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Editor in Chief: George Anastassiou

Department of Mathematical Sciences,

University of Memphis, Memphis, TN 38152-3240, U.S.A

ganastss@memphis.edu

<http://www.msci.memphis.edu/~ganastss/jocaaa>

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Francesco Altomare

Dipartimento di Matematica
Universita' di Bari
Via E.Orabona, 4
70125 Bari, ITALY
Tel+39-080-5442690 office
+39-080-3944046 home
+39-080-5963612 Fax
altomare@dm.uniba.it
Approximation Theory, Functional
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Ravi P. Agarwal

Department of Mathematics
Texas A&M University - Kingsville
700 University Blvd.
Kingsville, TX 78363-8202
tel: 361-593-2600
Agarwal@tamuk.edu
Differential Equations, Difference
Equations, Inequalities

George A. Anastassiou

Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152, U.S.A
Tel. 901-678-3144
e-mail: ganastss@memphis.edu
Approximation Theory, Real
Analysis,
Wavelets, Neural Networks,
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J. Marshall Ash

Department of Mathematics
De Paul University
2219 North Kenmore Ave.
Chicago, IL 60614-3504
773-325-4216
e-mail: mash@math.depaul.edu
Real and Harmonic Analysis

Dumitru Baleanu

Department of Mathematics and
Computer Sciences,
Cankaya University, Faculty of Art
and Sciences,
06530 Balgat, Ankara,
Turkey, dumitru@cankaya.edu.tr

Fractional Differential Equations
Nonlinear Analysis, Fractional
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Carlo Bardaro

Dipartimento di Matematica e
Informatica
Universita di Perugia
Via Vanvitelli 1
06123 Perugia, ITALY
TEL+390755853822
+390755855034
FAX+390755855024
E-mail carlo.bardaro@unipg.it
Web site:
<http://www.unipg.it/~bardaro/>
Functional Analysis and
Approximation Theory, Signal
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Analysis.

Martin Bohner

Department of Mathematics and
Statistics, Missouri S&T
Rolla, MO 65409-0020, USA
bohner@mst.edu
web.mst.edu/~bohner
Difference equations, differential
equations, dynamic equations on
time scale, applications in
economics, finance, biology.

Jerry L. Bona

Department of Mathematics
The University of Illinois at
Chicago
851 S. Morgan St. CS 249
Chicago, IL 60601
e-mail: bona@math.uic.edu
Partial Differential Equations,
Fluid Dynamics

Luis A. Caffarelli

Department of Mathematics
The University of Texas at Austin
Austin, Texas 78712-1082
512-471-3160
e-mail: caffarel@math.utexas.edu
Partial Differential Equations

George Cybenko

Thayer School of Engineering

Dartmouth College
8000 Cummings Hall,
Hanover, NH 03755-8000
603-646-3843 (X 3546 Secr.)
e-mail:george.cybenko@dartmouth.edu
Approximation Theory and Neural
Networks

Sever S. Dragomir

School of Computer Science and
Mathematics, Victoria University,
PO Box 14428,
Melbourne City,
MC 8001, AUSTRALIA
Tel. +61 3 9688 4437
Fax +61 3 9688 4050
sever.dragomir@vu.edu.au
Inequalities, Functional Analysis,
Numerical Analysis, Approximations,
Information Theory, Stochastics.

Oktay Duman

TOBB University of Economics and
Technology,
Department of Mathematics, TR-
06530,
Ankara, Turkey,
oduman@etu.edu.tr
Classical Approximation Theory,
Summability Theory, Statistical
Convergence and its Applications

Saber N. Elaydi

Department Of Mathematics
Trinity University
715 Stadium Dr.
San Antonio, TX 78212-7200
210-736-8246
e-mail: selaydi@trinity.edu
Ordinary Differential Equations,
Difference Equations

Christodoulos A. Floudas

Department of Chemical Engineering
Princeton University
Princeton, NJ 08544-5263
609-258-4595(x4619 assistant)
e-mail: floudas@titan.princeton.edu
Optimization Theory&Applications,
Global Optimization

J .A. Goldstein

Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152
901-678-3130
jgoldste@memphis.edu

Partial Differential Equations,
Semigroups of Operators

H. H. Gonska

Department of Mathematics
University of Duisburg
Duisburg, D-47048
Germany
011-49-203-379-3542
e-mail: heiner.gonska@uni-due.de
Approximation Theory, Computer
Aided Geometric Design

John R. Graef

Department of Mathematics
University of Tennessee at
Chattanooga
Chattanooga, TN 37304 USA
John-Graef@utc.edu
Ordinary and functional
differential equations, difference
equations, impulsive systems,
differential inclusions, dynamic
equations on time scales, control
theory and their applications

Weimin Han

Department of Mathematics
University of Iowa
Iowa City, IA 52242-1419
319-335-0770
e-mail: whan@math.uiowa.edu
Numerical analysis, Finite element
method, Numerical PDE, Variational
inequalities, Computational
mechanics

Tian-Xiao He

Department of Mathematics and
Computer Science
P.O. Box 2900, Illinois Wesleyan
University
Bloomington, IL 61702-2900, USA
Tel (309)556-3089
Fax (309)556-3864
the@iwu.edu
Approximations, Wavelet,
Integration Theory, Numerical
Analysis, Analytic Combinatorics

Margareta Heilmann

Faculty of Mathematics and Natural
Sciences, University of Wuppertal
Gaußstraße 20
D-42119 Wuppertal, Germany,
heilmann@math.uni-wuppertal.de

Approximation Theory (Positive
Linear Operators)

Xing-Biao Hu

Institute of Computational
Mathematics
AMSS, Chinese Academy of Sciences
Beijing, 100190, CHINA
hxb@lsec.cc.ac.cn
Computational Mathematics

Jong Kyu Kim

Department of Mathematics
Kyungnam University
Masan Kyungnam, 631-701, Korea
Tel 82-(55)-249-2211
Fax 82-(55)-243-8609
jongkyuk@kyungnam.ac.kr
Nonlinear Functional Analysis,
Variational Inequalities, Nonlinear
Ergodic Theory, ODE, PDE,
Functional Equations.

Robert Kozma

Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152, USA
rkozma@memphis.edu
Neural Networks, Reproducing Kernel
Hilbert Spaces,
Neural Percolation Theory

Mustafa Kulenovic

Department of Mathematics
University of Rhode Island
Kingston, RI 02881, USA
kulenm@math.uri.edu
Differential and Difference
Equations

Irena Lasiecka

Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152
PDE, Control Theory, Functional
Analysis, lasiecka@memphis.edu

Burkhard Lenze

Fachbereich Informatik
Fachhochschule Dortmund
University of Applied Sciences
Postfach 105018
D-44047 Dortmund, Germany
e-mail: lenze@fh-dortmund.de
Real Networks, Fourier Analysis,
Approximation Theory

Hrushikesh N. Mhaskar

Department Of Mathematics
California State University
Los Angeles, CA 90032
626-914-7002
e-mail: hmhaska@gmail.com
Orthogonal Polynomials,
Approximation Theory, Splines,
Wavelets, Neural Networks

Ram N. Mohapatra

Department of Mathematics
University of Central Florida
Orlando, FL 32816-1364
tel.407-823-5080
ram.mohapatra@ucf.edu
Real and Complex Analysis,
Approximation Th., Fourier
Analysis, Fuzzy Sets and Systems

Gaston M. N'Guerekata

Department of Mathematics
Morgan State University
Baltimore, MD 21251, USA
tel: 1-443-885-4373
Fax 1-443-885-8216
Gaston.N'Guerekata@morgan.edu
nguerekata@aol.com
Nonlinear Evolution Equations,
Abstract Harmonic Analysis,
Fractional Differential Equations,
Almost Periodicity & Almost
Automorphy

M.Zuhair Nashed

Department Of Mathematics
University of Central Florida
PO Box 161364
Orlando, FL 32816-1364
e-mail: znashed@mail.ucf.edu
Inverse and Ill-Posed problems,
Numerical Functional Analysis,
Integral Equations, Optimization,
Signal Analysis

Mubenga N. Nkashama

Department OF Mathematics
University of Alabama at Birmingham
Birmingham, AL 35294-1170
205-934-2154
e-mail: nkashama@math.uab.edu
Ordinary Differential Equations,
Partial Differential Equations

Vassilis Papanicolaou

Department of Mathematics

National Technical University of Athens
Zografou campus, 157 80
Athens, Greece
tel:: +30(210) 772 1722
Fax +30(210) 772 1775
papanico@math.ntua.gr
Partial Differential Equations,
Probability

Choonkil Park

Department of Mathematics
Hanyang University
Seoul 133-791
S. Korea, baak@hanyang.ac.kr
Functional Equations

Svetlozar (Zari) Rachev,
Professor of Finance, College of
Business, and Director of
Quantitative Finance Program,
Department of Applied Mathematics &
Statistics
Stonybrook University
312 Harriman Hall, Stony Brook, NY
11794-3775
tel: +1-631-632-1998,
svetlozar.rachev@stonybrook.edu

Alexander G. Ramm

Mathematics Department
Kansas State University
Manhattan, KS 66506-2602
e-mail: ramm@math.ksu.edu
Inverse and Ill-posed Problems,
Scattering Theory, Operator Theory,
Theoretical Numerical Analysis,
Wave Propagation, Signal Processing
and Tomography

Tomasz Rychlik

Polish Academy of Sciences
Instytut Matematyczny PAN
00-956 Warszawa, skr. poczt. 21
ul. Śniadeckich 8
Poland
trychlik@impan.pl
Mathematical Statistics,
Probabilistic Inequalities

Boris Shekhtman

Department of Mathematics
University of South Florida
Tampa, FL 33620, USA
Tel 813-974-9710
shekhtma@usf.edu

Approximation Theory, Banach
spaces, Classical Analysis

T. E. Simos

Department of Computer
Science and Technology
Faculty of Sciences and Technology
University of Peloponnese
GR-221 00 Tripolis, Greece
Postal Address:
26 Menelaou St.
Anfithea - Paleon Faliron
GR-175 64 Athens, Greece
tsimos@mail.ariadne-t.gr
Numerical Analysis

H. M. Srivastava

Department of Mathematics and
Statistics
University of Victoria
Victoria, British Columbia V8W 3R4
Canada
tel.250-472-5313; office,250-477-
6960 home, fax 250-721-8962
harimsri@math.uvic.ca
Real and Complex Analysis,
Fractional Calculus and Appl.,
Integral Equations and Transforms,
Higher Transcendental Functions and
Appl., q-Series and q-Polynomials,
Analytic Number Th.

I. P. Stavroulakis

Department of Mathematics
University of Ioannina
451-10 Ioannina, Greece
ipstav@cc.uoi.gr
Differential Equations
Phone +3-065-109-8283

Manfred Tasche

Department of Mathematics
University of Rostock
D-18051 Rostock, Germany
manfred.tasche@mathematik.uni-
rostock.de
Numerical Fourier Analysis, Fourier
Analysis, Harmonic Analysis, Signal
Analysis, Spectral Methods,
Wavelets, Splines, Approximation
Theory

Roberto Triggiani

Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152
PDE, Control Theory, Functional

Analysis, rtrggani@memphis.edu

Juan J. Trujillo

University of La Laguna
Departamento de Analisis Matematico
C/Astr.Fco.Sanchez s/n
38271. LaLaguna. Tenerife.
SPAIN
Tel/Fax 34-922-318209
Juan.Trujillo@ull.es
Fractional: Differential Equations-
Operators-Fourier Transforms,
Special functions, Approximations,
and Applications

Ram Verma

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1200 Dallas Drive #824 Denton,
TX 76205, USA
Verma99@msn.com
Applied Nonlinear Analysis,
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Theory

Xiang Ming Yu

Department of Mathematical Sciences
Southwest Missouri State University
Springfield, MO 65804-0094
417-836-5931
xmy944f@missouristate.edu
Classical Approximation Theory,
Wavelets

Lotfi A. Zadeh

Professor in the Graduate School
and Director, Computer Initiative,
Soft Computing (BISC)
Computer Science Division
University of California at
Berkeley
Berkeley, CA 94720
Office: 510-642-4959
Sec: 510-642-8271
Home: 510-526-2569
FAX: 510-642-1712
zadeh@cs.berkeley.edu
Fuzzyness, Artificial Intelligence,
Natural language processing, Fuzzy
logic

Richard A. Zalik

Department of Mathematics
Auburn University
Auburn University, AL 36849-5310

USA.

Tel 334-844-6557 office
678-642-8703 home
Fax 334-844-6555
zalik@auburn.edu
Approximation Theory, Chebychev
Systems, Wavelet Theory

Ahmed I. Zayed

Department of Mathematical Sciences
DePaul University
2320 N. Kenmore Ave.
Chicago, IL 60614-3250
773-325-7808
e-mail: azayed@condor.depaul.edu
Shannon sampling theory, Harmonic
analysis and wavelets, Special
functions and orthogonal
polynomials, Integral transforms

Ding-Xuan Zhou

Department Of Mathematics
City University of Hong Kong
83 Tat Chee Avenue
Kowloon, Hong Kong
852-2788 9708, Fax: 852-2788 8561
e-mail: mazhou@cityu.edu.hk
Approximation Theory, Spline
functions, Wavelets

Xin-long Zhou

Fachbereich Mathematik, Fachgebiet
Informatik
Gerhard-Mercator-Universitat
Duisburg
Lotharstr.65, D-47048 Duisburg,
Germany
e-mail: Xzhou@informatik.uni-
duisburg.de
Fourier Analysis, Computer-Aided
Geometric Design, Computational
Complexity, Multivariate
Approximation Theory, Approximation
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University of Memphis
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The University of Memphis
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Fuzzy analytical hierarchy process based on canonical representation on fuzzy numbers

Yong Deng^{a,b,c,*}

^a*School of Electronics and Information, Northwestern Polytechnical University, University, Xi'an, 710072, China*

^b*Big Data Decision Institute, Jinan University, Tianhe, Guangzhou, 510632, China*

^c*Department of Civil Engineering and Environment Science, School of Engineering, Vanderbilt University, Nashville, TN, 37235, USA*

Abstract

Fuzzy analytical hierarchy process (FAHP) is widely used in multi-criteria decision making (MCDM) under uncertain environments. Many works have been proposed. However, the existing methods are complex and time consuming. What's more, the conflict management in AHP is still an open issue. To solve these issues, a novel and simple FAHP method is proposed based on the canonical representation of multiplication operation on fuzzy numbers in this paper. We adopt the main idea of classical AHP, that is the weight of each criterion can be determined by its relative ratio. The relative ratio can be easily determined in the proposed method. In addition, the average method is adopted to handle conflicts in AHP. An example on supplier selection is used to illustrate the efficiency of our proposed method.

Keywords: Analytical Hierarchical Process, fuzzy numbers, fuzzy AHP, canonical representation of fuzzy numbers, supplier selection.

1. Introduction

Analytical Hierarchy Process (AHP) is a powerful tool for handling both qualitative and quantitative multi-criteria factors in decision-making problems, developed by Saaty [1] in the 1970s. This method has been extensively

*Corresponding author: Yong Deng, Department 801, School of Electronics and Information, Northwestern Polytechnical University, Xian, Shaanxi, 710072, China. Email: ydeng@nwpu.edu.cn; prof.deng@hotmail.com

studied and refined since then. It provides a comprehensive and rational framework for structuring a decision problem, for representing and quantifying its elements, for relating those elements to overall goals, and for evaluating alternative solutions. With this method, a complicated problem can be converted to an ordered hierarchical structure. AHP method has been widely applied to multi-criteria decision making situations[2], such as: web sites selection[3], tools' evaluation[4], e-business [5], drugs selection[6], group decision [7, 8]and so on[9, 10, 11, 12].

Multi-Criteria analysis problems require the decision maker to make qualitative assessments regarding the performance of the decision alternatives with respect to each independent criterion and the relative importance of each independent criterion with respect to the overall objective of the problem [13, 14]. As a result, uncertain subjective data are present which make the decision making process complex. Many math tools are developed. For example, evidence theory is heavily studied since it can fuse different data which make it widely used in multi-criteria decision making [15, 16, 17]. Due to the flexibility to handle linguistic information [18], the fuzzy sets theory is also widely used in many uncertain decision makings [19, 20, 21, 22, 23]. As a result, the classical AHP is extended to fuzzy AHP (FAHP) [24] and is applied to many MCDM applications under uncertain environment, such as environmental assessment and management[25, 26, 27], supplier management[28], group decision making [29], fuzzy MCDM[30], fuzzy MADM [31], and so on [32].

Two key issues should be solved in the application of fuzzy AHP. One issue is that how to determine the weight of each criterion when the elements of comparison matrix are fuzzy numbers. Unlike the classical AHP, the eigenvector of fuzzy comparison matrix cannot be obtained directly. Hence, some other steps are inevitable to get the final weights in most existing fuzzy AHP methods[24, 33], which makes the FAHP more completed to some degrees.

The other key problem when applying the AHP is to avoid rank reversal[34]. Due to the different preference and subjective and objective factors in decision making, evidence connected from different sources are often conflicting[35, 36, 37, 38]. How to deal with conflict and dependence in AHP is still an open issue [39, 40, 41, 42]. In classical AHP, a well known coefficient, called as Consistency index(CI), is used to measure the conflicting degree in decision making. In some application systems, the AHP model should be adjusted when the CI is higher than a certain threshold value. The problem still exists in fuzzy AHP. Many methods have been proposed to handle this

problem[43, 44]. In order to construct decision matrices of pairwise comparisons based on additive transitivity, Herrera-Viedma et al. propose consistent fuzzy preference relations[45]. In [43], the distance function between two linguistic preference relations is defined, then a new CI is defined based on the distance function. In [44, 46], a method is proposed to construct fuzzy linguistic preference relations, called as fuzzy LinPreRa method. However, it should be pointed out that is difficult to give a corresponding CI in fuzzy AHP.

To handel these two issues mentioned above, we propose a novel and simple FAHP in this paper. On the one hand, we use the canonical representation of multiplication operation on fuzzy numbers, presented in [47], to obtain the weigh of each criterion in a straight and easy manner. On the other hand, we suggest to use average method to deal with conflicts in AHP decision making. The numerical example on supplier selection shows the efficiency of our proposed method. The paper is organized as follows. Section 2 begins with a brief introduction to the basic theory used in the proposed method,including AHP, fuzzy set theory and genetic algorithm. A typical fuzzy AHP is also introduced in this section. The proposed methodology is detailed in section 3. In section 4, our proposed method is applied to supplier selection. Section 5 concludes the paper.

2. Preliminaries

2.1. Analytical Hierarchy Process[1]

The first step of AHP is to establish a hierarchical structure of the problem. Then, in each hierarchical level, use a nominal scale to construct pairwise comparison judgement matrix.

Definition 2.1. *Assuming $(E_1, \dots, E_i, \dots, E_n)$ are n decision elements, the pairwise comparison judgement matrix is denoted as $M_{n \times n} = [m_{ij}]$, which satisfies:*

$$m_{ij} = \frac{1}{m_{ji}} \quad (1)$$

where each element m_{ij} represents the judgment concerning the relative importance of decision element E_i over E_j .

With the matrix constructed, the third step is to calculate the eigenvector of the matrix.

Definition 2.2. *Eigenvector of $n \times n$ pairwise comparison judgement matrix can be denoted as: $\vec{w} = (w_1, \dots, w_i, \dots, w_n)^T$, which is calculated as follows:*

$$A\vec{w} = \lambda_{\max}\vec{w}, \quad \lambda_{\max} \geq n \tag{2}$$

where λ_{\max} is the maximum eigenvalue in the eigenvector \vec{w} of matrix $M_{n \times n}$.

Before we transform the eigenvector into the weights of elements, the consistency of the matrix should be checked.

Definition 2.3. *Consistency index(CI)[1] is used to measure the inconsistency within each pairwise comparison judgement matrix, which is formulated as follows:*

$$CI = \frac{\lambda_{\max} - n}{n - 1} \tag{3}$$

Accordingly, the consistency ratio(CR) can be calculated by using the following equation:

$$CR = \frac{CI}{RI} \tag{4}$$

where RI is the random consistency index. The value of RI is related to the dimension of the matrix, which is listed in Table 1.

Table 1: The value of RI(random consistency index)

dimension	1	2	3	4	5	6	7	8	9	10
RI	0	0	0.52	0.89	1.12	1.26	1.36	1.41	1.46	1.49

If the result of CR is less than 0.1, the consistency of the pairwise comparison matrix M is acceptable. Moreover, the eigenvector of pairwise comparison judgement matrix can be normalized as final weights of decision elements. Otherwise, the consistency is not passed and the elements in the matrix should be revised.

2.2. Fuzzy sets

In 1965, the notion of fuzzy sets was firstly introduced by Zadeh[18], providing a natural way of dealing with problems in which the source of imprecision is the absence of sharply defined criteria of class membership[48].

A brief introduction of Fuzzy sets are given as follows.

Definition 2.4. A fuzzy set A is defined on a universe X may be given as:

$$A = \{ \langle x, \mu_A(x) \rangle \mid x \in X \}$$

where $\mu_A : X \rightarrow [0, 1]$ is the membership function A . The membership value $\mu_A(x)$ describes the degree of belongingness of $x \in X$ in A .

For a finite set $A = \{x_1, \dots, x_i, \dots, x_n\}$, the fuzzy set (A, m) is often denoted by $\{ \mu_A(x_1)/x_1, \dots, \mu_A(x_i)/x_i, \dots, \mu_A(x_n)/x_n \}$.

In real application, the domain experts may give their opinions by fuzzy numbers. For example, in a new product price estimation, one expert may give his opinion as: the lowest price is 2 dollars, the most possibility price of the product may be 3 dollars, the highest price of this product will not be in excess of 4 dollars. Hence, we can use a triangular fuzzy number (2,3,4) to represent the expert's opinion. The triangular fuzzy numbers can be defined as follows.

Definition 2.5. A triangular fuzzy number \tilde{A} can be defined by a triplet (a, b, c) , where the membership can be determined as follows

A triangular fuzzy number $\tilde{A} = (a, b, c)$ can be shown in Fig.(1).

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a \leq x \leq b \\ \frac{c-x}{c-b}, & b \leq x \leq c \\ 0, & x > c \end{cases} \quad (5)$$

In Fig2. $N1, N3, N5, N7$ and $N9$ are used to represent the pairwise comparison of decision variables from *Equal to Absolutely preferred*, and TFNs $N2, N4, N6$ and $N8$ represent the middle preference values between them.

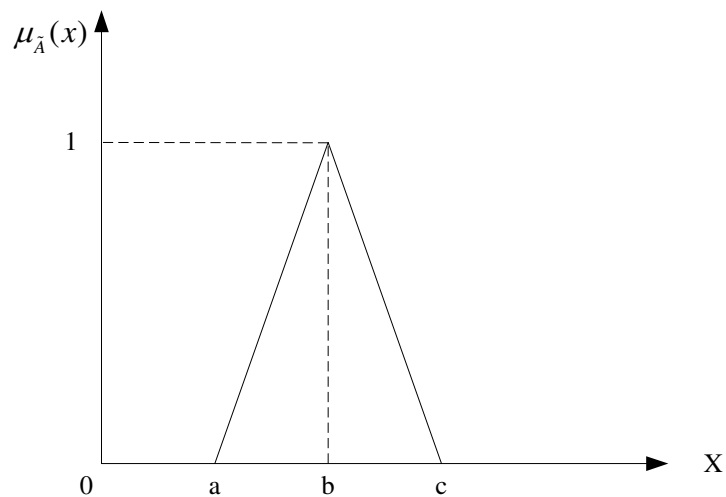


Figure 1: A triangular fuzzy number.

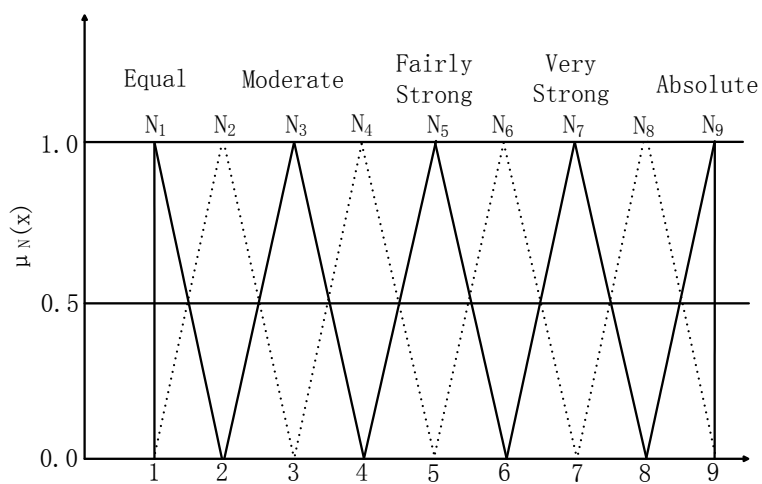


Figure 2: Nine fuzzy numbers

2.3. Canonical representation operation on fuzzy numbers

In this section, the canonical representation of operation on triangular fuzzy numbers which are based on the graded mean integration representation method [47], is used to obtain the weight of each criterion in a simple manner. The canonical representation operation on fuzzy numbers is applied to many decision makings [49, 50].

Definition 2.6. Given a triangular fuzzy number $\tilde{A} = (a_1, a_2, a_3)$, the graded mean integration representation of triangular fuzzy number \tilde{A} is defined as:

$$P(\tilde{A}) = \frac{1}{6}(a_1 + 4 \times a_2 + a_3) \tag{6}$$

Let $\tilde{A} = (a_1, a_2, a_3)$ and $\tilde{B} = (b_1, b_2, b_3)$ be two triangular fuzzy numbers. By applying Eq.(6), the graded mean integration representation of triangular fuzzy numbers \tilde{A} and \tilde{B} can be obtained, respectively, as follows:

$$\begin{aligned} P(\tilde{A}) &= \frac{1}{6}(a_1 + 4 \times a_2 + a_3) \\ P(\tilde{B}) &= \frac{1}{6}(b_1 + 4 \times b_2 + b_3) \end{aligned}$$

The representation of the addition operation \oplus on triangular fuzzy numbers \tilde{A} and \tilde{B} can be defined as :

$$P(\tilde{A} \oplus \tilde{B}) = P(\tilde{A}) + P(\tilde{B}) = \frac{1}{6}(a_1 + 4 \times a_2 + a_3) + \frac{1}{6}(b_1 + 4 \times b_2 + b_3) \tag{7}$$

The canonical representation of the multiplication operation on triangular fuzzy numbers \tilde{A} and \tilde{B} is defined as :

$$P(\tilde{A} \otimes \tilde{B}) = P(\tilde{A}) \times P(\tilde{B}) = \frac{1}{6}(a_1 + 4 \times a_2 + a_3) \times \frac{1}{6}(b_1 + 4 \times b_2 + b_3) \tag{8}$$

2.4. FAHP

In this section, we briefly introduce a typical FAHP method . For detailed information, please refer [51, 52].

In the first step,triangular fuzzy numbers are used for pair-wise comparisons. Then, by using extent analysis method the synthetic extent value Si of the pair-wise comparison is introduced and by applying the principle of the comparison of fuzzy numbers, the weight vectors with respect to each element under a certain criterion is calculated. The details of the methodology are presented in the following steps:

Let $X = \{x_1, x_2, \dots, x_n\}$ be an object set, and $U = \{u_1, u_2, \dots, u_m\}$ be a goal set. According to the method of Changs extent analysis, each object is taken and an extent analysis for each goal, g_i , is performed. Therefore, m extent analysis values for each object can be obtained, with the following signs:

$M_{g_i}^1, M_{g_i}^2, \dots, M_{g_i}^m, i = 1, 2, \dots, n$, where all the $M_{g_i}^j (j = 1, 2, \dots, m)$ are TFN's.

Step 1: The value of fuzzy synthetic extent with respect to the i th object is defined as

$$S_i = \sum_{j=1}^m M_{g_i}^j \otimes \left\{ \sum_{i=1}^n \sum_{j=1}^m M_{g_i}^j \right\}^{-1} \tag{9}$$

In order to obtain $\sum_{j=1}^m M_{g_i}^j$, perform the fuzzy addition operation of m extent analysis values for a particular matrix such that

$$\sum_{j=1}^m M_{g_i}^j = \left(\sum_{j=1}^m l_j, \sum_{j=1}^m m_j, \sum_{j=1}^m u_j \right) \tag{10}$$

To obtain $\left\{ \sum_{i=1}^n \sum_{j=1}^m M_{g_i}^j \right\}^{-1}$, perform the fuzzy addition operation of $M_{g_i}^j (j = 1, 2, \dots, m)$ values such that

$$\sum_{i=1}^n \sum_{j=1}^m M_{g_i}^j = \left(\sum_{i=1}^n l_i, \sum_{i=1}^n m_i, \sum_{i=1}^n u_i \right) \tag{11}$$

and then compute the inverse of the vector.

Step 2: The degree of possibility of $M_2 = (l_2, m_2, u_2) \geq M_1 = (l_1, m_1, u_1)$ is expressed as:

$$\begin{aligned} & V(M_2 \geq M_1) \\ &= hgt(M_1 \geq M_2) \\ &= \begin{cases} 1, & \text{if } m_2 \geq m_1 \\ \frac{(l_1 - u_2)}{((m_2 - u_2) - (m_1 - l_1))} & \text{otherwise} \\ 0, & \text{if } l_1 \geq u_2 \end{cases} \end{aligned} \tag{12}$$

To compare M_1 and M_2 both $V(M_2 \geq M_1)$ and $V(M_1 \geq M_2)$ are required.

Step 3: The degree of possibility for a convex fuzzy number to be greater than k convex fuzzy numbers $M_i(i = 1, 2, \dots, k)$ can be defined as:

$$V(M \geq M_1, M_2, \dots, M_k) = V[(M \geq M_1) \text{ and } (M \geq M_2) \text{ and } \dots \text{ and}$$

$$(M \geq M_k)] = \min V(M \geq M_i), i = 1, 2, \dots, k \tag{13}$$

Let $d'(A_i) = \min V(S_i \geq S_k)$, for $k = 1, 2, \dots, n; k \neq i$. Then the weight vector is given by:

$$W' = (d'(A_1), d'(A_2), \dots, d'(A_n))^T \tag{14}$$

Step 4: The weight vector obtained in step 3 is normalized to get the normalized weights.

3. The proposed methodology

One of the most key issue in fuzzy AHP is how to determine the weights given the fuzzy pairwise comparison judgement matrix. For example, given the linguistic data in Table 2, how can we get the weight of each criterion? In the following of this section, we solve the problem step by step.

Table 2: The Fuzzy evaluation of criteria with respect to the overall objective

	C1	C2	C3	C4	C5	W_C
C1	(1,1,1)	(3/2,2,5/2)	(3/2,2,5/2)	(5/2,3,7/2)	(5/2,3,7/2)	0.3283
C2	(2/5,1/2,2/3)	(1,1,1)	(3/2,2,5/2)	(5/2,3,7/2)	(5/2,3,7/2)	0.2839
C3	(2/5,1/2,2/3)	(2/5,1/2,2/3)	(1,1,1)	(3/2,2,5/2)	(3/2,2,5/2)	0.1798
C4	(2/7,1/3,2/5)	(2/7,1/3,2/5)	(2/5,1/2,2/3)	(1,1,1)	(3/2,2,5/2)	0.1262
C5	(2/7,1/3,2/5)	(2/7,1/3,2/5)	(2/5,1/2,2/3)	(2/5,1/2,2/3)	(1,1,1)	0.0818

3.1. Transformation of fuzzy comparison matrix

Let's consider the element in the comparison matrix classical AHP. The rating in the matrix means the relative importance of the criterion. For example, suppose only two criterion in a comparison matrix, listed as follows.

Example 3.1. *The comparison matrix is given as follows*

$$\begin{matrix} & C_1 & C_2 \\ C_1 & \begin{bmatrix} 1 & 3 \end{bmatrix} \\ C_2 & \begin{bmatrix} 1/3 & 1 \end{bmatrix} \end{matrix}$$

From the matrix, the element $C_{12} = 3$ means that weight of the second criterion C_2 is **three times** of that of the first criterion C_1 . In addition, the eigenvector of comparison matrix can be easily obtained as follows:

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix}$$

Two important points should be noticed: **First point**, the sum of the eigenvector of comparison matrix should be **ONE**. For example, $w_1 + w_2 = 0.75 + 0.25 = 1$. **Second point**, the ratio of the weight should be coincide with the corresponding element in comparison matrix. In Example 3.1, we can get $w_1/w_2 = 0.75/0.25 = 3 = C_{12}$.

This idea of AHP can be easily adopted in fuzzy AHP. For example, in the Table 2, the element $C_{12} = (3/2, 2, 5/2)$. According to the above analysis, we understand that the weight of the second criterion C_2 is **(3/2,2,5/2) times** of that of the first criterion C_1 (**Notice**: for the sake of simplicity, we suppose that $(3/2, 2, 5/2)$ is not a linguistic variable N_2 shown in Fig.2, but a simple fuzzy number to model the fuzzy variable "ABOUT 2"). The only difference between this case with Example 3.1 is that one is a crisp number 3 while the other is a fuzzy number $(3/2, 2, 5/2)$. How to represent the weight of the second criterion C_2 is **(3/2,2,5/2) times** of that of the first criterion C_1 in the canonical representation of multiplication operation on fuzzy numbers? According to the Eq.(8), we obtain the follow result.

$$\begin{aligned} & P(\tilde{A} \otimes \tilde{B}) \\ &= (1, 1, 1) \otimes (3/2, 2, 5/2) \\ &= \frac{1}{6}(1 + 4 \times 1 + 1) \times \frac{1}{6}(3/2 + 4 \times 2 + 5/2) \\ &= 1 \times 2 \\ &= 2 \end{aligned} \tag{15}$$

The Eq.(15) means that the the weight of the second criterion C_2 is **(3/2,2,5/2) times** of that of the first criterion C_1 could also be stated as "the weight of the second criterion C_2 is **2 times** of that of the first criterion C_1 under the canonical representation of multiplication operation on fuzzy numbers". The other element of the canonical representation of multiplication operation on fuzzy numbers can also be determined and shown in Table3.

Table 3: Evaluation of criteria with respect to the overall objective based on canonical representation of multiplication operation

	C1	C2	C3	C4	C5
C1	1	2	2	3	3
C2	46/90	1	2	3	3
C3	46/90	46/90	1	2	2
C4	212/630	212/630	46/90	1	2
C5	212/630	212/630	46/90	46/90	1

We call the matrix in Table3 the **comparison matrix with canonical representation of multiplication operation (CMCRMO)**

Let's us see the first row of Table 3. If we suppose that the relative weight of the first criterion is 1, then we get that: 1)both the the relative weight of the second and the third criterion is 2; 2)both the the relative weight of the fourth and the fifth criterion is 3. Then, a straight way to obtain the corresponding weight is with the simple normalization of these relative weights. The result can be shown as follows.

$$\begin{aligned}
 w_{C_1}^1 &= \frac{1}{1+2+2+3+3} = \frac{1}{11} \\
 w_{C_1}^2 &= \frac{2}{1+2+2+3+3} = \frac{2}{11} \\
 w_{C_1}^3 &= \frac{2}{1+2+2+3+3} = \frac{2}{11} \\
 w_{C_1}^4 &= \frac{3}{1+2+2+3+3} = \frac{3}{11} \\
 w_{C_1}^5 &= \frac{3}{1+2+2+3+3} = \frac{3}{11}
 \end{aligned}
 \tag{16}$$

In Eq.(16), the subscript C_1 means that the weight is obtained according to the criterion C_1 . The weight in Eq.(16) is not the final weight of each criterion since that there exists conflict in this situation, also called rank reversal[34, 43, 44, 46, 45]. This problem will be handled in the following part.

3.2. Conflict management with average method

It should also be mentioned that the **second point**, namely "the ratio of the weight should be coincide with the corresponding element in comparison matrix" can be satisfied on some ideal situations. However, the preference order will not be always keep coincided in the whole AHP process. In real application, the comparison matrix given by experts may not strictly obey the preference order as shown in Example3.2.

Example 3.2. The comparison matrix is given as follows

$$\begin{array}{c}
 C_1 \\
 C_2 \\
 C_3 \\
 C_4
 \end{array}
 \begin{bmatrix}
 C_1 & C_2 & C_3 & C_4 \\
 1 & 3 & 5 & 7 \\
 1/3 & 1 & 1/3 & 3 \\
 1/5 & 3 & 1 & 2 \\
 1/7 & 1/3 & 1/2 & 1
 \end{bmatrix}$$

From the first row of above comparison matrix, we can see that the importance ranking corresponding to C_1 is

$$C_1 < C_2 < C_3 < C_4$$

. However, from the second row of above comparison matrix, we can see that the importance ranking corresponding to C_2 is

$$C_1 < C_3 < C_2 < C_4$$

The consistency index index defined in Definition2.3 show the conflict in preference. In classical AHP, the CI is used to determine how consistence of the comparison matrix. If the value of CI is higher than a threshold, then some adjustments to deal with rank reversal should be made. Though many methods have been proposed on this filed, it is still an open issue. In decision making with fuzzy AHP, it is also inevitable. For example, see the first line of the Table 3, we get the following preference ranking order.

Table 4: The Fuzzy evaluation of criteria with respect to the overall objective

Preference ranking order	
C1	$C1 < C2 = C3 < C4 = C5$
C2	$C1 < C2 < C3 < C4 = C5$
C3	$C1 = C2 < C3 < C4 = C5$
C4	$C1 = C2 < C3 < C4 < C5$
C5	$C1 = C2 < C3 = C4 < C5$

As can be seen from Table4, for C_1 , the weight of C_2 is equal to C_3 . However, for C_2, C_3, C_4 and C_5 , the weight of C_2 is less than C_3 . There are many other conflicts in the ranking order. In this paper, we use average to decrease the conflict in the preference order. We average the weights of all criteria to get the final weight of each criterion. That is, if we get the the comparison matrix with canonical representation of multiplication operation (**CMCRMO**) shown in Table3, we can obtain the final weight of each criterion with the normalization of average weight of each criterion.

Example 3.3. Suppose we get the comparison matrix with canonical representation of multiplication operation (**CMCRMO**) shown in Table3, we can get the average weight of the five criteria, respectively as follows

$$\begin{aligned}
 w_{C_1}^{av} &= \frac{1}{5} \sum_i^5 w_{C_5}^{CR} = \frac{1}{5} (3 + 3 + 2 + 2 + 1) \\
 w_{C_2}^{av} &= \frac{1}{5} \sum_i^5 w_{C_4}^{CR} = \frac{1}{5} (3 + 3 + 2 + 1 + \frac{46}{90}) \\
 w_{C_3}^{av} &= \frac{1}{5} \sum_i^5 w_{C_3}^{CR} = \frac{1}{5} (2 + 2 + 1 + \frac{46}{90} + \frac{46}{90}) \\
 w_{C_4}^{av} &= \frac{1}{5} \sum_i^5 w_{C_2}^{CR} = \frac{1}{5} (2 + 1 + \frac{46}{90} + \frac{212}{630} + \frac{212}{630}) \\
 w_{C_5}^{av} &= \frac{1}{5} \sum_i^5 w_{C_1}^{CR} = \frac{1}{5} (1 + \frac{46}{90} + \frac{46}{90} + \frac{212}{630} + \frac{212}{630})
 \end{aligned}
 \tag{17}$$

Here, $w_{C_i}^{av}$ means the average weight of the i th's criterion, the superscript *av* denotes average. $w_{C_i}^{CR}$ means the canonical representation of multiplication operation of the i th's criterion, the superscript *CR* denotes canonical

representation. The final weight of the i th's criterion, $w_{C_i}^f$, can be obtained with the normalization of average weight of each criterion $w_{C_i}^{av}$ and listed as follows

$$\begin{aligned}
 w_{C_1}^f &= \frac{w_{C_1}^{av}}{w_{C_1}^{av} + w_{C_2}^{av} + w_{C_3}^{av} + w_{C_4}^{av} + w_{C_5}^{av}} = \frac{11}{\frac{1698}{630} + \frac{2636}{630} + \frac{3794}{630} + \frac{5992}{630} + 11} = 0.3283 \\
 w_{C_2}^f &= \frac{w_{C_2}^{av}}{w_{C_1}^{av} + w_{C_2}^{av} + w_{C_3}^{av} + w_{C_4}^{av} + w_{C_5}^{av}} = \frac{\frac{5992}{630}}{\frac{1698}{630} + \frac{2636}{630} + \frac{3794}{630} + \frac{5992}{630} + 11} = 0.2839 \\
 w_{C_3}^f &= \frac{w_{C_3}^{av}}{w_{C_1}^{av} + w_{C_2}^{av} + w_{C_3}^{av} + w_{C_4}^{av} + w_{C_5}^{av}} = \frac{\frac{3794}{630}}{\frac{1698}{630} + \frac{2636}{630} + \frac{3794}{630} + \frac{5992}{630} + 11} = 0.1798 \quad (18) \\
 w_{C_4}^f &= \frac{w_{C_4}^{av}}{w_{C_1}^{av} + w_{C_2}^{av} + w_{C_3}^{av} + w_{C_4}^{av} + w_{C_5}^{av}} = \frac{\frac{2636}{630}}{\frac{1698}{630} + \frac{2636}{630} + \frac{3794}{630} + \frac{5992}{630} + 11} = 0.1262 \\
 w_{C_5}^f &= \frac{w_{C_5}^{av}}{w_{C_1}^{av} + w_{C_2}^{av} + w_{C_3}^{av} + w_{C_4}^{av} + w_{C_5}^{av}} = \frac{\frac{1898}{630}}{\frac{1698}{630} + \frac{2636}{630} + \frac{3794}{630} + \frac{5992}{630} + 11} = 0.0818
 \end{aligned}$$

Note that to detail our proposed method in a easily understood way, we suppose that the fuzzy number $C_{12} = (3/2, 2, 5/2)$ means that the the weight of the second criterion C_2 is $(3/2, 2, 5/2)$ times of that of the first criterion C_1 .

However, according to the Figure2, the case is verse, where $C_{12} = (3/2, 2, 5/2)$ means that the the weight of the second criterion C_1 is $(3/2, 2, 5/2)$ times of that of the first criterion C_2 . As a result, if we use the linguistic variables shown in Figure2, the final weight of each criterion are shown in the right row in Table2.

3.3. The proposed fuzzy AHP algorithm

Here we detail the proposed fuzzy AHP algorithm to determine weight vector under uncertain environment step by step.

Step1: Construct the analytical hierarchy structure by domain experts. In this step, the experts will determine the objective of decision making, the relative criteria. In addition, the rating of the comparison matrix, modelled by fuzzy numbers can be given by experts through linguistic variables(for example, shown in Figure2), listed in Table2.

Step2: For each criterion, using the canonical representation of multiplication operation on fuzzy numbers to obtain the comparison matrix with

canonical representation of multiplication operation (**CMCRMO**), shown in Table3.

Step3: Determine the average weight of the i th's criterion $w_{C_i}^{av}$, respectively by Eq.19.

$$w_{C_i}^{av} = \frac{1}{N} \sum_i^N w_{C_i}^{CR} \tag{19}$$

where $w_{C_i}^{CR}$ means the canonical representation weight of multiplication operation of the i th's criterion, the superscript CR denotes canonical representation. In this average process, the conflict in preference is handled to achieve a consensus preference.

Step4: Determine the final weight of the i th's criterion, $w_{C_i}^f$, with the normalization of average weight of the i th's criterion $w_{C_i}^{av}$, respectively by Eq.20.

$$w_{C_i}^f = \frac{w_{C_i}^{av}}{\sum_i^N w_{C_i}^{av}} \tag{20}$$

4. Numerical Example

Decision making is widely used in supplier management and selection [51, 53, 54, 55, 56, 57, 58]. In this section, a numerical example originated from [51] is presented to illustrate the procedure of the proposed model.

Owing to the large number of factors affecting the supplier selection decision, an orderly sequence of steps should be required to tackle it. The problem taken here has four level of hierarchy, and the different decision criteria, attributes and the decision alternatives, will be further discussed. The main objective here is the selection of best global supplier for a manufacturing firm. Application of common criteria to all suppliers makes objective comparisons possible. The criteria which are considered here in selection of the global supplier are:

- (C1)Overall cost of the product
- (C2)Quality of the product
- (C3)Service performance of supplier
- (C4)Supplier profile
- (C5)Risk factor

The AHP model of supplier selection can be constructed as shown in Fig 3

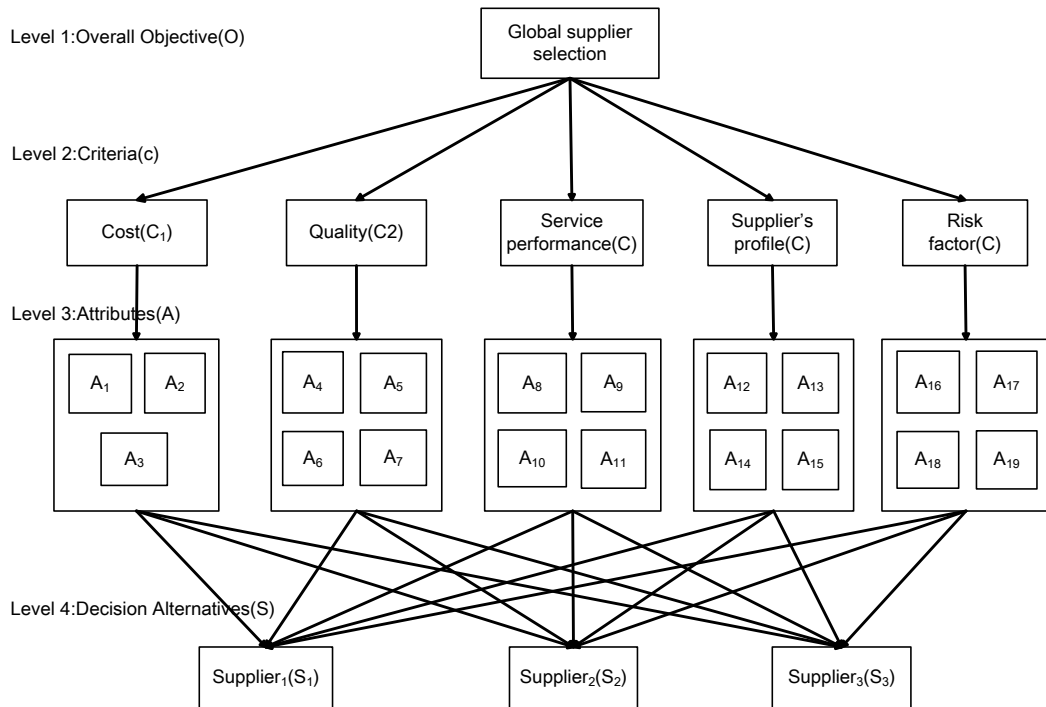


Figure 3: Hierarchy for the global supplier selection.

As been seen from Fig 3., the overall cost of the product (C1) has three factors (attributes):

- (A1) Product price ,
- (A2) Freight cost
- (A3) Tariff and custom duties .

The quality of the product (C2) has four factors:

- (A4) Rejection rate of the product ,
- (A5) Increased lead time ,
- (A6) Quality assessment
- (A7) Remedy for quality problems.

The service performance (C3) has four attributes:

- (A8) Delivery schedule ,

(A9) Technological and R&D support ,

(A10) Response to changes

(A11) Ease of communication .

The suppliers profile (C4) has four attributes:

(A12) Financial status ,

(A13) Customer base ,

(A14) Performance history

(A15) Production facility and capacity.

The Risk factor (C5) has four attributes:

(A16) Geographical location ,

(A17) Political stability ,

(A18) Economy

(A19) Terrorism.

Refer [51] for more detailed information about the attributes mentioned above.

After the construction of the decision hierarchy of supplier selection, the fuzzy evaluation matrix of the criteria is constructed by the pairwise comparison of the different criterion relevant to the overall objective using triangular fuzzy numbers, which is shown in Table 2.

The fuzzy evaluation of criteria with respect to the overall objective can be listed in Table 2. The final weights of each criteria can be determined by the GA method. The detailed calculation process is given in Section 3. The results are listed in right side of Table 2.

In a similar way, the The fuzzy evaluation of the attributes with respect to criterion C1 to C6 can be given by domain experts and there corresponding results based on GA are listed in Table 5 to Table 9, respectively.

For the criterion C1, the summary combination of priority weights can be listed in Table 10. Also, the others summary combination of priority weights of C2 to C5 are shown in Table 11 to Table 14.

The Fuzzy evaluation of criteria with respect to the overall objective can be shown in 15. As can be seen from Table 15 and Figure 4, the best supplier is S1, which is the same to the works in [51] using the commonly used fuzzy AHP method mentioned in Section 2.4.

5. Conclusions

In this paper, a novel and simple fuzzy AHP is proposed to handle M-CDM. In our new method, the weight of each criterion can be determined by

Table 5: The fuzzy evaluation of the attributes with respect to criterion C1

C1	A1	A2	A3	W_{C1}
A1	(1, 1, 1)	(3/2, 2, 5/2)	(3/2, 2, 5/2)	0.4747
A2	(2/5, 1/2, 2/3)	(1, 1, 1)	(3/2, 2, 5/2)	0.3333
A3	(2/5, 1/2, 2/3)	(2/5, 1/2, 2/3)	(1, 1, 1)	0.1920

Table 6: The fuzzy evaluation of the attributes with respect to criterion C2

C2	A4	A5	A6	A7	W_{C2}
A4	(1, 1, 1)	(3/2, 2, 5/2)	(2/3, 1, 3/2)	(5/2, 3, 7/2)	0.3703
A5	(2/5, 1/2, 2/3)	(1, 1, 1)	(2/3, 1, 3/2)	(3/2, 2, 5/2)	0.2391
A6	(2/3, 1, 3/2)	(2/3, 1, 3/2)	(1, 1, 1)	(3/2, 2, 5/2)	0.2663
A7	(2/7, 1/3, 2/5)	(2/5, 1/2, 2/3)	(2/5, 1/2, 2/3)	(1, 1, 1)	0.1243

Table 7: The fuzzy evaluation of the attributes with respect to criterion C3

C3	A8	A9	A10	A11	W_{C3}
A8	(1, 1, 1)	(3/2, 2, 5/2)	(5/2, 3, 7/2)	(7/2, 4, 9/2)	0.4264
A9	(2/5, 1/2, 2/3)	(1, 1, 1)	(5/2, 3, 7/2)	(5/2, 3, 7/2)	0.3274
A10	(2/7, 1/3, 2/5)	(2/7, 1/3, 2/5)	(1, 1, 1)	(3/2, 2, 5/2)	0.1566
A11	(2/9, 1/4, 2/7)	(2/7, 1/3, 2/5)	(2/5, 1/2, 2/3)	(1, 1, 1)	0.0895

Table 8: The fuzzy evaluation of the attributes with respect to criterion C4

C4	A12	A13	A14	A15	W_{C4}
A12	(1, 1, 1)	(3/2, 2, 5/2)	(3/2, 2, 5/2)	(7/2, 4, 9/2)	0.4880
A13	(2/5, 1/2, 2/3)	(1, 1, 1)	(2/5, 1/2, 2/3)	(3/2, 2, 5/2)	0.2030
A14	(2/5, 1/2, 2/3)	(2/7, 1/3, 2/5)	(1, 1, 1)	(3/2, 2, 5/2)	0.1942
A15	(2/9, 1/4, 2/7)	(2/5, 1/2, 2/3)	(2/5, 1/2, 2/3)	(1, 1, 1)	0.1148

Table 9: The fuzzy evaluation of the attributes with respect to criterion C5

C5	A16	A17	A18	A19	W_{C5}
A16	(1, 1, 1)	(2/3,1,3/2)	(2/3, 1, 3/2)	(2/3, 1, 3/2)	0.2331
A17	(2/3, 1, 3/2)	(1, 1, 1)	(3/2, 2, 5/2)	(3/2, 2, 5/2)	0.3438
A18	(2/3, 1, 3/2)	(2/5, 1/2, 2/3)	(1, 1, 1)	(3/2, 2, 5/2)	0.2741
A19	(2/5, 1/2, 2/3)	(2/5, 1/2, 2/3)	(2/5, 1/2, 2/3)	(1, 1, 1)	0.1489

Table 10: Summary combination of priority weights: attributes of criterion C1

	A1	A2	A3	Alternative priority weight
Weight	0.4747	0.3333	0.1920	
Alternatives				
S1	0.71	0.44	0.69	0.6217
S2	0.13	0.36	0.08	0.1920
S3	0.16	0.20	0.23	0.1862

Table 11: Summary combination of priority weights: attributes of criterion C2

Weight	A4	A5	A6	A7	Alternative priority weight
	0.3703	0.2391	0.2663	0.1243	
Alternatives					
S1	0.51	0.51	0.69	0.87	0.6027
S2	0.23	0.23	0.08	0.00	0.1615
S3	0.26	0.26	0.23	0.13	0.2359

Table 12: Summary combination of priority weights: attributes of criterion C3

Weight	A8	A9	A10	A11	Alternative priority weight
	0.4264	0.3274	0.1566	0.0895	
Alternatives					
S1	0.27	0.69	0.05	0.49	0.3927
S2	0.18	0.08	0.64	0.32	0.2318
S3	0.55	0.23	0.31	0.19	0.3754

Table 13: Summary combination of priority weights: attributes of criterion C4

Weight	A11	A12	A13	A14	Alternative priority weight
	0.4880	0.2030	0.1942	0.1148	
Alternatives					
S1	0.83	0.45	0.69	0.33	0.6683
S2	0.17	0.45	0.08	0.33	0.2277
S3	0.00	0.10	0.23	0.34	0.1040

Table 14: Summary combination of priority weights: attributes of criterion C5

Weight	A16	A17	A18	A19	Alternative priority weight
Alternatives					
S1	0.72	0.49	0.83	0.27	0.6040
S2	0.00	0.32	0.17	0.18	0.1834
S3	0.28	0.19	0.00	0.55	0.2125

Table 15: Summary combination of priority weights: main criteria of the overall objective

Weight	C1	C2	C3	C4	C5	Alternative priority weight
Alternatives						
S1	0.6217	0.6027	0.3927	0.6683	0.6040	0.5815
S2	0.1920	0.1615	0.2318	0.2277	0.1834	0.1938
S3	0.1862	0.2359	0.3754	0.1040	0.2125	0.2246

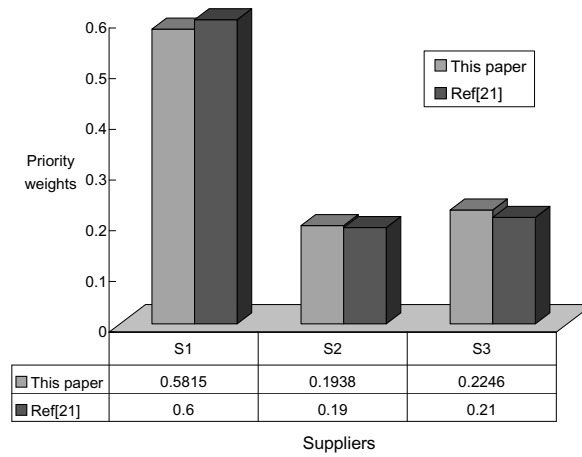


Figure 4: Comparison of proposed method with the previous work [21].

the the canonical representation of multiplication operation on fuzzy numbers. Instead of obtaining the eigenvector of the fuzzy comparison matrix, we get the weight simply by the ratio of each criterion. In addition, we get the final weight of each criterion by average method, which can deal with conflicts in an efficient manner. The proposed method is applied to supplier management under linguistic environment. The results show the efficiency of the proposed method. The method can be easily used in other fuzzy decision making problems.

disclosure

The author declares that there is no conflict of interests regarding the publication of this paper.

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A QUADRATURE RULE FOR THE FINITE HILBERT TRANSFORM VIA SIMPSON TYPE INEQUALITIES AND APPLICATIONS

SHUNFENG WANG, NA LU AND XINGYUE GAO

ABSTRACT. In this paper, a quadrature rule on an equidistant partition of the interval $[a, b]$ for the finite Hilbert Transform of different classes of absolutely continuous functions via Simpson type inequalities is given, which may have the better error bounds than those obtained via trapezoid type inequalities. Some numerical experiments for different divisions of the interval $[a, b]$ are also presented.

1. INTRODUCTION

The finite Hilbert transform plays an important role in scientific and engineering computing. Denote by $(Tf)(a, b, \cdot)$ the finite Hilbert transform of the function $f : [a, b] \rightarrow \mathbb{R}$, i.e., we recall it

$$(1.1) \quad (Tf)(a, b; t) = \frac{1}{\pi} PV \int_a^b \frac{f(\tau)}{\tau - t} d\tau := \frac{1}{\pi} \lim_{\varepsilon \rightarrow 0} \left[\int_a^{t-\varepsilon} + \int_{t+\varepsilon}^b \right] \frac{f(\tau)}{\tau - t} d\tau,$$

where PV has the usual meaning of the Cauchy principle value.

