z \in [Hz]_\alpha$. The point z is called a common α -fuzzy fixed point of G and H if $z \in [Gz]_\alpha \cap [Hz]_\alpha$.

1.2. Fixed point theory on metric spaces. Let (X, d) be a metric space, $B(X)$ and $CB(X)$ be the sets of all nonempty bounded and closed subsets of X , respectively. For $P, Q \in B(X)$ we define

$$\delta(P, Q) = \sup\{d(p, q) : p \in P, q \in Q\}$$

and

$$D(P, Q) = \inf\{d(p, q) : p \in P, q \in Q\}.$$

If $P = \{p\}$, we write $\delta(P, Q) = \delta(p, Q)$, and if $Q = \{q\}$, then $\delta(p, Q) = d(p, q)$. For P, Q, R in $B(X)$ one can easily prove the following properties.

$$\begin{aligned} \delta(P, Q) &= \delta(P, Q) \geq 0, \\ \delta(P, Q) &\leq \delta(P, R) + \delta(R, Q), \\ \delta(P, P) &= \sup\{d(p, r) : p, r \in P\} = \text{diam } P \\ \delta(P, Q) &= 0 \text{ implies that } P = Q = \{p\}. \end{aligned}$$

Let $\{A_n\}$ be a sequence in $B(X)$. Then the sequence $\{A_n\}$ converges to A if and only if

- (i) $a \in A$ implies that $a_n \rightarrow a$ for some sequence $\{a_n\}$ with $a_n \in A_n$ for $n \in N$,
- and
- (ii) for any $\varepsilon > 0$, there exist $n, m \in N$ with $n > m$ such that

$$A_n \subseteq A_\varepsilon = \{x \in X : d(x, a) < \varepsilon \text{ for some } a \in A\}.$$

See [10, 11].

The following results will be useful in the proof of our main result.

Lemma 1. [11] Let $\{A_n\}$ and $\{B_n\}$ be sequences in $B(X)$ and (X, d) be a complete metric space. If $A_n \rightarrow A \in B(X)$ and $B_n \rightarrow B \in B(X)$, then $\delta(A_n, B_n) \rightarrow \delta(A, B)$.

Lemma 2. [15] Let (X, d) be a complete metric space. If $\{A_n\}$ is a sequence of nonempty bounded subsets in (X, d) and if $\delta(A_n, y) \rightarrow 0$ for some $y \in X$, then $A_n \rightarrow \{y\}$.

Theorem 1. [21] Let (X, d) be a complete metric space and T be a φ -weak contraction on X ; that is, for each $x, y \in X$, there exists a function $\varphi : [0, \infty) \rightarrow [0, \infty)$ such that φ is positive on $(0, \infty)$ and $\varphi(0) = 0$, and

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)) \tag{1}$$

Also if φ is a continuous and nondecreasing function, then T has a unique fixed point.

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A weakly contractive mapping is a map satisfying the inequality (1) which was first defined by Alber and Guerre-Delabriere [2]. For more results on these mappings, see [5, 6, 12, 18, 23] and the related references therein. Zhang and Song [26] gave the following theorem.

Theorem 2. [26] *Let (X, d) be a complete metric space and $T, S : X \rightarrow X$ be two mappings such that for each $x, y \in X$,*

$$d(Tx, Sy) \leq m(x, y) - \varphi(m(x, y)),$$

where $\varphi : [0, \infty) \rightarrow [0, \infty)$ is a lower semi-continuous function with $\varphi(t) > 0$ for $t > 0$ and $\varphi(0) = 0$, and

$$m(x, y) = \max \left\{ d(x, y), d(x, Tx), d(y, Sy), \frac{1}{2} [d(y, Tx) + d(x, Sy)] \right\}$$

Then there exists a unique point $u \in X$ such that $u = Tu = Su$.

2. MAIN RESULTS

This section includes the main theorem of the paper. More precisely, we find out a common fixed point of fuzzy mappings which is also unique. Let (X, d) be a complete metric space. Then we define and use the following notations:

$$\begin{aligned} \xi^X &= \{A : A \text{ is the subset of } X\}, \\ B(\xi^X) &= \{A \in \xi^X : A \text{ is nonempty bounded}\}, \\ CB(\xi^X) &= \{A \in \xi^X : A \text{ is nonempty closed and bounded}\}. \end{aligned}$$

Theorem 3. *Let (X, d) be a complete metric space and $S, T : X \rightarrow \mathfrak{F}(X)$ and for $x \in X$, there exist $\alpha_S(x), \alpha_T(x) \in (0, 1]$ such that $[Sx]_{\alpha_S(x)}, [Tx]_{\alpha_T(x)} \in B(\xi^x)$, such that for all $x, y \in X$.*

$$\delta \left([Sx]_{\alpha_S(x)}, [Ty]_{\alpha_T(y)} \right) \leq M(x, y) - \varphi(M(x, y)) \tag{2}$$

where, $\varphi : [0, \infty) \rightarrow [0, \infty)$ is a lower semicontinuous function with $\varphi(t) > 0$ for $t \in (0, \infty)$ and $\varphi(0) = 0$ and

$$M(x, y) = \max \left\{ d(x, y), D(x, [Sx]_{\alpha_S(x)}), D(y, [Ty]_{\alpha_T(y)}), \frac{1}{2} \left[D(y, [Sx]_{\alpha_S(x)}) + D(x, [Ty]_{\alpha_T(y)}) \right] \right\} \tag{3}$$

Then there exists a unique $z \in [Sx]_{\alpha_S(x)}$ and $z \in [Ty]_{\alpha_T(y)}$.

Proof. Take $a_0 \in X$. According to the given condition, there exists $\alpha(a_0) \in (0, 1]$ such that $[Sa_0]_{\alpha(a_0)} \in CB(\xi^X)$. Let us denote $\alpha(a_0)$ by α_1 . We set $a_1 \in [Sa_0]_{\alpha_1}$, for this a_1 there exists $\alpha_2 \in (0, 1]$ such that, $[Ta_1]_{\alpha_2} \in CB(\xi^x)$. Iteratively, we shall construct a sequence $\{a_n\}$ in X in a way that

$$\begin{aligned} a_{2k+1} &\in [Sa_{2k}]_{\alpha_{2k+1}}, \\ a_{2k+2} &\in [Ta_{2k+1}]_{\alpha_{2k+2}} \end{aligned}$$

It is clear that if $M(a_n, a_{n+1}) = 0$, then the proof is completed. Consequently, throughout the proof, we suppose that

$$M(a_n, a_{n+1}) > 0 \text{ for all } n \geq 0. \tag{4}$$

We shall prove that

$$d(a_{2n+1}, a_{2n+2}) \leq d(a_{2n}, a_{2n+1}) \text{ for all } n \geq 0. \tag{5}$$

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Suppose, on the contrary, that there exists $\check{n} \geq 0$ such that

$$d(a_{2\check{n}+1}, a_{2\check{n}+2}) > d(a_{2\check{n}}, a_{2\check{n}+1}),$$

which yields that

$$M(a_{2\check{n}}, a_{2\check{n}+1}) \leq d(a_{2\check{n}+1}, a_{2\check{n}+2}).$$

Regarding (2), we derive that

$$\begin{aligned} d(a_{2\check{n}_0+1}, a_{2\check{n}+2}) &\leq \delta([Sa_{2\check{n}}]_{\alpha(a_{2\check{n}})}, [Ta_{2\check{n}+1}]_{\alpha(a_{2\check{n}+1})}) \\ &\leq M(a_{2\check{n}}, a_{2\check{n}+1}) - \varphi(M(a_{2\check{n}}, a_{2\check{n}+1})) \\ &\leq d(a_{2\check{n}}, a_{2\check{n}+1}) - \varphi(M(a_{2\check{n}}, a_{2\check{n}+1})). \end{aligned}$$

Consequently, we obtain that $\varphi(M(a_{2\check{n}}, a_{2\check{n}+1})) = 0$ and so we have $M(a_{2\check{n}}, a_{2\check{n}+1}) = 0$. This contradicts the observation (4). Hence we have the inequality (5). In an analogous way, one can conclude that

$$d(a_{2n+2}, a_{2n+3}) \leq d(a_{2n+1}, a_{2n+2}) \text{ for all } n \geq 0. \tag{6}$$

By combining (5) and (6), we get that

$$d(a_{n+1}, a_{n+2}) \leq d(a_n, a_{n+1}) \text{ for all } n \geq 0.$$

Hence we derive that the sequence $\{d(a_n, a_{n+1})\}$ is non-increasing and bounded below. Since (X, d) is complete, there exists $l \geq 0$ such that

$$\lim_{n \rightarrow \infty} d(a_n, a_{n+1}) = l. \tag{7}$$

Due to hypothesis, we observe that

$$\begin{aligned} &d(a_{2n}, a_{2n+1}) \leq M(a_{2n}, a_{2n+1}) \\ &= \max \left\{ d(a_{2n}, a_{2n+1}), D(a_{2n}, [Sa_{2n}]_{\alpha(a_{2n})}), D(a_{2n+1}, [Ta_{2n+1}]_{\alpha(a_{2n+1})}), \right. \\ &\quad \left. \frac{1}{2} [D(a_{2n+1}, [Sa_{2n}]_{\alpha(a_{2n})}) + D(a_{2n}, [Ta_{2n+1}]_{\alpha(a_{2n+1})})] \right\} \\ &\leq \max \left\{ d(a_{2n}, a_{2n+1}), d(a_{2n+1}, a_{2n+2}), \frac{1}{2} [d(a_{2n}, a_{2n+1}) + d(a_{2n+1}, a_{2n+2})] \right\}. \end{aligned}$$

Thus we have

$$l \leq \lim_{n \rightarrow \infty} M(a_{2n}, a_{2n+1}) \leq l.$$

Hence we get

$$\lim_{n \rightarrow \infty} M(a_{2n}, a_{2n+1}) = l. \tag{8}$$

Analogously, we have

$$\lim_{n \rightarrow \infty} M(a_{2n+1}, a_{2n+2}) = l. \tag{9}$$

By combining (7), (8) and (9), we derive that

$$\lim_{n \rightarrow \infty} d(a_n, a_{n+1}) = \lim_{n \rightarrow \infty} M(a_n, a_{n+1}) = l.$$

By the lower semi-continuity of φ , we find

$$\varphi(l) \leq \liminf_{n \rightarrow \infty} \varphi M(a_n, a_{n+1}).$$

Now we claim that $l = 0$. From (2), we have

$$\begin{aligned} d(a_{2n+1}, a_{2n+2}) &\leq \delta([Sa_{2n}]_{\alpha(a_{2n})}, [Ta_{2n+1}]_{\alpha(a_{2n+1})}) \\ &\leq M(a_{2n}, a_{2n+1}) - \varphi(M(a_{2n}, a_{2n+1})) \end{aligned}$$

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By letting the upper limit as $n \rightarrow \infty$ in the inequality above, we obtain

$$\begin{aligned} l &\leq l - \liminf_{n \rightarrow \infty} \varphi(M(a_{2n}, a_{2n+1})) \\ &\leq l - \varphi(l), \end{aligned}$$

that is, $\varphi(l) = 0$. Regarding the property of φ , we conclude that $l = 0$.

As a next step, we shall show that $\{a_n\}$ is Cauchy. For this purpose, it is sufficient to get that $\{a_{2n}\}$ is Cauchy. Suppose, on the contrary, that $\{a_{2n}\}$ is not Cauchy. Then there is an $\epsilon > 0$ such that for an even integer $2k$ there exist even integers $2m(k) > 2n(k) > 2k$ such that

$$d(a_{2n(k)}, a_{2m(k)}) > \epsilon. \tag{10}$$

For every even integer $2k$, let $2m(k)$ be the least positive integer exceeding $2n(k)$, satisfying (10), and such that

$$d(a_{2n(k)}, a_{2m(k)-2}) < \epsilon. \tag{11}$$

Now

$$\begin{aligned} \epsilon &\leq d(a_{2n(k)}, a_{2m(k)}) \\ &\leq d(a_{2n(k)}, a_{2m(k)-2}) + d(a_{2m(k)-2}, a_{2m(k)-1}) \\ &\quad + d(a_{2m(k)-1}, a_{2m(k)}). \end{aligned}$$

By (10) and (11), we get

$$\lim_{k \rightarrow \infty} d(a_{2n(k)}, a_{2m(k)}) = \epsilon. \tag{12}$$

Due to the triangle inequality, we have

$$|d(a_{2n(k)}, a_{2m(k)-1}) - d(a_{2n(k)}, a_{2m(k)})| < d(a_{2m(k)-1}, a_{2m(k)}).$$

By (12), we get

$$d(a_{2n(k)}, a_{2m(k)-1}) = \epsilon. \tag{13}$$

Now by (3) we observe that

$$\begin{aligned} d(a_{2n(k)}, a_{2m(k)-1}) &\leq M(a_{2n(k)}, a_{2m(k)-1}) \\ &= \max \left\{ \begin{aligned} &d(a_{2n(k)}, a_{2m(k)-1}), D(a_{2n(k)}, [Sa_{2n(k)}]_{\alpha(a_{2n(k)})}), \\ &D(a_{2m(k)-1}, [Ta_{2m(k)-1}]_{\alpha(a_{2m(k)-1})}), \\ &\frac{1}{2}D(a_{2m(k)-1}, [Sa_{2n(k)}]_{\alpha(a_{2n(k)})}) + D(a_{2n(k)}, [Ta_{2m(k)-1}]_{\alpha(a_{2m(k)-1})}) \end{aligned} \right\} \\ &\leq \max \left\{ \begin{aligned} &d(a_{2n(k)}, a_{2m(k)-1}), d(a_{2n(k)}, a_{2n(k)+1}), d(a_{2m(k)-1}, a_{2m(k)}) \\ &\frac{1}{2}[d(a_{2m(k)-1}, a_{2n(k)+1}) + d(a_{2n(k)}, a_{2m(k)})] \end{aligned} \right\} \\ &\leq \max \left\{ \begin{aligned} &d(a_{2n(k)}, a_{2m(k)-1}), d(a_{2n(k)}, a_{2n(k)+1}), d(a_{2m(k)-1}, a_{2m(k)}) \\ &\frac{1}{2}[d(a_{2m(k)-1}, a_{2n(k)}) + d(a_{2n(k)}, a_{2n(k)+1}) + d(a_{2n(k)}, a_{2m(k)})] \end{aligned} \right\}. \end{aligned}$$

By letting $k \rightarrow \infty$ in the inequality above and taking (12) and (13) into account, we conclude that

$$\epsilon \leq \lim_{k \rightarrow \infty} M(x_{2n(k)}, x_{2m(k)-1}) \leq \epsilon.$$

Consequently, we have

$$\lim_{k \rightarrow \infty} M(x_{2n(k)}, x_{2m(k)-1}) = \epsilon.$$

By the lower semi-continuity of φ , we derive that

$$\varphi(\epsilon) \leq \liminf_{k \rightarrow \infty} \varphi(M(x_{2n(k)}, x_{2m(k)-1})).$$

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Now by (2), we get

$$\begin{aligned} & d(x_{2n(k)}, x_{2m(k)}) \\ & \leq d(x_{2n(k)}, x_{2n(k)+1}) + \delta([Sx_{2n(k)}]_{\alpha(x_{2n(k)})}, [Tx_{2m(k)-1}]_{\alpha(x_{2m(k)-1})}) \\ & \leq d(x_{2n(k)}, x_{2n(k)+1}) + M(x_{2n(k)}, x_{2m(k)-1}) - \varphi(M(x_{2n(k)}, x_{2m(k)-1})). \end{aligned}$$

Letting the upper limit $k \rightarrow \infty$ in the inequality above, we have

$$\begin{aligned} \epsilon & \leq \epsilon - \liminf_{k \rightarrow \infty} \varphi(M(a_{2n(k)}, a_{2m(k)-1})) \\ & \leq \epsilon - \varphi(\epsilon), \end{aligned}$$

which is a contradiction. Hence we conclude that $\{a_{2n}\}$ is a Cauchy sequence. It follows from the completeness of X that there exists $c \in X$ such that $a_n \rightarrow c$ as $n \rightarrow \infty$. Furthermore, $a_{2n} \rightarrow c$ and $a_{2n+1} \rightarrow c$.

We shall prove that $c \in [Sc]_{\alpha_S(c)}$.

$$\begin{aligned} & D(c, [Sc]_{\alpha_S(c)}) \leq M(c, a_{2n-1}) \\ & = \max \left\{ d(c, a_{2n-1}), D(c, [Sc]_{\alpha_S(c)}), D(a_{2n-1}, [Ta_{2n-1}]_{\alpha_T(a_{2n-1})}), \right. \\ & \quad \left. \frac{1}{2}[D(a_{2n-1}, [Sc]_{\alpha_S(c)}) + D(c, [Ta_{2n-1}]_{\alpha_T(a_{2n-1})})] \right\} \\ & \leq \max \left\{ d(c, a_{2n-1}), D(c, [Sc]_{\alpha_S(c)}), d(a_{2n-1}, a_{2n}) \right\} \\ & \quad \left. \frac{1}{2}[D(a_{2n-1}, [Sc]_{\alpha_S(c)}) + d(c, a_{2n})] \right\} \end{aligned}$$

Letting $n \rightarrow \infty$, we have $\lim_{n \rightarrow \infty} M(c, a_{2n-1}) = D(c, [Sc]_{\alpha_S(c)})$. Due to the lower semi-continuity of φ , we have

$$\varphi(D(c, [Sc]_{\alpha_S(c)})) \leq \lim_{n \rightarrow \infty} \varphi(M(c, a_{2n-1})). \tag{14}$$

On the other hand, from (2)

$$\begin{aligned} \delta([Sc]_{\alpha_S(c)}, a_{2n}) & \leq \delta([Sc]_{\alpha_S(c)}, [Ta_{2n-1}]_{\alpha_T(a_{2n-1})}) \\ & \leq M(c, a_{2n-1}) - \varphi(M(c, a_{2n-1})) \end{aligned}$$

and letting $n \rightarrow \infty$, we have

$$\delta([Sc]_{\alpha_S(c)}, c) \leq D(c, [Sc]_{\alpha_S(c)}) - \lim_{n \rightarrow \infty} \varphi(M(c, a_{2n-1})). \tag{15}$$

This shows that $\lim_{n \rightarrow \infty} \varphi(M(c, a_{2n-1})) = 0$ and so from (14), we have $\varphi(D(c, [Sc]_{\alpha_S(c)})) = 0$, that is, $D(c, [Sc]_{\alpha_S(c)}) = 0$. This implies, from (15), that $\{c\} = [Sc]_{\alpha_S(c)}$. Now, from (3) it is easy to see that $M(c, c) = D(c, [Tc]_{\alpha_T(c)})$, and so from (2) we have

$$\begin{aligned} \delta(c, [Tc]_{\alpha_T(c)}) & \leq \delta([Sc]_{\alpha_S(c)}, [Tc]_{\alpha_T(c)}) \\ & \leq M(c, c) - \varphi(M(c, c)) \\ & = D(c, [Tc]_{\alpha_T(c)}) - \varphi(D(c, [Tc]_{\alpha_T(c)})). \end{aligned}$$

Therefore, we have $c \in [Tc]_{\alpha_T(c)}$ and so $\{c\} = [Tc]_{\alpha_T(c)}$. As a consequence, we have $\{c\} = [Sc]_{\alpha_S(c)} = [Tc]_{\alpha_T(c)}$, that is, c is a common fixed point of S and T . Now we will show that this common fixed point is unique. Assume that a and b are two common fixed points of S and T . Then $a \in [Sa]_{\alpha_S(a)}$, a

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$\in [Ta]_{\alpha_T(a)}$ and $b \in [Sb]_{\alpha_S(b)}, b \in [Tb]_{\alpha_T(b)}$. Therefore, from (3) we have $M(a, b) \leq d(a, b)$ and so from (2) we have

$$\begin{aligned} d(a, b) &\leq \delta([Sa]_{\alpha_S(a)}, [Tb]_{\alpha_T(b)}) \\ &\leq M(a, b) - \varphi(M(a, b)) \\ &\leq d(a, b) - \varphi(M(a, b)). \end{aligned}$$

This shows that $M(a, b) = 0$ and so $a = b$. □

Example 1. Let $X = [0, 1]$, $d(a, b) = |a - b|$, when $a, b \in X$ and let $G, H : X \rightarrow \mathfrak{F}(X)$ be fuzzy mappings defined as:

$$G(a)(t) = \begin{cases} 1 & \text{if } 0 \leq t < \frac{a}{6} \\ \frac{1}{2} & \text{if } \frac{a}{6} \leq t \leq \frac{a}{4} \\ \frac{1}{3} & \text{if } \frac{a}{4} \leq t < \frac{a}{3} \\ 0 & \text{if } \frac{a}{3} \leq t < \infty \end{cases}$$

$$H(a)(t) = \begin{cases} 1 & \text{if } 0 \leq t < \frac{a}{6} \\ \frac{1}{4} & \text{if } \frac{a}{6} \leq t \leq \frac{a}{3} \\ \frac{1}{6} & \text{if } \frac{a}{3} \leq t \leq \frac{a}{2} \\ 0 & \text{if } \frac{a}{2} < t < \infty \end{cases}$$

$$[Ga]_{\frac{1}{3}} = \left\{ t \in X : G(a)(t) \geq \frac{1}{3} \right\} = \left[0, \frac{a}{3} \right),$$

$$[Ha]_{\frac{1}{4}} = \left\{ t \in X : H(a)(t) \geq \frac{1}{4} \right\} = \left[0, \frac{a}{3} \right]$$

It is clear that $[Ga]_{\frac{1}{3}}$ and $[Ha]_{\frac{1}{4}}$ are nonempty bounded for all $a \in X$. We will show that the condition (2) of Theorem 3 is satisfied with $\varphi(t) = \frac{t}{2}$. Indeed, for all $a, b \in X$, we have

$$\begin{aligned} \delta([Ga]_{\frac{1}{3}}, [Hb]_{\frac{1}{4}}) &= \delta\left(\left[0, \frac{a}{3}\right), \left[0, \frac{b}{3}\right]\right) \\ &= \frac{b}{3} = \frac{1}{2} \frac{2b}{3} = \frac{1}{2} D\left(b, \left[0, \frac{b}{3}\right]\right) \\ &= \frac{1}{2} D\left(b, [Hb]_{\frac{1}{4}}\right) \leq \frac{1}{2} M(a, b) = M(a, b) - \frac{1}{2} M(a, b) \\ &= M(a, b) - \varphi(M(a, b)). \end{aligned}$$

All conditions of Theorem 3 are satisfied and so these mappings have a unique common fixed point in X .

