Volume 22, Number 6 June 1st, 2017
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of Computational Analysis and Applications

EUDOXUS PRESS,LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE

SCOPE OF THE JOURNAL

An international publication of Eudoxus Press, LLC
(fourteen times annually)

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A new result on the almost increasing sequences

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Abstract

In this paper, we have generalized a known theorem on $|\bar{N}, p_n|_k$ summability factors of infinite series to the $\varphi - |A, p_n|_k$ summability by using an almost increasing sequence. This new theorem also includes several new results.

1. INTRODUCTION

A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see [1]). Let $\sum a_n$ be a given infinite series with partial sums (s_n) and let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0, 1, ... \tag{1}$$

The series $\sum a_n$ is said to be summable $|A|_k$, $k \geq 1$, if (see [13])

$$\sum_{n=1}^{\infty} n^{k-1} \ \left|\bar{\Delta} A_n(s)\right|^k < \infty, \tag{2}$$

2010 AMS Subject Classification: 40D15, 40F05, 40G99.

Key Words: Summability factors, absolute matrix summability, almost increasing sequence, infinite series.

where

$$\bar{\Delta}A_n(s) = A_n(s) - A_{n-1}(s). \tag{3}$$

Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, i \geq 1). \tag{4}$$

The sequence-to-sequence transformation

$$u_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{5}$$

defines the sequence (u_n) of the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [8]). The series $\sum a_n$ is said to be summable $\mid \bar{N}, p_n \mid_k$, $k \geq 1$, if (see [2])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} \mid \Delta u_{n-1} \mid^k < \infty, \tag{6}$$

and it is said to be summable $|A, p_n|_k$, $k \geq 1$, if (see [12])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} \left| \bar{\Delta}A_n(s) \right|^k < \infty, \tag{7}$$

where

$$\bar{\Delta}A_n(s) = A_n(s) - A_{n-1}(s).$$

Let (φ_n) be any sequence of positive real numbers. The series $\sum a_n$ is summable $\varphi - |A, p_n|_k$, $k \geq 1$, if (see [11])

$$\sum_{n=1}^{\infty} \varphi_n^{k-1} |\bar{\Delta}A_n(s)|^k < \infty. \tag{8}$$

If we take $\varphi_n = \frac{P_n}{p_n}$, then $\varphi - |A, p_n|_k$ summability reduces to $|A, p_n|_k$ summability (see [10]). Also, if we take $\varphi_n = \frac{P_n}{p_n}$ and $a_{nv} = \frac{p_v}{P_n}$, then we get $|\bar{N}, p_n|_k$ summability. If we take $\varphi_n = n$ and $a_{nv} = \frac{p_v}{P_n}$, then we get $|R, p_n|_k$ summability (see [5]). Furthermore, if we take $\varphi_n = n$ and $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n, then $\varphi - |A, p_n|_k$ summability reduces to $|C, 1|_k$ summability (see [7]).

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In [6], Bor has proved the following theorem for $|\bar{N}, p_n|_k$ summability factors of infinite series.

Theorem 1.1. Let (X_n) be an almost increasing sequence and let there be sequences (β_n) and (λ_n) such that

$$| \Delta\lambda_n | \leq \beta_n, \tag{9}$$

$$\beta_n \to 0 \quad as \quad n \to \infty, \tag{10}$$

$$\sum_{n=1}^{\infty} n \mid \Delta\beta_n \mid X_n < \infty, \tag{11}$$

$$| \lambda_n | X_n = O(1) \tag{12}$$

and

$$\sum_{v=1}^{n} \frac{| t_v |^k}{v} = O(X_n) \quad as \quad n \to \infty, \tag{13}$$

where (t_n) is the n-th $(C, 1)$ mean of the sequence (na_n). Suppose further, the sequence (p_n) is such that

$$P_n = O(np_n), \tag{14}$$

$$P_n \Delta p_n = O(p_n p_{n+1}), \tag{15}$$

then the series $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{np_n}$ is summable $| \bar{N}, p_n |_k$, $k \geq 1$.

Remark 1.2. It should be noted that, from the hypotheses of the Theorem 1.1, (λ_n) is bounded and $\Delta\lambda_n = O(1/n)$ (see [3]).

2. THE MAIN RESULT

The aim of this paper is to generalize Theorem 1.1 for absolute matrix summability.

Before stating the main theorem we must first introduce some further notations. Given a normal matrix $A = (a_{nv})$, we associate two lover semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, ... \tag{16}$$

and

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, ... \tag{17}$$

It may be noted that $\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} \bar{a}_{nv} a_v \tag{18}$$

and

$$\bar{\Delta} A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv} a_v. \tag{19}$$

Now, we shall prove the following theorem.

Theorem 2.1. Let $A = (a_{nv})$ be a positive normal matrix such that

$$\bar{a}_{no} = 1, \ n = 0, 1, ..., \tag{20}$$

$$a_{n-1,v} \geq a_{nv}, \ for \ n \geq v + 1, \tag{21}$$

$$a_{nn} = O(\frac{p_n}{P_n}), \tag{22}$$

$$| \widehat{a}_{n,v+1} | = O(v \mid \Delta_v(\widehat{a}_{nv}) \mid) \tag{23}$$

Let (X_n) be an almost increasing sequence and $(\frac{\varphi_n p_n}{P_n})$ be a non-increasing sequence. If conditions (9)-(15) of the Theorem 1.1 and

$$\sum_{n=1}^{m} \varphi_n^{k-1} (\frac{p_n}{P_n})^k |t_n|^k = O(X_m) \quad as \quad m \to \infty \tag{24}$$

are satisfied, then the series $\sum_{n=1}^{\infty} a_n \frac{P_n\lambda_n}{np_n}$ is summable $\varphi - |A, p_n|_k$, $k \geq 1$.

We need the following lemmas for the proof of our theorem.

Lemma 2.2. **([9])** If (X_n) an almost increasing sequence, then under the conditions (10)-(11) we have that

$$nX_n\beta_n = O(1), \tag{25}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{26}$$

Lemma 2.3. **([4])** If the conditions (14) and (15) are satisfied, then $\Delta(P_n/p_n n^2) = O(1/n^2)$.

3. PROOF OF THEOREM 2.1

Let (T_n) denotes A-transform of the series $\sum_{n=1}^{\infty} \frac{a_n P_n \lambda_n}{np_n}$. Then we have by (18) and (19)

$$\bar{\Delta} T_n = \sum_{v=1}^{n} \hat{a}_{nv} \frac{a_v P_v \lambda_v}{vp_v}.$$

Applying Abel's transformation to this sum, we get that

$$\begin{aligned}
\bar{\Delta} T_n &= \sum_{v=1}^{n} \hat{a}_{nv} \frac{v a_v P_v \lambda_v}{v^2 p_v} \\
&= \sum_{v=1}^{n-1} \Delta_v \left(\frac{\hat{a}_{nv} P_v \lambda_v}{v^2 p_v}\right) \sum_{r=1}^{v} r a_r + \frac{\hat{a}_{nn} P_n \lambda_n}{n^2 p_n} \sum_{r=1}^{n} r a_r \\
&= \sum_{v=1}^{n-1} \Delta_v \left(\frac{\hat{a}_{nv} P_v \lambda_v}{v^2 p_v}\right)(v+1) t_v + \frac{a_{nn} P_n \lambda_n}{n^2 p_n}(n+1) t_n \\
&= \frac{a_{nn} P_n \lambda_n}{n^2 p_n}(n+1) t_n + \sum_{v=1}^{n-1} \Delta_v(\hat{a}_{nv}) \frac{(v+1)}{v^2} \frac{P_v \lambda_v}{p_v} t_v \\
&\quad + \sum_{v=1}^{n-1} \frac{\hat{a}_{n,v+1} P_v}{p_v} \Delta\lambda_v t_v \frac{(v+1)}{v^2} + \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \lambda_{v+1} \Delta\left(\frac{P_v}{v^2 p_v}\right) t_v (v+1) \\
&= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}, \quad say.
\end{aligned}$$

Since

$$|T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}|^k \leq 4^k (|T_{n,1}|^k + |T_{n,2}|^k + |T_{n,3}|^k + |T_{n,4}|^k)$$

to complete the proof of Theorem 2.1, it is sufficient to show that

$$\sum_{n=1}^{m} \varphi_n^{k-1} \mid T_{n,r} \mid^k < \infty, \quad for \quad r = 1, 2, 3, 4. \tag{27}$$

Firstly, by using Abel's transformation, we have that

$$\begin{aligned}
\sum_{n=1}^{m} \varphi_n^{k-1} \mid T_{n,1} \mid^k &= O(1) \sum_{n=1}^{m} \varphi_n^{k-1} a_{nn}^k (\frac{P_n}{p_n})^k |\lambda_n|^k \frac{|t_n|^k}{n^k} \\
&= O(1) \sum_{n=1}^{m} \varphi_n^{k-1} (\frac{p_n}{P_n})^k |\lambda_n|^{k-1} |\lambda_n| |t_n|^k \\
&= O(1) \sum_{n=1}^{m} \varphi_n^{k-1} (\frac{p_n}{P_n})^k |\lambda_n| |t_n|^k \\
&= O(1) \sum_{n=1}^{m-1} \Delta|\lambda_n| \sum_{v=1}^{n} \varphi_v^{k-1} (\frac{p_v}{P_v})^k |t_v|^k + O(1)|\lambda_m| \sum_{n=1}^{m} \varphi_n^{k-1} (\frac{p_n}{P_n})^k |t_n|^k \\
&= O(1) \sum_{n=1}^{m-1} |\Delta\lambda_n| X_n + O(1)|\lambda_m| X_m \\
&= O(1) \sum_{n=1}^{m-1} \beta_n X_n + O(1)|\lambda_m| X_m \\
&= O(1) \quad as \quad m \to \infty,
\end{aligned}$$

by virtue of the hypotheses of Theorem 2.1 and Lemma 2.2.

Now, using the fact that $P_v = O(vp_v)$ by (14), we have that

$$\sum_{n=1}^{m} \varphi_n^{k-1} \mid T_{n,2} \mid^k = O(1) \sum_{n=2}^{m+1} \varphi_n^{k-1} \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| \, |\lambda_v| \, |t_v| \right)^k$$

Now, applying Hölder's inequality with indices k and k', where $k > 1$ and $\frac{1}{k} + \frac{1}{k'} = 1$, as in $T_{n,1}$, we have that

$$\begin{aligned}
\sum_{n=1}^{m} \varphi_n^{k-1} \mid T_{n,2} \mid^k &= O(1) \sum_{n=2}^{m+1} \varphi_n^{k-1} \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| \, |\lambda_v|^k \, |t_v|^k \right) \\
&\quad \times (\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})|)^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{k-1} a_{nn}^{k-1} \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| \, |\lambda_v|^k \, |t_v|^k \right) \\
&= O(1) \sum_{n=2}^{m+1} (\frac{\varphi_n p_n}{P_n})^{k-1} \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| \, |\lambda_v|^k \, |t_v|^k \right)
\end{aligned}$$

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$$
\begin{aligned}
&= O(1)\sum_{v=1}^{m}(\frac{\varphi_v p_v}{P_v})^{k-1}|\lambda_v|^k|t_v|^k\sum_{n=v+1}^{m+1}|\Delta_v(\hat{a}_{nv})| \\
&= O(1)\sum_{v=1}^{m}(\frac{\varphi_v p_v}{P_v})^{k-1}|\lambda_v|^{k-1}|\lambda_v||t_v|^k a_{vv} \\
&= O(1)\sum_{v=1}^{m}\varphi_v^{k-1}(\frac{p_v}{P_v})^k\,|\lambda_v|\,|t_v|^k \\
&= O(1) \quad as \quad m\to\infty,
\end{aligned}
$$

by virtue of the hypotheses of Theorem 2.1 and Lemma 2.2.

Now, using Hölder's inequality we have that

$$
\begin{aligned}
\sum_{n=2}^{m+1}\varphi_n^{k-1}\mid T_{n,3}\mid^k &= O(1)\sum_{n=2}^{m+1}\varphi_n^{k-1}\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\Delta\lambda_v||t_v|\right)^k \\
&= O(1)\sum_{n=2}^{m+1}\varphi_n^{k-1}\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\beta_v|t_v|^k\right)\times\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\beta_v\right)^{k-1} \\
&= O(1)\sum_{n=2}^{m+1}\varphi_n^{k-1}a_{nn}^{k-1}(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\beta_v|t_v|^k) \\
&= O(1)\sum_{n=2}^{m+1}(\frac{\varphi_n p_n}{P_n})^{k-1}(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\beta_v|t_v|^k) \\
&= O(1)\sum_{v=1}^{m}\beta_v|t_v|^k\sum_{n=v+1}^{m+1}(\frac{\varphi_n p_n}{P_n})^{k-1}|\hat{a}_{n,v+1}| \\
&= O(1)\sum_{v=1}^{m}(\frac{\varphi_v p_v}{P_v})^{k-1}\beta_v|t_v|^k\sum_{n=v+1}^{m+1}|\hat{a}_{n,v+1}| \\
&= O(1)\sum_{v=1}^{m}\varphi_v^{k-1}(\frac{p_v}{P_v})^k v\beta_v|t_v|^k \\
&= O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)\sum_{r=1}^{v}\varphi_r^{k-1}(\frac{p_r}{P_r})^k|t_r|^k + O(1)m\beta_m\sum_{v=1}^{m}\varphi_v^{k-1}(\frac{p_v}{P_v})^k|t_v|^k \\
&= O(1)\sum_{v=1}^{m-1}|\Delta(v\beta_v)|X_v + O(1)m\beta_m X_m \\
&= O(1)\sum_{v=1}^{m-1}v|\Delta\beta_v|X_v + O(1)\sum_{v=1}^{m-1}\beta_{v+1}X_{v+1} + O(1)m\beta_m X_m \\
&= O(1) \quad as \quad m\to\infty,
\end{aligned}
$$

by virtue of the hypotheses of Theorem 2.1 and Lemma 2.2.

Finally, since $\Delta(\frac{P_v}{v^2 p_v}) = O(\frac{1}{v^2})$, as in $T_{n,1}$, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1} \varphi_n^{k-1} |T_n(4)|^k &= O(1) \sum_{n=2}^{m+1} \varphi_n^{k-1} \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}| |\lambda_{v+1}| \frac{|t_v|}{v} \right)^k \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{k-1} \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}| |\lambda_{v+1}|^k \frac{|t_v|^k}{v} \right) \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}| \frac{1}{v} \right)^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{k-1} \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}| |\lambda_{v+1}|^k \frac{|t_v|^k}{v} \right) \left(\sum_{v=1}^{n-1} |\Delta_v \hat{a}_{nv}| \right)^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{k-1} a_{nn}^{k-1} \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}| |\lambda_{v+1}|^{k-1} |\lambda_{v+1}| \frac{|t_v|^k}{v} \right) \\
&= O(1) \sum_{n=2}^{m+1} (\frac{\varphi_n p_n}{P_n})^{k-1} \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}| |\lambda_{v+1}| \frac{|t_v|^k}{v} \right) \\
&= O(1) \sum_{v=1}^{m} |\lambda_{v+1}| \frac{|t_v|^k}{v} \sum_{n=v+1}^{m+1} (\frac{\varphi_n p_n}{P_n})^{k-1} |\hat{a}_{n,v+1}| \\
&= O(1) \sum_{v=1}^{m} (\frac{\varphi_v p_v}{P_v})^{k-1} |\lambda_{v+1}| \frac{|t_v|^k}{v} \sum_{n=v+1}^{m+1} |\hat{a}_{n,v+1}| \\
&= O(1) \sum_{v=1}^{m} (\frac{\varphi_v p_v}{P_v})^{k-1} |\lambda_{v+1}| \frac{|t_v|^k}{v} \\
&= O(1) \sum_{v=1}^{m} \varphi_v^{k-1} (\frac{p_v}{P_v})^k |\lambda_{v+1}| |t_v|^k \\
&= O(1) \quad as \quad m \to \infty.
\end{aligned}$$

by virtue of hypotheses of Theorem 2.1 and Lemma 2.3

Therefore we get

$$\sum_{n=1}^{m} \varphi_n^{k-1} \mid T_{n,r} \mid^k = O(1) \quad as \quad m \to \infty, \quad for \quad r = 1, 2, 3, 4.$$

This completes the proof of Theorem 2.1

Corollary 3.1. If we take $\varphi_n = \frac{P_n}{p_n}$, then we get a theorem dealing with $|A, p_n|_k$ summability.

Corollary 3.2. If we take $\varphi_n = \frac{P_n}{p_n}$ and $a_{nv} = \frac{p_v}{P_n}$, then we get Theorem 1.1.

Corollary 3.3. If we take $a_{nv} = \frac{p_v}{P_n}$, then we have another a result dealing with $\varphi - |\bar{N}, p_n|_k$ summability.

Corollary 3.4. If we take $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n, then we get a result dealing with $\varphi - |C, 1|_k$ summability.

Corollary 3.5. If we take $\varphi_n = n$, $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n, then we get a result for $|C, 1|_k$ summability.

Corollary 3.6. If we take $k = 1$ and $a_{nv} = \frac{p_v}{P_n}$, then we get a result for $|\bar{N}, p_n|$ summability and in this case the condition "$\left(\frac{\varphi_n p_n}{P_n}\right)$ is a non-increasing sequence" is not needed.

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Certain Chebyshev type inequalities involving the generalized fractional integral operator

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Abstract: In this paper, we establish certain new Chebyshev type fractional integral inequalities involving the Gauss hypergeometric function. Several special cases as Chebyshev type fractional integral inequalities involving Saigo, Erdélyi-Kober, and Riemann-Liouville type fractional integral operators are presented. Furthermore, we also consider their relevance with other related known results. An example is also given to show the applications of obtained results.

Keywords: Chebyshev type inequalities; fractional integral inequalities; hypergeometric fractional integrals; synchronous (asynchronous) functions

2010 Mathematics Subject Classification: 26D10; 26A33; 33C05

1 Introduction and preliminaries

Due to the fact that the tools of fractional integral inequalities have many applications in establishing uniqueness of solutions in fractional boundary value problems and in fractional partial differential equations, fractional integral inequalities involving the fractional operators (like Saigo, Erdélyi-Kober, Riemann-Liouville type fractional integral operators, etc.) has gained considerable attention, attracting the interest of several researchers. For some recent developments on fractional integral inequalities, we refer the reader to [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12] and the references cited therein. Belarbi and Dahmani [13] gave the following integral inequality, using the Riemann-Liouville fractional integrals: if f and g are two synchronous functions (see Definition 1.4) on $C[0,\infty)$, then

$$J^\alpha(fg)(t) \geq \frac{\Gamma(\alpha+1)}{t^\alpha} J^\alpha f(t) J^\alpha g(t), \tag{1.1}$$

and

$$\frac{t^\alpha}{\Gamma(\alpha+1)} J^\beta(fg)(t) + \frac{t^\beta}{\Gamma(\beta+1)} J^\alpha(fg)(t) \geq J^\alpha f(t) J^\beta g(t) + J^\beta f(t) J^\alpha g(t), \tag{1.2}$$

for all $t > 0$, $\alpha > 0$, and $\beta > 0$. Öğünmez and Özkan [14], Chinchane and Pachpatte [15] and Purohit and Raina [16] obtained the Riemann-Liouville fractional q-integral inequalities, the Hadamard fractional integral inequalities and the Saigo fractional integral and q-integral inequalities similar to the inequalities (1.1) and (1.2), respectively.

Dahmani in [17] established the following fractional integral inequalities which are generalizations of the inequalities (1.1) and (1.2), by using the Riemann-Liouville fractional integrals. Let f and g be two synchronous functions on $[0,\infty)$ and let $u, v : [0,\infty) \to [0,\infty)$. Then

$$J^\alpha u(t) J^\alpha(vfg)(t) + J^\alpha v(t) J^\alpha(ufg)(t) \geq J^\alpha(uf)(t) J^\alpha(vg)(t) + J^\alpha(vf)(t) J^\alpha(ug)(t), \tag{1.3}$$

and

$$J^\alpha u(t) J^\beta(vfg)(t) + J^\beta v(t) J^\alpha(ufg)(t) \geq J^\alpha(uf)(t) J^\beta(vg)(t) + J^\beta(vf)(t) J^\alpha(ug)(t), \tag{1.4}$$

for all $t > 0$, $\alpha > 0$ and $\beta > 0$. Yang [18], Brahim and Taf [19] and Chinchane and Pachpatte [20] and Agarwal *et al.* [21] gave the fractional q-integral inequalities, the fractional integral inequalities with two parameters of deformation q_1 and q_2, the Hadamard fractional integral inequalities and generalized Erdélyi-Kober fractional q-integral inequalities similar to inequalities (1.3) and (1.4), respectively.

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Let us consider the celebrated Chebyshev functional (see [22])

$$T(f,g)=\frac{1}{b-a}\int_a^b f(x)g(x)dx-\frac{1}{b-a}\int_a^b f(x)dx\cdot\frac{1}{b-a}\int_a^b g(x)dx$$

where f and g are two integrable functions on $[a,b]$. In [23], Grüss proved the well known inequality:

$$|T(f,g)|\leq\frac{1}{4}(\Phi-\phi)(\Psi-\psi), \tag{1.5}$$

where f and g are two integrable functions on $[a,b]$ satisfying the conditions

$$\phi\leq f(x)\leq\Phi,\quad \psi\leq g(x)\leq\Psi,\quad \phi,\Phi,\psi,\Psi\in\mathbb{R},\quad x\in[a,b]. \tag{1.6}$$

The inequality (1.5) is known as Grüss inequality. By using the Riemann-Liouville fractional integral and q-integral operators, Dahmani *et al.* [26] and Zhu *et al.* [27] gave the fractional integral and q-integral inequality similar to inequality (1.5) satisfying the conditions (1.6), respectively. Wang *et al.* [29] and Baleanu [30] *et al.* obtained some q-integral inequality of Grüss type for the Saigo fractional q-integral operator, respectively.

Throughout the present paper, we shall investigate a fractional integral over the space C_λ introduced in [31] and defined as follows.

Definition 1.1. For each real number λ, let C_λ define the space of all functions $f:\mathbb{R}^+\to\mathbb{R}$ that can be represented in the form $f(x)=x^p f_1(x)$ with $p>\lambda$ and $f_1\in C[0,\infty)$, where $C[0,\infty)$ denotes the set of all continuous real functions defined in $[0,\infty)$.

We give the generalized fractional integral operator $K_t^{\alpha,\beta,\eta,\mu}$ associated with the Gauss hypergeometric function as follows.

Definition 1.2. *[28]* Consider $\lambda\in\mathbb{R}$ and $f\in C_\lambda$. For $\alpha>\max\{0,-(\mu+\eta+1)\}$, $\beta<1$, $\mu>-1$ and $\beta-1<\eta<0$, we define the fractional integral

$$K_t^{\alpha,\beta,\eta,\mu}f(x)=\frac{\Gamma(1-\beta)\Gamma(\alpha+\mu+\eta+1)}{\Gamma(\eta-\beta+1)\Gamma(\mu+1)}x^{\beta+\mu}I_t^{\alpha,\beta,\eta,\mu}\{f(x)\}, \tag{1.7}$$

where $I_t^{\alpha,\beta,\eta,\mu}$ is the Gauss hypergeometric fractional integral of order α and is defined in the following.

Definition 1.3. Let $\alpha>0$, $\mu>-1$, $\beta,\eta\in\mathbb{R}$. Then the generalized fractional integral $I_t^{\alpha,\beta,\eta,\mu}$ (in terms of the Gauss hypergeometric function) of order α for real-valued continuous function $f(t)$ is defined by [31] (see also [32])

$$I_t^{\alpha,\beta,\eta,\mu}\{f(x)\}=\frac{x^{-\alpha-\beta-2\mu}}{\Gamma(\alpha)}\int_0^x t^\mu(x-t)^{\alpha-1}{}_2F_1\left(\alpha+\beta+\mu,-\eta;\alpha;1-\frac{t}{x}\right)f(t)dt, \tag{1.8}$$

where the function ${}_2F_1(\cdot)$ appearing as a kernel for the operator (1.7) is the Gaussian hypergeometric function defined by

$${}_2F_1(a,b;c;t)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{t^n}{n!},$$

and $(a)_n$ is the Pochhammer symbol defined by

$$(a)_0=1;\quad (a)_n=a(a+1)\cdots(a+n-1),\quad \text{for}\ \ n\in\mathbb{N}.$$

Here $\mathbb{N}$ denotes the set of positive integers.

The above integral (1.8) has the following commutative property(see also [32, 33]):

$$I_t^{\alpha,\beta,\eta,\mu}I_t^{\gamma,\delta,\zeta,\nu}f(x)=I_t^{\gamma,\delta,\zeta,\nu}I_t^{\alpha,\beta,\eta,\mu}f(x).$$

Definition 1.4. Two functions f and g are said to be synchronous (asynchronous) functions on $[0,\infty)$ if

$$A(u,v)=(f(u)-f(v))(g(u)-g(v))\geq(\leq)0,\quad u,v\in[0,\infty).$$

In [31], Baleanu *et al.* obtained the following fractional integral inequalities involving the Gauss hypergeometric function: Let f and g be two synchronous functions on $[0,\infty)$. Then

$$I_t^{\alpha,\beta,\eta,\mu}\{f(t)g(t)\} \geq \frac{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)t^{\beta+\mu}}{\Gamma(1+\mu)\Gamma(1-\beta+\eta)} I_t^{\alpha,\beta,\eta,\mu}\{f(t)\} I_t^{\alpha,\beta,\eta,\mu}\{g(t)\},$$

for all $t>0$, where α,β,η,μ are real constants satisfying $\alpha>\max\{0,-\beta,-\mu\}$, $\beta<1$, $\mu>-1$ and $\beta-1<\eta<0$, and also

$$\frac{\Gamma(1+\mu)\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)t^{\beta+\mu}} I_t^{\gamma,\delta,\zeta,\nu}\{f(t)g(t)\} + \frac{\Gamma(1+\nu)\Gamma(1-\delta+\zeta)}{\Gamma(1-\delta)\Gamma(1+\nu+\gamma+\zeta)t^{\delta+\nu}} I_t^{\alpha,\beta,\eta,\mu}\{f(t)g(t)\}$$
$$\geq I_t^{\alpha,\beta,\eta,\mu}\{f(t)\} I_t^{\gamma,\delta,\zeta,\nu}\{g(t)\} + I_t^{\gamma,\delta,\zeta,\nu}\{f(t)\} I_t^{\alpha,\beta,\eta,\mu}\{g(t)\},$$

for all $t>0$, where α,β,η,μ satisfies the above inequalities and the constants γ,δ,ζ,ν satisfies $\gamma>\max\{0,-\delta,-\nu\}$, $\delta<1$, $\nu>-1$, $\delta-1<\zeta<0$.

In [28], Wang *et al.* gave the following integral inequalities by using the generalized fractional integral operator: Let f and g be two integrable functions with $f,g\in C_\lambda$ and satisfying the condition (1.6) on $[0,\infty)$. Thus we have

$$|K_t^{\alpha,\beta,\eta,\mu}(fg)(x) - K_t^{\alpha,\beta,\eta,\mu}f(x)K_t^{\alpha,\beta,\eta,\mu}g(x)| \leq \frac{1}{4}(\Phi-\phi)(\Psi-\psi),$$

for all $x\in[0,\infty)$, where α,β,η,μ are real constants with $\alpha>0$, $\mu>-1$, $\eta\leq 0$ and $\alpha+\beta+\mu\geq 0$. And Let f and g be two synchronous functions on $[0,\infty)$. Then the following inequality holds:

$$K_t^{\alpha,\beta,\eta,\mu}(fg)(x) \geq K_t^{\alpha,\beta,\eta,\mu}f(x)K_t^{\alpha,\beta,\eta,\mu}g(x),$$

for all $x\in[0,\infty)$, where α,β,η,μ are real constants such that $\alpha>0$, $\mu>-1$, $\eta\leq 0$ and $\alpha+\beta+\mu\geq 0$.

Motivated by the results mentioned above and using the generalized fractional integral operator, we establish certain new Chebyshev type fractional integral inequalities and some related inequalities. Furthermore, several special cases as Chebyshev type fractional integral inequalities involving Saigo, Erdélyi-Kober, and Riemann-Liouville type fractional integral operators are given. Then we present an example to show the applications of obtained results. At last, concluding remarks are also given.

2 Generalized fractional integral inequalities

In this section, we establish certain new Chebyshev type fractional integral inequalities and some related inequalities involving the generalized fractional integral operator.

For the sake of simplicity, we always assume that $K_t^{\alpha,\beta,\eta,\mu}u$ denotes $K_t^{\alpha,\beta,\eta,\mu}u(x)$ and all of the generalized fractional integral operator holds in this work.

Lemma 2.1. *Let f and g be two synchronous functions on $[0,\infty)$ and let u and v be two nonnegative functions on $[0,\infty)$. Then we have*

$$K_t^{\alpha,\beta,\eta,\mu}uK_t^{\alpha,\beta,\eta,\mu}(vfg) + K_t^{\alpha,\beta,\eta,\mu}vK_t^{\alpha,\beta,\eta,\mu}(ufg) \geq K_t^{\alpha,\beta,\eta,\mu}(vf)K_t^{\alpha,\beta,\eta,\mu}(ug) + K_t^{\alpha,\beta,\eta,\mu}(uf)K_t^{\alpha,\beta,\eta,\mu}(vg), \tag{2.1}$$

for all $x\in[0,\infty)$, and real constants α,β,η,μ with $\alpha>0$, $\mu>-1$, $\eta\leq 0$ and $\alpha+\beta+\mu\geq 0$.

Proof. Since f and g are two synchronous functions on $[0,\infty)$, for all $\tau>0$ and $\rho>0$, then we have

$$(f(\tau)-f(\rho))(g(\tau)-g(\rho)) \geq 0. \tag{2.2}$$

Rewriting (2.2), we obtain

$$f(\tau)g(\tau)+f(\rho)g(\rho) \geq f(\tau)g(\rho)+f(\rho)g(\tau). \tag{2.3}$$

Multiplying both side of (2.3) by $v(\tau)\frac{\tau^\mu(x-\tau)^{\alpha-1}}{\Gamma(\alpha)}{}_2F_1(\alpha+\mu+\beta,-\eta;\alpha;1-\frac{\tau}{x})$, where $x>0$ and $\tau\in(0,x)$, when we integrate the inequality with respect to τ from 0 to x, we obtain by Definition 1.2 that

$$K_t^{\alpha,\beta,\eta,\mu}(vfg)(x) + f(\rho)g(\rho)K_t^{\alpha,\beta,\eta,\mu}v(x) \geq g(\rho)K_t^{\alpha,\beta,\eta,\mu}(vf)(x) + f(\rho)K_t^{\alpha,\beta,\eta,\mu}(vg)(x). \tag{2.4}$$

Again, by multiplying both side of (2.4) by $u(\rho)\frac{\rho^{\mu}(x-\rho)^{\alpha-1}}{\Gamma(\alpha)}{}_2F_1(\alpha+\mu+\beta,-\eta;\alpha;1-\frac{\rho}{x})$, where $x>0$ and $\rho\in(0,x)$, and integrating the resulting identity with respect to ρ from 0 to x, and then applying Definition 1.2, we conclude

$$K_t^{\alpha,\beta,\eta,\mu}u(x)K_t^{\alpha,\beta,\eta,\mu}(vfg)(x)+K_t^{\alpha,\beta,\eta,\mu}v(x)K_t^{\alpha,\beta,\eta,\mu}(ufg)(x)$$
$$\geq K_t^{\alpha,\beta,\eta,\mu}(vf)(x)K_t^{\alpha,\beta,\eta,\mu}(ug)(x)+K_t^{\alpha,\beta,\eta,\mu}(uf)(x)K_t^{\alpha,\beta,\eta,\mu}(vg)(x),$$

which implies (2.1). □

Theorem 2.2. *Let f and g be two synchronous functions on $[0,\infty)$ and let p, q and r be three nonnegative functions on $[0,\infty)$. Then we have*

$$\begin{aligned}&2K_t^{\alpha,\beta,\eta,\mu}p\Big(K_t^{\alpha,\beta,\eta,\mu}qK_t^{\alpha,\beta,\eta,\mu}(rfg)+K_t^{\alpha,\beta,\eta,\mu}rK_t^{\alpha,\beta,\eta,\mu}(qfg)\Big)+2K_t^{\alpha,\beta,\eta,\mu}qK_t^{\alpha,\beta,\eta,\mu}rK_t^{\alpha,\beta,\eta,\mu}(pfg)\\&\geq K_t^{\alpha,\beta,\eta,\mu}p\Big(K_t^{\alpha,\beta,\eta,\mu}(qf)K_t^{\alpha,\beta,\eta,\mu}(rg)+K_t^{\alpha,\beta,\eta,\mu}(rf)K_t^{\alpha,\beta,\eta,\mu}(qg)\Big)+K_t^{\alpha,\beta,\eta,\mu}q\Big(K_t^{\alpha,\beta,\eta,\mu}(pf)K_t^{\alpha,\beta,\eta,\mu}(rg)\\&\quad+K_t^{\alpha,\beta,\eta,\mu}(rf)K_t^{\alpha,\beta,\eta,\mu}(pg)\Big)+K_t^{\alpha,\beta,\eta,\mu}r\Big(K_t^{\alpha,\beta,\eta,\mu}(pf)K_t^{\alpha,\beta,\eta,\mu}(qg)+K_t^{\alpha,\beta,\eta,\mu}(qf)K_t^{\alpha,\beta,\eta,\mu}(pg)\Big),\end{aligned}\tag{2.5}$$

for all $x\in[0,\infty)$, and real constants α,β,η,μ with $\alpha>0$, $\mu>-1$, $\eta\leq 0$ and $\alpha+\beta+\mu\geq 0$.

Proof. Putting $u=q$, $v=r$ and using Lemma 2.1, we can write

$$K_t^{\alpha,\beta,\eta,\mu}qK_t^{\alpha,\beta,\eta,\mu}(rfg)+K_t^{\alpha,\beta,\eta,\mu}rK_t^{\alpha,\beta,\eta,\mu}(qfg)\geq K_t^{\alpha,\beta,\eta,\mu}(rf)K_t^{\alpha,\beta,\eta,\mu}(qg)+K_t^{\alpha,\beta,\eta,\mu}(qf)K_t^{\alpha,\beta,\eta,\mu}(rg).\tag{2.6}$$

Multiplying both sides of (2.6) by $K_t^{\alpha,\beta,\eta,\mu}p$, we obtain

$$\begin{aligned}&K_t^{\alpha,\beta,\eta,\mu}p\Big(K_t^{\alpha,\beta,\eta,\mu}qK_t^{\alpha,\beta,\eta,\mu}(rfg)+K_t^{\alpha,\beta,\eta,\mu}rK_t^{\alpha,\beta,\eta,\mu}(qfg)\Big)\\&\geq K_t^{\alpha,\beta,\eta,\mu}p\Big(K_t^{\alpha,\beta,\eta,\mu}(rf)(x)K_t^{\alpha,\beta,\eta,\mu}(qg)+K_t^{\alpha,\beta,\eta,\mu}(qf)K_t^{\alpha,\beta,\eta,\mu}(rg)\Big).\end{aligned}\tag{2.7}$$

Putting $u=p$, $v=r$ and using Lemma 2.1, we can state that

$$K_t^{\alpha,\beta,\eta,\mu}pK_t^{\alpha,\beta,\eta,\mu}(rfg)+K_t^{\alpha,\beta,\eta,\mu}rK_t^{\alpha,\beta,\eta,\mu}(pfg)\geq K_t^{\alpha,\beta,\eta,\mu}(rf)K_t^{\alpha,\beta,\eta,\mu}(pg)+K_t^{\alpha,\beta,\eta,\mu}(pf)K_t^{\alpha,\beta,\eta,\mu}(rg).\tag{2.8}$$

Multiplying both sides of (2.8) by $I_{0,t}^{\alpha,\beta,\eta}y(t)$, one verifies that

$$\begin{aligned}&K_t^{\alpha,\beta,\eta,\mu}q\Big(K_t^{\alpha,\beta,\eta,\mu}pK_t^{\alpha,\beta,\eta,\mu}(rfg)+K_t^{\alpha,\beta,\eta,\mu}r(x)K_t^{\alpha,\beta,\eta,\mu}(pfg)\Big)\\&\geq K_t^{\alpha,\beta,\eta,\mu}q\Big(K_t^{\alpha,\beta,\eta,\mu}(rf)K_t^{\alpha,\beta,\eta,\mu}(pg)+K_t^{\alpha,\beta,\eta,\mu}(pf)K_t^{\alpha,\beta,\eta,\mu}(rg)\Big).\end{aligned}\tag{2.9}$$

With the same arguments as before, we can get

$$\begin{aligned}&K_t^{\alpha,\beta,\eta,\mu}r\Big(K_t^{\alpha,\beta,\eta,\mu}pK_t^{\alpha,\beta,\eta,\mu}(qfg)+K_t^{\alpha,\beta,\eta,\mu}q(x)K_t^{\alpha,\beta,\eta,\mu}(pfg)\Big)\\&\geq K_t^{\alpha,\beta,\eta,\mu}r\Big(K_t^{\alpha,\beta,\eta,\mu}(qf)K_t^{\alpha,\beta,\eta,\mu}(pg)+K_t^{\alpha,\beta,\eta,\mu}(pf)K_t^{\alpha,\beta,\eta,\mu}(qg)\Big).\end{aligned}\tag{2.10}$$

The required inequality (2.5) follows on adding the inequalities (2.7), (2.9) and (2.10). □

Lemma 2.3. *Let f and g be two synchronous functions on $[0,\infty)$ and let u and v be two nonnegative functions on $[0,\infty)$. Then we have*

$$\begin{aligned}&K_t^{\alpha,\beta,\eta,\mu}u(x)K_t^{\gamma,\delta,\zeta,\nu}(vfg)(x)+K_t^{\gamma,\delta,\zeta,\nu}v(x)K_t^{\alpha,\beta,\eta,\mu}(ufg)(x)\\&\geq K_t^{\alpha,\beta,\eta,\mu}(uf)(x)K_t^{\gamma,\delta,\zeta,\mu}(vg)(x)+K_t^{\gamma,\delta,\zeta,\mu}(vf)(x)K_t^{\alpha,\beta,\eta,\nu}(ug)(x),\end{aligned}\tag{2.11}$$

for all $x\in[0,\infty)$, and real constants $\alpha,\gamma,\beta,\delta,\eta,\zeta,\mu,\nu$ satisfying $\alpha,\gamma>0$, $\mu,\nu>-1$, $\eta,\zeta\leq 0$ and $\alpha+\beta+\mu,\gamma+\delta+\zeta\geq 0$.

Proof. Multiplying both sides of (2.3) by $v(\rho)\frac{\rho^{\nu}(x-\rho)^{\gamma-1}}{\Gamma(\gamma)}{}_2F_1(\gamma+\nu+\delta,-\zeta;\gamma;1-\frac{\rho}{x})$, where $x>0$ and $\rho\in(0,x)$, when we integrate the inequality with respect to ρ from 0 to x, we obtain by Definition 1.2 that

$$f(\tau)g(\tau)K_t^{\gamma,\delta,\zeta,\nu}v(x)+K_t^{\gamma,\delta,\zeta,\nu}(vfg)(x)\geq f(\tau)K_t^{\gamma,\delta,\zeta,\nu}(vg)(x)+g(\tau)K_t^{\gamma,\delta,\zeta,\nu}(vf)(x). \quad (2.12)$$

Again, by multiplying both side of (2.12) by $u(\tau)\frac{\tau^{\mu}(x-\tau)^{\alpha-1}}{\Gamma(\alpha)}{}_2F_1(\alpha+\mu+\beta,-\eta;\alpha;1-\frac{\tau}{x})$, where $x>0$ and $\tau\in(0,x)$, and integrating the resulting identity with respect to τ from 0 to x, and then applying Definition 1.2, we obtain

$$\begin{aligned}K_t^{\alpha,\beta,\eta,\mu}u(x)K_t^{\gamma,\delta,\zeta,\nu}(vfg)(x)+K_t^{\gamma,\delta,\zeta,\nu}v(x)K_t^{\alpha,\beta,\eta,\mu}(ufg)(x)&\\ \geq K_t^{\alpha,\beta,\eta,\mu}(uf)(x)K_t^{\gamma,\delta,\zeta,\mu}(vg)(x)+K_t^{\gamma,\delta,\zeta,\mu}(vf)(x)K_t^{\alpha,\beta,\eta,\nu}(ug)(x),&\end{aligned}$$

which implies (2.11). □

Theorem 2.4. *Let f and g be two synchronous functions on $[0,\infty)$ and let p, q and r be three nonnegative functions on $[0,\infty)$. Then we have*

$$\begin{aligned}&K_t^{\alpha,\beta,\eta,\mu}p\Big(K_t^{\alpha,\beta,\eta,\mu}rK_t^{\gamma,\delta,\zeta,\nu}(qfg)+2K_t^{\alpha,\beta,\eta,\mu}qK_t^{\gamma,\delta,\zeta,\nu}(rfg)+K_t^{\gamma,\delta,\zeta,\nu}rK_t^{\alpha,\beta,\eta,\mu}(qfg)\Big)\\&\quad+\Big(K_t^{\alpha,\beta,\eta,\mu}qI_{0,t}^{\gamma,\delta,\zeta}r+K_t^{\gamma,\delta,\zeta,\nu}qK_t^{\alpha,\beta,\eta,\mu}r\Big)K_t^{\alpha,\beta,\eta,\mu}(pfg)\\&\geq K_t^{\alpha,\beta,\eta,\mu}p\Big(K_t^{\alpha,\beta,\eta,\mu}(qf)K_t^{\gamma,\delta,\zeta,\nu}(rg)+K_t^{\gamma,\delta,\zeta,\nu}(rf)K_t^{\alpha,\beta,\eta,\mu}(qg)\Big)+K_t^{\gamma,\delta,\zeta,\nu}q\Big(K_t^{\alpha,\beta,\eta,\mu}(pf)K_t^{\gamma,\delta,\zeta,\nu}(rg)\\&\quad+K_t^{\gamma,\delta,\zeta,\nu}(rf)(t)K_t^{\alpha,\beta,\eta,\mu}(pg)\Big)+K_t^{\gamma,\delta,\zeta,\nu}r\Big(K_t^{\alpha,\beta,\eta,\mu}(pf)K_t^{\gamma,\delta,\zeta,\nu}(qg)+K_t^{\gamma,\delta,\zeta,\nu}(qf)K_t^{\alpha,\beta,\eta,\mu}(pg)\Big),\end{aligned} \quad (2.13)$$

for all $x\in[0,\infty)$, and real constants $\alpha,\gamma,\beta,\delta,\eta,\zeta,\mu,\nu$ satisfying $\alpha,\gamma>0$, $\mu,\nu>-1$, $\eta,\zeta\leq 0$ and $\alpha+\beta+\mu,\gamma+\delta+\zeta\geq 0$.

Proof. Putting $u=q$, $v=r$ and using Lemma 2.3, we can write

$$K_t^{\alpha,\beta,\eta,\mu}qK_t^{\gamma,\delta,\zeta,\nu}(rfg)+K_t^{\gamma,\delta,\zeta,\nu}rK_t^{\alpha,\beta,\eta,\mu}(qfg)\geq K_t^{\alpha,\beta,\eta,\mu}(qf)K_t^{\gamma,\delta,\zeta,\mu}(rg)+K_t^{\gamma,\delta,\zeta,\mu}(rf)K_t^{\alpha,\beta,\eta,\nu}(qg). \quad (2.14)$$

Multiplying both sides of (2.14) by $K_t^{\alpha,\beta,\eta,\mu}p$, we obtain

$$\begin{aligned}&K_t^{\alpha,\beta,\eta,\mu}p\Big(K_t^{\alpha,\beta,\eta,\mu}qK_t^{\gamma,\delta,\zeta,\nu}(rfg)+K_t^{\gamma,\delta,\zeta,\nu}rK_t^{\alpha,\beta,\eta,\mu}(qfg)\Big)\\&\qquad\geq K_t^{\alpha,\beta,\eta,\mu}p\Big(K_t^{\alpha,\beta,\eta,\mu}(qf)K_t^{\gamma,\delta,\zeta,\mu}(rg)+K_t^{\gamma,\delta,\zeta,\mu}(rf)K_t^{\alpha,\beta,\eta,\nu}(qg)\Big).\end{aligned} \quad (2.15)$$

Putting $u=p$, $v=r$ and using Lemma 2.3, we can state that

$$K_t^{\alpha,\beta,\eta,\mu}pK_t^{\gamma,\delta,\zeta,\nu}(rfg)+K_t^{\gamma,\delta,\zeta,\nu}rK_t^{\alpha,\beta,\eta,\mu}(pfg)\geq K_t^{\alpha,\beta,\eta,\mu}(pf)K_t^{\gamma,\delta,\zeta,\mu}(rg)+K_t^{\gamma,\delta,\zeta,\mu}(rf)K_t^{\alpha,\beta,\eta,\nu}(pg).$$

Multiplying both sides of (2.14) by $K_t^{\alpha,\beta,\eta,\mu}q$, one verifies that

$$\begin{aligned}&K_t^{\alpha,\beta,\eta,\mu}q\Big(K_t^{\alpha,\beta,\eta,\mu}pK_t^{\gamma,\delta,\zeta,\nu}(rfg)+K_t^{\gamma,\delta,\zeta,\nu}rK_t^{\alpha,\beta,\eta,\mu}(pfg)\Big)\\&\qquad\geq K_t^{\alpha,\beta,\eta,\mu}q\Big(K_t^{\alpha,\beta,\eta,\mu}(pf)K_t^{\gamma,\delta,\zeta,\mu}(rg)+K_t^{\gamma,\delta,\zeta,\mu}(rf)K_t^{\alpha,\beta,\eta,\nu}(pg)\Big).\end{aligned} \quad (2.16)$$

With the same arguments as before, we can get

$$\begin{aligned}&K_t^{\alpha,\beta,\eta,\mu}r\Big(K_t^{\alpha,\beta,\eta,\mu}qK_t^{\gamma,\delta,\zeta,\nu}(pfg)+K_t^{\gamma,\delta,\zeta,\nu}pK_t^{\alpha,\beta,\eta,\mu}(qfg)\Big)\\&\qquad\geq K_t^{\alpha,\beta,\eta,\mu}r\Big(K_t^{\alpha,\beta,\eta,\mu}(qf)K_t^{\gamma,\delta,\zeta,\mu}(pg)+K_t^{\gamma,\delta,\zeta,\mu}(pf)K_t^{\alpha,\beta,\eta,\nu}(qg)\Big).\end{aligned} \quad (2.17)$$

The required inequality (2.13) follows on adding the inequalities (2.15), (2.16) and (2.17). □

Remark 2.5. The inequalities (2.5) and (2.13) are reversed in the following cases: (a) The functions f and g are synchronous on $[0,\infty)$. (b) The functions p, q and r are negative on $[0,\infty)$. (c) Two of he functions p, q and r are positive and the third one is negative on $[0,\infty)$.

Theorem 2.6. *Let f, g and h be three synchronous functions on $[0,\infty)$ and let u be a nonnegative function on $[0,\infty)$. Then we have*

$$\begin{aligned} &K_t^{\alpha,\beta,\eta,\mu} u K_t^{\gamma,\delta,\zeta,\nu}(ufgh) + K_t^{\alpha,\beta,\eta,\mu}(uh) K_t^{\gamma,\delta,\zeta,\nu}(ufg) + K_t^{\alpha,\beta,\eta,\mu}(ufg) K_t^{\gamma,\delta,\zeta,\nu}(uh) \\ &\quad + K_t^{\alpha,\beta,\eta,\mu}(ufgh) K_t^{\gamma,\delta,\zeta,\nu} u \ge K_t^{\alpha,\beta,\eta,\mu}(uf) K_t^{\gamma,\delta,\zeta,\nu}(ugh) + K_t^{\alpha,\beta,\eta,\mu}(ug) K_t^{\gamma,\delta,\zeta,\nu}(ufh) \\ &\quad + K_t^{\alpha,\beta,\eta,\mu}(ugh) K_t^{\gamma,\delta,\zeta,\nu}(uf) + K_t^{\alpha,\beta,\eta,\mu}(ufh) K_t^{\gamma,\delta,\zeta,\nu}(ug), \end{aligned} \tag{2.18}$$

for all $x \in [0,\infty)$, and real constants $\alpha,\gamma,\beta,\delta,\eta,\zeta,\mu,\nu$ satisfying $\alpha,\gamma > 0$, $\mu,\nu > -1$, $\eta,\zeta \le 0$ and $\alpha+\beta+\mu, \gamma+\delta+\zeta \ge 0$.

Proof. Let f, g and h be three synchronous functions on $[0,\infty)$, Then, for all $\tau,\rho \ge 0$, we have

$$(f(\tau)-f(\rho))(g(\tau)-g(\rho))(h(\tau)+h(\rho)) \ge 0,$$

which implies that

$$\begin{aligned} &f(\tau)g(\tau)h(\tau) + f(\rho)g(\rho)h(\rho) + f(\tau)g(\tau)h(\rho) + f(\rho)g(\rho)h(\tau) \\ &\quad \ge f(\tau)g(\rho)h(\tau) + f(\tau)g(\rho)h(\rho) + f(\rho)g(\tau)h(\tau) + f(\rho)g(\tau)h(\rho). \end{aligned} \tag{2.19}$$

Multiplying both side of (2.19) by $u(\tau)\frac{\tau^\nu (x-\tau)^{\gamma-1}}{\Gamma(\gamma)} {}_2F_1(\gamma+\nu+\delta, -\zeta; \gamma; 1-\frac{\tau}{x})$, where $x>0$ and $\tau \in (0,x)$, and integrating the resulting identity with respect to τ from 0 to x, and then applying Definition 1.2, we obtain

$$\begin{aligned} &K_t^{\gamma,\delta,\zeta,\nu}(ufgh) + f(\rho)g(\rho)h(\rho) K_t^{\gamma,\delta,\zeta,\nu} u + h(\rho) K_t^{\gamma,\delta,\zeta,\nu}(ufg) + f(\rho)g(\rho) K_t^{\gamma,\delta,\zeta,\nu}(uh) \\ &\quad \ge g(\rho) K_t^{\gamma,\delta,\zeta,\nu}(ufh) + g(\rho)h(\rho) K_t^{\gamma,\delta,\zeta,\nu}(uf) + f(\rho) K_t^{\gamma,\delta,\zeta,\nu}(ugh) + f(\rho)h(\rho) K_t^{\gamma,\delta,\zeta,\nu}(ug). \end{aligned} \tag{2.20}$$

Again, by multiplying both sides of (2.20) by $u(\rho)\frac{\rho^\mu (x-\rho)^{\alpha-1}}{\Gamma(\alpha)} {}_2F_1(\alpha+\mu+\beta, -\eta; \alpha; 1-\frac{\rho}{x})$ where $x>0$ and $\rho \in (0,x)$, when we integrate the inequality with respect to ρ from 0 to x, we obtain by Definition 1.2 that

$$\begin{aligned} &K_t^{\alpha,\beta,\eta,\mu} u K_t^{\gamma,\delta,\zeta,\nu}(ufgh) + K_t^{\alpha,\beta,\eta,\mu}(uh) K_t^{\gamma,\delta,\zeta,\nu}(ufg) + K_t^{\alpha,\beta,\eta,\mu}(ufg) K_t^{\gamma,\delta,\zeta,\nu}(uh) \\ &\quad + K_t^{\alpha,\beta,\eta,\mu}(ufgh) K_t^{\gamma,\delta,\zeta,\nu} u \ge K_t^{\alpha,\beta,\eta,\mu}(uf) K_t^{\gamma,\delta,\zeta,\nu}(ugh) + K_t^{\alpha,\beta,\eta,\mu}(ug) K_t^{\gamma,\delta,\zeta,\nu}(ufh) \\ &\quad + K_t^{\alpha,\beta,\eta,\mu}(ugh) K_t^{\gamma,\delta,\zeta,\nu}(uf) + K_t^{\alpha,\beta,\eta,\mu}(ufh) K_t^{\gamma,\delta,\zeta,\nu}(ug), \end{aligned}$$

which implies (2.18). □

Theorem 2.7. *Let f, g and h be three synchronous functions on $[0,\infty)$ and let u and v be two nonnegative functions on $[0,\infty)$. Then we have*

$$\begin{aligned} &K_t^{\alpha,\beta,\eta,\mu} u K_t^{\gamma,\delta,\zeta,\nu}(vfgh) + K_t^{\alpha,\beta,\eta,\mu}(uh) K_t^{\gamma,\delta,\zeta,\nu}(vfg) + K_t^{\alpha,\beta,\eta,\mu}(ufg) K_t^{\gamma,\delta,\zeta,\nu}(vh) \\ &\quad + K_t^{\alpha,\beta,\eta,\mu}(ufgh) K_t^{\gamma,\delta,\zeta,\nu} v \ge K_t^{\alpha,\beta,\eta,\mu}(uf) K_t^{\gamma,\delta,\zeta,\nu}(vgh) + K_t^{\alpha,\beta,\eta,\mu}(ug) K_t^{\gamma,\delta,\zeta,\nu}(vfh) \\ &\quad + K_t^{\alpha,\beta,\eta,\mu}(ugh) K_t^{\gamma,\delta,\zeta,\nu}(vf) + K_t^{\alpha,\beta,\eta,\mu}(ufh) K_t^{\gamma,\delta,\zeta,\nu}(vg), \end{aligned} \tag{2.21}$$

for all $x \in [0,\infty)$, and real constants $\alpha,\gamma,\beta,\delta,\eta,\zeta,\mu,\nu$ satisfying $\alpha,\gamma > 0$, $\mu,\nu > -1$, $\eta,\zeta \le 0$ and $\alpha+\beta+\mu, \gamma+\delta+\zeta \ge 0$.

Proof. Multiplying both side of (2.19) by $v(\tau)\frac{\tau^\nu (x-\tau)^{\gamma-1}}{\Gamma(\gamma)} {}_2F_1(\gamma+\nu+\delta, -\zeta; \gamma; 1-\frac{\tau}{x})$, where $x>0$ and $\tau \in (0,x)$, and integrating the resulting identity with respect to τ from 0 to x, and then applying Definition 1.2, we obtain

$$\begin{aligned} &K_t^{\gamma,\delta,\zeta,\nu}(vfgh) + f(\rho)g(\rho)h(\rho) K_t^{\gamma,\delta,\zeta,\nu} v + h(\rho) K_t^{\gamma,\delta,\zeta,\nu}(vfg) + f(\rho)g(\rho) K_t^{\gamma,\delta,\zeta,\nu}(vh) \\ &\quad \ge g(\rho) K_t^{\gamma,\delta,\zeta,\nu}(vfh) + g(\rho)h(\rho) K_t^{\gamma,\delta,\zeta,\nu}(vf) + f(\rho) K_t^{\gamma,\delta,\zeta,\nu}(vgh) + f(\rho)h(\rho) K_t^{\gamma,\delta,\zeta,\nu}(vg). \end{aligned} \tag{2.22}$$

Again, by multiplying both sides of (2.22) by $u(\rho)\frac{\rho^{\mu}(x-\rho)^{\alpha-1}}{\Gamma(\alpha)}{}_2F_1(\alpha+\mu+\beta,-\eta;\alpha;1-\frac{\rho}{x})$ where $x>0$ and $\rho\in(0,x)$, when we integrate the inequality with respect to ρ from 0 to x, we obtain by Definition 1.2 that

$$\begin{aligned}K_t^{\alpha,\beta,\eta,\mu}uK_t^{\gamma,\delta,\zeta,\nu}(vfgh)+K_t^{\alpha,\beta,\eta,\mu}(ufgh)K_t^{\gamma,\delta,\zeta,\nu}v+K_t^{\alpha,\beta,\eta,\mu}(uh)K_t^{\gamma,\delta,\zeta,\nu}(vfg)\\+K_t^{\alpha,\beta,\eta,\mu}(ufg)K_t^{\gamma,\delta,\zeta,\nu}(vh)\geq K_t^{\alpha,\beta,\eta,\mu}(ug)K_t^{\gamma,\delta,\zeta,\nu}(vfh)+K_t^{\alpha,\beta,\eta,\mu}(ugh)K_t^{\gamma,\delta,\zeta,\nu}(vf)\\+K_t^{\alpha,\beta,\eta,\mu}(uf)K_t^{\gamma,\delta,\zeta,\nu}(vgh)+K_t^{\alpha,\beta,\eta,\mu}(ufh)K_t^{\gamma,\delta,\zeta,\nu}(vg),\end{aligned}$$

which implies (2.21). □

Remark 2.8. It may be noted that the inequalities in (2.18) and (2.21) are reversed if functions f,g and h are asynchronous. It is also easily seen that the special case $u=v$ of (2.21) in Theorem 2.7 reduces to Theorem 2.6.

Lemma 2.9. *Let f and u be two functions defined on $[0,\infty)$ satisfying the condition (1.6). Then we have*

$$\begin{aligned}K_t^{\alpha,\beta,\eta,\mu}uK_t^{\alpha,\beta,\eta,\mu}(uf^2)-\left(K_t^{\alpha,\beta,\eta,\mu}(uf)\right)^2=\left(\Phi K_t^{\alpha,\beta,\eta,\mu}u-K_t^{\alpha,\beta,\eta,\mu}(uf)\right)\left(K_t^{\alpha,\beta,\eta,\mu}(xf)(t)-\phi K_t^{\alpha,\beta,\eta,\mu}u\right)\\-K_t^{\alpha,\beta,\eta,\mu}uK_t^{\alpha,\beta,\eta,\mu}\left(u(x)(\Phi-f(x))(f(x)-\phi)\right),\quad(2.23)\end{aligned}$$

for all $x\in[0,\infty)$, and real constants α,β,η,μ with $\alpha>0$, $\mu>-1$, $\eta\leq0$ and $\alpha+\beta+\mu\geq0$.

Proof. Let f be a function defined on $[0,\infty)$ satisfying the condition (1.6) on $[0,\infty)$. For any $\rho,\tau\in[0,\infty)$, we have

$$\begin{aligned}(\Phi-f(\rho))(f(\tau)-\phi)+(\Phi-f(\tau))(f(\rho)-\phi)-(\Phi-f(\tau))(f(\tau)-\phi)\\-(\Phi-f(\rho))(f(\rho)-\phi)=f^2(\tau)+f^2(\rho)-2f(\rho)f(\tau).\quad(2.24)\end{aligned}$$

Multiplying both sides of (2.24) by $u(\rho)\frac{\rho^{\mu}(x-\rho)^{\alpha-1}}{\Gamma(\alpha)}{}_2F_1(\alpha+\mu+\beta,-\eta;\alpha;1-\frac{\rho}{x})$ where $x>0$ and $\rho\in(0,x)$, when we integrate the inequality with respect to ρ from 0 to x, we obtain by Definition 1.2 that

$$\begin{aligned}(f(\tau)-\phi)\left(\Phi K_t^{\alpha,\beta,\eta,\mu}u-K_t^{\alpha,\beta,\eta,\mu}(uf)\right)+(\Phi-f(\tau))\left(K_t^{\alpha,\beta,\eta,\mu}(uf)-\phi K_t^{\alpha,\beta,\eta,\mu}u\right)\\-(\Phi-f(\tau))(f(\tau)-\phi)K_t^{\alpha,\beta,\eta,\mu}u-K_t^{\alpha,\beta,\eta,\mu}\left(u(x)(\Phi-f(x))(f(x)-\phi)\right)\\=f^2(\tau)K_t^{\alpha,\beta,\eta,\mu}u+K_t^{\alpha,\beta,\eta,\mu}(uf^2)-2f(\tau)K_t^{\alpha,\beta,\eta,\mu}(uf).\quad(2.25)\end{aligned}$$

Again, by multiplying both sides of (2.25) by $u(\rho)\frac{\rho^{\mu}(x-\rho)^{\alpha-1}}{\Gamma(\alpha)}{}_2F_1(\alpha+\mu+\beta,-\eta;\alpha;1-\frac{\rho}{x})$ where $x>0$ and $\rho\in(0,x)$, when we integrate the inequality with respect to ρ from 0 to x, we obtain by Definition 1.2 that

$$\begin{aligned}\left(K_t^{\alpha,\beta,\eta,\mu}(uf)-\phi K_t^{\alpha,\beta,\eta,\mu}u\right)\left(\Phi K_t^{\alpha,\beta,\eta,\mu}u-K_t^{\alpha,\beta,\eta,\mu}(uf)\right)\\+\left(\Phi K_t^{\alpha,\beta,\eta,\mu}u-K_t^{\alpha,\beta,\eta,\mu}(uf)\right)\left(K_t^{\alpha,\beta,\eta,\mu}(uf)-\phi K_t^{\alpha,\beta,\eta,\mu}u\right)\\-K_t^{\alpha,\beta,\eta,\mu}\left(u(x)(\Phi-f(x))(f(x)-\phi)\right)K_t^{\alpha,\beta,\eta,\mu}u-K_t^{\alpha,\beta,\eta,\mu}uK_t^{\alpha,\beta,\eta,\mu}\left(u(x)(\Phi-f(x))(f(x)-\phi)\right)\\=K_t^{\alpha,\beta,\eta,\mu}(uf^2)K_t^{\alpha,\beta,\eta,\mu}u+K_t^{\alpha,\beta,\eta,\mu}uK_t^{\alpha,\beta,\eta,\mu}(uf^2)-2K_t^{\alpha,\beta,\eta,\mu}(uf)K_t^{\alpha,\beta,\eta,\mu}(uf),\end{aligned}$$

which gives (2.23) and proves the lemma. □

Theorem 2.10. *Let f and g be two functions defined satisfying the condition (1.6) on $[0,\infty)$ and let u be a nonnegative function on $[0,\infty)$. Then we have*

$$\left|K_t^{\alpha,\beta,\eta,\mu}uK_t^{\alpha,\beta,\eta,\mu}(ufg)-K_t^{\alpha,\beta,\eta,\mu}(uf)K_t^{\alpha,\beta,\eta,\mu}(ug)\right|\leq\frac{1}{4}(\Phi-\phi)(\Psi-\psi)\left(K_t^{\alpha,\beta,\eta,\mu}u\right)^2,\quad(2.26)$$

for all $x\in[0,\infty)$, and real constants α,β,η,μ with $\alpha>0$, $\mu>-1$, $\eta\leq0$ and $\alpha+\beta+\mu\geq0$.

Proof. Let f and g be two functions satisfying the conditions of Theorem 2.10. Let $H(\tau,\rho)$ be defined by

$$H(\tau,\rho)=(f(\tau)-f(\rho))(g(\tau)-g(\rho)),\quad \tau,\rho\in(0,x),\quad x>0. \tag{2.27}$$

Multiplying both sides of (2.27) by $u(\tau)F(x,\tau)u(\rho)F(x,\rho)$, where

$$F(x,\tau)=\frac{\Gamma(1-\beta)\Gamma(\alpha+\mu+\eta+1)}{\Gamma(\eta-\beta+1)\Gamma(\mu+1)}x^{\alpha+\beta}\frac{x^{-\alpha-\beta-2\mu}}{\Gamma(\alpha)}\tau^{\mu}(x-\tau)^{\alpha-1}{}_2F_1(\alpha+\mu+\beta,-\eta;\alpha;1-\frac{\tau}{x}), \tag{2.28}$$

where $x>0$ and $\tau\in(0,x)$, and integrating the resulting inequality obtained with respect to τ and ρ from 0 to x, we have

$$\int_0^x\int_0^x u(\tau)F(x,\tau)u(\rho)F(x,\rho)H(\tau,\rho)d\tau d\rho=2K_t^{\alpha,\beta,\eta,\mu}uK_t^{\alpha,\beta,\eta,\mu}(ufg)-2K_t^{\alpha,\beta,\eta,\mu}(uf)K_t^{\alpha,\beta,\eta,\mu}(ug). \tag{2.29}$$

Thanks to the weighted Cauchy-Schwartz integral inequality for double integrals, we can write that

$$\begin{aligned}&\left(\int_0^x\int_0^x u(\tau)F(x,\tau)u(\rho)F(x,\rho)H(\tau,\rho)d\tau d\rho\right)^2\\ &\le\left(\int_0^x\int_0^x u(\tau)F(x,\tau)u(\rho)F(x,\rho)(f(\tau)-f(\rho))d\tau d\rho\right)\left(\int_0^x\int_0^x u(\tau)F(x,\tau)u(\rho)F(x,\rho)(g(\tau)-g(\rho))d\tau d\rho\right)\\ &=4\left(K_t^{\alpha,\beta,\eta,\mu}uK_t^{\alpha,\beta,\eta,\mu}(uf^2)-\left(K_t^{\alpha,\beta,\eta,\mu}(uf)\right)^2\right)\left(K_t^{\alpha,\beta,\eta,\mu}uK_t^{\alpha,\beta,\eta,\mu}(ug^2)-\left(K_t^{\alpha,\beta,\eta,\mu}(ug)\right)^2\right).\end{aligned} \tag{2.30}$$

Since $(\Phi-f(\tau))(f(\tau)-\phi)\ge 0$ and $(\Psi-g(\tau))(g(\tau)-\psi)\ge 0$, we have

$$K_t^{\alpha,\beta,\eta,\mu}uK_t^{\alpha,\beta,\eta,\mu}\left(u(x)(\Phi-f(x))(f(x)-\phi)\right)\ge 0, \tag{2.31}$$

and

$$K_t^{\alpha,\beta,\eta,\mu}uK_t^{\alpha,\beta,\eta,\mu}\left(u(x)(\Psi-g(x))(g(x)-\psi)\right)\ge 0. \tag{2.32}$$

Thus, from (2.31), (2.32) and Lemma 2.9,we get

$$K_t^{\alpha,\beta,\eta,\mu}uK_t^{\alpha,\beta,\eta,\mu}(uf^2)-\left(K_t^{\alpha,\beta,\eta,\mu}(uf)\right)^2\le\left(\Phi K_t^{\alpha,\beta,\eta,\mu}u-K_t^{\alpha,\beta,\eta,\mu}(uf)\right)\left(K_t^{\alpha,\beta,\eta,\mu}(uf)-\phi K_t^{\alpha,\beta,\eta,\mu}u\right), \tag{2.33}$$

and

$$K_t^{\alpha,\beta,\eta,\mu}uK_t^{\alpha,\beta,\eta,\mu}(ug^2)-\left(K_t^{\alpha,\beta,\eta,\mu}(ug)\right)^2\le\left(\Psi K_t^{\alpha,\beta,\eta,\mu}u-K_t^{\alpha,\beta,\eta,\mu}(ug)\right)\left(K_t^{\alpha,\beta,\eta,\mu}(ug)-\phi K_t^{\alpha,\beta,\eta,\mu}u\right). \tag{2.34}$$

Combining (2.29), (2.30), (2.33) and (2.34), we deduce that

$$\begin{aligned}\left(K_t^{\alpha,\beta,\eta,\mu}uK_t^{\alpha,\beta,\eta,\mu}(ufg)-K_t^{\alpha,\beta,\eta,\mu}(uf)K_t^{\alpha,\beta,\eta,\mu}(ug)\right)^2&\le\left(\Phi K_t^{\alpha,\beta,\eta,\mu}u-K_t^{\alpha,\beta,\eta,\mu}(uf)\right)\\ \times\left(K_t^{\alpha,\beta,\eta,\mu}(uf)-\phi K_t^{\alpha,\beta,\eta,\mu}u\right)&\left(\Psi K_t^{\alpha,\beta,\eta,\mu}u-K_t^{\alpha,\beta,\eta,\mu}(ug)\right)\left(K_t^{\alpha,\beta,\eta,\mu}(ug)-\phi K_t^{\alpha,\beta,\eta,\mu}u\right).\end{aligned} \tag{2.35}$$

Now using the elementary inequality $4xy\le(x+y)^2$, $x,y\in\mathbb{R}$, we can state that

$$4\left(\Phi K_t^{\alpha,\beta,\eta,\mu}u-K_t^{\alpha,\beta,\eta,\mu}(uf)\right)\left(K_t^{\alpha,\beta,\eta,\mu}(uf)-\phi K_t^{\alpha,\beta,\eta,\mu}u\right)\le\left((\Phi-\phi)K_t^{\alpha,\beta,\eta,\mu}u\right)^2, \tag{2.36}$$

and

$$4\left(\Psi K_t^{\alpha,\beta,\eta,\mu}u-K_t^{\alpha,\beta,\eta,\mu}(ug)\right)\left(K_t^{\alpha,\beta,\eta,\mu}(ug)-\phi K_t^{\alpha,\beta,\eta,\mu}u\right)\le\left((\Psi-\psi)K_t^{\alpha,\beta,\eta,\mu}u\right)^2. \tag{2.37}$$

From (2.35)-(2.37), we abtain (2.26). This complete the proof of Theorem 2.10. □

Lemma 2.11. *Let f and g be two functions defined on $[0,\infty)$ and let u and v be two nonnegative functions on $[0,\infty)$. Then we have*

$$\left(K_t^{\alpha,\beta,\eta,\mu}uK_t^{\gamma,\delta,\zeta,\nu}(vfg)+K_t^{\gamma,\delta,\zeta,\nu}vK_t^{\alpha,\beta,\eta,\mu}(ufg)-K_t^{\alpha,\beta,\eta,\mu}(uf)K_t^{\gamma,\delta,\zeta,\nu}(vg)-K_t^{\gamma,\delta,\zeta,\nu}(vf)K_t^{\alpha,\beta,\eta,\mu}(ug)\right)^2$$
$$\leq\left(K_t^{\alpha,\beta,\eta,\mu}uK_t^{\gamma,\delta,\zeta,\nu}(vf^2)+K_t^{\gamma,\delta,\zeta,\nu}vK_t^{\alpha,\beta,\eta,\mu}(uf^2)-2K_t^{\alpha,\beta,\eta,\mu}(uf)K_t^{\gamma,\delta,\zeta,\nu}(vf)\right)$$
$$\times\left(K_t^{\alpha,\beta,\eta,\mu}uK_t^{\gamma,\delta,\zeta,\nu}(vg^2)+K_t^{\gamma,\delta,\zeta,\nu}uK_t^{\alpha,\beta,\eta,\mu}(ug^2)-2K_t^{\alpha,\beta,\eta,\mu}(ug)K_t^{\gamma,\delta,\zeta,\nu}(vg)\right),\quad(2.38)$$

for all $x\in[0,\infty)$, and real constants $\alpha,\gamma,\beta,\delta,\eta,\zeta,\mu,\nu$ satisfying $\alpha,\gamma>0$, $\mu,\nu>-1$, $\eta,\zeta\leq 0$ and $\alpha+\beta+\mu,\gamma+\delta+\zeta\geq 0$.

Proof. Multiplying (2.27) by $u(\tau)F(t,\tau)v(\rho)G(t,\rho)$, where $F(t,\tau)$ is defined by (2.28), and

$$G(x,\rho)=\frac{\Gamma(1-\delta)\Gamma(\gamma+\nu+\zeta+1)}{\Gamma(\zeta-\delta+1)\Gamma(\nu+1)}x^{\gamma+\delta}\frac{x^{-\gamma-\delta-2\nu}}{\Gamma(\gamma)}\rho^{\nu}(x-\rho)^{\gamma-1}{}_2F_1(\gamma+\nu+\delta,-\zeta;\gamma;1-\frac{\rho}{x}),\quad(2.39)$$

where $x>0$ and $\rho\in(0,x)$, and integrating the resulting inequality obtained with respect to τ and ρ from 0 to x, we have

$$\int_0^x\int_0^x u(\tau)F(x,\tau)v(\rho)G(t,\rho)H(\tau,\rho)d\tau d\rho=K_t^{\alpha,\beta,\eta,\mu}uK_t^{\gamma,\delta,\zeta,\nu}(vfg)+K_t^{\gamma,\delta,\zeta,\nu}vK_t^{\alpha,\beta,\eta,\mu}(ufg)$$
$$-K_t^{\alpha,\beta,\eta,\mu}(uf)K_t^{\gamma,\delta,\zeta,\nu}(vg)-K_t^{\gamma,\delta,\zeta,\nu}(vf)K_t^{\alpha,\beta,\eta,\mu}(ug).\quad(2.40)$$

Then, thanks to the weighted Cauchy-Schwartz integral inequality for double integrals, we can obtain (2.38). □

Lemma 2.12. *Let f be a function defined on $[0,\infty)$ and let u and v be two nonnegative functions on $[0,\infty)$. Then we have*

$$K_t^{\alpha,\beta,\eta,\mu}uK_t^{\gamma,\delta,\zeta,\nu}(vf^2)+K_t^{\gamma,\delta,\zeta,\nu}vK_t^{\alpha,\beta,\eta,\mu}(uf^2)-2K_t^{\gamma,\delta,\zeta,\nu}(vf)K_t^{\alpha,\beta,\eta,\mu}(uf)=\left(\Phi K_t^{\alpha,\beta,\eta,\mu}u-K_t^{\alpha,\beta,\eta,\mu}(uf)\right)$$
$$\times\left(K_t^{\gamma,\delta,\zeta,\nu}(vf)-\phi K_t^{\gamma,\delta,\zeta,\nu}v\right)+\left(K_t^{\alpha,\beta,\eta,\mu}(uf)-\phi K_t^{\alpha,\beta,\eta,\mu}u\right)\left(\Phi K_t^{\gamma,\delta,\zeta,\nu}v-K_t^{\gamma,\delta,\zeta,\nu}(vf)\right)$$
$$-K_t^{\alpha,\beta,\eta,\mu}uK_t^{\gamma,\delta,\zeta,\nu}\left(v(x)(\Phi-f(x))(f(x)-\phi)\right)-K_t^{\gamma,\delta,\zeta,\nu}vK_t^{\alpha,\beta,\eta,\mu}\left(u(x)(\Phi-f(x))(f(x)-\phi)\right),\quad(2.41)$$

for all $x\in[0,\infty)$, and real constants $\alpha,\gamma,\beta,\delta,\eta,\zeta,\mu,\nu$ satisfying $\alpha,\gamma>0$, $\mu,\nu>-1$, $\eta,\zeta\leq 0$ and $\alpha+\beta+\mu,\gamma+\delta+\zeta\geq 0$.

