Volume 31, Number 4 December, 2023 ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of Computational

Analysis and

Applications

 EUDOXUS PRESS,LLC

 Journal of Computational Analysis and Applications ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE SCOPE OF THE JOURNAL An international publication of Eudoxus Press, LLC (published quarterly) www.eudoxuspress.com. Editor in Chief: George Anastassiou Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152-3240, U.S.A [ganastss@memphis.edu,](mailto:ganastss@memphis.edu) ganastss2@gmail.com [http://web0.msci.memphis.edu/~ganastss/jocaaa/](http://web0.msci.memphis.edu/%7Eganastss/jocaaa/)

The main purpose of "J.Computational Analysis and Applications" is to publish high quality research articles from all subareas of Computational Mathematical Analysis and its many potential applications and connections to other areas of Mathematical Sciences. Any paper whose approach and proofs are computational,using methods from Mathematical Analysis in the broadest sense is suitable and welcome for consideration in our journal, except from Applied Numerical Analysis articles. Also plain word articles without formulas and proofs are excluded. The list of possibly connected mathematical areas with this publication includes, but is not restricted to: Applied Analysis, Applied Functional Analysis, Approximation Theory, Asymptotic Analysis, Difference Equations, Differential Equations, Partial Differential Equations, Fourier Analysis, Fractals, Fuzzy Sets, Harmonic Analysis, Inequalities, Integral Equations, Measure Theory, Moment Theory, Neural Networks, Numerical Functional Analysis, Potential Theory, Probability Theory, Real and Complex Analysis, Signal Analysis, Special Functions, Splines, Stochastic Analysis, Stochastic Processes, Summability, Tomography, Wavelets, any combination of the above, e.t.c.

 "J.Computational Analysis and Applications" is a peer-reviewed Journal. See the instructions for preparation and submission **of articles to JOCAAA.**

Journal of Computational Analysis and Applications(JoCAAA) is published by **EUDOXUS PRESS,LLC**,1424 Beaver Trail

Drive,Cordova,TN38016,USA,anastassioug@yahoo.com

http://www.eudoxuspress.com. **Annual Subscription Prices**:For USA and Canada,Institutional:Print \$500, Electronic OPEN ACCESS. Individual:Print \$250. For any other part of the world add \$150 more(handling and postages) to the above prices for Print. No credit card payments.

Copyright©2023 by Eudoxus Press,LLC,all rights reserved.JoCAAA is printed in USA. **JoCAAA is reviewed and abstracted by Elsevier-Scopus, available also via EBSCO publishing and EBSCO.**

It is strictly prohibited the reproduction and transmission of any part of JoCAAA and in any form and by any means without the written permission of the publisher.It is only allowed to educators to Xerox articles for educational purposes.The publisher assumes no responsibility for the content of published papers.

Editorial Board

Associate Editors of Journal of Computational Analysis and Applications

Francesco Altomare

Dipartimento di Matematica Universita' di Bari Via E.Orabona, 4 70125 Bari, ITALY Tel+39-080-5442690 office +39-080-3944046 home +39-080-5963612 Fax altomare@dm.uniba.it Approximation Theory, Functional Analysis, Semigroups and Partial Differential Equations, Positive Operators.

Ravi P. Agarwal

Department of Mathematics Texas A&M University - Kingsville 700 University Blvd. Kingsville, TX 78363-8202 tel: 361-593-2600 Agarwal@tamuk.edu Differential Equations, Difference Equations, Inequalities

George A. Anastassiou

Department of Mathematical Sciences The University of Memphis Memphis, TN 38152,U.S.A Tel.901-678-3144 e-mail: ganastss@memphis.edu Approximation Theory, Real Analysis, Wavelets, Neural Networks, Probability, Inequalities.

J. Marshall Ash

Department of Mathematics De Paul University 2219 North Kenmore Ave. Chicago, IL 60614-3504 773-325-4216 e-mail: mash@math.depaul.edu Real and Harmonic Analysis

Dumitru Baleanu

Department of Mathematics and Computer Sciences, Cankaya University, Faculty of Art and Sciences, 06530 Balgat, Ankara,

Turkey, dumitru@cankaya.edu.tr Fractional Differential Equations Nonlinear Analysis, Fractional Dynamics

Carlo Bardaro

Dipartimento di Matematica e Informatica Universita di Perugia Via Vanvitelli 1 06123 Perugia, ITALY TEL+390755853822 +390755855034 FAX+390755855024 E-mail carlo.bardaro@unipg.it Web site: http://www.unipg.it/~bardaro/ Functional Analysis and Approximation Theory, Signal Analysis, Measure Theory, Real Analysis.

Martin Bohner

Department of Mathematics and Statistics, Missouri S&T Rolla, MO 65409-0020, USA bohner@mst.edu web.mst.edu/~bohner Difference equations, differential equations, dynamic equations on time scale, applications in economics, finance, biology.

Jerry L. Bona

Department of Mathematics The University of Illinois at Chicago 851 S. Morgan St. CS 249 Chicago, IL 60601 e-mail:bona@math.uic.edu Partial Differential Equations, Fluid Dynamics

Luis A. Caffarelli

Department of Mathematics The University of Texas at Austin Austin, Texas 78712-1082 512-471-3160 e-mail: caffarel@math.utexas.edu Partial Differential Equations

George Cybenko

Thayer School of Engineering Dartmouth College 8000 Cummings Hall, Hanover, NH 03755-8000 603-646-3843 (X 3546 Secr.) e-mail:george.cybenko@dartmouth.edu Approximation Theory and Neural Networks

Sever S. Dragomir

School of Computer Science and Mathematics, Victoria University, PO Box 14428, Melbourne City, MC 8001, AUSTRALIA Tel. +61 3 9688 4437 Fax +61 3 9688 4050 e-mail: sever.dragomir@vu.edu.au Inequalities, Functional Analysis, Numerical Analysis, Approximations, Information Theory, Stochastics.

Oktay Duman

TOBB University of Economics and Technology, Department of Mathematics, TR-06530, Ankara, Turkey, e-mail: oduman@etu.edu.tr Classical Approximation Theory, Summability Theory, Statistical Convergence and its Applications

J .A. Goldstein

Department of Mathematical Sciences The University of Memphis Memphis, TN 38152 901-678-3130 e-mail: jgoldste@memphis.edu Partial Differential Equations, Semigroups of Operators

H. H. Gonska

Department of Mathematics University of Duisburg Duisburg, D-47048 Germany 011-49-203-379-3542 e-mail: heiner.gonska@uni-due.de Approximation Theory, Computer Aided Geometric Design

John R. Graef

Department of Mathematics University of Tennessee at Chattanooga Chattanooga, TN 37304 USA

e-mail: John-Graef@utc.edu Ordinary and functional differential equations, difference equations, impulsive systems, differential inclusions, dynamic equations on time scales, control theory and their applications

Weimin Han

Department of Mathematics University of Iowa Iowa City, IA 52242-1419 319-335-0770 e-mail: whan@math.uiowa.edu Numerical analysis, Finite element method, Numerical PDE, Variational inequalities, Computational mechanics

Tian-Xiao He

Department of Mathematics and Computer Science P.O. Box 2900, Illinois Wesleyan University Bloomington, IL 61702-2900, USA Tel (309)556-3089 Fax (309)556-3864 e-mail: the@iwu.edu Approximations, Wavelet, Integration Theory, Numerical Analysis, Analytic Combinatorics

Margareta Heilmann

Faculty of Mathematics and Natural Sciences, University of Wuppertal Gaußstraße 20 D-42119 Wuppertal, Germany, heilmann@math.uni-wuppertal.de Approximation Theory (Positive Linear Operators)

Xing-Biao Hu

Institute of Computational Mathematics AMSS, Chinese Academy of Sciences Beijing, 100190, CHINA e-mail: hxb@lsec.cc.ac.cn Computational Mathematics

Seda Karateke

Department of Computer Engineering, Faculty of Engineering, Istanbul Topkapi University, Istanbul, Zeytinburnu 34087, Turkey e-mail: sedakarateke@topkapi.edu.tr Approximation Theory, Neural Networks

Jong Kyu Kim

Department of Mathematics Kyungnam University Masan Kyungnam,631-701,Korea Tel 82-(55)-249-2211 Fax $82 - (55) - 243 - 8609$ e-mail: jongkyuk@kyungnam.ac.kr Nonlinear Functional Analysis, Variational Inequalities, Nonlinear Ergodic Theory, ODE, PDE, Functional Equations.

Robert Kozma

Department of Mathematical Sciences The University of Memphis Memphis, TN 38152, USA e-mail: rkozma@memphis.edu Neural Networks, Reproducing Kernel Hilbert Spaces, Neural Percolation Theory

Mustafa Kulenovic

Department of Mathematics University of Rhode Island Kingston, RI 02881, USA e-mail: kulenm@math.uri.edu Differential and Difference Equations

Burkhard Lenze

Fachbereich Informatik Fachhochschule Dortmund University of Applied Sciences Postfach 105018 D-44047 Dortmund, Germany e-mail: lenze@fh-dortmund.de D-44047 Dortmund, Germany
e-mail: lenze@fh-dortmund.de
Real Networks, Fourier Analysis,
Approximation Theory Approximation Theory

Alina Alb Lupas

Department of Mathematics and Computer Science Faculty of Informatics University of Oradea 2 Universitatii Street, 410087 Oradea, Romania e-mail: alblupas@gmail.com e-mail: dalb@uoradea.ro Complex Analysis, Topological Algebra, Mathematical Analysis

Razvan A. Mezei

Computer Science Department Hal and Inge Marcus School of Engineering Saint Martin's University Lacey, WA 98503, USA

e-mail: RMezei@stmartin.edu Numerical Approximation, Fractional Inequalities.

Hrushikesh N. Mhaskar

Department Of Mathematics California State University Los Angeles, CA 90032 626-914-7002 e-mail: hmhaska@gmail.com Orthogonal Polynomials, Approximation Theory, Splines, Wavelets, Neural Networks

Ram N. Mohapatra

Department of Mathematics University of Central Florida Orlando, FL 32816-1364 tel.407-823-5080 e-mail: ram.mohapatra@ucf.edu Real and Complex Analysis, Approximation Th., Fourier Analysis, Fuzzy Sets and Systems

Gaston M. N'Guerekata

Department of Mathematics Morgan State University Baltimore, MD 21251, USA tel: 1-443-885-4373 Fax 1-443-885-8216 Gaston.N'Guerekata@morgan.edu nguerekata@aol.com Nonlinear Evolution Equations, Abstract Harmonic Analysis, Fractional Differential Equations, Almost Periodicity & Almost Automorphy

M.Zuhair Nashed

Department Of Mathematics University of Central Florida PO Box 161364 Orlando, FL 32816-1364 e-mail: znashed@mail.ucf.edu Inverse and Ill-Posed problems, Numerical Functional Analysis, Integral Equations, Optimization, Signal Analysis

Mubenga N. Nkashama

Department OF Mathematics University of Alabama at Birmingham Birmingham, AL 35294-1170 205-934-2154 e-mail: nkashama@math.uab.edu Ordinary Differential Equations, Partial Differential Equations

Vassilis Papanicolaou

Department of Mathematics National Technical University of Athens Zografou campus, 157 80 Athens, Greece tel: +30(210) 772 1722 Fax +30(210) 772 1775 e-mail: papanico@math.ntua.gr Partial Differential Equations, Probability

Choonkil Park

Department of Mathematics Hanyang University Seoul 133-791 S. Korea, e-mail: baak@hanyang.ac.kr Functional Equations

Svetlozar (Zari) Rachev,

Professor of Finance, College of Business, and Director of Quantitative Finance Program, Department of Applied Mathematics & Statistics Stonybrook University 312 Harriman Hall, Stony Brook, NY 11794-3775 tel: +1-631-632-1998, svetlozar.rachev@stonybrook.edu

Alexander G. Ramm

Mathematics Department Kansas State University Manhattan, KS 66506-2602 e-mail: ramm@math.ksu.edu Inverse and Ill-posed Problems, Scattering Theory, Operator Theory, Theoretical Numerical Analysis, Wave Propagation, Signal Processing and Tomography

Tomasz Rychlik

Polish Academy of Sciences Instytut Matematyczny PAN 00-956 Warszawa, skr. poczt. 21 ul. Śniadeckich 8 Poland e-mail: trychlik@impan.pl Mathematical Statistics, Probabilistic Inequalities

Boris Shekhtman

Department of Mathematics University of South Florida Tampa, FL 33620, USA

Tel 813-974-9710 e-mail: shekhtma@usf.edu Approximation Theory, Banach spaces, Classical Analysis

T. E. Simos

Department of Computer Science and Technology Faculty of Sciences and Technology University of Peloponnese GR-221 00 Tripolis, Greece Postal Address: 26 Menelaou St. Anfithea - Paleon Faliron GR-175 64 Athens, Greece e-mail: tsimos@mail.ariadne-t.gr Numerical Analysis

Jagdev Singh

JECRC University, Jaipur, India jagdevsinghrathore@gmail.com Fractional Calculus, Mathematical Modelling, Special Functions, Numerical Methods

H. M. Srivastava

Department of Mathematics and Statistics University of Victoria Victoria, British Columbia V8W 3R4 Canada tel.250-472-5313; office,250-477- 6960 home, fax 250-721-8962 e-mail: harimsri@math.uvic.ca Real and Complex Analysis, Fractional Calculus and Appl., Integral Equations and Transforms, Higher Transcendental Functions and Appl.,q-Series and q-Polynomials, Analytic Number Th.

I. P. Stavroulakis

Department of Mathematics University of Ioannina 451-10 Ioannina, Greece e-mail: ipstav@cc.uoi.gr Differential Equations Phone +3-065-109-8283

Jessada Tariboon

Department of Mathematics King Mongut's University of Technology N. Bangkok 1518 Pracharat 1 Rd., Wongsawang, Bangsue, Bangkok, Thailand 10800 e-mail: jessada.t@sci.kmutnb.ac.th Time scales

Differential/Difference Equations, Fractional Differential Equations

Manfred Tasche

Department of Mathematics University of Rostock D-18051 Rostock, Germany manfred.tasche@mathematik.unirostock.de Numerical Fourier Analysis, Fourier Analysis, Harmonic Analysis, Signal Analysis, Spectral Methods, Wavelets, Splines, Approximation Theory

Juan J. Trujillo

University of La Laguna Departamento de Analisis Matematico C/Astr.Fco.Sanchez s/n 38271. LaLaguna. Tenerife. SPAIN Tel/Fax 34-922-318209 e-mail: Juan.Trujillo@ull.es Fractional: Differential Equations-Operators-Fourier Transforms, Special functions, Approximations, and Applications

Xiao-Jun Yang

State Key Laboratory for Geomechanics and Deep Underground Engineering, China University of Mining and Technology, Xuzhou 221116, China Local Fractional Calculus and Applications, Fractional Calculus and Applications, General Fractional Calculus and Applications, Variable-order Calculus and Applications, Viscoelasticity and Computational methods for Mathematical Physics.dyangxiaojun@163.com

Xiang Ming Yu

Department of Mathematical Sciences Southwest Missouri State University Springfield, MO 65804-0094 417-836-5931 e-mail: xmy944f@missouristate.edu Classical Approximation Theory, Wavelets

Richard A. Zalik

Department of Mathematics Auburn University Auburn University, AL 36849-5310 USA. Tel 334-844-6557 office Fax 334-844-6555 e-mail: zalik@auburn.edu Approximation Theory, Chebychev Systems, Wavelet Theory

Ahmed I. Zayed

Department of Mathematical Sciences DePaul University 2320 N. Kenmore Ave. Chicago, IL 60614-3250 773-325-7808 e-mail: azayed@condor.depaul.edu Shannon sampling theory, Harmonic analysis and wavelets, Special functions and orthogonal polynomials, Integral transforms

Ding-Xuan Zhou

Department Of Mathematics City University of Hong Kong 83 Tat Chee Avenue Kowloon, Hong Kong 852-2788 9708,Fax:852-2788 8561 e-mail: mazhou@cityu.edu.hk Approximation Theory, Spline functions, Wavelets

Xin-long Zhou

Fachbereich Mathematik, Fachgebiet Informatik Gerhard-Mercator-Universitat Duisburg Lotharstr.65, D-47048 Duisburg, Germany e-mail:Xzhou@informatik.uniduisburg.de Fourier Analysis, Computer-Aided Geometric Design, Computational Complexity, Multivariate Approximation Theory, Approximation and Interpolation Theory

Instructions to Contributors Journal of Computational Analysis and Applications

 An international publication of Eudoxus Press, LLC, of TN.

Editor in Chief: George Anastassiou

Department of Mathematical Sciences University of Memphis Memphis, TN 38152-3240, U.S.A.

1. Manuscripts files in Latex and PDF and in English, should be submitted via email to the Editor-in-Chief:

 Prof.George A. Anastassiou Department of Mathematical Sciences The University of Memphis Memphis,TN 38152, USA. Tel. 901.678.3144 e-mail: [ganastss@memphis.edu](mailto:ganastss@memphis.edu?subject=JCAAM%20inquirey)

Authors may want to recommend an associate editor the most related to the submission to possibly handle it.

 Also authors may want to submit a list of six possible referees, to be used in case we cannot find related referees by ourselves.

2. Manuscripts should be typed using any of TEX,LaTEX,AMS-TEX,or AMS-LaTEX and according to EUDOXUS PRESS, LLC. LATEX STYLE FILE. (Click [HERE](http://www.msci.memphis.edu/%7Eganastss/jcaam/EUDOXStyle.tex) to save a copy of the style file.)They should be carefully prepared in all respects. Submitted articles should be brightly typed (not dot-matrix), double spaced, in ten point type size and in 8(1/2)x11 inch area per page. Manuscripts should have generous margins on all sides and should not exceed 24 pages.

3. Submission is a representation that the manuscript has not been published previously in this or any other similar form and is not currently under consideration for publication elsewhere. A statement transferring from the authors(or their employers,if they hold the copyright) to Eudoxus Press, LLC, will be required before the manuscript can be accepted for publication.The Editor-in-Chief will supply the necessary forms for this transfer.Such a written transfer of copyright,which previously was assumed to be implicit in the act of submitting a manuscript,is necessary under the U.S.Copyright Law in order for the publisher to carry through the dissemination of research results and reviews as widely and effective as possible.

4. The paper starts with the title of the article, author's name(s) (no titles or degrees), author's affiliation(s) and e-mail addresses. The affiliation should comprise the department, institution (usually university or company), city, state (and/or nation) and mail code.

 The following items, 5 and 6, should be on page no. 1 of the paper.

5. An abstract is to be provided, preferably no longer than 150 words.

6. A list of 5 key words is to be provided directly below the abstract. Key words should express the precise content of the manuscript, as they are used for indexing purposes.

 The main body of the paper should begin on page no. 1, if possible.

7. All sections should be numbered with Arabic numerals (such as: 1. INTRODUCTION) .

Subsections should be identified with section and subsection numbers (such as 6.1. Second-Value Subheading).

If applicable, an independent single-number system (one for each category) should be used to label all theorems, lemmas, propositions, corollaries, definitions, remarks, examples, etc. The label (such as Lemma 7) should be typed with paragraph indentation, followed by a period and the lemma itself.

8. Mathematical notation must be typeset. Equations should be numbered consecutively with Arabic numerals in parentheses placed flush right, and should be thusly referred to in the text [such as Eqs.(2) and (5)]. The running title must be placed at the top of even numbered pages and the first author's name, et al., must be placed at the top of the odd numbed pages.

9. Illustrations (photographs, drawings, diagrams, and charts) are to be numbered in one consecutive series of Arabic numerals. The captions for illustrations should be typed double space. All illustrations, charts, tables, etc., must be embedded in the body of the manuscript in proper, final, print position. In particular, manuscript, source, and PDF file version must be at camera ready stage for publication or they cannot be considered.

 Tables are to be numbered (with Roman numerals) and referred to by number in the text. Center the title above the table, and type explanatory footnotes (indicated by superscript lowercase letters) below the table.

10. List references alphabetically at the end of the paper and number them consecutively. Each must be cited in the text by the appropriate Arabic numeral in square brackets on the baseline.

 References should include (in the following order): initials of first and middle name, last name of author(s) title of article,

name of publication, volume number, inclusive pages, and year of publication.

 Authors should follow these examples:

Journal Article

 1. H.H.Gonska,Degree of simultaneous approximation of bivariate functions by Gordon operators, (journal name in italics) *J. Approx. Theory***, 62,170-191(1990).**

Book

 2. G.G.Lorentz, (title of book in italics) *Bernstein Polynomials* **(2nd ed.), Chelsea,New York,1986.**

Contribution to a Book

 3. M.K.Khan, Approximation properties of beta operators,in(title of book in italics) *Progress in Approximation Theory* **(P.Nevai and A.Pinkus,eds.), Academic Press, New York,1991,pp.483-495.**

 11. All acknowledgements (including those for a grant and financial support) should occur in one paragraph that directly precedes the References section.

 12. Footnotes should be avoided. When their use is absolutely necessary, footnotes should be numbered consecutively using Arabic numerals and should be typed at the bottom of the page to which they refer. Place a line above the footnote, so that it is set off from the text. Use the appropriate superscript numeral for citation in the text.

 13. After each revision is made please again submit via email Latex and PDF files of the revised manuscript, including the final one.

 14. Effective 1 Nov. 2009 for current journal page charges, contact the Editor in Chief. Upon acceptance of the paper an invoice will be sent to the contact author. The fee payment will be due one month from the invoice date. The article will proceed to publication only after the fee is paid. The charges are to be sent, by money order or certified check, in US dollars, payable to Eudoxus Press, LLC, to the address shown on the Eudoxus [homepage.](http://www.eudoxuspress.com/)

 No galleys will be sent and the contact author will receive one (1) electronic copy of the journal issue in which the article appears.

 15. This journal will consider for publication only papers that contain proofs for their listed results.

Numerical Solutions of Fuzzy Two Coupled Nonlinear Differential Equations

K. Chellapriya^{1,*}, M. M. Shanmugapriya¹ 1,1^{*} Department of Mathematics, Karpagam Academy of Higher Education, Coimbatore, Tamilnadu, India. chellapriyavalliappan@gmail.com shanmugapriya.mm@kahedu.edu.in

December 26, 2022

Abstract

In this manuscript, we will use the new modified version of the Runge-Kutta method suitable for solving fuzzy two coupled systems of Nonlinear Ordinary Differential Equations (ODE). With the aid of a numerical example, we will demonstrate the accuracy of the $RK-4$ coupled method for solving these two coupled differential equations. To find the analytical solutions we use Laplace Adomian Decomposition Method since it is a semianalytical method used well in many existing studies on dynamical systems. In order to tell the accuracy, we use the error analysis technique. With the help of numerical simulations, we are able to show at what point of t , both $x(t)$ and $y(t)$ will interact in order to support the theory. 481 COMPUTATIONAL ANALYSIS AND APPLICATION SO C. 2023. AND A SURFACE CONDITION SO THE CONTRACT THE CONTRACT TO COMPROSITY THAT THE CONTRACT TH

Keywords: New theory of Numerical methods; Analytical Solution; Laplace Adomian Decomposition method; Runge-Kutta method; Two Coupled Differential Equations.

1 Introduction

The equivalence relations to a set of the non-crisp data set called fuzzy sets or fuzzy data set obtained by partisioning the existing relation that will not fail to satisfy the oprations satisfied by the crisp data set. The subsequent of differentiation as well as integration of fuzzy defined equations, and the ever existing theorems on existence and with it the uniqueness of FDE solutions in those space of quotients of fuzzy numbers are presented by various existing studies. The unique solution to the FDE's IVP will be well established if fuzzy normed f satisfies Lipschitz condition.

Many recent studies also developed the fuzzy methods and they have been implemented in so many grounds, such as optimization of multi-objective problems with various decision criteria. The development of mathematics has reached a very high level and is still available today.

The need of RK-4 method was very first arisen at the time of Euler methods to solve ODE numerically. Since it was clearly found very first time by the mathematicians Runge-Kutta, that the convergence of Euler method is only about $O(h^2)$ and error existence affects the coincidense of approximate solutions obtained by Euler with that of Exact solutions. $O(h^2)$ is not a good approximation order. So RK-4 methood was developed and found with $O(h^4)$ which provides the confidence

of least error and better approximation that coincdes to atleast four decimal places i.e., $O(h^4)$ for solving linear ode. It also helped the researchers to get the better approximate solutions to that of few non-linear problems like non-linear hybrid differential equations. But for two coupled system of differential equations there are still a research going on many fields like mathematical modelling in poplation dynamics, epidemiology etc.,

There are few noticable works are done on nonlinear epidemic modells and RK-4 methods have also been used but it is also to be mentioned that those modells are not completely three coupled differential equations. After this modell has been developed and if got published we hope strongely that it could be applied to get the solutions of three ccoupled or four coupled DE on the epidemic modells epidemic models. Also, The entire manuscript is brought up by the motivation of well established researches and some of the notable works are Allen, [1] gave his way of introduction mathematical biology. Abbasbandy extended a numerical method called Newtons method to deal with the nonlinear system of equations using modified Adomian Decomposition Method (ADM) in [2]. In [3] Bukley et al., researched on fuzzy differential equations (FDEs). Kermack et al., [4] mathematically analyzed theory of epidemics. Makinde et al., [5] applied ADM to a SIR epidemic model with uniform vaccination therapy. Farman [10] presented solution of SEIR epidemic model of meseales with non- integer time fractional derivatives by using LADM. Ongun [11], applied the LADM for solving a model for HIV infection of $CD4+T$ cells. Palese [12] analysed Variation of Influenza A, B, and C. Saberiad [15] applied of Homotopy Perturbation Method for solving Hybrid Fuzzy Differential Equations. Pederson et al., [19] numerically solved hybrid fuzzy differential equation IVPs by a characterisation theorem. [20] Kandel et al., studied Fuzzy dynamical systems and nature of their solutions. In [21], [22], Lakshmikantham et al., Impulsive hybrid systems and stability theory, Theory of fuzzy differential equations and inclusions. In [23] Seikkala, On the fuzzy initial value problem. [24] Sepahvandzadeh et al., applied Variational Iteration method (VIM) for solving Hybrid Fuzzy Differential Equations. Also there are many researchers who are working on different types fuzzy differential equations in his research on hybrid systems, delay systems, epidemic models etc., in [13, 14], [16],[17, 18], [6, 7], [8, 9]. The manuscript consists of preliminaries in 2, fuzzy-two-coupled non-linear differential equations in 3, Analytical Solution, Semi Analytical Solution in 4, modified Fuzzy RK-4 Algorithm in 5, and finally conclusion in 6 3 CONF view of the constraints of the state of the state of the state for the state of the

2 Preliminaries

Let E^1 represents the set of functions $q : \mathcal{R} \to [0,1]$ such that

$$
q(y) = \begin{cases} 4y - 3, & \text{if } y \in (0.75, 1], \\ -2y + 3, & \text{if } y \in (1, 1.5), \\ 0, & \text{if } y \notin (0.75, 1.5). \end{cases}
$$
(2.1)

The r-level set of q in (2.1) can be written as

$$
[q; r] = [0.75 + 0.25r, 1.5 - 0.5r]. \tag{2.2}
$$

We define $\hat{0} \in E^1$ as $\hat{0}(y) = 1$ if $y = 0$ and $\hat{0}(y) = 0$ if $y \neq 0$ for future reference.

From [23] of $y: I \to E^1$ where $I \subset \mathcal{R}$ is an interval. If $\tilde{y}(t) = [y(t; r), \overline{y}(t; r)]$ for all $t \in I$ and $r \in [0,1]$, then $\tilde{y}'(t) = [y'(t; r), \overline{y}'(t; r)]$, if $y'(t; r) \in E^1$. Following IVP,

$$
y'(t) = g(t, y(t)), \ y(0) = y_0,
$$
\n(2.3)

where $g : [0, \infty) \times \mathcal{R} \to \mathcal{R}$ is continuous. We would like to interpret (2.3) using the Seikkala's derivative and $y_0 \in E^1$. Let $\tilde{y}_0 = [y(0;r), \overline{y}(0;r)]$ and $\tilde{y}(t) =$ $[y(t; r), \overline{y}(t; r)].$

2.1 Definitions and Basic Results

This secion consits of important results considered from [25, 23, 3, 26] "Let $G_k(\mathscr{R}^n)$ represents the house of complete nonempty, compacted, convex collection of subsets of \mathscr{R}^n . Sum and product in $G_k(\mathscr{R}^n)$ are existing as usual. Let y be a point in \mathscr{R}^n and B be a non-empty sub set of \mathscr{R}^n . The distance $D(y, B)$ from y to B is defined by 3 COMPUTATIONAL ANALYSIS AND APPLICATIONS VOL. 31, NO. 4, 2023, COPYRIGHT 2023 JPM

where $g: [N, \infty] \times \Psi^* \rightarrow \mathcal{R}$ is constrained by the simulation of the interpret (2.3) value of the simulation of the simulation of the

$$
D(y, B) = \inf_{b \in B} \{ \|y - b\| \}
$$

Let M and N be two nonempty bounded subsets of \mathcal{R}^n . The Housdorff separation of M from N is defined by

$$
D_H^*(M, N) = \sup_{\mu \in M} \{d(\mu, \nu)\},
$$

The Housdorff separation of N from M is defined by

$$
D_H^*(N, M) = \sup_{\nu \in N} \{d(\nu, \mu)\},
$$

The distance of separation between M and N as understood by the Housdorff sense

$$
D_H(M, N) = \max \Big\{ \sup_{m \in M} \inf_{n \in N} ||m - n||, \sup_{n \in M} \inf_{m \in M} ||m - n|| \Big\},\,
$$

where $\|\cdot\|$ is the traditional Euclidean norm $\|\cdot\|$ in \mathcal{R}^n . Then it is clear that $(F_k(\mathscr{R}^n), D)$ becomes a complete metric space.

A fuzzy subset of \mathcal{R}^n is explained in terms of a membership arguments which coins to each point $x \in \mathcal{R}^n$, a grade of membership in the fuzzy set. Such a membership function $q : \mathcal{R}^n \to I \in [0,1]$ is used to denote the corresponding fuzzy set.

To every $r \in (0,1]$, the r- level set $[q]^r$ of a fuzzy set u is the subset of values $y \in \mathcal{R}^n$ with memberships $q(y)$ of r powers, that is $[q]^r = \{y \in \mathcal{R}^n : q(y) \geq r\}.$ The support $[q]$ ⁰ of a fuzzy set is then defined as the closure of the union of all its level sets, that is, $[q]$ ⁰ = $\boxed{)}$ $r \in (0,1]$ $[q]^r$. An inclusion result arrives spontaneously from

the above definitions.

Result 1

To every $0 \le r_1 \le r_2 \le 1$, $[q]^{r_2} \subseteq [q]^{r_1} \subseteq [q]^{0}$.

Universally, some level sets usually be null in a ordinary fuzzy set. Particularly, the triviality arise when $q(y) \equiv 0$ for all $y \in \mathcal{R}^n$, though the support is null: q is null fuzzy set in this sense. Here we shall pay focus only to the normal fuzzy sets which satisfy.

In view of Result 1. we have

Result 2

 $[q]^r$ is a compact subset of \mathcal{R}^n for all $r \in I$.

Result 3

"If u is fuzzy convex, then $[q]^r$ is convex for each $r \in I$.

Let $I = [0, 1] \subseteq R$ be as compact interval and let E^n denote the set of all $q: \mathcal{R}^n \to I$ such that q satisfies the following conditions.

(i) q is normal, that is, there exist an $q_0 \in \mathcal{R}^n$ such that $q_0 = 1$,

(ii) q is fuzzy convex,

(iii) q is upper semicontinuous,

(iv) $[q]^0 \equiv$ closure of $\{q \in \mathcal{R}^n : q(x) > 0\}$ is compact. Then, from $(1) - (4)$, it follows that the r-level set $[q]^r \in P_k(\mathcal{R}^1)$ for all $0 \le r \le 1$. If $g: \mathcal{R}^n \times \mathcal{R}^n \to \mathcal{R}^n$ is a function, then using Zadeh's extension principle we can extend g to $E^n \times E^n \to E^n$ by the equation"

$$
\tilde{g}(q_1, q_2)(z) = \sup_{z=g(x,y)} \min\{q_1(x), q_2(y)\}.
$$
\n(2.4)

It is well known that $[\tilde{g}(q_1, q_2)]^r = g([q_1]^r, [q_2]^r)$, for all $q_1, q_2 \in E^n$, $0 \le r \le 1$, and continuous function g. Further we have

$$
[q_1^r + q_2^r] = ([q_1]^r + [q_2]^r), \qquad (2.5)
$$

$$
[kq]^{r} = k[q]^{r}, \qquad (2.6)
$$

where $k \in \mathcal{R}$. The real numbers can be embedded in E^n by the rule $c \to \hat{c}(t)$ ", where,

$$
\hat{c}(t) = \begin{cases} 1 & \text{for } t = c, \\ 0 & \text{elsewhere.} \end{cases}
$$

3 Fuzzy-Two-Coupled Non-linear Differential Equations

For preliminary definitins of fuzzy differential equations authors are encouraged to go through [25, 26], [18], etc., Two Coupled differential Equations have wide range of applications in any mathematicall modell of physical phenomena in epidemiology, ecology,etc., By the application of fuzzy it is used to eliminate the randomness and vagueness that arises in any dynamics of the system. 484 J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC K. Chellapriya 481-489

$$
\begin{cases}\nx'(t) = c_1 x(t)y(t),\ t_0 \le t \le t_n \\
y'(t) = c_2 x(t)y(t),\ t_0 \le t \le t_n \\
x(t_0) = x_0, \\
y(t_0) = y_0\n\end{cases}
$$
\n(3.1)

where c_1, c_2 are numeric constants such that they are not equal to zero and also $c_1 \neq c_2$. By using the concept fuzzy, the equation (3.1) becomes,

$$
\begin{cases}\n\tilde{x}'(t) = c_1 \tilde{x}(t) \tilde{y}(t), t_0 \le t \le t_n \\
\tilde{y}'(t) = c_2 \tilde{x}(t) \tilde{y}(t), t_0 \le t \le t_n \\
\tilde{x}(t_0) = x_0, \\
\tilde{y}(t_0) = y_0\n\end{cases}
$$
\n(3.2)

Such that $\tilde{x}(t) = [\underline{x}(t; r), \overline{x}(t; r)]$. In the same way for $\tilde{y}(t) \tilde{x}'(t) \tilde{y}'(t)$ and also for $\tilde{x}_0, \tilde{y}(0)$

4 Analytical Solution, Semi Analytical Solution

The analytical Solution of the system (3.2) is given by

$$
\begin{array}{rcl}\n\tilde{x}(t) & = & \tilde{x}_0 e^{c_1 \int_0^t y(s) ds} \\
\tilde{y}(t) & = & \tilde{y}_0 e^{c_2 \int_0^t x(s) ds}\n\end{array} \tag{4.1}
$$

In order to obtain the semi analytical solution we are here by making use of well known Laplace Adomian Decomposition method (LADM). We prefer this method to compare the solutions of nonlinear coupled differential equations. The method is already defined and described in somany papers previously whereas the RK-4 algorith or method for nonlinear coupled differential equations is not defined clearly yet but found traces of the authors try over it in the literature. The method is taken since we are unable to process the analytical solutions even though its structure is expalined above.

4.1 Fuzzy Laplace Adomian Decomposition Method

$$
X(k+1) = L^{-1}(c_1/s \times L(A_k))
$$

\n
$$
Y(k+1) = L^{-1}(c_2/s^{\alpha 2} \times L(A_k))
$$
\n(4.2)

Where (A_k) is an Adomian polynomial defined by $A_k = \frac{1}{k!} \frac{d^k}{\lambda^k} \left(\sum_{l=0}^k (\lambda^l \cdot x_l \lambda^l \cdot y_l) \right)_{\lambda=0}$ i.e., $A_0 = x_0 y_0$ $A_1 = x_0y_1 + x_1y_0$ $A_2 = x_0y_2 + x_1y_1 + x_2y_0$ and so on. $x(t) = \sum_{k=0}^{\infty} (x(k))$ $y(t) = \sum_{k=0}^{\infty} (y(k))$

5 Modified Fuzzy RK-4 Algorithm:

We are at present sharing the new algorithm for novel RK-4 method for solving nonlinear coupled differential equations. In this section we are using the fourth order Runge-Kutta method (RK-4). We are finding the values of $\tilde{x}(t),\tilde{y}(t)$, at $h = 0.1$ for the best approximation. For $0 \le r \le 1$. To evaluate $x(t)$, and $y(t)$:

Consider,

$$
\begin{array}{rcl}\n\tilde{x}(t+1) & = & (\tilde{x}(t) + (1/6(A_1 + 2A_2 + 2A_3 + K_4))) \\
\tilde{y}(t+1) & = & (\tilde{y}(t) + (1/6(A_1 + 2B_2 + 2B_3 + B_4)))\n\end{array} \tag{5.1}
$$

To estimate (5.1), consider the following.

\n- 1. COMPIITADALA ANALYSIS AND APPLICATIONS, VOL. 31, NO. 4. 2023, COPYHSIET 2023 EUDOXUS PRESS, LC by a already defined and described in somary papers previously whereas in the RK-4 algorithm method for nonlinear coupling, it is to find the data for the total energy set in the differential equations. In method, it is is that some we use we must be to process the analytical solutions even though its structure is explained above.
\n- 4.1 Fuzzy Laplace Adominian Decomposition Method
$$
X(k+1) = L^{-1}(\alpha_1/\alpha \times L(A_k))
$$
\n
$$
Y(k+1) = L^{-1}(\alpha_2/\alpha^2 \times L(A_k))
$$
\n
$$
Y(k+1) = L^{-1}(\alpha_2/\alpha^2 \times L(A_k))
$$
\n
$$
Y(k+1) = L^{-1}(\alpha_2/\alpha^2 \times L(A_k))
$$
\n
$$
A_0 = x_0y_0
$$
\n
$$
A_1 = x_0y_1 + x_1y_0
$$
\n
$$
A_2 = x_0y_2 + x_1y_1 + x_2y_0
$$
\nand so on.

\n
$$
x(t) = \sum_{k=0}^{\infty} (x(k))
$$
\n
$$
y(t) = \sum_{k=0}^{\infty} (x(k))
$$
\n
\n- 5 Modified Fuzzy RK-4 Algorithm: We are at present, sharing the new algorithm for novel RK-4 method for solving nonlinear coupled differential equations. In this section, we are using the fourth order Rungentions with a method (KK-4); We are an important method (KK-4) is not a positive number of the best approximation. For 0 ≤ r ≤ 1.\nTo evaluate $x(t)$, and $y(t)$:

\n
$$
x(t+1) = (x(t) + (1/(6(A_1 + 2A_2 + 2A_3 + K_4)))
$$
\n
$$
y(t+1) = (y(t) + (1/(6(A_1 + 2A_2 + 2A_3 + K_4)))
$$
\n
$$
y(t+1) = (y(t) + (1/(6(A_1 + 2A_2 + 2A_3 + K_4)))
$$
\nTo estimate (5.1), consider the following:

\n
$$
\hat{A}_1 = h \times c_1((x(t)))(y(t) + (B_1/2))
$$
\n
$$
\hat{B}_2 = h \times c_2((x(t) + (A_2/2))(y(t) + (
$$

For $1 \leq p \leq 4$ and $0 \leq r \leq 1$, $\tilde{A}_p = \tilde{A}_p(t;r) = [\underline{A}_p(t;r),\overline{A}_p(t;r)],$ $\tilde{B}_p = \tilde{B}_p(t; r) = [\underline{B}_p(t; r), \overline{B}_p(t; r)],$ For $0 \le t \le n$, $n = 1, 2, 3, ...,$ and for $q = t, \overline{q} = t + 1, t = 0, 1, 2, 3, ...$ $\tilde{x}(q) = \tilde{x}(q)(t; r) = [\underline{x}_q(t; r), \overline{x}_q(t; r)],$ $\tilde{y}(q) = \tilde{y}(q)(t; r) = [\underline{y}_q(t; r), \overline{y}_q(t; r)],$ Where, $[f(t; r), \overline{f}(t; r)] = [0.75 + 0.25r, 1.125 - 0.125r]f(t)$.

t	$LADM-4$		Table 1: Approximate solution by RK-4 for non-fuzzy case $RK-4$		Error		
	$\mathbf{x}(\mathbf{t})$	y(t)	x(t)	y(t)	x(t)	y(t)	
θ	5	$\overline{3}$	5	$\overline{3}$	θ	$\overline{0}$	
0.1	4.991	$3.0045\,$	5.006	2.99325	0.015	0.01125	
0.2	4.98201	$3.009\,$	4.99701	2.99775	0.015	0.01125	
$\rm 0.3$	4.97301	3.01349	4.98802	3.00224	0.01501	0.01125	
0.4	4.96402	3.01799	4.97904	3.00673	0.01502	0.01126	
$0.5\,$	4.95503	3.02248	4.97006	3.01122	0.01503	0.01126	
0.6	4.94605	3.02697	4.96108	3.01571	0.01503	0.01126	
0.7	4.93707	3.03147	4.9521	3.0202	0.01503	0.01127	
0.8	4.92809	3.03595	4.94313	3.02469	0.01504	0.01126	
0.9	4.91912	3.04044	4.93416	3.02917	0.01504	0.01127	
1.0	4.91014	3.04493	4.92519	3.03366	0.01505	0.01127	
						Table 2: Approximate solution by RK-4 for fuzzy case	
r	x(t; r)		y(t; r)				
	min	$_{\mathrm{max}}$	\min	\max			
θ	3.6939	$5.625\,$	2.27524	3.41286			
0.1	3.81703	5.56917	2.35108	3.37494			
0.2 $\rm 0.3$	3.94016 4.06329	5.49671 5.42447	2.42692 2.50277	3.33702 3.2991			
0.4	4.18641	5.35246	2.57861	3.26118			
$0.5\,$	4.30954	5.28068	2.65445	3.22326			
0.6	4.43267	5.20913	2.73029	$3.18534\,$			
0.7	4.5558	5.13781	2.80613	3.14742			
0.8	4.67893	5.06671	2.88197	3.1095			
0.9	4.80206	4.99584	2.95781	3.07158			
$\mathbf{1}$	4.92519	4.92519	3.03366	3.03366			
							Let us consider the following problem and compare the results of the method LADM RKM-4 in both non-fuzzy and as well as fuzzy. In Table 1, we are presenting the values of $x(t)$, $y(t)$, in non fuzzy by means of LADM-4 and RK-4, and also the error analysis between them. In table 1, we have presented only the values for
	$t \in [0,1]$ but one can estimate the values for $t \in [0,100]$. In that way we found interaction between $x(t)$, and $y(t)$.						that at $t = 15.5$, $x(t) = 3.66502$, $y(t) = 3.66749$, i.e $x(t) \approx y(t)$, $t = 20$., there is a (5.3)
	$x(t; r)$ and $y(t; r)$ for $t \in [0, 1]$ and $r \in [0, 1]$.			$\left\{\begin{array}{l} x'(t)=-0.006\tilde{x}(t)\tilde{y}(t),\ 0\leq t\leq 1\\ \tilde{y}'(t)=0.003\tilde{x}(t)\tilde{y}(t),\ 0\leq t\leq 1\\ \tilde{x}(0)=5,\\ \tilde{y}(0)=3. \end{array}\right.$			We calculate error by means of $Error = (LADM - 4) - (RK - 4) $ Let us present below the table value of $x(t)$ and $y(t)$ in terms of fuzzy in Table 2. So that we have

Table 1: Approximate solution by RK-4 for non-fuzzy case

Table 2: Approximate solution by RK-4 for fuzzy case

r	x(t; r)		y(t; r)		
	mın	max	m ₁ n	max	
0	3.6939	5.625	2.27524	3.41286	
0.1	3.81703	5.56917	2.35108	3.37494	
0.2	3.94016	5.49671	2.42692	3.33702	
0.3	4.06329	5.42447	2.50277	3.2991	
0.4	4.18641	5.35246	2.57861	3.26118	
0.5	4.30954	5.28068	2.65445	3.22326	
0.6	4.43267	5.20913	2.73029	3.18534	
0.7	4.5558	5.13781	2.80613	3.14742	
0.8	4.67893	5.06671	2.88197	3.1095	
0.9	4.80206	4.99584	2.95781	3.07158	
1	4.92519	4.92519	3.03366	3.03366	

5.1 An Example

$$
\begin{cases}\n\tilde{x}'(t) = -0.006\tilde{x}(t)\tilde{y}(t), \ 0 \le t \le 1 \\
\tilde{y}'(t) = 0.003\tilde{x}(t)\tilde{y}(t), \ 0 \le t \le 1 \\
\tilde{x}(0) = 5, \\
\tilde{y}(0) = 3.\n\end{cases}
$$
\n(5.3)

5.2 Numerical Simlations

For non-Fuzzy coupled case of above example: For Fuzzy coupled case of above example:

Figure 1: Non-Fuzzy Nonlinear Two Coupled Differential Systems

By the above figures, Figure1 and 2 we are able to understand the travel of

Figure 2: Fuzzy Nonlinear Two Coupled Differential Systems

solutions in $t \in [0, 100]$ for non-fuzzy case and for $t \in [0, 1]$ and $r \in [0, 1]$ for fuzzy case respectively.

6 Conclusion

There are numerous numerical methods that one want to use other than Rung-Kutta method when it comes to the need to solve the function with non-linear

ODE especially for coupled differential equaions we are having so many applications but the methods like Laplace Adomian Decomposition method etc., are used as presented in earlier section. But now we had presented the new coupled form of RK-4 algorithm for solving any kind of nonlinear two coupled nonlinear ODE. We recommend this RK-4 algorithm since its accuracy is of about $O(h^1)$ or one decimal place when it is compared with semianalytical method like LADM. The important aspect is that one can easily see the interaction between $x(t)$ and $y(t)$ in figure 1 which tells us at $t = 15.5$, $x(t) \approx y(t)$. As a future work, we will present this approach on completely coupled fuzzy disease modelling problems. 4 COMPUTATIONAL ANALYSIS AND APPLICATIONS VOL. 31, NO. 4, 2023, COPYRIGHT 2023 LOOKUS PRESS, LLC CHELIGEN LIGHT AND A CONSULT AND A CONSU

References

- [1] L.J.S. Allen, An introduction to mathematical biology, NJ: Prentice Hall: 2007.
- [2] S. Abbasbandy Extended Newtons method for a system of nonlinear equations by modified Adomian Decomposition Method,Applied Mathematics and computation,170(2005), 648-656
- [3] J.J. Bukley and T. Feuring, Fuzzy differential equations, Fuzzy Sets and Systems, 110 (2000), 43-54.
- [4] W.O. Kermack and A.G. Mckendrick, Contribution to the Mathematical Theory of Epidemics, Proc.Roy.Soc.Lond.A, 115, 700-721 (1927).
- [5] O.D. Makinde, Adomian Decomposition approach to a SIR epidemic model with constant vaccination Strategy, Applied Mathematics and Computation, 184(2) (2007) 842-848.
- [6] P.B. Dhandapani, D. Baleanu, J. Thippan, and V. Sivakumar, Fuzzy Type RK4 Solutions to Fuzzy Hybrid Retarded Delay Differential Equations, Frontiers in Physics, 7:168, (2019), 1-6.
- [7] P.B. Dhandapni, D. Baleanu, J. Thippan, and V. Sivakumar, New Fuzzy Fractional Epidemic Model Involving Death Population, Computer Systems Science and Engineering, 37(3), (2021), 331-346.
- [8] P.B. Dhandapani, T. Jayakumar, and S. Vinoth, Existence and Uniqueness of Solutions for Fuzzy Mixed Type of Delay Differential Equations, Journal of Applied Nonlinear Dynamics, 10(1), (2021), 187-196.
- [9] P.B. Dhandapani, J. Thippan, C. Martin-Barreiro, V. Leiva, C. Chesneau, Numerical Solutions of a Differential System Considering a Pure Hybrid Fuzzy Neutral Delay Theory. Electronics 2022, 11, 1478.
- [10] Muhammad Farman, Muhammed Umer Saleem, Aqueel Ahmed, M.O. Ahamed, Analysis and Numerical solution of SEIR epidemic model of meseales with non- integer time fractional derivatives by using Laplace Adomian Decomposition Method, Ain Shams Engineering Journal, 9(2018) 3391-3397.
- [11] M.Y. Ongun, The Laplace Adomian Decomposition Method for solving a model for HIV infection of $CD4+T$ cells. Mathematical and Computer Modelling, 53(2011) 597-603.
- [12] P.Palese and J.F Young, Variation of Influenza A, B, and CScience, 215(4539), (1982) 1468-1474.
- [13] Prasantha Bharathi. D, Baleanu. D, Jayakumar. T, and Vinoth. S., On Stiff Fuzzy IRD-14 day average transmission model of COVID-19 pandemic disease, AIMS Bioengineering, 7(4), (2020),208-223.
- [14] D.P. Bharathi, T. Jayakumar, and S. Vinoth, Numerical Solution of Hybrid Fuzzy Mixed Delay Differential Equation by Fourth Order Runge-Kutta Method, Discontinuity, Nonlinearity and Complexity, 10(1), (2021), 77-86.
- [15] F. Saberiad, S.M. Karbassi, M. Heydari, S.M.M. Hosseini, Application of Homotopy Perturbation Methood for solving Hybrid Fuzzy Differential Equations, Journal of Mathematical Extension, 12(1) (2018), 113-145.
- [16] D.P. Bharathi, T. Jayakumar, and S. Vinoth, Numerical Solution of Fuzzy Mixed Delay Differential Equations Via Runge-Kutta Method of Order Four, International Journal of Applied Engineering Research, 14(3),(Special Issue) (2019), 70-74. 49 COMPUTATIONAL ANALYSIS AND APPLICATIONS VOL. 31, NO. 4, 2023, COPYRIGHT 2023 D. COPYRIGHT 2023 AND APPLICATIONS AND LOCATED IN THE STATIONAL AND APPLICATIONS CONTINUES THE TRANSMITTER (14) THE CHELIATION CONTINUES THE
	- [17] D.P. Bharathi, T. Jayakumar, and S. Vinoth, Numerical Solution of Fuzzy Multiple Hybrid Single Retarded Delay Differential Equations, International Journal of Recent Technology and Engineering, 8(3), (2019), 1946-1949.
	- [18] D.P. Bharathi, T. Jayakumar, and S. Vinoth, Numerical Solutions of Fuzzy Multiple Hybrid Single Neutral Delay Differential Equations, International Journal of Scientific & Technology Research, 8(09), (2019), 520-523.
	- [19] S.Pederson and M. Sambandham, Numerical Solution of hybrid fuzzy differential equation IVPs by a characterisation theorem, Information Sciences, 179 (2009), 319-328.
	- [20] A. Kandel, Fuzzy dynamical systems and the nature of their solutions, in: P.P. Wang, S.K. Chang(Eds.), Fuzzy Sets: Theory and Application to Policy Analysis and Information Systems, Plenum Press, New York, (1980), 93-122.
	- [21] V. Lakshmikantham and X.Z. Liu, Impulsive hybrid systems and stability theory, International Journal of Nonlinear Differential Equations, 5 (1999), 9-17.
	- [22] V. Lakshmikantham and R.N. Mohapatra, Theory of fuzzy differential equations and inclusions, Taylor and Francis, United Kingdom, 2003.
	- [23] S.Seikkala, On the fuzzy initial value problem, Fuzzy Sets and Systems, 24 (1987) 319-330.
	- [24] A. Sepahvandzadeh and B. Ghazanfari, Variational Iternation method for solving Hybri Fuzzy Differential Equations. Journal of Mathematical Extension,10 (4) (2016), 75-85.
	- [25] L.A. Zadeh, Fuzzy sets, Information and Control,8(1965), 338-353.
	- [26] R. Goetschel and W. Voxman, Elementary fuzzy calculus,Fuzzy Sets and Systems, 18(1986), 31-43.

Formatting of Funding Sources:

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

Parametrized hyperbolic tangent based Banach space valued multivariate multi layer neural network approximations

George A. Anastassiou Department of Mathematical Sciences University of Memphis Memphis, TN 38152, U.S.A. ganastss@memphis.edu

Seda Karateke

Department of Computer Engineering, Faculty of Engineering, Istanbul Topkapı University, Istanbul, Zeytinburnu 34087, Turkey. sedakarateke@topkapi.edu.tr

Abstract

Here we examine the multivariate quantitative approximations of Banach space valued continuous multivariate functions on a box or \mathbb{R}^N , $N \in \mathbb{N}$, by the multivariate normalized, quasi-interpolation, Kantorovich type and quadrature type neural network operators. We research also the case of approximation by iterated operators of the last four types, that is multi hidden layer approximations. These approximations are achieved by establishing multidimensional Jackson type inequalities involving the multivariate modulus of continuity of the engaged function or its high order Fréchet derivatives. Our multivariate operators are defined by using a multidimensional density function induced by a parametrized hyperbolic tangent sigmoid function. The approximations are pointwise, uniform and L_p . The related feed-forward neural networks are with one or multi hidden layers. 490 J. COMPUTATIONAL ANALYSIS AND APPLICATIONS VOL. 31, NO. 4, 2023, COPYRIGHT 2023 ET also control in the control i

2020 AMS Mathematics Subject Classification: 41A17, 41A25, 41A30, 41A36.

Keywords and Phrases: multi layer approximation, parametrized hyperbolic tangent sigmoid function, multivariate neural network approximation, quasi-interpolation operator, Kantorovich type operator, quadrature type operator, multivariate modulus of continuity, abstract approximation, iterated and L_p approximations.

1 Introduction

G.A. Anastassiou in [2] and [3], see chapters 2-5, was the first to establish neural network approximations to continuous functions with rates by very specifically defined neural network operators of Cardaliaguet-Euvrard and "Squashing" types, by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities. He treats there both the univariate and multivariate cases. The defining these operators "bell-shaped" and "squashing" functions are assumed to be of compact support. Also in [3] he gives the Nth order asymptotic expansion for the error of weak approximation of these two operators to a special natural class of smooth functions, see chapters 4-5 there. 1, COMPUTATIONAL ANALYSIS AND APPLICATIONS VOL. 31, NO. 4, 2023, COPYRIGHT 2023 LUCCOMUS PRESS, LLC

1. An Anastas in Figure 1, 2023, A specific and the specific and the specific and the specific and the specific and the

Motivations for this work are the article [22] of Z. Chen and F. Cao, and [4]-[19], [23], [24].

Here we perform a parametrized hyperbolic tangent sigmoid function based neural network multivariate approximation to continuous functions over boxes or over the whole \mathbb{R}^N , $N \in \mathbb{N}$, and also iterated, multi layer and L_p approximations. All convergences here are with rates expressed via the multivariate modulus of continuity of the involved function or its high order Fréchet derivative and given by very tight multidimensional Jackson type inequalities.

We come up with the "right" precisely defined multivariate normalized, quasi-interpolation neural network operators related to boxes or \mathbb{R}^N , as well as Kantorovich type and quadrature type related operators on \mathbb{R}^N . Our boxes are not necessarily symmetric to the origin. In preparation to prove our results we establish important properties of the basic multivariate density function induced by a parametrized hyperbolic tangent sigmoid function.

Feed-forward neural networks (FNNs) with one hidden layer, the only type of networks we deal with in this article, are mathematically expressed as

$$
N_n(x) = \sum_{j=0}^n c_j \sigma(\langle a_j \cdot x \rangle + b_j), \quad x \in \mathbb{R}^s, \quad s \in \mathbb{N},
$$

where for $0 \leq j \leq n$, $b_j \in \mathbb{R}$ are the thresholds, $a_j \in \mathbb{R}^s$ are the connection weights, $c_j \in \mathbb{R}$ are the coefficients, $\langle a_j \cdot x \rangle$ is the inner product of a_j and x, and σ is the activation function of the network. In many fundamental neural network models, the activation function is based on the hyperbolic tangent sigmoid function. About neural networks read [25]-[27].

2 Background

We consider here the generalized hyperbolic tangent function tanh $\lambda x, x \in \mathbb{R}$, $\lambda > 0$:

$$
\tanh \lambda x = \frac{e^{\lambda x} - e^{-\lambda x}}{e^{\lambda x} + e^{-\lambda x}}.
$$
\n(1)

It is $\tanh \lambda 0 = 0, -1 < \tanh \lambda x < 1, \forall x \in \mathbb{R}$, and $\tanh \lambda (-x) = -\tanh \lambda x$. Furthermore we have $\tanh \lambda (\infty) = 1$ and $\tanh \lambda (-\infty) = -1$, and $\tanh \lambda x$ is strictly increasing on R, with

$$
\frac{d}{dx}\tanh\lambda x = \frac{\lambda}{\cos^2\lambda x} > 0.
$$
 (2)

The induced activation function will be

$$
\theta(x) := \frac{1}{4} \left(\tanh \lambda \left(x + 1 \right) - \tanh \lambda \left(x - 1 \right) \right) > 0, \forall x \in \mathbb{R},\tag{3}
$$

with $\theta(x) = \theta(-x)$.

Clearly $\theta(x)$ is differentiable and thus it is continuous.

Proposition 1 $\theta(x)$ is strictly decrasing on $(0, \infty)$ and strictly increasing on $(-\infty, 0]$. We have that $\theta(-\infty) = \theta(\infty) = 0$. So that θ has the bell shape with horizontal asymptote the x-axis. The maximum of θ is

$$
\theta(0) = \frac{\tanh \lambda}{2}.\tag{4}
$$

We mention

Theorem 2 ([20]) It holds

$$
\sum_{i=-\infty}^{\infty} \theta(x - i) = 1, \ \forall \ x \in \mathbb{R}.
$$
 (5)

Theorem 3 ([20]) We have that

$$
\int_{-\infty}^{\infty} \theta(x) dx = 1.
$$
 (6)

So that θ is a density function on \mathbb{R} .

Theorem 4 ([20]) Let $0 < \alpha < 1$, $\lambda > 0$ and $n \in \mathbb{N}$. It holds

\n- \n**1. COMPUTATIONAL ANALYSIS AND APPLICATIONS.** VOL-31. NO. 4. 2023. COPYRIGHT 2023 EUDOXUS PRESS, LLC\n
	\n- \n**2. Background**\n
	\n- \n**3. a**\n
	$$
	\lambda x = \frac{e^{\lambda x} - e^{-\lambda x}}{e^{\lambda x} + e^{-\lambda x}}.
	$$
	\n
	\n- \n**4. b**\n $\lambda = 0:$ \n $\tan \lambda x = \frac{e^{\lambda x} - e^{-\lambda x}}{e^{\lambda x} + e^{-\lambda x}}.$ \n
	\n- \n**5. a**\n $\lambda = 0:$ \n $\tan \lambda x = \frac{e^{\lambda x} - e^{-\lambda x}}{e^{\lambda x} + e^{-\lambda x}}.$ \n
	\n- \n**6. a**\n $\lambda = 0:$ \n $\tan \lambda x = \frac{e^{\lambda x} - e^{-\lambda x}}{e^{\lambda x}}.$ \n
	\n- \n**7. b**\n $\alpha = \tan \lambda \ln \lambda (\alpha - \alpha) = -1$, and $\tan \lambda \ln \lambda x$ is strictly increasing on **R**, with $d(x) = 1$ and $\tan \lambda x = \frac{\lambda}{\cos^2 \lambda x} > 0.$ \n
	\n- \n**8. a**\n $\theta(x) := \frac{1}{4} \tanh \lambda (x + 1) - \tanh \lambda (x - 1) > 0, \forall x \in \mathbb{R},$ \n $\theta = \theta(-x).$ \n
	\n- \n**8. a**\n $\theta(x) = \theta(-x).$ \n $\theta(x)$ is differentiable and thus it is continuous.\n
	\n- \n**9. a**\n $\theta(x) = \frac{\tanh \lambda}{2}.$ \n
	\n- \n**10. a**\n $\theta = \frac{\tanh \lambda}{2}.$ \n
	\n- \n**11. b**\n $\theta = \frac{\tanh \lambda}{2}.$ \n
	\n- \n**2. a**\n<math display="inline

Denote by $|\cdot|$ the integral part of the number and by $\lceil \cdot \rceil$ the ceiling of the number.

Theorem 5 ([20]) Let $x \in [a, b] \subset \mathbb{R}$ and $n \in \mathbb{N}$, so that $[na] \leq |nb|$. Then

$$
\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \theta\left(nx-k\right)} < \frac{4}{\tanh 2\lambda} = \frac{1}{\theta\left(1\right)}.\tag{8}
$$

We make

Remark 6 (20)

(i) We have that

$$
\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \theta\left(nx - k\right) \neq 1,\tag{9}
$$

for at least some $x \in [a, b]$.

(ii) Let $[a, b] \subset \mathbb{R}$. For large n we always have $\lceil na \rceil \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $[na] \leq k \leq |nb|$.

In general it holds

$$
\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \theta\left(nx-k\right) \le 1. \tag{10}
$$

We introduce

$$
Z(x_1, ..., x_N) := Z(x) := \prod_{i=1}^{N} \theta(x_i), \quad x = (x_1, ..., x_N) \in \mathbb{R}^N, \ N \in \mathbb{N}.
$$
 (11)

It has the properties:

(i) Z (x) > 0, ∀ x ∈ R N , (ii) X∞ k=−∞ ^Z (^x [−] ^k) := ^X[∞] k1=−∞ X∞ k2=−∞ ... ^X[∞] k^N =−∞ Z (x¹ − k1, ..., x^N − k^N) = 1, (12) 493 J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC George A. Anastassiou 490-519

where $k := (k_1, ..., k_n) \in \mathbb{Z}^N, \forall x \in \mathbb{R}^N$, hence

(iii)

$$
\sum_{k=-\infty}^{\infty} Z(nx-k) = 1,
$$
\n(13)

 $\forall x \in \mathbb{R}^N; n \in \mathbb{N},$ and

 $\int_{\mathbb{R}^N} Z(x) dx = 1,$ (14)

that is Z is a multivariate density function.

Here denote $||x||_{\infty} := \max\{|x_1|, ..., |x_N|\}, x \in \mathbb{R}^N$, also set $\infty := (\infty, ..., \infty)$, $-\infty := (-\infty, ..., -\infty)$ upon the multivariate context, and

$$
[na] := ([na1], ..., [naN]),\n
$$
[nb] := ([nb1], ..., [nbN]),
$$
\n(15)
$$

where $a := (a_1, ..., a_N), b := (b_1, ..., b_N)$. We obviously see that

$$
\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k) = \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(\prod_{i=1}^N \theta(nx_i - k_i) \right) =
$$

$$
\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \dots \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} \left(\prod_{i=1}^N \theta(nx_i - k_i) \right) = \prod_{i=1}^N \left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \theta(nx_i - k_i) \right). \tag{16}
$$

For $0 < \beta < 1$ and $n \in \mathbb{N}$, a fixed $x \in \mathbb{R}^N$, we have that

1. COMPUTTONAL ANALYSIS AND APPLICATIONS. VOL-31. NO. 4. 2023. COPYRIGHT 2023 EUDOXUS PRESS, LLC
\nthat is *Z* is a multivariate density function.
\nHere denote
$$
||x||_{\infty}
$$
 := max { x_1 [x₁,..., [x_N]), $x ∈ ℝN$, also set ∞ := (∞, ..., ∞),
\n $-\infty := (-∞, ..., -∞)$ upon the multivariate context, and
\n $[na] := ([na_1], ..., [na_N]),$
\n $[nb] := ([na_1], ..., [nb_N]),$
\nwhere $a := (a_1,...a_N), b := (b_1,...b_N)$.
\nWe obviously see that
\n
$$
\sum_{k=1}^{|nb_1|} x_{k=1} = \sum_{k=1}^{|nb_1|} \left(\prod_{k=1}^N \theta(nx_k - k_1)\right) =
$$
\n
$$
\sum_{k=1}^{|nb_1|} x_{k=1} = [na_1] \left(\prod_{k=1}^{|nb_1|} \theta(nx_k - k_1)\right) =
$$
\n
$$
\sum_{k=1}^{|nb_1|} x_{k=1} = [na_2] \left(\prod_{k=1}^{|nb_1|} \theta(nx_k - k_1)\right) =
$$
\n
$$
\sum_{k=1}^{|nb_2|} Z(nx - k) +
$$
\n
$$
\sum_{k=1}^{|nb_2|} Z(nx - k) =
$$
\n
$$
\sum_{k=1}^{|nb_2|} x_{k} \leq \frac{1}{n^2}
$$
\n
$$
\sum_{k=1}^{|nb_1|} x_{k} = \frac{|a_1|}{n^2}
$$
\n
$$
\sum_{k=1}^{|nb_2|} Z(nx - k) +
$$
\n
$$
\sum_{k=1}^{|nb_2|} Z(nx - k) =
$$
\n
$$
\sum_{k=1}^{|nb_1|} x_{k} = \frac{1}{n^2}
$$
\n $$

In the last two sums the counting is over disjoint vector sets of k 's, because the condition $\left\|\frac{k}{n}-x\right\|_{\infty} > \frac{1}{n^{\beta}}$ implies that there exists at least one $\left|\frac{k_r}{n}-x_r\right| > \frac{1}{n^{\beta}}$, where $r \in \{1, ..., N\}$.

(v) As in, Theorem 4 we derive that

$$
\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k) \stackrel{(7)}{<} e^{4\lambda} e^{-2\lambda n^{(1-\beta)}}, \ 0 < \beta < 1, \ \lambda > 0. \tag{18}
$$
\n
$$
\left\{ \left\| \frac{k}{n} - x \right\|_{\infty} > \frac{1}{n^{\beta}}
$$

with $n \in \mathbb{N} : n^{1-\beta} > 2, x \in \prod_{i=1}^{N} [a_i, b_i].$ (vi) By Theorem 5 we get that

$$
0 < \frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z\left(nx-k\right)} < \left(\frac{4}{\tanh 2\lambda}\right)^N,\tag{19}
$$

 (iv)

$$
\lambda > 0, \forall x \in \left(\prod_{i=1}^{N} [a_i, b_i]\right), \quad n \in \mathbb{N}.
$$

It is also clear that
(vii)

$$
\sum_{i=1}^{\infty} Z(nx - k) < e^{4\lambda}e^{-\lambda}
$$

$$
\sum_{k=-\infty}^{\infty} Z(nx-k) < e^{4\lambda} e^{-2\lambda n^{(1-\beta)}}, \tag{20}
$$
\n
$$
\left\{ \left\| \frac{k}{n} - x \right\|_{\infty} > \frac{1}{n^{\beta}}
$$

 $\lambda > 0, 0 < \beta < 1, n \in \mathbb{N} : n^{1-\beta} > 2, x \in \mathbb{R}^N$. Furthermore it holds

$$
\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k) \neq 1,\tag{21}
$$

for at least some $x \in \left(\prod_{i=1}^N [a_i, b_i]\right)$.

Here $(X, \left\| \cdot \right\|_{\gamma})$ is a Banach space.

Let $f \in C\left(\prod_{i=1}^{N} [a_i, b_i], X\right)$, $x = (x_1, ..., x_N) \in \prod_{i=1}^{N} [a_i, b_i]$, $n \in \mathbb{N}$ such that $[na_i] \leq \lfloor nb_i \rfloor, i = 1, ..., N.$

We introduce and define the following multivariate linear normalized neural network operator $(x := (x_1, ..., x_N) \in \left(\prod_{i=1}^N [a_i, b_i] \right))$:

I. COMPUTIONAL ANALYSIS AND APPLICATIONS. VOL-31. NO. 4. 2023. COPYRIGHT 2023 EUDOXUS PRESS, LLC
\n
$$
\lambda > 0, \forall x \in \left(\prod_{i=1}^{N} [a_i, b_i]\right), n \in \mathbb{N}.
$$
\nIt is also clear that
\n(vii)
\n
$$
\sum_{i=1}^{N} Z(nx-k) < e^{4\lambda}e^{-2\lambda x^{1-\alpha i}},
$$
\n(20)
\n
$$
\begin{cases}\n k = -\infty \\
 \left\|\frac{k}{n} - x\right\|_{\infty} > \frac{1}{\sqrt{n}} \\
 \left\|\frac{k}{n} - x\right\|_{\infty} > \frac{1}{\sqrt{n}} \\
 \text{Furthermore it holds} \quad \lim_{n \to \infty} \sum_{k=1, n \neq i}^{[M]} Z(nx-k) \neq 1, \tag{21}\nfor at least some x ∈ $\left(\prod_{i=1}^{N} [a_i, b_i]\right).$
\nHere $\left(X, \left\|\cdot\right\|_{\infty}\right)$ is a Banach space.
\nLet $f \in C \left(\prod_{i=1}^{N} [a_i, b_i]\right), x = (x_1, ..., x_N) \in \prod_{i=1}^{N} [a_i, b_i], n \in \mathbb{N}$ such that $\left\{\pi a_i \right\} \subseteq [n\hat{b}_i], i = 1, ..., N$,
\nWe introduce and define the following multivariate linear normalized neural network operator $(x := [x_1, ..., x_N) \in \prod_{i=1}^{N} [a_i, b_i])$:
\n
$$
A_0(f, x_1, ..., x_N) := A_0(f, x) := \frac{\sum_{k=1}^{[N-1]} [a_i, b_i]}{\sum_{k=1}^{[N-1]} [a_i, b_i]} \times \frac{E_n(x_1 - k)}{2(x_1 - a_i)} = \frac{\sum_{k=1}^{[N-1]} [a_i, b_i]}{\sum_{k=1}^{[N-1]} [a_i, b_i]} \times \frac{E_n(x_1 - k)}{2(x_1 - a_i)} = \frac{\sum_{k=1}^{[N-1]} [a_i, b_i]}{\sum_{k=1}^{[N-1]} [a_i, b_i]} \times \frac{E_n(x_1 - x_0)}{2(x_1 - a_i)}\right).
$$
\nFor large enough n ∈
$$

For large enough $n \in \mathbb{N}$ we always obtain $[na_i] \leq |nb_i|, i = 1, ..., N$. Also $a_i \leq \frac{k_i}{n} \leq b_i$, iff $\lceil na_i \rceil \leq k_i \leq \lfloor nb_i \rfloor$, $i = 1, ..., N$.

When $g \in C\left(\prod_{i=1}^N [a_i, b_i]\right)$ we define the companion operator

$$
\widetilde{A}_n(g,x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} g\left(\frac{k}{n}\right) Z\left(nx-k\right)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z\left(nx-k\right)}.
$$
\n(23)

Clearly \widetilde{A}_n is a positive linear operator. We have that

$$
\widetilde{A}_n(1,x) = 1, \ \forall \ x \in \left(\prod_{i=1}^N [a_i, b_i]\right).
$$

Notice that $A_n(f) \in C\left(\prod_{i=1}^N [a_i, b_i], X\right)$ and $\widetilde{A}_n(g) \in C\left(\prod_{i=1}^N [a_i, b_i]\right)$.

Furthermore it holds

$$
\|A_n(f,x)\|_{\gamma} \le \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \|f\left(\frac{k}{n}\right)\|_{\gamma} Z\left(nx-k\right)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z\left(nx-k\right)} = \widetilde{A}_n \left(\|f\|_{\gamma},x\right),\tag{24}
$$

 $\forall x \in \prod_{i=1}^N [a_i, b_i].$ Clearly $||f||_{\gamma} \in C\left(\prod_{i=1}^{N} [a_i, b_i]\right)$. So, we have that

$$
\left\|A_n\left(f,x\right)\right\|_{\gamma} \le \widetilde{A}_n\left(\left\|f\right\|_{\gamma},x\right),\tag{25}
$$

 $\forall x \in \prod_{i=1}^{N} [a_i, b_i], \forall n \in \mathbb{N}, \forall f \in C \left(\prod_{i=1}^{N} [a_i, b_i], X\right).$ Let $c \in X$ and $g \in C\left(\prod_{i=1}^N [a_i, b_i]\right)$, then $cg \in C\left(\prod_{i=1}^N [a_i, b_i], X\right)$. Furthermore it holds

$$
A_n (cg, x) = c\widetilde{A}_n (g, x), \ \ \forall \ x \in \prod_{i=1}^N [a_i, b_i]. \tag{26}
$$

Since $\widetilde{A}_n(1) = 1$, we get that

$$
A_n(c) = c, \forall c \in X.
$$
\n
$$
(27)
$$

We call \widetilde{A}_n the companion operator of A_n .

For convenience we call

L. COMPUTATIONAL ANALYSIS AND APPLICATIONS. VOL-31. NO. 4. 2023. COPYRIGHT 2023 EUDOXUS PRESS, LLC
\nFurthermore it holds
\n
$$
||A_n(f,x)||_{\gamma} \le \frac{\sum_{k=1}^{\lfloor n b \rfloor} \left\|f\left(\frac{k}{n}\right)\right\|_{\gamma} Z(nx-k)}{\sum_{k=1}^{\lfloor n a \rfloor} \left\|f\left(\frac{k}{n}\right)\right\|_{\gamma} Z(nx-k)} = \tilde{A}_n \left(\|f\|_{\gamma}, x\right), \qquad (24)
$$
\n
$$
\forall x \in \prod_{k=1}^N [a_i, b_i].
$$
\nClearly $||f||_{\gamma} \in C\left(\prod_{k=1}^N [a_i, b_i]\right).$
\nSo, we have that
\n
$$
||A_n(f,x)||_{\gamma} \le \tilde{A}_n \left(\|f\|_{\gamma}, x\right), \qquad (25)
$$
\n
$$
\forall x \in \prod_{k=1}^N [a_i, b_i] \lor w \in \mathbb{N}, \forall f \in C\left(\prod_{k=1}^N [a_i, b_i], X\right).
$$
\nLet $c \in X$ and $g \in C\left(\prod_{k=1}^N [a_k, b_k], X\right).$
\nIntrikemore it holds
\n
$$
A_n(g,x) = c \tilde{A}_n(g,x), \forall x \in \prod_{k=1}^N [a_k, b_k].
$$
\nSince $\tilde{A}_n(t) = 1$, we get that
\n
$$
A_n(c) = c, \forall c \in X.
$$
\n
$$
\text{We call } \tilde{A}_n \text{ the companion operator of } A_n.
$$
\nFor convenience we call
\n
$$
A_n^*(f,x) := \sum_{k=1}^N \int \left(\frac{k}{n}\right) Z(nx-k) =
$$
\n
$$
\sum_{k=1}^N \sum_{k=1}^N [a_k, b_k].
$$
\n
$$
\forall x \in \left(\prod_{k=1}^N [a_k, b_k]\right).
$$
\nThat is
\n
$$
A_n(f,x) := \sum_{k=1}^N \sum_{k=1}^N \left\{ \frac{k}{n}, \dots, \frac{ky}{n} \right\} \left(\prod_{k=1}^N g(nx_k -
$$

 $\forall x \in \left(\prod_{i=1}^N [a_i, b_i]\right).$ That is

$$
A_n(f, x) := \frac{A_n^* (f, x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z (nx - k)},
$$
\n(29)

 $\forall x \in \left(\prod_{i=1}^N [a_i, b_i]\right), n \in \mathbb{N}.$ Hence

$$
A_n(f,x) - f(x) = \frac{A_n^*(f,x) - f(x) \left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)\right)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)}.
$$
 (30)

Consequently we derive

$$
\|A_n(f,x)-f(x)\|_{\gamma} \stackrel{(19)}{\leq} \left(\frac{4}{\tanh 2\lambda}\right)^N \left\|A_n^*(f,x)-f(x)\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k)\right\|_{\gamma},\tag{31}
$$

 $\forall x \in \left(\prod_{i=1}^N [a_i, b_i]\right).$

We will estimate the right hand side of (31) .

For the last and others we need

Definition 7 ([15], p. 274) Let M be a convex and compact subset of $\left(\mathbb{R}^N,\left\|\cdot\right\|_p\right)$, $p \in [1,\infty]$, and $\left(X,\left\|\cdot\right\|_{\gamma}\right)$ be a Banach space. Let $f \in C\left(M,X\right)$. We define the first modulus of continuity of f as

$$
\omega_1(f,\delta) := \sup_{\begin{subarray}{l} x, y \in M : \\ \|x - y\|_p \le \delta \end{subarray}} \|f(x) - f(y)\|_{\gamma}, \ \ 0 < \delta \le \operatorname{diam}(M). \tag{32}
$$

If $\delta > diam(M)$, then

$$
\omega_1(f,\delta) = \omega_1(f, diam(M)). \tag{33}
$$

Notice $\omega_1(f, \delta)$ is increasing in $\delta > 0$. For $f \in C_B(M, X)$ (continuous and bounded functions) $\omega_1(f, \delta)$ is defined similarly.

Lemma 8 ([15], p. 274) We have $\omega_1(f, \delta) \to 0$ as $\delta \downarrow 0$, iff $f \in C(M, X)$, where M is a convex compact subset of $(\mathbb{R}^N, \left\|\cdot\right\|_p), p \in [1, \infty]$.

Clearly we have also: $f \in C_U(\mathbb{R}^N, X)$ (uniformly continuous functions), iff $\omega_1(f,\delta) \to 0$ as $\delta \downarrow 0$, where ω_1 is defined similarly to (32). The space $C_B(\mathbb{R}^N, X)$ denotes the continuous and bounded functions on \mathbb{R}^N .

When $f \in C_B(\mathbb{R}^N, X)$ we define,

\n- \n**COMPUTIONAL ANALYSIS AND APPLICATIONS.** VOL-31.NO-4. 2023. COPYRIGHT 2023 EUDOXUS PRESS, LC Consequently we derive\n
$$
\|A_n(f, x) - f(x)\|_{\gamma} \stackrel{(19)}{\leq} \left(\frac{4}{\tanh 2\lambda}\right)^N \left\|A_n^*(f, x) - f(x)\sum_{k=|n\omega|}^{|\infty|} Z(nx - k)\right\|_{\gamma},
$$
\n
$$
\forall x \in \left(\prod_{i=1}^N [a_i, b_i]\right).
$$
\n We will estimate the right hand side of (31).\n For the last ad others are need

\n
\n- \n**Definition 7** (If 5), p. 274) Let M be a *conver and compact subset of* $\left\{\mathbb{R}^N, \|\cdot\|_{\gamma}\right\},$
$$
p \in [1, \infty], and \left\{\mathbf{X}, \|\cdot\|_{\gamma}\right\} be a Banach space. Let $f \in C(M, X)$. We define the first modulus of continuity of f as\n
$$
\omega_1(f, \delta) := \sup_{x,y \in M} \left\|f(x) - f(y)\right\|_{\gamma}, 0 < \delta \leq diam(M).
$$
\n (32)\n
$$
\|x - y\|_{\gamma} \leq \delta
$$
\n
$$
\n- \n If $\delta > diam(M), then$ \n
$$
\omega_1(f, \delta) := \omega_1(f, diam(M)).
$$
\n Notice\n
$$
\omega_1(f, \delta) := \omega_1(f, diam(M)).
$$
\n Notice\n
$$
\omega_1(f, \delta) := \omega_1(f, diam(M)).
$$
\n (33)\n
$$
\text{Notice } \omega_1(f, \delta) := \omega_1(f, diam(M)).
$$
\n (34)\n
$$
\text{Note: } \omega_1(f, \delta) = \omega_1(f, diam(M)).
$$
\n (35)\n
$$
\text{Note: } \omega_1(f, \delta) = \omega_1(f, diam(M)) = \omega_1(f, \delta) + \omega
$$

 $n \in \mathbb{N}, \forall x \in \mathbb{R}^N, N \in \mathbb{N}$, the multivariate quasi-interpolation neural network operator.

Also for $f \in C_B(\mathbb{R}^N, X)$ we define the multivariate Kantorovich type neural network operator

$$
C_n(f, x) := C_n(f, x_1, ..., x_N) := \sum_{k=-\infty}^{\infty} \left(n^N \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t) dt \right) Z(nx - k) =
$$

$$
\sum_{k_1 = -\infty}^{\infty} \sum_{k_2 = -\infty}^{\infty} \dots \sum_{k_N = -\infty}^{\infty} \left(n^N \int_{\frac{k_1}{n}}^{\frac{k_1 + 1}{n}} \int_{\frac{k_2}{n}}^{\frac{k_2 + 1}{n}} \dots \int_{\frac{k_N}{n}}^{\frac{k_N + 1}{n}} f(t_1, ..., t_N) dt_1 ... dt_N \right) \cdot \left(\prod_{i=1}^N \theta(nx_i - k_i) \right), \tag{35}
$$

 $n \in \mathbb{N}, \ \forall \ x \in \mathbb{R}^N.$

Again for $f \in C_B(\mathbb{R}^N, X)$, $N \in \mathbb{N}$, we define the multivariate neural network operator of quadrature type $D_n(f, x)$, $n \in \mathbb{N}$, as follows.

Let $\theta = (\theta_1, ..., \theta_N) \in \mathbb{N}^N$, $r = (r_1, ..., r_N) \in \mathbb{Z}_+^N$, $w_r = w_{r_1, r_2, ... r_N} \ge 0$, such that \sum^{θ} $\sum_{r=0}^{\theta} w_r = \sum_{r_1=0}^{\theta_1}$ $r_1=0$ $\frac{\theta_2}{\sum}$ $r_2=0$ \cdots $\sum_{N=1}^{\theta_{N}}$ $\sum_{r_N=0}^{N} w_{r_1,r_2,...r_N} = 1; k \in \mathbb{Z}^N$ and

$$
\delta_{nk}(f) := \delta_{n,k_1,k_2,...,k_N}(f) := \sum_{r=0}^{\theta} w_r f\left(\frac{k}{n} + \frac{r}{n\theta}\right) =
$$

$$
\sum_{r_1=0}^{\theta_1} \sum_{r_2=0}^{\theta_2} \dots \sum_{r_N=0}^{\theta_N} w_{r_1,r_2,...r_N} f\left(\frac{k_1}{n} + \frac{r_1}{n\theta_1}, \frac{k_2}{n} + \frac{r_2}{n\theta_2}, \dots, \frac{k_N}{n} + \frac{r_N}{n\theta_N}\right), \quad (36)
$$

where $\frac{r}{\theta} := \left(\frac{r_1}{\theta_1}, \frac{r_2}{\theta_2}, ..., \frac{r_N}{\theta_N}\right)$. We set

0. COMPUTIONAL ANALYSIS AND APPLICATIONS. VOL-31. NO. 4. 2023. COPYRIGHT 2023 EUDOXUS PRESS, LLC
\n
$$
\sum_{k_1=-\infty}^{\infty} \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_N=-\infty}^{\infty} \left(n^N \int_{\frac{k_1}{2}}^{\frac{k_1+1}{2}} \int_{\frac{k_2}{2}}^{\frac{k_2+1}{2}} f(t_1,...,t_N) dt_1...dt_N \right)
$$
\n
$$
\cdot \left(\prod_{i=1}^{N} \theta(nx_i - k_i) \right), \qquad (35)
$$
\n*n* ∈ R, ∀ x ∈ ℝ^N.
\nAgain for $f ∈ C_B (R^N, X), N ∈ R$, we define the multivariate neural network operator of quadratic
\n*x* of $\theta = (\theta_1,...,\theta_N) ∈ R^N, r = (r_1,...,r_N) ∈ Z^N, u_r = w_{r_1,r_2,...,r_N} ≥ 0$, such that $\sum_{i=0}^{n} w_i = \sum_{r_1=0}^{n} \sum_{r_2=0}^{\infty} \cdots \sum_{r_N=0}^{\infty} w_{r_1,r_2,...,r_N} = 1; k ∈ Z^N$ and
\n
$$
\delta_{nk}(f) := \delta_{n,k_1,k_2,...,k_N} (f) := \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} \cdots \frac{\ell}{n} + \frac{r}{n\theta} \cdot \frac{k_N}{n\theta} + \frac{r_N}{n\theta} \cdot \frac{r}{n\theta}.
$$
\nwhere $\hat{y} := (\frac{\hat{y}_1}{\hat{y}_1}, \frac{\hat{y}_2}{\hat{y}_2}, ..., \frac{\hat{y}_N}{\hat{y}_N}).$
\nWe set
\n
$$
D_{n_1}(f, x) := D_{n_2}(f, x_1,...,x_N) := \sum_{k=-\infty}^{\infty} \delta_{nk}(f) Z(nx - k) = (37)
$$
\n
$$
\sum_{k=1}^{\infty} \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} \cdots \sum_{k_N=-\infty}^{\infty} \delta_{nk}(f) Z(nx - k).
$$
\nIn this article we study the approximation

 $\forall x \in \mathbb{R}^N$.

In this article we study the approximation properties of A_n, B_n, C_n, D_n neural network operators and as well of their iterates. That is, the quantitative pointwise and uniform convergence of these operators to the unit operator I.

3 Multivariate Parametrized Hyperbolic Tangent Induced Banach Space Valued Network Approximations

Here we present several vectorial neural network approximations to Banach space valued functions given with rates.

We give

Theorem 9 Let $f\in C\left(\prod_{i=1}^N\left[a_i,b_i\right],X\right),\, 0<\beta<1,\, \lambda>0,\, x\in\left(\prod_{i=1}^N\left[a_i,b_i\right]\right), N,n\in\mathbb{Z}$ $\mathbb N$ with $n^{1-\beta} > 2$. Then 1)

$$
\|A_n(f,x) - f(x)\|_{\gamma} \le \left(\frac{4}{\tanh 2\lambda}\right)^N \left[\omega_1\left(f, \frac{1}{n^{\beta}}\right) + \frac{2e^{4\lambda} \left\|f\|_{\gamma}\right\|_{\infty}}{e^{2\lambda(n^{1-\beta})}}\right] =: \Omega_1(n),\tag{38}
$$

and

2)

$$
\left\| \left\| A_n \left(f \right) - f \right\|_{\gamma} \right\|_{\infty} \le \Omega_1 \left(n \right). \tag{39}
$$

We notice that $\lim_{n\to\infty} A_n(f) \stackrel{\|\cdot\|_1}{=} f$, pointwise and uniformly.