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Example 2. Let $X = [0, 1]$, $d(a, b) = |a - b|$, where $a, b \in X$, $\lambda, \mu \in (0, 1]$ and let $G, H : X \rightarrow \mathfrak{F}(X)$ be fuzzy mappings defined as:

$$\begin{aligned}
 & \text{if } a = 0, \\
 G(a)(t) &= \begin{cases} 1 & \text{if } t = 0 \\ \frac{1}{2} & \text{if } 0 < t \leq \frac{1}{100} \\ 0 & t > \frac{1}{100} \end{cases} & T(a)(t) = \begin{cases} 1 & \text{if } t = 0 \\ \frac{1}{3} & \text{if } 0 < t \leq \frac{1}{150} \\ 0 & t > \frac{1}{150} \end{cases} \\
 & \text{if } a \neq 0, \\
 G(a)(t) &= \begin{cases} \lambda & \text{if } 0 \leq t < \frac{a}{16} \\ \frac{\lambda}{2} & \text{if } \frac{a}{16} \leq t \leq \frac{a}{10} \\ \frac{\lambda}{3} & \text{if } \frac{a}{10} \leq t < a \\ 0 & \text{if } a \leq t < \infty \end{cases} & T(x)(t) = \begin{cases} \mu & \text{if } 0 \leq t < \frac{a}{16} \\ \frac{\mu}{4} & \text{if } \frac{a}{16} \leq t \leq \frac{a}{10} \\ \frac{\mu}{10} & \text{if } \frac{a}{10} \leq t < a \\ 0 & \text{if } a \leq t < \infty \end{cases}
 \end{aligned}$$

Note that

$$[G0]_{\lambda_S(0)} = [H0]_{\lambda_T(0)} = \{0\}, \text{ if } \lambda_G(0) = \lambda_H(0) = 1,$$

and for $a \neq 0$,

$$[Ga]_{\lambda} = \left[0, \frac{a}{16}\right) \text{ and } [Ha]_{\mu} = \left[0, \frac{a}{16}\right),$$

$$[Ga]_{\frac{\lambda}{2}} = \left[0, \frac{a}{10}\right] \text{ and } [Ha]_{\frac{\mu}{4}} = \left[0, \frac{a}{10}\right].$$

Since X is not linear and also $[Ga]_{\lambda}$ and $[Ha]_{\lambda}$ are not compact for each λ , all the previous fixed point results [4, 9, 15, 16] for fuzzy mappings on complete linear metric spaces are not applicable. However, G and H satisfy the conditions of Theorem 3.

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ON CARLITZ'S DEGENERATE EULER NUMBERS AND POLYNOMIALS

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ABSTRACT. In this paper, a p -adic measure is constructed by using the generalized distribution relation of degenerate Euler numbers and polynomials generalizing those satisfied by E_k and $E_k(x)$. Furthermore, a family of p -adic measures are obtained by regularizing that one.

1. INTRODUCTION

Let p be a fixed odd prime number. Throughout this paper, \mathbb{Z}_p , \mathbb{Q}_p and \mathbb{C}_p will denote the ring of p -adic integers, the field of p -adic rational numbers and the completion of the algebraic closure of \mathbb{Q}_p . Let $|\cdot|_p$ be the p -adic norm with $|p|_p = \frac{1}{p}$. In [2], Carlitz defined degenerate Euler numbers and polynomials and proved some properties generalizing those satisfied by E_k and $E_k(x)$. Recently, D. S. Kim and T. Kim gave some formulae and identities of degenerate Euler polynomials which are derived from the fermionic p -adic integrals on \mathbb{Z}_p (see [2, 4]). In this note, we use those properties of them, especially the distribution relation for the degenerate Euler polynomials, to construct p -adic measures.

For $\lambda, t \in \mathbb{C}_p$ with $|\lambda t|_p < p^{-\frac{1}{p-1}}$, the degenerate Euler polynomials are given by the generating function to be

$$(1.1) \quad \frac{2}{(1 + \lambda t)^{\frac{1}{\lambda}} + 1} (1 + \lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \mathcal{E}_n(x | \lambda) \frac{t^n}{n!}, \quad (\text{see [1, 2]}) .$$

Note that $\lim_{\lambda \rightarrow 0} \mathcal{E}_n(x | \lambda) = E_n(x)$, where $E_n(x)$ are the Euler polynomials defined by the generating function

$$(1.2) \quad \left(\frac{2}{e^t + 1} \right) e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad (\text{see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]})$$

When $x = 0$, $\mathcal{E}_n(\lambda) = \mathcal{E}_n(0 | \lambda)$ are called degenerate Euler numbers. From (1.1), we can derive the following equation:

$$(1.3) \quad \mathcal{E}_n(x | \lambda) = \sum_{l=0}^n \binom{n}{l} E_l(\lambda) (x | \lambda)_{n-l},$$

where $(x | \lambda)_n = x(x - \lambda)(x - 2\lambda) \cdots (x - (n - 1)\lambda)$.

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The degenerate Euler polynomials satisfy the following generalized distribution relation [4] :

$$(1.4) \quad \mathcal{E}_n(x | \lambda) = d^n \sum_{a=0}^{d-1} (-1)^a \mathcal{E}_n \left(\frac{a+x}{d} \middle| \frac{\lambda}{d} \right),$$

where $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$ and $n \in \mathbb{N} \cup \{0\}$.

2. DEGENERATE EULER MEASURES

Let $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$, and let p be a fixed odd prime number.

Proposition 2.1.

$$\begin{aligned} X_d &= \varprojlim_N \mathbb{Z}/d p^N \mathbb{Z}; \\ a + d p^N \mathbb{Z}_p &= \{x \in X_d \mid x \equiv a \pmod{d p^N}\}; \\ X_d^* &= \bigcup_{\substack{0 < a < d p \\ (a,p)=1}} a + d p \mathbb{Z}_p. \end{aligned}$$

We shall always take $0 \leq a < d p^N$ when we write $a + d p^N \mathbb{Z}_p$.

Theorem 2.2. For $k \geq 0$, let $\mu_{k,\varepsilon}$ be given by

$$(2.1) \quad \mu_{k,\varepsilon}(a + d p^N \mathbb{Z}_p) = (d p^N)^k (-1)^a \mathcal{E}_k \left(\frac{a}{d p^N} \middle| \frac{\lambda}{d p^N} \right).$$

Then $\mu_{k,\varepsilon}$ extends to a \mathbb{C}_p -valued measure on compact open sets $U \subset X_d$.

Proof. It is enough to show that

$$\sum_{i=0}^{p-1} \mu_{k,\varepsilon}(a + i d p^N + d p^{N+1} \mathbb{Z}_p) = \mu_{k,\varepsilon}(a + d p^N \mathbb{Z}_p).$$

From (2.1), we note that

$$\begin{aligned} &\sum_{i=0}^{p-1} \mu_{k,\varepsilon}(a + i d p^N + d p^{N+1} \mathbb{Z}_p) \\ &= (d p^{N+1})^k \sum_{i=0}^{p-1} (-1)^{a+i d p^N} \mathcal{E}_k \left(\frac{a + i d p^N}{d p^{N+1}} \middle| \frac{\lambda}{d p^{N+1}} \right) \\ &= (-1)^a (d p^N)^k p^k \sum_{i=0}^{p-1} (-1)^i \mathcal{E}_k \left(\frac{\frac{a}{d p^N} + i}{p} \middle| \frac{\lambda}{d p^N} \right) \\ &= (-1)^a (d p^N)^k \mathcal{E}_k \left(\frac{a}{d p^N} \middle| \frac{\lambda}{d p^N} \right) \\ &= \mu_{k,\varepsilon}(a + d p^N \mathbb{Z}_p). \end{aligned}$$

We easily see that $|\mu_{k,\varepsilon}| \leq M$ for some constant M . □

Definition 2.3. Let $\alpha \in X_d^*$, $\alpha \neq 1$, $k \geq 1$. For compact-open $U \subset X_d$, we define

$$\mu_{k,\alpha}(U) = \mu_{k,\varepsilon}(U) - \alpha^{-k} \mu_{k,\varepsilon}(\alpha U).$$

Remark. We note that $\mu_{k,\alpha}$ are (bounded) \mathbb{C}_p -valued measures on X_d for all $k \geq 0$.

For $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$, let χ be a Dirichlet character with conductor d . Then, we define the generalized degenerate Euler number attached to χ as follows:

$$(2.2) \quad \frac{2}{(1 + \lambda t)^{d/\lambda} + 1} \sum_{a=0}^{d-1} (-1)^a \chi(a) (1 + \lambda t)^{\frac{a}{\lambda}} = \sum_{n=0}^{\infty} \mathcal{E}_{n,\chi}(\lambda) \frac{t^n}{n!}.$$

Note that

$$(2.3) \quad \begin{aligned} & \sum_{n=0}^{\infty} \lim_{\lambda \rightarrow 0} \mathcal{E}_{n,\chi}(\lambda) \frac{t^n}{n!} \\ &= \lim_{\lambda \rightarrow 0} \frac{2}{(1 + \lambda t)^{d/\lambda} + 1} \sum_{a=0}^{d-1} (-1)^a \chi(a) (1 + \lambda t)^{\frac{a}{\lambda}} \\ &= \frac{2}{e^{dt} + 1} \sum_{a=0}^{d-1} (-1)^a \chi(a) e^{at} \\ &= \sum_{n=0}^{\infty} E_{n,\chi} \frac{t^n}{n!}, \end{aligned}$$

where $E_{n,\chi}$ are called the generalized Euler numbers attached to χ .

From (2.3), we have $\lim_{\lambda \rightarrow 0} \mathcal{E}_{n,\chi} = E_{n,\chi}$.

By (1.1) and (2.2), we get

$$(2.4) \quad \begin{aligned} & \sum_{n=0}^{\infty} \mathcal{E}_{n,\chi}(\lambda) \frac{t^n}{n!} \\ &= \sum_{a=0}^{d-1} (-1)^a \chi(a) \frac{2}{(1 + \lambda t)^{d/\lambda} + 1} (1 + \lambda t)^{\frac{a}{\lambda}} \\ &= \sum_{n=0}^{\infty} d^n \left(\sum_{a=0}^{d-1} (-1)^a \chi(a) \mathcal{E}_n \left(\frac{a}{d} \middle| \frac{\lambda}{d} \right) \right) \frac{t^n}{n!}. \end{aligned}$$

By comparing the coefficients of both sides of (2.4), we have

$$(2.5) \quad \mathcal{E}_{n,\chi}(\lambda) = d^n \sum_{a=0}^{d-1} (-1)^a \chi(a) \mathcal{E}_n \left(\frac{a}{d} \middle| \frac{\lambda}{d} \right).$$

The locally constant function χ on X_d can be integrated against the measure $\mu_{k,\mathcal{E}}$ defined by (2.1), and the result is given by

$$(2.6) \quad \begin{aligned} & \int_{X_d} \chi(x) d\mu_{k,\mathcal{E}}(x) \\ &= \lim_{N \rightarrow \infty} \sum_{a=0}^{dp^N-1} \chi(a) \mu_{k,\mathcal{E}}(a + dp^N \mathbb{Z}_p) \\ &= \lim_{N \rightarrow \infty} (dp^N)^k \sum_{a=0}^{dp^N-1} \chi(a) (-1)^a \mathcal{E}_k \left(\frac{a}{dp^N} \middle| \frac{\lambda}{dp^N} \right) \\ &= d^k \sum_{a=0}^{d-1} \chi(a) (-1)^a \lim_{N \rightarrow \infty} (p^N)^k \sum_{x=0}^{p^N-1} (-1)^x \mathcal{E}_k \left(\frac{\frac{a}{d} + x}{p^N} \middle| \frac{\lambda}{p^N} \right) \end{aligned}$$

$$\begin{aligned} &= d^k \sum_{a=0}^{d-1} \chi(a) (-1)^a \mathcal{E}_k \left(\frac{a}{d} \middle| \frac{\lambda}{d} \right) \\ &= \mathcal{E}_{k,\chi}(\lambda). \end{aligned}$$

Note that

$$\begin{aligned} (2.7) \quad & \int_{pX_d} \chi(x) d\mu_{k,\varepsilon}(x) \\ &= (pd)^k \sum_{a=0}^{d-1} \chi(pa) (-1)^{pa} \mathcal{E}_k \left(\frac{pa}{pd} \middle| \frac{\lambda}{pd} \right) \\ &= p^k \chi(p) d^k \sum_{a=0}^{d-1} \chi(a) (-1)^a \mathcal{E}_k \left(\frac{a}{d} \middle| \frac{\lambda}{d} \right) \\ &= p^k \chi(p) \mathcal{E}_{k,\chi} \left(\frac{\lambda}{p} \right), \end{aligned}$$

$$(2.8) \quad \int_{X_d} \chi(x) d\mu_{k,\varepsilon}(\alpha x) = \chi \left(\frac{1}{\alpha} \right) \mathcal{E}_{k,\chi}(\lambda),$$

and

$$(2.9) \quad \int_{pX_d} \chi(x) d\mu_{k,\varepsilon}(\alpha x) = p^k \chi \left(\frac{p}{\alpha} \right) \mathcal{E}_{k,\chi} \left(\frac{\lambda}{p} \right).$$

Hence, by definition of $\mu_{k,\alpha}$, we get

$$\begin{aligned} (2.10) \quad & \int_{X_d^*} \chi(x) d\mu_{k,\alpha}(x) \\ &= \mathcal{E}_{k,\chi}(\lambda) - p^k \chi(p) \mathcal{E}_{k,\chi} \left(\frac{\lambda}{p} \right) - \frac{1}{\alpha^k} \chi \left(\frac{1}{\alpha} \right) \mathcal{E}_{k,\chi}(\lambda) \\ & \quad + \frac{p^k}{\alpha^k} \chi \left(\frac{p}{\alpha} \right) \mathcal{E}_{k,\chi} \left(\frac{\lambda}{p} \right) \\ &= \left(1 - \alpha^{-k} \chi \left(\frac{1}{\alpha} \right) \right) \left(\mathcal{E}_{k,\chi}(\lambda) - p^k \chi(p) \mathcal{E}_{k,\chi} \left(\frac{\lambda}{p} \right) \right). \end{aligned}$$

Therefore, by (2.6), (2.7), (2.8), (2.9) and (2.10), we obtain the following theorem.

Theorem 2.4. *For $k \geq 0$, we have*

$$\begin{aligned} & \int_{X_d} \chi(x) d\mu_{k,\varepsilon}(x) = \mathcal{E}_{k,\chi}(\lambda), \\ & \int_{pX_d} \chi(x) d\mu_{k,\varepsilon}(x) = p^k \chi(p) \mathcal{E}_{k,\chi} \left(\frac{\lambda}{p} \right), \\ & \int_{X_d} \chi(x) d\mu_{k,\varepsilon}(\alpha x) = \chi \left(\frac{1}{\alpha} \right) \mathcal{E}_{k,\chi}(\lambda), \\ & \int_{pX_d} \chi(x) d\mu_{k,\varepsilon}(\alpha x) = p^k \chi \left(\frac{p}{\alpha} \right) \mathcal{E}_{k,\chi} \left(\frac{\lambda}{p} \right), \end{aligned}$$

and

$$\int_{X_d^*} \chi(x) d\mu_{k,\alpha}(x) = \left(1 - \alpha^{-k} \chi\left(\frac{1}{\alpha}\right)\right) \left(\mathcal{E}_{k,\chi}(\lambda) - p^k \chi(p) \mathcal{E}_{k,\chi}\left(\frac{\lambda}{p}\right)\right).$$

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Dynamics and Behavior of the Higher Order Rational Difference equation

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Abstract

The main objective of this paper is to study the periodic character and the global stability of the positive solutions of the difference equation

$$x_{n+1} = ax_n + bx_{n-k} + cx_{n-l} - \frac{dx_{n-s}}{ex_{n-s} - \alpha x_{n-t}}, \quad n = 0, 1, \dots,$$

where the parameters a, b, c, d, e and α are positive real numbers and the initial conditions $x_{-\sigma}, x_{-\sigma+1}, \dots, x_{-1}, x_0$ are positive real numbers where $\sigma = \max\{s, t, l, k\}$. Some numerical examples were given to illustrate our results.

Keywords: difference equations, stability, global stability, periodic solutions.

Mathematics Subject Classification: 39A10

1 Introduction

In this paper, we study the global stability character, the boundedness and the periodicity of the positive solutions of the nonlinear difference equation

$$x_{n+1} = ax_n + bx_{n-k} + cx_{n-l} - \frac{dx_{n-s}}{ex_{n-s} - \alpha x_{n-t}}, \quad n = 0, 1, \dots, \quad (1)$$

where the parameters a, b, c, d, e and α are positive real numbers and the initial conditions $x_{-\sigma}, x_{-\sigma+1}, \dots, x_{-1}, x_0$ are positive real numbers where $\sigma = \max\{s, t, l, k\}$. Here, we recall some notations and results, which will be useful in our investigation.

Let I be some interval of real numbers and let

$$F : I^{k+1} \rightarrow I, \quad k \in \mathbb{N}$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, \dots, x_0 \in I$, the difference equation

$$x_{n+1} = F(x_n, x_{n-1}, \dots, x_{n-k}), \quad n = 0, 1, \dots, \tag{2}$$

has a unique solution $\{x_n\}_{n=-k}^\infty$.

Definition 1 (*Equilibrium Point*)

A point $\bar{x} \in I$ is called an equilibrium point of Eq.(2) if

$$\bar{x} = F(\bar{x}, \bar{x}, \dots, \bar{x}).$$

That is, $x_n = \bar{x}$ for $n \geq 0$ is a solution of Eq.(2), or equivalently, \bar{x} is a fixed point of f .

Definition 2 (*Stability*)

(i) The equilibrium point \bar{x} of Eq.(2) is called locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, \dots, x_{-1}, x_0 \in I$ with

$$|x_{-k} - \bar{x}| + \dots + |x_{-1} - \bar{x}| + |x_0 - \bar{x}| < \delta,$$

we have

$$|x_n - \bar{x}| < \epsilon \quad \text{for all } n \geq -k.$$

(ii) The equilibrium point \bar{x} of Eq.(2) is called locally asymptotically stable if \bar{x} is locally stable solution of Eq.(2) and there exists $\gamma > 0$, such that for all $x_{-k}, \dots, x_{-1}, x_0 \in I$ with

$$|x_{-k} - \bar{x}| + \dots + |x_{-1} - \bar{x}| + |x_0 - \bar{x}| < \gamma,$$

we have

$$\lim_{n \rightarrow \infty} x_n = \bar{x}.$$

(iii) The equilibrium point \bar{x} of Eq.(2) is called global attractor if for all $x_{-k}, \dots, x_{-1}, x_0 \in I$, we have

$$\lim_{n \rightarrow \infty} x_n = \bar{x}.$$

(iv) The equilibrium point \bar{x} of Eq.(2) is called globally asymptotically stable if \bar{x} is locally stable, and \bar{x} is also a global attractor of Eq.(2).

(v) The equilibrium point \bar{x} of Eq.(2) is called unstable if \bar{x} is not locally stable.

Definition 3 (*Boundedness*)

A sequence $\{x_n\}_{n=-k}^\infty$ is said to be bounded and persisting if there exist positive constants m and M such that

$$m \leq x_n \leq M \quad \text{for all } n \geq -k.$$

Definition 4 (*Periodicity*)

A sequence $\{x_n\}_{n=-k}^\infty$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$. A sequence $\{x_n\}_{n=-k}^\infty$ is said to be periodic with prime period p if p is the smallest positive integer having this property.

Definition 5 The linearized equation of Eq.(2) about the equilibrium \bar{x} is the linear difference equation

$$y_{n+1} = \sum_{i=0}^k \frac{\partial F(\bar{x}, \bar{x}, \dots, \bar{x})}{\partial x_{n-i}} y_{n-i}. \tag{3}$$

Now, assume that the characteristic equation associated with (3) is

$$p(\lambda) = p_0\lambda^k + p_1\lambda^{k-1} + \dots + p_{k-1}\lambda + p_k = 0, \tag{4}$$

where

$$p_i = \frac{\partial F(\bar{x}, \bar{x}, \dots, \bar{x})}{\partial x_{n-i}}.$$

Theorem A [1]: Assume that $p_i \in R$, $i = 1, 2, \dots, k$ and k is non-negative integer. Then

$$\sum_{i=1}^k |p_i| < 1$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+k} + p_1x_{n+k-1} + \dots + p_kx_n = 0, \quad n = 0, 1, \dots .$$

Theorem B [2]: Let $g : [a, b]^{k+1} \rightarrow [a, b]$ be a continuous function, where k is a positive integer, and $[a, b]$ is an interval of real numbers and consider the difference equation

$$x_{n+1} = g(x_n, x_{n-1}, \dots, x_{n-k}), \quad n = 0, 1, \dots . \tag{5}$$

Suppose that g satisfies the following conditions:

- (i) For every integer i with $1 \leq i \leq k + 1$, the function $g(z_1, z_2, \dots, z_{k+1})$ is weakly monotonic in z_i , for fixed $z_1, z_2, \dots, z_{i-1}, z_{i+1}, \dots, z_{k+1}$.
- (ii) If (m, M) is a solution of the system

$$m = g(m_1, m_2, \dots, m_{k+1}) \text{ and } M = g(M_1, M_2, \dots, M_{k+1}),$$

then $m = M$, where for each $i = 1, 2, \dots, k + 1$, we set

$$m_i = \begin{cases} m & \text{if } g \text{ is non-decreasing in } z_i \\ M & \text{if } g \text{ is non-increasing in } z_i \end{cases},$$

and

$$M_i = \begin{cases} M & \text{if } g \text{ is non-decreasing in } z_i \\ m & \text{if } g \text{ is non-increasing in } z_i \end{cases}.$$

Then, there exists exactly one equilibrium point \bar{x} of the difference equation (5), and every solution of (5) converges to \bar{x} .

Many research have been done to study the global attractivity, boundedness character, periodicity and the solution form of nonlinear difference equations. For example, Agarwal et al. [3] investigated the global stability, periodicity character and gave the solution form of some special cases of the recursive sequence

$$x_{n+1} = ax_n + \frac{bx_n x_{n-3}}{cx_{n-2} + dx_{n-3}}, \quad n = 0, 1, \dots,$$

where a, b, c, d and the initial conditions $x_{-3}, x_{-2}, x_{-1}, x_0$ are positive real numbers.

Sun et al [4] studied the behavior of the solutions of the difference equation

$$x_{n+1} = p + \frac{x_{n-1}}{x_n}, \quad n = 0, 1, \dots$$

where initial values $x_{-1}, x_0 \in (0, \infty)$ and $0 < p < 1$, and obtain the set of all initial values $(x_{-1}, x_0) \in (0, +\infty) \times (0, +\infty)$ such that the positive solutions $\{x_n\}_{n=-1}^\infty$ are bounded.

Elsayed and El-Dessoky [5] studied the global convergence, boundedness, and periodicity of solutions of the difference equation

$$x_{n+1} = ax_{n-s} + \frac{bx_{n-l} + cx_{n-k}}{dx_{n-l} + ex_{n-k}}, \quad n = 0, 1, \dots,$$

where the parameters a, b, c, d and e are positive real numbers and the initial conditions $x_{-t}, x_{-t+1}, \dots, x_{-1}, x_0$ are positive real numbers where $t = \max\{s, l, k\}$.

Zayed [6] studied the global asymptotic properties of the solutions of the following difference equations

$$x_{n+1} = Ax_n + Bx_{n-k} + \frac{px_n + x_{n-k}}{q + x_{n-k}}.$$

Elsayed [7] studied the global stability character and the periodicity of solutions of the difference equation

$$x_{n+1} = ax_n + \frac{bx_{n-1} + cx_{n-2}}{dx_{n-1} + ex_{n-2}}, \quad n = 0, 1, \dots,$$

where the parameters a, b, c, d and e are positive real numbers and the initial conditions x_{-2}, x_{-1} and x_0 are positive real numbers.