Proof. Multiplying both sides of (2.25) by $v(\tau)G(t,\tau)$ ($G(t,\tau)$ defined by (2.39)), and integrating the resulting inequality obtained with respect to τ from 0 to x, we have

$$\left(K_t^{\gamma,\delta,\zeta,\nu}(vf)-\phi K_t^{\gamma,\delta,\zeta,\nu}v\right)\left(\Phi K_t^{\alpha,\beta,\eta,\mu}u-K_t^{\alpha,\beta,\eta,\mu}(uf)\right)$$
$$+\left(\Phi K_t^{\gamma,\delta,\zeta,\nu}v-K_t^{\gamma,\delta,\zeta,\nu}(vf)\right)\left(K_t^{\alpha,\beta,\eta,\mu}(uf)-\phi K_t^{\alpha,\beta,\eta,\mu}u\right)$$
$$-K_t^{\gamma,\delta,\zeta,\nu}\left(v(x)(\Phi-f(x))(f(x)-\phi)\right)K_t^{\alpha,\beta,\eta,\mu}u-K_t^{\gamma,\delta,\zeta,\nu}vK_t^{\alpha,\beta,\eta,\mu}\left(u(x)(\Phi-f(x))(f(x)-\phi)\right)$$
$$=K_t^{\gamma,\delta,\zeta,\nu}(vf^2)K_t^{\alpha,\beta,\eta,\mu}u+K_t^{\gamma,\delta,\zeta,\nu}vK_t^{\alpha,\beta,\eta,\mu}(uf^2)-2K_t^{\gamma,\delta,\zeta,\nu}(vf)K_t^{\alpha,\beta,\eta,\mu}(uf),\quad(2.42)$$

which gives (2.41) and proves the lemma. □

Theorem 2.13. *Let f and g be two functions satisfying the condition (1.6) on $[0,\infty)$ and let u and v be two nonnegative functions on $[0,\infty)$. Then we have*

$$\begin{aligned}&\left(K_t^{\alpha,\beta,\eta,\mu}uK_t^{\gamma,\delta,\zeta,\nu}(vfg)+K_t^{\gamma,\delta,\zeta,\nu}vK_t^{\alpha,\beta,\eta,\mu}(ufg)-K_t^{\alpha,\beta,\eta,\mu}(uf)K_t^{\gamma,\delta,\zeta,\nu}(vg)-K_t^{\gamma,\delta,\zeta,\nu}(vf)K_t^{\alpha,\beta,\eta,\mu}(ug)\right)^2\\&\leq\left[\left(\Phi K_t^{\alpha,\beta,\eta,\mu}u-K_t^{\alpha,\beta,\eta,\mu}(uf)\right)\left(K_t^{\gamma,\delta,\zeta,\nu}(vf)-\phi K_t^{\gamma,\delta,\zeta,\nu}v\right)+\left(K_t^{\alpha,\beta,\eta,\mu}(uf)-\phi K_t^{\alpha,\beta,\eta,\mu}u\right)\right.\\&\quad\left.\times\left(\Phi K_t^{\gamma,\delta,\zeta,\nu}v-K_t^{\gamma,\delta,\zeta,\nu}(vf)\right)\right]\left[\left(\Psi K_t^{\alpha,\beta,\eta,\mu}u-K_t^{\alpha,\beta,\eta,\mu}(ug)\right)\left(K_t^{\gamma,\delta,\zeta,\nu}(vg)-\psi K_t^{\gamma,\delta,\zeta,\nu}v\right)\right.\\&\quad\left.+\left(K_t^{\alpha,\beta,\eta,\mu}(ug)-\psi K_t^{\alpha,\beta,\eta,\mu}u\right)\left(\Psi K_t^{\gamma,\delta,\zeta,\nu}v-K_t^{\gamma,\delta,\zeta,\nu}(vg)\right)\right],\end{aligned}\tag{2.43}$$

for all $x\in[0,\infty)$, and real constants $\alpha,\gamma,\beta,\delta,\eta,\zeta,\mu,\nu$ satisfying $\alpha,\gamma>0$, $\mu,\nu>-1$, $\eta,\zeta\leq 0$ and $\alpha+\beta+\mu,\gamma+\delta+\zeta\geq 0$.

Proof. Since $(\Phi-f(\tau))(f(\tau)-\phi)\geq 0$ and $(\Psi-g(\tau))(g(\tau)-\psi)\geq 0$, we have

$$-K_t^{\alpha,\beta,\eta,\mu}uK_t^{\gamma,\delta,\zeta,\nu}\left(v(x)(\Phi-f(x))(f(x)-\phi)\right)-K_t^{\gamma,\delta,\zeta,\nu}vK_t^{\alpha,\beta,\eta,\mu}\left(u(x)(\Phi-f(x))(f(x)-\phi)\right)\leq 0,\tag{2.44}$$

and

$$-K_t^{\alpha,\beta,\eta,\mu}uK_t^{\gamma,\delta,\zeta,\nu}\left(v(x)(\Phi-g(x))(g(x)-\phi)\right)-K_t^{\gamma,\delta,\zeta,\nu}vK_t^{\alpha,\beta,\eta,\mu}\left(u(x)(\Phi-g(x))(g(x)-\phi)\right)\leq 0,\tag{2.45}$$

Applying Lemma 2.12 to f and g, and using Lemma 2.11 and the formulas (2.44), (2.45), we obtain (2.43). □

Theorem 2.14. *Let u be a nonnegative function on $[0,\infty)$ and let f,g and h be three functions defined on $[0,\infty)$, satisfying the following condition*

$$\phi\leq f(x)\leq\Phi,\quad \psi\leq g(x)\leq\Psi,\quad \omega\leq h(x)\leq\Omega,\quad \phi,\Phi,\psi,\Psi,\omega,\Omega\in\mathbb{R},\quad x\in[0,\infty).\tag{2.46}$$

Then we have

$$\begin{aligned}&\Big|K_t^{\alpha,\beta,\eta,\mu}(ufgh)K_t^{\gamma,\delta,\zeta,\nu}u+K_t^{\alpha,\beta,\eta,\mu}(uh)K_t^{\gamma,\delta,\zeta,\nu}(ufg)+K_t^{\alpha,\beta,\eta,\mu}(ug)K_t^{\gamma,\delta,\zeta,\nu}(ufh)\\&\quad+K_t^{\alpha,\beta,\eta,\mu}(uf)K_t^{\gamma,\delta,\zeta,\nu}(ugh)-K_t^{\alpha,\beta,\eta,\mu}(ugh)K_t^{\gamma,\delta,\zeta,\nu}(uf)-K_t^{\alpha,\beta,\eta,\mu}(ufh)K_t^{\gamma,\delta,\zeta,\nu}(ug)\\&\quad-K_t^{\alpha,\beta,\eta,\mu}(ufg)K_t^{\gamma,\delta,\zeta,\nu}(uh)-K_t^{\alpha,\beta,\eta,\mu}uK_t^{\gamma,\delta,\zeta,\nu}(ufgh)\Big|\leq K_t^{\alpha,\beta,\eta,\mu}uK_t^{\gamma,\delta,\zeta,\nu}u(\Phi-\phi)(\Psi-\psi)(\Omega-\omega),\end{aligned}$$

for all $x\in[0,\infty)$, and real constants $\alpha,\gamma,\beta,\delta,\eta,\zeta,\mu,\nu$ satisfying $\alpha,\gamma>0$, $\mu,\nu>-1$, $\eta,\zeta\leq 0$ and $\alpha+\beta+\mu,\gamma+\delta+\zeta\geq 0$.

Proof. From the condition (2.46), we have

$$|f(\tau)-f(\rho)|\leq\Phi-\phi,\quad |g(\tau)-g(\rho)|\leq\Psi-\psi,\quad |h(\tau)-h(\rho)|\leq\Omega-\omega,\quad \tau,\rho\in[0,\infty),$$

which implies that

$$|(f(\tau)-f(\rho))(g(\tau)-g(\rho))(h(\tau)-h(\rho))|\leq(\Phi-\phi)(\Psi-\psi)(\Omega-\omega).\tag{2.47}$$

Let us define a function

$$\begin{aligned}A(\tau,\rho)&=(f(\tau)-f(\rho))(g(\tau)-g(\rho))(h(\tau)-h(\rho))=f(\tau)g(\tau)h(\tau)+f(\rho)g(\rho)h(\tau)+f(\tau)g(\rho)h(\rho)\\&\quad+f(\rho)g(\tau)h(\rho)-f(\tau)g(\rho)h(\tau)-f(\rho)g(\rho)h(\rho)-f(\tau)g(\tau)h(\rho)-f(\rho)g(\tau)h(\tau).\end{aligned}\tag{2.48}$$

Multiplying (2.48) by $u(\tau)F(t,\tau)$, where $F(t,\tau)$ is defined by (2.28), and integrating the resulting inequality obtained with respect to τ from 0 to x, we have

$$\int_0^x u(\tau)F(x,\tau)A(\tau,\rho)d\tau = K_t^{\alpha,\beta,\eta,\mu}(ufgh) + f(\rho)g(\rho)K_t^{\alpha,\beta,\eta,\mu}(uh) + f(\rho)h(\rho)K_t^{\alpha,\beta,\eta,\mu}(ug) + g(\rho)h(\rho)K_t^{\alpha,\beta,\eta,\mu}(uf) - h(\rho)K_t^{\alpha,\beta,\eta,\mu}(ufg) - g(\rho)K_t^{\alpha,\beta,\eta,\mu}(ufh) - f(\rho)K_t^{\alpha,\beta,\eta,\mu}(ugh) - f(\rho)g(\rho)h(\rho)K_t^{\alpha,\beta,\eta,\mu}u. \quad (2.49)$$

Again, by multiplying (2.49) by $u(\rho)G(t,\rho)$, where $G(t,\tau)$ is defined by (2.39), and integrating the resulting inequality obtained with respect to ρ from 0 to x, we have

$$\int_0^x\int_0^x u(\tau)F(x,\tau)u(\rho)G(t,\rho)A(\tau,\rho)d\tau d\rho = K_t^{\alpha,\beta,\eta,\mu}(ufgh)K_t^{\gamma,\delta,\zeta,\nu}u + K_t^{\alpha,\beta,\eta,\mu}(uh)K_t^{\gamma,\delta,\zeta,\nu}(ufg) + K_t^{\alpha,\beta,\eta,\mu}(ug)K_t^{\gamma,\delta,\zeta,\nu}(ufh) + K_t^{\alpha,\beta,\eta,\mu}(uf)K_t^{\gamma,\delta,\zeta,\nu}(ugh) - K_t^{\alpha,\beta,\eta,\mu}(ugh)K_t^{\gamma,\delta,\zeta,\nu}(uf) - K_t^{\alpha,\beta,\eta,\mu}(ufh)K_t^{\gamma,\delta,\zeta,\nu}(ug) - K_t^{\alpha,\beta,\eta,\mu}(ufg)K_t^{\gamma,\delta,\zeta,\nu}(uh) - K_t^{\alpha,\beta,\eta,\mu}uK_t^{\gamma,\delta,\zeta,\nu}(ufgh). \quad (2.50)$$

Finally, by using (2.47) on to (2.50), we arrive at the desired result (**??**), involved in Theorem 2.14, after a little simplification. This concludes the proof. ☐

Theorem 2.15. *Let u and v be two nonnegative functions on $[0,\infty)$ and let f, g and h be three functions defined on $[0,\infty)$, satisfying the condition (2.46). Then we have*

$$\Big|K_t^{\alpha,\beta,\eta,\mu}(ufgh)K_t^{\gamma,\delta,\zeta,\nu}v + K_t^{\alpha,\beta,\eta,\mu}(uh)K_t^{\gamma,\delta,\zeta,\nu}(vfg) + K_t^{\alpha,\beta,\eta,\mu}(ug)K_t^{\gamma,\delta,\zeta,\nu}(vfh) + K_t^{\alpha,\beta,\eta,\mu}(uf)K_t^{\gamma,\delta,\zeta,\nu}(vgh) - K_t^{\alpha,\beta,\eta,\mu}(ugh)K_t^{\gamma,\delta,\zeta,\nu}(vf) - K_t^{\alpha,\beta,\eta,\mu}(ufh)K_t^{\gamma,\delta,\zeta,\nu}(vg) - K_t^{\alpha,\beta,\eta,\mu}(ufg)K_t^{\gamma,\delta,\zeta,\nu}(vh) - K_t^{\alpha,\beta,\eta,\mu}uK_t^{\gamma,\delta,\zeta,\nu}(vfgh)\Big| \leq K_t^{\alpha,\beta,\eta,\mu}uK_t^{\gamma,\delta,\zeta,\nu}v(\Phi-\phi)(\Psi-\psi)(\Omega-\omega), \quad (2.51)$$

for all $x \in [0,\infty)$, and real constants $\alpha,\gamma,\beta,\delta,\eta,\zeta,\mu,\nu$ satisfying $\alpha,\gamma > 0$, $\mu,\nu > -1$, $\eta,\zeta \leq 0$ and $\alpha+\beta+\mu, \gamma+\delta+\zeta \geq 0$.

Proof. Multiplying (2.49) by $v(\rho)G(t,\rho)$, where $G(t,\tau)$ is defined by (2.39), and integrating the resulting inequality obtained with respect to ρ from 0 to x, and then applying (2.47) on the resulting inequality, we get the desired result (2.51). This concludes the proof. ☐

Remark 2.16. It is not difficult to notice that the spacial case $u = v$ of (2.51) in Theorem 2.15 reduces to Theorem 2.14.

Theorem 2.17. *Let f and g be two integrable functions satisfying the condition M-g-Lipschitzian on $[0,\infty)$, i.e., $|f(x)-f(y)| \leq M|g(x)-g(y)|$, $M > 0$, $x,y \in \mathbb{R}$, and let u and v be two nonnegative continuous functions on $[0,\infty)$. Then we have*

$$\Big|K_t^{\alpha,\beta,\eta,\mu}uK_t^{\gamma,\delta,\zeta,\nu}(vfg) + K_t^{\gamma,\delta,\zeta,\nu}vK_t^{\alpha,\beta,\eta,\mu}(ufg) - K_t^{\alpha,\beta,\eta,\mu}(uf)K_t^{\gamma,\delta,\zeta,\nu}(yg) - K_t^{\gamma,\delta,\zeta,\nu}(vf)K_t^{\alpha,\beta,\eta,\mu}(xg)\Big| \leq M\left(K_t^{\alpha,\beta,\eta,\mu}uK_t^{\gamma,\delta,\zeta,\nu}(vg^2) + K_t^{\gamma,\delta,\zeta,\nu}vK_t^{\alpha,\beta,\eta,\mu}(ug^2) - 2K_t^{\alpha,\beta,\eta,\mu}(ug)K_t^{\gamma,\delta,\zeta,\nu}(vg)\right), \quad (2.52)$$

for all $x \in [0,\infty)$, and real constants $\alpha,\gamma,\beta,\delta,\eta,\zeta,\mu,\nu$ satisfying $\alpha,\gamma > 0$, $\mu,\nu > -1$, $\eta,\zeta \leq 0$ and $\alpha+\beta+\mu, \gamma+\delta+\zeta \geq 0$.

Proof. Let us define the following relations

$$|f(\tau)-f(\rho)| \leq M|g(\tau)-g(\rho)| \quad \tau,\rho \in [0,\infty), \quad (2.53)$$

which implies that

$$|H(\tau,\rho)| = |f(\tau)-f(\rho)||g(\tau)-g(\rho)| \leq M(g(\tau)-g(\rho))^2. \quad (2.54)$$

Multiplying (2.27) by $u(\tau)F(t,\tau)u(\rho)G(t,\rho)$, where $F(t,\tau)$ and $G(t,\rho)$ are defined by (2.28) and (2.39), respectively, and integrating the resulting inequality obtained with respect to τ and ρ from 0 to x, then applying (2.40) and (2.54) on the resulting inequality, we get the desired result (2.52). This concludes the proof of the theorem. □

Theorem 2.18. *Let u and v be two nonnegative functions on $[0,\infty)$ and let f and g be two Lipschitzian functions defined on $[0,\infty)$ with the constants L_1 and L_2, respectively. Then we have*

$$\left|K_t^{\alpha,\beta,\eta,\mu}uK_t^{\gamma,\delta,\zeta,\nu}(vfg)+K_t^{\gamma,\delta,\zeta,\nu}vK_t^{\alpha,\beta,\eta,\mu}(ufg)-K_t^{\alpha,\beta,\eta,\mu}(uf)K_t^{\gamma,\delta,\zeta,\nu}(yg)-K_t^{\gamma,\delta,\zeta,\nu}(vf)K_t^{\alpha,\beta,\eta,\mu}(xg)\right|$$
$$\leq L_1L_2\bigg(K_t^{\alpha,\beta,\eta,\mu}uK_t^{\gamma,\delta,\zeta,\nu}(x^2v(x))+K_t^{\gamma,\delta,\zeta,\nu}vK_t^{\alpha,\beta,\eta,\mu}(x^2u(x))-2K_t^{\alpha,\beta,\eta,\mu}(xu(x))K_t^{\gamma,\delta,\zeta,\nu}(xv(x))\bigg), \quad (2.55)$$

for all $x\in[0,\infty)$, and real constants $\alpha,\gamma,\beta,\delta,\eta,\zeta,\mu,\nu$ satisfying $\alpha,\gamma>0$, $\mu,\nu>-1$, $\eta,\zeta\leq 0$ and $\alpha+\beta+\mu,\gamma+\delta+\zeta\geq 0$.

Proof. From the conditions of Theorem 2.18, we have

$$|f(\tau)-f(\rho)|\leq L_1|\tau-\rho|,\quad |g(\tau)-g(\rho)|\leq L_2|\tau-\rho|,\quad \tau,\rho\in[0,\infty),$$

which implies that

$$|H(\tau,\rho)|=|f(\tau)-f(\rho)||g(\tau)-g(\rho)|\leq L_1L_2(\tau-\rho)^2. \quad (2.56)$$

Multiplying (2.27) by $u(\tau)F(t,\tau)v(\rho)G(t,\rho)$, where $F(t,\tau)$ and $G(t,\rho)$ are defined by (2.28) and (2.39), respectively, and integrating the resulting inequality obtained with respect to τ and ρ from 0 to x, then applying (2.40) and (2.56), on the resulting inequality, we get the desired result (2.55). This completes the proof. □

Corollary 2.19. *Let u and v be two nonnegative functions on $[0,\infty)$ and let f and g be two differentiable functions on $[0,\infty)$ with $\sup_{t\geq 0}|f'(t)|,\sup_{t\geq 0}|g'(t)|<\infty$. Then we have*

$$\left|K_t^{\alpha,\beta,\eta,\mu}uK_t^{\gamma,\delta,\zeta,\nu}(vfg)+K_t^{\gamma,\delta,\zeta,\nu}vK_t^{\alpha,\beta,\eta,\mu}(ufg)-K_t^{\alpha,\beta,\eta,\mu}(uf)K_t^{\gamma,\delta,\zeta,\nu}(yg)-K_t^{\gamma,\delta,\zeta,\nu}(vf)K_t^{\alpha,\beta,\eta,\mu}(xg)\right|$$
$$\leq \|f'\|_\infty\|g'\|_\infty\bigg(K_t^{\alpha,\beta,\eta,\mu}uK_t^{\gamma,\delta,\zeta,\nu}(x^2v(x))+K_t^{\gamma,\delta,\zeta,\nu}vK_t^{\alpha,\beta,\eta,\mu}(x^2u(x))-2K_t^{\alpha,\beta,\eta,\mu}(xu(x))K_t^{\gamma,\delta,\zeta,\nu}(xv(x))\bigg),$$

for all $x\in[0,\infty)$, and real constants $\alpha,\gamma,\beta,\delta,\eta,\zeta,\mu,\nu$ satisfying $\alpha,\gamma>0$, $\mu,\nu>-1$, $\eta,\zeta\leq 0$ and $\alpha+\beta+\mu,\gamma+\delta+\zeta\geq 0$.

Proof. We have $f(\tau)-f(\rho)=\int_\rho^\tau f'(t)dt$ and $g(\tau)-g(\rho)=\int_\rho^\tau g'(t)dt$. That is, $|f(\tau)-f(\rho)|\leq\|f'\|_\infty|\tau-\rho|$, $|g(\tau)-g(\rho)|\leq\|g'\|_\infty|\tau-\rho|$, $\tau,\rho\in[0,\infty)$, and the result follows from Theorem 2.18. This ends the proof. □

3 An example

In this section we present a way for constructing the four bounding functions, and use them to give some estimates of Chebyshev type inequalities involving the generalized fractional integral operator of two unknown functions.

For $0=x_0<x_1<x_2<\cdots<x_n<x_{n+1}=T$, we define a notation of sub-integrals of generalized fractional integral $I_x^{\alpha,\beta,\eta,\mu}$ as

$$I_{x_j,x_{j+1}}^{\alpha,\beta,\eta,\mu}\{f(T)\}=\frac{x^{-\alpha-\beta-2\mu}}{\Gamma(\alpha)}\int_{x_j}^{x_{j+1}}t^\mu(T-t)^{\alpha-1}{}_2F_1\left(\alpha+\beta+\mu,-\eta;\alpha;1-\frac{t}{T}\right)f(t)dt,\quad j=0,1,\ldots,n. \quad (3.1)$$

Note that

$$I_{0,T}^{\alpha,\beta,\eta,\mu}\{f(T)\}=\sum_{j=0}^{n}I_{x_j,x_{j+1}}^{\alpha,\beta,\eta,\mu}\{f(T)\}=\frac{x^{-\alpha-\beta-2\mu}}{\Gamma(\alpha)}\int_0^{x_1}t^{\mu}(T-t)^{\alpha-1}{}_2F_1\left(\alpha+\beta+\mu,-\eta;\alpha;1-\frac{t}{T}\right)f(t)dt$$
$$+\frac{x^{-\alpha-\beta-2\mu}}{\Gamma(\alpha)}\int_{x_1}^{x_2}t^{\mu}(T-t)^{\alpha-1}{}_2F_1\left(\alpha+\beta+\mu,-\eta;\alpha;1-\frac{t}{T}\right)f(t)dt+\cdots$$
$$+\frac{x^{-\alpha-\beta-2\mu}}{\Gamma(\alpha)}\int_{x_n}^{T}t^{\mu}(T-t)^{\alpha-1}{}_2F_1\left(\alpha+\beta+\mu,-\eta;\alpha;1-\frac{t}{T}\right)f(t)dt. \quad (3.2)$$

So, from (3.2), we can rewrite (1.7) as

$$K_{0,T}^{\alpha,\beta,\eta,\mu}f(T)=\frac{\Gamma(1-\beta)\Gamma(\alpha+\mu+\eta+1)}{\Gamma(\eta-\beta+1)\Gamma(\mu+1)}T^{\beta+\mu}\,I_{0,T}^{\alpha,\beta,\eta,\mu}\{f(T)\}$$
$$=\frac{\Gamma(1-\beta)\Gamma(\alpha+\mu+\eta+1)}{\Gamma(\eta-\beta+1)\Gamma(\mu+1)}T^{\beta+\mu}\sum_{j=0}^{n}I_{x_j,x_{j+1}}^{\alpha,\beta,\eta,\mu}\{f(T)\}=\frac{\Gamma(1-\beta)\Gamma(\alpha+\mu+\eta+1)}{\Gamma(\eta-\beta+1)\Gamma(\mu+1)}x^{\beta+\mu}$$
$$\times\left\{\frac{T^{-\alpha-\beta-2\mu}}{\Gamma(\alpha)}\int_0^{x_1}t^{\mu}(T-t)^{\alpha-1}{}_2F_1\left(\alpha+\beta+\mu,-\eta;\alpha;1-\frac{t}{T}\right)f(t)dt\right.$$
$$\frac{x^{-\alpha-\beta-2\mu}}{\Gamma(\alpha)}\int_{x_1}^{x_2}t^{\mu}(T-t)^{\alpha-1}{}_2F_1\left(\alpha+\beta+\mu,-\eta;\alpha;1-\frac{t}{T}\right)f(t)dt$$
$$\left.\cdots+\frac{x^{-\alpha-\beta-2\mu}}{\Gamma(\alpha)}\int_{x_n}^{T}t^{\mu}(T-t)^{\alpha-1}{}_2F_1\left(\alpha+\beta+\mu,-\eta;\alpha;1-\frac{t}{T}\right)f(t)dt\right\}. \quad (3.3)$$

Let φ be a unit step function defined by

$$\varphi(x)=\begin{cases}1, & x>0,\\ 0, & x\le 0,\end{cases}$$

and let $\varphi_a(x)$ the Heaviside unit step function defined by

$$\varphi_a(x)=\varphi(x-a)=\begin{cases}1, & x>a,\\ 0, & x\le a.\end{cases}$$

Let u be a piecewise continuous function on $[0,T]$ defined by

$$u(x)=U_1(\varphi_0(x)-\varphi_{x_1}(x))+U_2(\varphi_{x_1}(x)-\varphi_{x_2}(x))+U_3(\varphi_{x_2}(x)-\varphi_{x_3}(x))+\cdots+U_{m+1}\varphi_{x_m}(x)=U_1\varphi_0(x)$$
$$+(U_2-U_1)\varphi_{x_1}(x)+(U_3-U_2)\varphi_{x_2}(x)+\cdots+(U_{m+1}-U_m)\varphi_{x_m}(x)=\sum_{j=0}^{m}(U_{j+1}-U_j)\varphi_{x_j}(x), \quad (3.4)$$

where $U_0\equiv 0$ and $0=x_0<x_1<x_2<\cdots<x_m<x_{m+1}=T$. Similarly, we have

$$v(x)=\sum_{j=0}^{m}(V_{j+1}-V_j)\varphi_{x_j}(x). \quad (3.5)$$

where constants $U_0=V_0\equiv 0$.

Proposition 3.1. *Let f and g be two synchronous functions on $[0,T)$. Assume that let u and v defined by (3.4) and (3.5), respectively. Then for $\alpha>0,\mu>-1,\eta\le 0$ and $\alpha+\beta+\mu\ge 0$, the following inequality holds:*

$$\left(\sum_{j=0}^{m}U_{j+1}\right)\left(\sum_{j=0}^{m}V_{j+1}K_{x_j,x_{j+1}}^{\alpha,\beta,\eta,\mu}(fg)(T)\right)+\left(\sum_{j=0}^{m}V_{j+1}\right)\left(\sum_{j=0}^{m}U_{j+1}K_{x_j,x_{j+1}}^{\alpha,\beta,\eta,\mu}(fg)(T)\right)$$
$$\ge\left(\sum_{j=0}^{m}U_{j+1}K_{x_j,x_{j+1}}^{\alpha,\beta,\eta,\mu}g(T)\right)\left(\sum_{j=0}^{m}V_{j+1}K_{x_j,x_{j+1}}^{\alpha,\beta,\eta,\mu}f(T)\right)+\left(\sum_{j=0}^{m}V_{j+1}K_{x_j,x_{j+1}}^{\alpha,\beta,\eta,\mu}g(T)\right)\left(\sum_{j=0}^{m}U_{j+1}K_{x_j,x_{j+1}}^{\alpha,\beta,\eta,\mu}f(T)\right). \quad (3.6)$$

Proof. By using the definition (3.1) and (3.3), we have

$$K_{0,T}^{\alpha,\beta,\eta,\mu}u(T)=\sum_{j=0}^{m}U_{j+1}K_{x_j,x_{j+1}}^{\alpha,\beta,\eta,\mu}(1)(T)=\sum_{j=0}^{m}U_{j+1},$$

and

$$K_{0,T}^{\alpha,\beta,\eta,\mu}v(T)=\sum_{j=0}^{m}V_{j+1}K_{x_j,x_{j+1}}^{\alpha,\beta,\eta,\mu}(1)(T)=\sum_{j=0}^{m}V_{j+1},$$

where $K_{x_j,x_{j+1}}^{\alpha,\beta,\eta,\mu}(1)(T)=1$. Similarly, we have

$$K_{0,T}^{\alpha,\beta,\eta,\mu}(ufg)(T)=\sum_{j=0}^{m}U_{j+1}K_{x_j,x_{j+1}}^{\alpha,\beta,\eta,\mu}(fg)(T),\quad K_{0,T}^{\alpha,\beta,\eta,\mu}(vfg)(T)=\sum_{j=0}^{m}V_{j+1}K_{x_j,x_{j+1}}^{\alpha,\beta,\eta,\mu}(fg)(T),$$
$$K_{0,T}^{\alpha,\beta,\eta,\mu}(uf)(T)=\sum_{j=0}^{m}U_{j+1}K_{x_j,x_{j+1}}^{\alpha,\beta,\eta,\mu}f(T),\quad K_{0,T}^{\alpha,\beta,\eta,\mu}(vf)(T)=\sum_{j=0}^{m}V_{j+1}K_{x_j,x_{j+1}}^{\alpha,\beta,\eta,\mu}f(T),$$
$$K_{0,T}^{\alpha,\beta,\eta,\mu}(ug)(T)=\sum_{j=0}^{m}U_{j+1}K_{x_j,x_{j+1}}^{\alpha,\beta,\eta,\mu}g(T),\quad K_{0,T}^{\alpha,\beta,\eta,\mu}(vg)(T)=\sum_{j=0}^{m}V_{j+1}K_{x_j,x_{j+1}}^{\alpha,\beta,\eta,\mu}g(T),$$

By applying Lemma 2.1, the desired inequality (3.6) is established. □

4 Concluding remarks

In this section, we consider some consequences of the main results derived in the previous section. Following Curiel and Galue [33], the operator would reduce immediately to the extensively investigated Saigo, Erdélyi-Kober, and Riemann-Liouville type fractional integral operators, respectively, given by the following relationships (see also [32, 34]):

$$I_{0,x}^{\alpha,\beta,\eta}\{f(x)\}=I_{x}^{\alpha,\beta,\eta,0}\{f(x)\}=\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{t}{x}\right)f(\tau)dt,\quad(\alpha>0;\beta,\eta\in\mathbb{R}),\tag{4.1}$$

$$I^{\alpha,\eta}\{f(x)\}=I_{x}^{\alpha,0,\eta,0}\{f(x)\}=\frac{x^{-\alpha-\eta}}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}t^{\alpha-1}f(t)dt,\quad(\alpha>0;\eta\in\mathbb{R}),\tag{4.2}$$

and

$$J^{\alpha}\{f(x)\}=I_{x}^{\alpha,-\alpha,\eta,0}\{f(x)\}=\frac{1}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}f(t)dt,\quad(\alpha>0).\tag{4.3}$$

By setting $\mu=0$, $\mu=\beta=0$, and $\mu=0$ and $\beta=-\alpha$ in (1.7), Definition 1.2 would immediately reduce to the Saigo, Erdélyi-Kober, and Riemann-Liouville type fractional integral operators, respectively, given as follows:

$$K_{x}^{\alpha,\beta,\eta}f(x)=\frac{\Gamma(1-\beta)\Gamma(\alpha+\eta+1)}{\Gamma(\eta-\beta+1)}x^{\beta}I_{0,x}^{\alpha,\beta,\eta}\{f(x)\},\tag{4.4}$$

$$K_{x}^{\alpha,\eta}f(x)=\frac{\Gamma(\eta+\alpha+1)}{\Gamma(1+\eta)}I^{\alpha,\eta}\{f(x)\},\tag{4.5}$$

and

$$K_{x}^{\alpha}f(x)=\frac{\Gamma(\alpha+1)}{x^{\alpha}}J^{\alpha}\{f(x)\},\tag{4.6}$$

where $I_{0,x}^{\alpha,\beta,\eta}\{f(x)\}$, $I^{\alpha,\eta}\{f(x)\}$ and $J^{\alpha}\{f(x)\}$ are given by (4.1), (4.2), and (4.3), respectively.

Similar to main results in the preceding section, by using the fractional integral operators (4.1)-(4.6), we obtain various fractional integral inequalities involving such relatively more familiar fractional integral operators (4.1)-(4.6). Therefore, we omit the further details. For example, by (4.1), Theorem 2.2 and 2.4 yield the known results in [24, 25]. If we consider $u = v = 1$ and make use of fractional integral operator $I_x^{\alpha,\beta,\eta,\mu}\{f(x)\}$, Lemma 2.1 and 2.3 provides respectively, the known fractional integral inequalities due to Baleanu *et al.* [31].

Let $u = 1$, Theorem 2.10 corresponds to the known results due to Wang *et al.* [28]. Taking $u = 1$, $\mu = 0$ and $\beta = -\alpha$ in Theorem 2.10 yields the known result due to Dahmani *et al.* [26]. Make use of fractional integral operator (4.3), Lemma 2.1 and 2.3 provides respectively, the known fractional integral inequalities due to Dahmani [17]. At the end of this paper, generalized fractional integral inequalities obtained in the previous section are expected to find more applications, for example, applications for establishing the solutions in fractional differential equations and fractional integral equations boundary value problems.

Authors' contributions. ZL and WY equally participated in the design of the study and drafted the manuscript. PA gave an example to show the applications. All authors read and approved the final manuscript.

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Estimates for the Green's Function of 3D Elliptic Equations

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This article will first introduce the definition of the Green's function of 3D elliptic equations, which plays important roles in local superconvergence estimates for the finite element approximation. Then, using the weighted-norm methods, we derive some estimates for the 3D Green's function.

1 Introduction

It is well known that estimates for the Green's function play very important roles in the study of the superconvergence (especially, pointwise superconvergence) of the finite element method (see [1–9]). For dimensions three and up, we have obtained the estimates for discrete Green's functions and discrete derivative Green's functions, which were used to the global superconvergence estimates of the finite element approximation. However, the fact is that the high generalization conditions to the true solution is difficult to satisfy for the global superconvergence estimates. Thus the global superconvergence results is only theoretical. In order to study local superconvergence properties of the finite element approximation, we need to introduce a Green's function, which will play important roles in the study of local superconvergence properties.

we shall use the symbol C to denote a generic constant, which is independent from the discretization parameter h and which may not be the same in each occurrence and also use the standard notations for the Sobolev spaces and their norms.

In this article, we consider the following elliptic equation:

$$\mathcal{L}u \equiv -\sum_{i,j=1}^{3} \partial_j(a_{ij}\partial_i u) + a_0 u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where $\Omega \subset \mathcal{R}^3$ is a bounded polytopic domain. The weak formulation of (1.1) reads,

$$\begin{cases} \text{Find } u \in H_0^1(\Omega) \text{ satisfying} \\ a(u\,,\,v) = (f\,,\,v) \text{ for all } v \in H_0^1(\Omega), \end{cases}$$

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where

$$a(u\,,\,v)\equiv\int_\Omega(\sum_{i,j=1}^{3}a_{ij}\partial_i u\partial_j v+a_0uv)\,dxdydz,\ (f\,,\,v)\equiv\int_\Omega fv\,dxdydz.$$

We assume that the given functions $a_{ij}\in W^{1,\infty}(\Omega)$, $a_{ij}=a_{ji}$, $a_0\in L^\infty(\Omega)$, and $f\in L^2(\Omega)$. In addition, we write $\partial_1 u=\frac{\partial u}{\partial x}$, $\partial_2 u=\frac{\partial u}{\partial y}$, and $\partial_3 u=\frac{\partial u}{\partial z}$, which are usual partial derivatives. Let $\{\mathcal{T}^h\}$ be a regular family of partitions of $\bar\Omega$. Denote by $S^h(\Omega)$ a continuous finite elements space of degree $m(m\geq 1)$ regarding this kind of partitions and let $S_0^h(\Omega)=S^h(\Omega)\cap H_0^1(\Omega)$. Discretizing the above weak formulation using $S_0^h(\Omega)$ as approximating space means,

$$\begin{cases}\text{Find } u_h\in S_0^h(\Omega) \text{ satisfying}\\ a(u_h\,,\,v)=(f\,,\,v) \text{ for all } v\in S_0^h(\Omega).\end{cases}$$

For every $Z\in\Omega$, we define the discrete δ function $\delta_Z^h\in S_0^h(\Omega)$, the discrete derivative δ function $\partial_{Z,\ell}\delta_Z^h\in S_0^h(\Omega)$, the regularized Green's function $G_Z^*\in H^2(\Omega)\cap H_0^1(\Omega)$, the regularized derivative Green's function $\partial_{Z,\ell}G_Z^*\in H^2(\Omega)\cap H_0^1(\Omega)$, the discrete Green's function $G_Z^h\in S_0^h(\Omega)$, the discrete derivative Green's function $\partial_{Z,\ell}G_Z^h\in S_0^h(\Omega)$, and the L^2-projection $P_hu\in S_0^h(\Omega)$ such that (see [9])

$$(v,\,\delta_Z^h)=v(Z)\ \ \forall\,v\in S_0^h(\Omega), \tag{1.2}$$

$$(v,\,\partial_{Z,\ell}\delta_Z^h)=\partial_\ell v(Z)\ \ \forall\,v\in S_0^h(\Omega), \tag{1.3}$$

$$a(G_Z^*,\,v)=(\delta_Z^h,\,v)\ \ \forall\,v\in H_0^1(\Omega), \tag{1.4}$$

$$a(\partial_{Z,\ell}G_Z^*,\,v)=(\partial_{Z,\ell}\delta_Z^h,\,v)\ \ \forall\,v\in H_0^1(\Omega), \tag{1.5}$$

$$a(G_Z^h,\,v)=v(Z)\ \ \forall\,v\in S_0^h(\Omega), \tag{1.6}$$

$$a(\partial_{Z,\ell}G_Z^h,\,v)=\partial_\ell v(Z)\ \ \forall\,v\in S_0^h(\Omega), \tag{1.7}$$

$$(u-P_hu,\,v)=0\ \ \forall\,v\in S_0^h(\Omega). \tag{1.8}$$

Here, for any direction $\ell\in R^3$, $|\ell|=1$, $\partial_{Z,\ell}\delta_Z^h$, $\partial_{Z,\ell}G_Z^h$, and $\partial_\ell v(Z)$ stand for the following onesided directional derivatives, respectively.

$$\partial_{Z,\ell}\delta_Z^h=\lim_{|\Delta Z|\to 0}\frac{\delta_{Z+\Delta Z}^h-\delta_Z^h}{|\Delta Z|},\ \partial_{Z,\ell}G_Z^h=\lim_{|\Delta Z|\to 0}\frac{G_{Z+\Delta Z}^h-G_Z^h}{|\Delta Z|},$$

$$\partial_\ell v(Z)=\lim_{|\Delta Z|\to 0}\frac{v(Z+\Delta Z)-v(Z)}{|\Delta Z|},\ \Delta Z=|\Delta Z|\ell.$$

As for G_Z^*, $\partial_{Z,\ell}G_Z^*$, G_Z^h, and $\partial_{Z,\ell}G_Z^h$, we have obtained some optimal estimates (see [4–6]), which will be used in next section. From (1.4)–(1.7), we easily find G_Z^h and $\partial_{Z,\ell}G_Z^h$ are the finite element approximations to G_Z^* and $\partial_{Z,\ell}G_Z^*$, respectively.

For the L^2-projection operator P_h, we have (see [4])

Lemma 1.1. *For $P_h w$ the L^2-projection of $w \in L^p(\Omega)$, we have the following stability estimate:*

$$\|P_h w\|_{0,\,p,\,\Omega} \le C^t \|w\|_{0,\,p,\,\Omega}, \tag{1.9}$$

where $t = \left|1 - \frac{2}{p}\right|$, and $1 \le p \le \infty$.

Further, by Lemma 1.1, we easily obtain the following result:

$$\begin{aligned} \|w - P_h w\|_{0,\,p,\,\Omega} &\le (1 + C^t) \inf_{v \in S_0^h \Omega} \|w - v\|_{0,\,p,\,\Omega} \\ &\le C\|w - \Pi w\|_{0,\,p,\,\Omega} \le Ch^{m+1}\|w\|_{m+1,\,p,\,\Omega}, \end{aligned} \tag{1.10}$$

where $1 \le p \le \infty$.

In addition, we also assume the following a priori estimate holds.

Lemma 1.2. *For the true solution u of (1.1), there exists a $q_0 (1 < q_0 \le \infty)$ such that for every $1 < q < q_0$,*

$$\|u\|_{2,\,q,\,\Omega} \le C(q)\|\mathcal{L}u\|_{0,\,q,\,\Omega}. \tag{1.11}$$

2 Definition of the 3D Green's Function

For $Z \in \Omega$, we introduce the definition of the 3D Green's function G_Z as follows

$$a(G_Z, v) = v(Z) \ \ \forall\, v \in C_0^\infty(\Omega).$$

In the following, we will prove the existence and uniqueness of the Green's function.

Lemma 2.1. *For G_Z^* and G_Z^h defined by (1.4) and (1.6), respectively, we have*

$$\left\|G_Z^* - G_Z^h\right\|_{1,1} \le Ch\left|\ln h\right|^{\frac{2}{3}}. \tag{2.1}$$

This result can be seen in [4].

Theorem 2.1. *There exists a unique $G_Z \in W_0^{1,1}(\Omega)$ such that*

$$a(G_Z, v) = v(Z) \ \ \forall\, v \in W_0^{1,\infty}(\Omega). \tag{2.2}$$

Proof. We first prove the uniqueness of G_Z. Suppose there exists another Green's function $H_Z \in W_0^{1,1}(\Omega)$ satisfying (2.2). Set $E_Z = G_Z - H_Z$, thus

$$a(E_Z, v) = 0 \ \ \forall\, v \in W_0^{1,\infty}(\Omega). \tag{2.3}$$

Let $w \in W^{2,4}(\Omega) \cap W_0^{1,4}(\Omega)$ and $\mathcal{L}w = \mathrm{sgn}E_Z|E_Z|^{\frac{1}{4}}$. We have

$$\|E_Z\|_{0,\frac{5}{4}}^{\frac{5}{4}} = (E_Z, \mathrm{sgn}E_Z|E_Z|^{\frac{1}{4}}) = a(E_Z, w), \tag{2.4}$$

By the Sobolev Embedding Theorem [10], $W^{2,4}(\Omega) \hookrightarrow W^{1,\infty}$. Thus $w \in W_0^{1,\infty}(\Omega)$. From (2.3) and (2.4), $E_Z = 0$, i.e., $G_Z = H_Z$. The proof of the uniqueness is completed.

Next, we prove the existence of G_Z. We give a series of finite element spaces $S_0^{h_i}(\Omega)$, $i = 0, 1, 2, \cdots$ satisfying $S_0^{h_i}(\Omega) \subset S_0^{h_j}(\Omega)$ when $i < j$, where $h_0 \equiv h$ and $\frac{1}{4}h_{i-1} \le h_i \le \frac{1}{2}h_{i-1}$. Let $G^*_{Z,i}$ be the regularized Green's function for the finite element space $S_0^{h_i}(\Omega)$, and $G_Z^{h_i}$ the discrete Green's function. Their definitions can be seen in Section 1. Obviously, we have

$$a(G_Z^{h_i}, v) = v(Z),\ a(G^*_{Z,i+1}, v) = v(Z),\ \forall\, v \in S_0^{h_i}(\Omega).$$

Thus,

$$a(G^*_{Z,i+1} - G_Z^{h_i}, v) = 0\ \ \forall\, v \in S_0^{h_i}(\Omega). \tag{2.5}$$

Similar to the proof of Lemma 2.1, we have

$$\left\| G^*_{Z,i+1} - G_Z^{h_i} \right\|_{1,1} \le C h_i \left|\ln h_i\right|^{\frac{2}{3}}. \tag{2.6}$$

In addition, from (2.1),

$$\left\| G^*_{Z,i} - G_Z^{h_i} \right\|_{1,1} \le C h_i \left|\ln h_i\right|^{\frac{2}{3}}. \tag{2.7}$$

By (2.6), (2.7), and the triangular inequality, we immediately obtain

$$\left\| G^*_{Z,i+1} - G^*_{Z,i} \right\|_{1,1} \le C h_i \left|\ln h_i\right|^{\frac{2}{3}}.$$

Thus,

$$\sum_{i=0}^{\infty} \left\| G^*_{Z,i+1} - G^*_{Z,i} \right\|_{1,1} \le C \sum_{i=0}^{\infty} \frac{h}{2^i} \left|\ln \frac{h}{2^i}\right|^{\frac{2}{3}} \le C h \left|\ln h\right|^{\frac{2}{3}}. \tag{2.8}$$

Set

$$G_Z \equiv G_Z^* + \sum_{i=0}^{\infty} (G^*_{Z,i+1} - G^*_{Z,i}).$$

Thus we have $G_Z \in W_0^{1,1}(\Omega)$. From (2.8),

$$\left\| G_Z - G_Z^* \right\|_{1,1} \le C h \left|\ln h\right|^{\frac{2}{3}}. \tag{2.9}$$

Thus, we have

$$G^*_{Z,i} \longrightarrow G_Z \ \text{in}\, W^{1,1}(\Omega)\ \text{when}\, i \to \infty.$$

Hence, for $v \in W_0^{1,\infty}(\Omega)$, we have

$$a(G_Z, v) = \lim_{i\to\infty} a(G^*_{Z,i}, v) = \lim_{i\to\infty} P_{h_i} v(Z). \tag{2.10}$$

From (1.10),

$$\lim_{i\to\infty} P_{h_i} v(Z) = v(Z). \tag{2.11}$$

Combining (2.10) and (2.11) yields the result (2.2).

Finally, we show G_Z is independent of h. Suppose there exists a Green's function $\tilde{G}_Z$ for the mesh-size $\tilde{h}$. In addition, $\frac{1}{4}\tilde{h}_{i-1} \le \tilde{h}_i \le \frac{1}{2}\tilde{h}_{i-1}$ and $\tilde{h}_0 = \tilde{h}$. Thus, for every $f \in L^\infty(\Omega)$, we choose $v \in W^{2,\infty}(\Omega) \cap W_0^{1,\infty}(\Omega)$ such that $\mathcal{L}v = f$. Then we get $(G_Z, f) = a(G_Z, v) = v(Z)$ and $(\tilde{G}_Z, f) = a(\tilde{G}_Z, v) = v(Z)$. Thus, $(G_Z, f) = (\tilde{G}_Z, f)$, i.e., $(G_Z - \tilde{G}_Z, f) = 0$. So we get $G_Z = \tilde{G}_Z$. The proof of Theorem 2.1 is completed.

3 Estimates for the 3D Green's Function

Lemma 3.1. *Suppose* $1 < p < \min\{2, q_0\}$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *For* G_Z^*, $\partial_{Z,\ell} G_Z^*$, G_Z^h, *and* $\partial_{Z,\ell} G_Z^h$ *defined by (1.4)–(1.7), we have*

$$\left\| G_Z^* - G_Z^h \right\|_{0,q} + h \left\| \partial_{Z,\ell} G_Z^* - \partial_{Z,\ell} G_Z^h \right\|_{0,q} \le C h^{2-\frac{3}{p}}. \tag{3.1}$$

Proof. Obviously, by the interpolation error estimate and the a priori estimate (1.11), we have

$$\begin{aligned} \left\| G_Z^* - G_Z^h \right\|_1 &\le C \inf_{v \in S_0^h(\Omega)} \| G_Z^* - v \|_1 \le \| G_Z^* - \Pi G_Z^* \|_1 \\ &\le C h^{2.5-\frac{3}{p}} \| G_Z^* \|_{2,p} \le C h^{2.5-\frac{3}{p}} \left\| \delta_Z^h \right\|_{0,p}. \end{aligned} \tag{3.2}$$

For $\varphi \in L^p(\Omega)$, we choose $\Phi \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ such that $\mathcal{L}\Phi = \varphi$. Then we have

$$\begin{aligned} \left| (G_Z^* - G_Z^h, \varphi) \right| &= \left| a(G_Z^* - G_Z^h, \Phi) \right| = \left| a(G_Z^* - G_Z^h, \Phi - \Pi\Phi) \right| \\ &\le C \left\| G_Z^* - G_Z^h \right\|_1 \| \Phi - \Pi\Phi \|_1 . \end{aligned} \tag{3.3}$$

From (3.2), (3.3), and the interpolation error estimate, we get

$$\left| (G_Z^* - G_Z^h, \varphi) \right| \le C h^{5-\frac{6}{p}} \left\| \delta_Z^h \right\|_{0,p} \| \varphi \|_{0,p} . \tag{3.4}$$

Thus

$$\left\| G_Z^* - G_Z^h \right\|_{0,q} \le C h^{5-\frac{6}{p}} \left\| \delta_Z^h \right\|_{0,p} . \tag{3.5}$$

In addition, for $1 \le p \le \infty$, we easily prove

$$\left\| \delta_Z^h \right\|_{0,p} + h \left\| \partial_{Z,\ell} \delta_Z^h \right\|_{0,p} \le C h^{-3+\frac{3}{p}}. \tag{3.6}$$

From (3.5) and (3.6),

$$\left\| G_Z^* - G_Z^h \right\|_{0,q} \le C h^{2-\frac{3}{p}}.$$

Similarly, we have

$$\left\| \partial_{Z,\ell} G_Z^* - \partial_{Z,\ell} G_Z^h \right\|_{0,q} \le C h^{1-\frac{3}{p}}.$$

The result (3.1) is proved. We now introduce a weight function defined by

$$\phi \equiv \phi(X) = \left(|X - \bar{X}|^2 + \theta^2 \right)^{-\frac{3}{2}} \ \ \forall\, X \in \bar{\Omega},$$

where $\bar{X} \in \bar{\Omega}$ is a fixed point, $\theta = \gamma h$, and $\gamma \in [3, +\infty)$ is a suitable real number. As for the function ϕ, it is easy to prove the following properties hold.

$$\int_\Omega \phi^k(X) dX \le C(k-1)^{-1} \theta^{-3(k-1)} \ \ \forall\, k > 1, \tag{3.7}$$

$$\int_\Omega \phi^k(X) dX \le \frac{C}{1-k} \ \ \forall\, 0 < k < 1, \tag{3.8}$$

$$\int_\Omega \phi(X)dX \le C(\beta)|\ln\theta|, \ \ \theta \le \beta < 1. \tag{3.9}$$

Similar to the arguments of Lemma 2.4 in [4], we can get the following Lemma 3.2.

Lemma 3.2. *For δ_Z^h and $\partial_{Z,\ell}\delta_Z^h$, the discrete δ function and the discrete derivative δ function defined by (1.2) and (1.3), respectively, we have the following weighted-norm estimate:*

$$\left\|\delta_Z^h\right\|_{\phi^{-\alpha}} + h\left\|\nabla\delta_Z^h\right\|_{\phi^{-\alpha}} + h\left\|\partial_{Z,\ell}\delta_Z^h\right\|_{\phi^{-\alpha}} \le Ch^{\frac{3(\alpha-1)}{2}} \quad \forall\alpha > 0. \tag{3.10}$$

Lemma 3.3. *For δ_Z^h and G_Z^*, the discrete δ function and the regularized Green's function defined by (1.2) and (1.4), respectively, we have the following weighted-norm estimate:*

$$\|\nabla G_Z^*\|_{\phi^{-\alpha}} \le C\left\|\delta_Z^h\right\|_{\phi^{-\alpha-\frac{2}{3}}} + C\|G_Z^*\|_{\phi^{-\alpha+\frac{2}{3}}} \quad \forall\alpha \in R. \tag{3.11}$$

Proof. First, we find

$$\|\nabla G_Z^*\|^2_{\phi^{-\alpha}} \le a(G_Z^*\,,\, \phi^{-\alpha}G_Z^*) + C\|G_Z^*\|^2_{\phi^{-\alpha+\frac{2}{3}}}\,. \tag{3.12}$$

Moreover,

$$\begin{aligned} a(G_Z^*\,,\,\phi^{-\alpha}G_Z^*) &= (\delta_Z^h\,,\,\phi^{-\alpha}G_Z^*) \\ &\le \|\delta_Z^h\|_{\phi^{-\alpha-\frac{2}{3}}}\|G_Z^*\|_{\phi^{-\alpha+\frac{2}{3}}} \\ &\le \tfrac{1}{2}(\|\delta_Z^h\|^2_{\phi^{-\alpha-\frac{2}{3}}} + \|G_Z^*\|^2_{\phi^{-\alpha+\frac{2}{3}}}). \end{aligned} \tag{3.13}$$

Combining (3.12) and (3.13) immediately yields the result (3.11).

Theorem 3.1. *Suppose $q_0 > \frac{3}{2}$, $\frac{3}{2} < p < \min\{2, q_0\}$, and $\frac{1}{p} + \frac{1}{q} = 1$, then we have*

$$\|G_Z - G_Z^*\|_{0,q} \le Ch^{2-\frac{3}{p}} = Ch^{\frac{3-q}{q}}. \tag{3.14}$$

Remark 1. Similar to the arguments of (2.9) and with the result (3.1), we easily obtain the result (3.14). Obviously, we have $\max\{2, q_0'\} < q < 3$ and $\frac{1}{q_0} + \frac{1}{q_0'} = 1$.

Theorem 3.2. *Suppose $q_0 > \frac{3}{2}$. For G_Z, the Green's function defined by (2.2), and the weight function $\tau = |X - Z|^{-3}$, we have*

$$\|G_Z\|_{0,q} \le C(q), \ \ 1 \le q \le 3. \tag{3.15}$$

$$\|G_Z\|_{1,\tau^{-\epsilon}} \le C(\epsilon), \ \ \frac{1}{3} < \epsilon < \infty. \tag{3.16}$$

$$\|G_Z\|_{1,q} \le C(q), \ \ 1 \le q < \frac{3}{2}. \tag{3.17}$$

Proof. Obviously, from (3.14), $G_Z \in L^q(\Omega)$ and $1 \le q < 3$. In addition, we have proved $\|G_Z^*\|_{0,3} \le C$ in [4]. Moreover, $L^3(\Omega)$ is a reflexive space. Thus, $\{G_{Z,i}^*\}$

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is weakly convergent to $Q_Z \in L^3(\Omega) \subset L^q(\Omega)$, where $\max\{2, q_0'\} < q < 3$. From (3.14),

$$G_{Z,i}^* \longrightarrow G_Z \text{ in } L^q(\Omega) \text{ when } i \to \infty.$$

Thus $G_Z = Q_Z \in L^3(\Omega)$. So we have $G_Z \in L^q(\Omega)(1 \leq q \leq 3)$.

When $\max\{2, q_0'\} < q < 3$, we have $\frac{3}{2} < p < \min\{2, q_0\}$, where $\frac{1}{p} + \frac{1}{q} = 1$. For every $\varphi \in C_0^\infty(\Omega)$, we can find a function $\tilde{\varphi} \in C_0^\infty(\Omega)$ such that $\mathcal{L}\tilde{\varphi} = \varphi$. Moreover, by the Sobolev Embedding Theorem [10] and the a priori estimate (1.11), we get

$$(G_Z, \varphi) = a(G_Z, \tilde{\varphi}) = \tilde{\varphi}(Z) \leq \|\tilde{\varphi}\|_{0,\infty} \leq C(q) \|\tilde{\varphi}\|_{2,p} \leq C(q) \|\varphi\|_{0,p}.$$

Thus,

$$\|G_Z\|_{0,q} \leq C(q).$$

Since $\left\|G_{Z,i}^*\right\|_{0,3} \leq C$, and $\{G_{Z,i}^*\}$ is weakly convergent to $G_Z \in L^3(\Omega)$, thus, $\|G_Z\|_{0,3} \leq C$. In addition, when $1 \leq q \leq \max\{2, q_0'\}$, we have $\|G_Z\|_{0,q} \leq C(q) \|G_Z\|_{0,3} \leq C(q)$. Thus we have finished the proof of the result (3.15).

Now we prove the result (3.16). We have obtained the result $\|G_Z^*\|_{\phi^{\frac{1}{3}}} \leq C |\ln h|^{\frac{1}{6}}$ in [4]. When $0 < r < \frac{1}{3}$, we have by (3.8) and $\|G_Z^*\|_{0,3} \leq C$,

$$\|G_Z^*\|_{\phi^r}^2 = \int_\Omega \phi^r |G_Z^*|^2 \, dX \leq \left(\int_\Omega \phi^{3r} \, dX\right)^{\frac{1}{3}} \|G_Z^*\|_{0,\,3}^2 \leq C(r) \|G_Z^*\|_{0,\,3}^2 \leq C(r).$$

Namely, $\|G_Z^*\|_{\phi^r} \leq C(r) \ \forall\, 0 < r < \frac{1}{3}$. Obviously, when $s < t$, we have $\phi^s \leq C\phi^t$. Thus, $\|G_Z^*\|_{\phi^r} \leq C(r) \ \forall\, r \leq 0$. So we have

$$\|G_Z^*\|_{\phi^r} \leq C(r) \ \ \forall\, r < \frac{1}{3}. \tag{3.18}$$

From (3.10) and (3.11),

$$\begin{aligned} \|\nabla G_Z^*\|_{\phi^{-\epsilon}} &\leq C \left\|\delta_Z^h\right\|_{\phi^{-\epsilon-\frac{2}{3}}} + C \|G_Z^*\|_{\phi^{-\epsilon+\frac{2}{3}}} \\ &\leq Ch^{\frac{3\epsilon-1}{2}} + C \|G_Z^*\|_{\phi^{-\epsilon+\frac{2}{3}}}. \end{aligned} \tag{3.19}$$

Combining (3.18) and (3.19) yields

$$\|G_Z^*\|_{1,\phi^{-\epsilon}} \leq C(\epsilon) \ \ \forall\, \epsilon > \frac{1}{3}. \tag{3.20}$$

By the Hölder inequality, we have for $1 \leq q < \frac{3}{2}$

$$\|\nabla G_Z^*\|_{0,q}^q = \int_\Omega \phi^{\frac{q\epsilon}{2}} \phi^{-\frac{q\epsilon}{2}} |\nabla G_Z^*|^q \, dX \leq \left(\int_\Omega \phi^{\frac{q\epsilon}{2-q}} \, dX\right)^{\frac{2-q}{2}} \|\nabla G_Z^*\|_{\phi^{-\epsilon}}^q.$$

Choosing a suitable ϵ such that $\frac{q\epsilon}{2-q} < 1$, we have by (3.8) and (3.20),

$$\|\nabla G_Z^*\|_{0,q} \leq C(q). \tag{3.21}$$

Obviously,

$$\|G_Z^*\|_{1,\tau^{-\epsilon}} \leq \|G_Z^*\|_{1,\phi^{-\epsilon}} \leq C(\epsilon) \ \forall \epsilon > \frac{1}{3}. \tag{3.22}$$

Since G_Z^* is bounded according to the weighted-norm $\|\cdot\|_{1,\tau^{-\epsilon}}$, thus, $\{G_{Z,i}^*\}$ is weakly convergent to a function F_Z with $\|F_Z\|_{1,\tau^{-\epsilon}} < \infty$. Further, we have $\|F_Z\|_{1,1,\tau^{-\epsilon}} < \infty$. From (2.9),

$$\|G_Z - G_Z^*\|_{1,1,\tau^{-\epsilon}} \leq C(\epsilon) \|G_Z - G_Z^*\|_{1,1} \leq C(\epsilon) h \, |\ln h|^{\frac{2}{3}},$$

which shows $\{G_{Z,i}^*\}$ is convergent to the function G_Z with $\|G_Z\|_{1,1,\tau^{-\epsilon}} < \infty$. Thus, $F_Z = G_Z$. Namely,

$$\|G_Z\|_{1,\tau^{-\epsilon}} \leq C(\epsilon) \ \forall \epsilon > \frac{1}{3}.$$

Up to now, the result (3.16) is thoroughly proved. Similar to the arguments of (3.16), from (3.21), we can obtain the result (3.17).

Acknowledgments This work was supported by the National Natural Science Foundation of China Grant 11161039, the Zhejiang Provincial Natural Science Foundation Grant LY13A010007 and the Natural Science Foundation of Ningbo City Grant 2015A610163.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 22, NO.6, 2017, COPYRIGHT 2017 EUDOXUS PRESS, LLC

The structure of the zeros and fixed point for Genocchi polynomials

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Abstract We find the behavior of complex roots and fixed point for Genocchi polynomials by using numerical investigation. By means of numerical experiments, we display a remarkably regular structure of the complex roots and fixed point for the Genocchi polynomials.

2000 Mathematics Subject Classification - 11B83, 37N30, 41A10

Key words- Genocchi polynomials, Newton method, complex roots, fixed point

1. Introduction

Mathematicians have studied various kinds of the Euler, Bernoulli, Tangent, and Genocchi polynomials. Recently, many authors have studied the relations between these polynomials and Stirling numbers of the second kind(see [1-24]). Numerical experiments of Bernoulli, Euler, and Genocchi polynomials also have been made the subject of extensive research.

The computing environment will be making more and more rapid advance and this environment has been increasing the interest in solving mathematical problems with the aid of computers. The zeros of Genocchi polynomials $G_n(x)$ is very interesting a realistic study by using computer(see [2,16-20,23]).

The Genocchi numbers G_n and Genocchi polynomials $G_n(x)$ are usually defined by the following generating functions.

Definition 1.1.[5,14,17] Let $n \in \mathbb{N}_0$. Then we define

$$\sum_{n=0}^{\infty} G_n \frac{t^n}{n!} = \frac{2t}{e^t+1}, \qquad |t| < \pi,$$
$$\sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!} = \left(\frac{2t}{e^t+1}\right) e^{tx},$$

where we use the notation by replacing $G(x)^n$ by $G_n(x)$ symbolically. Clearly, $G_n = G_n(0)$. In general, it satisfies $G_3 = G_5 = G_7 = G_9 = \cdots = 0$, and even coefficients are given $G_n = 2nE_{2n-1} = 2(1-2^{2n})B_{2n}$, where E_n are the Euler numbers and B_n are the Bernoulli numbers(see [4-5, 6, 8, 12, 15]).

These polynomials and numbers play important roles in many different areas of mathematics such as combinatorics, number theory, special function and analysis, and numerous interesting results for them have been explored. The following elementary properties of Genocchi polynomials $G_n(x)$ are readily derived from the Definition 1.1. Therefore we choose to omit the details involved. More studies and results in this subject we may see references(see [5-6,14-20]).

Throughout this paper, we always make use of the following notations: $\mathbb{N} = \{1, 2, 3, \cdots\}$ denotes the set of natural numbers, $\mathbb{N}_0 = \{0, 1, 2, 3, \cdots\}$ denotes the set of nonnegative integers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{C}$ denotes the set of complex numbers, and $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$.

Theorem 1.2.[5,6,17,19] For $n \in \mathbb{N}_0$, we know

$$G_n(x) = \sum_{k=0}^{n} \binom{n}{k} G_k x^{n-k}.$$

Theorem 1.3.[5,6,15] Let $x \in \mathbb{N}_0$. Then we have

$$(G+1)^n + G_n = \begin{cases} 2 & \text{if } n = 1 \\ 0 & \text{if } n \neq 1 \end{cases}.$$

From the Theorem 1.2 and Theorem 1.3, it is easy to deduce that $G_n(x)$ are polynomials of degree n. The Genocchi polynomials are as follows.

$$\begin{aligned}
&G_1(x) = 1, \\
&G_2(x) = 2x - 1, \\
&G_3(x) = 3x^2 - 3x, \\
&G_4(x) = 4x^3 - 6x^2 + 1, \\
&G_5(x) = 5x^4 - 10x^3 + 5x, \\
&G_6(x) = 6x^5 - 15x^4 + 15x^2 - 3, \\
&G_7(x) = 7x^6 - 21x^5 + 35x^3 - 21x, \\
&G_8(x) = 8x^7 - 28x^6 + 70x^4 - 84x^2 + 17, \\
&\cdots.
\end{aligned}$$

Definition 1.4. Let $f : D \to D$ be a complex function, with D a subset of $\mathbb{C}$. We define the iterated maps of the complex function as the following:

$$f_n : z_0 \mapsto \underbrace{f(f(\cdots(f}_{n-\text{times}}(z_0)\cdots))$$

The iterates of f are the functions $f, f \circ f, f \circ f \circ f, ...$, which are denoted $f^1, f^2, f^3, ...$ If $z \in \mathbb{C}$, then the orbit of z_0 under f is the sequence $(z_0, f(z_0), f(f(z_0)), \cdots)$.

We consider the Newton's dynamical system as the follows:

$$\left\{ \mathbb{C}_\infty : R(x) = x - \frac{S(x)}{S'(x)} \right\}.$$

R is called the Newton iteration function of S. It can be shown that the fixed points of R are zeros of S and all fixed points of R are attracting. R may also have one or more attracting cycles(see [2, 23-24]).

This paper is organized as follows. In Section 2, we study some properties of zeros for Genocchi polynomials from Newtons'method. In section 3, we find some distributions and properties of fixed point for Genocchi polynomials by using iterating map.

2. The observation for scattering of zeros of the Genocchi polynomials

In this section, we can see the several conjecture from the Tables. we also find the approximate zeros of the Genocchi polynomials. Using the Mathematica software, we can see the structure of the zeros of the Genocchi polynomials in various viewpoints.

From the Definition of Genocchi polynomials, we get

$$\sum_{n=0}^{\infty} G_n(1-x)\frac{(-t)^n}{n!} = \frac{-2t}{e^{-t}+1}e^{-t(1-x)} = -\frac{2t}{e^t - 1}e^{tx} = -\sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}.$$

From the above equation, we find the following theorem.

Theorem 2.1.[14,-15,17,19-20]. For $n \in \mathbb{N}_0$, we have

$$G_n(x) = (-1)^{n+1} G_n(1-x).$$

Conjecture 2.2. $G_n(x) = 0$ has n distinct solutions.

We find a counterexample of the conjecture 2.2. When $n = 6$, there exist five numbers, $x_i (i = 1, 2, 3, 4, 5)$ such that $G_6(x_i) = 0$. That is, we can find $x_1 = \frac{1}{2}, x_2 = \frac{1}{2}\left(1-\sqrt{5}\right), x_3 = \frac{1}{2}\left(1-\sqrt{5}\right), x_4 = \frac{1}{2}\left(1+\sqrt{5}\right), x_5 = \frac{1}{2}\left(1+\sqrt{5}\right)$. Therefore, the conjecture 2.3 is not true for all n. Using computers, many more values of n have been checked. It still remains unknown if the conjecture fails or holds for any value $n \neq 6$.

See Table 1 for tabulated values of $R_{G_n(x)}$ and $C_{G_n(x)}$, where $R_{G_n(x)}$ denote the numbers of real zeros and $C_{G_n(x)}$ denotes the numbers of complex zeros. Our numerical results, that is the numbers of real and complex zeros of $G_n(x)$ for $29 \leq n \leq 60$ are displayed in the Table 1.

Table 1. Numbers of real and complex zeros of $G_n(x)$

degree n	$R_{G_n(x)}$	$C_{G_n(x)}$	degree n	$R_{G_n(x)}$	$C_{G_n(x)}$
29	8	20	45	12	32
30	9	20	46	13	32
31	10	20	47	14	32
32	11	20	48	15	32
33	8	24	49	12	36
34	9	24	50	13	36
35	10	24	51	14	36
36	11	24	52	15	36
37	12	24	53	16	36
38	9	28	54	13	40
39	10	28	55	14	40
40	11	28	56	15	40
41	12	28	57	16	40
42	13	28	58	17	40
43	11	32	59	14	44
44	11	32	60	15	44

If we consider $G_n(x)$ for $2 \le n \le 100$, we then find the Figure 1. The x-axis means the numbers of real zeros and the y-axis means the numbers of complex zeros in the Genocchi polynomials in Figure 1. From Table 1 and Figure 1, we can suggest a below conjecture.

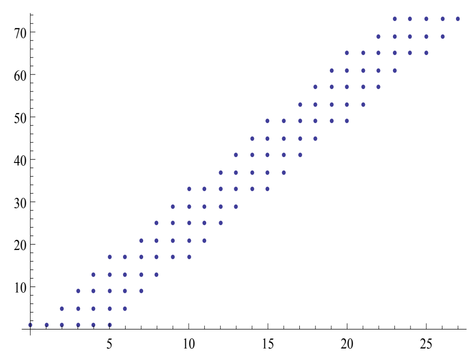

Figure 1: Numbers of real and complex zeros of $G_n(x)$ for $2 \le n \le 100$

Conjecture 2.3. When $Im(x) \neq 0$, we find that
(1) the numbers of $R_{G_n(x)}$ of $G_n(x)$:

$$R_{G_n(x)} = n - 1 - C_{G_n(x)}.$$

(2) the numbers of $C_{G_n(x)}$ of $G_n(x)$:

$$C_{G_n(x)} = 4\left[\frac{n-1-\alpha}{5}\right], \quad \alpha = \left[\frac{n+19}{21}\right],$$

4

where $[x]$ is the greatest integer not exceeding x.

By using the Theorem 2.1, we also have the following theorem.

Theorem 2.4. For $n \in \mathbb{N}_0$, if $n \equiv 0 \pmod 2$, then $G_n\left(\frac{1}{2}\right) = 0$.

By Theorem 2.4, we can know the center of the structure of zeros in Genocchi polynomials is $\frac{1}{2}$(see the Figure2). The forms of 3D structure which is stacks of zeros of $G_n(x)$ for $2 \leq n \leq 60$ are presented in the top-left of Figure 2. We can draw the top-right figure and bottom-left figure when we look at the top-left Figure 2 in the above position and left orthographic viewpoint, respectively.

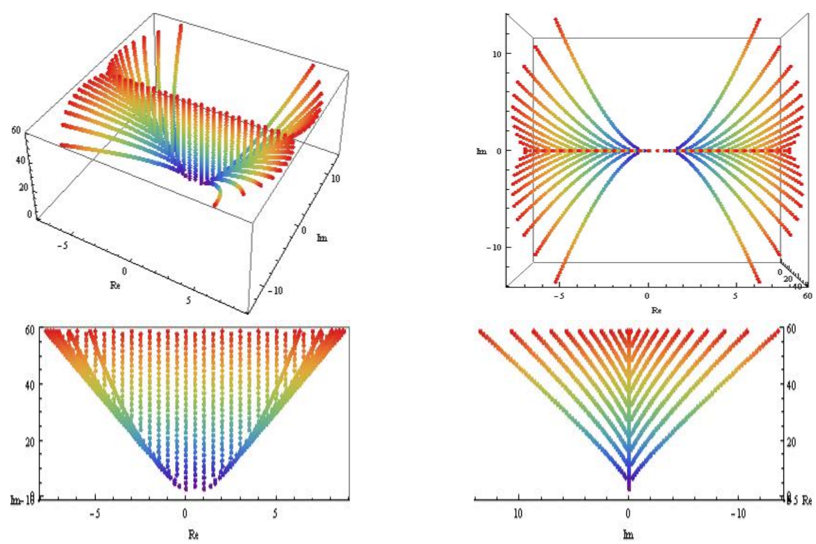

Figure 2: Stacks of zeros of $G_n(x)$ for $2 \leq n \leq 60$

From Definition of Genocchi polynomials, we get

$$\sum_{n=0}^{\infty} (G_n(x+1) + G_n(x)) \frac{t^n}{n!} = \frac{2t}{e^t+1} e^{t(x+1)} + \frac{2t}{e^t+1} e^{tx}$$
$$= 2te^{tx} = 2\sum_{n=0}^{\infty} (n+1)x^n \frac{t^n}{n!}.$$

By comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we find the theorem 2.5.

Theorem 2.5. For $n \in \mathbb{N}_0$ we find

$$G_n(x+1) + G_n(x) = 2nx^{n-1}.$$

Substituting $x = 0$ in the Theorem 2.5, we find the following corollary 2.6.

Corollary 2.6. For $n \in \mathbb{N}$, one has

$$G_n = -G_n(1).$$

We consider the Newton's dynamical system at numbers of roots in $G_{10}(x)$. We can obtain roots in the $G_{10}(x)$, that is,

$$\begin{aligned} &x_1 = -1.31362 - 0.876373i, && x_2 = -1.31362 + 0.876373i,\\ &x_3 = -1.21973, && x_4 = -0.50008,\\ &x_5 = 0.5, && x_6 = 1.50008,\\ &x_7 = 2.21973, && x_8 = 2.31362 - 0.876373i,\\ &x_9 = 2.31362 + 0.876373i. \end{aligned}$$

The orbit of x_0 from the Newton method appears by calculating until 30 iterations or the absolute difference value of the last two iterations is within 10^{-6}. We hope to determine whether the orbit of x_0 under the action of Newton's dynamical system converges to one of roots when it is given a point x_0 in the complex plane.

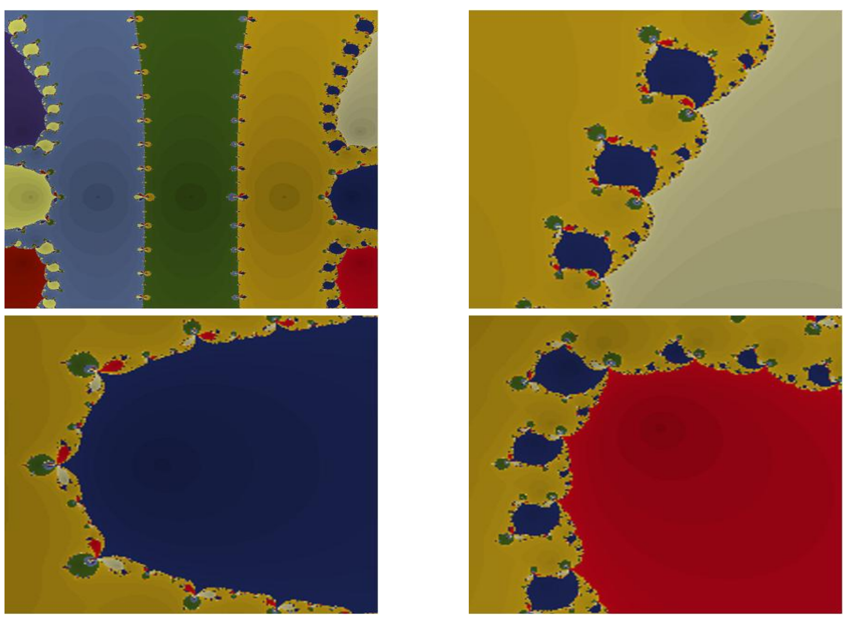

Figure 3: General structure of orbits for $\{-1.5 \leq x \leq 2.5\}, \{-1.5 \leq y \leq 2.5\}$

The output of Figure 3 is the orbit values by using the above method. We plot the blue, brown, yellow, skyblue, green, ocher, navy blue, red, or gray to x_0 in the Figure 3, when an orbit of x_0 converge to $-1.31362-0.876373i, -1.31362+0.876373i, -1.21973, -0.50008, 0.5, 1.50008, 2.21973, 2.31362-0.876373i, 2.31362+0.876373i$, respectively. From the top-left figure, we can observe general structure for $\{-1.5 \le x \le 2.5\}, \{-1.5 \le y \le 2.5\}$. Moreover, we can observe property of complex conjugate from the top-right figure and bottom-figures in the right part of general structure by narrowing range. The interesting result is the fact that each boundaries of range parts have every colors and self-similarity.

3. The fixed points of Genocchi polynomials

In this section, we present distributions of fixed points and period points from iterating map. From definition and property of fixed point, we find it and construct structure of this points in the complex plane. By expanding method of previous section we can discuss the fixed points and period points of the Genocchi polynomials.