There are some important approaches for evaluating finite Hilbert transforms, such as the Gaussian, Chebyshev, TANH, Iri-Moriguti-Takasawa, and double exponential quadrature methods. And for classical results on the finite Hilbert transform, see [4, 5, 6, 9, 11, 12, 13, 17].

In [5], by the use of trapezoid type rules taken on an equidistant partition of the interval $[a, b]$, Dragomir et al. proved the following inequalities for the finite Hilbert transform of different classes of absolutely continuous functions.

Theorem 1.1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable function such that its derivative f' is absolutely continuous on $[a, b]$. If*

$$(1.2) \quad T_n(f; t) = \frac{f'(t)(b-a) + f(b) - f(a)}{2\pi n} + \frac{b-a}{\pi n} \sum_{i=1}^{n-1} \left[f; t - \frac{t-a}{n} \cdot i, t + \frac{b-t}{n} \cdot i \right],$$

then we have the estimate

$$(1.3) \quad \left| (Tf)(a, b; t) - \frac{f(t)}{\pi} \ln \left(\frac{b-t}{t-a} \right) - T_n(f; t) \right| \leq \begin{cases} \frac{1}{4\pi n} \left[\frac{(b-a)^2}{4} + \left(t - \frac{a+b}{2} \right)^2 \right] \|f''\|_{[a,b],\infty}, & \text{if } f'' \in L_\infty[a, b]; \\ \frac{q}{2\pi n(q+1)^{1+\frac{1}{q}}} \left[(t-a)^{1+\frac{1}{q}} + (b-t)^{1+\frac{1}{q}} \right] \|f''\|_{[a,b],p}, & \text{if } f'' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{2\pi n} \left[\frac{1}{2}(b-a) + \left| t - \frac{a+b}{2} \right| \right] \|f''\|_{[a,b],1}, & \\ \frac{1}{8\pi n} (b-a)^2 \|f''\|_{[a,b],\infty}, & \text{if } f'' \in L_\infty[a, b]; \\ \frac{q}{2\pi n(q+1)^{1+\frac{1}{q}}} (b-a)^{1+\frac{1}{q}} \|f''\|_{[a,b],p}, & \text{if } f'' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{2\pi n} (b-a) \|f''\|_{[a,b],1}, & \end{cases}$$

for all $t \in (a, b)$, where $[f; c, d]$ denotes the divided difference $[f; c, d] := \frac{f(c)-f(d)}{c-d}$.

Key words and phrases. Finite Hilbert transform, Simpson type inequalities.

Theorem 1.2. Let $f : [a, b] \rightarrow \mathbb{R}$ be a twice differentiable function such that the second derivative f'' is absolutely continuous on $[a, b]$. Then

$$(1.4) \quad \left| (Tf)(a, b; t) - \frac{f(t)}{\pi} \ln \left(\frac{b-t}{t-a} \right) - T_n(f; t) \right|$$

$$\leq \begin{cases} \frac{1}{12n^2\pi} \left[\frac{(b-a)^2}{12} + \left(t - \frac{a+b}{2} \right)^2 \right] (b-a) \|f'''\|_{[a,b],\infty}, & \text{if } f''' \in L_\infty[a, b]; \\ \frac{q[B(q+1, q+1)]^{\frac{1}{q}}}{2(2q+1)n^2\pi} \left[(t-a)^{2+\frac{1}{q}} + (b-t)^{2+\frac{1}{q}} \right] \|f'''\|_{[a,b],p}, & \text{if } f''' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{8\pi n^2} \left[\frac{(b-a)^2}{4} + \left(t - \frac{a+b}{2} \right)^2 \right] \|f'''\|_{[a,b],1}, \end{cases}$$

$$\leq \begin{cases} \frac{(b-a)^3}{36\pi n^2} \|f'''\|_{[a,b],\infty}, & \text{if } f''' \in L_\infty[a, b]; \\ \frac{q[B(q+1, q+1)]^{\frac{1}{q}} (b-a)^{2+\frac{1}{q}}}{2\pi(2q+1)n^2} \|f'''\|_{[a,b],p}, & \text{if } f''' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{16\pi n^2} (b-a)^2 \|f'''\|_{[a,b],1}, \end{cases}$$

for all $t \in (a, b)$, where $T_n(f; t)$ is defined by (1.2).

An extensive literature such as [1, 2, 3, 7, 8, 10, 14, 15, 16, 18, 19, 20, 21, 22] deal with Simpson type inequalities.

In this paper, motivated by [5], by the use of Simpson type inequalities taken on an equidistant partition of the interval $[a, b]$, a quadrature formula for the Finite Hilbert transform of different classes of absolutely continuous functions is obtained. Estimates for some error bounds and some numerical examples for the obtained approximation will also be presented.

2. THE RESULTS

Lemma 2.1. Let $u : [a, b] \rightarrow \mathbb{R}$ be an absolutely continuous function on $[a, b]$. Then one has the inequalities:

$$(2.1) \quad \left| \int_a^b u(s)ds - \frac{u(a) + 4u\left(\frac{a+b}{2}\right) + u(b)}{6} (b-a) \right|$$

$$\leq \begin{cases} \frac{5(b-a)^2}{36} \|u'\|_{[a,b],\infty} & \text{if } u' \in L_\infty[a, b]; \\ \frac{2(b-a)^{1+\frac{1}{q}} \left(\frac{1}{6q+1} + \frac{1}{3q+1} \right)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \|u'\|_{[a,b],p}, & \text{if } u' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{b-a}{3} \|u'\|_{[a,b],1}. \end{cases}$$

A simple proof of this fact can be done by using the identity

$$(2.2) \quad \int_a^b u(s)ds - \frac{u(a) + 4u\left(\frac{a+b}{2}\right) + u(b)}{6} (b-a) = - \left[\int_a^{\frac{a+b}{2}} \left(s - \frac{5a+b}{6} \right) u'(s)ds \right. \\ \left. + \int_{\frac{a+b}{2}}^b \left(s - \frac{a+5b}{6} \right) u'(s)ds \right],$$

and we omit the details.

The following lemma holds.

Lemma 2.2. Let $u : [a, b] \rightarrow \mathbb{R}$ be an absolutely continuous function on $[a, b]$. Then for any $t, \tau \in (a, b), t \neq \tau$ and $n \in \mathbb{N}, n \geq 1$, we have the inequality:

$$\left| \frac{1}{\tau-t} \int_t^\tau u(s)ds - \frac{1}{6n} \sum_{i=0}^{n-1} \left[u \left(t + i \cdot \frac{\tau-t}{n} \right) + 4u \left(t + \left(i + \frac{1}{2} \right) \cdot \frac{\tau-t}{n} \right) + u \left(t + (i+1) \cdot \frac{\tau-t}{n} \right) \right] \right|$$

A QUADRATURE RULE FOR THE FINITE HILBERT TRANSFORM

$$\leq \begin{cases} \frac{5|\tau - t|}{36n} \|u'\|_{[t,\tau],\infty} & \text{if } u' \in L_\infty[a, b]; \\ \frac{2|\tau - t|^{\frac{1}{q}} \left(\frac{1}{6q+1} + \frac{1}{3q+1}\right)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}} n} \|u'\|_{[t,\tau],p}, & \text{if } u' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{3n} \|u'\|_{[t,\tau],1}, \end{cases}$$

where

$$\|u'\|_{[t,\tau],\infty} := \text{ess sup}_{s \in [t,\tau]} |u'(s)|, \text{ and } \|u'\|_{[t,\tau],p} := \left| \int_t^\tau |u'(s)|^p ds \right|^{\frac{1}{p}}, p \geq 1.$$

Proof. Consider the equidistant division of $[t, \tau]$ (if $t < \tau$) given by

$$E_n : x_i = t + i \cdot \frac{\tau - t}{n}, i = \overline{0, n}.$$

If we apply the inequality (2.1) on the interval $[x_i, x_{i+1}]$, we may write that:

$$\begin{aligned} & \left| \int_{x_i}^{x_{i+1}} u(s) ds - \frac{u\left(t + i \cdot \frac{\tau-t}{n}\right) + 4u\left(t + \left(i + \frac{1}{2}\right) \cdot \frac{\tau-t}{n}\right) + u\left(t + (i+1) \cdot \frac{\tau-t}{n}\right)}{6} \cdot \frac{\tau-t}{n} \right| \\ & \leq \begin{cases} \frac{5(\tau-t)^2}{36n^2} \|u'\|_{[x_i, x_{i+1}],\infty}, & \text{if } u' \in L_\infty[a, b]; \\ \frac{2|\tau-t|^{1+\frac{1}{q}} \left(\frac{1}{6q+1} + \frac{1}{3q+1}\right)^{\frac{1}{q}}}{n^{1+\frac{1}{q}} (q+1)^{\frac{1}{q}}} \|u'\|_{[x_i, x_{i+1}],p}, & \text{if } u' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{|\tau-t|}{3n} \|u'\|_{[x_i, x_{i+1}],1}, \end{cases} \end{aligned}$$

from which we get

$$\begin{aligned} & \left| \frac{1}{\tau-t} \int_{x_i}^{x_{i+1}} u(s) ds - \frac{1}{6n} \left[u\left(t + i \cdot \frac{\tau-t}{n}\right) + 4u\left(t + \left(i + \frac{1}{2}\right) \cdot \frac{\tau-t}{n}\right) + u\left(t + (i+1) \cdot \frac{\tau-t}{n}\right) \right] \right| \\ & \leq \begin{cases} \frac{5|\tau-t|}{36n^2} \|u'\|_{[x_i, x_{i+1}],\infty}, & \text{if } u' \in L_\infty[a, b]; \\ \frac{2|\tau-t|^{\frac{1}{q}} \left(\frac{1}{6q+1} + \frac{1}{3q+1}\right)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}} n^{1+\frac{1}{q}}} \|u'\|_{[x_i, x_{i+1}],p}, & \text{if } u' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{3n} \|u'\|_{[x_i, x_{i+1}],1}. \end{cases} \end{aligned}$$

Summing over i from 0 to $n - 1$ and using the generalised triangle inequality, we may write

$$\begin{aligned} & \left| \frac{1}{\tau-t} \int_t^\tau u(s) ds - \frac{1}{6n} \sum_{i=0}^{n-1} \left[u\left(t + i \cdot \frac{\tau-t}{n}\right) + 4u\left(t + \left(i + \frac{1}{2}\right) \cdot \frac{\tau-t}{n}\right) + u\left(t + (i+1) \cdot \frac{\tau-t}{n}\right) \right] \right| \\ & \leq \begin{cases} \frac{5|\tau-t|}{36n^2} \sum_{i=0}^{n-1} \|u'\|_{[x_i, x_{i+1}],\infty}, & \text{if } u' \in L_\infty[a, b]; \\ \frac{2|\tau-t|^{\frac{1}{q}} \left(\frac{1}{6q+1} + \frac{1}{3q+1}\right)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}} n^{1+\frac{1}{q}}} \sum_{i=0}^{n-1} \|u'\|_{[x_i, x_{i+1}],p}, & \text{if } u' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{3n} \sum_{i=0}^{n-1} \|u'\|_{[x_i, x_{i+1}],1}. \end{cases} \end{aligned}$$

However,

$$\sum_{i=0}^{n-1} \|u'\|_{[x_i, x_{i+1}],\infty} \leq n \|u'\|_{[t,\tau],\infty},$$

$$\sum_{i=0}^{n-1} \|u'\|_{[x_i, x_{i+1}], p} = \sum_{i=0}^{n-1} \left| \int_{x_i}^{x_{i+1}} |u'(s)|^p ds \right|^{\frac{1}{p}} \leq n^{\frac{1}{q}} \left[\left(\sum_{i=0}^{n-1} \left| \int_{x_i}^{x_{i+1}} |u'(s)|^p ds \right|^{\frac{1}{p}} \right)^p \right]^{\frac{1}{p}} = n^{\frac{1}{q}} \|u'\|_{[t, \tau], p},$$

and

$$\sum_{i=0}^{n-1} \|u'\|_{[x_i, x_{i+1}], 1} \leq \left| \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} |u'(s)| ds \right| = \left| \int_t^\tau |u'(s)| ds \right| = \|u'\|_{[t, \tau], 1},$$

and the lemma is proved. □

The following theorem in approximating the Hilbert transform of a differentiable function whose derivative is absolutely continuous holds.

Theorem 2.1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable function such that its derivative f' is absolutely continuous on $[a, b]$. If*

$$(2.3) \quad T_n(f; t) = \frac{f'(t)(b-a) + f(b) - f(a)}{6\pi n} + \frac{b-a}{3\pi n} \sum_{i=1}^{n-1} \left[f; t - \frac{t-a}{n} \cdot i, t + \frac{b-t}{n} \cdot i \right] \\ + \frac{2(b-a)}{3\pi n} \sum_{i=0}^{n-1} \left[f; t - \frac{t-a}{n} \cdot \left(i + \frac{1}{2}\right), t + \frac{b-t}{n} \cdot \left(i + \frac{1}{2}\right) \right],$$

then we have the estimate

$$(2.4) \quad \left| (Tf)(a, b; t) - \frac{f(t)}{\pi} \ln \left(\frac{b-t}{t-a} \right) - T_n(f; t) \right| \\ \leq \begin{cases} \frac{5}{36\pi n} \left[\frac{(b-a)^2}{4} + \left(t - \frac{a+b}{2} \right)^2 \right] \|f''\|_{[a, b], \infty}, & \text{if } f'' \in L_\infty[a, b]; \\ \frac{2q \left(\frac{1}{6^{q+1}} + \frac{1}{3^{q+1}} \right)^{\frac{1}{q}}}{\pi(q+1)^{\frac{1}{q}}} \left[(t-a)^{1+\frac{1}{q}} + (b-t)^{1+\frac{1}{q}} \right] \|f''\|_{[a, b], p}, & \text{if } f'' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{3\pi n} \left[\frac{1}{2}(b-a) + \left| t - \frac{a+b}{2} \right| \right] \|f''\|_{[a, b], 1}, \end{cases} \\ \leq \begin{cases} \frac{5}{72\pi n} (b-a)^2 \|f''\|_{[a, b], \infty}, & \text{if } f'' \in L_\infty[a, b]; \\ \frac{2q \left(\frac{1}{6^{q+1}} + \frac{1}{3^{q+1}} \right)^{\frac{1}{q}}}{\pi(q+1)^{\frac{1}{q}}} (b-a)^{1+\frac{1}{q}} \|f''\|_{[a, b], p}, & \text{if } f'' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{3\pi n} (b-a) \|f''\|_{[a, b], 1}, \end{cases}$$

for all $t \in (a, b)$.

Proof. Applying Lemma 2.2 for the function f' , we may write that

$$(2.5) \quad \left| \frac{f(\tau) - f(t)}{\tau - t} - \frac{1}{6n} \left[f'(t) + \sum_{i=1}^{n-1} f' \left(t + i \cdot \frac{\tau - t}{n} \right) + 4 \sum_{i=0}^{n-1} f' \left(t + \left(i + \frac{1}{2} \right) \cdot \frac{\tau - t}{n} \right) \right. \right. \\ \left. \left. + f'(\tau) + \sum_{i=0}^{n-2} f' \left(t + (i+1) \cdot \frac{\tau - t}{n} \right) \right] \right| \\ \leq \begin{cases} \frac{5|\tau - t|}{36n} \|f''\|_{[t, \tau], \infty}, & \text{if } f'' \in L_\infty[a, b]; \\ \frac{2|\tau - t| \left(\frac{1}{6^{q+1}} + \frac{1}{3^{q+1}} \right)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}} n} \|f''\|_{[t, \tau], p}, & \text{if } f'' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{3n} \|f''\|_{[t, \tau], 1}. \end{cases}$$

However,

$$\sum_{i=1}^{n-1} f' \left(t + i \cdot \frac{\tau - t}{n} \right) = \sum_{i=0}^{n-2} f' \left(t + (i + 1) \cdot \frac{\tau - t}{n} \right)$$

and then by (2.5), we may write:

$$(2.6) \quad \left| \frac{f(\tau) - f(t)}{\tau - t} - \left[\frac{f'(t) + f'(\tau)}{6n} + \frac{1}{3n} \sum_{i=1}^{n-1} f' \left(t + i \cdot \frac{\tau - t}{n} \right) + \frac{2}{3n} \sum_{i=0}^{n-1} f' \left(t + \left(i + \frac{1}{2} \right) \cdot \frac{\tau - t}{n} \right) \right] \right|$$

$$\leq \begin{cases} \frac{5|\tau - t|}{36n} \|f''\|_{[t,\tau],\infty}, \\ \frac{2|\tau - t|^{\frac{1}{q}} \left(\frac{1}{6q+1} + \frac{1}{3q+1} \right)^{\frac{1}{q}}}{(q + 1)^{\frac{1}{q}} n} \|f''\|_{[t,\tau],p}, \\ \frac{1}{3n} \|f''\|_{[t,\tau],1}, \end{cases}$$

for any $t, \tau \in [a, b], t \neq \tau$.

Consequently, we have

$$(2.7) \quad \left| \frac{1}{\pi} PV \int_a^b \frac{f(\tau) - f(t)}{\tau - t} d\tau - \frac{1}{\pi} PV \int_a^b \left[\frac{f'(t) + f'(\tau)}{6n} + \frac{1}{3n} \sum_{i=1}^{n-1} f' \left(t + i \cdot \frac{\tau - t}{n} \right) + \frac{2}{3n} \sum_{i=0}^{n-1} f' \left(t + \left(i + \frac{1}{2} \right) \cdot \frac{\tau - t}{n} \right) \right] d\tau \right|$$

$$\leq \begin{cases} \frac{5}{36\pi n} PV \int_a^b |\tau - t| \|f''\|_{[t,\tau],\infty} d\tau, \\ \frac{2 \left(\frac{1}{6q+1} + \frac{1}{3q+1} \right)^{\frac{1}{q}}}{(q + 1)^{\frac{1}{q}} \pi n} PV \int_a^b |\tau - t|^{\frac{1}{q}} \|f''\|_{[t,\tau],p} d\tau, \\ \frac{1}{3\pi n} PV \int_a^b \|f''\|_{[t,\tau],1} d\tau. \end{cases}$$

Since

$$PV \int_a^b \left[\frac{f'(t) + f'(\tau)}{6n} + \frac{1}{3n} \sum_{i=1}^{n-1} f' \left(t + i \cdot \frac{\tau - t}{n} \right) + \frac{2}{3n} \sum_{i=0}^{n-1} f' \left(t + \left(i + \frac{1}{2} \right) \cdot \frac{\tau - t}{n} \right) \right] d\tau$$

$$= \lim_{\varepsilon \rightarrow 0^+} \left(\int_a^{t-\varepsilon} + \int_{t+\varepsilon}^b \right) \left(\frac{f'(t) + f'(\tau)}{6n} + \frac{1}{3n} \sum_{i=1}^{n-1} f' \left(t + i \cdot \frac{\tau - t}{n} \right) + \frac{2}{3n} \sum_{i=0}^{n-1} f' \left(t + \left(i + \frac{1}{2} \right) \cdot \frac{\tau - t}{n} \right) \right) d\tau$$

$$= \frac{f'(t)(b - a) + f(b) - f(a)}{6n} + \frac{1}{3n} \sum_{i=1}^{n-1} \left[\lim_{\varepsilon \rightarrow 0^+} \left(\int_a^{t-\varepsilon} + \int_{t+\varepsilon}^b \right) \left(f' \left(t + i \cdot \frac{\tau - t}{n} \right) \right) \right] d\tau$$

$$+ \frac{2}{3n} \sum_{i=0}^{n-1} \left[\lim_{\varepsilon \rightarrow 0^+} \left(\int_a^{t-\varepsilon} + \int_{t+\varepsilon}^b \right) \left(f' \left(t + \left(i + \frac{1}{2} \right) \cdot \frac{\tau - t}{n} \right) \right) \right] d\tau$$

$$= \frac{f'(t)(b - a) + f(b) - f(a)}{6n} + \frac{1}{3n} \sum_{i=1}^{n-1} \left[\lim_{\varepsilon \rightarrow 0^+} \left[\frac{n}{i} \cdot f \left(t + i \cdot \frac{\tau - t}{n} \right) \left(\left|_a^{t-\varepsilon} + \left|_{t+\varepsilon}^b \right. \right) \right] \right]$$

$$+ \frac{2}{3n} \sum_{i=0}^{n-1} \left[\lim_{\varepsilon \rightarrow 0^+} \left[\frac{2n}{2i + 1} \cdot f \left(t + \left(i + \frac{1}{2} \right) \cdot \frac{\tau - t}{n} \right) \left(\left|_a^{t-\varepsilon} + \left|_{t+\varepsilon}^b \right. \right) \right] \right]$$

$$= \frac{f'(t)(b - a) + f(b) - f(a)}{6n} + \frac{1}{3n} \sum_{i=1}^{n-1} \frac{n}{i} \left[f \left(t + i \cdot \frac{b - t}{n} \right) - f \left(t + i \cdot \frac{a - t}{n} \right) \right]$$

$$\begin{aligned}
 & + \frac{2}{3n} \sum_{i=0}^{n-1} \frac{2n}{2i+1} \left[f \left(t + \left(i + \frac{1}{2} \right) \cdot \frac{b-t}{n} \right) - f \left(t + \left(i + \frac{1}{2} \right) \cdot \frac{a-t}{n} \right) \right] \\
 & = \frac{f'(t)(b-a) + f(b) - f(a)}{6n} + \frac{b-a}{3n} \sum_{i=1}^{n-1} \left[f; t + i \cdot \frac{b-t}{n}, t + i \cdot \frac{a-t}{n} \right] \\
 & + \frac{2(b-a)}{3n} \sum_{i=0}^{n-1} \left[f; t + \left(i + \frac{1}{2} \right) \cdot \frac{b-t}{n}, t + \left(i + \frac{1}{2} \right) \cdot \frac{a-t}{n} \right],
 \end{aligned}$$

and

$$\begin{aligned}
 PV \int_a^b |\tau - t| \|f''\|_{[t,\tau],\infty} d\tau & \leq \|f''\|_{[a,b],\infty} PV \int_a^b |\tau - t| d\tau = \|f''\|_{[a,b],\infty} \left[\frac{1}{4}(b-a)^2 + \left(t - \frac{a+b}{2} \right) \right], \\
 PV \int_a^b |\tau - t|^{\frac{1}{q}} \|f''\|_{[t,\tau],p} d\tau & \leq \|f''\|_{[a,b],p} PV \int_a^b |\tau - t|^{\frac{1}{q}} d\tau = \frac{q \|f''\|_{[a,b],p}}{q+1} \left[(t-a)^{1+\frac{1}{q}} + (b-t)^{1+\frac{1}{q}} \right], \\
 PV \int_a^b \|f''\|_{[t,\tau],1} d\tau & = PV \left[\int_a^t \|f''\|_{[\tau,t],1} d\tau + \int_t^b \|f''\|_{[t,\tau],1} d\tau \right] \\
 & \leq \|f''\|_{[a,t],1} (t-a) + \|f''\|_{[t,b],1} (b-t) \leq \left[\frac{1}{2}(b-a) + \left| t - \frac{a+b}{2} \right| \right] \|f''\|_{[a,b],1},
 \end{aligned}$$

then, by (2.7) we get

$$\begin{aligned}
 (2.8) \quad & \left| \frac{1}{\pi} PV \int_a^b \frac{f(\tau) - f(t)}{\tau - t} d\tau - \frac{f'(t)(b-a) + f(b) - f(a)}{6\pi n} - \frac{b-a}{3\pi n} \sum_{i=1}^{n-1} \left[f; t - \frac{t-a}{n} \cdot i, t + \frac{b-t}{n} \cdot i \right] \right. \\
 & \left. - \frac{2(b-a)}{3\pi n} \sum_{i=0}^{n-1} \left[f; t - \frac{t-a}{n} \cdot \left(i + \frac{1}{2} \right), t + \frac{b-t}{n} \cdot \left(i + \frac{1}{2} \right) \right] \right| \\
 & \leq \begin{cases} \frac{5 \|f''\|_{[a,b],\infty}}{36\pi n} \left[\frac{1}{4}(b-a)^2 + \left(t - \frac{a+b}{2} \right) \right], & \text{if } f'' \in L_\infty[a,b]; \\ \frac{2q \left(\frac{1}{6q+1} + \frac{1}{3q+1} \right)^{\frac{1}{q}} \|f''\|_{[a,b],p}}{\pi(q+1)^{1+\frac{1}{q}n}} \left[(t-a)^{1+\frac{1}{q}} + (b-t)^{1+\frac{1}{q}} \right], & \text{if } f'' \in L_p[a,b], p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{\|f''\|_{[a,b],1}}{3\pi n} \left[\frac{1}{2}(b-a) + \left| t - \frac{a+b}{2} \right| \right]. \end{cases}
 \end{aligned}$$

On the other hand, as for the function $f_0 : (a, b) \rightarrow \mathbb{R}, f_0(t) = 1$, we have

$$(T, f_0)(a, b; t) = \frac{1}{\pi} \ln \left(\frac{b-a}{t-a} \right), t \in (a, b),$$

then obviously

$$(Tf)(a, b; t) = \frac{1}{\pi} PV \int_a^b \frac{f(\tau) - f(t) + f(t)}{\tau - t} d\tau = \frac{1}{\pi} PV \int_a^b \frac{f(\tau) - f(t)}{\tau - t} d\tau + \frac{f(t)}{\pi} PV \int_a^b \frac{d\tau}{\tau - t},$$

from which we get the equality:

$$(2.9) \quad (Tf)(a, b; t) - \frac{f(t)}{\pi} \ln \left(\frac{b-t}{t-a} \right) = \frac{1}{\pi} PV \int_a^b \frac{f(\tau) - f(t)}{\tau - t} d\tau.$$

Finally, using (2.8) and (2.9), we deduce (2.4). □

Before we proceed with another estimate of the remainder in approximating the Hilbert Transform for functions whose second derivatives are absolutely continuous, we need the following lemma.

Lemma 2.3. *Let $u : [a, b] \rightarrow \mathbb{R}$ be a function such that its derivative is absolutely continuous on $[a, b]$. Then one has the inequalities:*

$$(2.10) \quad \left| \int_a^b u(s) ds - \frac{u(a) + 4u\left(\frac{a+b}{2}\right) + u(b)}{6} (b-a) \right|$$

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$$\leq \begin{cases} \frac{(b-a)^3}{81} \|u''\|_{[a,b],\infty}, & \text{if } u'' \in L_\infty[a, b]; \\ \frac{(b-a)^{2+\frac{1}{q}}}{2} \Lambda \|u''\|_{[a,b],p}, & \text{if } u'' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{(b-a)^2}{24} \|u''\|_{[a,b],1}, & \end{cases}$$

where

$$(2.11) \quad \Lambda = \left[\left(\int_0^{\frac{1}{3}} s^q \left(\frac{1}{3} - s \right)^q ds + \int_{\frac{1}{3}}^{\frac{1}{2}} s^q \left(s - \frac{1}{3} \right)^q ds \right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^{\frac{2}{3}} (1-s)^q \left(\frac{2}{3} - s \right)^q ds + \int_{\frac{2}{3}}^1 (1-s)^q \left(s - \frac{2}{3} \right)^q ds \right)^{\frac{1}{q}} \right].$$

A simple proof of the fact can be done by the use of the following identity:

$$(2.12) \quad \int_a^b u(s)ds - \frac{u(a) + 4u\left(\frac{a+b}{2}\right) + u(b)}{6}(b-a) = -\frac{1}{2} \int_a^{\frac{a+b}{2}} (s-a) \left(s - \frac{2a+b}{3} \right) u''(s)ds - \frac{1}{2} \int_{\frac{a+b}{2}}^b (s-b) \left(s - \frac{a+2b}{3} \right) u''(s)ds,$$

and we omit the details.

The following lemma also holds.

Lemma 2.4. *Let $u : [a, b] \rightarrow \mathbb{R}$ be a differentiable function such that $u' : [a, b] \rightarrow \mathbb{R}$ is absolutely continuous on $[a, b]$. Then for any $t, \tau \in (a, b), t \neq \tau$ and $n \in \mathbb{N}, n \geq 1$, we have the inequality:*

$$\left| \frac{1}{\tau-t} \int_t^\tau u(s)ds - \frac{1}{6n} \sum_{i=0}^{n-1} \left[u\left(t + i \cdot \frac{\tau-t}{n}\right) + 4u\left(t + \left(i + \frac{1}{2}\right) \cdot \frac{\tau-t}{n}\right) + u\left(t + (i+1) \cdot \frac{\tau-t}{n}\right) \right] \right| \leq \begin{cases} \frac{|\tau-t|^2}{81n^2} \|u''\|_{[t,\tau],\infty}, & \text{if } u'' \in L_\infty[a, b]; \\ \frac{|\tau-t|^{2+\frac{1}{q}}}{2n^2} \Lambda \|u''\|_{[t,\tau],p}, & \text{if } u'' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{|\tau-t|^2}{24n^2} \|u''\|_{[t,\tau],1}. & \end{cases}$$

Proof. Consider the equidistant division of $[t, \tau]$ (if $t < \tau$)

$$E_n : x_i = t + i \cdot \frac{\tau-t}{n}, i = \overline{0, n}.$$

If we apply the inequality (2.10), we may state that

$$\left| \int_{x_i}^{x_{i+1}} u(s)ds - \frac{u\left(t + i \cdot \frac{\tau-t}{n}\right) + 4u\left(t + \left(i + \frac{1}{2}\right) \cdot \frac{\tau-t}{n}\right) + u\left(t + (i+1) \cdot \frac{\tau-t}{n}\right)}{6} \cdot \frac{\tau-t}{n} \right| \leq \begin{cases} \frac{|\tau-t|^3}{81n^3} \|u''\|_{[x_i, x_{i+1}],\infty}, & \text{if } u'' \in L_\infty[a, b]; \\ \frac{|\tau-t|^{2+\frac{1}{q}}}{2n^{2+\frac{1}{q}}} \Lambda \|u''\|_{[x_i, x_{i+1}],p}, & \text{if } u'' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{|\tau-t|^2}{24n^2} \|u''\|_{[x_i, x_{i+1}],1}. & \end{cases}$$

Dividing by $|\tau-t| > 0$ and using a similar argument to the one in Lemma 2.2, we conclude that the desired inequality holds. □

The following theorem in approximating the Hilbert transform of a twice differentiable function whose second derivative f'' is absolutely continuous also holds.

Theorem 2.2. Let $f : [a, b] \rightarrow \mathbb{R}$ be a twice differentiable function such that the second derivative f'' is absolutely continuous on $[a, b]$. Then

$$(2.13) \quad \left| (Tf)(a, b; t) - \frac{f(t)}{\pi} \ln \left(\frac{b-t}{t-a} \right) - T_n(f; t) \right| \leq \begin{cases} \frac{1}{81n^2\pi} \left[\frac{(b-a)^2}{12} + \left(t - \frac{a+b}{2} \right)^2 \right] (b-a) \|f'''\|_{[a,b],\infty}, & \text{if } f''' \in L_\infty[a, b]; \\ \frac{q[(t-a)^{2+\frac{1}{q}} + (b-t)^{2+\frac{1}{q}}]\Lambda}{2(2q+1)n^2\pi} \|f'''\|_{[a,b],p}, & \text{if } f''' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{24n^2\pi} \left[\frac{(b-a)^2}{4} + \left(t - \frac{a+b}{2} \right)^2 \right] \|f'''\|_{[a,b],1}, & \end{cases}$$

$$\leq \begin{cases} \frac{(b-a)^3}{243n^2\pi} \|f'''\|_{[a,b],\infty}, & \text{if } f''' \in L_\infty[a, b]; \\ \frac{q(b-a)^{2+\frac{1}{q}}\Lambda}{2\pi(2q+1)n^2} \|f'''\|_{[a,b],p}, & \text{if } f''' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{(b-a)^2}{48n^2\pi} \|f'''\|_{[a,b],1}, & \end{cases}$$

for all $t \in (a, b)$, where $T_n(f; t)$ is defined by (2.3) and Λ is defined by (2.11).

Proof. Applying Lemma 2.4 for the function f' , we may write that (see also Theorem 2.1)

$$(2.14) \quad \left| \frac{f(\tau) - f(t)}{\tau - t} - \left[\frac{f'(t) + f'(\tau)}{6n} + \frac{1}{3n} \sum_{i=1}^{n-1} f' \left(t + i \cdot \frac{\tau - t}{n} \right) + \frac{2}{3n} \sum_{i=0}^{n-1} f' \left(t + \left(i + \frac{1}{2} \right) \cdot \frac{\tau - t}{n} \right) \right] \right| \leq \begin{cases} \frac{|\tau - t|^2}{81n^2} \|f'''\|_{[t,\tau],\infty}, & \text{if } f''' \in L_\infty[a, b]; \\ \frac{|\tau - t|^{1+\frac{1}{q}}}{2n^2} \Lambda \|f'''\|_{[t,\tau],p}, & \text{if } f''' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{|\tau - t|}{24n^2} \|f'''\|_{[t,\tau],1}, & \end{cases}$$

for any $t, \tau \in [a, b], t \neq \tau$. Consequently, we may write:

$$(2.15) \quad \left| \frac{1}{\pi} PV \int_a^b \frac{f(\tau) - f(t)}{\tau - t} d\tau - \frac{1}{\pi} PV \int_a^b \left[\frac{f'(t) + f'(\tau)}{6n} + \frac{1}{3n} \sum_{i=1}^{n-1} f' \left(t + i \cdot \frac{\tau - t}{n} \right) + \frac{2}{3n} \sum_{i=0}^{n-1} f' \left(t + \left(i + \frac{1}{2} \right) \cdot \frac{\tau - t}{n} \right) \right] d\tau \right| \leq \begin{cases} \frac{1}{81n^2\pi} PV \int_a^b |\tau - t|^2 \|f'''\|_{[t,\tau],\infty} d\tau, \\ \frac{\Lambda}{2n^2\pi} PV \int_a^b |\tau - t|^{1+\frac{1}{q}} \|f'''\|_{[t,\tau],p} d\tau, \\ \frac{1}{24n^2\pi} PV \int_a^b |\tau - t| \|f'''\|_{[t,\tau],1} d\tau. \end{cases}$$

since

$$PV \int_a^b |\tau - t|^2 \|f'''\|_{[t,\tau],\infty} d\tau \leq \|f'''\|_{[a,b],\infty} PV \int_a^b |\tau - t|^2 d\tau$$

$$= \|f'''\|_{[a,b],\infty} \left[\frac{(t-a)^3 + (b-t)^3}{3} \right] = \|f'''\|_{[a,b],\infty} \left[\frac{(b-a)^2}{12} + \left(t - \frac{a+b}{2} \right)^2 \right] (b-a),$$

$$PV \int_a^b |\tau - t|^{1+\frac{1}{q}} \|f'''\|_{[t,\tau],p} d\tau \leq \|f'''\|_{[a,b],p} PV \int_a^b |\tau - t|^{1+\frac{1}{q}} d\tau$$

A QUADRATURE RULE FOR THE FINITE HILBERT TRANSFORM

$$= \|f'''\|_{[a,b],p} \frac{(b-t)^{2+\frac{1}{q}} + (t-a)^{2+\frac{1}{q}}}{2+\frac{1}{q}} = \frac{q \|f'''\|_{[a,b],p}}{2q+1} \left[(b-t)^{2+\frac{1}{q}} + (t-a)^{2+\frac{1}{q}} \right]$$

and

$$PV \int_a^b |\tau - t| \|f'''\|_{[t,\tau],1} d\tau \leq \left[\frac{(b-a)^2}{4} + \left(t - \frac{a+b}{2} \right)^2 \right] \|f'''\|_{[a,b],1}.$$

Then by (2.15), we deduce the first part of (2.13). □

3. NUMERICAL EXPERIMENTS

For a function $f : [a, b] \rightarrow \mathbb{R}$, we may consider the quadrature formula

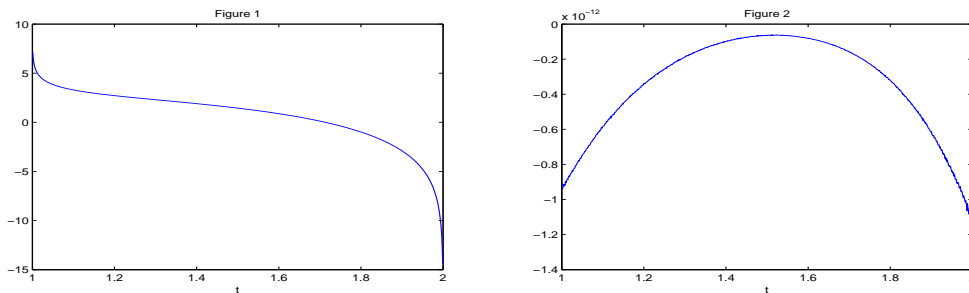
$$E_n(f; a, b, t) := \frac{f(t)}{\pi} \ln \left(\frac{b-t}{t-a} \right) + T_n(f; t), t \in [a, b].$$

As shown in the above section, $E_n(f; a, b, t)$ provides an approximation for the Finite Hilbert Transform $(Tf)(a, b; t)$ and the error estimate fulfils the bounds described in (2.4) and (2.15).

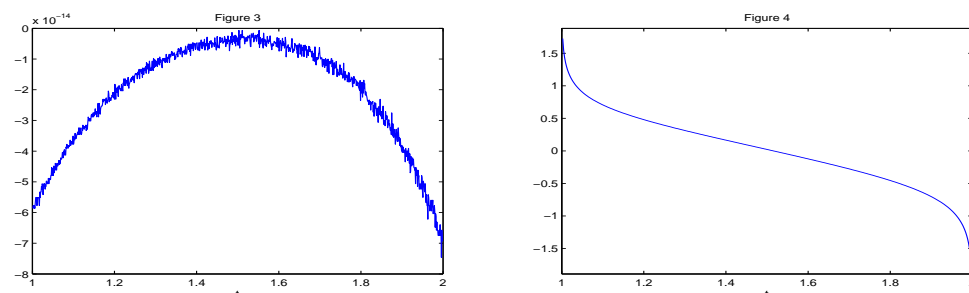
If we consider the function $f : [1, 2] \rightarrow \mathbb{R}$, $f(x) = \exp(x)$, the exact value of the Hilbert transform is

$$(Tf)(a, b; t) = \frac{\exp(t)Ei(2-t) - \exp(t)Ei(1-t)}{\pi}, t \in [1, 2].$$

and the plot of this function is embodied in Figure 1.



If we implement the quadrature formula provided by $E_n(f; a, b, t)$ using Matlab and chose the value of $n = 100$, the error $E_r(f; a, b, t) := (Tf)(a, b; t) - E_n(f; a, b, t)$ has the variation described in Figure 2.



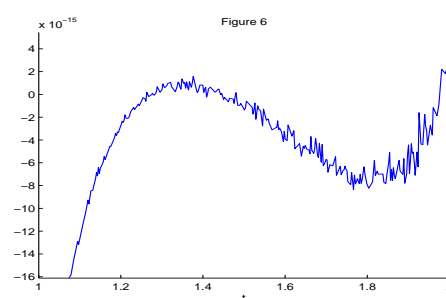
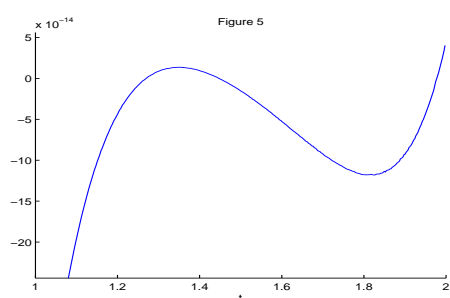
For $n = 200$, the plot of $E_r(f; a, b, t)$ is embodied in the following Figure 3, from which we can see that the precision of the error gets higher when n gets bigger.

Now, if we consider another function $f : [1, 2] \rightarrow \mathbb{R}$, $f(x) = \sin(x)$, then the exact value of Hilbert transform is

$$(Tf)(a, b; t) = \frac{-S_i(-2+t) \cos(t) + C_i(2-t) \sin(t) + S_i(t-1) \cos(t) - \sin(t)C_i(t-1)}{\pi}, t \in [1, 2]$$

having the plot embodied in Figure 4.

If we choose the value of $n = 50$, then the error $E_r(f; a, b, t) := (Tf)(a, b; t) - E_n(f; a, b, t)$ for the function $f(x) = \sin x$, $x \in [a, b]$ has the variation described in Figure 5. Moreover, for $n = 100$, the behaviour of $E_r(f; a, b, t)$ is plotted in Figure 6.



Remark 1. When $n = 100$, for function $f(x) = \exp(x)$, the precision of the error is 10^{-06} in [5], while the precision obtained here is 10^{-12} . When $n = 200$, we also have the higher precision. For function $f(x) = \sin(x)$, it's the same situation. Therefore, our results may have the better error bounds.

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(S. F. Wang, N. Lu and X. Y. Gao) COLLEGE OF MATHEMATICS AND STATISTICS, NANJING UNIVERSITY OF INFORMATION SCIENCE AND TECHNOLOGY, NANJING 210044, CHINA
E-mail address: wsfnuist@yeah.net

A QUADRATURE FORMULA IN APPROXIMATING THE FINITE HILBERT TRANSFORM VIA PERTURBED TRAPEZOID TYPE INEQUALITIES

SHUNFENG WANG, XINGYUE GAO AND NA LU

ABSTRACT. In this paper, we obtain the error estimation of a quadrature formula in approximating the finite Hilbert transform on an equidistant partition of the interval $[a, b]$. Some numerical examples for the obtained approximation are also presented.

1. INTRODUCTION

In the recent year, many authors tried to consider error inequalities for some known and some new quadrature rules. For example, the well-known trapezoid and midpoint quadrature rules were considered (see [1], [4], [6], [9], [11], [12], [14], [15], [18], [19] and [20]). In [5], the authors proved the following theorem:

Theorem 1.1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a mapping such the derivative $f^{(n-1)}$ ($n \geq 1$) is absolutely continuous on $[a, b]$. Then*

$$(1.1) \quad \int_a^b f(t)dt = \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b) \right] + \frac{1}{n!} \int_a^b (x-t)^n f^{(n)}(t)dt$$

for all $x \in [a, b]$.

Specially, we can obtain the following identity from (1.1) with $x = \frac{a+b}{2}$:

$$(1.2) \quad \int_a^b f(t)dt = \sum_{k=0}^{n-1} \frac{(b-a)^{k+1}}{2^{k+1}(k+1)!} \left[f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] + \frac{(-1)^n}{n!} \int_a^b \left(t - \frac{a+b}{2} \right)^n f^{(n)}(t)dt.$$

In (1.2), for $n = 1$, we obtain the trapezoid rule

$$(1.3) \quad \int_a^b f(t)dt = \frac{f(b) + f(a)}{2}(b-a) + \int_a^b \left(\frac{a+b}{2} - t \right) f'(t)dt.$$

The finite Hilbert transform of the function $f : (a, b) \rightarrow \mathbb{R}$ is defined as

$$T(f)(a, b; t) = \frac{1}{\pi} PV \int_a^b \frac{f(\tau)}{\tau-t} d\tau = \lim_{\varepsilon \rightarrow 0} \left[\int_a^{t-\varepsilon} + \int_{t+\varepsilon}^b \right] \frac{f(\tau)}{\pi(\tau-t)} d\tau$$

where PV has the usual meaning of the Cauchy principle value (see [3]).

In [7], the authors used the inequality (1.3) to approximate the finite Hilbert transform and obtain the following theorem:

Theorem 1.2. *Let $f : [a, b] \rightarrow \mathbb{R}$ be such that $f' : [a, b] \rightarrow \mathbb{R}$ is absolutely continuous on $[a, b]$. Then we have the bounds*

$$(1.4) \quad \left| T(f)(a, b; t) - \frac{f(t)}{\pi} \ln \left(\frac{b-t}{t-a} \right) - \frac{1}{2\pi} [f(b) - f(a) + f'(t)(b-a)] \right| \leq \begin{cases} \frac{\|f''\|_{\infty}}{4\pi} \left[\frac{(b-a)^2}{4} + \left(t - \frac{b+a}{2} \right)^2 \right], & f'' \in L^{\infty}[a, b]; \\ \frac{q \|f''\|_p}{2\pi(q+1)^{1+\frac{1}{q}}} \left[(t-a)^{1+\frac{1}{q}} + (b-t)^{1+\frac{1}{q}} \right], & p > 1, \frac{1}{p} + \frac{1}{q} = 1 \text{ and } f'' \in L^p[a, b]; \\ \frac{\|f''\|_1}{2\pi} (b-a), & f'' \in L^1[a, b], \end{cases}$$

for all $t \in (a, b)$, where $\|\cdot\|_p$ are the usual Lebesgue norms in $L^p[a, b]$ ($1 \leq p \leq \infty$).

Key words and phrases. perturbed trapezoid type inequality; numerical integration; finite Hilbert transform.

In [8], by the use of trapezoid type rules taken on an equidistant partition of the interval $[a, b]$, Dragomir et al. proved the following inequalities for the finite Hilbert transform of different classes of absolutely continuous functions.

Theorem 1.3. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable function such that its derivative f' is absolutely continuous on $[a, b]$. If*

$$(1.5) \quad T_n(f; t) = \frac{f'(t)(b-a) + f(b) - f(a)}{2\pi n} + \frac{b-a}{\pi n} \sum_{i=1}^{n-1} \left[f; t - \frac{t-a}{n} \cdot i, t + \frac{b-t}{n} \cdot i \right],$$

then we have the estimate

$$(1.6) \quad \left| (Tf)(a, b; t) - \frac{f(t)}{\pi} \ln \left(\frac{b-t}{t-a} \right) - T_n(f; t) \right| \leq \begin{cases} \frac{1}{4\pi n} \left[\frac{(b-a)^2}{4} + \left(t - \frac{a+b}{2} \right)^2 \right] \|f''\|_{[a,b],\infty}, & \text{if } f'' \in L_\infty[a, b]; \\ \frac{q}{2\pi n(q+1)^{1+\frac{1}{q}}} \left[(t-a)^{1+\frac{1}{q}} + (b-t)^{1+\frac{1}{q}} \right] \|f''\|_{[a,b],p}, & \text{if } f'' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{2\pi n} \left[\frac{1}{2}(b-a) + \left| t - \frac{a+b}{2} \right| \right] \|f''\|_{[a,b],1}, & \end{cases}$$

$$\leq \begin{cases} \frac{1}{8\pi n} (b-a)^2 \|f''\|_{[a,b],\infty}, & \text{if } f'' \in L_\infty[a, b]; \\ \frac{q}{2\pi n(q+1)^{1+\frac{1}{q}}} (b-a)^{1+\frac{1}{q}} \|f''\|_{[a,b],p}, & \text{if } f'' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{2\pi n} (b-a) \|f''\|_{[a,b],1}, & \end{cases}$$

for all $t \in (a, b)$, where $[f; c, d]$ denotes the divided difference $[f; c, d] := \frac{f(c)-f(d)}{c-d}$.

If we put $n = 2$ in (1.2), we can get the perturbed trapezoid rule

$$(1.7) \quad \int_a^b f(t)dt = \frac{f(b) + f(a)}{2}(b-a) - \frac{(b-a)^2}{8} [f'(b) - f'(a)] + \frac{1}{2} \int_a^b \left(t - \frac{a+b}{2} \right)^2 f''(t)dt,$$

Recently, Liu and Pan [16] proved the following inequalities for the finite Hilbert transform of different classes of absolutely continuous functions via the above rule (1.7) (see also [13] for other related results).

Theorem 1.4. *Let $f : [a, b] \rightarrow \mathbb{R}$ be such that $f'' : [a, b] \rightarrow \mathbb{R}$ is absolutely continuous on $[a, b]$. Then we have the bounds*

$$(1.8) \quad \left| T(f)(a, b; t) - \frac{f(t)}{\pi} \ln \left(\frac{b-t}{t-a} \right) - \frac{f'(t)}{2\pi} (b-a) - \frac{5}{8\pi} [f(b) - f(a)] + \frac{f'(b)}{8\pi} (b-t) + \frac{f''(t)}{16\pi} (a-b)(a+b-2t) - \frac{f'(a)}{8\pi} (t-a) \right| \leq \begin{cases} \frac{\|f'''\|_\infty}{24\pi} \left[(b-a) \left(t - \frac{b+a}{2} \right)^2 + \frac{(b-a)^3}{12} \right], & f''' \in L_\infty[a, b]; \\ \frac{q \|f'''\|_p}{8\pi(2q+1)^{1+\frac{1}{q}}} \left[(t-a)^{2+\frac{1}{q}} + (b-t)^{2+\frac{1}{q}} \right], & p > 1, \frac{1}{p} + \frac{1}{q} = 1 \text{ and } f''' \in L^p[a, b]; \\ \frac{\|f'''\|_1}{8\pi} \left[\left(t - \frac{a+b}{2} \right)^2 + \frac{(b-a)^2}{4} \right], & f''' \in L^1[a, b]; \end{cases}$$

for all $t \in (a, b)$, where $\|\cdot\|_p$ ($1 \leq p \leq \infty$) are the usual Lebesgue norms in $L^p[a, b]$.

In this paper, inspired by [8], we shall derive a quadrature formula in approximating the finite Hilbert transform of different classes of absolutely continuous functions. Some numerical examples for the obtained approximation will be presented in Section 3.

A QUADRATURE FORMULA IN APPROXIMATING THE FINITE HILBERT TRANSFORM

2. A QUADRATURE FORMULA FOR EQUIDISTANT DIVISIONS

Lemma 2.1. *Let $u : [a, b] \rightarrow \mathbb{R}$ be an absolutely continuous function on $[a, b]$. Then one has the inequalities:*

$$(2.1) \quad \left| \int_a^b u(s)ds - \frac{u(a) + u(b)}{2}(b - a) + \frac{(b - a)^2}{8}[u'(b) - u'(a)] \right| \leq \begin{cases} \frac{(b - a)^3}{24} \|u''\|_{[a,b],\infty} & \text{if } u'' \in L_\infty[a, b]; \\ \frac{(b - a)^{2+\frac{1}{q}}}{8(2q + 1)^{\frac{1}{q}}} \|u''\|_{[a,b],p}, & \text{if } u'' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{(b - a)^2}{8} \|u''\|_{[a,b],1}. \end{cases}$$

A simple proof of this fact can be done by using the identity

$$(2.2) \quad \int_a^b u(s)ds - \frac{u(a) + u(b)}{2}(b - a) + \frac{(b - a)^2}{8}[u'(b) - u'(a)] = \frac{1}{2} \int_a^b \left(s - \frac{a + b}{2} \right) u''(s)ds$$

and we omit the details.

The following lemma holds.

Lemma 2.2. *Let $u : [a, b] \rightarrow \mathbb{R}$ be an absolutely continuous function on $[a, b]$. Then for any $t, \tau \in (a, b), t \neq \tau$ and $n \in \mathbb{N}, n \geq 1$, we have the inequality:*

$$(2.3) \quad \left| \frac{1}{\tau - t} \int_t^\tau u(s)ds - \frac{1}{2n} \sum_{i=0}^{n-1} \left[u \left(t + i \cdot \frac{\tau - t}{n} \right) + u \left(t + (i + 1) \cdot \frac{\tau - t}{n} \right) \right] + \frac{\tau - t}{8n^2} \sum_{i=0}^{n-1} \left[u' \left(t + (i + 1) \cdot \frac{\tau - t}{n} \right) - u' \left(t + i \cdot \frac{\tau - t}{n} \right) \right] \right| \leq \begin{cases} \frac{|\tau - t|^2}{24n^2} \|u''\|_{[t,\tau],\infty} & \text{if } u'' \in L_\infty[a, b]; \\ \frac{|\tau - t|^{1+\frac{1}{q}}}{8n^2(2q + 1)^{\frac{1}{q}}} \|u''\|_{[t,\tau],p}, & \text{if } u'' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{|\tau - t|}{8n^2} \|u''\|_{[t,\tau],1}, \end{cases}$$

where

$$\|u''\|_{[t,\tau],\infty} := \text{ess sup}_{s \in [t,\tau]} |u''(s)|, \text{ and } \|u''\|_{[t,\tau],p} := \left| \int_t^\tau |u''(s)|^p ds \right|^{\frac{1}{p}}, p \geq 1.$$

Proof. Consider the equidistant division of $[t, \tau]$ (if $t < \tau$) given by

$$E_n : x_i = t + i \cdot \frac{\tau - t}{n}, i = \overline{0, n}.$$

If we apply the inequality (2.1) on the interval $[x_i, x_{i+1}]$, we may write that:

$$\left| \int_{x_i}^{x_{i+1}} u(s)ds - \frac{u \left(t + i \cdot \frac{\tau - t}{n} \right) + u \left(t + (i + 1) \cdot \frac{\tau - t}{n} \right)}{2} \cdot \frac{\tau - t}{n} + \frac{(\tau - t)^2}{8n^2} \left[u' \left(t + (i + 1) \cdot \frac{\tau - t}{n} \right) - u' \left(t + i \cdot \frac{\tau - t}{n} \right) \right] \right| \leq \begin{cases} \frac{|\tau - t|^3}{24n^3} \|u''\|_{[x_i, x_{i+1}],\infty}, & \text{if } u'' \in L_\infty[a, b]; \\ \frac{|\tau - t|^{2+\frac{1}{q}}}{8n^{2+\frac{1}{q}}(2q + 1)^{\frac{1}{q}}} \|u''\|_{[x_i, x_{i+1}],p}, & \text{if } u'' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{|\tau - t|^2}{8n^2} \|u''\|_{[x_i, x_{i+1}],1}, \end{cases}$$

from which we get

$$\begin{aligned} & \left| \frac{1}{\tau-t} \int_{x_i}^{x_{i+1}} u(s) ds - \frac{1}{2n} \left[u\left(t+i \cdot \frac{\tau-t}{n}\right) + u\left(t+(i+1) \cdot \frac{\tau-t}{n}\right) \right] \right. \\ & \quad \left. + \frac{(\tau-t)}{8n^2} \left[u'\left(t+(i+1) \cdot \frac{\tau-t}{n}\right) - u'\left(t+i \cdot \frac{\tau-t}{n}\right) \right] \right| \\ & \leq \begin{cases} \frac{|\tau-t|^2}{24n^3} \|u''\|_{[x_i, x_{i+1}], \infty}, & \text{if } u'' \in L_\infty[a, b]; \\ \frac{|\tau-t|^{1+\frac{1}{q}}}{8n^{2+\frac{1}{q}}(2q+1)^{\frac{1}{q}}} \|u''\|_{[x_i, x_{i+1}], p}, & \text{if } u'' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{|\tau-t|}{8n^2} \|u''\|_{[x_i, x_{i+1}], 1}, & \end{cases} \end{aligned}$$

Summing over i from 0 to $n-1$ and using the generalised triangle inequality, we may write

$$\begin{aligned} & \left| \frac{1}{\tau-t} \int_t^\tau u(s) ds - \frac{1}{2n} \sum_{i=0}^{n-1} \left[u\left(t+i \cdot \frac{\tau-t}{n}\right) + u\left(t+(i+1) \cdot \frac{\tau-t}{n}\right) \right] \right. \\ & \quad \left. + \frac{(\tau-t)}{8n^2} \sum_{i=0}^{n-1} \left[u'\left(t+(i+1) \cdot \frac{\tau-t}{n}\right) - u'\left(t+i \cdot \frac{\tau-t}{n}\right) \right] \right| \\ & \leq \sum_{i=0}^{n-1} \left| \frac{1}{\tau-t} \int_{x_i}^{x_{i+1}} u(s) ds - \frac{1}{2n} \left[u\left(t+i \cdot \frac{\tau-t}{n}\right) + u\left(t+(i+1) \cdot \frac{\tau-t}{n}\right) \right] \right. \\ & \quad \left. + \frac{(\tau-t)}{8n^2} \left[u'\left(t+(i+1) \cdot \frac{\tau-t}{n}\right) - u'\left(t+i \cdot \frac{\tau-t}{n}\right) \right] \right| \\ & \leq \begin{cases} \frac{|\tau-t|^2}{24n^3} \sum_{i=0}^{n-1} \|u''\|_{[x_i, x_{i+1}], \infty}, & \text{if } u'' \in L_\infty[a, b]; \\ \frac{|\tau-t|^{1+\frac{1}{q}}}{8n^{2+\frac{1}{q}}(2q+1)^{\frac{1}{q}}} \sum_{i=0}^{n-1} \|u''\|_{[x_i, x_{i+1}], p}, & \text{if } u'' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{|\tau-t|}{8n^2} \sum_{i=0}^{n-1} \|u''\|_{[x_i, x_{i+1}], 1}, & \end{cases} \end{aligned}$$

However,

$$\begin{aligned} & \sum_{i=0}^{n-1} \|u''\|_{[x_i, x_{i+1}], \infty} \leq n \|u''\|_{[t, \tau], \infty}, \\ & \sum_{i=0}^{n-1} \|u''\|_{[x_i, x_{i+1}], p} = \sum_{i=0}^{n-1} \left| \int_{x_i}^{x_{i+1}} |u''(s)|^p ds \right|^{\frac{1}{p}} \leq n^{\frac{1}{q}} \left[\left(\sum_{i=0}^{n-1} \left| \int_{x_i}^{x_{i+1}} |u''(s)|^p ds \right|^{\frac{1}{p}} \right)^p \right]^{\frac{1}{p}} = n^{\frac{1}{q}} \|u''\|_{[t, \tau], p}, \end{aligned}$$

and

$$\sum_{i=0}^{n-1} \|u''\|_{[x_i, x_{i+1}], 1} \leq \left| \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} |u''(s)| ds \right| = \left| \int_t^\tau |u''(s)| ds \right| = \|u''\|_{[t, \tau], 1},$$

and the lemma is proved. □

The following theorem in approximating the Hilbert transform of a differentiable function whose second derivative is absolutely continuous holds.