El-Moneam [8] investigated the periodicity, the boundedness and the global stability of the positive solutions of the following nonlinear difference equation

$$x_{n+1} = Ax_n + Bx_{n-k} + Cx_{n-l} + Dx_{n-s} + \frac{bx_{n-k}}{dx_{n-k} - ex_{n-l}}, \quad n = 0, 1, \dots,$$

where the coefficients $A, B, C, D, b, d, e \in (0, \infty)$, while k, l and s are positive integers. The initial conditions $x_{-s}, \dots, x_{-l}, \dots, x_{-k}, \dots, x_{-1}, x_0$ are arbitrary positive real numbers such that $k < l < s$.

Yalçinkaya [9] investigated the global behaviour of the difference equation

$$x_{n+1} = \alpha + \frac{x_{n-m}}{x_n^k}, \quad n = 0, 1, \dots,$$

where the parameter $\alpha, k \in (0, \infty)$ and the initial values are arbitrary positive real numbers.

Elabbasy et al. [10] studied the dynamics, the global stability, periodicity character and the solution of special case of the recursive sequence

$$x_{n+1} = ax_n - \frac{bx_n}{cx_n - dx_{n-1}}, \quad n = 0, 1, \dots,$$

where the initial conditions x_{-1}, x_0 are arbitrary real numbers and a, b, c, d are positive constants.

In [11] Berenhaut et al. studied the existence of positive prime periodic solutions of higher order for rational recursive equations of the form

$$y_{n+1} = A + \frac{y_{n-1}}{y_{n-m}}, \quad n = 0, 1, \dots,$$

with $y_{-m}, y_{-m+1}, \dots, y_{-1} \in (0, 1)$ and $m = \{2, 3, 4, \dots\}$.

Papaschinopoulos et al. [12] investigated the asymptotic behavior and the periodicity of the positive solutions of the nonautonomous difference equation:

$$x_{n+1} = A_n + \frac{x_{n-1}^p}{x_n^q}, \quad n = 0, 1, \dots,$$

where A_n is a positive bounded sequence, $p, q \in (0, \infty)$ and x_1, x_0 are positive numbers.

For some related results see [13-28].

2 Local Stability of the Equilibrium Point of Eq.(1)

In this section, we study the local stability character of the equilibrium point of Eq.(1).

Eq.(1) has an equilibrium point given by

$$\bar{x} = a\bar{x} + b\bar{x} + c\bar{x} - \frac{d\bar{x}}{e\bar{x} - \alpha\bar{x}},$$

and hence

$$(e - \alpha)(1 - a - b - c)\bar{x}^2 + d\bar{x} = 0.$$

Then if $a + b + c < 1$ and $\alpha > e$, the only positive equilibrium point of Eq.(1) is given by

$$\bar{x} = \frac{d}{(\alpha - e)(1 - a - b - c)}.$$

Theorem 1 *The equilibrium \bar{x} of Eq. (1) is locally asymptotically stable if*

$$\alpha - e > 2d.$$

Proof: Let $f : (0, \infty)^5 \rightarrow (0, \infty)$ be a continuous function defined by

$$f(u_1, u_2, u_3, u_4, u_5) = au_1 + bu_2 + cu_3 + \frac{du_4}{eu_4 - \alpha u_5}. \tag{6}$$

Therefore, it follows that

$$\begin{aligned} \frac{\partial f(u_1, u_2, u_3, u_4, u_5)}{\partial u_1} &= a, \\ \frac{\partial f(u_1, u_2, u_3, u_4, u_5)}{\partial u_2} &= b, \\ \frac{\partial f(u_1, u_2, u_3, u_4, u_5)}{\partial u_3} &= c, \\ \frac{\partial f(u_1, u_2, u_3, u_4, u_5)}{\partial u_4} &= \frac{\alpha du_5}{(eu_4 - \alpha u_5)^2}, \\ \frac{\partial f(u_1, u_2, u_3, u_4, u_5)}{\partial u_5} &= -\frac{\alpha du_4}{(eu_4 - \alpha u_5)^2}. \end{aligned}$$

So, we can write

$$\begin{aligned} \frac{\partial f(\bar{x}, \bar{x}, \bar{x}, \bar{x}, \bar{x})}{\partial u_1} &= a = p_1, \\ \frac{\partial f(\bar{x}, \bar{x}, \bar{x}, \bar{x}, \bar{x})}{\partial u_2} &= b = p_2, \\ \frac{\partial f(\bar{x}, \bar{x}, \bar{x}, \bar{x}, \bar{x})}{\partial u_3} &= c = p_3, \end{aligned}$$

$$\frac{\partial f(\bar{x}, \bar{x}, \bar{x}, \bar{x}, \bar{x})}{\partial u_4} = \frac{d(1 - a - b - c)}{(e - \alpha)} = p_4,$$

$$\frac{\partial f(\bar{x}, \bar{x}, \bar{x}, \bar{x}, \bar{x})}{\partial u_5} = -\frac{d(1 - a - b - c)}{(e - \alpha)} = p_5.$$

Then the linearized equation of Eq.(1) about \bar{x} is

$$y_{n+1} - p_1 y_n - p_2 y_{n-k} - p_3 y_{n-l} - p_4 y_{n-s} - p_5 y_{n-t} = 0.$$

It follows by Theorem A that, Eq.(1) is asymptotically stable if and only if

$$|p_1| + |p_2| + |p_3| + |p_4| + |p_5| < 1.$$

Thus,

$$|a| + |b| + |c| + \left| \frac{d(1 - a - b - c)}{(\alpha - e)} \right| + \left| \frac{d(1 - a - b - c)}{(\alpha - e)} \right| < 1,$$

and so

$$2 \left| \frac{d(1 - a - b - c)}{(\alpha - e)} \right| < 1 - b - a - c,$$

or

$$2d < \alpha - e.$$

The proof is complete.

Example 1. The solution of the difference equation (1) is local stability if $k = 2$, $l = 1$, $s = 3$, $t = 2$, $a = 0.23$, $b = 0.12$, $c = 0.3$, $d = 0.1$, $e = 0.6$ and $\alpha = 0.9$ and the initial conditions $x_{-3} = 11.1$, $x_{-2} = 1.1$, $x_{-1} = 1.4$ and $x_0 = 1.9$ (See Fig. 1).

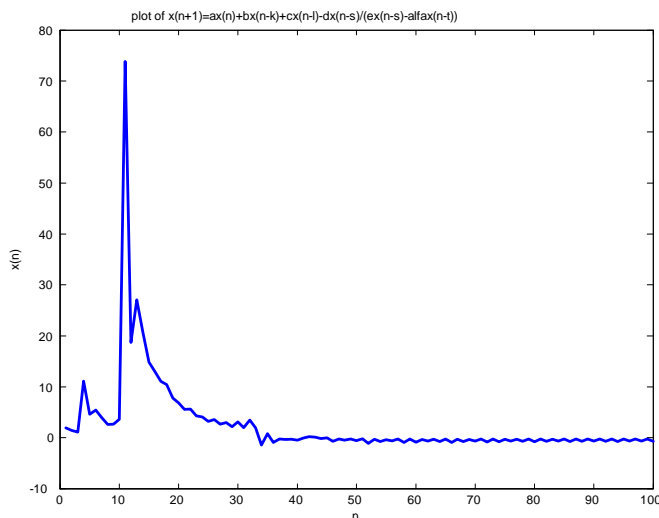


Figure 1. Plot the behavior of the solution of equation (1).

Example 2. The solution of the difference equation (1) if $k = 2, l = 1, s = 3, t = 2, a = 0.4, b = 0.2, c = 0.5, d = 0.1, e = 0.6$ and $\alpha = 0.9$ and the initial conditions $x_{-3} = 11.1, x_{-2} = 1.1, x_{-1} = 1.4$ and $x_0 = 1.9$ (See Fig. 2).

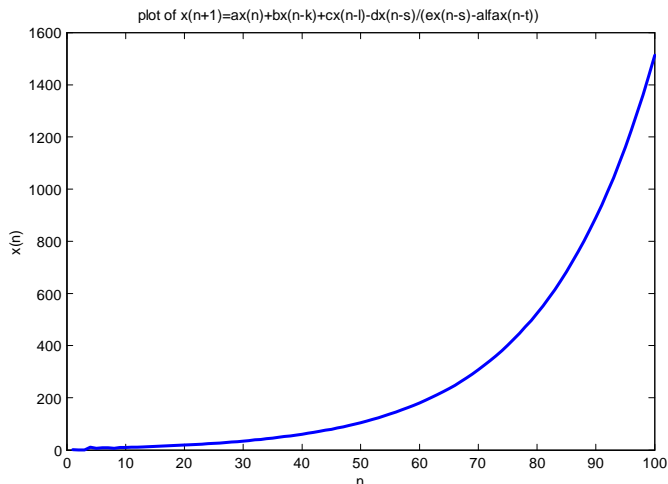


Figure 2. Plot the behavior of the solution of equation (1).

3 Global Attractivity of the Equilibrium Point of Eq.(1)

In this section, the global asymptotic stability of Eq.(1) will be studied.

Theorem 2 *The equilibrium point \bar{x} is a global attractor of Eq.(1) if $a + b + c < 1$.*

Proof: Suppose that ζ and η are real numbers and assume that $g : [\zeta, \eta]^5 \rightarrow [\zeta, \eta]$ is a function defined by

$$g(u_1, u_2, u_3, u_4, u_5) = au_1 + bu_2 + cu_3 - \frac{du_4}{eu_4 - \alpha u_5}.$$

Then

$$\begin{aligned} \frac{\partial g(u_1, u_2, u_3, u_4, u_5)}{\partial u_1} &= a, \quad \frac{\partial g(u_1, u_2, u_3, u_4, u_5)}{\partial u_2} = b, \\ \frac{\partial g(u_1, u_2, u_3, u_4, u_5)}{\partial u_3} &= c, \\ \frac{\partial g(u_1, u_2, u_3, u_4, u_5)}{\partial u_4} &= \frac{\alpha du_5}{(eu_4 - \alpha u_5)^2}, \\ \frac{\partial g(u_1, u_2, u_3, u_4, u_5)}{\partial u_5} &= -\frac{\alpha du_4}{(eu_4 - \alpha u_5)^2}. \end{aligned}$$

Now, we can see that the function $g(u_1, u_2, u_3, u_4, u_5)$ increasing in u_1, u_2, u_3, u_4 and decreasing in u_5 .

Let (m, M) be a solution of the system $M = g(M, M, M, M, m)$ and $m = g(m, m, m, m, M)$. Then from Eq.(1), we see that

$$M = aM + bM + cM - \frac{dM}{eM - \alpha m} \text{ and } m = am + bm + cm - \frac{dm}{em - \alpha M},$$

and then

$$M(1 - a - b - c) = -\frac{dM}{eM - \alpha m} \text{ and } m(1 - a - b - c) = -\frac{dm}{em - \alpha M},$$

thus

$$e(1 - a - b - c)M^2 - \alpha(1 - a - b - c)Mm = -dM$$

and

$$e(1 - a - b - c)m^2 - \alpha(1 - a - b - c)Mm = -dm.$$

Subtracting we obtain

$$e(1 - a - b - c)(M^2 - m^2) + d(M - m) = 0,$$

then

$$(M - m)\{e(1 - a - b - c)(M + m) + d\} = 0$$

under the condition $a + b + c < 1$, we see that

$$M = m.$$

It follows by Theorem B that \bar{x} is a global attractor of Eq.(1). This completes the proof.

Example 3. The solution of the difference equation (1) is global stability if $k = 2, l = 1, s = 3, t = 2, a = 0.2, b = 0.2, c = 0.5, d = 0.12, e = 0.6$ and $\alpha = 0.9$ and the initial conditions $x_{-3} = 11.1, x_{-2} = 1.1, x_{-1} = 1.4$ and $x_0 = 1.9$ (See Fig. 3).

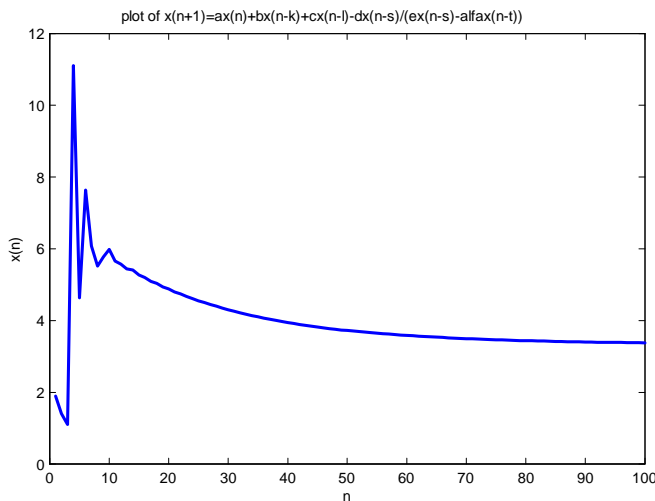


Figure 3. Plot the behavior of the solution of equation (1).

4 Existence of Periodic Solutions

In this section, we investigate the existence of periodic solutions of Eq.(1). The following theorem states the necessary and sufficient conditions for the Eq.(1) to be periodic solutions of prime period two.

Theorem 3 *Equation (1) has a prime period two solutions if and only if one of the following conditions satisfies*

- (i) $(e - 3\alpha)(a + b + c) + e + \alpha > 0$, $l, k, t - \text{even}$ and $s - \text{odd}$.
- (ii) $(e + \alpha)(a + b + c + 1) - 4\alpha > 0$, $l, k, s - \text{even}$ and $t - \text{odd}$.
- (iii) $(\alpha + e)(a + c - b + 1) - 4\alpha(a + c) > 0$, $l, t - \text{even}$ and $k, s - \text{odd}$.
- (iv) $(\alpha + e)(b - a - c - 1) - 4\alpha(b - 1) > 0$, $l, s - \text{even}$ and $k, t - \text{odd}$.
- (v) $(\alpha + e)(b - a - c - 1) - 4\alpha\alpha > 0$, $l, k, s - \text{odd}$, and $t - \text{even}$.
- (vi) $(\alpha + e)(b + c - a - 1) - 4\alpha(b + c - 1) > 0$, $l, k, t - \text{odd}$ and $s - \text{even}$.
- (vii) $(\alpha + e)(b + a + c - 1) - 4\alpha(c - 1) > 0$, $l, t - \text{odd}$ and $k, s - \text{even}$.
- (viii) $(\alpha + e)(b + a - c + 1) - 4\alpha(a + b) > 0$, $l, s - \text{odd}$ and $k, t - \text{even}$.

Proof: We prove first case when l, k and t are even and s odd (the other cases are similar and will be left to readers).

First suppose that there exists a prime period two solution

$$\dots p, q, p, q, \dots,$$

of Equation (1). We will prove that Inequality (i) holds.

We see from Equation (1) when l, k and t are even and s odd that

$$p = aq + bq + cq - \frac{dp}{ep - \alpha q},$$

and

$$q = ap + bp + cp - \frac{dq}{eq - \alpha p}.$$

Therefore,

$$ep^2 - \alpha pq = e(a + b + c)pq - \alpha(a + b + c)q^2 - dp, \tag{7}$$

and

$$eq^2 - \alpha pq = e(a + b + c)pq - \alpha(a + b + c)p^2 - dq. \tag{8}$$

Subtracting (8) from (7) gives

$$e(p^2 - q^2) - \alpha(a + b + c)(p^2 - q^2) + d(p - q) = 0,$$

then

$$(p - q) [e - \alpha(a + b + c)](p + q) + d = 0$$

Since $p \neq q$, then

$$(p + q) = \frac{d}{\alpha(a + b + c) - e}. \tag{9}$$

Again, adding (7) and (8) yields

$$e(q^2 + p^2) - 2\alpha pq = 2e(a + b + c)pq - \alpha(a + b + c)(q^2 + p^2) - d(q + p),$$

then

$$2(e(a + b + c) + \alpha)pq = (e + \alpha(a + b + c))(q^2 + p^2) + d(q + p). \tag{10}$$

By using (9), (10) and the relation

$$p^2 + q^2 = (p + q)^2 - 2pq \text{ for all } p, q \in R,$$

we obtain

$$\begin{aligned} (e + \alpha(a + b + c))((p + q)^2 - 2pq) + d(q + p) &= 2(e(a + b + c) - \alpha)pq, \\ 2[e(a + b + c) + \alpha + e + \alpha(a + b + c)]pq &= (e + \alpha(a + b + c))(p + q)^2 + d(q + p), \end{aligned}$$

$$2(e + \alpha)(a + b + c + 1)pq = \left(\frac{d}{\alpha(a + b + c) - e}\right)^2 (e + \alpha(a + b + c) + \alpha(a + b + c) - e),$$

$$2(e + \alpha)(a + b + c + 1)pq = 2\alpha(a + b + c) \left(\frac{d}{\alpha(a + b + c) - e}\right)^2.$$

Then,

$$pq = \left(\frac{\alpha(a + b + c)}{(a + b + c + 1)(e + \alpha)}\right) \left(\frac{d}{\alpha(a + b + c) - e}\right)^2. \tag{11}$$

Now it is obvious from Eq.(9) and Eq.(11) that p and q are the two distinct roots of the quadratic equation

$$\begin{aligned} t^2 - \frac{d}{\alpha(a + b + c) - e}t + \left(\frac{\alpha(a + b + c)}{(a + b + c + 1)(e + \alpha)}\right) \left(\frac{d}{\alpha(a + b + c) - e}\right)^2 &= 0, \\ (\alpha(a + b + c) - e)t^2 - dt + \frac{d^2\alpha(a + b + c)}{(a + b + c + 1)(e + \alpha)(\alpha(a + b + c) - e)} &= 0, \end{aligned} \tag{12}$$

and so

$$\begin{aligned} d^2 - \frac{4d^2\alpha(a + b + c)(\alpha(a + b + c) - e)}{(a + b + c + 1)(e + \alpha)(\alpha(a + b + c) - e)} &> 0, \\ (a + b + c + 1)(e + \alpha) - 4\alpha(a + b + c) &> 0, \\ e(a + b + c + 1) + \alpha - 3\alpha(a + b + c) &> 0, \end{aligned}$$

or

$$(e - 3\alpha)(a + b + c) + e + \alpha > 0.$$

For $\alpha(a + b + c) > e$ and $e > 3\alpha$ then the Inequality (i) holds.

Second suppose that Inequality (i) is true. We will show that Equation (1) has a prime period two solution.

Suppose that

$$p = \frac{d(1 + \zeta)}{2(\alpha A - e)} \text{ and } q = \frac{d(1 - \zeta)}{2(\alpha A - e)},$$

where $\zeta = \sqrt{1 - \frac{4\alpha A}{(A + 1)(e + \alpha)}}$ and $A = a + b + c$.

We see from the inequality (i) that

$$\begin{aligned} (e - 3\alpha)(a + b + c) + e + \alpha &> 0, \\ (a + b + c + 1)(e + \alpha) - 4\alpha(a + b + c) &> 0, \end{aligned}$$

which equivalent to

$$(A + 1)(e + \alpha) - 4\alpha A > 0.$$

Therefore p and q are distinct real numbers.

Set

$$x_{-l} = q, x_{-k} = q, x_{-s} = p, x_{-t} = q, \dots, x_{-3} = p, x_{-2} = q, x_{-1} = p, x_0 = q.$$

We would like to show that

$$x_1 = x_{-1} = p \text{ and } x_2 = x_0 = q.$$

It follows from Eq.(1) that

$$\begin{aligned} x_1 &= aq + bq + cq - \frac{dp}{ep - \alpha q}, \\ &= (a + b + c) \left(\frac{d(1 - \zeta)}{2(\alpha A - e)} \right) - \frac{d \left(\frac{d(1 + \zeta)}{2(\alpha A - e)} \right)}{e \left(\frac{d(1 + \zeta)}{2(\alpha A - e)} \right) - \alpha \left(\frac{d(1 - \zeta)}{2(\alpha A - e)} \right)}. \end{aligned}$$

Dividing the denominator and numerator by $2(\alpha A - e)$ we get

$$x_1 = (a + c + b) \left(\frac{d(1 - \zeta)}{2(\alpha A - e)} \right) - \frac{d(1 + \zeta)}{(e - \alpha) + (e + \alpha)\zeta}.$$

Multiplying the denominator and numerator of the right side by $(e - \alpha) - (e + \alpha)\zeta$

$$\begin{aligned} x_1 &= (a + c + b) \left(\frac{d(1 - \zeta)}{2(\alpha A - e)} \right) - \frac{d(1 + \zeta) ((e - \alpha) - (e + \alpha)\zeta)}{((e - \alpha) + (e + \alpha)\zeta) ((e - \alpha) - (e + \alpha)\zeta)}, \\ &= \frac{dA(1 - \zeta)}{2(\alpha A - e)} - \frac{d((e - \alpha) - 2\alpha\zeta - (e + \alpha)\zeta^2)}{(e - \alpha)^2 - (e + \alpha)^2\zeta^2} \\ &= \frac{Ad(1 - \zeta)}{2(\alpha A - e)} - \frac{d((A - 1) - \zeta(A + 1))}{2(A\alpha - e)}, \end{aligned}$$

$$x_1 = \frac{d(A - A\zeta - A + 1 + A\zeta + \zeta)}{2(\alpha A - e)} = \frac{d(1 + \zeta)}{2(\alpha A - e)} = p.$$

Similarly as before, it is easy to show that

$$x_2 = q.$$

Then by induction we get

$$x_{2n} = q \quad \text{and} \quad x_{2n+1} = p \quad \text{for all} \quad n \geq -2.$$

Thus Eq.(1) has the prime period two solution

$$\dots, p, q, p, q, \dots,$$

where p and q are the distinct roots of the quadratic equation (12) and the proof is complete.

Example 4. The solution of the difference equation (1) has a prime period two solution when $k = 4, l = 2, s = 3, t = 2, a = 0.3, b = 0.02, c = 0.01, d = 9, e = 3$ and $\alpha = 1.1$ and the initial conditions $x_{-5} = p, x_{-4} = q, x_{-3} = p, x_{-2} = q, x_{-1} = p$ and $x_0 = q$ since p and q as in the previous theorem (See Fig. 4).

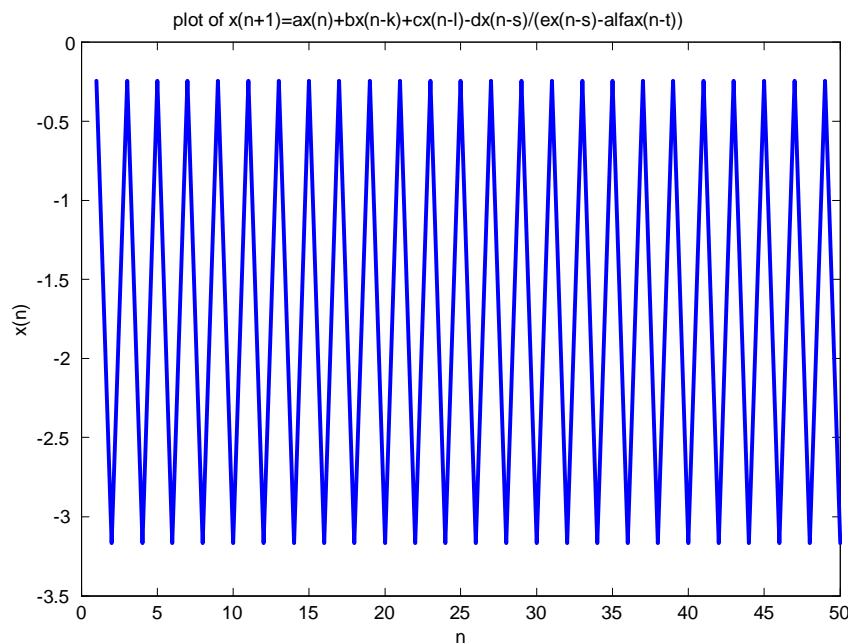


Figure 4. Plot the periodicity of the solution of equation (1).

