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Journal of Computational Analysis and Applications

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Contribution to a Book

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ON QUADRATIC ρ-FUNCTIONAL INEQUALITIES IN FUZZY NORMED SPACES

CHOONKIL PARK, SUN YOUNG JANG, AND SUNGSIK YUN*

ABSTRACT. In this paper, we solve the following quadratic ρ-functional inequalities

$$N\left(f(x+y)+f(x-y)-2f(x)-2f(y)-\rho\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right),t\right) \geq \frac{t}{t+\varphi(x,y)}, \tag{0.1}$$

where ρ is a fixed real number with $\rho \neq 2$, and

$$N\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)-\rho\left(f(x+y)+f(x-y)-2f(x)-2f(y)\right),t\right) \geq \frac{t}{t+\varphi(x,y)}, \tag{0.2}$$

where ρ is a fixed real number with $\rho \neq \frac{1}{2}$.

Using the direct method, we prove the Hyers-Ulam stability of the quadratic ρ-functional inequalities (0.1) and (0.2) in fuzzy Banach spaces.

1. INTRODUCTION AND PRELIMINARIES

Katsaras [14] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view [9, 16, 37]. In particular, Bag and Samanta [2], following Cheng and Mordeson [6], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [15]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [3].

We use the definition of fuzzy normed spaces given in [2, 19, 20] to investigate the Hyers-Ulam stability of quadratic ρ-functional inequalities in fuzzy Banach spaces.

Definition 1.1. [2, 19, 20, 21] Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0, 1]$ is called a *fuzzy norm* on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

(N_1) $N(x, t) = 0$ for $t \leq 0$;
(N_2) $x = 0$ if and only if $N(x, t) = 1$ for all $t > 0$;
(N_3) $N(cx, t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
(N_4) $N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\}$;
(N_5) $N(x, \cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x, t) = 1$.
(N_6) for $x \neq 0$, $N(x, \cdot)$ is continuous on $\mathbb{R}$.

The pair (X, N) is called a *fuzzy normed vector space.*

The properties of fuzzy normed vector spaces and examples of fuzzy norms are given in [18, 19].

Definition 1.2. [2, 19, 20, 21] Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to *be convergent* or *converge* if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$

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for all $t > 0$. In this case, x is called the *limit* of the sequence $\{x_n\}$ and we denote it by $N\text{-}\lim_{n\to\infty} x_n = x$.

Definition 1.3. [2, 19, 20, 21] Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called *Cauchy* if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well-known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space*.

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to x_0 in X, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be *continuous* on X (see [3]).

The stability problem of functional equations originated from a question of Ulam [36] concerning the stability of group homomorphisms.

The functional equation $f(x + y) = f(x) + f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [11] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [27] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [10] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach.

The functional equation $f(x+y)+f(x-y) = 2f(x)+2f(y)$ is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The stability of quadratic functional equation was proved by Skof [35] for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [7] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. Czerwik [8] proved the Hyers-Ulam stability of the quadratic functional equation. The functional equation $f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right) = \frac{1}{2}f(x)+\frac{1}{2}f(y)$ is called a *Jensen type quadratic equation*. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [5, 12, 13, 17, 24, 25, 26, 28, 29, **?**, 30, 31, 32, 33, 34]).

Park [22, 23] defined additive ρ-functional inequalities and proved the Hyers-Ulam stability of the additive ρ-functional inequalities in Banach spaces and non-Archimedean Banach spaces.

In Section 2, we solve the quadratic ρ-functional inequality (0.1) and prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (0.1) in fuzzy Banach spaces by using the direct method.

In Section 3, we solve the quadratic ρ-functional inequality (0.2) and prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (0.2) in fuzzy Banach spaces by using the direct method.

Throughout this paper, assume that X is a real vector space and (Y, N) is a fuzzy Banach space.

2. Quadratic ρ-functional inequality (0.1)

In this section, we prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (0.1) in fuzzy Banach spaces. Let ρ be a real number with $\rho \neq 2$. We need the following lemma to prove the main results.

Lemma 2.1. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$f(x+y) + f(x-y) - 2f(x) - 2f(y) = \rho\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right) \tag{2.1}$$

for all $x, y \in X$. Then $f : X \to Y$ is quadratic.

Proof. Replacing y by x in (2.1), we get $f(2x) - 4f(x) = 0$ and so $f(2x) = 4f(x)$ for all $x \in X$. Thus

$$\begin{aligned} f(x+y) + f(x-y) - 2f(x) - 2f(y) &= \rho\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right) \\ &= \frac{\rho}{2}(f(x+y) + f(x-y) - 2f(x) - 2f(y)) \end{aligned}$$

and so $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ for all $x, y \in X$, as desired. □

Theorem 2.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that*

$$\Phi(x, y) := \sum_{j=1}^{\infty} 4^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty \tag{2.2}$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$\begin{aligned} &N(f(x+y) + f(x-y) - 2f(x) - 2f(y) \\ &-\rho\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right), t) \geq \frac{t}{t + \varphi(x, y)} \end{aligned} \tag{2.3}$$

for all $x, y \in X$ and all $t > 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N(f(x) - Q(x), t) \geq \frac{t}{t + \frac{1}{4}\Phi(x, x)} \tag{2.4}$$

for all $x \in X$ and all $t > 0$.

Proof. Letting $y = x$ in (2.3), we get

$$N(f(2x) - 4f(x), t) \geq \frac{t}{t + \varphi(x, x)} \tag{2.5}$$

and so $N\left(f(x) - 4f\left(\frac{x}{2}\right), t\right) \geq \frac{t}{t+\varphi\left(\frac{x}{2}, \frac{x}{2}\right)}$ for all $x \in X$. Hence

$$\begin{aligned} &N\left(4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right), t\right) \\ &\geq \min\left\{N\left(4^l f\left(\frac{x}{2^l}\right) - 4^{l+1} f\left(\frac{x}{2^{l+1}}\right), t\right), \cdots, N\left(4^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 4^m f\left(\frac{x}{2^m}\right), t\right)\right\} \\ &= \min\left\{N\left(f\left(\frac{x}{2^l}\right) - 4f\left(\frac{x}{4^{l+1}}\right), \frac{t}{4^l}\right), \cdots, N\left(f\left(\frac{x}{2^{m-1}}\right) - 4f\left(\frac{x}{2^m}\right), \frac{t}{4^{m-1}}\right)\right\} \\ &\geq \min\left\{\frac{\frac{t}{4^l}}{\frac{t}{4^l} + \varphi\left(\frac{x}{2^{l+1}}, \frac{x}{2^{l+1}}\right)}, \cdots, \frac{\frac{t}{4^{m-1}}}{\frac{t}{4^{m-1}} + \varphi\left(\frac{x}{2^m}, \frac{x}{2^m}\right)}\right\} \\ &= \min\left\{\frac{t}{t + 4^l\varphi\left(\frac{x}{2^{l+1}}, \frac{x}{2^{l+1}}\right)}, \cdots, \frac{t}{t + 4^{m-1}\varphi\left(\frac{x}{2^m}, \frac{x}{2^m}\right)}\right\} \\ &\geq \frac{t}{t + \frac{1}{4}\sum_{j=l+1}^{m} 4^j \varphi\left(\frac{x}{2^j}, \frac{x}{2^j}\right)} \end{aligned} \tag{2.6}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$ and all $t > 0$. It follows from (2.2) and (2.6) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := N\text{-}\lim_{n\to\infty} 4^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.6), we get (2.4).

By (2.3),

$$N\left(4^n\left(f\left(\frac{x+y}{2^n}\right)+f\left(\frac{x-y}{2^n}\right)-2f\left(\frac{x}{2^n}\right)-2f\left(\frac{y}{2^n}\right)\right)\right.$$
$$\left.-\rho\left(4^n\left(2f\left(\frac{x+y}{2^{n+1}}\right)+2f\left(\frac{x-y}{2^{n+1}}\right)-f\left(\frac{x}{2^n}\right)-f\left(\frac{y}{2^n}\right)\right)\right),4^n t\right)\geq\frac{t}{t+\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. So

$$N\left(4^n\left(f\left(\frac{x+y}{2^n}\right)+f\left(\frac{x-y}{2^n}\right)-2f\left(\frac{x}{2^n}\right)-2f\left(\frac{y}{2^n}\right)\right)\right.$$
$$\left.-\rho\left(4^n\left(2f\left(\frac{x+y}{2^{n+1}}\right)+2f\left(\frac{x-y}{2^{n+1}}\right)-f\left(\frac{x}{2^n}\right)-f\left(\frac{y}{2^n}\right)\right)\right),t\right)$$
$$\geq\frac{\frac{t}{4^n}}{\frac{t}{4^n}+\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)}=\frac{t}{t+4^n\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty}\frac{t}{t+4^n\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$Q(x+y)+Q(x-y)-2Q(x)-2Q(y)=\rho\left(2Q\left(\frac{x+y}{2}\right)+2Q\left(\frac{x-y}{2}\right)-Q(x)-Q(y)\right)$$

for all $x, y \in X$. By Lemma 2.1, the mapping $Q : X \to Y$ is quadratic, as desired. □

Corollary 2.3. *Let $\theta \geq 0$ and let p be a real number with $p > 2$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$N\left(f(x+y)+f(x-y)-2f(x)-2f(y)-\rho\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right),t\right)$$
$$\geq\frac{t}{t+\theta(\|x\|^p+\|y\|^p)}\tag{2.7}$$

for all $x, y \in X$ and all $t > 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N\left(f(x)-Q(x),t\right)\geq\frac{(2^p-4)t}{(2^p-4)t+2\theta\|x\|^p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.2 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$, as desired. □

Theorem 2.4. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that*

$$\Phi(x,y):=\sum_{j=0}^{\infty}\frac{1}{4^j}\varphi\left(2^j x,2^j y\right)<\infty$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.3). Then $Q(x) := N$-$\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N(f(x) - Q(x), t) \geq \frac{1}{t + \frac{1}{4}\Phi(x, x)}$$

for all $x \in X$ and all $t > 0$.

Proof. It follows from (2.5) that $N\left(f(x) - \frac{1}{4}f(2x), \frac{1}{4}t\right) \geq \frac{t}{t+\varphi(x,x)}$ and so

$$N\left(f(x) - \frac{1}{4}f(2x), t\right) \geq \frac{4t}{4t + \varphi(x, x)} = \frac{t}{t + \frac{1}{4}\varphi(x, x)}$$

for all $x \in X$ and all $t > 0$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 2$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.7). Then $Q(x) := N$-$\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$N(f(x) - Q(x), t) \geq \frac{(4 - 2^p)t}{(4 - 2^p)t + 2\theta\|x\|^p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.4 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$, as desired. □

3. Quadratic ρ-functional inequality (0.2)

In this section, we prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (0.2) in fuzzy Banach spaces. Let ρ be a real number with $\rho \neq \frac{1}{2}$. We need the following lemma to prove the main results.

Lemma 3.1. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) = \rho\left(f(x+y) + f(x-y) - 2f(x) - 2f(y)\right) \tag{3.1}$$

for all $x, y \in X$. Then $f : X \to Y$ is quadratic.

Proof. Letting $y = 0$ in (3.1), we get $4f\left(\frac{x}{2}\right) - f(x) = 0$ and so $f(2x) = 4f(x)$ for all $x \in X$. Thus

$$\begin{aligned} \frac{1}{2}f(x+y) - \frac{1}{2}f(x-y) - f(x) - f(y) &= 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \\ &= \rho(f(x+y) + f(x-y) - 2f(x) - 2f(y)) \end{aligned}$$

and so $f(x + y) + f(x - y) = 2f(x) + 2f(y)$ for all $x, y \in X$, as desired. □

Theorem 3.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that*

$$\Phi(x, y) := \sum_{j=0}^{\infty} 4^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty \tag{3.2}$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$N\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) - \rho\left(f(x+y) + f(x-y) - 2f(x) - 2f(y)\right), t\right) \geq \frac{t}{t+\varphi(x,y)} \tag{3.3}$$

for all $x, y \in X$ and all $t > 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N\left(f(x) - Q(x), t\right) \geq \frac{t}{t+\Phi(x,0)} \tag{3.4}$$

for all $x \in X$ and all $t > 0$.

Proof. Letting $y = 0$ in (3.3), we get

$$N\left(f(x) - 4f\left(\frac{x}{2}\right), t\right) = N\left(4f\left(\frac{x}{2}\right) - f(x), t\right) \geq \frac{t}{t+\varphi(x,0)} \tag{3.5}$$

for all $x \in X$. Hence

$$\begin{aligned} & N\left(4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right), t\right) \qquad (3.6)\\ & \geq \min\left\{N\left(4^l f\left(\frac{x}{2^l}\right) - 4^{l+1} f\left(\frac{x}{2^{l+1}}\right), t\right), \cdots, N\left(4^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 4^m f\left(\frac{x}{2^m}\right), t\right)\right\} \\ & = \min\left\{N\left(f\left(\frac{x}{2^l}\right) - 4f\left(\frac{x}{2^{l+1}}\right), \frac{t}{4^l}\right), \cdots, N\left(f\left(\frac{x}{2^{m-1}}\right) - 4f\left(\frac{x}{2^m}\right), \frac{t}{4^{m-1}}\right)\right\} \\ & \geq \min\left\{\frac{\frac{t}{4^l}}{\frac{t}{4^l} + \varphi\left(\frac{x}{2^l}, 0\right)}, \cdots, \frac{\frac{t}{4^{m-1}}}{\frac{t}{4^{m-1}} + \varphi\left(\frac{x}{2^{m-1}}, 0\right)}\right\} \\ & = \min\left\{\frac{t}{t + 4^l \varphi\left(\frac{x}{2^l}, 0\right)}, \cdots, \frac{t}{t + 4^{m-1} \varphi\left(\frac{x}{2^{m-1}}, 0\right)}\right\} \\ & \geq \frac{t}{t + \sum_{j=l}^{m-1} 4^j \varphi\left(\frac{x}{2^j}, 0\right)} \end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$ and all $t > 0$. It follows from (3.2) and (3.6) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.6), we get (3.4).

By (3.2),

$$\begin{aligned} & N\left(4^n\left(2f\left(\frac{x+y}{2^{n+1}}\right) + 2f\left(\frac{x-y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right. \\ & \left.-\rho\left(4^n\left(f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right)\right)\right), 4^n t\right) \geq \frac{t}{t+\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)} \end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. So

$$\begin{aligned}
&N\left(4^n\left(2f\left(\frac{x+y}{2^{n+1}}\right)+2f\left(\frac{x-y}{2^{n+1}}\right)-f\left(\frac{x}{2^n}\right)-f\left(\frac{y}{2^n}\right)\right)\right.\\
&\left.-\rho\left(4^n\left(f\left(\frac{x+y}{2^n}\right)+f\left(\frac{x-y}{2^n}\right)-2f\left(\frac{x}{2^n}\right)-2f\left(\frac{y}{2^n}\right)\right)\right),t\right)\\
&\geq \frac{\frac{t}{4^n}}{\frac{t}{4^n}+\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)}=\frac{t}{t+4^n\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)}
\end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{t}{t+4^n\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$2Q\left(\frac{x+y}{2}\right)+2\left(\frac{x-y}{2}\right)-Q(x)-Q(y)=\rho\left(Q\left(x+y\right)+Q\left(x-y\right)-2Q(x)-2Q(y)\right)$$

for all $x, y \in X$=. By Lemma 3.1, the mapping $Q : X \to Y$ is quadratic, as desired. □

Corollary 3.3. *Let $\theta \geq 0$ and let p be a real number with $p > 2$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned}
&N(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y) \qquad (3.7)\\
&-\rho\left(f(x+y)+f(x-y)-2f(x)-2f(y)\right),t) \geq \frac{t}{t+\theta(\|x\|^p+\|y\|^p)}
\end{aligned}$$

for all $x, y \in X$ and all $t > 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N\left(f(x)-Q(x),t\right) \geq \frac{(2^p-4)t}{(2^p-4)t+2^p\theta\|x\|^p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.2 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$, as desired. □

Theorem 3.4. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that*

$$\Phi(x,y) := \sum_{j=1}^{\infty} \frac{1}{4^j}\varphi\left(2^jx, 2^jy\right) < \infty$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (3.3). Then $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N\left(f(x)-Q(x),t\right) \geq \frac{t}{t+\Phi(x,0)}$$

for all $x \in X$ and all $t > 0$.

Proof. It follows from (3.5) that $N\left(f(x) - \frac{1}{4}f(2x), \frac{t}{4}\right) \geq \frac{t}{t+\varphi(2x,0)}$ and so

$$N\left(f(x)-\frac{1}{4}f(2x),t\right) \geq \frac{4t}{4t+\varphi(2x,0)} = \frac{t}{t+\frac{1}{4}\varphi(2x,0)}$$

for all $x \in X$ and all $t > 0$.

The rest of the proof is similar to the proof of Theorem 3.2. □

Corollary 3.5. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 2$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (3.7). Then $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$N\left(f(x) - Q(x), t\right) \geq \frac{(4-2^p)t}{(4-2^p)t + 2^p\theta\|x\|^p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.4 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$, as desired. □

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ON A DOUBLE INTEGRAL EQUATION INCLUDING A SET OF TWO VARIABLES POLYNOMIALS SUGGESTED BY LAGUERRE POLYNOMIALS

M. ALİ ÖZARSLAN AND CEMALİYE KÜRT

ABSTRACT. In this paper, we introduce general classes of bivariate and Mittag-Leffler functions $E_{\gamma_1,\gamma_2}^{(\alpha,\beta,\eta,\xi,\lambda)}(x,y)$ and Laguerre polynomials $L_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)$. We investigate double fractional integrals and derivative properties of the above mentioned classes. We further obtain linear generating function for $L_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)$ in terms of $E_{\gamma_1,\gamma_2}^{(\alpha,\beta,\eta,\xi,\lambda)}(x,y)$. Finally, we calculate double Laplace transforms of the above mentioned classes and then we consider a general singular integral equation with $L_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)$ in the kernel and obtain the solution in terms of $E_{\gamma_1,\gamma_2}^{(\alpha,\beta,\eta,\xi,\lambda)}(x,y)$.

1. INTRODUCTION

The special function of the form [7]

$$E_{\alpha}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)} \tag{1.1}$$

$$(\alpha\in\mathbb{C},\mathrm{Re}(\alpha)>0,z\in\mathbb{C})$$

and more general function [12] of (1.1)

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)} \tag{1.2}$$

$$(\alpha,\beta\in\mathbb{C},\mathrm{Re}(\alpha),\mathrm{Re}(\beta)>0,z\in\mathbb{C})$$

are known as Mittag-Leffler functions the first of which was introduced by Swedish mathematician G. Mittag-Leffler and the second one by Wiman.

Setting $\alpha=\beta=1$, the equation (1.2) becomes the exponential function e^z. When $0<\alpha<1$, it bridges an interpolation between the pure exponential function e^z and a geometric function

$$\frac{1}{1-z}=\sum_{n=0}^{\infty}z^n.$$

$$(|z|<1)$$

Key words and phrases. Double fractional integrals and derivatives, Bivariate Mittag-Leffler function, Bivariate Laguerre polynomials, Double generating functions, Singular double integral equation, Double Laplace integral.

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A further generalization of (1.2) was introduced by Prabhakar (see [9]) as

$$E_{\alpha,\beta}^{\gamma}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!} \tag{1.3}$$
$$(\alpha,\beta,\gamma\in\mathbb{C},\mathrm{Re}(\alpha),\mathrm{Re}(\beta)>0)$$

where the Pochammer symbol [11], $(\gamma)_n$, is defined as

$$(\gamma)_n=\frac{\Gamma(\gamma+n)}{\Gamma(\gamma)}=\begin{cases}1 & ;n=0,\gamma\neq 0\\ \gamma(\gamma+1)\cdots(\gamma+n-1) & ;n=1,2,\cdots\end{cases}$$

In the special case, we have the polynomials $Z_n^{\alpha}(x;k)$ (see [6],[10]) which were defined by

$$Z_n^{\alpha}(x;k)=\frac{\Gamma(kn+\alpha+1)}{n!}E_{k,\alpha+1}^{-n}(x^k) \quad .$$
$$(\mathrm{Re}(\alpha)>0,k\in\mathbb{Z}_{0^+})$$

Note that, in [6] and [10], generating functions, integrals and recurrence relations were developed for the polynomials $Z_n^{\alpha}(x;k)$ of degree n in x^k, which form one set of the biorthogonal pair corresponding to the weight function $e^{-x}x^{\alpha}$ over the interval $(0,\infty)$.

For $k=1$, we have $Z_n^{\alpha}(x;1)=L_n^{\alpha}(x)$ where $L_n^{\alpha}(x)$ is the usual Laguerre polynomial which were given as follows

$$L_n^{\alpha}(x)=\frac{(1+\alpha)_n}{n!}\ {}_1F_1(-n;1+\alpha;x)$$

where

$${}_1F_1(-n;1+\alpha;x)=\sum_{k=0}^{n}\frac{(-n)_k}{(1+\alpha)_k}\frac{x^k}{k!}.$$

Very recently, a class of polynomials $Z_{n_1,\cdots,n_j}^{(\alpha)}\left(x_1,\cdots,x_j;\rho_1,\cdots,\rho_j\right)$ (see [8]) suggested by the multivariate Laguerre polynomials were defined by

(1.4)

$$\begin{aligned}&Z_{n_1,\cdots,n_j}^{(\alpha)}\left(x_1,\cdots,x_j;\rho_1,\cdots,\rho_j\right)\\&=\frac{\Gamma\left(\rho_1n_1+\cdots+\rho_jn_j+\alpha+1\right)}{n_1!\cdots n_j!}\sum_{k_1,\cdots,k_j=0}^{n_1,\cdots,n_j}\frac{(-n_1)_{k_1}\cdots(-n_j)_{k_j}\,x_1^{\rho_1k_1}\cdots x_j^{\rho_jk_j}}{\Gamma\left(\rho_1k_1+\cdots+\rho_jk_j+\alpha+1\right)k_1!\cdots k_j!}.\end{aligned}$$

$(\alpha,\rho_1,\cdots,\rho_j\in\mathbb{C},\ \mathrm{Re}(\rho_i)>0\ (i=1,\cdots,j))$

Obviously $Z_{n_1,\cdots,n_j}^{(\alpha)}\left(x_1,\cdots,x_j;\rho_1,\cdots,\rho_j\right)$ gives $L_{n_1,\cdots,n_j}^{(\alpha)}\left(x_1,\cdots,x_j\right)$ when $\rho_1=\cdots=\rho_j=1$, where $L_{n_1,\cdots,n_j}^{(\alpha)}\left(x_1,\cdots,x_j\right)$ is the multivariable Laguerre polynomial [2] given by

$$L_{n_1,\cdots,n_j}^{(\alpha)}\left(x_1,\cdots,x_j\right)=\frac{\Gamma\left(n_1+\cdots+n_j+\alpha+1\right)}{n_1!\cdots n_j!}\sum_{k_1,\cdots,k_j=0}^{n_1,\cdots,n_j}\frac{(-n_1)_{k_1}\cdots(-n_j)_{k_j}\,x_1^{k_1}\cdots x_j^{k_j}}{\Gamma\left(k_1+\cdots+k_j+\alpha+1\right)k_1!\cdots k_j!}.$$

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It is known that the multivariate Mittag-Leffler functions were defined by the multiple series as [13]

$$E^{(\gamma_1,\cdots,\gamma_j)}_{\rho_1,\cdots,\rho_j,\lambda}(x_1,\cdots,x_j)=\sum_{k_1,\cdots,k_j=0}^{\infty}\frac{(\gamma_1)_{k_1}\cdots(\gamma_j)_{k_j}x_1^{k_1}\cdots x_j^{k_j}}{\Gamma\left(\rho_1k_1+\cdots+\rho_jk_j+\lambda\right)k_1!\cdots k_j!}. \tag{1.5}$$

$$(\lambda,\rho_1,\cdots,\rho_j,\gamma_1,\cdots,\gamma_j\in\mathbb{C},\ \mathrm{Re}(\rho_i)>0\ (i=1,\cdots,j))$$

Note that the function in (1.5) is a special case of the generalized Lauricella series in several variables introduced and investigated by Srivastava and Daoust [16] (see also see [14],[15]). Also, when $j=1,\rho_1=\alpha,\lambda=\beta,\gamma_1=\gamma,x_1=z$, the function (1.5) reduces to (1.3).

The polynomials $Z^{(\alpha)}_{n_1,\cdots,n_j}\left(x_1,\cdots,x_j;\rho_1,\cdots,\rho_j\right)$ can be represented in terms of the multivariate Mittag-Leffler functions as follows (see [8]):

$$\begin{aligned}&Z^{(\alpha)}_{n_1,\cdots,n_j}\left(x_1,\cdots,x_j;\rho_1,\cdots,\rho_j\right)\\=\ &\frac{\Gamma\left(\rho_1n_1+\cdots+\rho_jn_j+\alpha+1\right)}{n_1!\cdots n_j!}E^{(-n_1,\cdots,-n_j)}_{\rho_1,\cdots,\rho_j,\alpha+1}(x_1^{\rho_1},\cdots,x_j^{\rho_j}).\end{aligned} \tag{1.6}$$

Clearly, setting $\rho_1=\rho_2=\cdots=\rho_j=1$ in (1.6) gives

$$L^{(\alpha)}_{n_1,\cdots,n_j}(x_1,\cdots,x_j)=\frac{\Gamma\left(n_1+\cdots+n_j+\alpha+1\right)}{n_1!\cdots n_j!}E^{(-n_1,\cdots,-n_j)}_{1,\cdots,1,\alpha+1}(x_1,\cdots,x_j).$$

Very recently, a slight motivated form of the multivariate Mittag-Leffler functions were introduced and investigated in [3].

On the other hand, a nontrivial two variables Mittag-Leffler functions were defined in [4] by

$$\begin{aligned}E_1(x,y)&=E_1\left(\begin{array}{c}\gamma_1,\alpha_1;\gamma_2,\beta_1\\ \delta_1,\alpha_2,\beta_2;\delta_2,\alpha_3;\delta_3,\beta_3\end{array}\ \middle|\ \begin{array}{c}x\\ y\end{array}\right)\\&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\gamma_1)_{\alpha_1m}(\gamma_2)_{\beta_1n}}{\Gamma(\delta_1+\alpha_2m+\beta_2n)}\frac{x^m}{\Gamma(\delta_2+\alpha_3m)}\frac{y^n}{\Gamma(\delta_3+\beta_3n)}.\end{aligned}$$

$$(\gamma_1,\gamma_2,\delta_1,\delta_2,\delta_3,x,y\in\mathbb{C},\min\{\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3\}>0\})$$

Motivated essentially by the above definitions and investigations, in this paper, we introduce a class of bivariate Mittag-Leffler function

$$E^{(\alpha,\beta,\eta,\xi,\lambda)}_{\gamma_1,\gamma_2}(x,y)=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{(\gamma_1)_{k_1}(\gamma_2)_{k_2}x^{k_1}y^{k_2}}{\Gamma(\alpha k_1+\beta k_2+\lambda)\Gamma(\eta k_2+\xi)k_1!k_2!} \tag{1.7}$$

where $\gamma_1,\gamma_2,\alpha,\beta,\lambda,\eta,\xi\in\mathbb{C},\mathrm{Re}(\alpha+\eta)>0$ and $\mathrm{Re}(\beta)>0$.

According to the convergence conditions investigated by Srivastava and Daoust ([15], p. 155) for the generalized Lauricella series in two variables, the series in (1.7) converges absolutely for $\mathrm{Re}(\alpha+\eta)>0$ and $\mathrm{Re}(\beta)>0$.

We also introduce a general class of bivariate Laguerre polynomials

$$\begin{aligned}&L^{(\alpha,\beta,\gamma,\eta,\xi)}_{n,m}(x,y)\\=\ &\frac{\Gamma(\alpha n+\beta m+\gamma+1)}{\Gamma(\xi+\eta m)}\sum_{k_1=0}^{n}\sum_{k_2=0}^{m}\frac{(-n)_{k_1}(-m)_{k_2}x^{\alpha k_1}y^{\beta k_2}}{\Gamma(\alpha k_1+\beta k_2+\gamma+1)\Gamma(\eta k_2+\xi)k_1!k_2!}\end{aligned} \tag{1.8}$$

where $\alpha,\beta,\gamma,\eta,\xi\in\mathbb{C},\mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(\eta),\mathrm{Re}(\xi)>0,\mathrm{Re}(\gamma)>-1$.

Comparing (1.7) and (1.8), we see that

$$L_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)=\frac{\Gamma(\alpha n+\beta m+\gamma+1)}{\Gamma(\xi+\eta m)}E_{-n,-m}^{(\alpha,\beta,\eta,\xi,\lambda)}(x^{\alpha},y^{\beta}). \tag{1.9}$$

This paper is organized as follows. In section 2, we calculate the double fractional integrals and derivatives of the above mentioned classes (1.7) and (1.8). Linear generating functions for $L_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)$ are given in terms of $E_{\gamma_1,\gamma_2}^{(\alpha,\beta,\eta,\xi,\lambda)}(x,y)$ in Section 3. In the last section, we first investigate double Laplace transforms of the above mentioned classes and then we consider a general singular integral equation with $L_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)$ in the kernel and obtain the solution by means of $E_{\gamma_1,\gamma_2}^{(\alpha,\beta,\eta,\xi,\lambda)}(x,y)$.

2. Fractional integrals and derivatives

This section aims to provide the fractional integral formulas of the functions $E_{\gamma_1,\gamma_2}^{(\alpha,\beta,\eta,\xi,\lambda)}(x,y)$ and $L_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)$. Throughout this section, we assume that $\operatorname{Re}(\alpha),\operatorname{Re}(\beta)>0,\operatorname{Re}(\mu),\operatorname{Re}(\lambda)>0,\operatorname{Re}(\gamma)>-1$.

Definition 2.1. *([1],[8])Let $\Omega=[a,b]$ be a finite interval of the real axis. The Riemann-Liouville fractional integral of order $\mu\in\mathbb{C}$ $(\operatorname{Re}(\mu)>0)$ is defined by*

$$ {}_xI_{a^+}^{\mu}[f]=\frac{1}{\Gamma(\mu)}\int_a^x\frac{f(t)\,dt}{(x-t)^{1-\mu}}\ .\quad (x>a,\operatorname{Re}(\mu)>0)$$

Similarly, the partial fractional integrals of a function $f(x,t)$, where $(x,t)\in\mathbb{R}\times\mathbb{R}$ is defined as follows:

$$ {}_xI_{a^+}^{\mu}f(x,t)=\frac{1}{\Gamma(\mu)}\int_a^x(x-\xi)^{\mu-1}f(\xi,t)d\xi\ ,\quad (x>a,\operatorname{Re}(\mu)>0)$$

$$ {}_tI_{a^+}^{\lambda}f(x,t)=\frac{1}{\Gamma(\lambda)}\int_b^t(t-\tau)^{\lambda-1}f(x,\tau)d\tau,\quad (t>b,\operatorname{Re}(\lambda)>0)$$

$$\begin{aligned}&{}_tI_{b^+}^{\lambda}{}_xI_{a^+}^{\mu}f(x,t)\\ =&\frac{1}{\Gamma(\mu)\Gamma(\lambda)}\int_b^t\int_a^x(t-\tau)^{\lambda-1}(x-\xi)^{\mu-1}f(\xi,\tau)d\xi d\tau\ .\quad (x>a,y>b,\operatorname{Re}(\lambda)>0,\operatorname{Re}(\mu)>0)\end{aligned}$$

Definition 2.2. *([1],[8])The Riemann-Liouville fractional derivative of order $\mu\in\mathbb{C}$ $(\operatorname{Re}(\mu)\geq 0)$ is defined by*

$$ {}_xD_{a^+}^{\mu}[f]=\left(\frac{d}{dx}\right)^n\frac{1}{\Gamma(n-\alpha)}\int_a^x(x-\xi)^{\alpha-n-1}f(\xi)d\xi\ ,\ (n=[\operatorname{Re}(\mu)]+1,\ x>a)$$

where, as usual, $[\operatorname{Re}(\mu)]$ means the integral part of $\operatorname{Re}(\mu)$.

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Similarly, the partial fractional derivatives of a function $f(x,t)$, where $(x,t) \in \mathbb{R}\times\mathbb{R}$ is defined as follows:

$$ {}_xD^{\mu}_{a^+}f(x,t) = \left(\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\mu)}\int_a^x (x-\xi)^{n-\mu-1}f(\xi,t)d\xi \ , \ (n=[\mathrm{Re}(\mu)]+1, \ x>a) $$

$$ {}_tD^{\lambda}_{b^+}f(x,t) = \left(\frac{d}{dt}\right)^m \frac{1}{\Gamma(m-\lambda)}\int_b^t (t-\tau)^{m-\lambda-1}f(x,\tau)d\tau \ , \ (m=[\mathrm{Re}(\lambda)]+1, \ t>b) $$

$$ \begin{aligned} &{}_tD^{\lambda}_{b^+}{}_xD^{\mu}_{a^+}f(x,t) \\ &= \left(\frac{d}{dt}\right)^m\left(\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\mu)}\frac{1}{\Gamma(m-\lambda)}\int_b^t\int_a^x (t-\tau)^{m-\lambda-1}(x-\xi)^{n-\mu-1}f(\xi,\tau)d\xi d\tau. \\ &(n=[\mathrm{Re}(\mu)]+1, m=[\mathrm{Re}(\lambda)]+1, \ t>b, \ x>a) \end{aligned} $$

Theorem 2.1. *We have for* $\mathrm{Re}(\alpha+\eta)>0$ *and* $\mathrm{Re}(\alpha)>0$ *and* $(\beta)>0$*, that*

$$ {}_yI^{\alpha}_{0^+}{}_xI^{\beta}_{0^+}\left[x^{\lambda-1}y^{\xi-1}E^{(\alpha,\beta,\eta,\xi,\lambda)}_{\gamma_1,\gamma_2}\left(x^{\alpha},x^{\beta}y^{\eta}\right)\right] = x^{\beta+\lambda-1}y^{\alpha+\xi-1}E^{(\alpha,\beta,\eta,\xi+\alpha,\lambda+\beta)}_{\gamma_1,\gamma_2}\left(x^{\alpha},x^{\beta}y^{\eta}\right) $$

Proof. Because of the hypothesis of the Theorem, we have a right to interchange of the order of series and fractional integral operators, which yields

$$ \begin{aligned} &{}_yI^{\alpha}_{0^+}{}_xI^{\beta}_{0^+}\left[x^{\lambda-1}y^{\xi-1}E^{(\alpha,\beta,\eta,\xi,\lambda)}_{\gamma_1,\gamma_2}\left(x^{\alpha},x^{\beta}y^{\eta}\right)\right] \\ &= \int_0^y\int_0^x \frac{(y-\tau)^{\alpha-1}(x-t)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}t^{\lambda-1}\tau^{\xi-1}E^{(\alpha,\beta,\eta,\xi,\lambda)}_{\gamma_1,\gamma_2}\left(t^{\alpha},t^{\beta}\tau^{\eta}\right)dtd\tau \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{(\gamma_1)_{k_1}(\gamma_2)_{k_2}}{\Gamma(\alpha k_1+\beta k_2+\lambda)\Gamma(\eta k_2+\xi)k_1!k_2!} \\ &\times \int_0^y (y-\tau)^{\alpha-1}\tau^{\eta k_2+\xi-1}d\tau\int_0^x (x-t)^{\beta-1}t^{\alpha k_1+\beta k_2+\lambda-1}dt \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{(\gamma_1)_{k_1}(\gamma_2)_{k_2}x^{\alpha k_1+\beta k_2+\beta+\lambda-1}y^{\eta k_2+\xi+\alpha-1}}{\Gamma(\alpha k_1+\beta k_2+\lambda+\beta)\Gamma(\eta k_2+\xi+\alpha)k_1!k_2!} \\ &= x^{\beta+\lambda-1}y^{\alpha+\xi-1}E^{(\alpha,\beta,\eta,\xi+\alpha,\lambda+\beta)}_{\gamma_1,\gamma_2}\left(x^{\alpha},x^{\beta}y^{\eta}\right) \end{aligned} $$

□

In a similar manner, we have the following corollary:

Corollary 2.2. *For* $\mathrm{Re}(\alpha)>0$ *and* $\mathrm{Re}(\beta)>0$*, that*

$$ \begin{aligned} &{}_yI^{\alpha}_{0^+}{}_xI^{\beta}_{0^+}\left[x^{\gamma}y^{\xi-1}L^{(\alpha,\beta,\gamma,\eta,\xi)}_{n,m}\left(x,xy^{\frac{\eta}{\beta}}\right)\right] \\ &= \frac{\Gamma(\alpha n+\beta m+\gamma+1)}{\Gamma(\xi+\eta m)}\frac{\Gamma(\alpha+\xi+\eta m)}{\Gamma(\alpha n+\beta m+\gamma+\beta+1)}x^{\beta+\gamma}y^{\alpha+\xi-1}L^{(\alpha,\beta,\gamma+\beta,\eta,\alpha+\xi)}_{n,m}\left(x,xy^{\frac{\eta}{\beta}}\right). \end{aligned} $$

Theorem 2.3. *For* $\mathrm{Re}(\alpha+\eta)>0$,$\mathrm{Re}(\alpha)\geqslant 0$ *and* $(\beta)>0$*, that*

$$ {}_yD^{\alpha}_{0^+}{}_xD^{\beta}_{0^+}\left[x^{\lambda-1}y^{\xi-1}E^{(\alpha,\beta,\eta,\xi,\lambda)}_{\gamma_1,\gamma_2}\left(x^{\alpha},x^{\beta}y^{\eta}\right)\right] = x^{\lambda-\beta-1}y^{\xi-\alpha-1}E^{(\alpha,\beta,\eta,\xi-\alpha,\lambda-\beta)}_{\gamma_1,\gamma_2}\left(x^{\alpha},x^{\beta}y^{\eta}\right). $$

Proof. Because of the hypothesis of the Theorem, we have a right to interchange of the order of series and fractional derivate operators, which yields

$$\begin{aligned}
&{}_yD^{\alpha}_{0^+ x}D^{\beta}_{0^+}\left[x^{\lambda-1}y^{\xi-1}E^{(\alpha,\beta,\eta,\xi,\lambda)}_{\gamma_1,\gamma_2}\left(x^{\alpha},x^{\beta}y^{\eta}\right)\right]\\
&= {}_yD^{\alpha}_{0^+ x}D^{\beta}_{0^+}\left[x^{\lambda-1}y^{\xi-1}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{(\gamma_1)_{k_1}(\gamma_2)_{k_2}x^{\alpha k_1}x^{\beta k_2}y^{\eta k_2}}{\Gamma(\alpha k_1+\beta k_2+\lambda)\Gamma(\eta k_2+\xi)k_1!k_2!}\right]\\
&= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{(\gamma_1)_{k_1}(\gamma_2)_{k_2 x}D^{\alpha}_{0^+ y}D^{\beta}_{0^+}[x^{\alpha k_1+\beta k_2+\lambda-1}y^{\eta k_2+\xi-1}]}{\Gamma(\alpha k_1+\beta k_2+\lambda)\Gamma(\eta k_2+\xi)k_1!k_2!}\\
&= (\frac{d}{dy})^m(\frac{d}{dx})^n\frac{1}{\Gamma(n-\beta)}\frac{1}{\Gamma(m-\alpha)}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{(\gamma_1)_{k_1}(\gamma_2)_{k_2}}{\Gamma(\alpha k_1+\beta k_2+\lambda)\Gamma(\eta k_2+\xi)k_1!k_2!}\\
&\times\int_0^y(y-\tau)^{m-\alpha-1}\tau^{\eta k_2+\xi-1}d\tau\int_0^x(x-\xi)^{n-\beta-1}\xi^{\alpha k_1+\beta k_2+\lambda-1}d\xi\\
&= x^{\lambda-\beta-1}y^{\xi-\alpha-1}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{(\gamma_1)_{k_1}(\gamma_2)_{k_2}x^{\alpha k_1+\beta k_2}y^{\eta k_2}}{\Gamma(\alpha k_1+\beta k_2+\lambda-\beta)\Gamma(\eta k_2+\xi-\alpha)k_1!k_2!}\\
&= x^{\lambda-\beta-1}y^{\xi-\alpha-1}E^{(\alpha,\beta,\eta,\xi-\alpha,\lambda-\beta)}_{\gamma_1,\gamma_2}\left(x^{\alpha},x^{\beta}y^{\eta}\right).
\end{aligned}$$

□

In a similar manner, we have the following corollary:

Corollary 2.4. *For* $\operatorname{Re}(\alpha+\eta)>0$ *and* $\operatorname{Re}(\beta)>0$, *that*

$$\begin{aligned}
&{}_yD^{\alpha}_{0^+ x}D^{\beta}_{0^+}[x^{\gamma}y^{\xi-1}L^{(\alpha,\beta,\gamma,\eta,\xi)}_{n,m}\left(x,xy^{\frac{\eta}{\beta}}\right)]\\
&=\frac{\Gamma(\alpha n+\beta m+\gamma+1)}{\Gamma(\xi+\eta m)}\frac{\Gamma(\xi-\alpha+\eta m)}{\Gamma(\alpha n+\beta m+\gamma-\beta+1)}x^{\gamma-\beta}y^{\xi-\alpha-1}L^{(\alpha,\beta,\gamma-\beta,\eta,\xi-\alpha)}_{n,m}\left(x,xy^{\frac{\eta}{\beta}}\right).
\end{aligned}$$

3. Linear generating function

In this section, we provide a linear generating function for the polynomials $L^{(\alpha,\beta,\gamma,\eta,\xi)}_{n,m}(x,y)$ by means of two variables analogue of Mittag-Leffler functions defined in (1.7).

Theorem 3.1. *For* $|t_1|<1$ *and* $|t_2|<1$, $\gamma_1,\gamma_2\in\mathbb{C}$ *and* $\alpha,\beta,\gamma,\xi,\eta\in\mathbb{C}$, *we have*

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\gamma_1)_n(\gamma_2)_mL^{(\alpha,\beta,\gamma,\eta,\xi)}_{n,m}(x,y)\,\Gamma(\xi+\eta m)}{\Gamma(\alpha n+\beta m+\gamma+1)n!m!}t_1^nt_2^m\\
&=(1-t_1)^{-\gamma_1}(1-t_2)^{-\gamma_2}E^{(\alpha,\beta,\eta,\xi,\gamma+1)}_{\gamma_1,\gamma_2}\left(\frac{-x^{\alpha}t_1}{1-t_1},\frac{-y^{\beta}t_2}{1-t_2}\right).
\end{aligned}$$

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Proof. Direct calculations yield that

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\gamma_1)_n(\gamma_2)_m L_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)\,\Gamma(\xi+\eta m)}{\Gamma(\alpha n+\beta m+\gamma+1)n!m!}t_1^n t_2^m$$
$$=\sum_{n,m=0}^{\infty}\sum_{k_1,k_2=0}^{n,m}\frac{(\gamma_1)_n(\gamma_2)_m(-n)_{k_1}(-m)_{k_2}x^{\alpha k_1}y^{\beta k_2}}{\Gamma(\alpha k_1+\beta k_2+\gamma+1)\Gamma(\xi+\eta k_2)k_1!k_2!n!m!}t_1^n t_2^m$$
$$=\sum_{n,m=0}^{\infty}\sum_{k_1,k_2=0}^{n,m}\frac{(-1)^{k_1+k_2}(\gamma_1)_n(\gamma_2)_m x^{\alpha k_1}y^{\beta k_2}}{\Gamma(\alpha k_1+\beta k_2+\gamma+1)\Gamma(\xi+\eta k_2)k_1!k_2!(n-k_1)!(m-k_2)!}t_1^n t_2^m.$$

Letting $n\to n+k_1$ and $m\to m+k_2$, we get

$$\sum_{n,m=0}^{\infty}\sum_{k_1,k_2=0}^{\infty}\frac{(-1)^{k_1+k_2}(\gamma_1)_{n+k_1}(\gamma_2)_{m+k_2}x^{\alpha k_1}y^{\beta k_2}}{\Gamma(\alpha k_1+\beta k_2+\gamma+1)\Gamma(\xi+\eta k_2)k_1!k_2!(n)!(m)!}t_1^{n+k_1}t_2^{m+k_2}.$$

Since $(\gamma_1)_{n+k_1}=(\gamma_1+k_1)_n(\gamma_1)_{k_1}$ and $(\gamma_2)_{m+k_2}=(\gamma_2+k_2)_m(\gamma_2)_{k2}$, we have

$$\sum_{k_1,k_2=0}^{\infty}\frac{(\gamma_1)_{k_1}(\gamma_2)_{k_2}(-x^{\alpha}t_1)^{k_1}(-y^{\beta}t_2)^{k_2}}{\Gamma(\alpha k_1+\beta k_2+\gamma+1)\Gamma(\xi+\eta k_2)k_1!k_2!}\sum_{n,m=0}^{\infty}\frac{(\gamma_1+k_1)_n(\gamma_2+k_2)_m}{(n)!(m)!}t_1^n t_2^m$$
$$=(1-t_1)^{-\gamma_1}(1-t_2)^{-\gamma_2}E_{\gamma_1,\gamma_2}^{(\alpha,\beta,\eta,\xi,\gamma+1)}\left(\frac{-x^{\alpha}t_1}{1-t_1},\frac{-y^{\beta}t_2}{1-t_2}\right).$$

Note that, because of the uniform converge of the series under the conditions $|t_1|<1$ and $|t_2|<1$, we have interchanged the order of summations. □

4. Singular double integral equation

In this section, we first obtain the double Laplace transform of the functions $E_{\gamma_1,\gamma_2}^{(\alpha,\beta,\eta,\xi,\lambda)}(x,y)$ and $L_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)$. Then, we compute the double integral involving the product of two $E_{\gamma_1,\gamma_2}^{(\alpha,\beta,\eta,\xi,\lambda)}(x,y)$ functions in the integrand. Finally, we solve a double integral equation with $L_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)$ in the kernel, in terms of the two $E_{\gamma_1,\gamma_2}^{(\alpha,\beta,\eta,\xi,\lambda)}(x,y)$ functions.

As usual [5],

$$\mathbb{L}_2[f(x,t)]=\int_0^{\infty}e^{-px}\int_0^{\infty}e^{-st}f(x,t)dtdx \quad (x,t>0,\quad p,s\in\mathbb{C}) \tag{4.1}$$

denotes the double Laplace transform of f.

Lemma 4.1. *For* $\mathrm{Re}(\lambda_1),\mathrm{Re}(\lambda_2),\mathrm{Re}(\alpha+\eta)>0,\mathrm{Re}(\beta)>0,\mathrm{Re}(s_1),\mathrm{Re}(s_2)>0$ *and* $\left|\frac{\lambda_1^{\alpha}}{s_1^{\alpha}}\right|,\left|\frac{\lambda_2^{\beta}}{s_1^{\beta}s_2^{\eta}}\right|<1$, *we have*

$$\mathbb{L}_2[x^{\lambda-1}y^{\xi-1}E_{\gamma_1,\gamma_2}^{(\alpha,\beta,\eta,\xi,\lambda)}((\lambda_1 x)^{\alpha},(\lambda_2^{\beta}x^{\beta}y^{\eta}))](s_1,s_2)=\frac{1}{s_1^{\lambda}}\frac{1}{s_2^{\xi}}(1-\frac{\lambda_1^{\alpha}}{s_1^{\alpha}})^{-\gamma_1}(1-\frac{\lambda_2^{\beta}}{s_1^{\beta}s_2^{\eta}})^{-\gamma_2}.$$

Proof. Using definition (4.1) and taking into account that $\left|\frac{\lambda_1^\alpha}{s_1^\alpha}\right| < 1$ and $\left|\frac{\lambda_2^\beta}{s_1^\beta s_2^\eta}\right| < 1$, we get

$$\begin{aligned}
&\mathbb{L}_2[x^{\lambda-1}y^{\xi-1}E_{\gamma_1,\gamma_2}^{(\alpha,\beta,\eta,\xi,\lambda)}((\lambda_1 x)^\alpha, (\lambda_2^\beta x^\beta y^\eta))](s_1,s_2)\\
&= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \frac{(\gamma_1)_{k_1}(\gamma_2)_{k_2}\lambda_1^{\alpha k_1}\lambda_2^{\beta k_2}}{\Gamma(\alpha k_1+\beta k_2+\lambda)\Gamma(\eta k_2+\xi)k_1!k_2!}\\
&\times \int_0^\infty x^{\alpha k_1+\beta k_2+\lambda-1}e^{-s_1 x}dx \int_0^\infty y^{\eta k_2+\xi-1}e^{-s_2 y}dy\\
&= \frac{1}{s_1^\lambda}\frac{1}{s_2^\xi}\sum_{k_1=0}^{\infty}\frac{(\gamma_1)_{k_1}}{k_1!}(\frac{\lambda_1^\alpha}{s_1^\alpha})^{k_1}\sum_{k_2=0}^{\infty}\frac{(\gamma_2)_{k_2}}{k_2!}(\frac{\lambda_2^\beta}{s_1^\beta s_2^\eta})^{k_2} = \frac{1}{s_1^\lambda}\frac{1}{s_2^\xi}(1-\frac{\lambda_1^\alpha}{s_1^\alpha})^{-\gamma_1}(1-\frac{\lambda_2^\beta}{s_1^\beta s_2^\eta})^{-\gamma_2}.
\end{aligned}$$

□

We deduce the following result from *Lemma* 4.1 by setting $\lambda - 1 = \gamma$ and using equation (1.9).

Corollary 4.2. *For* $\mathrm{Re}(\lambda_1), \mathrm{Re}(\lambda_2), \mathrm{Re}(\alpha), \mathrm{Re}(\beta), \mathrm{Re}(\lambda), \mathrm{Re}(s_1), \mathrm{Re}(s_2) > 0$ *and* $\left|\frac{\lambda_1^\alpha}{s_1^\alpha}\right|, \left|\frac{\lambda_2^\beta}{s_1^\beta s_2^\eta}\right| < 1,$ *we have*

$$\begin{aligned}
&\mathbb{L}_2[t^\gamma \tau^{\xi-1}L_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}((\lambda_1 t),(\lambda_2 t\tau^{\frac{\eta}{\beta}}))](s_1,s_2)\\
&= \frac{1}{s_1^{\gamma+1}}\frac{1}{s_2^\xi}\frac{\Gamma(\alpha n+\beta m+\gamma+1)}{\Gamma(\eta m+\xi)}(1-\frac{\lambda_1^\alpha}{s_1^\alpha})^n(1-\frac{\lambda_2^\beta}{s_1^\beta s_2^\eta})^m.
\end{aligned}$$

Theorem 4.3. *Let* $\lambda_1, \lambda_2 \in \mathbb{C}, \mathrm{Re}(\alpha+\eta) > 0$ *and* $\mathrm{Re}(\beta) > 0$. *Then*

$$\begin{aligned}
&\int_0^y\int_0^x \Big[(x-t)^{\lambda-1}(y-\tau)^{\xi-1}E_{\gamma_1,\gamma_2}^{(\alpha,\beta,\eta,\xi,\lambda)}(\lambda_1^\alpha(x-t)^\alpha, \lambda_2^\beta(x-t)^\beta(y-\tau)^\eta)\\
&\times t^{\gamma-1}\tau^{\zeta-1}E_{\gamma_3,\gamma_4}^{(\alpha,\beta,\eta,\zeta,\gamma)}(\lambda_1^\alpha t^\alpha, \lambda_2^\beta t^\beta \tau^\eta)dtd\tau\Big]\\
&= x^{\lambda+\gamma}y^{\xi+\zeta}E_{\gamma_1,\gamma_2}^{(\alpha,\beta,\eta,\xi,\lambda)}(\lambda_1^\alpha x^\alpha, \lambda_2^\beta x^\beta y^\eta)E_{\gamma_3,\gamma_4}^{(\alpha,\beta,\eta,\xi,\lambda)}(\lambda_1^\alpha x^\alpha, \lambda_2^\beta x^\beta y^\eta).
\end{aligned}$$

Proof. Using the convolution theorem for the Laplace transform we have,

$$\begin{aligned}
&\mathbb{L}_2\left[\int_0^y\int_0^x (x-t)^{\lambda-1}(y-\tau)^{\xi-1}E_{\gamma_1,\gamma_2}^{(\alpha,\beta,\eta,\xi,\lambda)}(\lambda_1^\alpha(x-t)^\alpha, \lambda_2^\beta(x-t)^\beta(y-\tau)^\eta)t^{\gamma-1}\tau^{\zeta-1}\right.\\
&\left.\times E_{\gamma_3,\gamma_4}^{(\alpha,\beta,\eta,\zeta,\gamma)}(\lambda_1^\alpha t^\alpha, \lambda_2^\beta t^\beta\tau^\eta)dtd\tau\right](s_1,s_2)\\
&= \mathbb{L}_2[x^{\lambda-1}y^{\xi-1}E_{\gamma_1,\gamma_2}^{(\alpha,\beta,\eta,\xi,\lambda)}(\lambda_1^\alpha x^\alpha, \lambda_2^\beta x^\beta y^\eta)]\mathbb{L}_2[x^{\gamma-1}y^{\zeta-1}E_{\gamma_3,\gamma_4}^{(\alpha,\beta,\eta,\zeta,\gamma)}(\lambda_1^\alpha t^\alpha, \lambda_2^\beta t^\beta\tau^\eta)](s_1,s_2)\\
&= \frac{1}{s_1^\lambda}\frac{1}{s_2^\xi}\left(1-\frac{\lambda_1^\alpha}{s_1^\alpha}\right)^{-\gamma_1}\left(1-\frac{\lambda_2^\beta}{s_1^\beta s_2^\eta}\right)^{-\gamma_2}\frac{1}{s_1^\gamma}\frac{1}{s_2^\zeta}\left(1-\frac{\lambda_1^\alpha}{s_1^\alpha}\right)^{-\gamma_3}\left(1-\frac{\lambda_2^\beta}{s_1^\beta s_2^\eta}\right)^{-\gamma_4}
\end{aligned}$$

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We have for $\mathrm{Re}(s_1), \mathrm{Re}(s_2) > 0$

$$\begin{aligned}(4.2)\quad & \mathbb{L}_2\left[\int_0^y\int_0^x (x-t)^{\lambda-1}(y-\tau)^{\xi-1}E_{\gamma_1,\gamma_2}^{(\alpha,\beta,\eta,\xi,\lambda)}(\lambda_1^\alpha(x-t)^\alpha, \lambda_2^\beta(x-t)^\beta(y-\tau)^\eta)\right.\\ & \left.\times t^{\gamma-1}\tau^{\zeta-1}E_{\gamma_1,\gamma_2}^{(\alpha,\beta,\eta,\zeta,\gamma)}(\lambda_1^\alpha t^\alpha, \lambda_2^\beta t^\beta\tau^\eta)dtd\tau\right](s_1,s_2)\\ = & \frac{1}{s_1^{\lambda+\gamma}}\frac{1}{s_2^{\xi+\zeta}}\left(1-\frac{\lambda_1^\alpha}{s_1^\alpha}\right)^{-\gamma_1}\left(1-\frac{\lambda_2^\beta}{s_1^\beta s_2^\eta}\right)^{-\gamma_2}\left(1-\frac{\lambda_1^\alpha}{s_1^\alpha}\right)^{-\gamma_3}\left(1-\frac{\lambda_2^\beta}{s_1^\beta s_2^\eta}\right)^{-\gamma_4}\\ = & \mathbb{L}_2\left[x^{\lambda+\gamma}y^{\xi+\zeta}E_{\gamma_1,\gamma_2}^{(\alpha,\beta,\eta,\xi,\lambda)}(\lambda_1^\alpha x^\alpha, \lambda_2^\beta x^\beta y^\eta)E_{\gamma_3,\gamma_4}^{(\alpha,\beta,\eta,\xi,\lambda)}(\lambda_1^\alpha x^\alpha, \lambda_2^\beta x^\beta y^\eta)\right](s_1,s_2).\end{aligned}$$

Taking inverse Laplace transform on both sides of (4.2), the result follows. □

The next assertion follows from *Theorem* 4.5 by letting $\lambda - 1 = \gamma$ and taking into account (1.9).

Corollary 4.4. *Let* $\lambda_1, \lambda_2 \in \mathbb{C}, \mathrm{Re}(\lambda), \mathrm{Re}(\xi), \mathrm{Re}(\gamma), \mathrm{Re}(\zeta) > 0$. *Then*

$$\begin{aligned}&\int_0^y\int_0^x (x-t)^{\gamma}(y-\tau)^{\xi-1}L_{n_1,m_1}^{(\alpha,\beta,\gamma_1,\eta,\xi_1)}(\lambda_1(x-t), \lambda_2(x-t)(y-\tau)^{\frac{\eta}{\beta}})\\ &\times t^\gamma\tau^{\xi-1}L_{n_2,m_2}^{(\alpha,\beta,\gamma_2,\eta,\xi_2)}(\lambda_1 t, \lambda_2 t\tau^{\frac{\eta}{\beta}})dtd\tau\\ &= x^{\gamma_1+\gamma_2+1}y^{\xi_1+\xi_2-1}L_{n_1,m_1}^{(\alpha,\beta,\gamma_1,\eta,\xi_1)}(\lambda_1 x, \lambda_2 xy^{\frac{\eta}{\beta}})L_{n_2,m_2}^{(\alpha,\beta,\gamma_2,\eta,\xi_2)}(\lambda_1 x, \lambda_2 xy^{\frac{\eta}{\beta}}).\end{aligned}$$

Now, we consider the following double convolution equation:

$$(4.3)\quad \int_0^y\int_0^x (x-t)^{\gamma-1}(y-\tau)^{\xi-1}L_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}((\lambda_1 x)^\alpha, (\lambda_2^\beta x^\beta y^\eta))\Phi(t,\tau)dtd\tau = \Psi(x,y)$$

where $\mathrm{Re}(\gamma) > -1$.

For the solution of the integral equation (4.3), we have the following theorem:

Theorem 4.5. *The singular double integral equation (4.3) admits a locally integrable solution*

$$\Phi(t,\tau) = \frac{\Gamma(\eta m+\xi)}{\Gamma(\alpha n+\beta m+\gamma+1)}$$

$$\times\int_0^y\int_0^x (x-t)^{\alpha_1-\gamma-2}(y-\tau)^{\alpha_2-\xi-1}E_{\gamma_1,\gamma_2}^{(\alpha,\beta,\eta,\xi,\lambda)}((\lambda_1 x)^\alpha, (\lambda_2^\beta x^\beta y^\eta))[I_{0^+}^{-\alpha_1}I_{0^+}^{-\alpha_2}\Psi(t,\tau)]dtd\tau.$$

Proof. Applying double Laplace transform on both sides of (4.3), then using double convolution theorem, we get

$$\frac{1}{s_1^{\gamma+1}}\frac{1}{s_2^\xi}\frac{\Gamma(\alpha n+\beta m+\gamma+1)}{\Gamma(\eta m+\xi)}\left(1-\frac{\lambda_1^\alpha}{s_1^\alpha}\right)^n\left(1-\frac{\lambda_2^\beta}{s_1^\beta s_2^\eta}\right)^m\mathbb{L}_2[\Phi(t,\tau)](s_1,s_2) = \mathbb{L}_2[\Psi(t,\tau)](s_1,s_2)$$

Therefore, we have,

$$\mathbb{L}_2[\Phi(t,\tau)](s_1,s_2) = \frac{\Gamma(\eta m+\xi)}{\Gamma(\alpha n+\beta m+\gamma+1)}$$

$$\times(s_1)^{\gamma-\alpha_1+1}(s_2)^{\xi-\alpha_2}\left(1-\frac{\lambda_1^\alpha}{s_1^\alpha}\right)^{-n}\left(1-\frac{\lambda_2^\beta}{s_1^\beta s_2^\eta}\right)^{-m}\{s_1^{\alpha_1}s_2^{\alpha_2}\mathbb{L}_2[\Psi(t,\tau)](s_1,s_2)\}$$

Finally taking inverse Laplace transform on both sides and using *Lemma* 3.2 of [1] and *Lemma* 4.1, we get

$$\Phi(t,\tau)=\frac{\Gamma(\eta m+\xi)}{\Gamma(\alpha n+\beta m+\gamma+1)}$$

$$\times\int_0^y\int_0^x(x-t)^{\alpha_1-\gamma-2}(y-\tau)^{\alpha_2-\xi-1}E_{\gamma_1,\gamma_2}^{(\alpha,\beta,\eta,\xi,\lambda)}((\lambda_1x)^{\alpha},(\lambda_2^{\beta}x^{\beta}y^{\eta}))[I_{0^+}^{-\alpha_1}I_{0^+}^{-\alpha_2}\Psi(t,\tau)]dtd\tau$$

and the proof is completed. □

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Generalized Inequalities of the type of Hermite-Hadamard-Fejer with Quasi-Convex Functions by way of k-Fractional Derivatives

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Abstract: In this article, Hermite-Hadamard-Fejer type inequalities are discussed with quasi-convex functions and obtained the generalized results of the type using k-fractional derivatives. And proposed some new bounds in terms of some special means.
Keywords: Hermite-Hadamard inequality, Hermite-Hadamard-Fejer inequality, quasi convex functions, k-Riemann-Liouville fractional derivatives, Hölder's integral inequality, Power mean inequality.

1. Introduction

The function $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ is said to be convex on I if for every $x, y \in I$ and $t \in [0, 1]$, we get

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

Let $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, f satisfies the following well-known Hermite-Hadamard type inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}.$$

Definition 1. *The function $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ is said to be quasi-convex if*

$$f(tx + (1-t)y) \leq max\left\{f(x), f(y)\right\},$$

for every $x, y \in I$ and $t \in [0, 1]$ (see [4]).

In [3] Mubeen and Habibullah introduced the following class of fractional derivatives.

Definition 2. *Let $f \in L[a,b]$, then k-Riemann-Liouville fractional derivatives ${}_kJ^{\alpha}_{a^+}f(x)$ and ${}_kJ^{\alpha}_{b^-}f(x)$ of order $\alpha > 0$ are defined by*

$$ {}_kJ^{\alpha}_{a^+}f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)dt, \qquad (0 \le a < x < b) $$

and

$$ {}_kJ^{\alpha}_{b^-}f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^b (t-x)^{\frac{\alpha}{k}-1} f(t)dt, \qquad (0 \le a < x < b) $$

respectively, where $k > 0$ and $\Gamma_k(\alpha)$ is the k-gamma function given as $\Gamma_k(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{\frac{-t^k}{k}} dt$. Furthermore $\Gamma_k(\alpha + k) = \alpha\Gamma_k(\alpha)$ and ${}_kJ^0_{a^+}f(x) = {}_kJ^0_{b^-}f(x) = f(x)$.

In [1] Fejér established the following inequality.

Lemma 1. *Let $f : [a,b] \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function, the inequality*

$$ f\left(\frac{a+b}{2}\right) \int_a^b g(x)dx \le \int_a^b f(x)g(x)dx \le \frac{f(a)+f(b)}{2} \int_a^b g(x)dx $$

holds, where $g : [a,b] \subset \mathbb{R} \longrightarrow \mathbb{R}$ is non-negative integrable and symmetric to $\frac{a+b}{2}$. This inequality is called Hermite-Hadamard-Fejer *inequality.*

Lemma 2. *(*[7]*) For $0 < t \le 1$ and $0 \le a < b$, we get*

$$ |a^t - b^t| \le (b-a)^t. $$

E. Set et al. established the following Lemma in [6].

Lemma 3. *Let $f : [a,b] \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable mapping on (a,b) and $g : [a,b] \subset \mathbb{R} \longrightarrow \mathbb{R}$. If $f', g \in L[a,b]$, the following identity for fractional derivatives holds*

$$ f\left(\frac{a+b}{2}\right)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}g(a) + J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}g(b)\right] - \left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}(fg)(a) + J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}(fg)(b)\right] = \frac{1}{\Gamma(\alpha)} \int_a^b m(t)f'(t)dt \tag{1.1} $$

where

$$ m(t) = \begin{cases} \int_a^t (s-a)^{\alpha-1} g(s)ds & t \in \left[a, \frac{a+b}{2}\right) \\ -\int_t^b (b-s)^{\alpha-1} g(s)ds & t \in \left[\frac{a+b}{2}, b\right]. \end{cases} $$

Iscan obtained the following lemma in [2].

Lemma 4. *Let $f : [a, b] \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable mapping on (a, b) and $f^{'} \in L[a, b]$. If $g : [a, b] \longrightarrow R$ is integrable and symmetric to $\frac{a+b}{2}$, the following identity for fractional derivatives holds*

$$\frac{f(a)+f(b)}{2}\left[J_{a^+}^{\alpha}g(b)+J_{b^-}^{\alpha}g(a)\right]-\left[J_{a^+}^{\alpha}(fg)(b)+J_{b^-}^{\alpha}(fg)(a)\right]$$
$$=\frac{1}{\Gamma(\alpha)}\int_a^b\left(\int_a^t(b-s)^{\alpha-1}g(s)ds-\int_t^b(s-a)^{\alpha-1}g(s)ds\right)f^{'}(t)dt \qquad (1.2)$$

where $\alpha > 0$.

In the present paper motivated by the recent results given in [5] we established some Hermite-Hadamard-Fejér type inequalities for quasi-convex functions via k-fractional derivatives.

2. Main Findings

Throughout this paper, let I be an interval on R and let $||g||_{[a,b],\infty} = \sup_{t\in[a,b]} g(t)$ for continuous function $g : [a, b]\mathbb{R} \longrightarrow \mathbb{R}$.

The following identity is the generalization of identity (1.1) in Lemma 3 for k-fractional derivatives.

Lemma 5. *Let $f : [a, b] \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable mapping on (a, b) and $g : [a, b] \subset \mathbb{R} \longrightarrow \mathbb{R}$. If $f^{'}, g \in L[a, b]$, the following identity for k-fractional derivatives holds*

$$f\left(\frac{a+b}{2}\right)\left[{}_kJ^{\alpha}_{\left(\frac{a+b}{2}\right)^-}g(a)+{}_kJ^{\alpha}_{\left(\frac{a+b}{2}\right)^+}g(b)\right]-\left[{}_kJ^{\alpha}_{\left(\frac{a+b}{2}\right)^-}(fg)(a)+{}_kJ^{\alpha}_{\left(\frac{a+b}{2}\right)^+}(fg)(b)\right]$$
$$=\frac{1}{\Gamma_k(\alpha)}\int_a^b m(t)f^{'}(t)dt$$

where

$$m(t)=\begin{cases}\int_a^t(s-a)^{\frac{\alpha}{k}-1}g(s)ds & t\in\left[a,\frac{a+b}{2}\right)\\ -\int_t^b(b-s)^{\frac{\alpha}{k}-1}g(s)ds & t\in\left[\frac{a+b}{2},b\right].\end{cases}$$

Here the identity (1.2) of Lemma 4 is also generalized for k-fractional derivatives.

Lemma 6. *Let $f : [a, b] \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable mapping on (a, b) and $f^{'} \in L[a, b]$. If $g : [a, b] \subset \mathbb{R} \longrightarrow \mathbb{R}$ is integrable and symmetric to $\frac{a+b}{2}$, the following for k-fractional* *derivatives holds*

$$\frac{f(a)+f(b)}{2}\left[{}_kJ^{\alpha}_{a^+}g(b)+{}_kJ^{\alpha}_{b^-}g(a)\right]-\left[{}_kJ^{\alpha}_{a^+}(fg)(b)+{}_kJ^{\alpha}_{b^-}(fg)(a)\right]$$
$$=\frac{1}{\Gamma_k(\alpha)}\int_a^b\left(\int_a^t(b-s)^{\frac{\alpha}{k}-1}g(s)ds-\int_t^b(s-a)^{\frac{\alpha}{k}-1}g(s)ds\right)f^{'}(t)dt$$

where $\frac{\alpha}{k}>0$.

Theorem 1. *Let* $f:I\subset\mathbb{R}\longrightarrow\mathbb{R}$ *be a differentiable mapping on* I^o *and* $f^{'}\in L[a,b]$ *and* $g:[a,b]\subset\mathbb{R}\longrightarrow\mathbb{R}$ *is continuous. If* $|f^{'}|^q$ *is quasi-convex function on* $[a,b], q\geq 1$*, the following inequality for* k*-fractional derivatives holds*

$$\left|f\left(\frac{a+b}{2}\right)\left[{}_kJ^{\alpha}_{\left(\frac{a+b}{2}\right)^-}g(a)+{}_kJ^{\alpha}_{\left(\frac{a+b}{2}\right)^+}g(b)\right]-\left[{}_kJ^{\alpha}_{\left(\frac{a+b}{2}\right)^-}(fg)(a)+{}_kJ^{\alpha}_{\left(\frac{a+b}{2}\right)^+}(fg)(b)\right]\right|$$
$$\leq\frac{(b-a)^{\frac{\alpha}{k}+1}||g||_{[a,b],\infty}}{2^{\frac{\alpha}{k}}\left(\frac{\alpha}{k}+1\right)\Gamma_k(\alpha+k)}\left(max\left\{|f^{'}(a)|^q,|f^{'}(b)|^q\right\}\right)^{\frac{1}{q}}$$

where $\frac{\alpha}{k}>0$.