Theorem 2.1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable function such that its derivative f' is absolutely continuous on $[a, b]$. If*

$$(2.4) \quad T_n(f; t) = \frac{f'(t)(b-a) + f(b) - f(a)}{2\pi n} + \frac{b-a}{\pi n} \sum_{i=1}^{n-1} \left[f; t - \frac{t-a}{n} \cdot i, t + \frac{b-t}{n} \cdot i \right]$$

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$$-\frac{1}{8n^2} \left[\frac{1}{2} f''(t)(a-b)(a+b-2t) - f(b) + f(a) + f'(b)(b-t) - f'(a)(a-t) \right],$$

then we have the estimate

$$(2.5) \quad \left| (Tf)(a, b; t) - \frac{f(t)}{\pi} \ln \left(\frac{b-t}{t-a} \right) - T_n(f; t) \right|$$

$$\leq \begin{cases} \frac{1}{24\pi n^2} \left[(b-a) \left(t - \frac{a+b}{2} \right)^2 + \frac{4}{3} (b-a)^3 \right] \|f'''\|_{[a,b],\infty}, & \text{if } f''' \in L_\infty[a, b]; \\ \frac{q}{8\pi n^2 (2q+1)^{1+\frac{1}{q}}} \left[(t-a)^{2+\frac{1}{q}} + (b-t)^{2+\frac{1}{q}} \right] \|f'''\|_{[a,b],p}, & \text{if } f''' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{8\pi n^2} \left[\frac{1}{4} (b-a)^2 + \left(t - \frac{a+b}{2} \right) \right] \|f'''\|_{[a,b],1}, \end{cases}$$

$$\leq \begin{cases} \frac{1}{18\pi n^2} (b-a)^3 \|f'''\|_{[a,b],\infty}, & \text{if } f''' \in L_\infty[a, b]; \\ \frac{q}{4\pi n^2 (2q+1)^{1+\frac{1}{q}}} \left(\frac{b-a}{2} \right)^{2+\frac{1}{q}} \|f'''\|_{[a,b],p}, & \text{if } f''' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{32\pi n^2} (b-a)^2 \|f'''\|_{[a,b],1}, \end{cases}$$

for all $t \in (a, b)$.

Proof. Applying Lemma 2.2 for the function f' , we may write that

$$(2.6) \quad \left| \frac{f(\tau) - f(t)}{\tau - t} - \frac{1}{2n} \left[f'(t) + \sum_{i=1}^{n-1} f' \left(t + i \cdot \frac{\tau - t}{n} \right) + \sum_{i=0}^{n-2} f' \left(t + (i+1) \cdot \frac{\tau - t}{n} \right) + f'(\tau) \right] \right.$$

$$\left. + \frac{\tau - t}{8n^2} \left[f''(\tau) + \sum_{i=0}^{n-2} f'' \left(t + (i+1) \cdot \frac{\tau - t}{n} \right) - \sum_{i=1}^{n-1} f'' \left(t + i \cdot \frac{\tau - t}{n} \right) - f''(t) \right] \right|$$

$$\leq \begin{cases} \frac{|\tau - t|^2}{24n^2} \|f'''\|_{[t,\tau],\infty} & \text{if } f''' \in L_\infty[a, b]; \\ \frac{|\tau - t|^{1+\frac{1}{q}}}{8n^2 (2q+1)^{\frac{1}{q}}} \|f'''\|_{[t,\tau],p}, & \text{if } f''' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{|\tau - t|}{8n^2} \|f'''\|_{[t,\tau],1}, \end{cases}$$

However,

$$\sum_{i=1}^{n-1} f' \left(t + i \cdot \frac{\tau - t}{n} \right) = \sum_{i=0}^{n-2} f' \left(t + (i+1) \cdot \frac{\tau - t}{n} \right), \sum_{i=1}^{n-1} f'' \left(t + i \cdot \frac{\tau - t}{n} \right) = \sum_{i=0}^{n-2} f'' \left(t + (i+1) \cdot \frac{\tau - t}{n} \right)$$

and then by (2.6), we may write:

$$(2.7) \quad \left| \frac{f(\tau) - f(t)}{\tau - t} - \left[\frac{f'(t) + f'(\tau)}{2n} + \frac{1}{n} \sum_{i=1}^{n-1} f' \left(t + i \cdot \frac{\tau - t}{n} \right) \right] + \frac{\tau - t}{8n^2} \left[f''(\tau) - f''(t) \right] \right|$$

$$\leq \begin{cases} \frac{|\tau - t|^2}{24n^2} \|f'''\|_{[t,\tau],\infty} \\ \frac{|\tau - t|^{1+\frac{1}{q}}}{8n^2 (2q+1)^{\frac{1}{q}}} \|f'''\|_{[t,\tau],p}, \\ \frac{|\tau - t|}{8n^2} \|f'''\|_{[t,\tau],1}, \end{cases}$$

for any $t, \tau \in [a, b], t \neq \tau$. Consequently, we have

$$(2.8) \quad \left| \frac{1}{\pi} PV \int_a^b \frac{f(\tau) - f(t)}{\tau - t} d\tau - \frac{1}{\pi} PV \int_a^b \left[\frac{f'(t) + f'(\tau)}{2n} + \frac{1}{n} \sum_{i=1}^{n-1} f' \left(t + i \cdot \frac{\tau - t}{n} \right) \right] d\tau \right|$$

$$\begin{aligned}
 & + \frac{1}{\pi} PV \int_a^b \frac{\tau - t}{8n^2} [f''(\tau) - f''(t)] d\tau \Big| \\
 \leq & \begin{cases} \frac{1}{24\pi n^2} PV \int_a^b |\tau - t|^2 \|f'''\|_{[t,\tau],\infty} d\tau, \\ \frac{1}{8\pi n^2 (2q+1)^{\frac{1}{q}}} PV \int_a^b |\tau - t|^{1+\frac{1}{q}} \|f'''\|_{[t,\tau],p} d\tau, \\ \frac{1}{8\pi n^2} PV \int_a^b |\tau - t| \|f'''\|_{[t,\tau],1} d\tau, \end{cases}
 \end{aligned}$$

Since

$$\begin{aligned}
 & PV \int_a^b \left[\frac{f'(t) + f'(\tau)}{2n} + \frac{1}{n} \sum_{i=1}^{n-1} f' \left(t + i \cdot \frac{\tau - t}{n} \right) \right] d\tau \\
 & = \frac{f'(t)(b-a) + f(b) - f(a)}{n} + \frac{b-a}{n} \sum_{i=1}^{n-1} \left[f; t + i \cdot \frac{b-t}{n}, t + i \cdot \frac{a-t}{n} \right],
 \end{aligned}$$

$$PV \int_a^b \frac{\tau - t}{8n^2} [f''(\tau) - f''(t)] d\tau = \frac{1}{8n^2} \left[\frac{1}{2} f''(t)(a-b)(a+b-2t) - f(b) + f(a) + f'(b)(b-t) - f'(a)(a-t) \right],$$

and

$$\begin{aligned}
 & PV \int_a^b |\tau - t|^2 \|f'''\|_{[t,\tau],\infty} d\tau \leq \|f'''\|_{[a,b],\infty} PV \int_a^b |\tau - t|^2 d\tau \\
 & = \|f'''\|_{[a,b],\infty} \frac{(b-t)^3 - (a-t)^3}{3} = \|f'''\|_{[a,b],\infty} \left[(b-a) \left(t - \frac{a+b}{2} \right)^2 + \frac{4}{3} (b-a)^3 \right],
 \end{aligned}$$

$$\begin{aligned}
 & PV \int_a^b |\tau - t|^{1+\frac{1}{q}} \|f'''\|_{[t,\tau],p} d\tau \leq \|f'''\|_{[a,b],p} PV \int_a^b |\tau - t|^{1+\frac{1}{q}} d\tau \\
 & = \|f'''\|_{[a,b],p} \left[\frac{(t-a)^{2+\frac{1}{q}} + (b-t)^{2+\frac{1}{q}}}{2 + \frac{1}{q}} \right] = \frac{q \|f'''\|_{[a,b],p}}{2q+1} \left[(t-a)^{2+\frac{1}{q}} + (b-t)^{2+\frac{1}{q}} \right],
 \end{aligned}$$

and

$$PV \int_a^b |\tau - t| \|f'''\|_{[t,\tau],1} d\tau \leq \|f'''\|_{[t,\tau],1} \left[\frac{1}{4} (b-a)^2 + \left(t - \frac{a+b}{2} \right)^2 \right],$$

then, by (2.8) we get

$$\begin{aligned}
 (2.9) \quad & \left| \frac{1}{\pi} PV \int_a^b \frac{f(\tau) - f(t)}{\tau - t} d\tau - \frac{f'(t)(b-a) + f(b) - f(a)}{2\pi n} - \frac{b-a}{\pi n} \sum_{i=1}^{n-1} \left[f; t - \frac{t-a}{n} \cdot i, t + \frac{b-t}{n} \cdot i \right] \right. \\
 & \left. + \frac{1}{8n^2} \left[\frac{1}{2} f''(t)(a-b)(a+b-2t) - f(b) + f(a) + f'(b)(b-t) - f'(a)(a-t) \right] \right| \\
 \leq & \begin{cases} \frac{\|f'''\|_{[a,b],\infty}}{24\pi n^2} \left[(b-a) \left(t - \frac{a+b}{2} \right)^2 + \frac{4}{3} (b-a)^3 \right], & \text{if } f''' \in L_\infty[a,b]; \\ \frac{q \|f'''\|_{[a,b],p}}{8\pi n^2 (2q+1)^{1+\frac{1}{q}}} \left[(t-a)^{2+\frac{1}{q}} + (b-t)^{2+\frac{1}{q}} \right], & \text{if } f''' \in L_p[a,b], p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{\|f'''\|_{[a,b],1}}{8\pi n^2} \left[\frac{1}{4} (b-a)^2 + \left(t - \frac{a+b}{2} \right)^2 \right], \end{cases}
 \end{aligned}$$

On the other hand, as for the function $f_0 : (a, b) \rightarrow \mathbb{R}, f_0(t) = 1$, we have

$$(T, f_0)(a, b; t) = \frac{1}{\pi} \ln \left(\frac{b-a}{t-a} \right), t \in (a, b),$$

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then obviously

$$(Tf)(a, b; t) = \frac{1}{\pi} PV \int_a^b \frac{f(\tau) - f(t) + f(t)}{\tau - t} d\tau = \frac{1}{\pi} PV \int_a^b \frac{f(\tau) - f(t)}{\tau - t} d\tau + \frac{f(t)}{\pi} PV \int_a^b \frac{d\tau}{\tau - t},$$

from which we get

$$(2.10) \quad (Tf)(a, b; t) - \frac{f(t)}{\pi} \ln \left(\frac{b-t}{t-a} \right) = \frac{1}{\pi} PV \int_a^b \frac{f(\tau) - f(t)}{\tau - t} d\tau.$$

Finally, using (2.9) and (2.10), we deduce (2.5). □

3. SOME NUMERICAL EXAMPLES

For a function $f : [a, b] \rightarrow \mathbb{R}$, we may consider the quadrature formula

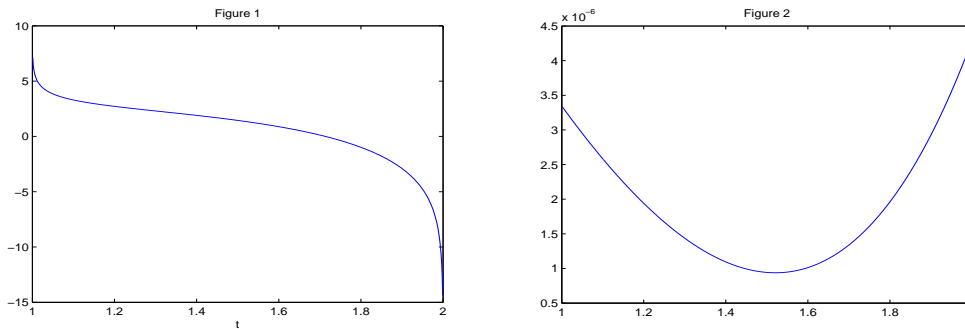
$$E_n(f; a, b, t) := \frac{f(t)}{\pi} \ln \left(\frac{b-t}{t-a} \right) + T_n(f; t), t \in [a, b].$$

As shown in the above section, $E_n(f; a, b, t)$ provides an approximation for the Finite Hilbert Transform $(Tf)(a, b; t)$ and the error estimate fulfils the bounds described in (2.5).

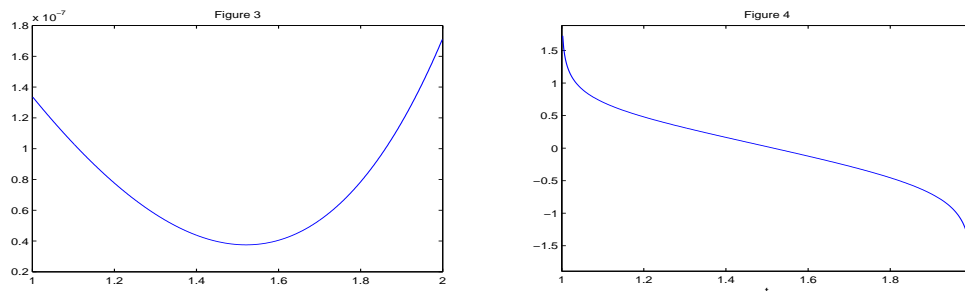
If we consider the function $f : [1, 2] \rightarrow \mathbb{R}$, $f(x) = \exp(x)$, the exact value of the Hilbert transform is

$$(Tf)(a, b; t) = \frac{\exp(t)Ei(2-t) - \exp(t)Ei(1-t)}{\pi}, t \in [1, 2].$$

and the plot of this function is embodied in Figure 1.



If we implement the quadrature formula provided by $E_n(f; a, b, t)$ using Matlab and chose the value of $n = 200$, the error $E_r(f; a, b, t) := (Tf)(a, b; t) - E_n(f; a, b, t)$ has the variation described in Figure 2.



For $n = 1000$, the plot of $E_r(f; a, b, t)$ is embodied in Figure 3, from which we can see that the precision of the error gets higher when n gets bigger.

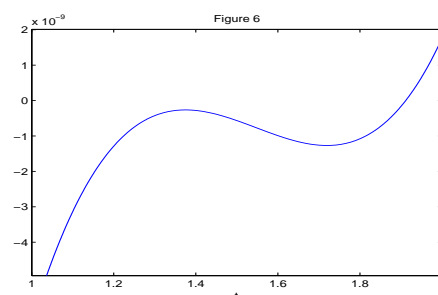
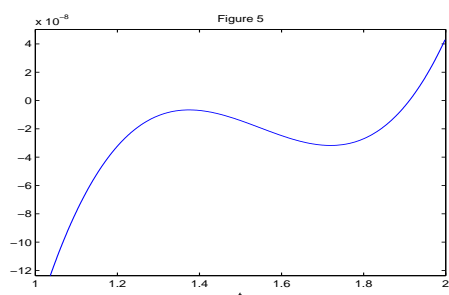
Now, if we consider another function $f : [1, 2] \rightarrow \mathbb{R}$, $f(x) = \sin(x)$, then

$$(Tf)(a, b; t) = \frac{-S_i(-2+t) \cos(t) + C_i(2-t) \sin(t) + S_i(t-1) \cos(t) - \sin(t)C_i(t-1)}{\pi}, t \in [1, 2]$$

having the plot embodied in Figure 4.

If we choose the value of $n = 200$, then the error $E_r(f; a, b, t) := (Tf)(a, b; t) - E_n(f; a, b, t)$ for the function $f(x) = \sin x$, $x \in [a, b]$ has the variation described in Figure 5. Moreover, for $n = 1000$, the behaviour of $E_r(f; a, b, t)$ is plotted in Figure 6.

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(S. F. Wang, X. Y. Gao and N. Lu) COLLEGE OF MATHEMATICS AND STATISTICS, NANJING UNIVERSITY OF INFORMATION SCIENCE AND TECHNOLOGY, NANJING 210044, CHINA

E-mail address: wsfnuist@yeah.net

Pointwise Superconvergence of the Displacement of the Six-Dimensional Finite Element

Yinsuo Jia* and Jinghong Liu†

In this article we first introduce definitions of the regularized Green's function, the discrete Green's function, the discrete δ function, and the L^2 -projection operator in six dimensions. Then the $W^{2,1}$ -seminorm estimates for the regularized Green's function and the discrete Green's function are derived. Finally, pointwise superconvergence of the displacement of the six-dimensional finite element is obtained.

1 Introduction

There have been many studies concerned with superconvergence of the finite element method for partial differential equations. Books and survey papers have been published. For the literature, we refer to [1–17] and references therein. It is well known that estimates for the Green's function play very important roles in the study of the superconvergence (especially, pointwise superconvergence) of the finite element method (see [4, 5, 8, 12, 13, 14, 17]). For one- and two-dimensional elliptic problems, one have obtained many optimal estimates for the Green's function (see [17]). Recently, for three-dimensional elliptic problems, the $W^{2,1}$ -seminorm optimal estimate with order $O(|\ln h|^{\frac{2}{3}})$ for the discrete Green's function was derived (see [12]).

In this article, we will discuss estimate for the the discrete Green's function based on the 6D Poisson equation.

we shall use the symbol C to denote a generic constant, which is independent from the discretization parameter h and which may not be the same in each occurrence and also use the standard notations for the Sobolev spaces and their norms.

We consider the following Poisson equation:

$$\mathcal{L}u \equiv -\Delta u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad (1.1)$$

where $\Omega \subset \mathcal{R}^6$ is a bounded polytopical domain. The weak formulation of (1.1)

*School of Mathematics and Computer Science, Shangrao Normal University, Shangrao 334001, China, email: jiyinsuo2002@sohu.com

†School of Mathematics and Computer Science, Shangrao Normal University, Shangrao 334001, China, email: ddliu1010@163.com

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reads,

$$\begin{cases} \text{Find } u \in H_0^1(\Omega) \text{ satisfying} \\ a(u, v) = (f, v) \text{ for all } v \in H_0^1(\Omega). \end{cases}$$

where

$$a(u, v) \equiv \int_{\Omega} \nabla u \cdot \nabla v \, dX,$$

and

$$(f, v) \equiv \int_{\Omega} f v \, dX.$$

Let $\{\mathcal{T}^h\}$ be a regular family of partitions of $\bar{\Omega}$. Denote by $S^h(\Omega)$ a continuous piecewise $m(m \geq 2)$ -degree (or tensor-product m -degree) polynomials space regarding this kind of partitions and let $S_0^h(\Omega) = S^h(\Omega) \cap H_0^1(\Omega)$. Discretizing the above weak formulation using $S_0^h(\Omega)$ as approximating space means,

$$\begin{cases} \text{Find } u_h \in S_0^h(\Omega) \text{ satisfying} \\ a(u_h, v) = (f, v) \text{ for all } v \in S_0^h(\Omega). \end{cases}$$

Thus, the following Galerkin orthogonality relation holds.

$$a(u - u_h, v) = 0 \quad \forall v \in S_0^h(\Omega). \tag{1.2}$$

For every $Z \in \Omega$, we define the discrete δ function $\delta_Z^h \in S_0^h(\Omega)$, the regularized Green's function $G_Z^* \in H^2(\Omega) \cap H_0^1(\Omega)$, the discrete Green's function $G_Z^h \in S_0^h(\Omega)$ and the L^2 -projection $P_h u \in S_0^h(\Omega)$ such that (see [17])

$$(v, \delta_Z^h) = v(Z) \quad \forall v \in S_0^h(\Omega), \tag{1.3}$$

$$a(G_Z^*, v) = (\delta_Z^h, v) \quad \forall v \in H_0^1(\Omega), \tag{1.4}$$

$$a(G_Z^* - G_Z^h, v) = 0 \quad \forall v \in S_0^h(\Omega), \tag{1.5}$$

$$(u - P_h u, v) = 0 \quad \forall v \in S_0^h(\Omega), \tag{1.6}$$

In this article, we will bound the terms $|G_Z^*|_{2,1}$ and $|G_Z^h|_{2,1}^h$. Here $|G_Z^h|_{2,1}^h = \sum_{e \in \mathcal{T}^h} |G_Z^h|_{2,1,e}$.

2 Estimates for the Regularized Green's Function

We first introduce the weight function defined by

$$\phi \equiv \phi(X) = (|X - \bar{X}|^2 + \theta^2)^{-3} \quad \forall X \in \bar{\Omega}, \tag{2.1}$$

where $\bar{X} \in \bar{\Omega}$ is a fixed point, $\theta = \gamma h$, and $\gamma \in [6, +\infty)$ is a suitable real number.

For every $\alpha \in \mathcal{R}$, we give the following notations:

$$|\nabla^n v|^2 = \sum_{|\beta|=n} |D^\beta v|^2, \quad |\nabla^n v|_{\phi^\alpha} = \left(\int_{\Omega} \phi^\alpha |\nabla^n v|^2 \, dX \right)^{\frac{1}{2}}, \quad \|v\|_{m, \phi^\alpha}^2 = \sum_{n=0}^m |\nabla^n v|_{\phi^\alpha}^2,$$

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where $\beta = (\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6)$, $|\beta| = \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_6$, and $\beta_i \geq 0$, $i = 1, \dots, 6$ are integers. In particular, for the case of $m = 0$, we write

$$\|v\|_{\phi^\alpha} = \left(\int_{\Omega} \phi^\alpha |v|^2 dX \right)^{\frac{1}{2}}.$$

We assume there exists a real number q_0 ($1 < q_0 \leq \infty$) such that

$$\|v\|_{2,q} \leq C(q) \|\mathcal{L}v\|_{0,q} \quad \forall v \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega), 1 < q < q_0, \quad (2.2)$$

which is the so-called a priori estimate (see [17]). As in the two-dimensional case (see [17]), we can obtain the following Lemma 2.1.

Lemma 2.1. *For ϕ the weight function defined by (2.1), we have the following estimates:*

$$|\nabla^n \phi^\alpha| \leq C(\alpha, n) \phi^{\alpha + \frac{n}{6}}, \quad \alpha \in \mathcal{R}, n = 1, 2, \quad (2.3)$$

$$\int_{\Omega} \phi dX \leq C(k) |\ln \theta|, \quad \theta \leq k < 1, \quad (2.4)$$

$$\int_{\Omega} \phi^\alpha dX \leq C(\alpha - 1)^{-1} \theta^{-6(\alpha - 1)} \quad \forall \alpha > 1. \quad (2.5)$$

For the L^2 -projection operator P_h and the discrete δ function δ_Z^h , similar to the arguments in the two-dimensional setting (see [17]), we have the following results (2.6)–(2.8).

Lemma 2.2. *For $P_h w$ the L^2 -projection of w , we have the following stability estimate:*

$$\|P_h w\|_{0,q} \leq C \|w\|_{0,q}, \quad 1 \leq q \leq \infty, \quad (2.6)$$

$$\|P_h w\|_{1,q} \leq C \|w\|_{1,q}, \quad 6 < q \leq \infty. \quad (2.7)$$

Lemma 2.3. *For δ_Z^h the discrete δ function defined by (1.3), we have the following estimate:*

$$|\delta_Z^h(X)| \leq Ch^{-6} e^{-Ch^{-1}|X-Z|}, \quad (2.8)$$

where $X, Z \in \bar{\Omega}$, and C is independent of h, X , and Z .

In addition, we have the following weighted-norm estimate for δ_Z^h .

Lemma 2.4. *For δ_Z^h the discrete δ function defined by (1.3) and ϕ defined by (2.1), we have the following estimate:*

$$\|\delta_Z^h\|_{\phi^{-1}} \leq C. \quad (2.9)$$

Proof. From (2.1) and (2.8),

$$\begin{aligned} \|\delta_Z^h\|_{\phi^{-1}}^2 &\leq C \int_{\Omega} (|X - Z|^2 + \theta^2)^3 h^{-12} e^{-Ch^{-1}|X-Z|} dX \\ &\leq C \int_0^\infty (r^2 + \theta^2)^3 h^{-12} e^{-Ch^{-1}r} r^5 dr. \end{aligned}$$

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Set $h^{-1}r = t$, then

$$\|\delta_Z^h\|_{\phi^{-1}}^2 \leq C \int_0^\infty (t^2 + \gamma^2)^3 e^{-Ct} t^5 dt \leq C,$$

which is the result (2.9).

Lemma 2.5. For G_Z^* the regularized Green's function defined by (1.4) and ϕ defined by (2.1), we have the following weighted-norm estimate:

$$\|G_Z^*\|_{\phi^{-\frac{1}{3}}} \leq C |\ln h|^{\frac{5}{6}}. \tag{2.10}$$

Proof. From (1.3), (1.4), (1.6), (2.2), (2.6), the inverse estimate, the Sobolev Embedding Theorem (see [18]), and the Poincaré inequality, we have

$$\begin{aligned} \|G_Z^*\|_{\phi^{-\frac{1}{3}}}^2 &= (G_Z^*, \phi^{-\frac{1}{3}} G_Z^*) = a(G_Z^*, w) = (\delta_Z^h, w) = P_h w(Z) \leq |P_h w|_{0, \infty} \\ &\leq Ch^{-\frac{6}{q}} |P_h w|_{0, q} \leq Ch^{-\frac{6}{q}} |w|_{0, q} \leq Ch^{-\frac{6}{q}} q^{\frac{5}{6}} \|w\|_{1, 6} \\ &\leq Ch^{-\frac{6}{q}} q^{\frac{5}{6}} \|w\|_{2, 3} \leq Ch^{-\frac{6}{q}} q^{\frac{5}{6}} \|\phi^{-\frac{1}{3}} G_Z^*\|_{0, 3} \\ &\leq Ch^{-\frac{6}{q}} q^{\frac{5}{6}} \left| \phi^{-\frac{1}{3}} G_Z^* \right|_1, \end{aligned} \tag{2.11}$$

where $w \in W^{2, 3}(\Omega) \cap W_0^{1, 6}(\Omega)$ and $\mathcal{L}w = \phi^{-\frac{1}{3}} G_Z^*$. Set $q = |\ln h|$ in (2.11), and by the Young inequality, we get

$$\|G_Z^*\|_{\phi^{-\frac{1}{3}}}^2 \leq C |\ln h|^{\frac{5}{6}} \left| \phi^{-\frac{1}{3}} G_Z^* \right|_1 \leq C(\varepsilon) |\ln h|^{\frac{5}{3}} + \varepsilon \left| \phi^{-\frac{1}{3}} G_Z^* \right|_1^2 \tag{2.12}$$

In addition, from (1.4) and (2.3),

$$\begin{aligned} \left| \phi^{-\frac{1}{3}} G_Z^* \right|_1^2 &\leq Ca \left(\phi^{-\frac{1}{3}} G_Z^*, \phi^{-\frac{1}{3}} G_Z^* \right) \\ &\leq C \left| a(G_Z^*, \phi^{-\frac{2}{3}} G_Z^*) \right| + C \left| \left((G_Z^*)^2, |\nabla(\phi^{-\frac{1}{3}})|^2 \right) \right| \\ &\leq C \left| a(G_Z^*, \phi^{-\frac{2}{3}} G_Z^*) \right| + C \|G_Z^*\|_{\phi^{-\frac{1}{3}}}^2 \\ &\leq \tilde{C} \|\delta_Z^h\|_{\phi^{-1}}^2 + \tilde{C} \|G_Z^*\|_{\phi^{-\frac{1}{3}}}^2. \end{aligned} \tag{2.13}$$

Combining (2.9), (2.12), (2.13), and choosing ε such that $\varepsilon \tilde{C} = \frac{1}{2}$, we immediately obtain the result (2.10).

Theorem 2.1. For G_Z^* the regularized Green's function defined by (1.4), we have the following $W^{2, 1}$ -seminorm estimate:

$$|G_Z^*|_{2, 1} \leq C |\ln h|^{\frac{4}{3}}. \tag{2.14}$$

Proof. Obviously,

$$|G_Z^*|_{2, 1}^2 \leq \int_\Omega \phi dX \cdot \|\nabla^2 G_Z^*\|_{\phi^{-1}}^2. \tag{2.15}$$

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Furthermore,

$$\begin{aligned}
 & \|\nabla^2 G_Z^*\|_{\phi^{-1}}^2 = \int_{\Omega} \phi^{-1} |\nabla^2 G_Z^*|^2 dX = \int_{\Omega} \left(\phi^{-\frac{1}{2}} |\nabla^2 G_Z^*| \right)^2 dX \\
 & \leq C \left(\int_{\Omega} |\nabla^2 (\phi^{-\frac{1}{2}} G_Z^*)|^2 dX + \int_{\Omega} |\nabla^2 \phi^{-\frac{1}{2}} G_Z^*|^2 dX \right. \\
 & \quad \left. + \int_{\Omega} |\nabla \phi^{-\frac{1}{2}}|^2 |\nabla G_Z^*|^2 dx \right) \\
 & \leq C \left(\|\nabla^2 (\phi^{-\frac{1}{2}} G_Z^*)\|_0^2 + \|G_Z^*\|_{\phi^{-\frac{1}{3}}}^2 + |G_Z^*|_{1, \phi^{-\frac{2}{3}}}^2 \right) \\
 & \leq C \left(\|\mathcal{L}(\phi^{-\frac{1}{2}} G_Z^*)\|_0^2 + \|G_Z^*\|_{\phi^{-\frac{1}{3}}}^2 + |G_Z^*|_{1, \phi^{-\frac{2}{3}}}^2 \right) \\
 & \leq C \left(\|\mathcal{L}G_Z^*\|_{\phi^{-1}}^2 + \|G_Z^*\|_{\phi^{-\frac{1}{3}}}^2 + |G_Z^*|_{1, \phi^{-\frac{2}{3}}}^2 \right) \\
 & \leq C \|\delta_Z^h\|_{\phi^{-1}}^2 + C \left| a(G_Z^*, \phi^{-\frac{2}{3}} G_Z^*) \right| + C \|G_Z^*\|_{\phi^{-\frac{1}{3}}}^2 \\
 & \leq C \|\delta_Z^h\|_{\phi^{-1}}^2 + C \left| (\delta_Z^h, \phi^{-\frac{2}{3}} G_Z^*) \right| + C \|G_Z^*\|_{\phi^{-\frac{1}{3}}}^2 \\
 & \leq C \|\delta_Z^h\|_{\phi^{-1}}^2 + C \|G_Z^*\|_{\phi^{-\frac{1}{3}}}^2,
 \end{aligned}$$

combined with (2.9) and (2.10), we have

$$\|\nabla^2 G_Z^*\|_{\phi^{-1}}^2 \leq C |\ln h|^{\frac{5}{3}}. \tag{2.16}$$

By (2.4), (2.15), and (2.16), we immediately obtain the result (2.14).

3 Estimates for the Discrete Green's Function

The definition (1.5) shows that G_Z^h is a finite element approximation to G_Z^* . In this section, we give the $W^{2,1}$ -seminorm estimate for G_Z^h .

Lemma 3.1. *For G_Z^* and G_Z^h , the regularized Green's function and the discrete Green's function, respectively, we have the following estimate:*

$$|G_Z^* - G_Z^h|_{1,1} \leq Ch |\ln h|^{\frac{4}{3}}. \tag{3.1}$$

Proof. Obviously,

$$|G_Z^* - G_Z^h|_{1,1}^2 \leq \int_{\Omega} \phi dX \cdot |G_Z^* - G_Z^h|_{1, \phi^{-1}}^2. \tag{3.2}$$

Similar to the proof of (2.43) in [13], and using (2.16), we have

$$\begin{aligned}
 |G_Z^* - G_Z^h|_{1, \phi^{-1}}^2 & \leq Ch^2 \|\nabla^2 G_Z^*\|_{\phi^{-1}}^2 + \hat{C} \|G_Z^* - G_Z^h\|_{\phi^{-\frac{2}{3}}}^2 \\
 & \leq Ch^2 |\ln h|^{\frac{5}{3}} + \hat{C} \|G_Z^* - G_Z^h\|_{\phi^{-\frac{2}{3}}}^2.
 \end{aligned} \tag{3.3}$$

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In addition

$$\begin{aligned} & \|G_Z^* - G_Z^h\|_{\phi^{-\frac{2}{3}}}^2 = (\phi^{-\frac{2}{3}}(G_Z^* - G_Z^h), G_Z^* - G_Z^h) = a(w, G_Z^* - G_Z^h) \\ & = a(w - \Pi w, G_Z^* - G_Z^h) \leq \varepsilon |G_Z^* - G_Z^h|_{1, \phi^{-1}}^2 + C(\varepsilon) |w - \Pi w|_{1, \phi}^2. \end{aligned} \tag{3.4}$$

where $\mathcal{L}w = \phi^{-\frac{2}{3}}(G_Z^* - G_Z^h)$ and Π is an interpolation operator. Using the weighted interpolation error estimate in (3.4) (similar to pp.110 Lemma 4 in [17]) yields

$$\|G_Z^* - G_Z^h\|_{\phi^{-\frac{2}{3}}}^2 \leq \varepsilon |G_Z^* - G_Z^h|_{1, \phi^{-1}}^2 + C(\varepsilon) h^2 |\nabla^2 w|_{\phi}^2. \tag{3.5}$$

Further, from the a priori estimate (2.2), (2.5), and the Sobolev Embedding Theorem [19],

$$\begin{aligned} |\nabla^2 w|_{\phi}^2 & \leq \|\phi\|_{0, \frac{3}{2}} |\nabla^2 w|_{0, 6}^2 \leq C\theta^{-2} \|w\|_{2, 6}^2 \leq C\theta^{-2} \left\| \phi^{-\frac{2}{3}}(G_Z^* - G_Z^h) \right\|_{0, 6}^2 \\ & \leq C\theta^{-2} \left\| \phi^{-\frac{2}{3}}(G_Z^* - G_Z^h) \right\|_2^2 \leq C\theta^{-2} \left\| \mathcal{L} \left(\phi^{-\frac{2}{3}}(G_Z^* - G_Z^h) \right) \right\|_0^2 \\ & \leq C\theta^{-2} \left(\left\| \phi^{-\frac{2}{3}} \delta_Z^h \right\|_0^2 + |G_Z^* - G_Z^h|_{1, \phi^{-1}}^2 + \|G_Z^* - G_Z^h\|_{\phi^{-\frac{2}{3}}}^2 \right). \end{aligned} \tag{3.6}$$

Similar to the proof of (2.9), we can obtain

$$\left\| \phi^{-\frac{2}{3}} \delta_Z^h \right\|_0^2 \leq Ch^2. \tag{3.7}$$

Combining (3.5)–(3.7) yields

$$\begin{aligned} \|G_Z^* - G_Z^h\|_{\phi^{-\frac{2}{3}}}^2 & \leq (\varepsilon + C(\varepsilon)\gamma^{-2}) |G_Z^* - G_Z^h|_{1, \phi^{-1}}^2 \\ & \quad + C(\varepsilon)\gamma^{-2} \|G_Z^* - G_Z^h\|_{\phi^{-\frac{2}{3}}}^2 + C(\varepsilon)\gamma^{-2} h^2. \end{aligned} \tag{3.8}$$

Choosing suitable ε and $\gamma \in [6, +\infty)$ such that $0 < (2\varepsilon + 1)\hat{C} < 1$ as well as $C(\varepsilon)\gamma^{-2} = \frac{1}{2}$. From (3.8),

$$\|G_Z^* - G_Z^h\|_{\phi^{-\frac{2}{3}}}^2 \leq (2\varepsilon + 1) |G_Z^* - G_Z^h|_{1, \phi^{-1}}^2 + h^2. \tag{3.9}$$

From (3.3) and (3.9),

$$|G_Z^* - G_Z^h|_{1, \phi^{-1}}^2 \leq Ch^2 |\ln h|^{\frac{5}{3}}. \tag{3.10}$$

The result (3.1) immediately follows the results (2.4), (3.2), and (3.10).

Theorem 3.1. For G_Z^h the discrete Green's function, we have the following estimate:

$$|G_Z^h|_{2, 1}^h \leq C |\ln h|^{\frac{4}{3}}. \tag{3.11}$$

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Proof. By the triangle inequality, the interpolation error estimate, and the inverse property, we have

$$\begin{aligned}
 |G_Z^h|_{2,1}^h &\leq |G_Z^* - G_Z^h|_{2,1}^h + |G_Z^*|_{2,1} \\
 &\leq |G_Z^*|_{2,1} + |G_Z^* - \Pi G_Z^*|_{2,1}^h + |\Pi G_Z^* - G_Z^h|_{2,1}^h \\
 &\leq C |G_Z^*|_{2,1} + Ch^{-1} |\Pi G_Z^* - G_Z^h|_{1,1} \\
 &\leq C |G_Z^*|_{2,1} + Ch^{-1} |G_Z^* - \Pi G_Z^*|_{1,1} \\
 &\quad + Ch^{-1} |G_Z^* - G_Z^h|_{1,1} \\
 &\leq C |G_Z^*|_{2,1} + Ch^{-1} |G_Z^* - G_Z^h|_{1,1}.
 \end{aligned} \tag{3.12}$$

Combining (2.14), (3.1), and (3.12) yields the result (3.11).

4 Superconvergence of the Displacement of the Finite Element

In this section, we give an application of the estimate for the discrete Green’s function in finite element superconvergence.

Let Πu and u_h be the interpolant and the finite element approximation to u , the solution of (1.1), respectively. Similar to the proof of [13], we can obtain the following lemma.

Lemma 4.1. *Let $S_0^h(\Omega)$ be the tensor-product m -degree finite element space. Suppose $v \in S_0^h(\Omega)$ and $u \in W^{m+2,\infty}(\Omega) \cap H_0^1(\Omega)$. Then we have the following weak estimate of the second type:*

$$|a(u - \Pi u, v)| \leq Ch^{m+2} \|u\|_{m+2,\infty} |v|_{2,1}^h, \quad m \geq 2, \tag{4.1}$$

where $|v|_{2,1}^h = \sum_{e \in \mathcal{T}^h} |v|_{2,1,e}$.

Finally, we give the following superconvergent estimate.

Theorem 4.1. *Let $\{\mathcal{T}^h\}$ be a regular family of partitions of $\bar{\Omega}$ and $u \in W^{m+2,\infty}(\Omega) \cap H_0^1(\Omega)$. For u_h and Πu , the tensor-product m -degree finite element approximation and the corresponding interpolant to u , respectively. Then we have the following superconvergent estimates:*

$$|u_h - \Pi u|_{0,\infty,\Omega} \leq Ch^{m+2} |\ln h|^{\frac{4}{3}} \|u\|_{m+2,\infty}, \quad m \geq 2. \tag{4.2}$$

Proof. For every $Z \in \Omega$, applying the definition of G_Z^h and the Galerkin orthogonality relation (1.2), we derive

$$(u_h - \Pi u)(Z) = a(u_h - \Pi u, G_Z^h) = a(u - \Pi u, G_Z^h). \tag{4.3}$$

From (3.11), (4.1), and (4.3), we immediately obtain the result (4.2).

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Estimates for Discrete Derivative Green's Function for Elliptic Equations in Dimensions Seven and Up

Jinghong Liu* and Yinsuo Jia[†]

This article will discuss estimates for discrete derivative Green's function for elliptic equations in dimensions seven and up. First, the definitions of some terms are given. Then the estimates for the regularized derivative Green's function are derived. Finally, using the triangular inequality, we obtain the estimates for discrete derivative Green's function. The results of the article play important roles in the research of superconvergence of finite element methods.

1 Introduction

It is well known that estimates for the Green's function play very important roles in the study of the superconvergence (especially, pointwise superconvergence) of the finite element method (see [1–8]). For one- and two-dimensional elliptic problems, one have obtained many optimal estimates for the Green's function (see [8]). Recently, for dimensions three to five, we have obtained some optimal estimates for the discrete Green's function (see [4–7]). At present, we also consider the six-dimensional discrete Green's function and its estimates, and some results have been submitted to some Journals. In this article, we will discuss estimates for the discrete derivative Green's function in dimensions seven and up.

we shall use the symbol C to denote a generic constant, which is independent from the discretization parameter h and which may not be the same in each occurrence and also use the standard notations for the Sobolev spaces and their norms.

We consider the following Poisson equation:

$$\mathcal{L}u \equiv -\Delta u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad (1.1)$$

where $\Omega \subset \mathcal{R}^d (d \geq 7)$ is a bounded polytopic domain. The weak formulation of

*Department of Fundamental Courses, Ningbo Institute of Technology, Zhejiang University, Ningbo 315100, China, email: jhliu1129@sina.com

[†]School of Mathematics and Computer Science, Shangrao Normal University, Shangrao 334001, China, email: jiyinsuo2002@sohu.com

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(1.1) reads,

$$\begin{cases} \text{Find } u \in H_0^1(\Omega) \text{ satisfying} \\ a(u, v) = (f, v) \text{ for all } v \in H_0^1(\Omega). \end{cases}$$

where

$$a(u, v) \equiv \int_{\Omega} \nabla u \cdot \nabla v \, dX,$$

and

$$(f, v) \equiv \int_{\Omega} f v \, dX.$$

Let $\{\mathcal{T}^h\}$ be a regular family of partitions of $\bar{\Omega}$. Denote by $S^h(\Omega)$ a continuous finite elements space regarding this kind of partitions and let $S_0^h(\Omega) = S^h(\Omega) \cap H_0^1(\Omega)$. Discretizing the above weak formulation using $S_0^h(\Omega)$ as approximating space means,

$$\begin{cases} \text{Find } u_h \in S_0^h(\Omega) \text{ satisfying} \\ a(u_h, v) = (f, v) \text{ for all } v \in S_0^h(\Omega). \end{cases}$$

For every $Z \in \Omega$, we define the discrete derivative δ function $\partial_{Z,\ell}\delta_Z^h \in S_0^h(\Omega)$ and the L^2 -projection $P_h u \in S_0^h(\Omega)$ such that

$$(v, \partial_{Z,\ell}\delta_Z^h) = \partial_{\ell}v(Z) \quad \forall v \in S_0^h(\Omega). \tag{1.2}$$

$$(u - P_h u, v) = 0 \quad \forall v \in S_0^h(\Omega). \tag{1.3}$$

Here, for any direction $\ell \in R^d$, $|\ell| = 1$, $\partial_{Z,\ell}\delta_Z^h$ and $\partial_{\ell}v(Z)$ stand for the following on-sided directional derivatives, respectively.

$$\partial_{Z,\ell}\delta_Z^h = \lim_{|\Delta Z| \rightarrow 0} \frac{\delta_{Z+\Delta Z}^h - \delta_Z^h}{|\Delta Z|}, \quad \partial_{\ell}v(Z) = \lim_{|\Delta Z| \rightarrow 0} \frac{v(Z + \Delta Z) - v(Z)}{|\Delta Z|}, \quad \Delta Z = |\Delta Z|\ell.$$

Let $\partial_{Z,\ell}G_Z^* \in H^2(\Omega) \cap H_0^1(\Omega)$ be the solution of the elliptic problem $-\Delta \partial_{Z,\ell}G_Z^* = \partial_{Z,\ell}\delta_Z^h$. We may call $\partial_{Z,\ell}G_Z^*$ the regularized derivative Green's function. Further, let the discrete derivative Green's function $\partial_{Z,\ell}G_Z^h \in S_0^h(\Omega)$ be the finite element approximation to $\partial_{Z,\ell}G_Z^*$. Thus,

$$a(\partial_{Z,\ell}G_Z^* - \partial_{Z,\ell}G_Z^h, v) = 0 \quad \forall v \in S_0^h(\Omega). \tag{1.4}$$

In this article, we will bound the terms $|\partial_{Z,\ell}G_Z^*|_{1,1}$ and $|\partial_{Z,\ell}G_Z^h|_{1,1}$.

2 Regularized Derivative Green's Function and Its Estimates

We first introduce the weight function defined by

$$\phi \equiv \phi(X) = (|X - \bar{X}|^2 + \theta^2)^{-\frac{d}{2}} \quad \forall X \in \bar{\Omega}, \tag{2.1}$$

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where $\bar{X} \in \bar{\Omega}$ is a fixed point, $\theta = \gamma h$, and $\gamma \in [d, +\infty)$ is a suitable real number.

For every $\alpha \in \mathcal{R}$, we give the following notations:

$$|\nabla^n v|^2 = \sum_{|\beta|=n} |D^\beta v|^2, |\nabla^n v|_{\phi^\alpha} = \left(\int_{\Omega} \phi^\alpha |\nabla^n v|^2 dX \right)^{\frac{1}{2}}, \|v\|_{m, \phi^\alpha}^2 = \sum_{n=0}^m |\nabla^n v|_{\phi^\alpha}^2,$$

where $\beta = (\beta_1, \beta_2, \dots, \beta_d)$, $|\beta| = \beta_1 + \beta_2 + \dots + \beta_d$, and $\beta_i \geq 0, i = 1, \dots, d$ are integers. In particular, for the case of $m = 0$, we write

$$\|v\|_{\phi^\alpha} = \left(\int_{\Omega} \phi^\alpha |v|^2 dX \right)^{\frac{1}{2}}.$$

We assume there exists a real number q_0 ($1 < q_0 \leq \infty$) such that

$$\|v\|_{2, q} \leq C(q) \|\mathcal{L}v\|_{0, q} \quad \forall v \in W^{2, q}(\Omega) \cap W_0^{1, q}(\Omega), 1 < q < q_0, \quad (2.2)$$

which is the so-called a priori estimate (see [8]). As in the two-dimensional case (see [8]), we can obtain the following Lemma 2.1.

Lemma 2.1. *For ϕ the weight function defined by (2.1), we have the following estimates:*

$$|\nabla^n \phi^\alpha| \leq C(\alpha, n) \phi^{\alpha + \frac{n}{d}}, \quad \alpha \in \mathcal{R}, n = 1, 2, \quad (2.3)$$

$$\int_{\Omega} \phi dX \leq C(k) |\ln \theta|, \quad \theta \leq k < 1, \quad (2.4)$$

$$\int_{\Omega} \phi^\alpha dX \leq C(\alpha - 1)^{-1} \theta^{-d(\alpha - 1)} \quad \forall \alpha > 1. \quad (2.5)$$

$$\int_{\Omega} \phi^\alpha dX \leq C(1 - \alpha)^{-1} \quad \forall 0 < \alpha < 1. \quad (2.6)$$

In addition, we also have the following Lemmas.

Lemma 2.2. *For $P_h w$ the L^2 -projection of w , we have the following stability estimate:*

$$\|P_h w\|_{0, q} \leq C \|w\|_{0, q}, \quad 1 \leq q \leq +\infty. \quad (2.7)$$

Lemma 2.3. *For $\partial_{Z, \ell} \delta_Z^h$ the discrete derivative δ function defined by (1.2), we have the following estimate:*

$$|\partial_{Z, \ell} \delta_Z^h(X)| \leq Ch^{-d-1} e^{-Ch^{-1}|X-Z|}, \quad (2.8)$$

where $X, Z \in \bar{\Omega}$, and C is independent of h, X , and Z .

As for $\partial_{Z, \ell} \delta_Z^h$, we have the following important estimate.

Lemma 2.4. *For $\partial_{Z, \ell} \delta_Z^h$ the discrete derivative δ function defined by (1.2) and ϕ defined by (2.1), when $\alpha > 0$, we have the following estimate:*

$$\|\partial_{Z, \ell} \delta_Z^h\|_{\phi^{-\alpha}} \leq Ch^{\frac{d(\alpha-1)-2}{2}}. \quad (2.9)$$

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Proof. From (2.1) and (2.8),

$$\begin{aligned} \|\partial_{Z,\ell}\delta_Z^h\|_{\phi^{-\alpha}}^2 &\leq C \int_{\Omega} (|X-Z|^2 + \theta^2)^{\frac{d\alpha}{2}} h^{-2d-2} e^{-Ch^{-1}|X-Z|} dX \\ &\leq C \int_0^\infty (r^2 + \theta^2)^{\frac{d\alpha}{2}} h^{-2d-2} e^{-Ch^{-1}r} r^{d-1} dr. \end{aligned}$$

Set $h^{-1}r = t$, then

$$\|\partial_{Z,\ell}\delta_Z^h\|_{\phi^{-\alpha}}^2 \leq Ch^{d(\alpha-1)-2}$$

which is the result (2.9).

Lemma 2.5. *Suppose $q_0 > 2$ and $0 < \varepsilon < 1$. For $\partial_{Z,\ell}G_Z^*$ the regularized derivative Green's function defined by (1.4) and ϕ defined by (2.1), we have the following weighted-norm estimate:*

$$\|\partial_{Z,\ell}G_Z^*\|_{\phi^{1-\varepsilon}} \leq Ch^{1-d+\frac{\varepsilon d}{2}}. \tag{2.10}$$

Proof. Set $r = \frac{1+\varepsilon}{1-\varepsilon}$, $r' = \frac{1+\varepsilon}{2\varepsilon}$, thus $\frac{1}{r} + \frac{1}{r'} = 1$. From (2.5),

$$\begin{aligned} \|\partial_{Z,\ell}G_Z^*\|_{\phi^{1-\varepsilon}}^2 &= \int_{\Omega} \phi^{1-\varepsilon} |\partial_{Z,\ell}G_Z^*|^2 dX \\ &\leq \left(\int_{\Omega} \phi^{1+\varepsilon} dX \right)^{\frac{1-\varepsilon}{1+\varepsilon}} \|\partial_{Z,\ell}G_Z^*\|_{0, \frac{1+\varepsilon}{\varepsilon}}^2 \\ &\leq C (\varepsilon^{-1}\theta^{-d\varepsilon})^{\frac{1-\varepsilon}{1+\varepsilon}} \|\partial_{Z,\ell}G_Z^*\|_{0, \frac{1+\varepsilon}{\varepsilon}}^2. \end{aligned}$$

Further,

$$\begin{aligned} \|\partial_{Z,\ell}G_Z^*\|_{0, \frac{1+\varepsilon}{\varepsilon}}^{\frac{1+\varepsilon}{\varepsilon}} &= \left(\partial_{Z,\ell}G_Z^*, |\partial_{Z,\ell}G_Z^*|^{\frac{1}{\varepsilon}} \operatorname{sgn} \partial_{Z,\ell}G_Z^* \right) \\ &= a(\partial_{Z,\ell}G_Z^*, w) = (\partial_{Z,\ell}\delta_Z^h, w) = \partial_{\ell}P_h w(Z) \\ &\leq |P_h w|_{1, \infty} \leq Ch^{-\frac{d}{q}-1} \|P_h w\|_{0, q} \\ &\leq Ch^{-\frac{d}{q}-1} \|w\|_{0, q}, \end{aligned}$$

where $q \geq 1$, and $w \in H_0^1(\Omega)$ satisfies

$$a(v, w) = \left(v, |\partial_{Z,\ell}G_Z^*|^{\frac{1}{\varepsilon}} \operatorname{sgn} \partial_{Z,\ell}G_Z^* \right) \quad \forall v \in H_0^1(\Omega).$$

Taking $q = \frac{d(1+\varepsilon)}{d-2(1+\varepsilon)} > 1$ and $\frac{1}{p} = \frac{1}{q} + \frac{2}{d}$, we have $p = 1 + \varepsilon < 2$. By the a priori estimate (2.2) and the Sobolev Embedding Theorem (see [9]), we get

$$\|w\|_{0, q} \leq C \|w\|_{2, p} \leq C \|\partial_{Z,\ell}G_Z^*\|_{0, \frac{1+\varepsilon}{\varepsilon}}^{\frac{1}{\varepsilon}}.$$

Thus

$$\|\partial_{Z,\ell}G_Z^*\|_{0, \frac{1+\varepsilon}{\varepsilon}}^2 \leq Ch^{-\frac{2d}{q}-2} = Ch^{2-\frac{2d}{1+\varepsilon}},$$

which results in

$$\|\partial_{Z,\ell}G_Z^*\|_{\phi^{1-\varepsilon}}^2 \leq Ch^{2-2d+\varepsilon d}.$$

The proof of the result (2.10) is completed.

Lemma 2.6. For $\partial_{Z,\ell}G_Z^*$ the regularized derivative Green's function defined by (1.4) and $\partial_{Z,\ell}\delta_Z^h$ the discrete derivative δ function defined by (1.2), we have the following weighted-norm estimate:

$$\|\nabla(\partial_{Z,\ell}G_Z^*)\|_{\phi^{-\alpha}}^2 \leq C \|\partial_{Z,\ell}\delta_Z^h\|_{\phi^{-\alpha-\frac{2}{d}}}^2 + C \|\partial_{Z,\ell}G_Z^*\|_{\phi^{-\alpha+\frac{2}{d}}}^2 \quad \forall \alpha \in R. \quad (2.11)$$

Proof. Obviously,

$$\|\nabla(\partial_{Z,\ell}G_Z^*)\|_{\phi^{-\alpha}}^2 \leq a(\partial_{Z,\ell}G_Z^*, \phi^{-\alpha}\partial_{Z,\ell}G_Z^*) + C \|\partial_{Z,\ell}G_Z^*\|_{\phi^{-\alpha+\frac{2}{d}}}^2. \quad (2.12)$$

Further,

$$\begin{aligned} a(\partial_{Z,\ell}G_Z^*, \phi^{-\alpha}\partial_{Z,\ell}G_Z^*) &= (\partial_{Z,\ell}\delta_Z^h, \phi^{-\alpha}\partial_{Z,\ell}G_Z^*) \\ &\leq \|\partial_{Z,\ell}\delta_Z^h\|_{\phi^{-\alpha-\frac{2}{d}}} \|\partial_{Z,\ell}G_Z^*\|_{\phi^{-\alpha+\frac{2}{d}}} \\ &\leq \frac{1}{2}(\|\partial_{Z,\ell}\delta_Z^h\|_{\phi^{-\alpha-\frac{2}{d}}}^2 + \|\partial_{Z,\ell}G_Z^*\|_{\phi^{-\alpha+\frac{2}{d}}}^2). \end{aligned} \quad (2.13)$$

Combining (2.12) and (2.13) immediately yields the result (2.11).

Lemma 2.7. Suppose $-\frac{2}{d} < \alpha < \frac{2}{d}$ and $q_0 > 2$. For $\partial_{Z,\ell}G_Z^*$ the regularized derivative Green's function defined by (1.4), we have the following weighted-norm estimate:

$$\|\nabla(\partial_{Z,\ell}G_Z^*)\|_{\phi^{-\alpha}} \leq Ch^{\frac{d(\alpha-1)}{2}} \quad (2.14)$$

Proof. From (2.9),

$$\|\partial_{Z,\ell}\delta_Z^h\|_{\phi^{-\alpha-\frac{2}{d}}} \leq Ch^{\frac{d(\alpha-1)}{2}}. \quad (2.15)$$

From (2.10),

$$\|\partial_{Z,\ell}G_Z^*\|_{\phi^{-\alpha+\frac{2}{d}}} \leq Ch^{\frac{d(\alpha-1)}{2}}. \quad (2.16)$$

Combining (2.11), (2.15) and (2.16) immediately yields the result (2.14).