Theorem 4 Equation (1) has no prime period two solutions if one

- (i) $1 + a + b + c \neq 0, l, k, s, t - \text{even.}$
- (ii) $1 + a - b - c \neq 0, l, k, s, t - \text{odd.}$
- (iii) $1 + a + c - b \neq 0, l, s, t - \text{even and } k - \text{odd.}$
- (iv) $1 + a + c + b \neq 0, l, k - \text{even and } t, s - \text{odd.}$
- (v) $1 + a + c - b \neq 0, l - \text{even and } k, s, t - \text{odd.}$
- (vi) $1 + a + b - c \neq 0, l, s, t - \text{odd and } k - \text{even.}$
- (vii) $1 + a - b - c \neq 0, l, k - \text{odd and } s, t - \text{even.}$
- (viii) $1 + a + b - c \neq 0, l - \text{odd and } k, s, t - \text{even.}$

Proof: We prove first case when l, k, s and t are both even positive integers (the other cases are similar and will be left to readers).

First suppose that there exists a prime period two solution

$$\dots p, q, p, q, \dots,$$

of Equation (1). We will prove that Inequality (i) holds.

We see from Equation (1) when l, k, s and t are both even positive integers that

$$p = aq + bq + cq - \frac{dq}{eq - \alpha q},$$

and

$$q = ap + bp + cp - \frac{dp}{ep - \alpha p}.$$

Therefore,

$$p - (-a - b - c)q = -\frac{d}{e - \alpha}, \tag{13}$$

and

$$q - (-a - b - c)p = -\frac{d}{e - \alpha}. \tag{14}$$

Subtracting (14) from (13) gives

$$(1 - a - b - c)(p - q) = 0.$$

Since $a + b + c \neq 1$, then $p = q$. This is a contradiction. Thus, the proof of (i) is now completed.

Example 5. Figure (5) shows the difference equation (1) has no period two solution when $k = 4, l = 2, s = 2, t = 4, a = 0.09, b = 0.2, c = 1, d = 9, e = 3$ and $\alpha = 2.1$

and the initial conditions $x_{-4} = 2, x_{-3} = 5, x_{-2} = 8, x_{-1} = 1.2$ and $x_0 = 5$.

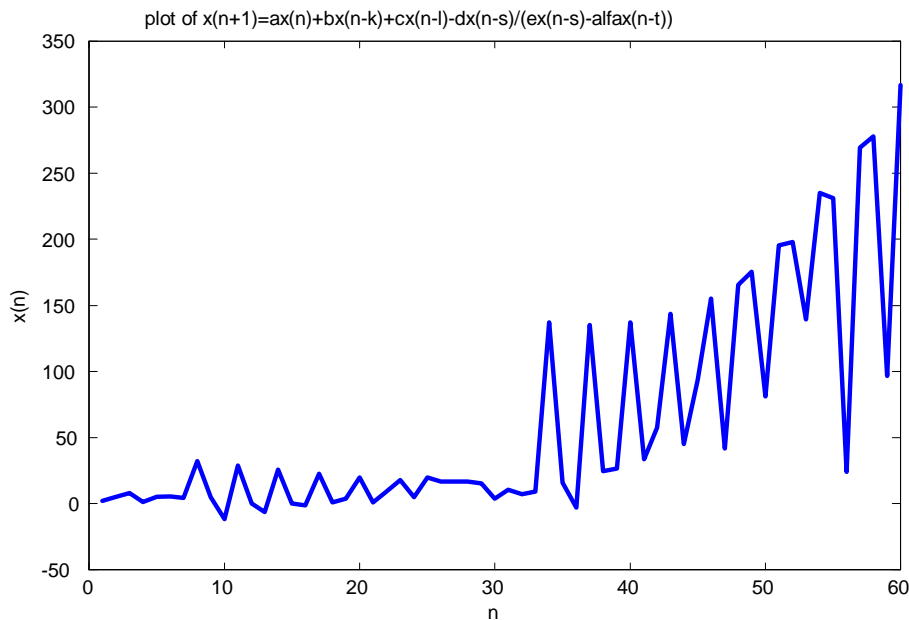


Figure 5. Plot of the solution of equation (1) has no periodic.

5 Existence of Bounded and Unbounded Solutions of Eq.(1)

In this section, we investigate the boundedness nature of the positive solutions of Eq.(1).

Theorem 5 Suppose $\{x_n\}$ be a solution of Eq.(1). Then the following statements are true:

(i) Let $d < e$ and for some $N \geq 0$, the initial conditions $x_{N-\sigma+1}, x_{N-\sigma+2}, \dots, x_{N-1}, x_N \in [\frac{d}{e}, 1]$, are valid, then for $d \neq \alpha$ and $e^2 \neq d\alpha$, we have the inequality

$$\frac{d}{e}(a + b + c) - \frac{d}{d - \alpha} \leq x_n \leq a + b + c - \frac{d^2}{e^2 - \alpha d}, \quad \text{for all } n \geq N. \quad (15)$$

(ii) Let $d > e$ and for some $N \geq 0$, the initial conditions $x_{N-\sigma+1}, x_{N-\sigma+2}, \dots, x_{N-1}, x_N \in [1, \frac{d}{e}]$, are valid, then for $d \neq \alpha, e^2 \neq d\alpha$ and $ex_{n-s} \neq \alpha x_{n-t}$, we have the inequality

$$a + b + c - \frac{d^2}{e^2 - \alpha d} \leq x_n \leq \frac{d}{e}(a + b + c) - \frac{d}{d - \alpha}, \quad \text{for all } n \geq N. \quad (16)$$

Proof: Let $\{x_n\}$ be a solution of Eq.(1). If for some $N \geq 0$, $\frac{d}{e} \leq x_n \leq 1$ and $d \neq \alpha$, we have

$$x_{n+1} = ax_n + bx_{n-k} + cx_{n-l} - \frac{dx_{n-s}}{ex_{n-s} - \alpha x_{n-t}} \leq a + b + c - \frac{dx_{n-s}}{ex_{n-s} - \alpha x_{n-t}}.$$

But, we can see that $ex_{n-s} - \alpha x_{n-t} \leq e - \alpha \left(\frac{d}{e}\right)$, $ex_{n-s} - \alpha x_{n-t} \leq \frac{e^2 - \alpha d}{e}$, $\frac{1}{ex_{n-s} - \alpha x_{n-t}} \geq \frac{e}{e^2 - \alpha d}$, $\frac{dx_{n-s}}{ex_{n-s} - \alpha x_{n-t}} \geq \frac{de\left(\frac{d}{e}\right)}{e^2 - \alpha d}$, $\frac{dx_{n-s}}{ex_{n-s} - \alpha x_{n-t}} \geq \frac{d^2}{e^2 - \alpha d}$, $-\frac{dx_{n-s}}{ex_{n-s} - \alpha x_{n-t}} \leq -\frac{d^2}{e^2 - \alpha d}$. Then for $\alpha d \neq e^2$, we get

$$x_{n+1} \leq a + b + c - \frac{d^2}{e^2 - \alpha d}. \tag{17}$$

Similarly, we can show that

$$x_{n+1} = ax_n + bx_{n-k} + cx_{n-l} - \frac{dx_{n-s}}{ex_{n-s} - \alpha x_{n-t}} \geq \frac{d}{e}(a + b + c) - \frac{dx_{n-s}}{ex_{n-s} - \alpha x_{n-t}}.$$

But, $ex_{n-s} - \alpha x_{n-t} \geq d - \alpha$, $\frac{1}{ex_{n-s} - \alpha x_{n-t}} \leq \frac{1}{d - \alpha}$, $\frac{dx_{n-s}}{ex_{n-s} - \alpha x_{n-t}} \leq \frac{d}{d - \alpha}$, $-\frac{dx_{n-s}}{ex_{n-s} - \alpha x_{n-t}} \geq -\frac{d}{d - \alpha}$. Then for $d \neq \alpha$, we get

$$x_{n+1} \geq \frac{d}{e}(a + b + c) - \frac{d}{d - \alpha}. \tag{18}$$

From (17) and (18), we get

$$\frac{d}{e}(a + b + c) - \frac{d}{d - \alpha} \leq x_{n+1} \leq a + b + c - \frac{d^2}{e^2 - \alpha d}, \quad \text{for all } n \geq N.$$

The proof of part (i) is completed.

Similarly, for some $N \geq 0$, $1 \leq x_n \leq \frac{d}{e}$, $d \neq \alpha$ and $e^2 \neq d\alpha$ we can prove part (ii) which is omitted here for convenience. Thus, the proof is now completed.

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A QUADRATIC FUNCTIONAL EQUATION IN INTUITIONISTIC FUZZY 2-BANACH SPACES

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ABSTRACT. In this paper, we define an intuitionistic fuzzy 2-normed space. Using the fixed point alternative approach, we investigate the Hyers-Ulam stability of the following quadratic functional equation

$$f(ax + by) + f(ax - by) = \frac{a}{2}f(x + y) + \frac{a}{2}f(x - y) + (2a^2 - a)f(x) + (2b^2 - a)f(y)$$

in intuitionistic fuzzy 2-Banach spaces.

1. INTRODUCTION

In 1940, Ulam [1] proposed the famous Ulam stability problem for a metric group homomorphism. In 1941, Hyers [2] solved this stability problem for additive mappings subject to the Hyers condition on approximately additive mappings in Banach spaces. In 1951, Bourgin [3] treated the Ulam stability problem for additive mappings. Subsequently the result of Hyers was generalized by Rassias [4] for linear mapping by considering an unbounded Cauchy difference.

The functional equation $f(x + y) + f(x - y) = 2f(x) + 2f(y)$ is called a quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a quadratic mapping. The Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [5] for mappings $f : \mathcal{X} \rightarrow \mathcal{Y}$, where \mathcal{X} is a normed space and \mathcal{Y} is a Banach space.

In 1984, Katrasas [6] defined a fuzzy norm on a linear space to construct a fuzzy vector topological structure on the space. Later, some mathematicians have defined fuzzy norms on a linear space from various points of view [7, 8]. In particular, in 2003, Bag and Samanta [9], following Cheng and Mordeson [10], gave an idea of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [11]. They also established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces. Recently, considerable attention has been increasing to the problem of fuzzy stability of functional equations. Several various fuzzy stability results concerning Cauchy, Jensen, simple quadratic, and cubic functional equations have been investigated [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22].

Quite recently, the stability results in the setting of intuitionistic fuzzy normed space were studied in [23, 24, 25, 26]; respectively, while the idea of intuitionistic fuzzy normed space was introduced in [27].

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2. PRELIMINARIES

Definition 2.1. Let \mathcal{X} be a real linear space of dimension greater than one and let $\|\cdot, \cdot\|$ be a real-valued function on $\mathcal{X} \times \mathcal{X}$ satisfying the following condition:

1. $\|x, y\| = \|y, x\|$ for all $x, y \in \mathcal{X}$
2. $\|x, y\| = 0$ if and only if x, y are linearly dependent.
3. $\|\alpha x, y\| = |\alpha| \|x, y\|$ for all $x, y \in \mathcal{X}$ and $\alpha \in \mathbb{R}$.
4. $\|x, y + z\| \leq \|x, y\| + \|x, z\|$ for all $x, y, z \in \mathcal{X}$.

Then the function $\|\cdot, \cdot\|$ is called a 2-norm on X and pair $(X, \|\cdot, \cdot\|)$ is called a 2-normed linear space.

Definition 2.2. A binary operation $* : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous t-norm if $*$ satisfies the following conditions:

1. $*$ is commutative and associative;
2. $*$ is continuous;
- 3 $a * 1 = a$ for all $a \in [0, 1]$;
- 4 $a * b \leq c * d$, whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Example 2.1. An example of continuous t-norm is

$$a * b = \min\{a, b\}$$

Definition 2.3. A binary operation $\diamond : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous t-conorm if \diamond satisfies the following conditions:

1. \diamond is commutative and associative;
2. \diamond is continuous;
- 3 $a \diamond 0 = a$ for all $a \in [0, 1]$;
- 4 $a \diamond b \leq c \diamond d$, whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Example 2.2. An example of continuous t-conorm is

$$a \diamond b = \max\{a, b\}$$

Definition 2.4. Let \mathcal{X} be a real linear space. A fuzzy subset μ of $\mathcal{X} \times \mathcal{X} \times \mathbb{R}$ is called a fuzzy 2-norm on \mathcal{X} if and only if for $x, y, z \in \mathcal{X}$, and $t, s, c \in \mathbb{R}$:

1. $\mu(x, y, t) = 0$ if $t \leq 0$.
2. $\mu(x, y, t) = 1$ if and only if x, y are linearly dependent, for all $t > 0$.
3. $\mu(x, y, t)$ is invariant under any permutation of x, y .
4. $\mu(x, cy, t) = \mu(x, y, \frac{t}{|c|})$ for all $t > 0$ and $c \neq 0$.
5. $\mu(x + z, y, t + s) \geq \mu(x, y, t) * \mu(z, y, s)$ for all $t, s > 0$.
6. $\mu(x, y, \cdot)$ is a non-decreasing function on \mathbb{R} and

$$\lim_{t \rightarrow \infty} \mu(x, y, t) = 1.$$

Then μ is said to be a fuzzy 2-norm on a linear space \mathcal{X} , and the pair (\mathcal{X}, μ) is called a fuzzy 2-normed linear space.

Example 2.3. Let $(\mathcal{X}, \|\cdot, \cdot\|)$ be a 2-normed linear space. Define

$$\mu(x, y, t) = \begin{cases} \frac{t}{t + \|x, y\|} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0 \end{cases}$$

where $x, y \in \mathcal{X}$ and $t \in \mathbb{R}$. Then (\mathcal{X}, μ) is a fuzzy 2-normed linear space.

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Definition 2.5. Let (\mathcal{X}, μ) be a fuzzy 2-normed linear space. Let $\{x_n\}$ be a sequence in \mathcal{X} . Then $\{x_n\}$ is said to be convergent if there exists $x \in \mathcal{X}$ such that

$$\lim_{n \rightarrow \infty} \mu(x_n - x, y, t) = 1$$

for all $t > 0$.

Definition 2.6. Let (\mathcal{X}, μ) be a fuzzy 2-normed linear space. Let $\{x_n\}$ be a sequence in \mathcal{X} . Then $\{x_n\}$ is said to be a Cauchy sequence if

$$\lim_{n \rightarrow \infty} \mu(x_{n+p} - x_n, y, t) = 1$$

for all $t > 0$ and $p = 1, 2, 3, \dots$.

Let (\mathcal{X}, μ) be a fuzzy 2-normed linear space and $\{x_n\}$ be a Cauchy sequence in \mathcal{X} . If $\{x_n\}$ is convergent in \mathcal{X} then (\mathcal{X}, μ) is said to be a fuzzy 2-Banach space.

Definition 2.7. Let \mathcal{X} be a real linear space. A fuzzy subset ν of $\mathcal{X} \times \mathcal{X} \times \mathbb{R}$ such that for all $x, y, z \in \mathcal{X}$, and $t, s, c \in \mathbb{R}$

1. $\nu(x, y, t) = 1$, for all $t \leq 0$.
2. $\nu(x, y, t) = 0$ if and only if x, y are linearly dependent, for all $t > 0$.
3. $\nu(x, y, t)$ is invariant under any permutation of x, y .
4. $\nu(x, cy, t) = \nu(x, y, \frac{t}{|c|})$ for all $t > 0, c \neq 0$.
5. $\nu(x, y + z, t + s) \leq \nu(x, y, t) \diamond \nu(x, z, s)$ for all $s, t > 0$
6. $\nu(x, y, \cdot)$ is a nonincreasing function and

$$\lim_{t \rightarrow \infty} \nu(x, y, t) = 0$$

Then ν is said to be an anti fuzzy 2-norm on a linear space \mathcal{X} and the pair (\mathcal{X}, ν) is called an anti fuzzy 2-normed linear space.

Definition 2.8. Let (\mathcal{X}, ν) be an anti fuzzy 2-normed linear space and $\{x_n\}$ be a sequence in \mathcal{X} . Then $\{x_n\}$ is said to be convergent if there exists $x \in \mathcal{X}$ such that

$$\lim_{n \rightarrow \infty} \nu(x_n - x, y, t) = 0$$

for all $t > 0$.

Definition 2.9. Let (\mathcal{X}, ν) be an anti fuzzy 2-normed linear space and $\{x_n\}$ be a sequence in \mathcal{X} . Then $\{x_n\}$ is said to be a Cauchy sequence if

$$\lim_{n \rightarrow \infty} \nu(x_{n+p} - x_n, y, t) = 0$$

for all $t > 0$ and $p = 1, 2, 3, \dots$.

Let (\mathcal{X}, ν) be an anti fuzzy 2-normed linear space and $\{x_n\}$ be a Cauchy sequence in \mathcal{X} . If $\{x_n\}$ is convergent in \mathcal{X} then (\mathcal{X}, ν) is said to be an anti fuzzy 2-Banach space.

The following lemma is easy to prove and we will omit it.

Lemma 2.1. Consider the set L^* and operation \leq_{L^*} defined by

$$L^* = \{(x_1, x_2) : (x_1, x_2) \in [0, 1]^2 \text{ and } x_1 + x_2 \leq 1\}$$

$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \iff x_1 \leq y_1, x_2 \geq y_2$$

for all $(x_1, x_2), (y_1, y_2) \in L^*$. Then (L^*, \leq_{L^*}) is a complete lattice.

Definition 2.10. A continuous t -norm τ on $L = [0, 1]^2$ is said to be continuous t -representable if there exist a continuous t -norm $*$ and a continuous t -conorm \diamond on $[0, 1]$ such that, for all $x = (x_1, x_2), y = (y_1, y_2) \in L$

$$\tau(x, y) = (x_1 * y_1, x_2 \diamond y_2).$$

Definition 2.11. Let \mathcal{X} be a set. A function $d : \mathcal{X} \times \mathcal{X} \rightarrow [0, \infty]$ is called a generalized metric on \mathcal{X} if and only if d satisfies:

- (M₁) $d(x, y) = 0 \Leftrightarrow x = y \ \forall x, y \in \mathcal{X}$
- (M₂) $d(x, y) = d(y, x) \ \forall x, y \in \mathcal{X}$
- (M₃) $d(x, z) \leq d(x, y) + d(y, z) \ \forall x, y, z \in \mathcal{X}$

Theorem 2.1. ([28]) Let (\mathcal{X}, d) be a complete generalized metric space and $\mathcal{J} : \mathcal{X} \rightarrow \mathcal{X}$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then, for all $x \in \mathcal{X}$, either

$$d(\mathcal{J}^n x, \mathcal{J}^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

- (a) $d(\mathcal{J}^n x, \mathcal{J}^{n+1} x) < \infty$ for all $n \geq n_0$;
- (b) the sequence $\{\mathcal{J}^n x\}$ converges to a fixed point y^* of \mathcal{J} ;
- (c) y^* is the unique fixed point of \mathcal{J} in the set $\mathcal{Y} = \{y \in \mathcal{X} : d(\mathcal{J}^{n_0} x, y) < \infty\}$;
- (d) $d(y, y^*) \leq \frac{1}{1-L} d(y, \mathcal{J}y)$ for all $y \in \mathcal{Y}$.

3. MAIN RESULTS

3.1. Intuitionistic fuzzy 2-normed spaces. In this subsection we define an intuitionistic fuzzy 2-normed space. Then in next subsection by the fixed point technique we investigate the Hyers-Ulam stability of a generalized quadratic functional equation in intuitionistic fuzzy 2-normed spaces.

Definition 3.1. A 3-tuple $(\mathcal{X}, \rho_{\mu, \nu}, \tau)$ is said to be an intuitionistic fuzzy 2-normed space (for short, IF2NS) if \mathcal{X} is a real linear space, and μ and ν are a fuzzy 2-norm and an anti fuzzy 2-norm, respectively, such that $\nu(x, y, t) + \mu(x, y, t) \leq 1$. τ is continuous t -representable, and

$$\begin{aligned} \rho_{\mu, \nu} : \mathcal{X} \times \mathcal{X} \times \mathbb{R} &\rightarrow L^* \\ \rho_{\mu, \nu}(x, y, t) &= (\mu(x, y, t), \nu(x, y, t)) \end{aligned}$$

is a function satisfying the following conditions, for all $x, y, z \in \mathcal{X}$, and $t, s, \alpha \in \mathbb{R}$

- (1) $\rho_{\mu, \nu}(x, y, t) = (0, 1) = 0_{L^*}$ for all $t \leq 0$.
- (2) $\rho_{\mu, \nu}(x, y, t) = (1, 0) = 1_{L^*}$ if and only if x, y are linearly dependent, for all $t > 0$.
- (3) $\rho_{\mu, \nu}(\alpha x, y, t) = \rho_{\mu, \nu}(x, y, \frac{t}{|\alpha|})$ for all $t > 0$ and $\alpha \neq 0$
- (4) $\rho_{\mu, \nu}(x, y, t)$ is invariant under any permutation of x, y .
- (5) $\rho_{\mu, \nu}(x + z, y, t + s) \geq_{L^*} \tau(\rho_{\mu, \nu}(x, y, t), \rho_{\mu, \nu}(z, y, s))$ for all $t, s > 0$.
- (6) $\rho_{\mu, \nu}(x, y, \cdot)$ is continuous and

$$\lim_{t \rightarrow 0} \rho_{\mu, \nu}(x, y, t) = 0_{L^*} \text{ and } \lim_{t \rightarrow \infty} \rho_{\mu, \nu}(x, y, t) = 1_{L^*}$$

Then $\rho_{\mu, \nu}$ is said to be an intuitionistic fuzzy 2-norm on a real linear space \mathcal{X} .

Example 3.1. Let $(\mathcal{X}, \|\cdot, \cdot\|)$ be a 2-normed space,

$$\tau(a, b) = (a_1 b_1, \min(a_2 + b_2, 1))$$

be continuous t -representable for all $a = (a_1, a_2), b = (b_1, b_2) \in L^*$ and μ, ν be a fuzzy and an anti fuzzy 2-norm, respectively. We define

$$\rho_{\mu,\nu}(x, y, t) = \left(\frac{t}{t + m\|x, y\|}, \frac{\|x, y\|}{t + m\|x, y\|} \right)$$

for all $t \in \mathbb{R}^+$ and $m > 1$. Then $(\mathcal{X}, \rho_{\mu,\nu}, \tau)$ is an IF2NS.

Definition 3.2. A sequence $\{x_n\}$ in an IF2NS $(\mathcal{X}, \rho_{\mu,\nu}, \tau)$ is said to be convergent to a point $x \in \mathcal{X}$ if

$$\lim_{n \rightarrow \infty} \rho_{\mu,\nu}(x_n - x, y, t) = 1_{L^*}$$

for every $t > 0$.