Proof. Since $|f^{'}|^q$ is quasi-convex on $[a,b]$, we know that for $t\in[a,b]$

$$|f^{'}(t)|^q=\left|f^{'}\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b\right)\right|^q\leq\max\left\{|f^{'}(a)|^q,|f^{'}(b)|^q\right\}.$$

Using lemma 5, power mean inequality and the fact that $|f^{'}|^q$ is quasi-convex function on $[a,b]$, it follows that

$$\left|f\left(\frac{a+b}{2}\right)\left[{}_kJ^{\alpha}_{\left(\frac{a+b}{2}\right)^-}g(a)+{}_kJ^{\alpha}_{\left(\frac{a+b}{2}\right)^+}g(b)\right]-\left[{}_kJ^{\alpha}_{\left(\frac{a+b}{2}\right)^-}(fg)(a)+{}_kJ^{\alpha}_{\left(\frac{a+b}{2}\right)^+}(fg)(b)\right]\right|$$
$$\leq\frac{1}{\Gamma_k(\alpha)}\left(\int_a^{\frac{a+b}{2}}\left|\int_a^t(s-a)^{\frac{\alpha}{k}-1}g(s)ds\right|dt\right)^{1-\frac{1}{q}}\left(\int_a^{\frac{a+b}{2}}\left|\int_a^t(s-a)^{\frac{\alpha}{k}-1}g(s)ds\right||f^{'}(t)|^qdt\right)^{\frac{1}{q}}$$
$$+\frac{1}{\Gamma_k(\alpha)}\left(\int_{\frac{a+b}{2}}^b\left|\int_t^b(b-s)^{\frac{\alpha}{k}-1}g(s)ds\right|dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{a+b}{2}}^b\left|\int_t^b(b-s)^{\frac{\alpha}{k}-1}g(s)ds\right||f^{'}(t)|^qdt\right)^{\frac{1}{q}}$$

$$\leq \frac{||g||_{[a,\frac{a+b}{2}],\infty}}{\Gamma_k(\alpha)} \left(\int_a^{\frac{a+b}{2}} \left|\int_a^t (s-a)^{\frac{\alpha}{k}-1} ds\right| dt\right)^{1-\frac{1}{q}} \left(\int_a^{\frac{a+b}{2}} \left|\int_a^t (s-a)^{\frac{\alpha}{k}-1} ds\right| |f^{'}(t)|^q dt\right)^{\frac{1}{q}}$$

$$+\frac{||g||_{[\frac{a+b}{2},b],\infty}}{\Gamma_k(\alpha)} \left(\int_{\frac{a+b}{2}}^b \left|\int_t^b (b-s)^{\frac{\alpha}{k}-1} ds\right| dt\right)^{1-\frac{1}{q}} \left(\int_{\frac{a+b}{2}}^b \left|\int_t^b (b-s)^{\frac{\alpha}{k}-1} ds\right| |f^{'}(t)|^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{1}{\Gamma_k(\alpha+k)} \left(\frac{(b-a)^{\frac{\alpha}{k}+1}}{2^{\frac{\alpha}{k}+1}\left(\frac{\alpha}{k}+1\right)}\right)^{1-\frac{1}{q}} \left(\frac{(b-a)^{\frac{\alpha}{k}+1}}{2^{\frac{\alpha}{k}+1}\left(\frac{\alpha}{k}+1\right)}\right)^{\frac{1}{q}} \left(||g||_{[a,\frac{a+b}{2}],\infty} + ||g||_{[\frac{a+b}{2},b],\infty}\right)$$

$$\left(\max\left\{|f^{'}(a)|^q, |f^{'}(b)|^q\right\}\right)^{\frac{1}{q}}$$

$$\leq \frac{(b-a)^{\frac{\alpha}{k}+1}||g||_{[a,b],\infty}}{2^{\frac{\alpha}{k}}\left(\frac{\alpha}{k}+1\right)\Gamma_k(\alpha+k)} \left(\max\left\{|f^{'}(a)|^q, |f^{'}(b)|^q\right\}\right)^{\frac{1}{q}}$$

where

$$\int_a^{\frac{a+b}{2}} \left|\int_a^t (s-a)^{\frac{\alpha}{k}-1} ds\right| dt = \int_{\frac{a+b}{2}}^b \left|\int_t^b (b-s)^{\frac{\alpha}{k}-1} ds\right| dt = \frac{(b-a)^{\frac{\alpha}{k}+1}}{2^{\frac{\alpha}{k}+1}\left(\frac{\alpha}{k}+1\right)\frac{\alpha}{k}}$$

Which completes the proof. □

Corollary 1. *If we choose* $g(x) = 1$ *and* $\alpha = k$ *in Theorem 1, we get*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{4}\left(\mathit{max}\left\{|f^{'}(a)|^q, |f^{'}(b)|^q\right\}\right)^{\frac{1}{q}}.$$

Theorem 2. *Let* $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ *be a differentiable mapping on* I^o *and* $f^{'} \in L[a,b]$ *and* $g : [a,b] \subset \mathbb{R} \longrightarrow \mathbb{R}$ *is continuous. If* $|f^{'}|^q$ *is quasi-convex function on* $[a,b], q > 1$*, the following inequality for* k*-fractional derivatives holds*

$$\left|f\left(\frac{a+b}{2}\right)\left[{}_kJ^{\alpha}_{\left(\frac{a+b}{2}\right)^-}g(a) + {}_kJ^{\alpha}_{\left(\frac{a+b}{2}\right)^+}g(b)\right] - \left[{}_kJ^{\alpha}_{\left(\frac{a+b}{2}\right)^-}(fg)(a) + {}_kJ^{\alpha}_{\left(\frac{a+b}{2}\right)^+}(fg)(b)\right]\right|$$
$$\leq \frac{(b-a)^{\frac{\alpha}{k}+1}||g||_{\infty}}{2^{\frac{\alpha}{k}}\left(\frac{\alpha}{k}p+1\right)^{\frac{1}{p}}\Gamma_k(\alpha+k)}\left(\mathit{max}\left\{|f^{'}(a)|^q, |f^{'}(b)|^q\right\}\right)^{\frac{1}{q}}$$

where $\frac{1}{p} + \frac{1}{q} = 1.$

Proof. Using Lemma 5, Hölder's inequality and the fact that $|f^{'}|^q$ is quasi-convex function on $[a,b]$, it follows that

$$\left| f\left(\frac{a+b}{2}\right)\left[{}_kJ^{\alpha}_{\left(\frac{a+b}{2}\right)^-}g(a)+{}_kJ^{\alpha}_{\left(\frac{a+b}{2}\right)^+}g(b)\right]-\left[{}_kJ^{\alpha}_{\left(\frac{a+b}{2}\right)^-}(fg)(a)+{}_kJ^{\alpha}_{\left(\frac{a+b}{2}\right)^+}(fg)(b)\right]\right|$$
$$\le \frac{1}{\Gamma_k(\alpha)}\left(\int_a^{\frac{a+b}{2}}\left|\int_a^t (s-a)^{\frac{\alpha}{k}-1}g(s)ds\right||f'(t)|dt\right.$$
$$\left.+\int_{\frac{a+b}{2}}^b\left|\int_t^b (b-s)^{\frac{\alpha}{k}-1}g(s)ds\right||f'(t)|dt\right)$$
$$\le \frac{1}{\Gamma_k(\alpha)}\left(\int_a^{\frac{a+b}{2}}\left|\int_a^t (s-a)^{\frac{\alpha}{k}-1}g(s)ds\right|^p dt\right)^{\frac{1}{p}}\left(\int_a^{\frac{a+b}{2}}|f'(t)|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{1}{\Gamma_k(\alpha)}\left(\int_{\frac{a+b}{2}}^b\left|\int_t^b (b-s)^{\frac{\alpha}{k}-1}g(s)ds\right|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{a+b}{2}}^b|f'(t)|^q dt\right)^{\frac{1}{q}}$$
$$=\frac{1}{\Gamma_k(\alpha)}\left(\int_a^{\frac{a+b}{2}}\left|\int_a^t (s-a)^{\frac{\alpha}{k}-1}g(s)ds\right|^p dt\right)^{\frac{1}{p}}$$
$$\left[\left(\int_a^{\frac{a+b}{2}}|f'(t)|^q dt\right)^{\frac{1}{q}}+\left(\int_{\frac{a+b}{2}}^b|f'(t)|^q dt\right)^{\frac{1}{q}}\right]$$
$$\le \frac{||g||_\infty}{\Gamma_k(\alpha)}\left(\frac{(b-a)^{\frac{\alpha}{k}p+1}}{2^{\frac{\alpha}{k}p+1}\left(\frac{\alpha}{k}p+1\right)\left(\frac{\alpha}{k}\right)^p}\right)^{\frac{1}{p}}$$
$$\left[\left(\int_a^{\frac{a+b}{2}}\max\left\{|f'(a)|^q,|f'(b)|^q\right\}dt\right)^{\frac{1}{q}}+\left(\int_{\frac{a+b}{2}}^b\max\left\{|f'(a)|^q,|f'(b)|^q\right\}dt\right)^{\frac{1}{q}}\right]$$
$$=\frac{(b-a)^{\frac{\alpha}{k}+1}||g||_\infty}{2^{\frac{\alpha}{k}}\left(\frac{\alpha}{k}p+1\right)^{\frac{1}{p}}\Gamma_k(\alpha+k)}\left(\max\left\{|f'(a)|^q,|f'(b)|^q\right\}\right)^{\frac{1}{q}}.$$

Where

$$\int_a^{\frac{a+b}{2}}\left|\int_a^t (s-a)^{\frac{\alpha}{k}-1}ds\right|^p dt=\frac{(b-a)^{\frac{\alpha}{k}p+1}}{2^{\frac{\alpha}{k}p+1}\left(\frac{\alpha}{k}p+1\right)\left(\frac{\alpha}{k}\right)^p}.$$

□

Corollary 2. *If we choose $g(x)=1$ and $\alpha=k$ in Theorem 2, then we get*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\right|\le\frac{b-a}{2(p+1)^{\frac{1}{p}}}\left(max\left\{|f'(a)|^q,|f'(b)|^q\right\}\right)^{\frac{1}{q}}.$$

Theorem 3. *Let $f:I\subset\mathbb{R}\longrightarrow\mathbb{R}$ be a differentiable mapping on I^o and $f'\in L[a,b]$. If $|f'|$ is quasi-convex function on $[a,b]$ and $g:[a,b]\subset\mathbb{R}\longrightarrow\mathbb{R}$ is continuous and symmetric to $\frac{a+b}{2}$, the following inequality for k-fractional derivatives holds*

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$$\left|\frac{f(a)+f(b)}{2}\left[{}_kJ^{\alpha}_{a^+}g(b)+{}_kJ^{\alpha}_{b^-}g(a)\right]-\left[{}_kJ^{\alpha}_{a^+}(fg)(b)+{}_kJ^{\alpha}_{b^-}(fg)(a)\right]\right|$$
$$\leq \frac{2(b-a)^{\frac{\alpha}{k}+1}||g||_{\infty}}{\left(\frac{\alpha}{k}+1\right)\Gamma_k(\alpha+k)}\left(1-\frac{1}{2^{\frac{\alpha}{k}}}\right) max\left\{|f^{'}(a)|,|f^{'}(b)|\right\}$$

where $\frac{\alpha}{k}>0$.

Proof. From Lemma 6, we get

$$\left|\frac{f(a)+f(b)}{2}\left[{}_kJ^{\alpha}_{a^+}g(b)+{}_kJ^{\alpha}_{b^-}g(a)\right]-\left[{}_kJ^{\alpha}_{a^+}(fg)(b)+{}_kJ^{\alpha}_{b^-}(fg)(a)\right]\right|$$
$$\leq \frac{1}{\Gamma_k(\alpha)}\int_a^b\left|\int_a^t(b-s)^{\frac{\alpha}{k}-1}g(s)ds-\int_t^b(s-a)^{\frac{\alpha}{k}-1}g(s)ds\right||f^{'}(t)|dt.$$

Since $|f^{'}|$ is quasi-convex on $[a,b]$, we know that for $t\in[a,b]$

$$|f^{'}(t)|=\left|f^{'}\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b\right)\right|\leq \max\left\{|f^{'}(a)|,|f^{'}(b)|\right\}$$

and since $g:[a,b]\subset\mathbb{R}\longrightarrow\mathbb{R}$ is continuous and symmetric to $\frac{a+b}{2}$ we can write

$$\int_t^b(s-a)^{\frac{\alpha}{k}-1}g(s)ds=\int_a^{a+b-t}(b-s)^{\frac{\alpha}{k}-1}g(a+b-s)ds$$
$$=\int_a^{a+b-t}(b-s)^{\frac{\alpha}{k}-1}g(s)ds$$

therefore we get

$$\left|\int_a^t(b-s)^{\frac{\alpha}{k}-1}g(s)ds-\int_t^b(s-a)^{\frac{\alpha}{k}-1}g(s)ds\right|$$
$$=\int_t^{a+b-t}(b-s)^{\frac{\alpha}{k}-1}g(s)ds$$

$$\leq\begin{cases}\int_t^{a+b-t}\left|(b-s)^{\frac{\alpha}{k}-1}g(s)\right|ds, & t\in\left[a,\frac{a+b}{2}\right]\\ \int_{a+b-t}^t\left|(b-s)^{\frac{\alpha}{k}-1}g(s)\right|ds, & t\in\left[\frac{a+b}{2},b\right].\end{cases}\tag{2.3}$$

Therefore we get

$$\left|\frac{f(a)+f(b)}{2}\left[{}_kJ^{\alpha}_{a^+}g(b)+{}_kJ^{\alpha}_{b^-}g(a)\right]-\left[{}_kJ^{\alpha}_{a^+}(fg)(b)+{}_kJ^{\alpha}_{b^-}(fg)(a)\right]\right|$$
$$\leq \frac{1}{\Gamma_k(\alpha)}\left[\int_a^{\frac{a+b}{2}}\int_t^{a+b-t}\left|(b-s)^{\frac{\alpha}{k}-1}g(s)\right|dsdt+\int_{\frac{a+b}{2}}^{b}\int_{a+b-t}^{t}\left|(b-s)^{\frac{\alpha}{k}-1}g(s)\right|dsdt\right]$$
$$\left(\max\left\{|f'(a)|,|f'(b)|\right\}\right)$$
$$\leq \frac{||g||_\infty}{\Gamma_k(\alpha+k)}\left(\int_a^{\frac{a+b}{2}}\left[(b-t)^{\frac{\alpha}{k}}-(t-a)^{\frac{\alpha}{k}}\right]dt+\int_{\frac{a+b}{2}}^{b}\left[(t-a)^{\frac{\alpha}{k}}-(b-t)^{\frac{\alpha}{k}}\right]dt\right)$$
$$\left(\max\left\{|f'(a)|,|f'(b)|\right\}\right)$$
$$=\frac{2(b-a)^{\frac{\alpha}{k}+1}||g||_\infty}{\left(\frac{\alpha}{k}+1\right)\Gamma_k(\alpha+k)}\left(1-\frac{1}{2^{\frac{\alpha}{k}}}\right)\left(\max\left\{|f'(a)|,|f'(b)|\right\}\right)$$

since

$$\int_a^{\frac{a+b}{2}}(b-t)^{\frac{\alpha}{k}}dt=\int_{\frac{a+b}{2}}^{b}(t-a)^{\frac{\alpha}{k}}dt=\frac{(b-a)^{\frac{\alpha}{k}+1}\left(2^{\frac{\alpha}{k}+1}-1\right)}{2^{\frac{\alpha}{k}+1}\left(\frac{\alpha}{k}+1\right)}$$

and

$$\int_a^{\frac{a+b}{2}}(t-a)^{\frac{\alpha}{k}}dt=\int_{\frac{a+b}{2}}^{b}(b-t)^{\frac{\alpha}{k}}dt=\frac{(b-a)^{\frac{\alpha}{k}+1}}{2^{\frac{\alpha}{k}+1}\left(\frac{\alpha}{k}+1\right)}.$$

□

Corollary 3. *In Theorem 3, if we take $g(x)=1$, we get the inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[{}_kJ^{\alpha}_{a^+}f(b)+{}_kJ^{\alpha}_{b^-}f(a)\right]\right|$$
$$\leq \frac{b-a}{\left(\frac{\alpha}{k}+1\right)}\left(1-\frac{1}{2^{\frac{\alpha}{k}}}\right)\, max\left\{|f'(a)|,|f'(b)|\right\}.$$

.

Theorem 4. *Let $f: I\subset\mathbb{R}\longrightarrow\mathbb{R}$ be a differentiable mapping on I^o and $f'\in L[a,b]$. If $|f'|^q, q\geq 1$ is quasi-convex function on $[a,b]$ and $g:[a,b]\subset\mathbb{R}\longrightarrow\mathbb{R}$ is continuous and symmetric to $\frac{a+b}{2}$, the following inequality for k-fractional derivatives holds*

$$\left|\frac{f(a)+f(b)}{2}\left[{}_kJ^{\alpha}_{a^+}g(b)+{}_kJ^{\alpha}_{b^-}g(a)\right]-\left[{}_kJ^{\alpha}_{a^+}(fg)(b)+{}_kJ^{\alpha}_{b^-}(fg)(a)\right]\right|$$
$$\leq \frac{2(b-a)^{\frac{\alpha}{k}+1}||g||_\infty}{\left(\frac{\alpha}{k}+1\right)\Gamma_k(\alpha+k)}\left(1-\frac{1}{2^{\frac{\alpha}{k}}}\right)\left(max\left\{|f'(a)|^q,|f'(b)|^q\right\}\right)^{\frac{1}{q}}$$

where $\frac{\alpha}{k}>0$.

Proof. From Lemma 6, power mean inequality, inequality (2.3) and the quasi-convexity of $|f^{'}|^q$, we get

$$\left|\frac{f(a)+f(b)}{2}\left[{}_kJ^{\alpha}_{a^+}g(b)+{}_kJ^{\alpha}_{b^-}g(a)\right]-\left[{}_kJ^{\alpha}_{a^+}(fg)(b)+{}_kJ^{\alpha}_{b^-}(fg)(a)\right]\right|$$
$$\leq \frac{1}{\Gamma_k(\alpha)}\left(\int_a^b\left|\int_t^{a+b-t}(b-s)^{\frac{\alpha}{k}-1}g(s)ds\right|dt\right)^{1-\frac{1}{q}}\left(\int_a^b\left|\int_t^{a+b-t}(b-s)^{\frac{\alpha}{k}-1}g(s)ds\right||f^{'}(t)|^q dt\right)^{\frac{1}{q}}$$
$$\leq \frac{1}{\Gamma_k(\alpha)}\left(\int_a^{\frac{a+b}{2}}\left[\int_t^{a+b-t}\left|(b-s)^{\frac{\alpha}{k}-1}g(s)\right|ds\right]dt+\int_{\frac{a+b}{2}}^b\left[\int_{a+b-t}^t\left|(b-s)^{\frac{\alpha}{k}-1}g(s)\right|ds\right]dt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_a^{\frac{a+b}{2}}\left[\int_t^{a+b-t}|(b-s)^{\frac{\alpha}{k}-1}g(s)|ds\right]|f^{'}(t)|^q dt+\int_{\frac{a+b}{2}}^b\left[\int_{a+b-t}^t|(b-s)^{\frac{\alpha}{k}-1}g(s)|ds\right]|f^{'}(t)|^q dt\right)^{\frac{1}{q}}$$
$$\leq \frac{2(b-a)^{\frac{\alpha}{k}+1}||g||_\infty}{\left(\frac{\alpha}{k}+1\right)\Gamma_k(\alpha+k)}\left(1-\frac{1}{2^{\frac{\alpha}{k}}}\right)\left(\max\left\{|f^{'}(a)|^q,|f^{'}(b)|^q\right\}\right)^{\frac{1}{q}}$$

where

$$\int_a^{\frac{a+b}{2}}\left[\int_t^{a+b-t}|(b-s)^{\frac{\alpha}{k}-1}g(s)|ds\right]dt+\int_{\frac{a+b}{2}}^b\left[\int_{a+b-t}^t|(b-s)^{\frac{\alpha}{k}-1}g(s)|ds\right]dt$$
$$=\frac{2(b-a)^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k}\left(\frac{\alpha}{k}+1\right)}\left(1-\frac{1}{2^{\frac{\alpha}{k}}}\right).$$

□

Theorem 5. *Let $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable mapping on I^o and $f^{'} \in L[a,b]$. If $|f^{'}|^q, q > 1$ is quasi-convex function on $[a,b]$, and $g : [a,b] \longrightarrow R$ is continuous and symmetric to $\frac{a+b}{2}$, the following inequality for k-fractional derivatives holds*

$$\left|\frac{f(a)+f(b)}{2}\left[{}_kJ^{\alpha}_{a^+}g(b)+{}_kJ^{\alpha}_{b^-}g(a)\right]-\left[{}_kJ^{\alpha}_{a^+}(fg)(b)+{}_kJ^{\alpha}_{b^-}(fg)(a)\right]\right|$$
$$\leq \frac{2^{\frac{1}{p}}(b-a)^{\frac{\alpha}{k}+1}||g||_\infty}{\left(\frac{\alpha}{k}p+1\right)^{\frac{1}{p}}\Gamma_k(\alpha+k)}\left(1-\frac{1}{2^{\frac{\alpha}{k}p}}\right)^{\frac{1}{p}}\left(max\left\{|f^{'}(a)|^q,|f^{'}(b)|^q\right\}\right)^{\frac{1}{q}},$$

where $\frac{\alpha}{k} > 0$.

Proof. From Lemma 6, Hölder's inequality, inequality (2.3) and the quasi-convexity of $|f^{'}|^q$, we get

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}\left[{}_kJ^{\alpha}_{a^+}g(b)+{}_kJ^{\alpha}_{b^-}g(a)\right]-\left[{}_kJ^{\alpha}_{a^+}(fg)(b)+{}_kJ^{\alpha}_{b^-}(fg)(a)\right]\right| \\
&\leq \frac{1}{\Gamma_k(\alpha)}\left(\int_a^b\left|\int_t^{a+b-t}(b-s)^{\frac{\alpha}{k}-1}g(s)ds\right|^p dt\right)^{\frac{1}{p}}\left(\int_a^b|f^{'}(t)|^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{||g||_\infty}{\Gamma_k(\alpha+k)}\left(\int_a^{\frac{a+b}{2}}\left[(b-t)^{\frac{\alpha}{k}}-(t-a)^{\frac{\alpha}{k}}\right]^p dt+\int_{\frac{a+b}{2}}^b\left[(t-a)^{\frac{\alpha}{k}}-(b-t)^{\frac{\alpha}{k}}\right]^p dt\right)^{\frac{1}{p}} \\
&\qquad\left(\int_a^b \max\left\{|f^{'}(a)|^q,|f^{'}(b)|^q\right\}dt\right)^{\frac{1}{q}} \\
&\leq \frac{||g||_\infty(b-a)^{\frac{\alpha}{k}+1}}{\Gamma_k(\alpha+k)}\left(\int_0^{\frac{1}{2}}\left[(1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}}\right]^p dt+\int_{\frac{1}{2}}^1\left[t^{\frac{\alpha}{k}}-(1-t)^{\frac{\alpha}{k}}\right]^p dt\right)^{\frac{1}{p}} \\
&\qquad\left(\max\left\{|f^{'}(a)|^q,|f^{'}(b)|^q\right\}\right)^{\frac{1}{q}} \\
&\leq \frac{||g||_\infty(b-a)^{\frac{\alpha}{k}+1}}{\Gamma_k(\alpha+k)}\left(\int_0^{\frac{1}{2}}\left[(1-t)^{\frac{\alpha}{k}p}-t^{\frac{\alpha}{k}p}\right] dt+\int_{\frac{1}{2}}^1\left[t^{\frac{\alpha}{k}p}-(1-t)^{\frac{\alpha}{k}p}\right] dt\right)^{\frac{1}{p}} \\
&\qquad\left(\max\left\{|f^{'}(a)|^q,|f^{'}(b)|^q\right\}\right)^{\frac{1}{q}} \\
&\leq \frac{2^{\frac{1}{p}}(b-a)^{\frac{\alpha}{k}+1}||g||_\infty}{\left(\frac{\alpha}{k}p+1\right)^{\frac{1}{p}}\Gamma_k(\alpha+k)}\left(1-\frac{1}{2^{\frac{\alpha}{k}p}}\right)^{\frac{1}{p}}\left(\max\left\{|f^{'}(a)|^q,|f^{'}(b)|^q\right\}\right)^{\frac{1}{q}}.
\end{aligned}$$

Where

$$\left[(1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}}\right]^p \leq (1-t)^{\frac{\alpha}{k}p}-t^{\frac{\alpha}{k}p}, \qquad \text{for } t\in\left[0,\frac{1}{2}\right]$$

and

$$\left[t^{\frac{\alpha}{k}}-(1-t)^{\frac{\alpha}{k}}\right]^p \leq t^{\frac{\alpha}{k}p}-(1-t)^{\frac{\alpha}{k}p}, \qquad \text{for } t\in\left[\frac{1}{2},1\right]$$

which follows from $(A-B)^q \leq A^q - B^q$, for any $A > B \geq 0$ and $q \geq 1$. Hence the proof is complete. □

Theorem 6. *Let $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable mapping on I^o and $f^{'} \in L[a,b]$. If $|f^{'}|^q, q > 1$ is quasi-convex function on $[a,b]$, and $g : [a,b] \subset \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and symmetric to $\frac{a+b}{2}$, the following inequality for k-fractional derivatives holds*

$$\left|\frac{f(a)+f(b)}{2}\left[{}_kJ^{\alpha}_{a^+}g(b)+{}_kJ^{\alpha}_{b^-}g(a)\right]-\left[{}_kJ^{\alpha}_{a^+}(fg)(b)+{}_kJ^{\alpha}_{b^-}(fg)(a)\right]\right|$$
$$\leq \frac{(b-a)^{\frac{\alpha}{k}+1}||g||_{\infty}}{\left(\frac{\alpha}{k}p+1\right)^{\frac{1}{p}}\Gamma_k(\alpha+k)}\left(max\left\{|f^{'}(a)|^q,|f^{'}(b)|^q\right\}\right)^{\frac{1}{q}}$$

where $0<\frac{\alpha}{k}\leq 1, \frac{1}{p}+\frac{1}{q}=1.$

Proof. The inequality can also be proved by using Lemma 6, Hölder's inequality, inequality (2.3), the quasi-convexity of $|f^{'}|^q$ and Lemma 2. □

3. Applications to Special Means

We now consider the means of arbitrary real numbers ξ,η $(\xi\neq\eta)$. We take

Arithmetic mean

$$A(\xi,\eta)=\frac{\xi+\eta}{2},\ \xi,\eta\in R.$$

Logarithmic mean

$$L(\xi,\eta)=\frac{\xi-\eta}{ln|\xi|-ln|\eta|},\ \xi,\eta\in R,\ \xi\neq\eta,\ |\xi|\neq|\eta|,\ \xi\eta\neq 0.$$

Genralised log-mean

$$L_n(\xi,\eta)=\left[\frac{\xi^{n+1}-\eta^{n+1}}{(n+1)(\xi-\eta)}\right]^{\frac{1}{n}},\ n\in Z\setminus\{-1,0\},\ \xi,\eta\in R,\ \xi\neq\eta.$$

Proposition 1. *Let* $a,b\in R\setminus\{0\}, a<b,$ *and* $n\in Z\setminus\{-1,0\}$*, then we have*

$$|A^n(a,b)A(a,b)-L^n_n(a,b)|\leq\frac{kb\,(b-a)}{4}\left(max\left\{|na^{n-1}|^q,|nb^{n-1}|^q\right\}\right)^{\frac{1}{q}}.$$

$$|A^n(a,b)A(a,b)-L^n_n(a,b)|\leq\frac{kb\,(b-a)}{2(p+1)^{\frac{1}{p}}}\left(max\left\{|na^{n-1}|^q,|nb^{n-1}|^q\right\}\right)^{\frac{1}{q}}.$$

Proof. The assertion follows from Theorem 1 and Theorem 2, applied to $f(x)=x^n, x\in R$, $g(x)=x$ and $\alpha=k$. □

Proposition 2. *Let* $a,b\in R, a<b,$ *and* $n\in Z\setminus\{-1,0\}$ *is odd, then for every* $q\geq 1$*,we have*

$$\left|A(a^n,b^n)[(a-A(a,b))^2+(b-A(a,b))^2]-\frac{2}{n+1}[(a-A(a,b))^{n+1}+(b-A(a,b))^{n+1}]\right|$$
$$\leq\frac{k|b-a|^3}{4}max\left\{|na^{n-1}|,|nb^{n-1}|\right\}$$

$$\left|A(a^n,b^n)[(a-A(a,b))^2+(b-A(a,b))^2]-\frac{2}{n+1}[(a-A(a,b))^{n+1}+(b-A(a,b))^{n+1}]\right|$$
$$\leq \frac{k|b-a|^3}{4}\left(max\left\{|na^{n-1}|^q,|nb^{n-1}|^q\right\}\right)^{\frac{1}{q}}$$

$$\left|A(a^n,b^n)[(a-A(a,b))^2+(b-A(a,b))^2]-\frac{2}{n+1}[(a-A(a,b))^{n+1}+(b-A(a,b))^{n+1}]\right|$$
$$\leq \frac{2^{\frac{1}{p}-1}k|b-a|^3}{(p+1)^{\frac{1}{p}}}\left(1-\frac{1}{2^p}\right)^{\frac{1}{p}}\left(max\left\{|na^{n-1}|^q,|nb^{n-1}|^q\right\}\right)^{\frac{1}{q}}$$

$$\left|A(a^n,b^n)[(a-A(a,b))^2+(b-A(a,b))^2]-\frac{2}{n+1}[(a-A(a,b))^{n+1}+(b-A(a,b))^{n+1}]\right|$$
$$\leq \frac{k|b-a|^3}{(p+1)^{\frac{1}{p}}}\left(max\left\{|na^{n-1}|^q,|nb^{n-1}|^q\right\}\right)^{\frac{1}{q}}$$

Proof. The assertion follows from Theorems 3, 4, 5 and 6 respectively, applied to $f(x)=x^n, x\in R$, $g(x)=|x-\frac{a+b}{2}|$ and $\alpha=k$. □

Note: If $n\in Z\setminus\{-1,0\}$ is even in Proposition 2, then the term $(a-A(a,b))^{n+1}$ in the left hand side of each of above inequalities will bear negative sign instead of positive.

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NONLINEAR DIFFERENTIAL POLYNOMIALS OF MEROMORPHIC FUNCTIONS WITH REGARD TO MULTIPLICITY SHARING A SMALL FUNCTION

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Abstract. We study the uniqueness problem of nonlinear differential polynomials of meromorphic functions that share one small function. A uniqueness result which related to multiplicity of meromorphic function is proved in this paper.

1. Introduction and main results

Let f be a nonconstant meromorphic function in the complex plane $\mathbb{C}$. We will assume that the reader is familiar with the standard notation of the Nevanlinna's theory of meromorphic functions, such as $T(r,f), m(r,f)$, $\overline{N}(r,f)$ and $N(r,f)$, see [9, 14, 16] for more details. The notation $S(r,f)$ is defined to be any quantity satisfying $S(r,f)=o(T(r,f))$ as $r\to\infty, r\notin E$, where E is a set of positive real number of finite linear measure, not necessarily the same at each occurrence. The notations $T(r)$ and $S(r)$ are defined respectively by

$$T(r)=\max\{T(r,f),T(r,g)\},\quad S(r)=o\left(T(r)\right)\text{ as }r\to\infty, r\notin E,$$

for any two nonconstant meromorphic functions f and g. A meromorphic function h is called a small function with respect to f, proved that $T(r,a)=S(r,f)$. Moreover, $\mathrm{GCD}(n_1,n_2,\ldots,n_k)$ denotes the greatest common divisor of positive integers $n_1,n_2,\ldots,n_k$.

Let f and g be two nonconstant meromorphic functions, and let $a\in\mathbb{C}$. We say that f and g share the value a CM (counting multiplicities), provided that $f-a$ and $g-a$ have the same zeros with the same multiplicities. If $f-a$ and $g-a$ have the same zeros, then we say that f and g share a IM (ignoring multiplicities). Similarly, we immediately get the definitions of f and g share h IM (or CM), where h is a small function of f and g. In addition, we also need the following notation, for any $a\in\overline{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$,

$$\Theta(a,f)=1-\limsup_{r\to\infty}\frac{\overline{N}(r,a;f)}{T(r,f)}.$$

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Hayman [10] and Clunie [5] proved the following result.

Theorem 1.1. *Let f be a transcendental entire function, $n \geq 1$ be a positive integer. Then $f^n f' = 1$ has infinitely many zeros.*

Remark 1. The similar result of Theorem 1.1 in which entire function is replaced with meromorphic function is proved in [2] and [4].

Fang and Hua [8], Yang and Hua [15] obtained a uniqueness theorem corresponding to Theorem 1.1.

Theorem 1.2. *Let f and g be two nonconstant entire (meromorphic) functions, and let $n \geq 6$ $(n \geq 11)$ be a positive integer. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$, or $f = tg$ for a constant t such that $t^{n+1} = 1$.*

Fang [7] considered the case of the kth derivative, and proved the following result.

Theorem 1.3. *Let f and g be two nonconstant entire functions, and let n, k be two positive integers with $n \geq 2k + 8$. If $(f^n(f-1))^{(k)}$ and $(g^n(g-1))^{(k)}$ share 1 CM, then $f = g$.*

Zhang et al. [18] considered some general differential polynomials. They proved the following results.

Theorem 1.4. *Let f and g be two nonconstant entire functions. Let n, k and m be three positive integers with $n \geq 3m+2k+5$ and let $P(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0$, where $a_0 \neq 0, a_1, \ldots, a_{m-1}, a_m \neq 0$ are complex constants. If $(f^n P(f))^{(k)}$ and $(g^n P(g))^{(k)}$ share 1 CM, then either $f = tg$ for a constant t such that $t^d = 1$, where $d =$GCD$(n+m, \ldots, n+m-i, \ldots, n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$, or f and g satisfying the algebraic function equation $R(f, g) = 0$, where $R(w_1, w_2) = w_1^n(a_m w_1^m + a_{m-1} w_1^{m-1} + \cdots + a_0) - w_2^n(a_m w_2^m + a_{m-1} w_2^{m-1} + \cdots + a_0)$.*

Theorem 1.5. *Let f and g be two nonconstant meromorphic functions, and $h(\not\equiv 0, \infty)$ be a small function with respect to f and g. Let n, k and m be three positive integers with $n > 3k + m + 8$ and $P(z)$ be defined as in Theorem 1.4. If $(f^n P(f))^{(k)}$ and $(g^n P(g))^{(k)}$ share $h(z)$ CM, then one of the following three cases holds:*

(i) *$f = tg$ for a constant t such that $t^d = 1$, where $d =$GCD$(n + m, \ldots, n+m-i, \ldots, n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$;*

(ii) *f and g satisfying the algebraic function equation $R(f,g)=0$, where* $R(w_1,w_2)=w_1^n(a_mw_1^m+a_{m-1}w_1^{m-1}+\cdots+a_0)-w_2^n(a_mw_2^m+a_{m-1}w_2^{m-1}+\cdots+a_0)$;

(iii) $(f^nP(f))^{(k)}(g^nP(g))^{(k)}=h^2$.

In 2011 Dyavanal [6] considered the uniqueness problem of meromorphic function related to the value sharing of two nonlinear differential polynomials in which the multiplicities of zeros and poles of f and g are taken into account. In 2013, Bhoosnurmath and Kabbur [3] proved the following uniqueness theorem by using the idea from Dyavanal [6].

Theorem 1.6. *Let f and g be two nonconstant meromorphic functions, whose zeros and poles are of multiplicities at least s, where s is a positive integer. Let n and m be two positive integers with $(n-m-1)s\geq\max\{10,2m+3\}$, and let $P(z)$ be defined as in Theorem 1.4. If $f^nP(f)f'$ and $g^nP(g)g'$ share 1 CM, then either $f=tg$ for a constant t such that $t^d=1$, where $d=$GCD$(n+m+1,\cdots,n+m+1-i,\cdots,n+1)$, $a_{m-i}\neq0$ for some $i=0,1,\cdots,m$ or f and g satisfy the algebraic function equation $R(f,g)=0$, where $R(x,y)=x^{n+1}(\frac{a_m}{n+m+1}x^m+\frac{a_{m-1}}{n+m}x^{m-1}+\cdots+\frac{a_0}{n+1})-y^{n+1}(\frac{a_m}{n+m+1}y^m+\frac{a_{m-1}}{n+m}y^{m-1}+\cdots+\frac{a_0}{n+1})$.*

Similar Theorem 1.5 in which a small function and kth derivative are considered, what can we say when the condition sharing 1 and the first derivative in Theorem 1.6 are replaced with sharing a small function and kth derivative respectively? In this paper, we will study the problem and establish the following uniqueness theorem.

Theorem 1.7. *Let f and g be two transcendental meromorphic functions, whose zeros and poles are of multiplicities at least s, where s is a positive integer. Let n and m be two positive integers with $n-m>\max\{2+\frac{2m}{s},\frac{(n+2)(k+4)}{ns}\}$, $\Theta(\infty,f)+\Theta(\infty,g)>\frac{4}{n}$ and let $P(z)$ be defined as in Theorem 1.4. If $(f^nP(f))^{(k)}$ and $(g^nP(g))^{(k)}$ share $h(z)$ CM, where $h(z)(\not\equiv0,\infty)$ is a small function of f and g, then one of the following three cases hold:*

(i) $(f^nP(f))^{(k)}(g^nP(g))^{(k)}=h^2$;

(ii) *$f=tg$ for a constant t such that $t^d=1$, where $d=$ GCD$(n+m,n+m-1,\ldots,n+m-i,\ldots,n+1,n)$, $a_{m-i}\neq0$ for $i=0,1,\ldots,m$;*

(iii) *f and g satisfy the algebraic equation $R(f,g)=0$, where $R(f,g)=f^nP(f)-g^nP(g)$.*

The possibility $(f^nP(f))^{(k)}(g^nP(g))^{(k)}=h^2$ does not occur for $k=1$.

2. Auxiliary results

For the proof of our result, we need the following lemmas and definitions.

Definition 2.1. [11] Let $a \in \overline{\mathbb{C}}$. We use $N(r,a;f|=1)$ to denote the counting function of simple $a-$points of f. For a positive integer p we denote by $N(r,a;f|\leq p)$ the counting function of those $a-$points of f (counted with proper multiplicities) whose multiplicities are not greater than p. By $\overline{N}(r,a;f|\leq p)$ we denote the corresponding reduced counting function. Similarly, we can define $N(r,a;f|\geq p)$ and $\overline{N}(r,a;f|\geq p)$.

Definition 2.2. [1] Let $a \in \overline{\mathbb{C}}$, and let k be a nonnegative integer. We denote by $N_k(r,\frac{1}{f-a})$ the counting function of $a-$points of f, where an $a-$point of multiplicity m is counted m times if $m\leq k$ and k times if $m>k$. Then

$$N_k(r,\frac{1}{f-a}) = \overline{N}(r,\frac{1}{f-a}) + \overline{N}(r,a;f|\geq 2) + \cdots + \overline{N}(r,a;f|\geq k). \tag{2.1}$$

Obviously $N_1(r,\frac{1}{f-a}) = \overline{N}(r,\frac{1}{f-a})$.

Lemma 2.1. [15] *Let f and g be two nonconstant meromorphic functions that share 1 CM. Then one of the following cases hold:*

(i) $T(r) \leq N_2(r,\frac{1}{f}) + N_2(r,\frac{1}{g}) + N_2(r,f) + N_2(r,g) + S(r)$;
(ii) $f=g$;
(iii) $fg=1$.

Lemma 2.2. [17] *Let f be a nonconstant meromorphic function, and p,k be positive integers. Then*

$$N_p(r,\frac{1}{f^{(k)}}) \leq T(r,f^{(k)}) - T(r,f) + N_{p+k}(r,\frac{1}{f}) + S(r,f), \tag{2.2}$$

$$N_p(r,\frac{1}{f^{(k)}}) \leq k\overline{N}(r,f) + N_{p+k}(r,\frac{1}{f}) + S(r,f). \tag{2.3}$$

Lemma 2.3. [13] *Let f be a nonconstant meromorphic function, and let $P_n(f) = \sum_{j=0}^{n} a_j f^j$ be a polynomial in f, where $a_n \neq 0$, $a_{n-1},\cdots,a_1,a_0$ satisfying $T(r,a_j)=S(r,f)$. Then*

$$T(r,P_n) = nT(r,f) + S(r,f). \tag{2.4}$$

Lemma 2.4. *Let f and g be two nonconstant meromorphic functions such that*

$$\Theta(\infty,f) + \Theta(\infty,g) > \frac{4}{n}$$

for all integer $n \geq 3$. *Then*

$$f^n(af+b) = g^n(ag+b)$$

implies $f = g$, *where* a *and* b *are two finite nonzero complex constants.*

Proof. By using similar way in [12], we can obtain the lemma. □

Lemma 2.5. *Let* f *and* g *be two nonconstant meromorphic functions, whose zeros and poles are of multiplicities at least* s, *where* s *is a positive integer and let* n, k *be two positive integers. Let* $F = (f^nP(f))^{(k)}$ *and* $G = (g^nP(g))^{(k)}$, *where* $P(z)$ *be defined as in Theorem 1.4. If there exist two nonzero constants* b_1 *and* b_2 *such that* $\overline{N}(r, \frac{1}{F}) = \overline{N}(r, \frac{1}{G-b_1})$ *and* $\overline{N}(r, \frac{1}{G}) = \overline{N}(r, \frac{1}{F-b_2})$, *then* $n - m \leq \frac{(k+1)(n+2)}{ns}$.

Proof. By the second fundamental theorem of Nevanlinna's theory,

$$\begin{aligned} T(r,F) &\leq \overline{N}(r,\frac{1}{F}) + \overline{N}(r,F) + \overline{N}(r,\frac{1}{F-b_2}) + S(r,F) \\ &\leq \overline{N}(r,\frac{1}{F}) + \overline{N}(r,F) + \overline{N}(r,\frac{1}{G}) + S(r,F). \end{aligned} \tag{2.5}$$

Combining (2.2), (2.3), (2.5) and Lemma 2.3, we get

$$\begin{aligned} (n+m)T(r,f) &\leq T(r,F) - \overline{N}(r,\frac{1}{F}) + N_{k+1}(r,\frac{1}{f^nP(f)}) + S(r,f) \\ &\leq \overline{N}(r,\frac{1}{G}) + \overline{N}(r,f) + N_{k+1}(r,\frac{1}{f^nP(f)}) + S(r,f) \\ &\leq N_{k+1}(r,\frac{1}{f^nP(f)}) + N_{k+1}(r,\frac{1}{g^nP(g)}) + \overline{N}(r,f) \\ &+ k\overline{N}(r,g) + S(r,f) + S(r,g) \\ &\leq (\frac{k+1+n}{ns} + m)T(r,f) + (\frac{k+1+nk}{ns} + m)T(r,g) \\ &+ S(r,f) + S(r,g) \\ &\leq (\frac{(k+1)(n+2)}{ns} + 2m)T(r) + S(r). \end{aligned} \tag{2.6}$$

Similarly, for the case of g,

$$(n+m)T(r,g) \leq (\frac{(k+1)(n+2)}{ns} + 2m)T(r) + S(r). \tag{2.7}$$

It follows from (2.6) and (2.7) that

$$(n - \frac{(k+1)(n+2)}{ns} - m)T(r) \leq S(r), \tag{2.8}$$

which gives $n - m \leq \frac{(k+1)(n+2)}{ns}$. This completes the proof. □

Lemma 2.6. *Let* f *and* g *be two transcendental meromorphic functions, whose zeros and poles are of multiplicities at least* s, *where* s *is a positive* *integer. Let $P(z)$ be defined as in Theorem 1.4, and n, m, k be three positive integers, and $h(\not\equiv 0, \infty)$ be small function of f and g. Then*

$$(f^nP(f))^{(k)}(g^nP(g))^{(k)} \not\equiv h^2 \tag{2.9}$$

holds for $k = 1$ and $(n + m - 2)p > 2m(1 + \frac{1}{s})$, where p is the number of distinct roots of $P(z) = 0$.

Proof. If (2.9) is possible for $k = 1$, i.e.,

$$(f^nP(f))'(g^nP(g))' = h^2.$$

Then

$$f^{n-1}Q(f)f'g^{n-1}Q(g)g' = h^2, \tag{2.10}$$

where $Q(z) = \sum_{j=0}^{m} b_jz^j$, $b_j = (n + j)a_j$, $j = 0, 1, \ldots, m$. Denote $Q(z)$ as

$$Q(z) = b_m(z - d_1)^{l_1}(z - d_2)^{l_2} \cdots (z - d_p)^{l_p},$$

where $\sum_{i=1}^{p} l_i = m$, $1 \le p \le m$, $d_i \ne d_j$, $i \ne j$, $1 \le i, j \le p$, d_i are nonzero constants and l_i are positive integers, $i = 1, 2, \ldots p$.

Suppose that $z_1 \notin S_0$ is a zero of f with multiplicity $s_1(\ge s)$, where S_0 is a set defined as

$$S_0 = \{z : h(z) = 0\} \cup \{z : h(z) = \infty\}.$$

Then z_1 is a pole of g with multiplicity $q_1(\ge s)$. We deduce from (2.10) that

$$ns_1 - 1 = (n + m)q_1 + 1$$

and so

$$mq_1 + 2 = n(s_1 - q_1). \tag{2.11}$$

From (2.11) we get $q_1 \ge \frac{n-2}{m}$, so

$$s_1 \ge \frac{n + m - 2}{m}.$$

Hence,

$$\overline{N}(r, \frac{1}{f}) \le \frac{m}{n + m - 2}N(r, \frac{1}{f}) + S(r, f). \tag{2.12}$$

Suppose that $z_2 \notin S_0$ is a zero of $Q(f)$ with multiplicity s_2 and is a zero of $f - d_i$ of order q_i, $i = 1, 2, \ldots p$. Then $s_2 = l_iq_i$, $i = 1, 2, \cdots p$. Then z_2 is a pole of g with multiplicity $q(\ge s)$. It follows from (2.10) that

$$q_il_i + q_i - 1 = (n + m)q + 1 \ge (n + m)s + 1.$$

So

$$q_i \geq \frac{(n+m)s+2}{l_i+1}, \quad i = 1, 2, \ldots p.$$

Hence,

$$\overline{N}(r, \frac{1}{f-d_i}) \leq \frac{l_i+1}{(n+m)s+2} N(r, d_i, f) + S(r, f), \quad i = 1, 2, \ldots p.$$

By this and the first fundamental theorem of Nevanlinna's theory, we have

$$\sum_{i=1}^{p} \overline{N}(r, \frac{1}{f-d_i}) \leq \frac{m+p}{(n+m)s+2} T(r, f) + S(r, f). \tag{2.13}$$

Suppose that $z_3 \notin S_0$ is a pole of f. Then we know that z_3 is either a zero of $g^{n-1}Q(g)$ or a zero of g' by (2.10). Therefore,

$$\begin{aligned} \overline{N}(r, f) &\leq \overline{N}(r, \frac{1}{g}) + \sum_{i=1}^{p} \overline{N}(r, \frac{1}{g-d_i}) + \overline{N_0}(r, \frac{1}{g'}) + S(r, f) + S(r, g) \\ &\leq (\frac{m}{n+m-2} + \frac{m+p}{(n+m)s+2}) T(r, g) + \overline{N_0}(r, \frac{1}{g'}) \\ &+ S(r, f) + S(r, g), \end{aligned} \tag{2.14}$$

where $\overline{N_0}(r, \frac{1}{g'})$ denote the reduce counting function of those zeros of g' which are not the zeros of $gQ(g)$.

By (2.12)-(2.14), and the second fundamental theorem of Nevanlinna's theory,

$$\begin{aligned} pT(r, f) &\leq \overline{N}(r, f) + \overline{N}(r, \frac{1}{f}) + \sum_{i=1}^{p} \overline{N}(r, \frac{1}{f-d_i}) - \overline{N_0}(r, \frac{1}{f'}) + S(r, f) \\ &\leq (\frac{m}{n+m-2} + \frac{m+p}{(n+m)s+2})(T(r, f) + T(r, g)) + \overline{N_0}(r, \frac{1}{g'}) \\ &- \overline{N_0}(r, \frac{1}{f'}) + S(r, f) + S(r, g). \end{aligned} \tag{2.15}$$

Similarly, for the case of g,

$$\begin{aligned} pT(r, g) &\leq (\frac{m}{n+m-2} + \frac{m+p}{(n+m)s+2})(T(r, f) + T(r, g)) + \overline{N_0}(r, \frac{1}{f'}) \\ &- \overline{N_0}(r, \frac{1}{g'}) + S(r, f) + S(r, g). \end{aligned} \tag{2.16}$$

It follows from (2.15) and (2.16) that

$$(p - \frac{2m}{n+m-2} - \frac{2(m+p)}{(n+m)s+2})(T(r, f) + T(r, g)) \leq S(r, f) + S(r, g).$$

This is a contradiction with our assumption that $(n+m-2)p > 2m(1+\frac{1}{s})$, and hence the proof is complete. $\square$

3. Proof of Theorem 1.7

Let $F = \dfrac{(f^nP(f))^{(k)}}{h}$, $G = \dfrac{(g^nP(g))^{(k)}}{h}$. Then F and G share 1 CM. Applying Lemma 2.3,

$$T(r,h) = o(T(r,F)) = S(r,f), \quad T(r,h) = o(T(r,G)) = S(r,g). \tag{3.1}$$

It follows from (2.2) and (3.1) that

$$\begin{aligned} N_2(r,\frac{1}{F}) &\le N_2(r,\frac{1}{(f^nP(f))^{(k)}}) + N_2(r,h) + S(r,f) \\ &\le N_2(r,\frac{1}{(f^nP(f))^{(k)}}) + S(r,f) \\ &\le T(r,(f^nP(f)) - (n+m)T(r,f) + N_{k+2}(r,\frac{1}{f^nP(f)}) + S(r,f) \\ &\le T(r,F) - (n+m)T(r,f) + N_{k+2}(r,\frac{1}{f^nP(f)}) + S(r,f). \end{aligned} \tag{3.2}$$

We deduce from (2.3) that

$$\begin{aligned} N_2(r,\frac{1}{F}) &\le N_2(r,\frac{1}{(f^nP(f))^{(k)}}) + S(r,f) \\ &\le k\overline{N}(r,f^nP(f))^{(k)}) + N_{k+2}(r,\frac{1}{f^nP(f)}) + S(r,f) \\ &\le k\overline{N}(r,f) + N_{k+2}(r,\frac{1}{f^nP(f)}) + S(r,f). \end{aligned} \tag{3.3}$$

It follows from (3.2) that

$$(n+m)T(r,f) \le T(r,F) + N_{k+2}(r,\frac{1}{f^nP(f)}) - N_2(r,\frac{1}{F}) + S(r,f). \tag{3.4}$$

Suppose that (i) of Lemma 2.1 holds, i.e.,

$$\begin{aligned} \max\{T(r,F),T(r,G)\} \le\ & N_2(r,\frac{1}{F}) + N_2(r,\frac{1}{G}) + N_2(r,F) + N_2(r,G) \\ & + S(r,f) + S(r,g). \end{aligned}$$

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Combining this, (3.3) and (3.4),

$$\begin{aligned}(n+m)T(r,f) &\leq T(r,F)+N_{k+2}(r,\frac{1}{f^nP(f)})-N_2(r,\frac{1}{F})+S(r,f)+S(r,g)\\ &\leq N_2(r,\frac{1}{G})+N_2(r,F)+N_2(r,G)+N_{k+2}(r,\frac{1}{f^nP(f)})\\ &+S(r,f)+S(r,g)\\ &\leq N_{k+2}(r,\frac{1}{f^nP(f)})+N_{k+2}(r,\frac{1}{g^nP(g)})+k\overline{N}(r,g)\\ &+2\overline{N}(r,f)+2\overline{N}(r,g)+S(r,f)+S(r,g)\\ &\leq (\frac{k+2+2n}{ns}+m)T(r,f)+(\frac{(k+2)(n+1)}{ns}+m)T(r,g)\\ &+S(r,f)+S(r,g)\\ &\leq (\frac{k(n+2)+4(n+1)}{ns}+2m)T(r)+S(r). \qquad (3.5)\end{aligned}$$

Similarly, for the case of g,

$$(n+m)T(r,g)\leq (\frac{k(n+2)+4(n+1)}{ns}+2m)T(r)+S(r). \qquad (3.6)$$

It follows from (3.5) and (3.6) that

$$(n+m)T(r)\leq (\frac{k(n+2)+4(n+1)}{ns}+2m)T(r)+S(r).$$

This implies that

$$(n-m-\frac{k(n+2)+4(n+1)}{ns})T(r)\leq S(r). \qquad (3.7)$$

This contradicts with our assumption that $(n-m)>\max\{2+\frac{2m}{s},\frac{(n+2)(k+4)}{ns}\}$. So, we conclude that either $FG=1$ or $F=G$ by Lemma 2.1. Suppose that $FG=1$, then

$$(f^nP(f))^{(k)}(g^nP(g))^{(k)}=h^2.$$

This is a contradiction when $k=1$ by Lemma 2.6. So $F=G$, this implies that

$$(f^nP(f))^{(k)}=(g^nP(g))^{(k)}. \qquad (3.8)$$

Integrating for (3.8), we have

$$(f^nP(f))^{(k-1)}=(g^nP(g))^{(k-1)}+b_{k-1}, \qquad (3.9)$$

where b_{k-1} is constant. If $b_{k-1}\neq 0$, we obtain $n-m\leq\frac{(k+1)(n+2)}{ns}<\frac{(k+4)(n+2)}{ns}$ by Lemma 2.5. This is a contradiction with our assumption that $(n-m)>\max\{2+\frac{2m}{s},\frac{(n+2)(k+4)}{ns}\}$. Thus $b_{k-1}=0$. By repeating $k-$times,

$$f^nP(f)=g^nP(g). \qquad (3.10)$$

If $m = 1$ in (3.10), then $f = g$ by Lemma 2.4. Suppose that $m \geq 2$ and $b = \frac{f}{g}$. If b is a constant, putting $f = bg$ in (3.10), we get

(3.11)
$$a_m g^{n+m}(b^{n+m} - 1) + a_{m-1} g^{n+m-1}(b^{n+m-1} - 1) + \cdots + a_0 g^n(b^n - 1) = 0,$$

which implies $b^d = 1$, where $d = \mathrm{GCD}(n+m, n+m-1, \ldots, n+1, n)$. Hence $f = tg$ for a constant t such that $t^d = 1$, $d = \mathrm{GCD}(n+m, n+m-1, \ldots, n+m-i, \ldots, n+1, n)$, $i = 0, 1, \ldots, m$.

If b is not a constant, then we can see that f and g satisfy the algebraic function equation $R(f, g) = 0$ by (3.10), where $R(f, g) = f^n P(f) - g^n P(g)$. This completes the proof of theorem.

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Impulsive hybrid fractional quantum difference equations

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Abstract

This paper is concerned with the existence of solutions for impulsive hybrid fractional q-difference equations involving a q-shifting operator of the type ${}_a\Phi_q(m) = qm + (1-q)a$. A hybrid fixed point theorem for two operators in a Banach algebra due to Dhage [29] is applied to obtain the existence result. An example illustrating the main result is also presented.

Key words and phrases: Quantum calculus; impulsive fractional q-difference equations; hybrid differential equations; existence; fixed point theorem
AMS (MOS) Subject Classifications: 34A08; 34A12; 34A37

1 Introduction

Fractional differential equations have been extensively investigated by several researchers in the recent years. The overwhelming interest in this branch of mathematics is due to the application of fractional-order operators in the mathematical modelling of several phenomena occurring in a variety of disciplines of applied sciences and engineering such as biomathematics, signal and image processing, control theory, dynamical systems, etc.

Hybrid fractional differential equations dealing with the fractional derivative of an unknown function hybrid with the nonlinearity depending on it is another interesting field of research. For some recent works on this topic, we refer the reader to a series of papers ([1]-[6]).

The subject of q-difference calculus or quantum calculus dates back to the beginning of the 20th century, when Jackson [7] introduced the concept of q-difference operator. Afterwards, this field of research flourished with the contributions of researchers from different parts of the world, for instance, see ([8]-[15]). The intensive development of fractional calculus motivated several investigators to consider fractional q-difference calculus. Now a great deal of work on initial and boundary value problems involving nonlinear fractional q-difference equations is available, for example, see [16]-[24] and the references therein.

The quantum calculus, known as the calculus without limits, provides a descent approach to study nondifferentiable functions in terms of difference operators. Quantum difference operators appear in different areas of mathematics such as orthogonal polynomials, basic hyper-geometric functions, combinatorics, the calculus of variations, mechanics and the theory of relativity. For the fundamental concepts of quantum calculus, we refer the reader to a text by Kac and Cheung [25].

More recently, the topic of q_k-calculus has also gained consideration attention. The notions of q_k-derivative and q_k-integral for a function $f : J_k := [t_k, t_{k+1}] \to \mathbb{R}$, together with their properties can be found in [26, 27]. In [28], new concepts of fractional quantum calculus were defined via a q-shifting operator of the form: ${}_a\Phi_q(m) = qm + (1-q)a$.

The purpose of the present work is to study the following impulsive hybrid fractional quantum difference equations:

$$\begin{cases} {}^{c}_{t_k}D^{\alpha_k}_{q_k}\left[\dfrac{x(t)}{f(t,x(t))}\right] = g(t,x(t)), \quad t \in J_k \subseteq [0,T], \ t \neq t_k, \\ \Delta x(t_k) = \varphi_k\left(x(t_k)\right), \quad k = 1,2,\ldots,m, \\ x(0) = \mu, \end{cases} \tag{1}$$

where $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, ${}^{c}_{t_k}D^{\alpha_k}_{q_k}$ denotes the Caputo fractional q_k-derivative of order α_k on intervals J_k, $J_0 = [0,t_1]$, $J_k = (t_k, t_{k+1}]$, $0 < \alpha_k \le 1$, $0 < q_k < 1$, $k = 0,1,\ldots,m$, $J = [0,T]$, $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, $\varphi_k \in C(\mathbb{R},\mathbb{R})$, $k = 1,2,\ldots,m$, $\mu \in \mathbb{R}$ and $\Delta x(t_k) = x(t_k^+) - x(t_k)$, $x(t_k^+) = \lim_{\theta\to 0^+} x(t_k+\theta)$, $k = 1,2,\ldots,m$. Here, we emphasize that the above initial value problem contains the new q-shifting operator ${}_a\Phi_q(m) = qm + (1-q)a$ [28].

The papers is organized as follows. In Section 2, we recall some preliminary concepts and present an auxiliary lemma which is used to convert the impulsive problem (1) into an equivalent integral equation. An existence result for the problem (1) obtained by means of a hybrid fixed point theorem due to Dhage [29] is presented in Section 3, which is well illustrated with the aid of an example.

2 Preliminaries

For the convenience of the reader, we recall some preliminary concepts from [28].

First of all, we define a q-shifting operator as

$${}_a\Phi_q(m) = qm + (1-q)a \tag{2}$$

such that

$${}_a\Phi_q^k(m) = {}_a\Phi_q^{k-1}\left({}_a\Phi_q(m)\right) \quad \text{and} \quad {}_a\Phi_q^0(m) = m,$$

for any positive integer k. The power law for q-shifting operator is

$${}_a(n-m)_q^{(0)} = 1, \quad {}_a(n-m)_q^{(k)} = \prod_{i=0}^{k-1}\left(n - {}_a\Phi_q^i(m)\right), \quad k \in \mathbb{N}\cup\{\infty\}.$$

In case $\gamma \in \mathbb{R}$, the above power law takes the form

$${}_a(n-m)_q^{(\gamma)} = n^{(\gamma)}\prod_{i=0}^{\infty}\frac{1 - {}_{\frac{a}{n}}\Phi_q^i(m/n)}{1 - {}_{\frac{a}{n}}\Phi_q^{\gamma+i}(m/n)}.$$

The q-derivative of a function h on interval $[a,b]$ is defined by

$$({}_aD_qh)(t) = \frac{h(t) - h({}_a\Phi_q(t))}{(1-q)(t-a)}, \quad t \neq a, \quad \text{and} \quad ({}_aD_qh)(a) = \lim_{t\to a}({}_aD_qh)(t),$$

while the higher order q-derivative is given by the formula

$$({}_aD_q^0f)(t) = f(t) \quad \text{and} \quad ({}_aD_q^kf)(t) = {}_aD_q^{k-1}({}_aD_qf)(t), \quad k \in \mathbb{N}.$$

The product and quotient formulas for q-derivative are

$${}_aD_q(h_1h_2)(t) = h_1(t){}_aD_qh_2(t) + h_2({}_a\Phi_q(t)){}_aD_qh_1(t) = h_2(t){}_aD_qh_1(t) + h_1({}_a\Phi_q(t)){}_aD_qh_2(t),$$

$${}_aD_q\left(\frac{h_1}{h_2}\right)(t) = \frac{h_2(t){}_aD_qh_1(t) - h_1(t){}_aD_qh_2(t)}{h_2(t)h_2({}_a\Phi_q(t))},$$

where h_1 and h_2 are well defined on $[a, b]$ with $h_2(t)h_2({}_a\Phi_q(t)) \neq 0$.

The q-integral of a function h defined on the interval $[a, b]$ is given by

$$({}_aI_q h)(t) = \int_a^t h(s)_a ds = (1-q)(t-a)\sum_{i=0}^{\infty} q^i h({}_a\Phi_{q^i}(t)), \quad t \in [a, b], \tag{3}$$

with

$$({}_aI_q^0 h)(t) = h(t) \;\text{ and }\; ({}_aI_q^k h)(t) = {}_aI_q^{k-1}({}_aI_q h)(t), \quad k \in \mathbb{N}.$$

The fundamental theorem of calculus applies to the operator ${}_aD_q$ and ${}_aI_q$, that is,

$$({}_aD_q {}_aI_q h)(t) = h(t),$$

and if h is continuous at $t = a$, then

$$({}_aI_q {}_aD_q h)(t) = h(t) - h(a).$$

The q-integration by parts formula on the interval $[a, b]$ is

$$\int_a^b f(s)_a D_q g(s)_a d_q s = (fg)(t)\Big|_a^b - \int_a^b g({}_a\Phi_q(s))_a D_q f(s)_a d_q s.$$

Let us now define Riemann-Liouville fractional q-derivative and q-integral on interval $[a, b]$ and outline some of their properties [28].

Definition 2.1 *The fractional q-derivative of Riemann-Liouville type of order $\nu \geq 0$ on interval $[a, b]$ is defined by $({}_aD_q^0 h)(t) = h(t)$ and*

$$({}_aD_q^\nu h)(t) = ({}_aD_q^l {}_aI_q^{l-\nu} h)(t), \quad \nu > 0,$$

where l is the smallest integer greater than or equal to ν.

Definition 2.2 *Let $\alpha \geq 0$ and h be a function defined on $[a, b]$. The fractional q-integral of Riemann-Liouville type is given by $({}_aI_q^0 h)(t) = h(t)$ and*

$$({}_aI_q^\alpha h)(t) = \frac{1}{\Gamma_q(\alpha)} \int_a^t {}_a(t - {}_a\Phi_q(s))_q^{(\alpha-1)} h(s)_a d_q s, \quad \alpha > 0, \quad t \in [a, b].$$

From [28], we have the following formulas

$${}_aD_q^\alpha (t-a)^\beta = \frac{\Gamma_q(\beta+1)}{\Gamma_q(\beta-\alpha+1)}(t-a)^{\beta-\alpha}, \tag{4}$$

$${}_aI_q^\alpha (t-a)^\beta = \frac{\Gamma_q(\beta+1)}{\Gamma_q(\beta+\alpha+1)}(t-a)^{\beta+\alpha}. \tag{5}$$

Lemma 2.3 *Let α, $\beta \in \mathbb{R}^+$ and f be a continuous function on $[a, b]$, $a \geq 0$. The Riemann-Liouville fractional q-integral has the following semi-group property*

$${}_aI_q^\beta {}_aI_q^\alpha h(t) = {}_aI_q^\alpha {}_aI_q^\beta h(t) = {}_aI_q^{\alpha+\beta} h(t).$$

Lemma 2.4 *Let h be a q-integrable function on $[a, b]$. Then the following equality holds*

$${}_aD_q^\alpha {}_aI_q^\alpha h(t) = h(t), \qquad \textit{for} \;\; \alpha > 0, \; t \in [a, b].$$

Lemma 2.5 *Let $\alpha > 0$ and p be a positive integer. Then for $t \in [a, b]$ the following equality holds*

$${}_aI_q^\alpha {}_aD_q^p h(t) = {}_aD_q^p {}_aI_q^\alpha h(t) - \sum_{k=0}^{p-1} \frac{(t-a)^{\alpha-p+k}}{\Gamma_q(\alpha+k-p+1)} {}_aD_q^k h(a).$$

We define Caputo fractional q-derivative as follows.

Definition 2.6 *The fractional q-derivative of Caputo type of order $\alpha \geq 0$ on interval $[a,b]$ is defined by $({}_a^cD_q^0 f)(t) = h(t)$ and*

$$({}_a^cD_q^\alpha h)(t) = ({}_aI_q^{n-\alpha}\,{}_aD_q^n h)(t), \quad \alpha > 0,$$

where n is the smallest integer greater than or equal to α.

Lemma 2.7 *Let $\alpha > 0$ and n be the smallest integer greater than or equal to α. Then for $t \in [a,b]$ the following equality holds*

$${}_aI_q^\alpha\,{}_a^cD_q^\alpha h(t) = h(t) - \sum_{k=0}^{n-1} \frac{(t-a)^k}{\Gamma_q(k+1)}\,{}_aD_q^k h(a).$$

Proof. From Lemma 2.5, for $\alpha = p = m$, where m is a positive integer, we have

$${}_aI_q^m\,{}_aD_q^m h(t) \;=\; {}_aD_q^m\,{}_aI_q^m h(t) - \sum_{k=0}^{m-1} \frac{(t-a)^k}{\Gamma_q(k+1)}\,{}_aD_q^k h(a) = h(t) - \sum_{k=0}^{m-1} \frac{(t-a)^k}{\Gamma_q(k+1)}\,{}_aD_q^k h(a).$$

Then, by Definition 2.6, we have

$${}_aI_q^\alpha\,{}_a^cD_q^\alpha h(t) = {}_aI_q^\alpha\,{}_aI_q^{n-\alpha}\,{}_aD_q^n h(t) = {}_aI_q^n\,{}_aD_q^n h(t) = h(t) - \sum_{k=0}^{n-1} \frac{(t-a)^k}{\Gamma_q(k+1)}\,{}_aD_q^k h(a).$$

□

Now we present a lemma which plays a pivotal role in the forthcoming analysis.