Theorem 2.1. Suppose $q_0 > 2$ and $d \geq 7$. For $\partial_{Z,\ell}G_Z^*$ the regularized derivative Green's function defined by (1.4), we have the following estimate:

$$|\partial_{Z,\ell}G_Z^*|_{1,1} \leq Ch^{\frac{2-d}{2}} \quad (2.17)$$

Proof. Obviously,

$$|\partial_{Z,\ell}G_Z^*|_{1,1} \leq \left(\int_{\Omega} \phi^{\alpha} dX \right)^{\frac{1}{2}} \|\nabla(\partial_{Z,\ell}G_Z^*)\|_{\phi^{-\alpha}}.$$

When $0 < \alpha < \frac{2}{d}$, we have by (2.6) and (2.14)

$$|\partial_{Z,\ell}G_Z^*|_{1,1} \leq C \inf_{\alpha} \frac{h^{\frac{d(\alpha-1)}{2}}}{1-\alpha} = Ch^{\frac{2-d}{2}},$$

which is the result (2.17).

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3 Discrete Derivative Green's Function and Its Estimates

In this section, we will consider the estimates for discrete derivative Green's function. Similar to the two-dimensional setting (see [8]), the following result holds.

Lemma 3.1. *Suppose $u_h \in S_0^h(\Omega)$ is the finite element approximation to u , we have the following estimate:*

$$\|u - u_h\|_{1, \phi^{-\alpha}}^2 \leq Ch^{2s} \|\nabla^{s+1} u\|_{\phi^{-\alpha}}^2 + C\gamma^{-2} \|u - u^h\|_{\phi^{-\alpha + \frac{2}{d}}}^2, \quad (3.1)$$

where $\gamma = \frac{\theta}{h}$. From the result (3.1), we get the following result.

Lemma 3.2. *Suppose $q_0 > 2$ and $0 < \alpha < \min\{\frac{4}{d}, 1 - \frac{2}{q_0} + \frac{2}{d}\}$, then we have*

$$\|\partial_{Z,\ell} G_Z^* - \partial_{Z,\ell} G_Z^h\|_{1, \phi^{-\alpha}} \leq Ch^{\frac{d(\alpha-1)}{2}}. \quad (3.2)$$

Proof. From (3.1),

$$\begin{aligned} & \|\partial_{Z,\ell} G_Z^* - \partial_{Z,\ell} G_Z^h\|_{1, \phi^{-\alpha}}^2 \\ & \leq Ch^2 \|\nabla^2(\partial_{Z,\ell} G_Z^*)\|_{\phi^{-\alpha}}^2 + C \|\partial_{Z,\ell} G_Z^* - \partial_{Z,\ell} G_Z^h\|_{\phi^{-\alpha + \frac{2}{d}}}^2 \\ & \leq \hat{C} \left(h^2 \|\nabla^2(\partial_{Z,\ell} G_Z^*)\|_{\phi^{-\alpha}}^2 + \|\partial_{Z,\ell} G_Z^* - \partial_{Z,\ell} G_Z^h\|_{\phi^{-\alpha + \frac{2}{d}}}^2 \right). \end{aligned}$$

Similar to the Lemma 6 in [8, Chapter 3], we obtain

$$\|\partial_{Z,\ell} G_Z^* - \partial_{Z,\ell} G_Z^h\|_{\phi^{-\alpha + \frac{2}{d}}}^2 \leq \frac{2}{3\hat{C}} \|\partial_{Z,\ell} G_Z^* - \partial_{Z,\ell} G_Z^h\|_{1, \phi^{-\alpha}}^2.$$

Then we have

$$\|\partial_{Z,\ell} G_Z^* - \partial_{Z,\ell} G_Z^h\|_{1, \phi^{-\alpha}}^2 \leq Ch^2 \|\nabla^2(\partial_{Z,\ell} G_Z^*)\|_{\phi^{-\alpha}}^2. \quad (3.3)$$

Similar to the arguments of the result (2.14), when $0 < \alpha < \frac{4}{d}$ and $q_0 > 2$, we can get

$$\|\nabla^2(\partial_{Z,\ell} G_Z^*)\|_{\phi^{-\alpha}} \leq Ch^{\frac{d(\alpha-1)-2}{2}}. \quad (3.4)$$

Combining (3.3) and (3.4) immediately yields the result (3.2).

Lemma 3.3. *Suppose $q_0 > \frac{2d}{d-2}$. For $\partial_{Z,\ell} G_Z^*$ and $\partial_{Z,\ell} G_Z^h$, the regularized derivative Green's function and the discrete derivative Green's function, respectively, we have the following estimate:*

$$|\partial_{Z,\ell} G_Z^* - \partial_{Z,\ell} G_Z^h|_{1,1} \leq Ch^{\frac{4-d}{2}}. \quad (3.5)$$

Proof. Obviously,

$$|\partial_{Z,\ell} G_Z^* - \partial_{Z,\ell} G_Z^h|_{1,1}^2 \leq \int_{\Omega} \phi^{\alpha} dX \cdot |\partial_{Z,\ell} G_Z^* - \partial_{Z,\ell} G_Z^h|_{1, \phi^{-\alpha}}^2. \quad (3.6)$$

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When $d \geq 7$ and $0 < \alpha < \min\{\frac{4}{d}, 1 - \frac{2}{q_0} + \frac{2}{d}\}$, from (2.6), (3.2) and (3.6),

$$|\partial_{Z,\ell}G_Z^* - \partial_{Z,\ell}G_Z^h|_{1,1}^2 \leq C(1 - \alpha)^{-1}h^{d(\alpha-1)}.$$

Since $q_0 > \frac{2d}{d-2}$, we have $\frac{4}{d} < 1 - \frac{2}{q_0} + \frac{2}{d}$. Thus,

$$|\partial_{Z,\ell}G_Z^* - \partial_{Z,\ell}G_Z^h|_{1,1}^2 \leq C \inf_{0 < \alpha < \frac{4}{d}} (1 - \alpha)^{-1}h^{d(\alpha-1)} = Ch^{4-d},$$

which shows the result (3.5) holds.

In the following, we give the estimate for the discrete derivative Green's function.

Theorem 3.1. *Suppose $q_0 > \frac{2d}{d-2}$ and $d \geq 7$. For $\partial_{Z,\ell}G_Z^h$ the discrete derivative Green's function defined by (1.4), we have the following estimate:*

$$|\partial_{Z,\ell}G_Z^h|_{1,1} \leq Ch^{\frac{2-d}{2}}. \tag{3.7}$$

Proof. By the triangular inequality,

$$|\partial_{Z,\ell}G_Z^h|_{1,1} \leq |\partial_{Z,\ell}G_Z^*|_{1,1} + |\partial_{Z,\ell}G_Z^* - \partial_{Z,\ell}G_Z^h|_{1,1}. \tag{3.8}$$

From (2.17), (3.5) and (3.8), we immediately obtain the result (3.7).

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Existence of Solutions to a Coupled System of Higher-order Nonlinear Fractional Differential Equations with Anti-periodic Boundary Conditions

Huina Zhang^{a,b}, Wenjie Gao^{b,*}

^aCollege of Science, China University of Petroleum (East China), Qingdao, 266555, P.R.China

^bInstitute of Mathematics, Jilin University, Changchun, 130012, P.R.China

Abstract

In this paper, the authors study a coupled system of nonlinear fractional differential equations of order $\alpha, \beta \in (4, 5)$, the differential operator is taken in the Caputo sense. By using the Schauder fixed point theorem and the contraction mapping principle, the existence and uniqueness of solutions to the system with anti-periodic boundary conditions are obtained. Two examples are given to demonstrate the feasibility of the results.

Keywords: Coupled system; Fractional differential equations; Anti-periodic boundary conditions; existence; uniqueness.

1. Introduction

Recently, fractional differential equations have proved to be valuable tools in various fields of science and engineering. Indeed, we can find numerous applications in control, porous media, fluid flows, chemical physics and many other branches of science, see[1–3]. As a result, there are many papers dealing with the existence and uniqueness of solutions to nonlinear fractional differential equations, see[4–10].

Anti-periodic boundary value problems arise in the mathematical modeling of a variety of physical process, many authors have paid much attention in such problems, for examples and details of anti-periodic boundary conditions, the interested readers may refer to [11–17]. On the other hand, the coupled systems of nonlinear fractional differential equations have been a subject of intensive studies [17–21].

It should be noted that in [18–21], the study objects are coupled systems, but not the case of Caputo fractional derivatives. In [11–16], the authors only studied the existence of solutions for anti-periodic boundary value problems of fractional differential equation but not the coupled system. Motivated by [17], we consider a coupled system of nonlinear fractional differential equations in the sense of Caputo with a nonlinear term containing the derivatives of unknown functions.

In this paper, we study the existence and uniqueness of solutions to the following coupled system of

*Corresponding author: +86 431 85166425

Email addresses: zhanghuina0521@163.com (Huina Zhang), wjgao@mail.jlu.edu.cn (Wenjie Gao)

nonlinear fractional differential equations

$$\begin{cases} {}^c D^\alpha x(t) + f(t, y(t), {}^c D^p y(t)) = 0, t \in [0, T], \\ {}^c D^\beta y(t) + g(t, x(t), {}^c D^q x(t)) = 0, t \in [0, T], \\ x^{(k)}(0) = -x^{(k)}(T), k = 0, 1, 2, 3, 4, \\ y^{(k)}(0) = -y^{(k)}(T), k = 0, 1, 2, 3, 4, \end{cases} \quad (1.1)$$

where $4 < \alpha, \beta < 5, \alpha - q \geq 1, \beta - p \geq 1$, ${}^c D^\alpha$ denotes the Caputo fractional derivative of order α , $f, g : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are given continuous functions.

This paper is organized as follows. In Section 2, we recall some basic definitions and preliminary results. In Section 3, we prove the existence of solutions to (1.1) by means of the Schauder fixed point theorem. Then, we obtain the uniqueness of solutions to the system by the contraction mapping principle. At the end, two examples are given to illustrate the applicability of our results.

2. Background Materials

For the convenience of the readers, we present here the necessary definitions and lemmas [2], which are used throughout this paper.

Definition 2.1. *The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y : (0, \infty) \rightarrow \mathbb{R}$ is given by*

$$I^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds,$$

provided the right hand side is pointwise defined on $(0, \infty)$.

Definition 2.2. *The Caputo fractional derivative of order $\alpha > 0$ of a continuous function $y : (0, \infty) \rightarrow \mathbb{R}$ is given by*

$${}^c D^\alpha y(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{y^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number α , provided that the right side is pointwise defined on $(0, \infty)$.

Lemma 2.3. *For any $y \in C[0, T]$, the unique solution of the boundary value problem*

$$\begin{cases} {}^c D^q x(t) = y(t), t \in [0, T], 4 < q \leq 5, \\ x^{(k)}(0) = -x^{(k)}(T), k = 0, 1, 2, 3, 4 \end{cases} \quad (2.1)$$

can be written as

$$x(t) = \int_0^T G(t, s) y(s) ds,$$

where $G(t, s)$ is the Green's function given by

$$G(t, s) = \begin{cases} \frac{2(t-s)^{q-1} - (T-s)^{q-1}}{2\Gamma(q)} + \frac{(T-2t)(T-s)^{q-2}}{4\Gamma(q-1)} + \frac{t(T-t)(T-s)^{q-3}}{4\Gamma(q-2)} \\ + \frac{(6Tt^2 - 4t^3 - T^3)(T-s)^{q-4}}{48\Gamma(q-3)} + \frac{(2Tt^3 - tT^3 - t^4)(T-s)^{q-5}}{48\Gamma(q-4)}, 0 < s < t < T, \\ -\frac{(T-s)^{q-1}}{2\Gamma(q)} + \frac{(T-2t)(T-s)^{q-2}}{4\Gamma(q-1)} + \frac{t(T-t)(T-s)^{q-3}}{4\Gamma(q-2)} \\ + \frac{(6Tt^2 - 4t^3 - T^3)(T-s)^{q-4}}{48\Gamma(q-3)} + \frac{(2Tt^3 - tT^3 - t^4)(T-s)^{q-5}}{48\Gamma(q-4)}, 0 < t < s < T. \end{cases}$$

Let

$$G_1(t, s) = \begin{cases} \frac{(t-s)^{\alpha-1} - \frac{1}{2}(T-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(T-2t)(T-s)^{\alpha-2}}{4\Gamma(\alpha-1)} + \frac{t(T-t)(T-s)^{\alpha-3}}{4\Gamma(\alpha-2)} \\ + \frac{(6Tt^2 - 4t^3 - T^3)(T-s)^{\alpha-4}}{48\Gamma(\alpha-3)} + \frac{(2Tt^3 - tT^3 - t^4)(T-s)^{\alpha-5}}{48\Gamma(\alpha-4)}, 0 < s < t < T, \\ -\frac{1}{2\Gamma(\alpha)}(T-s)^{\alpha-1} + \frac{(T-2t)(T-s)^{\alpha-2}}{4\Gamma(\alpha-1)} + \frac{t(T-t)(T-s)^{\alpha-3}}{4\Gamma(\alpha-2)} \\ + \frac{(6Tt^2 - 4t^3 - T^3)(T-s)^{\alpha-4}}{48\Gamma(\alpha-3)} + \frac{(2Tt^3 - tT^3 - t^4)(T-s)^{\alpha-5}}{48\Gamma(\alpha-4)}, 0 < t < s < T. \end{cases}$$

$$G_2(t, s) = \begin{cases} \frac{(t-s)^{\beta-1} - \frac{1}{2}(T-s)^{\beta-1}}{\Gamma(\beta)} + \frac{(T-2t)(T-s)^{\beta-2}}{4\Gamma(\beta-1)} + \frac{t(T-t)(T-s)^{\beta-3}}{4\Gamma(\beta-2)} \\ + \frac{(6Tt^2 - 4t^3 - T^3)(T-s)^{\beta-4}}{48\Gamma(\beta-3)} + \frac{(2Tt^3 - tT^3 - t^4)(T-s)^{\beta-5}}{48\Gamma(\beta-4)}, 0 < s < t < T, \\ -\frac{1}{2\Gamma(\beta)}(T-s)^{\beta-1} + \frac{(T-2t)(T-s)^{\beta-2}}{4\Gamma(\beta-1)} + \frac{t(T-t)(T-s)^{\beta-3}}{4\Gamma(\beta-2)} \\ + \frac{(6Tt^2 - 4t^3 - T^3)(T-s)^{\beta-4}}{48\Gamma(\beta-3)} + \frac{(2Tt^3 - tT^3 - t^4)(T-s)^{\beta-5}}{48\Gamma(\beta-4)}, 0 < t < s < T. \end{cases}$$

We call (G_1, G_2) Green's function for Problem (1.1).

Define the space

$$\mathcal{C} = \{x(t) : x(t) \in C^4[0, T], x^{(k)}(0) = -x^{(k)}(T), k = 0, 1, 2, 3, 4\},$$

and

$$X = \{x(t) : x(t) \in \mathcal{C} \text{ and } ({}^cD^q x)(t) \in C[0, T]\}$$

endowed with the norm

$$\|x\|_X = \max_{0 \leq i \leq 4} \max_{t \in [0, T]} |x^{(i)}(t)| + \max_{t \in [0, T]} |({}^cD^q x)(t)|,$$

where $i \in \mathbb{N}$.

Lemma 2.4. $(X, \|\cdot\|_X)$ is a Banach space.

Proof. Apparently X is a subspace of $C^4[0, T]$, so we only need to prove that X is closed. Let $x_n(t)$ be a sequence converging to some $x(t)$ in $(X, \|\cdot\|_X)$, then it is clear that $x_n(t)$ is a converging sequence in the space $C^4[0, T]$ and hence $x \in \mathcal{C}$. Furthermore, the uniform convergence of $({}^cD^q x_n)(t)$ to $({}^cD^q x)(t)$ implies that $({}^cD^q x)(t) \in C[0, T]$ and therefore $x(t) \in X$. The proof is complete.

Similarly, we can define the Banach space

$$Y = \{y(t) : y(t) \in \mathcal{C} \text{ and } ({}^cD^p y)(t) \in C[0, T]\}$$

endowed with the norm

$$\|y\|_Y = \max_{0 \leq i \leq 4} \max_{t \in [0, T]} |y^{(i)}(t)| + \max_{t \in [0, T]} |({}^cD^p y)(t)|,$$

where $i \in \mathbb{N}$.

For $(x, y) \in (X, Y)$, let

$$\|(x, y)\|_{X \times Y} = \max\{\|x\|_X, \|y\|_Y\}.$$

Then clearly $(X \times Y, \|\cdot\|_{X \times Y})$ is a Banach space.

Consider the following coupled system of integral equations:

$$\begin{cases} x(t) = \int_0^T G_1(t, s)f(s, y(s), {}^c D^p y(t))ds, \\ y(t) = \int_0^T G_2(t, s)g(s, x(s), {}^c D^q x(t))ds. \end{cases} \quad (2.2)$$

The following lemma states that Problem (1.1) is equivalent to Problem (2.2) and therefore the study of a system of differential equations is turn into the study of a system of integral equations.

Lemma 2.5. *Assume that $f, g : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions. Then $(x, y) \in (X, Y)$ is a solution of (1.1) if and only if $(x, y) \in (X, Y)$ is a solution of system (2.2).*

Proof. The proof is immediate from the discussion above, we omit the details here.

Let $F : X \times Y \rightarrow X \times Y$ be an operator defined as $F(x, y)(t) = (F_1 y(t), F_2 x(t))$, where

$$F_1 y(t) = \int_0^T G_1(t, s)f(s, y(s), {}^c D^p y(t))ds, \quad F_2 x(t) = \int_0^T G_2(t, s)g(s, x(s), {}^c D^q x(t))ds.$$

It is obvious that a fixed-point of the operator F is a solution of Problem (1.1).

Now we present the main results of this paper.

3. Main Results

In this section, we will discuss the existence and uniqueness of solutions to Problem (1.1).

Lemma 3.1. ^[17] *The Green's functions $G_1(t, s), G_2(t, s)$ satisfy the following estimates:*

$$\int_0^T |G_1(t, s)| ds \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \left(\frac{3}{2} + \frac{5\alpha^4 - 14\alpha^3 + 55\alpha^2 + 146\alpha}{768} \right) = U_1, t \in [0, T], \quad (3.1)$$

$$\int_0^T |G_2(t, s)| ds \leq \frac{T^\beta}{\Gamma(\beta+1)} \left(\frac{3}{2} + \frac{5\beta^4 - 14\beta^3 + 55\beta^2 + 146\beta}{768} \right) = U_2, t \in [0, T], \quad (3.2)$$

$$\int_0^T \left| \frac{\partial G_1(t, s)}{\partial t} \right| ds \leq \frac{T^{\alpha-1}}{\Gamma(\alpha)} \left(\frac{3}{2} + \frac{\alpha^3 - 3\alpha^2 + 14\alpha - 12}{48} \right) = U_3, t \in [0, T], \quad (3.3)$$

$$\int_0^T \left| \frac{\partial G_2(t, s)}{\partial t} \right| ds \leq \frac{T^{\beta-1}}{\Gamma(\beta)} \left(\frac{3}{2} + \frac{\beta^3 - 3\beta^2 + 14\beta - 12}{48} \right) = U_4, t \in [0, T]. \quad (3.4)$$

Theorem 3.2. *Assume that one of the following conditions is satisfied:*

(H₁) *there exist positive constants A, B and constants $b_i, c_i > 0, 0 < \rho_i, \theta_i < 1$ for $i = 1, 2$ such that*

$$|f(t, x, y)| \leq A + b_1 |x|^{\rho_1} + b_2 |y|^{\rho_2}, |g(t, x, y)| \leq B + c_1 |x|^{\theta_1} + c_2 |y|^{\theta_2};$$

(H₂) *there exist constants $l_i, k_i > 0, 0 < \gamma_i, \varphi_i < 1$ for $i = 1, 2$ such that*

$$|f(t, x, y)| \leq l_1 |x|^{\gamma_1} + l_2 |y|^{\gamma_2}, |g(t, x, y)| \leq k_1 |x|^{\varphi_1} + k_2 |y|^{\varphi_2};$$

(H₃) *there exist constants $d_i, \sigma_i > 0, \delta_i, \varepsilon_i > 1$ for $i = 1, 2$ such that*

$$|f(t, x, y)| \leq d_1 |x|^{\delta_1} + d_2 |y|^{\delta_2}, |g(t, x, y)| \leq \sigma_1 |x|^{\varepsilon_1} + \sigma_2 |y|^{\varepsilon_2};$$

then Problem (1.1) has a solution.

Before proving Theorem 3.2, we define a ball B in the Banach space $X \times Y$ as

$$B = \{(x(t), y(t)) | (x(t), y(t)) \in X \times Y, \|(x, y)\|_{X \times Y} \leq R, t \in [0, T]\},$$

where

$$R \geq \max\left\{3UA\lambda_1, (3Ub_1\lambda_1)^{\frac{1}{1-\rho_1}}, (3Ub_2\lambda_1)^{\frac{1}{1-\rho_2}}, 3KB\lambda_2, (3Kc_1\lambda_2)^{\frac{1}{1-\theta_1}}, (3Kc_2\lambda_2)^{\frac{1}{1-\theta_2}}\right\}.$$

$U = \max\{U_1, U_3, U_5, U_6, U_7\}$, where $U_5 = \frac{T^{\alpha-2}}{\Gamma(\alpha-1)}\left(\frac{3}{2} + \frac{\alpha^2 - \alpha - 2}{16}\right)$, $U_6 = \frac{T^{\alpha-3}(\alpha+3)}{4\Gamma(\alpha-2)}$, $U_7 = \frac{3T^{\alpha-4}}{2\Gamma(\alpha-3)}$, K is defined by the expression of U by replacing the corresponding α with β in each case, $\lambda_1 = \frac{\Gamma([q]-q+2)+T^{[q]-q+1}}{\Gamma([q]-q+2)}$, $\lambda_2 = \frac{\Gamma([p]-p+2)+T^{[p]-p+1}}{\Gamma([p]-p+2)}$.

Proof.

Part 1: Let (H_1) be valid.

Step 1 : $F : B \rightarrow B$.

$$\begin{aligned} \left| \int_0^T \frac{\partial G_1^2(t, s)}{\partial t^2} ds \right| &\leq \int_0^T \left| \frac{(t-s)^{\alpha-3}}{\Gamma(\alpha-2)} - \frac{(T-s)^{\alpha-3}}{2\Gamma(\alpha-2)} + \frac{(T-2t)(T-s)^{\alpha-4}}{4\Gamma(\alpha-3)} + \frac{t(T-t)(T-s)^{\alpha-5}}{4\Gamma(\alpha-4)} \right| ds \\ &\leq \frac{T^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{T^{\alpha-2}}{2\Gamma(\alpha-1)} + \frac{T^{\alpha-2}}{4\Gamma(\alpha-2)} + \frac{T^{\alpha-2}}{16\Gamma(\alpha-3)} \\ &= \frac{T^{\alpha-2}}{\Gamma(\alpha-1)} \left(\frac{3}{2} + \frac{\alpha^2 - \alpha - 2}{16} \right) = U_5, \end{aligned}$$

$$\begin{aligned} \left| \int_0^T \frac{\partial G_1^3(t, s)}{\partial t^3} ds \right| &\leq \int_0^T \left| \frac{(t-s)^{\alpha-4}}{\Gamma(\alpha-3)} - \frac{(T-s)^{\alpha-4}}{2\Gamma(\alpha-3)} + \frac{(T-2t)(T-s)^{\alpha-5}}{4\Gamma(\alpha-4)} \right| ds \\ &\leq \frac{T^{\alpha-3}}{\Gamma(\alpha-2)} + \frac{T^{\alpha-3}}{2\Gamma(\alpha-2)} + \frac{T^{\alpha-3}}{4\Gamma(\alpha-3)} \\ &= \frac{T^{\alpha-3}(\alpha+3)}{4\Gamma(\alpha-2)} = U_6, \end{aligned}$$

$$\begin{aligned} \left| \int_0^T \frac{\partial G_1^4(t, s)}{\partial t^4} ds \right| &\leq \int_0^T \left| \frac{(t-s)^{\alpha-5}}{\Gamma(\alpha-4)} - \frac{(T-s)^{\alpha-5}}{2\Gamma(\alpha-4)} \right| ds \\ &\leq \frac{T^{\alpha-4}}{\Gamma(\alpha-3)} + \frac{T^{\alpha-4}}{2\Gamma(\alpha-3)} \\ &= \frac{3T^{\alpha-4}}{2\Gamma(\alpha-3)} = U_7. \end{aligned}$$

Let $U = \max\{U_1, U_3, U_5, U_6, U_7\}$, when $k = 0, 1, 2, 3, 4$, we have

$$\begin{aligned} |(F_1 y)^{(k)}(t)| &= \left| \int_0^T \frac{\partial G_1^k(t, s)}{\partial t^k} f(s, y(s), {}^c D^p y(t)) ds \right| \\ &\leq \int_0^T \left| \frac{\partial G_1^k(t, s)}{\partial t^k} \right| (A + b_1 R^{\rho_1} + b_2 R^{\rho_2}) ds \\ &\leq U(A + b_1 R^{\rho_1} + b_2 R^{\rho_2}) = M. \end{aligned}$$

On the other hand, we can get

$$\begin{aligned} |{}^c D^q F_1 y(t)| &= \frac{1}{\Gamma([q] + 1 - q)} \int_0^t (t - s)^{[q]-q} |(F_1 y)^{([q]+1)}(s)| ds \\ &\leq \frac{M}{\Gamma([q] + 1 - q)} \int_0^t (t - s)^{[q]-q} ds \\ &\leq \frac{MT^{[q]-q+1}}{\Gamma([q] + 2 - q)}. \end{aligned}$$

As a result

$$\begin{aligned} \|F_1 y\|_X &= \max_{0 \leq i \leq 4} \max_{t \in [0, T]} |(F_1 y)^{(i)}(t)| + \max_{t \in [0, T]} |({}^c D^q F_1 y)(t)| \\ &\leq M + \frac{MT^{[q]-q+1}}{\Gamma([q] + 2 - q)} = M\lambda_1 \\ &= U(A + b_1 R^{\rho_1} + b_2 R^{\rho_2})\lambda_1 \\ &\leq \frac{R}{3} + \frac{R}{3} + \frac{R}{3} = R. \end{aligned}$$

Similarly

$$\begin{aligned} \|F_2 x\|_Y &= \max_{0 \leq i \leq 4} \max_{t \in [0, T]} |(F_2 x)^{(i)}(t)| + \max_{t \in [0, T]} |({}^c D^p F_2 x)(t)| \\ &\leq K(B + c_1 R^{\theta_1} + c_2 R^{\theta_2})\lambda_2 \leq R. \end{aligned}$$

Hence, we conclude that $\|F(x, y)\|_{X \times Y} = \max\{\|F_1 y\|_X, \|F_2 x\|_Y\} \leq R$, in consequence, $F : B \rightarrow B$.

Step 2: F is continuous. This follows easily from the continuity of $f, g, x(t), y(t)$ and $G_1(t, s), G_2(t, s)$.

Step 3: $F(B)$ is relatively compact. Let us set

$$M_1 = \max\{|f(t, y(t), {}^c D^p y(t))| : t \in [0, T], \|y\|_Y \leq R, \|{}^c D^p y\| \leq R\},$$

$$N_1 = \max\{|g(t, x(t), {}^c D^q x)| : t \in [0, T], \|x\|_X \leq R, \|{}^c D^q x\| \leq R\}.$$

$$\begin{aligned} |(F_1 y)'(t)| &= \left| \int_0^T \frac{\partial G_1(t, s)}{\partial t} f(s, y(s), {}^c D^p y(s)) ds \right| \\ &\leq M_1 \int_0^T \left| \frac{\partial G_1(t, s)}{\partial t} \right| ds \leq M_1 U_3. \end{aligned}$$

Hence, for $t_1, t_2 \in [0, T]$, we have

$$|(F_1 y)(t_2) - (F_1 y)(t_1)| \leq \int_{t_1}^{t_2} |(F_1 y)'(s)| ds \leq M_1 U_3 |t_2 - t_1|.$$

Similarly, we can get

$$|(F_2 x)(t_2) - (F_2 x)(t_1)| \leq \int_{t_1}^{t_2} |(F_2 x)'(s)| ds \leq N_1 U_4 |t_2 - t_1|.$$

By the Arzelà-Ascoli theorem, we can obtain that $F(B)$ is an equicontinuous set, the operator $F : B \rightarrow B$ is completely continuous. Thus, Problem (1.1) has one solution by the Schauder fixed-point theorem.

Part 2: Let (H_2) be valid. In this part, let

$$R \geq \max \left\{ (2U_1\lambda_1)^{\frac{1}{1-\gamma_1}}, (2U_2\lambda_1)^{\frac{1}{1-\gamma_2}}, (2Kk_1\lambda_2)^{\frac{1}{1-\varphi_1}}, (2Kk_2\lambda_2)^{\frac{1}{1-\varphi_2}} \right\}.$$

We can also get the result by repeating arguments similar to part 1.

Part 3: Let (H_3) be valid. In this part, let

$$0 \leq R \leq \min \left\{ (2Ud_1\lambda_1)^{-\frac{1}{\delta_1-1}}, (2Ud_2\lambda_1)^{-\frac{1}{\delta_2-1}}, (2K\sigma_1\lambda_2)^{-\frac{1}{\varepsilon_1-1}}, (2K\sigma_2\lambda_2)^{-\frac{1}{\varepsilon_2-1}} \right\}.$$

We can also get the result by repeating arguments similar to part 1. Here we omit it. This completes the proof.

Example 3.1. Consider the system

$$\begin{cases} {}^c D^{17/4}x(t) + \sin t + (y(t))^{2/3} + ({}^c D^{5/2}y(t))^{2/5} = 0, 0 < t < 1, \\ {}^c D^{9/2}y(t) + t^{1/2} + (x(t))^{1/3} + ({}^c D^{11/4}x(t))^{4/7} = 0, 0 < t < 1, \\ x^{(k)}(0) = -x^{(k)}(1), k = 0, 1, 2, 3, 4, \\ y^{(k)}(0) = -y^{(k)}(1), k = 0, 1, 2, 3, 4. \end{cases} \tag{3.5}$$

The system satisfies (H_1) and hence Theorem 3.2 implies the existence of the solution to system (3.5).

Theorem 3.3. Let f and g satisfy the following growth conditions :

(H_1) there exist four positive constants L_1, L_2, H_1, H_2 such that

$$\begin{aligned} |f(t, x_1, y_1) - f(t, x_2, y_2)| &\leq L_1|x_1 - x_2| + L_2|y_1 - y_2|, \\ |g(t, x_1, y_1) - g(t, x_2, y_2)| &\leq H_1|x_1 - x_2| + H_2|y_1 - y_2|, \\ t &\in [0, T], x_i, y_i \in \mathbb{R}, i = 1, 2. \end{aligned}$$

(H_2)

$$\max\{L_1, L_2\}U_1 = Q_1 < 1, \max\{H_1, H_2\}U_2 = Q_2 < 1.$$

Then Problem (1.1) has a unique solution.

Proof. Let $(x_1, y_1), (x_2, y_2) \in X \times Y$, then

$$\begin{aligned} |(F_1y_1 - F_1y_2)(t)| &= \left| \int_0^T G_1(t, s)f(s, y_1(s), {}^c D^p y_1(s))ds - \int_0^T G_1(t, s)f(s, y_2(s), {}^c D^p y_2(s))ds \right| \\ &\leq \int_0^T |G_1(t, s)| |f(s, y_1(s), {}^c D^p y_1(s)) - f(s, y_2(s), {}^c D^p y_2(s))| ds \\ &\leq U_1(L_1|y_1(s) - y_2(s)| + L_2|{}^c D^p y_1(s) - {}^c D^p y_2(s)|) \leq \max\{L_1, L_2\}U_1\|y_1 - y_2\|_Y. \end{aligned}$$

Analogously,

$$\begin{aligned} |(F_2x_1 - F_2x_2)(t)| &\leq \int_0^T |G_2(t, s)| |g(s, x_1(s), {}^c D^q x_1(s)) - g(s, x_2(s), {}^c D^q x_2(s))| ds \\ &\leq U_2(H_1|x_1(s) - x_2(s)| + H_2|{}^c D^q x_1(s) - {}^c D^q x_2(s)|) \leq \max\{H_1, H_2\}U_2\|x_1 - x_2\|_X. \end{aligned}$$

Thus,

$$\begin{aligned} \| F(x_1, y_1) - F(x_2, y_2) \|_{X \times Y} &= \| (F_1 y_1 - F_1 y_2, F_2 x_1 - F_2 x_2) \|_{X \times Y} \\ &= \max\{ \| F_1 y_1 - F_1 y_2 \|_X, \| F_2 x_1 - F_2 x_2 \|_Y \} \\ &\leq \max\{ Q_1 \| y_1 - y_2 \|_Y, Q_2 \| x_1 - x_2 \|_X \} \\ &\leq \max\{ Q_1, Q_2 \} \max\{ \| y_1 - y_2 \|_Y, \| x_1 - x_2 \|_X \} \\ &= \max\{ Q_1, Q_2 \} \| (x_1, y_1) - (x_2, y_2) \|_{X \times Y} . \end{aligned}$$

Hence, we conclude that Problem (1.1) has a unique solution by (H_2) and the contraction mapping principle, this ends the proof.

Example 3.2. Consider the system

$$\begin{cases} {}^c D^{17/4} x(t) + L_1 \sin y(t) + L_2 \frac{{}^c D^{5/2} y(t)}{1 + {}^c D^{5/2} y(t)} = 0, 0 < t < 1, \\ {}^c D^{9/2} y(t) + H_1 \arctan x(t) + H_2 \frac{{}^c D^{11/4} x(t)}{1 + {}^c D^{11/4} x(t)} = 0, 0 < t < 1, \\ x^{(k)}(0) = -x^{(k)}(1), k = 0, 1, 2, 3, 4, \\ y^{(k)}(0) = -y^{(k)}(1), k = 0, 1, 2, 3, 4. \end{cases} \tag{3.6}$$

Where $T = 1$, $f(t, y(t), {}^c D^p y(t)) = L_1 \sin y(t) + L_2 \frac{{}^c D^{5/2} y(t)}{1 + {}^c D^{5/2} y(t)}$, $g(t, x(t), {}^c D^q x(t)) = H_1 \arctan x(t) + H_2 \frac{{}^c D^{11/4} x(t)}{1 + {}^c D^{11/4} x(t)}$, $\alpha = \frac{17}{4}$, $\beta = \frac{9}{2}$, $p = 5/2$, $q = 11/4$ and $L_1, L_2, H_1, H_2 > 0$.

Noting that

$$|(\sin y)'| = |\cos y| \leq 1, |(\arctan x)'| = \frac{1}{1+x^2} \leq 1, |(\frac{v}{1+v})'| = \frac{1}{(1+v)^2} \leq 1,$$

we have

$$\begin{aligned} &| f(t, y_1(t), {}^c D^p y_1(t)) - f(t, y_2(t), {}^c D^p y_2(t)) | \\ &\leq L_1 | \sin y_1(t) - \sin y_2(t) | + L_2 \left| \frac{{}^c D^{5/2} y_1(t)}{1 + {}^c D^{5/2} y_1(t)} - \frac{{}^c D^{5/2} y_2(t)}{1 + {}^c D^{5/2} y_2(t)} \right| \\ &\leq L_1 | y_1(t) - y_2(t) | + L_2 | {}^c D^{5/2} y_1(t) - {}^c D^{5/2} y_2(t) | \\ &\leq \max\{L_1, L_2\} \| y_1 - y_2 \|_Y, \end{aligned}$$

$$\begin{aligned} &| g(t, x_1(t), {}^c D^q x_1(t)) - g(t, x_2(t), {}^c D^q x_2(t)) | \\ &\leq H_1 | \arctan x_1(t) - \arctan x_2(t) | + H_2 \left| \frac{{}^c D^{11/4} x_1(t)}{1 + {}^c D^{11/4} x_1(t)} - \frac{{}^c D^{11/4} x_2(t)}{1 + {}^c D^{11/4} x_2(t)} \right| \\ &\leq H_1 | x_1(t) - x_2(t) | + H_2 | {}^c D^{11/4} x_1(t) - {}^c D^{11/4} x_2(t) | \\ &\leq \max\{H_1, H_2\} \| x_1 - x_2 \|_X, \end{aligned}$$

$$U_1 = \frac{T^\alpha}{\Gamma(\alpha + 1)} \left(\frac{3}{2} + \frac{5\alpha^4 - 14\alpha^3 + 55\alpha^2 + 146\alpha}{768} \right) \approx 0.1229,$$

$$U_2 = \frac{T^\beta}{\Gamma(\beta + 1)} \left(\frac{3}{2} + \frac{5\beta^4 - 14\beta^3 + 55\beta^2 + 146\beta}{768} \right) \approx 0.0920.$$

as long as we let $\max\{L_1, L_2\} < \frac{1}{0.1229}$, $\max\{H_1, H_2\} < \frac{1}{0.0920}$, it will have $Q_1 < 1, Q_2 < 1$, then we can conclude from Theorem 3.3 that system (3.6) has a unique solution.

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Iteration Process for Pointwise Lipschitzian Type Mappings in Hyperbolic 2-uniformly Convex Metric Spaces

D. R. Sahu¹, Samir Dashputre² and Shin Min Kang^{3,*}

¹Department of Mathematics, Banaras Hindu University, Varansi 221005, India
e-mail: drsahudr@gmail.com

²Department of Applied Mathematics, Shri Shankaracharya Group of Institutions, Junwani, Bhilai 490020, India
e-mail: samir231973@gmail.com

³Department of Mathematics and RINS, Gyeongsang National University, Jinju 660-701, Korea
e-mail: smkang@gnu.ac.kr

Abstract

In this paper, we discuss the existence of fixed points for a class of Lipschitzian type mappings and asymptotic pointwise Lipschitz type mappings in hyperbolic 2-uniformly convex metric spaces. In the same space setting, we deal the problem of approximation of fixed points via modified Mann iteration process. Our result generalizes and extends the corresponding results of Dehaish et al. [7], Goebel and Kirk [8], Kirk and Xu [18] and Sahu et al. [24] and many others in this direction.

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1 Introduction

Let C be a nonempty subset of a metric space X and $T : C \rightarrow C$ be a mapping. Then T is called

- (1) *nonexpansive* if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in C$;
- (2) *asymptotically nonexpansive* [8] if for each $n \in \mathbb{N}$, there exists a constant $k_n \geq 1$ with $\lim_{n \rightarrow \infty} k_n = 1$ such that

$$d(T^n x, T^n y) \leq k_n d(x, y)$$

for all $x, y \in C$;

*Corresponding author

(3) a *pointwise contraction* [3] if there exists a function $\alpha : C \rightarrow [0, 1)$ such that

$$d(Tx, Ty) \leq \alpha(x)d(x, y)$$

for all $x, y \in C$;

(4) an *asymptotic pointwise contraction* [17] if for each $n \in \mathbb{N}$, there exists a function $\alpha_n : C \rightarrow [0, 1)$ such that

$$d(T^n x, T^n y) \leq \alpha_n(x)d(x, y)$$

for all $x, y \in C$, where $\alpha_n \rightarrow \alpha : C \rightarrow [0, 1)$ pointwise on C ;

(5) *pointwise asymptotically nonexpansive* [18] if there exists a sequence $\{\alpha_n\}$ for each integer $n \in \mathbb{N}$, a function exists a function $\alpha_n : C \rightarrow [1, \infty)$

$$d(T^n x, T^n y) \leq \alpha_n(x)d(x, y)$$

for all $x, y \in C$, where $\alpha_n(x) \rightarrow 1$ pointwise on C ;

(6) *asymptotically nonexpansive in the intermediate sense* [5] provided T is uniformly continuous and

$$\limsup_{n \rightarrow \infty} \sup_{x, y \in C} (d(T^n x, T^n y) - d(x, y)) \leq 0; \tag{1.1}$$

(7) *asymptotically nonexpansive type* [13, 16] if

$$\limsup_{n \rightarrow \infty} \sup_{y \in C} (d(T^n x, T^n y) - d(x, y)) \leq 0$$

for all $x \in C$.

There is a class of mappings which lies strictly between the class of contraction mappings and the class of nonexpansive mappings. The class of pointwise contractions was introduced in Belluce and Kirk [3] and later it was called generalized contraction in [12]. Banach’s celebrated contraction principle was extended to this larger class of mappings as follows:

Theorem 1.1. ([3, 12]) *Let C be a nonempty weakly compact convex subset of a Banach space and $T : C \rightarrow C$ a pointwise contraction. Then T has a unique fixed point, x^* , and $\{T^n x\}$ converges strongly to x^* for each $x \in C$.*

Kirk [17] combined ideas of pointwise contraction [3] and asymptotic contraction [15] and introduced the concept of an asymptotic pointwise contraction. He announced that an asymptotic pointwise contraction defined on closed convex and bounded subset of a super-reflexive Banach space has a fixed point.

In [18], Kirk and Xu introduced the concept of asymptotically pointwise and proved that every pointwise asymptotically nonexpansive mapping defined on a closed convex Banach space has a fixed point.

The class of asymptotically nonexpansive mappings in the intermediate sense which is essentially wider than that of asymptotically nonexpansive was introduced by Bruck et al. [5]. It is known that [16] if C is a nonempty closed convex bounded subset of a uniformly convex Banach space X and T is a self mapping of C which is asymptotically nonexpansive in the intermediate sense, then T has fixed point.

On the other hand, if $c_n = \max\{\sup_{x \in C}(d(T^n x, T^n y) - d(x, y)), 0\}$, then (1.1) reduces to relation

$$d(T^n x, T^n y) \leq d(x, y) + c_n \tag{1.2}$$

for all $x, y \in C$ and $n \in \mathbb{N}$. The classes of mappings more general than the class of mapping satisfying (1.2) were studied in Alber et al. [2] as the class of total asymptotically nonexpansive mappings and in Sahu [22] as the class of nearly Lipschitzian mappings.

Fix a sequence $\{a_n\}$ in $[0, \infty)$ with $a_n \rightarrow 0$. A mapping $T : C \rightarrow C$ is said to be *nearly Lipschitzian* with respect to $\{a_n\}$ ([22]) if for each $n \in \mathbb{N}$, there exists a constants $k_n > 0$ such that

$$d(T^n x, T^n y) \leq k_n(d(x, y) + a_n) \tag{1.3}$$

for all $x, y \in C$. The infimum of the constants k_n in (1.3) is called nearly Lipschitz constant and is denoted by $\eta(T^n)$. A nearly Lipschitzian mapping T with the sequence $\{a_n, \eta(T^n)\}$ is said to be

- (1) *nearly contraction* if $\eta(T^n) < 1$ for all $n \in \mathbb{N}$,
- (2) *nearly uniformly L-Lipschitzian* if $\eta(T^n) \leq L$ for all $n \in \mathbb{N}$,
- (3) *nearly uniformly k-contraction* if $\eta(T^n) \leq k < 1$ for all $n \in \mathbb{N}$,
- (4) *nearly nonexpansive* if $\eta(T^n) = 1$ for all $n \in \mathbb{N}$,
- (5) *nearly asymptotically nonexpansive* if $\eta(T^n) \geq 1$ for all $n \in \mathbb{N}$ with $\lim_{n \rightarrow \infty} \eta(T^n) = 1$.

The corresponding Lipschitzian type mappings (for instance, contraction type mappings) concerning asymptotically nonexpansive mappings in the intermediate sense and total asymptotically nonexpansive mappings are not defined in Bruck et al. [5] and Alber et al. [2]. The notion of nearly Lipschitzian mappings allows to define different classes of Lipschitzian types mappings, for example, nearly contraction, nearly nonexpansive, nearly asymptotically nonexpansive, nearly uniformly L - Lipschitzian etc. Therefore, the fixed point theory of nearly Lipschitzian mappings is of fundamental importance. Some properties and existence and convergence results for nearly Lipschitzian mappings are studied in [22, 23]. The perturbation of a nonexpansive mapping as a sequence of nearly nonexpansive mappings is studied and its applications are given in [26, 25].

Recently, Sahu et al. [24] introduced some new classes of pointwise nearly Lipschitz type mappings in Banach spaces and studied some existence theorems in Banach spaces. Inspired by the work of Kirk and Xu [18] and Sahu et al. [24] studied the existence of fixed points of pointwise nearly Lipschitzian mappings in Banach spaces. In [24], it is shown that the asymptotic center of every bounded orbit of a pointwise asymptotically nonexpansive mapping is fixed point of the mapping in a uniformly convex Banach space.

In [27], Schu considered modified Mann iterations for asymptotically nonexpansive mappings on a convex subset of a Banach space. Recently, Khan et al. [11] have introduced and studied the convergence of a general iteration scheme of asymptotically quasi-nonexpansive mappings in convex metric spaces and CAT(0) spaces.

Recently, Dehaish et al. [7] studied the existence of a fixed point of a single and a family of asymptotic pointwise nonexpansive mappings defined on uniformly convex hyperbolic

spaces. They also discussed the behavior of the following modified Mann iteration process associated with asymptotic pointwise nonexpansive mapping T :

$$x_{n+1} = t_n T^n(x_n) \oplus (1 - t_n)x_n, n \in \mathbb{N}, \tag{1.4}$$

where $\{t_n\} \subset [0, 1]$ be bounded away from 0 and 1 and $x_1 \in C$ is an arbitrary point.

The purpose of this paper is to extend the notion of the pointwise Lipschitzian type mappings introduced in [24] and establish existence and convergence theorems for fixed points for the class of pointwise nearly asymptotically nonexpansive mappings in the framework of hyperbolic 2-uniformly convex metric spaces. Our results generalize, extend and unify the corresponding results of Dehaish et al. [7], Goebel and Kirk [8], Kirk and Xu [18] and Sahu et al. [24] and many others in this direction..

2 Preliminaries

2.1 Uniformly convexity in metric spaces

Let (X, d) be a metric space. Suppose that there exists a family \mathcal{F} of metric space segments such that any two points $x, y \in X$ are end points of a unique metric segment $[x, y] \in \mathcal{F}$. Here $[x, y]$ is an isometric image of the real line interval $[0, d(x, y)]$. We shall denote by $tx \oplus (1 - t)y$ the unique point z of $[x, y]$ which satisfies

$$d(x, z) = (1 - t)d(x, y) \quad \text{and} \quad d(z, y) = td(x, y),$$

where $t \in [0, 1]$. Such metric spaces are usually called *convex metric spaces* [20]. Moreover, if

$$d(\alpha p \oplus (1 - \alpha)x, \alpha q \oplus (1 - \alpha)y) \leq \alpha d(p, q) + (1 - \alpha)d(x, y),$$

for all $p, q, x, y \in X$, and $\alpha \in [0, 1]$, then X is said to be a *hyperbolic metric space* (see [21]).

It is easy to see that normed linear spaces are hyperbolic spaces. As nonlinear examples, one can consider the Hadamard manifolds [6], the Hilbert open unit ball equipped with the hyperbolic metric [9], and the CAT(0) spaces [14, 16, 19] (see Example 2.8).

Definition 2.1. ([10]) A subset C of a hyperbolic metric space X is *convex* if $[x, y] \subset C$ whenever $x, y \in C$.

Definition 2.2. ([10]) Let (X, d) be a hyperbolic metric space. We say that X is *uniformly convex* if for any $a \in X$, for every $r > 0$, and for each $\varepsilon > 0$,

$$\delta(r, \varepsilon) = \inf \left\{ 1 - \frac{1}{r} d\left(\frac{1}{2}x \oplus \frac{1}{2}y, a\right) : d(x, a) \leq r, d(y, a) \leq r, d(x, y) \geq r\varepsilon \right\} > 0.$$

From now onward we assume that X is a hyperbolic metric space and if (X, d) is uniformly convex, then for every $s \geq 0, \varepsilon > 0$, there exists $\eta(s, \varepsilon) > 0$ depending on s and ε such that

$$\delta(r, \varepsilon) > \eta(s, \varepsilon) > 0 \quad \text{for any } r > s.$$

Remark 2.3. If (X, d) is uniformly convex, then we have the following:

- (1) $\delta(r, 0) = 0$ and $\delta(r, \varepsilon)$ is an increasing function of ε for every fixed r .
- (2) For $r_1 \leq r_2$, the following holds:

$$1 - \frac{r_2}{r_1} \left(1 - \delta \left(r_2, \varepsilon \frac{r_1}{r_2} \right) \right) \leq \delta(r_1, \varepsilon).$$

- (3) If (X, d) is uniformly convex, then (X, d) is strictly convex, that is, whenever

$$d(x, a) = d(y, a) = d \left(\frac{1}{2}x \oplus \frac{1}{2}y, a \right),$$

for any $x, y, a \in X$, then we must have $x = y$.

Recall that a hyperbolic metric space X is said to have *property (R)* [10] if any non-increasing sequence of nonempty, convex, bounded, and closed sets has a nonempty intersection.

The following theorem was proved by Khamsi and Khan [10].

Theorem 2.4. ([10]) *Assume that (X, d) is complete and uniformly convex. Let C be nonempty, convex, and closed. Then for any $x \in X$, there exists a unique best approximant of x in C , that is, a unique $x_0 \in C$ such that*

$$d(x, x_0) = d(x, C).$$

Note that any complete and uniformly convex metric space has the property (R) (see [10]).

We need the following results for our main results.

Lemma 2.5. ([10] Lemma 2.2) *Let (X, d) be uniformly convex. Assume that there exists $r \geq 0$ such that*

$$\limsup_{n \rightarrow \infty} d(x_n, a) \leq r, \quad \limsup_{n \rightarrow \infty} d(y_n, a) \leq r \quad \text{and} \quad \lim_{n \rightarrow \infty} d \left(a, \frac{1}{2}x_n \oplus \frac{1}{2}y_n \right) = r.$$

Then $\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$.

The following metric version of the parallelogram identity, also known as the inequality of Bruhat and Tits, has been established in [10].

Theorem 2.6. ([10]) *Let (X, d) be uniformly convex. Fix $a \in X$. For each $r > 0$ and for each $\varepsilon > 0$, denote*

$$\Psi(r, \varepsilon) = \inf \left\{ \frac{1}{2}d^2(a, x) + \frac{1}{2}d^2(a, y) - d^2 \left(a, \frac{1}{2}x \oplus \frac{1}{2}y \right) \right\},$$

where the infimum is taken over all $x, y \in X$ such that $d(a, x) \leq r, d(a, y) \leq r$ and $d(x, y) \geq r\varepsilon$. Then $\Psi(r, \varepsilon) > 0$ for any $r > 0$ and each $\varepsilon > 0$. Moreover, for a fixed $r > 0$, we have

- (i) $\Psi(r, 0) = 0$;
- (ii) $\Psi(r, \varepsilon)$ is non-decreasing function of ε ;
- (iii) if $\lim_{n \rightarrow \infty} \Psi(r, t_n) = 0$, then $\lim_{n \rightarrow \infty} t_n = 0$.

The notion of p -uniform convexity was studied extensively by Xu [28], its nonlinear version for $p = 2$ has been introduced by Khamsi and Khan [10] using the above function Ψ as follows.

Definition 2.7. ([10]) We say that (X, d) is 2-uniformly convex if

$$C_X = \inf \left\{ \frac{\Psi(r, \varepsilon)}{r^2 \varepsilon^2} : r > 0, \varepsilon > 0 \right\} > 0.$$

From the definition of C_X , we obtain the following inequality:

$$d^2 \left(a, \frac{1}{2}x \oplus \frac{1}{2}y \right) + C_X d^2(x, y) \leq \frac{1}{2}d^2(a, x) + \frac{1}{2}d^2(a, y)$$

for any $a \in X$ and $x, y \in X$.

Example 2.8. Let (X, d) be a metric space. A geodesic from x to y in X is a mapping c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$.

In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a geodesic (or metric) segment joining x and y . The space (X, d) is said to be a *geodesic space* if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$, which will be denoted by $[x, y]$, and called the segment joining x to y . A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points x_1, x_2, x_3 in X (the vertices of Δ) and a geodesic segment between each pair of vertices (the edges of Δ).

A comparison triangle for the geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\bar{\Delta}(x_1, x_2, x_3) := \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in \mathbb{R}^2 such that $d_{\mathbb{R}^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$ such triangle exists (see [4]).

A geodesic space is said to be a *CAT(0) space* if all geodesic triangles of appropriate size satisfy the following comparison axiom.

Let Δ be a geodesic triangle in X and let $\bar{\Delta} \subset \mathbb{R}^2$ be a comparison triangle for Δ . Then Δ is said to satisfy the CAT(0) inequality if

$$d(x, y) \leq d(\bar{x}, \bar{y}).$$

for all $x, y \in \Delta$ and all comparison points $\bar{x}, \bar{y} \in \bar{\Delta}$.

Complete CAT(0) spaces are often called *Hadamard spaces* (see [16]). If x, y_1, y_2 are points of a CAT(0) space and if y_0 is the midpoint of the segment $[y_1, y_2]$, which will be denoted by $\frac{y_1 \oplus y_2}{2}$, then the CAT(0) inequality implies

$$d^2 \left(x, \frac{1}{2}y_1 \oplus \frac{1}{2}y_2 \right) \leq \frac{1}{2}d^2(x, y_1) + \frac{1}{2}d^2(x, y_2) - \frac{1}{4}d^2(y_1, y_2). \quad (CN)$$

This inequality is the (CN) inequality of Bruhat and Tits [4]. As for the Hilbert space, the (CN) inequality implies the CAT(0) spaces are uniformly convex with

$$\delta(r, \varepsilon) = 1 - \sqrt{1 - \frac{\varepsilon^2}{4}}.$$

The (CN) inequality also implies that

$$\Psi(r, \varepsilon) = \frac{r^2 \varepsilon^2}{4}.$$

Thus, a CAT(0) space is 2-uniformly convex with $C_X = \frac{1}{4}$.

We need the following more general inequality for convergence of Mann iterations.

Theorem 2.9. ([7]) *Let (X, d) be 2-uniformly convex. Then, for any $\alpha \in (0, 1)$, there exists $C_X > 0$ such that*

$$d^2(a, \alpha x \oplus (1 - \alpha)y) + C_X \min\{\alpha^2, (1 - \alpha)^2\}d^2(x, y) \leq \alpha d^2(a, x) + (1 - \alpha)d^2(a, y)$$

for any $a, x, y \in X$.

Recall that $\Phi : X \rightarrow \mathbb{R}^+$ is called a *type* if there exists $\{x_n\}$ in X such that

$$\Phi(x) = \limsup_{n \rightarrow \infty} d(x, x_n).$$

Theorem 2.10. ([10, Theorem 2.4]) *Assume that (X, d) is a complete and uniformly convex. Let C be a nonempty closed bounded and convex subset of X . Let Φ be a type defined on C . Then any minimizing sequence of Φ is convergent. Its limit is independent of the minimizing sequence.*

In fact, if X is 2-uniformly convex, and Φ is a type defined on a nonempty closed convex bounded subset C of X , then there exists a unique $x_0 \in C$ such that

$$\Phi^2(x_0) + 2C_X d^2(x_0, x) \leq \Phi^2(x) \tag{2.1}$$

for any $x \in C$. In this inequality, one may find an analogy with Opial property used in the study of the fixed point property in Banach and metric spaces.

2.2 Pointwise Lipschitzian type mappings and fixed points

First, we extend some wider classes of nonlinear mappings studied by Sahu et al. [24] in a metric space setting.

Definition 2.11. ([24]) Let C be a nonempty subset of a metric space (X, d) . A mapping $T : C \rightarrow C$ is said to be

(1) *pointwise nearly Lipschitzian with sequence $\{(\alpha_n(\cdot), a_n)\}$* if, there exists a sequence $\{a_n\}$ in $[0, \infty)$ with $a_n \rightarrow 0$ and for each $n \in \mathbb{N}$, there exists a function $\alpha_n(\cdot) : C \rightarrow (0, \infty)$ such that

$$d(T^n x, T^n y) \leq \alpha_n(x)(d(x, y) + a_n)$$

for all $x, y \in C$;

(2) *pointwise nearly uniformly $\alpha(\cdot)$ -Lipschitzian with sequence $\{a_n\}$* if, there exists a sequence $\{a_n\}$ in $[0, \infty)$ with $a_n \rightarrow 0$ and there exists a function $\alpha(\cdot) : C \rightarrow (0, \infty)$ such that

$$d(T^n x, T^n y) \leq \alpha(x)(d(x, y) + a_n)$$

for all $x, y \in C$;

(3) *asymptotic pointwise nearly Lipschitzian with sequence $\{(\alpha_n(\cdot), a_n)\}$* if, there exists a sequence $\{a_n\}$ in $[0, \infty)$ with $a_n \rightarrow 0$ and for each $n \in \mathbb{N}$, there exists a function $\alpha_n(\cdot) : C \rightarrow (0, \infty)$ and with $\alpha_n \rightarrow \alpha : C \rightarrow (0, \infty)$ pointwise such that

$$d(T^n x, T^n y) \leq \alpha_n(x)(d(x, y) + a_n)$$

for all $x, y \in C$.