Definition 3.3. A sequence $\{x_n\}$ in an IF2NS $(\mathcal{X}, \rho_{\mu,\nu}, \tau)$ is said to be a Cauchy sequence if for any $0 < \epsilon < 1$ and $t > 0$, there exists $n_0 \in \mathcal{N}$ such that

$$\rho_{\mu,\nu}(x_n - x_m, y, t) \geq_{L^*} (1 - \epsilon, \epsilon)$$

for all $n, m \geq n_0$.

Definition 3.4. An IF2NS space $(\mathcal{X}, \rho_{\mu,\nu}, \tau)$ is said to be complete if every Cauchy sequence in $(\mathcal{X}, \rho_{\mu,\nu}, \tau)$ is convergent. A complete intuitionistic fuzzy 2-normed space is called an intuitionistic fuzzy 2-Banach space.

3.2. Hyers-Ulam stability of a generalized quadratic functional equation in IF2NS.

In this subsection, using the fixed point alternative approach, we prove the Hyers-Ulam stability of a generalized quadratic functional equation in intuitionistic fuzzy 2-Banach spaces.

Definition 3.5. Let \mathcal{X}, \mathcal{Y} be real linear spaces. For a given mapping $f : \mathcal{X} \rightarrow \mathcal{Y}$, we define

$$Df(x, y) := f(ax + by) + f(ax - by) - \frac{a}{2}f(x + y) - \frac{a}{2}f(x - y) - (2a^2 - a)f(x) - (2b^2 - a)f(y)$$

where $a, b \geq 1$, $a \neq 2b^2$ and $x, y \in \mathcal{X}$.

Theorem 3.1. Let \mathcal{X} be a real linear space, $(\mathcal{Z}, \rho'_{\mu,\nu}, \tau')$ an intuitionistic fuzzy 2-normed space and let $\phi : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{Z}, \varphi : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{Z}$ be mappings such that for some $0 < |\alpha| < a$

$$\rho'_{\mu,\nu}(\phi(ax, ay), \varphi(ax, ay), t) \geq_{L^*} \rho'_{\mu,\nu}\left(\frac{\alpha}{a^2}\phi(x, y), \varphi(x, y), t\right) \tag{3.1}$$

for all $x, y \in \mathcal{X}$ and $t \in \mathbb{R}^+$. Let $(\mathcal{Y}, \rho_{\mu,\nu}, \tau)$ be a complete intuitionistic fuzzy 2-normed space. If $\xi : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{Y}$ is a mapping such that $\xi(ax, ay) = \frac{1}{\alpha a^2}\xi(x, y)$ for all $x, y \in \mathcal{X}$ and $f : \mathcal{X} \rightarrow \mathcal{Y}$ is a mapping satisfying $f(0) = 0$ and

$$\rho_{\mu,\nu}(Df(x, y), \xi(x, y), t) \geq_{L^*} \rho'_{\mu,\nu}(\phi(x, y), \varphi(x, y), t) \tag{3.2}$$

for all $x, y \in \mathcal{X}, t > 0$, then there is a unique quadratic mapping $Q : \mathcal{X} \rightarrow \mathcal{Y}$ such that

$$\rho_{\mu,\nu}(f(x) - Q(x), \xi(x, 0), t) \geq_{L^*} \rho'_{\mu,\nu}(\phi(x, 0), \varphi(x, 0), (2(a^2 - \alpha^2)t)) \tag{3.3}$$

Proof. Putting $y = 0$ in (3.2), we have

$$\rho_{\mu,\nu}(2f(ax) - a^2f(x), \xi(x, 0), t) \geq_{L^*} \rho'_{\mu,\nu}(\phi(x, 0), \varphi(x, 0), t)$$

and so

$$\rho_{\mu,\nu}\left(\frac{1}{a^2}f(ax) - f(x), \xi(x, 0), t\right) \geq_{L^*} \rho'_{\mu,\nu}\left(\frac{1}{2a^2}\phi(x, 0), \varphi(x, 0), t\right) \tag{3.4}$$

for all $x \in \mathcal{X}$ and $t > 0$. Consider the set $E = \{g : \mathcal{X} \rightarrow \mathcal{Y}\}$ and define a generalized metric d on E by

$$d(g, h) = \inf \{c \in \mathbb{R}^+ : \rho_{\mu, \nu}(g(x) - h(x), \xi(x, 0), t) \geq_{L^*} (c\phi(x, 0), \varphi(x, 0), t)\}$$

for all $x \in \mathcal{X}$ and $t > 0$ with $\inf \emptyset = \infty$. It is easy to show that (E, d) is complete (see [29]). Define $J : \mathcal{X} \rightarrow \mathcal{X}$ by $Jg(x) = \frac{1}{a^2}g(ax)$ for all $x \in \mathcal{X}$. Now, we prove that J is strictly contractive mapping of E with the Lipschitz constant $\frac{\alpha^2}{a^2}$. Let $g, h \in E$ be given such that $d(g, h) < \epsilon$. Then

$$\rho_{\mu, \nu}(g(x) - h(x), \xi(x, 0), t) \geq_{L^*} \rho'_{\mu, \nu}(\epsilon\phi(x, 0), \varphi(x, 0), t)$$

for all $x \in \mathcal{X}$ and $t > 0$. So

$$\begin{aligned} \rho_{\mu, \nu}(Jg(x) - Jh(x), \xi(x, 0), t) &= \rho_{\mu, \nu}(g(ax) - h(ax), \xi(x, 0), a^2t) \\ &\geq_{L^*} \rho'_{\mu, \nu}\left(c\phi(ax, 0), \varphi(ax, 0), \frac{t}{\alpha}\right) \\ &=_{L^*} \rho'_{\mu, \nu}\left(\frac{\alpha^2}{a^2}c\phi(x, 0), \varphi(x, 0), t\right). \end{aligned}$$

Then $d(Jg, Jh) < \frac{\alpha^2}{a^2} d(g, h)$ for all $g, h \in E$. It follows from (3.4) that

$$d(f, Jf) \leq \frac{1}{2a^2} < \infty$$

It follows from Theorem 2.1 that there exists a mapping $Q : \mathcal{X} \rightarrow \mathcal{Y}$ satisfying the following

- (1) Q is a fixed point of J , that is ;

$$Q(ax) = a^2Q(x) \tag{3.5}$$

- (2) The mapping Q is a unique fixed point of J in the set

$$\Delta = \{h \in E : d(g, h) < \infty\}$$

This implies that Q is a unique mapping satisfying (3.5).

- (3) $d(J^n f, Q) \rightarrow 0$ as $n \rightarrow \infty$. This implies that

$$\lim_{n \rightarrow \infty} \frac{f(a^n x)}{a^{2n}} = Q(x)$$

for all $x \in \mathcal{X}$.

- (4) $d(f, Q) \leq \frac{1}{1-L} d(f, Jf)$ with $f \in \Delta$, which implies the inequality

$$d(f, Q) \leq \frac{1}{2(a^2 - \alpha^2)}$$

So

$$\rho_{\mu, \nu}(f(x) - Q(x), \xi(x, 0), t) \geq_{L^*} \rho'_{\mu, \nu}(\phi(x, 0), \varphi(x, 0), 2(a^2 - \alpha^2)t).$$

This implies that the inequality (3.2) holds.

It remains to show that Q is a quadratic mapping. Replacing x and y by $2^n x$ and $2^n y$ in (3.2), respectively, we get

$$\rho_{\mu, \nu}\left(\frac{1}{a^{2n}} Df(a^n x, a^n y), \xi(a^n x, a^n y), \frac{t}{a^{2n}}\right) \geq_{L^*} \rho'_{\mu, \nu}(\phi(a^n x, a^n y), \varphi(a^n x, a^n y), t).$$

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By the property of $\xi(x, y)$, we have

$$\begin{aligned} \rho_{\mu,\nu} \left(\frac{1}{a^{2n}} Df(a^n x, a^n y), \frac{1}{(\alpha a^2)^n} \xi(x, y), \frac{t}{a^{2n}} \right) \\ \geq_{L^*} \rho'_{\mu,\nu} (\phi(a^n x, a^n y), \varphi(a^n x, a^n y), t). \end{aligned}$$

Thus

$$\rho_{\mu,\nu} \left(\frac{1}{a^{2n}} Df(a^n x, a^n y), \xi(x, y), t \right) \geq_{L^*} \rho'_{\mu,\nu} \left(\phi(a^n x, a^n y), \varphi(a^n x, a^n y), \frac{t}{\alpha^n} \right).$$

By (3.1), we obtain

$$\begin{aligned} \rho_{\mu,\nu} \left(\frac{1}{a^{2n}} Df(a^n x, a^n y), \xi(x, y), t \right) &\geq_{L^*} \rho'_{\mu,\nu} \left(\frac{\alpha^n}{a^{2n}} \phi(x, y), \varphi(x, y), \frac{t}{\alpha^n} \right) \\ &= \rho'_{\mu,\nu} \left(\frac{\alpha^{2n}}{a^{2n}} \phi(x, y), \varphi(x, y), t \right). \end{aligned}$$

As $n \rightarrow \infty$, we have

$$\rho_{\mu,\nu}(DQ(x, y), \xi(x, y), t) \geq_{L^*} 1_{L^*}.$$

Thus Q is a quadratic mapping, as desired. □

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ON A q -ANALOGUE OF (h, q) -DAEHEE NUMBERS AND POLYNOMIALS OF HIGHER ORDER

JIN-WOO PARK

ABSTRACT. In this paper, we introduce a new q -analogue of the Daehee numbers and polynomials of the first kind and the second kind, and derive some new interesting identities.

1. INTRODUCTION

Let p be a fixed prime number. Throughout this paper, \mathbb{Z}_p , \mathbb{Q}_p , and \mathbb{C}_p will respectively denote the ring of p -adic rational integers, the field of p -adic rational numbers and the completions of algebraic closure of \mathbb{Q}_p . The p -adic norm is defined $|p|_p = \frac{1}{p}$.

When one talks of q -extension, q is variously considered as an indeterminate, a complex $q \in \mathbb{C}$, or p -adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume that $|q - 1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for each $x \in \mathbb{Z}_p$. Throughout this paper, we use the notation :

$$[x]_q = \frac{1 - q^x}{1 - q}.$$

Note that $\lim_{q \rightarrow 1} [x]_q = x$ for each $x \in \mathbb{Z}_p$.

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on \mathbb{Z}_p . For $f \in UD(\mathbb{Z}_p)$, the p -adic invariant integral on \mathbb{Z}_p is defined by Kim as follows :

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \rightarrow \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x, \text{ (see [8, 9, 10])}. \tag{1.1}$$

Let f_1 be the translation of f with $f_1(x) = f(x + 1)$. Then, by (1.1), we get

$$-qI_q(f_1) + I_q(f) = (1 - q)f(0) + \frac{1 - q}{\log q} f'(0), \text{ where } f'(0) = \left. \frac{df(x)}{dx} \right|_{x=0}. \tag{1.2}$$

As it is well-known fact, the *Stirling number of the first kind* is defined by

$$(x)_n = x(x - 1) \cdots (x - n + 1) = \sum_{l=0}^n S_1(n, l) x^l, \tag{1.3}$$

and the *Stirling number of the second kind* is given by the generating function to be

$$(e^t - 1)^m = m! \sum_{l=m}^{\infty} S_2(l, m) \frac{t^l}{l!}, \text{ (see [3, 17])}. \tag{1.4}$$

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Unsigned Stirling numbers of the first kind is given by

$$x^{(n)} = x(x+1)\cdots(x+n-1) = \sum_{l=0}^n |S_1(n, l)|x^l. \tag{1.5}$$

Note that if we replace x to $-x$ in (1.3), then

$$\begin{aligned} (-x)_n &= (-1)^n x^{(n)} = \sum_{l=0}^n S_1(n, l)(-1)^l x^l \\ &= (-1)^n \sum_{l=0}^n |S_1(n, l)|x^l. \end{aligned} \tag{1.6}$$

Hence $S_1(n, l) = |S_1(n, l)|(-1)^{n-l}$.

Recently, D. S. Kim and T. Kim introduced the *Daehee polynomials of the first kind of order r* are defined by the generating function to be

$$\left(\frac{\log(1+t)}{t}\right)^r (1+t)^x = \sum_{n=0}^{\infty} D_n^{(r)}(x) \frac{t^n}{n!}, \tag{1.7}$$

and the *Daehee polynomials of the second kind of order r* are given by

$$\left(\frac{\log(1+t)}{t+1}\right)^r (1+t)^x = \sum_{n=0}^{\infty} \widehat{D}_n^{(r)}(x) \frac{t^n}{n!} \text{ (see [5, 7, 9, 14, 16]),}$$

and Cho et. al. defined the *q -Daehee polynomials of order r* as follows.

$$\left(\frac{1-q + \frac{1-q}{\log q} \log(1+t)}{1-q-qt}\right)^r (1+t)^x = \sum_{n=0}^{\infty} D_{n,q}^{(r)}(x) \frac{t^n}{n!}, \text{ (see [2]).}$$

In recent years, Kim et. al. have studied the various generalization of Daehee polynomials (see [2, 6, 12, 14, 15, 16]), and in [1], authors give new q -analogue of Changhee numbers and polynomials.

In this paper, we introduce a new q -analogue of the Daehee numbers and polynomials of the first kind and the second kind of order r , which are called the Witt-type formula for the q -analogue of Daehee polynomials of order r . We can derive some new interesting identities related to the q -Daehee polynomials of order r .

2. ON A q -ANALOGUE OF DAEHEE NUMBERS AND POLYNOMIALS OF ORDER r

In this section, we assume that $t, q \in \mathbb{C}_p$ with $|t|_p < p^{-\frac{1}{p-1}}$. First, we consider the following integral representation associated with the Pochhammer symbol :

$$\underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r\text{-times}} q^{\sum_{i=1}^r h_i y_i} (x + y_1 + \cdots + y_r)_n d\mu_q(y_1) \cdots d\mu_q(y_r), \tag{2.1}$$

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where $n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, $h_1, \dots, h_r \in \mathbb{Z}$ and $r \in \mathbb{N}$. By (2.1),

$$\begin{aligned} & \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} q^{\sum_{i=1}^r h_i y_i} (x + y_1 + \dots + y_r)_n d\mu_q(y_1) \cdots d\mu_q(y_r) \frac{t^n}{n!} \\ &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{i=1}^r h_i y_i} \left(\sum_{n=0}^{\infty} \binom{x + y_1 + \dots + y_r}{n} t^n \right) d\mu_q(y_1) \cdots d\mu_q(y_r) \quad (2.2) \\ &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{i=1}^r h_i y_i} (1+t)^{x+y_1+\dots+y_r} d\mu_q(y_1) \cdots d\mu_q(y_r), \end{aligned}$$

where $t \in \mathbb{C}_p$ with $|t|_p < p^{-\frac{1}{p-1}}$. By (1.2) and (2.2), we have

$$\begin{aligned} & \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{i=1}^r h_i y_i} (x + y_1 + \dots + y_r)_n d\mu_q(y_1) \cdots d\mu_q(y_r) \frac{t^n}{n!} \\ &= \prod_{i=1}^r \left(\frac{q-1 + \frac{q-1}{\log q} (h_i \log q + \log(1+t))}{q^{h_i+1}(1+t) - 1} \right) (1+t)^x. \quad (2.3) \end{aligned}$$

If we put

$$F_q^{(h_1, \dots, h_r)}(x, t) = \prod_{i=1}^r \left(\frac{q-1 + \frac{q-1}{\log q} (h_i \log q + \log(1+t))}{q^{h_i+1}(1+t) - 1} \right) (1+t)^x,$$

then

$$\lim_{q \rightarrow 1} F_q^{(-1, \dots, -1)}(x, t) = \left(\frac{\log(1+t)}{t} \right)^r (1+t)^x.$$

Note that $F_q^{(h_1, \dots, h_r)}(x, t)$ seems to be a new q -extension of the generating function for the Daehee polynomials of the first kind of order r . Thus, by (1.7) and (2.2), we obtain the following definition.

Definition 2.1. A q -analogue of the n th (h, q) -Daehee polynomials of the first kind is defined by the generating function to be

$$\sum_{n=0}^{\infty} D_n^{(h_1, \dots, h_r)}(x|q) \frac{t^n}{n!} = \prod_{i=1}^r \left(\frac{q-1 + \frac{q-1}{\log q} (h_i \log q + \log(1+t))}{q^{h_i+1}(1+t) - 1} \right) (1+t)^x.$$

Moreover,

$$D_n^{(h_1, \dots, h_r)}(x|q) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{i=1}^r h_i y_i} (x + y_1 + \dots + y_r)_n d\mu_q(y_1) \cdots d\mu_q(y_r).$$

In the special case $x = 0$ in Definition 2.1, $D_n^{(h_1, \dots, h_r)}(0|q) = D_n^{(h_1, \dots, h_r)}(q)$ is called a q -analogue of the n th (h, q) -Daehee numbers of the first kind of order r . Note that, by (1.7) and Definition 2.1,

$$D_n^{(-1, \dots, -1)}(x|q) = \left(\frac{q-1}{\log q} \right)^r D_n^{(r)}(x). \quad (2.4)$$

The equation (2.4) shows that the q -analogue of the (h, q) -Daehee polynomials of the first kind of order r is closely related the n th Daehee polynomials of order r .

It is easy to show that

$$\begin{aligned} & \prod_{i=1}^r \left(\frac{q-1 + \frac{q-1}{\log q} (h_i \log q + \log(1+t))}{q^{h_i+1}(1+t) - 1} \right) (1+t)^x \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n \binom{n}{m} D_{n-m}^{(h_1, \dots, h_r)}(q)(x)_m \right) \frac{t^n}{n!}. \end{aligned} \tag{2.5}$$

By Definition 2.1 and (2.5), we have

$$\begin{aligned} D_n^{(h_1, \dots, h_r)}(x|q) &= \sum_{m=0}^n \binom{x}{m} D_{n-m}^{(h_1, \dots, h_r)}(q) \frac{n!}{n-m!} \\ &= \sum_{m=0}^n \binom{x}{n-m} D_m^{(h_1, \dots, h_r)}(q) \frac{n!}{m!}. \end{aligned} \tag{2.6}$$

Since

$$\begin{aligned} (x + y_1 + \dots + y_r)_n &= \sum_{l=0}^n S_1(n, l)(x + y_1 + \dots + y_r)^l \\ &= \sum_{l=0}^n S_1(n, l) \sum_{l_1 + \dots + l_r = l} y_1^{l_1} y_2^{l_2} \dots (x + y_r)^{l_r}, \end{aligned} \tag{2.7}$$

by Definition 2.1 and (2.6), we have

$$\begin{aligned} & D_n^{(h_1, \dots, h_r)}(x|q) \\ &= \int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} q^{\sum_{i=1}^r h_i y_i} \sum_{l=0}^n S_1(n, l) \sum_{l_1 + \dots + l_r = l} y_1^{l_1} y_2^{l_2} \dots (x + y_r)^{l_r} \\ &= \sum_{l=0}^n S_1(n, l) \sum_{l_1 + \dots + l_r = l} B_{l_1, q}^{(h_1)} \dots B_{l_{r-1}, q}^{(h_{r-1})} B_{l_r, q}^{(h_r)}(x), \end{aligned} \tag{2.8}$$

where $B_{n,q}^{(h)}(x)$ are the (h, q) -Bernoulli polynomials derived from

$$B_{n,q}^{(h)}(x) = \int_{\mathbb{Z}_p} q^{hy} (x+y)^n d\mu_q(y), \text{ (see [18]).}$$

Thus, by (2.6) and (2.8), we obtain the following theorem.

Theorem 2.2. For $n \geq 0$, we have

$$\begin{aligned} D_n^{(h_1, \dots, h_r)}(x|q) &= \sum_{m=0}^n \binom{x}{n-m} D_m^{(h_1, \dots, h_r)}(q) \frac{n!}{m!} \\ &= \sum_{l=0}^n \sum_{l_1 + \dots + l_r = l} S_1(n, l) B_{l_1, q}^{(h_1)} \dots B_{l_{r-1}, q}^{(h_{r-1})} B_{l_r, q}^{(h_r)}(x) \end{aligned}$$

Note that, by (1.1), the generating function of (h, q) -Bernoulli polynomials are

$$\begin{aligned} \sum_{n=0}^{\infty} B_{n,q}^{(h)}(x) \frac{t^n}{n!} &= \int_{\mathbb{Z}_p} q^{hy} e^{(x+y)t} d\mu_q(y) \\ &= \frac{q-1 + \frac{q-1}{\log q} (h \log q + t)}{q^{h+1} e^t - 1} e^{xt}. \end{aligned} \tag{2.9}$$

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By replacing t by $e^t - 1$ in Definition 2.1,

$$\begin{aligned} & \sum_{n=0}^{\infty} D_n^{(h_1, \dots, h_r)}(x|q) \frac{1}{n!} (e^t - 1)^n \\ &= \sum_{n=0}^{\infty} D_n^{(h_1, \dots, h_r)}(x|q) \frac{1}{n!} \sum_{l=n}^{\infty} S_2(l, n) \frac{t^l}{l!} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^n D_m^{(h_1, \dots, h_r)}(x|q) S_2(n, m) \frac{t^n}{n!}, \end{aligned} \tag{2.10}$$

and, by (2.9),

$$\begin{aligned} & \prod_{i=1}^r \left(\frac{q - 1 + \frac{q-1}{\log q} (h_i \log q + t)}{q^{h_i+1} e^t - 1} \right) e^{xt} \\ &= \left(\prod_{i=1}^{r-1} \left(\sum_{n=0}^{\infty} B_{n,q}^{(h_i)} \frac{t^n}{n!} \right) \right) \left(\sum_{n=0}^{\infty} B_{n,q}^{(h_r)}(x) \frac{t^n}{n!} \right) \\ &= \sum_{n=0}^{\infty} \sum_{l_1+\dots+l_r=n} \binom{n}{l_1, \dots, l_r} B_{l_1,q}^{(h_1)} \dots B_{l_{r-1},q}^{(h_{r-1})} B_{l_r,q}^{(h_r)}(x) \frac{t^n}{n!}. \end{aligned} \tag{2.11}$$

Thus, by (2.10) and (2.11), we obtain the following theorem.

Theorem 2.3. For $n \geq 0$, we have

$$\sum_{l_1+\dots+l_r=n} \binom{n}{l_1, \dots, l_r} B_{l_1,q}^{(h_1)} \dots B_{l_{r-1},q}^{(h_{r-1})} B_{l_r,q}^{(h_r)}(x) = \sum_{m=0}^n D_m^{(h_1, \dots, h_r)}(x|q) S_2(n, m).$$

Let us define the q -analogue of the n th (h, q) -Daehee polynomials of the second kind is defined as follows:

$$\widehat{D}_n^{(h_1, \dots, h_r)}(x|q) = \int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} q^{\sum_{i=1}^r h_i y_i} (-x - y_1 - \dots - y_r)_n d\mu_q(y_1) \dots d\mu_q(y_r) \tag{2.12}$$

where $n \in \mathbb{N} \cup \{0\}$. In particular, $\widehat{D}_n^{(h_1, \dots, h_r)}(0|q) = \widehat{D}_n^{(h_1, \dots, h_r)}(q)$ are called the q -analogue of the n th (h, q) -Daehee numbers of the second kind.