Definition 3.1. The orbit of the point $z_0 \in \mathbb{C}$ under the action of the function f is said to be bounded if there exists $M \in \mathbb{R}$ such that $|f^n(z_0)| < M$ for all $n \in \mathbb{N}$. If the orbit is not bounded, it is said to be unbounded.

Definition 3.2. Let $f : D \to D$ be a transformation on a metric space. A point $z_0 \in D$ such that $f(z_0) = z_0$ is called a fixed point of the transformation.

Suppose that the complex function f is analytic in a region D of $\mathbb{C}$, and f has a fixed point at $z_0 \in D$. Then z_0 is said to be:
an attracting fixed point if $|f'(z_0)| < 1$;
a repelling fixed point if $|f'(z_0)| > 1$;
a neutral fixed point if $|f'(z_0)| = 1$.

For example, $G_4(x) - 1.01 - 0.1i$ have three points satisfying $G_4(x) - 1.01 - 0.1i = x$. That is, $x_0 = -0.174314+0.0695883i, 0.0220059-0.0779681i, 1.65231+0.00837978i$. Since

$$\left|\frac{d}{dz}G_4(0.0220059 - 0.0779681i) - 1.01 - 0.1i\right| = 0.953792 < 1,$$

we obtain the following theorem.

Theorem 3.3. The Genocchi polynomials $G_4(x) - 1.01 - 0.1i$ has the only one attracting fixed point at

$$\alpha = 0.0220059 - 0.0779681i.$$

We can separate the numerical results for fixed point of $G_n(x)$ by using Mathematica software. In the Table 2, we can look for numbers of fixed points of $G_n(x)$ for $3 \le n \le 10$ and find property of their points.

Table 2. Numbers of attracting, repelling, and neutral fixed points of $G_n(x)$

degree n	attractor	repellor	neutral
3	0	2	0
4	0	3	0
5	0	4	0
6	0	5	0
7	0	6	0
8	0	7	0
9	0	8	0
10	0	9	0

Conjecture 3.4. The Genocchi polynomials $G_n(x)$ has no attracting and neutral fixed point except for infinity.

In the Table 3, we consider $G_4^r(x)$ by using iterating map. We can know the numbers of real roots of $G_4^r(x)$ using iterated function are less than 3^r. In addition, we observe the numbers of real roots will be $2^{r+1}-1$ for $r \geq 1$ and find there is no the real number which is related to fixed point.

Table 3. Numbers of roots and fixed points of $G_4^r(x)$ for $1 \leq r \leq 9$

r	numbers of real roots	numbers of real numbers in fixed points
$G_4^1(x)$	3	3
$G_4^2(x)$	7	5
$G_4^3(x)$	15	15
$G_4^4(x)$	31	51
$G_4^5(x)$	63	0
$G_4^6(x)$	127	0
$G_4^7(x)$	255	0
$G_4^8(x)$	511	0
$G_4^9(x)$	1023	0
$\cdots$	$\cdots$	$\cdots$

In the top-left Figure 4, we can see the forms of 3D structure which is related to stacks of fixed points of iterated $G_4^r(x)$ for $1 \leq r \leq 6$. We can draw the top-right figure when we look at the top-left Figure 4 in the below position. The bottom-left of Figure 4 represent that image and n axes are exist but there is no real axis. The bottom-right of Figure 4 is the right orthographic viewpoint for the top-left figure, that is, there exist real and n axes but don't exist image axis.

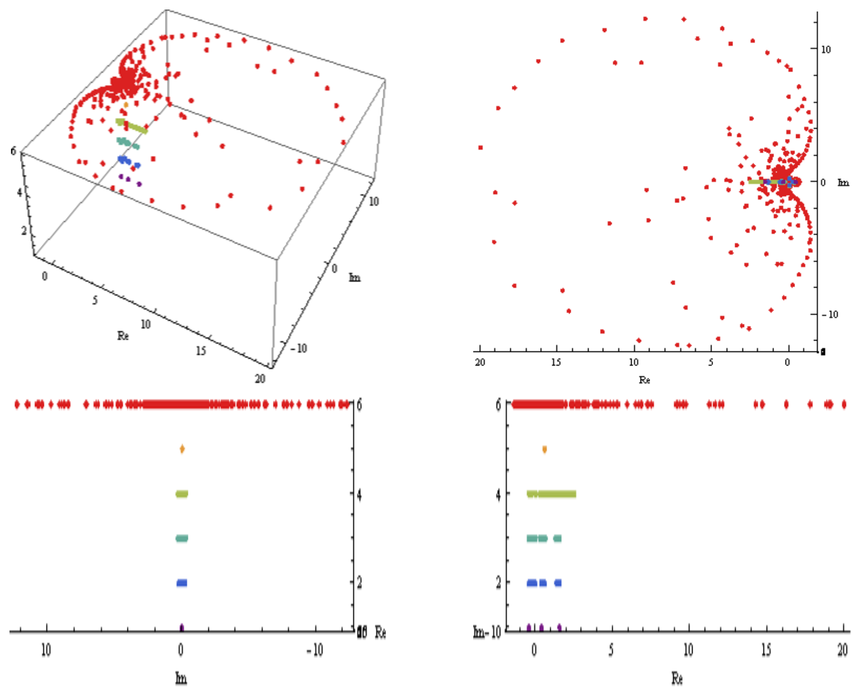

Figure 4: Stacks of fixed points of $G_4^r(x)$ for $1 \leq n \leq 6$

We consider $G_4^2(x)$ for $x \in \mathbb{C}$. This polynomial has nine distinct complex numbers, $a_i(i = 1,2,3,4,5,6,7,8,9)$ such that $G_4^2(a_i) = a_i$. We obtain $a_1 = -0.430403, a_2 = -0.244653, a_3 = -0.0322871-0.240632i, a_4 = -0.0322871+0.240632i, a_5 = 0.372949, a_6 = 0.582294, a_7 = 1.36347-0.0405081i, a_8 = 1.36347+0.0405081i, a_9 = 1.55745$. By combining Newton's method in the $G_4^2(x)$, we have

$$\left\{\mathbb{C}_\infty : \widetilde{R}(x) = x - \frac{G_4^2(x)}{(G_4^2(x))'}\right\}.$$

The general expectation is a typical orbit $\{\widetilde{R}(x)\}$ will converge to one of the fixed points of $G_4^2(x)$ for $x_0 \in \mathbb{C}$. If we choose x_0 close enough to a_i then it is readily proved that

$$\lim_{n\to\infty} \widetilde{R}(x_0) = a_i, \text{ for } i = 1,2,3,4,5,6,7,8,9.$$

Given a point x_0 in the complex plane, we want to find out if the orbit of x_0 under the action of $\widetilde{R}(x)$ does or does not converge to one of the fixed points, and if so, which one. When $\widetilde{R}(x)$ is applied to x_0, the orbit of x_0 under the action of $\widetilde{R}(x)$ is calculated until the absolute value of the last 2 iterations differs by an amount less than 10^{-6}, or until 30 iteration have been carried out.

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The Figure 5 is the last orbit value calculated. We construct a function which assigns one of nine colors to each point in the plane, according to the outcome of $\widetilde{R}$. We allocate the red, violet, yellow, skyblue, green, ocher, blue, navy blue, or gray to x_0 if its orbit converges to $-0.430403, -0.244653, -0.0322871 - 0.240632i, -0.0322871 + 0.240632i, 0.372949, 0.582294, 1.36347 - 0.0405081i, 1.36347 + 0.0405081i, 1.55745$, respectively. We make the range which is $\{(x, y) : -4 \leq x \leq 4, -4 \leq y \leq 4\}$. For example, the red region represents part of the attracting basin of $a_1 = -0.430403$

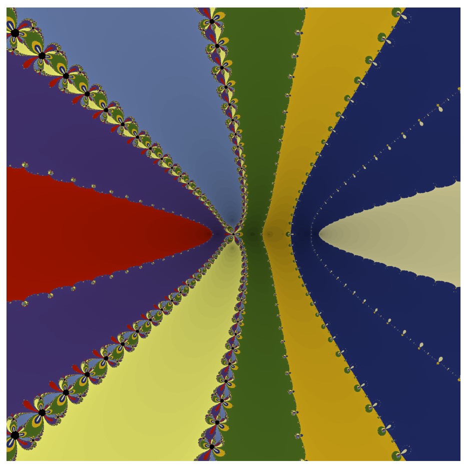

Figure 5: Orbit of x_0 under the action of $\widetilde{R}$ for $G_4^2(x)$

The Figure 6 express the coloring of the next Figure 7. Points which escape after 1 to 30 iterations are colored red to green.

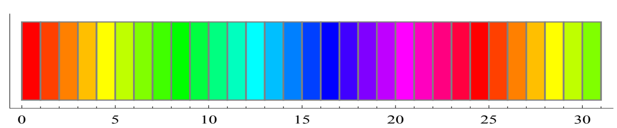

Figure 6: Palette for escaping points

In the Figure 7, the above-mentioned rapid change can be illustrated by applying the three-dimensional structure to the escape-time function. We consturct the range of left figure which is $\{(x, y) : -3 \leq x \leq 3, -3 \leq y \leq 3\}$ and the range of right figure which is $\{(x, y) : -4 \leq x \leq 4, -4 \leq y \leq 4\}$. From this figure, we can see the same color regions which are the orbit of point, z_0, approached an one of fixed points at the equivalent iterated step.

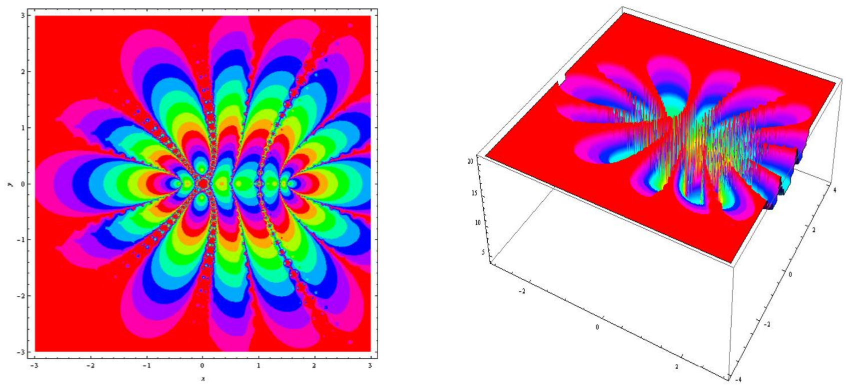

Figure 7: Escape-time map of $\widetilde{R}(x)$ for $G_4^2(x)$

Acknowledgements

This work was supported by NRF(National Research Foundation of Korea) Grant funded by the Korean Government(NRF-2013-Fostering Core Leaders of the Future Basic Science Program).

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ADDITIVE ρ-FUNCTIONAL EQUATIONS

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Abstract. In this paper, we solve the additive ρ-functional equations

$$f(x+y) - f(x) - f(y) = \rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right), \quad (0.1)$$

$$2f\left(\frac{x+y}{2}\right) - f(x) - f(y) = \rho\left(f(x+y) - f(x) - f(y)\right), \quad (0.2)$$

where ρ is a fixed non-Archimedean number or a fixed real or complex number with $\rho \neq 1$.

Using the direct method, we prove the Hyers-Ulam stability of the additive ρ-functional equations (0.1) and (0.2) in non-Archimedean Banach spaces and in Banach spaces.

1. Introduction and preliminaries

A *valuation* is a function $|\cdot|$ from a field K into $[0,\infty)$ such that 0 is the unique element having the 0 valuation, $|rs| = |r|\cdot|s|$ and the triangle inequality holds, i.e.,

$$|r+s| \le |r| + |s|, \qquad \forall r, s \in K.$$

A field K is called a *valued field* if K carries a valuation. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuations.

Let us consider a valuation which satisfies a stronger condition than the triangle inequality. If the triangle inequality is replaced by

$$|r+s| \le \max\{|r|, |s|\}, \qquad \forall r, s \in K,$$

then the function $|\cdot|$ is called a *non-Archimedean valuation*, and the field is called a *non-Archimedean field*. Clearly $|1| = |-1| = 1$ and $|n| \le 1$ for all $n \in \mathbb{N}$. A trivial example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except for 0 into 1 and $|0| = 0$.

Throughout this paper, we assume that the base field is a non-Archimedean field, hence call it simply a field.

2010 *Mathematics Subject Classification.* Primary 46S10, 39B62, 39B52, 47S10, 12J25.

Key words and phrases. Hyers-Ulam stability; additive ρ-functional equation; non-Archimedean normed space; Banach space.

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Definition 1.1. ([12]) Let X be a vector space over a field K with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\| : X \to [0, \infty)$ is said to be a *non-Archimedean norm* if it satisfies the following conditions:

(i) $\|x\| = 0$ if and only if $x = 0$;

(ii) $\|rx\| = |r|\|x\| \qquad (r \in K, x \in X)$;

(iii) the strong triangle inequality

$$\|x + y\| \le \max\{\|x\|, \|y\|\}, \qquad \forall x, y \in X$$

holds. Then $(X, \|\cdot\|)$ is called a *non-Archimedean normed space*.

Definition 1.2. (i) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X. Then the sequence $\{x_n\}$ is called *Cauchy* if for a given $\varepsilon > 0$ there is a positive integer N such that

$$\|x_n - x_m\| \le \varepsilon$$

for all $n, m \ge N$.

(ii) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X. Then the sequence $\{x_n\}$ is called *convergent* if for a given $\varepsilon > 0$ there are a positive integer N and an $x \in X$ such that

$$\|x_n - x\| \le \varepsilon$$

for all $n \ge N$. Then we call $x \in X$ a limit of the sequence $\{x_n\}$, and denote by $\lim_{n\to\infty} x_n = x$.

(iii) If every Cauchy sequence in X converges, then the non-Archimedean normed space X is called a *non-Archimedean Banach space*.

The stability problem of functional equations originated from a question of Ulam [17] concerning the stability of group homomorphisms.

The functional equation $f(x + y) = f(x) + f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [11] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [15] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [8] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. The functional equation $f\left(\frac{x+y}{2}\right) = \frac{1}{2}f(x) + \frac{1}{2}f(y)$ is called the *Jensen equation*. See [2, 3, 4, 5, 6, 7, 9, 10, 13, 14, 16, 18] for more information on functional equations.

In Section 2, we solve the additive ρ-functional equation (0.1) in vector spaces and prove the Hyers-Ulam stability of the additive ρ-functional equation (0.1) in non-Archimedean Banach spaces.

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In Section 3, we solve the additive ρ-functional equation (0.2) in vector spaces and prove the Hyers-Ulam stability of the additive ρ-functional equation (0.2) in non-Archimedean Banach spaces.

In Section 4, we prove the Hyers-Ulam stability of the additive ρ-functional equation (0.1) in Banach spaces.

In Section 5, we prove the Hyers-Ulam stability of the additive ρ-functional equation (0.2) in Banach spaces.

2. Additive ρ-functional equation (0.1) in non-Archimedean Banach spaces

Throughout Sections 2 and 3, assume that X is a non-Archimedean normed space and that Y is a non-Archimedean Banach space. Let $|2| \neq 1$ and let ρ be a fixed non-Archimedean number with $\rho \neq 1$.

We solve the additive ρ-functional equation (0.1) in vector spaces.

Lemma 2.1. *Let X and Y be vector spaces. If a mapping $f : X \to Y$ satisfies*

$$f(x+y) - f(x) - f(y) = \rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right) \tag{2.1}$$

for all $x, y \in X$, then $f : X \to Y$ is additive.

Proof. Assume that $f : X \to Y$ satisfies (2.1).

Letting $x = y = 0$ in (2.1), we get $-f(0) = 0$. So $f(0) = 0$.

Letting $y = x$ in (2.1), we get $f(2x) - 2f(x) = 0$ and so $f(2x) = 2f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{2}f(x) \tag{2.2}$$

for all $x \in X$.

It follows from (2.1) and (2.2) that

$$\begin{aligned} f(x+y) - f(x) - f(y) &= \rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right) \\ &= \rho(f(x+y) - f(x) - f(y)) \end{aligned}$$

and so

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$. □

We prove the Hyers-Ulam stability of the additive ρ-functional equation (2.1) in non-Archimedean Banach spaces.

Theorem 2.2. *Let $\varphi : X^2 \to [0,\infty)$ be a function and let $f : X \to Y$ be a mapping such that*

$$\Psi(x,y) := \sum_{j=1}^{\infty} |2|^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) \ < \ \infty, \tag{2.3}$$

$$\left\|f(x+y) - f(x) - f(y) - \rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right)\right\| \ \leq \ \varphi(x,y) \tag{2.4}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{1}{|2|}\Psi(x,x) \tag{2.5}$$

for all $x \in X$.

Proof. Letting $y = x$ in (2.4), we get

$$\|f(2x) - 2f(x)\| \leq \varphi(x,x) \tag{2.6}$$

for all $x \in X$. So

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| \leq \varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x \in X$. Hence

$$\begin{aligned} &\left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| \qquad (2.7)\\ &\leq \max\left\{\left\|2^l f\left(\frac{x}{2^l}\right) - 2^{l+1} f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, \left\|2^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\|\right\} \\ &\leq \max\left\{|2|^l \left\|f\left(\frac{x}{2^l}\right) - 2f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, |2|^{m-1}\left\|f\left(\frac{x}{2^{m-1}}\right) - 2f\left(\frac{x}{2^m}\right)\right\|\right\} \\ &\leq \sum_{j=l}^{\infty} |2|^j \varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right) \end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.7) that the sequence $\{2^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a non-Archimedean Banach space, the sequence $\{2^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.7), we get (2.5).

Now, let $T : X \to Y$ be another additive mapping satisfying (2.5). Then we have

$$\begin{aligned} \|A(x) - T(x)\| &= \left\|2^q A\left(\frac{x}{2^q}\right) - 2^q T\left(\frac{x}{2^q}\right)\right\| \\ &\leq \max\left\{\left\|2^q A\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\|, \left\|2^q T\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\|\right\} \\ &\leq |2|^{q-1}\Psi\left(\frac{x}{2^q}, \frac{x}{2^q}\right), \end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$. This proves the uniqueness of A.

It follows from (2.3) and (2.4) that

$$\begin{aligned}
&\left\|A(x+y) - A(x) - A(y) - \rho\left(2A\left(\frac{x+y}{2}\right) - A(x) - A(y)\right)\right\| \\
&= \lim_{n\to\infty} \left\|2^n\left(f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) - \rho\left(2f\left(\frac{x+y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right)\right\| \\
&\le \lim_{n\to\infty} |2|^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0
\end{aligned}$$

for all $x, y \in X$. So

$$A(x+y) - A(x) - A(y) = \rho\left(2A\left(\frac{x+y}{2}\right) - A(x) - A(y)\right)$$

for all $x, y \in X$. By Lemma 2.1, the mapping $A : X \to Y$ is additive. □

Corollary 2.3. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\left\|f(x+y) - f(x) - f(y) - \rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right)\right\| \le \theta(\|x\|^r + \|y\|^r) \quad (2.8)$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{2\theta}{|2|^r - |2|}\|x\|^r$$

for all $x \in X$.

Theorem 2.4. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying (2.4) and*

$$\Psi(x, y) := \sum_{j=0}^{\infty} \frac{1}{|2|^j}\varphi(2^j x, 2^j y) < \infty$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{1}{|2|}\Psi(x, x) \quad (2.9)$$

for all $x \in X$.

Proof. It follows from (2.6) that

$$\left\|f(x) - \frac{1}{2}f(2x)\right\| \le \frac{1}{|2|}\varphi(x, x)$$

for all $x \in X$. Hence

$$\begin{aligned}
&\left\|\frac{1}{2^l}f(2^lx)-\frac{1}{2^m}f(2^mx)\right\| \qquad (2.10)\\
&\le \max\left\{\left\|\frac{1}{2^l}f\left(2^lx\right)-\frac{1}{2^{l+1}}f\left(2^{l+1}x\right)\right\|,\cdots,\left\|\frac{1}{2^{m-1}}f\left(2^{m-1}x\right)-\frac{1}{2^m}f\left(2^mx\right)\right\|\right\}\\
&\le \max\left\{\frac{1}{|2|^l}\left\|f\left(2^lx\right)-\frac{1}{2}f\left(2^{l+1}x\right)\right\|,\cdots,\frac{1}{|2|^{m-1}}\left\|f\left(2^{m-1}x\right)-\frac{1}{2}f\left(2^mx\right)\right\|\right\}\\
&\le \sum_{j=l}^{\infty}\frac{1}{|2|^{j+1}}\varphi(2^jx,2^jx)
\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.10) that the sequence $\{\frac{1}{2^n}f(2^nx)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n}f(2^nx)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty}\frac{1}{2^n}f(2^nx)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.10), we get (2.9).

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying* (2.8). *Then there exists a unique additive mapping $h : X \to Y$ such that*

$$\|f(x)-h(x)\| \le \frac{2\theta}{|2|-|2|^r}\|x\|^r$$

for all $x \in X$.

3. Additive ρ-functional equation (0.2) in non-Archimedean Banach spaces

We solve the additive ρ-functional equation (0.2) in vector spaces.

Lemma 3.1. *Let X and Y be vector spaces. If a mapping $f : X \to Y$ satisfis $f(0) = 0$ and*

$$2f\left(\frac{x+y}{2}\right)-f(x)-f(y)=\rho\left(f(x+y)-f(x)-f(y)\right) \qquad (3.1)$$

for all $x, y \in X$, then $f : X \to Y$ is additive.

Proof. Assume that $f : X \to Y$ satisfies (3.1).

Letting $y = 0$ in (3.1), we get

$$2f\left(\frac{x}{2}\right)-f(x)=0 \qquad (3.2)$$

and so $f\left(\frac{x}{2}\right)=\frac{1}{2}f(x)$ for all $x \in X$.

It follows from (3.1) and (3.2) that

$$\begin{aligned} f(x+y)-f(x)-f(y) &= 2f\left(\frac{x+y}{2}\right)-f(x)-f(y) \\ &= \rho(f(x+y)-f(x)-f(y)) \end{aligned}$$

and so

$$f(x+y)=f(x)+f(y)$$

for all $x, y \in X$. □

Now, we prove the Hyers-Ulam stability of the additive ρ-functional equation (3.1) in non-Archimedean Banach spaces.

Theorem 3.2. *Let $\varphi : X^2 \to [0,\infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\Psi(x,y) := \sum_{j=0}^{\infty} |2|^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty,$$

$$\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y) - \rho\left(f(x+y)-f(x)-f(y)\right)\right\| \leq \varphi(x,y) \quad (3.3)$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \Psi(x,0) \quad (3.4)$$

for all $x \in X$.

Proof. Letting $y = 0$ in (3.3), we get

$$\left\|f(x)-2f\left(\frac{x}{2}\right)\right\| = \left\|2f\left(\frac{x}{2}\right)-f(x)\right\| \leq \varphi(x,0) \quad (3.5)$$

for all $x \in X$. So

$$\begin{aligned} &\left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| \quad (3.6) \\ &\leq \max\left\{\left\|2^l f\left(\frac{x}{2^l}\right) - 2^{l+1} f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, \left\|2^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\|\right\} \\ &\leq \max\left\{|2|^l \left\|f\left(\frac{x}{2^l}\right) - 2f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, |2|^{m-1} \left\|f\left(\frac{x}{2^{m-1}}\right) - 2f\left(\frac{x}{2^m}\right)\right\|\right\} \\ &\leq \sum_{j=l}^{\infty} |2|^j \varphi\left(\frac{x}{2^j}, 0\right) \end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.6) that the sequence $\{2^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a non-Archimedean Banach space, the sequence $\{2^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.6), we get (3.4).

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 3.3. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y) - \rho(f(x+y) - f(x) - f(y))\right\| \le \theta(\|x\|^r + \|y\|^r) \quad (3.7)$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{|2|^r\theta}{|2|^r - |2|}\|x\|^r$$

for all $x \in X$.

Theorem 3.4. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$, (3.3) and*

$$\Psi(x, y) := \sum_{j=1}^{\infty} \frac{1}{|2|^j}\varphi(2^j x, 2^j y) < \infty$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \Psi(x, 0) \quad (3.8)$$

for all $x \in X$.

Proof. It follows from (3.5) that

$$\left\|f(x) - \frac{1}{2}f(2x)\right\| \le \frac{1}{|2|}\varphi(2x, 0)$$

for all $x \in X$. Hence

$$\begin{aligned} &\left\|\frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x)\right\| \quad (3.9)\\ &\le \max\left\{\left\|\frac{1}{2^l}f\left(2^l x\right) - \frac{1}{2^{l+1}}f\left(2^{l+1}x\right)\right\|, \cdots, \left\|\frac{1}{2^{m-1}}f\left(2^{m-1}x\right) - \frac{1}{2^m}f\left(2^m x\right)\right\|\right\}\\ &\le \max\left\{\frac{1}{|2|^l}\left\|f\left(2^l x\right) - \frac{1}{2}f\left(2^{l+1}x\right)\right\|, \cdots, \frac{1}{|2|^{m-1}}\left\|f\left(2^{m-1}x\right) - \frac{1}{2}f\left(2^m x\right)\right\|\right\}\\ &\le \sum_{j=l+1}^{\infty} \frac{1}{|2|^j}\varphi(2^j x, 0) \end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.10) that the sequence $\{\frac{1}{2^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n}f(2^n x)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} \frac{1}{2^n}f(2^n x)$$

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for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.10), we get (3.9).

The rest of the proof is similar to the proof of Theorem 2.2. $\square$

Corollary 3.5. *Let $r > 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (3.7). Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \le \frac{|2|^r\theta}{|2| - |2|^r}\|x\|^r$$

for all $x \in X$.

4. Additive ρ-functional equation (0.1) in Banach spaces

Throughout Sections 4 and 5, assume that X is a normed space and that Y is a Banach space. Let ρ be a fixed real or complex number with $\rho \neq 1$.

We prove the Hyers-Ulam stability of the additive ρ-functional equation (2.1) in Banach spaces.

Theorem 4.1. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping such that*

$$\Psi(x, y) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty, \tag{4.1}$$

$$\left\|f(x+y) - f(x) - f(y) - \rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right)\right\| \le \varphi(x, y) \tag{4.2}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{1}{2}\Psi(x, x) \tag{4.3}$$

for all $x \in X$.

Proof. Letting $y = x$ in (4.2), we get

$$\|f(2x) - 2f(x)\| \le \varphi(x, x) \tag{4.4}$$

for all $x \in X$. So

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| \le \varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x \in X$. Hence

$$\begin{aligned} \left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| &\le \sum_{j=l}^{m-1} \left\|2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \\ &\le \sum_{j=l}^{m-1} 2^j \varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right) \end{aligned} \tag{4.5}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (4.5) that the sequence $\{2^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a Banach space, the sequence $\{2^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (4.5), we get (4.3).

Now, let $T : X \to Y$ be another additive mapping satisfying (4.3). Then we have

$$\begin{aligned} \|A(x) - T(x)\| &= \left\|2^q A\left(\frac{x}{2^q}\right) - 2^q T\left(\frac{x}{2^q}\right)\right\| \\ &\le \left\|2^q A\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\| + \left\|2^q T\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\| \\ &\le 2^q \Psi\left(\frac{x}{2^q}, \frac{x}{2^q}\right), \end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$. This proves the uniqueness of A.

It follows from (4.1) and (4.2) that

$$\begin{aligned} &\left\|A(x+y) - A(x) - A(y) - \rho\left(2A\left(\frac{x+y}{2}\right) - A(x) - A(y)\right)\right\| \\ &= \lim_{n\to\infty} \left\|2^n \left(f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) - \rho\left(2f\left(\frac{x+y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right)\right\| \\ &\le \lim_{n\to\infty} 2^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0 \end{aligned}$$

for all $x, y \in X$. So

$$A(x+y) - A(x) - A(y) = \rho\left(2A\left(\frac{x+y}{2}\right) - A(x) - A(y)\right)$$

for all $x, y \in X$. By Lemma 2.1, the mapping $A : X \to Y$ is additive. □

Corollary 4.2. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\left\|f(x+y) - f(x) - f(y) - \rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right)\right\| \le \theta(\|x\|^r + \|y\|^r) \quad (4.6)$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{2\theta}{2^r - 2}\|x\|^r$$

for all $x \in X$.

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Theorem 4.3. *Let $\varphi : X^2 \to [0,\infty)$ be a function and let $f : X \to Y$ be a mapping satisfying* (4.2) *and*

$$\Psi(x,y) := \sum_{j=0}^{\infty} \frac{1}{2^j}\varphi(2^j x, 2^j y) < \infty$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{1}{2}\Psi(x,x) \tag{4.7}$$

for all $x \in X$.

Proof. It follows from (4.4) that

$$\left\|f(x) - \frac{1}{2}f(2x)\right\| \le \frac{1}{2}\varphi(x,x)$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|\frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x)\right\| &\le \sum_{j=l}^{m-1}\left\|\frac{1}{2^j}f\left(2^j x\right) - \frac{1}{2^{j+1}}f\left(2^{j+1}x\right)\right\| \\ &\le \sum_{j=l}^{m-1}\frac{1}{2^{j+1}}\varphi(2^j x, 2^j x) \end{aligned}\tag{4.8}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (4.8) that the sequence $\{\frac{1}{2^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n}f(2^n x)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty}\frac{1}{2^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (4.8), we get (4.7).

The rest of the proof is similar to the proof of Theorem 4.1. □

Corollary 4.4. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying* (4.6). *Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \le \frac{2\theta}{2 - 2^r}\|x\|^r$$

for all $x \in X$.

5. Additive ρ-functional equation (0.2) in Banach spaces

In this section, we prove the Hyers-Ulam stability of the additive ρ-functional equation (3.1) in Banach spaces.

Theorem 5.1. *Let $\varphi : X^2 \to [0,\infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\Psi(x,y) := \sum_{j=0}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty,$$

$$\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y) - \rho\left(f(x+y) - f(x) - f(y)\right)\right\| \leq \varphi(x,y) \quad (5.1)$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \Psi(x,0) \quad (5.2)$$

for all $x \in X$.

Proof. Letting $y = 0$ in (5.1), we get

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| = \left\|2f\left(\frac{x}{2}\right) - f(x)\right\| \leq \varphi(x,0) \quad (5.3)$$

for all $x \in X$. So

$$\begin{aligned}\left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| &\leq \sum_{j=l}^{m-1} \left\|2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \\ &\leq \sum_{j=l}^{m-1} 2^j \varphi\left(\frac{x}{2^j}, 0\right) \quad (5.4)\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (5.4) that the sequence $\{2^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a Banach space, the sequence $\{2^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (5.4), we get (5.2).

The rest of the proof is similar to the proof of Theorem 4.1. □

Corollary 5.2. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y) - \rho(f(x+y) - f(x) - f(y))\right\| \leq \theta(\|x\|^r + \|y\|^r) \quad (5.5)$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{2^r\theta}{2^r - 2}\|x\|^r$$

for all $x \in X$.

Theorem 5.3. *Let $\varphi : X^2 \to [0,\infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$, (5.1) and*

$$\Psi(x,y) := \sum_{j=1}^{\infty} \frac{1}{2^j}\varphi(2^j x, 2^j y) < \infty$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \Psi(x,0) \tag{5.6}$$

for all $x \in X$.

Proof. It follows from (5.3) that

$$\left\|f(x) - \frac{1}{2}f(2x)\right\| \le \frac{1}{2}\varphi(2x,0)$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|\frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x)\right\| &\le \sum_{j=l+1}^{m}\left\|\frac{1}{2^j}f\left(2^j x\right) - \frac{1}{2^{j+1}}f\left(2^{j+1}x\right)\right\| \\ &\le \sum_{j=l+1}^{m}\frac{1}{2^j}\varphi(2^j x, 0) \end{aligned} \tag{5.7}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (5.7) that the sequence $\{\frac{1}{2^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n}f(2^n x)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty}\frac{1}{2^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (5.7), we get (5.6).

The rest of the proof is similar to the proof of Theorem 4.1. □

Corollary 5.4. *Let $r < 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (5.5). Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \le \frac{2^r\theta}{2 - 2^r}\|x\|^r$$

for all $x \in X$.

Acknowledgments

S. Y. Jang was supported by University of Ulsan, Research Program 2014.

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HYPERSTABILITY OF A GENERALIZED CAUCHY FUNCTIONAL EQUATION

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Abstract. The aim of this paper is to present some results concerning the hyperstability of the generalized Cauchy functional equation

$$f(ax+by) = Af(x) + Bf(y) + C$$

Namely, we show, under some assumptions, that a function satisfying the equation approximately must be actually a solution to it.

1. Introduction and preliminaries

Throughout the paper $\mathbb{F}$ and $\mathbb{K}$ denote the fields of real or complex numbers. Let X and Y be linear spaces over $\mathbb{F}$ and $\mathbb{K}$, respectively. In this paper we give some hyperstability results for the generalized Cauchy functional equation

$$f(ax+by) = Af(x) + Bf(y) + C \tag{1.1}$$

where $f : X \to Y$ and $a, b \in \mathbb{F}\backslash\{0\}, A, B \in \mathbb{K}, C \in Y$. In [10], Piszczek proved the hyperstability of the generalized Cauchy functional equation (1.1).

Theorem 1.1. [10] *Let X be a normed space over a field $\mathbb{F}$, Y be a Banach space over $\mathbb{K}$, $a, b \in \mathbb{F} \setminus \{0\}, A, B \in \mathbb{K}, p < 0$ and $g : X \to Y$ satisfy*

$$\|g(ax+by) - Ag(x) - Bg(y)\| \leqslant \varepsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in X \setminus \{0\}$. Then g satisfies

$$g(ax+by) = Ag(x) + Bg(y)$$

for all $x, y \in X \setminus \{0\}$.

The method of the proof used in Theorem 1.1 is based on a fixed point theorem in [3]. Let us recall that the study of stability problems of functional equations was motivated by a question of Ulam [15] asked in 1940. The first result of stability proved by Hyers [6] in 1941. For more details about various results concerning such problems the reader is referred to [4, 5, 8, 9, 11, 12, 13, 14].

It seems the first hyperstability result was published in [1] and concerned ring homomorphisms. However the term hyperstability was used for the first time in [7].

2000 *Mathematics Subject Classification.* Primary 39B82, 39B62; Secondary 47H14, 47H10.

Key words and phrases. Hyperstability, generalized Cauchy functional equation.

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2. Hyperstability results

In this part, we will prove a general version of Theorem 1.1. Let us start with a result. A version of the next result was proved in [2]. But we give another simple proof.

Proposition 2.1. *Assume that $\mathcal{X}$ and $\mathcal{Y}$ are linear spaces over $\mathbb{F}$ and $\mathbb{K}$, respectively. Let $a, b \in \mathbb{F} \setminus \{0\}$, $A, B \in \mathbb{K}$, $C \in Y$ and $f : \mathcal{X} \to \mathcal{Y}$ satisfy*

$$f(ax + by) = Af(x) + Bf(y) + C \tag{2.1}$$

for all $x, y \in \mathcal{X} \setminus \{0\}$. Then f satisfies $f(ax + by) = Af(x) + Bf(y) + C$ for all $x, y \in \mathcal{X}$.

Proof. Let $x \in \mathcal{X} \setminus \{0\}$. Then in view of (2.1), we get

$$\begin{aligned} f(0) &= Af(bx) + Bf(-ax) + C \\ &= A\Big[Af(2a^{-1}bx) + Bf(-x) + C\Big] + B\Big[Af(-2x) + Bf(ab^{-1}x) + C\Big] + C \\ &= A\Big[Af(2a^{-1}bx) + Bf(-2x) + C\Big] + B\Big[Af(-x) + Bf(ab^{-1}x) + C\Big] + C \\ &= Af(0) + Bf(0) + C. \end{aligned}$$

Therefore we have

$$f(0) = Af(bx) + Bf(-ax) + C \tag{2.2}$$

for all $x \in \mathcal{X}$. Consequently, by (2.1) and (2.2), we get

$$\begin{aligned} f(2a^2bx) &= Af(abx + b^2y) + Bf(a^2x - aby) + C \\ &= A\Big[Af(bx) + Bf(by) + C\Big] + B\Big[Af(ax) + Bf(-ay) + C\Big] + C \\ &= A\Big[Af(bx) + Bf(by) + C\Big] + B\Big[Af(ax) + f(0) - Af(by)\Big] + C \\ &= A\Big[Af(bx) + Bf(ax) + C\Big] + Bf(0) + C \\ &= Af(2abx) + Bf(0) + C \end{aligned}$$

Hence $f(2a^2bx) = Af(2abx) + Bf(0) + C$ for all $x \in \mathcal{X} \setminus \{0\}$. Replacing x by $(2ab)^{-1}x$, we infer that $f(ax) = Af(x) + Bf(0) + C$ holds for $x \in \mathcal{X}$ by (2.2). Similarly, one can prove that $f(by) = Af(0) + Bf(y) + C$ holds for $y \in \mathcal{X}$. Thus we have proved that f satisfies $f(ax + by) = Af(x) + Bf(y) + C$ for all $x, y \in \mathcal{X}$. □

In the following results we assume that X is a vector space over $\mathbb{F}$ and Y is a normed space over $\mathbb{K}$.

Theorem 2.2. *Let $a, b \in \mathbb{F} \setminus \{0\}$ and $\varphi : X \times X \to [0, +\infty)$ be a function such that*

$$\lim_{m\to\infty} \varphi(a^{-1}(m + 1)x, -b^{-1}mx) = 0, \quad \lim_{m\to\infty} \varphi(mx, my) = 0 \tag{2.3}$$

for all $x, y \in X \setminus \{0\}$. Let $A, B \in \mathbb{K}, C \in Y$ and $f : X \to Y$ satisfy

$$\|f(ax + by) - Af(x) - Bf(y) - C\| \leqslant \varphi(x, y) \tag{2.4}$$

for all $x, y \in X \setminus \{0\}$. Then f satisfies

$$f(ax + by) = Af(x) + Bf(y) + C, \tag{2.5}$$

for all $x, y \in X$. Moreover,

$$(A + B)f(0) = Af(x) + Bf(-ab^{-1}x) \tag{2.6}$$

for all $x \in X$.

Proof. Replacing x by $a^{-1}(m+1)x$ and y by $-b^{-1}mx$ in (2.4), we get

$$\left\|f(x) - Af(a^{-1}(m+1)x) - Bf(-b^{-1}mx) - C\right\| \leqslant \varphi(a^{-1}(m+1)x, -b^{-1}mx), \tag{2.7}$$

for all $x \in X \setminus \{0\}$ and positive integers m. Letting $m \to \infty$ in (2.7) and using (2.3), we obtain

$$f(x) = \lim_{m\to\infty} \Big[Af(a^{-1}(m+1)x) + Bf(-b^{-1}mx) + C\Big] \tag{2.8}$$

for all $x \in X \setminus \{0\}$. If $x \in X \setminus \{0\}$, then we get from (2.3) and (2.8)

$$\begin{aligned}
&\left\|(A+B)f(0) - Af(x) - Bf(-ab^{-1}x)\right\| \\
&= \lim_{m\to\infty} \Big\|(A+B)f(0) - A^2 f(a^{-1}(m+1)x) - ABf(-b^{-1}mx) - AC \\
&\qquad - ABf(-b^{-1}(m+1)x) - B^2 f(ab^{-2}mx) - BC\Big\| \\
&\leqslant |A| \lim_{m\to\infty} \left\|f(0) - Af(a^{-1}(m+1)x) - Bf(-b^{-1}(m+1)x) - C\right\| \\
&\quad + |B| \lim_{m\to\infty} \left\|f(0) - Af(-b^{-1}mx) - Bf(ab^{-2}mx) - C\right\| \\
&\leqslant |A| \lim_{m\to\infty} \varphi(a^{-1}(m+1)x, -b^{-1}(m+1)x) + |B| \lim_{m\to\infty} \varphi(-b^{-1}mx, ab^{-2}mx) = 0.
\end{aligned}$$

Hence we get

$$(A+B)f(0) = Af(x) + Bf(-ab^{-1}x)$$

for all $x \in X$. If we replace x by bmx and y by $-amx$ in (2.4), we get

$$\left\|f(0) - Af(bmx) - Bf(-amx) - C\right\| \leqslant \varphi(bmx, -amx), \tag{2.9}$$

for all $x \in X \setminus \{0\}$ and positive integers m. Thus

$$f(0) = \lim_{m\to\infty} \Big[Af(bmx) + Bf(-amx) + C\Big] \tag{2.10}$$

for all $x \in X \setminus \{0\}$. Replacing x by bmx in (2.9) and letting $m \to \infty$, we get from (2.10)

$$(1 - A - B)f(0) = C.$$

Therefore (2.8) holds for all $x \in X$.

To prove (2.5), let $x, y \in X \setminus \{0\}$. Then

$$\begin{aligned}
&\|f(ax+by) - Af(x) - Bf(y) - C\| \\
&= \lim_{m\to\infty} \Big\|Af(a^{-1}(m+1)(ax+by)) + Bf(-b^{-1}m(ax+by)) \\
&\qquad - A^2 f(a^{-1}(m+1)x) - ABf(-b^{-1}mx) - AC \\
&\qquad - ABf(a^{-1}(m+1)y) - B^2 f(-b^{-1}my) - BC\Big\| \\
&\leqslant |A| \lim_{m\to\infty} \left\|f(a^{-1}(m+1)(ax+by)) - Af(a^{-1}(m+1)x) - Bf(a^{-1}(m+1)y) - C\right\| \\
&\quad + |B| \lim_{m\to\infty} \left\|f(-b^{-1}m(ax+by)) - Af(-b^{-1}mx) - Bf(-b^{-1}my) - C\right\| \\
&\leqslant |A| \lim_{m\to\infty} \varphi(a^{-1}(m+1)x, -a^{-1}(m+1)y) + |B| \lim_{m\to\infty} \varphi(-b^{-1}mx, -b^{-1}my) = 0.
\end{aligned}$$

Therefore f satisfies (2.5) for all $x, y \in X \setminus \{0\}$. Hence f satisfies (2.5) for all $x, y \in X$ by Proposition 2.1. □

Remark 2.3. If f satisfies (2.4) with $A + B = 1$, then $C = 0$ and f satisfies $f(ax + by) = Af(x) + Bf(y)$ for all $x, y \in X$.

When X is a normed linear space, Theorem 1.1 is a corollary of Theorem 2.2. In the following results, we assume that X and Y are normed linear spaces.

Corollary 2.4. *Let $\varepsilon > 0$ and $p, q < 0$. If $a, b \in \mathbb{F} \setminus \{0\}, A, B \in \mathbb{K}, C \in Y$ and $f : X \to Y$ satisfies*

$$\|f(ax + by) - Af(x) - Bf(y) - C\| \leqslant \varepsilon(\|x\|^p + \|y\|^q)$$

for all $x, y \in X \setminus \{0\}$. Then f satisfies (2.5) and (2.6) for all $x, y \in X$.

Corollary 2.5. *Let $\varepsilon > 0$ and p, q be real numbers such that $p + q < 0$. If $a, b \in \mathbb{F} \setminus \{0\}, A, B \in \mathbb{K}, C \in Y$ and $f : X \to Y$ satisfies*

$$\|f(ax + by) - Af(x) - Bf(y) - C\| \leqslant \varepsilon\|x\|^p\|y\|^q$$

for all $x, y \in X \setminus \{0\}$. Then f satisfies (2.5) and (2.6) for all $x, y \in X$.

Corollary 2.6. *Let $\delta, \varepsilon \geqslant 0$, $p, q < 0$ and l, r, s be real numbers such that $l > 0$ and $r + s < 0$. If $a, b \in \mathbb{F} \setminus \{0\}, A, B \in \mathbb{K}, C \in Y$ and $f : X \to Y$ satisfies*

$$\|f(ax + by) - Af(x) - Bf(y) - C\| \leqslant \varepsilon(\|x\|^p + \|y\|^q)^l + \delta\|x\|^r\|y\|^s$$

for all $x, y \in X \setminus \{0\}$. Then f satisfies (2.5) and (2.6) for all $x, y \in X$.

Corollary 2.7. *Let $\theta, \delta, \varepsilon \geqslant 0$, $p, q < 0$ and r, s be real numbers such that $r + s < 0$. If $a, b \in \mathbb{F} \setminus \{0\}, A, B \in \mathbb{K}, C \in Y$ and $f : X \to Y$ satisfies*

$$\|f(ax + by) - Af(x) - Bf(y) - C\| \leqslant \varepsilon\|x + y\|^p + \delta\|x - y\|^q + \theta\|x\|^r\|y\|^s \tag{2.11}$$

for all $x, y \in X \setminus \{0\}$ with $x \pm y \neq 0$. Then we have

(i) if $a \neq \pm b$, then f satisfies (2.5) and (2.6) for all $x, y \in X$;

(ii) if $a = \pm b$ and $A, B \in \mathbb{K} \setminus \{0\}$, then f satisfies (2.5) for all $x, y \in X \setminus \{0\}$ with $x \pm y \neq 0$.

Proof. Let $\varphi(x, y) = \|x + y\|^p + \delta\|x - y\|^q + \theta\|x\|^r\|y\|^s$. If $a \neq \pm b$, then φ satisfies (2.3). Therefore the result follows from Theorem 2.2. If $a = \pm b$, then (2.11) implies that

$$Af(x) = \lim_{m\to\infty} \Big[f((a + bm)x) - Bf(mx) - C\Big]$$

for all $x \in X \setminus \{0\}$. Therefore

$$\begin{aligned}
&\Big\|f(ax + by) - Af(x) - Bf(y) - C\Big\| \\
&= |A|^{-1} \lim_{m\to\infty} \Big\|f((a + bm)(ax + by)) - Bf(m(ax + by)) - C \\
&\qquad - Af((a + bm)x) + ABf(mx) - Bf((a + bm)y) + B^2f(my) + BC\Big\| \\
&\leqslant |A|^{-1} \lim_{m\to\infty} \Big\|f((a + bm)(ax + by)) - Af((a + bm)x) - Bf((a + bm)y) - C\Big\| \\
&\quad + |B||A|^{-1} \lim_{m\to\infty} \Big\|f(m(ax + by)) - Af(mx) - Bf(my) - C\Big\| \\
&\leqslant |A|^{-1} \lim_{m\to\infty} \varphi((a + bm)x, (a + bm)y) + |B||A|^{-1} \lim_{m\to\infty} \varphi(mx, my) = 0.
\end{aligned}$$

Hence $f(ax + by) = Af(x) + Bf(y) + C$ for all $x, y \in X \setminus \{0\}$ with $x \pm y \neq 0$. □

In the next result we will derive from Theorem 2.2 a hyperstability result for the inhomogeneous version of the generalized Cauchy functional equation.

Theorem 2.8. *Let $a, b \in \mathbb{F} \setminus \{0\}$, $A, B \in \mathbb{K}$ and $\varphi : X \times X \to [0, +\infty)$ be a function satisfy (2.3) for all $x, y \in X \setminus \{0\}$. Assume that $d : X \times X \to Y$ and $f : X \to Y$ satisfy the inequality*

$$\|f(ax + by) - Af(x) - Bf(y) - d(x, y)\| \leqslant \varphi(x, y) \tag{2.12}$$

for all $x, y \in X \setminus \{0\}$. If the functional equation

$$g(ax + by) = Ag(x) + Bg(y) + d(x, y), \quad x, y \in X \tag{2.13}$$

has a solution $f_0 : X \to Y$, then f is a solution to (2.13).

Proof. It follows from (2.12) that $h := f - f_0$ satisfies (2.4) with $C = 0$. Consequently, Theorem 2.2 implies that h is a solution to (2.5) with $C = 0$, which means that f is a solution to (2.13). □

In the following results, we assume that $a, b \in \mathbb{F} \setminus \{0\}$, $A, B \in \mathbb{K}$, X and Y are normed linear spaces.

Corollary 2.9. *Let $\varepsilon > 0$ and $p, q < 0$. Assume that $d : X \times X \to Y$ and $f : X \to Y$ satisfy*

$$\|f(ax + by) - Af(x) - Bf(y) - d(x, y)\| \leqslant \varepsilon(\|x\|^p + \|y\|^q)$$

for all $x, y \in X \setminus \{0\}$. If the functional equation (2.13) has a solution $f_0 : X \to Y$, then f is a solution to (2.13).

Corollary 2.10. *Let $\varepsilon > 0$ and p, q be real numbers such that $p + q < 0$. Assume that $d : X \times X \to Y$ and $f : X \to Y$ satisfy*

$$\|f(ax + by) - Af(x) - Bf(y) - d(x, y)\| \leqslant \varepsilon\|x\|^p\|y\|^q$$

for all $x, y \in X \setminus \{0\}$. If the functional equation (2.13) has a solution $f_0 : X \to Y$, then f is a solution to (2.13).

Corollary 2.11. *Let $\delta, \varepsilon \geqslant 0$, $p, q < 0$ and l, r, s be real numbers such that $l > 0$ and $r + s < 0$. Assume that $d : X \times X \to Y$ and $f : X \to Y$ satisfy*

$$\|f(ax + by) - Af(x) - Bf(y) - d(x, y)\| \leqslant \varepsilon(\|x\|^p + \|y\|^q)^l + \delta\|x\|^r\|y\|^s$$

for all $x, y \in X \setminus \{0\}$. If the functional equation (2.13) has a solution $f_0 : X \to Y$, then f is a solution to (2.13).

Corollary 2.12. *Let $\theta, \delta, \varepsilon \geqslant 0$, $p, q < 0$ and r, s be real numbers such that $r + s < 0$. Assume that the functional equation (2.13) has a solution $f_0 : X \to Y$. Let $d : X \times X \to Y$ and $f : X \to Y$ satisfy*

$$\|f(ax + by) - Af(x) - Bf(y) - d(x, y)\| \leqslant \varepsilon\|x + y\|^p + \delta\|x - y\|^q + \theta\|x\|^r\|y\|^s$$

for all $x, y \in X \setminus \{0\}$ with $x \pm y \neq 0$. Then we have

- (*i*) *if $a \neq \pm b$, then f satisfies (2.13) for all $x, y \in X$;*
- (*ii*) *if $a = \pm b$ and $A, B \in \mathbb{K}\setminus\{0\}$, then f satisfies (2.13) for all $x, y \in X\setminus\{0\}$ with $x\pm y \neq 0$.*

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Stability analysis and optimal control of a cholera model with time delay

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Abstract

An optimal control method for cholera epidemic with time delay is developed in this paper. We first explore the local stability of both the disease-free and endemic equilibria of ODE model by analyzing the corresponding characteristic equations, whose global stability is established by constructing two suitable Lyapunov functionals. Furthermore, in order to, we use optimal control theory via the Pontryagin's Maximum Principle and genetic algorithm based on the forward and backward difference approximation to minimize the infected populations and the costs. Numerical simulations demonstrate that the time delay and multiple optimal controls can bring different effects on the dynamics behaviors of the proposed cholera model.

Cholera; optimal control; time delay; global asymptotical stability; Pontryagin's Maximum Principle.

1 Introduction

Cholera, a waterborne gastroenteric infection, caused by a number of types of Vibrio cholerae, remains a significant threat to public health for most of the developing countries in the past few years. Since 1961, cholera has become an acute disease throughout the world, according to the World Health Organization (WHO) report (2010), with an estimated 3-5 million cases worldwide and causes 58,000-130,000 deaths a year, children and the senior are being most affected. It was found in Congo (2008), in Iraq (2008), in Zimbabwe (2008-2009), in Vietnam (2009), in Kenya (2010), in Nigeria (2010), in Haiti (2010), in Mexico (2013), and most recently in South Sudan (2014). In the last few decades, enormous attention is being paid to the cholera disease and a number of mathematical models have been contributed to a better understanding of the transmission of cholera. In 2001, Codeço [1] put an emphasis on the decisive importance of the environmental component and proposed a $SIRB$ epidemic model in which B represents the V. cholerae concentration in water. Meanwhile, according to the laboratory results, Hartley Morris and Smith [2] in 2006 discovered a representitive hyperinfectious state of the pathogen-the explosive infectivity of freshly shed V. cholerae. Tien and Earn later [3] proposed a water-borne disease model with multiple transmission pathways, accounting both direct human-to-human and indirect water-to-human transmissions, they identified how these transmission routes influence disease dynamics. Mukandavire *et al.* [4] in 2011 simplified Hartley's model to understand transmission dynamics of cholera outbreak in Zimbabwe. Liao and Wang [5] conducted a dynamical analysis of the Hartley's model to study the stability of both the disease-free and endemic equilibria so as to explore the complex epidemic and endemic dynamics of the disease.

These epidemiological models above often take the form of a system of ordinary differential equations and ignore the time delay by assuming that the infectious process is instantaneous. However, it may make these models more biologically reasonable and mathematically challenging to consider incorporating suitable delay terms. Time delay plays an important role to reflect the real dynamical behaviors of models, many researchers have proposed and analyzed more realistic models including delays to model different mechanisms in the dynamics of epidemics. Wei *et al.* [6] considered a differential delay model of a vector-borne disease which has direct mode of transmission in addition to the vector-mediated transmission. The delay in their model accounts for the incubation time the vectors need to become infectious. They studied the effect of that delay on the stability of the equilibria and investigated that the introduction of a time delay in the host-to-vector transmission term can destabilize the system. McCluskey [7] in 2010 studied two $SIRS$ models with distributed delay and with discrete delay, respectively. They solved the global stability of the endemic equilibrium for larger delay when $R_0 > 1$. Misra *et al.* [8] in 2012 proposed a delay model to explore the dynamics of water borne diseases like cholera by using disinfectants to control the disease. Their analysis showed that under certain conditions, the cholera disease can be controlled by using disinfectants but a longer delay in their use may destabilize the system. Misra *et al.* [9] in 2013 analyzed a nonlinear delay mathematical model for the control of carrier-dependent infectious diseases, they suggested that as delay in using insecticides exceeds some critical value, the system loses its stability and Hopf-bifurcation occurs. Wang and Wei [10] investigated the global dynamics of a cholera model with delay to demonstrate the impact of the time lag.

Optimal control method [11] as a powerful tool has been applied to control various kinds of diseases [12–16]. Sunmi *et al.* [17] in 2010 studied a model for the transmission dynamics of influenza to evaluate the impact of isolation and/or antiviral drug delivery measures. They compared five control strategies to show the optimal control strategy involving antiviral treatment and/or isolation measures can reduce significantly the number of clinical cases of influenza. Ding *et al.* [18] studied the control problem of maximizing the total payoff in the conservation of a single species with a fixed amount of resource. The existence of an optimal control was established while its uniqueness and characterization was investigated as well. Okosun *et al.* [19] in 2011 derived and analyzed a deterministic model for the transmission of malaria disease with mass action form of infection. They obtained the conditions under which it is optimal to eradicate the disease and examined the impact of a possible combination of vaccination and treatment strategy on the disease transmission by using optimal control theory and the Pontryagin's Maximum Principle. Kar and Jana [20] in 2013 proposed an epidemic model and used the optimal control strategy to minimize both the infected populations and the associated costs. They compared the numerical results with no controls, with only vaccination control, with only treatment control and with both vaccination as well as treatment controls. It is observed that the best result comes out from the application of both vaccination and treatment controls in this case that the number of infected individuals would be the least in number. Wang and Modnak [21] presented a cholera epidemiological model with three control measures. Equilibrium analysis was conducted in the cases with constant controls and with optimal controls, respectively.

According to the above collection of works, an optimal control model including time delay in the context has been not completely understood yet. There are only few papers that tackle this problem. In recent years, Laarabi *et al.* [22] studied an epidemic model with optimal control strategies and time delay, the optimality system was numerically solved by using an algorithm based on the forward and backward difference approximation in their work. Mohamed *et al.* [23] investigated an optimally controlled SIR epidemic model with time delay in state and control variables, they used optimal control approach via Pontryagin's Maximum Principle to minimize the number of susceptible and infected individuals and to maximize the number of recovered individuals during the course of an epidemic.

In this paper, we will consider an optimally controlled cholera model with time delay based on the model originally suggested by Wang and Modnak [21], which involves both the environment-to-human and human-to-human transmission modes. Our main aim is to explore the role of time delay and optimal control on the spread of cholera in the model. Note most of the delay epidemic models mentioned above are only concerned with local stability of equilibria, we will pay attention to global stability of our model in this paper. The rest of the paper is organized as follows. In the next section, we formulate the mathematical model and determine the basic reproductive number R_0. Section 3 is devoted to the local and global stability analysis of both the disease-free and endemic equilibria of our model. The analysis of optimization problem is presented in Section 4. In Section 5 we present genetic algorithm based on the forward and backward difference approximation and carry out the numerical study of the model, which confirms our theoretical results. Finally, the conclusions are summarized in Section 6.

2 The model formulation

Cholera has been found in multiple transmission pathways including both direct human-to-human and indirect environment-to-human transmissions pathways, which distinct cholera from many other infectious diseases. It is important to notice that, it takes a period for the infected individual to affect the bacterial concentration of cholera, and its size may be very influential in controlling the outbreak of cholera. Thus the delay τ is used to describe the period during the person being infected to his pathogenic bacteria of V. cholera being given off to the aquatic environment. Motivated by the works of Wang and Modnak [21], the deterministic model is given by the following system of ODE:

$$\frac{dS}{dt} = \mu N - \beta_W \frac{SW}{\kappa + W} - \beta_I SI - \mu S - u_1 S, \tag{1}$$

$$\frac{dI}{dt} = \beta_W \frac{SW}{\kappa + W} - \beta_I SI - (\gamma + \mu) I - u_2 I, \tag{2}$$

$$\frac{dW}{dt} = \xi I(t-\tau) - \delta W - u_3 W, \tag{3}$$

$$\frac{dR}{dt} = \gamma I - \mu R + u_2 I + u_1 S. \tag{4}$$

In the equations above, let N be the total population which is divided into three epidemiological compartments, susceptible compartment S, infectious compartment I, recovered compartment R. Let W be the density of V. cholerae in the aquatic environment. The parameter κ is the concentration of vibrios in contaminated water in the environment,

β_W and β_I are rates of ingesting vibrios from the contaminated environment and through human-to-human interaction, respectively. μ represents the natural human birth/death rate, ξ the shedding rate, γ the recovery rate, δ the bacterial death rate. All the parameters are strictly positive constants. Intervention strategies are modeled by the control variables $u_i(t)$ $(i = 1, 2, 3)$, which are bounded, Lebesgue integrable functions. The control $u_1(t)$ represents the rate of vaccination, $u_2(t)$ represents the rate of therapeutic treatment, water sanitation leads to the death of vibrios at a rate $u_3(t)$. Based on biological assumption, we assume that for $\theta \in [-\tau, 0]$, $S(\theta)$, $I(\theta)$ and $R(\theta)$ are non negative real valued functions. Let $C = C([-\tau, 0], R^3)$ be the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into R^3 with the topology of uniform convergence. For ecological reasons, we assume that the initial conditions for system (1-4) satisfies:

$$S_0(\theta) \geq 0\,, I_0(\theta) \geq 0\,, R_0(\theta) \geq 0\,, \theta \in [-\tau, 0]. \tag{5}$$

In order to determine the dynamics of each class, we only need to study the first three equations in model (1-4), thereby reducing the order of the system through eliminating R to obtain the following system:

$$\frac{dS}{dt} = \mu N - \beta_W \frac{SW}{\kappa + W} - \beta_I SI - \mu S - u_1 S, \tag{6}$$

$$\frac{dI}{dt} = \beta_W \frac{SW}{\kappa + W} - \beta_I SI - (\gamma + \mu)I - u_2 I, \tag{7}$$

$$\frac{dW}{dt} = \xi I(t-\tau) - \delta W - u_3 W. \tag{8}$$

As the study of model system (1-4) is equivalent to study model system (6-8), so we study model system (6-8).

Based on the next-generation matrix approach [25], we define the basic reproduction number R_0, representing the average number of secondary infections that occurs when one infective is introduced into a completely susceptible host population, as:

$$R_0 = \frac{\mu N[\xi \beta_W + (\delta + u_3)\kappa \beta_I]}{\kappa(\mu + u_1)(\delta + u_3)(\gamma + \mu + u_2)}. \tag{9}$$

3 Mathematical analysis of the epidemic model

In particular, when the time delay is set to zero, i.e. $\tau = 0$, the above system (6-8) is reduced to the original model developed in Wang and Modnak [21]. Based on their work, the results below directly follows:

Theorem 1 *The disease-free equilibrium (DFE) of the model (6-8)* $E_0 = \left(\frac{\mu N}{\mu + u_1}\,,\ 0\,,\ 0\,,\ 0\right)^T$, *is both locally and globally asymptotically stable if* $R_0 < 1$ *with* $\tau = 0$.

Theorem 2 *The endemic equilibrium of the model (6-8)* $E^* = (S^*, I^*, W^*)$ *is locally asymptotically stable and globally asymptotically stable if* $R_0 > 1$ *with* $\tau = 0$.

3.1 The stability of the disease-free equilibrium

Our primary focus is on the stability analysis of the model when $\tau \neq 0$ in this section. First, we prove the local and global stability of the disease-free equilibrium E_0 with $\tau > 0$.

Theorem 3 *The disease-free equilibrium (DFE) of the model (6-8) is locally asymptotically stable if $R_0 < 1$ with $\tau > 0$.*

Proof After linearizing the ODE system (6-8) around the disease-free equilibrium E_0, we obtain one negative characteristic solution $\lambda = -\mu - u_1$ and the following transcendental characteristic equation is:

$$\lambda^2 + a_1\lambda + a_2 + b_1 e^{-\lambda\tau} = 0, \tag{10}$$

where

$$\begin{aligned} a_1 &= \delta + \gamma + \mu + u_2 + u_3 - \beta_I \frac{\mu N}{\mu + u_1}, \\ a_2 &= (\delta + u_3)(\gamma + \mu + u_2 - \beta_I \frac{\mu N}{\mu + u_1}), \\ b_1 &= -\frac{\xi\beta_W}{\kappa}\frac{\mu N}{\mu + u_1}. \end{aligned}$$

We can rearrange equation (10) in the form:

$$\begin{aligned} \lambda^2 + a_1\lambda &= (\delta + u_3)(\gamma + \mu + u_2)[(\frac{\mu N \kappa \beta_I}{\kappa(\mu + u_1)(\gamma + \mu + u_2)} - 1) \\ &\quad + \frac{\mu N \xi \beta_W}{\kappa(\mu + u_1)(\delta + u_3)(\gamma + \mu + u_2)} e^{-\lambda\tau}]. \end{aligned} \tag{11}$$

Let the left-hand side and right-hand side of equation (11) be $F(\lambda)$ and $H(\lambda)$, respectively. It is easy to see that $F(0) = 0$ and $lim_{\lambda\to\infty}F(\lambda) = \infty$, therefore, $F(\lambda)$ is an increasing function of λ. On the other hand, $H(\lambda)$ is a decreasing function of λ and $H(0) = (\delta + u_3)(\gamma + \mu + u_2)(R_0 - 1)$ is less than zero when $R_0 < 1$. Thus, equation (11) has no non-negative real roots. If equation (10) has roots with non-negative real parts, they must be complex and obtained from a pair of complex conjugate roots which cross the imaginary axis. As a result, a pair of purely imaginary solution may come out from the equation (10) for $\tau > 0$. Assume that $i\omega$ ($\omega > 0$) is the root of equation (10) and ω satisfies the following equation:

$$-\omega^2 + a_1 i\omega + a_2 + b_1(cos(\omega\tau) - isin(\omega\tau)) = 0. \tag{12}$$

Separating the real and imaginary parts of equation (12) gives

$$-\omega^2 + a_2 = -b_1 cos(\omega\tau), \qquad -a_1\omega = -b_1 sin(\omega\tau). \tag{13}$$

To eliminate the trigonometric functions, we add up the squares of equation (13) above, and obtain the following forth order equation in ω:

$$\omega^4 + (a_1^2 - 2a_2)\omega^2 + a_2^2 - b_1^2 = 0. \tag{14}$$

We can solve that

$$\omega^2 \;=\; \frac{1}{2}[-(a_1^2-2a_2)\pm\sqrt{(a_1^2-2a_2)^2-4(a_2^2-b_2^2)}]. \tag{15}$$

This implies equation (14) has no positive roots, which leads to the conclusion that there is no ω such that $i\omega$ is a solution of equation (10) for time delay $\tau > 0$. Based on Rouche's theorem [26], E_0 is locally asymptotically stable if $R_0 < 1$. ■ Next, we will analyze the global stability of the disease-free equilibrium of the model system (6-8) for time delay $\tau > 0$.

Theorem 4 *The disease-free equilibrium (DFE) of the model (6-8) is globally asymptotically stable with time delay $\tau > 0$ if $R_0 < 1$.*

Proof

Adding equations (1) and (2), we obtain

$$S^{'}+I^{'}=\mu N-(\mu+u_1)S-(\gamma+\mu+u_2)I\le \mu N-\eta(S+I), \tag{16}$$

and equation (3) yields

$$W^{'}=\xi I(t-\tau)-(\delta+u_3)W\le \xi\frac{\mu N}{\eta}-(\delta+u_3)W, \tag{17}$$

where $\eta=\min\{(\mu+u_1),(\gamma+\mu+u_2)\}$. These imply

$$\limsup_{t\to\infty} I(t) \;\le\; \frac{\mu N}{\eta}. \tag{18}$$

and

$$\limsup_{t\to\infty} W(t) \;\le\; \frac{\xi\mu N}{\eta(\delta+u_3)}. \tag{19}$$

We consider the following Lyapunov function:

$$V_1(t) \;=\; \xi[S(t)-\frac{\mu N}{\mu+u_1}ln\frac{S(t)}{\frac{\mu N}{\mu+u_1}}]+\xi I_t(0)+(\gamma+\mu+u_2)W(t)+\xi(\gamma+\mu+u_2)\int_{-\tau}^{0} I_t(\theta)d\theta. \tag{20}$$

Here, $I_t(\theta) = I(t+\theta)$ for $\theta \in [-\tau, 0]$, therefore, $I_t(0) = I(t)$ in this equation (20). Calculating the time derivative of $V_1(t)$ along solutions of system (6-8),

$$\begin{aligned}
\frac{dV_1(t)}{dt} &= \xi(S'(t) - \frac{\mu N}{\mu+u_1}\frac{S'(t)}{S(t)}) + \xi I'(t) + (\gamma+\mu+u_2)W'(t) + \xi(\gamma+\mu+u_2)[\int_{t-\tau}^{t} I(t)dS]' \\
&= \xi[\mu N - \beta_W \frac{S(t)W(t)}{\kappa+W(t)} - \beta_I S(t)I(t) - (\mu+u_1)S(t) \\
&\quad + \frac{\mu N}{\mu+u_1}(\frac{\beta_W W(t)}{\kappa+W(t)} + \beta_I I(t) + \mu + u_1 - \frac{\mu N}{S(t)})] + \xi\beta_W \frac{S(t)W(t)}{\kappa+W(t)} \\
&\quad + \xi\beta_I S(t)I(t) - \xi(\gamma+\mu+u_2)I(t) + (\gamma+\mu+u_2)\xi I(t-\tau) \\
&\quad - (\gamma+\mu+u_2)(\delta+u_3)W(t) + \xi(\gamma+\mu+u_2)I(t) - (\gamma+\mu+u_2)\xi I(t-\tau) \\
&= 2\xi\mu N - \xi(\mu+u_1)S(t) + \frac{\xi\mu N}{\mu+u_1}(\frac{\beta_W W(t)}{\kappa+W(t)} + \beta_I I(t) - \frac{\mu N}{S(t)}) \\
&\quad - (\gamma+\mu+u_2)(\delta+u_3)W(t) \\
&= \xi\mu N(2 - \frac{\mu N}{\mu+u_1}\frac{1}{S(t)} - \frac{\mu+u_1}{\mu N}S(t)) + [\frac{\xi\mu N}{\mu+u_1}(\frac{\beta_W W(t)}{\kappa+W(t)} + \beta_I I(t)) \\
&\quad - (\gamma+\mu+u_2)(\delta+u_3)W(t)].
\end{aligned} \tag{21}$$

Obviously, $2 - \frac{\mu N}{\mu+u_1}\frac{1}{S(t)} - \frac{\mu+u_1}{\mu N}S(t) \le 0$, thus, $\frac{dV_1(t)}{dt} = 0$ if and only if $S = \frac{\mu N}{\mu+u_1}$. In addition, if $R_0 < 1$, it is sufficient to verify that the second term of equation (21) is less than 0 by combining equations (18) and (19). Therefore, $\frac{dV_1(t)}{dt} \le 0$. This completes the proof. ■

3.2 The stability of the endemic equilibrium

To study the stability of the endemic equilibrium $E^*(S^*, I^*, W^*)$, we linearize the system (6-8) at the point E^* by Letting $S = S^* + s$, $I = I^* + i$, $W = W^* + w$, here s, i and w are small perturbations around the equilibrium E^*. To make the algebraic manipulation simpler, we set $P^* = \frac{\beta_W W^*}{\kappa+W^*} + \beta_I I^*$. When $\tau > 0$, the characteristic polynomial for linearized equation is obtained as:

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 + (b_1\lambda + b_2)e^{-\lambda\tau} = 0, \tag{22}$$

where

$$\begin{aligned}
a_1 &= -\beta_I S^* + P^* + \gamma + 2\mu + \delta + u_1 + u_2 + u_3, \\
a_2 &= (P^* + \mu + u_1)(-\beta_I S^* + \gamma + \mu + u_2) + P^* S^* \beta_I + (\delta + u_3)\times \\
&\quad (-\beta_I S^* + P^* + \gamma + 2\mu + u_1 + u_2), \\
a_3 &= (\delta + u_3)(P^* + \mu + u_1)(-\beta_I S^* + \gamma + \mu + u_2) + \beta_I(\delta + u_3)P^* S^*, \\
b_1 &= -\xi\beta_W S^* \frac{\kappa}{(\kappa + W^*)^2}, \\
b_2 &= -\xi(\mu + u_1)\beta_W S^* \frac{\kappa}{(\kappa + W^*)^2}.
\end{aligned}$$

Now we suppose λ is a root of equation (22), and substitute $\lambda = i\omega$ ($\omega > 0$) into equation (22), after separating real and imaginary parts, we finally obtain the following two transcendental equations:

$$-a_1\omega^2 + a_3 = -b_2cos(\omega\tau) - b_1\omega sin(\omega\tau), \quad (23)$$

$$-\omega^3 + a_2\omega = -b_1\omega cos(\omega\tau) + b_2 sin(\omega\tau). \quad (24)$$

By adding up the squares of both the equations (23) and (24), the following sixth degree equation for ω is obtained:

$$\omega^6 + \omega^4(a_1^2 - 2a_2) + \omega^2(a_2^2 - 2a_1a_3 - b_1^2) + a_3^2 - b_2^2 = 0. \quad (25)$$

Letting $\omega^2 = x$ gives:

$$F(x) = x^3 + B_1x^2 + B_2x + B_3 = 0, \quad (26)$$

where

$$B_1 = a_1^2 - 2a_2\,, B_2 = a_2^2 - 2a_1a_3 - b_1^2\,, B_3 = a_3^2 - b_2^2.$$

Here, we establish the following theorem.

Theorem 5 *When $R_0 > 1$, the endemic equilibrium E^* of ODE system (6-8) is locally asymptotically stable for the delay $\tau > 0$ if $B_1 \geq 0$, $B_3 \geq 0$ and $B_2 > 0$.*

Proof In order to show that the endemic equilibrium E^* is locally stable, we have to show that equation (26) does not have a positive real root. In fact, if we take the derivative of $F(x)$ with respect to x, $F'(x) = 3x^2 + 2B_1x + B_2$. The roots of equation $F'(x) = 0$ can be solved as $x_{1,2} = \frac{-B_1 \pm \sqrt{B_1^2 - 3B_2}}{3}$. If $B_2 > 0$, then $\sqrt{B_1^2 - 3B_2} < B_1$. Hence, neither x_1 nor x_2 is positive, it follows that equation $F'(x) = 0$ has no positive roots. Also, a simple assumption that $F(0) = B_3 \geq 0$, implies that equation (26) will have no positive real roots. Therefore, there is no ω such that $i\omega$ is an eigenvalue of the characteristic equation (22). By Rouch's theorem [26], the real parts of all the eigenvalues of (22) are negative for time delay $\tau \geq 0$. This completes the proof. ■

Next, we turn our attention to the global stability of the ODE system (6-8) if $R_0 > 1$ for all values of the delay $\tau > 0$.

Theorem 6 *When $R_0 > 1$, the positive endemic equilibrium E^* of ODE system (6-8) is globally asymptotically stable for all delay $\tau > 0$.*

Proof We consider the following Lyapunov function:

$$\begin{aligned} V_2(t) = {} & S^*(\frac{S(t)}{S^*} - 1 - ln\frac{S(t)}{S^*}) + I^*(\frac{I_t(0)}{I^*} - 1 - ln\frac{I_t(0)}{I^*}) + \frac{\gamma + \mu + u_2}{\xi}W^* \times \\ & (\frac{W(t)}{W^*} - 1 - ln\frac{W(t)}{W^*}) + (\gamma + \mu + u_2)I^*\int_{-\tau}^{0}(\frac{I_t(s)}{I^*} - 1 - ln\frac{I_t(s)}{I^*})ds. \end{aligned} \quad (27)$$

Differentiating $V_2(t)$ along solutions of (6-8), we can obtain:

$$\begin{aligned}
\frac{dV_2(t)}{dt} &= \mu N - \mu S(t) - u_1 S(t) - S^* \frac{\mu N}{S(t)} + S^* P + 2\mu S^* + 2u_1 S^* - \frac{\beta_W S^* S(t) W(t)}{\kappa + W(t)} \\
&\quad -\beta_I S(t) I^* + 2(\gamma + \mu + u_2) I^* - \frac{(\gamma + \mu + u_2)(\delta + u_3) W(t)}{\xi} \\
&\quad -\frac{(\gamma + \mu + u_2) W^* I(t-\tau)}{W(t)} + \frac{(\gamma + \mu + u_2)(\delta + u_3) W^*}{\xi} \\
&\quad +(\gamma + \mu + u_2) I^* (ln\frac{I(t-\tau)}{I^*} - ln\frac{I(t)}{I^*}) \\
&= \mu S^*(2 - \frac{S(t)}{S^*} - \frac{S^*}{S(t)}) + u_1 S^*(2 - \frac{S^*}{S(t)} - \frac{S(t)}{S^*}) + (\gamma + \mu + u_2) I^* \times \\
&\quad [(\frac{P(t)}{P^*} - 1)(1 - \frac{P^*}{P(t)}\frac{W(t)}{W^*})] - (\gamma + \mu + u_2) I^* (\frac{S^*}{S(t)} - 1 - ln\frac{S^*}{S(t)}) \\
&\quad -(\gamma + \mu + u_2) I^* [\frac{P(t)}{P^*}\frac{I^*}{S^*}\frac{S(t)}{I(t)} - 1 - ln(\frac{P(t)}{P^*}\frac{I^*}{S^*}\frac{S(t)}{I(t)})] \\
&\quad -(\gamma + \mu + u_2) I^* [\frac{W^*}{W(t)}\frac{I(t-\tau)}{I^*} - 1 - ln(\frac{W^*}{W(t)}\frac{I(t-\tau)}{I^*})]. \qquad (28)
\end{aligned}$$

Clearly, $2-\frac{S(t)}{S^*}-\frac{S^*}{S(t)} \leq 0$ for $S(t) > 0$. Furthermore, note that at the endemic equilibrium E^*, the right-hand side of equation (8) becomes 0, which yields $\xi I^* = (\delta + u_3)W^*$, and combine the facts (18) and (19), we can get $(\frac{P(t)}{P^*} - 1)(1 - \frac{P^*}{P(t)}\frac{W(t)}{W^*}) < 0$ if $R_0 > 1$. Also, for all $t \geq 0$, the function $g(t) = t - 1 - lnt$ is always non-negative, and $g(t) = 0$ if and only if $t = 1$, then the fourth term, the fifth term and the last term in (28) are non-negative. Therefore, we can finally show $\frac{dV_2(t)}{dt} \leq 0$. This completes the proof. ■

4 Optimal control analysis

In this section, we seek to minimize the objective functional defined by decreasing the number of infected and the costs of time-related controls,the method is described in [28]. We choose a linear function for the cost on infection I, and quadratic forms for the cost on the controls u_1, u_2 and u_3. The objective function subject to the differential equations (1-4) is constructed as follows:

$$J = \int_0^{t_f} (A_0 I + A_1 u_1^2 + A_2 u_2^2 + A_3 u_3^2) dt.$$

We assume t_f is the fixed final time, the parameters A_0, A_1, A_2 and A_3 are weight parameters describing the comparative importance of the all terms on control cost. The optimal control problem is that of finding optimal functions u_1^*, u_2^* and u_3^* such that

$$J(u_1^*, u_2^*, u_3^*) = \min_{u_1, u_2, u_3 \in \Theta} J(u_1, u_2, u_3), \qquad (29)$$

where Θ is measurable on $[0, 1]$ and $\Theta = \{u_i | 0 \leq u_i \leq 1\}$ for the controls.

