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Journal of Computational Analysis and Applications

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Numerical simulation of an electro-thermal model for superconducting nanowire single-photon detectors [1]

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Abstract

The electro-thermal model for Superconducting Nanowire Single-Photon Detectors is a nonlinear free boundary problem involving the temperature and the current, which are coupled together by a nonlinear parabolic interface equation and a second order ordinary differential equation. In this paper, we propose a novel method to numerically solve the preceding electro-thermal model. A series of numerical experiments are provided to demonstrate the effectiveness of the method proposed.

Keywords. Electro-thermal model, free boundary problem, finite difference method, shooting method

1 Introduction

In recent years, superconducting nanowires single photon detection (SNSPD) has emerged as a new and promising single photon detection technology and has received wide attention in the field of applied superconductivity (cf. [1, 8]). The corresponding device structures nanometer zigzag line on the ultra-thin superconducting material, and uses the highly sensitive response of superconducting nanowire to realize single-photon detection. As shown in Figure 1 (see [1]), the key step of SNSPD is to discover the variation of the photon-induced hotspot.

In 2007, some researchers in MIT (cf. [15]) proposed a relevant electro-thermal mechanism to account for the variation of the photon-induced hotspot in SNSPD, after a small resistive hotspot forms along the nanowire. In this model, the SNSPD is approximated as a one-dimensional structure, the thermal response is modeled by a one-dimensional nonlinear parabolic interface equation involving the current flowing through the nanowire, and the electrical response is modeled by a second order ordinary differential equation. The two equations are coupled together to form a free boundary problem (cf. [3]).

To be more precise, let L denote the length of the superconducting nanowire under discussion, d the wire thickness, and W the width of nanowire. The domain occupying the nanowire is simply written as $\tilde{\Omega} = (-L/2, L/2)$. Due to the symmetry of the physical process, it suffices for us to discuss the variation of physical quantities in the half part $\Omega = (0, L/2)$. This domain is further split into two regions, $\Omega_{norm}(t) = (0, l(t))$ and $\Omega_{super}(t) = (l(t), L/2)$, corresponding to the normal/resistive and superconducting states, respectively. The interface $x = l(t)$ is used to separate the two states at time t. Let $T(x, t)$ represent the temperature of the material in the point x at time t. Then, as given in [15], $T(x, t)$ is determined by the parabolic interface equation

$$J^2\rho + \kappa_n \frac{\partial^2 T}{\partial x^2} - \frac{\alpha}{d}(T - T_{sub}) = \frac{\partial C_n T}{\partial t}, \qquad 0 < x < l(t),\ t > 0, \tag{1.1}$$

$$\kappa_s \frac{\partial^2 T}{\partial x^2} - \frac{\alpha}{d}(T - T_{sub}) = \frac{\partial C_s T}{\partial t}, \qquad l(t) < x < L/2,\ t > 0, \tag{1.2}$$

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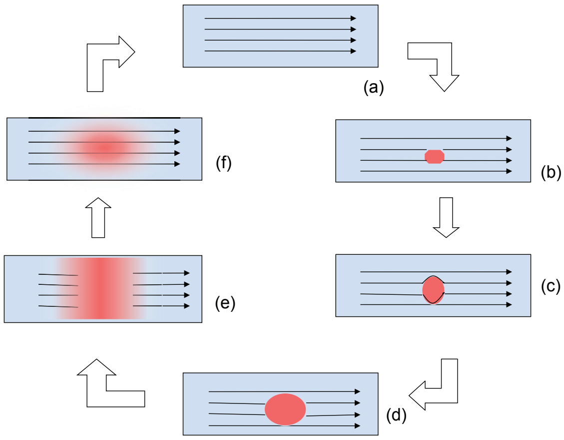

Figure 1: The variation of the photon-induced hotspot in SNSPD. (a) Bias direct current close to (but less than) its critical current, and set the nanowire temperature well below its superconducting critical temperature. (b) Form a small resistive hotspot. (c) The hotspot region forces the supercurrent to flow around the periphery of the hotspot, since the hotspot itself is not large enough to span the width of the nanowire. (d) Form a resistive barrier across the width of the nanowire, results in an easily measurable voltage pulse. (e) Resistive region is increased, the bias current is shunted by the external circuit. (f) The NbN nanowire becomes fully superconducting again.

with the initial condition

$$T(x,0) = T_0, \quad 0 \le x \le L/2.$$

Observe that $T(x,t)$ is symmetric about $x = 0$ with respect to x, and L is taken large enough such that the temperature at $x = L/2$ almost coincides with the substrate temperature. Then we impose the following boundary conditions:

$$\frac{\partial T(x,t)}{\partial x}\Big|_{x=0} = 0, \quad T(L/2,t) = T_{sub}, \quad t > 0. \tag{1.3}$$

Moreover, at the interface point $x = l(t)$ we impose the standard interface conditions:

$$T(x,t)|_{l^-} = T(x,t)|_{l^+}, \quad \kappa_n \frac{\partial T(x,t)}{\partial x}\Big|_{l^-} = \kappa_s \frac{\partial T(x,t)}{\partial x}\Big|_{l^+}, \quad t > 0, \tag{1.4}$$

as well as a phase transition condition

$$I_c(T) = I_c(0) \times (1 - (T/T_C)^2)^2. \tag{1.5}$$

Here, $J = \frac{I(t)}{Wd}$ is the current density through the nanowire, ρ is the electrical resistivity, κ_n and κ_s are the thermal conductivity coefficients, α is the thermal boundary conductance between the film and the substrate, T_{sub} is the substrate temperature (since the nanowire is thin enough), C_n and C_s are the heat capacity (per unit volume) of the superconducting film, $I_c(0)$ is the initial critical current, and T_c is the critical temperature. We mention that the transition condition is an empirical relation (see [15, p. 582]), which was obtained from an excellent fit with experimental measurements. Using this expression, one can determine

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a segment to be resistive when $I > I_c(T)$, where $T = T(x)$ is the temperature of a nanowire at the position x. The remaining part of the nanowire then belongs to the superconducting state.

On the other hand, using Kirchhoff's first law, we can find as in [3] that the current $I(t)$ through the nanowire satisfies an ordinary differential equation

$$I(t)R(t) + L_k \frac{dI(t)}{dt} = \frac{1}{C_{bt}} \int_0^t (I_{bias} - I(s))ds + (I_{bias} - I(t))Z_0, \quad t > 0, \tag{1.6}$$

with the initial condition

$$I(0) = I_0.$$

Differentiating (1.6) once with respect to t gives

$$C_{bt}\Big(\frac{d^2 L_k I}{dt^2} + \frac{d(I(t)R(t))}{dt} + Z_0 \frac{dI(t)}{dt}\Big) = I_{bias} - I(t), \quad t > 0, \tag{1.7}$$

where C_{bt} is the capacitor, an inductor L_k and a resistor $R(t)$ represent respectively the kinetic inductance of the superconducting nanowire and the time-dependent hotspot resistance, the time-dependent hotspot resistance respectively is given as $R(t) = 2\rho\frac{l(t)}{S} = 2\rho\frac{l(t)}{Wd}$, Z_0 is the impedance of the transmission line connecting the probe to RF amplifiers (cf. [9]), I_{bias} is the bias current of the SNSPD.

It is easy to see that the above electro-thermal model is a nonlinear free boundary problem with the interface $x = l(t)$ to be determined. Observe that the quantity J appearing in (1.1) satisfies that $J = \frac{I(t)}{Wd}$, and the quantity $R(t)$ appearing in (1.6) satisfies that $R(t) = 2\rho\frac{l(t)}{Wd}$. Hence, the temperature $T(x,t)$ and the current $I(t)$ are coupled together by the equations (1.1)-(1.2) and (1.7). Therefore, it is very challenging to devise an efficient method for numerically approximating the solution of this model. As far as we know, there is no work discussing numerical solution for the previous model systematically in the literature. The goal of this paper is intended to design some efficient algorithms for such a problem.

Before designing our algorithm, let us review some typical methods for numerically solving free boundary problems. First of all, front-tracking methods which use an explicit representation of the interface has always been a common way of solving moving boundary problems. Juric and Tryggvason presented in [5] a front-tracking method which use a fixed grid in space and explicit tracking of the liquid-solid interface, the method performs well in approximating the exact solution. The moving grid method can also be used to solve free boundary problems, which focuses on increasing the order of accuracy in discretization. For example, Javierr (cf. [4]) located the interface in the rth node and the grid should be adapted at each time step. Compared to the level set method, the accuracy of first-order convergence in the interface position was slightly higher. The level set method (cf. [2,6]) is also a widely used method for moving boundary problems. The main idea behind the method is that the interface position is represented by the zero level set, and it captures the interface position implicitly. Compared to the moving grid method, the level set has a main advantage that a fixed grid can be used, which avoids the mesh generation at every time step. Phase-field methods (cf. [4, 7]) have become increasingly popular for phase transition models over the past decade. These methods are based on phase field models, a free boundary arising from a phase field transition is assumed to have finite thickness, which differ from the classical model of a sharp interface. Phase-field methods present an advantage over front-tracking methods, because Phase-field methods only have an approximate representation of the front location. The main difference between the level set and phase-field methods is that

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the level set method can capture the front on a fixed grid, in order to apply discretizations that depend on the exact interface location. In contrast, in the phase-field model, the front is not being explicitly tracked, and thus near the front the discretization of the diffusion field is less accurate.

However, although there have developed many numerical methods for free boundary problems, it seems very difficult to simulate the above electro-thermal model effectively with these methods. Concretely speaking, since the moving grid method requires to introduce a transformation mapping to map a fictitious domain into the physical domain to form space grid points, it will lead to essential difficulty in discretization of the thermal equation, which is a nonlinear parabolic interface equation; for the level set method, it is inconvenient to establish a level set equation coupled with the original equations governing the variation of the temperature and current; for the phase-field method, it is very difficult to construct a relevant phase-field functional which involves very deep physical interpretation of the model. Hence, we develop a new approach to solve the electro-thermal model under discussion.

The main novelty of our method proposed here is that we determine the interface $x = l(t)$ at time t by means of the idea of the shooting method (cf. [11]) combined with the phase transition condition (1.5). We notice that the shooting method is often used in solving nonlinear two-point boundary value problems (cf. [11]). Our algorithm can be briefly described as follows. We use the finite difference method with fixed mesh to discretize the thermal equations (1.1)-(1.2) and the current equation (1.7). Assume the temperature T and the current I are available at time $t = t_n$. We then select a grid position $\tilde{l}$ as the guess of the interface position $x = l(t_{n+1})$ at $t = t_{n+1}$. Next, we compute the critical current $I_c(T)$ at $t = t_{n+1}$ in view of (1.5) at all grid points. If there exists a grid point $x = \tilde{l}_1$ such that the numerical current I_{n+1} is greater than the critical current at the left point of $x = \tilde{l}_1$, and less at the right side point, then we update the guess interface position $\tilde{l}$ as $\tilde{l}_1$. Repeat the above computation process until it converges. We present some numerical examples to show the computational performance of our method.

The rest of this paper is organized as follows. In section 2, we describe the Crank-Nicholson finite difference method and implicit-explicit scheme for the discretization of the thermal equation, and the trapezoidal rule for the discretization of the current equation. The algorithm for determination of the interface positions is given in section 3. A series of numerical results are given in section 4. In the final section, we present a short conclusion about our investigation in this paper.

2 Discretization of the governing equations

In order to numerically solve the electro-thermal model, we first partition the space region $[0, L/2]$ into N intervals with equal width Δx, to get the spatial nodes $0 = x_0 < x_1 < \cdots < x_N = L/2$ with $x_i = i\Delta x$, and then construct the time nodes $t_n = n\tau$ with $\tau > 0$ as the time stepsize, $n = 0, 1, \cdots$. We denote by T_i^n the approximate solution of the temperature T at a grid point (x_i, t_n) and denote by I_n the approximate solution of the current I at a grid point t_n. In this section, we will design effective finite difference methods for solving T_i^n and I_n, respectively.

2.1 Discretization of the thermal equation

2.1.1 The Crank-Nicholson method

Because the physical parameters rely on the temperature T itself, the thermal equations (1.1)-(1.2) are highly nonlinear. Hence, we use linearized schemes to carry out discretization, in order to avoid heavy cost in solving a nonlinear system of algebraic equations.

Let $x = l = l_0^{n+1} = x_j$ be the approximate interface position at the time $t = t_{n+1} = (n+1)\tau$. For a spatial point $x = x_i = i\Delta x$ in $(0, l)$, we view the physical parameters to be constant in the time interval $[t_n, t_{n+1}]$, equal to the ones corresponding to the temperature at $t = t_n$. Then we use the standard Crank-Nicholson finite difference method to discretize the equation (1.1) (cf. [12]), to get the following difference equation:

$$\begin{aligned}\Big(\frac{1}{Wd}\times\frac{I_{n+1}+I_n}{2}\Big)^2\rho+\kappa_n(T_i^n)\times\frac{1}{2}\Big(\frac{T_{i-1}^n-2T_i^n+T_{i+1}^n}{\Delta x^2}+\frac{T_{i-1}^{n+1}-2T_i^{n+1}+T_{i+1}^{n+1}}{\Delta x^2}\Big)\\-\frac{1}{d}\alpha(T_i^n)\times\Big(\frac{T_i^{n+1}+T_i^n}{2}-T_{sub}\Big)=M(T_i^n)\times\frac{T_i^{n+1}-T_i^n}{\tau},\end{aligned}\tag{2.1}$$

where $M(T) := C_n(T) + TC_n'(T)$ so that $\frac{\partial C_n T}{\partial t} = M(T)\frac{\partial T}{\partial t}$.

Similarly, for $x = x_i = i\Delta x$ in $(l, L/2)$ we can derive the following difference equation from (1.2):

$$\begin{aligned}\kappa_s(T_i^n)\times\frac{1}{2}\Big(\frac{T_{i-1}^n-2T_i^n+T_{i+1}^n}{\Delta x^2}+\frac{T_{i-1}^{n+1}-2T_i^{n+1}+T_{i+1}^{n+1}}{\Delta x^2}\Big)\\-\frac{1}{d}\alpha(T_i^n)\times\Big(\frac{T_i^{n+1}+T_i^n}{2}-T_{sub}\Big)=H(T_i^n)\times\frac{T_i^{n+1}-T_i^n}{\tau},\end{aligned}\tag{2.2}$$

where $H(T) := C_s(T) + TC_s'(T)$ so that $\frac{\partial C_s T}{\partial t} = H(T)\frac{\partial T}{\partial t}$.

Next, let us deal with discretization of the boundary conditions. The homogeneous Neumann boundary condition is imposed at the left boundary point $x = x_0$. To ensure second order accuracy of approximation, we use the ghost point method (cf. [12]). We introduce a ghost point $x_{-1} = -\Delta x$ outside the solution region $[0, L/2]$ and let T_{-1}^m denote the approximate solution of T at the grid point (x_{-1}, t_m) fictitiously. Then using the central difference scheme we have from (1.3) that

$$\frac{T_1^m - T_{-1}^m}{2\Delta x} = 0.\tag{2.3}$$

On the other hand, we assume the difference scheme (2.1) holds at $x = x_0$ to get

$$\begin{aligned}\Big(\frac{1}{Wd}\times\frac{I_{n+1}+I_n}{2}\Big)^2\rho+\kappa_n(T_0^n)\times\frac{1}{2}\Big(\frac{T_{-1}^n-2T_0^n+T_1^n}{\Delta x^2}+\frac{T_{-1}^{n+1}-2T_0^{n+1}+T_1^{n+1}}{\Delta x^2}\Big)\\-\frac{1}{d}\alpha(T_0^n)\times\Big(\frac{T_0^{n+1}+T_0^n}{2}-T_{sub}\Big)=M(T_0^n)\times\frac{T_0^{n+1}-T_0^n}{\tau}.\end{aligned}\tag{2.4}$$

From (2.3) we know $T_{-1}^n = T_1^n$ and $T_{-1}^{n+1} = T_1^{n+1}$, and plugging them into (2.4) we obtain

$$\begin{aligned}\Big(\frac{1}{Wd}\times\frac{I_{n+1}+I_n}{2}\Big)^2\rho+\kappa_n(T_0^n)\times\frac{1}{2}\Big(\frac{-2T_0^n+2T_1^n}{\Delta x^2}+\frac{-2T_0^{n+1}+2T_1^{n+1}}{\Delta x^2}\Big)\\-\frac{1}{d}\alpha(T_0^n)\times\Big(\frac{T_0^{n+1}+T_0^n}{2}-T_{sub}\Big)=H(T_0^n)\times\frac{T_0^{n+1}-T_0^n}{\tau}.\end{aligned}\tag{2.5}$$

The Dirichlet condition is imposed at the right boundary point $x = L/2$, so we directly have

$$T_N = T_{sub}. \tag{2.6}$$

To discretize the interface condition (1.4) at the interface point $x = x_j$, we use a backward (resp. forward) scheme to approximate $\frac{\partial T(x,t)}{\partial x}$ from the left (resp. right) at $x = x_j$. So we have from (1.4) that

$$\kappa_n \frac{T_j - T_{j-1}}{\Delta x} = \kappa_s \frac{T_{j+1} - T_j}{\Delta x}. \tag{2.7}$$

The combination of the difference equations (2.1), (2.2), (2.5)-(2.7) can uniquely determine the grid function $\{T_i^{n+1}\}_{i=0}^{N-1}$. Obviously the scheme is implicit, and can be expressed in matrix notation as a linear system with a tridiagonal coefficient matrix. So we can obtain $\{T_i^{n+1}\}_{i=0}^{N-1}$ in an efficient way.

2.1.2 The Implicit-Explicit (IMEX) method

In order to derive an efficient implicit-explicit scheme for solving the thermal model given before, we first make a reformulation for the equations (1.1) and (1.2). As a matter of fact, from some direct and routine manipulation, the two equations can be rewritten as follows.

$$\frac{J^2\rho}{M(T)} + G_n(T)\frac{\partial^2 T}{\partial x^2} + \frac{F(T)}{d}T - \frac{F(T)}{d}T_{sub} = \frac{\partial T}{\partial t}, \tag{2.8}$$

$$G_s(T)\frac{\partial^2 T}{\partial x^2} + \frac{E(T)}{d}T - \frac{E(T)}{d}T_{sub} = \frac{\partial T}{\partial t}. \tag{2.9}$$

where $G_n(T) = \frac{\kappa_n(T)}{M(T)}$, $G_s(T) = \frac{\kappa_s(T)}{H(T)}$, $F(T) = -\frac{\alpha(T)}{M(T)}$, $E(T) = -\frac{\alpha(T)}{H(T)}$.

Next, we choose a positive constant G_0, large enough, such that G_0 is no less than $G_n(T)$ and $G_s(T)$ at least. The constant can be obtained by some additional calculation in terms of the explicit form of the underlying function. In our numerical experiments developed in section 4, G_0 is taken such that

$$G_0 = \max_{T_{sub} \le T \le T_C} \{G_n(T), G_s(T)\}. \tag{2.10}$$

Therefore, the above equations can be reformulated further as

$$G_0\frac{\partial^2 T}{\partial x^2} + \frac{J^2\rho}{M(T)} + [G_n(T) - G_0]\frac{\partial^2 T}{\partial x^2} + \frac{F(T)}{d}T - \frac{F(T)}{d}T_{sub} = \frac{\partial T}{\partial t}, \tag{2.11}$$

$$G_0\frac{\partial^2 T}{\partial x^2} + [G_s(T) - G_0]\frac{\partial^2 T}{\partial x^2} + \frac{E(T)}{d}T - \frac{E(T)}{d}T_{sub} = \frac{\partial T}{\partial t}. \tag{2.12}$$

Hence, borrowing the same ideas to treat the variable coefficients as for the Crank-Nicholson method and using the technique that we discretize the partial derives of T with constant coefficients via implicit schemes and the other terms via explicit schemes (cf. [10]), we obtain from (2.11) that, at $x = x_i = i\Delta x$, $x \in (0, l)$ and $t = t_{n+1} = (n+1)\tau$, the difference equation for (1.1) reads

$$\begin{aligned} &G_0 \times \frac{T_{i-1}^{n+1} - 2T_i^{n+1} + T_{i+1}^{n+1}}{\Delta x^2} + \frac{1}{M(T_i^n)} \times \Big(\frac{1}{Wd} \times \frac{I^{n+1} + I^n}{2}\Big)^2 \rho \\ &+ [G_n(T_i^n) - G_0] \times \frac{T_{i-1}^n - 2T_i^n + T_{i+1}^n}{\Delta x^2} + \frac{F(T_i^n)}{d} \times (T_i^n - T_{sub}) = \frac{T_i^{n+1} - T_i^n}{\tau}. \end{aligned} \tag{2.13}$$

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Similarly, we have for $x = x_i = i\Delta x,\ x \in (l, L/2)$, the difference equation for (1.2) reads

$$G_0 \times \frac{T_{i-1}^{n+1} - 2T_i^{n+1} + T_{i+1}^{n+1}}{\Delta x^2} + [G_s(T_i^n) - G_0] \times \frac{T_{i-1}^n - 2T_i^n + T_{i+1}^n}{\Delta x^2} + \frac{E(T_i^n)}{d} \times (T_i^n - T_{sub}) = \frac{T_i^{n+1} - T_i^n}{\tau}. \tag{2.14}$$

Following the same ideas for construction of the Crank-Nicholson method mentioned above, we can derive the difference equations corresponding to the boundary conditions and the interface condition.

Compared to the Crank-Nicholson method, the present implicit-explicit (IMEX) scheme has an advantage. That is, if the generic constant G_0 is chosen feasibly, we only require to solve a linear system with the same coefficient matrix at different time nodes $t = t_n$. This will reduce the computational cost greatly, in particular, in high-dimensional case.

2.2 Discretization of the current equation

The current equation (1.7) is a second order ordinary differential equation, we rewrite it as a system of first-order equations and then carry out discretization. To this end, let $K(t) = \frac{1}{C_{bt}} \int_0^t (I_{bias} - I(s))ds + I_{bias}Z_0$. Hence, by some direct manipulation, (1.7) is equivalent to

$$K'(t) = \frac{I_{bias} - I(t)}{C_{bt}}, \tag{2.15}$$

$$L_k I'(t) = K(t) - (R(t) + Z_0)I(t), \tag{2.16}$$

where

$$R(t) = 2\rho\frac{l(t)}{S} = 2\rho\frac{l(t)}{Wd}.$$

The corresponding initial conditions are given by

$$I(0) = I_0, \quad K(0) = I_{bias}Z_0.$$

Integrating both sides of the equation (2.15) in the domain $[t_n, t_{n+1}]$ implies

$$\int_{t_n}^{t_{n+1}} K'(t)dt = \int_{t_n}^{t_{n+1}} \frac{I_{bias} - I(t)}{C_{bt}}dt,$$

and using the trapezoid method for numerical integration to the right side term we further have

$$K_{n+1} = K_n + \frac{\tau}{2C_{bt}}(2I_{bias} - I_n - I_{n+1}), \tag{2.17}$$

Similarly, integrating both sides of the equation (2.16) in the domain $[t_n, t_{n+1}]$, we have

$$\int_{t_n}^{t_{n+1}} L_k I'(t)dt = \int_{t_n}^{t_{n+1}} (K(t) - (R(t) + Z_0)I(t))dt,$$

which, in conjunction with the trapezoid method, implies

$$\Big(L_k + \frac{\tau}{2}R_{n+1} + \frac{\tau}{2}Z_0\Big)I_{n+1} = \Big(L_k - \frac{\tau}{2}R_n - \frac{\tau}{2}Z_0\Big)I_n + \frac{\tau}{2}K_n + \frac{\tau}{2}K_{n+1}. \tag{2.18}$$

where $R_n = R_n(t)$.

Making use of (2.17) and (2.18) immediately gives

$$aI_{n+1} = bI_n + \tau K_n + \frac{\tau^2}{2C_{bt}} I_{bias}, \tag{2.19}$$

where

$$a = L_k + \frac{\tau^2}{4C_{bt}} + \frac{\tau}{2}R_{n+1} + \frac{\tau}{2}Z_0, \quad b = L_k - \frac{\tau^2}{4C_{bt}} - \frac{\tau}{2}R_n - \frac{\tau}{2}Z_0,$$
$$R_n = 2\rho\frac{l_n}{Wd}, \quad R_{n+1} = 2\rho\frac{l_{n+1}}{Wd}.$$

We remark that in the real applications, the interface positions l_n and l_{n+1} at $t = t_n$ and $t = t_{n+1}$ should be replaced by their approximate values l_0^n and l_0^{n+1}, respectively. Therefore, it is clear that we can get the current I_{n+1} whenever the unknowns at the time $t = t_n$ and the interface position $x = l_0^{n+1}$ are available.

Observing that for the superconducting nanowire single-photon detector described in Figure 1, while the current drops below critical current and the resistive region subsides, the wire becomes fully superconducting again, the bias current through the wire returns to the original value. Thus, the time-dependent hotspot resistance $R_n(t) = 0$, and the current equation (1.7) becomes

$$\frac{d^2I(t)}{dt^2} + a_1\frac{dI(t)}{dt} + a_2I(t) = a_3, \tag{2.20}$$

where $a_1 = \frac{Z_0}{L_k}$, $a_2 = \frac{1}{C_{bt}L_k}$, $a_3 = \frac{I_{bias}}{C_{bt}L_k}$.

Since the equation (2.20) is a constant second order inhomogeneous linear equation, we can easily derive its closed form of the solution:

$$I(t) = c_1e^{\lambda_1 t} + c_2e^{\lambda_2 t} + \frac{a_3}{a_2}. \tag{2.21}$$

If the initial conditions are given by $I(\hat{t}) = a_4$ and $I'(\hat{t}) = a_5$, then we know by a direct manipulation that the undetermined coefficients in (2.21) are

$$\lambda_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_2}}{2}, \quad \lambda_2 = \frac{-a_1 - \sqrt{a_1^2 - 4a_2}}{2},$$
$$c_1 = \frac{a_4\lambda_2 - a_5 - \frac{a_3}{a_2}\lambda_2}{(\lambda_2 - \lambda_1)e^{\lambda_1\hat{t}}}, \quad c_2 = \frac{a_4\lambda_1 - a_5 - \frac{a_3}{a_2}\lambda_1}{(\lambda_1 - \lambda_2)e^{\lambda_2\hat{t}}}.$$

Therefore, if the nanowire returns to superconducting state again, we are able to get the current from the expression (2.21) explicitly, instead of the numerical solution. This will increase the computational efficiency greatly.

3 Determination of the interface position

Similar to the standard numerical method for solving evolutionary equations, we will conduct numerical simulation for the electro-thermal model along the time direction. That means, once the numerical results at $t = t_n$ are obtained, we will try to get the numerical results at $t = t_{n+1}$. From our discussion given in the above section, we easily know the key difficulty is to derive the interface position at this instant.

Our key points to overcome the above obstacle are as follows. First of all, we make the partition of the region $[0, L/2]$ fine enough, i.e. Δx is taken small enough, so that we

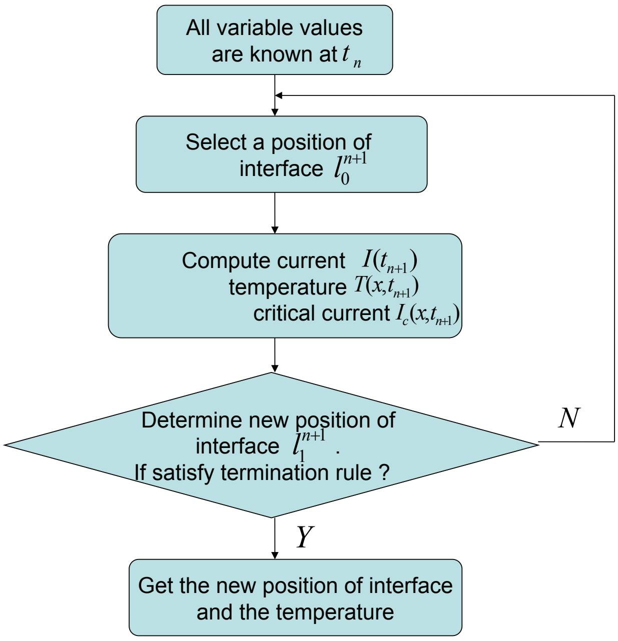

Figure 2: The flow chart of the iterative algorithm to form the interface position at different time nodes.

can assume that the interface positions always lie in the spatial grid points, approximately with desired accuracy. Next, we will use the shooting method to determine the interface position at $t = t_{n+1}$, in view of the idea of the shooting method (cf. [11]) combined with the phase transition condition (1.5). To be more precise, we choose $x = \tilde{l} = l^n$ as the initial guess of the interface position at $t = t_{n+1}$. Then, by means of the finite difference methods in section 2, we can derive the approximate temperature values T_i^{n+1} at all grid points as well as the approximate current I_{n+1}, and compute the critical current $I_c(T)$ at $t = t_{n+1}$ in view of (1.5) at all grid points. If the initial guess $x = \tilde{l}$ satisfies that the numerical current I_{n+1} is greater than the critical current at the left point of $x = \tilde{l}$, and less at the right side point, then we take $l_0^{n+1} = \tilde{l}$. Otherwise, we try to find a grid $x = \tilde{l}_1$ such that the numerical current I_{n+1} is greater than the critical current at the left point of $x = \tilde{l}$, and less at the right side point. And then replace the guess interface position $\tilde{l}$ by $\tilde{l}_1$. Repeat the above computation process until it converges, and choose the final result $\tilde{l}$ as l_0^{n+1}. In all the calculations presented in the following section, the termination rule is taken as $|\tilde{l}_1 - \tilde{l}| < tol$, with $tol = 1 \times 10^{-6}$.

For preciseness, the above algorithm is shown in a flow chart, described in Figure 2.

4 Numerical results

In this section, we give some numerical experiments to illustrate the performance and accuracy of our method introduced in sections 2 and 3, from which we can observe the evolution of interface positions, namely the growth of the normal region along the wire, the change in current through the wire, the change of resistance along the wire after a

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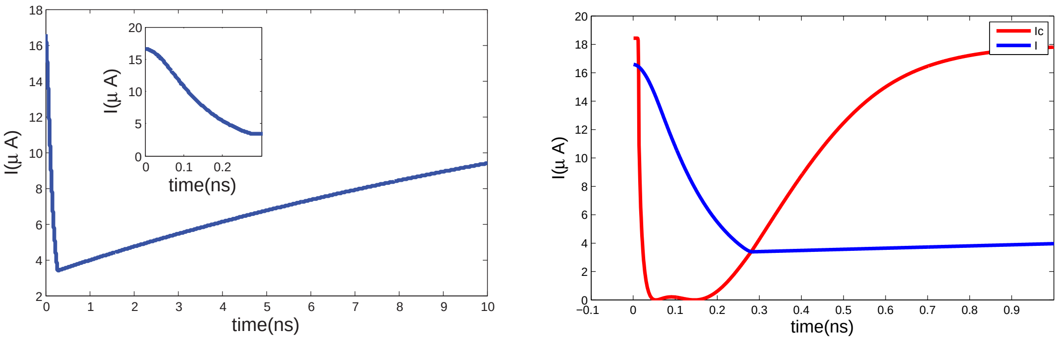

(a) (b)

Figure 3: The current variation in the electro-thermal model.(a) The calculated current through the wire vs. time. (b) The calculated current and critical current at $x = 100nm$ vs time.

small resistive hotspot is formed. Here the photon-induced resistive barrier forms at $t = 0$. Most of the physical parameters are taken from the monograph [13] about the theory of superconductivity, and the other ones are taken from the related literature. In particular, since the hotspot only forms and exists for several nanoseconds (cf. [1]), we choose in our numerical simulation the terminal time to be $t_{end} = 10ns$. Then we choose the stepsize in t to be $\tau = 1ps$, where $1ps = 10^{-3}ns$. If τ is taken a little larger, say $\tau = 5ps$, our algorithms will not converge.

In the normal/resistive state, the electrical resistivity $\rho = 2.4 \times 10^{-6}\Omega m$. According to the Wiedemann-Franz law, the ratio of the electronic contribution of the thermal conductivity κ_n to the electrical conductivity ρ of a metal, is proportional to the temperature T ($\kappa_n = \mathcal{L}\frac{T}{\rho}$, where $\mathcal{L} = 2.45 \times 10^{-8}W\Omega/K^2$ is the Lorenz number). The heat capacity (per unit volume) of the superconducting film C_n includes electron specific C_{en} and phonon specific heat C_{pn}, where C_{en} is proportional to the temperature T ($C_{en} = \gamma T$, where $\gamma = 240$), and C_{pn} is proportional to T^3 such that $C_{pn} = 9.8T^3$ (cf. [8]). The thermal boundary conductivity α between NbN and sapphire we used is obtained from [15], and we only considered its cubic dependence on temperature ($\alpha = BT^3$, where $B = 800$).

In the region of superconducting state, ρ is taken to be zero naturally. We express the thermal conductivity as $\kappa_s = \mathcal{L}\frac{T^2}{\rho T_c}$ (cf. [14]), where $T_c = 10K$ is the critical temperature. The heat capacity C_s also include two parts, the electron specific was calculated such that $C_{es} = Ae^{-\frac{3.5T_c}{T}}$ with $A = 1.93 \times 10^5$ (cf. [13]), and the phonon specific heat is state independent such that $C_{ps} = 9.8T^3$. The thermal boundary conductivity is given as $\alpha = BT^3$.

Some more data used in all the computations are given as follows. The length of superconducting nanowire $L = 2000nm$, the wire thickness $d = 4nm$, the width of nanowire $W = 100nm$, the substrate temperature $T_{sub} = 2K$, the initial critical current $I_c(0) = 20\mu A$, the capacitor $C_{bt} = 20 \times 10^{-9}F$, the kinetic inductance of the superconducting nanowire $L_k = 807.7nH$, the impedance of the transmission line connecting the probe to RF amplifiers $Z_0 = 50\Omega$, the current of the SNSPD $I_{bias} = 16.589\mu A$, the initial interface position $l_0 = 15nm$, the initial temperature $T_0 = 5K$ where the wire is normal and $T_0 = T_{sub} = 2K$ where the wire is superconducting.

For the Crank-Nicholson method and the IMEX method, we take $N = 1000$, so the space step is $\Delta x = L/2N = 1nm$. The time step is $\tau = 1ps$, as introduced at the beginning

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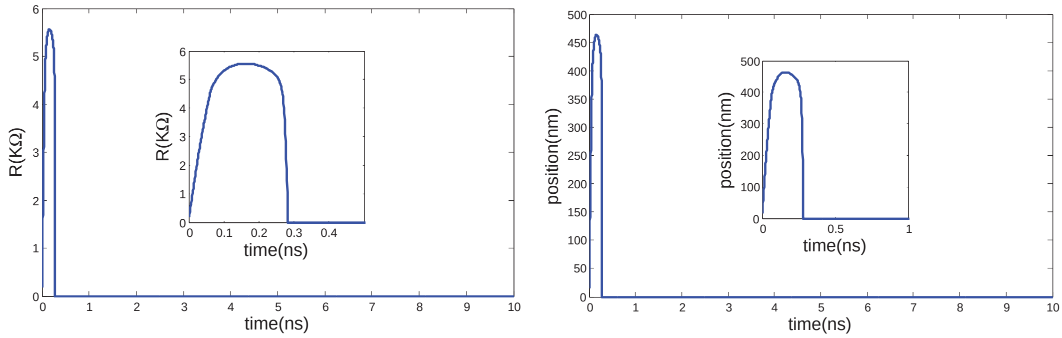
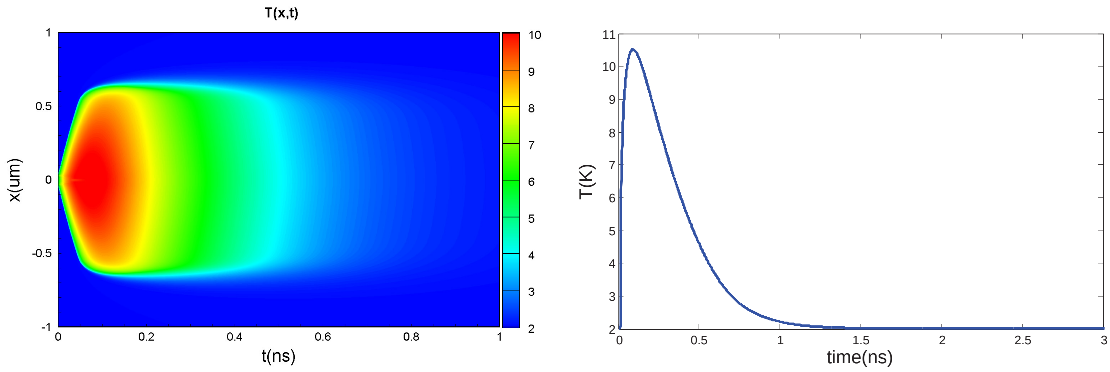


(a) (b)

Figure 4: The resistance variation in the electro-thermal model. (a) The calculated total normal state resistance vs. time, and the inset shows in greater detail the change of the resistance. (b) The interface position vs. time, and the inset shows in greater detail the change of the position.



(a) (b)

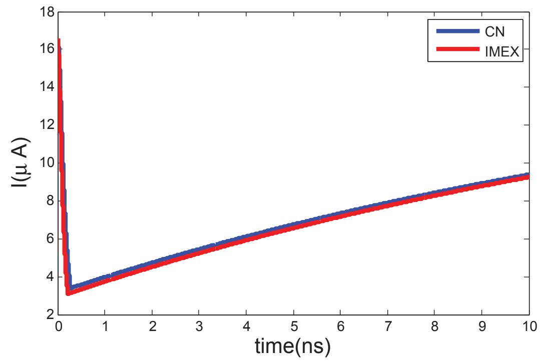
Figure 5: The temperature variation in the electro-thermal model. (a) The calculated temperature (shown using colors) at different positions along the wire and in time. (b) The calculated temperature history at $x = 100nm$.

of this section. Furthermore, for the IMEX method, we choose the parameter G_0 in view of the formulation (2.10) to get $G_0 = 2 * 10^{-5}$.

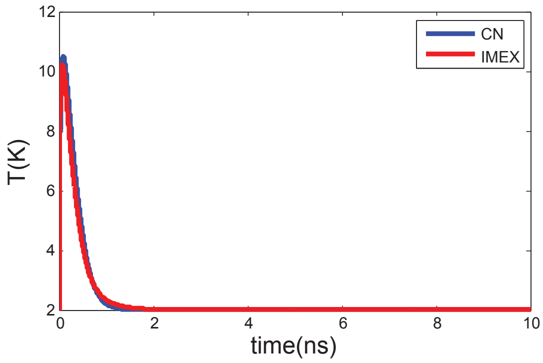
We first use the Crank-Nicholson method for solving the thermal equations, combined with the numerical method for solving the current equation and the algorithm in section 3 to search for interface positions, to implement numerical simulation.

It is shown in Figure 3(a) the calculated current through the SNSPD. We find the curve first forms a sharp decline within a short time period. Afterwards, the nanowire under consideration switches to superconducting state, and the calculated current increases at an exponential rate. It is shown in Figure 3(b) the calculated current and the critical current at $x = 100nm$ vs. time.

It is shown in Figure 4(a) the calculated total normal state resistance along the wire. It appears that the resistance increases gradually and then decreases to 0 sharply, and the inset shows in greater detail the change of the resistance. It is shown in Figure 4(b) the evolution of the interface position in the electro-thermal model. The initial interface position is taken as $l_0 = 15nm$. The interface point moves forward to about the position $x = 463nm$ at $t = 170ps$, where the resistance increases to a maximum value. Then the



(a)



(b)

Figure 6: Some numerical comparison between the Crank-Nicholson finite difference method and the IMEX method. (a) The calculated current through the wire vs. time. (b) The calculated temperature history at $x = 100nm$.

interface point returns gradually to the central position with $x = 0$, and the nanowire becomes superconducting state.

It is shown in Figure 5(a) the calculated temperature (shown using colors) at different positions along the wire and in time, the temperature at each segment show the segment under consideration switches into the normal state or remains superconducting. And it is shown in Figure 5(b) the calculated temperature history at $x = 100nm$. At that position, the initial temperature is $T = 5K$ where the wire is normal, then it increases to a maximum value of about $10.7K$, after that the temperature gradually returns to $2K$ and the position lies in superconducting state.

All the numerical results given above coincide with the physical phenomenon observed by experiments (cf. [15]).

We also compare the numerical results with the thermal equations numerically solved by the Crank-Nicholson method and the IMEX method, respectively. We observe from the numerical data in Figure 6 that the two methods which perform in the similar manners, can produce very similar numerical results.

5 Conclusions

In this paper, we propose two algorithms for numerically solving the electro-thermal model for Superconducting Nanowire Single-Photon Detectors. Such a model is governed by a nonlinear free boundary problem involving the temperature and the current, which are coupled together by a nonlinear parabolic interface equation and a second order ordinary differential equation (see the equations (1.1)-(1.2) and (1.7) for details). In our numerical experiments, for a fixed spatial size Δx, only if the stepsize in time τ is taken small enough, our numerical methods are convergent. Therefore, we only develop in this paper some initial but interesting results for the coupled system of equations (1.1)-(1.2) and (1.7). Due to the complexity of the model, it is very challenging to establish mathematical theory for this model and discuss convergence analysis of the methods proposed in this paper.

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On the Existence of Meromorphic Solutions of Some Nonlinear Differential-Difference Equations

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Abstract: In this paper, we investigate the conditions concerning the existence or non-existence of transcendental meromorphic or entire solutions of some kinds of differential-difference equations. We also give examples to illustrate the sharpness of our results.

Key words: Differential-difference equation, meromorphic solution, entire solution.

AMS Classification No. 39B32, 34M05, 30D35

1 Introduction and main results

Throughout this paper, we assume that $f(z)$ is a meromorphic function in the whole complex plane, and use standard notations, such as $m(r,f)$, $T(r,f)$, $N(r,f)$, in the Nevanlinna theory (see e.g. [3, 7, 8, 17]). And we also use $\sigma(f)$ and $\sigma_2(f)$ to denote respectively the order and the hyper order of $f(z)$. Moreover, we say that a meromorphic function $g(z)$ is small with respect to $f(z)$, if $T(r,g) = S(r,f)$, where $S(r,f)$ means any real quantity satisfying $S(r,f) = o(T(r,f))$ as $r \to \infty$ outside of a possible exceptional set of finite logarithmic measure.

Recently, with some establishments of difference analogues of the classic Nevanlinna theory (two typical and most important ones can be seen in [2, 4–6]), there has been a renewed interest in the properties of complex difference expressions and meromorphic solutions of complex difference equations (see e.g. [10–12, 18]). Further, Yang-Laine gave analogies between nonlinear difference and differential equations in [15] . From then on, some results concerning nonlinear differential-difference equations were found (see e.g. [13]).

In what follows, we use the defintion of the differential-difference polynomial in [15, 19]. A differential-difference polynomial is a polynomial in $f(z)$, its shifts, its derivatives and derivatives of its shifts, that is, an expression of the form

$$P(z,f) = \sum_{\lambda \in I} a_\lambda(z) f(z)^{\lambda_{0,0}} f'(z)^{\lambda_{0,1}} \cdots f^{(k)}(z)^{\lambda_{0,k}}$$
$$\cdot f(z+c_1)^{\lambda_{1,0}} f'(z+c_1)^{\lambda_{1,1}} \cdots f^{(k)}(z+c_1)^{\lambda_{1,k}} \cdots$$
$$\cdot f(z+c_l)^{\lambda_{l,0}} f'(z+c_l)^{\lambda_{l,1}} \cdots f^{(k)}(z+c_l)^{\lambda_{l,k}}$$

*Corresponding author.

$$= \sum_{\lambda \in I} a_\lambda(z) \prod_{i=0}^{l} \prod_{j=0}^{k} f^{(j)}(z+c_i)^{\lambda_{i,j}}, \tag{1.1}$$

where I is a finite set of multi-indices $\lambda = (\lambda_{0,0}, \cdots, \lambda_{0,k}, \lambda_{1,0}, \cdots, \lambda_{1,k}, \cdots, \lambda_{l,0}, \cdots, \lambda_{l,k})$, and $c_0(=0)$, $c_1, \cdots, c_l$ are distinct complex constants. And we assume that the meromorphic coefficients $a_\lambda(z), \lambda \in I$ of $P(z,f)$ are of growth $S(r,f)$. We denote the degree and the weight of the monomial $\prod\limits_{i=0}^{l} \prod\limits_{j=0}^{k} f^{(j)}(z+c_i)^{\lambda_{i,j}}$ of $P(z,f)$ respectively by

$$d(\lambda) = \sum_{i=0}^{l} \sum_{j=0}^{k} \lambda_{i,j} \quad \text{and} \quad w(\lambda) = \sum_{i=0}^{l} \sum_{j=0}^{k} (j+1)\lambda_{i,j}.$$

Then we denote the degree and the weight of $P(z,f)$ respectively by

$$d(P) = \max_{\lambda \in I}\{d(\lambda)\} \quad \text{and} \quad w(P) = \max_{\lambda \in I}\{w(\lambda)\}.$$

In the following, we assume $d(P) \geq 1$.

We recall the following result due to Wang-Li [13] by rewriting the original differential-difference polynomial in [13] as the one of the form (1.1).

Theorem A. Suppose that a nonlinear differential-difference equation is

$$f^n(z) + P(z,f) = p(z), \tag{1.2}$$

where $n \in \mathbb{N}$, $p(z)$ is a polynomial, and $P(z,f)$ is a differential-difference polynomial of the form (1.1) with polynomial coefficients. If

$$n > (s+1)d(P) - \sum_{\lambda \in I} d(\lambda), \tag{1.3}$$

where s is the number of components of I, then the equation (1.2) has no transcendental entire solutions of finite order.

Remark 1.1 Obviously, (1.3) results in

$$n > (s+1)d(P) - \sum_{\lambda \in I} d(\lambda) \geq (s+1)d(P) - sd(P) = d(P) \geq 1.$$

Then, our first main purpose is to improve Theorem A. On the one hand, we improve the restrict on n by introducing an important lemma of our own. On the other hand, we also consider the non-existence of meromorphic solutions of the equation (1.2). Our result is as follows.

Theorem 1.1 Consider the nonlinear differential-difference equation

$$f^n(z) + P(z,f) = c(z), \quad n \in \mathbb{N}, \tag{1.4}$$

where $P(z,f)$ is a differential-difference polynomial of the form (1.1) with meromorphic coefficients $a_\lambda(z), \lambda \in I$, and $c(z)$ is a meromorphic functions.

(i) If $n > d(P)$, then the equation (1.4) has no admissible transcendental entire solutions with hyper order less than 1.

(ii) If $n > w(P)$, then the equation (1.4) has no admissible transcendental meromorphic solutions with hyper order less than 1.

Remark 1.2 Here, a meromorphic or entire solution $f(z)$ of the equation (1.4) is called admissible, if $a_\lambda(z), \lambda \in I$ and $c(z)$ are small with respect to $f(z)$, that is, $T(r, a_\lambda) = S(r, f), \lambda \in I$ and $T(r, c) = S(r, f)$.

Wang-Li also investigated another kind of nonlinear differential-difference equation in [13] as follows.

Theorem B For two integers $n \geq 3$, $k > 0$ and a nonlinear differential-difference equation

$$f^n(z) + q(z)f^{(k)}(z+t) = ae^{ibz} + de^{-ibz}, \tag{1.5}$$

where $q(z)$ is a polynomial and t, a, b, d are complex numbers such that $|a| + |d| \neq 0$, $bt \neq 0$.

(i) Let $n = 3$. If $q(z)$ is nonconstant, then the equation (1.5) does not admit entire solutions of finite order. If $q = q(z)$ is constant, then the equation (1.5) admits three distinct transcendental entire solutions of finite order, provided that

$$bt = 3m\pi \ (m \neq 0, \ \text{if } q \neq 0), \quad q^3 = (-1)^{m+1}(\frac{3i}{b})^{3k} 27ad,$$

when k is even, or

$$bt = \frac{3\pi}{2} + 3m\pi \ (\text{if } q \neq 0), \quad q^3 = i(-1)^m(\frac{3i}{b})^{3k} 27ad,$$

when k is odd, for an integer m.

(ii) Let $n > 3$. If $ad \neq 0$, then the equation (1.5) does not admit entire solutions of finite order. If $ad = 0$, then the equation (1.5) admits n distinct transcentental entire solutions of finite order, provided that $q = q(z) \equiv 0$.

Moreover, they proposed a question in [13]: for the differential-difference equation of the form

$$f^n(z) + L(z, f) = ae^{ibz} + de^{-ibz}, \quad n \geq 3,$$

where $L(z, f)$ is some linear differential-difference polynomial of $f(z)$ with polynomial coefficients, what can we say considering Theorem B.

Then, our second main purpose is to give the following results, which answer the above question to some extent.

Theorem 1.2 Consider the nonlinear differential-difference equation

$$f^n(z) + \sum_{s=0}^{l}\sum_{t=0}^{k} A_{s,t}(z)f^{(t)}(z+c_s) = ae^{ibz} + de^{-ibz}, \quad n \in \mathbb{N}, n \geq 3, \tag{1.6}$$

where $c_0(= 0)$, $c_1, \cdots, c_l$ are distinct complex constants, $A_{s,t}(z), s = 0, 1, \cdots, l, t = 0, 1, \cdots, k$ are polynomials, and $a, b, d \in \mathbb{C}$ such that $b \neq 0$ and $|a| + |d| \neq 0$.

(i) Let $n = 3$. If

$$ad \neq 0 \quad \text{and} \quad \sum_{s=0}^{l}\sum_{t=0}^{k} A_{s,t}(z)\Big(e^{\frac{ibc_s}{3}}(\frac{ib}{3})^t - e^{\frac{-ibc_s}{3}}(\frac{-ib}{3})^t\Big) \not\equiv 0,$$

then the equation (1.6) has no transcendental entire solutions of finite order. If

$$\sum_{s=0}^{l}\sum_{t=0}^{k} A_{s,t}(z)e^{\frac{ibc_s}{3}}(\frac{ib}{3})^t \equiv 0 = d, \quad \text{or} \quad \sum_{s=0}^{l}\sum_{t=0}^{k} A_{s,t}(z)e^{\frac{-ibc_s}{3}}(\frac{-ib}{3})^t \equiv 0 = a,$$

or

$$\sum_{s=0}^{l}\sum_{t=0}^{k}A_{s,t}(z)e^{\frac{ibc_s}{3}}(\frac{ib}{3})^t \equiv \sum_{s=0}^{l}\sum_{t=0}^{k}A_{s,t}(z)e^{\frac{-ibc_s}{3}}(\frac{-ib}{3})^t \equiv -3d_1d_2, \quad d_1^3=a, d_2^3=d,$$

then the equation (1.6) has three transcendental entire solutions of finite order.

(ii) Let $n>3$. If $ad\neq 0$, or

$$ad=0 \quad \text{and} \quad \sum_{s=0}^{l}\sum_{t=0}^{k}A_{s,t}(z)e^{\frac{ibc_s}{n}}(\frac{ib}{n})^t \not\equiv 0,$$

then the equation (1.6) has no transcendental entire solutions of finite order. If

$$ad=0 \quad \text{and} \quad \sum_{s=0}^{l}\sum_{t=0}^{k}A_{s,t}(z)e^{\frac{ibc_s}{n}}(\frac{ib}{n})^t \equiv 0,$$

then the equation (1.6) has n transcendental entire solutions of finite order.

In particular, we obtain more concrete results for a special linear difference polynomial $L(z,f)$ as follows.

Theorem 1.3 Consider the nonlinear difference equation

$$f^n(z)+q(z)\triangle^m f(z)=ae^{ibz}+de^{-ibz}, \quad n,m\in\mathbb{N}, n\geq 3, \tag{1.7}$$

where $q(z)$ is a polynomial, $a,b,d\in\mathbb{C}$ such that $b\neq 0$ and $|a|+|d|\neq 0$.

(i) Let $n=3$. If $ad\neq 0$ and $q(z)$ is a nonconstant, then the equation (1.7) has no transcendental entire solutions of finite order. If $ad\neq 0$ and $q(z)$ is a constant q, then the equation (1.7) has three transcendental entire solutions of the form $f(z)=d_1e^{\frac{ibz}{3}}+d_2e^{\frac{-ibz}{3}}, d_1^3=a, d_2^3=d$, provided that

$$bc=6k\pi+3\pi+\frac{6s\pi}{m}\ (k\in\mathbb{Z}, s\in\{0,1,\cdots,m-1\}) \quad \text{and} \quad q^3=(-1)^{m+1}\frac{27ad}{(e^{\frac{2s\pi i}{m}}+1)^{3m}}.$$

If $ad=0$, then the equation (1.7) has three transcendental entire solutions of the form $f(z)=d_1e^{\frac{ibz}{3}}+d_2e^{\frac{-ibz}{3}}, d_1^3=a, d_2^3=d$, provided that $q(z)\equiv 0$ or $bc=6k\pi$, $k\in\mathbb{Z}$.

(ii) Let $n>3$. If $ad\neq 0$, then the equation (1.7) has no transcendental entire solutions of finite order. If $ad=0$, then the equation (1.7) has n transcendental entire solutions of the form $f(z)=d_1e^{\frac{ibz}{n}}+d_2e^{\frac{-ibz}{n}}, d_1^n=a, d_2^n=d$, provided that $q(z)\equiv 0$ or $bc=2kn\pi$, $k\in\mathbb{Z}$.

Remark 1.3 Here, the forward difference $\triangle^m f(z)$ for $m\in\mathbb{N}$ and $c\in\mathbb{C}\backslash\{0\}$ is defined in the standard way [14, p. 52] by

$$\triangle f(z)=f(z+c)-f(z),\ \triangle^m f(z)=\triangle(\triangle^{m-1}f(z))=\triangle^{m-1}f(z+c)-\triangle^{m-1}f(z),\ m\geq 2.$$

And it is shown as in [2] that

$$\triangle^m f(z)=\sum_{j=0}^{m}C_m^j(-1)^{m-j}f(z+jc), \quad f(z+mc)=\sum_{j=0}^{m}C_m^j\triangle^j f(z),$$

where $C_m^j, j=0,1,\cdots,m$ are the binomial coefficients.

2 Lemmas

Lemma 2.1. **([19])** Let $f(z)$ be a transcendental meromorphic function of $\sigma_2(f) < 1$, and $P(z,f)$ be a differential-difference polynomial of the form (1.1), then we have

$$m(r,P(z,f)) \leq d(P)m(r,f) + S(r,f).$$

Furthermore, if $f(z)$ also satisfies $N(r,f) = S(r,f)$, then we have

$$T(r,P(z,f)) \leq d(P)T(r,f) + S(r,f).$$

Lemma 2.2. **([6])** Let $T : [0,+\infty) \to [0,+\infty)$ be a non-decreasing continuous function and let $s \in (0,+\infty)$. If the hyper order of T is strictly less than one, i.e. $\overline{\lim\limits_{r\to\infty}} \frac{\log_2 T(r)}{\log r} = \zeta < 1$, and $\delta \in (0, 1-\zeta)$, then

$$T(r+s) = T(r) + o(\frac{T(r)}{r^\delta}),$$

where r runs to infinity outside of a set of finite logarithmic measure.

It is shown in [3, p.66] and [1, Lemma 1] that the inequality

$$(1+o(1))T(r-|c|,f) \leq T(r,f(z+c)) \leq (1+o(1))T(r+|c|,f)$$

holds for $c \neq 0$ and $r \to \infty$. And from its proof, the above relation is also true for counting function. By combining Lemma 2.2 and these inequalities, we immediately deduce the following lemma.

Lemma 2.3. Let $f(z)$ be a nonconstant meromorphic function of $\sigma_2(f) < 1$, and c be a nonzero complex constant. Then we have

$$T(r,f(z+c)) = T(r,f) + S(r,f),$$

$$N(r,f(z+c)) = N(r,f) + S(r,f), \quad N(r,\frac{1}{f(z+c)}) = N(r,\frac{1}{f}) + S(r,f).$$

Laine-Yang [9] gave a difference analogue of Clunie lemma as follows.

Lemma 2.4. **([9])** Let $f(z)$ be a transcendental finite order meromorphic solution of

$$U(z,f)P(z,f) = Q(z,f),$$

where $U(z,f), P(z,f), Q(z,f)$ are difference polynomials in $f(z)$ with small meromorphic coefficients, $\deg_f U = n$ and $\deg_f Q \leq n$. Moreover, we assume that $U(z,f)$ contains just one term of maximal total degree. Then

$$m(r,P(z,f)) = S(r,f).$$

Remark 2.1. Yang-Laine [15] also pointed out that Lemma 2.4 is also true if $P(z,f)$, $Q(z,f)$ are differential-difference polynomials in $f(z)$. Further, by a careful inspection of the proof of Lemma 2.4, we see that the same conclusion holds for the differential-difference case, if the coefficients $b_\mu(z)$ of $P(z,f), Q(z,f)$ satisfy $m(r,b_\mu) = S(r,f)$ instead of $T(r,b_\mu) = S(r,f)$.

Lemma 2.5. **([16])** Suppose that c is a nonzero complex constant, $\alpha(z)$ is a nonconstant meromorphic function. Then the differential equation

$$f^2(z) + (cf^{(n)}(z))^2 = \alpha(z)$$

has no transcendental meromorphic solutions satisfying $T(r,\alpha) = S(r,f)$.

3 Proofs of Theorems 1.1-1.3

Proof of Theorem 1.1. (i) Let $f(z)$ be an admissible transcendental entire solution of (1.4) with $\sigma_2(f) < 1$. By Lemma 2.1, we see that

$$m(r, P(z,f)) \le d(P)m(r,f) + S(r,f). \tag{3.1}$$

By (1.4) and (3.1), we obtain that

$$nT(r,f) = T(r,P(z,f)) + S(r,f) = m(r,P(z,f)) + S(r,f) \le d(P)T(r,f) + S(r,f). \tag{3.2}$$

Since $n > d(P)$, (3.2) is a contradiction. Thus, (1.4) has no admissible transcendental entire solutions with hyper order less than 1.

(ii) Let $f(z)$ be an admissible transcendental meromorphic solution of (1.4) with $\sigma_2(f) < 1$. We consider each pole of $P(z,f)$. Since each pole of $P(z,f)$ in $|z| < r$ comes from the poles of $f(z+c_i), i = 0, \cdots, l$ and $a_\lambda(z), \lambda \in I$ in $|z| < r$, and each pole of $f(z+c_i)$ with multiplicity p_i is a pole of $P(z,f)$ with multiplicity at most

$$p_i\lambda_{i,0}+(p_i+1)\lambda_{i,1}+\cdots+(p_i+k)\lambda_{i,k} \le p_i(\lambda_{i,0}+2\lambda_{i,1}+\cdots+(k+1)\lambda_{i,k}) = p_i\sum_{j=0}^{k}(j+1)\lambda_{i,j},$$

we have by Lemma 2.3 that

$$\begin{aligned} N(r,P(z,f)) &\le \max_{\lambda\in I}\{\sum_{i=0}^{l}\sum_{j=0}^{k}(j+1)\lambda_{i,j}N(r,f(z+c_i))\} + S(r,f) \\ &= \max_{\lambda\in I}\{\sum_{i=0}^{l}\sum_{j=0}^{k}(j+1)\lambda_{i,j}N(r,f)\} + S(r,f) \\ &= \max_{\lambda\in I} w(\lambda)N(r,f) + S(r,f) = w(P)N(r,f) + S(r,f). \end{aligned} \tag{3.3}$$

Clearly, (3.1) holds by Lemma 2.1 again. By (1.4), (3.1) and (3.3), we obtain that

$$\begin{aligned} nT(r,f) &= T(r,P(z,f)) + S(r,f) = m(r,P(z,f)) + N(r,P(z,f)) + S(r,f) \\ &\le d(P)m(r,f) + w(P)N(r,f) + S(r,f) \le w(P)T(r,f) + S(r,f). \end{aligned} \tag{3.4}$$

Since $n > w(P)$, (3.4) is a contradiction. Thus, (1.4) has no admissible transcendental meromorphic solutions with hyper order less than 1.□

Proof of Theorem 1.3. Suppose that $f(z)$ is a transcentental entire solution of finite order of (1.7). Differentiating (1.7), we have

$$\begin{aligned} nf^{n-1}(z)f'(z) + q'(z)\sum_{j=0}^{m}C_m^j(-1)^{m-j}f(z+jc) + q(z)\sum_{j=0}^{m}C_m^j(-1)^{m-j}f'(z+jc) \\ = ib(ae^{ibz} - de^{-ibz}). \end{aligned} \tag{3.5}$$

By combining (1.7) and (3.5), we have

$$\Big(nf^{n-1}(z)f'(z) + q'(z)\sum_{j=0}^{m}C_m^j(-1)^{m-j}f(z+jc) + q(z)\sum_{j=0}^{m}C_m^j(-1)^{m-j}f'(z+jc)\Big)^2$$

$$+b^2\Big(f^n(z)+q(z)\sum_{j=0}^{m}C_m^j(-1)^{m-j}f(z+jc)\Big)^2=4adb^2,$$

consequently,

$$f^{2n-2}(z)(b^2f^2(z)+n^2f'^2(z))=Q(z,f), \tag{3.6}$$

where $Q(z,f)$ is a differential-difference polynomial of $f(z)$ with the total degree at most $n+1$.

If $b^2f^2(z)+n^2f'^2(z)\equiv 0$, we differentiate it and obtain that

$$n^2f''(z)+b^2f(z)=0, \tag{3.7}$$

which implies the solution must be

$$f(z)=d_1e^{\frac{ibz}{n}}+d_2e^{\frac{-ibz}{n}}, \tag{3.8}$$

where d_1 and d_2 are arbitrary complex constants. If $b^2f^2(z)+n^2f'^2(z)\not\equiv 0$, we may apply Lemma 2.4 and Remark 2.1 to (3.6) and obtain that

$$T(r,b^2f^2+n^2f'^2)=m(r,b^2f^2+n^2f'^2)=S(r,f).$$

Thus, by Lemma 2.5, we see that $b^2f^2(z)+n^2f'^2(z)$ must be a constant M. Differentiating $b^2f^2(z)+n^2f'^2(z)=M$, we obtain (3.7) and (3.8) again.

Substituting (3.8) into (1.7) and denoting $w=w(z)=e^{\frac{ibz}{n}}$, we obtain that

$$d_1^nw^{2n}+C_n^1d_1^{n-1}d_2w^{2n-2}+C_n^2d_1^{n-2}d_2^2w^{2n-4}+\cdots+C_n^{n-2}d_1^2d_2^{n-2}w^4+C_n^{n-1}d_1d_2^{n-1}w^2+d_2^n$$

$$+d_1q(z)\sum_{j=0}^{m}C_m^j(-1)^{m-j}e^{\frac{ijbc}{n}}w^{n+1}+d_2q(z)\sum_{j=0}^{m}C_m^j(-1)^{m-j}e^{\frac{-ijbc}{n}}w^{n-1}=aw^{2n}+d. \tag{3.9}$$

(i) Let $n=3$, then (3.9) reduces into

$$a_6w^6+a_4w^4+a_2w^2+a_0=0,$$

where

$$\begin{cases} a_6=d_1^3-a, \\ a_4=3d_1^2d_2+d_1q(z)\sum\limits_{j=0}^{m}C_m^j(-1)^{m-j}e^{\frac{ijbc}{3}}=3d_1^2d_2+d_1q(z)(e^{\frac{ibc}{3}}-1)^m, \\ a_2=3d_1d_2^2+d_2q(z)\sum\limits_{j=0}^{m}C_m^j(-1)^{m-j}e^{\frac{-ijbc}{3}}=3d_1d_2^2+d_2q(z)(e^{\frac{-ibc}{3}}-1)^m, \\ a_0=d_2^3-d. \end{cases}$$

Since $w(z)$ is transcendental, we have

$$a_6=a_4=a_2=a_0=0.$$

If $ad\neq 0$, then $d_1^3=a\neq 0$, $d_2^3=d\neq 0$. It follows from $a_4=a_2=0$ that

$$3d_1d_2+q(z)(e^{\frac{ibc}{3}}-1)^m=3d_1d_2+q(z)(e^{\frac{-ibc}{3}}-1)^m=0. \tag{3.10}$$

If $q(z)$ is a nonconstant, then (3.10) results in $e^{\frac{ibc}{3}}-1=e^{\frac{-ibc}{3}}-1=0$, which implies a contradiction that $d_1=d_2=0$. Thus, (1.7) has no transcendental entire solutions of finite order for this case. If $q(z)$ is a constant q, then (3.10) results in $(e^{\frac{ibc}{3}}-1)^m=(e^{\frac{-ibc}{3}}-1)^m$. Denoting $v=e^{\frac{ibc}{3}}$, we have $(v-1)^m=(\frac{1}{v}-1)^m$, consequently,

$$v-1=u_s(\frac{1}{v}-1),\quad s=0,\cdots,m-1,$$

where $u_s=e^{\frac{2s\pi i}{m}}=\varepsilon^s,\varepsilon=e^{\frac{2\pi i}{m}},s=0,\cdots,m-1$. If $s=0$ (that is, $u_0=1$), then $v^2=e^{\frac{2ibc}{3}}=1$, that is, $bc=3k\pi$, where $k\in\mathbb{Z}$. Substituting it into (3.10), we deduce that

$$3d_1d_2+q((-1)^k-1)^m=0.$$

Then k is odd, and $q^3=(-1)^{m+1}\frac{27ad}{8^m}$. Thus, (1.7) has three distinct transcendental entire solutions of finite order for this case. If $s\in\{1,\cdots,m-1\}$ (that is, $u_s=\varepsilon^s$), then $v=1$ or $-\varepsilon^s$, that is, $bc=6k\pi$ or $bc=6k\pi+3\pi+\frac{6s\pi}{m}$, where $k\in\mathbb{Z}$. Substituting $bc=6k\pi$ into (3.10), we deduce that $d_1d_2=0$, which is a contradiction. Substituting $bc=6k\pi+3\pi+\frac{6s\pi}{m}$ into (3.10), we deduce that

$$3d_1d_2+q(-\varepsilon^s-1)^m=0.$$

Then $q^3=(-1)^{m+1}\frac{27ad}{(e^{\frac{2s\pi i}{m}}+1)^{3m}}$. Thus, (1.7) has three distinct transcendental entire solutions of finite order for this case.

If $a\neq 0$ and $d=0$, then $d_1^3=a\neq 0$ and $d_2=0$. If $q(z)\equiv 0$, then (1.7) has three distinct transcendental entire solutions of finite order for this case. If $q(z)\not\equiv 0$, it follows from $a_4=a_2=0$ that $e^{\frac{ibc}{3}}-1=e^{\frac{-ibc}{3}}-1=0$, that is, $bc=6k\pi$, where $k\in\mathbb{Z}$. Substituting $bc=6k\pi$ into (1.7), we see that (1.7) has three distinct transcendental entire solutions of finite order for this case.

If $a=0$ and $d\neq 0$, then $d_1=0$ and $d_2^3=d\neq 0$. We can deduce similar results as the above.

(ii) Let $n>3$ (which implies $2n-2>n+1$ and $2<n-1$), then we deduce from (3.9) that

$$a_{2n}w^{2n}+a_{2n-2}w^{2n-2}+\cdots+a_2w^2+a_0=0, \tag{3.11}$$

where

$$\begin{cases} a_{2n} &= d_1^n-a,\\ a_{2n-2} &= nd_1^{n-1}d_2,\\ \cdots & \cdots\\ a_2 &= nd_1d_2^{n-1},\\ a_0 &= d_2^n-d. \end{cases}$$

Since $w(z)$ is transcendental, we have

$$a_{2n}=a_{2n-2}=\cdots=a_2=a_0=0.$$

If $ad\neq 0$, then $d_1^n=a\neq 0$, $d_2^n=d\neq 0$. It follows from $a_{2n-2}=a_2=0$ that $d_1d_2=0$, which is a contradiction. Thus, (1.7) has no transcendental entire solutions of finite order for this case.

If $a \neq 0$ and $d = 0$, then $d_1^n = a \neq 0$ and $d_2 = 0$. If n is even, then $n+1$ is odd. Hence, the coefficient of w^{n+1} in (3.11) is

$$a_{n+1} = d_1 q(z) \sum_{j=0}^{m} C_m^j (-1)^{m-j} e^{\frac{ijbc}{n}} = d_1 q(z)(e^{\frac{ibc}{n}} - 1)^m.$$

Since $a_{n+1} = 0$, we have $q(z) \equiv 0$ or $e^{\frac{ibc}{n}} - 1 = 0$ (that is, $bc = 2kn\pi$, where $k \in \mathbb{Z}$). If n is odd, then $n+1$ is even. Hence, the coefficient of w^{n+1} in (3.11) is

$$a_{n+1} = C_n^{\frac{n-1}{2}} d_1^{\frac{n+1}{2}} d_2^{\frac{n-1}{2}} + d_1 q(z) \sum_{j=0}^{m} C_m^j (-1)^{m-j} e^{\frac{ijbc}{n}} = d_1 q(z)(e^{\frac{ibc}{n}} - 1)^m.$$

Hence, we deduce the same result as the above, that is, $q(z) \equiv 0$ or $bc = 2kn\pi$, where $k \in \mathbb{Z}$. Thus, (1.7) has n distinct transcendental entire solutions of finite order for this case.

If $a = 0$ and $d \neq 0$, then $d_1 = 0$ and $d_2^n = d \neq 0$. We can deduce similar results as the above.□

Proof of Theorem 1.2. The proof of Theorem 1.2 is similar as the one of Theorem 1.3.□

4 Examples

Example 4.1. In the following, we give examples to show the sharpness of Theorem 1.1.

Consider the nonlinear differential-difference equation

$$f^2(z) + P_1(z, f) = 1 + 4(z-\pi)^2, \tag{4.1}$$

where

$$P_1(z,f) = \frac{1}{4z^2} f'^2(z) + f'^2(z-\pi) + 4(z-\pi)^2 f^2(z-\pi) + f(z+\sqrt{\pi}) + f(z-\sqrt{\pi}) + 2\cos(2\sqrt{\pi}z) f(z).$$

Clearly, $n = 2 = d(P)$, and $f_1(z) = \sin z^2$ is an admissible transcendental entire solution of (4.1). This shows our assumption "$n > d(P)$" in Theorem 1.1(i) is sharp.

Consider the nonlinear differential-difference equation

$$f^4(z) + P_2(z, f) = 1 + z, \tag{4.2}$$

where

$$P_2(z,f) = 2f'(z) - f'^2(z+\pi) - z f(z) f(z + \frac{\pi}{2}).$$

Clearly, $n = 4 = w(P)$, and $f_2(z) = \tan z$ is an admissible transcendental meromorphic solution of (4.2). This shows our assumption "$n > w(P)$" in Theorem 1.1(ii) is sharp.

Example 4.2. In the following, we give examples to illustrate the existence of entire solutions of finite order of (1.7) under the assumptions in Theorem 1.3.

Denote $\varepsilon = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$, which is a cubit root of unity, and consider the nonlinear difference equation

$$f^3(z) + q \triangle^m f(z) = \pi^3 e^{ibz} - e^{-ibz}. \tag{4.3}$$

If $m = 2$, $q = \frac{3}{4}\pi$, $b = \frac{3}{2}\pi$, $c = 2$, then (4.3) has three solutions as follows.

$$f_1(z) = \pi e^{\frac{i\pi z}{2}} - e^{-\frac{i\pi z}{2}},$$

$$f_2(z) = \pi\varepsilon e^{\frac{i\pi z}{2}} - \varepsilon^2 e^{-\frac{i\pi z}{2}},$$

$$f_3(z) = \pi\varepsilon^2 e^{\frac{i\pi z}{2}} - \varepsilon e^{-\frac{i\pi z}{2}}.$$

If $m = 3$, $q = 3\pi$, $b = \pi$, $c = 1$, then (4.3) has three solutions as follows.

$$f_1(z) = \pi e^{\frac{i\pi z}{3}} - e^{-\frac{i\pi z}{3}},$$

$$f_2(z) = \pi\varepsilon e^{\frac{i\pi z}{3}} - \varepsilon^2 e^{-\frac{i\pi z}{3}},$$

$$f_3(z) = \pi\varepsilon^2 e^{\frac{i\pi z}{3}} - \varepsilon e^{-\frac{i\pi z}{3}}.$$

If $m = 4$, $q = -\frac{3}{4}\pi$, $b = -3\pi$, $c = \frac{1}{2}$, then (4.3) has three solutions as follows.

$$f_1(z) = \pi e^{-i\pi z} - e^{i\pi z},$$

$$f_2(z) = \pi\varepsilon e^{-i\pi z} - \varepsilon^2 e^{i\pi z},$$

$$f_3(z) = \pi\varepsilon^2 e^{-i\pi z} - \varepsilon e^{i\pi z}.$$

Consider the nonlinear difference equation

$$f^n(z) + q(z)\triangle^m f(z) = ie^{-3z}, \quad n, m \in \mathbb{N}, n \geq 3, \tag{4.4}$$

where $p(z)$ is a polynomial. If $q(z) \equiv 0$ or $c = -\frac{2kn\pi i}{3}$, $k \in \mathbb{Z}$, then (4.4) has n solutions as follows.

$$f_j(z) = d_j e^{-\frac{3z}{n}}, \quad j = 0, 1, \cdots, n-1,$$

where $d_j^n = i, \quad j = 0, 1, \cdots, n-1.$

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Asymptotic approximations of a stable and unstable manifolds of a two-dimensional quadratic map

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Abstract. We find the asymptotic approximations of the stable and unstable manifolds of the saddle equilibrium solutions and the saddle period-two solutions of the following difference equation $x_{n+1} = cx_{n-1}^2 + dx_n + 1$, where the parameters c and d are positive numbers and initial conditions x_{-1} and x_0 are arbitrary nonnegative numbers. These manifolds determine completely global dynamics of this equation.

Keywords. Basin of attraction, cooperative, difference equation, local stable manifold, local unstable manifold, monotonicity, period-two solutions;

AMS 2000 Mathematics Subject Classification: Primary: 37B25, 37D10 Secondary: 37M99, 65P40, 65Q30 .