By (1.3) and (2.12), it leads to

$$\begin{aligned} & \widehat{D}_n^{(h_1, \dots, h_r)}(x|q) \\ &= \int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} q^{\sum_{i=1}^r h_i y_i} (-x - y_1 - \dots - y_r)_n d\mu_q(y_1) \dots d\mu_q(y_r) \\ &= \int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} q^{\sum_{i=1}^r h_i y_i} (-1)^n (x + y_1 + \dots + y_r)^{(n)} d\mu_q(y_1) \dots d\mu_q(y_r) \\ &= \sum_{l=0}^n |S_1(n, l)| (-1)^n \sum_{l_1+\dots+l_r=l} B_{l_1,q}^{(h_1)} \dots B_{l_r,q}^{(h_r)}(x). \end{aligned} \tag{2.13}$$

Thus, we state the following theorem.

Theorem 2.4. For $n \geq 0$, we have

$$\widehat{D}_n^{(h_1, \dots, h_r)}(x|q) = \sum_{l=0}^n \sum_{l_1+\dots+l_r=l} |S_1(n, l)| (-1)^n B_{l_1,q}^{(h_1)} \dots B_{l_r,q}^{(h_r)}(x).$$

Let us now consider the generating function of the q -analogue of the (h, q) -Daehee polynomials of the second kind as follows:

$$\begin{aligned}
 & \sum_{n=0}^{\infty} \widehat{D}_n^{(h_1, \dots, h_r)}(x|q) \frac{t^n}{n!} \\
 &= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{i=1}^r h_i y_i} (x - y_1 - \cdots - y_r)_n d\mu_q(y_1) \cdots d\mu_q(y_r) \frac{t^n}{n!} \\
 &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{i=1}^r h_i y_i} \binom{x - y_1 - \cdots - y_r}{n} t^n d\mu_q(y_1) \cdots d\mu_q(y_r) \quad (2.14) \\
 &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{i=1}^r h_i y_i} (1+t)^{-x-y_1-\cdots-y_r} d\mu_q(y_1) \cdots d\mu_q(y_r) \\
 &= \left(\prod_{i=1}^r \frac{q-1 + \frac{q-1}{\log q} (h_i \log q - \log(1+t))}{q^{h_i+1} - 1 - t} \right) (1+t)^{r-x}.
 \end{aligned}$$

By replacing t by $e^t - 1$, we have

$$\begin{aligned}
 & \left(\prod_{i=1}^r \frac{q-1 + \frac{q-1}{\log q} (h_i \log q - t)}{q^{h_i+1} e^{-t} - 1} \right) e^{-xt} \\
 &= \sum_{n=0}^{\infty} \widehat{D}_n^{(h_1, \dots, h_r)}(x|q) \frac{(e^t - 1)^n}{n!} \\
 &= \sum_{n=0}^{\infty} \widehat{D}_n^{(h_1, \dots, h_r)}(x|q) \frac{1}{n!} \sum_{l=n}^{\infty} S_2(l, n) \frac{x^l}{l!} \\
 &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n \widehat{D}_m^{(h_1, \dots, h_r)}(x|q) S_2(n, m) \right) \frac{x^n}{n!}, \quad (2.15)
 \end{aligned}$$

and

$$\begin{aligned}
 & \left(\prod_{i=1}^r \frac{q-1 + \frac{q-1}{\log q} (h_i \log q - t)}{q^{h_i+1} e^{-t} - 1} \right) e^{-xt} \\
 &= \left(\prod_{i=1}^{r-1} \left(\sum_{n=0}^{\infty} (-1)^n B_{n,q}^{(h_i)} \frac{t^n}{n!} \right) \right) \left(\sum_{n=0}^{\infty} (-1)^n B_{n,q}^{(h_r)}(x) \frac{t^n}{n!} \right) \quad (2.16) \\
 &= \sum_{n=0}^{\infty} (-1)^n \left(\sum_{l_1+\cdots+l_r=n} \binom{n}{l_1, \dots, l_r} B_{l_1,q}^{(h_1)} \cdots B_{l_{r-1},q}^{(h_{r-1})} B_{l_r,q}^{(h_r)}(x) \right) \frac{t^n}{n!}.
 \end{aligned}$$

By (2.15) and (2.16), we obtain the following theorem.

Theorem 2.5. For $n \geq 0$, we have

$$\begin{aligned}
 & \sum_{m=0}^n \widehat{D}_m^{(h_1, \dots, h_r)}(x|q) S_2(n, m) \\
 &= (-1)^n \sum_{l_1+\cdots+l_r=n} \binom{n}{l_1, \dots, l_r} B_{l_1,q}^{(h_1)} \cdots B_{l_{r-1},q}^{(h_{r-1})} B_{l_r,q}^{(h_r)}(x).
 \end{aligned}$$

By Theorem 2.3 and Theorem 2.5, we obtain the following corollary.

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Corollary 2.6. For $n \geq 0$, we have

$$\sum_{m=0}^n D_m^{(h_1, \dots, h_r)}(x|q) S_2(n, m) = (-1)^n \sum_{m=0}^n \widehat{D}_m^{(h_1, \dots, h_r)}(x|q) S_2(n, m).$$

By Definition 2.1,

$$\begin{aligned} & (-1)^n \frac{D_n^{(h_1, \dots, h_r)}(x|q)}{n!} \\ &= (-1)^n \int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} q^{\sum_{i=1}^r h_i y_i} \binom{x + y_1 + \dots + y_r}{n} d\mu_q(y_1) \dots d\mu_q(y_r) \\ &= \int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} q^{\sum_{i=1}^r h_i y_i} \binom{-x - y_1 - \dots - y_r + n - 1}{n} d\mu_q(y_1) \dots d\mu_q(y_r) \\ &= \sum_{m=0}^n \binom{n-1}{n-m} \int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} q^{\sum_{i=1}^r h_i y_i} \binom{-x - y_1 - \dots - y_r}{m} d\mu_q(y_1) \dots d\mu_q(y_r) \\ &= \sum_{m=1}^n \binom{n-1}{m-1} \frac{\widehat{D}_m^{(h_1, \dots, h_r)}(x|q)}{m!}, \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} & (-1)^n \frac{\widehat{D}_n^{(h_1, \dots, h_r)}(x|q)}{n!} \\ &= (-1)^n \int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} q^{\sum_{i=1}^r h_i y_i} \binom{-x - y_1 - \dots - y_r}{n} d\mu_q(y_1) \dots d\mu_q(y_r) \\ &= \int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} q^{\sum_{i=1}^r h_i y_i} \binom{x + y_1 + \dots + y_r + n - 1}{n} d\mu_q(y_1) \dots d\mu_q(y_r) \\ &= \sum_{m=0}^n \binom{n-1}{n-m} \int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} q^{\sum_{i=1}^r h_i y_i} \binom{x + y_1 + \dots + y_r}{m} d\mu_q(y_1) \dots d\mu_q(y_r) \\ &= \sum_{m=1}^n \binom{n-1}{m-1} \frac{D_m^{(h_1, \dots, h_r)}(x|q)}{m!}. \end{aligned} \tag{2.18}$$

Therefore, by (2.17) and (2.18), we obtain the following theorem.

Theorem 2.7. For $n \geq 0$, we have

$$(-1)^n \frac{D_n^{(h_1, \dots, h_r)}(x|q)}{n!} = \sum_{m=1}^n \binom{n-1}{m-1} \frac{\widehat{D}_m^{(h_1, \dots, h_r)}(x|q)}{m!},$$

and

$$(-1)^n \frac{\widehat{D}_n^{(h_1, \dots, h_r)}(x|q)}{n!} = \sum_{m=1}^n \binom{n-1}{m-1} \frac{D_m^{(h_1, \dots, h_r)}(x|q)}{m!}.$$

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On blow-up of solutions for a semilinear damped wave equation with nonlinear dynamic boundary conditions

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Abstract: In this paper, we study a semilinear damped wave equation with nonlinear dynamic boundary conditions. Under certain assumptions, we extend the earlier exponentially growth result in Gerbi and Said-Houari (Adv. Differential Equations 13: 1051-1074, 2008) to a blow-up in finite time result with positive initial energy.

Keywords: damped wave equation; dynamic boundary conditions; blow-up

AMS Subject Classification (2010): 35L20; 35L71; 35B44

1. INTRODUCTION

In this work, we investigate the following semilinear damped wave equation with dynamic boundary conditions

$$\begin{cases} u_{tt} - \Delta u - \alpha \Delta u_t = |u|^{p-2}u, & x \in \Omega, \quad t > 0, \\ u(x, t) = 0, & x \in \Gamma_0, \quad t \geq 0, \\ u_{tt}(x, t) = - \left[\frac{\partial u}{\partial \nu}(x, t) + \alpha \frac{\partial u_t}{\partial \nu}(x, t) + r|u_t|^{m-2}u_t \right], & x \in \Gamma_1, \quad t > 0, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & x \in \Omega. \end{cases} \quad (1.1)$$

Here Ω is a regular and bounded domain in \mathbb{R}^N ($N \geq 1$) and $\partial\Omega = \Gamma_0 \cup \Gamma_1$, $\text{mes}(\Gamma_0) > 0$, $\Gamma_0 \cap \Gamma_1 = \emptyset$. We denote Δ the Laplacian operator with respect to the x variable and $\frac{\partial}{\partial \nu}$ the unit outer normal derivative, $m \geq 2$, $p > 2$, α, r are positive constants and u_0 and u_1 are given functions.

From the mathematical point of view, the boundary conditions that do not neglect the acceleration terms are usually called dynamic boundary conditions. Researches on these problems are very important in practical problems as well as in the theoretical fields.

For the cases of one dimension space, many results have been established (see [1, 2, 3, 11, 12, 13, 15, 24, 23, 35]). For example, Grobbelaar-van Dalsen [12] studied the following problem:

$$\begin{cases} u_{tt} - u_{xx} - u_{txx} = 0, & x \in (0, L), \quad t > 0, \\ u(0, t) = 0, & t > 0, \\ u_{tt}(L, t) = - [u_x + u_{tx}](L, t), & t > 0, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = v_0(x), & x \in (0, L), \\ u(L, 0) = \eta, \quad u_t(L, 0) = \mu & t > 0. \end{cases} \quad (1.2)$$

By using the theory of B-evolutions and the theory of fractional powers, the author proved that problem (1.2) gives rise to an analytic semigroup in an appropriate functional space and obtained the existence and the uniqueness of solutions. For a problem related to (1.2), an exponential

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decay result was obtained in [13], which describes the weakly damped vibrations of an extensible beam. Later, Zhang and Hu [35] considered (1.2) in a more general form and an exponential and polynomial decay rates for the energy were obtained by using the Nakao inequality. Pellicer and Solà-Morales [24] considered the linear wave equation with strong damping and dynamical boundary conditions as an alternative model for the classical spring-mass-damper ODE:

$$m_1 u''(t) + d_1 u'(t) + k_1(t) = 0. \tag{1.3}$$

Based on the semigroup theory, spectral perturbation analysis and dominant eigenvalues, they compared analytically these two approaches to the same physical system. Then, Pellicer [23] considered the same problem with a control acceleration $\varepsilon f\left(u(1, t), \frac{u_t(1, t)}{\sqrt{\varepsilon}}\right)$ as a model for a controlled spring-mass-damper system and established some results concerning its large time behavior. By applying invariant manifold theory, the author proved that the infinite dimensional system admits a two-dimensional attracting manifold where the equation is well represented by a classical nonlinear oscillations ODE, which can be exhibited explicitly.

For the multi-dimensional cases, we can cite [5, 6, 14, 21, 22, 30] for problems with the Dirichlet boundary conditions and [27, 28, 29] for the Cauchy problems. Recently, Gerbi and Said-Houari [7, 8] studied problem (1.1), in which the strong damping term $-\Delta u_t$ is involved. They showed in [7] that if the initial data are large enough then the energy and the L^p norm of the solution of problem (1.1) is unbounded and grows up exponentially as time goes to infinity. Later, they established in [8] the global existence and asymptotic stability of solutions starting in a stable set by combining the potential well method and the energy method. A blow-up result for the case $m = 2$ with initial data in the unstable set was also obtained. However, as indicated in [8], the blow-up of solutions in the presence of a strong damping and a nonlinear boundary damping (i.e., $m > 2$) at the same time is still an open problem. For other related works, we refer the readers to [4, 10, 9, 17, 18, 19, 20, 25, 26, 31, 32, 33, 34] and the references therein.

Motivated by the above works, in this article, we intend to extend the exponentially growth result in [7] to a blow-up result with positive initial energy. The main difficulty here is the simultaneous appearance of the strong damping term Δu_t , the nonlinear boundary damping term $r|u_t|^{m-2}u_t$, and the nonlinear source term $|u|^{p-2}u$. For our purpose, the functional like $L(t) = H(t) + \varepsilon F(t)$ in [7] is modified to $L(t) = H^{1-\alpha}(t) + \varepsilon F(t)$ for some $\alpha > 0$ in this paper. We also give a modified manner to estimate the term $\left| \int_{\Gamma_1} |u_t|^{m-2} u_t u d\sigma \right|$ so that the appearance of the form like $\gamma = RH^{-\sigma}(t)$ (for constants γ, R and σ) which has been used in many earlier works (for example in [10, 16, 21, 22]) can be avoided.

The paper is organized as follows. In Section 2 we present some notations and assumptions and state the main result. Section 3 is devoted to proof of the blow-up result - Theorem 2.2.

2. PRELIMINARIES AND MAIN RESULT

In this section, we first recall some notations and assumptions given in [7]. We denote

$$H_{\Gamma_0}^1(\Omega) = \{u \in H^1(\Omega) \mid u_{\Gamma_0} = 0\}$$

with the scalar product (\cdot, \cdot) in $L^2(\Omega)$ and we also mean by $\|\cdot\|_q$ the $L^q(\Omega)$ norm for $1 \leq q \leq \infty$ and by $\|\cdot\|_{q,\Gamma_1}$ the $L^q(\Gamma_1)$ norm. We will use the following embedding

$$H_{\Gamma_0}^1(\Omega) \hookrightarrow L^q(\Gamma_1), \quad 2 \leq q \leq \bar{q},$$

where

$$\bar{q} = \begin{cases} \frac{2(N-1)}{N-2}, & \text{if } N \geq 3, \\ +\infty, & \text{if } N = 1, 2. \end{cases} \quad (2.1)$$

We state the following local existence and uniqueness theorem established in [7].

Theorem 2.1. ([7, theorem 2.1]) *Let $2 \leq p \leq \bar{q}$ and $\max\left\{2, \frac{\bar{q}}{\bar{q}+1-q}\right\} \leq m \leq \bar{q}$. Then given $u_0 \in H_{\Gamma_0}^1(\Omega)$ and $u_1 \in L^2(\Omega)$, there exists $T > 0$ and a unique solution $u(t)$ of the problem (1.1) on $[0, T)$ such that*

$$u \in C(0, T; H_{\Gamma_0}^1(\Omega)) \cap C^1(0, T; L^2(\Omega)), \quad u_t \in L^2(0, T; H_{\Gamma_0}^1(\Omega)) \cap L^m(\Gamma_1 \times [0, T)).$$

We define the energy functional

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p}\|u\|_p^p + \frac{1}{2}\|u_t\|_{2,\Gamma_1}^2 \quad (2.2)$$

and set

$$\alpha_1 = B^{-p/(p-2)}, \quad E_1 = \left(\frac{1}{2} - \frac{1}{p}\right)\alpha_1^2, \quad (2.3)$$

where B is the best constant of the embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$. We can easily get

$$E'(t) = -\alpha\|\nabla u_t\|_2^2 - r\|u_t\|_{m,\Gamma_1}^m \leq 0. \quad (2.4)$$

Our main result reads as follows.

Theorem 2.2. *Suppose that $m < p$ with $2 < p \leq \bar{q}$ and that*

$$0 < \frac{N}{2} - \frac{N-1}{m} \leq \min\left\{\frac{p-2}{p}, \frac{2(p-m)}{mp}\right\} \quad (2.5)$$

holds. Assume that

$$E(0) < E_1, \quad \|\nabla u_0\|_2 > \alpha_1. \quad (2.6)$$

Then the solution of problem (1.1) blows up in a finite time T_0 , in the sense that

$$\lim_{t \rightarrow T_0^-} [\|u_t\|_2^2 + \|u_t\|_{2,\Gamma_1}^2 + \|\nabla u\|_2^2] = +\infty. \quad (2.7)$$

3. BLOW-UP OF SOLUTIONS

In this section, we prove our main result and use C to denote a generic positive constant. To this end, we need the following lemmas.

Lemma 3.1. ([7, Lemma 3.1]) *Let u be the solution of problem (1.1). Assume that $2 < p \leq \bar{q}$ and (2.6) holds. Then there exists a constant $\alpha_2 > \alpha_1$ such that*

$$\|\nabla u(\cdot, t)\|_2 \geq \alpha_2, \quad \forall t \geq 0, \quad (3.1)$$

and

$$\|u\|_p \geq B\alpha_2, \quad \forall t \geq 0. \quad (3.2)$$

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Lemma 3.2. *Let u be the solution of problem (1.1). Assume that $2 < p \leq \bar{q}$ and (2.6) holds. Then we have*

$$E_1 < \frac{p-2}{2p} \|u\|_p^p, \quad \forall t \geq 0. \tag{3.3}$$

Proof. Exploiting (2.3) and (3.2), we get

$$E_1 = \frac{p-2}{2p} \alpha_1^{2-p} \alpha_1^p = \frac{p-2}{2p} B^p \alpha_1^p < \frac{p-2}{2p} B^p \alpha_2^p \leq \frac{p-2}{2p} \|u\|_p^p.$$

Set

$$H(t) = E_1 - E(t), \tag{3.4}$$

then we have

$$0 < H(0) \leq H(t) < \frac{1}{p} \|u\|_p^p + \frac{p-2}{2p} \|u\|_p^p \leq \frac{1}{2} \|u\|_p^p. \tag{3.5}$$

As a result of (2.2) and (3.5), we can deduce as in [21, 22] the following lemma.

Lemma 3.3. *Let u be the solution of problem (1.1). Assume that $2 < p \leq \bar{q}$ and (2.6) holds. Then*

$$\|u\|_p^\kappa \leq C(\|\nabla u\|_2^2 + \|u\|_p^p) \leq C[-H(t) - \|u_t\|_2^2 - \|u_t\|_{2,\Gamma_1}^2 + \|u\|_p^p] \tag{3.6}$$

for any $2 \leq \kappa \leq p$.

Now, we are ready to prove our result.

Proof of Theorem 2.2. We assume by contradiction that (2.7) does not hold true. Then for $\forall T^* < +\infty$ and all $t \in [0, T^*]$, we have

$$\|u_t\|_2^2 + \|u_t\|_{2,\Gamma_1}^2 + \|\nabla u\|_2^2 \leq C_1, \tag{3.7}$$

where C_1 is a positive constant. Set

$$L(t) = H^{1-\theta}(t) + \varepsilon \int_{\Omega} u_t u dx + \varepsilon \int_{\Gamma_1} u_t u d\sigma + \frac{\varepsilon \alpha}{2} \|\nabla u\|_2^2 \tag{3.8}$$

for ε small to be chosen later and

$$\frac{s}{2} \leq \theta \leq \min \left\{ \frac{p-2}{2p}, \frac{1}{m} - \frac{1}{p} \right\}$$

with

$$0 < \frac{N}{2} - \frac{N-1}{m} \leq s \leq \min \left\{ \frac{p-2}{p}, \frac{2(p-m)}{mp} \right\}.$$

Taking a derivative of $L(t)$ in (3.8) and use (1.1) and (2.2), we get

$$\begin{aligned} L'(t) &= (1-\theta)H^{-\theta}(t)H'(t) + \varepsilon(\|u_t\|_2^2 + \|u_t\|_{2,\Gamma_1}^2 - \|\nabla u\|_2^2) + \varepsilon\|u\|_p^p - \varepsilon r \int_{\Omega} |u_t|^{m-2} u_t u d\sigma \\ &= (1-\theta)H^{-\theta}(t)H'(t) + 2\varepsilon(\|u_t\|_2^2 + \|u_t\|_{2,\Gamma_1}^2) - 2\varepsilon E_1 + 2\varepsilon H(t) \\ &\quad + \varepsilon \left(1 - \frac{2}{p}\right) \|u\|_p^p - \varepsilon r \int_{\Omega} |u_t|^{m-2} u_t u d\sigma. \end{aligned} \tag{3.9}$$

We now estimate the last term on the right-hand side of (3.9). By Hölder's inequality, we obtain

$$\left| \int_{\Gamma_1} |u_t|^{m-2} u_t u d\sigma \right| \leq \|u_t\|_{m,\Gamma_1}^{m-1} \|u\|_{m,\Gamma_1}. \tag{3.10}$$

As in [7], for $m \geq 1$ and $\frac{N}{2} - \frac{N-1}{m} \leq s \leq \min \left\{ \frac{p-2}{p}, \frac{2(p-m)}{mp} \right\} < 1$,

$$\|u\|_{m,\Gamma_1} \leq C\|u\|_{H^s(\Omega)} \leq C\|u\|_2^{1-s} \|\nabla u\|_2^s \leq C\|u\|_p^{1-s} \|\nabla u\|_2^s \leq C\|u\|_p^{\frac{p}{m}} \|u\|_p^{1-s-\frac{p}{m}} \|\nabla u\|_2^s. \tag{3.11}$$

Combining (3.10), (3.11), (3.5) and (3.8), and using Young's inequality we have

$$\begin{aligned} \left| \int_{\Gamma_1} |u_t|^{m-2} u_t u d\sigma \right| &\leq C C_1^{\frac{s}{2}} \|u_t\|_{m,\Gamma_1}^{m-1} \|u\|_p^{\frac{p}{m}} \|u\|_p^{1-s-\frac{p}{m}} \leq C C_1^{\frac{s}{2}} \|u_t\|_{m,\Gamma_1}^{m-1} \|u\|_p^{\frac{p}{m}} H^{-\theta_1}(t) \\ &\leq C C_1^{\frac{s}{2}} H^{-\theta_1}(t) \left(\frac{\beta^m}{m} \|u\|_p^p + \frac{(m-1)\beta^{-m}}{m} \|u_t\|_{m,\Gamma_1}^m \right), \end{aligned} \tag{3.12}$$

where $\theta_1 = \frac{1}{m} - \frac{1-s}{p} \geq \theta$ and $\beta > 0$ will be chosen later. Substituting (3.12) in (3.9) yields

$$\begin{aligned} L'(t) &\geq (1-\theta)H^{-\theta}(t)H'(t) + 2\varepsilon(\|u_t\|_2^2 + \|u_t\|_{2,\Gamma_1}^2) - 2\varepsilon E_1 + 2\varepsilon H(t) \\ &\quad + \varepsilon \left(1 - \frac{2}{p} - \frac{r C C_1^{\frac{s}{2}} \beta^m}{m} H^{-\theta_1}(t) \right) \|u\|_p^p - \frac{\varepsilon r C C_1^{\frac{s}{2}} (m-1)}{m \beta^m} H^{-\theta_1}(t) \|u_t\|_{m,\Gamma_1}^m. \end{aligned} \tag{3.13}$$

By virtue of (2.4), (3.4) and (3.5), we get

$$H'(t) \geq r \|u_t\|_{m,\Gamma_1}^m$$

and

$$H^{-\theta_1}(t) \leq H^{-\theta_1}(0), \quad H^{-\theta}(t) \leq H^{-(\theta_1-\theta)}(0)H^{-\theta}(t).$$

Furthermore, using (3.2), we have

$$-2\varepsilon E_1 \geq -2\varepsilon E_1 B^{-p} \alpha_2^{-p} \|u\|_p^p.$$

Therefore, (3.13) becomes

$$\begin{aligned} L'(t) &\geq r \left(1 - \theta - \frac{\varepsilon C C_1^{\frac{s}{2}} (m-1)}{m \beta^m} H^{-(\theta_1-\theta)}(0) \right) H^{-\theta}(t) \|u_t\|_{m,\Gamma_1}^m + 2\varepsilon(\|u_t\|_2^2 + \|u_t\|_{2,\Gamma_1}^2) \\ &\quad + \varepsilon \left(1 - \frac{2}{p} - \frac{r C C_1^{\frac{s}{2}} \beta^m}{m} H^{-\theta_1}(0) - 2E_1 B^{-p} \alpha_2^{-p} \right) \|u\|_p^p + 2\varepsilon H(t). \end{aligned} \tag{3.14}$$

Since $\alpha_2 > \alpha_1$ and combining the definition of E_1 , we have

$$1 - \frac{2}{p} - 2E_1 B^{-p} \alpha_2^{-p} = \frac{p-2}{p} \left[1 - \left(\frac{\alpha_1}{\alpha_2} \right)^p \right] > 0.$$

So, we can choose β small enough so that

$$1 - \frac{2}{p} - \frac{r C C_1^{\frac{s}{2}} \beta^m}{m} H^{-\theta_1}(0) - 2E_1 B^{-p} \alpha_2^{-p} > 0.$$

Once β is fixed, we choose ε small enough such that

$$L(0) = H^{1-\theta}(0) + \varepsilon \int_{\Omega} u_0 u_1 dx + \varepsilon \int_{\Gamma_1} u_0 u_1 d\sigma + \frac{\varepsilon \alpha}{2} \|\nabla u_0\|_2^2 > 0$$

and

$$1 - \theta - \frac{\varepsilon C C_1^{\frac{s}{2}} (m-1)}{m \beta^m} H^{-(\theta_1-\theta)}(0) > 0.$$

Hence, we have

$$L'(t) \geq \Lambda \varepsilon (\|u_t\|_2^2 + \|u_t\|_{2,\Gamma_1}^2 + \|u\|_p^p + H(t)) \tag{3.15}$$

for some positive constant Λ .