We say that, an asymptotic pointwise nearly Lipschitzian mapping is

(1) *pointwise nearly asymptotic nonexpansive* if $\alpha_n(x) \geq 1$ for all $n \in \mathbb{N}$ and $\alpha_n(x) \rightarrow 1$ pointwise,

(2) *pointwise asymptotically nonexpansive* [18] $a_n = 0$ and $\alpha_n(x) \geq 1$ for all $n \in \mathbb{N}$ and $\alpha_n(x) \rightarrow 1$ pointwise.

(3) a *asymptotic pointwise nearly contraction* if $\alpha_n \rightarrow \alpha$ pointwise and $\alpha(x) \leq k < 1$ for all $x \in C$.

A point $x \in C$ is called a *fixed point* of T if $T(x) = x$. The fixed point set of T is denoted by $Fix(T)$.

3 Existence theorem

First, we prove the existence of fixed point for a pointwise nearly asymptotically nonexpansive mapping in a 2-uniformly convex metric space.

Theorem 3.1. *Let C be nonempty closed convex and bounded subset of a complete hyperbolic 2-uniformly convex metric space (X, d) . Let $T : C \rightarrow C$ be a continuous pointwise nearly asymptotically nonexpansive mapping. Then T has a fixed point in C . Moreover, the set of fixed points is closed and convex.*

Proof. Fix $x \in C$. Define the function $\Phi(y) = \limsup_{n \rightarrow \infty} d(T^n(x), y)$ on C . By (2.1), there exists a unique $\omega \in C$ such that

$$\Phi^2(\omega) + 2C_X d^2(\omega, y) \leq \Phi^2(y)$$

for all $y \in C$. In particular, we have

$$\Phi^2(\omega) + 2C_X d^2(\omega, T^n(\omega)) \leq \Phi^2(T^n(\omega)) \tag{3.1}$$

for all $n \geq 1$. Observe that

$$\begin{aligned} \Phi(T^n(\omega)) &= \limsup_{m \rightarrow \infty} d(T^m(x), T^n(\omega)) \\ &\leq \limsup_{m \rightarrow \infty} d(T^n(T^{m-n}(x)), T^n(\omega)) \\ &\leq \limsup_{m \rightarrow \infty} [\alpha_n(\omega)(d(T^{m-n}(x), \omega) + a_n)] \\ &\leq \alpha_n(\omega)(\Phi(\omega) + a_n) \end{aligned} \tag{3.2}$$

for all $n \geq 1$. Hence, from (3.1) and (3.2), we have

$$\begin{aligned} \Phi^2(\omega) + 2C_X d^2(\omega, T^n(\omega)) &\leq \Phi^2(T^n(\omega)) \\ &\leq (\alpha_n(\omega)(\Phi(\omega) + a_n))^2 \\ &= \alpha_n^2(\omega)[\Phi^2(\omega) + a_n^2 + 2\Phi(\omega)a_n] \end{aligned}$$

for all $n \geq 1$. By the definition of T , $\alpha_n(\omega) \rightarrow 1$ pointwise and $a_n \rightarrow 0$ as $n \rightarrow \infty$. Thus, $\lim_{n \rightarrow \infty} d(\omega, T^n(\omega)) = 0$, i.e., $T^n(\omega) \rightarrow \omega$ as $n \rightarrow \infty$. By the continuity of T , we have

$$T(\omega) = T\left(\lim_{n \rightarrow \infty} T^n(\omega)\right) = \lim_{n \rightarrow \infty} T^{n+1}(\omega) = \omega.$$

CLOSEDNESS OF $Fix(T)$: Let $\{x_n\}$ be a sequence in $Fix(T)$ such that $\lim_{n \rightarrow \infty} x_n = x$ for some $x \in C$. Now it remains to show that $x \in Fix(T)$. Note that

$$d(T^n(x_n), T^n(x)) \leq \alpha_n(x)(d(x_n, x) + a_n),$$

which implies that

$$\lim_{n \rightarrow \infty} d(T^n(x), x_n) = 0.$$

Since

$$d(x, T^n(x)) \leq d(x, x_n) + d(x_n, T^n(x)),$$

we have, $\lim_{n \rightarrow \infty} d(x, T^n(x)) = 0$. By continuity of T , we have $Tx = x$.

CONVEXITY OF $Fix(T)$: Let $x, y \in Fix(T)$. We only need to prove that $z = \frac{x \oplus y}{2} \in Fix(T)$. Without loss of generality, we assume that $x \neq y$. Note that

$$\begin{aligned} d(x, T^n(z)) &= d(T^n(x), T^n(z)) \\ &\leq \alpha_n(x)(d(x, z) + a_n) \\ &\leq \alpha_n(x)\left(d\left(x, \frac{x \oplus y}{2}\right) + a_n\right) \\ &= \alpha_n(x)\left(\frac{1}{2}d(x, y) + a_n\right) \end{aligned}$$

for all $n \geq 1$. Similarly, we have

$$d(y, T^n(z)) \leq \alpha_n(y)\left(\frac{1}{2}d(x, y) + a_n\right)$$

for all $n \geq 1$. By triangular inequality, we have

$$\begin{aligned} d(x, y) &\leq d(x, T^n(z)) + d(T^n(z), y) \\ &\leq \alpha_n(x)\left(\frac{1}{2}d(x, y) + a_n\right) + \alpha_n(y)\left(\frac{1}{2}d(x, y) + a_n\right) \\ &= (\alpha_n(x) + \alpha_n(y))\left(\frac{1}{2}d(x, y) + a_n\right), \end{aligned}$$

it follows that

$$\lim_{n \rightarrow \infty} d(x, T^n(z)) = \lim_{n \rightarrow \infty} d(T^n(z), y) = d(x, y).$$

Note

$$\begin{aligned} d\left(x, \frac{z \oplus T^n(z)}{2}\right) &\leq \frac{1}{2} d(x, z) + \frac{1}{2} d(x, T^n(z)) \\ &\leq \frac{1}{2} d(x, z) + \frac{1}{2} \alpha_n(x) \left(\frac{1}{2} d(x, z) + a_n\right), \\ &= \left(\frac{1 + \alpha_n(x)}{2}\right) d(x, z) + \frac{\alpha_n(x) a_n}{2}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} d\left(y, \frac{z \oplus T^n(z)}{2}\right) &\leq \frac{1}{2} d(y, z) + \frac{1}{2} d(y, T^n(z)) \\ &\leq \frac{1}{2} d(y, z) + \frac{1}{2} \alpha_n(y) \left(\frac{1}{2} d(y, z) + a_n\right) \\ &= \left(\frac{1 + \alpha_n(y)}{2}\right) d(y, z) + \frac{\alpha_n(y) a_n}{2}. \end{aligned}$$

Thus,

$$\begin{aligned} d(x, y) &\leq d\left(x, \frac{z \oplus T^n(z)}{2}\right) + d\left(\frac{z \oplus T^n(z)}{2}, y\right) \\ &\leq \frac{1 + \alpha_n(x)}{2} \frac{d(x, y)}{2} + \frac{1 + \alpha_n(y)}{2} \frac{d(x, y)}{2} + \frac{\alpha_n(x) + \alpha_n(y)}{2} a_n. \end{aligned}$$

Taking limit as $n \rightarrow \infty$ both the sides, Hence, we have

$$\lim_{n \rightarrow \infty} d\left(x, \frac{z \oplus T^n(z)}{2}\right) = \lim_{n \rightarrow \infty} d\left(y, \frac{z \oplus T^n(z)}{2}\right) = \frac{d(x, y)}{2}.$$

Using Lemma 2.5, we conclude that $\lim_{n \rightarrow \infty} d(z, T^n(z)) = 0$. Therefore, we must have $T(z) = z$, i.e., $\frac{x \oplus y}{2} \in \text{Fix}(T)$. This completes the proof. \square

Remark 3.2. Theorem 3.1 is a natural generalization of Proposition 3.4 and Theorem 3.8 of Sahu et al. [24] in the framework of a hyperbolic 2-uniformly convex metric space. Theorem 3.1 extends the results of Dehaish et al. [7, Theorem 3.1], Goebel and Kirk [8, Theorem 1], and Kirk and Xu [18, Theorem 3.4] to the class of pointwise nearly Lipschitzian mappings which essentially wider than the class of mappings appearing in [7], [8] and [18].

4 Convergence of Mann iteration process

Lemma 4.1. *Let C be nonempty, closed, convex, and bounded subset of a complete hyperbolic 2-uniformly convex metric space (X, d) . Let $T : C \rightarrow C$ be a pointwise nearly asymptotically nonexpansive with sequence $\{(\alpha_n(\cdot), a_n)\}$ such that T is uniformly continuous. Assume that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty} (\alpha_n(p) - 1) < \infty$ for all $p \in \text{Fix}(T)$. Let $\{t_n\} \subset [0, 1]$ be bounded away from 0 and 1, i.e., there exist two real numbers a, b such*

that $0 < a \leq t_n \leq b < 1$. The modified Mann iteration process is defined by (1.4). Then we have the following:

- (a) $\lim_{n \rightarrow \infty} d(x_n, p)$ exists for all $p \in \text{Fix}(T)$.
- (b) $\lim_{n \rightarrow \infty} d(x_n, T^n(x_n)) = 0$ and $\lim_{n \rightarrow \infty} d(x_n, T^m(x_n)) = 0$ for all $m \geq 1$, provided that $L = \sup_{n \in \mathbb{N}} \sup_{x \in C} \alpha_n(x) < \infty$.

Proof. (a) Let $\delta(C) = \sup_{x, y \in C} d(x, y)$ be the diameter of C . Let $\omega \in \text{Fix}(T)$. Set $\delta_n := d(x_n, \omega)$, $\beta_n = \alpha_n(\omega)$ and $\gamma_n = a_n L$. From (1.4), we have

$$\begin{aligned} \delta_{n+1} &= d(x_{n+1}, \omega) \\ &= d(t_n T^n(x_n) \oplus (1 - t_n)x_n, \omega) \\ &\leq t_n d(T^n(x_n), \omega) + (1 - t_n)d(x_n, \omega) \\ &\leq t_n d(T^n(x_n), T^n(\omega)) + (1 - t_n)d(x_n, \omega) \\ &\leq t_n \alpha_n(\omega)(d(x_n, \omega) + a_n) + (1 - t_n)d(x_n, \omega) \\ &\leq t_n \alpha_n(\omega)d(x_n, \omega) + a_n t_n \alpha_n(\omega) + (1 - t_n)d(x_n, \omega) \\ &\leq \alpha_n(\omega)d(x_n, \omega) + \alpha_n(\omega)a_n \\ &\leq \beta_n \delta_n + \gamma_n \end{aligned}$$

for all $n \geq 1$. Noticing that $\sum_{n=1}^{\infty} (\beta_n - 1) < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$ for all $n \geq 1$. Therefore, from [1, Lemma 6.1.5], we conclude that $\lim_{n \rightarrow \infty} d(x_n, \omega)$ exists.

(b) First, we prove that $\lim_{n \rightarrow \infty} d(x_n, T^n(x_n)) = 0$. By Theorem 3.1, T has a fixed point $\omega \in C$. Lemma 4.1 implies that $\lim_{n \rightarrow \infty} d(x_n, \omega)$ exists. Set $r = \lim_{n \rightarrow \infty} d(x_n, \omega)$. Without loss of generality, we may assume $r > 0$. Note

$$\begin{aligned} \limsup_{n \rightarrow \infty} d(T^n(x_n), \omega) &= \limsup_{n \rightarrow \infty} d(T^n(x_n), T^n(\omega)) \\ &\leq \limsup_{n \rightarrow \infty} (\alpha_n(\omega)(d(x_n, \omega) + a_n)) = r. \end{aligned}$$

On the other hand, from (1.4), we have

$$d(x_{n+1}, \omega) \leq t_n d(T^n(x_n), \omega) + (1 - t_n)d(x_n, \omega)$$

for all $n \geq 1$. Let \mathcal{U} be a non-trivial ultrafilter over \mathbb{N} . Then $\lim_{\mathcal{U}} t_n = t \in [a, b]$. Hence

$$r = \lim_{\mathcal{U}} d(x_{n+1}, \omega) \leq t \lim_{\mathcal{U}} d(T^n(x_n), \omega) + (1 - t)r.$$

Since $t \neq 0$, we get $\lim_{\mathcal{U}} d(T^n(x_n), \omega) \geq r$. Hence

$$r \leq \liminf_{n \rightarrow \infty} d(T^n(x_n), \omega) \leq \limsup_{n \rightarrow \infty} d(T^n(x_n), \omega) \leq r.$$

So $\lim_{n \rightarrow \infty} d(T^n(x_n), \omega) = r$. Since X is 2-uniformly convex, Theorem 2.9 implies

$$\begin{aligned} C_X \min\{t_n^2, (1 - t_n)^2\} d^2(x_n, T^n(x_n)) &\leq t_n d^2(x_n, \omega) + (1 - t_n) d^2(T^n(x_n), \omega) \\ &\quad - d^2(x_{n+1}, \omega), \end{aligned}$$

where $C_X > 0$ depends only on X . Since

$$C_X \min\{t_n^2, (1 - t_n)^2\} \geq \min\{a^2, (1 - b)^2\} > 0,$$

and

$$\lim_{n \rightarrow \infty} \left\{ t_n d^2(x_n, \omega) + (1 - t_n) d^2(T^n(x_n), \omega) - d^2(x_{n+1}, \omega) \right\} = 0,$$

we get

$$\lim_{n \rightarrow \infty} d(x_n, T^n(x_n)) = 0,$$

which finish the prove of our claim.

Next, we prove that $\lim_{n \rightarrow \infty} d(x_n, T^m(x_n)) = 0$, for all $m \geq 1$. The uniform continuity of T implies that

$$\lim_{n \rightarrow \infty} d(Tx_n, T^{n+1}(x_n)) = 0.$$

From (1.4), we have

$$d(x_{n+1}, x_n) \leq d(x_n, T^n(x_n)) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Note that

$$\begin{aligned} d(x_n, T(x_n)) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1}(x_{n+1})) + d(T^{n+1}(x_{n+1}), T^{n+1}(x_n)) \\ &\quad + d(T^{n+1}(x_n), T(x_n)) \\ &\leq d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1}(x_{n+1})) + \alpha_{n+1}(x_n)(d(x_{n+1}, x_n) \\ &\quad + a_{n+1}) + d(T^{n+1}(x_n), T(x_n)) \\ &\leq d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1}(x_{n+1})) + L(d(x_{n+1}, x_n) + a_{n+1}) \\ &\quad + d(T^{n+1}(x_n), T(x_n)) \end{aligned}$$

for all $n \geq 1$. Hence, we get $\lim_{n \rightarrow \infty} d(x_n T(x_n)) = 0$. Again, from the uniform continuity of T , we have

$$\lim_{n \rightarrow \infty} d(T(x_n), T^2(x_n)) = 0,$$

it follows that

$$d(x_n, T^2(x_n)) \leq d(x_n, T(x_n)) + d(T(x_n), T^2(x_n)) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Inductively, we have

$$\lim_{n \rightarrow \infty} d(x_n, T^m(x_n)) = 0$$

for all $m \geq 1$. This completes the proof. □

We now establish main result of this section.

Theorem 4.2. *Let C be nonempty, closed, convex, and bounded subset of a complete hyperbolic 2-uniformly convex metric space (X, d) . Let $T : C \rightarrow C$ be a pointwise nearly asymptotically nonexpansive with sequence $\{(\alpha_n(\cdot), a_n)\}$ such that T is uniformly continuous and $\sup_{n \in \mathbb{N}} \sup_{x \in C} \alpha_n(x) < \infty$. Assume that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty} (\alpha_n(p) - 1) < \infty$ for all $p \in \text{Fix}(T)$. Let $\{t_n\} \subset [0, 1]$ be bounded away from 0 and 1, i.e., there exist two real numbers a, b such that $0 < a \leq t_n \leq b < 1$. The modified Mann iteration process is defined by (1.4). Consider the type $\Phi(x) = \limsup_{n \rightarrow \infty} d(x_n, x)$ on C . If ω is the minimum point of Φ , that is, $\Phi(\omega) = \inf\{\Phi(x) : x \in C\}$, then $T(\omega) = \omega$.*

Proof. Suppose that ω is the minimum point of Φ . For any $m, n \geq 1$, we have

$$d^2\left(x_n, \frac{\omega \oplus T^m(\omega)}{2}\right) + C_X d^2(\omega, T^m(\omega)) \leq \frac{1}{2}d^2(x_n, \omega) + \frac{1}{2}d^2(x_n, T^m(\omega)).$$

Letting limit as $n \rightarrow \infty$, we get

$$\Phi^2\left(\frac{\omega \oplus T^m(\omega)}{2}\right) + C_X d^2(\omega, T^m(\omega)) \leq \frac{1}{2}\Phi^2(\omega) + \frac{1}{2}\Phi^2(T^m(\omega)) \tag{4.1}$$

for any $m \geq 1$. Using Lemma 4.1, we get

$$\begin{aligned} \Phi(T^m(\omega)) &= \limsup_{n \rightarrow \infty} d(x_n, T^m(\omega)) \\ &\leq \limsup_{n \rightarrow \infty} \left[d(x_n, T^m(x_n)) + d(T^m(x_n), T^m(\omega)) \right], \\ &\leq \limsup_{n \rightarrow \infty} d(T^m(x_n), T^m(\omega)) \\ &\leq \limsup_{n \rightarrow \infty} (\alpha_m(\omega)(d(x_n, \omega) + a_m)) \\ &= \alpha_m(\omega)(\Phi(\omega) + a_m) \end{aligned}$$

for any $m \geq 1$. Since ω is the minimum point of Φ , we have

$$\Phi(\omega) \leq \Phi\left(\frac{\omega \oplus T^m(\omega)}{2}\right)$$

for any $m \geq 1$. From (4.1), we have

$$\begin{aligned} \Phi^2(\omega) + C_X d^2(\omega, T^m(\omega)) &\leq \Phi^2\left(\frac{\omega \oplus T^m(\omega)}{2}\right) + C_X d^2(\omega, T^m(\omega)) \\ &\leq \frac{1}{2}\Phi^2(\omega) + \frac{1}{2}\Phi^2(T^m(\omega)) \\ &\leq \frac{1}{2}\Phi^2(\omega) + \frac{1}{2}[\alpha_m(\omega)(\Phi(\omega) + a_m)]^2 \end{aligned}$$

for $m \geq 1$. Taking limit superior as $m \rightarrow \infty$, we get

$$\Phi^2(\omega) + C_X \limsup_{m \rightarrow \infty} d^2(\omega, T^m(\omega)) \leq \Phi^2(\omega).$$

This implies that $\lim_{m \rightarrow \infty} d(\omega, T^m(\omega)) = 0$. Therefore, $T(\omega) = \omega$, i.e., $\omega \in Fix(T)$. This completes the proof. \square

Remark 4.3. Theorem 4.2 extends the result of Dehaish et al. [7, Theorem 4.1] to pointwise nearly Lipschitzian mapping which essentially wider than the mapping appearing in [7].

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Regularity of the American Option Value Function in Jump-Diffusion Model

Sultan Hussain¹ * and Nasir Rehman²

¹Department of Mathematics, COMSATS Institute of Information Technology, Abbottabad Pakistan.

²Department of Mathematics and Statistics, Allama Iqbal Open University H-8, Islamabad Pakistan.

Abstract

This work is devoted to the regularity properties of the American options value function, when there are brusque variations in prices. We assume that there are finite number of jumps in each finite time interval and the asset price jumps in the proportions which are independent and identically distributed. These properties can be used to investigate the optimal hedging strategies, optimal exercise boundaries etc. for the options in jump-diffusion process.

Keywords: American Option, Jump-Diffusion Model, Poisson Process, Lipschitz Continuity, Weak Derivatives.

1 Introduction

The pricing of options and the corporate liabilities have been developed significantly after the classical paper by Black and Scholes (1973). Although several techniques for the calculation of the value of the European option have been proposed in closed-form, the American options are still open for further research and consideration, causing an extensive literature on numerical methods.

Recently, in Israel and Rincon (2008), the American options problem using inequality variational systems, and numerical methods based on finite elements and finite difference

*Corresponding author (Sultan Hussain) Tel.: +92 (0)333 9485988.
E-mail address: tausef775650@yahoo.co.in

techniques is solved properly. Indeed, as it has been proposed also in Jaillet et. al. (1990), the problem to find the value of a put American option can be equivalent to getting the solution of a system of variational inequalities provided that this formulation respects some necessary hypotheses, see also Isreal and Rincon (2008). Jaillet, Lamberon and Lapeyre (1990) rely on the link between the optimal stopping and variational inequality in order to exploit the theory of American options. Pham (1997) investigated the regularity of the value function of the put American option in jump-diffusion process using the properties of the optimal exercise boundary. For more detailed discussion on the value function of the American options we refer the readers to the papers by Chiarella and Kang (2011), El-Karoui, et. al. (1998), Elliot and Kopp (1990), Hussain and Shashiashvili (2010), Hussain and Rehman (2012), and books Glowinski, et. al. (1981), Karatzas and Shreve (1998), Lamberton and Lapeyre (1997), Shreve (2004) etc.

We assume the interest rate and volatility are Lipschitz functions of time, payoff is arbitrary bounded from below convex function, and use purely probabilistic approach to obtain rigorous estimates for the first and second order derivatives of the value function of the put American options in order to use these results in our next work to construct uniform approximations for the discrete time hedging strategies as well as for the investigation of the optimal exercise boundary of the put American options.

In Section 2, we set the basic notation and we formulate our model. Thus, we consider a financial market with two assets, i.e. the value of a money market account and the share of a stock whose price jumps proportionally at some times τ_j following the Poisson process, similarly as in Pham (1997). Note that in order to deal with this problem, following also the existing literature, we recall that the American options value function can be considered as the value of a function of an equivalent optimal stopping time problem. Thus, some preliminary results are presented here. Finally, in Section 3, the regularity properties of a put American option are derived solving a system of variational inequalities.

2 Notation - Preliminary Results

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space on which we define a standard Wiener process $W = (W_t)_{0 \leq t \leq T}$, a Poisson process $N = (N_t)_{0 \leq t \leq T}$ with intensity λ and a sequence $(U_j)_{j \geq 1}$ of independent, identically distributed random variables taking values in $(-1, \infty)$. Assume that the time horizon $T < \infty$ is finite and the σ -algebras generated respectively by $(W_t)_{0 \leq t \leq T}$, $(N_t)_{0 \leq t \leq T}$, and $(U_j)_{j \geq 1}$ are independent. Denote by $(\mathcal{F}_t)_{0 \leq t \leq T}$ the \mathbf{P} -completion of the natural filtration of (W_t) , (N_t) and $(U_j)_{j \leq N_t}$, $j \geq 1, 0 \leq t \leq T$.

On a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})_{0 \leq t \leq T}$ consider a financial market with two assets $m_t, 0 \leq t \leq T$, the price of a unit of a money market account at time t , and $S_t, 0 \leq t \leq T$, the value at time t of the share of a stock whose price jumps in the proportions U_1, U_2, \dots , at some times τ_1, τ_2, \dots , see also Pham (1997). We assume that the τ_j 's correspond to the jump times of a Poisson process.

The evolution of the assets m_t and S_t obeys the following ordinary and stochastic differential equations respectively,

$$dm_t = r(t)m_t dt, \quad m_0 = 1, \quad 0 \leq t \leq T,$$

$$dS_t = S_{t-} \left(b(t)dt + \sigma(t)dW_t + d \left(\sum_{j=1}^{N_t} U_j \right) \right).$$

We assume that $(b(t), \mathcal{F}_t)_{0 \leq t \leq T}$ is a certain, progressively measurable process; the deterministic time-varying interest rate $r(t)$ and the volatility $\sigma(t)$ are continuously differentiable functions of time, and the following requirements are satisfied:

$$0 \leq r(t) \leq \bar{r}, \quad 0 < \underline{\sigma} \leq \sigma(t) \leq \bar{\sigma}, \quad |b(t)| \leq \bar{r},$$

$$|r(t) - r(s)| + |\sigma(t) - \sigma(s)| \leq K|t - s|, \tag{1}$$

where $s, t \in [0, T]$ and $\bar{r}, \underline{\sigma}, \bar{\sigma}$ and K are some positive constants.

From the above stochastic differential equation, the dynamics of S_t can be described by:

$$S_t = S_0 \left(\prod_{j=1}^{N_t} (1 + U_j) \right) \exp \left[\int_0^t \left(b(u) - \frac{\sigma^2(u)}{2} \right) du + \int_0^t \sigma(u)dW_u \right].$$

It is known, see for instance Lamberton and Lapeyre (1997), that the discounted stock price $\tilde{S}_t = e^{-\int_0^t r(u)du} S_t$ is a *martingale* if and only if

$$\int_0^t b(u)du = \int_0^t r(u)du - \lambda t E(U_1). \tag{2}$$

In this brief paper, we investigate the *regularity* properties of the American option value function with a nonnegative, non-increasing convex payoff function $g(x), x \geq 0$. We assume that $g(0) = g(0+)$. Of course, a typical example of this family of functions is the *put American option* with payoffs $g(x) = (L - x)^+$ where L is the exercise price.

In the next paragraphs of this section, we present some necessary and preliminary results for the better understanding, and evaluation of our main outputs.

First, it is necessary to recall that the American option value function $v(t, x), x \geq 0, 0 \leq t \leq T$, can be considered as the value function of a relevant optimal stopping problem (see, for instance Karatzas and Shreve (1998), Section 2.5). In particular

$$v(t, x) = \sup_{\tau \in \mathcal{T}_{t,T}} E \left[\exp \left(- \int_t^\tau r(v)dv \right) g(S_\tau(t, x)) \right], \quad x \geq 0, 0 \leq t \leq T, \tag{3}$$

where $\mathcal{T}_{t,T}$ denotes the set of all stopping times τ such that $t \leq \tau \leq T$, and the stochastic process $S_u(t, x), t \leq u \leq T$ satisfies the same stochastic differential equation as above, i.e.

$$dS_u(t, x) = S_{u-}(t, x) \left(b(u)du + \sigma(u)dW_u + d \left(\sum_{j=1+N_t}^{N_u} U_j \right) \right), \quad t \leq u \leq T, \tag{4}$$

with the initial condition $S_t(t, x) = x, x \geq 0$.

The unique solution $(S_u(t, x), \mathcal{F}_u)_{t \leq u \leq T}$ of (4) is given by the exponential

$$S_u(t, x) = x \left(\prod_{j=1+N_t}^{N_u} (1 + U_j) \right) \exp \left[\int_t^u \left(b(u) - \frac{\sigma^2(u)}{2} \right) du + \int_t^u \sigma(u)dW_u \right].$$

Condition (2) leads to

$$S_u(t, x) = \exp \left[\ln x + \int_t^u \left(r(u) - \lambda E(U_1) - \frac{\sigma^2(u)}{2} \right) du + \int_t^u \sigma(u) dW_u + \sum_{j=N_t+1}^{N_u} \ln(1 + U_j) \right].$$

Now, we can introduce the new stochastic process $(X_u(t, x), \mathcal{F}_u)_{t \leq u \leq T}$

$$X_u(t, y) = y + \int_t^u \left(r(u) - \lambda E(U_1) - \frac{\sigma^2(v)}{2} \right) dv + \int_t^u \sigma(v) dW_v + \sum_{j=N_t+1}^{N_u} \ln(1 + U_j),$$

$$t \leq u \leq T, \quad -\infty < y < \infty, \quad U_j \in (-1, \infty), \quad j = 1, 2, \dots$$

Remark 2.1. *Profoundly,*

$$S_u(t, x) = \exp [X_u(t, \ln x)], \quad t \leq u \leq T \quad x > 0, \tag{5}$$

and for an arbitrary stopping time $\tau, t \leq \tau \leq T$, we obtain

$$g(S_\tau(t, x)) = \psi(X_\tau(t, \ln x)),$$

where $\psi(y) = g(e^y), -\infty < y < \infty$ is the new payoff function.

Now, it is clear that the corresponding optimal stopping time problem is derived straightforwardly by just substituting (5) into (3), having now

$$u(t, y) = \sup_{\tau \in \mathcal{T}_{t,T}} E \left[\exp \left(- \int_t^\tau r(v) dv \right) \psi(X_\tau(t, y)) \right], \tag{6}$$

with $0 \leq t \leq T$ and $-\infty < y < \infty$, then we obtain

$$v(t, x) = u(t, \ln x), \quad x > 0, \quad 0 \leq t \leq T.$$

In what follows, the next known result, from Hussain and Shashiashvili (2010), is needed

Lemma 2.2. *Let $g(x), x \geq 0$ be a nonnegative, non-increasing convex function. Then the new payoff function defined by $\psi(y) = g(e^y), -\infty < y < \infty$ is Lipschitz continuous, that is,*

$$|\psi(y_2) - \psi(y_1)| \leq g(0)|y_2 - y_1|, \quad y_1, y_2 \in \mathbb{R}.$$

Thus, using the *scaling* property of the Brownian motion we can express the value function $u(t, y)$ of the optimal stopping time problem (6), see Jaillet, et. al. (1990), as follows

$$u(t, y) = \sup_{\tau \in \mathcal{T}_{0,1}} E \left[\exp \left(- \int_t^{t+\tau(T-t)} r(v) dv \right) \psi \left(y + \int_t^{t+\tau(T-t)} \left(r(v) - \lambda E(U_1) - \frac{\sigma^2(v)}{2} \right) dv + \int_0^\tau \sqrt{T-t} \sigma(t + v(T-t)) dW_v + \sum_{j=1}^{N_{t+\tau(T-t)}} \ln(1 + U_j) \right) \right], \tag{7}$$

where $\mathcal{T}_{0,1}$ denotes the set of all stopping times τ with respect to the filtration $(\mathcal{F}_u)_{0 \leq u \leq 1}$ taking values in $[0, 1]$.

Finally, we conclude the preliminary results of this section by proving the following theorem.

Theorem 2.3. *The value function $u(t, y), 0 \leq t \leq T, -\infty < y < \infty$ of the optimal stopping problem (6) is Lipschitz continuous in the argument y and locally Lipschitz continuous in t i.e.*

$$|u(t, y) - u(t, z)| \leq g(0) |y - z|, \quad y, z \in \mathbb{R}, 0 \leq t \leq T, \tag{8}$$

$$|u(t, y) - u(s, y)| \leq \frac{A}{\sqrt{T-t}} |t - s|, \tag{9}$$

where A is some nonnegative constant depending on parameters $\bar{r}, \bar{\sigma}, g(0), \lambda, E(U_1), K$ and T .

Proof. Fixing any τ in $\mathcal{T}_{t,T}$ and $y, z \in \mathbb{R}$, and using Lemma 2.2, we take

$$\begin{aligned} & \left| E \exp\left(-\int_t^\tau r(v)dv\right) \psi(X_\tau(t, y)) - E \exp\left(-\int_t^\tau r(v)dv\right) \psi(X_\tau(t, z)) \right| \\ & \leq E |X_\tau(t, y) - X_\tau(t, z)| \\ & \leq g(0) |y - z|. \end{aligned}$$

Benefiting ourselves by the well-known property that the difference between supremums is less or equal than the supremum of difference leads to the result (8). To show the second part of the theorem, i.e. (9), we shall use the expression (7) for the value function $u(t, y)$.

Take any $\tau \in \mathcal{T}_{0,1}$ we can write

$$\begin{aligned} & \left| E e^{-\int_t^{t+\tau(T-t)} r(v)dv} \psi\left(y + \int_t^{t+\tau(T-t)} \left(r(v) - \lambda EU_1 - \frac{\sigma^2(v)}{2}\right) dv + \sqrt{T-t} \int_0^\tau \sigma(t + v(T-t)) dW_v \right. \right. \\ & \left. \left. + \sum_{j=1}^{N_{t+\tau(T-t)}} \ln(1 + U_j)\right) - E e^{-\int_s^{s+\tau(T-s)} r(v)dv} \psi\left(y + \int_s^{s+\tau(T-s)} \left(r(v) - \lambda EU_1 - \frac{\sigma^2(v)}{2}\right) dv \right. \right. \\ & \left. \left. + \sqrt{T-s} \int_0^\tau \sigma(s + v(T-s)) dW_v + \sum_{j=1}^{N_{s+\tau(T-s)}} \ln(1 + U_j)\right) \right| \\ & \leq E \left[\left| e^{-\int_t^{t+\tau(T-t)} r(v)dv} - e^{-\int_s^{s+\tau(T-s)} r(v)dv} \right| \psi\left(y + \int_t^{t+\tau(T-t)} \left(r(v) - \lambda EU_1 - \frac{\sigma^2(v)}{2}\right) dv \right. \right. \\ & \left. \left. + \sqrt{T-t} \int_0^\tau \sigma(t + v(T-t)) dW_v + \sum_{j=1}^{N_{t+\tau(T-t)}} \ln(1 + U_j)\right) + e^{-\int_s^{s+\tau(T-s)} r(v)dv} \times \right. \end{aligned}$$

$$\begin{aligned}
 & \left| \psi \left(y + \int_t^{t+\tau(T-t)} \left(r(v) - \lambda EU_1 - \frac{\sigma^2(v)}{2} \right) dv + \sqrt{T-t} \int_0^\tau \sigma(t+v(T-t)) dW_v \right. \right. \\
 & \left. \left. + \sum_{j=1}^{N_{t+\tau(T-t)}} \ln(1+U_j) \right) \right. \\
 & \left. - \psi \left(y + \int_s^{s+\tau(T-s)} \left(r(v) - \lambda EU_1 - \frac{\sigma^2(v)}{2} \right) dv + \sqrt{T-s} \int_0^\tau \sigma(s+v(T-s)) dW_v \right. \right. \\
 & \left. \left. + \sum_{j=1}^{N_{s+\tau(T-s)}} \ln(1+U_j) \right) \right| \\
 & \leq g(0) E \left[\left| e^{-\int_t^{t+\tau(T-t)} r(v) dv} - e^{-\int_s^{s+\tau(T-s)} r(v) dv} \right| + \left| \int_t^{t+\tau(T-t)} \left(r(v) - \lambda EU_1 - \frac{\sigma^2(v)}{2} \right) dv \right. \right. \\
 & \left. \left. - \int_s^{s+\tau(T-s)} \left(r(v) - \lambda EU_1 - \frac{\sigma^2(v)}{2} \right) dv \right| \right. \\
 & \left. + \left| \int_0^\tau \left(\sqrt{T-t} \sigma(t+v(T-t)) - \sqrt{T-s} \sigma(s+v(T-s)) \right) dW_v \right| \right. \\
 & \left. + \left| \sum_{j=N_{s+\tau(T-s)}+1}^{N_{t+\tau(T-t)}} \ln(1+U_j) \right| \right]. \tag{10}
 \end{aligned}$$

Let us denote $R(u) = \int_0^u r(v) dv, 0 \leq u \leq T$, and using the mean value theorem, we can write

$$\begin{aligned}
 \left| e^{-\int_t^{t+\tau(T-t)} r(v) dv} - e^{-\int_s^{s+\tau(T-s)} r(v) dv} \right| & \leq \left| \int_t^{t+\tau(T-t)} r(v) dv - \int_s^{s+\tau(T-s)} r(v) dv \right| \\
 & \leq |(R(t+\tau(T-t)) - R(t)) - (R(s+\tau(T-s)) - R(s))| \\
 & \leq 2 \bar{r} |t-s|. \tag{11}
 \end{aligned}$$

Similarly, we use the same arguments and obtain

$$\begin{aligned}
 \left| \int_t^{t+\tau(T-t)} \left(r(v) - \lambda E(U_1) - \frac{\sigma^2(v)}{2} \right) dv - \int_s^{s+\tau(T-s)} \left(r(v) - \lambda E(U_1) - \frac{\sigma^2(v)}{2} \right) dv \right| \\
 \leq 2 \left(\bar{r} + \lambda E|U_1| + \frac{\bar{\sigma}^2}{2} \right) |t-s|. \tag{12}
 \end{aligned}$$

Moreover, we fix $\tau, 0 \leq \tau \leq 1$, and $0 \leq s \leq t < T$, using the requirement (1), on $\sigma(t)$ we

write

$$\begin{aligned}
 & E \left| \int_0^\tau \left(\sqrt{T-t} \sigma(t+v(T-t)) - \sqrt{T-s} \sigma(s+v(T-s)) \right) dW_v \right|^2 \\
 & \leq E \int_0^\tau \left(\sqrt{T-t} \sigma(t+v(T-t)) - \sqrt{T-s} \sigma(s+v(T-s)) \right)^2 dv \\
 & \leq \int_0^1 \left(\sqrt{T-t} \sigma(t+v(T-t)) - \sqrt{T-s} \sigma(s+v(T-s)) \right)^2 dv \\
 & \leq 2 \int_0^1 (T-t) (\sigma(t+v(T-t)) - \sigma(s+v(T-s)))^2 dv \\
 & \quad + 2 \int_0^1 \left(\sqrt{T-t} - \sqrt{T-s} \right)^2 \sigma^2(t+v(T-t)) dv.
 \end{aligned}$$

From here we obtain

$$\begin{aligned}
 & E \left| \int_0^\tau \left(\sqrt{T-t} \sigma(t+v(T-t)) - \sqrt{T-s} \sigma(s+v(T-s)) \right) dW_v \right|^2 \\
 & \leq \frac{2K^2T^2 + \bar{\sigma}^2}{T-t} (t-s)^2.
 \end{aligned} \tag{13}$$

Since $(U_j)_{j \geq 1}$ be a sequence of independent, identically distributed, integrable random variables, therefore we can find

$$\begin{aligned}
 E \left| \sum_{j=N_{s+\tau(T-s)}+1}^{N_{t+\tau(T-t)}} \ln(1+U_j) \right| &= E \sum_{j=N_{s+\tau(T-s)}+1}^{N_{t+\tau(T-t)}} |\ln(1+U_j)| \\
 &= E \sum_{j=1}^{N_{t+\tau(T-t)}-N_{s+\tau(T-s)}} |\ln(1+U_j)| \\
 &= E \sum_{j=1}^{N_{(t-s)(1-\tau)}} |\ln(1+U_j)|.
 \end{aligned}$$

Since N_t is an increasing function of time and $\tau \leq 1$ so we can write

$$\begin{aligned}
 E \left| \sum_{j=N_{s+\tau(T-s)}+1}^{N_{t+\tau(T-t)}} \ln(1+U_j) \right| &\leq E \sum_{j=1}^{N_{(t-s)}} |\ln(1+U_j)| \\
 &= E(N_{t-s}) E|\ln(1+U_1)| \\
 &= \lambda E|\ln(1+U_1)|(t-s),
 \end{aligned} \tag{14}$$

Substituting (11)-(14) in (10) and using the fact that the difference between supremums is less or equal than difference supremum of the difference, we complete the proof. \square

In the next section, the main results of the paper are presented.

3 Variational Inequalities

In this section, the variational inequalities of the value function are developed in order to investigate the regularity results of the value function (3). Let $\tilde{S}_t = e^{-\int_0^t r(u)du} S_t$ is the

discounted stock price, then the discounted price function

$$\tilde{v}(t, x) = e^{-\int_0^t r(u)du} v(t, xe^{\int_0^t r(u)du}), \quad 0 \leq t \leq T, x > 0 \tag{15}$$

of the option at time t is C^2 on $[0, T) \times \mathbb{R}^+$ (see, Laberton and Lapeyre (1997)) and between the jump times, satisfies

$$\begin{aligned} \tilde{v}(t, \tilde{S}_t) &= v(0, S_0) + \int_0^t \frac{\partial \tilde{v}}{\partial u}(u, \tilde{S}_u) du + \int_0^t \frac{\partial \tilde{v}}{\partial x}(u, \tilde{S}_u) \tilde{S}_u (-\lambda E(U_1) du + \sigma(u) dW_u) \\ &\quad + \frac{1}{2} \int_0^t \frac{\partial^2 \tilde{v}}{\partial x^2}(u, \tilde{S}_u) \sigma^2(u) \tilde{S}_u^2 du + \sum_{j=1}^{N_t} \left(\tilde{v}(\tau_j, \tilde{S}_{\tau_j}) - \tilde{v}(\tau_j, \tilde{S}_{\tau_j-}) \right). \end{aligned} \tag{16}$$

The function $\tilde{v}(t, x)$ is Lipschitz of order 1 with respect to x and with $S_{\tau_j-} = S_{\tau_j}(1 + U_j)$, $j = 1, 2, \dots$

The process

$$M_t = \sum_{j=1}^{N_t} \left(\tilde{v}(\tau_j, \tilde{S}_{\tau_j}) - \tilde{v}(\tau_j, \tilde{S}_{\tau_j-}) \right) - \lambda \int_0^t \int \left(\tilde{v}(u, \tilde{S}_u(1+z)) - \tilde{v}(u, \tilde{S}_u) \right) d\nu(z) du \tag{17}$$

is a square integrable martingale, where $\nu(z)$ is the law of the process U .

Combining (16) and (17) we obtain that

$$\begin{aligned} \tilde{v}(t, \tilde{S}_t) &- \int_0^t \left[\frac{\partial \tilde{v}}{\partial u}(u, \tilde{S}_u) - \lambda E U_1 \tilde{S}_u \frac{\partial \tilde{v}}{\partial x}(u, \tilde{S}_u) + \frac{1}{2} \sigma^2(u) \tilde{S}_u^2 \frac{\partial^2 \tilde{v}}{\partial x^2}(u, \tilde{S}_u) \right. \\ &\quad \left. - \lambda \int \left(\tilde{v}(u, \tilde{S}_u(1+z)) - \tilde{v}(u, \tilde{S}_u) \right) d\nu(z) \right] du \end{aligned}$$

is a martingale, see Israel and Rincon (2008), and therefore

$$\begin{aligned} &\frac{\partial \tilde{v}}{\partial u}(u, \tilde{S}_u) - \lambda E U_1 \tilde{S}_u \frac{\partial \tilde{v}}{\partial x}(u, \tilde{S}_u) \\ &+ \frac{1}{2} \sigma^2(u) \tilde{S}_u^2 \frac{\partial^2 \tilde{v}}{\partial x^2}(u, \tilde{S}_u) - \lambda \int \left(\tilde{v}(u, \tilde{S}_u(1+z)) - \tilde{v}(u, \tilde{S}_u) \right) d\nu(z) \leq 0 \end{aligned} \tag{18}$$

a.e. in $[0, T) \times \mathbb{R}$.

From Pham (1997), we know that if the payoff function is convex and non-increasing then the price function of the put American contingent claim is a convex function of the stock.

Therefore, we can write

$$\frac{\partial^2 v(t, x)}{\partial x^2} \geq 0 \tag{19}$$

a.e. in $[0, T) \times \mathbb{R}$.

Theorem 3.1. *The mapping $\varsigma(t, x) = x v(t, x)$ is Lipschitz continuous in x and locally Lipschitz continuous in the argument of t , i.e.*

$$|\varsigma(t, x) - \varsigma(t, y)| \leq 2 g(0) |x - y|, \quad 0 \leq t \leq T, \quad 0 < x \leq y < \infty, \tag{20}$$

$$|\varsigma(t, x) - \varsigma(s, x)| \leq \frac{C x}{\sqrt{T-t}} |t - s|, \quad 0 \leq s \leq t < T, \quad x > 0, \tag{21}$$

where the constant C is the function of $\bar{r}, \bar{\sigma}, g(0), \lambda, E(U_1), K$ and T .

Proof. Consider that $v(t, x) = u(t, \ln x)$, $x > 0$, $0 \leq t \leq T$, we can write

$$\begin{aligned} |\varsigma(t, x) - \varsigma(t, y)| &= |x u(t, \ln x) - y u(t, \ln y)| \\ &\leq |x u(t, \ln x) - x u(t, \ln y)| + |x u(t, \ln y) - y u(t, \ln y)|. \end{aligned}$$

Using the bound (8) and the mean value theorem we arrive to (20).

The expression (21) derives using the same arguments as previously, and the bound (9). \square

Proposition 3.2. *The second order weak partial derivative $\frac{\partial^2 v(t, x)}{\partial x^2}$ of the value function (3) satisfies with respect to x the local Holder estimate*

$$x^2 \left| \frac{\partial^2 v(t, x)}{\partial x^2} \right| \leq \frac{D}{\sqrt{T-t}}, \quad x > 0, \quad 0 \leq t < T,$$

where D is a nonnegative constant depends on the parameters \bar{r} , $\bar{\sigma}$, $\underline{\sigma}$, $g(0)$, λ , $E|U_1|$, $E\left(\frac{|U_1|}{1+U_1}\right)$, K , T .

Proof. Using the expression (15), and from (18) and (19), we obtain the system of inequalities

$$\begin{cases} -r(t)v(t, x) + \frac{\partial v(t, x)}{\partial t} - \lambda x E U_1 e^{-\int_0^t r(u) du} \frac{\partial v(t, x)}{\partial x} + \frac{x^2}{2} \sigma^2(t) e^{-2\int_0^t r(u) du} \frac{\partial^2 v(t, x)}{\partial x^2} \\ -\lambda \int (v(t, x(1+z)) - v(t, x)) d\nu(z) \leq 0 \quad \text{a.e. in } [0, T) \times \mathbb{R}, \\ \frac{\partial^2 v(t, x)}{\partial x^2} \geq 0, \quad x > 0. \end{cases} \tag{22}$$

Also since $v(t, x) = u(t, \ln x)$, $x > 0$, $0 \leq t \leq T$, we have

$$\begin{cases} \frac{\partial v(t, x)}{\partial t} = \frac{\partial u(t, \ln x)}{\partial t}, \quad \frac{\partial v(t, x)}{\partial x} = \frac{1}{x} \frac{\partial u(t, \ln x)}{\partial y}, \\ \frac{\partial^2 v(t, x)}{\partial x^2} = \frac{1}{x^2} \frac{\partial^2 u(t, \ln x)}{\partial y^2} - \frac{1}{x^2} \frac{\partial u(t, \ln x)}{\partial y}, \quad 0 \leq t < T, x > 0. \end{cases} \tag{23}$$

Substituting the latter relations and using the results of the Theorem 2.3 in the system of inequalities (22), we have

$$\begin{aligned} &\left| \frac{\partial^2 v(t, x)}{\partial x^2} \right| \\ &\leq \frac{2 e^{2\int_0^t r(v) dv}}{x^2 \underline{\sigma}^2} \left[r(t)v(t, x) + \left| \frac{\partial v(t, x)}{\partial t} \right| + \lambda x E|U_1| \left| \frac{\partial v(t, x)}{\partial x} \right| + \lambda \left| \int (v(t, x(1+z)) - v(t, x)) d\nu(z) \right| \right] \\ &\leq \frac{2 e^{2\bar{r}T}}{x^2 \underline{\sigma}^2} \left[\bar{r}g(0) + \frac{A}{\sqrt{T-t}} + \lambda g(0) E|U_1| + \lambda g(0) E\left(\frac{|U_1|}{1+U_1}\right) \right] \\ &\leq \frac{D}{x^2 \sqrt{T-t}}. \end{aligned}$$

Thus, the required result is derived. \square

Before, we proceed with the main result of this section, we need to state and prove the following result.

Lemma 3.3. For the function $\gamma(t, y) = y \frac{\partial v(t, y)}{\partial y}$, $0 \leq t_1 \leq t_2 < T, y > 0$, of the value function (3) we have the following bound

$$\begin{aligned} |\gamma(t_2, y) - \gamma(t_1, y)| &\leq \frac{1}{h} \left[\int_y^{y+h} |\gamma(t_2, y) - \gamma(t_2, z)| dz + \int_y^{y+h} |\gamma(t_1, y) - \gamma(t_1, z)| dz \right. \\ &\quad + (y+h)|v(t_2, y+h) - v(t_1, y+h)| + y|v(t_2, y) - v(t_1, y)| \\ &\quad \left. + \int_y^{y+h} |v(t_2, z) - v(t_1, z)| dz \right], \end{aligned}$$

where $h > 0$.

Proof. We can express the difference

$$\gamma(t_2, y) - \gamma(t_1, y) = \gamma(t_2, y) - \gamma(t_2, z) + \gamma(t_2, z) - \gamma(t_1, z) + \gamma(t_1, z) - \gamma(t_1, y),$$

for any positive real number z .

Integrating both sides with respect to z over the interval $[y, y+h]$, we obtain

$$\begin{aligned} \gamma(t_2, y) - \gamma(t_1, y) &= \frac{1}{h} \left[\int_y^{y+h} (\gamma(t_2, y) - \gamma(t_2, z)) dz + \int_y^{y+h} (\gamma(t_2, z) - \gamma(t_1, z)) dz \right. \\ &\quad \left. + \int_y^{y+h} (\gamma(t_1, z) - \gamma(t_1, y)) dz \right]. \end{aligned} \tag{24}$$

Simplifying the second integral, we have

$$\begin{aligned} \int_y^{y+h} (\gamma(t_2, z) - \gamma(t_1, z)) dz &= \int_y^{y+h} z \left(\frac{\partial v(t_2, z)}{\partial z} - \frac{\partial v(t_1, z)}{\partial z} \right) dz \\ &= (y+h)(v(t_2, y+h) - v(t_1, y+h)) - y(v(t_2, y) - v(t_1, y)) \\ &\quad - \int_y^{y+h} (v(t_2, z) - v(t_1, z)) dz. \end{aligned}$$

Combining the latter expression with (24), the proof is complete. \square

In the next, a very interesting result for the value of a put American option is derived.

Theorem 3.4. The mapping $\gamma(t, x) = x \frac{\partial v(t, x)}{\partial x}$ satisfies with respect to time argument local Hölder estimate with exponent $\frac{1}{2}$, i.e.,

$$|\gamma(t, x) - \gamma(s, x)| \leq \frac{G + x H}{\sqrt{T-t}} |t - s|^{\frac{1}{2}}, \quad 0 \leq s \leq t < T, x > 0, \tag{25}$$

where G and H are positive constants depend on the parameters $\bar{r}, \bar{\sigma}, \underline{\sigma}, g(0), \lambda, E(U_1), E\left(\frac{|U_1|}{1+U_1}\right), K$ and T .

Proof. From the continuity of $\frac{\partial v(t, x)}{\partial x}$ and the relations (23), using Proposition 3.2 we can write

$$|\gamma(t, x) - \gamma(t, y)| = \left| x \frac{\partial v(t, x)}{\partial x} - y \frac{\partial v(t, y)}{\partial y} \right| \leq \frac{D}{\sqrt{T-t}} |x - y|, \quad 0 \leq t < T, 0 < x \leq y < \infty. \tag{26}$$

Application of the bounds (21) and (26) in Lemma 3.3 gives

$$\begin{aligned}
 |\gamma(t_2, y) - \gamma(t_1, y)| &\leq \frac{1}{h} \left[\int_y^{y+h} \frac{D}{\sqrt{T-t_2}}(z-y)dz + \int_y^{y+h} \frac{D}{\sqrt{T-t_1}}(z-y)dz \right. \\
 &\quad \left. + \frac{C(y+h)}{\sqrt{T-t_2}}|t_2-t_1| + \frac{C y}{\sqrt{T-t_2}}|t_2-t_1| + \int_y^{y+h} \frac{C}{\sqrt{T-t_2}}|t_2-t_1|dz \right] \\
 &= \frac{1}{h} \left[\frac{2 D}{\sqrt{T-t_2}}h^2 + \frac{2 C y}{\sqrt{T-t_2}}|t_2-t_1| + \frac{C h}{\sqrt{T-t_2}}|t_2-t_1| + \frac{C h}{\sqrt{T-t_2}}|t_2-t_1| \right].
 \end{aligned}$$

Let us choose $h = C^*|t_2 - t_1|^{\frac{1}{2}}$ from the latter estimate we get

$$\begin{aligned}
 |\gamma(t_2, y) - \gamma(t_1, y)| &\leq \frac{1}{\sqrt{T-t_2}} \left[\left(2 D C^* + \frac{2 C y}{C^*} \right) |t_2 - t_1|^{\frac{1}{2}} + 2 C |t_2 - t_1| \right] \\
 &\leq \frac{2}{\sqrt{T-t_2}} \left(2 D C^* + \frac{2 C y}{C^*} + 2 T C \right) |t_2 - t_1|^{\frac{1}{2}},
 \end{aligned}$$

the minimum of which is attained at the point $C^* = \sqrt{\frac{C y}{D}}$.

From here, the required result is derived. □

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On a summation boundary value problem for a second-order difference equations with resonance

Saowaluk Chasreechai and Thanin Sitthiwirattham ¹

Nonlinear Dynamic Analysis Research Center,
 Department of Mathematics, Faculty of Applied Science,
 King Mongkut's University of Technology North Bangkok, Thailand
E-mail: saowaluk.c@sci.kmutnb.ac.th, thanin.s@sci.kmutnb.ac.th

Abstract

In this paper, we obtain a sufficient condition for the existence of the solution for a second-order difference equation with summation boundary value problem at resonance, by using some properties of the Green's function, the Schaefer's fixed point theorem and intermediate value theorem. Finally, we present an example to show the importance of these result.

Keywords: boundary value problem; resonance; fixed point theorem; existence.

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1 Introduction

The study of the existence of solutions of boundary value problems for linear second-order ordinary differential and difference equations was initiated by Ilin and Moiseev [1]. Then Gupta [2] studied three-point boundary value problems for nonlinear second-order ordinary differential equations. Since then, nonlinear second-order three-point boundary value problems have also been studied by many authors, one may see [3-6] and references therein. Also, there are a lot of papers dealing with the resonant case for multi-point boundary value problems, see [7-11].

In [8], J.Liu, S.Wang and J.Zhang studied the existence of multiple solutions for boundary value problems of second-order difference equations with resonance:

$$\Delta^2 u(t-1) = g(t, u), \quad t \in \{1, 2, \dots, T\}, \quad (1.1)$$

$$u(0) = 0, \quad u(T+1) = 0. \quad (1.2)$$

Using Morse theory, critical point theory, minimax methods and bifurcation theory.

¹Corresponding author

In this paper, we study the existence of solutions of a second-order difference equation with summation boundary value problem at resonance

$$\Delta^2 u(t - 1) + f(t, u(t)) = 0, \quad t \in \{1, 2, \dots, T\}, \tag{1.3}$$

$$u(0) = 0, \quad u(T + 1) = \alpha \sum_{s=1}^{\eta} u(s), \tag{1.4}$$

where $\frac{2(T + 1)}{\alpha\eta(\eta + 1)} = 1$, $T \geq 3$, $\eta \in \{1, 2, \dots, T - 1\}$ and f is continuous function.

In this paper, we are interested in the existence of the solution for problem (1.3)-(1.4) under the condition $\frac{2(T+1)}{\alpha\eta(\eta+1)} = 1$, which is a resonant case. Using some properties of the Green's function $G(t, s)$, intermediate value theorems and Schaefer's fixed point theorem, we establish a sufficient condition for the existence of positive solutions of problem $\frac{2(T+1)}{\alpha\eta(\eta+1)} = 1$.

Let \mathbb{N} be a nonnegative integer, $\mathbb{N}_{i,j} = \{k \in \mathbb{N} \mid i \leq k \leq j\}$ and $\mathbb{N}_p = \mathbb{N}_{0,p}$.

Throughout this paper, we suppose the following conditions hold:

(H) $f(t, u) \in C(\mathbb{N}_{T+1} \times R, R)$ and there exist two positive continuous functions $p(t), q(t) \in C(\mathbb{N}_{T+1}, R^+)$ such that

$$|f(t, tu)| \leq p(t) + q(t)|u|^m, t \in \mathbb{N}_{T+1}, \tag{1.5}$$

where $0 \leq m \leq 1$. Furthermore, $\lim_{u \rightarrow \pm\infty} f(t, tu) = \infty$, for any $t \in \mathbb{N}_{1,T}$.

To accomplish this, we denote $C(\mathbb{N}_{T+1}, R)$, the Banach space of all function u with the norm defined by $\|u\| = \max\{u(t) \mid t \in \mathbb{N}_{T+1}\}$.

The proof of the main result is based upon an application of the following theorem.

Theorem 1.1. ([12]). *Let X be a Banach space and $T : X \rightarrow X$ be a continuous and compact mapping. If the set*

$$\{x \in X : x = \lambda T(x), \text{ for some } \lambda \in (0, 1)\}$$

is bounded, then T has a fixed point.

The plan of the paper is follows. In Section 2, we recall some lemmas. In Section 3, we prove our main result. Illustrate example is presented in Section 4.

2 Preliminaries

We now state and prove several lemmas before stating our main results.