Lemma 2.8 *Assume that the map $x \mapsto \dfrac{x}{f(t,x)}$ is injection for each $t \in J$. $x \in PC(J,\mathbb{R})$ is the solution of (1) if and only if x is a solution of the impulsive integral equation*

$$\begin{aligned} x(t) \;=\; & f(t,x(t))\Bigg(\frac{\mu}{f(0,\mu)}\prod_{i=1}^{k}\frac{f(t_i,x(t_i))}{f(t_i,x(t_i^+))} + \sum_{i=1}^{k}\prod_{i\le j\le k}{}_{t_{i-1}}I_{q_{i-1}}^{\alpha_{i-1}}g(t_i,x(t_i))\frac{f(t_j,x(t_j))}{f(t_j,x(t_j^+))} \\ & + \sum_{i=1}^{k}\prod_{i<j\le k}\frac{\varphi_i(x(t_i))}{f(t_i,x(t_i^+))}\cdot\frac{f(t_j,x(t_j))}{f(t_j,x(t_j^+))} + {}_{t_k}I_{q_k}^{\alpha_k}g(t,x(t))\Bigg), \end{aligned} \tag{6}$$

where $\sum_{b<a}(\cdot) = 0$, $\prod_{b<a}(\cdot) = 1$ for $b > a$ and for $t \in J_k$,

$${}_{t_k}I_{q_k}^{\alpha_k}g(t,x(t)) = \frac{1}{\Gamma_{q_k}(\alpha_k)}\int_{t_k}^{t}{}_{t_k}(t - {}_{t_k}\Phi_{q_k}(s))_{q_k}^{(\alpha_k-1)}g(s,x(s))\,{}_{t_k}d_{q_k}s. \tag{7}$$

Proof. Applying Riemann-Liouville fractional q_0-integral operator of order α_0 to both sides of the first equation of (1) for $t \in J_0$ and using Lemma 2.7, we get

$${}_{t_0}I_{q_0}^{\alpha_0}\,{}_{t_0}^{c}D_{q_0}^{\alpha_0}\left[\frac{x(t)}{f(t,x(t))}\right] = \frac{x(t)}{f(t,x(t))} - \frac{x(0)}{f(0,x(0))} = {}_{t_0}I_{q_0}^{\alpha_0}g(t,x(t)),$$

which, in view of the initial condition, takes the form

$$x(t) = f(t,x(t))\left[\frac{\mu}{f(0,\mu)} + {}_{t_0}I_{q_0}^{\alpha_0}g(t,x(t))\right].$$

At $t = t_1$, we have

$$x(t_1) = f(t_1,x(t_1))\left[\frac{\mu}{f(0,\mu)} + {}_{t_0}I_{q_0}^{\alpha_0}g(t_1,x(t_1))\right]. \tag{8}$$

For $t \in J_1$, operating the Riemann-Liouville fractional q_1-integral of order α_1 on (1) and using the above process together with impulsive condition, we obtain

$$\frac{x(t)}{f(t,x(t))} = \frac{x(t_1^+)}{f(t_1^+,x(t_1^+))} + {}_{t_1}I_{q_1}^{\alpha_1} g(t,x(t)) = \frac{x(t_1)+\varphi_1(x(t_1))}{f(t_1^+,x(t_1^+))} + {}_{t_1}I_{q_1}^{\alpha_1} g(t,x(t)). \tag{9}$$

By the continuity of f with respect to the variable t, the expression $f(t_1^+, x(t_1^+))$ can be written as $f(t_1, x(t_1^+))$. Substituting (8) into (9) yields

$$\begin{aligned} x(t) &= f(t,x(t))\Bigg(\frac{\mu}{f(0,\mu)}\cdot\frac{f(t_1,x(t_1))}{f(t_1,x(t_1^+))} + \frac{f(t_1,x(t_1))}{f(t_1,x(t_1^+))}{}_{t_0}I_{q_0}^{\alpha_0} g(t_1,x(t_1)) \\ &\quad + \frac{\varphi_1(x(t_1))}{f(t_1,x(t_1^+))} + {}_{t_1}I_{q_1}^{\alpha_1} g(t,x(t))\Bigg). \end{aligned}$$

Also, for $t \in J_2$, we have

$$\begin{aligned} x(t) &= f(t,x(t))\Bigg(\frac{\mu}{f(0,\mu)}\cdot\frac{f(t_1,x(t_1))}{f(t_1,x(t_1^+))}\cdot\frac{f(t_2,x(t_2))}{f(t_2,x(t_2^+))} \\ &\quad + \frac{f(t_1,x(t_1))}{f(t_1,x(t_1^+))}\cdot\frac{f(t_2,x(t_2))}{f(t_2,x(t_2^+))}{}_{t_0}I_{q_0}^{\alpha_0} g(t_1,x(t_1)) + \frac{f(t_2,x(t_2))}{f(t_2,x(t_2^+))}{}_{t_1}I_{q_1}^{\alpha_1} g(t_2,x(t_2)) \\ &\quad + \frac{\varphi_1(x(t_1))}{f(t_1,x(t_1^+))}\cdot\frac{f(t_2,x(t_2))}{f(t_2,x(t_2^+))} + \frac{\varphi_2(x(t_2))}{f(t_2,x(t_2^+))} + {}_{t_2}I_{q_2}^{\alpha_2} g(t,x(t))\Bigg). \end{aligned}$$

Repeating the above process, for $t \in J$, we obtain (6).

Conversely, we assume that $x(t)$ is a solution of (6). Dividing by $f(t,x(t))$ and applying ${}_{t_k}^{c}D_{q_k}^{\alpha_k}$ on both sides of (6) for $t \in J_k$, $t \neq t_k$ $k = 0,1,\ldots,m$, we get

$${}_{t_k}^{c}D_{q_k}^{\alpha_k}\left[\frac{x(t)}{f(t,x(t))}\right] = g(t,x(t)).$$

It is easy to see that $\Delta x(t_k) = x(t_k^+) - x(t_k) = \varphi_k(x(t_k))$. Since $f(0,x(0)) \neq 0$, and using the fact that the map $x \mapsto \dfrac{x}{f(t,x)}$ is injection for each $t \in J$, we have $x(0) = \mu$. This completes the proof. □

Now we state a hybrid fixed point theorem due to Dhage [29], which we need to prove our main existence result.

Lemma 2.9 *Let S be a nonempty, closed convex and bounded subset of the Banach algebra E and let $A : E \to E$ and $B : S \to E$ be two operators such that (a) A is Lipschitzian with Lipschitz constant δ; (b) B is completely continuous; (c) $x = AxBy \Rightarrow x \in S$ for all $y \in S$; (d) $\delta M < 1$, where $M = \|B(S)\| = \sup\{\|B(x)\| : x \in S\}$. Then the operator equation $x = AxBx$ has a solution in S.*

3 Main Result

Let $PC(J,\mathbb{R}) = \{x : J \to \mathbb{R} : x(t)$ is continuous everywhere except for some t_k at which $x(t_k^+)$ and $x(t_k^-)$ exist and $x(t_k^-) = x(t_k)$, $k = 1,2,\ldots,m\}$. Define a norm $\|\cdot\|$ and a multiplication in $PC(J,\mathbb{R})$ by $\|x\| = \sup_{t\in J}|x(t)|$ and $(xy)(t) = x(t)y(t)$, $\forall t \in J$.
Clearly $PC(J,\mathbb{R})$ is a Banach algebra with respect to above supremum norm and the multiplication in it.

Now, we are in the position to present the main existence result.

Theorem 3.1 *Assume that the map $x \mapsto \dfrac{x}{f(t,x)}$ is injection for each $t \in J$. In addition we suppose that:*

(H_1) *The function* $f : J \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ *is bounded, continuous and there exists a positive function* ϕ *with bound* $\|\phi\|$ *such that*

$$|f(t, x(t)) - f(t, y(t))| \le \phi(t)|x(t) - y(t)|, \ \text{for } t \in J \ \text{and} x, y \in \mathbb{R}. \tag{10}$$

(H_2) *There exist a function* $p \in C(J, \mathbb{R}^+)$ *and a continuous nondecreasing function* $\psi : [0, \infty) \to (0, \infty)$ *such that*

$$|g(t, x(t))| \le p(t)\psi(|x|), \qquad (t, x) \in J \times \mathbb{R}. \tag{11}$$

(H_3) *The functions* $\varphi_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2, \ldots, m$, *are bounded and continuous.*

(H_4) *There exists a number* $r > 0$ *such that*

$$r \ge \Omega_1 \Bigg(\frac{|\mu|}{|f(0,\mu)|} \left(\frac{\Omega_1}{\Omega_2}\right)^m + \|p\|\psi(r) \sum_{i=1}^{m+1} \frac{(t_i - t_{i-1})^{\alpha_{i-1}}}{\Gamma_{q_{i-1}}(\alpha_{i-1}+1)} \left(\frac{\Omega_1}{\Omega_2}\right)^{m+1-i} + \frac{\Omega_3}{\Omega_2} \sum_{i=1}^{m} \left(\frac{\Omega_1}{\Omega_2}\right)^{m-i} \Bigg), \tag{12}$$

and

$$\|\phi\| \Bigg(\frac{|\mu|}{|f(0,\mu)|} \left(\frac{\Omega_1}{\Omega_2}\right)^m + \|p\|\psi(r) \sum_{i=1}^{m+1} \frac{(t_i - t_{i-1})^{\alpha_{i-1}}}{\Gamma_{q_{i-1}}(\alpha_{i-1}+1)} \left(\frac{\Omega_1}{\Omega_2}\right)^{m+1-i} + \frac{\Omega_3}{\Omega_2} \sum_{i=1}^{m} \left(\frac{\Omega_1}{\Omega_2}\right)^{m-i} \Bigg) < 1,$$

where $\Omega_1 = \sup\{|f(t,x)| : (t,x) \in J \times \mathbb{R}\}$, $\Omega_2 = \inf\{|f(t,x)| : (t,x) \in J \times \mathbb{R}\}$ *and* $\Omega_3 = \max\{\sup|\varphi_i(x)| : x \in \mathbb{R}, i = 1, 2, \ldots, m\}$.

Then the impulsive initial value problem (1) *has at least one solution on* J.

Proof. Let us introduce a subset S of $PC(J, \mathbb{R})$ by

$$S = \{x \in PC(J, \mathbb{R}) : \|x\| \le r\},$$

where r satisfies inequality (12). Clearly S is closed, convex and bounded subset of the Banach space $PC(J, \mathbb{R})$. In view of Lemma 2.8, the problem (1) is equivalent to the integral equation (6). Let us define two operators $\mathcal{A} : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ by

$$\mathcal{A}x(t) = f(t, x(t)), \ \ t \in J, \tag{13}$$

and $\mathcal{B} : S \to PC(J, \mathbb{R})$ by

$$\begin{aligned} \mathcal{B}x(t) \;\; &= \;\; \frac{\mu}{f(0,\mu)} \prod_{i=1}^{k} \frac{f(t_i, x(t_i))}{f(t_i, x(t_i^+))} + \sum_{i=1}^{k} \prod_{i \le j \le k} {}_{t_{i-1}}I_{q_{i-1}}^{\alpha_{i-1}} g(t_i, x(t_i)) \frac{f(t_j, x(t_j))}{f(t_j, x(t_j^+))} \\ &\quad + \sum_{i=1}^{k} \prod_{i<j\le k} \frac{\varphi_i(x(t_i))}{f(t_i, x(t_i^+))} \cdot \frac{f(t_j, x(t_j))}{f(t_j, x(t_j^+))} + {}_{t_k}I_{q_k}^{\alpha_k} g(t, x(t)), \ t \in J. \end{aligned} \tag{14}$$

Then, the problem (1) is transformed into an operator equation as

$$x = \mathcal{A}x\mathcal{B}x. \tag{15}$$

Under our assumptions, we will show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Lemma 2.9. This will be achieved in a series of steps.

Step 1. *The operator* $\mathcal{A}$ *is Lipschitzian on* $PC(J, \mathbb{R})$.

Let $x, y \in PC(J, \mathbb{R})$. Then by (H_1), for $t \in J$, we have

$$|\mathcal{A}x(t) - \mathcal{A}y(t)| = |f(t, x(t)) - f(t, y(t))| \le \phi(t)|x(t) - y(t)|.$$

Taking supremum over t, we obtain $\|\mathcal{A}x - \mathcal{A}y\| \leq \|\phi\|\|x-y\|$ for all $x, y \in PC(J,\mathbb{R})$. This show that $\mathcal{A}$ is a Lipschitzian on $PC(J,\mathbb{R})$ with Lipschitz constant $\|\phi\|$.

Step 2. *The operator $\mathcal{B}$ is completely continuous on S.*

In this step, we first show that the operator $\mathcal{B}$ is continuous on S. Let $\{x_n\}$ be a sequence in S converging to a point $x \in S$. Then, for all $t \in J$, we have

$$\begin{aligned}
&\lim_{n\to\infty} \mathcal{B}x_n(t) \\
= &\lim_{n\to\infty} \frac{\mu}{f(0,\mu)} \prod_{i=1}^{k} \frac{f(t_i, x_n(t_i))}{f(t_i, x_n(t_i^+))} + \lim_{n\to\infty} \sum_{i=1}^{k} \prod_{i\leq j\leq k} {}_{t_{i-1}}I_{q_{i-1}}^{\alpha_{i-1}} g(t_i, x_n(t_i)) \frac{f(t_j, x_n(t_j))}{f(t_j, x_n(t_j^+))} \\
&+ \lim_{n\to\infty} \sum_{i=1}^{k} \prod_{i<j\leq k} \frac{\varphi_i(x_n(t_i))}{f(t_i, x_n(t_i^+))} \cdot \frac{f(t_j, x_n(t_j))}{f(t_j, x_n(t_j^+))} + \lim_{n\to\infty} {}_{t_k}I_{q_k}^{\alpha_k} g(t, x_n(t)) = \mathcal{B}x(t),
\end{aligned}$$

which implies that $\mathcal{B}x_n \to \mathcal{B}x$ point-wise on J. Further it can be shown that $\{\mathcal{B}x_n\}$ is an equicontinuous sequence of functions. So $\mathcal{B}x_n \to \mathcal{B}x$ uniformly and the operator $\mathcal{B}$ is continuous on S.

Next we will prove that $\mathcal{B}$ is a compact operator on S. It is enough to show that the set $\mathcal{B}(S)$ is uniformly bounded and equicontinuous in $PC(J,\mathbb{R})$. For any $x \in S$, on account of (5), we get

$$\begin{aligned}
|\mathcal{B}x(t)| \leq\ & \frac{|\mu|}{|f(0,\mu)|} \prod_{i=1}^{k} \frac{|f(t_i, x(t_i))|}{|f(t_i, x(t_i^+))|} + \sum_{i=1}^{k} \prod_{i\leq j\leq k} {}_{t_{i-1}}I_{q_{i-1}}^{\alpha_{i-1}} |g(t_i, x(t_i))| \frac{|f(t_j, x(t_j))|}{|f(t_j, x(t_j^+))|} \\
&+ \sum_{i=1}^{k} \prod_{i<j\leq k} \frac{|\varphi_i(x(t_i))|}{|f(t_i, x(t_i^+))|} \cdot \frac{|f(t_j, x(t_j))|}{|f(t_j, x(t_j^+))|} + {}_{t_k}I_{q_k}^{\alpha_k} |g(t, x(t))| \\
\leq\ & \frac{|\mu|}{|f(0,\mu)|} \prod_{i=1}^{m} \frac{|f(t_i, x(t_i))|}{|f(t_i, x(t_i^+))|} + \sum_{i=1}^{m} \prod_{i\leq j\leq m} {}_{t_{i-1}}I_{q_{i-1}}^{\alpha_{i-1}} |g(t_i, x(t_i))| \frac{|f(t_j, x(t_j))|}{|f(t_j, x(t_j^+))|} \\
&+ \sum_{i=1}^{m} \prod_{i<j\leq m} \frac{|\varphi_i(x(t_i))|}{|f(t_i, x(t_i^+))|} \cdot \frac{|f(t_j, x(t_j))|}{|f(t_j, x(t_j^+))|} + {}_{t_m}I_{q_m}^{\alpha_m} |g(T, x(T))| \\
\leq\ & \frac{|\mu|}{|f(0,\mu)|} \left(\frac{\Omega_1}{\Omega_2}\right)^m + \|p\|\psi(r) \sum_{i=1}^{m+1} \frac{(t_i - t_{i-1})^{\alpha_{i-1}}}{\Gamma_{q_{i-1}}(\alpha_{i-1}+1)} \left(\frac{\Omega_1}{\Omega_2}\right)^{m+1-i} \\
&+ \frac{\Omega_3}{\Omega_2} \sum_{i=1}^{m} \left(\frac{\Omega_1}{\Omega_2}\right)^{m-i} := K,
\end{aligned}$$

for all $t \in J$. Taking supremum over t, we have $\|\mathcal{B}x\| \leq K$ for all $x \in S$. This shows that $\mathcal{B}$ is uniformly bounded on S.

Further, we will show that $\mathcal{B}(S)$ is an equicontinuous set in $PC(J,\mathbb{R})$. Let $\tau_1, \tau_2 \in J$ with $\tau_1 < \tau_2$ and $x \in S$. Then we have

$$\begin{aligned}
&|\mathcal{B}x(\tau_2) - \mathcal{B}x(\tau_1)| \\
= &\Bigg| \frac{\mu}{f(0,\mu)} \prod_{i=1}^{k} \frac{f(t_i, x(t_i))}{f(t_i, x(t_i^+))} + \sum_{i=1}^{k} \prod_{i\leq j\leq k} {}_{t_{i-1}}I_{q_{i-1}}^{\alpha_{i-1}} g(t_i, x(t_i)) \frac{f(t_j, x(t_j))}{f(t_j, x(t_j^+))} \\
&+ \sum_{i=1}^{k} \prod_{i<j\leq k} \frac{\varphi_i(x(t_i))}{f(t_i, x(t_i^+))} \cdot \frac{f(t_j, x(t_j))}{f(t_j, x(t_j^+))} + {}_{t_k}I_{q_k}^{\alpha_k} g(\tau_2, x(\tau_2)) \\
&- \frac{\mu}{f(0,\mu)} \prod_{i=1}^{n} \frac{f(t_i, x(t_i))}{f(t_i, x(t_i^+))} - \sum_{i=1}^{n} \prod_{i\leq j\leq n} {}_{t_{i-1}}I_{q_{i-1}}^{\alpha_{i-1}} g(t_i, x(t_i)) \frac{f(t_j, x(t_j))}{f(t_j, x(t_j^+))} \\
&- \sum_{i=1}^{n} \prod_{i<j\leq n} \frac{\varphi_i(x(t_i))}{f(t_i, x(t_i^+))} \cdot \frac{f(t_j, x(t_j))}{f(t_j, x(t_j^+))} - {}_{t_n}I_{q_n}^{\alpha_n} g(\tau_1, x(\tau_1)) \Bigg|,
\end{aligned}$$

for some $n \leq k$, $n, k \in \{0, 1, 2, \ldots, m\}$. Further

$$\begin{aligned} |\mathcal{B}x(\tau_2) - \mathcal{B}x(\tau_1)| &= \left| {}_{t_k}I_{q_k}^{\alpha_k} g(\tau_2, x(\tau_2)) - {}_{t_k}I_{q_k}^{\alpha_k} g(\tau_1, x(\tau_1)) \right| \\ &\leq \|p\|\psi(r) \left| \frac{(\tau_2 - t_k)^{\alpha_k}}{\Gamma_{q_k}(\alpha_k + 1)} - \frac{(\tau_1 - t_k)^{\alpha_k}}{\Gamma_{q_k}(\alpha_k + 1)} \right| \to 0, \end{aligned}$$

independent of $x \in S$ as $\tau_1 \to \tau_2$. This shows that $\mathcal{B}(S)$ is an equicontinuous set in $PC(J, \mathbb{R})$. Therefore, it follows by the Arzelá-Ascoli theorem that $\mathcal{B}$ is a completely continuous operator on S.

Step 3. *The hypothesis* (c) *of Lemma 2.9 is satisfied.*

Let $x \in PC(J, \mathbb{R})$ and $y \in S$ be arbitrary elements such that $x = \mathcal{A}x\mathcal{B}y$. Then we have

$$\begin{aligned} & |x(t)| \\ \leq\ & |\mathcal{A}x(t)||\mathcal{B}y(t)| \\ \leq\ & |f(t, x(t))| \Bigg(\frac{|\mu|}{|f(0, \mu)|} \prod_{i=1}^{k} \frac{|f(t_i, y(t_i))|}{|f(t_i, y(t_i^+))|} + \sum_{i=1}^{k} \prod_{i \leq j \leq k} {}_{t_{i-1}}I_{q_{i-1}}^{\alpha_{i-1}} |g(t_i, y(t_i))| \frac{|f(t_j, y(t_j))|}{|f(t_j, y(t_j^+))|} \\ & + \sum_{i=1}^{k} \prod_{i < j \leq k} \frac{|\varphi_i(y(t_i))|}{|f(t_i, y(t_i^+))|} \cdot \frac{|f(t_j, y(t_j))|}{|f(t_j, y(t_j^+))|} + {}_{t_k}I_{q_k}^{\alpha_k} |g(t, y(t))| \Bigg) \\ \leq\ & \Omega_1 \Bigg(\frac{|\mu|}{|f(0, \mu|)} \prod_{i=1}^{m} \frac{|f(t_i, y(t_i))|}{|f(t_i, y(t_i^+))|} + \sum_{i=1}^{m} \prod_{i \leq j \leq m} {}_{t_{i-1}}I_{q_{i-1}}^{\alpha_{i-1}} |g(t_i, y(t_i))| \frac{|f(t_j, y(t_j))|}{|f(t_j, y(t_j^+))|} \\ & + \sum_{i=1}^{m} \prod_{i < j \leq m} \frac{|\varphi_i(y(t_i))|}{|f(t_i, y(t_i^+))|} \cdot \frac{|f(t_j, y(t_j))|}{|f(t_j, y(t_j^+))|} + {}_{t_m}I_{q_m}^{\alpha_m} |g(T, y(T))| \Bigg) \\ \leq\ & \Omega_1 \Bigg(\frac{|\mu|}{|f(0, \mu)|} \left(\frac{\Omega_1}{\Omega_2} \right)^m + \|p\|\psi(r) \sum_{i=1}^{m+1} \frac{(t_i - t_{i-1})^{\alpha_{i-1}}}{\Gamma_{q_{i-1}}(\alpha_{i-1} + 1)} \left(\frac{\Omega_1}{\Omega_2} \right)^{m+1-i} \\ & + \frac{\Omega_3}{\Omega_2} \sum_{i=1}^{m} \left(\frac{\Omega_1}{\Omega_2} \right)^{m-i} \Bigg). \end{aligned}$$

Taking supremum over t, we have

$$\begin{aligned} \|x\| \leq\ & \Omega_1 \Bigg(\frac{|\mu|}{|f(0, \mu)|} \left(\frac{\Omega_1}{\Omega_2} \right)^m + \|p\|\psi(r) \sum_{i=1}^{m+1} \frac{(t_i - t_{i-1})^{\alpha_{i-1}}}{\Gamma_{q_{i-1}}(\alpha_{i-1} + 1)} \left(\frac{\Omega_1}{\Omega_2} \right)^{m+1-i} \\ & + \frac{\Omega_3}{\Omega_2} \sum_{i=1}^{m} \left(\frac{\Omega_1}{\Omega_2} \right)^{m-i} \Bigg) \leq r. \end{aligned}$$

Thus we deduce that $x \in S$.

Step 4. *We show that the condition* (d) *of Lemma 2.9 holds.*

As

$$\begin{aligned} M &= \|\mathcal{B}(S)\| \\ &\leq \left(\frac{|\mu|}{|f(0, \mu)|} \left(\frac{\Omega_1}{\Omega_2} \right)^m + \|p\|\psi(r) \sum_{i=1}^{m+1} \frac{(t_i - t_{i-1})^{\alpha_{i-1}}}{\Gamma_{q_{i-1}}(\alpha_{i-1} + 1)} \left(\frac{\Omega_1}{\Omega_2} \right)^{m+1-i} + \frac{\Omega_3}{\Omega_2} \sum_{i=1}^{m} \left(\frac{\Omega_1}{\Omega_2} \right)^{m-i} \right), \end{aligned}$$

therefore, by (H_4), we have $\delta M < 1$ with $\delta = \|\phi\|$.

Thus all the conditions of Lemma 2.9 are satisfied and hence the operator equation $x = \mathcal{A}x\mathcal{B}x$ has a solution in S. In consequence, we infer that the problem (1) has a solution on J. This completes the proof. □

Example 3.2 *Consider the following impulsive hybrid fractional quantum difference equation with initial condition*

$$\begin{cases} {}^{c}_{t_k}D^{\frac{k^2+2k+1}{k^2+2k+3}}_{\frac{k^2+3k+1}{2k^2+3k+2}}\left[\dfrac{x(t)}{\frac{|x(t)|+30}{|x(t)|+35}+\frac{1}{25}\left(t-\frac{1}{2}\right)^2}\right] = \dfrac{1+\sin^2 t}{2(t+5)}\left(\dfrac{x^2(t)}{4(1+|x(t)|)}+\dfrac{e^{-|x(t)|}}{2}\right), \\ \qquad\qquad\qquad\qquad\qquad\qquad t\in[0,3/2]\setminus\{t_1,\ldots,t_5\}, \\ \Delta x(t_k) = \dfrac{|x(t_k)|+1}{(k+1)(|x(t_k)|+2)}, \qquad t_k=\dfrac{k}{4},\ k=1,\ldots,5, \\ x(0)=\dfrac{1}{3}. \end{cases} \tag{16}$$

Here $\alpha_k = (k^2+2k+1)/(k^2+2k+3)$, $q_k = (k^2+3k+1)/(2k^2+3k+2)$, $k = 0,1,\ldots,5$, $t_k = k/4$, $k = 1,2,\ldots,5$, $m = 5$, $T = 3/2$, $\mu = 1/3$, $f(t,x) = ((|x|+30)/(|x|+35)) + (1/25)(t-(1/2))^2$ and $g(t,x) = ((1+\sin^2 t)/(2(t+5)))((x^2/(4(1+|x|))) + (e^{-|x|}/2))$. With the given values, we find that $\Omega_1 = 26/25$, $\Omega_2 = 6/7$. Also, we have

$$|f(t,x)-f(t,y)| \le \frac{1}{245}|x-y|,\ |g(t,x)| \le \frac{1}{t+5}\left(\frac{|x|}{4}+\frac{1}{2}\right),\ |\varphi_k(x)| \le \frac{1}{(k+1)},\ k=1,2,\ldots,5.$$

Clearly $\|\phi\| = 1/245$, $\Omega_3 = 1/2$, $\|p\| = 1/5$ and $\psi(|x|) = (|x|/4)+(1/2)$. Hence, there exists a constant r such that $6.611569689 < r < 1092.541483$ satisfying (H_4). Thus all the conditions of Theorem 3.1 are satisfied. Therefore, the conclusion of Theorem 3.1 implies that the problem (16) has at least one solution on $[0,3/2]$.

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A FIXED POINT ALTERNATIVE TO THE STABILITY OF A QUADRATIC α-FUNCTIONAL EQUATION IN FUZZY BANACH SPACES

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Abstract. In this paper, we solve the following quadratic α-functional equation

$$N\left(2f(x)+2f(y)-f(x+y)-\alpha^{-2}f(\alpha(x-y)),t\right) \geq \frac{t}{t+\varphi(x,y)} \tag{0.1}$$

in fuzzy normed spaces, where ρ is a fixed real number with $\alpha^{-1} \neq \pm\sqrt{3}$.

Using the fixed point method, we prove the Hyers-Ulam stability of the quadratic α-functional equation (0.1) in fuzzy Banach spaces.

1. Introduction and preliminaries

Katsaras [22] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view [13, 26, 51]. In particular, Bag and Samanta [2], following Cheng and Mordeson [9], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [25]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [3].

We use the definition of fuzzy normed spaces given in [2, 30, 31] to investigate the Hyers-Ulam stability of additive ρ-functional inequalities in fuzzy Banach spaces.

Definition 1.1. [2, 30, 31, 32] Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0,1]$ is called a *fuzzy norm* on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

(N_1) $N(x,t) = 0$ for $t \leq 0$;

(N_2) $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$;

(N_3) $N(cx,t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;

(N_4) $N(x+y, s+t) \geq \min\{N(x,s), N(y,t)\}$;

(N_5) $N(x,\cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$.

(N_6) for $x \neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

The pair (X,N) is called a *fuzzy normed vector space.*

The properties of fuzzy normed vector spaces and examples of fuzzy norms are given in [29, 30].

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Definition 1.2. [2, 30, 31, 32] Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to *be convergent* or *converge* if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the *limit* of the sequence $\{x_n\}$ and we denote it by N-$\lim_{n\to\infty} x_n = x$.

Definition 1.3. [2, 30, 31, 32] Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called *Cauchy* if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well-known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space*.

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to x_0 in X, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be *continuous* on X (see [3]).

The stability problem of functional equations originated from a question of Ulam [50] concerning the stability of group homomorphisms.

The functional equation $f(x+y) = f(x) + f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [18] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [42] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [14] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [8, 12, 16, 17, 19, 21, 23, 24, 27, 35, 36, 37, 38, 39, 40, 43, 44, 45, 46, 47, 48, 49]).

We recall a fundamental result in fixed point theory.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 1.4. [5, 10] Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1}x) < \infty, \qquad \forall n \ge n_0$;
(2) the sequence $\{J^n x\}$ converges to a fixed point y^* of J;
(3) y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$;
(4) $d(y, y^*) \le \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

In 1996, G. Isac and Th.M. Rassias [20] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [4, 6, 7, 11, 15, 29, 33, 34, 40, 41]).

In this paper, we solve the quadratic α-functional equation (0.1) and prove the Hyers-Ulam stability of the quadratic α-functional equation(0.1) in fuzzy Banach spaces by using the fixed point method.

Throughout this paper, assume that X is a real vector space and (Y, N) is a fuzzy Banach space. Assume that α is a real number with $\alpha^{-1} \ne \pm\sqrt{3}$.

2. Quadratic α-functional equation (0.1)

In this section, we prove the Hyers-Ulam stability of the quadratic α-functional equation (0.1) in fuzzy Banach spaces.

We need the following lemma to prove the main results.

We solve the quadratic α-functional equation (0.1) in vector spaces.

Lemma 2.1. *Let X and Y be vector spaces. If a mapping $f : X \to Y$ satisfies*

$$2f(x) + 2f(y) = f(x+y) + \alpha^{-2} f(\alpha(x-y)) \tag{2.1}$$

for all $x, y \in X$, then $f : X \to Y$ is quadratic.

Proof. Assume that $f : X \to Y$ satisfies (2.1).

Letting $x = y = 0$ in (2.1), we get $3f(0) = \alpha^{-2} f(0)$. So $f(0) = 0$.

Letting $y = 0$ in (2.1), we get $f(x) = \alpha^{-2} f(\alpha x)$ and so $f(\alpha x) = \alpha^2 f(x)$ for all $x \in X$. Thus

$$2f(x) + 2f(y) = f(x+y) + \alpha^{-2} f(\alpha(x-y)) = f(x+y) + f(x-y)$$

for all $x, y \in X$, as desired. □

Using the fixed point method, we prove the Hyers-Ulam stability of the quadratic α-functional equation (2.1) in fuzzy Banach spaces.

Theorem 2.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \le \frac{L}{4} \varphi(2x, 2y)$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$N\left(2f(x) + 2f(y) - f(x+y) - \alpha^{-2} f(\alpha(x-y)), t\right) \ge \frac{t}{t + \varphi(x, y)} \tag{2.2}$$

for all $x, y \in X$ and all $t > 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N\left(f(x) - Q(x), t\right) \geq \frac{(4-4L)t}{(4-4L)t + L\varphi(x,x)} \tag{2.3}$$

for all $x \in X$ and all $t > 0$.

Proof. Letting $y = x$ in (2.2), we get

$$N\left(f\left(2x\right) - 4f(x), t\right) \geq \frac{t}{t + \varphi(x,x)} \tag{2.4}$$

for all $x \in X$.

Consider the set

$$S := \{g : X \to Y\}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\left\{\mu \in \mathbb{R}_+ : N(g(x) - h(x), \mu t) \geq \frac{t}{t+\varphi(x,x)},\ \forall x \in X, \forall t > 0\right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [28, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \geq \frac{t}{t+\varphi(x,x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\varepsilon t) &= N\left(4g\left(\frac{x}{2}\right) - 4h\left(\frac{x}{2}\right), L\varepsilon t\right) \\ &= N\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{L}{4}\varepsilon t\right) \\ &\geq \frac{\frac{Lt}{4}}{\frac{Lt}{4} + \varphi\left(\frac{x}{2}, \frac{x}{2}\right)} \geq \frac{\frac{Lt}{4}}{\frac{Lt}{4} + \frac{L}{2}\varphi(x,x)} \\ &= \frac{t}{t+\varphi(x,x)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g,h)$$

for all $g, h \in S$.

It follows from (2.4) that

$$N\left(f(x) - 4f\left(\frac{x}{2}\right), \frac{L}{4}t\right) \geq \frac{t}{t+\varphi(x,x)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq \frac{L}{4}$.

By Theorem 1.4, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q\left(\frac{x}{2}\right) = \frac{1}{4}Q(x) \tag{2.5}$$

for all $x \in X$. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (2.5) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N(f(x) - Q(x), \mu t) \geq \frac{t}{t + \varphi(x, x)}$$

for all $x \in X$;

(2) $d(J^n f, Q) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right) = Q(x)$$

for all $x \in X$;

(3) $d(f, Q) \leq \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, Q) \leq \frac{L}{4 - 4L}.$$

This implies that the inequality (2.3) holds.

By (2.2),

$$N\left(4^n\left(2f\left(\frac{x}{2^n}\right) + 2f\left(\frac{y}{2^n}\right) - f\left(\frac{x+y}{2^n}\right) - \alpha^{-2}f\left(\alpha\frac{x-y}{2^n}\right)\right), 4^n t\right) \geq \frac{t}{t + \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. So

$$N\left(4^n\left(2f\left(\frac{x}{2^n}\right) + 2f\left(\frac{y}{2^n}\right) - f\left(\frac{x+y}{2^n}\right) - \alpha^{-2}f\left(\alpha\frac{x-y}{2^n}\right)\right), t\right) \geq \frac{\frac{t}{4^n}}{\frac{t}{4^n} + \frac{L^n}{4^n}\varphi(x, y)}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{\frac{t}{4^n}}{\frac{t}{4^n} + \frac{L^n}{4^n}\varphi(x,y)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$2Q(x) + 2Q(y) - Q(x + y) - \alpha^{-2}Q(\alpha(x - y)) = 0$$

for all $x, y \in X$. By Lemma 2.1, the mapping $Q : X \to Y$ is quadratic, as desired. □

Corollary 2.3. *Let $\theta \geq 0$ and let p be a real number with $p > 2$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$N\left(2f(x) + 2f(y) - f(x + y) - \alpha^{-2}f(\alpha(x - y)), t\right) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)} \tag{2.6}$$

for all $x, y \in X$ and all $t > 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N(f(x) - Q(x), t) \geq \frac{(2^p - 4)t}{(2^p - 4)t + 2\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.2 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{2-p}$, and we get the desired result. □

Theorem 2.4. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \le 4L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.2). Then $Q(x) := N$-$\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N(f(x) - Q(x), t) \ge \frac{(4 - 4L)t}{(4 - 4L)t + \varphi(x, x)} \tag{2.7}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{4} g(2x)$$

for all $x \in X$.

It follows from (2.4) that

$$N\left(f(x) - \frac{1}{4}f(2x), \frac{1}{4}t\right) \ge \frac{t}{t + \varphi(x, x)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \le \frac{1}{4}$. Hence

$$d(f, A) \le \frac{1}{4 - 4L},$$

which implies that the inequality (2.7) holds.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let $\theta \ge 0$ and let p be a real number with $0 < p < 2$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.6). Then $Q(x) := N$-$\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$N(f(x) - Q(x), t) \ge \frac{(4 - 2^p)t}{(4 - 2^p)t + 2\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.4 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{p-2}$, and we get the desired result. □

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Four-point impulsive multi-orders fractional boundary value problems

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Abstract

Four-point boundary value problem for impulsive multi-orders fractional differential equation is studied. The existence and uniqueness results are obtained for impulsive multi-orders fractional differential equation with four-point fractional boundary conditions by applying standard fixed point theorems. An example for the illustration of the main result is presented.

Keywords: fractional differential equations, fixed point theorems, multi-orders, impulse.

1 Introduction

Impulsive differential equations have extensively been studied in the past two decades. Impulsive differential equations are used to describe the dynamics of processes in which sudden, discontinuous jumps occur. Such processes are naturally seen in harvesting, earthquakes, diseases, and so forth. Recently, fractional impulsive differential equations have attracted the attention of many researchers. For the general theory and applications of such equationswe refer the interested reader to see [1]-[18] and the references therein.

In [8], Kosmatov considered the following two impulsive problems:

$$\begin{cases} {}^{C}D^{\alpha}u\left(t\right)=f\left(t,u\left(t\right)\right),\ 1<\alpha<2,\ t\in\left[0,1\right]\setminus\left\{t_1,t_2,...,t_p\right\}, \\ {}^{C}D^{\gamma}u\left(t_k^{+}\right)-{}^{C}D^{\gamma}u\left(t_k^{-}\right)=I_k\left(u\left(t_k^{-}\right)\right),\quad t_k\in\left(0,1\right),\ k=1,...,p, \\ u\left(0\right)=u_0,\ u'\left(0\right)=u_0,\quad 0<\gamma<1, \end{cases}$$

and

$$\begin{cases} {}^{L}D^{\alpha}u\left(t\right)=f\left(t,u\left(t\right)\right),\ 0<\alpha<1,\ t\in\left[0,1\right]\setminus\left\{t_1,t_2,...,t_p\right\}, \\ {}^{L}D^{\gamma}u\left(t_k^{+}\right)-\ {}^{L}D^{\gamma}u\left(t_k^{-}\right)=I_k\left(u\left(t_k^{-}\right)\right),\quad t_k\in\left(0,1\right),\ k=1,...,p, \\ I^{1-\alpha}u\left(0\right)=u_0,\quad 0<\gamma<\alpha<1. \end{cases}$$

In [4], Feckan et al. studied the impulsive problem of the following form:

$$\begin{cases} {}^{C}D^{\alpha}u\left(t\right)=f\left(t,u\left(t\right)\right),\ 0<\alpha<1,\ t\in\left[0,1\right]\setminus\left\{t_1,t_2,...,t_p\right\}, \\ u\left(t_k^{+}\right)-u\left(t_k^{-}\right)=I_k\left(u\left(t_k^{-}\right)\right),\quad t_k\in\left(0,1\right),\ k=1,...,p, \\ u\left(0\right)=u_0,\ 0<\gamma<\alpha<1. \end{cases}$$

Wang et al. [17] obtained some existence and uniqueness results for the following impulsive multipoint fractional integral boundary value problem involving multi-orders fractional derivatives and deviating argument

$$\begin{cases} {}^{C}D_{t_k}^{\alpha_k}u\left(t\right)=f\left(t,u\left(t\right),u\left(\theta\left(t\right)\right)\right),\ 1<\alpha_k\leq 2,\ t\in\left[0,T\right]\setminus\left\{t_1,t_2,...,t_p\right\}, \\ \Delta u\left(t_k\right)=I_k\left(u\left(t_k^{-}\right)\right),\quad \Delta u'\left(t_k\right)=J_k\left(u\left(t_k^{-}\right)\right),\quad t_k\in\left(0,T\right),\ k=1,...,p, \\ u\left(0\right)=\sum_{k=0}^{p}\lambda_k I_{t_k}^{\beta_k}u\left(\eta_k\right),\quad t_k<\eta_k<t_{k+1}, \\ u'\left(0\right)=0. \end{cases}$$

Yukunthorn et.al. [18] studied the similar problem for multi-order Caputo–Hadamard fractional differential equations equipped with nonlinear integral boundary conditions.

Motivated by the above works, in this paper, we study the existence of solutions for the four-point nonlocal boundary value problems of nonlinear impulsive equations of fractional order

$$\begin{cases} {}^{C}D_{t_k}^{\alpha_k} u(t) = f(t, u(t), u'(t)), \ 1+\beta \leq \alpha_k \leq 2, \ t \in [t_k, t_{k+1}), \\ \Delta u(t_k) = I_k\left(u\left(t_k^-\right)\right), \quad \Delta u'(t_k) = J_k\left(u'\left(t_k^-\right)\right), \quad t_k \in (0,T), \ k = 1, ..., p, \\ u(0) + \mu_1 \ {}^{C}D_{0+}^{\beta} u(0) = \sigma_1 u(\eta_1), \quad 0 < \eta_1 < t_1 < T, \\ u(T) + \mu_2 \ {}^{C}D_{t_p}^{\beta} u(T) = \sigma_2 u(\eta_2), \quad 0 < t_p < \eta_2 < T, \ 0 < \beta < 1, \end{cases} \tag{1}$$

where ${}^{C}D_t^{\alpha}$ is the Caputo derivative, $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $I_k, J_k : \mathbb{R} \to \mathbb{R}$, $\Delta u(t_k) = u\left(t_k^+\right) - u\left(t_k^-\right)$, $\Delta u'(t_k) = u'\left(t_k^+\right) - u'\left(t_k^-\right)$, $u\left(t_k^+\right)$ and $u\left(t_k^-\right)$ represent the right hand limit and the left hand limit of the function $u(t)$ at $t = t_k$; and the sequence $\{t_k\}$ satisfies that $0 = t_0 < t_1 < ... < t_p < t_{p+1} = T$.

To the best of our knowledge, there is no paper that consider the four-point impulsive boundary value problem involving nonlinear differential equations of fractional order (1). The main difficulty of this problem is that the corresponding integral equation is very complex because of the impulse effects. In this paper, we study the existence and uniqueness of solutions for four-point impulsive boundary value problem (1). By use of Banach's fixed point theorem and Schauder's fixed point theorem, some existence and uniqueness results are obtained.

2 Preliminaries

Let $[0,T]^- = [0,T] \setminus \{t_1, t_2, ..., t_p\}$ and

$$\begin{aligned} PC([0,T], \mathbb{R}) = \{ & x : [0,T] \to \mathbb{R} : x(t) \text{ is continuous everywhere except for some} \\ & t_k \text{ at which } x\left(t_k^+\right), x\left(t_k^-\right) \text{ exist and } x\left(t_k^-\right) = x(t_k), \ k = 1, ..., p\}, \end{aligned}$$

and

$$\begin{aligned} PC^1([0,T], \mathbb{R}) = \{ & x \in PC([0,T], \mathbb{R}) : x'(t) \text{ is continuous everywhere except for some} \\ & t_k \text{ at which } x'\left(t_k^+\right), x'\left(t_k^-\right) \text{ exist and } x'\left(t_k^-\right) = x'(t_k), \ k = 1, ..., p\}. \end{aligned}$$

$PC([0,T], \mathbb{R})$ and $PC^1([0,T], \mathbb{R})$ are Banach spaces with the norms $\|x\|_{PC} = \sup\{|x(t)| : t \in [0,T]\}$ and $\|x\|_{PC^1} = \max\{\|x\|_{PC}, \|x'\|_{PC}\}$, respectively. Let $X = PC^1([0,T], \mathbb{R}) \cap C^2\left([0,T]^-, \mathbb{R}\right)$. A function $x \in X$ is called a solution of problem (1) if it satisfies (1).

Throughout the paper we will use the following notations.

$$\begin{aligned} \rho &= \sigma_1 \eta_1 + \left(T + \mu_2 \frac{T^{1-\beta}}{\Gamma(2-\beta)}\right)(1-\sigma_1), \\ A_0 &= \frac{\sigma_1}{1-\sigma_1} - \frac{\sigma_1}{\rho}\frac{\sigma_1\eta_1}{1-\sigma_1}, \quad B_0 = \frac{\sigma_1}{\rho}, \\ A_p &= \frac{(1-\sigma_1)}{\rho}\frac{\sigma_1\eta_1}{1-\sigma_1}, \quad B_p = \frac{1-\sigma_1}{\rho}. \end{aligned}$$

$$
\begin{aligned}
F_k\left(y,u,u'\right)(t) &= \frac{1}{\Gamma\left(\alpha_k\right)}\int_{t_k}^{t}(t-s)^{\alpha_k-1}y(s)\,ds \\
&\quad+\sum_{j=1}^{k}\frac{1}{\Gamma\left(\alpha_{j-1}\right)}\int_{t_{j-1}}^{t_j}\left(t_j-s\right)^{\alpha_{j-1}-1}y(s)\,ds+\sum_{j=1}^{k}I_j\left(u\left(t_j^-\right)\right) \\
&\quad+\sum_{j=1}^{k}\left(t-t_j\right)\frac{1}{\Gamma\left(\alpha_{j-1}-1\right)}\int_{t_{j-1}}^{t_j}\left(t_j-s\right)^{\alpha_{j-1}-2}y(s)\,ds+\sum_{j=1}^{k}\left(t-t_j\right)J_j\left(u'\left(t_j^-\right)\right), \\
G_k\left(y,u,u'\right)(t) &= \frac{1}{\Gamma\left(\alpha_k-\beta\right)}\int_{t_k}^{t}(t-s)^{\alpha_k-\beta-1}y(s)\,ds \\
&\quad+\frac{t^{1-\beta}}{\Gamma(2-\beta)}\sum_{j=1}^{k}\frac{1}{\Gamma\left(\alpha_{j-1}-1\right)}\int_{t_{j-1}}^{t_j}\left(t_j-s\right)^{\alpha_{j-1}-2}y(s)\,ds \\
&\quad+\frac{t^{1-\beta}}{\Gamma(2-\beta)}\sum_{j=1}^{k}J_j\left(u'\left(t_j^-\right)\right).
\end{aligned}
$$

$$
\begin{aligned}
F_k'\left(y,u,u'\right)(t) &= \frac{1}{\Gamma\left(\alpha_k-1\right)}\int_{t_k}^{t}(t-s)^{\alpha_k-2}y(s)\,ds \\
&\quad+\sum_{j=1}^{k}\frac{1}{\Gamma\left(\alpha_{j-1}-1\right)}\int_{t_{j-1}}^{t_j}\left(t_j-s\right)^{\alpha_{j-1}-2}y(s)\,ds+\sum_{j=1}^{k}J_j\left(u'\left(t_j^-\right)\right).
\end{aligned}
$$

Lemma 1 *Let $y\in C[0,T]$. A function $u\in PC^1[0,T]$ is a solution of the boundary value problem*

$$
\begin{cases}
{}^{C}D_{t_k}^{\alpha_k}u(t)=y(t), \ 1+\beta<\alpha_k\leq 2, \ t\in[0,T]\setminus\{t_1,t_2,...,t_p\}, \\
\Delta u\left(t_k\right)=I_k\left(u\left(t_k^-\right)\right), \quad \Delta u'\left(t_k\right)=J_k\left(u'\left(t_k^-\right)\right), \quad t_k\in(0,T), \ k=1,...,p, \\
u(0)+\mu_1\ {}^{C}D_{0+}^{\beta}u(0)=\sigma_1u\left(\eta_1\right), \quad 0<\eta_1<t_1<T, \\
u(T)+\mu_2\ {}^{C}D_{t_p}^{\beta}u(T)=\sigma_2u\left(\eta_2\right)=\sigma_2u\left(\eta_2\right), \quad 0<t_p<\eta_2<T, \ 0<\beta<1,
\end{cases}
\tag{2}
$$

if and only if

$$
\begin{aligned}
u(t) &= F_k(y,u)(t)+\frac{\sigma_1}{1-\sigma_1}F_0(y,u)\left(\eta_1\right) \\
&\quad-\frac{\sigma_1}{\rho}\left(\frac{\sigma_1\eta_1}{1-\sigma_1}+t\right)F_0(y,u)\left(\eta_1\right) \\
&\quad+\frac{\sigma_2\left(1-\sigma_1\right)}{\rho}\left(\frac{\sigma_1\eta_1}{1-\sigma_1}+t\right)F_p(y,u)\left(\eta_2\right) \\
&\quad-\frac{\left(1-\sigma_1\right)}{\rho}\left(\frac{\sigma_1\eta_1}{1-\sigma_1}+t\right)F_p(y,u)(T) \\
&\quad-\frac{\mu_2\left(1-\sigma_1\right)}{\rho}\left(\frac{\sigma_1\eta_1}{1-\sigma_1}+t\right)G_p(y,u)(T).
\end{aligned}
\tag{3}
$$

Proof. Suppose that u is a solution of (2). For $0\leq t\leq t_1$, we have

$$
u(t)=I_{0+}^{\alpha_0}y(t)-c_1-c_2t=\frac{1}{\Gamma\left(\alpha_0\right)}\int_0^t(t-s)^{\alpha_0-1}y(s)\,ds-c_1-c_2t, \quad c_1,c_2\in\mathbb{R}. \tag{4}
$$

Then differentiating (4), we get

$$
\begin{aligned}
D_{0+}^{\beta}u(t) &= \frac{1}{\Gamma\left(\alpha_0-\beta\right)}\int_0^t(t-s)^{\alpha_0-\beta-1}y(s)\,ds-c_2\frac{t^{1-\beta}}{\Gamma(2-\beta)}, \\
u'(t) &= \frac{1}{\Gamma\left(\alpha_0-1\right)}\int_0^t(t-s)^{\alpha_0-2}y(s)\,ds-c_2.
\end{aligned}
$$

If $t_1 < t \le t_2$, then for some $d_1, d_2 \in \mathbb{R}$, we have

$$u(t) = \frac{1}{\Gamma(\alpha_1)} \int_{t_1}^{t} (t-s)^{\alpha_1 - 1} y(s)\, ds - d_1 - d_2 (t - t_1),$$

$$u'(t) = \frac{1}{\Gamma(\alpha_1 - 1)} \int_{t_1}^{t} (t-s)^{\alpha_1 - 2} y(s)\, ds - d_2,$$

$$D_{t_1^+}^{\beta} u(t) = \frac{1}{\Gamma(\alpha_1 - \beta)} \int_{t_1}^{t} (t-s)^{\alpha_1 - \beta - 1} y(s)\, ds - d_2 \frac{t^{1-\beta}}{\Gamma(2-\beta)}.$$

Thus

$$u(t_1^-) = \frac{1}{\Gamma(\alpha_0)} \int_0^{t_1} (t_1 - s)^{\alpha_0 - 1} y(s)\, ds - c_1 - c_2 t_1, \quad u(t_1^+) = -d_1$$

$$u'(t_1^-) = \frac{1}{\Gamma(\alpha_0 - 1)} \int_0^{t_1} (t_1 - s)^{\alpha_0 - 2} y(s)\, ds - c_2, \quad u'(t_1^+) = -d_2.$$

In view of

$$u(t_1^+) - u(t_1^-) = I_1(u(t_1^-)), \quad u'(t_1^+) - u'(t_1^-) = J_1(u'(t_1^-)),$$

we find that

$$-d_1 = \frac{1}{\Gamma(\alpha_0)} \int_0^{t_1} (t_1 - s)^{\alpha_0 - 1} y(s)\, ds + I_1(u(t_1^-)) - c_1 - c_2 t_1,$$

$$-d_2 = \frac{1}{\Gamma(\alpha_0 - 1)} \int_0^{t_1} (t_1 - s)^{\alpha_0 - 2} y(s)\, ds + J_1(u'(t_1^-)) - c_2.$$

Hence we obtain for $t_1 < t \le t_2$

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha_1)} \int_{t_1}^{t} (t-s)^{\alpha_1 - 1} y(s)\, ds \\ &+ \frac{1}{\Gamma(\alpha_0)} \int_0^{t_1} (t_1 - s)^{\alpha_0 - 1} y(s)\, ds + I_1(u(t_1^-)) \\ &+ (t - t_1) \frac{1}{\Gamma(\alpha_0 - 1)} \int_0^{t_1} (t_1 - s)^{\alpha_0 - 2} y(s)\, ds + (t - t_1) J_1(u'(t_1^-)) \\ &- c_1 - c_2 t, \quad t_1 < t \le t_2. \end{aligned}$$

In a similar way, for $k = 1, 2, ..., p$ we can obtain

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha_k)} \int_{t_k}^{t} (t-s)^{\alpha_k - 1} y(s)\, ds + \sum_{j=1}^{k} \frac{1}{\Gamma(\alpha_{j-1})} \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha_{j-1} - 1} y(s)\, ds + \sum_{j=1}^{k} I_j(u(t_j^-)) \\ &+ \sum_{j=1}^{k} (t - t_j) \frac{1}{\Gamma(\alpha_{j-1} - 1)} \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha_{j-1} - 2} y(s)\, ds + \sum_{j=1}^{k} (t - t_j) J_j(u'(t_j^-)) \\ &- c_1 - c_2 t, \quad t_k < t \le t_{k+1}. \end{aligned} \tag{5}$$

Moreover,

$$\begin{aligned} {}^C D_{t_k}^{\beta} u(t) &= \frac{1}{\Gamma(\alpha_k - \beta)} \int_{t_k}^{t} (t-s)^{\alpha_k - \beta - 1} y(s)\, ds \\ &+ \frac{t^{1-\beta}}{\Gamma(2-\beta)} \sum_{j=1}^{k} \frac{1}{\Gamma(\alpha_{j-1} - 1)} \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha_{j-1} - 2} y(s)\, ds \\ &+ \frac{t^{1-\beta}}{\Gamma(2-\beta)} \sum_{j=1}^{k} J_j(u'(t_j^-)) - c_2 \frac{t^{1-\beta}}{\Gamma(2-\beta)}. \end{aligned}$$

Now applying the boundary conditions

$$\begin{aligned} &u(0)+\mu_1\ {}^{C}D_{0+}^{\beta}u(0)=\sigma_1 u(\eta_1), \quad 0<\eta_1<t_1<T, \\ &u(T)+\mu_2\ {}^{C}D_{t_p}^{\beta}u(T)=\sigma_2 u(\eta_2), \quad 0<t_p<\eta_2<T,\ 0<\beta<1, \end{aligned}$$

we get

$$-c_1=\sigma_1 F_0(y,u,u')(\eta_1)-\sigma_1 c_1-c_2\sigma_1\eta_1,$$

$$F_p(y,u,u')(T)-c_1-c_2T+\mu_2 G_p(y,u)(T)-\mu_2 c_2\frac{T^{1-\beta}}{\Gamma(2-\beta)}=\sigma_2 F_p(y,u,u')(\eta_2).$$

Solving this system for c_1, c_2 and inserting these values into (5) we get

$$\begin{aligned} u(t) &= F_k(y,u,u')(t)+\frac{\sigma_1}{1-\sigma_1}F_0(y,u,u')(\eta_1)-c_2\left(\frac{\sigma_1\eta_1}{1-\sigma_1}+t\right) \\ &= F_k(y,u,u')(t)+\frac{\sigma_1}{1-\sigma_1}F_0(y,u,u')(\eta_1)-\frac{\sigma_1}{\rho}\left(\frac{\sigma_1\eta_1}{1-\sigma_1}+t\right)F_0(y,u,u')(\eta_1) \\ &+\frac{\sigma_2(1-\sigma_1)}{\rho}\left(\frac{\sigma_1\eta_1}{1-\sigma_1}+t\right)F_p(y,u,u')(\eta_2) \\ &-\frac{(1-\sigma_1)}{\rho}\left(\frac{\sigma_1\eta_1}{1-\sigma_1}+t\right)F_p(y,u,u')(T) \\ &-\frac{\mu_2(1-\sigma_1)}{\rho}\left(\frac{\sigma_1\eta_1}{1-\sigma_1}+t\right)G_p(y,u,u')(T). \end{aligned}$$

Conversely, assume that u is a solution of the impulsive fractional integral equation (3). Then by a direct computation, it follows that the solution given by (3) satisfies (2). This completes the proof. ■

3 Existence and Uniqueness

In the sequel,we assume that

(A$_1$) $f:[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous function and such that

$$|f(t,x,x_1)-f(t,y,y_1)|\le l_f(|x-y|+|x_1-y_1|), \quad l_f>0,\ 0\le t\le T, \quad x,y,x_1,y_1\in\mathbb{R}.$$

(A$_2$) $I_k, J_k:\mathbb{R}\to\mathbb{R}$ are continuous functions and satisfy

$$\begin{aligned} &|I_k(x)-I_k(y)|\le l_1|x-y|, \\ &|J_k(x)-J_k(y)|\le l_2|x-y|, \quad l_1>0,\ l_2>0,\ 0\le t\le T, \quad x,y\in\mathbb{R}. \end{aligned}$$

For convenience, we will give some notations:

$$T^*=\max\{T^{\alpha_k}:0\le k\le p\}, \qquad \Gamma^*=\min\{\Gamma(\alpha_k):0\le k\le p\},$$

$$\Delta_1=\sum_{j=1}^{p}\frac{(t_j-t_{j-1})^{\alpha_{j-1}}}{\Gamma(\alpha_{j-1}+1)}, \quad \Delta_2=\sum_{j=1}^{p}\frac{(T-t_j)(t_j-t_{j-1})^{\alpha_{j-1}-1}}{\Gamma(\alpha_{j-1})},$$

$$\Delta_3=\frac{T^{1-\beta}}{\Gamma(2-\beta)}\sum_{j=1}^{p}\frac{(t_j-t_{j-1})^{\alpha_{j-1}-1}}{\Gamma(\alpha_{j-1})}, \quad \Delta_4=\sum_{j=1}^{p}\frac{(t_j-t_{j-1})^{\alpha_{j-1}-1}}{\Gamma(\alpha_{j-1})}.$$

$$\Lambda_F := l_f \frac{T^*}{\Gamma^*} + l_f \Delta_1 + l_f \Delta_2 + pl_1 + l_2 pT,$$
$$\Lambda_G := l_f \frac{T^*}{\Gamma^*} + l_f \Delta_1 \frac{T^{1-\beta}}{\Gamma(2-\beta)} + l_2 p \frac{T^{1-\beta}}{\Gamma(2-\beta)},$$
$$\Lambda_{F'} := l_f \frac{T^*}{\Gamma^*} + l_f \Delta_4 + l_2 p.$$

Lemma 2 *$F_k(f,u,u')$ and $G_k(f,u,u')$ are Lipschitzian operators.*

$$|F_k(f,u,u') - F_k(f,v,v')| \le \Lambda_F \|u-v\|_{PC^1}, \quad L_{F_k} > 0,$$
$$|G_k(f,u,u') - G_k(f,v,v')| \le \Lambda_G \|u-v\|_{PC^1}, \quad L_{G_k} > 0, \quad u,v \in PC^1([0,T],\mathbb{R}).$$

Proof. For $u,v \in PC^1([0,T],\mathbb{R})$, we have

$$\begin{aligned}
&|F_k(f,u,u')(t) - F_k(f,v,v')(t)| \\
&\le \frac{1}{\Gamma(\alpha_k)} \int_{t_k}^{t} (t-s)^{\alpha_k - 1} |f(s,u(s),u'(s)) - f(s,v(s),v'(s))| \, ds \\
&+ \sum_{j=1}^{k} \frac{1}{\Gamma(\alpha_{j-1})} \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha_{j-1}-1} |f(s,u(s),u'(s)) - f(s,v(s),v'(s))| \, ds \\
&+ \sum_{j=1}^{k} \left| I_j\left(u\left(t_j^-\right)\right) - I_j\left(v\left(t_j^-\right)\right) \right| \\
&+ \sum_{j=1}^{k} \frac{(t-t_j)}{\Gamma(\alpha_{j-1}-1)} \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha_{j-1}-2} |f(s,u(s),u'(s)) - f(s,v(s),v'(s))| \, ds \\
&+ \sum_{j=1}^{k} (t-t_j) \left| J_j\left(u'\left(t_j^-\right)\right) - J_j\left(v'\left(t_j^-\right)\right) \right|
\end{aligned}$$

$$\begin{aligned}
&\le l_f \frac{1}{\Gamma(\alpha_k)} \int_{t_k}^{t} (t-s)^{\alpha_k - 1} (|u(s) - v(s)| + |u'(s) - v'(s)|) \, ds \\
&+ l_f \sum_{j=1}^{k} \frac{1}{\Gamma(\alpha_{j-1})} \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha_{j-1}-1} (|u(s) - v(s)| + |u'(s) - v'(s)|) \, ds \\
&+ l_1 \sum_{j=1}^{k} \left| u\left(t_j^-\right) - v\left(t_j^-\right) \right| + l_f \sum_{j=1}^{k} \frac{1}{\Gamma(\alpha_{j-1}-1)} (t-t_j)
\end{aligned}$$

$$\begin{aligned}
&\times \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha_{j-1}-2} (|u(s) - v(s)| + |u'(s) - v'(s)|) \, ds \\
&+ l_2 \sum_{j=1}^{k} (t-t_j) \left| u'\left(t_j^-\right) - v'\left(t_j^-\right) \right| \\
&\le \Lambda_F \|u-v\|_{PC^1}.
\end{aligned}$$

Similarly,

$$
\begin{aligned}
&\left|G_k\left(f,u,u'\right)(t)-G_k\left(f,v,v'\right)(t)\right| \\
&\leq \frac{1}{\Gamma\left(\alpha_k-\beta\right)}\int_{t_k}^{t}(t-s)^{\alpha_k-\beta-1}\left|f\left(s,u\left(s\right),u'\left(s\right)\right)-f\left(s,v\left(s\right),v'\left(s\right)\right)\right|ds \\
&+\frac{T^{1-\beta}}{\Gamma\left(2-\beta\right)}\sum_{j=1}^{k}\frac{1}{\Gamma\left(\alpha_{j-1}-1\right)}\int_{t_{j-1}}^{t_j}\left(t_j-s\right)^{\alpha_{j-1}-2}\left|f\left(s,u\left(s\right),u'\left(s\right)\right)-f\left(s,v\left(s\right),v'\left(s\right)\right)\right|ds \\
&+\frac{t^{1-\beta}}{\Gamma\left(2-\beta\right)}\sum_{j=1}^{k}\left|J_j\left(u'\left(t_j^-\right)\right)-J_j\left(v'\left(t_j^-\right)\right)\right| \\
&\leq\left(l_f\frac{\left(T-t_k\right)^{\alpha_k-\beta}}{\Gamma\left(\alpha_k-\beta+1\right)}+l_f\frac{T^{1-\beta}}{\Gamma\left(2-\beta\right)}\sum_{j=1}^{p}\frac{\left(t_j-t_{j-1}\right)^{\alpha_{j-1}-1}}{\Gamma\left(\alpha_{j-1}\right)}+\frac{T^{1-\beta}}{\Gamma\left(2-\beta\right)}l_2\right)\left\|u-v\right\|_{PC^1} \\
&\leq\Lambda_G\left\|u-v\right\|_{PC^1}.
\end{aligned}
$$

Also, we have

$$
\left|F_k'\left(f,u,u'\right)(t)-F_k'\left(f,v,v'\right)(t)\right|\leq\Lambda_{F'}\left\|u-v\right\|_{PC^1}.
$$

■

In view of Lemma 1 we define an operator $\Theta : X \to X$ by

$$
\begin{aligned}
\left(\Theta u\right)(t) &= F_k\left(f,u\right)(t)-\left(A_0-B_0t\right)F_0\left(f,u\right)\left(\eta_1\right) \\
&+\sigma_2\left(A_p+B_pt\right)F_p\left(f,u\right)\left(\eta_2\right)-\left(A_p+B_pt\right)F_p\left(f,u\right)(T) \\
&-\mu_2\left(A_p+B_pt\right)G_p\left(f,u\right)(T),
\end{aligned}
$$

where

$$
\begin{aligned}
A_0 &= \frac{\sigma_1}{1-\sigma_1}-\frac{\sigma_1}{\rho}\frac{\sigma_1\eta_1}{1-\sigma_1}, \quad B_0=\frac{\sigma_1}{\rho}, \\
A_p &= \frac{\left(1-\sigma_1\right)}{\rho}\frac{\sigma_1\eta_1}{1-\sigma_1}, \quad B_p=\frac{1-\sigma_1}{\rho}.
\end{aligned}
$$

Let

$$
\Lambda_\Theta := \max\left\{\Lambda_F,\Lambda_G,\Lambda_{F'}\right\}.
$$

Theorem 3 *Suppose that the assumption* (A_1)*,* (A_2) *are satisfied. If*

$$
\begin{aligned}
\Lambda &:= \Lambda_\Theta \max\left\{\left(1+\left|A_0\right|+\left|B_0\right|T+\left(\left|\sigma_2\right|+\left|\mu_2\right|+1\right)\left(\left|A_p\right|+\left|B_p\right|T\right)\right)\right. \\
&\left.,\left(1+\left|B_0\right|+\left(\left|\sigma_2\right|+\left|\mu_2\right|+1\right)\left|B_p\right|\right)\right\}<1,
\end{aligned}
$$

then the boundary value problem (1) has a unique solution.

Proof. Let $u,v\in PC^1\left(\left[0,T\right],\mathbb{R}\right)$. For $u,v\in\left(t_k,t_{k+1}\right]$, $k=0,...,p$, we have

$$
\begin{aligned}
\left|\left(\Theta u\right)(t)-\left(\Theta v\right)(t)\right| &\leq \left|F_k\left(f,u,u'\right)(t)-F_k\left(f,v,v'\right)(t)\right| \\
&+\left|A_0-B_0t\right|\left|F_0\left(f,u,u'\right)\left(\eta_1\right)-F_0\left(f,v,v'\right)\left(\eta_1\right)\right| \\
&+\left|\sigma_2\right|\left|A_p+B_pt\right|\left|F_p\left(f,u,u'\right)\left(\eta_2\right)-F_p\left(f,v,v'\right)\left(\eta_2\right)\right| \\
&+\left|A_p+B_pt\right|\left|F_p\left(f,u,u'\right)(T)-F_p\left(f,v,v'\right)(T)\right| \\
&+\left|\mu_2\right|\left|A_p+B_pt\right|\left|G_p\left(f,u,u'\right)(T)-G_p\left(f,v,v'\right)(T)\right| \\
&\leq\Lambda_\Theta\left(1+\left|A_0\right|+\left|B_0\right|T+\left(\left|\sigma_2\right|+\left|\mu_2\right|+1\right)\left(\left|A_p\right|+\left|B_p\right|T\right)\right)\left\|u-v\right\|_{PC^1}.
\end{aligned}
$$

Similarly, for $u, v \in (t_k, t_{k+1}]$ we have

$$\begin{aligned}\left|(\Theta u)'(t) - (\Theta v)'(t)\right| &\leq \left|F_k'(f, u, u')(t) - F_k'(f, v, v')(t)\right| \\ &+ |B_0| \left|F_0(f, u, u')(\eta_1) - F_0(f, v, v')(\eta_1)\right| \\ &+ |\sigma_2| |B_p| \left|F_p(f, u, u')(\eta_2) - F_p(f, v, v')(\eta_2)\right| \\ &+ |B_p| \left|F_p(f, u, u')(T) - F_p(f, v, v')(T)\right| \\ &+ |\mu_2| |B_p| \left|G_p(f, u, u')(T) - G_p(f, v, v')(T)\right| \\ &\leq \Lambda_\Theta (1 + |B_0| + (|\sigma_2| + |\mu_2| + 1) |B_p|) \|u - v\|_{PC^1}.\end{aligned}$$

It follows that

$$\|\Theta u - \Theta v\|_{PC^1} \leq \Lambda \|u - v\|_{PC^1}.$$

Since $\Lambda < 1$, Θ is a contraction. According to the Banach fixed point theorem Θ has a unique fixed point, that is the problem (1) has a unique solution. ■

4 Existence

In this section, we assume that

(A$_3$) $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is continuous function and there exists $h \in C([0, T], \mathbb{R}^+)$ such that

$$|f(t, u, v)| \leq h(t) + b_1 |u|^\rho + b_2 |v|^\varrho, \quad (t, u, v) \in [0, T] \times \mathbb{R} \times \mathbb{R}, \quad 0 < \rho, \varrho < 1.$$

(A$_4$) $I_k, J_k : \mathbb{R} \to \mathbb{R}$ are continuous functions and there $L_2 > 0$, $L_3 > 0$ such that

$$|I_k(x)| \leq L_2, \quad |J_k(x)| \leq L_3, \quad x \in \mathbb{R}.$$

For convenience, we will give some notations:

$$\begin{aligned}C_1 &:= (1 + |A_0| + |B_0| T + (|\sigma_2| + 1)(|A_p| + |B_p| T))(pL_2 + pTL_3) \|h\| \\ &+ |\mu_2| (|A_p| + |B_p| T) \frac{T^{1-\beta}}{\Gamma(2-\beta)} L_3 \|h\|, \\ C_2 &:= (1 + |A_0| + |B_0| T + (|\sigma_2| + 1)(|A_p| + |B_p| T)) \left(\frac{T^*}{\Gamma^*} + \Delta_1 + \Delta_2\right) \\ &+ |\mu_2| (|A_p| + |B_p| T) \left(\frac{T^*}{\Gamma^*} + \Delta_3\right).\end{aligned}$$

Lemma 4 *If*

$$R \geq \max\left\{3C_1, (3b_1 C_2)^{\frac{1}{1-\rho}}, (3b_1 C_2)^{\frac{1}{1-\varrho}}\right\},$$

then Θ maps $B(0, R) := \left\{u \in PC^1([0, T], \mathbb{R}) : \|u\|_{PC^1} \leq R\right\}$ into itself.