On the other hand, we have

$$L^{\frac{1}{1-\theta}}(t) = \left(H^{1-\theta}(t) + \varepsilon \int_{\Omega} u_t u dx + \varepsilon \int_{\Gamma_1} u_t u d\sigma + \frac{\varepsilon \alpha}{2} \|\nabla u\|_2^2 \right)^{\frac{1}{1-\theta}}$$

$$\leq C \left(H(t) + \left| \int_{\Omega} u_t u dx \right|^{\frac{1}{1-\theta}} + \left| \int_{\Gamma_1} u_t u d\sigma \right|^{\frac{1}{1-\theta}} + \|\nabla u\|_2^{\frac{2}{1-\theta}} \right). \tag{3.16}$$

Using Hölder and Young inequalities, (3.7), (3.11) and Lemma 3.3, we get

$$\begin{aligned} \left| \int_{\Omega} u_t u dx \right|^{\frac{1}{1-\theta}} &\leq C (\|u\|_2 \|u_t\|_2)^{\frac{1}{1-\theta}} \leq C \|u\|_p^{\frac{1}{1-\theta}} \|u_t\|_2^{\frac{1}{1-\theta}} \leq C \left(\|u\|_p^{\frac{2}{1-2\theta}} + \|u_t\|_2^2 \right) \\ &\leq C (H(t) + \|u_t\|_2^2 + \|u_t\|_{2,\Gamma_1}^2 + \|u\|_p^p), \end{aligned} \tag{3.17}$$

$$\begin{aligned} \left| \int_{\Gamma_1} u_t u dx \right|^{\frac{1}{1-\theta}} &\leq C (\|u\|_{2,\Gamma_1} \|u_t\|_{2,\Gamma_1})^{\frac{1}{1-\theta}} \leq C \|u_t\|_{2,\Gamma_1}^{\frac{1}{1-\theta}} \|u\|_p^{\frac{1-s}{1-\theta}} \|\nabla u\|_2^{\frac{s}{1-\theta}} \\ &\leq C C_1^{\frac{s}{2(1-\theta)}} \left(\|u\|_p^{\frac{2(1-s)}{1-2\theta}} + \|u_t\|_{2,\Gamma_1}^2 \right) \leq C (H(t) + \|u_t\|_2^2 + \|u_t\|_{2,\Gamma_1}^2 + \|u\|_p^p), \end{aligned} \tag{3.18}$$

and

$$\|\nabla u\|_2^{\frac{2}{1-\theta}} \leq C_1^{\frac{1}{1-\theta}}. \tag{3.19}$$

Using the Poincaré’s inequality and (3.7), we have

$$\|u\|_p^p \leq B^p \|\nabla u\|_2^p \leq B^p C_1^{\frac{p}{2}}. \tag{3.20}$$

By virtue of (3.5) and (3.20), we know that $H(t)$ is bounded. There exists a positive constant C_2 such that

$$H(t) + C_1^{\frac{1}{1-\theta}} \leq C_2 H(t).$$

Therefore, we obtain

$$L^{\frac{1}{1-\theta}}(t) \leq C (\|u_t\|_2^2 + \|u_t\|_{2,\Gamma_1}^2 + \|u\|_p^p + H(t)). \tag{3.21}$$

A combining of (3.15) and (3.21) leads to

$$L'(t) \geq \frac{\varepsilon \Lambda}{C} L^{\frac{1}{1-\theta}}(t). \tag{3.22}$$

A simple integration of (3.22) over $[0, t]$ gives

$$L^{\frac{\theta}{1-\theta}}(t) \geq \frac{1}{L^{-\frac{\theta}{1-\theta}}(0) - \frac{\theta \Lambda \varepsilon}{C(1-\theta)} t}, \quad \forall t \geq 0. \tag{3.23}$$

This shows that $L(t)$ blows up in a finite time T_0 , where

$$T_0 \leq \frac{(1-\theta)C}{\Lambda \varepsilon \theta [L(0)]^{\theta/(1-\theta)}}.$$

If we choose $T^* \geq \frac{(1-\theta)C}{\Lambda \varepsilon \theta [L(0)]^{\theta/(1-\theta)}}$, then we obtain $T_0 \leq T^*$, which contradicts to our assumption. This completes the proof.

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UNIQUENESS OF MEROMORPHIC FUNCTIONS WITH THEIR DIFFERENCE OPERATORS

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ABSTRACT. This paper is devoted to considering sharing value problems for a meromorphic function $f(z)$ with its difference operator $\Delta_c f = f(z + c) - f(z)$, which improve some recent results in Chen and Yi in [2].

1. INTRODUCTION

In this paper a meromorphic function will mean meromorphic in the whole complex plane. We assume that the reader is familiar with the elementary Nevanlinna Theory, see, e.g. [8, 19]. In particular, we denote the order, exponent of convergence of zeros and poles of a meromorphic function $f(z)$ by $\sigma(f)$, $\lambda(f)$ and $\lambda(\frac{1}{f})$, respectively. As usual, the abbreviation CM stands for "counting multiplicities", while IM means "ignoring multiplicities".

The classical results in the uniqueness theory of meromorphic functions are the five-point, resp. four-point, theorems due to Nevanlinna [17]:

Theorem A. *If two meromorphic functions $f(z)$ and $g(z)$ share five distinct values $a_1, a_2, a_3, a_4, a_5 \in \mathbb{C} \cup \{\infty\}$ IM, then $f(z) = g(z)$.*

Theorem B. *If two meromorphic functions $f(z)$ and $g(z)$ share four distinct values $a_1, a_2, a_3, a_4 \in \mathbb{C} \cup \{\infty\}$ CM, then $f(z) = g(z)$ or $f(z) = T \circ g(z)$, where T is a Möbius transformation.*

It is well-known that 4 CM can not be improved to 4 IM, see [4]. Further, Gundersen [6, Theorem 1] has improved the assumption 4 CM to 2 CM+2 IM, while 1 CM+3 IM remains an open problem. Meanwhile, Gundersen [7], Mues and Stinmetz [16] got some uniqueness results on the case when $g(z)$ is the derivative of $f(z)$:

Theorem C. *If a meromorphic functions $f(z)$ and its derivative $f'(z)$ share two distinct values a_1, a_2 CM, then $f(z) = f'(z)$.*

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Gundersen [7] has given a counterexample to show that the conclusion of Theorem C is not valid if 2 CM is replaced by 1 CM + 1 IM. However, 2 CM can be replaced by 3 IM, see [5, 15].

In recent papers [9, 10], Heittokangas et al. started to consider the uniqueness of a finite order meromorphic function sharing values with its shift. They concluded that:

Theorem D. *Let $f(z)$ be a meromorphic function of finite order, let $c \in \mathbb{C}$, and let $a_1, a_2, a_3 \in S(f) \cup \{\infty\}$ be three distinct periodic functions with period c . If $f(z)$ and $f(z + c)$ share a_1, a_2 CM and a_3 IM, then $f(z) = f(z + c)$ for all $z \in \mathbb{C}$.*

Some improvements of Theorem D can be found in [2, 11, 12, 18]. The difference operator $\Delta_c f = f(z + c) - f(z)$ can be regarded as the difference counterpart of $f'(z)$. Therefore, some research results [9, 13] have been obtained for the problem that $\Delta_c f$ and $f(z)$ share one value a CM, which can be seen as difference analogues of Brück conjecture in [1]. Here, we just recall the following result in [2] as an example:

Theorem E. *Let $f(z)$ be a finite order transcendental entire function which has a finite Borel exceptional value a , and let $f(z)$ be not periodic of period c . If $\Delta_c f$ and $f(z)$ share a CM, then $a = 0$ and $\Delta_c f = \tau f(z)$, where τ is a non-zero constant.*

Zhang et al. gave some improvements of Theorem E, the reader is referred to [14, 20]. A natural question is: what is the uniqueness result on the case when $f(z)$ is meromorphic and $a(z)$ is a small function of $f(z)$ in Theorem E. Corresponding to this question, we get the following results:

Theorem 1.1. *Let $f(z)$ be a transcendental meromorphic function of finite order which has two Borel exceptional values a and ∞ , and let $f(z)$ be not periodic of period c . If $\Delta_c f$ and $f(z)$ share values a and ∞ CM, then $a = 0$ and $f(z) = Ae^{Bz}$, where A, B are non-zero constants.*

Theorem 1.2. *Let $f(z)$ be a transcendental meromorphic function of finite order which has a Borel exceptional value ∞ , and let $a(z)$ be a non-constant meromorphic function such that $\sigma(a) < \sigma(f)$ and $\lambda(f - a) < \sigma(f)$. If $\Delta_c f$ and $f(z)$ share values $a(z)$ and ∞ CM, then $f(z) = a(z) + Ce^{Dz}$ and $\sigma(a) < 1$, where C, D are non-zero constants, .*

2. SOME LEMMAS

Lemma 2.1. [3, Theorem 2.1] *Let $f(z)$ be a non-constant meromorphic function with finite order σ , and let c be a non-zero constant. Then, for each $\varepsilon > 0$, we have*

$$T(r, f(z + c)) = T(r, f(z)) + O(r^{\sigma-1+\varepsilon}) + O(\log r).$$

Lemma 2.2. [3, Theorem 8.2] *Let $f(z)$ be a meromorphic function of finite order σ , c be a non-zero constant. Let $\varepsilon > 0$ be a given real constant, then there exists a subset $E \subset (1, \infty)$ of finite logarithmic measure such that for all $|z| = r \notin [0, 1] \cup E$, we have*

$$\exp(-r^{\sigma-1+\varepsilon}) \leq \left| \frac{f(z+\eta)}{f(z)} \right| \leq \exp(r^{\sigma-1+\varepsilon}).$$

3. PROOF OF THEOREM 1.2

It follows by the assumption that

$$f(z) = a(z) + \frac{u(z)}{v(z)}e^{h(z)}, \tag{3.1}$$

where $u(z)$, $v(z)$ are two non-zero entire functions, $h(z)$ is a non-constant polynomial of degree m . Furthermore, we know $f(z)$ is of normal growth, and $a(z)$, $u(z)$, $v(z)$ satisfy:

$$\lambda(f-a) = \lambda(u) = \sigma(u) < \sigma(f) = m, \quad \lambda\left(\frac{1}{f}\right) = \lambda(v) = \sigma(v) < \sigma(f),$$

and

$$T(r, a) = S(r, f), \quad T(r, u) = S(r, e^{h(z)}), \quad T(r, v) = S(r, e^{h(z)}) = S(r, f).$$

From (3.1), we have

$$\begin{aligned} \Delta_c f &= \left(\frac{u(z+c)}{v(z+c)} e^{h(z+c)-h(z)} - \frac{u(z)}{v(z)} \right) e^{h(z)} + a(z+c) - a(z) \\ &= H(z)e^{h(z)} + a(z+c) - a(z). \end{aligned} \tag{3.2}$$

Applying Lemma 2.1 to equation (3.2), we conclude

$$\sigma(H) < m = \sigma(f),$$

which means that

$$T(r, H) = S(r, f).$$

By the sharing assumption, we obtain that

$$\frac{\Delta_c f - a(z)}{f(z) - a(z)} = e^{p(z)}, \tag{3.3}$$

where $p(z)$ is a polynomial. By combining Lemma 2.1 and (3.3), it follows that $\deg p(z) \leq \sigma(f) = m$. From (3.1), (3.2) and (3.3), we deduce that

$$H(z)e^{h(z)} + a(z+c) - 2a(z) = \frac{u(z)}{v(z)}e^{h(z)+p(z)}. \tag{3.4}$$

Case 1. Suppose that $a(z+c) - 2a(z) \not\equiv 0$. Then by (3.4) we get

$$N\left(r, \frac{1}{e^{h(z)} + \frac{a(z+c)-2a(z)}{H(z)}}\right) \leq N\left(r, \frac{1}{u(z)}\right) = S(r, e^{h(z)}),$$

when $H(z) \not\equiv 0$, which is a contradiction.

If $H(z) \equiv 0$, then it follows from (3.2) that

$$\frac{u(z+c)}{v(z+c)} e^{h(z+c)-h(z)} - \frac{u(z)}{v(z)} \equiv 0,$$

this gives

$$\frac{u(z+c)v(z)}{u(z)v(z+c)} e^{h(z+c)-h(z)} \equiv 1. \tag{3.5}$$

Denote

$$G(z) = \frac{u(z+c)v(z)}{u(z)v(z+c)}. \tag{3.6}$$

From equation (3.5), we know that $G(z)$ is a non-zero entire function. By Lemma 2.2, we see

$$\left| \frac{u(z+c)}{u(z)} \right| \leq \exp(r^{\sigma(u)-1+\varepsilon}), \quad \left| \frac{v(z)}{v(z+c)} \right| \leq \exp(r^{\sigma(v)-1+\varepsilon}),$$

which implies that

$$|G(z)| = \left| \frac{u(z+c)v(z)}{u(z)v(z+c)} \right| \leq \exp(2r^{\sigma-1+\varepsilon}),$$

where $\sigma = \max\{\sigma(u), \sigma(v)\} < \sigma(f) = m$, and $0 < \varepsilon < \frac{m-\sigma}{2}$. Hence,

$$T(r, G(z)) = m(r, G(z)) = 2r^{\sigma-1+\varepsilon},$$

that is

$$\sigma(G) \leq \sigma - 1 + \varepsilon < m - 1. \tag{3.7}$$

Consequently,

$$\sigma(f) = m = 1, \tag{3.8}$$

from equations (3.5) and (3.7). Thus,

$$\sigma(a) < 1, \quad \sigma(u) < 1, \quad \sigma(v) < 1. \tag{3.9}$$

Rewriting equation (3.5) as following

$$\frac{u(z+c)v(z)}{u(z)} e^{h(z+c)-h(z)} \equiv v(z+c), \tag{3.10}$$

$$\frac{u(z+c)v(z)}{v(z+c)} e^{h(z+c)-h(z)} \equiv u(z). \tag{3.11}$$

Next, we will prove that

$$u(z) \not\equiv 0, \quad v(z) \not\equiv 0. \tag{3.12}$$

In fact, suppose z_0 is a zero of $v(z)$, then from (3.10) and $u(z_0) \neq 0$, we get that $z_0 + c$ is also a zero of $v(z)$. By calculation, we know $v(z_0 + kc) = 0$ as well, where k is a positive integer. Hence, $\sigma(v) \geq 1$, which contradicts equation (3.9). Hence, $v(z) \not\equiv 0$. Similarly, we get $u(z) \not\equiv 0$. By combining equation (3.9) with (3.12), we affirm that $v(z)$ and $u(z)$ are two non-zero constants. Therefore, we conclude that $f(z) = a(z) + Ce^{Dz}$, where C, D are non-zero constants.

Case 2. Assume that

$$a(z + c) - 2a(z) \equiv 0. \tag{3.13}$$

Then we affirm that $\sigma(a) \geq 1$. In fact, if $\sigma(a) < 1$, then from (3.13) and Lemma 2.2, for any given ε such that $0 < \varepsilon < \frac{1-\sigma(a)}{2}$, there exists a subset $E \subset (1, \infty)$ of finite logarithmic measure such that for all $|z| = r \notin [0, 1] \cup E$, we deduce that

$$2 = \left| \frac{a(z + c)}{a(z)} \right| \leq \exp(r^{\sigma(a)-1+\varepsilon}) \rightarrow 0, \quad r \rightarrow \infty,$$

which is impossible. Therefore, we conclude that

$$\sigma(a) \geq 1.$$

(a). If $\deg p(z) = 0$, then $p(z) = p$, where p is a constant. From equations (3.1) and (3.3), it follows that

$$G(z)e^{h(z+c)-h(z)} = 1 + e^p,$$

where $G(z)$ denotes as equation (3.6). Using a similar way as Case 1, we know that $\sigma(f) = m = 1$, which contradicts the assumption that $\sigma(a) < \sigma(f)$.

(b). If $\deg p(z) \geq 1$, then by equations (3.1), (3.3) and (3.6), it follows that

$$G(z)e^{h(z+c)-h(z)} - 1 = e^{p(z)}. \tag{3.14}$$

Similarly as Case 1, we get equation (3.7) hold as well. Therefore, by equation (3.7), we obtain

$$\lambda(G(z)e^{h(z+c)-h(z)}) = \lambda(G(z)) \leq \sigma(G(z)) < m-1 = \sigma(G(z)e^{h(z+c)-h(z)}),$$

which means 0 is a Borel exceptional value. Clearly, we obtain that 1 and ∞ are two Borel exceptional values of $G(z)e^{h(z+c)-h(z)}$. Hence, we get the function $G(z)e^{h(z+c)-h(z)}$ have three Borel exceptional values, which is a contradiction. This completes the proof of Theorem 1.2.

4. PROOF OF THEOREM 1.1

From the assumption that $f(z)$ has two Borel exceptional values a and ∞ , $f(z)$ can be expressed as in the following form:

$$f(z) = a + \frac{U(z)}{V(z)}e^{\phi(z)}, \tag{4.1}$$

where $U(z)$, $V(z)$ are two non-zero entire functions, $\phi(z)$ is a non-constant polynomial of degree n . Similarly as Theorem 1.2, we get $U(z)$, $V(z)$ satisfy:

$$\lambda(f - a) = \lambda(U) = \sigma(U) < \sigma(f) = n, \quad \lambda\left(\frac{1}{f}\right) = \lambda(V) = \sigma(V) < \sigma(f),$$

and

$$T(r, U) = S(r, e^{\phi(z)}) = S(r, f), \quad T(r, V) = S(r, e^{\phi(z)}) = S(r, f).$$

Moreover, we get

$$\Delta_c f = \left(\frac{U(z+c)}{V(z+c)} e^{\phi(z+c)-\phi(z)} - \frac{U(z)}{V(z)} \right) e^{\phi(z)} = \psi(z) e^{\phi(z)}, \quad (4.2)$$

$$\frac{\Delta_c f - a}{f(z) - a} = e^{q(z)}, \quad (4.3)$$

and

$$\psi(z) e^{\phi(z)} - a = \frac{U(z)}{V(z)} e^{\phi(z)+q(z)},$$

where $q(z)$ is a polynomial.

Case 1. If $a \neq 0$, then by the above equation we obtain

$$N \left(r, \frac{1}{e^{\phi(z)} - \frac{a}{\psi(z)}} \right) \leq N \left(r, \frac{1}{U(z)} \right) = S(r, e^{\phi(z)})$$

when $\psi(z) \not\equiv 0$, which is a contradiction. If $\psi(z) \equiv 0$, then we have $\Delta_c f \equiv 0$ by (4.2), which contradicts the assumption $\Delta_c f \not\equiv 0$.

Case 2. If $a = 0$, then it follows that

$$\frac{\Delta_c f}{f} = \frac{U(z+c)V(z)}{U(z)V(z+c)} e^{\phi(z+c)-\phi(z)} - 1 = \omega(z) e^{\phi(z+c)-\phi(z)} - 1 = e^{q(z)}. \quad (4.4)$$

Similarly as Theorem 1.2, we conclude that $\sigma(\omega) < n-1 = \deg \phi(z) - 1$, which means that

$$T(r, \omega) = S(r, e^{\phi(z)}).$$

From equation (4.4), we have

$$N \left(r, \frac{1}{e^{q(z)} + 1} \right) = N \left(r, \frac{1}{\omega(z)} \right) = S(r, e^{\phi(z)}),$$

which is impossible when $q(z)$ is a non-constant polynomial. Hence, $q(z)$ is a constant. Let $q(z) = q$, then from equations (4.4), it follows that

$$\frac{U(z+c)V(z)}{U(z)V(z+c)} e^{\phi(z+c)-\phi(z)} = 1 + e^q.$$

Using a similar way as Case 1 of Theorem 1.2, we know that $\sigma(f) = 1$, and obtain $f(z) = Ae^{Bz}$ further. The conclusion follows.

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QUADRATIC ρ -FUNCTIONAL INEQUALITIES IN NON-ARCHIMEDEAN NORMED SPACES

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ABSTRACT. In this paper, we solve the quadratic ρ -functional inequalities

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \left\| \rho \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \right) \right\|, \quad (0.1)$$

where ρ is a fixed non-Archimedean number with $|\rho| < 1$, and

$$\left\| 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \right\| \leq \|\rho(f(x+y) + f(x-y) - 2f(x) - 2f(y))\|, \quad (0.2)$$

where ρ is a fixed non-Archimedean number with $|\rho| < \frac{1}{2}$.

Furthermore, we prove the Hyers-Ulam stability of the quadratic ρ -functional inequalities (0.1) and (0.2) in non-Archimedean Banach spaces and prove the Hyers-Ulam stability of quadratic ρ -functional equations associated with the quadratic ρ -functional inequalities (0.1) and (0.2) in non-Archimedean Banach spaces.