1 Introduction

In this paper we consider the difference equation

$$x_{n+1} = cx_{n-1}^2 + dx_n + 1, \tag{1}$$

where the parameters c and d are positive numbers and initial conditions x_{-1} and x_0 are arbitrary nonnegative numbers. Set

$$u_n = x_{n-1} \text{ and } v_n = x_n \text{ for } n = 0, 1, \ldots \tag{2}$$

and write Eq.(1) in the equivalent form:

$$\begin{aligned} u_{n+1} &= v_n \\ v_{n+1} &= cu_n^2 + dv_n + 1. \end{aligned} \tag{3}$$

Let T be the corresponding map defined by:

$$T\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ cu^2 + dv + 1 \end{pmatrix}. \tag{4}$$

It is easy to see that

$$T^2\begin{pmatrix} u \\ v \end{pmatrix} = T\left(T\begin{pmatrix} u \\ v \end{pmatrix}\right) = \begin{pmatrix} cu^2 + dv + 1 \\ d\left(cu^2 + dv + 1\right) + cv^2 + 1 \end{pmatrix}. \tag{5}$$

The local dynamics of the map T was derived in [1] where it was shown that the following holds:

Theorem 1 *If*

$$d < 1 \text{ and } (d-1)^2 - 4c \geq 0$$

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then Eq.(1) *has the equilibrium points* $\bar{x}_1$ *and* $\bar{x}_2$ *where*

$$\bar{x}_1 = \frac{1-d-\sqrt{(d-1)^2-4c}}{2c}, \quad \bar{x}_2 = \frac{1-d+\sqrt{(d-1)^2-4c}}{2c}$$

and the following holds:

i) $\bar{x}_1$ *is locally asymptotically stable if*

$$c < \frac{(d-1)^2}{4}.$$

ii) $\bar{x}_1$ *a non-hyperbolic point if*

$$c = \frac{(d-1)^2}{4}.$$

iii) $\bar{x}_2$ *is a repeller if*

$$c < \frac{(1-3d)(d+1)}{4}.$$

iv) $\bar{x}_2$ *is a saddle point if*

$$\frac{(1-3d)(d+1)}{4} < c < \frac{(d-1)^2}{4}$$

v) $\bar{x}_2$ *a non-hyperbolic point if*

$$c = \frac{(1-3d)(d+1)}{4} \text{ or } c = \frac{(d-1)^2}{4}.$$

Theorem 2 *If*

$$c < \frac{(1-3d)(d+1)}{4}$$

then Eq.(1) *has the minimal period-two solution*

$$P = \left\{ \frac{d+1-\sqrt{1-4c-d(3d+2)}}{2c}, \frac{d+1+\sqrt{1-4c-d(3d+2)}}{2c} \right\}$$

which is a saddle point.

The global dynamics of Eq.(1) is delicate and is described by the following theorem [1].

Theorem 3 *Consider Eq.*(1). *Then the following holds:*

(i) If $c < \frac{(1+d)(1-3d)}{4}$ *then Eq.*(1) *has two equilibrium solutions* $0 < \bar{x}_- < \bar{x}_+$, *where* x_- *is locally asymptotically stable,* $\bar{x}_+$ *is a repeller and the minimal period-two solution* $\ldots, \Phi, \Psi, \ldots,$ $\Phi < \Psi$ *is a saddle point. All non-equilibrium solutions* $\{x_n\}$ *converge to* x_-, *or to the period-two solution or are asymptotic to* ∞. *More precisely, there exist four continuous curves* $W^s(P_1), W^s(P_2)$ *(stable manifolds of* $P_1(\Phi, \Psi)$ *and* $P_2(\Psi, \Phi)$*),* $W^u(P_1), W^u(P_2)$, *(unstable manifolds of* P_1 *and* P_2*) where* $W^s(P_1), W^s(P_2)$ *are passing through the point* $E_+(\bar{x}_+, \bar{x}_+)$, *and are graphs of decreasing functions. The curves* $W^u(P_1), W^u(P_2)$ *are the graphs of increasing functions and are starting at* $E_-(\bar{x}_-, \bar{x}_-)$. *Every solution* $\{x_n\}$ *which starts below* $W^s(P_1) \cup W^s(P_2)$ *in North-east ordering converges to* $E_-(\bar{x}_-, \bar{x}_-)$ *and every solution* $\{x_n\}$ *which starts above* $W^s(P_1) \cup W^s(P_2)$ *in North-east ordering satisfies* $\lim x_n = \infty$.

(ii) If $c = \frac{(1+d)(1-3d)}{4}$ *then Eq.*(1) *has two equilibrium solutions* $0 < \bar{x}_- < \bar{x}_+$, *where* x_- *is locally asymptotically stable and* $\bar{x}_+$ *is the non-hyperbolic equilibrium solution. There exist the continuous decreasing curve* $W^s(E_+)$ *passing through the point* $E_+ = (\bar{x}_+, \bar{x}_+)$, *such that every solution* $\{x_n\}$ *which starts below* $W^s(E_+)$ *in North-east ordering converges to* $E_-(\bar{x}_-, \bar{x}_-)$ *and every solution* $\{x_n\}$ *which starts above* $W^s(E_+)$ *in North-east ordering satisfies* $\lim x_n = \infty$.

(iii) If $\frac{(1+d)(1-3d)}{4} < c < \frac{(1-d)^2}{4}$ *then Eq.*(1) *has two equilibrium solutions* $0 < \bar{x}_- < \bar{x}_+$ *and no minimal period-two solutions. If* $\bar{x}_+$ *is a saddle equilibrium solution, then there exist two continuous curves* $W^s(E_+)$ *and* $W^u(E_+)$, *both passing through the point* $E_+ = (\bar{x}_+, \bar{x}_+)$, *such that* $W^s(E_+)$ *is a graph of decreasing function and* $W^u(E_+)$ *is a graph of an increasing*

function. The first quadrant of initial condition $Q_1 = \{(x_{-1}, x_0) : x_{-1} \geq 0, x_0 \geq 0\}$ is is the union of three disjoint basins of attraction, namely

$$Q_1 = \mathcal{B}(E_-) \cup \mathcal{B}(E_+) \cup \mathcal{B}(E_\infty),$$

where E_- and E_∞ denote the points (x_-, x_-) and (∞, ∞) respectively, and $\mathcal{B}(E_+) = W^s(E_+)$,

$$\mathcal{B}(E_-) = \{(x,y)|(x,y) \preceq_{ne} (x_{E_+}, y_{E_+}) \quad \text{for some} \quad (x_{E_+}, y_{E_+}) \in W^s(E_+)\},$$

$$\mathcal{B}(E_\infty) = \{(x,y)|(x_{E_+}, y_{E_+}) \preceq_{ne} (x,y) \quad \text{for some} \quad (x_{E_+}, y_{E_+}) \in W^s(E_+)\}.$$

In addition, for every $(x_{-1}, x_0) \in Q_1 \setminus W^s(E_+)$ every solution is asymptotic to $W^u(E_+)$.

(iv) *If $c = \frac{(1-d)^2}{4}$ then Eq.(1) has one non-hyperbolic equilibrium solution $\bar{x}$ and there exists an invariant continuous curve $W^s(E)$, where $E(\bar{x}, \bar{x})$, which is the graph of a decreasing function, such that every solution $\{x_n\}$ of Eq.(1) for which $(x_{-1}, x_0) \in W^s(E)$ is attracted to E as well as every solution $\{x_n\}$ of Eq.(1) for which $(x_{-1}, x_0) \preceq_{ne} W^s(E)$.*

Every solution $\{x_n\}$ of Eq.(1) for which there exists $(x_W, y_W) \in W^s(E)$ such that $(x_W, y_W) \preceq_{ne} (x_{-1}, x_0), (x_{-1}, x_0) \notin W^s(E)$ satisfies $\lim x_n = \infty$.

(v) *If $c > \frac{(1-d)^2}{4}$ then Eq.(1) neither has an equilibrium solution nor the minimal period-two solution and every solution $\{x_n\}$ of Eq.(1) satisfies $\lim_{n\to\infty} x_n = \infty$.*

As one may see from Theorem 3 the boundaries of the basins of attraction of all attractors of Eq.(1) are the stable manifolds of either equilibrium points or of the period-two solution. In addition, by using the results from [9] one can see that the solutions which are asymptotic to the locally asymptotically stable equilibrium solutions are approaching the unstable manifolds of the neighboring saddle equilibrium points or period-two point. The monotonicity and smoothness of stable and unstable manifolds for the map T given with (4) is guaranteed by Theorems 4, 5, 6 of [9]. See [4, 7, 9, 12, 13] for related results about the stable manifolds for competitive maps. Our main goal here is to get the local asymptotic estimates for these manifolds for both equilibrium solutions and the period-two solutions. We will bring the considered map to the normal form around the equilibrium solutions and the period-two solutions and then use the method of undetermined coefficients to find the local approximations of the considered manifolds. Since the map T is cooperative, it is guaranteed that both stable and unstable manifolds are as smooth as the functions of the considered map and that are monotonic such that the stable manifold is decreasing and unstable manifold is increasing, see [2, 9]. See [4, 10, 14] for similar local approximations of stable and unstable manifolds. See [3, 5, 6, 11, 14] for basic results on stable ad unstable manifolds for general maps.

2 Preliminaries

In this section we present some basic results for the cooperative maps which describe the existence and the properties of their invariant manifolds.

A first order system of difference equations

$$\left\{ \begin{array}{lcl} x_{n+1} & = & f(x_n, y_n) \\ y_{n+1} & = & g(x_n, y_n) \end{array} \right. , \quad n = 0, 1, 2, \ldots, \quad (x_0, y_0) \in \mathcal{S}\,, \tag{6}$$

where $\mathcal{S} \subset \mathbb{R}^2$ is nonempty, $(f, g) : \mathcal{S} \to \mathcal{S}$, f, g are continuous functions is *cooperative* if $f(x, y)$ and $g(x, y)$ are non-decreasing in x and y. *Strongly cooperative* systems of difference equations or strongly cooperative maps are those for which the functions f and g are coordinate-wise strictly monotone.

If $\mathbf{v} = (u, v) \in \mathbb{R}^2$, we denote with $\mathcal{Q}_\ell(\mathbf{v})$, $\ell \in \{1, 2, 3, 4\}$, the four quadrants in $\mathbb{R}^2$ relative to $\mathbf{v}$, i.e., $\mathcal{Q}_1(\mathbf{v}) = \{(x, y) \in \mathbb{R}^2 : x \geq u,\ y \geq v\}$, $\mathcal{Q}_2(\mathbf{v}) = \{(x, y) \in \mathbb{R}^2 : x \leq u,\ y \geq v\}$, and so on. Define the *South-East* partial order $\preceq_{se}$ on $\mathbb{R}^2$ by $(x, y) \preceq_{se} (s, t)$ if and only if $x \leq s$ and $y \geq t$. Similarly, we define the *North-East* partial order $\preceq_{ne}$ on $\mathbb{R}^2$ by $(x, y) \preceq_{ne} (s, t)$ if and only if $x \leq s$ and $y \leq t$. For $\mathcal{A} \subset \mathbb{R}^2$ and $\mathrm{x} \in \mathbb{R}^2$, define the *distance from* x *to* $\mathcal{A}$ as $\mathrm{dist}(\mathrm{x}, \mathcal{A}) := \inf\{\|\mathrm{x} - \mathrm{y}\| : \mathrm{y} \in \mathcal{A}\}$. By $\mathrm{int}\,\mathcal{A}$ we denote the interior of a set $\mathcal{A}$.

It is easy to show that a map F is cooperative if it is non-decreasing with respect to the North-East partial order, that is if the following holds:

$$\begin{pmatrix} x^1 \\ y^1 \end{pmatrix} \preceq_{ne} \begin{pmatrix} x^2 \\ y^2 \end{pmatrix} \Rightarrow F\begin{pmatrix} x^1 \\ y^1 \end{pmatrix} \preceq_{ne} F\begin{pmatrix} x^2 \\ y^2 \end{pmatrix}. \tag{7}$$

The following five results were proved by Kulenović and Merino [8, 9] for competitive systems in the plane, when one of the eigenvalues of the linearized system at an equilibrium (hyperbolic or non-hyperbolic) is by absolute value smaller than 1 while the other has an arbitrary value. We give the analogue versions for cooperative maps.

A region $\mathcal{R} \subset \mathbb{R}^2$ is *rectangular* if it is the cartesian product of two intervals in $\mathbb{R}$.

Theorem 4 *Let T be a cooperative map on a rectangular region $\mathcal{R} \subset \mathbb{R}^2$. Let $\overline{\mathrm{x}} \in \mathcal{R}$ be a fixed point of T such that $\Delta := \mathcal{R} \cap \operatorname{int}(\mathcal{Q}_2(\overline{\mathrm{x}}) \cup \mathcal{Q}_4(\overline{\mathrm{x}}))$ is nonempty (i.e., $\overline{\mathrm{x}}$ is not the NE or SW vertex of $\mathcal{R}$), and T is strongly cooperative on Δ. Suppose that the following statements are true.*

a. The map T has a C^1 extension to a neighborhood of $\overline{\mathrm{x}}$.

b. The Jacobian matrix of T at $\overline{x}$ has real eigenvalues λ, μ such that $0 < |\lambda| < \mu$, where $|\lambda| < 1$, and the eigenspace E^λ associated with λ is not a coordinate axis.

Then there exists a curve $\mathcal{C} \subset \mathcal{R}$ through $\overline{\mathrm{x}}$ that is invariant and a subset of the basin of attraction of $\overline{\mathrm{x}}$, such that $\mathcal{C}$ is tangential to the eigenspace E^λ at $\overline{\mathrm{x}}$, and $\mathcal{C}$ is the graph of a strictly decreasing continuous function of the first coordinate on an interval. Any endpoints of $\mathcal{C}$ in the interior of $\mathcal{R}$ are either fixed points or minimal period-two points. In the latter case, the set of endpoints of $\mathcal{C}$ is a minimal period-two orbit of T.

Corollary 1 *If T has no fixed point nor periodic points of minimal period two in Δ, then the endpoints of $\mathcal{C}$ belong to $\partial\mathcal{R}$.*

For maps that are strongly cooperative near the fixed point, hypothesis (b). of Theorem 4 reduces just to $|\lambda| < 1$. This follows from a change of variables [13] that allows the Perron-Frobenius Theorem to be applied to give that at any point, the Jacobian matrix of a strongly cooperative map has two real and distinct eigenvalues, the larger one in absolute value being positive, and that corresponding eigenvectors may be chosen to point in the direction of the second and first quadrant, respectively. Also, one can show that in such a case no associated eigenvector is aligned with a coordinate axis.

Theorem 5 *Under the hypotheses of Theorem 4, suppose there exists a neighborhood $\mathcal{U}$ of $\overline{\mathrm{x}}$ in $\mathbb{R}^2$ such that T is of class C^k on $\mathcal{U} \cup \Delta$ for some $k \geq 1$, and that the Jacobian of T at each $\mathrm{x} \in \Delta$ is invertible. Then the curve $\mathcal{C}$ in the conclusion of Theorem 4 is of class C^k.*

The following result gives a description of the global stable and unstable manifolds of a saddle point of a cooperative map. The result is the modification of Theorem 5 from [7]. See also [8].

Theorem 6 *In addition to the hypotheses of Theorem 4, suppose that $\mu > 1$ and that the eigenspace E^μ associated with μ is not a coordinate axis. If the curve $\mathcal{C}$ of Theorem 4 has endpoints in $\partial\mathcal{R}$, then $\mathcal{C}$ is the global stable manifold $\mathcal{W}^s(\overline{\mathrm{x}})$ of $\overline{\mathrm{x}}$, and the global unstable manifold $\mathcal{W}^u(\overline{\mathrm{x}})$ is a curve in $\mathcal{R}$ that is tangential to E^μ at $\overline{\mathrm{x}}$ and such that it is the graph of a strictly increasing function of the first coordinate on an interval. Any endpoints of $\mathcal{W}^u(\overline{\mathrm{x}})$ in $\mathcal{R}$ are fixed points of T.*

Theorem 7 *Assume the hypotheses of Theorem 4, and let $\mathcal{C}$ be the curve whose existence is guaranteed by Theorem 4. If the endpoints of $\mathcal{C}$ belong to $\partial\mathcal{R}$, then $\mathcal{C}$ separates $\mathcal{R}$ into two connected components, namely*

$$\mathcal{W}_- := \{\mathrm{x} \in \mathcal{R} \setminus \mathcal{C} : \exists \mathrm{y} \in \mathcal{C} \text{ with } \mathrm{x} \preceq_{ne} \mathrm{y}\} \quad \text{and} \quad \mathcal{W}_+ := \{\mathrm{x} \in \mathcal{R} \setminus \mathcal{C} : \exists \mathrm{y} \in \mathcal{C} \text{with } \mathrm{y} \preceq_{ne} \mathrm{x}\}, \tag{8}$$

such that the following statements are true.

(i) $\mathcal{W}_-$ is invariant, and $\operatorname{dist}(T^n(\mathrm{x}), \mathcal{Q}_1(\overline{\mathrm{x}})) \to 0$ as $n \to \infty$ for every $\mathrm{x} \in \mathcal{W}_-$.

(ii) $\mathcal{W}_+$ is invariant, and $\operatorname{dist}(T^n(\mathrm{x}), \mathcal{Q}_3(\overline{\mathrm{x}})) \to 0$ as $n \to \infty$ for every $\mathrm{x} \in \mathcal{W}_+$.

If, in addition, $\overline{\mathrm{x}}$ is an interior point of $\mathcal{R}$ and T is C^2 and strongly cooperative in a neighborhood of $\overline{\mathrm{x}}$, then T has no periodic points in the boundary of $\mathcal{Q}_2(\overline{\mathrm{x}}) \cup \mathcal{Q}_4(\overline{\mathrm{x}})$ except for $\overline{\mathrm{x}}$, and the following statements are true.

(iii) For every $\mathrm{x} \in \mathcal{W}_-$ there exists $n_0 \in \mathbb{N}$ such that $T^n(\mathrm{x}) \in \operatorname{int} \mathcal{Q}_1(\overline{\mathrm{x}})$ for $n \geq n_0$.

(iv) For every $\mathrm{x} \in \mathcal{W}_+$ there exists $n_0 \in \mathbb{N}$ such that $T^n(\mathrm{x}) \in \operatorname{int} \mathcal{Q}_3(\overline{\mathrm{x}})$ for $n \geq n_0$.

Remark 1 The map T defined with (4) is strongly cooperative in the first quadrant of initial conditions. Theorems 4, 5 and 6 show that the stable and unstable manifolds of cooperative maps, which satisfies certain conditions, are simple monotonic curves which are as smooth as the functions of the map. Thus the assumed forms of these manifolds are justified. As is well-known the stable and unstable manifolds of general maps can have complicated structure consisting of many branches or being strange attractors, see [3, 5, 10, 14] for some examples of polynomial maps such as Henon with unstable manifold being a starnge attractor. Finally, see [13] for examples of competitive and so cooperative maps in the plane with chaotic attractors.

3 Invariant manifolds and Normal Forms

Let

$$\begin{pmatrix} \xi_{n+1} \\ \eta_{n+1} \end{pmatrix} = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix} \begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix} + \begin{pmatrix} g_1(\xi_n, \eta_n) \\ g_2(\xi_n, \eta_n) \end{pmatrix}, \tag{9}$$

where

$$g_1(0,0) = 0, \quad g_2(0,0) = 0, \quad Dg_1(0,0) = 0 \text{ and } Dg_2(0,0) = 0.$$

Suppose that $|\mu_1| < 1$ and $|\mu_2| > 1$. Then, there are two unique invariant manifolds $\mathcal{W}^s$ and $\mathcal{W}^u$ tangents to $(1,0)$ and $(0,1)$ at (0,0), which are graphs of the maps

$$\varphi : E_1 \to E_2 \text{ and } \psi : E_1 \to E_2,$$

such that

$$\varphi(0) = \psi(0) = 0 \text{ and } \varphi'(0) = \psi'(0) = 0.$$

See [4, 5, 10, 14]. Letting $\eta_n = \varphi(\xi_n)$ yields

$$\eta_{n+1} = \varphi(\xi_{n+1}) = \varphi(\mu_1 \xi_n + g_1(\xi_n, \varphi(\xi_n))). \tag{10}$$

On the other hand by (9)

$$\eta_{n+1} = \mu_2 \varphi(\xi_n) + g_2(\xi_n, \varphi(\xi_n)). \tag{11}$$

Equating equations (10) and (11) yields

$$\varphi(\mu_1 \xi_n + g_1(\xi_n, \varphi(\xi_n))) = \mu_2 \varphi(\xi_n) + g_2(\xi_n, \varphi(\xi_n)). \tag{12}$$

Similarly, letting $\xi_n = \psi(\eta_n)$ yields

$$\xi_{n+1} = \psi(\eta_{n+1}) = \psi(\mu_2 \eta_n + g_2(\psi(\eta_n), \eta_n)). \tag{13}$$

By using (9) we obtain

$$\xi_{n+1} = \mu_1 \psi(\eta_n) + g_1(\psi(\eta_n), \eta_n). \tag{14}$$

Equating equations (13) and (14) yields

$$\psi(\mu_2 \eta_n + g_2(\psi(\eta_n), \eta_n)) = \mu_1 \psi(\eta_n) + g_1(\psi(\eta_n), \eta_n). \tag{15}$$

Thus the functional equations (12) and (15), define the local stable manifold

$$\mathcal{W}^s = \{(\xi, \eta) \in \mathbb{R}^2 : \eta = \varphi(\xi)\},$$

and the local unstable manifold

$$\mathcal{W}^u = \{(\xi, \eta) \in \mathbb{R}^2 : \xi = \psi(\eta)\}.$$

Without loss generality, we can assume that solutions of the functional equations (12) and (15) take the forms

$$\psi(\eta) = \alpha_2 \eta^2 + \beta_2 \eta^3 + O(|\eta|^4)$$

and

$$\varphi(\xi) = \alpha_1 \xi^2 + \beta_1 \xi^3 + O(|\xi|^4),$$

where $\alpha_i, \beta_i, i = 1, 2$ are undetermined coefficients.

3.1 Normal form of the map T at $\bar{x}_2$

Put $y_n = x_n - \bar{x}_2$. Then Eq(1) becomes

$$y_{n+1} = c\left(\bar{x}_2 + y_{n-1}\right)^2 + d\left(\bar{x}_2 + y_n\right) - \bar{x}_2 + 1. \tag{16}$$

Set

$$u_n = y_{n-1} \text{ and } v_n = y_n \text{ for } n = 0, 1, \ldots \tag{17}$$

and write Eq(16) in the equivalent form:

$$\begin{aligned} u_{n+1} &= v_n \\ v_{n+1} &= c\left(\bar{x}_2 + u_n\right)^2 + d\left(\bar{x}_2 + v_n\right) - \bar{x}_2 + 1. \end{aligned} \tag{18}$$

Let F be the function defined by:

$$F\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ c\left(\bar{x}_2 + u\right)^2 + d\left(\bar{x}_2 + v\right) - \bar{x}_2 + 1 \end{pmatrix}. \tag{19}$$

Then F has the fixed point $(0,0)$ and maps $(-\bar{x}_2, \infty)^2$ into $(-\bar{x}_2, \infty)^2$. The Jacobian matrix of F is given by

$$Jac_F(u,v) = \begin{pmatrix} 0 & 1 \\ 2c\left(u + \bar{x}_2\right) & d \end{pmatrix}.$$

At $(0,0)$, $Jac_F(u,v)$ has the form

$$J_0 = Jac_F(0,0) = \begin{pmatrix} 0 & 1 \\ 2c\bar{x}_2 & d \end{pmatrix}. \tag{20}$$

The eigenvalues of (20) are $\mu_{1,2}$ where

$$\mu_1 = \frac{1}{2}\left(d - \sqrt{8c\bar{x}_2 + d^2}\right) \text{ and } \mu_2 = \frac{1}{2}\left(d + \sqrt{8c\bar{x}_2 + d^2}\right),$$

and the corresponding eigenvectors are given by

$$v_1 = \left(-\frac{d + \sqrt{8c\bar{x}_2 + d^2}}{4c\bar{x}_2}, 1\right)^T \text{ and } v_2 = \left(-\frac{d - \sqrt{8c\bar{x}_2 + d^2}}{4c\bar{x}_2}, 1\right)^T,$$

respectively.

Then we have that

$$F\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2c\bar{x}_2 & d \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f_1(u,v) \\ g_1(u,v) \end{pmatrix}, \tag{21}$$

where

$$f_1(u,v) = 0$$
$$g_1(u,v) = \bar{x}_2\left(c\bar{x}_2 + d - 1\right) + cu^2 + 1.$$

Then, the system (16) is equivalent to

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2c\bar{x}_2 & d \end{pmatrix}\begin{pmatrix} u_n \\ v_n \end{pmatrix} + \begin{pmatrix} f_1(u_n, v_n) \\ g_1(u_n, v_n) \end{pmatrix}. \tag{22}$$

Let

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = P \cdot \begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix}$$

where

$$P = \begin{pmatrix} -\frac{d+\sqrt{d^2+8c\bar{x}_2}}{4c\bar{x}_2} & -\frac{d-\sqrt{d^2+8c\bar{x}_2}}{4c\bar{x}_2} \\ 1 & 1 \end{pmatrix} \text{ and } P^{-1} = \begin{pmatrix} -\frac{2c\bar{x}_2}{\sqrt{d^2+8c\bar{x}_2}} & \frac{\sqrt{d^2+8c\bar{x}_2}-d}{2\sqrt{d^2+8c\bar{x}_2}} \\ \frac{2c\bar{x}_2}{\sqrt{d^2+8c\bar{x}_2}} & \frac{d+\sqrt{d^2+8c\bar{x}_2}}{2\sqrt{d^2+8c\bar{x}_2}} \end{pmatrix}.$$

Then system (22) is equivalent to

$$\begin{pmatrix} \xi_{n+1} \\ \eta_{n+1} \end{pmatrix} = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}\begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix} + P^{-1} \cdot H_1\left(P \cdot \begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix}\right), \tag{23}$$

where

$$H_1\begin{pmatrix}u\\v\end{pmatrix}:=\begin{pmatrix}f_1(u,v)\\g_1(u,v)\end{pmatrix}.$$

Let

$$G_1\begin{pmatrix}u\\v\end{pmatrix}:=\begin{pmatrix}\tilde{f}_1(u,v)\\\tilde{g}_1(u,v)\end{pmatrix}=P^{-1}\cdot H_1\left(P\cdot\begin{pmatrix}u\\v\end{pmatrix}\right).$$

By straightforward calculation we obtain that

$$\tilde{f}_1(u,v)=\frac{\Upsilon_1(u,v)\left(\sqrt{8c\bar{x}_2+d^2}-d\right)}{16c\bar{x}_2^2\sqrt{8c\bar{x}_2+d^2}},$$

$$\tilde{g}_1(u,v)=\frac{\Upsilon_1(u,v)\left(\sqrt{8c\bar{x}_2+d^2}+d\right)}{16c\bar{x}_2^2\sqrt{8c\bar{x}_2+d^2}},$$

where

$$\Upsilon_1(u,v)=8c^2\bar{x}_2^4+d\left(u^2-v^2\right)\sqrt{8c\bar{x}_2+d^2}+4c\bar{x}_2\left(2(d-1)\bar{x}_2^2+2\bar{x}_2+(u-v)^2\right)+d^2\left(u^2+v^2\right).$$

3.2 Stable and unstable manifolds corresponding to $\bar{x}_2$

Assume that $d<1$ and $(d-1)^2-4c\geq 0$. Then Eq.(1) has the equilibrium point $\bar{x}_2$ where

$$\bar{x}_2=\frac{1-d+\sqrt{(d-1)^2-4c}}{2c}$$

which is a saddle point if

$$\frac{(1-3d)(d+1)}{4}<c<\frac{(d-1)^2}{4}.$$

Let us assume that the local stable manifold is the graph of the map φ_1 of the form

$$\varphi_1(\xi)=\alpha_1\xi^2+\beta_1\xi^3+O(|\xi|^4),\quad \alpha_1,\beta_1\in\mathbb{R}.$$

Now we compute the constants α_1 and β_1. The function φ_1 must satisfy the stable manifold equation

$$\varphi_1\left(\mu_1\xi+\tilde{f}_1\left(\xi,\varphi_1(\xi)\right)\right)=\mu_2\varphi_1(\xi)+\tilde{g}_1\left(\xi,\varphi_1(\xi)\right),$$

This leads to the following polynomial equation

$$p_1\xi^2+p_2\xi^3+\cdots+p_{18}\xi^{18}=0$$

where the coefficients p_1 and p_2, obtain by using *Mathematica* are in appendix A. Substituting $\bar{x}_2$ into (42) and (43) and solving system $p_1=0$ and $p_2=0$, we obtain the values

$$\alpha_1=\frac{-8c^2}{\Upsilon_1(c,d)+\Upsilon_2(c,d)\sqrt{4\sqrt{(d-1)^2-4c}+d^2-4d+4}}$$

and

$$\beta_1=\frac{4\alpha_1c(4c+(d+1)(3d-1))}{\Upsilon_3(c,d)+\Upsilon_4(c,d)\sqrt{4\sqrt{(d-1)^2-4c}+d^2-4d+4}}$$

where

$$\begin{aligned}\Upsilon_1(c,d)=&\sqrt{(d-1)^2-4c}\left(d^6-4c\left(13d^2-4d+8\right)\right)\\&-4c\left(7d^4-12d^3+17d^2-12d+8\right)+(d-1)d^6+64c^2,\\\Upsilon_2(c,d)=&\sqrt{(d-1)^2-4c}\left(4c(5d+2)-d^5\right)+4c\left(5d^3-4d^2+3d+2\right)+(1-d)d^5,\\\Upsilon_3(c,d)=&\sqrt{(d-1)^2-4c}\left(-4c\left(5d^2-12d+16\right)-15d^4+27d^3-13d^2-32d+24\right)\\&+64c^2-4c\left(d^4-6d^3+7d^2-32d+28\right)-3d^6+16d^5-42d^4+40d^3+19d^2-56d+24,\\\Upsilon_4(c,d)=&\sqrt{(d-1)^2-4c}\left(4c(3d-2)+9d^3+d^2-3d+6\right)+3d^5-10d^4+8d^3+4d^2-9d+6\\&+4c\left(d^3-4d^2+5d-6\right).\end{aligned}\tag{24}$$

Since

$$\eta_n = \alpha_1 \xi_n^2 + \beta_1 \xi_n^3,$$
$$\begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix} = P^{-1} \cdot \begin{pmatrix} u_n \\ v_n \end{pmatrix},$$

and

$$u_n = x_{n-1} - \bar{x}_2 \text{ and } v_n = x_n - \bar{x}_2$$

we can approximate locally the local stable manifold $\mathcal{W}^s_{loc}(\bar{x}_2, \bar{x}_2)$ as the graph of the map $\tilde{\varphi}_1(u)$ such that $S(u, \tilde{\varphi}_1(u)) = 0$ where

$$\begin{aligned} S(u,v) :=& \alpha_1 \left(\frac{(v - \bar{x}_2)\left(\sqrt{8c\bar{x}_2 + d^2} - d\right)}{2\sqrt{8c\bar{x}_2 + d^2}} - \frac{2c\bar{x}_2 (u - \bar{x}_2)}{\sqrt{8c\bar{x}_2 + d^2}} \right)^2 \\ &+ \beta_1 \left(\frac{(v - \bar{x}_2)\left(\sqrt{8c\bar{x}_2 + d^2} - d\right)}{2\sqrt{8c\bar{x}_2 + d^2}} - \frac{2c\bar{x}_2 (u - \bar{x}_2)}{\sqrt{8c\bar{x}_2 + d^2}} \right)^3 \\ &- \frac{2c\bar{x}_2 (u - \bar{x}_2)}{\sqrt{8c\bar{x}_2 + d^2}} - \frac{(v - \bar{x}_2)\left(\sqrt{8c\bar{x}_2 + d^2} + d\right)}{2\sqrt{8c\bar{x}_2 + d^2}} \end{aligned} \tag{25}$$

and which satisfies

$$\tilde{\varphi}_1(\bar{x}_2) = \bar{x}_2 \text{ and } \tilde{\varphi}_1'(\bar{x}_2) = -\frac{4c\bar{x}_2}{\sqrt{8c\bar{x}_2 + d^2} + d}.$$

Let us assume that the local unstable manifold is the graph of the map ψ that has the form

$$\psi_1(\eta) = \alpha_2 \eta^2 + \beta_2 \eta^3 + O(|\eta|^4), \quad \alpha_1, \beta_1 \in \mathbb{R}.$$

Now we compute the constants α_2 and β_2. The function ψ_1 must satisfy the unstable manifold equation

$$\psi_1(\mu_2 \eta + \tilde{g}_1(\psi_1(\eta), \eta)) = \mu_1 \psi_1(\eta) + \tilde{f}_1(\psi_1(\eta), \eta),$$

This leads to the following polynomial equation

$$q_1 \eta^2 + q_2 \eta^3 + \cdots + q_{18} \eta^{18} = 0$$

where the coefficients q_1 and q_2 are in appendix A.

Substituting $\bar{x}_2$ into (44) and (45) and solving system $q_1 = 0$ and $q_2 = 0$, we obtain the values

$$\alpha_2 = \frac{-8c^2}{\Gamma_1(c,d) + \Gamma_2(c,d)\sqrt{4\sqrt{(d-1)^2 - 4c} + d^2 - 4d + 4}}$$

and

$$\beta_2 = \frac{\alpha_2 \Gamma_5(c,d)}{\Gamma_3(c,d) + \Gamma_4(c,d)\sqrt{4\sqrt{(d-1)^2 - 4c} + d^2 - 4d + 4}}$$

where

$$\begin{aligned} \Gamma_1(c,d) =& 64c^2 + \sqrt{(d-1)^2 - 4c}\left(d^6 - 4c\left(13d^2 - 4d + 8\right)\right) \\ &- 4c\left(7d^4 - 12d^3 + 17d^2 - 12d + 8\right) + (d-1)d^6, \\ \Gamma_2(c,d) =& \sqrt{(d-1)^2 - 4c}\left(d^5 - 4c(5d + 2)\right) - 4c\left(5d^3 - 4d^2 + 3d + 2\right) + (d-1)d^5, \\ \Gamma_3(c,d) =& 256c^2 + \sqrt{(d-1)^2 - 4c}\left(\left(d^2 - 8d + 8\right)^2\left(d^2 - 2d + 3\right) - 32c\left(3d^2 - 8d + 10\right)\right) \\ &- 4c\left(9d^4 - 72d^3 + 208d^2 - 304d + 176\right) - \left(d^2 - 8d + 8\right)^2\left(d^3 - 3d^2 + 5d - 3\right), \\ \Gamma_4(c,d) =& d\sqrt{(d-1)^2 - 4c}\left(-48c + d^4 - 14d^3 + 61d^2 - 88d + 40\right) \\ &- d\left(4c\left(7d^2 - 36d + 32\right) + \left(d^3 - 13d^2 + 48d - 40\right)(d-1)^2\right), \\ \Gamma_5(c,d) =& \sqrt{4\sqrt{(d-1)^2 - 4c} + d^2 - 4d + 4}\left(-2c(d-2)^2 - 8c\sqrt{(d-1)^2 - 4c}\right) \\ &- 4c\left(d^2 - 10d + 8\right)\sqrt{(d-1)^2 - 4c} + 2c\left(32c + 3d^3 - 22d^2 + 36d - 16\right). \end{aligned} \tag{26}$$

Since

$$\xi_n = \alpha_2 \eta_n^2 + \beta_2 \eta_n^3,$$
$$\begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix} = P^{-1} \cdot \begin{pmatrix} u_n \\ v_n \end{pmatrix},$$

and

$$u_n = x_{n-1} - \bar{x}_2 \text{ and } v_n = x_n - \bar{x}_2$$

we can approximate locally the local unstable manifold $\mathcal{W}^u_{loc}(\bar{x}_2, \bar{x}_2)$ as the graph of the map $\tilde{\psi}_1(u)$ such that $U(\tilde{\psi}_1(v), v) = 0$ where

$$\begin{aligned} U(u,v) :=& \alpha_2 \left(\frac{2c\bar{x}_2 (u - \bar{x}_2)}{\sqrt{8c\bar{x}_2 + d^2}} + \frac{(v - \bar{x}_2)\left(\sqrt{8c\bar{x}_2 + d^2} + d\right)}{2\sqrt{8c\bar{x}_2 + d^2}} \right)^2 \\ &+ \beta_2 \left(\frac{2c\bar{x}_2 (u - \bar{x}_2)}{\sqrt{8c\bar{x}_2 + d^2}} + \frac{(v - \bar{x}_2)\left(\sqrt{8c\bar{x}_2 + d^2} + d\right)}{2\sqrt{8c\bar{x}_2 + d^2}} \right)^3 \\ &+ \frac{2c\bar{x}_2 (u - \bar{x}_2)}{\sqrt{8c\bar{x}_2 + d^2}} - \frac{(v - \bar{x}_2)\left(\sqrt{8c\bar{x}_2 + d^2} - d\right)}{2\sqrt{8c\bar{x}_2 + d^2}} \end{aligned} \tag{27}$$

and which satisfies

$$\tilde{\psi}_1(\bar{x}_2) = \bar{x}_2 \text{ and } \tilde{\psi}_1'(\bar{x}_2) = \frac{4c\bar{x}_2}{\sqrt{8c\bar{x}_2 + d^2} - d}.$$

Thus we proved the following result

Theorem 8 *Consider Eq.*(1) *subject to the condition* $\frac{(1-3d)(d+1)}{4} < c < \frac{(d-1)^2}{4}$. *Then the local stable and unstable manifolds are given with the asymptotic expansions* (25) *and* (27) *respectively.*

3.3 Some numerical examples

In this section we provide some numerical examples and we compare visually the asymptotic approximations of stable and unstable manifolds, obtained by using *Mathematica*, with the boundaries of the basins of attraction obtained by using the software package *Dynamica* 3 [6].
For $c = 0.06$ and $d = 0.3$ we have that

$$\begin{aligned} S_1(u,v) =& -0.0000205931x^3 + x^2(0.0000492004y + 0.0322121) \\ &+ x\left(-0.0000391827y^2 - 0.0513067y + 0.411741\right) \\ &+ 0.0000104016y^3 + 0.0204302y^2 + 0.672096y - 10.9718 \end{aligned}$$

and for $c = 0.075$ and $d = 0.42$

$$\begin{aligned} S_2(u,v) =& y^2(0.0227887 - 0.000731949x) + 0.000814449(x - 62.2685)xy \\ &- 0.000302082(x - 105.844)x(x + 12.4415) \\ &+ 0.000219269y^3 + 0.642493y - 6.35325. \end{aligned}$$

Figures 1 and 2 show the graph of the functions $S_1(u,v) = 0$ and $S_2(u,v) = 0$ with the basins of attraction created with *Dynamica* 3. Figure 3 shows the graph of the functions $S_1(u,v) = 0$ and $S_2(u,v) = 0$ for different values of the parameters c and d.
For $c = 0.06$ and $d = 0.3$ we have that

$$\begin{aligned} U_1(u,v) =& (0.000113205x - 0.00441057)y^2 + 0.000108186(x - 77.922)xy \\ &+ 0.0000344633(x - 183.477)x(x + 66.5943) \\ &+ 0.0000394854y^3 + 0.559375y + 0.00870931 \end{aligned}$$

and for $c = 0.075$ and $d = 0.42$

$$\begin{aligned} U_2(u,v) =& (0.000446197x - 0.010175)y^2 + 0.000309085(x - 45.6077)xy \\ &+ 0.000071369(x - 111.063)x(x + 42.6512) \\ &+ 0.00021471y^3 + 0.511958y - 0.265003 \end{aligned}$$

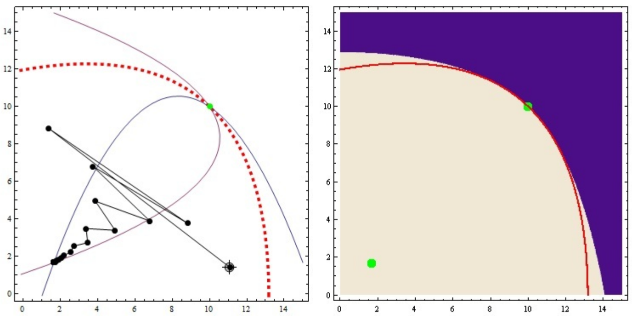

Figure 1: The graph of the function $S_1(u, v) = 0$ (red curve) for $c = 0.06$ and $d = 0.3$ with the basins of attraction generated by *Dynamica* 3.

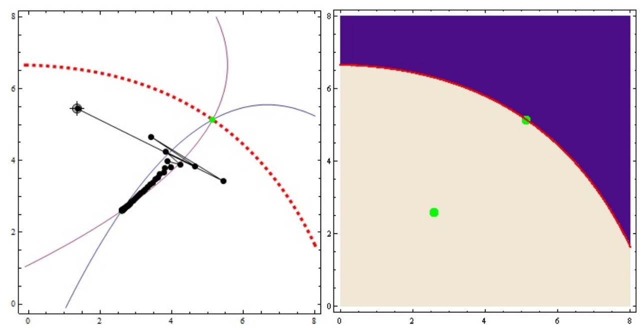

Figure 2: The graph of the function $S_2(u, v) = 0$ (red curve) for $c = 0.075$ and $d = 0.42$ with the basins of attraction generated by *Dynamica* 3.

3.4 Normal form of the map T^2 at the period-two solution

The period-two solution of (1) is given as

$$\bar{u}_1 = \frac{d+1-D}{2c} \text{ and } \bar{v}_1 = \frac{d+1+D}{2c},$$

where

$$D = \sqrt{1-4c-2d-3d^2}.$$

First we transform the period two solution $(\bar{u}_1, \bar{v}_1)$ of (1) to the origin by the translation

$$\tilde{u} = u - \bar{u}_1 \text{ and } \tilde{v} = v - \bar{v}_1$$

under which the corresponding map (5) becomes

$$\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \to \tilde{F}\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = T^2\begin{pmatrix} \tilde{u}+\bar{u}_1 \\ \tilde{v}+\bar{v}_1 \end{pmatrix} - \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = \begin{pmatrix} c\tilde{u}^2 + \tilde{u}(d-D+1) + d\tilde{v} \\ \left(c\tilde{u}^2 - D\tilde{u} + \tilde{u} + \tilde{v}\right) + \tilde{v}(c\tilde{v}+D+1) + d^2(\tilde{u}+\tilde{v}) \end{pmatrix}. \tag{28}$$

Then $\tilde{F}$ has the fixed point at $(0, 0)$. The Jacobian matrix of $\tilde{F}$ is given by

$$Jac_{\tilde{F}}(\tilde{u}, \tilde{v}) = \begin{pmatrix} d-D+2c\tilde{u}+1 & d \\ d^2+(-D+2c\tilde{u}+1)d & d^2+d+D+2c\tilde{v}+1 \end{pmatrix}.$$

At $(0, 0)$, $Jac_{\tilde{F}}(\tilde{u}, \tilde{v})$ has the form

$$J_0 = Jac_{\tilde{F}}(0,0) = \begin{pmatrix} d-D+1 & d \\ d^2+(1-D)d & d^2+d+D+1 \end{pmatrix}. \tag{29}$$

The eigenvalues of (29) are

$$\nu_1 = \frac{1}{2}(d(d+2)+2-C) \text{ and } \nu_2 = \frac{1}{2}(d(d+2)+2+C),$$

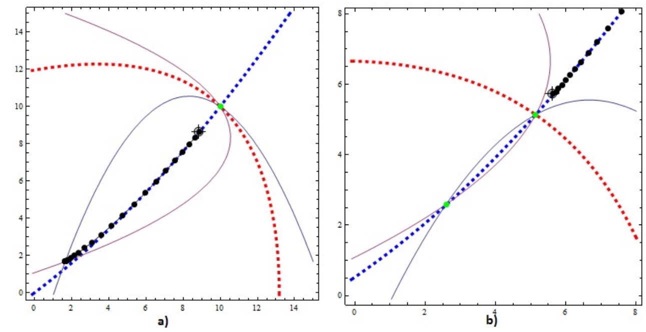

Figure 3: a) The graph of the functions $S_1(u,v)=0$ (red curve) and $U_1(u,v)=0$ (blue curve) for $c=0.06$ and $d=0.3$. b) The graph of the functions $S_2(u,v)=0$ (red curve) and $U_2(u,v)=0$ (blue curve) for $c=0.075$ and $d=0.42$.

where

$$C=\sqrt{-16c+(d-2)d(d(d+6)+4)+4}.$$

The eigenvectors corresponding to the eigenvalues $\nu_{1,2}$ are given by

$$\mathbf{v}_1=\left(\frac{2d}{-C+d^2+2D},1\right)^T \text{ and } \mathbf{v}_2=\left(\frac{2d}{C+d^2+2D},1\right)^T,$$

respectively.

Then we have that

$$\begin{pmatrix}\tilde{u}\\ \tilde{v}\end{pmatrix}\to\tilde{F}\begin{pmatrix}\tilde{u}\\ \tilde{v}\end{pmatrix}=\begin{pmatrix} d-D+1 & d\\ d^2+(1-D)d & d^2+d+D+1\end{pmatrix}\begin{pmatrix}\tilde{u}\\ \tilde{v}\end{pmatrix}+\begin{pmatrix}f_2(\tilde{u},\tilde{v})\\ g_2(\tilde{u},\tilde{v})\end{pmatrix}, \tag{30}$$

where

$$f_2(\tilde{u},\tilde{v})=c\tilde{u}^2,\quad g_2(\tilde{u},\tilde{v})=c(d\tilde{u}^2+\tilde{v}^2).$$

Let

$$\begin{pmatrix}\tilde{u}\\ \tilde{v}\end{pmatrix}=P\cdot\begin{pmatrix}\xi\\ \eta\end{pmatrix}$$

where

$$P=\begin{pmatrix}\frac{2d}{d^2-C+2D} & \frac{2d}{d^2+C+2D}\\ 1 & 1\end{pmatrix} \text{ and } P^{-1}=\begin{pmatrix}\frac{\left(d^2+2D\right)^2-C^2}{4Cd} & -\frac{d^2-C+2D}{2C}\\ \frac{C^2-\left(d^2+2D\right)^2}{4Cd} & \frac{d^2+C+2D}{2C}\end{pmatrix}.$$

Then (30) leads to the corresponding normal form

$$\begin{pmatrix}\xi\\ \eta\end{pmatrix}\to\begin{pmatrix}\nu_1 & 0\\ 0 & \nu_2\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}+P^{-1}\cdot H_2\left(P\cdot\begin{pmatrix}\xi\\ \eta\end{pmatrix}\right), \tag{31}$$

where

$$H_2\begin{pmatrix}u\\ v\end{pmatrix}:=\begin{pmatrix}f_2(u,v)\\ g_2(u,v)\end{pmatrix}.$$

Let

$$G_2\begin{pmatrix}u\\ v\end{pmatrix}=\begin{pmatrix}\tilde{f}_2(u,v)\\ \tilde{g}_2(u,v)\end{pmatrix}=P^{-1}\cdot H_2\left(P\cdot\begin{pmatrix}u\\ v\end{pmatrix}\right).$$

By straightforward calculation we obtain that

$$\tilde{f}_2(u,v)=c\Upsilon_2(u,v)\left(\left(d^2+2D\right)^2-C^2\right),$$

$$\tilde{g}_2(u,v)=-c\Upsilon_2(u,v)\left(\left(d^2+2D\right)^2-C^2\right),$$

where

$$\Upsilon_2(u,v)=\frac{1}{2C}\left(2d\left(\frac{u}{-C+d^2+2D}+\frac{v}{C+d^2+2D}\right)^2-\frac{4d^3\left(\frac{u}{-C+d^2+2D}+\frac{v}{C+d^2+2D}\right)^2+(u+v)^2}{C+d^2+2D}\right).$$

3.5 Stable and unstable manifolds corresponding to the saddle period-two solution

If $c<\frac{(1-3d)(d+1)}{4}$ then Eq.(1) has minimal period-two solution $\{P_1(\bar{u}_1,\bar{v}_1),P_2(\bar{v}_1,\bar{u}_1)\}$ which is a saddle point.

In view of the fact that the T^2 is cooperative map if T is cooperative map, we can assume that the stable manifold W^s_{loc} at the period-two solution $(0,0)$, which corresponding to $(\bar{u}_1,\bar{v}_1)$, is the graph of the map

$$\varphi_2(\xi)=\alpha_3\xi^2+\beta_3\xi^3,\quad \alpha_3,\beta_3\in\mathbb{R}.$$

Now we compute the constants α_3 and β_3. The function φ_2 must satisfy the stable manifold equation

$$\varphi_2\left(\mu_1\xi+\tilde{g}_1\left(\xi,\varphi_2(\xi)\right)\right)=\mu_2\varphi_2(\xi)+\tilde{g}_2\left(\xi,\varphi_2(\xi)\right),$$

This leads to the following polynomial equation

$$\tilde{p}_1\xi^2+\tilde{p}_2\xi^3+\cdots+\tilde{p}_{18}\xi^{18}=0$$

where

$$\tilde{p}_1=\frac{1}{4}\alpha_3(-C+d(d+2)+2)^2+\frac{1}{2}\alpha_3(-C-d(d+2)-2)+\frac{cd\left(\left(d^2+2D\right)^2-C^2\right)}{C\left(-C+d^2+2D\right)^2}-\frac{c\left(\left(d^2+2D\right)^2-C^2\right)}{2C\left(-C+d^2+2D\right)}-\frac{2cd^3\left(\left(d^2+2D\right)^2-C^2\right)}{C\left(-C+d^2+2D\right)^3}\tag{32}$$

and

$$\tilde{p}_2=\frac{1}{8}\beta_3\left(-C^3+3C^2(d(d+2)+2)-C(3d(d+2)(d(d+2)+4)+16)+d(d+2)(d(d+2)+2)(d(d+2)+4)\right)+\frac{4\alpha_3c(-d(d+2)-4)\left((d+2)d^3+4(d-1)dD+4D^2\right)}{2C\left(-C+d^2+2D\right)}+\frac{4\alpha_3c\left(C^3-C^2(d(3d+4)+4D)+C\left(3d^4+8d^3+8d^2(D+1)+4dD+4D(D+2)\right)\right)}{2C\left(-C+d^2+2D\right)}.\tag{33}$$

By solving system $\tilde{p}_1=0$ and $\tilde{p}_2=0$, we obtain the values

$$\alpha_3=\frac{2c\left(C+d^2+2D\right)\left(d^2(4D-2C)+2d(C-2D)+(C-2D)^2+d^4+2d^3\right)}{C\left(C^2-2C(d(d+2)+3)+d(d+2)(d(d+2)+2)\right)\left(-C+d^2+2D\right)^2}$$

and

$$\beta_3=\frac{\alpha_3\Phi_1(c,d)}{\Phi_2(c,d)},$$

where

$$\Phi_1(c,d)=4c\left(-C^3+C^2(d(3d+4)+4D)-C\left(3d^4+8d^3+8d^2(D+1)+4dD+4D(D+2)\right)+(d(d+2)+4)\left((d+2)d^3+4(d-1)dD+4D^2\right)\right),\tag{34}$$

$$\Phi_2(c,d)=4C\left(-C^3+3C^2(d(d+2)+2)-C(3d(d+2)(d(d+2)+4)+16)+d(d+2)(d(d+2)+2)(d(d+2)+4)\right)\left(-C+d^2+2D\right).\tag{35}$$

Let us assume that the unstable manifold at the period two solution $(0,0)$, which corresponding to $(\bar{u}_1,\bar{v}_1)$, is the graph of the map

$$\psi_2(\eta)=\alpha_4\eta^2+\beta_4\eta^3,\quad \alpha_3,\beta_3\in\mathbb{R}.$$

Now we compute the constants α_4 and β_4. The function ψ_2 must satisfy the stable manifold equation

$$\psi_2(\mu_2\eta + \tilde{g}_2(\psi_2(\eta),\eta)) = \mu_1\psi_2(\eta) + \tilde{f}_2(\psi_2(\eta),\eta),$$

This leads to the following polynomial equation

$$\tilde{q}_1\eta^2 + \tilde{q}_2\eta^3 + \cdots + \tilde{q}_{18}\eta^{18} = 0$$

where

$$\begin{aligned}\tilde{q}_1 = {} & \frac{1}{4}A(C+d(d+2)+2)^2 + \frac{1}{2}A(C-d(d+2)-2)\\ & - \frac{cd\left(\left(d^2+2D\right)^2-C^2\right)}{C\left(C+d^2+2D\right)^2} + \frac{c\left(\left(d^2+2D\right)^2-C^2\right)}{2C\left(C+d^2+2D\right)} + \frac{2cd^3\left(\left(d^2+2D\right)^2-C^2\right)}{C\left(C+d^2+2D\right)^3} \end{aligned} \tag{36}$$

and

$$\begin{aligned}\tilde{q}_2 = {} & \frac{1}{8}B\left(C^3+3C^2(d(d+2)+2)+C(3d(d+2)(d(d+2)+4)+16)\right.\\ & \left.+d(d+2)(d(d+2)+2)(d(d+2)+4)\right) + \frac{\alpha_3 c(d(d+2)+4)\left((d+2)d^3+4(d-1)dD+4D^2\right)}{2C\left(C+d^2+2D\right)}\\ & + \frac{Ac\left(C^3+C^2(d(3d+4)+4D)+C\left(3d^4+8d^3+8d^2(D+1)+4dD+4D(D+2)\right)\right)}{2C\left(C+d^2+2D\right)}. \end{aligned} \tag{37}$$

By solving system $\tilde{q}_1 = 0$ and $\tilde{q}_2 = 0$, we obtain the values

$$\alpha_4 = -\frac{2c\left(-C+d^2+2D\right)\left(2d^2(C+2D)-2d(C+2D)+(C+2D)^2+d^4+2d^3\right)}{C\left(C^2+2C(d(d+2)+3)+d(d+2)(d(d+2)+2)\right)\left(C+d^2+2D\right)^2}$$

and

$$\beta_4 = \frac{\alpha_4\tilde{\Phi}_1(c,d)}{\tilde{\Phi}_2(c,d)},$$

where

$$\begin{aligned}\tilde{\Phi}_1(c,d) = {} & 4c\left(-C^3-C^2(d(3d+4)+4D)\right.\\ & -C\left(3d^4+8d^3+8d^2(D+1)+4dD+4D(D+2)\right)+(-d(d+2)-4)\\ & \left.\left((d+2)d^3+4(d-1)dD+4D^2\right)\right), \end{aligned} \tag{38}$$

$$\begin{aligned}\tilde{\Phi}_2(c,d) = {} & C\left(C^3+3C^2(d(d+2)+2)\right.\\ & \left.+C(3d(d+2)(d(d+2)+4)+16)+d(d+2)(d(d+2)+2)(d(d+2)+4)\right)\left(C+d^2+2D\right). \end{aligned} \tag{39}$$

As in the case of the saddle point equilibrium, one can show that we can approximate locally local stable manifold $\mathcal{W}^s_{loc}(\bar{u}_1,\bar{v}_1)$ and local unstable manifold $\mathcal{W}^u_{loc}(\bar{u}_1,\bar{v}_1)$ as the graph of the maps $\tilde{\varphi}_2(u)$ and $\tilde{\psi}_2(u)$ such that $\tilde{S}(u,\tilde{\varphi}_2(u)) = 0$ and $\tilde{U}(\tilde{\psi}_2(v),v) = 0$ hold, where

$$\begin{aligned}\tilde{S}(u,v) := {} & \alpha_3\left(\frac{(u-\bar{u}_1)\left(\left(d^2+2D\right)^2-C^2\right)}{4Cd} - \frac{(v-\bar{v}_1)\left(-C+d^2+2D\right)}{2C}\right)^2\\ & + \beta_3\left(\frac{(u-\bar{u}_1)\left(\left(d^2+2D\right)^2-C^2\right)}{4Cd} - \frac{(v-\bar{v}_1)\left(-C+d^2+2D\right)}{2C}\right)^3\\ & - \frac{(u-\bar{u}_1)\left(C^2-\left(d^2+2D\right)^2\right)}{4Cd} - \frac{(v-\bar{v}_1)\left(C+d^2+2D\right)}{2C}, \end{aligned} \tag{40}$$

$$\tilde{U}(u,v) := \alpha_4 \left(\frac{(u-\bar{u}_1)\left(C^2-\left(d^2+2D\right)^2\right)}{4Cd} + \frac{(v-\bar{v}_1)\left(C+d^2+2D\right)}{2C} \right)^2$$
$$+ \beta_4 \left(\frac{(u-\bar{u}_1)\left(C^2-\left(d^2+2D\right)^2\right)}{4Cd} + \frac{(v-\bar{v}_1)\left(C+d^2+2D\right)}{2C} \right)^3$$
$$- \frac{(u-\bar{u}_1)\left(\left(d^2+2D\right)^2-C^2\right)}{4Cd} + \frac{(v-\bar{v}_1)\left(-C+d^2+2D\right)}{2C} \quad (41)$$

Thus we proved the following result

Theorem 9 *Consider Eq.*(1) *subject to the condition* $c < \frac{(1-3d)(d+1)}{4}$. *Then the local stable and local unstable manifolds of the unique period-two solution are given with the asymptotic expansions* (40) *and* (41) *respectively.*

3.6 Some numerical examples

For $c = 0.09$ and $d = 0.23$ we have that

$$\begin{aligned}\tilde{S}_1(u,v) =& 0.147835(0.207875(v-7.64414) - 0.422497(u-6.02253))^3 \\ &- 0.418625(0.207875(v-7.64414) - 0.422497(u-6.02253))^2 \\ &- 0.422497(u-6.02253) - 0.792125(v-7.64414)\end{aligned}$$

and

$$\begin{aligned}\tilde{U}_1(u,v) =& -0.00042726(0.422497(u-6.02253) + 0.792125(v-7.64414))^3 \\ &+ 0.00729364(0.422497(u-6.02253) + 0.792125(v-7.64414))^2 \\ &+ 0.422497(u-6.02253) - 0.207875(v-7.64414).\end{aligned}$$

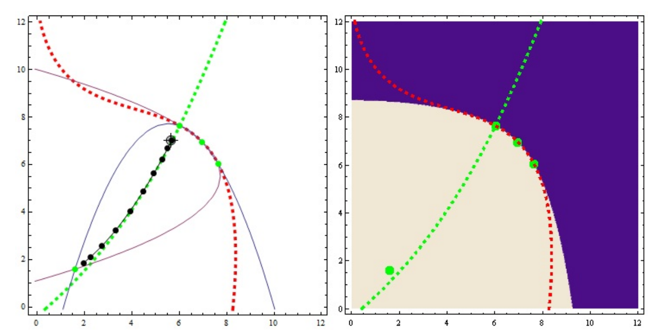

Figure 4: The graphs of the functions $\tilde{S}_1(u,v) = 0$ (red curve) and $\tilde{U}_1(u,v) = 0$ (green curve) for $c = 0.09$ and $d = 0.23$ with the basins of attraction generated by *Dynamica* 3.

For $c = 0.03$ and $d = 0.22$ we have that

$$\begin{aligned}\tilde{S}_2(u,v) =& 0.0912789(0.0236741(v-29.3826) - 0.125096(u-11.2841))^3 \\ &- 0.458495(0.0236741(v-29.3826) - 0.125096(u-11.2841))^2 \\ &- 0.125096(u-11.2841) - 0.976326(v-29.3826)\end{aligned}$$

and

$$\begin{aligned}\tilde{U}_2(u,v) =& 3.7\times 10^{-6}(0.125096(u-11.2841) + 0.976326(v-29.3826))^3 \\ &+ 0.000212577(0.125096(u-11.2841) + 0.976326(v-29.3826))^2 \\ &+ 0.125096(u-11.2841) - 0.0236741(v-29.3826)\end{aligned}$$

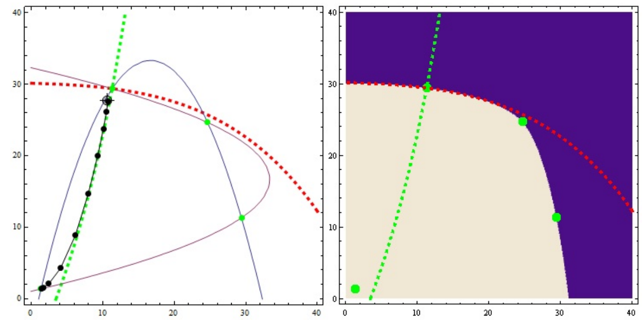

Figure 5: The graphs of the functions $\tilde{S}_2(u,v) = 0$ (red curve) and $\tilde{U}_2(u,v) = 0$ (green curve) for $c = 0.03$ and $d = 0.22$ with the basins of attraction generated by *Dynamica* 3.

Figures 4 and 5 show the graph of the functions $\tilde{S}_2(u,v) = 0$ and $\tilde{U}_2(u,v) = 0$ with the basins of attraction created with *Dynamica* 3. for different values of the parameters c and d.

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A Values of coefficients p_1, p_2, q_1 and q_2

$$
\begin{aligned}
p_1 =& -12c\beta_1\bar{x}_2^3d^6 - 12c^2\beta_1\bar{x}_2^4d^5 + 12c\beta_1\bar{x}_2^3d^5 - 4c\alpha_1\bar{x}_2^2d^5 - 12c\beta_1\bar{x}_2^2d^5 - d^5 - 168c^2\beta_1\bar{x}_2^4d^4\\
&- 4c\alpha_1\bar{x}_2^2d^4 - 168c^3\beta_1\bar{x}_2^5d^3 + 156c^2\beta_1\bar{x}_2^4d^3 - 64c^2\alpha_1\bar{x}_2^3d^3 - 168c^2\beta_1\bar{x}_2^3d^3 - 14c\bar{x}_2d^3\\
&- 600c^3\beta_1\bar{x}_2^5d^2 + 24c^2\beta_1\bar{x}_2^4d^2 - 64c^2\alpha_1\bar{x}_2^3d^2 - 24c^2\beta_1\bar{x}_2^3d^2 - 588c^4\beta_1\bar{x}_2^6d + 600c^3\beta_1\bar{x}_2^5d\\
&- 256c^3\alpha_1\bar{x}_2^4d - 600c^3\beta_1\bar{x}_2^4d - 12c^2\beta_1\bar{x}_2^4d + 24c^2\beta_1\bar{x}_2^3d - 48c^2\bar{x}_2^2d - 12c^2\beta_1\bar{x}_2^2d - 256c^3\alpha_1\bar{x}_2^4\\
&+ \sqrt{d^2+8c\bar{x}_2}\,\big(204c^4\beta_1\bar{x}_2^6 - 216c^3\beta_1\bar{x}_2^5 + 120c^3d^2\beta_1\bar{x}_2^5 + 216c^3d\beta_1\bar{x}_2^5 + 144c^3\alpha_1\bar{x}_2^4 + 12c^2d^4\beta_1\bar{x}_2^4\\
&+ 216c^3\beta_1\bar{x}_2^4 + 120c^2d^3\beta_1\bar{x}_2^4 + 12c^2\beta_1\bar{x}_2^4 - 108c^2d^2\beta_1\bar{x}_2^4 - 24c^2d\beta_1\bar{x}_2^4 - 16c^2\alpha_1\bar{x}_2^3\\
&+ 48c^2d^2\alpha_1\bar{x}_2^3 - 16c^2d\alpha_1\bar{x}_2^3 + 12cd^5\beta_1\bar{x}_2^3 - 12cd^4\beta_1\bar{x}_2^3 - 24c^2\beta_1\bar{x}_2^3 + 120c^2d^2\beta_1\bar{x}_2^3 + 24c^2d\beta_1\bar{x}_2^3\\
&-16c^2\bar{x}_2^2 + 4cd^4\alpha_1\bar{x}_2^2 - 4cd^3\alpha_1\bar{x}_2^2 + 16c^2\alpha_1\bar{x}_2^2 + 12cd^4\beta_1\bar{x}_2^2 + 12c^2\beta_1\bar{x}_2^2 - 10cd^2\bar{x}_2 - d^4\big),
\end{aligned}
\tag{42}
$$

$$
\begin{aligned}
p_2 =& -\beta_1\bar{x}_2d^6 + 3\alpha_1\beta_1\bar{x}_2^2d^5 + 6c\alpha_1\beta_1\bar{x}_2^3d^4 - 18c\beta_1\bar{x}_2^2d^4 - 6\alpha_1\beta_1\bar{x}_2^2d^4 + 2\alpha_1^2\bar{x}_2d^4 + 6\alpha_1\beta_1\bar{x}_2d^4\\
&- \beta_1\bar{x}_2d^4 + 3c^2\alpha_1\beta_1\bar{x}_2^4d^3 + 12c\alpha_1\beta_1\bar{x}_2^3d^3 + 2\alpha_1^2d^3 + 2c\alpha_1^2\bar{x}_2^2d^3 - 6c\beta_1\bar{x}_2^2d^3 + 6c\alpha_1\beta_1\bar{x}_2^2d^3\\
&+ 3\alpha_1\beta_1\bar{x}_2^2d^3 + \alpha_1d^3 + 3\alpha_1\beta_1d^3 - 2\alpha_1^2\bar{x}_2d^3 - 6\alpha_1\beta_1\bar{x}_2d^3 + 36c^2\alpha_1\beta_1\bar{x}_2^4d^2 - 102c^2\beta_1\bar{x}_2^3d^2\\
&- 36c\alpha_1\beta_1\bar{x}_2^3d^2 + 16c\alpha_1^2\bar{x}_2^2d^2 - 10c\beta_1\bar{x}_2^2d^2 + 36c\alpha_1\beta_1\bar{x}_2^2d^2 - 4c\alpha_1\bar{x}_2d^2 - 6c\beta_1\bar{x}_2d^2\\
&+ 18c^3\alpha_1\beta_1\bar{x}_2^5d - 36c^2\alpha_1\beta_1\bar{x}_2^4d + 16c^2\alpha_1^2\bar{x}_2^3d - 48c^2\beta_1\bar{x}_2^3d + 36c^2\alpha_1\beta_1\bar{x}_2^3d + 18c\alpha_1\beta_1\bar{x}_2^3d\\
&- 16c\alpha_1^2\bar{x}_2^2d - 36c\alpha_1\beta_1\bar{x}_2^2d + 16c\alpha_1^2\bar{x}_2d + 8c\alpha_1\bar{x}_2d + 18c\alpha_1\beta_1\bar{x}_2d - 176c^3\beta_1\bar{x}_2^4 - 16c^2\beta_1\bar{x}_2^3\\
&- 32c^2\alpha_1\bar{x}_2^2 - 48c^2\beta_1\bar{x}_2^2 + \sqrt{d^2+8c\bar{x}_2}\,\big(\beta_1\bar{x}_2d^5 - 3\alpha_1\beta_1\bar{x}_2^2d^4 - 6c\alpha_1\beta_1\bar{x}_2^3d^3 + 14c\beta_1\bar{x}_2^2d^3\\
&+ 6\alpha_1\beta_1\bar{x}_2^2d^3 - 2\alpha_1^2\bar{x}_2d^3 - 6\alpha_1\beta_1\bar{x}_2d^3 - \beta_1\bar{x}_2d^3 - 3c^2\alpha_1\beta_1\bar{x}_2^4d^2 - 2\alpha_1^2d^2 - 2c\alpha_1^2\bar{x}_2^2d^2\\
&+ 6c\beta_1\bar{x}_2^2d^2 - 6c\alpha_1\beta_1\bar{x}_2^2d^2 - 3\alpha_1\beta_1\bar{x}_2^2d^2 + \alpha_1d^2 - 3\alpha_1\beta_1d^2 + 2\alpha_1^2\bar{x}_2d^2 + 6\alpha_1\beta_1\bar{x}_2d^2\\
&- 12c^2\alpha_1\beta_1\bar{x}_2^4d + 54c^2\beta_1\bar{x}_2^3d + 12c\alpha_1\beta_1\bar{x}_2^3d - 8c\alpha_1^2\bar{x}_2^2d - 14c\beta_1\bar{x}_2^2d - 12c\alpha_1\beta_1\bar{x}_2^2d\\
&+ 6c\beta_1\bar{x}_2d - 6c^3\alpha_1\beta_1\bar{x}_2^5 + 12c^2\alpha_1\beta_1\bar{x}_2^4 - 8c^2\alpha_1^2\bar{x}_2^3 - 12c^2\alpha_1\beta_1\bar{x}_2^3 - 6c\alpha_1\beta_1\bar{x}_2^3 + 8c\alpha_1^2\bar{x}_2^2\\
&+12c\alpha_1\beta_1\bar{x}_2^2 - 8c\alpha_1^2\bar{x}_2 + 8c\alpha_1\bar{x}_2 - 6c\alpha_1\beta_1\bar{x}_2\big)
\end{aligned}
\tag{43}
$$

$$
\begin{aligned}
q_1 =&12\beta_2c\bar{x}_2^3d^7 + 12\beta_2c^2\bar{x}_2^4d^6 - 12\beta_2c\bar{x}_2^3d^6 + 4\alpha_2c\bar{x}_2^2d^6 + 12\beta_2c\bar{x}_2^2d^6 - d^6 + 216\beta_2c^2\bar{x}_2^4d^5 - 4\alpha_2c\bar{x}_2^2d^5\\
&+ 216\beta_2c^3\bar{x}_2^5d^4 - 204\beta_2c^2\bar{x}_2^4d^4 + 80\alpha_2c^2\bar{x}_2^3d^4 + 216\beta_2c^2\bar{x}_2^3d^4 - 18c\bar{x}_2d^4 + 1176\beta_2c^3\bar{x}_2^5d^3\\
&- 24\beta_2c^2\bar{x}_2^4d^3 - 48\alpha_2c^2\bar{x}_2^3d^3 + 24\beta_2c^2\bar{x}_2^3d^3 + 1164\beta_2c^4\bar{x}_2^6d^2 - 1080\beta_2c^3\bar{x}_2^5d^2 + 528\alpha_2c^3\bar{x}_2^4d^2\\
&+ 1176\beta_2c^3\bar{x}_2^4d^2 + 12\beta_2c^2\bar{x}_2^4d^2 - 16\alpha_2c^2\bar{x}_2^3d^2 - 24\beta_2c^2\bar{x}_2^3d^2 + 16\alpha_2c^2\bar{x}_2^2d^2 + 12\beta_2c^2\bar{x}_2^2d^2\\
&- 96c^2\bar{x}_2^2d^2 + 1728\beta_2c^4\bar{x}_2^6d - 192\beta_2c^3\bar{x}_2^5d - 128\alpha_2c^3\bar{x}_2^4d + 192\beta_2c^3\bar{x}_2^4d + 1632\beta_2c^5\bar{x}_2^7\\
&- 1728\beta_2c^4\bar{x}_2^6 + 1152\alpha_2c^4\bar{x}_2^5 + 1728\beta_2c^4\bar{x}_2^5 + 96\beta_2c^3\bar{x}_2^5 - 128\alpha_2c^3\bar{x}_2^4 - 192\beta_2c^3\bar{x}_2^4 + 128\alpha_2c^3\bar{x}_2^3\\
&+ 96\beta_2c^3\bar{x}_2^3 - 128c^3\bar{x}_2^3 + \sqrt{d^2+8c\bar{x}_2}\,\big(12\beta_2c\bar{x}_2^3d^6 + 12\beta_2c^2\bar{x}_2^4d^5 - 12\beta_2c\bar{x}_2^3d^5 + 4\alpha_2c\bar{x}_2^2d^5\\
&+ 12\beta_2c\bar{x}_2^2d^5 + d^5 + 168\beta_2c^2\bar{x}_2^4d^4 + 4\alpha_2c\bar{x}_2^2d^4 + 168\beta_2c^3\bar{x}_2^5d^3 - 156\beta_2c^2\bar{x}_2^4d^3 + 64\alpha_2c^2\bar{x}_2^3d^3\\
&+ 168\beta_2c^2\bar{x}_2^3d^3 + 14c\bar{x}_2d^3 + 600\beta_2c^3\bar{x}_2^5d^2 - 24\beta_2c^2\bar{x}_2^4d^2 + 64\alpha_2c^2\bar{x}_2^3d^2 + 24\beta_2c^2\bar{x}_2^3d^2\\
&+ 588\beta_2c^4\bar{x}_2^6d - 600\beta_2c^3\bar{x}_2^5d + 256\alpha_2c^3\bar{x}_2^4d + 600\beta_2c^3\bar{x}_2^4d + 12\beta_2c^2\bar{x}_2^4d - 24\beta_2c^2\bar{x}_2^3d\\
&+12\beta_2c^2\bar{x}_2^2d + 48c^2\bar{x}_2^2d + 256\alpha_2c^3\bar{x}_2^4\big),
\end{aligned}
\tag{44}
$$

16

$$
\begin{aligned}
q_2 =&\beta_2\bar{x}_2d^6 - 3\alpha_2\beta_2\bar{x}_2^2d^5 - 6c\alpha_2\beta_2\bar{x}_2^3d^4 + 18c\beta_2\bar{x}_2^2d^4 + 6\alpha_2\beta_2\bar{x}_2^2d^4 - 2\alpha_2^2\bar{x}_2d^4 - 6\alpha_2\beta_2\bar{x}_2d^4\\
&+ \beta_2\bar{x}_2d^4 - 3c^2\alpha_2\beta_2\bar{x}_2^4d^3 - 12c\alpha_2\beta_2\bar{x}_2^3d^3 - 2\alpha_2^2d^3 - 2c\alpha_2^2\bar{x}_2^2d^3 + 6c\beta_2\bar{x}_2^2d^3 - 6c\alpha_2\beta_2\bar{x}_2^2d^3\\
&- 3\alpha_2\beta_2\bar{x}_2^2d^3 - \alpha_2d^3 - 3\alpha_2\beta_2d^3 + 2\alpha_2^2\bar{x}_2d^3 + 6\alpha_2\beta_2\bar{x}_2d^3 - 36c^2\alpha_2\beta_2\bar{x}_2^4d^2 + 102c^2\beta_2\bar{x}_2^3d^2\\
&+ 36c\alpha_2\beta_2\bar{x}_2^3d^2 - 16c\alpha_2^2\bar{x}_2^2d^2 + 10c\beta_2\bar{x}_2^2d^2 - 36c\alpha_2\beta_2\bar{x}_2^2d^2 + 4c\alpha_2\bar{x}_2d^2 + 6c\beta_2\bar{x}_2d^2\\
&- 18c^3\alpha_2\beta_2\bar{x}_2^5d + 36c^2\alpha_2\beta_2\bar{x}_2^4d - 16c^2\alpha_2^2\bar{x}_2^3d + 48c^2\beta_2\bar{x}_2^3d - 36c^2\alpha_2\beta_2\bar{x}_2^3d - 18c\alpha_2\beta_2\bar{x}_2^3d\\
&+ 16c\alpha_2^2\bar{x}_2^2d + 36c\alpha_2\beta_2\bar{x}_2^2d - 16c\alpha_2^2\bar{x}_2d - 8c\alpha_2\bar{x}_2d - 18c\alpha_2\beta_2\bar{x}_2d + 176c^3\beta_2\bar{x}_2^4 + 16c^2\beta_2\bar{x}_2^3\\
&+ 32c^2\alpha_2\bar{x}_2^2 + 48c^2\beta_2\bar{x}_2^2 + \sqrt{d^2+8c\bar{x}_2}\,\big(\beta_2\bar{x}_2d^5 - 3\alpha_2\beta_2\bar{x}_2^2d^4 - 6c\alpha_2\beta_2\bar{x}_2^3d^3 + 14c\beta_2\bar{x}_2^2d^3\\
&+ 6\alpha_2\beta_2\bar{x}_2^2d^3 - 2\alpha_2^2\bar{x}_2d^3 - 6\alpha_2\beta_2\bar{x}_2d^3 - \beta_2\bar{x}_2d^3 - 3c^2\alpha_2\beta_2\bar{x}_2^4d^2 - 2\alpha_2^2d^2 - 2c\alpha_2^2\bar{x}_2^2d^2\\
&+ 6c\beta_2\bar{x}_2^2d^2 - 6c\alpha_2\beta_2\bar{x}_2^2d^2 - 3\alpha_2\beta_2\bar{x}_2^2d^2 + \alpha_2d^2 - 3\alpha_2\beta_2d^2 + 2\alpha_2^2\bar{x}_2d^2 + 6\alpha_2\beta_2\bar{x}_2d^2\\
&- 12c^2\alpha_2\beta_2\bar{x}_2^4d + 54c^2\beta_2\bar{x}_2^3d + 12c\alpha_2\beta_2\bar{x}_2^3d - 8c\alpha_2^2\bar{x}_2^2d - 14c\beta_2\bar{x}_2^2d - 12c\alpha_2\beta_2\bar{x}_2^2d\\
&+ 6c\beta_2\bar{x}_2d - 6c^3\alpha_2\beta_2\bar{x}_2^5 + 12c^2\alpha_2\beta_2\bar{x}_2^4 - 8c^2\alpha_2^2\bar{x}_2^3 - 12c^2\alpha_2\beta_2\bar{x}_2^3 - 6c\alpha_2\beta_2\bar{x}_2^3 + 8c\alpha_2^2\bar{x}_2^2\\
&+12c\alpha_2\beta_2\bar{x}_2^2 - 8c\alpha_2^2\bar{x}_2 + 8c\alpha_2\bar{x}_2 - 6c\alpha_2\beta_2\bar{x}_2\big)
\end{aligned}
\tag{45}
$$

Fractional differential equations with integral and ordinary-fractional flux boundary conditions

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Abstract

We investigate the existence of solutions for a coupled system of fractional differential equations with integral and ordinary-fractional flux boundary conditions. The existence results are derived via Schauder's fixed point theorem and Leray-Schauder's alternative, while the uniqueness of solutions is established by applying Banach's contraction principle. Several new results appear as a special case of the present work with appropriate choice of the parameters involved in the problem at hand.