Proof. Assume that

$$R \geq \max\left\{3C_1, (3b_1 C_2)^{\frac{1}{1-\rho}}, (3b_1 C_2)^{\frac{1}{1-\varrho}}\right\}.$$

Then for $t \in (t_k, t_{k+1}]$, $k = 0, ..., p$, we have

$$\begin{aligned}&|F_k(f, u, u')(t)| \\ &\leq \frac{1}{\Gamma(\alpha_k)} \int_{t_k}^{t} (t-s)^{\alpha_k - 1} |f(s, u(s), u'(s))| ds \\ &+ \sum_{j=1}^{k} \frac{1}{\Gamma(\alpha_{j-1})} \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha_{j-1}-1} |f(s, u(s), u'(s))| ds + \sum_{j=1}^{k} \left|I_j\left(u\left(t_j^-\right)\right)\right| \\ &+ \sum_{j=1}^{k} \frac{(t - t_j)}{\Gamma(\alpha_{j-1} - 1)} \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha_{j-1}-2} |f(s, u(s), u'(s))| ds + \sum_{j=1}^{k} (t - t_j) \left|J_j\left(v'\left(t_j^-\right)\right)\right|,\end{aligned}$$

$$
\begin{aligned}
&\left|F_k\left(f,u,u'\right)(t)\right| \\
&\leq \frac{1}{\Gamma\left(\alpha_k\right)}\int_{t_k}^{t}(t-s)^{\alpha_k-1}\left(h(s)+b_1\left|u(s)\right|^{\rho}+b_2\left|u'(s)\right|^{\varrho}\right)ds \\
&+\sum_{j=1}^{k}\frac{1}{\Gamma\left(\alpha_{j-1}\right)}\int_{t_{j-1}}^{t_j}\left(t_j-s\right)^{\alpha_{j-1}-1}\left(h(s)+b_1\left|u(s)\right|^{\rho}+b_2\left|u'(s)\right|^{\varrho}\right)ds \\
&+\sum_{j=1}^{k}\left|I_j\left(u\left(t_j^-\right)\right)\right| \\
&+\sum_{j=1}^{k}\frac{\left(t-t_j\right)}{\Gamma\left(\alpha_{j-1}-1\right)}\int_{t_{j-1}}^{t_j}\left(t_j-s\right)^{\alpha_{j-1}-2}\left(h(s)+b_1\left|u(s)\right|^{\rho}+b_2\left|v(s)\right|^{\varrho}\right)ds \\
&+\sum_{j=1}^{k}\left(t-t_j\right)\left|J_j\left(u\left(t_j^-\right)\right)\right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{T^{\alpha_k}}{\Gamma\left(\alpha_k+1\right)}\left(\|h\|+b_1\|u\|^{\rho}+b_2\|u'\|^{\varrho}\right) \\
&+\sum_{j=1}^{p}\frac{\left(t_j-t_{j-1}\right)^{\alpha_{j-1}}}{\Gamma\left(\alpha_{j-1}+1\right)}\left(\|h\|+b_1\|u\|^{\rho}+b_2\|u'\|^{\varrho}\right)+pL_2 \\
&+\sum_{j=1}^{p}\frac{\left(t-t_j\right)\left(t_j-t_{j-1}\right)^{\alpha_{j-1}-1}}{\Gamma\left(\alpha_{j-1}\right)}\left(\|h\|+b_1\|u\|^{\rho}+b_2\|u'\|^{\varrho}\right)+pTL_3 \\
&\leq \left(\frac{T^*}{\Gamma^*}+\Delta_1+\Delta_2\right)\left(\|h\|+b_1\|u\|^{\rho}+b_2\|u'\|^{\varrho}\right)+pL_2+pTL_3,
\end{aligned}
$$

$$
\begin{aligned}
\left|G_k\left(y,u,u'\right)(t)\right| &\leq \frac{T^{\alpha_k-\beta}}{\Gamma\left(\alpha_k-\beta+1\right)}\left(\|h\|+b_1\|u\|^{\rho}+b_2\|u'\|^{\varrho}\right) \\
&+\frac{t^{1-\beta}}{\Gamma(2-\beta)}\sum_{j=1}^{p}\frac{\left(t_j-t_{j-1}\right)^{\alpha_{j-1}-1}}{\Gamma\left(\alpha_{j-1}\right)}\left(\|h\|+b_1\|u\|^{\rho}+b_2\|u'\|^{\varrho}\right)+\frac{t^{1-\beta}}{\Gamma(2-\beta)}L_3 \\
&\leq \left(\frac{T^*}{\Gamma^*}+\Delta_3\right)\left(\|h\|+b_1\|u\|^{\rho}+b_2\|u'\|^{\varrho}\right)+\frac{t^{1-\beta}}{\Gamma(2-\beta)}L_3,
\end{aligned}
$$

$$
\begin{aligned}
\left|F_k'\left(y,u,u'\right)(t)\right| \leq &\left(\frac{T^{\alpha_k-1}}{\Gamma\left(\alpha_k\right)}+\sum_{j=1}^{k}\frac{\left(t_j-t_{j-1}\right)^{\alpha_{j-1}-1}}{\Gamma\left(\alpha_{j-1}\right)}\right)\left(\|h\|+b_1\|u\|^{\rho}+b_2\|u'\|^{\varrho}\right) \\
&+L_3.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\left|(\Theta u)(t)\right| &\leq \left(1+\left|A_0\right|+\left|B_0\right|T+\left(\left|\sigma_2\right|+1\right)\left(\left|A_p\right|+\left|B_p\right|T\right)\right) \\
&\times\left(\left(\frac{T^*}{\Gamma^*}+\Delta_1+\Delta_2\right)\left(\|h\|+b_1\|u\|^{\rho}+b_2\|u'\|^{\varrho}\right)+pL_2+pTL_3\right) \\
&+\left|\mu_2\right|\left(\left|A_p\right|+\left|B_p\right|T\right)\left(\left(\frac{T^*}{\Gamma^*}+\Delta_3\right)\left(\|h\|+b_1\|u\|^{\rho}+b_2\|u'\|^{\varrho}\right)+\frac{T^{1-\beta}}{\Gamma(2-\beta)}L_3\right) \\
&\leq C_1+C_2b_1\|u\|^{\rho}+C_2b_2\|u'\|^{\varrho},
\end{aligned}
$$

and

$$\begin{aligned}\left|(\Theta u)'(t)\right| &\leq \left(\frac{T^{\alpha_k-1}}{\Gamma(\alpha_k)}+\sum_{j=1}^{k}\frac{(t_j-t_{j-1})^{\alpha_{j-1}-1}}{\Gamma(\alpha_{j-1})}\right)\left(\|h\|+b_1\|u\|^{\rho}+b_2\|u'\|^{\varrho}\right)+L_3\\ &+\left(|B_0|+|\sigma_2||B_p|+|B_p|\right)\left(\frac{T^*}{\Gamma^*}+\Delta_1+\Delta_2\right)\left(\|h\|+b_1\|u\|^{\rho}+b_2\|u'\|^{\varrho}\right)+pL_2+pTL_3\\ &+|\mu||B_p|\left(\frac{T^*}{\Gamma^*}+\Delta_3\right)\left(\|h\|+b_1\|u\|^{\rho}+b_2\|u'\|^{\varrho}\right)+\frac{T^{1-\beta}}{\Gamma(2-\beta)}L_3\\ &\leq C_1+C_2b_1\|u\|^{\rho}+C_2b_2\|u'\|^{\varrho}.\end{aligned}$$

Thus

$$\|(\Theta u)\|_{PC^1}\leq C_1+C_2b_1R^{\rho}+C_2b_2R^{\varrho}\leq\frac{R}{3}+\frac{R}{3}+\frac{R}{3}=R.$$

■

Theorem 5 *Assume that the conditions (A_3) and (A_4) are satisfied. Then the problem (1) has at least one solution.*

Proof. Firstly, we prove that Θ: $PC^1([0,T],R)\to PC^1([0,T],R)$ is completely continuous operator. It is clear that, the continuity of functions $f, I_k$and J_k implies the continuity of the operator Θ.

Let $\Omega\subset PC^1([0,T],R)$ be bounded. Then there exist positive constants such that

$$|f(t,u,u')|\leq L_1,\quad |I_k(u)|\leq L_2,\quad |J_k(u)|\leq L_3,$$

for all $u\in\Omega$. Thus, for any $u\in\Omega$,we have

$$\left|F_k(f,u,u')\right|\leq L_1\left(\frac{T^*}{\Gamma^*}+\Delta_1+\Delta_2\right)+pL_2+L_3pT,$$

Similarly,

$$\left|G_k\left(f,u,u'\right)(t)\right|\leq L_1\frac{T^*}{\Gamma^*}+L_1\Delta_1\frac{T^{1-\beta}}{\Gamma(2-\beta)}+\frac{T^{1-\beta}}{\Gamma(2-\beta)}pL_3.$$

It follows that

$$|(\Theta u)(t)|\leq\Lambda_{\Theta}^{1}(\text{constant}).$$

In a like manner,

$$\left|F_k'\left(f,u,u'\right)(t)\right|\leq L_1\left(\frac{T^*}{\Gamma^*}+\Delta_4\right)+L_3p.$$

It follows that

$$\begin{aligned}\left|(\Theta u)'(t)\right|&\leq L_1\left(\frac{T^*}{\Gamma^*}+\Delta_4\right)+L_3p\\ &+\left(|\sigma_0|+|\sigma_2||B_p|+|B_p|\right)\Lambda_F+|\mu_2||B_p|\Lambda_G=:\Lambda_{\Theta}^{2}\end{aligned}$$

Thus

$$\|\Theta u\|_{PC^1}\leq\Lambda_{\Theta}^{1}+\Lambda_{\Theta}^{2}\ =constant.$$

On the other hand, for τ_1 , $\tau_2\in[t_k,t_{k+1}]$ with $\tau_1\leq\tau_2$ and we have

$$|(\Theta u)(\tau_1)-(\Theta u)\tau_2|\leq\int_{\tau_1}^{\tau_2}\left|(\Theta u)'(s)\right|ds\leq\Lambda_{\Theta}(\tau_2-\tau_1).$$

Similarly
$$(\Theta u)^{'}(\tau_2) - (\Theta u)^{'}(\tau_1) \leq \Pi_{\Theta}(\tau_2 - \tau_1),$$
where Π_{Θ} is a constant. This implies that Θu is equicontinuous on all $(t_k, t_{k+1}]$, $k = 0, 1, ..., p$. Consequently, Arzela-Ascoli theorem ensures us that the operator Θ is a completely continuous operator and by Lemma 4 Θ: $B(0, R) \to B(0, R)$. Hence, we conclude that Θ: $B(0, R) \to B(0, R)$ is completely continuous. It follows from the Schauder fixed point theorem that Θhas at least one fixed point. That is problem (1) has at least one solution. ■

Example 1. For $p = 1$, $t_1 = \frac{1}{4}$, $T = 1$, $\beta = \frac{1}{2}$, $\mu_1 = 2$, $\sigma_1 = \frac{1}{2}$, $\mu_2 = 3$, $\sigma_1 = \frac{1}{10}$, $\eta_1 = \frac{1}{5}$, $\eta_2 = \frac{2}{3}$, $\alpha_0 = \frac{3}{2}$, $\alpha_k = \frac{3}{2}$, we consider the following impulsive multi-orders fractional differential equation:

$$\begin{cases} {}^{C}D_{t_k}^{\alpha_k} u(t) = \frac{1}{100} \cos u(t) + \frac{|u'(t)|}{|u'(t)|+100} + t, & 0 < t < 1, \ t \neq \frac{1}{4}, \\ \Delta u\left(\frac{1}{4}\right) = \frac{|u(\frac{1}{4})|}{|u(\frac{1}{4})|+50}, \quad \Delta u'\left(\frac{1}{4}\right) = \frac{|u'(\frac{1}{4})|}{|u'(\frac{1}{4})|+70}, \\ u(0) + 2\ {}^{C}D_{0^+} u(0) = \frac{1}{2} u\left(\frac{1}{5}\right), \\ u(1) + 2\ {}^{C}D_{0^+} u(1) = \frac{1}{2} u\left(\frac{2}{3}\right). \end{cases} \tag{6}$$

It is clear that
$$|f(t, x, x_1) - f(t, y, y_1)| \leq 0.02 \left(|x - y| + |x_1 - y_1|\right), \ 0 \leq t \leq 1, \quad x, y, x_1, y_1 \in \mathbb{R}.$$
One can easily calculate that
$$\Lambda = 0.2178 < 1.$$
Therefore, all the assumptions of Theorem 3 hold. Thus, by Theorem 3, the impulsive multi-orders fractional boundary value problem (6) has a unique solution on $[0, 1]$.

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Convergence of modification of the Kantorovich-type q-Bernstein-Schurer operators

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Abstract. In this paper, we introduce a new modification of Kantorovich-type Bernstein-Schurer operators $K^*_{n,p,q}(f;x)$ based on the concept of q-integers. We investigate statistical approximation properties, establish a local approximation theorem, give a convergence theorem for the Lipschitz continuous functions and obtain a Voronovskaja-type theorem. Furthermore, we also give some illustrative graphics and some numerical examples for comparisons for the convergence of operators to some function.

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1 Introduction

In 2015, Agrawal, Finta and Kumar [1] introduced a new Kantorovich-type generalization of the q-Bernstein-Schurer operators, they gave the basic convergence theorem, obtained the local direct results, estimated the rate of convergence and so on. The operators are defined as

$$K_{n,p,q}(f;x) = [n+1]_q \sum_{k=0}^{n+p} b_{n+p,k}(q;x) q^{-k} \int_{\frac{[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} f(t) d_q^R t, \tag{1}$$

where $b_{n+p,k}(q;x)$ is defined by

$$b_{n+p,k}(q;x) = \begin{bmatrix} n+p \\ k \end{bmatrix}_q x^k (1-x)_q^{n+p-k}. \tag{2}$$

They obtained the following lemma of the moments.

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Lemma 1.1. *(See [1], Lemma 2.1) The following equalities hold*

$$K_{n,p,q}(1;x) = 1; \tag{3}$$

$$K_{n,p,q}(t;x) = \frac{2q[n+p]_q}{[2]_q[n+1]_q}x + \frac{1}{[2]_q[n+1]_q}; \tag{4}$$

$$K_{n,p,q}\left(t^2;x\right) = \frac{q^2(1+q+4q^2)[n+p]_q[n+p-1]_q}{[2]_q[3]_q[n+1]_q^2}x^2 + \frac{q(3+5q+4q^2)[n+p]_q}{[2]_q[3]_q[n+1]_q^2}x + \frac{1}{[3]_q[n+1]_q^2}. \tag{5}$$

Apparently, these operators reproduce only constant functions. In present paper, we will introduce a new modification of Kantorovich-type q-Bernstein-Schurer operators $K^*_{n,p,q}(f;x)$ which will be defined in (7). The advantage of these new operators is that they reproduce not only constant functions but also linear functions. We will investigate statistical approximation properties, establish a local approximation theorem, give a convergence theorem for the Lipschitz continuous functions and obtain a Voronovskaja-type theorem. Furthermore, we will give some illustrative graphics and some numerical examples for comparisons for the convergence of operators to some function. We may observe that the new operators $K^*_{n,p,q}(f;x)$ give a better approximation to $f(x)$ than $K_{n,p,q}(f;x)$.

Before introducing the operators, we mention certain definitions based on q-integers, detail can be found in [4, 5]. For any fixed real number $0 < q \le 1$ and each nonnegative integer k, we denote q-integers by $[k]_q$, where

$$[k]_q = \begin{cases} \frac{1-q^k}{1-q}, & q \ne 1; \\ k, & q = 1. \end{cases}$$

Also q-factorial and q-binomial coefficients are defined as follows:

$$[k]_q! = \begin{cases} [k]_q[k-1]_q...[1]_q, & k = 1,2,...; \\ 1, & k = 0, \end{cases}, \quad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}, \quad (n \ge k \ge 0).$$

For $x \in [0,1]$ and $n \in \mathbb{N}_0$, we recall that

$$(1-x)_q^n = \begin{cases} 1, & n = 0; \\ \prod_{j=0}^{n-1}\left(1-q^jx\right) = (1-x)(1-qx)...\left(1-q^{n-1}x\right), & n = 1,2,... \end{cases}.$$

The Riemann-type q-integral is defined by

$$\int_a^b f(t)d_q^Rt = (1-q)(b-a)\sum_{j=0}^{\infty} f\left(a+(b-a)q^j\right)q^j, \tag{6}$$

where the real numbers a, b and q satisfy that $0 \le a < b$ and $0 < q < 1$.

For $f \in C(I)$, $I = [0, 1+p]$, $p \in \mathbb{N}_0$, $q \in (0,1)$ and $n \in \mathbb{N}$, we introduce the modification of Kantorovich-type q-Bernstein-Schurer operators as follows:

$$K^*_{n,p,q}(f;x) = [n+1]_q \sum_{k=0}^{n+p} b_{n+p,k}\left(q;u(x)\right)q^{-k}\int_{\frac{[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} f(t)d_q^Rt, \tag{7}$$

where $b_{n+p,k}(q;x)$ is defined by (2), and

$$u(x)=\frac{[2]_q[n+1]_qx-1}{2q[n+p]_q},\quad \left(\frac{1}{[2]_q[n+1]_q}\le x\le\frac{1+2q[n+p]_q}{[2]_q[n+1]_q}\right). \tag{8}$$

2 Auxiliary Results

In order to obtain the approximation properties, We need the following lemmas:

Lemma 2.1. *For the modification of Kantorovich-type q-Bernstein-Schurer operators* (7), *using lemma 1.1, by some easily computations we have*

$$K^*_{n,p,q}(1;x) = 1, \tag{9}$$
$$K^*_{n,p,q}(t;x) = x, \tag{10}$$
$$\begin{aligned} K^*_{n,p,q}\left(t^2;x\right) &= \frac{[2]_q\left(1+q+4q^2\right)[n+p-1]_qx^2}{4[3]_q[n+p]_q}+\frac{\left(3+5q+4q^2\right)x}{2[3]_q[n+1]_q}\\ &-\frac{\left(1+q+4q^2\right)[n+p-1]_qx}{2[3]_q[n+1]_q[n+p]_q}+\frac{\left(1+q+4q^2\right)[n+p-1]_q}{4[2]_q[3]_q[n+1]_q^2[n+p]_q}\\ &-\frac{3+5q+4q^2}{2[2]_q[3]_q[n+1]_q^2}+\frac{1}{[3]_q[n+1]_q^2}. \end{aligned}\tag{11}$$

Remark 2.2. *Let $\{q_n\}$ denotes a sequence such that $0<q_n<1$. Then, by Bohman and Korovkin Theorem, for any $f\in C(I)$, operators $K^*_{n,p,q}(f;x)$ converge uniformly to $f(x)$, if and only if $\lim_{n\to\infty}q_n=1$.*

Lemma 2.3. *For the modification of Kantorovich-type q-Bernstein-Schurer operators* (7), *we have*

$$K^*_{n,p,q}(t-x;x) = 0, \tag{12}$$
$$K^*_{n,p,q}\left((t-x)^2;x\right) \le \frac{\left(q^2+4q^3-2q-3\right)x^2}{4[3]_q}+\frac{(1+2q)x}{[3]_q[n+1]_q}+\frac{(3+5q+4q^2)x}{2[3]_q[n+1]_q[n+p]_q} \tag{13}$$
$$\le \frac{(1+2q)x}{[3]_q[n+1]_q}+\frac{(3+5q+4q^2)x}{2[3]_q[n+1]_q[n+p]_q}, \tag{14}$$
$$K^*_{n,p,q}\left((t-x)^4;x\right) \le O\left(\frac{1}{[n]_q^2}\right). \tag{15}$$

Proof. By (9) and (10), we get (12). Using (10), (11) and some computations, we have

$$\begin{aligned} &K^*_{n,p,q}\left((t-x)^2;x\right)\\ =\ & K^*_{n,p,q}(t^2;x)-2xK^*_{n,p,q}(t;x)+x^2\\ \le\ & \frac{\left(q^2+4q^3-2q-3\right)x^2}{4[3]_q}+\frac{(1+2q)[n+p-1]_qx}{[3]_q[n+1]_q[n+p]_q}+\frac{q^{n+p-1}(3+5q+4q^2)x}{2[3]_q[n+1]_q[n+p]_q}\\ \le\ & \frac{\left(q^2+4q^3-2q-3\right)x^2}{4[3]_q}+\frac{(1+2q)x}{[3]_q[n+1]_q}+\frac{(3+5q+4q^2)x}{2[3]_q[n+1]_q[n+p]_q}. \end{aligned}$$

Indeed, using the similar method for estimate $K_{n,p,q}\left((t-x)^4;x\right)$ in [1, P. 229], we have

$$\begin{aligned}
&K^*_{n,p,q}\left((t-x)^4;x\right)\\
&\le 64\sum_{k=0}^{n+p} b_{n+p,k}(q;u(x))\left(\frac{[k]_q}{[n+1]_q}-\frac{[k]_q}{[n+p]_q}\right)^4+64\sum_{k=0}^{n+p} b_{n+p,k}(q;u(x))\left(\frac{[k]_q}{[n+p]_q}-x\right)^4\\
&\quad+\frac{8}{[5]_q}\sum_{k=0}^{n+p} b_{n+p,k}(q;u(x))\left(\frac{q^k}{[n+1]_q}\right)^4\\
&\le 64\sum_{k=0}^{n+p} b_{n+p,k}(q;u(x))\left(\frac{[k]_q}{[n+p]_q}\right)^4\left[\frac{q^n([p]_q-1)}{[n+1]_q}\right]^4\\
&\quad+64\sum_{k=0}^{n+p} b_{n+p,k}(q;u(x))\left(\frac{[k]_q}{[n+p]_q}-\frac{[2]_q[n+1]_qx-1}{2q[n+p]_q}+\frac{[2]_q[n+1]_qx-1}{2q[n+p]_q}-x\right)^4\\
&\quad+\frac{8}{[5]_q}\sum_{k=0}^{n+p} b_{n+p,k}(q;u(x))\left(\frac{q^k}{[n+1]_q}\right)^4\\
&\le C_1\frac{([p]_q-1)^4}{[n]_q^2}+512\sum_{k=0}^{n+p} b_{n+p,k}(q;u(x))\left((t-u(x))^4;x\right)\\
&\quad+512\sum_{k=0}^{n+p} b_{n+p,k}(q;u(x))\left(\frac{[2]_q[n+1]_qx-1}{2q[n+p]_q}-x\right)^4+\frac{C_2}{[n]_q^2},
\end{aligned}$$

where $u(x)$ is defined in (8), C_1 and C_2 are some positive constants. Thus,

$$\begin{aligned}
&K^*_{n,p,q}\left((t-x)^4;x\right)\\
&\le C_1\frac{([p]_q-1)^4}{[n]_q^2}+512\frac{C_3}{[n]_q^2}+512\left[\frac{[2]_q\left([n]_q+q^n\right)x-1-2q\left([n]_q+q^n[p]_q\right)x}{2q[n+p]_q}\right]^4+\frac{C_2}{[n]_q^2}\\
&= C_1\frac{([p]_q-1)^4}{[n]_q^2}+512\frac{C_3}{[n]_q^2}+512\left[\frac{\left(1+q^{n+1}-2q^{n+1}[p]_q\right)x-1}{2q[n+p]_q}\right]^4+\frac{C_2}{[n]_q^2}=O\left(\frac{1}{[n]_q^2}\right),
\end{aligned}$$

where C_3 is a positive constant, lemma 2.3 is proved. □

3 Statistical approximation properties

In this section, we present the statistical approximation properties of the operator $K^*_{n,p,q}(f;x)$.

Let K be a subset of $\mathbb{N}$, the set of all natural numbers. The density of K is defined by $\delta(K) := \lim_n \frac{1}{n}\sum_{k=1}^{n}\chi_K(k)$ provided the limit exists, where χ_K is the characteristic function of K. A sequence $x := \{x_n\}$ is called statistically convergent to a number L if, for every $\varepsilon > 0$, $\delta\{n\in\mathbb{N} : |x_n - L| \ge \varepsilon\} = 0$. Let $A := (a_{jn}), j, n = 1, 2, ...$ be an infinite summability matrix. For a given sequence $x := \{x_n\}$, the $A-$transform of x, denoted by $Ax := ((Ax)_j)$, is given by $(Ax)_j = \sum_{k=1}^{\infty} a_{jn}x_n$ provided the series converges for each

j. We say that A is regular if $\lim_n (Ax)_j = L$ whenever $\lim x = L$. Assume that A is a non-negative regular summability matrix. A sequence $x = \{x_n\}$ is called A-statistically convergent to L provided that for every $\varepsilon > 0$, $\lim_j \sum_{n:|x_n - L|\geq\varepsilon} a_{jn} = 0$. We denote this limit by $st_A - \lim_n x_n = L$. For $A = C_1$, the Cesàro matrix of order one, A-statistical convergence reduces to statistical convergence. It is easy to see that every convergent sequence is statistically convergent but not conversely.

We consider a sequence $q := \{q_n\}$ for $0 < q_n < 1$ satisfying

$$st_A - \lim_n q_n = 1. \tag{16}$$

If $e_i = t^i,\ t \in \mathbb{R}^+,\ i = 0, 1, 2, ...$ stands for the ith monomial, then we have

Theorem 3.1. *Let $A = (a_{nk})$ be a non-negative regular summability matrix and $q := \{q_n\}$ be a sequence satisfying* (16)*, then for all $f \in C(I)$, $x \in [0, 1]$, we have*

$$st_A - \lim_n ||K^*_{n,p,q} f - f||_{C(I)} = 0. \tag{17}$$

Proof. Obviously

$$st_A - \lim_n ||K^*_{n,p,q_n}(e_i) - e_i||_{C(I)} = 0. \quad (i = 0, 1) \tag{18}$$

By (11) and (13), we have

$$\left|K^*_{n,p,q_n}(e_2; x) - e_2(x)\right| \leq \frac{1 + 2q_n}{[3]_{q_n}[n + 1]_{q_n}} + \frac{3 + 5q_n + 4q_n^2}{2[3]_{q_n}[n + 1]_{q_n}[n + p]_{q_n}}.$$

Now for a given $\varepsilon > 0$, let us define the following sets:

$$U := \left\{k : \left|\left|K^*_{n,p,q_k}(e_2) - e_2\right|\right|_{C(I)} \geq \varepsilon\right\},\ U_1 := \left\{k : \frac{1 + 2q_k}{[3]_{q_k}[n + 1]_{q_k}} \geq \frac{\varepsilon}{2}\right\},$$
$$U_2 := \left\{k : \frac{3 + 5q_k + 4q_k^2}{2[3]_{q_k}[n + 1]_{q_k}[n + p]_{q_k}} \geq \frac{\varepsilon}{2}\right\}.$$

Then one can see that $U \subseteq U_1 \cup U_2$, so we have

$$\begin{aligned}\delta\left\{k \leq n : ||K^*_{n,p,q_k}(e_2) - e_2||_{C(I)}\right\} \quad &\leq \quad \delta\left\{k \leq n : \frac{1 + 2q_k}{[3]_{q_k}[n + 1]_{q_k}} \geq \frac{\varepsilon}{2}\right\} \\ &\quad + \delta\left\{k \leq n : \frac{3 + 5q_k + 4q_k^2}{2[3]_{q_k}[n + 1]_{q_k}[n + p]_{q_k}} \geq \frac{\varepsilon}{2}\right\},\end{aligned}$$

since $st_A - \lim_n q_n = 1$, we have

$$st_A - \lim_n \frac{1 + 2q_n}{[3]_{q_n}[n + 1]_{q_n}} = 0, \quad st_A - \lim_n \frac{3 + 5q_n + 4q_n^2}{2[3]_{q_n}[n + 1]_{q_n}[n + p]_{q_n}} = 0,$$

which implies that the right-hand side of the above inequality is zero, thus we have

$$st_A - \lim_n ||K^*_{n,p,q_n}(e_2) - e_2||_{C(I)} = 0. \tag{19}$$

Combining (18) and (19), theorem 3.1 follows from the Korovkin-type statistical approximation theorem established in [3], the proof is completed. □

4 Local approximation properties

Let $f \in C(I)$, endowed with the norm $||f|| = \sup_{x\in I} |f(x)|$. The Peetre's $K-$functional is defined by

$$K_2(f;\delta) = \inf_{g\in C^2} \left\{||f-g|| + \delta||g''||\right\},$$

where $\delta > 0$ and $C^2 = \{g \in C(I) : g', g'' \in C(I)\}$. By [2, p.177, Theorem 2.4], there exits an absolute constant $C > 0$ such that

$$K_2(f;\delta) \le C\omega_2(f;\sqrt{\delta}), \tag{20}$$

where

$$\omega_2(f;\delta) = \sup_{0<h\le\delta} \sup_{x,x+h,x+2h\in I} |f(x+2h) - 2f(x+h) + f(x)|$$

is the second order modulus of smoothness of $f \in C(I)$.

Now we give a direct local approximation theorem for the operators $K^*_{n,p,q}(f,x)$.

Theorem 4.1. *For $q \in (0,1)$, $x \in [0,1]$ and $f \in C(I)$, we have*

$$\left|K^*_{n,p,q}(f,x) - f(x)\right| \le C\omega_2\left(f; \sqrt{\frac{\left(q^2+4q^3-2q-3\right)x^2}{8[3]_q} + \frac{(1+2q)x}{2[3]_q[n+1]_q} + \frac{(3+5q+4q^2)x}{4[3]_q[n+1]_q[n+p]_q}}\right), \tag{21}$$

where C is a positive constant.

Proof. Let $g \in C^2$. By Taylor's expansion

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)du,$$

and (12), we get

$$K^*_{n,p,q}(g;x) = g(x) + K^*_{n,p,q}\left(\int_x^t (t-u)g''(u)du; x\right).$$

Hence, by (13), we have

$$\begin{aligned}
& \left|K^*_{n,p,q}(g;x) - g(x)\right| \\
\le & \left|K^*_{n,p,q}\left(\int_x^t (t-u)g''(u)du; x\right)\right| \\
\le & K^*_{n,p,q}\left(\left|\int_x^t (t-u)|g''(u)|du\right|; x\right) \\
\le & K^*_{n,p,q}\left((t-x)^2; x\right)||g''|| \\
\le & \left[\frac{\left(q^2+4q^3-2q-3\right)x^2}{4[3]_q} + \frac{(1+2q)x}{[3]_q[n+1]_q} + \frac{(3+5q+4q^2)x}{2[3]_q[n+1]_q[n+p]_q}\right]||g''||. \qquad (22)
\end{aligned}$$

On the other hand, using lemma 2.1, we have

$$\left|K^*_{n,p,q}(f;x)\right| \leq [n+1]_q \sum_{k=0}^{n+p} b_{n+p,k}\left(q;u(x)\right) q^{-k} \int_{\frac{[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} |f(t)| d_q^R t \leq ||f||. \tag{23}$$

Now (22) and (23) imply

$$\begin{aligned}
& \left|K^*_{n,p,q}(f;x) - f(x)\right| \\
\leq\ & \left|K^*_{n,p,q}(f-g;x) - (f-g)(x)\right| + \left|K^*_{n,p,q}(g;x) - g(x)\right| \\
\leq\ & 2||f-g|| + \left[\frac{\left(q^2+4q^3-2q-3\right)x^2}{4[3]_q} + \frac{(1+2q)x}{[3]_q[n+1]_q} + \frac{(3+5q+4q^2)x}{2[3]_q[n+1]_q[n+p]_q}\right] ||g''||.
\end{aligned}$$

Hence taking infimum on the right hand side over all $g \in C^2$, we get

$$\left|K^*_{n,p,q}(f;x) - f(x)\right| \leq 2K_2\left(f; \frac{\left(q^2+4q^3-2q-3\right)x^2}{8[3]_q} + \frac{(1+2q)x}{2[3]_q[n+1]_q} + \frac{(3+5q+4q^2)x}{4[3]_q[n+1]_q[n+p]_q}\right).$$

By (20), for every $q \in (0,1)$, we have

$$|K^*_{n,p,q}(f;x) - f(x)| \leq C\omega_2\left(f; \sqrt{\frac{\left(q^2+4q^3-2q-3\right)x^2}{8[3]_q} + \frac{(1+2q)x}{2[3]_q[n+1]_q} + \frac{(3+5q+4q^2)x}{4[3]_q[n+1]_q[n+p]_q}}\right).$$

This completes the proof of theorem 4.1. □

Remark 4.2. *For any fixed $x \in [0,1]$, $p \in \mathbb{N}_0$ and $n \in \mathbb{N}$, let $q := \{q_n\}$ be a sequence satisfying $0 < q_n < 1$ and $\lim_n q_n = 1$, we have*

$$\lim_{n\to\infty}\left[\frac{\left(q_n^2+4q_n^3-2q_n-3\right)x^2}{8[3]_{q_n}} + \frac{(1+2q_n)x}{2[3]_{q_n}[n+1]_{q_n}} + \frac{(3+5q_n+4q_n^2)x}{4[3]_{q_n}[n+1]_{q_n}[n+p]_{q_n}}\right] = 0.$$

These gives us a rate of pointwise convergence of the operators $K^*_{n,p,q_n}(f;x)$ to $f(x)$.

Next we study the rate of convergence of the operators $K^*_{n,p,q}(f;x)$ with the help of functions of Lipschitz class $Lip_M(\alpha)$, where $M > 0$ and $0 < \alpha \leq 1$. A function f belongs to $Lip_M(\alpha)$ if

$$|f(y) - f(x)| \leq M|y-x|^\alpha \quad (y, x \in \mathbb{R}). \tag{24}$$

We have the following theorem.

Theorem 4.3. *Let $q := \{q_n\}$ be a sequence satisfying $0 < q_n < 1$, $\lim\limits_n q_n = 1$ and $f \in Lip_M(\alpha)$, $0 < \alpha \leq 1$. Then we have*

$$\left|K^*_{n,p,q}(f;x) - f(x)\right| \leq M\left[\frac{\left(q^2+4q^3-2q-3\right)x^2}{4[3]_q} + \frac{(1+2q)x}{[3]_q[n+1]_q} + \frac{(3+5q+4q^2)x}{2[3]_q[n+1]_q[n+p]_q}\right]^{\frac{\alpha}{2}}. \tag{25}$$

Proof. Since $K^*_{n,p,q}$ is a linear positive operator and $f \in Lip_M(\alpha)$ $(0<\alpha\le 1)$, we have

$$\begin{aligned}
& |K^*_{n,p,q}(f;x)-f(x)| \\
\le\ & K^*_{n,p,q}\left(|f(t)-f(x)|;x\right) \\
=\ & [n+1]_q\sum_{k=0}^{n+p} b_{n+p,k}(q;u(x))q^{-k}\int_{\frac{[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} |f(t)-f(x)|d_q^R t \\
\le\ & M[n+1]_q\sum_{k=0}^{n+p} b_{n+p,k}(q;u(x))q^{-k}\int_{\frac{[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} |t-x|^{\alpha} d_q^R t \\
\le\ & M[n+1]_q\sum_{k=0}^{n+p} b_{n+p,k}(q;u(x))q^{-k}\left(\int_{\frac{[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} \left[|t-x|^{\alpha}\right]^{\frac{2}{\alpha}} d_q^R t\right)^{\frac{\alpha}{2}}\left(\int_{\frac{[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} d_q^R t\right)^{\frac{2-\alpha}{2}} \\
=\ & M[n+1]_q\sum_{k=0}^{n+p} b_{n+p,k}(q;u(x))q^{-k}\left(\int_{\frac{[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} (t-x)^2 d_q^R t\right)^{\frac{\alpha}{2}}\left(\frac{q^k}{[n+1]_q}\right)^{\frac{2-\alpha}{2}} \\
=\ & M\sum_{k=0}^{n+p} b_{n+p,k}(q;u(x))\left(\int_{\frac{[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} (t-x)^2 d_q^R t\right)^{\frac{\alpha}{2}}\left(\frac{[n+1]_q}{q^k}\right)^{\frac{\alpha}{2}} \\
=\ & M\sum_{k=0}^{n+p} \left[b_{n+p,k}(q;u(x))\right]^{\frac{2-\alpha}{2}}\left([n+1]_q b_{n+p,k}(q;u(x))q^{-k}\int_{\frac{[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} (t-x)^2 d_q^R t\right)^{\frac{\alpha}{2}}.
\end{aligned}$$

Applying Hölder's inequality for sums, we obtain

$$\begin{aligned}
& |K^*_{n,p,q}(f;x)-f(x)| \\
\le\ & M\left(\sum_{k=0}^{n+p} b_{n+p,k}(q;u(x))\right)^{\frac{2-\alpha}{2}}\left(\sum_{k=0}^{n+p}[n+1]_q b_{n+p,k}(q;u(x))q^{-k}\int_{\frac{[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} (t-x)^2 d_q^R t\right)^{\frac{\alpha}{2}} \\
=\ & M\left[K^*_{n,p,q}\left((t-x)^2;x\right)\right]^{\frac{\alpha}{2}}.
\end{aligned}$$

Thus, theorem 4.3 is proved. □

Now, we give a Voronovskaja-type asymptotic formula for $K^*_{n,p,q}(f;x)$ by means of the second and fourth central moments.

Theorem 4.4. *Let $q := \{q_n\}$ be a sequence satisfying $0<q_n<1$, $\lim_n q_n = 1$. For $f\in C^2(I)$, ($f(x)$ is a twice differentiable function in I), the following equality holds*

$$\lim_{n\to\infty}[n]_q\left(K^*_{n,p,q}(f;x)-f(x)\right)=\frac{f''(x)}{2}x. \tag{26}$$

Proof. Let $x\in[0,1]$ be fixed. By the Taylor formula, we may write

$$f(t)=f(x)+f'(x)(t-x)+\frac{1}{2}f''(x)(t-x)^2+r(t;x)(t-x)^2, \tag{27}$$

where $r(t;x)$ is the Peano form of the remainder, $r(t;x) \in C(I)$, using L'Hopital's rule, we have

$$\begin{aligned} \lim_{t\to x} r(t;x) &= \lim_{t\to x} \frac{f(t)-f(x)-f'(x)(t-x)-\frac{1}{2}f''(x)(t-x)^2}{(t-x)^2} \\ &= \lim_{t\to x} \frac{f'(t)-f'(x)-f''(x)(t-x)}{2(t-x)} = \lim_{t\to x} \frac{f''(t)-f''(x)}{2} = 0. \end{aligned}$$

Since (12), applying $K^*_{n,p,q}(f;x)$ to (27), we obtain

$$[n]_q \left(K^*_{n,p,q}(f;x) - f(x)\right) = \frac{1}{2}[n]_q f''(x) K^*_{n,p,q}\left((t-x)^2;x\right) + [n]_q K^*_{n,p,q}\left(r(t;x)(t-x)^2;x\right).$$

By the Cauchy-Schwarz inequality, we have

$$K^*_{n,p,q}\left(r(t;x)(t-x)^2;x\right) \le \sqrt{K^*_{n,p,q}\left(r^2(t;x);x\right)}\sqrt{K^*_{n,p,q}\left((t-x)^4;x\right)}. \tag{28}$$

Since $r^2(x;x)=0$, then it is obtained easily that $\lim_n K_{n,p,q}\left(r^2(t;x);x\right) = r^2(x;x) = 0$ by remark 2.2. Now, from (15), (28) and (14), we get immediately

$$\lim_{n\to\infty} [n]_q K^*_{n,p,q}\left(r(t;x)(t-x)^2;x\right) = 0, \quad \lim_{n\to\infty} \frac{1}{2}[n]_q f''(x) K^*_{n,p,q}\left((t-x)^2;x\right) = \frac{f''(x)}{2}x.$$

Thus, theorem 4.4 is proved. □

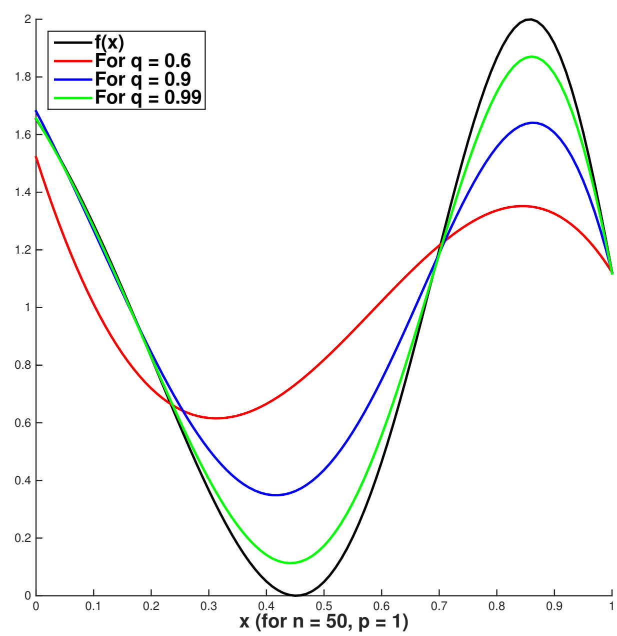

Figure 1: Convergence of $K^*_{n,p,q}(f;x)$ for $n=50$, $p=1$ and different values of q.

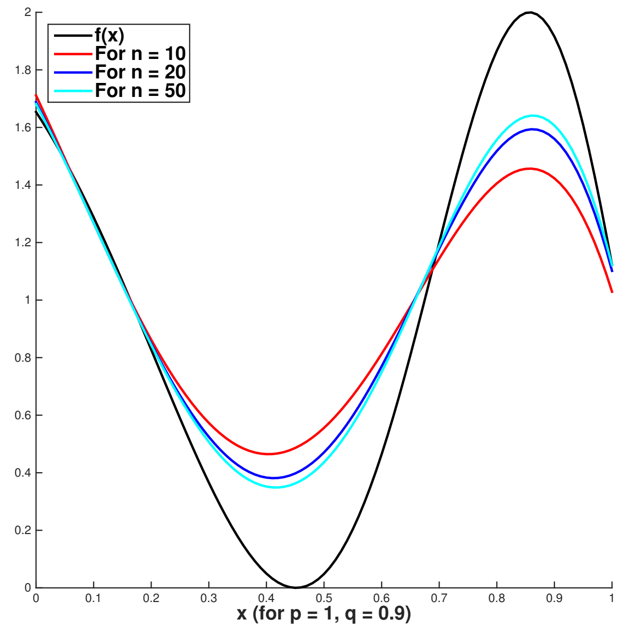

Figure 2: Convergence of $K^*_{n,p,q}(f;x)$ for $p = 1$, $q = 0.9$ and different values of n.

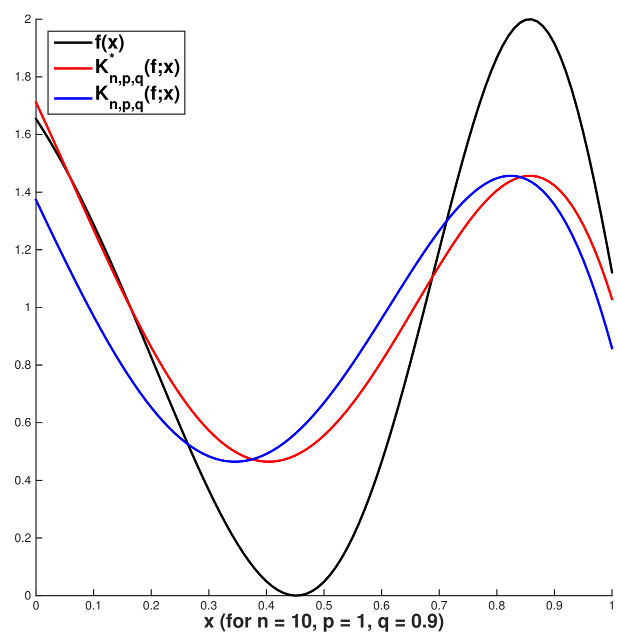

Figure 3: The graphs of $K^*_{n,p,q}(f;x)$ (red) and $K_{n,p,q}(f;x)$ (blue) for $n = 10$, $p = 1$ and $q = 0.9$.

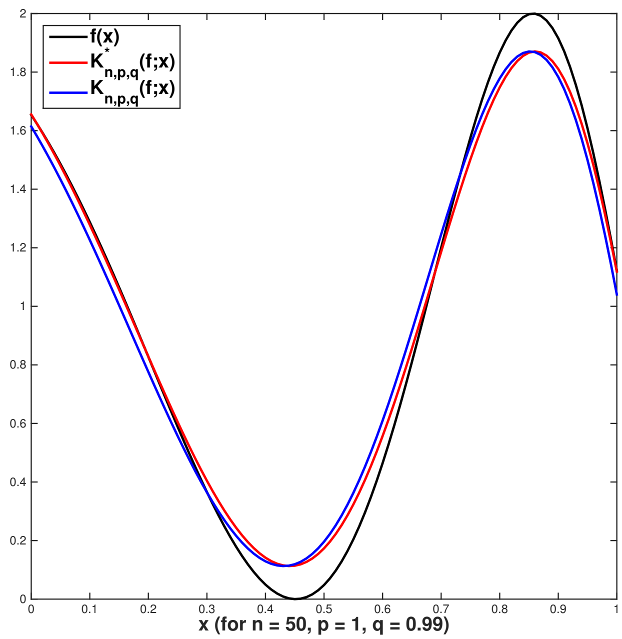

Figure 4: The graphs of $K^*_{n,p,q}(f;x)$ (red) and $K_{n,p,q}(f;x)$ (blue) for $n = 50$, $p = 1$ and $q = 0.99$.

5 Graphical and numerical examples analysis

In this section, we give several graphs and numerical examples to show the convergence of $K^*_{n,p,q}(f;x)$ to $f(x)$ with different values of n and q, and also compare the operators $K^*_{n,p,q}(f;x)$ with $K_{n,p,q}(f;x)$.

Let $f(x) = 1 - cos(4e^x)$, for $n = 50$ and $p = 1$, the graphs of $K^*_{n,p,q}(f;x)$ with different values of q are shown in figure 1. Moreover, for $p = 1$ and $q = 0.9$, the graphs of $K^*_{n,p,q}(f;x)$ with different values of n are shown in figure 2.

Figure 3 shows the graphs of $K^*_{n,p,q}(f;x)$ (red) and $K_{n,p,q}(f;x)$ (blue) for $n = 10$, $p = 1$ and $q = 0.9$. In figure 4, the values of n and q are replaced by 50 and 0.99, respectively.

In Table 1, we give the errors of the approximation to $f(x)$ of $K^*_{n,p,q}(f;x)$ and $K_{n,p,q}(f;x)$ with different values of q and n. We may observe that operators $K^*_{n,p,q}(f;x)$ give a better estimate than $K_{n,p,q}(f;x)$.

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Table 1: The errors of the approximation to $f(x)$ of $K^*_{n,p,q}(f;x)$ and $K_{n,p,q}(f;x)$.

$q = 1 - 1/m$	$\|f - K^*_{n,p,q}(f)\|_\infty$		$\|f - K_{n,p,q}(f)\|_\infty$	
	$n = 10$	$n = 50$	$n = 10$	$n = 50$
$m = 100$	0.4890	0.1318	0.5628	0.1587
$m = 200$	0.4856	0.1201	0.5638	0.1471
$m = 300$	0.4844	0.1163	0.5642	0.1436
$m = 400$	0.4838	0.1145	0.5645	0.1419
$m = 500$	0.4835	0.1134	0.5646	0.1409
$m = 600$	0.4832	0.1126	0.5647	0.1402
$m = 700$	0.4831	0.1121	0.5648	0.1397
$m = 800$	0.4829	0.1117	0.5648	0.1394
$m = 900$	0.4829	0.1114	0.5649	0.1391
$m = 1000$	0.4828	0.1112	0.5649	0.1389

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BARNES-TYPE DEGENERATE BERNOULLI AND EULER MIXED-TYPE POLYNOMIALS

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Abstract. In this paper, we consider the Barnes-type degenerate Bernoulli and Euler mixed-type polynomials. We present several explicit formulas and recurrence relations for these polynomials. Also, we establish a connection between our polynomials and several known families of polynomials.

1. Introduction

The goals of this paper are to use umbral calculus to obtain several new and interesting identities of Barnes-type degenerate Bernoulli and Euler mixed-type polynomials. The use of umbral calculus technique has been very attractive in numerous problems of mathematics (for example, see [1, 6, 8, 14, 18–21, 24]) and used in different areas of physics (for example, see [4, 5, 19]).

Let $r, s \in \mathbb{Z}_{>0}$. Throughout the paper we assume that $\mathbf{a} = a_1, \dots, a_r$ and $\mathbf{b} = b_1, b_2, \dots, b_s$. The *Barnes-type degenerate Bernoulli and Euler mixed-type polynomials* $\beta\mathcal{E}_n(\lambda, x|\mathbf{a}; \mathbf{b})$ with $a_1, \dots, a_r; b_1, \dots, b_s \neq 0$ are defined by the generating function

$$\prod_{i=1}^{r}\left(\frac{t}{(1+\lambda t)^{a_i/\lambda}-1}\right)\prod_{i=1}^{s}\left(\frac{2}{(1+\lambda t)^{b_i/\lambda}+1}\right)(1+\lambda t)^{x/\lambda} = \sum_{n\geq 0}\beta\mathcal{E}_n(\lambda, x|\mathbf{a}; \mathbf{b})\frac{t^n}{n!}. \tag{1}$$

If $x = 0$, $\beta\mathcal{E}_n(\lambda|\mathbf{a}; \mathbf{b}) = \beta\mathcal{E}_n(\lambda, 0|\mathbf{a}; \mathbf{b})$ are called the *Barnes-type degenerate Bernoulli and Euler mixed-type numbers*. Here, we recall that the polynomial $\beta_n(\lambda, x|\mathbf{a})$ with $a_1, \dots, a_r \neq 0$ are given by

$$\prod_{i=1}^{r}\left(\frac{t}{(1+\lambda t)^{a_i/\lambda}-1}\right)(1+\lambda t)^{x/\lambda} = \sum_{n\geq 0}\beta_n(\lambda, x|\mathbf{a})\frac{t^n}{n!} \tag{2}$$

are called the *Barnes-type degenerate Bernoulli polynomials* and studied in [7]. We note here that

$$\lim_{\lambda\to 0}\beta_n(\lambda, x|\mathbf{a}) = B_n(x|\mathbf{a}),$$
$$\lim_{\lambda\to\infty}\lambda^{-n}\beta_n(\lambda, \lambda x|\mathbf{a}) = (a_1a_2\cdots a_r)^{-1}b_n^{(r)}(x),$$

where $B_n(x|\mathbf{a})$ are the *Barnes-type Bernoulli polynomials* given by $\prod_{i=1}^{r}\left(\frac{t}{e^{a_it}-1}\right)e^{tx} = \sum_{n\geq 0}B_n(x|\mathbf{a})\frac{t^n}{n!}$ and $b_n^{(r)}(x)$ are the *Bernoulli polynomials of the second kind of order*

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r given by $\left(\frac{t}{\log(1+t)}\right)^r (1+t)^x = \sum_{n\geq 0} b_n^{(r)}(x)\frac{t^n}{n!}$ (see [12, 22]). Also, we recall that the polynomial $\mathcal{E}_n(\lambda, x|\mathbf{b})$ with $b_1, \ldots, b_s \neq 0$ are given by

$$\prod_{i=1}^{s}\left(\frac{2}{(1+\lambda t)^{b_i/\lambda}+1}\right)(1+\lambda t)^{x/\lambda} = \sum_{n\geq 0}\mathcal{E}_n(\lambda, x|\mathbf{b})\frac{t^n}{n!} \tag{3}$$

are called the *Barnes-type degenerate Euler polynomials* and studied in [11, 17, 25]. We denote $\mathcal{E}_n(\lambda, 0|\mathbf{b})$ by $\mathcal{E}_n(\lambda|\mathbf{b})$. We note here that

$$\lim_{\lambda\to 0}\mathcal{E}_n(\lambda, x|\mathbf{b}) = E_n(x|\mathbf{b}),$$
$$\lim_{\lambda\to\infty}\lambda^{-n}\mathcal{E}_n(\lambda, \lambda x|\mathbf{a}) = (x)_n = x(x-1)\cdots(x-n+1),$$

where $E_n(x|\mathbf{a})$ are the *Barnes-type Euler polynomials* given by (see [3])

$$\prod_{i=1}^{s}\left(\frac{2}{e^{b_i t}+1}\right)e^{tx} = \sum_{n\geq 0}E_n(x|\mathbf{b})\frac{t^n}{n!}.$$

In order to study the Barnes-type degenerate Bernoulli and Euler mixed-type polynomials, we use the umbral calculus technique. We denote the algebra of polynomials in a single variable x over $\mathbb{C}$ by Π. Let Π^* be the vector space of all linear functionals on Π. Let $\langle L|p(x)\rangle$ be the action of a linear functional $L \in \Pi^*$ on a polynomial $p(x)$, where we extend it as $\langle cL + c'L'|p(x)\rangle = c\langle L|p(x)\rangle + c'\langle L'|p(x)\rangle$, where $c, c' \in \mathbb{C}$ (see [22, 23]). Define

$$\mathcal{H} = \left\{ f(t) = \sum_{k\geq 0} a_k \frac{t^k}{k!} \mid a_k \in \mathbb{C} \right\} \tag{4}$$

to be the algebra of formal power series in a single variable t. The formal power series in the variable t defines a linear functional on Π by setting $\langle f(t)|x^n\rangle = a_n$, for all $n \geq 0$ (see [22, 23]). By (4), we have

$$\langle t^k|x^n\rangle = n!\delta_{n,k}, \text{ for all } n, k \geq 0, \text{ (see [22, 23])}, \tag{5}$$

where $\delta_{n,k}$ is the Kronecker's symbol. For $f_L(t) = \sum_{n\geq 0}\langle L|x^n\rangle\frac{t^n}{n!}$, by (5), we have that $\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle$. Thus, the map $L \mapsto f_L(t)$ is a vector space isomorphism from Π^* onto $\mathcal{H}$, namely $\mathcal{H}$ is thought of as set of both formal power series and linear functionals. We call $\mathcal{H}$ the *umbral algebra*. The *umbral calculus* is the study of umbral algebra.

The *order* $O(f(t))$ of the non-zero power series $f(t)$ is the smallest integer ℓ for which the coefficient of t^ℓ does not vanish (see [22, 23]). If $O(f(t)) = 1$ ($O(f(t)) = 0$) then $f(t)$ is called a *delta* (an *invertable*) series. If $O(f(t)) = 1$ and $O(g(t)) = 0$, then there exists a unique sequence $s_n(x)$ of polynomials such that $\langle g(t)(f(t))^k|s_n(x)\rangle = n!\delta_{n,k}$, where $n, k \geq 0$. The sequence $s_n(x)$ is called the *Sheffer sequence* for $(g(t), f(t))$, and we write $s_n(x) \sim (g(t), f(t))$ (see [22, 23]). For $f(t) \in \mathcal{H}$ and $p(x) \in \Pi$, we have that $\langle e^{yt}|p(x)\rangle = p(y)$, $\langle f(t)g(t)|p(x)\rangle = \langle g(t)|f(t)p(x)\rangle$, $f(t) = \sum_{n\geq 0}\langle f(t)|x^n\rangle\frac{t^n}{n!}$ and $p(x) = \sum_{n\geq 0}\langle t^n|p(x)\rangle\frac{x^n}{n!}$. Thus,

$$\langle t^k|p(x)\rangle = p^{(k)}(0), \quad \langle 1|p^{(k)}(x)\rangle = p^{(k)}(0), \tag{6}$$

where $p^{(k)}(0)$ denotes the k-th derivative of $p(x)$ with respect to x at $x=0$. So, by (6), we get that $t^k p(x) = p^{(k)}(x) = \frac{d^k}{dx^k}p(x)$, for all $k \geq 0$, (see [22, 23]). Let $s_n(x) \sim (g(t), f(t))$. Then we have

$$\frac{1}{g(\bar{f}(t))}e^{y\bar{f}(t)} = \sum_{n\geq 0} s_n(y)\frac{t^n}{n!}, \tag{7}$$

for all $y \in \mathbb{C}$, where $\bar{f}(t)$ is the compositional inverse of $f(t)$ (see [22, 23]). For $s_n(x) \sim (g(t), f(t))$ and $r_n(x) \sim (h(t), \ell(t))$, let $s_n(x) = \sum_{k=0}^{n} c_{n,k} r_k(x)$, then we have

$$c_{n,k} = \frac{1}{k!}\left\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))}(\ell(\bar{f}(t)))^k | x^n \right\rangle, \tag{8}$$

(see [22, 23]).

By the theory of Sheffer sequences, it is immediate that the Barnes-type degenerate Bernoulli and Euler mixed-type polynomial is the Sheffer sequence for the pair $g(t) = \left(\frac{\lambda}{e^{\lambda t}-1}\right)^r \prod_{i=1}^{r}\left(e^{a_i t}-1\right)\prod_{i=1}^{s}\left(\frac{e^{b_i t}+1}{2}\right)$ and $f(t) = \frac{1}{\lambda}(e^{\lambda t}-1)$. Thus

$$\beta\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) \sim \left(\left(\frac{\lambda}{e^{\lambda t}-1}\right)^r \prod_{i=1}^{r}(e^{a_i t}-1)\prod_{i=1}^{s}\left(\frac{e^{b_i t}+1}{2}\right), \frac{1}{\lambda}(e^{\lambda t}-1)\right). \tag{9}$$

The aim of the present paper is to present several new identities for Barnes-type degenerate Bernoulli and Euler mixed-type polynomials by the use of umbral calculus.

2. Explicit Expressions

In this section we suggest several explicit formulas for the Barnes-type degenerate Bernoulli and Euler mixed-type polynomials. To do so, we recall that the Stirling numbers $S_1(n,m)$ of the first kind are defined as $(x)_n = \sum_{m=0}^{n} S_1(n,m)x^m \sim (1, e^t - 1)$ or $\frac{1}{j!}(\log(1+t))^j = \sum_{\ell\geq j} S_1(\ell, j)\frac{t^\ell}{\ell!}$. Also, we recall that the Stirling numbers $S_2(n,m)$ of the second kind are defined by $\frac{(e^t-1)^k}{k!} = \sum_{\ell\geq k} S_2(\ell,k)\frac{t^\ell}{\ell!}$. Define $(x|\lambda)_n = \lambda^n (x/\lambda)_n$ to be $(x|\lambda)_n = x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda)$ with $(x|\lambda)_0 = 1$. Also, we define

$$P_{r,s}(t) = \prod_{i=1}^{r}\left(\frac{t}{(1+\lambda t)^{a_i/\lambda}-1}\right)\prod_{i=1}^{s}\left(\frac{2}{(1+\lambda t)^{b_i/\lambda}+1}\right)$$

and

$$Q_{r,s}(t) = \prod_{i=1}^{r}\left(\frac{t}{e^{a_i t}-1}\right)\prod_{i=1}^{s}\left(\frac{2}{e^{b_i t}+1}\right).$$

Theorem 2.1. *For all $n \geq 0$,*

$$\beta\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) = \sum_{j=0}^{n}\left(\sum_{\ell=j}^{n}\binom{n}{\ell}S_1(\ell,j)\lambda^{\ell-j}\beta\mathcal{E}_{n-\ell}(\lambda|\mathbf{a};\mathbf{b})\right)x^j.$$

Proof. By applying the conjugation representation for $s_n(x) \sim (g(t), f(t))$, that is,

$$s_n(x) = \sum_{j=0}^{n} \frac{1}{j!} \langle g(\bar{f}(t))^{-1} \bar{f}(t)^j | x^n \rangle x^j,$$

(see [22, 23]) we obtain

$$\langle g(\bar{f}(t))^{-1} \bar{f}(t)^j | x^n \rangle = \left\langle P_{r,s}(t) \left(\frac{\log(1+\lambda t)}{\lambda} \right)^j | x^n \right\rangle = \lambda^{-j} \left\langle P_{r,s}(t) | \left(\log(1+\lambda t) \right)^j x^n \right\rangle$$
$$= \lambda^{-j} \left\langle P_{r,s}(t) | j! \sum_{\ell \geq j} S_1(\ell, j) \lambda^\ell \frac{t^\ell}{\ell!} x^n \right\rangle.$$

Thus,

$$\frac{1}{j!} \langle g(\bar{f}(t))^{-1} \bar{f}(t)^j | x^n \rangle = \sum_{\ell=j}^{n} \binom{n}{\ell} S_1(\ell, j) \lambda^{\ell-j} \left\langle P_{r,s}(t) | x^{n-\ell} \right\rangle$$
$$= \sum_{\ell=j}^{n} \binom{n}{\ell} S_1(\ell, j) \lambda^{\ell-j} \left\langle \sum_{m \geq 0} \beta\mathcal{E}_m(\lambda | \mathbf{a}; \mathbf{b}) \frac{t^m}{m!} | x^{n-\ell} \right\rangle$$
$$= \sum_{\ell=j}^{n} \binom{n}{\ell} S_1(\ell, j) \lambda^{\ell-j} \beta\mathcal{E}_{n-\ell}(\lambda | \mathbf{a}; \mathbf{b}),$$

which completes the proof. □

Theorem 2.2. *For all* $n \geq 0$,

$$\beta\mathcal{E}_n(\lambda, x | \mathbf{a}; \mathbf{b}) = \lambda^n \sum_{m=0}^{n} \sum_{k=0}^{m} \frac{\binom{m}{k}}{\binom{k+r}{r}} S_1(n, m) S_2(k+r, r) \lambda^{k-m} BE_{m-k}(x | \mathbf{a}; \mathbf{b}),$$

where $BE_n(x|\mathbf{a}; \mathbf{b})$ *are the Barnes-type Bernoulli and Euler mixed-type polynomials with*

$$\prod_{i=1}^{r} \left(\frac{t}{e^{a_i t} - 1} \right) \prod_{i=1}^{s} \left(\frac{2}{e^{b_i t} + 1} \right) e^{xt} = \sum_{n=0}^{\infty} BE_n(x | \mathbf{a}; \mathbf{b}) \frac{t^n}{n!}$$

(see [26]).

Proof. By (9), we have

$$\left(\frac{\lambda}{e^{\lambda t} - 1} \right)^r \prod_{i=1}^{r} \left(e^{a_i t} - 1 \right) \prod_{i=1}^{s} \left(\frac{e^{b_i t} + 1}{2} \right) \beta\mathcal{E}_n(\lambda, x | \mathbf{a}; \mathbf{b}) \sim \left(1, \frac{1}{\lambda} (e^{\lambda t} - 1) \right), \tag{10}$$

which implies

$$\begin{aligned}
\beta\mathcal{E}_n(\lambda,x|\mathbf{a};\mathbf{b}) &= \sum_{m=0}^{n} S_1(n,m)\lambda^{n-m}Q_{r,s}(t)\left(\frac{e^{\lambda t}-1}{\lambda t}\right)^r x^m \\
&= \sum_{m=0}^{n} S_1(n,m)\lambda^{n-m}Q_{r,s}(t)\left(r!\sum_{k\ge 0} S_2(k+r,r)\frac{\lambda^k}{(k+r)!}t^k\right)x^m \\
&= \lambda^n \sum_{m=0}^{n}\sum_{k=0}^{m}\frac{\binom{m}{k}}{\binom{k+r}{r}} S_1(n,m)S_2(k+r,r)\lambda^{k-m}Q_{r,s}(t)x^{m-k} \\
&= \lambda^n \sum_{m=0}^{n}\sum_{k=0}^{m}\frac{\binom{m}{k}}{\binom{k+r}{r}} S_1(n,m)S_2(k+r,r)\lambda^{k-m}BE_{m-k}(x|\mathbf{a};\mathbf{b}),
\end{aligned}$$

which completes the proof. □

Theorem 2.3. *For all $n \ge 1$,*

$$\beta\mathcal{E}_n(\lambda,x|\mathbf{a};\mathbf{b}) = \sum_{\ell=0}^{n-1}\sum_{k=0}^{n-\ell}\frac{\binom{n-1}{\ell}\binom{n-\ell}{k}}{\binom{k+r}{r}}\lambda^{k+\ell}S_2(k+r,r)B_\ell^{(n)}BE_{n-\ell-k}(x|\mathbf{a};\mathbf{b}).$$

Proof. We proceed the proof by invoking the following transfer formula (see (7) and (8)): for $p_n(x) \sim (1, f(t))$ and $q_n(x) \sim (1, g(t))$, then $q_n(x) = x\left(\frac{f(t)}{g(t)}\right)^n x^{-1}p_n(x)$, for all $n \ge 1$. In our case, by $x^n \sim (1,t)$ and (10), we have

$$\begin{aligned}
&\left(\frac{\lambda}{e^{\lambda t}-1}\right)^r \prod_{i=1}^{r}(e^{a_i t}-1)\prod_{i=1}^{s}\left(\frac{e^{b_i t}+1}{2}\right)\beta\mathcal{E}_n(\lambda,x|\mathbf{a};\mathbf{b}) \\
&= x\left(\frac{\lambda t}{e^{\lambda t}-1}\right)^n x^{n-1} = x\sum_{\ell\ge 0} B_\ell^{(n)}\frac{\lambda^\ell}{\ell!}t^\ell x^{n-1} = \sum_{\ell=0}^{n-1}\binom{n-1}{\ell}\lambda^\ell B_\ell^{(n)}x^{n-\ell}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\beta\mathcal{E}_n(\lambda,x|\mathbf{a};\mathbf{b}) &= \sum_{\ell=0}^{n-1}\binom{n-1}{\ell}\lambda^\ell B_\ell^{(n)}\left(Q_{r,s}(t)\left(\frac{e^{\lambda t}-1}{\lambda t}\right)^r x^{n-\ell}\right) \\
&= \sum_{\ell=0}^{n-1}\binom{n-1}{\ell}\lambda^\ell B_\ell^{(n)}\left(Q_{r,s}(t)\sum_{k\ge 0}S_2(k+r,r)\frac{r!\lambda^k t^k}{(k+r)!}x^{n-\ell}\right) \\
&= \sum_{\ell=0}^{n-1}\sum_{k=0}^{n-\ell}\frac{\binom{n-1}{\ell}\binom{n-\ell}{k}}{\binom{k+r}{r}}\lambda^{k+\ell}S_2(k+r,r)B_\ell^{(n)}Q_{r,s}(t)x^{n-\ell-k} \\
&= \sum_{\ell=0}^{n-1}\sum_{k=0}^{n-\ell}\frac{\binom{n-1}{\ell}\binom{n-\ell}{k}}{\binom{k+r}{r}}\lambda^{k+\ell}S_2(k+r,r)B_\ell^{(n)}BE_{n-\ell-k}(x|\mathbf{a};\mathbf{b}),
\end{aligned}$$

as claimed. □

Note that the *Barnes-type Daehee polynomials with* λ*-parameter* $D_{n,\lambda}(x|\mathbf{a})$ with $a_1,\ldots,a_r\neq 0$ was defined as

$$\prod_{i=1}^{r}\frac{\log(1+\lambda t)}{\lambda(1+\lambda t)^{a_i/\lambda}-1}(1+\lambda t)^{x/\lambda}=\sum_{n\geq 0}D_{n,\lambda}(x|\mathbf{a})\frac{t^n}{n!}, \tag{11}$$

see [15, 16]. When $x=0$ we write $D_{n,\lambda}(\mathbf{a})=D_{n,\lambda}(0|\mathbf{a})$; the *Barnes-type Daehee numbers.*

Theorem 2.4. *For all* $n\geq 0$,

$$\begin{aligned}\beta\mathcal{E}_n(\lambda,x|\mathbf{a};\mathbf{b})&=\sum_{\ell=0}^{n}\sum_{k=0}^{n-\ell}\binom{n}{\ell}\binom{n-\ell}{k}\lambda^\ell b_\ell^{(r)}(x/\lambda)D_{k,\lambda}(\mathbf{a})\mathcal{E}_{n-\ell-k}(\lambda|\mathbf{b})\\&=\sum_{\ell=0}^{n}\sum_{k=0}^{n-\ell}\binom{n}{\ell}\binom{n-\ell}{k}\lambda^\ell b_\ell^{(r)}D_{k,\lambda}(x|\mathbf{a})\mathcal{E}_{n-\ell-k}(\lambda|\mathbf{b})\\&=\sum_{\ell=0}^{n}\sum_{k=0}^{n-\ell}\binom{n}{\ell}\binom{n-\ell}{k}\lambda^\ell b_\ell^{(r)}D_{k,\lambda}(\mathbf{a})\mathcal{E}_{n-\ell-k}(x,\lambda|\mathbf{b}).\end{aligned}$$

Proof. By (9) we have

$$\begin{aligned}&\beta\mathcal{E}_n(\lambda,x|\mathbf{a};\mathbf{b})=\left\langle P_{r,s}(t)(1+\lambda t)^{x/\lambda}|x^n\right\rangle\\&=\left\langle\prod_{i=1}^{r}\left(\frac{\log(1+\lambda t)}{\lambda((1+\lambda t)^{a_i/\lambda}-1)}\right)\prod_{i=1}^{s}\left(\frac{2}{(1+\lambda t)^{b_i/\lambda}+1}\right)\left(\frac{\lambda t}{\log(1+\lambda t)}\right)^r(1+\lambda t)^{x/\lambda}|x^n\right\rangle\\&=\left\langle\prod_{i=1}^{r}\left(\frac{\log(1+\lambda t)}{\lambda((1+\lambda t)^{a_i/\lambda}-1)}\right)\prod_{i=1}^{s}\left(\frac{2}{(1+\lambda t)^{b_i/\lambda}+1}\right)|\sum_{\ell\geq 0}b_\ell^{(r)}(x/\lambda)\frac{\lambda^\ell}{\ell!}t^\ell x^n\right\rangle\\&=\sum_{\ell=0}^{n}\binom{n}{\ell}b_\ell^{(r)}(x/\lambda)\lambda^\ell\left\langle\prod_{i=1}^{r}\left(\frac{\log(1+\lambda t)}{\lambda((1+\lambda t)^{a_i/\lambda}-1)}\right)\prod_{i=1}^{s}\left(\frac{2}{(1+\lambda t)^{b_i/\lambda}+1}\right)|x^{n-\ell}\right\rangle.\end{aligned}$$

Thus, by (11), we obtain

$$\begin{aligned}\beta\mathcal{E}_n(\lambda,x|\mathbf{a};\mathbf{b})&=\sum_{\ell=0}^{n}\binom{n}{\ell}b_\ell^{(r)}(x/\lambda)\lambda^\ell\left\langle\prod_{i=1}^{s}\left(\frac{2}{(1+\lambda t)^{b_i/\lambda}+1}\right)|\sum_{k\geq 0}D_{k,\lambda}(\mathbf{a})\frac{t^k}{k!}x^{n-\ell}\right\rangle\\&=\sum_{\ell=0}^{n}\sum_{k=0}^{n-\ell}\binom{n}{\ell}\binom{n-\ell}{k}\lambda^\ell b_\ell^{(r)}(x/\lambda)D_{k,\lambda}(\mathbf{a})\left\langle\prod_{i=1}^{s}\left(\frac{2}{(1+\lambda t)^{b_i/\lambda}+1}\right)|x^{n-\ell-k}\right\rangle\\&=\sum_{\ell=0}^{n}\sum_{k=0}^{n-\ell}\binom{n}{\ell}\binom{n-\ell}{k}\lambda^\ell b_\ell^{(r)}(x/\lambda)D_{k,\lambda}(\mathbf{a})\mathcal{E}_{n-\ell-k}(\lambda|\mathbf{b}),\end{aligned}$$

which proves the first formula. Similar techniques show the second and the third formulas. □

3. Recurrence relations

In this section, we present several recurrence relations for the Barnes-type degenerate Bernoulli and Euler mixed-type polynomials.