The Lagrangian of this object is given by

$$L(I, u_1, u_2, u_3) \;=\; A_0 I + A_1 u_1^2 + A_2 u_2^2 + A_3 u_3^2, \tag{30}$$

and the Hamiltonian H for the control problem is:

$$H(S, I, W, R, u_1, u_2, u_3, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = L + \lambda_1(t)\frac{dS}{dt} + \lambda_2(t)\frac{dI}{dt} + \lambda_3(t)\frac{dW}{dt} + \lambda_4(t)\frac{dR}{dt}, \tag{31}$$

where $\lambda_i(t)$ for $i = 1, 2, 3, 4$ are the adjoint variables, which determine the adjoint system, and can be solved by the following system:

$$\begin{aligned}
\dot{\lambda_1}(t) &= -\frac{\partial H}{\partial S} - \chi_{[0,t_f-\tau]}\frac{\partial H}{\partial S_\tau}(t+\tau) \\
&= \lambda_1(\frac{\beta_W W}{\kappa + W} + \beta_I + \mu + u_1) - \lambda_2(\frac{\beta_W W}{\kappa + W} + \beta_I) - \lambda_4\mu, && (32) \\
\dot{\lambda_2}(t) &= -\frac{\partial H}{\partial I} - \chi_{[0,t_f-\tau]}\frac{\partial H}{\partial I_\tau}(t+\tau) \\
&= -A_0 + \lambda_1\beta_I S - \lambda_2[\beta_I S - (\gamma + \mu + u_2)] - \lambda_4(\gamma + u_2) - \lambda_2(t+h)\xi, && (33) \\
\dot{\lambda_3}(t) &= -\frac{\partial H}{\partial W} - \chi_{[0,t_f-\tau]}\frac{\partial H}{\partial W_\tau}(t+\tau) \\
&= \lambda_1\frac{\beta_W S\kappa}{(\kappa + W)^2} - \lambda_2\frac{\beta_W S\kappa}{(K + W)^2} + \lambda_3(\delta + u_3), && (34) \\
\dot{\lambda_4}(t) &= -\frac{\partial H}{\partial R} - \chi_{[0,t_f-\tau]}\frac{\partial H}{\partial R_\tau}(t+\tau) \\
&= \lambda_4\mu. && (35)
\end{aligned}$$

Satisfying the transversality conditions:

$$\lambda_i(t_f) = 0, \qquad\qquad i = 1, 2, 3, 4. \tag{36}$$

The combination of the ODE system (1-4) and the state system (32-35) is the optimality system, which describes how the system behaves minimize J under the control applications. By applying Pontryagin's Maximum theory and the existence result for the optimal control [27], we thus establish the following theorem:

Theorem 7 *There is a triplet of optimal control* (u_1^*, u_2^*, u_3^*) *such that* $J(u_1^*, u_2^*, u_3^*) = \min_{u_1,u_2,u_3\in\Theta} J(u_1, u_2, u_3)$ *subject to the optimality control system.*

Theorem 8 *There is a triplet of optimal control* (u_1^*, u_2^*, u_3^*) *which minimizes* J *over the region* Θ *given by*

$$u_1^* = \min\{\max\{0, u_1\}, 1\}\,,\ u_2^* = \min\{\max\{0, u_2\}, 1\}\,,\ u_3^* = \min\{\max\{0, u_3\}, 1\}, \tag{37}$$

where

$$u_1 = \frac{(\lambda_1(t) - \lambda_4(t))S^*}{2A_1}\,,\ u_2 = \frac{(\lambda_2(t) - \lambda_4(t))I^*}{2A_2}\,,\ u_3 = \frac{\lambda_3(t)W^*}{2A_3}. \tag{38}$$

Proof The optimal controls u_1^*, u_2^* and u_3^* can be solved by setting the partial derivatives of H equal to zero,

$$\frac{\partial H}{\partial u_1} = 2A_1u_1 - \lambda_1(t)S^* + \lambda_4(t)S^* = 0, \tag{39}$$

$$\frac{\partial H}{\partial u_2} = 2A_2u_2 - \lambda_2(t)I^* + \lambda_4(t)I^* = 0, \tag{40}$$

$$\frac{\partial H}{\partial u_3} = 2A_3u_3 - \lambda_3(t)W^* = 0. \tag{41}$$

After a simple manipulation, the optimal control pair (u_1^*, u_2^*, u_3^*) is characterized as (37) and (38). ■

By standard control arguments involving the bounds on the controls, we conclude

$$u_1^* = \begin{cases} \frac{(\lambda_1(t)-\lambda_4(t))S^*}{2A_1} & \text{if } 0 < \frac{(\lambda_1(t)-\lambda_4(t))S^*}{2A_1} < 1, \\ 0 & \text{if } \frac{(\lambda_1(t)-\lambda_4(t))S^*}{2A_1} \le 0, \\ 1 & \text{if } \frac{(\lambda_1(t)-\lambda_4(t))S^*}{2A_1} \ge 1. \end{cases}$$

$$u_2^* = \begin{cases} \frac{(\lambda_2(t)-\lambda_4(t))I^*}{2A_2} & \text{if } 0 < \frac{(\lambda_2(t)-\lambda_4(t))I^*}{2A_2} < 1, \\ 0 & \text{if } \frac{(\lambda_2(t)-\lambda_4(t))I^*}{2A_2} \le 0, \\ 1 & \text{if } \frac{(\lambda_2(t)-\lambda_4(t))I^*}{2A_2} \ge 1 \end{cases}$$

$$u_3^* = \begin{cases} \frac{\lambda_3(t)W^*}{2A_3} & \text{if } 0 < \frac{\lambda_3(t)W^*}{2A_3} < 1, \\ 0 & \text{if } \frac{\lambda_3(t)W^*}{2A_3} \le 0, \\ 1 & \text{if } \frac{\lambda_3(t)W^*}{2A_3} \ge 1. \end{cases}$$

5 Numerical results

In this section, we work out the optimality system which is combined by the ODE system (1-4) and the adjoint system (32-35) by using the data regarding the course of the cholera in Zimbabwe (2008-2009). It began in August 2008, not only swept to all of Zimbabwe's ten provinces but also spread to Botswana, Mozambique, South Africa and Zambia quickly. The principal cause of the outbreak was the collapse of Zimbabwe's public health system. By the end of November 2008, three of Zimbabwe's four major hospitals had shut down, and many places had no basic drugs, medicines and water supply for such a long enough period during the outbreak period. On 4 December 2008, the Zimbabwe government declared the outbreak to be a national emergency. By March 2009, the World Health Organization (WHO) estimated that 4,011 people had succumbed to this waterborne disease and 91,164 cases were infected. The total population in Zimbabwe is 12,347,240, in order to make the calculation simpler, we scale down all data numbers by a factor of 1,200. All epidemiological parameter values for cholera in literature are given as $N = 10000$, $\mu = 0.000442$, $\gamma = 1.4$, $\xi = 70$, $\delta = 0.023$, $\beta_W = 0.12$, $\beta_I = 0.00075$. We use the initial values as $S_0 = 9999$, $I_0 = 1$, $W_0 = 0$, $R_0 = 0$. The weight constants are set as $A_0 = A_1 = A_2 = A_3 = 10$.

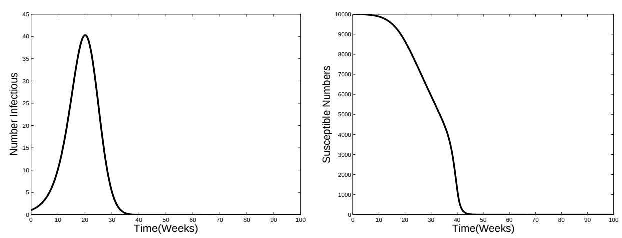
We note that the optimality system is a two-point boundary value problem, with separated boundary conditions at initial time $t = 0$ and final time $t = t_f$. Solving this optimality system requires an iterative scheme which is combination of forward and backward difference approximation developed by [22, 24], we show this procedure in the following algorithm. In the programming, let there exist a uniform step size $h > 0$ and $(n, m) \in N^2$, $\tau = mh$ and $t_f = nh$. We can obtain the following partition by setting m knots to left of 0 and right of t_f.

$$\Delta = (t_{-m} = -\tau < \cdots < t_{-1} < 0 < t_1 < \cdots < t_n = t_f < \cdots < t_{n+m}).$$

Therefore, $t_i = ih(-m \leq i \leq n+m)$. The state and adjoint variables and control variables, such as $S(t)$, $I(t)$, $W(t)$, $R(t)$, λ_i and u_i in terms of nodal points S_i, I_i, W_i, R_i, λ_i^i and u_i.

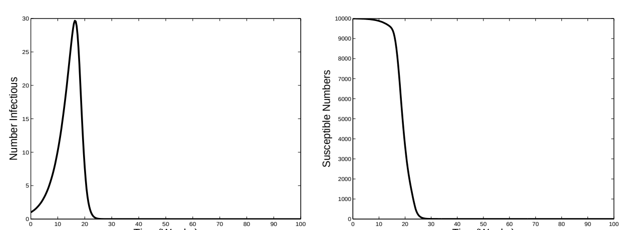
Fig.1 (a) represents the number of infected individuals as a function of time when $\tau = 5$, epidemic outbreak increases rapidly and reaches the peak at $t = 22$ weeks with value 40, the controls take some time to react with the infected individuals, it then starts to gradually drop to almost zero, meaning the disease is gradually eradicated from the population. Fig.1 (b) shows the susceptible population S vs. time (weeks), we observe that there is a significant decrease in the number of susceptible after around 40 weeks.

In order to clearly see the effect of the time lag on the dynamical behavior of the system, we take a smaller time delay as $\tau = 1$ in Fig.2. By comparison with Fig.1, we can observe the smaller the time delay, the shorter it takes the equilibrium points to settle to their state value, which implies that the disease will be more serious if the delay lag is shorter.



(a) (b)

Figure 1: (a)The plot shows the infected population I vs. time (weeks) for time delay $\tau = 5$. (b)The plot shows the susceptible population S vs. time (weeks) for time delay $\tau = 5$.



(a) (b)

Figure 2: (a)The plot shows the infected population I vs. time (weeks) for time delay $\tau = 1$. (b)The plot shows the susceptible population S vs. time (weeks) for time delay $\tau = 1$.

We have plotted the controls $u_i(t)$ $(i = 1, 2, 3)$ as a function of time in Fig.3, representing the optimal controls in blocking new infection and inhibiting viral production under two

Algorithm

Step1

for $i = -m, ..., 0,$ **do**

$S_i = S(0),\ I_i = I(0),\ W_i = W(0),\ R_i = R(0),\ u_1^i = 0,\ u_2^i = 0,\ u_3^i = 0,$

end for

for $i = n, ..., n+m,$ **do** $\lambda_1^i = 0,\ \lambda_2^i = 0,\ \lambda_3^i = 0,$

end for

Step2

for $i = 0, ..., n-1,$ **do**

$S_{i+1} = \frac{S_i + h\mu N}{1 + h(\frac{\beta_W W_{i+1}}{\kappa + W_{i+1}} + \beta_I I_{i+1} + \mu + u_1)},$

$I_{i+1} = \frac{I_i + h\beta_W \frac{S_{i+1} W_{i+1}}{\kappa + W_{i+1}}}{1 + h(\gamma + \mu + u_2 - \beta_I S_{i+1})},$

$W_{i+1} = \frac{W_i + h\xi I_{i-m}}{1 + h(\delta + u_3)},$

$R_{i+1} = \frac{R_i + h(\gamma I_{i+1} + u_2 I_{i+1} + u_1 S_{i+1})}{1 + h\mu},$

$\lambda_1^{n-i-1} = \frac{\lambda_1^{n-i} + h(\frac{\beta_W W_{i+1}}{\kappa + W_{i+1}} + \beta_I I_{i+1})\lambda_2^{n-i} + h\mu\lambda_4^{n-i}}{1 + h(\frac{\beta_W W_{i+1}}{\kappa + W_{i+1}} + \beta_I I_{i+1} + \mu + u_1)},$

$\lambda_2^{n-i-1} = \frac{\lambda_2^{n-i} + h - h\lambda_1^{n-i-1}\beta_I S_{i+1} + h\lambda_4^{n-i}(\gamma + u_2) + h\lambda_2^{n-i+m}\chi_{[0, t_f - \tau]}(t_{n-i})\xi}{1 + h[\beta_I S_{i+1} - (\gamma + \mu + u_2)]},$

$\lambda_3^{n-i-1} = \frac{\lambda_3^{n-i} - h\lambda_1^{n-i-1}\frac{\beta_W \kappa S_{i+1}}{(\kappa + W_{i+1})^2} + h\lambda_2^{n-i-1}\frac{\beta_W \kappa S_{i+1}}{(\kappa + W_{i+1})^2}}{1 + h(\delta + u_3)},$

$\lambda_4^{n-i-1} = \frac{\lambda_4^{n-i}}{1 + h\mu},$

$T_1^{i+1} = \frac{(\lambda_1^{n-i} - \lambda_4^{n-i})S_{i+1}}{2A_1},$

$T_2^{i+1} = \frac{(\lambda_2^{n-i} - \lambda_4^{n-i})I_{i+1}}{2A_2},$

$T_3^{i+1} = \frac{\lambda_3^{n-i} W_{i+1}}{2A_1},$

$u_1^{i+1} = min(max(0, T_1^{i+1}), 1),$

$u_2^{i+1} = min(max(0, T_2^{i+1}), 1),$

$u_2^{i+1} = min(max(0, T_3^{i+1}), 1),$

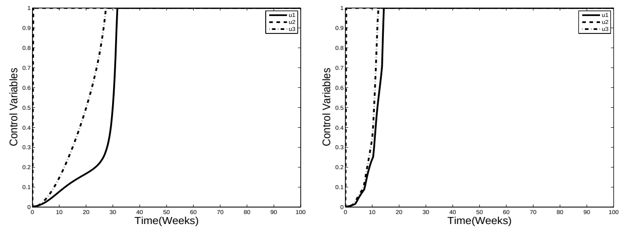
Step3

for $i = 0, ..., n,$ **write**

$S^*(t_i) = S_i, I^*(t_i) = I_i, W^*(t_i) = W_i, R^*(t_i) = R_i, u_1^*(t_i) = u_1^i, u_2^*(t_i) = u_2^i, u_3^*(t_i) = u_3^i,$

end for

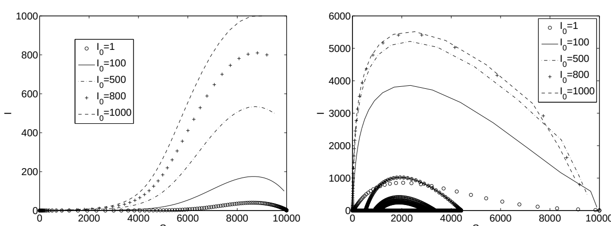
different cases: $\tau = 6$ and $\tau = 3$, respectively. From Fig.3, it is apparent that a larger value of optimal control variables is necessary in case of smaller time delay. It is also clear to see that the control u_2 in both cases always needs to be the maximal while the other two controls u_1 and u_3, which need not to be the maximal at very first, increase gradually and reach the maximal until certain weeks. Hence, we can firstly apply more of the therapeutic treatment in order to effectively reduce the number of infectious individuals.



(a) (b)

Figure 3: (a)The plot represents the controls u_1, u_2 and u_3 vs. time (weeks) for time delay $\tau = 6$. (b)The plot represents the controls u_1, u_2 and u_3 vs. time (weeks) for time delay $\tau = 3$.

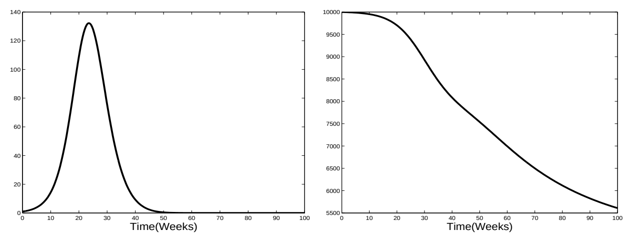
To verify the global asymptotic stability of the ODE system analyzed in Sections 3, we pick five different initial conditions with $I(0) = 1, 100, 500, 800, 1000$, respectively, and plot these five solution curves by the phase plane portrait of I vs. S in Fig. 4. We clearly see that all these five orbits converge to the disease-free equilibrium E_0 when $R_0 < 1$ in Fig. 4(a) and converge to endemic equilibrium E^* when $R_0 > 1$ in Fig. 4(b), respectively.



(a) (b)

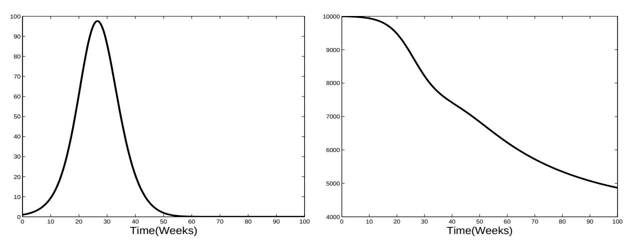
Figure 4: (a)The phase plane portrait of I vs. S for $R_0 < 1$, all these orbits converge to the disease-free equilibrium E_0. (b)The phase plane portrait of I vs. S for $R_0 > 1$, all these orbits converge to the endemic equilibrium E^*.

In order to illustrate the impacts of the different optimal control strategies, we investigate and compare numerical results in the following four strategies for the control of the disease: (1)when the objective function J is optimized through the control u_1, while u_2 and u_3 are set to be zero; (2)when the objective function J is optimized through the control u_2, while u_1 and u_3 are set to be zero; (3)when the objective function J is optimized through the control u_3, while u_1 and u_2 are set to be zero; (4)without any controls, while u_1, u_2 and u_3 are all set to be zero. We observe from Fig.5, as can be expected, there is a significant increase in the number of infected individuals and susceptible individuals controlled compared with optimal controlled, so that the infected population is affected very much due to the lack of all the three controls. Compared with Fig.6, Fig.7 and Fig.8, the number of infectious does not differ significantly by applying either the strategies with control u_1 only or with control u_3 only, but does make greater significance when only treatment control u_2 is employed, thus the application of therapeutic treatment control gives better result than the application of u_1 or u_3 only. This simulation indicates that therapeutic treatment is more effective in reducing the infection level, which highlights the effectiveness of treatment measure in controlling the diseases. In a word, the use of a single optimal control method does not make a significant impact, while the use of multi-strategies is more efficient. However, if the budget is limited, it is much better to apply the treatment well before the occurrence of the outbreak.



(a) (b)

Figure 5: (a)The plot shows the infected population I vs. time (weeks) for time delay $\tau = 5$ if there are no controls. (b)The plot shows the susceptible population S vs. time (weeks) for time delay $\tau = 5$ if there are no controls.

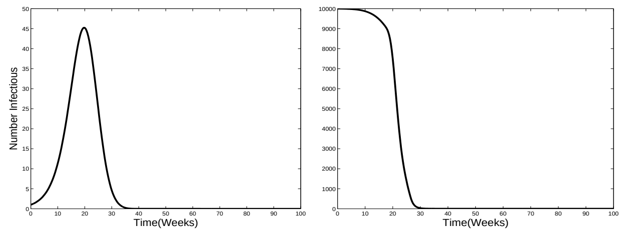


(a) (b)

Figure 6: (a)The plot shows the infected population I vs. time (weeks) for time delay $\tau = 5$ if there is only control u_1. (b)The plot shows the susceptible population S vs. time (weeks) for time delay $\tau = 5$ if there is only control u_1.

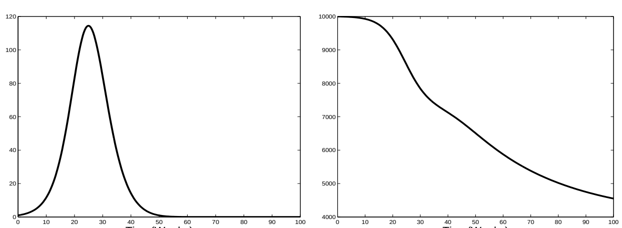
6 Conclusions and discussions

In this paper, we have presented a cholera epidemiological model by incorporating three types of intervention strategies and time delay inspired by the work in Wang and Modnak [21]. We have mainly investigated that by applying both an optimal control and a time delay to a



(a) (b)

Figure 7: (a)The plot shows the infected population I vs. time (weeks) for time delay $\tau = 5$ if there is only control u_2. (b)The plot shows the susceptible population S vs. time (weeks) for time delay $\tau = 5$ if there is only control u_2.



(a) (b)

Figure 8: (a)The plot shows the infected population I vs. time (weeks) for time delay $\tau = 5$ if there is only control u_3. (b)The plot shows the susceptible population S vs. time (weeks) for time delay $\tau = 5$ if there is only control u_3.

cholera model in order to eliminate the infectious disease. First of all, both the disease-free equilibrium E_0 and endemic equilibrium E^* of the model were obtained. By analyzing the corresponding characteristic equations, the local stability of E_0 and E^* was investigated. In particular, we have established the global stability analysis of the disease-free and endemic equilibria of ODE system by constructing two suitable Lyapunov functionals. Moreover, we used the Pontryagins Maximum Principle with delay to characterize optimal controls and derived the optimality system at the same time. Finally, we presented an efficient numerical simulation based on a specific algorithm to show that the optimal control strategy is much more effective for reducing the number of infected individuals than using of any single control, which highlights the effectiveness of treatment measure in controlling the diseases. However, if the budget is limited, it is much better to apply the therapeutic treatment well before the occurrence of the outbreak.

Since the choice of the weights A_i reflects the different scales of the costs for different controls, it is important to notice that the ideal weights are very difficult to obtain in the real world. We only use theoretical weights to propose the simulations in this paper, thus the appropriate data is a difficult problem and it still remains for our further work. We also need to pay attention to that different choices of final time t_f lead to different results, because there is an opposite time orientations for the optimality system when we carry out the simulations. Mathematically speaking, the control is very sensitive to the final time. In the work of [19] in 2011, it was mentioned that the shorter the period of control programme is, the smaller the marginal cost of control will be.

7 Acknowledgments

This work was partially supported by the Natural Science Foundation of China (NO.11271388, NO. 11401059), National Social Science Foundation of China (NO.13CTJ016), Natural Science Foundation of CQ (NO. cstc2015jcyjA00024).

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Effect of antibodies and latently infected cells on HIV dynamics with differential drug efficacy in cocirculating target cells

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Abstract

In this paper, we investigate the qualitative behaviors of three viral infection models with two types of cocirculating target cells. The models take into account both antibodies and latently infected cells. The incidence rate is represented by bilinear, saturation and general function. For the first two models, we have derived two threshold parameters, R_0 and R_1 which completely determined the global properties of the models. Lyapunov functions are constructed and LaSalle's invariance principle is applied to prove the global asymptotic stability of all equilibria of the models. For the third model, we have established a set of conditions on the general incidence rate function which are sufficient for the global stability of the equilibria of the model. Theoretical results have been checked by numerical simulations.
Keywords: Virus infection; Global stability; Latently infected cells; cocirculating target cells; Lyapunov function.

1 Introduction

Mathematical modeling and model analysis of virus infection in vivo have attracted the interests of mathematicians during the recent years. Such virus infection models can be very useful in the control of epidemic diseases and provide insights into the dynamics of viral load in vivo. Therefore, mathematical analysis of the virus infection models can play a significant role in the development of a better understanding of diseases and various drug therapy strategies.Many authors have formulated mathematical models to describe the population dynamics of several viruses such as, human immunodeficiency virus (HIV) (see e.g. [1]-[10]), hepatitis B virus (HBV) [11]-[13], hepatitis C virus (HCV) [14]-[15], human T cell leukemia HTLV [16] and dengue virus [17], etc. During viral infections, the host immune system reacts with antigen-specific immune response. The immune system has two main responses to viral infections. The first is based on the Cytotoxic T Lymphocyte (CTL) cells which are responsible to attack and kill the infected cells. The second immune response is based on the antibodies that are produced by the B cells. The function of the antibodies is to attack the viruses [1]. In some infections such as in malaria, the CTL immune response is less effective than the antibody immune response [18]. Several mathematical models have been proposed to consider the antibody immune response into the viral infection models ([19]-[24]). The basic model of viral infection with antibody immune response has been

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introduced by Murase et. al. [19] and Wang and Zou [21] as:

$$\dot{x} = \lambda - dx - \bar{\beta}xv, \tag{1}$$
$$\dot{y} = \bar{\beta}xv - ay, \tag{2}$$
$$\dot{v} = ky - cv - rzv, \tag{3}$$
$$\dot{z} = gzv - \mu z, \tag{4}$$

where x, y, v and z represent, respectively, the concentrations of uninfected cells, infected cells, free viruses and the antibody immune cells. Parameters λ, k and g represent respectively, the rate of new uninfected cells that are generated from sources within the body, the rate of free virus production and the proliferation rate constant of the antibody immune cells. Parameters d, a, c and μ are the natural death rate constant of uninfected cells, infected cells, free virus particles and the antibody immune cells respectively. Parameter $\bar{\beta}$ is the infection rate constant at which a target cell becomes infected via contacting with virus and r is the removal rate constant of the virus due to the antibodies. Model (1)-(4) is based on the assumption that the infection could occur and that the viruses are produced from infected cells instantaneously, once the uninfected cells are contacted by the virus particles. Other accurate models incorporate the latently infected cells which are due to the delay between the time of infection and the time when the infected cell becomes active to produce infectious viruses. In [26], model (1)-(4) was extended to take into consideration both latently and actively infected cells as:

$$\dot{x} = \lambda - dx - \bar{\beta}xv, \tag{5}$$
$$\dot{w} = (1-\alpha)\bar{\beta}xv - (e+b)w, \tag{6}$$
$$\dot{y} = \alpha\bar{\beta}xv + bw - ay, \tag{7}$$
$$\dot{v} = ky - cv - rvz, \tag{8}$$
$$\dot{z} = gvz - \mu z, \tag{9}$$

where w and y are the concentrations of latently infected and actively infected cells, respectively. Eq. (6) describes the population dynamics of the latently infected cells and show that they are converted to actively infected cells with rate constant b. The parameters e and a are the death rate constants of the latently and actively infected cells, respectively. The fractions $(1-\alpha)$ where, $0 < \alpha < 1$ are the probabilities that upon infection, an uninfected cell will become either latently infected or actively infected. Model (5)-(9) it have been assumed that, the HIV has one class of target cells, CD4$^+$T cells. However, Perelson et al. in [25] have shown that, HIV infects the macrophages in addition to the CD4$^+$T cells. Recently, many efforts have been devoted to study various mathematical models of HIV dynamics with two classes of target cells (see e.g. [3]).

Our primary goal of the present paper is to propose the global stability analysis of three viral infection models with two types of target cells, CD4$^+$T cells and macrophages taking into consideration the latently, actively infected cells and antibody immune response. The infection rate is represented by bilinear incidence and saturated incidence in the first and the second models, respectively, while it is given by a general function in the third one. The global stability of the three models is established using Lyapunov functionals.

2 HIV model with bilinear incidence rate

In this section, we introduce an HIV dynamics model which describes two cocirculation populations of target cells, CD4$^+$ T cells and macrophages and takes into account the antibody immune response. We consider two types of infected cells, the latently infected and actively infected cells.

$$\dot{x}_i = \lambda_i - d_i x_i - \beta_i x_i v, \qquad i = 1, 2, \tag{10}$$
$$\dot{w}_i = (1-\alpha_i)\beta_i x_i v - (e_i + b_i)w_i, \qquad i = 1, 2, \tag{11}$$
$$\dot{y}_i = \alpha_i \beta_i x_i v + b_i w_i - a_i y_i, \qquad i = 1, 2, \tag{12}$$
$$\dot{v} = \sum_{i=1}^{2} k_i y_i - cv - rvz, \tag{13}$$
$$\dot{z} = gvz - \mu z. \tag{14}$$

Here $i = 1, 2$ correspond to the CD4$^+$ T cells and macrophages and $\beta_1 = (1-\varepsilon)\bar{\beta}_1,\ \beta_2 = (1-\varepsilon f)\bar{\beta}_2$. The model incorporates RTI drug therapy where in the CD4$^+$T cells, the drug efficacy is ε and $0 \leqslant \varepsilon < 1$, while in the macrophages the drug efficacy εf is reduced by a factor f and $0 < f < 1$. All the parameters and variables of the model have the same meanings as given in (5)-(9).

2.1 Properties of solutions

One can easily show that the non-negative orthant $\mathbb{R}^8 \geq 0$ by model (10)-(14).

Proposition 1. There exist positive numbers $L_j,\ j = 1, 2, 3, 4$ such that the compact set $\Omega = \left\{(x_i, w_i, y_i, v, z) \in \mathbb{R}^8 \geq 0 : 0 \leq x_i, w_i, y_i \leq L_i,\ 0 \leq v \leq L_3,\ 0 \leq z \leq L_4,\ i = 1, 2\right\}$ is positively invariant.

Proof. To show the boundedness of the solutions of system (10)-(14) we let $T_i(t) = x_i(t) + w_i(t) + y_i(t)$, then

$$\dot{T}_i(t) = \lambda_i - d_i x_i(t) - e_i w_i(t) - a_i y_i(t) \leq \lambda_i - \rho_i T_i(t),$$

where $\rho_i = \min\{d_i, a_i, e_i\}, i = 1, 2$. Hence $T_i(t) \leq L_i$, if $T_i(0) \leq L_i$, where $L_i = \dfrac{\lambda_i}{\rho_i}$. Since $x_i(t)$, $w_i(t)$ and $y(t)$ are all non-negative, then $0 \leq x_i(t),\ w_i(t),\ y_i(t) \leq L_i$, for all $t \geq 0$, if $0 \leq x_i(0) + w_i(0) + y_i(0) \leq L_i$, $i = 1, 2$. On the other hand, let $G(t) = v(t) + \dfrac{r}{g} z(t)$, then

$$\dot{G}(t) = \sum_{i=1}^{2} k_i y_i - cv - \frac{r\mu}{g} z \leq \sum_{i=1}^{2} k_i L_i - \delta\left(v + \frac{r}{g} z\right) = \sum_{i=1}^{2} k_i L_i - \delta G(t),$$

where $\delta = \min\{c, \mu\}$. Hence $G(t) \leq L_3$, if $G(0) \leq L_3$, where $L_3 = \dfrac{1}{\delta}\sum_{i=1}^{2} k_i L_i$. Since $v(t) \geq 0$ and $z(t) \geq 0$, then $0 \leq v(t) \leq L_3$ and $0 \leq z(t) \leq L_4$ if $0 \leq v(0) + \dfrac{r}{g} z(0) \leq L_3$, where $L_4 = \dfrac{gL_3}{r}$.

2.2 Equilibria and biological thresholds

Let $\overset{\circ}{\Omega}$ be the interior of Ω.

Lemma 1. For system (10)-(14) we have (i) There exist only one uninfected equilibrium $E_0 = (x_1^0, x_2^0, 0, 0, 0, 0, 0, 0) \in \Omega$, when $R_0 \leq 1$.

(ii) There exist E_0 and a chronic-infection equilibrium without antibody immune response $E_1 = (\tilde{x}_1, \tilde{x}_2, \tilde{w}_1, \tilde{w}_2, \tilde{y}_1, \tilde{y}_2, \tilde{v}, 0,) \in \Omega$, when $R_1 \leq 1 < R_0$.

(iii) There exist E_0, E_1 and a chronic-infection equilibrium with antibody immune response $E_2 = (\bar{x}_1, \bar{x}_2, \bar{w}_1, \bar{w}_2, \bar{y}_1, \bar{y}_2, \bar{v}, \bar{z}) \in \overset{\circ}{\Omega}$, when $R_1 > 1$.

Proof. The equilibria of (10)-(14) satisfy the following equations:

$$\lambda_i - d_i x_i - \beta_i x_i v = 0, \tag{15}$$

$$(1-\alpha_i)\beta_i x_i v - (e_i + b_i) w_i = 0, \tag{16}$$

$$\alpha_i \beta_i x_i v + b_i w_i - a_i y_i = 0, \tag{17}$$

$$\sum_{i=1}^{2} k_i y_i - cv - rvz = 0, \tag{18}$$

$$gvz - \mu z = 0. \tag{19}$$

Eq. (19) has two possible solutions $z = 0$ or $v = \dfrac{\mu}{g}$. If $z = 0$, then from Eqs.(15)-(17) we get

$$x_i = \frac{x_i^0}{(1 + \eta_i v)}, \qquad w_i = \frac{(1-\alpha_i)\beta_i x_i^0}{(e_i + b_i)(1 + \eta_i v)} v, \qquad y_i = \frac{(e_i \alpha_i + b_i)\beta_i x_i^0}{a_i (e_i + b_i)(1 + \eta_i v)} v, \tag{20}$$

where $x_i^0 = \dfrac{\lambda_i}{d_i},\ \eta_i = \dfrac{\beta_i}{d_i},\ i = 1, 2$. From Eq. (18) we obtain

$$\left(\sum_{i=1}^{2} \frac{(e_i \alpha_i + b_i) k_i \beta_i x_i^0}{a_i c (e_i + b_i)(1 + \eta_i v)} - 1\right) cv = 0. \tag{21}$$

We note that $v = 0$ is a solution for Eq. (21) which leads to the disease-free equilibrium $E_0 = (x_1^0, x_2^0, 0, 0, 0, 0, 0, 0)$. If $v \neq 0$, we have

$$\sum_{i=1}^{2} \frac{\Phi_i}{1 + \eta_i v} = 1. \tag{22}$$

where $\Phi_i = \dfrac{(e_i\alpha_i + b_i)k_i\beta_i x_i^0}{a_i c(e_i + b_i)}$. Equation (22) can be written as:

$$Av^2 + Bv - C = 0, \tag{23}$$

where

$$A = \eta_1\eta_2, \quad B = \eta_1\Phi_1 + \eta_2\Phi_2 + (1 - \Phi_1 - \Phi_2)(\eta_1 + \eta_2), \quad C = \Phi_1 + \Phi_2 - 1$$

The solutions of Eq. (23) is given by

$$v^{\pm} = \frac{-B \pm \sqrt{B^2 + 4AC}}{2A}.$$

We have $A > 0$, therefore if $C > 0$, then $v^+ > 0$ and $v^- < 0$. Let $\tilde{v} = v^+$, then from Eq. (20) we get

$$\tilde{x}_i = \frac{x_i^0}{1 + \eta_i\tilde{v}}, \quad \tilde{w}_i = \frac{(1 - \alpha_i)\beta_i x_i^0}{(e_i + b_i)(1 + \eta_i\tilde{v})}\tilde{v}, \quad \tilde{y}_i = \frac{(e_i\alpha_i + b_i)\beta_i x_i^0}{a_i(e_i + b_i)(1 + \eta_i\tilde{v})}\tilde{v}, \quad i = 1, 2. \tag{24}$$

Therefore, a chronic-infection equilibrium without antibody immune response $E_1 = (\tilde{x}_1, \tilde{x}_2, \tilde{w}_1, \tilde{w}_2, \tilde{y}_1, \tilde{y}_2, \tilde{v}, 0)$ exists when $C > 0$ or $(\Phi_1 + \Phi_2 > 1)$. Now we are ready to define the basic infection reproduction number R_0 as

$$R_0 = \Phi_1 + \Phi_2 = \sum_{i=1}^{2} R_{0i} = \sum_{i=1}^{2} \frac{k_i\beta_i x_i^0(e_i\alpha_i + b_i)}{a_i c(e_i + b_i)}.$$

If $v = \dfrac{\mu}{g}$, then we obtain the chronic-infection equilibrium with antibody immune response $E_2 = (\bar{x}_1, \bar{x}_2, \bar{w}_1, \bar{w}_2, \bar{y}_1, \bar{y}_2, \bar{v}, \bar{z})$, where

$$\bar{x}_i = \frac{g\lambda_i}{gd_i + \mu\beta_i}, \quad \bar{w}_i = \frac{(1 - \alpha_i)\lambda_i\beta_i\mu}{(e_i + b_i)(gd_i + \mu\beta_i)}, \quad \bar{y}_i = \frac{(e_i\alpha_i + b_i)\lambda_i\beta_i\mu}{a_i(e_i + b_i)(gd_i + \mu\beta_i)}, \quad i = 1, 2,$$

$$\bar{v} = \frac{\mu}{g}, \qquad \bar{z} = \frac{c}{r}\left(\sum_{i=1}^{2} \frac{gk_i\beta_i\lambda_i(e_i\alpha_i + b_i)}{a_i c(e_i + b_i)(gd_i + \mu\beta_i)} - 1\right).$$

We note that E_2 exists when $\sum_{i=1}^{2} \dfrac{gk_i\beta_i\lambda_i(e_i\alpha_i + b_i)}{a_i c(e_i + b_i)(gd_i + \mu\beta_i)} > 1$. Let us define the antibody immune response activation number as

$$R_1 = \sum_{i=1}^{2} \frac{gk_i\beta_i\lambda_i(e_i\alpha_i + b_i)}{a_i c(e_i + b_i)(gd_i + \mu\beta_i)} = \sum_{i=1}^{2} \frac{R_{0i}}{1 + \dfrac{\mu\beta_i}{gd_i}},$$

which determines whether or not a persistent antibody immune response can be established. Then we can write $\bar{z} = \dfrac{c}{r}(R_1 - 1)$. Clearly $R_1 < R_0$.

Now, we show that E_0, $E_1 \in \Omega$ and $E_2 \in \mathring{\Omega}$. Clearly, $E_0 \in \Omega$. Let $R_0 > 1$, then from Eq. (20) we have $\tilde{x}_i < x_i^0$, then

$$0 < \tilde{x}_i < \frac{\lambda_i}{d_i} \leq \frac{\lambda_i}{\rho_i} = L_i.$$

From Eqs. (10)-(12), we get

$$\lambda_i = d_i\tilde{x}_i + e_i\tilde{w}_i + a_i\tilde{y}_i.$$

Thus,

$$0 < \tilde{w}_i < \frac{\lambda_i}{e_i} \leq \frac{\lambda_i}{\rho_i} = L_i, \qquad 0 < \tilde{y}_i < \frac{\lambda_i}{a_i} \leq \frac{\lambda_i}{\rho_i} = L_i.$$

Also, $\tilde{v} = \frac{1}{c}\sum_{i=1}^{2} k_i\tilde{y}_i < \frac{1}{c}\sum_{i=1}^{2} k_iL_i \leq \frac{1}{\delta}\sum_{i=1}^{2} k_iL_i = L_3$. Moreover, $\tilde{z} = 0$, and then, $E_1 \in \Omega$. Let $R_1 > 1$, then one can show that $0 < \bar{x}_i < L_i$, $0 < \bar{w}_i < L_i$ and $0 < \bar{y}_i < L_i$. Now we show that $0 < \bar{v} < L_3$ and $0 < \bar{z} < L_4$. From Eq. (13), we have $c\bar{v} + r\bar{v}\bar{z} = \sum_{i=1}^{2} k_i\bar{y}_i$. Then

$$c\bar{v} < \sum_{i=1}^{2} k_i\bar{y}_i \Rightarrow 0 < \bar{v} < \frac{1}{c}\sum_{i=1}^{2} k_iL_i \leqslant \frac{1}{\delta}\sum_{i=1}^{2} k_iL_i = L_3,$$

$$r\bar{v}\bar{z} < \sum_{i=1}^{2} k_i\bar{y}_i \Rightarrow 0 < \bar{z} < \frac{g}{r\mu}\sum_{i=1}^{2} k_i\bar{y}_i < \frac{g}{r\delta}\sum_{i=1}^{2} k_iL_i = \frac{gL_3}{r} = L_4.$$

It follows that, $E_2 \in \mathring{\Omega}$.

2.3 Global stability

Let us define the function $F(s) = s - 1 - \ln s$.

Theorem 1. The infection-free equilibrium E_0 of system (10)-(14) is GAS when $R_0 \leq 1$.

Proof. Define a Lyapunov function W_0 as follows:

$$W_0 = \sum_{i=1}^{2} \gamma_i \left[x_i^0 F\left(\frac{x_i}{x_i^0}\right) + \frac{b_i}{e_i\alpha_i + b_i}w_i + \frac{e_i + b_i}{e_i\alpha_i + b_i}y_i\right] + v + \frac{r}{g}z, \tag{25}$$

where $\gamma_i = \dfrac{k_i(e_i\alpha_i + b_i)}{a_i(e_i + b_i)}$, $\quad i = 1, 2$. The time derivative of W_0 along the trajectories of (10)-(14) satisfies

$$\begin{aligned}\frac{dW_0}{dt} &= \sum_{i=1}^{2} \gamma_i \left[\left(1 - \frac{x_i^0}{x_i}\right)(\lambda_i - d_ix_i - \beta_ix_iv) + \frac{b_i}{e_i\alpha_i + b_i}\left((1-\alpha_i)\beta_ix_iv - (e_i + b_i)w_i\right)\right. \\ &\left. + \frac{e_i + b_i}{e_i\alpha_i + b_i}\left(\alpha_i\beta_ix_iv + b_iw_i - a_iy_i\right)\right] + \sum_{i=1}^{2} k_iy_i - cv - rvz + \frac{r}{g}(gvz - \mu z).\end{aligned} \tag{26}$$

Collecting terms of Eq. (26) we get

$$\begin{aligned}\frac{dW_0}{dt} &= \sum_{i=1}^{2} \gamma_i \left[d_i\left(1 - \frac{x_i^0}{x_i}\right)(x_i^0 - x_i) + \beta_ix_i^0v\right] - cv - \frac{r\mu}{g}z \\ &= -\sum_{i=1}^{2} \gamma_id_i\frac{(x_i - x_i^0)^2}{x_i} + \sum_{i=1}^{2} \frac{k_i(e_i\alpha_i + b_i)}{a_i(e_i + b_i)}\beta_ix_i^0v - cv - \frac{r\mu}{g}z \\ &= -\sum_{i=1}^{2} \gamma_id_i\frac{(x_i - x_i^0)^2}{x_i} + \left(\sum_{i=1}^{2} \frac{k_i\beta_ix_i^0(e_i\alpha_i + b_i)}{a_ic(e_i + b_i)} - 1\right)cv - \frac{r\mu}{g}z \\ &= -\sum_{i=1}^{2} \gamma_id_i\frac{(x_i - x_i^0)^2}{x_i} + (R_0 - 1)cv - \frac{r\mu}{g}z.\end{aligned} \tag{27}$$

If $R_0 \leq 1$ then $\frac{dW_0}{dt} \leq 0$ for all $x_i, v, z > 0$. Thus, the solutions of system (10)-(14) converge to Ω, the largest invariant subset of $\left\{\frac{dW_0}{dt} = 0\right\}$ [27]. Clearly, it follows from Eq. (26) that $\frac{dW_0}{dt} = 0$ if and only if $x_i = x_i^0$, $v = 0$ and $z = 0$. The set Ω is invariant and for any element belongs to Ω satisfies $v = 0$ and $z = 0$, then $\dot{v} = 0$. We can see from Eq. (13) that $0 = \dot{v} = \sum_{i=1}^{2} k_iy_i$, and thus $y_i = 0$. Moreover, from Eq. (12) we get $w_i = 0$. Hence $\frac{dW_0}{dt} = 0$ occurs at E_0. From LaSalle's invariance principle, E_0 is GAS.

Theorem 2. The chronic-infection equilibrium without antibody immune response E_1 of system (10)-(14) is GAS when $R_1 \leq 1 < R_0$.

Proof. We construct the following Lyapunov function

$$W_1=\sum_{i=1}^{2}\gamma_i\left[\tilde{x}_iF(\frac{x_i}{\tilde{x}_i})+\frac{b_i}{e_i\alpha_i+b_i}\tilde{w}_iF\left(\frac{w_i}{\tilde{w}_i}\right)+\frac{e_i+b_i}{e_i\alpha_i+b_i}\tilde{y}_iF\left(\frac{y_i}{\tilde{y}_i}\right)\right]+\tilde{v}F\left(\frac{v}{\tilde{v}}\right)+\frac{r}{g}z.$$

Calculating $\frac{dW_1}{dt}$ along the trajectories of (10)-(14) we get

$$\frac{dW_1}{dt}=\sum_{i=1}^{2}\gamma_i\left[\left(1-\frac{\tilde{x}_i}{x_i}\right)(\lambda_i-d_ix_i-\beta_ix_iv)+\frac{b_i}{e_i\alpha_i+b_i}\left(1-\frac{\tilde{w}_i}{w_i}\right)((1-\alpha_i)\beta_ix_iv-(e_i+b_i)w_i)\right.$$
$$\left.+\frac{e_i+b_i}{e_i\alpha_i+b_i}\left(1-\frac{\tilde{y}_i}{y_i}\right)(\alpha_i\beta_ix_iv+b_iw_i-a_iy_i)\right]+\left(1-\frac{\tilde{v}}{v}\right)\left(\sum_{i=1}^{2}k_iy_i-cv-rvz\right)+\frac{r}{g}\left(gvz-\mu z\right). \quad (28)$$

Collecting terms of Eq. (28) we get

$$\frac{dW_1}{dt}=\sum_{i=1}^{2}\gamma_i\left[\left(1-\frac{\tilde{x}_i}{x_i}\right)(\lambda_i-d_ix_i)+\beta_i\tilde{x}_iv-\frac{b_i(1-\alpha_i)}{e_i\alpha_i+b_i}\frac{\beta_ix_iv\tilde{w}_i}{w_i}+\frac{e_i+b_i}{e_i\alpha_i+b_i}b_i\tilde{w}_i-\frac{\alpha_i\left(e_i+b_i\right)}{e_i\alpha_i+b_i}\frac{\beta_ix_iv\tilde{y}_i}{y_i}\right.$$
$$\left.-\frac{b_i\left(e_i+b_i\right)}{e_i\alpha_i+b_i}\frac{w_i\tilde{y}_i}{y_i}+\frac{e_i+b_i}{e_i\alpha_i+b_i}a_i\tilde{y}_i\right]-cv-\frac{\tilde{v}}{v}\sum_{i=1}^{2}k_iy_i+c\tilde{v}+r\tilde{v}z-\frac{r\mu}{g}z. \quad (29)$$

Using the value of $\tilde{x}_i$ given in Eq. (24) we get $\left(\sum_{i=1}^{2}\gamma_i\beta_i\tilde{x}_i-c\right)v=0$. Applying $\lambda_i=\ d_i\tilde{x}_i+\beta_i\tilde{x}_i\tilde{v}$, we obtain

$$\frac{dW_1}{dt}=\sum_{i=1}^{2}\gamma_i\left[\left(1-\frac{\tilde{x}_i}{x_i}\right)(d_i\tilde{x}_i-d_ix_i)+\beta_i\tilde{x}_i\tilde{v}\left(1-\frac{\tilde{x}_i}{x_i}\right)-\frac{b_i(1-\alpha_i)}{e_i\alpha_i+b_i}\frac{\beta_ix_iv\tilde{w}_i}{w_i}+\frac{e_i+b_i}{e_i\alpha_i+b_i}b_i\tilde{w}_i\right.$$
$$\left.-\frac{\alpha_i\left(e_i+b_i\right)}{e_i\alpha_i+b_i}\frac{\beta_ix_iv\tilde{y}_i}{y_i}-\frac{b_i\left(e_i+b_i\right)}{e_i\alpha_i+b_i}\frac{w_i\tilde{y}_i}{y_i}+\frac{e_i+b_i}{e_i\alpha_i+b_i}a_i\tilde{y}_i\right]-\frac{\tilde{v}}{v}\sum_{i=1}^{2}k_iy_i+c\tilde{v}+r\tilde{v}z-\frac{r\mu}{g}z. \quad (30)$$

Using the equilibrium condition for E_1

$$(1-\alpha_i)\beta_i\tilde{x}_i\tilde{v}=(e_i+b_i)\tilde{w}_i,\quad \alpha_i\beta_i\tilde{x}_i\tilde{v}+b_i\tilde{w}_i=a_i\tilde{y}_i,\quad c\tilde{v}=\sum_{i=1}^{2}k_i\tilde{y}_i=\sum_{i=1}^{2}\gamma_i\beta_i\tilde{x}_i\tilde{v},$$

$$\frac{e_i+b_i}{e_i\alpha_i+b_i}a_i\tilde{y}_i=\beta_i\tilde{x}_i\tilde{v}=\frac{b_i(1-\alpha_i)}{e_i\alpha_i+b_i}\beta_i\tilde{x}_i\tilde{v}+\frac{(e_i+b_i)\alpha_i}{e_i\alpha_i+b_i}\beta_i\tilde{x}_i\tilde{v}.$$

we have

$$\frac{dW_1}{dt}=\sum_{i=1}^{2}\gamma_i\left[-d_i\frac{(x_i-\tilde{x}_i)^2}{x_i}+\beta_i\tilde{x}_i\tilde{v}\left(1-\frac{\tilde{x}_i}{x_i}\right)\left(\frac{b_i(1-\alpha_i)}{e_i\alpha_i+b_i}+\frac{(e_i+b_i)\alpha_i}{e_i\alpha_i+b_i}\right)-\frac{b_i(1-\alpha_i)}{e_i\alpha_i+b_i}\beta_i\tilde{x}_i\tilde{v}\frac{x_i\tilde{w}_iv}{\tilde{x}_iw_i\tilde{v}}\right.$$
$$+\frac{b_i(1-\alpha_i)}{e_i\alpha_i+b_i}\beta_i\tilde{x}_i\tilde{v}-\frac{(e_i+b_i)\alpha_i}{e_i\alpha_i+b_i}\beta_i\tilde{x}_i\tilde{v}\frac{x_i\tilde{y}_iv}{\tilde{x}_iy_i\tilde{v}}-\frac{b_i(1-\alpha_i)}{e_i\alpha_i+b_i}\beta_i\tilde{x}_i\tilde{v}\frac{w_i\tilde{y}_i}{\tilde{w}_iy_i}+\frac{b_i(1-\alpha_i)}{e_i\alpha_i+b_i}\beta_i\tilde{x}_i\tilde{v}+\frac{(e_i+b_i)\alpha_i}{e_i\alpha_i+b_i}\beta_i\tilde{x}_i\tilde{v}$$
$$\left.-\left(\frac{b_i(1-\alpha_i)}{e_i\alpha_i+b_i}+\frac{(e_i+b_i)\alpha_i}{e_i\alpha_i+b_i}\right)\beta_i\tilde{x}_i\tilde{v}\frac{y_i\tilde{v}}{\tilde{y}_iv}+\left(\frac{b_i(1-\alpha_i)}{e_i\alpha_i+b_i}+\frac{(e_i+b_i)\alpha_i}{e_i\alpha_i+b_i}\right)\beta_i\tilde{x}_i\tilde{v}\right]+(\tilde{v}-\bar{v})\,rz.$$
$$=\sum_{i=1}^{2}\gamma_i\left[-d_i\frac{(x_i-\tilde{x}_i)^2}{x_i}+\frac{b_i(1-\alpha_i)}{e_i\alpha_i+b_i}\beta_i\tilde{x}_i\tilde{v}\left(4-\frac{\tilde{x}_i}{x_i}-\frac{x_i\tilde{w}_iv}{\tilde{x}_iw_i\tilde{v}}-\frac{y_i\tilde{v}}{\tilde{y}_iv}-\frac{w_i\tilde{y}_i}{\tilde{w}_iy_i}\right)\right.$$
$$\left.+\frac{(e_i+b_i)\alpha_i}{e_i\alpha_i+b_i}\beta_i\tilde{x}_i\tilde{v}\left(3-\frac{\tilde{x}_i}{x_i}-\frac{y_i\tilde{v}}{\tilde{y}_iv}-\frac{x_i\tilde{y}_iv}{\tilde{x}_iy_i\tilde{v}}\right)\right]+(\tilde{v}-\bar{v})\,rz. \quad (31)$$

We have $x_i,w_i,y_i,v>0$ when $R_0>1$. Since the geometrical mean is less than or equal to the arithmetical mean, the second and the third terms are less than or equal to zero. Now we show that if $R_1\leq 1$ then $\tilde{v}\leq\frac{\mu}{g}=\bar{v}$.

Using the steady state conditions for E_1 we have $\sum_{i=1}^{2}\dfrac{k_i\beta_i\lambda_i(e_i\alpha_i+b_i)}{a_icd_i(e_i+b_i)(1+\eta_i\tilde{v})}=1$, then

$$R_1-1=\sum_{i=1}^{2}\frac{gk_i\beta_i\lambda_i(e_i\alpha_i+b_i)}{a_ic(e_i+b_i)(gd_i+\mu\beta_i)}-\sum_{i=1}^{2}\frac{k_i\beta_i\lambda_i(e_i\alpha_i+b_i)}{a_id_ic(e_i+b_i)(1+\eta_i\tilde{v})}$$
$$=\sum_{i=1}^{2}\frac{k_i\beta_i\lambda_i(e_i\alpha_i+b_i)}{a_id_ic(e_i+b_i)(1+\eta_i\bar{v})}-\sum_{i=1}^{2}\frac{k_i\beta_i\lambda_i(e_i\alpha_i+b_i)}{a_id_ic(e_i+b_i)(1+\eta_i\tilde{v})}=(\tilde{v}-\bar{v})\chi, \tag{32}$$

where $\chi=\sum_{i=1}^{2}\dfrac{k_i\beta_i\lambda_i\eta_i(e_i\alpha_i+b_i)}{a_id_ic(e_i+b_i)(1+\eta_i\bar{v})(1+\eta_i\tilde{v})}$. It follows that, if $R_1\leq 1$ then $\frac{dW_1}{dt}\leq 0$ for all $x_i,w_i,y_i,v,z>0$. Thus, the solutions of system (10)-(14) limit to Ω, the largest invariant subset of $\left\{\frac{dW_1}{dt}=0\right\}$ [27]. It can be seen that, $\frac{dW_1}{dt}=0$ occurs at E_1. Applying LaSalle's invariance principle we obtain that E_1 is GAS.

Theorem 3. The chronic-infection equilibrium with antibody immune response E_2 of system (10)-(14) is GAS when $R_1>1$.

Proof. Consider the following Lyapunov function

$$W_2=\sum_{i=1}^{2}\gamma_i\left[\bar{x}_iF(\frac{x_i}{\bar{x}_i})+\frac{b_i}{e_i\alpha_i+b_i}\bar{w}_iF\left(\frac{w_i}{\bar{w}_i}\right)+\frac{e_i+b_i}{e_i\alpha_i+b_i}\bar{y}_iF\left(\frac{y_i}{\bar{y}_i}\right)\right]+\bar{v}F\left(\frac{v}{\bar{v}}\right)+\frac{r}{g}\bar{z}F\left(\frac{z}{\bar{z}}\right).$$

Calculating the derivative of W_2 along the trajectories of (10)-(14) we get

$$\frac{dW_2}{dt}=\sum_{i=1}^{2}\gamma_i\left[\left(1-\frac{\bar{x}_i}{x_i}\right)(\lambda_i-d_ix_i-\beta_ix_iv)+\frac{b_i}{e_i\alpha_i+b_i}\left(1-\frac{\bar{w}_{\bar{\imath}}}{w_i}\right)((1-\alpha_i)\beta_ix_iv-(e_i+b_i)w_i)\right.$$
$$\left.+\frac{e_i+b_i}{e_i\alpha_i+b_i}\left(1-\frac{\bar{y}_i}{y_i}\right)(\alpha_i\beta_ix_iv+bw_i-a_iy_i)\right]+\left(1-\frac{\bar{v}}{v}\right)(\sum_{i=1}^{2}k_iy_i-cv-rvz)+\frac{r}{g}\left(1-\frac{\bar{z}}{z}\right)(gvz-\mu z). \tag{33}$$

Collecting terms of Eq. (33) we get

$$\frac{dW_2}{dt}=\sum_{i=1}^{2}\gamma_i\left[\left(1-\frac{\bar{x}_i}{x_i}\right)(\lambda_i-d_ix_i)+\beta_i\bar{x}_iv-\frac{b_i(1-\alpha_i)}{e_i\alpha_i+b_i}\frac{\beta_ix_iv\bar{w}_{\bar{\imath}}}{w_i}+\frac{e_i+b_i}{e_i\alpha_i+b_i}b_i\bar{w}_{\bar{\imath}}\right.$$
$$\left.-\frac{\alpha_i\left(e_i+b_i\right)}{e_i\alpha_i+b_i}\frac{\beta_ix_iv\bar{y}_i}{y_i}-\frac{b_i\left(e_i+b_i\right)}{e_i\alpha_i+b_i}\frac{w_i\bar{y}_i}{y_i}+\frac{e_i+b_i}{e_i\alpha_i+b_i}a_i\bar{y}_i\right]-cv-\frac{\bar{v}}{v}\sum_{i=1}^{2}k_iy_i+c\bar{v}-rv\bar{z}+\frac{r\mu}{g}\bar{z}. \tag{34}$$

Applying $\lambda_i=\ d_i\bar{x}_i+\beta\bar{x}_i\bar{v}$, we get

$$\frac{dW_2}{dt}=\sum_{i=1}^{2}\gamma_i\left[\left(1-\frac{\bar{x}_i}{x_i}\right)(d_i\bar{x}_i-d_ix_i)+\beta_i\bar{x}_i\bar{v}\left(1-\frac{\bar{x}_i}{x_i}\right)+\beta_i\bar{x}_iv-\frac{b_i(1-\alpha_i)}{e_i\alpha_i+b_i}\frac{\beta_ix_iv\bar{w}_{\bar{\imath}}}{w_i}+\frac{e_i+b_i}{e_i\alpha_i+b_i}b_i\bar{w}_{\bar{\imath}}\right.$$
$$\left.-\frac{\alpha_i\left(e_i+b_i\right)}{e_i\alpha_i+b_i}\frac{\beta_ix_iv\bar{y}_i}{y_i}-\frac{b_i\left(e_i+b_i\right)}{e_i\alpha_i+b_i}\frac{w_i\bar{y}_i}{y_i}+\frac{e_i+b_i}{e_i\alpha_i+b_i}a_i\bar{y}_i\right]-cv-\frac{\bar{v}}{v}\sum_{i=1}^{2}k_iy_i+c\bar{v}-rv\bar{z}+\frac{r\mu}{g}\bar{z}. \tag{35}$$

Using the equilibrium conditions for E_2

$$(1-\alpha_i)\beta_i\bar{x}_i\bar{v}=(e_i+b_i)\bar{w}_i,\quad \alpha_i\beta_i\bar{x}_i\bar{v}+b_i\bar{w}_i=a_i\bar{y}_i,\quad c\bar{v}+r\bar{v}\bar{z}=\sum_{i=1}^{2}k_i\bar{y}_i=\sum_{i=1}^{2}\gamma_i\beta_i\bar{x}_i\bar{v},$$
$$\frac{e_i+b_i}{e_i\alpha_i+b_i}a_i\bar{y}_i=\beta_i\bar{x}_i\bar{v}=\frac{b_i(1-\alpha_i)}{e_i\alpha_i+b_i}\beta_i\bar{x}_i\bar{v}+\frac{(e_i+b_i)\alpha_i}{(e_i\alpha_i+b_i)}\beta_i\bar{x}_i\bar{v},\quad \sum_{i=1}^{2}\gamma_i\beta_i\bar{x}_i\bar{v}-c\bar{v}-r\bar{v}\bar{z}=0,$$

we have

$$\begin{aligned}\frac{dW_2}{dt} &= \sum_{i=1}^{2}\gamma_i\Bigg[-d_i\frac{(x_i-\bar{x}_i)^2}{x_i}+\beta_i\bar{x}_i\bar{v}\left(1-\frac{\bar{x}_i}{x_i}\right)\left(\frac{b_i(1-\alpha_i)}{e_i\alpha_i+b_i}+\frac{(e_i+b_i)\alpha_i}{e_i\alpha_i+b_i}\right)-\frac{b_i(1-\alpha_i)}{e_i\alpha_i+b_i}\beta_i\bar{x}_i\bar{v}\frac{x_i\bar{w}_iv}{\bar{x}_iw_i\bar{v}}\\
&+\frac{b_i(1-\alpha_i)}{e_i\alpha_i+b_i}\beta_i\bar{x}_i\bar{v}-\frac{(e_i+b_i)\alpha_i}{e_i\alpha_i+b_i}\beta_i\bar{x}_i\bar{v}\frac{x_i\bar{y}_iv}{\bar{x}_iy_i\bar{v}}-\frac{b_i(1-\alpha_i)}{e_i\alpha_i+b_i}\beta_i\bar{x}_i\bar{v}\frac{w_i\bar{y}_i}{\bar{w}_iy_i}+\frac{b_i(1-\alpha_i)}{e_i\alpha_i+b_i}\beta_i\bar{x}_i\bar{v}+\frac{(e_i+b_i)\alpha_i}{e_i\alpha_i+b_i}\beta_i\bar{x}_i\bar{v}\\
&-\left(\frac{b_i(1-\alpha_i)}{e_i\alpha_i+b_i}+\frac{(e_i+b_i)\alpha_i}{e_i\alpha_i+b_i}\right)\beta_i\bar{x}_i\bar{v}\frac{y_i\bar{v}}{\bar{y}_iv}+\left(\frac{b_i(1-\alpha_i)}{e_i\alpha_i+b_i}+\frac{(e_i+b_i)\alpha_i}{e_i\alpha_i+b_i}\right)\beta_i\bar{x}_i\bar{v}\Bigg]\\
&= \sum_{i=1}^{2}\gamma_i\Bigg[-d_i\frac{(x_i-\bar{x}_i)^2}{x_i}+\frac{b_i(1-\alpha_i)}{e_i\alpha_i+b_i}\beta_i\bar{x}_i\bar{v}\left[4-\frac{\bar{x}_i}{x_i}-\frac{x_i\bar{w}_iv}{\bar{x}_iw_i\bar{v}}-\frac{y_i\bar{v}}{\bar{y}_iv}-\frac{w_i\bar{y}_i}{\bar{w}_iy_i}\right]\\
&+\frac{(e_i+b_i)\alpha_i}{(e_i\alpha_i+b_i)}\beta_i\bar{x}_i\bar{v}\left[3-\frac{\bar{x}_i}{x_i}-\frac{y_i\bar{v}}{\bar{y}_iv}-\frac{x_i\bar{y}_iv}{\bar{x}_iy_i\bar{v}}\right]\Bigg].\end{aligned}$$

Thus, if $R_1 > 1$, then $\bar{x}_i, \bar{w}_i, \bar{y}_i, \bar{v}, \bar{z} > 0$. Using the relation between arithmetical and geometrical means, we get $\frac{dW_2}{dt} \le 0$. Clearly, $\frac{dW_2}{dt} = 0$ if and only if $x_i = \bar{x}_i$, $w_i = \bar{w}_i$, $y_i = \bar{y}_i$ and $v = \bar{v}$. If $v = \bar{v}$, then $\dot{v} = 0$ and from Eq. (13) we have $0 = \sum_{i=1}^{2} k_i\bar{y}_i - c\bar{v} - r\bar{v}\bar{z}$, which give $z = \bar{z}$. Therefore, $\frac{dW_2}{dt}$ equal to zero at E_2. The global stability of E_2 follows from LaSalle's invariance principle.

3 Model with saturation functional response

In this section, we modify model (10)-(14) by taking into account the saturation functional response as:

$$\dot{x}_i = \lambda_i - d_ix_i - \frac{\beta_ix_iv}{1+\sigma_iv}, \qquad i = 1, 2, \tag{36}$$

$$\dot{w}_i = \frac{(1-\alpha_i)\beta_ix_iv}{1+\sigma_iv} - (e_i+b_i)w_i, \qquad i = 1, 2, \tag{37}$$

$$\dot{y}_i = \frac{\alpha_i\beta_ix_iv}{1+\sigma_iv} + b_iw_i - a_iy_i, \qquad i = 1, 2, \tag{38}$$

$$\dot{v} = \sum_{i=1}^{2} k_iy_i - cv - rvz, \tag{39}$$

$$\dot{z} = gvz - \mu z, \tag{40}$$

where $\sigma_i > 0, i = 1, 2$, is the saturation constant, and all the variables and parameters of the model have the same definition as given in (10)-(14). We mention that the compact set Ω given in Section 2 is also positively invariant with respect to system (36)-(40).

3.1 Equilibria

Lemma 2. For system (36)-(40) we have (i) There exist only one uninfected equilibrium $E_0 = (x_1^0, x_2^0, 0, 0, 0, 0, 0, 0) \in \Omega$, when $R_0 \le 1$.

(ii) There exist E_0 and a chronic-infection equilibrium without antibody immune response $E_1 = (\tilde{x}_1, \tilde{x}_2, \tilde{w}_1, \tilde{w}_2, \tilde{y}_1, \tilde{y}_2, \tilde{v}, 0,) \in \Omega$, when $R_1 \le 1 < R_0$.

(iii) There exist E_0, E_1 and a chronic-infection equilibrium with antibody immune response $E_2 = (\bar{x}_1, \bar{x}_2, \bar{w}_1, \bar{w}_2, \bar{y}_1, \bar{y}_2, \bar{v}, \bar{z}) \in \mathring{\Omega}$, when $R_1 > 1$.

Proof. We let the right-hand side of Eqs.(36)-(40) equal zero, then we obtain the following: Eq. (40) has two possible solutions $z = 0$ or $v = \frac{\mu}{g}$.

If $z = 0$, then from Eqs.(36)-(38) we have

$$x_i = \frac{x_i^0(1+\sigma_iv)}{(1+\xi_iv)}, \qquad w_i = \frac{(1-\alpha_i)\beta_ix_i^0}{(e_i+b_i)(1+\xi_iv)}v, \qquad y_i = \frac{(e_i\alpha_i+b_i)\beta_ix_i^0}{a_i(e_i+b_i)(1+\xi_iv)}v, \tag{41}$$

where $x_i^0 = \dfrac{\lambda_i}{d_i}$, $\xi_i = \sigma_i + \dfrac{\beta_i}{d_i}$, $i = 1,2$. From Eq. (39) we find

$$\left(\sum_{i=1}^{2} \frac{(e_i\alpha_i + b_i)k_i\beta_i x_i^0}{a_i c(e_i + b_i)(1 + \xi_i v)} - 1 \right) cv = 0. \tag{42}$$

Eq. (42) has also two possible solutions $v = 0$ or $\sum_{i=1}^{2} \dfrac{(e_i\alpha_i + b_i)k_i\beta_i x_i^0}{a_i c(e_i + b_i)(1 + \xi_i v)} - 1 = 0$.

If $v = 0$, then substituting it in Eq. (41) we get the disease-free equilibrium $E_0 = (x_1^0, x_2^0, 0, 0, 0, 0, 0, 0)$.

If $v \neq 0$, we have

$$\sum_{i=1}^{2} \frac{\Psi_i}{(1 + \xi_i v)} = 1. \tag{43}$$

where $\Psi_i = \dfrac{(e_i\alpha_i + b_i)k_i\beta_i x_i^0}{a_i c(e_i + b_i)}$. Eq. (43) can be written as:

$$A_1 v^2 + B_1 v - C_1 = 0 \tag{44}$$

where

$$A_1 = \xi_1\xi_2, \quad B_1 = \xi_1\Psi_1 + \xi_2\Psi_2 + (1 - \Psi_1 - \Psi_2)(\xi_1 + \xi_2), \quad C_1 = \Psi_1 + \Psi_2 - 1$$

The solutions of Eq. (23) is given by:

$$v^{\pm} = \frac{-B_1 \pm \sqrt{B_1^2 + 4A_1C_1}}{2A}.$$

We have $A_1 > 0$, therefore $v^+ > 0$ and $v^- < 0$ when $C_1 > 0$. Let $\tilde{v} = v^+$, then from Eq. (41) we get

$$\tilde{x}_i = \frac{x_i^0(1 + \sigma_i\tilde{v})}{(1 + \xi_i\tilde{v})} > 0, \qquad \tilde{w}_i = \frac{(1 - \alpha_i)\beta_i x_i^0}{(e_i + b_i)(1 + \xi_i\tilde{v})}\tilde{v} > 0, \qquad \tilde{y}_i = \frac{(e_i\alpha_i + b_i)\beta_i x_i^0}{a_i(e_i + b_i)(1 + \xi_i\tilde{v})}\tilde{v} > 0, \quad i = 1,2.$$

Therefore, an endemic equilibrium $E_1 = (\tilde{x}_1, \tilde{x}_2, \tilde{w}_1, \tilde{w}_2, \tilde{y}_1, \tilde{y}_2, \tilde{v}, 0,)$ exists when $C_1 > 0$ or $(\Psi_1 + \Psi_2 > 1)$.

Now we are ready to define the basic reproduction number R_0 as

$$R_0 = \sum_{i=1}^{2} R_{0i} = \sum_{i=1}^{2} \Psi_i = \sum_{i=1}^{2} \frac{(e_i\alpha_i + b_i)k_i\beta_i x_i^0}{a_i c(e_i + b_i)}.$$

If $v = \dfrac{\mu}{g}$, then we obtain the chronic-infection equilibrium with antibody immune response $E_2 = (\bar{x}_1, \bar{x}_2, \bar{w}_1, \bar{w}_2, \bar{y}_1, \bar{y}_2, \bar{v}, \bar{z})$, where

$$\bar{x}_i = \frac{(g + \mu\sigma_i)x_i^0}{g + \mu\xi_i}, \quad \bar{w}_i = \frac{(1 - \alpha_i)\beta_i\mu x_i^0}{(e_i + b_i)(g + \mu\xi_i)}, \quad \bar{y}_i = \frac{(e_i\alpha_i + b_i)\beta_i\mu x_i^0}{a_i(e_i + b_i)(g + \mu\xi_i)}, \quad i = 1,2,$$

$$\bar{v} = \frac{\mu}{g}, \qquad \bar{z} = \frac{c}{r}\left(\sum_{i=1}^{2} \frac{(e_i\alpha_i + b_i)k_i\beta_i g x_i^0}{a_i c(e_i + b_i)(g + \mu\xi_i)} - 1 \right).$$

We note that E_2 exists when $\sum_{i=1}^{2} \dfrac{(e_i\alpha_i + b_i)k_i\beta_i g x_i^0}{a_i c(e_i + b_i)(g + \mu\xi_i)} > 1$. This equilibrium represents the state that both the viruses and antibodies are present. Let us define the antibody immune response activation number as

$$R_1 = \sum_{i=1}^{2} \frac{(e_i\alpha_i + b_i)k_i\beta_i g x_i^0}{a_i c(e_i + b_i)(g + \mu\xi_i)} = \sum_{i=1}^{2} \frac{R_{0i}}{\left(1 + \frac{\mu}{g}\xi_i\right)},$$

which determines whether a persistent antibody immune response can be established. Then we can write $\bar{z} = \dfrac{c}{r}(R_1 - 1)$. Clearly $R_1 < R_0$. Similar to Section 2.2, one can show that, E_0, $E_1 \in \Omega$ and $E_2 \in \mathring{\Omega}$

3.2 Global stability

Theorem 4. The disease-free equilibrium E_0 of system (36)-(40) is GAS when $R_0 \leq 1$.

Proof. We define a Lyapunov function W_0 as:

$$W_0 = \sum_{i=1}^{2} \gamma_i \left[x_i^0 F\left(\frac{x_i}{x_i^0}\right) + \frac{b_i}{e_i\alpha_i + b_i} w_i + \frac{e_i + b_i}{e_i\alpha_i + b_i} y_i \right] + v + \frac{r}{g} z. \tag{45}$$

We calculate $\frac{dW_0}{dt}$ along the trajectories of (36)-(40)

$$\frac{dW_0}{dt} = \sum_{i=1}^{2} \gamma_i \Bigg[\left(1 - \frac{x_i^0}{x_i}\right)\left(\lambda_i - d_i x_i - \frac{\beta_i x_i v}{1+\sigma_i v}\right) + \frac{b_i}{e_i\alpha_i + b_i}\left(\frac{(1-\alpha_i)\beta_i x_i v}{1+\sigma_i v} - (e_i + b_i) w_i\right)$$
$$+ \frac{e_i + b_i}{e_i\alpha_i + b_i}\left(\frac{\alpha_i\beta_i x_i v}{1+\sigma_i v} + b_i w_i - a_i y_i\right)\Bigg] + \sum_{i=1}^{2} k_i y_i - cv - rvz + \frac{r}{g}(gvz - \mu z). \tag{46}$$

Collecting terms of Eq. (46) we get

$$\begin{aligned}
\frac{dW_0}{dt} &= \sum_{i=1}^{2} \gamma_i \left[d_i \left(1 - \frac{x_i^0}{x_i}\right)(x_i^0 - x_i) + \frac{\beta_i x_i^0 v}{1+\sigma_i v}\right] - cv - \frac{r\mu}{g} z \\
&= -\sum_{i=1}^{2} \gamma_i d_i \frac{(x_i - x_i^0)^2}{x_i} + \sum_{i=1}^{2} \frac{(e_i\alpha_i + b_i) k_i \beta_i x_i^0}{a_i(e_i + b_i)(1+\sigma_i v)} v - cv - \frac{r\mu}{g} z \\
&= -\sum_{i=1}^{2} \gamma_i d_i \frac{(x_i - x_i^0)^2}{x_i} + \left(\sum_{i=1}^{2} \frac{R_{0i}}{(1+\sigma_i v)} - 1\right) cv - \frac{r\mu}{g} z \\
&= -\sum_{i=1}^{2} \gamma_i d_i \frac{(x_i - x_i^0)^2}{x_i} + (R_0 - 1) cv - \sum_{i=1}^{2} \frac{c\sigma_i R_{0i} v^2}{(1+\sigma_i v)} - \frac{r\mu}{g} z.
\end{aligned} \tag{47}$$

If $R_0 \leq 1$ then $\frac{dW_0}{dt} \leq 0$ for all $x_i, v, z > 0$. Similar to the proof of Theorem 1, one can easily show that $\frac{dW_0}{dt} = 0$ at E_0. Then using LaSalle's invariance principle, we can show the global stability of E_0.