Above ω_1 is with respect to $p = \infty$ and the speed of convergence is $\max\left(\frac{1}{n^{\beta}}, \frac{1}{e^{2\lambda n^{1/\beta}}}\right)$ $\frac{1}{e^{2\lambda n^{(1-\beta)}}}\right) =$ $\frac{1}{n^{\beta}}$.

Proof. We observe that

1. COMPUTATIONAL ANALYSIS AND APPLICATIONS. VOL-31. NO. 4. 2023. COPYRIGHT 2023 EUDOXUS PRESS, LLC
\n**THEOREM 9** Let
$$
f \in C \left(\prod_{i=1}^{N} [a_i, b_i], X \right)
$$
, $0 < \beta < 1$, $\lambda > 0$, $x \in \left(\prod_{i=1}^{N} [a_i, b_i] \right)$, $N, n \in$
\n**N** with $n^{1-\beta} > 2$. Then
\n
$$
||A_n(f, x) - f(x)||_{\gamma} \leq \left(\frac{4}{\tanh 2\lambda} \right)^N \left[\omega_1 \left(f, \frac{1}{n^{\beta}} \right) + \frac{2e^{t\lambda_1} ||f||_{\gamma} ||_{\infty} }{e^{2\lambda_1(n^2 - t)} \omega} \right] =: \Omega_1(n),
$$
\n(38)
\nand
\n
$$
||A_n(f) - f||_{\gamma} ||_{\infty} \leq \Omega_1(n). \qquad (39)
$$
\nWe notice that $\lim_{\lambda \to \infty} A_n(f) \frac{||\cdot||_{\gamma}}{||\cdot||_{\gamma}}$, pointwise and uniformly.
\n
$$
A \text{for } \omega_1 \text{ is with respect to } p = \infty \text{ and the speed of convergence is } \max \left(\frac{1}{n^{\beta}}, \frac{1}{\epsilon^{2\lambda_0(1-\beta)}} \right) = \frac{1}{n^{\alpha}},
$$
\n**Proof.** We observe that
\n
$$
\Delta(x) := A_n^*(f, x) - f(x) \sum_{k=1/n\alpha}^{[n\alpha]} Z(nx - k) =
$$
\n
$$
\sum_{k=1/n\alpha}^{[n\alpha]} \left\{ \int \left(\frac{k}{n} \right) - f(x) \right\} Z(nx - k).
$$
\nThus
\n
$$
||\Delta(x)||_{\gamma} \leq \sum_{k=1/n\alpha}^{[n\alpha]} \left\| f \left(\frac{k}{n} \right) - f(x) \right\|_{\gamma} Z(nx - k) =
$$
\n
$$
\left\{ \left\| \frac{k}{n} - x \right\|_{\infty} \geq \frac{1}{n^{\beta}} \right\}
$$
\n
$$
\left\{ \left\| \left\
$$

Thus

$$
\|\Delta(x)\|_{\gamma} \leq \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\| f\left(\frac{k}{n}\right) - f(x) \right\|_{\gamma} Z(nx-k) =
$$

$$
\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\| f\left(\frac{k}{n}\right) - f(x) \right\|_{\gamma} Z(nx-k) +
$$

$$
\left\{ \left\| \frac{k}{n} - x \right\|_{\infty} \leq \frac{1}{n^{\beta}} \right\}
$$

$$
\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\| f\left(\frac{k}{n}\right) - f(x) \right\|_{\gamma} Z(nx-k) \leq
$$

$$
\left\{ \left\| \frac{k}{n} - x \right\|_{\infty} > \frac{1}{n^{\beta}} \right\}
$$

0. COMPUTATIONAL ANALYSIS AND APPLICATIONS. VOL-31. NO. 4. 2023. COPYRIGHT 2023 EUDOXUS PRESS. LLC
\n
$$
\omega_1 \left(f, \frac{1}{n^{\beta}} \right) + 2 \left\| \|f\|_{\gamma} \right\|_{\infty} \sum_{k=1}^{\lfloor n\alpha \rfloor} Z(nx-k) \frac{185}{\xi}
$$
\n
$$
\omega_1 \left(f, \frac{1}{n^{\beta}} \right) + 2 \left\| \|f\|_{\gamma} \right\|_{\infty} e^{i\lambda} e^{-2\lambda n^{(1-\beta)}}, \ 0 < \beta < 1, \lambda > 0. \tag{41}
$$
\nSo that
\n
$$
\|\Delta(x)\|_{\gamma} \le \omega_1 \left(f, \frac{1}{n^{\beta}} \right) + \frac{2e^{i\lambda}}{e^{i\lambda} x^{(1-\beta)}}, \ 0 < \beta < 1, \lambda > 0. \tag{41}
$$
\nNow using (31) we finish the proof. We make
\nRemark 10 (15), pp. 263–266) Let $\left(\mathbb{R}^N, \|\cdot\|_{\beta} \right), N \in \mathbb{N}; \text{ where } \|\cdot\|_{\beta} \text{ is the } L_p$ -
\n
$$
norm = \left(\mathbb{E} \rho \le \infty, \mathbb{R}^N \text{ is an Banach space, and } \left[\mathbb{R}^N \right)^2 \text{ denotes the } j \text{-parting. } \left[\frac{1}{2} \rho \le \frac{1}{2} \rho \le \frac{1}{2} \rho \log \rho \right] \text{ and } \rho \
$$

So that

$$
\left\|\Delta\left(x\right)\right\|_{\gamma} \leq \omega_1 \left(f, \frac{1}{n^{\beta}}\right) + \frac{2e^{4\lambda} \left\|f\right\|_{\gamma}}{e^{2\lambda n^{(1-\beta)}}}.
$$
\n(42)

Now using (31) we finish the proof. \blacksquare

We make

Remark 10 ([15], pp. 263-266) Let $(\mathbb{R}^N, \|\cdot\|_p)$, $N \in \mathbb{N}$; where $\|\cdot\|_p$ is the L_p norm, $1 \leq p \leq \infty$. \mathbb{R}^N is a Banach space, and $(\mathbb{R}^N)^j$ denotes the j-fold product space $\mathbb{R}^N \times ... \times \mathbb{R}^N$ endowed with the max-norm $||x||_{(\mathbb{R}^N)^j} := \max_{1 \leq \rho \leq j} ||x_\rho||_p$, where $x := (x_1, ..., x_j) \in (\mathbb{R}^N)^j$.

Let $\left(X,\left\|\cdot\right\|_{\gamma}\right)$ be a general Banach space. Then the space $V_j:=V_j\left(\left(\mathbb{R}^N\right)^j;X\right)$ of all j-multilinear continuous maps $g: (\mathbb{R}^N)^j \to X$, $j = 1, ..., m$, is a Banach space with norm

$$
||g|| := ||g||_{V_j} := \sup_{\left(\|x\|_{\left(\mathbb{R}^N\right)^j} = 1\right)} \left||g(x)||_{\gamma} = \sup \frac{\|g(x)\|_{\gamma}}{\|x_1\|_{p} \dots \|x_j\|_{p}}.\right)
$$
(43)

∥g (x)∥^γ

Let M be a non-empty convex and compact subset of \mathbb{R}^N and $x_0 \in M$ is fixed.

Let O be an open subset of \mathbb{R}^N : $M \subset O$. Let $f: O \to X$ be a continuous function, whose Fréchet derivatives (see [28]) $f^{(j)}: O \to V_j = V_j \left((\mathbb{R}^N)^j; X \right)$ exist and are continuous for $1 \le j \le m$, $m \in \mathbb{N}$.

Call $(x - x_0)^j := (x - x_0, ..., x - x_0) \in (\mathbb{R}^N)^j, x \in M$.

We will work with $f|_M$.

Then, by Taylor's formula $([21]), ([28], p. 124),$ we get

$$
f(x) = \sum_{j=0}^{m} \frac{f^{(j)}(x_0)(x - x_0)^j}{j!} + R_m(x, x_0), \quad all \ x \in M,
$$
 (44)

where the remainder is the Riemann integral

$$
R_m(x, x_0) := \int_0^1 \frac{(1-u)^{m-1}}{(m-1)!} \left(f^{(m)}(x_0 + u(x-x_0)) - f^{(m)}(x_0) \right) (x-x_0)^m du,
$$
\n(45)

here we set $f^{(0)}(x_0)(x-x_0)^0 = f(x_0)$. We consider

$$
\omega := \omega_1 \left(f^{(m)}, h \right) := \sup_{\substack{x, y \in M:\\ \|x - y\|_p \le h}} \left\| f^{(m)}(x) - f^{(m)}(y) \right\|, \tag{46}
$$

 $h > 0$.

We obtain

 f (m) (x⁰ + u (x − x0)) − f (m) (x0) (x − x0) m γ ≤ f (m) (x⁰ + u (x − x0)) − f (m) (x0) · ∥^x [−] ^x0[∥] m ^p ≤ ω ∥x − x0∥ m p u ∥x − x0∥^p h , (47) 501 J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC George A. Anastassiou 490-519

by Lemma 7.1.1, [1], p. 208, where $\lceil \cdot \rceil$ is the ceiling. Therefore for all $x \in M$ (see [1], pp. 121-122):

$$
||R_{m}(x, x_{0})||_{\gamma} \le \omega ||x - x_{0}||_{p}^{m} \int_{0}^{1} \left[\frac{u ||x - x_{0}||_{p}}{h} \right] \frac{(1 - u)^{m-1}}{(m - 1)!} du
$$

$$
= \omega \Phi_{m} \left(||x - x_{0}||_{p} \right) \tag{48}
$$

by a change of variable, where

$$
\Phi_{m}(t) := \int_{0}^{|t|} \left[\frac{s}{h} \right] \frac{(|t| - s)^{m-1}}{(m-1)!} ds = \frac{1}{m!} \left(\sum_{j=0}^{\infty} (|t| - jh)_{+}^{m} \right), \ \ \forall \ t \in \mathbb{R}, \tag{49}
$$

is a (polynomial) spline function, see $\begin{bmatrix} 1 \end{bmatrix}$, p. 210-211.

Also from there we get

$$
\Phi_m(t) \le \left(\frac{|t|^{m+1}}{(m+1)!h} + \frac{|t|^m}{2m!} + \frac{h|t|^{m-1}}{8(m-1)!} \right), \quad \forall \ t \in \mathbb{R},\tag{50}
$$

with equality true only at $t = 0$.

Therefore it holds

$$
\left\|R_m\left(x,x_0\right)\right\|_{\gamma} \le \omega \left(\frac{\left\|x-x_0\right\|_p^{m+1}}{(m+1)!h} + \frac{\left\|x-x_0\right\|_p^m}{2m!} + \frac{h\left\|x-x_0\right\|_p^{m-1}}{8\left(m-1\right)!}\right), \quad \forall \ x \in M. \tag{51}
$$

We have found that

$$
\left\| f(x) - \sum_{j=0}^{m} \frac{f^{(j)}(x_0) (x - x_0)^j}{j!} \right\|_{\gamma} \le
$$

$$
\omega_1\left(f^{(m)},h\right)\left(\frac{\|x-x_0\|_p^{m+1}}{(m+1)!h} + \frac{\|x-x_0\|_p^m}{2m!} + \frac{h\|x-x_0\|_p^{m-1}}{8(m-1)!}\right) < \infty,\tag{52}
$$

 $\forall x, x_0 \in M$.

Here $0 < \omega_1(f^{(m)}, h) < \infty$, by M being compact and $f^{(m)}$ being continuous on M.

One can rewrite (52) as follows:

$$
\left\| f(\cdot) - \sum_{j=0}^{m} \frac{f^{(j)}(x_0) \left(\cdot - x_0 \right)^j}{j!} \right\|_{\gamma} \le
$$

$$
\omega_1 \left(f^{(m)}, h \right) \left(\frac{\left\| \cdot - x_0 \right\|_p^{m+1}}{(m+1)!h} + \frac{\left\| \cdot - x_0 \right\|_p^m}{2m!} + \frac{h \left\| \cdot - x_0 \right\|_p^{m-1}}{8(m-1)!} \right), \ \forall \ x_0 \in M, \ (53)
$$

a pointwise functional inequality on M.

Here $(-x_0)^j$ maps M into $(\mathbb{R}^N)^j$ and it is continuous, also $f^{(j)}(x_0)$ maps $(\mathbb{R}^N)^j$ into X and it is continuous. Hence their composition $f^{(j)}(x_0)(-x_0)^j$ is continuous from M into X.

Clearly
$$
f(\cdot) - \sum_{j=0}^{m} \frac{f^{(j)}(x_0)(-x_0)^j}{j!} \in C(M, X)
$$
, hence $||f(\cdot) - \sum_{j=0}^{m} \frac{f^{(j)}(x_0)(-x_0)^j}{j!}||_{\gamma} \in C(M)$.

Let $\left\{\widetilde{S}_N\right\}$ be a sequence of positive linear operators mapping $C(M)$ into $N \in \mathbb{N}$ $C(M)$.

Therefore we obtain

0. COMPUTATIONAL ANALYSIS AND APPLICATIONS. VOL-31. NO. 4. 2023. COPYRIGHT 2023 EUDOXUS PRESS. LLC
\n
$$
\omega_1 \left(f^{(m)}, h \right) \left(\frac{||x-x_0||_n^{m+1}}{(m+1)!h} + \frac{||x-x_0||_n^{m}}{2m!} + \frac{h ||x-x_0||_n^{m-1}}{8(m-1)!} \right) < \infty, \quad (52)
$$
\n
$$
\forall x, x_0 \in M.
$$
\nHere 0 < $\omega_1 (f^{(m)}, h) < \infty$, by M being compact and $f^{(m)}$ being continuous on M.\n
$$
M.
$$
\nOne can rewrite (52) as follows:\n
$$
\left\| f\left(\cdot\right) - \sum_{j=0}^m \frac{f^{(j)}(x_0) \left(\cdot - x_0\right)^j}{j!} \right\|_2 \le
$$
\n
$$
\omega_1 \left(f^{(m)}, h \right) \left(\frac{||x-x_0||_n^{m+1}}{(m+1)!h} + \frac{||x-x_0||_n^{m}}{2m!} + \frac{h ||x-x_0||_n^{m-1}}{8(m-1)!} \right), \forall x_0 \in M, \quad (53)
$$
\na pointwise functional inequality on M.\n
$$
H.
$$
\nHere $(-x_0)^j$ maps of M into $(\mathbb{R}^N)^j$ and it is continuous, hence the *compostation* of $(0, \mathbb{R}^N)$ (or) maps continuous from M into $(0, \mathbb{R}^N)^j$ into
$$
H.
$$
\n
$$
H.
$$
\nLet $\left\{ \overline{S_N} \right\}_{N \in \mathbb{N}^N}$ be a sequence of positive linear operators mapping $G(M)$ into $W.$ \nLet $\left\{ \overline{S_N} \right\}_{N \in \mathbb{N}^N}$ be a sequence of positive linear operators mapping $G(M)$ into $W.$ \n
$$
H.
$$
\n
$$
H.
$$
\n
$$
G(M).
$$
\nLet $\left\{ \overline{S_N} \right\}_{N \in \mathbb{N}^N}$ be a sequence of positive linear operators mapping $G(M)$ into $W.$

 $\forall\ N\in\mathbb{N},\, \forall\ x_0\in M.$

Clearly (54) is valid when $M = \prod_{n=1}^{N}$ $\prod_{i=1} [a_i, b_i]$ and $S_n = A_n$, see (23).

All the above is preparation for the following theorem, where we assume Fréchet differentiability of functions.

This will be a direct application of Theorem 10.2, [15], pp. 268-270. The operators A_n , A_n fulfill its assumptions, see (22), (23), (25), and (26).

We present the following high order approximation results.

Theorem 11 Let O open subset of $(\mathbb{R}^N, \|\cdot\|_p)$, $p \in [1, \infty]$, such that \prod^N $\prod_{i=1} [a_i, b_i] \subset$ $O\subseteq\mathbb{R}^N,$ and let $\Big(X,\left\|\cdot\right\|_{\gamma}\Big)$ be a general Banach space. Let $m\in\mathbb{N}$ and $f\in$ $C^m(O, X)$, the space of m-times continuously Fréchet differentiable functions $from\ O\ into\ X\ .\ \ We\ study\ the\ approximation\ of\ f|_{\textstyle \prod\limits_{i=1}^{N}[a_i,b_i]}\ .\ Let\ x_0\in \biggl(\prod\limits_{i=1}^{N}[b_i,b_i] \biggr)\ .$ $\prod\limits_{i=1}^N \left[a_i, b_i\right]\bigg)$ $\frac{1}{2}$ $\frac{1}{2}$

$$
and r > 0. Then
$$

$$
1)
$$

$$
\left\| (A_n(f))(x_0) - \sum_{j=0}^m \frac{1}{j!} \left(A_n \left(f^{(j)}(x_0) \left(\cdot - x_0 \right)^j \right) \right) (x_0) \right\|_{\gamma} \le
$$

$$
\frac{\omega_1\left(f^{(m)}, r\left(\left(\tilde{A}_n\left(\left\|\cdot-x_0\right\|_p^{m+1}\right)\right)(x_0)\right)^{\frac{1}{m+1}}\right)}{rm!}\left(\left(\tilde{A}_n\left(\left\|\cdot-x_0\right\|_p^{m+1}\right)\right)(x_0)\right)^{\left(\frac{m}{m+1}\right)}
$$
\n
$$
\left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8}\right],\tag{55}
$$

2) additionally if $f^{(j)}(x_0) = 0$, $j = 1, ..., m$, we have

$$
\| (A_n(f))(x_0) - f(x_0) \|_{\gamma} \le
$$

$$
\frac{\omega_1 \left(f^{(m)}, r \left(\left(\tilde{A}_n \left(\left\| \cdot - x_0 \right\|_p^{m+1} \right) \right) (x_0) \right)^{\frac{1}{m+1}} \right)}{rm!} \left(\left(\tilde{A}_n \left(\left\| \cdot - x_0 \right\|_p^{m+1} \right) \right) (x_0) \right)^{\left(\frac{m}{m+1} \right)}
$$

$$
\left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8} \right],
$$

$$
\frac{\omega_1 \left(\left(\frac{1}{m+1} + \frac{r}{2} + \frac{mr^2}{8} \right) \right)}{2 \left(\frac{1}{m+1} + \frac{r}{2} + \frac{mr^2}{8} \right)}
$$

1. COMPUTATIONAL ANALYSIS AND APPLICATIONS. VOL-31. No. 4. 2023. COPYRIGHT 2023 EUDOXUS PRESS, LLC
\n**Theorem 11** Let O open subset of
$$
(\mathbb{R}^N, ||\cdot||_p)
$$
, $p \in [1, \infty]$, such that $\prod_{i=1}^N [a_i, b_i] \in$
\n $O \subseteq \mathbb{R}^N$, and let $(X, ||\cdot||_p)$ be a general Banach space. Let $m \in \mathbb{N}$ and $f \in$
\n $C^m(O, X)$, the space of m -times continuously Fréchet differentiable functions
\nfrom O into X. We study the approximation of $f|_{\prod_{i=1}^N [a_i, b_i]}$
\nand $r > 0$. Then
\n
$$
|_{(A_n(f))(x_0)} = \sum_{j=0}^m \frac{1}{j!} (A_n (f^{(j)}(x_0)(\cdot - x_0)^j)) (x_0) |_{\infty} \le
$$
\n
$$
\omega_1 (f^{(m)} \cdot r ((\widetilde{A}_n (||\cdot - x_0||_p^{m+1})) (x_0)) \xrightarrow{m+1} (\widetilde{A}_n (||\cdot - x_0||_p^{m+1})) (x_0))^{(\frac{m}{m+1})}
$$
\n
$$
= \omega_1 (f^{(m)} \cdot r ((\widetilde{A}_n (||\cdot - x_0||_p^{m+1})) (x_0)) \xrightarrow{m+1} (\widetilde{A}_n (||\cdot - x_0||_p^{m+1})) (x_0))^{(\frac{m}{m+1})}
$$
\n
$$
= \omega_1 (f^{(m)} \cdot r ((\widetilde{A}_n (||\cdot - x_0||_p^{m+1})) (x_0)) \xrightarrow{m+1} (\widetilde{A}_n (||\cdot - x_0||_p^{m+1})) (x_0))^{(\frac{m}{m+1})}
$$
\n
$$
= \omega_1 (f^{(m)} \cdot r ((\widetilde{A}_n (||\cdot - x_0||_p^{m+1}))(x_0)) \xrightarrow{m+1} (\widetilde{A}_n (||\cdot - x_0||_p^{m+1})) (x_0))^{(\frac{m}{m+1})}
$$
\n

I. COMPUTATIONAL ANALYSIS AND APPLICATIONS. VOL 31. NO. 4. 2023. COPYRIGHT 2023 EUDOXUS PRESS, LLC
\n
$$
\sum_{j=1}^{m} \frac{1}{j!} \left\| \left\| \left(A_n \left(f^{(j)}(x_0) \left(\cdot - x_0 \right)^j \right) \right) (x_0) \right\|_{\infty, x_0 \in \prod_{i=1}^{N} [a_i, b_i]} + \frac{\omega_1 \left(f^{(m)}, r \left| \left(\tilde{A}_n \left(\left\| \cdot - x_0 \right\|_p^{m+1} \right) \right) (x_0) \right|_{\infty, x_0 \in \prod_{i=1}^{N} [a_i, b_i]} - \frac{\omega_1 \left(f^{(m)}, r \left| \left(\tilde{A}_n \left(\left\| \cdot - x_0 \right\|_p^{m+1} \right) \right) (x_0) \right| \Big|_{\infty, x_0 \in \prod_{i=1}^{N} [a_i, b_i]} - \frac{\omega_1 \left(\frac{n}{\sqrt{n}} \right)}{\left[(m+1) + \frac{r}{2} + \frac{n\omega^2}{8} \right] \right\}} \right\}
$$
\nWe need
\n**Lemma 12** The function $\left(\tilde{A}_n \left(\left\| \cdot - x_0 \right\|_p^{m} \right) \right) (x_0) \text{ is continuous in } x_0 \in \left(\prod_{i=1}^{N} [a_i, b_i] \right),$
\n*m* ∈ ℕ.
\n**Proof.** By Lemma 10.3, [15], p. 272.
\n**Remark** 13. By Remark 10.4, [15], p. 273, we get that
\n
$$
\left\| \left(\tilde{A}_n \left(\left\| \cdot - x_0 \right\|_p^{b} \right) \right) (x_0) \right\|_{\infty, x_0 \in \prod_{i=1}^{N} [a_i, b_i]} \leq \left\| \left(\tilde{A}_n \left(\left\| \cdot - x_0 \right\|_p^{m+1} \right) \right) (x_0) \right\|_{\infty, x_0 \in \prod_{i=1}^{N} [a_i, b_i]} \right\}
$$

\nfor all

We need

Lemma 12 The function $\left(\widetilde{A}_n\left(\left\|\cdot-x_0\right\|_p^m\right)\right)(x_0)$ is continuous in $x_0 \in \left(\prod_{i=1}^N\right)$ $\prod_{i=1}^N\left[a_i,b_i\right]\bigg),$ $m \in \mathbb{N}$.

Proof. By Lemma 10.3, [15], p. 272.

Remark 13 By Remark 10.4 [15], p.273, we get that

$$
\left\| \left(\widetilde{A}_n \left(\left\| \cdot - x_0 \right\|_p^k \right) \right) (x_0) \right\|_{\infty, x_0 \in \prod_{i=1}^N [a_i, b_i]} \leq \left\| \left(\widetilde{A}_n \left(\left\| \cdot - x_0 \right\|_p^{m+1} \right) \right) (x_0) \right\|_{\infty, x_0 \in \prod_{i=1}^N [a_i, b_i]}^{(\frac{k}{m+1})},\tag{59}
$$

for all $k = 1, ..., m$.

We give

1)

Corollary 14 (to Theorem 11, case of $m = 1$) Then

$$
\| (A_n(f)) (x_0) - f (x_0) \|_{\gamma} \le \| \left(A_n \left(f^{(1)} (x_0) \left(\cdot - x_0 \right) \right) \right) (x_0) \|_{\gamma} +
$$

$$
\frac{1}{2r} \omega_1 \left(f^{(1)}, r \left(\left(\tilde{A}_n \left(\left\| \cdot - x_0 \right\|_p^2 \right) \right) (x_0) \right)^{\frac{1}{2}} \right) \left(\left(\tilde{A}_n \left(\left\| \cdot - x_0 \right\|_p^2 \right) \right) (x_0) \right)^{\frac{1}{2}} \quad (60)
$$

$$
\left[1 + r + \frac{r^2}{4} \right],
$$

and 2)

$$
\left\| \left\| (A_n(f)) - f \right\|_{\gamma} \right\|_{\infty, \prod\limits_{i=1}^N [a_i, b_i]} \le
$$

$$
\left\| \left\| \left(A_n \left(f^{(1)} \left(x_0 \right) \left(\cdot - x_0 \right) \right) \right) (x_0) \right\|_{\gamma} \right\|_{\infty, x_0 \in \prod_{i=1}^N [a_i, b_i]} +
$$

$$
\frac{1}{2r} \omega_1 \left(f^{(1)}, r \left\| \left(\widetilde{A}_n \left(\left\| \cdot - x_0 \right\|_p^2 \right) \right) (x_0) \right\|_{\infty, x_0 \in \prod_{i=1}^N [a_i, b_i]}^{\frac{1}{2}} \right)
$$

$$
\left\| \left(\widetilde{A}_n \left(\left\| \cdot - x_0 \right\|_p^2 \right) \right) (x_0) \right\|_{\infty, x_0 \in \prod_{i=1}^N [a_i, b_i]}^{\frac{1}{2}} \left[1 + r + \frac{r^2}{4} \right],
$$
(61)

 $r > 0.$

We make

Remark 15 We estimate $0 < \alpha < 1$, $\lambda > 0$, $m, n \in \mathbb{N} : n^{1-\alpha} > 2$,

L. COMPUTATIONAL ANALYSIS AND APPLICATIONS. VOL. 31. NO. 4. 2023. COPYRIGHT 2023 EUDOXUS PRESS. LLC
\n
$$
\left\| \left\| \left(A_n \left(f^{(1)}(x_0) \left(\cdot - x_0 \right) \right) \right) (x_0) \right\|_{\infty} \right\|_{\infty, x_0 \in \prod_{i=1}^N [a_i, b_i]}
$$
\n
$$
\frac{1}{2r} \omega_1 \left(f^{(1)} \cdot r \left\| \left(\tilde{A}_n \left(\left\| \cdot - x_0 \right\|_p^2 \right) \right) (x_0) \right\|_{\infty, x_0 \in \prod_{i=1}^N [a_i, b_i]}
$$
\n
$$
\left\| \left(\tilde{A}_n \left(\left\| \cdot - x_0 \right\|_p^2 \right) \right) (x_0) \right\|_{\infty, x_0 \in \prod_{i=1}^N [a_i, b_i]}
$$
\n
$$
r > 0.
$$
\nWe make
\nRemark 15 We estimate $0 < \alpha < 1, \lambda > 0, m, n \in \mathbb{N} : n^{1-\alpha} > 2,$
\n
$$
\tilde{A}_n \left(\left\| \cdot - x_0 \right\|_{\infty}^{m+1} \right) (x_0) = \frac{\sum_{k=1}^N [a_k]}{[a_k]} \left\| \frac{k}{n} - x_0 \right\|_{\infty}^{m+1} Z \left(nx_0 - k \right)
$$
\n
$$
\left(\frac{4}{\tanh 2\lambda} \right)^N \left\{ \sum_{k=1}^N \left\| \frac{k}{n} - x_0 \right\|_{\infty}^{m+1} Z \left(nx_0 - k \right) = \frac{\left(\frac{a_0}{2} \right)}{2 \cdot \left\| \frac{k}{n} - x_0 \right\|_{\infty}^{m+1} Z \left(nx_0 - k \right) + \left\{ \frac{1}{\tanh 2\lambda} \right\}^N \left\{ \sum_{k=1}^N \left\| \frac{k}{n} - x_0 \right\|_{\infty}^{m+1} Z \left(nx_0 - k \right) + \left\{ \frac{1}{\tanh 2\lambda} \right\}^N \left\{ \sum
$$

 $(where b - a = (b_1 - a_1, ..., b_N - a_N)).$ We have proved that $(\forall x_0 \in \prod^N$ $\prod_{i=1} [a_i, b_i]$

$$
\widetilde{A}_n \left(\left\| \cdot - x_0 \right\|_{\infty}^{m+1} \right) (x_0) < \left(\frac{4}{\tanh 2\lambda} \right)^N \left\{ \frac{1}{n^{\alpha(m+1)}} + \frac{e^{4\lambda} \left\| b - a \right\|_{\infty}^{m+1}}{e^{2\lambda(n^{1-\beta})}} \right\} =: \Lambda_1(n) \tag{64}
$$

 $(0 < \alpha < 1, m, n \in \mathbb{N} : n^{1-\alpha} > 2, \lambda > 0).$ And, consequently it holds

$$
\left\| \widetilde{A}_n \left(\left\| \cdot - x_0 \right\|_{\infty}^{m+1} \right) (x_0) \right\|_{\infty, x_0 \in \prod_{i=1}^N [a_i, b_i]} \left(\frac{4}{\tanh 2\lambda} \right)^N \left\{ \frac{1}{n^{\alpha(m+1)}} + \frac{e^{4\lambda} \left\| b - a \right\|_{\infty}^{m+1}}{e^{2\lambda(n^{1-\beta})}} \right\} = \Lambda_1(n) \to 0, \text{ as } n \to +\infty.
$$
\n(65)

So, we have that $\Lambda_1(n) \to 0$, as $n \to +\infty$. Thus, when $p \in [1,\infty]$, from Theorem 11 we have the convergence to zero in the right hand sides of parts (1) , (2).

Next we estimate
$$
\left\| \left(\widetilde{A}_n \left(f^{(j)}(x_0) \left(\cdot - x_0 \right)^j \right) \right) (x_0) \right\|_{\gamma}
$$
.
We have that

$$
\left(\widetilde{A}_{n}\left(f^{(j)}\left(x_{0}\right)\left(\cdot-x_{0}\right)^{j}\right)\right)\left(x_{0}\right)=\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}f^{(j)}\left(x_{0}\right)\left(\frac{k}{n}-x_{0}\right)^{j}Z\left(nx_{0}-k\right)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}Z\left(nx_{0}-k\right)}.
$$
\n(66)

When $p = \infty$, $j = 1, ..., m$, we obtain

$$
\left\| f^{(j)}\left(x_0\right) \left(\frac{k}{n} - x_0\right)^j \right\|_{\gamma} \le \left\| f^{(j)}\left(x_0\right) \right\| \left\| \frac{k}{n} - x_0 \right\|_{\infty}^j. \tag{67}
$$

We further have that

0. COMPUTTONAL ANALYSIS AND APPLICATIONS. VOL-31. NO. 4. 2023. COPYRIGHT 2023 EUDOXUS PRESS. LIC
\n
$$
(0 < \alpha < 1, m, n \in \mathbb{N} : n^{1-\alpha} > 2, \lambda > 0).
$$
\nAnd, consequently it holds
\n
$$
\left\| \widetilde{A}_n \left(\left\| \cdots x_0 \right\|_{\infty}^{m+1} \right) (x_0) \right\|_{\infty, x_0 \in \prod_{i=1}^{\infty} [a_i, b_i]} \n\left(\frac{4}{\tanh 2\lambda} \right)^N \left\{ \frac{1}{n^{\alpha(m+1)}} + \frac{e^{4\lambda} |b-a|_{\infty}^{m+1}}{e^{2\lambda(n-2)}} \right\} = \Lambda_1(n) \rightarrow 0, \text{ as } n \rightarrow +\infty.
$$
\nSo, we have that $\Lambda_1(n) \rightarrow 0$, as $n \rightarrow +\infty$. Thus, when $p \in [1, \infty]$, from Theorem II we have the convergence to zero in the right hand sides of parts (1), (2).
\n*Next we estimate* $\left\| (\widetilde{A}_n \left(f^{(j)}(x_0) (-x_0)^j) \right) (x_0) \right\|_{\infty}$.
\n*When* $p = \infty$, $j = 1, ..., m$, we obtain
\n
$$
\left\| f^{(j)}(x_0) \left(\frac{k}{n} - x_0 \right)^j \right\|_{\infty} \le \left\| f^{(j)}(x_0) \left(\frac{k}{n} - x_0 \right)^j \frac{z}{\alpha} (x_0 - k) \right\}.
$$
\n*When* $p = \infty$, $j = 1, ..., m$, we obtain
\n
$$
\left\| f^{(j)}(x_0) \left(\frac{k}{n} - x_0 \right)^j \right\|_{\infty} \le \left\| f^{(j)}(x_0) \right\| \left\| \frac{k}{n} - x_0 \right\|_{\infty}^j.
$$
\n*(67)*
\n*We further have that*
\n
$$
\left\| \left(\widetilde{A}_n \left(f^{(j)}(x_0) (-x_0)^j \right) \right)
$$
1. COMPUTATIONAL ANALYSIS AND APPLICATIONS. VOL-31. NO. 4. 2023. COPYRIGHT 2023 EUDOXUS PRESS, LLC
\n
$$
\left\{\n\begin{array}{l}\n\text{1. } k = |ma| \\
\text{2. } |a| & k = |na| \\
\text{3. } |a| & k = |na| \\
\text{4. } |a| & k = |na| \\
\text{5. } |a| & k = |na| \\
\text{7. } |a| & k = |na| \\
\text{8. } |a| & k = |na| \\
\text{9. } |a| & k = |na| \\
\text{10. } |a| & k = |na| \\
\text{11. } |a| & k = |na| \\
\text{12. } |a| & k = |na| \\
\text{13. } |a| & k = |na| \\
\text{14. } |a| & k = |na| \\
\text{15. } |a| & k = |na| \\
\text{16. } |a| & k = |na| \\
\text{17. } |a| & k = |na| \\
\text{18. } |a| & k = |na| \\
\text{18. } |a| & k = |na| \\
\text{19. } |a| & k = |na| \\
\text{10. } |a| & k = |na| \\
\text{11. } |a| & k = |na| \\
\text{12. } |a| & k = |na| \\
\text{13. } |a| & k = |na| \\
\text{14. } |a| & k = |na| \\
\text{15. } |a| & k = |na| \\
\text{16. } |a| & k = |na| \\
\text{17. } |a| & k = |na| \\
\text{18. } |a| & k = |na| \\
\text{19. } |a| & k = |na| \\
\text{10. } |a| & k = |na| \\
\text{11. } |a| & k = |na| \\
\text{12. } |a| & k = |na| \\
\text{13. } |a| & k = |a| \\
\text{14. } |a| & k = |a| \\
\text{15. } |a| & k = |a| \\
\text{16. } |a| & k = |a| \\
\text{17. } |a| & k = |a| \\
\text{18. } |a| & k = |a| \\
\text{19. } |a| & k = |a| \\
\text{10. } |a| & k = |a| \\
\text{11. } |a| & k = |a| \\
\text{12. } |a|
$$

That is

$$
\left\|\left(\widetilde{A}_n\left(f^{(j)}\left(x_0\right)(\cdot-x_0)^j\right)\right)(x_0)\right\|_{\gamma}\to 0, \text{ as } n\to\infty.
$$

Therefore when $p = \infty$, for $j = 1, ..., m$, we have proved:

$$
\left\| \left(\widetilde{A}_n \left(f^{(j)} \left(x_0 \right) (\cdot - x_0)^j \right) \right) (x_0) \right\|_{\gamma} <
$$

$$
\left(\frac{4}{\tanh 2\lambda} \right)^N \left\| f^{(j)} \left(x_0 \right) \right\| \left\{ \frac{1}{n^{\alpha j}} + \frac{e^{4\lambda} \left\| b - a \right\|_{\infty}^j}{e^{2\lambda(n^{1-\beta})}} \right\} \le
$$

$$
\left(\frac{4}{\tanh 2\lambda} \right)^N \left\| f^{(j)} \left(x_0 \right) \right\|_{\infty} \left\{ \frac{1}{n^{\alpha j}} + \frac{e^{4\lambda} \left\| b - a \right\|_{\infty}^j}{e^{2\lambda(n^{1-\beta})}} \right\} =: \Lambda_{2j} \left(n \right) < \infty, \tag{70}
$$

and converges to zero, as $n \to \infty$.

We conclude:

In Theorem 11, the right hand sides of (57) and (58) converge to zero as $n \to \infty$, for any $p \in [1, \infty]$.

Also in Corollary 14, the right hand sides of (60) and (61) converge to zero as $n \to \infty$, for any $p \in [1, \infty]$.

Conclusion 16 We have proved that the left hand sides of (55), (56), (57), (58) and (60), (61) converge to zero as $n \to \infty$, for $p \in [1,\infty]$. Consequently $A_n \to I$ (unit operator) pointwise and uniformly, as $n \to \infty$, where $p \in [1, \infty]$. In the presence of initial conditions we achieve a higher speed of convergence, see (56). Higher speed of convergence happens also to the left hand side of (55).

We further give

Corollary 17 (to Theorem 11) Let O open subset of $(\mathbb{R}^N, \|\cdot\|_{\infty})$, such that \prod^N $\prod_{i=1}^N [a_i, b_i] \subset O \subseteq \mathbb{R}^N$, and let $(X, \|\cdot\|_{\gamma})$ be a general Banach space. Let $m \in \mathbb{N}$ and $f \in C^m(0,X)$, the space of m-times continuously Fréchet differentiable functions from O into X. We study the approximation of $f|_{\prod_{i} [a_i, b_i]}$. Let $x_0 \in$ $i=1$

L. COMPUTATIONAL ANALYSIS AND APPLICATIONS. VOL. 31. No. 4. 2023. COPYRIGHT 2023 EUDOXUS PRESS, LLC
\n
$$
\left(\prod_{n=1}^{N} |a_{i}, b_{i}|\right) and r > 0. Here \Delta_{1}(n) as in (65) and \Delta_{2j}(n) as in (70), where
$$
\n
$$
n \in \mathbb{N} : n^{1-\alpha} > 2, 0 < \alpha < 1, \lambda > 0, j = 1, ..., m. Then
$$
\n
$$
1)
$$
\n
$$
\left\| (A_{n}(f))(x_{0}) - \sum_{n=0}^{m} \frac{1}{j!} \left(A_{n}\left(f^{(j)}(x_{0})(\cdot - x_{0})^{j}\right) \right) (x_{0}) \right\|_{1} \leq
$$
\n
$$
\frac{\omega_{1}\left(f^{(m)}, r(\Delta_{1}(n))^{\frac{1}{m+1}}\right)}{rm} \left(\Delta_{1}(n) \right)^{(\frac{m}{m+1})} \left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^{2}}{8} \right], \qquad (71)
$$
\n2) additionally, if $f^{(j)}(x_{0}) = 0, j = 1, ..., m$, we have
\n
$$
\left\| (A_{n}(f))(x_{0}) - f(x_{0}) \right\|_{1} \leq
$$
\n
$$
\frac{\omega_{1}\left(f^{(m)}, r(\Delta_{1}(n))^{\frac{1}{m+1}}\right)}{rm} \left(\Delta_{1}(n) \right)^{(\frac{m}{m+1})} \left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^{2}}{8} \right]. \qquad (72)
$$
\n3)
\n
$$
\left\| \|A_{n}(f) - f\|_{r} \right\|_{\infty, \prod_{i=1}^{m} (a_{i}, b_{i})} \leq \sum_{j=1}^{m} \frac{\Delta_{2j}(n)}{j!} +
$$
\n
$$
\frac{\omega_{1}\left(f^{(m)}, r(\Delta_{1}(n))^{\frac{1}{m+1}}\right)}{rm} \left(\Delta_{1}(n) \right)^{(\frac{m}{m+1})} \left(\frac{1}{m+1} + \frac{r}{2} + \frac{mr^{2}}{8} \right]. \qquad (
$$

2) additionally, if $f^{(j)}(x_0) = 0, j = 1, ..., m$, we have

$$
\left\| (A_n(f))(x_0) - f(x_0) \right\|_{\gamma} \le
$$

$$
\frac{\omega_1\left(f^{(m)}, r\left(\Lambda_1\left(n\right)\right)^{\frac{1}{m+1}}\right)}{rm!}\left(\Lambda_1\left(n\right)\right)^{\left(\frac{m}{m+1}\right)}\left[\frac{1}{\left(m+1\right)} + \frac{r}{2} + \frac{mr^2}{8}\right],\tag{72}
$$

3)

$$
\left\| \|A_n(f) - f\|_{\gamma} \right\|_{\infty, \prod_{i=1}^N [a_i, b_i]} \le \sum_{j=1}^m \frac{\Lambda_{2j}(n)}{j!} + \frac{\omega_1 \left(f^{(m)}, r \left(\Lambda_1(n) \right)^{\frac{1}{m+1}} \right)}{rm!} (\Lambda_1(n))^{\left(\frac{m}{m+1} \right)}
$$

$$
\left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8} \right] =: \Lambda_3(n) \to 0, \text{ as } n \to \infty. \tag{73}
$$

We continue with

Theorem 18 Let $f \in C_B(\mathbb{R}^N, X)$, $0 < \beta < 1$, $\lambda > 0$, $x \in \mathbb{R}^N, N, n \in \mathbb{N}$ with $n^{1-\beta} > 2$, ω_1 is for $p = \infty$. Then 1)

$$
\|B_n(f,x) - f(x)\|_{\gamma} \le \omega_1 \left(f, \frac{1}{n^{\beta}}\right) + \frac{e^{4\lambda} \|f\|_{\gamma}}{e^{2\lambda(n^{1-\beta})}} =: \Omega_2(n),\tag{74}
$$

2)

$$
\left\| \left\| B_n \left(f \right) - f \right\|_{\gamma} \right\|_{\infty} \le \Omega_2 \left(n \right). \tag{75}
$$

Given that $f \in (C_U(\mathbb{R}^N, X) \cap C_B(\mathbb{R}^N, X)),$ we obtain $\lim_{n \to \infty} B_n(f) = f$, uniformly. The speed of convergence above is $\max\left(\frac{1}{n^{\beta}}, \frac{1}{e^{2\lambda n^{0}}}\right)$ $\frac{1}{e^{2\lambda n(1-\beta)}}\Big) = \frac{1}{n^{\beta}}.$

Proof. We have that

$$
B_n(f, x) - f(x) \stackrel{(13)}{=} \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) Z(nx - k) - f(x) \sum_{k=-\infty}^{\infty} Z(nx - k) = (76)
$$

$$
\sum_{k=-\infty}^{\infty} \left(f\left(\frac{k}{n}\right) - f(x) \right) Z(nx - k).
$$

Hence

L. COMPUTATIONAL ANALYSIS AND APPLICATIONS. VOL-31. NO. 4. 2223. COPYRIGHT 2223 EUDOXUS PRESS, LLC
\nProof. We have that
\n
$$
B_n(f,x) - f(x) \left(\frac{320}{2}\right) \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) Z(nx-k) - f(x) \sum_{k=-\infty}^{\infty} Z(nx-k) = (76)
$$
\n
$$
\sum_{k=-\infty}^{\infty} \left(f\left(\frac{k}{n}\right) - f(x) \right) Z(nx-k).
$$
\nHence
\n
$$
||B_n(f,x) - f(x)||_{\gamma} \le \sum_{k=-\infty}^{\infty} \left||f\left(\frac{k}{n}\right) - f(x)\right||_{\gamma} Z(nx-k) =
$$
\n
$$
\left\{ \left|\frac{k}{n} - x\right|_{\infty} \le \frac{1}{n^3}
$$
\n
$$
\left|\int_{1}^{k} \left|\frac{k}{n}\right| - f(x) \right|\right|_{\gamma} Z(nx-k) \right\} \le \left\{ \left|\int_{1}^{k} - x\right|_{\infty} \le \frac{1}{n^3}
$$
\n
$$
\omega_1 \left(f, \frac{1}{n^3}\right) + 2 \left|\left||f\right|_{\gamma}\right|\right|_{\infty} \sum_{k=-\infty}^{\infty} Z(nx-k) \stackrel{(33)}{\leq}
$$
\n
$$
\omega_1 \left(f, \frac{1}{n^3}\right) + 2 \left|\left||f\right|_{\gamma}\right|_{\infty} \sum_{k=-\infty}^{\infty} Z(nx-k) \stackrel{(35)}{\leq}
$$
\n
$$
\omega_1 \left(f, \frac{1}{n^3}\right) + 2 \left|\left|\left||f\right|_{\gamma}\right|_{\infty} \sum_{k=-\infty}^{\infty} Z(nx-k) \stackrel{(36)}{\leq}
$$
\n
$$
\omega_1 \left(f, \frac{1}{n^3}\right) + 2 \left|\left|\left|\left|f\right|_{\gamma}\right|\right|_{\infty},
$$
\n
$$
\omega_1 \left(f, \frac{1}{n^3}\right) + \frac{2e^{4\lambda}}{e^{3\lambda(n^{1-3})}} ||f\right|_{\gamma}\right|_{\infty},
$$
\n
$$
\
$$

proving the claim. \blacksquare

We give

Theorem 19 Let $f \in C_B(\mathbb{R}^N, X)$, $0 < \beta < 1$, $x \in \mathbb{R}^N$, $\lambda > 0, N, n \in \mathbb{N}$ with $n^{1-\beta} > 2$, ω_1 is for $p = \infty$. Then 1)

$$
\|C_n(f,x) - f(x)\|_{\gamma} \le \omega_1 \left(f, \frac{1}{n} + \frac{1}{n^{\beta}}\right) + \frac{2e^{4\lambda} \|f\|_{\gamma}}{e^{2\lambda(n^{1-\beta})}} =: \Omega_3(n), \quad (78)
$$

2)

$$
\left\| \left\| C_n \left(f \right) - f \right\|_{\gamma} \right\|_{\infty} \le \Omega_3 \left(n \right). \tag{79}
$$

Given that $f \in (C_U(\mathbb{R}^N, X) \cap C_B(\mathbb{R}^N, X))$, we obtain $\lim_{n \to \infty} C_n(f) = f$, uniformly.

Proof. We notice that

$$
\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t) dt = \int_{\frac{k_1}{n}}^{\frac{k_1+1}{n}} \int_{\frac{k_2}{n}}^{\frac{k_2+1}{n}} \dots \int_{\frac{k_N}{n}}^{\frac{k_N+1}{n}} f(t_1, t_2, ..., t_N) dt_1 dt_2...dt_N =
$$

$$
\int_0^{\frac{1}{n}} \int_0^{\frac{1}{n}} \dots \int_0^{\frac{1}{n}} f\left(t_1 + \frac{k_1}{n}, t_2 + \frac{k_2}{n}, ..., t_N + \frac{k_N}{n}\right) dt_1...dt_N = \int_0^{\frac{1}{n}} f\left(t + \frac{k}{n}\right) dt.
$$
(80)

Thus it holds (by (35))

$$
C_n(f,x) = \sum_{k=-\infty}^{\infty} \left(n^N \int_0^{\frac{1}{n}} f\left(t + \frac{k}{n}\right) dt \right) Z\left(nx - k\right). \tag{81}
$$

We observe that

$$
J. \text{ COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL-31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC}
$$
\nProof. We notice that\n
$$
\int_{\frac{1}{u}}^{\frac{h+1}{u}} f(t) dt = \int_{\frac{1}{u}}^{\frac{h+1}{u}} \int_{\frac{1}{u}}^{\frac{h+1}{u}} ... \int_{\frac{1}{u}}^{\frac{h+1}{u}} f(t_1, t_2, ..., t_N) dt_1 dt_2 ... dt_N =
$$
\n
$$
\int_{0}^{\frac{1}{u}} \int_{0}^{\frac{1}{u}} ... \int_{0}^{\frac{1}{u}} f(t_1 + \frac{k_1}{u}, t_2 + \frac{k_2}{u}, ..., t_N + \frac{k_N}{u}) dt_1 ... dt_N = \int_{0}^{\frac{1}{u}} f(t_1 + \frac{k}{u}) dt,
$$
\nThus it holds (by (35))\n
$$
C_n (f, x) = \sum_{k=-\infty}^{\infty} \left(n^N \int_{0}^{\frac{1}{u}} f(t_1 + \frac{k}{u}) dt \right) Z(nx - k).
$$
\nWe observe that\n
$$
||C_n (f, x) - f(x)||_{\gamma} =
$$
\n
$$
\left\| \sum_{k=-\infty}^{\infty} \left(n^N \int_{0}^{\frac{1}{u}} f(t_1 + \frac{k}{u}) dt \right) - f(x) \right) Z(nx - k) \right\|_{\gamma} =
$$
\n
$$
\left\| \sum_{k=-\infty}^{\infty} \left(n^N \int_{0}^{\frac{1}{u}} f(t_1 + \frac{k}{u}) dt \right) - f(x) \right) dt
$$
\n
$$
Z(nx - k) =
$$
\n
$$
\left\| \sum_{k=-\infty}^{\infty} \left(n^N \int_{0}^{\frac{1}{u}} \left| f(t_1 + \frac{k}{u}) - f(x) \right|_{\gamma} dt \right) Z(nx - k) +
$$
\n
$$
\left\{ \left\| \frac{k}{u} - x \right\|_{\infty} \leq \frac{1}{u^2}.
$$
\n
$$
\sum_{k=-\infty}^{\infty} \left(n^N \int_{0}^{\frac{1}{u}} \left| f(t_1 + \frac{k}{u}) - f(x) \right|
$$

$$
2 \left\| \|f\|_{\gamma} \right\|_{\infty} \left(\sum_{\begin{subarray}{c}k = -\infty \\ \left\{\left\|\frac{k}{n} - x\right\|_{\infty} > \frac{1}{n^{\beta}}\end{subarray}\right\}} Z\left(\left|nx - k\right|\right) \right) \le
$$

$$
\omega_{1} \left(f, \frac{1}{n} + \frac{1}{n^{\beta}} \right) + \frac{2e^{4\lambda} \left\| \|f\|_{\gamma} \right\|_{\infty}}{e^{2\lambda(n^{1-\beta})}},
$$
(83)

proving the claim. \quadblacksquare

We also present

Theorem 20 Let $f \in C_B(\mathbb{R}^N, X)$, $0 < \beta < 1$, $x \in \mathbb{R}^N$, $\lambda > 0, N, n \in \mathbb{N}$ with $n^{1-\beta} > 2$, ω_1 is for $p = \infty$. Then 1)

$$
||D_n(f, x) - f(x)||_{\gamma} \le \omega_1 \left(f, \frac{1}{n} + \frac{1}{n^{\beta}}\right) + \frac{2e^{4\lambda} ||f||_{\gamma} ||_{\infty}}{e^{2\lambda(n^{1-\beta})}} =: \Omega_4(n), \quad (84)
$$

2)

$$
\left\| \left\| D_n \left(f \right) - f \right\|_{\gamma} \right\|_{\infty} \le \Omega_4 \left(n \right). \tag{85}
$$

Given that $f \in (C_U (\mathbb{R}^N, X) \cap C_B (\mathbb{R}^N, X))$, we obtain $\lim_{n \to \infty} D_n (f) = f$, uniformly.

Proof. We have that (by (37))

0. COMPUTTONAL ANALYSIS AND APPLICATIONS. VOL. 31. NO. 4. 2023. COPYRIGHT 2023 EUDOXUS PRESS. LLC
\n
$$
2 ||||f||_x||_{\infty} \left(\sum_{\substack{k=-\infty \\ |\frac{n}{n}-x|_{\infty} > \frac{1}{n^{\alpha}}}} Z(|nx-k|) \right) \le
$$
\n
$$
2 ||||f||_x||_{\infty} \left(\sum_{\substack{l=-\infty \\ |\frac{n}{n}-x|_{\infty} > \frac{1}{n^{\alpha}}}} Z(|nx-k|) \right) \le
$$
\n
$$
= \sum_{\substack{k=-\infty \\ |\frac{n}{n}-\frac{n}{n} > 0}} \omega_1 \left(f, \frac{1}{n} + \frac{1}{n^{\beta}} \right) + \frac{2e^{4\lambda} |||f||_x||_{\infty}}{e^{2\lambda(n^2-\alpha)}} , \qquad (83)
$$
\nproving the claim. **■**
\nWe also present
\n**Theorem 20** Let $f \in C_R (\mathbb{R}^N, X), 0 < \beta < 1, x \in \mathbb{R}^N, \lambda > 0, N, n \in \mathbb{N}$ with $n^{1-\beta} > \omega_1$ is a for $p = \infty$. Then
\n
$$
|D_{\alpha}(f, x) - f(x)|_{\alpha} \leq \omega_1 \left(f, \frac{1}{n} + \frac{1}{n^{\beta}} \right) + \frac{2e^{4\lambda} |||f||_x||_{\infty}}{e^{2\lambda(n^2-\alpha^2)}} =: \Omega_4(n), \qquad (84)
$$
\n
$$
= \sum_{\substack{k=-\infty \\ \text{uniformly.}}} \left(|D_{\alpha}(f) - f|_{\alpha} \right) \left| \sum_{\substack{k=-\infty \\ |\frac{n}{n} - \infty}} \delta_{nk}(f) 2 \left(nx - k \right) - \sum_{k=-\infty}^{\infty} f(x) 2 \left(nx - k \right) \right|_{\alpha} =
$$
\n
$$
\left\| \sum_{k=-\infty}^{\infty} \left(\sum_{k=0}^{\infty} \omega_k \left| f \left(\frac{k}{n} + \frac{r}{n\theta} \right) - f(x) \right|_{\alpha} \right) 2 \left
$$

0. COMPUTIONAL ANALYSIS AND APPLICATIONS. VOL-31. NO. 4. 2023. COPYRIGHT 2023 EUDOXUS PRESS, LLC
\n
$$
\left\{ \begin{aligned}\n &\sum_{k=-\infty}^{\infty} \left(\sum_{r=-\infty}^{\theta} \omega_r \left\| f \left(\frac{k}{n} + \frac{r}{n\theta} \right) - f(x) \right\|_{\gamma} \right) Z(nx - k) \leq \\
 &\sum_{k=-\infty}^{\infty} \left(\sum_{r=-\infty}^{\theta} \omega_r \left\| f \left(\frac{k}{n} + \frac{r}{n\theta} \right) - f(x) \right\|_{\gamma} \right) Z(nx - k) + \\
 &\sum_{k=-\infty}^{\infty} \left(\sum_{r=-\infty}^{\theta} \omega_r \left\| f \left(\frac{k}{n} + \frac{r}{n\theta} \right) - f(x) \right\|_{\gamma} \right) Z(nx - k) + \\
 &\sum_{k=-\infty}^{\infty} \left\| f \left(\frac{k}{n} - x \right\|_{\infty} \leq \frac{1}{n^2}\n\right\}\n\end{aligned}
$$
\n
$$
\omega_1 \left(f, \frac{1}{n} + \frac{1}{n^{\beta}} \right) + \frac{2e^{2\lambda}}{e^{2\lambda} \left\| f \left(f \right\|_{\infty} \right\|_{\infty}} = \Omega_1(n),
$$
\n\nproving the claim. **■**
\nNext we perform multi layer neural network approximations.
\nWe make
\n**Definition 21** Let $f \in C_B(\mathbb{R}^N, X)$, $N \in \mathbb{N}$, where $\left(X, \|\cdot\|_{\gamma} \right)$ is a Banach
\nspace. We define the general network operator
\n $F_n(f, x) := \sum_{k=-\infty}^{\infty} I_{nk}(f) \times (nx - k) =$
\n
$$
\begin{cases}\n B_n(f, x) & \text{if } I_{nk}(f) = f \left(\frac{k}{n} \right) \dots \\
 C_n(f, x) & \text{if } I_{nk}(f) = \frac{1}{n^{\beta}} \sum_{k=1}^{\beta} f(t) dt, \\
 D_n(f, x) & \text{if } I_{nk}(f) = \frac{1}{n^{\beta}} \sum_{k=1}^{\
$$

proving the claim.

Next we perform multi layer neural network approximations. We make

Definition 21 Let $f \in C_B(\mathbb{R}^N, X)$, $N \in \mathbb{N}$, where $(X, \left\| \cdot \right\|_{\gamma})$ is a Banach space. We define the general neural network operator

$$
F_n(f, x) := \sum_{k=-\infty}^{\infty} l_{nk}(f) Z(nx - k) =
$$

$$
\begin{cases} B_n(f, x), & \text{if } l_{nk}(f) = f\left(\frac{k}{n}\right), \\ C_n(f, x), & \text{if } l_{nk}(f) = n^N \int_{\frac{k}{n}}^{\frac{k}{n}} f(t) dt, \\ D_n(f, x), & \text{if } l_{nk}(f) = \delta_{nk}(f). \end{cases}
$$
(86)

Clearly l_{nk} (f) is an X-valued bounded linear functional such that $||l_{nk}(f)||_{\gamma} \le$ $\left\| \|f\|_{\gamma} \right\|_{\infty}$.

Hence $F_n(f)$ is a bounded linear operator with $\left\| \|F_n(f)\|_{\gamma} \right\|_{\infty} \le \left\| \|f\|_{\gamma} \right\|_{\infty}$. We need

Theorem 22 Let $f \in C_B(\mathbb{R}^N, X)$, $N \ge 1$. Then $F_n(f) \in C_B(\mathbb{R}^N, X)$.

Proof. Lengthy and similar to the proof of Theorem 11 of [18], as such is omitted. \blacksquare

L. COMPUTATIONAL ANALYSIS AND APPLICATIONS. VOL. 31. NO. 4. 2023. COPYRIGHT 2023 EUDOXUS PRESS, LLC
\nRemark 23. *By* (22) it is obvious that
$$
|| ||A_{\alpha}(f)||_{\infty} \le || ||f||_{\infty} ||_{\infty} < \infty
$$
, and
\n
$$
A_{\alpha}(f) \in C \left(\prod_{i=1}^{\infty} [a_i, b_i], X \right), given that $f \in C \left(\prod_{i=1}^{\infty} [a_i, b_i], X \right)$.
\n*Clearly* K_{α} only of the operators A_n, B_n, C_n, D_n ¹
\n*Clearly* Q $|| ||X_{\alpha}^{\alpha}(f)||_{\infty} ||_{\infty} = || ||K_{\alpha}(f)||_{\infty} ||_{\infty} \le || ||f||_{\infty} ||_{\infty} \le || ||f||_{\infty} ||_{\infty}$, (ST)
\netc.
\nTherefore, the expression property.
\nAlso we see that
\n $|| ||K_{\alpha}^{\alpha}(f)||_{\infty} || \le || ||f||_{\infty} ||_{\infty} \le || ||X_{\alpha}(f)||_{\infty} ||_{\infty} \le || ||f||_{\infty} \le ||S||_{\infty}$, (SS)
\nthere exists an bounded linear operators.
\nNotation 24. Here $N \in \mathbb{N}, 0 < \beta < 1$. Denote by
\n
$$
c_N := \begin{cases} \frac{1}{(\frac{1}{1000})} N_{\alpha} & \text{if } N_n = A_n, \\ 1, & \text{if } N_n = B_n, C_n, D_n, \end{cases}
$$
\n
$$
\Lambda(n) := \begin{cases} \frac{1}{n^2}, & \text{if } N_n = A_n, B_n, \\ 1, & \text{if } N_n = C_n, D_n, \end{cases}
$$
\n
$$
\Gamma := \begin{cases} C \left(\prod_{i=1}^{\infty} [a_i, b_i], X \right), & \text{if } K_n = A_n, \\ C_B \left(R^N, X \right), & \text{if } K_n = A_n, \\ C_B \left(R^N, X \right), & \text{if } K_n = B_n, C_n, D_n, \end{cases}
$$
\n $$
$$

$$
\left\| \left\| K_n^2(f) \right\|_{\gamma} \right\|_{\infty} = \left\| \left\| K_n \left(K_n \left(f \right) \right) \right\|_{\gamma} \right\|_{\infty} \le \left\| \left\| K_n \left(f \right) \right\|_{\gamma} \right\|_{\infty} \le \left\| \left\| f \right\|_{\gamma} \right\|_{\infty}, \tag{87}
$$

etc.

Therefore we get

$$
\left\| \left\| K_n^k(f) \right\|_{\gamma} \right\|_{\infty} \le \left\| \left\| f \right\|_{\gamma} \right\|_{\infty}, \ \forall \ k \in \mathbb{N}, \tag{88}
$$

the contraction property.

Also we see that

$$
\left\| \left\| K_n^k(f) \right\|_{\gamma} \right\|_{\infty} \le \left\| \left\| K_n^{k-1}(f) \right\|_{\gamma} \right\|_{\infty} \le \dots \le \left\| \left\| K_n(f) \right\|_{\gamma} \right\|_{\infty} \le \left\| \left\| f \right\|_{\gamma} \right\|_{\infty} . \tag{89}
$$

Here K_n^k are bounded linear operators.

Notation 24 Here $N \in \mathbb{N}$, $0 < \beta < 1$. Denote by

$$
c_N := \begin{cases} \left(\frac{4}{\tanh 2\lambda}\right)^N, & \text{if } K_n = A_n, \\ 1, & \text{if } K_n = B_n, C_n, D_n, \end{cases}
$$
(90)

$$
\Lambda(n) := \begin{cases} \frac{1}{n^{\beta}}, & \text{if } K_n = A_n, B_n, \\ \frac{1}{n} + \frac{1}{n^{\beta}}, & \text{if } K_n = C_n, D_n, \end{cases}
$$
(91)

$$
\Gamma := \begin{cases}\nC\left(\prod_{i=1}^{N} [a_i, b_i], X\right), & \text{if } K_n = A_n, \\
C_B\left(\mathbb{R}^N, X\right), & \text{if } K_n = B_n, C_n, D_n,\n\end{cases} \tag{92}
$$

and

$$
Y := \begin{cases} \prod_{i=1}^{N} [a_i, b_i], & \text{if } K_n = A_n, \\ \mathbb{R}^N, & \text{if } K_n = B_n, C_n, D_n. \end{cases}
$$
 (93)

We give the condensed

Theorem 25 Let $f \in \Gamma$, $0 < \beta < 1$, $x \in Y$; $n, \lambda > 0$; $N \in \mathbb{N}$ with $n^{1-\beta} > 2$. Then (i)

$$
\|K_n(f,x) - f(x)\|_{\gamma} \le c_N \left[\omega_1(f,\Lambda(n)) + \frac{2e^{4\lambda} \|f\|_{\gamma}\|_{\infty}}{e^{2\lambda(n^{1-\beta})}}\right] =: \tau(n), \quad (94)
$$

where ω_1 is for $p = \infty$, and (ii)

$$
\left\| \left\| K_n \left(f \right) - f \right\|_{\gamma} \right\|_{\infty} \le \tau \left(n \right) \to 0, \text{ as } n \to \infty. \tag{95}
$$

For f uniformly continuous and in Γ we obtain

$$
\lim_{n\to\infty}K_n(f)=f,
$$

pointwise and uniformly.

Proof. By Theorems 9, 18, 19, 20. \blacksquare

Next we do iterated, multi layer neural network approximation. (see also $[10]$.

We make

Remark 26 Let $r \in \mathbb{N}$ and K_n as above. We observe that

$$
K_n^r f - f = (K_n^r f - K_n^{r-1} f) + (K_n^{r-1} f - K_n^{r-2} f) +
$$

$$
(K_n^{r-2} f - K_n^{r-3} f) + \dots + (K_n^2 f - K_n f) + (K_n f - f).
$$

Then

1. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL-31, NO. 4. 2023, COPYRIGHT 2023 EUDOXUS PRESS, LC

\nwhere
$$
\omega_1
$$
 is for $p = \infty$, and

\n(i)

\n
$$
\left\| \| K_n(f) - f \|_{\infty} \right\|_{\infty} \leq \tau(n) \to 0, \text{ as } n \to \infty.
$$
\n(95)

\nFor f uniformly continuous and in Γ we obtain

\n
$$
\lim_{n \to \infty} K_n(f) = f,
$$

\npointwise and uniformly.

\nProof. By Theorem 9, 18, 19, 20. ■

\nNext we do iterated, multi layer neural network approximation. (see also [10]).

\nWe make

\nRemark 26 Let $r \in \mathbb{N}$ and K_n as above. We observe that

\n
$$
K_n^r f = f = (K_n^r f - K_n^{r-1} f) + (K_n^{r-1} f - K_n^{r-2} f) + (K_n^{r-2} f - K_n^{r-2} f) + (K_n^r - f - K_n^{r-2} f) + \dots + (K_n^2 f - K_n f) + (K_n^2 f - K_n f
$$

$$
\left\| \|K_n^r f - f\|_{\gamma} \right\|_{\infty} \le r \left\| \|K_n f - f\|_{\gamma} \right\|_{\infty}.
$$
\n(96)

We give

Theorem 27 All here as in Theorem 25 and $r \in \mathbb{N}$, $\tau(n)$ as in (94). Then

$$
\left\| \left\| K_n^r f - f \right\|_{\gamma} \right\|_{\infty} \leq r \tau(n). \tag{97}
$$

So that the speed of convergence to the unit operator of K_n^r is not worse than of K_n .

Proof. As similar to [18] is omitted. \blacksquare

Remark 28 Let $m_1, m_2, ..., m_r \in \mathbb{N} : m_1 \leq m_2 \leq ... \leq m_r, 0 < \beta < 1, \lambda > 0,$ $f \in \Gamma$. Then

$$
\Lambda(m_1) \ge \Lambda(m_2) \ge \dots \ge \Lambda(m_r), \quad \Lambda \text{ as in (91).}
$$

Therefore

$$
\omega_1(f, \Lambda(m_1)) \geq \omega_1(f, \Lambda(m_2)) \geq \ldots \geq \omega_1(f, \Lambda(m_r)).
$$

Assume further that $m_i^{(1-\beta)} > 2$, $i = 1, ..., r$. Then

$$
\frac{e^{4\lambda}}{e^{2\lambda m_1^{(1-\beta)}}}\geq \frac{e^{4\lambda}}{e^{\lambda m_2^{(1-\beta)}}}\geq \ldots \geq \frac{e^{4\lambda}}{e^{\lambda m_r^{(1-\beta)}}}.
$$

Let K_{m_i} as above, $i = 1, ..., r$, all of the same kind. We write

$$
K_{m_r} (K_{m_{r-1}} (...K_{m_2} (K_{m_1 f}))) - f =
$$

\n
$$
K_{m_r} (K_{m_{r-1}} (...K_{m_2} (K_{m_1 f}))) - K_{m_r} (K_{m_{r-1}} (...K_{m_2} f)) +
$$

\n
$$
K_{m_r} (K_{m_{r-1}} (...K_{m_2} f)) - K_{m_r} (K_{m_{r-1}} (...K_{m_3} f)) +
$$

\n
$$
K_{m_r} (K_{m_{r-1}} (...K_m f)) - K_{m_r} (K_{m_{r-1}} (...K_{m_4} f)) + ... +
$$

\n
$$
K_{m_r} (K_{m_{r-1}} f) - K_{m_r} f + K_{m_r} f - f =
$$

 $K_{m_r}\left(K_{m_{r-1}}\left(...K_{m_2}\right)\right)\left(K_{m_1}f - f\right) + K_{m_r}\left(K_{m_{r-1}}\left(...K_{m_3}\right)\right)\left(K_{m_2}f - f\right) +$ $K_{m_r}\left(K_{m_{r-1}}\left(...K_{m_4}\right)\right)\left(K_{m3}f - f\right) + ... + K_{m_r}\left(K_{m_{r-1}}f - f\right) + K_{m_r}f - f.$ Hence by the triangle inequality of $\left\|\|\cdot\|_{\gamma}\right\|_{\infty}$ we get

I. COMPUTATIONAL ANALYSIS AND APPLICATIONS. VOL-31. NO. 4. 2023. COPYRIGHT 2023 EUDOXUS PRESS, LLC
\nRemark 28 Let
$$
m_1, m_2, ..., m_r
$$
 ∈ N : $m_1 ≤ m_2 ≤ ... ≤ m_r, 0 < \beta < 1, λ > 0$,
\n $f ∈ Γ$. Then
\n
$$
\Lambda(m_1) ≥ Λ(m_2) ≥ ... ≥ Λ(m_r), Λ as in (91).
$$
\nTherefore
\n
$$
\omega_1(f, Λ(m_1)) ≥ \omega_1(f, Λ(m_2)) ≥ ... ≥ \omega_1(f, Λ(m_r)).
$$
\nAssume further that $m_1^{(1-\beta)} > 2$, $i = 1, ..., T$ then
\n
$$
\frac{e^{t\lambda}}{e^{2\lambda m_1^{(1-\beta)}}} ≥ \frac{e^{t\lambda}}{e^{\lambda m_2^{(1-\beta)}}} ≥ ... ≥ \frac{e^{t\lambda}}{e^{\lambda m_1^{(1-\beta)}}}.
$$
\nLet K_{m_n} as above, $i = 1, ..., r$, all of the same kind. We write
\n
$$
K_{m_r}
$$
 $(K_{m_{r-1}}(...K_{m_2}f)) - K_{m_r}$ $(K_{m_{r-1}}(...K_{m_2}f)) +$ \n
$$
K_{m_r}
$$
 $(K_{m_{r-1}}(...K_{m_2}f)) - K_{m_r}$ $(K_{m_{r-1}}(...K_{m_2}f)) + ... +$ \n
$$
K_{m_r}
$$
 $(K_{m_{r-1}}(...K_{m_2}f)) - K_{m_r}$ $(K_{m_{r-1}}(...K_{m_2}f)) + ... +$ \n
$$
K_{m_r}
$$
 $(K_{m_{r-1}}(...K_{m_2}f)) - K_{m_r}$ $(K_{m_{r-1}}(...K_{m_2}f)) + ... +$ \n
$$
K_{m_r}
$$
 $(K_{m_{r-1}}(...K_{m_2}f)) - K_{m_r}$ $(K_{m_{r-1}}(...K_{m_2}f)) + ... +$ \n
$$
K_{m_r}
$$
 $(K_{m_{r-1}}(...K_{m_1}f)) - K_{m_r}$ $(K_{m_{r-$

(repeatedly applying (87))

$$
\left\| \|K_{m_1}f - f\|_{\gamma} \right\|_{\infty} + \left\| \|K_{m_2}f - f\|_{\gamma} \right\|_{\infty} + \left\| \|K_{m_3}f - f\|_{\gamma} \right\|_{\infty} + \dots +
$$

$$
\left\| \|K_{m_{r-1}}f - f\|_{\gamma} \right\|_{\infty} + \left\| \|K_{m_2}f - f\|_{\gamma} \right\|_{\infty} + \left\| \|K_{m_3}f - f\|_{\gamma} \right\|_{\infty} + \dots +
$$

$$
\left\| \left\| K_{m_{r-1}} f - f \right\|_{\gamma} \right\|_{\infty} + \left\| \left\| K_{m_r} f - f \right\|_{\gamma} \right\|_{\infty} = I_{i=1}^{r} \left\| \left\| K_{m_i} f - f \right\|_{\gamma} \right\|_{\infty}.
$$

That is, we proved

$$
\left\| \left\| K_{m_r} \left(K_{m_{r-1}} \left(\ldots K_{m_2} \left(K_{m_1} f \right) \right) \right) - f \right\|_{\gamma} \right\|_{\infty} \leq_{i=1}^r \left\| \left\| K_{m_i} f - f \right\|_{\gamma} \right\|_{\infty} . \tag{98}
$$

We also present

Theorem 29 Let $f \in \Gamma$; m, N, $m_1, m_2, ..., m_r \in \mathbb{N} : m_1 \leq m_2 \leq ... \leq m_r$ $0 < \beta < 1, \lambda > 0; m_i^{(1-\beta)} > 2, i = 1, ..., r, x \in Y, and let (K_{m_1}, ..., K_{m_r})$ as $(A_{m_1},...,A_{m_r})$ or $(B_{m_1},...,B_{m_r})$ or $(C_{m_1},...,C_{m_r})$ or $(D_{m_1},...,D_{m_r}),$ $p = \infty$. Then

L. COMPUTIONAL ANALYSIS AND APPLICATIONS. VOL. 31. No. 4. 2023. COPYRIGHT 2023 EUDOXUS PRESS, LLC
\nThat is, we proved
\n
$$
\left\|\left\|K_{m_r}\left(f-f\right\|_{\eta}\right\|_{\infty} + \left\|{\left\|K_{m_r}f - f\right\|_{\eta}\right\|_{\infty} - \frac{r}{r-1}\left\|{\left\|K_{m_r}f - f\right\|_{\eta}\right\|_{\infty}}.
$$
\n(98)
\nWe also present
\nTheorem 29 Let $f \in \Gamma_r$, $N, m_1, m_2, ..., m_r \in \mathbb{N}$; $m_1 \leq m_2 \leq ... \leq m_r$,
\n $0 < \beta < 1, \lambda > 0$, $m_1/2, \cdots, n_r$, $\beta < 1, m_2, ..., m_r \in \mathbb{N}$; $m_1 \leq m_2 \leq ... \leq m_r$,
\n $0 < \beta < 1, \lambda > 0$, $m_1/2, \cdots, \beta < ... > 0$ or $(D_{m_1}, ..., D_{m_r})$, $p = \infty$.
\nThen
\n
$$
\left\|K_{m_r}(K_{m_{r-1}}\left(\ldots K_{m_2}(K_{m_r}f))\right)(x) - f(x)\right\|_{\infty} \leq
$$
\n
$$
\left\|K_{m_r}(K_{m_{r-1}}\left(\ldots K_{m_2}(K_{m_r}f))\right)(x) - f(x)\right\|_{\infty} \leq
$$
\n
$$
r \leq \sum_{i=1}^r \left\|W_{m_i}f - f\|_{\gamma}\right\|_{\infty} \leq
$$
\n
$$
r \leq \sum_{i=1}^r \left\|W_{m_i}f - f\|_{\gamma}\right\|_{\infty} \leq
$$
\n
$$
r \leq \sum_{i=1}^r \left\|M(f, \lambda(m_1)) + \frac{2e^{i\lambda}}{e^{2\lambda m_1^{1+\beta}}}\right\|_{\infty} \right\}.
$$
\n(99)
\nClearly, we notice that the speed of convergence to the unit operator of the multiple
\n
$$
\left\|W_{\in
$$

Clearly, we notice that the speed of convergence to the unit operator of the multiply iterated multi layer operator is not worse than the speed of K_{m_1} .

Proof. As similar to [18] is omitted. \blacksquare

We continue with

Theorem 30 Let all as in Corollary 17, and $r \in \mathbb{N}$. Here $\Lambda_3(n)$ is as in (73). Then

$$
\left\| \|A_n^r f - f\|_{\gamma} \right\|_{\infty} \le r \left\| \|A_n f - f\|_{\gamma} \right\|_{\infty} \le r \Lambda_3(n). \tag{100}
$$

Proof. As similar to [18] is omitted. \blacksquare

Next we present some $L_{p_1}, p_1 \geq 1$, approximation related results.

Theorem 31 Let $p_1 \geq 1$, $f \in C \left(\prod\limits_{i=1}^n \right)$ $\prod_{i=1}^n [a_i, b_i], X$, $0 < \beta < 1, \lambda > 0; N, n \in \mathbb{N}$ with $n^{1-\beta} > 2$, and $\Omega_1(n)$ as in (38), ω_1 is for $p = \infty$. Then

$$
\left\| \|A_n f - f\|_{\gamma} \right\|_{p_1, \prod_{i=1}^n [a_i, b_i]} \leq \Omega_1(n) \left(\prod_{i=1}^n (b_i - a_i) \right)^{\frac{1}{p_1}}.
$$
 (101)

We notice that $\lim_{n\to\infty}$ $|||A_nf-f||_{\gamma}$ $||_{p_1, \prod_{i=1}^n [a_i,b_i]}=0.$

Proof. Obvious, by integrating (38) , etc. It follows

Theorem 32 Let $p_1 \geq 1$, $f \in C_B(\mathbb{R}^N, X)$, $0 < \beta < 1, \lambda > 0$; $N, n \in \mathbb{N}$ with $n^{1-\beta} > 2$, and ω_1 is for $p = \infty$; $\Omega_2(n)$ as in (74) and P a compact set of \mathbb{R}^N . Then **517 J. Comparison Analysis And Applications** $\mathbf{v}(\mathbf{x}) = \mathbf{0} \mathbf{x} \mathbf{z} + \mathbf{z} \mathbf{z} \mathbf{z} + \mathbf{z} \mathbf{z} \mathbf{z} \mathbf{z}$
 Proced. Obvious, by harpyoning (88), res.

Theorems 32, $\mathbf{Z} \mathbf{z} \mathbf{z} \mathbf{z} \mathbf{z} + \mathbf{z} \math$

$$
\left\| \|B_n f - f\|_{\gamma} \right\|_{p_1, P} \le \Omega_2(n) |P|^{\frac{1}{p_1}}, \tag{102}
$$

where $|P| < \infty$, is the Lebesgue measure of P. We notice that $\lim_{n\to\infty} || ||B_nf-f||_{\gamma} ||_{p_1,P}$ $0 \text{ for } f \in (C_U \left(\mathbb{R}^N, X \right) \cap C_B \left(\mathbb{R}^N, X \right)).$

Proof. By integrating (74) , etc. Next come

Theorem 33 All as in Theorem 32, but we use $\Omega_3(n)$ of (78). Then

$$
\left\| \|C_n f - f\|_{\gamma} \right\|_{p_1, P} \le \Omega_3(n) |P|^{\frac{1}{p_1}}.
$$
 (103)

We have that $\lim_{n\to\infty}$ $\left\| \|C_nf-f\|_{\gamma}\right\|_{p_1, P} = 0$ for $f \in (C_U (\mathbb{R}^N, X) \cap C_B (\mathbb{R}^N, X))$.

Proof. By (78). \blacksquare

Theorem 34 All as in Theorem 32, but we use $\Omega_4(n)$ of (84). Then

$$
\left\| |D_n f - f| \right\|_{p_1, P} \le \Omega_4(n) |P|^{\frac{1}{p_1}}.
$$
 (104)

We have that $\lim_{n\to\infty}$ $|||D_nf-f||_{\gamma}$ $||_{p_1, P} = 0$ for $f \in (C_U (\mathbb{R}^N, X) \cap C_B (\mathbb{R}^N, X))$.

Proof. By (84) .

Application 35 A typical application of all of our results is when $(X, \left\|\cdot\right\|_{\gamma}) =$ $(\mathbb{C}, \lvert \cdot \rvert)$, where $\mathbb C$ is the set of the complex numbers.

References

- [1] G.A. Anastassiou, Moments in Probability and Approximation Theory, Pitman Research Notes in Math., Vol. 287, Longman Sci. & Tech., Harlow, U.K., 1993.
- [2] G.A. Anastassiou, Rate of convergence of some neural network operators to the unit-univariate case, J. Math. Anal. Appli. 212 (1997), 237-262.
- [3] G.A. Anastassiou, Quantitative Approximations, Chapman&Hall/CRC, Boca Raton, New York, 2001.
- [4] G.A. Anastassiou, Intelligent Systems: Approximation by Artificial Neural Networks, Intelligent Systems Reference Library, Vol. 19, Springer, Heidelberg, 2011.
- [5] G.A. Anastassiou, Univariate hyperbolic tangent neural network approximation, Mathematics and Computer Modelling, 53(2011), 1111-1132.
- [6] G.A. Anastassiou, Multivariate hyperbolic tangent neural network approximation, Computers and Mathematics 61(2011), 809-821.
- [7] G.A. Anastassiou, Multivariate sigmoidal neural network approximation, Neural Networks 24(2011), 378-386.
- [8] G.A. Anastassiou, Univariate sigmoidal neural network approximation, J. of Computational Analysis and Applications, Vol. 14, No. 4, 2012, 659-690.
- [9] G.A. Anastassiou, Fractional neural network approximation, Computers and Mathematics with Applications, 64 (2012), 1655-1676.
- [10] G.A. Anastassiou, Approximation by neural networks iterates, Advances in Applied Mathematics and Approximation Theory, pp. 1-20, Springer Proceedings in Math. & Stat., Springer, New York, 2013, Eds. G. Anastassiou, O. Duman. 5 COMPUTATIONAL ANALYSIS AND APPLICATIONS VOL. 31, NOL. 2023, COPYRIGHT 2023 LOCANS PRESS, LLC (CA Anastassious, Quantitudes, Applications, Depressionless, Markyleis M , Northern Entire Kenting and Care Methods (Separate
	- [11] G.A. Anastassiou, Intelligent Systems II: Complete Approximation by Neural Network Operators, Springer, Heidelberg, New York, 2016.
	- [12] G.A. Anastassiou, Strong Right Fractional Calculus for Banach space valued functions, 'Revista Proyecciones, Vol. 36, No. 1 (2017), 149-186.
	- [13] G.A. Anastassiou, Vector fractional Korovkin type Approximations, Dynamic Systems and Applications, 26(2017), 81-104.
	- [14] G.A. Anastassiou, A strong Fractional Calculus Theory for Banach space valued functions, Nonlinear Functional Analysis and Applications (Korea), 22(3) (2017), 495-524.
	- [15] G.A. Anastassiou, Intelligent Computations: Abstract Fractional Calculus, Inequalities, Approximations, Springer, Heidelberg, New York, 2018.
	- [16] G.A. Anastassiou, Nonlinearity: Ordinary and Fractional Approximations by Sublinear and Max-Product Operators, Springer, Heidelberg, New York, 2018.
- [17] G.A. Anastassiou, Algebraic function based Banach space valued ordinary and fractional neural network approximations, New Trends in Mathematical Sciences, 10 special issue (1) (2022), 100-125. 519 J. C. Analysis and the state of the
	- [18] G.A. Anastassiou, General multimariate arctangent function activated neural network approximations, Journal of Numerical Analysis and Approximation Theory 51(1) (2022), 37-66.
	- [19] G.A. Anastassiou, Abstract multivariate algebraic function activated neural network approximations, accepted to Journal of Computational Analysis and Applications, 2022.
	- [20] G.A. Anastassiou and S. Karateke, Parametrized hyperbolic tangent induced Banach space valued ordinary and fractional neural network approximation, submitted, 2022.
	- [21] H. Cartan, Differential Calculus, Hermann, Paris, 1971.
	- [22] Z. Chen and F. Cao, The approximation operators with sigmoidal functions, Computers and Mathematics with Applications, 58 (2009), 758-765.
	- [23] D. Costarelli and R. Spigler, Approximation results for neural network operators activated by sigmoidal functions, Neural Networks 44 (2013), 101-106.
	- [24] D. Costarelli and R. Spigler, Multivariate neural network operators with sigmoidal activation functions, Neural Networks 48 (2013), 72-77.
	- [25] S. Haykin, Neural Networks: A Comprehensive Foundation (2 ed.), Prentice Hall, New York, 1998.
	- [26] W. McCulloch and W. Pitts, A logical calculus of the ideas immanent in nervous activity, Bulletin of Mathematical Biophysics, 7 (1943), 115-133.
	- [27] T.M. Mitchell, Machine Learning, WCB-McGraw-Hill, New York, 1997.
	- [28] L.B. Rall, Computational Solution of Nonlinear Operator Equations, John Wiley & Sons, New York, 1969.

General sigmoid based Banach space valued neural network approximation

George A. Anastassiou Department of Mathematical Sciences University of Memphis Memphis, TN 38152, U.S.A. ganastss@memphis.edu

Abstract

Here we study the univariate quantitative approximation of Banach space valued continuous functions on a compact interval or all the real line by quasi-interpolation Banach space valued neural network operators. We perform also the related Banach space valued fractional approximation. These approximations are derived by establishing Jackson type inequalities involving the modulus of continuity of the engaged function or its Banach space valued high order derivative or fractional derivaties. Our operators are defined by using a density function induced by a general sigmoid function. The approximations are pointwise and with respect to the uniform norm. The related Banach space valued feed-forward neural networks are with one hidden layer. We finish with a convergence analysis. 520 J. CONFUTATIONAL ANNEVISIS AND APPLICATIONS, VOL. 31, NO. 4, 2022, COPYRIGHT 2023 EUDOXUS PRESS, LLC (Separation contents).
 \blacksquare Deparation of A. Anastassion in Equation (Separation of Mathematical Sciences (Separa

2020 AMS Mathematics Subject Classification: 26A33, 41A17, 41A25, 41A30, 46B25.

Keywords and Phrases: general sigmoid function, Banach space valued neural network approximation, Banach space valued quasi-interpolation operator, modulus of continuity, Banach space valued Caputo fractional derivative, Banach space valued fractional approximation.

1 Introduction

The author in [1] and [2], see Chapters 2-5, was the first to establish neural network approximation to continuous functions with rates by very specifically defined neural network operators of Cardaliagnet-Euvrard and "Squashing" types, by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities. He treats there both the univariate and multivariate cases. The defining these operators "bellshaped" and "squashing" functions are assumed to be of compact suport. Also in [2] he gives the Nth order asymptotic expansion for the error of weak approximation of these two operators to a special natural class of smooth functions, see Chapters 4-5 there.

The author inspired by [14], continued his studies on neural networks approximation by introducing and using the proper quasi-interpolation operators of sigmoidal and hyperbolic tangent type which resulted into $[3]$, $[4]$, $[5]$, $[6]$, [7], by treating both the univariate and multivariate cases. He did also the corresponding fractional case [8].

In this article we are greatly inspired by the related works [15], [16].

The author here performs general sigmoid function based neural network approximations to continuous functions over compact intervals of the real line or over the whole $\mathbb R$ with values to an arbitrary Banach space $(X, \|\cdot\|)$. Finally he treats completely the related X-valued fractional approximation. All convergences here are with rates expressed via the modulus of continuity of the involved function or its X -valued high order derivative, or X -valued fractional derivatives and given by very tight Jackson type inequalities. LOOKUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC has the state of the control of th

Our compact intervals are not necessarily symmetric to the origin. Some of our upper bounds to error quantity are very áexible and general. In preparation to prove our results we establish important properties of the basic density function defining our operators which is induced by a general sigmoid function.