Key words and phrases: Fractional differential systems; nonlocal boundary conditions; integral boundary conditions; fixed point theorem.
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1 Introduction

In this paper, we study a coupled system of Caputo type fractional differential equations:

$$\begin{cases} {}^cD^{\alpha}x(t) = f(t, x(t), y(t)), & t \in [0,1], \quad 1 < \alpha \leq 2 \\ {}^cD^{\beta}y(t) = h(t, x(t), y(t)), & t \in [0,1], \quad 1 < \beta \leq 2, \end{cases} \tag{1}$$

supplemented with integral and ordinary-fractional flux boundary conditions:

$$\begin{cases} x(0) + x(1) = a \displaystyle\int_0^1 x(s)ds, & x'(0) = b \; {}^cD^{\gamma}x(1), \quad 0 < \gamma \leq 1, \\ y(0) + y(1) = a_1 \displaystyle\int_0^1 y(s)ds, & y'(0) = b_1 \; {}^cD^{\delta}y(1), \quad 0 < \delta \leq 1, \end{cases} \tag{2}$$

where ${}^cD^{\alpha}, {}^cD^{\beta}$ denote the Caputo fractional derivatives of orders α and β respectively, $f, h : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions and a, b, a_1, b_1 are real constants.

Fractional differential equations arise in the mathematical modeling of systems and processes occurring in many engineering and scientific disciplines such as biophysics, blood flow phenomena, control theory, aerodynamics, electrodynamics of complex medium, polymer rheology, signal and image processing to name a few [1]-[4]. The popularity of fractional order operators owes to their ability to describe the hereditary properties of various materials and processes. With this distinguished capability, fractional order models have become more realistic and practical than the corresponding classical integer order models. For some recent development on the topic, see [5]-[15] and the references therein. The investigation of coupled systems of fractional order differential equations is also very significant as such systems appear in a variety of problems of applied nature, especially in biosciences. For details and examples, the reader is referred to the papers [16]-[23] and the references cited therein.

The paper is organized as follows. In Section 2, we recall some basic definitions of fractional calculus and present some auxiliary lemmas. The main results are presented in Section 3. We give two existence results relying on Leray-Schauder's alternative and Schauder's fixed point theorem, while the uniqueness result is established by means of Banach's contraction mapping principle. It is worthwhile to note that our results are not only new in the present configuration but also correspond to some new special results for different values of the parameters involved in the given problem.

2 Preliminaries

Before presenting an auxiliary lemma, we recall some basic definitions of fractional calculus [1, 2].

Definition 2.1 *For $(n-1)-$times absolutely continuous function $g:[0,\infty)\to\mathbb{R}$, the Caputo derivative of fractional order q is defined as*

$$^cD^q g(t)=\frac{1}{\Gamma(n-q)}\int_0^t (t-s)^{n-q-1}g^{(n)}(s)ds,\quad n-1<q<n, n=[q]+1,$$

where $[q]$ denotes the integer part of the real number q.

Definition 2.2 *The Riemann-Liouville fractional integral of order q is defined as*

$$I^q g(t)=\frac{1}{\Gamma(q)}\int_0^t \frac{g(s)}{(t-s)^{1-q}}ds,\quad q>0,$$

provided the integral exists.

Lemma 2.3 *(see [1], [2]) (i) If $\alpha>0,\beta>0,\beta>\alpha, f\in L(0,1)$ then*

$$I^\alpha I^\beta f(t)=I^{\alpha+\beta}f(t),\ D^\alpha I^\alpha f(t)=f(t),\ D^\alpha I^\beta f(t)=I^{\beta-\alpha}f(t).$$

$$(ii)\quad {}^cD^\alpha t^{\lambda-1}=\frac{\Gamma(\lambda)}{\Gamma(\lambda-\alpha)}t^{\lambda-\alpha-1},\quad \lambda>[\alpha]\ \ and\ \ {}^cD^\alpha t^{\lambda-1}=0,\quad \lambda<[\alpha].$$

To define the solution for the problem (1)-(2), we use the following lemma.

Lemma 2.4 *Let $a\neq 2$ and $\Gamma(2-\beta)\neq b$. For $\phi\in C([0,1],\mathbb{R})$, the integral solution of the linear problem*

$$\begin{cases} {}^cD^\alpha x(t)=\phi(t), & t\in[0,1],\quad 1<q\le 2,\\ x(0)+x(1)=a\displaystyle\int_0^1 x(s)ds, & x'(0)=b\ {}^cD^\gamma x(1),\quad 0<\gamma\le 1,\end{cases}\tag{3}$$

is given by

$$\begin{aligned} x(t)= & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\phi(s)ds+\frac{b(2t-1)\Gamma(2-\gamma)}{2(\Gamma(2-\gamma)-b)}\int_0^1\frac{(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}\phi(s)ds\\ & -\frac{1}{2-a}\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\phi(s)ds+\frac{a}{2-a}\int_0^1\frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)}\phi(s)ds.\end{aligned}\tag{4}$$

Proof. As argued in [2], the general solution of the fractional differential equation in (3) can be written as

$$x(t)=c_0+c_1t+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(q)}\phi(s)ds,\tag{5}$$

where $c_0,c_1\in\mathbb{R}$ are arbitrary constants.

Using the boundary condition $x'(0)=b\ {}^cD^\gamma x(1)$ in (5), we find that

$$c_1=\frac{b\Gamma(2-\beta)}{\Gamma(2-\beta)-b}\int_0^1\frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\phi(s)ds.$$

In view of the condition $x(0)+x(1)=a\int_0^1 x(s)ds$, (5) yields

$$2c_0+c_1+\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\phi(s)ds=a\int_0^1\int_0^s\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}\phi(u)du+\frac{ac_1}{2}+ac_0,$$

which, on inserting the value of c_1 and using the composition law of Riemann-Liouville integration, gives

$$\begin{aligned} c_0 &= -\frac{1}{2}\frac{b\Gamma(2-\beta)}{[\Gamma(2-\beta)-b]}\int_0^1\frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\phi(s)ds \\ &\quad +\frac{a}{2-a}\int_0^1\frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)}\phi(s)ds-\frac{1}{2-a}\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\phi(s)ds. \end{aligned}$$

Substituting the values of c_0, c_1 in (5) yields (4). This completes the proof. □

3 Main Results

Let us introduce the space $X_i=\{u_i(t)|u_i(t)\in C([0,1])\}$ endowed with the norm $\|u_i\|=\sup\{|u_i(t)|, t\in[0,1]\}$, $i=1,2$. Obviously $(X_i,\|\cdot\|)$ is a Banach space. In consequence, the product space $(X_1\times X_2,\|(u_1,u_2)\|)$ is also a Banach space with norm $\|(u_1,u_2)\|=\|u_1\|+\|u_2\|$.

In view of Lemma 2.4, we define an operator $T: X_1\times X_2\to X_1\times X_2$ by

$$T(u,v)(t)=\begin{pmatrix} T_1(u,v)(t) \\ T_2(u,v)(t)\end{pmatrix},$$

where

$$\begin{aligned} T_1(u,v)(t) &= \int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s),v(s))ds+\frac{b(2t-1)\Gamma(2-\gamma)}{2(\Gamma(2-\gamma)-b)}\int_0^1\frac{(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}f(s,u(s),v(s))ds \\ &\quad +\frac{1}{2-a}\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s),v(s))ds-\frac{a}{2-a}\int_0^1\frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)}f(s,u(s),v(s))ds, \end{aligned}$$

and

$$\begin{aligned} T_2(u,v)(t) &= \int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\alpha)}h(s,u(s),v(s))ds+\frac{b_1(2t-1)\Gamma(2-\delta)}{2(\Gamma(2-\delta)-b_1)}\int_0^1\frac{(1-s)^{\beta-\delta-1}}{\Gamma(\beta-\delta)}h(s,u(s),v(s))ds \\ &\quad +\frac{1}{2-a_1}\int_0^1\frac{(1-s)^{\beta-1}}{\Gamma(\beta)}h(s,u(s),v(s))ds-\frac{a_1}{2-a_1}\int_0^1\frac{(1-s)^{\beta}}{\Gamma(\beta+1)}h(s,u(s),v(s))ds. \end{aligned}$$

For the sake of convenience, let us set

$$M_1=\frac{1+|2-a|}{|2-a|\Gamma(\alpha+1)}+\frac{|b|\Gamma(2-\gamma)}{2|\Gamma(2-\gamma)-b|\Gamma(\alpha-\gamma+1)}+\frac{|a|}{|2-a|\Gamma(\alpha+2)}, \tag{6}$$

$$M_2=\frac{1+|2-a_1|}{|2-a_1|\Gamma(\beta+1)}+\frac{|b_1|\Gamma(2-\delta)}{2|\Gamma(2-\delta)-b_1|\Gamma(\beta-\delta+1)}+\frac{|a_1|}{|2-a_1|\Gamma(\beta+2)}. \tag{7}$$

We need the following known theorems in the sequel.

Lemma 3.1 *(Schauder's fixed point theorem) [24]. Let U be a closed, convex and nonempty subset of a Banach space X. Let $P: U\to U$ be a continuous mapping such that $P(U)$ is a relatively compact subset of X. Then P has at least one fixed point in U.*

Lemma 3.2 *(Leray-Schauder alternative) ([24] p. 4.) Let $F: E\to E$ be a completely continuous operator (i.e., a map that restricted to any bounded set in E is compact). Let*

$$\mathcal{E}(F)=\{x\in E: x=\lambda F(x) \text{ for some } 0<\lambda<1\}.$$

Then either the set $\mathcal{E}(F)$ is unbounded, or F has at least one fixed point.

3.1 Existence results

Here we study the existence of solutions for the system (1)-(2) by means of Schauder's fixed point theorem and Leray-Schauder alternative.

Theorem 3.3 *Assume that there exist positive constants $c_i, d_i, e_i \in (0,\infty)$, $i = 1,2$ such that the following condition holds:*

(H_1) $|f(t,x,y)| \le c_1|x|^{\rho_1} + d_1|y|^{\sigma_1} + e_1$, *and*
$|h(t,x,y)| \le c_2|x|^{\rho_2} + d_2|y|^{\sigma_2} + e_2, \quad 0 < \rho_i, \sigma_i < 1, \ i = 1,2.$

Then the system (1)-(2) has at least one solution on $[0,1]$.

Proof. Define a ball in Banach space $X_1 \times X_2$ as $B_R = \{(u,v) : (u,v) \in X_1 \times X_2, \|(u,v)\| \le R\}$, where

$$R \ge \max\{(6M_ic_i)^{\frac{1}{1-\rho_i}}, (6M_id_i)^{\frac{1}{1-\sigma_i}}, 6M_ie_i, \}, \ i = 1,2. \tag{8}$$

Obviously B_R is a closed, bounded and convex subset of the Banach space $X_1 \times X_2$. In the first step, we show that $T : B_R \to B_R$. For $(u,v) \in B_R$. For that we have

$$\begin{aligned}
|T_1(u,v)(t)| \ \le\ & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u(s),v(s))|ds \\
&+\frac{|b(2t-1)|\Gamma(2-\gamma)}{2|\Gamma(2-\gamma)-b|}\int_0^1 \frac{(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}|f(s,u(s),v(s))|ds \\
&+\frac{1}{|2-a|}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u(s),v(s))|ds + \frac{|a|}{|2-a|}\int_0^1 \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)}|f(s,u(s),v(s))|ds \\
\le\ & (c_1R^{\rho_1} + d_1R^{\sigma_1} + e_1)\Bigg\{\frac{1}{\Gamma(\alpha+1)} + \frac{|b|\Gamma(2-\gamma)}{2|\Gamma(2-\gamma)-b|\Gamma(\alpha-\gamma+1)} \\
&+\frac{1}{|2-a|\Gamma(\alpha+1)} + \frac{|a|}{|2-a|\Gamma(\alpha+2)}\Bigg\},
\end{aligned}$$

which implies that

$$\|T_1(u,v)\| \le M_1(c_1R^{\rho_1} + d_1R^{\sigma_1} + e_1) \le \frac{R}{6} + \frac{R}{6} + \frac{R}{6} = \frac{R}{2}.$$

Similarly, we can obtain

$$\|T_2(u,v)\| \le M_2(c_2R^{\rho_2} + d_2R^{\sigma_2} + e_2) \le \frac{R}{6} + \frac{R}{6} + \frac{R}{6} = \frac{R}{2}.$$

Clearly

$$\|T(u,v)\| = \|T_1(u,v)\| + \|T_2(u,v)\| \le R,$$

and in consequence we get $T : B_R \to B_R$.

Observe that continuity of f, h implies that T is continuous. Next, we shall show that for every bounded subset B_R of $X_1 \times X_2$ the family $F(B_R)$ is equicontinuous. Since f, g are continuous, we can assume that $|f(t,u(t),v(t)| \le N_1$ and $|h(t,u(t),v(t)| \le N_2$ for any $u, v \in B_R$ and $t \in [0,1]$.

Now let $0 \le t_1 < t_2 \le 1$. Then we have

$$\begin{aligned}
& |T_1(u,v)(t_2) - T_1(u,v)(t_1)| \\
\le\ & \left|\frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}f(s,u(s),v(s))ds - \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}f(s,u(s),v(s))ds\right| \\
&+\frac{2|b|\Gamma(2-\gamma)|t_2-t_1|}{2|\Gamma(2-\gamma)-b|}\int_0^1 \frac{(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}|f(s,u(s),v(s))|ds
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{N_1}{\Gamma(\alpha)}\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]ds+\frac{N_1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds\\
&\quad+\frac{2N_1|b|\Gamma(2-\gamma)|t_2-t_1|}{2|\Gamma(2-\gamma)-b|}\int_0^1\frac{(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}ds\\
&\leq \frac{N_1}{\Gamma(\alpha+1)}(t_2^\alpha-t_1^\alpha)+\frac{2N_1|b|\Gamma(2-\gamma)|t_2-t_1|}{2|\Gamma(2-\gamma)-b|\Gamma(\alpha-\gamma+1)}.
\end{aligned}$$

Analogously, we can have

$$|T_2(u,v)(t_2)-T_2(u,v)(t_1)|\leq\frac{N_2}{\Gamma(\beta+1)}(t_2^\alpha-t_1^\alpha)+\frac{2N_2|b|\Gamma(2-\delta)|t_2-t_1|}{2|\Gamma(2-\delta)-b|\Gamma(\beta-\delta+1)}.$$

So

$$\|T_1(u,v)(t_2)-T_1(u,v)(t_1)\|\to 0,\quad \|T_2(u,v)(t_2)-T_2(u,v)(t_1)\|\to 0,\quad \text{as } t_1\to t_2.$$

Therefore it follows that the operator $T: B_R\to B_R$ is equicontinuous and uniformly bounded. Hence, by Arzelá-Ascoli theorem, T is completely continuous operator. Thus all the conditions of Theorem 3.1 are satisfied, which in turn, implies that the problem (1) has at least one solution. This completes the proof. □

Remark 3.4 *For $\rho_i,\sigma_i>1$ $(i=1,2)$ in the condition (H_1), the conclusion of Theorem 3.6 remains true with a modified value of R given by (8).*

Theorem 3.5 *Assume that:*

(H_2) There exist real constants $k_i,\ \lambda_i\geq 0$ $(i=1,2)$ and $k_0>0,\lambda_0>0$ such that $\forall x_i\in\mathbb{R},\ i=1,2,$ we have

$$|f(t,x_1,x_2)|\leq k_0+k_1|x_1|+k_2|x_2|,\ |h(t,x_1,x_2)|\leq\lambda_0+\lambda_1|x_1|+\lambda_2|x_2|.$$

Then the system (1)-(2) has at least one solution, provided

$$M_1k_1+M_2\lambda_1<1 \quad\text{and}\quad M_1k_2+M_2\lambda_2<1,$$

where M_1 and M_2 are given by (6) and (7) respectively.

Proof. First we show that the operator $T: X_1\times X_2\to X_1\times X_2$ is completely continuous. By continuity of functions f and h, the operator T is continuous.

Let $\Omega\subset X_1\times X_2$ be bounded. Then there exist positive constants L_1 and L_2 such that

$$|f(t,u(t),v(t)|\leq L_1,\quad |h(t,u(t),v(t)|\leq L_2,\quad \forall (u,v)\in\Omega.$$

Then for any $(u,v)\in\Omega$, we have

$$\begin{aligned}
|T_1(u,v)(t)| &\leq \int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u(s),v(s))|ds\\
&\quad+\frac{|b(2t-1)|\Gamma(2-\gamma)}{2|\Gamma(2-\gamma)-b|}\int_0^1\frac{(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}|f(s,u(s),v(s))|ds\\
&\quad+\frac{1}{|2-a|}\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u(s),v(s))|ds+\frac{|a|}{|2-a|}\int_0^1\frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)}|f(s,u(s),v(s))|ds\\
&\leq L_1\left\{\frac{1}{\Gamma(\alpha+1)}+\frac{|b|\Gamma(2-\gamma)}{2|\Gamma(2-\gamma)-b|\Gamma(\alpha-\gamma+1)}+\frac{1}{|2-a|\Gamma(\alpha+1)}+\frac{|a|}{|2-a|\Gamma(\alpha+2)}\right\},
\end{aligned}$$

which implies that

$$\|T_1(u,v)\| \leq L_1\left\{\frac{1}{\Gamma(\alpha+1)}+\frac{|b|\Gamma(2-\gamma)}{2|\Gamma(2-\gamma)-b|\Gamma(\alpha-\gamma+1)}\right.$$

$$+\frac{1}{|2-a|\Gamma(\alpha+1)}+\frac{|a|}{|2-a|\Gamma(\alpha+2)}\Bigg\}=L_1M_1.$$

Similarly, we can get

$$\begin{aligned}\|T_2(u,v)\| &\le L_2\Bigg\{\frac{1}{\Gamma(\beta+1)}+\frac{|b_1|\Gamma(2-\delta)}{2|\Gamma(2-\delta)-b_1|\Gamma(\beta-\delta+1)}\\ &\quad+\frac{1}{|2-a_1|\Gamma(\beta+1)}+\frac{|a_1|}{|2-a_1|\Gamma(\beta+2)}\Bigg\}=L_2M_2.\end{aligned}$$

Thus, it follows from the above inequalities that the operator T is uniformly bounded.

Next, we show that T is equicontinuous. Let $t_1,t_2\in[0,1]$ with $t_1<t_2$. Then we have

$$\begin{aligned}&|T_1(u(t_2),v(t_2))-T_1(u(t_1),v(t_1))|\\ \le\ & L_1\left|\frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}ds-\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}ds\right|\\ &+L_1\frac{2|b|\Gamma(2-\gamma)|t_2-t_1|}{2|\Gamma(2-\gamma)-b|}\int_0^1\frac{(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}ds\\ \le\ & \frac{L_1}{\Gamma(\alpha)}\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]ds+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds\\ &+\frac{2|b|L_1\Gamma(2-\gamma)|t_2-t_1|}{2|\Gamma(2-\gamma)-b|}\int_0^1\frac{(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}ds\\ \le\ & \frac{L_1}{\Gamma(\alpha+1)}(t_2^\alpha-t_1^\alpha)+\frac{2L_1|b|\Gamma(2-\gamma)|t_2-t_1|}{2|\Gamma(2-\gamma)-b|\Gamma(\alpha-\gamma+1)}.\end{aligned}$$

Analogously, we can obtain

$$|T_2(u(t_2),v(t_2))-T_2(u(t_1),v(t_1))|\le\frac{L_2}{\Gamma(\alpha+1)}(t_2^\alpha-t_1^\alpha)+\frac{2L_2|b_1|\Gamma(2-\delta)|t_2-t_1|}{2|\Gamma(2-\delta)-b_1|\Gamma(\beta-\delta+1)}.$$

Therefore, the operator $T(u,v)$ is equicontinuous, and thus the operator $T(u,v)$ is completely continuous.

Finally, it will be verified that the set $\mathcal{E}=\{(u,v)\in X_1\times X_2|(u,v)=\lambda T(u,v), 0\le\lambda\le1\}$ is bounded. Let $(u,v)\in\mathcal{E}$, then $(u,v)=\lambda T(u,v)$. For any $t\in[0,1]$, we have

$$u(t)=\lambda T_1(u,v)(t),\quad v(t)=\lambda T_2(u,v)(t).$$

Then

$$|u(t)|\le\left\{\frac{1+|2-a|}{|2-a|\Gamma(\alpha+1)}+\frac{|b|\Gamma(2-\gamma)}{2|\Gamma(2-\gamma)-b|\Gamma(\alpha-\gamma+1)}+\frac{|a|}{|2-a|\Gamma(\alpha+2)}\right\}(k_0+k_1\|u\|+k_2\|v\|),$$

and

$$|v(t)|\le\left\{\frac{1+|2-a_1|}{|2-a_1|\Gamma(\alpha+1)}+\frac{|b|\Gamma(2-\delta)}{2|\Gamma(2-\delta)-b_1|\Gamma(\beta-\delta+1)}+\frac{|a|}{|2-a|\Gamma(\beta+2)}\right\}(\lambda_0+\lambda_1\|u\|+\lambda_2\|v\|).$$

Hence we have

$$\|u\|\le M_1(k_0+k_1\|u\|+k_2\|v\|),\ \|v\|\le M_2(\lambda_0+\lambda_1\|u\|+\lambda_2\|v\|),$$

which imply that

$$\|u\|+\|v\|=(M_1k_0+M_2\lambda_0)+(M_1k_1+M_2\lambda_1)\|u\|+(M_1k_2+M_2\lambda_2)\|v\|.$$

Consequently,

$$\|(u,v)\| \le \frac{M_1 k_0 + M_2 \lambda_0}{M_0},$$

for any $t \in [0,1]$, where

$$M_0 = \min\{1-(M_1k_1+M_2\lambda_1),\ 1-(M_1k_2+M_2\lambda_2)\},\quad k_i,\ \lambda_i \ge 0\ (i=1,2).$$

This shows that $\mathcal{E}$ is bounded. Thus, by Lemma 3.2, the operator T has at least one fixed point. Hence the problem (1)-(2) has at least one solution. This completes the proof. □

3.2 Uniqueness of solutions

In this subsection, we prove the uniqueness of solutions for the system (1)-(2) via Banach's contraction mapping principle.

Theorem 3.6 *Assume that*

(H_3) $f,h:[0,1]\times\mathbb{R}^2 \to \mathbb{R}$ are continuous functions and there exist constants m_i, n_i $(i=1,2)$ such that for all $t\in[0,1]$ and $u_i, v_i \in \mathbb{R}, i=1,2$,

$$|f(t,u_1,u_2)-f(t,v_1,v_2)| \le m_1|u_1-v_1|+m_2|u_2-v_2|$$

and

$$|h(t,u_1,u_2)-h(t,v_1,v_2)| \le n_1|u_1-v_1|+n_2|u_2-v_2|.$$

Then the system (1)-(2) has a unique solution if $M_1(m_1+m_2)+M_2(n_1+n_2)<1$, where M_1 and M_2 are given by (6) and (7) respectively.

Proof. Let us fix $\sup_{t\in[0,1]} f(t,0,0)=\zeta_1<\infty$ and $\sup_{t\in[0,1]} h(t,0,0)=\zeta_2<\infty$ such that

$$r \ge \frac{\zeta_1 M_1+\zeta_2 M_2}{1-M_1(m_1+m_2)-M_2(n_1+n_2)}.$$

As a first step, we show that $TB_r \subset B_r$, where $B_r=\{(u,v)\in X\times Y : \|(u,v)\|\le r\}$. For $(u,v)\in B_r$, we have

$$\begin{aligned}
|T_1(u,v)(t)| \le\ & \sup_{t\in[0,1]}\Bigg\{\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u(s),v(s))|ds \\
&+\frac{|b(2t-1)|\Gamma(2-\gamma)}{2|\Gamma(2-\gamma)-b|}\int_0^1 \frac{(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}|f(s,u(s),v(s))|ds \\
&+\frac{1}{|2-a|}\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u(s),v(s))|ds+\frac{|a|}{|2-a|}\int_0^1\frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)}|f(s,u(s),v(s))|ds\Bigg\} \\
\le\ & \sup_{t\in[0,1]}\Bigg\{\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(|f(s,u(s),v(s))-f(s,0,0)|+|f(s,0,0)|)ds \\
&+\frac{|b|\Gamma(2-\gamma)}{2|\Gamma(2-\gamma)-b|}\int_0^1\frac{(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}(|f(s,u(s),v(s))-f(s,0,0)|+|f(s,0,0)|)ds \\
&+\frac{1}{|2-a|}\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}(|f(s,u(s),v(s))-f(s,0,0)|+|f(s,0,0)|)ds \\
&+\frac{|a|}{|2-a|}\int_0^1\frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)}|(f(s,u(s),v(s))-f(s,0,0)|+|f(s,0,0)|)ds\Bigg\} \\
\le\ & \Bigg\{\frac{1}{\Gamma(\alpha+1)}+\frac{|b|\Gamma(2-\gamma)}{2|\Gamma(2-\gamma)-b|\Gamma(\alpha-\gamma+1)}
\end{aligned}$$

$$
\begin{aligned}
& \left.+\frac{1}{|2-a|\Gamma(\alpha+1)}+\frac{|a|}{|2-a|\Gamma(\alpha+2)}\right\}(m_1\|u\|+m_2\|v\|+\zeta_1) \\
= \;& M_1[(m_1+m_2)r+\zeta_1].
\end{aligned}
$$

Hence

$$\|T_1(u,v)(t)\| \leq M_1[(m_1+m_2)r+\zeta_1].$$

In the same way, we can obtain that

$$\|T_2(u,v)(t)\| \leq M_2[(n_1+n_2)r+\zeta_2].$$

Consequently, it follows that $\|T(u,v)(t)\| \leq r$.

Now for $(u_2,v_2),(u_1,v_1) \in X_1 \times X_2$, and for any $t \in [0,1]$, we get

$$
\begin{aligned}
& |T_1(u_2,v_2)(t) - T_1(u_1,v_1)(t)| \\
\leq \;& \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u_2(s),v_2(s)) - f(s,u_1(s),v_1(s))|ds \\
& +\frac{|b(2t-1)|\Gamma(2-\gamma)}{2|\Gamma(2-\gamma)-b|}\int_0^1 \frac{(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}|f(s,u_2(s),v_2(s)) - f(s,u_1(s),v_1(s))|ds \\
& +\frac{1}{|2-a|}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u_2(s),v_2(s)) - f(s,u_1(s),v_1(s))|ds \\
& +\frac{|a|}{|2-a|}\int_0^1 \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)}|f(s,u_2(s),v_2(s)) - f(s,u_1(s),v_1(s))|ds \\
\leq \;& \left\{\frac{1}{\Gamma(\alpha+1)}+\frac{|b|\Gamma(2-\gamma)}{2|\Gamma(2-\gamma)-b|\Gamma(\alpha-\gamma+1)}\right. \\
& \left.+\frac{1}{|2-a|\Gamma(\alpha+1)}+\frac{|a|}{|2-a|\Gamma(\alpha+2)}\right\}(m_1\|u_2-u_1\|+m_2\|v_2-v_1\|) \\
= \;& M_1(m_1\|u_2-u_1\|+m_2\|v_2-v_1\|) \\
\leq \;& M_1(m_1+m_2)(\|u_2-u_1\|+\|v_2-v_1\|),
\end{aligned}
$$

and consequently we obtain

$$\|T_1(u_2,v_2)(t) - T_1(u_1,v_1)\| \leq M_1(m_1+m_2)(\|u_2-u_1\|+\|v_2-v_1\|). \tag{9}$$

Similarly, we can get

$$\|T_2(u_2,v_2)(t) - T_2(u_1,v_1)\| \leq M_2(n_1+n_2)(\|u_2-u_1\|+\|v_2-v_1\|). \tag{10}$$

Clearly it follows from (9) and (10) that

$$\|T(u_2,v_2)(t) - T(u_1,v_1)(t)\| \leq [M_1(m_1+m_2)+M_2(n_1+n_2)](\|u_2-u_1\|+\|v_2-v_1\|).$$

Since $M_1(m_1+m_2)+M_2(n_1+n_2) < 1$, therefore T is a contraction operator. So, by Banach's fixed point theorem, the operator T has a unique fixed point, which is the unique solution of problem (1)-(2). This completes the proof. □

Example 3.7 *Consider the following problem*

$$
\begin{cases}
{}^cD^{3/2}x(t) = \dfrac{1}{4(t+2)^2}\dfrac{|x(t)|}{1+|x(t)|}+1+\dfrac{1}{32}\sin^2 y(t), \quad t \in [0,1], \\[2ex]
{}^cD^{5/3}y(t) = \dfrac{1}{32\pi}\sin(2\pi x(t))+\dfrac{|y(t)|}{16(1+|y(t)|)}+\dfrac{1}{2}, \quad t \in [0,1], \\[2ex]
x(0)+x(1) = 4\displaystyle\int_0^1 x(s)ds, \quad x'(0) = \dfrac{1}{2}\;{}^cD^{1/2}x(1), \\[2ex]
y(0)+y(1) = \dfrac{1}{5}\displaystyle\int_0^1 y(s)ds, \quad y'(0) = 3\;{}^cD^{1/3}y(1).
\end{cases} \tag{11}
$$

Here $\alpha = 3/2,\ \beta = 5/3,\ \gamma = 1/2,\ \delta = 1/3,\ a = 4,\ a_1 = 1/5,\ b = 1/2,\ b_1 = 3$. Using the given data, it is found that $M_1 \approx 5.6166715,\ M_2 \approx 1.6038591$,

$$|f(t,u_1,u_2) - f(t,v_1,v_2)| \le \frac{1}{16}|u_1 - u_2| + \frac{1}{16}|v_1 - v_2|,$$

$$|h(t,u_1,u_2) - h(t,v_1,v_2)| \le \frac{1}{16}|u_1 - u_2| + \frac{1}{16}|v_1 - v_2|,$$

and $M_1(m_1 + m_2) + M_2(n_1 + n_2) \approx 0.9025662 < 1$. As all the conditions of Theorem 3.6 are satisfied, therefore its conclusion applies to the problem (11).

3.3 Special cases

We obtain some special cases of the results obtained in this paper by fixing the parameters involved in the problem (1)-(2) which are listed below.

- If $b = b_1 = 0$, then our results correspond to the boundary conditions: $x(0)+x(1) = a\int_0^1 x(s)ds,\ x'(0) = 0;\ y(0) + y(1) = a_1 \int_0^1 y(s)ds,\ y'(0) = 0$.
- We can get the results for the boundary data: $x(0)+x(1) = 0,\ x'(0) = 0;\ y(0)+y(1) = 0,\ y'(0) = 0$ by fixing $a = 0, a_1 = 0, b = 0, b_1 = 0$.
- In case we choose $a = 0, a_1 = 0, b \neq 0, b_1 \neq 0$, we get the results for the boundary conditions: $x(0) + x(1) = 0,\ x'(0) = b\ {}^cD^{\gamma}x(1);\quad y(0) + y(1) = 0,\ y'(0) = b_1\ {}^cD^{\delta}y(1),\quad 0 < \gamma, \delta \le 1$.
- By taking $\gamma = \delta = 1$ with $b \neq 1 \neq b_1$, our results reduce to the ones for a given system of fractional differential equations with boundary conditions: $x(0) + x(1) = a\int_0^1 x(s)ds,\ x'(0) = bx'(1); y(0) + y(1) = a_1 \int_0^1 y(s)ds,\ y'(0) = b_1 y'(1)$.

We emphasize that all the results obtained for different values of the parameters are new.

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Solutions of the nonlinear evolution equation via the generalized Riccati equation mapping together with the (G'/G)-expansion method

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Abstract In this article, we investigate the combined KdV-MKdV equation to obtain new exact traveling wave solutions via the generalized Riccati equation mapping together with the(G'/G)-expansion method. In this method, $G'(\theta) = h + f\,G(\varphi) + g\,G^2(\theta)$ is used with constant coefficients, as the auxiliary equation and called the generalized Riccati equation. By using this method, we obtain twenty seven exact traveling wave solutions including solitons and periodic solutions and solutions are expressed in the hyperbolic, the trigonometric and the rational functions. It is found that one of our solutions is in good agreement for a special case with the published results which validates our other results.

Keywords: The generalized Riccati equation, (G'/G)-expansion method, Exp-function method, traveling wave solutions, nonlinear evolution equations.

Mathematics Subject Classification: 35K99, 35P99, 35P05.

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1. Introduction

The enormous analysis of exact solutions of the nonlinear partial differential equations (PDEs) is one of the important and amazing research fields in all areas in science and engineering, such as, plasma physics, fluid mechanics, chemical

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physics, optical fibers, solid state physics, chemistry, biology, plasma physics and many others [1-41]. In recent years, many researchers used various methods to different nonlinear partial differential equations for constructing traveling wave solutions, for instance, the Backlund transformation [1], the inverse scattereing [2,3], the Jacobi elliptic function expansion [4,5], the tanh function [6,7], the variational iteration [8], the Hirota's bilinear transformation [9], the direct algebraic [10], the Cole-Hopf transformation [11], the Exp-function [12-18] and others [19-25].

Recently, Wang *et al.* [26] presented a method, called the (G'/G)-expansion method. By using this method, they constructed exact traveling wave solutions for the nonlinear evolution equations (NLEEs). In this method, the second order linear ordinary differential equation with constant coefficients $G''(\theta)+\lambda G'(\theta)+\mu G(\theta)=0$ is used, as an auxiliary equation. Afterwards, many researchers applied the (G'/G)-expansion method to obtain exact traveling wave solutions for the NLEEs. For example, Ozis and Aslan [27] investigated the Kawahara type equations using symbolic computation via this method. In Ref. [28] Gepreel employed this method and found exact solutions for nonlinear PDEs with variable coefficients in mathematical physics whilst Zayed and Al-Joudi [29] studied nonlinear partial differential equations by applying the same method to construct solutions. Naher *et al.* [30] investigated the Caudrey-Dodd-Gibbon equation by using the useful (G'/G)-expansion method and obtained abundant exact traveling wave solutions. Feng et al. [31] applied the method to the Kolmogorov-Petrovskii-Piskunov equation for constructing traveling wave solutions. In Ref. [32], Zhao *et al.* concerned about this method to obtain exact solutions for the variant Boussinesq equations while Nofel *et al.* [33] implemented the same method to the higher order KdV equation to get exact traveling wave solutions and so on.

Zhu [34] introduced the generalized Riccati equation mapping to solve the (2+1)-dimensional Boiti-Leon-Pempinelle equation. In this generalized Riccati equation mapping, he employed $G'(\theta)=h+f\,G(\varphi)+g\,G^2(\theta)$ with constants coefficients, as the auxiliary equation. In Ref. [35], Li et al. used the Riccati equation expansion method to solve the higher dimensional NLEEs. Bekir and Cevikel [36] investigated nonlinear coupled equation in mathematical physics by applying the tanh-coth method combined with the Riccati equation. Guo *et al.* [37] studied the diffusion-reaction and the mKdV equation with variable coefficient via the extended Riccati equation mapping method whilst Li and Dai [38] implemented the generalized Riccati equation mapping with the (G'/G)-expansion method to construct traveling wave solutions for the higher dimensional Jimbo-Miwa equation. In Ref. [39,40] Salas used the projective Riccati equation method to obtain some exact solutions for the Caudrey-Dodd-Gibbon equation and the generalized Sawada-Kotera equations respectively.

Many researchers utilized different methods for investigating the combined KdV-MKdV equation to construct exact traveling wave solutions, such as, Liu et al. [41] studied the equation by applying the (G'/G)-expansion method to obtain traveling wave solutions. In the (G'/G)-expansion method, they used the second order linear ordinary differential equation (LODE) with constant coefficients, as an auxiliary equation. To the best of our knowledge, the combined KdV-MKdV equation is not examined by applying the generalized Riccati equation mapping together with the (G'/G)-expansion method.

In this article, we construct twenty seven exact traveling wave solutions including solitons, periodic, and rational solutions of the combined KdV-MKdV equation involving parameters via the generalized Riccati equation mapping together with the (G'/G)-expansion method and Exp-function method.

2. The generalized Riccati equation mapping together with the(G'/G)-expansion method

Suppose the general nonlinear partial differential equation:

$$H\left(v, v_t, v_x, v_{xt}, v_{t\,t}, v_{x\,x}, ...\right) = 0, \tag{1}$$

where $v = v\left(x, t\right)$ is an unknown function, H is a polynomial in $v = v\left(x, t\right)$and the subscripts indicate the partial derivatives.

The most important steps of the generalized Riccati equation mapping together with the (G'/G)-expansion method [26,34] are as follows:

Step 1. Consider the traveling wave variable:

$$v\left(x,t\right)=r\left(\theta\right),\qquad \theta=x-B\,t, \tag{2}$$

where $B$is the wave speed. Now using Eq. (2), Eq. (1) is converted into an ordinary differential equation for $r\left(\theta\right)$:

$$F\left(r,r',r'',r''',...\right)=0, \tag{3}$$

where the superscripts stand for the ordinary derivatives with respect to θ.

Step 2. Eq. (3) integrates term by term one or more times according to possibility, yields constant(s) of integration. The integral constant(s) may be zero for simplicity.

Step 3. Suppose that the traveling wave solution of Eq. (3) can be expressed in the form [26,34]:

$$r\left(\theta\right)=\sum_{j\,=\,0}^{n}e_j\left(\frac{G'}{G}\right)^j \tag{4}$$

where e_j $(j=0,\,1,\,2,...,\,n)$and $e_n\neq 0$, with $G=G\left(\theta\right)$ is the solution of the generalized Riccati equation:

$$G'=h+f\,G+g\,G^2, \tag{5}$$

where f,g,h are arbitrary constants and $g\neq 0$.

Step 4. To decide the positive integer n, consider the homogeneous balance between the nonlinear terms and the highest order derivatives appearing in Eq. (3).

Step 5. Substitute Eq. (4) along with Eq. (5) into the Eq. (3), then collect all the coefficients with the same order, the left hand side of Eq. (3) converts into polynomials in $G^{k}(\theta)$ and $G^{-k}(\theta)$, $(k = 0, 1, 2, ...)$. Then equating each coefficient of the polynomials to zero, yield a set of algebraic equations for $f_j \ (j = 0,\, 1,\, 2, ..., n)\,, f, g, h$ and B.

Step 6. Solve the system of algebraic equations which are found in Step 5 with the aid of algebraic software Maple and we obtain values for $f_j \ (j = 0,\, 1, 2, ..., n)$ and B. Then, substitute obtained values in Eq. (4) along with Eq. (5) with the value of n, we obtain exact solutions of Eq. (1).

In the following, we have twenty seven solutions including four different families of Eq. (5).

Family 1: When $f^2 - 4gh > 0$ and $f\,g \neq 0$ or $g\,h \neq 0$, the solutions of Eq. (5) are:

$$G_1 = \frac{-1}{2g}\left(f + \sqrt{f^2 - 4gh}\ \tanh\left(\frac{\sqrt{f^2 - 4gh}}{2}\,\theta\right)\right),$$

$$G_2 = \frac{-1}{2g}\left(f + \sqrt{f^2 - 4gh}\ \coth\left(\frac{\sqrt{f^2 - 4gh}}{2}\,\theta\right)\right),$$

$$G_3 = \frac{-1}{2g}\left(f + \sqrt{f^2 - 4gh}\left(\tanh\left(\sqrt{f^2 - 4gh}\,\theta\right) \pm i\,\mathrm{sec}\,h\left(\sqrt{f^2 - 4gh}\,\theta\right)\right)\right),$$

$$G_4 = \frac{-1}{2g}\left(f + \sqrt{f^2 - 4gh}\left(\coth\left(\sqrt{f^2 - 4gh}\,\theta\right) \pm \mathrm{csc}\,h\left(\sqrt{f^2 - 4gh}\,\theta\right)\right)\right),$$

$$G_5 = \frac{-1}{4g}\left(2f + \sqrt{f^2 - 4gh}\left(\tanh\left(\frac{\sqrt{f^2 - 4gh}}{4}\,\theta\right) + \cot h\left(\frac{\sqrt{f^2 - 4gh}}{4}\,\theta\right)\right)\right),$$

$$G_6 = \frac{1}{2g}\left(-f + \frac{\sqrt{(X^2 + Y^2)(f^2 - 4gh)}\ - X\sqrt{f^2 - 4gh}\ \cosh\left(\sqrt{f^2 - 4gh}\,\theta\right)}{X\sinh\left(\sqrt{f^2 - 4gh}\,\theta\right) + Y}\right),$$

$$G_7 = \frac{1}{2g}\left(-f - \frac{\sqrt{(Y^2 - X^2)(f^2 - 4gh)}\ + X\sqrt{f^2 - 4gh}\ \sinh\left(\sqrt{f^2 - 4gh}\,\theta\right)}{X\cosh\left(\sqrt{f^2 - 4gh}\,\theta\right) + Y}\right),$$

where X and Y are two non-zero real constants and satisfies $Y^2 - X^2 > 0$.

$$G_8 = \frac{2\,h\,\cosh\left(\frac{\sqrt{f^2-4gh}}{2}\,\theta\right)}{\sqrt{f^2 - 4gh}\sinh\left(\frac{\sqrt{f^2-4gh}}{2}\,\theta\right) - f\cosh\left(\frac{\sqrt{f^2-4gh}}{2}\,\theta\right)},$$

$$G_9 = \frac{-2\,h\sinh\left(\frac{\sqrt{f^2-4gh}}{2}\,\theta\right)}{f\sinh\left(\frac{\sqrt{f^2-4gh}}{2}\,\theta\right) - \sqrt{f^2-4gh}\cosh\left(\frac{\sqrt{f^2-4gh}}{2}\,\theta\right)},$$

$$G_{10} = \frac{2\,h\,\cosh\left(\sqrt{f^2-4gh}\,\theta\right)}{\sqrt{f^2-4gh}\sinh\left(\sqrt{f^2-4gh}\,\theta\right) - f\cosh\left(\sqrt{f^2-4gh}\,\theta\right)\ \pm\ i\sqrt{f^2-4gh}},$$

$$G_{11} = \frac{2\,h\,\sinh\left(\sqrt{f^2-4gh}\,\theta\right)}{-f\sinh\left(\sqrt{f^2-4gh}\,\theta\right) + \sqrt{f^2-4gh}\cosh\left(\sqrt{f^2-4gh}\,\theta\right)\ \pm\ \sqrt{f^2-4gh}},$$

$$G_{12} = \frac{4\,h\sinh\left(\frac{\sqrt{f^2-4gh}}{4}\,\theta\right)\cosh\left(\frac{\sqrt{f^2-4gh}}{4}\,\theta\right)}{-2f\sinh\left(\frac{\sqrt{f^2-4gh}}{4}\theta\right)\cosh\left(\frac{\sqrt{f^2-4gh}}{4}\,\theta\right) + 2\sqrt{f^2-4gh}\cosh^2\left(\frac{\sqrt{f^2-4gh}}{4}\,\theta\right) - \sqrt{f^2-4gh}},$$

Family 2: When $f^2-4gh<0$ and $fg \neq 0$ or $gh \neq 0$, the solutions of Eq. (5) are:

$$G_{13} = \frac{1}{2g}\left(-f + \sqrt{4gh-f^2}\ \tan\left(\frac{\sqrt{4gh-f^2}}{2}\,\theta\right)\right),$$

$$G_{14} = \frac{-1}{2g}\left(f + \sqrt{4gh-f^2}\ \cot\left(\frac{\sqrt{4gh-f^2}}{2}\,\theta\right)\right),$$

$$G_{15} = \frac{1}{2g}\left(-f + \sqrt{4gh-f^2}\left(\tan\left(\sqrt{4gh-f^2}\,\theta\right) \pm \sec\left(\sqrt{4gh-f^2}\,\theta\right)\right)\right),$$

$$G_{16} = \frac{-1}{2g}\left(f + \sqrt{4gh-f^2}\left(\cot\left(\sqrt{4gh-f^2}\,\theta\right) \pm \csc\left(\sqrt{4gh-f^2}\,\theta\right)\right)\right),$$

$$G_{17} = \frac{1}{4g}\left(-2f + \sqrt{4gh-f^2}\left(\tan\left(\frac{\sqrt{4gh-f^2}}{4}\,\theta\right) - \cot\left(\frac{\sqrt{4gh-f^2}}{4}\,\theta\right)\right)\right),$$

$$G_{18} = \frac{1}{2g}\left(-f + \frac{\pm\sqrt{(X^2-Y^2)\,(4gh-f^2)}\ - X\sqrt{4gh-f^2}\ \cos\left(\sqrt{4gh-f^2}\,\theta\right)}{X\sin\left(\sqrt{4gh-f^2}\,\theta\right)+Y}\right),$$

$$G_{19} = \frac{1}{2g}\left(-f - \frac{\pm\sqrt{(X^2-Y^2)\,(4gh-f^2)}\ + X\sqrt{4gh-f^2}\ \cos\left(\sqrt{4gh-f^2}\,\theta\right)}{X\sin\left(\sqrt{4gh-f^2}\,\theta\right)+Y}\right),$$

where X and Y are two non-zero real constants and satisfies $X^2 - Y^2 > 0$.

$$G_{20} = \frac{-2\,h\cos\left(\frac{\sqrt{4gh-f^2}}{2}\,\theta\right)}{\sqrt{4gh-f^2}\sin\left(\frac{\sqrt{4gh-f^2}}{2}\,\theta\right) + f\cos\left(\frac{\sqrt{4gh-f^2}}{2}\,\theta\right)},$$

$$G_{21} = \frac{2\,h\sin\left(\frac{\sqrt{4gh-f^2}}{2}\,\theta\right)}{-f\sin\left(\frac{\sqrt{4gh-f^2}}{2}\,\theta\right) + \sqrt{4gh-f^2}\cos\left(\frac{\sqrt{4gh-f^2}}{2}\,\theta\right)},$$

$$G_{22} = \frac{-2\,h\,\cos\left(\sqrt{4gh-f^2}\,\theta\right)}{\sqrt{4gh-f^2}\,\sin\left(\sqrt{4gh-f^2}\,\theta\right) + f\cos\left(\sqrt{4gh-f^2}\,\theta\right) \pm \sqrt{4gh-f^2}\,\theta},$$

$$G_{23} = \frac{2\,h\sin\left(\sqrt{4gh-f^2}\,\theta\right)}{-f\sin\left(\sqrt{4gh-f^2}\theta\right) + \sqrt{4gh-f^2}\,\cos\left(\sqrt{4gh-f^2}\,\theta\right) \pm \sqrt{4gh-f^2}},$$

$$G_{24} = \frac{4\,h\sin\left(\frac{\sqrt{4gh-f^2}}{4}\,\theta\right)\;\cos\left(\frac{\sqrt{4gh-f^2}}{4}\,\theta\right)}{-2f\sin\left(\frac{\sqrt{4gh-f^2}}{4}\,\theta\right)\;\cos\left(\frac{\sqrt{4gh-f^2}}{4}\,\theta\right) + 2\sqrt{4gh-f^2}\cos^2\left(\frac{\sqrt{4gh-f^2}}{4}\,\theta\right) - \sqrt{4gh-f^2}},$$

Family 3: when $h = 0$ and $fg \neq 0$, the solution Eq. (5) becomes:

$$G_{25} = \frac{-f\,b_1}{g\,(b_1 + \cosh(f\theta) - \sinh(f\theta))},$$

$$G_{26} = \frac{-f\,(\cosh(f\theta) + \sinh(f\theta))}{g\,(b_1 + \cosh(f\theta) + \sinh(f\theta))},$$

where b_1 is an arbitrary constant.

Family 4: when $g \neq 0$ and $h = f = 0$, the solution of Eq. (5) becomes:

$$G_{27} = \frac{-1}{g\,\theta + u_1},$$

where u_1 is an arbitrary constant.

2.1. Exp-function Method

Consider the general nonlinear partial differential equation of the type (1)

Using the transformation (2) in equation (1) we have equation of the type (3).

According to the Exp-function method, developed by He and Wu [12-18],we assume that the wave solutions can be expressed in the following form

$$u(\eta) = \frac{\sum_{n=-c}^{d} a_n exp(n\eta)}{\sum_{m=-p}^{q} b_m exp(m\eta)} \tag{6}$$

where p, q, c and d are positive integers which are to be further determined,a_n and b_m are unknown constants. We can rewrite equation (6) in the following equivalent form

$$u\left(\eta\right) = \frac{a_c \exp\left(c\eta\right) + ... + a_{-d} \exp\left(-d\eta\right)}{b_p \exp\left(p\eta\right) + ... + b_{-q} \exp\left(-q\eta\right)}. \tag{7}$$

To determine the value of c and$p$, we balance the linear term of highest order of equation (3) with the highest order nonlinear term. Similarly, to determine the value of $d$and$q$, we balance the linear term of lowest order of equation (3) with lowest order non linear term.

3. Solution procedure

By using Exp-function method and the generalized Riccati equation mapping together with the(G'/G)-expansion method, we construct new exact traveling wave solutions for the combined KdV-MKdV equation (Gardner equation) in this section.

3.1 The combined KdV-MKdV equation (Gardner equation)

We consider the combined KdV-MKdV equation with parameters followed by Liu et al. [41]:

$$u_t + p\, u\, u_x + q\, u^2\, u_x - s\, u_{xxx} = 0, \tag{8}$$

where $p,\ s$ are free parameters and $q \neq 0$.

Now, we use the transformation Eq. (2) into the Eq. (8), which yields:

$$-\ B\, r' + prr' + qr^2 r' - sr''' = 0, \tag{9}$$

Eq. (9) is integrable, therefore, integrating with respect θ once yields:

$$-B\,r + \frac{p}{2}r^2 + \frac{q}{3}r^3 - sr'' + K = 0, \tag{10}$$

where $K$is an integral constant which is to be determined later.

Taking the homogeneous balance between $r''$and r^3 in Eq. (10), we obtain $n = 1$.

Therefore, the solution of Eq. (10) is of the form:

$$r\,(\theta) = e_1\,(G'/G) + e_0,\ \ e_1 \neq 0. \tag{11}$$

Using Eq. (5), Eq. (11) can be re-written as:

$$r\,(\theta) = e_1\left(f + h\,G^{-1} + g\,G\right) + e_0, \tag{12}$$

where f, g and $h$are free parameters.

By substituting Eq. (12) into Eq. (10), collecting all coefficients of G^k and $G^{-\,k}$ $(k = 0, 1, 2, ...)$ and setting them equal to zero, we obtain a set of algebraic equations for e_0, e_1, f, g, h, K and B (algebraic equations are not shown, for simplicity). Solving the system of algebraic equations with the help of algebraic software Maple, we obtain

$$e_0 = \frac{\mp p\sqrt{\frac{6s}{q}} - 6sf}{\pm\,2q\sqrt{\frac{6s}{q}}},\ \ e_1 = \pm\sqrt{\frac{6s}{q}},\ \ B = \frac{2sqf^2 - p^2 + 16sqgh}{4q},$$
$$K = \frac{-48psqgh\left(\pm\sqrt{\frac{6s}{q}}\right) + 6psqf^2\left(\pm\sqrt{\frac{6s}{q}}\right) - p^3\left(\pm\sqrt{\frac{6s}{q}}\right) + 288s^2qfgh}{24q^2\left(\pm\sqrt{\frac{6s}{q}}\right)},$$

where p, s are free parameters and $q \neq 0$.

Family 1: The soliton and soliton-like solutions of Eq. (6) (when $f^2 - 4gh > 0$ and $f\,g \neq 0$ or $g\,h \neq 0$) are:

$$r_1 = e_1 \frac{\Delta^2 \sec h^2\left(\frac{\Delta}{2}\,\theta\right)}{2\left(f + \Delta\ \tanh\left(\frac{\Delta}{2}\,\theta\right)\right)} + e_0,$$

where $\Delta = \sqrt{f^2 - 4gh}$, $\Delta^2 = f^2 - 4gh$, $e_0 = \frac{\mp p\sqrt{\frac{6s}{q}} - 6sf}{\pm 2q\sqrt{\frac{6s}{q}}}$, $e_1 = \pm\sqrt{\frac{6s}{q}}$ and $\theta = x - \left(\frac{2sqf^2 - p^2 + 16sqgh}{4q}\right)t$.

$$r_2 = e_1\,\frac{-\,\Delta^2 \csc h^2\left(\frac{\Delta}{2}\,\theta\right)}{2\left(f + \Delta\ \coth\left(\frac{\Delta}{2}\,\theta\right)\right)} + e_0,$$

$$r_3 = e_1\,\frac{\Delta^2\left(\sec h^2\left(\Delta\,\theta\right) \mp i\tanh\left(\Delta\theta\right)\sec h\ \left(\Delta\,\theta\right)\right)}{f + \sqrt{f^2 - 4gh}\ \left(\tanh\left(\Delta\ \theta\ \right)\ \pm\ \ i\sec h\ \left(\Delta\ \theta\right)\right)} + e_0,$$

$$r_4 = e_1\frac{-\,\Delta^2\left(\csc h^2\left(\Delta\,\theta\right) \pm \coth\left(\Delta\theta\right)\csc h\ \left(\Delta\,\theta\,\right)\right)}{f + \Delta\ \left(\coth\left(\Delta\theta\right)\ \pm\ \csc h\ \left(\Delta\,\theta\right)\right)} + e_0,$$

$$r_5 = e_1\,\frac{\Delta^{\,2}\left(\sec h^2\left(\frac{\Delta}{4}\theta\ \right) - \csc h^2\left(\frac{\Delta}{4}\ \theta\right)\right)}{8\,f + 4\Delta\ \left(\tanh\left(\frac{\Delta}{4}\theta\right)\ +\ \coth\left(\frac{\Delta}{4}\ \theta\right)\right)} + e_0,$$

$$r_6 = e_1\,\frac{-X\left(f^2X - \sinh\left(\Delta\,\theta\right)f^2Y - 4ghX + 4ghY\sinh\left(\Delta\,\theta\right) - \Delta^2\sqrt{\left(X^2 + Y^2\right)}\cosh\left(\Delta\,\theta\right)\right)}{\left(X\sinh\left(\Delta\,\theta\right) + Y\right)\left(f\,X\sinh\left(\Delta\,\theta\right) + f\,Y - \Delta\,\sqrt{\left(X^2 + Y^2\right)}\ + X\Delta\ \cosh\left(\Delta\,\theta\right)\right)} + e_0,$$

$$r_7 = e_1\,\frac{X\left(f^2Y\cosh\left(\Delta\,\theta\right)f^2Y - 4ghY\cosh\left(\Delta\,\theta\right) - \Delta^2\sqrt{-\left(X^2 - Y^2\right)}\sinh\left(\Delta\,\theta\right) + f^2X - 4ghX\right)}{\left(X\cosh\left(\Delta\,\theta\right) + Y\right)\left(fX\cosh\left(\Delta\,\theta\right) + fY + \Delta\,\sqrt{-\left(X^2 - Y^2\right)}\ + X\Delta\ \sinh\left(\Delta\,\theta\right)\right)} + e_0,$$

where X and Y are two non-zero real constants and satisfies $Y^2 - X^2 > 0$.

$$r_8 = e_1\,\frac{-\,\Delta^{\,2}}{2\cosh\left(\frac{\Delta}{2}\theta\right)\left(\Delta\ \sinh\left(\frac{\Delta}{2}\theta\right) - f\cosh\left(\frac{\Delta}{2}\theta\right)\right)} + e_0,$$

$$r_9 = e_1\frac{\Delta^{\,2}}{2\sinh\left(\frac{\Delta}{2}\theta\right)\left(-f\sinh\left(\frac{\Delta}{2}\theta\right) + \Delta\ \cosh\left(\frac{\Delta}{2}\theta\right)\right)}\ + e_0,$$

$$r_{10} = e_1\,\frac{-\Delta^2 + i\,f^2\sinh\left(\Delta\,\theta\right) - i4gh\sinh\left(\Delta\,\theta\right)}{\Delta\ \sinh\left(\Delta\,\theta\right) - f\cosh\left(\Delta\,\theta\right) + i\Delta\ \cosh\left(\Delta\,\theta\right)}\ + e_0,$$

$$r_{11} = e_1\frac{\Delta^2 + f^2\cosh\left(\Delta\,\theta\right) - 4gh\cosh\left(\Delta\,\theta\right)}{\left(-f\sinh\left(\Delta\,\theta\right) + \Delta\ \cosh\left(\Delta\,\theta\right) + \Delta\ \right)\sinh\left(\Delta\,\theta\right)} + e_0,$$

$$r_{12} = e_1\,\frac{\Delta^2}{4\sinh\left(\frac{\Delta}{4}\theta\right)\cosh\left(\frac{\Delta}{4}\theta\right)\left(-2f\sinh\left(\frac{\Delta}{4}\theta\right)\cosh\left(\frac{\Delta}{4}\theta\right) + 2\Delta\ \cosh^2\left(\frac{\Delta}{4}\theta\right) - \Delta\ \right)} + e_0,$$

Family 2: The periodic form solutions of Eq. (8) (when $f^2 - 4gh < 0$ and

$fg \neq 0$ or $gh \neq 0$) are:

$$r_{13} = e_1 \frac{\Omega^2}{2\cos\left(\frac{\Omega}{2}\theta\right)\left(-f\cos\left(\frac{\Omega}{2}\theta\right) + \Omega\sin\left(\frac{\Omega}{2}\theta\right)\right)} + e_0,$$

where $\Omega = \sqrt{-f^2 + 4gh}$, $\Omega^2 = 4gh - f^2$,$e_0 = \frac{\mp p\sqrt{\frac{6s}{q}} - 6sf}{\pm 2q\sqrt{\frac{6s}{q}}}$, $e_1 = \pm\sqrt{\frac{6s}{q}}$ and $\theta = x - \left(\frac{2sqf^2 - p^2 + 16sqgh}{4q}\right)t$.

$$r_{14} = e_1 \frac{\Omega^2}{2\left(-1 + \cos^2\left(\frac{\Omega}{2}\theta\right)\right)\left(f + \Omega\cot\left(\frac{\Omega}{2}\theta\right)\right)} + e_0,$$

$$r_{15} = e_1 \frac{\Omega^2\left(1 + \sin\left(\Omega\theta\right)\right)}{\cos\left(\Omega\theta\right)\left(-f\cos\left(\Omega\theta\right) + \Omega\sin\left(\Omega\theta\right) + \Omega\right)} + e_0,$$

$$r_{16} = e_1 \frac{\Omega^2\sin\left(\Omega\theta\right)}{\cos\left(\Omega\theta\right)f\sin\left(\Omega\theta\right) + \Omega\cos^2\left(\Omega\theta\right) - f\sin\left(\Omega\theta\right) - \Omega} + e_0,$$

$$r_{17} = e_1 \frac{-\Omega^2}{4\cos^2\left(\frac{\Omega}{4}\theta\right)\left(-1 + \cos^2\left(\frac{\Omega}{4}\theta\right)\right)\left(-2f + \Omega\left(\tan\left(\frac{\Omega}{4}\theta\right) - \cot\left(\frac{\Omega}{4}\theta\right)\right)\right)} + e_0,$$

$$r_{18} = e_1 \frac{X.N_1}{\left(-X^2 + X^2\cos^2\left(\Omega\theta\right) - 2XY\sin\left(\Omega\theta\right) - Y^2\right)\left(-f + \frac{\Omega\sqrt{(X^2-Y^2)}}{X\sin(\Omega\theta)+Y} - X\Omega\cos\left(\Omega\theta\right)\right)} + e_0, \tag{13}$$

where

$$N_1 = -4ghX - 4ghY\sin\left(\Omega\theta\right) + f^2X + f^2Y\sin\left(\Omega\theta\right) + 4gh\sqrt{(X^2 - Y^2)}\cos\left(\Omega\theta\right) - f^2\sqrt{(X^2 - Y^2)}\cos\left(\Omega\theta\right)$$

$$r_{19} = e_1 \frac{X\left(-4ghX - 4ghY\sin(\Omega\theta) + f^2X + f^2Y\sin(\Omega\theta) - 4gh\sqrt{(X^2-Y^2)}\cos(\Omega\theta) + f^2\sqrt{(X^2-Y^2)}\cos(\Omega\theta)\right)}{\left(-X^2 + X^2\cos^2(\Omega\theta) - 2XY\sin(\Omega\theta) - Y^2\right)\left(-f - \frac{\Omega\sqrt{(X^2-Y^2)}}{X\sin(\Omega\theta)+Y} + X\Omega\cos(\Omega\theta)\right)} + e_0,$$

where X and Y are two non-zero real constants and satisfies $X^2 - Y^2 > 0$.

$$r_{20} = e_1 \frac{-\Omega^2\sec\left(\frac{\Omega}{2}\theta\right)\left(\Omega\sin\left(\frac{\Omega}{2}\theta\right) + f\cos\left(\frac{\Omega}{2}\theta\right)\right)}{2\left(4gh - 4gh\cos^2\left(\frac{\Omega}{2}\theta\right) - f^2 + 2f^2\cos^2\left(\frac{\Omega}{2}\theta\right) + 2f\Omega\sin\left(\frac{\Omega}{2}\theta\right)\cos\left(\frac{\Omega}{2}\theta\right)\right)} + e_0,$$

$$r_{21} = e_1 \frac{-\Omega^2\left(-f\sin\left(\frac{\Omega}{2}\theta\right) + \Omega\cos\left(\frac{\Omega}{2}\theta\right)\right)}{2\sin\left(\frac{\Omega}{2}\theta\right)\left(-f^2 + 2f^2\cos^2\left(\frac{\Omega}{2}\theta\right) + 2f\Omega\sin\left(\frac{\Omega}{2}\theta\right)\cos\left(\frac{\Omega}{2}\theta\right) - 4gh\cos^2\left(\frac{\Omega}{2}\theta\right)\right)} + e_0,$$

$$r_{22} = \frac{\frac{1}{2}e_1\sec\left(\Omega\theta\right)\left(-\Omega^2 - 4gh\sin\left(\Omega\theta\right) + f^2\sin\left(\Omega\theta\right)\right)\left(\Omega\sin\left(\Omega\theta\right) + f\cos\left(\Omega\theta\right) + \Omega\right)}{N_2} + e_0,$$

where

$$N_2 = 4gh - 2gh\cos^2(\Omega\theta) - f^2 + f^2\cos^2(\Omega\theta) + \Omega f\sin(\Omega\theta)\cos(\Omega\theta) + 4gh\sin(\Omega\theta) - f^2\sin(\Omega\theta) + f\Omega\cos(\Omega\theta)$$

$$r_{23} = e_1\,\frac{-\Omega^2\left(-f\sin(\Omega\theta) + \Omega\cos(\Omega\theta) + \Omega\right)}{2\sin(\Omega\theta)\left(-2gh\cos(\Omega\theta) + f^2\cos(\Omega\theta) + f\Omega\sin(\Omega\theta) - 2gh\right)} + e_0,$$

$$q_{24} = \frac{\frac{-\Omega^2}{4}e_1\csc\left(\frac{\Omega}{4}\theta\right)\sec\left(\frac{\Omega}{4}\theta\right)\left(-2f\sin\left(\frac{\Omega}{4}\theta\right)\cos\left(\frac{\Omega}{4}\theta\right) + 2\Omega\cos^2\left(\frac{\Omega}{4}\theta\right) - \Omega\right)}{N_3} + e_0,$$

where

$$N_3 = -8f^2\cos^2\left(\frac{\Omega}{4}\theta\right) + 8f^2\cos^4\left(\frac{\Omega}{4}\theta\right) + 8\Omega f\cos^3\left(\frac{\Omega}{4}\theta\right)\sin\left(\frac{\Omega}{4}\theta\right) - 4f\Omega\sin\left(\frac{\Omega}{4}\theta\right)\cos\left(\frac{\Omega}{4}\theta\right) - 16gh\cos^4\left(\frac{\Omega}{4}\theta\right) + 16gh\cos^2\left(\frac{\Omega}{4}\theta\right) - \Omega^2$$

Family 3: The soliton and soliton-like solutions of Eq. (6) (when $h = 0$ and $f\,g \neq 0$) are:

$$r_{25} = e_1\frac{f\left(\cosh(f\theta) - \sinh(f\theta)\right)}{b_1 + \cosh(f\theta) - \sinh(f\theta)} + e_0,$$

$$r_{26} = e_1\,\frac{f\,b_1}{b_1 + \cosh(f\theta) + \sinh(f\theta)} + e_0,$$

where b_1 is an arbitrary constant, $e_0 = \frac{\mp p\sqrt{\frac{6s}{q}} - 6sf}{\pm 2q\sqrt{\frac{6s}{q}}}$, $e_1 = \pm\sqrt{\frac{6s}{q}}$ and $\theta = x - \left(\frac{2sqf^2 - p^2 + 16sqgh}{4q}\right)t$.

Family 4: The rational function solution (when $g \neq 0$ and $h = f = 0$) is:

$$r_{27} = \frac{-e_1 g}{g\,\theta + u_1},$$

where u_1 is an arbitrary constant , $e_1 = \pm\sqrt{\frac{6s}{q}}$ and $\theta = x - \left(\frac{2sqf^2 - p^2 + 16sqgh}{4q}\right)t$.

3.2 The combined KdV-MKdV equation (Gardner equation) using Exp-function method

We consider the combined KdV-MKdV equation (8) with parameters followed by Liu et al. [41]:

Now, we use the transformation Eq. (2) into the Eq. (8), which yields (9).

Using Exp-function Method we have following solution sets satisfy the given combined KdV-MKdV equation (8)

1^{st} Solution set:

$$\left\{\begin{array}{l} B=B, b_{-1}=\frac{1}{4}\frac{\left(p^2+4Bq+-2sq\right)}{b_1\left(p^2+4Bq+4sq\right)}, b_0=b_0, b_1=b_1, a_{-1}=-\frac{1}{8}\frac{b_0^2\left(p+\sqrt{p^2+4Bq+4sq}\right)\left(4Bq+p^2-2sq\right)}{qb_1\left(p^2+4Bq+4sq\right)}, \\ a_0=\frac{b_0\left(4s-2B-\frac{1}{2}p\left(\frac{p+\sqrt{p^2+4Bq+4sq}}{q}\right)\right)}{\sqrt{p^2+4Bq+4sq}}, a_1=-\frac{1}{2}\left(\frac{p+\sqrt{p^2+4Bq+4sq}}{q}\right)b_1 \end{array}\right\}$$

We therefore, obtained the following generalized solitary solution

$$U(\eta)=\frac{-\frac{1}{8}\frac{b_0^2\left(p+\sqrt{p^2+4Bq+4sq}\right)\left(4Bq+p^2-2sq\right)}{qb_1\left(p^2+4Bq+4sq\right)}e^{-\eta}+\frac{b_0\left(4s-2B-\frac{1}{2}p\left(\frac{p+\sqrt{p^2+4Bq+4sq}}{q}\right)\right)}{\sqrt{p^2+4Bq+4sq}} -\frac{1}{2}\left(\frac{p+\sqrt{p^2+4Bq+4sq}}{q}\right)b_1e^{\eta}}{\frac{1}{4}\frac{\left(p^2+4Bq+-2sq\right)}{b_1\left(p^2+4Bq+4sq\right)}e^{-\eta}+b_0+b_1e^{\eta}}$$

$$U(x,t)=\frac{-\frac{1}{8}\frac{b_0^2\left(p+\sqrt{p^2+4Bq+4sq}\right)\left(4Bq+p^2-2sq\right)}{qb_1\left(p^2+4Bq+4sq\right)}e^{-(x-Bt)}+\frac{b_0\left(4s-2B-\frac{1}{2}p\left(\frac{p+\sqrt{p^2+4Bq+4sq}}{q}\right)\right)}{\sqrt{p^2+4Bq+4sq}} -\frac{1}{2}\left(\frac{p+\sqrt{p^2+4Bq+4sq}}{q}\right)b_1e^{(x-Bt)}}{\frac{1}{4}\frac{\left(p^2+4Bq+-2sq\right)}{b_1\left(p^2+4Bq+4sq\right)}e^{-(x-Bt)}+b_0+b_1e^{(x-Bt)}}$$

2nd Solution set:

$$\left\{\begin{array}{l} B=B, b_{-1}=\frac{1}{4}\frac{\left(p^2+4Bq+-2sq\right)b_0^2}{b_1\left(p^2+4Bq+4sq\right)}, b_0=b_0, b_1=b_1, a_{-1}=-\frac{1}{8}\frac{b_0^2\left(-p+\sqrt{p^2+4Bq+4sq}\right)\left(4Bq+p^2-2sq\right)}{qb_1\left(p^2+4Bq+4sq\right)}, \\ a_0=-\frac{b_0\left(4s-2B-\frac{1}{2}p\left(\frac{p+\sqrt{p^2+4Bq+4sq}}{q}\right)\right)}{\sqrt{p^2+4Bq+4sq}}, a_1=\frac{1}{2}\left(\frac{-p+\sqrt{p^2+4Bq+4sq}}{q}\right)b_1 \end{array}\right\}$$

We therefore, obtained the following generalized solitary solution

$$U(\eta)=\frac{-\frac{1}{8}\frac{b_0^2\left(-p+\sqrt{p^2+4Bq+4sq}\right)\left(4Bq+p^2-2sq\right)}{qb_1\left(p^2+4Bq+4sq\right)}e^{-\eta}-\frac{b_0\left(4s-2B-\frac{1}{2}p\left(\frac{p+\sqrt{p^2+4Bq+4sq}}{q}\right)\right)}{\sqrt{p^2+4Bq+4sq}} -\frac{1}{2}\left(\frac{-p+\sqrt{p^2+4Bq+4sq}}{q}\right)b_1e^{\eta}}{\frac{1}{4}\frac{\left(p^2+4Bq+-2sq\right)b_0^2}{b_1\left(p^2+4Bq+4sq\right)}e^{-\eta}+b_0+b_1e^{\eta}}$$

$$U(x,t)=\frac{-\frac{1}{8}\frac{b_0^2\left(-p+\sqrt{p^2+4Bq+4sq}\right)\left(4Bq+p^2-2sq\right)}{qb_1(p^2+4Bq+4sq)}e^{-(x-Bt)}-\frac{b_0\left(4s-2B-\frac{1}{2}p\left(\frac{p+\sqrt{p^2+4Bq+4sq}}{q}\right)\right)}{\sqrt{p^2+4Bq+4sq}}-\frac{1}{2}\left(\frac{-p+\sqrt{p^2+4Bq+4sq}}{q}\right)b_1e^{(x-Bt)}}{\frac{1}{4}\frac{(p^2+4Bq+-2sq)b_0^2}{b_1(p^2+4Bq+4sq)}e^{-(x-Bt)}+b_0+b_1e^{(x-Bt)}}$$

4. Results and discussion

It is significance mentioning that our solution $q_{27}$is coincided with $u_{3,4}(x,t)$ in example 1 of section 4 of Liu et al. [41] for $s=1$, $q=1, p=2$ and $u_1=0$. Moreover, it is showing that our solution $q_{27}$is coincided with $u_{3,4}(x,t)$ in example 2 of section 4 of Liu et al. [41] for $s=-1$, $q=1, p=2$ and $u_1=0$. In addition, we construct many new exact traveling wave solutions for the combined KdV-MKdV equation in this work, which have not been found in the previous literature. Furthermore, the graphical demonstrations of some of them are depicted in the following subsection in figures below.