Theorem 3.1. *For all $n \geq 0$,*

$$\beta\mathcal{E}_n(\lambda, x+y|\mathbf{a};\mathbf{b}) = \sum_{j=0}^{n}\binom{n}{j}\beta\mathcal{E}_j(\lambda, x|\mathbf{a};\mathbf{b})(y|\lambda)_{n-j}.$$

Proof. By (9) we have $\left(\frac{\lambda t}{e^{\lambda t}-1}\right)^r \frac{1}{Q_{r,s}(t)}\beta\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) = (x|\lambda)_n \sim \left(1, \frac{e^{\lambda t}-1}{\lambda}\right)$, which implies the result. □

Theorem 3.1 with $x = 0$, gives the following result.

Corollary 3.2. *For all $n \geq 0$,*

$$\beta\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) = \sum_{j=0}^{n}\binom{n}{j}\beta\mathcal{E}_{n-j}(\lambda|\mathbf{a};\mathbf{b})(x|\lambda)_j.$$

Theorem 3.3. *For all $n \geq 1$,*

$$\beta\mathcal{E}_n(\lambda, x+\lambda|\mathbf{a};\mathbf{b}) = \beta\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) + n\lambda\beta\mathcal{E}_{n-1}(\lambda, x|\mathbf{a};\mathbf{b}).$$

Proof. By (7) we have that $f(t)s_n(x) = ns_{n-1}(x)$ when $s_n(x) \sim (g(t), f(t))$. In our case, from (9), we have

$$\frac{e^{\lambda t}-1}{\lambda}\beta\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) = n\beta\mathcal{E}_{n-1}(\lambda, x|\mathbf{a};\mathbf{b}),$$

which implies that $\beta\mathcal{E}_n(\lambda, x+\lambda|\mathbf{a};\mathbf{b}) - \beta\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) = n\lambda\beta\mathcal{E}_{n-1}(\lambda, x|\mathbf{a};\mathbf{b})$, as required. □

Theorem 3.4. *For all $n \geq 1$,*

$$\begin{aligned}\frac{d}{dx}\beta\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) &= n!\sum_{\ell=0}^{n-1}\frac{(-\lambda)^{n-1-\ell}}{\ell!(n-\ell)}\beta\mathcal{E}_\ell(\lambda, x|\mathbf{a};\mathbf{b})\\ &= n\lambda^{n-1}\sum_{\ell=0}^{n-1}S_1(n-1,\ell)\lambda^{-\ell}BE_\ell(x|\mathbf{a};\mathbf{b}).\end{aligned}$$

Proof. By (7) we have $\frac{d}{dx}s_n(x) = \sum_{\ell=0}^{n-1}\binom{n}{\ell}\langle\bar{f}(t)|x^{n-\ell}\rangle s_\ell(x)$ when $s_n(x) \sim (g(t), f(t))$. In our case, from (9), we have

$$\begin{aligned}\langle\bar{f}(t)|x^{n-\ell}\rangle &= \langle\frac{1}{\lambda}\log(1+\lambda t)|x^{n-\ell}\rangle = \lambda^{-1}\langle\sum_{m\geq1}\frac{(-1)^{m-1}\lambda^m t^m}{m}|x^{n-\ell}\rangle\\ &= \lambda^{-1}(-1)^{n-\ell-1}\lambda^{n-\ell}(n-\ell-1)! = (-\lambda)^{n-\ell-1}(n-\ell-1)!.\end{aligned}$$

Thus $\frac{d}{dx}\beta\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) = n!\sum_{\ell=0}^{n-1}\frac{(-\lambda)^{n-1-\ell}}{\ell!(n-\ell)}\beta\mathcal{E}_\ell(\lambda, x|\mathbf{a};\mathbf{b})$, as required.

To show the second formula, we note that $(x|\lambda)_n = \sum_{\ell=0}^{n}S_1(n,\ell)\lambda^{n-\ell}x^\ell \sim \left(1, \frac{e^{\lambda t}-1}{\lambda}\right)$, which shows that $\frac{e^{\lambda t}-1}{\lambda}(x|\lambda)_n = n(x|\lambda)_{n-1}$. Thus $\left(\frac{e^{\lambda t}-1}{\lambda}\right)^r(x|\lambda)_n = (n)_r(x|\lambda)_{n-r}$, for all

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$n \geq r$. Thus, by (10), we have

$$\begin{aligned}\frac{d^r}{dx^r}\beta\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) &= t^r\beta\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) = Q_{r,s}(t)\left(\frac{e^{\lambda t}-1}{\lambda}\right)^r (x|\lambda)_n \\ &= (n)_r Q_{r,s}(t)(x|\lambda)_{n-r} = (n)_r\lambda^{n-r}\sum_{m=0}^{n-r} S_1(n-r,m)\lambda^{-m} BE_m(x|\mathbf{a};\mathbf{b}),\end{aligned}$$

which completes the proof. □

Theorem 3.5. *For all* $n \geq 1$,

$$\begin{aligned}(1-r/n)\beta\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) &= \left(x - \sum_{i=1}^r a_i - \sum_{j=1}^s b_j\right)\beta\mathcal{E}_{n-1}(\lambda, x-\lambda|\mathbf{a};\mathbf{b}) \\ &\quad - \frac{1}{n}\sum_{i=1}^r a_i\beta\mathcal{E}_n(\lambda, x-\lambda|a_i,\mathbf{a};\mathbf{b}) + \frac{1}{2}\sum_{i=1}^s b_i\beta\mathcal{E}_{n-1}(\lambda, x-\lambda|\mathbf{a};b_i,\mathbf{b}).\end{aligned}$$

Proof. Let $n \geq 1$. By (9), we have

$$\begin{aligned}\beta\mathcal{E}_n(\lambda, y|\mathbf{a};\mathbf{b}) &= \left\langle P_{r,s}(t)(1+\lambda t)^{y/\lambda}|x^n\right\rangle \\ &= \left\langle \frac{d}{dt}\left(\prod_{i=1}^r\left(\frac{t}{(1+\lambda t)^{a_i/\lambda}-1}\right)\prod_{i=1}^s\left(\frac{2}{(1+\lambda t)^{b_i/\lambda}+1}\right)(1+\lambda t)^{y/\lambda}\right)|x^{n-1}\right\rangle \\ &= \left\langle \frac{d}{dt}\prod_{i=1}^r\left(\frac{t}{(1+\lambda t)^{a_i/\lambda}-1}\right)\prod_{i=1}^s\left(\frac{2}{(1+\lambda t)^{b_i/\lambda}+1}\right)(1+\lambda t)^{y/\lambda}|x^{n-1}\right\rangle \quad (12)\\ &\quad + \left\langle \prod_{i=1}^r\left(\frac{t}{(1+\lambda t)^{a_i/\lambda}-1}\right)\frac{d}{dt}\prod_{i=1}^s\left(\frac{2}{(1+\lambda t)^{b_i/\lambda}+1}\right)(1+\lambda t)^{y/\lambda}|x^{n-1}\right\rangle \quad (13)\\ &\quad + \left\langle \prod_{i=1}^r\left(\frac{t}{(1+\lambda t)^{a_i/\lambda}-1}\right)\prod_{i=1}^s\left(\frac{2}{(1+\lambda t)^{b_i/\lambda}+1}\right)\frac{d}{dt}(1+\lambda t)^{y/\lambda}|x^{n-1}\right\rangle. \quad (14)\end{aligned}$$

The term in (14) is given by

$$y\left\langle P_{r,s}(t)(1+\lambda t)^{y/\lambda-1}|x^{n-1}\right\rangle = y\beta\mathcal{E}_{n-1}(\lambda, y-\lambda|\mathbf{a};\mathbf{b}). \quad (15)$$

In order to find the first term, namely (12), we note that

$$\begin{aligned}&\frac{d}{dt}\prod_{i=1}^r\left(\frac{t}{(1+\lambda t)^{a_i/\lambda}-1}\right) \\ &= \prod_{i=1}^r\left(\frac{t}{(1+\lambda t)^{a_i/\lambda}-1}\right)\sum_{i=1}^r\left(-\frac{a_i}{1+\lambda t}+\frac{1}{t}\left(1-\frac{a_i}{1+\lambda t}\frac{t}{(1+\lambda t)^{a_i/\lambda}-1}\right)\right),\end{aligned}$$

where the order of $1-\frac{a_i}{1+\lambda t}\frac{t}{(1+\lambda t)^{a_i/\lambda}-1}$ is at least 1. Thus the term in (12) is given by

$$-\sum_{i=1}^{r} a_i \left\langle P_{r,s}(t)(1+\lambda t)^{y/\lambda-1}|x^{n-1}\right\rangle + \left\langle P_{r,s}(t)(1+\lambda t)^{y/\lambda}|\frac{1}{t}\sum_{i=1}^{r}\left(1-\frac{a_i}{1+\lambda t}\frac{t}{(1+\lambda t)^{a_i/\lambda}-1}\right)x^{n-1}\right\rangle$$

which equals

$$\begin{aligned}&-\sum_{i=1}^{r} a_i\beta\mathcal{E}_{n-1}(\lambda,y-\lambda|\mathbf{a};\mathbf{b})+\frac{r}{n}\left\langle P_{r,s}(t)(1+\lambda t)^{y/\lambda}|x^n\right\rangle\\ &\qquad -\frac{1}{n}\sum_{i=1}^{r} a_i\left\langle \frac{t}{(1+\lambda t)^{a_i/\lambda}-1}P_{r,s}(t)(1+\lambda t)^{y/\lambda-1}|x^n\right\rangle\\ &=-\sum_{i=1}^{r} a_i\beta\mathcal{E}_{n-1}(\lambda,y-\lambda|\mathbf{a};\mathbf{b})+\frac{r}{n}\beta\mathcal{E}_n(\lambda,y|\mathbf{a};\mathbf{b})-\frac{1}{n}\sum_{i=1}^{r} a_i\beta\mathcal{E}_n(\lambda,y-\lambda|a_i,\mathbf{a};\mathbf{b}).\end{aligned} \tag{16}$$

In order to find the second term, namely (13), we note that

$$\begin{aligned}&\frac{d}{dt}\prod_{i=1}^{s}\left(\frac{2}{(1+\lambda t)^{b_i/\lambda}+1}\right)\\ &=\prod_{i=1}^{s}\left(\frac{2}{(1+\lambda t)^{b_i/\lambda}+1}\right)\sum_{i=1}^{s}\left(-\frac{b_i}{1+\lambda t}+\frac{b_i}{2(1+\lambda t)}\frac{2}{(1+\lambda t)^{b_i/\lambda}+1}\right).\end{aligned}$$

Thus the term in (13) is given by

$$\begin{aligned}&\sum_{i=1}^{s} b_i\left\langle\left(-1+\frac{1}{(1+\lambda t)^{b_i/\lambda}+1}\right)P_{r,s}(t)(1+\lambda t)^{y/\lambda-1}|x^{n-1}\right\rangle\\ &=-\sum_{i=1}^{s} b_i\beta\mathcal{E}_{n-1}(\lambda,y-\lambda|\mathbf{a};\mathbf{b})+\frac{1}{2}\sum_{i=1}^{s} b_i\beta\mathcal{E}_{n-1}(\lambda,y-\lambda|\mathbf{a};b_i,\mathbf{b}).\end{aligned} \tag{17}$$

Altogether, namely by (15), (16) and (17), we complete the proof. □

Theorem 3.6. *For $n \geq 0$,*

$$\begin{aligned}&\beta\mathcal{E}_{n+1}(\lambda,x|\mathbf{a};\mathbf{b})=x\beta\mathcal{E}_n(\lambda,x-\lambda|\mathbf{a};\mathbf{b})\\ &-\lambda^n\sum_{m=0}^{n}\sum_{k=0}^{m}\sum_{\ell=0}^{k}\frac{\lambda^{-k}S_1(n,m)S_2(m-k+r,r)\binom{m}{k}\binom{k}{\ell}}{\binom{m-k+r}{r}}\cdot\\ &\cdot\left(\frac{B_{k-\ell+1}(1)}{k-\ell+1}\left(\sum_{i=1}^{r}a_i^{k-\ell+1}-r\lambda^{k-\ell+1}\right)+\frac{E_{k-\ell}(1)}{2}\left(\sum_{j=1}^{r}b_j^{k-\ell+1}\right)\right)BE_\ell(x-\lambda|\mathbf{a};\mathbf{b}).\end{aligned}$$

Proof. By (7), we have that $s_{n+1}(x) = \left(x - \frac{g'(t)}{g(t)}\right) \frac{1}{f'(t)} s_n(x)$ when $s_n(x) \sim (g(t), f(t))$. In our case, see (9), we have

$$\beta\mathcal{E}_{n+1}(\lambda, x|\mathbf{a};\mathbf{b}) = x\beta\mathcal{E}_n(\lambda, x-\lambda|\mathbf{a};\mathbf{b}) - e^{-\lambda t}\frac{g'(t)}{g(t)}\beta\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}),$$

where

$$\begin{aligned}
\frac{g'(t)}{g(t)} &= (\log g(t))' \\
&= \left(r\log\lambda - r\log(e^{\lambda t}-1) + \sum_{i=1}^{r}\log(e^{a_i t}-1) + \sum_{j=1}^{s}\log(e^{b_j t}+1) - s\log 2\right)' \\
&= -\frac{r\lambda e^{\lambda t}}{e^{\lambda t}-1} + \sum_{i=1}^{r}\frac{a_i e^{a_i t}}{e^{a_i t}-1} + \sum_{j=1}^{s}\frac{b_j e^{b_j t}}{e^{b_j t}+1} \\
&= \frac{1}{t}\left(-\frac{r\lambda t e^{\lambda t}}{e^{\lambda t}-1} + \sum_{i=1}^{r}\frac{a_i t e^{a_i t}}{e^{a_i t}-1}\right) + \frac{1}{2}\sum_{j=1}^{s}\frac{2b_j e^{b_j t}}{e^{b_j t}+1} \\
&= \frac{1}{t}\left(-r\sum_{\ell\ge 0}B_\ell(1)\frac{\lambda^\ell t^\ell}{\ell!} + \sum_{i=1}^{r}\sum_{\ell\ge 0}B_\ell(1)\frac{a_i^\ell t^\ell}{\ell!}\right) + \frac{1}{2}\sum_{j=1}^{s}\sum_{\ell\ge 0}E_\ell(1)\frac{b_j^{\ell+1}t^\ell}{\ell!} \\
&= \sum_{\ell\ge 0}\frac{B_{\ell+1}(1)}{(\ell+1)!}\left(\sum_{i=1}^{r}a_i^{\ell+1} - r\lambda^{\ell+1}\right)t^\ell + \frac{1}{2}\sum_{\ell\ge 0}\left(\frac{E_\ell(1)}{\ell!}\sum_{j=1}^{s}b_j^{\ell+1}\right)t^\ell.
\end{aligned}$$

Therefore, by Theorem 2.2, we obtain

$$\begin{aligned}
&\frac{g'(t)}{g(t)}\beta\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) \\
&= \lambda^n\sum_{m=0}^{n}\sum_{k=0}^{m}\frac{\lambda^{k-m}S_1(n,m)S_2(k+r,r)\binom{m}{k}}{\binom{k+r}{r}}\sum_{\ell=0}^{m-k}\frac{B_{\ell+1}(1)}{(\ell+1)!}\left(\sum_{i=1}^{r}a_i^{\ell+1} - r\lambda^{\ell+1}\right)t^\ell BE_{m-k}(x|\mathbf{a};\mathbf{b}) \\
&+ \frac{\lambda^n}{2}\sum_{m=0}^{n}\sum_{k=0}^{m}\frac{\lambda^{k-m}S_1(n,m)S_2(k+r,r)\binom{m}{k}}{\binom{k+r}{r}}\sum_{\ell=0}^{m-k}\frac{E_\ell(1)}{\ell!}\left(\sum_{j=1}^{s}b_j^{\ell+1}\right)t^\ell BE_{m-k}(x|\mathbf{a};\mathbf{b}) \\
&= \lambda^n\sum_{m=0}^{n}\sum_{k=0}^{m}\sum_{\ell=0}^{k}\frac{\lambda^{-k}S_1(n,m)S_2(m-k+r,r)\binom{m}{k}\binom{k}{\ell}}{\binom{m-k+r}{r}}\cdot \\
&\qquad\cdot\left(\frac{B_{k-\ell+1}(1)}{k-\ell+1}\left(\sum_{i=1}^{r}a_i^{k-\ell+1} - r\lambda^{k-\ell+1}\right) + \frac{E_{k-\ell}(1)}{2}\left(\sum_{j=1}^{r}b_j^{k-\ell+1}\right)\right)BE_\ell(x|\mathbf{a};\mathbf{b}).
\end{aligned}$$

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Thus,

$$\begin{aligned}
&\beta\mathcal{E}_{n+1}(\lambda, x|\mathbf{a};\mathbf{b})\\
&= x\beta\mathcal{E}_n(\lambda, x-\lambda|\mathbf{a};\mathbf{b})\\
&-\lambda^n \sum_{m=0}^{n}\sum_{k=0}^{m}\sum_{\ell=0}^{k} \frac{\lambda^{-k}S_1(n,m)S_2(m-k+r,r)\binom{m}{k}\binom{k}{\ell}}{\binom{m-k+r}{r}}\cdot\\
&\cdot\left(\frac{B_{k-\ell+1}(1)}{k-\ell+1}\left(\sum_{i=1}^{r} a_i^{k-\ell+1} - r\lambda^{k-\ell+1}\right) + \frac{E_{k-\ell}(1)}{2}\left(\sum_{j=1}^{r} b_j^{k-\ell+1}\right)\right) BE_\ell(x-\lambda|\mathbf{a};\mathbf{b}),
\end{aligned}$$

as claimed. □

4. Connections with families of polynomials

The Bernoulli polynomials $B_n^{(\alpha)}(x)$ of order α are defined by the generating function

$$\left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt} = \sum_{n\geq 0} B_n^{(\alpha)}(x)\frac{t^n}{n!},$$

equivalently, $B_n^{(\alpha)}(x) \sim \left(\left(\frac{e^t-1}{t}\right)^{\alpha}, t\right)$ (see [3, 9, 10]). In the next result, we express our polynomials $\beta\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b})$ in terms of Bernoulli polynomials of order α.

Theorem 4.1. *For $n \geq 0$,*

$$\beta\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) = \sum_{m=0}^{n} \lambda^{-m} d_{n,m} B_m^{(\alpha)}(x),$$

where

$$\begin{aligned}
d_{n,m} = \sum_{\ell=m}^{n}\sum_{k=0}^{n-\ell}\Bigg[&\binom{n}{\ell}\binom{n-\ell}{k}\lambda^{k+\ell}S_1(\ell,m)b_k^{(\alpha)}\cdot\\
&\cdot\sum_{q=0}^{n-\ell-k}\sum_{p=0}^{q}\frac{\binom{n-\ell-k}{q}}{\binom{q+\alpha}{\alpha}}S_1(q+\alpha, q-p+\alpha)S_2(q-p+\alpha,\alpha)\lambda^p\beta\mathcal{E}_{n-\ell-k-q}(\lambda|\mathbf{a};\mathbf{b})\Bigg].
\end{aligned}$$

Proof. Let $\beta\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) = \sum_{m=0}^{n} c_{n,m}B_m^{(\alpha)}(x)$. By (8) and (9), we have

$$\begin{aligned}
c_{n,m} &= \frac{1}{m!\lambda^m}\left\langle P_{r,s}(t)\left(\frac{(1+\lambda t)^{1/\lambda}-1}{t}\right)^{\alpha}\left(\frac{\lambda t}{\log(1+\lambda t)}\right)^{\alpha}(\log(1+\lambda t))^m \Big| x^n\right\rangle\\
&= \frac{1}{m!\lambda^m}\left\langle P_{r,s}(t)\left(\frac{(1+\lambda t)^{1/\lambda}-1}{t}\right)^{\alpha}\left(\frac{\lambda t}{\log(1+\lambda t)}\right)^{\alpha}\Big| m!\sum_{\ell\geq m}S_1(\ell,m)\frac{\lambda^\ell t^\ell}{\ell!}x^n\right\rangle\\
&= \lambda^{-m}\sum_{\ell=m}^{n}\binom{n}{\ell}\lambda^\ell S_1(\ell,m)\left\langle P_{r,s}(t)\left(\frac{(1+\lambda t)^{1/\lambda}-1}{t}\right)^{\alpha}\Big|\left(\frac{\lambda t}{\log(1+\lambda t)}\right)^{\alpha}x^{n-\ell}\right\rangle\\
&= \lambda^{-m}\sum_{\ell=m}^{n}\sum_{k=0}^{n-\ell}\binom{n}{\ell}\binom{n-\ell}{k}\lambda^{k+\ell}S_1(\ell,m)b_k^{(\alpha)}\left\langle P_{r,s}(t)\Big|\left(\frac{(1+\lambda t)^{1/\lambda}-1}{t}\right)^{\alpha}x^{n-\ell-k}\right\rangle.
\end{aligned}$$

Before proceeding further, we note that

$$\left(\frac{(1+\lambda t)^{1/\lambda}-1}{t}\right)^{\alpha} = \left(\frac{e^{\frac{1}{\lambda}\log(1+\lambda t)}-1}{t}\right)^{\alpha}$$
$$= \sum_{q\geq 0}\sum_{p=0}^{q}\binom{q+\alpha}{\alpha}^{-1} S_1(q+\alpha,q-p+\alpha)S_2(q-p+\alpha,\alpha)\frac{\lambda^p t^q}{q!}.$$

Thus,

$$c_{n,m} = \lambda^{-m}\sum_{\ell=m}^{n}\sum_{k=0}^{n-\ell}\Bigg[\binom{n}{\ell}\binom{n-\ell}{k}\lambda^{k+\ell}S_1(\ell,m)b_k^{(\alpha)}\cdot$$
$$\cdot\sum_{q=0}^{n-\ell-k}\sum_{p=0}^{q}\frac{\binom{n-\ell-k}{q}}{\binom{q+\alpha}{\alpha}}S_1(q+\alpha,q-p+\alpha)S_2(q-p+\alpha,\alpha)\lambda^p\left\langle P_{r,s}(t)|x^{n-\ell-k-q}\right\rangle\Bigg],$$

which gives

$$c_{n,m} = \lambda^{-m}\sum_{\ell=m}^{n}\sum_{k=0}^{n-\ell}\Bigg[\binom{n}{\ell}\binom{n-\ell}{k}\lambda^{k+\ell}S_1(\ell,m)b_k^{(\alpha)}\cdot$$
$$\cdot\sum_{q=0}^{n-\ell-k}\sum_{p=0}^{q}\frac{\binom{n-\ell-k}{q}}{\binom{q+\alpha}{\alpha}}S_1(q+\alpha,q-p+\alpha)S_2(q-p+\alpha,\alpha)\lambda^p\beta\mathcal{E}_{n-\ell-k-q}(\lambda|\mathbf{a};\mathbf{b})\Bigg],$$

which completes the proof. □

The degenerate Bernoulli polynomials $\beta_n^{(\alpha)}(\lambda,x)$ of order α are defined by the generating function

$$\left(\frac{t}{(1+\lambda t)^{1/\lambda}-1}\right)^{\alpha}(1+\lambda t)^{x/\lambda} = \sum_{n\geq 0}\beta_n^{(\alpha)}(\lambda,x)\frac{t^n}{n!},$$

equivalently, $B_n^{(\alpha)}(\lambda,x) \sim \left(\left(\frac{\lambda(e^t-1)}{e^{\lambda t}-1}\right)^{\alpha}, \frac{1}{\lambda}(e^{\lambda t}-1)\right)$. Then by using similar arguments as in the proof of Theorem 4.1, we obtain the following result.

Theorem 4.2. *For $n \geq 0$,*

$$\beta\mathcal{E}_n(\lambda,x|\mathbf{a};\mathbf{b}) = \sum_{m=0}^{n}\binom{n}{m}d_{n,m}\beta_m^{(\alpha)}(\lambda,x),$$

where

$$d_{n,m} = \sum_{q=0}^{n-m}\sum_{p=0}^{q}\frac{\binom{n-m}{q}}{\binom{q+\alpha}{\alpha}}S_1(q+\alpha,q-p+\alpha)S_2(q-p+\alpha,\alpha)\lambda^p\beta\mathcal{E}_{n-m-q}(\lambda|\mathbf{a};\mathbf{b}).$$

The *Frobenius-Euler polynomials* of order α are defined by the generating function

$$\left(\frac{1-\mu}{e^t-\mu}\right)^{\alpha}e^{xt} = \sum_{n\geq 0}H_n^{(\alpha)}(x|\mu)\frac{t^n}{n!},$$

equivalently, $H_n^{(\alpha)}(x|\mu) \sim \left(\left(\frac{e^t-\mu}{1-\mu}\right)^{\alpha}, t\right)$ (see [2, 13]). In the next result, we express our polynomials $\beta\mathcal{E}_n(\lambda,x|\mathbf{a};\mathbf{b})$ in terms of Frobenius-Euler polynomials.

Theorem 4.3. *For $n \geq 0$,*

$$\beta\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) = (1-\mu)^{-\alpha} \sum_{m=0}^{n} \lambda^{-m} d_{n,m} H_m^{(\alpha)}(x|\mu),$$

where

$$d_{n,m} = \sum_{\ell=m}^{n} \sum_{k=0}^{n-\ell} \sum_{p=0}^{\alpha} \binom{n}{\ell}\binom{n-\ell}{k}\binom{\alpha}{p} S_1(\ell,m)\lambda^{\ell}(-\mu)^{\alpha-p}\beta\mathcal{E}_k(\lambda|\mathbf{a};\mathbf{b})(p|\lambda)_{n-\ell-k}.$$

Proof. Let $\beta\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) = \sum_{m=0}^{n} c_{n,m} H_m^{(\alpha)}(x|\mu)$. Then

$$\begin{aligned}
c_{n,m} &= \frac{1}{m!(1-\mu)^{\alpha}\lambda^m} \left\langle ((1+\lambda t)^{1/\lambda} - \mu)^{\alpha} P_{r,s}(t) | (\log(1+\lambda t))^m x^n \right\rangle \\
&= \frac{1}{m!(1-\mu)^{\alpha}\lambda^m} \left\langle ((1+\lambda t)^{1/\lambda} - \mu)^{\alpha} P_{r,s}(t) | m! \sum_{\ell \geq m} S_1(\ell,m) \frac{\lambda^{\ell} t^{\ell}}{\ell!} x^n \right\rangle \\
&= \frac{1}{(1-\mu)^{\alpha}\lambda^m} \sum_{\ell=m}^{n} \binom{n}{\ell} \lambda^{\ell} S_1(\ell,m) \left\langle ((1+\lambda t)^{1/\lambda} - \mu)^{\alpha} | P_{r,s}(t) x^{n-\ell} \right\rangle \\
&= \frac{1}{(1-\mu)^{\alpha}\lambda^m} \sum_{\ell=m}^{n} \sum_{k=0}^{n-\ell} \binom{n}{\ell}\binom{n-\ell}{k} \lambda^{\ell} S_1(\ell,m) \beta\mathcal{E}_k(\lambda|\mathbf{a};\mathbf{b}) \left\langle ((1+\lambda t)^{1/\lambda} - \mu)^{\alpha} | x^{n-\ell-k} \right\rangle.
\end{aligned}$$

Note that

$$\begin{aligned}
\left\langle ((1+\lambda t)^{1/\lambda} - \mu)^{\alpha} | x^{n-\ell-k} \right\rangle &= \left\langle \sum_{p=0}^{\alpha} \binom{\alpha}{p} (-\mu)^{\alpha-p} (1+\lambda t)^{p/\lambda} | x^{n-\ell-k} \right\rangle \\
&= \sum_{p=0}^{\alpha} \binom{\alpha}{p} (-\mu)^{\alpha-p} \left\langle \sum_{q \geq 0} (p|\lambda)_q \frac{t^q}{q!} | x^{n-\ell-k} \right\rangle \\
&= \sum_{p=0}^{\alpha} \binom{\alpha}{p} (-\mu)^{\alpha-p} (p|\lambda)_{n-\ell-k}.
\end{aligned}$$

Thus,

$$c_{n,m} = \frac{1}{(1-\mu)^{\alpha}\lambda^m} \sum_{\ell=m}^{n} \sum_{k=0}^{n-\ell} \sum_{p=0}^{\alpha} \binom{n}{\ell}\binom{n-\ell}{k}\binom{\alpha}{p} \lambda^{\ell}(-\mu)^{\alpha-p} S_1(\ell,m)\beta\mathcal{E}_k(\lambda|\mathbf{a};\mathbf{b})(p|\lambda)_{n-\ell-k},$$

which completes the proof. □

The degenerate Euler polynomials $\mathcal{E}_n^{(\alpha)}(\lambda, x)$ of order α are defined by the generating function

$$\left(\frac{2}{(1+\lambda t)^{1/\lambda}+1}\right)^{\alpha} (1+\lambda t)^{x/\lambda} = \sum_{n \geq 0} \mathcal{E}_n^{(\alpha)}(\lambda, x) \frac{t^n}{n!},$$

equivalently, $E_n^{(\alpha)}(\lambda, x) \sim \left(\left(\frac{e^t+1}{2}\right)^{\alpha}, \frac{e^{\lambda t}-1}{\lambda}\right)$. Then by using similar arguments as in the proof of Theorems 4.1 and 4.3, we obtain the following result.

Theorem 4.4. *For $n \geq 0$,*

$$\beta\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) = 2^{-\alpha}\sum_{m=0}^{n}\binom{n}{m}d_{n,m}\mathcal{E}_m^{(\alpha)}(\lambda, x),$$

where

$$d_{n,m} = \sum_{q=0}^{n-m}\sum_{p=0}^{\alpha}\binom{n-m}{q}\binom{\alpha}{p}\beta\mathcal{E}_{n-m-q}(\lambda|\mathbf{a};\mathbf{b})(p|\lambda)_q.$$

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Ground state solutions for second order nonlinear p-Laplacian difference equations with periodic coefficients

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Abstract

We study the existence of homoclinic solutions for nonlinear p-Laplacian difference equations with periodic coefficients. The proof of the main result is based on the critical point theory in combination with the Nehari manifold approach. Under rather weaker conditions, we obtain the existence of ground state solutions and considerably improve some existing ones even for some special cases.

Key words: P-Laplacian Difference equations; Nehari manifold; Ground state solutions; Critical point theory.

1 Introduction

Difference equations represent the discrete counterpart of ordinary differential equations, have been widely used in many fields such as computer science, economics, neural network, ecology, cybernetics, etc. In the past decades, the existence of homoclinic solutions for difference equations with p-Laplacian has been extensively studied, The classical method used is fixed point theory, to mention a few, see [1–3] and references therein for details. As it is well known, the critical point theory is used to deal with the existence of solutions of difference equations [4–10]. Here we mention the works of Cabada, Iannizzotto and Tersian [4], Jiang and Zhou [5], Long and Shi [6]. In these papers, critical point theory is applied on bound discrete intervals, which leads to the study of critical points of an energy functional defined on a finite-dimensional Banach space. For unbounded discrete intervals such as the whole set of integers $\mathbb{Z}$, Ma and Guo used critical point theory in combination with periodic approximation to deal with such problems [7]. In the present paper, under convenient assumption,

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without periodic approximation and without verifying Palais-Smale condition, we not only prove the existence of homoclinic solution, but also obtain the ground state solution. we extend [11] to the case of the p-lapacian difference equation with periodic coefficients.

In this paper, our work focus on the existence of homoclinic solution for the following second order nonlinear difference equations with p-Laplacian

$$-\Delta[a(k)\phi_p(\Delta u(k-1))] + b(k)\phi_p(u(k)) = f(k, u(k)), \quad k \in \mathbb{Z}, \tag{1.1}$$

where $\phi_p(t) = |t|^{p-2}t$ for all $t \in \mathbb{R}$, $p > 1$. $a(k), b(k)$ are positive and $T-$periodic sequences, T is a fixed positive integer. $f(k,u) : \mathbb{Z}\times\mathbb{R} \to \mathbb{R}$ is a continuous function on u and $T-$periodic on k. The forward difference operator Δ is defined by

$$\Delta u(k-1) = u(k) - u(k-1), \qquad \text{for all } k \in \mathbb{Z}.$$

where $\mathbb{Z}$ and $\mathbb{R}$ denote the set of all integers and real numbers, respectively.

In addition, we are interested in the existence of nontrivial homoclinic solution for (1.1), that is, solutions that are not equal to 0 identically. We call that a solution $u = \{u(k)\}$ of (1.1) is homoclinic (to 0) if

$$\lim_{|k|\to\infty} u(k) = 0. \tag{1.2}$$

Throughout this paper, we always suppose that the following conditions hold.

(A) $a(k) > 0$ and $a(k+T) = a(k)$ for all $k \in \mathbb{Z}$.

(B) $b(k) > 0$ and $b(k+T) = b(k)$ for all $k \in \mathbb{Z}$.

(f_1) $f \in C(\mathbb{Z}\times\mathbb{R}, \mathbb{R})$, and there exist $C > 0$, $q \in (p, \infty)$ such that

$$|f(k,u)| \le C(1 + |u|^{q-1}), \quad \text{for all } k \in \mathbb{Z}, u \in \mathbb{R}.$$

(f_2) $\lim_{|u|\to 0} f(k,u)/|u|^{p-1} = 0$ uniformly for $k \in \mathbb{Z}$.

(f_3) $\lim_{|u|\to\infty} F(k,u)/|u|^p = +\infty$ uniformly for $k \in \mathbb{Z}$, where $F(k,u)$ is the primitive function of $f(k,u)$, *i.e.*,

$$F(k,u) = \int_0^u f(k,s)ds.$$

(f_4) $u \mapsto f(k,u)/|u|^{p-1}$ is strictly increasing on $(-\infty, 0)$ and $(0, \infty)$.

The main result in this paper is the following theorem:

Theorem 1.1. *Suppose conditions* $(A), (B)$ *and* $(f_1)-(f_4)$ *are satisfied. Then equation* (1.1) *has at least a nontrivial ground state solution.*

Remark 1.1. In [7], Ma and Guo considered the special case of (1.1) with $p = 2$ and obtained the following theorem:

Theorem A *Suppose conditions* $(A), (B)$, (f_2) *and the following generalized Ambrosetti-Rabinowitz superlinear condition are satisfied:*

(GAR) *there exists a constant* $\mu > p$ *such that*

$$0 < \mu F(k, u) \leq f(k, u)u, \quad \text{for all } k \text{ and } u \neq 0, \tag{1.3}$$

Then equation (1.1) *has a nontrivial ground state solution.*

It is easy to see that (1.3) implies (f_3). There exists a p-superlinear function, such as

$$f(k, u) = |u|^{p-2}u \ln(1 + |u|),$$

does not satisfy (1.3). However, it satisfies the condition $(f_1) - (f_4)$. So our conditions are weaker than conditions in [7]. And we do not need periodic approximation technique to obtain homoclinic solutions. Furthermore, we obtain the existence of a ground state solution. Therefore, our result not only extends the main result in [7] to difference equations with p-Laplacian but also improves it.

Remark 1.2. In [12], the authors considered the following second order nonlinear difference equations with p-Laplacian

$$-\Delta\phi_p(\Delta u(k-1)) + b(k)\phi_p(u(k)) = f(k, u(k)), \quad k \in \mathbb{Z}, \tag{1.4}$$

without any periodic assumption, they obtained the homoclinic solutions of the equation. However, PS condition need to be proved in [12], in this paper, we only prove the coercive condition (below Lemma 3.2) is satisfied.

Example 1.1. Let

$$f(k, u) = \begin{cases} 0, & u = 0, \\ |u|^{p-2}u \ln(1 + |u|), & u \neq 0, \end{cases}$$

for all $k \in \mathbb{Z}$, If (A) and (B) are satisfied, then it is easy to check that all the conditions of our Theorem 1.1 are satisfied. Therefore, the nontrivial homoclinic solution is obtained at once.

The rest of the paper is organized as follows: In Section 2, we establish the variational framework associated with (1.1), then present the main results of this paper. Section 3 is devoted to prove the main result.

2 Preliminaries

In this section, we shall establish the corresponding variational framework associated with (1.1). We are going to define a suitable space E and an energy functional $J \in E$, such that critical points of J in E are exactly solutions of (1.1).

Consider the real sequence spaces

$$l^p \equiv l^p(\mathbb{Z}) = \left\{ u = \{u(k)\}_{k\in\mathbb{Z}} : \forall\ k \in \mathbb{Z}, u(k) \in \mathbb{R}, \|u\|_{l^p} = \left(\sum_{k\in\mathbb{Z}} |u(k)|^p \right)^{\frac{1}{p}} < \infty \right\}. \quad (2.1)$$

Then the following embedding between l^p spaces holds,

$$l^q \subset l^p, \|u\|_{l^p} \leq \|u\|_{l^q}, 1 \leq q \leq p \leq \infty. \quad (2.2)$$

Define the space

$$E := \{\ u \in l^p : \sum_{k\in\mathbb{Z}} [a(k)|\Delta u(k-1)|^p + b(k)|u(k)|^p] < \infty\}.$$

Then E is a Hilbert space equipped with the norm

$$\|u\|^p = \sum_{k\in\mathbb{Z}} [a(k)|\Delta u(k-1)|^p + b(k)|u(k)|^p] .$$

$|\cdot|$ is the usual absolute value in $\mathbb{R}$.

Now we consider the variational functional J defined on E by

$$\begin{aligned} J(u) &= \frac{1}{p} \sum_{k\in\mathbb{Z}} [a(k)|\Delta u(k-1)|^p + b(k)|u(k)|^p] - \sum_{k\in\mathbb{Z}} F(k, u(k)) \\ &= \frac{1}{p}\|u\|^p - \sum_{k\in\mathbb{Z}} F(k, u(k)). \end{aligned}$$

Then $J \in C^1(E, \mathbb{R})$ with for all $v \in E$,

$$\begin{aligned} (J'(u), v) &= \lim_{t\to 0} \frac{J(u + tv) - J(u)}{t} \\ &= \sum_{k\in\mathbb{Z}} [a(k)\phi_p(\Delta u(k-1))\Delta v(k-1) + b(k)\phi_p(u(k))v(k)] \\ &\quad - \sum_{k\in\mathbb{Z}} f(k, u(k))v(k). \end{aligned}$$

and

$$\frac{\partial J(u)}{\partial u(k)} = -a(k)\Delta\phi_p(\Delta u(k-1)) + b(k)\phi_p(u(k)) - f(k, u(k)), \quad k \in \mathbb{Z}.$$

Thus, u is a critical point of J on E only if u is a homoclinic solutions of equation (1.1).

Let

$$c_{min} = \inf\{J(u) : J'(u) = 0, u \in E\backslash\{0\}\}.$$

Then $u_0 \neq 0$ with $J(u_0) = c_{min}$ is said to be a ground state solution of (1.1).

3 Proofs of main result

We define the Nehari manifold

$$\mathcal{N} = \{u \in E \setminus \{0\} : (J'(u), u) = 0\}.$$

To prove the main results, we need some lemmas.

Lemma 3.1. *Assume that* (B), $(f_1) - (f_4)$ *are satisfied. Then for each* $w \in E \setminus \{0\}$, *there exists a unique* $s_w > 0$ *such that* $s_w w \in \mathcal{N}$.

Proof. Let $I(u) = \sum\limits_{k\in\mathbb{Z}} F(k, u(k))$. By (f_2), we have

$$I'(u) = o(\|u\|^{p-1}) \quad \text{as} \quad u \to 0. \tag{3.1}$$

From (f_4), for all $u \neq 0$ and $s > 0$, we have

$$s \mapsto I'(su)u/s^{p-1} \quad \text{is strictly increasing.} \tag{3.2}$$

Let $W \subset E \setminus \{0\}$ be a weakly compact subset and $s > 0$, we claim that

$$I(su)/s^p \to \infty \text{ uniformly for } u \text{ on } W, \text{ as } s \to \infty. \tag{3.3}$$

Indeed, let $\{u_n\} \subset W$. It suffices to show that

$$\text{if } s_n \to \infty, \quad I(s_n u_n)/(s_n)^p \to \infty.$$

as $n \to \infty$. Passing to a subsequence if necessary, $u_n \rightharpoonup u \in E \setminus \{0\}$ and $u_n(k) \to u(k)$ for every k, as $n \to \infty$.

Note that from (f_2) and (f_4), it is easy to get that

$$F(k, u) > 0, \quad \text{for all } u \neq 0. \tag{3.4}$$

Since $|s_n u_n(k)| \to \infty$ and $u_n \neq 0$, by (f_3) and (3.4), we have

$$\frac{I(s_n u_n)}{(s_n)^p} = \sum_{k\in\mathbb{Z}} \frac{F(k, s_n u_n(k))}{|s_n u_n(k)|^p} |u_n(k)|^p \to \infty \quad \text{as } n \to \infty.$$

Therefore, (3.3) holds.

Let $g(s) := J(sw), \ s > 0$. Then

$$g'(s) = J'(sw)w = s^{p-1}(\|w\|^p - s^{1-p} I'(sw)w),$$

from (3.1)-(3.3), then there exists a unique s_w, such that $g'(s) > 0$ whenever $0 < s < s_w$, $g'(s) < 0$ whenever $s > s_w$ and $g'(s_w) = J'(s_w w)w = 0$. So $s_w w \in \mathcal{N}$. □

Lemma 3.2. *J is coercive on $\mathcal{N}$, i.e., $J(u) \to \infty$ as $\|u\| \to \infty$, $u \in \mathcal{N}$.*

Proof. Suppose by contradiction, there exists a sequence $\{u_n\} \subset \mathcal{N}$ such that $\|u_n\| \to \infty$ and $J(u_n) \leq d$. Let $v_n = \frac{u_n}{\|u_n\|}$, then there exists a subsequence, still denoted by the same notation, such that $v_n \rightharpoonup v$ and $v_n(k) \to v(k)$ for every k, as $n \to \infty$.

First we know that there exist $\delta > 0$ and $k_j \in \mathbb{Z}$ such that

$$|v_n(k_j)| \geq \delta. \tag{3.5}$$

Indeed, if not, then $v_n \to 0$ in l^∞ as $n \to \infty$. For $r > p$,

$$\|v_n\|_{l^r}^r \leq \|v_n\|_{l^\infty}^{r-p} \|v_n\|_{l^p}^p$$

we have $v_n \to 0$ in all $l^r,\ r > p$.

Note that by (f_1) and (f_2), for any $\varepsilon > 0$, there exists $c_\varepsilon > 0$ such that

$$|f(k,u)| \leq \varepsilon |u|^{p-1} + c_\varepsilon |u|^{q-1} \quad \text{and} \quad |F(k,u)| \leq \varepsilon |u|^p + c_\varepsilon |u|^q. \tag{3.6}$$

Then for each $s > 0$, we have

$$\sum_{k\in\mathbb{Z}} F(k, sv_n(k)) \leq \varepsilon s^p \|v_n\|_{l^p}^p + c_\varepsilon s^q \|v_n\|_{l^q}^q$$

which implies that $\sum\limits_{k\in\mathbb{Z}} F(k, sv_n(k)) \to 0$ as $n \to \infty$. So

$$d \geq J(u_n) \geq J(sv_n) = \frac{s^p}{p}\|v^{(k)}\|^p - \sum_{k\in\mathbb{Z}} F(k, sv_n(k)) \to \frac{s^p}{p}, \tag{3.7}$$

as $n \to \infty$. This is a contradiction with $s > \sqrt[p]{pd}$.

Due to periodicity of coefficients, we know J and $\mathcal{N}$ are both invariant under T-translation. Making such shifts, we can assume that $1 \leq k_j \leq T-1$ in (3.5). Moreover, passing to a subsequence, we can assume that $k_j = k_0$ is independent of j.

Next we may extract a subsequence, still denoted by $\{v_n\}$, such that $v_n(k) \to v(k)$ for all $k \in \mathbb{Z}$. Specially, for $k = k_0$, inequality (3.5) shows that $|v(k_0)| \geq \delta$, so $v \neq 0$. Since $|u_n(k)| \to \infty$ as $n \to \infty$, it follows again from (f_3) that

$$0 \leq \frac{J(u_n)}{\|u_n\|^p} = \frac{1}{p} - \sum_{k\in\mathbb{Z}} \frac{F(k, u_n(k))}{(u_n(k))^p} (v_n(k))^p \to -\infty \qquad \textit{as } n \to \infty,$$

a contradiction again. □

Proof of Theorem 1.1.

The proof consists of five steps. The proof of step 1-3 is similar to [12], for readers' convenience, we give the proof.

step 1. we claim that $\mathcal{N}$ is homeomorphic to the unit sphere S in E.

By (3.1) and (3.3), $g(s) > 0$ for $s > 0$ small and $g(s) < 0$ for $s > 0$ large. So s_w is a unique maximum of $g(s)$ and $s_w w$ is the unique point on the ray $s \mapsto sw$ $(s > 0)$ which intersects $\mathcal{N}$. That is, $u \in \mathcal{N}$ is the unique maximum of J on the ray. Therefore, by Lemma 3.1, we may define the mapping $\hat{m} : E \setminus \{0\} \to \mathcal{N}$ by setting

$$\hat{m}(w) := s_w w.$$

Next we show the mapping $\hat{m}$ is continuous. Indeed, suppose $w_n \to w \neq 0$. Since $\hat{m}(tu) = \hat{m}(u)$ for each $t > 0$, we may assume $w_n \in S$ for all n. Write $\hat{m}(w_n) = s_{w_n} w_n$. Then $\{s_{w_n}\}$ is bounded. If not, $s_{wn} \to \infty$ as $n \to \infty$.

Note that by (f_4), for all $u \neq 0$,

$$\begin{aligned} \frac{1}{p} f(k,u)u - F(k,u) &= \frac{1}{p} f(k,u)u - \int_0^u f(k,s)ds \\ &> \frac{1}{p} f(k,u)u - \frac{f(k,u)}{u^{p-1}} \int_0^u s^{p-1} ds \\ &= 0. \end{aligned}$$

Therefore, for all $u \in \mathcal{N}$, we have

$$J(u) = J(u) - \frac{1}{p} J'(u)u = \sum_{k\in\mathbb{Z}} \left(\frac{1}{p} f(k,u(k))u(k) - F(k,u(k)) \right) > 0. \tag{3.8}$$

Combining with (f_3) and Lemma 3.1, we have

$$0 < \frac{J(s_{wn} w)}{(s_{wn})^p} = \frac{1}{p} \|w\|^p - \sum_{k\in\mathbb{Z}} \frac{F(k, s_{wn} w(k))}{|s_{wn} w(k)|^p} |w(k)|^p \to -\infty, \quad \text{as } n \to \infty,$$

this is a contradiction. Therefore, $s_{w_n} \to s > 0$ after passing to a subsequence if needed. Since $\mathcal{N}$ is closed and $\hat{m}(w_n) = s_{w_n} w_n \to sw, sw \in \mathcal{N}$. Hence $sw = s_w w = \hat{m}(w)$ by the uniqueness of s_w of Lemma 3.1. Therefore, $\hat{m}$ is continuous.

Then we define a mapping $m : S \to \mathcal{N}$ by setting $m := \hat{m}|_S$, then m is a homeomorphism between S and $\mathcal{N}$, and the inverse of m is given by $m^{-1}(u) = \frac{u}{\|u\|}$.

step 2. now we define the functional $\hat{\Psi} : E \setminus \{0\} \to \mathbb{R}$ and $\Psi : S \to \mathbb{R}$ by

$$\hat{\Psi}(w) := J(\hat{m}(w)) \quad \text{and} \quad \Psi(w) := \hat{\Psi}|_S.$$

Then we have

$\hat{\Psi} \in C^1(E \setminus \{0\}, \mathbb{R})$ and $\Psi \in C^1(S, \mathbb{R})$. Moreover,

$$\hat{\Psi}'(w)z = \frac{\|\hat{m}(w)\|}{\|w\|} J'(\hat{m}(w))z \quad \text{for all } w, z \in E, w \neq 0. \tag{3.9}$$

$$\Psi'(w)z = \|m(w)\| J'(m(w))z \quad \text{for all } z \in T_w(S) = \{v \in E : (w,v) = 0\}. \tag{3.10}$$

In fact, let $w \in E \setminus \{0\}$ and $z \in E$. By Lemma 3.1 and the mean value theorem, we obtain

$$\begin{aligned}\hat{\Psi}(w+tz) - \hat{\Psi}(w) &= J(s_{w+tz}(w+tz)) - J(s_w w) \\ &\leq J(s_{w+tz}(w+tz)) - J(s_{w+tz}(w)) \\ &= J'(s_{w+tz}(w+\tau_t tz))s_{w+tz}tz,\end{aligned}$$

where $|t|$ is small enough and $\tau_t \in (0,1)$. Similarly,

$$\begin{aligned}\hat{\Psi}(w+tz) - \hat{\Psi}(w) &= J(s_{w+tz}(w+tz)) - J(s_w w) \\ &\geq J(s_w(w+tz)) - J(s_w(w)) \\ &= J'(s_w(w+\eta_t tz))s_w tz,\end{aligned}$$

where $\eta_t \in (0,1)$. Combining these two inequalities and the continuity of function $w \mapsto s_w$, we have

$$\lim_{t\to 0} \frac{\hat{\Psi}(w+tz) - \hat{\Psi}(w)}{t} = s_w J'(s_w w)z = \frac{\|\hat{m}(w)\|}{\|w\|} J'(\hat{m}(w))z.$$

Hence the Gâteaux derivative of $\hat{\Psi}$ is bounded linear in z and continuous in w. It follows that $\hat{\Psi}$ is a class of C^1 and (3.9) holds. Note only that since $w \in S$, $m(w) = \hat{m}(w)$, so (3.10) is clear.

step 3. $\{w_n\}$ is a Palais-Smale sequence for Ψ if and only if $\{m(w_n)\}$ is a Palais-Smale sequence for J.

Let $\{w_n\}$ be a Palais-Smale sequence for Ψ, and let $u_n = m(w_n) \in \mathcal{N}$. Since for every $w_n \in S$ we have an orthogonal splitting $E = T_{w_n}S \oplus \mathbb{R}w_n$, we have

$$\|\Psi'(w_n)\| = \sup_{\substack{z\in T_{w_n}S \\ \|z\|=1}} \Psi'(w_n)z = \|m(w_n)\| \sup_{\substack{z\in T_{w_n}S \\ \|z\|=1}} J'(m(w_n))z = \|u_n\| \sup_{\substack{z\in T_{w_n}S \\ \|z\|=1}} J'(u_n)z.$$

Then

$$\begin{aligned}\|\Psi'(w_n)\| &\leq \|u_n\|\|J'(u_n)\| = \|u_n\| \sup_{\substack{z\in T_{w_n}S,\ t\in\mathbb{R} \\ z+tw\neq 0}} \frac{J'(u_n)(z+tw)}{\|z+tw\|} \\ &\leq \|u_n\| \sup_{z\in T_{w_n}S\setminus\{0\}} \frac{J'(u_n)(z)}{\|z\|} = \|\Psi'(w_n)\|,\end{aligned}$$

Therefore

$$\|\Psi'(w_n)\| = \|u_n\|\|J'(u_n)\|. \tag{3.11}$$

By (3.8), for $u_n \in \mathcal{N}$, $J(u_n) > 0$, so there exists a constant $c_0 > 0$ such that $J(u_n) > c_0$. And since $c_0 \leq J(u_n) = \frac{1}{p}\|u_n\|^p - I(u_n) \leq \frac{1}{p}\|u_n\|^p$, $\|u_n\| \geq \sqrt[p]{pc_0}$. Together with Lemma 3.2, $\sqrt[p]{pc_0} \leq \|u_n\| \leq \sup_n \|u_n\| < \infty$. Hence $\{w_n\}$ is a Palais-Smale sequence for Ψ if and only if $\{u_n\}$ is a Palais-Smale sequence for J.

step 4. by (3.11), $\Psi'(w) = 0$ if and only if $J'(m(w)) = 0$. So w is a critical point of Ψ if and only if $m(w)$ is a nontrivial critical point of J. Moreover, the corresponding values of Ψ and J coincide and $\inf_S \Psi = \inf_{\mathcal{N}} J$.

If $u_0 \in \mathcal{N}$ satisfies $J(u_0) = c := \inf_{u\in\mathcal{N}} J(u)$, then $m^{-1}(u_0) \in S$ is a minimizer of Ψ and therefore a critical point of Ψ, so u_0 is a critical point of J. It remains to show that there exists a minimizer $u \in \mathcal{N}$ of $J|_{\mathcal{N}}$.

Let $\{w_n\} \subset S$ be a minimizing sequence for Ψ. By Ekeland's variational principle we may assume $\Psi(w_n) \to c$, $\Psi'(w_n) \to 0$ as $n \to \infty$, hence $J(u_n) \to c, J'(u_n) \to 0$ as $n \to \infty$, where $u_n := m(w_n) \in \mathcal{N}$.

We know that $\{u_n\}$ is bounded in $\mathcal{N}$ by Lemma 3.2, then there exists a subsequence, still denoted by the same notation, such that u_n weakly converges to some $u \in E$. We claim that there exist $\delta > 0$ and $k_j \in \mathbb{Z}$ such that

$$|u_n(k_j)| \geq \delta. \tag{3.12}$$

Indeed, if not, then $u_n \to 0$ in l^∞ as $n \to \infty$. From the simple fact that, for $r > p$,

$$\|u_n\|_{l^r}^r \leq \|u_n\|_{l^\infty}^{r-p}\|u_n\|_{l^p}^p$$

we have $u_n \to 0$ in all l^r, $r > p$. By (3.6), we know

$$\begin{aligned}\sum_{k\in\mathbb{Z}} f(k, u_n(k))u_n(k) &\leq \varepsilon \sum_{k\in\mathbb{Z}} |u_n(k)|^{p-1} \cdot |u_n(k)| + c_\varepsilon \sum_{k\in\mathbb{Z}} |u_n(k)|^{q-1} \cdot |u_n(k)| \\ &\leq \varepsilon\|u_n\|_{l^p}^p + c_\varepsilon\|u_n\|_{l^q}^{q-1}\end{aligned}$$

which implies that $\sum\limits_{k\in\mathbb{Z}} f(k, u_n(k))u_n(k) = o(\|u_n\|)$ as $n \to \infty$. Then

$$o(\|u_n\|) = (J'(u_n), u_n) = \|u_n\|^p - \sum_{k\in\mathbb{Z}} f(k, u_n(k))u_n(k) = \|u_n\|^p - o(\|u_n\|).$$

So $\|u_n\|^p \to 0$, as $n \to \infty$, which contradicts with $u_n \in \mathcal{N}$.

Since J and J' are both invariant under T-translation. Making such shifts, we can assume that $1 \leq k_j \leq T-1$ in (3.12). Moreover passing to a subsequence, we can assume that $k_j = k_0$ is independent of j. Extract a subsequence, still denoted by $\{u_n\}$, we have $u_n \rightharpoonup u$ and $u_n(k) \to u(k)$ for all $k \in \mathbb{Z}$. Specially, for $k = k_0$, inequality (3.12) shows that $|u(k_0)| \geq \delta$, so $u \neq 0$. Hence $u \in \mathcal{N}$.

step 5. we need to show that $J(u) = c$. By Fatou's lemma, we have

$$\begin{aligned} c &= \lim_{n\to\infty}\left(J(u_n) - \frac{1}{2}J'(u_n)u_n\right) = \lim_{n\to\infty}\sum_{k\in\mathbb{Z}}\left(\frac{1}{2}f(k, u_n(k))u_n(k) - F(k, u_n(k))\right) \\ &\geq \sum_{k\in\mathbb{Z}}\left(\frac{1}{2}f(k, u(k))u(k) - F(k, u(k))\right) = J(u) - \frac{1}{2}J'(u)u = J(u) \geq c.\end{aligned}$$

Hence $J(u) = c$. The proof of Theorem 1.1 is completed. □

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On a solutions of fourth order rational systems of difference equations

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ABSTRACT

In this paper, we get the form of the solutions of the following difference equation systems of order four

$$x_{n+1} = \frac{y_n x_{n-2}}{y_n + y_{n-3}}, \quad y_{n+1} = \frac{x_n y_{n-2}}{\pm x_n \pm x_{n-3}}, \quad n = 0, 1, 2, \cdots ,$$

where the initial conditions $x_{-3},\ x_{-2},\ x_{-1},\ x_0, y_{-3},\ y_{-2},\ y_{-1},\ y_0$ are arbitrary non zero real numbers.

Keywords: difference equations, recursive sequences, system of difference equations, stability, periodicity, boundedness.

Mathematics Subject Classification: 39A10.

1. INTRODUCTION

Difference equations enter as approximations of continuous problems and as models describing life situations in many directions. Recently there has been a great interest in studying difference equations, see, for instance [4], [11], [30] and references cited therein, as well as in studying systems of difference equations (see, e.g. [1], [3], [6], [8]-[10]).

Some of the systems of difference equations that are of considerable interest nowadays are symmetric or those obtained from symmetric ones by modifications of their parameters or the sequence coefficients appearing in them (for the case of nonautonomous systems of difference equations). Such systems are studied, for example, in the following papers: Clark et al. [2] has investigated the global stability properties and asymptotic behavior of solutions of the system

$$x_{n+1} = \frac{x_n}{a + cy_n}, \ \ y_{n+1} = \frac{y_n}{b + dx_n}.$$

Din and Elsayed [5] investigated the boundedness character, persistence, local and global behavior of positive solutions of following two directional interactive and invasive species model

$$x_{n+1} = \alpha + \beta x_n + \gamma x_{n-1} e^{-y_n}, \ y_{n+1} = \delta + \epsilon y_n + \zeta y_{n-1} e^{-x_n}.$$

Halim et al. [13] deal with the form of the solutions of the two following systems of rational difference equations

$$x_{n+1} = \frac{y_n(x_{n-2} + y_{n-3})}{y_{n-3} + x_{n-2} - y_n}, \ y_{n+1} = \frac{x_{n-1}(x_{n-1} + y_{n-2})}{2x_{n-1} + y_{n-2}},$$

$$x_{n+1} = \frac{(y_{n-3} - x_{n-2}) y_n}{y_{n-3} - x_{n-2} + y_n}, \ y_{n+1} = \frac{(y_{n-2} - x_{n-1}) x_{n-1}}{y_{n-2}}.$$

Kurbanli [21] investigated the behavior of the solution of the difference equation system

$$x_{n+1}=\frac{x_{n-1}}{x_{n-1}y_n-1},\ y_{n+1}=\frac{y_{n-1}}{y_{n-1}x_n-1},\quad z_{n+1}=\frac{1}{z_ny_n}.$$

The authors in [27] have got the form of the solutions of some systems of the following rational difference equations

$$x_{n+1}=\frac{x_{n-1}}{\alpha-x_{n-1}y_n},\ y_{n+1}=\frac{y_{n-1}}{\beta+\gamma y_{n-1}x_n}.$$

In [29] Papaschinnopoulos and Schinas studied the oscillatory behavior, the boundedness of the solutions, and the global asymptotic stability of the positive equilibrium of the system of nonlinear difference equations

$$x_{n+1}=A+\frac{y_n}{x_{n-p}},\ \ y_{n+1}=A+\frac{x_n}{y_{n-q}}.$$

In [39], Yalcinkaya et al. studied the periodic character of the following two systems of difference equations

$$x^{(1)}_{n+1}=\frac{x^{(2)}_n}{x^{(2)}_n-1},x^{(2)}_{n+1}=\frac{x^{(3)}_n}{x^{(3)}_n-1},\ldots,x^{(k)}_{n+1}=\frac{x^{(1)}_n}{x^{(1)}_n-1},$$

and

$$x^{(1)}_{n+1}=\frac{x^{(k)}_n}{x^{(k)}_n-1},x^{(2)}_{n+1}=\frac{x^{(1)}_n}{x^{(1)}_n-1},\ldots,x^{(k)}_{n+1}=\frac{x^{(k-1)}_n}{x^{(k-1)}_n-1},$$

where the initial values are nonzero real numbers for $x^{(1)}_0,x^{(2)}_0,\ldots,x^{(k)}_0\neq 1$.

In [42]-[43] Zhang et al. studied the boundedness, the persistence and global asymptotic stability of the positive solutions of the systems of difference equations

$$x_{n+1}=A+\frac{y_{n-m}}{x_n},\qquad y_{n+1}=A+\frac{x_{n-m}}{y_n},$$

and

$$x_n=A+\frac{1}{y_{n-p}},\qquad y_n=A+\frac{y_{n-1}}{x_{n-r}y_{n-s}}.$$

Similar to difference equations and nonlinear systems of rational difference equations were investigated see [12]-[45].

In this paper, we obtain the expressions of the solutions of the following nonlinear systems of difference equations

$$x_{n+1}=\frac{y_nx_{n-2}}{y_n+y_{n-3}},\qquad y_{n+1}=\frac{x_ny_{n-2}}{\pm x_n\pm x_{n-3}},\quad n=0,1,2,\cdots,$$

where the initial values $x_{-3},\ x_{-2},\ x_{-1},\ x_0,\ y_{-3},\ y_{-2},\ y_{-1},\ y_0$ are arbitrary non zero real numbers, moreover, we take some numerical examples for the equation to illustrate the results.

2. ON THE SYSTEM $X_{N+1}=\frac{Y_NX_{N-2}}{Y_N+Y_{N-3}},\quad Y_{N+1}=\frac{X_NY_{N-2}}{X_N+X_{N-3}}$

In this section,we study the solutions of the following system of difference equations

$$x_{n+1}=\frac{y_nx_{n-2}}{y_n+y_{n-3}},\qquad y_{n+1}=\frac{x_ny_{n-2}}{x_n+x_{n-3}},\tag{1}$$

where the initial values $x_{-3},\ x_{-2},\ x_{-1},\ x_0,\ y_{-3},\ y_{-2},\ y_{-1},\ y_0$ are arbitrary nonzero real numbers.

Theorem 1. Suppose that $\{x_n,y_n\}$ are solutions of the system (1). Then for $n=0,1,2,\cdots$, we have the following formula

$$x_{6n-3}=\frac{ad^nh^n}{\prod_{i=0}^{n-1}(e+(6i+3)h)(a+(6i)d)},\qquad x_{6n-2}=\frac{bd^nh^n}{\prod_{i=0}^{n-1}(e+(6i+1)h)(a+(6i+4)d)},$$

$$x_{6n-1}=\frac{cd^nh^n}{\prod_{i=0}^{n-1}(e+(6i+5)h)(a+(6i+2)d)},\qquad x_{6n}=\frac{d^{n+1}h^n}{\prod_{i=0}^{n-1}(e+(6i+3)h)(a+(6i+6)d)},$$

$$x_{6n+1} = \frac{bd^nh^{n+1}}{(e+h)\prod_{i=0}^{n-1}(e+(6i+7)h)(a+(6i+4)d)}, \quad x_{6n+2} = \frac{cd^{n+1}h^n}{(a+2d)\prod_{i=0}^{n-1}(e+(6i+5)h)(a+(6i+8)d)},$$
$$y_{6n-3} = \frac{ed^nh^n}{\prod_{i=0}^{n-1}(e+(6i)h)(a+(6i+3)d)}, \quad y_{6n-2} = \frac{fd^nh^n}{\prod_{i=0}^{n-1}(e+(6i+4)h)(a+(6i+1)d)},$$
$$y_{6n-1} = \frac{gd^nh^n}{\prod_{i=0}^{n-1}(e+(6i+2)h)(a+(6i+5)d)}, \quad y_{6n} = \frac{d^nh^{n+1}}{\prod_{i=0}^{n-1}(e+(6i+6)h)(a+(6i+3)d)},$$
$$y_{6n+1} = \frac{fd^{n+1}h^n}{(a+d)\prod_{i=0}^{n-1}(e+(6i+4)h)(a+(6i+7)d)}, \quad y_{6n+2} = \frac{gd^nh^{n+1}}{(e+2h)\prod_{i=0}^{n-1}(e+(6i+8)h)(a+(6i+5)d)},$$

where $x_{-3}=a,\ x_{-2}=b,\ x_{-1}=c,\ x_0=d,\ y_{-3}=e,\ y_{-2}=f,\ y_{-1}=g,\ y_0=h.$

Proof. By using mathematical induction. The result holds for $n=0$. Suppose that the result holds for $n-1$

$$x_{6n-7} = \frac{cd^{n-1}h^{n-1}}{\prod_{i=0}^{n-2}(e+(6i+5)h)(a+(6i+2)d)}, \quad x_{6n-6} = \frac{d^nh^{n-1}}{\prod_{i=0}^{n-2}(e+(6i+3)h)(a+(6i+6)d)},$$
$$x_{6n-5} = \frac{bd^{n-1}h^n}{(e+h)\prod_{i=0}^{n-2}(e+(6i+7)h)(a+(6i+4)d)}, \quad x_{6n-4} = \frac{cd^nh^{n-1}}{(a+2d)\prod_{i=0}^{n-2}(e+(6i+5)h)(a+(6i+8)d)},$$
$$y_{6n-7} = \frac{gd^{n-1}h^{n-1}}{\prod_{i=0}^{n-2}(e+(6i+2)h)(a+(6i+5)d)}, \quad y_{6n-6} = \frac{d^{n-1}h^n}{\prod_{i=0}^{n-2}(e+(6i+6)h)(a+(6i+3)d)},$$
$$y_{6n-5} = \frac{fd^nh^{n-1}}{(a+d)\prod_{i=0}^{n-2}(e+(6i+4)h)(a+(6i+7)d)}, \quad y_{6n-4} = \frac{gd^{n-1}h^n}{(e+2h)\prod_{i=0}^{n-2}(e+(6i+8)h)(a+(6i+5)d)}.$$

From system (1) we can prove as follow

$$\begin{aligned} x_{6n-3} &= \frac{y_{6n-4}x_{6n-6}}{y_{6n-4}+y_{6n-7}} = \frac{\left(\frac{gd^{n-1}h^n}{(e+2h)\prod_{i=0}^{n-2}(e+(6i+8)h)(a+(6i+5)d)}\right)\left(\frac{d^nh^{n-1}}{\prod_{i=0}^{n-2}(e+(6i+3)h)(a+(6i+6)d)}\right)}{\left(\frac{gd^{n-1}h^n}{(e+2h)\prod_{i=0}^{n-2}(e+(6i+8)h)(a+(6i+5)d)}\right)+\left(\frac{gd^{n-1}h^{n-1}}{\prod_{i=0}^{n-2}(e+(6i+2)h)(a+(6i+5)d)}\right)} \\ &= \frac{d^nh^n}{\prod_{i=0}^{n-2}(e+(6i+3)h)(a+(6i+6)d)\left(h+\frac{(e+2h)\prod_{i=0}^{n-2}(e+(6i+8)h)}{\prod_{i=0}^{n-2}(e+(6i+2)h)}\right)} \\ &= \frac{d^nh^n}{(e+(6n-3)h)\prod_{i=0}^{n-2}(e+(6i+3)h)(a+(6i+6)d)} = \frac{ad^nh^n}{\prod_{i=0}^{n-1}(e+(6i+3)h)(a+(6i)d)}, \end{aligned}$$

$$\begin{aligned} y_{6n-3} &= \frac{x_{6n-4}y_{6n-6}}{x_{6n-4}+x_{6n-7}} = \frac{\frac{cd^nh^{n-1}}{(a+2d)\prod_{i=0}^{n-2}(e+(6i+5)h)(a+(6i+8)d)}\frac{d^{n-1}h^n}{\prod_{i=0}^{n-2}(e+(6i+6)h)(a+(6i+3)d)}}{\left(\frac{cd^nh^{n-1}}{(a+2d)\prod_{i=0}^{n-2}(e+(6i+5)h)(a+(6i+8)d)}+\frac{cd^{n-1}h^{n-1}}{\prod_{i=0}^{n-2}(e+(6i+5)h)(a+(6i+2)d)}\right)} \\ &= \frac{d^nh^n}{(a+2d)\prod_{i=0}^{n-2}(a+(6i+8)d)(e+(6i+6)h)(a+(6i+3)d)\left(\frac{d}{(a+2d)\prod_{i=0}^{n-2}(a+(6i+8)d)}+\frac{1}{\prod_{i=0}^{n-2}(a+(6i+2)d)}\right)} \\ &= \frac{d^nh^n}{\prod_{i=0}^{n-2}(e+(6i+6)h)(a+(6i+3)d)\left(d+\frac{(a+2d)\prod_{i=0}^{n-2}(a+(6i+8)d)}{\prod_{i=0}^{n-2}(a+(6i+2)d)}\right)} \\ &= \frac{d^nh^n}{\prod_{i=0}^{n-2}(e+(6i+6)h)(a+(6i+3)d)\left(d+\frac{(a+2d)\prod_{i=0}^{n-2}(a+(6i+8)d)}{\prod_{i=0}^{n-2}(a+(6i+2)d)}\right)} \\ &= \frac{d^nh^n}{\prod_{i=0}^{n-2}(e+(6i+6)h)(a+(6i+3)d)(a+(6n-3)d)} = \frac{ed^nh^n}{\prod_{i=0}^{n-1}(e+(6i)h)(a+(6i+3)d)}. \end{aligned}$$

The other relations can be proved similarly, this completes the proof.