1. INTRODUCTION AND PRELIMINARIES

A *valuation* is a function $|\cdot|$ from a field K into $[0, \infty)$ such that 0 is the unique element having the 0 valuation, $|rs| = |r| \cdot |s|$ and the triangle inequality holds, i.e.,

$$|r + s| \leq |r| + |s|, \quad \forall r, s \in K.$$

A field K is called a *valued field* if K carries a valuation. The usual absolute values of \mathbb{R} and \mathbb{C} are examples of valuations.

Let us consider a valuation which satisfies a stronger condition than the triangle inequality. If the triangle inequality is replaced by

$$|r + s| \leq \max\{|r|, |s|\}, \quad \forall r, s \in K,$$

then the function $|\cdot|$ is called a *non-Archimedean valuation*, and the field is called a *non-Archimedean field*. Clearly $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$. A trivial example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except for 0 into 1 and $|0| = 0$.

Throughout this paper, we assume that the base field is a non-Archimedean field, hence call it simply a field.

Definition 1.1. ([12]) Let X be a vector space over a field K with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\| : X \rightarrow [0, \infty)$ is said to be a *non-Archimedean norm* if it satisfies the following conditions:

- (i) $\|x\| = 0$ if and only if $x = 0$;
- (ii) $\|rx\| = |r|\|x\|$ ($r \in K, x \in X$);
- (iii) the strong triangle inequality

$$\|x + y\| \leq \max\{\|x\|, \|y\|\}, \quad \forall x, y \in X$$

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holds. Then $(X, \| \cdot \|)$ is called a *non-Archimedean normed space*.

Definition 1.2. (i) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X . Then the sequence $\{x_n\}$ is called *Cauchy* if for a given $\varepsilon > 0$ there is a positive integer N such that

$$\|x_n - x_m\| \leq \varepsilon$$

for all $n, m \geq N$.

(ii) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X . Then the sequence $\{x_n\}$ is called *convergent* if for a given $\varepsilon > 0$ there are a positive integer N and an $x \in X$ such that

$$\|x_n - x\| \leq \varepsilon$$

for all $n \geq N$. Then we call $x \in X$ a limit of the sequence $\{x_n\}$, and denote by $\lim_{n \rightarrow \infty} x_n = x$.

(iii) If every Cauchy sequence in X converges, then the non-Archimedean normed space X is called a *non-Archimedean Banach space*.

The stability problem of functional equations originated from a question of Ulam [25] concerning the stability of group homomorphisms. Hyers [11] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [17] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [8] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

The functional equation $f(x + y) + f(x - y) = 2f(x) + 2f(y)$ is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The stability of quadratic functional equation was proved by Skof [24] for mappings $f : E_1 \rightarrow E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [5] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. The functional equation $2f\left(\frac{x+y}{2}\right) + 2\left(\frac{x-y}{2}\right) = f(x) + f(y)$ is called a *Jensen type quadratic equation*. The stability problems of various functional equations have been extensively investigated by a number of authors (see [1, 3, 4, 15, 16, 19, 20, 21, 22, 23, 26, 27]).

In [9], Gilányi showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(xy^{-1})\| \leq \|f(xy)\| \tag{1.1}$$

then f satisfies the Jordan-von Neumann functional equation $2f(x) + 2f(y) = f(xy) + f(xy^{-1})$. See also [18]. Gilányi [10] and Fechner [7] proved the Hyers-Ulam stability of the functional inequality (1.1). Park, Cho and Han [14] proved the Hyers-Ulam stability of additive functional inequalities. The stability problems of functional equations and inequalities have also been treated by many authors ([6, 13]).

In Section 2, we solve the quadratic ρ -functional inequality (0.1) and prove the Hyers-Ulam stability of the quadratic ρ -functional inequality (0.1) in non-Archimedean Banach spaces. We moreover prove the Hyers-Ulam stability of a quadratic ρ -functional equation associated with the quadratic ρ -functional inequality (0.1) in non-Archimedean Banach spaces.

In Section 3, we solve the quadratic ρ -functional inequality (0.2) and prove the Hyers-Ulam stability of the quadratic ρ -functional inequality (0.2) in non-Archimedean Banach spaces. We moreover prove the Hyers-Ulam stability of a quadratic ρ -functional equation associated with the quadratic ρ -functional inequality (0.2) in non-Archimedean Banach spaces.

Throughout this paper, assume that X is a non-Archimedean normed space and that Y is a non-Archimedean Banach space. Let $|2| \neq 1$.

2. QUADRATIC ρ -FUNCTIONAL INEQUALITY (0.1) IN NON-ARCHIMEDEAN NORMED SPACES

Throughout this section, assume that ρ is a fixed non-Archimedean number with $|\rho| < 1$.

QUADRATIC ρ -FUNCTIONAL INEQUALITIES

In this section, we solve the quadratic ρ -functional inequality (0.1) in non-Archimedean normed spaces.

Lemma 2.1. *A mapping $f : X \rightarrow Y$ satisfies*

$$\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \leq \left\| \rho \left(2f \left(\frac{x + y}{2} \right) + 2f \left(\frac{x - y}{2} \right) - f(x) - f(y) \right) \right\| \quad (2.1)$$

for all $x, y \in X$ if and only if $f : X \rightarrow Y$ is quadratic.

Proof. Assume that $f : X \rightarrow Y$ satisfies (2.1).

Letting $x = y = 0$ in (2.1), we get $\|2f(0)\| \leq |\rho| \|2f(0)\|$. So $f(0) = 0$.

Letting $y = x$ in (2.1), we get $\|f(2x) - 4f(x)\| \leq 0$ and so $f(2x) = 4f(x)$ for all $x \in X$. Thus

$$f \left(\frac{x}{2} \right) = \frac{1}{4} f(x) \quad (2.2)$$

for all $x \in X$.

It follows from (2.1) and (2.2) that

$$\begin{aligned} \|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| &\leq \left\| \rho \left(2f \left(\frac{x + y}{2} \right) + 2f \left(\frac{x - y}{2} \right) - f(x) - f(y) \right) \right\| \\ &= \frac{|\rho|}{2} \|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \end{aligned}$$

and so

$$f(x + y) + f(x - y) = 2f(x) + 2f(y)$$

for all $x, y \in X$.

The converse is obviously true. □

Corollary 2.2. *A mapping $f : X \rightarrow Y$ satisfies*

$$f(x + y) + f(x - y) - 2f(x) - 2f(y) = \rho \left(2f \left(\frac{x + y}{2} \right) + 2f \left(\frac{x - y}{2} \right) - f(x) - f(y) \right) \quad (2.3)$$

for all $x, y \in X$ if and only if $f : X \rightarrow Y$ is quadratic.

The functional equation (2.3) is called a *quadratic ρ -functional equation*.

We prove the Hyers-Ulam stability of the quadratic ρ -functional inequality (2.1) in non-Archimedean Banach spaces.

Theorem 2.3. *Let $r < 2$ and θ be nonnegative real numbers, and let $f : X \rightarrow Y$ be a mapping such that*

$$\begin{aligned} &\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \\ &\leq \left\| \rho \left(2f \left(\frac{x + y}{2} \right) + 2f \left(\frac{x - y}{2} \right) - f(x) - f(y) \right) \right\| + \theta(\|x\|^r + \|y\|^r) \end{aligned} \quad (2.4)$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $h : X \rightarrow Y$ such that

$$\|f(x) - h(x)\| \leq \frac{2\theta}{|2|^r} \|x\|^r \quad (2.5)$$

for all $x \in X$.

Proof. Letting $x = y = 0$ in (2.4), we get $\|2f(0)\| \leq |\rho| \|2f(0)\|$. So $f(0) = 0$.

Letting $y = x$ in (2.4), we get

$$\|f(2x) - 4f(x)\| \leq 2\theta \|x\|^r \quad (2.6)$$

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for all $x \in X$. So $\|f(x) - 4f(\frac{x}{2})\| \leq \frac{2}{|2|^r} \theta \|x\|^r$ for all $x \in X$. Hence

$$\begin{aligned} & \left\| 4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right) \right\| \tag{2.7} \\ & \leq \max \left\{ \left\| 4^l f\left(\frac{x}{2^l}\right) - 4^{l+1} f\left(\frac{x}{2^{l+1}}\right) \right\|, \dots, \left\| 4^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 4^m f\left(\frac{x}{2^m}\right) \right\| \right\} \\ & = \max \left\{ |4|^l \left\| f\left(\frac{x}{2^l}\right) - 4f\left(\frac{x}{2^{l+1}}\right) \right\|, \dots, |4|^{m-1} \left\| f\left(\frac{x}{2^{m-1}}\right) - 4f\left(\frac{x}{2^m}\right) \right\| \right\} \\ & \leq \max \left\{ \frac{|4|^l}{|2|^{rl+1}}, \dots, \frac{|4|^{m-1}}{|2|^{r(m-1)+1}} \right\} 2\theta \|x\|^r = \frac{2\theta}{|2|^{(r-2)l+1}} \|x\|^r \end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.7) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $h : X \rightarrow Y$ by

$$h(x) := \lim_{n \rightarrow \infty} 4^n f\left(\frac{x}{2^n}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \rightarrow \infty$ in (2.7), we get (2.5).

It follows from (2.4) that

$$\begin{aligned} & \|h(x+y) + h(x-y) - 2h(x) - 2h(y)\| \\ & = \lim_{n \rightarrow \infty} |4|^n \left\| f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right) \right\| \\ & \leq \lim_{n \rightarrow \infty} |4|^n |\rho| \left\| 2f\left(\frac{x+y}{2^{n+1}}\right) + 2f\left(\frac{x-y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) \right\| + \lim_{n \rightarrow \infty} \frac{|4|^n \theta}{|2|^{nr}} (\|x\|^r + \|y\|^r) \\ & = |\rho| \left\| 2h\left(\frac{x+y}{2}\right) + 2h\left(\frac{x-y}{2}\right) - h(x) - h(y) \right\| \end{aligned}$$

for all $x, y \in X$. So

$$\|h(x+y) + h(x-y) - 2h(x) - 2h(y)\| \leq \left\| \rho \left(2h\left(\frac{x+y}{2}\right) + 2h\left(\frac{x-y}{2}\right) - h(x) - h(y) \right) \right\|$$

for all $x, y \in X$. By Lemma 2.1, the mapping $h : X \rightarrow Y$ is quadratic.

Now, let $T : X \rightarrow Y$ be another quadratic mapping satisfying (2.5). Then we have

$$\begin{aligned} \|h(x) - T(x)\| & = \left\| 4^q h\left(\frac{x}{2^q}\right) - 4^q T\left(\frac{x}{2^q}\right) \right\| \\ & \leq \max \left\{ \left\| 4^q h\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right) \right\|, \left\| 4^q T\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right) \right\| \right\} \leq \frac{2\theta}{|2|^{(r-2)q+1}} \|x\|^r, \end{aligned}$$

which tends to zero as $q \rightarrow \infty$ for all $x \in X$. So we can conclude that $h(x) = T(x)$ for all $x \in X$. This proves the uniqueness of h . Thus the mapping $h : X \rightarrow Y$ is a unique quadratic mapping satisfying (2.5). \square

Theorem 2.4. *Let $r > 2$ and θ be positive real numbers, and let $f : X \rightarrow Y$ be a mapping satisfying (2.4). Then there exists a unique quadratic mapping $h : X \rightarrow Y$ such that*

$$\|f(x) - h(x)\| \leq \frac{2\theta}{|4|} \|x\|^r \tag{2.8}$$

for all $x \in X$.

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Proof. It follows from (2.6) that

$$\left\| f(x) - \frac{1}{4}f(2x) \right\| \leq \frac{2\theta}{|4|} \|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned} & \left\| \frac{1}{4^l}f(2^l x) - \frac{1}{4^m}f(2^m x) \right\| & (2.9) \\ & \leq \max \left\{ \left\| \frac{1}{4^l}f(2^l x) - \frac{1}{4^{l+1}}f(2^{l+1} x) \right\|, \dots, \left\| \frac{1}{4^{m-1}}f(2^{m-1} x) - \frac{1}{4^m}f(2^m x) \right\| \right\} \\ & = \max \left\{ \frac{1}{|4|^l} \left\| f(2^l x) - \frac{1}{4}f(2^{l+1} x) \right\|, \dots, \frac{1}{|4|^{m-1}} \left\| f(2^{m-1} x) - \frac{1}{4}f(2^m x) \right\| \right\} \\ & \leq \max \left\{ \frac{|2|^{rl}}{|4|^{l+1}}, \dots, \frac{|2|^{r(m-1)}}{|4|^{(m-1)+1}} \right\} 2\theta \|x\|^r = \frac{2\theta}{|2|^{(2-r)l+2}} \|x\|^r \end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.9) that the sequence $\{\frac{1}{4^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n}f(2^n x)\}$ converges. So one can define the mapping $h : X \rightarrow Y$ by

$$h(x) := \lim_{n \rightarrow \infty} \frac{1}{4^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \rightarrow \infty$ in (2.9), we get (2.8).

The rest of the proof is similar to the proof of Theorem 2.3. □

Let $A(x, y) := f(x + y) + f(x - y) - 2f(x) - 2f(y)$ and

$$B(x, y) := \rho \left(2f \left(\frac{x + y}{2} \right) + 2f \left(\frac{x - y}{2} \right) - f(x) - f(y) \right)$$

for all $x, y \in X$.

For $x, y \in X$ with $\|A(x, y)\| \leq \|B(x, y)\|$,

$$\|A(x, y)\| - \|B(x, y)\| \leq \|A(x, y) - B(x, y)\|.$$

For $x, y \in X$ with $\|A(x, y)\| > \|B(x, y)\|$,

$$\begin{aligned} \|A(x, y)\| &= \|A(x, y) - B(x, y) + B(x, y)\| \\ &\leq \max\{\|A(x, y) - B(x, y)\|, \|B(x, y)\|\} \\ &= \|A(x, y) - B(x, y)\| \\ &\leq \|A(x, y) - B(x, y)\| + \|B(x, y)\|, \end{aligned}$$

since $\|A(x, y)\| > \|B(x, y)\|$. So we have

$$\begin{aligned} & \|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| - \left\| \rho \left(2f \left(\frac{x + y}{2} \right) + 2f \left(\frac{x - y}{2} \right) - f(x) - f(y) \right) \right\| \\ & \leq \left\| f(x + y) + f(x - y) - 2f(x) - 2f(y) - \rho \left(2f \left(\frac{x + y}{2} \right) + 2f \left(\frac{x - y}{2} \right) - f(x) - f(y) \right) \right\|. \end{aligned}$$

As corollaries of Theorems 2.3 and 2.4, we obtain the Hyers-Ulam stability results for the quadratic ρ -functional equation (2.3) in non-Archimedean Banach spaces.

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Corollary 2.5. *Let $r < 2$ and θ be nonnegative real numbers, and let $f : X \rightarrow Y$ be a mapping such that*

$$\begin{aligned} & \|f(x+y) + f(x-y) - 2f(x) - 2f(y) \\ & - \rho \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \right) \| \leq \theta(\|x\|^r + \|y\|^r) \end{aligned} \tag{2.10}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $h : X \rightarrow Y$ satisfying (2.5).

Corollary 2.6. *Let $r > 2$ and θ be positive real numbers, and let $f : X \rightarrow Y$ be a mapping satisfying (2.10). Then there exists a unique quadratic mapping $h : X \rightarrow Y$ satisfying (2.8).*

3. QUADRATIC ρ -FUNCTIONAL INEQUALITY (0.2)

Throughout this section, assume that ρ is a fixed non-Archimedean number with $|\rho| < \frac{1}{2}$.

In this section, we solve the quadratic ρ -functional inequality (0.2) in non-Archimedean normed spaces.

Lemma 3.1. *A mapping $f : X \rightarrow Y$ satisfies*

$$\left\| 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \right\| \leq \|\rho(f(x+y) + f(x-y) - 2f(x) - 2f(y))\| \tag{3.1}$$

for all $x, y \in X$ if and only if $f : X \rightarrow Y$ is quadratic.

Proof. Assume that $f : X \rightarrow Y$ satisfies (3.1).

Letting $x = y = 0$ in (3.1), we get $\|2f(0)\| \leq |\rho|\|2f(0)\|$. So $f(0) = 0$.

Letting $y = 0$ in (3.1), we get

$$\left\| 4f\left(\frac{x}{2}\right) - f(x) \right\| \leq 0 \tag{3.2}$$

and so $f\left(\frac{x}{2}\right) = \frac{1}{4}f(x)$ for all $x \in X$.

It follows from (3.1) and (3.2) that

$$\begin{aligned} \frac{1}{2}\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| &= \left\| 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \right\| \\ &\leq |\rho|\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \end{aligned}$$

and so

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in X$.

The converse is obviously true. □

Corollary 3.2. *A mapping $f : X \rightarrow Y$ satisfies*

$$2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) = \rho(f(x+y) + f(x-y) - 2f(x) - 2f(y)) \tag{3.3}$$

for all $x, y \in X$ and only if $f : X \rightarrow Y$ is quadratic.

The functional equation (3.3) is called a *quadratic ρ -functional equation*.

We prove the Hyers-Ulam stability of the quadratic ρ -functional inequality (3.1) in non-Archimedean Banach spaces.

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Theorem 3.3. *Let $r < 2$ and θ be nonnegative real numbers, and let $f : X \rightarrow Y$ be a mapping such that*

$$\begin{aligned} & \|2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\| \\ & \leq \|\rho(f(x+y) + f(x-y) - 2f(x) - 2f(y))\| + \theta(\|x\|^r + \|y\|^r) \end{aligned} \tag{3.4}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $h : X \rightarrow Y$ such that

$$\|f(x) - h(x)\| \leq \theta\|x\|^r \tag{3.5}$$

for all $x \in X$.

Proof. Letting $x = y = 0$ in (3.4), we get $\|2f(0)\| \leq |\rho|\|2f(0)\|$. So $f(0) = 0$.

Letting $y = 0$ in (3.4), we get

$$\left\|4f\left(\frac{x}{2}\right) - f(x)\right\| \leq \theta\|x\|^r \tag{3.6}$$

for all $x \in X$. So

$$\begin{aligned} & \left\|4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\| \\ & \leq \max \left\{ \left\|4^l f\left(\frac{x}{2^l}\right) - 4^{l+1} f\left(\frac{x}{2^{l+1}}\right)\right\|, \dots, \left\|4^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\| \right\} \\ & = \max \left\{ |4|^l \left\|f\left(\frac{x}{2^l}\right) - 4f\left(\frac{x}{2^{l+1}}\right)\right\|, \dots, |4|^{m-1} \left\|f\left(\frac{x}{2^{m-1}}\right) - 4f\left(\frac{x}{2^m}\right)\right\| \right\} \\ & \leq \max \left\{ \frac{|4|^l}{|2|^{rl}}, \dots, \frac{|4|^{m-1}}{|2|^{r(m-1)}} \right\} \theta\|x\|^r = \frac{\theta}{|2|^{(r-2)l}} \|x\|^r \end{aligned} \tag{3.7}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.7) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $h : X \rightarrow Y$ by

$$h(x) := \lim_{n \rightarrow \infty} 4^n f\left(\frac{x}{2^n}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \rightarrow \infty$ in (3.7), we get (3.5).

The rest of the proof is similar to the proof of Theorem 2.3. □

Theorem 3.4. *Let $r > 2$ and θ be positive real numbers, and let $f : X \rightarrow Y$ be a mapping satisfying (3.4). Then there exists a unique quadratic mapping $h : X \rightarrow Y$ such that*

$$\|f(x) - h(x)\| \leq \frac{|2|^r \theta}{|4|} \|x\|^r \tag{3.8}$$

for all $x \in X$.

Proof. It follows from (3.6) that

$$\left\|f(x) - \frac{1}{4}f(2x)\right\| \leq \frac{|2|^r \theta}{|4|} \|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned} & \left\| \frac{1}{4^l} f(2^l x) - \frac{1}{4^m} f(2^m x) \right\| \tag{3.9} \\ & \leq \max \left\{ \left\| \frac{1}{4^l} f(2^l x) - \frac{1}{4^{l+1}} f(2^{l+1} x) \right\|, \dots, \left\| \frac{1}{4^{m-1}} f(2^{m-1} x) - \frac{1}{4^m} f(2^m x) \right\| \right\} \\ & = \max \left\{ \frac{1}{|4|^l} \left\| f(2^l x) - \frac{1}{4} f(2^{l+1} x) \right\|, \dots, \frac{1}{|4|^{m-1}} \left\| f(2^{m-1} x) - \frac{1}{4} f(2^m x) \right\| \right\} \\ & \leq \max \left\{ \frac{|2|^{rl}}{|4|^{l+1}}, \dots, \frac{|2|^{r(m-1)}}{|4|^{(m-1)+1}} \right\} |2|^r \theta \|x\|^r = \frac{|2|^r \theta}{|2|^{(2-r)l+2}} \|x\|^r \end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.9) that the sequence $\{\frac{1}{4^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n} f(2^n x)\}$ converges. So one can define the mapping $h : X \rightarrow Y$ by

$$h(x) := \lim_{n \rightarrow \infty} \frac{1}{4^n} f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \rightarrow \infty$ in (3.9), we get (3.8).

The rest of the proof is similar to the proofs of Theorems 2.3 and 3.3. □

Let $A(x, y) := 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)$ and

$$B(x, y) := \rho(f(x+y) + f(x-y) - 2f(x) - 2f(y))$$

for all $x, y \in X$.

For $x, y \in X$ with $\|A(x, y)\| \leq \|B(x, y)\|$,

$$\|A(x, y)\| - \|B(x, y)\| \leq \|A(x, y) - B(x, y)\|.$$

For $x, y \in X$ with $\|A(x, y)\| > \|B(x, y)\|$,

$$\begin{aligned} \|A(x, y)\| &= \|A(x, y) - B(x, y) + B(x, y)\| \\ &\leq \max\{\|A(x, y) - B(x, y)\|, \|B(x, y)\|\} \\ &= \|A(x, y) - B(x, y)\| \\ &\leq \|A(x, y) - B(x, y)\| + \|B(x, y)\|, \end{aligned}$$

since $\|A(x, y)\| > \|B(x, y)\|$. So we have

$$\begin{aligned} & \left\| 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \right\| - \left\| \rho(f(x+y) + f(x-y) - 2f(x) - 2f(y)) \right\| \\ & \leq \left\| 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) - \rho(f(x+y) + f(x-y) - 2f(x) - 2f(y)) \right\|. \end{aligned}$$

As corollaries of Theorems 3.3 and 3.4, we obtain the Hyers-Ulam stability results for the quadratic ρ -functional equation (3.3) in non-Archimedean Banach spaces.

Corollary 3.5. *Let $r < 2$ and θ be nonnegative real numbers, and let $f : X \rightarrow Y$ be a mapping such that*

$$\begin{aligned} & \left\| 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \right. \\ & \quad \left. - \rho(f(x+y) + f(x-y) - 2f(x) - 2f(y)) \right\| \leq \theta(\|x\|^r + \|y\|^r) \end{aligned} \tag{3.10}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $h : X \rightarrow Y$ satisfying (3.5).

Corollary 3.6. *Let $r > 2$ and θ be positive real numbers, and let $f : X \rightarrow Y$ be a mapping satisfying (3.10). Then there exists a unique quadratic mapping $h : X \rightarrow Y$ satisfying (3.8).*

QUADRATIC ρ -FUNCTIONAL INEQUALITIES

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