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Lemma 2.1. *The problem (1.3)-(1.4) is equivalent to the following*

$$u(t) = \sum_{s=1}^T G(t, s)f(s, u(s)) + \frac{u(T + 1)}{T + 1}t, \tag{2.1}$$

where

$$G(t, s) = \frac{1}{(T + 1)(\alpha - 1)} \begin{cases} \alpha t(T + 1 - s) - \frac{1}{2}\alpha t(\eta - s)(\eta - s + 1) \\ \quad - (T + 1)(\alpha - 1)(t - s), & s \in \mathbb{N}_{1, t-1} \cap \mathbb{N}_{1, \eta-1} \\ \alpha t(T + 1 - s) - \frac{1}{2}\alpha t(\eta - s)(\eta - s + 1), & s \in \mathbb{N}_{t, \eta-1} \\ \alpha t(T + 1 - s) - (T + 1)(\alpha - 1)(t - s), & s \in \mathbb{N}_{\eta, t-1} \\ \alpha t(T + 1 - s), & s \in \mathbb{N}_{t, T} \cap \mathbb{N}_{\eta, T} \end{cases} \tag{2.2}$$

Proof. Assume that $u(t)$ is a solution of problem (1.3)-(1.4), then it satisfies the following equation:

$$u(t) = C_1 + C_2t - \sum_{s=1}^{t-1} (t - s)f(s, u(s)),$$

where C_1, C_2 are constants. By the boundary value condition (1.3), we obtain $C_1 = 0$.

So,

$$u(t) = C_2t - \sum_{s=1}^{t-1} (t - s)f(s, u(s)). \tag{2.3}$$

From (2.3),

$$\begin{aligned} \sum_{s=1}^{\eta} u(s) &= \frac{\eta(\eta + 1)}{2}C_2 - \sum_{s=1}^{\eta-1} \sum_{l=1}^{\eta-s} ly(s) \\ &= \frac{\eta(\eta + 1)}{2}C_2 - \frac{1}{2} \sum_{s=1}^{\eta-1} (\eta - s)(\eta - s + 1)y(s). \end{aligned}$$

From the second boundary condition, we have

$$(2T + 2 - \alpha\eta(\eta + 1))C_2 = 2 \sum_{s=1}^T (T + 1 - s)f(s, u(s)) + \alpha \sum_{s=1}^{\eta-1} (\eta - s)(\eta - s + 1)f(s, u(s)). \tag{2.4}$$

Since $\frac{2(T+1)}{\alpha\eta(\eta+1)} = 1$, then (2.4) is solvable if and only if

$$\sum_{s=1}^T (T + 1 - s)f(s, u(s)) = \frac{\alpha}{2} \sum_{s=1}^{\eta-1} (\eta - s)(\eta - s + 1)f(s, u(s)).$$

Note that

$$u(T + 1) - \sum_{s=1}^{\eta} u(s) = (T + 1)C_2 - \sum_{s=1}^T (T + 1 - s)f(s, u(s)) - \frac{\eta(\eta + 1)}{2}C_2 + \frac{1}{2} \sum_{s=1}^{\eta-1} (\eta - s)(\eta + 1 - s)f(s, u(s)),$$

and then

$$\begin{aligned} C_2 &= \frac{2}{2T + 2 - \eta(\eta + 1)} \left[u(T + 1) - \sum_{s=1}^{\eta} u(s) + \sum_{s=1}^T (T + 1 - s)f(s, u(s)) - \frac{1}{2} \sum_{s=1}^{\eta-1} (\eta - s)(\eta + 1 - s)f(s, u(s)) \right] \\ &= \frac{\alpha}{(T + 1)(\alpha - 1)} \left[u(T + 1) - \sum_{s=1}^{\eta} u(s) + \sum_{s=1}^T (T + 1 - s)f(s, u(s)) - \frac{1}{2} \sum_{s=1}^{\eta-1} (\eta - s)(\eta + 1 - s)f(s, u(s)) \right]. \end{aligned}$$

We now use that $u(T + 1) = \frac{2(T + 1)}{\eta(\eta + 1)} \sum_{s=1}^{\eta} u(s)$ to get

$$\frac{\alpha}{(T + 1)(\alpha - 1)} \left[u(T + 1) - \sum_{s=1}^{\eta} u(s) \right] = \frac{u(T + 1)}{T + 1},$$

and

$$\begin{aligned} C_2 &= \frac{\alpha}{(T + 1)(\alpha - 1)} \left[\sum_{s=1}^T (T + 1 - s)f(s, u(s)) - \frac{1}{2} \sum_{s=1}^{\eta-1} (\eta - s)(\eta + 1 - s)f(s, u(s)) \right] \\ &\quad + \frac{u(T + 1)}{T + 1}. \end{aligned}$$

Hence the solution of (1.3)-(1.4) is given, implicitly as

$$\begin{aligned} u(t) &= \frac{\alpha t}{(T + 1)(\alpha - 1)} \left[\sum_{s=1}^T (T + 1 - s)f(s, u(s)) - \frac{1}{2} \sum_{s=1}^{\eta-1} (\eta - s)(\eta + 1 - s)f(s, u(s)) \right] \\ &\quad - \sum_{s=1}^{t-1} (t - s)f(s, u(s)) + \frac{u(T + 1)}{T + 1}t. \end{aligned} \tag{2.5}$$

According to (2.5) it is easy to show that (2.1) holds. Therefore, problem (1.3)-(1.4) is equivalent to the equation (2.1) with the function $G(t, s)$ defined in (2.2). The proof is completed. \square

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Lemma 2.2. For any $(t, s) \in \mathbb{N}_{T+1} \times \mathbb{N}_{T+1}$, $G(t, s)$ is continuous, and $G(t, s) > 0$ for any $(t, s) \in \mathbb{N}_{1,T} \times \mathbb{N}_{1,T}$.

Proof. The continuity of $G(t, s)$ for any $(t, s) \in \mathbb{N}_{T+1} \times \mathbb{N}_{T+1}$, is obvious. Let

$$g_1(t, s) = \alpha t(T + 1 - s) - \frac{1}{2}\alpha t(\eta - s)(\eta - s + 1) - (T + 1)(\alpha - 1)(t - s),$$

where $s \in \mathbb{N}_{1,t-1} \cap \mathbb{N}_{1,\eta-1}$.

Here we only need to prove that $g_1(t, s) > 0$ for $s \in \mathbb{N}_{1,t-1} \cap \mathbb{N}_{1,\eta-1}$, the rest of the proof is similar. So, from the definition of $g_1(t, s)$, $\eta \in \mathbb{N}_{1,T-1}$ and the resonant condition $\frac{2(T+1)}{\alpha\eta(\eta+1)} = 1$, we have

$$\begin{aligned} g_1(t, s) &= \alpha t(T + 1 - s) - \frac{1}{2}\alpha t(\eta - s)(\eta - s + 1) - (T + 1)(\alpha - 1)(t - s) \\ &= (T + 1)(t - s) + \alpha s(T + 1 - t) - \frac{1}{2}\alpha t(\eta - s)(\eta - s + 1) \\ &> (T + 1)(t - s) - \frac{\alpha}{2} [t\eta(\eta + 1) - 2s(T + 1 - t)] \\ &> (T + 1)(t - s) - \frac{\alpha}{2} \\ &> (T + 1)(t - s) - \frac{T + 1}{\eta(\eta + 1)} \\ &> (T + 1)(t - s - 1) \\ &\geq 0, \end{aligned}$$

for $s \in \mathbb{N}_{1,t-1} \cap \mathbb{N}_{1,\eta-1}$. Since $t > s$ and $\eta(\eta + 1) \geq 2(T + 1 - t)$ where $T \geq 3$. The proof is completed. □

Let

$$G^*(t, s) = \frac{1}{t}G(t, s). \tag{2.6}$$

Then

$$G^*(t, s) = \frac{1}{(T + 1)(\alpha - 1)} \begin{cases} \alpha(T + 1 - s) - \frac{1}{2}\alpha(\eta - s)(\eta - s + 1) - \frac{1}{t}(T + 1)(\alpha - 1)(t - s), & s \in \mathbb{N}_{1,t-1} \cap \mathbb{N}_{1,\eta-1} \\ \alpha(T + 1 - s) - \frac{1}{2}\alpha(\eta - s)(\eta - s + 1), & s \in \mathbb{N}_{t,\eta-1} \\ \alpha(T + 1 - s) - \frac{1}{t}(T + 1)(\alpha - 1)(t - s), & s \in \mathbb{N}_{\eta,t-1} \\ \alpha(T + 1 - s), & s \in \mathbb{N}_{t,T} \cap \mathbb{N}_{\eta,T}. \end{cases} \tag{2.7}$$

Thus, problem (1.3)-(1.4) is equivalent to the following equation:

$$u(t) = \sum_{s=1}^T tG^*(t, s)f(s, u(s)) + \frac{u(T + 1)}{T + 1}t. \tag{2.8}$$

By a simple computation, the new Green's function $G^*(t, s)$ has the following properties.

Lemma 2.3. For any $(t, s) \in \mathbb{N}_{T+1} \times \mathbb{N}_{T+1}$, $G^*(t, s)$ is continuous, and $G^*(t, s) > 0$ for any $(t, s) \in \mathbb{N}_{1,T} \times \mathbb{N}_{1,T}$. Furthermore,

$$\begin{aligned} \lim_{t \rightarrow 0} G^*(t, s) &:= G^*(0, s) \\ &= \frac{1}{(T+1)(\alpha-1)} \begin{cases} \alpha(T+1-s) - \frac{1}{2}\alpha(\eta-s)(\eta-s+1), & s \in \mathbb{N}_{1,\eta-1} \\ \alpha(T+1-s), & s \in \mathbb{N}_{\eta,T}. \end{cases} \end{aligned} \tag{2.9}$$

Lemma 2.4. For any $s \in \mathbb{N}_{1,T}$, $G^*(t, s)$ is nonincreasing with respect to $t \in \mathbb{N}_{T+1}$, and for any $s \in \mathbb{N}_{T+1}$, $\frac{\Delta_t G^*(t,s)}{\Delta t} < 0$, and $\frac{\Delta_t G^*(t,s)}{\Delta t} = 0$ for $t \in \mathbb{N}_s$. That is, $G^*(T+1, s) \leq G^*(t, s) \leq G^*(s, s)$ where

$$\begin{aligned} G^*(t, s) &\leq G^*(s, s) \\ &= \frac{1}{(T+1)(\alpha-1)} \begin{cases} \alpha(T+1-s) - \frac{1}{2}\alpha(\eta-s)(\eta-s+1), & s \in \mathbb{N}_{1,\eta-1} \\ \alpha(T+1-s), & s \in \mathbb{N}_{\eta,T} \end{cases} \end{aligned} \tag{2.10}$$

$$\begin{aligned} G^*(t, s) &\geq G^*(T+1, s) \\ &= \frac{1}{(T+1)(\alpha-1)} \begin{cases} (T+1)(T+1-s) - \frac{1}{2}\alpha(\eta-s)(\eta-s+1), & s \in \mathbb{N}_{1,\eta-1} \\ (T+1)(T+1-s), & s \in \mathbb{N}_{\eta,T}. \end{cases} \end{aligned} \tag{2.11}$$

Let

$$u(t) = tw(t). \tag{2.12}$$

Then $u(T+1) = (T+1)w(T+1)$, and equation (2.8) gives

$$w(t) = \sum_{s=1}^T G^*(t, s)f(s, sw(s)) + w(T+1). \tag{2.13}$$

Now we have

$$y(t) = w(t) - w(T+1). \tag{2.14}$$

Then $y(T+1) = w(T+1) - w(T+1) = 0$, and equation (2.13) gives

$$y(t) = \frac{1}{T+1} \sum_{s=1}^T G^*(t, s)f(s, s(y(s) + w(T+1))). \tag{2.15}$$

We replace $w(T+1)$ by any real number λ , then (2.15) can be rewritten as

$$y(t) = \frac{1}{T+1} \sum_{s=1}^T G^*(t, s)f(s, s(y(s) + \lambda)). \tag{2.16}$$

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The following result is based on the Schaefer’s fixed point theorem. We define an operator T on the set $\Omega = C(\mathbb{N}_{T+1})$ as follows:

$$Ty(t) = \frac{1}{T+1} \sum_{s=1}^T G^*(t, s) f(s, s(y(s) + \lambda)). \tag{2.17}$$

Lemma 2.5. Assume that $f \in C(\mathbb{N}_{T+1} \times R, R)$, $\sum_{s=1}^T G^*(t, s)q(s) < T + 1$ and (1.5) holds. Then the equation (2.16) has at least one solution for any real number λ .

Proof. We divide the proof into four steps.

Step I. T maps bounded sets into bounded sets in Ω . Let us prove that for any $R > 0$, there exists a positive constant L such that for each $y \in B_R = \{y \in C(\mathbb{N}_{T+1} \times R) : \|y\| \leq R\}$, we have $\|(Ty)(t)\| \leq L$. Indeed, for any $y \in B_R$, we obtain

$$\begin{aligned} |(Ty)(t)| &= \left| \frac{1}{T+1} \sum_{s=1}^T G^*(t, s) f(s, s(y(s) + \lambda)) \right| \\ &\leq \frac{1}{T+1} \sum_{s=1}^T G^*(t, s) p(s) + \frac{1}{T+1} \sum_{s=1}^T G^*(t, s) |q(s) + \lambda|^m \\ &\leq \frac{1}{T+1} \sum_{s=1}^T G^*(t, s) p(s) + \frac{1}{T+1} \sum_{s=1}^T G^*(t, s) q(s) (\|y(s)\| + \|\lambda\|)^m \\ &\leq \frac{1}{T+1} \sum_{s=1}^T G^*(s, s) p(s) + \frac{(R + \|\lambda\|)^m}{T+1} \sum_{s=1}^T G^*(s, s) q(s) \\ &:= L. \end{aligned} \tag{2.18}$$

Step II. *Continuity of T .* Let $\epsilon > 0$, there exists $\delta > 0$ such that for all $t \in \mathbb{N}_{T+1}$ and for all $x, y \in B_R$ with $|(t, t(x(t) + \lambda) - (t, t(y(t) + \lambda))| < \delta$, we have

$$|f(t, t(x(t) + \lambda) - f(t, t(y(t) + \lambda))| < \epsilon.$$

Then, we obtain

$$\begin{aligned} |(Tx)(t) - (Ty)(t)| &\leq \left| \frac{1}{T+1} \sum_{s=1}^T G^*(t, s) [f(s, s(x(s) + \lambda)) - f(s, s(y(s) + \lambda))] \right| \\ &\leq \frac{\epsilon}{T+1} \left| \sum_{s=1}^T G^*(t, s) \right| = \epsilon. \end{aligned}$$

This means that T is continuous in Ω .

Step III. $T(B_R)$ is equicontinuous with B_R defined as in Step II. Since B_R is bounded, then there exists $M > 0$ such that $|f| \leq M$.

For any $\varepsilon > 0$, there exists $\delta > 0$ such that for $t_1, t_2 \in \mathbb{N}_{T+1}$

$$|G^*(t_2, s) - G^*(t_1, s)| \leq \frac{\varepsilon}{M}.$$

Then we have

$$\begin{aligned} |(Ty)(t_2) - (Ty)(t_1)| &\leq \left| \frac{1}{T+1} \sum_{s=1}^T |G^*(t_2, s) - G^*(t_1, s)| |f(s, s(y(s) + \lambda))| \right| \\ &\leq \frac{M}{T+1} \sum_{s=1}^T |G^*(t_2, s) - G^*(t_1, s)| \\ &= M \cdot \frac{\varepsilon}{M} \leq \varepsilon. \end{aligned}$$

This means that the set $T(B_R)$ is an equicontinuous set. As a consequence of Steps I to III together with the Arzela'-Ascoli theorem, we get that T is completely continuous in Ω .

Step IV. A priori bounds. We show that the set

$$E = \{y \in C(\mathbb{N}_{T+1}, \mathbb{R}) / y = \mu Ty \text{ for some } \mu \in (0, 1)\} \text{ is bounded.}$$

By Lemma 2.1, assume that there exist $y \in \partial B_R$ with $\|y(t)\| = R$ and $\mu \in (0, 1)$ such that $y = \mu Ty$. It follows that

$$\begin{aligned} |y(t)| &= \frac{\mu}{T+1} \left| \sum_{s=1}^T G^*(t, s) f(s, s(y(s) + \lambda)) \right| \\ &\leq \frac{\mu}{T+1} \sum_{s=1}^T G^*(s, s) |f(s, s(y(s) + \lambda))| \\ &< \frac{1}{T+1} \left[\sum_{s=1}^T G^*(s, s) p(s) + \sum_{s=1}^T G^*(s, s) q(s) (\|y(s)\| + \|\lambda\|)^m \right] \\ &\leq \frac{1}{T+1} \sum_{s=1}^T G^*(s, s) p(s) + \frac{(R + \|\lambda\|)^m}{T+1} \sum_{s=1}^T G^*(s, s) q(s) \\ &:= L. \end{aligned} \tag{2.19}$$

This shows that the set E is bounded. By the Schaefer's fixed point theorem, we conclude that T has a fixed point which is a solution of problem (1.1). \square

3 Main Results

In this section, we prove our result by using Lemmas 2.5-2.7 and the intermediate value theorem.

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Theorem 3.1. Assume that (H1) holds. If $\sum_{s=1}^T G^*(s, s)q(s) < 1$, then the problem (1.3)-(1.4) has at least one solution, where

$$G^*(s, s) = \frac{1}{(T+1)(\alpha-1)} \begin{cases} \alpha(T+1-s) - \frac{1}{2}\alpha(\eta-s)(\eta-s+1), & s \in \mathbb{N}_{1,\eta-1} \\ \alpha(T+1-s), & s \in \mathbb{N}_{\eta,T} \end{cases}$$

Proof. Since (2.19) is continuously dependent on the parameter λ . So, we should only investigate λ such that $y(T+1) = 0$ in order that $u(T+1) = \lambda$.

Equation (2.16) is rewrite as

$$y_\lambda(t) = \frac{1}{T+1} \sum_{s=1}^T G^*(t, s)f(s, s(y_\lambda(s) + \lambda)), \quad t \in \mathbb{N}_{T+1}. \tag{3.1}$$

where λ is any given real number.

Equation(3.1) show that there exists λ such that

$$L(\lambda) := y_\lambda(T+1) = \frac{1}{T+1} \sum_{s=1}^T G^*(T+1, s)f(s, s(y_\lambda(s) + \lambda)) \tag{3.2}$$

and we can observe that, $y_\lambda(T+1)$ is continuously dependent on the parameter λ .

To prove that there exists λ^* such that $y_{\lambda^*}(T+1) = 0$, we must to show that $\lim_{\lambda \rightarrow \infty} L(\lambda) = \infty$ and $\lim_{\lambda \rightarrow -\infty} L(\lambda) = -\infty$.

Firstly, we prove that $\lim_{\lambda \rightarrow \infty} L(\lambda) = \infty$ by supposing that $\lim_{\lambda \rightarrow \infty} L(\lambda) < \infty$ as a contradiction. Therefore there exists a sequence $\{\lambda_n\}$ with $\lim_{n \rightarrow \infty} L(\lambda) = \infty$ such that $\lim_{\lambda_n \rightarrow \infty} L(\lambda_n) < \infty$. This implies that the sequence $\{L(\lambda_n)\}$ is bounded. Since the function $f(t, ty)$ is continuous with respect to $t \in \mathbb{N}_{T+1}$ and $y \in R$, we have

$$f(t, t(y_{\lambda_n}(t) + \lambda_n)) \geq 0, \quad t \in \mathbb{N}_{T+1} \tag{3.3}$$

where λ_n is large enough, Assuminh that (3.3) is true and using (3.1), we have

$$y_\lambda \geq 0, \quad t \in \mathbb{N}_{T+1}. \tag{3.4}$$

Therefore,

$$\lim_{\lambda_n \rightarrow \infty} f(t, t(y_{\lambda_n}(t) + \lambda_n)) = \infty, \quad t \in \mathbb{N}_{T+1}. \tag{3.5}$$

From (H), we get

$$\lim_{\lambda \rightarrow \infty} f(t, tu) = \infty, \quad t \in \mathbb{N}_{T+1}. \tag{3.6}$$

From (3.2),(3.5) and (3.6), we have

$$\lim_{\lambda_n \rightarrow \infty} y_{\lambda_n}(T+1) = \lim_{\lambda_n \rightarrow \infty} \sum_{s=1}^T G^*(T+1, s)f(s, s(y_{\lambda_n}(s) + \lambda_n)) \tag{3.7}$$

$$\begin{aligned} &\geq \lim_{\lambda_n \rightarrow \infty} \sum_{s=\frac{1}{4}(T-1)}^{\frac{3}{4}(T-1)} G^*(T+1, s) f(s, s(y_{\lambda_n}(s) + \lambda_n)) \\ &= \infty, \end{aligned} \tag{3.8}$$

we find that this result contradicts our assumption.

We define

$$S_n = \{t \in \mathbb{N}_{T+1} \mid f(t, t(y_{\lambda_n}(t) + \lambda_n)) < 0\}.$$

where λ_n is large. Note that S_n is not empty.

Secondly, we divide the set S_n into set \tilde{S}_n and set \hat{S}_n as follows:

$$\tilde{S}_n = \{t \in S_n \mid y_{\lambda_n} + \lambda_n > 0\} \quad \text{and} \quad \hat{S}_n = \{t \in S_n \mid y_{\lambda_n} + \lambda_n \leq 0\}$$

where $\tilde{S}_n \cap \hat{S}_n = \emptyset$, $\tilde{S}_n \cup \hat{S}_n = S_n$. So, we have from (H) that \hat{S}_n is not empty.

In addition, we find from (H) that the function $f(t, tu)$ is bounded below by a constant for $t \in \mathbb{N}_{T+1}$ and $\lambda \in [0, \infty)$. Thus, there exists a constant $M (< 0)$ which is independent of t and λ_n , such that

$$f(t, t(y_{\lambda_n}(t) + \lambda_n)) \geq M, \quad t \in \tilde{S}_n, \tag{3.9}$$

Let $h(\lambda_n) = \min_{t \in S_n} y_{\lambda_n}(t)$ and using the definitions of \tilde{S}_n and set \hat{S}_n , we have

$$h(\lambda_n) = \min_{t \in \hat{S}_n} y_{\lambda_n}(t) = -\|y_{\lambda_n}(t)\|_{\hat{S}_n}.$$

It follows that $h(\lambda_n) \rightarrow -\infty$ as $\lambda_n \rightarrow \infty$ since if $h(\lambda_n)$ is bounded below by a constant as $\lambda_n \rightarrow \infty$, then (3.7) holds. Therefore, we can choose large λ_{n_1} such that

$$h(\lambda_n) < \frac{1}{T+1} \max \left\{ -1, \frac{M \sum_{s=1}^T G^*(s, s) - \sum_{s=1}^T G^*(s, s)p(s)}{1 - \sum_{s=1}^T G^*(s, s)q(s)} \right\} \tag{3.10}$$

for $n > n_1$. Employing (H), (3.1), (3.8), (3.9), the definitions of \tilde{S}_n , and set \hat{S}_n , for any $\lambda_n > \lambda_{n_1}$, we have

$$\begin{aligned} y_{\lambda_n}(t) &\geq \frac{1}{T+1} \sum_{s \in S_n} G^*(s, s) f(s, s(y_{\lambda_n}(s) + \lambda_n)) \\ &\geq \frac{1}{T+1} \sum_{s \in \tilde{S}_n} G^*(s, s) f(s, s(y_{\lambda_n}(s) + \lambda_n)) \\ &\quad + \frac{1}{T+1} \sum_{s \in \hat{S}_n} G^*(s, s) (-p(s) - q(s)|y_{\lambda_n}(s) + \lambda_n|^m) \end{aligned}$$

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$$\geq \frac{1}{T+1} \left[M \sum_{s \in \tilde{S}_n} G^*(s, s) - \sum_{s \in \tilde{S}_n} G^*(s, s)p(s) - \sum_{s \in \tilde{S}_n} G^*(s, s)q(s) \|y_{\lambda_n}(s) + \lambda_n\|^m \right].$$

It follows that

$$\begin{aligned} y_{\lambda_n}(t) &\geq \frac{1}{T+1} \left[M \sum_{s=1}^T G^*(s, s) - \sum_{s=1}^T G^*(s, s)p(s) - \sum_{s=1}^T G^*(s, s)q(s) \|y_{\lambda_n}(s) + \lambda_n\|^m \right], \\ &\geq \frac{1}{T+1} \left[M \sum_{s=1}^T G^*(s, s) - \sum_{s=1}^T G^*(s, s)p(s) - \sum_{s=1}^T G^*(s, s)q(s)h(\lambda_n) \right], \quad t \in S_n, \end{aligned}$$

which implies that

$$h(\lambda_n) \geq \frac{1}{T+1} \left[\frac{M \sum_{s=1}^T G^*(s, s) - \sum_{s=1}^T G^*(s, s)p(s)}{1 - \sum_{s=1}^T G^*(s, s)q(s)} \right].$$

This result contradicts (3.9). Thus, the proof that $\lim_{\lambda \rightarrow \infty} L(\lambda) = \infty$ is done. using a similar method, we can prove that $\lim_{\lambda \rightarrow -\infty} L(\lambda) = -\infty$.

Notice that $L(\lambda)$ is continuous with respect to $\lambda \in (-\infty, \infty)$. From the intermediate value theorem, there exists $\lambda^* \in (-\infty, \infty)$ such that $L(\lambda^*) = 0$, that is, $y(T+1) = y_{\lambda^*}(T+1) = 0$, which satisfies the second boundary value condition of (1.2). The proof is completed. □

4 Example

In this section, we give an example to illustrate our result.

Example Consider the BVP

$$\Delta^2 u(t-1) + t^2 + \frac{1}{2}u(t) = 0, \quad t \in N_{1,4}, \tag{4.1}$$

$$u(0) = 0, \quad u(5) = \frac{5}{6} \sum_{s=1}^2 u(s). \tag{4.2}$$

Set $\alpha = \frac{5}{6}$, $\eta = 2$, $T = 4$, $f(t, u) = t^2 + \frac{1}{2}u(t)$. So we have

$$\frac{\alpha\eta(\eta+1)}{2(T+1)} = 1 \quad \text{and} \quad f(t, tu) = t^2 + \frac{t}{2}u(t).$$

Now we take $q(t) = \frac{t}{5}$. It is easy to check that

$$\lim_{u \rightarrow \pm\infty} f(t, tu) = \pm\infty \quad \text{and} \quad \sum_{s=1}^4 G^*(s, s)q(s) \leq \frac{1}{25} \sum_{s=1}^4 (5-s)s = \frac{4}{5} < 1.$$

Thus the conditions of Theorem 3.1 are satisfied. Therefore problem (4.1)-(4.2) has at least a nontrivial solution. □

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Fuzzy quadratic mean operators and their use in group decision making

Jin Han Park, Seung Mi Yu
Department of Applied Mathematics, Pukyong National University,
Pusan 608-737, South Korea
jihpark@pknu.ac.kr (J.H. Park), sweetymi12@naver.com (S.M. Yu)

Young Chel Kwun*
Department of Mathematics, Dong-A University,
Pusan 604-714, South Korea
yckwun@dau.ac.kr

Abstract

Quadratic mean in statistics is a statistical measure defined as the square root of the mean of the squares of a sample. In this paper, we investigate the situations in which the input data are expressed in fuzzy values and develop some fuzzy quadratic mean operators, such as fuzzy weighted quadratic mean operator, fuzzy ordered weighted quadratic mean operator, and fuzzy hybrid quadratic mean operator. Especially, all these operators can reduce to aggregate interval or real numbers. Then based on the developed operators, we present an approach to group decision making and illustrate it with a practical example.

1 Introduction

Information aggregation is an essential process of gathering relevant information from multiple sources by using a proper aggregation technique. Many techniques, such as the weighted average operator [5], the weighted geometric mean operator [1], harmonic mean operator [2], weighted harmonic mean (WHM) operator [2], ordered weighted average (OWA) operator [17], ordered weighted geometric operator [3, 13], weighted OWA operator [8], induced OWA operator [21], induced ordered weighted geometric operator [15], uncertain OWA operator [14], hybrid aggregation operator [10] and so on, have been developed to aggregate data information. However, yet most of existing aggregation operators do not take into account the information about the relationship between the values being fused. Yager [18] introduced a tool to provide more versatility in the

*Corresponding author: yckwun@dau.ac.kr

information aggregation process, i.e., developed a power-average (PA) operator and a power OWA (POWA) operator. In some situations, however, these two operators are unsuitable to deal with the arguments taking the forms of multiplicative variables because of lack of knowledge, or data, and decision makers' limited expertise related to the problem domain. Based on this tool, Xu and Yager [16] developed additional new geometric aggregation operators, including the power-geometric (PG) operator, weighted PG operator and power-ordered weighted geometric (POWG) operator, whose weighting vectors depend upon the input arguments and allow values being aggregated to support and reinforce each other.

Quadratic mean in statistics is a statistical measure defined as the square root of the mean of the squares of a sample, which is a conservative average to be used to provide for aggregation lying between the max and min operators. Consider that, in the existing literature, the quadratic mean is generally considered as a fusion technique of numerical data, in the real-life situations, the input data sometimes cannot be obtained exactly, but fuzzy data can be given. Therefore, how to aggregate fuzzy data by using the quadratic mean? is an interesting research topic and is worth paying attention to. In this paper, we develop some fuzzy quadratic mean (FQM) operators. To do so, the remainder of this paper is arranged in the following sections. Section 2 reviews some basic aggregation operators. Section 3 develops some FQM operators, such as fuzzy weighted quadratic mean (FWQM) operator, fuzzy ordered weighted quadratic mean (FOWQM) operator, fuzzy hybrid quadratic mean (FHQM) operator, and so on. Section 4 presents an approach to multiple attribute group decision making based on the developed operators. Section 5 illustrates the presented approach with a practical example. Section 6 ends the paper with some concluding remarks.

2 Basic aggregation operators

We review some basic aggregation techniques and some of their fundamental characteristics.

Definition 2.1 [5] Let $WAA : R^n \rightarrow R$, if

$$WAA(a_1, a_2, \dots, a_n) = \sum_{j=1}^n w_j a_j, \tag{1}$$

where R is the set of real numbers, a_j ($j = 1, 2, \dots, n$) is a collection of positive real numbers, and $w = (w_1, w_2, \dots, w_n)^T$ is the weight vector of a_j ($j = 1, 2, \dots, n$), with $w_j \geq 0$ and $\sum_{j=1}^n w_j = 1$, then WAA is called the weighted arithmetic averaging (WAA) operator. Especially, if $w_i = 1$, $w_j = 0$, $j \neq i$, then $WAA(a_1, a_2, \dots, a_n) = a_i$; if $w = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})^T$, then the WAA

operator is reduced to the arithmetic averaging (AA) operator, i.e.,

$$AA(a_1, a_2, \dots, a_n) = \frac{1}{n} \sum_{j=1}^n a_j. \tag{2}$$

Definition 2.2 [2] Let $WQM : (R^+)^n \rightarrow R^+$, if

$$WQM(a_1, a_2, \dots, a_n) = \left(\sum_{j=1}^n w_j a_j^2 \right)^{\frac{1}{2}}, \tag{3}$$

where R^+ is the set of all positive real numbers, a_j ($j = 1, 2, \dots, n$) is a collection of positive real numbers, and $w = (w_1, w_2, \dots, w_n)^T$ is the weight vector of a_j ($j = 1, 2, \dots, n$), with $w_j \geq 0$ and $\sum_{j=1}^n w_j = 1$, then WQM is called the weighted quadratic mean (WQM) operator. Especially, if $w_i = 1, w_j = 0, j \neq i$, then $WQM(a_1, a_2, \dots, a_n) = a_i$; if $w = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})^T$, then the WQM operator is reduced to the quadratic mean (QM) operator, i.e.,

$$QM(a_1, a_2, \dots, a_n) = \left(\frac{\sum_{j=1}^n a_j^2}{n} \right)^{\frac{1}{2}}. \tag{4}$$

The WAA and WQM operators first weight all the given data, and then aggregate all these weighted data into a collective one. Yager [17] introduced and studied the OWA operator that weights the ordered positions of the data instead of weighting the data themselves.

Definition 2.3 [17] An OWA operator of dimension n is a mapping $OWA : R^n \rightarrow R$ that has an associated vector $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ such that $\omega_j \geq 0$ and $\sum_{j=1}^n \omega_j = 1$. Furthermore,

$$OWA(a_1, a_2, \dots, a_n) = \sum_{j=1}^n \omega_j b_j, \tag{5}$$

where b_j is the j th largest of a_i ($i = 1, 2, \dots, n$). Especially, if $w_i = 1, w_j = 0, j \neq i$, then $b_n \leq OWA(a_1, a_2, \dots, a_n) = b_i \leq b_1$; if $w = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})^T$, then

$$OWA(a_1, a_2, \dots, a_n) = \frac{1}{n} \sum_{j=1}^n b_j = \frac{1}{n} \sum_{j=1}^n a_j = AA(a_1, a_2, \dots, a_n). \tag{6}$$

3 Fuzzy quadratic mean operators

The above aggregation techniques can only deal with the situation that the arguments are represented by the exact numerical values, but are invalid if the aggregation information is given in other forms, such as triangular fuzzy number [9], which is a widely used tool to deal with uncertainty and fuzzyness, described as follows:

Definition 3.1 [9] A triangular fuzzy number \hat{a} can be defined by a triplet $[a^L, a^M, a^U]$. The membership function $\mu_{\hat{a}}(x)$ is defined as:

$$\mu_{\hat{a}}(x) = \begin{cases} 0, & x < a^L; \\ \frac{x-a^L}{a^M-a^L}, & a^L \leq x \leq a^M; \\ \frac{x-a^U}{a^M-a^U}, & a^M \leq x \leq a^U; \\ 0, & x > a^U, \end{cases}$$

where $a^U \geq a^M \geq a^L \geq 0$, a^L and a^U stand for the lower and upper values of \hat{a} , respectively, and a^M stands for the modal value [9]. Especially, if and two of a^L, a^M and a^U are equal, then \hat{a} is reduced to an interval number; if all a^L, a^M and a^U are equal, then \hat{a} is reduced to a real number. For convenience, we let Ω be the set of all triangular fuzzy numbers.

Let $\hat{a} = [a^L, a^M, a^U]$ and $\hat{b} = [b^L, b^M, b^U]$ be two triangular fuzzy numbers, then some operational laws defined as follows [9]:

- (1) $\hat{a} + \hat{b} = [a^L, a^M, a^U] + [b^L, b^M, b^U] = [a^L + b^L, a^M + b^M, a^U + b^U]$;
- (2) $\lambda \hat{a} = \lambda[a^L, a^M, a^U] = [\lambda a^L, \lambda a^M, \lambda a^U]$;
- (3) $\hat{a} \times \hat{b} = [a^L, a^M, a^U] \times [b^L, b^M, b^U] = [a^L b^L, a^M b^M, a^U b^U]$
- (4) $\frac{1}{\hat{a}} = \frac{1}{[a^L, a^M, a^U]} = [\frac{1}{a^U}, \frac{1}{a^M}, \frac{1}{a^L}]$.

In order to compare two triangular fuzzy numbers, Xu [12] provided the following definition:

Definition 3.2 [12] Let $\hat{a} = [a^L, a^M, a^U]$ and $\hat{b} = [b^L, b^M, b^U]$ be two triangular fuzzy numbers, then the degree of possibility of $\hat{a} \geq \hat{b}$ is defined as follows:

$$p(\hat{a} \geq \hat{b}) = \delta \max \left\{ 1 - \max \left(\frac{b^M - a^L}{a^M - a^L + b^M - b^L}, 0 \right), 0 \right\} + (1 - \delta) \max \left\{ 1 - \max \left(\frac{b^U - a^M}{a^U - a^M + b^U - b^M}, 0 \right), 0 \right\}, \delta \in [0, 1] \quad (7)$$

which satisfies the following properties:

$$0 \leq p(\hat{a} \geq \hat{b}) \leq 1, \quad p(\hat{a} \geq \hat{a}) = 0.5, \quad p(\hat{a} \geq \hat{b}) + p(\hat{b} \geq \hat{a}) = 1. \quad (8)$$

Here, δ reflects the decision maker's risk-bearing attitude. If $\delta > 0.5$, then the decision maker is risk lover; If $\delta = 0.5$, then the decision maker is neutral to risk; If $\delta < 0.5$, then the decision maker is risk avertor.

In the following, we shall give a simple procedure for ranking of the triangular fuzzy numbers \hat{a}_i ($i = 1, 2, \dots, n$). First, by using Eq. (7), we compare each \hat{a}_i with all the \hat{a}_j ($j = 1, 2, \dots, n$), for simplicity, let $p_{ij} = p(\hat{a}_i \geq \hat{a}_j)$, then we develop a possibility matrix [14] as

$$P = \begin{pmatrix} p_{11} & p_{12} & \dots & p_{1n} \\ p_{21} & p_{22} & \dots & p_{2n} \\ & & \vdots & \\ p_{n1} & p_{n2} & \dots & p_{nn} \end{pmatrix}, \quad (9)$$

where $p_{ij} \geq 0$, $p_{ij} + p_{ji} = 1$, $p_{ii} = \frac{1}{2}$, $i, j = 1, 2, \dots, n$.

Summing all elements in each line of matrix P, we have $p_i = \sum_{j=1}^n p_{ij}$, $i = 1, 2, \dots, n$. Then, in accordance with the values of p_i ($i = 1, 2, \dots, n$), we rank the \hat{a}_i ($i = 1, 2, \dots, n$) in descending order.

Now, based on operational laws, we extend the WQM operator (3) to fuzzy environment:

Definition 3.3 Let $\hat{a}_j = [a_j^L, a_j^M, a_j^U]$ ($j = 1, 2, \dots, n$) be a collection of triangular fuzzy numbers, and let FWQM : $\Omega^n \rightarrow \Omega$, if

$$FWQM(\hat{a}_1, \hat{a}_2, \dots, \hat{a}_n) = \left(\sum_{j=1}^n w_j \hat{a}_j^2 \right)^{\frac{1}{2}}, \tag{10}$$

where $w = (w_1, w_2, \dots, w_n)^T$ be the weight vector of \hat{a}_j ($j = 1, 2, \dots, n$), with $w_j \geq 0$ and $\sum_{j=1}^n w_j = 1$, then FWQM is called a fuzzy weighted quadratic mean (FWQM) operator.

Especially, if $w_i = 1$, $w_j = 0$, $j \neq i$, then $FWQM(\hat{a}_1, \hat{a}_2, \dots, \hat{a}_n) = \hat{a}_i$; if $w = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})^T$, then the FWQM operator is reduced to the fuzzy quadratic mean (FQM) operator:

$$FQM(\hat{a}_1, \hat{a}_2, \dots, \hat{a}_n) = \left(\frac{\sum_{j=1}^n \hat{a}_j^2}{n} \right)^{\frac{1}{2}}. \tag{11}$$

By the operational laws and Eq. (10), we have

$$\begin{aligned} FWQM(\hat{a}_1, \hat{a}_2, \dots, \hat{a}_n) &= \left(\sum_{j=1}^n w_j \hat{a}_j^2 \right)^{\frac{1}{2}} = \left(\sum_{j=1}^n w_j [a_j^L, a_j^M, a_j^U]^2 \right)^{\frac{1}{2}} \\ &= \left[\left(\sum_{j=1}^n w_j (a_j^L)^2 \right)^{\frac{1}{2}}, \left(\sum_{j=1}^n w_j (a_j^M)^2 \right)^{\frac{1}{2}}, \left(\sum_{j=1}^n w_j (a_j^U)^2 \right)^{\frac{1}{2}} \right] \end{aligned} \tag{12}$$

and then by Eq. (12), we have

$$\begin{aligned} FQM(\hat{a}_1, \hat{a}_2, \dots, \hat{a}_n) &= \left[\left(\frac{\sum_{j=1}^n (a_j^L)^2}{n} \right)^{\frac{1}{2}}, \left(\frac{\sum_{j=1}^n (a_j^M)^2}{n} \right)^{\frac{1}{2}}, \left(\frac{\sum_{j=1}^n (a_j^U)^2}{n} \right)^{\frac{1}{2}} \right]. \end{aligned} \tag{13}$$

Especially, if the triangular fuzzy numbers $\hat{a}_j = [a_j^L, a_j^M, a_j^U]$ ($j = 1, 2, \dots, n$) are reduced to the interval numbers $\tilde{a}_j = [a_j^L, a_j^U]$ ($j = 1, 2, \dots, n$), then the

FWQM operator is reduced to the uncertain weighted quadratic mean(UWQM) operator:

$$\begin{aligned} \text{UWQM}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) &= \left(\sum_{j=1}^n w_j \tilde{a}_j^2 \right)^{\frac{1}{2}} \\ &= \left[\left(\sum_{j=1}^n w_j (a_j^L)^2 \right)^{\frac{1}{2}}, \left(\sum_{j=1}^n w_j (a_j^U)^2 \right)^{\frac{1}{2}} \right]. \end{aligned} \quad (14)$$

If $w = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})^T$, then the UWQM operator is reduced to the uncertain quadratic mean(UQM) operator:

$$\begin{aligned} \text{UQM}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) &= \left(\frac{\sum_{j=1}^n (\tilde{a}_j^2)}{n} \right)^{\frac{1}{2}} \\ &= \left[\left(\frac{\sum_{j=1}^n (a_j^L)^2}{n} \right)^{\frac{1}{2}}, \left(\frac{\sum_{j=1}^n (a_j^U)^2}{n} \right)^{\frac{1}{2}} \right]. \end{aligned} \quad (15)$$

If $a_j^L = a_j^U = a_j$, for all $j = 1, 2, \dots, n$, then Eqs. (14) and (15) are, respectively, reduced to the WQM operator (3) and the QM operator (4).

Example 3.4 Given a collection of triangular fuzzy numbers: $\hat{a}_1 = [2, 3, 4]$, $\hat{a}_2 = [1, 2, 4]$, $\hat{a}_3 = [2, 4, 6]$, $\hat{a}_4 = [1, 3, 5]$, let $w = (0.3, 0.1, 0.2, 0.4)^T$ be the weight vector of \hat{a}_i ($i = 1, 2, 3, 4$), then by Eq. (12), we have

$$\begin{aligned} \text{FWQM}(\hat{a}_1, \hat{a}_2, \hat{a}_3, \hat{a}_4) &= \left[\left(\sum_{j=1}^n w_j (a_j^L)^2 \right)^{\frac{1}{2}}, \left(\sum_{j=1}^n w_j (a_j^M)^2 \right)^{\frac{1}{2}}, \left(\sum_{j=1}^n w_j (a_j^U)^2 \right)^{\frac{1}{2}} \right] \\ &= [1.5811, 3.1464, 4.8580]. \end{aligned}$$

Based on the OWA and FQM operators and Definition 3.2, we define a fuzzy ordered weighted quadratic mean (FOWQM) operator as below:

Definition 3.5 Let $\hat{a}_j = [a_j^L, a_j^M, a_j^U]$ ($j = 1, 2, \dots, n$) be a collection of triangular fuzzy numbers. A FOWQM operator of dimension n is a mapping $\text{FOWQM} : \Omega^n \rightarrow \Omega$, that has an associated vector $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ such that $\omega_j \geq 0$ and $\sum_{j=1}^n \omega_j = 1$. Furthermore,

$$\begin{aligned} \text{FOWQM}(\hat{a}_1, \hat{a}_2, \dots, \hat{a}_n) &= \left(\sum_{j=1}^n \omega_j \hat{a}_{\sigma(j)}^2 \right)^{\frac{1}{2}} \\ &= \left[\left(\sum_{j=1}^n \omega_j (a_{\sigma(j)}^L)^2 \right)^{\frac{1}{2}}, \left(\sum_{j=1}^n \omega_j (a_{\sigma(j)}^M)^2 \right)^{\frac{1}{2}}, \left(\sum_{j=1}^n \omega_j (a_{\sigma(j)}^U)^2 \right)^{\frac{1}{2}} \right], \end{aligned} \quad (16)$$

where $a_{\sigma(j)} = [a_{\sigma(j)}^L, a_{\sigma(j)}^M, a_{\sigma(j)}^U]$ ($j = 1, 2, \dots, n$), and $(\sigma(1), \sigma(2), \dots, \sigma(n))$ is a permutation of $(1, 2, \dots, n)$ such that $\hat{a}_{\sigma(j-1)} \geq \hat{a}_{\sigma(j)}$ for all j .

However, if there is a tie between \hat{a}_i and \hat{a}_j by their average $(\hat{a}_i + \hat{a}_j)/2$ in process of aggregation. If k items are tied, then we replace these by k replicas of their average. The weighting vector $w = (w_1, w_2, \dots, w_n)^T$ can be determined by using some weight determining methods like the normal distribution based method, see Refs [11, 20] for more details.

Similarly to the OWA operator, the FOWQM operator has the following properties:

Theorem 3.6 Let $\hat{a}_j = [a_j^L, a_j^M, a_j^U]$ ($j = 1, 2, \dots, n$) be a collection of triangular fuzzy numbers, the following are valid:

(1) **Idempotency:** If all \hat{a}_j ($j = 1, 2, \dots, n$) are equal, i.e., $\hat{a}_j = \hat{a}$, for all i , then

$$\text{FOWQM}(\hat{a}_1, \hat{a}_2, \dots, \hat{a}_n) = \hat{a}.$$

(2) **Boundedness:** Let $\hat{a}^- = [\min_j(a_j^L), \min_j(a_j^M), \min_j(a_j^U)]$ and $\hat{a}^+ = [\max_j(a_j^L), \max_j(a_j^M), \max_j(a_j^U)]$, then

$$\hat{a}^- \leq \text{FOWQM}(\hat{a}_1, \hat{a}_2, \dots, \hat{a}_n) \leq \hat{a}^+.$$

(3) **Monotonicity:** Let $\hat{a}_j^* = [a_j^{L*}, a_j^{M*}, a_j^{U*}]$ ($j = 1, 2, \dots, n$) be a collection of triangular fuzzy numbers, then if $a_j^L \leq a_j^{L*}$, $a_j^M \leq a_j^{M*}$ and $a_j^U \leq a_j^{U*}$ for all j , then

$$\text{FOWQM}(\hat{a}_1, \hat{a}_2, \dots, \hat{a}_n) \leq \text{FOWQM}(\hat{a}_1^*, \hat{a}_2^*, \dots, \hat{a}_n^*).$$

(4) **Commutativity:** Let $\hat{a}'_j = [a_j^{L'}, a_j^{M'}, a_j^{U'}]$ ($j = 1, 2, \dots, n$) be a collection of triangular fuzzy numbers, then

$$\text{FOWQM}(\hat{a}_1, \hat{a}_2, \dots, \hat{a}_n) = \text{FOWQM}(\hat{a}'_1, \hat{a}'_2, \dots, \hat{a}'_n),$$

where $(\hat{a}'_1, \hat{a}'_2, \dots, \hat{a}'_n)$ is any permutation of $(\hat{a}_1, \hat{a}_2, \dots, \hat{a}_n)$.

Especially, if $w = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})^T$, then the FOWQM operator is reduced to the FQM operator; if the triangular fuzzy numbers $\hat{a}_j = [a_j^L, a_j^M, a_j^U]$ ($j = 1, 2, \dots, n$) are reduced to the interval numbers $\tilde{a}_j = [a_j^L, a_j^U]$ ($j = 1, 2, \dots, n$), then the FOWQM operator is reduced to the uncertain ordered weighted quadratic mean (UOWQM) operator:

$$\begin{aligned} \text{UOWQM}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) &= \left(\sum_{j=1}^n \omega_j \tilde{a}_{\sigma(j)}^2 \right)^{\frac{1}{2}} \\ &= \left[\left(\sum_{j=1}^n \omega_j (a_{\sigma(j)}^L)^2 \right)^{\frac{1}{2}}, \left(\sum_{j=1}^n \omega_j (a_{\sigma(j)}^U)^2 \right)^{\frac{1}{2}} \right], \end{aligned} \tag{17}$$

where $\tilde{a}_\sigma(j) = [a_{\sigma(j)}^L, a_{\sigma(j)}^U]$, $(\sigma(1), \sigma(2), \dots, \sigma(n))$ is a permutation of $(1, 2, \dots, n)$ such that $\tilde{a}_{\sigma(j-1)} \geq \tilde{a}_{\sigma(j)}$ for all j . If there is a tie between \tilde{a}_i and \tilde{a}_j , then we replace each of \tilde{a}_i and \tilde{a}_j by their average $(\tilde{a}_i + \tilde{a}_j)/2$ in process of aggregation. If k items are tied, then we replace these by k replicas of their average.

If $a_i^L = a_i^U = a_i$, for all $i = 1, 2, \dots, n$, then the UOWQM operator is reduced to the ordered weighted quadratic mean (OWQM) operator:

$$\text{OWQM}(a_1, a_2, \dots, a_n) = \left(\sum_{j=1}^n \omega_j b_j^2 \right)^{\frac{1}{2}}, \tag{18}$$

where b_j is the j th largest of $a_j (j = 1, 2, \dots, n)$. The OWQM operator (18) has some special cases:

(1) If $\omega = (1, 0, \dots, 0)^T$, then

$$\text{OWQM}(a_1, a_2, \dots, a_n) = \max\{a_i\} = b_1. \tag{19}$$

(2) If $\omega = (0, 0, \dots, 1)^T$, then

$$\text{OWQM}(a_1, a_2, \dots, a_n) = \min\{a_i\} = b_n. \tag{20}$$

(3) If $\omega_j = 1, w_i = 0, i \neq j$, then

$$b_n \leq \text{OWQM}(a_1, a_2, \dots, a_n) = b_j \leq b_1. \tag{21}$$

(4) If $\omega = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})^T$, then

$$\begin{aligned} \text{OWQM}(a_1, a_2, \dots, a_n) &= \left(\frac{\sum_{j=1}^n b_j^2}{n} \right)^{\frac{1}{2}} = \left(\frac{\sum_{j=1}^n a_j^2}{n} \right)^{\frac{1}{2}} \\ &= \text{QM}(a_1, a_2, \dots, a_n). \end{aligned} \tag{22}$$

Clearly, the fundamental characteristic of the FWQM operator is that it considers the importance of each given triangular fuzzy number, whereas the fundamental characteristic of the FOWQM operator is the reordering step, and it weights all the ordered positions of the triangular fuzzy numbers instead of weighing the given triangular fuzzy numbers themselves. By combining the advantages of the FWQM and FOWQM operators, in the following, we develop a fuzzy hybrid quadratic mean (FHQM) operator that weights both the given triangular fuzzy numbers and their ordered positions.

Definition 3.7 Let $\hat{a}_j = [a_j^L, a_j^M, a_j^U]$ ($j = 1, 2, \dots, n$) be a collection of triangular fuzzy numbers. A FHQM operator of dimension n is a mapping $\text{FHQM} : \Omega^n \rightarrow \Omega$, which has an associated vector $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ with $\omega_j \geq 0$ and $\sum_{j=1}^n \omega_j = 1$, such that

$$\text{FHQM}(\hat{a}_1, \hat{a}_2, \dots, \hat{a}_n) = \left(\sum_{j=1}^n \omega_j \hat{a}_{\sigma(j)}^2 \right)^{\frac{1}{2}}$$

$$= \left[\left(\sum_{j=1}^n \omega_j (\dot{a}_{\sigma(j)}^L)^2 \right)^{\frac{1}{2}}, \left(\sum_{j=1}^n \omega_j (\dot{a}_{\sigma(j)}^M)^2 \right)^{\frac{1}{2}}, \left(\sum_{j=1}^n \omega_j (\dot{a}_{\sigma(j)}^U)^2 \right)^{\frac{1}{2}} \right], \quad (23)$$

where $\dot{a}_{\sigma(j)} = [\dot{a}_{\sigma(j)}^L, \dot{a}_{\sigma(j)}^M, \dot{a}_{\sigma(j)}^U]$ is the j th largest of the weighted triangular fuzzy numbers \dot{a}_j ($\dot{a}_j = n w_j \hat{a}_j$, $j = 1, 2, \dots, n$), $w = (w_1, w_2, \dots, w_n)^T$ is the weight vector of \hat{a}_j ($j = 1, 2, \dots, n$) with $w_j \geq 0$ and $\sum_{j=1}^n w_j = 1$, and n is the balancing coefficient.

Especially, if $w = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})^T$, then $\dot{a}_j = \hat{a}_j$, $j = 1, 2, \dots, n$, in this case, the FHQM operator is reduced to the FOWQM operator; if $w = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})^T$, then

$$\begin{aligned} \text{FHQM}(\hat{a}_1, \hat{a}_2, \dots, \hat{a}_n) &= \left(\sum_{j=1}^n w_j \hat{a}_{\sigma(j)}^2 \right)^{\frac{1}{2}} \\ &= \left[\left(\sum_{j=1}^n n w_j^2 (a_{\sigma(j)}^L)^2 \right)^{\frac{1}{2}}, \left(\sum_{j=1}^n n w_j^2 (a_{\sigma(j)}^M)^2 \right)^{\frac{1}{2}}, \left(\sum_{j=1}^n n w_j^2 (a_{\sigma(j)}^U)^2 \right)^{\frac{1}{2}} \right] \end{aligned} \quad (24)$$

which we call the generalized fuzzy weighted quadratic mean (GFWQM) operator.

Moreover, if the triangular fuzzy numbers $\hat{a}_j = [a_j^L, a_j^M, a_j^U]$ ($j = 1, 2, \dots, n$) are reduced to the interval numbers $\tilde{a}_j = [a_j^L, a_j^U]$ ($j = 1, 2, \dots, n$), then the FHQM operator is reduced to the uncertain hybrid quadratic mean (UHQM) operator:

$$\begin{aligned} \text{UHQM}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) &= \left(\sum_{j=1}^n \omega_j \dot{a}_{\sigma(j)}^2 \right)^{\frac{1}{2}} \\ &= \left[\left(\sum_{j=1}^n n w_j^2 (a_{\sigma(j)}^L)^2 \right)^{\frac{1}{2}}, \left(\sum_{j=1}^n n w_j^2 (a_{\sigma(j)}^U)^2 \right)^{\frac{1}{2}} \right], \end{aligned} \quad (25)$$

where $\dot{a}_{\sigma(j)}$ is the j th largest of the weighted interval numbers \dot{a}_j ($\dot{a}_j = n w_j \tilde{a}_j$, $j = 1, 2, \dots, n$), $w = (w_1, w_2, \dots, w_n)^T$ is the weight vector of \tilde{a}_j ($j = 1, 2, \dots, n$) with $w_j \geq 0$ and $\sum_{j=1}^n w_j = 1$, and n is the balancing coefficient. Especially, if $w = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})^T$, then $\dot{a}_j = \tilde{a}_j$, $j = 1, 2, \dots, n$, in this case, the UHQM operator is reduced to the UOWQM operator.

If $a_i^L = a_i^U = a_i$, for all $i = 1, 2, \dots, n$, then the UHQM operator is reduced to the hybrid quadratic mean (HQM) operator:

$$\text{HQM}(a_1, a_2, \dots, a_n) = \left(\sum_{j=1}^n \omega_j a_{\sigma(j)}^2 \right)^{\frac{1}{2}}, \quad (26)$$

where $\hat{a}_{\sigma(j)}$ is the j th largest of the weighted interval numbers \hat{a}_j ($\hat{a}_j = nw_j \tilde{a}_j, j = 1, 2, \dots, n$), $w = (w_1, w_2, \dots, w_n)^T$ is the weight vector of a_j ($j = 1, 2, \dots, n$) with $w_j \geq 0$ and $\sum_{j=1}^n w_j = 1$, and n is the balancing coefficient. Especially, if $w = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})^T$, then $\hat{a}_j = a_j, j = 1, 2, \dots, n$, in this case, the HQM operator is reduced to the OWQM operator.