4.1 Graphical representations of the solutions

The graphical depictions of the solutions are shown in the figures with the help of Maple:

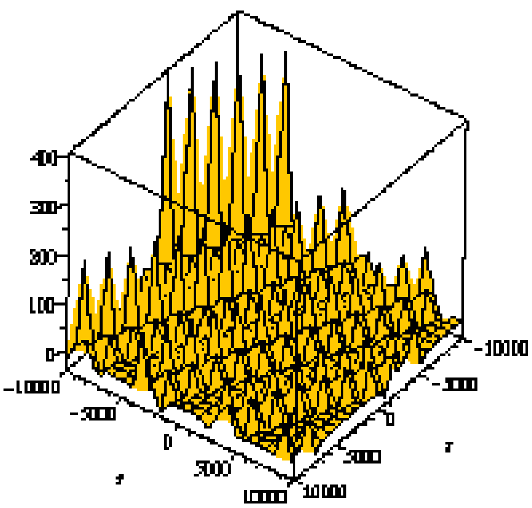

Fig. 1: Periodic solutions for $f=5, g=4, h=3, p=3, s=2, q=5$

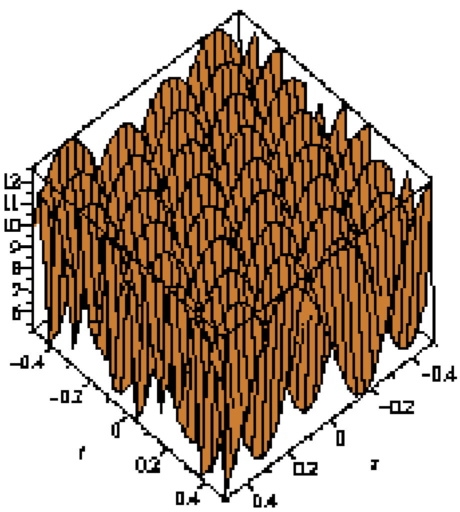

Fig. 2: Periodic solutions for $f=9, g=8, h=0, p=8, s=6, q=7, b_1=8$

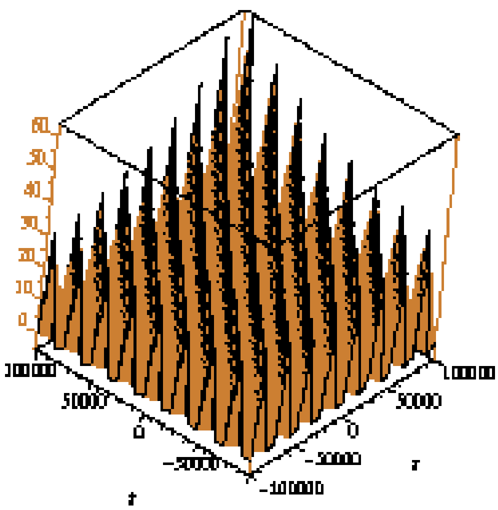

Fig. 3: Periodic solutions for
$f = 5, g = 4, h = 3, p = 2, s = 5, q = 9$

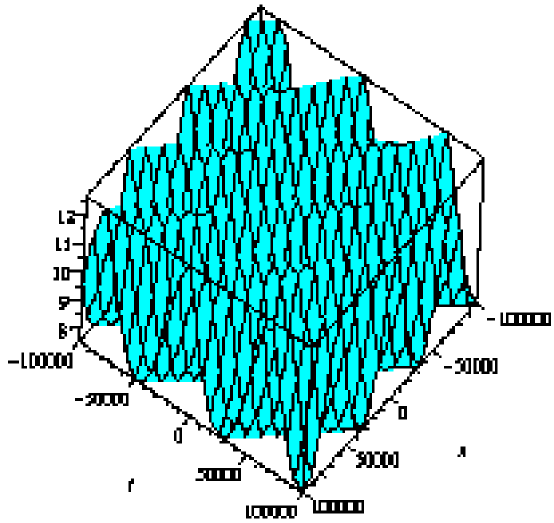

Fig. 4: Periodic solutions for
$f = 3, g = 4, h = 0, p = 1, s = 3, q = 4, b_1 = 4$

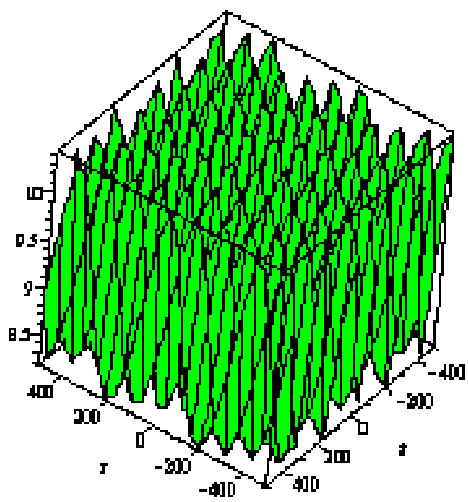

Fig. 5: Periodic solutions for
$f = 3, g = 4, h = 0, p = 2, s = 5, q = 7, b_1 = 8$

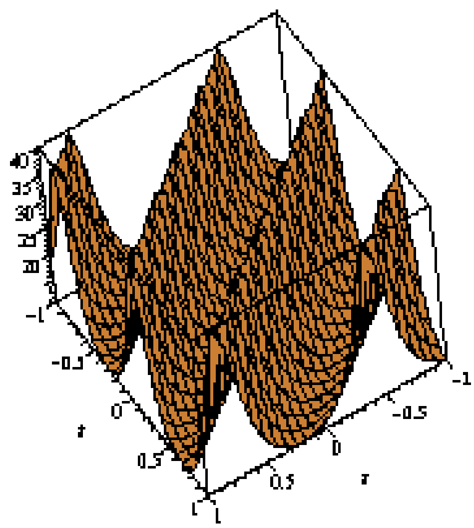

Fig. 6: Periodic solutions for
$f = 5, g = 7, h = 0, p = 5, s = 5, q = 7, b_1 = 2$

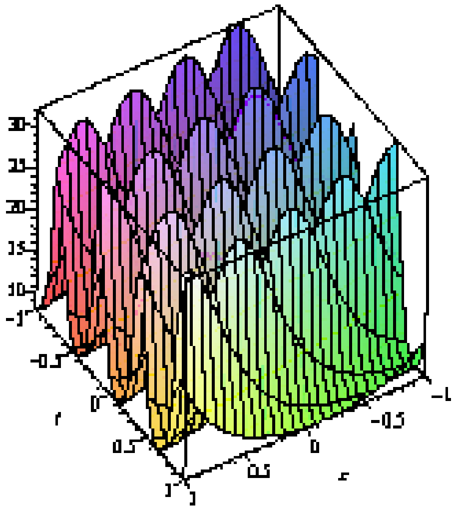

Fig. 7: Periodic solutions for
$f = 2, g = 4, h = 0, p = 1, s = 3, q = 1, b_1 = 2$

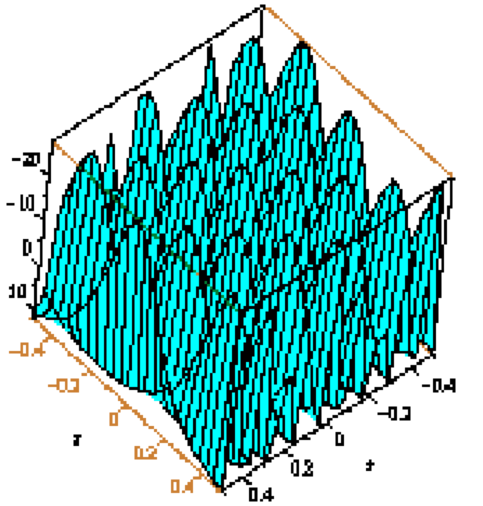

Fig. 8: Periodic solutions for
$f = 5, g = 4, h = 0, p = 3, s = 4, q = 3, b_1 = 2$

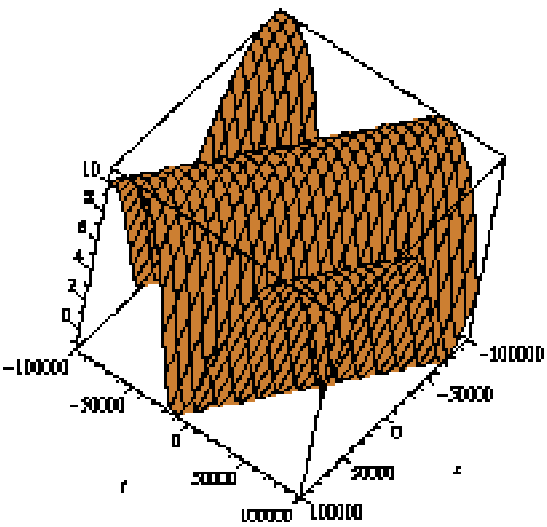

Fig. 9: Periodic solutions for
$f = 7, g = 15, h = 0, p = 3, s = 4, q = 5, b_1 = 4$

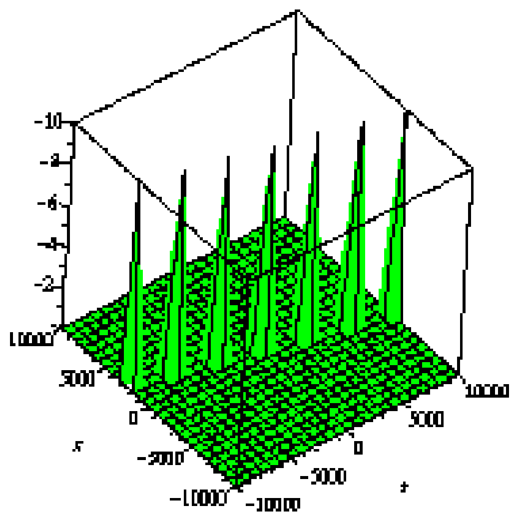

Fig. 10: Periodic solutions for
$f = 0, g = 11, h = 0, p = 3, s = 5, q = 9, b_1 = 2$

5. Conclusions

In this article, we apply the Exp-function method and generalized Riccati equation mapping together with the(G'/G)-expansion method to solve the combined KdV-MKdV equation. In (G'/G)-expansion method, the generalized Riccati equation $G'(\theta) = h + f\,G(\varphi) + g\,G^2(\theta)$ is used with constant coefficients, as the auxiliary equation, instead of the second order linear ordinary differential equation with constant coefficients. By applying these methods, we obtain abundant exact traveling wave solutions including solitons and periodic solutions and solutions are expressed in terms of the hyperbolic, the trigonometric and the rational functions. The correctness of the obtained solutions is verified to compare with the published results. We hope that these useful and powerful methods can be effectively used to solve many nonlinear evolution equations which are arising in technical arena.

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STABILITY OF A LATTICE PRESERVING FUNCTIONAL EQUATION ON RIESZ SPACE: FIXED POINT ALTERNATIVE

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Abstract. The aim of this paper is to investigate Hyers-Ulam stability of the following lattice preserving functional equation on Riesz space with fixed point method:

$$\|F(\tau x \vee \eta y) - \tau F(x) \vee \eta F(y)\| \leq \varphi(\tau x \vee \eta y, \tau x \wedge \eta y),$$

where $\mathcal{X}$ is a Banach lattice and $\varphi : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ is a mapping such that

$$\varphi(x, y) \leq (\tau\eta)^{\frac{\alpha}{2}} \varphi\left(\frac{x}{\tau}, \frac{y}{\eta}\right)$$

for all $\tau, \eta \geq 1$ and $\alpha \in [0, \frac{1}{2})$.

1. Introduction

In 1940 Ulam [1] proposed the famous Ulam stability problem: When is it true that a function which satisfies some functional equation approximately must be close to one satisfying the equation exactly?. If the answer is affirmative, we would say that the equation is stable. In 1941, Hyers [2] solved this stability problem for additive mappings subject to the Hyers condition on approximately additive mappings. The result of Hyers was generalized by Rassias [3] for linear mapping by considering an unbounded Cauchy difference.

In 1996, Isac and Rassias [4] were the first authors to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. Some authors have considered the Hyers-Ulam stability of quadratic functional equations in random normed spaces [5, 6, 7, 8, 9, 10, 11, 12, 13]. By the fixed point method, the stability problems of several functional equations have been extensively investigated by a number of authors (see [14, 15]). We generalize the Agbeko's theorem [16] and prove it by fixed point method.

A non-empty set $\mathcal{M}$ with a relation "$\leq$" is said to be an order set whenever the following conditions are satisfied:

1. $x \leq x$ for every $x \in \mathcal{M}$;
2. $x \leq y$ and $y \leq x$ implies that $x = y$;
3. $x \leq y$ and $y \leq z$ implies that $x \leq z$.

If, in addition, for two elements $x, y \in \mathcal{M}$ either $x \leq y$ or $y \leq x$, then $\mathcal{M}$ is called a totally ordered set. Let $\mathcal{A}$ be a subset of an ordered set $\mathcal{M}$. $x \in \mathcal{M}$ is called an upper bound of $\mathcal{A}$ if $y \leq x$ for all $y \in \mathcal{A}$. $z \in \mathcal{M}$ is called a lower bound of $\mathcal{A}$ if $y \geq z$ for all $y \in \mathcal{A}$. Moreover, if there is an upper bound of $\mathcal{A}$, then $\mathcal{A}$ is said to be bounded from above. If there is an lower bound of $\mathcal{A}$, then $\mathcal{A}$ is said to be bounded from below. If $\mathcal{A}$ is bounded from above and from below, then we will briefly say that $\mathcal{A}$ is order bounded.

An order set $(\mathcal{M}, \leq)$ is called a lattice if any two elements $x, y \in \mathcal{M}$ have a least upper bound denoted by $x \vee y = \sup(x, y)$ and a greatest lower bound denoted by $x \wedge y = \inf(x, y)$.

A real vector space E which is also an order set is called an order vector space if the order and the vector space structure are compatible in the following sense:

1. if $x, y \in E$ such that $x \leq y$ then $x + z \leq y + z$ for all $z \in E$;

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2. if $x, y \in E$ such that $x \leq y$ then $\alpha x \leq \alpha y$ for all $\alpha \geq 0$.
$(E, \leq)$ is called a **Riesz space** if $(E, \leq)$ is a lattice and order vector space.
A norm ρ on Riesz space E, is called a lattice norm if $\rho(x) \leq \rho(y)$ whenever $|x| \leq |y|$. In the latter case $(E, \|.\|)$ is called a normed Riesz space.
$(E, \|.\|)$ is called a **Banach lattice** if $(E, \|.\|)$ is a Banach space, E is Riesz space and $|x| \leq |y|$ implies that $\|x\| \leq \|y\|$ for all $x, y \in E$.

Example 1.1. *Suppose that $\mathcal{X}$ is a compact Hausdorff space. We denote by $C(K)$ the Banach space of all real-valued continuous functions on $\mathcal{X}$. Let "$\leq$" be a point-wise order on $C(K)$, and $f \leq g$ if and only if $f(t) \leq g(t)$ for all $t \in K$. It is easy to show that $(C(K), \leq)$ is a Banach lattice.*

Let E be a Riesz space, and let the positive cone E^+ of E consist of all $\in E$ such that $x \geq 0$. For every $x \in E$ let

$$x^+ = x \vee 0 \quad x^- = -x \vee 0 \quad |x| = x \vee -x.$$

Let E be a Riesz space. For all $x, y, z \in E$ and $a \in \mathcal{R}$, the following assertions hold:
1. $x + y = x \vee y + x \wedge y$, $-(x \vee y) = -x \wedge y$.
2. $x + (y \vee z) = (x + y) \vee (x + z)$, $x + (y \wedge z) = (x + y) \wedge (x + z)$.
3. $|x| = x^+ + x^-$, $|x + y| \leq |x| + |y|$.
4. $x \leq y$ is equivalent to $x^+ \leq y^+$ and $y^- \leq x^-$
5. $(x \vee y) \wedge z = (x \wedge y) \vee (y \wedge z)$, $(x \wedge y) \vee z = (x \vee y) \wedge (y \vee z)$

A Riesz space E is called **Archimedean** if $x \leq 0$ holds whenever the set $\{nx : n \in \mathcal{N}\}$ is bounded from above.

Theorem 1.1. *Let E be a normed Riesz space. The following assertions hold:*
1. *the lattice operations is continuous;*
2. *the positive cone E^+ is closed;*
3. $\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathcal{N}\}$.

Definition 1.1. *Let $\mathcal{X}, \mathcal{Y}$ be Banach lattices. A mapping $T : \mathcal{X} \to \mathcal{Y}$ is called positive if $T(\mathcal{X}^+) = \{T(|x|) : x \in \mathcal{X}\} \subset \mathcal{Y}^+$.*

Definition 1.2. *Let $\mathcal{X}, \mathcal{Y}$ be Banach lattices and let $T : \mathcal{X} \to \mathcal{Y}$ be a positive mapping. We define*
P_1) *lattice homomorphism:*

$$T(|x| \vee |y|) = T(|x|) \vee T(|y|);$$

P_2) *semi-homogeneity: for all $x \in \mathcal{X}$ and all $\alpha \in R^+$*

$$T(\alpha|x|) = \alpha T(|x|);$$

P_3) *continuity from below on the positive cone: for all increasing sequences $x_n \subset \mathcal{X}^+$*

$$\lim_{n\to\infty} T(x_n) = T(\lim_{n\to\infty} x_n).$$

Observe that every lattice homomorphism $T : \mathcal{X} \to \mathcal{Y}$ is necessarily a positive operator. Indeed, if $x \in E^+$ then

$$T(x) = T(x \vee 0) = T(x) \vee T(0) = T(x)^+ \geq 0$$

holds in $\mathcal{Y}$. Also it is important to note that the range of a lattice homomorphism is a Riesz subspace.

Theorem 1.2. *For an operator $T : \mathcal{X} \to \mathcal{Y}$ between two Riesz spaces, the following statements are equivalent:*
1. *T is a lattice homomorphism;*
2. *$T(x^+) = T(x)^+$ for all $x \in \mathcal{X}$;*
3. *$T(x \wedge y) = T(x) \wedge T(y)$;*
4. *if $x \wedge y = 0$ in $\mathcal{X}$, then $T(x) \wedge T(y) = 0$ holds in $\mathcal{Y}$;*
5. *$T(|x|) = |T(x)|$.*

Definition 1.3. *Let $\mathcal{X}$ be a set. A function $d : \mathcal{X} \times \mathcal{X} \to [0, \infty]$ is called a generalized metric on $\mathcal{X}$ if d satisfies the following conditions:*
(a) *$d(x, y) = 0$ if and only if $x = y$ for all $x, y \in \mathcal{X}$;*
(b) *$d(x, y) = d(y, x)$ for all $x, y \in \mathcal{X}$;*
(c) *$d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in \mathcal{X}$.*

Note that the only substantial difference of the generalized metric from the metric is that the range of generalized metric includes the infinity.

Theorem 1.3. *Let $(\mathcal{X},d)$ be a complete generalized metric space and $\mathcal{J} : \mathcal{X} \to \mathcal{X}$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then, for all $x \in \mathcal{X}$, either*

$$d(\mathcal{J}^n x, \mathcal{J}^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that
(a) *$d(\mathcal{J}^n x, \mathcal{J}^{n+1} x) < \infty$ for all $n \geq n_0$;*
(b) *the sequence $\{\mathcal{J}^n x\}$ converges to a fixed point y^* of $\mathcal{J}$;*
(c) *y^* is the unique fixed point of $\mathcal{J}$ in the set $\mathcal{Y} = \{y \in \mathcal{X} : d(\mathcal{J}^{n_0} x, y) < \infty\}$;*
(d) *$d(y, y^*) \leq \frac{1}{1-L} d(y, \mathcal{J} y)$ for all $y \in \mathcal{Y}$.*

2. Main results

Using the fixed point method, we prove the Hyers-Ulam stability of lattice homomorphisms in Banach lattices.

Theorem 2.1. *Let $\mathcal{X}, \mathcal{Y}$ be Banach lattices. Consider a positive operator $\mathcal{F} : \mathcal{X} \to \mathcal{Y}$ such that*

$$\|F(\tau x \vee \eta y) - \tau F(x) \vee \eta F(y)\| \leq \varphi(\tau x \vee \eta y, \tau x \wedge \eta y), \tag{2.1}$$

where $\varphi : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ is a mapping such that

$$\varphi(x, y) \leq (\tau\eta)^{\frac{\alpha}{2}} \varphi\left(\frac{x}{\tau}, \frac{y}{\eta}\right)$$

for all $x, y \in \mathcal{X}$, $\tau, \eta \geq 1$ and for which there is a real number $\alpha \in [0, \frac{1}{2})$ Then there is a unique positive operator $T : \mathcal{X} \to \mathcal{Y}$ satisfying the properties P_1, P_2 and the inequality

$$\|T(x) - F(x)\| \leq \frac{\tau^\alpha}{\tau - \tau^\alpha}$$

for all $x \in \mathcal{X}$.

Proof. Putting $\tau = \eta$ and $x = y$ in (2.1), we get

$$\|F(\tau x) - \tau F(x)\| \leq \varphi(\tau x, \tau y).$$

Then

$$\left\|\frac{1}{\tau} F(\tau x) - F(x)\right\| \leq \quad \tfrac{1}{\tau}\varphi(\tau x, \tau x) \leq \quad \tau^{\alpha-1}\varphi(x, x). \tag{2.2}$$

Consider the set

$$\Delta = \{g \mid g : \mathcal{X} \to \mathcal{Y}\ g(0) = 0\}$$

and introduce the generalized metric on Δ

$$d(g, h) = \inf\left\{c \in \mathbb{R}^+, \|g(x) - h(x)\| \leq c\ \varphi(x, x) \text{ for all } x \in \mathcal{X}\right\},$$

where as usual, $\inf \emptyset = \infty$. It is easy to show that (Δ, d) is complete generalized metric space. Now we define the operator $J : \Delta \to \Delta$ by

$$Jg(x) = \frac{1}{\tau} g(\tau x)$$

for all $x \in \mathcal{X}$. Given $g, h \in \Delta$, let $c \in [0, \infty]$ be an arbitrary constant with $d(g, h) \leq c$, that is,

$$\|g(x) - h(x)\| \leq c\ \varphi(x, x).$$

So we have

$$\begin{aligned} \|Jg(x) - Jh(x)\| &= \frac{1}{\tau}\ \|g(\tau x) - h(\tau x)\| \leq \frac{1}{\tau}\ c\ \varphi(\tau x, \tau x) \\ &\leq \frac{1}{\tau}\ c\ \tau^{\alpha}\varphi(x,x) = \tau^{\alpha-1}\ c\ \varphi(x,x) \end{aligned}$$

for all $x \in \mathcal{X}$, that is, $d(Jg, Jh) < \tau^{\alpha-1}\ c$. Thus we have

$$d(Jg, Jh) \leq \tau^{\alpha-1} d(g,h)$$

for all $g, h \in \Delta$. So J is a strictly contractive mapping with constant $\tau^{\alpha-1} < 1$ on Δ, For all $g, h \in \Delta$ and $\alpha \in [0, \frac{1}{2})$. By (2.2), we have

$$d(JF, F) \leq \tau^{\alpha-1} < \infty.$$

By Theorem 1.3, there exists a mapping $T : \mathcal{X} \to \mathcal{Y}$ satisfying the following:
1. T is a fixed point of J, i.e.,

$$T(\tau x) = \tau T(x)$$

for all $x \in \mathcal{X}$. Also the mapping T is a unique fixed point of J in the set

$$M = \{g \in \Delta : d(g, h < \infty)\}.$$

This implies that P_2 holds.
2. $d(J^n F, T) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{1}{\tau^n} F(2^n x) = T(x)$$

for all $x \in \mathcal{X}$.
3. $d(F, T) \leq \frac{1}{1-L} d(F, JF)$, which implies the inequality

$$\|T(x) - F(x)\| \leq \frac{\tau^{\alpha-1}}{1 - \tau^{\alpha-1}} = \frac{\tau^{\alpha}}{\tau - \tau^{\alpha}},$$

which implies that the inequality (2.1) holds.
Now we show that T satisfies P_1. Putting $\tau = \eta = \tau^n$ in (2.1), we get

$$\|F(\tau^n(x \vee y)) - \tau^n F(x) \vee \tau^n F(y)\| \leq \tau^{2n\alpha}\varphi(x \vee y, x \wedge y). \tag{2.3}$$

Replacing x, y by $\tau^n x$ and $\tau^n y$ in (2.3), respectively, we get

$$\begin{aligned} \left\|F(\tau^{2n}(x \vee y)) - \tau^n F(\tau^n x) \vee \tau^n F(\tau^n y)\right\| &\leq \tau^{2n\alpha}\varphi(\tau^n x \vee \tau^n y, \tau^n x \wedge \tau^n y) \\ &= \tau^{4n\alpha}(\varphi(x \vee y, x \wedge y). \end{aligned}$$

Then

$$\left\|\frac{1}{\tau^{2n}} F(\tau^{2n}(x \vee y)) - \frac{1}{\tau^n} F(\tau^n x) \vee \frac{1}{\tau^n} F(\tau^n y)\right\| \leq \left(\tau^{2n(2\alpha-1)}.\varphi(x \vee y, x \wedge y)\right).$$

Since $\alpha \in [0, \frac{1}{2})$, when $n \to \infty$, we have

$$\|T(x \vee y) - T(x) \vee T(y)\| \leq 0.$$

and so

$$T(x \vee y) = T(x) \vee T(y)$$

for all $x, y \in \mathcal{X}$. Note that the lattice operation is continuous. □

Theorem 2.2. *Let $\mathcal{X}, \mathcal{Y}$ be Banach lattices and let a continuous function $p : [0, \infty) \to [0, \infty)$ be given. Consider a positive $T : \mathcal{X} \to \mathcal{Y}$ for which there are real numbers $\nu \in (0, \infty)$ and $0 \leq r < 1$ such that*

$$\left\|T(\alpha|x| \vee \beta|y|) - \frac{\alpha p(\alpha) T(|x|) \vee \beta p(\beta) T(|y|)}{p(\alpha) \vee p(\beta)}\right\| \leq \nu\left(\|x\|^r + \|y\|^r\right) \tag{2.4}$$

for all $x, y \in \mathcal{X}$ and $\alpha, \beta \in \mathcal{R}^+$. Then there exist a unique positive mapping $F : \mathcal{X} \to \mathcal{Y}$ which satisfies the properties P_1, P_2 and the inequality

$$\|F(|x|) - T(|x|)\| \leq \frac{2\nu}{2-2^r}$$

for all $x \in \mathcal{X}$.

Proof. Putting $\alpha = \beta = 2$ and $x = y$ in (2.4), we get

$$\left\| T(2|x| \vee 2|x|) - \frac{2p(2)T(|x|) \vee 2p(2)T(|x|)}{p(2) \vee p(2)} \right\| \leq 2\nu \|x\|^r$$

for all $x \in \mathcal{X}$ and $r \in [0, 1)$. Thus

$$\|T(2|x|) - 2T(|x|)\| \leq 2\nu \|x\|^r$$

and so

$$\left\| \frac{1}{2} T(2|x| - T(|x|)) \right\| \leq \nu \|x\|^r \tag{2.5}$$

for all $x \in \mathcal{X}$ and $\alpha \in [0, 1)$. Consider the set

$$\Delta := \{S : S : \mathcal{X} \to \mathcal{Y}\ ,\ S(0) = 0\}$$

and introduce the generalized metric on Δ

$$d(S, H) = \inf\{c \in \mathcal{R}^+, \|S(x) - H(x)\| \leq c\|x\|^r,\ \forall x \in \mathcal{X}\},$$

where, as usual, $\inf \emptyset = \infty$. It is know that (Δ, d) is complete. Now we define the mapping $J : \Delta \to \Delta$ by

$$JS(|x|) = \frac{1}{2} S(2|x|)$$

for all $x \in \mathcal{X}$. First we assert that J is strictly contractive with constant 2^{r-1} on Δ. Given $S, H \in \Delta$, let $c \in [0, \infty]$ be an arbitrary constant with $d(S, H) < c$, that is,

$$\|S(|x|) - H(|x|)\| \leq c\|x\|^r.$$

So we have

$$\|JS(x) - JH(x)\| = \frac{1}{2}\|S(2|x|) - H(2|x|)\| \leq \frac{1}{2} c\|2x\|^r = 2^{r-1} c\|x\|^r$$

for all $x \in \mathcal{X}$, that is, $d(JS, JH) \leq 2^{r-1}c$. Thus we have

$$d(JS, JH) \leq 2^{r-1} d(S, H)$$

for all $S, H \in \Delta$ and so J is strictly contractive with constant $2^{r-1} < 1$ on Δ. For all $S, H \in \Delta$ and $r \in [0, 1]$. By (2.5) we have

$$d(JF, F) \leq \nu < \infty$$

By Theorem 1.3, there exists a mapping $F : \mathcal{X} \to \mathcal{Y}$ satisfying the following:

1. F is a fixed point J i. e.

$$F(2|x|) = 2F(|x|)$$

for all $x \in \mathcal{X}$. Also the mapping F is a unique fixed point of J in the set

$$M = \{S \in \Delta : d(S, H) < \infty\}.$$

2. $d(J^n T, F) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{1}{2^n} T(2^n x) = F(x) \tag{2.6}$$

for all $x \in \mathcal{X}$.

3. $d(T, F) \leq \dfrac{1}{1-L} d(T, JT)$, which implies the inequality

$$\|F(|x|) - T(|x|)\| \leq \frac{\nu}{1 - 2^{r-1}} = \frac{2\nu}{2 - 2^r}.$$

This implies that the inequality (2.2) holds.

Now, we show that F is a lattice homomorphism. Putting $\alpha = \beta = 2^n$ in (2.4),

$$\|T\left(2^n(|x| \vee |y|)\right) - 2^n(T(|x|) \vee T(|y|))\| \leq \nu(\|x\|^r + \|y\|^r). \tag{2.7}$$

Replacing x and y by 2^x and $2^n y$ in (2.7), respectively, we obtain

$$\|T\left(4^n(|x| \vee |y|)\right) - 2^n(T(2^n|x|) \vee T(2^n|y|))\| \leq 2^{nr}\nu(\|x\|^r + \|y\|^r)$$

and so

$$\left\|\frac{1}{4^n}T\left(4^n(|x| \vee |y|)\right) - \frac{1}{2^n}(T(2^n|x|) \vee T(2^n|y|))\right\| \leq 2^{n(r-2)}\nu(\|x\|^r + \|y\|^r).$$

As $n \to \infty$, we have

$$\|F(|x| \vee |y|) - F(|x|) \vee F(|y|)\| \leq 0.$$

and so

$$F(|x| \vee |y|) = F(|x|) \vee F(|y|)$$

for all $x, y \in \mathcal{X}$. Next we show that $T(\alpha|x|) = \alpha T(|x|)$ for all $x \in \mathcal{X}$ and all real numbers $\alpha \in [0, \infty)$. Letting $\alpha = \beta, y = 0$ and replacing α by $2^n\alpha$ in (2.4), we get $F(0) = 0$ and so F satisfies P_1. So $T(0) = 0$ with (2.6) and

$$\|T(2^n\alpha|x|) - 2^n\alpha T(|x|)\| \leq \nu\|x\|^r \tag{2.8}$$

for all $x \in \mathcal{X}$ and all real numbers $\alpha \in [0, \infty)$. Replacing x by $2^n x$ in (2.8),

$$\|T(4^n\alpha|x|) - 2^n\alpha T(2^n|x|)\| \leq \nu\ 2^{nr}\ \|x\|^r$$

and so

$$\left\|\frac{T(4^n\alpha|x|)}{4^n} - \alpha\frac{T(2^n(|x|))}{2^n}\right\| \leq \nu\ 2^{n(r-2)}\ \|x\|^r$$

for all $x \in \mathcal{X}$. As $n \to \infty$, we obtain

$$\|F(\alpha|x| - \alpha F(|x|))\| \leq 0$$

and so

$$F(\alpha|x| = \alpha F(|x|).$$

for all $x \in \mathcal{X}$ and $\alpha \in [0, \infty)$. □

Corollary 2.1. *Let $\mathcal{X}, \mathcal{Y}$ be Banach lattices. Consider a positive operator $T : \mathcal{X} \to \mathcal{Y}$ for which there are real numbers $\nu \in (0, \infty)$ and $0 \leq r < 1$ such that*

$$\|T(\alpha|x| \vee \beta|y|) - \alpha T(|x|) \vee \beta T(|y|)\| \leq \nu(\|x\|^r + \|y\|^r)$$

for all $x, y \in \mathcal{X}$ and $\alpha, \beta \in \mathcal{R}^+$. Then there exists a unique positive mapping $F : \mathcal{X} \to \mathcal{Y}$ which satisfies the properties P_1, P_2 and the inequality

$$\|F(|x|) - T(|x|)\| \leq \frac{2\nu}{2 - 2^r}.$$

for all $x \in \mathcal{X}$.

Corollary 2.2. *Let $\mathcal{X}, \mathcal{Y}$ be Banach lattices. Consider a positive operator $T : \mathcal{X} \to \mathcal{Y}$ for which there are real numbers $\nu \in (0, \infty)$ and $0 \leq r < 1$ such that*

$$\left\|T(\alpha|x| \vee \beta|y|) - \frac{\alpha^2 T(|x|) \vee \beta^2 T(|y|)}{\alpha \vee \beta}\right\| \leq \nu(\|x\|^r + \|y\|^r)$$

for all $x, y \in \mathcal{X}$ and $\alpha, \beta \in \mathcal{R}^+$. Then there exists a unique positive mapping $F : \mathcal{X} \to \mathcal{Y}$ which satisfies the properties P_1, P_2 and the inequality

$$\|F(|x|) - T(|x|)\| \leq \frac{2\nu}{2 - 2^r}$$

for all $x \in \mathcal{X}$.

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UNIQUENESS THEOREM OF MEROMORPHIC FUNCTIONS AND THEIR k-TH DERIVATIVES SHARING SET

JUNFENG XU AND FENG LÜ

Abstract. In this paper, due to the theories of normal family and complex differential equation, we consider a uniqueness problem of meromorphic functions share set $S = \{a,\ b\}$ with their k-th derivatives.

1. Introduction and main results

Let $\mathcal{F}$ be a family of meromorphic functions defined in D. $\mathcal{F}$ is said to be normal in D, in the sense of Montel, if for any sequence $f_n \in \mathcal{F}$, there exists a subsequence f_{nj} , such that f_{nj} converges spherically locally uniformly in D, to a meromorphic function or ∞ (see, [18]).

Let f and g be two meromorphic functions in a domain D, and let a be a complex number. If $g(z) = a$ whenever $f(z) = a$, we write $f(z) = a \Rightarrow g(z) = a$. If $f(z) = a \Rightarrow g(z) = a$ and $g(z) = a \Rightarrow f(z) = a$, we write $f(z) = a \Leftrightarrow g(z) = a$ and say that f and g share the value a IM (ignoring multiplicity). If $f - a$ and $g - a$ have the same zeros with the same multiplicities, we write $f(z) = a \rightleftharpoons g(z) = a$ and say that f and g share the value a CM (counting multiplicity). Let S be a set of complex numbers. Provide that $f(z) \in S$ if and only if $g(z) \in S$ in a domain D, then we say f and g share the set S in D. It is assumed that the reader is familiar with the standard symbols and fundamental results of Nevanlinna theory, as found in [4, 21].

In the theory of normal family, it is meaningful to find sufficient conditions for normality(see. [1, 7, 8, 9, 10, 11, 15, 17, 20]). Recently, Y. Li [7] obtained a normal family of holomorphic functions share set with their k-th derivatives as follows.

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Theorem A. *Let $\mathcal{F}$ be a family of holomorphic functions in a domain D, let $k(\geq 2)$ be a positive integer, and let a, b be two distinct finite complex numbers. If for each $f \in \mathcal{F}$, all the zeros of f are of multiplicity at least k, and f and f' share the set $S = \{a, b\}$, then $\mathcal{F}$ is normal in D.*

Remark 1. In fact, for the case $ab \neq 0$, the conclusion of Theorem A still holds if the condition f and $f^{(k)}$ share the set $S = \{a, b\}$ CM is replaced by

$$f(z) \in S \Rightarrow f^{(k)}(z) \in S.$$

See Section 3.

In the uniqueness theory, an important subtopic that a meromorphic function and it's derivative share some values or functions or set is well investigated. Due to Theorem A, Y. Li [7] obtained a uniqueness theorem of entire functions.

Theorem B. *Let $k(\geq 2)$ be a positive integer, and let a, b be two distinct finite complex numbers, and let f be a non-constant entire function. If all the zeros of f are of multiplicity at least k, and f and $f^{(k)}$ share the set $S = \{a, b\}$ CM, then*

(1). *$f(z) = Ce^{Dz}$, where $C \neq 0$ and D are two constants with $D^k = \pm 1$,*

(2). *$f = -f^{(k)} + a + b$.*

In [7], Y. Li also gave an example to show that the case (2) can not omitted.

Example 1. Let $f(z) = \cos^2 \frac{z}{2}$. Then f and f'' share set $\{0, \frac{1}{2}\}$ CM and all zeros of f are of multiplicity at least 2. Obviously, $f = -f'' + \frac{1}{2}$.

After considering Theorem B and Example 1, we naturally ask the following questions.

Question 1. What happens if f is a meromorphic function?

Question 2. Note that $k = 2$ in **Example** 1. Naturally, we ask whether Case (2) occurs for $k \neq 2$ or not?

Question 3. What's the specific form of f in Case (2)?

In the work, we focus on the above questions. Basing on the idea of Y. Li in [7] and due to the theories of normal family and complex differential equation, we further study the uniqueness problem of meromorphic functions of finitely many poles sharing a set CM with their derivatives.

Theorem 1.1. *Let $k(\geq 2)$ be a positive integer, and let a, b be two distinct finite complex numbers, and let f be a non-constant meromorphic function with finitely many poles. If all the zeros of f are of multiplicity at least k, and f and $f^{(k)}$ share the set $S = \{a, b\}$ CM, then one of the following cases must occur:*

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(1). $f(z) = Ce^{Dz}$, where $C \neq 0$ and D are two constants with $D^k = 1$, and $f = f^{(k)}$;

(2). $f(z) = Ce^{Dz}$, where $C \neq 0$ and D are two constants with $D^k = -1$, $f = -f^{(k)}$ and $S = \{a, -a\}$;

(3). $f(z) = A_1e^{iz} + A_2e^{-iz} + a + b$, where A_1 and A_2 are two nonzero constants with $(a+b)^2 = 4A_1A_2$, $f = -f^{''} + a + b$, and k must be 2.

Remark 2. For the special case that $A_1 = A_2 = \frac{1}{4}$, $a = 0$ and $b = \frac{1}{2}$, then Case (3) becomes **Example** 1.

Remark 3. We answer the **Questions** 2 and find out the case (2) occurs only for $k = 2$ in Theorem B. We also answer the **Question** 3 and give the form of f. We partial answer the **Question** 1.

In 2008, we considered the case of $k = 1$ and obtained a normal criteria theorem and a uniqueness theorem[10].

Theorem C. *Let $\mathcal{F}$ be a family of functions holomorphic in a domain, let a and b be two distinct finite complex numbers with $a + b \neq 0$. If for all $f \in \mathcal{F}$, f and f' share $S = \{a,\ b\}$ CM, then $\mathcal{F}$ is normal in D.*

Theorem D. *Let a and b be two distinct complex numbers with $a + b \neq 0$, and let $f(z)$ be a nonconstant entire function. If f and f' share the set $\{a,\ b\}$ CM, then one and only one of the following conclusions holds: (i) $f = Ae^z$ or (ii) $f = Ae^{-z} + a + b$, where A is a nonzero constant.*

By the same way to Theorem 1.1, we can obtain the following.

Theorem 1.2. *Let a and b be two distinct complex numbers with $a+b \neq 0$, and let $f(z)$ be a nonconstant meromorphic function with finite poles. If f and f' share the set $\{a,\ b\}$ CM, then one and only one of the following conclusions holds: (i) $f = Ae^z$ or (ii) $f = Ae^{-z} + a + b$, where A is a nonzero constant.*

2. Some Lemmas

Lemma 2.1. [15] *Let $\mathcal{F}$ be a family of functions holomorphic on a domain D, all of whose zeros have multiplicity at least k , and suppose that there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$. Then if $\mathcal{F}$ is not normal at $z_0 \in D$, for each $0 \leq \alpha \leq k$, there exist,*

(a) a number $0 < r < 1$;

(b) points $z_n \to z_0$;

(c) functions $f_n \in \zeta$, and

(d) positive number $\rho_n \to \infty$ such that $\rho_n^{-\alpha} f_n(z_n + \rho_n\xi) = g_n(\xi) \to g(\xi)$ locally uniformly, where g is a nonconstant entire function on C with order at most 1, all of whose zeros have multiplicity at least k, such that $g^\sharp(\xi) \leq g^\sharp(0) = kA + 1$.

Here, as usual,

$$g^{\sharp}(\xi) = \frac{|g'(\xi)|}{1+|g(\xi)|^2}$$

is the spherical derivative.

Lemma 2.2. [3, 13] *Let f be an entire (resp. meromorphic) function, and let M be a positive number. If $f^{\sharp}(z) \le M$ for any $z \in C$, then f is of order at most 1 (resp. 2).*

It is well known that it is very important of the Wiman-Valiron theory[5, 6] to investigate the property of the entire solutions of differential equations. In 1999, Zong-Xuan Chen[2] has extented the Wiman-Valiron theory from entire functions to meromorphic functions with infinitely many poles. Here we show the following form given by Jun Wang and Wei-Ran Lü[19].

Lemma 2.3. *Let $f(z) = \frac{g(z)}{d(z)}$ be a meromorphic function with $\rho(f) = \rho$, where g, d are entire functions satisfying one of the following conditions:*
(i) g being transcendental and d being polynomial;
(ii) g, d all being transcendental and $\lambda(d) = \rho(d) = \beta < \rho(g) = \rho$.
Then there exists one sequence $\{r_k\}(r_k \to \infty)$ such that

$$\frac{f^{(n)}(z)}{f(z)} = (\frac{\nu(r_k, g)}{z})^n(1+o(1)), \quad n \in \mathbb{N}$$

holds for enough large r_k as $|z| = r_k$ and $|g(z)| = M(r_k, g)$, where $\nu(r_k, g)$ denotes the central index of g.

Lemma 2.4. [14] *Let f be an entire function of order at most 1 and k be a positive integer, then*

$$m(r, \frac{f^{(k)}}{f}) = o(\log\ r), \ \text{as } r \to \infty.$$

Lemma 2.5. [21] *Let f be a nonconstant meromorphic function, and a_j $(j = 1, \cdots q)$ be q (≥ 3) distinct constant (one of them may be ∞), then*

$$(q-2)T(r,f) \le \sum_{j=1}^{q} \overline{N}(r, \frac{1}{f-a_j}) + S(r,f),$$

where

$$S(r,f) = m(r, \frac{f'}{f}) + m(r, \sum_{j=1}^{q} \frac{f'}{f-a_j}) + O(1).$$

Combining Lemmas 2.4 and 2.5, we have the following special case of the Nevanlinna's second fundamental theorem.

Lemma 2.6. *Let f be a nonconstant entire function of order at most 1, and a_j $(j = 1, \cdots q)$ be q (≥ 3) distinct constants (one of them may be ∞), then*

$$(q-2)T(r,f) \le \sum_{j=1}^{q} \overline{N}(r, \frac{1}{f-a_j}) + o(\log\ r).$$

3. Proof of Theorem 1.1

Firstly, we will prove that the meromorphic (resp. entire) function f is of order at most 2 (resp. 1).

Suppose that the spherical derivative of f is bounded. Then by Lemma 2.2, we have meromorphic (resp. entire) function f is of order at most 2 (resp. 1). Now, we assume that the spherical derivative of f is unbounded. Then there exist a sequence $\{w_n\}$ such that $w_n \to \infty$, $f^\sharp(w_n) \to +\infty$ as $n \to \infty$.

Define $D = \{z : |z| < 1\}$ and

$$F_n(z) = f(w_n + z).$$

Since f only has finitely many poles, we can assume that all $F_n(z)$ are analytic in D. Furthermore, $F_n^\sharp(0) = f^\sharp(w_n) \to \infty$ as $n \to \infty$. It follows from Marty's criterion that $(F_n)_n$ is not normal at $z = 0$.

Obviously, for each n, F_n has zeros with multiplicities at least k, F_n and $F_n^{(k)}$ share S CM. Thus, from Theorem A, we derive that $(F_n)_n$ is normal at $z = 0$, a contradiction.

Thus, we prove that the meromorphic (resp. entire) function f is of order at most 2 (resp. 1).

Since f and $f^{(k)}$ share S CM and f has finitely many poles, we have

$$\frac{(f^{(k)} - a)(f^{(k)} - b)}{(f - a)(f - b)} = \frac{e^Q}{P}, \tag{3.1}$$

where P, Q are two polynomials. Rewrite (3.1) as follows.

$$Q = \log P \frac{(\frac{f^{(k)}}{f} - \frac{a}{f})(\frac{f^{(k)}}{f} - \frac{b}{f})}{(1 - \frac{a}{f})(1 - \frac{b}{f})}, \tag{3.2}$$

where log h is the principle branch of *Log h*.

If $f(z) = \frac{g(z)}{d(z)}$ is a transcendental meromorphic function, where $g(z)$ is a transcendental entire function and $d(z)$ is a polynomial. Then by Lemma 2.3, we get

$$\frac{f^{(k)}(z)}{f(z)} = (\frac{\nu(r_k, g)}{z})^k(1 + o(1)), \tag{3.3}$$

holds for enough large r_k as $|z| = r_k$ and $|g(z)| = M(r_k, g)$. Note that f is transcendental, we have $\frac{a}{f}|_{z_r} \to 0$ and $\frac{b}{f}|_{z_r} \to 0$ as $r \to \infty$. It follows from the fact g is of finite order that $\log \nu(r,g) = O(\log r)$. Then, we deduce that

$$|Q(z)| = |\log P\frac{(\frac{f^{(k)}}{f}-\frac{a}{f})(\frac{f^{(k)}}{f}-\frac{b}{f})}{(1-\frac{a}{f})(1-\frac{b}{f})}|_{z_r} = O(\log r),$$

for enough large r_k as $|z| = r_k$ and $|g(z)| = M(r_k, g)$. It implies that Q is a constant.

If $f(z)$ is a rational function, then by (3.1) we know that Q must be a constant.

Without loss of generality, we rewrite (3.1) as

$$\frac{1}{P} = \frac{(f^{(k)}-a)(f^{(k)}-b)}{(f-a)(f-b)}.$$

Next, we will prove that P is also a constant. On the contrary, suppose that P is not a constant. We know any zero of P comes from the pole of f, so $d = \deg P \geq 2k$.

From the above equation, we get

$$1 = P\frac{(\frac{f^{(k)}}{f}-\frac{a}{f})(\frac{f^{(k)}}{f}-\frac{b}{f})}{(1-\frac{a}{f})(1-\frac{b}{f})}.$$

In a similar way as the above, we get

$$1 = |P(z_r)(\frac{\nu(r,g)}{z_r})^{2k}(1+o(1))| = |\nu(r,g)^2|z_r|^{d-2k}| = \nu(r,g)^2 r^{d-2k},$$

possibly outside a finite logarithmic measure set E, where $|g(z_r)| = M(r,g)$ and $|z| = r_k$. Since $d = \deg P \geq 2k$, it implies that $\nu(r,g)$ is bound, a contradiction. Hence, P is also a constant.

Thus, we prove that

$$A = \frac{(f^{(k)}-a)(f^{(k)}-b)}{(f-a)(f-b)},$$

where A is a nonzero constant. From the above equation, we see that f is an entire function, so the order of f is at most 1.

Set $F = f - \frac{a+b}{2}$ and $G = f^{(k)} - \frac{a+b}{2}$. Then $\frac{G^2-\frac{(a-b)^2}{4}}{F^2-\frac{(a-b)^2}{4}} = A$. Set $h_1 = G - \sqrt{A}F$ and $h_2 = G + \sqrt{A}F$, then we have

$$h_1 h_2 = \frac{(a-b)^2}{4}(1-A).$$

We consider two cases.

Case 1. $A \neq 1$.

Obviously, h_1, h_2 has no zeros and poles. Then we set $h_1(z) = A_1e^{Bz}$ and $h_2(z) = A_2e^{-Bz}$, where $A_1,\ A_2,\ B$ are constants. Furthermore, we have

$$f(z) = \frac{a+b}{2} + \frac{1}{2\sqrt{A}}(-A_1e^{Bz} + A_2e^{-Bz}),$$

$$f^{(k)}(z) = \frac{a+b}{2} + \frac{1}{2}(A_1e^{Bz} + A_2e^{-Bz}),$$

$$f'(z) = \frac{B}{2\sqrt{A}}(-A_1e^{Bz} - A_2e^{-Bz}).$$

The above part is based on the idea in [7]. Now, we consider two subcases again.

Case 1.1. $A_1A_2 \neq 0$.

It follows from the form of f that f has infinitely many zeros. Noting that the zeros of f has multiplicities at least k, we have $f^{(s)}$ $(s = 0, \cdots k-2)$ has multiple zeros. Clearly, f' just has simple zeros. Then, $k - 2 \leq 0$, so k must equal to 2. By differentiating f' one time, we have

$$f''(z) = \frac{B^2}{2\sqrt{A}}(-A_1e^{Bz} + A_2e^{-Bz}).$$

Comparing it to the above form of $f^{(k)}$, we have

$$\frac{a+b}{2} + \frac{1}{2}(A_1e^{Bz} + A_2e^{-Bz}) = \frac{B^2}{2\sqrt{A}}(-A_1e^{Bz} + A_2e^{-Bz}),$$

which means that either A_1 or A_2 is zero, a contradiction.

Case 1.2. $A_1A_2 = 0$.

Without loss of generality, we assume that $A_2 = 0$. Then we have

$$f(z) = \frac{a+b}{2} - \frac{1}{2\sqrt{A}}A_1e^{Bz}.$$

From the form of f, it is easy to see that if f has zeros, then f just has simple zeros. It contradicts with the fact f has zeros of multiplicities at least k. So, f has no zeros and $a + b = 0$. Thus, we can set

$$f(z) = Ce^{Dz},$$

where $C,\ D$ are two constants. By differentiating f k-times, we have $f^{(k)}(z) = CD^ke^{Dz}$. From f and $f^{(k)}$ share S CM, we have $D^k = \pm 1$.

If $D^k = 1$, then $f = f^{(k)}$, and f and $f^{(k)}$ share $a,\ b$ CM.
If $D^k = -1$, then $f = -f^{(k)}$, and $b = -a$.

Case 2. $A = 1$.

Then it is easy to see that $f = f^{(k)}$ or $f^{(k)} + f = a + b$.

Suppose that $f = f^{(k)}$. Noting that f equals to $f^{(k)}$, so they share 0 CM. Moreover, from the fact that all the zeros of f has multiplicities at least k, we derive f has no zeros. Then, by the same way in **Case 1.2**, we get the same results.

Finally, by the similar way in [12], we will discuss the case of $f^{(k)} + f = a + b$.

Solving the differential equation, we have

$$f(z) = \sum_{j=0}^{k-1} C_j \exp^{\lambda_j z} + a + b, \tag{3.4}$$

where $\lambda_j = \exp^{\frac{2j\pi+\pi}{k}i}$ and C_j are constants. Since f is a non-constant, then there exist $C_j \in \{C_0,\ C_1,\ \cdots,\ C_{k-1}\}$ such that $C_j \neq 0$. Denote the non-zero constants in $\{C_j\}$ by C_{j_m} $0 \le j_m \le k-1$ and $m = 0,\ 1,\ \cdots,\ s,\ s \le k-1$. Thus, rewrite (3.4) as

$$f(z) = \sum_{m=0}^{s} C_{j_m} \exp^{\lambda_{j_m} z} + a + b. \tag{3.5}$$

Differentiating (3.5) t-times yields

$$f^{(t)}(z) = \sum_{m=0}^{s} C_{j_m} \lambda_{j_m}^t \exp^{\lambda_{j_m} z},\ (t = 1,\ 2\ \cdots,\ k-1). \tag{3.6}$$

Suppose that f has finitely many zeros, then we can set $f(z) = P_1(z)e^{\lambda z}$, where P_1 is a polynomial. By differentiating it k times, we have

$$f^{(k)}(z) = [\lambda^k P_1 + \lambda^{k-1} P_1' + H(P_1'',\ P_1''',\ \cdots,\ P_1^{(k)})]e^{\lambda z},$$

where $H(P_1'',\ P_1''',\ \cdots,\ P_1^{(k)})$ is the linear combination of $P_1'',\ P_1''',\ \cdots,\ P_1^{(k)}$. Substituting the above forms of f and $f^{(k)}$ into $f + f^{(k)} = a + b$, we derive that

$$P_1 + \lambda^k P_1 + \lambda^{k-1} P_1' + H(P_1'',\ P_1''',\ \cdots,\ P_1^{(k)}) = 0,$$

which implies that $\lambda^k = -1$ and $P_1' = 0$. Thus, P_1 is a constant and f has no zeros. By the same way in **Subcase 1.2**, we derive the desired results.

Thus, in what follows, we assume that f has infinitely many zeros, say $z_n = r_n e^{\theta_n}$, where $0 \le \theta_n < 2\pi$. Without loss of generality, we may assume that $\theta_n \to \theta_0$ and $r_n \to \infty$ as $n \to \infty$. Substituting z_n into (3.5) and (3.6), we have

$$f(z_n) = \sum_{m=0}^{s} C_{j_m} \exp^{\lambda_{j_m} z_n} = -(a+b) \tag{3.7}$$

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and

$$f^{(t)}(z_n) = \sum_{m=0}^{s} C_{j_m}(\lambda_{j_m})^t \exp^{\lambda_{j_m} z_n} = 0, \ (t = 1,\ 2\ \cdots,\ k-1). \tag{3.8}$$

We consider two cases again.

Subcase 2.1. $s = k - 1$.

From (3.7) and (3.8), we have

$$\begin{pmatrix} a+b \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} C_{j_0} & C_{j_1} & \cdots & C_{j_{k-1}} \\ C_{j_0}\lambda_{j_0} & C_{j_1}\lambda_{j_1} & \cdots & C_{j_{k-1}}\lambda_{j_{k-1}} \\ \vdots & & & \\ C_{j_0}(\lambda_{j_0})^{k-1} & C_{j_1}(\lambda_{j_1})^{k-1} & \cdots & C_{j_{k-1}}(\lambda_{j_{k-1}})^{k-1} \end{pmatrix} \begin{pmatrix} \exp^{\lambda_{j_0} z_n} \\ \exp^{\lambda_{j_1} z_n} \\ \vdots \\ \exp^{\lambda_{j_{k-1}} z_n} \end{pmatrix}.$$

We know

$$\begin{aligned} & det \begin{pmatrix} C_{j_0} & C_{j_1} & \cdots & C_{j_{k-1}} \\ C_{j_0}\lambda_{j_0} & C_{j_1}\lambda_{j_1} & \cdots & C_{j_{k-1}}\lambda_{j_{k-1}} \\ \vdots & & & \\ C_{j_0}(\lambda_{j_0})^{k-1} & C_{j_1}(\lambda_{j_1})^{k-1} & \cdots & C_{j_{k-1}}(\lambda_{j_{k-1}})^{k-1} \end{pmatrix} \\ & = C_{j_0}C_{j_1}\cdots C_{j_{k-1}} det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_{j_0} & \lambda_{j_1} & \cdots & \lambda_{j_{k-1}} \\ \vdots & & & \\ (\lambda_{j_0})^{k-1} & (\lambda_{j_1})^{k-1} & \cdots & (\lambda_{j_{k-1}})^{k-1} \end{pmatrix} \\ & = C_{j_0}C_{j_1}\cdots C_{j_{k-1}} \prod_{0\le q<p\le k-1} (\lambda_{j_p} - \lambda_{j_q}). \end{aligned}$$

It's is a Vandermonde determinant.

Noting that $\lambda_{j_p} \neq \lambda_{j_q}$ $(0 \le q < p \le k-1)$, we obtain that the system of linear equations of $\exp^{\lambda_{j_0} z_n}$, $\exp^{\lambda_{j_1} z_n}$, $\cdots$, $\exp^{\lambda_{j_{k-1}} z_n}$ has a unique solution. A routine calculation leads to the solution that

$$\exp^{\lambda_{j_p} z_n} = D_p, \ (0 \le p \le k-1). \tag{3.9}$$

where D_p is a constant and is of independent with z_n.

If $a + b = 0$, we see that $D_p = 0$, a contradiction. Then, we assume that $a + b \neq 0$.

Thus, as $n \to \infty$, by (3.9) we can deduce that

$$\cos(\theta_0 + \frac{2j_p\pi + \pi}{k}) = 0, (0 \le p \le k-1). \tag{3.10}$$

Otherwise, we have $\cos(\theta_0 + \frac{2j_p\pi+\pi}{k}) > 0$ or $\cos(\theta_0 + \frac{2j_p\pi+\pi}{k}) < 0$.

If $\cos(\theta_0 + \frac{2j_p\pi+\pi}{k}) > 0$, then we can assume (for n large enough) $\cos(\theta_n + \frac{2j_p\pi+\pi}{k}) > \delta$, here δ is a small positive number. Thus, as $n \to \infty$, by (3.9)

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we have

$$|D_p| = \exp^{r_n \cos(\theta_n + \frac{2j_p\pi+\pi}{k})} \to \infty,$$

a contradiction.

If $\cos(\theta_0 + \frac{2j_p\pi+\pi}{k}) < 0$, then we can assume (for n large enough) $\cos(\theta_n + \frac{2j_p\pi+\pi}{k}) < -\delta$, here δ is a small positive number. Thus, as $n \to \infty$, by (3.9) we have

$$|D_p| = \exp^{r_n \cos(\theta_n + \frac{2j_p\pi+\pi}{k})} \to 0,$$

a contradiction.

Observing that $0 \leq j_p,\ j_q \leq k-1$, by (3.10), we deduce

$$|\frac{2j_p\pi+\pi}{k} - \frac{2j_q\pi+\pi}{k}| = \pi, (0 \leq p \neq q \leq k-1). \tag{3.11}$$

Let $j_p = 0$ and $j_q = k-1$. Substitute them into (3.11), we have

$$2(k-1) = k,$$

that is $k = 2$. Thus, k must be 2.

Now we discuss the equation $f + f^{''} = a + b$. From the above discussion, we can obtain $\lambda_0 = i$, $\lambda_1 = -i$. Then, we have

$$f(z) = A_1 e^{iz} + A_2 e^{-iz} + a + b.$$

Noting that f has zeros of multiplicity at leat 2, Then

$$(a+b)^2 = 4A_1A_2.$$

Then, we finish the proof of this subcase.

Subcase 2.2. $s < k-1$.

Then, by (3.8), we can choose $t = 1,\ 2,\ \cdots,\ s+1$. Then they form a system of linearly equation of $\exp^{\lambda_{j_0} z_n},\ \exp^{\lambda_{j_1} z_n},\ \cdots,\ \exp^{\lambda_{j_s} z_n}$. By solving it, we have

$$\exp^{\lambda_{j_p} z_n} = 0, \tag{3.12}$$

a contradiction.

Hence, we complete the proof of this theorem.

4. Proof of Theorem 1.2

If Theorem A is replaced by Theorem C, by the same way to the proof of Theorem 1.1, we can also obtain the

$$A = \frac{(f'-a)(f'-b)}{(f-a)(f-b)},$$

where A is a nonzero constant. From the above equation, we see that f is an entire function. Hence we can get the conclusion by Theorem D.

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Compact adaptive aggregation multigrid method for Markov chains

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Abstract

A new adaptive aggregation-based multigrid scheme is presented for the calculation of the stationary probability vector of an irreducible Markov chain. By exploiting the experimental observation that components of vectors converge nonuniformly, we develop a new algorithm to speed up the on-the-fly adaptive multigrid method proposed by Treiter and Yavney [On-the-fly adaptive smoothed aggregation multigrid for Markov chains, *SIAM J. Sci. Comput.*, 33(2011): 2927-2949]. In our algorithm, the converged components are collected and compacted into one aggregate on the finest level, which is able to cut down the cost of coarsen operators construction and the total amount of work. In addition, we present a technique to delete the possible weak-links introduced in the process of aggregation. Several types of test cases are calculated, and experiment results show that the new adaptive method can improve the on-the-fly algorithm in terms of total execution time.

Key words: Adaptive aggregation multigrid; on-the-fly adaptive method; Markov chains; converged components

1 Introduction

This paper is concerned with a new adaptive multigrid method for the numerical calculation of the stationary probability vector of irreducible, large and sparse Markov matrices. Let $B \in \mathbb{R}^{n\times n}$ be a sparse column-stochastic matrix, which means $\mathbf{1}^T B = \mathbf{1}^T$, where $\mathbf{1}$ is the column vector of all ones, and $0 \le b_{ij} \le 1 \forall i, j$. We seek a vector $\mathbf{x} \in \mathbb{R}^n$ that satisfies

$$B\mathbf{x} = \mathbf{x}, \| \mathbf{x} \|_1 = 1, x_i \ge 0 \forall i. \tag{1.1}$$

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Matrix B represents the transition matrix of a Markov chain and $\mathbf{x}$ is a stationary probability vector of this Markov chain. If B is irreducible, that is, there exists a path from each vertex i to each vertex j in its directed graph, then according to the Perron-Frobenius theorem for nonnegative matrices [1], the equation (1.1) has a unique solution $\mathbf{x}$, with $x_i > 0 \forall i$. This problem (1.1) is equivalent to the singular linear system

$$A\mathbf{x} = \mathbf{0}, \| \mathbf{x} \|_1 = 1, x_i \geq 0 \forall i. \tag{1.2}$$

where $A := I - B$, by $\mathbf{1}^T B = \mathbf{1}^T$, we have $\mathbf{1}^T A = \mathbf{0}$,which means the vector we seek, $\mathbf{x}$, is the only left null-vector of the matrix A.

Algebraic multigrid method (AMG) was developed and applied widely due to its efficiency for solving large problems arising from partial differential equations and M-matrices. Compared with geometric multigrid methods, AMG constructs the multigrid hierarchy only using the information of the given matrix, which extends the application of multigrid methods. However, it leads to the inefficiency and the lack of robustness, because the operators of these multigrid methods are constructed based on the unsatisfied assumptions made on the near null spaces of the matrices. To overcome this disadvantage, several adaptive algebraic multigrid methods were developed in [4, 26, 5]. The basic idea of these adaptive approaches was of improving multigrid methods by updating interpolation and coarsen operators to fit the slow-to-converge components of the vector. The idea was further developed in adaptive AMG [23] and adaptive SA [24, 25], where slow-to-converge components were exposed through multiscale development instead of relaxation on finest-level.

The Markov chains solver which was outlined in [13] was actually another form of adaptive AMG, because they share the same concept of updating operator to get more accurate approximation of the near null space of A. With the same idea, a multilevel adaptive aggregation [7] was suggested with aggregates updated in each step of the iteration. Based on this algorithm, a collection of Markov chains solvers were proposed recently: adaptive aggregation multigrid for Markov chains (AGG) [7], smoothed aggregation multigrid (SA) [6], AMG for Markov chains (MCAMG) [8]. Several accelerated methods were developed in [18, 10]. While all these adaptive approaches improved the algorithms robustness and accuracy by adapting coarsen operators in every cycle, they also suffered from considerable computation time for calculating the coarsen matrix [27]. The on-the-fly adaptive multigrid hierarchy for Markov chains which was developed in [19] significantly cut the cost of constructing the coarse-level operators. Here, the classical solution cycles are preferred over the adaptive cycles, under the assumption that the former is comparatively cheaper but it needs the operators provided by the latter.

The algorithm presented in this paper is inspired by the following experimental observation: when applying aggregation multigrid V-cycle to obtain approximation of stationary probability vector, the elements of the stationary probability vector do not converge uniformly. Based on this observation, we propose a compressed on-the-fly adaptive scheme to save the cost on constructing coarsen operators. The main idea is to compact the converged components into a single aggregate and rescale the coarsen operators . Also we develop a new technique that deletes weak-links introduced by the above procedure. As the improvement of the on-the-fly adaptive aggregation method, the new algorithm adopts the same adaptive hierarchy as on-the-fly method does. It differs, however, in that the on-the-fly method uses operators supplied by SET cycles without any amendment, whereas in new algorithm, the coarsen operators are rescaled to smaller size to fit the not-converged-yet components. It is shown numerically that the new algorithm can reduce the total execution time of the on-the-fly adaptive multigrid method. New algorithm can also be applied to various adaptive multigrid Markov solvers. In this paper we apply it to the aggregation-based algebraic multigrid solver (AGG), with unsmoothed interpolation and prolongation operators.

In the next section, we give a brief description of multilevel aggregation multigrid method for Markov problems. Then we recall the on-the-fly adaptive framework in Section 3, which the new algorithm is based on. In Section 4, we outline the experimental observation as the stage for the introduction of new algorithm, and we compare the proposed algorithm with compatible relaxation method as well. Numerical tests are presented in Section 5.

2 Classical aggregation multigrid for Markov chains

In this section, we briefly recall the aggregation-based multigrid methods for Markov chains from [13, 7, 6]. The interpolation operators of aggregation multigrid are often smoothed to overcome the instinct difficulties produced by aggregation [6, 14]. In our work, we stay with the unsmoothed coarsen operators.

First, we define the multiplicative error $\mathbf{e}_i$ by $\mathbf{x} = diag(\mathbf{x}_i)\mathbf{e}_i$, where $\mathbf{x}_i$ is the current approximate at ith iterate. Thus we have

$$Adiag(\mathbf{x}_i)\mathbf{e}_i = \mathbf{0}. \tag{2.1}$$

It is necessary to assume that all components of $\mathbf{x}_i$ are nonzero. At convergence, $\mathbf{x}_i = \mathbf{x}$ and the fine-level error $\mathbf{e}_i = \mathbf{1}$, where $\mathbf{1}$ is the column vector with all ones.

Note that the aggregation technique used in this paper is the same as that used in [7], which is based on strength of connection in the scaled matrix $\widetilde{A} = Adiag(\mathbf{x}_i)$, the benefit of using the scaled matrix $\widetilde{A}$ instead of original matrix A is that the former gives more appropriate notion of weak and strong links than the latter, more details are in [7]. We consider node i is strongly connected to node j if

$$-\tilde{a}_{ij} \geq \theta \max_{k \neq i}\{-\tilde{a}_{ik}\}. \tag{2.2}$$

where $\theta \in [0, 1]$ is a strength threshold parameter, we choose $\theta = 0.8$. Aggregates based on the strength of connection are then constructed by the following procedure: choose point i with the largest value in current proximation $\mathbf{x}_i$ from the unassigned points as the seed point of a new aggregate, then add all unassigned points j satisfies (4) to the new

3

aggregates. Repeat this procedure until all points are assigned to aggregates. Assuming that the n fine-level points are aggregated into m groups, then the aggregation matrix $Q \in \mathbb{R}^{n\times m}$ are formed, where $q_{ij} = 1$ indicates that fine-level point i belongs to aggregate j and $q_{ij} = 0$ the opposite[6]. Then the coarse version of (3) is given by

$$Q^T Adiag(\mathbf{x}_i)Q\mathbf{e}_c = 0, \tag{2.3}$$

where $\mathbf{e}_c$ is the coarse-level approximation of the fine-level error $\mathbf{e}_i$, with $\mathbf{e}_i \approx Q\mathbf{e}_c$.

The restriction and prolongation operators, R and P are defined as follows:

$$R = Q^T, P = diag(\mathbf{x}_i)Q. \tag{2.4}$$

Then (5) can be rewrited as

$$RAP\mathbf{e}_c = 0. \tag{2.5}$$

Same as the definition of fine-level multiplicative error $\mathbf{x}_i$, the coarse-level error $\mathbf{x}_c$ is given by

$$\mathbf{x}_c = diag(R\mathbf{x}_i)e_c. \tag{2.6}$$

Notice that $P^T\mathbf{1} = R\mathbf{x}_i$,thus (3) can be rewrited as

$$RAPdiag(P^T\mathbf{1})^{-1}\mathbf{x}_c = 0. \tag{2.7}$$

Then the coarse-level error equation (5) is equivalent to coarse-level probability equation $A_c\mathbf{x}_c = 0$, with coarsen matrix A_c defined by

$$A_c = RAPdiag(P^T\mathbf{1})^{-1}. \tag{2.8}$$

When the coarsen solution $\mathbf{x}_c$ is obtained, the next iterate, $\mathbf{x}_{i+1}$ can be calculated according to the coarse-level correction

$$\mathbf{x}_{i+1} = P\mathbf{e}_c = Pdiag(P^T\mathbf{1})^{-1}\mathbf{x}_c. \tag{2.9}$$

In this paper we use weighted Jacobi method for all relaxation procedure, at each coarser level we perform v_1 pre-relaxation and v_2 post-relaxations. One iteration of weighted Jacobi relaxation applied to problem $A\mathbf{x} = b$ is given by

$$\mathbf{x} \leftarrow \mathbf{x} + \omega D^{-1}(b - A\mathbf{x}). \tag{2.10}$$

where D is the diagonal part of A, its relaxation parameter $\omega = 0.7$. On coarsest level we perform direct solver described in [8]. The procedure above is described in Algorithm 1, which is originally presented in [13]. The multilevel aggregation method is obtained by recursively applying Algorithm 1 to step 5.

Algorithm 1: Two-level aggregation for Markov chains $\mathbf{x} \leftarrow \text{AGG}(A,\mathbf{x},\mu,\nu_1,\nu_2)$

Input*:Initial vector:* $\mathbf{x} \in \mathbb{R}^n$*, operator:* $A \in \mathbb{R}^{n\times n}$*, cycle index:* μ*,*
number of pre-relaxations: ν_1*, number of post-relaxations:* ν_2*.*
Output*: New approximation to the solution of* $A\mathbf{x} = \mathbf{0}$*.*
Algorithm*:*
if *not at coarsest level*
1. $\mathbf{x} \leftarrow Relax(A,\mathbf{x},\mathbf{0})$ ν_1 *times*
2. *Build* Q *based on* A *and* $\mathbf{x}$
3. *Set* $R \leftarrow Q^T$, $P \leftarrow diag(\mathbf{x}_i)Q$
4. *Set* $\mathbf{x}_c \leftarrow R\mathbf{x}$*, and repeat* $\mu \geq 1$ *times:*
$$\mathbf{x}_c \leftarrow AGG(RAPdiag(P^T\mathbf{1})^{-1}, \mathbf{x}_c, \mu, \nu_1, \nu_2)$$
5. *Coarse-grid correction:* $\mathbf{x} \leftarrow Pdiag(P^T\mathbf{1})^{-1}\mathbf{x}_c$
6. $\mathbf{x} \leftarrow Relax(A,\mathbf{x},\mathbf{0})$ ν_2 *times*

else
7. *Direct solve of* $A\mathbf{x} = 0$

end

3 On-the-fly aggregation multigrid for Markov chains

In this section, we briefly describe the on-the-fly multigrid method developed recently in [19]. The main idea of this method is reducing the cost of expensive SET cycle such as Algorithm 1, which updating the whole multigrid hierarchy of operators in every cycle, by using classical algebraic multigrid cycles (Algorithm 2) instead, as the two algorithms are actually equivalent. In the approach, SET cycle provide classical cycle with improved operators, while classical cycle use them without adaptation and then offer SET cycle with better approximation of vector. It is obvious that the classical cycle with frozen operators are much more cheaper than the SET cycle, the advantage of this scheme is, by combining the two algorithms neatly, it speeds up the multigrid methods without sacrificing the convergence rate.

Classical algebraic multigrid method for linear systems are generally based on the following basic idea. Given the linear system

$$A\mathbf{x} = \mathbf{b}, \tag{3.1}$$

where $A \in \mathbb{R}^{n\times n}$ is a positive definite matrix. Traditional one-level iterative method for calculating $\mathbf{x}$, such as Power method or weighted Jocobi relaxation, converge very slowly due to only a relatively small number of components in the error, known as algebraically smooth, that approximately satisfy $A\mathbf{e} = \mathbf{0}$. To eliminate the algebraic smoothed errors, classical multigrid methods solve this problem on a coarse level with smaller size, referred to as coarse-grid correction process. It is noted that on the coarse grid, the smooth error appears to be relatively higher in frequency, which means relaxations are more effective

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on coarser grid [3]. Algorithm 2 gives a typical two-level classical multigid cycle [5], a multilevel V-cycle is obtained by recursively applying the algorithm in step 4.

Algorithm 2: Two-level additive cycle

Input:*Initial vector:* $\mathbf{x} \in \mathbb{R}^n$, *Right-hand-side vector:* $\mathbf{b} \in \mathbb{R}^n$, *operator:* $A \in \mathbb{R}^{n\times n}$, $P \in \mathbb{R}^{n\times n_c}$,$R \in \mathbb{R}^{n_c\times n}$,$A_c \in \mathbb{R}^{n_c\times n_c}$.

Output: *New approximation to the solution of* $A\mathbf{x} = \mathbf{b}$.

Algorithm:

1. *Apply pre-relaxations:* $\mathbf{x} \leftarrow Relax(A,\mathbf{x},\mathbf{b})$
2. *Define the residual:* $\mathbf{r} \leftarrow \mathbf{b} - A\mathbf{x}$
3. *Restrict the residual:* $\mathbf{r}_c \leftarrow R\mathbf{r}$
4. *Define* $\mathbf{e}_c$ *as the solution of the coarse-grid problem:* $A_c\mathbf{e}_c = \mathbf{r}_c$
5. *Prolong* $\mathbf{e}_c$ *and apply coarse-grid correction:* $\mathbf{x} \leftarrow \mathbf{x} + P\mathbf{e}_c$
6. *Apply post-relaxations:* $\mathbf{x} \leftarrow Relax(A,\mathbf{x},\mathbf{b})$

The difference between Algorithm 1 and Algorithm 2 is that, on the coarse-grid, the correction scheme of two-level additive cycle approximates the error $\mathbf{e}$ rather than the exact solution $\mathbf{x}$. Moreover, the classical algorithm requires the whole hierarchy of coarsen operators in advance, while the setup schemes calculate them in every cycle. In spite of that, Algorithm 2 can be written as the form of Algorithm 1 equivalently. For the problem (2) in which $\mathbf{b} = \mathbf{0}$, the residual $\mathbf{r}$ in step 2 and $\mathbf{r}_c$ in step 3 of algorithm 2 are given as $\mathbf{r} = -A\mathbf{x}$ and $\mathbf{r}_c = -RA\mathbf{x}$, the coarse-grid problem then is given by

$$A_c\mathbf{e}_c = RAP\mathbf{e}_c = -RA\mathbf{x}, \tag{3.2}$$

then we obtain

$$RA(P\mathbf{e}_c + \mathbf{x}) = 0. \tag{3.3}$$

Since the approximation $\mathbf{x}$ is in the range of P, there exists a vector $\mathbf{x}_c$ satisfies $\mathbf{x} = P\mathbf{x}_c$. Then the equation above can be rewritten as $RAP(\mathbf{e}_c + \mathbf{x}_c) = 0$. Note that the $\mathbf{x}_c$ we mentioned above is not necessary the same as $\mathbf{x}_c$ in Step 5 in Algorithm 1. We define $\mathbf{z}_c = \mathbf{e}_c + \mathbf{x}_c$, thus $A_c\mathbf{z}_c = \mathbf{0}$, which is equivalent to the coarse-grid problem in SET cycle of Algorithm 1.

In the on-the-fly approach, an initial SET cycle is performed, followed by a SOL cycle which freezes the operators the SET cycle provided. If the convergence speed of SOL cycle is acceptable, another SOL cycle is performed. Conversely, a SET cycle is performed to yield more accurate operators. This procedure is described as follows.