Feed-forward X-valued neural networks (FNNs) with one hidden layer, the only type of networks we deal with in this article, are mathematically expressed as

$$
N_n(x) = \sum_{j=0}^n c_j \sigma\left(\langle a_j \cdot x \rangle + b_j\right), \ \ x \in \mathbb{R}^s, \ s \in \mathbb{N},
$$

where for $0 \leq j \leq n$, $b_j \in \mathbb{R}$ are the thresholds, $a_j \in \mathbb{R}^s$ are the connection weights, $c_j \in X$ are the coefficients, $\langle a_j \cdot x \rangle$ is the inner product of a_j and x, and σ is the activation function of the network. In many fundamental neural network models, the activation function is derived from various specific sigmoid functions. Here we work for a general sigmoid function. About neural networks in general read [17], [18], [20]. See also [9] for a complete study of real valued approximation by neural network operators.

2 Basics

Let $h : \mathbb{R} \to [-1, 1]$ be a general sigmoid function, such that it is strictly increasing, $h(0) = 0$, $h(-x) = -h(x)$, $h(+\infty) = 1$, $h(-\infty) = -1$. Also h is strictly convex over $(-\infty, 0]$ and striclty concave over $[0, +\infty)$, with $h^{(2)} \in$ $C\left(\mathbb{R}\right).$

We consider the activation function

$$
\psi(x) := \frac{1}{4} \left(h \left(x + 1 \right) - h \left(x - 1 \right) \right), \ \ x \in \mathbb{R}, \tag{1}
$$

As in [13], p. 285, we get that $\psi(-x) = \psi(x)$, thus ψ is an even function. Since $x + 1 > x - 1$, then $h(x + 1) > h(x - 1)$, and $\psi(x) > 0$, all $x \in \mathbb{R}$.

We see that

$$
\psi(0) = \frac{h(1)}{2}.\tag{2}
$$

Let $x > 1$, we have that

$$
\psi'(x) = \frac{1}{4} \left(h'(x+1) - h'(x-1) \right) < 0,
$$

by h' being strictly decreasing over $[0, +\infty)$.

Let now $0 < x < 1$, then $1-x > 0$ and $0 < 1-x < 1+x$. It holds $h'(x-1) = h'(1-x) > h'(x+1)$, so that again $\psi'(x) < 0$. Consequently ψ is stritly decreasing on $(0, +\infty)$. 2. CONFUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC

We consider the automation function
 $x = 1.3$ and $y = 1.5$ and $y = 1.5$ and $y = 1.5$ and $y = 1.5$ and $y = 0.5$
 $x = 1.$

Clearly, ψ is strictly increasing on $(-\infty, 0)$, and $\psi'(0) = 0$. See that

$$
\lim_{x \to +\infty} \psi(x) = \frac{1}{4} (h(+\infty) - h(+\infty)) = 0,
$$
\n(3)

and

$$
\lim_{x \to -\infty} \psi(x) = \frac{1}{4} (h(-\infty) - h(-\infty)) = 0.
$$
 (4)

That is the x-axis is the horizontal asymptote on ψ .

Conclusion, ψ is a bell symmetric function with maximum

$$
\psi(0) = \frac{h(1)}{2}.
$$

We need

Theorem 1 We have that

$$
\sum_{i=-\infty}^{\infty} \psi(x-i) = 1, \ \forall \ x \in \mathbb{R}.
$$
 (5)

Proof. As exactly the same as in [13], p. 286 is omitted. \blacksquare

Theorem 2 It holds

$$
\int_{-\infty}^{\infty} \psi(x) dx = 1.
$$
 (6)

Proof. Similar to [13], p. 287. It is omitted. \blacksquare Thus $\psi(x)$ is a density function on R. We give

Theorem 3 Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$. It holds

$$
\sum_{\begin{cases}\nk = -\infty \\
\colon |nx - k| \ge n^{1-\alpha}\n\end{cases}}^{\infty} \psi(nx - k) < \frac{\left(1 - h\left(n^{1-\alpha} - 2\right)\right)}{2}.\tag{7}
$$

Notice that

$$
\lim_{n \to +\infty} \frac{\left(1 - h\left(n^{1-\alpha} - 2\right)\right)}{2} = 0.
$$

Proof. Let $x \geq 1$. That is $0 \leq x - 1 < x + 1$. Applying the mean value theorem we get

$$
\psi(x) \stackrel{(1)}{=} \frac{1}{4} \cdot 2 \cdot h'(\xi) = \frac{h'(\xi)}{2},
$$
\n(8)

for some $x - 1 < \xi < x + 1$.

Since h' is strictly decreasing we obtain $h'(\xi) < h'(x-1)$ and

$$
\psi(x) < \frac{h'(x-1)}{2}, \ \forall \ x \ge 1. \tag{9}
$$

Therefore we have

X¹ 8 < : k = 1 : jnx kj n 1 (nx ^k) = ^X¹ 8 < : k = 1 : jnx kj n 1 (jnx kj) < 1 2 X¹ 8 < : k = 1 : jnx kj n 1 h 0 (jnx kj 1) 1 2 Z ⁺¹ (n11) h 0 (x 1) d (x 1) = 1 2 h (x 1)j +1 (n11) = 1 2 h (+1) h n ¹ ² = 1 2 1 h n ¹ ² : (10) 523 J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC George A. Anastassiou 520-534

The claim is proved. \blacksquare

Denote by $\lfloor \cdot \rfloor$ the integral part of the number and by $\lfloor \cdot \rfloor$ the ceiling of the number.

We further give

Theorem 4 Let $x \in [a, b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $[na] \leq [nb]$. It holds

$$
\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \psi\left(nx-k\right)} < \frac{1}{\psi\left(1\right)}, \quad \forall \ x \in [a, b] \,. \tag{11}
$$

Proof. As similar to [13], p. 289 is omitted. \blacksquare

Remark 5 We have that

$$
\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k) \neq 1,\tag{12}
$$

for at least some $x \in [a, b]$.

See [13], p. 290, same reasoning.

Note 6 For large enough n we always obtain $[na] \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, if $\lceil na \rceil \leq k \leq \lceil nb \rceil$. In general it holds (by (5))

$$
\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi\left(nx-k\right) \le 1. \tag{13}
$$

Let $(X, \|\cdot\|)$ be a Banach space.

Definition 7 Let $f \in C([a, b], X)$ and $n \in \mathbb{N} : [na] \leq |nb|$. We introduce and define the X -valued linear neural network operators

$$
A_n(f, x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \psi\left(nx - k\right)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi\left(nx - k\right)}, \quad x \in [a, b]. \tag{14}
$$

Clearly here $A_n(f, x) \in C([a, b], X)$. For convenience we use the same A_n for real valued function when needed. We study here the pointwise and uniform convergence of $A_n(f, x)$ to $f(x)$ with rates.

For convenience also we call

$$
A_n^*\left(f,x\right) := \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \psi\left(nx-k\right),\tag{15}
$$

(similarly A_n^* can be defined for real valued function) that is

$$
A_n(f, x) = \frac{A_n^*(f, x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi\left(nx - k\right)}.
$$
 (16)

So that

1. COMPUTATIONAL ANALYSIS AND APPLICATIONS, Vol. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC

\nRemark 5 We have that

\n
$$
\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{[nb]}\psi(nx-k) \neq 1,
$$
\nfor at least some $x \in [a, b]$.

\nSee [13], p. 299, same reasoning.

\nNote 6 For large enough n we always obtain $\lceil na \rceil \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff

\n
$$
\lceil \nu a \rceil \leq k \leq \lfloor nb \rfloor
$$
. In general it holds (by (5))\n
$$
\qquad \qquad \sum_{k=\lceil na \rceil}^{[nb]}\psi(nx-k) \leq 1.
$$
\nLet $(X, ||\cdot||)$ be a Banach space.

\nDefinition *T Let* $f \in C([a, b], X)$ and $n \in \mathbb{N}$; $\lceil na \rceil \leq \lfloor nb \rfloor$. We introduce and define the X-valued linear neural network operators.

\nAs $a_n(f, x) := \frac{\sum_{k=\lceil na \rceil}^{[nb]}\psi(nx-k)}{\sum_{k=\lceil na \rceil}^{[nb]}\psi(nx-k)}, x \in [a, b].$

\nClearly $a_n(f, x) := \frac{\sum_{k=\lceil na \rceil}^{[nb]}\psi(nx-k)}{\sum_{k=\lceil na \rceil}^{[nb]}\psi(nx-k)}, x \in [a, b].$

\n(14) Clearly $\text{free} A_n(f, x) := \sum_{k=\lceil na \rceil}^{[nb]}\psi(nx-k)$, for convenience we use the same A_n for real valued function when needed. We study here the pointwise and uniform convergence of $A_n(f, x)$ to $f(x)$ with rates.

\nFor constructive also we call

\n
$$
A_n(f, x) := \sum_{k=\lceil na \rceil}^{[nb]}\psi(nx-k).
$$
\n(35) that

\n
$$
A_n(f, x) = f(x) = \frac{A_n^*(f, x)}{\sum_{k=\lceil na \rceil}^{[nb]}\psi(nx-k)} - f(x)
$$
\n
$$
= \frac{A_n^*(f, x
$$

Consequently we derive

$$
||A_n(f, x) - f(x)|| \le \frac{1}{\psi(1)} \left||A_n^*(f, x) - f(x) \left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k)\right) \right||. \tag{18}
$$

That is

3. COMPUTATIONAL ANALYSIS AND APPLICATIONS, Vol. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\nThat is
\n
$$
||A_n(f, x) - f(x)|| \leq \frac{1}{\psi(1)} \left\| \sum_{k=[n\alpha]}^{[n\beta]} \left(f\left(\frac{k}{n}\right) - f(x) \right) \psi(nx - k) \right\|.
$$
\n(19)
\nWe will estimate the right hand side of (19).
\nFor that we need, for $f \in C_0$ (β, β) + the first modulus of continuity
\n
$$
\omega_1(f, \delta)_{[a, b]} := \omega_1(f, \delta) := \sup_{x,y \in [a, N]} ||f(x) - f(y)||, \delta > 0.
$$
\n(20)
\nSimilarly, it is defined ω_1 for $f \in C_0$, [R, X) (uniformly continuous and bounded
\nfunctions from R into X), for $f \in C_0$ (R, X) (continuous)
\n
$$
\text{The fact } f \in C_0
$$
 (R, X) (uniformly continuous).
\nThe fact $f \in C_0$ (R, X), or $f \in C_2$ (R, X), we define
\n
$$
\overline{A}_0(f, x) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) \psi(nx - k), \quad n \in \mathbb{N}, x \in \mathbb{R},
$$
\n(21)
\n**Definition 8** When $f \in C_{\alpha} \mathbb{R}, X$), or $f \in C_2 \mathbb{R}, X$), we define
\n
$$
\overline{A}_0(f, x) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) \psi(nx - k), \quad n \in \mathbb{N}, x \in \mathbb{R},
$$
\n(21)
\nthe X-valued quasi-intergulation neural network operator.
\nRemark 9 We have that
\n
$$
||f\left(\frac{k}{n}\right)|| \leq ||f||_{\infty, \mathbb{R}} \iff \psi(nx - k),
$$
\n(22)
\nand
\n
$$
\sum_{k=-\infty}^{\infty} ||f\left(\frac{k}{n}\right) || \psi(nx - k) \leq ||f||_{\infty, \mathbb{R}} \left(\sum
$$

We will estimate the right hand side of (19).

For that we need, for $f \in C([a, b], X)$ the first modulus of continuity

$$
\omega_1(f, \delta)_{[a,b]} := \omega_1(f, \delta) := \sup_{\substack{x, y \in [a, b] \\ |x - y| \le \delta}} ||f(x) - f(y)||, \ \ \delta > 0. \tag{20}
$$

Similarly, it is defined ω_1 for $f \in C_{uB} (\mathbb{R}, X)$ (uniformly continuous and bounded functions from $\mathbb R$ into X), for $f \in C_B(\mathbb R, X)$ (continuous and bounded Xvalued) and for $f \in C_u (\mathbb{R}, X)$ (uniformly continuous).

The fact $f \in C([a, b], X)$ or $f \in C_u (\mathbb{R}, X)$, is equivalent to $\lim_{\delta \to 0} \omega_1(f, \delta) = 0$, see [11].

Definition 8 When $f \in C_{uB} (\mathbb{R}, X)$, or $f \in C_B (\mathbb{R}, X)$, we define

$$
\overline{A}_{n}(f,x) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) \psi\left(nx-k\right), \quad n \in \mathbb{N}, \ x \in \mathbb{R}, \tag{21}
$$

the X-valued quasi-interpolation neural network operator.

Remark 9 We have that

$$
\left\|f\left(\frac{k}{n}\right)\right\| \leq \|f\|_{\infty,\mathbb{R}} < +\infty,
$$

and

$$
\left\| f\left(\frac{k}{n}\right) \right\| \psi\left(nx-k\right) \le \|f\|_{\infty, \mathbb{R}} \psi\left(nx-k\right),\tag{22}
$$

and

$$
\sum_{k=-\lambda}^{\lambda} \left\| f\left(\frac{k}{n}\right) \right\| \psi(nx-k) \leq \|f\|_{\infty, \mathbb{R}} \left(\sum_{k=-\lambda}^{\lambda} \psi(nx-k) \right),
$$

and Önally

$$
\sum_{k=-\infty}^{\infty} \left\| f\left(\frac{k}{n}\right) \right\| \psi\left(nx-k\right) \le \|f\|_{\infty,\mathbb{R}},\tag{23}
$$

a convergent in R series.

So the series $\sum_{k=-\infty}^{\infty} \underline{f\left(\frac{k}{n}\right)} \psi\left(nx-k\right)$ is absolutely convergent in X, hence it is convergent in X and $A_n(f, x) \in X$.

We denote by $||f||_{\infty} := \sup_{x \in [a,b]} ||f(x)||$, for $f \in C([a,b],X)$, similarly is defined for $f \in C_B (\mathbb{R}, X)$.

3 Main Results

We present a series of X -valued neural network approximations to a function given with rates.

We first give

Theorem 10 Let $f \in C([a, b], X)$, $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $x \in [a, b]$. Then

i)

$$
||A_n(f,x) - f(x)|| \le \frac{1}{\psi(1)} \left[\omega_1 \left(f, \frac{1}{n^{\alpha}} \right) + \left(1 - h \left(n^{1-\alpha} - 2 \right) \right) ||f||_{\infty} \right] =: \rho,
$$
\n(24)

and

ii)

$$
\left\|A_n\left(f\right) - f\right\|_{\infty} \le \rho. \tag{25}
$$

We notice $\lim_{n \to \infty} A_n(f) = f$, pointwise and uniformly.

The speed of convergence is $\max\left(\frac{1}{n^{\alpha}},\left(1-h\left(n^{1-\alpha}-2\right)\right)\right)$.

Proof. As similar to [13], p. 293 is omitted. \blacksquare Next we give

Theorem 11 Let $f \in C_B(\mathbb{R}, X)$, $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $x \in \mathbb{R}$. Then i)

$$
\left\|\overline{A}_{n}(f,x)-f\left(x\right)\right\| \leq \omega_{1}\left(f,\frac{1}{n^{\alpha}}\right)+\left(1-h\left(n^{1-\alpha}-2\right)\right)\left\|f\right\|_{\infty}=: \mu,\qquad(26)
$$

and

ii)

$$
\left\| \overline{A}_n \left(f \right) - f \right\|_{\infty} \le \mu. \tag{27}
$$

For $f \in C_{uB}(\mathbb{R}, X)$ we get $\lim_{n \to \infty} A_n(f) = f$, pointwise and uniformly.

The speed of convergence is $\max\left(\frac{1}{n^{\alpha}},\left(1-h\left(n^{1-\alpha}-2\right)\right)\right)$.

Proof. As similar to [13], p. 294 is omitted. \blacksquare

In the next we discuss high order neural network X -valued approximation by using the smoothness of f.

Theorem 12 Let $f \in C^N([a, b], X)$, $n, N \in \mathbb{N}$, $0 < \alpha < 1$, $x \in [a, b]$ and $n^{1-\alpha} > 2$. Then i)

3. COMPUTATORAL ANALYSIS AND APPLICATIONS, Vol. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\nWe present a series of X-valued neural network approximations to a function
\ngiven with rates.
\nWe first give
\n**Theorem 10** Let
$$
f \in C([a, b], X)
$$
, $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $x \in [a, b]$.
\nThen
\n
$$
||A_n(f, x) - f(x)|| \le \frac{1}{\psi(1)} \left[\omega_1 \left(f, \frac{1}{n^{\alpha}} \right) + (1 - h(n^{1-\alpha} - 2)) ||f||_{\infty} \right] =: \rho,
$$
\nand
\n
$$
||A_n(f, x) - f(x)|| \le \frac{1}{\psi(1)} \left[\omega_1 \left(f, \frac{1}{n^{\alpha}} \right) + (1 - h(n^{1-\alpha} - 2)) ||f||_{\infty} \right] =: \rho,
$$
\n(23)
\nWe notice $\lim_{n \to \infty} A_n(f) = f$, pointwise and uniformly.
\nThe speed of convergence is $\lim_{n \to \infty} (\frac{1}{n^{\alpha}}, (1 - h(n^{1-\alpha} - 2)))$.
\nProvef. As similar to [13], p. 293 is omitted.
\n**Theorem 11** Let $f \in C_n(R, X)$, $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $x \in \mathbb{R}$. Then
\n
$$
||\overline{A}_n(f, x) - f(x)|| \le \omega_1 \left(f, \frac{1}{n^{\alpha}} \right) + (1 - h(n^{1-\alpha} - 2)) ||f||_{\infty} =: \mu,
$$
\n(26)
\nand
\n
$$
||\overline{A}_n(f) - f||_{\infty} \le \mu.
$$
\n(27)
\nFor $f \in C_{n,0}(\mathbb{R}, X)$ we get $\lim_{n \to \infty} \overline{A}_n(f) = f$, pointwise and uniformly.
\nThe speed of convergence is $\lim_{n \to \infty} \left(\frac{1}{n^{\alpha}}, (1 - h(n^{1-\alpha} - 2)) \right)$.
\n

$$
\left[\omega_1\left(f^{(N)},\frac{1}{n^{\alpha}}\right)\frac{1}{n^{\alpha N}N!}+\frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)\left\|f^{(N)}\right\|_{\infty}\left(b-a\right)^N}{N!}\right]\right\}
$$

ii) assume further $f^{(j)}(x_0) = 0, j = 1, ..., N$, for some $x_0 \in [a, b]$, it holds

$$
||A_n(f, x_0) - f(x_0)|| \le \frac{1}{\psi(1)}
$$

$$
\left\{ \omega_1 \left(f^{(N)}, \frac{1}{n^{\alpha}} \right) \frac{1}{n^{\alpha N} N!} + \frac{\left(1 - h \left(n^{1 - \alpha} - 2 \right) \right) ||f^{(N)}||_{\infty} (b - a)^N}{N!} \right\}, \quad (29)
$$

and

$$
iii)
$$

0. COMPUTATIONAL ANALYSIS AND APPLICATIONS. VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\n
$$
\left[\omega_1\left(f^{(N)}, \frac{1}{n^N}\right) \frac{1}{n^{aN}N!} + \frac{\left(1 - h\left(n^{1-\alpha} - 2\right)\right) ||f^{(N)}||_{\infty} (b - a)^N}{N!}\right]\right\}
$$
\n*ii) assume further* $f^{(j)}(a) = 0, j = 1,..., N, for some $x_0 \in [a, b]$, *it holds*
\n $||A_n(f, x_0) - f(x_0)|| \le \frac{1}{\psi(1)}$
\n
$$
\left\{\omega_1\left(f^{(N)}, \frac{1}{n^{\alpha}}\right) \frac{1}{n^{\alpha N}N!} + \frac{\left(1 - h\left(n^{1-\alpha} - 2\right)\right) ||f^{(N)}||_{\infty} (b - a)^N}{N!}\right\}, (29)
$$
\n*and*
\n*iii*)
\n $||A_n(f) - f||_{\infty} \le \frac{1}{\psi(1)} \left\{\sum_{j=1}^N \frac{||f^{(j)}||_{\infty}}{j!} \left[\frac{1}{n^{c,j}} + \frac{\left(1 - h\left(n^{1-\alpha} - 2\right)\right) (b - a)^j}{2!} \right] + \frac{1}{\psi(1)} \left[\omega_1\left(f^{(N)}, \frac{1}{n^{\alpha}}\right) \frac{1}{n^{c/N}N!} + \frac{\left(1 - h\left(n^{1-\alpha} - 2\right)\right) ||f^{(N)}||_{\infty} (b - a)^N}{N!} \right]\right\}.$ \n(30)
\n*Again, we obtain* $\lim_{n \to \infty} \omega_n(f) = f$, *pointwise and uniformly.*
\nProof. As similar to [13], pp. 296-301 is omitted.
\nAll integrals from now on are of Bochner type [19].
\nWe need to find the Cambridge, for *n*th column, *n*th column,$

Again we obtain $\lim_{n \to \infty} A_n(f) = f$, pointwise and uniformly.

Proof. As similar to [13], pp. 296-301 is omitted. \blacksquare All integrals from now on are of Bochner type [19]. We need

Definition 13 ([12]) Let $[a, b] \subset \mathbb{R}$, X be a Banach space, $\alpha > 0$; $m = [\alpha] \in \mathbb{N}$, ([iii] is the ceiling of the number), $f : [a, b] \rightarrow X$. We assume that $f^{(m)} \in$ $L_1([a, b], X)$. We call the Caputo-Bochner left fractional derivative of order α :

$$
\left(D_{*a}^{\alpha}f\right)(x) := \frac{1}{\Gamma(m-\alpha)} \int_{a}^{x} \left(x-t\right)^{m-\alpha-1} f^{(m)}\left(t\right) dt, \quad \forall \ x \in [a,b]. \tag{31}
$$

If $\alpha \in \mathbb{N}$, we set $D_{*a}^{\alpha} f := f^{(m)}$ the ordinary X-valued derivative (defined similar to numerical one, see [21], p. 83), and also set $D_{*a}^0 f := f$.

By [12], $(D_{*a}^{\alpha} f)(x)$ exists almost everywhere in $x \in [a, b]$ and $D_{*a}^{\alpha} f \in$ $L_1([a, b], X)$.

If $||f^{(m)}||_{L_{\infty}([a,b],X)} < \infty$, then by [12], $D_{*a}^{\alpha} f \in C([a,b],X)$, hence $||D_{*a}^{\alpha} f|| \in$ $C([a, b])$.

We mention

Lemma 14 ([11]) Let $\alpha > 0$, $\alpha \notin \mathbb{N}$, $m = [\alpha]$, $f \in C^{m-1}([a, b], X)$ and $f^{(m)} \in L_{\infty}([a, b], X)$. Then $D_{*a}^{\alpha} f(a) = 0$.

We mention

Definition 15 ([10]) Let $[a, b] \subset \mathbb{R}$, X be a Banach space, $\alpha > 0$, $m := [\alpha]$. We assume that $f^{(m)} \in L_1([a, b], X)$, where $f : [a, b] \to X$. We call the Caputo-Bochner right fractional derivative of order α :

$$
\left(D_{b-}^{\alpha}f\right)(x) := \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (z-x)^{m-\alpha-1} f^{(m)}(z) dz, \quad \forall \ x \in [a,b]. \tag{32}
$$

We observe that $(D_{b-}^m f)(x) = (-1)^m f^{(m)}(x)$, for $m \in \mathbb{N}$, and $(D_{b-}^0 f)(x) =$ $f(x)$.

By [10], $(D_b^{\alpha} f)(x)$ exists almost everywhere on $[a, b]$ and $(D_b^{\alpha} f) \in L_1([a, b], X)$. If $||f^{(m)}||_{L_{\infty}([a,b],X)} < \infty$, and $\alpha \notin \mathbb{N}$, by [10], $D_{b-}^{\alpha} f \in C([a,b],X)$, hence $\left\|D_{b-}^{\alpha}f\right\| \in C\left(\left[a,b\right]\right).$ We need 528 J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC George A. Anastassiou 520-534

Lemma 16 ([11]) Let $f \in C^{m-1}([a, b], X)$, $f^{(m)} \in L_{\infty}([a, b], X)$, $m = [\alpha]$, $\alpha > 0, \ \alpha \notin \mathbb{N}.$ Then $D_{b-}^{\alpha} f(b) = 0.$

Convention 17 We assume that

$$
D_{*x_0}^{\alpha} f(x) = 0, \text{ for } x < x_0,\tag{33}
$$

and

$$
D_{x_0-}^{\alpha} f(x) = 0, \text{ for } x > x_0,
$$
\n(34)

for all $x, x_0 \in [a, b]$.

We mention

Proposition 18 ([11]) Let $f \in C^n([a, b], X)$, $n = [\nu], \nu > 0$. Then $D_{*a}^{\nu} f(x)$ is continuous in $x \in [a, b]$.

Proposition 19 ([11]) Let $f \in C^m([a, b], X)$, $m = [\alpha], \alpha > 0$. Then $D_{b-}^{\nu} f(x)$ is continuous in $x \in [a, b]$.

We also mention

Proposition 20 ([11]) Let $f \in C^{m-1}([a, b], X)$, $f^{(m)} \in L_{\infty}([a, b], X)$, $m =$ $\lceil \alpha \rceil$, $\alpha > 0$ and

$$
D_{*x_0}^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_{x_0}^{x} (x-t)^{m-\alpha-1} f^{(m)}(t) dt,
$$
 (35)

for all $x, x_0 \in [a, b] : x \geq x_0$. Then $D_{\ast x_0}^{\alpha} f(x)$ is continuous in x_0 .

Proposition 21 ([11]) Let $f \in C^{m-1}([a, b], X)$, $f^{(m)} \in L_{\infty}([a, b], X)$, $m =$ $\lceil \alpha \rceil$, $\alpha > 0$ and

$$
D_{x_0-}^{\alpha}f\left(x\right) = \frac{\left(-1\right)^m}{\Gamma\left(m-\alpha\right)} \int_x^{x_0} \left(\zeta - x\right)^{m-\alpha-1} f^{(m)}\left(\zeta\right) d\zeta,\tag{36}
$$

for all $x, x_0 \in [a, b] : x_0 \geq x$.

Then $D_{x_0-}^{\alpha} f(x)$ is continuous in x_0 .

Corollary 22 ([11]) Let $f \in C^m([a, b], X)$, $m = [\alpha]$, $\alpha > 0$, $x, x_0 \in [a, b]$. Then $D_{*x_0}^a f(x)$, $D_{x_0}^a f(x)$ are jointly continuous functions in (x, x_0) from $[a, b]^2$ into X, X is a Banach space.

We need

Theorem 23 ([11]) Let $f : [a, b]^2 \rightarrow X$ be jointly continuous, X is a Banach space. Consider

$$
G(x) = \omega_1(f(\cdot, x), \delta, [x, b]), \qquad (37)
$$

 $\delta > 0, x \in [a, b]$.

Then G is continuous on $[a, b]$.

Theorem 24 ([11]) Let $f : [a, b]^2 \rightarrow X$ be jointly continuous, X is a Banach space. Then

$$
H(x) = \omega_1 \left(f(\cdot, x), \delta, [a, x] \right), \tag{38}
$$

 $x \in [a, b]$, is continuous in $x \in [a, b]$, $\delta > 0$.

We present the following X -valued fractional approximation result by neural networks.

Theorem 25 Let $\alpha > 0$, $N = [\alpha]$, $\alpha \notin \mathbb{N}$, $f \in C^N([a, b], X)$, $0 < \beta < 1$, $x \in [a, b], n \in \mathbb{N} : n^{1-\beta} > 2.$ Then i)

3. COMPUTATIONAL ANALYSIS AND APPLICATIONS, Vol. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\nProposition 21 (111) Let
$$
f \in C^{m-1}([a, b], X)
$$
, $f^{(m)} \in L_{\infty}([a, b], X)$, $m = [a]$, $\alpha > 0$ and
\n
$$
D_{x_0}^{\alpha} - f(x) = \frac{(-1)^m}{\Gamma(m - \alpha)} \int_x^{x_0} (\zeta - x)^{m - \alpha - 1} f^{(m)}(\zeta) d\zeta,
$$
\n(36)
\nfor all $x, x_0 \in [a, b]$: $x_0 \ge x$.
\nThen $D_{x_0}^{\alpha} - f(x)$ is continuous in x_0 .
\nCorollary 22 (111) Let $f \in C^m([a, b], X)$, $m = [a]$, $\alpha > 0$, $x, x_0 \in [a, b]$.
\nThen $D_{x_0}^{\alpha} f(x)$, $D_{x_0}^{\alpha} - f(x)$ are jointly continuous functions in (x, x_0) from
\n $[a, b]^2$ and αX is a Banach space.
\nWe need
\n**Theorem 23** (111) Let $f : [a, b]^2 \rightarrow X$ be jointly continuous, X is a Banach
\nspace. Consider
\n $G(x) = \omega_1(f(\cdot, x), \delta, [x, b])$,
\nThen G is continuous on $[a, b]$.
\n**Theorem 24** (111) Let $f : [a, b]^2 \rightarrow X$ be jointly continuous, X is a Banach
\nspace. Then
\n $H(x) = \omega_2(f(\cdot, x), \delta, [a, x])$, (38)
\n $x \in [a, b]$, is continuous in $x \in [a, b]$, $\delta > 0$.
\nWe present the following X -valued fractional approximation result by neural
\nnetwork.
\n**Theorem 25** Let $\alpha > 0$, $N = [\alpha]$, $\alpha \notin \mathbb{N}$, $f \in C^N([a, b], X)$, $$

ii) if
$$
f^{(j)}(x) = 0
$$
, for $j = 1, ..., N - 1$, we have
\n
$$
||A_n(f, x) - f(x)|| \leq \frac{(\psi(1))^{-1}}{\Gamma(\alpha + 1)}
$$
\n
$$
\left\{ \frac{(\omega_1 (D_{x}^{\alpha} f, \frac{1}{n^{\beta}})_{[a,x]} + \omega_1 (D_{*x}^{\alpha} f, \frac{1}{n^{\beta}})_{[x,b]}}{n^{\alpha \beta}} + \frac{1}{n^{\alpha \beta}} \right\}
$$
\n
$$
\left(\frac{1 - h(n^{1-\beta} - 2)}{2} \right) \left(||D_{x}^{\alpha} f||_{\infty, [a,x]} (x - a)^{\alpha} + ||D_{*x}^{\alpha} f||_{\infty, [x,b]} (b - x)^{\alpha} \right) \right\},
$$
\n(iii)

$$
||A_{n}(f, x) - f (x)|| \leq (\psi(1))^{-1}
$$

$$
\left\{ \sum_{j=1}^{N-1} \frac{||f^{(j)}(x)||}{j!} \left\{ \frac{1}{n^{\beta j}} + (b-a)^{j} \left(\frac{1-h(n^{1-\beta}-2)}{2} \right) \right\} + \frac{1}{\Gamma(\alpha+1)} \left\{ \frac{(\omega_{1} (D_{x}^{\alpha}f, \frac{1}{n^{\beta}})_{[a,x]} + \omega_{1} (D_{*x}^{\alpha}f, \frac{1}{n^{\beta}})_{[x,b]}}{n^{\alpha \beta}} + \frac{1}{\Gamma(\alpha+1)} \left\{ \frac{(\omega_{1} (D_{x}^{\alpha}f, \frac{1}{n^{\beta}})_{[a,x]} + \omega_{1} (D_{*x}^{\alpha}f, \frac{1}{n^{\beta}})_{[x,b]}}{n^{\alpha \beta}} + \frac{1}{\Gamma(\alpha+1)} \left\{ \frac{(\omega_{1} (D_{x}^{\alpha}f, \frac{1}{n^{\beta}})_{[a,x]} + \omega_{1} (D_{*x}^{\alpha}f, \frac{1}{n^{\beta}})_{[x,b]}}{n^{\alpha \beta}} + \frac{1}{\Gamma(\alpha+1)} \left\{ \frac{(\omega_{1} (D_{x}^{\alpha}f, \frac{1}{n^{\beta}})_{[a,x]} + \omega_{1} (D_{*x}^{\alpha}f, \frac{1}{n^{\beta}})_{[x,b]}}{n^{\alpha \beta}} \right\} \right\}
$$

 $\forall\,\,x\in\left[a,b\right] ,$ and iv)

1. COMPUTATIONAL ANALYSIS AND APPLICATIONS. VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\nii) if
$$
f^{(j)}
$$
 (x) = 0, for $j = 1, ..., N - 1$, we have
\n
$$
||A_n (f, x) - f (x)|| \leq \frac{\left(\psi(1)\right)^{-1}}{\Gamma(\alpha + 1)}
$$
\n
$$
\left\{ \frac{\left(\omega_1 (D_{x-}^{ij} f, \frac{1}{n^{2j}})_{[0,a]} + \omega_1 (D_{x-}^{ij} f, \frac{1}{n^{2j}})_{[x,b]}\right)}{\pi^{\alpha \beta}} + \frac{\left(1 - h (n^{1 - \beta} - 2) \right) \left(||D_{x-}^{n} f||_{\infty, [a,x]} (x - a)^{\alpha - 1} ||D_{xx}^{n} f||_{\infty, [x,b]} (b - x)^{\alpha} \right) \right\},\n(40)
$$
\niii)
\n
$$
||A_n (f, x) - f (x)|| \leq (\psi(1))^{-1}
$$
\n
$$
\left\{ \sum_{j=1}^{N-1} \frac{||f^{(j)}(x)||}{j!} \left\{ \frac{1}{n^{2j}} + (b - a)^{j} \left(\frac{1 - h (n^{1 - \beta} - 2)}{2} \right) \right\} + \frac{1}{\Gamma(\alpha + 1)} \left\{ \left(\frac{\omega_1 (D_{x-}^{n} f, \frac{1}{n^{2j}})_{[a,a]} + \omega_1 (D_{xx}^{n} f, \frac{1}{n^{2j}})_{[x,b]} \right)}{\Gamma(\alpha + 1)} \right\}
$$
\n
$$
\left\{ \frac{1 - h (n^{1 - \beta} - 2)}{2} \right) \left(||D_{x-}^{n} f||_{\infty, [a,a]} (x - a)^{\alpha} + ||D_{xx}^{n} f||_{\infty, [a,b]} (b - x)^{\alpha} \right) \right\},\n\forall x \in [a, b],
$$
\nand
\n
$$
||A_n f - f||_{\infty} \leq (\psi(1))^{-1}
$$
\n
$$
\left\{ \sum_{j=1}^{N-1} \frac{||f^{(j)}||_{\infty}}{j!} \left\{ \frac{1}{n^{2
$$

Above, when $N=1$ the sum $\sum_{j=1}^{N-1}$ $\cdot = 0$.

As we see here we obtain X-valued fractionally type pointwise and uniform convergence with rates of $A_n \to I$ the unit operator, as $n \to \infty$.

Proof. It is very lengthy, as similar to [13], pp. 305-316, is omitted. \blacksquare Next we apply Theorem 25 for $N = 1$.

Theorem 26 Let $0 < \alpha, \beta < 1, f \in C^1([a, b], X), x \in [a, b], n \in \mathbb{N} : n^{1-\beta} > 2$. Then i)

$$
||A_{n}(f, x) - f(x)|| \le
$$

$$
\frac{(\psi(1))^{-1}}{\Gamma(\alpha+1)} \left\{ \frac{\left(\omega_{1} \left(D_{x}^{\alpha} f, \frac{1}{n^{\beta}}\right)_{[a,x]} + \omega_{1} \left(D_{*x}^{\alpha} f, \frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}} + \frac{\left(\frac{1 - h\left(n^{1-\beta} - 2\right)}{2}\right) \left(\left\|D_{x}^{\alpha} f\right\|_{\infty,[a,x]} (x - a)^{\alpha} + \left\|D_{*x}^{\alpha} f\right\|_{\infty,[x,b]} (b - x)^{\alpha}\right)\right\},\right\}
$$
(43)

and ii)

3. COMPUTATIONAL ANALYSIS AND APPLICATIONS, Vol. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\nProof. It is very lengthy, as similar to [13], pp. 305-316, is omitted. ■
\nNext we apply Theorem 25 for
$$
N = 1
$$
.
\n**Thororm** 26 Let $0 < \alpha, \beta < 1$, $f \in C^1([a, b], N, x \in [a, b], n \in \mathbb{N} : n^{1-\beta} > 2$.
\n*Then*
\n*if*
\n*if*

When $\alpha = \frac{1}{2}$ we derive

Corollary 27 Let $0 < \beta < 1$, $f \in C^{1}([a, b], X)$, $x \in [a, b]$, $n \in \mathbb{N} : n^{1-\beta} > 2$. Then

i)
\n
$$
||A_n(f, x) - f(x)|| \le
$$
\n
$$
\frac{2(\psi(1))^{-1}}{\sqrt{\pi}} \left\{ \frac{\left(\omega_1 \left(D_{x-}^{\frac{1}{2}} f, \frac{1}{n^{\beta}}\right)_{[a,x]} + \omega_1 \left(D_{*x}^{\frac{1}{2}} f, \frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\frac{\beta}{2}}} + \frac{\left(\frac{1 - h\left(n^{1-\beta} - 2\right)}{2}\right) \left(\left\|D_{x-}^{\frac{1}{2}} f\right\|_{\infty,[a,x]} \sqrt{(x-a)} + \left\|D_{*x}^{\frac{1}{2}} f\right\|_{\infty,[x,b]} \sqrt{(b-x)}\right)\right\},\tag{45}
$$

and

ii)

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS. VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\n
$$
\iint_{\mathcal{H}}\iint_{\mathcal{H}}\int_{\math
$$

We finish with

Remark 28 Some convergence analysis follows:

Let $0 < \beta < 1$, $f \in C^1([a, b], X)$, $x \in [a, b]$, $n \in \mathbb{N} : n^{1-\beta} > 2$. We elaborate on (46). Assume that

$$
\omega_1 \left(D_{x-}^{\frac{1}{2}} f, \frac{1}{n^{\beta}} \right)_{[a,x]} \le \frac{K_1}{n^{\beta}},\tag{47}
$$

and

$$
\omega_1 \left(D_{*x}^{\frac{1}{2}} f, \frac{1}{n^{\beta}} \right)_{[x,b]} \le \frac{K_2}{n^{\beta}},\tag{48}
$$

 $\forall x \in [a, b], \forall n \in \mathbb{N}, where K_1, K_2 > 0.$

Then it holds

$$
\frac{\left[\sup_{x \in [a,b]} \omega_1 \left(D_{x-}^{\frac{1}{2}}f, \frac{1}{n^{\beta}}\right)_{[a,x]} + \sup_{x \in [a,b]} \omega_1 \left(D_{*x}^{\frac{1}{2}}f, \frac{1}{n^{\beta}}\right)_{[x,b]}\right]}{n^{\frac{\beta}{2}}} \le \frac{\frac{(K_1 + K_2)}{n^{\frac{\beta}{2}}} = \frac{(K_1 + K_2)}{n^{\frac{3\beta}{2}}} = \frac{K}{n^{\frac{3\beta}{2}}},\tag{49}
$$

where $K := K_1 + K_2 > 0$.

The other summand of the right hand side of (46) , for large enough n, converges to zero at the speed $\left(\frac{1-h(n^{1-\beta}-2)}{2}\right)$ $\overline{ }$:

Then, for large enough $n \in \mathbb{N}$, by (46) and (49) and the last comment, we obtain that

$$
||A_n f - f||_{\infty} \le M \max\left(\frac{1}{n^{\frac{3\beta}{2}}}, \left(\frac{1 - h\left(n^{1-\beta} - 2\right)}{2}\right)\right),\tag{50}
$$

where $M > 0$. If $\frac{1}{n^{\frac{3\beta}{2}}} \geq$ $\left(\frac{1-h\left(n^{1-\beta }-2\right) }{2}\right)$), then $\frac{1}{n^{\beta}} \geq$ $\left(\frac{1-h\left(n^{1-\beta }-2\right) }{2}\right)$ $\overline{}$, and consequently $||A_nf - f||_{\infty}$ in (50) converges to zero faster than in Theorem 10. This because the differentiability of f . 5 CONFUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 31, NO. 4, 2023, COPYRIGHT 2023 CLOCOUS PRESS, LLC

The $\frac{1}{N} \geq \left(-\frac{1}{2} \left(\alpha + \frac{1}{2}\right)^2\right)$, $R/m_1 \leq \frac{1}{2} \left(-\frac{1}{2} \left(\alpha + \frac{1}{2}\right)^2\right)$, and convergenently
 $\left[\frac{1}{2$

References

- [1] G.A. Anastassiou, Rate of convergence of some neural network operators to the unit-univariate case, J. Math. Anal. Appl, 212 (1997), 237-262.
- [2] G.A. Anastassiou, Quantitative Approximations, Chapman & Hall / CRC, Boca Raton, New York, 2001.
- [3] G.A. Anastassiou, Univariate hyperbolic tangent neural network approximation, Mathematics and Computer Modelling, 53 (2011), 1111-1132.
- [4] G.A. Anastassiou, Multivariate hyperbolic tangent neural network approximation, Computers and Mathematics, 61 (2011), 809-821.
- [5] G.A. Anastassiou, Multivariate sigmoidal neural network approximation, Neural Networks, 24 (2011), 378-386.
- [6] G.A. Anastassiou, *Inteligent Systems: Approximation by Artificial Neural* Networks, Intelligent Systems Reference Library, Vol. 19, Springer, Heidelberg, 2011.
- [7] G.A. Anastassiou, Univariate sigmoidal neural network approximation, J. of Computational Analysis and Applications, Vol. 14, No. 4, 2012, 659-690.
- [8] G.A. Anastassiou, Fractional neural network approximation, Computers and Mathematics with Applications, 64 (2012), 1655-1676.
- [9] G.A. Anastassiou, Intelligent Systems II: Complete Approximation by Neural Network Operators, Springer, Heidelberg, New York, 2016.
- [10] G.A. Anastassiou, Strong Right Fractional Calculus for Banach space valued functions, ëRevista Proyecciones, Vol. 36, No. 1 (2017), 149-186.
- [11] G.A. Anastassiou, Vector fractional Korovkin type Approximations, Dynamic Systems and Applications, 26 (2017), 81-104.
- [12] G.A. Anastassiou, A strong Fractional Calculus Theory for Banach space valued functions, Nonlinear Functional Analysis and Applications (Korea), 22(3)(2017), 495-524.
- [13] G.A. Anastassiou, Intelligent Computations: Abstract Fractional Calculus, Inequalities, Approximations, Springer, Heidelberg, Neq York, 2018.
- [14] Z. Chen and F. Cao, The approximation operators with sigmoidal functions, Computers and Mathematics with Applications, 58 (2009), 758-765. 534 J. Computational Analysis And APPLICATIONS, VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC (For a statistic of the statistic
	- [15] D. Costarelli, R. Spigler, Approximation results for neural network operators activated by sigmoidal functions, Neural Networks 44 (2013), 101-106.
	- [16] D. Costarelli, R. Spigler, Multivariate neural network operators with sigmoidal activation functions, Neural Networks 48 (2013), 72-77.
	- [17] S. Haykin, Neural Networks: A Comprehensive Foundation (2 ed.), Prentice Hall, New York, 1998.
	- [18] W. McCulloch and W. Pitts, A logical calculus of the ideas immanent in nervous activity, Bulletin of Mathematical Biophysis, 7 (1943), 115-133.
	- [19] J. Mikusinski, The Bochner integral, Academic Press, New York, 1978.
	- [20] T.M. Mitchell, Machine Learning, WCB-McGraw-Hill, New York, 1997.
	- [21] G.E. Shilov, Elementary Functional Analysis, Dover Publications, Inc., New York, 1996.

Approximation of Time Separating Stochastic Processes by Neural Networks

George A. Anastassiou

Department of Mathematical Sciences University of Memphis, Memphis, TN 38152, U.S.A. ganastss@memphis.edu

Dimitra Kouloumpou

Section of Mathematics Hellenic Naval Academy, Piraeus, 18539, Greece dimkouloumpou@hna.gr

Abstract

Here we study the univariate quantitative approximation of time separating stochastic process over a compact interval or all the real line by quasi-interpolation neural network operators. We perform also the related fractional approximation. These approximations are derived by establishing Jackson type inequalities involving the modulus of continuity of the engaged stochastic function or its high order derivative or fractional derivatives. Our operators are defined by using a density function induced by a general sigmoid function. The approximations are pointwise and with respect to the uniform norm. The feed-forward neural networks are with one hidden layer. We finish with a lot of interesting applications.

2020 AMS Subject Classification: 26A33, 41A17, 41A25

Keywords and Phrases: general sigmoid function, time separating stochastic process, neural network approximation, quasi-interpolation operator, modulus of continuity, Caputo fractional derivative, fractional approximation.

1 Introduction

The first author in [1] and [2], see Chapters 2-5, was the first to establish neural network approximation to continuous functions with rates by very specifically defined neural network operators of Cardaliaguet-Euvrard and "Squashing" types, by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities. He treats there both the univariate and multivariate cases. The defining these operators "bell-shaped" and "squashing" functions are assumed to be of compact suport. Also in [2] he gives the Nth order asymptotic expansion for the error of weak approximation of these two operators to a special natural class of smooth functions, see Chapters 4-5 there. The first author inspired by [15], continued his studies on neural networks approximation by introducing and using the proper quasi interpolation operators of sigmoidal and hyperbolic tangent type which resulted into EXERCISE THE SPECIES AND APPLICATIONS, VOL. 31, NO. 4, 2023, COPYRIGHT 2023 COOKSPRESS, LLC (DEVICATION OF THE SPECIESTING SUCH ASSESS). THE SPECIES COOK CONTINUES COOK CONTINUES COOK CONTINUES COOK CONTINUES COOK CONTINU

[3]-[7], by treating both the univariate and multivariate cases. He did also the corresponding fractional case [8]. In this article we are also inspired by the related works [16], [17]. The authors here use general sigmoid function based neural network quantitative approximations to continuous functions over compact intervals of the real line or over the whole $\mathbb R$ with values to $\mathbb R$. All convergences here are with rates expressed via the modulus of continuity of the involved function or its high order derivative, or fractional derivatives and given by very tight Jackson type inequalities. More precisely, here we perform quantitative approximations of time separating stochastic processes by neural networks. We give plenty of varied and interesting applications. Specific motivations came by: 2. COMPUTATIONAL ANNEWSK ORD APPLICATIONS, VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC

7. Sp Testing, both the microchite and militionize cones. He did also the corresponding fractional cone

for the variety

1. Stationary Gaussian processes with an explicit representation such as

$$
X_t = \cos(\alpha t)\,\xi_1 + \sin(\alpha t)\,\xi_2, \alpha \in \mathbb{R},
$$

where ξ_1, ξ_2 are independent random variables with the standard normal distribution, see [19],

2. by the "Fourier model" of a stationary process, see [20].

Feed-forward neural networks (FNNs) with one hidden layer, the only type of networks we deal with in this article, are mathematically expressed as

$$
N_n(x) = \sum_{j=0}^n c_j \sigma(\langle a_j \cdot x \rangle + b_j), \ \ x \in \mathbb{R}^s, s \in \mathbb{N},
$$

where for $0 \le j \le n$, $b_j \in \mathbb{R}$ are the thresholds, $a_j \in \mathbb{R}^s$ are the connection weights, $c_j \in X$ are the coefficients, $\langle a_i \cdot x \rangle$ is the inner product of a_j and x, and σ is the activation function of the network. In many fundamental neural network models, the activation function is derived from various specific sigmoid functions. Here we work for a general sigmoid function. About neural networks in general read [18], [21],[23]. See also [9] for a complete study of real valued approximation by neural network operators.

2 Background

Here we follow [14].

2.1 Basics on Neural Network

Let $h : \mathbb{R} \to [-1,1]$ be a general sigmoid function, such that it is strictly increasing, $h(0) = 0, h(-x) = -h(x)$ for every $x \in \mathbb{R}$, $h(+\infty) = 1$, $h(-\infty) = -1$. Also h is strictly convex over $(-\infty, 0]$ and strictly concave over $[0, +\infty)$, with $h^{(2)} \in C(\mathbb{R})$.

Some examples of related sigmoid functions follow: $\frac{1}{1+e^{-x}}$; $\tanh x$; $\frac{2}{\pi} \arctan\left(\frac{\pi}{2}x\right)$; $\frac{x}{2\sqrt[n]{1+x^{2m}}}$, $m \in \mathbb{N}$; $\frac{2}{\pi}gd(x)$; \boldsymbol{x} $\frac{x}{(1+|x|^{\lambda})^{\frac{1}{\lambda}}}, \lambda \text{ is odd }; erf\left(\frac{\sqrt{\pi}}{2}\right)$ $\left(\frac{\sqrt{\pi}}{2}x\right)$; $\frac{1}{1+e^{-\mu x}}$; tanh (μx) , $\mu > 0$ for all $x \in \mathbb{R}$

We consider the activation function

$$
\psi(x) := \frac{1}{4} \left(h(x+1) - h(x-1) \right), x \in \mathbb{R},
$$
\n(1)

As in [13], p.285, we get that

$$
\psi(-x) = \psi(x), \text{ for every } x \in \mathbb{R}.
$$

Thus ψ is an even function.

Since $x + 1 > x - 1$, then $h(x + 1) > h(x - 1)$, and $\psi(x) > 0$, for all $x \in \mathbb{R}$. We see that

$$
\psi(0) = \frac{h(1)}{2}.\tag{2}
$$

Let $x > 1$, we have that

$$
\psi'(x) = \frac{1}{4} \left(h'(x+1) - h'(x-1) \right) < 0,\tag{3}
$$

by h' be strictly decreasing on $[0, +\infty)$.

Let now $0 < x < 1$, then $1 - x > 0$ and $0 < 1 - x < 1 + x$. It holds $h'(x - 1) = h'(1 - x) > h'(x + 1)$, so that again $\psi'(x) < 0$. Consequently ψ is strictly decreasing on $(0, +\infty)$. Clearly ψ is strictly increasing on $(-\infty, 0)$ and $\psi'(0) = 0$. See that

3. COMPUTATIONAL AVALYSIS AND APPLICATIONS. Vol. 31, NO. 4, 2023, OPYRIGHT 2023 EUDOXUS PRESS, LC
\nus φ is an even function.
\n
$$
we x + 1 > x - 1, then h(x + 1) > h(x - 1), and φ(x) > 0, for all x ∈ ℝ.
$$
\n
$$
w'(0) = \frac{h(1)}{2}.
$$
\n
$$
x > 1, we have that
$$
\n
$$
v'(x) = \frac{1}{4} (h'(x + 1) - h'(x - 1)) < 0,
$$
\n
$$
h' \text{ be strictly decreasing on } [0, +\infty).
$$
\n
$$
h'' \text{ be strictly decreasing on } [0, +\infty).
$$
\n
$$
w(0) < x < 1, \text{ then } 1 - x > 0 \text{ and } 0 < 1 - x < 1 + x.
$$
\nIt holds $h'(x - 1) = h'(1 - x) > h'(x + 1),$ so
\n
$$
h \text{ again } \psi'(x) = \frac{1}{4} (h(+\infty) - h(+\infty)) = 0,
$$
\nand\n
$$
\lim_{x \to +\infty} \psi(x) = \frac{1}{4} (h(+\infty) - h(-\infty)) = 0.
$$
\nAt is the x-axis is the horizontal asymptote on ψ .\n
$$
= \lim_{x \to +\infty} \psi(x) = \frac{1}{4} (h(-\infty) - h(-\infty)) = 0.
$$
\nAt the $\text{at } x = 1, \text{ and } y = 0$.\n
$$
= \lim_{x \to +\infty} \psi(x - i) = 1, \text{ for every } x \in \mathbb{R}.
$$
\n
$$
= \lim_{x \to +\infty} \psi(x) \text{ is a density function on R}.
$$
\n
$$
\text{From 2. } (f4)/f \text{ It holds}
$$
\n
$$
\int_{-\infty}^{+\infty} \psi(x) dx = 1.
$$
\n
$$
\text{From 3. } (f4)/f \text{ Let } 0 < \alpha < 1, \text{ and } n \in \mathbb{N} \text{ with } n^{1-\alpha} > 2.
$$
\n
$$
f \text{ holds}
$$
\n
$$
\int_{-\infty}^{+\in
$$

That is the x-axis is the horizontal asymptote on ψ . Conclusion, ψ is a bell symmetric function with maximum $\psi(0) = \frac{h(1)}{2}$. We need

Theorem 1. *([14]) We have that*

$$
\sum_{i=-\infty}^{+\infty} \psi(x-i) = 1, \text{ for every } x \in \mathbb{R}.
$$
 (5)

Theorem 2. *([14]) It holds*

$$
\int_{-\infty}^{+\infty} \psi(x)dx = 1.
$$
 (6)

Thus, $\psi(x)$ is a density function on R. We give

Theorem 3. *([14])* Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$. It holds

$$
\sum_{k=-\infty}^{\infty} \psi(nx-k) < \frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)}{2}.
$$
\n
$$
\left\{\n\begin{array}{l}\nk = -\infty \\
\vdots \\
\left|nx - k\right| \ge n^{1-\alpha}\n\end{array}\n\right.
$$
\n(7)

Notice that

$$
\lim_{n \to +\infty} \frac{\left(1 - h\left(n^{1-\alpha} - 2\right)\right)}{2} = 0.
$$

Denote by $|\cdot|$ the integral part of the number and by $[\cdot]$ the ceiling of the number. We further give

Theorem 4. $([14])$ Let $x \in [a, b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $[na] \leq [nb]$. It holds

$$
\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi\left(nx-k\right)} < \frac{1}{\psi\left(1\right)}, \quad \forall \ x \in [a, b]. \tag{8}
$$

Remark 5. *We have that*

$$
\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi\left(nx-k\right) \neq 1,\tag{9}
$$

for at least some $x \in [a, b]$. *See* [13], p. 290, same reasoning.

Note 6. For large enough n we always obtain $\lceil na \rceil \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, if and only if $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$. *In general it holds (by (5))*

$$
\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi\left(nx-k\right) \le 1. \tag{10}
$$

Let $(X, \|\cdot\|)$ be a Banach space.

Definition 7. *([14])* Let $f \in C([a, b], X)$ and $n \in \mathbb{N} : [na] \leq \lfloor nb \rfloor$. We introduce and define the X-valued *linear neural network operators*

$$
A_n(f, x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \psi\left(nx - k\right)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi\left(nx - k\right)}, \quad x \in [a, b].
$$
\n(11)

Clearly here $A_n(f, x) \in C([a, b], X)$. For convenience we use the same A_n for real valued function when needed. We mention here the pointwise and uniform convergence of $A_n(f, x)$ to $f(x)$ with rates. For convenience also we call $n \times n$

$$
A_n^*(f, x) := \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \psi\left(nx - k\right),\tag{12}
$$

(similarly A_n^* can be defined for real valued function) that is

$$
A_n(f, x) = \frac{A_n^*(f, x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi\left(nx - k\right)}.\tag{13}
$$

So that

J. COMPUTATIONAL AVALYSIS AND APPLICATIONS. VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUB PRESS, LIC
\nmark 5. We have that
\n
$$
\lim_{n \to \infty} \sum_{k=|\alpha|}^{|\alpha|} \psi(nx-k) \neq 1,
$$
\n(d least some $x \in [a, b]$. See [13], p. 290, some reasoning.
\nthe 6. For large enough n we always obtain [na] ≤ [nb]. Also $a \leq \frac{5}{n} \leq b$, if and only if $[na] \leq k \leq [nb]$.
\nLet $(X, ||+||)$ be a Banach space.
\n**Hint** $T = \sum_{k=|\alpha|}^{|\alpha|} \psi(nx-k) \leq 1.$ (10)
\nLet $(X, ||+||)$ be a Banach space.
\n**Hint** $T = \sum_{k=|\alpha|}^{|\alpha|} \psi(nx-k)$ and $n \in \mathbb{N}: |na| \leq |nb|$. We introduce and define the X-valued
\narr neural network operators
\n $A_n (f, x) \in C([a, b], X)$. For convenience we use the same A_n for real valued function
\nin model. We mention here the pointwise and uniform convergence of $A_n (f, x)$ to $f(x)$ with rates. For
\nvertices also we call
\n
$$
A_n (f, x) := \sum_{k=|\alpha|}^{|\alpha|} \int_{\alpha}^{k} \psi(nx-k)
$$
\n
$$
A_n (f, x) = \sum_{k=|\alpha|}^{|\alpha|} \int_{\alpha}^{k} \psi(nx-k)
$$
\n
$$
A_n (f, x) = \sum_{k=|\alpha|}^{|\alpha|} \psi(nx-k).
$$
 (12)
\n
$$
A_n (f, x) = \sum_{k=|\alpha|}^{|\alpha|} \psi(nx-k)
$$
 (13)
\n
$$
A_n (f, x) = f(x) = \frac{A_n^*(f, x)}{\sum_{k=|\alpha|}^{|\alpha|} \psi(nx-k)} - f(x)
$$
 (14)
\n
$$
A_n (f, x) = f(x) = \frac{A_n^*(f, x) - f(x) \left(\sum_{k=|\alpha|}^{|\alpha|} \psi(nx-k) - f(x) \right)}{\sum_{k=|\alpha|}^{|\alpha|} \psi
$$

Consequently we derive

$$
||A_n(f, x) - f(x)|| \le \frac{1}{\psi(1)} \left||A_n^*(f, x) - f(x) \left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k)\right) \right||. \tag{15}
$$

That is

$$
||A_n(f,x) - f(x)|| \le \frac{1}{\psi(1)} \left\| \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(f\left(\frac{k}{n}\right) - f(x) \right) \psi(nx-k) \right\|. \tag{16}
$$

We will estimate the right hand side of (16).

For that we need, for $f \in C([a, b], X)$ the first modulus of continuity

$$
\omega_1(f, \delta)_{[a,b]} := \omega_1(f, \delta) := \sup_{\begin{array}{l} x, y \in [a, b] \\ |x - y| \le \delta \end{array}} ||f(x) - f(y)||, \quad \delta > 0. \tag{17}
$$

Similarly, it is defined ω_1 for $f \in C_{uB}(\mathbb{R}, X)$ (uniformly continuous and bounded functions from $\mathbb R$ into X), for $f \in C_B(\mathbb{R}, X)$ (continuous and bounded X-valued) and for $f \in C_u(\mathbb{R}, X)$ (uniformly continuous). The fact $f \in C([a, b], X)$ or $f \in C_u(\mathbb{R}, X)$, is equivalent to $\lim_{\delta \to 0} \omega_1(f, \delta) = 0$, see [11]. 1. COMPUTATIONAL ANNEXTRO AND APPLICATIONS, VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
 $\int \mathcal{L} \mathcal$

Definition 8. *([14])* When $f \in C_{uB}(\mathbb{R}, X)$, or $f \in C_B(\mathbb{R}, X)$, we define

$$
\overline{A}_{n}(f,x) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) \psi\left(nx-k\right), \quad n \in \mathbb{N}, \ x \in \mathbb{R},\tag{18}
$$

the X*-valued quasi-interpolation neural network operator.*

Remark 9. *([14]) We have that the series*

$$
\sum_{k=-\infty}^{+\infty} f\left(\frac{k}{n}\right) \psi\left(nx-k\right)
$$

is absolutely convergent in X*, hence it is convergent in* X and $\overline{A}_n(f, x) \in X$.

We denote by $||f||_{\infty} := \sup_{x \in [a,b]} ||f(x)||$, for $f \in C([a,b],X)$, similarly is defined for $f \in C_B(\mathbb{R},X)$. We mention a series of X -valued neural network approximations to a function given with rates. We first give

Theorem 10. $(|14|)$. Let $f \in C([a, b], X)$, $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $x \in [a, b]$. Then

i)

$$
||A_n(f,x) - f(x)|| \le \frac{1}{\psi(1)} \left[\omega_1 \left(f, \frac{1}{n^{\alpha}} \right) + \left(1 - h \left(n^{1-\alpha} - 2 \right) \right) ||f||_{\infty} \right] =: \rho,
$$
\n(19)

and

ii)

$$
\|A_n\left(f\right) - f\|_{\infty} \le \rho. \tag{20}
$$

We notice $\lim_{n \to \infty} A_n(f) = f$, pointwise and uniformly. The speed of convergence is max $\left(\frac{1}{n^{\alpha}}, \left(1 - h\left(n^{1-\alpha} - 2\right)\right)\right)$.

Next we give

Theorem 11. *([14]).* Let $f \in C_B(\mathbb{R}, X)$, $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $x \in \mathbb{R}$. Then

i)

$$
\left\|\overline{A}_{n}\left(f,x\right)-f\left(x\right)\right\| \leq \omega_{1}\left(f,\frac{1}{n^{\alpha}}\right)+\left(1-h\left(n^{1-\alpha}-2\right)\right)\left\|f\right\|_{\infty}=: \mu,
$$
\n(21)

and

ii)

$$
\left\| \overline{A}_n \left(f \right) - f \right\|_{\infty} \le \mu. \tag{22}
$$

For $f \in C_{uB}(\mathbb{R}, X)$ *we get* $\lim_{n \to \infty} \overline{A}_n(f) = f$, pointwise and uniformly. The speed of convergence is

$$
\max\left(\frac{1}{n^{\alpha}},\left(1-h\left(n^{1-\alpha}-2\right)\right)\right).
$$

In the next we discuss high order neural network X-valued approximation by using the smoothness of f . The X-valued derivatives are as the numerical ones, see $([24])$.

Theorem 12. $([14])$ Let $f \in C^N([a, b], X)$, $n, N \in \mathbb{N}$, $0 < \alpha < 1$, $x \in [a, b]$ and $n^{1-\alpha} > 2$. Then

i)

J. COMPUTATIONAL AVALYSIS AND APPLICATIONS. VOL. 31, NO. 4, 2023, OPYRIGHT 2023 EUDOXUS PRESS, LIC
\nIn the next we discuss high order neural network X-valued approximation by using the smoothness of f.
\nX-valued derivatives are as the numerical ones, see ([24]).
\neorem 12.
$$
([I_{d}])Let f \in C^{N}([a, b], X), n, N \in \mathbb{N}, 0 < \alpha < 1, x \in [a, b] \text{ and } n^{1-\alpha} > 2. Then
$$
\n
$$
|A_{n}(f, x) - f(x)| \leq \frac{1}{\psi(1)} \left\{ \sum_{i=1}^{N} \frac{||f^{(j)}(x)||}{j!} \left[\frac{1}{n^{\alpha_{j}}} + \frac{(1-h(n^{1-\alpha}-2))}{2} (b-a)^{j} \right] + \frac{1}{(23)}
$$
\n
$$
|u_{1}(f^{(N)}, \frac{1}{n^{\alpha}}) \frac{1}{n^{\alpha_{N}} N!} + \frac{(1-h(n^{1-\alpha}-2)) ||f^{(N)}||_{\infty} (b-a)^{N}}{N!} \right\}.
$$
\n(ii) Assume further $f^{(j)}(x_{0}) = 0$, $j = 1, ..., N$, for some $x_{0} \in [a, b],$ it holds
\n
$$
||A_{n}(f, x_{0}) - f(x_{0})|| \leq \frac{1}{\psi(1)}
$$
\n
$$
\left\{ \omega_{1} \left(f^{(N)}, \frac{1}{n^{\alpha}} \right) \frac{1}{n^{\alpha_{N}} N!} + \frac{(1-h(n^{1-\alpha}-2)) ||f^{(N)}||_{\infty} (b-a)^{N}}{N!} \right\}.
$$
\n(i)
$$
||A_{n}(f) - f||_{\infty} \leq \frac{1}{\psi(1)} \left\{ \sum_{j=1}^{N} \frac{||f^{(j)}||_{\infty}}{j!} \left[\frac{1}{n^{\alpha_{j}}} + \frac{(1-h(n^{1-\alpha}-2))}{2} (b-a)^{j} \right] + \frac{1}{\psi(1)} \left[\omega_{1} \left(f^{(N)}, \frac{1}{n^{\alpha}} \right) \frac{1}{n^{\alpha_{N}} N!} + \frac{(1-h(n^{1-\alpha}-2)) ||f^{(N)}||_{\in
$$

ii) Assume further $f^{(j)}(x_0) = 0$, $j = 1, ..., N$, for some $x_0 \in [a, b]$, it holds

$$
||A_n(f, x_0) - f(x_0)|| \le \frac{1}{\psi(1)}
$$

$$
\left\{ \omega_1 \left(f^{(N)}, \frac{1}{n^{\alpha}} \right) \frac{1}{n^{\alpha N} N!} + \frac{\left(1 - h \left(n^{1 - \alpha} - 2 \right) \right) ||f^{(N)}||_{\infty} (b - a)^N}{N!} \right\},
$$
(24)

and

iii)

$$
||A_n(f) - f||_{\infty} \le \frac{1}{\psi(1)} \left\{ \sum_{j=1}^N \frac{||f^{(j)}||_{\infty}}{j!} \left[\frac{1}{n^{\alpha j}} + \frac{(1 - h(n^{1-\alpha} - 2))}{2} (b - a)^j \right] + \left[\omega_1 \left(f^{(N)}, \frac{1}{n^{\alpha}} \right) \frac{1}{n^{\alpha N} N!} + \frac{(1 - h(n^{1-\alpha} - 2)) ||f^{(N)}||_{\infty} (b - a)^N}{N!} \right] \right\}.
$$
 (25)

Again we obtain $\lim_{n \to \infty} A_n(f) = f$, pointwise and uniformly.

All integrals from now on are of Bochner type [22]. We need

Definition 13. *([12])* Let $[a, b] \subset \mathbb{R}$, X be a Banach space, $\alpha > 0$; $m = [\alpha] \in \mathbb{N}$, ([·] is the ceiling of the *number*), $f : [a, b] \to X$. We assume that $f^{(m)} \in L_1([a, b], X)$. We call the Caputo-Bochner left fractional *derivative of order* α*:*

$$
\left(D_{*a}^{\alpha}f\right)(x) := \frac{1}{\Gamma\left(m-\alpha\right)} \int_{a}^{x} \left(x-t\right)^{m-\alpha-1} f^{(m)}\left(t\right) dt, \quad \forall \ x \in [a,b]. \tag{26}
$$

 $If \alpha \in \mathbb{N}, we set D_{*a}^{\alpha}f := f^{(m)}$ the ordinary X-valued derivative (defined similar to numerical one, see [24], *p. 83*), and also set $D_{*a}^{0}f := f$.

By [12], $(D_{*a}^{\alpha} f)(x)$ exists almost everywhere in $x \in [a, b]$ and $D_{*a}^{\alpha} f \in L_1([a, b], X)$.

If $||f^{(m)}||_{L_{\infty}([a,b],X)} < \infty$, then by [12], $D_{*a}^{\alpha} f \in C([a,b],X)$, hence $||D_{*a}^{\alpha} f|| \in C([a,b])$.

Definition 14. *([10])* Let $[a, b] \subset \mathbb{R}$, X be a Banach space, $\alpha > 0$, $m := [\alpha]$. We assume that $f^{(m)} \in$ $L_1([a, b], X)$ *, where* $f : [a, b] \to X$ *. We call the Caputo-Bochner right fractional derivative of order* α *:*

$$
\left(D_{b-}^{\alpha}f\right)(x) := \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (z-x)^{m-\alpha-1} f^{(m)}(z) dz, \quad \forall \ x \in [a,b]. \tag{27}
$$

We observe that $(D_{b-}^{m} f)(x) = (-1)^{m} f^{(m)}(x)$, *for* $m \in \mathbb{N}$, *and* $(D_{b-}^{0} f)(x) = f(x)$.

By [10], $(D_{b-}^{\alpha}f)(x)$ exists almost everywhere on $[a, b]$ and $(D_{b-}^{\alpha}f) \in L_1([a, b], X)$. If $||f^{(m)}||_{L_{\infty}([a, b], X)} <$ ∞ , and $\alpha \notin \mathbb{N}$, by [10], $D_{b-}^{\alpha} f \in C([a, b], X)$, hence $||D_{b-}^{\alpha} f|| \in C([a, b])$.

In the next $\omega_1(f, \delta)_{[a,b]}$ refers to a modulus of continuity. ω_1 defined over $[a, b]$.

We mention the following X−valued fractional approximation result by neural networks.

Theorem 15. *([14]).* Let $\alpha > 0$, $N = [\alpha]$, $\alpha \notin \mathbb{N}$, $f \in C^N([a, b], X)$, $0 < \beta < 1$, $x \in [a, b]$, $n \in \mathbb{N} : n^{1-\beta} > 2$. *Then*

i)

$$
\left\| A_n(f, x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} A_n \left((\cdot - x)^j \right) (x) - f(x) \right\| \le
$$

$$
\frac{(\psi(1))^{-1}}{\Gamma(\alpha + 1)} \left\{ \frac{\left(\omega_1 \left(D_{x-}^{\alpha} f, \frac{1}{n^{\beta}} \right)_{[a,x]} + \omega_1 \left(D_{*x}^{\alpha} f, \frac{1}{n^{\beta}} \right)_{[x,b]} \right)}{n^{\alpha \beta}} + \left(\frac{1 - h \left(n^{1-\beta} - 2 \right)}{2} \right) \left(\left\| D_{x-}^{\alpha} f \right\|_{\infty, [a,x]} (x - a)^{\alpha} + \left\| D_{*x}^{\alpha} f \right\|_{\infty, [x,b]} (b - x)^{\alpha} \right) \right\},
$$

(28)

ii) if $f^{(j)}(x) = 0$ *, for* $j = 1, ..., N - 1$ *, we have*

$$
||A_{n}(f,x) - f(x)|| \leq \frac{(\psi(1))^{-1}}{\Gamma(\alpha+1)}
$$

$$
\left\{ \frac{\left(\omega_{1}\left(D_{x}^{\alpha} f, \frac{1}{n^{\beta}}\right)_{[a,x]} + \omega_{1}\left(D_{*x}^{\alpha} f, \frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}} + \frac{\left(\omega_{1}\left(D_{x}^{\alpha} f, \frac{1}{n^{\beta}}\right)_{[a,x]} + \omega_{1}\left(D_{*x}^{\alpha} f, \frac{1}{n^{\beta}}\right)_{[x,b]}}{n^{\alpha\beta}}\right)}{\left(\left\|D_{x}^{\alpha} f\right\|_{\infty, [a,x]} (x-a)^{\alpha} + \left\|D_{*x}^{\alpha} f\right\|_{\infty, [x,b]} (b-x)^{\alpha}\right)}\right\},
$$
(29)

iii)

^kAⁿ (f, x) [−] ^f (x)k ≤ (^ψ (1))[−]¹ N X−1 j=1 f (j) (x) j! (1 ⁿβj + (^b [−] ^a) j 1 − h n ¹−^β [−] ² 2 !) ⁺ 1 Γ (α + 1) ω1 D^α ^x−f, ¹ n^β [a,x] + ω¹ D^α ∗x f, ¹ n^β [x,b] ⁿαβ ⁺ 1 − h n ¹−^β [−] ² 2 ! ^D^α ^x−f ∞,[a,x] (x − a) ^α ⁺ ^kD^α ∗x ^fk∞,[x,b] (b − x) α)) , (30) ∀ x ∈ [a, b] , 541 J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC George A. Anastassiou 535-556

and

iv)

$$
||A_n f - f||_{\infty} \le (\psi(1))^{-1}
$$

$$
\left\{ \sum_{j=1}^{N-1} \frac{||f^{(j)}||_{\infty}}{j!} \left\{ \frac{1}{n^{\beta j}} + (b-a)^j \left(\frac{1-h(n^{1-\beta}-2)}{2} \right) \right\} + \right\}
$$

$$
\frac{1}{\Gamma(\alpha+1)} \left\{ \frac{\left(\sup_{x \in [a,b]} \omega_1 \left(D_{x-}^{\alpha}f, \frac{1}{n^{\beta}}\right)_{[a,x]} + \sup_{x \in [a,b]} \omega_1 \left(D_{*x}^{\alpha}f, \frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha \beta}} + \frac{\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)(b-a)^{\alpha} \left(\sup_{x \in [a,b]} \left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]} + \sup_{x \in [a,b]} \left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}\right)\right\} \right\}.
$$
\n(31)

Above, when $N = 1$ *the sum* $\sum_{j=1}^{N-1}$ $\cdot = 0$.

As we see here we obtain X*-valued fractionally type pointwise and uniform convergence with rates of* $A_n \to I$ *the unit operator, as* $n \to \infty$.

Next we apply Theorem 15 for $N = 1$.

Corollary 16. *([14])* Let $0 < \alpha, \beta < 1$, $f \in C^1([a, b], X)$, $x \in [a, b]$, $n \in \mathbb{N} : n^{1-\beta} > 2$. Then

i)

$$
||A_{n}(f,x) - f(x)|| \le
$$

$$
\frac{(\psi(1))^{-1}}{\Gamma(\alpha+1)} \left\{ \frac{\left(\omega_{1} \left(D_{x-}^{\alpha} f, \frac{1}{n^{\beta}}\right)_{[a,x]} + \omega_{1} \left(D_{*x}^{\alpha} f, \frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}} + \frac{\left(\omega_{1} \left(D_{x-}^{\alpha} f, \frac{1}{n^{\beta}}\right)_{[a,x]} + \omega_{1} \left(D_{*x}^{\alpha} f, \frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}} + \frac{\left(\omega_{1} \left(D_{x-}^{\alpha} f, \frac{1}{n^{\beta}}\right)_{[a,x]} + \omega_{1} \left(D_{*x}^{\alpha} f, \frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}} + \frac{\left(\omega_{1} \left(D_{x-}^{\alpha} f, \frac{1}{n^{\beta}}\right)_{[a,x]} + \omega_{1} \left(D_{*x}^{\alpha} f, \frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{(32)}
$$

and

ii)

J. COMPUTAIAA AAA'YSS AND APPUCATIONS, VOL. 31, NO. 4, 2023, OPYRIGHT 2023 EUDOXUS PRESS, LIC
\n
$$
\frac{1}{\Gamma(\alpha+1)} \left\{ \frac{\left(\sup_{x \in [0,b]} \omega_1 \left(D_{x-}^{\alpha} f, \frac{1}{n^{\beta}}\right)_{\{x \in [0,b]}\right.}{n^{\alpha\beta}} + \sup_{x \in [0,b]} \omega_1 \left(D_{x+}^{\alpha} f, \frac{1}{n^{\beta}}\right)_{\{x \in [a,b]}\right.}{n^{\alpha\beta}} + \frac{\left(\frac{1 - h\left(n^{1-\beta} - 2\right)}{2}\right) \left(b - a\right)^{\alpha} \left(\sup_{x \in [0,b]} \|D_x^{\alpha} f\|_{\infty, [a,b]} + \sup_{x \in [0,b]} \|D_{xx}^{\alpha} f\|_{\infty, [x,b]}\right)\right\} \right\}.
$$
\n(J1)
\n*we, when N = 1 the sum $\sum_{i=1}^{N-1} = 0$.*
\n*As we see here we obtain X -valued fractionally type pointwise and uniform convergence with rates of $\rightarrow I$ the unit operator, as $n \rightarrow \infty$.
\nNext we apply Theorem 15 for $N = 1$.
\n
$$
\left\{ \frac{|A_0(f, x) - f(x)| \le}{2} \left(\frac{\omega_1 (D_{x-}^{\alpha} f, \frac{1}{n^{\beta}})_{\omega, \alpha\beta} + \omega_1 (D_{xx}^{\alpha} f, \frac{1}{n^{\beta}})_{\{x, b\}}\right) + \frac{\left(\frac{1 - h\left(n^{1-\beta} - 2\right)}{2}\right) \left(\|D_x^{\alpha} f\|_{\infty, [\alpha, \alpha]} (x - a)^{\alpha} + \|D_{xx}^{\alpha} f\|_{\infty, [\alpha, b]} (b - x)^{\alpha}\right)\right\},
$$
\n*and*
\n*and**

When $\alpha = \frac{1}{2}$ we derive

Corollary 17. *([14])* Let $0 < \beta < 1$, $f \in C^1([a, b], X)$, $x \in [a, b]$, $n \in \mathbb{N} : n^{1-\beta} > 2$. Then

i)

$$
||A_{n}(f,x) - f(x)|| \le
$$

$$
\frac{2(\psi(1))^{-1}}{\sqrt{\pi}} \left\{ \frac{\left(\omega_{1} \left(D_{x}^{\frac{1}{2}}f, \frac{1}{n^{\beta}}\right)_{[a,x]} + \omega_{1} \left(D_{*x}^{\frac{1}{2}}f, \frac{1}{n^{\beta}}\right)_{[x,b]}}{n^{\frac{\beta}{2}}} + \frac{\left(\omega_{1} \left(D_{x}^{\frac{1}{2}}f, \frac{1}{n^{\beta}}\right)_{[a,x]} + \omega_{1} \left(D_{*x}^{\frac{1}{2}}f, \frac{1}{n^{\beta}}\right)_{[x,b]}}{n^{\frac{\beta}{2}}} \right\}
$$

$$
\left(\frac{1 - h\left(n^{1-\beta} - 2\right)}{2}\right) \left(\left\|D_{x}^{\frac{1}{2}}f\right\|_{\infty, [a,x]} \sqrt{(x-a)} + \left\|D_{*x}^{\frac{1}{2}}f\right\|_{\infty, [x,b]} \sqrt{(b-x)}\right)\right\},
$$
(34)
and

ii)

J. COMPUTATIONAL AVALYSIS AND APPLICATIONS. VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUB PRESS, LLC
\nand
\n
$$
|A_n f - f||_{\infty} \leq \frac{2(\psi(1))^{-1}}{\sqrt{\pi}}
$$
\n
$$
\left\{ \left(\frac{\sup_{x \in [n,k]} \omega_1 \left(D_{x}^{\frac{1}{2}}, f, \frac{1}{\psi^2} \right)_{[n,k]} + \sup_{x \in [n,k]} \omega_1 \left(D_{x}^{\frac{1}{2}}, f, \frac{1}{\psi^2} \right)_{[n,k]} \right) + \frac{2}{n^2} \left(\frac{\left(\frac{\psi_1}{2} \left(D_{x}^{\frac{1}{2}}, f, \frac{1}{\psi^2} \right) \right)_{[n,k]} - \left(\frac{\psi_1}{2} \left(D_{x}^{\frac{1}{2}}, f, \frac{1}{\psi^2} \right) \right)_{[n,k]} \right) + \frac{2}{n^2} \left(\frac{\left(\frac{\psi_1}{2} \left(D_{x}^{\frac{1}{2}}, f, \frac{1}{\psi^2} \right) \right)_{[n,k]} - \left(\frac{\psi_1}{2} \left(D_{x}^{\frac{1}{2}}, f, \frac{1}{\psi^2} \right) \right) \right) < \infty.
$$
\n(35)
\nFrom now on we set $X = \mathbb{R}$.
\n**Time Separating Stochastic Processes**
\n (Ω, \mathcal{F}, P) be a probability space, $\omega \in \Omega; Y_1, Y_2, ..., Y_m, m \in \mathbb{N}$, be real-valued random variables on Ω with
\ninfinite length interval of R, usually $I = \mathbb{R}$ or $I = \mathbb{R}$, where I is an infinite subset of R. Typically here I is
\nonly, then
\n $Y(t, \omega) := \sum_{i=1}^{m} h_i(t) E(Y_i(\omega), t \in I, \mathbb{R})$
\nquite common stochastic process separating time.
\ncan assume that $h_i \in C'(I), i = 1, 2, ..., m; r \in \mathbb{N}$. Consequently, we have that the expectation
\n $(kY)(t) = \sum_{i=1}^{m} h_i(t) E(Y_i(\omega), t \in I, \math$

From now on we set $X = \mathbb{R}$.

2.2 Time Seperating Stochastic Processes

Let (Ω, \mathcal{F}, P) be a probability space, $\omega \in \Omega; Y_1, Y_2, \ldots, Y_m, m \in \mathbb{N}$, be real-valued random variables on Ω with finite expectations, and $h_1(t), h_2(t), \ldots h_m(t) : I \to \mathbb{R}$, where I is an infinite subset of \mathbb{R} . Typically here I is an infinite length interval of \mathbb{R} , usualy $I = \mathbb{R}$ or $I = \mathbb{R}_+$. Clearly, then

$$
Y(t,\omega) := \sum_{i=1}^{m} h_i(t) Y_i(\omega), t \in I,
$$
\n(36)

is a quite common stochastic process separating time.

We can assume that $h_i \in C^r(I), i = 1, 2, ..., m; r \in \mathbb{N}$. Consequently, we have that the expectation

$$
(EY)(t) = \sum_{i=1}^{m} h_i(t) EY_i \in C(I) \text{ or } C^{r}(I). \tag{37}
$$

A classical example of a stochastic process separating time is

$$
(\sin t) Y_1(\omega) + (\cos t) Y_2(\omega), t \in I.
$$

Notice that $|\sin t| \leq 1$ and $|\cos t| \leq 1$.

Another typical example is

$$
\sinh(t)Y_1(\omega) + \cosh(t)Y_2(\omega), t \in I. \tag{38}
$$

In this article we will apply the main results of section 2.1, to $f(t) = (EY)(t)$. We will finish with several applications. See the related [19], [20].

3 Main Results

We present the following general approximation of the seperating stochastic processes by neural network operators.

Theorem 18. *Let* $(EY)(t)$ *as in (37),* $t \in [t_1, t_2]$ *, where* $t_1, t_2 \in \mathbb{R}$ *, with* $t_1 < t_2, h_i \in C([t_1, t_2])$ *for every* $i = 1, 2, ..., m, 0 < \alpha < 1, n \in \mathbb{N} : n^{1-\alpha} > 2$. Then

i)

$$
|A_n\left(\left(EY\right),t\right)-\left(EY\right)\left(t\right)| \leq \frac{1}{\psi\left(1\right)} \left[\omega_1\left(EY,\frac{1}{n^{\alpha}}\right)+\left(1-h\left(n^{1-\alpha}-2\right)\right) \|EY\|_{\infty}\right] =:\rho,\tag{39}
$$

and

ii)

$$
\|A_n\left(EY\right) - EY\|_{\infty} \le \rho. \tag{40}
$$

We have that $\lim_{n \to \infty} A_n$ (*EY*) = *EY*, pointwise and uniformly. The speed of convergence is max $\left(\frac{1}{n^{\alpha}}, \left(1 - h\left(n^{1-\alpha} - 2\right)\right)\right)$.

Proof. Notice that when $h_i \in C([t_1, t_2])$ for every $i = 1, 2, ..., m$, then $(EY)(t) \in C([t_1, t_2])$. Thus, the conclusion comes from Theorem 10. \Box We continue with,

Theorem 19. Let $(EY)(t)$ as in (37), $h_i \in C_B(\mathbb{R})$ for every $i = 1, 2, ..., m, 0 < \alpha < 1, n \in \mathbb{N} : n^{1-\alpha} > 2$, $t \in \mathbb{R}$. *Then*

i)

$$
\left|\overline{A}_{n}\left(EY,t\right)-\left(EY\right)\left(t\right)\right| \leq \omega_{1}\left(EY,\frac{1}{n^{\alpha}}\right)+\left(1-h\left(n^{1-\alpha}-2\right)\right)\left\|EY\right\|_{\infty}=: \mu,\tag{41}
$$

and

ii)

$$
\left\| \overline{A}_n \left(EY \right) - EY \right\|_{\infty} \le \mu. \tag{42}
$$

 $For EY \in C_{uB}(\mathbb{R})$ *we get* $\lim_{n \to \infty} \overline{A}_n(EY) = EY$, pointwise and uniformly. The speed of convergence is $\max\left(\frac{1}{n^{\alpha}},\left(1-h\left(n^{1-\alpha}-2\right)\right)\right).$

Proof. Since that $h_i \in C_B(\mathbb{R})$ for every $i = 1, 2, ..., m$ and $t \in \mathbb{R}$, then $EX \in C_B(\mathbb{R})$. Therefore the results come from Theorem 11. \Box Furthermore, we have

Theorem 20. Let $(EY)(t)$ as in (37), $t \in [t_1, t_2]$, where $t_1, t_2 \in \mathbb{R}$, with $t_1 < t_2, h_i \in C^N([t_1, t_2])$ for every $i = 1, 2, ..., m, n, N \in \mathbb{N}, 0 < \alpha < 1, and n^{1-\alpha} > 2.$ Then

i)

J. COMPUTATIONAL AVALYSIS AND APPLICATIONS. VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LIC
\nij
\n
$$
|A_n((EY),t) - (EY)(t)| \leq \frac{1}{\psi(1)} \left[\omega_1 \left(EY, \frac{1}{n^{\alpha 1}} \right) + (1 - h (n^{1 - \alpha} - 2)) ||EY||_{\infty} \right] =: \rho, \qquad (39)
$$
\n*in*
\n*have that* $\lim_{n \to \infty} A_n (EY) = EY$, pointwise and uniformly.
\n*space of convergence is* max $(\frac{1}{n^{\alpha}}, (1 - h (n^{1 - \alpha} - 2)))$.
\nof. Notice that when $h_i \in C([t_1,t_2])$ for every $i = 1, 2, ..., m$, then $(EY)(t) \in C([t_1,t_2])$. Thus, the chain comes from Theorem 10. \Box
\ncoefficient with
\ncoefficient with
\ncoefficient with
\n \Box In. Let $(EY)(t)$ as in (37), $h_i \in C_B(\mathbb{R})$ for every $i = 1, 2, ..., m$, then $(EY)(t) \in C([t_1,t_2])$. Thus, the chain of
\n
$$
|\overline{A}_n(EY, t) - (EY)(t)| \leq \omega_1 \left(EY, \frac{1}{n^{\alpha}} \right) + (1 - h (n^{1 - \alpha} - 2)) ||EY||_{\infty} =: \mu, \tag{41}
$$
\n*if*
\n $|EY| \in C_{aB}(R)$ *we get* $\lim_{n \to \infty} \overline{A}_n (EY) = EY$, pointwise and uniformly. The speed of convergence is
\nthe form Theorem 11. \Box
\n \Box In the $(\frac{1}{n^{\alpha}}, (1 - h (n^{1 - \alpha} - 2)))$.
\n \Box In the $(\frac{1}{n^{\alpha}}, \Box f)$ for every $i = 1, 2, ..., m$ and $t \in \mathbb{R}$, then $EX \in C_B(\mathbb{R})$. Therefore

ii) Assume further $(EY)^{(j)}(t_0) = 0$, $j = 1, ..., N$, for some $t_0 \in [t_1, t_2]$, it holds

$$
|A_n (EY, t_0) - (EY) (t_0)| \le \frac{1}{\psi(1)}
$$

$$
\left\{\omega_{1}\left((EY)^{(N)},\frac{1}{n^{\alpha}}\right)\frac{1}{n^{\alpha N}N!}+\frac{(1-h(n^{1-\alpha}-2))\left\|(EY)^{(N)}\right\|_{\infty}(t_{2}-t_{1})^{N}}{N!}\right\},\tag{44}
$$

and

iii)

$$
||A_n(EY) - EY||_{\infty} \le \frac{1}{\psi(1)} \left\{ \sum_{j=1}^N \frac{\left\| (EY)^{(j)} \right\|_{\infty}}{j!} \left[\frac{1}{n^{\alpha j}} + \frac{\left(1 - h\left(n^{1-\alpha} - 2\right)\right)}{2} \left(t_2 - t_1\right)^j \right] + \left[\omega_1 \left((EY)^{(N)}, \frac{1}{n^{\alpha}} \right) \frac{1}{n^{\alpha N} N!} + \frac{\left(1 - h\left(n^{1-\alpha} - 2\right)\right) \left\| (EY)^{(N)} \right\|_{\infty} \left(t_2 - t_1\right)^N}{N!} \right] \right\}.
$$
\n(45)

Again we obtain $\lim_{n \to \infty} A_n$ (EY) = EY, pointwise and uniformly.