Example 3.8 Given a collection of triangular fuzzy numbers: $\hat{a}_1 = [2, 4, 5]$, $\hat{a}_2 = [1, 3, 4]$, $\hat{a}_3 = [2, 3, 5]$, $\hat{a}_4 = [3, 4, 5]$, and $\hat{a}_5 = [2, 5, 8]$, and $w = (0.20, 0.25, 0.15, 0.25, 0.15)^T$ be the weight vector of \hat{a}_j ($j = 1, 2, 3, 4, 5$). Then we get the weighted triangular fuzzy numbers:

$$\begin{aligned} \hat{a}_1 &= [2, 4, 5], \hat{a}_2 = [1.25, 3.75, 5], \hat{a}_3 = [1.5, 2.25, 3.75], \\ \hat{a}_4 &= [3.75, 5, 6.25], \hat{a}_5 = [1.5, 3.75, 6]. \end{aligned}$$

By using Eq. (9) (without loss of generality, set $\delta = 0.5$), we construct the following matrix:

$$P = \begin{pmatrix} 0.5000 & 0.5833 & 0.9545 & 0.0385 & 0.4864 \\ 0.4167 & 0.5000 & 0.8462 & 0 & 0.4154 \\ 0.0455 & 0.1538 & 0.5000 & 0 & 0.1250 \\ 0.9615 & 1 & 1 & 0.5000 & 0.8571 \\ 0.5136 & 0.5846 & 0.8750 & 0.1429 & 0.5000 \end{pmatrix}.$$

Summing all elements in each line of matrix P , we have

$$p_1 = 2.5628, \quad p_2 = 2.1782, \quad p_3 = 0.8243, \quad p_4 = 4.3187, \quad p_5 = 2.6160$$

and then we rank the triangular fuzzy number \hat{a}_i ($i = 1, 2, 3, 4, 5$) in descending order in accordance with the values of p_i ($i = 1, 2, 3, 4, 5$):

$$\hat{a}_{\sigma(1)} = \hat{a}_4, \quad \hat{a}_{\sigma(2)} = \hat{a}_5, \quad \hat{a}_{\sigma(3)} = \hat{a}_1, \quad \hat{a}_{\sigma(4)} = \hat{a}_2, \quad \hat{a}_{\sigma(5)} = \hat{a}_3.$$

Suppose that $\omega = (0.1117, 0.2365, 0.3036, 0.3265, 0.1117)^T$ is the weighting vector of the FHQM operator (derived by the normal distribution based method [11]), then by Eq. (23), we get

$$\begin{aligned} \text{FHQM}(\hat{a}_1, \hat{a}_2, \hat{a}_3, \hat{a}_4, \hat{a}_5) &= \left(\sum_{j=1}^n \omega_j \hat{a}_{\sigma(j)}^2 \right)^{\frac{1}{2}} \\ &= \left[\left(\sum_{j=1}^n \omega_j (\hat{a}_{\sigma(j)}^L)^2 \right)^{\frac{1}{2}}, \left(\sum_{j=1}^n \omega_j (\hat{a}_{\sigma(j)}^M)^2 \right)^{\frac{1}{2}}, \left(\sum_{j=1}^n \omega_j (\hat{a}_{\sigma(j)}^U)^2 \right)^{\frac{1}{2}} \right] \\ &= [2.0196, 4.0166, 5.4955]. \end{aligned}$$

4 Approaches to multiple attribute group decision making with triangular fuzzy information

For a group decision making with triangular fuzzy information, let $X = \{x_1, x_2, \dots, x_n\}$ be a discrete set of n alternatives, and $G = \{G_1, G_2, \dots, G_m\}$ be the set of

m attributes, whose weight vector is $w = (w_1, w_2, \dots, w_m)^T$ with $w_i \geq 0$ and $\sum_{i=1}^m w_i = 1$, and let $D = \{d_1, d_2, \dots, d_s\}$ be the set of decision makers, whose weight vector is $v = (v_1, v_2, \dots, v_s)^T$, where $v_k \geq 0$ and $\sum_{k=1}^s v_k = 1$. Suppose that $A^{(k)} = (\hat{a}_{ij}^{(k)})_{m \times n}$ is the decision matrix, where $\hat{a}_{ij}^{(k)} = [a_{ij}^{L(k)}, a_{ij}^{M(k)}, a_{ij}^{U(k)}]$ is an attribute value, which takes the form of triangular fuzzy number, of the alternative $x_j \in X$ with respect to the attribute $G_i \in G$.

Then, we utilize the FWQM and FHQM operators to propose an approach to multiple attribute group decision making with triangular fuzzy information, which involves the following steps:

Step 1. Normalize each attribute value $\hat{a}_{ij}^{(k)}$ in the matrix $A^{(k)}$ into a corresponding element in the matrix $R^{(k)} = (\hat{r}_{ij}^{(k)})_{m \times n}$ ($\hat{r}_{ij}^{(k)} = [r_{ij}^{L(k)}, r_{ij}^{M(k)}, r_{ij}^{U(k)}]$) using the following formulas:

$$\hat{r}_{ij}^{(k)} = \frac{\hat{a}_{ij}^{(k)}}{\sum_{j=1}^n \hat{a}_{ij}^{(k)}} = \left[\frac{a_{ij}^{L(k)}}{\sum_{j=1}^n a_{ij}^{U(k)}}, \frac{a_{ij}^{M(k)}}{\sum_{j=1}^n a_{ij}^{M(k)}}, \frac{a_{ij}^{U(k)}}{\sum_{j=1}^n a_{ij}^{L(k)}} \right],$$

for benefit attribute G_i ,

(27)

$$\hat{r}_{ij}^{(k)} = \frac{1/\hat{a}_{ij}^{(k)}}{\sum_{j=1}^n (1/\hat{a}_{ij}^{(k)})}$$

$$= \left[\frac{1/a_{ij}^{U(k)}}{\sum_{j=1}^n (1/a_{ij}^{L(k)})}, \frac{1/a_{ij}^{M(k)}}{\sum_{j=1}^n (1/a_{ij}^{M(k)})}, \frac{1/a_{ij}^{L(k)}}{\sum_{j=1}^n (1/a_{ij}^{U(k)})} \right],$$

for cost attribute G_i ,

(28)

where $i = 1, 2, \dots, m, j = 1, 2, \dots, n, k = 1, 2, \dots, s$.

Step 2. Utilize the FWQM operator:

$$\hat{r}_j^{(k)} = \text{FWQM}(\hat{r}_{1j}^{(k)}, \hat{r}_{2j}^{(k)}, \dots, \hat{r}_{mj}^{(k)}) = \left(\sum_{i=1}^m w_i (\hat{r}_{ij}^{(k)})^2 \right)^{\frac{1}{2}}$$

$$= \left[\left(\sum_{i=1}^m w_i (\hat{r}_{ij}^{L(k)})^2 \right)^{\frac{1}{2}}, \left(\sum_{i=1}^m w_i (\hat{r}_{ij}^{M(k)})^2 \right)^{\frac{1}{2}}, \left(\sum_{i=1}^m w_i (\hat{r}_{ij}^{U(k)})^2 \right)^{\frac{1}{2}} \right]$$
(29)

to aggregate all the elements in the j th column of $R^{(k)}$ and get the overall attribute value $\hat{r}_j^{(k)}$ of the alternative x_j corresponding to the decision maker d_k .

Step 3. Utilize the FHQM operator:

$$\hat{r}_j = \text{FHQM}(\hat{r}_j^{(1)}, \hat{r}_j^{(2)}, \dots, \hat{r}_j^{(s)}) = \left(\sum_{k=1}^s \omega_k (\hat{r}_j^{(\sigma(k))})^2 \right)^{\frac{1}{2}}$$

$$= \left[\left(\sum_{k=1}^s \omega_k (\hat{r}_j^{L(\sigma(k))})^2 \right)^{\frac{1}{2}}, \left(\sum_{k=1}^s \omega_k (\hat{r}_j^{M(\sigma(k))})^2 \right)^{\frac{1}{2}}, \left(\sum_{k=1}^s \omega_k (\hat{r}_j^{U(\sigma(k))})^2 \right)^{\frac{1}{2}} \right]$$
(30)

to aggregate the overall attribute values $\hat{r}_j^{(k)}$ ($k = 1, 2, \dots, s$) corresponding to the decision maker d_k ($k = 1, 2, \dots, s$) and get the collective overall attribute value \hat{r}_j , where $\hat{r}_j^{(\sigma(k))} = [\hat{r}_j^{L(\sigma(k))}, \hat{r}_j^{M(\sigma(k))}, \hat{r}_j^{U(\sigma(k))}]$ is the k th largest of the weighted data $\hat{r}_j^{(k)}$ ($\hat{r}_j^{(k)} = sv_k \hat{r}_j^{(k)}$, $k = 1, 2, \dots, s$), $\omega = (\omega_1, \omega_2, \dots, \omega_s)^T$ is the weighting vector of the FHQM operator, with $\omega_k \geq 0$ and $\sum_{k=1}^s \omega_k = 1$.

Step 4. Compare each \hat{r}_j with all \hat{r}_i ($i = 1, 2, \dots, n$) by using Eq. (9), and let $p_{ij} = p(\hat{r}_i \geq \hat{r}_j)$, and then construct a possibility matrix $P = (p_{ij})_{n \times n}$, where $p_{ij} \geq 0$, $p_{ij} + p_{ji} = 1$, $p_{ii} = 0.5$, $i, j = 1, 2, \dots, n$. Summing all elements in each line of matrix P , we have $p_i = \sum_{j=1}^n p_{ij}$, $i = 1, 2, \dots, n$, and then reorder \hat{r}_j ($j = 1, 2, \dots, n$) in descending order in accordance with the values of p_j ($j = 1, 2, \dots, n$).

Step 5. Rank all the alternatives x_j ($j = 1, 2, \dots, n$) by the ranking of \hat{r}_j ($j = 1, 2, \dots, n$), and then select the most desirable one.

Step 6. End.

5 Illustrative example

In this section, we use a multiple attribute group decision making problem of determining what kind of air-conditioning systems should be installed in a library(adopted from [6, 7, 12, 22]) to illustrate the proposed approach.

A city is planning to build a municipal library. One of the problems facing the city development commissioner is to determine what kind of air-conditioning systems should be installed in the library. The contractor offers five feasible alternatives, which might be adapted to the physical structure of the library. The alternatives x_j ($j = 1, 2, 3, 4, 5$) are to be evaluated using triangular fuzzy numbers by the three decision makers d_k ($k = 1, 2, 3$) (whose weight vector is $v = (0.4, 0.3, 0.3)^T$) under three major impacts: economic, functional, and operational. Two monetary attributes and six nonmonetary attributes (that is, G_1 : owning cost ($\$/ft^2$), G_2 : operating cost ($\$/ft^2$), G_3 : performance (*), G_4 : noise level (Db), G_5 : maintainability (*), G_6 : reliability (%), G_7 : flex-

Table 1: Triangular fuzzy number decision matrix $A^{(1)}$

	x_1	x_2	x_3	x_4	x_5
G_1	[3.5, 4.0, 4.7]	[1.7, 2.0, 2.3]	[3.5, 3.8, 4.2]	[3.5, 3.8, 4.5]	[3.3, 3.8, 4.0]
G_2	[5.5, 6.0, 6.5]	[4.8, 5.1, 5.5]	[4.5, 5.2, 5.5]	[4.5, 4.7, 5.0]	[5.5, 5.7, 6.0]
G_3	[0.7, 0.8, 0.9]	[0.5, 0.56, 0.6]	[0.5, 0.6, 0.7]	[0.7, 0.85, 0.9]	[0.6, 0.7, 0.8]
G_4	[35, 40, 45]	[70, 73, 75]	[65, 68, 70]	[40, 42, 45]	[50, 55, 60]
G_5	[0.4, 0.45, 0.5]	[0.4, 0.44, 0.6]	[0.7, 0.76, 0.8]	[0.9, 0.97, 1.0]	[0.5, 0.54, 0.6]
G_6	[95, 98, 100]	[70, 73, 75]	[80, 83, 90]	[90, 93, 95]	[85, 90, 95]
G_7	[0.3, 0.35, 0.5]	[0.7, 0.75, 0.8]	[0.8, 0.9, 1.0]	[0.6, 0.75, 0.8]	[0.4, 0.5, 0.6]
G_8	[0.7, 0.74, 0.8]	[0.5, 0.53, 0.6]	[0.6, .68, 0.7]	[0.7, 0.8, 0.9]	[0.8, .85, 0.9]

ibility (*), G_8 : safety (*), where * unit is from 0 – 1 scale, three attributes G_1 , G_2 , and G_4 are cost attributes, and the other five attributes are benefit attributes, suppose that the weight vector of the attributes G_i ($i = 1, 2, \dots, 8$) is $w = (0.05, 0.08, 0.14, 0.12, 0.18, 0.21, 0.05, 0.17)^T$ emerged from three impacts is Tables 1-3.

Table 2: Triangular fuzzy number decision matrix $A^{(2)}$

	x_1	x_2	x_3	x_4	x_5
G_1	[4.0, 4.3, 4.5]	[2.1, 2.2, 2.4]	[5.0, 5.1, 5.2]	[4.3, 4.4, 4.5]	[3.0, 3.3, 3.5]
G_2	[6.0, 6.3, 6.5]	[5.0, 5.1, 5.2]	[4.5, 4.7, 5.0]	[5.0, 5.1, 5.3]	[7.0, 7.5, 8.0]
G_3	[0.7, 0.8, 0.9]	[0.4, 0.5, 0.6]	[0.5, .55, 0.6]	[0.7, 0.75, 0.8]	[0.7, 0.8, 0.9]
G_4	[37, 38, 39]	[70, 73, 75]	[65, 66, 67]	[40, 42, 45]	[50, 52, 55]
G_5	[0.4, 0.5, 0.6]	[0.5, 0.55, 0.6]	[0.8, 0.85, 0.9]	[0.8, 0.95, 1.0]	[0.4, 0.44, 0.5]
G_6	[92, 93, 95]	[70, 75, 80]	[83, 84, 85]	[90, 91, 92]	[90, 93, 95]
G_7	[0.4, 0.45, 0.5]	[0.8, 0.85, 0.9]	[0.7, 0.73, 0.8]	[0.7, 0.85, 0.9]	[0.4, 0.45, 0.5]
G_8	[0.6, 0.7, 0.8]	[0.6, 0.65, 0.7]	[0.5, 0.6, 0.7]	[0.7, 0.76, 0.8]	[0.7, 0.8, 0.9]

Table 3: Triangular fuzzy number decision matrix $A^{(3)}$

	x_1	x_2	x_3	x_4	x_5
G_1	[4.3, 4.4, 4.6]	[2.2, 2.4, 2.5]	[4.5, 4.8, 5.0]	[4.7, 4.9, 5.0]	[3.1, 3.2, 3.4]
G_2	[6.4, 6.7, 7.0]	[5.0, 5.2, 5.5]	[4.7, 4.8, 4.9]	[5.5, 5.7, 6.0]	[6.0, 6.5, 7.0]
G_3	[0.8, 0.85, 0.9]	[0.5, 0.6, 0.7]	[0.6, 0.7, 0.8]	[0.7, 0.8, 0.9]	[0.7, 0.75, 0.8]
G_4	[36, 38, 40]	[72, 73, 75]	[67, 68, 70]	[45, 48, 50]	[55, 57, 60]
G_5	[0.4, 0.46, 0.5]	[0.4, 0.45, 0.6]	[0.8, 0.95, 1.0]	[0.8, 0.85, 0.9]	[0.5, 0.55, 0.6]
G_6	[93, 94, 95]	[77, 78, 80]	[85, 87, 90]	[90, 94, 95]	[90, 96, 100]
G_7	[0.4, 0.5, 0.6]	[0.8, 0.9, 1.0]	[0.8, 0.86, 0.9]	[0.6, 0.7, 0.8]	[0.5, 0.57, 0.6]
G_8	[0.7, 0.78, 0.8]	[0.5, 0.55, 0.6]	[0.6, 0.68, 0.7]	[0.8, 0.85, 0.9]	[0.8, 0.85, 0.9]

To select the best air-conditioning system, we first utilize the approach based on the FWQM and FHQM operators, the main steps are as follows:

Step 1. By using Eqs. (27) and (28), we normalize each attribute value $\hat{a}_{ij}^{(k)}$ in the matrices $A^{(k)}$ ($k = 1, 2, 3$) into the corresponding element in the matrices $R^{(k)} = (\hat{r}_{ij})_{8 \times 5}$ ($k = 1, 2, 3$) (Tables 4-6):

Step 2. Utilize the FWQM operator (29) to aggregate all elements in the j th column $R^{(K)}$ and get the overall attribute value $\hat{r}_j^{(k)}$:

$$\begin{aligned} \hat{r}_1^{(1)} &= [0.1736, 0.2029, 0.2436], \hat{r}_2^{(1)} = [0.1473, 0.1751, 0.2167], \\ \hat{r}_3^{(1)} &= [0.1689, 0.1985, 0.2354], \hat{r}_4^{(1)} = [0.2043, 0.2422, 0.2759], \\ \hat{r}_5^{(1)} &= [0.1687, 0.1991, 0.2370], \\ \hat{r}_1^{(2)} &= [0.1770, 0.2044, 0.2417], \hat{r}_2^{(2)} = [0.1622, 0.1878, 0.2191], \end{aligned}$$

Table 4: Normalized triangular fuzzy number decision matrix $R^{(1)}$

	x_1	x_2	x_3	x_4	x_5
G_1	[0.12, 0.16, 0.21]	[0.25, 0.32, 0.43]	[0.14, 0.17, 0.21]	[0.13, 0.17, 0.21]	[0.14, 0.17, 0.22]
G_2	[0.15, 0.18, 0.21]	[0.18, 0.21, 0.24]	[0.18, 0.20, 0.25]	[0.20, 0.23, 0.25]	[0.16, 0.19, 0.21]
G_3	[0.18, 0.23, 0.30]	[0.13, 0.16, 0.20]	[0.13, 0.17, 0.23]	[0.18, 0.24, 0.30]	[0.15, 0.20, 0.27]
G_4	[0.22, 0.26, 0.32]	[0.13, 0.14, 0.16]	[0.14, 0.15, 0.17]	[0.22, 0.25, 0.28]	[0.16, 0.19, 0.23]
G_5	[0.11, 0.14, 0.17]	[0.11, 0.14, 0.21]	[0.20, 0.24, 0.28]	[0.26, 0.31, 0.34]	[0.14, 0.17, 0.21]
G_6	[0.21, 0.22, 0.24]	[0.15, 0.17, 0.18]	[0.18, 0.19, 0.21]	[0.20, 0.21, 0.23]	[0.19, 0.21, 0.23]
G_7	[0.08, 0.11, 0.18]	[0.19, 0.23, 0.29]	[0.22, 0.28, 0.36]	[0.16, 0.23, 0.29]	[0.11, 0.15, 0.21]
G_8	[0.18, 0.21, 0.24]	[0.13, 0.15, 0.18]	[0.15, 0.19, 0.21]	[0.18, 0.22, 0.27]	[0.21, 0.24, 0.27]

Table 5: Normalized triangular fuzzy number decision matrix $R^{(2)}$

	x_1	x_2	x_3	x_4	x_5
G_1	[0.15, 0.16, 0.19]	[0.28, 0.32, 0.36]	[0.13, 0.14, 0.15]	[0.15, 0.16, 0.17]	[0.19, 0.21, 0.25]
G_2	[0.17, 0.18, 0.19]	[0.21, 0.22, 0.23]	[0.21, 0.24, 0.26]	[0.20, 0.22, 0.23]	[0.13, 0.15, 0.17]
G_3	[0.18, 0.24, 0.30]	[0.11, 0.15, 0.20]	[0.13, 0.16, 0.20]	[0.18, 0.22, 0.27]	[0.18, 0.24, 0.30]
G_4	[0.25, 0.27, 0.29]	[0.13, 0.14, 0.15]	[0.15, 0.15, 0.16]	[0.22, 0.24, 0.27]	[0.18, 0.20, 0.21]
G_5	[0.11, 0.15, 0.21]	[0.14, 0.17, 0.21]	[0.22, 0.26, 0.31]	[0.22, 0.29, 0.34]	[0.11, 0.13, 0.17]
G_6	[0.21, 0.21, 0.22]	[0.16, 0.17, 0.19]	[0.19, 0.19, 0.20]	[0.20, 0.21, 0.22]	[0.20, 0.21, 0.22]
G_7	[0.11, 0.14, 0.17]	[0.22, 0.26, 0.30]	[0.19, 0.22, 0.27]	[0.19, 0.26, 0.30]	[0.19, 0.14, 0.17]
G_8	[0.15, 0.20, 0.26]	[0.15, 0.19, 0.23]	[0.13, 0.17, 0.23]	[0.18, 0.22, 0.26]	[0.18, 0.23, 0.29]

Table 6: Normalized triangular fuzzy number decision matrix $R^{(3)}$

	x_1	x_2	x_3	x_4	x_5
G_1	[0.15, 0.17, 0.18]	[0.28, 0.30, 0.35]	[0.14, 0.15, 0.17]	[0.14, 0.15, 0.16]	[0.20, 0.23, 0.25]
G_2	[0.16, 0.17, 0.19]	[0.20, 0.22, 0.24]	[0.22, 0.24, 0.25]	[0.18, 0.20, 0.22]	[0.16, 0.17, 0.20]
G_3	[0.20, 0.23, 0.27]	[0.12, 0.16, 0.21]	[0.15, 0.19, 0.24]	[0.17, 0.22, 0.27]	[0.17, 0.20, 0.24]
G_4	[0.26, 0.28, 0.31]	[0.14, 0.15, 0.16]	[0.15, 0.16, 0.17]	[0.21, 0.22, 0.25]	[0.17, 0.19, 0.20]
G_5	[0.11, 0.14, 0.17]	[0.11, 0.14, 0.21]	[0.20, 0.24, 0.28]	[0.26, 0.31, 0.34]	[0.14, 0.17, 0.21]
G_6	[0.21, 0.22, 0.24]	[0.15, 0.17, 0.18]	[0.18, 0.19, 0.21]	[0.20, 0.21, 0.23]	[0.19, 0.21, 0.23]
G_7	[0.08, 0.11, 0.18]	[0.19, 0.23, 0.29]	[0.22, 0.28, 0.36]	[0.16, 0.23, 0.29]	[0.11, 0.15, 0.21]
G_8	[0.18, 0.21, 0.24]	[0.13, 0.15, 0.18]	[0.15, 0.19, 0.21]	[0.18, 0.22, 0.27]	[0.21, 0.24, 0.27]

$$\begin{aligned} \hat{r}_3^{(2)} &= [0.1744, 0.1974, 0.2314], \hat{r}_4^{(2)} = [0.1977, 0.2342, 0.2676], \\ \hat{r}_5^{(2)} &= [0.1717, 0.1979, 0.2333], \\ \hat{r}_1^{(3)} &= [0.0714, 0.0795, 0.0892], \hat{r}_2^{(3)} = [0.0573, 0.0638, 0.0772], \\ \hat{r}_3^{(3)} &= [0.0699, 0.0831, 0.0959], \hat{r}_4^{(3)} = [0.0782, 0.0879, 0.1004], \\ \hat{r}_5^{(3)} &= [0.0704, 0.0781, 0.0890]. \end{aligned}$$

Step 3. Utilize the FHQM operator (30) (suppose that its weight vector is $w = (0.243, 0.514, 0.243)^T$ determined by using the normal distribution based method [11], let $\sigma = 0.5$) to aggregate the overall attribute value $\hat{r}_j^{(k)}$ ($k = 1, 2, 3$) corresponding to the decision maker d_k ($k = 1, 2, 3$), and get the collective overall

attribute value \hat{r}_j :

$$\begin{aligned} \hat{r}_1 &= [0.1568, 0.1818, 0.2160], \hat{r}_2 = [0.1385, 0.1619, 0.1939], \\ \hat{r}_3 &= [0.1536, 0.1771, 0.2086], \hat{r}_4 = [0.1791, 0.2119, 0.2417], \\ \hat{r}_5 &= [0.1523, 0.1771, 0.2095]. \end{aligned}$$

Step 4. Compare each \hat{r}_j with all \hat{r}_i ($i = 1, 2, 3, 4, 5$) by using Eq. (9) (without loss of generality, set $\delta = 0.5$), and let $p_{ij} = p(\hat{r}_i \geq \hat{r}_j)$, and then construct a possibility matrix:

$$P = \begin{pmatrix} 0.5 & 0.8558 & 0.5869 & 0.0553 & 0.5882 \\ 0.1442 & 0.5 & 0.2209 & 0 & 0.2301 \\ 0.4131 & 0.7791 & 0.5 & 0 & 0.5031 \\ 0.9447 & 1 & 1 & 0.5 & 1 \\ 0.4118 & 0.7699 & 0.4969 & 0 & 0.5 \end{pmatrix}.$$

Summing all elements in each line of matrix P , we have

$$p_1 = 2.5861, p_2 = 1.0952, p_3 = 2.1953, p_4 = 4.4447, p_5 = 2.1786$$

and then we reorder \hat{r}_j ($j = 1, 2, 3, 4, 5$) in descending order in accordance with the values of p_j ($j = 1, 2, 3, 4, 5$):

$$\hat{r}_4 > \hat{r}_1 > \hat{r}_3 > \hat{r}_5 > \hat{r}_2.$$

Step 5. Rank all the alternatives x_j ($j = 1, 2, 3, 4, 5$) by the ranking of \hat{r}_j ($j = 1, 2, 3, 4, 5$):

$$x_4 \succ x_1 \succ x_3 \succ x_5 \succ x_2$$

and thus the most desirable alternative is x_4 .

Table 7: Comparison of the proposed approach with other approaches

	Xu's approach [12]	Park et al.'s approach [7]	Proposed approach
Solution method			
Aggregation stage	FWHM operator	FWCHM operator	FWQM operator
Exploitation stage	FHHM operator	FHCHM operator	FHQM operator
Ranking of alternatives	$x_4 \succ x_5 \succ x_3 \succ x_1 \succ x_2$	$x_4 \succ x_1 \succ x_3 \succ x_5 \succ x_2$	$x_4 \succ x_1 \succ x_3 \succ x_5 \succ x_2$

From the above analysis, the results obtained by the proposed approach are slightly different to the ones obtained Xu's [12] approach but the same with Park et al. [7] approach (see Table 7). It perfectly depends on how we look at things, and not on how they are themselves. Therefore, depending on aggregation operators used, the results may lead to different decisions. However, the best alternative is x_4 .

6 Conclusions

In this paper, we have extended the traditional quadratic mean to fuzzy environments and introduced the FWQM operator. Based on the FWQM operator and Yager's OWA operator [17], we have developed the FOWQM operator and the FHQM operator. It has been shown that both the FOWQM and FWQM operators are the special cases of the FHQM operator. It has also been pointed out that if all the input fuzzy data are reduced to the interval or numerical data, then the FHQM operator is reduced to the UHQM operator and the HQM operator, respectively. In these situations, the WQM operator and the OWQM operator are the two special cases of the HQM operator; the UWQM operator and the UOWQM operator are the two special cases of the UHQM operator. Then, based on the FWQM and FHQM operators, we present an approach to multiple attribute group decision making with triangular fuzzy information and illustrate it with a practical example.

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Sensitivity Analysis for General Nonlinear Nonconvex Set-Valued Variational Inequalities in Banach Spaces

Jong Kyu Kim

Department of Mathematics Education, Kyungnam University,
Changwon, Gyeongnam 51767, Korea
e-mail: jongkyuk@kyungnam.ac.kr

Abstract. In this paper, we show that the parametric general nonlinear nonconvex set-valued variational inequality is equivalent to the parametric general Wiener-Hopf equations. We used the equivalence formulation to study the sensitivity analysis for general nonlinear nonconvex set-valued variational inequalities without assuming the differentiability of the given data.

Keywords: Sensitivity analysis, general nonlinear nonconvex variational set-valued inequalities, fixed point, general Wiener-Hopf equations, relaxed φ -accretive mapping, locally Lipschitz continuous mappings, uniformly r -prox regular sets, uniformly smooth Banach spaces.

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1 Introduction

Variational inequality theory has become a very effective and powerful tools for studying a wide range of problems arising in pure and applied sciences which include the work on differential equations, mechanics, control problems in elasticity, general equilibrium problems in economics and transportation, obstacle, moving, and free boundary problems (see [1,3,5,8-10]).

Sensitivity analysis for the solutions of variational inequalities with single-valued mappings has been studied by many authors by quite different techniques. By using the projection methods, Anastassiou *et al.* [2], Agarwal *et al.* [4], Dafermos [6], Faraj and Salahuddin [7], Kim *et al.* [11], Kyparisis [12], Khan and Salahuddin [13], Liu [14], Lee and Salahuddin [15], Noor and Noor [16], Qiu and Magnanti [18], Salahuddin [19,20], Yen and Lee [23], and Verma [24] studied the sensitivity analysis for the solutions of some variational inequalities with single-valued mappings in finite dimensional spaces, Hilbert spaces and Banach spaces.

Noor and Noor [16] introduced and considered a new class of variational inequalities on the uniformly prox regular sets which are called the general nonlinear nonconvex variational inequalities. We note that the uniformly prox regular sets are nonconvex and include the convex sets as a special cases (see [5,17]).

In this paper, we developed the general framework of sensitivity analysis for the general nonlinear nonconvex set-valued variational inequalities. For this, we established the equivalence between the parametric general nonlinear nonconvex set-valued variational inequalities and parametric general Wiener-Hopf equations by using the

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projection techniques (see [11,21]). This fixed point formulation is obtained by a suitable and approximate rearrangement of the parametric general Wiener-Hopf equations. We would like to point out that the Wiener-Hopf equations technique is quite general unified flexible and provides us with new approach to study the sensitivity analysis of general nonlinear nonconvex set-valued variational inequalities and related optimization problems. We used this equivalence to develop the sensitivity analysis for general nonlinear nonconvex set-valued variational inequalities without assuming the differentiability of the given data.

2 Preliminaries

Let X be a real Banach space with dual space X^* , $\langle \cdot, \cdot \rangle$ be the dual pairing between X and X^* , and $CB(X)$ denotes the family of all nonempty closed bounded subsets of X . The generalized duality mapping $J_q : X \rightarrow 2^{X^*}$ is defined by

$$J_q(u) = \{f^* \in X^* : \langle u, f^* \rangle = \|u\|^q, \|f^*\| = \|u\|^{q-1}\}, \forall u \in X,$$

where $q > 1$ is a constant. In particular J_2 is a usual normalized duality mapping. It is known that in general $J_q(u) = \|u\|^{q-2} J_2(u)$ for all $u \neq 0$ and J_q is single-valued if X^* is strictly convex. In the sequel, we always assume that X is a real Banach space such that J_q is a single-valued. If X is a Hilbert space then J_q becomes the identity mapping on \cdot . The modulus of smoothness of X is the function $\rho_X : [0, \infty) \rightarrow [0, \infty)$ is defined by

$$\rho_X(t) = \sup \left\{ \frac{1}{2} (\|u + v\| + \|u - v\|) - 1 : \|u\| \leq 1, \|v\| \leq t \right\}.$$

A Banach space X is called uniformly smooth if

$$\lim_{t \rightarrow 0} \frac{\rho_X(t)}{t} = 0.$$

X is called q -uniformly smooth if there exists a constant $c > 0$ such that

$$\rho_X(t) < ct^q, q > 1.$$

It is well known that the Hilbert spaces, L_p (or l_p) spaces, $1 < p < \infty$ and the Sobolev spaces $W^{m,p}$, $1 < p < \infty$ are all q -uniformly smooth. Note that J_q is single-valued if X is uniformly smooth. Concerned with the characteristic inequalities in q -uniformly smooth Banach spaces. Xu [22] proved the following results.

Lemma 2.1. [22] *The real Banach space X is q -uniformly smooth if and only if there exists a constant $c_q > 0$ such that for all $u, v \in X$,*

$$\|u + v\|^q \leq \|u\|^q + q\langle v, J_q(u) \rangle + c_q \|v\|^q.$$

Let \mathcal{K} be a nonempty closed subsets of X and we denote $d_{\mathcal{K}}(\cdot)$ or $d(\cdot, \mathcal{K})$ the usual distance function to the subset \mathcal{K} , that is,

$$d_{\mathcal{K}}(u) = \inf_{v \in \mathcal{K}} \|u - v\|.$$

The set of all projections of u onto \mathcal{K} is given by

$$P_{\mathcal{K}}(u) = \{v \in \mathcal{K} : d_{\mathcal{K}}(u) = \|u - v\|\}.$$

Definition 2.2. The proximal normal cone of \mathcal{K} at a point $u \in X$ is given by

$$N_{\mathcal{K}}^P(u) = \{\zeta \in X : u \in P_{\mathcal{K}}(u + \alpha\zeta) \text{ for some } \alpha > 0\}.$$

Lemma 2.3. [5] Let \mathcal{K} be a nonempty closed subset of X . Then $\zeta \in N_{\mathcal{K}}^P(u)$ if and only if there exists a constant $\alpha = \alpha(\zeta, u) > 0$ such that

$$\langle \zeta, v - u \rangle \leq \alpha \|v - u\|^2, \quad \forall v \in \mathcal{K}.$$

Lemma 2.4. [5] Let \mathcal{K} be a nonempty closed and convex subset in X . Then $\zeta \in N_{\mathcal{K}}^P(u)$ if and only if

$$\langle \zeta, v - u \rangle \leq 0, \quad \forall v \in \mathcal{K}.$$

Definition 2.5. Let $f : X \rightarrow R$ be a locally Lipschitz continuous mapping with constant τ near a given point $u \in X$, i.e., for some $\epsilon > 0$,

$$|f(v) - f(w)| \leq \tau \|v - w\|, \quad \forall v, w \in B(u; \epsilon),$$

where $B(u; \epsilon)$ denotes the open ball of radius $r > 0$ and centered at u . The generalized directional derivative of f at u in the direction z , denoted by $f^o(u; z)$ is defined as follows:

$$f^o(u; z) = \limsup_{v \rightarrow u, t \downarrow 0} \frac{f(v + tz) - f(v)}{t},$$

where v is a vector in X and t is a positive scalar.

Definition 2.6. The tangent cone $T_{\mathcal{K}}(u)$ to \mathcal{K} at a point $u \in \mathcal{K}$ is defined as follows:

$$T_{\mathcal{K}}(u) = \{v \in X : d_{\mathcal{K}}^o(u; v) = 0\}.$$

The normal cone of \mathcal{K} at u by polarity with $T_{\mathcal{K}}(u)$ is defined as follows:

$$N_{\mathcal{K}}(u) = \{\zeta : \langle \zeta, v \rangle \leq 0, \quad \forall v \in T_{\mathcal{K}}(u)\}.$$

The Clarke normal cone $N_{\mathcal{K}}^C(u)$ is given by

$$N_{\mathcal{K}}^C(u) = \overline{\text{co}}\{N_{\mathcal{K}}^P(u)\},$$

where $\overline{\text{co}}(S)$ is the closure of the convex hull of S .

It is clear that $N_{\mathcal{K}}^P(u) \subseteq N_{\mathcal{K}}^C(u)$. The converse is not true in general. Note that $N_{\mathcal{K}}^C(u)$ is always closed and convex, where as $N_{\mathcal{K}}^P(u)$ is always convex but may not be closed (see [5,17]).

Definition 2.7. [17] For any $r \in (0, +\infty]$, a subset \mathcal{K}_r of X is said to be normalized uniformly r -prox regular (or uniformly r -prox regular) if every nonzero proximal normal to \mathcal{K}_r can be realized by an r -ball, that is, for all $u \in \mathcal{K}_r$ and all $0 \neq \zeta \in N_{\mathcal{K}_r}^P(u)$ with $\|\zeta\| = 1$,

$$\langle \zeta, v - u \rangle \leq \frac{1}{2r} \|v - u\|^2, \quad \forall v \in \mathcal{K}_r.$$

Proposition 2.8. [17] *Let $r > 0$ and \mathcal{K}_r be a nonempty closed and uniformly r -prox regular subset of X . Set*

$$\mathcal{U}(r) = \{u \in X : 0 \leq d_{\mathcal{K}_r}(u) < r\}.$$

Then we have the following statements:

- (i) *For all $u \in \mathcal{U}(r)$, we have $P_{\mathcal{K}_r}(u) \neq \emptyset$;*
- (ii) *For all $r' \in (0, r)$, $P_{\mathcal{K}_r}$ is Lipschitz continuous with constant $\delta = \frac{r}{r-r'}$ on $\mathcal{U}(r')$;*
- (iii) *The proximal normal cone is closed as a set-valued mapping.*

Assume that $T : X \rightarrow 2^{X^*}$ is a set-valued mapping and $h : X \rightarrow X$ is a nonlinear single-valued mapping. For any constant $\rho > 0$, we consider the problem of finding $u \in X, x \in T(u)$ such that $h(u) \in \mathcal{K}_r$ and

$$\langle \rho x + h(u) - u, v - h(u) \rangle + \frac{1}{2r} \|v - h(u)\|^2 \geq 0, \quad \forall v \in \mathcal{K}_r. \tag{2.1}$$

The equation (2.1) is called a general nonlinear nonconvex set-valued variational inequality.

Now we consider the problem of solving general Wiener-Hopf equations. To be more precise, let $Q_{\mathcal{K}_r} = I - h^{-1}P_{\mathcal{K}_r}$ where $P_{\mathcal{K}_r}$ is the projection operator, h^{-1} is the inverse of nonlinear operator h and I is an identity operator. For given nonlinear operators, $z, u \in X, x \in T(u)$ such that

$$TP_{\mathcal{K}_r}z + \rho^{-1}Q_{\mathcal{K}_r}z = 0 \tag{2.2}$$

is a called general Wiener-Hopf equation.

Lemma 2.9. [17] *$u \in X, x \in T(u), h(u) \in \mathcal{K}_r$ is a solution of (2.1) if and only if $u \in X, x \in T(u), h(u) \in \mathcal{K}_r$ satisfies the relation*

$$h(u) = P_{\mathcal{K}_r}[u - \rho x], \tag{2.3}$$

where $P_{\mathcal{K}_r}$ is the projection of X onto the uniformly r -prox regular set \mathcal{K}_r .

Lemma 2.9 implies that the general nonlinear nonconvex set-valued variational inequality (2.1) is equivalent to the fixed point problem (2.3).

Now, we consider the parametric version of equations (2.1) and (2.2). To formulate the problem, let Γ be an open subset of X in which parameter λ takes values. Let $x_\lambda(u) \in T_\lambda(u)$ be a given operator defined on $X \times \Gamma$ and takes values in $X \times X$. From now, we denote $x_\lambda(u) \in T_\lambda(u)$ unless otherwise specified. The parametric general nonlinear nonconvex set-valued variational inequality problem is to find $(u, \lambda) \in X \times \Gamma, x_\lambda(u) \in T_\lambda(u)$ such that

$$\langle \rho x_\lambda(u) + h_\lambda(u) - u, v - h_\lambda(u) \rangle \geq 0, \quad \forall v \in \mathcal{K}_r. \tag{2.4}$$

We also assume that for some $\bar{\lambda} \in \mathcal{B}$, problem (2.4) has a unique solution \bar{u} . Related to parametric general nonlinear nonconvex set-valued variational inequality problem (2.4), we consider the parametric general Wiener-Hopf equation. We consider the problem of finding $(z, \lambda) \in X \times \Gamma, x_\lambda(u) \in T_\lambda(u)$ such that

$$T_\lambda P_{\mathcal{K}_r}z + \rho^{-1}Q_{\mathcal{K}_r}z = 0, \tag{2.5}$$

where $\rho > 0$ is a constant and $Q_{\mathcal{K}_r} z$ is defined on the set (z, λ) with $\lambda \in \Gamma$ and takes values in \mathcal{B} . Equation (2.5) is called a parametric general Wiener-Hopf equation.

Lemma 2.10. *Let X be a real Banach space. Then the following two statements are equivalent:*

- (i) *An element $u \in X, x_\lambda(u) \in T_\lambda(u)$ is a solution of (2.4),*
- (ii) *The mapping*

$$F_\lambda(u) = u - h_\lambda(u) + P_{\mathcal{K}_r}[u - \rho x_\lambda(u)]$$

has a fixed point.

One can established the equivalence relation between inequality (2.4) and equation (2.5) by using the projection techniques.

Lemma 2.11. *Parametric general nonlinear nonconvex set-valued variational inequality (2.4) has a solution $(u, \lambda) \in X \times \Gamma, x_\lambda(u) \in T_\lambda(u)$ if and only if parametric general Wiener-Hopf equation (2.5) has a solution $(z, \lambda) \in X \times \Gamma, x_\lambda(u) \in T_\lambda(u)$, where*

$$h_\lambda(u) = P_{\mathcal{K}_r} z \tag{2.6}$$

and

$$z = u - \rho x_\lambda(u). \tag{2.7}$$

From Lemma 2.11, we know that Parametric general nonlinear nonconvex set-valued variational inequality (2.4) and parametric general Wiener-Hopf equation (2.5) are equivalent.

We used these equivalence to study the sensitivity analysis of general nonlinear nonconvex set-valued variational inequalities. We assume that for some $\bar{\lambda} \in \Gamma$, problem (2.5) has a solution \bar{z} and \mathcal{B} is a closure of a ball in X centered at \bar{z} . We want to investigate those condition under which for each λ in a neighbourhood of $\bar{\lambda}$, then (2.5) has a unique solution $z(\lambda)$ near \bar{z} and the function $z(\lambda)$ is (Lipschitz) continuous and differentiable.

Definition 2.12. Let $T : X \times \Gamma \rightarrow 2^{X^*}$ be a set-valued mapping. Then the operator $T_\lambda(\cdot)$ is said to be locally relaxed φ -accretive if there exists a constant $\varphi > 0$ such that

$$\langle x_\lambda(u) - x_\lambda(v), j_q(u - v) \rangle \geq -\varphi \|u - v\|^q, \forall u, v \in X, \lambda \in \Gamma,$$

and locally D -Lipschitz continuous if there exists a constant $\beta > 0$ such that

$$\|x_\lambda(u) - x_\lambda(v)\| \leq D(T_\lambda(u), T_\lambda(v)) \leq \beta \|u - v\|,$$

where $D : 2^{X^*} \times 2^{X^*} \rightarrow (-\infty, \infty) \cup \{+\infty\}$ is the Hausdorff metric *i.e.*,

$$D(A, B) = \left\{ \sup_{u \in A} \inf_{v \in B} \|u - v\|, \sup_{u \in B} \inf_{v \in A} \|u - v\| \right\}, \forall A, B \in 2^{X^*}.$$

Definition 2.13. A single-valued mapping $h : X \times \Omega \rightarrow X$ is said to be locally Lipschitz continuous if there exists a constant $\gamma > 0$ such that

$$\|h_\lambda(u) - h_\lambda(v)\| \leq \gamma \|u - v\|, \forall u, v \in X,$$

and locally strongly accretive if there exists a constant $\xi > 0$ such that

$$\langle h_\lambda(u) - h_\lambda(v), j_q(u - v) \rangle \geq \xi \|u - v\|^q, \forall u, v \in X, \lambda \in \Gamma.$$

3 Main Results

In this section, we derive the main results of this paper. We consider the case when the solutions of the parametric general Wiener-Hopf equation (2.5) lies in the interior of \mathcal{B} .

We consider the map: for all $(z, \lambda) \in X \times \Gamma, x_\lambda(u) \in T_\lambda(u)$,

$$F_\lambda(z) = P_{\mathcal{K}_r}z - \rho x_\lambda(u) = u - \rho x_\lambda(u), \tag{3.1}$$

where

$$h_\lambda(u) = P_{\mathcal{K}_r}z. \tag{3.2}$$

We have to show that the map $F_\lambda(z)$ has a fixed point, which is a solution of parametric general Wiener-Hopf equation (2.5). First of all we prove the map $F_\lambda(z)$ defined by (3.1) is contractive with respect to z uniformly in $\lambda \in \Gamma$.

Lemma 3.1. *Let $P_{\mathcal{K}_r}$ be a locally Lipschitz continuous operator with constant $\delta = \frac{r}{r-r'}$. Let $h : X \times \Gamma \rightarrow X$ be a locally Lipschitz continuous with constant $\gamma > 0$ and locally strongly accretive mapping with respect to the constant $\xi > 0$. Let $T : \Gamma \times X \rightarrow 2^{X^*}$ be a locally D -Lipschitz continuous with respect to the constant $\beta > 0$ and locally relaxed φ -accretive mapping with respect to the constant $\varphi > 0$. Then for all $z_1, z_2 \in X$ and $\lambda \in \Gamma$, we have*

$$\|F_\lambda(z_1) - F_\lambda(z_2)\| \leq \theta \|z_1 - z_2\|, \tag{3.3}$$

where

$$\theta = \frac{\delta \sqrt[q]{1 + q\rho\varphi + c_q\rho^q\beta^q}}{1 - \kappa}, \quad \kappa = \sqrt[q]{1 - q\xi + c_q\gamma^q} \tag{3.4}$$

for

$$\sqrt[q]{\rho^q c_q \beta^q + \rho q \varphi + 1} < \frac{1 - \kappa}{\delta}. \tag{3.5}$$

Proof. For all $z_1, z_2 \in \mathcal{B}, \lambda \in \Gamma$, from (3.1) we have

$$\|F_\lambda(z_1) - F_\lambda(z_2)\| = \|u - v - \rho(x_\lambda(u) - x_\lambda(v))\|. \tag{3.6}$$

Since $T_\lambda(\cdot)$ is a locally D -Lipschitz continuous mapping, we have

$$\|x_\lambda(u) - x_\lambda(v)\| \leq D(T_\lambda(u), T_\lambda(v)) \leq \beta \|u - v\|. \tag{3.7}$$

Using the locally relaxed φ -accretivity and locally D -Lipschitz continuity of $T_\lambda(\cdot)$, we have

$$\begin{aligned} \|u - v - \rho(x_\lambda(u) - x_\lambda(v))\|^q &\leq \|u - v\|^q - q\rho \langle x_\lambda(u) - x_\lambda(v), j_q(u - v) \rangle \\ &\quad + c_q \rho^q \|x_\lambda(u) - x_\lambda(v)\|^q \\ &\leq \|u - v\|^q - q\rho(-\varphi \|u - v\|^q) + c_q \rho^q \beta^q \|u - v\|^q \\ &\leq (1 + q\rho\varphi + c_q \rho^q \beta^q) \|u - v\|^q. \end{aligned} \tag{3.8}$$

From (3.6) and (3.8), we have

$$\|F_\lambda(z_1) - F_\lambda(z_2)\| \leq \sqrt[q]{1 + q\rho\varphi + c_q \rho^q \beta^q} \|u - v\|. \tag{3.9}$$

Also from (3.2) and locally Lipschitz continuity of projection operator $P_{\mathcal{K}_r}$ with constant δ , we have

$$\begin{aligned} \|u - v\| &\leq \|u - v - (h_\lambda(u) - h_\lambda(v))\| + \|P_{\mathcal{K}_r}(z_1) - P_{\mathcal{K}_r}(z_2)\| \\ &\leq \|u - v - (h_\lambda(u) - h_\lambda(v))\| + \delta\|z_1 - z_2\|. \end{aligned} \tag{3.10}$$

Since h_λ is a locally Lipschitz continuous with constant $\gamma > 0$ and locally strongly accretive mapping with constant $\xi > 0$, we have

$$\begin{aligned} \|u - v - (h_\lambda(u) - h_\lambda(v))\|^q &\leq \|u - v\|^q - q(h_\lambda(u) - h_\lambda(v), j_q(u - v)) \\ &\quad + c_q \|h_\lambda(u) - h_\lambda(v)\|^q \\ &\leq \|u - v\|^q - q\xi \|u - v\|^q + c_q \gamma^q \|u - v\|^q \\ &\leq (1 - q\xi + c_q \gamma^q) \|u - v\|^q. \end{aligned}$$

It implies that

$$\|u - v - (h_\lambda(u) - h_\lambda(v))\| \leq \sqrt[q]{1 - q\xi + c_q \gamma^q} \|u - v\|. \tag{3.11}$$

From (3.10) and (3.11), we have

$$\|u - v\| \leq \kappa \|u - v\| + \delta \|z_1 - z_2\|,$$

where $\kappa = \sqrt[q]{1 - q\xi + c_q \gamma^q}$. From which we have

$$\|u - v\| \leq \frac{\delta}{1 - \kappa} \|z_1 - z_2\|. \tag{3.12}$$

Combining (3.9),(3.12) and (3.3), we have

$$\begin{aligned} \|F_\lambda(z_1) - F_\lambda(z_2)\| &\leq (1 - \alpha)\|z_1 - z_2\| + \alpha\delta \frac{\sqrt[q]{1 + q\rho\varphi + c_q\rho^q\beta^q}}{1 - \kappa} \|z_1 - z_2\| \\ &= (1 - \alpha)\|z_1 - z_2\| + \alpha\theta\|z_1 - z_2\|. \end{aligned} \tag{3.13}$$

It follows from (3.4) that $\theta < 1$. Hence the mapping $F_\lambda(z)$ defined by (3.1) is contractive and has a fixed point $z(\lambda)$ which is the solution of parametric general Wiener-Hopf equation (2.5).

Remark 3.2. From Lemma 3.1, we see that the map $F_\lambda(z)$ defined by (2.4) has a unique fixed point $z(\lambda)$, that is, $z(\lambda) = F_\lambda(z)$. Also by assumptions, the function \bar{z} for $\lambda = \bar{\lambda}$ is a solutions of parametric general Wiener-Hopf equation (2.5). Again by Lemma 3.1, we know that \bar{z} for $\lambda = \bar{\lambda}$ is a fixed point of $F_\lambda(z)$ and it is also a fixed point of $F_{\bar{\lambda}}(z)$. Consequently, we conclude that

$$z(\bar{\lambda}) = \bar{z} = F_{\bar{\lambda}}(z(\bar{\lambda})).$$

Using Lemma 3.1, we can prove the continuity of the solution $z(\lambda)$ of parametric general Wiener-Hopf equation (2.5). However for the sake of completeness and to convey the idea of the technique involved, we give the proof.

Lemma 3.3. *Assume that the operator $T_\lambda(\cdot)$ is locally D -Lipschitz continuous with respect to the parameter λ and $h_\lambda(\cdot)$ is a locally Lipschitz continuous mapping. If $T_\lambda(\cdot)$ is a locally Lipschitz continuous mapping and the maps $\lambda \rightarrow P_{\mathcal{K}_r,\lambda}z, \lambda \rightarrow h_\lambda(u), \lambda \rightarrow T_\lambda(u)$ are continuous (or Lipschitz continuous), then the function $z(\lambda)$ satisfying (3.3) is (Lipschitz) continuous at $\lambda = \bar{\lambda}$.*

Proof. For all $\lambda \in \Gamma$ invoking Lemma 3.1 and the triangle inequality, we have

$$\begin{aligned} \|z(\lambda) - z(\bar{\lambda})\| &\leq \|F_\lambda(z(\lambda)) - F_\lambda(z(\bar{\lambda}))\| + \|F_\lambda(z(\bar{\lambda})) - F_{\bar{\lambda}}(z(\bar{\lambda}))\| \quad (3.14) \\ &\leq \theta \|z(\lambda) - z(\bar{\lambda})\| + \|F_\lambda(z(\bar{\lambda})) - F_{\bar{\lambda}}(z(\bar{\lambda}))\|. \end{aligned}$$

From (3.1) and the fact that the operator $T_\lambda(\cdot)$ is locally D -Lipschitz continuous with respect to the parameter λ , we have

$$\begin{aligned} \|F_\lambda(z(\bar{\lambda})) - F_{\bar{\lambda}}(z(\bar{\lambda}))\| &= \|u(\bar{\lambda}) - u(\bar{\lambda}) - \rho(T_\lambda(u(\bar{\lambda})) - T_{\bar{\lambda}}(u(\bar{\lambda})))\| \quad (3.15) \\ &\leq \rho\beta \|\lambda - \bar{\lambda}\|. \end{aligned}$$

Combining (3.14) and (3.15), we obtain

$$\|z(\lambda) - z(\bar{\lambda})\| \leq \frac{\rho\beta}{1-\theta} \|\lambda - \bar{\lambda}\|, \quad \forall \lambda, \bar{\lambda} \in \Gamma.$$

This completes the proof.

Now, we are in a position to state and prove the main result of this paper.

Theorem 3.4. *Let \bar{u} be a solution of parametric general nonlinear nonconvex set-valued variational inequality (2.4) and \bar{z} be a solution of parametric general Wiener-Hopf equation (2.5) for $\lambda = \bar{\lambda}$. Let $h_\lambda(u)$ be a locally strongly accretive and locally Lipschitz continuous mapping. Let $T_\lambda(u)$ be a locally D -Lipschitz continuous and locally relaxed φ -accretive mapping with respect to $\varphi > 0$ for all $u \in \mathcal{B}$. If the maps $\lambda \rightarrow P_{\mathcal{K}_r}, \lambda \rightarrow h_\lambda(u), \lambda \rightarrow T_\lambda(u)$ are Lipschitz (continuous) at $\lambda = \bar{\lambda}$, then there exists a neighbourhood \mathcal{M} of Γ of $\bar{\lambda}$ such that for $\lambda \in \mathcal{M}$, parametric general Wiener-Hopf equation (2.5) has a unique solution $z(\lambda)$ in the interior of \mathcal{B} , $z(\bar{\lambda}) = \bar{z}$ and $z(\lambda)$ is (Lipschitz) continuous at $\lambda = \bar{\lambda}$.*

Proof. The proof follows from Lemma 3.1, 3.3 and Remark 3.2.

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Common Fixed Point Theorems for Non-compatible Self-mappings in b -Fuzzy Metric Spaces

Jong Kyu Kim¹, Shaban Sedghi^{2,3}, Nabi Shobe³
and Hassan Sadati⁴

¹Department of Mathematics Education, Kyungnam University,
Changwon, Gyeongnam 51767, Korea
e-mail: jongkyuk@kyungnam.ac.kr

^{2,4}Department of Mathematics, Qaemshahr Branch,
Islamic Azad University, Qaemshahr, Iran
e-mail: sedghi_gh@yahoo.com
e-mail: sadati_s@yahoo.com

³Department of Mathematics, Babol BranchIs,
Islamic Azad University, Babol, Iran
e-mail: nabi_shobe@yahoo.com

Abstract. By using R -weak commutativity of type (A_g) and non-compatible conditions of self-mapping pairs in a b -fuzzy metric space, without the conditions for the completeness of space and the continuity of mappings, we establish some new common fixed point theorems for two self-mappings. An example is provided to support our new result.

Keywords: b -fuzzy metric space, common fixed point theorem, R -weakly commuting mappings of type (A_g) , non-compatible mapping pairs.

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1 Introduction and Preliminaries

The concept of fuzzy sets was introduced initially by Zadeh [17] in 1965. Since then, to use this concept in topology and analysis, many authors have expansively developed the theory of fuzzy sets and applications. George and Veeramani [5], Kramosil and Michalek [7] have introduced the concept of fuzzy topological spaces induced by fuzzy metric which have very important applications in quantum particle physics, particularly in connections with both string and E -infinity theory which were given and studied by El Naschie [1-4]. Many authors [6,9,10,13-15] have proved fixed point theorems in fuzzy (probabilistic) metric spaces.

Definition 1.1. A binary operation $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous t -norm if it satisfies the following conditions:

- (1) $*$ is associative and commutative,
- (2) $*$ is continuous,
- (3) $a * 1 = a$, for all $a \in [0, 1]$,

⁰Corresponding author: Jong Kyu Kim(jongkyuk@kyungnam.ac.kr)

(4) $a * b \leq c * d$, whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0, 1]$.

Two typical examples of a continuous t -norm are $a * b = ab$ and $a * b = \min(a, b)$.

Definition 1.2. [11] A 3-tuple $(X, M, *)$ is called a *fuzzy metric space* if X is an arbitrary (non-empty) set, $*$ is a continuous t -norm and M is a fuzzy set on $X^2 \times (0, \infty)$, satisfying the following conditions for each $x, y, z \in X$ and $t, s > 0$,

- (1) $M(x, y, t) > 0$,
- (2) $M(x, y, t) = 1$ if and only if $x = y$,
- (3) $M(x, y, t) = M(y, x, t)$,
- (4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
- (5) $M(x, y, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous.

Definition 1.3. [11] A 3-tuple $(X, M, *)$ is called a b -fuzzy metric space for $b \geq 1$ if X is an arbitrary nonempty set, $*$ is a continuous t -norm and M is a fuzzy set on $X^2 \times (0, \infty)$, satisfying the following conditions for each $x, y, z \in X$ and $t, s > 0$,

- (1) $M(x, y, t) > 0$,
- (2) $M(x, y, t) = 1$ if and only if $x = y$,
- (3) $M(x, y, t) = M(y, x, t)$,
- (4) $M(x, y, \frac{t}{b}) * M(y, z, \frac{s}{b}) \leq M(x, z, t + s)$,
- (5) $M(x, y, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous.

It should be noted that, the class of b -fuzzy metric spaces is effectively larger than that of fuzzy metric spaces, since a b -fuzzy metric is a fuzzy metric when $b = 1$.

We present an example shows that a b -fuzzy metric on X need not be a fuzzy metric on X .