Next, we show that the endemic equilibrium E_1 is GAS.

Theorem 5. The chronic-infection equilibrium without antibody immune response E_1 of system (36)-(40) is GAS when $R_1 \leq 1 < R_0$.

Proof. We consider the following Lyapunov function

$$W_1 = \sum_{i=1}^{2} \gamma_i \left[\tilde{x}_i F\left(\frac{x_i}{\tilde{x}_i}\right) + \frac{b_i}{e_i\alpha_i + b_i} \tilde{w}_i F\left(\frac{w_i}{\tilde{w}_i}\right) + \frac{e_i + b_i}{e_i\alpha_i + b_i} \tilde{y}_i F\left(\frac{y_i}{\tilde{y}_i}\right) \right] + \tilde{v} F\left(\frac{v}{\tilde{v}}\right) + \frac{r}{g} z.$$

Calculating $\frac{dW_1}{dt}$ along the solutions of (36)-(40) we get

$$\frac{dW_1}{dt} = \sum_{i=1}^{2} \gamma_i \Bigg[\left(1 - \frac{\tilde{x}_i}{x_i}\right)\left(\lambda_i - d_i x_i - \frac{\beta_i x_i v}{1+\sigma_i v}\right) + \frac{b_i}{e_i\alpha_i + b_i}\left(1 - \frac{\tilde{w}_i}{w_i}\right)\left(\frac{(1-\alpha_i)\beta_i x_i v}{1+\sigma_i v} - (e_i + b_i) w_i\right)$$
$$+ \frac{e_i + b_i}{e_i\alpha_i + b_i}\left(1 - \frac{\tilde{y}_i}{y_i}\right)\left(\frac{\alpha_i\beta_i x_i v}{1+\sigma_i v} + b_i w_i - a_i y_i\right)\Bigg] + \left(1 - \frac{\tilde{v}}{v}\right)\left(\sum_{i=1}^{2} k_i y_i - cv - rvz\right) + \frac{r}{g}(gvz - \mu z). \tag{48}$$

Collecting terms of Eq. (48) we have:

$$\frac{dW_1}{dt} = \sum_{i=1}^{2} \gamma_i \Bigg[\left(1 - \frac{\tilde{x}_i}{x_i}\right)(\lambda_i - d_i x_i) + \frac{\beta_i \tilde{x}_i v}{1+\sigma_i v} + \frac{b_i}{e_i\alpha_i + b_i}\left(-\frac{(1-\alpha_i)\beta_i x_i v \tilde{w}_i}{(1+\sigma_i v) w_i} + (e_i + b_i)\tilde{w}_i\right)$$
$$+ \frac{e_i + b_i}{e_i\alpha_i + b_i}\left(-\frac{\alpha_i\beta_i x_i v \tilde{y}_i}{(1+\sigma_i v) y_i} + \frac{b_i w_i \tilde{y}_i}{y_i} + a_i \tilde{y}_i\right)\Bigg] - cv - \frac{\tilde{v}}{v}\sum_{i=1}^{2} k_i y_i + c\tilde{v} + r\tilde{v}z - \frac{\mu r}{g} z.$$

Using the equilibrium condition for E_1:

$$\lambda_i = d_i\tilde{x}_i + \frac{\beta\tilde{x}_i\tilde{v}}{1+\sigma_i\tilde{v}}, \quad \frac{(1-\alpha_i)\beta_i\tilde{x}_i\tilde{v}}{1+\sigma_i\tilde{v}} = (e_i+b_i)\tilde{w}_i, \quad a_i\tilde{y}_i = \frac{\alpha_i\beta_i\tilde{x}_i\tilde{v}}{1+\sigma_i\tilde{v}} + b_i\tilde{w}_i = \frac{e_i\alpha_i+b_i}{e_i+b_i}\frac{\beta_i\tilde{x}_i\tilde{v}}{1+\sigma_i\tilde{v}},$$

$$c\tilde{v} = \sum_{i=1}^{2}k_i\tilde{y}_i = \sum_{i=1}^{2}\gamma_i\frac{\beta_i\tilde{x}_i\tilde{v}}{1+\sigma_i\tilde{v}}, \quad \frac{\tilde{v}}{v}\sum_{i=1}^{2}k_iy_i = \sum_{i=1}^{2}\gamma_i\frac{\beta_i\tilde{x}_i\tilde{v}}{1+\sigma_i\tilde{v}}\frac{y_i\tilde{v}}{\tilde{y}_iv}, \quad cv = \frac{v}{\tilde{v}}\sum_{i=1}^{2}\gamma_i\frac{\beta_i\tilde{x}_i\tilde{v}}{1+\sigma_i\tilde{v}},$$

$$\frac{\beta_i\tilde{x}_i\tilde{v}}{1+\sigma_i\tilde{v}} = \frac{b_i(1-\alpha_i)}{(e_i\alpha_i+b_i)}\frac{\beta_i\tilde{x}_i\tilde{v}}{(1+\sigma_i\tilde{v})} + \frac{(e_i+b_i)\alpha_i}{(e_i\alpha_i+b_i)}\frac{\beta_i\tilde{x}_i\tilde{v}}{(1+\sigma_i\tilde{v})},$$

we obtain

$$\begin{aligned}
\frac{dW_1}{dt} &= \sum_{i=1}^{2}\gamma_i\Bigg[\left(1-\frac{\tilde{x}_i}{x_i}\right)\left(d_i\tilde{x}_i + \frac{\beta_i\tilde{x}_i\tilde{v}}{1+\sigma_i\tilde{v}} - d_ix_i\right) + \frac{\beta_i\tilde{x}_iv}{1+\sigma_iv} + \frac{b_i}{e_i\alpha_i+b_i}\left(-\frac{(1-\alpha_i)\beta_ix_iv\tilde{w}_i}{(1+\sigma_iv)w_i} + \frac{(1-\alpha_i)\beta_i\tilde{x}_i\tilde{v}}{1+\sigma_i\tilde{v}}\right) \\
&\quad + \frac{e_i+b_i}{e_i\alpha_i+b_i}\left(-\frac{\alpha_i\beta_ix_iv\tilde{y}_i}{(1+\sigma_iv)y_i} + \frac{b_iw_i\tilde{y}_i\tilde{w}_i}{y_i\tilde{w}_i} + \frac{e_i\alpha_i+b_i}{e_i+b_i}\frac{\beta_i\tilde{x}_i\tilde{v}}{1+\sigma_i\tilde{v}}\right) - \frac{y_iv}{\tilde{y}_i\tilde{v}}\frac{\beta_i\tilde{x}_i\tilde{v}}{1+\sigma_i\tilde{v}} - \frac{v}{\tilde{v}}\frac{\beta_i\tilde{x}_i\tilde{v}}{1+\sigma_i\tilde{v}} + \frac{\beta_i\tilde{x}_i\tilde{v}}{1+\sigma_i\tilde{v}}\Bigg] + r\tilde{v}z - \frac{\mu r}{g}z. \\
&= \sum_{i=1}^{2}\gamma_i\Bigg[-d_i\frac{(x_i-\tilde{x}_i)^2}{x_i} + \frac{\beta_i\tilde{x}_i\tilde{v}}{1+\sigma_i\tilde{v}}\left(-1 + \frac{v(1+\sigma_i\tilde{v})}{\tilde{v}(1+\sigma_iv)} - \frac{v}{\tilde{v}} + \frac{1+\sigma_iv}{1+\sigma_i\tilde{v}}\right) \\
&\quad + \frac{b_i(1-\alpha_i)}{(e_i\alpha_i+b_i)}\frac{\beta_i\tilde{x}_i\tilde{v}}{(1+\sigma_i\tilde{v})}\left(5 - \frac{\tilde{x}_i}{x_i} - \frac{x_i\tilde{w}_iv(1+\sigma_i\tilde{v})}{\tilde{x}_iw_i\tilde{v}(1+\sigma_iv)} - \frac{y_i\tilde{v}}{\tilde{y}_iv} - \frac{w_i\tilde{y}_i}{\tilde{w}_iy_i} - \frac{1+\sigma_iv}{1+\sigma_i\tilde{v}}\right) \\
&\quad + \frac{(e_i+b_i)\alpha_i}{(e_i\alpha_i+b_i)}\frac{\beta_i\tilde{x}_i\tilde{v}}{(1+\sigma_i\tilde{v})}\left(4 - \frac{\tilde{x}_i}{x_i} - \frac{x_i\tilde{y}_iv(1+\sigma_i\tilde{v})}{\tilde{x}_iy_i\tilde{v}(1+\sigma_iv)} - \frac{y_i\tilde{v}}{\tilde{y}_iv} - \frac{1+\sigma_iv}{1+\sigma_i\tilde{v}}\right)\Bigg] + \left(\tilde{v} - \frac{\mu}{g}\right)rz \\
&= \sum_{i=1}^{2}\gamma_i\Bigg[-d_i\frac{(x_i-\tilde{x}_i)^2}{x_i} - \frac{\beta_i\tilde{x}_i\tilde{v}}{(1+\sigma_i\tilde{v})}\frac{\sigma_i(v-\tilde{v})^2}{(1+\sigma_iv)(1+\sigma_i\tilde{v})\tilde{v}} \\
&\quad + \frac{b_i(1-\alpha_i)}{(e_i\alpha_i+b_i)}\frac{\beta_i\tilde{x}_i\tilde{v}}{(1+\sigma_i\tilde{v})}\left(5 - \frac{\tilde{x}_i}{x_i} - \frac{x_i\tilde{w}_iv(1+\sigma_i\tilde{v})}{\tilde{x}_iw_i\tilde{v}(1+\sigma_iv)} - \frac{y_i\tilde{v}}{\tilde{y}_iv} - \frac{w_i\tilde{y}_i}{\tilde{w}_iy_i} - \frac{1+\sigma_iv}{1+\sigma_i\tilde{v}}\right) \\
&\quad + \frac{(e_i+b_i)\alpha_i}{(e_i\alpha_i+b_i)}\frac{\beta_i\tilde{x}_i\tilde{v}}{(1+\sigma_i\tilde{v})}\left(4 - \frac{\tilde{x}_i}{x_i} - \frac{x_i\tilde{y}_iv(1+\sigma_i\tilde{v})}{\tilde{x}_iy_i\tilde{v}(1+\sigma_iv)} - \frac{y_i\tilde{v}}{\tilde{y}_iv} - \frac{1+\sigma_iv}{1+\sigma_i\tilde{v}}\right)\Bigg] + \left(\tilde{v} - \frac{\mu}{g}\right)rz. \qquad (49)
\end{aligned}$$

As the same proof of Eq. (32) we can show that $(\tilde{v} - \bar{v}) = \frac{1}{\omega}(R_1 - 1)$, where $\omega = \sum_{i=1}^{2}\frac{k_i\beta_i\lambda_i\xi_i(e_i\alpha_i+b_i)}{a_id_ic(e_i+b_i)(1+\xi_i\bar{v})(1+\xi_i\tilde{v})}$. So, if $R_1 \le 1$ then $\tilde{v} \le \frac{\mu}{g} = \bar{v}$. We have $x_i, w_i, y_i, v > 0$ when $R_0 > 1$. Since the geometrical mean is less than or equal to the arithmetical mean, then the third and fourth terms of Eq. (49) are less than or equal zero, then if $R_1 \le 1$ then $\frac{dW_1}{dt} \le 0$ for all $x_i, w_i, y_i, v, z > 0$. Clearly, $\frac{dW_1}{dt} = 0$ occurs at E_1. LaSalle's invariance principle implies global stability of E_1.

Theorem 6. The chronic-infection equilibrium with antibody immune response E_2 of system (36)-(40) is GAS when $R_1 > 1$.

Proof. Define Lyapunov function W_2 as:

$$W_2 = \sum_{i=1}^{2}\gamma_i\left[\bar{x}_iF\left(\frac{x_i}{\bar{x}_i}\right) + \frac{b_i}{e_i\alpha_i+b_i}\bar{w}_iF\left(\frac{w_i}{\bar{w}_i}\right) + \frac{e_i+b_i}{e_i\alpha_i+b_i}\bar{y}_iF\left(\frac{y_i}{\bar{y}_i}\right)\right] + \bar{v}F\left(\frac{v}{\bar{v}}\right) + \frac{r}{g}\bar{z}F\left(\frac{z}{\bar{z}}\right).$$

The time derivative of W_2 along the trajectories of (36)-(40) is given by

$$\begin{aligned}
\frac{dW_2}{dt} &= \sum_{i=1}^{2}\gamma_i\Bigg[\left(1-\frac{\bar{x}_i}{x_i}\right)\left(\lambda_i - d_ix_i - \frac{\beta_ix_iv}{1+\sigma_iv}\right) + \frac{b_i}{e_i\alpha_i+b_i}\left(1-\frac{\bar{w_i}}{w_i}\right)\left(\frac{(1-\alpha_i)\beta_ix_iv}{1+\sigma_iv} - (e_i+b_i)w_i\right) \\
&\quad + \frac{e_i+b_i}{e_i\alpha_i+b_i}\left(1-\frac{\bar{y}_i}{y_i}\right)\left(\frac{\alpha_i\beta_ix_iv}{1+\sigma_iv} + bw_i - a_iy_i\right)\Bigg] + \left(1-\frac{\bar{v}}{v}\right)\left(\sum_{i=1}^{2}k_iy_i - cv - rvz\right) + \frac{r}{g}\left(1-\frac{\bar{z}}{z}\right)(gvz - \mu z).
\end{aligned} \qquad (50)$$

Collecting terms of Eq. (50) and using the equilibrium condition for E_2

$$\lambda_i = d_i\bar{x}_i + \frac{\beta\bar{x}_i\bar{v}}{1+\sigma_i\bar{v}}, \ \frac{(1-\alpha_i)\beta_i\bar{x}_i\bar{v}}{1+\sigma_i\bar{v}} = (e_i+b_i)\bar{w}_i, \ \frac{\alpha_i\beta_i\bar{x}_i\bar{v}}{1+\sigma_i\bar{v}} + b_i\bar{w}_i = a_i\bar{y}_i, \ c\bar{v} + r\bar{v}\bar{z} = \sum_{i=1}^{2}k_i\bar{y}_i,$$

$$\frac{e_i+b_i}{e_i\alpha_i+b_i}a_i\bar{y}_i=\frac{\beta_i\bar{x}_i\bar{v}}{1+\sigma_i\bar{v}}=\frac{b_i(1-\alpha_i)}{(e_i\alpha_i+b_i)}\frac{\beta_i\bar{x}_i\bar{v}}{(1+\sigma_i\bar{v})}+\frac{(e_i+b_i)\alpha_i}{(e_i\alpha_i+b_i)}\frac{\beta_i\bar{x}_i\bar{v}}{(1+\sigma_i\bar{v})}$$

Eq. (50) becomes

$$\begin{aligned}\frac{dW_2}{dt}&=\sum_{i=1}^{2}\gamma_i\Bigg[-d_i\frac{(x_i-\bar{x}_i)^2}{x_i}-\frac{\beta_i\bar{x}_i\bar{v}}{(1+\sigma_i\bar{v})}\frac{\sigma_i(v-\bar{v})^2}{\bar{v}(1+\sigma_i v)(1+\sigma_i\bar{v})}\\&+\frac{b_i(1-\alpha_i)}{(e_i\alpha_i+b_i)}\frac{\beta_i\bar{x}_i\bar{v}}{(1+\sigma_i\bar{v})}\left(5-\frac{\bar{x}_i}{x_i}-\frac{x_i\bar{w}_i v(1+\sigma_i\bar{v})}{\bar{x}_i w_i\bar{v}(1+\sigma_i v)}-\frac{y_i\bar{v}}{\bar{y}_i v}-\frac{w_i\bar{y}_i}{\bar{w}_i y_i}-\frac{1+\sigma_i v}{1+\sigma_i\bar{v}}\right)\\&+\frac{(e_i+b_i)\alpha_i}{(e_i\alpha_i+b_i)}\frac{\beta_i\bar{x}_i\bar{v}}{(1+\sigma_i\bar{v})}\left(4-\frac{\bar{x}_i}{x_i}-\frac{x_i\bar{y}_i v(1+\sigma_i\bar{v})}{\bar{x}_i y_i\bar{v}(1+\sigma_i v)}-\frac{y_i\bar{v}}{\bar{y}_i v}-\frac{1+\sigma_i v}{1+\sigma_i\bar{v}}\right)\Bigg]\end{aligned}$$

Thus, if $R_1>1$ then x_i, w_i, y_i, v and $z>0$. Similar to the proof of Theorem 3, one can show that E_2 is GAS.

4 Model with general incidence rate

In this section, we propose a viral infection model with latently infected cells and antibody immune response. The incidence rate of infection is represented by a general function of the populations of the uninfected target cells and free viruses.

$$\dot{x}_i=\lambda_i-d_i x_i-f_i(x_i,v),\qquad i=1,2, \tag{51}$$

$$\dot{w}_i=(1-\alpha_i)f_i(x_i,v)-(e_i+b_i)w_i,\qquad i=1,2, \tag{52}$$

$$\dot{y}_i=\alpha_i f_i(x_i,v)+b_i w_i-a_i y_i,\qquad i=1,2, \tag{53}$$

$$\dot{v}=\sum_{i=1}^{2}k_i y_i-cv-rvz, \tag{54}$$

$$\dot{z}=gvz-\mu z, \tag{55}$$

where the function $f_i(x_i,v)$ represents the rate of the uninfected target cells to be infected by the viruses.

Assumption A1 For $i=1,2$, function f_i satisfies:

(i) $f_i(x_i,v)$ is positive, continuous, and differentiable,

(ii) $\dfrac{\partial f_i(x_i,v)}{\partial v}>0$ and $\dfrac{\partial f_i(x_i,v)}{\partial x_i}>0$ for any x_i,$v>0$. Furthermore, $\dfrac{\partial f_i(x_i,0)}{\partial v}>0$ for any $x_i>0$,

(iii) $f_i(x_i,0)=f_i(0,v)=0$, for all $x_i>0$ and $v>0$.

Assumption A2 For $i=1,2$, function f_i satisfies:

(i) $f_i(x_i,v)\le v\dfrac{\partial f_i(x_i,0)}{\partial v}$, for all $v>0$.

(ii) $\frac{d}{dx_i}\left(\dfrac{\partial f_i(x_i,0)}{\partial v}\right)>0$

4.1 Equilibria and biological thresholds

We define the basic infection reproduction number of system (51)-(55) as:

$$R_0=\sum_{i=1}^{2}\frac{k_i(e_i\alpha_i+b_i)}{a_i c(e_i+b_i)}\frac{\partial f_i(x_i^0,0)}{\partial v}.$$

The equilibria of (51)-(55) satisfy the following equations:

$$\lambda_i-d_i x_i-f_i(x_i,v)=0, \tag{56}$$

$$(1-\alpha_i)f_i(x_i,v)-(e_i+b_i)w_i=0, \tag{57}$$

$$\alpha_i f_i(x_i,v)+b_i w_i-a_i y_i=0, \tag{58}$$

$$\sum_{i=1}^{2}k_i y_i-cv-rvz=0, \tag{59}$$

$$(gv-\mu)z=0. \tag{60}$$

Equation (60) has two possible solutions, $z = 0$ or $v = \mu/g$. When $z = 0$, we obtain two equilibria, the infection-free equilibrium $E_0 = (x_1^0, x_2^0, 0, 0, 0, 0, 0, 0)$, where $x_i^0 = \dfrac{\lambda_i}{d_i}$, $i = 1, 2$ and the infected steady state without antibody immune response $E_1 = (\tilde{x}_1, \tilde{x}_2, \tilde{w}_1, \tilde{w}_2, \tilde{y}_1, \tilde{y}_2, \tilde{v}, 0,)$, where the coordinates satisfy the equalities:

$$\lambda_i = d_i\tilde{x}_i + f_i(\tilde{x}_i, \tilde{v}), \quad (1-\alpha_i) f_i(\tilde{x}_i, \tilde{v}) = (e_i + b_i)\tilde{w}_i, \; \alpha_i f_i(\tilde{x}_i, \tilde{v}) + b_i\tilde{w}_i = a_i\tilde{y}_i, \; \sum_{i=1}^{2} k_i\tilde{y}_i = c\tilde{v}. \tag{61}$$

The other possibility of Eq. (60) $z \neq 0$ leads to $\bar{v} = \dfrac{\mu}{g}$. Substitute the value of $\bar{v}$ in Eq. (56) and let

$$\Pi(x_i) = \lambda_i - d_i x_i - f_i(x_i, \bar{v}) = 0.$$

According to Assumptions A1, Π is a strictly decreasing function of x_i. Besides, $\Pi(0) = \lambda_i > 0$ and $\Pi(x_i^0) = -f_i(x_i^0, \bar{v}) < 0$. Thus, there exists a unique $\bar{x}_i \in (0, x_i^0)$ such that $\Pi(\bar{x}_i) = 0$. From Eqs. (57)-(59) we have

$$\bar{w}_i = \frac{(1-\alpha_i) f_i(\bar{x}_i, \bar{v})}{(e_i + b_i)}, \quad \bar{y}_i = \frac{(e_i\alpha_i + b_i) f_i(\bar{x}_i, \bar{v})}{a_i(e_i + b_i)}, \quad \bar{z} = \frac{c}{r}\left[\sum_{i=1}^{2} \frac{k_i(e_i\alpha_i + b_i) f_i(\bar{x}_i, \bar{v})}{a_i c(e_i + b_i)\bar{v}} - 1\right].$$

Thus $\bar{w}_i > 0$ and $\bar{y}_i > 0$, moreover, $\bar{z} > 0$ when $\sum_{i=1}^{2} \dfrac{k_i(e_i\alpha_i + b_i) f_i(\bar{x}_i, \bar{v})}{a_i c(e_i + b_i)\bar{v}} > 1$. Now we define the antibody immune response activation number as:

$$R_1 = \sum_{i=1}^{2} \frac{k_i(e_i\alpha_i + b_i) f_i(\bar{x}_i, \bar{v})}{a_i c(e_i + b_i)\bar{v}}.$$

Hence, $\bar{z}$ can be rewritten as $\bar{z} = \frac{c}{r}(R_1 - 1)$. It follows that, there exists a chronic-infection equilibrium with antibody immune response $E_2 = (\bar{x}_1, \bar{w}_1, \bar{y}_1, \bar{x}_2, \bar{w}_2, \bar{y}_2, \bar{v}, \bar{z})$ when $R_1 > 1$. Clearly from **Assumptions A1** and **A2**, we have

$$R_1 = \sum_{i=1}^{2} \frac{k_i(e_i\alpha_i + b_i) f_i(\bar{x}_i, \bar{v})}{a_i c(e_i + b_i)\bar{v}} < \sum_{i=1}^{2} \frac{k_i(e_i\alpha_i + b_i)}{a_i c(e_i + b_i)\bar{v}} \frac{\partial f_i(\bar{x}_i, 0)}{\partial \bar{v}}\bar{v} < \sum_{i=1}^{2} \frac{k_i(e_i\alpha_i + b_i)}{a_i c(e_i + b_i)} \frac{\partial f_i(x_i^0, 0)}{\partial v} = R_0.$$

5 Global stability analysis

Theorem 7. Let Assumptions A1-A2 be hold true and $R_0 \leq 1$, then the infection-free equilibrium E_0 for system (51)-(55) is GAS.

Proof. Define a Lyapunov functional W_0 as follows:

$$W_0 = \sum_{i=1}^{2} \gamma_i \left[x_i - x_i^0 - \int_{x_i^0}^{x_i} \lim_{v \to 0^+} \frac{f_i(x_i^0, v)}{f_i(s_i, v)} ds_i + \frac{b_i}{e_i\alpha_i + b_i} w_i + \frac{e_i + b_i}{e_i\alpha_i + b_i} y_i \right] + v + \frac{r}{g} z.$$

Calculating $\dfrac{dW_0}{dt}$ along the trajectories of (51)-(55) as:

$$\begin{aligned}
\frac{dW_0}{dt} &= \sum_{i=1}^{2} \gamma_i \Bigg[\left(1 - \lim_{v \to 0^+} \frac{f_i(x_i^0, v)}{f_i(x_i, v)}\right) (\lambda_i - d_i x_i - f_i(x_i, v)) + \frac{b_i}{e_i\alpha_i + b_i} ((1-\alpha_i) f_i(x_i, v) - (e_i + b_i) w_i) \\
&\quad + \frac{e_i + b_i}{e_i\alpha_i + b_i} (\alpha_i f_i(x_i, v) + b_i w_i - a_i y_i) \Bigg] + \sum_{i=1}^{2} k_i y_i - cv - rvz + \frac{r}{g}(gvz - \mu z) \\
&= \sum_{i=1}^{2} \gamma_i \lambda_i \left(1 - \frac{\partial f_i(x_i^0, 0)/\partial v}{\partial f_i(x_i, 0)/\partial v}\right) \left(1 - \frac{x_i}{x_i^0}\right) + (R_0 - 1) cv - \frac{r\mu}{g} z.
\end{aligned} \tag{62}$$

Based on Assumption A2, the first term of Eq. (62) is less than or equal zero. Therefore if $R_0 \leq 1$, then $\frac{dW_0}{dt} \leq 0$ for all $x_i, v, z > 0$. Similar to the previous sections, one can show that E_0 is GAS.

Now we need to the following Assumption to proof that, E_1 and E_2 for the system (51)-(55) are GAS.

Assumption A3 Function $f_i(x_i, v)$ satisfies the following:

$$\left(\frac{f_i(x_i,v)}{f_i(x_i,\tilde{v})}-\frac{v}{\tilde{v}}\right)\left(1-\frac{f_i(x_i,\tilde{v})}{f_i(x_i,v)}\right)\le 0, \quad \left(\frac{f_i(x_i,v)}{f_i(x_i,\bar{v})}-\frac{v}{\bar{v}}\right)\left(1-\frac{f_i(x_i,\bar{v})}{f_i(x_i,v)}\right)\le 0, \quad x_i, v>0,$$

Theorem 8. Suppose that Assumptions A1-A3 are satisfied, E_1 exists and $R_1 \le 1$, then E_1 for system (51)-(55) is GAS.

Proof. We construct the following Lyapunov functional

$$W_1=\sum_{i=1}^{2}\gamma_i\left[x_i-\tilde{x}_i-\int_{\tilde{x}_i}^{x_i}\frac{f_i(\tilde{x}_i,\tilde{v})}{f_i(s_i,\tilde{v})}ds_i+\frac{b_i}{e_i\alpha_i+b_i}\tilde{w}_iF\left(\frac{w_i}{\tilde{w}_i}\right)+\frac{e_i+b_i}{e_i\alpha_i+b_i}\tilde{y}_iF\left(\frac{y_i}{\tilde{y}_i}\right)\right]+\tilde{v}F\left(\frac{v}{\tilde{v}}\right)+\frac{r}{g}z.$$

The time derivative of W_1 along the trajectories of (51)-(55) is given by

$$\begin{aligned}\frac{dW_1}{dt}=&\sum_{i=1}^{2}\gamma_i\Bigg[\left(1-\frac{f_i(\tilde{x}_i,\tilde{v})}{f_i(x_i,\tilde{v})}\right)(\lambda_i-d_ix_i-f_i(x_i,v))+\frac{b_i}{e_i\alpha_i+b_i}\left(1-\frac{\tilde{w}_i}{w_i}\right)((1-\alpha_i)f_i(x_i,v)-(e_i+b_i)w_i)\\&+\frac{e_i+b_i}{e_i\alpha_i+b_i}\left(1-\frac{\tilde{y}_i}{y_i}\right)(\alpha_if_i(x_i,v)+b_iw_i-a_iy_i)\Bigg]+\left(1-\frac{\tilde{v}}{v}\right)(\left(\sum_{i=1}^{2}k_iy_i-cv-rvz\right)+\frac{r}{g}(gvz-\mu z).\end{aligned} \tag{63}$$

Collecting terms of Eq. (63) we get

$$\begin{aligned}\frac{dW_1}{dt}=&\sum_{i=1}^{2}\gamma_i\Bigg[\left(1-\frac{f_i(\tilde{x}_i,\tilde{v})}{f_i(x_i,\tilde{v})}\right)(\lambda_i-d_ix_i)+f_i(x_i,v)\frac{f_i(\tilde{x}_i,\tilde{v})}{f_i(x_i,\tilde{v})}-\frac{b_i(1-\alpha_i)}{e_i\alpha_i+b_i}f_i(x_i,v)\frac{\tilde{w}_i}{w_i}\\&+\frac{(e_i+b_i)}{e_i\alpha_i+b_i}b_i\tilde{w}_i-\frac{(e_i+b_i)\alpha_i}{e_i\alpha_i+b_i}f_i(x_i,v)\frac{\tilde{y}_i}{y_i}-\frac{(e_i+b_i)b_iw_i}{e_i\alpha_i+b_i}\frac{\tilde{y}_i}{y_i}-\frac{e_i+b_i}{e_i\alpha_i+b_i}a_i\tilde{y}_i\Bigg]\\&-cv-\sum_{i=1}^{2}k_iy_i\frac{\tilde{v}}{v}+c\tilde{v}+r\tilde{v}z-\frac{r\mu}{g}z.\end{aligned}$$

Using the equilibrium condition for E_1:

$$\lambda_i=d_i\tilde{x}_i+f_i(\tilde{x}_i,\tilde{v}),\quad (1-\alpha_i)f_i(\tilde{x}_i,\tilde{v})=(e_i+b_i)\tilde{w}_i,\quad a_i\tilde{y}_i=\alpha_if_i(\tilde{x}_i,\tilde{v})+b_i\tilde{w}_i,\quad cv=\frac{v}{\tilde{v}}\sum_{i=1}^{2}\gamma_if_i(\tilde{x}_i,\tilde{v}),$$

$$c\tilde{v}=\sum_{i=1}^{2}k_i\tilde{y}_i=\sum_{i=1}^{2}\gamma_if_i(\tilde{x}_i,\tilde{v}),\quad \frac{e_i+b_i}{e_i\alpha_i+b_i}a_i\tilde{y}_i=f_i(\tilde{x}_i,\tilde{v})=\frac{b_i(1-\alpha_i)}{(e_i\alpha_i+b_i)}f_i(\tilde{x}_i,\tilde{v})+\frac{(e_i+b_i)\alpha_i}{(e_i\alpha_i+b_i)}f_i(\tilde{x}_i,\tilde{v}),$$

we obtain

$$\begin{aligned}\frac{dW_1}{dt}=&\sum_{i=1}^{2}\gamma_i\Bigg[d_i\tilde{x}_i\left(1-\frac{f_i(\tilde{x}_i,\tilde{v})}{f_i(x_i,\tilde{v})}\right)\left(1-\frac{x_i}{\tilde{x}_i}\right)+\left(1-\frac{f_i(x_i,\tilde{v})}{f_i(x_i,v)}\right)\left(\frac{f_i(x_i,v)}{f_i(x_i,\tilde{v})}-\frac{v}{\tilde{v}}\right)\\&+\frac{b_i(1-\alpha_i)}{(e_i\alpha_i+b_i)}f_i(\tilde{x}_i,\tilde{v})\left(5-\frac{f_i(\tilde{x}_i,\tilde{v})}{f_i(x_i,\tilde{v})}-\frac{\tilde{w}_if_i(x_i,v)}{w_if_i(\tilde{x}_i,\tilde{v})}-\frac{w_i\tilde{y}_i}{\tilde{w}_iy_i}-\frac{y_i\tilde{v}}{\tilde{y}_iv}-\frac{vf_i(x_i,\tilde{v})}{\tilde{v}f_i(x_i,v)}\right)\\&+\frac{(e_i+b_i)\alpha_i}{(e_i\alpha_i+b_i)}f_i(\tilde{x}_i,\tilde{v})\left(4-\frac{f_i(\tilde{x}_i,\tilde{v})}{f_i(x_i,\tilde{v})}-\frac{\tilde{y}_if_i(x_i,v)}{y_if_i(\tilde{x}_i,\tilde{v})}-\frac{y_i\tilde{v}}{\tilde{y}_iv}-\frac{vf_i(x_i,\tilde{v})}{\tilde{v}f_i(x_i,v)}\right)\Bigg]+r\left(\tilde{v}-\frac{\mu}{g}\right)z.\end{aligned} \tag{64}$$

From **Assumptions A1 and A3**, we get that the first and second terms of Eq. (64) are less than or equal zero. Because the geometrical mean is less than or equal to the arithmetical mean, then the third and fourth terms of Eq. (64) are less than or equal zero. Now we show that if $R_1 \le 1$ then $\tilde{v} \le \frac{\mu}{r} = \bar{v}$. This can be achieved if we show that

$$sgn(\bar{x}_i-\tilde{x}_i)=sgn(\tilde{v}-\bar{v})=sgn(R_1-1).$$

Applying **Assumptions A1-A2**, we have

$$(f_i(\bar{x}_i,\tilde{v})-f_i(\tilde{x}_i,\tilde{v}))(\bar{x}_i-\tilde{x}_i)>0, \tag{65}$$

$$(f_i(\tilde{x}_i,\bar{v})-f_i(\tilde{x}_i,\tilde{v}))(\bar{v}-\tilde{v})>0, \quad (f_i(\bar{x}_i,\bar{v})-f_i(\bar{x}_i,\tilde{v}))(\bar{v}-\tilde{v})>0. \tag{66}$$

Using **Assumption A3** with $x_i=\tilde{x}_i$ and $v=\bar{v}$, we get

$$(f_i(\tilde{x}_i,\bar{v})\tilde{v}-f_i(\tilde{x}_i,\tilde{v})\bar{v})(f_i(\tilde{x}_i,\bar{v})-f_i(\tilde{x}_i,\tilde{v}))\le 0$$

It follows from inequality (66) that

$$((f_i(\tilde{x}_i,\bar{v})\tilde{v}-f_i(\tilde{x}_i,\tilde{v})\bar{v}))(\tilde{v}-\bar{v})>0. \tag{67}$$

Suppose that, $sgn(\bar{x}_i-\tilde{x}_i)=sgn(\bar{v}-\tilde{v})$. Using the conditions of the equilibria E_1 and E_2 we have

$$(\lambda_i-d_i\bar{x}_i)-(\lambda_i-d_i\tilde{x}_i)=f_i(\bar{x}_i,\bar{v})-f_i(\tilde{x}_i,\tilde{v})=f_i(\bar{x}_i,\bar{v})-f_i(\bar{x}_i,\tilde{v})+f_i(\bar{x}_i,\tilde{v})-f_i(\tilde{x}_i,\tilde{v}),$$

and from inequalities (65) and (66) we get $sgn(\tilde{x}_i-\bar{x}_i)=sgn(\bar{x}_i-\tilde{x}_i)$, which leads to contradiction. Thus, $sgn(\bar{x}_i-\tilde{x}_i)=sgn(\tilde{v}-\bar{v})$. Using the equilibrium conditions for E_1 we have $\sum_{i=1}^{2}\frac{k_i(e_i\alpha_i+b_i)f_i(\tilde{x}_i,\tilde{v})}{a_ic(e_i+b_i)\tilde{v}}=1$, then

$$\begin{aligned} R_1-1&=\sum_{i=1}^{2}\frac{k_i(e_i\alpha_i+b_i)}{a_ic(e_i+b_i)}\left(\frac{f_i(\bar{x}_i,\bar{v})}{\bar{v}}-\frac{f_i(\tilde{x}_i,\tilde{v})}{\tilde{v}}\right)\\ &=\sum_{i=1}^{2}\frac{k_i(e_i\alpha_i+b_i)}{a_ic(e_i+b_i)}\left(\frac{1}{\bar{v}}(f_i(\bar{x}_i,\bar{v})-f_i(\tilde{x}_i,\bar{v}))+\frac{1}{\tilde{v}\bar{v}}(f_i(\tilde{x}_i,\bar{v})\tilde{v}-f_i(\tilde{x}_i,\tilde{v})\bar{v})\right). \end{aligned}$$

From inequalities (65) and (67) we get $sgn(R_1-1)=sgn(\tilde{v}-\bar{v})$. It follows that, if $R_1\le 1$ then $\tilde{v}\le\frac{\mu}{r}=\bar{v}$. Therefore, if $R_1\le 1$ then $\frac{dW_1}{dt}\le 0$ for all $x_i,w_i,y_i,v,z>0$, where the equality occurs at the equilibrium E_1. LaSalle's invariance principle implies the global stability of E_1.

Theorem 9. Let **Assumptions A1-A3** be hold true and $R_1>1$, then chronic-infection equilibrium with antibody immune response E_2 for system (51)-(55) is GAS.

Proof. We construct the following Lyapunov functional

$$W_2=\sum_{i=1}^{2}\gamma_i\left[x_i-\bar{x}_i-\int_{\bar{x}_i}^{x_i}\frac{f_i(\bar{x}_i,\bar{v})}{f_i(s_i,\bar{v})}ds_i+\frac{b_i}{e_i\alpha_i+b_i}\bar{w}_iF\left(\frac{w_i}{\bar{w}_i}\right)+\frac{e_i+b_i}{e_i\alpha_i+b_i}\bar{y}_iF\left(\frac{y_i}{\bar{y}_i}\right)\right]+\bar{v}F\left(\frac{v}{\bar{v}}\right)+\frac{r}{g}\bar{z}F\left(\frac{z}{\bar{z}}\right).$$

We calculate the time derivative of W_2 along the trajectories of (51)-(55) as:

$$\begin{aligned}\frac{dW_2}{dt}&=\sum_{i=1}^{2}\gamma_i\left[\left(1-\frac{f_i(\bar{x}_i,\bar{v})}{f_i(x_i,\bar{v})}\right)(\lambda_i-d_ix_i-f_i(x_i,v))+\frac{b_i}{e_i\alpha_i+b_i}\left(1-\frac{\bar{w}_i}{w}\right)((1-\alpha_i)f_i(x_i,v)-(e_i+b_i)w_i)\right.\\ &\left.+\frac{e_i+b_i}{e_i\alpha_i+b_i}\left(1-\frac{\bar{y}_i}{y}\right)(\alpha_if_i(x_i,v)+b_iw_i-a_iy_i)\right]+\left(1-\frac{\bar{v}}{v}\right)(\sum_{i=1}^{2}k_iy_i-cv-rvz)+\frac{r}{g}\left(1-\frac{\bar{z}}{z}\right)(gvz-\mu z).\end{aligned} \tag{68}$$

Collecting terms of Eq. (68) and using the equilibrium conditions for E_2

$$\lambda_i=d_i\bar{x}_i+f_i(\bar{x}_i,\bar{v}),\quad (1-\alpha_i)f_i(\bar{x}_i,\bar{v})=(e_i+b_i)\bar{w}_i,\quad a_i\bar{y}_i=\alpha_if_i(\bar{x}_i,\bar{v})+b_i\bar{w}_i,\quad c\bar{v}=\sum_{i=1}^{2}\gamma_if_i(\bar{x}_i,\bar{v})-r\bar{v}\bar{z},$$

$$cv=\frac{v}{\bar{v}}\sum_{i=1}^{2}\gamma_if_i(\bar{x}_i,\bar{v})-rv\bar{z},\quad \frac{e_i+b_i}{e_i\alpha_i+b_i}a_i\bar{y}_i=f_i(\bar{x}_i,\bar{v})=\frac{b_i(1-\alpha_i)}{(e_i\alpha_i+b_i)}f_i(\bar{x}_i,\bar{v})+\frac{(e_i+b_i)\alpha_i}{(e_i\alpha_i+b_i)}f_i(\bar{x}_i,\bar{v}),$$

we get

$$\frac{dW_2}{dt} = \sum_{i=1}^{2}\gamma_i\Bigg[d_i\bar{x}_i\left(1-\frac{f_i(\bar{x}_i,\bar{v})}{f_i(x_i,\bar{v})}\right)(1-\frac{x_i}{\bar{x}_i}) + f_i(\bar{x}_i,\bar{v})\left(1-\frac{f_i(x_i,\bar{v})}{f_i(x_i,v)}\right)\left(\frac{f_i(x_i,v)}{f_i(x_i,\bar{v})}-\frac{v}{\bar{v}}\right)$$
$$+\frac{b_i(1-\alpha_i)}{(e_i\alpha_i+b_i)}f_i(\bar{x}_i,\bar{v})\left(5-\frac{f_i(\bar{x}_i,\bar{v})}{f_i(x_i,\bar{v})}-\frac{\bar{w}_i f_i(x_i,v)}{w_i f_i(\bar{x}_i,\bar{v})}-\frac{\bar{y}_i w_i}{y_i\bar{w}_i}-\frac{y_i\bar{v}}{\bar{y}_i v}-\frac{v f_i(x_i,\bar{v})}{\bar{v} f_i(x_i,v)}\right)$$
$$+\frac{(e_i+b_i)\alpha_i}{(e_i\alpha_i+b_i)}f_i(\bar{x}_i,\bar{v})\left(4-\frac{f_i(\bar{x}_i,\bar{v})}{f_i(x_i,\bar{v})}-\frac{\bar{y}_i f_i(x_i,v)}{y_i f_i(\bar{x}_i,\bar{v})}-\frac{y_i\bar{v}}{\bar{y}_i v}-\frac{v f_i(x_i,\bar{v})}{\bar{v} f_i(x_i,v)}\right)\Bigg] \tag{69}$$

Thus, if $R_1 > 1$ then $\bar{x}_i, \bar{w}_i, \bar{y}_i, \bar{v}$ and $\bar{z} > 0$. From Assumptions A1 and A3, we get that the first and second terms of Eq. (69) are less than or equal zero. Since the arithmetical mean is greater than or equal to the geometrical mean, then $\frac{dW_2}{dt} \le 0$. It can be seen that, $\frac{dW_2}{dt} = 0$ if and only if $x_i = \bar{x}_i$, $w_i = \bar{w}_i$ and $v = \bar{v}$. From Eq. (54), if $v = \bar{v}$ and $y_i = \bar{y}_i$ then $\dot{v} = 0$ and $0 = \sum_{i=1}^{2} k\bar{y}_i - c\bar{v} - r\bar{v}\bar{z} = 0$, which yields $z = \bar{z}$ and hence $\frac{dW_2}{dt}$ equal to zero at E_2. LaSalle's invariance principle implies global stability of E_2.

5.1 Special forms of the incidence rate

By using the Lyapunov direct method, we have established a set of conditions on $f_i(x_i, v),\ i = 1, 2$ ensuring the global asymptotic stability of the equilibria of model (51)-(55). Now we introduce some forms of the incidence rate and verify A1-A3.

(1) Bilinear incidence rate: $f_i(x_i, v) = \beta_i x_i v$,
(2) Saturation functional response: $f_i(x_i, v) = \frac{\beta_i x_i v}{1+\eta_i v}$,
(3) Beddington-DeAngelis functional response: $f_i(x_i, v) = \frac{\beta_i x_i v}{1+\gamma_i x_i+\eta_i v}$,
(4) Crowley-Martin functional response: $f_i(x_i, v) = \frac{\beta_i x_i v}{(1+\gamma_i x_i)(1+\eta_i v)}$,
(5) Hill type incidence rate: $f_i(x_i, v) = \frac{\beta_i x_i^n v}{\gamma_i^n + x_i^n}$, where $\beta_i, \gamma_i, n > 0$.

One can easily show that A1-A3 for the functions $f_i,\ i = 1, 2$ given above.

Now we verify Assumptions A1-A3 for the function $f_i(x_i, v) = \frac{\beta_i x_i^n v}{\gamma_i^n + x_i^n}$, $i = 1, 2$. We have $f_i(x_i, v) > 0$ for all $x_i > 0$, $v > 0$, $f_i(0, v) = f_i(x_i, 0) = 0$ and

$$\frac{\partial f_i(x_i,v)}{\partial x_i} = \frac{n\beta_i\gamma_i^n x_i^{n-1} v}{(\gamma_i^n + x_i^n)^2}, \qquad \frac{\partial f_i(x_i,v)}{\partial v} = \frac{\beta_i x_i^n}{\gamma_i^n + x_i^n} = \frac{\partial f_i(x_i,0)}{\partial v}.$$

Then, for all $x_i > 0, v > 0$, we have $\frac{\partial f_i(x_i,v)}{\partial x_i} > 0$, $\frac{\partial f_i(x_i,v)}{\partial v} > 0$ and $\frac{\partial f_i(x_i,0)}{\partial v} > 0$ if $n > 0$. Therefore Assumptions A1 is satisfied. We have also

$$f_i(x_i, v) = \frac{\beta_i x_i^n v}{\gamma_i^n + x_i^n} = v\frac{\beta_i x_i^n}{\gamma_i^n + x_i^n} = v\frac{\partial f_i(x_i,0)}{\partial v},$$
$$\frac{d}{dx_i}\left(\frac{\partial f_i(x_i^0,0)/\partial v}{\partial f_i(x_i,0)/\partial v}\right) = -\frac{n\gamma_i^n (x_i^0)^n}{(\gamma_i^n + (x_i^0)^n)x_i^{n+1}} < 0,$$

then, Assumptions A2 is satisfied. Moreover,

$$\left(\frac{f_i(x_i,v)}{f_i(x_i,\tilde{v})} - \frac{v}{\tilde{v}}\right)\left(1 - \frac{f_i(x_i,\tilde{v})}{f_i(x_i,v)}\right) = \left(\frac{v}{\tilde{v}} - \frac{v}{\tilde{v}}\right)\left(1 - \frac{\tilde{v}}{v}\right) = 0.$$

Thus, Assumptions A3 is satisfied. In this case, R_0 and R_1 are given by

$$R_0 = \sum_{i=1}^{2}\frac{k_i(e_i\alpha_i+b_i)}{a_i c(e_i+b_i)}\frac{\partial f_i(x_i^0,0)}{\partial v} = \sum_{i=1}^{2}\frac{k_i(e_i\alpha_i+b_i)}{a_i c(e_i+b_i)}\frac{\beta_i (x_i^0)^n}{\gamma_i^n + (x_i^0)^n},$$
$$R_1 = \sum_{i=1}^{2}\frac{k_i(e_i\alpha_i+b_i) f_i(\bar{x}_i,\bar{v})}{a_i c(e_i+b_i)\bar{v}} = \sum_{i=1}^{2}\frac{k_i(e_i\alpha_i+b_i)}{a_i c(e_i+b_i)}\frac{\beta_i\bar{x}_i^n}{\gamma_i^n + \bar{x}_i^n}.$$

6 Numerical simulations

In this section, we will perform some numerical simulations to confirm our theoretical results. Let us consider model (51)-(55) with the incidence rate $f_i(x_i, v) = \frac{\beta_i x_i^n v}{\gamma_i^n + x_i^n}$, $i = 1, 2$. In Table 1, we provide the values of some parameters of model (51)-(55) with the incidence rate given by the function f_i. The effect of the parameter ε on the dynamical behavior of the system will be discussed below in details.In order to investigate the theoretical

Table 1: The values of the parameters of model (51)-(55).

$Parameter$	λ_1	λ_2	$\bar{\beta}_1$	$\bar{\beta}_2$	d_1	d_2	α_1	α_2	e_1	e_2	b_1	b_2	γ_1
$Value$	6.03198	0.03198	0.05	0.08	0.01	0.01	0.5	0.5	0.02	0.02	0.2	0.2	0.1
$Parameter$	γ_2	k_1	k_2	a_1	a_2	f	r	c	μ	g	n	ε	
$Value$	0.5	10	5	0.3	0.1	0.3	0.5	3	0.07	0.1	1	Varied	

results involved in Theorems 7-9, we shall study the following cases:

Case (I): $R_0 \leq 1$. Choosing $\varepsilon = 0.85$ and using the data in Table 1, we have $R_0 = 0.899$ and $R_1 = 0.641$. Since $R_0 < 1$, then according to Theorem 7, the infection-free equilibrium E_0 is GAS. Evidently, Figures 1-8 show that, the numerical results are consistent with the theoretical results of Theorem 7. We can see that, the concentration of uninfected target cells tends to its normal value $\frac{\lambda_1}{d_1} = 603.198$, $\frac{\lambda_2}{d_2} = 3.198$, respectively, while the concentrations of latently infected cells, actively infected cells, free virus particles and antibody immune cells are decreasing and tend to zero. In this case, the treatment succeeded to eliminate the HIV viruses from the blood.

Case (II): $R_1 \leq 1$. By taking $\varepsilon = 0.40$, we have $R_1 = 0.915 < 1$ and E_1 exists where $E_1 = (601.504, 0.780, 0.038, 0.055, 0.054, 0.231, 0.565, 0.000)$. Based on Theorem 8, E_1 is GAS. Figures 1-8 show that the numerical simulations confirm our theoretical result presented in Theorem 8. We observe that, the trajectory of the system will converge to the chronic-infection equilibrium without antibody immune response E_1. In such situation, the infection becomes chronic but without antibody immune response.

Case (III): $R_1 > 1$.We choose, $\varepsilon = 0.0$. Then, we calculate $R_0 = 1.631$ and $R_1 = 1.149 > 1$, this means that, E_2 exists and it is GAS. From Figures 1-8, we can see that, our simulation results are consistent with the theoretical results of Theorem 9. We observe that, the trajectory of the system tend to the chronic-infection equilibrium with antibody immune response $E_2 = (599.699, 0.474, 0.079, 0.062, 0.111, 0.260, 0.700, 0.896)$. In this case, the infection becomes chronic but with persistent antibody immune response. Figures 1 and 7 demonstrate that, when $R_1 > 1$, the antibody immune response is activated and it reduces the concentration of free virus particles and increases the concentration of uninfected cells. In case (i) we calculate the critical drug efficacy (i.e, the efficacy needed in order stabilize the system around the disease-free equilibrium). For system (51)-(55), E_0 is GAS when $R_0 \leq 1$ i.e.

$$\varepsilon_1^{crit} \leq \varepsilon < 1, \qquad \varepsilon_1^{crit} = \max\left\{0, \frac{\bar{R}_0 - 1}{\bar{R}_{01} + f\bar{R}_{02}}\right\},$$

where, $\bar{R}_0 = R_0\,|_{\varepsilon=0}$ and $\bar{R}_{0i} = R_{0i}\,|_{\varepsilon=0}$, $i = 1, 2$. Using the data in Table 1, we have $\varepsilon_1^{crit} = 0.7332$. Also, in case (ii) we can calculate the critical drug efficacy $\varepsilon_2^{crit} = 0.2566$.

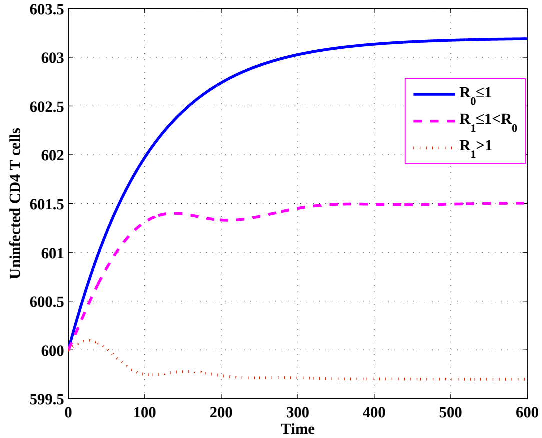

Figure 1: The evolution of uninfected CD4+T cells for model (51)-(55).

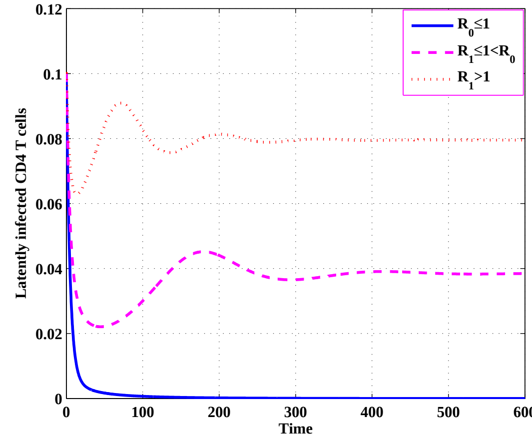

Figure 2: The evolution of uninfected macrophage cells for model (51)-(55).

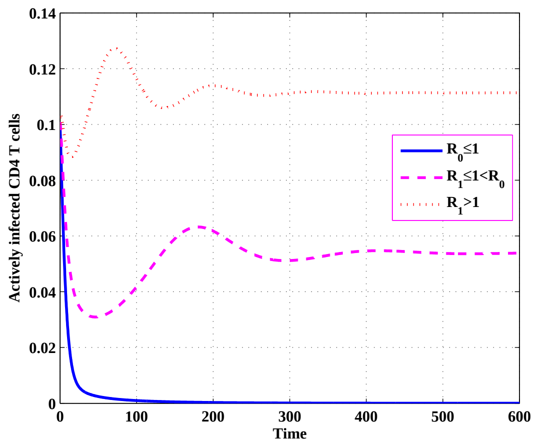

Figure 3: The evolution of actively infected CD4+T cells for model (51)-(55).

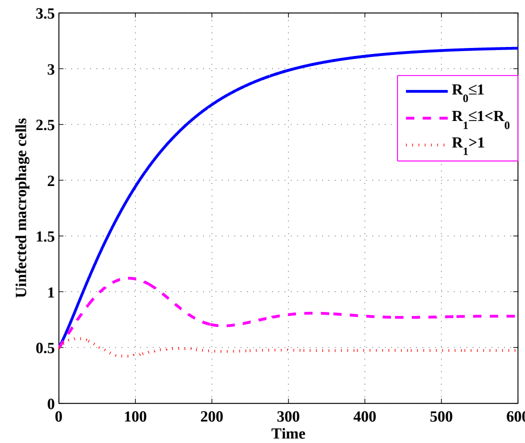

Figure 4: The evolution of uninfected macrophage cells for model (51)-(55).

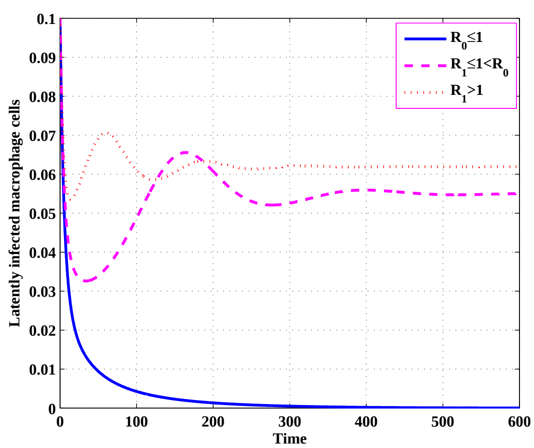

Figure 5: The evolution of latently infected macrophage cells for model (51)-(55).

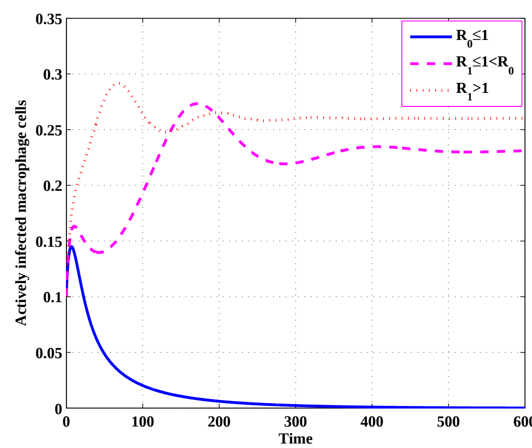

Figure 6: The evolution of actively infected macrophage cells for model (51)-(55).

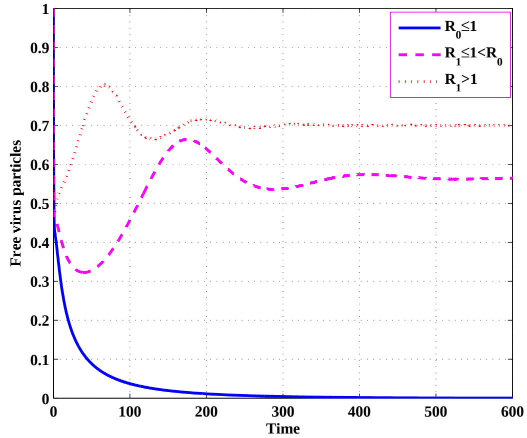

Figure 7: The evolution of free virus particles for model (51)-(55).

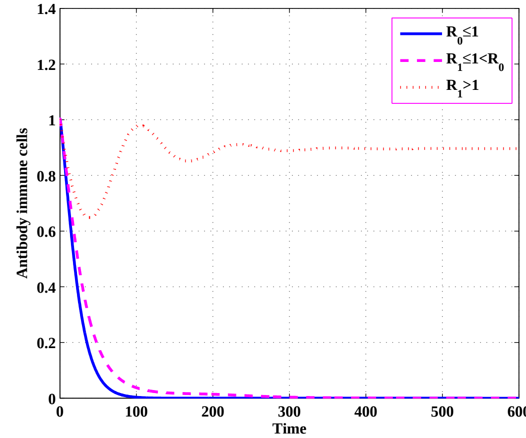

Figure 8: The evolution of antibody immune cells for model (51)-(55).

7 Acknowledgment

This article was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah. The authors, therefore, acknowledge with thanks DSR technical and financial support.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 22, NO.6, 2017, COPYRIGHT 2017 EUDOXUS PRESS, LLC

A New Implicit Midpoint Iterative Scheme Involving Asymptotically Nonexpansive Mappings in Abstract Spaces

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Abstract

We establish the convergence properties of the implicit midpoint iterative scheme for solving the nonlinear equation $T\varrho = \varrho$ for asymptotically nonexpansive mappings in Hilbert and more general uniformly convex Banach spaces.

2010 Mathematics Subject Classification: 47J25, 65J15.

Key words and phrases: asymptotically nonexpansive mappings, iterative scheme, Hilbert spaces, Banach spaces.

1 Introduction

In 2001, Xu and Ori [7] introduced the following implicit iteration process for a finite family of nonexpansive mappings $\{T_i : i \in I\}$ (here $I = \{1, 2, \ldots, N\}$), with $\{t_n\}$ a real

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sequence in $(0,1)$, and an initial point $\varrho_0 \in K \subset X$, where X is an arbitrary Banach space:

$$\begin{aligned}
\varrho_1 &= (1-t_1)\varrho_0 + t_1 T_1 \varrho_1, \\
\varrho_2 &= (1-t_2)\varrho_1 + t_2 T_2 \varrho_2, \\
&\vdots \\
\varrho_N &= (1-t_N)\varrho_{N-1} + t_N T_N \varrho_N, \\
\varrho_{N+1} &= (1-t_{N+1})\varrho_N + t_{N+1} T_{N+1} \varrho_{N+1}, \\
&\vdots,
\end{aligned}$$

which can be written in the following compact form:

$$\varrho_n = (1-t_n)\varrho_{n-1} + t_n T_n \varrho_n, \quad n \geq 1,$$

where $T_n = T_{n\,(mod\,N)}$ (here the $mod\,N$ function takes values in I). Xu and Ori [7] proved the weak convergence of this process to a common fixed point of the finite family defined in a Hilbert space.

Let H be the Hilbert space and T is, in general, a nonlinear operator. Recently Alghamdi et al. [1] defined the following algorithm:

Algorithm 1.1. *Initialize $\varrho_0 \in H$ arbitrarily and define*

$$\varrho_{n+1} = (1-t_n)\varrho_n + t_n T\left(\frac{\varrho_n + \varrho_{n+1}}{2}\right), \quad n \geq 0,$$

where $t_n \in (0,1)$ for all n.

For the approximation of fixed points of nonexpansive mappings under the setting of Hilbert spaces. They proved the following results:

Lemma 1.2. ([1]) *Let $\{\varrho_n\}$ be the sequence generated by Algorithm 1.1. Then*

(i) $\|\varrho_{n+1} - p\| \leq \|\varrho_n - p\|$ *for all $n \geq 0$ and $p \in Fi\varrho(T)$,*

(ii) $\sum_{n=1}^{\infty} t_n \|\varrho_n - \varrho_{n+1}\|^2 < \infty$,

(iii) $\sum_{n=1}^{\infty} t_n(1-t_n)\|\varrho_n - T(\frac{\varrho_n+\varrho_{n+1}}{2})\|^2 < \infty$.

Lemma 1.3. ([1]) *Let $\{\varrho_n\}$ be the sequence generated by Algorithm I. Suppose that $t_{n+1}^2 \leq at_n$ for all $n \geq 0$ and $a > 0$. Then*

$$\lim_{n\to\infty} \|\varrho_{n+1} - \varrho_n\| = 0.$$

Lemma 1.4. ([1]) *Assume that,*

(i) $t_{n+1}^2 \leq at_n$ *for all $n \geq 0$ and $a > 0$,*

(ii) $\liminf_{n\to\infty} t_n > 0$.

Then the sequence $\{\varrho_n\}$ generated by Algorithm 1.1 satisfies the property

$$\lim_{n\to\infty} \|\varrho_n - T\varrho_n\| = 0.$$

Theorem 1.5. ([1]) *Let H be a Hilbert space and $T : H \to H$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. Assume that $\{\varrho_n\}$ is generated by Algorithm 1.1, where the sequence $\{t_n\}$ of parameters satisfies the conditions:*

(i) $t_{n+1}^2 \leq at_n$ *for all* $n \geq 0$ *and* $a > 0$,

(ii) $\limsup_{n\to\infty} t_n > 0$.

Then $\{\varrho_n\}$ converges weakly to a fixed point of T.

We establish the convergence properties of the implicit midpoint iterative scheme for solving the nonlinear equation $T\varrho = \varrho$ for asymptotically nonexpansive mappings in Hilbert and more general uniformly convex Banach, spaces.

2 Preliminaries

Throughout this section we always assume that H is a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$ and that $T : H \to H$ is a nonexpansive mapping with a fixed point. We use $Fix(T)$ to denote the set of fixed points of T.

We establish the strong convergence of a new implicit midpoint iterative scheme for nonexpansive mappings under the setting of Hilbert and more general uniformly convex Banach spaces.

We need the following well known results:

Lemma 2.1. ([5]) *Let $\{\sigma_n\}$ and $\{\beta_n\}$ be sequences of nonnegative real numbers satisfying the following inequality*

$$\beta_{n+1} \leq (1 + \sigma_n)\,\beta_n, \quad n \geq 0.$$

If $\sum_{n=1}^{\infty} \sigma_n < \infty$, then $\lim_{n\to\infty} \beta_n$ exists.

Lemma 2.2. ([3]) *For all $\varrho, \varsigma \in H$ and $\lambda \in [0, 1]$, the following well-known identity holds:*

$$\|(1-\lambda)\varrho + \lambda\varsigma\|^2 = (1-\lambda)\|\varrho\|^2 + \lambda\|\varsigma\|^2 - \lambda(1-\lambda)\|\varrho - \varsigma\|^2.$$

For every ε with $0 \leq \varepsilon \leq 2$, we define the modulus $\delta(\varepsilon)$ of convexity of E by

$$\delta(\varepsilon) = \inf\left\{1 - \frac{\|\varrho + \varsigma\|}{2} : \|\varrho\| \leq 1, \|\varsigma\| \leq 1, \|\varrho - \varsigma\| \geq \varepsilon,\ \varrho, \varsigma \in E\right\}.$$

The space E is said to be *uniformly convex* if

$$\delta(\varepsilon) > 0$$

for every $\varepsilon > 0$.

If E is uniformly convex, then for each r, ε with $r \geq \varepsilon > 0$, we have $\delta(\frac{\varepsilon}{r}) > 0$ and

$$\left\|\frac{\varrho + \varsigma}{2}\right\| \leq r\left(1 - \delta\left(\frac{\varepsilon}{r}\right)\right)$$

for every ϱ, $\varsigma \in E$ with $\|\varrho\| \leq r$, $\|\varsigma\| \leq r$ and $\|\varrho - \varsigma\| \geq \varepsilon$.

The space E is said to be *strictly convex* if

$$\left\|\frac{\varrho + \varsigma}{2}\right\| < 1$$

for every ϱ, $\varsigma \in E$ with $\|\varrho\| = \|\varsigma\| = 1$ and $\varrho \neq \varsigma$.

Lemma 2.3. ([6]) *Let X be the arbitrary Banach space and $p > 1$, $r > 0$ be two fixed numbers. Then X is uniformly convex if and only if there exists a continuous, strictly increasing and convex function $g : [0, \infty) \to [0, \infty)$, $g(0) = 0$, such that*

$$\|\lambda\varrho + (1-\lambda)\varsigma\|^p \leq \lambda \|\varrho\|^p + (1-\lambda) \|\varsigma\|^p - w_p(\lambda) g (\|\varrho - \varsigma\|)$$

for all ϱ, ς in $B_r = \{\varrho \in X : \|\varrho\| \leq r\}$, $\lambda \in [0, 1]$, where $w_p(\lambda) = \lambda(1-\lambda)^p + \lambda^p(1-\lambda)$.

3 Main results

Algorithm 3.1. *Initialize $\varrho_0 \in H$ arbitrarily and define*

$$\varrho_n = (1-t_n)\frac{\varrho_{n-1} + \varrho_n}{2} + t_n T^n \left(\frac{\varrho_{n-1} + \varrho_n}{2}\right), \quad n \geq 0,$$

where $t_n \in (0, 1)$ for all n,

and T is asymptotically nonexpansive, that is,

$$\|T^n\varrho - T^n\varsigma\| \leq k_n\|\varrho - \varsigma\|, \quad \varrho, \varsigma \in H;$$

$\{k_n\} \in [0, \infty)$ satisfying $\sum_{n=1}^{\infty}(k_n - 1) < \infty$.

Remark 3.2. The Algorithm 3.1 can be rewritten as

$$\varrho_n = e_n\varrho_{n-1} + (1-e_n)T^n \left(\frac{\varrho_{n-1} + \varrho_n}{2}\right), \quad n \geq 0,$$

where $e_n = \frac{1-t_n}{1+t_n}$.

Remark 3.3. The Algorithm 3.1 is well defined.

Indeed, for each fixed $u \in H$ and $t \in (0, 1)$, the mapping

$$\varrho \mapsto T_u\varrho = tu + (1-t)T^n \left(\frac{u + \varrho}{2}\right), \quad n \geq 0,$$

is asymptotically nonexpansive with coefficient $\frac{1-t}{2}k_n \in [0, \infty)$. That is,

$$\begin{aligned} \|T_u\varrho - T_u\varsigma\| &= (1-t) \left\|T^n \left(\frac{u+\varrho}{2}\right) - T^n \left(\frac{u+\varsigma}{2}\right)\right\| \\ &\leq \frac{1-t}{2}k_n\|\varrho - \varsigma\|, \quad \varrho, \varsigma \in H. \end{aligned}$$

Remark 3.4. Since $k_n \geq 1$, it is obvious that for any $q > 0$, $\sum_{n=1}^{\infty}(k_n^q - 1) < \infty$ implies $\sum_{n=1}^{\infty}(k_n - 1) < \infty$.

Now we prove our main results.

Lemma 3.5. *The sequence $\{\varrho_n\}$ defined by the Algorithm 3.1, where $\{t_n\} \in (0, 1)$ satisfying $\{t_n\} \in [\delta, 1-\delta]$, is bounded.*

Proof. For $\varrho^* \in Fix(T)$, consider

$$\begin{aligned}
\|\varrho_n - \varrho^*\| &= \left\|(1-t_n)\frac{\varrho_{n-1}+\varrho_n}{2} + t_n T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right) - \varrho^*\right\| \\
&= \left\|(1-t_n)\left(\frac{\varrho_{n-1}+\varrho_n}{2} - \varrho^*\right) + t_n\left(T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right) - \varrho^*\right)\right\| \\
&\le (1-t_n)\left\|\frac{\varrho_{n-1}+\varrho_n}{2} - \varrho^*\right\| + t_n\left\|T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right) - \varrho^*\right\| \\
&\le (1-t_n)\left\|\frac{\varrho_{n-1}+\varrho_n}{2} - \varrho^*\right\| + t_n k_n\left\|\frac{\varrho_{n-1}+\varrho_n}{2} - \varrho^*\right\| \\
&= (1-t_n+t_nk_n)\left\|\frac{\varrho_{n-1}+\varrho_n}{2} - \varrho^*\right\| \\
&= (1-t_n+t_nk_n)\left\|\frac{1}{2}(\varrho_{n-1}-\varrho^*) + \frac{1}{2}(\varrho_n - \varrho^*)\right\| \\
&\le (1-t_n+t_nk_n)\left(\frac{1}{2}\|\varrho_{n-1}-\varrho^*\| + \frac{1}{2}\|\varrho_n-\varrho^*\|\right),
\end{aligned}$$

which implies that

$$\|\varrho_n - \varrho^*\| \le \frac{\frac{1}{2}(1-t_n+t_nk_n)}{1-\frac{1}{2}(1-t_n+t_nk_n)}\|\varrho_{n-1}-\varrho^*\|.$$

Let

$$\begin{aligned}
\frac{\frac{1}{2}(1-t_n+t_nk_n)}{1-\frac{1}{2}(1-t_n+t_nk_n)} &= 1 + \frac{t_n(k_n-1)}{1-\frac{1}{2}(1-t_n+t_nk_n)} \\
&= 1 + \frac{2t_n(k_n-1)}{1-t_n(k_n-1)}.
\end{aligned}$$

By $\sum_{n=1}^{\infty}(k_n-1) < \infty$, there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$, $k_n - 1 \le 1$ and

$$1 - t_n(k_n-1) \ge \delta,$$

which implies that

$$\frac{1}{1-t_n(k_n-1)} \le \frac{1}{\delta}.$$

Thus

$$\|\varrho_n - \varrho^*\| \le \left(1 + 2\frac{\delta}{1-\delta}(k_n-1)\right)\|\varrho_{n-1}-\varrho^*\|.$$

Hence according to Lemma 2.1, the sequence $\{\varrho_n\}$ is bounded. This completes the proof. □

Lemma 3.6. *Let $\{\varrho_n\}$ be the sequence generated by Algorithm 3.1 where $\{t_n\} \in (0,1)$ satisfying $\{t_n\} \in [\delta, 1-\delta]$. Then*

(i) $\lim_{n\to\infty}\|\varrho_{n-1} - \varrho_n\| = 0$,

(ii) $\lim_{n\to\infty}\left\|\frac{\varrho_{n-1}+\varrho_n}{2} - T^n(\frac{\varrho_{n-1}+\varrho_n}{2})\right\| = 0$.