Proof. By Theorem 12. \Box

We also present

Theorem 21. Let $\alpha > 0, N = [\alpha], \alpha \notin \mathbb{N}, 0 < \beta < 1, t \in [t_1, t_2]$ where $t_1, t_2 \in \mathbb{R}$, with $t_1 < t_2$, $n \in \mathbb{N}: n^{1-\beta} > 2$. *Then*

i)

J. COMPUTATIONAL AVALYSIS AND APPLICATIONS. VOL. 31. NO. 4. 2023, OPYRIGHT 2923 EUDOXUS PRESS. LC
\n
$$
\left\{\omega_{1}\left((EY)^{(N)}, \frac{1}{n^{\alpha}}\right) \frac{1}{n^{\alpha N} N!} + \frac{(1-h(n^{1-\alpha}-2))||(EY)^{(N)}||_{\infty}(t_{2}-t_{1})^{N}}{N!}\right\},
$$
\n(44)
\nII
\nii)
$$
\|A_{n}(EY) - EY\|_{\infty} \leq \frac{1}{\psi(1)} \left\{ \sum_{j=1}^{N} \frac{||(EY)^{(j)}||_{\infty}}{j!} \left[\frac{1}{n^{\alpha j}} + \frac{(1-h(n^{1-\alpha}-2))||(EY)^{(N)}||_{\infty}(t_{2}-t_{1})^{j}}{N!} \right] + \frac{1}{\psi(1)} \left[\frac{\omega_{1}((EY)^{(N)}, \frac{1}{n^{\alpha}})}{n^{\alpha N} N!} + \frac{(1-h(n^{1-\alpha}-2))||(EY)^{(N)}||_{\infty}(t_{2}-t_{1})^{N}}{N!} \right] \right\}.
$$
\n(45)
\nand we obtain $\lim_{n \to \infty} \mu_{n}(EY) = EY$, pointwise and uniformly.
\nalso present 12. Let $\alpha > 0, N = [\alpha], \alpha \notin \mathbb{N}, 0 < \beta < 1, t \in [t_{1}, t_{2}]$ where $t_{1}, t_{2} \in \mathbb{R}$, with $t_{1} < t_{2}$,
\nalso present 12. Let $\alpha > 0, N = [\alpha], \alpha \notin \mathbb{N}, 0 < \beta < 1, t \in [t_{1}, t_{2}]$ where $t_{1}, t_{2} \in \mathbb{R}$, with $t_{1} < t_{2}$,
\n
$$
\left| \frac{A_{n}(EY,t) - \sum_{j=1}^{N-1} \frac{(EY)^{(j)}(t)}{j!} A_{n} \left((-t)^{j} \right) (t) - (EY) (t) \right| \leq \frac{(\psi(1))^{-1}}{1 - (\alpha + 1)^{\alpha} + 1} \left\{ \frac{(\omega(1))^{-1}}{1 - (\alpha + 1
$$

ii) if $(EY)^{(j)}(t) = 0$ *, for* $j = 1, ..., N - 1$ *, we have*

$$
|A_{n}(EY,t) - (EY)(t)| \leq \frac{(\psi(1))^{-1}}{\Gamma(\alpha+1)}
$$

$$
\left\{ \frac{\left(\omega_{1}\left(D_{t-}^{\alpha}(EY), \frac{1}{n^{\beta}}\right)_{[t_{1},t]} + \omega_{1}\left(D_{\ast t}^{\alpha}(EY), \frac{1}{n^{\beta}}\right)_{[t,t_{2}]}\right)}{n^{\alpha\beta}} + \left(\frac{1 - h\left(n^{1-\beta} - 2\right)}{2}\right) \left(\left\|D_{t-}^{\alpha}(EY)\right\|_{\infty,[t_{1},t]}(t - t_{1})^{\alpha} + \left\|D_{\ast t}^{\alpha}(EY)\right\|_{\infty,[t,t_{2}]}(t_{2} - t)^{\alpha}\right)\right\},
$$
(47)

iii)

$$
||A_{n}(EY,t) - (EY) (t)|| \leq (\psi (1))^{-1}
$$

$$
\left\{ \sum_{j=1}^{N-1} \frac{\left| (EY)^{(j)}(t) \right|}{j!} \left\{ \frac{1}{n^{\beta j}} + (t_{2} - t_{1})^{j} \left(\frac{1 - h (n^{1-\beta} - 2)}{2} \right) \right\} + \frac{1}{\Gamma(\alpha+1)} \left\{ \frac{\left(\omega_{1} \left(D_{t-}^{\alpha}(EY), \frac{1}{n^{\beta}} \right)_{[t_{1},t]} + \omega_{1} \left(D_{*t}^{\alpha}(EY), \frac{1}{n^{\beta}} \right)_{[t,t_{2}]} \right)}{n^{\alpha \beta}} + \frac{1}{\Gamma(\alpha+1)} \left\{ \frac{\left(\omega_{1} \left(D_{t-}^{\alpha}(EY), \frac{1}{n^{\beta}} \right)_{[t_{1},t]} + \omega_{1} \left(D_{*t}^{\alpha}(EY), \frac{1}{n^{\beta}} \right)_{[t_{1},t_{2}]} \right)}{n^{\alpha \beta}} + \frac{1}{\Gamma(\alpha+1)} \left\{ \frac{\left(\omega_{1} \left(D_{t-}^{\alpha}(EY), \frac{1}{n^{\beta}} \right)_{[t_{1},t]} + \omega_{1} \left(D_{*t}^{\alpha}(EY), \frac{1}{n^{\beta}} \right)_{[t_{1},t_{2}]} \right)}{n^{\alpha \beta}} \right\} \right\}
$$

$$
\left\{ \frac{1 - h\left(n^{1-\beta} - 2\right)}{2} \right) \left(\left\| D_{t-}^{\alpha} \left(EY \right) \right\|_{\infty, [t_1, t]} (t - t_1)^{\alpha} + \left\| D_{*t}^{\alpha} \left(EY \right) \right\|_{\infty, [t, t_2]} (t_2 - t)^{\alpha} \right) \right\},
$$
(48)

and

iv)

3. COMPUTATIONAL ANALYSIS AND APPLICATIONS. VOL. 31, NO. 4, 2023, OPTRIGHT 2023 EUDOXUS PRESS, LLC
\n
$$
\left(\frac{1-h(n^{1-\beta}-2)}{2}\right) \left(\|D_{L}^{\alpha}(EY)\|_{\infty,[t_1,t_1]}(t-t_1)^{\alpha} + \|D_{\text{wt}}^{\alpha}(EY)\|_{\infty,[t_2,t_2]}(t_2-t)^{\alpha}\right)\right\}, \qquad (48)
$$
\n
$$
\forall t \in [t_1, t_2],
$$
\n*and*\n*iv*)
\n
$$
\left\{\sum_{j=1}^{N-1} \frac{\left|\left(RY\right)^{(j)}\right|_{\infty}\left\{\frac{1}{n^{\beta}} + (t_2 - t_1)^j \left(\frac{1-h(n^{1-\beta}-2)}{2}\right)\right\} + \frac{1}{\Gamma(\alpha+1)}\left\{\sum_{j=1}^{N-1} \left(\frac{\sup_{\pi} \omega_1(\left[D_{L}^{\alpha}(EY), \frac{1}{n^{\beta}}\right)\left(\mu_1 + \frac{\sup_{\pi} \omega_1(\left[D_{\text{wt}}^{\alpha}(EY), \frac{1}{n^{\beta}}\right)\mu_1\right)}{\mu^{\alpha\beta}} + \frac{1}{\Gamma(\alpha+1)}\left\{\sum_{j=1}^{N-1} \left(\frac{\sup_{\pi} \omega_1(\left[D_{L}^{\alpha}(EY), \frac{1}{n^{\beta}}\right)\mu_1\right)\left(\mu_1\right)}{\mu^{\alpha\beta}} + \frac{\sup_{\pi} \omega_1(\left[D_{\text{wt}}^{\alpha}(EY), \frac{1}{n^{\beta}}\right)\mu_1\right)}{\mu^{\alpha\beta}} + \frac{1}{\Gamma(\alpha+1)}\left\{\sum_{j=1}^{N-1} \left(\frac{\sup_{\pi} \omega_1(\left[D_{L}^{\alpha}(EY), \frac{1}{n^{\beta}}\right)\mu_1\right)\left(\mu_2\left(EY\right)\right)\left(\mu_2\left(EY\right)\right)\right\}\right\}. \qquad (49)
$$
\n*Also we, show N = 1 to*
\n*so the see here we obtair i and* <

As we see here we obtain t-valued fractionally type pointwise and uniform convergence with rates of $A_n \to I$ *the unit operator, as* $n \to \infty$.

Proof. By Theorem 15. \Box Next we apply Theorem 21 for $N = 1$.

Corollary 22. Let $(EY)(t)$ as in (37), $0 < \alpha, \beta < 1$, $t \in [t_1, t_2]$, where $t_1, t_2 \in \mathbb{R}$, with $t_1 < t_2, n \in \mathbb{N}$: $n^{1-\beta} > 2$. and $h_i \in C([t_1, t_2])$ for every $i = 1, 2, ..., m$. Then

i)

$$
|A_{n}(EY,t) - (EY)(t)| \le
$$

$$
\frac{(\psi(1))^{-1}}{\Gamma(\alpha+1)} \left\{ \frac{\left(\omega_{1}\left(D_{t-}^{\alpha}(EY), \frac{1}{n^{\beta}}\right)_{[t_{1},t]} + \omega_{1}\left(D_{*t}^{\alpha}(EY), \frac{1}{n^{\beta}}\right)_{[t,t_{2}]}\right)}{n^{\alpha\beta}} + \frac{\left(\omega_{1}\left(D_{t-}^{\alpha}(EY), \frac{1}{n^{\beta}}\right)_{[t_{1},t]} + \omega_{1}\left(D_{*t
$$

and

ii)

$$
||A_{n}(EY) - (EY)||_{\infty} \leq \frac{(\psi(1))^{-1}}{\Gamma(\alpha+1)}
$$

$$
\left\{ \frac{\left(\sup_{t \in [t_{1}, t_{2}]} \omega_{1} \left(D_{t-}^{\alpha}(EY), \frac{1}{n^{\beta}}\right)_{[t_{1}, t]} + \sup_{t \in [t_{1}, t_{2}]} \omega_{1} \left(D_{\ast t}^{\alpha}(EY), \frac{1}{n^{\beta}}\right)_{[t, t_{2}]} \right)}{n^{\alpha \beta}} + \frac{\left(\frac{1 - h\left(n^{1-\beta} - 2\right)}{2}\right)(t_{2} - t_{1})^{\alpha} \left(\sup_{t \in [t_{1}, t_{2}]} ||D_{t-}^{\alpha}(EY)||_{\infty, [t_{1}, t]} + \sup_{x \in [t_{1}, t_{2}]} ||D_{\ast t}^{\alpha}(EY)||_{\infty, [t, t_{2}]} \right)\right\}.
$$
(51)

When $\alpha = \frac{1}{2}$ we derive

Corollary 23. Assume again $(EY)(t)$ as in (37). Let $0 < \beta < 1$, $t \in [t_1, t_2]$, where $t_1, t_2 \in \mathbb{R}$, with $t_1 < t_2$, $n \in \mathbb{N} : n^{1-\beta} > 2$ and $h_i \in C([t_1, t_2])$ for every $i = 1, 2, ..., m$. Then

i)

$$
|A_{n}(EY,t) - (EY)(t)| \le
$$

$$
\frac{2(\psi(1))^{-1}}{\sqrt{\pi}} \left\{ \frac{\left(\omega_{1}\left(D_{t-}^{\frac{1}{2}}(EY), \frac{1}{n^{\beta}}\right)_{[t_{1},t]} + \omega_{1}\left(D_{*t}^{\frac{1}{2}}(EY), \frac{1}{n^{\beta}}\right)_{[t,t_{2}]}\right)}{n^{\frac{\beta}{2}}} + \frac{\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\left(\left\|D_{t-}^{\frac{1}{2}}(EY)\right\|_{\infty,[t_{1},t]}\sqrt{(t-t_{1})} + \left\|D_{*t}^{\frac{1}{2}}(EY)\right\|_{\infty,[t,t_{2}]}\sqrt{(t_{2}-t)}\right)\right\},
$$
(52)

and

ii)

J. COMPUTATIONAL AVALYSIS AND APPLICATIONS. VOL. 31, NO. 4, 2023, OPYRIGHT 2023 EUDOXUS PRESS, LLC
\nWhen
$$
\alpha = \frac{1}{2}
$$
 we derive
\nrollary 23. Assume again (FY)(t) as in (97). Let $0 < \beta < 1$, $t \in [t_1, t_2]$, where $t_1, t_2 \in \mathbb{R}$, with $t_1 < t_2$,
\nN: $n^{1-9} > 2$ and $h_i \in C([t_1, t_2])$ for every $i = 1, 2, ..., m$. Then
\n
$$
\begin{aligned}\nA_n(EY, t) - (EY) (t) | \le \\
&\frac{2(e(1))^{-1}}{\sqrt{\pi}} \left\{ \left. \frac{\left(\omega_1 \left(D_{t-}^{\frac{1}{2}}(EY), \frac{1}{h^2} \right)_{[t_1, t]} + \omega_1 \left(D_{t+}^{\frac{1}{2}}(EY), \frac{1}{h^2} \right)_{[t_1, t]} + \omega_1 \left(D_{t+}^{\frac{1}{2}}(EY), \frac{1}{h^2} \right)_{[t_1, t]} + \omega_1 \left(D_{t+}^{\frac{1}{2}}(EY), \frac{1}{h^2} \right)_{[t_1, t]} + \omega_1 \left(D_{t+}^{\frac{1}{2}}(EY) \right) \right|_{\infty,[t_1, t_2]} \sqrt{(t_2 - t_1)} \right\}, \qquad (52)\n\end{aligned}
$$
\n
$$
\begin{aligned}\n\text{and} \\
\frac{\left(1 - h \left(n^{1-9} - 2 \right)}{2} \right) \left(\left\| D_{t-}^{\frac{1}{2}}(EY) \right\|_{\infty,[t_1, t]} + \left\| D_{t+}^{\frac{1}{2}}(EY) \right\|_{\infty,[t_1, t_2]} \sqrt{(t_2 - t_1)} \right\} + \omega_1 \left(\frac{\sin \alpha_1}{\sqrt{\pi}} \right) \left(D_{t+}^{\frac{1}{2}}(EY) \right) \left\|_{\infty,[t_1, t_2]} + \omega_1 \left\| D_{t+}^{\frac{1}{2}}(EY)
$$

4 Applications

For the next applications we consider (Ω, F, P) be a probability space and Y_0, Y_1, Y_2 be real valued random variables on Ω with finite expectations. We consider the stochastic processes $Z_i(t, \omega)$ for $i = 1, 2, ..., 9$, where $t\in\mathbb{R}$ and $\omega\in\Omega$ as follows:

$$
Z_1(t,\omega) = \left[(t - t_0)^{\mu+1} + 1 \right] Y_0(\omega), \tag{54}
$$

where $t_0 \in \mathbb{R}$ and $\mu \in \mathbb{N}$ are fixed;

$$
Z_2(t,\omega) = \sin\left(\xi t\right) Y_1(\omega) + \cos\left(\xi t\right) Y_2(\omega),\tag{55}
$$

where $\xi > 0$ is fixed;

$$
Z_3(t,\omega) = \sinh(\mu t) Y_1(\omega) + \cosh(\mu t) Y_2(\omega),\tag{56}
$$

where $\mu > 0$ is fixed;

$$
Z_4(t,\omega) = sech(\mu t) Y_1(\omega) + \tanh(\mu t) Y_2(\omega), \qquad (57)
$$

where $\mu > 0$ is fixed. Here $sechx := \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}, x \in \mathbb{R}$.

$$
Z_5(t,\omega) = e^{-\ell_1 t} Y_1(\omega) + e^{-\ell_2 t} Y_2(\omega),\tag{58}
$$

where $\ell_1, \ell_2 > 0$ are fixed;

$$
Z_6(t,\omega) = \frac{1}{1 + e^{-\ell_1 t}} Y_1(\omega) + \frac{1}{1 + e^{-\ell_2 t}} Y_2(\omega),\tag{59}
$$

where $\ell_1, \ell_2 > 0$ are fixed;

$$
Z_7(t,\omega) = e^{-e^{-\mu_1 t}} Y_1(\omega) + e^{-e^{-\mu_2 t}} Y_2(\omega),
$$
\n(60)

where $\mu_1, \mu_2 > 0$ are fixed;

$$
Z_8(t,\omega) = P_m(\ell_1 t) Y_1(\omega) + P_m(\ell_2 t) Y_2(\omega),
$$
\n(61)

where $\ell_1,\ell_2>0$ and $m\in\mathbb{N}$ are fixed.

Here $P_m(x)$ is the Legendre Polynomial of degree $m \in \mathbb{N}$, i.e

$$
P_m : [-1,1] \longrightarrow [-1,1]
$$

such that,

3. COMPUTATIONAL ANALYSIS AND APPLICATIONS. VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\nare
$$
ℓ_1, ℓ_2 > 0
$$
 are fixed;
\n
$$
Z_6(t, \omega) = \frac{1}{1 + e^{-\ell_{11}} Y_1(\omega) + \frac{1}{1 + e^{-\ell_{21}} Y_2(\omega)},
$$
\n(59)
\n10. (60)
\n11. (a) $z_7(t, \omega) = e^{-e^{-\ell_{11}} Y_1(\omega) + e^{-e^{-\ell_{12}} Y_2(\omega)},$ \n(61)
\n12. (a) $z_8(t, \omega) = P_{in} (\ell_2 t) Y_1(\omega) + P_{in} (\ell_2 t) Y_2(\omega),$ \n(62)
\n13. (a) $z_8(t, \omega) = P_{in} (\ell_2 t) Y_1(\omega) + P_{in} (\ell_2 t) Y_2(\omega),$ \n(63)
\n14. (a) $P_{in} : [-1, 1] \rightarrow [-1, 1]$
\n15. (b) $P_{in} : [-1, 1] \rightarrow [-1, 1]$
\n16. (c) $P_{in} : [-1, 1] \rightarrow [-1, 1]$
\n17. (a) $P_{in} : [-1, 1] \rightarrow [-1, 1]$
\n(b) $P_{in} : [-1, 1] \rightarrow [-1, 1]$
\n18. (a) $P_{in} : [-1, 1] \rightarrow [-1, 1]$
\n(b) $P_{in} : [-1, 1] \rightarrow [-1, 1]$
\n19. (a) $P_{in} : [-1, 1] \rightarrow [-1, 1]$
\n(b) $P_{in} : [-1, 1] \rightarrow [-1, 1]$
\n(c) $P_{in} : [-1, 1] \rightarrow [-1, 1]$
\n10. (a) $P_{in} : (-1, 1] \rightarrow 1]$
\n(b) $P_{in} : (-1, 1] \rightarrow 1]$
\n(c) $P_{in} : (-1, 1] \rightarrow 1]$
\n11. (a) $P_{in} : (-1, 1] \rightarrow 1$
\n(b) $P_{$

To define the stochastic process $Z_9(t, \omega)$, we consider the Cauchy function

$$
\hat{f}(x) = \begin{cases}\ne^{-\frac{1}{x^2}}, & x \neq 0 \\
0, & x = 0\n\end{cases}.
$$

Notice that, $\hat{f} \in C^{\infty}(\mathbb{R})$ and it has $\hat{f}^{(i)}(0) = 0$, for all $i = 1, 2, ..., \infty$. We set,

$$
Z_9(t,\omega) = \hat{f}(t)Y_0(\omega),\tag{62}
$$

The expectations of Z_i , $i = 1, 2, ..., 9$ are

$$
(EZ1) (t) = [(t - t0)μ+1 + 1] E(Y0), \t(63)
$$

$$
(EZ_2)(t) = \sin(\xi t) E(Y_1) + \cos(\xi t) E(Y_2),
$$
\n(64)

$$
(EZ_3)(t) = \sinh(\mu t) E(Y_1) + \cosh(\mu t) E(Y_2),
$$
\n(65)

$$
(EZ_4)(t) = sech(\mu t) E(Y_1) + \tanh(\mu t) E(Y_2), \tag{66}
$$

$$
(EZ_5)(t) = e^{-\ell_1 t} E(Y_1) + e^{-\ell_2 t} E(Y_2),
$$
\n(67)

$$
(EZ_6)(t) = \frac{1}{1 + e^{-\ell_1 t}} E(Y_1) + \frac{1}{1 + e^{-\ell_2 t}} E(Y_2),
$$
\n(68)

$$
(EZ_7)(t) = e^{-e^{-\mu_1 t}} E(Y_1) + e^{-e^{-\mu_2 t}} E(Y_2),
$$
\n(69)

$$
(EZ_8)(t) = P_m(\ell_1 t) E(Y_1) + P_m(\ell_2 t) E(Y_2), \tag{70}
$$

$$
(EZ_9)(t) = \hat{f}(t)E(Y_0),
$$
\n(71)

For the next $(EZ_i)(t)$, $i = 1, 2, ..., 9$ are as defined in relations between (63) and (71) respectively. We present the following result.

Proposition 24. Let $t \in [t_1, t_2]$, where $t_1, t_2 \in \mathbb{R}$, with $t_1 < t_2, 0 < \alpha < 1, n \in \mathbb{N} : n^{1-\alpha} > 2$. Then for $i = 1, 2, \ldots, 9$

i)

$$
|A_n\left(\left(EZ_i\right),t\right)-\left(EZ_i\right)(t)| \leq \frac{1}{\psi\left(1\right)} \left[\omega_1\left(EZ_i,\frac{1}{n^{\alpha}}\right)+\left(1-h\left(n^{1-\alpha}-2\right)\right) \|EZ_i\|_{\infty}\right] =:\rho,\tag{72}
$$

and

ii)

$$
||A_n(EZ_i) - EZ_i||_{\infty} \le \rho.
$$
\n(73)

We have that $\lim_{n\to\infty} A_n(EZ_i) = EZ_i$ *, pointwise and uniformly.* The speed of convergence is max $\left(\frac{1}{n^{\alpha}}, \left(1 - h\left(n^{1-\alpha} - 2\right)\right)\right)$.

Proof. From Theorem 18. \Box

In the cases of stochastic processes $Z_i(t,\omega)$, for $i = 2, 4, 6, 7$ we have the next

Proposition 25. *Let* $i \in \{2, 4, 6, 7\}$, $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $t \in \mathbb{R}$. *Then*

i)

$$
\left|\overline{A}_{n}\left(EZ_{i},t\right)-\left(EZ_{i}\right)\left(t\right)\right| \leq \omega_{1}\left(EZ_{i},\frac{1}{n^{\alpha}}\right)+\left(1-h\left(n^{1-\alpha}-2\right)\right)\left\|EZ_{i}\right\|_{\infty}=: \mu,\tag{74}
$$

and

ii)

$$
\left\| \overline{A}_n \left(EZ_i \right) - EZ_i \right\|_{\infty} \le \mu. \tag{75}
$$

For $EZ_i \in C_{uB}(\mathbb{R})$ *we get* $\lim_{n \to \infty} \overline{A}_n(EZ_i) = EZ_i$ *, pointwise and uniformly.* The speed of convergence is max $\left(\frac{1}{n^{\alpha}}, \left(1 - h\left(n^{1-\alpha} - 2\right)\right)\right)$.

Proof. Notice that for every $t \in \mathbb{R}$ we have that:

for
$$
Z_2(t, \omega)
$$
, $|\sin(\xi t)| \le 1$ and $|\cos(\xi t)| \le 1$,
for $Z_4(t, \omega)$, $|\text{sech}(\mu t)| \le 1$ and $|\tanh(\mu t)| \le 1$,
for $Z_6(t, \omega)$, $0 < \frac{1}{1 + e^{-\ell_1 t}} < 1$ and $0 < \frac{1}{1 + e^{-\ell_2 t}} < 1$,
for $Z_7(t, \omega)$, $0 < e^{-e^{-\mu_1 t}} < 1$ and $0 < e^{-e^{-\mu_2 t}} < 1$.

Thus, the results come from Theorem 19. \Box Moreover, we present the next

Proposition 26. *Let* $i = 1, 2, ..., 9, t \in [t_1, t_2]$, where $t_1, t_2 \in \mathbb{R}$, with $t_1 < t_2, 0 < \alpha < 1$, and $n^{1-\alpha} > 2$. Then

i)

J. COMPUTAIANA: AVALYSS AND APPLICATIONS, VOL. 31, NO. 4, 2023, OPYRIGHT 2023 EUDOXUS PRESS, LIC
\nposition 24. Let
$$
t \in [t_1, t_2]
$$
, where $t_1, t_2 \in \mathbb{R}$, with $t_1 < t_2, 0 < \alpha < 1, n \in \mathbb{N}$: $n^{1-\alpha} > 2$. Then for
\n $1, 2, ..., 9$
\n $|A_n((EZ), t) - (EZ), (t)| \le \frac{1}{\psi(1)} \left[\omega_1 \left(EZ_n \frac{1}{n^{\alpha}}\right) + (1 - h(n^{1-\alpha} - 2)) ||EZ||_{\infty}\right] =: \rho,$ (72)
\n t have that $\lim_{x \to 0} A_n(EZ_i) = EZ_i$, pointwise and uniformly.
\n $Q_n(x) = EZ_n$ for $|\text{nontrivial, } [X_n(x)] = X_n$ for $i = 2, 4, 6, 7$ we have the next
\n*speed of convergence* is max $\left(\frac{1}{n^{\alpha}}, (1 - h(n^{1-\alpha} - 2))\right)$.
\n $\text{of. From Theorem 18. L1}$
\n $\text{the cases of stochastic processes } Z_i(t, \omega), \text{ for } i = 2, 4, 6, 7$ we have the next
\nposition 25. Let $i \in \{2, 4, 6, 7\}$, $0 < \alpha < 1, n \in \mathbb{N}$; $n^{1-\alpha} > 2, t \in \mathbb{R}$. Then
\n j
\n $|\overline{A}_n(EZ_i) - (EZ_i)(t)| \le \omega_1 \left(EZ_i \frac{1}{n^{\alpha}}\right) + (1 - h(n^{1-\alpha} - 2)) ||EZ_i||_{\infty} =: \mu,$ (74)
\n1
\n11)
\n12.
\n13.
\n14.
\n15. $EZ_i \in C_{i,R}(\mathbb{R})$ we get $\lim_{n \to \infty} \overline{A}_n(EZ_i) = EZ_i$, pointwise and uniformly.
\n16. $Z_i \in C_{i,R}(\mathbb{R})$ we get \lim_{n

ii) Assume further
$$
(EZ_i)^{(j)}
$$
 $(t_a) = 0, j = 1, ..., N$, for some $t_a \in [t_1, t_2]$, it holds
\n
$$
|A_n (EZ_i, t_a) - (EZ_i) (t_a)| \le \frac{1}{\psi(1)}
$$
\n
$$
\left\{ \omega_1 \left((EZ_i)^{(N)}, \frac{1}{n^{\alpha}} \right) \frac{1}{n^{\alpha N} N!} + \frac{(1 - h (n^{1 - \alpha} - 2)) ||(EZ_i)^{(N)} ||_{\infty} (t_2 - t_1)^N}{N!} \right\},
$$
\n(77)

and

iii)

$$
||A_n(EZ_i) - EZ_i||_{\infty} \le \frac{1}{\psi(1)} \left\{ \sum_{j=1}^N \frac{\left\| (EZ_i)^{(j)} \right\|_{\infty}}{j!} \left[\frac{1}{n^{\alpha j}} + \frac{\left(1 - h\left(n^{1-\alpha} - 2\right)\right)}{2} (t_2 - t_1)^j \right] + \left[\omega_1 \left((EZ_i)^{(N)}, \frac{1}{n^{\alpha}} \right) \frac{1}{n^{\alpha N} N!} + \frac{\left(1 - h\left(n^{1-\alpha} - 2\right)\right) \left\| (EZ_i)^{(N)} \right\|_{\infty} (t_2 - t_1)^N}{N!} \right] \right\}.
$$
 (78)

Again we obtain $\lim_{n \to \infty} A_n (EZ_i) = EZ_i$, pointwise and uniformly.

Proof. By Theorem 20. \Box We also present

Proposition 27. *Let* $i = 1, 2, ..., 9, \alpha > 0, N = \lceil \alpha \rceil, \alpha \notin \mathbb{N}, 0 < \beta < 1, t \in [t_1, t_2]$ where $t_1, t_2 \in \mathbb{R}$, with $t_1 < t_2, n \in \mathbb{N} : n^{1-\beta} > 2$. *Then*

i)

$$
\left| A_n(EZ_i, t) - \sum_{j=1}^{N-1} \frac{(EZ_i)^{(j)}(t)}{j!} A_n \left((\cdot - t)^j \right) (t) - (EZ_i) (t) \right| \le
$$

$$
\frac{(\psi(1))^{-1}}{\Gamma(\alpha+1)} \left\{ \frac{\left(\omega_1 \left(D_{t-}^{\alpha} (EZ_i), \frac{1}{n^{\beta}} \right)_{[t_1, t]} + \omega_1 \left(D_{*t}^{\alpha} (EZ_i), \frac{1}{n^{\beta}} \right)_{[t, t_2]} \right)}{n^{\alpha \beta}} + \left(\frac{1 - h \left(n^{1-\beta} - 2 \right)}{2} \right) \left(\left\| D_{t-}^{\alpha} (EZ_i) \right\|_{\infty, [t_1, t]} (t - t_1)^{\alpha} + \left\| D_{*t}^{\alpha} (EZ_i) \right\|_{\infty, [t, t_2]} (t_2 - t)^{\alpha} \right) \right\},
$$
(79)

ii) if $(EZ_i)^{(j)}(t) = 0$ *, for* $j = 1, ..., N - 1$ *, we have*

|Aⁿ (EZⁱ , t) [−] (EZi) (t)| ≤ (^ψ (1))[−]¹ Γ (α + 1) ω1 D^α ^t[−] (EZi), 1 n^β [t1,t] + ω¹ D^α ∗t (EZi), 1 n^β [t,t2] ⁿαβ ⁺ 1 − h n ¹−^β [−] ² 2 ! ^D^α ^t[−] (EZi) ∞,[t1,t] (t − t1) ^α ⁺ ^kD^α ∗t (EZi)k∞,[t,t2] (t² − t) α) , (80) kAⁿ (EZⁱ , t) [−] (EZi) (t)k ≤ (^ψ (1))[−]¹ 550 J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC George A. Anastassiou 535-556

iii)

$$
n(EZ_i, t) - (EZ_i)(t) \leq (\psi(1))
$$

$$
\left\{\sum_{j=1}^{N-1} \frac{\left|(EZ_i)^{(j)}(t)\right|}{j!} \left\{\frac{1}{n^{\beta j}} + (t_2 - t_1)^j \left(\frac{1 - h\left(n^{1-\beta} - 2\right)}{2}\right)\right\} + \frac{1}{\Gamma(\alpha+1)} \left\{\frac{\left(\omega_1 \left(D_{t-}^{\alpha} (EZ_i), \frac{1}{n^{\beta}}\right)_{[t_1, t]} + \omega_1 \left(D_{*t}^{\alpha} (EZ_i), \frac{1}{n^{\beta}}\right)_{[t, t_2]}\right)}{n^{\alpha\beta}} + \frac{\left(\frac{1 - h\left(n^{1-\beta} - 2\right)}{2}\right) \left(\left\|D_{t-}^{\alpha} (EZ_i)\right\|_{\infty, [t_1, t]} (t - t_1)^{\alpha} + \left\|D_{*t}^{\alpha} (EZ_i)\right\|_{\infty, [t, t_2]} (t_2 - t)^{\alpha}\right)\right\},\n\tag{81}
$$

and

iv)

J. COMPUTATIONAL AVALYSIS AND APPLICATIONS. Vol. 31, NO. 4, 2023, OPYRISH 7823 EUDOXUS PRESS, LIC
\n
$$
\left\{\sum_{j=1}^{N-1} \left|\frac{(EZ_{i})^{(j)}(t)}{j!} \left\{\frac{1}{n^{\beta j}} + (t_{2} - t_{1})^{j} \left(\frac{1 - h(n^{1 - \beta} - 2)}{2}\right)\right\} + \frac{1}{\Gamma(\alpha + 1)} \left\{\frac{(\omega_{1}(D_{i_{-}}^{n}(EZ_{i}), \frac{1}{n^{\beta}})_{[i_{+},i_{+}] + \omega_{1}(D_{i_{+}}^{n}(EZ_{i}), \frac{1}{n^{\beta}})_{[i_{+},i_{+}]}}{n^{\alpha \beta}} + \frac{1}{\Gamma(\alpha + 1)} \left\{\frac{1 - h(n^{1 - \beta} - 2)}{2}\right) \left(\|D_{i_{-}}^{n}(EZ_{i})\|_{\infty,[t_{+},i_{+}]}\left(t - t_{1}\right)^{\alpha} + \|D_{i_{0}}^{n}(EZ_{i})\|_{\infty,[t_{+},i_{+}]}\left(t_{2} - t_{1}\right)^{\alpha}\right)\right\}, \qquad (81)
$$
\nand\n
$$
\left\{\sum_{j=1}^{N-1} \frac{\left|\left(EZ_{i}\right)^{(j)}\right|_{\infty}\left\{\frac{1}{n^{\beta j}} + (t_{2} - t_{1})^{j} \left(\frac{1 - h(n^{1 - \beta} - 2)}{2}\right)\right\} + \frac{1}{\Gamma(\alpha + 1)} \left\{\frac{1}{\sqrt{\pi^{(\alpha + 1)}}\left\{\frac{1}{n^{\beta j}} + (t_{2} - t_{1})^{j} \left(\frac{1 - h(n^{1 - \beta} - 2)}{2}\right)\right\} + \frac{1}{\Gamma(\alpha + 1)} \left\{\frac{1}{\sqrt{\pi^{(\alpha + 1)}}\left\{\frac{1}{n^{\beta j}} + (t_{2} - t_{1})^{j} \left(\frac{1 - h(n^{1 - \beta} - 2)}{2}\right)\right\} + \frac{1}{\Gamma(\alpha + 1)} \left\{\frac{1}{n^{\beta \beta}}\left[\frac{1}{n^{\beta \beta}} + (t_{2} - t_{1})^{j} \left(\frac{1 - h(n^{1 - \beta} - 2)}{2}\right
$$

Above, when $N = 1$ *the sum* $\sum_{j=1}^{N-1}$ $\cdot = 0$.

As we see here we obtain t-valued fractionally type pointwise and uniform convergence with rates of $A_n \to I$ *the unit operator, as* $n \to \infty$.

Proof. By Theorem 21. \Box

Next we apply Proposition 27 for $N = 1$.

Corollary 28. Let $i = 1, 2, ..., 9, 0 < \alpha, \beta < 1, t \in [t_1, t_2]$, where $t_1, t_2 \in \mathbb{R}$, with $t_1 < t_2, n \in \mathbb{N} : n^{1-\beta} > 2$. *Then*

i)

$$
|A_{n}(EZ_{i},t) - (EZ_{i})(t)| \leq
$$

$$
\frac{(\psi(1))^{-1}}{\Gamma(\alpha+1)} \left\{ \frac{\left(\omega_{1}\left(D_{t-}^{\alpha}(EZ_{i}), \frac{1}{n^{\beta}}\right)_{[t_{1},t]} + \omega_{1}\left(D_{*t}^{\alpha}(EZ_{i}), \frac{1}{n^{\beta}}\right)_{[t,t_{2}]}\right)}{n^{\alpha\beta}} + \frac{\left(\omega_{1}\left(D_{t-}^{\alpha}(EZ_{i}), \frac{1}{n^{\beta}}\right)_{[t_{1},t]} + \omega_{1}\left(D_{*t}^{\alpha}(EZ_{i}), \frac{1}{n^{\beta}}\right)_{[t,t_{2}]}\right)}{n^{\alpha\beta}} + \frac{\left(\omega_{1}\left(D_{t-}^{\alpha}(EZ_{i})\right)_{[t_{1},t]} + \omega_{1}\left(D_{*t}^{\alpha}(EZ_{i}), \frac{1}{n^{\beta}}\right)_{[t_{1},t_{2}]}\right)}{n^{\alpha\beta}} + \cdots \right\}, \tag{83}
$$

and

ii)

$$
||A_n(EZ_i) - (EZ_i)||_{\infty} \le \frac{(\psi(1))^{-1}}{\Gamma(\alpha+1)}
$$

$$
\left\{\frac{\left(\sup_{t\in[t_{1},t_{2}]}\omega_{1}\left(D_{t-}^{\alpha}(EZ_{i}),\frac{1}{n^{\beta}}\right)_{[t_{1},t]} + \sup_{t\in[t_{1},t_{2}]}\omega_{1}\left(D_{*t}^{\alpha}(EZ_{i}),\frac{1}{n^{\beta}}\right)_{[t,t_{2}]}\right)}{n^{\alpha\beta}} + \frac{\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)(t_{2}-t_{1})^{\alpha}\left(\sup_{t\in[t_{1},t_{2}]}\left\|D_{t-}^{\alpha}(EZ_{i})\right\|_{\infty,[t_{1},t]} + \sup_{x\in[t_{1},t_{2}]}\left\|D_{*t}^{\alpha}(EZ_{i})\right\|_{\infty,[t,t_{2}]}\right)\right\}.
$$
\n(84)

When $\alpha = \frac{1}{2}$ we derive

Corollary 29. Assume $i = 1, 2, ..., 9$. Let $0 < \beta < 1$, $t \in [t_1, t_2]$, where $t_1, t_2 \in \mathbb{R}$, with $t_1 < t_2$, and $n \in \mathbb{N}: n^{1-\beta} > 2$. Then

i)

$$
|A_{n}(EZ_{i},t) - (EZ_{i})(t)| \le
$$

$$
\frac{2(\psi(1))^{-1}}{\sqrt{\pi}} \left\{ \frac{\left(\omega_{1}\left(D_{t-}^{\frac{1}{2}}(EZ_{i}), \frac{1}{n^{\beta}}\right)_{[t_{1},t]} + \omega_{1}\left(D_{*t}^{\frac{1}{2}}(EZ_{i}), \frac{1}{n^{\beta}}\right)_{[t,t_{2}]}\right)}{n^{\frac{\beta}{2}}} + \frac{\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\left(\left\|D_{t-}^{\frac{1}{2}}(EZ_{i})\right\|_{\infty,[t_{1},t]}\sqrt{(t-t_{1})} + \left\|D_{*t}^{\frac{1}{2}}(EZ_{i})\right\|_{\infty,[t,t_{2}]}\sqrt{(t_{2}-t)}\right)\right\},\qquad(85)
$$

and

ii)

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS. VOJ. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LIC
\n
$$
\left\{\frac{\left(\sup_{t\in\{t_1,t_2\}}\omega_1\left(D_{t-}^{\alpha}(EZ_1),\frac{1}{n^2}\right)_{[t_1,t_2]} + \sup_{t\in\{t_1,t_2\}}\omega_1\left(D_{t-}^{\alpha}(EZ_1),\frac{1}{n^2}\right)_{[t_1,t_2]}}{n^{\alpha\beta}}\right\} + \left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)(t_2-t_1)^{\alpha}\left(\sup_{t\in\{t_1,t_2\}}\left||D_{t-}^{\alpha}(EZ_1)\right||_{\infty,[t_1,t_1]} + \sup_{t\in\{t_1,t_2\}}\left||D_{\tau t}^{\alpha}(EZ_1)\right||_{\infty,[t_1,t_2]}\left||D_{\tau t}^{\alpha}(EZ_1)\right||_{\infty,[t_1,t_2]}\right)\right\},
$$
\n(84)
\nWhen $\alpha = \frac{1}{2}$ we derive
\n
$$
\text{volume } \alpha = \frac{1}{2}
$$
 we derive
\n
$$
\text{volume } \alpha = 1, 2, ..., 9.
$$
 Let $0 < \beta < 1$, $t \in [t_1, t_2]$, where $t_1, t_2 \in \mathbb{R}$, with $t_1 < t_2$, t_3 , and
\n
$$
\beta = \frac{2(\psi(1))^{-1}}{\sqrt{\pi}}\left\{\frac{\left(\omega_1\left(D_{t-}^{\frac{1}{\alpha}}(EZ_1),\frac{1}{n^{\alpha}}\right)_{[t_1,t_1]} + \omega_1\left(D_{t-}^{\frac{1}{\alpha}}(EZ_1),\frac{1}{n^{\alpha}}\right)_{[t_1,t_2]} + \omega_1\left(D_{t-}^{\frac{1}{\alpha}}(EZ_1),\frac{1}{n^{\alpha}}\right)_{[t_1,t_2]}\right)}{n^{\frac{\alpha}{2}}}\right\}
$$
\nand
\n
$$
\left\{\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\left(\left||D_{t-}^{\frac
$$

5 Specific Applications

Let (Ω, \mathcal{F}, P) , where Ω is the set of non-negative integers, be a probability space, $Y_{1,1}, Y_{2,1}$ be real-valued random variables on Ω following Poisson distributions with parameters $\lambda_1, \lambda_2 \in (0, \infty)$ respectively. We consider the stochastic processes $Z_{i,1}(t,\omega)$ for $i = 1, 2, 3, 5$, where $t \in \mathbb{R}$ and $\omega \in \Omega$ as follows:

$$
Z_{1,1}(t,\omega) = \left[(t - t_0)^{\mu+1} + 1 \right] Y_{1,1}(\omega), \tag{87}
$$

where $t_0 \in \mathbb{R}$ and $\mu \in \mathbb{N}$ are fixed;

$$
Z_{2,1}(t,\omega) = \sin(\xi t) Y_{1,1}(\omega) + \cos(\xi t) Y_{2,1}(\omega), \tag{88}
$$

where $\xi > 0$ is fixed;

$$
Z_{3,1}(t,\omega) = \sinh(\mu t) Y_{1,1}(\omega) + \cosh(\mu t) Y_{2,1}(\omega), \tag{89}
$$

where $\mu > 0$ is fixed;

$$
Z_{5,1}(t,\omega) = e^{-\ell_1 t} Y_{1,1}(\omega) + e^{-\ell_2 t} Y_{2,1}(\omega),
$$
\n(90)

where $\ell_1, \ell_2 > 0$ are fixed.

Since $E\left(Y_{1,1}\right)=\lambda_{1}$ and $E\left(Y_{2,1}\right)=\lambda_{2}$, the expectations of $Z_{i,1}, i=1,2,3,5,$ are

$$
(EZ_{1,1})(t) = \lambda_1 \left[(t - t_0)^{\mu+1} + 1 \right],
$$
\n(91)

$$
(EZ_{2,1})(t) = \lambda_1 \sin\left(\xi t\right) + \lambda_2 \cos\left(\xi t\right),\tag{92}
$$

$$
(EZ_{3,1})(t) = \lambda_1 \sinh(\mu t) + \lambda_2 \cosh(\mu t), \qquad (93)
$$

$$
(EZ_{5,1})(t) = \lambda_1 e^{-\ell_1 t} + \lambda_2 e^{-\ell_2 t}, \tag{94}
$$

For the next we consider (Ω, \mathcal{F}, P) , where $\Omega = \mathbb{R}$, be a probability space, $Y_{1,2}, Y_{2,2}$ be real-valued random variables on Ω following Gaussian distributions with expectations $\hat{\mu}_1, \hat{\mu}_2 \in \mathbb{R}$ respectively.

We consider the stochastic processes $Z_{i,2}(t,\omega)$ for $i = 1,2,3,5$, where $t \in \mathbb{R}$ and $\omega \in \Omega$ as follows:

$$
Z_{1,2}(t,\omega) = \left[(t - t_0)^{\mu+1} + 1 \right] Y_{1,2}(\omega), \tag{95}
$$

where $t_0 \in \mathbb{R}$ and $\mu \in \mathbb{N}$ are fixed;

$$
Z_{2,2}(t,\omega) = \sin(\xi t) Y_{1,2}(\omega) + \cos(\xi t) Y_{2,2}(\omega), \tag{96}
$$

where $\xi > 0$ is fixed;

$$
Z_{3,2}(t,\omega) = \sinh(\mu t) Y_{1,2}(\omega) + \cosh(\mu t) Y_{2,2}(\omega), \tag{97}
$$

where $\mu > 0$ is fixed;

$$
Z_{5,2}(t,\omega) = e^{-\ell_1 t} Y_{1,2}(\omega) + e^{-\ell_2 t} Y_{2,2}(\omega),
$$
\n(98)

where $\ell_1, \ell_2 > 0$ are fixed.

Since $E(Y_{1,2}) = \hat{\mu}_1$ and $E(Y_{2,2}) = \hat{\mu}_2$, The expectations of $Z_{i,2}$, $i = 1, 2, 3, 5$ are

$$
(EZ_{1,2}) (t) = \hat{\mu}_1 \left[(t - t_0)^{\mu + 1} + 1 \right], \tag{99}
$$

$$
(EZ_{2,2})(t) = \hat{\mu}_1 \sin(\xi t) + \hat{\mu}_2 \cos(\xi t), \qquad (100)
$$

$$
(EZ3,2)(t) = \hat{\mu}_1 \sinh(\mu t) + \hat{\mu}_2 \cosh(\mu t), \qquad (101)
$$

$$
(EZ_{5,2})(t) = \hat{\mu}_1 e^{-\ell_1 t} + \hat{\mu}_2 e^{-\ell_2 t}.
$$
\n(102)

Furthermore, we consider (Ω, \mathcal{F}, P) , where $\Omega = [0, \infty)$, be a probability space, $Y_{1,3}, Y_{2,3}$ be real-valued random variables on Ω following Weibull distributions with scale parameters 1 and shape parameters $\gamma_1, \gamma_2 \in (0,\infty)$ respectively. LOONTINTONAL ANNEWS AND APPLICATIONS, VOL. 31, NO 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
 $\forall x \in \mathbb{R}$. Comparison $\mathbb{E}_x(x) = 1$, the experimentation $\mathbb{E}_x(x) = 1$, $2, 3, 5, 5$ and
 $\forall x \in \mathbb{R}$. The simulation

We consider the stochastic processes $Z_{i,3}(t,\omega)$ for $i = 1, 2, 3, 5$, where $t \in \mathbb{R}$ and $\omega \in \Omega$ as follows:

$$
Z_{1,3}(t,\omega) = \left[(t - t_0)^{\mu+1} + 1 \right] Y_{1,3}(\omega), \tag{103}
$$

where $t_0 \in \mathbb{R}$ and $\mu \in \mathbb{N}$ are fixed;

$$
Z_{2,3}(t,\omega) = \sin(\xi t) Y_{1,3}(\omega) + \cos(\xi t) Y_{2,3}(\omega), \qquad (104)
$$

where $\xi > 0$ is fixed;

$$
Z_{3,3}(t,\omega) = \sinh(\mu t) Y_{1,3}(\omega) + \cosh(\mu t) Y_{2,3}(\omega), \qquad (105)
$$

where $\mu > 0$ is fixed;

$$
Z_{5,3}(t,\omega) = e^{-\ell_1 t} Y_{1,3}(\omega) + e^{-\ell_2 t} Y_{2,3}(\omega),\tag{106}
$$

where $\ell_1, \ell_2 > 0$ are fixed. Since $E(Y_{1,3}) = \Gamma\left(1+\frac{1}{\gamma_1}\right)$ and $E(Y_{2,3}) = \Gamma\left(1+\frac{1}{\gamma_2}\right)$, where $\Gamma\left(\cdot\right)$ is the Gamma function, The expectations of $Z_{i,3}, i = 1, 2, 3, 5$, are 1. COMPUTATIONAL ANNEWS AND APPLICATIONS, VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
 $\exp(x\Delta x) \le 0$ Are fixed:
 $\exp(x\Delta x) = 1/2 + \frac{1}{\sqrt{2}} \sin(kx\Delta y) = 1/2 + \frac{1}{\sqrt{2}} \sin(kx\Delta y) + \frac{1}{2}$. (a)
 $\sin(kx\Delta y) = 1/2 + \frac{1}{\sqrt{2$

$$
(EZ_{1,3}) (t) = \Gamma \left(1 + \frac{1}{\gamma_1} \right) \left[(t - t_0)^{\mu + 1} + 1 \right], \tag{107}
$$

$$
(EZ_{2,3})(t) = \Gamma\left(1 + \frac{1}{\gamma_1}\right)\sin\left(\xi t\right) + \Gamma\left(1 + \frac{1}{\gamma_2}\right)\cos\left(\xi t\right),\tag{108}
$$

$$
(EZ_{3,3}) (t) = \Gamma\left(1 + \frac{1}{\gamma_1}\right) \sinh\left(\mu t\right) + \Gamma\left(1 + \frac{1}{\gamma_2}\right) \cosh\left(\mu t\right),\tag{109}
$$

$$
(EZ_{5,3})(t) = \Gamma\left(1 + \frac{1}{\gamma_1}\right)e^{-\ell_1 t} + \Gamma\left(1 + \frac{1}{\gamma_2}\right)e^{-\ell_2 t},\tag{110}
$$

We present the following result.

Proposition 30. Let $t \in [t_1, t_2]$, where $t_1, t_2 \in \mathbb{R}$, with $t_1 < t_2, 0 < \alpha < 1, n \in \mathbb{N} : n^{1-\alpha} > 2$. Then for $i = 1, 2, 3, 5$ *and* $k = 1, 2, 3$

$$
i)
$$

$$
|A_n\left(\left(EZ_{i,k}\right),t\right)-\left(EZ_{i,k}\right)(t)| \leq \frac{1}{\psi\left(1\right)} \left[\omega_1\left(EZ_{i,k},\frac{1}{n^{\alpha}}\right)+\left(1-h\left(n^{1-\alpha}-2\right)\right) \|EZ_{i,k}\|_{\infty}\right]=:\rho,\qquad(111)
$$

and

ii)

$$
||A_n(EZ_{i,k}) - EZ_{i,k}||_{\infty} \le \rho.
$$
\n(112)

We have that $\lim_{n\to\infty} A_n(EZ_{i,k}) = EZ_{i,k}$, pointwise and uniformly. The speed of convergence is max $\left(\frac{1}{n^{\alpha}}, \left(1 - h\left(n^{1-\alpha} - 2\right)\right)\right)$.

Proof. From Proposition 24. □

In the cases of stochastic processes $Z_{2,k}$ (t,ω) , for $k = 1,2,3$ we have the next

Proposition 31. *Let* $k \in \{1, 2, 3\}$, $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $t \in \mathbb{R}$. *Then*

i)

$$
\left|\overline{A}_{n}\left(EZ_{2,k},t\right)-\left(EZ_{2,k}\right)(t)\right| \leq \omega_{1}\left(EZ_{2,k},\frac{1}{n^{\alpha}}\right)+\left(1-h\left(n^{1-\alpha}-2\right)\right)\left\|EZ_{2,k}\right\|_{\infty}=: \mu,\tag{113}
$$

and

ii)

$$
\left\| \overline{A}_n \left(EZ_{2,k} \right) - EZ_{2,k} \right\|_{\infty} \le \mu. \tag{114}
$$

For $EZ_{2,k} \in C_{uB}(\mathbb{R})$ *we get* $\lim_{n \to \infty} \overline{A}_n(EZ_{2,k}) = EZ_{2,k}$, pointwise and uniformly. The speed of convergence is max $\left(\frac{1}{n^{\alpha}}, \left(1 - h\left(n^{1-\alpha} - 2\right)\right)\right)$.

Proof. The results come from Proposition 25. \Box Moreover, we present the next

Corollary 32. Assume $i = 1, 2, 3, 5$ and $k = 1, 2, 3$. Let $0 < \beta < 1$, $t \in [t_1, t_2]$, where $t_1, t_2 \in \mathbb{R}$, with $t_1 < t_2$, $and n \in \mathbb{N}: n^{1-\beta} > 2.$ Then

i)

$$
|A_{n}(EZ_{i,k},t) - (EZ_{i,k})(t)| \le
$$

$$
\frac{2(\psi(1))^{-1}}{\sqrt{\pi}} \left\{ \frac{\left(\omega_{1}\left(D_{t-}^{\frac{1}{2}}(EZ_{i,k}), \frac{1}{n^{\beta}}\right)_{[t_{1},t]} + \omega_{1}\left(D_{*t}^{\frac{1}{2}}(EZ_{i,k}), \frac{1}{n^{\beta}}\right)_{[t,t_{2}]}\right)}{n^{\frac{\beta}{2}}} + \frac{\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\left(\left\|D_{t-}^{\frac{1}{2}}(EZ_{i,k})\right\|_{\infty,[t_{1},t]}\sqrt{(t-t_{1})} + \left\|D_{*t}^{\frac{1}{2}}(EZ_{i,k})\right\|_{\infty,[t,t_{2}]}\sqrt{(t_{2}-t)}\right)\right\},\qquad(115)
$$

and

ii)

J. COMPUTATIONAL AVALYSIS AND APPLICATIONS. VOL. 31. NO. 4, 2023, COPYRIGFT 2023 EUDOXUS PRESS, LLC
\nrollary 32. Assume
$$
i = 1, 2, 3, 5
$$
 and $k = 1, 2, 3$. Let $0 < \beta < 1$, $t \in [t_1, t_2]$, where $t_1, t_2 \in \mathbb{R}$, with $t_1 < t_2$,
\n
$$
y
$$
\n
$$
|A_n(EZ_{i,k},t) - (EZ_{i,k})(t)| \le
$$
\n
$$
\frac{2(\psi(1))^{-1}}{\sqrt{\pi}} \left\{ \frac{\left(\omega_1 \left(D_{t-}^{\frac{1}{2}}(EZ_{i,k})\frac{1}{n^2}\right)_{[t_1,t_1]} + \omega_1 \left(D_{t+}^{\frac{1}{2}}(EZ_{i,k})\frac{1}{n^2}\right)_{[t_1,t_2]} + \omega_1 \left(D_{t+}^{\frac{1}{2}}(EZ_{i,k})\frac{1}{n^2}\right)}{n^{\frac{3}{2}}} + \frac{\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\left(\left\|D_{t-}^{\frac{1}{2}}(EZ_{i,k})\right\|_{\infty,[t_1,t_1]} \sqrt{(t-t_1)} + \left\|D_{t+}^{\frac{1}{2}}(EZ_{i,k})\right\|_{\infty,[t_2]}\sqrt{(t_2-t)}\right)\right\}, \qquad (115)
$$
\n
$$
= |A_n(EZ_{i,k}) - (EZ_{i,k})|_{\infty} \leq \frac{2(\psi(1))^{-1}}{\sqrt{\pi}}
$$
\n
$$
\left\{ \frac{\left(\sup_{t \in [t_1,t_2]} \omega_1 \left(D_{t-}^{\frac{1}{2}}(EZ_{i,k})\frac{1}{n^2} + \sum_{t \in [t_1,t_2]} \omega_1 \left(D_{t+}^{\frac{1}{2}}(EZ_{i,k})\frac{1}{n^2}\right)_{[t_1,t_2]} \left(D_{t+}^{\frac{1}{2}}(EZ_{i,k})\frac{1}{n^2}\right) + \frac{\left(1-h\left(n^{1-\beta}-2\right)}{2}\right)\sqrt{(t_2-t_1)} \left(\sup_{t \in [t_1,t_
$$

Proof. From Corollary 29. \Box

References

- [1] G.A. Anastassiou, *Rate of convergence of some neural network operators to the unit-univariate case*, J. Math. Anal. Appl, 212 (1997), 237-262.
- [2] G.A. Anastassiou, *Quantitative Approximations*, Chapman & Hall / CRC, Boca Raton, New York, 2001.
- [3] G.A. Anastassiou, *Univariate hyperbolic tangent neural network approximation*, Mathematics and Computer Modelling, 53 (2011), 1111-1132.
- [4] G.A. Anastassiou, *Multivariate hyperbolic tangent neural network approximation*, Computers and Mathematics, 61 (2011), 809-821.
- [5] G.A. Anastassiou, *Multivariate sigmoidal neural networkapproximation*, Neural Networks, 24 (2011), 378-386.
- [6] G.A. Anastassiou, *Inteligent Systems: Approximation by Artificial Neural Networks*, Intelligent Systems Reference Library, Vol. 19, Springer, Heidelberg, 2011.
- [7] G.A. Anastassiou, *Univariate sigmoidal neural network approximation*, J. of Computational Analysis and Applications, Vol. 14, No.4, 2012, 659-690.
- [8] G.A. Anastassiou, *Fractional neural network approximation*, Computers and Mathematics with Applications, 64 (2012), 1655-1676.
- [9] G.A. Anastassiou, *Intelligent Systems II: Complete Approximation by Neural Network Operators*, Springer, Heidelberg, New York, 2016.
- [10] G.A. Anastassiou, *Strong Right Fractional Calculus for Banach space valued functions*, Revista Proyecciones, Vol. 36, No. 1 (2017), 149-186.
- [11] G.A. Anastassiou, *Vector fractional Korovkin type Approximations*, Dynamic Systems and Applications, 26 (2017), 81-104.
- [12] G.A. Anastassiou, *A strong Fractional Calculus Theory for Banach space valued functions*, Nonlinear Functional Analysis and Applications (Korea), 22(3)(2017), 495-524.
- [13] G.A. Anastassiou, *Intelligent Computations: Abstract Fractional Calculus, Inequalities, Approximations*, Springer, Heidelberg, New York, 2018. 5. COMPUTATIONAL ANNEWSE AND APPLICATIONS, VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC

G.A. Anastassion, Participate System. I.E. Complete Approximation by Premier Systems. One of Figure A. Analysis and the Co
- [14] G.A. Anastassiou,*General sigmoid based Banach space valued neural network approximation*, Journal of Computational Analysis and Applications, Accepted, 2022.
- [15] Z. Chen and F. Cao, *The approximation operators with sigmoidal functions*, Computers and Mathematics with Applications, 58 (2009), 758-765.
- [16] D. Costarelli, R. Spigler, *Approximation results for neural network operators activated by sigmoidal functions*, Neural Networks 44 (2013), 101-106.
- [17] D. Costarelli, R. Spigler, *Multivariate neural network operators with sigmoidal activation functions*, Neural Networks 48 (2013), 72-77.
- [18] S. Haykin, *Neural Networks: A Comprehensive Foundation* (2 ed.), Prentice Hall, New York, 1998.
- [19] M. Kac , A.J.F. Siegert, *An explicit representation of a stationary Gaussian process*, The Annals of Mathematical Statistics, 18 (3) (1947), 438-442.
- [20] Yuriy Kozachenko et al, *Simulation of stochastic processes with given accuracy and reliability*, Elsevier (2016), pp.71-104.
- [21] W. McCulloch and W. Pitts, *A logical calculus of the ideas immanent in nervous activity*, Bulletin of Mathematical Biophysis, 7 (1943), 115-133.
- [22] J. Mikusinski, *The Bochner integral*, Academic Press, New York, 1978.
- [23] T.M. Mitchell, *Machine Learning*, WCB-McGraw-Hill, New York, 1997.
- [24] G.E. Shilov, *Elementary Functional Analysis*, Dover Publications, Inc., New York, 1996.

Multivariate Gudermannian function based neural network approximation

George A. Anastassiou Department of Mathematical Sciences University of Memphis Memphis, TN 38152, U.S.A. ganastss@memphis.edu

Abstract

Here we present multivariate quantitative approximations of Banach space valued continuous multivariate functions on a box or \mathbb{R}^N , $N \in \mathbb{N}$, by the multivariate normalized, quasi-interpolation, Kantorovich type and quadrature type neural network operators. We examine also the case of approximation by iterated operators of the last four types. These approximations are achieved by establishing multidimensional Jackson type inequalities involving the multivariate modulus of continuity of the engaged function or its high order Fréchet derivatives. Our multivariate operators are defined by using a multidimensional density function induced by the Gudermannian sigmoid function. The approximations are pointwise and uniform. The related feed-forward neural network is with one hidden layer. 557 J. CONSULTATIONAL ANNEXT COMPRESS, VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC

1991 Terrari and the three contracts of the state of the sta

2020 AMS Mathematics Subject Classification: $41A17$, $41A25$, $41A30$, 41A36.

Keywords and Phrases: Gudermannian sigmoid function, multivariate neural network approximation, quasi-interpolation operator, Kantorovich type operator, quadrature type operator, multivariate modulus of continuity, abstract approximation, iterated approximation.

1 Introduction

G.A. Anastassiou in $[2]$ and $[3]$, see chapters 2-5, was the first to establish neural network approximations to continuous functions with rates by very specifically defined neural network operators of Cardaliaguet-Euvrard and "Squashing" types, by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities. He treats there both the univariate and multivariate cases. The defining these operators îbell-shapedî and îsquashingî functions are assumed to be of compact support. Also in [3] he gives the Nth order asymptotic expansion for the error of weak approximation of these two operators to a special natural class of smooth functions, see chapters 4-5 there.

Motivations for this work are the article [17] of Z. Chen and F. Cao, and [4], [5], [6], [7], [8], [9], [10], [11], [12], [14], [15], [18], [19].

Here we perform multivariate Gudermannian sigmoid function based neural network approximations to continuous functions over boxes or over the whole \mathbb{R}^N , $N \in \mathbb{N}$, and also iterated approximations. All convergences here are with rates expressed via the multivariate modulus of continuity of the involved function or its high order Fréchet derivative and given by very tight multidimensional Jackson type inequalities. 5 CONFUTATIONAL ANNEWSES AND APPLICATIONS, VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC COMPUTATIONS TO A THE CONFUTATIONS AND THE CONFUTATIONS CONFUTATIONS CONFUTATIONS CONFUTATIONS CONFUTATIONS CONFUTATIONS C

We come up with the "right" precisely defined multivariate normalized, quasi-interpolation neural network operators related to boxes or \mathbb{R}^N , as well as Kantorovich type and quadrature type related operators on \mathbb{R}^N . Our boxes are not necessarily symmetric to the origin. In preparation to prove our results we establish important properties of the basic multivariate density function induced by Gudermannian sigmoid function and defining our operators.

Feed-forward neural networks (FNNs) with one hidden layer, the only type of networks we deal with in this article, are mathematically expressed as

$$
N_n(x) = \sum_{j=0}^n c_j \sigma\left(\langle a_j \cdot x \rangle + b_j\right), \quad x \in \mathbb{R}^s, \quad s \in \mathbb{N},
$$

where for $0 \leq j \leq n$, $b_j \in \mathbb{R}$ are the thresholds, $a_j \in \mathbb{R}^s$ are the connection weights, $c_j \in \mathbb{R}$ are the coefficients, $\langle a_j \cdot x \rangle$ is the inner product of a_j and x, and σ is the activation function of the network. In many fundamental network models, the activation function is the Gudermannian sigmoid function. About neural networks see [20], [21], [22].

2 Background

See also [13], [24].

Here we consider $gd(x)$ the Gudermannian function [24], which is a sigmoid function, as a generator function:

$$
\sigma(x) = 2 \arctan\left(\tanh\left(\frac{x}{2}\right)\right) = \int_0^x \frac{dt}{\cosh t} =: gd(x), x \in \mathbb{R}.
$$
 (1)

Let the normalized generator sigmoid function

$$
f(x) := \frac{2}{\pi}\sigma(x) = \frac{2}{\pi}\int_0^x \frac{dt}{\cosh t} = \frac{4}{\pi}\int_0^x \frac{1}{e^t + e^{-t}}dt, \ \ x \in \mathbb{R}.
$$
 (2)

Here

$$
f'(x) = \frac{2}{\pi \cosh x} > 0, \quad \forall \ x \in \mathbb{R},
$$

hence f is strictly increasing on R.

Notice that $tanh(-x) = -\tanh x$ and $arctan(-x) = -\arctan x$, $x \in \mathbb{R}$. So, here the neural network activation function will be:

$$
W(x) = \frac{1}{4} [f(x+1) - f(x-1)], x \in \mathbb{R}.
$$
 (3)

By [3], we get that

$$
W(x) = W(-x), \quad \forall \ x \in \mathbb{R}, \tag{4}
$$

i.e. it is even and symmetric with respect to the y-axis. Here we have $f(+\infty) =$ 1, $f(-\infty) = -1$ and $f(0) = 0$. Clearly it is

$$
f(-x) = -f(x), \quad \forall x \in \mathbb{R}, \tag{5}
$$

an odd function, symmetric with respect to the origin. Since $x + 1 > x - 1$, and $f(x+1) > f(x-1)$, we obtain $W(x) > 0, \forall x \in \mathbb{R}$. 5 CONFUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC

15 There $f'(x) = \frac{2}{\pi}x + 0$. V. $x \in \mathbb{R}$.

Notice that the incidence are not a set of $x \in \mathbb{R}$.

Note that the inc

By [13], we have that

$$
W(0) = \frac{1}{\pi}gd(1) \cong 0.2757.
$$
 (6)

By [13] W is strictly decreasing on $(0, +\infty)$, and strictly increasing on $(-\infty, 0)$, and $W'(0) = 0$.

Also we have that

$$
\lim_{x \to +\infty} W(x) = \lim_{x \to -\infty} W(x) = 0,\tag{7}
$$

that is the x -axis is the horizontal asymptote for W .

Conclusion, W is a bell shaped symmetric function with maximum $W(0) \cong$ 0:2757.

We need

Theorem 1 ($\left(13\right)$) It holds that

$$
\sum_{i=-\infty}^{\infty} W(x-i) = 1, \ \forall \ x \in \mathbb{R}.
$$
 (8)

Theorem 2 ([13]) We have that

$$
\int_{-\infty}^{\infty} W(x) dx = 1.
$$
 (9)

So $W(x)$ is a density function.

Theorem 3 ([13]) Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$. It holds

$$
\sum_{k=-\infty}^{\infty} W(nx-k) < \frac{2}{\pi e^{(n^{1-\alpha}-2)}} = \frac{2e^2}{\pi e^{n^{1-\alpha}}}.
$$
\n(10)

\n
$$
\left\{ \left. \frac{k=-\infty}{|nx-k| \ge n^{1-\alpha}} \right. \right.
$$

Denote by $\lfloor \cdot \rfloor$ the integral part of the number and by $\lfloor \cdot \rfloor$ the ceiling of the number.

Theorem 4 ([13]) Let $[a, b] \subset \mathbb{R}$ and $n \in \mathbb{N}$, so that $[na] \leq [nb]$. It holds

$$
\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} W\left(nx-k\right)} < \frac{2\pi}{gd\left(2\right)} \cong 4.824,\tag{11}
$$

 $\forall x \in [a, b]$.

We make

Remark 5 $([13])$ (i) We have that

$$
\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} W(nx-k) \neq 1,\tag{12}
$$

for at least some $x \in [a, b]$.

(ii) Let $[a, b] \subset \mathbb{R}$. For large n we always have $\lceil na \rceil \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $[na] \leq k \leq \lfloor nb \rfloor$.

In general it holds

$$
\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} W(nx-k) \le 1.
$$
 (13)

We introduce

$$
Z(x_1, ..., x_N) := Z(x) := \prod_{i=1}^{N} W(x_i), \quad x = (x_1, ..., x_N) \in \mathbb{R}^N, \ N \in \mathbb{N}. \tag{14}
$$

It has the properties:

1. COMPUTATIONAL ANALYSIS AND APPLICATIONS, Vol. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC

\n2. Theorem 3 ([13]) Let
$$
0 < \alpha < 1
$$
, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$. It holds

\n
$$
\sum_{k=-\infty}^{\infty} W(nx-k) < \frac{2}{\pi e^{(n^{1-\alpha}-2)}} - \frac{2n^2}{\pi e^{n^{1-\alpha}}}, \qquad (10)
$$
\n
$$
\begin{cases}\n k = -\infty \\
 i \mid nx-k \mid \geq n^{1-\alpha}\n\end{cases}
$$
\nDenote by $\lfloor \cdot \rfloor$ the integral part of the number and by $\lceil \cdot \rceil$ the ceiling of the number.

\nTherorm 4 ([13]) Let $[a,b] \in \mathbb{R}$ and $n \in \mathbb{N}$, so that $\lceil na \rceil \leq \lfloor nb \rfloor$. It holds

\n
$$
\frac{1}{\frac{[nb]}{k-1}W(nx-k)} < \frac{2\pi}{\pi d(2)} \cong 4.824, \qquad (11)
$$
\n
$$
\forall x \in [a,b].
$$
\nWe make

\nRemark 5 ([13])

\n(i) We have that

\n
$$
\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{[nb]} W(nx-k) \neq 1, \qquad (12)
$$
\nfor at least some $x \in [a,b].$

\n(ii) Let $[a,b] \in \mathbb{R}$. For large n we always have $\lceil na \rceil \leq \lceil nb \rceil$. Also $a \leq \frac{k}{n} \leq b$, if $\lceil na \rceil \leq k \leq \lceil nb \rceil$.

\n(iii) Let $[a,b] \in \mathbb{R}$. For large n we always have $\lceil na \rceil \leq \lceil nb \rceil$. Also <

3. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC

\nwhere
$$
k := (k_1, ..., k_n) \in \mathbb{Z}^N
$$
, $\forall x \in \mathbb{R}^N$, hence

\n(ii)

\n
$$
\sum_{k=-\infty}^{\infty} Z(nx-k) = 1,
$$
\nand

\n(iv)

\n
$$
\int_{\mathbb{R}^N} Z(x) dx = 1,
$$
\nand

\n(iv)

\n
$$
\int_{\mathbb{R}^N} Z(x) dx = 1,
$$
\nand

\n(iv)

\n
$$
\int_{-\infty}^{\infty} \left(-\infty, ..., -\infty\right) \text{ upon the multivariate const, and}
$$
\n[na] := ([na_1], ..., [na_N]),

\n
$$
[nb] := ([nb_1], ..., [nb_N]),
$$
\nwhere $a := (a_1, ..., a_N)$, $b := (b_1, ..., b_N)$,

\nwhere

\n
$$
a := (a_1, ..., a_N)
$$
, $b := (b_1, ..., b_N)$,\nwhere

\n
$$
\sum_{k=[na_1]}^{\infty} \left(-\frac{b_1}{b_1} + \frac{b_2}{b_2} + \frac{b_3}{b_3} + \frac{b_4}{b_4} + \frac{b_5}{b_5} + \frac{b_6}{b_6} + \frac{b_7}{b_6} + \frac{b_7}{b_6} + \frac{b_8}{b_6} + \frac{b_9}{b_6} + \frac{b_9}{b_6} + \frac{b_9}{b_6} + \frac{b_9}{b_6} + \frac{b_1}{b_6} + \frac{b_1}{b_6} + \frac{b_1}{b_6} + \frac{b_1}{b_6} + \frac{b_1}{b_6} + \frac{b_2}{b_6} + \frac{b_1}{b_6} + \frac{b_1}{b_6} + \frac{b_1}{b_6} + \frac{b_1}{b_6} + \frac{b_2}{b_6} + \frac{b_1}{b_6} + \frac{b_
$$

$$
\int_{\mathbb{R}^N} Z(x) dx = 1,
$$
\n(17)

that is Z is a multivariate density function.

Here denote $||x||_{\infty} := \max\{|x_1|, ..., |x_N|\}, x \in \mathbb{R}^N$, also set $\infty := (\infty, ..., \infty)$, $-\infty := (-\infty, ..., -\infty)$ upon the multivariate context, and

$$
[na] := ([na1], ..., [naN]),\n
$$
[nb] := ([nb1], ..., [nbN]),
$$
\n(18)
$$

where $a := (a_1, ..., a_N), b := (b_1, ..., b_N)$.

We obviously see that

$$
\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k) = \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(\prod_{i=1}^{N} W(nx_i - k_i) \right) =
$$

$$
\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \dots \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} \left(\prod_{i=1}^{N} W(nx_i - k_i) \right) = \prod_{i=1}^{N} \left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} W(nx_i - k_i) \right). \tag{19}
$$

For $0 < \beta < 1$ and $n \in \mathbb{N}$, a fixed $x \in \mathbb{R}^N$, we have that

$$
\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k) =
$$
\n
$$
\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k) + \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)
$$

$$
\sum_{\begin{cases}\nk = \lceil na \rceil \\
\left\|\frac{k}{n} - x\right\|_{\infty} \le \frac{1}{n^{\beta}}\n\end{cases}} Z(nx - k) + \sum_{\begin{cases}\nk = \lceil na \rceil \\
\left\|\frac{k}{n} - x\right\|_{\infty} > \frac{1}{n^{\beta}}\n\end{cases}} Z(nx - k). \tag{20}
$$

In the last two sums the counting is over disjoint vector sets of k 's, because the condition $\left\|\frac{k}{n}-x\right\|_{\infty} > \frac{1}{n^{\beta}}$ implies that there exists at least one $\left|\frac{k_r}{n}-x_r\right| > \frac{1}{n^{\beta}}$, where $r \in \{1, ..., N\}$.

(v) As in [10], pp. 379-380, we derive that

$$
\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k) \stackrel{(10)}{<} \frac{2e^2}{\pi e^{n^{1-\beta}}}, \ 0 < \beta < 1, \ m \in \mathbb{N}, \qquad (21)
$$
\n
$$
\left\{ \left\| \frac{k}{n} - x \right\|_{\infty} > \frac{1}{n^{\beta}}
$$

with $n \in \mathbb{N} : n^{1-\beta} > 2, x \in \prod_{i=1}^{N} [a_i, b_i]$.

(vi) By Theorem 4 we get that

$$
0 < \frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z\left(nx-k\right)} < \left(\frac{2\pi}{gd\left(2\right)}\right)^N \cong \left(4.824\right)^N,\tag{22}
$$

 $\forall x \in \left(\prod_{i=1}^N [a_i, b_i]\right), \ \ n \in \mathbb{N}.$

It is also clear that

(vii)

$$
\sum_{k=-\infty}^{\infty} Z(nx-k) < \frac{2e^2}{\pi e^{n^{1-\beta}}},\tag{23}
$$
\n
$$
\left\{ \left\| \frac{k}{n} - x \right\|_{\infty} > \frac{1}{n^{\beta}} \right\}
$$

 $0 < \beta < 1, n \in \mathbb{N}: n^{1-\beta} > 2, x \in \mathbb{R}^N, m \in \mathbb{N}.$

Furthermore it holds

$$
\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k) \neq 1,\tag{24}
$$

for at least some $x \in \left(\prod_{i=1}^N [a_i, b_i]\right)$.

Here $(X, \left\| \cdot \right\|_{\gamma})$ is a Banach space.

Let $f \in C\left(\prod_{i=1}^N [a_i, b_i], X\right)$, $x = (x_1, ..., x_N) \in \prod_{i=1}^N [a_i, b_i]$, $n \in \mathbb{N}$ such that $\lceil na_i \rceil \leq \lfloor nb_i \rfloor, i = 1, ..., N.$

We introduce and define the following multivariate linear normalized neural network operator $(x := (x_1, ..., x_N) \in \left(\prod_{i=1}^N [a_i, b_i] \right))$:

3. COMPUTATIONAL ANALYSIS AND APPLICATIONS, Vol. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\n(v) As in [10], pp. 370-380, we derive that
\n
$$
\sum_{k=-1}^{[100]} Z(nx-k)^{-1/2} \sum_{k=0}^{k-2} x^2
$$
\nwith $n \in \mathbb{N}: n!^{-\beta} > 2, x \in \prod_{k=1}^{N} [a_k, b_k].$
\n(vi) By Theorem 4 we get that
\n
$$
0 < \sum_{k=-1}^{[100]} Z(nx-k) \left(\frac{2\pi}{gd(2)} \right)^N \approx (4.824)^N,
$$
\n(22)
\n
$$
\forall x \in \left(\prod_{k=1}^{N} [a_k, b_k] \right), n \in \mathbb{N}.
$$
\nIt is also clear that
\n(vi)
\n
$$
\sum_{k=-\infty}^{\infty} Z(nx-k) \left(\frac{2e^2}{gd(2)} \right)^N \approx (4.824)^N,
$$
\n(23)
\n
$$
\begin{cases}\n k = -\infty \\
 k = -\infty\n\end{cases}
$$
\n
$$
0 < \beta < 1, n \in \mathbb{N}: n^{1-\beta} > 2, x \in \mathbb{R}^N, m \in \mathbb{N}.
$$
\nFurthermore it holds
\n
$$
\lim_{n \to \infty} \sum_{k=-\infty}^{\lfloor n\beta \rfloor} Z(nx-k) \neq 1,
$$
\n(24)
\nfor at least some $x \in \left(\prod_{k=1}^{N} [a_k, b_k] \right), x = (x_1, ..., x_N) \in \prod_{k=1}^{\infty} [a_k, b_k], n \in \mathbb{N}$ such that $[na_k] = \{a_k, b_k\} \Rightarrow n \in \mathbb{N}.$
\nLet $f \in C \left(\prod_{k=1}^{\infty} [a_k, b_k] \right), x = (x_1, ..., x_N) \in \prod_{k=1}^{\infty} [a_k, b_k], n \in \mathbb{N}$ such that $[na_k] = \{a_k, b_k\} \Rightarrow n \in \mathbb{N}.$
\n
$$
\text{that } [na_k] \leq \frac
$$

For large enough $n \in \mathbb{N}$ we always obtain $[na_i] \leq [nb_i]$, $i = 1, ..., N$. Also $a_i \leq \frac{k_i}{n} \leq b_i$, iff $\lceil na_i \rceil \leq k_i \leq \lfloor nb_i \rfloor$, $i = 1, ..., N$.

When $g \in C\left(\prod_{i=1}^N [a_i, b_i]\right)$ we define the companion operator

$$
\widetilde{A}_n(g,x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} g\left(\frac{k}{n}\right) Z\left(nx-k\right)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z\left(nx-k\right)}.
$$
\n(26)

Clearly \widetilde{A}_n is a positive linear operator. We have that

$$
\widetilde{A}_n(1,x) = 1, \ \forall \ x \in \left(\prod_{i=1}^N [a_i, b_i]\right).
$$

Notice that $A_n(f) \in C\left(\prod_{i=1}^N [a_i, b_i], X\right)$ and $\widetilde{A}_n(g) \in C\left(\prod_{i=1}^N [a_i, b_i]\right)$. Furthermore it holds

$$
\|A_n(f,x)\|_{\gamma} \le \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \|f\left(\frac{k}{n}\right)\|_{\gamma} Z\left(nx-k\right)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z\left(nx-k\right)} = \widetilde{A}_n\left(\|f\|_{\gamma},x\right),\tag{27}
$$

 $\forall x \in \prod_{i=1}^N [a_i, b_i].$ Clearly $||f||_{\gamma} \in C\left(\prod_{i=1}^{N} [a_i, b_i]\right)$. So, we have that

$$
\|A_n(f,x)\|_{\gamma} \le \widetilde{A}_n\left(\|f\|_{\gamma},x\right),\tag{28}
$$

 $\forall x \in \prod_{i=1}^N [a_i, b_i], \forall n \in \mathbb{N}, \forall f \in C \left(\prod_{i=1}^N [a_i, b_i], X \right).$ Let $c \in X$ and $g \in C\left(\prod_{i=1}^N [a_i, b_i]\right)$, then $cg \in C\left(\prod_{i=1}^N [a_i, b_i], X\right)$. Furthermore it holds

$$
A_n (cg, x) = c\widetilde{A}_n (g, x), \ \ \forall \ x \in \prod_{i=1}^N [a_i, b_i]. \tag{29}
$$

Since $\widetilde{A}_n(1) = 1$, we get that

$$
A_n(c) = c, \forall c \in X.
$$
\n(30)

We call \widetilde{A}_n the companion operator of A_n .

For convinience we call

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, Vol. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\n
$$
\tilde{A}_{n}(g,x) := \frac{\sum_{k=1}^{|N|} \alpha_{k}(\theta_{k}^{k})}{\sum_{k=1}^{|N|} \alpha_{k}(\theta_{k}^{k})} \times (nx - k)
$$
\n(26)
\nClearly \tilde{A}_{n} is a positive linear operator. We have that
\n
$$
\tilde{A}_{n}(1,x) = 1, \forall x \in \left(\prod_{k=1}^{N} [a_{k}, b_{k}]\right).
$$
\nNotice that $A_{n}(f) \in C\left(\prod_{k=1}^{N} [a_{k}, b_{k}], X\right)$ and $\tilde{A}_{n}(g) \in C\left(\prod_{k=1}^{N} [a_{k}, b_{k}]\right).$
\nNotice that $A_{n}(f) \in C\left(\prod_{k=1}^{N} [a_{k}, b_{k}], X\right)$ and $\tilde{A}_{n}(g) \in C\left(\prod_{k=1}^{N} [a_{k}, b_{k}]\right).$
\nFurthermore it holds
\n
$$
||A_{n}(f,x)||_{\gamma} \leq \frac{\sum_{k=1}^{|N|} \alpha_{k}[\alpha_{k}]}{2\sum_{k=1}^{|N|} \alpha_{k}} \times (nx - k) = \tilde{A}_{n}\left(\|f\|_{\gamma}, x\right),
$$
\n(27)
\n
$$
\forall x \in \prod_{k=1}^{N} [a_{k}, b_{k}], \forall n \in N, \forall f \in C\left(\prod_{k=1}^{N} [a_{k}, b_{k}], X\right).
$$

\nNow that
\n
$$
||A_{n}(f,x)||_{\gamma} \leq \tilde{A}_{n}([|f|]_{\gamma}, x),
$$
\n(28)
\n
$$
\forall x \in \prod_{k=1}^{N} [a_{k}, b_{k}]; \forall n \in \mathbb{N}, \forall f \in C\left(\prod_{k=1}^{N} [a_{k}, b_{k}], X\right).
$$

\n
$$
L_{n} \in C \times \text{ and } g \in C\left(\prod_{k=1}^{N} [a_{k}, b_{k}], \text{ then } cg \in C\left(\prod_{k=1
$$

1. COMPUTATIONAL ANALYSIS AND APPLICATIONS. Vol. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC

\nY
$$
x \in (\prod_{k=1}^{N} [a_k, b_k])
$$
.

\n1. That is

\n2. $A_n(f, x) := \sum_{k=1}^{|n_k|} Z(nx - k)$

\n2. $x \in (\prod_{k=1}^{N} [a_k, b_k])$, $n \in \mathbb{N}$.

\n2. $A_n(f, x) - f(x) = \frac{A_n^*(f, x) - f(x) \left(\sum_{k=1 \text{ real}}^{10k} Z(nx - k)\right)}{\sum_{k=1 \text{ real}}^{10k} Z(nx - k)}$.

\n3.3 $A_n(f, x) - f(x) \big|_{\infty} \stackrel{20}{\leq} (4.824)^N \left\| A_n^*(f, x) - f(x) \sum_{k=1 \text{ real}}^{10k} Z(nx - k) \right\|_{\infty}$.

\n3.4 $\forall x \in (\prod_{k=1}^{N} [a_k, b_k])$.

\n3.5 $\forall x \in (\prod_{k=1}^{N} [a_k, b_k])$.

\n4.6 $\forall x \in \prod_{k=1}^{N} [a_k, b_k]$.

\n4.7 $x \in \prod_{k=1}^{N} [a_k, b_k]$.

\n5.8 $\forall x \in \prod_{k=1}^{N} [a_k, b_k]$.

\n6.9 $\forall x \in \prod_{k=1}^{N} [a_k, b_k]$.

\n7. $\forall x \in \prod_{k=1}^{N} [a_k, b_k]$.

\n8.7 $\forall x \in \prod_{k=1}^{N} [a_k, b_k]$.

\n9. $\forall x \in \prod_{k=1}^{N} [a_k, b_k]$.

\n10. $\forall x \in \prod_{k=1}^{N} [a_k, b_k]$.