Lemma 1. Every positive solution of system (1) is bounded and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0.$

Proof: It follows from system (1), that

$$x_{n+1} = \frac{y_n x_{n-2}}{y_n + y_{n-3}} < \frac{y_n x_{n-2}}{y_n} = x_{n-2}, \qquad y_{n+1} = \frac{x_n y_{n-2}}{x_n + x_{n-3}} < \frac{x_n y_{n-2}}{x_n} = y_{n-2}.$$

Then the subsequences $\{x_{3n-2}\}_{n=0}^{\infty}$, $\{x_{3n-1}\}_{n=0}^{\infty}$, $\{x_{3n}\}_{n=0}^{\infty}$, $\{y_{3n-2}\}_{n=0}^{\infty}$, $\{y_{3n-1}\}_{n=0}^{\infty}$, $\{y_{3n}\}_{n=0}^{\infty}$ are decreasing and so are bounded from above by M, N respectively since $M = \max\{x_{-3}, x_{-2}, x_{-1}, x_0\}$, $N = \max\{y_{-3}, y_{-2}, y_{-1}, y_0\}$.

3. ON THE SYSTEM $X_{N+1} = \frac{Y_N X_{N-2}}{Y_N + Y_{N-3}}, \quad Y_{N+1} = \frac{X_N Y_{N-2}}{X_N - X_{N-3}}$

We study, in this section, the solutions formulas of the system of rational difference equations

$$x_{n+1} = \frac{y_n x_{n-2}}{y_n + y_{n-3}}, \qquad y_{n+1} = \frac{x_n y_{n-2}}{x_n - x_{n-3}}, \tag{2}$$

where the initial values $x_{-3},\ x_{-2},\ x_{-1},\ x_0,\ y_{-3},\ y_{-2},\ y_{-1},\ y_0$ are arbitrary nonzero real numbers.

Theorem 2. Assume that $\{x_n, y_n\}$ are solutions of system (2) with $x_{-3} \neq x_0,\ x_{-3} \neq 2x_0$ and $y_{-3} \neq \pm y_0$. Then for $n = 0, 1, 2, ...,$

$$\begin{aligned}
x_{12n-3} &= \frac{d^{2n}h^{2n}}{a^{n-1}(h+e)^n(h-e)^n(2d-a)^n}, & y_{12n-3} &= \frac{h^{2n}d^{2n}}{e^{2n-1}(d-a)^{2n}},\\
x_{12n-2} &= \frac{bd^{2n}h^{2n}}{a^{n}(h+e)^n(h-e)^n(2d-a)^n}, & y_{12n-2} &= \frac{fh^{2n}d^{2n}}{e^{2n}(d-a)^{2n}},\\
x_{12n-1} &= \frac{cd^{2n}h^{2n}}{a^{n}(h+e)^n(h-e)^n(2d-a)^n}, & y_{12n-1} &= \frac{gh^{2n}d^{2n}}{e^{2n}(d-a)^{2n}},\\
x_{12n} &= \frac{d^{2n+1}h^{2n}}{a^{n}(h+e)^n(h-e)^n(2d-a)^n}, & y_{12n} &= \frac{h^{2n+1}d^{2n}}{e^{2n}(d-a)^{2n}},\\
x_{12n+1} &= \frac{bd^{2n}h^{2n+1}}{a^{n}(h+e)^{n+1}(h-e)^n(2d-a)^n}, & y_{12n+1} &= \frac{fh^{2n}d^{2n+1}}{e^{2n}(d-a)^{2n+1}},\\
x_{12n+2} &= \frac{cd^{2n+1}h^{2n}}{a^{n}(h+e)^n(h-e)^n(2d-a)^{n+1}}, & y_{12n+2} &= \frac{-gh^{2n+1}d^{2n}}{e^{2n+1}(d-a)^{2n}},\\
x_{12n+3} &= \frac{d^{2n+1}h^{2n+1}}{a^{n}(h+e)^n(h-e)^{n+1}(2d-a)^n}, & y_{12n+3} &= \frac{-h^{2n+1}d^{2n+1}}{e^{2n}(d-a)^{2n+1}},\\
x_{12n+4} &= \frac{bd^{2n+1}h^{2n+1}}{a^{n+1}(h+e)^{n+1}(h-e)^n(2d-a)^n}, & y_{12n+4} &= \frac{fh^{2n+1}d^{2n+1}}{e^{2n+1}(d-a)^{2n+1}},\\
x_{12n+5} &= \frac{cd^{2n+1}h^{2n+1}}{a^{n}(h+e)^{n+1}(h-e)^n(2d-a)^{n+1}}, & y_{12n+5} &= \frac{-gh^{2n+1}d^{2n+1}}{e^{2n+1}(d-a)^{2n+1}},\\
x_{12n+6} &= \frac{d^{2n+2}h^{2n+1}}{a^{n}(h+e)^n(h-e)^{n+1}(2d-a)^{n+1}}, & y_{12n+6} &= \frac{h^{2n+2}d^{2n+1}}{e^{2n+1}(d-a)^{2n+1}},\\
x_{12n+7} &= \frac{bd^{2n+1}h^{2n+2}}{a^{n+1}(h+e)^{n+1}(h-e)^{n+1}(2d-a)^n}, & y_{12n+7} &= \frac{-fh^{2n+1}d^{2n+2}}{e^{2n+1}(d-a)^{2n+2}},\\
x_{12n+8} &= \frac{cd^{2n+2}h^{2n+1}}{a^{n+1}(h+e)^{n+1}(h-e)^n(2d-a)^{n+1}}, & y_{12n+8} &= \frac{-gh^{2n+2}d^{2n+1}}{e^{2n+2}(d-a)^{2n+1}}.
\end{aligned}$$

Proof. By using mathematical induction. The result holds for $n=0$. Suppose that the result holds for $n-1$

$$x_{12n-7} = \frac{cd^{2n-1}h^{2n-1}}{a^{n-1}(h+e)^{n}(h-e)^{n-1}(2d-a)^{n}}, \qquad y_{12n-7} = \frac{-gh^{2n-1}d^{2n-1}}{e^{2n-1}(d-a)^{2n-1}},$$
$$x_{12n-6} = \frac{d^{2n}h^{2n-1}}{a^{n-1}(h+e)^{n-1}(h-e)^{n}(2d-a)^{n}}, \qquad y_{12n-6} = \frac{h^{2n}d^{2n-1}}{e^{2n-1}(d-a)^{2n-1}},$$
$$x_{12n-5} = \frac{bd^{2n-1}h^{2n}}{a^{n}(h+e)^{n}(h-e)^{n}(2d-a)^{n-1}}, \qquad y_{12n-5} = \frac{-fh^{2n-1}d^{2n}}{e^{2n-1}(d-a)^{2n}},$$
$$x_{12n-4} = \frac{cd^{2n}h^{2n-1}}{a^{n}(h+e)^{n}(h-e)^{n-1}(2d-a)^{n}}, \qquad y_{12n-4} = \frac{-gh^{2n}d^{2n-1}}{e^{2n}(d-a)^{2n-1}},$$

From system (2) we have

$$x_{12n-3} = \frac{y_{12n-4}x_{12n-6}}{y_{12n-4}+y_{12n-7}} = \frac{\frac{-gh^{2n}d^{2n-1}}{e^{2n}(d-a)^{2n-1}}\frac{d^{2n}h^{2n-1}}{a^{n-1}(h+e)^{n-1}(h-e)^{n}(2d-a)^{n}}}{\frac{-gh^{2n}d^{2n-1}}{e^{2n}(d-a)^{2n-1}}+\frac{-gh^{2n-1}d^{2n-1}}{e^{2n-1}(d-a)^{2n-1}}}$$
$$= \frac{h^{2n}d^{2n}}{a^{n-1}(h+e)^{n-1}(h-e)^{n}(2d-a)^{n}(h+e)} = \frac{d^{2n}h^{2n}}{a^{n-1}(h+e)^{n}(h-e)^{n}(2d-a)^{n}},$$
$$y_{12n-3} = \frac{x_{12n-4}y_{12n-6}}{x_{12n-4}-x_{12n-7}} = \frac{\frac{cd^{2n}h^{2n-1}}{a^{n}(h+e)^{n}(h-e)^{n-1}(2d-a)^{n}}\frac{h^{2n}d^{2n-1}}{e^{2n-1}(d-a)^{2n-1}}}{\frac{cd^{2n}h^{2n-1}}{a^{n}(h+e)^{n}(h-e)^{n-1}(2d-a)^{n}}-\frac{cd^{2n-1}h^{2n-1}}{a^{n-1}(h+e)^{n}(h-e)^{n-1}(2d-a)^{n}}} = \frac{h^{2n}d^{2n}}{e^{2n-1}(d-a)^{2n}},$$
$$x_{12n-2} = \frac{y_{12n-3}x_{12n-5}}{y_{12n-3}+y_{12n-6}}$$
$$= \frac{d^{2n}h^{2n}bd^{2n-1}h^{2n}}{e^{2n-1}(d-a)^{2n}a^{n}(h+e)^{n}(h-e)^{n}(2d-a)^{n-1}\left[\frac{d^{2n}h^{2n}}{e^{2n}(d-a)^{2n}}+\frac{d^{2n-1}h^{2n}}{e^{2n-1}(d-a)^{2n-1}}\right]} = \frac{bd^{2n}h^{2n}}{a^{n}(h+e)^{n}(h-e)^{n}(2d-a)^{n}},$$
$$y_{12n-2} = \frac{x_{12n-3}y_{12n-5}}{x_{12n-3}-x_{12n-6}}$$
$$= \frac{-h^{2n}d^{2n}fd^{2n}h^{2n-1}}{a^{n-1}(h+e)^{n}(h-e)^{n}(2d-a)^{n}e^{2n-1}(d-a)^{2n}\left[\frac{h^{2n}d^{2n}}{a^{n-1}(h+e)^{n}(h-e)^{n}(2d-a)^{n}}-\frac{h^{2n-1}d^{2n}}{a^{n-1}(h+e)^{n-1}(h-e)^{n}(2d-a)^{n}}\right]} = \frac{fd^{2n}h^{2n}}{e^{2n}(d-a)^{2n}}.$$

So, we can prove the other relations and the proof is completed.

Lemma 2. Every positive solution of the equation $x_{n+1} = \frac{y_n x_{n-2}}{y_n + y_{n-3}}$ is bounded and $\lim_{n\to\infty} x_n = 0$.

The following cases can be proved similarly.

4. ON THE SYSTEM $X_{N+1} = \frac{Y_N X_{N-2}}{Y_N+Y_{N-3}}$, $Y_{N+1} = \frac{X_N Y_{N-2}}{-X_N+X_{N-3}}$

In this section, we study the solutions of the system of the difference equations

$$x_{n+1} = \frac{y_n x_{n-2}}{y_n + y_{n-3}}, \qquad y_{n+1} = \frac{x_n y_{n-2}}{-x_n + x_{n-3}}, \tag{3}$$

where the initial values $x_{-3},\ x_{-2},\ x_{-1},\ x_0,\ y_{-3},\ y_{-2},\ y_{-1},\ y_0$ are arbitrary nonzero real numbers with $x_{-3} \neq x_0$, and $y_{-3} \neq -y_0$.

Theorem 3. Let $\{x_n, y_n\}_{n=-3}^{+\infty}$ be solutions of system (3). Then for $n = 0, 1, 2, ...,$

$$x_{6n-3} = \frac{h^n d^n}{a^{n-1}(h+e)^n}, \qquad y_{6n-3} = \frac{h^n d^n}{e^{n-1}(-d+a)^n},$$
$$x_{6n-2} = \frac{bh^n d^n}{a^n(h+e)^n}, \qquad y_{6n-2} = \frac{fh^n d^n}{e^n(-d+a)^n},$$
$$x_{6n-1} = \frac{ch^n d^n}{a^n(h+e)^n}, \qquad y_{6n-1} = \frac{gh^n d^n}{e^n(-d+a)^n},$$

$$x_{6n} = \frac{h^n d^{n+1}}{a^n(h+e)^n}, \qquad y_{6n} = \frac{d^n h^{n+1}}{e^n(-d+a)^n},$$
$$x_{6n+1} = \frac{bd^n h^{n+1}}{a^n(h+e)^{n+1}}, \qquad y_{6n+1} = \frac{fh^n d^{n+1}}{e^n(-d+a)^{n+1}},$$
$$x_{6n+2} = \frac{ch^n d^{n+1}}{a^{n+1}(h+e)^n}, \qquad y_{6n+2} = \frac{gd^n h^{n+1}}{e^{n+1}(-d+a)^n},$$

Lemma 3. The system (3) has a periodic solutions of period 6 iff $hd = e(a-d) = a(h+e)$.

Proof. First, if $hd = e(a-d) = a(h+e)$, then from the form of the solutions of system (3), we see that

$$x_{6n-3} = a^n(h+e)^n a^{n-1}(h+e)^n = a \ , \quad x_{6n-2} = b, \quad x_{6n-1} = c, \ x_{6n} = d, \quad x_{6n+1} = hh+e, \quad x_{6n+2} = cda,$$
$$y_{6n-3} = e, \quad y_{6n-2} = f, \quad y_{6n-1} = g, \ y_{6n} = h, \quad y_{6n+1} = fda - d, \quad y_{6n+2} = hge.$$

Thus system (3) has a periodic solution with period 6. Second:if we have a period 6 then

$$x_{6n-3} = \frac{h^n d^n}{a^{n-1}(h+e)^n} = x_{-3} = a, \ x_{6n-2} = \frac{bh^n d^n}{a^n(h+e)^n} = x_{-2} = b, \ x_{6n-1} = \frac{ch^n d^n}{a^n(h+e)^n} = x_{-1} = c,$$
$$x_{6n} = \frac{h^n d^{n+1}}{a^n(h+e)^n} = x_0 = d, \ x_{6n+1} = \frac{bd^n h^{n+1}}{a^n(h+e)^{n+1}} = x_1 = \frac{bh}{h+e}, \ x_{6n+2} = \frac{ch^n d^{n+1}}{a^{n+1}(h+e)^n} = x_2 = \frac{cd}{a},$$
$$y_{6n-3} = \frac{h^n d^n}{e^{n-1}(-d+a)^n} = y_{-3} = e, \qquad y_{6n-2} = \frac{fh^n d^n}{e^n(-d+a)^n} = y_{-2} = f,$$
$$y_{6n-1} = \frac{gh^n d^n}{e^n(-d+a)^n} = y_{-1} = g, \qquad y_{6n} = \frac{d^n h^{n+1}}{e^n(-d+a)^n} = y_0 = h,$$
$$y_{6n+1} = \frac{fh^n d^{n+1}}{e^n(-d+a)^{n+1}} = y_1 = \frac{fd}{a-d}, \ \ y_{6n+2} = \frac{gd^n h^{n+1}}{e^{n+1}(-d+a)^n} = y_2 = \frac{gh}{e},$$

Then we get $hd = a(h+e)$, $\qquad hd = e(a-d)$, and the proof is completed.

5. ON THE SYSTEM $X_{N+1} = \frac{Y_N X_{N-2}}{Y_N + Y_{N-3}}, \quad Y_{N+1} = \frac{X_N Y_{N-2}}{-X_N - X_{N-3}}$

In this section,we study the solutions of the system of the difference equations

$$x_{n+1} = \frac{y_n x_{n-2}}{y_n + y_{n-3}}, \qquad y_{n+1} = \frac{x_n y_{n-2}}{-x_n - x_{n-3}}, \tag{4}$$

where the initial values $x_{-3}, \ x_{-2}, \ x_{-1}, \ x_0, y_{-3}, \ y_{-2}, \ y_{-1}, \ y_0$ are arbitrary nonzero real numbers.

Theorem 4. If $\{x_n, y_n\}$ are solutions of difference equation system (4). Then for $n = 0, 1, 2, ...,$ we have

$$x_{12n-3} = \frac{d^{2n} h^{2n}}{a^{2n-1}(h+e)^{2n}}, \qquad y_{12n-3} = \frac{(-1)^n d^{2n} h^{2n}}{e^{n-1}(d+a)^n(d-a)^n(2h+e)^n},$$
$$x_{12n-2} = \frac{bd^{2n} h^{2n}}{a^{2n}(h+e)^{2n}}, \qquad y_{12n-2} = \frac{(-1)^n f d^{2n} h^{2n}}{e^n(d+a)^n(d-a)^n(2h+e)^n},$$
$$x_{12n-1} = \frac{cd^{2n} h^{2n}}{a^{2n}(h+e)^{2n}}, \qquad y_{12n-1} = \frac{(-1)^n g d^{2n} h^{2n}}{e^n(d+a)^n(d-a)^n(2h+e)^n},$$
$$x_{12n} = \frac{d^{2n+1} h^{2n}}{a^{2n}(h+e)^{2n}}, \qquad y_{12n} = \frac{(-1)^n d^{2n} h^{2n+1}}{e^n(d+a)^n(d-a)^n(2h+e)^n},$$

$$\begin{aligned}
x_{12n+1} &= \frac{bd^{2n}h^{2n+1}}{a^{2n}(h+e)^{2n+1}}, & y_{12n-3} &= \frac{(-1)^{n+1}fd^{2n+1}h^{2n}}{e^{n}(d+a)^{n+1}(d-a)^{n}(2h+e)^{n}},\\
x_{12n+2} &= \frac{-cd^{2n+1}h^{2n}}{a^{2n+1}(h+e)^{2n}}, & y_{12n+2} &= \frac{(-1)^{n+1}gd^{2n}h^{2n+1}}{e^{n}(d+a)^{n}(d-a)^{n}(2h+e)^{n+1}},\\
x_{12n+3} &= \frac{-d^{2n+1}h^{2n+1}}{a^{2n}(h+e)^{2n+1}}, & y_{12n+3} &= \frac{(-1)^{n+1}d^{2n+1}h^{2n+1}}{e^{n}(d+a)^{n}(d-a)^{n+1}(2h+e)^{n}},\\
x_{12n+4} &= \frac{bd^{2n+1}h^{2n+1}}{a^{2n+1}(h+e)^{2n+1}}, & y_{12n+4} &= \frac{(-1)^{n+1}fd^{2n+1}h^{2n+1}}{e^{n+1}(d+a)^{n+1}(d-a)^{n}(2h+e)^{n}},\\
x_{12n+5} &= \frac{-cd^{2n+1}h^{2n+1}}{a^{2n+1}(h+e)^{2n+1}}, & y_{12n+5} &= \frac{(-1)^{n}gd^{2n+1}h^{2n+1}}{e^{n}(d+a)^{n+1}(d-a)^{n}(2h+e)^{n+1}},\\
x_{12n+6} &= \frac{d^{2n+2}h^{2n+1}}{a^{2n+1}(h+e)^{2n+1}}, & y_{12n+6} &= \frac{(-1)^{n}d^{2n+1}h^{2n+2}}{e^{n}(d+a)^{n}(d-a)^{n+1}(2h+e)^{n+1}},\\
x_{12n+7} &= \frac{-bd^{2n+1}h^{2n+2}}{a^{2n+1}(h+e)^{2n+2}}, & y_{12n+7} &= \frac{(-1)^{n}fd^{2n+2}h^{2n+1}}{e^{n+1}(d+a)^{n+1}(d-a)^{n+1}(2h+e)^{n}},\\
x_{12n+8} &= \frac{-cd^{2n+2}h^{2n+1}}{a^{2n+2}(h+e)^{2n+1}}, & y_{12n+8} &= \frac{(-1)^{n}gd^{2n+1}h^{2n+2}}{e^{n+1}(d+a)^{n+1}(d-a)^{n}(2h+e)^{n+1}}.
\end{aligned}$$

6. NUMERICAL EXAMPLES

Here, we consider interesting numerical examples in order to illustrate the results of the previous sections and to support our theoretical discussions.

Example 1. We consider numerical example for the difference system (1) with the initial conditions $x_{-3} = 2,\ x_{-2} = 14,\ x_{-1} = 6,\ x_0 = 7,\ y_{-3} = 5,\ y_{-2} = 9,\ y_{-1} = 7$ *and* $y_0 = -8$. (See Fig. 1).

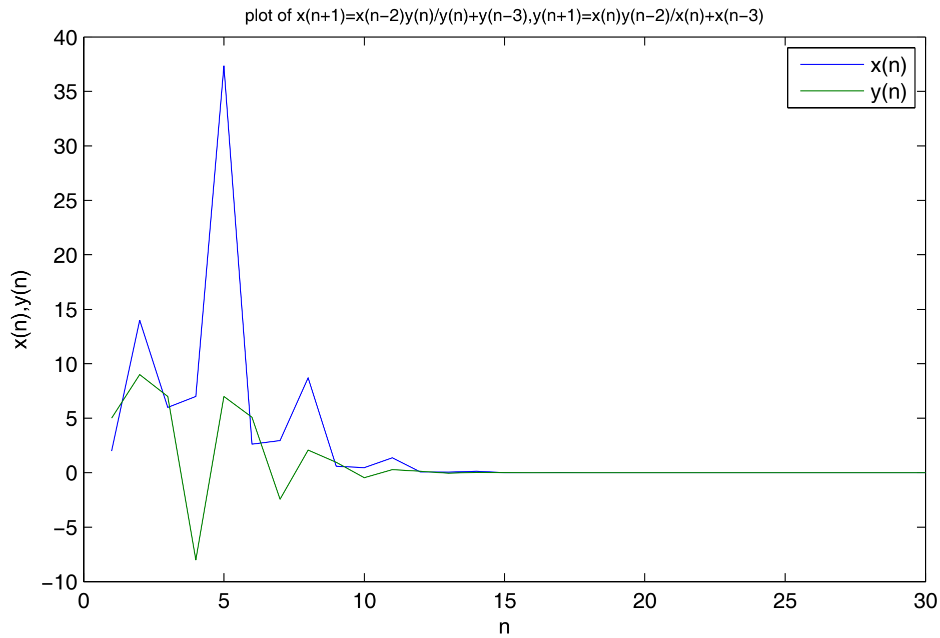

Figure 1.

Example 2. Assume for the system (2) with the initial conditions $x_{-3} = 4,\ x_{-2} = 5,\ x_{-1} = 6, x_0 = 3,\ y_{-3} = 1.8,\ y_{-2} = 9,\ y_{-1} = 2$ *and* $y_0 = 1.9$. See Figure (2).

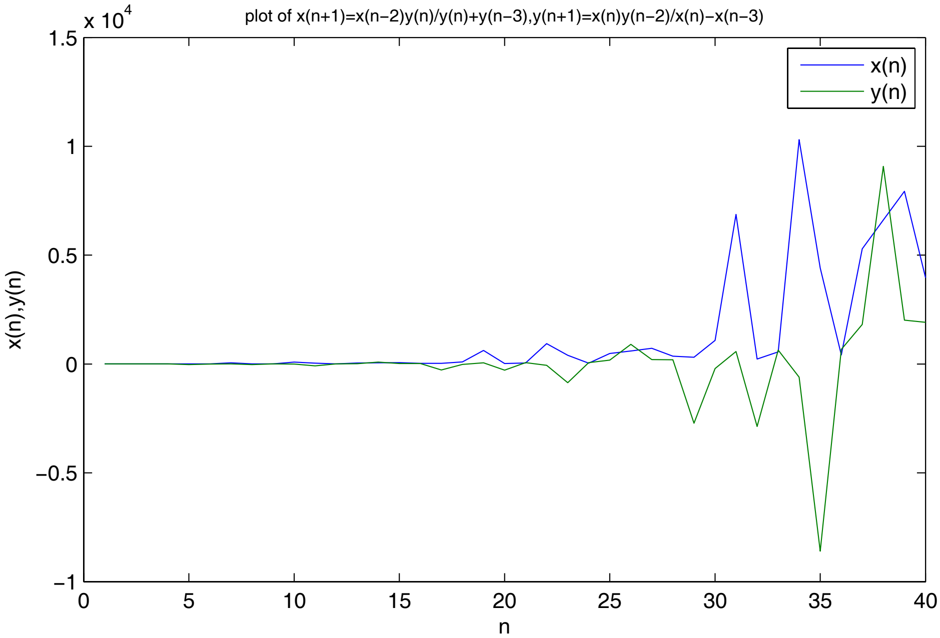

Figure 2.

Example 3. Figure (3) shows the behavior of the solution of the difference system (3) with the initial conditions $x_{-3} = 4,\ x_{-2} = 5,\ x_{-1} = 6, x_0 = 10,\ y_{-3} = 8, y_{-2} = 9,\ y_{-1} = 2$ *and* $y_0 = 2$.

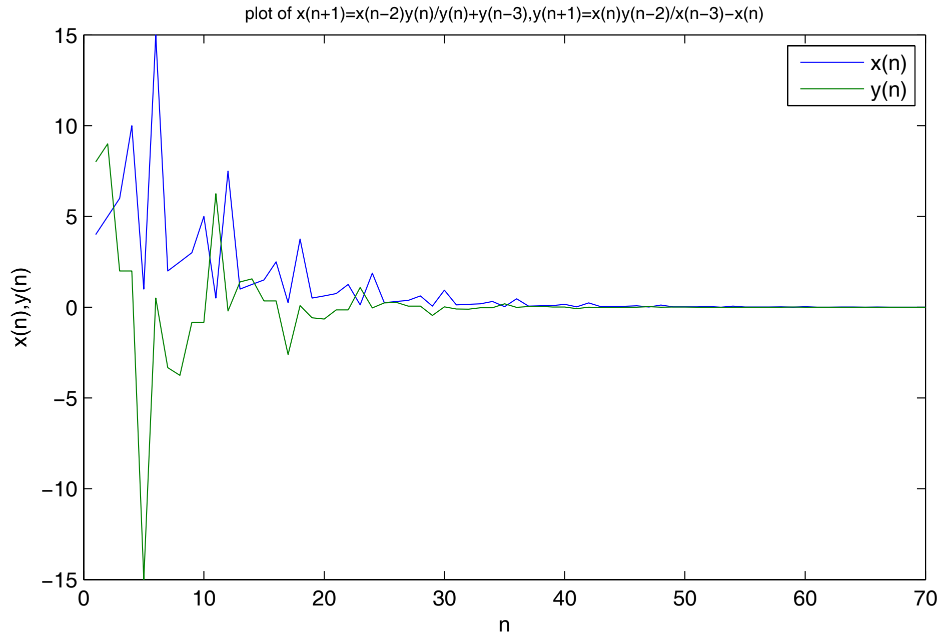

Figure 3.

Example 4. We take the initial conditions, for the system (4), as follows $x_{-3} = 3,\ x_{-2} = 5,\ x_{-1} = -9,$ $x_0 = 6,\ y_{-3} = 2, y_{-2} = 1.7,\ y_{-1} = 2.8$ *and* $y_0 = 4$. See Figure (4).

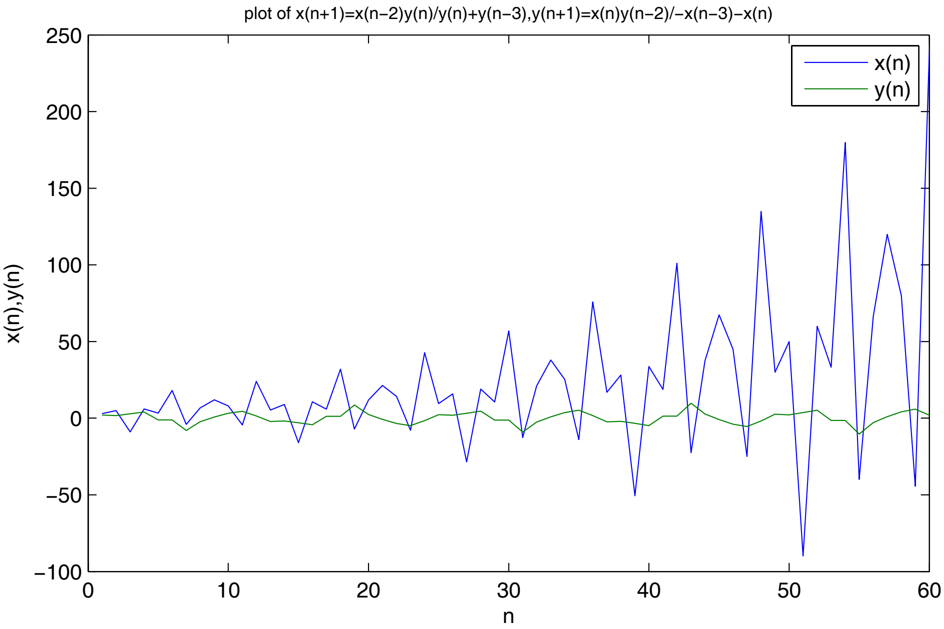

Figure 4.

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On the dynamics of higher Order difference equations $x_{n+1} = ax_n + \frac{\alpha x_n x_{n-l}}{\beta x_n + \gamma x_{n-k}}$

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Abstract

The main objective of this paper is to study the global stability of the positive solutions and the periodic character of the difference equation

$$x_{n+1} = ax_n + \frac{\alpha x_n x_{n-l}}{\beta x_n + \gamma x_{n-k}}, \qquad n = 0,\ 1,\ ...,$$

where the parameters $\alpha,\ \beta,\ \gamma$ and a are positive real numbers and the initial conditions $x_{-t},\ x_{-t+1}\ ...,\ x_{-1}$ and x_0 are positive real numbers where $t = max\{l,\ k\}$. Numerical examples to the difference equation are given to explain our results.

Keywords: difference equations, stability, global stability, boundedness, periodic solutions.
Mathematics Subject Classification: 39A10

1 Introduction and Preliminaries

Our object in this paper is to study some qualitative behavior of the positive solutions of the difference equation

$$x_{n+1} = ax_n + \tfrac{\alpha x_n x_{n-l}}{\beta x_n + \gamma x_{n-k}}, \quad n = 0,\ 1,\ ..., \tag{1}$$

where the parameters $\alpha,\ \beta,\ \gamma$ and δ are positive real numbers and the initial conditions $x_{-t},\ x_{-t+1}\ ...,\ x_{-1}$ and x_0 are positive real numbers where $t = max\{l,\ k\}$. In addition, we obtain the solutions of some special cases of this equation.

Many researchers have Many researchers have studied the behavior of the solution of difference equations for example: Kalabušić et al. [1] studied the global character of the solution of the nonlinear rational difference equation

$$x_{n+1} = \frac{\beta x_{n-l}+\delta x_{n-k}}{Bx_{n-l}+Dx_{n-k}}, \qquad n = 0,\ 1,\ ...,$$

with positive parameters and non-negative initial conditions.

Cinar [2] studied the solutions of the following difference equation

$$x_{n+1} = \frac{ax_{n-1}}{1+bx_n x_{n-1}}, \qquad n = 0,\ 1,\ ...,$$

where $a,\ b,\ x_{-1}$ and x_0 are non-negative real numbers.

Yang et al. [3] studied the invariant intervals, the asymptotic behavior of the solutions, and the global attractivity of equilibrium points of the recursive sequence

$$x_{n+1} = \frac{ax_{n-1}+bx_{n-2}}{c+dx_{n-1}x_{n-2}}, \qquad n = 0,\ 1,\ ...,$$

where $a \geq 0,\ b,\ c,\ d > 0$.

In [4] kenneth et al. got the global asymptotic stability for positive solutions to the difference equation

$$y_{n+1} = \frac{y_{n-k}\ +\ y_{n-m}}{1+y_{n-k}y_{n-m}}, \qquad n = 0,\ 1,\ ...,$$

with $y_{-m},\ y_{-m+1},\ ...,\ y_{-1} \in (0,\ \infty)$ and $1 \leq k \leq m$.

Raafat [5] investigated the global asymptotic stability of all solutions of the difference equation

$$x_{n+1} = \frac{Ax_{n-2}}{B+Cx_n x_{n-1}x_{n-2}}, \qquad n = 0,\ 1,\ ...,$$

where $A,\ B,\ C$ are positive real numbers and the initial conditions $x_{-2},\ x_{-1},\ x_0$ are real numbers.

Also, Raafat [6] introduced an explicit formula and discuss the global behavior of solutions of the difference equation

$$x_{n+1} = \frac{ax_{n-3}}{b+cx_{n-1}x_{n-3}}, \qquad n = 0,\ 1,\ ...,$$

where $a,\ b,\ c$ are positive real numbers and the initial conditions $x_{-3},\ x_{-2},\ x_{-1},\ x_0$ are real numbers.

In [7] Elsayed studied the behavior of the solutions of the difference equation

$$x_{n+1} = ax_{n-1} + \frac{bx_n x_{n-1}}{cx_n+dx_{n-2}}, \qquad n = 0,\ 1,\ ...,$$

where $a,\ b,\ c$ are positive constant and the initial conditions $x_{-2},\ ,\ x_{-1},\ x_0$ are arbitrary positive real numbers.

Zayed et al. [8] investigated some qualitative behavior of the solutions of the difference equation,

$$x_{n+1} = \gamma x_{n-k} + \frac{ax_n + bx_{n-k}}{cx_n - dx_{n-k}}, \qquad n = 0, 1, ...,$$

where the coefficients $\gamma,\ a,\ b, c$ and d are positive constants and the initial conditions $x_{-k},\ ...,\ x_{-1},\ x_0$ are arbitrary positive real numbers, while k is a positive integer number.

Other related results on rational difference equations can be found in refs. [11] - [24].

Let I be some interval of real numbers and let

$$F : I^{t+1} \to I,$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-t}, x_{-t+1}, ..., x_0 \in I$, the difference equation

$$x_{n+1} = F(x_n, x_{n-1}, ..., x_{n-t}), \quad n = 0, 1, ..., \tag{2}$$

has a unique solution $\{x_n\}_{n=-t}^{\infty}$.

Definition 1 *The linearized equation of the difference equation (2) about the equilibrium $\overline{x}$ is the linear difference equation*

$$y_{n+1} = \sum_{i=0}^{t} \frac{\partial F(\overline{x}, \overline{x}, ..., \overline{x})}{\partial x_{n-i}} y_{n-i}. \tag{3}$$

Now, assume that the characteristic equation associated with (3) is

$$p(\lambda) = p_0\lambda^t + p_1\lambda^{t-1} + ... + p_{t-1}\lambda + p_t = 0, \tag{4}$$

where

$$p_i = \frac{\partial F(\overline{x}, \overline{x}, ..., \overline{x})}{\partial x_{n-i}}.$$

Theorem 1 *[9]: Assume that $p_i \in R,\ i = 1, 2, ..., t$ and t is non-negative integer. Then*

$$\sum_{i=1}^{t} |p_i| < 1,$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+t} + p_1 x_{n+t-1} + ... + p_t x_n = 0, \quad n = 0, 1, ... \ .$$

Theorem 2 *[10, 11]: Let $g : [a,b]^{t+1} \rightarrow [a,b]$, be a continuous function, where t is a positive integer, and where $[a,b]$ is an interval of real numbers. Consider the difference equation*

$$x_{n+1} = g(x_n, x_{n-1}, ..., x_{n-t}), \qquad n = 0, 1, ... \, . \tag{5}$$

Suppose that g satisfies the following conditions.

(1) For each integer i with $1 \leq i \leq t+1$; the function $g(z_1, z_2, ..., z_{t+1})$ is weakly monotonic in z_i for fixed $z_1, z_2, ..., z_{i-1}, z_{i+1}, ..., z_{t+1}$.

(2) If m, M is a solution of the system

$$m = g(m_1, m_2, ..., m_{t+1}), \qquad M = g(M_1, M_2, ..., M_{t+1}),$$

then $m = M$, where for each $i = 1, 2, ..., t+1$, we set

$$m_i = \begin{Bmatrix} m, & \text{if } g \text{ is non-decreasing in } z_i, \\ M, & \text{if } g \text{ is non-increasing in } z_i, \end{Bmatrix},$$

and

$$M_i = \begin{Bmatrix} M, & \text{if } g \text{ is non-decreasing in } z_i, \\ m, & \text{if } g \text{ is non-increasing in } z_i. \end{Bmatrix}.$$

Then there exists exactly one equilibrium point $\bar{x}$ of Equation (5), and every solution of Equation (5) converges to $\bar{x}$.

2 Stability of the Equilibrium Point of Eq. (1)

2.1 Local stability

In this subsection, we study the local stability character of the equilibrium point of Eq. (1).

Eq. (1) has equilibrium point and is given by

$$\bar{x} = a\bar{x} + \tfrac{\alpha\bar{x}^2}{\beta\bar{x}+\gamma\bar{x}}, \qquad or \qquad \left((1-a)(\beta+\gamma) - \alpha\right)\bar{x}^2 = 0,$$

if $(1-a)(\beta+\gamma) \neq \alpha$, then the unique equilibrium point is $\overline{x} = 0$.

Theorem 3 *Assume that $a + \frac{2\alpha}{\beta+\gamma} < 1$, then equilibrium $\overline{x}$ of Eq. (1) is locally asymptotically stable.*

Proof: Let $f : (0, \ \infty)^3 \longrightarrow (0, \ \infty)$ be a continuous function defined by

$$f(v_0, \ v_1, \ v_2) = av_0 + \frac{\alpha v_0 v_1}{\beta v_0 + \gamma v_2}. \tag{6}$$

Therefore, it follows that

$$\begin{aligned}\frac{\partial f(v_0,\ v_1,\ v_2)}{\partial v_0} &= a+\frac{\alpha v_1(\beta v_0+\gamma v_2)-\alpha\beta v_0 v_1}{(\beta v_0+\gamma v_2)^2} = a+\frac{\alpha\beta v_1^2}{(\beta v_0+\gamma v_2)^2},\\ \frac{\partial f(v_0,\ v_1,\ v_2)}{\partial v_1} &= \frac{\alpha v_0}{\beta v_0+\gamma v_2},\\ \frac{\partial f(v_0,\ v_1,\ v_2)}{\partial v_2} &= \frac{-\alpha v_0 v_1}{(\beta v_0+\gamma v_2)^2} = -\frac{\alpha\gamma v_0 v_1}{(\beta v_0+\gamma v_2)^2}.\end{aligned}$$

Then, we see that

$$\frac{\partial f(\bar{x},\ \bar{x},\ \bar{x})}{\partial v_0} = a+\frac{\alpha\beta}{(\beta+\gamma)^2},\quad \frac{\partial f(\bar{x},\ \bar{x},\ \bar{x})}{\partial v_1} = \frac{\alpha}{\beta+\gamma},\quad \frac{\partial f(\bar{x},\ \bar{x},\ \bar{x})}{\partial v_2} = -\frac{\alpha\gamma}{(\beta+\gamma)^2}.$$

and the linearized equation of Eq. (1) about $\bar{x}$, is

$$y_{n+1} = \left(a+\frac{\alpha\beta}{(\beta+\gamma)^2}\right)y_n+\left(\frac{\alpha}{\beta+\gamma}\right)y_{n-l}+\left(\frac{-\alpha\gamma}{(\beta+\gamma)^2}\right)y_{n-k},$$

Under the conditions, we get

$$\left|a+\frac{\alpha\beta}{(\beta+\gamma)^2}\right|+\left|\frac{\alpha}{\beta+\gamma}\right|+\left|\frac{-\alpha\gamma}{(\beta+\gamma)^2}\right|<1,$$

and so

$$a+\frac{2\alpha}{\beta+\gamma}<1.$$

According to Theorem 1, the proof is complete.

Example 1. The solution of the difference equation (1) is local stability if $l = 2, k = 3,\ \alpha = 0.1,\ \beta = 0.2,\ \gamma = 1,\ a = 0.2$ and the initial conditions $x_{-3} = 0.6$, $x_{-2} = 0.3,\ x_{-1} = 0.4$ and $x_0 = 0.8$ (See Fig. 1).

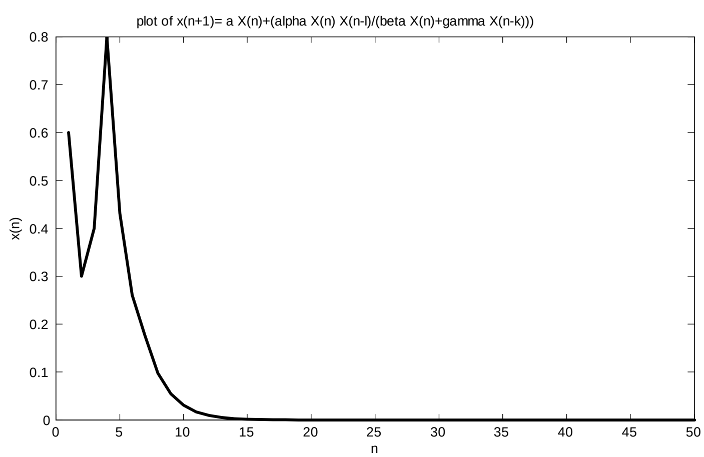

Fig. 1. Plot the behavior of the solution of equation (1).

Example 2. See Figure (2) when we take the difference equation (1) with $l = 2, k = 3,\ \alpha = 1,\ \beta = 0.2,\ \gamma = 0.4,\ a = 0.5$ and the initial conditions $x_{-3} = 0.6$, $x_{-2} = 0.3,\ x_{-1} = 0.4$ and $x_0 = 0.8$.

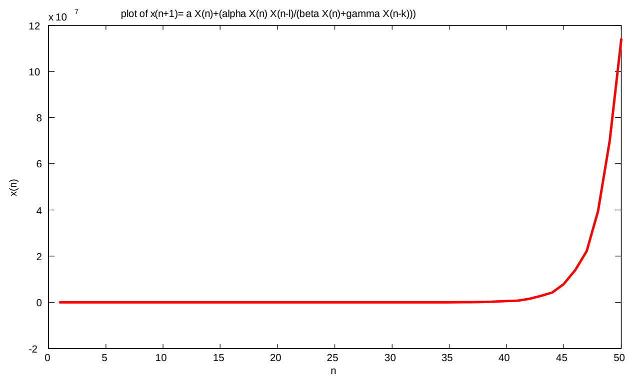

Fig. 2. Draw the behavior of the solution of equation (1).

2.2 Global Stability

In this subsection we study the global stability of the positive solutions of Eq. (1).

Theorem 4 *The equilibrium point $\bar{x}$ is a global attractor of equation (1) if*

$$(1-a)(\beta-\gamma) \neq \alpha.$$

Proof. Let r, s be nonnegative real numbers and assume that $h : [r, s]^3 \to [r, s]$ be a function defined by

$$h(v_0,\ v_1,\ v_2) = av_0 + \frac{\alpha v_0 v_1}{\beta v_0 + \gamma v_2}.$$

Then

$$\frac{\partial h(v_0,\ v_1,\ v_2)}{\partial v_0} = a + \frac{\alpha\beta v_1^2}{(\beta v_0+\gamma v_2)^2},\ \frac{\partial h(v_0,\ v_1,\ v_2)}{\partial v_1} = \frac{\alpha v_0}{\beta v_0+\gamma v_2} \text{ and } \frac{\partial h(v_0,\ v_1,\ v_2)}{\partial v_2} = -\frac{\alpha\gamma v_0 v_1}{(\beta v_0+\gamma v_2)^2}.$$

We can see that the function $h(v_0,\ v_1,\ v_2)$ increasing in $v_0,\ v_1$ and decreasing in v_2.

Suppose that (m, M) is a solution of the system

$$M = h(M,\ M,\ m) \quad \text{and} \quad m = h(m,\ m,\ M).$$

Then from Equation (1), we see that

$$M = aM + \frac{\alpha M^2}{\beta M+\gamma m},\ m = am + \frac{\alpha m^2}{\beta m+\gamma M},$$

then

$$\begin{aligned} \beta(1-a)M + \gamma(1-a)m &= \alpha M, \\ \beta(1-a)m + \gamma(1-a)M &= \alpha m, \end{aligned}$$

6

Subtracting this two equations, we obtain

$$((1-a)(\beta-\gamma)-\alpha)(M-m)=0,$$

under the condition $(1-a)(\beta-\gamma)\neq\alpha$, we see that $M=m$. It follows from Theorem 2 that $\bar{x}$ is a global attractor of Equation (1).

Example 3. The solution of the difference equation (1) is global stability if $l=2, k=3,\ \alpha=0.01,\ \beta=0.2,\ \gamma=0.4,\ a=0.1$ and the initial conditions $x_{-3}=0.6$, $x_{-2}=0.3,\ x_{-1}=0.4$ and $x_0=0.8$ (See Fig. 3).

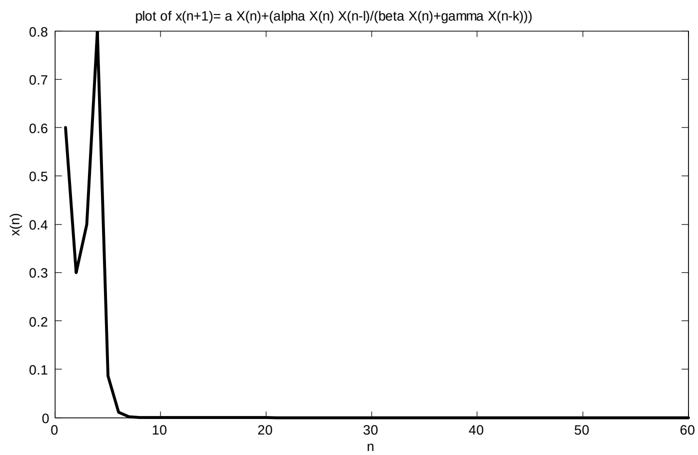

Fig. 3. Sketch the behavior of the solution of Eq. (1).

3 Boundedness of Solutions of Equation (1)

In this section we investigate the boundedness nature of the solutions of Equation (1).

Theorem 5 *Every solution of Equation (1) is bounded if* $a<1$.

Proof. Let $\{x_n\}_{n=-m}^{\infty}$ be a solution of Equation (1). It follows from Equation (1) that

$$x_{n+1}=ax_n+\tfrac{\alpha x_n x_{n-l}}{\beta x_n+\gamma x_{n-k}}\leq ax_n+\tfrac{\alpha x_n x_{n-l}}{\beta x_n}=ax_n+\left(\tfrac{\alpha}{\beta}\right)x_{n-l}.$$

By using a comparison, we can right hand side as follows

$$t_{n+1}=at_n+\left(\tfrac{\alpha}{\beta}\right)t_{n-l}.$$

and this equation is locally asymptotically stable if $a<1$, and converges to the equilibrium point $\overline{t}=0$. Therefore

$$\lim_{n\to\infty}\sup x_n\leqslant 0.$$

Example 4. Figure (4) shows that $l = 4, k = 3,\ \alpha = 0.1,\ \beta = 0.2,\ \gamma = 0.4,\ a = 1.3$, the solution of the difference equation (1) with initial conditions $x_{-3} = 0.6$, $x_{-2} = 0.3,\ x_{-1} = 0.4$ and $x_0 = 0.8$ is unbounded.

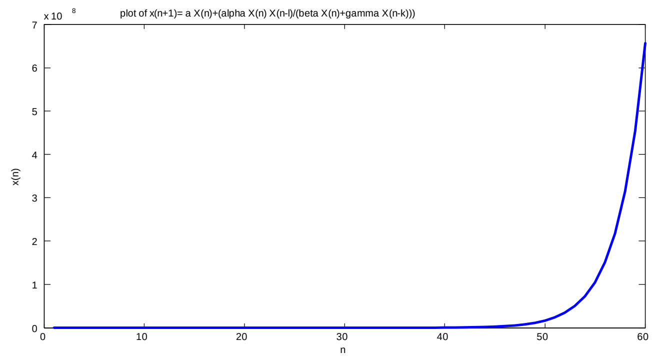

Fig. 4. Polt the behavior of the solution of equation (1) when $a > 1$.

4 Existence of Periodic Solutions

In this section we investigate the existence of periodic solutions of Eq. (5).

Theorem 6 *Equation (1) has no prime period two solutions if l and k are even when $a + \alpha \neq 0$ and $\beta + \gamma \neq 0$.*

Proof. Suppose that there exists a prime period two solution $...p, q, p, q, ...,$ of Equation (1). We see from Equation (1) when l and k are even that

$$p = aq + \frac{\alpha q^2}{\beta q + \gamma q}, \qquad q = ap + \frac{\alpha p^2}{\beta p + \gamma p}.$$

$$(\beta + \gamma)\, pq = a(\beta + \gamma)\, q^2 + \alpha q^2, \tag{7}$$

$$(\beta + \gamma)\, pq = a(\beta + \gamma)\, p^2 + \alpha p^2 \tag{8}$$

Subtracting (7) from (8) gives

$$(a + \alpha)(\beta + \gamma)(p^2 - q^2) = 0,$$

Since $a + \alpha \neq 0$ and $\beta + \gamma \neq 0$, then $p = q$. This is a contradiction. Thus, the proof is completed.

Theorem 7 *Equation (1) has no prime period two solutions if l and k are odd when $\gamma \neq a\beta$.*

Theorem 8 *Equation (1) has no prime period two solutions if l is an even and k is an odd when $\alpha + \gamma \neq a\beta$.*

Theorem 9 *Equation (1) has no prime period two solutions if l is an odd and k is an even when $a(\beta + \gamma) \neq 0$.*

5 Special Cases of Equation (1)

5.1 First Equation When $l = k = 1,\ a = 0$ and $\alpha = \beta = \gamma = 1.$

In this subsection we study the following special case of Eq. (1)

$$x_{n+1} = \frac{x_n x_{n-1}}{x_n + x_{n-1}}, \tag{9}$$

where the initial conditions are arbitrary non zero real numbers.

Theorem 10 *Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq. (9). Then for $n = 0,\ 1,\ 2,\ ...$*

$$x_n = \frac{cb}{f_n b + f_{n+1} c},$$

where $x_{-1} = c,\ x_0 = b,\ \{f_n\}_{n=1}^{\infty} = \{1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ ...\}\ f_0 = 0$ and $f_{-1} = 1.$

Proof: For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n-1$ and n. Now, it follows

$$x_{n-2} = \frac{cb}{f_{n-2}b + f_{n-1}c} \text{ and } x_{n-1} = \frac{cb}{f_{n-1}b + f_n c}.$$

Now, it follows from Eq. (9) that

$$\begin{aligned} x_{n+1} &= \frac{x_n x_{n-1}}{x_n + x_{n-1}} = \frac{\left(\frac{cb}{f_n b + f_{n+1}c}\right)\left(\frac{cb}{f_{n-1}b + f_n c}\right)}{\left(\frac{cb}{f_n b + f_{n+1}c}\right) + \left(\frac{cb}{f_{n-1}b + f_n c}\right)} = \frac{\left(\frac{c^2b^2}{(f_n b + f_{n+1}c)(f_{n-1}b + f_n c)}\right)}{\left(\frac{cb(f_{n-1}b + f_n c) + cb(f_n b + f_{n+1}c)}{(f_n b + f_{n+1}c)(f_{n-1}b + f_n c)}\right)} \\ &= \frac{c^2b^2}{cb(f_{n-1}b + f_n c) + cb(f_n b + f_{n+1}c)} = \frac{cb}{(f_{n-1} + f_n)b + (f_n + f_{n+1})c} = \frac{cb}{f_{n+1}b + f_{n+2}c}. \end{aligned}$$

Thus, the proof is completed.

5.2 Second Equation When $l = k = 1,\ a = 0,\quad \alpha = \beta = 1$ and $\gamma = -1.$

In this subsection we study the following special case of Eq. (1)

$$x_{n+1} = \frac{x_n x_{n-1}}{x_n - x_{n-1}}, \tag{10}$$

where the initial conditions are arbitrary non zero real numbers.

Theorem 11 *Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq. (10). Then for $n = 0,\ 1,\ 2,\ ...$*

$$x_n = \frac{(-1)^{n+1}cb}{f_n b - f_{n+1}c},$$

where $x_{-1} = c,\ x_0 = b,$ and $\{f_n\}_{n=-1}^{\infty} = \{1,\ 0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ ...\}.$

Proof: For $n=0$ the result holds. Now suppose that $n>0$ and that our assumption holds for $n-1$ and n. Now, it follows

$$x_{n-2}=\frac{(-1)^{n-1}cb}{f_{n-2}b-f_{n-1}c} \text{ and } x_{n-1}=\frac{(-1)^{n}cb}{f_{n-1}b-f_{n}c}.$$

Now, it follows from Eq. (10) that

$$\begin{aligned} x_{n+1} &= \frac{x_n x_{n-1}}{x_n-x_{n-1}}=\frac{\left(\frac{(-1)^{n+1}cb}{f_nb-f_{n+1}c}\right)\left(\frac{(-1)^{n}cb}{f_{n-1}b-f_nc}\right)}{\left(\frac{(-1)^{n+1}cb}{f_nb-f_{n+1}c}\right)-\left(\frac{(-1)^{n}cb}{f_{n-1}b-f_nc}\right)}=\frac{\left(\frac{(-1)^{2n+1}c^2b^2}{(f_nb-f_{n+1}c)(f_{n-1}b-f_nc)}\right)}{\left(\frac{-cb(f_{n-1}b-f_nc)-cb(f_nb-f_{n+1}c)}{(f_nb-f_{n+1}c)(f_{n-1}b-f_nc)}\right)} \\ &= \frac{(-1)^{2n+2}c^2b^2}{cb(f_{n-1}b-f_nc)+cb(f_nb-f_{n+1}c)}=\frac{(-1)^{n+2}cb}{(f_{n-1}+f_n)b-(f_{n+1}+f_n)c}=\frac{(-1)^{n+2}cb}{f_{n+1}b-f_{n+2}c}. \end{aligned}$$

Thus, the proof is completed.

5.3 Third Equation When $l=k=1,\ a=0,\quad \alpha=\gamma=1$ and $\beta=-1$.

In this subsection we study the following special case of Eq. (1)

$$x_{n+1}=\frac{x_nx_{n-1}}{-x_n+x_{n-1}}, \tag{11}$$

where the initial conditions are arbitrary non zero real numbers.

Theorem 12 *Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq. (11). Then for $n=0,\ 1,\ 2,\ ...$*

$$x_{3n-1}=(-1)^n c,\quad x_{3n}=(-1)^n b,\ \text{and } x_{3n+1}=\frac{(-1)^{n+1}bc}{b-c},$$

where $x_{-1}=c,\ x_0=b$.

Proof: For $n=0$ the result holds. Now suppose that $n>0$ and that our assumption holds for $n-1$ and n. Now, it follows

$$x_{3n-4}=(-1)^{n-1}c,\quad x_{3n-3}=(-1)^{n-1}b,\ \text{and } x_{3n-2}=\frac{(-1)^{n}bc}{b-c}.$$

Now, it follows from Eq. (11) that

$$\begin{aligned} x_{3n+2} &= \frac{x_{3n+1}x_{3n}}{-x_{3n+1}+x_{3n}}=\frac{\left(\frac{(-1)^{n+1}bc}{b-c}\right)((-1)^nb)}{-\left(\frac{(-1)^{n+1}bc}{b-c}\right)+(-1)^nb}=\frac{(-1)^{n+1}\left(\frac{b^2c}{b-c}\right)}{\left(\frac{bc}{b-c}+b\right)}=\frac{(-1)^{n+1}b^2c}{b^2}=(-1)^{n+1}c, \\ x_{3n} &= \frac{x_{3n-1}x_{3n-2}}{-x_{3n-1}+x_{3n-2}}=\frac{((-1)^nc)\left(\frac{(-1)^nbc}{b-c}\right)}{-(-1)^nc+\left(\frac{(-1)^nbc}{b-c}\right)}=\frac{(-1)^n\left(\frac{bc^2}{b-c}\right)}{\left(-c+\frac{bc}{b-c}\right)}=\frac{(-1)^nbc^2}{c^2}=(-1)^nb, \end{aligned}$$

and

$$x_{3n+4}=\frac{x_{3n}x_{3n-1}}{-x_{3n}+x_{3n-1}}=\frac{((-1)^nb)((-1)^nc)}{-(-1)^nb+(-1)^nc}=\frac{(-1)^nbc}{-(b-c)}=\frac{(-1)^{n+1}bc}{b-c}.$$

Thus, the proof is completed.

Theorem 13 *Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq. (11). Then every solution of Eq. (11) is a periodic with period six. Moreover $\{x_n\}_{n=-1}^{\infty}$ takes the form form*

$$\{x_n\} = \left\{c,\ b,\ -\tfrac{bc}{b-c},\ -c,\ -b,\ \tfrac{bc}{b-c},\ c,\ b,\ -\tfrac{bc}{b-c},\ -c,\ -b,\ \tfrac{bc}{b-c},\ ...\right\}.$$

where $x_{-1} = c,\ x_0 = b$.

Example 5. Figure (5) shows the solution of Eq. (11) when the initial conditions $x_{-1} = 0.3$ and $x_0 = 0.6$.

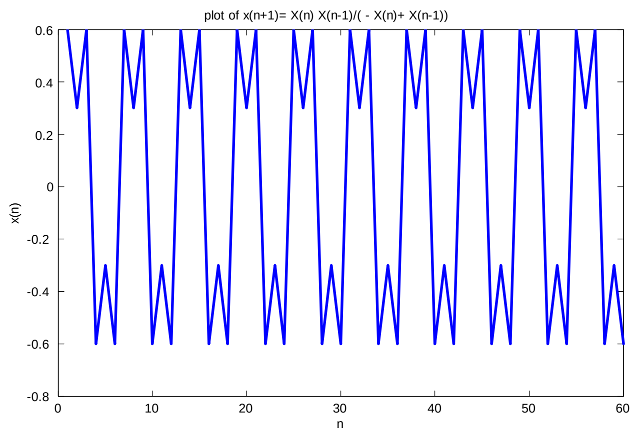

Fig. 5. Draw the solution of equation (11) has a periodic with period six.

5.4 Fourth Equation When $l = k = 1,\ a = 0,\quad \beta = \gamma = 1$ and $\alpha = -1$.

In this subsection we study the following special case of Eq. (1)

$$x_{n+1} = -\frac{x_n x_{n-1}}{x_n + x_{n-1}}, \tag{12}$$

where the initial conditions are arbitrary non zero real numbers.

Theorem 14 *Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq. (12). Then for $n = 0,\ 1,\ 2,\ ...$*

$$x_{3n-1} = c,\quad x_{3n} = b,\ \text{and}\ x_{3n+1} = -\frac{bc}{b+c},$$

where $x_{-1} = c,\ x_0 = b$.

Proof: For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n-1$ and n. Now, it follows

$$x_{3n-4} = c,\quad x_{3n-3} = b,\ \text{and}\ x_{3n-2} = -\frac{bc}{b+c}.$$

Now, it follows from Eq. (12) that

$$x_{3n+2} = -\frac{x_{3n+1}x_{3n}}{x_{3n+1}+x_{3n}} = -\frac{\left(-\frac{bc}{b+c}\right)(b)}{\left(-\frac{bc}{b+c}\right)+b} = -\frac{\left(-\frac{b^2c}{b+c}\right)}{\left(\frac{-bc+b^2+bc}{b+c}\right)} = \frac{b^2c}{b^2} = c,$$

$$x_{3n} = -\frac{x_{3n-1}x_{3n-2}}{x_{3n-1}+x_{3n-2}} = -\frac{(c)\left(-\frac{bc}{b-c}\right)}{c+\left(\frac{-bc}{b-c}\right)} = \frac{\left(\frac{bc^2}{b-c}\right)}{\left(\frac{bc+c^2-bc}{b-c}\right)} = \frac{bc^2}{c^2} = b,$$

and

$$x_{3n+4} = \frac{x_{3n}x_{3n-1}}{-x_{3n}+x_{3n-1}} = -\frac{bc}{b+c}.$$

Thus, the proof is completed.

Theorem 15 *Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq. (12). Then every solution of Eq. (12) is a periodic with period three. Moreover $\{x_n\}_{n=-1}^{\infty}$ takes the form form*

$$\{x_n\} = \left\{c,\ b,\ -\frac{bc}{b+c},\ c,\ b,\ -\frac{bc}{b+c}, c,\ b,\ -\frac{bc}{b+c}, ...\right\},$$

where $x_{-1} = c,\ x_0 = b.$

Example 6. The solution of Eq. (12) when the initial conditions $x_{-1} = 0.3$ and $x_0 = 0.6$ (See Fig. 6).

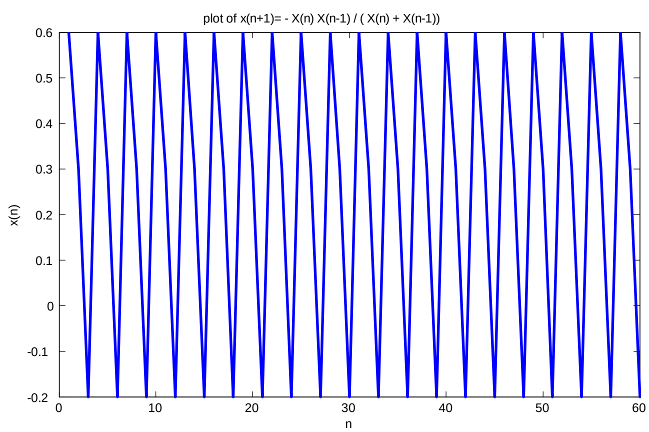

Fig. 6. Polt the solution of equation (12) has a periodic with period three.

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Applications of soft sets in BF-algebras

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Abstract. The aim of this article is to lay a foundation for providing a soft algebraic tool in considering many problems that contain uncertainties. In order to provide these soft algebraic structures, the notion of an intersectional soft subalgebra and an intersectional soft normal subalgebra of a BF-algebra are introduced, and related properties are investigated. A quotient structure of a BF-algebra using an intersectional soft normal subalgebra is constructed. The fundamental homomorphism of a quotient BF-algebra is established.

1. Introduction

The real world is inherently uncertain, imprecise and vague. Various problems in system identification involve characteristics which are essentially non-probabilistic in nature [14]. In response to this situation Zadeh [15] introduced *fuzzy set theory* as an alternative to probability theory. Uncertainty is an attribute of information. In order to suggest a more general framework, the approach to uncertainty is outlined by Zadeh [16]. To solve complicated problem in economics, engineering, and environment, we can't successfully use classical methods because of various uncertainties typical for those problems. There are three theories: theory of probability, theory of fuzzy sets, and the interval mathematics which we can consider as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties. Uncertainties can't be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as probability theory, theory of (intuitionistic) fuzzy sets, theory of vague sets, theory of interval mathematics, and theory of rough sets. However, all of these theories have their own difficulties which are pointed out in [11]. Maji et al. [10] and Molodtsov [11] suggested that one reason for these difficulties may be due to the inadequacy of the parametrization tool of the theory. To overcome these difficulties, Molodtsov [11] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. Worldwide, there has been a rapid growth in interest in soft set theory and its applications in recent years. Evidence of this can be found in the increasing number of high-quality articles on soft sets and related topics that have been published in a variety of international journals, symposia, workshops, and international conferences in recent years.

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Maji et al. [10] described the application of soft set theory to a decision making problem. Maji et al. [9] also studied several operations on the theory of soft sets. Aktaş and Çağman [2] studied the basic concepts of soft set theory, and compared soft sets to fuzzy and rough sets, providing examples to clarify their differences. They also discussed the notion of soft groups. Jun [7] discussed the union soft sets with applications in BCK/BCI-algebras. We refer the reader to the papers [1, 3, 5, 6, 13] for further information regarding algebraic structures/properties of soft set theory.

In this paper, we discuss applications of the an intersectional soft sets in a (normal) subalgebra of a BF-algebra. We introduce the notion of an intersectional (normal) soft subalgebra of a BF-algebra, and investigated related properties. We consider a new construction of a quotient BF-algebra induced by an int-soft normal subalgebra. Also we establish the fundamental homomorphism of a quotient BF-algebra.

2. Preliminaries

We review some definitions and properties that will be useful in our results (see [12]).

By a *BF-algebra* we mean an algebra $(X, *, 0)$ of type (2,0) satisfying the following conditions:

(B1) $x * x = 0$,

(B2) $x * 0 = x$,

(B3) $0 * (x * y) = y * x$

for all $x, y \in X$.

A BF-algebra $(X, *, 0)$ is called a BF_1-algebra if it satisfies the following identity:

(BG) $x = (x * y) * (0 * y)$ for all $x, y \in X$.

A BF-algebra $(X, *, 0)$ is called a BF_2-algebra if it satisfies the following identity:

(BH) $x * y = y * x = 0$ imply $x = y$ for all $x, y \in X$.

For brevity, we also call X a BF-algebra. If we can define a binary operation " $\leq$ " by $x \leq y$ if and only if $x * y = 0$. A non-empty subset A of a BF-algebra X is called a *subalgebra* of X if $x * y \in A$ for any $x, y \in A$. A non-empty subset A of a BF-algebra X is said to be *normal* (or *normal subalgebra*) ([8]) of X if $(x * a) * (y * b) \in A$ for any $x * y, a * b \in A$. Note that any normal subalgebra A of a BF-algebra X is a subalgebra of X, but the converse need not be true. A mapping $f : X \to Y$ of BF-algebras is called a *homomorphism* if $f(x*y) = f(x)*f(y)$ for all $x, y \in X$.

Lemma 2.1. *If X is a BF-algebra, then*

(i) $0 * (0 * x) = x$, *for all* $x \in X$.

(ii) $0 * x = 0 * y$ *implied* $x = y$ *for any* $x, y \in X$.

(iii) *if* $x * y = 0$, *then* $y * x = 0$ *for any* $x, y \in X$.

Lemma 2.2. *Let X be a BF-algebra and let N be a subalgebra of X. If $x * y \in N$ for any $x, y \in N$, then $y * x \in N$.*

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A BG-algebra $(X;*,0)$ is an algebra of type $(2,0)$ satisfying (B1), (B2) and (BG).

Theorem 2.3 *Let X be a BF_1-algebra. Then*

(i) *X is a BG-algebra.*

(ii) *$x*y=0$ implies $x=y$ for any $x,y\in X$.*

(iii) *The right cancellation law holds in X, i.e., if $x*y=z*y$, then $x=z$ for any $x,y,z\in X$.*

(iv) *The left cancellation law holds in X, i.e., if $y*x=y*z$, then $x=z$ for any $x,y,z\in X$.*

Molodtsov [11] defined the soft set in the following way: Let U be an initial universe set and let E be a set of parameters. We say that the pair (U,E) is a *soft universe.* Let $\mathscr{P}(U)$ denotes the power set of U and $A,B,C,\cdots\subseteq E$.

A fair $(\tilde{f},A)$ is called a *soft set* over U, where $\tilde{f}$ is a mapping given by $\tilde{f}:X\to\mathscr{P}(U)$.

In other words, a soft set over U is parameterized family of subsets of the universe U. For $\varepsilon\in A$, $\tilde{f}(\varepsilon)$ may be considered as the set of ε-approximate elements of the set $(\tilde{f},A)$. A soft set over U can bd represented by the set of ordered pairs:

$$(\tilde{f},A)=\{(x,\tilde{f}(x))|x\in A,\tilde{f}(x)\in\mathscr{P}(U)\},$$

where $\tilde{f}:X\to\mathscr{P}(U)$ such that $\tilde{f}(x)=\emptyset$ if $x\notin A$. Clearly, a soft set is not a set.

For a soft set $(\tilde{f},A)$ of X and a subset γ of U, the *γ-inclusive set* of $(\tilde{f},A)$, defined to be the set

$$i_A(\tilde{f};\gamma):=\{x\in A|\gamma\subseteq\tilde{f}(x)\}.$$

3. Intersectional soft subalgebras

In what follows let X denote a BF-algebra X unless otherwise specified.

Definition 3.1. A soft set $(\tilde{f},X)$ over U is called an *intersectional soft subalgebra* (briefly, *int-soft subalgebra* of X if it satisfies:

(3.1) $\tilde{f}(x)\cap\tilde{f}(y)\subseteq\tilde{f}(x*y)$ for all $x,y\in X$.

Proposition 3.2. *Every int-soft subalgebra $(\tilde{f},X)$ of a BF-algebra X satisfies the following inclusion:*

(3.2) *$\tilde{f}(x)\subseteq\tilde{f}(0)$ for all $x\in X$.*

Proof. Using (3.1) and (B1), we have $\tilde{f}(x)=\tilde{f}(x)\cap\tilde{f}(x)\subseteq\tilde{f}(x*x)=\tilde{f}(0)$ for all $x\in X$. □

Example 3.3. Let $(U=\mathbb{Z},X)$ where $X=\{0,1,2,3\}$ is a BF-algebra ([12]) with the following Cayley table:

*	0	1	2	3
0	0	1	2	3
1	1	0	1	1
2	2	1	0	1
3	3	1	1	0

Let $(\tilde{f},X)$ be a soft set over U defined as follows:

$$\tilde{f} : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \mathbb{Z} & \text{if } x = 0 \\ 2\mathbb{Z} & \text{if } x \in \{1, 2\} \\ 3\mathbb{Z} & \text{if } x = 3. \end{cases}$$

It is easy to check that $(\tilde{f}, X)$ is an int-soft subalgebra over U.

Theorem 3.4. *A soft set $(\tilde{f}, X)$ of a BF-algebra X over U is an int-soft subalgebra of X over U if and only if the γ-inclusive set $i_X(\tilde{f}; \gamma)$ is a subalgebra of X for all $\gamma \in \mathscr{P}(U)$ with $i_X(\tilde{f}; \gamma) \neq \emptyset$.*

The subalgebra $i_X(\tilde{f}; \gamma)$ in Theorem 3.4 is called the *inclusive subalgebra* of X.