Procedure: try-SOL-else-SET(γ)

1. *Try a solution cycle:* $\mathbf{y} = V_{sol}(\mathbf{x})$
2. *If* $q(\mathbf{y}) > q(\mathbf{x})$ *do* $\mathbf{x} \leftarrow V_{set}(\mathbf{x})$ *and return*
3. *If* $q(\mathbf{y}) > \gamma q(\mathbf{x})$ *then* $\mathbf{x} = \mathbf{y}$, *else* $\mathbf{x} \leftarrow V_{set}(\mathbf{y})$

In above procedure, V_{sol} represents a SOL cycle (Algorithm 2), V_{set} represents a SET cycle (Algorithm 1) and $\gamma \in [0, 1]$ is the scalar threshold for acceptable convergence speed of the SOL cycles. We use

$$q(\mathbf{x}) = \frac{\| A\mathbf{x} \|_1}{\| \mathbf{x} \|_1}, \tag{3.4}$$

which means the convergence factor is measured by the l_1 residual norm. The criteria $q(\mathbf{y}) > q(\mathbf{x})$ indicates that the SOL cycle increases the error and should be abandoned. The criteria $q(\mathbf{y}) > \gamma q(\mathbf{x})$ indicates that if the convergence factor of SOL cycle is better than the scalar threshold, then accept it, otherwise, perform a SET cycle instead. The on-the-fly adaptive algorithm is described as in Algorithm 3.

Algorithm 3: On-the-fly adaptive multigrid method

Input*:Initial tolerance: ε_α, convergence parameter: γ, operator: $A \in \mathbb{R}^{n\times n}$, initial guess $\mathbf{x}_0$.*

Output*: New approximation to the solution of $A\mathbf{x} = \mathbf{0}$.*

Algorithm:

1. ***Initial Setup:***
   Apply a few relaxations to smooth $\mathbf{x}_0$
   Do an initial Setup cycle: $\mathbf{x} \leftarrow V_{set}(\mathbf{x}_0)$
   if $\| A\mathbf{x}_0 \|_1 < \varepsilon_\alpha$, ***goto*** *Step 4*
2. ***Improve Solution Cycle:***
   while $\| A\mathbf{x}_0 \|_1 > \varepsilon_\alpha$ ***do*** *try-SOL-else-SET(γ)*
3. ***Finalize Setup cycle:***
   $\mathbf{x} \leftarrow V_{set}(\mathbf{x})$
4. ***Solution:***
   Apply $\mathbf{x} \leftarrow V_{sol}(\mathbf{x})$ *until convergence*

4 Compact adaptive aggregation multigrid

In this section, we show how compacting the converged points into an aggregate, coupled with deleting the weak-links between them, can lead to better performance of on-the-fly method for Markov chains.

4.1 Experimental observation

We define a point has already converged as in [15]:

$$|x_i^{(\nu+1)} - x_i^{(\nu)}|/|x_i^{(\nu)}| < \tau_p, \tag{4.1}$$

where x_i denotes the ith element of the vector, $x_i^{(\nu)}$ denotes ith element at νth iterate, and τ_p is the convergence parameter. In [15], it is noted that the convergence patterns of the stationary probability vector of web matrix in the power method have a nonuniform distribution. Additional theoretical analysis in [17] has confirmed this conclusion recently. During the application of AGG on Markov problems, we have seen the similar convergence behavior that some points converge quickly while some others need more iterations before convergence. It is shown in Figure 1 that the number of the converged points increased gradually as iteration number increased.

To exploit this observation, the method outlined in [15] is that the converged components won't be recomputed so that computation cost can be reduced. The basic idea

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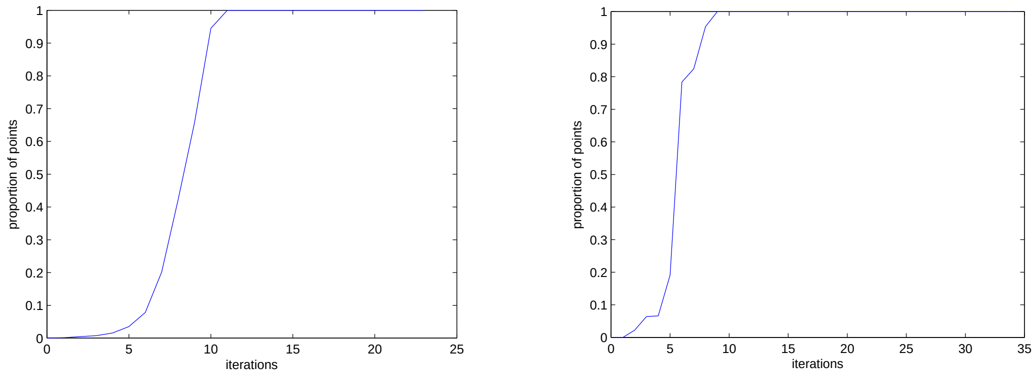

(a) Tandem queueing network with n=1600 (b) Uniform 2D lattice with n=4096

Fig. 1: (a) Tandem queueing network with n=1600, (b) Uniform 2D lattice with n=4096, where x-axis represents iterations and y-axis represents the proportion of the points that satisfy the equation (4.1).

developed there has three steps: splitting the vector into converged and not-yet-converged components, setting the submatrix $A_N \in \mathbb{R}^{m\times n}$ which corresponds to the not-yet-converged components as target matrix, and then applying the power method until convergence without recomputing converged components. More details can be seen in [15]. However, as A_N is not a $n \times n$ matrix, many algorithms including AMG can not be applied to this method. For this reason, with the similar principle but different procedures, we propose a new algorithm in this paper.

4.2 Compact adaptive aggregation multigrid

The main idea of our algorithm is reducing the computational cost by reducing the size of the coarse levels as well as the time spent on the coarse matrix construction. The new algorithm follows the same framework as the on-the-fly adaptive multigrid method does.

Consider that we have executed a setup cycle, then an approximation $\mathbf{x}$ and the aggregation matrix Q are constructed in this cycle. Perform the try-SOL-else-SET procedure until the number of converged points meets $m > \zeta n$, where m is the number of the converged points, n is the size of the problem, $\zeta \in (0, 1)$ is the threshold parameter. The reason why we set this standard will be addressed in the following paragraphs. Let C as set of the converged points whose elements are positive integers between 1 and n, and N as set of the points have not converged yet.

Partitioning the finest-level matrix as

$$\hat{A} = \begin{pmatrix} A_{NN} & A_{NC} \\ A_{CN} & A_{CC} \end{pmatrix}. \tag{4.2}$$

Similarly, the current approximation $\mathbf{x}$ and its multiplicative error, $\hat{\mathbf{e}}$ are reordered as

$$\hat{\mathbf{x}} = \begin{pmatrix} \mathbf{x}_N \\ \mathbf{x}_C \end{pmatrix}, \tag{4.3}$$

$$\hat{\mathbf{e}} = \begin{pmatrix} \mathbf{e}_N \\ \mathbf{e}_C \end{pmatrix}, \tag{4.4}$$

respectively. To reduce the time cost of coarse matrix construction, on-the-fly method proposed that the SOL cycle use the aggregation matrix Q which is offered by SET cycle without any modification [19]. Whereas in our method, we modify the aggregation matrix Q before we perform a SOL cycle. As to modifications, we keep the non-converged points in their aggregates and collect the converged points into a new aggregate. Then a further standard solution cycle is performed with amended operators and smaller scales.

The motivation is that we try to speed up the multigrid solvers by cutting down the cost on coarsen operators construction as well as reducing the size of coarse operators.

Now we show how to construct the new aggregation matrix $\hat{Q}$ by modifying the aggregation matrix Q from the setup cycle. We first delete the rows of Q which belongs to C, then check for those columns with all zero elements and delete them, finally, construct $\hat{\mathbf{Q}}$ as given in Algorithm 3, where the length of $\mathbf{1}$ equals to that of C. The procedure is simple and inexpensive:

Procedure: Construct compact aggregation matrix $\hat{Q}$

1. *Delete* $Q(i,:)$, $i \in C$
2. *Delete* $Q(:,j)$ *if* $Q(:,j) = \mathbf{0}$
3. $\hat{Q} \leftarrow \begin{pmatrix} Q & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$, *where* $\mathbf{1}$ *is the column vector of all ones, with length equals that of C*

Now we constructed coarse operators based on aggregation matrix $\hat{Q}$. As the same definition in the classical AMG, the restriction and prolongation operators, R and P, are given by

$$\hat{R} = \hat{Q}^T, \tag{4.5}$$

$$\hat{P} = diag(\hat{\mathbf{x}}_C)\hat{Q}, \tag{4.6}$$

respectively. The coarse-level operator $\hat{A}_c$ is given by

$$\hat{A}_c = \hat{R}\hat{A}\hat{P}. \tag{4.7}$$

Thus we obtain the complete hierarchy of multigrid operators the SOL cycle required, then we perform a standard SOL cycle as the final step to finish the new solution cycle, as described in Algorithm 4.

Algorithm 4: Compact Solution Cycle(C-SOL)

Input*:Approximate vector:* $\mathbf{x} \in \mathbb{R}^n$, *operator:* $A \in \mathbb{R}^{n\times n}$, *the converged points set* C. *aggregation matrix* $Q \in \mathbb{R}^{n_c \times n}$

Output*: New approximation to the solution of* $A\mathbf{x} = \mathbf{0}$.

Initial setup*:*

1. *Set* $\hat{A} \leftarrow \begin{pmatrix} A_{NN} & A_{NC} \\ A_{CN} & A_{CC} \end{pmatrix}$, $\hat{\mathbf{x}} \leftarrow \begin{pmatrix} \mathbf{x}_N \\ \mathbf{x}_C \end{pmatrix}$
2. *Construct compactive aggregation matrix* $\hat{Q}$ *based on* Q *and* C.
3. *Set* $\hat{R} = \hat{Q}^T$, $\hat{P} = diag(\hat{\mathbf{x}}_C)\hat{Q}$, *Set* $\hat{A}_c = \hat{R}\hat{A}\hat{P}$

Apply solution cycle*:*

4. *Do a standard solution cycle described in Algorithm 2*

In new method we prefer C-SOL cycle over SOL cycle if the former's error reduction is acceptable. The underlying assumption is that C-SOL cycles are considerably cheaper with satisfied convergence rate. However, if the ratio of m above n is too small or too big, this assumption will be ruined.

On the one hand, for most of test cases, when we put a small number ($m < 0.1n$) of the converged points into an aggregate, the C-SOL cycle is more expensive than the SOL cycle. This is because the cost on SET process in Algorithm 4 cannot be balanced out by the time saved by cutting scales of coarse-levels. On the other hand, if a large number of the converged points are compacted into an aggregate, it may lead to quite inaccurate operators in coarse-levels. Numerical experiments confirm that the resulting algorithm performs worse than the original on-the-fly method or leads to divergence for most problems. For the above reasons, we introduce the restriction for the number of converged points m: if $m < \zeta n$ we perform the procedure try-SOL-else-SET(γ), elsewhere we perform try-CSOL-else-SET(γ) instead.

Similar with on-the-fly method, the goal of our method is to fall off the time cost on reaching the accuracy $\| A\mathbf{x}_0 \|_1 < \varepsilon_\alpha$. Whereas the most distinguished difference of the new algorithm from on-the-fly adaptive multigrid is in Step 2. At Step 2 in new algorithm we initially perform the procedure try-SOL-else-SET until the number of the converged points meets the compactive condition $m \geq \zeta n$. With the converged points set C supplied by the process above and the aggregation matrix Q provided by SET cycle, we construct the C-SOL cycle, then we repeat the procedure try-CSOL-else-SET with the until the residual norm of the approximation reduced to ε_α. It is noted that in C-SOL cycle we frozen the coarsen operators as well as the converged points. The algorithm is described in Algorithm 5.

Algorithm 5: Compactive On-the-fly adaptive multigrid method

Input*:Initial tolerance:* ε_α*, convergence criterion:* τ_p*,convergence factor:* γ*,*
  size control parameter: ζ*, operator:* $A \in \mathbb{R}^{n\times n}$*,initial guess* $\mathbf{x}_0$*,*
Output*: New approximation to the solution of* $A\mathbf{x} = \mathbf{0}$.
Algorithm*:*
1. ***Initial Setup****:*
  Apply a few relaxations to smooth $\mathbf{x}_0$
  Do an initial Setup cycle: $\mathbf{x} \leftarrow V_{set}(\mathbf{x}_0)$
  if $\| A\mathbf{x}_0 \|_1 < \varepsilon_\alpha$ ***goto*** *Step 5*
2. ***Improve Solution Cycle****:*
  $[N, C] \leftarrow$ *Detect-converged-points(*$\mathbf{x}^{(1)}, \mathbf{x}^{(0)}, \tau_p$*)*
  While $\| A\mathbf{x}_0 \|_1 > \varepsilon_\alpha$ ***do***
    While $m < \zeta n$ ***do***
      try-SOL-else-SET(γ*)*
      $[N, C] \leftarrow$ *Detect-converged-points(*$\mathbf{x}^{(\nu+1)}, \mathbf{x}^{(\nu)}, \tau_p$*)*
    end
    try-CSOL-else-SOL(γ*)*
  end
4. ***Finalize Setup cycle****:*
  $\mathbf{x} \leftarrow V_{set}(\mathbf{x})$
5. ***Solution****:*
  Apply $\mathbf{x} \leftarrow V_{sol}(\mathbf{x})$ *until convergence*

As mentioned above, compacting the converged points into an aggregate may lead to a single aggregate with a large number of points that are not strongly connected to each other. As is shown in [6], the aggregate of points that are weakly connected may result in very poor convergence of the multilevel method. The reason is that if the link between two points is weak compared to the other links in the same aggregate, the differences in the error of these two points can neither be eliminated efficiently by relaxation, nor smoothed out by coarse-level correction. Thus, although we have made the restriction for the size of the aggregate, it may still induce unsatisfied convergence.

Our next work is trying to define and delete the weak links in the aggregation of converged points to avoid the poor convergency.

4.3 Compactive on-the-fly method with correction

We illustrate with a simple example. In C-SOL cycles, the converged points are compacted into a single aggregation. Figure 2 is an example of such an aggregation. Links between the converged points and not-converged-yet points are not presented in this figure. We assume that the converged points set as C=[4,5,9,10,14,17,18,38].

To capture the weak links in this aggregate, we need to determine what is meant by weak links. In the classical AMG, the strong connection is defined by formula (2.2), which indicates that if the size of the transition probability from i to j timed with the

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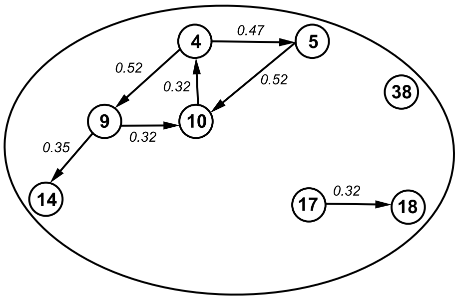

Figure 2: *Single aggregate of converged points with fine-level transitions. The converged points are indicated by numbers in cycles, and the transitions are indicated by arrows with strength based on the scaled matrix $A_{CC}diag(\mathbf{x_C})$. Connections between the converged points and those have not converged yet are not presented in this figure.*

probability of residing in i is comparative large, then it is a strong link. Rather than the connection strength between two points, our attention is turned to the overall connection strength between a point and the rest points in the same aggregate, which is used to measure the importance of a point in its aggregate.

We define the connection strength of point i based on scaled matrix $A_{CC}diag(\mathbf{x_C})$ with elements $\tilde{a}_{il}$ by

$$S_i = -\sum_{l \neq i} (\tilde{a}_{il} + \tilde{a}_{li}). \tag{4.8}$$

This definition has an simple intuitive interpretation that the overall connection strength of a point is measured not only by the probability from other points to it but also by the probability from it to others. If a point's overall connection strength is comparative small, it cannot contribute efficiently to the elimination of errors but may lead to poor convergence. In the view of the above, we define a point is weakly connected to the others if

$$S_i \leq \delta \bar{S}_i, \tag{4.9}$$

where $\bar{S}_i$ is the mean value of all S_i $(i \in C)$, and δ is a fixed threshold parameter, whose function is as the same as θ in (2.2). Choosing $\delta > 1$ may set down all points as "not important" points especially when the number of points strong connected to the others is large. Meanwhile, it should not be taken much smaller than 1 because this may leave weak-links staying in the aggregate. The numerical results indicate that choosing $\delta < 1$ but close to 1 results in the best convergence properties for the new method. In generally we take $\delta = 0.8$.

It is easy to calculate and conclude that the points 14,17,18,38 in figure 1 are weakly connected to the other points in the aggregate, thus we have the new $\bar{C} = [4, 5, 9, 10]$ to replace the original C.

5 Numerical results and discussion

In this section, we demonstrate the performance of the new algorithm for several test problems. The algorithm is applied to the two-level classical aggregation multigrid method, without smoothing operators. We compare the results of original on-the-fly adaptive aggregation multigrid algorithms (OTF) and the compactive on-the-fly adaptive aggregation multigrid algorithms (C-OTF). We start with an initial guess of unit vector with its elements all equal to 1/n, in which n is the length of the vector. All setup cycles employ (4,1) cycles, with four prerelaxations and one postrelaxation on each level, while all compactive solution cycles and original solution cycles use (2,1) cycles. We use the stopping criteria

$$stop\ if \quad \nu > maxit \quad or \quad \frac{\| A\mathbf{x}_\nu \|_1}{\| \mathbf{x}_\nu \|_1} < \tau \| A\mathbf{x}_0 \|_1. \tag{5.1}$$

proposed in [9],where $maxit$ is the upper limit of the number of iterations the algorithm will be allowed to perform, ν is the current νth iteration. Here we use $maxit = 200$ and $\tau = 10^{-8}$. We also say the problem has reached global convergence if this criterion has met. Several threshold value τ_p are tested in the experiments. Through extensive simulations, we found that $\tau_p = 10^{-3}$ achieved the best performance among any others for most of test cases. For OTF algorithm we use scalar threshold $\gamma = 0.8$ and $\varepsilon_\alpha = 10^{-5}$, while for C-OTF algorithm we use $\gamma = 0.8$ and various choices of ε_α are presented in the following table. As to the AGG part in algorithms we use the aggregation strategy based on scaled matrix proposed in [7], with the strength threshold parameter $\theta = 0.25$.

In the following tables, we show the operator complexity C_{OP} and the work units WU which is defined as the cost of a single $V_{sol}(2, 1)$ [19]. WU is calculated as follows: for each problem and its size, averaging the execution time of a $V_{sol}(2, 1)$ by calculating the mean value of last five solution cycles in step 4 in Algorithm 3, the work units are the total execution time of the algorithm divided by this time. The motivation is that the execution time of the algorithm is susceptible to MATLAB's compilation time. V_{set}, V_{sol}, V_{csol} are the number of SET cycles, SOL cycles and C-SOL cycles, respectively. The experiments were performed using MATLAB R2010a with an Intel core i3 CPU with 4 GB of RAM memory.

5.1 Uniform chain

The first three test problems are generated by graphs with weighted edges[6, 20]. Their transition probabilities are determined by weights of the edge: if node i transforms to j with p weights and then its probability p_{ji} is obtained by p divided by the sum of the weights of all outgoing edges from node i. Our first test problem is the 1D uniform chain, generated by linear graphs in which each of two connected points has one outgoing edge with weight 1. The stencil of the matrix of uniform chain is given by

$$H_{UniformChain} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}. \tag{5.2}$$

Table 5.1 shows the results for the uniform chain problem using OTF-AGG algorithm (Algorithm 3) and C-OTF-AGG algorithm (Algorithm 5). When we set $\tau_p = 10^{-3}$ the new algorithm achieves the much better performance compared with $\tau_p = 10^{-4}$ and $\tau_p = 10^{-2}$.

Various choices of weak-links parameter δ are tested and it does not make too much difference when $\delta \leq 0.9$. We set $\delta = 0.8$ and the size control parameter $\zeta = 0.45$ for this test case. The experiments show that the SET cycle is significantly more expensive than the SOL cycle, while the C-SOL cycle is cheaper than SOL when the number of converged points meets $m > 0.1n$. Comparing the C-OTF and the corrected C-OTF under the same parameters, we observe a decrease in work units and the number of cycles. The results also indicate that a sufficiently small ε_α enhance the opportunities of executing C-SOL cycles, as is shown in the table 5.1, so that reduce the total execution time.

Table 5.1. Uniform chain results. t_{sol} is the average timing of a single $V_{sol}(2,1)$ solution cycle, ε_α is the threshold parameter for performing the on-the-fly procedure at step 3 in algorithm 5, τ_p is the threshold parameter to explore the converged points in equation (12), C_{OP} is the operator complexity, $V_{set}, V_{sol}, V_{csol}$ are the number of SET cycles, SOL cycles and C-SOL cycles, respectively. WU is the work units defined as the cost of a single $V_{sol}(2,1)$ solution cycle. $Iter$ is the number of overall iterations.

n	t_{sol}	Algorithm (ε_α,τ_p)	$V_{set}, V_{sol}, V_{csol}$	C_{OP}	WU	Iter
961	0.05s	OTF (10^{-4}, -)	2,18,0	1.50	81	20
		C-OTF (10^{-5},10^{-3})	2,16,4	1.22	80	> 20
		C-OTF(-*cor*) (10^{-5},10^{-3})	2,15,3	1.27	80	20
		C-OTF(-*cor*) (10^{-6},10^{-3})	2,12,6	1.05	73	20
4096	2.67s	OTF (10^{-4}, -)	2,18,0	1.50	45	20
		C-OTF(10^{-8},10^{-3})	2,15,16	0.77	42	> 21
		C-OTF(-*cor*) (10^{-8},10^{-3})	2,9,10	0.79	36	21
		C-OTF(-*cor*) (10^{-10},10^{-3})	2,3,16	0.35	30	21
13225	339.25s	OTF (10^{-4}, -)	2,18,0	1.54	54	20
		C-OTF (10^{-8},10^{-3})	2,18,11,	0.82	56	> 20
		C-OTF(-*cor*) (10^{-8},10^{-3})	2,9,9	0.82	44	20
		C-OTF(-*cor*) (10^{-10},10^{-3})	2,4,15	0.43	37	21

5.2 Uniform chain with two weak links

The next test problem is a chain with uniform weights, except for two weak links with weight ϵ in the middle of the chain [6]. The stencil matrix is given by

$$H_{TwoWeakLinks} = \begin{pmatrix} \frac{1}{2} & \frac{1}{1+\epsilon} & 0 & \frac{\epsilon}{1+\epsilon} & \frac{1}{2} \end{pmatrix}. \tag{5.3}$$

where $\epsilon = 10^{-3}$ same as in [6]. As the same as the first case, we set the weak-links parameter $\delta = 0.8$ and the size control parameter $\zeta = 0.45$ here. The experiments show that the convergence criterion parameter $\tau_p = 10^{-3}$ is the best choice for this case. Results in Table 5.2 show again that the corrected C-OTF method is competitive compared with OTF and C-OTF without corrections.

5.3 Uniform 2D lattice

The next test problem is a 2D lattice with uniform weights [6, 20]. The stencil matrix is given by

$$H_{Uniform2D} = \frac{1}{4}\begin{pmatrix} & 1 & \\ 1 & 0 & 1 \\ & 1 & \end{pmatrix}. \tag{5.4}$$

We set the weak-links parameter $\delta = 0.8$ and the size control parameter $\zeta = 0.45$ for this test case. Table 5.3 shows numerical results for this problem.

For the small scale $n = 4096$ of this case, the choice of $\tau_p = 10^{-2}$ performs better than $\tau_p = 10^{-3}$ because the components of the prototype vector converge comparative slowly. In the larger case $n = 13225$, when we set $\tau_p = 10^{-3}$, the new algorithm fails to reduce the work units of OTF, largely due to the poor convergency of C-SOL cycles. To be specific, if the convergence rate of C-SOL cycle is unacceptable, we perform a SOL cycle instead. This procedure costs more time than a single SOL cycle and thus results in the worse performance than that of OTF.

Table 5.2. Uniform chain with two weak links results.

n	t_{sol}	Algorithm (ε_α,τ_p)	$V_{set}, V_{sol}, V_{csol}$	C_{OP}	WU	Iter
962	0.05	OTF (10^{-4}, -)	2,18,0	1.50	79	20
		C-OTF (10^{-5},10^{-3})	2,17,2	1.40	80	21
		C-OTF(-*cor*) (10^{-5},10^{-3})	2,17,2	1.41	80	21
		C-OTF(-*cor*) (10^{-6},10^{-3})	2,13,5	1.26	78	21
4096	2.63s	OTF (10^{-4}, -)	2,20,0	1.50	47	22
		C-OTF(10^{-8},10^{-4})	2,15,11	0.91	42	> 22
		C-OTF(-*cor*) (10^{-8},10^{-3})	2,8,11	0.71	36	21
		C-OTF(-*cor*) (10^{-10},10^{-3})	2,14,17	0.31	29	21
13224	426.50	OTF (10^{-4}, -)	2,19,0	1.54	53	21
		C-OTF (10^{-8},10^{-3})	2,19,12,	0.77	43	> 21
		C-OTF(-*cor*) (10^{-8},10^{-3})	2,9,10,	0.79	35	21
		C-OTF(-*cor*) (10^{-10},10^{-3})	2,3,16	0.35	26	21

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Table 5.3. Uniform 2D lattice results.

n	t_{sol}	Algorithm $(\varepsilon_\alpha,\tau_p)$	$V_{set}, V_{sol}, V_{csol}$	C_{OP}	WU	cycles
961	0.05s	OTF (10^{-4}, -)	2,31,0	1.63	97	33
		C-OTF (10^{-5},10^{-2})	2,32,5	1.35	98	> 33
		C-OTF(-*cor*) (10^{-5},10^{-2})	2,29,8	1.30	94	> 33
		C-OTF(-*cor*) (10^{-6},10^{-2})	2,31,12	1.15	98	> 33
4096	2.48s	OTF (10^{-4}, -)	2,33,0	1.66	61	35
		C-OTF(10^{-8},10^{-3})	2,41,15	0.99	60	> 35
		C-OTF(-*cor*) (10^{-8},10^{-2})	2,28,24	0.82	51	> 35
		C-OTF(-*cor*) (10^{-6},10^{-2})	2,22,16	1.06	53	> 35
13225	522.11s	OTF (10^{-4}, -)	2,24,0	1.70	44	26
		C-OTF (10^{-8},10^{-3})	2,37,23	0.96	49	> 26
		C-OTF(-*cor*) (10^{-8},10^{-3})	2,36,21	0.97	48	> 26
		C-OTF(-*cor*) (10^{-10},10^{-3})	2,37,25	0.92	48	> 26

Table 5.4. Tandem queueing network results.

n	t_{sol}	Algorithm $(\varepsilon_\alpha,\tau_p)$	$V_{set}, V_{sol}, V_{csol}$	C_{OP}	WU	Iter
961	0.04	OTF (10^{-4}, -)	2,23,0	1.57	97	25
		C-OTF (10^{-5},10^{-3})	2,22,0	1.57	99	24
		C-OTF(-*cor*) (10^{-5},10^{-3})	2,22,0	1.57	99	24
		C-OTF(-*cor*) (10^{-6},10^{-3})	2,22,2	1.42	97	26
4096	2.40s	OTF (10^{-4}, -)	2,23,0	1.60	52	25
		C-OTF(10^{-8},10^{-3})	2,15,5	1.20	50	> 25
		C-OTF(-*cor*) (10^{-6},10^{-3})	2,21,1	1.59	51	24
		C-OTF(-*cor*) (10^{-8},10^{-3})	2,15,5	1.22	50	> 25
13225	570.38s	OTF (10^{-4}, -)	2,23,0	1.66	49	25
		C-OTF (10^{-8},10^{-3})	2,25,3,	1.30	47	> 25
		C-OTF(-*cor*) (10^{-8},10^{-3})	2,25,3	1.30	44	> 25
		C-OTF(-*cor*) (10^{-10},10^{-3})	2,26,6	1.10	36	>25

5.4 Tandem queueing network

The next test problem is a tandem queueing network appeared in [2, 6, 9, 20], which has two finite single-server queues placed in tandem. Customers arrive in Poisson distribution with rate μ, and two server stations' service time distribution is Poisson with rates μ_1 and μ_2 respectively. The stencil matrix of tandem queueing work is given by

$$H_{TandemQueue} = \frac{1}{\mu + \mu_1 + \mu_2} \begin{pmatrix} & & \mu_1 \\ \mu & 0 & \\ & \mu_2 & \end{pmatrix}, \tag{5.5}$$

where we use $\mu = 10, \mu_1 = 11, \mu_2 = 10$ as in [2, 6, 9, 20]. Table 5.4 shows numerical results for this problem.

In this case we set the weak-links parameter $\delta = 0.8$, the size control parameter $\zeta = 0.45$ and convergence parameter $\tau_p = 10^{-3}$. We also try using more strict convergency parameter $\tau_p = 10^{-4}$. Results show that the algorithm fails to expose the converged points and the number of C-SOL cycle is equal to 0. Similar with the previous problems, several choices of ε_α are tested. Experiments show that with a sufficiently small ε_α, new algorithm improves the performance of OTF in terms of the total execution time, but suffers from an unsatisfied convergence rate, which increases the number of iterations. For the reason that the operators of C-SOL cycles are less accurate than that of SOL cycles, they have a probability to lead to poor convergence rate. To achieve the same accuracy ε_α, more C-SOL cycles are needed. Whereas, the total execution time is reduced because C-SOL cycles are comparative cheaper than SOL cycles.

Table 5.5. Random walk on unstructured planar graph results.

n	t_{sol}	Algorithm (ε_α,τ_p)	$V_{set}, V_{sol}, V_{csol}$	C_{OP}	WU	Iter
961	2	OTF (10^{-4}, -)	2,27,0	1.20	182	29
		C-OTF (10^{-5},10^{-3})	2,29,3	1.07	186	> 29
		C-OTF(*-cor*) (10^{-5},10^{-3})	2,29,3	1.12	186	> 29
		C-OTF(*-cor*) (10^{-6},10^{-3})	2,40,2	0.95	183	> 29
4096	0.23s	OTF (10^{-4}, -)	2,29,0	1.21	162	31
		C-OTF(10^{-6},10^{-3})	2,32,6	0.99	162	> 31
		C-OTF(*-cor*) (10^{-6},10^{-3})	2,18,6	0.99	157	> 31
		C-OTF(*-cor*) (10^{-8},10^{-3})	2,18,6	0.94	158	> 31
13225	6.16s	OTF (10^{-4}, -)	2,28,0	1.21	122	30
		C-OTF (10^{-6},10^{-3})	2,29,8,	0.98	130	> 30
		C-OTF(*-cor*) (10^{-6},10^{-3})	2,30,5	0.94	120	> 30
		C-OTF(*-cor*) (10^{-9},10^{-3})	2,41,20	0.57	116	> 30

5.5 Random walk on unstructured planar graph

The next test problem is random walks on graphs, which have significant applications in many fields, one of the well-known examples is Google's pagerank algorithm. Here we consider an unstructured planar graph, which is generated by choosing n random points in the unit square, and triangulating them by Delaunay triangulation. The transition probability from point i to point j is given by the reciprocal of the number of egdes incident on point i.

In this test case, when $m < 0.6n$, a single C-SOL cycle costs more time than SOL cycle does, thus we use the size control parameter $\zeta = 0.6$ here. We set the weak-links parameter $\delta = 0.8$. Experiments show that convergence parameter $\tau_p = 10^{-3}$ is the best choice among any others. The performance of corrected C-OTF method is moderate. However, the work units, which indicates the total execution time, is still smaller than that of OTF and C-OTF without corrections. Table 5.5 shows numerical results for this problem.

6 Conclusions

This paper proposes a compact on-the-fly adaptive aggregation multigrid method for Markov chain problems. As is known, adaptive multigrid methods suffer from the common defect that considerable computation cost is spent on coarsen operators construction. The reason is that they update the entire multigrid hierarchy of operators in every cycles. We consider distributing the converged points into an aggregate and reducing the scale of the coarsen operators to decrease this cost. Meanwhile, a simple technique is proposed to delete the possible weak-links introduced by the procedure above. According to numerical results, for most of test cases, the corrected algorithm leads to better performance than on-the-fly adaptive aggregation multigrid algorithm in terms of total execution time. New algorithm can also be applied to various adaptive multigrid Markov solvers. One future work may be to study how to improve the convergence rate of compacted solution cycles.

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Travelling Solitary Wave Solutions for Stochastic Kadomtsev-Petviashvili Equation.

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Abstract. In this paper, generalized Wick-type stochastic Kadomtsev-Petviashvili equations are investigated. Abundant white noise functional solutions for Wick-type generalized stochastic Kadomtsev-Petviashvili equations are obtained. By using white noise analysis, Hermite transform, modified Riccati equation and modified tanh-coth method many exact travelling wave solutions are given. Detailed computations and implemented examples for the investigated model are explicitly provided .

Keywords: White noise; Stochastic ; Wick product; Kadomtsev-Petviashvili equations.

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1 Introduction

In this paper we investigate the generalized variable coefficient Kadomtsev-Petviashvili (KP) equation:

$$u_t + \frac{\partial}{\partial x}(\phi(t)u\frac{\partial u}{\partial x} + \psi(t)\frac{\partial^3 u}{\partial x^3}) + \theta(t)\frac{\partial^2 u}{\partial y^2} = 0, \quad (x,y,t) \in \mathbb{R}^2 \times \mathbb{R}_+ \tag{1.1}$$

where u is a stochastic process on $\mathbb{R}^2 \times \mathbb{R}_+$ and $\phi(t), \psi(t)$ and $\theta(t)$ are bounded measurable or integrable functions on $\mathbb{R}_+$. Equation (1.1) plays a significant role in many scientific applications such as solid state physics, nonlinear optics, chemical kinetics, etc. The KP equations[1-2] are universal models(normal forms) for the propagation of long, dispersive, weakly nonlinear waves that travel predominantly in the x direction, with weak transverse effects. The notion of well-posed-ness will be the usual one in the context of nonlinear dispersive equations, that is, it includes existence, uniqueness, persistence property, and continuous dependence upon the data. Recently, many researchers pay more attention to study of random waves, which are important subjects of stochastic partial differential equation (SPDE). Wadati [3] first answered the interesting question, How does external noise affect the motion of solitons? and studied the diffusion of soliton of the KdV equation under Gaussian noise, which satisfies a diffusion equation in transformed coordinates.

Wadati and Akutsu also studied the behaviors of solitons under the Gaussian white noise of the stochastic KdV equations with and without damping [4]. Wadati [3] first answered the interesting question, "How does external noise affect the motion of solitons?" and studied the diffusion of soliton of the KdV equation under Gaussian noise, which satisfies a diffusion equation in transformed coordinates. The stochastic PDEs was discussed by many authors, e.g., de Bouard and Debussche [6, 7], Debussche and Printems [8, 9], Printems [17] and Ghany and Hyder [13]. On the basis of white noise functional analysis [5], Ghany et al. [10-16] studied more intensely the white noise functional solutions for some nonlinear stochastic PDEs. This paper is mainly concerned to investigate the white noise functional solutions for the generalized Wick-type stochastic Kadomstev-Petviashvili (KP) equation:

$$U_t + \Phi(t) \diamond U_x \diamond U_x + \Psi(t) \diamond U \diamond U_{xx} + \Psi(t) \diamond U_{xxxx} + \Theta(t) \diamond U_{yy} = 0. \tag{1.2}$$

where "$\diamond$" is the Wick product on the Kondratiev distribution space $(\mathcal{S})_{-1}$ and $\Phi(t), \Psi(t)$ and $\Theta(t)$ are $(\mathcal{S})_{-1}$-valued functions [5]. It is well known that the solitons are stable against mutual collisions and behave like particles. In this sense, it is very important to study the nonlinear equations in random environment. However, variable coefficients nonlinear equations, as well as constant coefficients equations, cannot describe the realistic physical phenomena exactly. The rest of this paper is organized as follows: In Section 2, we recall the definition and some properties of white noise analysis. In Section 3, we apply some method to explore exact travelling wave solutions for Eq.(1.1). In Section 4, we use the Hermite transform and [5,Theorem 4.1.1] to obtain white noise functional solutions for Eq.(1.2). In Section 5, we give illustrative examples for the investigated model. The last section is devoted to summary and discussion.

2 Preliminaries

Suppose that $S(\mathbb{R}^d)$ and $S'(\mathbb{R}^d)$ are the Hida test function space and the Hida distribution space on $\mathbb{R}^d$, respectively. Let $h_n(x)$ be Hermite polynomials and put

$$\zeta_n = e^{-x^2} h_n(\sqrt{2}x)/((n-1)!\pi)^{\frac{1}{2}}, \quad n \geqslant 1. \tag{2.1}$$

then, the collection $\{\zeta_n\}_{n\geqslant 1}$ constitutes an orthogonal basis for $L_2(\mathbb{R})$.

Let $\alpha = (\alpha_1, \alpha_2, ..., \alpha_d)$ denote d-dimensional multi-indices with $\alpha_1, \alpha_2, ..., \alpha_d \in \mathbb{N}$. The family of tensor products

$$\zeta_\alpha := \zeta_{(\alpha_1, \alpha_2, ..., \alpha_d)} = \zeta_{\alpha_1} \otimes \zeta_{\alpha_2} \otimes ... \otimes \zeta_{\alpha_d} \tag{2.2}$$

forms an orthogonal basis for $L_2(\mathbb{R}^d)$.

Suppose that $\alpha^{(i)} = (\alpha_1^{(i)}, \alpha_2^{(i)}, ..., \alpha_d^{(i)})$ is the i-th multi-index number in some fixed ordering of all d-dimensional multi-indices α. We can, and will, assume that this ordering has the property that

$$i < j \Rightarrow \alpha_1^{(i)} + \alpha_2^{(i)} + ... + \alpha_d^{(i)} < \alpha_1^{(j)} + \alpha_2^{(j)} + ... + \alpha_d^{(j)} \tag{2.3}$$

i.e., the $\{\alpha^{(j)}\}_{j=1}^{\infty}$ occurs in an increasing order. Now

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Define

$$\eta_i := \zeta_{\alpha_1^{(i)}} \otimes \zeta_{\alpha_2^{(i)}} \otimes ... \otimes \zeta_{\alpha_d^{(i)}}, \quad i \geqslant 1. \tag{2.4}$$

We need to consider multi-indices of arbitrary length. For simplification of notation, we regard multi-indices as elements of the space $(\mathbb{N}_0^{\mathbb{N}})_c$ of all sequences $\alpha = (\alpha_1, \alpha_2, ..., \alpha_d)$ with elements $\alpha_i \in \mathbb{N}_0$ and with compact support, i.e., with only finitely many $\alpha_i \neq 0$. We write $J = (\mathbb{N}_0^{\mathbb{N}})_c$, for $\alpha \in J$,

Define

$$H_\alpha(\omega) := \prod_{i=1}^{\infty} h_{\alpha_i}(< \omega, \eta_i >), \quad \omega = (\omega_1, \omega_2, ..., \omega_d) \in S'(\mathbb{R}^d) \tag{2.5}$$

For a fixed $n \in \mathbb{N}$ and for all $k \in \mathbb{N}$, suppose the space $(S)_1^n$ consists of those $f(\omega) = \sum_\alpha c_\alpha H_\alpha(\omega) \in \bigoplus_{k=1}^n L_2(\mu)$ with $c_\alpha \in \mathbb{R}^n$ such that

$$||f||_{1,k}^2 = \sum_\alpha c_\alpha^2 (\alpha!)^2 (2\mathbb{N})^{k\alpha} < \infty \tag{2.6}$$

where, $c_\alpha^2 = |c_\alpha|^2 = \sum_{k=1}^n (c_\alpha^{(k)})^2$ if $c_\alpha = (c_\alpha^{(1)}, c_\alpha^{(2)}, ..., c_\alpha^{(n)}) \in \mathbb{R}^n$ and μ is the white noise measure on $(S'(\mathbb{R}), B(S'(\mathbb{R})))$, $\alpha! = \prod_{k=1}^\infty \alpha_k!$ and $(2\mathbb{N})^\alpha = \prod_j (2j)^{\alpha_j}$ for $\alpha \in J$.

The space $(S)_{-1}^n$ consists of all formal expansions $F(\omega) = \sum_\alpha b_\alpha H_\alpha(\omega)$ with $b_\alpha \in \mathbb{R}^n$ such that $||f||_{-1,-q} = \sum_\alpha b_\alpha^2 (2\mathbb{N})^{-q\alpha} < \infty$ for some $q \in \mathbb{N}$. The family of seminorms $||f||_{1,k}, k \in \mathbb{N}$ gives rise to a topology on $(S)_1^n$, and we can regard $(S)_{-1}^n$ as the dual of $(S)_1^n$ by the action

$$< F, f >= \sum_\alpha (b_\alpha, c_\alpha)\alpha! \tag{2.7}$$

where (b_α, c_α) is the inner product in $\mathbb{R}^n$.

The Wick product $f \diamond F$ of two elements $f = \sum_\alpha a_\alpha H_\alpha, F = \sum_\beta b_\beta H_\beta \in (S)_{-1}^n$ with $a_\alpha, b_\beta \in \mathbb{R}^n$, is defined by

$$f \diamond F = \sum_{\alpha,\beta} (a_\alpha, b_\beta) H_{\alpha+\beta} \tag{2.8}$$

The spaces $(S)_1^n, (S)_{-1}^n, S(\mathbb{R}^d)$ and $S'(\mathbb{R}^d)$ are closed under Wick products.

For $F = \sum_\alpha b_\alpha H_\alpha \in (S)_{-1}^n$, with $b_\alpha \in \mathbb{R}^n$, the Hermite transformation of F , is defined by

$$\mathcal{H}F(z) = \widetilde{F}(z) = \sum_\alpha b_\alpha z^\alpha \in \mathbb{C}^N \tag{2.9}$$

where $z = (z_1, z_2, ...) \in \mathbb{C}^N$ (the set of all sequences of complex numbers) and $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} ... z_n^{\alpha_n}$, if $\alpha \in J$, where $z_j^0 = 1$.

For $F, G \in (S)_{-1}^n$ we have

$$\widetilde{F \diamond G}(z) = \widetilde{F}(z).\widetilde{G}(z) \tag{2.10}$$

for all z such that $\widetilde{F}(z)$ and $\widetilde{G}(z)$ exist. The product on the right-hand side of the above formula is the complex bilinear product between two elements of $\mathbb{C}^N$ defined by $(z_1^1, z_2^1, ..., z_n^1).(z_1^2, z_2^2, ..., z_n^2) = \sum_{k=1}^n z_k^1 z_k^2$.

Let $X = \sum_\alpha a_\alpha H_\alpha$, then the vector $c_0 = \widetilde{X}(0) \in \mathbb{R}^N$ is called the generalized expectation

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of X which denoted by $E(X)$. Suppose that $g : U \longrightarrow \mathbb{C}^M$ is an analytic function, where U is a neighborhood of $E(X)$. Assume that the Taylor series of g around $E(X)$ have coefficients in $\mathbb{R}^M$. Then the Wick version $g^{\diamond}(X) = \mathcal{H}^{-1}(g \circ \widetilde{X}) \in (S)_{-1}^M$. In other words, if g has the power series expansion $g(z) = \sum a_\alpha (z - E(X))^\alpha$, with $a_\alpha \in \mathbb{R}^M$, then $g^{\diamond}(z) = \sum a_\alpha (z - E(X))^{\diamond\alpha} \in (S)_{-1}^M$.

3 Exact travelling wave solutions

In this section, we will give exact solutions of Eq.(1.1). Taking the Hermite transform of Eq.(1.2), we get:

$$\widetilde{U_t}(t,x,y,z) + \widetilde{\Phi(t)}.\widetilde{U_x}(t,x,y,z).\widetilde{U_x}(t,x,y,z) + \widetilde{\Psi(t)}.\widetilde{U}(t,x,y,z).\widetilde{U_{xx}}(t,x,y,z)$$

$$+ \widetilde{\Psi(t)}.\widetilde{U_{xxxx}}(t,x,y,z) + \widetilde{\Theta(t)}.\widetilde{U_{yy}}(t,x,y,z) = 0 \tag{3.1}$$

where $z = (z_1, z_2, ...) \in C^{\mathbb{N}}$ is a parameter. To look for the travelling wave solution of Eq.(3.1),we make the transformations $u(t,x,y,z) := \widetilde{U}(t,x,y,z) = \varphi(\xi(t,x,y,z))$ with

$$\xi(t,x,y,z) := k_1 x + +k_2 y + s\int_0^t l(\tau,z)d\tau + c$$

where k_1, k_2, s, c are arbitrary constants which satisfy $k_1 k_2 s \neq 0$, $l(\tau,z)$ is a non zero functions of indicated variables to be determined. So, Eq.(3.1) can be changing into the form:

$$slu'(t,x,z) + k_1^2 \Phi u'(t,x,z)u'(t,x,z) + k_1^2 \Psi u(t,x,z)u''(t,x,z) +$$

$$k_1^4 \Psi u''''(t,x,z) + k_2^2 \Theta u''(t,x,z) = 0 \tag{3.2}$$

The solution can be proposed by the tanh method as a finite power series in Y in the form:

$$u(\mu\zeta) = S(Y) = \sum_{k=0}^{M} a_k Y^k, \tag{3.3}$$

limiting them to solitary and shock wave profiles. However, the extended tanh method admits the use of the finite expansion

$$u(\mu\zeta) = S(Y) = \sum_{k=0}^{M} a_k Y^k + \sum_{k=1}^{M} b_k Y^{-k}, \tag{3.4}$$

where M is a positive integer, in most cases, that will be determined. Expansion (3.4) reduces to the standard tanh method [4-6], where $Y(\xi)$ satisfies the Riccati equation

$$Y' = c_0 + c_1 Y + c_2 Y^2, \tag{3.5}$$

and c_0, c_1, c_2 are constant to be prescribed later. By virtue of (3.3) and (3.4) with observation of the linear independence of $Y^n (n = -6, -5, ..., 6)$ and using Mathematica Eqn.(3.2) implies the following nonlinear algebraic system of equations:

$$\begin{cases} sl\alpha_{1,0} + k_1^4\Psi\alpha_{4,0} + k_2^2\Theta\alpha_{2,0} + k_1^2[\Phi(\alpha_{1,0}^2 + 2\alpha_{1,1}\alpha_{1,-1} + 2\alpha_{1,2}\alpha_{1,-2} + 2\alpha_{1,3}\alpha_{1,-3}) \\ +\Psi(a_0\alpha_{2,0} + a_1\alpha_{2,-1} + a_2\alpha_{2,-2} + b_1\alpha_{2,1} + b_2\alpha_{2,2})] = 0, \\ sl\alpha_{1,1} + k_1^4\Psi\alpha_{4,1} + k_2^2\Theta\alpha_{2,1} + k_1^2[\Phi(2\alpha_{1,0}\alpha_{1,1} + 2\alpha_{1,-1}\alpha_{1,2} + 2\alpha_{1,-2}\alpha_{1,3}) \\ +\Psi(a_0\alpha_{2,1} + a_1\alpha_{2,0} + a_2\alpha_{2,-1} + b_1\alpha_{2,2} + b_2\alpha_{2,3})] = 0, \\ sl\alpha_{1,-1} + k_1^4\Psi\alpha_{4,-1} + k_2^2\Theta\alpha_{2,-1} + k_1^2[\Phi(2\alpha_{1,0}\alpha_{1,-1} + 2\alpha_{1,1}\alpha_{1,-2} + 2\alpha_{1,2}\alpha_{1,-3}) \\ +\Psi(a_0\alpha_{2,-1} + a_1\alpha_{2,0-2} + a_2\alpha_{2,-3} + b_1\alpha_{2,0} + b_2\alpha_{2,1})] = 0, \\ sl\alpha_{1,2} + k_1^4\Psi\alpha_{4,2} + k_2^2\Theta\alpha_{2,2} + k_1^2[\Phi(\alpha_{1,1}^2 + 2\alpha_{1,0}\alpha_{1,2} + 2\alpha_{1,-1}\alpha_{1,3}) \\ +\Psi(a_0\alpha_{2,2} + a_1\alpha_{2,1} + a_2\alpha_{2,0} + b_1\alpha_{2,3} + b_2\alpha_{2,4})] = 0, \\ sl\alpha_{1,-2} + k_1^4\Psi\alpha_{4,-2} + k_2^2\Theta\alpha_{2,-2} + k_1^2[\Phi(\alpha_{1,-1}^2 + 2\alpha_{1,0}\alpha_{1,-2} + 2\alpha_{1,1}\alpha_{1,-3}) \\ +\Psi(a_0\alpha_{2,-2} + a_1\alpha_{2,-3} + a_2\alpha_{2,-4} + b_1\alpha_{2,-1} + b_2\alpha_{2,0})] = 0, \\ sl\alpha_{1,3} + k_1^4\Psi\alpha_{4,3} + k_2^2\Theta\alpha_{2,3} + k_1^2[\Phi(2\alpha_{1,0}\alpha_{1,3} + 2\alpha_{1,1}\alpha_{1,2}) \\ +\Psi(a_0\alpha_{2,3} + a_1\alpha_{2,2} + a_2\alpha_{2,1} + b_1\alpha_{2,4})] = 0, \\ sl\alpha_{1,-3} + k_1^4\Psi\alpha_{4,-3} + k_2^2\Theta\alpha_{2,-3} + k_1^2[\Phi(2\alpha_{1,0}\alpha_{1,-3} + 2\alpha_{1,-1}\alpha_{1,-2}) \\ +\Psi(a_0\alpha_{2,-3} + a_1\alpha_{2,-4} + b_1\alpha_{2,-2} + b_2\alpha_{2,-1})] = 0, \\ k_1^4\Psi\alpha_{4,4} + k_2^2\Theta\alpha_{2,4} + k_1^2[\Phi(2\alpha_{1,1}\alpha_{1,3} + \alpha_{1,2}^2) + \Psi(a_0\alpha_{2,4} + a_1\alpha_{2,3} + a_2\alpha_{2,2})] = 0, \\ k_1^4\Psi\alpha_{4,-4} + k_2^2\Theta\alpha_{2,-4} + k_1^2[\Phi(2\alpha_{1,-1}\alpha_{1,-3} + \alpha_{1,-2}^2) + \Psi(a_0\alpha_{2,-4} + b_1\alpha_{2,-3} + b_2\alpha_{2,-2})] = 0, \\ k_1^4\Psi\alpha_{4,5} + 2k_1^2\alpha_{1,2}\alpha_{1,3} + k_1^2(a_1\alpha_{2,4} + a_2\alpha_{2,3}) = 0, \\ k_1^4\Psi\alpha_{4,-5} + 2k_1^2\alpha_{1,-2}\alpha_{1,-3} + k_1^2(b_1\alpha_{2,-4} + b_2\alpha_{2,-3}) = 0, \\ k_1^4\Psi\alpha_{4,6} + k_1^2\Phi\alpha_{1,3}^2 + k_1^2\Psi a_2\alpha_{2,4} = 0, \\ k_1^4\Psi\alpha_{4,-6} + k_1^2\Phi\alpha_{1,-3}^2 + k_1^2\Psi b_2\alpha_{2,-4} = 0, \end{cases} \tag{3.6}$$

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where

$$\begin{cases} \alpha_{1,0} = a_1c_0 - b_1c_2, \quad \alpha_{1,1} = a_1c_1 + 2a_2c_0, \quad \alpha_{1,2} = a_1c_2 + 2a_2c_1, \quad \alpha_{1,3} = 2a_2c_2, \\ \alpha_{1,-2} = -(b_1c_0 + 2b_2c_1), \quad \alpha_{1,-1} = -(b_1c_1 + 2b_2c_2), \quad \alpha_{2,0} = \alpha_{1,1}c_0 - \alpha_{1,-1}c_2, \\ \alpha_{1,-3} = -2b_2c_0, \quad \alpha_{2,1} = \alpha_{1,1}c_1 + 2\alpha_{1,2}c_0, \quad \alpha_{2,2} = \alpha_{1,1}c_2 + 2\alpha_{1,2}c_1 + 3\alpha_{1,3}c_0, \\ \alpha_{2,3} = 2\alpha_{1,2}c_2 + 3\alpha_{1,3}c_1, \quad \alpha_{2,4} = 3\alpha_{1,3}c_2, \quad \alpha_{2,-1} = -(\alpha_{1,-1}c_1 + 2\alpha_{1,-2}c_2), \\ \alpha_{2,-2} = -(\alpha_{1,-1}c_0 + 2\alpha_{1,-2}c_1 + 3\alpha_{1,-3}c_2), \quad \alpha_{2,-3} = -(2\alpha_{1,-2}c_0 + 3\alpha_{1,-3}c_1), \\ \alpha_{2,-4} = -3\alpha_{1,-3}c_0, \quad \alpha_{3,0} = \alpha_{2,1}c_0 - \alpha_{2,-1}c_2, \quad \alpha_{3,1} = \alpha_{2,1}c_1 + 2\alpha_{2,2}c_0, \\ \alpha_{3,2} = \alpha_{2,1}c_0 + 2\alpha_{2,2}c_1 + 3\alpha_{2,3}c_2, \quad \alpha_{3,3} = 2\alpha_{2,2}c_2 + 3\alpha_{2,3}c_1 + 4\alpha_{2,4}c_0, \\ \alpha_{3,4} = 3\alpha_{2,3}c_2 + 4\alpha_{2,4}c_1, \quad \alpha_{3,5} = 4\alpha_{2,4}c_2, \quad \alpha_{3,-1} = -(\alpha_{2,-1}c_1 + 2\alpha_{2,-2}c_2), \\ \alpha_{3,-2} = -(\alpha_{2,-1}c_0 + 2\alpha_{2,-2}c_1 + \alpha_{2,-3}c_2), \quad \alpha_{3,-4} = -(3\alpha_{2,-3}c_0 + 4\alpha_{2,-4}c_1), \\ \alpha_{3,-3} = -(2\alpha_{2,-2}c_0 + 3\alpha_{2,-3}c_1 + 4\alpha_{2,-4}c_2), \quad \alpha_{3,-5} = -(4\alpha_{2,-4}c_0), \\ \alpha_{4,0} = \alpha_{3,1}c_0 - \alpha_{3,-1}c_2, \quad \alpha_{4,1} = \alpha_{3,1}c_1 + 2\alpha_{3,2}c_0, \quad \alpha_{4,6} = 5\alpha_{3,5}c_2, \\ \alpha_{4,2} = \alpha_{3,1}c_2 + 2\alpha_{3,2}c_1 + 3\alpha_{3,3}c_0, \quad \alpha_{4,3} = 2\alpha_{3,2}c_2 + 3\alpha_{3,3}c_1 + 4\alpha_{3,4}c_0, \\ \alpha_{4,4} = 3\alpha_{3,3}c_2 + 4\alpha_{3,4}c_1 + 5\alpha_{3,5}c_0, \quad \alpha_{4,5} = 4\alpha_{3,4}c_2 + 5\alpha_{3,5}c_0, \\ \alpha_{4,-1} = -(\alpha_{3,-1}c_1 + 2\alpha_{3,-2}c_2), \quad \alpha_{4,-3} = -(2\alpha_{3,-2}c_0 + 3\alpha_{3,-3}c_1 + 4\alpha_{3,-4}c_2), \\ \alpha_{4,-2} = -(\alpha_{3,-1}c_0 + 2\alpha_{3,-2}c_1 + 3\alpha_{3,-3}c_2), \quad \alpha_{4,-6} = -5\alpha_{3,-5}c_0, \\ \alpha_{4,-4} = -(3\alpha_{3,-3}c_0 + 4\alpha_{3,-4}c_1 + 5\alpha_{3,-5}c_2), \quad \alpha_{4,-5} = -(4\alpha_{3,-4}c_0 + 5\alpha_{3,-5}c_2). \end{cases}$$

At the rest of this section we will discuss and solve our problem for some particular cases for the Riccati equation as follows:

A. $c_0 = c_1 = 1, c_2 = 0$.

For this choice of the constants, the Riccati equation has the solution:

$$Y_1(\xi) = exp(\xi) - 1 \tag{3.7}$$

By the aid of Mathematica, the above system of equations (3.6) can be solved for the following cases:

Case 1:

$a_1 = a_2 = 0, \alpha_{i,j} = 0$ for all $i, j > 0$; $a_0 = \frac{1}{k_1^2\widetilde{\Psi}}\{sl - k_1^4\widetilde{\Psi} - k_2^2\widetilde{\Theta}\}$; $b_1 = 12k_1^2\frac{3\Psi}{5\Phi}$; $b_2 = -12k_1^2\frac{\Psi}{\Phi}$.
According to (3.2),(3.6) and (3.7), Eq.(3.1) has the solution

$$u_1(t,x,y,z) = \frac{1}{k_1^2\widetilde{\Psi}}\{sl - k_1^4\widetilde{\Psi} - k_2^2\widetilde{\Theta}\} + \frac{36k_1^2\widetilde{\Psi}}{5\widetilde{\Phi}}(exp(\xi)-1)^{-1} - \frac{12k_1^2\widetilde{\Psi}}{\widetilde{\Phi}}(exp(\xi)-1)^{-2}. \tag{3.8}$$

where,

$$\xi = k_1x + k_2y - 11.4k_1^4\int_0^t \widetilde{\Psi}(\tau, z)d\tau \tag{3.9}$$

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Case 2:

$a_2 = b_2 = 0$; $a_0 = 25k_1^2 \frac{\widetilde{\Phi}+3\widetilde{\Psi}}{\widetilde{\Phi}+2\widetilde{\Psi}} - 25k_1^2 - (\frac{k_1^2}{k_2^2})^2 \frac{\widetilde{\Theta}}{\widetilde{\Psi}}$; $b_1 = -50k_1^2 \frac{\widetilde{\Psi}}{\widetilde{\Phi}+2\widetilde{\Psi}}$; $a_1 = -2k_1^2 \frac{\widetilde{\Psi}}{\widetilde{\Phi}+\widetilde{\Psi}}$.
According to (3.2),(3.6) and (3.7), Eq.(3.1) has the solution

$$u_2(t,x,y,z) = 25k_1^2 \frac{\widetilde{\Phi}+3\widetilde{\Psi}}{\widetilde{\Phi}+2\widetilde{\Psi}} - 25k_1^2 - (\frac{k_2^2}{k_1^2})^2 \frac{\widetilde{\Theta}}{\widetilde{\Psi}} - \frac{2k_1^2\widetilde{\Psi}}{\widetilde{\Phi}+\widetilde{\Psi}}(exp(\xi)-1) - \frac{50k_1^2\widetilde{\Psi}}{\widetilde{\Phi}+2\widetilde{\Psi}}(exp(\xi)-1)^{-1} \tag{3.10}$$

where,

$$\xi = k_1x + k_2y + k_1^4 \int_0^t \frac{11\widetilde{\Psi^2}(\tau,z) - 12\widetilde{\Phi^2}(\tau,z) + 12\widetilde{\Phi}(\tau,z)\widetilde{\Psi}(\tau,z)}{(\widetilde{\Phi}(\tau,z)+\widetilde{\Psi}(\tau,z))(\widetilde{\Phi}(\tau,z)+3\widetilde{\Psi}(\tau,z))}\widetilde{\Psi}(\tau,z)d\tau \tag{3.11}$$

B. $c_0 = -c_2 = 0.5, c_1 = 0$.

For this choice of the constants, the Riccati equation has the solution:

$$Y_2(\xi) = tanh(\xi) \pm isech(\xi) \tag{3.12}$$

or

$$Y_3(\xi) = coth(\xi) \pm csch(\xi) \tag{3.13}$$

By the aid of Mathematica, the above system of equations (3.6) can be solved for the following case:

Case 3:

$a_2 = b_1 = b_2 = 0$; $a_0 = 1.25k_1^2 - (\frac{k_2}{k_1})^2 \frac{\widetilde{\Theta}}{\widetilde{\Psi}} - 7.5k_1^2 \frac{\widetilde{\Phi}}{2\widetilde{\Phi}+3\widetilde{\Psi}} - 3.75k_1^2 \frac{\widetilde{\Psi}}{2\widetilde{\Phi}+3\widetilde{\Psi}}$; $a_2 = -15k_1^2 \frac{\widetilde{\Psi}}{2\widetilde{\Phi}+3\widetilde{\Psi}}$.
According to (3.2),(3.6) and (3.7), Eq.(3.1) has the solution

$$u_i(t,x,y,z) = 1.25k_1^2 - (\frac{k_2}{k_1})^2 \frac{\widetilde{\Theta}}{\widetilde{\Psi}} - 7.5k_1^2 \frac{\widetilde{\Phi}}{2\widetilde{\Phi}+3\widetilde{\Psi}} - 3.75k_1^2 \frac{\widetilde{\Psi}}{2\widetilde{\Phi}+3\widetilde{\Psi}} -$$

$$15k_1^2 \frac{\widetilde{\Psi}}{2\widetilde{\Phi}+3\widetilde{\Psi}} Y_{i-1}^2(\xi), \quad i = 3,4. \tag{3.14}$$

where,

$$\xi = k_1x + k_2y. \tag{3.15}$$

C. $c_2 = 4c_0 = 1, c_1 = 0$.

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For this choice of the constants, the Riccati equation has the solution:

$$Y_4(\xi) = 0.5tan(2\xi) \tag{3.16}$$

or

$$Y_5(\xi) = 0.5cot(2\xi) \tag{3.17}$$

By the aid of Mathematica, the above system of equations (3.6) can be solved for the following case:

Case 4:

$a_1 = a_2 = b_1 = 0$; a $_0 = -16k_1^2 - (\frac{k_2}{k_1})^2 \frac{\widetilde{\Theta}}{\widetilde{\Psi}}$; $b_2 = -120k_1^2 \frac{\widetilde{\Psi}}{4\widetilde{\Phi}+6\widetilde{\Psi}}$.
According to (3.2),(3.6) and (3.7), Eq.(3.1) has the solution

$$u_i(t,x,y,z) = -16k_1^2 - (\frac{k_2}{k_1})^2 \frac{\widetilde{\Theta}}{\widetilde{\Psi}} - 120k_1^2 \frac{\widetilde{\Psi}}{4\widetilde{\Phi}+6\widetilde{\Psi}} Y_{i-1}^{-2}(\xi), \quad i = 5, 6. \tag{3.18}$$

where,

$$\xi = k_1 x + k_2 y. \tag{3.19}$$

At the end of this section we should remark that, there exists infinitely number of solutions for Eqn.(1.1) these solution coming from solving the system (3.6) with regarding the Riccati equation (3.5). The above mentioned cases are just to clarify how far my technique is applicable.

4 White noise functional solutions

The main aim of the rest of this paper is to obtain white noise functional solutions of Eqs.(1.2). As pointed out from Xie [16], we will use Theorem 2.1 of for $d = 2$. The properties of hyperbolic functions yield that there exists a bounded open set $\mathbf{S} \subset \mathbb{R}_+ \times \mathbb{R}^2, m > 0$ and $n > 0$ such that $u(x,y,t,z), u_{xt}(x,y,t,z)$ are uniformally bounded for all $(t,x,y,z) \in \mathbf{S} \times \mathbb{K}_m(n)$, continuous with respect to $(t,x,y) \in \mathbf{S}$ for all $z \in \mathbb{K}_m(n)$ and analytic with respect to $z \in \mathbb{K}_m(n)$ for all $(t,x,y) \in \mathbf{S}$. Using Theorem 2.1 of Xie [16], there exists a stochastic process $U(t,x,y)$ such that the Hermite transformation of $U(t,x,y)$ is $u(t,x,y,z)$ for all $\mathbf{S} \times \mathbb{K}_m(n)$, and $U(t,x,y)$ is the solution of (1.2). This implies that $U(t,x,y)$ is the inverse Hermite transformation of $u(t,x,y,z)$. Hence, for $\Phi(t)\Psi(t)\Theta(t) \neq 0$ the white noise functional solutions of Eqs.(1.2) can be written as follows:

$$\begin{aligned} U_1(t,x,y) = {} & \frac{1}{k_1^2\Psi(t)}\{sl - k_1^4\Psi(t) - k_2^2\Theta(t)\} + \frac{36k_1^2\Psi(t)}{5\Phi(t)(exp^{\diamond}(\Xi_1(t,x,y)) - 1)} \\ & - \frac{12k_1^2\Psi(t)}{\Phi(t)(exp^{\diamond}(\Xi_1(t,x,y)) - 1)^{\diamond 2}} \end{aligned} \tag{4.1}$$

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where,

$$\Xi_1 = k_1 x + k_2 y - 11.4 k_1^4 \int_0^t \Psi(\tau) d\tau \tag{4.2}$$

and,

$$U_2(t,x,y) = 25k_1^2 \frac{\Phi(t)+3\Psi(t)}{\Phi(t)+2\Psi(t)} - 25k_1^2 - (\frac{k_2}{k_1})^2 \frac{\Theta(t)}{\Psi(t)} - 2k_1^2 \frac{\Psi(t)}{\Phi(t)+\Psi(t)} Y_1^{\diamond}(\Xi_2(t,x,y))$$

$$- 50k_1^2 \frac{\Psi(t)}{\Phi(t)+2\Psi(t)} Y_1^{-\diamond}(\Xi_2(t,x,y)) \tag{4.3}$$

where,

$$Y_1^{\diamond}(\Xi_2(t,x,y)) = exp^{\diamond}(\Xi_2(t,x,y)) - 1$$

and,

$$\Xi_2 = k_1 x + k_2 y + k_1^4 \int_0^t \frac{11\Psi^2(\tau) - 12\Phi^2(\tau) + 12\Phi(\tau)\Psi(\tau)}{(\Phi(\tau)+\Psi(\tau))(\Phi(\tau)+3\Psi(\tau))} \Psi(\tau) d\tau \tag{4.4}$$

$$U_i(t,x,y) = 1.25k_1^2 - (\frac{k_2}{k_1})^2 \frac{\Theta(t)}{\Psi(t)} - 7.5k_1^2 \frac{\Phi(t)}{2\Phi(t)+3\Psi(t)} - 3.75k_1^2 \frac{\Psi(t)}{2\Phi(t)+3\Psi(t)}$$

$$- 15k_1^2 \frac{\Psi(t)}{2\Phi(t)+3\Psi(t)} Y_{i-1}^{\diamond 2}(\Xi_3(x,y)), \quad i = 3, 4. \tag{4.5}$$

where

$$Y_2^{\diamond}(\Xi_3(x,y)) = tanh^{\diamond}(\Xi_3(x,y)) \pm isech^{\diamond}(\Xi_3(x,y))$$

or

$$Y_3^{\diamond}(\Xi_3(x,y)) = coth^{\diamond}(\Xi_3(x,y)) \pm csch^{\diamond}(\Xi_3(x,y))$$

$$U_i(t,x,y) = -16k_1^2 - (\frac{k_2}{k_1})^2 \frac{\Theta(t)}{\Psi(t)} - 120k_1^2 \frac{\Psi(t)}{(4\Phi(t)+6\Psi(t)) Y_{i-1}^{\diamond 2}(\Xi_3(x,y))}, \quad i = 5, 6. \tag{4.6}$$

where,

$$Y_4^{\diamond}(\Xi_3(x,y)) = 0.5tan^{\diamond}(2\Xi_3(x,y))$$

or

$$Y_5^{\diamond}(\Xi_3(x,y)) = 0.5cot^{\diamond}(2\Xi_3(x,y))$$

and,

$$\Xi_3(x,y) = k_1 x + k_2 y \tag{4.7}$$

5 Discussions

Our first interest in present work being in implementing the extended tanh-coth method, Hermite transform and white noise analysis to stress its power in handling nonlinear equations so that one can apply it to models of various types of nonlinearity. The next interest is in the determination of exact travelling wave solutions for modified KP equations. Also, we have presented Riccati equation expansion method and applied it to the modified KP equations. As a result, some new exact travelling wave solutions of the modified KP equation are obtained because of more special solutions of Eq.(2.1). The method which we have proposed in this letter is standard, direct and computerized method, which allow us to do complicated and tedious algebraic calculation. It is shown that the algorithm can be also applied to other NLPDEs in mathematical physics such as KdV-Burgers, Modified KdV-Burgers, Zhiber-Shabat equations (specially: Liouville equation, Sinh-Gordon equation, Dodd-Bullough-Mikhailov equation, Dodd-Bullough-Mikhailov equation and Tzitzeica-Dodd-Bullough equation) and Benjamin-Bona-Mahony equations. Also, we remark that, since the Riccati equation has other solution if select other values of c_0, c_1 and c_2 , there are many other exact solutions of variable coefficient and wick-type stochastic modified KP equation.

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Global Dynamics and Bifurcations of Two Quadratic Fractional Second Order Difference Equations

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Abstract. We investigate the bifurcations and the global asymptotic stability of the following two difference equation

$$\begin{aligned} x_{n+1} &= \frac{\beta x_n x_{n-1} + \gamma x_{n-1}}{Ax_n^2 + Bx_n x_{n-1}}, \quad x_0 + x_{-1} > 0, \quad A + B > 0 \\ x_{n+1} &= \frac{\alpha x_n^2 + \beta x_n x_{n-1} + \gamma x_{n-1}}{Ax_n^2}, \quad x_0 > 0, \quad A > 0 \end{aligned}$$

where all parameters and initial conditions are positive.

Keywords. asymptotic stability, attractivity, bifurcation, difference equation, global, local stability, period two;

AMS 2000 Mathematics Subject Classification: 39A10, 39A28, 39A30.

1 Introduction and Preliminaries

We investigate global behavior of the equations:

$$x_{n+1} = \frac{\beta x_n x_{n-1} + \gamma x_{n-1}}{Ax_n^2 + Bx_n x_{n-1}}, \quad n = 0, 1, \ldots \tag{1}$$

$$x_{n+1} = \frac{\alpha x_n^2 + \beta x_n x_{n-1} + \gamma x_{n-1}}{Ax_n^2}, \quad n = 0, 1, \ldots \tag{2}$$

where the parameters $\alpha, \beta, \gamma, A, B$ and the initial conditions x_{-1}, x_0 are positive numbers. Equations (1), (2)) are the special cases of equations

$$x_{n+1} = \frac{\alpha x_n^2 + \beta x_n x_{n-1} + \gamma x_{n-1}}{Ax_n^2 + Bx_n x_{n-1} + Cx_{n-1}}, \quad n = 0, 1, 2, ... \tag{3}$$

and

$$x_{n+1} = \frac{Ax_n^2 + Bx_n x_{n-1} + Cx_{n-1}^2 + Dx_n + Ex_{n-1} + F}{ax_n^2 + bx_n x_{n-1} + cx_{n-1}^2 + dx_n + ex_{n-1} + f}, \quad n = 0, 1, 2, ... \tag{4}$$

Some special cases of equation (4) have been considered in the series of papers [3, 4, 12, 13, 20, 22]. Some special second order quadratic fractional difference equations have appeared in analysis of competitive and anti-competitive systems of linear fractional difference equations in the plane, see [5, 8, 7, 9, 18, 19]. Local stability analysis of the equilibrium solutions of equation (3) was performed in [11].

Describing the global dynamics of equation (4) is a formidable task as this equation contains as a special cases many equations with complicated dynamics, such as the linear fractional difference equation

$$x_{n+1} = \frac{Dx_n + Ex_{n-1} + F}{dx_n + ex_{n-1} + f}, \quad n = 0, 1, 2, ... \tag{5}$$

The special cases considered so far shows that all kind of dynamics are possible including conservative and non-conservative chaos, Naimark-Sacker bifurcation, period-doubling bifurcation, exchange of stability bifurcation, etc. In this paper we use the theory of monotone maps developed in [16, 17] to describe precisely the basins of attraction of all attractors of this equation as well as bifurcations. Equations (1) and (2) exhibit essentially one period doubling bifurcation with different outcomes. Equation (1) allows the coexistence of the unique minimal period-two solution, which is a saddle point and the equilibrium but only the equilibrium solution and the degenerate period-two solution $(0, \infty)$ and $(\infty, 0)$ have substantial basins of attraction. In one region of parameters, Equation (2) also allows the coexistence of the unique minimal period-two solution, which is locally asymptotically stable and the equilibrium, but the period-two solution attracts all solutions outside the global stable manifold of the equilibrium. In the complementary region of parameters every solution is either attracted to the equilibrium or to the degenerate period-two solution $(1, \infty)$ and $(\infty, 1)$.

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Our results will be based on the following theorem for a general second order difference equation

$$x_{n+1} = f(x_n, x_{n-1}), \qquad n = 0, 1, 2, ..., \tag{6}$$

see [2].

Theorem 1 *Let I be a set of real numbers and $f : I \times I \to I$ be a function which is non-increasing in the first variable and non-decreasing in the second variable. Then, for ever solution $\{x_n\}_{n=-1}^{\infty}$ of the equation*

$$x_{n+1} = f(x_n, x_{n-1}), \qquad x_{-1}, x_0 \in I, \ n = 0, 1, 2, ... \tag{7}$$

the subsequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n-1}\}_{n=0}^{\infty}$ of even and odd terms of the solution do exactly one of the following:
(i) Eventually they are both monotonically increasing.
(ii) Eventually they are both monotonically decreasing.
(iii) One of them is monotonically increasing and the other is monotonically decreasing.

The consequence of Theorem 1 is that every bounded solution of (7) converges to either equilibrium or period-two solution or to the point on the boundary, and most important question becomes determining the basins of attraction of these solutions as well as the unbounded solutions. The answer to this question follows from an application of theory of monotone maps in the plane which will be presented for the sake of completeness.

We now give some basic notions about monotone maps in the plane.

Consider a partial ordering $\preceq$ on $\mathbb{R}^2$. Two points $x, y \in \mathbb{R}^2$ are said to be related if $x \preceq y$ or $x \preceq y$. Also, a strict inequality between points may be defined as $x \prec y$ if $x \preceq y$ and $x \neq y$. A stronger inequality may be defined as $x = (x_1, x_2) \ll y = (y_1, y_2)$ if $x \preceq y$ with $x_1 \neq y_1$ and $x_2 \neq y_2$.

A map T on a nonempty set $\mathcal{R} \subset \mathbb{R}^2$ is a continuous function $T : \mathcal{R} \to \mathcal{R}$. The map T is monotone if $x \preceq y$ implies $T(x) \preceq T(y)$ for all $x, y \in \mathcal{R}$, and it is strongly monotone on $\mathcal{R}$ if $x \prec y$ implies that $T(x) \ll T(y)$ for all $x, y \in \mathcal{R}$. The map is strictly monotone on $\mathcal{R}$ if $x \prec y$ implies that $T(x) \prec T(y)$ for all $x, y \in \mathcal{R}$. Clearly, being related is invariant under iteration of a strongly monotone map.