\n11. $\forall x \in \$

 $\forall x \in \left(\prod_{i=1}^N [a_i, b_i]\right), n \in \mathbb{N}.$ Hence

$$
A_n(f,x) - f(x) = \frac{A_n^*(f,x) - f(x) \left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k) \right)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)}.
$$
 (33)

Consequently we derive

$$
\|A_{n}(f,x)-f(x)\|_{\gamma} \stackrel{(22)}{\leq} (4.824)^{N} \left\|A_{n}^{*}(f,x)-f(x)\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z\left(nx-k\right)\right\|_{\gamma}, \tag{34}
$$

 $\forall x \in \left(\prod_{i=1}^N [a_i, b_i]\right).$

We will estimate the right hand side of (34). For the last and others we need

Definition 6 ([11], p. 274) Let M be a convex and compact subset of $(\mathbb{R}^N, \left\|\cdot\right\|_p)$, $p \in [1,\infty]$, and $(X, \left\|\cdot\right\|_{\gamma})$ be a Banach space. Let $f \in C(M,X)$. We define the first modulus of continuity of f as

$$
\omega_1(f,\delta) := \sup_{\begin{subarray}{l} x, y \in M \\ \|x - y\|_p \le \delta \end{subarray}} \|f(x) - f(y)\|_{\gamma}, \ \ 0 < \delta \le \operatorname{diam}(M). \tag{35}
$$

If $\delta > diam(M)$, then

$$
\omega_1(f,\delta) = \omega_1(f, diam(M)). \tag{36}
$$

Notice $\omega_1(f,\delta)$ is increasing in $\delta > 0$. For $f \in C_B(M,X)$ (continuous and bounded functions) $\omega_1(f, \delta)$ is defined similarly.

Lemma 7 ([11], p. 274) We have $\omega_1(f, \delta) \to 0$ as $\delta \downarrow 0$, iff $f \in C(M, X)$, where M is a convex compact subset of $(\mathbb{R}^N, \|\cdot\|_p), p \in [1, \infty]$.

Clearly we have also: $f \in C_U(\mathbb{R}^N, X)$ (uniformly continuous functions), iff $\omega_1(f,\delta) \to 0$ as $\delta \downarrow 0$, where ω_1 is defined similarly to (35). The space $C_B(\mathbb{R}^N, X)$ denotes the continuous and bounded functions on \mathbb{R}^N .

When $f \in C_B(\mathbb{R}^N, X)$ we define,

$$
B_n(f, x) := B_n(f, x_1, ..., x_N) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) Z(nx - k) :=
$$

$$
\sum_{k_1 = -\infty}^{\infty} \sum_{k_2 = -\infty}^{\infty} \dots \sum_{k_N = -\infty}^{\infty} f\left(\frac{k_1}{n}, \frac{k_2}{n}, ..., \frac{k_N}{n}\right) \left(\prod_{i=1}^{N} W(nx_i - k_i)\right), \qquad (37)
$$

 $n \in \mathbb{N}, \forall x \in \mathbb{R}^N, N \in \mathbb{N}$, the multivariate quasi-interpolation neural network operator.

Also for $f \in C_B(\mathbb{R}^N, X)$ we define the multivariate Kantorovich type neural network operator

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\nWhen
$$
f \in C_B (\mathbb{R}^N, X)
$$
 we define,
\n
$$
B_n (f, x) := B_n (f, x_1, ..., x_N) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) Z(nx - k) :=
$$
\n
$$
\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \sum_{k_3=-\infty}^{\infty} \sum_{k_4=-\infty}^{\infty} \int \left(\frac{k_1}{n}, \frac{k_2}{n}, ..., \frac{k_N}{n}\right) \left(\prod_{i=1}^{N} W(nx_i - k_i)\right), \qquad (37)
$$
\n $n \in \mathbb{N}, \forall x \in \mathbb{R}^N, N \in \mathbb{N}$, the multivariate quasi-intryption neural network operator
\n
$$
D_n (f, x) := C_n (f, x_1, ..., x_N) := \sum_{k=-\infty}^{\infty} \left(n^N \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t) dt\right) Z(nx - k) =
$$
\n
$$
\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \sum_{k_3=-\infty}^{\infty} \left(n^N \int_{\frac{k_1}{n}}^{\frac{k+1}{n}} \int_{\frac{k_2}{n}}^{t_2 + \frac{1}{n}} f(t_1, ..., t_N) dt_1 ... dx_N\right)
$$
\n
$$
= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \sum_{k_3=-\infty}^{\infty} \left(n^N \int_{\frac{k_1}{n}}^{\frac{k+1}{n}} \int_{\frac{k_2}{n}}^{t_2 + \frac{1}{n}} f(t_1, ..., t_N) dt_1 ... dx_N\right)
$$
\n
$$
= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \sum_{k_3=-\infty}^{\infty} \left(n^N \int_{\frac{k_1}{n}} f(t) dt\right).
$$
\n
$$
= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\in
$$

 $n \in \mathbb{N}, \ \forall \ x \in \mathbb{R}^N.$

Again for $f \in C_B(\mathbb{R}^N, X)$, $N \in \mathbb{N}$, we define the multivariate neural network operator of quadrature type $D_n(f, x)$, $n \in \mathbb{N}$, as follows.

Let $\theta = (\theta_1, ..., \theta_N) \in \mathbb{N}^N$, $r = (r_1, ..., r_N) \in \mathbb{Z}_+^N$, $w_r = w_{r_1, r_2, ..., r_N} \ge 0$, such that \sum^{θ} $\sum_{r=0}^{\theta} w_r = \sum_{r_1=0}^{\theta_1}$ $r_1=0$ $\frac{\theta_2}{\sum}$ $r_2=0$ \ldots $\sum_{N}^{\theta_{N}}$ $\sum_{r_N=0}^{\infty} w_{r_1,r_2,...r_N} = 1$; $k \in \mathbb{Z}^N$ and

$$
\delta_{nk}(f) := \delta_{n,k_1,k_2,...,k_N}(f) := \sum_{r=0}^{\theta} w_r f\left(\frac{k}{n} + \frac{r}{n\theta}\right) =
$$

$$
\sum_{r_1=0}^{\theta_1} \sum_{r_2=0}^{\theta_2} \dots \sum_{r_N=0}^{\theta_N} w_{r_1, r_2, \dots r_N} f\left(\frac{k_1}{n} + \frac{r_1}{n\theta_1}, \frac{k_2}{n} + \frac{r_2}{n\theta_2}, \dots, \frac{k_N}{n} + \frac{r_N}{n\theta_N}\right), \quad (39)
$$

where $\frac{r}{\theta} := \left(\frac{r_1}{\theta_1}, \frac{r_2}{\theta_2}, ..., \frac{r_N}{\theta_N}\right)$ $\big).$ We set

$$
D_n(f, x) := D_n(f, x_1, \dots, x_N) := \sum_{k=-\infty}^{\infty} \delta_{nk}(f) Z(nx - k) = \qquad (40)
$$

$$
\sum_{k=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \left(\frac{N}{n-k} \right)
$$

$$
\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\dots\sum_{k_N=-\infty}^{\infty}\delta_{n,k_1,k_2,\dots,k_N}\left(f\right)\left(\prod_{i=1}^NW\left(nx_i-k_i\right)\right),\,
$$

 $\forall x \in \mathbb{R}^N.$

In this article we study the approximation properties of A_n, B_n, C_n, D_n neural network operators and as well of their iterates. That is, the quantitative pointwise and uniform convergence of these operators to the unit operator I.

3 Multivariate general Neural Network Approximations

Here we present several vectorial neural network approximations to Banach space valued functions given with rates.

We give

Theorem 8 Let $f \in C\left(\prod_{i=1}^N [a_i, b_i], X\right),\ 0 < \beta < 1,\ x \in \left(\prod_{i=1}^N [a_i, b_i]\right),$ $m, N, n \in \mathbb{N}$ with $n^{1-\beta} > 2$. Then 1)

$$
\left\| A_n \left(f, x \right) - f \left(x \right) \right\|_{\gamma} \le \left(4.824 \right)^N \left[\omega_1 \left(f, \frac{1}{n^{\beta}} \right) + \frac{4e^2 \left\| \| f \|_{\gamma} \right\|_{\infty}}{\pi e^{n^{1-\beta}}} \right] =: \lambda_1 \left(n \right),\tag{41}
$$

and

2)

$$
\left\| \left\| A_n \left(f \right) - f \right\|_{\gamma} \right\|_{\infty} \leq \lambda_1 \left(n \right). \tag{42}
$$

We notice that $\lim_{n\to\infty} A_n(f) \stackrel{\|\cdot\|_{\mathcal{A}}}{=} f$, pointwise and uniformly. Above ω_1 is with respect to $p = \infty$.

Proof. We observe that

3. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\n
$$
\forall x \in \mathbb{R}^N.
$$
\nIn this article we study the approximation properties of *A_n, D_n, C_n, D_n*
\nment network operators and as well of their iterates. That is, the quantitative
\npointwise and uniform convergence of these operators to the unit operator *I*.
\n3. Multivariate general Neural Network Approx-
\nimations
\nHere we present several vectorial neural network approximations to Banach
\nspace valued functions given with rates.
\nWe give
\n**Therorm S** Let $f \in C \left(\prod_{i=1}^N |a_i, b_i|, X \right), 0 < \beta < 1, x \in \left(\prod_{i=1}^N |a_i, b_i| \right),$
\n $m, N, n \in \mathbb{N}$ with $n^{1-\beta} > 2$. Then
\n
$$
||A_n(f, x) - f(x)||_{\gamma} \le (4.824)^N \left[\omega_1 \left(f, \frac{1}{n^{\beta}} \right) + \frac{4n^2}{\pi e^{n^2}} \right] ||f||_{\gamma}||_{\infty} \right] =: \lambda_1(n),
$$
\nand
\n
$$
||A_n(f) - f||_{\gamma} ||_{\infty} \le \lambda_1(n).
$$
\n(42)
\nWe notice that $\lim_{A \to 0} A_0(f) \frac{||\cdot||_{\gamma}}{||\cdot||_{\gamma}}$, pointwise and uniformly.
\nAbove ω_1 is valid, respect to $p = \infty$.
\nProof. We observe that
\n
$$
\Delta(x) := A_n^*(f, x) - f(x) \sum_{k=\lceil na \rceil}^{[nb]} f(x) Z(nx - k) =
$$
\n
$$
\sum_{k=\lceil na \rceil}^{[nb]} \left(f \left(\frac{k}{n} \right) Z(nx - k) - \sum_{k=\lceil na \rceil}^{[nb]} f(x) Z(nx - k) \right)
$$
\nThus
\n
$$
||\Delta(x)||_{\gamma} \le \sum_{k=\lceil na \rceil}^{[nb]} |f \left(\frac{k}{n} \right) - f(x) ||_{\gamma} Z(nx - k) =
$$
\n10
\n566
\n660, A Antstassion 587-82

Thus

$$
\left\|\Delta\left(x\right)\right\|_{\gamma} \leq \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\|f\left(\frac{k}{n}\right) - f\left(x\right)\right\|_{\gamma} Z\left(nx-k\right) =
$$

3. COMPUTATIONAL ANALYSIS AND APPLICATIONS, Vol. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\n
$$
\begin{aligned}\n&\left\{\begin{aligned}\n&\left\|\frac{k}{n}-x\right\|_{\infty} &\leq \frac{1}{n^{\beta}} \\
&\left\|\frac{k}{n}-x\right\|_{\infty} &\leq \frac{1}{n^{\beta}}\n\end{aligned}\right\|f\left(\frac{k}{n}\right)-f(x)\left\|_{\gamma} Z(nx-k)\right\| \leq \\
&\left\{\begin{aligned}\n&\left\|\frac{k}{n}-x\right\|_{\infty} &\leq \frac{1}{n^{\beta}} \\
&\left\|\frac{k}{n}-x\right\|_{\infty} &\geq \frac{1}{n^{\beta}}\n\end{aligned}\right\} \left\{f\left(\frac{k}{n}\right)-f(x)\left\|\frac{2}{n}\left\langle nx-k\right\rangle\right\| \leq \\
&\left\{\begin{aligned}\n&\left\|\frac{k}{n}-x\right\|_{\infty} &\geq \frac{1}{n^{\beta}} \\
&\left\|\frac{k}{n}-x\right\|_{\infty} &\geq \frac{1}{n^{\beta}}\n\end{aligned}\right\} \left\{ \begin{aligned}\n&\left\|\frac{k}{n}-x\right\|_{\infty} &\geq \frac{1}{n^{\beta}} \\
&\leq \left\|\frac{k}{n^{\beta}}-x\right\|_{\infty} &\geq \frac{1}{n^{\beta}}\n\end{aligned}\right\}. \tag{44}\right\} \text{So that} \\
&\left\|\Delta(x)\right\|_{\gamma} \leq \omega_{1} \left(f, \frac{1}{n^{\beta}}\right) + \frac{4e^{2}}{n e^{n^{\beta}-n}} \left\|\frac{f\|\mathbf{1}\|_{\gamma}}{n e^{n^{\beta}-n}}\right\|_{\infty}. \tag{45}\n\nA formula for $||\mathbf{1}||_{\infty} \leq \omega_{1} ||\mathbf{1}||_{\infty} ||\mathbf{1}||_{\infty} \leq \omega_{2} ||\mathbf{1}||_{\infty} ||\mathbf{1}||_{\infty} \text{ and } \mathbf{1} ||\mathbf{1}||_{\infty} \leq \omega_{2} ||\mathbf{1}||_{\infty} \text{ and$
$$

So that

$$
\left\|\Delta\left(x\right)\right\|_{\gamma} \leq \omega_1 \left(f, \frac{1}{n^{\beta}}\right) + \frac{4e^2 \left\|\|f\|_{\gamma}\right\|_{\infty}}{\pi e^{n^{1-\beta}}}.
$$
\n(45)

 $\overline{11}$

 $\overline{11}$

Now using (34) we finish the proof. \blacksquare

We make

Remark 9 ([11], pp. 263-266) Let $(\mathbb{R}^N, \|\cdot\|_p)$, $N \in \mathbb{N}$; where $\|\cdot\|_p$ is the L_p norm, $1 \leq p \leq \infty$. \mathbb{R}^N is a Banach space, and $(\mathbb{R}^N)^j$ denotes the j-fold product space $\mathbb{R}^N \times \ldots \times \mathbb{R}^N$ endowed with the max-norm $||x||_{(\mathbb{R}^N)^j} := \max_{1 \leq \lambda \leq \lambda}$ $\max_{1 \leq \lambda \leq j} ||x_{\lambda}||_p$, where $x := (x_1, ..., x_j) \in (\mathbb{R}^N)^j$.

Let $\left(X,\left\|\cdot\right\|_{\gamma}\right)$ be a general Banach space. Then the space $L_j:=L_j\left(\left(\mathbb{R}^N\right)^j;X\right)$ of all j-multilinear continuous maps $g: (\mathbb{R}^N)^j \to X$, $j = 1, ..., m$, is a Banach space with norm

$$
\|g\| := \|g\|_{L_j} := \sup_{\left(\|x\|_{\left(\mathbb{R}^N\right)^j} = 1\right)} \|g\left(x\right)\|_{\gamma} = \sup \frac{\|g\left(x\right)\|_{\gamma}}{\|x_1\|_{p} \dots \|x_j\|_{p}}. \tag{46}
$$

Let M be a non-empty convex and compact subset of \mathbb{R}^N and $x_0 \in M$ is fixed.

Let O be an open subset of \mathbb{R}^N : $M \subset O$. Let $f: O \to X$ be a continuous function, whose Fréchet derivatives (see [23]) $f^{(j)}: O \to L_j = L_j \left((\mathbb{R}^N)^j; X \right)$ exist and are continuous for $1 \le j \le m$, $m \in \mathbb{N}$.

Call $(x - x_0)^j := (x - x_0, ..., x - x_0) \in (\mathbb{R}^N)^j, x \in M.$

We will work with $f|_M$. Then, by Taylor's formula $([16]), ([23], p. 124),$ we get

$$
f(x) = \sum_{j=0}^{m} \frac{f^{(j)}(x_0)(x - x_0)^j}{j!} + R_m(x, x_0), \quad all \ x \in M,
$$
 (47)

where the remainder is the Riemann integral

$$
R_m(x, x_0) := \int_0^1 \frac{(1-u)^{m-1}}{(m-1)!} \left(f^{(m)}(x_0 + u(x - x_0)) - f^{(m)}(x_0) \right) (x - x_0)^m du,
$$
\n(48)

here we set $f^{(0)}(x_0)(x-x_0)^0 = f(x_0)$. We consider

$$
w := \omega_1 \left(f^{(m)}, h \right) := \sup_{\substack{x, y \in M:\\ \|x - y\|_p \le h}} \left\| f^{(m)}(x) - f^{(m)}(y) \right\|, \tag{49}
$$

 $h > 0.$

We obtain

3. COMPUTATIONAL ANALYSIS AND APPLICATIONS. VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\nWe will work with
$$
f|_M
$$
.
\nThen, by Taylor's formula (116)), (123), p. 124), we get
\n
$$
f(x) = \sum_{j=0}^{m} \frac{f^{(j)}(x_0) (x-x_0)^j}{j!} + R_m(x,x_0), \quad all \ x \in M,
$$
\n(47)
\nwhere the remainder is the Riemann integral
\n
$$
R_m(x,x_0) := \int_0^1 \frac{(1-u)^{m-1}}{(m-1)!} (f^{(m)}(x_0 + u (x-x_0)) - f^{(m)}(x_0)) (x-x_0)^m dx,
$$
\n(48)
\nherez we set $f^{(0)}(x_0)(x-x_0)^0 = f(x_0)$.
\nWe consider
\n
$$
w := \omega_1 (f^{(m)}, h) := \sup_{x,y \in M_1^*} ||f^{(m)}(x) - f^{(m)}(y)||,
$$
\n(49)
\nh
$$
= \sum_{|x-x|_0 \le 2^k} |f^{(m)}(x_0 + u (x-x_0)) - f^{(m)}(x_0)|| ||z - x_0||^m \le
$$
\n
$$
= \left\| f^{(m)}(x_0 + u (x-x_0)) - f^{(m)}(x_0)|| ||z - x_0||^m \right\|
$$
\n(50)
\nby Lemma 7.1.1, if, p. 208, where [1, is the *exists*],
\n
$$
||h_m(x,x_0)||_x \le m ||x - x_0||_p^m \int_0^1 \frac{u ||x - x_0||_p}{h} \Big|.
$$
\n(50)
\n
$$
|H_m(x,x_0)||_x \le m ||x - x_0||_p^m \int_0^1 \frac{u ||x - x_0||_p}{h} \Big|_x^1 \frac{(1 - u)^{m-1}}{(m-1)!} du
$$
\n
$$
= w \Phi_m (||x - x_0||_p)
$$
\nby a change of variable, where
\n
$$
\Phi_m(t) := \int_0^t \left| \frac{K}{h} \left| \frac{((k-x)^{m-1}}{(m-1)!} ds
$$

by Lemma 7.1.1, [1], p. 208, where $\lceil \cdot \rceil$ is the ceiling. Therefore for all $x \in M$ (see [1], pp. 121-122):

$$
\|R_m(x, x_0)\|_{\gamma} \le w \|x - x_0\|_p^m \int_0^1 \left[\frac{u \|x - x_0\|_p}{h} \right] \frac{(1 - u)^{m-1}}{(m-1)!} du
$$

$$
= w \Phi_m \left(\|x - x_0\|_p \right) \tag{51}
$$

by a change of variable, where

$$
\Phi_m(t) := \int_0^{|t|} \left[\frac{s}{h} \right] \frac{(|t| - s)^{m-1}}{(m-1)!} ds = \frac{1}{m!} \left(\sum_{j=0}^{\infty} (|t| - jh)_+^m \right), \ \ \forall \ t \in \mathbb{R}, \tag{52}
$$

is a (polynomial) spline function, see $[1]$, p. 210-211.

Also from there we get

$$
\Phi_m(t) \le \left(\frac{|t|^{m+1}}{(m+1)!h} + \frac{|t|^m}{2m!} + \frac{h\,|t|^{m-1}}{8\,(m-1)!} \right), \quad \forall \ t \in \mathbb{R},\tag{53}
$$

with equality true only at $t = 0$. Therefore it holds

$$
\left\|R_m\left(x,x_0\right)\right\|_{\gamma} \le w \left(\frac{\left\|x-x_0\right\|_p^{m+1}}{(m+1)!h} + \frac{\left\|x-x_0\right\|_p^m}{2m!} + \frac{h\left\|x-x_0\right\|_p^{m-1}}{8\left(m-1\right)!}\right), \quad \forall \ x \in M. \tag{54}
$$

We have found that

3. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\n*minb* equality true only at t = 0.
\nTherefore it holds
\n
$$
||R_{vn}(x, x_0)||_y \leq w \left(\frac{||x - x_0||_p^{m+1}}{(m+1)!h} + \frac{||x - x_0||_p^{m}}{2ml} + \frac{h||x - x_0||_p^{m-1}}{8(m-1)!} \right), \forall x \in M.
$$
\n(54)
\nWe have found that
\n
$$
||f(x) = \sum_{j=0}^{n} \frac{f^{(j)}(x_0) (x - x_0)^j}{j!} \Big|_y \leq
$$
\n
$$
\omega_1 \left(f^{(m)}, h \right) \left(\frac{||x - x_0||_p^{m+1}}{(m+1)!h} + \frac{||x - x_0||_p^{m}}{2m!} + \frac{h||x - x_0||_p^{m-1}}{8(m-1)!} \right) < \infty,
$$
\n(55)
\n
$$
\forall x, x_0 \in M.
$$
\n*Here* 0 $\lt \sim x_1 (f^{(m)}, h) < \infty$, ∞ , by M being compared and $f^{(m)}$ being continuous
\non M.
\n*One can rewrite* (55) as follows:
\n
$$
||f(x) = \sum_{j=0}^{m} \frac{f^{(j)}(x_0) (x - x_0)^j}{j!} \Big|_x \leq
$$
\n
$$
\omega_1 \left(f^{(m)}, h \right) \left(\frac{||x - x_0||_p^{m+1}}{(m+1)!h} + \frac{||x - x_0||_p^{m}}{2m!} + \frac{h||x - x_0||_p^{m-1}}{8(m-1)!} \right), \forall x_0 \in M,
$$
\n(56)
\na pointwise functional inequality on M.
\n*Here* $(x - x_0)^j$ maps M into $f^{(m)}$ and it is continuous.
\n($\forall x$)^j and X and it is continuous.
\n
$$
|F(x) = \frac{e^{-e^{-e^{\lambda t}}}}{e^{-e^{\lambda t}}}
$$
\n*Let*

 $\forall x, x_0 \in M.$

Here $0 < \omega_1(f^{(m)}, h) < \infty$, by M being compact and $f^{(m)}$ being continuous on M.

One can rewrite (55) as follows:

$$
\left\| f(\cdot) - \sum_{j=0}^{m} \frac{f^{(j)}(x_0) (\cdot - x_0)^j}{j!} \right\|_{\gamma} \le
$$

$$
\omega_1 \left(f^{(m)}, h \right) \left(\frac{\left\| \cdot - x_0 \right\|_p^{m+1}}{(m+1)!h} + \frac{\left\| \cdot - x_0 \right\|_p^m}{2m!} + \frac{h \left\| \cdot - x_0 \right\|_p^{m-1}}{8(m-1)!} \right), \ \forall \ x_0 \in M, \ (56)
$$

a pointwise functional inequality on M.

Here $(-x_0)^j$ maps M into $(\mathbb{R}^N)^j$ and it is continuous, also $f^{(j)}(x_0)$ maps $(\mathbb{R}^N)^j$ into X and it is continuous. Hence their composition $f^{(j)}(x_0)(-x_0)^j$ is continuous from M into X.

Clearly
$$
f(\cdot) - \sum_{j=0}^{m} \frac{f^{(j)}(x_0)(\cdot - x_0)^j}{j!} \in C(M, X)
$$
, hence $||f(\cdot) - \sum_{j=0}^{m} \frac{f^{(j)}(x_0)(\cdot - x_0)^j}{j!}||_{\gamma} \in C(M)$.

Let $\left\{ \widetilde{L}_{N}\right\}$ $N\in\mathbb{N}$ be a sequence of positive linear operators mapping $C(M)$ into $C(M)$.

Therefore we obtain

$$
\left(\widetilde{L}_N\left(\left\|f\left(\cdot\right)-\sum_{j=0}^m\frac{f^{(j)}\left(x_0\right)\left(\cdot-x_0\right)^j}{j!}\right\|_{\gamma}\right)\right)(x_0) \le
$$

$$
\omega_1\left(f^{(m)},h\right)\left[\frac{\left(\widetilde{L}_N\left(\left\|\cdot-x_0\right\|_p^{m+1}\right)\right)(x_0)}{(m+1)!h}+\frac{\left(\widetilde{L}_N\left(\left\|\cdot-x_0\right\|_p^m\right)\right)(x_0)}{2m!}\right)
$$

$$
+\frac{h\left(\widetilde{L}_N\left(\left\|\cdot-x_0\right\|_p^{m-1}\right)\right)(x_0)}{8\left(m-1\right)!}\right],\tag{57}
$$

 $\forall N \in \mathbb{N}, \forall x_0 \in M.$

Clearly (57) is valid when $M = \prod_{i=1}^{N}$ $\prod_{i=1} [a_i, b_i]$ and $L_n = A_n$, see (26).

All the above is preparation for the following theorem, where we assume Fréchet differentiability of functions.

This will be a direct application of Theorem 10.2, [11], pp. 268-270. The operators A_n , A_n fulfill its assumptions, see (25), (26), (28), (29) and (30).

We present the following high order approximation results.

Theorem 10 Let O open subset of $(\mathbb{R}^N, \|\cdot\|_p)$, $p \in [1, \infty]$, such that $\prod_{i=1}^N$ $\prod_{i=1} [a_i, b_i] \subset$ $O\subseteq\mathbb{R}^N$, and let $\Big(X,\left\|\cdot\right\|_{\gamma}\Big)$ be a general Banach space. Let $m\in\mathbb{N}$ and $f\in$ $C^m(0,X)$, the space of \overline{m} -times continuously Fréchet differentiable functions from O into X. We study the approximation of $f|_{\prod_{i=1}^{N}[a_i,b_i]}$. Let $x_0 \in \left(\prod_{i=1}^{N}\right)$ $i=1$ $\prod_{i=1} [a_i, b_i]$ \setminus

and $r > 0$. Then 1)

3. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\n
$$
+ \frac{h(\tilde{L}_{N}([||-x_{0}||_{T}^{m-1})))(x_{0})}{8(m-1)!},
$$
\n(57)
\n $\forall N \in \mathbb{N}, \forall x_{0} \in M.$
\nClearly (57) is valid when $M = \prod_{i=1}^{N} [a_{i}, b_{i}]$ and $\tilde{L}_{n} = \tilde{A}_{n}$, see (26).
\nAll the above is preparation for the following theorem, where we assume
\nFriedet differentiability of functions.
\nThis will be a direct application of Theorem 10.2, [11], pp. 268-270. The
\noperators A_{n} , \tilde{A}_{n} fulfill its assumption sets.
\nWe present the following high order approximation results.
\nTheorem 10 Let *O* open subset of $[\mathbb{R}^{N}, ||||_{p}]$, $p \in [1, \infty]$, such that $\prod_{i=1}^{N} [a_{i}, b_{i}] \subset$
\n*O* ⊆ R^N, and let $(X, ||||_{\infty})$ be a general Banach space. Let $m \in \mathbb{N}$ and f
\n $C^{m}(O, X)$, the space of m -times continuously Fréchet differentiable functions
\nfrom *O* into *X*. We study the approximation of $f\Big|_{\tilde{A}_{n}}^{X}[a_{i}, b_{i}]$. Let $x_{0} \in (\prod_{i=1}^{N} [a_{i}, b_{i}])$
\nand $r > 0$. Then
\n
$$
\left|\begin{pmatrix} A_{n}(f))(x_{0}) - \sum_{j=0}^{m} \frac{1}{j!} (A_{n}(f^{(j)}(x_{0})(-x_{0})^{j})) (x_{0}) \end{pmatrix}\right|_{\infty} \le
$$
\n
$$
\frac{\omega_{1} (f^{(m)}, r((\tilde{A}_{n}([||-x_{0}||_{T}^{m+1}))(x_{0}))^{\frac{1}{m+1}})}{rm!} ((\tilde{A}_{n}([||-x_{0}||_{T}^{m+1}))(x_{0}))^{\frac{1
$$

2) additionally if $f^{(j)}(x_0) = 0, j = 1, ..., \overline{m}$, we have

$$
\left\|\left(A_n\left(f\right)\right)(x_0)-f\left(x_0\right)\right\|_{\gamma}\le
$$

 $n+1$

$$
\frac{\omega_1\left(f^{(m)}, r\left(\left(\widetilde{A}_n\left(\|\cdot-x_0\|_p^{m+1}\right)\right)(x_0)\right)^{\frac{1}{m+1}}\right)}{rm!}\left(\left(\widetilde{A}_n\left(\|\cdot-x_0\|_p^{m+1}\right)\right)(x_0)\right)^{\left(\frac{m}{m+1}\right)}\tag{59}
$$

3)

$$
\| (A_n(f)) (x_0) - f (x_0) \|_{\gamma} \leq \sum_{j=1}^m \frac{1}{j!} \left\| \left(A_n \left(f^{(j)} (x_0) \left(\cdot - x_0 \right)^j \right) \right) (x_0) \right\|_{\gamma} +
$$

$$
\frac{\omega_1\left(f^{(m)}, r\left(\left(\widetilde{A}_n\left(\left\|\cdot-x_0\right\|_p^{m+1}\right)\right)(x_0)\right)^{\frac{1}{m+1}}\right)}{rm!}\left(\left(\widetilde{A}_n\left(\left\|\cdot-x_0\right\|_p^{m+1}\right)\right)(x_0)\right)^{\left(\frac{m}{m+1}\right)}\right)
$$
\n
$$
\left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8}\right],\tag{60}
$$

and 4)

3. COMPUTATORAL ANALYSIS AND APPLICATIONS. VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\n
$$
\omega_1 \left(f^{(m)}, r \left(\left(\tilde{A}_n \left(\left\| \cdot - x_0 \right\|_p^{m+1} \right) \right) (x_0) \right)^{\frac{1}{m+1}} \right) \left(\left(\tilde{A}_n \left(\left\| \cdot - x_0 \right\|_p^{m+1} \right) \right) (x_0) \right)^{\left(\frac{m}{m+1} \right)} \right)
$$
\n
$$
\begin{aligned}\n\text{and} \\
\downarrow \left\{\n\begin{aligned}\n\text{and} \\
\text{and} \\
\text
$$

We need

Lemma 11 The function $\left(\widetilde{A}_n\left(\left\|\cdot-x_0\right\|_p^m\right)\right)(x_0)$ is continuous in $x_0 \in \left(\prod_{i=1}^N\right)$ $\prod_{i=1} [a_i, b_i]$ $\overline{ }$, $m \in \mathbb{N}$.

Proof. By Lemma 10.3, [11], p. 272. ■ We give

Corollary 12 (to Theorem 10, case of $m = 1$) Then 1)

$$
\left\| \left(A_n\left(f\right)\right)(x_0) - f\left(x_0\right) \right\|_{\gamma} \le \left\| \left(A_n\left(f^{(1)}\left(x_0\right)(\cdot - x_0)\right)\right)(x_0) \right\|_{\gamma} +
$$

$$
\frac{1}{2r}\omega_1\left(f^{(1)}, r\left(\left(\widetilde{A}_n\left(\left\|\cdot - x_0\right\|_p^2\right)\right)(x_0)\right)^{\frac{1}{2}}\right) \left(\left(\widetilde{A}_n\left(\left\|\cdot - x_0\right\|_p^2\right)\right)(x_0)\right)^{\frac{1}{2}} \quad (62)
$$

$$
\left[1 + r + \frac{r^2}{4}\right],
$$

and

2)

$$
\left\| \left\| (A_n(f)) - f \right\|_{\gamma} \right\|_{\infty, \prod_{i=1}^{N} [a_i, b_i]} \le
$$

$$
\left\| \left\| \left(A_n\left(f^{(1)}\left(x_0 \right) (\cdot - x_0) \right) \right) (x_0) \right\|_{\gamma} \right\|_{\infty, x_0 \in \prod_{i=1}^{N} [a_i, b_i]} +
$$

$$
\frac{1}{2r} \omega_1 \left(f^{(1)}, r \left\| \left(\widetilde{A}_n \left(\left\| \cdot - x_0 \right\|_p^2 \right) \right) (x_0) \right\|_{\infty, x_0 \in \prod_{i=1}^{N} [a_i, b_i]}^{\frac{1}{2}} \right)
$$

$$
\left\| \left(\widetilde{A}_n \left(\left\| \cdot - x_0 \right\|_p^2 \right) \right) (x_0) \right\|_{\infty, x_0 \in \prod_{i=1}^{N} [a_i, b_i]}^{\frac{1}{2}} \left[1 + r + \frac{r^2}{4} \right],
$$
 (63)

 $r > 0.$

We make

Remark 13 We estimate $0 < \alpha < 1$, $m, n \in \mathbb{N} : n^{1-\alpha} > 2$,

3. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\n
$$
\left\| \left\| \left(A_n \left(f^{(1)}(x_0) \left(\cdot - x_0 \right) \right) \right) (x_0) \right\|_{\infty} \right\|_{\infty, \sum_{i=1}^N [n_i, b_i]} \le
$$
\n
$$
\left\| \left| \left(A_n \left(f^{(1)}(x_0) \left(\cdot - x_0 \right) \right) \right) (x_0) \right\|_{\infty, \max} \in \prod_{i=1}^N [n_i, b_i] + \frac{1}{2r} \omega_1 \left(f^{(1)}, r \left\| \left(\tilde{A}_n \left(\left\| \cdot - x_0 \right\|_p^2 \right) \right) (x_0) \right\|_{\infty, \max \in \prod_{i=1}^N [n_i, b_i]}^2 \right)
$$
\n
$$
\left\| \left(\tilde{A}_n \left(\left\| \cdot - x_0 \right\|_p^2 \right) \right) (x_0) \right\|_{\infty, \max \in \prod_{i=1}^N [n_i, b_i]} \left[1 + r + \frac{r^2}{4} \right], \qquad (63)
$$
\n
$$
r > 0.
$$
\nWe make\n
\nRemark 13 We estimate $0 < \alpha < 1, m, n \in \mathbb{N} : n^{1-\alpha} > 2$,\n
$$
\tilde{A}_n \left(\left\| \cdot - x_0 \right\|_{\infty}^{m+1} \right) (x_0) = \frac{\sum_{k=1}^{\lfloor n \rfloor} \omega_1 \left\| \frac{k}{n} - x_0 \right\|_{\infty}^{m+1} Z \left(nx_0 - k \right)}{ \sum_{k=1}^{\lfloor n \rfloor} \omega_1 \left\| \frac{n+1}{2} \left(nx_0 - k \right) - \frac{1}{2} \left(64 \right) \right\|_{\infty}^2} \right\}
$$
\n
$$
(4.824)^N \left\{ \left\{ \left\| \frac{k}{n} - x_0 \right\|_{\infty}^{m+1} Z \left(nx_0 - k \right) \right\} \right\}
$$

(where $b - a = (b_1 - a_1, ..., b_N - a_N)$)

We have proved that
$$
(\forall x_0 \in \prod_{i=1}^N [a_i, b_i])
$$

$$
\widetilde{A}_n \left(\left\| \cdot - x_0 \right\|_{\infty}^{m+1} \right) (x_0) < (4.824)^N \left\{ \frac{1}{n^{\alpha(m+1)}} + \frac{2e^2 \left\| b - a \right\|_{\infty}^{m+1}}{\pi e^{n^{1-\alpha}}} \right\} =: \varphi_1(n) \tag{66}
$$

 $(0 < \alpha < 1, m, n \in \mathbb{N} : n^{1-\alpha} > 2).$

And, consequently it holds

$$
\left\|\widetilde{A}_n\left(\|\cdot-x_0\|_{\infty}^{m+1}\right)(x_0)\right\|_{\infty,x_0\in \prod\limits_{i=1}^N[a_i,b_i]} <
$$

$$
(4.824)^N \left\{ \frac{1}{n^{\alpha(m+1)}} + \frac{2e^2 \left\| b - a \right\|_{\infty}^{m+1}}{\pi e^{n^{1-\alpha}}} \right\} = \varphi_1(n) \to 0, \text{ as } n \to +\infty. \tag{67}
$$

So, we have that $\varphi_1(n) \to 0$, as $n \to +\infty$. Thus, when $p \in [1,\infty]$, from Theorem 10 we have the convergence to zero in the right hand sides of parts (1), (2).

Next we estimate \parallel $\left(\widetilde{A}_n\left(f^{(j)}\left(x_0\right)\left(\cdot-x_0\right)^j\right)\right)(x_0)\right\|_{\gamma}.$ We have that

$$
\left(\widetilde{A}_{n}\left(f^{(j)}\left(x_{0}\right)\left(\cdot-x_{0}\right)^{j}\right)\right)\left(x_{0}\right)=\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}f^{(j)}\left(x_{0}\right)\left(\frac{k}{n}-x_{0}\right)^{j}Z\left(nx_{0}-k\right)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}Z\left(nx_{0}-k\right)}.
$$
\n(68)

When $p = \infty$, $j = 1, ..., m$, we obtain

$$
\left\| f^{(j)}\left(x_0\right) \left(\frac{k}{n} - x_0\right)^j \right\|_{\gamma} \le \left\| f^{(j)}\left(x_0\right) \right\| \left\| \frac{k}{n} - x_0 \right\|_{\infty}^j. \tag{69}
$$

We further have that

J. COMPUTATORAL ANALYSIS AND APPLICATIONS, VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\nWe have proved that
$$
(\forall x_0 \in \prod_{i=1}^{N} [a_i, b_i])
$$

\n
$$
\widetilde{A}_0([1-x_0]\max_{i=1}^{m+1})(x_0) \le (4.824)^N \left\{ \frac{1}{n^{\alpha(n+1)}} + \frac{2e^2 ||b-a||_{\infty}^{m+1}}{n^{\alpha n^{1-\alpha}}} \right\} =: \varphi_1(n)
$$
\n
$$
(0 < a < 1, m, n \in \mathbb{N}: n^{1-\alpha} > 2).
$$
\nAnd, consequently it holds
\n
$$
||\widetilde{A}_n([1-x_0]\max_{i=1}^{m+1})(x_0)||_{\infty,\alpha \in \prod_{i=1}^{N} [a_i, b_i]} \le
$$
\n
$$
(4.824)^N \left\{ \frac{1}{n^{\alpha(n+1)}} + \frac{2e^2 ||b-a||_{\infty}^{m+1}}{n^{\alpha(n+1)}} \right\} = \varphi_1(n) \to 0, \text{ as } n \to +\infty. \text{ (67)}
$$
\nSo, we have that $\varphi_1(n) \to 0, a \in n \to +\infty$. Thus, when $p \in [1, \infty]$, from Theorem 10 we have the convergence to zero in the right hand side of parts (1),
\n(2).
\nNext we estimate $\left\| \left(\widetilde{A}_n \left(f^{(j)}(x_0)(-x_0)^j \right) \right)(x_0) \right\|_2.$
\nWe have that
\n
$$
\left(\widetilde{A}_n \left(f^{(j)}(x_0)(-x_0)^j \right) \right) (x_0) = \frac{\sum_{k=1}^{N} [a_k]}{n^{\alpha(n)}} \frac{f^{(j)}(x_0)(\frac{k}{n} - x_0)^j}{2^{\alpha(n)}} \frac{Z(x_0 - k)}{Z(x_0 - k)}.
$$
\n(88)
\nWhen $p = \infty, j = 1, ..., m$, we obtain
\n
$$
\left\| \left| f^{(j)}(x_0)(\frac{k}{n} - x_0)^j \right|_2 \le \left\| f^{(j)}
$$

3. COMPUTATIONAL ANALYSIS AND APPLICATIONS. VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\n
$$
(4.824)^N ||f^{(j)}(x_0)|| \left\{ \begin{aligned} &\sum_{k=-\lceil n/a \rceil}^{n b \rceil} \left\| \frac{k}{n} - x_0 \right\|_{\infty}^{j} Z \left(nx_0 - k \right) \\ &+ \sum_{k=-\lceil n/a \rceil}^{n b \rceil} \left\| \frac{k}{n} - x_0 \right\|_{\infty}^{j} Z \left(nx_0 - k \right) \right\}^{(2j)} \leq (71) \\ &+ \sum_{k=-\lceil n/a \rceil}^{k b \rceil} \left\| \frac{k}{n} - x_0 \right\|_{\infty}^{j} Z \left(nx_0 - k \right) \right\}^{(2j)} \leq (71) \\ &+ \sum_{k=-\lceil n/a \rceil}^{k b \rceil} \left\| \frac{k}{n} - x_0 \right\|_{\infty}^{j} \left\| \frac{k}{n^2} - x_0 \right\|_{\infty}^{j} Z \left(nx_0 - k \right) \right\}^{(2j)} \leq (71) \\ &+ (4.824)^N ||f^{(j)}(x_0) \left(-x_0)^j \right) \left\| \left\| \left(\frac{1}{n^0} + \frac{2e^2 ||b-a||_{\infty}^j}{\pi e^{n^{1-\alpha}}} \right) - 0, \text{ as } n \to \infty. \end{aligned}
$$
\nTherefore when $p = \infty$, for $j = 1, ..., m$, we have proved:
\n
$$
|| \left(\tilde{A}_n \left(f^{(j)}(x_0) \left(-x_0 \right)^j \right) \left\| \left\| x_0 \right\| \right\|_{\infty} \n+ (4.824)^N ||f^{(j)}||_{\infty} \left\{ \frac{1}{n^{5/2}} + \frac{2e^2 ||b-a||_{\infty}^j}{\pi e^{n^{1-\alpha}}} \right\} =: \varphi_{2j}(n) < \infty,
$$
\nand converges to zero, as $n \to \infty$.
\nWe conclude:
\nIn Theorem 10, the right hand sides of (60) and (61) converge to zero as

That is

$$
\left\|\left(\widetilde{A}_n\left(f^{(j)}\left(x_0\right)(\cdot-x_0)^j\right)\right)(x_0)\right\|_{\gamma}\to 0, \text{ as } n\to\infty.
$$

Therefore when $p = \infty$, for $j = 1, ..., m$, we have proved:

7

$$
\left\| \left(\widetilde{A}_n \left(f^{(j)} \left(x_0 \right) (\cdot - x_0)^j \right) \right) (x_0) \right\|_{\gamma} <
$$
\n
$$
(4.824)^N \left\| f^{(j)} \left(x_0 \right) \right\| \left\{ \frac{1}{n^{\alpha j}} + \frac{2e^2 \left\| b - a \right\|_{\infty}^j}{\pi e^{n^{1-\alpha}}} \right\} \le
$$
\n
$$
(4.824)^N \left\| f^{(j)} \right\|_{\infty} \left\{ \frac{1}{n^{\alpha j}} + \frac{2e^2 \left\| b - a \right\|_{\infty}^j}{\pi e^{n^{1-\alpha}}} \right\} =: \varphi_{2j} \left(n \right) < \infty,
$$
\n
$$
(72)
$$

and converges to zero, as $n \to \infty$.

We conclude:

In Theorem 10, the right hand sides of (60) and (61) converge to zero as $n \to \infty$, for any $p \in [1,\infty]$.

Also in Corollary 12, the right hand sides of (62) and (63) converge to zero as $n \to \infty$, for any $p \in [1,\infty]$.

Conclusion 14 We have proved that the left hand sides of (58) , (59) , (60) , (61) and (62), (63) converge to zero as $n \to \infty$, for $p \in [1,\infty]$. Consequently $A_n \to I$ (unit operator) pointwise and uniformly, as $n \to \infty$, where $p \in [1,\infty]$. In the presence of initial conditions we achieve a higher speed of convergence, see (59). Higher speed of convergence happens also to the left hand side of (58).

We further give

Corollary 15 (to Theorem 10) Let O open subset of $(\mathbb{R}^N, \|\cdot\|_{\infty})$, such that \prod $\prod_{i=1}^N [a_i, b_i] \subset O \subseteq \mathbb{R}^N$, and let $(X, \|\cdot\|_{\gamma})$ be a general Banach space. Let $m \in \mathbb{N}$ and $f \in C^m (O, X)$, the space of \overline{m} -times continuously Fréchet differentiable functions from O into X. We study the approximation of $f|_{\prod_{i=1}^N [a_i, b_i]}$. Let $x_0 \in$ \setminus

 $\left(\begin{array}{c}N\\ \prod\end{array}\right)$ $\prod_{i=1} [a_i, b_i]$ and $r > 0$. Here $\varphi_1(n)$ as in (67) and $\varphi_{2j}(n)$ as in (72), where $n \in \mathbb{N} : n^{1-\alpha} > 2, 0 < \alpha < 1, j = 1, ..., m.$ Then

1)

$$
\left\| (A_n(f))(x_0) - \sum_{j=0}^m \frac{1}{j!} \left(A_n \left(f^{(j)}(x_0) \left(\cdot - x_0 \right)^j \right) \right) (x_0) \right\|_{\gamma} \le
$$

$$
\frac{\omega_1 \left(f^{(m)}, r \left(\varphi_1(n) \right)^{\frac{1}{m+1}} \right)}{rm!} \left(\varphi_1(n) \right)^{\left(\frac{m}{m+1} \right)} \left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8} \right], \qquad (73)
$$

2) additionally, if $f^{(j)}(x_0) = 0$, $j = 1, ..., \overline{m}$, we have

$$
\| (A_n(f)) (x_0) - f (x_0) \|_{\gamma} \le
$$

$$
\frac{\omega_1 \left(f^{(m)}, r (\varphi_1(n))^{\frac{1}{m+1}} \right)}{rm!} (\varphi_1(n))^{\left(\frac{m}{m+1}\right)} \left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8} \right],
$$
 (74)

3)

3. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\nCorollary 15 (to Theorem 10) Let O open subset of
$$
(\mathbb{R}^N, ||\cdot||_{\infty})
$$
, such that
\n $\sum_{i=1}^{N} [\alpha_i, b_i] \subset O \subseteq \mathbb{R}^N$, and let $(X, ||\cdot||_{\infty})$ be a general Banach space. Let $m \in \mathbb{N}$
\nand $f \in C^m (O, X)$, the space of \overline{m} -times continuously Prechet differentiable
\nfuncations from O into X. We study the approximation of f $\left| \prod_{i=1}^{\infty} [a_i, b_i] \right|$. Let $x_0 \in$
\n $\left| \prod_{i=1}^{\infty} [a_i, b_i] \right\rangle$ and $r > 0$. Here $\varphi_1(n)$ as in (07) and $\varphi_{2j}(n)$ as in (72), where
\n $n \in \mathbb{N}$: $n^{1-\alpha} > 2$, $0 < \alpha < 1$, $j = 1, ..., m$. Then
\n
$$
\left\| (A_n(f))(x_0) - \sum_{j=0}^m \frac{1}{j!} \left(A_n \left(f^{(j)}(x_0) \left(\cdot - x_0 \right)^j \right) \right) (x_0) \right\|_{\infty} \le
$$
\n
$$
\frac{\omega_1 \left(f^{(m)}, r(\varphi_1(n)) \frac{1}{m+1} \right)}{\pi m!} (\varphi_1(n)) \left(\frac{\omega_1}{m+1} \right) + \frac{r}{2} + \frac{mr^2}{8} \right],
$$
\n(73)
\n2) additionally, if $f^{(j)}(x_0) = 0$, $j = 1, ..., \overline{m}$, we have
\n $|(A_n(f))(x_0) - f(x_0)||_{\infty} \le$
\n
$$
\frac{\omega_1 \left(f^{(m)}, r(\varphi_1(n)) \frac{1}{m+1} \right)}{\pi n!} (\varphi_1(n)) \left(\frac{\omega_1}{m+1} \right) + \frac{r}{2} + \frac
$$

We continue with

Theorem 16 Let $f \in C_B(\mathbb{R}^N, X)$, $0 < \beta < 1$, $x \in \mathbb{R}^N$, $m, N, n \in \mathbb{N}$ with $n^{1-\beta} > 2$, ω_1 is for $p = \infty$. Then 1)

$$
\|B_n(f,x) - f(x)\|_{\gamma} \le \omega_1 \left(f, \frac{1}{n^{\beta}}\right) + \frac{4e^2 \left\|f\|_{\gamma}\right\|_{\infty}}{\pi e^{n^{1-\beta}}} =: \lambda_2(n),\qquad(76)
$$

2)

$$
\left\| \left\| B_n \left(f \right) - f \right\|_{\gamma} \right\|_{\infty} \leq \lambda_2 \left(n \right). \tag{77}
$$

Given that $f \in (C_U(\mathbb{R}^N, X) \cap C_B(\mathbb{R}^N, X)),$ we obtain $\lim_{n \to \infty} B_n(f) = f$, uniformly.

Proof. We have that

$$
B_n(f, x) - f(x) \stackrel{(16)}{=} \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) Z(nx - k) - f(x) \sum_{k=-\infty}^{\infty} Z(nx - k) = (78)
$$

$$
\sum_{k=-\infty}^{\infty} \left(f\left(\frac{k}{n}\right) - f(x) \right) Z(nx - k).
$$

Hence

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, Vol. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\n*Given that*
$$
f \in (C_U (\mathbb{R}^N, X) \cap C_B (\mathbb{R}^N, X)), we obtain $\lim_{n \to \infty} B_n(f) = f$, unit.
\n**Proof.** We have that
\n
$$
B_n(f, x) - f(x) \stackrel{(16)}{=} \sum_{k = -\infty}^{\infty} f\left(\frac{k}{n}\right) Z(nx - k) - f(x) \sum_{k = -\infty}^{\infty} Z(nx - k) = (78)
$$
\n
$$
\sum_{k = -\infty}^{\infty} \left(f\left(\frac{k}{n}\right) - f(x) \right) Z(nx - k).
$$
\nHence
\n
$$
||B_n(f, x) - f(x)||_2 \le \sum_{k = -\infty}^{\infty} ||f\left(\frac{k}{n}\right) - f(x)||_2 Z(nx - k) +
$$
\n
$$
\left\{ ||\frac{k}{n} - x||_{\infty} \le \frac{1}{n^{\sigma}} \right\| f\left(\frac{k}{n}\right) - f(x) \right\|_2 Z(nx - k) +
$$
\n
$$
\sum_{k = -\infty}^{\infty} ||f\left(\frac{k}{n}\right) - f(x)||_2 Z(nx - k) \stackrel{(16)}{\le}
$$
\n
$$
\left\{ \left\| \frac{k}{n} - x \right\|_{\infty} \le \frac{1}{n^{\sigma}} \right\}
$$
\n
$$
\omega_1 \left(f, \frac{1}{n^{\beta}} \right) + 2 ||||f||_2 \right\|_{\infty} \sum_{k = -\infty}^{\infty} Z(nx - k) \stackrel{(23)}{\le}
$$
\n
$$
\omega_1 \left(f, \frac{1}{n^{\beta}} \right) + 2 ||||f||_2 \right\|_{\infty} \sum_{k = -\infty}^{\infty} Z(nx - k) \stackrel{(33)}{\le}
$$
\n
$$
\omega_1 \left(f, \frac{1}{n^{\beta}} \right) + 2 ||||f||_2 \right\|_{\infty} \sum_{n = n^{1 - \beta}}^{\infty} Z(nx - k) \stackrel{(33)}{\le}
$$
\n
$$
\omega_1 \left(f, \
$$
$$

proving the claim. \quadblacksquare

We give

Theorem 17 Let $f \in C_B(\mathbb{R}^N, X)$, $0 < \beta < 1$, $x \in \mathbb{R}^N$, $m, N, n \in \mathbb{N}$ with $n^{1-\beta} > 2$, ω_1 is for $p = \infty$. Then 1)

$$
\|C_n(f, x) - f(x)\|_{\gamma} \le \omega_1 \left(f, \frac{1}{n} + \frac{1}{n^{\beta}}\right) + \frac{4e^2 \left\| \|f\|_{\gamma} \right\|_{\infty}}{\pi e^{n^{1-\beta}}} =: \lambda_3(n), \qquad (80)
$$

2)

$$
\left\| \left\| C_n \left(f \right) - f \right\|_{\gamma} \right\|_{\infty} \leq \lambda_3 \left(n \right). \tag{81}
$$

Given that $f \in (C_U(\mathbb{R}^N, X) \cap C_B(\mathbb{R}^N, X))$, we obtain $\lim_{n \to \infty} C_n(f) = f$, uniformly.

Proof. We notice that

$$
\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t) dt = \int_{\frac{k_1}{n}}^{\frac{k_1+1}{n}} \int_{\frac{k_2}{n}}^{\frac{k_2+1}{n}} \dots \int_{\frac{k_N}{n}}^{\frac{k_N+1}{n}} f(t_1, t_2, ..., t_N) dt_1 dt_2...dt_N =
$$

$$
\int_0^{\frac{1}{n}} \int_0^{\frac{1}{n}} \dots \int_0^{\frac{1}{n}} f\left(t_1 + \frac{k_1}{n}, t_2 + \frac{k_2}{n}, ..., t_N + \frac{k_N}{n}\right) dt_1...dt_N = \int_0^{\frac{1}{n}} f\left(t + \frac{k}{n}\right) dt.
$$
(82)

Thus it holds (by (38))

$$
C_n(f,x) = \sum_{k=-\infty}^{\infty} \left(n^N \int_0^{\frac{1}{n}} f\left(t + \frac{k}{n} \right) dt \right) Z\left(nx - k \right). \tag{83}
$$

We observe that

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUB PRESS, LLC
\nProof. We notice that
\n
$$
\int_{\frac{1}{6}}^{\frac{1}{6}+\frac{1}{2}} f(t) dt = \int_{\frac{1}{4}}^{\frac{1}{6}+\frac{1}{2}} \int_{\frac{1}{2}}^{\frac{1}{6}+\frac{1}{2}} ... \int_{\frac{1}{2}}^{\frac{1}{6}+\frac{1}{2}} f(t_1, t_2, ..., t_N) dt_1 dt_2...dt_N =
$$
\n
$$
\int_{0}^{\frac{1}{6}} \int_{0}^{\frac{1}{6}} ... \int_{0}^{\frac{1}{6}} f\left(t_1 + \frac{k_1}{n}, t_2 + \frac{k_2}{n}, ..., t_N + \frac{k_N}{n}\right) dt_1...dx_N = \int_{0}^{\frac{1}{6}} f\left(t + \frac{k}{n}\right) dt.
$$
\nThus it holds (by (38))
\n
$$
C_n \left(f, x\right) = \sum_{k=-\infty}^{\infty} \left(n^N \int_{0}^{\frac{1}{6}} f\left(t + \frac{k}{n}\right) dt \right) Z \left(nx - k \right).
$$
\n(83)
\nWe observe that
\n
$$
\left\| C_n \left(f, x \right) - f(x) \right\|_{\gamma} =
$$
\n
$$
\left\| \sum_{k=-\infty}^{\infty} \left(n^N \int_{0}^{\frac{1}{6}} f\left(t + \frac{k}{n}\right) dt \right) - f(x) \right) Z \left(nx - k \right) \right\|_{\gamma} =
$$
\n
$$
\left\| \sum_{k=-\infty}^{\infty} \left(n^N \int_{0}^{\frac{1}{6}} \left| f\left(t + \frac{k}{n}\right) - f(x) \right|_{\gamma} dt \right) Z \left(nx - k \right) \right\|_{\gamma} =
$$
\n
$$
\sum_{k=-\infty}^{\infty} \left(n^N \int_{0}^{\frac{1}{6}} \left| f\left(t + \frac{k}{n}\right) - f(x) \right|_{\gamma} dt \right) Z \left(nx - k \right) =
$$
\n
$$
\sum_{k=-\infty}^{\infty} \left(n^N \int_{0}^{\
$$

3. COMPUTATIONAL ANALYSIS AND APPLICATIONS, Vol. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\n
$$
2 ||||f||_2 ||_{\infty} \left(\sum_{\substack{k=-\infty \\ \left|\frac{k}{n} - x\right|_{\infty} \leq \frac{1}{n^3}} Z(|nx - k|) \right) \leq
$$
\n
$$
\omega_1 \left(f, \frac{1}{n} + \frac{1}{n^2} \right) + \frac{4e^2 |||f||_2 ||_{\infty}}{n e^{n^2 - r}}. \tag{85}
$$
\nproving the claim. ■
\nWe also present
\nTheorem 18 Let $f \in C_B (\mathbb{R}^N, X)$, $0 < \beta < 1$, $x \in \mathbb{R}^N$, $m, N, n \in \mathbb{N}$ with $n^{1-\beta} > 2$, ω_1 is f for $y = \infty$. Then
\n
$$
||D_{\alpha}(f, x) - f(x)||_2 \leq \omega_1 \left(f, \frac{1}{n} + \frac{1}{n^2} \right) + \frac{4e^2 |||f||_2 ||_{\infty}}{n e^{n^{1-\beta}}} = \lambda_1(n), \qquad \text{(86)}
$$
\n
$$
2)
$$
\n
$$
|||D_{\alpha}(f) - f||_2 ||_{\infty} \leq \lambda_4(n). \tag{87}
$$
\n
$$
Given that $f \in (C_U (\mathbb{R}^N, X) \cap C_D (\mathbb{R}^N, X)),$ we obtain $\lim_{n \to \infty} D_{\alpha}(f) = f,$
\n
$$
|D_{\alpha}(f, x) - f(x)||_2 \leq \lambda_4(n). \tag{87}
$$
\n
$$
Proof. Similar to the proof of Theorem 17, as such is omitted. ■\n
$$
For S. Similar to the proof of Theorem 18, as such is omitted.
$$
\n
$$
D_{\alpha} = 1
$$
\n
$$
P_{\alpha} = 1
$$
\n
$$
\int_{\alpha} B_{\alpha}(f, x) \cdot \int_{\alpha} C_{\alpha} \left[\frac{g_1}{g_2} \right]_{\alpha} f_1 \
$$
$$
$$

proving the claim.

We also present

Theorem 18 Let $f \in C_B(\mathbb{R}^N, X)$, $0 < \beta < 1$, $x \in \mathbb{R}^N$, $m, N, n \in \mathbb{N}$ with $n^{1-\beta} > 2$, ω_1 is for $p = \infty$. Then 1)

$$
\|D_n(f,x) - f(x)\|_{\gamma} \le \omega_1 \left(f, \frac{1}{n} + \frac{1}{n^{\beta}}\right) + \frac{4e^2 \left\| \|f\|_{\gamma} \right\|_{\infty}}{\pi e^{n^{1-\beta}}} = \lambda_4(n), \quad (86)
$$

2)

$$
\left\| \left\| D_n\left(f\right) - f \right\|_{\gamma} \right\|_{\infty} \leq \lambda_4(n). \tag{87}
$$

Given that $f \in (C_U(\mathbb{R}^N, X) \cap C_B(\mathbb{R}^N, X))$, we obtain $\lim_{n \to \infty} D_n(f) = f$, uniformly.

Proof. Similar to the proof of Theorem 17, as such is omitted. ■ We make

Definition 19 Let $f \in C_B(\mathbb{R}^N, X)$, $N \in \mathbb{N}$, where $(X, \left\| \cdot \right\|_{\gamma})$ is a Banach space. We define the general neural network operator

$$
F_n(f, x) := \sum_{k=-\infty}^{\infty} l_{nk}(f) Z(nx - k) =
$$

$$
\begin{cases} B_n(f, x), & \text{if } l_{nk}(f) = f\left(\frac{k}{n}\right), \\ C_n(f, x), & \text{if } l_{nk}(f) = n^N \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t) dt, \\ D_n(f, x), & \text{if } l_{nk}(f) = \delta_{nk}(f). \end{cases}
$$
(88)

Clearly $l_{nk} (f)$ is an X-valued bounded linear functional such that $||l_{nk} (f)||_{\gamma} \le$ $\left\| \left\| f \right\|_{\gamma} \right\|_{\infty}.$

Hence $F_n(f)$ is a bounded linear operator with $\left\| \|F_n(f)\|_{\gamma} \right\|_{\infty} \leq$ $\left\| \|f\|_{\gamma}\right\|_{\infty}$. We need
Theorem 20 Let $f \in C_B(\mathbb{R}^N, X)$, $N \ge 1$. Then $F_n(f) \in C_B(\mathbb{R}^N, X)$.

Proof. Lengthy and similar to the proof of Theorem 21 of [14], as such is omitted. \blacksquare

1. COMPUTATIONAL ANALYSIS AND APPLICATIONS, Vol. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC

\n2. Theorem 20 Let
$$
f \in C_B
$$
 (R^N, X), $N \geq 1$. Then $F_n(f) \in C_B$ (R^N, X).

\nProof. Lengthly and similar to the proof of Theorem 21 of [14], as such is omitted.

\n1. By (25) it is obvious that $\left\| \|A_n(f)\|_w \right\|_{\infty} \leq \left\| \|f\|_w \right\|_{\infty} < \infty$, and $A_n(f) \in C \left(\prod_{i=1}^N [a_i, b_i], X \right)$, given that $f \in C \left(\prod_{i=1}^N [a_i, b_i], X \right)$.

\n1. Clearly, the operators A_n, B_n, C_n, D_n .

\n2. Clearly, the operators A_n, B_n, C_n, D_n .

\n3. Let E_n and E_n and E_n are $\left\| \|L_n^k(f) \|_w \|_{\infty} \leq \| \|f\|_w \right\|_{\infty} < \mathbb{R} \mathbb{N}$, then the contradiction property.

\nAns. Also, we see that

\n3. Show, we have that

\n4. Show, we see that

\n5. Now, we have:

\n7. Show, we have:

\n7. Show, we have:

\n8. Show, we have:

\n9. Show, we have:

\n1. Show, we have:

\n

$$
\left\| \left\| L_n^2(f) \right\|_{\gamma} \right\|_{\infty} = \left\| \left\| L_n\left(L_n\left(f \right) \right) \right\|_{\gamma} \right\|_{\infty} \le \left\| \left\| L_n\left(f \right) \right\|_{\gamma} \right\|_{\infty} \le \left\| \left\| f \right\|_{\gamma} \right\|_{\infty}, \tag{89}
$$

etc.

Therefore we get

$$
\left\| \left\| L_n^k(f) \right\|_{\gamma} \right\|_{\infty} \le \left\| \left\| f \right\|_{\gamma} \right\|_{\infty}, \ \ \forall \ k \in \mathbb{N}, \tag{90}
$$

the contraction property.

Also we see that

$$
\left\| \left\| L_n^k(f) \right\|_{\gamma} \right\|_{\infty} \le \left\| \left\| L_n^{k-1}(f) \right\|_{\gamma} \right\|_{\infty} \le \dots \le \left\| \left\| L_n(f) \right\|_{\gamma} \right\|_{\infty} \le \left\| \left\| f \right\|_{\gamma} \right\|_{\infty} . \tag{91}
$$

Here L_n^k are bounded linear operators.

Notation 22 Here $N \in \mathbb{N}$, $0 < \beta < 1$. Denote by

$$
c_N := \begin{cases} (4.824)^N, & \text{if } L_n = A_n, \\ 1, & \text{if } L_n = B_n, C_n, D_n, \end{cases}
$$
 (92)

$$
\varphi(n) := \begin{cases} \frac{1}{n^{\beta}}, & \text{if } L_n = A_n, B_n, \\ \frac{1}{n} + \frac{1}{n^{\beta}}, & \text{if } L_n = C_n, D_n, \end{cases}
$$
(93)

$$
\Omega := \begin{cases}\nC\left(\prod_{i=1}^{N} [a_i, b_i], X\right), & \text{if } L_n = A_n, \\
C_B\left(\mathbb{R}^N, X\right), & \text{if } L_n = B_n, C_n, D_n,\n\end{cases} \tag{94}
$$

and

$$
Y := \begin{cases} \prod_{i=1}^{N} [a_i, b_i], & \text{if } L_n = A_n, \\ \mathbb{R}^N, & \text{if } L_n = B_n, C_n, D_n. \end{cases}
$$
 (95)

We give the condensed

Theorem 23 Let $f \in \Omega$, $0 < \beta < 1$, $x \in Y$; $n, m, N \in \mathbb{N}$ with $n^{1-\beta} > 2$. Then (i)

$$
\|L_n(f, x) - f(x)\|_{\gamma} \le c_N \left[\omega_1(f, \varphi(n)) + \frac{4e^2 \left\| \|f\|_{\gamma} \right\|_{\infty}}{\pi e^{n^{1-\beta}}} \right] =: \tau(n), \quad (96)
$$

where ω_1 is for $p = \infty$,

and

(ii)

$$
\left\| \left\| L_n\left(f\right) - f \right\|_{\gamma} \right\|_{\infty} \le \tau(n) \to 0, \text{ as } n \to \infty. \tag{97}
$$

For f uniformly continuous and in Ω we obtain

$$
\lim_{n\to\infty}L_n(f)=f,
$$

pointwise and uniformly.

Proof. By Theorems 8, 16, 17, 18. \blacksquare

Next we talk about iterated neural network approximation (see also [9]). We give

Theorem 24 All here as in Theorem 23 and $r \in \mathbb{N}$, $\tau(n)$ as in (96). Then

$$
\left\| \left\| L_n^r f - f \right\|_{\gamma} \right\|_{\infty} \leq r \tau(n). \tag{98}
$$

So that the speed of convergence to the unit operator of L_n^r is not worse than of L_n .

Proof. As similar to [14] is omitted. \blacksquare We also present

Theorem 25 Let $f \in \Omega$; m, N, $m_1, m_2, ..., m_r \in \mathbb{N} : m_1 \leq m_2 \leq ... \leq m_r$, $0 <$ $\beta < 1; m_i^{1-\beta} > 2, i = 1, ..., r, x \in Y$, and let $(L_{m_1}, ..., L_{m_r})$ as $(A_{m_1}, ..., A_{m_r})$ or $(B_{m_1},...,B_{m_r})$ or $(C_{m_1},...,C_{m_r})$ or $(D_{m_1},...,D_{m_r}),$ $p = \infty$. Then

3. COMPUTATIONAL ANALYSIS AND APPLICATIONS, Vol. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\nThoorern 23 Let
$$
f \in \Omega
$$
, $0 < \beta < 1$, $x \in Y$; *n*, *m*, *N* ∈ N with $n^{1-3} > 2$. *Then* (*i*)
\n
$$
||L_n (f, x) - f(x)||_{\gamma} \le c_N \left[\omega_1 (f, \varphi(n)) + \frac{4e^2 ||||f||_{\gamma}||_{\infty}}{n e^{n(1-\beta)}} \right] =: \tau(n), \qquad (96)
$$
\nwhere x_i is for $p = \infty$,
\nand
\n(*ii*)
\n
$$
|||L_n (f) - f||_{\gamma}||_{\infty} \le \tau(n) \to 0, \text{ as } n \to \infty. \qquad (97)
$$
\nFor *f* uniformly continuous and in Ω we obtain
\n
$$
\lim_{n \to \infty} L_n (f) = f,
$$
\npointwise and uniformly.
\nProof. By Theorems 8, 16, 17, 18. ■
\nNext we talk about iterated neural network approximation (see also [9]).
\nWe give
\n**Theorem 24** All here as in Theorem 23 and $r \in \mathbb{N}$, $\tau(n)$ as in (96). Then
\n
$$
||L_n^r f - f||_{\gamma}||_{\infty} \le r \tau(n).
$$
\n(98)
\nSo that the speed of convergence to the unit operator of L_n^r is not worse than of
\n L_n .
\nProof. As similar to [14] is omitted. ■
\nWe also present
\n**Theorem 25** Let $f \in \Omega$; *m*, *N*, *m*₁, *m*₂, ..., *m*_n ∈ N : *m*₁ ≤ *m*₂ ≤ ... ≤ *m*_n, 0
\n $\beta < 1$; *m*₁² → 2, *i* = 1, ..., *r*, *m*₂

$$
rc_N\left[\omega_1\left(f,\varphi\left(m_1\right)\right)+\frac{4e^2\left\| \left\| f\right\|_{\gamma}\right\|_{\infty}}{\pi e^{m_1^{1-\beta}}}\right].
$$
\n
$$
(99)
$$

Clearly, we notice that the speed of convergence to the unit operator of the multiply iterated operator is not worse than the speed of L_{m_1} .

Proof. As similar to [14] is omitted. \blacksquare We also give

Theorem 26 Let all as in Corollary 15, and $r \in \mathbb{N}$. Here $\varphi_3(n)$ is as in (75). Then

$$
\left\| \|A_n^r f - f\|_{\gamma} \right\|_{\infty} \le r \left\| \|A_n f - f\|_{\gamma} \right\|_{\infty} \le r \varphi_3(n). \tag{100}
$$

Proof. As similar to [14] is omitted. \blacksquare

Application 27 A typical application of all of our results is when $(X, \left\| \cdot \right\|_{\gamma}) =$ $(\mathbb{C}, \lvert \cdot \rvert)$, where $\mathbb C$ are the complex numbers.

References

- [1] G.A. Anastassiou, Moments in Probability and Approximation Theory, Pitman Research Notes in Math., Vol. 287, Longman Sci. & Tech., Harlow, U.K., 1993.
- [2] G.A. Anastassiou, Rate of convergence of some neural network operators to the unit-univariate case, J. Math. Anal. Appli. 212 (1997), 237-262.
- [3] G.A. Anastassiou, Quantitative Approximations, Chapman&Hall/CRC, Boca Raton, New York, 2001.
- [4] G.A. Anastassiou, *Inteligent Systems: Approximation by Artificial Neural* Networks, Intelligent Systems Reference Library, Vol. 19, Springer, Heidelberg, 2011. 5 Conservations, and the set of th
	- [5] G.A. Anastassiou, Univariate hyperbolic tangent neural network approximation, Mathematics and Computer Modelling, 53(2011), 1111-1132.
	- [6] G.A. Anastassiou, Multivariate hyperbolic tangent neural network approximation, Computers and Mathematics 61(2011), 809-821.
	- [7] G.A. Anastassiou, Multivariate sigmoidal neural network approximation, Neural Networks 24(2011), 378-386.
	- [8] G.A. Anastassiou, Univariate sigmoidal neural network approximation, J. of Computational Analysis and Applications, Vol. 14, No. 4, 2012, 659-690.
- [9] G.A. Anastassiou, Approximation by neural networks iterates, Advances in Applied Mathematics and Approximation Theory, pp. 1-20, Springer Proceedings in Math. & Stat., Springer, New York, 2013, Eds. G. Anastassiou, O. Duman. 5 COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC George A. Analysis Algorithmost and Applications in Density A. Society A. Analysis Algorithmost and Applications in Densit
	- [10] G.A. Anastassiou, Intelligent Systems II: Complete Approximation by Neural Network Operators, Springer, Heidelberg, New York, 2016.
	- [11] G.A. Anastassiou, Intelligent Computations: Abstract Fractional Calculus, Inequalities, Approximations, Springer, Heidelberg, New York, 2018.
	- [12] G.A. Anastassiou, Algebraic function based Banach space valued ordinary and fractional neural network approximations, New Trends in Mathematical Sciences, 10 special issue (1) (2022), 100-125.
	- [13] G.A. Anastassiou, Gudermannian function activated Banach space valued ordinary and fractional neural network approximation, Advances in Nonlinear Variational Inequalities, 25 (2) (2022), 27-64.
	- [14] G.A. Anastassiou, General multimariate arctangent function activated neural network approximations, submitted, 2022.
	- [15] G.A. Anastassiou, Abstract multivariate algebraic function activated neural network approximations, submitted, 2022.
	- [16] H. Cartan, *Differential Calculus*, Hermann, Paris, 1971.
	- [17] Z. Chen and F. Cao, The approximation operators with sigmoidal functions, Computers and Mathematics with Applications, 58 (2009), 758-765.
	- [18] D. Costarelli, R. Spigler, Approximation results for neural network operators activated by sigmoidal functions, Neural Networks 44 (2013), 101-106.
	- [19] D. Costarelli, R. Spigler, Multivariate neural network operators with sigmoidal activation functions, Neural Networks 48 (2013), 72-77.
	- [20] S. Haykin, Neural Networks: A Comprehensive Foundation (2 ed.), Prentice Hall, New York, 1998.
	- [21] W. McCulloch and W. Pitts, A logical calculus of the ideas immanent in nervous activity, Bulletin of Mathematical Biophysics, 7 (1943), 115-133.
	- [22] T.M. Mitchell, Machine Learning, WCB-McGraw-Hill, New York, 1997.
	- [23] L.B. Rall, Computational Solution of Nonlinear Operator Equations, John Wiley & Sons, New York, 1969.
	- [24] E.W. Weisstein, Gudermannian, MathWorld.

p-Schatten norm generalized Canavati fractional Ostrowski, Opial and Grüss type inequalities involving several functions **583** D. CONFUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC

(Strewords) **C**, Copyind and Gritiss type inequalities
 EVALUATIONS

(Strewords) **C**, Copyind A. Anastasion 583-

George A. Anastassiou Department of Mathematical Sciences University of Memphis Memphis, TN 38152, U.S.A. ganastss@memphis.edu

Abstract

Using generalized Canavati fractional left and right vectorial Taylor formulae we establish generalized fractional Ostrowski, Opial and Grüss type inequalities for several functions that take values in the von Neumann-Schatten class $\mathcal{B}_p(H)$, $1 \leq p < \infty$. The estimates are with respect to all p-Schatten norms, $1 \leq p < \infty$. We finish with applications.

2020 Mathematics Subject Classification : 26A33, 26D10, 26D15, 47A60, 47A63.

Keywords and Phrases: p-Schatten norms, von Neumann-Schatten class, Ostrowski, Opial and Grüss inequalities, generalized Canavati fractional derivative, generalized Canavati fractional inequalities.

1 Introduction

The following results inspire our work.

Theorem 1 (1938, Ostrowski [16]) Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous on [a, b] and differentiable on (a, b) whose derivative $f' : (a, b) \to \mathbb{R}$ is bounded on (a, b) , *i.e.*, $||f'||_{\infty}^{\sup} := \sup_{t \in (a,b)} |f'(t)| < +\infty$. Then

$$
\left|\frac{1}{b-a}\int_{a}^{b} f(t) dt - f(x)\right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^{2}}{\left(b-a\right)^{2}}\right] \left(b-a\right) \left\|f'\right\|_{\infty}^{\sup} ,\tag{1}
$$

for any $x \in [a, b]$. The constant $\frac{1}{4}$ is the best possible.

Ostrowski type inequalities have great applications to integral approximations in Numerical Analysis.

We mention

Theorem 2 (1882, Cebyšev [8]) Let $f, g : [a, b] \rightarrow \mathbb{R}$ be absolutely continuous functions with $f', g' \in L_{\infty}([a, b])$. Then

$$
\left| \frac{1}{b-a} \int_{a}^{b} f(x) g(x) dx - \left(\frac{1}{b-a} \int_{a}^{b} f(x) dx \right) \left(\frac{1}{b-a} \int_{a}^{b} g(x) dx \right) \right|
$$

$$
\leq \frac{1}{12} (b-a)^{2} ||f'||_{\infty} ||g'||_{\infty}.
$$
 (2)

The above integrals are assumed to exist.

The related Grüss type inequalities have many applications to Probability Theory. We presented also $([3], Ch. 8,9)$ mixed fractional Ostrowski and Grüss-Cebysev type inequalities for several functions, acting to all possible directions. The estimates involve the left and right Caputo fractional derivatives. See also the monographs written by the author [1], Chapters 24-26 and [2], Chapters 2-6.

We are motivated also by S. Dragomir [11] recent work:

An operator $A \in \mathcal{B}(H)$ is said to belong to the von Neumann-Schatten class $\mathcal{B}_p(H)$, $1 \leq p < \infty$ if the *p*-Schatten norm is finite

$$
||A||_p := [tr(|A|^p)]^{\frac{1}{p}} < \infty.
$$

Assume that $A : [a, b] \to B_p(H)$, $B : [a, b] \to B_q(H)$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, are continuous and B is strongly differentiable on (a, b) , then

3. COMPUTATIONAL ANALYSIS AND APPLICATIONS. VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\nOstrowski type inequalities have great applications to integral approximations in Numerisma-
\ntions in Numerical Analysis.
\nWe mention
\nTheorem 2 (1882, CchySet: [8]) Let f, g : [a, b] → ℝ be absolutely continuous
\nfuncations with f', g' ∈ L_{CC} ([a, b]). Then
\n
$$
\left| \frac{1}{b-a} \int_a^b f(x) g(x) dx - \left(\frac{1}{b-a} \int_a^b f(x) dx \right) \left(\frac{1}{b-a} \int_a^b g(x) dx \right) \right|
$$
\n
$$
\leq \frac{1}{12} (b-a)^2 ||f'||_{\infty} ||g'||_{\infty}.
$$
\n(2)
\nThe above integrals are assumed to exist.
\nThe related Gräks type inequalities have many applications to Probability
\nTheco-
\nCheysive type inequalities for several functions, setting to all possible directions.
\nThe estimated SNUS (B], C.B. (9. 8.9) mixed fractional derivatives. See also
\nthe monographs written by the author [1], Chapter 24-26 and [2], Chapters
\nWe are motivated also by S. Dragomin [11] recent work:
\nSo, the nonperator A ∈ B (If) is said to belong to the von Neumann-Schatten class
\n $B_p(H)$, $1 \leq p < \infty$ if the p-Schatten norm is finite
\n
$$
||A||_p := |tr (||A|^p)|^{\frac{1}{p}} < \infty.
$$

\nAssume that A : [a, b] → B_p(H), B : [a, b] = B₀(H), p, q > 1 with $\frac{1}{p} + \frac{1}{q} = 1$,
\nare continuous and B is strongly different in our Neumann-Schatten class
\n
$$
||B|_p = |E_p(H), B : [a, b] = B_{\delta}(H), 1 |B|_p = \frac{1}{q} \left(\int_a^b A(s) ds \right) B(u) \Big|_1 \leq
$$

\n
$$
||\int_a^b A(t) B(t) dt - \left(\int_a^b A(s) ds \right) B(u) \Big|_1 \leq
$$

\n
$$
||\int_a^b A(t) B(t) dt - \left(\int_a^b A(s) ds \right) B(u) \Big|_1 \leq
$$

for all $u \in [a, b]$, an Ostrowski type inequality.

Further inspiration comes from S. Dragomir [12] recent work on Grüss inequalities:

For two continuous functions $A, B : [a, b] \to \mathcal{B}(H)$ we define the noncommutative Cebysev fractional

$$
D(A, B) := (b - a) \int_{a}^{b} A(t) B(t) dt - \int_{a}^{b} A(t) dt \int_{a}^{b} B(t) dt.
$$

If $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, let $A : [a, b] \rightarrow \mathcal{B}_p(H)$, $B : [a, b] \rightarrow \mathcal{B}_q(H)$ be strongly differentiable functions on the interval (a, b) , then

$$
\|D(A,B)\|_{1} \le D\left(\int_{a}^{b} \|A'(u)\|_{p} du, \int_{a}^{b} \|B'(u)\|_{q} du\right) \le (4)
$$

$$
\frac{1}{4} (b-a)^2 \int_a^b \|A'(u)\|_p du \int_a^b \|B'(u)\|_q du.
$$

We are also inspired by Z. Opial [15], 1960, famous inequality.

Theorem 3 Let $x(t) \in C^1([0, h])$ be such that $x(0) = x(h) = 0$, and $x(t) > 0$ in $(0, h)$. Then

$$
\int_{0}^{h} |x(t) x'(t)| dt \leq \frac{h}{4} \int_{0}^{h} (x'(t))^{2} dt.
$$
 (5)

In (5), the constant $\frac{h}{4}$ is the best possible. Inequality (5) holds as equality for the optimal function

$$
x(t) = \begin{cases} ct, & 0 \le t \le \frac{h}{2}, \\ c(h-t), & \frac{h}{2} \le t \le h, \end{cases}
$$
 (6)

where $c > 0$ is an arbitrary constant.

Opial-type inequalities are used a lot in proving uniqueness of solutions to differential equations and also to give upper bounds to their solutions.

For an extensive study about fractional Opial inequalities see the author's monograph [1].

In this article we generalize [3], Ch. 8,9 for several Banach algebra $\mathcal{B}_p(H)$ valued functions, in the sense of developing fractional Ostrowski, Opial and Grüss type inequalities. Now our left and right generalized Canavati fractional derivatives are for Banach space valued functions and our integrals are of Bochner type [13]. Applications finish the article. 5 CONFUTATIONAL ANNEWSIS AND APPLICATIONS, VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC

The search constrained free class α , $B : [\alpha, \beta] \rightarrow B(T)$ and define the reasonstitute constrained for the search constrain

2 Background on Vectorial generalized Canavati fractional calculus

All in this section come from [5], pp. 109-115 and [4].

Let $g : [a, b] \to \mathbb{R}$ be a strictly increasing function. such that $g \in C^1([a, b]),$ and $g^{-1} \in C^n([g(a), g(b)]), n \in \mathbb{N}, (X, \|\cdot\|)$ is a Banach space. Let $f \in$ $C^{n}([a,b], X)$, and call $l := f \circ g^{-1} : [g(a), g(b)] \to X$. It is clear that $l, l', ..., l^{(n)}$ are continuous functions from $[g(a), g(b)]$ into $f([a, b]) \subseteq X$. LOOKETATIONAL ANNEWS AND APPLICATIONS, VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC $\sum_{i=1}^n \{x_i\} \in \mathbb{C}^n \setminus \{y_i\} \cup \$

Let $\nu \geq 1$ such that $[\nu] = n, n \in \mathbb{N}$ as above, where $[\cdot]$ is the integral part of the number.

Clearly when $0 < \nu < 1$, $[\nu] = 0$.

I) Let $h \in C([g (a), g (b)], X)$, we define the left Riemann-Liouville Bochner fractional integral as

$$
\left(J_{\nu}^{z_0}h\right)(z) := \frac{1}{\Gamma(\nu)} \int_{z_0}^{z} \left(z - t\right)^{\nu - 1} h\left(t\right) dt,\tag{7}
$$

for $g(a) \le z_0 \le z \le g(b)$, where Γ is the gamma function; $\Gamma(\nu) = \int_0^\infty e^{-t} t^{\nu-1} dt$. We set $J_0^{z_0}h = h$.

Let $\alpha := \nu - [\nu]$ $(0 < \alpha < 1)$. We define the subspace $C_{g(x_0)}^{\nu}([g(a), g(b)], X)$ of $C^{[\nu]}([g(a), g(b)], X)$, where $x_0 \in [a, b]$ as:

$$
C_{g(x_0)}^{\nu}([g(a), g(b)], X) =
$$

$$
\left\{ h \in C^{[\nu]}([g(a), g(b)], X) : J_{1-\alpha}^{g(x_0)}h^{([\nu])} \in C^1([g(x_0), g(b)], X) \right\}.
$$
 (8)

So let $h \in C_{g(x_0)}^{\nu}([g(a), g(b)], X)$, we define the left g-generalized X-valued fractional derivative of h of order ν , of Canavati type, over $[g(x_0), g(b)]$ as

$$
D_{g(x_0)}^{\nu}h := \left(J_{1-\alpha}^{g(x_0)}h^{([\nu])}\right)'.
$$
 (9)

Clearly, for $h \in C_{g(x_0)}^{\nu}([g(a), g(b)], X)$, there exists

$$
\left(D_{g(x_0)}^{\nu}h\right)(z) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dz}\int_{g(x_0)}^{z} (z-t)^{-\alpha}h^{(\nu)}(t)dt,\tag{10}
$$

for all $g(x_0) \leq z \leq g(b)$.

In particular, when $f \circ g^{-1} \in C_{g(x_0)}^{\nu}([g(a), g(b)], X)$, we have that

$$
\left(D_{g(x_0)}^{\nu}\left(f \circ g^{-1}\right)\right)(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_{g(x_0)}^{z} \left(z-t\right)^{-\alpha} \left(f \circ g^{-1}\right)^{([\nu])}(t) \, dt, \tag{11}
$$

for all $g(x_0) \leq z \leq g(b)$. We have that $D_{g(x_0)}^n (f \circ g^{-1}) = (f \circ g^{-1})^{(n)}$ and $D_{g(x_0)}^0$ $(f \circ g^{-1}) = f \circ g^{-1}$, see [4].

By [4], we have for $f \circ g^{-1} \in C_{g(x_0)}^{\nu}([g(a), g(b)], X)$, where $x_0 \in [a, b]$ the following left generalized g -fractional, of Canavati type, X-valued Taylor's formula:

Theorem 4 Let $f \circ g^{-1} \in C_{g(x_0)}^{\nu}([g(a), g(b)], X)$, where $x_0 \in [a, b]$ is fixed. (i) If $\nu > 1$, then

3. COMPUTATIONAL ANALYSIS AND APPLICATIONS. Vol. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\n**Thocorern 4** Let
$$
f \circ g^{-1} \in C_{g(n)}^{\omega} ([g (a), g (b)], X), where x_0 \in [a, b] is fixed.\n(i) If $\nu \ge 1$, then
\n
$$
f(x) = f(x_0) = \sum_{k=1}^{\infty} \frac{(f \circ g^{-1})^{(k)} (g(x_0))}{k!} (g(x) - g(x_0))^k + \frac{1}{\Gamma(\nu)} \int_{g(x_0)}^{g(x_0)} (g(x) - t)^{\nu-1} (D_{g(n)}^{\nu} (f \circ g^{-1})) (t) dt,
$$
\n(12)
\nfor all $x_0 \le x \le b$.
\n(ii) If $0 < \nu < 1$, we get
\n
$$
f(x) = \frac{1}{\Gamma(\nu)} \int_{g(x_0)}^{g(x)} (g(x) - t)^{\nu-1} (D_{g(x_0)}^{\nu} (f \circ g^{-1})) (t) dt,
$$
\n(13)
\nfor all $x_0 \le x \le b$.
\nII) Let $h \in C([g (a), g (b)], X)$, we define the right Riemann-Liouville Bochner
\nfractional integral as
\n
$$
(J_{n_0}^{\nu} - h) (z) := \frac{1}{\Gamma(\nu)} \int_{z}^{\infty} (t - z)^{\nu-1} h(t) dt,
$$
\n(14)
\nfor $g(a) \le z \le x_0 \le g(b)$. We set $J_{n_0}^0 - h = h$.
\nLet $\alpha := \nu - |\nu| (0 < \alpha < x)$, where $x_0 \in [a, b]$ as:
\n $C_{g(x_0)}^{\omega} - ([g (a), g (b)], X) :=$
\n
$$
\begin{cases} h \in C^{[p]}([g (n), g (b)], X) : J_{n_0}^{\omega} - h = h, \\ \text{Let } \alpha := \nu - |\nu| (0 < \alpha < x) \quad W_0^{\omega} - h = h. \end{cases}
$$

\n
$$
C_{g(x_0)}^{\omega} - ([g (a), g (b)], X) :=
$$

\n $$
$$

for all $x_0 \leq x \leq b$.