Proof. Assume that $(\tilde{f}, X)$ is an int-soft subalgebra over U. Let $x, y \in X$ and $\gamma \in \mathscr{P}(U)$ be such that $x, y \in i_X(\tilde{f}; \gamma)$. Then $\gamma \subseteq \tilde{f}(x)$ and $\gamma \subseteq \tilde{f}(y)$. It follows from (3.1) that $\gamma \subseteq \tilde{f}(x) \cap \tilde{f}(y) \subseteq \tilde{f}(x*y)$ Hence $x*y \in i_X(\tilde{f}; \gamma)$. Thus $i_X(\tilde{f}, X)$ is a subalgebra of X.

Conversely, suppose that $i_X(\tilde{f}; \gamma)$ is a subalgebra X for all $\gamma \in \mathscr{P}(U)$ with $i_X(; \gamma) \neq \emptyset$. Let $x, y \in X$, be such that $\tilde{f}(x) = \gamma_x$ and $\tilde{f}(y) = \gamma_y$. Take $\gamma = \gamma_x \cap \gamma_y$. Then $x, y \in i_X(\tilde{f}; \gamma)$ and so $x * y \in i_X(\tilde{f}; \gamma)$ by assumption. Hence $\tilde{f}(x) \cap \tilde{f}(y) = \gamma_x \cap \gamma_y = \gamma \subseteq \tilde{f}(x * y)$. Thus $(\tilde{f}, X)$ is an int-soft subalgebra over U. □

Theorem 3.5. *Every subalgebra of a BF-algebra can be represented as a γ-inclusive set of an int-soft subalgebra.*

Proof. Let A be a subalgebra of a BF-algebra X. For a subset γ of U, define a soft set $(\tilde{f}, X)$ over U by

$$\tilde{f} : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \gamma & \text{if } x \in A \\ \emptyset & \text{if } x \notin A \end{cases}$$

Obviously, $A = i_X(\tilde{f}; \gamma)$. We now prove that $(\tilde{f}; \gamma)$ is an int-soft subalgebra over U. Let $x, y \in X$. If $x, y \in A$, then $x * y \in A$ because A is a subalgebra of X. Hence $\tilde{f}(x) = \tilde{f}(y) = \tilde{f}(x * y) = \gamma$, and so $\tilde{f}(x) \cap \tilde{f}(y) \subseteq \tilde{f}(x * y)$. If $x \in A$ and $y \notin A$, then $\tilde{f}(x) = \gamma$ and $\tilde{f}(y) = \emptyset$ which imply that $\tilde{f}(x) \cap \tilde{f}(y) = \gamma \cap \emptyset = \emptyset \subseteq \tilde{f}(x * y)$. Similarly, if $x \notin A$ and $y \in A$, then $\tilde{f}(x) \cap \tilde{f}(y) \subseteq \tilde{f}(x * y)$. Obviously, if $x \notin A$ and $y \notin A$, then $\tilde{f}(x) \cap \tilde{f}(y) \subseteq \tilde{f}(x * y)$. Therefore $(\tilde{f}, X)$ is an int-soft subalgebra over U. □

Any subalgebra of a BF-algebra X may not be represented as a γ-inclusive set of an int-soft subalgebra $(\tilde{f}, X)$ over U in general (see the following example).

Example 3.6. Let $E = X$ be the set of parameters, and let $U = X$ be the initial universe set where where $X = \{0, 1, 2, 3\}$ is a BF-algebra ([12]) with the following Cayley table:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	0
2	2	3	0	2
3	3	0	2	0

Consider a soft set $(\tilde{f}, X)$ which is given by

$$\tilde{f} : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \{0,3\} & \text{if } x = 0 \\ \{3\} & \text{if } x \in \{1,2,3\} \end{cases}$$

Then $(\tilde{f}, X)$ is an int-soft subalgebra over U. The γ-inclusive set of $(\tilde{f}, X)$ are described as follows:

$$i_X(\tilde{f};\gamma) = \begin{cases} X & \text{if } \gamma \in \{\emptyset, \{3\}\} \\ \{0\} & \text{if } \gamma \in \{\{0\}, \{0,3\}\} \\ \emptyset & \text{otherwise.} \end{cases}$$

The subalgebra $\{0, 2\}$ cannot be a γ-inclusive set $i_X(\tilde{f};\gamma)$ since there is no $\gamma \subseteq U$ such that $i_X(\tilde{f};\gamma) = \{0, 2\}$.

We make a new int-soft subalgebra from old one.

Theorem 3.7. *Let $(\tilde{f}, X)$ be a soft set of a BF-algebra X over U. Define a soft set $(\tilde{f^*}, X)$ of X over U by*

$$\tilde{f^*} : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \tilde{f}(x) & \text{if } x \in i_X(\tilde{f};\gamma) \\ \emptyset & \text{otherwise} \end{cases}$$

where γ is a non-empty subset subset of U. If $(\tilde{f}, X)$ is an int-soft subalgebra of X, then so is $(\tilde{f^*}, X)$.

Proof. If $(\tilde{f}, X)$ is an int-soft subalgebra over U, then $i_X(\tilde{f};\gamma)$ is a subalgebra of X for all $\gamma \subseteq U$ by Theorem 3.6. Let $x, y \in X$. If $x, y \in i_X(\tilde{f};\gamma)$, then $x * y \in i_X(\tilde{f};\gamma)$. Hence we have

$$\tilde{f^*}(x) \cap \tilde{f^*}(y) = \tilde{f}(x) \cap \tilde{f}(y) \subseteq \tilde{f}(x * y) = \tilde{f^*}(x * y).$$

If $x \notin i_X(\tilde{f};\gamma)$ or $y \notin i_X(\tilde{f};\gamma)$, then $\tilde{f^*}(x) = \emptyset$ or $\tilde{f^*}(y) = \emptyset$. Thus

$$\tilde{f^*}(x) \cap \tilde{f^*}(y) = \emptyset \subseteq \tilde{f^*}(x) * \tilde{f^*}(y).$$

Therefore $(\tilde{f^*}, X)$ is an int-soft subalgebra over U. □

Definition 3.8. A soft set $(\tilde{f}, X)$ over U is called an *intersectional soft normal subalgebra* (briefly, *int-soft normal subalgebra* of X if it satisfies:

(3.3) $\tilde{f}(x * y) \cap \tilde{f}(a * b) \subseteq \tilde{f}((x * a) * (y * b))$ for all $x, y, a, b \in X$.

Proposition 3.9. *Every int-soft subalgebra $(\tilde{f}, X)$ of a BF-algebra X satisfies the following inclusion:*

(3.4) $\tilde{f}(x * y) \subseteq \tilde{f}(y * x)$ *for all* $x, y \in X$.

Proof. Using (B3), (3.1) and (3.2), we have

$$\tilde{f}(y * x) = \tilde{f}(0 * (x * y)) \supseteq \tilde{f}(0) \cap \tilde{f}(x * y) = \tilde{f}(x * y),\ \forall x, y \in X.$$

□

Proposition 3.10. *Every int-soft normal subalgebra $(\tilde{f}, X)$ of a BF-algebra X is an int-soft subalgebra of X.*

Proof. Put $y := 0, b := 0$ and $a := y$ in (3.3). Then $\tilde{f}(x * 0) \cap \tilde{f}(y * 0) \subseteq \tilde{f}((x * y) * (0 * 0))$ for any $x, y \in X$. Using (B2) and (B1), we have $\tilde{f}(x) \cap \tilde{f}(y) \subseteq \tilde{f}(x * y)$. Hence $(\tilde{f}, X)$ is an int-soft subalgebra of X. □

The converse of Proposition 3.10 may not be true in general (see Example 3.11).

Example 3.11 Let $E = X$ be the set of parameters where where $X = \{0, 1, 2, 3\}$ is a BF-algebra with the following Cayley table:

$*$	0	1	2	3
0	0	2	1	3
1	1	0	1	1
2	2	2	0	2
3	3	2	1	0

Let $(\tilde{f}, X)$ be a soft set over U defined as follows:

$$\tilde{f} : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \gamma_3 & \text{if } x = 0 \\ \gamma_2 & \text{if } x = 3 \\ \gamma_1 & \text{if } x \in \{1, 2\}. \end{cases}$$

where γ_1, γ_2 and γ_3 are subsets of U with $\gamma_1 \subsetneq \gamma_2 \subsetneq \gamma_3$. It is easy to check that $(\tilde{f}, X)$ is an int-soft normal subalgebra over U.

Let $(\tilde{g}, X)$ be a soft set over U defined as follows:

$$\tilde{g} : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \alpha_3 & \text{if } x = 0 \\ \alpha_2 & \text{if } x \in \{1, 2\} \\ \alpha_1 & \text{if } x = 3. \end{cases}$$

where α_1, α_2 and α_3 are subsets of U with $\alpha_1 \subsetneq \alpha_2 \subsetneq \alpha_3$. It is easy to check that $(\tilde{f}, X)$ is an int-soft subalgebra over U. But it is not an int-soft normal subalgebra over U since $\tilde{g}(2 * 3) \cap \tilde{g}(2 * 0) = \tilde{g}(2) \cap \tilde{g}(2) = \alpha_2 \nsubseteq \alpha_1 = \tilde{g}(3) = \tilde{g}((2 * 2) * (3 * 0))$.

Theorem 3.12. *A soft set $(\tilde{f}, X)$ of X over U is an int-soft normal subalgebra of X over U if and only if the γ-inclusive set $i_X(\tilde{f}; \gamma)$ is a normal subalgebra of X for all $\gamma \in \mathscr{P}(U)$ with $i_X(\tilde{f}; \gamma) \neq \emptyset$.*

Proof. Similar to Theorem 3.4. □

The normal subalgebra $i_X(\tilde{f}; \gamma)$ in Theorem 3.12 is called the *inclusive normal subalgebra* of X.

4. Quotient BF-algebras induces by soft sets

Let $(\tilde{f}, X)$ be an int-soft normal subalgebra of a BF-algebra X. For any $x, y \in X$, we define a binary operation "$\sim^{\tilde{f}}$" on X as follows:

$$x \sim^{\tilde{f}} y \Leftrightarrow \tilde{f}(x * y) = \tilde{f}(0).$$

Lemma 4.1. *The operation $\sim^{\tilde{f}}$ is an equivalence relation on a BF-algebra X.*

Proof. Obviously, it is reflexive. Let $x \sim^{\tilde{f}} y$. Then $\tilde{f}(x * y) = \tilde{f}(0)$. It follows from (3.4) and (3.2) that $\tilde{f}(0) = \tilde{f}(x * y) \subseteq \tilde{f}(y * x) \subseteq \tilde{f}(0)$. Hence $\tilde{f}(y * x) = \tilde{f}(0)$. Hence $\sim^{\tilde{f}}$ is symmetric. Let $x, y, z \in X$ be such that

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$x \sim^{\tilde{f}} y$ and $y \sim^{\tilde{f}} z$. Then $\tilde{f}(x*y) = \tilde{f}(0)$ and $\tilde{f}(y*z) = \tilde{f}(0)$. Using (3.4), (3.3), (B1), (B2) and (3.2), we have

$$\begin{aligned}\tilde{f}(0) = \tilde{f}(x*y) \cap \tilde{f}(y*z) \subseteq \tilde{f}(x*y) \cap \tilde{f}(z*y) \\ \subseteq \tilde{f}((x*z)*(y*y)) \\ = \tilde{f}((x*z)*0) = \tilde{f}(x*z) \subseteq \tilde{f}(0).\end{aligned}$$

Hence $\tilde{f}(x*z) = \tilde{f}(0)$, i.e., $\sim^{\tilde{f}}$ is transitive. Therefore " $\sim^{\tilde{f}}$ " is an equivalence relation on X. □

Lemma 4.2. *For any $x, y, p, q \in X$, if $x \sim^{\tilde{f}} y$ and $p \sim^{\tilde{f}} q$, then $x*p \sim^{\tilde{f}} y*q$.*

Proof. Let $x, y, p, q \in X$ be such that $x \sim^{\tilde{f}} y$ and $p \sim^{\tilde{f}} q$. Then $\tilde{f}(x*y) = \tilde{f}(y*x) = \tilde{f}(0)$ and $\tilde{f}(p*q) = \tilde{f}(q*p) = \tilde{f}(0)$. Using (3.3) and (3.2), we have

$$\begin{aligned}\tilde{f}(0) = \tilde{f}(x*y) \cap \tilde{f}(p*q) \\ \subseteq \tilde{f}((x*p)*(y*q)) \subseteq \tilde{f}(0).\end{aligned}$$

Hence $\tilde{f}((x*p)*(y*q)) = \tilde{f}(0)$. By similar way, we get $\tilde{f}((y*q)*(x*p)) = \tilde{f}(0)$. Therefore $x*p \sim^{\tilde{f}} y*q$. Thus " $\sim^{\tilde{f}}$ " is a congruence relation on X. □

Denote $\tilde{f}_x$ and $X/\tilde{f}$ the set of all equivalence classes containing x and the set of all equivalence classes of X, respectively, i.e.,

$$\tilde{f}_x := \{y \in X | y \sim^{\tilde{f}} x\} \text{ and } X/\tilde{f} := \{\tilde{f}_x | x \in X\}.$$

Define a binary relation $\bullet$ on $X/\tilde{f}$ as follows:

$$\tilde{f}_x \bullet \tilde{f}_y = \tilde{f}_{x*y}$$

for all $\tilde{f}_x, \tilde{f}_y \in X/\tilde{f}$. Then this operation is well-defined by Lemma 4.2.

Theorem 4.3. *If $(\tilde{f}, X)$ is an int-soft normal subalgebra of a BF-algebra X, then the quotient $X/\tilde{f} := (X/\tilde{f}, \bullet, \tilde{f}_0)$ is a BF-algebra.*

Proof. Let $\tilde{f}_x, \tilde{f}_y, \tilde{f}_z \in X/\tilde{f}$. Then we have $\tilde{f}_x \bullet \tilde{f}_x = \tilde{f}_{x*x} = \tilde{f}_0$, $\tilde{f}_x \bullet \tilde{f}_0 = \tilde{f}_{x*0} = \tilde{f}_x$, $\tilde{f}_0 \bullet (\tilde{f}_x \bullet \tilde{f}_y) = \tilde{f}_{0*(x*y)} = \tilde{f}_{y*x} = \tilde{f}_y \bullet \tilde{f}_x$. Therefore $X/\tilde{f} = (X/\tilde{f}, \bullet, \tilde{f}_0)$ is a BF-algebra. □

Corollary 4.4. *If $(\tilde{f}, X)$ is an int-soft normal subalgebra of a BF_2-algebra X, then the quotient $X/\tilde{f} := (X/\tilde{f}, \bullet, \tilde{f}_0)$ is a BF_2-algebra.*

Proof. It is enough to show that $X/\tilde{f}$ satisfies (BH). If $\tilde{f}_x \bullet \tilde{f}_y = \tilde{f}_0$ and $\tilde{f}_y \bullet \tilde{f}_x = \tilde{f}_0$ for any $\tilde{f}_x, \tilde{f}_y \in X/\tilde{f}$, then $\tilde{f}_{x*y} = \tilde{f}_0 = \tilde{f}_{y*x}$. Hence $\tilde{f}(x*y) = \tilde{f}(0) = \tilde{f}(y*x)$ and so $x \sim^{\tilde{f}} y$. Hence $\tilde{f}_x = \tilde{f}_y$. Therefore $X/\tilde{f} = (X/\tilde{f}, \bullet, \tilde{f}_0)$ is a BF_2-algebra. □

Proposition 4.5. *Let $\mu : X \to Y$ be a homomorphism of BF-algebras. If $(\tilde{f}, Y)$ is an int-soft normal subalgebra of Y, then $(\tilde{f} \circ \mu, X)$ is an int-soft normal subalgebra of X.*

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Proof. For any $x, y, a, b \in X$, we have

$$\begin{aligned}(\tilde{f} \circ \mu)((x * a) * (y * b)) =& \tilde{f}(\mu((x * a) * (y * b)) \\ =& \tilde{f}((\mu(x) * \mu(a)) * (\mu(y) * \mu(b))) \\ \supseteq& \tilde{f}(\mu(x) * \mu(y)) \cap \tilde{f}(\mu(a) * \mu(b)) \\ =& \tilde{f}(\mu(x * y)) \cap \tilde{f}(\mu(a * b)) \\ =& (\tilde{f} \circ \mu)(x * y) \cap (\tilde{f} \circ \mu)(a * b).\end{aligned}$$

Hence $\tilde{f} \circ \mu$ is an int-soft normal subalgebra. Therefore $(\tilde{f} \circ \mu, X)$ is an int-soft normal subalgebra of X. □

Theorem 4.6. *Let $X := (X; *_X, 0_X)$ be a BF-algebra and $Y := (Y; *_Y, 0_Y)$ be a BF_2-algebra and let $\mu : X \to Y$ be an epimorphism. If $(\tilde{f}, Y)$ is an int-soft normal subalgebra of Y, then the quotient algebra $X/(\tilde{f} \circ \mu) := (X/(\tilde{f} \circ \mu), \bullet_X, (\tilde{f} \circ \mu)_{0_X})$ is isomorphic to the quotient algebra $Y/\tilde{f} := (Y/\tilde{f}, \bullet_Y, \tilde{f}_{0_Y})$.*

Proof. By Theorem 4.3, Corollary 4.4, and Proposition 4.5, $X/\tilde{f} \circ \mu : (X/(\tilde{f} \circ \mu), \bullet_X, (\tilde{f} \circ \mu)_{0_X})$ is a BF-algebra and $Y/\tilde{f} := (Y/\tilde{f}, \bullet_Y, \tilde{f}_{0_Y})$ is a BF_2-algebra. Define a map

$$\eta : X/(\tilde{f} \circ \mu) \to Y/\tilde{f},\ (\tilde{f} \circ \mu)_x \mapsto \tilde{f}_{\mu(x)}$$

for all $x \in X$. Then the function η is well-defined. In fact, assume that $(\tilde{f} \circ \mu)_x = (\tilde{f} \circ \mu)_y$ for all $x, y \in X$. Then we have

$$\begin{aligned}\tilde{f}(\mu(x) *_Y \mu(y)) =& \tilde{f}(\mu(x *_X y)) = (\tilde{f} \circ \mu)(x *_X y) \\ =& (\tilde{f} \circ \mu)(0_X) = \tilde{f}(\mu(0_X)) = \tilde{f}(0_Y)\end{aligned}$$

and

$$\begin{aligned}\tilde{f}(\mu(y) *_Y \mu(x)) =& \tilde{f}(\mu(y *_X x)) = (\tilde{f} \circ \mu)(y *_X x) \\ =& (\tilde{f} \circ \mu)(0_X) = \tilde{f}(\mu(0_X)) = \tilde{f}(0_Y).\end{aligned}$$

Hence $\tilde{f}_{\mu(x)} = \tilde{f}_{\mu(y)}$.

For any $(\tilde{f} \circ \mu)_x, (\tilde{f} \circ \mu)_X \in X/(\tilde{f} \circ \mu)$, we have

$$\begin{aligned}\eta((\tilde{f} \circ \mu)_x \bullet_X (\tilde{f} \circ \mu)_y) =& \eta((\tilde{f} \circ \mu)_{x*y}) = \tilde{f}_{\mu(x *_X y)} \\ =& \tilde{f}_{\mu(x) *_Y \mu(y)} = \tilde{f}_{\mu(x)} \bullet \tilde{f}_{\mu(y)} \\ =& \eta((\tilde{f} \circ \mu)_x) \bullet_Y \eta((\tilde{f} \circ \mu)_y)).\end{aligned}$$

Therefore η is a homomorphism.

Let $\tilde{f}_a \in Y/\tilde{f}$. Then there exists $x \in X$ such that $\mu(x) = a$ since μ is surjective. Hence $\eta((\tilde{f} \circ \mu)_X) = \tilde{f}_{\mu(x)} = \tilde{f}_a$ and so η is surjective.

Let $x, y \in X$ be such that $\tilde{f}_{\mu(x)} = \tilde{f}_{\mu(y)}$. Then we have

$$\begin{aligned}(\tilde{f} \circ \mu)(x *_X y) =& \tilde{f}(\mu(x *_X y)) = \tilde{f}(\mu(x) *_Y \mu(y)) \\ =& \tilde{f}(0_Y) = \tilde{f}(\mu(0_X)) = (\tilde{f} \circ \mu)(0_X)\end{aligned}$$

and

$$\begin{aligned}(\tilde{f}\circ\mu)(y *_X x) =&\tilde{f}(\mu(y *_X x)) = \tilde{f}(\mu(y) *_Y \mu(x))\\ =&\tilde{f}(0_Y) = \tilde{f}(\mu(0_X)) = (\tilde{f}\circ\mu)(0_X).\end{aligned}$$

It follows that $(\tilde{f}\circ\mu)_X = (\tilde{f}\circ\mu)_y$. Thus η is injective. This completes. □

The homomorphism $\pi : X \to X/\tilde{f}$, $x \to \tilde{f}_X$, is called the *natural homomorphism* of X onto $X/\tilde{f}$. In Theorem 4.6, if we define natural homomorphisms $\pi_X : X \to X/\tilde{f}\circ\mu$ and $\pi_Y : Y \to Y/\tilde{f}$ then it is easy to show that $\eta\circ\pi_X = \pi_Y\circ\mu$, i.e., the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{\mu} & Y \\ {\scriptstyle\pi_X}\downarrow & & {\scriptstyle\pi_Y}\downarrow \\ X/(\tilde{f}\circ\mu) & \xrightarrow{\eta} & Y/\tilde{f}. \end{array}$$

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Symmetric solutions for hybrid fractional differential equations

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Abstract

In this paper we introduce a new class of symmetric functions and study the existence of symmetric solutions for hybrid Caputo fractional differential equations. A fixed point theorem in Banach algebra for two operators is used. An example is presented to illustrate our result.

Keywords: Caputo fractional derivative; hybrid fractional differential equation; symmetric solution; fixed point theorem
2010 Mathematics Subject Classifications: 34A08; 34A12.

1 Introduction

The aim of this manuscript is to study the existence at least one symmetric solution for hybrid Caputo fractional differential equation subject to initial and symmetric conditions

$$\begin{cases} D^{\alpha}\left[\dfrac{x(t)}{f(t,x(t))}\right] + g(t,x(t)) = 0, \quad t \in J := [0,T], \\ x(0) = \beta, \qquad x(t) = x(T-t), \end{cases} \tag{1.1}$$

where D^{α} denotes the Caputo fractional derivative of order α, $1 < \alpha \leq 2$, $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, $g \in C(J \times \mathbb{R}, \mathbb{R})$, $\beta \in \mathbb{R}$. A function $x \in C([0,T], \mathbb{R})$ satisfying the relation $x(t) = x(T-t)$, $t \in [0,T]$, is called *symmetric* on $[0,T]$.

Fractional differential equations have been of great interest recently. It is caused both by the intensive development of the theory of fractional calculus itself and by the applications of such constructions in various science such as physics, mechanics, chemistry, and engineering. There have appeared lots of works, in which fractional derivatives are used for a better description of considered material properties. For details, and some recent results on the subject we refer to [1]-[17] and references cited therein.

Recently, many authors have focused on the existence of symmetric solutions for ordinary differential equation boundary value problems; for example, see [18]-[21] and the references therein. In [22]

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the existence and uniqueness of symmetric solutions for a boundary value problem for nonlinear fractional differential equations with multi-order fractional integral boundary conditions was studied, by using a variety of fixed point theorems (such as Banach contraction principle, nonlinear contractions, Krasnoselskii fixed point theorem and Leray-Schauder nonlinear alternative).

Hybrid fractional differential equations have also been studied by several researchers. This class of equations involves the fractional derivative of an unknown function hybrid with the nonlinearity depending on it. Some recent results on hybrid differential equations can be found in a series of papers ([23]-[28]).

In this paper we prove the existence of symmetric solutions for the hybrid Caputo fractional boundary value problem (1.1). One new result is proved by using a hybrid fixed point theorem for two operators in a Banach algebra due to Dhage [29].

The rest of this paper is organized as follows: In Section 2 we present some preliminary notations, definitions and lemmas that we need in the sequel. Also we introduce a new class of symmetric functions and prove some interesting properties, which are used to establish the Green function. In Section 3 we establish the existence of symmetric solutions for the boundary value problem (1.1). An example illustrating the obtained result is also presented.

2 Preliminaries

In this section, we introduce some notations and definitions of fractional calculus [1, 2] and present preliminary results needed in our proofs later. In addition, a new definition of α-symmetric function is presented and also some properties are proved.

Definition 2.1 *The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $g : (0,\infty) \to \mathbb{R}$ is defined by*

$$I^\alpha g(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s)ds,$$

provided the right-hand side is point-wise defined on $(0,\infty)$, where Γ is the Gamma function.

Definition 2.2 *The Caputo fractional derivative of order $\alpha > 0$ for an at least n-times differentiable function $g : (0,\infty) \to \mathbb{R}$ is defined by*

$$D^\alpha g(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{g^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds, \quad n-1 < \alpha < n,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of real number α.

From the definition of the Caputo fractional derivative, we can obtain the following lemmas.

Lemma 2.3 *(see [1]) Let $\alpha > 0$, the general solution of the fractional differential equation $D^\alpha y(t) = 0$ is given by*

$$y(t) = c_0 + c_1 t + \cdots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, $n = [\alpha] + 1$.

Definition 2.4 *A function $y \in C^2(J,\mathbb{R})$ is called symmetric, if it satisfies the relation $y(t) = y(T-t)$.*

From Definition 2.4 we have $y'(t) = -y'(T-t)$, $y''(t) = y''(T-t)$ and

$$\int_0^{T-t} y(s)ds = \int_0^T y(s)ds - \int_0^t y(s)ds. \tag{2.1}$$

Lemma 2.5 *Let $f \in L^2(J,\mathbb{R})$ be symmetric function. Then we have*

$$I^1 f(T) = \frac{2}{T} I^2 f(T). \tag{2.2}$$

Proof. Since f is symmetric on $[0,T]$, we have

$$\begin{aligned} I^1 f(T) = \int_0^T f(s)ds &= \frac{1}{T}\int_0^T (T-s+s)f(s)ds \\ &= \frac{1}{T}\int_0^T (T-s)f(s)ds + \frac{1}{T}\int_0^T sf(s)ds \\ &= \frac{2}{T}\int_0^T (T-s)f(s)ds = \frac{2}{T}I^2 f(T). \end{aligned}$$

Therefore, (2.2) holds. □

Now, we define a new class of symmetric functions as follows:

Definition 2.6 *A function $f \in C^1(J,\mathbb{R})$ is called α-symmetric if $D^{2-\alpha}f(t)$ is symmetric function on $[0,T]$, where $1<\alpha\le 2$.*

Example 2.7 *Let $f:[0,1]\to\mathbb{R}$ be defined as*

$$f(t) = \frac{4}{3\sqrt{\pi}}t^{\frac{3}{2}}\left(1-\frac{4}{5}t\right).$$

It easy to verify that

$$\begin{aligned} D^{2-\frac{3}{2}}f(t) &= D^{\frac{1}{2}}f(t) \\ &= \frac{4}{3\sqrt{\pi}}D^{\frac{1}{2}}t^{\frac{3}{2}} - \frac{16}{15\sqrt{\pi}}D^{\frac{1}{2}}t^{\frac{5}{2}} \\ &= t(1-t). \end{aligned}$$

Therefore, f is $\frac{3}{2}$-symmetric function.

Remark 2.8 *If $\alpha = 2$, then the class of α-symmetric functions is reduced to the class of usual symmetric functions.*

Lemma 2.9 *Let $z \in C^1(J,\mathbb{R})$ be an α-symmetric function. Then the symmetric solution of linear fractional differential equation*

$$D^\alpha y(t) = z(t), \quad 1<\alpha\le 2,\ t\in J, \tag{2.3}$$
$$y(t) = y(T-t), \tag{2.4}$$

is given by

$$y(t) = I^\alpha z(t) - \frac{t}{T}I^\alpha z(T) + c_0, \tag{2.5}$$

where $c_0 \in \mathbb{R}$.

Proof. By Lemma 2.3, we have

$$y(t) = I^\alpha z(t) + c_1 t + c_0, \tag{2.6}$$

where $c_0, c_1 \in \mathbb{R}$. We apply symmetric condition to obtain

$$I^\alpha z(t) + c_1 t + c_0 = I^\alpha z(T-t) + c_1(T-t) + c_0. \tag{2.7}$$

Evidently, (2.7) becomes

$$\begin{aligned} c_1(2t-T) &= I^\alpha z(T-t) - I^\alpha z(t) \\ &= \int_0^{T-t}\frac{(T-t-s)^{\alpha-1}}{\Gamma(\alpha)}z(s)ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}z(s)ds. \end{aligned} \tag{2.8}$$

Taking the first-order usual derivative with respect to t in (2.8), we get

$$\begin{aligned} 2c_1 &= -I^{\alpha-1}z(T-t) - I^{\alpha-1}z(t) \\ &= -I^1\left(D^{2-\alpha}z\right)(T-t) - I^1\left(D^{2-\alpha}z\right)(t). \end{aligned}$$

Since $D^{2-\alpha}z(t)$ is symmetric on J, and z is symmetric, by (2.1), we have

$$I^1\left(D^{2-\alpha}z\right)(T-t) = I^1\left(D^{2-\alpha}z\right)(T) - I^1\left(D^{2-\alpha}z\right)(t),$$

which leads to

$$\begin{aligned} 2c_1 &= -I^1\left(D^{2-\alpha}z\right)(T) \\ &= -\frac{2}{T}I^2\left(D^{2-\alpha}z\right)(T), \end{aligned}$$

by using Lemma 2.5.
Therefor, we obtain the constant c_1 as

$$c_1 = -\frac{1}{T}I^\alpha z(T) = -\frac{1}{T}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}z(s)ds.$$

Substituting the constant c_1 in (2.6), we get the result in (2.5) as desired. □

In the following we present the Green function of the hybrid fractional boundary value problem (1.1).

Lemma 2.10 *Let $h \in C^1(J,\mathbb{R})$ be the α-symmetric function and $f \in C(J\times\mathbb{R},\mathbb{R}\setminus\{0\})$. Then the unique solution of*

$$D^\alpha\left[\frac{x(t)}{f(t,x(t))}\right] + h(t) = 0, \quad t \in J, \tag{2.9}$$

$$x(0) = \beta, \qquad x(t) = x(T-t), \tag{2.10}$$

is given by

$$x(t) = f(t,x(t))\left(\int_0^T G(t,s)h(s)ds + \frac{\beta}{f(0,\beta)}\right), \tag{2.11}$$

where

$$G(t,s) = \begin{cases} \dfrac{t\,(T-s)^{\alpha-1} - T\,(t-s)^{\alpha-1}}{T\Gamma(\alpha)}, & 0 \le s \le t \le T, \\ \dfrac{t\,(T-s)^{\alpha-1}}{T\Gamma(\alpha)}, & 0 \le t \le s \le T. \end{cases} \tag{2.12}$$

Proof. Applying Lemma 2.9, the equation (2.9) can be written as

$$\frac{x(t)}{f(t,x(t))} = -I^\alpha h(t) + \frac{t}{T}I^\alpha h(T) + c_0, \tag{2.13}$$

where $c_0 \in \mathbb{R}$. The condition $x(0) = 0$ implies that

$$c_0 = \frac{\beta}{f(0,\beta)}.$$

Therefore, the unique solution of problem (2.9)-(2.10) is

$$x(t) = f(t,x(t))\bigg(-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds$$

$$
\begin{aligned}
&+\frac{t}{T\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}h(s)ds\Big) + \frac{\beta}{f(0,\beta)}f(t,x(t)) \\
= \;& f(t,x(t))\int_0^T G(t,s)h(s)ds + \frac{\beta}{f(0,\beta)}f(t,x(t)).
\end{aligned}
$$

The proof is completed. □

Remark 2.11 *The Green's function $G(t,s)$ defined by (2.12), is not positive for all $t,s \in J$. For example, if $T=5$, $t=2$, $s=1$ and $\alpha=3/2$, then we have $G(2,1) = -2/(5\sqrt{\pi})$.*

Lemma 2.12 *The Green's function $G(t,s)$ in (2.12) satisfies the following inequalities*

$$G(t,s) \le G(s,s) \le \frac{((\alpha-1)T)^{\alpha-1}}{\alpha^{\alpha-1}\Gamma(\alpha+1)} \quad \text{for all} \quad s,t \in J. \tag{2.14}$$

Proof. Let us define two functions by

$$g_1(t,s) = t\,(T-s)^{\alpha-1} - T\,(t-s)^{\alpha-1}, \;\; 0 \le s \le t \le T,$$

and

$$g_2(t,s) = t\,(T-s)^{\alpha-1}, \;\; 0 \le t \le s \le T.$$

Obviously, for $0 \le t \le s \le T$, the function $g_2(t,s)$ satisfies

$$g_2(t,s) \le g_2(s,s) = s(T-s)^{\alpha-1}.$$

Let $s \in [0,T)$ be fixed. Differentiating with respect to t the function $g_1(t,s)$, we have

$$\frac{\partial}{\partial t}g_1(t,s) = (T-s)^{\alpha-1} - (\alpha-1)T\,(t-s)^{\alpha-2}, \;\; s<t.$$

We can find that $\partial g_1/\partial t = 0$ if and only if

$$t = t^* = s + \frac{(T-s)^{\frac{\alpha-1}{\alpha-2}}}{((\alpha-1)T)^{\frac{1}{\alpha-2}}}.$$

It follows from $\partial g_1/\partial t > 0$ on $(0,t^*)$ and $\partial g_1/\partial t < 0$ on (t^*,T) that

$$g_1(t,s) \le g_1(t^*,s).$$

Simplifying the above inequality, we get

$$
\begin{aligned}
g_1(t,s) &\le g_1(t^*,s) \\
&= s(T-s)^{\alpha-1} - (2-\alpha)T\cdot\frac{(T-s)^{\frac{(\alpha-1)^2}{\alpha-2}}}{((\alpha-1)T)^{\frac{\alpha-1}{\alpha-2}}} \\
&\le s(T-s)^{\alpha-1} = g_1(s,s),
\end{aligned}
$$

which implies the first inequality.
Next, we will prove the second inequality. Taking the first derivative for $g_2(s,s)$ with respect to s on $[0,T)$, we have

$$g_2'(s,s) = (T-s)^{\alpha-2}(T-\alpha s).$$

Thus $g_2'(s,s)$ has a unique zero at the point $s = s^* = T/\alpha$ such that $s^* \in (0,T)$. Observe that $g_2'(s,s) > 0$ on $(0,s^*)$ and $g_2'(s,s) < 0$ on (s^*,T). Hence

$$g_2(s,s) \le g_2\left(\frac{T}{\alpha},\frac{T}{\alpha}\right) = \frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha}T^\alpha.$$

Then the second inequality is proved. □

Let $E = C([0,T],\mathbb{R})$ be the Banach space endowed with the supremum norm $\|\cdot\|$. Define a multiplication in E by

$$(xy)(t) = x(t)y(t), \ \ \forall t \in J.$$

Clearly E is a Banach algebra with respect to above supremum norm and the multiplication in it. The main result is based on the following fixed point theorem for two operators in Banach algebra due to Dhage [29].

Lemma 2.13 *Let S be a non-empty, closed convex and bounded subset of the Banach algebra E, let $A : E \to E$ and $B : S \to E$ be two operators such that:*

(a) *A is Lipschitzian with a Lipschitz constant δ,*

(b) *B is completely continuous,*

(c) *$x = AxBy \Rightarrow x \in S$ for all $y \in S$, and*

(d) *$M\delta < 1$, where $M = \|B(S)\| = \sup\{\|B(x)\| : x \in S\}$.*

Then the operator equation $x = AxBx$ has a solution in S.

3 Main Result

Now, we are in the position to prove the existence of symmetric solutions for hybrid fractional problem (1.1).

Theorem 3.1 *Assume that the following conditions are satisfied:*

(H_1) *The functions $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $g \in C^1(J \times \mathbb{R}, \mathbb{R})$ are symmetric and α-symmetric on J, respectively.*

(H_2) *There exists a bounded function $\phi(t)$, with bound $\|\phi\|$, such that*

$$|f(t,x) - f(t,y)| \le \|\phi\| \cdot |x - y|$$

for $t \in J$ and $x, y \in \mathbb{R}$.

(H_3) *There exist a function $p \in C(J, \mathbb{R}^+)$ and a continuous nondecreasing function $\Psi : [0,\infty) \to (0,\infty)$ such that*

$$|g(t,x)| \le p(t)\Psi(|x|), \qquad (t,x) \in J \times \mathbb{R}.$$

(H_4) *There exist a number $r > 0$ such that*

$$r \ge \frac{F_0 \left[\dfrac{(\alpha-1)^{\alpha-1}T^\alpha}{\alpha^{\alpha-1}\Gamma(\alpha+1)} \|p\|\Psi(r) + \dfrac{|\beta|}{|f(0,\beta)|} \right]}{1 - \|\phi\| \left[\dfrac{(\alpha-1)^{\alpha-1}T^\alpha}{\alpha^{\alpha-1}\Gamma(\alpha+1)} \|p\|\Psi(r) + \dfrac{|\beta|}{|f(0,\beta)|} \right]} \tag{3.1}$$

where $F_0 = \sup_{t\in J} |f(t,0)|$ and

$$\|\phi\| \left[\frac{(\alpha-1)^{\alpha-1}T^\alpha}{\alpha^{\alpha-1}\Gamma(\alpha+1)} \|p\|\Psi(r) + \frac{|\beta|}{|f(0,\beta)|} \right] < 1. \tag{3.2}$$

Then the problem (1.1) has at least one symmetric solution on J.

Proof. To prove our main result, we first define a subset S of E by

$$S = \{x \in E : \|x\| \leq r\},$$

where r satisfies (3.1). Clearly S is closed, convex and bounded subset of the Banach space E. By Lemma 2.10, we define two operators $\mathcal{A} : E \to E$ by

$$\mathcal{A}x(t) = f(t, x(t)), \qquad t \in J, \tag{3.3}$$

and

$$\mathcal{B}x(t) = \int_0^T G(t,s)g(s,x(s))ds + \frac{\beta}{f(0,\beta)}, \qquad t \in J. \tag{3.4}$$

Hence, the problem (1.1) is transformed into an operator equation as

$$x = \mathcal{A}x\mathcal{B}x. \tag{3.5}$$

Next, we shall show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Lemma 2.13 under our assumptions. This will be achieved in the series of following steps.

Step 1. *We first show that $\mathcal{A}$ is Lipschitzian on E.*

Let $x, y \in E$. Then by (H_2), for $t \in J$ we have

$$\begin{aligned} |\mathcal{A}x(t) - \mathcal{A}y(t)| &= |f(t,x(t)) - f(t,y(t))| \\ &\leq \phi(t)|x(t) - y(t)| \\ &\leq \|\phi\|\|x - y\|, \end{aligned}$$

which implies that $\|\mathcal{A}x - \mathcal{A}y\| \leq \|\phi\|\|x - y\|$ for all $x, y \in E$. Therefore, $\mathcal{A}$ is a Lipschitzian on E with Lipschitz constant $\delta = \|\phi\|$.

Step 2. *The operator $\mathcal{B}$ is completely continuous on S.*

We first show that the operator $\mathcal{B}$ is continuous on S. Let $\{x_n\}$ be a sequence in S converging to a point $x \in S$. Then by Lebesgue dominated convergence theorem, for all $t \in J$, we have

$$\begin{aligned} \lim_{n\to\infty} \mathcal{B}x_n(t) &= \lim_{n\to\infty} \int_0^T G(t,s)g(s,x_n(s))ds + \frac{\beta}{f(0,\beta)} \\ &= \int_0^T G(t,s) \lim_{n\to\infty} g(s,x_n(s))ds + \frac{\beta}{f(0,\beta)} \\ &= \int_0^T G(t,s)g(s,x(s))ds + \frac{\beta}{f(0,\beta)} \\ &= \mathcal{B}x(t). \end{aligned}$$

This shows that $\{\mathcal{B}x_n\}$ converges to $\mathcal{B}x$ pointwise on J.

Next, we will show that $\{\mathcal{B}x_n\}$ is an equicontinuous sequence of functions in S. Let $\tau_1, \tau_2 \in J$ be arbitrary with $\tau_1 < \tau_2$. Then

$$\begin{aligned} |\mathcal{B}x_n(\tau_2) - \mathcal{B}x_n(\tau_1)| &= \left| \int_0^T G(\tau_2,s)g(s,x_n(s))ds - \int_0^T G(\tau_1,s)g(s,x_n(s))ds \right| \\ &\leq \|p\|\Psi(r) \left| \int_0^T G(\tau_2,s)ds - \int_0^T G(\tau_1,s)ds \right| \\ &\leq \|p\|\Psi(r) \left| \int_0^{\tau_2} \frac{\tau_2(T-s)^{\alpha-1} - T(\tau_2-s)^{\alpha-1}}{T\Gamma(\alpha)} ds \right. \end{aligned}$$

$$
\begin{aligned}
& + \int_{\tau_2}^{T} \frac{\tau_2(T-s)^{\alpha-1}}{T\Gamma(\alpha)} ds - \int_{\tau_1}^{T} \frac{\tau_1(T-s)^{\alpha-1}}{T\Gamma(\alpha)} ds \\
& - \int_0^{\tau_1} \frac{\tau_1(T-s)^{\alpha-1} - T(\tau_1-s)^{\alpha-1}}{T\Gamma(\alpha)} ds \Bigg| \\
= \ & \|p\|\Psi(r) \Bigg| \int_0^{T} \frac{\tau_2(T-s)^{\alpha-1}}{T\Gamma(\alpha)} ds - \int_0^{\tau_2} \frac{(\tau_2-s)^{\alpha-1}}{\Gamma(\alpha)} ds \\
& - \int_0^{T} \frac{\tau_1(T-s)^{\alpha-1}}{T\Gamma(\alpha)} ds + \int_0^{\tau_1} \frac{(\tau_1-s)^{\alpha-1}}{\Gamma(\alpha)} ds \Bigg| \\
\leq \ & \|p\|\Psi(r) \int_0^{T} \frac{(\tau_2-\tau_1)(T-s)^{\alpha-1}}{T\Gamma(\alpha)} ds \\
& + \|p\|\Psi(r) \int_0^{\tau_1} \frac{(\tau_2-s)^{\alpha-1} - (\tau_1-s)^{\alpha-1}}{\Gamma(\alpha)} ds \\
& + \|p\|\Psi(r) \int_{\tau_1}^{\tau_2} \frac{(\tau_2-s)^{\alpha-1}}{\Gamma(\alpha)} ds.
\end{aligned}
$$

Consequently

$$|\mathcal{B}x_n(\tau_2) - \mathcal{B}x_n(\tau_1)| \to 0 \quad \text{as} \quad \tau_2 \to \tau_1$$

uniformly for all $n \in \mathbb{N}$. This shows that the convergence $\mathcal{B}x_n \to \mathcal{B}x$ is uniformly and hence $\mathcal{B}$ is a continuous operator on S.

Now we will prove that the set $\mathcal{B}(S)$ is a uniformly bounded in S. For any $x \in S$ and using Lemma 2.12, we have

$$
\begin{aligned}
|\mathcal{B}x(t)| &= \left| \int_0^{T} G(t,s) g(s,x(s)) ds + \frac{\beta}{f(0,\beta)} \right| \\
&\leq \int_0^{T} |G(t,s)| p(s) \Psi(r) ds + \frac{|\beta|}{|f(0,\beta)|} \\
&\leq \frac{(\alpha-1)^{\alpha-1} T^{\alpha}}{\alpha^{\alpha-1}\Gamma(\alpha+1)} \|p\|\Psi(r) + \frac{|\beta|}{|f(0,\beta)|} := K_1,
\end{aligned}
$$

for all $t \in J$. Therefore, $\|\mathcal{B}x\| \leq K_1$ which shows that $\mathcal{B}$ is uniformly bounded on S.

Next, we will show that $\mathcal{B}(S)$ is an equicontinuous set in E. Let $\tau_1, \tau_2 \in J$ with $\tau_1 < \tau_2$ and $x \in S$. Then, as above, we have

$$
\begin{aligned}
|\mathcal{B}x(\tau_2) - \mathcal{B}x(\tau_1)| &= \left| \int_0^{T} G(\tau_2,s) g(s,x(s)) ds - \int_0^{T} G(\tau_1,s) g(s,x(s)) ds \right| \\
&\leq \|p\|\Psi(r) \left| \int_0^{T} G(\tau_2,s) ds - \int_0^{T} G(\tau_1,s) ds \right| \\
&\leq \|p\|\Psi(r) \int_0^{T} \frac{(\tau_2-\tau_1)(T-s)^{\alpha-1}}{T\Gamma(\alpha)} ds \\
&\quad + \|p\|\Psi(r) \int_0^{\tau_1} \frac{(\tau_2-s)^{\alpha-1} - (\tau_1-s)^{\alpha-1}}{\Gamma(\alpha)} ds \\
&\quad + \|p\|\Psi(r) \int_{\tau_1}^{\tau_2} \frac{(\tau_2-s)^{\alpha-1}}{\Gamma(\alpha)} ds,
\end{aligned}
$$

which is independent of $x \in S$. As $\tau_1 \to \tau_2$, the right-hand side of the above inequality tends to zero. Therefore, it follows from the Arzelá-Ascoli theorem that $\mathcal{B}$ is a completely continuous operator on S.

Step 3. *The hypothesis (c) of Lemma 2.13 is satisfied.*

Let $x \in E$ and $y \in S$ be arbitrary elements such that $x = \mathcal{A}x\mathcal{B}y$. Then we have

$$\begin{aligned} |x(t)| &\leq |\mathcal{A}x(t)||\mathcal{B}y(t)| \\ &= |f(t,x(t))| \left| \int_0^T G(t,s)g(s,y(s))ds + \frac{\beta}{f(0,\beta)} \right| \\ &\leq (|f(t,x(t)) - f(t,0)| + |f(t,0)|) \left(\frac{(\alpha-1)^{\alpha-1}T^\alpha}{\alpha^{\alpha-1}\Gamma(\alpha+1)} \|p\|\Psi(r) + \frac{|\beta|}{|f(0,\beta)|} \right) \\ &\leq (|x(t)| \cdot \|\phi\| + F_0) \left(\frac{(\alpha-1)^{\alpha-1}T^\alpha}{\alpha^{\alpha-1}\Gamma(\alpha+1)} \|p\|\Psi(r) + \frac{|\beta|}{|f(0,\beta)|} \right), \end{aligned}$$

which leads to

$$\begin{aligned} \|x\| &\leq (\|x\| \cdot \|\phi\| + F_0) \left(\frac{(\alpha-1)^{\alpha-1}T^\alpha}{\alpha^{\alpha-1}\Gamma(\alpha+1)} \|p\|\Psi(r) + \frac{|\beta|}{|f(0,\beta)|} \right) \\ &\leq r. \end{aligned}$$

Therefore, $x \in S$.

Step 4. *Finally we show that $\delta M < 1$, that is (d) of Lemma 2.13 holds.*

Since

$$\begin{aligned} M &= \|\mathcal{B}(S)\| \\ &= \sup_{x \in S} \left\{ \sup_{t \in J} |\mathcal{B}x(t)| \right\} \\ &\leq \frac{(\alpha-1)^{\alpha-1}T^\alpha}{\alpha^{\alpha-1}\Gamma(\alpha+1)} \|p\|\Psi(r) + \frac{|\beta|}{|f(0,\beta)|}, \end{aligned} \tag{3.6}$$

by (3.2) we have

$$\delta\|M \leq \|\phi\| \left(\frac{(\alpha-1)^{\alpha-1}T^\alpha}{\alpha^{\alpha-1}\Gamma(\alpha+1)} \|p\|\Psi(r) + \frac{|\beta|}{|f(0,\beta)|} \right) < 1,$$

with $\delta = \|\phi\|$.

Thus all the conditions of Lemma 2.13 are satisfied and hence the operator equation $x = \mathcal{A}x\mathcal{B}x$ has a solution in S. In consequence, the problem (1.1) has a symmetric solution on J. This completes the proof. □

Next, we present an example to illustrate our result.

Example 3.2 *Consider the following hybrid fractional differential equation with initial and symmetric conditions*

$$\begin{cases} D^{\frac{3}{2}} \left[\dfrac{x(t)}{\dfrac{x^2(t) + 2|x(t)|}{2(5 + (t-1)^2)(|x(t)|+1)} + \dfrac{1}{2}} \right] + \dfrac{1}{24} \left(1 + \sin^2 \left(\dfrac{8t^{\frac{3}{2}}}{3\sqrt{\pi}} \left(1 - \dfrac{2}{5}t \right) \right) \right) \\ \qquad\qquad \times \left(\dfrac{x^2(t)}{1 + |x(t)|} + 1 \right) = 0, \quad t \in [0,2], \\ x(0) = \dfrac{1}{3}, \qquad x(t) = x(2-t). \end{cases} \tag{3.7}$$

Here $\alpha = 3/2$, $T = 2$ and $\beta = 1/3$. Since $(t-1)^2$ is symmetric on $[0,2]$ and $(8t^{\frac{3}{2}})(1-(2/5)t)/(3\sqrt{\pi})$ is 3/2-symmetric by

$$D^{\frac{1}{2}} \left(\frac{8t^{\frac{3}{2}}}{3\sqrt{\pi}} \left(1 - \frac{2}{5}t \right) \right) = t(2-t), \;\; t \in [0,2],$$

then we get that $f(t,\cdot)$ and $g(t,\cdot)$ are symmetric and 3/2-symmetric functions on $[0,2]$, respectively. With the above information, we find that

$$\begin{aligned} |f(t,x)-f(t,y)| &= \left|\frac{x^2+2|x|}{2(5+(t-1)^2)(|x|+1)} - \frac{y^2+2|y|}{2(5+(t-1)^2)(|y|+1)}\right| \\ &\leq \frac{1}{5+(t-1)^2}|x-y|, \end{aligned}$$

and

$$|g(t,x)| \leq \frac{1}{12}(|x|+1),$$

and $F_0 = \sup_{t\in[0,2]} |f(t,0)| = 1/2$. Choosing $\phi(t) = 1/(5+(t-1)^2)$, $p(t) = 1/12$, we have $\|\phi\| = 1/5$ and $\|p\| = 1/12$. Setting $\Psi(|x|) = |x|+1$, we can find that there exists $0.06962115393 < r < 45.01973321$ which is satisfied (3.1)-(3.2). Thus all the conditions of Theorem 3.1 are satisfied. Therefore, by the conclusion of Theorem 3.1, the problem (3.7) has at least one symmetric solution on $[0,2]$.

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ON THE k-TH DEGENERATION OF THE GENOCCHI POLYNOMIALS

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Abstract. Jeong-Rim-Kim(2015) studied the degenerate Cauchy numbers and polynomials and investigated some properties of these k-times degenerate Cauchy numbers and polynomials. In this paper, we define the degenerate Genocchi polynomials and the k-th degeneration of Genocchi polynomials , and investigate some properties of these polynomials.

1. Introduction

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will, respectively, denote the rings of p-adic integers, the field of p-adic rational numbers, and the completion of algebraic closure of $\mathbb{Q}_p$. The p-adic norm $|\cdot|$ is normalized by $|p|_p = \frac{1}{p}$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the fermionic p-adic integral on $\mathbb{Z}_p$ is defined by Kim as

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x \tag{1}$$

(see [7,8,11,13,14,16,17,20,22]). Then, by (1), we get

$$I_{-1}(f) = -I_{-1}(f) + 2f(0), \tag{2}$$

where $f_1(x) = f(x+1)$.

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From (2), we can derive the following integral equation

$$I_{-1}(f_n) = (-1)^n I_{-1}(f) + 2\sum_{l=0}^{n-1}(-1)^{n-1-l}f(l), \tag{3}$$

where $f_n(x) = f(x+n)$, $(n \in \mathbb{N})$.

As is well known, the Euler polynomials are also defined by the generating function to be

$$\left(\frac{2}{e^t+1}\right)e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} \qquad (\text{see } [1,2,4-22]). \tag{4}$$

When $x = 0$, $E_n = E_n(0)$ are called the Euler numbers.

The degenerate Euler polynomials are also defined by the degenerating function to be

$$\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} E_n(\lambda, x)\frac{t^n}{n!} \qquad (\text{see } [1,4,8,11,13,14,16,17,20,22]). \tag{5}$$

When $x = 0, E_n(\lambda) = E_n(\lambda, 0)$ are called the degenerate Euler number.

Note that $\lim_{x\to 0} E_n(\lambda, x) = E_n(x)$. We recall that the Genocchi polynomials are defined by the generating function to be

$$\frac{2}{e^t+1} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!} \qquad (\text{see } [18,20,22]). \tag{6}$$

In recent years, many researchers have studied various types of special polynomials, for examples, Barnes-type degenerate Euler polynomials, the degenerate Frobenius-Euler polynomials, the degenerate Frobenius-Genocchi polynomials, and degenerate Bernoulli polynomials (see [2,3,6,9,10,12,13,15]).

In particular, recently, Jeong-Rim-Kim ([5]) studied finite times degenerate Cauchy polynomials and investigated some properties of them.

Thus, our motivation in this paper is to define the degenerate Genocchi polynomials, to define the k-th degeneration of Genocchi polynomials, and to investigate some properties of these k-th degeneration of Genocchi polynomials.

2. The k-th degeneration of Genocchi polynomials

In this section, we define the degenerate Genocchi polynomials which are defined by the generating function to be

$$\frac{2t}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=1}^{\infty} g_n^{(0)}(x|\lambda)\frac{t^n}{n!}. \tag{7}$$

When $x = 0, g_n^{(0)}(\lambda) = g_n^{(0)}(0|\lambda)$ are called the degenerate Genocchi number.

From (2), we easily obtain

$$\frac{2t}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}} = t\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{x+y}{\lambda}}d\mu_{-1}(y). \tag{8}$$

We note that the Stirling number of the first kind is defined as

$$(x)_n = \sum_{l=0}^{n} S_1(n,l)x^l \qquad (n \geq 0) \tag{9}$$

where $(x)_n = x(x-1)\cdots(x-n+1)$ and $(x)_0 = 1$, and

$$(\log(1+t))^n = n! \sum_{m=n}^{\infty} S_1(m,n)\frac{t^m}{m!} \tag{10}$$

and the Stirling number of the second kind is defined as

$$(e^t - 1)^n = n! \sum_{l=n}^{\infty} S_2(l,n)\frac{t^l}{l!}. \tag{11}$$

From (7), we get

$$\begin{aligned} \sum_{n=1}^{\infty} g_n^{(0)}(x|\lambda)\frac{t^n}{n!} &= \sum_{n=0}^{\infty} g_{n+1}^{(0)}(x|\lambda)\frac{t^{n+1}}{(n+1)!} \\ &= t\sum_{n=0}^{\infty} \frac{g_{n+1}^{(0)}(x|\lambda)}{n+1}\frac{t^n}{n!}. \end{aligned} \tag{12}$$

From (8), we get

$$\begin{aligned} \frac{2t}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}} &= t\int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{x+y}{\lambda}} d\mu_{-1}(y) \\ &= \sum_{n=0}^{\infty} \lambda^{-n} \int_{\mathbb{Z}_p} \left(\frac{x+y}{\lambda}\right)_n d\mu_{-1}\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} (x+y|\lambda)_n d\mu_{-1}\frac{t^n}{n!}. \end{aligned} \tag{13}$$

Thus, by (7), (12), and (13), we get

$$\frac{g_{n+1}^{(0)}(x|\lambda)}{(n+1)} = \int_{\mathbb{Z}_p} (x+y|\lambda)_n d\mu_{-1}. \tag{14}$$

In the viewpoint of (7), we consider the first degeneration of Genocchi polynomials which are defined by the generating function to be

$$\frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\log(1+\lambda t))^{\frac{1}{\lambda}}+1}(1+\log(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{m=1}^{\infty} g_m^{(1)}(x|\lambda)\frac{t^m}{m!}. \tag{15}$$

By replacing t by $\log(1+\lambda t)^{\frac{1}{\lambda}}$ in (8), we get

$$\begin{aligned} &\frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\log(1+\lambda t))^{\frac{1}{\lambda}}+1}(1+\log(1+\lambda t)^{\frac{x}{\lambda}} \\ = &\ \frac{1}{\lambda}\log(1+\lambda t)\int_{\mathbb{Z}_p} (1+\log(1+\lambda t))^{\frac{x+y}{\lambda}} d\mu_{-1}(y) \\ = &\ \frac{1}{\lambda}\log(1+\lambda t)\sum_{n=0}^{\infty}\frac{1}{n!}\int_{\mathbb{Z}_p} \lambda^{-n}(x+y|\lambda)_n d\mu_{-1}(\log(1+\lambda t))^n \\ = &\ \sum_{n=0}^{\infty}\frac{1}{n!}\int_{\mathbb{Z}_p} (x+y|\lambda)_n \lambda^{-n-1} d\mu_{-1}(\log(1+\lambda t))^{n+1} \end{aligned}$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}(x+y|\lambda)_n\lambda^{-n-1}d\mu_{-1}\frac{(n+1)!}{n!}\sum_{m=n+1}^{\infty}\lambda^m S_1(m,n+1)\frac{t^m}{m!}\\
&= \sum_{m=1}^{\infty}\sum_{n=0}^{m-1}(n+1)\lambda^{m-n-1}S_1(m,n+1)\int_{\mathbb{Z}_p}(x+y|\lambda)_n d\mu_{-1}(y)\frac{t^m}{m!}. \qquad (16)
\end{aligned}$$

Thus, by (14), (15), and (16), we get

$$\begin{aligned}
g_m^{(1)}(x|\lambda) &= \sum_{n=0}^{m-1}(n+1)\lambda^{m-n-1}S_1(m,n+1)\int_{\mathbb{Z}_p}(x+y|\lambda)_n d\mu_{-1}(y)\\
&= \sum_{n=0}^{m-1}\lambda^{-m-n-1}S_1(m,n+1)g_{n+1}^{(0)}(x|\lambda). \qquad (17)
\end{aligned}$$

By (17), we obtain the following theorem.

Theorem 2.1. *Let $m \in \mathbb{N}$. Then we have*

$$g_m^{(1)}(x|\lambda) = \sum_{n=0}^{m-1}\lambda^{m-n-1}S_1(m,n+1)g_{n+1}^{(0)}(x|\lambda). \qquad (18)$$

Now, we consider the second degeneration of Genocchi polynomials as follows:

$$\begin{aligned}
&\frac{2\log(1+\log(1+\lambda t))^{\frac{1}{\lambda}}}{(1+\log(1+\log(1+\lambda t)))^{\frac{1}{\lambda}}+1}(1+\log(1+\log(1+\lambda t)))^{\frac{1}{\lambda}}\\
&= \sum_{m=1}^{\infty}g_m^{(2)}(x|\lambda)\frac{t^m}{m!}. \qquad (19)
\end{aligned}$$

From (19), we get

$$\begin{aligned}
&\frac{\frac{2}{\lambda}\log(1+\log(1+\lambda t))}{(1+\log(1+\log(1+\lambda t)))^{\frac{1}{\lambda}}+1}(1+\log(1+\log(1+\lambda t)))^{\frac{x}{\lambda}}\\
&= \frac{1}{\lambda}\log(1+\log(1+\lambda t))\int_{\mathbb{Z}_p}(1+\log(1+\log(1+\lambda t)))^{\frac{x+y}{\lambda}}d\mu_{-1}(y). \qquad (20)
\end{aligned}$$

From (20), we get

$$\begin{aligned}
&\frac{1}{\lambda}\log(1+\log(1+\lambda t))\int_{\mathbb{Z}_p}(1+\log(1+\log(1+\lambda t)))^{\frac{x+y}{\lambda}}d\mu_{-1}(y)\\
&= \frac{1}{\lambda}\log(1+\log(1+\lambda t))\sum_{n=0}^{\infty}\frac{1}{n!}\int_{\mathbb{Z}_p}\lambda^{-n}(x+y|\lambda)_n d\mu_{-1}(y)(\log(1+\log(1+\lambda t)))^n\\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\int_{\mathbb{Z}_p}(x+y|\lambda)_n\lambda^{-n-1}d\mu_{-1}(\log(1+\log(1+\lambda t)))^n\\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\int_{\mathbb{Z}_p}(x+y|\lambda)_n\lambda^{-n-1}d\mu_{-1}(y)(n+1)!\sum_{m=n+1}^{\infty}S_1(m,n+1)\frac{(\log(1+\lambda t))^m}{m!}\\
&= \sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}(x+y|\lambda)_n\lambda^{-n-1}d\mu_{-1}(y)(n+1)\sum_{m=n+1}^{\infty}S_1(m,n+1)\sum_{l=m}^{\infty}S_1(l,m)\lambda^l\frac{t^l}{l!}\\
&= \sum_{n_3=0}^{\infty}\sum_{n_2=0}^{n_3}\sum_{n_1=0}^{n_2-1}\lambda^{n_3-n_1-1}S_1(n_3,n_2)S_1(n_2,n_1+1)g_{n_1+1}^{(0)}(x|\lambda)\frac{t^{n_3}}{n_3!}. \qquad (21)
\end{aligned}$$

From (20) and (21), we obtain the following theorem.

Theorem 2.2. *Let $n_3 \in \mathbb{N}$. Then we have*

$$g_{n_3}^{(2)}(x|\lambda) = \sum_{n_2=0}^{n_3}\sum_{n_1=0}^{n_2-1} \lambda^{n_3-n_1-1} S_1(n_3,n_1)S_1(n_2,n_1+1)g_{n_1+1}^{(0)}(x|\lambda). \tag{22}$$

Inductively, we have the k-th degeneration of Genocchi polynomials as follows:

Theorem 2.3. *Let $k, n_k \in \mathbb{N}$. Then we have*

$$g_{n_k}^{(k-1)}(x|\lambda) = \sum_{n_{k-1}=0} \cdots \sum_{n_1-0}^{n_2-1} \lambda^{n_k-n_1-1} S_1(n_k,n_{k-1})\cdots S_1(n_2,n_1+1)g_{n_1+1}^{(0)}(x|\lambda). \tag{23}$$

By replacing t by $\frac{1}{\lambda}(e^{\lambda t}-1)$ in (19) and (20).
We have

$$\begin{aligned}\sum_{n=1}^{\infty} g_n^{(2)}(x|\lambda)\frac{1}{\lambda^n}\frac{(e^{\lambda t}-1)^n}{n!} &= \sum_{n=1}^{\infty} g_n^{(2)}(x|\lambda)\sum_{l=n}^{\infty}\lambda^{l-n}S_2(l,n)\frac{t^l}{l!} \\ &= \sum_{l=0}^{\infty}(\sum_{n=0}^{l} g_n^{(2)}(x|\lambda)\lambda^{l-n}S_2(l,n))\frac{t^l}{l!}. \end{aligned} \tag{24}$$

By (14) and (23), we obtain the following theorem.

Theorem 2.4. *Let $l \in \mathbb{N}$. Then we have*

$$g_l^{(p)}(x|\lambda) = \sum_{n=0}^{l} g_n^{(2)}(x|\lambda)\lambda^{l-n}S_2(l,n). \tag{25}$$

We note the Daehee polynomials of order r is defined by the generating function to be

$$\left(\frac{\log(1+t)}{t}\right)^r (1+t)^x = \sum_{n=0}^{\infty} D_n^{(r)}(x)\frac{t^n}{n!} \tag{26}$$

When $x=0$, $D_n^{(r)} = D_n^{(r)}(0)$ are called the Daehee numbers of order r.
By replacing t by $\log(1+\lambda t)^{\frac{1}{\lambda}}$ in (7), we get

$$\begin{aligned}\sum_{n=1}^{\infty} g_n^{(0)}(x|\lambda)\frac{(\log(1+\lambda t)^{\frac{1}{\lambda}})^n}{n!} &= \frac{2(\log(1+\lambda t)^{\frac{1}{\lambda}})}{(1+\log(1+\lambda t))^{\frac{1}{\lambda}}+1} - (1+\log(1+\lambda t)^{\frac{x}{\lambda}} \\ &= \sum_{n=1}^{\infty} g_n^{(1)}(\lambda|x)\frac{t^n}{n!}. \end{aligned} \tag{27}$$

and

$$\begin{aligned}\sum_{n=1}^{\infty} g_n^{(0)}(x|\lambda)\left(\frac{\log(1+\lambda t)}{t}\right)^n \lambda^{-n}\frac{t^n}{n!} &= \sum_{n=1}^{\infty} g_n^{(0)}(x|\lambda)\left(\sum_{l=0}^{\infty} D_l^{(n)}\frac{t^l}{l!}\right)\lambda^{-n}\frac{t^n}{n!} \\ &= \sum_{n=1}^{\infty}\sum_{l=0}^{\infty} g_n^{(0)}(x|\lambda)D_l^{(n)}\lambda^{-n}\frac{t^{l+n}}{l!n!} \\ &= \sum_{m=1}^{\infty}\left(\sum_{n=0}^{m} g_n^{(0)}(x|\lambda)D_{m-n}^{(n)}\lambda^{-n}\binom{n}{m}\right)\frac{t^m}{m!}. \end{aligned} \tag{28}$$

Thus, by (27) and (28), we obtain the following theorem.

Theorem 2.5. *Let $m \in \mathbb{N}$. Then we have*

$$g_m^{(1)}(x|\lambda) = \sum_{n=0}^{m}\binom{m}{n}\lambda^{-n}g_n^{(0)}(x|\lambda)D_{m-n}^{(n)}. \tag{29}$$

By replacing t by $\frac{1}{\lambda}(e^{\lambda t}-1)$ in (15), we get

$$\begin{aligned}\sum_{m=1}^{\infty} g_m^{(1)}(x|\lambda)\frac{\left(\frac{e^{\lambda t}-1}{\lambda}\right)^m}{m!} &= \frac{2t}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}} \\ &= \sum_{l=1}^{\infty} g_l^{(0)}(x|\lambda)\frac{t^l}{l!}. \end{aligned} \tag{30}$$

and

$$\begin{aligned}\sum_{m=1}^{\infty} g_m^{(1)}(x|\lambda)\lambda^{-m}\frac{\left(e^{\lambda t}-1\right)^m}{m!} &= \sum_{m=1}^{\infty} g_m^{(1)}(x|\lambda)\lambda^{-m}\sum_{l=m}^{\infty} S_2(l,m)\frac{(\lambda t)^l}{l!} \\ &= \sum_{m=1}^{\infty}\left(\sum_{l=m}^{\infty}\lambda^{l-m}g_m^{(1)}(x|\lambda)S_2(l,m)\right)\frac{t^l}{l!} \\ &= \sum_{l=0}^{\infty}\left(\sum_{m=1}^{l}\lambda^{l-m}g_m^{(1)}(x|\lambda)S_2(l,m)\right)\frac{t^l}{l!}. \end{aligned} \tag{31}$$

Thus, by (30) and (31), we obtain the following theorem.

Theorem 2.6. *Let $l \in \mathbb{N}$. Then we have*

$$g_l^{(0)}(x|\lambda) = \sum_{m=1}^{l}\lambda^{l-m}g_m^{(1)}(x|\lambda)S_2(l,m). \tag{32}$$

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Regularization solutions of ill-posed Helmholtz-type equations with fuzzy mixed boundary value[†]

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Abstract In this study, we discuss the solutions of fuzzy Helmholtz-type equations(FHTEs) and their ill-posedness. A regularization method is required to recover the numerical stability. Moreover, the error estimates and convergence of the method is considered. To support our study, one numerical example is illustrated.
Keywords: Fuzzy numbers; Ill-posed problem; Helmholtz equation; Regularization method; Convergence estimate.

1. Introduction

The study of fuzzy partial differential equations (FPDEs) provides a suitable setting for the mathematical modeling of real-world problems that include uncertainty or vagueness. As a new and powerful mathematical tool, FPDEs have been studied using several approaches. The first definition of an FPDE was presented by Buckley and Feuring in [1]. In [2], the authors considered the application of FPDEs obtained using fuzzy rule-based systems. Furthermore, Oberguggenberger described weak and fuzzy solutions for FPDEs [3] and Chen et al. presented a new inference method with applications to FPDEs [4]. In [5], an interpretation was provided of the use of FPDEs for modeling hydrogeological systems. Studies of heat, wave, and Poisson equations with uncertain parameters were provided in [6]. Fuzzy solutions for heat equations based on generalized Hukuhara differentiability were considered in [7]. Several numerical methods have been proposed to solve FPDEs. Such as Allahviranloo ([8]) proposed a difference method for solving FPDEs. The Adomian decomposition method was studied for finding the approximate solution of the fuzzy heat equation in [9]. Solving FPDEs by the differential transformation method was considered in [10].

In this paper, we proposed a numerical method to solve ill-posed problems for the fuzzy Helmholtz-type equation (FHTEs). The Helmholtz equation arises in many areas, especially in practical physical applications, such as acoustic, wave propagation and scattering, vibration of the structure, electromagnetic scattering and so on, see [11, 12, 13, 14]. The direct problems, i.e. Dirichlet, Neumann or mixed boundary value problems for the Helmholtz equation have been studied extensively in the past century. However, in some practical problems, the boundary data on the whole boundary cannot be obtained. We only know the noisy data on a part of the boundary or at some interior points of the concerned domain, which will lead to some inverse problems and severely ill-posed problems. In 1923, Hadamard [15] introduced the concept of a well-posed problem from philosophy where the mathematical model of a physical problem must have properties where the solution exhibits uniqueness, existence, and stability. If one of the properties fails to hold, the problem is known as ill-posed. Numerical computation is difficult due to the ill-posedness of the problem. That means the solution does not depend continuously on the given Cauchy data and any small perturbation in the given data may cause large change to the solution [15, 16, 17]. The present study addresses two issues. First, we consider the ill-posedness of FHTEs using the decomposition theorem. Second, we use the regularization method to recover the numerical stability.