Throughout this paper we shall use the *North-East ordering* (NE) for which the positive cone is the first quadrant, i.e. this partial ordering is defined by $(x_1, y_1) \preceq_{ne} (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \leq y_2$ and the South-East (SE) ordering defined as $(x_1, y_1) \preceq_{se} (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \geq y_2$.

A map T on a nonempty set $\mathcal{R} \subset \mathbb{R}^2$ which is monotone with respect to the North-East ordering is called *cooperative* and a map monotone with respect to the South-East ordering is called *competitive*.

If T is differentiable map on a nonempty set $\mathcal{R}$, a sufficient condition for T to be strongly monotone with respect to the SE ordering is that the Jacobian matrix at all points x has the sign configuration

$$\text{sign}(J_T(\mathbf{x})) = \begin{bmatrix} + & - \\ - & + \end{bmatrix}, \tag{8}$$

provided that $\mathcal{R}$ is open and convex.

For $x \in \mathbb{R}^2$, define $Q_\ell(x)$ for $\ell = 1, \dots, 4$ to be the usual four quadrants based at x and numbered in a counterclockwise direction, for example, $Q_1(x) = \{y \in R^2 : \ x_1 \leq y_1, \ x_2 \leq y_2\}$. Basin of attraction of a fixed point $(\bar{x}, \bar{y})$ of a map T, denoted as $\mathcal{B}((\bar{x}, \bar{y}))$, is defined as the set of all initial points (x_0, y_0) for which the sequence of iterates $T^n((x_0, y_0))$ converges to $(\bar{x}, \bar{y})$. Similarly, we define a basin of attraction of a periodic point of period p. The next five results, from [17, 16], are useful for determining basins of attraction of fixed points of competitive maps. Related results have been obtained by H. L. Smith in [21, 22].

Theorem 2 *Let T be a competitive map on a rectangular region $\mathcal{R} \subset \mathbb{R}^2$. Let $\overline{\mathrm{x}} \in \mathcal{R}$ be a fixed point of T such that $\Delta := \mathcal{R} \cap int\,(Q_1(\overline{\mathrm{x}}) \cup Q_3(\overline{\mathrm{x}}))$ is nonempty (i.e., $\overline{\mathrm{x}}$ is not the NW or SE vertex of $\mathcal{R}$), and T is strongly competitive on Δ. Suppose that the following statements are true.*

a. The map T has a C^1 extension to a neighborhood of $\overline{\mathrm{x}}$.

b. The Jacobian $J_T(\overline{\mathrm{x}})$ of T at $\overline{\mathrm{x}}$ has real eigenvalues λ, μ such that $0 < |\lambda| < \mu$, where $|\lambda| < 1$, and the eigenspace E^λ associated with λ is not a coordinate axis.

Then there exists a curve $\mathcal{C} \subset \mathcal{R}$ through $\overline{\mathrm{x}}$ that is invariant and a subset of the basin of attraction of $\overline{\mathrm{x}}$, such that $\mathcal{C}$ is tangential to the eigenspace E^λ at $\overline{\mathrm{x}}$, and $\mathcal{C}$ is the graph of a strictly increasing continuous function of the first coordinate on an interval. Any endpoints of $\mathcal{C}$ in the interior of $\mathcal{R}$ are either fixed points or minimal period-two points. In the latter case, the set of endpoints of $\mathcal{C}$ is a minimal period-two orbit of T.

We shall see in Theorem 4 that the situation where the endpoints of $\mathcal{C}$ are boundary points of $\mathcal{R}$ is of interest. The following result gives a sufficient condition for this case.

Theorem 3 *For the curve $\mathcal{C}$ of Theorem 2 to have endpoints in $\partial\mathcal{R}$, it is sufficient that at least one of the following conditions is satisfied.*

i. The map T has no fixed points nor periodic points of minimal period two in Δ.

ii. The map T has no fixed points in Δ, $\det J_T(\overline{\mathrm{x}}) > 0$, and $T(x) = \overline{\mathrm{x}}$ has no solutions $x \in \Delta$.

iii. The map T has no points of minimal period-two in Δ, $\det J_T(\overline{\mathrm{x}}) < 0$, and $T(x) = \overline{\mathrm{x}}$ has no solutions $x \in \Delta$.

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For maps that are strongly competitive near the fixed point, hypothesis b. of Theorem 2 reduces just to $|\lambda| < 1$. This follows from a change of variables [22] that allows the Perron-Frobenius Theorem to be applied. Also, one can show that in such case no associated eigenvector is aligned with a coordinate axis. The next result is useful for determining basins of attraction of fixed points of competitive maps.

Theorem 4 *Assume the hypotheses of Theorem 2, and let $\mathcal{C}$ be the curve whose existence is guaranteed by Theorem 2. If the endpoints of $\mathcal{C}$ belong to $\partial\mathcal{R}$, then $\mathcal{C}$ separates $\mathcal{R}$ into two connected components, namely*

$$\mathcal{W}_- := \{x \in \mathcal{R} \setminus \mathcal{C} : \exists y \in \mathcal{C} \text{ with } \mathrm{x} \preceq_{se} y\} \quad \text{and} \quad \mathcal{W}_+ := \{x \in \mathcal{R} \setminus \mathcal{C} : \exists y \in \mathcal{C} \text{ with } y \preceq_{se} x\}, \tag{9}$$

such that the following statements are true.

(i) $\mathcal{W}_-$ is invariant, and $\mathrm{dist}(T^n(x), Q_2(\overline{\mathrm{x}})) \to 0$ as $n \to \infty$ for every $x \in \mathcal{W}_-$.

(ii) $\mathcal{W}_+$ is invariant, and $\mathrm{dist}(T^n(x), Q_4(\overline{\mathrm{x}})) \to 0$ as $n \to \infty$ for every $x \in \mathcal{W}_+$.

(B) If, in addition to the hypotheses of part (A), $\overline{\mathrm{x}}$ is an interior point of $\mathcal{R}$ and T is C^2 and strongly competitive in a neighborhood of $\overline{\mathrm{x}}$, then T has no periodic points in the boundary of $Q_1(\overline{\mathrm{x}}) \cup Q_3(\overline{\mathrm{x}})$ except for $\overline{\mathrm{x}}$, and the following statements are true.

(iii) For every $x \in \mathcal{W}_-$ there exists $n_0 \in \mathbb{N}$ such that $T^n(x) \in int\, Q_2(\overline{\mathrm{x}})$ for $n \geq n_0$.

(iv) For every $x \in \mathcal{W}_+$ there exists $n_0 \in \mathbb{N}$ such that $T^n(x) \in int\, Q_4(\overline{\mathrm{x}})$ for $n \geq n_0$.

If T is a map on a set $\mathcal{R}$ and if $\overline{\mathrm{x}}$ is a fixed point of T, the stable set $\mathcal{W}^s(\overline{\mathrm{x}})$ of $\overline{\mathrm{x}}$ is the set $\{x \in \mathcal{R} : T^n(x) \to \overline{\mathrm{x}}\}$ and unstable set $\mathcal{W}^u(\overline{\mathrm{x}})$ of $\overline{\mathrm{x}}$ is the set

$$\left\{ x \in \mathcal{R} : \text{there exists } \{x_n\}_{n=-\infty}^{0} \subset \mathcal{R} \text{ s.t. } T(x_n) = x_{n+1},\ x_0 = x, \text{ and } \lim_{n \to -\infty} x_n = \overline{\mathrm{x}} \right\}$$

When T is non-invertible, the set $\mathcal{W}^s(\overline{\mathrm{x}})$ may not be connected and made up of infinitely many curves, or $\mathcal{W}^u(\overline{\mathrm{x}})$ may not be a manifold. The following result gives a description of the stable and unstable sets of a saddle point of a competitive map. If the map is a diffeomorphism on $\mathcal{R}$, the sets $\mathcal{W}^s(\overline{\mathrm{x}})$ and $\mathcal{W}^u(\overline{\mathrm{x}})$ are the stable and unstable manifolds of $\overline{x}$.

Theorem 5 *In addition to the hypotheses of part (B) of Theorem 4, suppose that $\mu > 1$ and that the eigenspace E^μ associated with μ is not a coordinate axis. If the curve $\mathcal{C}$ of Theorem 2 has endpoints in $\partial\mathcal{R}$, then $\mathcal{C}$ is the stable set $\mathcal{W}^s(\overline{\mathrm{x}})$ of $\overline{\mathrm{x}}$, and the unstable set $\mathcal{W}^u(\overline{\mathrm{x}})$ of $\overline{x}$ is a curve in $\mathcal{R}$ that is tangential to E^μ at $\overline{\mathrm{x}}$ and such that it is the graph of a strictly decreasing function of the first coordinate on an interval. Any endpoints of $\mathcal{W}^u(\overline{\mathrm{x}})$ in $\mathcal{R}$ are fixed points of T.*

Remark 1 We say that $f(u, v)$ is strongly decreasing in the first argument and strongly increasing in the second argument if it is differentiable and has first partial derivative $D_1 f$ negative and first partial derivative $D_2 f$ positive in a considered set. The connection between the theory of monotone maps and the asymptotic behavior of equation (7) follows from the fact that if f is strongly decreasing in the first argument and strongly increasing in the second argument, then the second iterate of a map associated to equation (7) is a strictly competitive map on $I \times I$, see [17].

Set $x_{n-1} = u_n$ and $x_n = v_n$ in Eq.(7) to obtain the equivalent system

$$\begin{array}{l} u_{n+1} = v_n \\ v_{n+1} = f(v_n, u_n) \end{array}, \quad n = 0, 1, \ldots.$$

Let $T(u, v) = (v, f(v, u))$. The second iterate T^2 is given by

$$T^2(u, v) = (f(v, u),\ f(f(v, u), v))$$

and it is strictly competitive on $I \times I$, see [17].

Remark 2 The characteristic equation of Eq.(7) at an equilibrium point $(\bar{x}, \bar{x})$:

$$\lambda^2 - D_1 f(\bar{x}, \bar{x})\lambda - D_2 f(\bar{x}, \bar{x}) = 0, \tag{10}$$

has two real roots λ, μ which satisfy $\lambda < 0 < \mu$, and $|\lambda| < \mu$, whenever f is strictly decreasing in first and increasing in second variable. Thus the applicability of Theorems 2-5 depends on the nonexistence of minimal period-two solution.

There are several global attractivity results for Eq. (7). Some of these results give the sufficient conditions for all solutions to approach a unique equilibrium and they were used efficiently in [14].

The next result is from [6]. See also [1].

Theorem 6 *Consider Eq. (7) where $f : I \times I \to I$ is a continuous function and f is decreasing in the first argument and increasing in the second argument. Assume that $\overline{x}$ is a unique equilibrium point which is locally asymptotically stable and assume that (φ, ψ) and (ψ, φ) are minimal period-two solutions which are saddle points such that*

$$(\varphi, \psi) \preceq_{se} (\overline{x}, \overline{x}) \preceq_{se} (\psi, \varphi).$$

Then, the basin of attraction $\mathcal{B}((\overline{x}, \overline{x}))$ of $(\overline{x}, \overline{x})$ is the region between the global stable sets $\mathcal{W}^s((\varphi, \psi))$ and $\mathcal{W}^s((\psi, \varphi))$. More precisely

$$\mathcal{B}((\overline{x}, \overline{x})) = \{(x, y) : \exists y_u, y_l : y_u < y < y_l, (x, y_l) \in \mathcal{W}^s((\varphi, \psi)), (x, y_u) \in \mathcal{W}^s((\psi, \varphi))\}.$$

The basins of attraction $\mathcal{B}((\varphi, \psi)) = \mathcal{W}^s((\varphi, \psi))$ and $\mathcal{B}((\psi, \varphi)) = \mathcal{W}^s((\psi, \varphi))$ are exactly the global stable sets of (φ, ψ) and (ψ, φ).

If $(x_{-1}, x_0) \in \mathcal{W}_+((\psi, \varphi))$ or $(x_{-1}, x_0) \in \mathcal{W}_-((\varphi, \psi))$, then $T^n((x_{-1}, x_0))$ converges to the other equilibrium point or to the other minimal period-two solutions or to the boundary of the region $I \times I$.

2 Equation $x_{n+1} = \frac{\beta x_n^2 + \gamma x_{n-1}}{Ax_n^2 + Bx_n x_{n-1}}$

In this section we present the global dynamics of Eq. (11).

2.1 Local stability analysis

By substitution $x_n = \frac{\beta}{A} y_n$, this equation is reduced to the equation

$$y_{n+1} = \frac{y_n y_{n-1} + \frac{\gamma A}{\beta^2} y_{n-1}}{y_n^2 + \frac{B}{A} y_n y_{n-1}},\ n = 0, 1, ...$$

Thus we consider the following equation

$$x_{n+1} = \frac{x_n x_{n-1} + \gamma x_{n-1}}{x_n^2 + Bx_n x_{n-1}},\ n = 0, 1, ... \tag{11}$$

Equation (11) has the unique positive equilibrium $\bar{x}$ given by

$$\bar{x} = \frac{1+\sqrt{1+4\gamma(1+B)}}{2(1+B)}.$$

The partial derivatives associated to the Eq(11) at equilibrium $\bar{x}$ are

$$f'_x = \left.\frac{-x^2 y - 2\gamma xy - B\gamma y^2}{(x^2 + Bxy)^2}\right|_{\bar{x}} = \frac{-2(1+2(1+B)(2+B)\gamma\sqrt{1+4(1+B)\gamma})}{(1+B)(1+\sqrt{1+4(1+B)\gamma})^2}, \quad f'_y = \left.\frac{x+\gamma}{(x+By)^2}\right|_{\bar{x}} = \frac{1}{1+B}.$$

Characteristic equation associated to the Eq.(11) at equilibrium is

$$\lambda^2 + \frac{2(1+2(1+B)(2+B)\gamma\sqrt{1+4(1+B)\gamma})}{(1+B)(1+\sqrt{1+4(1+B)\gamma})^2}\lambda - \frac{1}{1+B} = 0.$$

By applying the linearized stability Theorem [14, 15] we obtain the following result.

Theorem 7 *The unique positive equilibrium point $\bar{x} = \frac{1+\sqrt{1+4\gamma(1+B)}}{2(1+B)}$ of equation (11) is:*

i) locally asymptotically stable when $B > 4\gamma + 1$;

ii) a saddle point when $B < 4\gamma + 1$;

ii) a nonhyperbolic point (with eigenvalues $\lambda_1 = -1$ and $\lambda_2 = \frac{1}{2+4\gamma}$) when $B = 4\gamma + 1$.

Lemma 1 *If*

$$B > 1 + 4\gamma$$

then Eq.(11) possesses a unique minimal period-two solution $\{P(\phi, \psi), Q(\psi, \phi)\}$ where

$$\phi = \frac{1}{2} - \frac{\sqrt{B-1-4\gamma}}{2\sqrt{B-1}} \text{ and } \psi = \frac{1}{2} + \frac{\sqrt{B-1-4\gamma}}{2\sqrt{B-1}}.$$

The minimal period-two solution $\{P(\phi, \psi), Q(\psi, \phi)\}$ is a saddle point.

Proof. Periodic solution $\phi, \psi, \phi, \psi, \ldots$ is the positive solution of the following system

$$\begin{cases} (B-1)y - \gamma = 0 \\ -xy + y = 0. \end{cases} \tag{12}$$

where $\phi + \psi = x$ and $\phi\psi = y$. We have that solution of system (12) is

$$x = 1 \text{ and } y = \frac{\gamma}{B-1}.$$

Since

$$x^2 - 4y = \frac{B-1-4\gamma}{B-1} > 0$$

if and only if $B > 1 + 4\gamma$, we have a unique minimal period-two solution $\{P(\phi, \psi), Q(\psi, \phi)\}$ where

$$\phi = \frac{1}{2} - \frac{\sqrt{B-1-4\gamma}}{2\sqrt{B-1}} \text{ and } \psi = \frac{1}{2} + \frac{\sqrt{B-1-4\gamma}}{2\sqrt{B-1}}.$$

Set

$$u_n = x_{n-1} \text{ and } v_n = x_n,\ \text{for } n = 0, 1, ...$$

and write equation (11) in the equivalent form

$$\begin{aligned} u_{n+1} &= v_n \\ v_{n+1} &= \frac{u_n v_n + \gamma u_n}{v_n^2 + B u_n v_n}, \ n = 0, 1, \dots . \end{aligned}$$

Let T be the function on $(0,\infty) \times (0,\infty)$ defined by

$$T \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ \frac{uv+\gamma u}{v^2+Buv} \end{pmatrix}.$$

By a straightforward calculation we find that

$$T^2 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} g(u,v) \\ h(u,v) \end{pmatrix}$$

where

$$g(u,v) = \frac{uv + \gamma u}{v^2 + Buv}, \quad h(u,v) = \frac{v^2(Bu+v)(v^2\gamma + u(v+\gamma+Bv\gamma))}{u(v+\gamma)(Bv^3 + u(v+B^2v^2+\gamma))}.$$

We have

$$J_{T^2} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} g'_u(\phi,\psi) & g'_v(\phi,\psi) \\ h'_u(\phi,\psi) & h'_v(\phi,\psi) \end{pmatrix},$$

where

$$\begin{aligned} g'_u &= \tfrac{v+\gamma}{(Bu+v)^2}, \\ g'_v &= -\tfrac{u(v^2+B\gamma u+2\gamma v)}{v^2(Bu+v)^2}, \end{aligned}$$

$$h'_u = -\tfrac{v^3(B\gamma v^5 + 2u\gamma v^2(v+\gamma+B^2v^2) + u^2(v^2+v(2+Bv(2+B^2v))\gamma + (1+2Bv)\gamma^2))}{u^2(v+\gamma)(Bv^3+u(v+B^2v^2+\gamma))^2},$$

$$\begin{aligned} h'_v &= \tfrac{v(B^4u^3v^3\gamma^2 + B^3u^2v^4\gamma(v+4\gamma) + B(v+2\gamma)(v^6\gamma + 4u^2v^2\gamma(v+\gamma) + u^3(v+\gamma)^2)}{u^2(v+\gamma)(Bv^3+u(v+B^2v^2+\gamma))^2} \\ &+ \tfrac{B^2uv\gamma(u^2(v+\gamma)(v+3\gamma) + v^4(2v+5\gamma)) + uv(v+\gamma)(u(v+\gamma)(2v+3\gamma) + v^2\gamma(3v+5\gamma)))}{u^2(v+\gamma)(Bv^3+u(v+B^2v^2+\gamma))^2}. \end{aligned}$$

Set

$$\mathcal{S} = g'_u(\phi,\psi) + h'_v(\phi,\psi), \quad \mathcal{D} = g'_u(\phi,\psi)h'_v(\phi,\psi) - g'_v(\phi,\psi)h'_u(\phi,\psi).$$

After some lengthy calculation one can see that

$$\mathcal{S} = \frac{1 + 6\gamma + B(-3 - 6\gamma + B(2+\gamma))}{(B-1)(B+(B-1)\gamma)} \quad \text{and} \quad \mathcal{D} = \frac{\gamma}{(B-1)(B+(B-1)\gamma)}.$$

We have that

$$|\mathcal{S}| > |1 + \mathcal{D}| \quad \text{if and only if} \quad B > 1 + 4\gamma.$$

By applying the linearized stability Theorem we obtain that a unique prime period-two solution $\{P(\phi,\psi), Q(\psi,\phi)\}$ of Eq.(11) is a saddle point if and only if $B > 1 + 4\gamma$. ■

2.2 Global results and basins of attraction

In this section we present global dynamics results for equation (11).

Theorem 8 *If $B > 4\gamma + 1$ then equation (11) has a unique equilibrium point $E(\overline{x},\overline{x})$ which is locally asymptotically stable and there exists the minimal period-two solution $\{P(\phi,\psi), Q(\psi,\phi)\}$, where*

$$\phi = \frac{1}{2} - \tfrac{\sqrt{B-1-4\gamma}}{2\sqrt{B-1}} \ \textit{and} \ \psi = \frac{1}{2} + \tfrac{\sqrt{B-1-4\gamma}}{2\sqrt{B-1}}$$

which is a saddle point.

Furthermore, the global stable manifold of the periodic solution $\{P,Q\}$ is given by $\mathcal{W}^s(\{P,Q\}) = \mathcal{W}^s(P) \cup \mathcal{W}^s(Q)$ where $\mathcal{W}^s(P)$ and $\mathcal{W}^s(Q)$ are continuous increasing curves, that divide the first quadrant into two connected components, namely

$\mathcal{W}_1^+ := \{x \in \mathcal{R} \setminus \mathcal{W}^s(P) : \exists y \in \mathcal{W}^s(P)$ with $y \preceq_{se} x\}$, $\mathcal{W}_1^- := \{x \in \mathcal{R} \setminus \mathcal{W}^s(P) : \exists y \in \mathcal{W}^s(P)$ with $x \preceq_{se} y\}$, and

$\mathcal{W}_2^+ := \{x \in \mathcal{R} \setminus \mathcal{W}^s(Q) : \exists y \in \mathcal{W}^s(Q)$ with $y \preceq_{se} x\}$, $\mathcal{W}_2^- := \{x \in \mathcal{R} \setminus \mathcal{W}^s(Q) : \exists y \in \mathcal{W}^s(Q)$ with $x \preceq_{se} y\}$ respectively such that the following statements are true.

i) If $(u_0, v_0) \in \mathcal{W}^s(P)$ then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ is attracted to P and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ is attracted to Q.

ii) If $(u_0, v_0) \in \mathcal{W}^s(Q)$ then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ is attracted to Q and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ is attracted to P.

iii) *If* $(u_0, v_0) \in \mathcal{W}_1^-$ *(the region above* $\mathcal{W}^s(P)$*) then the subsequence of even-indexed terms* $\{(u_{2n}, v_{2n})\}$ *tends to* $(0, \infty)$ *and the subsequence of odd-indexed terms* $\{(u_{2n+1}, v_{2n+1})\}$ *tends to* $(\infty, 0)$.

iv) *If* $(u_0, v_0) \in \mathcal{W}_2^+$ *(the region below* $\mathcal{W}^s(Q)$*) then the subsequence of even-indexed terms* $\{(u_{2n}, v_{2n})\}$ *tends to* $(\infty, 0)$ *and the subsequence of odd-indexed terms* $\{(u_{2n+1}, v_{2n+1})\}$ *tends to* $(0, \infty)$.

v) *If* $(u_0, v_0) \in \mathcal{W}_1^+ \cap \mathcal{W}_2^-$ *(the region between* $\mathcal{W}^s(P)$ *and* $\mathcal{W}^s(Q)$*) then the sequence* $\{(u_n, v_n)\}$ *is attracted to* $E(\overline{x}, \overline{x})$.

Proof. From Theorem 7 Eq.(11) has a unique equilibrium point $E(\overline{x}, \overline{x})$, which is locally asymptotically stable. Theorem 1 implies that the periodic solution $\{P, Q\}$ is a saddle point. The map $T^2(u, v) = T(T(u, v))$ is competitive on $\mathcal{R} = \mathbb{R}^2 \setminus \{(0, 0)\}$ and strongly competitive on $int(\mathcal{R})$. It follows from the Perron-Frobenius Theorem and a change of variables that at each point the Jacobian matrix of a strongly competitive map has two real and distinct eigenvalues, the larger one in absolute value being positive and that corresponding eigenvectors may be chosen to point in the direction of the second and first quadrant, respectively, see [16, 17]. Also, as is well known [16, 17] if the map is strongly competitive then no eigenvector is aligned with a coordinate axis.

i) By Theorem 4 we have that if $(u_0, v_0) \in \mathcal{W}^s(P)$ then $(u_{2n}, v_{2n}) = T^{2n}(u_0, v_0) \to P$ as $n \to \infty$, which implies that $(u_{2n+1}, v_{2n+1}) = T(T^{2n}(u_0, v_0)) \to T(P) = Q$ as $n \to \infty$, which implies the statement i).

ii) The proof of the statement ii) is similar to the proof of the statement i) and will be ommitted.

iii) A straightforward calculation shows that $(\phi, \psi) \preceq_{se} (\overline{x}, \overline{x}) \preceq_{se} (\psi, \phi)$. Since Eq.(11) has no the other equilibrium point or the other minimal-period two solution from Theorem 6 we have if $(x_{-1}, x_0) \in \mathcal{W}_1^-$, then

$$(u_{2n}, v_{2n}) = T^{2n}((u_0, v_0)) \to (0, \infty) \text{ and } (u_{2n+1}, v_{2n+1}) = T^{2n+1}((u_0, v_0)) \to (\infty, 0).$$

and hence if $(x_{-1}, x_0) \in \mathcal{W}_1^-$, then

$$\lim_{n\to\infty} x_{2n} = \infty \text{ and } \lim_{n\to\infty} x_{2n+1} = 0.$$

iv) If $(x_{-1}, x_0) \in \mathcal{W}_2^+$, then

$$(u_{2n}, v_{2n}) = T^{2n}((u_0, v_0)) \to (\infty, 0) \text{ and } (u_{2n+1}, v_{2n+1}) = T^{2n+1}((u_0, v_0)) \to (0, \infty).$$

and hence if $(x_{-1}, x_0) \in \mathcal{W}_2^+$, then

$$\lim_{n\to\infty} x_{2n} = 0 \text{ and } \lim_{n\to\infty} x_{2n+1} = \infty.$$

v) If $(x_{-1}, x_0) \in \mathcal{W}_1^+ \cap \mathcal{W}_2^-$, then

$$\lim_{n\to\infty} x_n = \frac{1+\sqrt{1+4\gamma(1+B)}}{2(1+B)}.$$

■

Theorem 9 *If* $B < 4\gamma + 1$ *then equation (11) has a unique equilibrium point* $E(\overline{x}, \overline{x})$ *which is a saddle point.*
The global stable manifold $\mathcal{W}^s(E)$ *which is a continuous increasing curve divides the first quadrant such that the following holds:*

i) *Every initial point* (u_0, v_0) *in* $\mathcal{W}^s(E)$ *is attracted to* E.

ii) *If* $(u_0, v_0) \in \mathcal{W}^+(E)$ *(the region below* $\mathcal{W}^s(E)$*) then the subsequence of even-indexed terms* $\{(u_{2n}, v_{2n})\}$ *tends to* $(\infty, 0)$ *and the subsequence of odd-indexed terms* $\{(u_{2n+1}, v_{2n+1})\}$ *tends to* $(0, \infty)$.

iii) *If* $(u_0, v_0) \in \mathcal{W}^-(E)$ *(the region above* $\mathcal{W}^s(E)$*) then the subsequence of even-indexed terms* $\{(u_{2n}, v_{2n})\}$ *tends to* $(0, \infty)$ *and the subsequence of odd-indexed terms* $\{(u_{2n+1}, v_{2n+1})\}$ *tends to* $(\infty, 0)$.

Proof. From Theorem 7 equation (11) has a unique equilibrium point $E(\overline{x}, \overline{x})$, which is a saddle point. The map T has no fixed points or periodic points of minimal period-two in $\Delta = \mathcal{R} \cap int(Q_1(\overline{x}) \cup Q_3(\overline{x}))$. It is immediate to see that $det J_T(E) < 0$ and $T(x) = \overline{x}$ only for $x = \overline{x}$. Since the map T is anti-competitive, see [10] and T^2 is strongly competitive we have that all conditions of Theorem 10 in [10] are satisfied from which the proof follows. ■

Theorem 10 *If* $B = 4\gamma + 1$ *then Eq.(11) has a unique equilibrium point* $E(\overline{x}, \overline{x}) = (\frac{1}{2}, \frac{1}{2})$ *which is a nonhyperbolic point.*
There exists a continuous increasing curve $\mathcal{C}_E$ *which is a subset of the basin of attraction of* E *and it divides the first quadrant such that the following holds:*

i) *Every initial point* (u_0, v_0) *in* $\mathcal{C}_E$ *is attracted to* E.

ii) *If* $(u_0, v_0) \in \mathcal{W}^-(E)$ *(the region above* $\mathcal{C}_E$*) then the subsequence of even-indexed terms* $\{(u_{2n}, v_{2n})\}$ *tends to* $(0, \infty)$ *and the subsequence of odd-indexed terms* $\{(u_{2n+1}, v_{2n+1})\}$ *tends to* $(\infty, 0)$.

iii) *If* $(u_0, v_0) \in \mathcal{W}^+(E)$ *(the region below* $\mathcal{C}_E$*) then the subsequence of even-indexed terms* $\{(u_{2n}, v_{2n})\}$ *tends to* $(\infty, 0)$ *and the subsequence of odd-indexed terms* $\{(u_{2n+1}, v_{2n+1})\}$ *tends to* $(0, \infty)$.

Proof. From Theorem 7 equation (11) has a unique equilibrium point $E(\overline{x}, \overline{x}) = (\frac{1}{2}, \frac{1}{2})$, which is nonhyperbolic. All conditions of Theorem 4 are satisfied, which yields the existence of a continuous increasing curve $\mathcal{C}_E$ which is a subset of the basin of attraction of E and for every $x \in \mathcal{W}^-(E)$ there exists $n_0 \in \mathbb{N}$ such that $T^n(x) \in intQ_2(\bar{x})$ for $n \geq n_0$ and for every $x \in \mathcal{W}^+(E)$ there exists $n_0 \in \mathbb{N}$ such that $T^n(x) \in intQ_4(\bar{x})$ for $n \geq n_0$.

Set

$$U(t) = \frac{1 - (4\gamma + 1)t + \sqrt{(1 - (4\gamma + 1)t)^2 + 4\gamma}}{2}.$$

It is easy to see that $(t, U(t)) \preceq_{se} E$ if $t < \overline{x}$ and $E \preceq_{se} (t, U(t))$ if $t > \overline{x}$. One can show that

$$T^2(t, U(t)) = \left(t, \frac{2\gamma(t + \gamma)}{t(-t + t^2 + 2\gamma + 4t^2\gamma + 8\gamma^2 + t\sqrt{4\gamma + (-1 + t + 4t\gamma)^2})} \right).$$

Now we have that

$$T^2(t, U(t)) \preceq_{se} (t, U(t)) \text{ if } t < \overline{x}$$

and

$$(t, U(t)) \preceq_{se} T^2(t, U(t)) \text{ if } t > \overline{x}.$$

By monotonicity if $t < \overline{x}$ we obtain that $T^{2n}(t, U(t)) \to (0, \infty)$ as $n \to \infty$ and if $t > \overline{x}$ then we have that $T^{2n}(t, U(t)) \to (\infty, 0)$ as $n \to \infty$.

If $(u', v') \in intQ_2(\overline{x})$ then there exists t_1 such that $(u', v') \preceq_{se} (t_1, U(t_1)) \preceq_{se} E$. By monotonicity of the map T^2 we obtain that $T^{2n}(u', v') \preceq_{se} T^{2n}(t_1, U(t_1)) \preceq_{se} E$ which implies that $T^{2n}(u', v') \to (0, \infty)$ and $T^{2n+1}(u', v') \to T(0, \infty) = (\infty, 0)$ as $n \to \infty$ which proves the statement ii).

If $(u'', v'') \in intQ_4(\overline{x})$ then there exists t_2 such that $E \preceq_{se} (t_2, U(t_2)) \preceq_{se} (u'', v'')$. By monotonicity of the map T^2 we obtain that $E \preceq_{se} T^{2n}(t_2, U(t_2)) \preceq_{se} T^{2n}(u'', v'')$ which implies that $T^{2n}(u'', v'') \to (\infty, 0)$ and $T^{2n+1}(u'', v'') \to T(\infty, 0) = (0, \infty)$ as $n \to \infty$ which proves the statement iii). This completes the proof of Theorem. ∎

Remark 3 Theorems 8, 9 and 10 show new type of period doubling bifurcation. When $B \leq 4\gamma + 1$ all solutions outside the global stable manifold are asymptotic to $(0, \infty)$ or to $(\infty, 0)$, and when $B > 4\gamma + 1$ all solutions are either asymptotic to $(0, \infty)$ or to $(\infty, 0)$ or to the minimal period-two solution $\{P, Q\}$ or a unique equilibrium E. In the second case each attractor has a substantial basin of attraction.

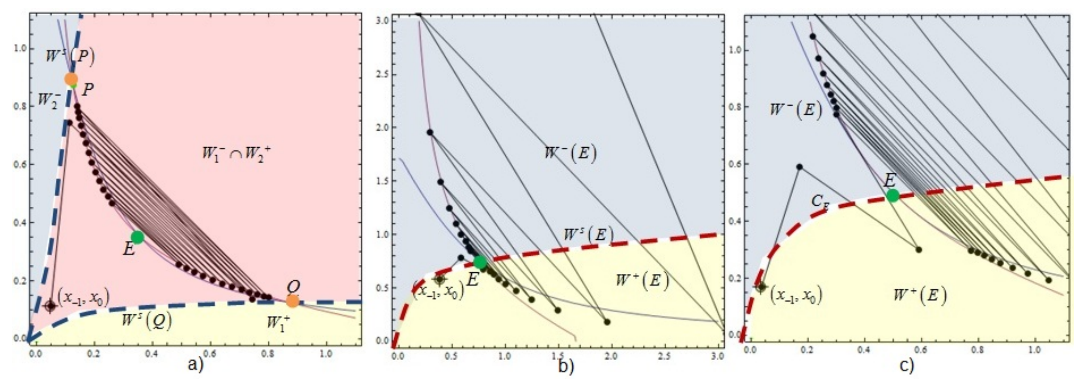

Figure 1: Visual illustration of Theorems 8, 9 and 10 . Figures are generated by *Dynamica* 3, [15].

3 Equation $x_{n+1} = \frac{\alpha x_n^2 + \beta x_n x_{n-1} + \gamma x_{n-1}}{A x_n^2}$

In this section we present the global dynamics and bifurcation analysis of Equation (13).

3.1 Local stability analysis

This equation is reduced to the equation

$$x_{n+1} = \frac{x_n^2 + \beta x_n x_{n-1} + \gamma x_{n-1}}{x_n^2}, \ n = 0, 1, ... \tag{13}$$

Equation (13) has the unique positive equilibrium $\bar{x}$ given by

$$\bar{x} = \frac{1+\beta+\sqrt{(1+\beta)^2+4\gamma}}{2}.$$

The partial derivatives associated to equation (13) at equilibrium $\bar{x}$ are

$$f'_x = \left.\frac{-xy\beta-2\gamma y}{x^3}\right|_{\bar{x}} = -\frac{2(4\gamma+\beta(1+\beta+\sqrt{(1+\beta)^2+4\gamma}))}{(1+\beta+\sqrt{(1+\beta)^2+4\gamma})^2}, \quad f'_y = \left.\frac{\beta x+\gamma}{x^2}\right|_{\bar{x}} = \frac{2(2\gamma+\beta(1+\beta+\sqrt{(1+\beta)^2+4\gamma}))}{(1+\beta+\sqrt{(1+\beta)^2+4\gamma})^2}.$$

Characteristic equation associated to the Eq.(13) at equilibrium is

$$\lambda^2 + \frac{2(4\gamma+\beta(1+\beta+\sqrt{(1+\beta)^2+4\gamma}))}{(1+\beta+\sqrt{(1+\beta)^2+4\gamma})^2}\lambda - \frac{2(2\gamma+\beta(1+\beta+\sqrt{(1+\beta)^2+4\gamma}))}{(1+\beta+\sqrt{(1+\beta)^2+4\gamma})^2} = 0.$$

By applying the linearized stability Theorem we obtain the following result.

Theorem 11 *The unique positive equilibrium point* $\bar{x} = \frac{1+\beta+\sqrt{(1+\beta)^2+4\gamma}}{2}$ *of equation (13) is*

i) locally asymptotically stable when $4\gamma + 2\beta + \beta^2 < 3$;

ii) a saddle point when $4\gamma + 2\beta + \beta^2 > 3$;

iii) a nonhyperbolic point (with eigenvalues $\lambda_1 = -1$ *and* $\lambda_2 = \frac{\beta+1}{\beta+3}$*) when* $4\gamma + 2\beta + \beta^2 = 3$.

Lemma 2 *Equation (13) has the minimal period-two solution* $\{P(\phi,\psi), Q(\psi,\phi)\}$ *where*

$$\phi = \frac{-\gamma+\beta\gamma-\gamma\sqrt{-3+2\beta+\beta^2+4\gamma}}{2(-1+\beta+\gamma)} \text{ and } \psi = \frac{-\gamma+\beta\gamma+\gamma\sqrt{-3+2\beta+\beta^2+4\gamma}}{2(-1+\beta+\gamma)}$$

if and only if

$$\beta < 1 \text{ and } \frac{3-2\beta-\beta^2}{4} < \gamma < 1-\beta.$$

The minimal period-two solution $\{P(\phi,\psi), Q(\psi,\phi)\}$ *is locally asymptotically stable.*

Proof. Period-two solution is a positive solution of the following systems

$$\left\{ \begin{array}{c} x - y - \gamma = 0 \\ x^2 - xy + (\beta-1)y = 0. \end{array} \right. \tag{14}$$

where $\phi + \psi = x$ and $\phi\psi = y$. We have that only one solution of system (14) is

$$x = \frac{(\beta-1)\gamma}{\beta+\gamma-1}, \quad y = \frac{-\gamma^2}{\beta+\gamma-1},$$

Since

$$x^2 - 4y = \frac{\gamma^2(-3+2\beta+\beta^2+4\gamma)}{(\beta+\gamma-1)^2} > 0$$

if and only if

$$\frac{3-2\beta-\beta^2}{4} < \gamma$$

and $x, y > 0$ if and only if $\beta < 1$ and $\gamma < 1-\beta$, we have that ϕ and ψ are solution of the equation

$$t^2 - \frac{(\beta-1)\gamma}{\beta+\gamma-1}t + \frac{-\gamma^2}{\beta+\gamma-1} = 0$$

if and only if

$$\beta < 1 \text{ and } \frac{3-2\beta-\beta^2}{4} < \gamma < 1-\beta.$$

The second iterate of the map T is

$$T^2\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} g(u,v) \\ h(u,v) \end{pmatrix}$$

where

$$g(u,v) = \frac{v^2+\beta uv+\gamma u}{v^2}, \quad h(u,v) = \frac{v^4\left(1+v(\beta+\gamma)+\frac{u(2+v\beta)(v\beta+\gamma)}{v^2}+\frac{u^2(v\beta+\gamma)^2}{v^4}\right)}{(v^2+\beta uv+\gamma u)^2}.$$

We have

$$J_{T^2}\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} g'_u(\phi,\psi) & g'_v(\phi,\psi) \\ h'_u(\phi,\psi) & h'_v(\phi,\psi) \end{pmatrix}$$

where

$$\begin{aligned} g'_u &= \frac{v\beta+\gamma}{v^2}, \\ g'_v &= -\frac{u(v\beta+2\gamma)}{v^3}, \end{aligned}$$

$$h'_u = -\frac{v^3(v\beta+\gamma)(uv\beta^2+u\beta\gamma+v^2(\beta+2\gamma))}{(v^2+\beta uv+\gamma u)^3},$$

$$h_v' = \frac{v^2(5u^2v\beta^2\gamma+3u^2\beta\gamma^2+v^4(\beta+\gamma)+3uv^3\beta(\beta+\gamma)+uv^2(2u\beta^3+\gamma(u\beta+5\gamma)))}{(v^2+\beta uv+\gamma u)^3}.$$

Set

$$\mathcal{S} = g_u'(\phi,\psi)+h_v'(\phi,\psi) \quad \text{and} \quad \mathcal{D} = g_u'(\phi,\psi)h_v'(\phi,\psi)-g_v'(\phi,\psi)h_u'(\phi,\psi).$$

After some lengthy calculation one can see that

$$\mathcal{S} = \frac{4+\beta(-6+\beta+\beta^2)-9\gamma+\beta(7+\beta)\gamma+6\gamma^2}{\gamma^2}, \quad \mathcal{D} = \frac{(-1+\gamma)(-1+\beta+\gamma)}{\gamma^2}.$$

Applying the linearized stability Theorem we obtain that a unique prime period-two solution $\{P(\phi,\psi), Q(\psi,\phi)\}$ of Eq(13) is locally asymptotically stable when

$$\beta < 1 \text{ and } \frac{3-2\beta-\beta^2}{4} < \gamma < 1-\beta.$$

■

3.2 Global results and basins of attraction

In this section we present global dynamics results for Eq.(13).

Theorem 12 *If $4\gamma+2\beta+\beta^2<3$ then Eq. (13) has a unique equilibrium point $E(\overline{x},\overline{x})$ which is globally asymptotically stable.*

Proof. From Theorem 11 equation (13) has a unique equilibrium point $E(\overline{x},\overline{x})$, which is locally asymptotically stable. Every solution of equation (13) is bounded from above and from below by positive constants. If $4\gamma+2\beta+\beta^2<3$ then $\beta+\gamma<1$ and we have

$$x_{n+1} = \frac{x_n^2+\beta x_n x_{n-1}+\gamma x_{n-1}}{x_n^2} \geq 1$$

and

$$\begin{aligned}
x_{n+1} &= 1+\frac{\beta x_{n-1}}{x_n}+\frac{\gamma x_{n-1}}{x_n^2} \leq 1+\beta x_{n-1}+\gamma x_{n-1} = 1+(\beta+\gamma)x_{n-1}. \\
x_{2n} &\leq 1+(\beta+\gamma)[1+(\beta+\gamma)x_{2n-4}] \leq ... \\
&\leq 1+(\beta+\gamma)+(\beta+\gamma)^2+...+(\beta+\gamma)^n x_0, \\
&< \frac{1}{1-\alpha-\beta}+(\beta+\gamma)^n x_0, \\
x_{2n-1} &\leq 1+(\beta+\gamma)[1+(\beta+\gamma)x_{2n-5}] \leq ... \\
&\leq 1+(\beta+\gamma)+(\beta+\gamma)^2+...+(\beta+\gamma)^n x_{-1} \\
&< \frac{1}{1-\alpha-\beta}+(\beta+\gamma)^n x_{-1}.
\end{aligned}$$

Thus $x_n \leq \frac{1}{1-\alpha-\beta}+\varepsilon$, for some $\varepsilon>0$ and $n\geq N$ and so every solution is bounded. Equation (13) has no other equilibrium points or period two points and using Theorem 1 we have that equilibrium point $E(\overline{x},\overline{x})$ is globally asymptotically stable. ■

Theorem 13 *If $4\gamma+2\beta+\beta^2>3$ and $\beta+\gamma<1$ then equation (13) has a unique equilibrium point $E(\overline{x},\overline{x})$ which is a saddle point and the minimal period-two solution $\{P(\phi,\psi),Q(\psi,\phi)\}$ which is locally asymptotically stable, where*

$$\phi = \frac{-\gamma+\beta\gamma-\gamma\sqrt{-3+2\beta+\beta^2+4\gamma}}{2(-1+\beta+\gamma)}, \ \psi = \frac{-\gamma+\beta\gamma+\gamma\sqrt{-3+2\beta+\beta^2+4\gamma}}{2(-1+\beta+\gamma)}.$$

The global stable manifold $\mathcal{W}^s(E)$ which is a continuous increasing curve, divides the first quadrant such that the following holds:

i) Every initial point (u_0,v_0) in $\mathcal{W}^s(E)$ is attracted to E.

ii) If $(u_0,v_0)\in\mathcal{W}^+(E)$ (the region below $\mathcal{W}^s(E)$) then the subsequence of even-indexed terms $\{(u_{2n},v_{2n})\}$ is attracted to Q and the subsequence of odd-indexed terms $\{(u_{2n+1},v_{2n+1})\}$ is attracted to P.

iii) If $(u_0,v_0)\in\mathcal{W}^-(E)$ (the region above $\mathcal{W}^s(E)$) then the subsequence of even-indexed terms $\{(u_{2n},v_{2n})\}$ is attracted to P and the subsequence of odd-indexed terms $\{(u_{2n+1},v_{2n+1})\}$ is attracted to Q.

Proof. From Theorem 11 equation (13) has a unique equilibrium point $E(\overline{x},\overline{x})$, which is a saddle point. The map T has no fixed points or periodic points of minimal period-two in $\Delta=\mathcal{R}\cap int(Q_1(\overline{x})\cup Q_3(\overline{x}))$. A straightforward calculation shows that $detJ_T(E)<0$ and $T(x)=\overline{x}$ only for $x=\overline{x}$. Since the map T is anti-competitive and T^2 is strongly competitive we have that all conditions of Theorem 10 in [10] are satisfied from which the proof follows. ■

Theorem 14 *If $4\gamma+2\beta+\beta^2>3$ and $\beta+\gamma\geq1$ then Eq. (13) has a unique equilibrium point $E(\overline{x},\overline{x}\ )$ which is a saddle point.*

The global stable manifold $\mathcal{W}^s(E)$, which is a continuous increasing curve divides the first quadrant such that the following holds:

i) Every initial point (u_0, v_0) in $\mathcal{W}^s(E)$ is attracted to E.

ii) If $(u_0, v_0) \in \mathcal{W}^+(E)$ (the region below $\mathcal{W}^s(E)$) then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ tends to $(\infty, 1)$ and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ tends to $(1, \infty)$.

iii) If $(u_0, v_0) \in \mathcal{W}^-(E)$ (the region above $\mathcal{W}^s(E)$) then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ tends to $(1, \infty)$ and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ tends to $(\infty, 1)$.

Proof. The proof is similar to the proof of the previous theorem using the fact that every solution of equation (13) is bounded from below by 1. ■

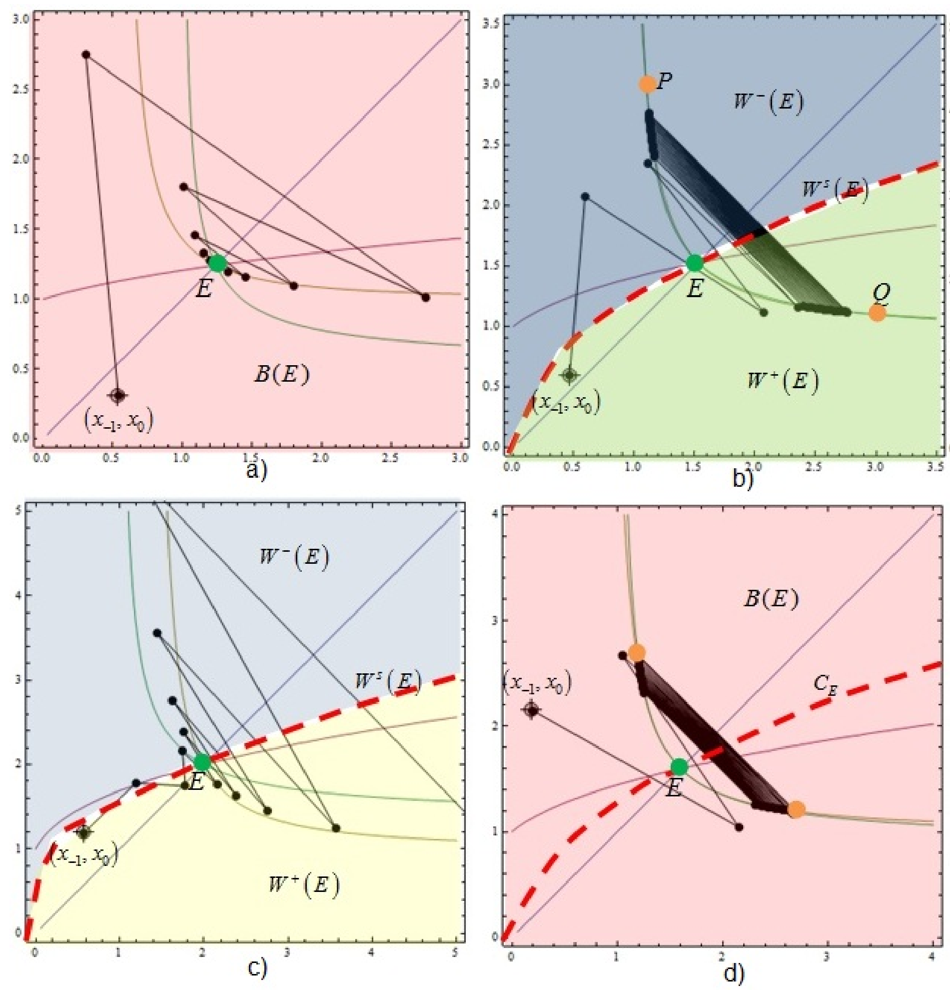

Figure 2: Visual illustration of Theorems 12, 13, 14 and 15 . Figures are generated by *Dynamica* 3, [15].

Theorem 15 *If $4\gamma + 2\beta + \beta^2 = 3$ then Eq. (13) has a unique equilibrium point $E(\overline{x}, \overline{x}\ )$ which is a nonhyperbolic point and a global attractor.*

Proof. From Theorem 11 Eq.(13) has a unique equilibrium point $E(\overline{x}, \overline{x})$, which is non-hyperbolic. All conditions of Theorem 4 are satisfied, which yields the existence a continuous increasing curve $\mathcal{C}_E$ which is a subset of the basin of attraction of E and for every $x \in \mathcal{W}^-(E)$ (the region above $\mathcal{C}_E$) there exists $n_0 \in \mathbb{N}$ such that $T^n(x) \in intQ_2(\overline{x})$ for $n \geq n_0$ and for every $x \in \mathcal{W}^+(E)$ (the region below $\mathcal{C}_E$) there exists $n_0 \in \mathbb{N}$ such that $T^n(x) \in intQ_4(\overline{x})$ for $n \geq n_0$.

Set

$$U(t) = \frac{\beta t + \sqrt{\beta^2 t^2 + (3 - 2\beta - \beta^2)(t^2 - t)}}{2(t-1)}.$$

It is easy to see that $(t, U(t)) \preceq_{se} E$ if $t < \overline{x}$ and $E \preceq_{se} (t, U(t))$ if $t > \overline{x}$. One can show that

$$T^2(t, U(t)) = \left(t, \frac{(t\beta + s)^4 (8t^2 + \frac{t\beta + (3 + (-2 + 4t - \beta)\beta)}{t-1} + \frac{(3 + (-2 + 4t - \beta)\beta)s}{t-1}}{8t^4(-3 + t(3 + (-2 + 4t - \beta)\beta) + \beta(2 + \beta + 2s))^2} \right),$$

where

$$s = \sqrt{t(t(3 - 2\beta) + (\beta - 1)(\beta + 3)}.$$

Now we have that

$$T^2(t, U(t)) \preceq_{se} (t, U(t)) \text{ if } t > \overline{x}$$

and

$$(t, U(t)) \preceq_{se} T^2(t, U(t)) \text{ if } t < \overline{x}$$

since

$$\frac{(t\beta + s)^4 (8t^2 + \frac{t\beta + (3 + (-2 + 4t - \beta)\beta)}{t-1} + \frac{(3 + (-2 + 4t - \beta)\beta)s}{t-1}}{8t^4(-3 + t(3 + (-2 + 4t - \beta)\beta) + \beta(2 + \beta + 2s))^2} - \frac{\beta t + \sqrt{\beta^2 t^2 + (3 - 2\beta - \beta^2)(t^2 - t)}}{2(t-1)} > 0,$$

if and only if $t > \overline{x}$. By monotonicity if $t < \overline{x}$ then we obtain that $T^{2n}(t, U(t)) \to E$ as $n \to \infty$ and if $t > \overline{x}$ then we have that $T^{2n}(t, U(t)) \to E$ as $n \to \infty$.

If $(u', v') \in intQ_2(\bar{x})$ then there exists t_1 such that $(t_1, U(t_1)) \preceq_{se} (u', v') \preceq_{se} E$. By monotonicity of the map T^2 we obtain that $T^{2n}(t_1, U(t_1)) \preceq_{se} T^{2n}(u', v') \preceq_{se} E$ which implies that $T^{2n}(u', v') \to E$ and $T^{2n+1}(u', v') \to T(E) = E$, as $n \to \infty$ which proves the statement ii).

If $(u'', v'') \in intQ_4(\overline{x})$ then there exists t_2 such that $E \preceq_{se} (u'', v'') \preceq_{se} (t_2, U(t_2))$. By monotonicity of the map T^2 we obtain that $E \preceq_{se} T^{2n}(u'', v'') \preceq_{se} T^{2n}(t_2, U(t_2))$ which implies that $T^{2n}(u'', v'') \to E$ and $T^{2n+1}(u'', v'') \to T(E) = E$ as $n \to \infty$ which proves the statement iii), which completes the proof of the Theorem. ■

Remark 4 Theorems 12, 13, 14 and 15 show another type of period doubling bifurcation. When $4\gamma + 2\beta + \beta^2 \leq 3$ all solutions are asymptotic to the unique equilibrium E. When $4\gamma + 2\beta + \beta^2 > 3$ and $\beta + \gamma < 1$ all solutions which starts off the global stable manifold of the unique equilibrium E are asymptotic to the unique minimal period-two solution $\{P, Q\}$. Finally, when $4\gamma + 2\beta + \beta^2 > 3$ and $\beta + \gamma \geq 1$ all solutions which starts off the global stable manifold of the unique equilibrium E are asymptotic to $(1, \infty)$ or to $(\infty, 1)$.

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Some New Inequalities of Hermite–Hadamard Type for Geometrically Mean Convex Functions on the Co-ordinates

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Abstract

In the paper, the authors introduce a new concept geometrically mean convex function on co-ordinates and establish some new integral inequalities of Hermite–Hadamard type for geometrically mean convex functions of two variables on the co-ordinates.

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1 Introduction

The following definitions are well known in the literature.

Definition 1.1 ([3, 4])**.** A function $f:\Delta=[a,b]\times[c,d]\subseteq\mathbb{R}^2\to\mathbb{R}$ is said to be convex on the co-ordinates on Δ with $a<b$ and $c<d$ if the partial mappings

$$f_y:[a,b]\to\mathbb{R},\quad f_y(u)=f(u,y)\quad\text{and}\quad f_x:[c,d]\to\mathbb{R},\quad f_x(v)=f(x,v)$$

are convex for all $x\in(a,b)$ and $y\in(c,d)$.

Definition 1.2 ([3, 4])**.** A function $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ is said to be convex on the co-ordinates on Δ with $a<b$ and $c<d$ if

$$f(tx+(1-t)z,\lambda y+(1-\lambda)w)\le t\lambda f(x,y)+t(1-\lambda)f(x,w)+(1-t)\lambda f(z,y)+(1-t)(1-\lambda)f(z,w)$$

holds for all $t,\lambda\in[0,1],(x,y),(z,w)\in\Delta$.

Definition 1.3 ([1])**.** A function $f:\Delta=[a,b]\times[c,d]\subseteq\mathbb{R}^2\to\mathbb{R}_+$ is called co-ordinated log-convex on Δ with $a<b$ and $c<d$ for all $t,\lambda\in[0,1]$ and $(x,y),(z,w)\in\Delta$, if

$$f(tx+(1-t)z,\lambda y+(1-\lambda)w)\le[f(x,y)]^{t\lambda}[f(x,w)]^{t(1-\lambda)}[f(z,y)]^{(1-t)\lambda}[f(z,w)]^{(1-t)(1-\lambda)}.$$

In recent years, the following integral inequalities of Hermite–Hadamard type for the above kinds of convex functions were published.

Theorem 1.1 ([3, Theorem 2.2] and [4, Theorem 2.2]). *Let $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ be convex on the co-ordinates on Δ with $a<b$ and $c<d$. Then*

$$\begin{aligned}
f\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)\,\mathrm{d}\,x+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)\,\mathrm{d}\,y\right]\\
&\le \frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y\\
&\le \frac{1}{4}\left[\frac{1}{b-a}\left(\int_a^b f(x,c)\,\mathrm{d}\,x+\int_a^b f(x,d)\,\mathrm{d}\,x\right)+\frac{1}{d-c}\left(\int_c^d f(a,y)\,\mathrm{d}\,y+\int_c^d f(b,y)\,\mathrm{d}\,y\right)\right]\\
&\le \frac{1}{4}\big[f(a,c)+f(b,c)+f(a,d)+f(b,d)\big].
\end{aligned}$$

Theorem 1.2 ([7, Theorem 2.3]). *Let $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ be a partial differentiable function on Δ. If $\left|\frac{\partial^2 f}{\partial x\partial y}\right|$ is convex on the co-ordinates on Δ, then*

$$\begin{aligned}
&\Bigg|\frac{1}{9}\left[f\left(a,\frac{c+d}{2}\right)+f\left(b,\frac{c+d}{2}\right)+4f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)\right]\\
&\quad+\frac{1}{36}\big[f(a,c)+f(a,d)+f(b,c)+f(b,d)\big]+\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y-A\Bigg|\\
&\qquad\le\left(\frac{5}{72}\right)^2(b-a)(d-c)\left\{\left|\frac{\partial^2 f(a,c)}{\partial x\partial y}\right|+\left|\frac{\partial^2 f(a,d)}{\partial x\partial y}\right|+\left|\frac{\partial^2 f(b,c)}{\partial x\partial y}\right|+\left|\frac{\partial^2 f(b,d)}{\partial x\partial y}\right|\right\},
\end{aligned}$$

where

$$\begin{aligned}
A=\frac{1}{b-a}\int_a^b\left\{\frac{1}{6}\left[f(x,c)+4f\left(x,\frac{c+d}{2}\right)+f(x,d)\right]\right\}\mathrm{d}\,x\\
+\frac{1}{d-c}\int_c^d\left\{\frac{1}{6}\left[f(a,y)+4f\left(\frac{a+b}{2},y\right)+f(b,y)\right]\right\}\mathrm{d}\,y.
\end{aligned}$$

Theorem 1.3 ([6, Theorem 2]). *Let $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ be a partial differentiable function on Δ. If $\left|\frac{\partial^2 f}{\partial x\partial y}\right|$ is convex on the co-ordinates on Δ, then*

$$\begin{aligned}
&\left|\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-A\right|\\
&\qquad\le\frac{(b-a)(d-c)}{16}\left[\frac{\left|\frac{\partial^2 f(a,c)}{\partial x\partial y}\right|+\left|\frac{\partial^2 f(a,d)}{\partial x\partial y}\right|+\left|\frac{\partial^2 f(b,c)}{\partial x\partial y}\right|+\left|\frac{\partial^2 f(b,d)}{\partial x\partial y}\right|}{4}\right],
\end{aligned}$$

where

$$A=\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)\mathrm{d}\,x+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)\mathrm{d}\,y.$$

Theorem 1.4 ([1, Corollary 3.1]). *Suppose that $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}_+$ is log-convex on the co-ordinates on Δ. Then*

$$\begin{aligned}\ln f\left(\frac{a+b}{2},\frac{c+d}{2}\right)&\le\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b\ln f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y\\&\le\frac{\ln f(a,c)+\ln f(b,c)+\ln f(a,d)+\ln f(b,d)}{4}.\end{aligned}$$

In the papers [2, 5, 8, 9, 10, 11], there are also some new results on this topic.

2 A definition and lemmas

In this section, we introduce the notion "geometrically mean convex function" and establish an integral identity.

Definition 2.1. A function $f:\Delta=[a,b]\times[c,d]\subseteq\mathbb{R}_+^2\to\mathbb{R}_+$ is said to be geometrically mean convex on the co-ordinates on Δ with $a<b$ and $c<d$, if

$$f\big(x^tz^{1-t},y^\lambda w^{1-\lambda}\big)\le[f(x,y)]^{[t+\lambda]/4}[f(x,w)]^{[t+(1-\lambda)]/4}[f(z,y)]^{[(1-t)+\lambda]/4}[f(z,w)]^{[(1-t)+(1-\lambda)]/4}$$

holds for all $t,\lambda\in[0,1]$ and $(x,y),(z,w)\in\Delta$.

In order to prove our main results, we need the following integral identity.

Lemma 2.1. *Let $f:\Delta=[a,b]\times[c,d]\subseteq\mathbb{R}_+^2\to\mathbb{R}$ have partial derivatives of the second order with $a<b$ and $c<d$. If $\frac{\partial^2 f}{\partial x\partial y}\in L_1(\Delta)$, where $L_1(\Delta)$ denotes the set of all Lebesgue integrable functions on Δ, then*

$$\begin{aligned}S(f)\triangleq{}&\frac{4}{(\ln b-\ln a)(\ln d-\ln c)}\bigg[\frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{4}\\&-A+\frac{1}{(\ln b-\ln a)(\ln d-\ln c)}\int_c^d\int_a^b\frac{f(x,y)}{xy}\,\mathrm{d}\,x\,\mathrm{d}\,y\bigg]\\={}&\int_0^1\int_0^1 t\lambda a^tb^{1-t}c^\lambda d^{1-\lambda}\frac{\partial^2}{\partial x\partial y}f\big(a^tb^{1-t},c^\lambda d^{1-\lambda}\big)\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\\&-\int_0^1\int_0^1 t\lambda a^tb^{1-t}c^{1-\lambda}d^\lambda\frac{\partial^2}{\partial x\partial y}f\big(a^tb^{1-t},c^{1-\lambda}d^\lambda\big)\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\\&-\int_0^1\int_0^1 t\lambda a^{1-t}b^tc^\lambda d^{1-\lambda}\frac{\partial^2}{\partial x\partial y}f\big(a^{1-t}b^t,c^\lambda d^{1-\lambda}\big)\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\\&+\int_0^1\int_0^1 t\lambda a^{1-t}b^tc^{1-\lambda}d^\lambda\frac{\partial^2}{\partial x\partial y}f\big(a^{1-t}b^t,c^{1-\lambda}d^\lambda\big)\,\mathrm{d}\,t\,\mathrm{d}\,\lambda,\end{aligned}$$

where

$$A=\frac{1}{2(\ln b-\ln a)}\int_a^b\left[\frac{f(x,c)}{x}+\frac{f(x,d)}{x}\right]\mathrm{d}\,x+\frac{1}{2(\ln d-\ln c)}\int_c^d\left[\frac{f(a,y)}{y}+\frac{f(b,y)}{y}\right]\mathrm{d}\,y.$$

Proof. Let $x = a^t b^{1-t}$ and $y = c^\lambda d^{1-\lambda}$ for $0 \le t, \lambda \le 1$. Using integration by parts, we have

$$\begin{aligned}
&\int_0^1 \int_0^1 t\lambda a^t b^{1-t} c^\lambda d^{1-\lambda} \frac{\partial^2}{\partial x \partial y} f\big(a^t b^{1-t}, c^\lambda d^{1-\lambda}\big) \,\mathrm{d}\,t \,\mathrm{d}\,\lambda \\
&= \frac{1}{\ln a - \ln b} \int_0^1 \lambda c^\lambda d^{1-\lambda} \left[t \frac{\partial}{\partial y} f\big(a^t b^{1-t}, c^\lambda d^{1-\lambda}\big)\Big|_0^1 - \int_0^1 \frac{\partial}{\partial y} f\big(a^t b^{1-t}, c^\lambda d^{1-\lambda}\big) \,\mathrm{d}\,t \right] \mathrm{d}\,\lambda \\
&= \frac{1}{\ln a - \ln b} \left[\int_0^1 \lambda c^\lambda d^{1-\lambda} \frac{\partial}{\partial y} f\big(a, c^\lambda d^{1-\lambda}\big) \,\mathrm{d}\,\lambda - \int_0^1 \int_0^1 \lambda c^\lambda d^{1-\lambda} \frac{\partial}{\partial y} f\big(a^t b^{1-t}, c^\lambda d^{1-\lambda}\big) \,\mathrm{d}\,t \,\mathrm{d}\,\lambda \right] \\
&= \frac{1}{(\ln b - \ln a)(\ln d - \ln c)} \left[f(a,c) - \int_0^1 f\big(a, c^\lambda d^{1-\lambda}\big) \,\mathrm{d}\,\lambda \right. \\
&\quad \left. - \int_0^1 f\big(a^t b^{1-t}, c\big) \,\mathrm{d}\,t + \int_0^1 \int_0^1 f\big(a^t b^{1-t}, c^\lambda d^{1-\lambda}\big) \,\mathrm{d}\,t \,\mathrm{d}\,\lambda \right] \\
&= \frac{1}{(\ln b - \ln a)(\ln d - \ln c)} \left[f(a,c) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x,c)}{x} \,\mathrm{d}\,x \right. \\
&\quad \left. - \frac{1}{\ln d - \ln c} \int_c^d \frac{f(a,y)}{y} \,\mathrm{d}\,y + \frac{1}{(\ln b - \ln a)(\ln d - \ln c)} \int_c^d \int_a^b \frac{f(x,y)}{xy} \,\mathrm{d}\,x \,\mathrm{d}\,y \right].
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\int_0^1 \int_0^1 t\lambda a^t b^{1-t} c^{1-\lambda} d^\lambda \frac{\partial^2}{\partial x \partial y} f\big(a^t b^{1-t}, c^{1-\lambda} d^\lambda\big) \,\mathrm{d}\,t \,\mathrm{d}\,\lambda \\
&= -\frac{1}{(\ln b - \ln a)(\ln d - \ln c)} \left[f(a,d) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x,d)}{x} \,\mathrm{d}\,x \right. \\
&\quad \left. - \frac{1}{\ln d - \ln c} \int_c^d \frac{f(a,y)}{y} \,\mathrm{d}\,y + \frac{1}{(\ln b - \ln a)(\ln d - \ln c)} \int_c^d \int_a^b \frac{f(x,y)}{xy} \,\mathrm{d}\,x \,\mathrm{d}\,y \right], \\
&\int_0^1 \int_0^1 t\lambda a^{1-t} b^t c^\lambda d^{1-\lambda} \frac{\partial^2}{\partial x \partial y} f\big(a^{1-t} b^t, c^\lambda d^{1-\lambda}\big) \,\mathrm{d}\,t \,\mathrm{d}\,\lambda \\
&= -\frac{1}{(\ln b - \ln a)(\ln d - \ln c)} \left[f(b,c) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x,c)}{x} \,\mathrm{d}\,x \right. \\
&\quad \left. - \frac{1}{\ln d - \ln c} \int_c^d \frac{f(b,y)}{y} \,\mathrm{d}\,y + \frac{1}{(\ln b - \ln a)(\ln d - \ln c)} \int_c^d \int_a^b \frac{f(x,y)}{xy} \,\mathrm{d}\,x \,\mathrm{d}\,y \right],
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 \int_0^1 t\lambda a^{1-t} b^t c^{1-\lambda} d^\lambda \frac{\partial^2}{\partial x \partial y} f\big(a^{1-t} b^t, c^{1-\lambda} d^\lambda\big) \,\mathrm{d}\,t \,\mathrm{d}\,\lambda \\
&= \frac{1}{(\ln b - \ln a)(\ln d - \ln c)} \left[f(b,d) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x,d)}{x} \,\mathrm{d}\,x \right. \\
&\quad \left. - \frac{1}{\ln d - \ln c} \int_c^d \frac{f(b,y)}{y} \,\mathrm{d}\,y + \frac{1}{(\ln b - \ln a)(\ln d - \ln c)} \int_c^d \int_a^b \frac{f(x,y)}{xy} \,\mathrm{d}\,x \,\mathrm{d}\,y \right].
\end{aligned}$$

Lemma 2.1 is proved. □

4

Lemma 2.2. *Let $u, v > 0$. Then*

$$F(u,v) \triangleq \int_0^1 t u^t v^{1-t} \,\mathrm{d}\, t = \begin{cases} \dfrac{L(u,v)-u}{\ln v - \ln u}, & u \ne v, \\ \dfrac{1}{2} u, & u = v, \end{cases} \tag{2.1}$$

where $L(u,v)$ is logarithmic mean defined by

$$L(u,v) \triangleq \int_0^1 u^t v^{1-t} \,\mathrm{d}\, t = \begin{cases} \dfrac{v-u}{\ln v - \ln u}, & u \ne v, \\ u, & u = v. \end{cases}$$

Proof. The proof is straightforward. □

3 Some integral inequalities of Hermite–Hadamard type

In this section, we prove some new inequalities of Hermite–Hadamard type for geometrically mean convex functions.

Theorem 3.1. *Let $f : \Delta = [a,b] \times [c,d] \subseteq \mathbb{R}_+^2 \to \mathbb{R}$ be a partial differentiable function on Δ with $a < b$, $c < d$ and $\frac{\partial^2 f}{\partial x \partial y} \in L_1(\Delta)$. If $\left|\frac{\partial^2 f}{\partial x \partial y}\right|^q$ is geometrically mean convex functions on the co-ordinates on Δ for $q \ge 1$, then*

$$\begin{aligned} |S(f)| \le{}& \big[F(a,b)F(c,d)\big]^{1-1/q} \big[F\big(M_q(a,a), M_q(b,b)\big) F\big(N_q(c,c), N_q(d,d)\big)\big]^{1/q} \\ &+ \big[F(a,b)F(d,c)\big]^{1-1/q} \big[F\big(M_q(a,a), M_q(b,b)\big) F\big(N_q(d,d), N_q(c,c)\big)\big]^{1/q} \\ &+ \big[F(b,a)F(c,d)\big]^{1-1/q} \big[F\big(M_q(b,b), M_q(a,a)\big) F\big(N_q(c,c), N_q(d,d)\big)\big]^{1/q} \\ &+ \big[F(b,a)F(d,c)\big]^{1-1/q} \big[F\big(M_q(b,b), M_q(a,a)\big) F\big(N_q(d,d), N_q(c,c)\big)\big]^{1/q}, \end{aligned} \tag{3.1}$$

where $F(u,v)$ is defined by (2.1),

$$M_q(u^r, u) = u^r \left[\left|\frac{\partial^2 f(u,c)}{\partial x \partial y}\right| \left|\frac{\partial^2 f(u,d)}{\partial x \partial y}\right|\right]^{q/4}, \quad N_q(v^r, v) = v^r \left[\left|\frac{\partial^2 f(a,v)}{\partial x \partial y}\right| \left|\frac{\partial^2 f(b,v)}{\partial x \partial y}\right|\right]^{q/4} \tag{3.2}$$

for $r \ge 0$.