Proof. According to Lemma 2.2,

$$\begin{aligned}
\|\varrho_n - \varrho^*\|^2 &= \left\|(1-t_n)\frac{\varrho_{n-1}+\varrho_n}{2} + t_n T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right) - \varrho^*\right\|^2 \\
&= \left\|(1-t_n)\left(\frac{\varrho_{n-1}+\varrho_n}{2} - \varrho^*\right) + t_n\left(T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right) - \varrho^*\right)\right\|^2 \\
&= (1-t_n)\left\|\frac{\varrho_{n-1}+\varrho_n}{2} - \varrho^*\right\|^2 + t_n\left\|T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right) - \varrho^*\right\|^2 \\
&\quad - t_n(1-t_n)\left\|\frac{\varrho_{n-1}+\varrho_n}{2} - T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right)\right\|^2 \\
&\le (1-t_n)\left\|\frac{\varrho_{n-1}+\varrho_n}{2} - \varrho^*\right\|^2 + t_n k_n^2\left\|\frac{\varrho_{n-1}+\varrho_n}{2} - \varrho^*\right\|^2 \\
&\quad - t_n(1-t_n)\left\|\frac{\varrho_{n-1}+\varrho_n}{2} - T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right)\right\|^2 \\
&= (1-t_n+t_nk_n^2)\left\|\frac{\varrho_{n-1}+\varrho_n}{2} - \varrho^*\right\|^2 \\
&\quad - t_n(1-t_n)\left\|\frac{\varrho_{n-1}+\varrho_n}{2} - T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right)\right\|^2 \\
&\le (1-t_n+t_nk_n^2)\left(\frac{1}{2}\|\varrho_{n-1}-\varrho^*\|^2 + \frac{1}{2}\|\varrho_n-\varrho^*\|^2 - \frac{1}{4}\|\varrho_{n-1}-\varrho_n\|^2\right) \\
&\quad - t_n(1-t_n)\left\|\frac{\varrho_{n-1}+\varrho_n}{2} - T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right)\right\|^2,
\end{aligned}$$

which implies that

$$\begin{aligned}
\|\varrho_n - \varrho^*\|^2 \le{}& \frac{\frac{1}{2}(1-t_n+t_nk_n^2)}{1-\frac{1}{2}(1-t_n+t_nk_n^2)}\|\varrho_{n-1}-\varrho^*\|^2 \\
&- \frac{1}{4}\frac{(1-t_n+t_nk_n^2)}{1-\frac{1}{2}(1-t_n+t_nk_n^2)}\|\varrho_{n-1}-\varrho_n\|^2 \\
&- \frac{t_n(1-t_n)}{1-\frac{1}{2}(1-t_n+t_nk_n^2)}\left\|\frac{\varrho_{n-1}+\varrho_n}{2} - T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right)\right\|^2.
\end{aligned}$$

Let us assume that

$$\begin{aligned}
\frac{\frac{1}{2}(1-t_n+t_nk_n^2)}{1-\frac{1}{2}(1-t_n+t_nk_n^2)} &= 1 + \frac{t_n(k_n^2-1)}{1-\frac{1}{2}(1-t_n+t_nk_n^2)} \\
&= 1 + \frac{2t_n(k_n^2-1)}{1-t_n(k_n^2-1)}.
\end{aligned}$$

By $\sum_{n=1}^{\infty}(k_n^2-1) < \infty$, there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$, $k_n^2 - 1 \le 1$ and

$$1 - t_n(k_n^2-1) \ge \delta,$$

which implies that

$$\frac{1}{1-t_n(k_n^2-1)} \le \frac{1}{\delta}.$$

Also

$$1 - t_n + t_n k_n^2 = 1 + t_n(k_n^2 - 1) \geq 1$$

and

$$\begin{aligned} 1 - \frac{1}{2}(1 - t_n + t_n k_n^2) &= 1 - \frac{1}{2}\left(1 + t_n(k_n^2 - 1)\right) \\ &= \frac{1}{2}\left(1 - t_n(k_n^2 - 1)\right) \\ &\leq \frac{1}{2}, \end{aligned}$$

which yields that

$$\frac{1}{1 - \frac{1}{2}(1 - t_n + t_n k_n^2)} \geq 2.$$

Thus for $M > 0$,

$$\begin{aligned} \|\varrho_n - \varrho^*\|^2 \leq& \left(1 + 2\frac{\delta}{1-\delta}(k_n^2 - 1)\right)\|\varrho_{n-1} - \varrho^*\|^2 - \frac{1}{2}\|\varrho_{n-1} - \varrho_n\|^2 \\ &- 2\delta^2\left\|\frac{\varrho_{n-1} + \varrho_n}{2} - T^n\left(\frac{\varrho_{n-1} + \varrho_n}{2}\right)\right\|^2 \\ \leq& \|\varrho_{n-1} - \varrho^*\|^2 + 2M^2\frac{\delta}{1-\delta}(k_n^2 - 1) - \frac{1}{2}\|\varrho_{n-1} - \varrho_n\|^2 \\ &- 2\delta^2\left\|\frac{\varrho_{n-1} + \varrho_n}{2} - T^n\left(\frac{\varrho_{n-1} + \varrho_n}{2}\right)\right\|^2, \end{aligned}$$

which implies that

$$\begin{aligned} &\frac{1}{2}\|\varrho_{n-1} - \varrho_n\|^2 + 2\delta^2\left\|\frac{\varrho_{n-1} + \varrho_n}{2} - T(\frac{\varrho_{n-1} + \varrho_n}{2})\right\|^2 \\ &\leq \|\varrho_{n-1} - \varrho^*\|^2 - \|\varrho_n - \varrho^*\|^2 + 2M^2\frac{\delta}{1-\delta}(k_n^2 - 1). \end{aligned}$$

Thus

$$\begin{aligned} &\frac{1}{2}\sum_{j=1}^{m}\|\varrho_{j-1} - \varrho_j\|^2 + 2\delta^2\sum_{j=1}^{m}\left\|\frac{\varrho_{j-1} + \varrho_j}{2} - T^n\left(\frac{\varrho_{j-1} + \varrho_j}{2}\right)\right\|^2 \\ &\leq \sum_{j=1}^{m}\left(\|\varrho_{j-1} - \varrho^*\|^2 - \|\varrho_j - \varrho^*\|^2 + 2M^2\frac{\delta}{1-\delta}(k_j^2 - 1)\right). \end{aligned}$$

Hence

$$\sum_{j=1}^{\infty}\|\varrho_{n-1} - \varrho_n\|^2 < +\infty$$

and

$$\sum_{j=1}^{\infty}\left\|\frac{\varrho_{n-1} + \varrho_n}{2} - T^n\left(\frac{\varrho_{n-1} + \varrho_n}{2}\right)\right\|^2 < +\infty.$$

It implies that

$$\lim_{n\to\infty}\|\varrho_{n-1} - \varrho_n\| = 0$$

and

$$\lim_{n\to\infty}\left\|\frac{\varrho_{n-1}+\varrho_n}{2}-T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right)\right\|=0.$$

This completes the proof. □

Lemma 3.7. *Let $\{\varrho_n\}$ be the sequence generated by Algorithm 3.1, where $\{t_n\}\in(0,1)$ satisfying $\{t_n\}\in[\delta,1-\delta]$. Then $\lim_{n\to\infty}\|\varrho_n-T\varrho_n\|=0$.*

Proof. Consider

$$\begin{aligned}\|\varrho_n-T^n\varrho_n\| &\le \left\|\varrho_n-\frac{\varrho_{n-1}+\varrho_n}{2}\right\|+\left\|\frac{\varrho_{n-1}+\varrho_n}{2}-T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right)\right\|\\ &\quad+\left\|T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right)-T^n\varrho_n\right\|\\ &\le \left\|\varrho_n-\frac{\varrho_{n-1}+\varrho_n}{2}\right\|+\left\|\frac{\varrho_{n-1}+\varrho_n}{2}-T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right)\right\|\\ &\quad+k_n\left\|\varrho_n-\frac{\varrho_{n-1}+\varrho_n}{2}\right\|\\ &=(1+k_n)\left\|\varrho_n-\frac{\varrho_{n-1}+\varrho_n}{2}\right\|+\left\|\frac{\varrho_{n-1}+\varrho_n}{2}-T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right)\right\|\\ &=\frac{1+k_n}{2}\|\varrho_{n-1}-\varrho_n\|+\left\|\frac{\varrho_{n-1}+\varrho_n}{2}-T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right)\right\|\\ &\to 0 \quad \text{as } n\to\infty\end{aligned}$$

and

$$\begin{aligned}\|\varrho_n-T\varrho_n\| &\le \|\varrho_n-T^n\varrho_n\|+\|T^n\varrho_n-T^n\varrho_{n-1}\|+\|T^n\varrho_{n-1}-T\varrho_n\|\\ &\le \|\varrho_n-T^n\varrho_n\|+k_n\|\varrho_n-\varrho_{n-1}\|+k_1\|T^{n-1}\varrho_{n-1}-\varrho_n\|\\ &\le \|\varrho_n-T^n\varrho_n\|+k_n\|\varrho_n-\varrho_{n-1}\|\\ &\quad+k_1\left(\|T^{n-1}\varrho_{n-1}-\varrho_{n-1}\|+\|\varrho_{n-1}-\varrho_n\|\right)\\ &\to 0 \quad \text{as } n\to\infty.\end{aligned}$$

This completes the proof. □

Theorem 3.8. *Let $T:H\to H$ be asymptotically nonexpansive. For arbitrary $\varrho_0\in K$, generate the sequence $\{\varrho_n\}$ by the Algorithm 3.1. If T is completely continuos, then $\{\varrho_n\}$ converges strongly to some fixed point of T in H.*

Proof. From Lemma 3.7, $\lim\limits_{n\to\infty}\|\varrho_n-T\varrho_n\|=0$. Therefore, there exists a subsequence $\{\varrho_{n_j}\}$ of $\{\varrho_n\}$ such that $\lim_{j\to\infty}\left\|\varrho_{n_j}-T\varrho_{n_j}\right\|=0$. Since $\{\varrho_{n_j}\}$ is bounded and T is completely continuous, then $\{T\varrho_{n_j}\}$ has a subsequence $\{T\varrho_{n_{j_k}}\}$ which converges strongly. Hence $\{\varrho_{n_{j_k}}\}$ converges strongly. Let $\lim_{j\to\infty}\varrho_{n_{j_k}}=p$. Then $\lim_{j\to\infty}T\varrho_{n_{j_k}}=Tp$. Thus we have $\lim_{j\to\infty}\left\|\varrho_{n_{j_k}}-T\varrho_{n_{j_k}}\right\|=\|p-Tp\|=0$. Hence $p\in F(T)$. From Lemma 2.1 and Lemma 3.7 it follows that $\lim_{n\to\infty}\|\varrho_n-p\|=0$. This completes the proof. □

Lemma 3.9. *Let E be the uniformly convex Banach space and $T : E \to E$ be asymptotically nonexpansive mapping. Let $\{\varrho_n\} \in E$ be the sequence generated by Algorithm 3.1 and $\{t_n\} \in (0,1)$ satisfying $\{t_n\} \in [\delta, 1-\delta]$. Then*

(i) $\lim_{n\to\infty} \|\varrho_{n-1} - \varrho_n\| = 0$,

(ii) $\lim_{n\to\infty} \left\|\frac{\varrho_{n-1}+\varrho_n}{2} - T^n(\frac{\varrho_{n-1}+\varrho_n}{2})\right\| = 0$.

Proof. According to Lemma 2.3,

$$\begin{aligned}
\|\varrho_n - \varrho^*\|^p &= \left\|(1-t_n)\frac{\varrho_{n-1}+\varrho_n}{2} + t_n T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right) - \varrho^*\right\|^p \\
&= \left\|(1-t_n)\left(\frac{\varrho_{n-1}+\varrho_n}{2} - \varrho^*\right) + t_n\left(T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right) - \varrho^*\right)\right\|^p \\
&\le (1-t_n)\left\|\frac{\varrho_{n-1}+\varrho_n}{2} - \varrho^*\right\|^p + t_n\left\|T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right) - \varrho^*\right\|^p \\
&\quad - w_p(t_n)g\left(\left\|\frac{\varrho_{n-1}+\varrho_n}{2} - T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right)\right\|\right) \\
&\le (1-t_n)\left\|\frac{\varrho_{n-1}+\varrho_n}{2} - \varrho^*\right\|^p + t_n k_n^p\left\|\frac{\varrho_{n-1}+\varrho_n}{2} - \varrho^*\right\|^p \\
&\quad - w_p(t_n)g\left(\left\|\frac{\varrho_{n-1}+\varrho_n}{2} - T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right)\right\|\right) \\
&= (1-t_n+t_n k_n^p)\left\|\frac{\varrho_{n-1}+\varrho_n}{2} - \varrho^*\right\|^p \\
&\quad - w_p(t_n)g\left(\left\|\frac{\varrho_{n-1}+\varrho_n}{2} - T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right)\right\|\right),
\end{aligned}$$

where

$$\begin{aligned}
\left\|\frac{\varrho_{n-1}+\varrho_n}{2} - \varrho^*\right\|^p &= \left\|\frac{1}{2}(\varrho_{n-1} - \varrho^*) + \frac{1}{2}(\varrho_n - \varrho^*)\right\|^p \\
&\le \left[\frac{1}{2}\|\varrho_{n-1} - \varrho^*\| + \frac{1}{2}\|\varrho_n - \varrho^*\|\right]^p \\
&\le \frac{1}{2}\|\varrho_{n-1} - \varrho^*\|^p + \frac{1}{2}\|\varrho_n - \varrho^*\|^p.
\end{aligned}$$

Thus

$$\begin{aligned}
\|\varrho_n - \varrho^*\|^p \le{}& (1-t_n+t_n k_n^p)\left(\frac{1}{2}\|\varrho_{n-1} - \varrho^*\|^p + \frac{1}{2}\|\varrho_n - \varrho^*\|^p\right) \\
&- w_p(t_n)g\left(\left\|\frac{\varrho_{n-1}+\varrho_n}{2} - T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right)\right\|\right),
\end{aligned}$$

which implies that

$$\begin{aligned}
\|\varrho_n - \varrho^*\|^p \le{}& \frac{\frac{1}{2}(1-t_n+t_n k_n^p)}{1-\frac{1}{2}(1-t_n+t_n k_n^p)}\|\varrho_{n-1} - \varrho^*\|^p \\
&- \frac{w_p(t_n)}{1-\frac{1}{2}(1-t_n+t_n k_n^p)}g\left(\left\|\frac{\varrho_{n-1}+\varrho_n}{2} - T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right)\right\|\right).
\end{aligned}$$

Let us assume that

$$\frac{\frac{1}{2}(1-t_n+t_nk_n^p)}{1-\frac{1}{2}(1-t_n+t_nk_n^p)} = 1+\frac{t_n(k_n^p-1)}{1-\frac{1}{2}(1-t_n+t_nk_n^p)}$$
$$= 1+\frac{2t_n(k_n^p-1)}{1-t_n(k_n^p-1)}.$$

By $\sum_{n=1}^{\infty}(k_n^p-1)<\infty$, there exists $n_0\in\mathbb{N}$ such that for all $n\geq n_0$, $k_n^p-1\leq 1$, and

$$1-t_n(k_n^p-1)\geq\delta,$$

which implies that

$$\frac{1}{1-t_n(k_n^p-1)}\leq\frac{1}{\delta}.$$

Also

$$1-\frac{1}{2}(1-t_n+t_nk_n^p) = 1-\frac{1}{2}\left(1+t_n(k_n^p-1)\right)$$
$$= \frac{1}{2}\left(1-t_n(k_n^p-1)\right)$$
$$\leq\frac{1}{2},$$

which yields that

$$\frac{1}{1-\frac{1}{2}(1-t_n+t_nk_n^p)}\geq 2.$$

Hence

$$\|\varrho_n-\varrho^*\|^p \leq \left(1+2\frac{\delta}{1-\delta}(k_n^p-1)\right)\|\varrho_{n-1}-\varrho^*\|^p$$
$$-4\delta^{p+1}g\left(\left\|\frac{\varrho_{n-1}+\varrho_n}{2}-T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right)\right\|\right).$$

For $M>0$,

$$\|\varrho_n-\varrho^*\|^p \leq \|\varrho_{n-1}-\varrho^*\|^p+2M^p\frac{\delta}{1-\delta}(k_n^p-1)$$
$$-4\delta^{p+1}g\left(\left\|\frac{\varrho_{n-1}+\varrho_n}{2}-T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right)\right\|\right),$$

which implies that

$$4\delta^{p+1}g\left(\left\|\frac{\varrho_{n-1}+\varrho_n}{2}-T^n(\frac{\varrho_{n-1}+\varrho_n}{2})\right\|\right)$$
$$\leq\|\varrho_{n-1}-\varrho^*\|^p-\|\varrho_n-\varrho^*\|^p+2M^p\frac{\delta}{1-\delta}(k_n^p-1).$$

Thus

$$4\delta^{p+1}\sum_{j=1}^{m}g\left(\left\|\frac{\varrho_{n-1}+\varrho_n}{2}-T^n(\frac{\varrho_{n-1}+\varrho_n}{2})\right\|\right)$$
$$\leq\sum_{j=1}^{m}\left(\|\varrho_{j-1}-\varrho^*\|^p-\|\varrho_j-\varrho^*\|^p+2M^p\frac{\delta}{1-\delta}(k_n^p-1)\right).$$

Hence

$$\sum_{j=1}^{\infty} g\left(\left\|\frac{\varrho_{n-1}+\varrho_n}{2} - T^n(\frac{\varrho_{n-1}+\varrho_n}{2})\right\|\right) < +\infty.$$

It implies that

$$\lim_{n\to\infty}\left\|\frac{\varrho_{n-1}+\varrho_n}{2} - T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right)\right\| = 0.$$

From this, it can be easily see that

$$\lim_{n\to\infty}\|\varrho_{n-1}-\varrho_n\| = 0.$$

This completes the proof. □

Lemma 3.10. *Let E and T as in Lemma 3.9. Let $\{\varrho_n\}$ be the sequence generated by Algorithm 3.1, where $\{t_n\} \in (0,1)$ satisfying $\{t_n\} \in [\delta, 1-\delta]$. Then* $\lim_{n\to\infty}\|\varrho_n - T\varrho_n\| = 0$.

Theorem 3.11. *Let E and T as in Lemma 3.9. For arbitrary $\varrho_0 \in K$, generate the sequence $\{\varrho_n\}$ by the Algorithm 3.1. If T is completely continuos, then $\{\varrho_n\}$ converges strongly to some fixed point of T in E.*

Acknowledgment

This study was supported by research funds from Dong-A University.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 22, NO.6, 2017, COPYRIGHT 2017 EUDOXUS PRESS, LLC

Hesitant fuzzy filters in lattice implication algebras

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Abstract. The notion of hesitant fuzzy filters in lattice implication algebras is introduced, and several properties are investigated. Characterizations of hesitant fuzzy filters are discussed.

1. **Introduction**

In the field of many-valued logic, lattice-valued logic plays an important role for two aspects: One is that it extends the chain-type truth-value field of some well-known presented logic [1] to some relatively general lattices. The other is that the incompletely comparable property of truth value characterized by general lattice can more efficiently reflect the uncertainty of people's thinking, judging and decision. Hence, lattice-valued logic is becoming a research field which strongly influences the development of Algebraic Logic, Computer Science and Artificial Intelligence Technology. Therefore Goguen, Novak and Pavelka researched on this lattice-valued logic formal systems (see [2, 10, 11]). In order to research the logical system whose propositional value is given in a lattice, Xu [12] proposed the concept of lattice implication algebras, and discussed their some properties. For the general development of lattice implication algebras, filter theory and its fuzzification play an important role. Xu and Qin [14] introduced the notion of (implicative) filters in a lattice implication algebra, and investigated their properties. Jun (together with Xu and Qin) [3, 9] discussed positive implicative and associative filters of a lattice implication algebra, and Jun [4] considered the fuzzification of positive implicative and associative filters of a lattice implication algebra. In [13], Xu and Qin considered the fuzzification of (implicative) filters.

Torra [16] introduced the hesitant fuzzy set which is a useful generalization of the fuzzy set that is designed for situations in which it is difficult to determine the membership of an element to a set owing to ambiguity between a few different values. The hesitant fuzzy set permits the

[0] **2010 Mathematics Subject Classification**: 03G10; 06B10; 06D72.
[0]**Keywords**: hesitant fuzzy filter; hesitant level set.
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membership degree of an element to a set to be represented by a set of possible values between 0 and 1 (see [16] and [17]). Jun et al. applied the notion of hesitant fuzzy sets to semigroups, MTL-algebras and EQ-algebras (see [5, 6, 7, 8]).

In this paper, we apply the notion of hesitant fuzzy sets to the filter theory in lattice implication algebras. We introduce the concept of hesitant fuzzy filters in lattice implication algebras, and investigate several properties. We discuss characterizations of hesitant fuzzy filters.

2. **Preliminaries**

By a *lattice implication algebra* we mean a bounded lattice $L := (L, \vee, \wedge, 0, 1)$ with order-reversing involution " $'$ " and a binary operation " $\rightarrow$ " satisfying the following axioms:

(I1) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,
(I2) $x \rightarrow x = 1$,
(I3) $x \rightarrow y = y' \rightarrow x'$,
(I4) $x \rightarrow y = y \rightarrow x = 1 \Rightarrow x = y$,
(I5) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$,
(L1) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$,
(L2) $(x \wedge y) \rightarrow z = (x \rightarrow z) \vee (y \rightarrow z)$,

for all $x, y, z \in L$. We define a relation $\leq$ on a lattice implication algebra L by $x \leq y$ if and only if $x \rightarrow y = 1$.

In a lattice implication algebra L, the following hold (see [12]):

(a1) $0 \rightarrow x = 1$, $1 \rightarrow x = x$ and $x \rightarrow 1 = 1$.
(a2) $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$.
(a3) $x \leq y$ implies $y \rightarrow z \leq x \rightarrow z$ and $z \rightarrow x \leq z \rightarrow y$.
(a4) $x' = x \rightarrow 0$.
(a5) $x \vee y = (x \rightarrow y) \rightarrow y$.
(a6) $((y \rightarrow x) \rightarrow y')' = x \wedge y = ((x \rightarrow y) \rightarrow x')'$.
(a7) $x \leq (x \rightarrow y) \rightarrow y$

where $x \leq y$ means $x \rightarrow y = 1$.

A subset F of a lattice implication algebra L is called a *filter* of L (see [14]) if it satisfies:

(F1) $1 \in F$,
(F2) $x \in F$ and $x \rightarrow y \in F$ imply $y \in F$

for all $x, y \in L$.

Let L be a reference set. Then we define hesitant fuzzy set on L in terms of a function $\mathcal{H}$ that when applied to X returns a subset of $[0, 1]$ (see [16]).

For a hesitant fuzzy set $\mathcal{H}$ on L and $x, y, z \in L$, we use the notations $\mathcal{H}_x := \mathcal{H}(x)$, $\mathcal{H}_x^y := \mathcal{H}(x) \cap \mathcal{H}(y)$, $\mathcal{H}_x(\varepsilon) := \mathcal{H}(x) \cap \varepsilon$ and $\mathcal{H}_x^y(\varepsilon) := \mathcal{H}(x) \cap \mathcal{H}(y) \cap \varepsilon$ where $\varepsilon \in \mathscr{P}([0,1])$. It is clear that $\mathcal{H}_x^y = \mathcal{H}_y^x$, $\mathcal{H}_x^y(\varepsilon) \subseteq \mathcal{H}_x(\varepsilon)$ and

$$\mathcal{H}_x = \mathcal{H}_y \;\Leftrightarrow\; \mathcal{H}_x \subseteq \mathcal{H}_y,\; \mathcal{H}_y \subseteq \mathcal{H}_x$$

for all $x, y \in L$.

For a hesitant fuzzy set $\mathcal{H}$ on L and a subset ε of $[0,1]$, the set

$$L(\mathcal{H};\varepsilon) := \{x \in L \mid \varepsilon \subseteq \mathcal{H}_x\},$$

is called the hesitant level set of $\mathcal{H}$.

3. **Hesitant fuzzy filters**

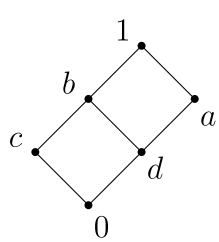
In what follows, we take a lattice implication algebra L as a reference set unless otherwise specified.

Definition 3.1. A hesitant fuzzy set $\mathcal{H}$ on L is a hesitant fuzzy filter of L if it satisfies the following assertions.

$$(\forall x \in L)\,(\mathcal{H}_1 \supseteq \mathcal{H}_x)\,, \tag{3.1}$$

$$(\forall x, y \in L)\,\big(\mathcal{H}_y \supseteq \mathcal{H}_{x\to y)}^x\big)\,. \tag{3.2}$$

Example 3.2. Let $L = \{0, a, b, c, d, 1\}$ be a set with the following Hasse diagram and Cayley tables:



x	x'
0	1
a	c
b	d
c	a
d	b
1	0

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	b	c	b	1
b	d	a	1	b	a	1
c	a	a	1	1	a	1
d	b	1	1	b	1	1
1	0	a	b	c	d	1

Then L is a lattice implication algebra (see [15]). Let $\mathcal{H}$ be a hesitant fuzzy set on L which is given as follows:

$$\mathcal{H} : L \to \mathscr{P}([0,1]),\; x \mapsto \begin{cases} [0.2, 0.8] & \text{if } x \in \{a, 1\}, \\ [0.3, 0.7] & \text{otherwise.} \end{cases}$$

Then $\mathcal{H}$ is a hesitant fuzzy filter of L.

Theorem 3.3. *A hesitant fuzzy set $\mathcal{H}$ on L is a hesitant fuzzy filter of L if and only if the hesitant level set $L(\mathcal{H};\varepsilon)$ of $\mathcal{H}$ is a filter of L for all $\varepsilon \in \mathscr{P}([0,1])$ with $L(\mathcal{H};\varepsilon) \neq \emptyset$.*

Proof. Assume that $\mathcal{H}$ is a hesitant fuzzy filter of L. Let $\varepsilon \in \mathscr{P}([0,1])$ be such that $L(\mathcal{H};\varepsilon) \neq \emptyset$. Then there exists $a \in L(\mathcal{H};\varepsilon)$, and so $\mathcal{H}_a \supseteq \varepsilon$. It follows from (3.1) that $\mathcal{H}_1 \supseteq \mathcal{H}_a \supseteq \varepsilon$ and so that $1 \in L(\mathcal{H};\varepsilon)$. Let $x, y \in L$ be such that $x \in L(\mathcal{H};\varepsilon)$ and $x \to y \in L(\mathcal{H};\varepsilon)$. Then $\varepsilon \subseteq \mathcal{H}_x$ and $\varepsilon \subseteq \mathcal{H}_{x\to y}$. Using (3.2), we get $\mathcal{H}_y \supseteq \mathcal{H}^x_{x\to y} \supseteq \varepsilon$. Thus $y \in L(\mathcal{H};\varepsilon)$, and hence $L(\mathcal{H};\varepsilon)$ is a filter of L for all $\varepsilon \in \mathscr{P}([0,1])$ with $L(\mathcal{H};\varepsilon) \neq \emptyset$.

Conversely, suppose that the nonempty hesitant level set $L(\mathcal{H};\varepsilon)$ of $\mathcal{H}$ is a filter of L for all $\varepsilon \in \mathscr{P}([0,1])$. For any $x \in L$, let $\mathcal{H}_x = \varepsilon_x$. Then $x \in L(\mathcal{H};\varepsilon_x)$, and so $L(\mathcal{H};\varepsilon_x) \neq \emptyset$. Hence $1 \in L(\mathcal{H};\varepsilon_x)$, and thus $\mathcal{H}_1 \supseteq \varepsilon_x = \mathcal{H}_x$ for all $x \in L$. For any $x, y \in L$, let $\mathcal{H}^x_{x\to y} = \delta$. Then $\mathcal{H}_x \supseteq \delta$ and $\mathcal{H}_{x\to y} \supseteq \delta$, that is, $x \in L(\mathcal{H};\delta)$ and $x \to y \in L(\mathcal{H};\delta)$. It follows from (F2) that $y \in L(\mathcal{H};\delta)$ and so that $\mathcal{H}_y \supseteq \delta = \mathcal{H}^x_{x\to y}$ for all $x, y \in L$. Therefore $\mathcal{H}$ is a hesitant fuzzy filter of L. □

Proposition 3.4. *Every hesitant fuzzy filter $\mathcal{H}$ of L satisfies:*

$$(\forall x, y \in L)\,(x \leq y \;\Rightarrow\; \mathcal{H}_x \subseteq \mathcal{H}_y)\,. \tag{3.3}$$

Proof. Let $x, y \in L$ satisfy $x \leq y$. Then $x \to y = 1$, and so

$$\mathcal{H}_y \supseteq \mathcal{H}^x_{x\to y} = \mathcal{H}^x_1 = \mathcal{H}_x$$

by (3.2) and (3.1). □

Theorem 3.5. *A hesitant fuzzy set $\mathcal{H}$ on L is a hesitant fuzzy filter of L if and only if it satisfies* (3.1) *and*

$$(\forall x, y, z \in L)\left(\mathcal{H}_{x\to z} \supseteq \mathcal{H}^{x\to y}_{y\to z}\right). \tag{3.4}$$

Proof. Assume that $\mathcal{H}$ is a hesitant fuzzy filter of L. Since $x \to y \leq (y \to z) \to (x \to z)$ for all $x, y, z \in L$, it follows from (3.3) that $\mathcal{H}_{x\to y} \subseteq \mathcal{H}_{(y\to z)\to(x\to z)}$ and so from (3.2) that

$$\mathcal{H}_{x\to z} \supseteq \mathcal{H}^{y\to z}_{(y\to z)\to(x\to z)} \supseteq \mathcal{H}^{y\to z}_{x\to y}$$

for all $x, y, z \in L$.

Conversely, let $\mathcal{H}$ satisfy (3.1) and (3.4). Taking $x = 1$ in (3.4) and using (a1), we have

$$\mathcal{H}_z = \mathcal{H}_{1\to z} \supseteq \mathcal{H}^{1\to y}_{y\to z} = \mathcal{H}^y_{y\to z}$$

for all $y, z \in L$. Therefore $\mathcal{H}$ is a hesitant fuzzy filter of L. □

Theorem 3.6. *For any hesitant fuzzy set $\mathcal{H}$ on L, the following assertions are equivalent.*

(1) *$\mathcal{H}$ is a hesitant fuzzy filter of L.*
(2) $(\forall x, y, z \in L)\left(x \leq y \to z \;\Rightarrow\; \mathcal{H}_z \supseteq \mathcal{H}^x_y\right).$

Proof. Suppose that $\mathcal{H}$ is a hesitant fuzzy filter of L. Let $x, y, z \in L$ satisfy $x \leq y \to z$. Using (3.2) and (3.3) implies that $\mathcal{H}_z \supseteq \mathcal{H}^y_{y\to z} \supseteq \mathcal{H}^y_x$.

Assume that the second condition is valid. Since $x \leq x \to 1$ for all $x \in L$, we have $\mathcal{H}_1 \supseteq \mathcal{H}^x_x = \mathcal{H}_x$ for all $x \in L$. Note that $y \leq (y \to x) \to x$ for all $x, y \in L$. Hence $\mathcal{H}_x \supseteq \mathcal{H}^y_{y\to x}$ for all $x, y \in L$. Therefore $\mathcal{H}$ is a hesitant fuzzy filter of L. □

Theorem 3.7. *A hesitant fuzzy set $\mathcal{H}$ on L is a hesitant fuzzy filter of L if and only if it satisfies* (3.1), (3.3) *and*

$$(\forall x, y \in L) \left(\mathcal{H}_{(x\to y')'} \supseteq \mathcal{H}^x_y\right). \tag{3.5}$$

Proof. Assume that $\mathcal{H}$ is a hesitant fuzzy filter of L. Then the conditions (3.1) and (3.3) are valid by Definition 3.1 and Proposition 3.4. Using (3.1), (3.2) and (I2), we have

$$\begin{aligned}\mathcal{H}_{(x\to y')'} &\supseteq \mathcal{H}^y_{y\to(x\to y')'} \supseteq \mathcal{H}^y_x(x \to (y \to (x \to y')')) \\ &= \mathcal{H}^y_x((x \to y')' \to (x \to y')') \\ &= \mathcal{H}^x_y(1) = \mathcal{H}^x_y\end{aligned}$$

for all $x, y \in L$. Hence (3.5) is valid.

Conversely, let $\mathcal{H}$ satisfy conditions (3.1), (3.3) and (3.5). Note that

$$(x \to (x \to y)')' \leq y$$

for all $x, y \in L$. It follows from (3.3) and (3.5) that

$$\mathcal{H}_y \supseteq \mathcal{H}_{(x\to(x\to y)')'} \supseteq \mathcal{H}^x_{x\to y}$$

for all $x, y \in L$. Therefore $\mathcal{H}$ is a hesitant fuzzy filter of L by Theorem 3.3. □

Theorem 3.8. *A hesitant fuzzy set $\mathcal{H}$ on L is a hesitant fuzzy filter of L if and only if it satisfies* (3.1) *and*

$$(\forall x, y, z \in L) \left(\mathcal{H}_{z\to x} \supseteq \mathcal{H}^y_{(z\to y)\to x}\right). \tag{3.6}$$

Proof. Suppose that $\mathcal{H}$ is a hesitant fuzzy filter of L. Let $x, y, z \in L$. Since $x \leq z \to x$ and $y \leq z \to y$, we have

$$(z \to y) \to x \leq (z \to y) \to (z \to x) \leq y \to (z \to x).$$

It follows from (3.2) and (3.3) that

$$\mathcal{H}_{z\to x} \supseteq \mathcal{H}^y_{y\to(z\to x)} \supseteq \mathcal{H}^y_{(z\to y)\to x}.$$

Hence (3.6) is valid.

Conversely, let $\mathcal{H}$ satisfy conditions (3.1) and (3.6). If we take $z = 1$ in (3.6) and use (a1), then

$$\mathcal{H}_x = \mathcal{H}_{1\to x} \supseteq \mathcal{H}^y_{(1\to y)\to x} = \mathcal{H}^y_{y\to x}$$

for all $x, y \in L$. Therefore $\mathcal{H}$ is a hesitant fuzzy filter of L. □

Let $\mathcal{H}$ be a hesitant fuzzy set on L and $a \in L$. We consider the set

$$\mathcal{H}_a^{\to} := \{x \in L \mid \mathcal{H}_a \subseteq \mathcal{H}_x\}.$$

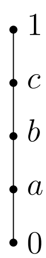
Obviously, $a \in \mathcal{H}_a^{\to}$. If $\mathcal{H}$ is a hesitant fuzzy filter of L , then $1 \in \mathcal{H}_a^{\to}$ since $\mathcal{H}_1 \supseteq \mathcal{H}_x$ for all $x \in L$.

Let $\mathcal{H}$ satisfy the condition (3.1). Then there exists $a \in L$ such that $\mathcal{H}_a^{\to}$ is not a filter of L as seen in the following example.

Example 3.9. Consider the set $L = \{a_i \mid i = 1, 2, \cdots, n\}$. For any $1 \leq j, k \leq n$, define

$$\begin{aligned} &a_j \vee a_k = a_{\max\{j,k\}}, \\ &a_j \wedge a_k = a_{\min\{j,k\}}, \\ &(a_j)' = a_{n-j+1}, \\ &a_j \to a_k = a_{\min\{n-j+k,n\}}. \end{aligned}$$

Then $(L, \vee, \wedge, ', \to)$ is a lattice implication algebra which is called the Łukasiewicz implication algebra (of order n) (see [15]). The Łukasiewicz implication algebra $L = \{0, a, b, c, 1\}$ of order 5 is represented by



x	x'
0	1
a	c
b	b
c	a
1	0

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	b	c	1	1	1
c	a	b	c	1	1
1	0	a	b	c	1

Let $\mathcal{H}$ be a hesitant fuzzy set on L defined by

$$\mathcal{H} : L \to \mathscr{P}([0,1]),\ x \mapsto \begin{cases} (0.2, 0.3) \cup (0.6, 0.8] & \text{if } x \in \{0, c\}, \\ [0.1, 0.3) \cup (0.5, 0.9) & \text{if } x = a, \\ [0.2, 0.3) \cup [0.6, 0.9) & \text{if } x = b, \\ [0.1, 0.3] \cup [0.5, 0.9] & \text{if } x = 1. \end{cases}$$

Then $\mathcal{H}_b^{\to} = \{a, b, 1\}$ is not a filter of L since $a \to c = 1 \in \mathcal{H}_b^{\to}$ and $a \in \mathcal{H}_b^{\to}$, but $c \notin \mathcal{H}_b^{\to}$.

We provide conditions for the set $\mathcal{H}_a^{\to}$ to be a filter of L for $a \in L$.

Theorem 3.10. *Let $a \in L$. If $\mathcal{H}$ is a hesitant fuzzy filter of L , then $\mathcal{H}_a^{\to}$ is a filter of L.*

Proof. Obviously $1 \in \mathcal{H}_a^{\to}$ by (3.1). Let $x, y \in L$ satisfy $x \to y \in \mathcal{H}_a^{\to}$ and $x \in \mathcal{H}_a^{\to}$. Then $\mathcal{H}_{x\to y} \supseteq \mathcal{H}_a$ and $\mathcal{H}_x \supseteq \mathcal{H}_a$. It follows from (3.2) that

$$\mathcal{H}_y \supseteq \mathcal{H}_{x\to y}^x \supseteq \mathcal{H}_a.$$

Thus $y \in \mathcal{H}_a^{\to}$ and $\mathcal{H}_a^{\to}$ is a filter of L. □

Theorem 3.11. *For any $a \in L$ and a hesitant fuzzy set $\mathcal{H}$ on L, we have the following assertions:*

(1) *If $\mathcal{H}_a^{\to}$ is a filter of L, then $\mathcal{H}$ satisfies the following implication.*

$$(\forall x, y \in L)\left(\mathcal{H}_a \subseteq \mathcal{H}_{x\to y}^x \;\Rightarrow\; \mathcal{H}_a \subseteq \mathcal{H}_y\right). \tag{3.7}$$

(2) *If $\mathcal{H}$ satisfies (3.1) and (3.7), then $\mathcal{H}_a^{\to}$ is a filter of L.*

Proof. (1) Assume that $\mathcal{H}_a^{\to}$ is a filter of L for $a \in L$. Let $x, y \in L$ be such that

$$\mathcal{H}_a \subseteq \mathcal{H}_{x\to y}^x.$$

Then $x \to y \in \mathcal{H}_a^{\to}$ and $x \in \mathcal{H}_a^{\to}$. Since $\mathcal{H}_a^{\to}$ is a filter of L, it follows that $y \in \mathcal{H}_a^{\to}$, that is, $\mathcal{H}_a \subseteq \mathcal{H}_y$.

(2) Suppose that $\mathcal{H}$ satisfies (3.1) and (3.7). Let $x, y \in L$ be such that $x \to y \in \mathcal{H}_a^{\to}$ and $x \in \mathcal{H}_a^{\to}$. Then $\mathcal{H}_a \subseteq \mathcal{H}_{x\to y}$ and $\mathcal{H}_a \subseteq \mathcal{H}_x$, which implies that $\mathcal{H}_a \subseteq \mathcal{H}_{x\to y}^x$. It follows from (3.7) that $\mathcal{H}_a \subseteq \mathcal{H}_y$, i.e., $y \in \mathcal{H}_a^{\to}$. Since $\mathcal{H}$ satisfies (3.1), we have $1 \in \mathcal{H}_a^{\to}$. Therefore $\mathcal{H}_a^{\to}$ is a filter of L. □

For a fixed element $a \in L$ and a hesitant fuzzy set $\mathcal{H}$ on L, let $[a\mathcal{H}]$ be a hesitant fuzzy set on L given as follows:

$$[a\mathcal{H}] : L \to \mathscr{P}([0,1]),\; x \mapsto \begin{cases} \varepsilon_1 & \text{if } a \le x, \\ \varepsilon_2 & \text{otherwise} \end{cases}$$

where $\varepsilon_1, \varepsilon_2 \in \mathscr{P}([0,1])$ with $\varepsilon_1 \supsetneq \varepsilon_2$.

Let $L = \{0, a, b, c, 1\}$ be the lattice implication algebra in Example 3.9. For $b \in L$, the hesitant fuzzy set $[b\mathcal{H}]$ on L which is given by

$$[b\mathcal{H}] : L \to \mathscr{P}([0,1]),\; x \mapsto \begin{cases} [0.2, 0.7] & \text{if } b \le x, \\ [0.3, 0.6] & \text{otherwise} \end{cases}$$

is not a hesitant fuzzy filter of L since $[b\mathcal{H}]_a = [0.3, 0.6] \not\supseteq [0.2, 0.7] = [b\mathcal{H}]_{c\to a}^c$.

Given $a \in L$, we provide conditions for the hesitant fuzzy set $[a\mathcal{H}]$ to be a hesitant fuzzy filter of L.

Theorem 3.12. *Given $a \in L$, the hesitant fuzzy set $[a\mathcal{H}]$ is a hesitant fuzzy filter of L if and only if the following assertion is valid.*

$$(\forall x, y \in L)\,(a \le y \to x,\; a \le y \;\Rightarrow\; a \le x). \tag{3.8}$$

Proof. Suppose that $[a\mathcal{H}]$ is a hesitant fuzzy filter of L and let $x, y \in L$ satisfy $a \leq y \to x$ and $a \leq y$. Then $[a\mathcal{H}]_{y\to x} = \varepsilon_1 = [a\mathcal{H}]_y$, and so $[a\mathcal{H}]_x \supseteq [a\mathcal{H}]_{y\to x}^y = \varepsilon_1$. Thus $a \leq x$, which satisfies the condition (3.8).

Conversely, assume that the condition (3.8) is valid. Note that

$$L([a\mathcal{H}];\varepsilon) = \begin{cases} L & \text{if } \varepsilon \subseteq \varepsilon_2, \\ \{x \in L \mid a \leq x\} & \text{if } \varepsilon_2 \subsetneq \varepsilon \subseteq \varepsilon_1, \\ \emptyset & \text{otherwise} \end{cases}$$

For the case of $\varepsilon_2 \subsetneq \varepsilon \subseteq \varepsilon_1$, obviously $1 \in L([a\mathcal{H}];\varepsilon)$. Let $x, y \in L$ be such that $x \in L([a\mathcal{H}];\varepsilon)$ and $x \to y \in L([a\mathcal{H}];\varepsilon)$. Then $a \leq x$ and $a \leq x \to y$, which imply from the hypothesis that $a \leq y$, that is, $y \in L([a\mathcal{H}];\varepsilon)$. Hence $L([a\mathcal{H}];\varepsilon)$ is a filter of L whenever it is nonempty. Therefore $[a\mathcal{H}]$ is a hesitant fuzzy filter of L. □

Theorem 3.13. *For a subset J of L, let $\mathcal{G}$ be a hesitant fuzzy set on L given as follows:*

$$\mathcal{G} : L \to \mathscr{P}([0,1]),\ x \mapsto \begin{cases} \varepsilon_1 & \text{if } x \in J, \\ \varepsilon_2 & \text{otherwise} \end{cases}$$

where $\varepsilon_1, \varepsilon_2 \in \mathscr{P}([0,1])$ with $\varepsilon_1 \supsetneq \varepsilon_2$. Then $\mathcal{G}$ is a hesitant fuzzy filter of L if and only if the following assertion is valid.

$$(\forall x, y \in J)(\forall z \in L)\left(x, y \in J,\ y \leq x \to z \ \Rightarrow\ z \in J\right). \tag{3.9}$$

Proof. Note that

$$L(\mathcal{G};\varepsilon) = \begin{cases} L & \text{if } \varepsilon \subseteq \varepsilon_2, \\ J & \text{if } \varepsilon_2 \subsetneq \varepsilon \subseteq \varepsilon_1, \\ \emptyset & \text{otherwise} \end{cases}$$

Assume that $\mathcal{G}$ is a hesitant fuzzy filter of L. Then $J = L(\mathcal{G};\varepsilon)$ for $\varepsilon_2 \subsetneq \varepsilon \subseteq \varepsilon_1$, and J is a filter of L. Let $x, y, z \in L$ be such that $x, y \in J$ and $y \leq x \to z$. Then $y \to (x \to z) = 1 \in J$, and so $z \in J$.

Conversely, let $\mathcal{G}$ be a hesitant fuzzy set on L and suppose that (3.9) is valid. Since $y \leq 1 = x \to 1$ for all $x, y \in L$, we have $1 \in J$ by (3.9), and so $1 \in L(\mathcal{G};\varepsilon)$ for $\varepsilon_2 \subsetneq \varepsilon \subseteq \varepsilon_1$. Let $x, y \in L$ be such that $y \in J = L(\mathcal{G};\varepsilon)$ and $y \to x \in J = L(\mathcal{G};\varepsilon)$ for $\varepsilon_2 \subsetneq \varepsilon \subseteq \varepsilon_1$. Since $y \leq (y \to x) \to x$, it follows from (3.9) that $x \in J = L(\mathcal{G};\varepsilon)$. Hence $L(\mathcal{G};\varepsilon)$ is a filter of L for all $\varepsilon \in \mathscr{P}([0,1])$ with $L(\mathcal{G};\varepsilon) \neq \emptyset$. Therefore $\mathcal{G}$ is a hesitant fuzzy filter of L. □

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3D Green's Function and Its Finite Element Error Estimates

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In our previous article, we introduced the definition of the 3D Green's function, and gave some estimates for this function. In this article, we will give the finite element approximation to the 3D Green's function. Moreover, some error estimates between 3D Green's function and its finite element approximation are derived, which will be used to the local superconvergence analysis.

1 Introduction

Superconvergence study is still an important topic in the finite element method, and the Green's function plays very important roles in the study of the superconvergence (especially, pointwise superconvergence) of the finite element method (see [1–9]). As for the global superconvergence, we know that the discrete Green's function and the discrete derivative Green's function are usually used. However, as for the local superconvergence, we need to use the Green's function which is independent of the mesh-size h. In our recent articles, we have introduced the definition of the 3D Green's function and its some estimates. This article will focus on the finite element approximation to the 3D Green's function.

we shall use the symbol C to denote a generic constant, which is independent of the mesh-size h and which may not be the same in each occurrence and also use the standard notations for the Sobolev spaces and their norms.

In this article, we consider the following Poisson equation:

$$\mathcal{L}u \equiv -\Delta u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

where $\Omega \subset \mathcal{R}^3$ is a bounded polytopic domain. The weak formulation of the above equation reads,

$$\begin{cases} \text{Find } u \in H_0^1(\Omega) \text{ satisfying} \\ a(u\,,\,v) = (f\,,\,v) \text{ for all } v \in H_0^1(\Omega), \end{cases} \tag{1.1}$$

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where

$$a(u\,,\,v) \equiv \int_\Omega \nabla u \cdot \nabla v \, dX,\ \ (f\,,\,v) \equiv \int_\Omega f v \, dX.$$

Let $\{\mathcal{T}^h\}$ be a regular family of partitions of $\bar{\Omega}$. Denote by $S^h(\Omega)$ a continuous finite elements space of degree $m(m \geq 1)$ regarding this kind of partitions and let $S_0^h(\Omega) = S^h(\Omega) \cap H_0^1(\Omega)$.

For every $Z \in \bar{\Omega}$, we define the discrete δ function $\delta_Z^h \in S_0^h(\Omega)$, the discrete derivative δ function $\partial_{Z,\ell}\delta_Z^h \in S_0^h(\Omega)$, the regularized Green's function $G_Z^* \in H^2(\Omega) \cap H_0^1(\Omega)$, the regularized derivative Green's function $\partial_{Z,\ell}G_Z^* \in H^2(\Omega) \cap H_0^1(\Omega)$, the discrete Green's function $G_Z^h \in S_0^h(\Omega)$, the discrete derivative Green's function $\partial_{Z,\ell}G_Z^h \in S_0^h(\Omega)$, and the L^2-projection $P_h u \in S_0^h(\Omega)$ such that (see [9])

$$(v,\,\delta_Z^h) = v(Z) \ \ \forall\, v \in S_0^h(\Omega), \tag{1.2}$$

$$(v,\,\partial_{Z,\ell}\delta_Z^h) = \partial_\ell v(Z) \ \ \forall\, v \in S_0^h(\Omega), \tag{1.3}$$

$$a(G_Z^*,\,v) = (\delta_Z^h,\,v) \ \ \forall\, v \in H_0^1(\Omega), \tag{1.4}$$

$$a(\partial_{Z,\ell}G_Z^*,\,v) = (\partial_{Z,\ell}\delta_Z^h,\,v) \ \ \forall\, v \in H_0^1(\Omega), \tag{1.5}$$

$$a(G_Z^h,\,v) = v(Z) \ \ \forall\, v \in S_0^h(\Omega), \tag{1.6}$$

$$a(\partial_{Z,\ell}G_Z^h,\,v) = \partial_\ell v(Z) \ \ \forall\, v \in S_0^h(\Omega), \tag{1.7}$$

$$(u - P_h u,\,v) = 0 \ \ \forall\, v \in S_0^h(\Omega). \tag{1.8}$$

Here, for any direction $\ell \in R^3$, $|\ell| = 1$, $\partial_{Z,\ell}\delta_Z^h$, $\partial_{Z,\ell}G_Z^h$, and $\partial_\ell v(Z)$ stand for the following onesided directional derivatives, respectively.

$$\partial_{Z,\ell}\delta_Z^h = \lim_{|\Delta Z|\to 0} \frac{\delta_{Z+\Delta Z}^h - \delta_Z^h}{|\Delta Z|},\ \partial_{Z,\ell}G_Z^h = \lim_{|\Delta Z|\to 0} \frac{G_{Z+\Delta Z}^h - G_Z^h}{|\Delta Z|},$$

$$\partial_\ell v(Z) = \lim_{|\Delta Z|\to 0} \frac{v(Z+\Delta Z) - v(Z)}{|\Delta Z|},\ \Delta Z = |\Delta Z|\ell.$$

As for G_Z^*, $\partial_{Z,\ell}G_Z^*$, G_Z^h, and $\partial_{Z,\ell}G_Z^h$, we have obtained some optimal estimates (see [4–6]), which will be used in next section. From (1.4)–(1.7), we easily find G_Z^h and $\partial_{Z,\ell}G_Z^h$ are the finite element approximations to G_Z^* and $\partial_{Z,\ell}G_Z^*$, respectively.

For the L^2-projection operator P_h, we have (see [4])

Lemma 1.1. *For $P_h w$ the L^2-projection of $w \in L^p(\Omega)$, we have the following stability estimate:*

$$\|P_h w\|_{0,\,p,\,\Omega} \leq C^t \|w\|_{0,\,p,\,\Omega}, \tag{1.9}$$

where $t = \left|1 - \frac{2}{p}\right|$, and $1 \leq p \leq \infty$.

Further, by Lemma 1.1, we easily obtain the following result:

$$\|w - P_h w\|_{0,\,p,\,\Omega} \leq (1 + C^t) \inf_{v \in S_0^h \Omega} \|w - v\|_{0,\,p,\,\Omega}, \tag{1.10}$$

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where $1 \le p \le \infty$. Using the result (1.10), we easily obtain

$$\|P_h w\|_{1,\,p,\,\Omega} \le C\|w\|_{1,\,p,\,\Omega},\ \text{for } 3 < p \le \infty. \tag{1.11}$$

In addition, we also assume the following a priori estimate holds.

Lemma 1.2. *For the true solution u of (1.1), there exists a $q_0 (1 < q_0 \le \infty)$ such that for every $1 < q < q_0$,*

$$\|u\|_{2,\,q,\,\Omega} \le C(q)\|\mathcal{L}u\|_{0,\,q,\,\Omega}. \tag{1.12}$$

2 Regularized Green's Function and Its Finite Element Approximation

We introduce two weight functions defined by

$$\phi = \left(|X - Z|^2 + \theta^2\right)^{-\frac{3}{2}} \ \text{and}\ \tau = |X - Z|^{-3}\ \ \forall\, X \in \bar{\Omega},$$

where $Z \in \bar{\Omega}$ is a fixed point, $\theta = \gamma h$, and $\gamma \in [3, +\infty)$ is a suitable real number. They will be used in this section and next section.

In [4], we derived the following Lemma 2.1 (see (2.62) and (2.63) in [4]).

Lemma 2.1. *Suppose $q_0 > 3$. For G_Z^* and G_Z^h defined by (1.4) and (1.6), respectively, we have*

$$\left\|G_Z^* - G_Z^h\right\|_{1,\phi^{-1}} \le Ch\left|\nabla^2 G_Z^*\right|_{\phi^{-1}} \le Ch\left|\ln h\right|^{\frac{1}{6}}. \tag{2.1}$$

Lemma 2.2. *For G_Z^* and G_Z^h defined by (1.4) and (1.6), respectively, we have*

$$\left\|G_Z^* - G_Z^h\right\|_{1,\phi^{-\alpha}} \le C(\alpha)h \begin{cases} \forall\, 1 < \alpha < \frac{5}{3} - \frac{2}{q_0} \ \text{when } 3 < q_0 < 6, \\ \forall\, 1 < \alpha < \frac{4}{3}\ \text{when } q_0 \ge 6. \end{cases} \tag{2.2}$$

Proof. Similar to the proof of the result (2.43) in [4], we have

$$\left\|G_Z^* - G_Z^h\right\|_{1,\,\phi^{-\alpha}}^2 \le Ch^2\left\|\nabla^2 G_Z^*\right\|_{\phi^{-\alpha}}^2 + C\left\|G_Z^* - G_Z^h\right\|_{\phi^{-\alpha+\frac{2}{3}}}^2. \tag{2.3}$$

We easily obtain

$$\begin{aligned} &\left\|G_Z^* - G_Z^h\right\|_{\phi^{-\alpha+\frac{2}{3}}}^2 = \left(\phi^{-\alpha+\frac{2}{3}}(G_Z^* - G_Z^h),\, G_Z^* - G_Z^h\right) \\ =\ & a(v,\, G_Z^* - G_Z^h) = a(v - \Pi v,\, G_Z^* - G_Z^h) \\ \le\ & \left|G_Z^* - G_Z^h\right|_{1,\,\phi^{-\alpha}} \cdot |v - \Pi v|_{1,\,\phi^{\alpha}} \\ \le\ & \varepsilon\left|G_Z^* - G_Z^h\right|_{1,\,\phi^{-\alpha}}^2 + C(\varepsilon)\,|v - \Pi v|_{1,\,\phi^{\alpha}}^2 \\ \le\ & \varepsilon\left|G_Z^* - G_Z^h\right|_{1,\,\phi^{-\alpha}}^2 + C(\varepsilon)h^2\left|\nabla^2 v\right|_{\phi^{\alpha}}^2 \\ \le\ & \varepsilon\left|G_Z^* - G_Z^h\right|_{1,\,\phi^{-\alpha}}^2 + C(\varepsilon)h^2\theta^{-2}\left|\nabla(\phi^{-\alpha+\frac{2}{3}}(G_Z^* - G_Z^h))\right|_{\phi^{\alpha-\frac{4}{3}}}^2, \end{aligned} \tag{2.4}$$

where $\mathcal{L}v = \phi^{-\alpha+\frac{2}{3}}(G_Z^* - G_Z^h)$.

Note that the result $\left|\nabla^2 v\right|^2_{\phi^\alpha} \le C\theta^{-2}\left|\nabla(\phi^{-\alpha+\frac{2}{3}}(G^*_Z - G^h_Z))\right|^2_{\phi^{\alpha-\frac{4}{3}}}$ in (2.4) should satisfy one of the following two conditions: (1) $1 < \alpha < \frac{5}{3} - \frac{2}{q_0}$ when $3 < q_0 < 6$; (2) $1 < \alpha < \frac{4}{3}$ when $q_0 \ge 6$. In addition,

$$\begin{aligned}
&\left|\nabla(\phi^{-\alpha+\frac{2}{3}}(G^*_Z - G^h_Z))\right|^2_{\phi^{\alpha-\frac{4}{3}}}\\
= &\int_\Omega \phi^{\alpha-\frac{4}{3}}\left|\nabla\phi^{-\alpha+\frac{2}{3}}\cdot(G^*_Z - G^h_Z) + \phi^{-\alpha+\frac{2}{3}}\cdot\nabla(G^*_Z - G^h_Z)\right|^2 dX\\
\le & C\int_\Omega \phi^{\alpha-\frac{4}{3}}\left(|\nabla\phi^{-\alpha+\frac{2}{3}}|^2|G^*_Z - G^h_Z|^2 + (\phi^{-\alpha+\frac{2}{3}})^2|\nabla(G^*_Z - G^h_Z)|^2\right) dX\\
\le & C\left(\left|G^*_Z - G^h_Z\right|^2_{1,\,\phi^{-\alpha}} + \left\|G^*_Z - G^h_Z\right\|^2_{\phi^{-\alpha+\frac{2}{3}}}\right).
\end{aligned}$$

Combining (2.4) and the above result, we have

$$\begin{aligned}
\left\|G^*_Z - G^h_Z\right\|^2_{\phi^{-\alpha+\frac{2}{3}}} \le\ & \varepsilon\left|G^*_Z - G^h_Z\right|^2_{1,\,\phi^{-\alpha}}\\
& + C(\varepsilon)h^2\theta^{-2}\left(\left|G^*_Z - G^h_Z\right|^2_{1,\,\phi^{-\alpha}} + \left\|G^*_Z - G^h_Z\right\|^2_{\phi^{-\alpha+\frac{2}{3}}}\right)\\
=\ & \varepsilon\left|G^*_Z - G^h_Z\right|^2_{1,\,\phi^{-\alpha}}\\
& + C(\varepsilon)\gamma^{-2}\left(\left|G^*_Z - G^h_Z\right|^2_{1,\,\phi^{-\alpha}} + \left\|G^*_Z - G^h_Z\right\|^2_{\phi^{-\alpha+\frac{2}{3}}}\right).
\end{aligned} \tag{2.5}$$

Choosing $\gamma \in [3, +\infty)$ in (2.5) such that $0 < C(\varepsilon)\gamma^{-2} < \min(\varepsilon, \frac{1}{2})$, we have

$$\left\|G^*_Z - G^h_Z\right\|^2_{\phi^{-\alpha+\frac{2}{3}}} \le 4\varepsilon\left|G^*_Z - G^h_Z\right|^2_{1,\,\phi^{-\alpha}}. \tag{2.6}$$

Taking a suitable $\varepsilon \in (0, +\infty)$, from (2.3) and (2.6), we obtain

$$\left\|G^*_Z - G^h_Z\right\|_{1,\,\phi^{-\alpha}} \le Ch\left\|\nabla^2 G^*_Z\right\|_{\phi^{-\alpha}}. \tag{2.7}$$

We can prove

$$\left\|\nabla^2 G^*_Z\right\|_{\phi^{-\alpha}} \le C\left\|\delta^h_Z\right\|_{\phi^{-\alpha}} + C\left\|G^*_Z\right\|_{\phi^{-\alpha+\frac{4}{3}}} \le Ch^{\frac{3(\alpha-1)}{2}} + C\left\|G^*_Z\right\|_{\phi^{-\alpha+\frac{4}{3}}}. \tag{2.8}$$

Further, from (1.4), (1.8), (1.9), (1.12), and the Sobolev Embedding Theorem [10], we have

$$\begin{aligned}
&\left\|G^*_Z\right\|^2_{\phi^{-\alpha+\frac{4}{3}}} = (G^*_Z, \phi^{-\alpha+\frac{4}{3}}G^*_Z) = a(G^*_Z, w)\\
= & P_h w(Z) \le \|P_h w\|_{0,\infty} \le C\|w\|_{0,\infty} \le C\|w\|_{2,p} \le C\left\|\phi^{-\alpha+\frac{4}{3}}G^*_Z\right\|_{0,p}\\
= & C\left(\int_\Omega \phi^{(\frac{4}{3}-\alpha)p}|G^*_Z|^p\,dX\right)^{\frac{1}{p}} \le C\left(\int_\Omega \phi^{\frac{(\frac{4}{3}-\alpha)p}{2-p}}\,dX\right)^{\frac{2-p}{2p}}\left\|G^*_Z\right\|_{\phi^{-\alpha+\frac{4}{3}}}.
\end{aligned}$$

Here we choose p such that $\frac{3}{2} < p < \frac{6}{7-3\alpha} < 2$ and $0 < \frac{(\frac{4}{3}-\alpha)p}{2-p} < 1$. It is easy to prove

$$\int_\Omega \phi^{\frac{(\frac{4}{3}-\alpha)p}{2-p}}\,dX \le C(\alpha).$$

Thus we have

$$\|G_Z^*\|_{\phi^{-\alpha+\frac{4}{3}}} \le C(\alpha). \tag{2.9}$$

From (2.7)–(2.9), the result (2.2) is obtained.

Lemma 2.3. *For $\partial_{Z,\ell}G_Z^*$ and $\partial_{Z,\ell}G_Z^h$ defined by (1.5) and (1.7), respectively, we have*

$$\left\|\partial_{Z,\ell}G_Z^* - \partial_{Z,\ell}G_Z^h\right\|_{1,\phi^{-\alpha}} \le Ch^{\frac{3(\alpha-1)}{2}}\left|\ln h\right|^{\frac{4-3\alpha}{6}}, \tag{2.10}$$

where $1<\alpha<\frac{5}{3}-\frac{2}{q_0}$ *when* $3<q_0<6$ *and* $1<\alpha<\frac{4}{3}$ *when* $q_0\ge 6$.

Proof. Similar to the result (2.7), we have

$$\left\|\partial_{Z,\ell}G_Z^* - \partial_{Z,\ell}G_Z^h\right\|_{1,\,\phi^{-\alpha}} \le Ch\left\|\nabla^2\partial_{Z,\ell}G_Z^*\right\|_{\phi^{-\alpha}}. \tag{2.11}$$

In addition

$$\begin{aligned}\left\|\nabla^2\partial_{Z,\ell}G_Z^*\right\|_{\phi^{-\alpha}} &\le C\left\|\partial_{Z,\ell}\delta_Z^h\right\|_{\phi^{-\alpha}} + C\left\|\partial_{Z,\ell}G_Z^*\right\|_{\phi^{-\alpha+\frac{4}{3}}}\\ &\le Ch^{\frac{3\alpha-5}{2}} + C\left\|\partial_{Z,\ell}G_Z^*\right\|_{\phi^{-\alpha+\frac{4}{3}}}.\end{aligned} \tag{2.12}$$

Further, from (1.5), (1.8), (1.11), (1.12), the inverse inequality, the Sobolev Embedding Theorem [10], and the Hölder inequality, we have

$$\begin{aligned}&\left\|\partial_{Z,\ell}G_Z^*\right\|^2_{\phi^{-\alpha+\frac{4}{3}}} = (\partial_{Z,\ell}G_Z^*, \phi^{-\alpha+\frac{4}{3}}\partial_{Z,\ell}G_Z^*) = a(\partial_{Z,\ell}G_Z^*, w) = \partial_{Z,\ell}P_h w(Z)\\ \le\ & |P_h w|_{1,\infty} \le Ch^{-\frac{3}{q}}|P_h w|_{1,q} \le Ch^{-\frac{3}{q}}|w|_{1,q} \le Ch^{-\frac{3}{q}}\|w\|_{2,s}\\ \le\ & Ch^{-\frac{3}{q}}\left\|\phi^{\frac{4}{3}-\alpha}\partial_{Z,\ell}G_Z^*\right\|_{0,s} = Ch^{-\frac{3}{q}}\left(\int_\Omega \phi^{(\frac{4}{3}-\alpha)s}|\partial_{Z,\ell}G_Z^*|^s\,dX\right)^{\frac{1}{s}}\\ \le\ & Ch^{-\frac{3}{q}}\left(\int_\Omega \phi^{\frac{(\frac{4}{3}-\alpha)s}{2-s}}\,dX\right)^{\frac{2-s}{2s}}\left\|\partial_{Z,\ell}G_Z^*\right\|_{\phi^{-\alpha+\frac{4}{3}}}.\end{aligned}$$

Here we choose $s=\frac{6}{7-3\alpha}$ and $\frac{1}{q}=\frac{1}{s}-\frac{1}{3}$. Obviously,

(A) $\frac{3}{2}<s<\frac{3q_0}{3+q_0}$ and $3<q<q_0$ when $3<q_0<6$.

(B) $\frac{3}{2}<s<2$ and $3<q<6$ when $q_0\ge 6$.

In the meantime, we have $\frac{(\frac{4}{3}-\alpha)s}{2-s}=1$. By the result (2.14) in [4], we then get

$$\left(\int_\Omega \phi^{\frac{(\frac{4}{3}-\alpha)s}{2-s}}\,dX\right)^{\frac{2-s}{2s}} \le C\left|\ln h\right|^{\frac{4-3\alpha}{6}}.$$

Thus we have

$$\left\|\partial_{Z,\ell}G_Z^*\right\|_{\phi^{-\alpha+\frac{4}{3}}} \le Ch^{\frac{3\alpha-5}{2}}\left|\ln h\right|^{\frac{4-3\alpha}{6}}. \tag{2.13}$$

From (2.11)–(2.13), $\left\|\partial_{Z,\ell}G_Z^* - \partial_{Z,\ell}G_Z^h\right\|_{1,\,\phi^{-\alpha}} \le Ch^{\frac{3(\alpha-1)}{2}}\left|\ln h\right|^{\frac{4-3\alpha}{6}}$. The proof of the result (2.10) is completed.

Lemma 2.4. *For G_Z^* and G_Z^h defined by (1.4) and (1.6), respectively, we have*

$$\left\|G_Z^* - G_Z^h\right\|_{1,p} \le \begin{cases} Ch^{\frac{3-2p}{p}}\left|\ln h\right|^{\frac{1}{6}}, & 1<p<\frac{3}{2},\\ Ch\left|\ln h\right|^{\frac{2}{3}}, & p=1.\end{cases} \tag{2.14}$$

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Proof. When $p = 1$, the result can be seen in [4]. Thus we only need to prove the case of $1 < p < \frac{3}{2}$. By the Hölder inequality, we have

$$\left\|G_Z^* - G_Z^h\right\|_{1,p} \leq \left(\int_\Omega \phi^{\frac{p}{2-p}}\, dX\right)^{\frac{2-p}{2p}} \left\|G_Z^* - G_Z^h\right\|_{1,\phi^{-1}}. \tag{2.15}$$

From (2.13) in [4],

$$\int_\Omega \phi^{\frac{p}{2-p}}\, dX \leq Ch^{\frac{6-6p}{2-p}}. \tag{2.16}$$

Combining (2.1), (2.15), and (2.16) yields $\left\|G_Z^* - G_Z^h\right\|_{1,p} \leq Ch^{\frac{3-2p}{p}} \left|\ln h\right|^{\frac{1}{6}}$. The proof of the result (2.14) is completed.

3 Finite Element Approximation to the 3D Green's Function

In this section, we discuss the 3D Green's function and its finite element approximation. We call G_Z Green's function which satisfies the following Theorem 3.1.

Theorem 3.1. *There exists a unique* $G_Z \in W_0^{1,p}(\Omega)$ $(1 \leq p < \frac{3}{2})$ *such that*

$$a(G_Z, v) = v(Z) \;\; \forall\, v \in W_0^{1,p'}(\Omega), \;\; \frac{1}{p} + \frac{1}{p'} = 1. \tag{3.1}$$

Proof. We first prove the uniqueness of G_Z. Suppose there exists another Green's function $G'_Z \in W_0^{1,p}(\Omega)$ satisfying (3.1). Set $E_Z = G_Z - G'_Z$, thus

$$a(E_Z, v) = 0 \;\; \forall\, v \in W_0^{1,p'}(\Omega). \tag{3.2}$$

When $1 < p < \frac{3}{2}$, for each $\varphi \in L^{p'}(\Omega)$, there exists a $w \in W^{2,p'} \cap W_0^{1,p'}(\Omega)$ such that $\mathcal{L}w = \varphi$. Obviously, $\text{sgn}E_Z|E_Z|^{p-1} \in L^{p'}(\Omega)$, thus we can find $w \in W^{2,p'} \cap W_0^{1,p'}(\Omega)$ such that $\mathcal{L}w = v$. Then we have

$$\|E_Z\|_{0,p}^p = (E_Z, \text{sgn}E_Z|E_Z|^{p-1}) = a(E_Z, w), \tag{3.3}$$

From (3.2) and (3.3), $\|E_Z\|_{0,p} = 0$, i.e., $G_Z = G'_Z$. Similarly, when $p = 1$, we can also prove $G_Z = G'_Z$. Thus we have completed the proof of the uniqueness.

Next, we prove the existence of G_Z. We give a series of finite element spaces $S_0^{h_i}(\Omega)$, $i = 0, 1, 2, \cdots$ satisfying $S_0^{h_i}(\Omega) \subset S_0^{h_j}(\Omega)$ when $i < j$, where $h_0 \equiv h$ and $\frac{1}{4}h_{i-1} \leq h_i \leq \frac{1}{2}h_{i-1}$. Let $G_{Z,i}^*$ be the regularized Green's function for the finite element space $S_0^{h_i}(\Omega)$, and $G_Z^{h_i}$ the discrete Green's function. Their definitions can be seen in Section 1. Obviously, we have $a(G_{Z,i+1}^* - G_Z^{h_i}, v) = 0 \;\forall v \in S_0^{h_i}(\Omega)$. Similar to the proof of the result (2.14), we have for $1 < p < \frac{3}{2}$

$$\left\|G_{Z,i+1}^* - G_Z^{h_i}\right\|_{1,p} \leq Ch_i^{\frac{3-2p}{p}} \left|\ln h_i\right|^{\frac{1}{6}},$$

which combined with (2.14), we get

$$\left\|G^*_{Z,i+1} - G^*_{Z,i}\right\|_{1,p} \leq C h_i^{\frac{3-2p}{p}} \left|\ln h_i\right|^{\frac{1}{6}}. \tag{3.4}$$

Thus

$$\sum_{i=0}^{\infty} \left\|G^*_{Z,i+1} - G^*_{Z,i}\right\|_{1,p} \leq C \sum_{i=0}^{\infty} \left(\frac{h}{2^i}\right)^{\frac{3-2p}{p}} \left|\ln \frac{h}{2^i}\right|^{\frac{1}{6}} \leq C h^{\frac{3-2p}{p}} \left|\ln h\right|^{\frac{1}{6}}. \tag{3.5}$$

Set

$$G_Z \equiv G^*_Z + \sum_{i=0}^{\infty} (G^*_{Z,i+1} - G^*_{Z,i}).$$

Thus we have $G_Z \in W^{1,p}_0(\Omega)$. From (3.5),

$$\|G_Z - G^*_Z\|_{1,p} \leq C h^{\frac{3-2p}{p}} \left|\ln h\right|^{\frac{1}{6}}. \tag{3.6}$$

Similarly, when $p = 1$, we have

$$\|G_Z - G^*_Z\|_{1,1} \leq C h \left|\ln h\right|^{\frac{2}{3}}. \tag{3.7}$$

Therefore, for $1 \leq p < \frac{3}{2}$, we have $G^*_{Z,i} \longrightarrow G_Z$ in $W^{1,p}(\Omega)$ when $i \to \infty$. Using (1.10) and the interpolation error estimate, we obtain

$$\|v - P_h v\|_{0,\infty,\Omega} \leq C \|v - \Pi v\|_{0,\infty,\Omega} \leq C h^{1-\frac{3}{p'}} \|v\|_{1,p',\Omega}, \tag{3.8}$$

where $3 < p' \leq \infty$. Thus, for every $v \in W^{1,p'}_0(\Omega)$, we have by (3.6)–(3.8)

$$a(G_Z, v) = \lim_{i\to\infty} a(G^*_{Z,i}, v) = \lim_{i\to\infty} P_{h_i} v(Z) = v(Z).$$

The proof of Theorem 3.1 is completed. Now we show G_Z is independent of h. Suppose there exists a Green's function $\tilde{G}_Z$ for the mesh-size $\tilde{h}$. In addition, $\frac{1}{4}\tilde{h}_{i-1} \leq \tilde{h}_i \leq \frac{1}{2}\tilde{h}_{i-1}$ and $\tilde{h}_0 = \tilde{h}$. Thus, for every $f \in L^{p'}(\Omega)$, we choose $v \in W^{2,p'}(\Omega) \cap W^{1,p'}_0(\Omega)$ such that $\mathcal{L}v = f$. Then we get $(G_Z, f) = a(G_Z, v) = v(Z)$ and $(\tilde{G}_Z, f) = a(\tilde{G}_Z, v) = v(Z)$. Thus, $(G_Z, f) = (\tilde{G}_Z, f)$, i.e., $(G_Z - \tilde{G}_Z, f) = 0$. So we get $G_Z = \tilde{G}_Z$. Namely, G_Z is independent of h.

In addition, we find

$$a(G_Z, v) = v(Z) \quad \forall\, v \in S^h_0(\Omega) \subset W^{1,p'}(\Omega). \tag{3.9}$$

Combining (1.6) and (3.9), we have $a(G_Z - G^h_Z, v) = 0 \ \forall\, v \in S^h_0(\Omega)$. Thus G^h_Z is the finite element approximation to G_Z. Further, we have the following error estimates.

Theorem 3.2. *For G_Z and G^h_Z defined by (3.1) and (1.6), respectively, we have*

$$\left\|G_Z - G^h_Z\right\|_{1,p} \leq \begin{cases} C h^{\frac{3-2p}{p}} \left|\ln h\right|^{\frac{1}{6}}, & 1 < p < \frac{3}{2}, \\ C h \left|\ln h\right|^{\frac{2}{3}}, & p = 1, \end{cases} \tag{3.10}$$

where C is independent of h and Z.

Proof. From (2.14), (3.6), (3.7), and the triangular inequality, we immediately obtain the result (3.10).

Theorem 3.3. *Suppose $q_0 = \infty$, for G_Z and G_Z^h defined by (3.1) and (1.6), respectively, we have*

$$\left\|G_Z - G_Z^h\right\|_{0,1} \leq Ch^2 \left|\ln h\right|^{\frac{5}{3}}, \tag{3.11}$$

where C is independent of h and Z.

Proof. For every $\varphi \in L^\infty(\Omega)$, there exists a unique $v \in W^{2,\infty}(\Omega) \cap H_0^1(\Omega)$ such that $\mathcal{L}v = \varphi$ and

$$(G_Z - G_Z^h, \varphi) = a(G_Z - G_Z^h, v) = a(G_Z, v - v_h) = v(Z) - v_h(Z), \tag{3.12}$$

where v_h is the finite element approximation to v. From (1.10),

$$|v(Z) - P_h v(Z)| \leq \|v - P_h v\|_{0,\infty} \leq C\,\|v - \Pi v\|_{0,\infty} \leq Ch^{2-\frac{3}{q}} \|v\|_{2,q}, \tag{3.13}$$

where $1 < q < q_0$. In addition, by (2.14), the Hölder inequality, and the interpolation error estimate, we have

$$\begin{aligned} & |P_h v(Z) - v_h(Z)| = |a(v - v_h, G_Z^*)| = |a(v - v_h, G_Z^* - G_Z^h)| \\ = \;& |a(v - \Pi v, G_Z^* - G_Z^h)| \leq C\left\|G_Z^* - G_Z^h\right\|_{1,1} \|v - \Pi v\|_{1,\infty} \\ \leq \;& Ch^{2-\frac{3}{q}} \left|\ln h\right|^{\frac{2}{3}} \|v\|_{2,q}. \end{aligned} \tag{3.14}$$

From (3.12)–(3.14), and the triangular inequality,

$$|(G_Z - G_Z^h, \varphi)| = |v(Z) - v_h(Z)| \leq Ch^{2-\frac{3}{q}} \left|\ln h\right|^{\frac{2}{3}} \|v\|_{2,q}.$$

From (1.12),

$$|(G_Z - G_Z^h, \varphi)| \leq C(q)h^{2-\frac{3}{q}} \left|\ln h\right|^{\frac{2}{3}} \|\varphi\|_{0,q}. \tag{3.15}$$

Because of $q_0 = \infty$, we can take $q = |\ln h| < q_0$ in (3.15), and we have $C(q) \leq Cq$. Thus,

$$|(G_Z - G_Z^h, \varphi)| \leq Ch^2 \left|\ln h\right|^{\frac{5}{3}} \|\varphi\|_{0,\infty}. \tag{3.16}$$

From (3.16), we know the result (3.11) holds. So, the proof of the result (3.11) is completed.

Theorem 3.4. *For G_Z and G_Z^h defined by (3.1) and (1.6), respectively, we have*

$$\left\|G_Z - G_Z^h\right\|_{1,\tau^{-1}} \leq Ch \left|\ln h\right|^{\frac{1}{6}}, \tag{3.17}$$

$$\left\|G_Z - G_Z^h\right\|_{1,\tau^{-\alpha}} \leq C(\alpha)h \begin{cases} \forall\, 1 < \alpha < \frac{5}{3} - \frac{2}{q_0} & \text{when } 3 < q_0 < 6, \\ \forall\, 1 < \alpha < \frac{4}{3} & \text{when } q_0 \geq 6, \end{cases} \tag{3.18}$$

where C is independent of h and Z.