(ii) If $0 < \nu < 1$, we get

$$
f(x) = \frac{1}{\Gamma(\nu)} \int_{g(x_0)}^{g(x)} (g(x) - t)^{\nu - 1} \left(D_{g(x_0)}^{\nu} \left(f \circ g^{-1} \right) \right) (t) dt, \tag{13}
$$

for all $x_0 \leq x \leq b$.

II) Let $h \in C([g (a), g (b)], X)$, we define the right Riemann-Liouville Bochner fractional integral as

$$
\left(J_{z_0}^{\nu} - h\right)(z) := \frac{1}{\Gamma(\nu)} \int_{z}^{z_0} \left(t - z\right)^{\nu - 1} h\left(t\right) dt,\tag{14}
$$

for $g(a) \le z \le z_0 \le g(b)$. We set $J_{z_0}^0 - h = h$.

Let $\alpha := \nu - [\nu]$ $(0 < \alpha < 1)$. We define the subspace $C_{g(x_0)-}^{\nu}([g(a), g(b)], X)$ of $C^{[\nu]}([g(a), g(b)], X)$, where $x_0 \in [a, b]$ as:

$$
C_{g(x_0)-}^{\nu}([g(a), g(b)], X) :=
$$

$$
\left\{ h \in C^{[\nu]}([g(a), g(b)], X) : J_{g(x_0)-}^{1-\alpha}h^{([\nu])} \in C^1([g(a), g(x_0)], X) \right\}.
$$
 (15)

So let $h \in C'_{g(x_0)-}([g(a), g(b)], X)$, we define the right g-generalized Xvalued fractional derivative of h of order ν , of Canavati type, over $[g(a), g(x_0)]$ as

$$
D_{g(x_0)-}^{\nu}h := (-1)^{n-1} \left(J_{g(x_0)-}^{1-\alpha}h^{([\nu])} \right)'.
$$
 (16)

Clearly, for $h \in C_{g(x_0)-}^{\nu}([g(a), g(b)], X)$, there exists

$$
\left(D_{g(x_0)}^{\nu} - h\right)(z) = \frac{(-1)^{n-1}}{\Gamma(1-\alpha)} \frac{d}{dz} \int_{z}^{g(x_0)} (t-z)^{-\alpha} h^{([\nu])}(t) dt, \tag{17}
$$

for all $g(a) \leq z \leq g(x_0) \leq g(b)$.

In particular, when $f \circ g^{-1} \in C_{g(x_0)-}^{\nu}([g(a), g(b)], X)$, we have that

$$
\left(D_{g(x_0)-}^{\nu}(f\circ g^{-1})\right)(z) = \frac{(-1)^{n-1}}{\Gamma(1-\alpha)}\frac{d}{dz}\int_{z}^{g(x_0)}(t-z)^{-\alpha}(f\circ g^{-1})^{([{\nu}])}(t)\,dt,\tag{18}
$$

for all $g(a) \leq z \leq g(x_0) \leq g(b)$. We get that

> $\left(D_{g(x_0)-}^n(f\circ g^{-1})\right)(z)=(-1)^n\left(f\circ g^{-1}\right)^{(n)}$ (19)

and $\left(D_{g(x_0)-}^0(f\circ g^{-1})\right)(z) = (f\circ g^{-1})(z)$, all $z \in [g(a), g(b)]$, see [4].

By [4], we have for $f \circ g^{-1} \in C_{g(x_0) - 1}^{\nu}([g(a), g(b)], X)$, where $x_0 \in [a, b]$ is fixed, the following right generalized g -fractional, of Canavati type, X-valued Taylor's formula:

Theorem 5 Let $f \circ g^{-1} \in C_{g(x_0)-}^{\nu}([g(a), g(b)], X)$, where $x_0 \in [a, b]$ is fixed. (i) If $\nu > 1$, then

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, Vol. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\nfor all
$$
g(a) \le z \le g(x_0) \le g(b)
$$
.
\nWe get that
\n
$$
\left(D_{g(x_0)}^n - (f \circ g^{-1})\right)(z) = (-1)^n (f \circ g^{-1})^{(n)}(z)
$$
\nand
$$
\left(D_{g(x_0)}^n - (f \circ g^{-1})\right)(z) = (f \circ g^{-1}) (z), \text{ all } z \in [g(a), g(b)], \text{ see [4]}.
$$
\nBy [4], we have for $f \circ g^{-1} \in C_{g(x_0)}^n - (g(a), g(b)), X)$, where $x_0 \in [a, b]$ is fixed, the following right generalized g -fractional, of Canavait type, X-enhord
\nTubor's formula:
\n**Theorem 5** Let $f \circ g^{-1} \in C_{g(x_0)}^n - (g(a), g(b)), X)$, where $x_0 \in [a, b]$ is fixed.
\n
$$
\left(i \frac{\partial f}{\partial t} \ge 1, \text{ then}
$$
\n
$$
f(x) = f(x_0) = \sum_{k=1}^{\lfloor n \rfloor} \frac{(f \circ g^{-1})^{(k)}(g(x_0))}{k!} (g(x) - g(x_0))^k + \frac{1}{\Gamma(\nu)} \int_{g(x_0)}^{g(x_0)} (t - g(x))^{\nu-1} \left(D_{g(x_0)}^n - (f \circ g^{-1})\right)(t) dt,
$$
\n(20)
\nfor all $a \le x \le x_0$,
\n
$$
\left(i \right) \frac{f}{g(x)} = D_{g(x_0)}^{\nu} \frac{f}{g(x_0)}, \quad (t - g(x))^{\nu-1} \left(D_{g(x_0)}^{\nu} - (f \circ g^{-1})\right)(t) dt,
$$
\n(21)
\nand $a \le x \le x_0$.
\nIII) Denote by
\n
$$
D_{g(x_0)}^{\nu} = D_{g(x_0)}^{\nu} D_{g(x_0)}^{\nu} - D_{g(x_0)}^{\nu} \left(m + \text{times})
$$
, $n \in \mathbb{N}$.
\n22)
\nWe mention the

for all $a \leq x \leq x_0$,

(ii) If $0 < \nu < 1$, we get

$$
f(x) = \frac{1}{\Gamma(\nu)} \int_{g(x)}^{g(x_0)} (t - g(x))^{\nu - 1} \left(D_{g(x_0) -}^{\nu} (f \circ g^{-1}) \right) (t) dt, \qquad (21)
$$

all $a \leq x \leq x_0$.

III) Denote by

$$
D_{g(x_0)}^{m\nu} = D_{g(x_0)}^{\nu} D_{g(x_0)}^{\nu} \dots D_{g(x_0)}^{\nu} \quad (m\text{-times}), \ m \in \mathbb{N}.
$$
 (22)

We mention the following modified and generalized left X -valued fractional Taylor's formula of Canavati type:

Theorem 6 Let $f \in C^1([a, b], X)$, $g \in C^1([a, b])$, strictly increasing: $g^{-1} \in$ $C^{1}([g(a),g(b)]).$ Assume that $(D_{g(x_0)}^{i\nu}(f\circ g^{-1}))\in C_{g(x_0)}^{\nu}([g(a),g(b)],X),$ $0 < \nu < 1, x_0 \in [a, b],$ for $i = 0, 1, ..., m$. Then

$$
f(x) = \frac{1}{\Gamma((m+1)\nu)} \int_{g(x_0)}^{g(x)} (g(x) - z)^{(m+1)\nu - 1} \left(D_{g(x_0)}^{(m+1)\nu} (f \circ g^{-1}) \right) (z) dz,
$$
\n(23)

all $x_0 \leq x \leq b$.

IV) Denote by

$$
D_{g(x_0)-}^{m\nu} = D_{g(x_0)-}^{\nu} D_{g(x_0)-}^{\nu} \dots D_{g(x_0)-}^{\nu} \quad (m \text{ times}), \, m \in \mathbb{N}.\tag{24}
$$

We mention the following modified and generalized right X -valued fractional Taylor's formula of Canavati type:

Theorem 7 Let $f \in C^1([a, b], X)$, $g \in C^1([a, b])$, strictly increasing: $g^{-1} \in$ $C^{1}([g(a), g(b)])$. Assume that $(D_{g(x_0)-}^{i\nu}(f\circ g^{-1}))\in C_{g(x_0)-}^{\nu}([g(a), g(b)], X)$, $0 < \nu < 1, x_0 \in [a, b],$ for all $i = 0, 1, ..., m$. Then 5 DONEUTATIONAL ANNEWSIS AND APPLICATIONS, VOL. 31, NO. 4, 2023, COPYRIGHT 2023 CLOCOXUS PRESS, LLC

1979 Thereon by

The mean on $T = H_{\text{SUSY}} = T_{\text{SUSY}} = T_{\text{SUSY}}$ (or thereon, $\phi \leq \mathbf{R}$

Theoretical Properties of Col

$$
f(x) = \frac{1}{\Gamma((m+1)\nu)} \int_{g(x)}^{g(x_0)} (z - g(x))^{(m+1)\nu - 1} \left(D_{g(x_0)-}^{(m+1)\nu} (f \circ g^{-1}) \right) (z) dz,
$$
\n(25)

all $a \leq x \leq x_0 \leq b$.

3 Basic Banach Algebras background

All here come from [17].

We need

Definition 8 ([17], p. 245) A complex algebra is a vector space A over the complex filed $\mathbb C$ in which a multiplication is defined that satisfies

$$
x(yz) = (xy)z,
$$
\n⁽²⁶⁾

$$
(x + y) z = xz + yz, \ \ x(y + z) = xy + xz,
$$
\n(27)

and

$$
\alpha (xy) = (\alpha x) y = x (\alpha y), \qquad (28)
$$

for all x, y and z in A and for all scalars α .

Additionally if A is a Banach space with respect to a norm that satisfies the multiplicative inequality

$$
||xy|| \le ||x|| \, ||y|| \quad (x \in A, \ y \in A)
$$
\n(29)

and if A contains a unit element e such that

$$
xe = ex = x \quad (x \in A)
$$
\n⁽³⁰⁾

and

$$
||e|| = 1,\t\t(31)
$$

then A is called a Banach algebra.

A is commutative iff $xy = yx$ for all $x, y \in A$.

We make

Remark 9 Commutativity of A is explicited stated when needed. There exists at most one $e \in A$ that satisfies (30).

Inequality (29) makes multiplication to be continuous, more precisely left and right continuous, see [17], p. 246.

Multiplication in A is not necessarily the numerical multiplication, it is something more general and it is defined abstractly, that is for $x, y \in A$ we have $xy \in A$, e.g. composition or convolution, etc.

For nice examples about Banach algebras see [17], p. $247-248$, § 10.3.

We also make

Remark 10 Next we mention about integration of A-valued functions, see [17], p. 259, $§ 10.22$:

If A is a Banach algebra and f is a continuous A-valued function on some compact Hausdorff space Q on which a complex Borel measure μ is defined, then $\int f d\mu$ exists and has all the properties that were discussed in Chapter 3 of [17], simply because A is a Banach space. However, an additional property can be added to these, namely: If $x \in A$, then **EXERCTATIONAL ANALYSIS AND APPLICATIONS, VOL. 31, NO.** 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
 Straight 0 Convention of A statistically of \vec{A} at a stabilistical dislocity and

The model of exercise of A statis

$$
x\int_{Q} f d\mu = \int_{Q} xf(p) d\mu(p) \tag{32}
$$

and

$$
\left(\int_{Q} f \ d\mu\right) x = \int_{Q} f(p) \ x \ d\mu(p). \tag{33}
$$

The Bochner integrals we will involve in our article follow (32) and (33). Also, let $f \in C([a, b], X)$, where $[a, b] \subset \mathbb{R}$, $(X, \|\cdot\|)$ is a Banach space. By [5], p. 3, f is Bochner integrable.

4 p-Schatten norms background

In this advanced section all come from [11].

Let $(H, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on H. If ${e_i}_{i\in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is of trace class if

$$
||A||_1 := \sum_{i \in I} \langle |A|e_i, e_i \rangle < \infty. \tag{34}
$$

The definition of $||A||_1$ does not depend on the choice of the orthornormal basis ${e_i}_{i \in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

We define the trace of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$
tr(A) := \sum_{i \in I} \langle Ae_i, e_i \rangle, \qquad (35)
$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (35) converges absolutely and it is independent from the choice of basis. CONFUTATIONAL ANNEVISIS AND APPLICATIONS, VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC

We define the trunc of a many close of H . A convention of $\in \mathcal{B}_1(H)$ the convention of $\in \mathcal{B}_2(H)$ and the conventi

The following result collects some properties of the trace:

Theorem 11 We have:

(i) If $A \in \mathcal{B}_1(H)$ then $A^* \in \mathcal{B}_1(H)$ and

$$
tr(A^*) = \overline{tr(A)};
$$
\n(36)

(ii) If $A \in \mathcal{B}_1(H)$ and $T \in \mathcal{B}(H)$, then AT , $TA \in \mathcal{B}_1(H)$ and

$$
tr\left(AT\right) = tr\left(TA\right) \quad and \quad \left|tr\left(AT\right)\right| \le \left\|A\right\|_1 \left\|T\right\|; \tag{37}
$$

- (iii) $tr(\cdot)$ is a bounded linear functional on $\mathcal{B}_1(H)$ with $||tr|| = 1$;
- (iv) If $A, B \in \mathcal{B}_2(H)$ then $AB, BA \in \mathcal{B}_1(H)$ and $tr(AB) = tr(BA)$;
- (v) $\mathcal{B}_{fin}(H)$ (finite rank operators) is a dense subspace of $\mathcal{B}_1(H)$.

An operator $A \in \mathcal{B}(H)$ is said to belong to the von Neumann-Schatten class $\mathcal{B}_p(H)$, $1 \leq p < \infty$ if the p-Schatten norm is finite [19, p. 60-64]

$$
||A||_p := [tr (|A|^p)]^{\frac{1}{p}} < \infty,
$$

 $\vert A\vert^p$ is an operator notation and not a power.

For $1 < p < q < \infty$ we have that

$$
\mathcal{B}_1(H) \subset \mathcal{B}_p(H) \subset \mathcal{B}_q(H) \subset \mathcal{B}(H) \tag{38}
$$

and

$$
||A||_1 \ge ||A||_p \ge ||A||_q \ge ||A||. \tag{39}
$$

For $p \ge 1$ the functional $\left\| \cdot \right\|_p$ is a norm on the \ast -ideal $\mathcal{B}_p(H)$, which is a Banach algebra, and $(\mathcal{B}_{p}(H), \left\|\cdot\right\|_{p})$ is a Banach space.

Also, see for instance [19, p. 60-64], for $p \geq 1$,

$$
||A||_p = ||A^*||_p, \ \ A \in \mathcal{B}_p(H) \tag{40}
$$

$$
||AB||_p \le ||A||_p ||B||_p, \quad A, B \in \mathcal{B}_p(H)
$$
\n⁽⁴¹⁾

and

$$
||AB||_p \le ||A||_p ||B||, ||BA||_p \le ||B|| ||A||_p, A \in \mathcal{B}_p(H), B \in \mathcal{B}(H).
$$
 (42)

This implies that

$$
||CAB||_p \le ||C|| ||A||_p ||B||, \quad A \in \mathcal{B}_p(H), \ B, C \in \mathcal{B}(H). \tag{43}
$$

In terms of p-Schatten norm we have the Hölder inequality for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$:

$$
(|tr(AB)| \le ||AB||_1 \le ||A||_p ||B||_q, \quad A \in \mathcal{B}_p(H), B \in \mathcal{B}_q(H). \tag{44}
$$

For the theory of trace functionals and their applications the interested reader is referred to [18] and [19].

For some classical trace inequalities see [9], [10] and [14], which are continuations of the work of Bellman [7].

5 Main Results

We start with 1-Schatten norm weighted mixed generalized Canavati fractional Ostrowski type inequalities involving several functions taking values in the Banach algebra $\mathcal{B}_2(H) \subset \mathcal{B}(H)$:

Theorem 12 Let the \ast -ideal $\mathcal{B}_2(H)$, which $(\mathcal{B}_2(H), \|\cdot\|_2)$ is a Banach algebra; $x_0 \in [a, b] \subset \mathbb{R}, \nu \ge 1, n = [\nu]; f_i \in C^n([a, b], \mathcal{B}_2(H)), i = 1, ..., r \in \mathbb{N}$ $\{1\}; g \in C^1([a, b]),$ strictly increasing such that $g^{-1} \in C^n([g(a), g(b)]),$ with $(f_i \circ g^{-1})^{(k)} (g(x_0)) = 0, \ \ k = 1, ..., n-1; \ i = 1, ..., r.$ Assume further that $f_i \circ f$ $g^{-1} \in C_{g(x_0)-}^{\nu}([g(a), g(b)], \mathcal{B}_2(H)) \cap C_{g(x_0)}^{\nu}([g(a), g(b)], \mathcal{B}_2(H)), i = 1, ..., r.$ Denote by

$$
K(f_1, ..., f_r)(x_0) :=
$$

$$
\sum_{i=1}^r \left[\int_a^b \left(\prod_{\substack{j=1 \ j \neq i}}^r f_j(x) \right) f_i(x) dx - \left(\int_a^b \left(\prod_{\substack{j=1 \ j \neq i}}^r f_j(x) \right) dx \right) f_i(x_0) \right].
$$
 (45)

Then

3. COMPUTATIONAL ANALYSIS AND APPLICATIONS, Vol. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\nThis implies that
\n
$$
||CAB||_p \le ||C|| ||A||_p ||B||, A ∈ B_p(H), B, C ∈ B(H).
$$
\n(43)
\nIn terms of p-Schatten norm we have the Hölder inequality for p, q > 1 with
\n
$$
\frac{1}{p} + \frac{1}{q} = 1:
$$
\n($(tr(AH))| ≤ ||A||_p ||B||_q, A ∈ B_p(H), B ∈ B_q(H).$ \n(44)
\nFor the theory of trace functionals and their applications the interested reader
\nis referred to [18] and [19].
\nFor the M. 1-60: for trace functionals and their applications the interested reader
\nis referred to [18] and [19].
\nFor the W. 67 Bellman [7].
\n**5 Main Results**
\nWe start with 1-8dature in normal with a given expression
\nOutrows (32) and (49).
\nSubstituting several functions taking values in the Ba-
\nand algebra $B_2(H) ≤ B(H)$:
\nTheorem 12 Let the *i*-ideal $B_3(H)$, which $(B_3(H), ||A||_p)$ is a Banach algebra;
\n $x_0 ∈ [a, b] ≤ B, p ≥ 1, n = [b]; f ∈ C^*(a, B, 1B), j = 1, ..., r = N =$
\n{ $(f_1 \circ g^{-1})^{(k)} (g(x_0)) = 0, k = 1, ..., n - 1; i = 1, ..., r$. Assume further that $f_i \circ g^{-1} (x_0, y_0) = (g(a), g(b)), B_2(H))$, $i = 1, ..., r$.
\n $(f_1 \circ g^{-1})^{(k)} (g(x_0)) = 0, k = 1, ..., n - 1; i = 1, ..., r$. Assume further that $f_i \circ g^{-1} (x_0, y_0) = (g(a), g(b)), B_2(H))$, $i = 1, ..., r$.
\n
$$
E(f_1, ..., f_r) (x_0) :=
$$
\n
$$
\sum_{i=1}^r \left[\int_a^b \left(\prod_{j=1}^r f_j(x) \right) f_i(x) dx - \left(\int_a^b \left(\prod_{j=1}
$$

:

Proof. Since $(f_i \circ g^{-1})^{(k)}(g(x_0)) = 0, k = 1, ..., [\nu] - 1; i = 1, ..., r;$ we have by Theorem 4 that

$$
f_i(x) - f_i(x_0) = \frac{1}{\Gamma(\nu)} \int_{g(x_0)}^{g(x)} (g(x) - t)^{\nu - 1} \left(D_{g(x_0)}^{\nu} \left(f_i \circ g^{-1} \right) \right) (t) dt, \quad (47)
$$

 $\forall x \in [x_0, b]$,

and by Theorem 5 that

$$
f_i(x) - f_i(x_0) = \frac{1}{\Gamma(\nu)} \int_{g(x)}^{g(x_0)} (t - g(x))^{\nu - 1} \left(D_{g(x_0) -}^{\nu} (f_i \circ g^{-1}) \right) (t) dt, \tag{48}
$$

 $\forall x \in [a, x_0]$, for all $i = 1, ..., r$.

Left multiplying (47) and (48) with $\left(\prod_{\substack{j=1 \ j \neq i}}^{r} \right)$ $f_j\left(x\right)$ \setminus we get, respectively,

$$
\left(\prod_{\substack{j=1 \ j \neq i}}^{r} f_j(x)\right) f_i(x) - \left(\prod_{\substack{j=1 \ j \neq i}}^{r} f_j(x)\right) f_i(x_0) =
$$
\n
$$
\frac{\left(\prod_{\substack{j=1 \ j \neq i}}^{r} f_j(x)\right)}{\Gamma(\nu)} \int_{g(x_0)}^{g(x)} \left(g(x) - t\right)^{\nu-1} \left(D_{g(x_0)}^{\nu}\left(f_i \circ g^{-1}\right)\right)(t) dt, \qquad (49)
$$

 $\forall x \in [x_0, b],$

and

$$
\left(\prod_{\substack{j=1 \ j\neq i}}^r f_j(x)\right) f_i(x) - \left(\prod_{\substack{j=1 \ j\neq i}}^r f_j(x)\right) f_i(x_0) =
$$
\n
$$
\left(\prod_{\substack{j=1 \ j\neq i}}^r f_j(x)\right) \int_{g(x)}^{g(x_0)} (t - g(x))^{\nu - 1} \left(D_{g(x_0) -}^{\nu}(f_i \circ g^{-1})\right)(t) dt,
$$
\n(50)

 $\forall x \in [a, x_0]$, for all $i = 1, ..., r$.

Adding (49) and (50) as separate groups, we obtain

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\nProof. Since
$$
(f_i \circ g^{-1})^{(k)}
$$
 $(g(x_0)) = 0, k = 1, ..., [p]-1; i = 1, ..., r$; we have
\nby Theorem 4 that
\n $f_i(x) = f_i(x_0) = \frac{1}{\Gamma(\nu)} \int_{g(x_0)}^{g(x)} (g(x) - t)^{\nu - 1} (D'_{g(x_0)} (f_i \circ g^{-1})) (t) dt$, (47)
\n $\forall x \in [x_0, b],$
\nand by Theorem 5 that
\n $f_i(x) = f_i(x_0) = \frac{1}{\Gamma(\nu)} \int_{g(x_0)}^{g(x_0)} (t - g(x))^{v-1} (D'_{g(x_0)} - (f_i \circ g^{-1})) (t) dt$, (48)
\n $\forall x \in [a, x_0]$, for all $i = 1, ..., r$.
\nLet multiplying (47) and (48) with $(\prod_{\substack{j=1 \ j \neq i}}^{r} f_j(x))$ we get, respectively,
\n $\left(\prod_{\substack{j=1 \ j \neq i}}^{r} f_j(x)\right) f_i(x) = \left(\prod_{\substack{j=1 \ j \neq i}}^{r} f_j(x)\right) f_i(x_0) =$
\n $\left(\prod_{\substack{j=1 \ j \neq i}}^{r} f_j(x)\right) \int_{g(x_0)}^{g(x)} (g(x) - t)^{\nu - 1} (D'_{g(x_0)} (f_i \circ g^{-1})) (t) dt$, (49)
\n $\forall x \in [x_0, b],$
\nand
\n $\left(\prod_{\substack{j=1 \ j \neq i}}^{r} f_j(x)\right) \int_{g(x_0)}^{g(x)} (t - g(x))^{v-1} (D'_{g(x_0)} (f_i \circ g^{-1})) (t) dt$, (49)
\n $\forall x \in [x_0, b],$
\n $\left(\prod_{\substack{j=1 \ j \neq i}}^{r} f_j(x)\right) \int_{g(x_0)}^{g(x_0)} (t - g(x))^{v-1} (D'_{g(x_0)} (f_i \circ g^{-1})) (t) dt$,
\

$$
\forall x \in [x_0, b],
$$

\nand
\n
$$
\sum_{i=1}^r \left(\prod_{\substack{j=1 \ j \neq i}}^r f_j(x) \right) f_i(x) - \sum_{i=1}^r \left(\prod_{\substack{j=1 \ j \neq i}}^r f_j(x) \right) f_i(x_0) =
$$

\n
$$
\frac{1}{\Gamma(\nu)} \sum_{i=1}^r \left(\prod_{\substack{j=1 \ j \neq i}}^r f_j(x) \right) \int_{g(x)}^{g(x_0)} (t - g(x))^{\nu-1} \left(D_{g(x_0)-}^{\nu} (f_i \circ g^{-1}) \right) (t) dt, (52)
$$

 $\forall x \in [a, x_0]$.

Next, we integrate (51) and (52) with respect to $x \in [a, b]$. We have

$$
\sum_{i=1}^{r} \int_{x_0}^{b} \left(\prod_{\substack{j=1 \ j \neq i}}^{r} f_j(x) \right) f_i(x) dx - \sum_{i=1}^{r} \left(\int_{x_0}^{b} \left(\prod_{\substack{j=1 \ j \neq i}}^{r} f_j(x) \right) dx \right) f_i(x_0) =
$$

$$
\frac{1}{\Gamma(\nu)} \sum_{i=1}^{r} \left[\int_{x_0}^{b} \left(\prod_{\substack{j=1 \ j \neq i}}^{r} f_j(x) \right) \left(\int_{g(x_0)}^{g(x)} (g(x) - t)^{\nu-1} \left(D_{g(x_0)}^{\nu} (f_i \circ g^{-1}) \right) (t) dt \right) dx \right],
$$

(53)

and

$$
\sum_{i=1}^{r} \int_{a}^{x_{0}} \left(\prod_{\substack{j=1 \ j \neq i}}^{r} f_{j}(x) \right) f_{i}(x) dx - \sum_{i=1}^{r} \left(\int_{a}^{x_{0}} \left(\prod_{\substack{j=1 \ j \neq i}}^{r} f_{j}(x) \right) dx \right) f_{i}(x_{0}) =
$$
\n
$$
\frac{1}{\Gamma(\nu)} \sum_{i=1}^{r} \left[\int_{a}^{x_{0}} \left(\prod_{\substack{j=1 \ j \neq i}}^{r} f_{j}(x) \right) \left(\int_{g(x)}^{g(x_{0})} (t - g(x))^{\nu-1} \left(D_{g(x_{0})-}^{\nu} (f_{i} \circ g^{-1}) \right) (t) dt \right) dx \right],
$$
\n(54)

Finally, adding (53) and (54) we obtain the useful identity

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS. VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\nand
\n
$$
\sum_{i=1}^{r} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) f_i(x) - \sum_{i=1}^{r} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) f_i(x) = \frac{1}{\prod_{j=1}^{r} f_j(x) f_j(x) + \frac{1}{\prod_{j=1}^{r} f_j(x) f_j(x) + \frac{1}{\prod_{j=1}^{r} f_j(x) f_j(x) + \frac{1}{\prod_{j=1}^{r} f_j(x) + \frac{1
$$

$$
+ \left[\int_{x_0}^b \left(\prod_{\substack{j=1 \ j \neq i}}^r f_j(x) \right) \left(\int_{g(x_0)}^{g(x)} \left(g(x) - t \right)^{\nu-1} \left(D_{g(x_0)}^{\nu} \left(f_i \circ g^{-1} \right) \right) (t) dt \right) dx \right] \Bigg] \Bigg] \, .
$$

Therefore, we get that

J. COMPUTATIONAL AVALYSIS AND APPLICATIONS, Vol. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\n
$$
+ \left[\int_{x_0}^{b} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) \left(\int_{g(x_0)}^{g(x)} (g(x) - t)^{v-1} \left(D_{g(x_0)}^v (f_i \circ g^{-1}) \right) (t) dt \right) dx \right],
$$
\nTherefore, we get that
\n
$$
\left\| \sum_{i=1}^{r} \left[\int_{a}^{b} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) f_i(x) dx - \left(\int_{a}^{b} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) dx \right) f_i(x_0) \right\|_1 =
$$
\n
$$
\sum_{i=1}^{r} \left[\left\| \left[\int_{a}^{b} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) \left(\int_{g(x)}^{g(x_0)} (t - g(x))^{v-1} \left(D_{g(x_0)}^v (f_i \circ g^{-1}) \right) (t) dt \right) dx \right]_1 \right\|_1
$$
\n
$$
+ \left\| \left[\int_{x_0}^{b} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) \left(\int_{g(x_0)}^{g(x)} (g(x) - t)^{v-1} \left(D_{g(x_0)}^v (f_i \circ g^{-1}) \right) (t) dt \right) dx \right]_1 \right\|_1 =
$$
\n
$$
+ \left[\int_{a}^{b} \left\| \left[\int_{x_0}^{r} f_j(x) \right] \left(\int_{g(x_0)}^{g(x)} (g(x) - t)^{v-1} \left(D_{g(x_0)}^v (f_i \circ g^{-1}) \right) (t) dt \right) dx \right\|_1 =
$$
\n
$$
+ \left[\int_{a_0}^{b} \left\| \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) \left(\int_{g(x_0)}^{g(x)} (g(x) - t)^{v-1} \left(D
$$

Hence it holds

$$
||K(f_1, ..., f_r)(x_0)||_1 \leq (*).
$$
 (59)

We have that

$$
(*) \leq \frac{1}{\Gamma(\nu+1)}
$$

3. COMPUTATIONAL AVALYSIS AND APPLICATIONS, VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\n
$$
\sum_{i=1}^{v} \left[\left[\left\| \left\| \left(D_{g(x_0)}^{\nu} - (f_i \circ g^{-1}) \right) \right\|_2 \right\|_{\infty, [g(x_0), g(x_0)]} \int_x^{\pi_a} \left(\prod_{j=1}^{v} ||f_j(x)||_2 \right) (g(x_0) - g(x))^{\nu} dx \right] \right]
$$
\n+
$$
\left[\left\| \left\| \left(D_{g(x_0)}^{\nu} (f_i \circ g^{-1}) \right) \right\|_2 \right\|_{\infty, [g(x_0), g(x_0)]} \int_x^b \left(\prod_{j=1}^{r} ||f_j(x)||_2 \right) (g(x) - g(x_0))^{\nu} dx \right] \right] \le
$$
\n
$$
\frac{1}{\Gamma(\nu + 1)} \sum_{i=1}^{v} \left[\left[\left[\left\| \left(D_{g(x_0)}^{\nu} (f_i \circ g^{-1}) \right) \right\|_2 \right\|_{\infty, [g(x_0), g(x_0)]} \right) dx \right] + \left[\left\| \left\| \left(D_{g(x_0)}^{\nu} (f_i \circ g^{-1}) \right) \right\|_2 \right\|_{\infty, [g(x_0), g(x_0)]} (g(b) - g(x_0))^{\nu} \left(\int_{x_0}^b \left(\prod_{j=1}^{r} ||f_j(x)||_2 \right) dx \right) \right] \right],
$$
\n(61)
\n
$$
\left[\left\| \left\| \left(D_{g(x_0)}^{\nu} (f_i \circ g^{-1}) \right) \right\|_2 \right\|_{\infty, [g(x_0), g(x_0)]} (g(b) - g(x_0))^{\nu} \left(\int_{x_0}^b \left(\prod_{j=1}^{r} ||f_j(x)||_2 \right) dx \right) \right] \right],
$$
\n(62)
\n
$$
\sum_{i=1}^{r} \left[\left[\left\| \left\| \left(D_{g(x_0)}^{\nu} (f_i \circ g^{-1}) \right) \right\|_2 \right\|_{\infty, [g(x_0), g
$$

proving (46) .

Next comes an \mathcal{L}_1 estimate.

Theorem 13 All as in Theorem 12. Then

$$
\|K(f_1, ..., f_r)(x_0)\|_1 \leq \frac{1}{\Gamma(\nu)}
$$

\n
$$
\sum_{i=1}^r \left[\left[\left\| \left\| \left(D_{g(x_0) -}^{\nu}(f_i \circ g^{-1}) \right) \right\|_2 \right\|_{L_1([g(a), g(x_0)])} \int_a^{x_0} \left(\prod_{\substack{j=1 \ j \neq i}}^r \|f_j(x)\|_2 \right) (g(x_0) - g(x))^{\nu-1} dx \right] \right]
$$

\n
$$
+ \left[\left\| \left\| \left(D_{g(x_0)}^{\nu}(f_i \circ g^{-1}) \right) \right\|_2 \right\|_{L_1([g(x_0), g(b)])} \int_{x_0}^b \left(\prod_{\substack{j=1 \ j \neq i}}^r \|f_j(x)\|_2 \right) (g(x) - g(x_0))^{\nu-1} dx \right] \right].
$$

\n(62)

1

Proof. We observe that (by (58) , (59))

$$
(*) \leq \frac{1}{\Gamma(\nu)}
$$

$$
\sum_{i=1}^{r} \left[\left[\left\| \left\| \left(D_{g(x_0) -}^{\nu}(f_i \circ g^{-1}) \right) \right\|_2 \right\|_{L_1([g(a), g(x_0)])} \int_a^{x_0} \left(\prod_{\substack{j=1 \ j \neq i}}^{r} \left\| f_j(x) \right\|_2 \right) (g(x_0) - g(x))^{\nu-1} dx \right] \right]
$$

(63)

$$
+ \left[\left. \left\| \left\| \left(D_{g(x_0)}^{\nu} \left(f_i \circ g^{-1} \right) \right) \right\|_2 \right\|_{L_1([g(x_0), g(b)])} \int_{x_0}^b \left(\prod_{\substack{j=1 \\ j \neq i}}^r \left\| f_j \left(x \right) \right\|_2 \right) \left(g \left(x \right) - g \left(x_0 \right) \right)^{\nu-1} dx \right] \right],
$$

proving (62). \blacksquare

An L_p estimate follows.

Theorem 14 All as in Theorem 12. Let now $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then

$$
||K(f_1, ..., f_r)(x_0)||_1 \leq \frac{1}{(p(\nu - 1) + 1)^{\frac{1}{p}} \Gamma(\nu)}
$$

$$
\sum_{i=1}^r \left[\left[\left\| \left\| \left(D_{g(x_0) -}^{\nu}(f_i \circ g^{-1}) \right) \right\|_2 \right\|_{q, [g(a), g(x_0)]} \left(\int_a^{x_0} (g(x_0) - g(x))^{\nu - \frac{1}{q}} \left(\prod_{\substack{j=1 \ j \neq i}}^r \|f_j(x)\|_2 \right) dx \right) \right] \right]
$$

+
$$
\left[\left\| \left\| \left(D_{g(x_0)}^{\nu}(f_i \circ g^{-1}) \right) \right\|_2 \right\|_{q, [g(x_0), g(b)]} \left(\int_{x_0}^b (g(x) - g(x_0))^{\nu - \frac{1}{q}} \left(\prod_{\substack{j=1 \ j \neq i}}^r \|f_j(x)\|_2 \right) dx \right) \right] \right].
$$

Proof. We have that $(by (58), (59))$

J. COMPUTIONAL ANALYSIS AND APPLICATIONS. VOL. 51. NO. 4. 2023. COPYRIGHT 2023 EUDOXUS FRESS. LIC
\n
$$
+ \left[\left\|\left\|\left(D_{g(x_0)}^{\nu}(f_i \circ g^{-1})\right)\right\|_2\right\|_{L_1([g(x_0),g(b)])} \int_{x_0}^b \left(\prod_{\substack{j=1 \ j \neq i}}^r \|f_j(x)\|_2\right) (g(x) - g(x_0))^{\nu-1} dx\right\|_1^b,
$$
\n\nproving (62).
$$
=
$$
\nAn L_p estimate follows.
\n**Theorem 14** All as in Theorem 22. Let now $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then\n\n
$$
\|K(f_1,...,f_r)(x_0)\|_1 \leq \frac{1}{(p(\nu-1)+1)^{\frac{1}{2}}\Gamma(\nu)}
$$
\n
$$
\sum_{i=1}^r \left[\left[\left\|\left\|\left(D_{g(x_i)}^{\nu}(f_i \circ g^{-1})\right)\right\|_2\right\|_{q,[g(x_i),g(x_i)]} \left(\int_a^b (g(x_0) - g(x_0))^{\nu-\frac{1}{q}} \left(\prod_{\substack{j=1 \ j \neq i}}^r \|f_j(x)\|_2\right) dx\right)\right]\right].
$$
\n**Proof.** We have that (by (58), (59))
\n
$$
(*) \leq \frac{1}{\Gamma(\nu)} \sum_{i=1}^r \left[\left[\int_a^{n_0} \left(\prod_{\substack{j=1 \ j \neq i}}^r \|f_j(x)\|_2\right) \left(\int_{g(x)}^{g(x_0)} (t - g(x_0))^{\nu(\nu-1)} dt\right)^{\frac{1}{2}} + \left(\int_{g(x_0)}^{g(x_0)} \left[\left(D_{g(x_0)}^{\nu}(f(x_0) - f)^{\nu(\nu-1)} dt\right)^{\frac{1}{2}} dt\right] + \left(\int_{g(x_0)}^{g(x_0)} \left[\left(D_{g(x_0)}^{\nu}(f(x_0) - f)^{\nu(\nu-1)} dt\right)^{\frac{1}{2}} dt\right] + \left(\int_{g(x_0)}^{g(x_0)} \left[\left(D_{g(x_0)}^{\nu}(
$$

3. COMPUTATIONAL AVALYSIS AND APPLICATIONS, Vol. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\n
$$
+ \left[\int_{x_0}^{x} \left(\prod_{j \neq k}^{r} ||f_j(x)||_2 \right) \frac{(g(x) - g(x_0))^{v-1 + \frac{1}{p}}}{(p(\nu - 1) + 1)^{\frac{1}{p}}} \left\| \left\| \left(D_{g(x_0)}^{\nu}(f_1 \circ g^{-1}) \right) (z) \right\|_2 \right\|_{g, [g(x_0), g(b)]} dx \right\} \right]
$$
\n
$$
= \frac{1}{(p(\nu - 1) + 1)^{\frac{1}{p}} \Gamma(\nu)}
$$
\n
$$
\sum_{i=1}^{r} \left[\left\| \left\| \left(D_{g(x_0)}^{\nu}(f_i \circ g^{-1}) \right) \right\|_2 \right\|_{q, [g(x_0), g(b)]} \left(\int_{x_0}^{x_0} (g(x_0) - g(x))^{v-\frac{1}{q}} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} ||f_j(x)||_2 \right) dx \right) \right]
$$
\n
$$
+ \left\| \left\| \left(D_{g(x_0)}^{\nu}(f_i \circ g^{-1}) \right) \right\|_2 \right\|_{q, [g(x_0), g(b)]} \left(\int_{x_0}^{x} (g(x) - g(x_0))^{v-\frac{1}{q}} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} ||f_j(x)||_2 \right) dx \right) \right\},
$$
\n
$$
|W(\text{FOMI, } x) \in [0, 1] \times [
$$

proving (64) .

We continue with γ -Schatten norm related Ostrowski fractional inequalities:

Theorem 15 Let $\gamma \geq 1$, the \ast -ideal $\mathcal{B}_{\gamma}(H)$, which $\left(\mathcal{B}_{\gamma}(H), \left\|\cdot\right\|_{\gamma}\right)$ is a Banach algebra; $x_0 \in [a, b] \subset \mathbb{R}, \nu \ge 1, n = [\nu]; f_i \in C^n([a, b], \mathcal{B}_{\gamma}(H)), i = 1, ..., r \in$ $\mathbb{N} - \{1\}; g \in C^1([a, b]),$ strictly increasing such that $g^{-1} \in C^n([g(a), g(b)]),$ with $(f_i \circ g^{-1})^{(k)} (g(x_0)) = 0, \quad k = 1, ..., n-1; \quad i = 1, ..., r.$ Assume further that $f_i \circ g^{-1} \in C_{g(x_0)-}^{\nu}([g(a), g(b)], \mathcal{B}_{\gamma}(H)) \cap C_{g(x_0)}^{\nu}([g(a), g(b)], \mathcal{B}_{\gamma}(H)), i =$ $1, ..., r.$

Here $K(f_1, ..., f_r)(x_0)$ is as in (45). Then

$$
||K(f_1, ..., f_r)(x_0)||_{\gamma} \leq \frac{1}{\Gamma(\nu+1)} \sum_{i=1}^r \left[\left[\left\| \left\| D_{g(x_0) -}^{V}(f_i \circ g^{-1}) \right\|_{\gamma} \right\|_{\infty, [g(a), g(x_0)]} \right] \right]
$$

$$
(g(x_0) - g(a))^{\nu} \left(\int_a^{x_0} \left(\prod_{\substack{j=1 \ j \neq i}}^r \|f_j(x)\|_{\gamma} \right) dx \right) + \tag{67}
$$

$$
\left[\left\| \left\| D_{g(x_0)}^{\nu}(f_i \circ g^{-1}) \right\|_{\gamma} \right\|_{\infty, [g(x_0), g(b)]} (g(b) - g(x_0))^{\nu} \left(\int_{x_0}^b \left(\prod_{\substack{j=1 \ j \neq i}}^r \|f_j(x)\|_{\gamma} \right) dx \right) \right] \right]
$$

Proof. As similar to Theorem 12 is omitted. Use of (41). \blacksquare An L_1 estimate follows:

:

Theorem 16 All as in Theorem 15. Then

$$
\|K(f_1, ..., f_r)(x_0)\|_{\gamma} \leq \frac{1}{\Gamma(\nu)}
$$

$$
\sum_{i=1}^r \left[\left[\left\| \left\| \left(D_{g(x_0)-}^{\nu}(f_i \circ g^{-1}) \right) \right\|_{\gamma} \right\|_{L_1([g(a),g(x_0)])} \int_a^{x_0} \left(\prod_{\substack{j=1 \ j \neq i}}^r \|f_j(x)\|_{\gamma} \right) (g(x_0) - g(x))^{\nu-1} dx \right] \right]
$$

$$
+ \left[\left\| \left\| \left(D_{g(x_0)}^{\nu}(f_i \circ g^{-1}) \right) \right\|_{\gamma} \right\|_{L_1([g(x_0),g(b)])} \int_{x_0}^b \left(\prod_{\substack{j=1 \ j \neq i}}^r \|f_j(x)\|_{\gamma} \right) (g(x) - g(x_0))^{\nu-1} dx \right] \right].
$$

(68)

Proof. As similar to Theorem 13 is omitted. \blacksquare An L_p estimate follows.

Theorem 17 All as in Theorem 15. Let now $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then

J. COMPUTATIONAL AVALYSIS AND APPLICATIONS, VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LIC
\n
$$
\sum_{i=1}^{r} \left[\left[\left\| \left\| \left(D_{g(x_0)}^{\nu}(f_i \circ g^{-1}) \right) \right\|_{1} \right\|_{L_1([g(x_0),g(x_0)])} \int_{x_0}^{x_0} \left(\prod_{j=1}^{r} \left\| f_j(x) \right\|_{\infty} \right) (g(x_0) - g(x))^{v-1} dx \right] \right]
$$
\n(68)
\n
$$
+ \left[\left[\left\| \left\| \left(D_{g(x_0)}^{\nu}(f_i \circ g^{-1}) \right) \right\|_{1} \right\|_{L_1([g(x_0),g(b)])} \int_{x_0}^{x_0} \left(\prod_{j=1}^{r} \left\| f_j(x) \right\|_{\infty} \right) (g(x) - g(x_0))^{v-1} dx \right] \right].
$$
\nPROC. As similar to Theorem 13 is omitted.
\nAn L_p estimate follows.
\n
$$
\text{Theorem 17. All as in Theorem 15. Let now } p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1. Then
$$
\n
$$
\| K (f_1,...,f_r) (x_0) \|_{\infty} \leq \frac{1}{(p(\nu - 1) + 1)^{\frac{1}{p}} \Gamma(\nu)}
$$
\n
$$
\sum_{i=1}^{r} \left[\left[\left\| \left\| \left(D_{g(x_0)}^{\nu}(f_i \circ g^{-1}) \right) \right\|_{\infty} \right\|_{g(x_0),g(x_0)} \right| \left(\int_{x_0}^{x_0} (g(x_0) - g(x))^{v-\frac{1}{q}} \left(\prod_{j=1}^{r} \left\| f_j(x) \right\|_{\infty} \right) dx \right) \right] \right].
$$
\nProof. As similar to Theorem 14 is omitted.
\nWhen $r = 2$ we derive the following p-Schatten norm operator related Os-
\ntowski type Caavasti fractional inequalities.
\n**Theorem 18. Let $g(x_0) = g(x_0)^{v-1} \left(\prod_{j=1}^{r}$**

Proof. As similar to Theorem 14 is omitted. \blacksquare

When $r = 2$ we derive the following p-Schatten norm operator related Ostrowski type Canavati fractional inequalities.

Theorem 18 Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, and let the *-ideals $\mathcal{B}_p(H)$, $\mathcal{B}_q(H)$, for which $\left(B_p(H), \left\|\cdot\right\|_p\right), \left(\mathcal{B}_q(H), \left\|\cdot\right\|_q\right)$ are Banach algebras; $x_0 \in [a, b] \subset \mathbb{R}, \alpha \geq 0$ 1, $n = [\alpha]$; $A_1 \in C^n([a, b], \mathcal{B}_p(H))$, $A_2 \in C^n([a, b], \mathcal{B}_q(H))$; $g \in C^1([a, b]),$ strictly increasing, such that $g^{-1} \in C^n ([g (a), g (b)])$, with $(A_i \circ g^{-1})^{(k)} (g (x_0)) =$ $0, k = 1, ..., n-1; i = 1, 2$. Assume further that $A_1 \circ g^{-1} \in C^{\alpha}_{g(x_0)-}([g(a), g(b)], \mathcal{B}_p(H)) \cap$ $C_{g(x_0)}^{\alpha}([g(a),g(b)], \mathcal{B}_p(H)), \text{ and } A_2 \circ g^{-1} \in C_{g(x_0)-}^{\alpha}([g(a),g(b)], \mathcal{B}_q(H)) \cap$ $C_{g(x_0)}^{\alpha}([g(a),g(b)],\mathcal{B}_q(H))$. Then

1) it holds

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS. VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\n
$$
\Phi(A_1, A_2) (x_0) := \int_a^b A_2(x) A_1(x) + \int_a^b A_1(x) A_2(x) dx -
$$
\n
$$
\left(\int_a^b A_2(x) dx \right) A_1(x_0) - \left(\int_a^b A_1(x) dx \right) A_2(x_0) =
$$
\n
$$
\frac{1}{\Gamma(\alpha)} \left\{ \left[\int_a^{x_0} A_2(x) \left(\int_{g(x)}^{g(x_0)} (x - a(x))^{a-1} (D_{g(x_0)}^a - (A_1 \circ g^{-1})) (x) dx \right) dx \right] +
$$
\n
$$
\left[\int_a^b A_2(x) \left(\int_{g(x)}^{g(x_0)} (g(x) - z)^{a-1} (D_{g(x_0)}^b - (A_2 \circ g^{-1})) (z) dx \right) dx \right] +
$$
\n
$$
\left[\int_a^b A_1(x) \left(\int_{g(x)}^{g(x_0)} (g(x) - z)^{a-1} (D_{g(x_0)}^b - (A_2 \circ g^{-1})) (z) dx \right) dx \right] +
$$
\n
$$
\left[\int_a^b A_1(x) \left(\int_{g(x_0)}^{g(x_0)} (g(x) - z)^{a-1} (D_{g(x_0)}^b - (A_2 \circ g^{-1})) (z) dx \right) dx \right] +
$$
\n
$$
\left\{ \left[\left\| \left\| D_{g(x_0)}^b (A_1 \circ g^{-1}) \right\|_{x} \right\|_{x \in [g(x_0), g(x_0)]} \int_a^{x_0} \|A_2(x) \|_{x} (g(x_0) - g(x))^{a-2} dx \right] +
$$
\n
$$
\left\{ \left[\left\| \left\| D_{g(x_0)}^b (A_1 \circ g^{-1}) \right\|_{x} \right\|_{x \in [g(x_0), g(x_0)]} \int_a^{x_0} \|A_2(x) \|_{x} (g(x_0) - g(x))^{a-2} dx \right] +
$$
\n
$$
\left[\left\| \left\| D_{g(x_0)}^b (A_1 \circ g^{-1})
$$

2) for
$$
\gamma
$$
, $\delta > 1 : \frac{1}{\gamma} + \frac{1}{\delta} = 1$, we have that

$$
\|\Phi(A_{1}, A_{2})(x_{0})\|_{1} \leq \frac{1}{\Gamma(\alpha) (\gamma(\alpha - 1) + 1)^{\frac{1}{\gamma}}}
$$

$$
\left\{\left[\left\|\left\|D_{g(x_{0})-}^{\alpha}(A_{1} \circ g^{-1})\right\|_{p}\right\|_{\delta, [g(a), g(x_{0})]} \int_{a}^{x_{0}} \|A_{2}(x)\|_{q} (g(x_{0}) - g(x))^{\alpha - \frac{1}{\delta}} dx\right]+\right.
$$

$$
\left[\left\|\left\|D_{g(x_{0})}^{\alpha}(A_{1} \circ g^{-1})\right\|_{p}\right\|_{\delta, [g(x_{0}), g(b)]} \int_{x_{0}}^{b} \|A_{2}(x)\|_{q} (g(x) - g(x_{0}))^{\alpha - \frac{1}{\delta}} dx\right]+\right.
$$

$$
\left[\left\|\left\|D_{g(x_{0})-}^{\alpha}(A_{2} \circ g^{-1})\right\|_{q}\right\|_{\delta, [g(a), g(x_{0})]} \int_{a}^{x_{0}} \|A_{1}(x)\|_{p} (g(x_{0}) - g(x))^{\alpha - \frac{1}{\delta}} dx\right]+\right.
$$

$$
\left[\left\|\left\|D_{g(x_{0})}^{\alpha}(A_{2} \circ g^{-1})\right\|_{q}\right\|_{\delta, [g(x_{0}), g(b)]} \int_{x_{0}}^{b} \|A_{1}(x)\|_{p} (g(x) - g(x_{0}))^{\alpha - \frac{1}{\delta}} dx\right]\right\},
$$

$$
g) we also obtain
$$

$$
\|\Phi(A_{1}, A_{2})(x_{0})\|_{1} \leq \frac{1}{\Gamma(\alpha)}
$$

$$
\left\{ \left[\left\| \left\| D_{g(x_0)-}^{\alpha}(A_1 \circ g^{-1}) \right\|_p \right\|_{L_1([g(a),g(x_0)])} \int_a^{x_0} \left\| A_2(x) \right\|_q (g(x_0) - g(x))^{\alpha-1} dx \right] + \right\}
$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS. VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\n
$$
\left[\left\|\left\|D_{g(x_0)}^{\alpha}(A_1 \circ g^{-1})\right\|_{g}\right\|_{L_1([g(x_0),g(x_0)])} \int_{x_0}^{x_0} \|A_2(x)\|_{g}(g(x)-g(x_0))^{\alpha-1} dx\right]+\left[\left\|\left\|D_{g(x_0)}^{\alpha}(A_2 \circ g^{-1})\right\|_{g}\right\|_{L_1([g(x_0),g(x_0)])} \int_{x_0}^{x_0} \|A_1(x)\|_{p}(g(x_0)-g(x))^{\alpha-1} dx\right]+\left[\left\|D_{g(x_0)}^{\alpha}(A_2 \circ g^{-1})\right\|_{g}\right\|_{L_1([g(x_0),g(x_0)])} \int_{x_0}^{x_0} \|A_1(x)\|_{p}(g(x)-g(x_0))^{\alpha-1} dx\right]+\left[\left\|\left\|D_{g(x_0)}^{\alpha}(A_2 \circ g^{-1})\right\|_{p}\right\|_{\infty,[g(x),g(x_0)]} \int_{x_0}^{x_0} \|A_2(x)\|_{q}(g(x_0)-g(x))^{\alpha} dx\right]+\left[\left\|\left\|D_{g(x_0)}^{\alpha}(A_1 \circ g^{-1})\right\|_{p}\right\|_{\infty,[g(x),g(x_0)]} \int_{x_0}^{x_0} \|A_2(x)\|_{q}(g(x)-g(x_0))^{\alpha} dx\right]+\left[\left\|\left\|D_{g(x_0)}^{\alpha}(A_1 \circ g^{-1})\right\|_{p}\right\|_{\infty,[g(x),g(x_0)]} \int_{x_0}^{x_0} \|A_1(x)\|_{p}(g(x)-g(x_0))^{\alpha} dx\right]+\left[\left\|\left\|D_{g(x_0)}^{\alpha}(A_2 \circ g^{-1})\right\|_{q}\right\|_{\infty,[g(x),g(x_0)]} \int_{x_0}^{x_0} \|A_1(x)\|_{p}(g(x)-g(x_0))^{\alpha} dx\right]+\right]+\left[\left\|\left\|D_{g(x_0)}^{\alpha}(A_2 \circ g^{-1})\right\|_{q}\right\|_{\infty,[g(x),g(x_0)]} \int_{x_0}^{x_0} \|A_1(x)\|_{p
$$

Proof. Here we have that (acting as in the proof of Theorem 12 for $r = 2$)

$$
\Phi (A_1, A_2) (x_0) := \int_a^b A_2(x) A_1(x) + \int_a^b A_1(x) A_2(x) dx -
$$
\n
$$
\left(\int_a^b A_2(x) dx \right) A_1(x_0) - \left(\int_a^b A_1(x) dx \right) A_2(x_0) \stackrel{(55)}{=}
$$
\n
$$
\frac{1}{\Gamma(\alpha)} \left\{ \left[\int_a^{x_0} A_2(x) \left(\int_{g(x)}^{g(x_0)} (z - g(x))^{\alpha - 1} \left(D_{g(x_0) -}^{\alpha} (A_1 \circ g^{-1}) \right) (z) dz \right) dx \right] +
$$
\n
$$
\left[\int_{x_0}^b A_2(x) \left(\int_{g(x_0)}^{g(x)} (g(x) - z)^{\alpha - 1} \left(D_{g(x_0)}^{\alpha} (A_1 \circ g^{-1}) \right) (z) dz \right) dx \right] +
$$
\n
$$
\left[\int_a^{x_0} A_1(x) \left(\int_{g(x)}^{g(x_0)} (z - g(x))^{\alpha - 1} \left(D_{g(x_0) -}^{\alpha} (A_2 \circ g^{-1}) \right) (z) dz \right) dx \right] +
$$
\n
$$
\left[\int_{x_0}^b A_1(x) \left(\int_{g(x_0)}^{g(x)} (g(x) - z)^{\alpha - 1} \left(D_{g(x_0)}^{\alpha} (A_2 \circ g^{-1}) \right) (z) dz \right) dx \right] \right\}.
$$
\n(74)

Therefore it holds by taking the 1-Schatten norm that

J. COMPUTATIONAL AVALYSIS AND APPLICATIONS, VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\nTherefore it holds by taking the L-Schatten norm that
\n
$$
\|\Phi(A_1, A_2)(x_0)\|_1 = \left\|\int_a^b A_2(x) A_1(x) + \int_a^b A_1(x) A_2(x) dx - \left(\int_a^b A_2(x) dx\right) A_1(x_0) - \left(\int_a^b A_1(x) dx\right) A_2(x_0)\right\|_1 \le
$$
\n
$$
\frac{1}{\Gamma(\alpha)} \left\{\left[\left\|\int_a^{x_0} A_2(x) \left(\int_{g(x)}^{g(x_0)} (x - y)^{\alpha - 1} \left(D_{g(x_0)}^{\alpha} - (A_1 \circ g^{-1})\right)(z) dz\right) dx\right\}\right]_1 + \left\{\left\|\int_{x_0}^{x_0} A_1(x) \left(\int_{g(x_0)}^{g(x_0)} (y(x - z)^{\alpha - 1} \left(D_{g(x_0)}^{\alpha} - (A_1 \circ g^{-1})\right)(z) dz\right) dx\right\|_1\right\} + \left\{\left\|\int_{x_0}^{x_0} A_1(x) \left(\int_{g(x_0)}^{g(x_0)} (y - z)^{\alpha - 1} \left(D_{g(x_0)}^{\alpha} - (A_2 \circ g^{-1})\right)(z) dz\right) dx\right\|_1\right\} + \left\{\left\|\int_{x_0}^{x_0} A_1(x) \left(\int_{g(x_0)}^{g(x_0)} (x - z)^{\alpha - 1} \left(D_{g(x_0)}^{\alpha} - (A_2 \circ g^{-1})\right)(z) dz\right) dx\right\|_1\right\} \le
$$
\n
$$
\frac{1}{\Gamma(\alpha)} \left\{\left[\int_{x_0}^{x_0} \left\|A_2(x) \left(\int_{g(x_0)}^{g(x_0)} (x - z)^{\alpha - 1} \left(D_{g(x_0)}^{\alpha} - (A_2 \circ g^{-1})\right)(z) dz\right)\right\|_1 dx\right\} + \left\{\int_{x_0}^{x_0} \left\|A_1(x) \left(\int_{g(x_0)}^{g(x_0)} (x - z)^{\alpha - 1} \left(D_{g(x_0)}^{\alpha} - (A_2 \
$$

(by using the *p*-Schatten norm and Hölder's type inequality (44) for $p, q > 1$: $\frac{1}{p} + \frac{1}{q} = 1$

$$
\frac{1}{\Gamma(\alpha)} \left\{ \left[\int_{a}^{x_{0}} \left\| A_{2} (x) \right\|_{q} \left\| \left(\int_{g(x)}^{g(x_{0})} (z - g (x))^{\alpha - 1} \left(D_{g(x_{0}) -}^{\alpha} (A_{1} \circ g^{-1}) \right) (z) dz \right) \right\|_{p} dx \right] +
$$

$$
\left[\int_{x_{0}}^{b} \left\| A_{2} (x) \right\|_{q} \left\| \left(\int_{g(x_{0})}^{g(x)} (g (x) - z)^{\alpha - 1} \left(D_{g(x_{0})}^{\alpha} (A_{1} \circ g^{-1}) \right) (z) dz \right) \right\|_{p} dx \right] +
$$

20

2 4 Z ^x⁰ a kA¹ (x)k^p Z ^g(x0) g(x) (^z ^g (x))¹ D g(x0) A² g 1 (z) dz! q dx 3 5 + 2 4 Z ^b x⁰ kA¹ (x)k^p Z ^g(x) g(x0) (g (x) z) 1 D g(x0) A² g 1 (z) dz! q dx 3 5 9 = ; (77) 1 () ("Z ^x⁰ a kA² (x)k^q Z ^g(x0) g(x) (^z ^g (x))¹ D g(x0) A¹ g 1 (z) p dz! dx# + "Z ^b x⁰ kA² (x)k^q Z ^g(x) g(x0) (g (x) z) 1 D g(x0) A¹ g 1 (z) p dz! dx# + "Z ^x⁰ a kA¹ (x)k^p Z ^g(x0) g(x) (^z ^g (x))¹ D g(x0) A² g 1 (z) q dz! dx# + "Z ^b x⁰ kA¹ (x)k^p Z ^g(x) g(x0) (g (x) z) 1 D g(x0) A² g 1 (z) q dz! dx#) : (78) 603 J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC George A. Anastassiou 583-621

We have proved, so far, that

$$
\|\Phi(A_{1}, A_{2}) (x_{0})\|_{1} \le
$$
\n
$$
\frac{1}{\Gamma(\alpha)} \left\{ \left[\int_{a}^{x_{0}} \|A_{2} (x)\|_{q} \left(\int_{g(x)}^{g(x_{0})} (z - g (x))^{\alpha - 1} \left\| \left(D_{g(x_{0})-}^{\alpha} (A_{1} \circ g^{-1}) \right) (z) \right\|_{p} dz \right) dx \right] + \left[\int_{x_{0}}^{b} \|A_{2} (x)\|_{q} \left(\int_{g(x_{0})}^{g(x)} (g (x - z)^{\alpha - 1} \left\| \left(D_{g(x_{0})}^{\alpha} (A_{1} \circ g^{-1}) \right) (z) \right\|_{p} dz \right) dx \right] + \left[\int_{a}^{x_{0}} \|A_{1} (x)\|_{p} \left(\int_{g(x)}^{g(x_{0})} (z - g (x))^{\alpha - 1} \left\| \left(D_{g(x_{0})-}^{\alpha} (A_{2} \circ g^{-1}) \right) (z) \right\|_{q} dz \right) dx \right] + \left[\int_{x_{0}}^{b} \|A_{1} (x)\|_{p} \left(\int_{g(x_{0})}^{g(x)} (g (x - z)^{\alpha - 1} \left\| \left(D_{g(x_{0})}^{\alpha} (A_{2} \circ g^{-1}) \right) (z) \right\|_{q} dz \right) dx \right] \right\} =: (\lambda).
$$
\n(79)

Let now $\gamma, \delta > 1$ such that $\frac{1}{\gamma} + \frac{1}{\delta} = 1$, and we apply the usual Hölder's inequality in (79). Then we have that

$$
\|\Phi(A_1, A_2)(x_0)\|_1 \leq (\lambda) \leq \frac{1}{\Gamma(\alpha) (\gamma (\alpha - 1) + 1)^{\frac{1}{\gamma}}}
$$

$$
\left\{ \left[\int_a^{x_0} \|A_2(x)\|_q (g(x_0) - g(x))^{\frac{\gamma (\alpha - 1) + 1}{\gamma}} \left(\int_{g(x)}^{g(x_0)} \left\| \left(D_{g(x_0)}^{\alpha} - (A_1 \circ g^{-1})\right)(z)\right\|_p^{\delta} dz \right)^{\frac{1}{\delta}} dx \right] + \right\}
$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS. VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\n
$$
\left[\int_{x_0}^{b} ||A_2(x)||_q(g(x) - g(x_0))^{\frac{2(n-1)+1}{2}} \left(\int_{g(x_0)}^{g(x_0)} ||(D_{g(x_0)}^{\alpha} - (A_2 \circ g^{-1})) (z)||_q^5 dz\right)^{\frac{1}{3}} dx\right] +
$$
\n
$$
\left[\int_{a}^{x_0} ||A_1(x)||_p(g(x_0) - g(x_0))^{\frac{2(n-1)+1}{2}} \left(\int_{g(x)}^{g(x_0)} ||(D_{g(x_0)}^{\alpha} - (A_2 \circ g^{-1})) (z)||_q^5 dz\right)^{\frac{1}{3}} dz\right] +
$$
\n
$$
\left[\int_{x_0}^{b} ||A_1(x)||_p(g(x) - g(x_0))^{\frac{2(n-1)+1}{2}} \left(\int_{g(x_0)}^{g(x_0)} ||(D_{g(x_0)}^{\alpha} - (A_2 \circ g^{-1})) (z)||_q^5 dz\right)^{\frac{1}{2}} dx\right]\right]
$$
\n
$$
\leq \frac{1}{\Gamma(\alpha)\left(\gamma(\alpha - 1) + 1\right)^{\frac{1}{2}}}
$$
\n
$$
\left\{\left[\left|\left||D_{g(x_0)}^{\alpha} - (A_1 \circ g^{-1})\right|\right|_{x}\right|\bigg|_{\mathcal{L}_2(g(x_0), g(x_0))}\int_{x_0}^{x_0} ||A_2(x)||_q(g(x) - g(x))^{\alpha - \frac{1}{2}} dx\right]+\right]
$$
\n
$$
\left[\left|\left||D_{g(x_0)}^{\alpha} (A_1 \circ g^{-1})\right|\right|_{x}\bigg|_{\mathcal{L}_2(g(x_0), g(x_0))}\int_{x_0}^{x_0} ||A_1(x)||_p(g(x) - g(x))^{\alpha - \frac{1}{2}} dx\right]+\right]
$$
\n
$$
\left[\left|\left||D_{g(x_0)}^{\alpha} (A_2 \circ g^{-1})\right|\right|_{x}\bigg|_{\mathcal{L}_2(g(x_0), g(x_0))}\int_{x_0}^{x_0} ||A_1(x)||_p(g(x) - g(x_0))^{\alpha - \frac{1}{2}} dx\right]+\
$$

proving (71).

$$
\hbox{\it We also obtain} \\
$$

$$
\|\Phi(A_1, A_2)(x_0)\|_1 \leq (\lambda) \leq \frac{1}{\Gamma(\alpha)}
$$
\n
$$
\left\{ \left[\left\| \left\| D_{g(x_0)}^{\alpha} - (A_1 \circ g^{-1}) \right\|_p \right\|_{L_1([g(a), g(x_0)])} \int_a^{x_0} \|A_2(x)\|_q (g(x_0) - g(x))^{\alpha - 1} dx \right] + \left[\left\| \left\| D_{g(x_0)}^{\alpha} (A_1 \circ g^{-1}) \right\|_p \right\|_{L_1([g(x_0), g(b)])} \int_{x_0}^b \|A_2(x)\|_q (g(x) - g(x_0))^{\alpha - 1} dx \right] + \left[\left\| \left\| D_{g(x_0)-}^{\alpha} (A_2 \circ g^{-1}) \right\|_q \right\|_{L_1([g(a), g(x_0)])} \int_a^{x_0} \|A_1(x)\|_p (g(x_0) - g(x))^{\alpha - 1} dx \right] + \left[\left\| \left\| D_{g(x_0)}^{\alpha} (A_2 \circ g^{-1}) \right\|_q \right\|_{L_1([g(x_0), g(b)])} \int_{x_0}^b \|A_1(x)\|_p (g(x) - g(x_0))^{\alpha - 1} dx \right] \right\},\tag{82}
$$
\n\nproving (72).

proving (72).

At last we derive

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS. VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\n
$$
\left\| \Phi(A_1, A_2) (x_0) \|_1 \leq (\lambda) \leq \frac{1}{\Gamma(\alpha + 1)}
$$
\n
$$
\left\{ \left[\left\| \left\| D_{g(x_0)}^{\alpha} - (A_1 \circ g^{-1}) \right\|_p \right\|_{\infty, [g(x_0), g(x_0)]} \int_x^{\pi_0} \|A_2(x) \|_q (g(x_0) - g(x))^{\alpha} dx \right\} + \left\| \left\| D_{g(x_0)}^{\alpha} (A_1 \circ g^{-1}) \right\|_p \right\|_{\infty, [g(x_0), g(x_0)]} \int_x^{\pi_0} \|A_1(x) \|_p (g(x) - g(x_0))^{\alpha} dx \right\} + \left\{ \left\| \left\| D_{g(x_0)}^{\alpha} (A_2 \circ g^{-1}) \right\|_q \right\|_{\infty, [g(x_0), g(x_0)]} \int_x^{\pi_0} \|A_1(x) \|_p (g(x) - g(x_0))^{\alpha} dx \right\} + \left\| \left\| D_{g(x_0)}^{\alpha} (A_2 \circ g^{-1}) \right\|_q \right\|_{\infty, [g(x_0), g(x_0)]} \int_x^{\pi_0} \|A_1(x) \|_p (g(x) - g(x_0))^{\alpha} dx \right\} + \left\| \left\| D_{g(x_0)}^{\alpha} (3) \right\|_{\infty}
$$
\nProving (73).

\nThe theorem is proved.

\nNow present *p*-Schutten left and right generalized Camavati fractional Opiity to the present *p*-Schutten (64 and right) σ of π of $[0, 1]$, π

proving (73).

The theorem is proved. ■

Next we present p-Schatten left and right generalized Canavati fractional Opial type inequalities:

Theorem 19 Let the \ast -ideal $\mathcal{B}_2(H)$, which $(\mathcal{B}_2(H), \|\cdot\|_2)$ is a Banach algebra; $x_0 \in [a, b] \subset \mathbb{R}, \nu \geq 1, n = [\nu]; f \in C^n([a, b], \mathcal{B}_2(H)), g \in C^1([a, b]), \text{ strictly}$ increasing such that $g^{-1} \in C^n([g(a), g(b)])$, with $(f \circ g^{-1})^{(k)}(g(x_0)) = 0$, $k = 0, 1, ..., n - 1$. Assume further that $f \circ g^{-1} \in C_{g(x_0)}^{\nu}([g(a), g(b)], \mathcal{B}_2(H))$. Let also $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then

$$
\int_{g(x_0)}^{z} \left\| \left(\left(f \circ g^{-1} \right) (w) \right) \left(\left(D_{g(x_0)}^{\nu} \left(f \circ g^{-1} \right) \right) (w) \right) \right\|_1 dw \leq \tag{84}
$$

 $2^{-\frac{1}{q}}(z-g(x_0))^{\nu+\frac{1}{p}-\frac{1}{q}}$ $\Gamma(\nu) [(p(\nu-1)+1) (p(\nu-1)+2)]^{\frac{1}{p}}$ $\int f^z$ $g(x_0)$ $\biggl\| \biggr\|$ $\left(D_{g{\left(x_{0}\right) }^{\nu }}^{\nu }\left(f\circ g^{-1}\right) \right) \left(w\right) \bigg\Vert$ q $\left(\begin{array}{c} q \\ 2 \end{array}\right)^{\frac{2}{q}}$; for all $q(x_0) \leq z \leq q(b)$.

Proof. Very similar to the proof of Theorem 13 of [6]. Use of (44) for $p = q = 2.$

A similar result comex next:

Theorem 20 Let $\gamma \geq 1$, the \ast -ideal $\mathcal{B}_{\gamma}(H)$, which $\left(\mathcal{B}_{\gamma}(H), \left\|\cdot\right\|_{\gamma}\right)$ is a Banach algebra; $x_0 \in [a, b] \subset \mathbb{R}, \nu \ge 1, n = [\nu]; f \in C^n ([a, b], \mathcal{B}_{\gamma}(H)), g \in C^1 ([a, b]),$ strictly increasing such that $g^{-1} \in C^n ([g (a), g (b)]),$ with $(f \circ g^{-1})^{(k)} (g (x_0)) =$ 0, $k = 0, 1, ..., n-1$. Assume further that $f \circ g^{-1} \in C_{g(x_0)}^{\nu}([g(a), g(b)], \mathcal{B}_{\gamma}(H))$. Let also $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then

$$
\int_{g(x_0)}^{z} \left\| \left(\left(f \circ g^{-1} \right) (w) \right) \left(\left(D_{g(x_0)}^{\nu} \left(f \circ g^{-1} \right) \right) (w) \right) \right\|_{\gamma} dw \leq \tag{85}
$$

$$
\frac{2^{-\frac{1}{q}}(z-g(x_0))^{\nu+\frac{1}{p}-\frac{1}{q}}}{\Gamma(\nu)[(p(\nu-1)+1)(p(\nu-1)+2)]^{\frac{1}{p}}}\left(\int_{g(x_0)}^z \left\|\left(D_{g(x_0)}^{\nu}(f\circ g^{-1})\right)(w)\right\|_{\gamma}^q dw\right)^{\frac{2}{q}},
$$

for all $g(x_0) \leq z \leq g(y)$.

Proof. Very similar to the proof of Theorem 13 of [6]. Use of (41) for $p = \gamma$.

It follows the corresponding right side fractional Opial type inequalities:

Theorem 21 All as in Theorem 19, however now it is $f \circ g^{-1} \in C_{g(x_0)-}^{\nu}([g(a), g(b)], \mathcal{B}_2(H))$. Then $ea(x_0)$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, Vol. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\nLet also
$$
p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1
$$
. Then
\n
$$
\int_{g(x_0)}^{z} ||((f \circ g^{-1})(w)) \left(\left(D'_{g(x_0)}(f \circ g^{-1})(w) \right) \Big| \Big|_{\infty} dw \leq (85)
$$
\n
$$
\frac{2^{-\frac{1}{q}} (z - g(x_0))^{w + \frac{1}{p} - \frac{1}{q}}}{\Gamma(v)[(p(v - 1) + 1)(p(v - 1) + 2)]^{\frac{1}{p}}} \left(\int_{g(x_0)}^{e} ||(D''_{g(x_0)}(f \circ g^{-1})(w)||_{\infty}^{q} dw) \right)^{\frac{1}{q}},
$$
\nfor all $g(x_0) \leq \varepsilon \leq g(b)$.
\nProof. Very similar to the proof of Theorem 13 of [6]. Use of (41) for $p = \gamma$.
\nIt follows the corresponding right side fractional Opial type inequalities:
\n**Theorem 21** All as in Theorem 19, however non if $is \log^{-1} \in C''_{g(x_0) -} ([g(a), g(b)], B_2(H))$.
\nThen
\n
$$
\int_{z}^{g(x_0)} ||((f \circ g^{-1})(w)) \left(\left(D''_{g(x_0)} - (f \circ g^{-1})(w) \right) \Big| \Big|_{1} dw \leq
$$
\n
$$
\frac{2^{-\frac{1}{q}} (g(x_0) - z)^{w + \frac{1}{p} - \frac{1}{q}}}{\Gamma(v)([p(v - 1) + 1)(p(v - 1) + 2])^{\frac{1}{p}}} \left(\int_{z}^{g(x_0)} ||(D''_{g(x_0)} - (f \circ g^{-1})(v)||_{2}^{2} dt) \right)^{\frac{2}{q}},
$$
\nfor all $g(a) \leq \varepsilon \leq g(x_0)$.
\nProof. Based on (20), and as similar to the proof of Theorem 19 is omitted.
\nNext comes another right side fractional Opial type inequality:
\n**Theorem 22** All as in Theorem 20, however now it is $f \circ g^{-1} \in C''_{g(x_0)} - ([g(a$

Proof. Based on (20) , and as similar to the proof of Theorem 19 is omitted.

Next comes another right ride fractional Opial type inequality:

Theorem 22 All as in Theorem 20, however now it is $f \circ g^{-1} \in C_{g(x_0)-}^{\nu}([g(a), g(b)], \mathcal{B}_{\gamma}(H)).$ Then $ea(x_0)$

$$
\int_{z}^{g(x_{0})} \left\| \left(\left(f \circ g^{-1} \right) (w) \right) \left(\left(D_{g(x_{0})-}^{\nu} \left(f \circ g^{-1} \right) \right) (w) \right) \right\|_{\gamma} dw \le
$$
\n
$$
\frac{2^{-\frac{1}{q}} \left(g(x_{0}) - z \right)^{\nu + \frac{1}{p} - \frac{1}{q}}}{\Gamma \left(\nu \right) \left[\left(p(\nu - 1) + 1 \right) \left(p(\nu - 1) + 2 \right) \right]^{\frac{1}{p}}} \left(\int_{z}^{g(x_{0})} \left\| \left(D_{g(x_{0})-}^{\nu} \left(f \circ g^{-1} \right) \right) (t) \right\|_{\gamma}^{q} dt \right)^{\frac{2}{q}},
$$
\n
$$
\text{for all } z(\alpha) \le \alpha \le \alpha(\alpha).
$$
\n(87)

for all $g(a) \leq z \leq g(x_0)$.

Proof. Based on (20) , and as similar to the proof of Theorem 19 is omitted.

It follows the modified generalized left $\mathcal{B}_2(H)$ -valued fractional Opial inequality:

Theorem 23 All as in Theorem 6, where $X = \mathcal{B}_2(H)$ and let $p, q > 1 : \frac{1}{p} + \frac{1}{q} =$ 1. Here we assume that $\frac{1}{(m+1)q} < \nu < 1$. Then

$$
\int_{g(x_0)}^{z} \left\| \left(\left(f \circ g^{-1} \right) (w) \right) \left(\left(D_{g(x_0)}^{(m+1)\nu} \left(f \circ g^{-1} \right) \right) (w) \right) \right\|_1 dw \le
$$
\n
$$
\frac{2^{-\frac{1}{q}} (z - g(x_0))^{(m+1)\nu + \frac{1}{p} - \frac{1}{q}}}{\Gamma \left((m+1) \nu \right) \left[(p \left((m+1) \nu - 1 \right) + 1 \right) \left(p \left((m+1) \nu - 1 \right) + 2 \right) \right]^{\frac{1}{p}}}
$$
\n
$$
\left(\int_{g(x_0)}^{z} \left\| \left(D_{g(x_0)}^{(m+1)\nu} \left(f \circ g^{-1} \right) \right) (t) \right\|_2^q dt \right)^{\frac{2}{q}},
$$
\n
$$
(88)
$$

for all $g(x_0) \leq z \leq g(b)$.

Proof. As in Theorem 19. ■

Next comes another modified generalized left $\mathcal{B}_{\gamma}(H)$ -valued fractional Opial inequality:

Theorem 24 All as in Theorem 6, where $X = \mathcal{B}_{\gamma}(H)$ and let $p, q > 1 : \frac{1}{p} + \frac{1}{q} =$ 1. Here we assume that $\frac{1}{(m+1)q} < \nu < 1$. Then

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, Vol. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\n**Theorem 23** All as in Theorem 6, where
$$
X = B_2(H)
$$
 and let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Here we assume that $\frac{1}{(m+1)q} < p < 1$. There is $\int_{g(x_0)}^{g(x_0)} \left\| \left(\int (f \circ g^{-1}) (w) \right) \left(\left(D_{g(x_0)}^{(m+1)\nu} (f \circ g^{-1}) \right) (w) \right) \right\|_1 dw \leq$ (88)
\n
$$
\frac{2^{-\frac{1}{2}} (z - g(x_0))^{(m+1)\nu + \frac{1}{p} - \frac{1}{q}}}{\Gamma((m+1)\nu)\Gamma((m+1)\nu - 1) + 1) (p((m+1)\nu - 1) + 2)]^{\frac{1}{p}}}
$$
\nfor all $g(x_0) \leq z \leq g(b)$.
\n**Proof.** As in Theorem 19. **Example**
\nNext, one as *m* theorem 19. **Example**
\nNext, one is $\frac{1}{(g(x_0))} \left\| \left(D_{g(x_0)}^{(m+1)\nu} (f \circ g^{-1}) \right) (b) \right\|_2^2 dk \right\}^{\frac{2}{q}}$,
\n f or all $g(x_0) \leq z \leq g(b)$.
\n**Proof.** As in Theorem 19. **Example**
\nNext, one is *m* Theorem 21. **Example**
\n
$$
\int_{g(x_0)}^{x} \left\| \left((f \circ g^{-1}) (w) \right) \left(\left(D_{g(x_0)}^{(m+1)\nu} (f \circ g^{-1}) \right) (w) \right) \right\|_2 dw \leq
$$
 (89)
\n
$$
\frac{2^{-\frac{1}{2}} (z - g(x_0))^{(m+1)\nu + \frac{1}{p} - \frac{1}{q}}}{\Gamma((m+1)\nu)\Gamma((m+1)\nu - 1) + 2) \Gamma^{\frac{1}{2}}}
$$
\n
$$
\
$$

for all $g(x_0) \leq z \leq g(b)$.

Proof. As in Theorem 19. ■

The corresponding modified generalized right $\mathcal{B}_2(H)$ -valued fractional Opial inequality comes next:

Theorem 25 All as in Theorem 7, where $X = \mathcal{B}_2(H)$ and let $p, q > 1 : \frac{1}{p} + \frac{1}{q} =$ 1. Here we assume that $\frac{1}{(m+1)q} < \nu < 1$. Then

$$
\int_{z}^{g(x_{0})} \left\| \left(\left(f \circ g^{-1} \right) (w) \right) \left(\left(D_{g(x_{0})-}^{(m+1)\nu} \left(f \circ g^{-1} \right) \right) (w) \right) \right\|_{1} dw \leq
$$
\n
$$
\frac{2^{-\frac{1}{q}} \left(g(x_{0}) - z \right)^{(m+1)\nu + \frac{1}{p} - \frac{1}{q}}}{\Gamma \left((m+1)\nu \right) \left[(p \left((m+1)\nu - 1 \right) + 1 \right) \left(p \left((m+1)\nu - 1 \right) + 2 \right) \right]^{\frac{1}{p}}}
$$
\n(90)

$$
\left(\int_{z}^{g(x_0)} \left\| \left(D_{g(x_0)-}^{(m+1)\nu} (f \circ g^{-1})\right)(t)\right\|_2^q dt\right)^{\frac{2}{q}},
$$

for all $g(a) \leq z \leq g(x_0)$.

Proof. As in Theorem 19. \blacksquare

The corresponding modified generalized right $\mathcal{B}_{\gamma}(H)$ -valued fractional Opial inequality comes next:

Theorem 26 All as in Theorem 7, where $X = \mathcal{B}_{\gamma}(H)$ and let $p, q > 1 : \frac{1}{p} + \frac{1}{q} =$ 1. Here we assume that $\frac{1}{(m+1)q} < \nu < 1$. Then

1. COMPUTATIONAL ANALYSIS AND APPLICATIONS, Vol. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC

\n
$$
\int_{0}^{f(x_0)} \left\| \left(D_{g(x_0)}^{(m+1)\nu} (f \circ g^{-1}) \right) (t) \right\|_2^2 dt \right)^{\frac{2}{3}},
$$
\nfor all $g(a) \le z \le g(x_0)$.

\nProof. As in Theorem 19. ■
\nThe corresponding modified generalized right $B_{\gamma}(H)$ -valued fractional Opial
\ninequality comes next:

\nTheorem 26 All as in Theorem 7, where $X = B_{\gamma}(H)$ and let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$.

\nHere we assume that $\frac{1}{(m+1)q} < p < 1$. Then

\n
$$
\int_{z}^{g(x_0)} \left\| \left((f \circ g^{-1}) (w) \right) \left(\left(D_{(g(x_0)-z)}^{(m+1)\nu+1} (f \circ g^{-1}) \right) (w) \right) \right\|_{\gamma}^2 dw \leq (91)
$$
\n
$$
\frac{2^{-\frac{1}{3}} (g(x_0) - z)^{(m+1)\nu + \frac{1}{p} - \frac{1}{q}}}{\Gamma((m+1)\nu) \left\| p \left((m+1)\nu - 1 \right) + 1 \right\| Q \left((m+1)\nu - 1 \right) + 2 \right\|^{2}}
$$
\nfor all $g(a) \le z \le g(x_0)$.

\nProof. As in Theorem 19. ■
\n**Remark 27** (*a* or Theorem 19)

\nWe make

\n
$$
\frac{\sup}{\left\| \left(\int_{z_0}^{g(x_0)} \left| \left(D_{g(x_0)}^{(m+1)\nu} (f \circ g^{-1}) \right) (t) \right| \right\|_{\gamma}^{\gamma} dw \right\|_{\gamma}^{\gamma} < +\infty.
$$
\nThus, $\left\{ \sup_{z \in [0,1]} \left\| \left[D_{g(x_0)}^{\nu}(f \circ g^{-1}) \right]_{\gamma} \right\|_{\infty, [g(x_$

for all $g(a) \leq z \leq g(x_0)$.