Proof. Since $\left|\frac{\partial^2 f}{\partial x \partial y}\right|^q$ is geometrically mean convex on coordinates Δ, using Lemma 2.1 and by Hölder's integral inequality, we have

$$\begin{aligned} |S(f)| \le{}& \int_0^1 \int_0^1 t\lambda a^t b^{1-t} c^\lambda d^{1-\lambda} \left|\frac{\partial^2}{\partial x \partial y} f\big(a^t b^{1-t}, c^\lambda d^{1-\lambda}\big)\right| \mathrm{d}\, t \,\mathrm{d}\, \lambda \\ &+ \int_0^1 \int_0^1 t\lambda a^t b^{1-t} c^{1-\lambda} d^\lambda \left|\frac{\partial^2}{\partial x \partial y} f\big(a^t b^{1-t}, c^{1-\lambda} d^\lambda\big)\right| \mathrm{d}\, t \,\mathrm{d}\, \lambda \\ &+ \int_0^1 \int_0^1 t\lambda a^{1-t} b^t c^\lambda d^{1-\lambda} \left|\frac{\partial^2}{\partial x \partial y} f\big(a^{1-t} b^t, c^\lambda d^{1-\lambda}\big)\right| \mathrm{d}\, t \,\mathrm{d}\, \lambda \\ &+ \int_0^1 \int_0^1 t\lambda a^{1-t} b^t c^{1-\lambda} d^\lambda \left|\frac{\partial^2}{\partial x \partial y} f\big(a^{1-t} b^t, c^{1-\lambda} d^\lambda\big)\right| \mathrm{d}\, t \,\mathrm{d}\, \lambda \end{aligned}$$

5

$$
\begin{aligned}
&\le \left(\int_0^1\int_0^1 t\lambda a^t b^{1-t} c^{\lambda} d^{1-\lambda}\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\right)^{1-1/q}\left[\int_0^1\int_0^1 t\lambda a^t b^{1-t} c^{\lambda} d^{1-\lambda}\left|\frac{\partial^2 f(a,c)}{\partial x\partial y}\right|^{q[t+\lambda]/4}\right.\\
&\quad\left.\times\left|\frac{\partial^2 f(a,d)}{\partial x\partial y}\right|^{q[t+(1-\lambda)]/4}\left|\frac{\partial^2 f(b,c)}{\partial x\partial y}\right|^{q[(1-t)+\lambda]/4}\left|\frac{\partial^2 f(b,d)}{\partial x\partial y}\right|^{q[(1-t)+(1-\lambda)]/4}\mathrm{d}\,t\,\mathrm{d}\,\lambda\right]^{1/q}\\
&\quad+\left(\int_0^1\int_0^1 t\lambda a^t b^{1-t} c^{1-\lambda} d^{\lambda}\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\right)^{1-1/q}\left[\int_0^1\int_0^1 t\lambda a^t b^{1-t} c^{1-\lambda} d^{\lambda}\left|\frac{\partial^2 f(a,c)}{\partial x\partial y}\right|^{q[t+(1-\lambda)]/4}\right.\\
&\quad\left.\times\left|\frac{\partial^2 f(a,d)}{\partial x\partial y}\right|^{q[t+\lambda]/4}\left|\frac{\partial^2 f(b,c)}{\partial x\partial y}\right|^{q[(1-t)+(1-\lambda)]/4}\left|\frac{\partial^2 f(b,d)}{\partial x\partial y}\right|^{q[(1-t)+\lambda]/4}\mathrm{d}\,t\,\mathrm{d}\,\lambda\right]^{1/q}\\
&\quad+\left(\int_0^1\int_0^1 t\lambda a^{1-t} b^{t} c^{\lambda} d^{1-\lambda}\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\right)^{1-1/q}\left[\int_0^1\int_0^1 t\lambda a^{1-t} b^{t} c^{\lambda} d^{1-\lambda}\left|\frac{\partial^2 f(a,c)}{\partial x\partial y}\right|^{q[(1-t)+\lambda]/4}\right.\\
&\quad\left.\times\left|\frac{\partial^2 f(a,d)}{\partial x\partial y}\right|^{q[(1-t)+(1-\lambda)]/4}\left|\frac{\partial^2 f(b,c)}{\partial x\partial y}\right|^{q[t+\lambda]/4}\left|\frac{\partial^2 f(b,d)}{\partial x\partial y}\right|^{q[t+(1-\lambda)]/4}\mathrm{d}\,t\,\mathrm{d}\,\lambda\right]^{1/q}\\
&\quad+\left(\int_0^1\int_0^1 t\lambda a^{1-t} b^{t} c^{1-\lambda} d^{\lambda}\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\right)^{1-1/q}\left[\int_0^1\int_0^1 t\lambda a^{1-t} b^{t} c^{1-\lambda} d^{\lambda}\left|\frac{\partial^2 f(a,c)}{\partial x\partial y}\right|^{q[(1-t)+(1-\lambda)]/4}\right.\\
&\quad\left.\times\left|\frac{\partial^2 f(a,d)}{\partial x\partial y}\right|^{q[(1-t)+\lambda]/4}\left|\frac{\partial^2 f(b,c)}{\partial x\partial y}\right|^{q[t+(1-\lambda)]/4}\left|\frac{\partial^2 f(b,d)}{\partial x\partial y}\right|^{q[t+\lambda]/4}\mathrm{d}\,t\,\mathrm{d}\,\lambda\right]^{1/q}.
\end{aligned}
\tag{3.3}
$$

Also by Lemma 2.2, we have

$$
\int_0^1\int_0^1 t\lambda a^t b^{1-t} c^{\lambda} d^{1-\lambda}\,\mathrm{d}\,t\,\mathrm{d}\,\lambda = F(a,b)F(c,d) \tag{3.4}
$$

and

$$
\begin{aligned}
&\int_0^1\int_0^1 t\lambda a^t b^{1-t} c^{\lambda} d^{1-\lambda}\left|\frac{\partial^2 f(a,c)}{\partial x\partial y}\right|^{q[t+\lambda]/4}\left|\frac{\partial^2 f(a,d)}{\partial x\partial y}\right|^{q[t+(1-\lambda)]/4}\\
&\quad\times\left|\frac{\partial^2 f(b,c)}{\partial x\partial y}\right|^{q[(1-t)+\lambda]/4}\left|\frac{\partial^2 f(b,d)}{\partial x\partial y}\right|^{q[(1-t)+(1-\lambda)]/4}\mathrm{d}\,t\,\mathrm{d}\,\lambda\\
&=\int_0^1\int_0^1 t\lambda[M_q(a,a)]^t[M_q(b,b)]^{1-t}[N_q(c,c)]^{\lambda}[N_q(d,d)]^{1-\lambda}\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\\
&=F\big(M_q(a,a),M_q(b,b)\big)F\big(N_q(c,c),N_q(d,d)\big).
\end{aligned}
$$

By simple computation,

$$
\begin{aligned}
&\int_0^1\int_0^1 t\lambda a^t b^{1-t} c^{1-\lambda} d^{\lambda}\,\mathrm{d}\,t\,\mathrm{d}\,\lambda = F(a,b)F(d,c),\quad \int_0^1\int_0^1 t\lambda a^{1-t} b^{t} c^{\lambda} d^{1-\lambda}\,\mathrm{d}\,t\,\mathrm{d}\,\lambda = F(b,a)F(c,d),\\
&\int_0^1\int_0^1 t\lambda a^{1-t} b^{t} c^{1-\lambda} d^{\lambda}\,\mathrm{d}\,t\,\mathrm{d}\,\lambda = F(b,a)F(d,c),\\
&\int_0^1\int_0^1 t\lambda a^t b^{1-t} c^{1-\lambda} d^{\lambda}\left|\frac{\partial^2 f(a,c)}{\partial x\partial y}\right|^{q[t+(1-\lambda)]/4}\left|\frac{\partial^2 f(a,d)}{\partial x\partial y}\right|^{q[t+\lambda]/4}\left|\frac{\partial^2 f(b,c)}{\partial x\partial y}\right|^{q[(1-t)+(1-\lambda)]/4}
\end{aligned}
$$

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$$\times\left|\frac{\partial^2 f(b,d)}{\partial x\partial y}\right|^{q[(1-t)+\lambda]/4}\mathrm{d}\,t\,\mathrm{d}\,\lambda=F\big(M_q(a,a),M_q(b,b)\big)F\big(N_q(d,d),N_q(c,c)\big),$$

$$\int_0^1\int_0^1 t\lambda a^{1-t}b^t c^{\lambda}d^{1-\lambda}\left|\frac{\partial^2 f(a,c)}{\partial x\partial y}\right|^{q[(1-t)+\lambda]/4}\left|\frac{\partial^2 f(a,d)}{\partial x\partial y}\right|^{q[(1-t)+(1-\lambda)]/4}\left|\frac{\partial^2 f(b,c)}{\partial x\partial y}\right|^{q[t+\lambda]/4}$$
$$\times\left|\frac{\partial^2 f(b,d)}{\partial x\partial y}\right|^{q[t+(1-\lambda)]/4}\mathrm{d}\,t\,\mathrm{d}\,\lambda=F\big(M_q(b,b),M_q(a,a)\big)F\big(N_q(c,c),N_q(d,d)\big)$$

and

$$\int_0^1\int_0^1 t\lambda a^{1-t}b^t c^{1-\lambda}d^{\lambda}\left|\frac{\partial^2 f(a,c)}{\partial x\partial y}\right|^{q[(1-t)+(1-\lambda)]/4}\left|\frac{\partial^2 f(a,d)}{\partial x\partial y}\right|^{q[(1-t)+\lambda]/4}\left|\frac{\partial^2 f(b,c)}{\partial x\partial y}\right|^{q[t+(1-\lambda)]/4}$$
$$\times\left|\frac{\partial^2 f(b,d)}{\partial x\partial y}\right|^{q[t+\lambda]/4}\mathrm{d}\,t\,\mathrm{d}\,\lambda=F\big(M_q(b,b),M_q(a,a)\big)F\big(N_q(d,d),N_q(c,c)\big).\quad(3.5)$$

Substituting equalities (3.4) to (3.5) into the inequality (3.3) and rearranging yield the inequality (3.1). Theorem 3.1 is proved. □

Corollary 3.1.1. *Under the conditions of Theorem 3.1, when $q=1$, we have*

$$\begin{aligned}|S(f)|\le{}& F(M_1(a,a),M_1(b,b))F(N_1(c,c),N_1(d,d))+F(M_1(a,a),M_1(b,b))F(N_1(d,d),N_1(c,c))\\&+F\big(M_1(b,b),M_1(a,a)\big)F(N_1(c,c),N_1(d,d))+F\big(M_1(b,b),M_1(a,a)\big)F(N_1(d,d),N_1(c,c)),\end{aligned}$$

where $F(u,v)$ is defined by (2.1), and $M_q(u^r,u)$ and $N_q(v^r,v)$ are defined by (3.2).

Theorem 3.2. *Let $f:\Delta=[a,b]\times[c,d]\subseteq\mathbb{R}_+^2\to\mathbb{R}$ be a partial differentiable function on Δ with $a<b$, $c<d$ and $\frac{\partial^2 f}{\partial x\partial y}\in L_1(\Delta)$. If $\left|\frac{\partial^2 f}{\partial x\partial y}\right|^q$ is geometrically mean convex functions on the co-ordinates on Δ for $q>1$ and $q\ge r\ge 0$, then*

$$\begin{aligned}|S(f)|\le{}&\big[F(a^{(q-r)/(q-1)},b^{(q-r)/(q-1)})F(c^{(q-r)/(q-1)},d^{(q-r)/(q-1)})\big]^{1-1/q}\\&\times\big[F\big(M_q(a^r,a),M_q(b^r,b)\big)F\big(N_q(c^r,c),N_q(d^r,d)\big)\big]^{1/q}\\&+\big[F(a^{(q-r)/(q-1)},b^{(q-r)/(q-1)})F(d^{(q-r)/(q-1)},c^{(q-r)/(q-1)})\big]^{1-1/q}\\&\times\big[F\big(M_q(a^r,a),M_q(b^r,b)\big)F\big(N_q(d^r,d),N_q(c^r,c)\big)\big]^{1/q}\\&+\big[F(b^{(q-r)/(q-1)},a^{(q-r)/(q-1)})F(c^{(q-r)/(q-1)},d^{(q-r)/(q-1)})\big]^{1-1/q}\\&\times\big[F\big(M_q(b^r,b),M_q(a^r,a)\big)F\big(N_q(c^r,c),N_q(d^r,d)\big)\big]^{1/q}\\&+\big[F(b^{(q-r)/(q-1)},a^{(q-r)/(q-1)})F(d^{(q-r)/(q-1)},c^{(q-r)/(q-1)})\big]^{1-1/q}\\&\times\big[F\big(M_q(b^r,b),M_q(a^r,a)\big)F\big(N_q(d^r,d),N_q(c^r,c)\big)\big]^{1/q},\end{aligned}$$

where $F(u,v)$ is defined by (2.1), and $M_q(u^r,u)$ and $N_q(v^r,v)$ are defined by (3.2).

Proof. From Lemma 2.1, we have

$$
\begin{aligned}
|S(f)| \le & \int_0^1\int_0^1 t\lambda a^t b^{1-t} c^\lambda d^{1-\lambda}\left|\frac{\partial^2}{\partial x\partial y} f\big(a^t b^{1-t}, c^\lambda d^{1-\lambda}\big)\right| \mathrm{d}\,t\,\mathrm{d}\,\lambda \\
& + \int_0^1\int_0^1 t\lambda a^t b^{1-t} c^{1-\lambda} d^{\lambda}\left|\frac{\partial^2}{\partial x\partial y} f\big(a^t b^{1-t}, c^{1-\lambda} d^{\lambda}\big)\right| \mathrm{d}\,t\,\mathrm{d}\,\lambda \\
& + \int_0^1\int_0^1 t\lambda a^{1-t} b^{t} c^\lambda d^{1-\lambda}\left|\frac{\partial^2}{\partial x\partial y} f\big(a^{1-t} b^{t}, c^\lambda d^{1-\lambda}\big)\right| \mathrm{d}\,t\,\mathrm{d}\,\lambda \\
& + \int_0^1\int_0^1 t\lambda a^{1-t} b^{t} c^{1-\lambda} d^{\lambda}\left|\frac{\partial^2}{\partial x\partial y} f\big(a^{1-t} b^{t}, c^{1-\lambda} d^{\lambda}\big)\right| \mathrm{d}\,t\,\mathrm{d}\,\lambda.
\end{aligned} \tag{3.6}
$$

Using Hölder's integral inequality, and by the geometrically mean convexity of $\left|\frac{\partial^2 f}{\partial x\partial y}\right|^q$ on Δ and Lemma 2.2, it is easy to observe that

$$
\begin{aligned}
& \int_0^1\int_0^1 t\lambda a^t b^{1-t} c^\lambda d^{1-\lambda}\left|\frac{\partial^2}{\partial x\partial y} f\big(a^t b^{1-t}, c^\lambda d^{1-\lambda}\big)\right| \mathrm{d}\,t\,\mathrm{d}\,\lambda \\
\le & \left(\int_0^1\int_0^1 t\lambda\big(a^t b^{1-t} c^\lambda d^{1-\lambda}\big)^{(q-r)/(q-1)}\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\right)^{1-1/q} \\
& \times\left[\int_0^1\int_0^1 t\lambda a^{rt} b^{r(1-t)} c^{r\lambda} d^{r(1-\lambda)}\left|\frac{\partial^2 f(a,c)}{\partial x\partial y}\right|^{q[t+\lambda]/4}\left|\frac{\partial^2 f(a,d)}{\partial x\partial y}\right|^{q[t+(1-\lambda)]/4}\right. \\
& \times\left.\left|\frac{\partial^2 f(b,c)}{\partial x\partial y}\right|^{q[(1-t)+\lambda]/4}\left|\frac{\partial^2 f(b,d)}{\partial x\partial y}\right|^{q[(1-t)+(1-\lambda)]/4}\mathrm{d}\,t\,\mathrm{d}\,\lambda\right]^{1/q} \\
= & \left[F\big(a^{(q-r)/(q-1)}, b^{(q-r)/(q-1)}\big)F\big(c^{(q-r)/(q-1)}, d^{(q-r)/(q-1)}\big)\right]^{1-1/q} \\
& \times\left[F\big(M_q(a^r,a), M_q(b^r,b)\big)F\big(N_q(c^r,c), N_q(d^r,d)\big)\right]^{1/q}.
\end{aligned} \tag{3.7}
$$

Similarly, we can show that

$$
\begin{aligned}
& \int_0^1\int_0^1 t\lambda a^t b^{1-t} c^{1-\lambda} d^{\lambda}\left|\frac{\partial^2}{\partial x\partial y} f\big(a^t b^{1-t}, c^{1-\lambda} d^{\lambda}\big)\right| \mathrm{d}\,t\,\mathrm{d}\,\lambda \\
\le & \left[F\big(a^{(q-r)/(q-1)}, b^{(q-r)/(q-1)}\big)F\big(d^{(q-r)/(q-1)}, c^{(q-r)/(q-1)}\big)\right]^{1-1/q} \\
& \times\left[F\big(M_q(a^r,a), M_q(b^r,b)\big)F\big(N_q(d^r,d), N_q(c^r,c)\big)\right]^{1/q}, \\
& \int_0^1\int_0^1 t\lambda a^{1-t} b^{t} c^\lambda d^{1-\lambda}\left|\frac{\partial^2}{\partial x\partial y} f\big(a^{1-t} b^{t}, c^\lambda d^{1-\lambda}\big)\right| \mathrm{d}\,t\,\mathrm{d}\,\lambda \\
= & \left[F\big(b^{(q-r)/(q-1)}, a^{(q-r)/(q-1)}\big)F\big(c^{(q-r)/(q-1)}, d^{(q-r)/(q-1)}\big)\right]^{1-1/q} \\
& \times\left[F\big(M_q(b^r,b), M_q(a^r,a)\big)F\big(N_q(c^r,c), N_q(d^r,d)\big)\right]^{1/q},
\end{aligned}
$$

and

$$
\begin{aligned}
& \int_0^1\int_0^1 t\lambda a^{1-t} b^{t} c^{1-\lambda} d^{\lambda}\left|\frac{\partial^2}{\partial x\partial y} f\big(a^{1-t} b^{t}, c^{1-\lambda} d^{\lambda}\big)\right| \mathrm{d}\,t\,\mathrm{d}\,\lambda \\
\le & \left[F\big(b^{(q-r)/(q-1)}, a^{(q-r)/(q-1)}\big)F\big(d^{(q-r)/(q-1)}, c^{(q-r)/(q-1)}\big)\right]^{1-1/q} \\
& \times\left[F\big(M_q(b^r,b), M_q(a^r,a)\big)F\big(N_q(d^r,d), N_q(c^r,c)\big)\right]^{1/q}.
\end{aligned} \tag{3.8}
$$

Using the inequalities (3.7) to (3.8) in the inequality (3.6), we conclude the required inequality. The proof is completed. □

Corollary 3.2.1. *Under the conditions of Theorem 3.2,*

1. *when $r = 0$, we deduce*

$$\begin{aligned}|S(f)| \le{}& \big[F(a^{q/(q-1)}, b^{q/(q-1)})F(c^{q/(q-1)}, d^{q/(q-1)})\big]^{1-1/q} \\ &\times \big[F\big(M_q(1,a), M_q(1,b)\big)F\big(N_q(1,c), N_q(1,d)\big)\big]^{1/q} \\ &+ \big[F(a^{q/(q-1)}, b^{q/(q-1)})F(d^{q/(q-1)}, c^{q/(q-1)})\big]^{1-1/q} \\ &\times \big[F\big(M_q(1,a), M_q(1,b)\big)F\big(N_q(1,d), N_q(1,c)\big)\big]^{1/q} \\ &+ \big[F(b^{q/(q-1)}, a^{q/(q-1)})F(c^{q/(q-1)}, d^{q/(q-1)})\big]^{1-1/q} \\ &\times \big[F\big(M_q(1,b), M_q(1,a)\big)F\big(N_q(1,c), N_q(1,d)\big)\big]^{1/q} \\ &+ \big[F(b^{q/(q-1)}, a^{q/(q-1)})F(d^{q/(q-1)}, c^{q/(q-1)})\big]^{1-1/q} \\ &\times \big[F\big(M_q(1,b), M_q(1,a)\big)F\big(N_q(1,d), N_q(1,c)\big)\big]^{1/q};\end{aligned}$$

2. *when $r = q$, we have*

$$\begin{aligned}|S(f)| \le{}& \left(\frac{1}{4}\right)^{1-1/q}\Big\{\big[F\big(M_q(a^q,a), M_q(b^q,b)\big)F\big(N_q(c^q,c), N_q(d^q,d)\big)\big]^{1/q} \\ &+ \big[F\big(M_q(a^q,a), M_q(b^q,b)\big)F\big(N_q(d^q,d), N_q(c^q,c)\big)\big]^{1/q} \\ &+ \big[F\big(M_q(b^q,b), M_q(a^q,a)\big)F\big(N_q(c^q,c), N_q(d^q,d)\big)\big]^{1/q} \\ &+ \big[F\big(M_q(b^q,b), M_q(a^q,a)\big)F\big(N_q(d^q,d), N_q(c^q,c)\big)\big]^{1/q}\Big\},\end{aligned}$$

where $F(u, v)$ is defined by (2.1), and $M_q(u^r, u)$ and $N_q(v^r, v)$ are defined by (3.2).

Proof. This follows from letting $r = 0$ and $r = q$ respectively in Theorem 3.2. □

Theorem 3.3. *Let $f : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}_+^2 \to \mathbb{R}_+$ be integrable on Δ with $a < b$, $c < d$. If f is geometrically mean convex on Δ, then*

$$\begin{aligned}f\big(\sqrt{ab}, \sqrt{cd}\big) &\le \frac{1}{(\ln b - \ln a)(\ln d - \ln c)} \int_c^d \int_a^b \frac{\big[f(x,y)f\big(x, \frac{cd}{y}\big)f\big(\frac{ab}{x}, y\big)f\big(\frac{ab}{x}, \frac{cd}{y}\big)\big]^{1/4}}{xy} \,\mathrm{d}\,x\,\mathrm{d}\,y \\ &\le \big[f(a,c)f(a,d)f(b,c)f(b,d)\big]^{1/4}.\end{aligned}$$

Proof. Taking $x = a^t b^{1-t}$ and $y = c^\lambda d^{1-\lambda}$ for $0 \le t, \lambda \le 1$ and using the geometrically mean convexity of f, we have

$$f\big(\sqrt{ab}, \sqrt{cd}\big) = \int_0^1 \int_0^1 f\big([a^t b^{1-t}]^{1/2}[a^{1-t}b^t]^{1/2}, [c^\lambda d^{1-\lambda}]^{1/2}[c^{1-\lambda}d^\lambda]^{1/2}\big) \,\mathrm{d}\,t\,\mathrm{d}\,\lambda$$

$$
\begin{aligned}
&\le \int_0^1 \int_0^1 \Big[f\big(a^t b^{1-t}, c^\lambda d^{1-\lambda}\big) f\big(a^t b^{1-t}, c^{1-\lambda} d^\lambda\big) f\big(a^{1-t} b^t, c^\lambda d^{1-\lambda}\big) f\big(a^{1-t} b^t, c^{1-\lambda} d^\lambda\big)\Big]^{1/4} \,\mathrm{d}\,t \,\mathrm{d}\,\lambda \\
&\qquad = \frac{1}{(\ln b - \ln a)(\ln d - \ln c)} \int_c^d \int_a^b \frac{\big[f(x,y) f\big(x, \frac{cd}{y}\big) f\big(\frac{ab}{x}, y\big) f\big(\frac{ab}{x}, \frac{cd}{y}\big)\big]^{1/4}}{xy} \,\mathrm{d}\,x \,\mathrm{d}\,y
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^1 \int_0^1 \big[f\big(a^t b^{1-t}, c^\lambda d^{1-\lambda}\big) f\big(a^t b^{1-t}, c^{1-\lambda} d^\lambda\big) f\big(a^{1-t} b^t, c^\lambda d^{1-\lambda}\big) f\big(a^{1-t} b^t, c^{1-\lambda} d^\lambda\big)\big]^{1/4} \,\mathrm{d}\,t \,\mathrm{d}\,\lambda \\
&\le \int_0^1 \int_0^1 \big\{[f(a,c)]^{t+\lambda} [f(a,d)]^{t+(1-\lambda)} [f(b,c)]^{(1-t)+\lambda} [f(b,d)]^{(1-t)+(1-\lambda)} \\
&\quad \times [f(a,d)]^{t+\lambda} [f(a,c)]^{t+(1-\lambda)} [f(b,d)]^{(1-t)+\lambda} [f(b,c)]^{(1-t)+(1-\lambda)} \\
&\quad \times [f(b,c)]^{t+\lambda} [f(b,d)]^{t+(1-\lambda)} [f(a,c)]^{(1-t)+\lambda} [f(a,d)]^{(1-t)+(1-\lambda)} \\
&\quad \times [f(b,d)]^{t+\lambda} [f(b,c)]^{t+(1-\lambda)} [f(a,d)]^{(1-t)+\lambda} [f(a,c)]^{(1-t)+(1-\lambda)}\big\}^{1/16} \,\mathrm{d}\,t \,\mathrm{d}\,\lambda \\
&= \int_0^1 \int_0^1 \big[f(a,c) f(a,d) f(b,c) f(b,d)\big]^{1/4} \,\mathrm{d}\,t \,\mathrm{d}\,\lambda = \big[f(a,c) f(a,d) f(b,c) f(b,d)\big]^{1/4}.
\end{aligned}
$$

The proof of Theorem 3.3 is complete. □

Theorem 3.4. *Let $f : \Delta = [a,b] \times [c,d] \subseteq \mathbb{R}_+^2 \to \mathbb{R}_+$ be integrable on Δ with $a < b$, $c < d$. If f is geometrically mean convex on Δ, then*

$$
\begin{aligned}
&\frac{1}{(\ln b - \ln a)(\ln d - \ln c)} \int_c^d \int_a^b \frac{f(x,y)}{xy} \,\mathrm{d}\,x \,\mathrm{d}\,y \\
&\qquad \le L\big([f(a,c) f(a,d)]^{1/4}, [f(b,c) f(b,d)]^{1/4}\big) L\big([f(a,c) f(b,c)]^{1/4}, [f(a,d) f(b,d)]^{1/4}\big),
\end{aligned}
$$

where $L(u,v)$ is the logarithmic mean.

Proof. Putting $x = a^t b^{1-t}$ and $y = c^\lambda d^{1-\lambda}$ for $0 \le t, \lambda \le 1$, from the geometrically mean convexity of f, we obtain

$$
\begin{aligned}
&\frac{1}{(\ln b - \ln a)(\ln d - \ln c)} \int_c^d \int_a^b \frac{f(x,y)}{xy} \,\mathrm{d}\,x \,\mathrm{d}\,y = \int_0^1 \int_0^1 f\big(a^t b^{1-t}, c^\lambda d^{1-\lambda}\big) \,\mathrm{d}\,t \,\mathrm{d}\,\lambda \\
&\qquad \le \int_0^1 \int_0^1 \big\{[f(a,c)]^{t+\lambda} [f(a,d)]^{t+(1-\lambda)} [f(b,c)]^{(1-t)+\lambda} [f(b,d)]^{(1-t)+(1-\lambda)}\big\}^{1/4} \,\mathrm{d}\,t \,\mathrm{d}\,\lambda \\
&\qquad\quad = L\big([f(a,c) f(a,d)]^{1/4}, [f(b,c) f(b,d)]^{1/4}\big) L\big([f(a,c) f(b,c)]^{1/4}, [f(a,d) f(b,d)]^{1/4}\big).
\end{aligned}
$$

The proof of Theorem 3.4 is complete. □

Theorem 3.5. *Let $f : \Delta = [a,b] \times [c,d] \subseteq \mathbb{R}_+^2 \to \mathbb{R}_+$ be integrable on Δ with $a < b$, $c < d$. If f is co-ordinated geometrically mean convex on Δ, then*

$$
\begin{aligned}
&\frac{1}{(\ln b - \ln a)(\ln d - \ln c)} \int_c^d \int_a^b f(x,y) \,\mathrm{d}\,x \,\mathrm{d}\,y \\
&\qquad \le L\big(a[f(a,c) f(a,d)]^{1/4}, b[f(b,c) f(b,d)]^{1/4}\big) L\big(c[f(a,c) f(b,c)]^{1/4}, d[f(a,d) f(b,d)]^{1/4}\big),
\end{aligned}
$$

where $L(u,v)$ is the logarithmic mean.

Proof. Similar to the proof of Theorem 3.4, by the geometrically mean convexity of f, we drive

$$\begin{aligned}\frac{1}{(\ln b-\ln a)(\ln d-\ln c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y &= \int_0^1\int_0^1 a^t b^{1-t}c^\lambda d^{1-\lambda} f\big(a^t b^{1-t}, c^\lambda d^{1-\lambda}\big)\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\\
&\le \int_0^1\int_0^1 a^t b^{1-t}c^\lambda d^{1-\lambda}\big\{[f(a,c)]^{t+\lambda}[f(a,d)]^{t+(1-\lambda)}[f(b,c)]^{(1-t)+\lambda}[f(b,d)]^{(1-t)+(1-\lambda)}\big\}^{1/4}\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\\
&= L\big(a[f(a,c)f(a,d)]^{1/4}, b[f(b,c)f(b,d)]^{1/4}\big)L\big(c[f(a,c)f(b,c)]^{1/4}, d[f(a,d)f(b,d)]^{1/4}\big).\end{aligned}$$

The proof of Theorem 3.5 is complete. □

We proceed similarly as in the proof of Theorem 3.3 to Theorem 3.5, we can get

Theorem 3.6. *Let $f,\ g:\Delta=[a,b]\times[c,d]\subseteq\mathbb{R}_+^2\to\mathbb{R}_+$ be integrable on Δ with $a<b$, $c<d$. If f and g are co-ordinated geometrically mean convex on Δ, then*

$$\begin{aligned}f\big(\sqrt{ab},\sqrt{cd}\big)g\big(\sqrt{ab},\sqrt{cd}\big) &\le \frac{1}{(\ln b-\ln a)(\ln d-\ln c)}\\
&\quad\times\int_c^d\int_a^b \frac{\big[f(x,y)g(x,y)f\big(x,\frac{cd}{y}\big)g\big(x,\frac{cd}{y}\big)f\big(\frac{ab}{x},y\big)g\big(\frac{ab}{x},y\big)f\big(\frac{ab}{x},\frac{cd}{y}\big)g\big(\frac{ab}{x},\frac{cd}{y}\big)\big]^{1/4}}{xy}\,\mathrm{d}\,x\,\mathrm{d}\,y\\
&\le \big[f(a,c)g(a,c)f(a,d)g(a,d)f(b,c)g(b,c)f(b,d)g(b,d)\big]^{1/4}.\end{aligned}$$

Theorem 3.7. *Under the conditions of Theorem 3.6, we have*

$$\begin{aligned}\frac{1}{(\ln b-\ln a)(\ln d-\ln c)}\int_c^d\int_a^b \frac{f(x,y)g(x,y)}{xy}\,\mathrm{d}\,x\,\mathrm{d}\,y &\le L\big([f(a,c)g(a,c)f(a,d)g(a,d)]^{1/4}, [f(b,c)g(b,c)f(b,d)g(b,d)]^{1/4}\big)\\
&\quad\times L\big([f(a,c)g(a,c)f(b,c)g(b,c)]^{1/4}, [f(a,d)g(a,d)f(b,d)g(b,d)]^{1/4}\big)\end{aligned}$$

and

$$\begin{aligned}\frac{1}{(\ln b-\ln a)(\ln d-\ln c)}\int_c^d\int_a^b f(x,y)g(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y &\le L\big(a[f(a,c)g(a,c)f(a,d)g(a,d)]^{1/4}, b[f(b,c)g(b,c)f(b,d)g(b,d)]^{1/4}\big)\\
&\quad\times L\big(c[f(a,c)g(a,c)f(b,c)g(b,c)]^{1/4}, d[f(a,d)g(a,d)f(b,d)g(b,d)]^{1/4}\big).\end{aligned}$$

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STABILITY OF THE GENERALAIZED VERSION OF EULER-LAGRANGE TYPE QUADRATIC EQUATION

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Abstract. In this paper, we consider the generalized Hyers-Ulam stability for the following quadratic functional equation.

$$f(ax+by)+f(ax-by)+G_f(x,y)=2a^2f(x)+2b^2f(y)$$

Here G_f is a functional operator of f. We consider some sufficient conditions on G_f which can be applied easily for the generalized Hyers-Ulam stability, and illustrate some new functional equations by using them.

1. Introduction

In 1940, Ulam proposed the following stability problem (See [17]):

"Let G_1 be a group and G_2 a metric group with the metric d. Given a constant $\delta > 0$, does there exist a constant $c > 0$ such that if a mapping $f : G_1 \longrightarrow G_2$ satisfies $d(f(xy), f(x)f(y)) < c$ for all $x, y \in G_1$, then there exists a unique homomorphism $h : G_1 \longrightarrow G_2$ with $d(f(x), h(x)) < \delta$ for all $x \in G_1$?"

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In 1941, Hyers [7] answered this problem under the assumption that the groups are Banach spaces. Aoki [1] and Rassias [13] generalized the result of Hyers. Th. M. Rassias [13] solved the generalized Hyers-Ulam stability of the functional inequality

$$\|f(x+y)-f(x)-f(y)\| \leq \epsilon(\|x\|^p+\|y\|^p)$$

for some $\epsilon \geq 0$ and p with $p<1$ and for all $x, y \in X$, where $f: X \longrightarrow Y$ is a function between Banach spaces. The paper of Rassias [13] has provided a lot of influence in the development of what we call *the generalized Hyers-Ulam stability or Hyers-Ulam-Rassias stability* of functional equations. A generalization of the Rassias' theorem was obtained by Găvruta [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

is called *a quadratic functional equation* and a solution of a quadratic functional equation is called *quadratic*. A generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [16] for mappings $f: X \longrightarrow Y$, where X is a normed space and Y is a Banach space. Cholewa [2] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [3] proved the generalized Hyers-Ulam stability for the quadratic functional equation and Park [12] proved the generalized Hyers-Ulam stability of the quadratic functional eqution in Banach modules over a C^*-algebra.

Rassias [14] investigated the following Euler-Lagrange functional equation

$$f(ax+by)+f(bx-ay)=2(a^2+b^2)[f(x)+f(y)]$$

and Gordji and Khodaei [6] investigated other Euler-Lagrange functional equations

$$\begin{aligned} f(ax+by)+f(ax-by) &= \frac{b(a+b)}{2}f(x+y) \\ +\frac{b(a+b)}{2}f(x-y) &+(2a^2-ab-b^2)f(x)+(b^2-ab)f(y) \end{aligned} \tag{1.2}$$

for fixed integers a, b with $b \neq a, -a, -3a$ and

$$f(ax+by)+f(ax-by)=2a^2f(x)+2b^2f(y) \tag{1.3}$$

for fixed integers a, b with $a^2 \neq b^2$ and $ab \neq 0$.

In this paper, we are interested in what kind of terms can be added to the quadratic functional equation

$$f(ax + by) + f(ax - by) = 2a^2 f(x) + 2b^2 f(y)$$

while the generalized Hyers-Ulam stability still holds for the new functional equation.

We denote the added term by $G_f(x, y)$ which can be regarded as a functional operator depending on the variables x, y, and function f. Then the new functional equation can be written as

$$f(ax + by) + f(ax - by) + G_f(x, y) = 2a^2 f(x) + 2b^2 f(y)$$

for some rational numbers a, b with $ab \neq 0$ and $a^2 \neq b^2$. The precise definition of G_f is given in section 2. In fact, the functional operator $G_f(x, y)$ was introduced and considered in the case of the additive functional equations with somewhat different point of view by the authors([11]).

The new observation in this article makes possible to prove many previous problems on quadratic functional equations more easily and provides methods to construct new ones. So we can have a larger class of functional equations related with quadratic functions for the generalized Hyers-Ulam stability. We illustrate some new functional equations in section 3 in order to see how our observation works for the generalized Hyers-Ulam stability.

2. Quadratic functional equations with general terms

Let X be a real normed linear space and Y a real Banach space. For given $l \in \mathbb{N}$ and any $i \in \{1, 2, \cdots, l\}$, let $\sigma_i : X \times X \longrightarrow X$ be a binary operation such that

$$\sigma_i(rx, ry) = r\sigma_i(x, y)$$

for all $x, y \in X$ and all $r \in \mathbb{R}$. It is clear that $\sigma_i(0, 0) = 0$.

Also let $F : Y^l \longrightarrow Y$ be a linear, continuous function. For a map $f : X \longrightarrow Y$, define

$$G_f(x, y) = F(f(\sigma_1(x, y)), f(\sigma_2(x, y)), \cdots, f(\sigma_l(x, y))).$$

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Here, G_f is a functional operator on the function space $\{f|f : X \longrightarrow Y\}$. In this paper, for an appropriate function $\phi : X^2 \longrightarrow [0, \infty)$, we consider the functional inequalty

$$\|f(ax+by)+f(ax-by)+G_f(x,y)-2a^2f(x)-2b^2f(y)\| \le \phi(x,y) \tag{2.1}$$

for fixed non-zero rational numbers a, b with $a^2 \neq b^2$, where the functional operator G_f satisfies

$$G_f(x,0) \equiv \lambda[f(ax)-a^2f(x)] \tag{2.2}$$

for some $\lambda(\lambda \neq -2)$. Here, $\equiv$ means that $G_f(x,0) = \lambda[f(ax)-a^2f(x)]$ holds for all $x \in X$ and all $f : X \longrightarrow Y$.

In fact, as we shall see in Theorem 2.2, for a function f with $f(0) = 0$ satisfying the equation

$$f(ax+by)+f(ax-by)+G_f(x,y) = 2a^2f(x)+2b^2f(y), \tag{2.3}$$

f is quadratic if and only if $G_f(x,0) = \lambda[f(ax)-a^2f(x)]$ and $G_f(x,y) = G_f(y,x)$. So the condition (2.2) is reasonable for the stability problem of (2.1). From now on, we assume that the functional operator G_f satisfies the condition (2.2) unless otherwise stated. We deonte

$$H_f(x,y) = f(x+y)+f(x-y)-2f(x)-2f(y).$$

The following lemma is proved in the authors' previous paper [10].

Lemma 2.1. [10] *Consider the following functional equation.*

$$f(ax+by)+f(ax-by)+cH_f(x,y) = 2a^2f(x)+2b^2f(y) \tag{2.4}$$

for fixed non-zero rational numbers a, b with $a^2 \neq b^2$ and a real number c. Then if $f : X \longrightarrow Y$ satisfies (2.4) and $f(0) = 0$, f is quadratic.

By using Lemma 2.1, we can examine the properties of a solution function of the equation (2.3).

Theorem 2.2. *Suppose the equation (2.3) holds. Then the following coditions are equivalent :*

(1) *f is quadratic.*

(2) *$G_f(x,y) = G_f(y,x)$ for all $x, y \in X$, and $f(0) = 0$*

(3) *There are non-zero rational numbers m, n, δ such that $a^2m^2 \neq b^2n^2$ and*

$$G_f(mx, ny) = \delta H_f(x, y), \ \ f(mx) = m^2 f(x), \ \ f(nx) = n^2 f(x) \tag{2.5}$$

for all $x, y \in X$.

Proof. We prove the theorem by showing (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1)

(1) $\Rightarrow$ (2) Since f is quadratic, $f(0) = 0$ and $G_f(x, y) = 0$ for all $x, y \in X$. So $G_f(x, y)$ satisfies (2) with $\lambda = 0$ in (2.2).

(2) $\Rightarrow$ (3) Putting $y = 0$ in (2.3) and by (2.2), we have

$$(2 + \lambda)[f(ax) - a^2 f(x)] = 0 \tag{2.6}$$

for all $x \in X$ and since $f(0) = 0$, $G_f(x, 0) = 2\left(a^2 f(x) - f(ax)\right) = 0$. From the condition $G_f(x, 0) = G_f(0, x)$, we have

$$f(bx) + f(-bx) = 2b^2 f(x)$$

for all $x \in X$ and so we have

$$b^2 f(x) = b^2 f(-x) \tag{2.7}$$

for all $x \in X$. Since $b \neq 0$, by (2.7), f is even and hence $f(bx) = b^2 f(x)$ for all $x \in X$. Thus (2.3) becomes

$$H_f(ax, by) + G_f(x, y) = 0,$$

and from the condition $G_f(x, y) = G_f(y, x)$ we have

$$G_f(x, y) = -H_f(ay, bx) \tag{2.8}$$

for all $x, y \in X$. Replacing x and y by ax and by respectively in (2.8), we have

$$\begin{aligned} G_f(ax, by) &= -H_f(aby, abx) \\ &= -a^2b^2 H_f(y, x) \\ &= -a^2b^2 H_f(x, y) \end{aligned}$$

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for all $x, y \in X$. The last equality comes from the fact that f is even. Note that $a^4 \neq b^4$. So we have (3) with $m = a$, $n = b$, $\delta = -a^2b^2$.

(3) $\Rightarrow$ (1) By (2.3) and (3),

$$\begin{aligned}
&\delta H_f(x, y) - G_f(mx, ny)\\
&= \delta H_f(x, y) + f(amx + bny) + f(amx - bny) - 2a^2 f(mx) - 2b^2 f(ny)\\
&= \delta H_f(x, y) + f(amx + bny) + f(amx - bny) - 2a^2m^2 f(x) - 2b^2n^2 f(y)\\
&= 0
\end{aligned}$$

for all $x, y \in X$. Since $a^2m^2 \neq b^2n^2$, by Lemma 2.1, f is quadratic. □

Now we prove the following stability theorem.

Theorem 2.3. *Let $\phi : X^2 \longrightarrow [0, \infty)$ be a function such that*

$$\sum_{n=0}^{\infty} a^{-2n}\phi(a^n x, a^n y) < \infty \tag{2.9}$$

for all $x, y \in X$. Assume that $G_f(x, y)$ satisfies one of the conditions in Thorem 2.2 when the equation (2.3) holds, and let $f : X \longrightarrow Y$ be a mapping such that

$$\|f(ax + by) + f(ax - by) + G_f(x, y) - 2a^2 f(x) - 2b^2 f(y)\| \leq \phi(x, y) \tag{2.10}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \longrightarrow Y$ such that

$$\|Q(x) - f(x) - f(0)\| \leq \frac{1}{|\lambda + 2|}\sum_{n=0}^{\infty} a^{-2(n+1)}\phi(a^n x, 0) \tag{2.11}$$

for all $x \in X$.

Proof. By the standard argument, we may assume that $f(0) = 0$.

Setting $y = 0$ in (2.10), we have

$$\|f(ax) + 2^{-1}G_f(x, 0) - a^2 f(x)\| \leq 2^{-1}\phi(x, 0)$$

for all $x \in X$ and by (2.2), we have

$$\|f(x) - a^{-2}f(ax)\| \le \frac{1}{|\lambda+2|}a^{-2}\phi(x,0) \tag{2.12}$$

for all $x \in X$. Replacing x by $a^n x$ in (2.12) and dividing (2.12) by a^{2n}, we have

$$\|a^{-2n}f(a^n x) - a^{-2(n+1)}f(a^{n+1}x)\| \le \frac{1}{|\lambda+2|}a^{-2(n+1)}\phi(a^n x,0)$$

for all $x \in X$ and all non-negative integer n. For $m, n \in \mathbb{N}\cup\{0\}$ with $0 \le m < n$,

$$\begin{aligned}
&\|a^{-2m}f(a^m x) - a^{-2n}f(a^n x)\| \\
&= a^{-2m}\|f(a^m x) - a^{-2(n-m)}f(a^{n-m}(a^m x))\| \\
&\le \frac{1}{|\lambda+2|}\sum_{k=m}^{n-1} a^{-2(k+1)}\phi(a^k x,0)
\end{aligned} \tag{2.13}$$

for all $x \in X$. By (2.13), $\{a^{-2n}f(a^n x)\}$ is a Cauchy sequence in Y and since Y is a Banach space, there exists a mapping $Q : X \longrightarrow Y$ such that

$$Q(x) = \lim_{n\longrightarrow\infty} a^{-2n}f(a^n x)$$

for all $x \in X$ and

$$\|Q(x) - f(x)\| \le \frac{1}{|\lambda+2|}\sum_{n=0}^{\infty} a^{-2(n+1)}\phi(a^n x,0)$$

for all $x \in X$. Replacing x and y by $a^n x$ and $a^n y$ respectively in (2.10) and dividing (2.10) by a^{2n}, we have

$$\begin{aligned}
&\|a^{-2n}f(a^n(ax+by)) + a^{-2n}f(a^n(ax-by)) + a^{-2n}G_f(a^n x, a^n y) \\
&- 2\cdot a^2\cdot a^{-2n}f(a^n x) - 2\cdot b^2\cdot a^{-2n}f(a^n y)\| \le a^{-2n}\phi(a^n x, a^n y)
\end{aligned}$$

for all $x, y \in X$ and letting $n \to \infty$ in the above inequality, we have

$$\begin{aligned}
&Q(ax+by) + Q(ax-by) \\
&+ \lim_{n\longrightarrow\infty} a^{-2n}G_f(a^n x, a^n y) - 2a^2Q(x) - 2b^2Q(y) = 0
\end{aligned} \tag{2.14}$$

for all $x, y \in X$. Since F is continuous, we have

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$$
\begin{aligned}
&\lim_{n\longrightarrow\infty} a^{-2n}G_f(a^n x, a^n y)\\
&= \lim_{n\longrightarrow\infty} F(a^{-2n}f(a^n\sigma_1(x,y)),\, a^{-2n}f(a^n\sigma_2(x,y)),\, \cdots,\, a^{-2n}f(a^n\sigma_l(x,y)))\\
&= F(Q(\sigma_1(x,y)),\, Q(\sigma_2(x,y)),\, \cdots,\, Q(\sigma_l(x,y)))\\
&= G_Q(x,y)
\end{aligned}
$$

for all $x, y \in X$. Hence by (2.14), we have

$$Q(ax+by) + Q(ax-by) + G_Q(x,y) = 2a^2Q(x) + 2b^2Q(y) \tag{2.15}$$

for all $x, y \in X$. Since Q satisfies (2.3), Q is quadratic by Theorem 2.2.

Now, we show the uniqueness of Q. Suppose that Q_0 is a quadratic mapping with (2.11). Then we have

$$
\begin{aligned}
&\|Q(x) - Q_0(x)\|\\
&= a^{-2k}\|Q(a^k x) - Q_0(a^k x)\|\\
&\le \frac{2}{|\lambda+2|}\sum_{n=k}^{\infty} a^{-2(n+1)}\phi(a^n x, 0)
\end{aligned}
$$

for all $x \in X$. Hence, letting $k \to \infty$ in the above inequality, we have

$$Q(x) = Q_0(x)$$

for all $x \in X$. □

Theorem 2.4. *Assume that G_f satisfies all of the conditions in Theorem 2.3. Let $\phi : X^2 \longrightarrow [0,\infty)$ be a function such that*

$$\sum_{n=0}^{\infty} a^{2n}\phi(a^{-n}x, a^{-n}y) < \infty$$

for all $x, y \in X$. Let $f : X \longrightarrow Y$ be a mapping such that

$$\|f(ax+by) + f(ax-by) + G_f(x,y) - 2a^2f(x) - 2b^2f(y)\| \le \phi(x,y)$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \longrightarrow Y$ such that

$$\|Q(x) - f(x) - f(0)\| \le \frac{1}{|\lambda+2|}\sum_{n=0}^{\infty} a^{2(n+1)}\phi(a^{-n}x, 0)$$

for all $x \in X$.

As examples of $\phi(x, y)$ in Theorem 2.3 and Theorem 2.4, we can take $\phi(x, y) = \epsilon(\|x\|^p\|y\|^p + \|x\|^{2p} + \|y\|^{2p})$ which is appeared in [11]. Then we can formulate the following corollary

Corollary 2.5. *Assume that all of the conditions in Theorem 2.3 are satisfied. Let* p *be a real number with* $p \neq 1$. *Let* $f : X \longrightarrow Y$ *be a mapping such that*

$$\begin{aligned} &\|f(ax + by) + f(ax - by) - 2a^2 f(x) - 2b^2 f(y) + G_f(x, y)\| \\ \le\ &\epsilon(\|x\|^p\|y\|^p + \|x\|^{2p} + \|x\|^{2p}) \end{aligned}$$

for fixed non-zero rational numbers a, b *with* $a^2 \neq b^2$, *a fixed positive real number* ϵ, *and all* $x, y \in X$. *Then there exists a unique quadratic mapping* $Q : X \longrightarrow Y$ *such that*

$$\|Q(x) - f(x) - f(0)\| \le \frac{\epsilon\|x\|^{2p}}{a^2|\lambda + 2|[1 - a^{2(p-1)}]}$$
$$(p < 1 \ \ and \ \ |a| > 1, \ \ or \ \ p > 1 \ \ and \ \ |a| < 1)$$

and

$$\|Q(x) - f(x) - f(0)\| \le \frac{a^2\epsilon\|x\|^{2p}}{|\lambda + 2|[1 - a^{2(1-p)}]}$$
$$(p > 1 \ \ and \ \ |a| > 1, \ \ or \ \ p < 1 \ \ and \ \ |a| < 1)$$

for all $x \in X$.

3. Applications

In this section we illustrate how the theorems in section 2 work well for the generalized Hyers-Ulam stability of various quadratic functional equations. By applying the results in this article, we can construct many concrete members in our calss of functional equations easily.

First, we consider the following functional equation related with Theorem 2.3.

$$\begin{aligned} &f(ax + by) + f(ax - by) + f(x + y) + f(x - y) \\ &+ f(y - x) - f(-x) - f(-y) = 2(a^2 + 1)f(x) + 2(b^2 + 1)f(y) \end{aligned} \tag{3.1}$$

for fixed non-zero rational numbers a, b with $a^2 \neq b^2$.

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Using Theorem 2.3, we can prove the stability for (3.1).

Theorem 3.1. *Let $\phi : X^2 \longrightarrow [0,\infty)$ be a function with (2.9) and $f : X \longrightarrow Y$ a mapping such that*

$$\begin{aligned} &\|f(ax+by)+f(ax-by)+f(x+y)+f(x-y)+f(y-x) \\ &-f(-x)-f(-y)-2(a^2+1)f(x)-2(b^2+1)f(y)\| \le \phi(x,y). \end{aligned} \tag{3.2}$$

for fixed non-zero rational numbers a, b with $a^2 \neq b^2$, and all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \longrightarrow Y$ such that Q satisfies (3.1) and

$$\|Q(x)-f(x)-f(0)\| \le \frac{1}{2}\sum_{n=0}^{\infty} a^{-2(n+1)}\phi(a^n x, 0) \tag{3.3}$$

for all $x \in X$.

Proof. In this case, $G_f(x,y) = f(x+y)+f(x-y)+f(y-x)-f(-x)-f(-y)-2f(x)-2f(y)$. So $G_f(x,0) = 0$. Hence G_f and f satisfies all the conditions in Theorem 2.3. and the functional inequality (3.2) can be rewritten as the functional inequality

$$\|f(ax+by)+f(ax-by)+G_f(x,y)-2a^2f(x)-2b^2f(y)\| \le \phi(x,y).$$

By Theorem 2.3, we get the result. □

When the equation (2.3) holds, $G_f(x,y)$ can be represented as different forms. In some cases, these forms together help us to analyze a solution. Especially the following case happens often in some interesting equations. We will give an example later.

Lemma 3.2. *Suppose when the equation (2.3) with $a^2 \neq b^4$ holds, $G_f(x,y)$ can be represented as both of the followings.*

$$G_f(0,y) = k[f(y)-f(-y)] \tag{3.4}$$

$$G_f(x,by) = k_1 H_f(x,y) \tag{3.5}$$

for all $x, y \in X$ and a fixed positive real number k with $k \neq b^2$, and a fixed real number k_1.

Then G_f satisfies the condition (3) in Theorem 2.2.

Proof. Suppose that (2.3) holds. Then we have

$$G_f(0, y) - G_f(0, -y) = 2b^2[f(y) - f(-y)]$$

for all $y \in X$. Also by (3.4), we have

$$G_f(0, y) - G_f(0, -y) = 2k[f(y) - f(-y)]$$

Since $k \neq b^2$, f is even and so $f(bx) = b^2 f(x)$. By (3.5) and $a^2 \neq b^4$, we get the result with $m = 1$, $n = b$, and $k_1 = \delta$. □

Note that Lemma 3.2 is still valid if we does not impose the condition (2.2) on G_f. By Lemma 3.2 and Theorem 2.3, we can formulate the following proposition.

Proposition 3.3. *Let ϕ be a function in Theorem 2.2 and suppose that $G_f(x, y)$ satisfies the condition in Lemma 3.2 when the equation (2.3) holds. Then there exists a unique quadratic mapping $Q : X \longrightarrow Y$ such that*

$$\|Q(x) - f(x) - f(0)\| \leq \frac{1}{|\lambda + 2|} \sum_{n=0}^{\infty} a^{-2(n+1)} \phi(a^n x, 0)$$

for all $x \in X$.

Now, we consider the following functional equation related with Proposition 3.3.

$$\begin{aligned} & f(ax + by) + f(ax - by) - f(bx + y) - f(bx - y) + 2f(bx) \\ = \; & 2a^2 f(x) + 2(b^2 - 1) f(y) \end{aligned} \tag{3.6}$$

for fixed non-zero rational numbers a, b with $a^2 \neq b^2$ and $a^2 \neq b^4$.

Theorem 3.4. *Let $\phi : X^2 \longrightarrow [0, \infty)$ be a function such that*

$$\sum_{n=0}^{\infty} a^{-2n} \phi((a^n x, a^n y) < \infty$$

for all $x, y \in X$. Let $f : X \longrightarrow Y$ be a mapping such that

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$$\begin{aligned}&\|f(ax+by)+f(ax-by)-f(bx+y)-f(bx-y)\\&+2f(bx)-2a^2f(x)-2(b^2-1)f(y)\|\leq \phi(x,y).\end{aligned} \tag{3.7}$$

Then there exists a unique quadratic mapping $Q : X \longrightarrow Y$ such that Q satisfies (3.6) and

$$\|Q(x)-f(x)-f(0)\| \leq \frac{1}{2}\sum_{n=0}^{\infty} a^{-2(n+1)}\phi(a^n x, 0)$$

for all $x \in X$.

Proof. It is enough to show with the condition $f(0) = 0$. In this case, $G_f(x, y) = -f(bx + y) - f(bx - y) + 2f(bx) + 2f(y)$. First we can check $G_f(x, 0) = 0$ as a functional operator. Now suppose that f satisfies (3.6). Then $G_f(0, y) = f(y) - f(-y)$ for all $y \in X$ and hence $f(by) = b^2 f(y)$. So $G_f(x, by) = -b^2 H_f(x, y)$. Since all the conditions in Proposition 3.3 are satisfied, we have the result. □

Similar to Proposition 3.3, we have the following proposition :

Proposition 3.5. *Suppose that $f(0) = 0$ and $G_f(x, y)$ satisfies*

$$G_f(ax, y) = k_2 H_f(x, y) \tag{3.8}$$

for all $x, y \in X$ and a fixed real number k_2 when the equation (2.3) holds. Let ϕ be a function in Theorem 2.3. Then there exists a unique quadratic mapping $Q : X \longrightarrow Y$ such that

$$\|Q(x)-f(x)-f(0)\| \leq \frac{1}{|\lambda+2|}\sum_{n=0}^{\infty} a^{-2(n+1)}\phi(a^n x, 0)$$

for all $x \in X$.

Finally, we consider the following functional equation related with Proposition 3.5.

$$2f(2x+y)+f(2x-y)+f(x-2y) = 13f(x)+6f(y)+f(-y) \tag{3.9}$$

for all $x, y \in X$.

Theorem 3.6. *Let $\phi : X^2 \longrightarrow [0,\infty)$ be a function such that*

$$\sum_{n=0}^{\infty} a^{-2n}\phi(a^n x, a^n y) < \infty$$

for all $x, y \in X$. Let $f : X \longrightarrow Y$ be a mapping such that

$$\begin{aligned}&\|2f(2x+y)+f(2x-y)+f(x-2y)-13f(x)-6f(y)-(-y)\| \\ &\leq \phi(x,y)\end{aligned} \tag{3.10}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \longrightarrow Y$ such that Q satisfies (3.6) and

$$\|Q(x) - f(x) - f(0)\| \leq \frac{1}{3}\sum_{n=0}^{\infty} a^{-2(n+1)}\phi(a^n x, 0)$$

for all $x \in X$.

Proof. In this case $G_f(x,y) = f(2x+y)+f(x-2y)-5f(x)-4f(y)-f(-y)$, so $G_f(x,0) = f(2x)-4f(x)$ under the condition $f(0)=0$. Now suppose f satisfies (3.9). Since $G_f(0,y) = 3[f(-y)-f(y)]$ for all $y \in X$, by the argument in Lemma 3.2, f is even. Aso, the functional equation (3.9) implies

$$f(2x+y)+f(2x-y)+\bar{G}_f(x,y) = 8f(x)+2f(y), \tag{3.11}$$

where $\bar{G}_f(x,y) = \frac{1}{3}f(x+2y)+\frac{1}{3}f(x-2y)-\frac{2}{3}f(x)-\frac{1}{3}f(y)-\frac{7}{3}f(-y)$.

Since f is even, we have $\bar{G}_f(2x,y) = \frac{4}{3}H_f(x,y)$ for all $x, y \in X$. By (3.10), we have

$$\begin{aligned}&\|f(2x+y)+f(2x-y)+\bar{G}_f(x,y)-8f(x)-2f(y)\| \\ &\leq \frac{1}{3}[\phi(x,y)+\phi(x,-y)]\end{aligned} \tag{3.12}$$

for all $x, y \in X$ and so by Proposition 3.5, we have the result. □

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A FIXED POINT APPROACH TO THE STABILITY OF EULER-LAGRANGE SEXTIC (a,b)-FUNCTIONAL EQUATIONS IN ARCHIMEDEAN AND NON-ARCHIMEDEAN BANACH SPACES

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ABSTRACT. In this paper, we present a fixed point method to prove the Hyers-Ulam stability of the system of Euler-Lagrange quadratic-quartic functional equations

$$\begin{cases} f(ax_1+bx_2,y)+f(bx_1+ax_2,y)+abf(x_1-x_2,y) \\ \qquad\qquad = (a^2+b^2)[f(x_1,y)+f(x_2,y)]+4abf(\frac{x_1+x_2}{2},y), \\ f(x,ay_1+by_2)+f(x,by_1+ay_2)+\frac{1}{2}ab(a-b)^2f(x,y_1-y_2) \\ \qquad\qquad = (a^2-b^2)^2[f(x,y_1)+f(x,y_2)]+8abf(x,\frac{y_1+y_2}{2}) \end{cases} \tag{0.1}$$

for all numbers a and b with $a+b\notin\{0,\pm1\}$, $ab+2\neq 2(a+b)^2$ and $ab(a-b)^2+4\neq 4(a+b)^4$ in Archimedean and non-Archimedean Banach spaces and we show that the approximation in non-Archimedean Banach spaces is better than the approximation in (Archimedean) Banach spaces.

1. **Introduction**

The *stability problem* of functional equations started with the following question concerning stability of group homomorphisms proposed by Ulam [69] during a talk before a Mathematical Colloquium at the University of Wisconsin. In 1941, Hyers [32] gave a first affirmative answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [3] for additive mappings and by Rassias [61] for linear mappings by considering an unbounded Cauchy difference, respectively. In 1994, a generalization of the Rassias theorem was obtained by Găvruta [28] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. For more details about the results concerning such problems, the reader refer to [2, 5, 8, 11, 14, 15, 24, 27, 29, 33, 34, 35, 36, 40, 41, 42, 44], [52]–[67] and [71, 72, 73].

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

is related to a symmetric bi-additive mapping [1, 43]. It is natural that this equation is called a *quadratic functional equation*. In particular, every solution of the quadratic equation (1.1) is called a *quadratic mapping*. The Hyers-Ulam stability problem for the quadratic functional equation was solved by Skof [68]. In [14], Czerwik proved the Hyers-Ulam stability of the equation (1.1). Eshaghi Gordji and Khodaei [25] obtained the general solution and the Hyers-Ulam stability of the following quadratic functional equation: for all $a,b\in\mathbb{Z}\backslash\{0\}$ with $a\neq\pm1,\pm b$,

$$f(ax+by)+f(ax-by)=2a^2f(x)+2b^2f(y). \tag{1.2}$$

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Jun and Kim [38] introduced the following cubic functional equation:

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x) \tag{1.3}$$

and they established the general solution and the Hyers-Ulam stability for the functional equation (1.3). Jun et al. [39] investigated the solution and the Hyers-Ulam stability for the cubic functional equation

$$f(ax+by)+f(ax-by)=ab^2(f(x+y)+f(x-y))+2a(a^2-b^2)f(x), \tag{1.4}$$

where $a,b\in\mathbb{Z}\backslash\{0\}$ with $a\neq\pm1,\pm b$. For other cubic functional equations, see [50].

Lee et. al. [48] considered the following functional equation:

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y) \tag{1.5}$$

In fact, they proved that a mapping f between two real vector spaces X and Y is a solution of the equation (1.5) if and only if there exists a unique symmetric bi-quadratic mapping $B_2 : X\times X\to Y$ such that $f(x)=B_2(x,x)$ for all $x\in X$. The bi-quadratic mapping B_2 is given by

$$B_2(x,y)=\frac{1}{12}(f(x+y)+f(x-y)-2f(x)-2f(y)).$$

Obviously, the function $f(x)=cx^4$ satisfies the functional equation (1.5), which is called the *quartic functional equation*. For other quartic functional equations, see [13].

Ebadian et al. [16] proved the Hyers-Ulam stability of the following systems of the additive-quartic functional equation:

$$\begin{cases} f(x_1+x_2,y)=f(x_1,y)+f(x_2,y),\\ f(x,2y_1+y_2)+f(x,2y_1-y_2)\\ \qquad =4f(x,y_1+y_2)+4f(x,y_1-y_2)+24f(x,y_1)-6f(x,y_2)\end{cases} \tag{1.6}$$

and the quadratic-cubic functional equation:

$$\begin{cases} f(x,2y_1+y_2)+f(x,2y_1-y_2)\\ \qquad =2f(x,y_1+y_2)+2f(x,y_1-y_2)+12f(x,y_1),\\ f(x,y_1+y_2)+f(x,y_1-y_2)=2f(x,y_1)+2f(x,y_2).\end{cases} \tag{1.7}$$

For more details about the results concerning mixed type functional equations, the readers refer to [18, 20, 21] and [23].

Recently, Ghaemi et. al. [30] and Cho et. al. [10] investigated the the stability of the following systems of the quadratic-cubic functional equation:

$$\begin{cases} f(ax_1+bx_2,y)+f(ax_1-bx_2,y)=2a^2f(x_1,y)+2b^2f(x_2,y),\\ f(x,ay_1+by_2)+f(x,ay_1-by_2)\\ \qquad =ab^2(f(x,y_1+y_2)+f(x,y_1-y_2))+2a(a^2-b^2)f(x,y_1)\end{cases} \tag{1.8}$$

and the additive-quadratic-cubic functional equation:

$$\begin{cases} f(ax_1+bx_2,y,z)+f(ax_1-bx_2,y,z)=2af(x_1,y,z),\\ f(x,ay_1+by_2,z)+f(x,ay_1-by_2,z)=2a^2f(x,y_1,z)+2b^2f(x,y_2,z),\\ f(x,y,az_1+bz_2)+f(x,y,az_1-bz_2)\\ \qquad =ab^2(f(x,y,z_1+z_2)+f(x,y,z_1-z_2))+2a(a^2-b^2)f(x,y,z_1)\end{cases} \tag{1.9}$$

in PN-spaces and PM-spaces, where $a, b \in \mathbb{Z}\backslash\{0\}$ with $a \neq \pm 1, \pm b$. The function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $f(x, y) = cx^2y^3$ is a solution of the system (1.8). In particular, letting $y = x$, we get a quintic function $g : \mathbb{R} \to \mathbb{R}$ in one variable given by $g(x) := f(x, x) = cx^5$.

In this paper, we present a fixed point method to prove the Hyers-Ulam stability of the following system of the Euler-Lagrange quadratic-quartic (a, b)-functional equation:

$$\begin{cases} f(ax_1 + bx_2, y) + f(bx_1 + ax_2, y) + abf(x_1 - x_2, y) \\ \qquad\qquad = (a^2 + b^2)[f(x_1, y) + f(x_2, y)] + 4abf(\frac{x_1+x_2}{2}, y), \\ f(x, ay_1 + by_2) + f(x, by_1 + ay_2) + \frac{1}{2}ab(a - b)^2 f(x, y_1 - y_2) \\ \qquad\qquad = (a^2 - b^2)^2[f(x, y_1) + f(x, y_2)] + 8abf(x, \frac{y_1+y_2}{2}) \end{cases} \tag{1.10}$$

for all numbers a and b with $a + b \notin \{0, \pm 1\}$, $ab + 2 \neq 2(a + b)^2$ and $ab(a - b)^2 + 4 \neq 4(a + b)^4$ in Archimedean and non-Archimedean Banach spaces. For details about the results concerning such problems in non-Archimedean normed spaces, the reader refer to [9, 12, 17, 20, 26, 37, 46, 47, 55, 72]. It is easy to see that the function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $f(x, y) = cx^2y^4$ is a solution of the system (1.10). In particular, letting $x = y$, we get a sextic function $h : \mathbb{R} \to \mathbb{R}$ in one variable given by $h(x) := f(x, x) = cx^6$.

The proof of the following propositions is evident.

Proposition 1.1. *Let X and Y be real linear spaces. If a mapping $f : X \times X \to Y$ satisfies the system* (1.10)*, then $f(\lambda x, \mu y) = \lambda^2\mu^4 f(x, y)$ for all $x, y \in X$ and rational numbers λ, μ.*

In this paper, we investigate the Hyers-Ulam stability of a sextic mapping from linear spaces into Archimedean and non-Archimedean Banach spaces. Hensel [31] has introduced a normed space which does not have the Archimedean property. During the last three decades theory of non-Archimedean spaces has gained the interest of physicists for their research in particular in problems coming from quantum physics, p–adic strings and superstrings [45]. Although many results in the classical normed space theory have a non–Archimedean counterpart, their proofs are different and require a rather new kind of intuition [4, 22, 51, 54, 70]. One may note that $|n| \leq 1$ in each valuation field, every triangle is isosceles and there may be no unit vector in a non-Archimedean normed space; cf. [51]. These facts show that the non-Archimedean framework is of special interest.

Definition 1.2. Let $\mathbb{K}$ be a field. A valuation mapping on $\mathbb{K}$ is a function $|\cdot| : \mathbb{K} \to \mathbb{R}$ such that for any $a, b \in \mathbb{K}$ we have

(i) $|a| \geq 0$ and equality holds if and only if $a = 0$,
(ii) $|ab| = |a||b|$,
(iii) $|a + b| \leq |a| + |b|$.

A field endowed with a valuation mapping will be called a valued field. If the condition (iii) in the definition of a valuation mapping is replaced with

$$(iii)' \ |a + b| \leq \max\{|a|, |b|\}$$

then the valuation $|\cdot|$ is said to be non-Archimedean. The condition $(iii)'$ is called the strict triangle inequality. By (ii), we have $|1| = |-1| = 1$. Thus, by induction, it follows from $(iii)'$ that $|n| \leq 1$ for each integer n. We always assume in addition that $|\cdot|$ is non trivial, i.e., that there is an $a_0 \in \mathbb{K}$ such that $|a_0| \notin \{0, 1\}$.The most important examples of non-Archimedean spaces are p-adic numbers.

Example 1.3. Let p be a prime number. For any non–zero rational number $a = p^r \frac{m}{n}$ such that m and n are coprime to the prime number p, define the p-adic absolute value $|a|_p = p^{-r}$. Then $|\cdot|$ is a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to $|\cdot|$ is denoted by $\mathbb{Q}_p$ and is called the p-adic number field.

Definition 1.4. Let X be a linear space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$. A function $\|\cdot\| : X \to \mathbb{R}$ is a non-Archimedean norm (valuation) if it satisfies the following conditions:

$(NA1)$ $\|x\| = 0$ if and only if $x = 0$;

$(NA2)$ $\|rx\| = |r|\|x\|$ for all $r \in \mathbb{K}$ and $x \in X$;

$(NA3)$ the strong triangle inequality (ultrametric); namely,
$$\|x + y\| \le \max\{\|x\|, \|y\|\} \qquad (x, y \in X).$$

Then $(X, \|\cdot\|)$ is called a non-Archimedean normed space.

Definition 1.5. (i) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X. Then the sequence $\{x_n\}$ is called Cauchy if for a given $\varepsilon > 0$ there is a positive integer N such that
$$\|x_n - x_m\| < \varepsilon$$
for all $n, m \ge N$.

(ii) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X. Then the sequence $\{x_n\}$ is called convergent if for a given $\varepsilon > 0$ there are a positive integer N and an $x \in X$ such that
$$\|x_n - x\| < \varepsilon$$
for all $n \ge N$. Then we call $x \in X$ a limit of the sequence $\{x_n\}$, and denote by $\lim_{n\to\infty} x_n = x$.

(iii) If every Cauchy sequence in X converges, then the non-Archimedean normed space X is called a non-Archimedean Banach space.

In 2003, Radu [60] proposed a new method for obtaining the existence of exact solutions and error estimations, based on the fixed point alternative (see also [6, 7]). Our aim is based on the following fixed point approach:
Let (X, d) be a generalized metric space. An operator $T : X \to X$ satisfies a Lipschitz condition with Lipschitz constant L if there exists a constant $L \ge 0$ such that $d(Tx, Ty) \le Ld(x, y)$ for all $x, y \in X$. If the Lipschitz constant L is less than 1, then the operator T is called a strictly contractive operator. Note that the distinction between the generalized metric and the usual metric is that the range of the former is permitted to include the infinity. We recall the following theorem by Margolis and Diaz.

Theorem 1.6. ([49, 60]) *Suppose that we are given a complete generalized metric space* (Ω, d) *and a strictly contractive mapping* $T : \Omega \to \Omega$ *with Lipschitz constant* L*. Then for each given* $x \in \Omega$*, either*
$$d(T^m x, T^{m+1} x) = \infty \quad \textit{for all } m \ge 0,$$
or there exists a natural number m_0 *such that*

- $d(T^m x, T^{m+1} x) < \infty$ *for all* $m \ge m_0$;

- *the sequence* $\{T^m x\}$ *is convergent to a fixed point* y^* *of* T*;*
- y^* *is the unique fixed point of* T *in* $\Lambda = \{y \in \Omega : d(T^{m_0}x, y) < \infty\}$*;*
- $d(y, y^*) \leq \frac{1}{1-L} d(y, Ty)$ *for all* $y \in \Lambda$.

2. Sextic functional inequalities in non-Archimedean Banach spaces

Throughout this section, we will assume that X is a non-Archimedean Banach space. In this section, we establish the conditional stability of sextic functional equations in non–Archimedean Banach spaces.

Theorem 2.1. *Let* $s \in \{-1, 1\}$ *be fixed. Let* E *be a real or complex linear space and let* X *be a non–Archimedean Banach space. Suppose* $f : E\times E \to X$ *satisfies the condition* $f(x, 0) = f(0, y) = 0$ *and inequalities of the form*

$$\begin{aligned} &\|f(ax_1 + bx_2, y) + f(bx_1 + ax_2, y) + abf(x_1 - x_2, y) \\ &\quad -(a^2 + b^2)[f(x_1, y) + f(x_2, y)] - 4abf(\frac{x_1 + x_2}{2}, y)\| \leq \phi(x_1, x_2, y), \end{aligned} \tag{2.1}$$

$$\begin{aligned} &\|f(x, ay_1 + by_2) + f(x, by_1 + ay_2) + \frac{1}{2}ab(a - b)^2 f(x, y_1 - y_2) \\ &\quad -(a^2 - b^2)^2[f(x, y_1) + f(x, y_2)] - 8abf(x, \frac{y_1 + y_2}{2})\| \leq \psi(x, y_1, y_2), \end{aligned} \tag{2.2}$$

where $\phi, \psi : E \times E \times E \to [0, \infty)$ *is given functions such that*

$$\begin{aligned} \phi((a + b)^s x_1, (a + b)^s x_2, (a + b)^s y) &\leq |a + b|^{6s} L\phi(x_1, x_2, y), \\ \psi((a + b)^s x, (a + b)^s y_1, (a + b)^s y_2) &\leq |a + b|^{6s} L\psi(x, y_1, y_2), \end{aligned} \tag{2.3}$$

and have the properties

$$\begin{aligned} \lim_{n\to\infty} \left|(a + b)^{-6sn}\right| \phi\big((a + b)^{sn} x_1, (a + b)^{sn} x_2, (a + b)^{sn} y\big) = 0, \\ \lim_{n\to\infty} \left|(a + b)^{-6sn}\right| \psi\big((a + b)^{sn} x, (a + b)^{sn} y_1, (a + b)^{sn} y_2\big) = 0, \end{aligned} \tag{2.4}$$

for all $x, x_1, x_2, y, y_1, y_2 \in E$ *and a constant* $0 < L < 1$*. Then there exists a unique sextic mapping* $T : E \times E \to X$ *satisfying the system* (1.10) *and*

$$\|T(x, y) - f(x, y)\| \leq \frac{1}{1 - L}\Phi(x, y), \tag{2.5}$$

where

$$\begin{aligned} \Phi(x, y) \;:=\; &\left|\frac{1}{2}\right| \max\Big\{ \left|(a + b)^{-3s+1}\right| \phi\left((a + b)^{\frac{s-1}{2}} x, (a + b)^{\frac{s-1}{2}} x, (a + b)^{\frac{s-1}{2}} y\right), \\ &\left|(a + b)^{-3s-3}\right| \psi\left((a + b)^{\frac{s+1}{2}} x, (a + b)^{\frac{s-1}{2}} y, (a + b)^{\frac{s-1}{2}} y\right) \Big\} \end{aligned}$$

for all $x, y \in E$.

Proof. We denote $A := a + b$. Putting $x_1 = x_2 = x$ in (2.1), we get

$$\|f(Ax, y) - A^2 f(x, y)\| \leq \left|\frac{1}{2}\right| \phi(x, x, y) \tag{2.6}$$

for all $x, y \in E$. Putting $y_1 = y_2 = y$ and replacing x by Ax in (2.2), we get

$$\|f(Ax, Ay) - A^4 f(Ax, y)\| \leq \left|\frac{1}{2}\right| \psi(Ax, y, y) \tag{2.7}$$

for all $x, y \in E$. Thus by (2.6) and (2.7) we have

$$\|f(Ax, Ay) - A^6 f(x,y)\| \leq \left|\frac{1}{2}\right| \max\left\{\left|A^4\right| \phi(x,x,y), \psi(Ax,y,y)\right\},$$

for all $x, y \in E$. By last inequality we get

$$\|A^{-6} f(Ax, Ay) - f(x,y)\| \leq \left|\frac{1}{2}\right| \max\left\{\left|A^{-2}\right| \phi(x,x,y), \left|A^{-6}\right| \psi(Ax,y,y)\right\}, \tag{2.8}$$

$$\|A^6 f\left(\frac{x}{A}, \frac{y}{A}\right) - f(x,y)\| \leq \left|\frac{1}{2}\right| \max\left\{\left|A^4\right| \phi\left(\frac{x}{A}, \frac{x}{A}, \frac{y}{A}\right), \psi\left(x, \frac{y}{A}, \frac{y}{A}\right)\right\}, \tag{2.9}$$

for all $x, y \in E$. Therefore

$$\left\|\frac{1}{A^{6s}} f(A^s x, A^s y) - f(x,y)\right\| \leq \Phi(x,y), \tag{2.10}$$

for all $x, y \in E$. We now consider the set

$$\mathcal{S} = \{h : E \times E \to X, \quad h(x,0) = h(0,x) = 0 \text{ for all } x \in E\}$$

and introduce the generalized metric on $\mathcal{S}$ as follows:

$$d(h,k) = \inf\left\{\alpha \in \mathbb{R}^+ : \|h(x,y) - k(x,y)\| \leq \alpha \Phi(x,y),\ \forall x, y \in E\right\}$$

where, as usual, $\inf \emptyset = +\infty$. The proof of the fact that $(\mathcal{S}, d)$ is a complete generalized metric space, can be found in [6]. Now we consider the mapping $J : \mathcal{S} \to \mathcal{S}$ defined by

$$Jh(x,y) := A^{-6s} h(A^s x, A^s y)$$

for all $h \in \mathcal{S}$ and $x, y \in E$. Let $f, g \in \mathcal{S}$ such that $d(f,g) < \varepsilon$. Then

$$\begin{aligned}
\|Jg(x,y) - Jf(x,y)\| &= \|A^{-6s} g(A^s x, A^s y) - A^{-6s} f(A^s x, A^s y)\| \\
&= |A^{-6s}| \|g(A^s x, A^s y) - f(A^s x, A^s y)\| \\
&\leq |A^{-6s}| \varepsilon \Phi(A^s x, A^s y) \\
&\leq L\varepsilon \Phi(x,y),
\end{aligned}$$

that is, if $d(f,g) < \varepsilon$, then we have $d(Jf, Jg) \leq L\varepsilon$. This means that

$$d(Jf, Jg) \leq L d(f,g)$$

for all $f, g \in \mathcal{S}$, that is, J is a strictly contractive self-mapping on $\mathcal{S}$ with the Lipschitz constant L. It follows from (2.10) that

$$\|Jf(x,y) - f(x,y)\| \leq \Phi(x,y)$$

for all $x, y \in E$ which implies that $d(Jf, f) \leq 1$. Due to Theorem 1.6, there exists a unique mapping $T : E \times E \to X$ such that T is a fixed point of J, i.e., $T(A^s x, A^s y) = A^{6s} T(x,y)$ for all $x, y \in E$. Also, $d(J^m f, T) \to 0$ as $m \to \infty$, which implies the equality

$$\lim_{m \to \infty} A^{-6sm} f(A^{sm} x, A^{sm} y) = T(x,y)$$

for all $x, y \in E$.