Proof. Obviously, $\tau^{-k} < \phi^{-k}$ when $k > 0$. Thus from (2.1) and (2.2),

$$\left\|G_Z^* - G_Z^h\right\|_{1,\tau^{-1}} \leq Ch \left|\ln h\right|^{\frac{1}{6}}, \tag{3.19}$$

$$\left\|G_Z^* - G_Z^h\right\|_{1,\tau^{-\alpha}} \le C(\alpha)h \begin{cases} \forall\, 1 < \alpha < \frac{5}{3} - \frac{2}{q_0} \text{ when } 3 < q_0 < 6, \\ \forall\, 1 < \alpha < \frac{4}{3} \text{ when } q_0 \ge 6, \end{cases} \tag{3.20}$$

Similar to the arguments of Theorem 3.1, we can obtain the results (3.17) and (3.18). Obviously,

$$\left\|\partial_{Z,\ell} G_Z^* - \partial_{Z,\ell} G_Z^h\right\|_{1,\tau^{-\alpha}} \le \left\|\partial_{Z,\ell} G_Z^* - \partial_{Z,\ell} G_Z^h\right\|_{1,\phi^{-\alpha}} \le Ch^{\frac{3(\alpha-1)}{2}} \left|\ln h\right|^{\frac{4-3\alpha}{6}}, \tag{3.21}$$

where $1 < \alpha < \frac{5}{3} - \frac{2}{q_0}$ when $3 < q_0 < 6$ and $1 < \alpha < \frac{4}{3}$ when $q_0 \ge 6$. Adopting the techniques in the proof of Theorem 3.1, we can derive by (3.21)

$$\sum_{i=0}^{\infty} \left\|\partial_{Z,\ell} G_{Z,i+1}^* - \partial_{Z,\ell} G_{Z,i}^*\right\|_{1,\tau^{-\alpha}} \le Ch^{\frac{3(\alpha-1)}{2}} \left|\ln h\right|^{\frac{4-3\alpha}{6}}.$$

Set

$$F \equiv \partial_{Z,\ell} G_Z^* + \sum_{i=0}^{\infty} (\partial_{Z,\ell} G_{Z,i+1}^* - \partial_{Z,\ell} G_{Z,i}^*).$$

Here, $\|F\|_{1,\tau^{-\alpha}} < \infty$ and $\partial_{Z,\ell} G_{Z,i}^* = \lim_{|\Delta Z| \to 0} \frac{G_{Z+\Delta Z,i}^* - G_{Z,i}^*}{|\Delta Z|}$, $\Delta Z = |\Delta Z|\ell$. By the arguments of Theorem 3.1,

$$\begin{aligned} G_{Z+\Delta Z} &\equiv G_{Z+\Delta Z}^* + \sum_{i=0}^{\infty} (G_{Z+\Delta Z,i+1}^* - G_{Z+\Delta Z,i}^*), \\ G_Z &\equiv G_Z^* + \sum_{i=0}^{\infty} (G_{Z,i+1}^* - G_{Z,i}^*). \end{aligned}$$

Thus we have $F = \lim_{|\Delta Z| \to 0} \frac{G_{Z+\Delta Z} - G_Z}{|\Delta Z|} = \partial_{Z,\ell} G_Z$. Namely,

$$\partial_{Z,\ell} G_Z = \partial_{Z,\ell} G_Z^* + \sum_{i=0}^{\infty} (\partial_{Z,\ell} G_{Z,i+1}^* - \partial_{Z,\ell} G_{Z,i}^*), \; \|\partial_{Z,\ell} G_Z\|_{1,\tau^{-\alpha}} < \infty. \tag{3.22}$$

We write $W_\beta(\Omega) = \{v : v|_{\partial\Omega} = 0, \|v\|_{1,\tau^\beta} < \infty\}$. From (3.22), $\partial_{Z,\ell} G_Z \in W_{-\alpha}(\Omega)$. Further, we can obtain the following Theorem 3.5.

Theorem 3.5. *There exists a unique $\partial_{Z,\ell} G_Z \in W_{-\alpha}(\Omega)$ such that*

$$a(\partial_{Z,\ell} G_Z, v) = \partial_\ell v(Z) \;\; \forall\, v \in W_\alpha(\Omega) \cap C_0^\infty(\Omega), \tag{3.23}$$

where $1 < \alpha < \frac{5}{3} - \frac{2}{q_0}$ when $3 < q_0 < 6$ and $1 < \alpha < \frac{4}{3}$ when $q_0 \ge 6$.

Proof. From (3.22),

$$\|\partial_{Z,\ell} G_Z - \partial_{Z,\ell} G_Z^*\|_{1,\tau^{-\alpha}} \le Ch^{\frac{3(\alpha-1)}{2}} \left|\ln h\right|^{\frac{4-3\alpha}{6}}. \tag{3.24}$$

Namely, $\partial_{Z,\ell} G_Z^* \longrightarrow \partial_{Z,\ell} G_Z$ in $W_{-\alpha}(\Omega)$ when $h \to 0$. Then we have by (1.3), (1.5), and (1.8)

$$a(\partial_{Z,\ell} G_Z, v) = \lim_{h \to 0} a(\partial_{Z,\ell} G_Z^*, v) = \lim_{h \to 0} \partial_\ell P_h v(Z). \tag{3.25}$$

From (1.11), $\|v - P_h v\|_{1,\infty} \le C\|v - \Pi v\|_{1,\infty} \le Ch\|v\|_{2,\infty}$. That is

$$\|v - P_h v\|_{1,\infty} \longrightarrow 0 \text{ when } h \to 0. \tag{3.26}$$

Combining (3.25) and (3.26) yields

$$a(\partial_{Z,\ell} G_Z, v) = \partial_\ell v(Z). \tag{3.27}$$

The uniqueness of $\partial_{Z,\ell} G_Z$ satisfying (3.27) can be similarly proved as that of G_Z in (3.1).

By (3.21), (3.24), and the triangular inequality, we immediately obtain the following result (3.28).

Theorem 3.6. *For $\partial_{Z,\ell} G_Z$ and $\partial_{Z,\ell} G_Z^h$ defined by (3.23) and (1.7), respectively, we have*

$$\left\|\partial_{Z,\ell} G_Z - \partial_{Z,\ell} G_Z^h\right\|_{1,\tau^{-\alpha}} \le Ch^{\frac{3(\alpha-1)}{2}} |\ln h|^{\frac{4-3\alpha}{6}}, \tag{3.28}$$

where $1 < \alpha < \frac{5}{3} - \frac{2}{q_0}$ when $3 < q_0 < 6$ and $1 < \alpha < \frac{4}{3}$ when $q_0 \ge 6$.

Acknowledgments This work was supported by the National Natural Science Foundation of China Grant 11161039, the Zhejiang Provincial Natural Science Foundation Grant LY13A010007 and the Natural Science Foundation of Ningbo City Grant 2015A610163.

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Hermite–Hadamard Type Inequalities for s-Convex Functions via Riemann–Liouville Fractional Integrals

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Received on October 7, 2015; accepted on 25 January 2016

Abstract

In the paper, by establishing a Riemann–Liouville fractional integral identity involving an n-times differentiable function, the authors present some Hermite–Hadamard type inequalities involving Riemann–Liouville fractional integrals for s-convex functions.

2010 Mathematics Subject Classification: Primary 26A33; Secondary 26D15, 26E60, 41A55.

Key words and phrases: Riemann–Liouville fractional integral; Hermite–Hadamard type inequality; s-convex function.

1 Introduction

Throughout this paper, let $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{R}_0 = [0, \infty)$, use $I \subseteq \mathbb{R}$ and I° to denote an interval and the interior of I respectively, and utilize $\mathbb{N}$ to denote the set of all positive integers.

The following definition is well known in the literature.

Definition 1.1. A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if

$$f(\lambda x + (1 - \lambda)y) \le \lambda f(x) + (1 - \lambda)f(y)$$

holds for $x, y \in I$ and $\lambda \in [0, 1]$. If this inequality reverses, then f is said to be concave on I.

The most important inequality in the theory of convex functions, Hermite–Hadamard's inequality, may be stated as follows. If f is a convex function on $[a, b]$, then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a} \int_a^b f(x) \,\mathrm{d}\, x \le \frac{f(a) + f(b)}{2}. \tag{1.1}$$

If f is concave on $[a, b]$, then the inequality (1.1) is reversed. See [6], for example.

The inequality (1.1) has been generalized in many articles. Some of them may be recited as follows.

Theorem 1.1 ([2, Theorem 2.2]). *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° and $a, b \in I^\circ$ with $a < b$. If $|f'(x)|$ is convex on $[a, b]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{(b-a)(|f'(a)|+|f'(b)|)}{8}.$$

Theorem 1.2 ([7, Theorem 1]). *If f is differentiable on $[a, b]$ such that $|f'(x)|^q$ is a convex function on $[a, b]$ for $q \ge 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{b-a}{4}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{1/q}.$$

Theorem 1.3 ([5, Theorem 2.3]). *Let $f : I \to \mathbb{R}$ be differentiable on I°, $a, b \in I^\circ$ with $a < b$, and $p > 1$. If $|f'(x)|^{p/(p-1)}$ is convex on $[a, b]$, then*

$$\begin{aligned}\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| &\le \frac{b-a}{16}\left(\frac{4}{p+1}\right)^{1/p} \\ &\times \left\{\left[|f'(a)|^{p/(p-1)}+3|f'(b)|^{p/(p-1)}\right]^{1-1/p} + \left[3|f'(a)|^{p/(p-1)}+|f'(b)|^{p/(p-1)}\right]^{1-1/p}\right\}.\end{aligned}$$

For more information, please refer to [2, 5, 6, 7] and references therein.

In addition to the classical convex functions, the class of functions which are s-convex has been introduced in [4] as follows.

Definition 1.2 ([4, p. 100]). A function $f : \mathbb{R}_0 \to \mathbb{R}$ is said to be s-convex for some fixed $s \in (0, 1]$ if $f(tx + (1-t)y) \le t^s f(x) + (1-t)^s f(y)$ holds for all $x, y \in \mathbb{R}_0$ and $t \in [0, 1]$.

It is obvious that when $s = 1$, the so-called s-convexity reduces to the ordinary convexity of functions defined on $\mathbb{R}_0$.

Some inequalities of Hermite–Hadamard type for s-convex functions may be narrated as follows.

Theorem 1.4 ([3]). *Suppose that $f : \mathbb{R}_0 \to \mathbb{R}_0$ is a s-convex function for $s \in (0, 1)$ and let $a, b \in \mathbb{R}_0$ and $a < b$. If $f' \in L_1([a, b])$, then*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x \le \frac{f(a)+f(b)}{s+1}. \tag{1.2}$$

The constant $\frac{1}{s+1}$ is the best possible in the right hand side inequality in (1.2).

Theorem 1.5 ([1]). *Let $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L_1([a, b])$, where $a, b \in I$ and $a < b$. If $|f'|^q$ is s-convex on $[a, b]$ for some fixed $s \in (0, 1]$, $q > 1$, and $p = \frac{q}{q-1}$, and if $|f'(x)| \le M$, then*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}\,t\right| \le \frac{M}{(1+p)^{1/p}}\left(\frac{2}{s+1}\right)^{1/q}\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right], \quad x \in [a, b].$$

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For more results about s-convex functions, one can see [1, 3, 4, 8] and references therein.

Definition 1.3 ([9]). Let $f \in L_1([a,b])$. The Riemann–Liouville integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ with $b > a > 0$ are defined by

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\,\mathrm{d}\,t \quad \text{and} \quad J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)\,\mathrm{d}\,t$$

for $x \in (a,b)$ respectively, where Γ is the classical Euler gamma function defined for $\mathfrak{Re}(z) > 0$ by $\Gamma(z) = \int_0^\infty e^{-u}u^{z-1}\,\mathrm{d}\,u$. Moreover, define $J^{0}_{b-}f(x) = J^{0}_{a+}f(x) = f(x)$.

In the case $\alpha = 1$, the fractional integral reduces to the classical and usual integral.

Very recently, Hermite–Hadamard's inequality was extended in [9] to the case of Riemann–Liouville fractional integrals.

Theorem 1.6 ([9, Theorem 2]). *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \le a < b$ and $x \in [a,b]$. If f is a convex function on $[a,b]$, then*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{a+}f(b) + J^{\alpha}_{b-}f(a)\right] \le \frac{f(a)+f(b)}{2}, \quad \alpha > 0.$$

Theorem 1.7 ([9, Theorem 3]). *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) and $a < b$. If $|f'|$ is convex on $[a,b]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{a+}f(b) + J^{\alpha}_{b-}f(a)\right]\right| \le \frac{b-a}{2(\alpha+1)}\left(1 - \frac{1}{2^{\alpha}}\right)\left[|f'(a)| + |f'(b)|\right], \quad \alpha > 0.$$

Theorem 1.8 ([10, Theorem 7]). *Let $f : [a,b] \subseteq \mathbb{R}_0 \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$ such that $f' \in L_1([a,b])$. If $|f'|$ is s-convex on $[a,b]$ for some fixed $s \in (0,1]$ and $|f'(x)| \le M$, then*

$$\left|\frac{(x-a)^{\alpha} + (b-x)^{\alpha}}{b-a} f(x) - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{x+}f(b) + J^{\alpha}_{x-}f(a)\right]\right|$$
$$\le \frac{M}{b-a}\left[1 + \frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+1)}\right]\frac{(x-a)^{\alpha+1} + (b-x)^{\alpha+1}}{\alpha+s+1}, \quad \alpha > 0, \quad x \in [a,b].$$

For recent development on fractional calculus, one can see the monographs [9, 10, 11] and the references therein.

Motivated by the above results, we establish a Riemann–Liouville fractional integral identity involving a n-times differentiable mapping and give some new Hermite–Hadamard type inequalities involving Riemann–Liouville fractional integrals for s-convex functions.

2 A lemma

In order to obtain our main results, we need the following lemma.

Lemma 2.1. *For $n \in \mathbb{N}$ and $a < b$, let $f : [a,b] \subseteq \mathbb{R}_0 \to \mathbb{R}$ be an n-times differentiable mapping on (a,b) and $\alpha > 0$. If $f^{(n)} \in L_1([a,b])$, then*

$$\frac{\Gamma(\alpha+n)}{2(b-a)^\alpha}\big[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\big]=\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\big[f^{(k)}(a)+(-1)^k f^{(k)}(b)\big] - \frac{(b-a)^n}{2}\int_0^1\big[(-1)^{n-1}(1-t)^{\alpha+n-1}-t^{\alpha+n-1}\big]f^{(n)}(ta+(1-t)b)\,\mathrm{d}\,t.$$

Proof. When $n=1$, by integrating by part in the right-hand side of (2.1), we have

$$\frac{b-a}{2}\int_0^1\big[(1-t)^\alpha-t^\alpha\big]f'(ta+(1-t)b)\,\mathrm{d}\,t = \frac{f(a)+f(b)}{2}-\frac{\alpha}{2}\int_0^1\big[(1-t)^{\alpha-1}+t^{\alpha-1}\big]f(ta+(1-t)b)\,\mathrm{d}\,t, \quad (2.1)$$

where

$$\alpha\int_0^1(1-t)^{\alpha-1}f(ta+(1-t)b)\,\mathrm{d}\,t = \frac{\alpha}{b-a}\int_a^b\left(\frac{x-a}{b-a}\right)^{\alpha-1}f(x)\,\mathrm{d}\,x = \frac{\Gamma(\alpha+1)}{(b-a)^\alpha}J_{b-}^\alpha f(a) \quad (2.2)$$

and

$$\alpha\int_0^1 t^{\alpha-1}f(ta+(1-t)b)\,\mathrm{d}\,t = \frac{\alpha}{b-a}\int_a^b\left(\frac{b-x}{b-a}\right)^{\alpha-1}f(x)\,\mathrm{d}\,x = \frac{\Gamma(\alpha+1)}{(b-a)^\alpha}J_{a+}^\alpha f(b). \quad (2.3)$$

Substituting (2.2) and (2.3) into (2.1) yields the identity (2.1) for $n=1$.

When $n=m-1$ and $m\ge 2$, suppose that the identity (2.1) is valid. When $n=m$, by the hypothesis, we have

$$\begin{aligned}
&\frac{(b-a)^m}{2}\int_0^1\big[(-1)^{m-1}(1-t)^{\alpha+m-1}-t^{\alpha+m-1}\big]f^{(m)}(ta+(1-t)b)\,\mathrm{d}\,t\\
&=\frac{(b-a)^{m-1}}{2}\bigg\{\big[f^{(m-1)}(a)+(-1)^{m-1}f^{(m-1)}(b)\big]\\
&\quad+(\alpha+m-1)\int_0^1\big[(-1)^{m-2}(1-t)^{\alpha+m-2}-t^{\alpha+m-2}\big]f^{(m-1)}(ta+(1-t)b)\,\mathrm{d}\,t\bigg\}\\
&=\frac{(b-a)^{m-1}}{2}\big[f^{(m-1)}(a)+(-1)^{m-1}f^{(m-1)}(b)\big]\\
&\quad+\frac{(\alpha+m-1)(b-a)^{m-1}}{2}\int_0^1\big[(-1)^{m-2}(1-t)^{\alpha+m-2}-t^{\alpha+m-2}\big]f^{(m-1)}(ta+(1-t)b)\,\mathrm{d}\,t\\
&=\frac{(b-a)^{m-1}}{2}\big[f^{(m-1)}(a)+(-1)^{m-1}f^{(m-1)}(b)\big]\\
&\quad+\sum_{k=0}^{m-2}\frac{(\alpha+m-1)\Gamma(\alpha+m-1)(b-a)^k}{2\Gamma(\alpha+k+1)}\big[f^{(k)}(a)+(-1)^k f^{(k)}(b)\big]\\
&\quad-\frac{(\alpha+m-1)\Gamma(\alpha+m-1)}{2(b-a)^\alpha}\big[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\big]\\
&=\sum_{k=0}^{m-1}\frac{\Gamma(\alpha+m)(b-a)^k}{2\Gamma(\alpha+k+1)}\big[f^{(k)}(a)+(-1)^k f^{(k)}(b)\big]-\frac{\Gamma(\alpha+m)}{2(b-a)^\alpha}\big[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\big].
\end{aligned}$$

Therefore, when $n = m$, the identity (2.1) holds. By induction, the proof of Lemma 2.1 is complete. □

Remark 2.1. When $n = 1$ in (2.1), we obtain the identity

$$\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\big[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\big]=\frac{b-a}{2}\int_0^1\big[(1-t)^{\alpha}-t^{\alpha}\big]f'(ta+(1-t)b)\,\mathrm{d}\,t,$$

which is the identity established in [9].

3 Hermite–Hadamard type inequalities involving Riemann–Liouville fractional integrals

Now we start out to establish some new Hermite–Hadamard type inequalities involving Riemann–Liouville fractional integrals for s-convex functions.

Theorem 3.1. *For $n \in \mathbb{N}$ and $a, b \in \mathbb{R}_0$ with $a < b$, let $f : \mathbb{R}_0 \to \mathbb{R}$ be an n-times differentiable function on $\mathbb{R}_0$ such that $f^{(n)} \in L_1([a, b])$. If $|f^{(n)}|^q$ is s-convex on $[a, b]$ for $q \ge 1$ and some fixed $s \in (0, 1]$, then*

$$\begin{aligned}\Bigg|\frac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}\big[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\big]-\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\big[f^{(k)}(a)+(-1)^kf^{(k)}(b)\big]\Bigg|\\ \le\frac{(b-a)^n}{2(\alpha+n)^{1-1/q}}\Bigg\{\Bigg[B(s+1,\alpha+n)\big|f^{(n)}(a)\big|^q+\frac{1}{\alpha+n+s}\big|f^{(n)}(b)\big|^q\Bigg]^{1/q}\\ +\Bigg[\frac{1}{\alpha+n+s}\big|f^{(n)}(a)\big|^q+B(s+1,\alpha+n)\big|f^{(n)}(b)\big|^q\Bigg]^{1/q}\Bigg\},\end{aligned}$$

where $\alpha > 0$ and B is the classical Beta function which may be defined for $\mathfrak{Re}(x) > 0$ and $\mathfrak{Re}(y) > 0$ by $B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}\,t$.

Proof. By Lemma 2.1, s-convexity of $|f^{(n)}|^q$, and Hölder's inequality, we obtain

$$\begin{aligned}&\Bigg|\frac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}\big[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\big]-\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\big[f^{(k)}(a)+(-1)^kf^{(k)}(b)\big]\Bigg|\\ &\le\frac{(b-a)^n}{2}\Bigg[\int_0^1(1-t)^{\alpha+n-1}\big|f^{(n)}(ta+(1-t)b)\big|\mathrm{d}\,t+\int_0^1t^{\alpha+n-1}\big|f^{(n)}(ta+(1-t)b)\big|\mathrm{d}\,t\Bigg]\\ &\le\frac{(b-a)^n}{2}\Bigg\{\Bigg[\int_0^1(1-t)^{\alpha+n-1}\,\mathrm{d}\,t\Bigg]^{1-1/q}\Bigg[\int_0^1(1-t)^{\alpha+n-1}\big|f^{(n)}(ta+(1-t)b)\big|^q\mathrm{d}\,t\Bigg]^{1/q}\\ &\quad+\Bigg[\int_0^1t^{\alpha+n-1}\,\mathrm{d}\,t\Bigg]^{1-1/q}\Bigg[\int_0^1t^{\alpha+n-1}\big|f^{(n)}(ta+(1-t)b)\big|^q\mathrm{d}\,t\Bigg]^{1/q}\Bigg\}\\ &\le\frac{(b-a)^n}{2(\alpha+n)^{1-1/q}}\Bigg\{\Bigg[\int_0^1\Big((1-t)^{\alpha+n-1}t^s\big|f^{(n)}(a)\big|^q+(1-t)^{\alpha+n+s-1}\big|f^{(n)}(b)\big|^q\Big)\mathrm{d}\,t\Bigg]^{1/q}\end{aligned}$$

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$$+\left[\int_0^1\left(t^{\alpha+n+s-1}\left|f^{(n)}(a)\right|^q+t^{\alpha+n-1}(1-t)^s\left|f^{(n)}(b)\right|^q\right)\mathrm{d}\,t\right]^{1/q}\Bigg\}$$
$$=\frac{(b-a)^n}{2(\alpha+n)^{1-1/q}}\left\{\left[B(s+1,\alpha+n)\left|f^{(n)}(a)\right|^q+\frac{1}{\alpha+n+s}\left|f^{(n)}(b)\right|^q\right]^{1/q}\right.$$
$$\left.+\left[\frac{1}{\alpha+n+s}\left|f^{(n)}(a)\right|^q+B(s+1,\alpha+n)\left|f^{(n)}(b)\right|^q\right]^{1/q}\right\}.$$

Theorem 3.1 is proved. □

Corollary 3.1.1. *Under the assumptions of Theorem 3.1,*

1. *when $s=1$, we have*

$$\left|\frac{\Gamma(\alpha+n)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right]-\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\left[f^{(k)}(a)+(-1)^k f^{(k)}(b)\right]\right|$$
$$\le\frac{(b-a)^n}{2(\alpha+n)(\alpha+n+1)^{1/q}}\left\{\left[\left|f^{(n)}(a)\right|^q+(\alpha+n)\left|f^{(n)}(b)\right|^q\right]^{1/q}\right.$$
$$\left.+\left[(\alpha+n)\left|f^{(n)}(a)\right|^q+\left|f^{(n)}(b)\right|^q\right]^{1/q}\right\};$$

2. *when $n=1$, we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right]\right|\le\frac{b-a}{2(\alpha+1)^{1-1/q}}\left\{\left[B(s+1,\alpha+1)\left|f'(a)\right|^q\right.\right.$$
$$\left.\left.+\frac{1}{\alpha+s+1}\left|f'(b)\right|^q\right]^{1/q}+\left[\frac{1}{\alpha+s+1}\left|f'(a)\right|^q+B(s+1,\alpha+1)\left|f'(b)\right|^q\right]^{1/q}\right\};$$

3. *when $q=1$, we have*

$$\left|\frac{\Gamma(\alpha+n)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right]-\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\left[f^{(k)}(a)+(-1)^k f^{(k)}(b)\right]\right|$$
$$\le\frac{(b-a)^n}{2}\left[B(s+1,\alpha+n)+\frac{1}{\alpha+n+s}\right]\left[\left|f^{(n)}(a)\right|+\left|f^{(n)}(b)\right|\right];$$

4. *when $s=n=q=1$, we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right]\right|\le\frac{b-a}{2(\alpha+1)}\left[\left|f'(a)\right|+\left|f'(b)\right|\right].$$

Theorem 3.2. *For $n\in\mathbb{N}$ and $a,b\in\mathbb{R}_0$ with $a<b$, let $f:\mathbb{R}_0\to\mathbb{R}$ be an n-times differentiable function on $\mathbb{R}_0$ such that $f^{(n)}\in L_1([a,b])$. If $|f^{(n)}|^q$ is s-convex on $[a,b]$ for $q>1$ and some fixed*

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$s \in (0,1]$, *then*

$$\left|\frac{\Gamma(\alpha+n)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b)+J_{b^-}^\alpha f(a)\right]-\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\left[f^{(k)}(a)+(-1)^k f^{(k)}(b)\right]\right|$$
$$\le \frac{(b-a)^n}{2}\left[\frac{q-1}{q(\alpha+n)-r-1}\right]^{1-1/q}\left\{\left[B(s+1,r+1)\left|f^{(n)}(a)\right|^q+\frac{1}{r+s+1}\left|f^{(n)}(b)\right|^q\right]^{1/q}\right.$$
$$\left.+\left[\frac{1}{r+s+1}\left|f^{(n)}(a)\right|^q+B(s+1,r+1)\left|f^{(n)}(b)\right|^q\right]^{1/q}\right\}$$

for $\alpha>0$ *and* $0\le r\le q(\alpha+n-1)$.

Proof. From Lemma 2.1, s-convexity of $|f^{(n)}|^q$, and the Hölder's inequality, it follows that

$$\left|\frac{\Gamma(\alpha+n)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b)+J_{b^-}^\alpha f(a)\right]-\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\left[f^{(k)}(a)+(-1)^k f^{(k)}(b)\right]\right|$$
$$\le \frac{(b-a)^n}{2}\left[\int_0^1(1-t)^{\alpha+n-1}\left|f^{(n)}(ta+(1-t)b)\right|\,\mathrm{d}\,t+\int_0^1 t^{\alpha+n-1}\left|f^{(n)}(ta+(1-t)b)\right|\,\mathrm{d}\,t\right]$$
$$\le \frac{(b-a)^n}{2}\left\{\left[\int_0^1(1-t)^{[q(\alpha+n-1)-r]/(q-1)}\,\mathrm{d}\,t\right]^{1-1/q}\left[\int_0^1(1-t)^r\left|f^{(n)}(ta+(1-t)b)\right|^q\,\mathrm{d}\,t\right]^{1/q}\right.$$
$$\left.+\left[\int_0^1 t^{[q(\alpha+n-1)-r]/(q-1)}\,\mathrm{d}\,t\right]^{1-1/q}\left[\int_0^1 t^r\left|f^{(n)}(ta+(1-t)b)\right|^q\,\mathrm{d}\,t\right]^{1/q}\right\}$$
$$\le \frac{(b-a)^n}{2}\left[\frac{q-1}{q(\alpha+n)-r-1}\right]^{1-1/q}\left\{\left[\int_0^1\left((1-t)^r t^s\left|f^{(n)}(a)\right|^q+(1-t)^{r+s}\left|f^{(n)}(b)\right|^q\right)\mathrm{d}\,t\right]^{1/q}\right.$$
$$\left.+\left[\int_0^1\left(t^{r+s}\left|f^{(n)}(a)\right|^q+t^r(1-t)^s\left|f^{(n)}(b)\right|^q\right)\mathrm{d}\,t\right]^{1/q}\right\}$$
$$=\frac{(b-a)^n}{2}\left[\frac{q-1}{q(\alpha+n)-r-1}\right]^{1-1/q}\left\{\left[B(s+1,r+1)\left|f^{(n)}(a)\right|^q+\frac{1}{r+s+1}\left|f^{(n)}(b)\right|^q\right]^{1/q}\right.$$
$$\left.+\left[B(1,r+s+1)\left|f^{(n)}(a)\right|^q+\frac{1}{r+s+1}\left|f^{(n)}(b)\right|^q\right]^{1/q}\right\}.$$

Theorem 3.2 is proved. □

Corollary 3.2.1. *Under the assumptions of Theorem 3.2,*

1. *if* $s=1$, *then*

$$\left|\frac{\Gamma(\alpha+n)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b)+J_{b^-}^\alpha f(a)\right]-\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\left[f^{(k)}(a)+(-1)^k f^{(k)}(b)\right]\right|$$
$$\le \frac{(b-a)^n}{2\big((r+1)(r+2)\big)^{1/q}}\left[\frac{q-1}{q(\alpha+n)-r-1}\right]^{1-1/q}$$
$$\times\left\{\left[\left|f^{(n)}(a)\right|^q+(r+1)\left|f^{(n)}(b)\right|^q\right]^{1/q}+\left[(r+1)\left|f^{(n)}(a)\right|^q+\left|f^{(n)}(b)\right|^q\right]^{1/q}\right\};$$

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2. if $n = 1$, then

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right]\right| \le \frac{b-a}{2}\left[\frac{q-1}{q(\alpha+1)-r-1}\right]^{1-1/q} \times \left\{\left[B(s+1,r+1)\left|f'(a)\right|^q + \frac{1}{r+s+1}\left|f'(b)\right|^q\right]^{1/q} + \left[\frac{1}{r+s+1}\left|f'(a)\right|^q + B(s+1,r+1)\left|f'(b)\right|^q\right]^{1/q}\right\};$$

3. is $s = n = 1$, then

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right]\right| \le \frac{b-a}{2\left[(r+1)(r+2)\right]^{1/q}} \times \left[\frac{q-1}{q(\alpha+1)-r-1}\right]^{1-1/q}\left\{\left[\left|f'(a)\right|^q + (r+1)\left|f'(b)\right|^q\right]^{1/q} + \left[(r+1)\left|f'(a)\right|^q + \left|f'(b)\right|^q\right]^{1/q}\right\}.$$

Corollary 3.2.2. *Under the assumptions of Theorem 3.2,*

1. when $r = 0$, we have

$$\left|\frac{\Gamma(\alpha+n)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right] - \sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\left[f^{(k)}(a) + (-1)^k f^{(k)}(b)\right]\right| \le \frac{(b-a)^n}{(s+1)^{1/q}}\left[\frac{q-1}{q(\alpha+n)-1}\right]^{1-1/q}\left[\left|f^{(n)}(a)\right|^q + \left|f^{(n)}(b)\right|^q\right]^{1/q};$$

2. when $r = 0$ and $s = n = 1$, we have

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right]\right| \le (b-a)\left[\frac{q-1}{q(\alpha+1)-1}\right]^{1-1/q}\left[\frac{\left|f'(a)\right|^q + \left|f'(b)\right|^q}{2}\right]^{1/q};$$

3. when $r = q$, we have

$$\left|\frac{\Gamma(\alpha+n)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right] - \sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\left[f^{(k)}(a) + (-1)^k f^{(k)}(b)\right]\right| \le \frac{(b-a)^n}{2}\left[\frac{q-1}{q(\alpha+n-1)-1}\right]^{1-1/q}\left\{\left[B(s+1,q+1)\left|f^{(n)}(a)\right|^q + \frac{1}{q+s+1}\left|f^{(n)}(b)\right|^q\right]^{1/q} + \left[\frac{1}{q+s+1}\left|f^{(n)}(a)\right|^q + B(s+1,q+1)\left|f^{(n)}(b)\right|^q\right]^{1/q}\right\};$$

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4. when $r = q$ and $s = n = 1$, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b)+J_{b^-}^\alpha f(a)\right]\right|\le\frac{b-a}{2\left[(q+1)(q+2)\right]^{1/q}}\left(\frac{q-1}{q\alpha-1}\right)^{1-1/q}$$
$$\times\left\{\left[\left|f'(a)\right|^q+(q+1)\left|f'(b)\right|^q\right]^{1/q}+\left[(q+1)\left|f'(a)\right|^q+\left|f'(b)\right|^q\right]^{1/q}\right\};$$

5. when $r = q(\alpha+n-1)$, we have

$$\left|\frac{\Gamma(\alpha+n)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b)+J_{b^-}^\alpha f(a)\right]-\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\left[f^{(k)}(a)+(-1)^k f^{(k)}(b)\right]\right|$$
$$\le\frac{(b-a)^n}{2}\left\{\left[B(s+1,q(\alpha+n-1)+1)\left|f^{(n)}(a)\right|^q+\frac{1}{q(\alpha+n-1)+s+1}\left|f^{(n)}(b)\right|^q\right]^{1/q}\right.$$
$$\left.+\left[\frac{1}{q(\alpha+n-1)+s+1}\left|f^{(n)}(a)\right|^q+B(s+1,q(\alpha+n-1)+1)\left|f^{(n)}(b)\right|^q\right]^{1/q}\right\};$$

6. when $r = q(\alpha+n-1)$ and $s = n = 1$, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b)+J_{b^-}^\alpha f(a)\right]\right|\le\frac{b-a}{2\left[(q\alpha+1)(q\alpha+2)\right]^{1/q}}$$
$$\times\left\{\left[\left|f'(a)\right|^q+(q\alpha+1)\left|f'(b)\right|^q\right]^{1/q}+\left[(q\alpha+1)\left|f'(a)\right|^q+\left|f'(b)\right|^q\right]^{1/q}\right\}.$$

Theorem 3.3. *For $n\in\mathbb{N}$ and $a,b\in\mathbb{R}_0$ with $a<b$, let $f:\mathbb{R}_0\to\mathbb{R}$ be an n-times differentiable function on $\mathbb{R}_0$ such that $f^{(n)}\in L_1([a,b])$. If $|f^{(n)}|^q$ is s-concave on $[a,b]$ for $q>1$ and some fixed $s\in(0,1]$, then*

$$\left|\frac{\Gamma(\alpha+n)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b)+J_{b^-}^\alpha f(a)\right]-\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\left[f^{(k)}(a)+(-1)^k f^{(k)}(b)\right]\right|$$
$$\le\frac{(b-a)^n}{2^{(1-s)/q}}\left[\frac{q-1}{q(\alpha+n)-1}\right]^{1-1/q}\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|,\quad\alpha>0.$$

Proof. Using Lemma 2.1 and the well-known Hölder's inequality yields

$$\left|\frac{\Gamma(\alpha+n)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b)+J_{b^-}^\alpha f(a)\right]-\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\left[f^{(k)}(a)+(-1)^k f^{(k)}(b)\right]\right|$$
$$\le\frac{(b-a)^n}{2}\left[\int_0^1(1-t)^{\alpha+n-1}\left|f^{(n)}(ta+(1-t)b)\right|\,\mathrm{d}\,t+\int_0^1 t^{\alpha+n-1}\left|f^{(n)}(ta+(1-t)b)\right|\,\mathrm{d}\,t\right]$$
$$\le\frac{(b-a)^n}{2}\left\{\left[\int_0^1(1-t)^{q(\alpha+n-1)/(q-1)}\,\mathrm{d}\,t\right]^{1-1/q}\left[\int_0^1\left|f^{(n)}(ta+(1-t)b)\right|^q\,\mathrm{d}\,t\right]^{1/q}\right.$$

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$$+\left[\int_0^1 t^{q(\alpha+n-1)/(q-1)}\,\mathrm{d}\,t\right]^{1-1/q}\left[\int_0^1\left|f^{(n)}(ta+(1-t)b)\right|^q\,\mathrm{d}\,t\right]^{1/q}\Bigg\}$$
$$=(b-a)^n\left[\frac{q-1}{q(\alpha+n)-1}\right]^{1-1/q}\left[\int_0^1\left|f^{(n)}(ta+(1-t)b)\right|^q\,\mathrm{d}\,t\right]^{1/q}.$$

Since $|f^{(n)}|^q$ is s-concave, we have

$$\int_0^1\left|f^{(n)}(ta+(1-t)b)\right|^q\,\mathrm{d}\,t\le 2^{s-1}\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q.$$

Combining the above two inequalities yields (3.3). The proof of Theorem 3.3 is complete. ☐

Corollary 3.3.1. *Under the assumptions of Theorem 3.3,*

1. *if* $s=1$, *then*

$$\left|\frac{\Gamma(\alpha+n)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right]-\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\left[f^{(k)}(a)+(-1)^k f^{(k)}(b)\right]\right|$$
$$\le(b-a)^n\left[\frac{q-1}{q(\alpha+n)-1}\right]^{1-1/q}\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|;$$

2. *if* $n=1$, *then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right]\right|\le\frac{b-a}{2^{(1-s)/q}}\left[\frac{q-1}{q(\alpha+1)-1}\right]^{1-1/q}\left|f'\left(\frac{a+b}{2}\right)\right|;$$

3. *if* $s=n=1$, *then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right]\right|\le(b-a)\left[\frac{q-1}{q(\alpha+1)-1}\right]^{1-1/q}\left|f'\left(\frac{a+b}{2}\right)\right|.$$

Acknowledgements

This work was partially supported by the National Natural Science Foundation of China under Grant No. 11361038 and by the Inner Mongolia Autonomous Region Natural Science Foundation Project under Grant No. 2015MS0123, China.

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A Monotone Hybrid Projection Algorithm for Solving Fixed Point and Equilibrium Problems in a Banach Space

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Abstract. In this paper, an uncountable infinite family of nonlinear mappings are investigated. Strong convergence theorems of common solutions are established in a strictly convex and uniformly smooth Banach space which also has the Kadec-Klee property. The results obtained in this paper unify and improve many corresponding results announced recently.

Keywords: quasi-ϕ-nonexpansive mapping; equilibrium problem; fixed point; projection.

2010 AMS Subject Classification: 65J15, 90C33.

1 Introduction

Recently, common solution problems have been intensively investigated based on iterative methods. The so called common solution problems which capture lots of applications in multi-disciplines such as image restoration, and radiation therapy treatment planning are to find a special point in the intersection of a family of convex sets, which are usually considered as solution sets of nonlinear problems; see [1]-[15] and the references therein. Mean-valued iterative processes, in particular, Mann iterative process and Ishikawa iterative process, are efficient and powerful for studying fixed points of Lipschitz continuous nonlinear operators. However, in the framework of infinite-dimensional Hilbert spaces,

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they are only weakly convergent; see [16], [17] and the references therein. In many modern disciplines, including image recovery, economics, control theory, and quantum physics, problems arises in the framework of infinite dimension spaces. In such nonlinear problems, strong convergence is often much more desirable than the weak convergence; see [18] and the references therein. To guarantee the strong convergence of mean-valued iteration processes, many authors use different regularization methods. The projection method which was first introduced by Haugazeau [19] has been considered for the approximation of fixed points of nonexpansive mappings. The advantage of projection methods is that strong convergence of iterative sequences can be guaranteed without compact restrictions imposed on operators.

In this paper, we study a common solution problem via projection methods. Strong convergence theorems of common solutions are established with the aid of a generalized projection in a Banach space. The results obtained in this paper mainly unify and improve the corresponding results in [20]-[30].

2 Preliminaries

Let E be a real Banach space and let E^* be the dual space of E. Let B_E be the unit sphere of E. Recall that E is said to be a strictly convex space if for all $x, y \in B_E$ and $x \neq y$, $\|x + y\| < 2$. It is said to be uniformly convex if for any $\epsilon \in (0, 2]$ there exists $\delta > 0$ such that for any $x, y \in B_E$,

$$\|x - y\| \geq \epsilon \quad \text{implies} \quad \|x + y\| \leq 2 - 2\delta.$$

It is known that a uniformly convex Banach space is reflexive and strictly convex; see [31] and the references therein.

Recall that E is said to have a Gâteaux differentiable norm if for all $x, y \in B_E$. $\lim_{t\to 0}(\|\frac{x}{t} + y\| - \|\frac{x}{t}\|)$. In this case, we also say that E is a smooth space. E is said to have a uniformly Gâteaux differentiable norm if for each $y \in B_E$, the limit is attained uniformly for all $x \in B_E$. E is also said to have a uniformly Fréchet differentiable norm if the above limit is attained uniformly for $x, y \in B_E$. In this case, we say that E is uniformly smooth. It is known that a uniformly smooth Banach space is reflexive and smooth.

Recall that normalized duality mapping J from E to 2^{E^*} is defined by

$$Jx = \{y \in E^* : \|x\|^2 = \langle x, y\rangle = \|y\|^2\}.$$

It is known if E is uniformly smooth, then J is uniformly norm-to-norm continuous on every bounded subset of E; if E is a strictly convex Banach space, then

J is strictly monotone; if E is a smooth Banach space, then J is single-valued and demicontinuous, i.e.,continuous from the strong topology of E to the weak star topology of E; if E is a reflexive and strictly convex Banach space with a strictly convex dual E^* and $J^* : E^* \to E$ is the normalized duality mapping in E^*, then $J^{-1} = J^*$; if E is a smooth, strictly convex and reflexive Banach space, then J is single-valued, one-to-one and onto.

Recall that E has the Kadec-Klee Property (KKP) if $\lim_{m\to\infty} \|x_m - x\| = 0$, for any sequence $\{x_m\} \subset E$, and $x \in E$ with $\{x_n\}$ converges weakly to x, and $\{\|x_n\|\}$ converges strongly to $\|x\|$. It is known that every uniformly convex Banach space has the KKP; see [31] and the references therein.

Let C be a nonempty closed and convex subset of E and let $B : C \times C \to \mathbb{R}$ be a function. Recall that the following equilibrium problem in the terminology of Blum and Oettli [32]. Find $\bar{x} \in C$ such that $B(\bar{x}\, y) \geq 0$, $\forall y \in C$. We use $Sol(B)$ to denote the solution set of the equilibrium problem. That is, $Sol(B) = \{x \in C : B(x, y) \geq 0, \forall y \in C\}$.

The following restrictions are essential for solving the equilibrium problem in this paper.

(R-1) $B(a, a) \equiv 0, \forall a \in C$;

(R-2) $B(b, a) + B(a, b) \leq 0, \forall a, b \in C$;

(R-3) $B(a, b) \geq \limsup_{t\downarrow 0} B(tc + (1 - t)a, b)$, $\forall a, b, c \in C$;

(R-4) $b \mapsto B(a, b)$ is convex and weakly lower semi-continuous, $\forall a \in C$.

Let T be a self mapping on C. T is said to be closed if for any sequence $\{x_n\} \subset C$ such that $\lim_{n\to\infty} x_n = \bar{x}$ and $\lim_{m\to\infty} Tx_n = \bar{y}$, then $\bar{y} = T\bar{x}$. Let B be a bounded subset of C. Recall that T is said to be uniformly asymptotically regular on C if and only if $\limsup_{n\to\infty} \sup_{x\in B}\{\|T^n x - T^{n+1}x\|\} = 0$. From now on, we use $\to$ and $\rightharpoonup$ to stand for the strong convergence and weak convergence, respectively. and use $Fix(T)$ to denote the fixed point set of mapping T.

Recall that a point p is said to be an asymptotic fixed point of mapping T if and only if subset C contains a sequence $\{x_m\}$ which converges weakly to p such that $\lim_{m\to\infty} \|Tx_m - x_m\| = 0$. We use $\widetilde{Fix}(T)$ to stand for the asymptotic fixed point set in this paper.

Next, we assume that E is a smooth Banach space which means duality mapping J is single-valued. Study the functional

$$\phi(x, y) := \|x\|^2 + \|y\|^2 - 2\langle x, Jy\rangle, \quad \forall x, y \in E.$$

In [33], Alber studied a generalized projection $Proj_C : E \to C$, which is a mapping assigning to an arbitrary point $x \in E$ the minimum point of $\phi(x, y)$, which implies from the definition of ϕ $\phi(x, y)+2\|x\|\|y\| \geq \|x\|^2+\|y\|^2, \forall x, y \in E$.

T is said to be relatively nonexpansive iff

$$\phi(p, x) \geq \phi(p, Tx), \quad \forall x \in C, \forall p \in \widetilde{Fix}(T) = Fix(T) \neq \emptyset.$$

T is said to be relatively asymptotically nonexpansive iff

$$\phi(p, x) + \xi_n \phi(p, x) \geq \phi(p, T^n x), \quad \forall x \in C, \forall p \in Fix(T) = \widetilde{Fix}(T) \neq \emptyset, \forall n \geq 1,$$

where $\{\xi_n\} \subset [0, \infty)$ is a sequence such that $\mu_n \to 0$ as $n \to \infty$.

Remark 2.1. The class of relatively asymptotically nonexpansive mappings, which was first considered in [34], covers the class of relatively nonexpansive mappings [35].

T is said to be quasi-ϕ-nonexpansive iff

$$\phi(p, x) \geq \phi(p, Tx), \quad \forall x \in C, \forall p \in Fix(T) \neq \emptyset.$$

T is said to be asymptotically quasi-ϕ-nonexpansive if and only if there exists a sequence $\{\xi_n\} \subset [0, \infty)$ with $\mu_n \to 0$ as $n \to \infty$ such that

$$\phi(p, x) + \xi_n \phi(p, x) \geq \phi(p, T^n x), \quad \forall x \in C, \forall p \in Fix(T) \neq \emptyset, \forall n \geq 1.$$

Remark 2.2. The class of quasi-ϕ-nonexpansive mappings [26] and the class of asymptotically quasi-ϕ-nonexpansive mappings [27] cover the class of relatively nonexpansive mappings and the class of relatively asymptotically nonexpansive mappings. Quasi-ϕ-nonexpansive mappings and asymptotically quasi-ϕ-nonexpansive mappings do not require the strong restriction that the fixed point set equals the asymptotic fixed point set.

Remark 2.3. The class of quasi-ϕ-nonexpansive mappings and the class of asymptotically quasi-ϕ-nonexpansive mappings are generalizations of the class of quasi-nonexpansive mappings and the class of asymptotically quasi-nonexpansive mappings in Banach spaces because of $\sqrt{\phi(x, y)} = \|x - y\|$.

The following lemmas also play an important role in this paper.

Lemma 2.4. [33] *Let E be a strictly convex, reflexive, and smooth Banach space and let C be a nonempty, closed, and convex subset of E. Let $x \in E$. Then*

$$\phi(y, \Pi_C x) \leq \phi(y, x) - \phi(\Pi_C x, x), \quad \forall y \in C,$$

$\langle y - x_0, Jx - Jx_0 \rangle \leq 0$, $\forall y \in C$ *if and only if* $x_0 = \Pi_C x$.

Lemma 2.5. ([26], [32]) *Let* E *be a strictly convex, smooth, and reflexive Banach space and let* C *be a closed convex subset of* E. *Let* B *be a function with the restrictions (R-1), (R-2), (R-3) and (R-4), from* $C \times C$ *to* $\mathbb{R}$. *Let* $x \in E$ *and let* $r > 0$. *Then there exists* $z \in C$ *such that* $rB(z, y) + \langle z - y, Jz - Jx \rangle \leq 0$, $\forall y \in C$ *Define a mapping* $K^{B,r}$ *by*

$$K^{B,r}x = \{z \in C : rB(z, y) + \langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C\}.$$

The following conclusions hold:

(1) $K^{B,r}$ *is single-valued quasi-*ϕ*-nonexpansive;*

(2) $Sol(B) = Fix(K^{B,r})$ *is closed and convex.*

Lemma 2.6 [36] *Let* E *be a strictly convex and uniformly smooth Banach space which also has the KKP. Let* C *be a convex and closed subset of* E *and let* T *be an asymptotically quasi-*ϕ*-nonexpansive mapping on* C. $Fix(T)$ *is convex.*

Lemma 2.7 [37] *Let* r *be a positive real number and let* E *be uniformly convex. Then there exists a convex, strictly increasing and continuous function* cof : $[0, 2r] \to \mathbb{R}$ *such that* $cof(0) = 0$ *and*

$$t\|a\|^2 + (1 - t)\|b\|^2 \geq \|(1 - t)b + ta\|^2 + t(1 - t)cof(\|b - a\|)$$

for all $t \in [0, 1]$ *and* $a, b \in B^r := \{a \in E : \|a\| \leq r\}$.

3 Main results

Theorem 3.1. *Let* E *be a strictly convex and uniformly smooth Banach space which also has the KKP. Let* C *be a convex and closed subset of* E *and let* Λ *be an arbitrary index set. Let* B_i *be a bifunction with (R-1), (R-2), (R-3) and (R-4). Let* T_i *be an asymptotically quasi-*ϕ*-nonexpansive mapping on* C *for every* $i \in \Lambda$. *Assume that* T_i *is uniformly asymptotically regular and closed for every* $i \in \Lambda$ *and* $\cap_{i\in\Lambda} Sol(B_i) \bigcap \cap_{i\in\Lambda} Fix(T_i)$ *is nonempty and bounded. Let* $\{x_j\}$ *be a* *sequence generated by*

$$\begin{cases} x_0 \in E \textit{ chosen arbitrarily}, \\ C_{(1,i)} = C, \forall i \in \Lambda, \\ C_1 = \cap_{i\in\Lambda} C_{(1,i)}, x_1 = Proj_{C_1} x_0, \\ Jy_{(j,i)} = \alpha_{(j,i)} JT_i^j x_j + (1-\alpha_{(j,i)}) Ju_{(j,i)}, \\ C_{(j+1,i)} = \{z \in C_{(j,i)} : \phi(z, y_{(j,i)}) - \phi(z, x_j) \le \alpha_{(j,i)} \xi_{(j,i)} D_{(j,i)}\}, \\ C_{j+1} = \cap_{i\in\Lambda} C_{(j+1,i)}, x_{j+1} = Proj_{C_{j+1}} x_1, \end{cases}$$

where $u_{(j,i)}$ *is such that* $r_{(j,i)} B_i(u_{(j,i)}, \mu) \ge \langle u_{(j,i)} - \mu, Ju_{(j,i)} - Jx_j \rangle$, $\forall \mu \in C_j$, $D_{(j,i)} = \sup\{\phi(z, x_j) : z \in \cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(B_i)\}$, $\{\alpha_{(j,i)}\}$ *is a real sequence in* $(0,1)$ *such that* $\liminf_{j\to\infty} \alpha_{(j,i)}(1-\alpha_{(j,i)}) > 0$ *and* $\{r_{(j,i)}\} \subset [r, \infty)$ *is a real sequence, where* r *is some positive real number. Then* $\{x_j\}$ *converges strongly to* $Proj_{\cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(B_i)} x_1$.

Proof. First, we prove $\cap_{i\in\Lambda} Sol(B_i) \bigcap \cap_{i\in\Lambda} Fix(T_i)$ is convex and closed. Using Lemma 2.5 and 2.6, we find that $Sol(B_i)$ is convex and closed and $Fix(T_i)$ is convex for every $i \in \Lambda$. Since T_i is closed, we find that $Fix(T_i)$ is also closed. So, $Proj_{\cap_{i\in\Lambda} Sol(B_i) \bigcap \cap_{i\in\Lambda} Fix(T_i)} x$ is well defined, for any element x in E.

Next, we prove that C_j is convex and closed. It is obvious that $C_{(1,i)} = C$ is convex and closed. Assume that $C_{(m,i)}$ is convex and closed for some $m \ge 1$. Let $p_1, p_2 \in C_{(m+1,i)}$. It follows that $p = sp_1 + (1-s)p_2 \in C_{(m,i)}$, where $s \in (0,1)$. Notice that $\phi(p_1, y_{(m,i)}) - \phi(p_1, x_m) \le \alpha_{(m,i)} \xi_{(m,i)} D_{(m,i)}$, and $\phi(p_2, y_{(m,i)}) - \phi(p_2, x_m) \le \alpha_{(m,i)} \xi_{(m,i)} D_{(m,i)}$. Hence, one has

$$2\langle p_1, Jx_m - Jy_{(m,i)} \rangle - \|x_m\|^2 + \|y_{(m,i)}\|^2 \le \alpha_{(m,i)} \xi_{(m,i)} D_{(m,i)},$$

and

$$2\langle p_2, Jx_m - Jy_{(m,i)} \rangle - \|x_m\|^2 + \|y_{(m,i)}\|^2 \le \alpha_{(m,i)} \xi_{(m,i)} D_{(m,i)}.$$

Using the above two inequalities, one has $\phi(p, y_{(m,i)}) - \phi(p, x_m) \le \alpha_{(m,i)} \xi_{(m,i)} D_{(m,i)}$. This shows that $C_{(m+1,i)}$ is closed and convex. Hence, $C_j = \cap_{i\in\Lambda} C_{(j,i)}$ is a convex and closed set. This proves that $Proj_{C_{j+1}} x_1$ is well defined.

On the other hand, we find that $\cap_{i\in\Lambda} Sol(B_i) \bigcap \cap_{i\in\Lambda} Fix(T_i) \subset C_1 = C$ is clear. Suppose that $\cap_{i\in\Lambda} Sol(B_i) \bigcap \cap_{i\in\Lambda} Fix(T_i) \subset C_{(m,i)}$ for some positive integer m. For any $w \in \cap_{i\in\Lambda} Sol(B_i) \bigcap \cap_{i\in\Lambda} Fix(T_i) \subset C_{(m,i)}$, we see that

$$\begin{aligned}
\phi(z, y_{(m,i)}) &= \|z\|^2 + \|\alpha_{(m,i)} JT_i^m x_m + (1-\alpha_{(m,i)})Ju_{(m,i)}\|^2 \\
&\quad - 2\langle z, \alpha_{(m,i)} JT_i^m x_m + (1-\alpha_{(m,i)})Ju_{(m,i)}\rangle \\
&\le \|z\|^2 + \alpha_{(m,i)}\|T_i^m x_m\|^2 + (1-\alpha_{(m,i)})\|u_{(m,i)}\|^2 \\
&\quad - 2\alpha_{(m,i)}\langle z, JT_i^m x_m\rangle - 2(1-\alpha_{(m,i)})\langle z, Ju_{(m,i)}\rangle \\
&\le \phi(z, x_m) + \alpha_{(m,i)}\xi_{(m,i)} D_{(m,i)},
\end{aligned}$$

where $D_{(m,i)} = \sup\{\phi(z, x_m) : z \in \cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(B_i)\}$. This shows that $z \in C_{(m+1,i)}$. This implies that $\cap_{i\in\Lambda} Sol(B_i) \bigcap \cap_{i\in\Lambda} Fix(T_i) \subset \cap_{i\in\Lambda} C_{(j,i)} = C_j$. Using Lemma 2.4, one has $\langle z - x_j, Jx_1 - Jx_j\rangle \le 0$, for any $z \in C_j$. It follows that

$$\langle z - x_j, Jx_1 - Jx_j\rangle \le 0, \quad \forall z \in \cap_{i\in\Lambda} Sol(B_i) \bigcap \cap_{i\in\Lambda} Fix(T_i) \subset C_j. \tag{3.1}$$

Using Lemma 2.4 yields that

$$\begin{aligned}
\phi(x_j, x_1) &\le \phi(Proj_{\cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(B_i)} x_1, x_1) \\
&\quad - \phi(Proj_{\cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(B_i)} x_1, x_j),
\end{aligned}$$

which shows that $\{\phi(x_j, x_1)\}$ is bounded. Hence, $\{x_j\}$ is also bounded. Without loss of generality, we assume $x_j \rightharpoonup \bar{x} \in C_j$. Hence $\phi(x_j, x_1) \le \phi(\bar{x}, x_1)$. This implies that

$$\phi(\bar{x}, x_1) \le \liminf_{j\to\infty}(\|x_j\|^2 + \|x_1\|^2 - 2\langle x_j, Jx_1\rangle) = \limsup_{j\to\infty} \phi(x_j, x_1) \le \phi(\bar{x}, x_1).$$

It follows that $\lim_{j\to\infty} \phi(x_j, x_1) = \phi(\bar{x}, x_1)$. Hence, we have $\lim_{j\to\infty} \|x_j\| = \|\bar{x}\|$. Using the KKP, one obtains that $\{x_j\}$ converges strongly to $\bar{x}$ as $j \to \infty$. On the other hand, we find that $\phi(x_{j+1}, x_1) \ge \phi(x_j, x_1)$, which shows that $\{\phi(x_j, x_1)\}$ is nondecreasing. It follows that $\lim_{j\to\infty} \phi(x_j, x_1)$ exists. Since $\phi(x_{j+1}, x_1) - \phi(x_j, x_1) \ge \phi(x_{j+1}, x_j)$, one has $\lim_{j\to\infty} \phi(x_{j+1}, x_j) = 0$. Since $x_{j+1} \in C_{j+1}$, one sees that $\phi(x_{j+1}, y_{(j,i)}) - \phi(x_{j+1}, x_j) \le \alpha_{(j,i)}\xi_{(j,i)} D_{(j,i)}$. It follows that $\lim_{j\to\infty} \phi(x_{j+1}, y_{(j,i)}) = 0$. Hence, one has $\lim_{j\to\infty}(\|y_{(j,i)}\| - \|x_{j+1}\|) = 0$. This implies that $\lim_{j\to\infty} \|Jy_{(j,i)}\| = \lim_{j\to\infty} \|y_{(j,i)}\| = \|\bar{x}\| = \|J\bar{x}\|$. This implies that $\{Jy_{(j,i)}\}$ is bounded. Without loss of generality, we assume that $\{Jy_{(j,i)}\}$ converges weakly to $y^{(*,i)} \in E^*$. In view of the reflexivity of E, we see that $J(E) = E^*$. This shows that there exists an element $y^i \in E$ such that $Jy^i = y^{(*,i)}$. It follows that $\phi(x_{j+1}, y_{(j,i)}) + 2\langle x_{j+1}, Jy_{(j,i)}\rangle = \|x_{j+1}\|^2 + \|Jy_{(j,i)}\|^2$. Taking $\liminf_{j\to\infty}$, one has $0 \ge \|\bar{x}\|^2 - 2\langle \bar{x}, y^{(*,i)}\rangle + \|y^{(*,i)}\|^2 = \|\bar{x}\|^2 + \|Jy^i\|^2 - 2\langle \bar{x}, Jy^i\rangle = \phi(\bar{x}, y^i) \ge 0$. That is, $\bar{x} = y^i$, which in turn implies that $J\bar{x} = y^{(*,i)}$.

Hence, $Jy_{(j,i)} \rightharpoonup J\bar{x} \in E^*$. Since E^* is uniformly convex. Hence, it has the KKP, we obtain $\lim_{i\to\infty} Jy_{(j,i)} = J\bar{x}$. Since $J^{-1}: E^* \to E$ is demi-continuous and E has the KKP, one gets that $y_{(j,i)} \to \bar{x}$, as $j \to \infty$. Using the fact

$$\phi(z, x_j) - \phi(z, y_{(j,i)}) \leq (\|x_j\| + \|y_{(j,i)}\|)\|y_{(j,i)} - x_j\| + 2\langle z, Jy_{(j,i)} - Jx_j\rangle,$$

we find

$$\lim_{j\to\infty} \big(\phi(z, x_j) - \phi(z, y_{(j,i)})\big) = 0. \tag{3.2}$$

On the other hand, one sees from Lemma 2.7

$$\begin{aligned}
\phi(z, y_{(j,i)}) &= \|z\|^2 + \|\alpha_{(j,i)} JT_i^j x_j + (1-\alpha_{(j,i)}) Ju_{(j,i)}\|^2 \\
&\quad - 2\langle z, \alpha_{(j,i)} JT_i^j x_j + (1-\alpha_{(j,i)}) Ju_{(j,i)}\rangle \\
&\leq \|z\|^2 + \alpha_{(j,i)}\|T_i^j x_j\|^2 + (1-\alpha_{(j,i)})\|u_{(j,i)}\|^2 \\
&\quad - \alpha_{(j,i)}(1-\alpha_{(j,i)}) cof(\||Ju_{(j,i)} - JT_i^j x_j\|) \\
&\quad - 2\alpha_{(j,i)}\langle z, JT_i^j x_j\rangle - 2(1-\alpha_{(j,i)})\langle z, Ju_{(j,i)}\rangle \\
&\leq \phi(z, x_j) + \alpha_{(j,i)}\xi_{(j,i)} D_{(j,i)} - \alpha_{(j,i)}(1-\alpha_{(j,i)}) cof(\||Ju_{(j,i)} - JT_i^j x_j\|).
\end{aligned}$$

This implies

$$\begin{aligned}
&\alpha_{(j,i)}(1-\alpha_{(j,i)}) cof(\||Ju_{(j,i)} - JT_i^j x_j\|) \\
&\leq \phi(z, x_j) - \phi(z, y_{(j,i)}) + \alpha_{(j,i)}\xi_{(j,i)} D_{(j,i)}.
\end{aligned}$$

Using the restriction imposed on the sequence $\{\alpha_{(j,i)}\}$ and (3.2), one has

$$\lim_{j\to\infty} \||Ju_{(j,i)} - JT_i^j x_j\| = 0.$$

It follows that $JT_i^j x_j \to J\bar{x}$ as $j \to \infty$. Since $J^{-1}: E^* \to E$ is demi-continuous, one has $T_i^j x_j \rightharpoonup \bar{x}$. Using the fact $\||T_i^j x_j\| - \|\bar{x}\|| = \||JT_i^j x_j\| - \|J\bar{x}\|| \leq \|JT_i^j x_j - J\bar{x}\|$, one has $\|T_i^j x_j\| \to \|\bar{x}\|$ as $j \to \infty$. Since E has the KKP, one has $\lim_{j\to\infty} \||\bar{x} - T_i^j x_j\| = 0$. Since T_i is also uniformly asymptotically regular, one has $\lim_{j\to\infty} \|\bar{x} - T_i^{j+1} x_j\| = 0$. That is, $T_i(T_i^j x_j) \to \bar{x}$. Using the closedness of T_i, we find $T_i\bar{x} = \bar{x}$. This proves $\bar{x} \in Fix(T_i)$, that is, $\bar{x} \in \cap_{i\in\Lambda} Fix(T_i)$.

Next, we show that $\bar{x} \in \cap_{i\in\Lambda} Sol(B_i)$. Since B_i is monotone, we find that

$$r_{(j,i)} B_i(\mu, u_{(j,i)}) \leq \|\mu - u_{(j,i)}\|\|Ju_{(j,i)} - Jx_j\|.$$

Therefore, one sees $B_i(\mu, \bar{x}) \leq 0$. For $0 < t_i < 1$, define $\mu_{(t,i)} = (1-t_i)\bar{x} + t_i\mu$. This implies that $0 \geq B_i(\mu_{(t,i)}, \bar{x})$. Hence, we have $0 = B_i(\mu_{(t,i)}, \mu_{(t,i)}) \leq t_i B_i(\mu_{(t,i)}, \mu)$. It follows that $B_i(\bar{x}, \mu) \geq 0$, $\forall \mu \in C$. This implies that $\bar{x} \in Sol(B_i)$ for every $i \in \Lambda$.

Finally, we prove $\bar{x} = Proj_{\cap_{i\in\Lambda}(Fix(T_i)\cap Sol(B_i))}x_1$. Using (3.1), one has $\langle \bar{x} - z, Jx_1 - J\bar{x}\rangle \geq 0$ $z \in \cap_{i\in\Lambda}(Fix(T_i) \cap Sol(B_i))$. Using Lemma 2.4, we find that $\bar{x} = Proj_{\cap_{i\in\Lambda}(Fix(T_i)\cap Sol(B_i))}x_1$. This completes the proof.

For the class of quasi-ϕ-nonexpansive mappings, the boundedness of the common solution set is not required. Indeed, we have the following result.

Corollary 3.2. *Let E be a strictly convex and uniformly smooth Banach space which also has the KKP. Let C be a convex and closed subset of E and let Λ be an arbitrary index set. Let B_i be a bifunction with (R-1), (R-2), (R-3) and (R-4). Let T_i be a quasi-ϕ-nonexpansive mapping on C for every $i \in \Lambda$. Assume that T_i is closed for every $i \in \Lambda$ and $\cap_{i\in\Lambda}Sol(B_i) \bigcap \cap_{i\in\Lambda}Fix(T_i)$ is nonempty. Let $\{x_j\}$ be a sequence generated by*

$$\begin{cases} x_0 \in E \textit{ chosen arbitrarily}, \\ C_{(1,i)} = C, \forall i \in \Lambda, \\ C_1 = \cap_{i\in\Lambda}C_{(1,i)}, x_1 = Proj_{C_1}x_0, \\ Jy_{(j,i)} = \alpha_{(j,i)}JT_ix_j + (1-\alpha_{(j,i)})Ju_{(j,i)}, \\ C_{(j+1,i)} = \{z \in C_{(j,i)} : \phi(z, y_{(j,i)}) \leq \phi(z, x_j)\}, \\ C_{j+1} = \cap_{i\in\Lambda}C_{(j+1,i)}, x_{j+1} = Proj_{C_{j+1}}x_1, \end{cases}$$

where $u_{(j,i)}$ is such that $r_{(j,i)}B_i(u_{(j,i)}, \mu) \geq \langle u_{(j,i)} - \mu, Ju_{(j,i)} - Jx_j\rangle$, $\forall \mu \in C_j$, $D_{(j,i)} = \sup\{\phi(z, x_j) : z \in \cap_{i\in\Lambda}Fix(T_i) \bigcap \cap_{i\in\Lambda}Sol(B_i)\}$, $\{\alpha_{(j,i)}\}$ is a real sequence in $(0,1)$ such that $\liminf_{j\to\infty} \alpha_{(j,i)}(1-\alpha_{(j,i)}) > 0$ and $\{r_{(j,i)}\} \subset [r, \infty)$ is a real sequence, where r is some positive real number. Then $\{x_j\}$ converges strongly to $Proj_{\cap_{i\in\Lambda}Fix(T_i)\bigcap\cap_{i\in\Lambda}Sol(B_i)}x_1$.

From Theorem 3.1, we also have the following result.

Corollary 3.3. *Let E be a strictly convex and uniformly smooth Banach space which also has the KKP. Let C be a convex and closed subset of E and let B be a bifunction with (R-1), (R-2), (R-3) and (R-4). Let T be an asymptotically quasi-ϕ-nonexpansive mapping on C. Assume that T is uniformly asymptotically regular and closed and $Sol(B) \cap Fix(T)$ is nonempty and bounded. Let $\{x_j\}$ be* *a sequence generated by*

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ C_1 = C, x_1 = Proj_{C_1} x_0, \\ Jy_j = \alpha_j JT^j x_j + (1-\alpha_j) Ju_j, \\ C_{j+1} = \{z \in C_j : \phi(z, y_j) - \phi(z, x_j) \leq \alpha_j \xi_j D_j\}, \\ x_{j+1} = Proj_{C_{j+1}} x_1, \end{cases}$$

where u_j *is such that* $r_j B(u_j, \mu) \geq \langle u_j - \mu, Ju_j - Jx_j \rangle$, $\forall \mu \in C_j$, $D_j = \sup\{\phi(z, x_j) : z \in Fix(T) \cap Sol(B)\}$, $\{\alpha_j\}$ *is a real sequence in* $(0,1)$ *such that* $\liminf_{j\to\infty} \alpha_j(1-\alpha_j) > 0$ *and* $\{r_j\} \subset [r, \infty)$ *is a real sequence, where* r *is some positive real number. Then* $\{x_j\}$ *converges strongly to* $Proj_{Fix(T)\cap Sol(B)} x_1$.

4 Applications

In this section, we consider common solutions of a family of variational inequalities in the framework Banach spaces. we give some deduced results of our main results in the framework of Hilbert spaces.

Let $A : C \to E^*$ be a single valued monotone operator which is continuous along each line segment in C with respect to the weak* topology of E^* (hemicontinuous). Recall the the following variational inequality. Finding a point $x \in C$ such that $\langle x - y, Ax \rangle \leq 0$, $\forall y \in C$. The symbol $Nc(x)$ stand for the normal cone for C at a point $x \in C$; that is, $Nc(x) = \{x^* \in E^* : \langle x - y, x^* \rangle \geq 0, \ \forall y \in C\}$. From now on, we use $VI(C, A)$ to denote the solution set of the variational inequality.

Theorem 4.1. *Let* E *be a strictly convex and uniformly smooth Banach space which also has the KKP. Let* C *be a convex and closed subset of* E*. Let* Λ *be an index set and let* $A_i : C \to E^*$ *be a single valued, monotone and hemicontinuous operator. Let* B_i *be a bifunction with (R-1), (R-2), (R-3) and (R-4). Assume that* $\cap_{i\in\Lambda} VI(C, A_i)$ *is not empty. Let* $\{x_n\}$ *be a sequence generated in the* *following process.*

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ C_{(1,i)} = C, \forall i \in \Lambda, \\ C_1 = \cap_{i\in\Lambda} C_{(1,i)}, x_1 = Proj_{C_1} x_0, \\ u_{(n,i)} = VI(C, A_i + \frac{1}{r_i}(J - Jx_n)), \\ Jy_{(j,i)} = \alpha_{(j,i)} Jx_j + (1-\alpha_{(j,i)}) Ju_{(j,i)}, \\ C_{(j+1,i)} = \{z \in C_{(j,i)} : \phi(z, y_{(j,i)}) \le \phi(z, x_j)\}, \\ C_{j+1} = \cap_{i\in\Lambda} C_{(j+1,i)}, x_{j+1} = Proj_{C_{j+1}} x_1, \end{cases}$$

where $\{\alpha_{(j,i)}\}$ *is a real sequence in* $(0,1)$ *such that* $\liminf_{j\to\infty} \alpha_{(j,i)}(1-\alpha_{(j,i)}) > 0$. *Then* $\{x_j\}$ *converges strongly to* $Proj_{\cap_{i\in\Lambda} VI(C,A_i)} x_1$.

Proof. Define a new operator M_i by $M_i x = A_i x + Nc(x)$, $x \in C$, $M_i x = \emptyset$, $x \notin C$. Hence, M_i is maximal monotone and $M_i^{-1}(0) = VI(C, A_i)$, where $M_i^{-1}(0)$ stand for the zero point set of M_i. For each $r_i > 0$, and $x \in E$, we see that there exists a unique x_{r_i} in the domain of M_i such that $Jx \in Jx_{r_i} + r_i M_i(x_{r_i})$, where $x_{r_i} = (J + r_i M_i)^{-1} Jx$. Notice that $u_{j,i} = VI(C, \frac{1}{r_i}(J - Jx_j) + A_i)$, which is equivalent to $\langle u_{j,i} - y, A_i z_{j,i} + \frac{1}{r_i}(Jz_{j,i} - Jx_j)\rangle \le 0$, $\forall y \in C$, that is, $\frac{1}{r_i}\big(Jx_j - Ju_{j,i}\big) \in Nc(u_{j,i}) + A_i z_{j,i}$. This implies that $u_{j,i} = (J + r_i M_i)^{-1} Jx_j$. From [26], we find that $(J + r_i M_i)^{-1} J$ is closed quasi-ϕ-nonexpansive with $Fix((J + r_i M_i)^{-1} J) = M_i^{-1}(0)$. Using Theorem 3.1, we find the desired conclusion immediately.

Theorem 4.2. *Let E be a Hilbert. Let C be a convex and closed subset of E and let Λ be an arbitrary index set. Let B_i be a function with (R-1), (R-2), (R-3) and (R-4). Let T_i be an asymptotically quasi-nonexpansive mapping on C for every $i \in \Lambda$. Assume that T_i is uniformly asymptotically regular and closed for every $i \in \Lambda$ and $\cap_{i\in\Lambda} Sol(B_i) \bigcap \cap_{i\in\Lambda} Fix(T_i)$ is nonempty and bounded. Let $\{x_j\}$ be a sequence generated by*

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ C_{(1,i)} = C, \forall i \in \Lambda, \\ C_1 = \cap_{i\in\Lambda} C_{(1,i)}, x_1 = P_{C_1} x_0, \\ y_{(j,i)} = \alpha_{(j,i)} T_i^j x_j + (1-\alpha_{(j,i)}) u_{(j,i)}, \\ C_{(j+1,i)} = \{z \in C_{(j,i)} : \|z - y_{(j,i)}\|^2 - \|z - x_j\|^2 \le \alpha_{(j,i)} \xi_{(j,i)} D_{(j,i)}\}, \\ C_{j+1} = \cap_{i\in\Lambda} C_{(j+1,i)}, x_{j+1} = P_{C_{j+1}} x_1, \end{cases}$$

where $u_{(j,i)}$ *is such that* $r_{(j,i)} B_i(u_{(j,i)}, \mu) \ge \langle u_{(j,i)} - \mu, u_{(j,i)} - x_j\rangle$, $\forall \mu \in C_j$, $D_{(j,i)} = \sup\{\|z - x_j\|^2 : z \in \cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(B_i)\}$, $\{\alpha_{(j,i)}\}$ *is a real* *sequence in* $(0,1)$ *such that* $\liminf_{j\to\infty} \alpha_{(j,i)}(1-\alpha_{(j,i)}) > 0$ *and* $\{r_{(j,i)}\} \subset [r,\infty)$ *is a real sequence, where* r *is some positive real number. Then* $\{x_j\}$ *converges strongly to* $P_{\cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(B_i)} x_1$.

Proof. In the framework of Hilbert spaces, we see that $\sqrt{\phi(x,y)} = \|x-y\|$, $\forall x,y \in E$. The generalized projection is reduced to the metric projection and the asymptotically-ϕ-nonexpansive mapping is reduced to the asymptotically quasi-nonexpansive mapping. Using Theorem 3.1, we find the desired conclusion immediately.

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Inner-outer factorization on Besov-type spaces

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Abstract. In this paper, motivated by some results of Dyakonov, we give an inner-outer factorization on Besov-type spaces.

MSC 2000: 30H25, 30J05.

Keywords: Inner function, outer function, $BMOA$ space, Besov-type s-paces.

1 Introduction

We denote the unit disc $\{z \in \mathbb{C} : |z| < 1\}$ by $\mathbb{D}$ and its boundary by $\partial\mathbb{D}$. Let $H(\mathbb{D})$ be the space of all analytic functions in $\mathbb{D}$. For $0 < p < \infty$, the Hardy space H^p is the set of $f \in H(\mathbb{D})$ for which

$$\|f\|_{H^p}^p = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

As usual, H^∞ is the set of $f \in H(\mathbb{D})$ with $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)| < \infty$ (see [5]).

For $0 < p, q < \infty$ and $0 < s < 1$, the Besov-type space, denoted by B^s_{pq}, is the set of functions $f \in L^p(\partial\mathbb{D})$ such that

$$\int_0^\infty \frac{\omega_p(t,f)^q dt}{|t|^{sq+1}} < \infty,$$

where

$$\omega_p(t,f)^p := \sup_{-t\le h\le t} \int_{\partial\mathbb{D}} |f(e^{ih}\zeta) - f(\zeta)|^p dm(\zeta), \quad 0 \le t \le \pi$$

and

$$\omega_p(t,f) := \omega_p(\pi, f) \quad \text{when} \quad \pi < t < \infty.$$

Here dm is the normalized Lebesgue measure on $\partial\mathbb{D}$.

The analytic Besov space, denoted by $AB^s_{pq} = B^s_{pq} \cap H^p$, is the space of functions $f \in H^p$ such that

$$\int_0^1 (1-r)^{(1-s)q-1} \left(\int_{\partial\mathbb{D}} |f'(r\zeta)|^p dm(\zeta) \right)^{\frac{q}{p}} dr < \infty.$$

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We refer the reader to [2], [3], [4] and [10]. For the simplicity of notation, we denote B^s_{pp} and AB^s_{pp} by B^s_p and AB^s_p, respectively.