Proof. As in Theorem 19. \blacksquare We make

Remark 27 (to Theorem 12)

Case of inequality (46) : Call and assume

$$
M_1(f_1, ..., f_r) := \qquad (92)
$$

$$
\max_{i=1,...,r} \left\{ \sup_{x_0 \in [a,b]} \left\| \left\| D_{g(x_0)-}^{\nu}(f_i \circ g^{-1}) \right\|_2 \right\|_{\infty, [g(a), g(x_0)]},
$$

$$
\sup_{x_0 \in [a,b]} \left\| \left\| D_{g(x_0)}^{\gamma}(f_i \circ g^{-1}) \right\|_2 \right\|_{\infty, [g(x_0), g(b)]} \right\} < +\infty.
$$

Then

$$
||K(f_1, ..., f_r)(x_0)||_1 \leq Right \; hand \; side \; (46) \; \leq
$$

$$
\frac{M_1(f_1, ..., f_r) (g (b) - g (a))^{\nu}}{\Gamma(\nu + 1)} \sum_{i=1}^r \left(\int_a^b \left(\prod_{\substack{j=1 \ j \neq i}}^r \|f_j(x)\|_2 \right) dx \right). \tag{93}
$$

We make

Remark 28 (to Theorem 13) Case of inequality (62):

Call and assume

M² (f1; :::; fr) := (94) max ⁱ=1;:::;r (sup x02[a;b] D g(x0) fⁱ g 1 2 L1([g(a);g(x0)]) ; sup x02[a;b] D g(x0) fⁱ g 1 2 ^L1([g(x0);g(b)])) < +1: 609 J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC George A. Anastassiou 583-621

Then

$$
||K(f_1, ..., f_r)(x_0)||_1 \leq Right \; hand \; side \; (62) \leq
$$

$$
\frac{M_2(f_1, ..., f_r)(g(b) - g(a))^{\nu - 1}}{\Gamma(\nu)} \sum_{i=1}^r \left(\int_a^b \left(\prod_{\substack{j=1 \ j \neq i}}^r ||f_j(x)||_2 \right) dx \right). \tag{95}
$$

We make

Remark 29 (to Theorem 14)

Case of inequality (64): Call and assume $(p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1)$:

$$
M_3(f_1, \ldots, f_r) := \tag{96}
$$

$$
\max_{i=1,\ldots,r} \left\{ \sup_{x_0 \in [a,b]} \left\| \left\| D_{g(x_0)-}^{\nu}(f_i \circ g^{-1}) \right\|_2 \right\|_{q,([g(a),g(x_0)])}, \sup_{x_0 \in [a,b]} \left\| \left\| D_{g(x_0)}^{\nu}(f_i \circ g^{-1}) \right\|_2 \right\|_{q,([g(x_0),g(b)])} \right\} < +\infty.
$$

Then

$$
||K(f_1,...,f_r)(x_0)||_1 \leq Right \; hand \; side \; (64) \leq
$$

$$
\frac{M_3(f_1, ..., f_r) (g (b) - g (a))^{\nu - \frac{1}{q}}}{(p(\nu - 1) + 1)^{\frac{1}{p}} \Gamma(\nu)} \sum_{i=1}^r \left(\int_a^b \left(\prod_{\substack{j=1 \ j \neq i}}^r \|f_j(x)\|_2 \right) dx \right). \tag{97}
$$

We make

Remark 30 (to Theorem 15) (
$$
\gamma \ge 1
$$
)
Case of inequality (67):
Call and assume

$$
M_1^{\gamma}(f_1,...,f_r) :=
$$
(98)

$$
\max_{i=1,\ldots,r} \left\{ \sup_{x_0 \in [a,b]} \left\| \left\| D_{g(x_0)-}^{\nu}(f_i \circ g^{-1}) \right\|_{\gamma} \right\|_{\infty, [g(a), g(x_0)]}, \sup_{x_0 \in [a,b]} \left\| \left\| D_{g(x_0)}^{\gamma}(f_i \circ g^{-1}) \right\|_{\gamma} \right\|_{\infty, [g(x_0), g(b)]} \right\} < +\infty.
$$

Then

 $\|K(f_1, ..., f_r)(x_0)\|_{\gamma} \leq Right \; hand \; side \; (67) \; \leq$

$$
\frac{M_1^{\gamma}(f_1, ..., f_r) (g (b) - g (a))^{\nu}}{\Gamma(\nu + 1)} \sum_{i=1}^r \left(\int_a^b \left(\prod_{\substack{j=1 \ j \neq i}}^r \|f_j(x)\|_{\gamma} \right) dx \right). \tag{99}
$$

We make

Remark 31 *(to Theorem 16)* $(\gamma \geq 1)$

Case of inequality (68): Call and assume:

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, Vol. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\n
$$
\max_{i=1,...,r} \left\{ \sup_{x_0 \in [a,b]} \left\| \left\| D_{g(x_0)}^{\gamma}(f_i \circ g^{-1}) \right\|_{\gamma} \right\|_{\infty, [g(x_0), g(b)]} \right\} < +\infty.
$$
\nThen\n
$$
\left\| K(f_1,...,f_r) (x_0) \right\|_{\gamma} \leq Kighk \tanh \text{side } (\theta 7) \leq
$$
\n
$$
\frac{M_1^{\gamma}(f_1,...,f_r) (g(b) - g(a))^{\gamma}}{\Gamma(\nu + 1)} \sum_{i=1}^{r} \left(\int_a^b \left(\prod_{j=1}^r \|f_j(x)\|_{\gamma} \right) dx \right). \qquad (99)
$$
\nWe make\nRemark 31 (to Theorem 16) $(\gamma \geq 1)$ \nCase of inequality (68):\nCall and assume:\n
$$
\left\| K \left(f_1,...,f_r \right) := \left\| D_{g(x_0)}^{\gamma}(f_1,...,f_r) := \left\| D_{g(x_0)}^{\gamma}(f_1,...,f_r) \right\|_{\gamma} \right\} = \left\| D_{g(x_0)}^{\gamma}(f_1,...,f_r) \right\|_{\gamma}.
$$
\nThen\n
$$
\left\| K(f_1,...,f_r) (x_0) \right\|_{\gamma} \leq Righk \tanh \text{side } (\theta 8) \leq
$$
\n
$$
\frac{M_2^{\gamma}(f_1,...,f_r) (g(b) - g(a))^{\gamma-1}}{\Gamma(\nu)} \sum_{i=1}^{r} \left(\int_a^b \left(\prod_{j=i}^r \|f_j(x)\|_{\gamma} \right) dx \right). \qquad (100)
$$
\n
$$
\text{The number 32 (for Theorem 17) $(\gamma \geq 1)$ \n
$$
\text{We make}
$$
\nRemark 32 (to Theorem 17) $(\gamma \geq 1)$ \n
$$
\text{Case of inequality } (69):
$$
\n
$$
\left\| K(f_1,...,f_r) (g(b) - g(a))^{\gamma-1} \sum_{i=1}^{r} \left(
$$
$$

Then

$$
\|K(f_1, ..., f_r)(x_0)\|_{\gamma} \leq Right \text{ hand side } (68) \leq
$$

$$
\frac{M_2^{\gamma}(f_1, ..., f_r)(g(b) - g(a))^{\nu - 1}}{\Gamma(\nu)} \sum_{i=1}^r \left(\int_a^b \left(\prod_{\substack{j=1 \ j \neq i}}^r \|f_j(x)\|_{\gamma} \right) dx \right). \tag{101}
$$

We make

Remark 32 (to Theorem 17)
$$
(\gamma \ge 1)
$$

Case of inequality (69):
Call and assume $(p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1)$:

$$
M_3^{\gamma} (f_1, ..., f_r) := \tag{102}
$$

$$
\max_{i=1,\ldots,r} \left\{ \sup_{x_0 \in [a,b]} \left\| \left\| D_{g(x_0)-}^{\nu}(f_i \circ g^{-1}) \right\|_{\gamma} \right\|_{q,([g(a),g(x_0)])}, \sup_{x_0 \in [a,b]} \left\| \left\| D_{g(x_0)}^{\nu}(f_i \circ g^{-1}) \right\|_{\gamma} \right\|_{q,([g(x_0),g(b)])} \right\} < +\infty.
$$

Then

$$
\|K(f_1, ..., f_r)(x_0)\|_{\gamma} \leq Right \; hand \; side \; (69) \leq
$$

$$
\frac{M_3^{\gamma}(f_1, ..., f_r)(g(b) - g(a))^{\nu - \frac{1}{q}}}{(p(\nu - 1) + 1)^{\frac{1}{p}} \Gamma(\nu)} \sum_{i=1}^r \left(\int_a^b \left(\prod_{\substack{j=1 \ j \neq i}}^r \|f_j(x)\|_{\gamma} \right) dx \right). \tag{103}
$$

Remark 33 (to Theorem 18)

i) for $\gamma, \delta > 1$: $\frac{1}{\gamma} + \frac{1}{\delta} = 1$, case of inequality (71): Call and assume $N_1(A_1, A_2) :=$

J. COMPUTIONAL ANALYSIS AND APPLICATIONS. VOL. 31, NO. 4. 2023, ODYRIGHT 2023 EUDOXUS PRESS, LLC
\n
$$
\frac{M_3^2 (f_1,...,f_r) (g_1)|_{\gamma} \leq Right \tanh d \, side \, (69) \leq
$$
\n
$$
\frac{M_3^2 (f_1,...,f_r) (g_1)|_{\gamma} \leq Right \tanh d \, side \, (69) \leq
$$
\n
$$
\frac{M_3^2 (f_1,...,f_r) (g_1)|_{\gamma} \cdot \gamma \cdot \delta > 1 : \frac{1}{\gamma} + \frac{1}{\gamma} - 1, \text{ case of inequality } (71):
$$
\nCall and assume\n
$$
\frac{1}{\gamma} \int_{\gamma} \int_{\
$$

Then

$$
\|\Phi(A_1, A_2)(x_0)\|_1 \le \text{right hand side (71)} \le
$$

$$
\frac{N_1(A_1, A_2)(g(b) - g(a))^{\alpha - \frac{1}{\delta}}}{\Gamma(\alpha) (\gamma (\alpha - 1) + 1)^{\frac{1}{\gamma}}} \left[\int_a^b \|A_1(x)\|_p dx + \int_a^b \|A_2(x)\|_q dx \right].
$$
 (105)
ii) case of inequality (72):

Call and assume

$$
N_2(A_1, A_2) := (106)
$$

$$
\max \left\{ \sup_{x_0 \in [a,b]} \left\| \left\| D_{g(x_0) -}^{\nu}\left(A_1 \circ g^{-1}\right) \right\|_p \right\|_{L_1([g(a),g(x_0)])}, \sup_{x_0 \in [a,b]} \left\| \left\| D_{g(x_0)}^{\nu}\left(A_1 \circ g^{-1}\right) \right\|_p \right\|_{L_1([g(x_0),g(b)])}, \sup_{x_0 \in [a,b]} \left\| \left\| D_{g(x_0)}^{\nu}\left(A_1 \circ g^{-1}\right) \right\|_p \right\|_{L_1([g(x_0),g(b)])}, \sup_{x_0 \in [a,b]} \left\| \left\| D_{g(x_0)}^{\nu}\left(A_2 \circ g^{-1}\right)^{-1} \right\|_q \right\|_{L_1([g(x_0),g(b)])} \right\} < +\infty.
$$

Then

$$
\|\Phi(A_1, A_2)(x_0)\|_1 \le \text{right hand side (72)} \le
$$

$$
\frac{N_2(A_1, A_2) (g (b) - g (a))^{\alpha - 1}}{\Gamma(\alpha)} \left[\int_a^b \|A_1 (x)\|_p dx + \int_a^b \|A_2 (x)\|_q dx \right].
$$
 (107)

iii) case of inequality (73):

Call and assume

$$
N_3(A_1, A_2) := (108)
$$

$$
\max \left\{\sup_{x_0 \in [a,b]} \left\| \left\| D_{g(x_0)-}^{\nu}\left(A_1 \circ g^{-1}\right) \right\|_p \right\|_{\infty, [g(a), g(x_0)]}, \sup_{x_0 \in [a,b]} \left\| \left\| D_{g(x_0)}^{\nu}\left(A_1 \circ g^{-1}\right) \right\|_p \right\|_{\infty, [g(x_0), g(b)]},
$$

$$
\sup_{x_0 \in [a,b]} \left\| \left\| D_{g(x_0) -}^{\nu} \left(A_2 \circ g^{-1} \right) \right\|_{q} \right\|_{\infty, [g(a), g(x_0)]}, \sup_{x_0 \in [a,b]} \left\| \left\| D_{g(x_0)}^{\nu} \left(A_2 \circ g^{-1} \right)^{-1} \right\|_{q} \right\|_{\infty, [g(x_0), g(b)]} \right\} < +\infty.
$$

Then

$$
\|\Phi(A_1, A_2)(x_0)\|_1 \le \text{right hand side (73)} \le
$$

$$
\frac{N_3(A_1, A_2)(g(b) - g(a))^{\alpha}}{\Gamma(\alpha + 1)} \left[\int_a^b \|A_1(x)\|_p \, dx + \int_a^b \|A_2(x)\|_q \, dx \right]. \tag{109}
$$

We need

Remark 34 (i) This is regarding Theorems 12-17. Here $K(f_1, ..., f_r)(x_0)$, $x_0 \in [a, b],$ is as in (45). Next we denote and have (case of $1 \leq \nu < 2$):

$$
\Delta(f_1, ..., f_r) := \int_a^b K(f_1, ..., f_r) (x_0) dx_0 =
$$

$$
\sum_{i=1}^r \left[(b-a) \int_a^b \left(\prod_{\substack{j=1 \ j \neq i}}^r f_j(x) \right) f_i(x) dx - \left(\int_a^b \left(\prod_{\substack{j=1 \ j \neq i}}^r f_j(x) \right) dx \right) \left(\int_a^b f_i(x) dx \right) \right],
$$
(110)

(ii) This is regarding Theorem 18. Here $\Phi(A_1, A_2)(x_0)$, $x_0 \in [a, b]$, is as in (70). Next we denote and have (case of $1 \leq \alpha < 2$):

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS. VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\n
$$
\sup_{x_0 \in [0, N]} \left\| \left\| D_{g(x_0)}^V - \left(A_2 \circ g^{-1} \right) \right\|_q \right\|_{\infty, [g(x_0), g(x_0)]}, \sup_{x_0 \in [0, N]} \left\| \left\| D_{g(x_0)}^V (A_2 \circ g^{-1})^{-1} \right\|_q \right\|_{\infty, [g(x_0), g(b)]} \right\} \nThen\n
$$
\left\| \Phi(A_1, A_2) (x_0) \right\|_1 \leq \operatorname{right} \left\| A_1 (x) \right\|_q dx + \int_0^1 \|A_2 (x) \right\|_q dx \Big\}. \qquad (100)
$$
\nWe need
\nRemark 84 (i) This is regarding Theorems 12-17. Here K $(f_1, ..., f_r)$ (x₀),
\n $x_0 \in [a, b],$ is as in (45). Next we denote and have (case of $1 \le \nu < 2$):
\n
$$
\Delta(f_1, ..., f_r) := \int_a^b K(f_1, ..., f_r) (x_0) dx_0 =
$$
\n
$$
\sum_{i=1}^r \left[(b-a) \int_a^b \left(\prod_{j=i}^r f_j (x) \right) f_i (x) dx - \left(\int_a^b \left(\prod_{j=i}^r f_j (x) \right) dx \right) \left(\int_a^b f_i (x) dx \right) \right],
$$
\n(iii) This is regarding Theorem 18. Here $\Phi(A_1, A_2)$ (x₀), $x_0 \in [a, b],$ is as in (70). Next we denote and have (case of $1 \le \alpha < 2$):
\n
$$
\Delta (A_1, A_2) := \int_a^b \Phi(A_1, A_2) (x_0) dx_0 =
$$
\n
$$
(b-a) \left(\int_a^b A_2 (x) A_1 (x) dx \right) - \left(\int_a^b A_1 (x) A_2 (x) dx \right) - \left(\frac{111}{1111111111111
$$
$$

$$
\left\| \Delta \left(f_1, ..., f_r \right) \right\|_{\gamma} \le \int_a^b \left\| K \left(f_1, ..., f_r \right) (x) \right\|_{\gamma} dx, \tag{112}
$$

and

$$
\left\| \Delta \left(A_1, A_2 \right) \right\|_1 \le \int_a^b \left\| \Phi \left(A_1, A_2 \right) (x) \right\|_1 dx. \tag{113}
$$

We give the following set of γ -Schatten norm generalized Canavati type fractional Grüss type inequalities involving several functions over $\mathcal{B}_{\gamma}(H)$, $\gamma \geq 1$.

Theorem 35 All as in Theorem 12, with $1 \leq \nu < 2$ (i.e. $n = 1$). Then i)

$$
\|\Delta(f_1, ..., f_r)\|_1 \le \frac{M_1(f_1, ..., f_r) (g (b) - g (a))^{\nu} (b - a)^2}{\Gamma(\nu + 1)}
$$

$$
\sum_{i=1}^r \left(\left\| \prod_{\substack{j=1 \ j \neq i}}^r \|f_j(x)\|_2 \right\|_{\infty, [a, b]}\right),
$$
(114)

where $M_1(f_1, ..., f_r)$ is as in (92), ii)

$$
\|\Delta(f_1, ..., f_r)\|_1 \le \frac{M_2(f_1, ..., f_r) (g (b) - g (a))^{\nu - 1} (b - a)^2}{\Gamma(\nu)}
$$

$$
\sum_{i=1}^r \left(\left\| \prod_{\substack{j=1 \ j \neq i}}^r \|f_j(x)\|_2 \right\|_{\infty, [a, b]} \right), \tag{115}
$$

where $M_2(f_1, ..., f_r)$ is as in (94) ,

iii) when $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, we have

1. COMPUTATIONAL ANALYSIS AND APPLICATIONS, Vol. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC

\n1. Theorem 35

\n
$$
||\Delta(f_1,...,f_r)||_1 \leq \frac{M_1(f_1,...,f_r) (g(b) - g(a))^{\nu} (b-a)^2}{\Gamma(\nu+1)}
$$
\n
$$
\sum_{i=1}^r \left(\left\| \prod_{\substack{j=1 \\ j \neq i}}^r \|f_j(x)\|_a \right\|_{\infty, [a,b]} \right),
$$
\n(114)\nwhere M₁ (f₁,...,f_r) is as in (92),

\nii)
$$
||\Delta(f_1,...,f_r) \text{ is as in (92),}
$$
\n
$$
\sum_{i=1}^r \left(\left\| \prod_{\substack{j=1 \\ j \neq i}}^r \|f_j(x)\|_a \right\|_{\infty, [a,b]} \right),
$$
\n(115)

\nwhere M₂ (f₁,...,f_r) is as in (94),

\niii) when $p, q > 1$: $\frac{1}{p} + \frac{1}{q} = 1$, we have

\n
$$
||\Delta(f_1,...,f_r)||_1 \leq \frac{M_3(f_1,...,f_r) (g(b) - g(a))^{\nu-1} (b-a)^2}{(p(\nu-1)+1)^{\frac{1}{2}} \Gamma(\nu)}
$$
\n
$$
\sum_{i=1}^r \left(\left\| \prod_{\substack{j=1 \\ j \neq i}}^r \|f_j(x)\|_a \right\|_{\infty, [a,b]} \right),
$$
\n(116)\nwhere M₃ (f₁,...,f_r) is as in (96).

\nProof. By Remarks 34, 27-29 and that

\n
$$
\int_a^b \left(\prod_{\substack{j=1 \\ j \neq i}}^r \|f_j(x)\|_2 \right) dx \leq (b-a) \left\| \prod_{\substack{j=1 \\ j \neq i}}^r \|f_j(x)\|_2 \right\|_{\infty, [a,b]}
$$
\nWe continue with

\nTheorem 36

where $M_3(f_1, ..., f_r)$ is as in (96).

Proof. By Remarks 34, 27-29 and that

$$
\int_{a}^{b} \left(\prod_{\substack{j=1 \ j \neq i}}^{r} \left\| f_{j} \left(x \right) \right\|_{2} \right) dx \leq (b-a) \left\| \prod_{\substack{j=1 \ j \neq i}}^{r} \left\| f_{j} \left(x \right) \right\|_{2} \right\|_{\infty, [a,b]}
$$

We continue with

Theorem 36 All as in Theorem 15, with $1 \leq \nu < 2$ (i.e. $n = 1$), $\gamma \geq 1$. Then i)

$$
\left\|\Delta(f_1, ..., f_r)\right\|_{\gamma} \le \frac{M_1^{\gamma}(f_1, ..., f_r)(g(b) - g(a))^{\nu}(b - a)^2}{\Gamma(\nu + 1)}
$$

:

$$
\sum_{i=1}^{r} \left(\left\| \prod_{\substack{j=1 \ j \neq i}}^{r} \left\| f_j \left(x \right) \right\|_{\gamma} \right\|_{\infty, [a,b]} \right), \tag{117}
$$

where $M_1^{\gamma}(f_1, ..., f_r)$ is as in (98), ii)

$$
\|\Delta(f_1, ..., f_r)\|_{\gamma} \le \frac{M_2^{\gamma}(f_1, ..., f_r) (g (b) - g (a))^{\nu - 1} (b - a)^2}{\Gamma(\nu)}
$$

$$
\sum_{i=1}^r \left(\left\| \prod_{\substack{j=1 \ j \neq i}}^r \|f_j (x)\|_{\gamma} \right\|_{\infty, [a, b]}\right),
$$
(118)

where $M_2^{\gamma}(f_1, ..., f_r)$ is as in (100), *iii*) when $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, we have

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\n
$$
\sum_{i=1}^{v} \left(\left\| \prod_{\substack{j=1 \\ j \neq i}}^{r} \|f_j(x)\|_v \right\|_{\infty, [a, b]}\right),
$$
\n(117)
\nwhere $M_1^{\gamma}(f_1, ..., f_r)$ is as in (98),
\n
$$
||\Delta(f_1, ..., f_r)||_{\gamma} \leq \frac{M_2^{\gamma}(f_1, ..., f_r)(g(b) - g(a))^{\nu - 1}(b - a)^2}{\Gamma(\nu)}
$$
\nwhere $M_2^{\gamma}(f_1, ..., f_r)$ is as in (100),
\n
$$
||\Delta(f_1, ..., f_r)||_{\gamma} \leq \frac{M_3^{\gamma}(f_1, ..., f_r)(g(b) - g(a))^{\nu - \frac{1}{2}}(b - a)^2}{(p(\nu - 1) + 1)^{\frac{1}{2}} \Gamma(\nu)}
$$
\nwhere $||\Delta(f_1, ..., f_r)||_{\gamma} \leq \frac{M_3^{\gamma}(f_1, ..., f_r)(g(b) - g(a))^{\nu - \frac{1}{2}}(b - a)^2}{(p(\nu - 1) + 1)^{\frac{1}{2}} \Gamma(\nu)}$
\n
$$
\sum_{i=1}^{r} \left(\left\| \prod_{\substack{j=1 \\ j \neq i}}^{r} \|f_j(x)\|_v \right\|_{\infty, [a, b]}\right),
$$
\n(119)
\nwhere $M_3^{\gamma}(f_1, ..., f_r)$ is as in (102).
\nProof. By Remarks 34, 30-32 and that
\n
$$
\int_a^b \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \|f_j(x)\|_v \right) dx \leq (b - a) \left\| \prod_{\substack{j=1 \\ j \neq i}}^{r} \|f_j(x)\|_v \right\|_{\infty, [\alpha, b]}.
$$
\nTherefore, we have: $(r = 2 \text{ case of } p\text{-Solution from Grits inequalities})$
\n**Thororm 37** All as in Theorem 18, with $1 \leq \alpha < 2$ (i.e. $|\alpha| = 1$

where $M_3^{\gamma}(f_1, ..., f_r)$ is as in (102).

Proof. By Remarks 34, 30-32 and that

$$
\int_{a}^{b} \left(\prod_{\substack{j=1 \ j \neq i}}^{r} \left\| f_{j} \left(x \right) \right\|_{\gamma} \right) dx \leq (b-a) \left\| \prod_{\substack{j=1 \ j \neq i}}^{r} \left\| f_{j} \left(x \right) \right\|_{\gamma} \right\|_{\infty, [a,b]}.
$$

Furthermore we have $(r = 2 \text{ case of } p\text{-Schatten norm}$ Grüss inequalities)

Theorem 37 All as in Theorem 18, with $1 \le \alpha < 2$ (i.e. $[\alpha] = 1$). Then i) for $\gamma, \delta > 1 : \frac{1}{\gamma} + \frac{1}{\delta} = 1$, we have

$$
\|\Delta (A_1, A_2)\|_1 \le \frac{N_1 (A_1, A_2) (g (b) - g (a))^{\alpha - \frac{1}{\delta}} (b - a)}{\Gamma (\alpha) (\gamma (\alpha - 1) + 1)^{\frac{1}{\gamma}}}
$$

$$
\left[\int_a^b \|A_1 (x)\|_p \, dx + \int_a^b \|A_2 (x)\|_q \, dx \right],
$$
(120)
where $N_1(A_1, A_2)$ is as in (104) , ii)

$$
\|\Delta (A_{1}, A_{2})\|_{1} \leq \frac{N_{2} (A_{1}, A_{2}) (g (b) - g (a))^{\alpha - 1} (b - a)}{\Gamma (\alpha)}
$$

$$
\left[\int_{a}^{b} \|A_{1} (x)\|_{p} dx + \int_{a}^{b} \|A_{2} (x)\|_{q} dx\right],
$$
(121)

where $N_2(A_1, A_2)$ is as in (106),

and iii)

$$
\|\Delta (A_1, A_2)\|_1 \le \frac{N_3 (A_1, A_2) (g (b) - g (a))^{\alpha} (b - a)}{\Gamma (\alpha + 1)}
$$

$$
\left[\int_a^b \|A_1 (x)\|_p \, dx + \int_a^b \|A_2 (x)\|_q \, dx \right],
$$
(122)

where $N_3(A_1, A_2)$ is as in (108).

Proof. By Remarks 34, 33. \blacksquare

6 Applications

We start with applications on Ostrowski type inequalities:

Corollary 38 (to Theorems 12-14) All as in Theorem 12 for $g(t) = t$. Then i) 1

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\n
$$
\lim_{n \to \infty} N_1(A_1, A_2) \text{ is as in } (104),
$$
\n
$$
\|\Delta(A_1, A_2)\|_1 \le \frac{N_2(A_1, A_2)(g(b) - g(a))^{\alpha - 1}(b - a)}{\Gamma(\alpha)}
$$
\nand\n
$$
\lim_{n \to \infty} \left[\int_a^b \|A_1(x)\|_p dx + \int_a^b \|A_2(x)\|_q dx \right],
$$
\n(121)
\n
$$
\lim_{n \to \infty} \|\Delta(A_1, A_2)\|_3 \text{ is as in } (106),
$$
\n
$$
\lim_{n \to \infty} \left[\int_a^b \|A_1(x)\|_p dx + \int_a^b \|A_2(x)\|_q dx \right],
$$
\n(122)
\nwhere $N_3(A_1, A_2)$ is as in $(108).$
\nProof. By Remarks 34, 33. \blacksquare
\n**6** Applications
\nWe start with applications on Octrowski type inequalities:
\nCorollary 38 (to Theorems 12-14) All as in Theorem 12 for $g(t) = t$. Then
\n
$$
\|\langle K(t_1, ..., f_r)(x_0)\|_1 \le \frac{1}{\Gamma(\nu + 1)}
$$
\n
$$
\sum_{i=1}^r \left[\left[\left\| \left\| (D_{x_0}^{\nu} - f_i) \right\|_2 \right\|_{\infty, [a, x_0]} (x_0 - a)^{\nu} \left(\int_a^x \left(\prod_{j=1}^x \|f_j(x)\|_2 \right) dx \right) \right] \right] + (123)
$$
\n
$$
\left\| \left\| (D_{x_0}^{\nu} f_i) \right\|_2 \left\|_{\infty, [a, b]} (b - a_0)^{\nu} \left(\int_a^x \left(\prod_{j=1}^r \|f_j(x)\|_2 \right) dx \right) \right\| \right\|_1,
$$
\n
$$
\|\langle D_{x_0}^{\nu} - f_i \rangle \|_2 \left\|_{\infty, [a, b]} (b -
$$

$$
\left[\left\| \left\| \left(D_{x_0}^{\nu} f_i \right) \right\|_2 \right\|_{L_1([x_0,b])} \int_{x_0}^b \left(\prod_{\substack{j=1 \ j \neq i}}^r \left\| f_j \left(x \right) \right\|_2 \right) \left(x - x_0 \right)^{\nu - 1} dx \right] \right], \tag{124}
$$

iii) when $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, we have

$$
||K(f_1, ..., f_r)(x_0)||_1 \leq \frac{1}{(p(\nu - 1) + 1)^{\frac{1}{p}} \Gamma(\nu)}
$$

$$
\sum_{i=1}^r \left[\left[\left\| || \left(D_{x_0}^{\nu} - f_i \right) ||_2 \right\|_{q,[a,x_0]} \left(\int_a^{x_0} (x_0 - x)^{\nu - \frac{1}{q}} \left(\prod_{\substack{j=1 \ j \neq i}}^r \| f_j(x) \|_2 \right) dx \right) \right] +
$$

$$
\left[\left\| || \left(D_{x_0}^{\nu} f_i \right) ||_2 \right\|_{q,[x_0,b]} \left(\int_{x_0}^b (x - x_0)^{\nu - \frac{1}{q}} \left(\prod_{\substack{j=1 \ j \neq i}}^r \| f_j(x) \|_2 \right) dx \right) \right] \right].
$$
 (125)

It follows:

Corollary 39 (to Theorems 15-17) All as in Theorem 15 for $g(t) = t$, $\gamma \ge 1$. Then i)

J. COMPUTATORAL AVALYSIS AND APPLICATIONS, Vol. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\n
$$
\left[\left\| \left\| \left(D_{xy}^{\nu} f_{1} \right) \right\|_{2} \right\|_{L_{1}([y_{1},b])} \int_{x_{0}}^{b} \left(\prod_{\substack{j=1 \\ j \neq i}}^{\nu} \left\| f_{j}(x) \right\|_{2} \right) (x - x_{0})^{\nu - 1} dx \right] \right], \qquad (124)
$$
\n
$$
\sum_{i=1}^{r} \left[\left[\left\| \left\| \left((D_{xy}^{\nu} f_{1}) \right) \right\|_{2} \right\|_{q_{i}[x_{2},x_{1}]} \left(\int_{x_{0}}^{x_{0}} (x_{0} - x)^{\nu - \frac{1}{q}} \left(\prod_{\substack{j=1 \\ j \neq i}}^{\nu} \left\| f_{j}(x) \right\|_{2} \right) dx \right) \right] + \left\{ \left\| \left\| \left\| \left(D_{xy}^{\nu} f_{1} \right) \right\|_{2} \right\|_{q_{i}[x_{2},x_{1}]} \left(\int_{x_{0}}^{x_{0}} (x_{0} - x)^{\nu - \frac{1}{q}} \left(\prod_{\substack{j=1 \\ j \neq i}}^{\nu} \left\| f_{j}(x) \right\|_{2} \right) dx \right) \right] \right] + \left\{ \left\| \left\| \left(D_{xy}^{\nu} f_{1} \right) \right\|_{2} \right\|_{q_{i}[x_{2},x_{1}]} \left(\int_{x_{0}}^{b} (x - x_{0})^{\nu - \frac{1}{q}} \left(\prod_{\substack{j=1 \\ j \neq i}}^{\nu} \left\| f_{j}(x) \right\|_{2} \right) dx \right) \right] \right]. \qquad (125)
$$
\n
$$
\text{Corollary 39} \quad (to \text{ Theorems 15-17) } \text{All as in Theorem 15 for } g(t) = t, \gamma \ge 1.
$$
\n
$$
T_{Rex}
$$
\n
$$
\left\| K(f_{1},...,f_{r}) (x_{0}) \right\|_{\gamma} \le \frac{1}{\Gamma(\nu +
$$

iii) when $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, we have

$$
\|K(f_1, ..., f_r)(x_0)\|_{\gamma} \leq \frac{1}{(p(\nu - 1) + 1)^{\frac{1}{p}} \Gamma(\nu)}
$$

$$
\sum_{i=1}^r \left[\left[\left\| \| (D_{x_0}^{\nu} - f_i) \|_{\gamma} \right\|_{q,[a,x_0]} \left(\int_a^{x_0} (x_0 - x)^{\nu - \frac{1}{q}} \left(\prod_{\substack{j=1 \ j \neq i}}^r \| f_j(x) \|_{\gamma} \right) dx \right) \right] +
$$

$$
\left[\left\| \| (D_{x_0}^{\nu} f_i) \|_{\gamma} \right\|_{q,[x_0,b]} \left(\int_{x_0}^b (x - x_0)^{\nu - \frac{1}{q}} \left(\prod_{\substack{j=1 \ j \neq i}}^r \| f_j(x) \|_{\gamma} \right) dx \right) \right] \right].
$$
 (128)

We continue with

Corollary 40 (to Theorem 18) All as in Theorem 18, with $g(t) = e^t$. Then i) for $\gamma, \delta > 1$: $\frac{1}{\gamma} + \frac{1}{\delta} = 1$, we have

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS. VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\niii) when
$$
p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1
$$
, we have
\n
$$
||K (f_1,...,f_r) (x_0)||_{\gamma} \leq \frac{1}{(p(\nu - 1) + 1)^{\frac{1}{2}} \Gamma(\nu)}
$$
\n
$$
\sum_{i=1}^{r} \left[\left[\left| || (D_{x_0}^{\nu} f_i) ||_{\gamma} ||_{\gamma} ||_{q,[x_0,t]} \right| \left(\int_{x_0}^{x_0} (x - x)^{\nu - \frac{1}{q}} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} ||f_j(x)||_{\gamma} \right) dx \right) \right] +
$$
\n
$$
\left[\left| || (D_{x_0}^{\nu} f_i) ||_{\gamma} ||_{q,[x_0,t]} \right| \left(\int_{x_0}^{x_0} (x - x_0)^{\nu - \frac{1}{q}} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} ||f_j(x)||_{\gamma} \right) dx \right) \right] \right]. \qquad (128)
$$
\nWe continue with
\nCorollary 40 (to Theorem 18) All as in Theorem 18, with $g(t) = e^t$. Then
\n
$$
||\Phi(A_1, A_2) (x_0)||_{1} \leq \frac{1}{\Gamma(\alpha)} (\gamma(\alpha - 1) + 1)^{\frac{1}{2}}
$$
\n
$$
\left\{ \left[\left| || D_{x_0}^{\nu} - (A_1 \circ \log) ||_{\mu} ||_{\delta, [e^{\mu}, e^{\mu}]} \right| \int_{x_0}^{x_0} ||A_2(x)||_{q} (e^{\nu_0} - e^{\nu})^{\alpha - \frac{1}{2}} dx \right] +
$$
\n
$$
\left[|| [D_{x_0}^{\nu} - (A_2 \circ \log) ||_{\mu} ||_{\delta, [e^{\mu}, e^{\mu}]} \int_{x_0}^{x_0} ||A_1(x)||_{p} (e^{\nu} - e^{\nu})^{\alpha - \frac{1}{2}} dx \right] +
$$
\n
$$
\left[|| [D_{x_0}^{\nu}
$$

$$
\|\Phi(A_1, A_2)(x_0)\|_1 \le \frac{1}{\Gamma(\alpha)}
$$

$$
\left\{ \left[\left\| \|D^{\alpha}_{e^{x_0} -}(A_1 \circ \log)\|_p \right\|_{L_1([e^a, e^{x_0}])} \int_a^{x_0} \|A_2(x)\|_q (e^{x_0} - e^x)^{\alpha - 1} dx \right] + \left\| \|D^{\alpha}_{e^{x_0}}(A_1 \circ \log)\|_p \right\|_{L_1([e^{x_0}, e^b])} \int_{x_0}^b \|A_2(x)\|_q (e^x - e^{x_0})^{\alpha - 1} dx \right\} + \left\{ \left\| \|D^{\alpha}_{e^{x_0} -}(A_2 \circ \log)\|_q \right\|_{L_1([e^a, e^{x_0}])} \int_a^{x_0} \|A_1(x)\|_p (e^{x_0} - e^x)^{\alpha - 1} dx \right\} + \tag{130}
$$

$$
\left[\left\| \|D^{\alpha}_{e^{x_0}}(A_2 \circ \log)\|_{q} \right\|_{L_1([e^{x_0}, e^b])} \int_{x_0}^b \|A_1(x)\|_{p} (e^x - e^{x_0})^{\alpha - 1} dx \right] \right\},
$$

and

iii)

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS. VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\n
$$
\left[\left\|\|D_{e^{a_0}}^{\alpha}(A_2 \circ \log)\|_q\right\|_{L_1([e^{a_0},e^a])}\int_{x_0}^b \|A_1(x)\|_p (e^x - e^{x_0})^{\alpha - 1} dx\right\},\
$$
\n
$$
\|\Phi(A_1, A_2) (x_0)\|_1 \le \frac{1}{\Gamma(\alpha + 1)}
$$
\n
$$
\left\{\left[\left\|\|D_{e^{a_0}}^{\alpha} - (A_1 \circ \log)\|_p\right\|_{\infty, [e^{a},e^{a_0}]}\int_{x_0}^b \|A_2(x)\|_q (e^{x_0} - e^{x_0})^{\alpha} dx\right] + \left[\left\|\|D_{e^{a_0}}^{\alpha} - (A_1 \circ \log)\|_p\right\|_{\infty, [e^{a},e^{a_0}]}\int_{x_0}^b \|A_2(x)\|_q (e^x - e^{x_0})^{\alpha} dx\right] + \left[\left\|\|D_{e^{a_0}}^{\alpha} - (A_2 \circ \log)\|_q\right\|_{\infty, [e^{a_0},e^{a_0}]}\int_{x_0}^a \|A_1(x)\|_p (e^{x_0} - e^{x_0})^{\alpha} dx\right] + \left[\left\|\|D_{e^{a_0}}^{\alpha} (A_2 \circ \log)\|_q\|_{\infty, [e^{a_0},e^{a_0}]}\int_{x_0}^a \|A_1(x)\|_p (e^{x_0} - e^{x_0})^{\alpha} dx\right]\right\}.
$$
\n11. (31) We continue with applications on Opial inequalities

\nCorollary 44. (to Theorem 19) All as in Theorem 19 with $g(t) = t$. Let $p, q > 1: \frac{1}{p} + \frac{1}{q} = 1$. Then

\n
$$
\int_{x_0}^a \|f(w) (D_{x_0}^{\nu} f) (w) \|_{1}^a dw \le
$$
\n
$$
\frac{2^{-\frac{1}{4}}(z - z_0)^{\nu + \frac{1}{p} - \frac{1}{4}}}{\Gamma(\nu
$$

We continue with applications on Opial inequalities

Corollary 41 (to Theorem 19) All as in Theorem 19 with $g(t) = t$. Let $p, q > 0$ $1: \frac{1}{p} + \frac{1}{q} = 1$. Then

$$
\int_{x_0}^{z} \left\| f(w) \left(D_{x_0}^{\nu} f \right)(w) \right\|_1 dw \le
$$
\n
$$
\frac{2^{-\frac{1}{q}} \left(z - x_0 \right)^{\nu + \frac{1}{p} - \frac{1}{q}}}{\Gamma(\nu) \left[(p(\nu - 1) + 1) \left(p(\nu - 1) + 2 \right) \right]^{\frac{1}{p}}} \left(\int_{x_0}^{z} \left\| \left(D_{x_0}^{\nu} f \right)(w) \right\|_2^q dw \right)^{\frac{2}{q}}, \quad (132)
$$

for all $x_0 \leq z \leq b$.

It follows:

Corollary 42 (to Theorem 20) All as in Theorem 20, $\gamma \geq 1$, with $g(t) = e^t$. Let also $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then

$$
\int_{e^{x_0}}^{z} \left\| \left((f \circ \log)(w) \right) \left((D_{e^{x_0}}^{\nu} (f \circ \log))(w) \right) \right\|_{\gamma} dw \le
$$
\n
$$
\frac{2^{-\frac{1}{q}} (z - e^{x_0})^{\nu + \frac{1}{p} - \frac{1}{q}}}{\Gamma(\nu) \left[(p(\nu - 1) + 1) \left(p(\nu - 1) + 2 \right) \right]^{\frac{1}{p}}} \left(\int_{e^{x_0}}^{z} \left\| \left(D_{e^{x_0}}^{\nu} (f \circ \log)(w) \right\|_{\gamma}^{q} dw \right)^{\frac{2}{q}},
$$
\n(133)

for all $e^{x_0} \leq z \leq e^b$.

We finish with applications on Grüss inequalities: $% \alpha$

Corollary 43 (to Theorem 35) All as in Theorem 35 with $g(t) = t$ $(1 \le \nu < 2)$. Then i)

$$
\left\| \Delta(f_1, ..., f_r) \right\|_1 \le \frac{M_1(f_1, ..., f_r)(b-a)^{\nu+2}}{\Gamma(\nu+1)} \sum_{i=1}^r \left(\left\| \prod_{\substack{j=1 \ j \neq i}}^r \|f_j(x)\|_2 \right\|_{\infty, [a,b]} \right),
$$
\n(134)

where M_1 $(f_1, ..., f_r)$ is as in (92), ii)

$$
\left\| \Delta(f_1, ..., f_r) \right\|_1 \le \frac{M_2(f_1, ..., f_r)(b-a)^{\nu+1}}{\Gamma(\nu)} \sum_{i=1}^r \left(\left\| \prod_{\substack{j=1 \ j \neq i}}^r \|f_j(x)\|_2 \right\|_{\infty,[a,b]} \right),\tag{135}
$$

where $M_2(f_1, ..., f_r)$ is as in (94) ,

iii) when $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, we have

$$
\left\|\Delta\left(f_{1},...,f_{r}\right)\right\|_{1} \leq \frac{M_{3}\left(f_{1},...,f_{r}\right)\left(b-a\right)^{\nu+1+\frac{1}{p}}}{\left(p\left(\nu-1\right)+1\right)^{\frac{1}{p}}\Gamma\left(\nu\right)}\sum_{i=1}^{r}\left(\left\|\prod_{\substack{j=1\\j\neq i}}^{r}\left\|f_{j}\left(x\right)\right\|_{2}\right\|_{\infty,[a,b]}\right),\tag{136}
$$

where $M_3(f_1, ..., f_r)$ is as in (96).

It follows $(r = 2 \text{ case})$

Corollary 44 (to Theorem 37) All as in Theorem 37, with $[a, b] \subset \mathbb{R}_+ - \{0\}$, and $g(t) = \log t$. Then

i) for $\gamma, \delta > 1 : \frac{1}{\gamma} + \frac{1}{\delta} = 1$, we have

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, Vol. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\n*There*
\n
$$
\|\Delta(f_1,...,f_r)\|_1 \leq \frac{M_1(f_1,...,f_r)}{\Gamma(\nu+1)} \sum_{i=1}^r \left(\left\| \prod_{j=1}^r \|f_j(x)\|_2 \right\|_2 \right),
$$
\nwhere $M_1(f_1,...,f_r)$ is as in (92),
\n
$$
\|\Delta(f_1,...,f_r)\|_1 \leq \frac{M_2(f_1,...,f_r)}{\Gamma(\nu+1)} \sum_{i=1}^r \left(\left\| \prod_{j=1}^r \|f_j(x)\|_2 \right\|_2 \right),
$$
\nwhere $M_2(f_1,...,f_r)$ is as in (94),
\nwhere $M_2(f_1,...,f_r)$ is as in (94),
\nwhere $M_3(f_1,...,f_r)$ is as in (94),
\n
$$
\|\Delta(f_1,...,f_r)\|_1 \leq \frac{M_2(f_1,...,f_r)}{(\nu-\nu+1)^{\frac{1}{2}} \Gamma(\nu)} \sum_{i=1}^r \left(\left\| \prod_{j=1}^r \|f_j(x)\|_2 \right\|_2 \right),
$$
\nwhere $M_3(f_1,...,f_r)$ is as in (96).
\nIt follows (r = 2 case)
\nCorollary 44 (to Theorem 37) All as in Theorem 37, with $[a, b] \subset \mathbb{R}_+ - \{0\},$
\nand $g(t) = \log t$. Then
\n
$$
\|\Delta(A_1,A_2)\|_1 \leq \frac{N_1(A_1,A_2) \left(\log \frac{k}{2}\right)^{\alpha-\frac{1}{2}} \left(\log \frac{1}{2}\right),
$$
\nwhere $N_1(A_1,A_2)$ is as in (166).
\n
$$
\|\Delta(A_1,A_2)\|_1 \leq \frac{N_1(A_1,A_2) \left(\log \frac{k}{2}\right)^{\alpha-\frac{1}{2}} \left(\delta - \alpha\right)}{\Gamma(\alpha) (\gamma (\alpha-1)+1)^{\frac{1}{2}}}
$$
\nwhere $N_1(A_1,A_2)$ is as in (

where $N_1(A_1, A_2)$ is as in (104) ,

ii)

$$
\left\|\Delta\left(A_1, A_2\right)\right\|_1 \le \frac{N_2\left(A_1, A_2\right)\left(\log\frac{b}{a}\right)^{\alpha-1}\left(b-a\right)}{\Gamma\left(\alpha\right)}
$$

$$
\left[\int_{a}^{b} \|A_{1}(x)\|_{p} dx + \int_{a}^{b} \|A_{2}(x)\|_{q} dx\right],
$$
\n(138)

where $N_2(A_1, A_2)$ is as in (106), and

iii)

\n- **J. COMPUTATIONAL ANALYSIS AND APPLICATIONS.** VOL-31, NO.4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\n- $$
\left[\int_{a}^{b} ||A_1(x)||_p dx + \int_{a}^{b} ||A_2(x)||_q dx\right],
$$
\n(138)

\nwhere $N_2(A_1, A_2)$ is as in (106),

\nand

\niii.)

\n
$$
||\Delta(A_1, A_2)||_1 \leq \frac{N_3(A_1, A_2)(\log \frac{b}{a})^{\alpha} (b - a)}{\Gamma(\alpha + 1)}
$$
\n
$$
\left[\int_{a}^{b} ||A_1(x)||_p dx + \int_{a}^{b} ||A_2(x)||_q dx\right],
$$
\n(139)

\nwhere $N_3(A_1, A_2)$ is as in (108).

\n**References**

\n
	\n- [1] G.A. Annstassion, *Partional Differentiation Inequalities*, Research Monograph, Springer, New York, 2001.
	\n- [2] G.A. Annstassion, *Intelligent Comparisons* and generalized fractional calculus for Ban-3 and *theoremitized independent independent*

where $N_3(A_1, A_2)$ is as in (108).

References

- [1] G.A. Anastassiou, *Fractional Differentiation Inequalities*, Research Monograph, Springer, New York, 2009.
- [2] G.A. Anastassiou, Advances on Fractional Inequalities, Research Monograph, Springer, New York, 2011.
- [3] G.A. Anastassiou, Intelligent Comparisons: Analytic Inequalities, Springer, Heidelberg, New York, 2016.
- [4] G.A. Anastassiou, Strong mixed and generalized fractional calculus for Banach space valued functions, Mat. Vesnik, 69(3) (2017), 176-191.
- [5] G.A. Anastassiou, Intelligent Computations: Abstract Fractional Calculus, Inequalities, Approximations, Springer, Heidelberg, New York, 2018.
- [6] G.A. Anastassiou, Generalized Canavati Fractional Ostrowski, Opial and Grüss type inequalities for Banach algebra valued functions, submitted, 2021.
- [7] R. Bellman, Some inequalities for positive definite matrices, in E.F. Beckenbach (Ed.), General Inequalities 2, Proceedings of the 2nd International Conference on General Inequalities, Birkhauser, Basel, 1980, 89-90.
- $[8]$ Cebyšev, Sur les expressions approximatives des intégrales définies par les aures prises entre les mêmes limites, Proc. Math. Soc. Charkov, 2(1882), 93-98.
- [9] D. Chang, A matrix trace inequality for products of Hermitian matrices, J. Math. Anal. Appl., 237 (1999), 721-725.
- [10] I.D. Coop, On matrix trace inequalities and related topics for products of Hermitian matrix, J. Math. Anal. Appl. 188 (1994), 999-1001.
- [11] S.S. Dragomir, p-Schatten norm inequalities of Ostrowski's type, RGMIA Res. Rep. Coll. 24 (2021), Art. 108, 19 pp. 6268471810664. ANALYSIS AND APPLICATIONS, VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC (147). Comps. On another, J. Alsah. Anal. Anyt. 128 (1994), 994-1011.

[13] S.S. Denomine, p. Scheding, news, torquebities
	- [12] S.S. Dragomir, p -Schatten norm inequalities of Grüss type, RGMIA Res. Rep. Coll. 24 (2021), Art. 115, 16 pp.
	- [13] J. Mikusinski, The Bochner integral, Academic Press, New York, 1978.
	- [14] H. Neudecker, A matrix trace inequality, J. Math. Anal. Appl., 166 (1992), 302-303.
	- [15] Z. Opial, Sur une inegalite, Ann. Polon. Math. 8(1960), 29-32.
	- [16] A. Ostrowski, Über die Absolutabweichung einer differentiabaren Funcktion von ihrem Integralmittelwert, Comment. Math. Helv., 10 (1938), 226-227.
	- [17] W. Rudin, Functional Analysis, Second Edition, McGraw-Hill, Inc., New York, 1991.
	- [18] B. Simon, Trace ideals and Their Applications, Cambridge University Press, Cambridge, 1979.
	- [19] V.A. Zagrebvov, Gibbs Semigroups, Operator Theory: Advances and Applications, Volume 273, Birkhauser, 2019.

Abstract multivariate algebraic function activated neural network approximations

George A. Anastassiou, Robert Kozma Department of Mathematical Sciences University of Memphis Memphis, TN 38152, U.S.A. ganastss@memphis.edu

Abstract

Here we exhibit multivariate quantitative approximations of Banach space valued continuous multivariate functions on a box or \mathbb{R}^N , $N \in \mathbb{N}$, by the multivariate normalized, quasi-interpolation, Kantorovich type and quadrature type neural network operators. We study also the case of approximation by iterated operators of the last four types. These approximations are achieved by establishing multidimensional Jackson type inequalities involving the multivariate modulus of continuity of the engaged function or its high order Fréchet derivatives. Our multivariate operators are defined by using a multidimensional density function induced by the algebraic sigmoid function. The approximations are pointwise and uniform. The related feed-forward neural network is with one hidden layer. 623 CONFUTATIONAL ANNEXTS AND APPLICATIONS, VOL. 31, NO. 4, 2022, COPYRIGHT 2022 EUDOXUS PRESS, LLC (Groups A. Anastassion, Relater Known (Distributed Schemer Californization Content Analysis (Groups A. Analysis and Europ

2020 AMS Mathematics Subject Classification: $41A17$, $41A25$, $41A30$, 41A36.

Keywords and Phrases: algebraic sigmoid function, multivariate neural network approximation, quasi-interpolation operator, Kantorovich type operator, quadrature type operator, multivariate modulus of continuity, abstract approximation, iterated approximation.

1 Introduction

G.A. Anastassiou in [2] and [3], see chapters 2-5, was the first to establish neural network approximations to continuous functions with rates by very specifically defined neural network operators of Cardaliagnet-Euvrard and "Squashingî types, by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities. He treats there both the univariate and multivariate cases. The defining these operators "bell-shaped" and "squashing" functions are assumed to be of compact support. Also in [3] he gives the Nth order asymptotic expansion for the error of weak approximation of these two operators to a special natural class of smooth functions, see chapters 4-5 there.

Motivations for this work are the article [15] of Z. Chen and F. Cao, and [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [16], [17].

Here we perform multivariate algebraic sigmoid function based neural network approximations to continuous functions over boxes or over the whole \mathbb{R}^N , $N \in \mathbb{N}$, and also iterated approximations. All convergences here are with rates expressed via the multivariate modulus of continuity of the involved function or its high order Fréchet derivative and given by very tight multidimensional Jackson type inequalities.

We come up with the "right" precisely defined multivariate normalized, quasi-interpolation neural network operators related to boxes or \mathbb{R}^N , as well as Kantorovich type and quadrature type related operators on \mathbb{R}^N . Our boxes are not necessarily symmetric to the origin. In preparation to prove our results we establish important properties of the basic multivariate density function induced by algebraic sigmoid function and defining our operators. **CONFUTATIONAL ANEXYSIS AND APPLICATIONS, VOL. 31, NO. 4, 2023, COPYRIGHT 2023 COOXUS PRESS, LLC George A. Anastas III (Application 2023) AND APPLICATIONS CONFIDENTIAL CONFIDENTIAL CONFIDENTIAL CONFIDENTIAL CONFIDENTIAL C**

Feed-forward neural networks (FNNs) with one hidden layer, the only type of networks we deal with in this article, are mathematically expressed as

$$
N_n(x) = \sum_{j=0}^n c_j \sigma(\langle a_j \cdot x \rangle + b_j), \quad x \in \mathbb{R}^s, \quad s \in \mathbb{N},
$$

where for $0 \leq j \leq n$, $b_j \in \mathbb{R}$ are the thresholds, $a_j \in \mathbb{R}^s$ are the connection weights, $c_j \in \mathbb{R}$ are the coefficients, $\langle a_j \cdot x \rangle$ is the inner product of a_j and x, and σ is the activation function of the network. In many fundamental network models, the activation function is the algebrai sigmoid function. About neural networks see [18], [19], [20].

2 Basic

Here see also [12].

We consider the generator algebraic function

$$
\varphi(x) = \frac{x}{\sqrt[2m]{1 + x^{2m}}}, \quad m \in \mathbb{N}, \, x \in \mathbb{R}, \tag{1}
$$

which is a sigmoid type of function and is a strictly increasing function.

We see that $\varphi(-x) = -\varphi(x)$ with $\varphi(0) = 0$. We get that

$$
\varphi'(x) = \frac{1}{(1+x^{2m})^{\frac{2m+1}{m}}} > 0, \ \forall \ x \in \mathbb{R},
$$
\n(2)

proving φ as strictly increasing over $\mathbb{R}, \varphi'(x) = \varphi'(-x)$. We easily find that $\lim_{x \to +\infty} \varphi(x) = 1, \varphi(+\infty) = 1, \text{ and } \lim_{x \to -\infty} \varphi(x) = -1, \varphi(-\infty) = -1.$

We consider the activation function

$$
\Phi(x) = \frac{1}{4} \left[\varphi(x+1) - \varphi(x-1) \right]. \tag{3}
$$

Clearly it is $\Phi(x) = \Phi(-x), \forall x \in \mathbb{R}$, so that Φ is an even function and symmetric with respect to the y-axis. Clearly $\Phi(x) > 0, \forall x \in \mathbb{R}$.

Also it is

$$
\Phi(0) = \frac{1}{2^{2m}\sqrt{2}}.\tag{4}
$$

By [12], we have that $\Phi'(x) < 0$ for $x > 0$. That is Φ is strictly decreasing over $(0, +\infty)$.

Clearly, Φ is strictly increasing over $(-\infty, 0)$ and $\Phi'(0) = 0$. Furthermore we obtain that

$$
\lim_{x \to +\infty} \Phi(x) = \frac{1}{4} \left[\varphi(+\infty) - \varphi(+\infty) \right] = 0,\tag{5}
$$

and

$$
\lim_{x \to -\infty} \Phi(x) = \frac{1}{4} \left[\varphi(-\infty) - \varphi(-\infty) \right] = 0.
$$
 (6)

That is the x-axis is the horizontal asymptote of Φ .

Conclusion, Φ is a bell shape symmetric function with maximum

$$
\Phi(0) = \frac{1}{2^{2m}\sqrt{2}}, \quad m \in \mathbb{N}.
$$
\n(7)

We need

Theorem 1 ([12]) We have that

$$
\sum_{i=-\infty}^{\infty} \Phi(x - i) = 1, \ \forall \ x \in \mathbb{R}.
$$
 (8)

Theorem 2 ([12]) It holds

$$
\int_{-\infty}^{\infty} \Phi(x) dx = 1.
$$
 (9)

Theorem 3 ([12]) Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$. It holds

3. COMPUTATIONAL ANALYSIS AND APPLICATIONS, Vol. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\n
$$
\lim_{x \to +\infty} \varphi(x) = 1, \varphi(+\infty) = 1, \text{ and } \lim_{x \to +\infty} \varphi(x) = -1, \varphi(-\infty) = -1.
$$
\nWe consider the activation function
\n
$$
\Phi(x) = \frac{1}{4} [\varphi(x + 1) - \varphi(x - 1)].
$$
\n(3)
\nClearly it is 0 (x) = 0 (-(x), y (x ∈ R), so that 0 is an even function and
\nsymmetric with respect to the *p*-axis. Clearly $\Phi(x) > 0, \forall x \in R$.
\nAlso it is
\n
$$
\Phi(0) = \frac{1}{2} \frac{1}{\sqrt{2}}.
$$
\nBy [12], we have that $\Phi'(x) < 0$ for $x > 0$. That is Φ is strictly decreasing over
\n(0, +*∞*).
\nClearly, Φ is strictly increasing over (-∞, 0) and $\Phi'(0) = 0$.
\nFurthermore we obtain that
\n
$$
\lim_{x \to +\infty} \Phi(x) = \frac{1}{4} [\varphi(+\infty) - \varphi(+\infty)] = 0,
$$
\n(5)
\nand
\n
$$
\lim_{x \to +\infty} \Phi(x) = \frac{1}{4} [\varphi(-\infty) - \varphi(-\infty)] = 0.
$$
\n(6)
\nThat is the x-axis is the horizontal asymptote of θ.
\nConclusion, θ is a bell shape symmetric function with maximum
\n
$$
\Phi(0) = \frac{1}{2} \frac{1}{\sqrt{2}}, \quad m \in \mathbb{N}.
$$
\n(7)
\nWe need
\n**Theorem 1** (112*)*) We have that
\n
$$
\sum_{i=-\infty}^{\infty} \Phi(x - i) = 1, \forall x \in \mathbb{R}.
$$
\n(8)
\n**Theorem 2** (112*) It holds
\n
$$
\int_{-\infty}^{\infty} \Phi(x) dx = 1.
$$
\n(9)
\n**Theorem 3** (112*) Let 0 < \infty < 1*, and $n \in \$*

Denote by $\lfloor \cdot \rfloor$ the integral part of the number and by $\lceil \cdot \rceil$ the ceiling of the number.

We need

Theorem 4 ([12]) Let $[a, b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $[na] \leq [nb]$. It holds

$$
\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}} < 2\left(\sqrt[2m]{1+4^m}\right),\tag{11}
$$

 $\forall x \in [a, b], m \in \mathbb{N}.$

Note 5 1) By $\left|12\right|$ we have that

$$
\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi\left(nx-k\right) \neq 1,\tag{12}
$$

for at least some $x \in [a, b]$.

2) Let $[a, b] \subset \mathbb{R}$. For large $n \in \mathbb{N}$ we always have $[na] \leq |nb|$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$.

In general it holds that

$$
\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi\left(nx-k\right) \le 1. \tag{13}
$$

We introduce

$$
Z(x_1, ..., x_N) := Z(x) := \prod_{i=1}^{N} \Phi(x_i), \quad x = (x_1, ..., x_N) \in \mathbb{R}^{N}, \ N \in \mathbb{N}.
$$
 (14)

It has the properties:

(i) Z (x) > 0, 8 x 2 R N ; (ii) X¹ k=1 ^Z (^x ^k) := ^X¹ k1=1 X¹ k2=1 ::: ^X¹ k^N =1 Z (x¹ k1; :::; x^N k^N) = 1; (15) where k := (k1; :::; kn) 2 Z ^N , ⁸ ^x ² ^R N ; hence (iii) 625 J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC George A. Anastassiou 622-653

$$
\sum_{k=-\infty}^{\infty} Z\left(nx-k\right) = 1,\tag{16}
$$

 $\forall x \in \mathbb{R}^N; n \in \mathbb{N},$

and (iv)

$$
\int_{\mathbb{R}^N} Z(x) dx = 1,\tag{17}
$$

that is \boldsymbol{Z} is a multivariate density function.

Here denote $||x||_{\infty} := \max\{|x_1|, ..., |x_N|\}, x \in \mathbb{R}^N$, also set $\infty := (\infty, ..., \infty)$, $-\infty := (-\infty, ..., -\infty)$ upon the multivariate context, and

$$
[na] := ([na1], ..., [naN]),\n
$$
[nb] := ([nb1], ..., [nbN]),
$$
\n(18)
$$

where $a := (a_1, ..., a_N), b := (b_1, ..., b_N)$.

We obviously see that

0. COMPUTATIONAL ANALYSIS AND APPLICATIONS, Vol. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\nthat is *Z* is a multivariate density function.
\nHerc denote
$$
||x||_{\infty}
$$
 := max { $||x_1|, ..., |x_N|$ }, $x \in \mathbb{R}^N$, also set $\infty := (\infty, ..., \infty)$,
\n $-\infty := (-\infty, ..., -\infty)$ upon the multivariate context, and
\n $||na| := (||na_1|, ..., |na_N|)$,
\n $||nb|| := (||nb_1|, ..., |nb_N|)$,
\nwhere $a := (a_1, ..., a_N)$, $b := (b_1, ..., b_N)$.
\nWe obviously set that
\n
$$
\sum_{k=|n\alpha|}^{|\alpha|} z (nx - k) = \sum_{k=|n\alpha|}^{|\alpha|} \left(\prod_{k=1}^{N} \Phi(nx_i - k_i)\right) =
$$
\n
$$
\sum_{k=|n\alpha|}^{|\alpha|} \left(\prod_{k=1}^{N} \Phi(nx_i - k_i)\right) = \sum_{k=|n\alpha|}^{|\alpha|} \left(\prod_{k=1}^{N} \Phi(nx_i - k_i)\right) =
$$
\n
$$
\sum_{k=|n\alpha|}^{|\alpha|} \left(\prod_{k=1}^{N} \Phi(nx_i - k_i)\right) = \sum_{k=|n\alpha|}^{|\alpha|} \left(\sum_{k=1}^{|\alpha|} \Phi(nx_i - k_i)\right)
$$
.(19)
\n
$$
\sum_{k=|n\alpha|}^{|\alpha|} z (nx - k) =
$$
\n
$$
\sum_{k=|n\alpha|}^{|\alpha|} z (nx - k) =
$$
\n
$$
\sum_{k=|n\alpha|}^{|\alpha|} z (nx - k) =
$$
\n
$$
\sum_{k=|n\alpha|}^{|\alpha|} z (nx - k) =
$$
\n
$$
\sum_{k=|n\alpha|}^{|\alpha|} \left(\sum_{k=1}^{k} \sum_{k=1}^{|\alpha|} \sum_{k=1}^{|\alpha|} \sum_{k=1}^{|\alpha|} \sum_{k=1}^{|\alpha|} \sum_{k=1}^{|\alpha|} \sum_{k=1}^{|\alpha|} \sum_{k=1}^{|\alpha
$$

For $0 < \beta < 1$ and $n \in \mathbb{N}$, a fixed $x \in \mathbb{R}^N$, we have that

$$
\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k) =
$$

$$
\sum_{\begin{cases}\n|k = \lceil na\n\end{cases}}^{\lfloor nb\rfloor} Z(nx-k) + \sum_{\begin{cases}\n|k = \lceil na\n\end{cases}}^{\lfloor nb\rfloor} Z(nx-k). \tag{20}
$$
\n
$$
\left\{ \left\| \frac{k}{n} - x \right\|_{\infty} \le \frac{1}{n^{\beta}} \right\} \left\{ \left\| \frac{k}{n} - x \right\|_{\infty} > \frac{1}{n^{\beta}}
$$

In the last two sums the counting is over disjoint vector sets of k 's, because the condition $\left\|\frac{k}{n}-x\right\|_{\infty} > \frac{1}{n^{\beta}}$ implies that there exists at least one $\left|\frac{k_r}{n}-x_r\right| > \frac{1}{n^{\beta}}$, where $r \in \{1, ..., N\}$.

(v) As in [10], pp. 379-380, we derive that

$$
\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k) \stackrel{(10)}{<} \frac{1}{4m(n^{1-\beta}-2)^{2m}}, \ 0 < \beta < 1, \ m \in \mathbb{N}, \ (21)
$$
\n
$$
\left\{ \| \frac{k}{n} - x \|_{\infty} > \frac{1}{n^{\beta}}
$$

with $n \in \mathbb{N} : n^{1-\beta} > 2, x \in \prod_{i=1}^{N} [a_i, b_i]$.

(vi) By Theorem 4 we get that

$$
0 < \frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z\left(nx-k\right)} < \frac{1}{\left(\Phi\left(1\right)\right)^N} \cong \left[2\left(\sqrt[2m]{1+4^m}\right)\right]^N, \tag{22}
$$

$$
\forall x \in \left(\prod_{i=1}^{N} [a_i, b_i]\right), \ n \in \mathbb{N}.
$$

It is also clear that
(vii)

 $\sum_{i=1}^{\infty}$ $\begin{cases} k = -\infty \\ \left\| \frac{k}{n} - x \right\|_{\infty} > \end{cases}$ $\left\|\frac{k}{n} - x\right\|_{\infty} > \frac{1}{n^{\beta}}$ $Z(nx-k) < \frac{1}{4m(n^{1-\beta}-2)^{2m}},$ (23)

 $0 < \beta < 1, n \in \mathbb{N}: n^{1-\beta} > 2, x \in \mathbb{R}^N, m \in \mathbb{N}.$

Furthermore it holds

 \int

$$
\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k) \neq 1,\tag{24}
$$

for at least some $x \in \left(\prod_{i=1}^N [a_i, b_i]\right)$.

Here $(X, \left\| \cdot \right\|_{\gamma})$ is a Banach space.

Let $f \in C\left(\prod_{i=1}^N [a_i, b_i], X\right)$, $x = (x_1, ..., x_N) \in \prod_{i=1}^N [a_i, b_i]$, $n \in \mathbb{N}$ such that $\lceil na_i \rceil \leq \lfloor nb_i \rfloor, i = 1, ..., N.$

We introduce and define the following multivariate linear normalized neural network operator $(x := (x_1, ..., x_N) \in \left(\prod_{i=1}^N [a_i, b_i] \right))$:

3. COMPUTATIONAL ANALYSIS AND APPLICATIONS, Vol. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\n(vi) By Theorem 4 we get that
\n
$$
0 < \frac{1}{\sum_{k=1}^{100} (2k-12)} (2n\pi - k) < \frac{1}{(9(1))^N} \approx [2(^2\sqrt{1+4^m})]^N, \qquad (22)
$$
\n
$$
\forall x \in (\prod_{k=1}^{N} [a_i, b_i)], n \in \mathbb{N}.
$$
\nIt is also clear that
\n(vii)
\n
$$
\sum_{k=-\infty}^{\infty} Z(nx-k) < \frac{1}{4m(n^{1-\beta}-2)^{2m}}, \qquad (23)
$$
\n
$$
\begin{cases}\n\left\| \frac{k}{n} - x \right\|_{\infty} > \frac{1}{n^2} \\
0 < \beta < 1, n \in \mathbb{N} : n^{1-\beta} > 2, \pi \in \mathbb{R}^N, m \in \mathbb{N}. \n\end{cases}
$$
\nFurthermore it holds
\n
$$
\lim_{n \to \infty} \sum_{k=1}^{\lfloor n\beta \rfloor} Z(nx-k) \neq 1, \qquad (24)
$$
\nfor at least some $x \in (\prod_{k=1}^{N} [a_i, b_i])$.
\nHere $(X, [\cdot | I_0])$ is a Banach space.
\nLet $f \in G(\prod_{k=1}^{N} [a_i, b_i], X) \cdot x = (x_1, ..., x_N) \in \prod_{k=1}^{N} [a_i, b_i], n \in \mathbb{N}$ such that $[na_i] \leq [n\beta_i], i = 1, ..., N$.
\nWe introduce and define the following multivariate linear normalized neural network operator (x := (x_1, ..., x_N) = (1<sup>{N_{\text{min}}}_{\infty} [a_i, b_i])):\n
$$
A_n(f, x_1, ..., x_N) := A_n(f, x) := \frac{\sum_{k=1}^{\lfloor n\beta \rfloor} [a_i, b_i]}{\sum_{k=1}^{\lfloor n\beta \rfloor} [a_i, b_i]} \times (nx - k) =
$$
\n
$$
\sum_{k=1}^{\lfloor n\beta \rfloor} \sum_{k=1}^{\lfloor n\beta \rfloor}
$$</sup>

For large enough $n \in \mathbb{N}$ we always obtain $[na_i] \leq |nb_i|, i = 1, ..., N$. Also $a_i \leq \frac{k_i}{n} \leq b_i$, iff $\lceil na_i \rceil \leq k_i \leq \lfloor nb_i \rfloor$, $i = 1, ..., N$.

When $g \in C\left(\prod_{i=1}^N [a_i, b_i]\right)$ we define the companion operator

$$
\widetilde{A}_n(g,x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} g\left(\frac{k}{n}\right) Z\left(nx-k\right)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z\left(nx-k\right)}.
$$
\n(26)

Clearly \widetilde{A}_n is a positive linear operator. We have that

$$
\widetilde{A}_n(1,x) = 1, \ \forall \ x \in \left(\prod_{i=1}^N [a_i, b_i]\right).
$$

Notice that $A_n(f) \in C\left(\prod_{i=1}^N [a_i, b_i], X\right)$ and $\widetilde{A}_n(g) \in C\left(\prod_{i=1}^N [a_i, b_i]\right)$. Furthermore it holds

$$
\|A_n(f,x)\|_{\gamma} \le \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \|f\left(\frac{k}{n}\right)\|_{\gamma} Z\left(nx-k\right)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z\left(nx-k\right)} = \widetilde{A}_n\left(\|f\|_{\gamma},x\right),\tag{27}
$$

 $\forall x \in \prod_{i=1}^N [a_i, b_i].$ Clearly $||f||_{\gamma} \in C\left(\prod_{i=1}^{N} [a_i, b_i]\right)$. So, we have that

$$
\|A_n(f,x)\|_{\gamma} \le \widetilde{A}_n\left(\|f\|_{\gamma},x\right),\tag{28}
$$

 $\forall x \in \prod_{i=1}^N [a_i, b_i], \forall n \in \mathbb{N}, \forall f \in C \left(\prod_{i=1}^N [a_i, b_i], X \right).$ Let $c \in X$ and $g \in C\left(\prod_{i=1}^N [a_i, b_i]\right)$, then $cg \in C\left(\prod_{i=1}^N [a_i, b_i], X\right)$. Furthermore it holds

$$
A_n (cg, x) = c\widetilde{A}_n (g, x), \ \ \forall \ x \in \prod_{i=1}^N [a_i, b_i]. \tag{29}
$$

Since $\widetilde{A}_n(1) = 1$, we get that

$$
A_n(c) = c, \ \forall \ c \in X. \tag{30}
$$

We call \widetilde{A}_n the companion operator of A_n .

For convinience we call

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, Vol. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\nClearly
$$
\tilde{A}_n
$$
 is a positive linear operator. We have that
\n
$$
\tilde{A}_n (1, x) = 1, \forall x \in \left(\prod_{i=1}^N [a_i, b_i]\right).
$$
\nNotice that $A_n (f) \in C \left(\prod_{i=1}^N [a_i, b_i]\right)$, X and $\tilde{A}_n (g) \in C \left(\prod_{i=1}^N [a_i, b_i]\right)$.
\nFurthermore it holds
\n
$$
||A_n (f, x)||_{\infty} \leq \frac{\sum_{k=1}^{\lfloor n/2 \rfloor} [a_k^k] ||_{\infty} Z (nx - k)}{\sum_{k=1}^{\lfloor n/2 \rfloor} [Z (nx - k)]} = \tilde{A}_n \left(\|f\|_{\infty}, x\right), \qquad (27)
$$
\n
$$
\forall x \in \prod_{i=1}^N [a_i, b_i],
$$
\nSo, we have that
\n
$$
||A_n (f, x)||_{\infty} \in G \left(\prod_{i=1}^N [a_i, b_i]\right).
$$
\nSo, we have that
\n
$$
||A_n (f, x)||_{\infty} \leq \tilde{A}_n \left(\|f\|_{\infty}, x\right), \qquad (28)
$$
\n
$$
\forall x \in \prod_{i=1}^N [a_i, b_i], \forall n \in \mathbb{N}, \forall f \in C \left(\prod_{i=1}^N [a_i, b_i], x\right),
$$
\nLet $c \in X$ and $g \in C \left(\prod_{i=1}^N [a_i, b_i]\right)$, then $c g \in C \left(\prod_{i=1}^N [a_i, b_i], X\right)$.
\nFurthermore it holds
\n
$$
A_n (rg, x) = c \tilde{A}_n (gx, x), \forall x \in \prod_{i=1}^N [a_i, b_i].
$$
\n(29)
\nSince $\tilde{A}_n (1) = 1$, we get that
\n
$$
A_n (c) = c, \forall c \in X.
$$
\n(30)
\nWe call \tilde{A}_n the companion operator of

 $\forall x \in \left(\prod_{i=1}^N [a_i, b_i]\right), n \in \mathbb{N}.$ Hence

$$
A_n(f,x) - f(x) = \frac{A_n^*(f,x) - f(x) \left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)\right)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)}.
$$
 (33)

Consequently we derive

$$
\|A_{n}(f,x)-f(x)\|_{\gamma} \leq \left[2\left(\sqrt[2m]{1+4^{m}}\right)\right]^{N} \left\|A_{n}^{*}(f,x)-f(x)\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k)\right\|_{\gamma},
$$
\n(34)

 $\forall x \in \left(\prod_{i=1}^N [a_i, b_i]\right).$

We will estimate the right hand side of (34). For the last and others we need

Definition 6 ([11], p. 274) Let M be a convex and compact subset of $(\mathbb{R}^N, \left\|\cdot\right\|_p)$, $p \in [1,\infty]$, and $(X, \left\|\cdot\right\|_{\gamma})$ be a Banach space. Let $f \in C(M,X)$. We define the first modulus of continuity of f as

$$
\omega_1(f,\delta) := \sup_{\begin{subarray}{l} x, y \in M : \\ \|x - y\|_p \le \delta \end{subarray}} \|f(x) - f(y)\|_{\gamma}, \ \ 0 < \delta \le \operatorname{diam}(M). \tag{35}
$$

If $\delta > diam(M)$, then

$$
\omega_1(f,\delta) = \omega_1(f, diam(M)). \tag{36}
$$

Notice ω_1 (f, δ) is increasing in $\delta > 0$. For $f \in C_B (M, X)$ (continuous and bounded functions) ω_1 (f, δ) is defined similarly.

Lemma 7 ([11], p. 274) We have $\omega_1(f, \delta) \to 0$ as $\delta \downarrow 0$, iff $f \in C(M, X)$, where M is a convex compact subset of $(\mathbb{R}^N, \|\cdot\|_p), p \in [1, \infty]$.

Clearly we have also: $f \in C_U(\mathbb{R}^N, X)$ (uniformly continuous functions), iff $\omega_1(f,\delta) \to 0$ as $\delta \downarrow 0$, where ω_1 is defined similarly to (35). The space $C_B(\mathbb{R}^N, X)$ denotes the continuous and bounded functions on \mathbb{R}^N .

When $f \in C_B(\mathbb{R}^N, X)$ we define,

3. COMPUTATIONAL ANALYSIS AND APPLICATIONS, Vol. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\n
$$
\forall x \in (\prod_{k=1}^{N} [a_k, b_k]), n \in \mathbb{N}.
$$
\nHence
\n
$$
A_n (f, x) - f (x) = \frac{A_n^{\pi} (f, x) - f (x) (\sum_{k=1}^{10} [a_k - k])}{\sum_{k=1}^{10} [2 \cdot (n x - k)]}.
$$
\n(33)
\nConsequently we derive
\n
$$
||A_n (f, x) - f (x)||_2 \stackrel{(22)}{=} [2 (\sqrt[n]{1 + 4^m})]^{N} ||A_n^{\star} (f, x) - f (x) \sum_{k=1}^{10} [2 \cdot (n x - k)]_1,
$$
\n(34)
\n
$$
\forall x \in (\prod_{k=1}^{N} [a_k, b_k]),
$$
\nWe will estimate the right hand side of (34).
\nFor the last and others we need
\nDefinition 6 (iii), p. 274) Let M be a connect and compact subset of $[(\mathbb{R}^N, ||\cdot||_p),$
\n
$$
p \in [1, \infty], and (X, ||\cdot||_q) be a Banach space. Let f \in C (M, X). We define the first modulas of continuity of f as\n
$$
\omega_1 (f, \delta) := \sup_{x,y \in \mathbb{N}} ||f(x) - f(y)||_q, 0 < \delta \leq \text{diam}(M).
$$
\n(36)
\nNotice $\omega_1 (f, \delta) := \sup_{x,y \in \mathbb{N}} ||f(x) - f(y)||_q, 0 < \delta \leq \text{diam}(M).$ \n(36)
\nNotice $\omega_1 (f, \delta) = \omega_1 (f, \text{diam}(M)).$
\nNotice $\omega_1 (f, \delta) = \text{arcsin}(x) \text{ for } \epsilon \in \mathbb{Q}_2 (M$
$$

 $n \in \mathbb{N}, \forall x \in \mathbb{R}^N, N \in \mathbb{N}$, the multivariate quasi-interpolation neural network operator.

Also for $f \in C_B(\mathbb{R}^N, X)$ we define the multivariate Kantorovich type neural network operator

0. COMPUTATIONAL ANALYSIS AND APPLICATIONS. VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUB PRESS, LLC
\n
$$
n \in N, \forall x \in \mathbb{R}^N, N \in \mathbb{N},
$$
 the multivariate quasi-interpolation normal network operator.
\n
$$
C_n(f, x) := C_n(f, x_1, ..., x_N) := \sum_{k=-\infty}^{\infty} \left(n^N \int_{\frac{k}{k}}^{\frac{k+1}{k}} f(t) dt \right) Z(nx - k) =
$$
\n
$$
\sum_{k=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \left(n^N \int_{\frac{k}{k}}^{\frac{k+1}{k}} \int_{\frac{k}{k}}^{\frac{k+1}{k}} f(t_1, ..., t_N) dt_1 ... dx_N \right)
$$
\n
$$
= \sum_{k=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \left(n^N \int_{\frac{k}{k}}^{\frac{k+1}{k}} \int_{\frac{k}{k}}^{\frac{k+1}{k}} ... \int_{\frac{k}{k}}^{\frac{k+1}{k}} f(t_1, ..., t_N) dt_1 ... dx_N \right)
$$
\n
$$
+ \left(\prod_{k=1}^{N} \Phi(nx_i - k_i) \right), \qquad (38)
$$
\n
$$
n \in \mathbb{N}, \forall x \in \mathbb{R}^N.
$$
\nAgain for $f \in C_R$ (R^N, Y, Y) ∈ R, we define the multivariate neural network operator of quadrature type $D_n(f, x)$, n ∈ N, as follows.
\n
$$
L_n e \theta = (\theta_1, ..., \theta_N) ∈ \mathbb{N}^N, r = (r_1, ..., r_N) ∈ \mathbb{Z}_+^N, m_r = w_{r_1, r_2, ..., r_N} \ge 0, \text{ such that } \sum_{k=0}^{\infty} w_r = \sum_{r=0}^{\infty} \sum_{r_1, r_2, ..., r_N}^{\infty} w_{r_1, r_2, ..., r_N} = 1; k \in \mathbb{Z}^N \text{ and}
$$
\n
$$
\delta_{nk}(f) := \delta_{nk,k,k,...,k_N}(f) := \sum_{r=0}^{\infty} w_r f\left(\frac{k}{n} + \frac{r}{n\theta} \right) =
$$
\n<

 $n \in \mathbb{N}, \ \forall \ x \in \mathbb{R}^N.$

Again for $f \in C_B(\mathbb{R}^N, X)$, $N \in \mathbb{N}$, we define the multivariate neural network operator of quadrature type $D_n(f, x)$, $n \in \mathbb{N}$, as follows.

Let $\theta = (\theta_1, ..., \theta_N) \in \mathbb{N}^N$, $r = (r_1, ..., r_N) \in \mathbb{Z}_+^N$, $w_r = w_{r_1, r_2, ..., r_N} \ge 0$, such that $\sum_{i=1}^{n}$ $\sum_{r=0}^{\theta} w_r = \sum_{r_1=0}^{\theta_1}$ $r_1=0$ $\frac{\theta_2}{\sum}$ $r_2=0$ $\ldots \sum_{N} \frac{\theta_N}{N}$ $\sum_{r_N=0}^{N} w_{r_1,r_2,...r_N} = 1$; $k \in \mathbb{Z}^N$ and

$$
\delta_{nk}(f) := \delta_{n,k_1,k_2,...,k_N}(f) := \sum_{r=0}^{\theta} w_r f\left(\frac{k}{n} + \frac{r}{n\theta}\right) =
$$

$$
\sum_{r_1=0}^{\theta_1} \sum_{r_2=0}^{\theta_2} \dots \sum_{r_N=0}^{\theta_N} w_{r_1,r_2,...r_N} f\left(\frac{k_1}{n} + \frac{r_1}{n\theta_1}, \frac{k_2}{n} + \frac{r_2}{n\theta_2}, \dots, \frac{k_N}{n} + \frac{r_N}{n\theta_N}\right), \quad (39)
$$

ere $\frac{r}{\theta} := \left(\frac{r_1}{\theta_1}, \frac{r_2}{\theta_2}, \dots, \frac{r_N}{\theta_N}\right).$

where $\frac{r}{\theta} := \left(\frac{r_1}{\theta_1}, \frac{r_2}{\theta_2}, ..., \frac{r_N}{\theta_N}\right)$ We set

$$
D_n(f, x) := D_n(f, x_1, ..., x_N) := \sum_{k=-\infty}^{\infty} \delta_{nk}(f) Z(nx - k) = (40)
$$

$$
\sum_{k_1 = -\infty}^{\infty} \sum_{k_2 = -\infty}^{\infty} ... \sum_{k_N = -\infty}^{\infty} \delta_{n, k_1, k_2, ..., k_N}(f) \left(\prod_{i=1}^{N} \Phi(nx_i - k_i) \right),
$$

 $\forall x \in \mathbb{R}^N.$

In this article we study the approximation properties of A_n, B_n, C_n, D_n neural network operators and as well of their iterates. That is, the quantitative pointwise and uniform convergence of these operators to the unit operator I.

3 Multivariate general Neural Network Approximations

Here we present several vectorial neural network approximations to Banach space valued functions given with rates.

We give

Theorem 8 Let $f \in C\left(\prod_{i=1}^N [a_i, b_i], X\right),\ 0 < \beta < 1,\ x \in \left(\prod_{i=1}^N [a_i, b_i]\right),$ $m, N, n \in \mathbb{N}$ with $n^{1-\beta} > 2$. Then 1)

$$
\left\| A_n \left(f, x \right) - f \left(x \right) \right\|_{\gamma} \leq \left[2 \left(\sqrt[2m]{1 + 4^m} \right) \right]^N \left[\omega_1 \left(f, \frac{1}{n^{\beta}} \right) + \frac{\left\| \| f \|_{\gamma} \right\|_{\infty}}{2m \left(n^{1 - \beta} - 2 \right)^{2m}} \right] =: \lambda_1 \left(n \right), \tag{41}
$$

and

2)

$$
\left\| \left\| A_n \left(f \right) - f \right\|_{\gamma} \right\|_{\infty} \leq \lambda_1 \left(n \right). \tag{42}
$$

We notice that $\lim_{n\to\infty} A_n(f) \stackrel{\|\cdot\|_{\mathcal{A}}}{=} f$, pointwise and uniformly. Above ω_1 is with respect to $p = \infty$.

Proof. We observe that

3 Multivariate and a
Pluctatorions, Vol. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LC
limits
\nHere we present several vectorial neural network approximations to Banach
\nspace valued functions given with rates.
\nWe give
\n**Theorem 8** Let
$$
f \in C \left(\prod_{i=1}^{N} [a_i, b_i], X \right), 0 < \beta < 1, x \in \left(\prod_{i=1}^{N} [a_i, b_i] \right),
$$

\n $m, N, n \in \mathbb{N}$ with $n^{1-\beta} > 2$. Then
\n
\n
$$
||A_n(f, x) - f(x)||_{\gamma} \leq [2(\sqrt[n]{1 + 4^m})]^N \left[\omega_1 \left(f, \frac{1}{n^2} \right) + \frac{||f||_{\gamma}||_{\infty}}{2m (n^{1-\beta} - 2)^{2m}} \right] =: \lambda_1(n),
$$

\nand
\n
$$
||A_n(f, x) - f(x)||_{\gamma} \leq [2(\sqrt[n]{1 + 4^m})]^N \left[\omega_1 \left(f, \frac{1}{n^2} \right) + \frac{||f||_{\gamma}||_{\infty}}{2m (n^{1-\beta} - 2)^{2m}} \right] =: \lambda_1(n),
$$

\nand
\n
$$
||A_n(f) - f||_{\gamma} ||_{\infty} \leq \lambda_1(n).
$$

\n(42)
\nWe notice that $\lim_{n \to \infty} A_n(f) \frac{\|\cdot\|_{\gamma}}{\|\cdot\|_{\gamma}}$ pointwise and uniformly.
\n $Abwe \omega_1$ is similar respect to $p = \infty$.
\n**Proof.** We observe that
\n
$$
\Delta(x) := A_n^*(f, x) - f(x) \sum_{k = \lceil n\alpha \rceil}^{k+1} f(x) Z(nx - k) =
$$

\n
$$
\sum_{k = \lceil n\alpha \rceil}^{k+1} \left(f \left(\frac{k}{n} \right) - f(x) \right) Z(nx - k).
$$

\nThus
\n
$$
||\Delta(x)||_{\gamma} \leq \sum_{k = \lceil n\alpha \rceil}^{k+1} ||f \left(\frac{k}{n} \right) - f(x) ||_{\gamma} Z(nx - k) =
$$

Thus

$$
\|\Delta(x)\|_{\gamma} \le \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\|f\left(\frac{k}{n}\right) - f(x)\right\|_{\gamma} Z(nx-k) =
$$

$$
\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\|f\left(\frac{k}{n}\right) - f(x)\right\|_{\gamma} Z(nx-k) +
$$

$$
\left\{\left\|\frac{k}{n} - x\right\|_{\infty} \le \frac{1}{n^{\beta}}
$$

3. COMPUTATIONAL AVALYSIS AND APPLICATIONS, Vol. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\n
$$
\begin{aligned}\n&\left\{\begin{aligned}\n&\left\|\frac{k}{n}-x\right\|_{\infty} > \frac{1}{n^3} \\
&\omega_1\left(f,\frac{1}{n^3}\right)+2\left\||f\right\|_{\infty}\right\|_{\infty} > \frac{1}{n^4} \\
&\omega_2\left(f,\frac{1}{n^3}\right)+2\left\||f\right\|_{\infty}\right\|_{\infty} > \frac{1}{n^3} \\
&\omega_3\left(f,\frac{1}{n^2}\right)+2\left\||f\right\|_{\infty}\left\{\begin{aligned}\n&\left\|\frac{k}{n}-\ln a\right\|_{\infty} > \frac{1}{n^3} \\
&\omega_2\left(f,\frac{1}{n^3}\right)+\frac{2\left\||f\right\|_{\infty}\right\|_{\infty} > \frac{1}{n^3} \\
&\omega_3\left(\left\|\frac{k}{n}-\pi\right\|_{\infty} > \frac{1}{n^3}\n\end{aligned}\right.\n\end{aligned}
$$
\nSo that
\n
$$
\left\|\Delta(x)\right\|_{\infty} \leq \omega_1\left(f,\frac{1}{n^3}\right)+\frac{2\left\||f\right\|_{\infty}\right\|_{\infty}}{2m(n^{1-\beta}-2)^{2m}}.\n\tag{44}
$$
\nSo, what
\n
$$
\omega_2\left(f,\frac{1}{n^3}\right)+\frac{2\left\||f\right\|_{\infty}\right\|_{\infty}}{2m(n^{1-\beta}-2)^{2m}}.\n\tag{45}
$$
\n
$$
\text{Now using (34) we finish the proof. } \blacksquare
$$
\n
$$
\text{Remark 9: } \begin{aligned}\n&\text{Now, if } \Omega\left(\frac{2N}{n^3}\right) < \omega_1\left(\frac{2N}{n^3}\right) + \frac{2\left\||f\right\|_{\infty}}{2m(n^{1-\beta}-2)^{2m}}.\n\end{aligned}
$$
\n
$$
x:=\left(x_1,...,x_n\right) \in (\mathbb{R}^N)^{\frac{1}{2}}.\n\
$$

So that

$$
\left\|\Delta\left(x\right)\right\|_{\gamma} \leq \omega_1 \left(f, \frac{1}{n^{\beta}}\right) + \frac{\left\|\left\|f\right\|_{\gamma}\right\|_{\infty}}{2m\left(n^{1-\beta}-2\right)^{2m}}.\tag{45}
$$

Now using (34) we finish the proof. \blacksquare

We make

Remark 9 ([11], pp. 263-266) Let $(\mathbb{R}^N, \|\cdot\|_p)$, $N \in \mathbb{N}$; where $\|\cdot\|_p$ is the L_p norm, $1 \leq p \leq \infty$. \mathbb{R}^N is a Banach space, and $(\mathbb{R}^N)^j$ denotes the j-fold product space $\mathbb{R}^N \times \ldots \times \mathbb{R}^N$ endowed with the max-norm $||x||_{(\mathbb{R}^N)^j} := \max_{1 \leq \lambda \leq \lambda}$ $\max_{1 \leq \lambda \leq j} ||x_{\lambda}||_p$, where

 $x := (x_1, ..., x_j) \in (\mathbb{R}^N)^j$. Let $\left(X,\left\|\cdot\right\|_{\gamma}\right)$ be a general Banach space. Then the space $L_j:=L_j\left(\left(\mathbb{R}^N\right)^j;X\right)$ of all j-multilinear continuous maps $g: (\mathbb{R}^N)^j \to X$, $j = 1, ..., m$, is a Banach

space with norm

$$
||g|| := ||g||_{L_j} := \sup_{\left(\|x\|_{\left(\mathbb{R}^N\right)^j} = 1\right)} ||g(x)||_{\gamma} = \sup_{\left(\|x_1\|_{p} \dots \|x_j\|_{p}\right)} \frac{||g(x)||_{\gamma}}{||x_1||_{p} \dots ||x_j||_{p}}.
$$
 (46)

Let M be a non-empty convex and compact subset of \mathbb{R}^N and $x_0 \in M$ is fixed.