It follows from (2.1), (2.2) and (2.4) that

$$\begin{aligned}
&\|T(ax_1+bx_2,y)+T(bx_1+ax_2,y)+abT(x_1-x_2,y)-(a^2+b^2)[T(x_1,y)+T(x_2,y)]\\
&-4abT(\frac{x_1+x_2}{2},y)\|=\\
&\lim_{n\to\infty}\|\frac{f(A^{ns}(ax_1+bx_2),A^{ns}y)}{A^{6ns}}+\frac{f(A^{ns}(bx_1+ax_2),A^{ns}y)}{A^{6ns}}+ab\frac{f(A^{ns}(x_1-x_2),A^{ns}y)}{A^{6ns}}\\
&-(a^2+b^2)\frac{f(A^{ns}x_1,A^{ns}y)+f(A^{ns}x_2,A^{ns}y)}{A^{6ns}}-4ab\frac{f(A^{ns}\{x_1+x_2/2\},A^{ns}y)}{A^{6ns}}\|=\\
&\lim_{n\to\infty}|A^{-6ns}|\|f(A^{ns}(ax_1+bx_2),A^{ns}y)+f(A^{ns}(bx_1+ax_2),A^{ns}y)+abf(A^{ns}(x_1-x_2),A^{ns}y)\\
&\qquad-(a^2+b^2)[f(A^{ns}x_1,A^{ns}y)+f(A^{ns}x_2,A^{ns}y)]-4abf(A^{ns}\{x_1+x_2/2\},A^{ns}y)\|\\
&\le\lim_{n\to\infty}|A^{-6ns}|\phi(A^{ns}x_1,A^{ns}x_2,A^{ns}y)=0,
\end{aligned}\tag{2.11}$$

for all $x_1,x_2,y\in E$, and

$$\begin{aligned}
&\|T(x,ay_1+by_2)+T(x,by_1+ay_2)+\frac{1}{2}ab(a-b)^2T(x,y_1-y_2)-(a^2-b^2)^2[T(x,y_1)+T(x,y_2)]\\
&-8abT(x,\frac{y_1+y_2}{2})\|=\\
&\lim_{n\to\infty}\|\frac{f(A^{ns}x,A^{ns}(ay_1+by_2))}{A^{6ns}}+\frac{f(A^{ns}x,A^{ns}(by_1+ay_2))}{A^{6ns}}\\
&+\frac{1}{2}ab(a-b)^2\frac{f(A^{ns}x,A^{ns}(y_1-y_2))}{A^{6ns}}\|=\\
&-(a^2-b^2)^2\frac{f(A^{ns}x,A^{ns}y_1)+f(A^{ns}x,A^{ns}y_2)}{A^{6ns}}-8ab(a+b)^2\frac{f(A^{ns}x,A^{ns}\{y_1+y_2/2\})}{A^{6ns}}\|\\
&\le\lim_{n\to\infty}|A^{6ns}|\psi(A^{ns}x,A^{ns}y_1,A^{ns}y_2)=0,
\end{aligned}\tag{2.12}$$

for all $x,y_1,y_2\in E$. It follows from (2.11)) and (2.12) that T satisfies (1.10), that is, T is sextic.

According to the fixed point alternative, since T is the unique fixed point of J in the set $\Omega=\{g\in\mathcal{S}:d(f,g)<\infty\}$, T is the unique mapping such that

$$\|f(x,y)-T(x,y)\|\le\Phi(x,y)$$

for all $x,y\in E$. Using the fixed point alternative, we obtain that

$$d(f,T)\le\frac{1}{1-L}d(f,Jf)\le\frac{1}{1-L}\Phi(x,y),$$

for all $x,y\in E$, which implies the inequality (2.5). □

Corollary 2.2. *Let $s\in\{-1,1\}$ be fixed. Let E be a normed space and let F be a is a non-Archimedean Banach space. Suppose $f:E\times E\to F$ is a mapping with $f(x,0)=f(0,y)=0$ and there exist constants θ, $\vartheta\ge 0$ and non-negative real number p such that $ps<6s$ and*

$$\begin{aligned}
&\|f(ax_1+bx_2,y)+f(bx_1+ax_2,y)+abf(x_1-x_2,y)\\
&\quad-(a^2+b^2)[f(x_1,y)+f(x_2,y)]-4abf(\frac{x_1+x_2}{2},y)\| \le \theta(\|x_1\|^p+\|x_2\|^p+\|y\|^p),\\
&\|f(x,ay_1+by_2)+f(x,by_1+ay_2)+\frac{1}{2}ab(a-b)^2f(x,y_1-y_2)\\
&\quad-(a^2-b^2)^2[f(x,y_1)+f(x,y_2)]-8abf(x,\frac{y_1+y_2}{2})\| \le \vartheta(\|x\|^p+\|y_1\|^p+\|y_2\|^p),
\end{aligned}$$

for all $x, x_1, x_2, y, y_1, y_2 \in E$, where norms of the left-hand side of last inequalities is the non–Archimedean norm on F. Then there exists a unique sextic mapping $T : E \times E \to F$ such that

$$\|f(x,y) - T(x,y)\| \leq \max\Big\{\frac{\theta(2\|x\|^p + \|y\|^p)}{2s|a+b|^2 - 2s|a+b|^{p-4}}, \frac{\vartheta(\||a+b|x\|^p + 2\|y\|^p)}{2s|a+b|^6 - 2s|a+b|^p}\Big\},$$

for all $x, y \in E$.

Proof. Defining

$$\phi(x_1,x_2,y) = \theta(\|x_1\|^p + \|x_2\|^p + \|y\|^p), \quad \psi(x,y_1,y_2) = \vartheta(\|x\|^p + \|y_1\|^p + \|y_2\|^p),$$

and applying Theorem 2.1, we get the desired result. □

3. Sextic functional inequalities in (Archimedean) Banach spaces

Throughout this section, we will assume that X is a (Archimedean) Banach space. In this section, we establish the conditional stability of sextic functional equations.

Theorem 3.1. *Let $s \in \{-1, 1\}$ be fixed. Let E be a real or complex linear space and let X be a (Archimedean) Banach space. Suppose $f : E \times E \to X$ satisfies the condition $f(x, 0) = f(0, y) = 0$ and inequalities of (2.1) and (2.2), where $\phi, \psi : E \times E \times E \to [0, \infty)$ are given functions which satisfy (2.3) and have the properties (2.4) for all $x, x_1, x_2, y, y_1, y_2 \in E$ and a constant $0 < L < 1$. Then there exists a unique sextic mapping $T : E \times E \to X$ satisfying the system* (1.10) *and*

$$\|T(x,y) - f(x,y)\| \leq \frac{1}{1-L}\tilde{\Phi}(x,y), \tag{3.1}$$

where

$$\begin{aligned}\tilde{\Phi}(x,y) \quad := \quad & \left|\frac{1}{2}\right| \Big\{ \left|(a+b)^{-3s+1}\right| \phi\left((a+b)^{\frac{s-1}{2}}x, (a+b)^{\frac{s-1}{2}}x, (a+b)^{\frac{s-1}{2}}y\right) \\ & + \left|(a+b)^{-3s-3}\right| \psi\left((a+b)^{\frac{s+1}{2}}x, (a+b)^{\frac{s-1}{2}}y, (a+b)^{\frac{s-1}{2}}y\right) \Big\}\end{aligned}$$

for all $x, y \in E$.

Proof. We denote $A := a + b$. Putting $x_1 = x_2 = x$ in (2.1), we get

$$\|f(Ax,y) - A^2 f(x,y)\| \leq \left|\frac{1}{2}\right| \phi(x,x,y) \tag{3.2}$$

for all $x, y \in E$. Putting $y_1 = y_2 = y$ and replacing x by Ax in (2.2), we get

$$\|f(Ax,Ay) - A^4 f(Ax,y)\| \leq \left|\frac{1}{2}\right| \psi(Ax,y,y) \tag{3.3}$$

for all $x, y \in E$. Thus by (3.2) and (3.3) we have

$$\|f(Ax,Ay) - A^6 f(x,y)\| \leq \left|\frac{1}{2}\right| \left\{\left|A^4\right| \phi(x,x,y) + \psi(Ax,y,y)\right\},$$

for all $x, y \in E$. By last inequality we get

$$\|A^{-6} f(Ax,Ay) - f(x,y)\| \leq \left|\frac{1}{2}\right| \left\{\left|A^{-2}\right| \phi(x,x,y) + \left|A^{-6}\right| \psi(Ax,y,y)\right\}, \tag{3.4}$$

$$\|A^6 f\left(\frac{x}{A}, \frac{y}{A}\right) - f(x,y)\| \leq \left|\frac{1}{2}\right| \left\{\left|A^4\right| \phi\left(\frac{x}{A}, \frac{x}{A}, \frac{y}{A}\right) + \psi\left(x, \frac{y}{A}, \frac{y}{A}\right)\right\}, \tag{3.5}$$

for all $x, y \in E$. Therefore

$$\left\| \frac{1}{A^{6s}} f(A^s x, A^s y) - f(x,y) \right\| \leq \tilde{\Phi}(x,y),$$

for all $x, y \in E$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 3.2. *Let $s \in \{-1, 1\}$ be fixed. Let E be a normed space and let F be a (Archimedean) Banach space. Suppose $f : E \times E \to F$ is a mapping with $f(x, 0) = f(0, y) = 0$ and there exist constants θ, $\vartheta \geq 0$ and non-negative real number p such that $ps < 6s$ and*

$$\begin{aligned} &\|f(ax_1 + bx_2, y) + f(bx_1 + ax_2, y) + abf(x_1 - x_2, y) \\ &\quad -(a^2 + b^2)[f(x_1, y) + f(x_2, y)] - 4abf(\frac{x_1 + x_2}{2}, y)\| \leq \theta(\|x_1\|^p + \|x_2\|^p + \|y\|^p), \\ &\|f(x, ay_1 + by_2) + f(x, by_1 + ay_2) + \frac{1}{2}ab(a - b)^2 f(x, y_1 - y_2) \\ &\quad -(a^2 - b^2)^2[f(x, y_1) + f(x, y_2)] - 8abf(x, \frac{y_1 + y_2}{2})\| \leq \vartheta(\|x\|^p + \|y_1\|^p + \|y_2\|^p), \end{aligned}$$

for all $x, x_1, x_2, y, y_1, y_2 \in E$. Then there exists a unique sextic mapping $T : E \times E \to F$ such that

$$\|f(x,y) - T(x,y)\| \leq \frac{\theta(2\|x\|^p + \|y\|^p)}{2s|a + b|^2 - 2s|a + b|^{p-4}} + \frac{\vartheta(\||a + b|x\|^p + 2\|y\|^p)}{2s|a + b|^6 - 2s|a + b|^p},$$

for all $x, y \in E$.

Proof. Defining

$$\phi(x_1, x_2, y) = \theta(\|x_1\|^p + \|x_2\|^p + \|y\|^p), \quad \psi(x, y_1, y_2) = \vartheta(\|x\|^p + \|y_1\|^p + \|y_2\|^p),$$

and applying Theorem 3.1, we get the desired result. □

Remark 3.3. Comparison of (2.5) and (3.1) shows that the approximation in non-Archimedean Banach spaces is better than the approximation in (Archimedean) Banach spaces.

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Dynamical behaviors of a nonlinear virus infection model with latently infected cells and immune response

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Abstract

In this paper, we study the global stability of a mathematical model that describes the virus dynamics under the effect of antibody immune response. The model is a modification of some of the existing virus dynamics models by considering the latently infected cells and nonlinear incidence rate for virus infections. We show that the global dynamics of the model is completely determined by two threshold values R_0, the corresponding reproductive number of viral infection and R_1, the corresponding reproductive number of antibody immune response, respectively. Using Lyapunov method, we have proven that, if $R_0 \leq 1$, then the uninfected steady state is globally asymptotically stable (GAS), if $R_1 \leq 1 < R_0$, then the infected steady state without antibody immune response is GAS, and if $R_1 > 1$, then the infected steady state with antibody immune response is GAS.

Keywords: Virus infection; Global stability; Immune response; Lyapunov function; nonlinear infection rate.

1 Introduction

Recently, mathematical modeling and analysis of viral infections such as hepatitis C virus (HCV) [1]-[3], hepatitis B virus (HBV) [4]-[5], human immunodeficiency virus (HIV) [6]-[15] human T cell leukemia (HTLV) [16] have attracted the interest several researchers. In 1996, Nowak and Bangham [7] has proposed the basic viral infection model which contains three compartments, the uninfected target cells, infected cells and free virus particles. This model does not take into consideration the latently infected cells which is due to the delay between the moment of infection and the moment when the infected cell becomes active to produce infectious viruses. Latently infected cells have been incorporated into the basic viral infection model in several papers (see e.g. [18], [19] and [20]). The basic viral infection model which takes into account the latently infected cells is given by [20]:

$$\dot{x} = \rho - dx - \eta xv, \tag{1}$$

$$\dot{w} = \eta xv - (e+b)w, \tag{2}$$

$$\dot{y} = bw - ay, \tag{3}$$

$$\dot{v} = ky - cv, \tag{4}$$

where x, w, y and v representing the populations of the uninfected target cells, latently infected cells, actively infected cells and free virus particles, respectively. Parameters ρ and k represent, respectively, the rate at which new uninfected cells are generated from the source within the body, and the generation rate constant of free viruses produced from the actively infected cells. Parameters d, e, a and c are the natural death rate constants of the uninfected cells, latently infected cells, actively infected cells and free virus particles, respectively. Parameter η is the infection rate constant. Eq. (2) describes the population dynamics of the latently infected cells and show that they are converted to actively infected cells with rate constant b. All the parameters given in model (1)-(4) are positive.

We observe that in model (1)-(4), the immune response has been neglected. To provide more accurate modelling for the viral infection, the effect of immune response has to be considered. The antibody immune response which is based on the antibodies that are produced by the B cells plays an important role in controlling the disease [17]. In the literature, several mathematical models have been formulated to consider the antibody immune response into the viral infection models (see e.g., [21]-[24]). However, in [21]-[24], it was assumed that all the infected cells are active which is an unrealistic assumption. The aim of this paper is to propose a viral infection model with antibody immune response taking into consideration both latently and actively infected cells and investigate its basic and global properties. The incidence rate is given by nonlinear function which is more general than the bilinear incidence rate given in model (1)-(4). Using Lyapunov functions, we prove that the global dynamics of the model is determined by two threshold parameters, the basic reproductive number of viral infection R_0 and the the basic reproductive number of antibody immune response R_1. If $R_0 \leq 1$, then the infection-free steady state is globally asymptotically stable (GAS), if $R_1 \leq 1 < R_0$, then the chronic-infection steady state without antibody immune response is GAS, and if $R_1 > 1$, then the chronic-infection steady state with antibody immune response is GAS.

2 The mathematical model

In this section, we propose a viral infection model with latently infected cells and antibody immune response. The incidence rate is given by a nonlinear infection rate.

$$\dot{x} = \rho - dx - \frac{\eta x^n v}{(\delta^n + x^n)(\theta + v)}, \tag{5}$$

$$\dot{w} = (1-\xi)\frac{\eta x^n v}{(\delta^n + x^n)(\theta + v)} - (e+b)w, \tag{6}$$

$$\dot{y} = \xi \frac{\eta x^n v}{(\delta^n + x^n)(\theta + v)} + bw - ay, \tag{7}$$

$$\dot{v} = ky - cv - rvz, \tag{8}$$

$$\dot{z} = gvz - \mu z, \tag{9}$$

where, z is the population of the antibody immune cells. Here δ, θ and n are positive constants. The fractions $(1-\xi)$ and ξ with $0 < \xi < 1$ are the probabilities that upon infection, an uninfected cell will become either latently infected or actively infected. Parameters r, g and μ are the removal rate constant of the virus due to the antibodies, the proliferation rate constant of antibody immune cells and the natural death rate constant of the antibody immune cells, respectively.

2.1 Positive invariance

We note that, model (5)-(9) is biologically acceptable in the sense that no population goes negative. It is straightforward to check the positive invariance of the non-negative orthant $\mathbb{R}^5_{\geq 0}$ by model (5)-(9). In the following, we show the boundednes of the solution of model (5)-(9).

Proposition 1. There exist positive numbers $L_i, i = 1,2,3$ such that the compact set $\Omega = \left\{(x,w,y,v,z) \in \mathbb{R}^5_{\geq 0} : 0 \leq x,w,y \leq L_1, 0 \leq v \leq L_2, 0 \leq z \leq L_3\right\}$
is positively invariant.

Proof. Let $X_1(t) = x(t) + w(t) + y(t)$, then

$$\dot{X}_1 = \rho - dx - ew - ay \leq \rho - s_1 X_1,$$

where $s_1 = \min\{d, a, e\}$. Hence $X_1(t) \leq L_1$, if $X_1(0) \leq L_1$, where $L_1 = \dfrac{\rho}{s_1}$. Since $x(t) > 0$, $w(t) \geq 0$ and $y(t) \geq 0$, then $0 \leq x(t)$, $w(t)$, $y(t) \leq L_1$ if $0 \leq x(0) + w(0) + y(0) \leq L_1$. On the other hand, let $X_2(t) = v(t) + \frac{r}{g}z(t)$, then

$$\dot{X}_2 = ky - cv - \frac{r\mu}{g}z \leq kL_1 - s_2\left(v + \frac{r}{g}z\right) = kL_1 - s_2 X_2,$$

where $s_2 = \min\{c, \mu\}$. Hence $X_2(t) \leq L_2$, if $X_2(0) \leq L_2$, where $L_2 = \dfrac{kL_1}{s_2}$. Since $v(t) \geq 0$ and $z(t) \geq 0$, then $0 \leq v(t) \leq L_2$ and $0 \leq z(t) \leq L_3$ if $0 \leq v(0) + \frac{r}{g}z(0) \leq L_2$, where $L_3 = \frac{gL_2}{r}$.

2.2 Steady states

In this subsection, we calculate the steady states of model (5)-(9) and derive two thresholds parameters. The steady states of model (5)-(9) satisfy the following equations:

$$\rho - dx - \frac{\eta x^n v}{(\delta^n + x^n)(\theta + v)} = 0, \tag{10}$$

$$(1-\xi)\frac{\eta x^n v}{(\delta^n + x^n)(\theta + v)} - (e+b)w = 0, \tag{11}$$

$$\xi \frac{\eta x^n v}{(\delta^n + x^n)(\theta + v)} + bw - ay = 0, \tag{12}$$

$$ky - cv - rvz = 0, \tag{13}$$

$$(gv - \mu)z = 0. \tag{14}$$

Equation (14) has two possible solutions, $z = 0$ or $v = \mu/g$. If $z = 0$, then from Eqs. (11) and (12) we obtain w and y as:

$$w = \frac{(1-\xi)}{e+b}\frac{\eta x^n v}{(\delta^n + x^n)(\theta + v)}, \qquad y = \frac{(e\xi + b)}{a(e+b)}\frac{\eta x^n v}{(\delta^n + x^n)(\theta + v)}. \tag{15}$$

Substituting Eq. (15) into Eq. (13), we obtain

$$\frac{k(e\xi + b)}{a(e+b)}\frac{\eta x^n v}{(\delta^n + x^n)(\theta + v)} - cv = 0. \tag{16}$$

Equation (16) has two possibilities, $v = 0$ or $v \neq 0$. If $v = 0$, then $w = y = 0$ and $x = \frac{\rho}{d}$ which leads to the uninfected steady state $E_0 = (x_0, 0, 0, 0, 0)$, where $x_0 = \frac{\rho}{d}$. If $v \neq 0$, then from Eqs. (10) and (16) we obtain

$$v = \frac{k(e\xi + b)}{ac(e+b)}\frac{\eta x^n v}{(\delta^n + x^n)(\theta + v)} = \frac{k(e\xi + b)(\rho - dx)}{ac(e+b)} \tag{17}$$

$$\Rightarrow x = x_0 - \frac{ac(e+b)}{dk(e\xi + b)}v. \tag{18}$$

From Eq. (18) into Eq. Eq. (16) we get

$$\frac{k(e\xi + b)}{a(e+b)}\frac{\eta\left(x_0 - \frac{ac(e+b)}{dk(e\xi+b)}v\right)^n}{\delta^n + \left(x_0 - \frac{ac(e+b)}{dk(e\xi+b)}v\right)^n}\frac{v}{(\theta + v)} - cv = 0.$$

Let us define a function Ψ_1 as

$$\Psi_1(v) = \frac{k(e\xi + b)}{a(e+b)}\frac{\eta\left(x_0 - \frac{ac(e+b)}{dk(e\xi+b)}v\right)^n}{\delta^n + \left(x_0 - \frac{ac(e+b)}{dk(e\xi+b)}v\right)^n}\frac{v}{(\theta + v)} - cv = 0.$$

It is clear that, $\Psi_1(0) = 0$, and when $v = \overline{v} = \frac{x_0 dk(e\xi+b)}{ac(e+b)} > 0$, then $\Psi_1(\overline{v}) = -c\overline{v} < 0$. Since $\Psi_1(v)$ is continuous for all $v \geq 0$, then we have

$$\Psi_1'(0) = c\left(\frac{k(e\xi + b)}{ac\theta(e+b)}\frac{\eta x_0^n}{\delta^n + x_0^n} - 1\right).$$

Therefore, if $\Psi_1'(0) > 0$ i.e.

$$\frac{k(e\xi + b)}{ac\theta(e + b)} \frac{\eta x_0^n}{\delta^n + x_0^n} > 1,$$

then there exist a $v_1 \in (0, \overline{v})$ such that $\Psi_1(v_1) = 0$. From Eq. (13) we obtain $y_1 = \frac{c}{k} v_1 > 0$ and from Eq. (10) we define a function Ψ_2 as:

$$\Psi_2(x) = \rho - dx - \frac{\eta x^n v_1}{(\delta^n + x^n)(\theta + v_1)} = 0.$$

We have $\Psi_2(0) = \rho > 0$ and $\Psi_2(x_0) = -\frac{\eta x_0^n v_1}{(\delta^n + x_0^n)(\theta + v_1)} < 0$. Since $f(x) = \frac{x^n}{\delta^n + x^n}$ is a strictly increasing function of x, for all $n, \delta > 0$, then Ψ_2 is a strictly decreasing function of x, and there exist a unique $x_1 \in (0, x_0)$ such that $\Psi_2(x_1) = 0$. It follows that, $w_1 = \frac{(1-\xi)}{e+b} \frac{\eta x_1^n v_1}{(\delta^n + x_1^n)(\theta + v_1)} > 0$ and $y_1 = \frac{(e\xi + b)}{a(e+b)} \frac{\eta x_1^n v_1}{(\delta^n + x_1^n)(\theta + v_1)} > 0$. It means that, an infected steady state without antibody immune response $E_1(x_1, w_1, y_1, v_1, 0)$ exists when $\frac{k(e\xi+b)}{ac\theta(e+b)} \frac{\eta x_0^n}{\delta^n + x_0^n} > 1$. Then we can define the basic reproductive number of viral infection as:

$$R_0 = \frac{k(e\xi + b)}{ac\theta(e + b)} \frac{\eta x_0^n}{\delta^n + x_0^n}.$$

The parameter R_0 determines whether a chronic-infection can be established.

The other possibility of Eq. (14) is $v_2 = \dfrac{\mu}{g}$. Inserting v_2 in Eq. (10) and defining a function Ψ_3 as:

$$\Psi_3(x) = \rho - dx - \frac{\eta x^n v_2}{(\delta^n + x^n)(\theta + v_2)} = 0.$$

Note that, Ψ_3 is a strictly decreasing function of x. Clearly, $\Psi_3(0) = \rho > 0$ and $\Psi_3(x_0) = -\frac{\eta x_0^n v_2}{(\delta^n + x_0^n)(\theta + v_2)} < 0$. Thus, there exists a unique $x_2 \in (0, x_0)$ such that $\Psi_3(x_2) = 0$. It follows from Eqs. (11)-(13) that,

$$w_2 = \frac{(1 - \xi)}{e + b} \frac{\eta x_2^n v_2}{(\delta^n + x_2^n)(\theta + v_2)}, \quad y_2 = \frac{(e\xi + b)}{a(e + b)} \frac{\eta x_2^n v_2}{(\delta^n + x_2^n)(\theta + v_2)},$$

$$z_2 = \frac{c}{r} \left[\frac{k(e\xi + b)}{ac(e + b)} \frac{\eta x_2^n}{(\delta^n + x_2^n)(\theta + v_2)} - 1 \right].$$

Thus $w_2, y_2 > 0$, and if $\frac{k(e\xi+b)}{ac(e+b)} \frac{\eta x_2^n}{(\delta^n + x_2^n)(\theta + v_2)} > 1$, then $z_2 > 0$. Now we define the basic reproductive number of antibody immune response:

$$R_1 = \frac{k(e\xi + b)}{ac(e + b)} \frac{\eta x_2^n}{(\delta^n + x_2^n)(\theta + v_2)},$$

which determines whether a persistent antibody immune response can be established. Hence, z_2 can be rewritten as $z_2 = \frac{c}{r}(R_1 - 1)$. It follows that, there exists an infected steady state with antibody immune response $E_2(x_2, w_2, y_2, v_2, z_2)$ when $R_1 > 1$. Since $x_1 < x_0$ and $v_2 > 0$, then

$$R_1 = \frac{k(e\xi + b)}{ac(e + b)} \frac{\eta x_2^n}{(\delta^n + x_2^n)(\theta + v_2)} < \frac{k(e\xi + b)}{ac\theta(e + b)} \frac{\eta x_0^n}{\delta^n + x_0^n} = R_0.$$

From above we have the following result.

Lemma 1 (i) if $R_0 \leq 1$, then there exists only one positive steady state E_0,

(ii) if $R_1 \leq 1 < R_0$, then there exist two positive steady states E_0 and E_1, and

(iii) if $R_1 > 1$, then there exist three positive steady states E_0, E_1 and E_2.

3 Main results

In this section, we investigate the global stability of steady states E_0, E_1 and E_2 employing the direct Lyapunov method and LaSalle's invariance principle.

3.1 Global stability of the uninfected steady state E_0

Theorem 1. If $R_0 \leq 1$, then E_0 is globally asymptotically stable (GAS).

Proof. Define a Lyapunov functional W_0 as follows:

$$W_0 = x - x_0 - \int_{x_0}^{x} \frac{x_0^n(\delta^n + s^n)}{s^n(\delta^n + x_0^n)} ds + \frac{b}{e\xi + b} w + \frac{e+b}{e\xi + b} y + \frac{a(e+b)}{k(e\xi + b)} v + \frac{ar(e+b)}{kg(e\xi + b)} z.$$

Calculating $\frac{dW_0}{dt}$ along the trajectories of (5)-(9) as:

$$\begin{aligned}
\frac{dW_0}{dt} &= \left(1 - \frac{x_0^n(\delta^n + x^n)}{x^n(\delta^n + x_0^n)}\right)\left(\rho - dx - \frac{\eta x^n v}{(\delta^n + x^n)(\theta + v)}\right) + \frac{b}{e\xi + b}\left((1-\xi)\frac{\eta x^n v}{(\delta^n + x^n)(\theta + v)} - (e+b)w\right) \\
&+ \frac{e+b}{e\xi + b}\left(\xi \frac{\eta x^n v}{(\delta^n + x^n)(\theta + v)} + bw - ay\right) + \frac{a(e+b)}{k(e\xi + b)}(ky - cv - rvz) + \frac{ar(e+b)}{kg(e\xi + b)}(gvz - \mu z) \\
&= \rho\left(1 - \frac{x_0^n(\delta^n + x^n)}{x^n(\delta^n + x_0^n)}\right)\left(1 - \frac{x}{x_0}\right) + \frac{\eta x_0^n v}{(\delta^n + x_0^n)(\theta + v)} - \frac{ac(e+b)}{k(e\xi + b)} v - \frac{ar\mu(e+b)}{kg(e\xi + b)} z \\
&= \rho\left(1 - \frac{x_0^n(\delta^n + x^n)}{x^n(\delta^n + x_0^n)}\right)\left(1 - \frac{x}{x_0}\right) + \frac{ac(e+b)}{k(e\xi + b)}\left(\frac{k(e\xi + b)}{ac(e+b)} \frac{\eta x_0^n}{(\delta^n + x_0^n)(\theta + v)} - 1\right) v - \frac{ar\mu(e+b)}{kg(e\xi + b)} z \\
&= \rho\left(1 - \frac{x_0^n(\delta^n + x^n)}{x^n(\delta^n + x_0^n)}\right)\left(1 - \frac{x}{x_0}\right) + \frac{ac(e+b)}{k(e\xi + b)}\left(R_0 \frac{\theta}{\theta + v} - 1\right) v - \frac{ar\mu(e+b)}{kg(e\xi + b)} z \\
&= \rho\left(1 - \frac{x_0^n(\delta^n + x^n)}{x^n(\delta^n + x_0^n)}\right)\left(1 - \frac{x}{x_0}\right) + \frac{ac(e+b)}{k(e\xi + b)}\left(R_0 \frac{\theta}{\theta + v} - 1\right) v - \frac{ar\mu(e+b)}{kg(e\xi + b)} z \\
&= \frac{d\delta^n(x^n - x_0^n)(x_0 - x)}{x^n(\delta^n + x_0^n)} + \frac{ac(e+b)}{k(e\xi + b)}(R_0 - 1)v - \frac{ac(e+b)R_0}{k(e\xi + b)} \frac{v^2}{\theta + v} - \frac{ar\mu(e+b)}{kg(e\xi + b)} z.
\end{aligned} \tag{19}$$

We have $(x^n - x_0^n)(x_0 - x) \leq 0$ for all $x, n > 0$. Thus if $R_0 \leq 1$ then $\frac{dW_0}{dt} \leq 0$ for all $x, v, z > 0$. Thus, the solutions of system (5)-(9) limited to M, the largest invariant subset of $\left\{\frac{dW_0}{dt} = 0\right\}$ [25]. Clearly, it follows from Eq. (19) that $\frac{dW_0}{dt} = 0$ if and only if $x(t) = x_0$, $v(t) = 0$ and $z(t) = 0$. The set M is invariant and for any element belongs to M satisfies $v(t) = 0$ and $z(t) = 0$, then $\dot{v}(t) = 0$. We can see from Eq. (8) that, $0 = \dot{v}(t) = ky(t)$, and thus $y(t) = 0$. Moreover, from Eq. (7) we get $w(t) = 0$. Hence, $\frac{dW_0}{dt} = 0$ if and only if $x(t) = x_0$, $w(t) = 0$, $y(t) = 0$, $v(t) = 0$ and $z(t) = 0$. From LaSalle's invariance principle, E_0 is GAS.

3.2 Global stability of the infected steady state without antibody immune response E_1

Theorem 2. If $R_1 \leq 1 < R_0$, then E_1 is GAS.

Proof. We construct the following Lyapunov functional

$$W_1 = x - x_1 - \int_{x_1}^{x} \frac{x_1^n(\delta^n + s^n)}{s^n(\delta^n + x_1^n)} ds + \frac{b}{e\xi + b} w_1 H\left(\frac{w}{w_1}\right) + \frac{e+b}{e\xi + b} y_1 H\left(\frac{y}{y_1}\right) + \frac{a(e+b)}{k(e\xi + b)} v_1 H\left(\frac{v}{v_1}\right) + \frac{ar(e+b)}{kg(e\xi + b)} z.$$

The time derivative of W_1 along the trajectories of (5)-(9) is given by

$$\begin{aligned}\frac{dW_1}{dt} &= \left(1 - \frac{x_1^n(\delta^n + x^n)}{x^n(\delta^n + x_1^n)}\right)\left(\rho - dx - \frac{\eta x^n v}{(\delta^n + x^n)(\theta + v)}\right) \\ &+ \frac{b}{e\xi + b}\left(1 - \frac{w_1}{w}\right)\left((1-\xi)\frac{\eta x^n v}{(\delta^n + x^n)(\theta + v)} - (e+b)w\right) \\ &+ \frac{e+b}{e\xi + b}\left(1 - \frac{y_1}{y}\right)\left(\xi \frac{\eta x^n v}{(\delta^n + x^n)(\theta + v)} + bw - ay\right) \\ &+ \frac{a(e+b)}{k(e\xi + b)}\left(1 - \frac{v_1}{v}\right)(ky - cv - rvz) + \frac{ar(e+b)}{kg(e\xi + b)}(gvz - \mu z). \end{aligned} \tag{20}$$

Applying $\rho = dx_1 + \frac{\eta x_1^n v_1}{(\delta^n + x_1^n)(\theta + v_1)}$ and collecting terms of Eq. (20) we get

$$\begin{aligned}\frac{dW_1}{dt} &= \left(1 - \frac{x_1^n(\delta^n + x^n)}{x^n(\delta^n + x_1^n)}\right)(dx_1 - dx) + \frac{\eta x_1^n v}{(\delta^n + x_1^n)(\theta + v)} \\ &+ \frac{\eta x_1^n v_1}{(\delta^n + x_1^n)(\theta + v_1)}\left(1 - \frac{x_1^n(\delta^n + x^n)}{x^n(\delta^n + x_1^n)}\right) \\ &- \frac{b(1-\xi)}{e\xi + b}\frac{\eta x^n v}{(\delta^n + x^n)(\theta + v)}\frac{w_1}{w} + \frac{b(e+b)}{e\xi + b}w_1 - \frac{(e+b)\xi}{e\xi + b}\frac{\eta x^n v}{(\delta^n + x^n)(\theta + v)}\frac{y_1}{y} \\ &- \frac{(e+b)b}{e\xi + b}\frac{y_1 w}{y} + \frac{e+b}{e\xi + b}ay_1 - \frac{ac(e+b)}{k(e\xi + b)}v - \frac{a(e+b)}{(e\xi + b)}\frac{yv_1}{v} + \frac{ac(e+b)}{k(e\xi + b)}v_1 \\ &+ \frac{ar(e+b)}{k(e\xi + b)}v_1 z - \frac{ar\mu(e+b)}{kg(e\xi + b)}z. \end{aligned}$$

Using the equilibrium conditions for E_1:

$$(1-\xi)\frac{\eta x_1^n v_1}{(\delta^n + x_1^n)(\theta + v_1)} = (e+b)w_1, \; y_1 = \frac{(e\xi + b)}{a(e+b)}\frac{\eta x_1^n v_1}{(\delta^n + x_1^n)(\theta + v_1)}, \; cv_1 = ky_1,$$

we obtain

$$\begin{aligned}\frac{dW_1}{dt} &= dx_1\left(1 - \frac{x_1^n(\delta^n + x^n)}{x^n(\delta^n + x_1^n)}\right)\left(1 - \frac{x}{x_1}\right) + \frac{\eta x_1^n v_1}{(\delta^n + x_1^n)(\theta + v_1)}\left[\frac{v(\theta + v_1)}{v_1(\theta + v)} - \frac{v}{v_1}\right] \\ &+ \left(\frac{b(1-\xi)}{e\xi + b} + \frac{(e+b)\xi}{e\xi + b}\right)\frac{\eta x_1^n v_1}{(\delta^n + x_1^n)(\theta + v_1)}\left(1 - \frac{x_1^n(\delta^n + x^n)}{x^n(\delta^n + x_1^n)}\right) \\ &- \frac{b(1-\xi)}{e\xi + b}\frac{\eta x_1^n v_1}{(\delta^n + x_1^n)(\theta + v_1)}\frac{x^n(\delta^n + x_1^n)(\theta + v_1)vw_1}{x_1^n(\delta^n + x^n)(\theta + v)v_1 w} \\ &+ \frac{b(1-\xi)}{e\xi + b}\frac{\eta x_1^n v_1}{(\delta^n + x_1^n)(\theta + v_1)} - \frac{(e+b)\xi}{e\xi + b}\frac{\eta x_1^n v_1}{(\delta^n + x_1^n)(\theta + v_1)}\frac{x^n(\delta^n + x_1^n)(\theta + v_1)vy_1}{x_1^n(\delta^n + x^n)(\theta + v)v_1 y} \\ &- \frac{b(1-\xi)}{e\xi + b}\frac{\eta x_1^n v_1}{(\delta^n + x_1^n)(\theta + v_1)}\frac{y_1 w}{yw_1} + \left(\frac{b(1-\xi)}{e\xi + b} + \frac{(e+b)\xi}{e\xi + b}\right)\frac{\eta x_1^n v_1}{(\delta^n + x_1^n)(\theta + v_1)} \\ &- \left(\frac{b(1-\xi)}{e\xi + b} + \frac{(e+b)\xi}{e\xi + b}\right)\frac{\eta x_1^n v_1}{(\delta^n + x_1^n)(\theta + v_1)}\frac{yv_1}{y_1 v} \\ &+ \left(\frac{b(1-\xi)}{e\xi + b} + \frac{(e+b)\xi}{e\xi + b}\right)\frac{\eta x_1^n v_1}{(\delta^n + x_1^n)(\theta + v_1)} + \frac{ar(e+b)}{k(e\xi + b)}\left(v_1 - \frac{\mu}{g}\right)z. \end{aligned}$$

Collecting terms we get

$$\begin{aligned}\frac{dW_1}{dt} &= -\frac{d\delta^n(x^n - x_1^n)(x - x_1)}{x^n(\delta^n + x_1^n)} - \frac{\eta x_1^n \theta (v - v_1)^2}{(\delta^n + x_1^n)(\theta + v)(\theta + v_1)^2} + \frac{ar(e+b)}{k(e\xi + b)}\left(v_1 - \frac{\mu}{g}\right) z \\ &+ \frac{b(1-\xi)}{(e\xi + b)} \frac{\eta x_1^n v_1}{(\delta^n + x_1^n)(\theta + v_1)} \left[5 - \frac{x_1^n(\delta^n + x^n)}{x^n(\delta^n + x_1^n)} - \frac{x^n(\delta^n + x_1^n)(\theta + v_1) v w_1}{x_1^n(\delta^n + x^n)(\theta + v) v_1 w} - \frac{y_1 w}{y w_1} - \frac{y v_1}{y_1 v} - \frac{\theta + v}{\theta + v_1}\right] \\ &+ \frac{(e+b)\xi}{(e\xi + b)} \frac{\eta x_1^n v_1}{(\delta^n + x_1^n)(\theta + v_1)} \left[4 - \frac{x_1^n(\delta^n + x^n)}{x^n(\delta^n + x_1^n)} - \frac{x^n(\delta^n + x_1^n)(\theta + v_1) v y_1}{x_1^n(\delta^n + x^n)(\theta + v) v_1 y} - \frac{y v_1}{y_1 v} - \frac{\theta + v}{\theta + v_1}\right].\end{aligned} \tag{21}$$

Clearly, the first two terms of Eq. (21) are less than or equal zero. Because the geometrical mean is less than or equal to the arithmetical mean, then the last two terms of Eq. (21) are less than or equal zero. Now we show that if $R_1 \le 1$ then $v_1 \le \frac{\mu}{r} = v_2$. This can be achieved if we show that

$$sgn\,(x_2 - x_1) = sgn\,(v_1 - v_2) = sgn\,(R_1 - 1)\,.$$

We have

$$(x_2^n - x_1^n)(x_2 - x_1) > 0, \quad \text{for all } n > 0 \tag{22}$$

Suppose that, $sgn\,(x_2 - x_1) = sgn\,(v_2 - v_1)$. Using the conditions of the steady states E_1 and E_2 we have

$$\begin{aligned}(\rho - dx_2) - (\rho - dx_1) &= \frac{\eta x_2^n v_2}{(\delta^n + x_2^n)(\theta + v_2)} - \frac{\eta x_1^n v_1}{(\delta^n + x_1^n)(\theta + v_1)} \\ &= \frac{\eta x_2^n v_2}{(\delta^n + x_2^n)(\theta + v_2)} - \frac{\eta x_2^n v_1}{(\delta^n + x_2^n)(\theta + v_1)} + \frac{\eta x_2^n v_1}{(\delta^n + x_2^n)(\theta + v_1)} - \frac{\eta x_1^n v_1}{(\delta^n + x_1^n)(\theta + v_1)} \\ &= \frac{\eta x_2^n}{\delta^n + x_2^n} \frac{\theta(v_2 - v_1)}{(\theta + v_2)(\theta + v_1)} + \frac{\eta v_1}{\theta + v_1}\left(\frac{\delta^n(x_2^n - x_1^n)}{(\delta^n + x_2^n)(\delta^n + x_1^n)}\right)\end{aligned}$$

and from inequalities (22) we get:

$$sgn\,(x_1 - x_2) = sgn\,(x_2 - x_1)\,,$$

which leads to contradiction. Thus, $sgn\,(x_2 - x_1) = sgn\,(v_1 - v_2)$. Using the steady state conditions for E_1 we have $\frac{k(e\xi+b)}{ac(e+b)} \frac{\eta x_1^n}{(\delta^n + x_1^n)(\theta + v_1)} = 1$, then

$$\begin{aligned}R_1 - 1 &= \frac{k(e\xi + b)}{ac(e+b)}\left[\frac{\eta x_2^n}{(\delta^n + x_2^n)(\theta + v_2)} - \frac{\eta x_1^n}{(\delta^n + x_1^n)(\theta + v_1)}\right] \\ &= \frac{k(e\xi + b)}{ac(e+b)}\left[\frac{\eta x_2^n}{(\delta^n + x_2^n)(\theta + v_2)} - \frac{\eta x_2^n}{(\delta^n + x_2^n)(\theta + v_1)}\right. \\ &\left. + \frac{\eta x_2^n}{(\delta^n + x_2^n)(\theta + v_1)} - \frac{\eta x_1^n}{(\delta^n + x_1^n)(\theta + v_1)}\right] \\ &= \frac{k(e\xi + b)}{ac(e+b)}\left[\frac{\eta x_2^n(v_1 - v_2)}{(\delta^n + x_2^n)(\theta + v_1)(\theta + v_2)} + \frac{\eta \delta^n(x_2^n - x_1^n)}{(\delta^n + x_2^n)(\delta^n + x_1^n)(\theta + v_1)}\right].\end{aligned}$$

From inequality (22) we get:

$$sgn\,(R_1 - 1) = sgn\,(v_1 - v_2)\,.$$

It follows that, if $R_1 \leq 1$ then $v_1 \leq \frac{\mu}{r} = v_2$. Therefore, if $R_1 \leq 1$ then $\frac{dW_1}{dt} \leq 0$ for all $x, w, y, v, z > 0$, where the equality occurs at the steady state E_1. LaSalle's invariance principle implies the global stability of E_1.

3.3 Global stability of the infected steady state with antibody immune response E_2

Theorem 3. If $R_1 > 1$, then E_2 is GAS.

Proof. We construct the following Lyapunov functional

$$W_2 = x - x_2 - \int_{x_2}^{x} \frac{x_2^n(\delta^n + s^n)}{s^n(\delta^n + x_2^n)} ds + \frac{b}{e\xi + b} w_2 H\left(\frac{w}{w_2}\right) + \frac{e+b}{e\xi + b} y_2 H\left(\frac{y}{y_2}\right) + \frac{a(e+b)}{k(e\xi + b)} v_2 H\left(\frac{v}{v_2}\right) + \frac{ar(e+b)}{kg(e\xi + b)} z_2 H\left(\frac{z}{z_2}\right).$$

We calculate the time derivative of W_2 along the trajectories of (5)-(9) as:

$$\begin{aligned}
\frac{dW_2}{dt} &= \left(1 - \frac{x_2^n(\delta^n + x^n)}{x^n(\delta^n + x_2^n)}\right)\left(\rho - dx - \frac{\eta x^n v}{(\delta^n + x^n)(\theta + v)}\right) \\
&+ \frac{b}{e\xi + b}\left(1 - \frac{w_2}{w}\right)\left((1-\xi)\frac{\eta x^n v}{(\delta^n + x^n)(\theta + v)} - (e+b)w\right) \\
&+ \frac{e+b}{e\xi + b}\left(1 - \frac{y_2}{y}\right)\left(\xi\frac{\eta x^n v}{(\delta^n + x^n)(\theta + v)} + bw - ay\right) \\
&+ \frac{a(e+b)}{k(e\xi + b)}\left(1 - \frac{v_2}{v}\right)(ky - cv - rvz) + \frac{ar(e+b)}{kg(e\xi + b)}\left(1 - \frac{z_2}{z}\right)(gvz - \mu z).
\end{aligned} \tag{23}$$

Applying $\rho = dx_2 + \frac{\eta x_2^n v_2}{(\delta^n + x_2^n)(\theta + v_2)}$ and collecting terms of Eq. (23) we get

$$\begin{aligned}
\frac{dW_2}{dt} &= \left(1 - \frac{x_2^n(\delta^n + x^n)}{x^n(\delta^n + x_2^n)}\right)(dx_2 - dx) + \frac{\eta x_2^n v}{(\delta^n + x_2^n)(\theta + v)} \\
&+ \frac{\eta x_2^n v_2}{(\delta^n + x_2^n)(\theta + v_2)}\left(1 - \frac{x_2^n(\delta^n + x^n)}{x^n(\delta^n + x_2^n)}\right) - \frac{b(1-\xi)}{e\xi + b}\frac{\eta x^n v}{(\delta^n + x^n)(\theta + v)}\frac{w_2}{w} + \frac{b(e+b)}{e\xi + b}w_2 \\
&- \frac{(e+b)\xi}{e\xi + b}\frac{\eta x^n v}{(\delta^n + x^n)(\theta + v)}\frac{y_2}{y} - \frac{(e+b)b}{e\xi + b}\frac{y_2 w}{y} + \frac{e+b}{e\xi + b}ay_2 - \frac{ac(e+b)}{k(e\xi + b)}v - \frac{a(e+b)}{(e\xi + b)}\frac{yv_2}{v} \\
&+ \frac{ac(e+b)}{k(e\xi + b)}v_2 + \frac{ar(e+b)}{k(e\xi + b)}v_2 z - \frac{ar\mu(e+b)}{kg(e\xi + b)}z - \frac{ar(e+b)}{k(e\xi + b)}z_2 v + \frac{ar\mu(e+b)}{kg(e\xi + b)}z_2.
\end{aligned}$$

Using the steady state conditions for E_2

$$(1-\xi)\frac{\eta x_2^n v_2}{(\delta^n + x_2^n)(\theta + v_2)} = (e+b)w_2,\ \xi\frac{\eta x_2^n v_2}{(\delta^n + x_2^n)(\theta + v_2)} + bw_2 = ay_2, ky_2 = cv_2 + rv_2 z_2,\ \mu = gv_2,$$

we get

$$\begin{aligned}
\frac{dW_2}{dt} &= dx_2\left(1-\frac{x_2^n(\delta^n+x^n)}{x^n(\delta^n+x_2^n)}\right)\left(1-\frac{x}{x_2}\right)+\frac{\eta x_2^n v_2}{(\delta^n+x_2^n)(\theta+v_2)}\left[\frac{v(\theta+v_2)}{v_2(\theta+v)}-\frac{v}{v_2}\right] \\
&+\left(\frac{b(1-\xi)}{e\xi+b}+\frac{(e+b)\xi}{e\xi+b}\right)\frac{\eta x_2^n v_2}{(\delta^n+x_2^n)(\theta+v_2)}\left(1-\frac{x_2^n(\delta^n+x^n)}{x^n(\delta^n+x_2^n)}\right) \\
&-\frac{b(1-\xi)}{e\xi+b}\frac{\eta x_2^n v_2}{(\delta^n+x_2^n)(\theta+v_2)}\frac{x^n(\delta^n+x_2^n)(\theta+v_2)vw_2}{x_2^n(\delta^n+x^n)(\theta+v)v_2w} \\
&+\frac{b(1-\xi)}{e\xi+b}\frac{\eta x_2^n v_2}{(\delta^n+x_2^n)(\theta+v_2)}-\frac{(e+b)\xi}{e\xi+b}\frac{\eta x_2^n v_2}{(\delta^n+x_2^n)(\theta+v_2)}\frac{x^n(\delta^n+x_2^n)(\theta+v_2)vy_2}{x_2^n(\delta^n+x^n)(\theta+v)v_2y} \\
&-\frac{b(1-\xi)}{e\xi+b}\frac{\eta x_2^n v_2}{(\delta^n+x_2^n)(\theta+v_2)}\frac{y_2w}{yw_2}+\left(\frac{b(1-\xi)}{e\xi+b}+\frac{(e+b)\xi}{e\xi+b}\right)\frac{\eta x_2^n v_2}{(\delta^n+x_2^n)(\theta+v_2)} \\
&-\left(\frac{b(1-\xi)}{e\xi+b}+\frac{(e+b)\xi}{e\xi+b}\right)\frac{\eta x_2^n v_2}{(\delta^n+x_2^n)(\theta+v_2)}\frac{yv_2}{y_2v}+\left(\frac{b(1-\xi)}{e\xi+b}+\frac{(e+b)\xi}{e\xi+b}\right)\frac{\eta x_2^n v_2}{(\delta^n+x_2^n)(\theta+v_2)} \\
&= -\frac{d\delta^n(x^n-x_2^n)(x-x_2)}{x^n(\delta^n+x_2^n)}-\frac{\eta x_2^n\theta(v-v_2)^2}{(\delta^n+x_2^n)(\theta+v)(\theta+v_2)^2} \\
&+\frac{b(1-\xi)}{(e\xi+b)}\frac{\eta x_2^n v_2}{(\delta^n+x_2^n)(\theta+v_2)}\left[5-\frac{x_2^n(\delta^n+x^n)}{x^n(\delta^n+x_2^n)}-\frac{x^n(\delta^n+x_2^n)(\theta+v_2)vw_2}{x_2^n(\delta^n+x^n)(\theta+v)v_2w}-\frac{y_2w}{yw_2}-\frac{yv_2}{y_2v}-\frac{\theta+v}{\theta+v_2}\right] \\
&+\frac{(e+b)\xi}{(e\xi+b)}\frac{\eta x_2^n v_2}{(\delta^n+x_2^n)(\theta+v_2)}\left[4-\frac{x_2^n(\delta^n+x^n)}{x^n(\delta^n+x_2^n)}-\frac{x^n(\delta^n+x_2^n)(\theta+v_2)vy_2}{x_2^n(\delta^n+x^n)(\theta+v)v_2y}-\frac{yv_2}{y_2v}-\frac{\theta+v}{\theta+v_2}\right].
\end{aligned} \tag{24}$$

Thus, if $R_1 > 1$ then x_2, w_2, y_2, v_2 and $z_2 > 0$. Clearly, $\frac{dW_2}{dt} \leq 0$ and $\frac{dW_2}{dt} = 0$ if and only if $x(t) = x_2, w(t) = w_2$ and $v(t) = v_2$. From Eq. (8), if $v(t) = v_2$ and $y(t) = y_2$, then $\dot{v}(t) = 0$ and $0 = ky_2 - cv_2 - rv_2z(t)$, which yields $z(t) = z_2$ and hence $\frac{dW_2}{dt}$ equal to zero at E_2. LaSalle's invariance principle implies global stability of E_2.

4 Conclusion

In this paper, we have proposed and analyzed a virus dynamics model with antibody immune response. The model is a five dimensional that describe the interaction between the uninfected target cells, latently infected cells, actively infected cells, free virus particles and antibody immune cells. The incidence rate has been represented by nonlinear function. We have derived two threshold parameters, the basic reproductive number of viral infection R_0 and the basic reproductive number of antibody immune response R_1 which completely determined the basic and global properties of the virus dynamics model. Using Lyapunov method and applying LaSalle's invariance principle we have proven that, if $R_0 \leq 1$, then the uninfected steady state is GAS, if $R_1 \leq 1 < R_0$, then the infected steady state without antibody immune response is GAS, and if $R_1 > 1$, then the infected steady state with antibody immune response is GAS.

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On the symmetric properties for the generalized twisted (h, q)-tangent numbers and polynomials associated with p-adic integral on $\mathbb{Z}_p$

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Abstract : In this paper, we study the symmetry for the generalized twisted (h, q)-tangent numbers $T^{(h)}_{n,\chi,q,\zeta}$ and polynomials $T^{(h)}_{n,\chi,q,\zeta}(x)$. We obtain some interesting identities of the power sums and the generalized twisted polynomials $T^{(h)}_{n,\chi,q,\zeta}(x)$ using the symmetric properties for the p-adic invariant integral on $\mathbb{Z}_p$.

Key words : Symmetric properties, power sums, the tangent numbers and polynomials, the generalized twisted (h, q)-tangent numbers and polynomials, p-adic integral on $\mathbb{Z}_p$.

2000 Mathematics Subject Classification : 11B68, 11S40, 11S80.

1. Introduction

Recently, many mathematicians have studies different kinds of the Euler, Bernoulli, Genocchi, Tangent numbers and polynomials(see [1-10]). These numbers and polynomials play important roles in many different areas of mathematics such as number theory, combinatorics, special function and analysis. The purpose of this paper is to obtain some interesting identities of the power sums and generalized twisted (h, q)-tangent polynomials $T^{(h)}_{n,\chi,q,\zeta}(x)$ using the symmetric properties for the p-adic invariant integral on $\mathbb{Z}_p$.

Throughout this paper, we always make use of the following notations: $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, $\mathbb{C}$ denotes the set of complex numbers, $\mathbb{Z}_p$ denotes the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q - 1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$.

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable function on $\mathbb{Z}_p$. For $g \in UD(\mathbb{Z}_p)$ the fermionic p-adic invariant q-integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x, \text{ see } [1, 2, 3] .$$

Note that

$$\lim_{q\to 1} I_{-q}(g) = I_{-1}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x). \tag{1.1}$$

If we take $g_n(x) = g(x + n)$ in (1.1), then we see that

$$I_{-1}(g_n) = (-1)^n I_{-1}(g) + 2 \sum_{l=0}^{n-1} (-1)^{n-1-l} g(l). \tag{1.2}$$

Let a fixed positive integer d with $(p,d)=1$, set

$$X = X_d = \varprojlim_N (\mathbb{Z}/dp^N\mathbb{Z}), \quad X_1 = \mathbb{Z}_p,$$
$$X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} a + dp\mathbb{Z}_p,$$
$$a + dp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^N}\},$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \le a < dp^N$.

It is easy to see that

$$I_{-1}(g) = \int_X g(x) d\mu_{-1}(x) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x). \tag{1.3}$$

We assume that $h \in \mathbb{Z}$. Let $T_p = \cup_{N \ge 1} C_{p^N} = \lim_{N \to \infty} C_{p^N}$, where $C_{p^N} = \{\zeta | w^{p^N} = 1\}$ is the cyclic group of order p^N. For $\zeta \in T_p$, we denote by $\phi_\zeta : \mathbb{Z}_p \to \mathbb{C}_p$ the locally constant function $x \longmapsto \zeta^x$.

First, we introduce the tangent numbers and tangent polynomials. In [5], we investigated the zeros of the tangent polynomials $T_n(x)$. The tangent numbers T_n are defined by the generating function:

$$F(t) = \frac{2}{e^{2t}+1} = \sum_{n=0}^{\infty} T_n \frac{t^n}{n!} \quad (|t| < \frac{\pi}{2}), \text{ cf. [5]}$$

where we use the technique method notation by replacing T^n by $T_n (n \ge 0)$ symbolically. We consider the tangent polynomials $T_n(x)$ as follows:

$$F(x,t) = \left(\frac{2}{e^{2t}+1}\right) e^{xt} = \sum_{n=0}^{\infty} T_n(x) \frac{t^n}{n!}. \tag{1.4}$$

Note that $T_n(x) = \sum_{k=0}^{n} \binom{n}{k} T_k x^{n-k}$. In the special case $x=0$, we define $T_n(0) = T_n$.

In [8], we introduced the generalized twisted (h,q)-tangent numbers $T_{n,\chi,q,\zeta}^{(h)}$ and polynomials $T_{n,\chi,q,\zeta}^{(h)}(x)$ attached to χ. Let χ be Dirichlet's character with conductor $d \in \mathbb{N}$ with $d \equiv 1 (\mathrm{mod}\ 2)$. The generalized twisted (h,q)-tangent numbers $T_{n,\chi,q,\zeta}^{(h)}$ attached to χ are defined by the generating function:

$$\frac{2\sum_{a=0}^{d-1} \chi(a)(-1)^a \zeta^a q^{ha} e^{2at}}{\zeta^d q^{hd} e^{2dt}+1} = \sum_{n=0}^{\infty} T_{n,\chi,q,\zeta}^{(h)} \frac{t^n}{n!}, \text{ cf. [8]}. \tag{1.5}$$

We consider the generalized twisted (h,q)-tangent polynomials $T_{n,\chi,q,w}^{(h)}(x)$ attached to χ as follows:

$$\left(\frac{2\sum_{a=0}^{d-1} \chi(a)(-1)^a \zeta^a q^{ha} e^{2at}}{\zeta^d q^{hd} e^{2dt}+1}\right) e^{xt} = \sum_{n=0}^{\infty} T_{n,\chi,q,\zeta}^{(h)}(x) \frac{t^n}{n!}. \tag{1.6}$$

Let $g(y) = \chi(y)\phi_\zeta(y) q^{hy} e^{(2y+x)t}$. By (1.3), we derive

$$\begin{aligned} I_{-1}\left(\chi(y)\phi_\zeta(y) q^{hy} e^{(2y+x)t}\right) &= \int_X \chi(y)\phi_\zeta(y) q^{hy} e^{(2y+x)t} d\mu_{-1}(y) \\ &= \left(\frac{2\sum_{a=0}^{d-1} \chi(a)(-1)^a \zeta^a q^{ha} e^{2at}}{\zeta^d q^{hd} e^{2dt}+1}\right) e^{xt} \\ &= \sum_{n=0}^{\infty} T_{n,\chi,q,\zeta}^{(h)}(x) \frac{t^n}{n!}. \end{aligned} \tag{1.7}$$

By using Taylor series of $e^{(2y+x)t}$ in the above equation (1.7), we obtain

$$\sum_{n=0}^{\infty} \left(\int_X \chi(y)\phi_\zeta(y) q^{hy} (2y+x)^n d\mu_{-1}(y)\right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} T_{n,\chi,q,\zeta}^{(h)}(x) \frac{t^n}{n!}.$$

By comparing coefficients of $\frac{t^n}{n!}$ in the above equation, we have the Witt formula for the generalized twisted (h,q)- tangent polynomials attached to χ as follows:

Theorem 1. For positive integers n, $\zeta \in T_p$, and $h \in \mathbb{Z}$, we have

$$T^{(h)}_{n,\chi,q,\zeta}(x) = \int_X \chi(y)\phi_\zeta(y)q^{hy}(2y+x)^n d\mu_{-1}(y). \tag{1.8}$$

If we take $x = 0$ in Theorem 1, we also have the following corollary.

Corollary 2. For positive integers n, $\zeta \in T_p$, and $h \in \mathbb{Z}$, we have

$$T^{(h)}_{n,\chi,q,\zeta} = \int_X \chi(y)\phi_\zeta(y)q^{hy}(2y)^n d\mu_{-1}(y). \tag{1.9}$$

By (1.8) and (1.9), we have the following theorem.

Theorem 3. For positive integers n, $\zeta \in T_p$, and $h \in \mathbb{Z}$, we have

$$T^{(h)}_{n,\chi,q,\zeta}(x) = \sum_{l=0}^{n} \binom{n}{l} T^{(h)}_{l,\chi,q,\zeta} x^{n-l}.$$

2. Symmetry for the generalized twisted (h,q)-tangent polynomials

In this section, we assume that $q \in \mathbb{C}_p$ and $\zeta \in T_p$. By using the symmetric properties for the p-adic invariant integral on $\mathbb{Z}_p$, we obtain some interesting identities of the power sums and the generalized twisted polynomials $T^{(h)}_{n,\chi,q,\zeta}(x)$. If n is odd from (1.2), we obtain

$$I_{-1}(g_n) + I_{-1}(g) = 2\sum_{k=0}^{n-1}(-1)^k g(k) \text{ (see [1], [2], [3], [5]).} \tag{2.1}$$

It will be more convenient to write (2.1) as the equivalent integral form

$$\int_{\mathbb{Z}_p} g(x+n)d\mu_{-1}(x) + \int_{\mathbb{Z}_p} g(x)d\mu_{-1}(x) = 2\sum_{k=0}^{n-1}(-1)^k g(k). \tag{2.2}$$

Substituting $g(x) = \chi(x)\zeta^x q^{hx} e^{2xt}$ into the above, we obtain

$$\begin{aligned}&\int_X \chi(x+n)\zeta^{x+n}q^{h(x+n)}e^{(2x+2n)t}d\mu_{-1}(x) + \int_X \chi(x)\zeta^x q^{hx}e^{2xt}d\mu_{-1}(x)\\ &= 2\sum_{j=0}^{n-1}(-1)^j\chi(j)\zeta^j q^{hj}e^{2jt}.\end{aligned} \tag{2.3}$$

For $k \in \mathbb{Z}_+$, let us define the power sums $\mathcal{T}^{(h)}_{k,\chi,q,\zeta}(n)$ as follows:

$$\mathcal{T}^{(h)}_{k,\chi,q,\zeta}(n) = \sum_{l=0}^{n}(-1)^l\chi(l)\zeta^l q^{hl}(2l)^k. \tag{2.4}$$

After some elementary calculations, we have

$$\begin{aligned}&\int_X \chi(x)\zeta^x q^{hx}e^{2xt}d\mu_{-1}(x) = \frac{2\sum_{a=0}^{d-1}\chi(a)(-1)^a\zeta^a q^{ha}e^{2at}}{\zeta^d q^{hd}e^{2dt}+1},\\ &\int_X \chi(x)\zeta^{x+n}q^{h(x+n)}e^{(2x+2n)t}d\mu_{-1}(x) = \zeta^n q^{hn}e^{2nt}\frac{2\sum_{a=0}^{d-1}\chi(a)(-1)^a\zeta^a q^{ha}e^{2at}}{\zeta^d q^{hd}e^{2dt}+1}.\end{aligned} \tag{2.5}$$

By using (2.5), we have

$$\begin{aligned}&\int_X \chi(x)\zeta^{x+nd}q^{h(x+nd)}e^{(2x+2nd)t}d\mu_{-1}(x)+\int_X \chi(x)\zeta^{x}q^{hx}e^{2xt}d\mu_{-1}(x)\\&=\left(1+\zeta^{nd}q^{hnd}e^{2ndt}\right)\frac{2\sum_{a=0}^{d-1}\chi(a)(-1)^a\zeta^a q^{ha}e^{2at}}{\zeta^d q^{hd}e^{2dt}+1}.\end{aligned}$$

From the above, we get

$$\begin{aligned}&\int_X \chi(x)\zeta^{x+nd}q^{h(x+nd)}e^{(2x+2nd))t}d\mu_{-1}(x)+\int_X \chi(x)\zeta^{x}q^{hx}e^{2xt}d\mu_{-1}(x)\\&=\frac{2\int_X \chi(x)\zeta^x q^{hx}e^{2xt}d\mu_{-1}(x)}{\int_X \zeta^{ndx}q^{hndx}e^{2ndtx}d\mu_{-1}(x)}.\end{aligned}\tag{2.6}$$

By substituting Taylor series of e^{2xt} into (2.3), we obtain

$$\begin{aligned}&\sum_{m=0}^{\infty}\left(\int_X \chi(x)\zeta^{x+nd}q^{h(x+nd)}(2x+2nd)^m d\mu_{-1}(x)+\int_X \chi(x)\zeta^x q^{hx}(2x)^m d\mu_{-1}(x)\right)\frac{t^m}{m!}\\&=\sum_{m=0}^{\infty}\left(2\sum_{j=0}^{nd-1}(-1)^j\chi(j)\zeta^j q^{hj}(2j)^m\right)\frac{t^m}{m!}\end{aligned}$$

By comparing coefficients $\dfrac{t^m}{m!}$ in the above equation, we obtain

$$\begin{aligned}&\zeta^{nd}q^{hnd}\sum_{k=0}^{m}\binom{m}{k}(2nd)^{m-k}\int_X \chi(x)\zeta^x q^{hx}(2x)^k d\mu_{-1}(x)+\int_X \chi(x)\zeta^x q^{hx}(2x)^m d\mu_{-1}(x)\\&=2\sum_{j=0}^{nd-1}(-1)^j\chi(j)\zeta^j q^{hj}(2j)^m\end{aligned}$$

By using (2.4), we have

$$\begin{aligned}&\zeta^{nd}q^{hnd}\sum_{k=0}^{m}\binom{m}{k}(2nd)^{m-k}\int_X \chi(x)\zeta^x q^{hx}(2x)^k d\mu_{-1}(x)+\int_X \chi(x)\zeta^x q^{hx}(2x)^m d\mu_{-1}(x)\\&=2\mathcal{T}^{(h)}_{m,\chi,q,\zeta}(nd-1).\end{aligned}\tag{2.7}$$

By using (2.6) and (2.7), we arrive at the following theorem:

Theorem 4. Let n be odd positive integer. Then we obtain

$$\frac{2\int_X \chi(x)\zeta^x q^{hx}e^{2xt}d\mu_{-1}(x)}{\int_X \zeta^{ndx}q^{hndx}e^{2ndtx}d\mu_{-1}(x)}=\sum_{m=0}^{\infty}\left(2\mathcal{T}^{(h)}_{m,\chi,q,\zeta}(nd-1)\right)\frac{t^m}{m!}.$$

Let w_1 and w_2 be odd positive integers. Then we set

$$\begin{aligned}&S(w_1,w_2)=\\&\frac{\int_X\int_X \chi(x_1)\chi(x_2)\zeta^{(w_1x_1+w_2x_2)}q^{h(w_1x_1+w_2x_2)}e^{(2w_1x_1+2w_2x_2+w_1w_2x)t}d\mu_{-1}(x_1)d\mu_{-1}(x_2)}{\int_X \zeta^{w_1w_2dx}q^{hw_1w_2dx}e^{2w_1w_2dxt}d\mu_{-1}(x)}.\end{aligned}\tag{2.8}$$

By Theorem 4 and (2.8), after elementary calculations, we obtain

$$\begin{aligned}S(w_1,w_2)&=\left(\frac{1}{2}\int_X \chi(x_1)\zeta^{w_1x_1}q^{hw_1x_1}e^{(2w_1x_1+w_1w_2x)t}d\mu_{-1}(x_1)\right)\\&\quad\times\left(\frac{2\int_X \chi(x_2)\zeta^{w_2x_2}q^{hw_2x_2}e^{2x_2w_2t}d\mu_{-1}(x_2)}{\int_X \zeta^{w_1w_2dx}q^{hw_1w_2dx}e^{2w_1w_2dtx}d\mu_{-1}(x)}\right)\\&=\left(\frac{1}{2}\sum_{m=0}^{\infty}T^{(h)}_{m,\chi,q^{w_1},\zeta^{w_1}}(w_2x)w_1^m\frac{t^m}{m!}\right)\left(2\sum_{m=0}^{\infty}\mathcal{T}^{(h)}_{m,\chi,q^{w_2},\zeta^{w_2}}(w_1d-1)w_2^m\frac{t^m}{m!}\right).\end{aligned}\tag{2.9}$$

By using Cauchy product in the above, we have

$$S(w_1,w_2)=\sum_{m=0}^{\infty}\left(\sum_{j=0}^{m}\binom{m}{j}T^{(h)}_{j,\chi,q^{w_1},\zeta^{w_1}}(w_2x)w_1^j\mathcal{T}^{(h)}_{m-j,\chi,q^{w_2},\zeta^{w_2}}(w_1d-1)w_2^{m-j}\right)\frac{t^m}{m!}. \tag{2.10}$$

From the symmetry of $S(w_1,w_2)$ in w_1 and w_2, we also see that

$$\begin{aligned}S(w_1,w_2)&=\left(\frac{1}{2}\int_X\chi(x_2)\zeta^{w_2x_2}q^{hw_2x_2}e^{(2w_2x_2+w_1w_2x)t}d\mu_{-1}(x_2)\right)\\&\quad\times\left(\frac{2\int_X\chi(x_1)\zeta^{w_1x_1}q^{hw_1x_1}e^{2x_1w_1t}d\mu_{-1}(x_1)}{\int_X\zeta^{w_1w_2dx}q^{hw_1w_2dx}e^{2w_1w_2dtx}d\mu_{-1}(x)}\right)\\&=\left(\frac{1}{2}\sum_{m=0}^{\infty}T^{(h)}_{m,\chi,q^{w_2},\zeta^{w_2}}(w_1x)w_2^m\frac{t^m}{m!}\right)\left(2\sum_{m=0}^{\infty}\mathcal{T}^{(h)}_{m,\chi,q^{w_1},\zeta^{w_1}}(w_2d-1)w_1^m\frac{t^m}{m!}\right).\end{aligned}$$

Thus we have

$$S(w_1,w_2)=\sum_{m=0}^{\infty}\left(\sum_{j=0}^{m}\binom{m}{j}T^{(h)}_{j,\chi,q^{w_2},\zeta^{w_2}}(w_1x)w_2^j\mathcal{T}^{(h)}_{m-j,\chi,q^{w_1},\zeta^{w_1}}(w_2d-1)w_1^{m-j}\right)\frac{t^m}{m!} \tag{2.11}$$

By comparing coefficients $\frac{t^m}{m!}$ in the both sides of (2.10) and (2.11), we arrive at the following theorem:

Theorem 5. Let w_1 and w_2 be odd positive integers. Then we obtain

$$\begin{aligned}&\sum_{j=0}^{m}\binom{m}{j}w_1^{m-j}w_2^jT^{(h)}_{j,\chi,q^{w_2},\zeta^{w_2}}(w_1x)\mathcal{T}^{(h)}_{m-j,\chi,q^{w_1},\zeta^{w_1}}(w_2d-1)\\&=\sum_{j=0}^{m}\binom{m}{j}w_1^jw_2^{m-j}T^{(h)}_{j,\chi,q^{w_1},\zeta^{w_1}}(w_2x)\mathcal{T}^{(h)}_{m-j,\chi,q^{w_2},\zeta^{w_2}}(w_1d-1),\end{aligned}$$

where $T^{(h)}_{k,\chi,q,\zeta}(x)$ and $\mathcal{T}^{(h)}_{m,\chi,q,\zeta}(k)$ denote the generalized twisted (h,q)-tangent polynomials and the alternating sums of powers of consecutive (h,q)-integers, respectively.

By Theorem 3, we have the following corollary.

Corollary 6. Let w_1 and w_2 be odd positive integers. Then we obtain

$$\begin{aligned}&\sum_{j=0}^{m}\sum_{k=0}^{j}\binom{m}{j}\binom{j}{k}w_1^{m-k}w_2^jx^{j-k}T^{(h)}_{k,\chi,q^{w_2},\zeta^{w_2}}\mathcal{T}^{(h)}_{m-j,\chi,q^{w_1},\zeta^{w_1}}(w_2d-1)\\&=\sum_{j=0}^{m}\sum_{k=0}^{j}\binom{m}{j}\binom{j}{k}w_1^jw_2^{m-k}x^{j-k}T^{(h)}_{k,\chi,q^{w_1},\zeta^{w_1}}\mathcal{T}^{(h)}_{m-j,\chi,q^{w_2},\zeta^{w_2}}(w_1d-1).\end{aligned}$$

Now we will derive another interesting identities for the generalized twisted (h,q)-tangent polynomials using the symmetric property of $S(w_1,w_2)$.

$$
\begin{aligned}
S(w_1,w_2) &= \left(\frac{1}{2}\int_X \chi(x_1)\zeta^{w_1x_1}q^{hw_1x_1}e^{(2w_1x_1+w_1w_2x)t}d\mu_{-1}(x_1)\right) \\
&\quad\times\left(\frac{2\int_X \chi(x_2)\zeta^{w_2x_2}q^{hw_2x_2}e^{2x_2w_2t}d\mu_{-1}(x_2)}{\int_X \zeta^{w_1w_2dx}q^{hw_1w_2dx}e^{2w_1w_2dtx}d\mu_{-1}(x)}\right) \\
&= \left(\frac{1}{2}e^{w_1w_2xt}\int_X \chi(x_1)\zeta^{w_1x_1}q^{hw_1x_1}e^{2x_1w_1t}d\mu_{-1}(x_1)\right) \\
&\quad\times\left(2\sum_{j=0}^{w_1d-1}(-1)^j\chi(j)\zeta^{w_2j}q^{w_2hj}e^{2jw_2t}\right) \\
&= \sum_{j=0}^{w_1d-1}(-1)^j\chi(j)\zeta^{w_2j}q^{w_2hj}\int_X \chi(x_1)\zeta^{w_1x_1}q^{hw_1x_1}e^{\left(2x_1+w_2x+\frac{2jw_2}{w_1}\right)(w_1t)}d\mu_{-1}(x_1) \\
&= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{w_1d-1}(-1)^j\chi(j)\zeta^{w_2j}q^{w_2hj}T^{(h)}_{n,\chi,q^{w_1},\zeta^{w_1}}\left(w_2x+\frac{2jw_2}{w_1}\right)w_1^n\right)\frac{t^n}{n!}.
\end{aligned}
\tag{2.12}
$$

By using the symmetry property in (2.12), we also have

$$
\begin{aligned}
S(w_1,w_2) &= \left(\frac{1}{2}e^{w_1w_2xt}\int_X \chi(x_2)\zeta^{w_2x_2}q^{hw_2x_2}e^{2x_2w_2t}d\mu_{-1}(x_2)\right) \\
&\quad\times\left(\frac{2\int_X \chi(x_1)\zeta^{w_1x_1}q^{hw_1x_1}e^{2x_1w_1t}d\mu_{-1}(x_1)}{\int_X \zeta^{w_1w_2dx}q^{hw_1w_2dx}e^{2w_1w_2dtx}d\mu_{-1}(x)}\right) \\
&= \left(\frac{1}{2}e^{w_1w_2xt}\int_X \chi(x_2)\zeta^{w_2x_2}q^{hw_2x_2}e^{2x_2w_2t}d\mu_{-1}(x_2)\right) \\
&\quad\times\left(2\sum_{j=0}^{w_2d-1}(-1)^j\chi(j)\zeta^{w_1j}q^{w_1hj}e^{2jw_1t}\right) \\
&= \sum_{j=0}^{w_2d-1}(-1)^j\chi(j)\zeta^{w_1j}q^{w_1hj}\int_X \chi(x_2)\zeta^{w_2x_2}q^{hw_2x_2}e^{\left(2x_2+w_1x+\frac{2jw_1}{w_2}\right)(w_2t)}d\mu_{-1}(x_1) \\
&= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{w_2-1}(-1)^j\chi(j)\zeta^{w_1j}q^{w_1hj}T^{(h)}_{n,\chi,q^{w_2},\zeta^{w_2}}\left(w_1x+\frac{2jw_1}{w_2}\right)w_2^n\right)\frac{t^n}{n!}.
\end{aligned}
\tag{2.13}
$$

By comparing coefficients $\frac{t^n}{n!}$ in the both sides of (2.12) and (2.13), we have the following theorem.

Theorem 7. Let w_1 and w_2 be odd positive integers. Then we obtain

$$
\begin{aligned}
&\sum_{j=0}^{w_1d-1}(-1)^j\chi(j)\zeta^{w_2j}q^{w_2hj}T^{(h)}_{n,\chi,q^{w_1},\zeta^{w_1}}\left(w_2x+\frac{2jw_2}{w_1}\right)w_1^n \\
&=\sum_{j=0}^{w_2d-1}(-1)^j\chi(j)\zeta^{w_1j}q^{w_1hj}T^{(h)}_{n,\chi,q^{w_2},\zeta^{w_2}}\left(w_1x+\frac{2jw_1}{w_2}\right)w_2^n.
\end{aligned}
\tag{2.14}
$$

If we take $x = 0$ in Theorem 7, we also derive the interesting identity for the generalized twisted (h,q)-tangent numbers as follows:

$$
\begin{aligned}
&\sum_{j=0}^{w_1d-1}(-1)^j\chi(j)\zeta^{w_2j}q^{w_2hj}T^{(h)}_{n,\chi,q^{w_1},\zeta^{w_1}}\left(\frac{2jw_2}{w_1}\right)w_1^n \\
&=\sum_{j=0}^{w_2d-1}(-1)^j\chi(j)\zeta^{w_1j}q^{w_1hj}T^{(h)}_{n,\chi,q^{w_2},\zeta^{w_2}}\left(\frac{2jw_1}{w_2}\right)w_2^n.
\end{aligned}
$$

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