Let $0 < p, s < \infty$, $-2 < q < \infty$. An $f \in H(\mathbb{D})$ is said to belong to $F(p,q,s)$ if (see [24])

$$\|f\|^p_{p,q,s} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q g^s(z,a) dA(z) < \infty,$$

where $g(z,a) = \log \frac{1}{|\varphi_a(z)|}$, $z, a \in \mathbb{D}, z \neq a$, $\varphi_a(z) = \frac{a-z}{1-\overline{a}z}$, $dA(z) = \frac{1}{\pi} dxdy$. $F(p,q,s)$ is called general function space because it can get many function spaces if it takes special parameters of p, q, s. For example, when $s > 1$, $F(p,q,s) = \mathcal{B}^{\frac{q+2}{p}}$, which is called the Bloch-type space; $F(2,0,s) = Q_s$ (see [23]); $F(2,0,1) = BMOA$, the space of analytic functions in the Hardy space $H^1(\mathbb{D})$ whose boundary functions have bounded mean oscillation (see [13, 14, 19]). It is easy to see that $F(p,p-2,s)$ is a Möbius invariant Besov-type space. In fact, from [17], we know that $f \in F(p,p-2,s)$ if and only if

$$\sup_{a\in\mathbb{D}} \|f\circ\varphi_a - f(a)\|_{AB_p^{\frac{1-s}{p}}} < \infty$$

when $0 < p, s < \infty$ and $F(p,p-2,s) \subseteq BMOA$ when $1 \leq p < \infty$ and $0 < s < 1$.

For a sequence $\{z_n\}$ in $\mathbb{D}$ with $\sum_{n=1}^{\infty}(1-|z_n|) < \infty$, the Blaschke product is defined by

$$B(z) = \prod_{n=1}^{\infty} \frac{-\overline{z_n}}{|z_n|} \frac{z - \overline{z_n}}{1 - z\overline{z_n}}.$$

If for every bounded sequence of complex numbers $\{a_n\}$, there exists an $f \in H^\infty$ such that $f(z_n) = a_n$ for every n, then both the sequence $\{z_n\}$ and the Blaschke product B are called interpolating. A Blaschke product B is called Carleson-Newman if B is a product of finitely many interpolating Blaschke products. Products of finitely many interpolating Blaschke products is an important tool in the study of H^∞, see [13].

An $f \in H(\mathbb{D})$ is called an inner function if it is bounded and has boundary values of modulus 1 almost everywhere on $\partial\mathbb{D}$. It is obvious that every Blaschke product is an inner function. For an inner function θ and $\epsilon \in (0,1)$, define the level set of order ϵ of θ as

$$\Omega(\theta,\epsilon) = \{z \in \mathbb{D} : |\theta(z)| < \epsilon\}.$$

We refer to [1, 12, 15, 16, 20] for more information about inner function.

A function $g \in H(\mathbb{D})$ is said to be an outer function if there exists a positive function h with $\log h \in L^1(\partial\mathbb{D})$ and a complex number C with $|C| = 1$ such that

$$g(z) := C \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log h(e^{it}) \frac{e^{it}+z}{e^{it}-z} dt\right).$$

Moreover, for almost all $\zeta \in \partial\mathbb{D}$, $h(\zeta) = |g(\zeta)|$.

It is well known that each $f \in H^p$ has a unique factorization θg, where θ is an inner function and g is an outer function. Hence if we fix a function $f \in H^p$, there must have some relationship between θ and g. Dyakonov obtained many results on inner-outer factorization and characterized the moduli of analytic functions in $\mathbb{D}$ whose boundary values belong to certain smoothness classes. For many nice results about this topic, we refer to [6, 7, 9, 11, 22]. The following result can be found in [7, Theorem 1].

Theorem A. *If $f \in BMOA$ and θ is an inner function, then the following conditions are equivalent:*

(1) $f\theta \in BMOA$;

(2) $\sup_{z\in\mathbb{D}} |f(z)|^2(1-|\theta(z)|^2) < \infty$;

(3) $\sup_{z\in\Omega(\theta,\ \epsilon)} |f(z)| < \infty$, *for every* ϵ, $0 < \epsilon < 1$;

(4) $\sup_{z\in\Omega(\theta,\ \epsilon)} |f(z)| < \infty$, *for some* ϵ, $0 < \epsilon < 1$.

In this paper, we extend Theorem A from $BMOA$ to a more general spaces $F(p, p-2, s)$ and give the similar theorem as Theorem A.

Theorem 1. *Let $1 \le p < \infty$ and $0 < s < 1$. If $f \in F(p, p-2, s)$ and $\theta \in F(p, p-2, s)$ is an inner function, then the following statements are equivalent:*

(1) $f\theta \in F(p, p-2, s)$;

(2) $\sup_{z\in\mathbb{D}} |f(z)|^2(1-|\theta(z)|^2) < \infty$;

(3) $\sup_{z\in\Omega(\theta,\ \epsilon)} |f(z)| < \infty$, *for every* ϵ, $0 < \epsilon < 1$;

(4) $\sup_{z\in\Omega(\theta,\ \epsilon)} |f(z)| < \infty$, *for some* ϵ, $0 < \epsilon < 1$.

For more general Besov space, we have the following result.

Theorem 2. *Suppose that $2 \le p < \infty$, $0 < q < \infty$ and $0 < s < \frac{1}{2}$. If $f \in AB^s_{pq} \cap BMOA$ and $\theta \in AB^s_{pq}$ is an inner function, then the following statements are equivalent:*

(1) $f\theta \in AB^s_{pq} \cap BMOA$;

(2) $\sup_{z\in\mathbb{D}} |f(z)|^2(1-|\theta(z)|^2) < \infty$;

(3) $\sup_{z\in\Omega(\theta,\ \epsilon)} |f(z)| < \infty$, *for every* ϵ, $0 < \epsilon < 1$;

(4) $\sup_{z\in\Omega(\theta,\ \epsilon)} |f(z)| < \infty$, *for some* ϵ, $0 < \epsilon < 1$.

Throughout this paper, for two functions f and g, $f \asymp g$ means that $g \lesssim f \lesssim g$, that is, there are positive constants C_1 and C_2, such that $C_1 g \le f \le C_2 g$.

2 Proof of main results

In this section, we will give the proof of main results in this paper. To prove Theorem 1, we need the following lemmas.

Lemma 1. ([21, Theorem 1.4]) *Let $0 < s < 1$. Then an inner function belongs to the Möbius invariant Besov-type space $F(p, p-2, s)$ for all $p > \max\{s, 1-s\}$ if and only if it is the Blaschke product associated with a sequence $\{a_k\}_{k=1}^{\infty}$ which satisfies*

$$\sup_{a\in\mathbb{D}} \sum_{k=1}^{\infty} (1 - |\varphi_a(a_k)|^2)^s < \infty.$$

Lemma 2. ([18, Lemma 21]) *Let $\{a_k\}_{k=1}^{\infty}$ be a sequence in $\mathbb{D}$. Then the measure $d\mu_{a_k} = \sum_{k=1}^{\infty}(1-|a_k|^2)\delta_{a_k}$ is a Carleson measure, i.e.*

$$\sup_{a\in\mathbb{D}} \sum_{k=1}^{\infty} (1 - |\varphi_a(a_k)|^2) < \infty,$$

if and only if $\{a_k\}_{k=1}^{\infty}$ is a finite union of uniformly separated sequences.

Lemma 3. *Let $1 \leq p < \infty$, $0 < s < 1$, $f \in F(p, p-2, s)$ and B be a Carleson-Newman Blaschke product with a sequence of zeros $\{a_k\}_{k=1}^{\infty}$. Then $fB \in F(p, p-2, s)$ if and only if*

$$\sup_{a\in\mathbb{D}} \sum_{k=1}^{\infty} |f(a_k)|^p (1 - |\varphi_a(a_k)|^2)^s < \infty.$$

Proof. Necessity. The proof is similar to the proof of [17, Lemma 2.6].

Sufficiency. Let B be a Carleson-Newman Blaschke products with zeros $\{a_k\}_{k=1}^{\infty}$. Suppose that $B = \prod_{i=1}^{n} B_i$, B_i is an interpolating Blaschke products with zeros $\{a_{i,k}\}_{k=1}^{\infty}$ and

$$\{a_k\}_{k=1}^{\infty} = \bigcup_{i=1}^{n} \{a_{i,k}\}_{k=1}^{\infty}.$$

It is easy to see that

$$\sup_{a\in\mathbb{D}} \sum_{k=1}^{\infty} |f(a_{i,k})|^p (1 - |\varphi_a(a_{i,k})|^2)^s \leq \sup_{a\in\mathbb{D}} \sum_{k=1}^{\infty} |f(a_k)|^p (1 - |\varphi_a(a_k)|^2)^s < \infty.$$

Since $f \in F(p, p-2, s)$, $\rho(w, z) = \rho(\varphi_a(w), \varphi_a(z))$, $B_i \circ \varphi_a$ is an interpolating Blachke products with zeros $\{\varphi_a(a_{i,k})\}_{k=1}^{\infty}$. By [8, Theorem 8] and its remark (1), we have

$$\begin{aligned}\sup_{a\in\mathbb{D}}\|P_-\Big((f\circ\varphi_a)\cdot\overline{B_i\circ\varphi_a}\Big)\|^p_{B_p^{\frac{1-s}{p}}} &\lesssim \sup_{a\in\mathbb{D}}\sum\frac{|f\circ\varphi_a(\varphi_a(a_{i,k}))|^p}{(1-|\varphi_a(a_{i,k})|^2)^{\frac{1-s}{p}p-1}}\\ &=\sup_{a\in\mathbb{D}}\sum_{k=1}^{\infty}|f(a_{i,k})|^p(1-|\varphi_a(a_{i,k})|^2)^s.\end{aligned}$$

Combine with [20, Theorem 5], we get

$$\begin{aligned}&\sup_{a\in\mathbb{D}}\|f\circ\varphi_a-f(a)\|^p_{AB_p^{\frac{1-s}{p}}}+\sup_{a\in\mathbb{D}}\|(fB_i)\circ\varphi_a-f(a)B_i(a)\|^p_{AB_p^{\frac{1-s}{p}}}\\ \approx&\sup_{a\in\mathbb{D}}\|f\circ\varphi_a-f(a)\|^p_{B_p^{\frac{1-s}{p}}}+\sup_{a\in\mathbb{D}}\|(fB_i)\circ\varphi_a-f(a)B_i(a)\|^p_{B_p^{\frac{1-s}{p}}}\\ \approx&\sup_{a\in\mathbb{D}}\|f\circ\varphi_a-f(a)\|^p_{B_p^{\frac{1-s}{p}}}+\sup_{a\in\mathbb{D}}\|P_-\Big((f\circ\varphi_a)\cdot\overline{B_i\circ\varphi_a}\Big)\|^p_{B_p^{\frac{1-s}{p}}}\\ \lesssim&\sup_{a\in\mathbb{D}}\|f\circ\varphi_a-f(a)\|^p_{B_p^{\frac{1-s}{p}}}+\sup_{a\in\mathbb{D}}\sum_{k=1}^{\infty}|f(a_{i,k})|^p(1-|\varphi_a(a_{i,k})|^2)^s\\ \approx&\sup_{a\in\mathbb{D}}\|f\circ\varphi_a-f(a)\|^p_{AB_p^{\frac{1-s}{p}}}+\sup_{a\in\mathbb{D}}\sum_{k=1}^{\infty}|f(a_{i,k})|^p(1-|\varphi_a(a_{i,k})|^2)^s.\end{aligned}$$

Thus,

$$\begin{aligned}&\sup_{a\in\mathbb{D}}\|(fB_i)\circ\varphi_a-f(a)B_i(a)\|^p_{AB_p^{\frac{1-s}{p}}}\\ \lesssim&\sup_{a\in\mathbb{D}}\sum_{k=1}^{\infty}|f(a_{i,k})|^p(1-|\varphi_a(a_{i,k})|^2)^s+\sup_{a\in\mathbb{D}}\|f\circ\varphi_a-f(a)\|^p_{AB_p^{\frac{1-s}{p}}}.\end{aligned}$$

Since $f\in F(p,p-2,s)$, by Lemma 2.1 in [17], we have

$$fB_i\in F(p,p-2,s),\qquad i=1,...,n.$$

By inductive, we have

$$(fB)'(z)=\sum_{j=1}^{n}(fB_j)'(z)\prod_{i=1,i\neq j}^{n}B_i(z)-(n-1)f'(z)\prod_{i=1}^{n}B_i(z).$$

Hence,

$$|(fB)'(z)|\le\sum_{j=1}^{n}|(fB_j)'(z)|+(n-1)|f'(z)|,\qquad z\in\mathbb{D}.$$

Notice that $f\in F(p,p-2,s)$, $fB_i\in F(p,p-2,s)$, combine with p-inequality, we obtain $fB\in F(p,p-2,s)$. The proof is complete.

Proof of Theorem 1. (1) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (2). Since $f \in F(p, p-2, s) \subseteq BMOA$, $f\theta \in F(p, p-2, s) \subseteq BMOA$. From Theorem A, we easily get our result.

(2) $\Rightarrow$ (1). Assume that (2) holds. Since $\theta \in F(p, p-2, s)$, by Lemma 1, we see that θ is a Blaschke product with zeros $\{a_k\}_{k=1}^{\infty}$, and

$$\sup_{a\in\mathbb{D}} \sum_{k=1}^{\infty} (1-|\varphi_a(a_k)|^2)^s < \infty, \quad 0 < s < 1,$$

which implies that

$$\sup_{a\in\mathbb{D}} \sum_{k=1}^{\infty} (1-|\varphi_a(a_k)|^2) < \infty.$$

From Lemma 2, we get that θ is a Carleson-Newman Blaschke product. Since $f \in F(p, p-2, s) \subseteq BMOA$, by the assumption that $\sup_{z\in\mathbb{D}} |f(z)|^2(1-|\theta(z)|^2) < \infty$ and Theorem A, we see that $f\theta \in BMOA$. Theorem A gives

$$\sup_{z\in\Omega(\theta,\ \epsilon)} |f(z)| < \infty, \quad 0 < \epsilon < 1,$$

which implies that $\sup_k |f(a_k)| < \infty$. Thus,

$$\begin{aligned} &\sup_{a\in\mathbb{D}} \sum_{k=1}^{\infty} |f(a_k)|^p (1-|\varphi_a(a_k)|^2)^s \\ \le\ & \sup_k |f(a_k)|^p \sup_{a\in\mathbb{D}} \sum_{k=1}^{\infty} (1-|\varphi_a(a_k)|^2)^s < \infty. \end{aligned}$$

Applying Lemma 3, we see that $f\theta \in F(p, p-2, s)$. The proof is complete.

Proof of Theorem 2. (1) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (2). The proof is similar to Theorem 1 and hence we omit the details.

(2) $\Rightarrow$ (1). Suppose that $f \in AB_{pq}^s \cap BMOA$ and $\theta \in AB_{pq}^s$. Since θ is bounded, if we want to prove $f\theta \in AB_{pq}^s$, we only need to prove

$$\int_0^1 (1-r)^{(1-s)q-1} \left(\int_{\partial\mathbb{D}} |f(r\zeta)\theta'(r\zeta)|^p dm(\zeta) \right)^{\frac{q}{p}} dr < \infty.$$

Using the well known Schwarz's Lemma, we have

$$|\theta'(z)| \le \frac{1-|\theta(z)|^2}{1-|z|^2}.$$

Therefore

$$\begin{aligned} &\int_0^1 (1-r)^{(1-s)q-1} \left(\int_{\partial\mathbb{D}} |f(r\zeta)\theta'(r\zeta)|^p dm(\zeta) \right)^{\frac{q}{p}} dr \\ \lesssim &\int_0^1 (1-r)^{(1-s)q-1} \left(\int_{\partial\mathbb{D}} |f(r\zeta)|^p \left| \frac{1-|\theta(r\zeta)|^2}{1-r^2} \right|^p dm(\zeta) \right)^{\frac{q}{p}} dr. \end{aligned}$$

From [10, Theorem 3.2], we known that $\theta \in AB^s_{pq}$ if and only if

$$\int_0^1 \left(\int_{\partial\mathbb{D}} (1 - |\theta(r\zeta)|)^{\frac{p}{2}} dm(\zeta) \right)^{\frac{q}{p}} \frac{dr}{(1-r)^{sq+1}} < \infty.$$

Thus, combine with the assumption that $\sup_{z\in\mathbb{D}} |f(z)|^2(1 - |\theta(z)|^2) < \infty$, we deduce that

$$\int_0^1 (1-r)^{(1-s)q-1} \left(\int_{\partial\mathbb{D}} |f(r\zeta)\theta'(r\zeta)|^p dm(\zeta) \right)^{\frac{q}{p}} dr < \infty,$$

which implies that $f\theta \in AB^s_{pq}$. In addition, by Theorem A, we see that $f\theta \in BMOA$. Hence $f\theta \in AB^s_{pq} \cap BMOA$. The proof is complete.

Acknowledgement. This work was supported by NSF of China (No.11471143).

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GENERALIZED RATIONAL CONTRACTIONS ENDOWED WITH A GRAPH AND AN APPLICATION TO A SYSTEM OF INTEGRAL EQUATIONS

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ABSTRACT. In the present paper, we introduce the notion of generalized rational contraction including admissible mappings and establish coincidence point and common fixed point results for this class of mappings defined on ordinary as well as ordered metric spaces. Our results extend, generalize and unify comparable results in the existing literature. Applying these results, we deduce fixed point results on metric spaces endowed with graph. An example and application to obtain the existence of common solution for a system of integral equations are also given in order to illustrate the effectiveness of the offered results.

1. INTRODUCTION AND PRELIMINARIES

Fixed point theory is one of the most powerful and effective tools in mathematics which has enormous applications within as well as outside mathematics. One of the most fundamental fixed point theorems is the Banach contraction principle [8] which gives an answer on the existence and uniqueness of a solution of an operator equation $Fx = x$. Since then, there is a great number of generalizations of this fundamental principle (for example, see [1]-[7], [9]-[29]).

Recently, Samet et al. [28] first introduced α-admissible mappings and then α-ψ-contractive type mappings to obtain some interesting generalizations of Banach contraction principle. For more results in this direction, we refer to [3, 5, 6, 11, 15, 17, 21, 23, 25, 27, 22] and references mentioned therein.

Definition 1 ([28]). *Let X be a nonempty set and $\alpha : X \times X \longrightarrow [0, +\infty)$. A self-mapping T on X is called α-admissible mapping if*

$$x, y \in X, \qquad \alpha(x, y) \geq 1 \text{ implies } \alpha(Tx, Ty) \geq 1.$$

Afterward, Patel et al. [25] extended the definition of α-admissible mapping to a pair of two mappings to obtain common fixed point results as follows:

Definition 2 ([25]). *Let f, g, S and T be four self-mappings of a non-empty set X, and let $\alpha : S(X) \cup T(X) \times S(X) \cup T(X) \to [0, +\infty)$. Then the pair (f, g) is called α-admissible with respect to S and T (in short, α_{ST}-admissible) if for all $x, y \in X$,*

$$\alpha(Sx, Ty) \geq 1 \text{ or } \alpha(Tx, Sy) \geq 1 \Longrightarrow \alpha(fx, gy) \geq 1 \text{ and } \alpha(gx, fy) \geq 1.$$

If we take $S = T = I_X$ (identity mapping on X) in above definition, then we have:

2000 *Mathematics Subject Classification.* Primary 47H10, Secondary 54H25.

Key words and phrases. Point of coincidence, common fixed point, admissible mappings, rational contractions, weakly compatible mappings, integral equations.

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Definition 3 ([3])**.** *Let f and g be self-mappings of a non-empty set X and $\alpha : X \times X \to [0, +\infty)$. Then the pair (f, g) is called α-admissible if for all $x, y \in X$,*

$$\alpha(x, y) \geq 1 \Longrightarrow \alpha(fx, gy) \geq 1 \quad and \quad \alpha(gx, fy) \geq 1.$$

Definition 4 ([19])**.** *A pair (f, T) of self-mappings on a set X is said to be* weakly compatible *if f and T commute at their coincidence point (i.e. $fTx = Tfx$, $x \in X$ whenever $fx = Tx$).*

A point $y \in X$ is called a *point of coincidence* of two self-mappings f and T on X if there exists a point $x \in X$ such that $y = fx = Tx$. Also, $x \in X$ is called a *common fixed point* of mappings f and T if $x = fx = Tx$.

The notations $\mathcal{F}(f, T)$ and $\mathcal{C}(f, T)$ stand for the set of all common fixed point and the set of all coincidence points of f and T, respectively. In the sequel, we will indicate the set of all real numbers, the set of all non-negative real numbers and the set of all natural numbers by the letters $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$, respectively.

On the other side, Khan et al. [20] introduced and employed the notion of altering distance function to obtain some interesting fixed point results in metric spaces. Note that altering distance functions are continuous whereas Su [29] defined generalized altering distance function, not necessarily continuous, as follows:

Definition 5 ([29])**.** *A mapping $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is called generalized altering distance function if*

(a) *ψ is non-decreasing,*
(b) *$\psi(t) = 0$ iff $t = 0$.*

We set $\Psi = \{\psi : \mathbb{R}^+ \to \mathbb{R}^+ : \psi$ is a generalized altering distance function$\}$ and $\Phi = \{\varphi : \mathbb{R}^+ \to \mathbb{R}^+ : \varphi$ is a nondecreasing and right upper semi-continuous function and we have $\psi(t) > \varphi(t)$ for all $t > 0$ where $\psi \in \Psi\}$.

We now introduce generalized rational contraction mappings as follows:

Definition 6. *Let f, g, S and T be selfmaps of a metric space (X, d), and (f, g) be an α_{ST}-admissible pair. We say that (f, g) is a generalized $(\alpha, \psi, \varphi)_{(S,T)}$-rational contraction if*

$$\alpha(Sx, Ty) \geq 1 \text{ implies } \psi(d(fx, gy)) \leq \varphi(M(x, y)) \tag{1.1}$$

for all $x, y \in X$, where $\psi \in \Psi$, $\varphi \in \Phi$ and

$$\begin{aligned} M(x, y) = \max\Big(& d(Sx, Ty), d(Sx, fx), d(Ty, gy), \frac{d(Sx, gy) + d(fx, Ty)}{2}, \\ & \frac{d(Ty, gy)[1 + d(Sx, fx)]}{1 + d(Sx, Ty)}, \frac{d(fx, Ty)[1 + d(Sx, gy)]}{1 + d(Sx, Ty)}\Big). \end{aligned}$$

In this paper, we prove some common fixed point results of generalized $(\alpha, \psi, \varphi)_{(S,T)}$-rational contractions for a quadruple of self-mappings defined on ordinary as well as ordered metric spaces. Our results extend, generalize and unify comparable results in the existing literature. Applying these results, we deduce fixed point results on metric spaces endowed with graph. An example is presented to support the results obtained herein. As an application of offered results, the existence of the common solution for a system of integral equations are also investigated.

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2. Main Results

We start with the following first result.

Theorem 1. *Let f, g, S and T be selfmaps of a complete metric space (X, d) with $f(X) \subset T(X)$, $g(X) \subset S(X)$ and (f, g) be a generalized $(\alpha, \psi, \varphi)_{(S,T)}$-rational contraction pair. Suppose that:*

- (a) *there exists $x_0 \in X$ such that $\alpha(Sx_0, fx_0) \geq 1$;*
- (b) *$\alpha(Sx_n, Tx_{n+1}) \geq 1$ for all n even implies that $\alpha(Sx_n, Tx_j) \geq 1$ for all n even and $j > n$ odd;*
- (c) *$\alpha(Sx_n, Tx_{n+1}) \geq 1$ for all n even and, Sx_n and Tx_{n+1} converge to an $x \in X$ as $n \to \infty$ implies that $\alpha(Sx_n, x) \geq 1$ and $\alpha(x, Tx_{n+1}) \geq 1$ for all n even.*

Then the pairs (f, S) and (g, T) have a point of coincidence in X. Moreover, if

- (i) *$\{f, S\}$ and $\{g, T\}$ are weakly compatible,*
- (ii) *$\alpha(Su, Tv) \geq 1$ whenever $u \in \mathcal{C}(f, S)$ and $v \in \mathcal{C}(g, T)$.*

Then f, g, S and T have a common fixed point.

Proof. Let $x_0 \in X$ such that $\alpha(Sx_0, fx_0) \geq 1$. Since $fX \subset TX$, there exists an $x_1 \in X$ such that $fx_0 = Tx_1$. Again since $gX \subset SX$, there exists an $x_2 \in X$ such that $gx_1 = Tx_2$. Continuing this process, we can construct the sequences $\{x_n\}$ and $\{y_n\}$ in X defined by

$$y_{2n} = fx_{2n} = Tx_{2n+1}, \quad y_{2n+1} = gx_{2n+1} = Sx_{2n+2}, \quad n \in \mathbb{N}_0, \tag{2.1}$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. As (f, g) is an α_{ST}-admissible pair and $\alpha(Sx_0, fx_0) = \alpha(Sx_0, Tx_1) \geq 1$, we have $\alpha(fx_0, gx_1) \geq 1$ and $\alpha(gx_0, fx_1) \geq 1$ which implies that $\alpha(Tx_1, Sx_2) \geq 1$. Again, since $\alpha(Tx_1, Sx_2) \geq 1$, we have $\alpha(fx_1, gx_2) \geq 1$ and $\alpha(gx_1, fx_2) \geq 1$ which gives that $\alpha(Sx_2, Tx_3) \geq 1$. Continuing this way, we obtain

$$\alpha(Sx_{2n}, Tx_{2n+1}) \geq 1 \quad \text{and} \quad \alpha(Tx_{2n+1}, Sx_{2n+2}) \geq 1 \quad \text{for all } n \in \mathbb{N}_0. \tag{2.2}$$

Suppose that $y_{2n} \neq y_{2n+1}$ for all $n \in \mathbb{N}_0$. Now we show that

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0. \tag{2.3}$$

Putting $x = x_{2n}$ and $y = x_{2n+1}$ in (1.1) and using (2.1) and (2.2), we get

$$\begin{aligned} \psi(d(y_{2n}, y_{2n+1})) &= \psi(d(fx_{2n}, gx_{2n+1})) \\ &\leq \varphi(M(x_{2n}, x_{2n+1})), \end{aligned} \tag{2.4}$$

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where

$$\begin{aligned}
M(x_{2n},x_{2n+1}) &= \max\Big(d(Sx_{2n},Tx_{2n+1}),d(Sx_{2n},fx_{2n}),d(Tx_{2n+1},gx_{2n+1}),\\
&\quad \frac{d(Sx_{2n},gx_{2n+1})+d(fx_{2n},Tx_{2n+1})}{2},\\
&\quad \frac{d(Tx_{2n+1},gx_{2n+1})[1+d(Sx_{2n},fx_{2n})]}{1+d(Sx_{2n},Tx_{2n+1})},\\
&\quad \frac{d(fx_{2n},Tx_{2n+1})[1+d(Sx_{2n},gx_{2n+1})]}{1+d(Sx_{2n},Tx_{2n+1})}\Big)\\
&= \max\Big(d(y_{2n-1},y_{2n}),d(y_{2n-1},y_{2n}),d(y_{2n},y_{2n+1}),\\
&\quad \frac{d(y_{2n-1},y_{2n+1})+d(y_{2n},y_{2n})}{2},\\
&\quad \frac{d(y_{2n},y_{2n+1})[1+d(y_{2n-1},y_{2n})]}{1+d(y_{2n-1},y_{2n})},\\
&\quad \frac{d(y_{2n},y_{2n})[1+d(y_{2n-1},y_{2n+1})]}{1+d(y_{2n-1},y_{2n})}\Big)\\
&\le \max\Big(d(y_{2n-1},y_{2n}),d(y_{2n},y_{2n+1}),\frac{d(y_{2n-1},y_{2n})+d(y_{2n},y_{2n+1})}{2}\Big)\\
&= \max(d(y_{2n-1},y_{2n}),d(y_{2n},y_{2n+1})).
\end{aligned}$$

If $d(y_{2n-1},y_{2n})\le d(y_{2n},y_{2n+1})$ for some $n\in\mathbb{N}$, then by (2.4), we have

$$\psi(d(y_{2n},y_{2n+1}))\le\varphi(d(y_{2n},y_{2n+1})),$$

a contradiction to the fact that $y_{2n}\neq y_{2n+1}$. So for all $n\in\mathbb{N}$, we have $d(y_{2n},y_{2n+1})< d(y_{2n-1},y_{2n})$.

From (2.4), we also obtain

$$\psi(d(y_{2n},y_{2n+1}))\le\varphi(d(y_{2n-1},y_{2n})). \tag{2.5}$$

Again, putting $x=x_{2n-1}$ and $y=x_{2n}$ in (1.1) and following arguing similar to those given above, we get

$$\psi(d(y_{2n-1},y_{2n}))\le\varphi(d(y_{2n-2},y_{2n-1})). \tag{2.6}$$

From (2.5) and (2.6), we conclude

$$\psi(d(y_n,y_{n+1}))\le\varphi(d(y_{n-1},y_n)). \tag{2.7}$$

It follows that the sequence $\{d(y_n,y_{n+1})\}$ is decreasing and bounded below. Hence, there exists $r\ge 0$ such that $\lim_{n\to\infty}d(y_n,y_{n+1})=r$. If $r>0$, then taking limit as $n\to\infty$ on both sides of (2.7), we have

$$\begin{aligned}
\psi(r) &\le \lim_{n\to\infty}\psi(d(y_n,y_{n+1}))\\
&\le \lim_{n\to\infty}\varphi(d(y_{n-1},y_n))\le\varphi(r),
\end{aligned}$$

a contradiction and hence $r=0$, that is, the equation (2.3) holds.

Now, we prove that $\{y_n\}$ is a Cauchy sequence. To this end, it is sufficient to verify that $\{y_{2n}\}$ is a Cauchy sequence. Suppose, to the contrary, that $\{y_{2n}\}$ is not a Cauchy sequence. Then, there exists an $\varepsilon>0$ for which we can find two

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subsequences $\{y_{2m_k}\}$ and $\{y_{2n_k}\}$ of $\{y_{2n}\}$ such that m_k is the smallest index for which $m_k > n_k > k$ and

$$d(y_{2m_k}, y_{2n_k}) \geq \varepsilon \quad \text{and} \quad d(y_{2m_k-1}, y_{2n_k}) < \varepsilon. \tag{2.8}$$

Using the triangular inequality and (2.8), we have

$$\begin{aligned} \varepsilon &\leq d(y_{2m_k}, y_{2n_k}) \leq d(y_{2m_k}, y_{2m_k-1}) + d(y_{2m_k-1}, y_{2n_k}) \\ &< d(y_{2m_k}, y_{2m_k-1}) + \varepsilon. \end{aligned}$$

Taking $k \to \infty$ on both sides of above inequality and using (2.3), we obtain

$$\lim_{k\to\infty} d(y_{2m_k}, y_{2n_k}) = \varepsilon. \tag{2.9}$$

Again, using the triangular inequality, we get

$$|d(y_{2n_k}, y_{2m_k+1}) - d(y_{2n_k}, y_{2m_k})| \leq d(y_{2m_k}, y_{2m_k+1}).$$

Letting $k \to \infty$ in the above inequality and using (2.3) and (2.9), we have

$$\lim_{k\to\infty} d(y_{2n_k}, y_{2m_k+1}) = \varepsilon. \tag{2.10}$$

Similarly, one can easily show that

$$\lim_{k\to\infty} d(y_{2n_k-1}, y_{2m_k}) = \lim_{k\to\infty} d(y_{2n_k-1}, y_{2m_k+1}) = \varepsilon. \tag{2.11}$$

Since $\alpha(Sx_{2n_k}, Tx_{2m_k+1}) \geq 1$ from (2.2) and the hypothesis (b), putting $x = x_{2n_k}$ and $y = x_{2m_k+1}$ in (1.1), we get

$$\begin{aligned} \psi(d(y_{2n_k}, y_{2m_k+1})) &= \psi(d(fx_{2n_k}, gx_{2m_k+1})) \\ &\leq \varphi(M(x_{2n_k}, x_{2m_k+1})), \end{aligned} \tag{2.12}$$

where

$$\begin{aligned} M(x_{2n_k}, x_{2m_k+1}) &= \max\Bigg(d(Sx_{2n_k}, Tx_{2m_k+1}), d(Sx_{2n_k}, fx_{2n_k}), d(Tx_{2m_k+1}, gx_{2m_k+1}), \\ &\quad \frac{d(Sx_{2n_k}, gx_{2m_k+1}) + d(fx_{2n_k}, Tx_{2m_k+1})}{2}, \\ &\quad \frac{d(Tx_{2m_k+1}, gx_{2m_k+1})[1 + d(Sx_{2n_k}, fx_{2n_k})]}{1 + d(Sx_{2n_k}, Tx_{2m_k+1})}, \\ &\quad \frac{d(fx_{2n_k}, Tx_{2m_k+1})[1 + d(Sx_{2n_k}, gx_{2m_k+1})]}{1 + d(Sx_{2n_k}, Tx_{2m_k+1})}\Bigg) \\ &= \max\Bigg(d(y_{2n_k-1}, y_{2m_k}), d(y_{2n_k-1}, y_{2n_k}), d(y_{2m_k}, y_{2m_k+1}), \\ &\quad \frac{d(y_{2n_k-1}, y_{2m_k+1}) + d(y_{2n_k}, y_{2m_k})}{2}, \\ &\quad \frac{d(y_{2m_k}, y_{2m_k+1})[1 + d(y_{2n_k-1}, y_{2n_k})]}{1 + d(y_{2n_k-1}, y_{2m_k})}, \\ &\quad \frac{d(y_{2n_k}, y_{2m_k})[1 + d(y_{2n_k-1}, y_{2m_k+1})]}{1 + d(y_{2n_k-1}, y_{2m_k})}\Bigg). \end{aligned}$$

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Now, from the properties of ψ and φ and using (2.3), (2.9), (2.10) and (2.11) as $k \to \infty$ in (2.12), we obtain

$$\begin{aligned} \psi(\varepsilon) &\leq \lim_{k\to\infty} \psi\left(d\left(y_{2n_k}, y_{2m_k+1}\right)\right) \\ &\leq \lim_{k\to\infty} \varphi\left(M\left(x_{2n_k}, x_{2m_k+1}\right)\right) \\ &\leq \varphi\left(\max\left(\varepsilon, 0, 0, \varepsilon, 0, \varepsilon\right)\right) = \varphi(\varepsilon), \end{aligned}$$

which implies that $\varepsilon = 0$, a contradiction with $\varepsilon > 0$. Thus $\{y_{2n}\}$ is a Cauchy sequence in X and hence $\{y_n\}$ is a Cauchy sequence. From the completeness of (X, d), there exists $z \in X$ such that

$$\lim_{n\to\infty} y_n = z. \tag{2.13}$$

From (2.1) and (2.13), we get

$$fx_{2n} \to z, \quad Tx_{2n+1} \to z, \quad gx_{2n+1} \to z, \quad Sx_{2n+2} \to z \quad \text{as } n \to \infty. \tag{2.14}$$

Now we shall prove that z is a common fixed point of f, g, S and T.

Since $g(X) \subset S(X)$, we can choose a point u in X such that $z = Su$. Suppose that $d(z, fu) \neq 0$.

By (2.2), (2.14) and the condition (c), we have $\alpha\left(Su, Tx_{2n+1}\right) \geq 1$. Then, substituting $x = u$ and $y = x_{2n+1}$ in (1.1), we deduce

$$\psi\left(d\left(fu, gx_{2n+1}\right)\right) \leq \varphi\left(M\left(u, x_{2n+1}\right)\right), \tag{2.15}$$

where

$$\begin{aligned} M\left(u, x_{2n+1}\right) = \max\Bigg(& d\left(Su, Tx_{2n+1}\right), d\left(Su, fu\right), d\left(Tx_{2n+1}, gx_{2n+1}\right), \\ & \frac{d\left(Su, gx_{2n+1}\right) + d\left(fu, Tx_{2n+1}\right)}{2}, \\ & \frac{d\left(Tx_{2n+1}, gx_{2n+1}\right)\left[1 + d\left(Su, fu\right)\right]}{1 + d\left(Su, Tx_{2n+1}\right)}, \\ & \frac{d\left(fu, Tx_{2n+1}\right)\left[1 + d\left(Su, gx_{2n+1}\right)\right]}{1 + d\left(Su, Tx_{2n+1}\right)}\Bigg). \end{aligned}$$

Letting $k \to \infty$ in (2.15), we have

$$\begin{aligned} \psi\left(d\left(fu, z\right)\right) &\leq \lim_{n\to\infty} \psi\left(d\left(fu, gx_{2n+1}\right)\right) \\ &\leq \lim_{n\to\infty} \varphi\left(M\left(u, x_{2n+1}\right)\right) \\ &\leq \varphi\left(\max\left(0, d\left(z, fu\right), 0, \frac{d\left(fu, z\right)}{2}, 0, d\left(fu, z\right)\right)\right) \\ &= \varphi\left(d\left(fu, z\right)\right), \end{aligned}$$

a contradiction and hence $d(fu, z) = 0$, that is $fu = z$, and so $u \in \mathcal{C}(f, S)$.

Similarly, since $f(X) \subset T(X)$, we can choose a point v in X such that $z = Tv$. Suppose that $d(z, gv) \neq 0$.

By (2.2), (2.14) and the condition (c), we have $\alpha\left(Sx_{2n}, Tv\right) \geq 1$. Then, putting $x = x_{2n}$ and $y = v$ in (1.1), we obtain

$$\psi\left(d\left(fx_{2n}, gv\right)\right) \leq \varphi\left(M\left(x_{2n}, v\right)\right), \tag{2.16}$$

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where

$$\begin{aligned} M(x_{2n}, v) &= \max\Big(d(Sx_{2n}, Tv), d(Sx_{2n}, fx_{2n}), d(Tv, gv), \\ &\quad \frac{d(Sx_{2n}, gv) + d(fx_{2n}, Tv)}{2}, \\ &\quad \frac{d(Tv, gv)[1 + d(Sx_{2n}, fx_{2n})]}{1 + d(Sx_{2n}, Tv)}, \\ &\quad \frac{d(fx_{2n}, Tv)[1 + d(Sx_{2n}, gv)]}{1 + d(Sx_{2n}, Tv)} \Big). \end{aligned}$$

Taking limit on (2.16), we get

$$\begin{aligned} \psi(d(z, gv)) &\le \lim_{n\to\infty} \psi(d(fx_{2n}, gv)) \\ &\le \lim_{n\to\infty} \varphi(M(x_{2n}, v)) \\ &\le \varphi\left(\max\left(0, 0, d(z, gv), \frac{d(z, gv)}{2}, d(z, gv), 0\right)\right) \\ &= \varphi(d(z, gv)), \end{aligned}$$

a contradiction and hence $d(z, gv) = 0$, that is $z = gv$, and so $v \in \mathcal{C}(g, T)$.

Thus, $z = fu = Su = gv = Tv$. By the weak compatibility of the pairs (f, S) and (g, T), we deduce that $fz = Sz$ and $gz = Tz$.

Since $z \in \mathcal{C}(f, S)$ and $v \in \mathcal{C}(g, T)$, by (ii), we have $\alpha(Sz, Tv) \ge 1$ and so, from (1.1)

$$\psi(d(fz, z)) = \psi(d(fz, gv)) \le \varphi(M(z, v)), \tag{2.17}$$

where

$$\begin{aligned} M(z, v) &= \max\Big(d(Sz, Tv), d(Sz, fz), d(Tv, gv), \\ &\quad \frac{d(Sz, gv) + d(fz, Tv)}{2}, \frac{d(Tv, gv)[1 + d(Sz, fz)]}{1 + d(Sz, Tv)}, \\ &\quad \frac{d(fz, Tv)[1 + d(Sz, gv)]}{1 + d(Sz, Tv)} \Big) \\ &= \max(d(fz, z), 0, 0, d(fz, z), 0, d(fz, z)) = d(fz, z) \end{aligned}$$

By (2.17), we get

$$\psi(d(fz, z)) \le \varphi(d(fz, z)),$$

which implies that $z = fz$, and so $z = fz = Sz$. Similarly, it can be shown that $z = gz = Tz$. This completes the proof. □

Corollary 1. *Let f, g, S and T be selfmaps of a complete metric space (X, d) with $f(X) \subset T(X)$, $g(X) \subset S(X)$ and (f, g) be an α_{ST}-admissible pair such that*

$$\alpha(Sx, Ty)\psi(d(fx, gy)) \le \varphi(M(x, y)), \tag{2.18}$$

for all $x, y \in X$, where $\psi \in \Psi$ and $\varphi \in \Phi$. Assume that the following conditions are satisfied:

- (a) *there exists $x_0 \in X$ such that $\alpha(Sx_0, fx_0) \ge 1$;*
- (b) *$\alpha(Sx_n, Tx_{n+1}) \ge 1$ for all n even implies that $\alpha(Sx_n, Tx_j) \ge 1$ for all n even and $j > n$ odd;*

(c) *$\alpha(Sx_n, Tx_{n+1}) \geq 1$ for all n even and, Sx_n and Tx_{n+1} converge to an $x \in X$ as $n \to \infty$ implies that $\alpha(Sx_n, x) \geq 1$ and $\alpha(x, Tx_{n+1}) \geq 1$ for all n even.*

Then the pairs (f, S) and (g, T) have a point of coincidence in X. Moreover, if

(i) *$\{f, S\}$ and $\{g, T\}$ are weakly compatible,*

(ii) *$\alpha(Su, Tv) \geq 1$ whenever $u \in \mathcal{C}(f, S)$ and $v \in \mathcal{C}(g, T)$.*

Then f, g, S and T have a common fixed point.

Proof. Let $\alpha(Sx, Ty) \geq 1$ for $x, y \in X$. Then by (2.18), we have

$$\psi(d(fx, gy)) \leq \varphi(M(x, y)).$$

This implies that the inequality (1.1) holds. Therefore, the proof follows from Theorem 1. □

If we take $\alpha(Sx, Ty) = 1$ in Corollary 1, we have a generalized version of Theorem 2.3 in [29]:

Theorem 2. *Let f, g, S and T be selfmaps of a complete metric space (X, d) with $f(X) \subset T(X)$ and $g(X) \subset S(X)$. Suppose that*

$$\psi(d(fx, gy)) \leq \varphi(M(x, y)), \tag{2.19}$$

for all $x, y \in X$, where $\psi \in \Psi$ and $\varphi \in \Phi$. Then the pairs (f, S) and (g, T) have a point of coincidence in X. Moreover, if $\{f, S\}$ and $\{g, T\}$ are weakly compatible, then f, g, S and T have a common fixed point.

If we take $\psi(t) = t$ in Corollary 1, we have a generalized version of Theorem 2.2 in [28]:

Theorem 3. *Let f, g, S and T be selfmaps of a complete metric space (X, d) with $f(X) \subset T(X)$, $g(X) \subset S(X)$ and (f, g) be an α_{ST}-admissible pair such that*

$$\alpha(Sx, Ty)\, d(fx, gy) \leq \varphi(M(x, y)), \tag{2.20}$$

for all $x, y \in X$, where $\varphi \in \Phi$. Assume that the following conditions are satisfied:

(a) *there exists $x_0 \in X$ such that $\alpha(Sx_0, fx_0) \geq 1$;*

(b) *$\alpha(Sx_n, Tx_{n+1}) \geq 1$ for all n even implies that $\alpha(Sx_n, Tx_j) \geq 1$ for all n even and $j > n$ odd;*

(c) *$\alpha(Sx_n, Tx_{n+1}) \geq 1$ for all n even and, Sx_n and Tx_{n+1} converge to an $x \in X$ as $n \to \infty$ implies that $\alpha(Sx_n, x) \geq 1$ and $\alpha(x, Tx_{n+1}) \geq 1$ for all n even.*

Then the pairs (f, S) and (g, T) have a point of coincidence in X. Moreover, if

(i) *$\{f, S\}$ and $\{g, T\}$ are weakly compatible,*

(ii) *$\alpha(Su, Tv) \geq 1$ whenever $u \in \mathcal{C}(f, S)$ and $v \in \mathcal{C}(g, T)$.*

Then f, g, S and T have a common fixed point.

If we take $\varphi(t) = \psi(t) - \phi(t)$ in Corollary 1, we have the following result.

Corollary 2. *Let f, g, S and T be selfmaps of a complete metric space (X, d) with $f(X) \subset T(X)$, $g(X) \subset S(X)$ and (f, g) be an α_{ST}-admissible pair such that*

$$\alpha(Sx, Ty)\, \psi(d(fx, gy)) \leq \psi(M(x, y)) - \phi(M(x, y)), \tag{2.21}$$

for all $x, y \in X$, where $\psi \in \Psi$ and $\phi \in \Phi$. Assume that the following conditions are satisfied:

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(a) *there exists* $x_0 \in X$ *such that* $\alpha(Sx_0, fx_0) \geq 1$;
(b) $\alpha(Sx_n, Tx_{n+1}) \geq 1$ *for all* n *even implies that* $\alpha(Sx_n, Tx_j) \geq 1$ *for all* n *even and* $j > n$ *odd;*
(c) $\alpha(Sx_n, Tx_{n+1}) \geq 1$ *for all* n *even and,* Sx_n *and* Tx_{n+1} *converge to an* $x \in X$ *as* $n \to \infty$ *implies that* $\alpha(Sx_n, x) \geq 1$ *and* $\alpha(x, Tx_{n+1}) \geq 1$ *for all* n *even.*

Then the pairs (f, S) *and* (g, T) *have a point of coincidence in* X. *Moreover, if*
(i) $\{f, S\}$ *and* $\{g, T\}$ *are weakly compatible,*
(ii) $\alpha(Su, Tv) \geq 1$ *whenever* $u \in \mathcal{C}(f, S)$ *and* $v \in \mathcal{C}(g, T)$.

Then f, g, S *and* T *have a common fixed point.*

Let us give the following hypothesis for the uniqueness of the common fixed point in Theorem 1.

(H) For all $x, y \in \mathcal{F}(f, g, S, T)$, we have $\alpha(Sx, Ty) \geq 1$.

Theorem 4. *Adding condition* (H) *to the hypotheses of Theorem 1, we obtain the uniqueness of the common fixed point of* f, g, S *and* T.

Proof. Suppose that $x = fx = gx = Sx = Tx$ and $y = fy = gy = Sy = Ty$. Then, from (H), we have $\alpha(Sx, Ty) \geq 1$. Then, applying (1.1), we obtain

$$\psi(d(x, y)) = \psi(d(fx, gy)) \leq \varphi(M(x, y)), \tag{2.22}$$

where

$$\begin{aligned} M(x, y) &= \max\Bigg(d(Sx, Ty), d(Sx, fx), d(Ty, gy), \\ &\quad \frac{d(Sx, gy) + d(fx, Ty)}{2}, \frac{d(Ty, gy)[1 + d(Sx, fx)]}{1 + d(Sx, Ty)}, \\ &\quad \frac{d(fx, Ty)[1 + d(Sx, gy)]}{1 + d(Sx, Ty)}\Bigg) \\ &= \max(d(x, y), 0, 0, d(x, y), 0, d(x, y)) = d(x, y). \end{aligned}$$

From (2.22), we have

$$\psi(d(x, y)) \leq \varphi(d(x, y)),$$

which implies that $d(x, y) = 0$, that is, $x = y$. □

Remark 1. *Adding condition* (H) *to the hypotheses of Corollaries 1 and 2, we obtain the uniqueness of the common fixed point.*

If we choose $S = T = I_X$ in Corollary 1, we have the following corollary.

Corollary 3. *Let* f *and* g *be selfmaps of a complete metric space* (X, d) *and* (f, g) *be an* α*-admissible pair such that*

$$\alpha(x, y)\psi(d(fx, gy)) \leq \varphi(M_{fg}(x, y)), \tag{2.23}$$

for all $x, y \in X$, *where* $\psi \in \Psi$, $\varphi \in \Phi$ *and*

$$\begin{aligned} M_{fg}(x, y) &= \max\Bigg(d(x, y), d(x, fx), d(y, gy), \frac{d(x, gy) + d(fx, y)}{2}, \\ &\quad \frac{d(y, gy)[1 + d(x, fx)]}{1 + d(x, y)}, \frac{d(fx, y)[1 + d(x, gy)]}{1 + d(x, y)}\Bigg). \end{aligned}$$

Assume that the following conditions are satisfied:

(*a*) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, fx_0) \geq 1$;
(*b*) $\alpha(x_n, x_{n+1}) \geq 1$ *for all* n *implies that* $\alpha(x_n, x_j) \geq 1$ *for all* $j > n$;
(*c*) $\alpha(x_n, x_{n+1}) \geq 1$ *for all* n *and,* $x_n \to x \in X$ *as* $n \to \infty$ *implies that* $\alpha(x_n, x) \geq 1$ *for all* n.

Then f *and* g *have a common fixed point. Moreover, if* $\alpha(x, y) \geq 1$ *whenever* $x, y \in \mathcal{F}(f, g)$, *then* f *and* g *have a unique common fixed point.*

Now, we furnish the following example which illustrates Theorem 1 as well as Theorem 4.

Example 1. *Let* $X = \mathbb{R}^+$ *with the usual metric* $d(x, y) = |x - y|$ *for all* $x, y \in X$ *and* $\psi, \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *be defined by* $\psi(t) = t$ *and* $\varphi(t) = \frac{t}{2}$. *Define the mappings* f, g, S *and* T *on* X *by*

$$fx = \begin{cases} \frac{x}{6} & \text{if } x \in [0,1], \\ 3x & \text{if } x > 1, \end{cases} \quad \text{and} \quad gx = \begin{cases} \frac{x}{4} & \text{if } x \in [0,1], \\ 6x & \text{if } x > 1, \end{cases}$$

$$Sx = \begin{cases} \frac{x}{2} & \text{if } x \in [0,1], \\ 3x & \text{if } x > 1, \end{cases} \quad \text{and} \quad Tx = \begin{cases} \frac{x}{3} & \text{if } x \in [0,1], \\ 2x & \text{if } x > 1. \end{cases}$$

Note that $f(X) \subset T(X)$ *and* $g(X) \subset S(X)$, $\{f, S\}$ *and* $\{g, T\}$ *are weakly compatible.*

Also, we define the mapping $\alpha : S(X) \cup T(X) \times S(X) \cup T(X) \to \mathbb{R}^+$ *by*

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in \left[0, \frac{1}{2}\right], \\ 0 & \text{otherwise.} \end{cases}$$

Now, let $x, y \in X$ *such that* $\alpha(Sx, Ty) \geq 1$. *Then* $Sx, Ty \in \left[0, \frac{1}{2}\right]$ *and this implies that* $x, y \in [0, 1]$. *By the definitions of* f, g *and* α, *we have* $fx, gy \in \left[0, \frac{1}{2}\right]$ *and* $gx, fy \in \left[0, \frac{1}{2}\right]$ *which implies that* $\alpha(fx, gy) \geq 1$ *and* $\alpha(gx, fy) \geq 1$.

In case of $\alpha(Tx, Sy) \geq 1$, *analogously to the above proof, one can easily obtain that* $\alpha(fx, gy) \geq 1$ *and* $\alpha(gx, fy) \geq 1$.

Then (f, g) *is* α_{ST}*-admissible. Moreover, the condition* $\alpha(Sx_0, fx_0) \geq 1$ *is satisfied with* $x_0 = 0$.

Let $\{x_n\}$ *be a sequence in* X *such that* $\alpha(Sx_n, Tx_{n+1}) \geq 1$ *for all* n *even. Then, by the definition of* α, *we get* $x_n \in [0, 1]$ *for all* n *even. Thus,* $x_j \in [0, 1]$ *for all* $j > n$ *odd, and so* $\alpha(Sx_n, Tx_j) \geq 1$.

Similarly, if $\{x_n\}$ *is any sequence in* X *such that* $\alpha(Sx_n, Tx_{n+1}) \geq 1$ *for all* n *even and,* Sx_n *and* Tx_{n+1} *converge to an* $x \in X$ *as* $n \to \infty$, *then by the definition of* α, *we have* $Sx_n \in \left[0, \frac{1}{2}\right]$ *and* $Tx_{n+1} \in \left[0, \frac{1}{2}\right]$ *for all* n *even and so* $x \in \left[0, \frac{1}{2}\right]$ *which implies that* $\alpha(Sx_n, x) \geq 1$ *and* $\alpha(x, Tx_{n+1}) \geq 1$.

Now, we prove that (f, g) *is a generalized* $(\alpha, \psi, \varphi)_{(S,T)}$*-rational contraction. Let* $\alpha(Sx, Ty) \geq 1$. *Then,* $x, y \in [0, 1]$, *and so*

$$\begin{aligned} \psi(d(fx, gy)) &= |fx - gy| = \left|\frac{x}{6} - \frac{y}{4}\right| \\ &\leq \frac{x}{6} = \frac{1}{2}|Sx - fx| \\ &\leq \frac{1}{2}M(x, y) = \varphi(M(x, y)). \end{aligned}$$

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Obviously, assumption (ii) *of Theorem 1 and condition* (H) *are satisfied. Consequently, by Theorems 1 and 4,* f, g, S *and* T *have a unique common fixed point which is* 0.

3. Fixed Point Results on Partially Ordered Metric Spaces

The existence of fixed points of nonlinear contraction mappings in metric spaces endowed with a partial ordering has been considered recently by Ran and Reurings [26] in order to obtain a solution of a matrix equation in 2004. Nieto and Lopez [24] extended the results in [26] by removing the continuity condition of the mapping. They applied their result to get a solution of a boundary value problem (see also [4, 13, 14] and references mentioned therein).

Let X be a non-empty set. If d is a complete metric on X and $\preceq$ is a partial order on the set X, then $(X, d, \preceq)$ is called complete partially ordered metric space. Let $(X, \preceq)$ be a partially ordered set and f, g, S and T be self-mappings on X. Then, (f, g) is called a (S, T)-nondecreasing mapping pair if $fx \preceq gy$ and $gx \preceq fy$ whenever $Sx \preceq Ty$ or $Tx \preceq Sy$ for all $x, y \in X$.

From Theorem 1, in the setting of complete partially ordered metric spaces, we obtain the following theorem.

Theorem 5. *Let* $(X, d, \preceq)$ *be a complete partially ordered metric space and let* f, g, S *and* T *be self-mappings on* X *such that* $f(X) \subset T(X)$, $g(X) \subset S(X)$. *Let* (f, g) *be a* (S, T)*-nondecreasing pair such that*

$$\psi\left(d\left(fx, gy\right)\right) \leq \varphi\left(M\left(x, y\right)\right), \tag{3.1}$$

for all $x, y \in X$ *such that* $Sx \preceq Ty$, *where* $\psi \in \psi$ *and* $\varphi \in \Phi$.

Assume that the following conditions are satisfied:

- (a) *there exists* $x_0 \in X$ *such that* $Sx_0 \preceq fx_0$;
- (b) $Sx_n \preceq Tx_{n+1}$ *for all* n *even implies that* $Sx_n \preceq Tx_j$ *for all* n *even and* $j > n$ *odd;*
- (c) $Sx_n \preceq Tx_{n+1}$ *for all* n *even and,* Sx_n *and* Tx_{n+1} *converge to an* $x \in X$ *as* $n \to \infty$ *implies that* $Sx_n \preceq x$ *and* $x \preceq Tx_{n+1}$ *for all* n *even.*

Then the pairs (f, S) *and* (g, T) *have point of coincidence in* X. *Moreover, if*

- (i) $\{f, S\}$ *and* $\{g, T\}$ *are weakly compatible,*
- (ii) $Su \preceq Tv$ *whenever* $u \in \mathcal{C}(f, S)$ *and* $v \in \mathcal{C}(g, T)$.

Then f, g, S *and* T *have common fixed point. Moreover, if* $Sx \preceq Ty$ *whenever* $x, y \in \mathcal{F}(f, g, S, T)$, *then* f, g, S *and* T *have a unique common fixed point.*

Proof. Define the function $\alpha : X \times X \to \mathbb{R}^+$ by

$$\alpha\left(x, y\right) = \begin{cases} 1 & \text{if } x \preceq y, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\alpha\left(Sx, Ty\right) \geq 1$. Then

$$Sx \preceq Ty. \tag{3.2}$$

From (3.1), we obtain that

$$\psi\left(d\left(fx, gy\right)\right) \leq \varphi\left(M\left(x, y\right)\right).$$

Also, since (f, g) is (S, T)-nondecreasing, by (3.2) we have $fx \preceq gy$ and $gx \preceq fy$, which gives us that $\alpha(fx, gy) \geq 1$ and $\alpha(gx, fy) \geq 1$. Then (f, g) is α_{ST}-admissible.

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On the other hand, one can easily show that the hypotheses $(a),(b),(c)$ and (ii) imply the conditions $(a),(b),(c)$ and (ii) of Theorem 1.

Now, let $x,y\in\mathcal{F}(f,g,S,T)$. Then, $Sx\preceq Ty$ and so $\alpha(Sx,Ty)\geq 1$. Therefore, the uniqueness of the common fixed point follows from condition (H). □

If we take $\varphi(t)=\psi(t)-\eta(t)$ in Theorem 5, we have the following result.

Corollary 4. *Let $(X,d,\preceq)$ be a complete partially ordered metric space and let f,g,S and T be self-mappings on X such that $f(X)\subset T(X)$, $g(X)\subset S(X)$. Let (f,g) be a (S,T)-nondecreasing pair such that*

$$\psi(d(fx,gy))\leq\psi(M(x,y))-\eta(M(x,y)), \tag{3.3}$$

for all $x,y\in X$ such that $Sx\preceq Ty$, where $\psi\in\psi$ and $\varphi\in\Phi$.

Assume that the following conditions are satisfied:

- (a) *there exists $x_0\in X$ such that $Sx_0\preceq fx_0$;*
- (b) *$Sx_n\preceq Tx_{n+1}$ for all n even implies that $Sx_n\preceq Tx_j$ for all n even and $j>n$ odd;*
- (c) *$Sx_n\preceq Tx_{n+1}$ for all n even and, Sx_n and Tx_{n+1} converge to an $x\in X$ as $n\to\infty$ implies that $Sx_n\preceq x$ and $x\preceq Tx_{n+1}$ for all n even.*

Then the pairs (f,S) and (g,T) have point of coincidence in X. Moreover, if

- (i) *$\{f,S\}$ and $\{g,T\}$ are weakly compatible,*
- (ii) *$Su\preceq Tv$ whenever $u\in\mathcal{C}(f,S)$ and $v\in\mathcal{C}(g,T)$.*

Then f,g,S and T have common fixed point. Moreover, if $Sx\preceq Ty$ whenever $x,y\in\mathcal{F}(f,g,S,T)$, then f,g,S and T have a unique common fixed point.

If we take $\psi(t)=t$ and $\eta(t)=(1-k)t$ in Corollary 4, we have the following result.

Corollary 5. *Let $(X,d,\preceq)$ be a complete partially ordered metric space and let f,g,S and T be self-mappings on X such that $f(X)\subset T(X)$, $g(X)\subset S(X)$. Let (f,g) be a (S,T)-nondecreasing pair such that*

$$d(fx,gy)\leq kM(x,y), \tag{3.4}$$

for all $x,y\in X$ such that $Sx\preceq Ty$, where $k\in[0,1)$.

Assume that the following conditions are satisfied:

- (a) *there exists $x_0\in X$ such that $Sx_0\preceq fx_0$;*
- (b) *$Sx_n\preceq Tx_{n+1}$ for all n even implies that $Sx_n\preceq Tx_j$ for all n even and $j>n$ odd;*
- (c) *$Sx_n\preceq Tx_{n+1}$ for all n even and, Sx_n and Tx_{n+1} converge to an $x\in X$ as $n\to\infty$ implies that $Sx_n\preceq x$ and $x\preceq Tx_{n+1}$ for all n even.*

Then the pairs (f,S) and (g,T) have point of coincidence in X. Moreover, if

- (i) *$\{f,S\}$ and $\{g,T\}$ are weakly compatible,*
- (ii) *$Su\preceq Tv$ whenever $u\in\mathcal{C}(f,S)$ and $v\in\mathcal{C}(g,T)$.*

Then f,g,S and T have common fixed point. Moreover, if $Sx\preceq Ty$ whenever $x,y\in\mathcal{F}(f,g,S,T)$, then f,g,S and T have a unique common fixed point.

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4. Some Results for Graphic Contractions

Consistent with Jachymski [18], let (X,d) be a metric space and let $\Delta := \{(x,x) : x \in X\}$ be a diagonal of the Cartesian product $X \times X$. Consider a graph G such that the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of its edges contains all loops; that is, $E(G) \supseteq \Delta$. We assume G has no parallel edges, so we can identify G with the pair $(V(G), E(G))$. Moreover, we may treat G as a weighted graph by assigning to each edge the distance between its vertices. If x and y are vertices in a graph G, then a path in G from x to y of length N ($N \in \mathbb{N}$) is a sequence $\{x_i\}_{i=0}^{N}$ of $N+1$ vertices such that $x_0 = x$, $x_N = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, \ldots, N$. A graph G is connected if there is a path between any two vertices. G is weakly connected if $\widetilde{G}$ is connected (see for more details [2, 9, 10]).

In this section, we give the existence and uniqueness of fixed point theorems on a metric space endowed with graph. Before presenting our results, we give the following notions and definitions.

Definition 7 ([18])**.** *Let (X,d) be a metric space endowed with a graph G and $T : X \to X$ be a mapping. One says that T preserves edges of G if*

$$\forall x, y \in X, \qquad (x,y) \in E(G) \Rightarrow (Tx, Ty) \in E(G). \tag{4.1}$$

Definition 8. *Let f, g, S and T be selfmaps of a metric space (X,d) endowed with a graph G. One says that (f,g) preserves edges of G with respect to (S,T) if for all $x, y \in X$,*

$$(Sx, Ty) \in E(G) \Rightarrow (fx, gy) \in E(G) \text{ and } (gx, fy) \in E(G). \tag{4.2}$$

Definition 9. *Let (X,d) be a metric space endowed with a graph G and f, g, S and T be selfmaps on X such that (f,g) preserves edges of G with respect to (S,T). We say that (f,g) is a generalized $(\alpha,\psi,\varphi)_{(S,T)}$-graphic contraction involving rational expressions if*

$$\psi(d(fx, gy)) \leq \varphi(M(x,y)), \tag{4.3}$$

for all $x, y \in X$ for which $(Sx, Ty) \in E(G)$, where $\psi \in \Psi$, $\varphi \in \Phi$ and

$$\begin{aligned} M(x,y) &= \max\Bigg(d(Sx,Ty), d(Sx,fx), d(Ty,gy), \frac{d(Sx,gy)+d(fx,Ty)}{2}, \\ &\qquad \frac{d(Ty,gy)[1+d(Sx,fx)]}{1+d(Sx,Ty)}, \frac{d(fx,Ty)[1+d(Sx,gy)]}{1+d(Sx,Ty)}\Bigg). \end{aligned}$$

Theorem 6. *Let f, g, S and T be selfmaps of a metric space (X,d) endowed with a graph G, and $f(X) \subset T(X)$, $g(X) \subset S(X)$ and (f,g) be a generalized $(\alpha,\psi,\varphi)_{(S,T)}$-graphic contraction involving rational expressions. Assume that the following conditions are satisfied:*

- *(a) there exists $x_0 \in X$ such that $(Sx_0, fx_0) \in E(G)$;*
- *(b) $(Sx_n, Tx_{n+1}) \in E(G)$ for all n even implies that $(Sx_n, Tx_j) \in E(G)$ for all n even and $j > n$ odd;*
- *(c) $(Sx_n, Tx_{n+1}) \in E(G)$ for all n even and, Sx_n and Tx_{n+1} converge to an $x \in X$ as $n \to \infty$ implies that $(Sx_n, x) \in E(G)$ and $(x, Tx_{n+1}) \in E(G)$ for all n even.*

Then the pairs (f,S) and (g,T) have a point of coincidence in X. Moreover, if

- *(i) $\{f,S\}$ and $\{g,T\}$ are weakly compatible,* (*ii*) $(Su, Tv) \in E(G)$ *whenever* $u \in \mathcal{C}(f, S)$ *and* $v \in \mathcal{C}(g, T)$.

Then f, g, S *and* T *have common fixed point. Moreover, if* $(Sx, Ty) \in E(G)$ *whenever* $x, y \in \mathcal{F}(f, g, S, T)$, *then* f, g, S *and* T *have a unique common fixed point.*

Proof. Define the function $\alpha : X \times X \to \mathbb{R}^+$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } (x, y) \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

Let $\alpha(Sx, Ty) \geq 1$. Then

$$(Sx, Ty) \in E(G). \tag{4.4}$$

From (4.3), we obtain that

$$\psi(d(fx, gy)) \leq \varphi(M(x, y)).$$

Also, since (f, g) preserves edges of G with respect to (S, T), by (4.4) we have $(fx, gy) \in E(G)$ and $(gx, fy) \in E(G)$, which gives us that $\alpha(fx, gy) \geq 1$ and $\alpha(gx, fy) \geq 1$. Then (f, g) is α_{ST}-admissible.

On the other hand, it is easy to see that the hypotheses $(a), (b), (c)$ and (ii) imply the conditions $(a), (b), (c)$ and (ii) of Theorem 1.

Now, let $x, y \in \mathcal{F}(f, g, S, T)$. Then, $(Sx, Ty) \in E(G)$ and so $\alpha(Sx, Ty) \geq 1$. Therefore, the uniqueness of the common fixed point follows from condition (H). □

If we take $\varphi(t) = \psi(t) - \phi(t)$ in Theorem 6, we have the following result.

Corollary 6. *Let* f, g, S *and* T *be selfmaps of a metric space* (X, d) *endowed with a graph* G, *and* $f(X) \subset T(X)$, $g(X) \subset S(X)$. *Assume that* (f, g) *preserves edges of* G *with respect to* (S, T) *such that*

$$\psi(d(fx, gy)) \leq \psi(M(x, y)) - \phi(M(x, y)), \tag{4.5}$$

for all $x, y \in X$ *for which* $(Sx, Ty) \in E(G)$, *where* $\psi \in \Psi$ *and* $\phi \in \Phi$.

Suppose also that the following conditions are satisfied:

(*a*) *there exists* $x_0 \in X$ *such that* $(Sx_0, fx_0) \in E(G)$;

(*b*) $(Sx_n, Tx_{n+1}) \in E(G)$ *for all* n *even implies that* $(Sx_n, Tx_j) \in E(G)$ *for all* n *even and* $j > n$ *odd;*

(*c*) $(Sx_n, Tx_{n+1}) \in E(G)$ *for all* n *even and,* Sx_n *and* Tx_{n+1} *converge to an* $x \in X$ *as* $n \to \infty$ *implies that* $(Sx_n, x) \in E(G)$ *and* $(x, Tx_{n+1}) \in E(G)$ *for all* n *even.*

Then the pairs (f, S) *and* (g, T) *have a point of coincidence in* X. *Moreover, if*

(*i*) $\{f, S\}$ *and* $\{g, T\}$ *are weakly compatible and,*

(*ii*) $(Su, Tv) \in E(G)$ *whenever* $u \in \mathcal{C}(f, S)$ *and* $v \in \mathcal{C}(g, T)$.

Then f, g, S *and* T *have common fixed point. Moreover, if* $(Sx, Ty) \in E(G)$ *whenever* $x, y \in \mathcal{F}(f, g, S, T)$, *then* f, g, S *and* T *have a unique common fixed point.*

If we take $\psi(t) = t$ and $\phi(t) = (1 - k)t$ in Corollary 6, we have the following result.

Corollary 7. *Let* f, g, S *and* T *be selfmaps of a metric space* (X, d) *endowed with a graph* G, *and* $f(X) \subset T(X)$, $g(X) \subset S(X)$. *Assume that* (f, g) *preserves edges of* G *with respect to* (S, T) *such that*

$$d(fx, gy) \leq kM(x, y), \tag{4.6}$$

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for all $x, y \in X$ for which $(Sx, Ty) \in E(G)$, where $\psi \in \Psi$ and $\phi \in \Phi$.

Suppose also that the following conditions are satisfied:

- (a) *there exists $x_0 \in X$ such that $(Sx_0, fx_0) \in E(G)$;*
- (b) *$(Sx_n, Tx_{n+1}) \in E(G)$ for all n even implies that $(Sx_n, Tx_j) \in E(G)$ for all n even and $j > n$ odd;*
- (c) *$(Sx_n, Tx_{n+1}) \in E(G)$ for all n even and, Sx_n and Tx_{n+1} converge to an $x \in X$ as $n \to \infty$ implies that $(Sx_n, x) \in E(G)$ and $(x, Tx_{n+1}) \in E(G)$ for all n even.*

Then the pairs (f, S) and (g, T) have a point of coincidence in X. Moreover, if

- (i) *$\{f, S\}$ and $\{g, T\}$ are weakly compatible and,*
- (ii) *$(Su, Tv) \in E(G)$ whenever $u \in \mathcal{C}(f, S)$ and $v \in \mathcal{C}(g, T)$.*

Then f, g, S and T have common fixed point. Moreover, if $(Sx, Ty) \in E(G)$ whenever $x, y \in \mathcal{F}(f, g, S, T)$, then f, g, S and T have a unique common fixed point.

5. An Application

Consider the following integral equations:

$$x(s) = \int_a^b H_1(s, r, x(r))\, dr, \tag{5.1}$$

and

$$x(s) = \int_a^b H_2(s, r, x(r))\, dr, \tag{5.2}$$

where $s, r \in I = [a, b]$, $H_1, H_2 : I \times I \times \mathbb{R} \to \mathbb{R}$ and $b > a \geq 0$.

In this section, we present an existence and uniqueness theorem for a common solution to (5.1) and (5.2) that belongs to $X := C(I, \mathbb{R})$ (the set of continuous functions defined on I) by using the obtained result in Corollary 3.

We consider the operators $f, g : X \to X$ given by for all $x \in X$

$$fx(s) = \int_a^b H_1(s, r, x(r))\, dr, \quad s \in I,$$

and

$$gx(s) = \int_a^b H_2(s, r, x(r))\, dr, \quad s \in I.$$

Then the existence of a common solution to (5.1) and (5.2) are equivalent to the existence of a common fixed point of f and g.

Meanwhile, X endowed with the metric d defined by

$$d(x, y) = \sup_{s \in I} |x(s) - y(s)|$$

for all $x, y \in X$, is a complete metric space.

Suppose that the following conditions hold.

- $(A1)$ $H_1, H_2 : I \times I \times \mathbb{R} \to \mathbb{R}$ are continuous;
- $(A2)$ there exist $\xi : X \times X \to \mathbb{R}$ such that if $\xi(x, y) \geq 0$ for all $x, y \in X$, then for every $s, r \in I$, we have

$$|H_1(s, r, x(r)) - H_2(s, r, y(r))|^2 \leq \gamma(s, r) \ln\left(1 + |x(r) - y(r)|^2\right)$$

where $\gamma : I \times I \to \mathbb{R}^+$ is a continuous function satisfying $\sup_{s\in I} \int_a^b \gamma(s,r) \leq 1/(b-a)$;

$(A3)$ for every $s \in I$ there exist $x_0 \in X$ such that $\xi(x_0(s), fx_0(s)) \geq 0$;

$(A4)$ for all $s \in I$ and $x, y \in X$,

$$\xi(x(s), y(s)) \geq 0 \ \Rightarrow \ \xi(fx(s), gy(s)) \geq 0 \text{ and } \xi(gx(s), fy(s)) \geq 0,$$

$(A5)$ $\xi(x_n(s), x_{n+1}(s)) \geq 0$ for all n and $s \in I$ implies that $\xi(x_n(s), x_j(s)) \geq 0$ for all $j > n$;

$(A6)$ $\xi(x_n(s), x_{n+1}(s)) \geq 0$ for all n and $s \in I$ and, $x_n \to x \in X$ as $n \to \infty$ implies that $\xi(x_n(s), x(s)) \geq 0$ for all n.

Theorem 7. *Assume that the conditions $(A1)-(A6)$ are satisfied. Then, integral equations (5.1) and (5.2) have a common solution in X.*

Proof. Let $x, y \in X$ such that $\xi(x,y) \geq 0$. Then, by $(A2)$, for all $s, r \in I$, we deduce

$$\begin{aligned}
|fx(s) - gy(s)|^2 &\leq \left(\int_a^b |H_1(s,r,x(r)) - H_2(s,r,y(r))|\, dr\right)^2 \\
&\leq \int_a^b 1^2 dr \int_a^b |H_1(s,r,x(r)) - H_2(s,r,y(r))|^2\, dr \\
&\leq (b-a) \int_a^b \gamma(s,r) \ln\left(1 + |x(r) - y(r)|^2\right) dr \\
&\leq (b-a) \int_a^b \gamma(s,r) \ln\left(1 + d(x,y)^2\right) dr \\
&= (b-a) \left(\int_a^b \gamma(s,r)\, dr\right) \ln\left(1 + d(x,y)^2\right) \\
&\leq \ln\left(1 + d(x,y)^2\right) \leq \ln\left(1 + M_{fg}(x,y)^2\right),
\end{aligned}$$

where

$$\begin{aligned}
M_{fg}(x,y) = \max\Bigg(& d(x(s), y(s)), d(x(s), fx(s)), d(y(s), gy(s)), \\
& \frac{d(x(s), gy(s)) + d(fx(s), y(s))}{2}, \\
& \frac{d(y(s), gy(s))\,[1 + d(x(s), fx(s))]}{1 + d(x(s), y(s))}, \\
& \frac{d(fx(s), y(s))\,[1 + d(x(s), gy(s))]}{1 + d(x(s), y(s))}\Bigg).
\end{aligned}$$

Therefore, we obtain

$$\left(\sup_{s\in I} |fx(s) - gy(s)|\right)^2 \leq \ln\left(1 + M_{fg}(x,y)^2\right).$$

Now, define $\alpha : X \times X \to \mathbb{R}^+$ by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } \xi(x,y) \geq 0 \text{ where } x, y \in X, \\ 0 & \text{otherwise.} \end{cases}$$

Also, define $\psi, \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\psi(t) = t^2$ and $\varphi(t) = \ln(1 + t^2)$. Therefore, using the last inequality, we have

$$\alpha(x, y)\, \psi(d(fx, gy)) \leq \varphi(M_{fg}(x, y)).$$

It easily shows that all the hypotheses of Corollary 3 are satisfied. Therefore f and g have a common fixed point, that is, integral equations (5.1) and (5.2) have a common solution. □

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