The remainder of this paper is organized as follows. In Section2, we briefly introduce the necessary notions related to fuzzy numbers and differentiability properties for fuzzy set-valued mappings. In Section3, we define the solution and ill-posedness of FHTEs. The regularization method and convergence

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estimates for the initial-boundary value problems of FHTEs are considered in Section4. In Section5, we present some numerical results and our conclusions are given in Section6.

2. Definitions and preliminaries

Let $P_k(R^n)$ denote the family of all nonempty compact convex subset of R^n and define the addition and scalar multiplication in $P_k(R^n)$ as usual. Let A and B be two nonempty bounded subset of R^n. The distance between A and B is defined by the Hausdorff metric

$$d_H(A,B) = \max\Big\{\sup_{a\in A}\inf_{b\in B}||a-b||, \sup_{b\in B}\inf_{a\in A}||b-a||\Big\}, \tag{2.1}$$

where $||\cdot||$ denotes the usual Euclidean norm in R^n [18]. Then $(P_k(R^n); d_H)$ is a metric space.

Denote

$$E^n = \{u : R^n \to [0,1] | u \text{ satisfies (1)-(4) below}\}$$

is a fuzzy number space, where

(1) u is normal, i.e. there exists an $x_0 \in R^n$ such that $u(x_0) = 1$,

(2) u is fuzzy convex, i.e. $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$ for any $x, y \in R^n$ and $0 \leq \lambda \leq 1$,

(3) u is upper semi-continuous,

(4) $[u]_0 = cl\{x \in R^n | u(x) > 0\}$ is compact.

Here, $cl(X)$ denotes the closure of set X. For $0 < \alpha \leq 1$, the α-level set of u (or simply the α-cut) is defined by $[u]_\alpha = \{x \in R^n | u(x) \geq \alpha\}$. The core of u is the set of elements of R^n having membership grade 1, i.e., $[u]_1 = \{x | x \in R^n, u(x) = 1\}$. Then from above (1)-(4), it follows that the α-level set $[u]_\alpha \in P_k(R^n)$ for all $0 < \alpha \leq 1$. According to Zadeh's extension principle, we have addition and scalar multiplication in fuzzy number space E^n as follows:

$$[u+v]_\alpha = [u]_\alpha + [v]_\alpha = \{x + y | x \in [u]_\alpha, y \in [v]_\alpha\},$$

$$[ku]_\alpha = k[u]_\alpha = \{kx | x \in [u]_\alpha\}, [0]_\alpha = \{0\}.$$

where $u, v \in E^n$ and $0 < \alpha \leq 1$. The distance between two fuzzy numbers u and v is defined by

$$D(u,v) = \sup_{\alpha\in[0,1]} d_H([u]_\alpha, [v]_\alpha). \tag{2.2}$$

We recall some differentiability properties for fuzzy set-valued mappings.

Definition 2.1 (See[19]) Let K_C denote the family of all bounded closed intervals in R, the generalized Hukuhara difference of two intervals $A, B \in K_C$ (gH-difference, for short), denoted by $A \ominus_{gH} B$, is defined by

$$A \ominus_{gH} B = C \Longleftrightarrow \begin{cases} (i) A = B + C; \\ or (ii) B = A + (-C). \end{cases} \tag{2.3}$$

Definition 2.2 (See[20]) The generalized Hukuhara difference of two fuzzy numbers $u, v \in E^1$ (gH-difference, for short) is the fuzzy number ω, if it exists, such that

$$u \ominus_{gH} v = \omega \Longleftrightarrow \begin{cases} (i) u = v + \omega; \\ or (ii) v = u + (-\omega). \end{cases} \tag{2.4}$$

It is easy to show that (i) and (ii) are both valid if and only if w is a crisp number.

It may happen that the gH-difference of two fuzzy numbers does not exist (see, for example, [21]). In order to overcome this shortcoming, in [20, 21], a new difference between fuzzy numbers was proposed, which always exists.

Henceforth, T=]a, b[denotes an open interval in R. A function $F : T \to F_C$ is said to be a fuzzy function. For each $\alpha \in [0, 1]$, associated to F, we define the family of interval-valued functions $F_\alpha : T \to K_C$ given by $F_\alpha(x) = [F(x)]^\alpha$, for $x \in T$. For any $\in [0, 1]$, we denote

$$F_\alpha(x) = [\underline{f}_\alpha(x), \overline{f}_\alpha(x)]. \tag{2.5}$$

Here, for each $\alpha \in [0, 1]$, the endpoint functions $\overline{f}_\alpha, \underline{f}_\alpha : T \to R$ are called upper and lower functions of F, respectively. Next, we present the concept of gH-differentiable fuzzy functions based on the gH-difference of fuzzy intervals.

Definition 2.3 (See [21]) The gH-derivative of a fuzzy function $F : T \to F_C$ at $x_0 \in T$ is defined as

$$F'(x_0) = \lim_{h\to 0} \frac{1}{h}[F(x_0 + h) \ominus_{gH} F(x_0)]. \tag{2.6}$$

If $F(x_0) \in F_C$ satisfying (2.5) exists, we say that F is generalized Hukuhara differentiable (gH-differentiable, for short) at x_0.

Theorem 2.1(See [22]) If $F : T \to F_C$ is gH-differentiable at $x_0 \in T$, then F_α is gH-differentiable at x_0 uniformly in $\alpha \in [0, 1]$ and

$$F'_\alpha(x_0) = [F'(x_0)]^\alpha, \tag{2.7}$$

for all $\alpha \in [0, 1]$.

Theorem 2.2 (See [22]) Let F: $T \to F_C$ be a fuzzy function and $x \in T$. Then F is gH-differentiable at x if and only if one of the following four cases holds:

(a) $\underline{f}_\alpha$ and $\overline{f}_\alpha$ are differentiable at x, uniformly in $\alpha \in [0, 1], (\underline{f}_\alpha)'(x)$ is monotonic increasing and $(\overline{f}_\alpha)'(x)$ is monotonic decreasing as functions of α and $(\underline{f}_\alpha)'(x) \le (\overline{f}_\alpha)'(x)$. In this case,

$$F'_\alpha(x) = [(\underline{f}_\alpha)'(x), (\overline{f}_\alpha)'(x)]. \qquad (2.8),$$

for all $\alpha \in [0, 1]$.

(b) $\underline{f}_\alpha$ and $\overline{f}_\alpha$ are differentiable at x, uniformly in $\alpha \in [0, 1], (\underline{f}_\alpha)'(x)$ is monotonic increasing and $(\overline{f}_\alpha)'(x)$ is monotonic decreasing as functions of α and $(\overline{f}_\alpha)'(x) \le (\underline{f}_\alpha)'(x)$. In this case,

$$F'_\alpha(x) = [\overline{f}_\alpha)'(x), (\underline{f}_\alpha)'(x)]. \qquad (2.9),$$

for all $\alpha \in [0, 1]$.

(c) $(\underline{f}_\alpha)'_{+/-}(x)$ and $(\overline{f}_\alpha)'_{+/-}(x)$ exist uniformly in $\alpha \in [0, 1], (\underline{f}_\alpha)'_+(x) = (\overline{f}_\alpha)'_-(x)$ is monotonic increasing and $(\overline{f}_\alpha)'_+(x) = (\underline{f}_\alpha)'_-(x)$ is monotonic decreasing as functions of α and $(\underline{f}_\alpha)'_+(x) \le (\overline{f}_\alpha)'_+(x)$. In this case,

$$F'_\alpha(x) = [(\underline{f}_\alpha)'_+(x), \overline{f}_\alpha)'_+(x)] = [\overline{f}_\alpha)'_-(x), (\underline{f}_\alpha)'_-(x)]. \qquad (2.10),$$

for all $\alpha \in [0, 1]$.

(d) $(\underline{f}_\alpha)'_{+/-}(x)$ and $(\overline{f}_\alpha)'_{+/-}(x)$ exist uniformly in $\alpha \in [0, 1], (\underline{f}_\alpha)'_+(x) = (\overline{f}_\alpha)'_-(x)$ is monotonic increasing and $(\overline{f}_\alpha)'_+(x) = (\underline{f}_\alpha)'_-(x)$ is monotonic decreasing as functions of α and $(\overline{f}_\alpha)'_+(x) \le (\underline{f}_\alpha)'_+(x)$. In this case,

$$F'_\alpha(x) = [\overline{f}_\alpha)'_+(x), (\underline{f}_\alpha)'_+(x)] = [(\underline{f}_\alpha)'_-(x), \overline{f}_\alpha)'_-(x)]. \qquad (2.11),$$

for all $\alpha \in [0, 1]$.

Theorem 2.3(**Decomposition Theorem**[23]) If $u \in E^n$, then

$$u = \bigcup_{\alpha\in[0,1]} (\alpha \cdot [u]_\alpha). \tag{2.3}$$

The following well-known characterization theorem makes the connection between a fuzzy interval and its LU-representation.

Theorem 2.4 (See[24]) Let u be a fuzzy number. Then the functions $\underline{u}$, $\overline{u}$: $[0,1] \to R$, defining the endpoints of the α-level sets of u, satisfy the following conditions:

(i) $\underline{u}$ is a bounded, non-decreasing, left-continuous function in $(0,1]$ and it is right-continuous at 0.
(ii) $\overline{u}$ is a bounded, non-increasing, left-continuous function in $(0,1]$ and it is right-continuous at 0.
(iii) $u(1) \le u(1)$.

Reciprocally, given two functions that satisfy the above conditions, they uniquely determine a fuzzy number.

3. Solutions of FHTEs and Ill-posedness

Now, we consider a Cauchy problem for the Helmholtz-type equation with fuzzy initial-boundary value in a rectangle domain as follows

$$\begin{cases} \dfrac{\partial^2 \widetilde{u}}{\partial x^2} + \dfrac{\partial^2 \widetilde{u}}{\partial y^2} + k^2 \widetilde{u} = \widetilde{0}, & 0 < x < \pi, 0 < y < 1, \\ \widetilde{u}(x,0) = \widetilde{\varphi}(x), & 0 \le x \le \pi, \\ \dfrac{\partial \widetilde{u}}{\partial y}(x,0) = \widetilde{0}, & 0 \le x \le \pi, \\ \widetilde{u}(0,y) = \widetilde{u}(\pi,y) = \widetilde{0}, & 0 \le y \le 1, \end{cases} \tag{3.1}$$

where where constant $k > 0$ is the wave number. $\widetilde{u}, \frac{\partial^2 \widetilde{u}}{\partial x^2}, \frac{\partial^2 \widetilde{u}}{\partial y^2}, \frac{\partial \widetilde{u}}{\partial y}, \widetilde{\varphi}(x), \widetilde{0}$ are fuzzy-number-valued functions and their α-cut sets are shown as follows:

$$[\widetilde{u}(x,y)]_\alpha = [\underline{u}(x,y,\alpha),\ \overline{u}(x,y,\alpha)],$$
$$[\frac{\partial^2 \widetilde{u}}{\partial x^2}(x,y)]_\alpha = [\frac{\partial^2 \underline{u}}{\partial x^2}(x,y,\alpha),\ \frac{\partial^2 \overline{u}}{\partial x^2}(x,y,\alpha)],$$
$$[\frac{\partial^2 \widetilde{u}}{\partial y^2}(x,y)]_\alpha = [\frac{\partial^2 \underline{u}}{\partial y^2}(x,y,\alpha),\ \frac{\partial^2 \overline{u}}{\partial y^2}(x,y,\alpha)],$$
$$[\frac{\partial \widetilde{u}}{\partial y}(x,y)]_\alpha = [\frac{\partial \underline{u}}{\partial y}(x,y,\alpha),\ \frac{\partial \overline{u}}{\partial y}(x,y,\alpha)],$$
$$[\widetilde{\varphi}(x)]_\alpha = [\underline{\varphi}(x,\alpha),\ \overline{\varphi}(x,\alpha)], \quad [\widetilde{0}]_\alpha = [\underline{0}(\alpha),\ \overline{0}(\alpha)].$$

From Theorem 2.1 and Theorem 2.2, in order to investigate the solution of (3.1), we consider the following two systems of two partial differential equations

$$\begin{cases} \dfrac{\partial^2 \underline{u}}{\partial x^2}(x,y,\alpha) + \dfrac{\partial^2 \underline{u}}{\partial y^2}(x,y,\alpha) + k^2 \underline{u}(x,y,\alpha) = \underline{0}, & 0 < x < \pi, 0 < y < 1, \\ \underline{u}(x,0) = \underline{\varphi}(x,\alpha), & 0 \le x \le \pi, \\ \dfrac{\partial \underline{u}}{\partial y}(x,0,\alpha) = \underline{0}(\alpha), & 0 \le x \le \pi, \\ \underline{u}(0,y,\alpha) = \underline{u}(\pi,y) = \underline{0}(\alpha), & 0 \le y \le 1, \end{cases} \tag{3.2}$$

$$\begin{cases} \frac{\partial^2 \overline{u}}{\partial x^2}(x,y,\alpha) + \frac{\partial^2 \overline{u}}{\partial y^2}(x,y,\alpha) + k^2\overline{u}(x,y,\alpha) = \overline{0}, & 0 < x < \pi, 0 < y < 1, \\ \overline{u}(x,0,\alpha) = \overline{\varphi}(x,\alpha), & 0 \le x \le \pi, \\ \frac{\partial \overline{u}}{\partial y}(x,0,\alpha) = \overline{0}(\alpha), & 0 \le x \le \pi, \\ \overline{u}(0,y) = \overline{u}(\pi,y,\alpha) = \overline{0}(\alpha), & 0 \le y \le 1, \end{cases} \tag{3.3}$$

Definition 3.1 (see [1]) Let $\underline{u}(x,y,\alpha)$ and $\overline{u}(x,y,\alpha)$ be solutions of equations (3.2) and (3.3), respectively. If $[\underline{u}(x,y,\alpha),\overline{u}(x,y,\alpha)]$ defines the α-cut of a fuzzy number, for all $(x,y) \in [0,\ \pi] \times [0,\ 1]$, then $\widetilde{u}(x,y)$ is a solution for (3.1).

By the method of separation of variables, it is easy to derive a solution of the direct problem (3.2) and (3.3), respectively as follows:

$$\underline{u}(x,y,\alpha) = \sum_{n=1}^{[k]} \underline{c}_n \sin(nx)\cos(\sqrt{k^2-n^2}y) + \sum_{n=[k]+1}^{\infty} \underline{c}_n \sin(nx)\cosh(\sqrt{n^2-k^2}y) \tag{3.4}$$

where

$$\underline{c}_n = \frac{2}{\pi}\int_0^{\pi} \underline{\varphi}(x,\alpha)\sin(nx)dx \tag{3.5}$$

$$\overline{u}(x,y,\alpha) = \sum_{n=1}^{[k]} \overline{c}_n \sin(nx)\cos(\sqrt{k^2-n^2}y) + \sum_{n=[k]+1}^{\infty} \overline{c}_n \sin(nx)\cosh(\sqrt{n^2-k^2}y) \tag{3.6}$$

where

$$\overline{c}_n = \frac{2}{\pi}\int_0^{\pi} \overline{\varphi}(x,\alpha)\sin(nx)dx \tag{3.7}$$

Obviously, for the solutions $\underline{u}(x,y,\alpha)$ of the equations (3.2) and the solutions $\overline{u}(x,y,\alpha)$ of the equations (3.3), $[\underline{u}(x,y,\alpha),\overline{u}(x,y,\alpha)]$ satisfies the conditions of Theorem 2.2, $[\underline{u}(x,y,\alpha),\overline{u}(x,y,\alpha)]$ determines a solution of problem (3.1) as follows:

$$u = \bigcup_{\alpha\in[0,1]} (\alpha \cdot [\underline{u}(x,y,\alpha),\overline{u}(x,y,\alpha)]). \tag{3.8}$$

Remark 3.1 If $0 < k < 1$, the first term in Equations (3.4) and (3.6) is vanished.

In the following, we discuss the ill-posedness of problem (3.1).

Definition 3.2 (Hadamard's definition of well-posedness [15]) If a deterministic solution problem of FPDE satisfies the following properties (3.9-3.11), then it is well-posed.

For all admissible date, a solution exists. (3.9)

For all admissible date, the solution is unique. (3.10)

The solution depends continuously on the date. (3.11)

Conversely, if one of the properties (3.9-3.11) does not satisfy for a deterministic solution problem of FPDE, then it is ill-posed.

Next, we are always suppose that (3.9) and (3.10) hold for the convenience of discussion, (3.11) does not hold.

Definition 3.3 Problem of FHTEs (3.1) is said to be ill-posed if both problems of PDE (3.2) and PDE (3.3) are ill-posed.

The the systems of PDE (3.2) and (3.3) are highly ill-posed, see[16]. Thus, the systems (3.1) is ill-posed.

Ill-posed problem means the solution does not depend continuously on the given Cauchy data and any small perturbation in the given data may cause large change to the solution. Thus regularization techniques are required to stabilize numerical computations. In general terms, regularization is the approximation of an ill-posed problem by a family neighbouring well-posed problems.

4. Regularization and Convergence estimates

In this section, we use the solution of perturbation problems to approach the solution of problems (3.2) and (3.3). Thus the regularization solution of problems (3.1) be derived by (3.4).

For $0 < k < 1$, we consider the following problem

$$\begin{cases} \Delta\underline{v}(x,y) + k^2\underline{v}(x,y) = \underline{0}, & 0 < x < \pi, 0 < y < 1, \\ \underline{v}(x,0) + \beta\underline{v}(x,1) = \underline{\varphi}^{\delta_1}(x,\alpha), & 0 \le x \le \pi, \\ \underline{v}_y(x,0) = 0, & 0 \le x \le \pi, \\ \underline{v}(0,y) = \underline{v}(\pi,y) = \underline{0}, & 0 \le y \le 1, \end{cases} \tag{4.1}$$

$$\begin{cases} \Delta\overline{v}(x,y) + k^2\overline{v}(x,y) = \overline{0}, & 0 < x < \pi, 0 < y < 1, \\ \overline{v}(x,0) + \beta\overline{v}(x,1) = \overline{\varphi}^{\delta_2}(x,\alpha), & 0 \le x \le \pi, \\ \overline{v}_y(x,0) = 0, & 0 \le x \le \pi, \\ \overline{v}(0,y) = \overline{v}(\pi,y) = \overline{0}, & 0 \le y \le 1, \end{cases} \tag{4.2}$$

where $0 < \alpha \le 1$ is α-level set parameter, and $\beta > 0$ is a regularization parameter. The measured data of equations (3.1) is fuzzy-number-valued function $\widetilde{\varphi}(x)$, and its α-level set is defined as $[\widetilde{\varphi}(x)]_\alpha = [\underline{\varphi}(x,\alpha),\ \overline{\varphi}(x,\alpha)]$. $\underline{\varphi}^{\delta_1} \in L^2(0,\pi)$, $\overline{\varphi}^{\delta_2} \in L^2(0,\pi)$ satisfies

$$\|\underline{\varphi}^{\delta_1} - \underline{\varphi}\| \le \delta_1, \tag{4.3}$$

$$\|\overline{\varphi}^{\delta_2} - \overline{\varphi}\| \le \delta_2, \tag{4.4}$$

in which the constant $\delta_1 > 0$ and $\delta_2 > 0$ is called an error level and $\|\cdot\|$ denotes the L^2-norm. Further assume that there exists a constant $E > 0$ such that the following a priori bound exists

$$\|u(\cdot,1)\| \le E. \tag{4.5}$$

By the method of separation of variables, it is easy to derive a solution of direct problem (4.1) and (4.2) as follows, respectively

$$\underline{v}(x,y,\alpha) = \sum_{n=1}^{\infty} \underline{c}_n^{\delta_1} \frac{\cosh(\sqrt{n^2-k^2}y)}{1+\beta\cosh(\sqrt{n^2-k^2})} \sin(nx), \tag{4.6}$$

where

$$\underline{c}_n^{\delta_1} = \frac{2}{\pi}\int_0^\pi \underline{\varphi}^{\delta_1}(x,\alpha)\sin(nx)dx. \tag{4.7}$$

$$\overline{v}(x,y,\alpha) = \sum_{n=1}^{\infty} \overline{c}_n^{\delta_2} \frac{\cosh(\sqrt{n^2-k^2}y)}{1+\beta\cosh(\sqrt{n^2-k^2})} \sin(nx), \tag{4.8}$$

where

$$\overline{c}_n^{\delta_2} = \frac{2}{\pi}\int_0^\pi \overline{\varphi}^{\delta_2}(x,\alpha)\sin(nx)dx. \tag{4.9}$$

For $k \geq 1$, we define a regularized solution v as follows:

$$\underline{v}(x,y,\alpha) = \sum_{n=1}^{[k]} \underline{c}_n^{\delta_1} \cosh(\sqrt{n^2-k^2}y) + \sum_{n=[k]+1}^{\infty} \underline{c}_n^{\delta_1} \frac{\cosh(\sqrt{n^2-k^2}y)}{1+\beta\cosh(\sqrt{n^2-k^2})} \sin(nx), \tag{4.10}$$

where $\underline{c}_n^{\delta_1}$ is defined by Equation (4.7).

$$\overline{v}(x,y,\alpha) = \sum_{n=1}^{[k]} \overline{c}_n^{\delta_2} \cosh(\sqrt{n^2-k^2}y) + \sum_{n=[k]+1}^{\infty} \overline{c}_n^{\delta_2} \frac{\cosh(\sqrt{n^2-k^2}y)}{1+\beta\cosh(\sqrt{n^2-k^2})} \sin(nx), \tag{4.11}$$

where $\overline{c}_n^{\delta_2}$ is defined by Equation (4.9).

Remark 4.1 (see[25]) For $k \geq 1$, the regularized solution (4.10) and (4.11) be not an exact solution of the problem (4.1) and (4.2), respectively, but a modified solution. This is done to avoid the case $1+\beta\cos(\sqrt{n^2-k^2}) = 0$ for $k \geq 1$ and $n < k$ and prove a convergence result.

In the following results shall show that the regularization solution $\underline{v}$ given by Equation (4.6)and(4.10), and $\overline{v}$ given by Equation (4.8) and (4.11) are a stable approximation to the exact solution $\underline{u}$ and $\overline{u}$ given by Equation (3.4) and (3.6),respectively. The regularization solution $\underline{v}$ and $\overline{v}$ depends continuously on the measured data $\underline{\varphi}^{\delta_1}$ and $\overline{\varphi}^{\delta_2}$ for a fixed parameter $\beta > 0$, respectively.

Theorem 4.1 (see[25]) Suppose that $\underline{u}$ and $\overline{u}$ is defined by Equation (3.4) and (3.6) with the exact data $\underline{\varphi}$ and $\overline{\varphi}$, respectively. Suppose that $\underline{v}$ is defined by Equation (4.6) for the case $0 < k < 1$ or Equation (4.10)for the case $k \geq 1$ with the measured data $\underline{\varphi}^{\delta_1}$, $\overline{v}$ is defined by Equation (4.8) for the case $0 < k < 1$ or Equation (4.11) for the case $k \geq 1$ with the measured data $\overline{\varphi}^{\delta_2}$. Let the measured data $\underline{\varphi}^{\delta_1}$ and $\overline{\varphi}^{\delta_2}$ satisfy Equation (4.3) and (4.4), respectively. Let the exact solution u at $y = 1$ satisfy Equation (4.5). If the regularization parameter β is chosen as, respectively

$$\beta = \frac{\delta_1}{E}, \tag{4.11}$$

$$\beta = \frac{\delta_2}{E}, \tag{4.12}$$

then for fixed $0 < y < 1$, we have the following convergence estimate

$$\|\underline{v}(\cdot,y) - \underline{u}(\cdot,y)\| \leq \delta_1 + 2C_y E^y \delta_1^{1-y}. \tag{4.13}$$

$$\|\overline{v}(\cdot,y) - \overline{u}(\cdot,y)\| \leq \delta_2 + 2C_y E^y \delta_2^{1-y}. \tag{4.14}$$

where $C_y = \frac{1-y}{(\frac{2y}{(1-y)})^y}$.

However, the convergence estimate in Equation (4.13) and (4.14) is not useful for $y = 1$. In order to obtain a convergence rate at $y = 1$, we need a stronger a priori assumption

$$\|\frac{\partial^p u(\cdot,1)}{\partial y^p}\| \leq E, \tag{4.15}$$

where $p \geq 1$ is an integer. We have the following convergence estimate.

Theorem 4.2 (see[25]) Suppose that $\underline{u}$ and $\overline{u}$ is defined by Equation (3.4) and (3.6) with the exact data $\underline{\varphi}$ and $\overline{\varphi}$, respectively. Suppose that $\underline{v}$ is defined by Equation (4.6) for the case $0 < k < 1$ or Equation (4.10) for the case $k \geq 1$ with the measured data $\underline{\varphi}^{\delta_1}$, $\overline{v}$ is defined by Equation (4.8) for the case $0 < k < 1$ or Equation (4.11) for the case $k \geq 1$ with the measured data $\overline{\varphi}^{\delta_2}$. Let the measured data $\underline{\varphi}^{\delta_1}$ and $\overline{\varphi}^{\delta_2}$

satisfy Equation (4.3) and (4.4), respectively. Let the exact solution u at $y = 1$ satisfy Equation (4.15). If the regularization parameter β is chosen as, respectively

$$\beta = \frac{\delta_1}{E}, \tag{4.16}$$

$$\beta = \frac{\delta_2}{E}, \tag{4.17}$$

then we have the following convergence estimate at $y = 1$,

$$\|\underline{v}(\cdot,1) - \underline{u}(\cdot,1)\| \leq \delta_1 + \sqrt{\delta_1 E} + \frac{2E}{1-e^{-2k}} \max\{K^{-p}(\frac{\delta_1}{E})^{\frac{1}{3}}, 2(\frac{1}{6}\ln\frac{E}{\delta_1})^{-p}\}. \tag{4.18}$$

$$\|\overline{v}(\cdot,1) - \overline{u}(\cdot,1)\| \leq \delta_2 + \sqrt{\delta_2 E} + \frac{2E}{1-e^{-2k}} \max\{K^{-p}(\frac{\delta_2}{E})^{\frac{1}{3}}, 2(\frac{1}{6}\ln\frac{E}{\delta_2})^{-p}\}. \tag{4.19}$$

where $K = \sqrt{([k]+1)^2 - k^2}$ and $[\cdot]$ denotes the integer part of a real number.

Theorem 4.3 Suppose that $\widetilde{u}$ defined by Equation (3.8) is a solution of problem (3.1) and $\widetilde{v}$ is its regularization solution. If $\underline{u}$ is defined by Equation (3.4) and $\underline{v}$ is its regularization solution defined by Equation (4.6) for the case $0 < k < 1$ or Equation (4.10) for the case $k \geq 1$, while $\overline{u}$ is defined by Equation (3.6) and $\overline{v}$ is its regularization solution defined by Equation (4.8) for the case $0 < k < 1$ or Equation (4.11) for the case $k \geq 1$. then $\widetilde{v}$ is a stable approximation to $\widetilde{u}$, where

$$\widetilde{v} = \bigcup_{\alpha\in[0,1]} (\alpha \cdot [\underline{v}(x,y,\alpha), \overline{v}(x,y,\alpha)]). \tag{4.20}$$

Proof By Equation(2.2), since

$$\begin{aligned} D(\widetilde{u},\widetilde{v}) &= \sup_{\alpha\in[0,1]} d_H([\widetilde{u}]_\alpha, [\widetilde{v}]_\alpha) \\ &= \sup_{\alpha\in[0,1]} \max\{|\underline{u}(\alpha) - \underline{v}(\alpha)|, |\overline{u}(\alpha) - \overline{v}(\alpha)|\}, \end{aligned} \tag{4.21}$$

from Theorem 4.1 and 4.2, $\underline{v}(\alpha)$ is a stable approximation to $\underline{u}(\alpha)$ and $\overline{v}(\alpha)$ is a stable approximation to $\overline{u}(\alpha)$. Hence, From (4.21) we have that $\widetilde{v}$ is a stable approximation to $\widetilde{u}$. The proof is complete.

5. Numerical examples

Consider the following direct problem for the Helmholtz equation with fuzzy mixed boundary value

$$\begin{cases} \dfrac{\partial^2\widetilde{u}}{\partial x^2} + \dfrac{\partial^2\widetilde{u}}{\partial y^2} + k^2\widetilde{u} = \widetilde{0}, & 0 < x < \pi, 0 < y < 1, \\ \widetilde{u}(x,1) = \widetilde{f}(x), & 0 \leq x \leq \pi, \\ \dfrac{\partial\widetilde{u}}{\partial y}(x,0) = \widetilde{0}, & 0 \leq x \leq \pi, \\ \widetilde{u}(0,y) = \widetilde{u}(\pi,y) = \widetilde{0}, & 0 \leq y \leq 1, \end{cases} \tag{5.1}$$

in which $\widetilde{f} : [0,\pi] \to E^1$.

$$\widetilde{f} = \widetilde{v} \cdot 2x(\pi - x), x \in [0,\pi]. \tag{5.2}$$

where $\widetilde{v} \in E^1$ is given by a triangular fuzzy number

$$\widetilde{v}(t) = \begin{cases} t+1, & t \in (-1,0), \\ -t+1, & t \in (0,1), \\ 0, & t \in (-\infty,-1] \cup [1,+\infty). \end{cases} \tag{5.3}$$

The α-cut set of $\widetilde{f}(x)$ is given by

$$[\widetilde{f}(x)]_\alpha = [2x(\pi - x)\underline{v}(t,\alpha), 2x(\pi - x)\overline{v}(t,\alpha)]$$
$$= [2x(\pi - x)(\alpha - 1), 2x(\pi - x)(1 - \alpha)]. \tag{5.4}$$

In order to investigate the numerical solution of (5.1), we consider the following two systems of two partial differential equations

$$\begin{cases} \dfrac{\partial^2 \underline{u}}{\partial x^2} + \dfrac{\partial^2 \underline{u}}{\partial y^2} + k^2 \underline{u} = \underline{0}, & 0 < x < \pi, 0 < y < 1, \\ \underline{u}(x,1) = 2x(\pi - x)(\alpha - 1), & 0 \le x \le \pi, \\ \dfrac{\partial \underline{u}}{\partial y}(x,0) = \underline{0}, & 0 \le x \le \pi, \\ \underline{u}(0,y) = \underline{u}(\pi,y) = \underline{0}, & 0 \le y \le 1, \end{cases} \tag{5.5}$$

$$\begin{cases} \dfrac{\partial^2 \overline{u}}{\partial x^2} + \dfrac{\partial^2 \overline{u}}{\partial y^2} + k^2 \overline{u} = \overline{0}, & 0 < x < \pi, 0 < y < 1, \\ \overline{u}(x,1) = 2x(\pi - x)(1 - \alpha), & 0 \le x \le \pi, \\ \dfrac{\partial \overline{u}}{\partial y}(x,0) = \overline{0}, & 0 \le x \le \pi, \\ \overline{u}(0,y) = \overline{u}(\pi,y) = \overline{0}, & 0 \le y \le 1, \end{cases} \tag{5.6}$$

By the method of separation of variables, the solution of the direct problem (5.5) and (5.6) can be obtained as follows, respectively.

$$\underline{u}(x,y,\alpha) = \sum_{n=1}^{[k]} \underline{c}_n \sin(nx)\cos(\sqrt{k^2 - n^2}y) + \sum_{n=[k]+1}^{\infty} \underline{c}_n \sin(nx)\cosh(\sqrt{n^2 - k^2}y), \tag{5.7}$$

$$\overline{u}(x,y,\alpha) = \sum_{n=1}^{[k]} \overline{c}_n \sin(nx)\cos(\sqrt{k^2 - n^2}y) + \sum_{n=[k]+1}^{\infty} \overline{c}_n \sin(nx)\cosh(\sqrt{n^2 - k^2}y), \tag{5.8}$$

where $\underline{\varphi}_n = \frac{2}{\pi\cosh(n)} d_n, d_n = \int_0^\pi 2x(\pi - x)(\alpha - 1)\sin(nx)dx$, $\overline{\varphi}_n = \frac{2}{\pi\cosh(n)} d_n, d_n = \int_0^\pi 2x(\pi - x)(1 - \alpha)\sin(nx)dx$, and they can be computed by the Simpson formulation, respectively.

Remark 5.1 If $0 < k < 1$, the first term in Equations (5.7) and (5.8) is vanished.

Then we choose the initial data $\underline{\varphi}(x)$ for equation (3.2) and $\overline{\varphi}(x)$ for equation (3.3) as follows,

$$\underline{\varphi}(x) = u(x,0) \approx \sum_{n=1}^{25} \underline{\varphi}_n \sin(nx). \tag{5.9}$$

$$\overline{\varphi}(x) = u(x,0) \approx \sum_{n=1}^{25} \overline{\varphi}_n \sin(nx). \tag{5.10}$$

The measured data $\underline{\varphi}_{\delta_1}$ and $\overline{\varphi}_{\delta_2}$is given by $\underline{\varphi}^{\delta_1}(x_i) = \varphi(x_i) + \varepsilon \cdot \mathrm{rand}(i)$, and $\overline{\varphi}^{\delta_2}(x_i) = \varphi(x_i) + \varepsilon \cdot \mathrm{rand}(i)$, respectively, where ε is an error level,

$$\delta_1 := \|\underline{\varphi}_{\delta_1} - \underline{\varphi}\|_{l_2} = \left(\frac{1}{N_1} \sum_{i=1}^{N_1} \left| \underline{\varphi}_{\delta_1}(x_i) - \underline{\varphi}(x_i) \right|^2 \right)^{1/2}. \tag{5.11}$$

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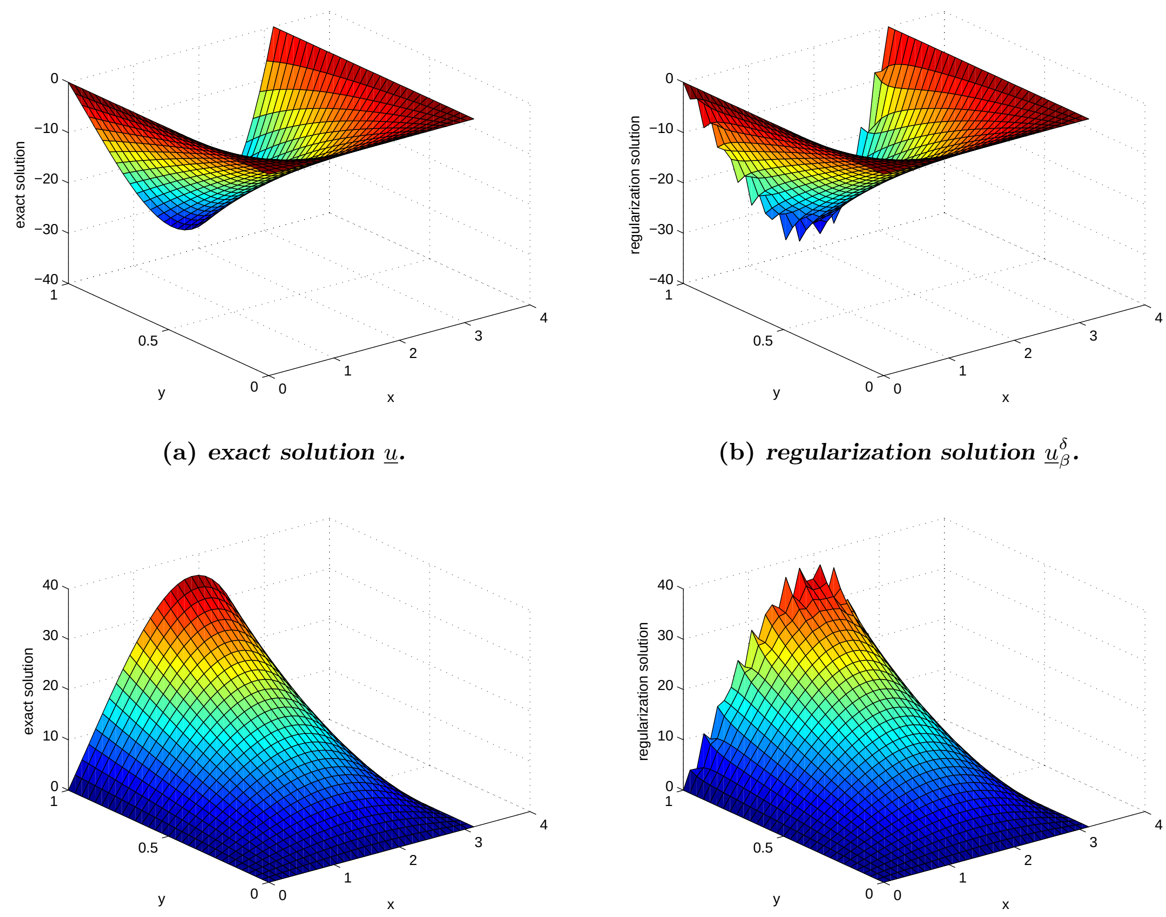
$$\delta_2 := \|\overline{\varphi}_{\delta_2} - \overline{\varphi}\|_{l_2} = \left(\frac{1}{N_1} \sum_{i=1}^{N_1} \left| \overline{\varphi}_{\delta_2}(x_i) - \overline{\varphi}(x_i) \right|^2 \right)^{1/2}. \tag{5.12}$$

The function rand$(\cdot)$ denotes a random number uniformly distributed in the interval $[0,1]$. In our numerical computations, we always take $N_1 = 31$. The regularization parameter β is chosen by (4.10),(4.11) and (4.15),(4.16) respectively.

The numerical results for $u(\cdot, y)$ and $u_\beta^\delta(\cdot, y)$ with $k = \frac{1}{2}, \varepsilon = 0.0001, \alpha = \frac{1}{2}$ are shown in Figure1.



(a) ***exact solution*** $\underline{u}$**.**

(b) ***regularization solution*** $\underline{u}_\beta^\delta$**.**

(c) ***exact solution*** $\overline{u}$**.**

(d) ***regularization solution*** $\overline{u}_\beta^\delta$**.**

Figure 1: $\varepsilon = 1 \times 10^{-4}, \alpha = \frac{1}{2}, k = \frac{1}{2}$.

6. Conclusion

In this paper, we investigate a new numerical method of solution for inverse problem of FHTEs. We defined the ill-posedness for deterministic solution problem of FHTEs and the regularization method is proposed to solve a Cauchy problem for the ill-posed FHTEs. The convergence and stability estimates for $0 < y < 1$, $y = 1$ have been obtained under a-priori bound assumptions for the exact solution. Finally, one example shows that our proposed numerical method is effective.

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Behavior of the Difference Equations $x_{n+1} = x_n\, x_{n-1} - 1$

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Abstract

In this paper, the behavior of solutions of a kind of nonlinear difference equations was studied. According to the first initial value, the regions of the second initial values was partitioned by zeroes of auxiliary functions such that the asymptotical behavior of the equation was determined, which was convergent or unbounded.

Key words: Nonlinear difference equations; Convergent; Unbounded

AMS 2000 Subject Classification: 39A10, 39A11

1 Introduction

In 2011, Kosmala[1] proposed a kind of nonlinear difference equations

$$x_{n+1} = x_{n-k}\, x_{n-l} - 1, \qquad n = 1, 2, \dots \tag{1}$$

with k, $l \in N$ and the initial values being real numbers. It stems from investigating periodic difference equations.

Stević and Iričanin [2] presented the first general result on the behavior of solutions of (1), by describing the long-term behavior of the solutions of (1) for all values of parameters k and l, where the initial values satisfy a special condition.

Moreover, some particular cases of (1) were investigated in [3–7]. Paper [3] investigated the case where $k = 1$, $l = 2$; paper [4] and [7] investigated the case where $k = 0$, $l = 1$; paper [5] investigated the case where $k = 0$, $l = 2$; paper [6] investigated the case where $k = 0$, $l = 3$.

The relatively simple appearance of (1) is deceiving in that its behavior changes significantly for different choices of k and l. These results of (1) were mainly about the periodicity, unboundedness and stability for particular choices of k and l.

In this paper, we consider a special case of (1), which was investigated in [4] and [7],

$$x_{n+1} = x_n\, x_{n-1} - 1, \qquad n = 0, 1, 2, \dots \tag{2}$$

with the initial values x_{-1} and x_0 being real numbers. Note that the equilibria $\bar{x}$ of (2) are

$$\bar{x}_1 = \frac{1-\sqrt{5}}{2}, \qquad \bar{x}_2 = \frac{1+\sqrt{5}}{2}.$$

Furthermore, $\bar{x}_1$ was locally asymptotically stable and $\bar{x}_2$ is unstable[4].

We first summarize the main results[4, 7] on the solutions of (2).

(1) (C) If $-1 < x_{-1}, x_0 < 0$, then every solution of (2) converges to $\bar{x}_1$.

(2) (UB) If one of the following holds, then the solution of (2) is unbounded.

$$\begin{array}{lll} (i) & x_{-1} > \bar{x}_2, & x_0 > \bar{x}_2; \\ (ii) & x_{-1} < -1, & x_0 < -1; \\ (iii) & x_{-1} < 0, & x_0 > 0; \\ (iv) & 0 < x_{-1} < 1, & 0 < x_0 < 1, \\ & x_0^2 x_{-1}^2 - 2x_0 x_{-1} + 1 - x_{-1} > 0. & \end{array}$$

(3) (UB or C)

- If $1 < x_{-1}, x_0 < \bar{x}_2$, then one of the following occurs.
  - (i) The solution of (2) is unbounded.
  - (ii) There exists $n_0 \geq 1$ such that $x_n \in (-1, 0)$ for all $n \geq n_0$.
- If $x_{-1} > 0$, $x_0 < 0$, then the solution of (2) in certain cases is bounded and in other cases is unbounded.
- If $0 < x_{-1}, x_0 < \bar{x}_2$, then the solutions of (2) exhibit somewhat chaotic behavior relative to the initial values. A little change in the initial conditions can cause a drastic difference in the long-term behavior of the solutions.

For simplicity, we show them in Figure 1. For the initial values (x_{-1}, x_0) in different colored regions, the solution of (2) is of three kinds: being convergent(C) and being unbounded(UB), being unbounded or convergent(UB or C).

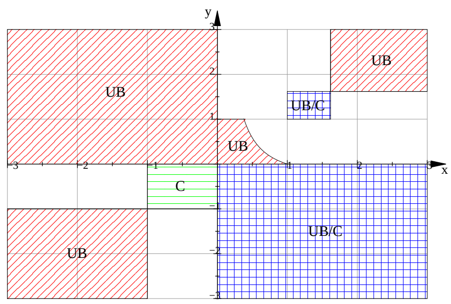

Figure 1: Different regions of the initial values of (2)

From the above results, one can see that these regions were presented from the perspective of the relation of two initial values of (2). For the initial values in the green regions, the corresponding solution is bounded and convergent. For the initial values in the red regions, the solution is unbounded. As far as the initial values in the blue regions is concerned, the solution was either unbounded or convergent and such a conclusion was not concise.

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Specially, for the initial values in the blank regions, the behavior of the solution is unknown.

It is interesting to investigate the evolution of the solution according to the initial values in the plane. In the following, we try to use a new method to consider the behavior of (2). Different from the method in [4], we construct auxiliary functions and then use the zeroes of them to create new partitions of the second initial value. In this way, for the first initial value which is arbitrarily chosen, the corresponding solution is convergent only for the second initial value in some intervals which are determined by the zeroes of auxiliary functions. And the lengths of these intervals are decreasing to zero.

2 Main Results

In this section, we present the main result by investigating the behavior of solutions of (2).

First of all, from the results in [4], we made a little generalization.

Theorem 2.1.

(I) If there is an $N \geq 0$ *such that* $-1 < x_{N-1},\ x_N < 0$*, then* $\{x_n\}$ *of* (2) *converges to* $\bar{x}_1$.

(II) If there is an $N \geq 0$ *such that one of the following five conditions holds, then the solution of* (2) *is unbounded.*

1) $x_{N-1} > \bar{x}_2, \quad x_N > \bar{x}_2;$
2) $x_{N-1} < -1, \quad x_N < -1;$
3) $x_{N-1} < 0, \quad x_N > 0;$
4) $0 < x_{N-1} < 1, \quad 0 < x_N < 1,$
$x_N^2 x_{N-1}^2 - 2x_N x_{N-1} + 1 - x_{N-1} > 0;$
5) $x_{N-1} > 0, \quad x_N < -1.$

It is worth pointing out that the last case *5)* is a direct result of the case *2)* and it is crucial for our main result.

Thus, the behavior of solutions of (2) depends on the location of its two consecutive terms of x_{N-1} and x_N being less than -1, greater than $\bar{x}_2$ or in the interval $(-1, 0)$. However, it is still complicated in terms of the boundedness of solutions of (2) for other cases.

By Remark 2.6 in [4], if the solution of (2) is not periodic or eventually periodic with minimal period three, then the solution is either bounded, while inside $(-1, 0)$, or unbounded.

Now, we present a necessary and sufficient condition on the existence of eventually prime period-three solutions of (2).

Lemma 2.1. *The solution* $\{x_n\}$ *of (2) is an eventually prime period-three solution if and only if there is an* $N \geq 1$ *such that* $x_N = 0$.

Proof. By Theorem 2.1 in [4], if the solution $\{x_n\}$ is an eventually prime period-three solution, then there is an $N \geq 1$ such that $x_N = 0$.

On the other hand, if there is an $N \geq 1$ such that $x_N = 0$, then we have $x_{N+1} = -1$ and $x_{N+2} = -1$ from (2). Thus, it is an eventually prime period-three solution. □

In the following, letting the first initial value x_{-1} being fixed, we consider the behavior of the solution for the second initial value x_0, mainly on the convergence and unboundedness of the corresponding solution of (2).

For simplicity, we could assume that $x_{-1} = a$ and $x_0 = b$, where a and b are real numbers.

Now, we introduce auxiliary functions $F_i(b) = x_i$ for $i \geq 1$, from (2), which are

$$\begin{aligned} F_1(b) &= ab - 1, && (3)\\ F_2(b) &= b\,F_1(b) - 1 \;=\; ab^2 - b - 1, && (4)\\ F_3(b) &= F_2(b)\,F_1(b) - 1 \;=\; b\,(F_1^2(b) - a), && (5)\\ F_4(b) &= F_3(b)\,F_2(b) - 1 \;=\; F_1(b)\,(F_2^2(b) - b), && (6)\\ F_5(b) &= F_4(b)\,F_3(b) - 1 \;=\; F_2(b)\,(F_3^2(b) - F_1(b)), && (7) \end{aligned}$$

and by induction, for $i \geq 5$, we have

$$F_{i+1}(b) = F_i(b)\,F_{i-1}(b) - 1 \;=\; F_{i-2}(b)(F_{i-1}^2(b) - F_{i-3}(b)), \tag{8}$$

from which we know that $F_i(b)$ is a higher-degree polynomial of b.

By listing the roots of $F_i(b) = 0$ for each $i \geq 1$, we consider the behavior of $F_i(b)$ with b in the intervals between these adjacent roots, which describes the long term behavior of the solution of (2) with the second initial value x_0 in such intervals for the first one x_{-1} being fixed.

In the following, we investigate the roots of $F_i(b) = 0$ step by step.

It is obvious that $r_{11} = 1/a$ is the root of $F_1(b) = 0$ if $a \neq 0$.

If $a \geq -0.25$ and $a \neq 0$, then $F_2(b)=0$ has two roots which are

$$r_{21} = \frac{1 - \sqrt{1+4a}}{2a}, \qquad r_{22} = \frac{1 + \sqrt{1+4a}}{2a}$$

and they satisfy $r_{21} < r_{11} < r_{22}$ for $a > 0$.

It is noted that 0 is always a root of $F_3(b)=0$ (for convenience, denoted by itself) and for $a > 0$, the other two roots are

$$r_{31} = \frac{1 - \sqrt{a}}{a}, \qquad r_{32} = \frac{1 + \sqrt{a}}{a}$$

satisfying $0 < r_{31} < r_{11} < r_{22} < r_{32}$ for $0 < a < 1$ and $r_{31} < 0 < r_{11} < r_{22} < r_{32}$ for $a > 1$.

From (6), we know that $F_4(b)=0$ is equivalent to $F_1(b)=0$ or $F_2^2(b)=b$. Thus r_{11} is always a root of $F_4(b)=0$. From $F_2^2(b) = b$, in view of the strict monotonicity of $F_2(b)$ for $b>r_{11}$, there are only two roots of $F_4(b)=0$, satisfying $r_{41} < r_{22} < r_{42}$ for $a>0$ and $b>0$.

Similarly, the other two roots of $F_5(b) = 0$ satisfy $r_{51} < r_{32} < r_{52}$ for $a > 0$ and $b > r_{11}$, which are different from r_{21} and r_{22}.

Here and after, we only focus on these "new" roots of $F_i(b) = 0$, which have not been labeled by other smaller indices.

Now, we conclude the existence of two roots of $F_{i+1}(b)=0$ for $i\geq 5$.

Lemma 2.2. *$F_{i+1}(b)=0$ has only two roots for $a>0$ and $b>r_{(i-3)2}$ for $i\geq 5$.*

Proof. Letting r_{ij} be the roots of $F_i(b)=0$ for $i>1$ and $j=1$, 2, from (8), we have

$$F'_{i+1}(b)=F'_i(b)F_{i-1}(b) + F_i(b)F'_{i-1}(b) > 0 \tag{9}$$

for $b>r_{i2}$ and thus $F_{i+1}(b)$ is strictly increasing for $b>r_{i2}$.

From (8), we have $F_{i-1}^2(b)=F_{i-3}(b)$ for $i\geq 5$. Hence, in view of the monotonicity of $F_{i-1}(b)$ for $b>r_{(i-2)2}$ and the positivity of $F_{i-3}(b)$ for $b>r_{(i-3)2}$, by induction, there are only two roots of $F_{i+1}(b)=0$ satisfying $r_{(i+1)1} < r_{(i-1)2} < r_{(i+1)2}$ for $a>0$ and $b>r_{(i-3)2}$. □

Furthermore, we could conclude that $\{r_{i1}\}_{i=2}^{+\infty}$ and $\{r_{i2}\}_{i=2}^{+\infty}$ are convergent.

Lemma 2.3.

$$\lim_{i\to+\infty} r_{i1} = \lim_{i\to+\infty} r_{i2}. \tag{10}$$

Proof. First, from the strict monotonicity of $F_{i+1}(b)$ for $b>r_{i2}$, we have $r_{i2} < r_{(i+1)2}$ and thus $r_{22}<r_{32}<r_{42}<\ldots$. The convergence of $\{r_{i2}\}_{i=2}^{+\infty}$ is guaranteed by $F_i(b)$ being a higher-degree polynomial of b. Similarly, $\{r_{i1}\}_{i=2}^{+\infty}$ is convergent.

Denote

$$\lim_{i\to+\infty} r_{i1} = \underline{b} = \underline{b}(a), \qquad \lim_{i\to+\infty} r_{i2} = \hat{b} = \hat{b}(a).$$

In order to prove (10), we suppose that $\underline{b} < \hat{b}$. Then there exists N such that $r_{N2} > \underline{b}$. From the above, there exists a root $r_{(N+2)1}$ of $F_{N+2}(b) = 0$ near r_{N2}. In view of $\{r_{i2}\}_{i=2}^{+\infty}$ being increasing, it is enough to choose such a r_{N2} that the corresponding $r_{(N+2)1} > \underline{b}$, which contradicts the convergence of $\{r_{i1}\}_{i=2}^{+\infty}$. The case of $\hat{b} < \underline{b}$ is similar. In fact, we can find such an M that $\hat{b} < r_{M1} < r_{M2}$ and they are roots of $F_M(b)=0$, which contradicts the convergence of $\{r_{i2}\}_{i=2}^{+\infty}$. Hence, (10) is true. □

From the above, for the first initial value $x_{-1} = a$ being fixed, we have obtained that $F_i(b)=0$ has only two "new" roots for $i>3$ and the sequences $\{r_{i1}\}_{i=2}^{+\infty}$ and $\{r_{i2}\}_{i=2}^{+\infty}$ converge to a same number.

To investigate the behavior of the solution of (2) with initial values in these intervals which are partitioned by the adjacent roots r_{ij}, we consider three cases.

Case 1 $\quad a = 0$

If $-1<b<0$, it follows that both $F_2(b)=-b-1$ and $F_3(b)=b$ are in the interval $(-1,\ 0)$. Thus $\{x_n\}$ of (2) converges to $\bar{x}_1$ by Theorem 2.1.

Case 2 $\quad a < 0$

From $r_{11}=1/a < 0$ and $F_2(b) = a\,b^2 - b - 1$, we have that $F_2(r_{11})=F_2(0)=-1$. Hence, only two cases in the following are needed.

(i) If $a<-0.25$ and $b \in (r_{11},\ 0)$, then $-1 < F_1(b),\ F_2(b) < 0$. Thus, by Theorem 2.1, $\{x_n\}$ of (2) converges to $\bar{x}_1$.

(ii) If $-0.25 \le a < 0$, then $F_2(b) = 0$ has two roots satisfying $r_{11} < r_{21} \le r_{22} < 0$. For $b \in (r_{11},\ r_{21}) \cup (r_{22},\ 0)$, we have $-1 < F_1(b),\ F_2(b) < 0$ and $\{x_n\}$ of (2) converges to $\bar{x}_1$ by Theorem 2.1. For $b\in(-\infty,\ r_{11})$, we have $F_2(b)<-1$ and $F_3(b)<-1$ and thus $\{x_n\}$ of (2) is unbounded by Theorem 2.1. For $b\in(r_{21},\ r_{22})$, we have $F_1(b)<0$ and $F_2(b)>0$ and thus $\{x_n\}$ of (2) is unbounded by Theorem 2.1.

Generally speaking, for $a< 0$, the solution $\{x_n\}$ of (2) converges to $\bar{x}_1$ only for two cases: one is $a<-0.25$ and $b \in (r_{11},\ 0)$, the other is $-0.25 \le a< 0$ and $b \in (r_{11},\ r_{21}) \cup (r_{22},\ 0)$. Hence , for $a\le 0$ and $b>0$, $\{x_n\}$ of (2) is unbounded. Thus, the dynamics of (2) is clear.

Case 3 $\quad a > 0$

For this case, it is complicated to arrange these roots r_{ij}. We divide it into three cases.

3.1 $\quad 0 < a < 1$

In this case, we prove that

$$r_{21} < 0 < r_{31} < r_{41} < r_{11} < r_{22} < r_{51} < r_{61} < r_{32} < r_{42} < r_{52} < r_{62}. \tag{11}$$

From the above, we only need to show $r_{31} < r_{41} < r_{11}$ and $r_{22} < r_{51} < r_{61} < r_{32}$.
First, from $r_{41} < r_{22} < r_{42}$, in view of $F_2(r_{31}) < 0$ and $F_3'(r_{31}) < 0$, it follows that

$$F_4(r_{31}) = -1, \quad F_4(r_{11}) = 0, \quad F_4'(r_{31}) > 0, \quad F_4'(r_{11}) = a-1 < 0 \tag{12}$$

for $0 < a < 1$. Thus, $r_{31} < r_{41} < r_{11}$.
Second, in order to prove $r_{22} < r_{51} < r_{61} < r_{32}$, we only need to show $r_{22} < r_{51}$ and $r_{61} < r_{32}$. Thus, the key is to compare r_{i2} with r_{j1} for $i \geq 2$ and $j = i+3$.
From $r_{51} < r_{32} < r_{52}$, we could conclude that $r_{22} < r_{51}$.
In fact, in view of $F_5(r_{22}) = 0$ and $F_5(r_{32}) = -1$, we have

$$F_5'(r_{32}) = -F_3'(r_{32}) < 0, \tag{13}$$

$$F_5'(r_{22}) = F_2'(r_{22})\,(1 - F_1(r_{22})) > 0 \tag{14}$$

which is guaranteed by

$$1 - F_1(r_{22}) = \frac{2(2-a)}{3+\sqrt{1+4a}} > 0 \tag{15}$$

for $0 < a < 2$. Thus, the conclusion is true.
In a similar way, from $r_{61} < r_{42} < r_{62}$, we conclude $r_{51} < r_{61} < r_{32}$.
In fact, in view of $F_6(r_{32}) = 0$ and $F_6(r_{51}) = -1$, we have

$$\begin{aligned} F_6'(r_{51}) &= F_5'(r_{51})F_4(r_{51}) > 0, \\ F_6'(r_{32}) &= F_3'(r_{22})\,(1 - F_2(r_{32})) < 0 \end{aligned} \tag{16}$$

which is guaranteed by

$$1 - F_2(r_{32}) = \frac{a-1}{a+\sqrt{a}} < 0 \tag{17}$$

for $0 < a < 1$. Thus, $r_{51} < r_{61} < r_{32}$ holds.
Hence, from the above, it is proved that (11) holds for $0 < a < 1$.
Third, we analyze the behavior of $\{x_n\}$ of (2) with x_0 being in the intervals partitioned by these adjacent roots.

(1) For $b \in (-\infty, r_{21})$, we have $F_1(b) < 0,\ F_2(b) > 0$. Thus, $\{x_n\}$ is unbounded by Theorem 2.1. It is also true for $b \in (0, r_{31}) \cup (r_{41}, r_{11}) \cup (r_{22}, r_{51}) \cup (r_{61}, r_{32})$ which are listed in Table 1.

(2) For $b \in (r_{21}, 0)$, we have $-1 < F_2(b),\ F_3(b) < 0$. Thus, $\{x_n\}$ converges to $\bar{x}_1$ by Theorem 2.1. It is also true for $b \in (r_{31}, r_{41}) \cup (r_{11}, r_{22}) \cup (r_{51}, r_{61})$ which are listed in Table 2.

Table 1: Intervals of x_0 such that $\{x_n\}$ is unbounded for $0 < x_{-1} < 1$

Intervals of x_0	Reasons	$\{x_n\}$ is
$(-\infty,\ r_{21})$	$F_1(b) < 0,\ F_2(b) > 0$	unbounded
$(0,\ r_{31})$	$F_2(b) < 0,\ F_3(b) > 0$	unbounded
$(r_{41},\ r_{11})$	$F_3(b) < 0,\ F_4(b) > 0$	unbounded
$(r_{22},\ r_{51})$	$F_4(b) < 0,\ F_5(b) > 0$	unbounded
$(r_{61},\ r_{32})$	$F_5(b) < 0,\ F_6(b) > 0$	unbounded

Table 2: Intervals of x_0 such that $\{x_n\}$ is convergent for $0 < x_{-1} < 1$

Intervals of x_0	Reasons	$\{x_n\}$
$(r_{21},\ 0)$	$-1<F_2(b),\ F_3(b)<0$	converges to $\bar{x}_1$
$(r_{31},\ r_{41})$	$-1<F_3(b),\ F_4(b)<0$	converges to $\bar{x}_1$
$(r_{11},\ r_{22})$	$-1<F_4(b),\ F_5(b)<0$	converges to $\bar{x}_1$
$(r_{51},\ r_{61})$	$-1<F_5(b),\ F_6(b)<0$	converges to $\bar{x}_1$

3.2 $1 \le a < 2$

In this case, we prove that

$$r_{21} < r_{31} \le 0 < r_{11} \le r_{41} < r_{22} < r_{51} < r_{32} \le r_{61} < r_{42} < r_{52} < r_{62}. \tag{18}$$

In fact, for $a = 1$, in view of their expressions and $F_4(b) = (b-1)^2\,(b^3 - b^3 - 2b - 1)$, we have $r_{31}=0,\ \ r_{32}=2=r_{61},\ r_{11}=1=r_{41}$.

For $1 < a < 2$, it is apparent that $r_{21} < r_{31} < 0 < r_{11} < r_{22} < r_{32}$. From (12), (16) and (17), we have

$$\begin{array}{rcl} F_4'(r_{11}) & = & a-1 > 0, \\ F_6'(r_{32}) & = & F_3'(r_{22})\frac{a-1}{a+\sqrt{a}} > 0. \end{array} \tag{19}$$

It follows that $r_{11} < r_{41}$ and $r_{32} < r_{61}$.

And $r_{22} < r_{51}$ follows from (14) and (15). Thus, (18) holds for $1 \le x_{-1} < 2$.

It is worth pointing out that $\{x_n\}$ of (2) converges to $\bar{x}_1$ for $1 \le x_{-1} < 2$ and $x_0 \in (r_{21}, r_{31}) \cup (0, r_{11}) \cup (r_{41}, r_{22}) \cup (r_{51}, r_{32})$ which are listed in Table 3.

Table 3: Intervals of x_0 such that $\{x_n\}$ is convergent for $1 \le x_{-1} < 2$

Intervals of x_0	Reasons	$\{x_n\}$
$(r_{21},\ r_{31})$	$-1<F_2(b),\ F_3(b)<0$	converges to $\bar{x}_1$
$(0,\ \ r_{11})$	$-1<F_3(b),\ F_4(b)<0$	converges to $\bar{x}_1$
$(r_{41},\ r_{22})$	$-1<F_4(b),\ F_5(b)<0$	converges to $\bar{x}_1$
$(r_{51},\ r_{32})$	$-1<F_5(b),\ F_6(b)<0$	converges to $\bar{x}_1$

3.3 $a \ge 2$

In this case, we prove that

$$r_{21} < r_{31} < 0 < r_{11} < r_{41} < r_{51} \le r_{22} < r_{32} < r_{61} < r_{42} < r_{52} < r_{62}. \tag{20}$$

Compared with (18), we only need to prove $r_{22} \ge r_{51}$ for $a \ge 2$. In fact, from (14) and (15), for $a \ge 2$, we have that $r_{22} \ge r_{51}$.

It is worth pointing out that $\{x_n\}$ of (2) converges to $\bar{x}_1$ for $x_{-1} \ge 2$ and $x_0 \in (r_{21}, r_{31}) \cup (0, r_{11}) \cup (r_{41}, r_{51}) \cup (r_{22}, r_{32})$ which are listed in Table 4.

From the above, we derive such intervals of x_0 for x_{-1} such that $\{x_n\}$ of (2) is convergent. It is worth pointing out that we couldn't continue such a procedure because there are no explicit expressions of r_{42} and so on. From the above procedures, we know that the key is how to compare r_{i2} with r_{j1} where $j=i+3$ for $i \ge 4$.

In fact, for such an interval $I_i = (r_{i2},\ r_{j1})$ (or $(r_{j1},\ r_{i2})$) where $j=i+3$ for $i \ge 4$, in view of auxiliary functions $F_j(b)$, we have $F_{j-1}(b) < 0$ and $F_j(b) > 0$. Thus, for x_{-1} being fixed and $x_0 \in \bigcup I_i$ (the union of I_i for $i \ge 4$), $\{x_n\}$ of (2) is unbounded by Theorem 2.1.

Table 4: Intervals of x_0 such that $\{x_n\}$ is convergent for $x_{-1} \geq 2$

Intervals of x_0	Reasons	$\{x_n\}$
$(r_{21},\ r_{31})$	$-1 < F_2(b),\ F_3(b) < 0$	converges to $\bar{x}_1$
$(0,\ \ r_{11})$	$-1 < F_3(b),\ F_4(b) < 0$	converges to $\bar{x}_1$
$(r_{41},\ r_{51})$	$-1 < F_4(b),\ F_5(b) < 0$	converges to $\bar{x}_1$
$(r_{22},\ r_{32})$	$-1 < F_5(b),\ F_6(b) < 0$	converges to $\bar{x}_1$

In view of Lemma 2.3, we obtain that the lengths of these open intervals I_i for $i \geq 4$ tend to zero as i tends to $+\infty$. For $x_0 > \hat{b}$, the increasing property of $\{x_n\}$ of (2) leads to its divergence.

Therefore, we generalize the above results into the following theorem.

Theorem 2.2. *The solution $\{x_n\}$ of (2) is unbounded only for its second initial value x_0 in such open intervals depending on the first initial valuex_0, which are listed in Table 5, where the endpoints r_{ij} are the roots of auxiliary functions $F_i(b) = x_i = 0$ with $x_0 = b$ and $x_{-1} = a$ for $i \geq 1$. And $\{x_n\}$ of (2) is an eventually prime period-three solution just at $x_0 = r_{ij}$ or $x_0 = 0$. For x_0 belongs to the complementary set of such intervals except those endpoints, $\{x_n\}$ of (2) is convergent to the negative equilibrium $\bar{x}_1$.*

Table 5: Intervals of x_0 for x_{-1} such that $\{x_n\}$ is unbounded

x_{-1}	Intervals of x_0
$(-\infty,\ -0.25)$	$(-\infty,\ r_{11}) \cup (0,\ +\infty)$
$[-0.25,\ 0)$	$(-\infty,\ r_{11}) \cup (r_{21}, r_{22}) \cup (0,\ +\infty)$
0	$(-\infty,\ -1) \cup (0,\ +\infty)$
$(0,\ 1)$	$(-\infty,\ r_{21}) \cup (0,\ r_{31}) \cup (r_{41},\ r_{11}) \cup (\hat{b},\ +\infty) \cup (\bigcup I_i)$
$[1,\ +\infty)$	$(-\infty,\ r_{21}) \cup (0,\ r_{31}) \cup (r_{41},\ r_{11}) \cup (\hat{b},\ +\infty) \cup (\bigcup I_i)$

From Theorem 2.2 and Table 5, for x_{-1} and x_0 greater than zero, solutions of (2) would exhibit somewhat chaotic behavior[4], that is, $\{x_n\}$ is either unbounded or convergent alternately for x_0 depending on x_{-1}, which is more concise from Table 5.

Now, we give some examples for particular x_{-1} which are listed in Table 6. Here, we only present the former six intervals of x_0 such that the solution $\{x_n\}$ of (2) is convergent. It is noted that the numerical values of these endpoints of these intervals are approximated to the values of the solutions of the auxiliary equations $F_i(b) = 0$.

From Table 6, for $x_{-1} = 1.5$, it is shown the former six intervals of x_0 such that the solution $\{x_n\}$ of (2) is convergent, which are on both sides of zero. If $x_0 = 1.6$ in $(1.4975,\ 1.6073)$, then the solution of (2) enters and then remains in the interval $(-1,\ 0)$, and hence is bounded and convergent. Whereas if $x_0 = 1.61$, then the solution is unbounded. It is clear for the third case that the solution is UB or C.

3 Conclusion

The existence of prime period-three solutions of (2) is proved in [4] and the convergence of (2) in its invariant interval $(-1,\ 0)$ is proved in [7]. In this paper, we present a new method to partition the intervals of x_0 depending on x_{-1} to describe the behavior of solutions of (2) and explain in detail that the solution of (2) exhibits somewhat chaotic behavior relative to the

Table 6: Intervals of x_0 for $x_{-1} > 0$ such that $\{x_n\}$ is convergent

x_{-1}	Intervals of x_0
0.1	$(-0.9161,\ 0)$, $(6.8377,\ 7.6946)$, $(10,\ 10.9161)$, $(12.4540,\ 12.8553)$, $(13.1623,\ 13.4675)$, $(13.6396,\ 13.7755)$
0.618	$(-0.6985,\ 0)$, $(0.3461,\ 1.4048)$, $(1.6181,\ 2.3166)$, $(2.5350,\ 2.8614)$, $(2.8902,\ 3.0690)$, $(3.0996,\ 3.1756)$
1	$(-0.618,\ 0)$, $(0,\ 1)$, $(1,\ 1.6180)$, $(1.7121,\ 2)$, $(2,\ 2.1479)$, $(2.1637,\ 2.2237)$
1.5	$(-0.5486,\ -0.1498)$, $(0,\ 0.6667)$, $(0.7717,\ 1.2153)$, $(1.2447,\ 1.4832)$, $(1.4975,\ 1.6073)$, $(1.6149,\ 1.6633)$
2.5	$(-0.4633,\ -0.2325)$, $(0,\ 0.4)$, $(0.5711,\ 0.8476)$, $(0.8633,\ 1.0325)$, $(1.0558,\ 1.1302)$, $(1.1316,\ 1.1680)$
10	$(-0.2702,\ -0.2162)$, $(0,\ 0.1)$, $(0.2740,\ 0.3327)$, $(0.3702,\ 0.4162)$, $(0.4383,\ 0.4596)$, $(0.4630,\ 0.4752)$

initial values. Compared with the known results[4], our results are much more accurate and easy to obtain by computers to describe the evolution of (2) for the initial values in the plane.

We conclude that the solution of (2) is bounded and convergent only for x_0 in particular intervals depending on x_{-1}, which are partitioned by the zeroes of auxiliary functions presented in this paper. Specially, it is unbounded only for x_0 in such open intervals listed in Table 5 which depend on x_{-1}.

It is of great interest to continue the investigation of the monotonicity, periodicity, and boundedness nature of solutions of (1) for different choices of parameters k and l and other equations presented in [4]. We believe that prime-period solutions and the negative equilibrium are crucial for the dynamics of difference equations (1). The future work is to extend our study to a more generalized equation (1).

Conflict of Interests

The authors declare that they have no competing interests.

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