Let O be an open subset of \mathbb{R}^N : $M \subset O$. Let $f: O \to X$ be a continuous function, whose Fréchet derivatives (see [21]) $f^{(j)}: O \to L_j = L_j \left((\mathbb{R}^N)^j; X \right)$ exist and are continuous for $1 \leq j \leq \overline{m}$, $\overline{m} \in \mathbb{N}$.

Call $(x - x_0)^j := (x - x_0, ..., x - x_0) \in (\mathbb{R}^N)^j, x \in M.$

We will work with $f|_M$.

Then, by Taylor's formula (14) , (21) , p. 124), we get

$$
f(x) = \sum_{j=0}^{\overline{m}} \frac{f^{(j)}(x_0)(x - x_0)^j}{j!} + R_{\overline{m}}(x, x_0), \quad all \ x \in M,
$$
 (47)

where the remainder is the Riemann integral

$$
R_{\overline{m}}(x,x_0) := \int_0^1 \frac{(1-u)^{\overline{m}-1}}{(\overline{m}-1)!} \left(f^{(\overline{m})} (x_0 + u (x-x_0)) - f^{(\overline{m})} (x_0) \right) (x-x_0)^{\overline{m}} du,
$$
\n(48)

here we set $f^{(0)}(x_0)(x-x_0)^0 = f(x_0)$.

We consider

$$
w := \omega_1 \left(f^{(\overline{m})}, h \right) := \sup_{\substack{x, y \in M:\\ \|x - y\|_p \le h}} \left\| f^{(\overline{m})}(x) - f^{(\overline{m})}(y) \right\|, \tag{49}
$$

 $h > 0.$

We obtain

1. COMPUTATIONAL ANALYSIS AND APPLICATIONS, Vol. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC

\nwhere we set
$$
f(0)
$$
 (a, b)

\nHere we set $f(0)$ (a, b)

\nhere we set $f(0)$ (a, b)

\nwhere u du du

\nwhere u du

\

by Lemma 7.1.1, [1], p. 208, where $\lceil \cdot \rceil$ is the ceiling. Therefore for all $x \in M$ (see [1], pp. 121-122):

$$
||R_{\overline{m}}(x, x_0)||_{\gamma} \le w ||x - x_0||_p^{\overline{m}} \int_0^1 \left[\frac{u ||x - x_0||_p}{h} \right] \frac{(1 - u)^{\overline{m} - 1}}{(\overline{m} - 1)!} du
$$

= $\overline{w} \Phi_{\overline{m}} \left(||x - x_0||_p \right)$ (51)

by a change of variable, where

$$
\Phi_{\overline{m}}(t) := \int_0^{|t|} \left[\frac{s}{h} \right] \frac{(|t| - s)^{\overline{m} - 1}}{(\overline{m} - 1)!} ds = \frac{1}{\overline{m}!} \left(\sum_{j=0}^\infty (|t| - jh)^{\overline{m}}_+ \right), \ \ \forall \ t \in \mathbb{R}, \tag{52}
$$

is a (polynomial) spline function, see [1], p. 210-211.

Also from there we get

$$
\Phi_{\overline{m}}\left(t\right) \leq \left(\frac{\left|t\right|^{\overline{m}+1}}{(\overline{m}+1)!h} + \frac{\left|t\right|^{\overline{m}}}{2\overline{m}!} + \frac{h\left|t\right|^{\overline{m}-1}}{8(\overline{m}-1)!}\right), \quad \forall \ t \in \mathbb{R},\tag{53}
$$

with equality true only at $t = 0$.

Therefore it holds

$$
||R_{\overline{m}}(x,x_0)||_{\gamma} \le w \left(\frac{||x-x_0||_p^{\overline{m}+1}}{(\overline{m}+1)!h} + \frac{||x-x_0||_p^{\overline{m}}}{2\overline{m}!} + \frac{h ||x-x_0||_p^{\overline{m}-1}}{8(\overline{m}-1)!} \right), \quad \forall \ x \in M.
$$
\n(54)

We have found that

$$
\left\| f(x) - \sum_{j=0}^{\overline{m}} \frac{f^{(j)}(x_0)(x - x_0)^j}{j!} \right\|_{\gamma} \le
$$

$$
\omega_1 \left(f^{(\overline{m})}, h \right) \left(\frac{\|x - x_0\|_p^{\overline{m}+1}}{(\overline{m} + 1)!h} + \frac{\|x - x_0\|_p^{\overline{m}}}{2\overline{m}!} + \frac{h\|x - x_0\|_p^{\overline{m}-1}}{8(\overline{m} - 1)!} \right) < \infty, \quad (55)
$$

 $\forall x, x_0 \in M.$

Here $0 < \omega_1(f^{(\overline{m})}, h) < \infty$, by M being compact and $f^{(\overline{m})}$ being continuous on M.

One can rewrite (55) as follows:

$$
\left\| f\left(\cdot\right)-\sum_{j=0}^{\overline{m}}\frac{f^{\left(j\right)}\left(x_{0}\right)\left(\cdot-x_{0}\right)^{j}}{j!}\right\|_{\gamma}\leq
$$

$$
\omega_1\left(f^{(\overline{m})},h\right)\left(\frac{\|\cdot - x_0\|_p^{\overline{m}+1}}{(\overline{m}+1)!h} + \frac{\|\cdot - x_0\|_p^{\overline{m}}}{2\overline{m}!} + \frac{h\|\cdot - x_0\|_p^{\overline{m}-1}}{8(\overline{m}-1)!}\right), \ \forall \ x_0 \in M, \ (56)
$$

a pointwise functional inequality on M.

Here $(-x_0)^j$ maps M into $(\mathbb{R}^N)^j$ and it is continuous, also $f^{(j)}(x_0)$ maps $(\mathbb{R}^N)^j$ into X and it is continuous. Hence their composition $f^{(j)}(x_0)(-x_0)^j$ is continuous from M into X.

Clearly $f(\cdot) - \sum_{j=0}^{\overline{m}} \frac{f^{(j)}(x_0)(-x_0)^j}{j!}$ $\frac{f^{(1)}(x-1)^j}{j!} \in C(M, X)$, hence $||f(\cdot) - \sum_{j=0}^{\overline{m}} \frac{f^{(j)}(x_0)(-x_0)^j}{j!}$ j! $\big\|_{\gamma} \in$ $C(M).$

Let $\left\{ \widetilde{L}_{N}\right\}$ $N\in\mathbb{N}$ be a sequence of positive linear operators mapping $C(M)$ into $C(M)$.

Therefore we obtain

3. COMPUTATIONAL ANALYSIS AND APPLICATIONS. VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\nWe have found that
\n
$$
\left\|f(x) - \sum_{j=0}^{m} \frac{f^{(j)}(x_0)(x - x_0)^j}{j!} \right\|_{\gamma} \le
$$
\n
$$
\omega_1 \left(f^{(m)}, h\right) \left(\frac{\|x - x_0\|_p^{\frac{m+1}{m}} + \|\cdot x - x_0\|_p^{\frac{m}{m}} + h\|\cdot x - x_0\|_p^{\frac{m-1}{m}}\right)}{8(m-1)!} < \infty, \quad (55)
$$
\n
$$
\forall x, x_0 \in M.
$$
\nHere $0 < \omega_1 \left(f^{(m)}, h\right) < \infty, by M being compact and $f^{(m)}$ being continuous on M.$
\nOne can rewrite (55) as follows:
\n
$$
\left\|f\left(\cdot\right) - \sum_{j=0}^{m} \frac{f^{(j)}(x_0)(\cdot - x_0)^j}{j!} \right\|_{\gamma}
$$
\n
$$
\omega_2 \left(f^{(m)}, h\right) \left(\frac{\|-\overline{x}_0\|_p^{\frac{m+1}{m}} + \|\cdot - x_0\|_p^{\frac{m}{m}} + h\|\cdot - x_0\|_p^{\frac{m}{m}-1}}{8(m-1)!}\right), \forall x_0 \in M, \quad (56)
$$
\na pointwise function at inequality on M.
\nHere $(-x_0)^j$ maps M into $(\mathbb{R}^N)^j$ and it is continuous, also $f^{(j)}(x_0)$ maps
\n
$$
(\mathbb{R}^N)^j
$$
 and M is continuous. Hence their composition $f^{(j)}(x_0)(\cdot - x_0)^j$
\nis continuous from M into K.
\nClearly $f\left(\cdot\right) - \sum_{j=0}^{m} \frac{f^{(j)}(x_0)\left(\frac{1}{j} - x_0\right)^j}{j!} \in C(M, X)$, hence $\left\|f\left(\cdot\right) - \sum_{j=0}^{m} \frac{f^{(j)}(x_0)\left(\frac{1}{j} - x_0\right)^j}{j!} \right\|_{\gamma}$ <

 $\forall N \in \mathbb{N}, \forall x_0 \in M.$

Clearly (57) is valid when $M = \prod_{i=1}^{N}$ $\prod_{i=1} [a_i, b_i]$ and $L_n = A_n$, see (26).

All the above is preparation for the following theorem, where we assume Fréchet differentiability of functions.

This will be a direct application of Theorem 10.2, [11], pp. 268-270. The operators A_n , A_n fulfill its assumptions, see (25), (26), (28), (29) and (30).

We present the following high order approximation results.

Theorem 10 Let O open subset of $(\mathbb{R}^N, \|\cdot\|_p)$, $p \in [1, \infty]$, such that $\prod_{i=1}^N$ $\prod_{i=1} [a_i, b_i] \subset$ $O\subseteq\mathbb{R}^N$, and let $\Big(X,\left\|\cdot\right\|_{\gamma}\Big)$ be a general Banach space. Let $\overline{m}\in\mathbb{N}$ and $f\in$ $C^{\overline{m}}(O, X)$, the space of \overline{m} -times continuously Fréchet differentiable functions from O into X. We study the approximation of $f|_{\prod_{i=1}^N [a_i,b_i]}$. Let $x_0 \in \left(\prod_{i=1}^N a_i\right)$ $\prod_{i=1} [a_i, b_i]$ \setminus and $r > 0$. Then

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\nClearly (57) is valid when
$$
M = \prod_{i=1}^{N} [a_i, b_i]
$$
 and $\tilde{L}_n = \tilde{A}_n$, see (26).
\nAll the above is preparation for the following theorem, where we assume
\nFried the differentiation of Theorem 10.2, [11], pp. 268-270. The
\noperators A_n , \tilde{A}_n fulfill its assumptions, see (25), (26), (28), (29) and (30).
\nWe present the following high order approximation results.
\n**Theorem 10** Let O open subset of $\left(\mathbb{R}^N, ||\cdot||_p\right)$, $p \in [1, \infty]$, such that $\prod_{i=1}^{N} [a_i, b_i] \subset$
\n $O \subseteq \mathbb{R}^N$, *in the set of vertices continuous* $P(\mathbb{R}^N, ||\cdot||_p)$, $p \in [1, \infty]$, such that $\prod_{i=1}^{N} [a_i, b_i] \subset$
\n $O \subseteq \mathbb{R}^N$, *in the set of vertices continuous* $P(\mathbb{R}^N, ||\cdot||_p)$, $p \in [1, \infty]$, such that $\prod_{i=1}^{N} [a_i, b_i] \subset$
\n $O \subseteq \mathbb{R}^N$, *in the set of vertices continuous* $P(\mathbb{R}^N, ||\cdot||_p)$, $p \in [1, \infty]$, $|\mathbb{R}^N \in \mathbb{R}^N \in \mathbb{R}^N$
\n $P(\mathbb{R}^N, \mathbb{R}^N) = \mathbb{R}^N$
\n $P(\mathbb{R}^N, \mathbb{R}^N) = \mathbb{R}^N$
\n $\left|\left(A_n(f)\right)(x_0) - \sum_{j=0}^{m} \frac{1}{j!} \left(A_n\left(f^{(j)}(x_0) \left(-x_0\right)^j\right)\right)(x_0)\right)\right|_{\gamma} \leq$
\n $\frac{\omega_1\left(f^{(m)}, r\left(\$

2) additionally if $f^{(j)}(x_0) = 0, j = 1, ..., \overline{m}$, we have

$$
\| (A_n(f))(x_0) - f(x_0) \|_{\gamma} \le
$$

$$
\frac{\omega_1 \left(f^{(\overline{m})}, r \left(\left(\tilde{A}_n \left(\left\| \cdot - x_0 \right\|_p^{\overline{m}+1} \right) \right) (x_0) \right)^{\frac{1}{\overline{m}+1}} \right)}{r \overline{m}!} \left(\left(\tilde{A}_n \left(\left\| \cdot - x_0 \right\|_p^{\overline{m}+1} \right) \right) (x_0) \right)^{\left(\frac{\overline{m}}{\overline{m}+1}\right)}
$$

$$
\left[\frac{1}{(\overline{m}+1)} + \frac{r}{2} + \frac{\overline{m}r^2}{8} \right],
$$

3)

$$
\left\| \left(A_n\left(f\right)\right)(x_0) - f\left(x_0\right) \right\|_{\gamma} \le \sum_{j=1}^{\overline{m}} \frac{1}{j!} \left\| \left(A_n\left(f^{(j)}\left(x_0\right)(\cdot - x_0)^j\right)\right)(x_0) \right\|_{\gamma} + \frac{\omega_1\left(f^{(\overline{m})}, r\left(\left(\tilde{A}_n\left(\left\|\cdot - x_0\right\|_p^{\overline{m}+1}\right)\right)(x_0)\right)^{\frac{1}{\overline{m}+1}}\right)}{r\overline{m}!} \left(\left(\tilde{A}_n\left(\left\|\cdot - x_0\right\|_p^{\overline{m}+1}\right)\right)(x_0)\right)^{\left(\frac{\overline{m}}{\overline{m}+1}\right)} (60)
$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, Vol. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\nand
\n
$$
\left\{\frac{1}{(m+1)} + \frac{r}{2} + \frac{\overline{m}r^2}{8}\right\},
$$
\nand
\n
$$
\left\{\left\|\left\|A_n\left(f\right) - f\right\|_{\infty} \sum_{i=1}^n \mu_{(s_i, b_i)} \right\|^2 \le \sum_{j=1}^{\overline{m}} \frac{1}{j!} \left\|\left\|\left(A_n\left(f^{(j)}(x_0) \left(1 - x_0\right)^j\right)\right)(x_0)\right\|_{\infty} \right\|_{\infty, x_0 \in \bigcup_{i=1}^N [a_i, b_i]} + \frac{\omega_1\left(f^{(\overline{m})}, r\right) \left|\left(\overline{A}_n\left(\left\|1 - x_0\right\|_p^{\overline{m}+1}\right)\right)(x_0)\right| \Big\|_{\infty, x_0 \in \bigcup_{i=1}^N [a_i, b_i]}^{\frac{1}{m+1}}}{\left\|\left(\overline{A}_n\left(\left\|1 - x_0\right\|_p^{\overline{m}}\right)\right)(x_0)\right)\Big\|_{\infty, x_0 \in \bigcup_{i=1}^N [a_i, b_i]}^{\frac{1}{m+1}}}
$$
\nWe need
\nLemma 11. The function
\n
$$
\left[\frac{1}{(\overline{m+1})} + \frac{r}{2} + \frac{\overline{m}r^2}{8}\right].
$$
\nWe need
\n**Lemma 11. The function** $\left(\overline{A}_n\left(1 - x_0\right|_p^{\overline{m}}\right)(x_0) \text{ is continuous in } x_0 \in \left(\bigcup_{i=1}^N [a_i, b_i]\right),$
\n $m \in \mathbb{N}.$
\nProof. By Lemma 10.3, [11], p. 272.
\nWe give
\nCorollary 12. (to Theorem
\n1) $\left\|\left(A_n\left(f\right)(x_0) \left(-x_0\right)\right)\right\|^2$ $\left\|\left(A_n\left(f^{(1)}(x_0) \left(-x_0\right)\right)\right)($

We need

4)

Lemma 11 The function $\left(\widetilde{A}_n\left(\|\cdot-x_0\|_p^{\overline{m}}\right)\right)(x_0)$ is continuous in $x_0 \in \left(\prod_{i=1}^N x_i\right)$ $\prod_{i=1} [a_i, b_i]$ $\overline{ }$, $\overline{m}\in \mathbb{N}.$

Proof. By Lemma 10.3, [11], p. 272. We give

Corollary 12 (to Theorem 10, case of $\overline{m} = 1$) Then 1)

$$
\left\| \left(A_n\left(f\right)\right)(x_0) - f\left(x_0\right) \right\|_{\gamma} \le \left\| \left(A_n\left(f^{(1)}\left(x_0\right)(\cdot - x_0)\right)\right)(x_0) \right\|_{\gamma} +
$$

$$
\frac{1}{2r}\omega_1\left(f^{(1)}, r\left(\left(\tilde{A}_n\left(\left\|\cdot - x_0\right\|_p^2\right)\right)(x_0)\right)^{\frac{1}{2}}\right) \left(\left(\tilde{A}_n\left(\left\|\cdot - x_0\right\|_p^2\right)\right)(x_0)\right)^{\frac{1}{2}} \quad (62)
$$

$$
\left[1 + r + \frac{r^2}{4}\right],
$$

and 2)

$$
\left\| \left\| \left(A_n\left(f \right) \right) - f \right\|_\gamma \right\|_{\infty, \prod\limits_{i=1}^N [a_i,b_i]} \le
$$

$$
\left\| \left\| \left(A_n \left(f^{(1)} \left(x_0 \right) \left(\cdot - x_0 \right) \right) \right) (x_0) \right\|_{\gamma} \right\|_{\infty, x_0 \in \prod\limits_{i=1}^N [a_i, b_i]} +
$$

$$
\frac{1}{2r} \omega_1 \left(f^{(1)}, r \left\| \left(\widetilde{A}_n \left(\left\| \cdot - x_0 \right\|_p^2 \right) \right) (x_0) \right\|_{\infty, x_0 \in \prod\limits_{i=1}^N [a_i, b_i]}^{\frac{1}{2}} \right)
$$

$$
\left\| \left(\widetilde{A}_n \left(\left\| \cdot - x_0 \right\|_p^2 \right) \right) (x_0) \right\|_{\infty, x_0 \in \prod\limits_{i=1}^N [a_i, b_i]}^{\frac{1}{2}} \left[1 + r + \frac{r^2}{4} \right],
$$
 (63)

 $r > 0.$

We make

Remark 13 We estimate $0 < \alpha < 1$, $m, \overline{m}, n \in \mathbb{N} : n^{1-\alpha} > 2$,

3. COMPUTATIONAL ANALYSIS AND APPLICATIONS. VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\n
$$
\left\| \left\| \left(A_n \left(f^{(1)}(x_0) \langle \cdot - x_0 \rangle \right) \right) (x_0) \right\|_{\infty}^{\frac{1}{2}} \right\|_{\infty, \omega_0 \in \prod_{i=1}^{N} [\alpha_i, \beta_i]}
$$
\n
$$
\left\| \left(\tilde{A}_n \left(\left\| \cdot - x_0 \right\|_{\rho}^2 \right) (x_0) \right\|_{\infty, \omega_0 \in \prod_{i=1}^{N} [\alpha_i, \beta_i]}^{\frac{1}{2}} \right\|_{\infty, \beta_0}
$$
\n7 > 0.
\nWe make
\nRemark 13. We estimate $0 < \alpha < 1$, $m, \overline{m}, n \in \mathbb{N}$: $n^{1-\alpha} > 2$,
\n
$$
\tilde{A}_n \left(\left\| \cdot - x_0 \right\|_{\infty}^{\frac{1}{2}} \right) (x_0) = \frac{\sum_{k=1}^{|k|} [\alpha_k]}{\sum_{k=1}^{|k|} [\alpha_k]} \frac{1}{2} (x_0 - k) \frac{(x_0)}{2}
$$
\n
$$
\left[2 \left(\sqrt[2]{1 + 4^m} \right) \right]^{N} \sum_{k=1}^{|k|} \left\| \frac{k}{n} - x_0 \right\|_{\infty}^{\frac{1}{2}} \frac{1}{2} (x_0 - k) = (64)
$$
\n
$$
\left[2 \left(\sqrt[2]{1 + 4^m} \right) \right]^{N} \left\{ \sum_{k=1}^{|k|} \left\| \frac{k}{n} - x_0 \right\|_{\infty}^{\frac{1}{2}} \frac{1}{2} (x_0 - k) = (64)
$$
\n
$$
\left\{ \left\| \frac{2}{n} \left\| \frac{x_0}{n} - x_0 \right\|_{\infty}^{\frac{1}{2}} \leq (nx_0 - k) + \left\{ \left\| \frac{k}{n} - x_0 \right\|_{\infty}^{\frac{1}{2}} \leq (nx_0 - k) + \left\{ \
$$

(where $b - a = (b_1 - a_1, ..., b_N - a_N)$). We have proved that $(\forall x_0 \in \prod_{i=1}^N$ $\prod_{i=1} [a_i, b_i]$

$$
\widetilde{A}_n \left(\left\| \cdot - x_0 \right\|_{\infty}^{\overline{m} + 1} \right) (x_0) < \left[2 \left(\sqrt[2m]{1 + 4^m} \right) \right]^N \left\{ \frac{1}{n^{\alpha(\overline{m} + 1)}} + \frac{\left\| b - a \right\|_{\infty}^{\overline{m} + 1}}{4m \left(n^{1 - \alpha} - 2 \right)^{2m}} \right\} =: \varphi_1(n) \tag{66}
$$

 $(0 < \alpha < 1, m, \overline{m}, n \in \mathbb{N} : n^{1-\alpha} > 2).$ And, consequently it holds

$$
\left\|\widetilde{A}_{n}\left(\left\|\cdot-x_{0}\right\|_{\infty}^{\overline{m}+1}\right)(x_{0})\right\|_{\infty,x_{0}\in\prod_{i=1}^{N}[a_{i},b_{i}]}\left[2\left(\sqrt[2^{n}]{1+4^{m}}\right)\right]^{N}\left\{\frac{1}{n^{\alpha(\overline{m}+1)}}+\frac{\left\|b-a\right\|_{\infty}^{\overline{m}+1}}{4m\left(n^{1-\alpha}-2\right)^{2m}}\right\}=\varphi_{1}(n)\to 0, \quad as\ n\to+\infty.
$$
\n(67)

So, we have that $\varphi_1(n) \to 0$, as $n \to +\infty$. Thus, when $p \in [1,\infty]$, from Theorem 10 we have the convergence to zero in the right hand sides of parts (1) , (2).

Next we estimate
$$
\left\| \left(\widetilde{A}_n \left(f^{(j)}(x_0) \left(\cdot - x_0 \right)^j \right) \right) (x_0) \right\|_{\gamma}
$$
.
We have that

$$
\left(\widetilde{A}_{n}\left(f^{(j)}\left(x_{0}\right)\left(\cdot-x_{0}\right)^{j}\right)\right)\left(x_{0}\right)=\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}f^{(j)}\left(x_{0}\right)\left(\frac{k}{n}-x_{0}\right)^{j}Z\left(nx_{0}-k\right)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}Z\left(nx_{0}-k\right)}.
$$
\n(68)

When $p = \infty$, $j = 1, ..., \overline{m}$, we obtain

$$
\left\| f^{(j)}\left(x_0\right) \left(\frac{k}{n} - x_0\right)^j \right\|_{\gamma} \le \left\| f^{(j)}\left(x_0\right) \right\| \left\| \frac{k}{n} - x_0 \right\|_{\infty}^j. \tag{69}
$$

We further have that

3. COMPUTATIONAL ANALYSIS AND APPLICATIONS, Vol. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\n
$$
\begin{aligned}\n\left(0 < \alpha < 1, m, \overline{m}, n \in \mathbb{N}: n^{1-\alpha} > 2.\right. \\
\left. \left. \left. \frac{1}{4} \left(1 - x_0 \right) \right|_{\infty}^{\infty} \right) \left. \left(1 - \frac{1}{4} \right) \left(1 - \frac{1}{4} \right)
$$

3. COMPUTATIONAL AVALYSIS AND APPLICATIONS. VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\n
$$
\left\{\begin{array}{l} \left. k = \lceil na \rceil \right. \\ \left. k = \lceil na \rceil \right. \end{array} \right\} \left\{\begin{array}{l} \left. \left. \frac{1}{n} - x_0 \right| \right|_{\infty} > \frac{1}{n^2} \\ \left. \left. \frac{1}{n} - x_0 \right| \right|_{\infty} > \frac{1}{n^2} \end{array} \right\} \left\{\begin{array}{l} \frac{2l}{n^2} < \\ \left. \frac{2}{n} \right. \\ \left. \frac{2}{n} \left(2 \sqrt{1 + 4^m} \right) \right]^N \left| f^{(j)}(x_0) \right| \left\{ \frac{1}{n^{\alpha j}} + \frac{\lVert b - a \rVert_{\infty}^j}{4m(n^{1 - \alpha} - 2)^{2m}} \right\} \right. \rightarrow 0, \text{ as } n \rightarrow \infty. \right\}
$$
\nThat is
\n
$$
\left\| \left(\tilde{A}_n \left(f^{(j)}(x_0) \left(\cdot - x_0 \right)^j \right) \left(x_0 \right) \right\|_{\infty} \rightarrow 0, \text{ as } n \rightarrow \infty. \right\}
$$
\nTherefore when $p = \infty$, for $j = 1, ..., \pi$, we have prnerat:
\n
$$
\left\| \left(\tilde{A}_n \left(f^{(j)}(x_0) \left(\cdot - x_0 \right)^j \right) \right) (x_0) \right\|_{\infty} \left\{ \left. \left. \left. \frac{1}{n^{\alpha j}} + \frac{\lVert b - a \rVert_{\infty}^j}{4m(n^{1 - \alpha} - 2)^{2m}} \right. \right\} \right\} \leq \qquad (72)
$$
\n[2 (2°√1 + 4^m)]^N ||f^{(j)}(x_0) || \left\{ \frac{1}{n^{\alpha j}} + \frac{\lVert b - a \rVert_{\infty}^j}{4m(n^{1 - \alpha} - 2)^{2m}} \right\} =: \varphi_{2j}(n

 $\overline{ }$

That is

 $\sqrt{2}$

$$
\left\| \left(\widetilde{A}_n \left(f^{(j)} \left(x_0 \right) \left(\cdot - x_0 \right)^j \right) \right) (x_0) \right\|_{\gamma} \to 0, \text{ as } n \to \infty.
$$

Therefore when $p = \infty$, for $j = 1, ..., \overline{m}$, we have proved:

$$
\left\| \left(\widetilde{A}_n \left(f^{(j)} \left(x_0 \right) (\cdot - x_0)^j \right) \right) (x_0) \right\|_{\gamma} <
$$

$$
\left[2 \left(\sqrt[2m]{1 + 4^m} \right) \right]^N \left\| f^{(j)} \left(x_0 \right) \right\| \left\{ \frac{1}{n^{\alpha j}} + \frac{\|b - a\|_{\infty}^j}{4m \left(n^{1 - \alpha} - 2 \right)^{2m}} \right\} \le \q (72)
$$

$$
\left[2 \left(\sqrt[2m]{1 + 4^m} \right) \right]^N \left\| f^{(j)} \right\|_{\infty} \left\{ \frac{1}{n^{\alpha j}} + \frac{\|b - a\|_{\infty}^j}{4m \left(n^{1 - \alpha} - 2 \right)^{2m}} \right\} =: \varphi_{2j} \left(n \right) < \infty,
$$

and converges to zero, as $n \to \infty$.

We conclude:

In Theorem 10, the right hand sides of (60) and (61) converge to zero as $n \to \infty$, for any $p \in [1,\infty]$.

Also in Corollary 12, the right hand sides of (62) and (63) converge to zero as $n \to \infty$, for any $p \in [1,\infty]$.

Conclusion 14 We have proved that the left hand sides of (58) , (59) , (60) , (61) and (62), (63) converge to zero as $n \to \infty$, for $p \in [1,\infty]$. Consequently $A_n \to I$ (unit operator) pointwise and uniformly, as $n \to \infty$, where $p \in [1,\infty]$. In the presence of initial conditions we achieve a higher speed of convergence, see (59). Higher speed of convergence happens also to the left hand side of (58).

We further give

Corollary 15 (to Theorem 10) Let O open subset of $(\mathbb{R}^N, \|\cdot\|_{\infty})$, such that \prod $\prod_{i=1}^N [a_i, b_i] \subset O \subseteq \mathbb{R}^N$, and let $(X, \|\cdot\|_{\gamma})$ be a general Banach space. Let $\overline{m} \in \mathbb{N}$ and $f \in C^{\overline{m}}(O,X)$, the space of \overline{m} -times continuously Fréchet differentiable functions from O into X. We study the approximation of $f|_{\prod_{i=1}^N [a_i, b_i]}$. Let $x_0 \in$

3. COMPUTATIONAL ANALYSIS AND APPLICATIONS, Vol. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\n
$$
\left(\prod_{i=1}^{N} [a_i, b_i]\right) and r > 0. Here \varphi_1(n) as in (66) and \varphi_{2j}(n) as in (72), where
$$
\n
$$
n \in \mathbb{N}: n^{1-\alpha} > 2, 0 < \alpha < 1, j = 1, ..., \overline{m}. Then
$$
\n
$$
\left|\left(A_n (f) \right)(x_0) - \sum_{j=0}^{\infty} \frac{1}{j!} \left(A_n (f^{(j)} (x_0) \left(\cdot - x_0 \right)^j \right) \right) (x_0) \right| \le
$$
\n
$$
\frac{\omega_1 \left(f^{(\overline{m})}, r(\varphi_1(n)) \right)^{\frac{1}{m-1}}}{r^{\overline{m}!}} \left(\varphi_1(n) \right)^{\left(\frac{m}{m+1}\right)} \left[\frac{1}{(\overline{m}+1)} + \frac{r}{2} + \frac{\overline{m}r^2}{8} \right],
$$
\n(73)
\n2) additionally, if $f^{(j)} (x_0) = 0, j = 1, ..., \overline{m}, w$ have
\n
$$
\left| (A_n (f)) (x_0) - f(x_0) \right| \le
$$
\n
$$
\frac{\omega_1 \left(f^{(\overline{m})}, r(\varphi_1(n)) \right)^{\frac{1}{m-1}}}{r^{\overline{m}!}} \left(\varphi_1(n) \right)^{\left(\frac{m}{m+1}\right)} \left[\frac{1}{(\overline{m}+1)} + \frac{r}{2} + \frac{\overline{m}r^2}{8} \right],
$$
\n(74)
\n3)
\n
$$
\left| \left| A_n (f) - f \right| \right|_{\infty, \prod_{j=1}^{\infty} [a_j, b_j]} \leq \sum_{j=1}^{\infty} \frac{\varphi_{2j}(n)}{j!} +
$$
\n
$$
\frac{\omega_1 \left(f^{(\overline{m})}, r(\varphi_1(n)) \right)^{\frac{1}{m-1}}}{r^{\overline{m}!}} \left(\varphi_1(n) \right)^{\left(\frac{
$$

2) additionally, if $f^{(j)}(x_0) = 0$, $j = 1, ..., \overline{m}$, we have

$$
||(A_n(f))(x_0) - f(x_0)||_{\gamma} \le
$$

$$
\frac{\omega_1\left(f^{(\overline{m})}, r\left(\varphi_1\left(n\right)\right)^{\frac{1}{\overline{m}+1}}\right)}{r\overline{m}!}\left(\varphi_1\left(n\right)\right)^{\left(\frac{\overline{m}}{\overline{m}+1}\right)}\left[\frac{1}{\left(\overline{m}+1\right)}+\frac{r}{2}+\frac{\overline{m}r^2}{8}\right],\tag{74}
$$

3)

$$
\left\| |A_n(f) - f||_{\gamma} \right\|_{\infty, \prod_{i=1}^N [a_i, b_i]} \le \sum_{j=1}^m \frac{\varphi_{2j}(n)}{j!} + \frac{\omega_1 \left(f^{\left(\overline{m}\right)}, r \left(\varphi_1(n)\right)^{\frac{1}{m+1}} \right)}{r \overline{m}!} \left(\varphi_1(n)\right)^{\left(\frac{\overline{m}}{m+1}\right)} \tag{75}
$$
\n
$$
\left[\frac{1}{\left(\overline{m} + 1\right)} + \frac{r}{2} + \frac{\overline{m}r^2}{8} \right] =: \varphi_3(n) \to 0, \text{ as } n \to \infty.
$$

 \equiv

We continue with

Theorem 16 Let $f \in C_B(\mathbb{R}^N, X)$, $0 < \beta < 1$, $x \in \mathbb{R}^N$, $m, N, n \in \mathbb{N}$ with $n^{1-\beta} > 2$, ω_1 is for $p = \infty$. Then 1)

$$
\|B_n(f, x) - f(x)\|_{\gamma} \le \omega_1 \left(f, \frac{1}{n^{\beta}}\right) + \frac{\left\|f\|_{\gamma}\right\|_{\infty}}{2m\left(n^{1-\beta} - 2\right)^{2m}} =: \lambda_2(n),\tag{76}
$$

(2)

$$
\left\| \left\| B_n \left(f \right) - f \right\|_{\gamma} \right\|_{\infty} \leq \lambda_2 \left(n \right). \tag{77}
$$

Given that $f \in (C_U(\mathbb{R}^N, X) \cap C_B(\mathbb{R}^N, X)),$ we obtain $\lim_{n \to \infty} B_n(f) = f$, uniformly.

Proof. We have that

$$
B_n(f, x) - f(x) \stackrel{(16)}{=} \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) Z(nx - k) - f(x) \sum_{k=-\infty}^{\infty} Z(nx - k) = (78)
$$

$$
\sum_{k=-\infty}^{\infty} \left(f\left(\frac{k}{n}\right) - f(x) \right) Z(nx - k).
$$

Hence

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, Vol. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\nProof. We have that
\n
$$
B_n(f, x) - f(x) \stackrel{(16)}{=} \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) Z(nx-k) - f(x) \sum_{k=-\infty}^{\infty} Z(nx-k) = (78)
$$
\n
$$
\sum_{k=-\infty}^{\infty} \left(f\left(\frac{k}{n}\right) - f(x) \right) Z(nx-k).
$$
\nHence
\n
$$
||B_n(f, x) - f(x)||_{\gamma} \le \sum_{k=-\infty}^{\infty} \left||f\left(\frac{k}{n}\right) - f(x)\right||_{\gamma} Z(nx-k) +
$$
\n
$$
\left\{ \left|\frac{k}{n} - x\right|_{\infty} \le \frac{1}{n^2}
$$
\n
$$
\left\| f\left(\frac{k}{n}\right) - f(x) \right\|_{\gamma} Z(nx-k) \stackrel{(16)}{\leq}
$$
\n
$$
\left\{ \left|\frac{k}{n} - x\right|_{\infty} \le \frac{1}{n^2}
$$
\n
$$
\omega_1 \left(f, \frac{1}{n^2}\right) + 2 \left\| |f| \right\|_{\gamma} \right\|_{\infty} \sum_{k=-\infty}^{\infty} Z(nx-k) \stackrel{(26)}{\leq}
$$
\n
$$
\left\{ \left|\frac{k}{n} - x\right|_{\infty} > \frac{1}{n^3}
$$
\n
$$
\omega_1 \left(f, \frac{1}{n^2}\right) + 2 \left\| |f| \right\|_{\gamma} \right\|_{\infty} \sum_{k=-\infty}^{\infty} Z(nx-k) \stackrel{(36)}{\leq}
$$
\n
$$
\omega_1 \left(f, \frac{1}{n^2}\right) + 2 \left\| |f| \right\|_{\infty} \sum_{k=-\infty}^{\infty} Z(nx-k) \stackrel{(36)}{\leq}
$$
\n
$$
\omega_1 \left(f, \frac{1}{n^2}\right) + 2 \left\| |f| \right\|_{\infty} \sum_{k=-\infty}^{\infty} Z(nx-k) \stackrel{(36)}{\leq}
$$
\n
$$
\omega_1 \left(f
$$

proving the claim. We give

Theorem 17 Let $f \in C_B(\mathbb{R}^N, X)$, $0 < \beta < 1$, $x \in \mathbb{R}^N$, $m, N, n \in \mathbb{N}$ with $n^{1-\beta} > 2$, ω_1 is for $p = \infty$. Then 1)

$$
\|C_n(f,x) - f(x)\|_{\gamma} \le \omega_1 \left(f, \frac{1}{n} + \frac{1}{n^{\beta}}\right) + \frac{\left\|f\|_{\gamma}\right\|_{\infty}}{2m\left(n^{1-\beta} - 2\right)^{2m}} =: \lambda_3(n), \quad (80)
$$

2)

$$
\left\| \left\| C_n \left(f \right) - f \right\|_{\gamma} \right\|_{\infty} \leq \lambda_3 \left(n \right). \tag{81}
$$

Given that $f \in (C_U(\mathbb{R}^N, X) \cap C_B(\mathbb{R}^N, X))$, we obtain $\lim_{n \to \infty} C_n(f) = f$, uniformly.

Proof. We notice that

$$
\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t) dt = \int_{\frac{k_1}{n}}^{\frac{k_1+1}{n}} \int_{\frac{k_2}{n}}^{\frac{k_2+1}{n}} \dots \int_{\frac{k_N}{n}}^{\frac{k_N+1}{n}} f(t_1, t_2, ..., t_N) dt_1 dt_2...dt_N =
$$

$$
\int_0^{\frac{1}{n}} \int_0^{\frac{1}{n}} \dots \int_0^{\frac{1}{n}} f\left(t_1 + \frac{k_1}{n}, t_2 + \frac{k_2}{n}, ..., t_N + \frac{k_N}{n}\right) dt_1...dt_N = \int_0^{\frac{1}{n}} f\left(t + \frac{k}{n}\right) dt.
$$
(82)

Thus it holds (by (38))

$$
C_n(f,x) = \sum_{k=-\infty}^{\infty} \left(n^N \int_0^{\frac{1}{n}} f\left(t + \frac{k}{n} \right) dt \right) Z\left(nx - k \right). \tag{83}
$$

We observe that

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUB PRESS, LLC
\nProof. We notice that
\n
$$
\int_{\frac{1}{6}}^{\frac{1}{6}+\frac{1}{2}} f(t) dt = \int_{\frac{1}{4}}^{\frac{1}{6}+\frac{1}{2}} \int_{\frac{1}{2}}^{\frac{1}{6}+\frac{1}{2}} ... \int_{\frac{1}{2}}^{\frac{1}{6}+\frac{1}{2}} f(t_1, t_2, ..., t_N) dt_1 dt_2...dt_N =
$$
\n
$$
\int_{0}^{\frac{1}{6}} \int_{0}^{\frac{1}{6}} ... \int_{0}^{\frac{1}{6}} f\left(t_1 + \frac{k_1}{n}, t_2 + \frac{k_2}{n}, ..., t_N + \frac{k_N}{n}\right) dt_1...dx_N = \int_{0}^{\frac{1}{6}} f\left(t + \frac{k}{n}\right) dt.
$$
\nThus it holds (by (38))
\n
$$
C_n \left(f, x\right) = \sum_{k=-\infty}^{\infty} \left(n^N \int_{0}^{\frac{1}{6}} f\left(t + \frac{k}{n}\right) dt \right) Z \left(nx - k \right).
$$
\n(83)
\nWe observe that
\n
$$
\left\| C_n \left(f, x \right) - f(x) \right\|_{\gamma} =
$$
\n
$$
\left\| \sum_{k=-\infty}^{\infty} \left(n^N \int_{0}^{\frac{1}{6}} f\left(t + \frac{k}{n}\right) dt \right) - f(x) \right) Z \left(nx - k \right) \right\|_{\gamma} =
$$
\n
$$
\left\| \sum_{k=-\infty}^{\infty} \left(n^N \int_{0}^{\frac{1}{6}} \left| f\left(t + \frac{k}{n}\right) - f(x) \right|_{\gamma} dt \right) Z \left(nx - k \right) \right\|_{\gamma} =
$$
\n
$$
\sum_{k=-\infty}^{\infty} \left(n^N \int_{0}^{\frac{1}{6}} \left| f\left(t + \frac{k}{n}\right) - f(x) \right|_{\gamma} dt \right) Z \left(nx - k \right) =
$$
\n
$$
\sum_{k=-\infty}^{\infty} \left(n^N \int_{0}^{\
$$

$$
2\left\| \|f\|_{\gamma} \right\|_{\infty} \left(\sum_{\begin{subarray}{c}k=-\infty\\ \left\{\left\|\frac{k}{n}-x\right\|_{\infty} > \frac{1}{n^{\beta}}\end{subarray}\right\}}^{\infty} Z\left(|nx-k|\right) \right) \leq
$$

$$
\omega_{1}\left(f, \frac{1}{n} + \frac{1}{n^{\beta}}\right) + \frac{\left\| \|f\|_{\gamma} \right\|_{\infty}}{2m\left(n^{1-\beta}-2\right)^{2m}},
$$
(85)

proving the claim. \blacksquare

We also present

Theorem 18 Let $f \in C_B(\mathbb{R}^N, X)$, $0 < \beta < 1$, $x \in \mathbb{R}^N$, $m, N, n \in \mathbb{N}$ with $n^{1-\beta} > 2$, ω_1 is for $p = \infty$. Then 1)

$$
\|D_n(f, x) - f(x)\|_{\gamma} \le \omega_1 \left(f, \frac{1}{n} + \frac{1}{n^{\beta}}\right) + \frac{\left\|f\|_{\gamma}\right\|_{\infty}}{2m\left(n^{1-\beta} - 2\right)^{2m}} = \lambda_4(n), \quad (86)
$$

2)

$$
\left\| \left\| D_n\left(f\right) - f \right\|_{\gamma} \right\|_{\infty} \leq \lambda_4(n). \tag{87}
$$

Given that $f \in (C_U(\mathbb{R}^N, X) \cap C_B(\mathbb{R}^N, X))$, we obtain $\lim_{n \to \infty} D_n(f) = f$, uniformly.

Proof. We have that $(by (40))$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\n
$$
2 \left\| ||f||_2 \right\|_{\infty} \left(\sum_{\substack{k=-\infty \\ k \equiv -\infty}}^{\infty} Z(|nx - k|) \right) \le
$$
\n
$$
\omega_1 \left(f, \frac{1}{n} + \frac{1}{n^{\beta}} \right) + \frac{||f||_2 ||_{\infty}}{2m(n^{1-\beta} - 2)^{2m}}, \qquad (85)
$$
\nproving the claim. \blacksquare
\nWe also present
\nTheorem 18 Let $f \in C_B(\mathbb{R}^N, X)$, $0 < \beta < 1$, $x \in \mathbb{R}^N$, $m, N, n \in \mathbb{N}$ with $n^{1-\beta} > 2$, ω_1 is for $p = \infty$. Then
\n
$$
y
$$
\n
$$
||D_n(f, x) - f(x)||_2 \le \omega_1 \left(f, \frac{1}{n} + \frac{1}{n^{\beta}} \right) + \frac{||f||_2 ||_{\infty}}{2m(n^{1-\beta} - 2)^{2m}} = \lambda_4(n), \qquad (86)
$$
\n
$$
z)
$$
\n
$$
Girez, that \quad f \in (C_U(\mathbb{R}^N, X) \cap C_D(\mathbb{R}^N, X)), we obtain $\lim_{n \to \infty} D_n(f) = f$,
\nuniformly.
\nProof. We have that $(\text{by } (40))$
\n
$$
||D_n(f, x) - f(x)||_2 = \left\| \sum_{k=-\infty}^{\infty} \delta_{nk}(f) Z(nx - k) - \sum_{k=-\infty}^{\infty} f(x) Z(nx - k) \right\|_2 =
$$
\n
$$
\left\| \sum_{k=-\infty}^{\infty} \left(\sum_{r=0}^{\delta} w_r \left(f \left(\frac{k}{n} + \frac{r}{n\theta} \right) - f(x) \right) \right) Z(nx - k) \right\|_2 =
$$
\n
$$
\sum_{k=-\infty}^{\infty} \left(\sum_{r=0}^{\delta} w_r \
$$
$$

0. COMPUTATIONAL ANALYSIS AND APPLICATIONS. VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\n
$$
\left\{\begin{aligned}\n&\sum_{k=-\infty}^{\infty} \left(\sum_{r=0}^{d} w_r \left\|f\left(\frac{k}{n} + \frac{r}{n\theta}\right) - f(x)\right\|_{\infty}\right) Z(nx - k) \leq \\
&\sum_{k=-\infty}^{\infty} \left(\sum_{r=0}^{d} w_r \omega_1 \left(f, \left\|\frac{k}{n} - x\right\|_{\infty} + \left\|\frac{r}{n\theta}\right\|_{\infty}\right)\right) Z(nx - k) + \\
&\sum_{k=-\infty}^{\infty} \left(\sum_{r=0}^{d} w_r \omega_1 \left(f, \left\|\frac{k}{n} - x\right\|_{\infty} + \left\|\frac{r}{n\theta}\right\|_{\infty}\right)\right) Z(nx - k) + \\
&\sum_{k=-\infty}^{\infty} 2 \left\|f\left|f\right|_{\infty}\left(\sum_{k=-\infty}^{\infty} \frac{z(nx - k)}{\sqrt{n}}\right)\right\| \leq \\
&\sum_{k=-\infty}^{\infty} 4 \left(f, \frac{1}{n} + \frac{1}{n\theta}\right) + \frac{\left\|f\left|\frac{r}{n}\right\|_{\infty}\right\|_{\infty}}{\left\|\left\|\frac{k}{n} - x\right\|_{\infty} \leq \frac{1}{n\pi}}, \n\end{aligned}
$$
\n(89)
\nproving the claim. ■
\nWe make
\nDefinition 19 Let $f \in C_R(\mathbb{R}^N, X)$, $N \in \mathbb{N}$, where $\left(X, ||\cdot||_q\right)$ is a Banach
\nspace. We define the general normal network operator $\left\{f\left|\frac{r}{n}\right|\right\}$ is a Banach
\nspace. We define the general normal network operator $\left\{f\left|\frac{r}{n}\right|\right\}$ is $f(t) dt$, (90)
\nClearly $l_{nk}(f)$ is an X-valued bounded linear function such that $\left\|l_{nk}(f)\right|\right\|_{\infty} \leq$
\nHence $F_n(f)$ is a bounded linear operator with $\left\|[F_n(f)|]$

proving the claim.

We make

Definition 19 Let $f \in C_B(\mathbb{R}^N, X)$, $N \in \mathbb{N}$, where $(X, \left\| \cdot \right\|_{\gamma})$ is a Banach space. We define the general neural network operator

$$
F_n(f, x) := \sum_{k=-\infty}^{\infty} l_{nk}(f) Z(nx - k) =
$$

$$
\begin{cases} B_n(f, x), & \text{if } l_{nk}(f) = f\left(\frac{k}{n}\right), \\ C_n(f, x), & \text{if } l_{nk}(f) = n^N \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t) dt, \\ D_n(f, x), & \text{if } l_{nk}(f) = \delta_{nk}(f). \end{cases}
$$
(90)

Clearly $l_{nk} (f)$ is an X-valued bounded linear functional such that $||l_{nk} (f)||_{\gamma} \le$ $\left\| \left\| f \right\|_{\gamma} \right\|_{\infty}.$

Hence $F_n(f)$ is a bounded linear operator with $\left\| \|F_n(f)\|_{\gamma} \right\|_{\infty} \leq$ $\left\Vert \left\Vert f\right\Vert _{\gamma}\right\Vert _{\infty}$. We need

Theorem 20 Let $f \in C_B(\mathbb{R}^N, X)$, $N \ge 1$. Then $F_n(f) \in C_B(\mathbb{R}^N, X)$.

Proof. Clearly $F_n(f)$ is a bounded function.

Next we prove the continuity of $F_n(f)$. Notice for $N = 1, Z = \Phi$ by (14). We will use the generalized Weierstrass M test: If a sequence of positive constants M_1, M_2, M_3, \ldots , can be found such that in some interval

(a) $||u_n(x)||_{\gamma} \leq M_n, \ \ n = 1, 2, 3, ...$

(b)
$$
\sum M_n
$$
 converges,

then $\sum u_n (x)$ is uniformly and absolutely convergent in the interval. Also we will use:

If $\{u_n(x)\}\$, $n = 1, 2, 3, \dots$ are continuous in [a, b] and if $\sum u_n(x)$ converges uniformly to the sum $S(x)$ in [a, b], then $S(x)$ is continuous in [a, b]. I.e. a uniformly convergent series of continuous functions is a continuous function. First we prove claim for $N = 1$. **LOOSEVINTONAL ANNEYSIS AND APPLICATIONS, VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC

(b)** $\sum_{i=1}^{n} \lambda_i$ on the space of the spa

We will prove that $\sum_{k=-\infty}^{\infty} l_{nk} (f) \Phi (nx - k)$ is continuous in $x \in \mathbb{R}$. There always exists $\lambda \in \mathbb{N}$ such that $nx \in [-\lambda, \lambda]$.

Since $nx \leq \lambda$, then $-nx \geq -\lambda$ and $k - nx \geq k - \lambda \geq 0$, when $k \geq \lambda$. Therefore

$$
\sum_{k=\lambda}^{\infty} \Phi(nx-k) = \sum_{k=\lambda}^{\infty} \Phi(k-nx) \le \sum_{k=\lambda}^{\infty} \Phi(k-\lambda) = \sum_{k'=0}^{\infty} \Phi(k') \le 1.
$$
 (91)

So for $k \geq \lambda$ we get

$$
||l_{nk}(f)||_{\gamma} \Phi (nx - k) \leq ||||f||_{\gamma} ||_{\infty} \Phi (k - \lambda),
$$

and

$$
\left\| \left\| f \right\|_{\gamma} \right\|_{\infty} \sum_{k=\lambda}^{\infty} \Phi(k-\lambda) \leq \left\| \left\| f \right\|_{\gamma} \right\|_{\infty}.
$$

Hence by the generalized Weierstrass M test we obtain that $\sum_{k=\lambda}^{\infty} l_{nk} (f) \Phi (nx - k)$ is uniformly and absolutely convergent on $\left[-\frac{\lambda}{n},\frac{\lambda}{n}\right]$.

Since $l_{nk}(f) \Phi(nx-k)$ is continuous in x, then $\sum_{k=\lambda}^{\infty} l_{nk}(f) \Phi(nx-k)$ is continuous on $\left[-\frac{\lambda}{n}, \frac{\lambda}{n}\right]$.

Because $nx \ge -\lambda$, then $-nx \le \lambda$, and $k - nx \le k + \lambda \le 0$, when $k \le -\lambda$. Therefore

$$
\sum_{k=-\infty}^{-\lambda} \Phi(nx-k) = \sum_{k=-\infty}^{-\lambda} \Phi(k-nx) \le \sum_{k=-\infty}^{-\lambda} \Phi(k+\lambda) = \sum_{k'=-\infty}^{0} \Phi(k') \le 1.
$$

So for $k \leq -\lambda$ we get

$$
\|l_{nk}(f)\|_{\gamma} \Phi(nx-k) \le \| \|f\|_{\gamma} \|_{\infty} \Phi(k+\lambda), \qquad (92)
$$

and

$$
\left\| \left\| f \right\|_{\gamma} \right\|_{\infty} \sum_{k=-\infty}^{-\lambda} \Phi\left(k + \lambda \right) \le \left\| \left\| f \right\|_{\gamma} \right\|_{\infty}.
$$

Hence by Weierstrass M test we obtain that $\sum_{k=-\infty}^{-\lambda} l_{nk} (f) \Phi (nx - k)$ is uniformly and absolutely convergent on $\left[-\frac{\lambda}{n},\frac{\lambda}{n}\right]$.

Since $l_{nk}(f) \Phi(nx-k)$ is continuous in x, then $\sum_{k=-\infty}^{-\lambda} l_{nk}(f) \Phi(nx-k)$ is continuous on $\left[-\frac{\lambda}{n}, \frac{\lambda}{n}\right]$.

So we proved that $\sum_{k=\lambda}^{\infty} l_{nk} (f) \Phi (nx - k)$ and $\sum_{k=-\infty}^{-\lambda} l_{nk} (f) \Phi (nx - k)$ are continuous on \mathbb{R} . Since $\sum_{k=-\lambda+1}^{\lambda-1} l_{nk}(f) \Phi(nx-k)$ is a finite sum of continuous functions on R, it is also a continuous function on R.

Writing

$$
\sum_{k=-\infty}^{\infty} l_{nk}(f) \Phi(nx-k) = \sum_{k=-\infty}^{-\lambda} l_{nk}(f) \Phi(nx-k) +
$$

$$
\sum_{k=-\lambda+1}^{\lambda-1} l_{nk}(f) \Phi(nx-k) + \sum_{k=\lambda}^{\infty} l_{nk}(f) \Phi(nx-k) \qquad (93)
$$

we have it as a continuous function on R. Therefore $F_n(f)$, when $N = 1$, is a continuous function on R.

When $N = 2$ we have

$$
F_n(f, x_1, x_2) = \sum_{k_1 = -\infty}^{\infty} \sum_{k_2 = -\infty}^{\infty} l_{nk}(f) \Phi(nx_1 - k_1) \Phi(nx_2 - k_2) =
$$

$$
\sum_{k_1 = -\infty}^{\infty} \Phi(nx_1 - k_1) \left(\sum_{k_2 = -\infty}^{\infty} l_{nk}(f) \Phi(nx_2 - k_2) \right)
$$

(there always exist $\lambda_1, \lambda_2 \in \mathbb{N}$ such that $nx_1 \in [-\lambda_1, \lambda_1]$ and $nx_2 \in [-\lambda_2, \lambda_2]$)

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS. VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\nSince
$$
l_{nk}(f) \Phi(nx-k)
$$
 is continuous in x , then $\sum_{k=-\infty}^{\infty} l_{nk}(f) \Phi(nx-k)$
\nis continuous on $[\mathbb{R}, \sum_{k=-\infty}^{\infty} l_{nk}(f) \Phi(nx-k)$ and $\sum_{k=-\infty}^{\infty} l_{nk}(f) \Phi(nx-k)$
\nare continuous on $\mathbb{R}, \sum_{k=-\infty}^{\infty} l_{nk}(f) \Phi(nx-k)$ and $\sum_{k=-\infty}^{\infty} l_{nk}(f) \Phi(nx-k)$
\nare continuous functions on $\mathbb{R}, \sum_{k=-\infty}^{\infty} l_{nk}(f) \Phi(nx-k) = \sum_{k=-\infty}^{\infty} l_{nk}(f) \Phi(nx-k) +$
\nWriting
\n
$$
\sum_{k=-\infty}^{\infty} l_{nk}(f) \Phi(nx-k) = \sum_{k=-\infty}^{\infty} l_{nk}(f) \Phi(nx-k) +
$$
\n
$$
\sum_{k=-\infty}^{\infty} l_{nk}(f) \Phi(nx-k) + \sum_{k=-\infty}^{\infty} l_{nk}(f) \Phi(nx-k)
$$
\n(93)
\nwe have it as a continuous function on \mathbb{R} . Therefore $F_n(f)$, when $N = 1$, is a continuous function on \mathbb{R} .
\nWhen $N = 2$ we have
\n
$$
F_n(f, x_1, x_2) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} l_{nk}(f) \Phi(nx_2 - k_2) =
$$
\n
$$
\sum_{k_2=-\infty}^{\infty} \Phi(nx_2 - k_1) \left(\sum_{k_2=-\infty}^{\infty} l_{nk}(f) \Phi(nx_2 - k_2) \right)
$$
\n(there always exist $\lambda_1, \lambda_2 \in \mathbb{N}$ such that $nx_1 \in [-\lambda_1, \lambda_1]$ and $nx_2 \in [-\lambda_2, \lambda_2]$)
\n
$$
= \sum_{k_1=-\infty}^{\infty}
$$

(For convenience call

$$
F(k_1, k_2, x_1, x_2) := l_{nk}(f) \Phi(nx_1 - k_1) \Phi(nx_2 - k_2).
$$

Thus

3. COMPUTATORAL ANALYSIS AND APPLICATIONS, Vol. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\nThus
\n
$$
(*) = \sum_{k_1 = -\infty}^{3} \sum_{k_2 = -\infty}^{-\infty} F^{k}(k_1, k_2, x_1, x_2) + \sum_{k_1 = -\infty}^{3} \sum_{k_2 = -\infty}^{-\infty} F^{k}(k_1, k_2, x_1, x_2) + \sum_{k_1 = -\infty}^{3} \sum_{k_2 = -\infty}^{-\infty} F^{k}(k_1, k_2, x_1, x_2) + \sum_{k_1 = -\infty}^{3} \sum_{k_2 = -\infty}^{-\infty} F^{k}(k_1, k_2, x_1, x_2) + \sum_{k_1 = -\infty}^{-\infty} \sum_{k_2 = -\infty}^{-\infty} F^{k}(k_1, k_2, x_1, x_2) + \sum_{k_1 = -\infty}^{-\infty} \sum_{k_2 = -\infty}^{-\infty} F^{k}(k_1, k_2, x_1, x_2) + \sum_{k_2 = -\infty}^{-\infty} \sum_{k_2 = -\infty}^{-\infty} F^{k}(k_1, k_2, x_1, x_2) + \sum_{k_2 = -\infty}^{-\infty} \sum_{k_2 = -\infty}^{-\infty} F^{k}(k_1, k_2, x_1, x_2) + \sum_{k_2 = -\infty}^{-\infty} \sum_{k_2 = -\infty}^{-\infty} F^{k}(k_1, k_2, x_1, x_2) + \sum_{k_2 = -\infty}^{-\infty} \sum_{k_2 = -\infty}^{-\infty} F^{k}(k_1, k_2, x_1, x_2).
$$
\nNotice that the finite sum of continuous functions $F^{k}(k_1, k_2, x_1, x_2)$.
\nNotice that the function sum of continuous functions $F^{k}(k_1, k_2, x_1, x_2)$.
\n
$$
\sum_{k_1 = -\infty}^{-\infty} \sum_{k_1 = -\infty}^{-\infty} \sum_{k_2 = -\infty}^{-\infty} \sum_{k_2 = -\infty}^{-\infty} k_2 \text{ of } (k_1 =
$$

Notice that the finite sum of continuous functions $F(k_1, k_2, x_1, x_2)$, $\sum_{k_1=-\lambda_1+1}^{\lambda_1-1} \sum_{k_2=-\lambda_2+1}^{\lambda_2-1} F(k_1, k_2, x_1, x_2)$ is a continuous function.

The rest of the summands of $F_n(f, x_1, x_2)$ are treated all the same way and similarly to the case of $N = 1$. The method is demonstrated as follows.

We will prove that $\sum_{k_1=\lambda_1}^{\infty} \sum_{k_2=-\infty}^{-\lambda_2} l_{nk}(f) \Phi(nx_1 - k_1) \Phi(nx_2 - k_2)$ is continuous in $(x_1, x_2) \in \mathbb{R}^2$.

The continuous function

$$
\|l_{nk}(f)\|_{\gamma} \Phi\left(nx_1 - k_1\right) \Phi\left(nx_2 - k_2\right) \le \left\| \left\| f \right\|_{\gamma} \right\|_{\infty} \Phi\left(k_1 - \lambda_1\right) \Phi\left(k_2 + \lambda_2\right),
$$

and

$$
\left\| \|f\|_{\gamma} \right\|_{\infty} \sum_{k_1 = \lambda_1}^{\infty} \sum_{k_2 = -\infty}^{-\lambda_2} \Phi(k_1 - \lambda_1) \Phi(k_2 + \lambda_2) =
$$

$$
\left\| \|f\|_{\gamma} \right\|_{\infty} \left(\sum_{k_1 = \lambda_1}^{\infty} \Phi(k_1 - \lambda_1) \right) \left(\sum_{k_2 = -\infty}^{-\lambda_2} \Phi(k_2 + \lambda_2) \right) \le
$$

$$
\left\| \|f\|_{\gamma} \right\|_{\infty} \left(\sum_{k_1' = 0}^{\infty} \Phi(k_1') \right) \left(\sum_{k_2' = -\infty}^{0} \Phi(k_2') \right) \le \left\| \|f\|_{\gamma} \right\|_{\infty}.
$$

So by the Weierstrass M test we get that

 $\sum_{k_1=\lambda_1}^{\infty} \sum_{k_2=-\infty}^{-\lambda_2} l_{nk}(f) \Phi(nx_1 - k_1) \Phi(nx_2 - k_2)$ is uniformly and absolutely convergent. Therefore it is continuous on \mathbb{R}^2 .

Next we prove continuity on \mathbb{R}^2 of $\sum_{k_1=-\lambda_1+1}^{\lambda_1-1} \sum_{k_2=-\infty}^{-\lambda_2} l_{nk}(f) \Phi(nx_1-k_1) \Phi(nx_2-k_2).$ Notice here that

$$
\|l_{nk}(f)\|_{\gamma} \Phi(nx_1 - k_1) \Phi(nx_2 - k_2) \le \left\| \|f\|_{\gamma} \right\|_{\infty} \Phi(nx_1 - k_1) \Phi(k_2 + \lambda_2)
$$

$$
\le \left\| \|f\|_{\gamma} \right\|_{\infty} \Phi(0) \Phi(k_2 + \lambda_2) = \frac{1}{2^{\frac{2m}{2}} \cdot} \left\| \|f\|_{\gamma} \right\|_{\infty} \Phi(k_2 + \lambda_2),
$$

and

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, Vol. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\nNotice here that
\n
$$
||f_{nk}(f)||_{\gamma} \Phi(nx_1 - k_1) \Phi(nx_2 - k_2) \le |||f||_{\gamma}||_{\infty} \Phi(nx_1 - k_1) \Phi(k_2 + \lambda_2)
$$
\n
$$
\le |||f||_{\gamma}||_{\infty} \Phi(0) \Phi(k_2 + \lambda_2) = \frac{1}{2 \sqrt[3]{2}} \cdot |||f||_{\gamma}||_{\infty} \Phi(k_2 + \lambda_2),
$$
\nand
\n
$$
\frac{1}{2 \sqrt[3]{2}} \cdot |||f||_{\gamma}||_{\infty} (2\lambda_1 - 1) \left(\sum_{k_2 = -\infty}^{5} \Phi(k_2')\right) \le \frac{1}{2 \sqrt[3]{2}} \cdot (2\lambda_1 - 1) |||f||_{\gamma}||_{\infty}.
$$
\n(95)
\nSo the double series under consideration is uniformly convergent and continuous.
\nClearly, $F_{nk}(f, x_2)$ is proved to be continuous on \mathbb{R}^2 , for any $N \ge 1$. We choose to omit
\nthis $F_{nk}(f, x_1, x_2)$ is continuous on \mathbb{R}^N , for any $N \ge 1$. We choose to omit
\nthis $F_{nk}(f, x_1, x_2)$ is continuous for all $||A_n(f)||_{\infty} = |||f||_{\gamma}||_{\infty} < \infty$, and
\n
$$
A_n(f) \in C\left(\prod_{i=1}^{N} |a_i, b_i|, X\right), given that $f \in C\left(\prod_{i=1}^{N} |a_i, b_i|, X\right).$
\n
$$
C(x_1 + y_1, y_2) \text{ is continuous for all } ||A_n(f)||_{\infty} = |||f||_{\infty} ||f||_{\infty} < \infty, \text{ and }
$$

\n
$$
A_n(f) \in C\left(\prod_{i=1}^{N} |a_i, b_i|, X\right), given that $f \in C\left(\prod_{i=1}^{N} |a_i, b_i|, X\right).$
\n
$$
C(x_1 + y_1 + y_2 +
$$
$$
$$

So the double series under consideration is uniformly convergent and continuous. Clearly $F_n(f, x_1, x_2)$ is proved to be continuous on \mathbb{R}^2 .

Similarly reasoning one can prove easily now, but with more tedious work, that F_n $(f, x_1, ..., x_N)$ is continuous on \mathbb{R}^N , for any $N \geq 1$. We choose to omit this similar extra work. \blacksquare

Remark 21 By (25) it is obvious that $\left\| \|A_n(f)\|_{\gamma} \right\|_{\infty} \le$ $\left\| \|f\|_{\gamma} \right\|_{\infty} < \infty$, and $A_n(f) \in C\left(\prod_{i=1}^N\right)$ $\prod_{i=1}^{N} [a_i, b_i], X$, given that $f \in C \left(\prod_{i=1}^{N} \right)$ $\prod_{i=1}^N [a_i, b_i], X$. Call L_n any of the operators A_n, B_n, C_n, D_n . Clearly then

$$
\left\| \left\| L_n^2(f) \right\|_{\gamma} \right\|_{\infty} = \left\| \left\| L_n\left(L_n\left(f \right) \right) \right\|_{\gamma} \right\|_{\infty} \le \left\| \left\| L_n\left(f \right) \right\|_{\gamma} \right\|_{\infty} \le \left\| \left\| f \right\|_{\gamma} \right\|_{\infty}, \tag{96}
$$

etc.

Therefore we get

$$
\left\| \left\| L_n^k \left(f \right) \right\|_{\gamma} \right\|_{\infty} \le \left\| \left\| f \right\|_{\gamma} \right\|_{\infty}, \ \ \forall \ k \in \mathbb{N}, \tag{97}
$$

the contraction property.

Also we see that

$$
\left\| \left\| L_n^k(f) \right\|_{\gamma} \right\|_{\infty} \le \left\| \left\| L_n^{k-1}(f) \right\|_{\gamma} \right\|_{\infty} \le \dots \le \left\| \left\| L_n(f) \right\|_{\gamma} \right\|_{\infty} \le \left\| \left\| f \right\|_{\gamma} \right\|_{\infty} . \tag{98}
$$

Here L_n^k are bounded linear operators.

Notation 22 Here $N \in \mathbb{N}$, $0 < \beta < 1$. Denote by

$$
c_N := \begin{cases} \left[2\left(\sqrt[2m]{1+4^m}\right)\right]^N, & \text{if } L_n = A_n, \\ 1, & \text{if } L_n = B_n, C_n, D_n, \end{cases}
$$
 (99)

$$
\varphi(n) := \begin{cases} \frac{1}{n^{\beta}}, & \text{if } L_n = A_n, B_n, \\ \frac{1}{n} + \frac{1}{n^{\beta}}, & \text{if } L_n = C_n, D_n, \end{cases}
$$
(100)

$$
\Omega := \begin{cases}\nC\left(\prod_{i=1}^{N} [a_i, b_i], X\right), & \text{if } L_n = A_n, \\
C_B\left(\mathbb{R}^N, X\right), & \text{if } L_n = B_n, C_n, D_n,\n\end{cases} \tag{101}
$$

and

$$
Y := \begin{cases} \prod_{i=1}^{N} [a_i, b_i], & \text{if } L_n = A_n, \\ \mathbb{R}^N, & \text{if } L_n = B_n, C_n, D_n. \end{cases}
$$
 (102)

We give the condensed

Theorem 23 Let $f \in \Omega$, $0 < \beta < 1$, $x \in Y$; $n, m, N \in \mathbb{N}$ with $n^{1-\beta} > 2$. Then (i)

$$
\|L_n(f, x) - f(x)\|_{\gamma} \le c_N \left[\omega_1(f, \varphi(n)) + \frac{\left\|f\|_{\gamma}\right\|_{\infty}}{2m\left(n^{1-\beta} - 2\right)^{2m}}\right] =: \tau(n), \tag{103}
$$

where ω_1 is for $p = \infty$,

and

(ii)

$$
\left\| \left\| L_n(f) - f \right\|_{\gamma} \right\|_{\infty} \le \tau(n) \to 0, \text{ as } n \to \infty. \tag{104}
$$

For f uniformly continuous and in Ω we obtain

$$
\lim_{n\to\infty}L_n(f)=f,
$$

pointwise and uniformly.

Proof. By Theorems 8, 16, 17, 18. \blacksquare

Next we do iterated neural network approximation (see also [9]). We make

Remark 24 Let $r \in \mathbb{N}$ and L_n as above. We observe that

$$
L_n^r f - f = (L_n^r f - L_n^{r-1} f) + (L_n^{r-1} f - L_n^{r-2} f) +
$$

$$
(L_n^{r-2} f - L_n^{r-3} f) + \dots + (L_n^2 f - L_n f) + (L_n f - f).
$$

Then

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\n
$$
\varphi(n) := \begin{cases}\n\frac{1}{n}v, & \text{if } L_n = A_n, B_n, \\
\frac{1}{n} + \frac{1}{n^2}, & \text{if } L_n = C_n, D_n, \\
C_B\left(\prod_{i=1}^N [a_i, b_i], X\right), & \text{if } L_n = A_n, \\
C_B\left(\mathbb{R}^N, X\right), & \text{if } L_n = A_n, \\
C_B\left(\mathbb{R}^N, X\right), & \text{if } L_n = A_n, \\
\sum_{i=1}^N [a_i, b_i], & \text{if } L_n = A_n, \\
\sum_{i=1}^N [a_i, b_i], & \text{if } L_n = A_n, \\
\sum_{i=1}^N [a_i, b_i], & \text{if } L_n = B_n, C_n, D_n.\n\end{cases}
$$
\nWe give the condensed
\n**Theorem 23** Let $f \in \Omega, 0 \le \beta \le 1, x \in V; n, m, N \in \mathbb{N}$ with $n^{1-\beta} > 2$. Then
\n(i)
\n
$$
||L_n(f, x) - f(x)||_1 \le c_N \begin{bmatrix} \omega_1(f, \varphi(n)) + \frac{||f||_n||_n}{2m(n^{1-\beta} - 2)^{2m}} \end{bmatrix} =: \tau(n), (103)
$$
\nwhere ω_1 is for $p = \infty$,
\n(i)
\n(ii)
\n
$$
|||L_n(f) - f||_n||_{\infty} \le \tau(n) \to 0, \text{ as } n \to \infty.
$$
\n(104)
\nFor f uniformly continuous and in Ω we obtain
\n
$$
\lim_{n \to \infty} L_n(f) = f,
$$
\npointwise and uniformly.
\n**Proof.** By Theorems 8, 16, 17, 18. **Example**
\n**Remark 24** Let $r \in \mathbb{N}$ and L_n as above. We observe that
\n $L_n^T f = (L_n^T f - L_n^{T-1} f) + (L_n^{T-1} f - L_n^{T-2} f) + (L_n$

$$
\left\| \left\| L_{n}^{r-1} \left(L_{n} f - f \right) \right\|_{\gamma} \right\|_{\infty} + \left\| \left\| L_{n}^{r-2} \left(L_{n} f - f \right) \right\|_{\gamma} \right\|_{\infty} + \left\| \left\| L_{n}^{r-3} \left(L_{n} f - f \right) \right\|_{\gamma} \right\|_{\infty}
$$

+...+
$$
\left\| \left\| L_{n} \left(L_{n} f - f \right) \right\|_{\gamma} \right\|_{\infty} + \left\| \left\| L_{n} f - f \right\|_{\gamma} \right\|_{\infty} \leq r \left\| \left\| L_{n} f - f \right\|_{\gamma} \right\|_{\infty} . \tag{105}
$$

That is

$$
\left\| \left\| L_n^r f - f \right\|_{\gamma} \right\|_{\infty} \le r \left\| \left\| L_n f - f \right\|_{\gamma} \right\|_{\infty}.
$$
 (106)

We give

Theorem 25 All here as in Theorem 23 and $r \in \mathbb{N}$, $\tau(n)$ as in (104). Then

$$
\left\| \left\| L_n^r f - f \right\|_{\gamma} \right\|_{\infty} \leq r \tau(n). \tag{107}
$$

So that the speed of convergence to the unit operator of L_n^r is not worse than of L_n .

Proof. By (106) and (104). \blacksquare We make

Remark 26 Let $m, m_1, ..., m_r \in \mathbb{N} : m_1 \leq m_2 \leq ... \leq m_r, 0 < \beta < 1, f \in \Omega$. Then $\varphi(m_1) \geq \varphi(m_2) \geq ... \geq \varphi(m_r)$, φ as in (100).

Therefore

$$
\omega_1(f, \varphi(m_1)) \ge \omega_1(f, \varphi(m_2)) \ge \dots \ge \omega_1(f, \varphi(m_r)). \tag{108}
$$

Assume further that $m_i^{1-\beta} > 2$, $i = 1, ..., r$. Then

$$
\frac{1}{4m\left(m_1^{1-\beta}-2\right)^{2m}} \ge \frac{1}{4m\left(m_2^{1-\beta}-2\right)^{2m}} \ge \dots \ge \frac{1}{4m\left(m_r^{1-\beta}-2\right)^{2m}}.\tag{109}
$$

Let L_{m_i} as above, $i = 1, ..., r$, all of the same kind.

We write

0. COMPUTATIONAL ANALYSIS AND APPLICATIONS. VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\n
$$
\left\| ||L_n^{-1} (L_n f - f)||_2 \right\|_{\infty} + \left\| ||L_n^{-2} (L_n f - f)||_2 \right\|_{\infty} + \left\| ||L_n f - f||_2 \right\|_{\infty}.
$$
\n(105)
\nThat is
\n
$$
\left\| ||L_n (L_n f - f)||_2 \right\|_{\infty} + \left\| ||L_n f - f||_2 \right\|_{\infty} \le r \left\| ||L_n f - f||_2 \right\|_{\infty}.
$$
\n(106)
\n*That is*
\n
$$
\left\| ||L_n^* f - f||_2 \right\|_{\infty} \le r \left\| ||L_n f - f||_2 \right\|_{\infty}.
$$
\n(107)
\nWe give
\n**Theorem 25** All here as in Theorem 23 and r ∈ N, τ (n) as in (104). Then
\n
$$
\left\| ||L_n^* f - f||_2 \right\|_{\infty} \le r \left[n \right].
$$
\n(108)
\nSo that the speed of convergence to the unit operator of L_n^* is not worse than of
\n L_n .
\n**Proof.** By (106) and (104). ■
\nWe make
\n**Remark 26** Let $m, m_1, ..., m_r ∈ N : m_1 ≤ m_2 ≤ ... ≤ m_r, 0 < \beta < 1, f ∈ \Omega$.
\n*There of (m_1) ≥ ϕ (m_2) ≥ ... ≥ υ (m_r), ψ as in (100).
\n*Assume further that m_1^{1−β} > 2, i = 1, ..., r. Then*
\n
$$
\frac{1}{4m (m_1^{1−θ} - 2)^{2m}} ≥ \frac{1}{4m (m_2^{1−θ} - 2)^{2m}} ≥ ... ≥ \frac{1}{4m (m_2^{1−θ} - 2)^{2m}}. \quad (109)
$$
\n*Assume further that m_1^{1−β} > 2, i = 1, ..., r. Then*
\n
$$
\frac{1}{4m (m_1^{1−θ} -
$$*

$$
L_{m_r}\left(L_{m_{r-1}}\left(\ldots L_{m_4} \right)\right)\left(L_{m_3}f - f\right) + \ldots + L_{m_r}\left(L_{m_{r-1}}f - f\right) + L_{m_r}f - f.
$$

Hence by the triangle inequality property of $\|\|\cdot\|_{\gamma}$ $\Big\|_\infty$ we get

$$
\left\|\left\|L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_2}\left(L_{m_1}f\right)\right)\right)-f\right\|_{\gamma}\right\|_{\infty}\leq
$$
$$
\| \| L_{m_r} (L_{m_{r-1}} (... L_{m_2})) (L_{m_1} f - f) \|_{\gamma} \|_{\infty} +
$$

$$
\| \| L_{m_r} (L_{m_{r-1}} (... L_{m_3})) (L_{m_2} f - f) \|_{\gamma} \|_{\infty} +
$$

$$
\| \| L_{m_r} (L_{m_{r-1}} (... L_{m_4})) (L_{m_3} f - f) \|_{\gamma} \|_{\infty} + ... +
$$

$$
\| \| L_{m_r} (L_{m_{r-1}} f - f) \|_{\gamma} \|_{\infty} + \| \| L_{m_r} f - f \|_{\gamma} \|_{\infty}
$$

(repeatedly applying (96))

$$
\leq \left\| \|L_{m_1}f - f\|_{\gamma} \right\|_{\infty} + \left\| \|L_{m_2}f - f\|_{\gamma} \right\|_{\infty} + \left\| \|L_{m_3}f - f\|_{\gamma} \right\|_{\infty} + \dots +
$$

$$
\left\| \|L_{m_{r-1}}f - f\|_{\gamma} \right\|_{\infty} + \left\| \|L_{m_r}f - f\|_{\gamma} \right\|_{\infty} = \sum_{i=1}^r \left\| \|L_{m_i}f - f\|_{\gamma} \right\|_{\infty} . \tag{111}
$$

That is, we proved

$$
\left\| \left\| L_{m_r} \left(L_{m_{r-1}} \left(... L_{m_2} \left(L_{m_1} f \right) \right) \right) - f \right\|_{\gamma} \right\|_{\infty} \le \sum_{i=1}^r \left\| \left\| L_{m_i} f - f \right\|_{\gamma} \right\|_{\infty} . \tag{112}
$$

We give

Theorem 27 Let $f \in \Omega$; m, N, $m_1, m_2, ..., m_r \in \mathbb{N} : m_1 \leq m_2 \leq ... \leq m_r$, $0 <$ $\beta < 1; m_i^{1-\beta} > 2, i = 1, ..., r, x \in Y$, and let $(L_{m_1}, ..., L_{m_r})$ as $(A_{m_1}, ..., A_{m_r})$ or $(B_{m_1},...,B_{m_r})$ or $(C_{m_1},...,C_{m_r})$ or $(D_{m_1},...,D_{m_r}),$ $p = \infty$. Then

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, Vol. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC
\n
$$
\left\| \left\| L_{m_r} \left(L_{m_{r-1}} \left(\ldots L_{m_n} \right) \right) \left(L_{m_r} f - f \right) \right\|_{\gamma} \right\|_{\infty} +
$$
\n
$$
\left\| \left\| L_{m_r} \left(L_{m_{r-1}} \left(\ldots L_{m_n} \right) \right) \left(L_{m_r} f - f \right) \right\|_{\gamma} \right\|_{\infty} +
$$
\n
$$
\left\| \left\| L_{m_r} \left(L_{m_{r-1}} \left(\ldots L_{m_n} \right) \right) \left(L_{m_p} f - f \right) \right\|_{\gamma} \right\|_{\infty} + \dots +
$$
\n
$$
\left\| \left\| L_{m_r} f - f \right\|_{\gamma} \right\|_{\infty} + \left\| \left\| L_{m_r} f - f \right\|_{\gamma} \right\|_{\infty} + \left\| \left\| L_{m_r} f - f \right\|_{\gamma} \right\|_{\infty} + \dots +
$$
\n
$$
\left\| \left\| L_{m_r} f - f \right\|_{\gamma} \right\|_{\infty} + \left\| \left\| L_{m_r} f - f \right\|_{\gamma} \right\|_{\infty} + \left\| \left\| L_{m_r} f - f \right\|_{\gamma} \right\|_{\infty} + \dots +
$$
\n
$$
\left\| \left\| L_{m_r} f - f \right\|_{\gamma} \right\|_{\infty} + \left\| \left\| L_{m_r} f - f \right\|_{\gamma} \right\|_{\infty} = \sum_{i=1}^{r} \left\| \left\| L_{m_i} f - f \right\|_{\gamma} \right\|_{\infty} + \dots +
$$
\n
$$
\left\| L_{m_r} \left(L_{m_{r-1}} \left(\ldots L_{m_2} \left(L_{m_1} f \right) \right) \right) - f \right\|_{\gamma} \right\|_{\infty} \leq \sum_{i=1}^{r} \left\| \left\| L_{m_i} f - f \right\|_{\gamma} \
$$

Clearly, we notice that the speed of convergence to the unit operator of the multiply iterated operator is not worse than the speed of L_{m_1} .

Proof. Using (112), (108), (109) and (103), (104). \blacksquare We continue with

Theorem 28 Let all as in Corollary 15, and $r \in \mathbb{N}$. Here $\varphi_3(n)$ is as in (75). Then

$$
\left\| \left\| A_n^r f - f \right\|_{\gamma} \right\|_{\infty} \le r \left\| \left\| A_n f - f \right\|_{\gamma} \right\|_{\infty} \le r \varphi_3 \left(n \right). \tag{114}
$$

Proof. By (106) and (75). \blacksquare

Application 29 A typical application of all of our results is when $(X, \left\| \cdot \right\|_{\gamma}) =$ $(\mathbb{C}, \lvert \cdot \rvert)$, where $\mathbb C$ are the complex numbers.

References

- [1] G.A. Anastassiou, Moments in Probability and Approximation Theory, Pitman Research Notes in Math., Vol. 287, Longman Sci. & Tech., Harlow, U.K., 1993.
- [2] G.A. Anastassiou, Rate of convergence of some neural network operators to the unit-univariate case, J. Math. Anal. Appli. 212 (1997), 237-262.
- [3] G.A. Anastassiou, Quantitative Approximations, Chapman&Hall/CRC, Boca Raton, New York, 2001.
- [4] G.A. Anastassiou, *Inteligent Systems: Approximation by Artificial Neural* Networks, Intelligent Systems Reference Library, Vol. 19, Springer, Heidelberg, 2011.
- [5] G.A. Anastassiou, Univariate hyperbolic tangent neural network approximation, Mathematics and Computer Modelling, 53(2011), 1111-1132.
- [6] G.A. Anastassiou, Multivariate hyperbolic tangent neural network approximation, Computers and Mathematics 61(2011), 809-821.
- [7] G.A. Anastassiou, Multivariate sigmoidal neural network approximation, Neural Networks 24(2011), 378-386.
- [8] G.A. Anastassiou, Univariate sigmoidal neural network approximation, J. of Computational Analysis and Applications, Vol. 14, No. 4, 2012, 659-690.
- [9] G.A. Anastassiou, Approximation by neural networks iterates, Advances in Applied Mathematics and Approximation Theory, pp. 1-20, Springer Proceedings in Math. & Stat., Springer, New York, 2013, Eds. G. Anastassiou, O. Duman. CONFUTATIONAL ANNEWSIS AND APPLICATIONS, VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC

Theorem 26 Jet als et on Consideration (\vec{B}_i and $\vec{r} \in \mathbb{Z}$ H, $||A_{ij}f - f||_{ij}||_{ij} \leq r ||\vec{r}||_{ij} \leq r ||\vec{r}||_{ij} \leq r ||\$
	- [10] G.A. Anastassiou, Intelligent Systems II: Complete Approximation by Neural Network Operators, Springer, Heidelberg, New York, 2016.
- [11] G.A. Anastassiou, Intelligent Computations: Abstract Fractional Calculus, Inequalities, Approximations, Springer, Heidelberg, New York, 2018.
- [12] G.A. Anastassiou, Algebraic function based Banach space valued ordinary and fractional neural network approximations, New Trends in Mathematical Sciences, 10 special issue (1) (2022), 100-125. 6 COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 31, NO. 4, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC (1) C.A. Americanism, And Explications, Findeling, and the state of factorization (and the control of the state of the sta
	- [13] G.A. Anastassiou, General multimariate arctangent function activated neural network approximations, submitted, 2022.
	- [14] H. Cartan, *Differential Calculus*, Hermann, Paris, 1971.
	- [15] Z. Chen and F. Cao, The approximation operators with sigmoidal functions, Computers and Mathematics with Applications, 58 (2009), 758-765.
	- [16] D. Costarelli, R. Spigler, Approximation results for neural network operators activated by sigmoidal functions, Neural Networks 44 (2013), 101-106.
	- [17] D. Costarelli, R. Spigler, Multivariate neural network operators with sigmoidal activation functions, Neural Networks 48 (2013), 72-77.
	- [18] S. Haykin, Neural Networks: A Comprehensive Foundation (2 ed.), Prentice Hall, New York, 1998.
	- [19] W. McCulloch and W. Pitts, A logical calculus of the ideas immanent in nervous activity, Bulletin of Mathematical Biophysics, 7 (1943), 115-133.
	- [20] T.M. Mitchell, Machine Learning, WCB-McGraw-Hill, New York, 1997.
	- [21] L.B. Rall, Computational Solution of Nonlinear Operator Equations, John Wiley & Sons, New York, 1969.

TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 31, NO. 4, 2023