Example 1.4. Let $M(x, y, t) = e^{-\frac{|x-y|^p}{t}}$, where $p > 1$ is a real number. We show that M is a b -fuzzy metric with $b = 2^{p-1}$. In fact, obviously conditions (1),(2),(3) and (5) of definition 1.3 are satisfied. Let $f(x) = x^p$ ($x > 0$). Then we know that it is a convex function, for $1 < p < \infty$. So, we have

$$\left(\frac{a+c}{2}\right)^p \leq \frac{1}{2}(a^p + c^p),$$

it implies that $(a+c)^p \leq 2^{p-1}(a^p + c^p)$. Therefore, we have

$$\begin{aligned} \frac{|x-y|^p}{t+s} &\leq 2^{p-1} \frac{|x-z|^p}{t+s} + 2^{p-1} \frac{|z-y|^p}{t+s} \\ &\leq 2^{p-1} \frac{|x-z|^p}{t} + 2^{p-1} \frac{|z-y|^p}{s} \\ &= \frac{|x-z|^p}{t/2^{p-1}} + \frac{|z-y|^p}{s/2^{p-1}}. \end{aligned}$$

Thus, for each $x, y, z \in X$ we obtain

$$\begin{aligned}
 M(x, y, t + s) &= e^{-\frac{|x-y|^p}{t+s}} \\
 &\geq M(x, z, \frac{t}{2^{p-1}}) * M(z, y, \frac{s}{2^{p-1}}),
 \end{aligned}$$

where $a * c = ac$ for all $a, c \in [0, 1]$. So condition (4) of definition 1.3 is hold and M is a b -fuzzy metric.

It should be noted that in preceding example, for $p = 2$ it is easy to see that $(X, M, *)$ is not a fuzzy metric space.

Example 1.5. Let $M(x, y, t) = e^{-\frac{d(x,y)}{t}}$ or $M(x, y, t) = \frac{t}{t+d(x,y)}$, where d is a b -metric on X and $a * c = ac$ for all $a, c \in [0, 1]$. Then it is easy to show that M is a b -fuzzy metric. In fact, obviously conditions (1),(2),(3) and (5) of definition 1.3 are satisfied. Since d is a b -metric, for each $x, y, z \in X$ we have

$$d(x, y) \leq b[d(x, z) + d(z, y)].$$

Therefore, we obtain

$$\begin{aligned}
 M(x, y, t + s) &= e^{-\frac{d(x,y)}{t+s}} \\
 &\geq e^{-b\frac{d(x,z)+d(z,y)}{t+s}} \\
 &= \left(e^{-b\frac{d(x,z)}{t+s}} \right) \left(e^{-b\frac{d(z,y)}{t+s}} \right) \\
 &\geq \left(e^{-\frac{d(x,z)}{t/b}} \right) \left(e^{-\frac{d(z,y)}{s/b}} \right) \\
 &= M(x, z, \frac{t}{b}) * M(z, y, \frac{s}{b}).
 \end{aligned}$$

So condition (4) of definition 1.3 is hold and M is a b -fuzzy metric. Similarly, we can show that $M(x, y, t) = \frac{t}{t+d(x,y)}$ is also a b -fuzzy metric.

Next, we need the following definitions and propositions in b -metric spaces for our main theorems.

Definition 1.6. Let $f : R \rightarrow R$ be a function. Then f is called b -nondecreasing, if $x > by$ implies that $f(x) \geq f(y)$ for each $x, y \in R$.

Lemma 1.7. [11] *Let $(X, M, *)$ be a b -fuzzy metric space. Then $M(x, y, t)$ is b -nondecreasing with respect to t , for all x, y in X . Also,*

$$M(x, y, b^n t) \geq M(x, y, t), \forall n \in N.$$

Let $(X, M, *)$ be a b -fuzzy metric space. For $t > 0$, the open ball $B(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$$

We recall the notions of convergence and completeness in a b -fuzzy metric space.

Let $(X, M, *)$ be a b -fuzzy metric space. Let τ be the set of all $A \subset X$ with $x \in A$ if and only if there exists $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subset A$. Then

τ is a topology on X (induced by the b -fuzzy metric M). A sequence $\{x_n\}$ in X converges to x if and only if $M(x_n, x, t) \rightarrow 1$ as $n \rightarrow \infty$, for each $t > 0$. It is called a Cauchy sequence if for each $0 < \varepsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$, for each $n, m \geq n_0$. The b -fuzzy metric space $(X, M, *)$ is said to be complete if every Cauchy sequence is convergent. A subset A of X is said to be F -bounded if there exists $t > 0$ and $0 < r < 1$ such that $M(x, y, t) > 1 - r$, for all $x, y \in A$.

Lemma 1.8. [11] *Let $(X, M, *)$ be a b -fuzzy metric space. Then the following assertions hold:*

- (i) *If sequence $\{x_n\} \subset X$ converges to x , then x is unique,*
- (ii) *The convergent sequence $\{x_n\} \subset X$ is Cauchy.*

We have the following propositions in a b -fuzzy metric space.

Proposition 1.9. [11] *Let $(X, M, *)$ be a b -fuzzy metric space and suppose that $\{x_n\}$ and $\{y_n\}$ are convergent to x, y respectively. Then we have*

$$M(x, y, \frac{t}{b^2}) \leq \limsup_{n \rightarrow \infty} M(x_n, y_n, t) \leq M(x, y, b^2t)$$

and

$$M(x, y, \frac{t}{b^2}) \leq \liminf_{n \rightarrow \infty} M(x_n, y_n, t) \leq M(x, y, b^2t).$$

Proposition 1.10. [12] *Let $(X, M, *)$ be a b -fuzzy metric space and suppose that $\{x_n\}$ is convergent to x . Then, for all $y \in X$ we have*

$$M(x, y, \frac{t}{b}) \leq \limsup_{n \rightarrow \infty} M(x_n, y, t) \leq M(x, y, bt)$$

and

$$M(x, y, \frac{t}{b}) \leq \liminf_{n \rightarrow \infty} M(x_n, y, t) \leq M(x, y, bt).$$

Lemma 1.11. *A b -fuzzy metric is not continuous in general.*

Throughout, in this paper we assume that $\lim_{t \rightarrow \infty} M(x, y, t) = 1$.

Lemma 1.12. *Let $(X, M, *)$ be a b -fuzzy metric space and suppose that $M(x, y, kt) \geq M(x, y, t)$, for all $x, y \in X$, $0 < k < 1$ and $t > 0$. Then $x = y$.*

Proof. Since, $M(x, y, kt) \geq M(x, y, t)$, it follows that

$$M(x, y, t) \geq M(x, y, \frac{t}{k}) \geq \dots \geq M(x, y, \frac{t}{k^n}).$$

Hence, we can get $M(x, y, t) \geq \lim_{n \rightarrow \infty} M(x, y, \frac{t}{k^n}) = 1$, therefore, $x = y$.

In 2010, Vats et al. [16] introduced the concept of weakly compatible. Also, in 2010, Manro et al. [8] introduced the concepts of weakly commuting, R -weakly commuting mappings, and R -weakly commuting mappings of type (P) , (A_f) , and (A_g) in a G -metric space.

We will introduce these concepts in a b -fuzzy metric space.

Definition 1.13. The self-mappings f and g of a b -fuzzy metric space $(X, M, *)$ are said to be compatible if

$$\lim_{n \rightarrow \infty} M(fgx_n, gfx_n, t) = 1$$

and

$$\lim_{n \rightarrow \infty} M(gfx_n, gfx_n, t) = 1,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = z$, for some $z \in X$.

Definition 1.14. A pair of self-mappings (f, g) of a b -fuzzy metric space $(X, M, *)$ is said to be

- (1) weakly commuting if $M(fgx, gfx, t) \geq M(fx, gx, t)$, for all $x \in X$.
- (2) R -weakly commuting if there exists some positive real number R such that $M(fgx, gfx, t) \geq M(fx, gx, \frac{t}{R})$, $\forall x \in X$.

Remark 1.15. If $R \leq 1$, then R -weakly commuting mappings are weakly commuting.

Definition 1.16. A pair of self-mappings (f, g) of a b -fuzzy metric space $(X, M, *)$ are said to be

- (1) R -weakly commuting mappings of type (A_f) if there exists some positive real number R such that $M(fgx, ggx, t) \geq M(fx, gx, \frac{t}{R})$, for all $x \in X$.
- (2) R -weakly commuting mappings of type (A_g) if there exists some positive real number R such that $M(gfx, ffx, t) \geq M(gx, fx, \frac{t}{R})$, for all $x \in X$.
- (3) R -weakly commuting mappings of type (P) if there exists some positive real number R such that $M(ffx, ggx, t) \geq M(fx, gx, \frac{t}{R})$, for all $x \in X$.

Remark 1.17. The self-mapping f of a b -fuzzy metric space $(X, M, *)$ is said to be b -continuous at $x \in X$ if and only if it is b -sequentially continuous at x , that is, whenever $\{x_n\}$ is b -convergent to x , $\{f(x_n)\}$ is b -convergent to $f(x)$.

Example 1.18. Let $M(x, y, t) = e^{-\frac{|x-y|^2}{t}}$, $fx = 1$ and

$$gx = \begin{cases} 1, & x \in Q, \\ -1, & \text{otherwise,} \end{cases}$$

for each $x, y \in R$, where $a * c = ac$. Then it is easy to see that a pair of self-mappings (f, g) of a b -fuzzy metric space is weakly commuting, R -weakly commuting, and R -weakly commuting of type (P) , (A_f) , and (A_g) .

2 The Main Results

Now we are in a position to introduce the main results of this paper.

Theorem 2.1. Let $(X, M, *)$ be a b -fuzzy metric space and (f, g) be a pair of non-compatible self-mappings with $\overline{fX} \subseteq gX$ (\overline{fX} denotes the closure of fX). Assume

that the following condition is satisfied:

$$M(fx, fy, kt) \geq \min\{M(gx, gy, b^2t), M(fx, gx, b^2t), M(fy, gy, b^2t)\}, \quad (2.1)$$

for all $x, y \in X$ and $0 < k < 1$. If (f, g) is a pair of R -weakly commuting mappings of type (A_g) , then f and g have a unique common fixed point (say z) and both f and g are not b -continuous at z .

Proof. Since f and g are non-compatible mappings, there exists a sequence $\{x_n\} \subset X$, such that

$$\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = z, \quad z \in X,$$

but either $\lim_{n \rightarrow \infty} M(fgx_n, gfx_n, t)$ or $\lim_{n \rightarrow \infty} M(gfx_n, fgx_n, t)$ does not exist or exists and is different from 1. Since $z \in \overline{fX} \subset gX$, there must exist a $u \in X$ satisfying $z = gu$. We can assert that $fu = gu$. If not, from condition (2.1) and Proposition 1.10, we obtain

$$\begin{aligned} & M(fu, gu, bkt) \\ & \geq \limsup_{n \rightarrow \infty} M(fu, fx_n, kt) \\ & \geq \limsup_{n \rightarrow \infty} \min\{M(gu, gx_n, b^2t), M(fu, gx_n, b^2t), M(fx_n, gu, b^2t)\} \\ & \geq \min\{M(gu, gu, bt), M(fu, gu, bt), M(fu, gu, bt)\} \\ & = M(fu, gu, bt), \end{aligned}$$

that is, $M(fu, gu, kt) \geq M(fu, gu, t)$. Hence, by Lemma 1.12, we get $fu = gu$. Since (f, g) is a pair of R -weakly commuting mappings of type (A_g) , we have

$$M(gfu, ffu, t) \geq M(gu, fu, \frac{t}{R}) = 1.$$

It means that $ffu = gfu$. Next, we prove $ffu = fu$. From condition (2.1), $fu = gu$ and $ffu = gfu$, we have

$$\begin{aligned} M(fu, ffu, kt) & \geq \min\{M(gu, gfu, b^2t), M(fu, gfu, b^2t), M(gu, ffu, b^2t)\} \\ & = M(fu, ffu, b^2t) \\ & \geq M(fu, ffu, t). \end{aligned}$$

From Lemma 1.12, we have $fu = ffu$, which implies that $fu = ffu = gfu$, and so $z = fu$ is a common fixed point of f and g .

Next we prove that the common fixed point z is unique. Actually, suppose that w is also a common fixed point of f and g . Then using the condition (2.1), we have

$$\begin{aligned} M(z, w, kt) & = M(fz, fw, kt) \\ & \geq \min\{M(gz, gw, b^2t), M(fz, gw, b^2t), M(fw, gz, b^2t)\} \\ & = M(z, w, b^2t) \\ & \geq M(z, w, t), \end{aligned}$$

which implies that $z = w$, so that uniqueness is proved.

Now, we prove that f and g are not b -continuous at z . In fact, if f is b -continuous at z , then for the b -convergent sequence $\{x_n\}$ to x , we have

$$\lim_{n \rightarrow \infty} ffx_n = fz = z \text{ and } \lim_{n \rightarrow \infty} fgx_n = fz = z.$$

Since f and g are R -weakly commuting mappings of type (A_g) , we get

$$M(gfx_n, ffx_n, t) \geq M(gx_n, fx_n, \frac{t}{R}).$$

Hence, by Proposition 1.9, we have

$$\begin{aligned} M(\lim_{n \rightarrow \infty} gfx_n, z, b^2t) &\geq \limsup_{n \rightarrow \infty} M(gfx_n, ffx_n, t) \\ &\geq \limsup_{n \rightarrow \infty} M(gx_n, fx_n, \frac{t}{R}) \\ &\geq M(z, z, \frac{t}{Rb^2}) = 1, \end{aligned}$$

it follows that $\lim_{n \rightarrow \infty} gfx_n = z$. Hence, we can get

$$\lim_{n \rightarrow \infty} M(fgfx_n, gfx_n, t) \geq M(z, z, \frac{t}{b^2}) = 1.$$

Therefore, we have

$$\lim_{n \rightarrow \infty} M(fgfx_n, gfx_n, t) = 1.$$

This contradicts with f and g being non-compatible. So f is not b -continuous at z . If g is b -continuous at z , then for the b -convergent sequence $\{x_n\}$ to x , we have

$$\lim_{n \rightarrow \infty} gfx_n = gz = z \text{ and } \lim_{n \rightarrow \infty} ggx_n = gz = z.$$

Since f and g are R -weakly commuting mappings of type (A_g) , we get

$$M(gfx_n, ffx_n, t) \geq M(gx_n, fx_n, \frac{t}{R}).$$

Hence, we have

$$\begin{aligned} M(z, \lim_{n \rightarrow \infty} ffx_n, b^2t) &\geq \limsup_{n \rightarrow \infty} M(gfx_n, ffx_n, t) \\ &\geq \limsup_{n \rightarrow \infty} M(gx_n, fx_n, \frac{t}{R}) \\ &\geq M(z, z, \frac{t}{Rb^2}) = 1, \end{aligned}$$

it implies that

$$\lim_{n \rightarrow \infty} ffx_n = z = fz.$$

This contradicts with f being not b -continuous at z , which implies that g is not b -continuous at z . This completes the proof.

For the case $b = 1$ in Theorem 2.1, we have the following corollary.

Corollary 2.2. *Let $(X, M, *)$ be a fuzzy metric space and (f, g) be a pair of non-compatible selfmappings with $fX \subseteq gX$. Assume that the following condition is satisfied:*

$$M(fx, fy, kt) \geq \min \{M(gx, gy, t), M(fx, gx, t), M(fy, gy, t)\}, \tag{2.2}$$

for all $x, y \in X$ and $0 < k < 1$. If (f, g) is a pair of R -weakly commuting mappings of type (A_g) , then f and g have a unique common fixed point (say z) and both f and g are not b -continuous at z .

3 Example

Next, we give an example to support for the main Theorem 2.1.

Example 3.1. Let $X = [2, 20]$ and $a * c = ac$, for all $a, c \in [0, 1]$ and let M be a fuzzy set on $X \times X \times (0, +\infty)$ defined as follows:

$$M(x, y, t) = e^{-\frac{|x-y|}{t}},$$

for all $t \in R^+$. Then $(X, M, *)$ is a fuzzy metric space. We define mappings f and g on X by

$$fx = \begin{cases} 2, & x = 2 \text{ or } x \in (5, 20], \\ 6, & x \in (2, 5] \end{cases}$$

and

$$gx = \begin{cases} 2, & x = 2, \\ 18, & x \in (2, 5], \\ \frac{x+1}{3}, & x \in (5, 20]. \end{cases}$$

Clearly, from the above definitions, we know that $\overline{f(X)} \subseteq g(X)$, and (f, g) is a pair of non-compatible self-mappings. To see that f and g are non-compatible, consider a sequence $\{x_n\} = \{5 + \frac{1}{n}\}$. Then we have $fx_n \rightarrow 2, gx_n \rightarrow 2, fgx_n \rightarrow 6$ and $gfx_n \rightarrow 2$. Thus

$$\lim_{n \rightarrow \infty} M(gfx_n, gfx_n, t) = e^{-\frac{4}{t}} \neq 1.$$

On the other hand, there exists $R = 1$ such that

$$M(gfx, gfx, t) = \begin{cases} e^{-\frac{(2-2)}{t}}, & x = 2, \\ e^{-\frac{(\frac{7}{3}-2)}{t}}, & x \in (2, 5], \\ e^{-\frac{(2-2)}{t}}, & x \in (5, 20] \end{cases}$$

and

$$M(fx, gx, t) = \begin{cases} e^{-\frac{(2-2)}{t}}, & x = 2, \\ e^{-\frac{(18-6)}{t}}, & x \in (2, 5], \\ e^{-\frac{(\frac{x+1}{3}-2)}{t}}, & x \in (5, 20], \end{cases}$$

for all $x \in X$. Hence, it is easy to see that in every cases, we have

$$M(gfx, gfx, t) \geq M(gx, fx, t).$$

That is, (f, g) is a pair of R -weakly commuting mappings of type (A_g) .

Now we prove that the mappings f and g satisfy the condition (2.1) of Theorem 2.1 with $k = \frac{1}{2}$. To do this, we consider the following cases:

Case (1) If $x, y \in \{2\} \cup (5, 20]$, then we have

$$\begin{aligned} M(fx, fy, kt) &= M(2, 2, kt) = 1 \\ &\geq \min \{M(gx, gy, t), M(fx, gx, t), M(fy, gy, t)\}, \end{aligned}$$

and hence (2.1) is obviously satisfied.

Case (2) If $x, y \in (2, 5]$, then we have

$$\begin{aligned} M(fx, fy, kt) &= M(6, 6, kt) = 1 \\ &\geq \min \{M(gx, gy, t), M(fx, gx, t), M(fy, gy, t)\}, \end{aligned}$$

and hence (2.1) is obviously satisfied.

Case (3) If $x \in \{2\} \cup (5, 20]$ and $y \in (2, 5]$, then we have

$$M(fx, fy, kt) = M(2, 6, kt) = e^{-\frac{4}{kt}}$$

and

$$M(gx, gy, t) = \begin{cases} e^{-\frac{|2-18|}{t}}, & x = 2, \\ e^{-\frac{|\frac{x+1}{3}-18|}{t}}, & x \in (5, 20]. \end{cases}.$$

Thus we obtain

$$M(fx, fy, t) \geq \min \{M(gx, gy, t), M(fx, gx, t), M(fy, gy, t)\},$$

for all x, y in X . Thus all the conditions of Theorem 2.1 are satisfied and 2 is a unique common fixed point of f and g .

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On hesitant fuzzy filters in BE -algebras

Young Bae Jun¹ and Sun Shin Ahn^{2,*}

¹*Department of Mathematics Education, Gyeongsang National University, Jinju 52828, Korea*

¹*Department of Mathematics Education, Dongguk University, Seoul 04620, Korea*

Abstract. The notions of hesitant fuzzy subalgebras and hesitant fuzzy filters are introduced and related properties are investigated. Relations between a hesitant fuzzy subalgebras and a hesitant fuzzy filters are discussed. The problem of classifying hesitant fuzzy filters by their γ -inclusive filter will be solved. Given a special set, we provide conditions for this set to be a hesitant fuzzy filter.

1. Introduction

In 2007, Kim and Kim [4] introduced the notion of a BE -algebra, and investigated several properties. In [1], Ahn and So introduced the notion of ideals in BE -algebras. They gave several descriptions of ideals in BE -algebras. Song et al. [7] considered the fuzzification of ideals in BE -algebras. They introduced the notion of fuzzy ideals in BE -algebras, and investigated related properties. They gave characterizations of a fuzzy ideal in BE -algebras.

The notions of Atanassov's intuitionistic fuzzy sets, type 2 fuzzy sets and fuzzy multisets etc. are a generalization of fuzzy sets. As another generalization of fuzzy sets, Torra [8] introduced the notion of hesitant fuzzy sets which are a very useful to express peoples hesitancy in daily life. The hesitant fuzzy set is a very useful tool to deal with uncertainty, which can be accurately and perfectly described in terms of the opinions of decision makers. Also, hesitant fuzzy set theory is used in decision making problem etc. (see [6, 10, 11, 12, 13, 14]), and is applied to residuated lattices and MTL -algebras (see [3, 5]).

In this paper, we introduce the notions of hesitant fuzzy subalgebras and hesitant fuzzy filters of BE -algebras, and investigate their relations and properties. We consider characterizations of hesitant fuzzy subalgebras and hesitant fuzzy filters of BE -algebras. Given a special set, we provide conditions for this set to be a hesitant fuzzy filter. Given a special set, we provide conditions for this set to be a hesitant fuzzy filter.

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* The corresponding author. Tel.: +82 2 2260 3410, Fax: +82 2 2266 3409 (S. S. Ahn).

⁰**E-mail:** skywine@gmail.com (Y. B. Jun); sunshine@dongguk.edu (S. S. Ahn).

2. Preliminaries

Let $K(\tau)$ be the class of all algebras of type $\tau = (2, 0)$. By a *BE-algebra* we mean a system $(X; *, 1) \in K(\tau)$ in which the following axioms hold (see [4]):

$$(\forall x \in X) (x * x = 1), \tag{2.1}$$

$$(\forall x \in X) (x * 1 = 1), \tag{2.2}$$

$$(\forall x \in X) (1 * x = x), \tag{2.3}$$

$$(\forall x, y, z \in X) (x * (y * z) = y * (x * z)). \quad (\text{exchange}) \tag{2.4}$$

A relation “ \leq ” on a BE-algebra X is defined by

$$(\forall x, y \in X) (x \leq y \iff x * y = 1). \tag{2.5}$$

A BE-algebra $(X; *, 1)$ is said to be *transitive* (see [1]) if it satisfies:

$$(\forall x, y, z \in X) (y * z \leq (x * y) * (x * z)). \tag{2.6}$$

A BE-algebra $(X; *, 1)$ is said to be *self distributive* (see [4]) if it satisfies:

$$(\forall x, y, z \in X) (x * (y * z) = (x * y) * (x * z)). \tag{2.7}$$

Every self distributive BE-algebra $(X; *, 1)$ satisfies the following properties:

$$(\forall x, y, z \in X) (x \leq y \Rightarrow z * x \leq z * y \text{ and } y * z \leq x * z), \tag{2.8}$$

$$(\forall x, y \in X) (x * (x * y) = x * y), \tag{2.9}$$

$$(\forall x, y, z \in X) (x * y \leq (z * x) * (z * y)). \tag{2.10}$$

Note that every self distributive BE-algebra is transitive, but the converse is not true in general (see [1]).

Definition 2.1. ([4]) Let $(X; *, 1)$ be a BE-algebra and let F be a non-empty subset of X . Then F is a *filter* of X if

$$(F1) \quad 1 \in F;$$

$$(F2) \quad (\forall x, y \in X) (x * y, x \in F \Rightarrow y \in F).$$

3. Hesitant fuzzy filters

Definition 3.1. ([8]) Let E be a reference set. A *hesitant fuzzy set* on E is defined in terms of a function that when applied to E returns a subset of $[0, 1]$, which can be viewed as the following mathematical representation:

$$H_E := \{(e, h_E(e)) | e \in E\}$$

where $h_E : E \rightarrow \mathcal{P}([0, 1])$.

Definition 3.2. Given a non-empty subset A of X , a hesitant fuzzy set

$$H_X := \{(x, h_X(x)) | x \in X\}$$

on satisfying the following condition:

$$h_X(x) = \emptyset \text{ for all } x \notin A \tag{3.1}$$

is called a *hesitant fuzzy set related to A* (briefly, *A -hesitant fuzzy set*) on X , and is represented by $H_A := \{(x, h_A(x)) | x \in X\}$, where h_A is a mapping from X to $\mathcal{P}([0, 1])$ with $h_A(x) = \emptyset$ for all $x \notin A$.

Definition 3.3. Given a non-empty subset (subalgebra as much as possible) A of X , let $H_A := \{(x, h_A(x)) | x \in X\}$ be an A -hesitant fuzzy set on X . Then $H_A := \{(x, h_A(x)) | x \in X\}$ is called a *hesitant fuzzy subalgebra of X related to A* (briefly, *A -hesitant fuzzy subalgebra of X*) if it satisfies the following condition:

$$(\forall x, y \in A) (h_A(x * y) \supseteq h_A(x) \cap h_A(y)). \tag{3.2}$$

An A -hesitant fuzzy subalgebra of X with $A = X$ is called a *hesitant fuzzy subalgebra of X* .

Example 3.4. Let $X = \{0, 1, a, b, c\}$ be a *BE*-algebra with the following Cayley table:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	a
b	1	1	1	a
c	1	1	a	1

For a subalgebra $A = \{1, a, b\}$ of X , let $H_A := \{(x, h_A(x)) | x \in X\}$ be an A -hesitant fuzzy set on X defined by

$$H_A = \{(1, [0, 1]), (a, (0, \frac{1}{2}]), (b, (\frac{1}{4}, \frac{3}{4}]), (c, \emptyset)\}$$

Then H_A is an A -hesitant fuzzy subalgebra of X .

Definition 3.5. Given a non-empty subset (subalgebra as much as possible) A of X , let $H_A := \{(x, h_A(x)) | x \in X\}$ be an A -hesitant fuzzy set on X . Then $H_A := \{(x, h_A(x)) | x \in X\}$ is called a *hesitant fuzzy filter of X related to A* (briefly, *A -hesitant fuzzy filter of X*) if it satisfies the following condition:

$$(\forall x \in A) (h_A(x) \subseteq h_A(1)), \tag{3.3}$$

$$(\forall x, y \in A) (h_A(x * y) \cap h_A(x) \subseteq h_A(y)). \tag{3.4}$$

An A -hesitant fuzzy filter of X with $A = X$ is called a *hesitant fuzzy filter of X* .

Example 3.6. (1) Consider a *BE*-algebra $X = \{1, a, b, c\}$ as in Example 3.4. Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by

$$H_X = \{(1, [0, 1]), (a, (0, \frac{1}{8})), (b, (\frac{1}{4}, \frac{3}{4})), (c, (0, \frac{1}{4}))\}$$

Then H_X is a hesitant fuzzy subalgebra of X , but not a hesitant fuzzy filter of X since $h_A(b * a) \cap h_A(b) = h_A(1) \cap h_A(b) = [0, 1] \cap (\frac{1}{4}, \frac{3}{4}) \not\subseteq h_A(a) = (0, \frac{1}{8})$.

(2) Let $X = \{0, 1, a, b, c\}$ be a *BE*-algebra with the following Cayley table:

*	1	a	b	c
1	1	a	b	c
a	1	1	a	a
b	1	1	1	a
c	1	a	a	1

Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set defined by

$$H_X = \{(1, [0, 1]), (a, (0, \frac{1}{4})), (b, (0, \frac{1}{4})), (c, (0, \frac{1}{2}))\}$$

It is routine to verify that $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy filter of X .

Proposition 3.7. Let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be an *A*-hesitant fuzzy filter of X where A is a subalgebra of X . Then the following assertions are valid.

- (i) $(\forall x, y \in A)(x \leq y \Rightarrow h_A(x) \subseteq h_A(y))$,
- (ii) $(\forall x, y, z \in A)(h_A(x * (y * z)) \cap h_A(y) \subseteq h_A(x * z))$,
- (iii) $(\forall a, x \in A)(h_A(a) \subseteq h_A((a * x) * x))$.

Proof. Let $x, y \in A$ be such that $x \leq y$. Then $x * y = 1$. It follows from (3.3) and (3.4) that $h_A(x) = h_A(1) \cap h_A(x) = h_A(x * y) \cap h_A(x) \subseteq h_A(y)$.

(ii) Using (3.4) and (2.4), we have $h_A(x * z) \supseteq h_A(y * (x * z)) \cap h_A(y) = h_A(x * (y * z)) \cap h_A(y)$ for all $x, y, z \in A$.

(iii) Take $y := (a * x) * x$ and $x := a$ in (3.4). Then we have

$$\begin{aligned} h_A((a * x) * x) &\supseteq h_A(a * ((a * x) * x)) \cap h_A(a) \\ &= h_A((a * x) * (a * x)) \cap h_A(a) \\ &= h_A(1) \cap h_A(a) = h_A(a) \end{aligned}$$

by using (2.4), (2.1) and (3.3). □

Corollary 3.8. Every hesitant fuzzy filter $H_X := \{(x, h_X(x)) \mid x \in X\}$ of X satisfies the following properties:

- (i) $(\forall x, y \in X)(x \leq y \Rightarrow h_X(x) \subseteq h_X(y))$,
- (ii) $(\forall x, y, z \in X)(h_X(x * (y * z)) \cap h_X(y) \subseteq h_X(x * z))$,

(iii) $(\forall a, x \in X)(h_A(a) \subseteq h_A((a * x) * x))$.

We provide conditions for a hesitant fuzzy set to be a hesitant filter.

Theorem 3.9. *Let A be a subalgebra of a BE -algebra X . Every A -hesitant fuzzy set satisfies (3.3) and Proposition 3.7(ii). Then it is an A -hesitant fuzzy filter of X .*

Proof. Taking $x := 1$ in Proposition 3.7(ii) and using (2.3), we obtain $h_A(z) = h_A(1 * z) \supseteq h_A(1 * (y * z)) \cap h_A(y) = h_A(y * z) \cap h_A(y)$ for all $y, z \in A$. Hence $H_A = \{(x, h_A(x)) | x \in X\}$ is an A -hesitant fuzzy filter of X . \square

Corollary 3.10. *Let $H_A := \{(x, h_A(x)) | x \in X\}$ be an A -hesitant fuzzy set for a subalgebra A of X . Then h_A is an A -hesitant fuzzy filter of X if and only if it satisfies (3.3) and Proposition 3.7(ii).*

Theorem 3.11. *An hesitant fuzzy set H_A of X , where A is a subalgebra of X , is an A -hesitant fuzzy filter of X if and only if it satisfies the following conditions:*

- (i) $(\forall x, y \in A)(h_A(y * x) \supseteq h_A(x))$,
- (ii) $(\forall x, a, b \in A)(h_A((a * (b * x)) * x) \supseteq h_A(a) \cap h_A(b))$.

Proof. Assume that $H_A := \{(x, h_A(x)) | x \in X\}$ is an A -hesitant fuzzy filter of X . Using (3.3), (3.4), (2.4), (2.1) and (2.2), we get $h_A(y * x) \supseteq h_A(x * (y * x)) \cap h_A(x) = h_A(1) \cap h_A(x) = h_A(x)$ for all $x, y \in A$. It follows from Proposition 3.7 that $h_A((a * (b * x)) * x) \supseteq h_A((a * (b * x)) * (b * x)) \cap h_A(b) \supseteq h_A(a) \cap h_A(b)$ for all $x, a, b \in X$.

Conversely, let $H_A(X) = \{(x, h_A(x)) | x \in A\}$ be an A -hesitant fuzzy set of X satisfying conditions (i) and (ii). If we take $y := x$ in (i), then $h_A(1) = h_A(x * x) \supseteq h_A(x)$ for all $x \in A$. Using (ii), we obtain $h_A(y) = h_A(1 * y) = h_A(((x * y) * (x * y)) * y) \supseteq h_A(x * y) \cap h_A(x)$ for all $x, y \in A$. Hence H_A is an A -hesitant fuzzy filter of X . \square

Proposition 3.12. *Let $H_X := \{(x, h_X(x)) | x \in X\}$ be a hesitant fuzzy set on X . Then H_X is a hesitant fuzzy filter of X if and only if*

$$(\forall x, y, z \in X)(z \leq x * y \Rightarrow h_X(y) \supseteq h_X(x) \cap h_X(z)). \tag{3.5}$$

Proof. Assume that H_X is a hesitant fuzzy filter of X . Let $x, y, z \in X$ be such that $z \leq x * y$. By Proposition 3.7 and Definition 3.5, we have $h_X(y) \supseteq h_X(x * y) \cap h_X(x) \supseteq h_X(z) \cap h_X(x)$.

Conversely, suppose that H_X satisfies (3.5). By (2.2), we have $x \leq x * 1 = 1$. Using (3.5), we have $h_X(1) \supseteq h_X(x)$ for all $x \in X$. It follows from (2.1) and (2.4) that $x \leq (x * y) * y$ for all $x, y \in X$. Using (3.5), we have $h_X(y) \supseteq h_X(x * y) \cap h_X(x)$. Therefore H_X is a hesitant fuzzy filter of X . \square

As a generalization of Proposition 3.12, we have the following results.

Theorem 3.13. *Let $H_X := \{(x, h_X(x)|x \in X)\}$ be a hesitant fuzzy filter of X . Then*

$$\prod_{i=1}^n w_i * x = 1 \Rightarrow h_X(x) \supseteq \bigcap_{i=1}^n h_X(w_i) \tag{3.6}$$

for all $x, w_1, \dots, w_n \in X$, where $\prod_{i=1}^n w_i * x = w_n * (w_{n-1} * (\dots w_1 * x) \dots)$.

Proof. The proof is by induction on n . Let H_X be a hesitant fuzzy filter of X . By Proposition 3.7(i) and (3.6), we know that the condition (3.6) is true for $n = 1, 2$. Assume that H_X satisfies the condition (3.6) for $n = k$, i.e., $\prod_{i=1}^k w_i * x = 1 \Rightarrow \bigcap_{i=1}^k h_X(w_i)$ for all $x, w_1, \dots, w_k \in X$. Suppose that $\prod_{i=1}^{k+1} w_i * x = 1$ for all $x, w_1, \dots, w_k, w_{k+1} \in X$. Then

$$h_X(w_1 * x) \supseteq \bigcap_{i=2}^{k+1} h_X(w_i).$$

Since H_X is a hesitant fuzzy filter of X , it follows from (3.4) that

$$\begin{aligned} h_X(x) &\supseteq h_X(w_1 * x) \cap h_X(w_1) \\ &\supseteq (\bigcap_{i=2}^{k+1} h_X(w_i)) \cap h_X(w_1) \\ &= \bigcap_{i=1}^{k+1} h_X(w_i). \end{aligned}$$

This completes the proof. □

Theorem 3.14. *Let $H_X = \{(x, h_X(x)|x \in X)\}$ be a hesitant fuzzy set of a BE-algebra satisfying (3.6). Then H_X is a hesitant fuzzy filter of X .*

Proof. Let $x, y, z \in X$ be such that $z \leq x * y$. Then $z * (x * y) = 1$ and so $h_X(y) \supseteq h_X(x) \cap h_X(z)$ by (3.6). Using Proposition 3.12, H_X is a hesitant fuzzy filter of X . □

Theorem 3.15. *A hesitant fuzzy set $H_X := \{(x, h_X(x)|x \in X)\}$ of a BE-algebra X is a hesitant fuzzy filter of X if and only if the set $H_X(\gamma) := \{x \in X|h_X(x) \supseteq \gamma\}$ is a filter of X for all $\gamma \in \mathcal{P}([0, 1])$ whenever it is nonempty.*

Proof. Suppose that H_X is a hesitant fuzzy filter of X . Let $x, y \in X$ and $\gamma \in \mathcal{P}([0, 1])$ be such that $x * y \in H_X(\gamma)$ and $x \in H_X(\gamma)$. Then $h_X(x * y) \supseteq \gamma$ and $h_X(x) \supseteq \gamma$. It follows from (3.3) and (3.4) that $h_X(1) \supseteq h_X(y) \supseteq h_X(x * y) \cap h_X(x) \supseteq \gamma$. Hence $1 \in H_X(\gamma)$ and $y \in H_X(\gamma)$, and therefore $H_X(\gamma)$ is a filter of X .

Conversely, assume that $H_X(\gamma)$ is a filter of X for all $\gamma \in \mathcal{P}([0, 1])$ with $H_X(\gamma) \neq \emptyset$. For any $x \in X$, let $h_X(x) = \gamma$. Then $x \in H_X(\gamma)$. Since $H_X(\gamma)$ is a filter of X , we have $1 \in h_X(\gamma)$ and so $h_X(x) = \gamma \subseteq h_X(1)$. For any $x, y \in X$, let $h_X(x * y) = \gamma_{x*y}$ and $h_X(x) = \gamma_x$. Take $x * y \in H_X(\gamma)$ and $x \in H_X(\gamma)$ which imply that $y \in H_X(\gamma)$. Hence $h_X(y) \supseteq \gamma = \gamma_{x*y} \cap \gamma_x = h_X(x * y) \cap h_X(x)$. Thus H_X is a hesitant fuzzy filter of X . □

The filter $H_X(\gamma)$ in Theorem 3.15 is called the *hesitant γ -inclusive set* of $H_X := \{(x, h_X(x)) | x \in X\}$.

We make a new hesitant fuzzy filter from old one.

Theorem 3.16. *Let $H_X := \{(x, h_X(x)) | x \in X\}$ be a hesitant fuzzy set on a BE-algebra X . Define a hesitant fuzzy set H_X^* on X by*

$$h_X^* : X \rightarrow \mathcal{P}([0, 1]), \quad x \mapsto \begin{cases} h_X(x) & \text{if } x \in H_X(\gamma) \\ \delta & \text{otherwise} \end{cases}$$

where γ is any subset of $[0, 1]$ and δ is a subset of $[0, 1]$ satisfying $\delta \subsetneq \cap_{x \notin H_X(\gamma)} h_X(x)$. If H_X is a hesitant fuzzy filter of X , then so is H_X^* .

Proof. Assume that H_X is a hesitant fuzzy filter of X . Then $H_X(\gamma)$ is a filter of X for all $\gamma \in \mathcal{P}([0, 1])$ by Theorem 3.15. Hence $1 \in H_X(\gamma)$ and so $h_X^*(1) = h_X(1) \supseteq h_X(x) \supseteq h_X^*(x)$ for all $x \in X$. Let $x, y \in X$. If $x * y \in H_X(\gamma)$ and $x \in H_X(\gamma)$, then $y \in H_X(\gamma)$. Hence $h_X^*(y) = h_X(y) \supseteq h_X(x * y) \cap h_X(x) = h_X^*(x * y) \cap h_X^*(x)$. If $x * y \notin H_X(\gamma)$ or $x \notin H_X(\gamma)$, then $h_X^*(x * y) = \delta$ or $h_X^*(x) = \delta$. Thus $h_X^*(y) \supseteq \delta = h_X^*(x * y) \cap h_X^*(x)$. Therefore H_X^* is a hesitant fuzzy filter of X . □

For two elements a and B of X , consider a hesitant fuzzy set $H_X^{a,b} = \{(x, h_X(x)) | x \in X\}$ where

$$h_X^{a,b} : X \rightarrow \mathcal{P}([0, 1]), \quad x \mapsto \begin{cases} \gamma_1 & \text{if } a * (b * x) = 1 \\ \gamma_2 & \text{otherwise} \end{cases}$$

where γ_1 and γ_2 are subsets of $[0, 1]$ with $\gamma_2 \subsetneq \gamma_1$. In the following example, we know that there exist $a, b \in X$ such that $H_X^{a,b}$ is not a hesitant fuzzy filter of X .

Example 3.17. Let $X = \{0, 1, a, b, c\}$ be a BE-algebra with the following Cayley table:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	c
b	1	1	1	c
c	1	a	b	1

Let $H_X := \{(x, h_X(x)) | x \in X\}$ be a hesitant fuzzy set defined by

$$H_X = \left\{ (1, [0, 1]), (a, (0, \frac{1}{4})), (b, (\frac{1}{4}, \frac{3}{4})), (c, (\frac{6}{8}, \frac{7}{8})) \right\}$$

Then $H_X^{1,a}$ is not a hesitant fuzzy filter of X since $h_X^{1,a}(a * b) \cap h_X^{1,a}(a) = [0, 1] \not\subseteq h_X^{1,a}(b) = (0, \frac{1}{4})$.

Now we provide a condition for the hesitant fuzzy set $H_X^{a,b}$ to be a hesitant fuzzy filter of X for all $a, b \in X$.

Theorem 3.18. *If X is a self distributive BE-algebra, then the hesitant fuzzy set $H_X^{a,b}$ is a hesitant fuzzy filter of X for all $a, b \in X$.*

Proof. Let $a, b \in X$. Obviously, $h_X^{a,b}(1) \supseteq h_X^{a,b}(x)$ for all $x \in X$. Let $x, y \in X$ be such that $a * (b * (x * y)) \neq 1$ or $a * (b * x) \neq 1$. Then $h_X^{a,b}(x * y) = \gamma_2$ or $h_X^{a,b}(x) = \gamma_2$. Hence $h_X^{a,b}(x * y) \cap h_X^{a,b}(x) = \gamma_2 \subseteq h_X^{a,b}(y)$. Assume that $a * (b * (x * y)) = 1$ and $a * (b * x) = 1$. Then

$$\begin{aligned} 1 &= a * (b * (x * y)) \\ &= a * ((b * x) * (b * y)) \\ &= (a * (b * x)) * (a * (b * y)) \\ &= 1 * (a * (b * y)) \\ &= a * (b * y) \end{aligned}$$

and so $h_X^{a,b}(x * y) \cap h_X^{a,b}(x) = \gamma_1 = h_X^{a,b}(y)$. Therefore $H_X^{a,b}$ is a hesitant fuzzy filter of X for all $a, b \in X$. □

Theorem 3.19. *Every filter of a BE-algebra can be represented as γ -inclusive set of a hesitant fuzzy filter.*

Proof. Let F be a filter of a BE-algebra X . For a subset γ of $[0, 1]$, define a hesitant set H_X by

$$h_X : X \rightarrow \mathcal{P}([0, 1]), \quad x \mapsto \begin{cases} \gamma & \text{if } x \in F \\ \emptyset & \text{if } x \notin F \end{cases}$$

Obviously, $F = H_X(\gamma)$. We now prove that H_X is a hesitant fuzzy filter of X . Since $1 \in H_X(\gamma)$, we have $H_X(1) = \gamma \supseteq h_X(x)$ for all $x \in X$. Let $x, y \in X$. If $x * y, x \in F$, then $y \in F$ since F is a filter of X . Hence $h_X(x * y) = h_X(x) = h_X(y) = \gamma$ and so $h_X(x * y) \cap h_X(x) \subseteq h_X(y)$. If $x * y \in F$ and $x \notin F$, then $h_X(x * y) = \gamma$ and $h_X(x) = \emptyset$ which imply that $h_X(x * y) \cap h_X(x) = \gamma \cap \emptyset = \emptyset \subseteq h_X(y)$. Similarly, if $x * y \notin F$ and $x \in F$, then $h_X(x * y) \cap h_X(x) \subseteq h_X(y)$. Obviously, if $x * y \notin F$ and $x \notin F$, then $h_X(x * y) \cap h_X(x) \subseteq h_X(y)$. Therefore H_X is a hesitant fuzzy filter of X . □

Let $H_X = \{(x, h(x)) | x \in X\}$ be a hesitant fuzzy set on X . For any $a, b \in X$ and $k \in \mathbb{N}$, consider the set

$$h_X[a^k; b] := \{x \in X | h_X(a^k * (b * x)) = h_X(1)\}$$

where $h_X(a * (a * (\dots * (a * (a * x)) \dots)))$ in which a appears k -times. Note that $1, a, b \in H_X[a^k; b]$ for all $a, b \in X$ and $k \in \mathbb{N}$.

Proposition 3.20. *Let $H_X := \{(x, h_x(x)) | x \in X\}$ be a hesitant fuzzy set on X satisfying (3.3) and $h_X(x * y) = h_X(x) \cup h_X(y)$ for all $x, y \in X$. For any $a, b \in X$ and $k \in \mathbb{N}$, if $x \in h_X[a^k; b]$, then $y * x \in h_X[a^k; b]$ for all $y \in X$.*

Proof. Assume that $x \in h_X[a^k; b]$. Then $h_X(a^k * (b * x)) = h_X(1)$ and so

$$\begin{aligned} h_X(a^k * (b * (y * x))) &= h_X(a^k * (y * b * x)) \\ &= h_X(y * (a^k * (b * x))) \\ &= h_X(y) \cup h_X(a^k * (b * x)) \\ &= h_X(y) \cup h_X(1) = h_X(1) \end{aligned}$$

for all $y \in X$ by (2.4). Hence $y * x \in h_X[a^k; b]$ for all $y \in X$. □

Proposition 3.21. *Let $H_X := \{(x, h_X)\} | x \in X\}$ be a hesitant fuzzy set on a BE-algebra X . If an element $a \in X$ satisfies $a * x = 1$ for all $x \in X$, then $h_X[a^k; b] = X = [b^k; a]$ for all $b \in X$ and $k \in \mathbb{N}$.*

Proof. For any $x \in X$, we have

$$\begin{aligned} h_X(a^k * (b * x)) &= h_X(a^{k-1} * (a * (b * x))) \\ &= h_X(a^{k-1} * (b * (a * x))) \\ &= h_X(a^{k-1} * (b * 1)) \\ &= h_X(1), \end{aligned}$$

and so $x \in h_X[a^k; b]$. Similarly, $x \in h_X[b^k; a]$. □

Proposition 3.22. *Let X be a self distributive BE-algebra and let $H_X := \{(x, h_X(x)) | x \in X\}$ be a order reversing hesitant fuzzy set of X with the property (3.3). If $b \leq c$ in X , then $h_X[a^k; c] \subseteq h_X[a^k; b]$ for all $a \in X$ and $k \in \mathbb{N}$.*

Proof. Let $a, b, c \in X$ be such that $b \leq c$. For any $k \in \mathbb{N}$, if $x \in h_X[a^k; c]$, then

$$\begin{aligned} h_X(1) &= h_X(a^k * (c * x)) \\ &= h_X(c * (a^k * x)) \\ &\subseteq h_X(b * (a^k * x)) \\ &= h_X(a^k * (b * x)) \end{aligned}$$

by (2.4) and (2.8). Hence $h_X(a^k * (b * x)) = h_X(1)$. Thus $x \in h_X[a^k; b]$, which completes the proof. □

The following example shows that there exists a hesitant fuzzy set H_X of X , $a, b \in X$ and $k \in \mathbb{N}$ such that $h_X[a^k; b]$ is not a filter of X .

Example 3.23. Let $X = \{0, 1, a, b, c\}$ be a BE-algebra with the following Cayley table:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	a
b	1	1	1	a
c	1	a	a	1

Let $H_X := \{(x, h_X(x)) | x \in X\}$ be a hesitant fuzzy set defined by

$$H_X = \{(1, [0, 1]), (a, (\frac{1}{4}, \frac{3}{4})), (b, (\frac{3}{4}, \frac{1}{2})), (c, (\frac{6}{8}, \frac{7}{8}))\}$$

Then $h_X[c; b] = \{x \in X | h_X(c * (b * x)) = h_X(1)\} = \{1, a, b\}$ is not a filter, since $a * c = a \in h_X[c; b]$ and $c \notin h_X[c; b]$.

Theorem 3.24. $H_X := \{(x, h_X(x)) | x \in X\}$ be a hesitant fuzzy set on a self distributive BE-algebra X in which h_X is injective. Then $h_X[a^k; b]$ is a filter of X for all $a, b \in X$ and $k \in \mathbb{N}$.

Proof. Assume that X is a self distributive BE-algebra and h_X is injective. Obviously, $1 \in h_X[a^k; b]$. Let $a, b, x, y \in X$ and $k \in \mathbb{N}$ be such that $x * y \in h_X[a^k; b]$ and $x \in h_X[a^k; b]$. Then $h_X(a^k * (b * x)) = h_X(1)$ which implies that $a^k * (b * x) = 1$, since h_X is injective. Using (2.7), we have

$$\begin{aligned} h_X(1) &= h_X(a^k * (b * (x * y))) \\ &= h_X(a^{k-1} * (a * (b * (x * y)))) \\ &= h_X(a^{k-1} * (a * ((b * x) * (b * y)))) \\ &= \dots \\ &= h_X((a^k * (b * x)) * (a^k * (b * y))) \\ &= h_X(1 * (a^k * (b * y))) \\ &= h_X(a^k * (b * y)) \end{aligned}$$

which imply that $y \in h_X[a^k; b]$. Therefore $h_X[a^k; b]$ is a filter of X for all $a, b \in X$ and $k \in \mathbb{N}$. \square

Theorem 3.25. $H_X := \{(x, h_X(x)) | x \in X\}$ be a hesitant fuzzy set of a self distributive BE-algebra X satisfying the condition (3.3) and $h_X(x * y) = h_X(x) \cap h_X(y)$, for all $x, y \in X$. Then $h_X[a^k; b]$ is a filter of X for all $a, b \in X$ and $k \in \mathbb{N}$.

Proof. Let $a, b \in X$ and $k \in \mathbb{N}$. Obviously, $1 \in h_X[a^k; b]$. Let $x, y \in X$ be such that $x * y \in h_X[a^k; b]$ and $x \in h_X[a^k; b]$. Then $h_X(a^k * (b * (x * y))) = h_X(1)$ and $h_X(a^k * (b * x)) = h_X(1)$, which implies

from the hypothesis that

$$\begin{aligned}
 h_X(1) &= h_X(a^k * (b * (x * y))) \\
 &= h_X(a^{k-1} * (a * (b * (x * y)))) \\
 &= h_X(a^{k-1} * (a * ((b * x) * (b * y)))) \\
 &= \dots \\
 &= h_X((a^k * (b * x)) * (a^k * (b * y))) \\
 &= h_X(a^k * (b * x)) \cap h_X(a^k * (b * y)) \\
 &= h_X(1) \cap h_X(a^k * (b * y)) \\
 &= h_X(a^k * (b * y)).
 \end{aligned}$$

Hence $y \in h_X[a^k; b]$ and therefore $h_X[a^k; b]$ is a filter of X for all $a, b \in X$ and $k \in \mathbb{N}$. □

Proposition 3.26. $H_X := \{(x, h_X(x)) | x \in X\}$ be a hesitant fuzzy set of a BE-algebra X in which h_X is injective. If F is a filter of X , then the following holds.

$$(\forall a, b \in F)(\forall k \in \mathbb{N})(h_X[a^k; b] \subseteq F). \tag{3.7}$$

Proof. Assume that F is a filter of X and let $a, b \in F$ and $k \in \mathbb{N}$. If $x \in h_X[a^k; b]$, then $h_X(a * (a^{k-1} * (b * x))) = h_X(a^k * (b * x)) = h_X(1)$ and so $a * (a^{k-1} * (b * x)) = 1 \in F$ since h_X is injective. Since F is a filter of X , it follows from (F2) that $a^{k-1} * (b * x) \in F$. Continuing this process, we obtain $b * x \in F$ and so $x \in F$. Therefore $h_X[a^k; b] \subseteq F$ for all $a, b \in F$ and $k \in \mathbb{N}$. □

Theorem 2.27. $H_X := \{(x, h_X(x)) | x \in X\}$ be a hesitant fuzzy set of a BE-algebra X . For any subset F of X , if the condition (3.7) holds, then F is a filter of X .

Proof. Suppose that the condition (3.7) holds. Obviously, $1 \in h_X[a^k; b] \subseteq F$. Let $x, y \in X$ be such that $x * y \in F$ and $x \in F$. Then

$$\begin{aligned}
 h_X(x^k * ((x * y) * y)) &= h_X(x^{k-1} * (x * ((x * y) * y))) \\
 &= h_X(x^{k-1} * ((x * y) * (x * y))) \\
 &= h_X(x^{k-1} * 1) = h_X(1)
 \end{aligned}$$

and hence $y \in h_X[a^k; b] \subseteq F$, where $b = x * y$. Therefore F is a filter of X . □

Theorem 3.28. $H_X := \{(x, h_X(x)) | x \in X\}$ be a hesitant fuzzy set of a BE-algebra X . If F is a filter of X , then

$$(\forall k \in \mathbb{N})(F = \cup\{h_X[a^k; b] | a, b \in F\}).$$

Proof. Let F is a filter of X . By Proposition 3.26, the inclusion $\cup\{h_X[a^k; b] | a, b \in F\} \subseteq F$ holds. Let $x \in F$. Since $x \in h_X[1^k; x]$ for all $k \in \mathbb{N}$, it follows that

$$F \subseteq \cup \{h_X[1^k; x] | x \in F\} \\ \subseteq \cup \{h_X[a^k; b] | a, b \in F\}.$$

This completes the proof. □

Theorem 3.29. If $H_X := \{(x, h_X(x)) | x \in X\}$ is a hesitant filter of X , then the set

$$H_a := \{x \in X | h_X(a) \subseteq h_X(x)\}$$

is a filter of X for all $a \in X$.

Proof. Let $x, y \in X$ be such that $x * y \in H_a$ and $x \in H_a$. Then $h_X(a) \subseteq h_X(x * y)$ and $h_X(a) \subseteq h_X(y)$. By (3.3) and (3.4), we have $h_X(a) \subseteq h_X(x * y) \cap h_X(x) \subseteq H_X(y) \subseteq h_X(1)$ and so $1 \in H_a$ and $y \in H_a$. Therefore H_a is a filter of X . □

Theorem 3.30. Let $a \in X$ and $H_X := \{(x, h_X(x)) | x \in X\}$ be a hesitant fuzzy set on X . Then the following properties are valid:

(i) if H_a is a filter of X , then $H_X := \{(x, h_X(x)) | x \in X\}$ satisfies:

$$(\forall x, y \in X)(h_X(a) \subseteq h_X(x * y) \cap h_X(x) \Rightarrow h_X(a) \subseteq h_X(y)). \tag{3.8}$$

(ii) if $H_X := \{(x, h_X(x)) | x \in X\}$ satisfies the condition (3.3) and (3.8), then H_a is a filter of X .

Proof. (i) Assume that H_a is a filter of X and let $x, y \in X$ be such that $h_X(a) \subseteq H_X(x * y) \cap H_X(x)$. Then $x * y \in H_a$ and $y \in H_a$. Since H_a is a filter of X , we obtain $x \in H_a$. Therefore $h_X(a) \subseteq h_X(y)$.

(ii) Let $H_X := \{(x, h_X(x)) | x \in X\}$ be a hesitant fuzzy set on X in which the conditions (3.3) and (3.8) hold. Then $1 \in H_a$. Let $x, y \in X$ be such that $x * y \in H_a$ and $x \in H_a$. Then $h_X(a) \subseteq h_X(x * y)$ and $h_X(a) \subseteq h_X(x)$. Hence $H_X(a) \subseteq h_X(x * y) \cap h_X(x)$. Using (3.8), we have $h_X(a) \subseteq h_X(y)$, i.e., $y \in H_a$. Thus H_a is a filter of X . □

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A fixed point approach to the stability of nonic functional equation in non-Archimedean spaces*

Tian-Zhou Xu^{1,†}, Yali Ding¹, John Michael Rassias²

(1. School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, P. R. China)

(2. Pedagogical Department E.E., Section of Mathematics and Informatics, National and Capodistrian University of Athens, 4, Agamemnonos Str., Aghia Paraskevi, Athens 15342, Greece)

Abstract In this paper, a new nonic functional equation is introduced. The solution of this functional equation can also be determined in certain type of groups using two important results due to Székelyhidi. Using the fixed point theorems due to Brzdęk and Ciepliński, we give some Ulam–Hyers stability results for the nonic functional equation in non-Archimedean spaces.

Keywords Ulam–Hyers stability; nonic functional equation; non-Archimedean space; fixed point method.

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1 Introduction and preliminaries

In this paper \mathbb{R} and \mathbb{N} denote the sets of reals and positive integers, respectively. Moreover, $\mathbb{R}_+ := [0, \infty)$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

A valuation is a function $|\cdot|$ from a field \mathbb{K} into \mathbb{R}_+ such that 0 is the unique element having the 0 valuation, $|rs| = |r| \cdot |s|$ and the triangle inequality holds, i.e.,

$$|r + s| \leq |r| + |s|, \quad \forall r, s \in \mathbb{K}.$$

A field \mathbb{K} is called a valued field if \mathbb{K} carries a valuation. The usual absolute values of \mathbb{R} and \mathbb{C} are examples of valuations.

Let us consider a valuation which satisfies a stronger condition than the triangle inequality.

Let \mathbb{K} be a field. A non-Archimedean valuation on \mathbb{K} is a function $|\cdot| : \mathbb{K} \rightarrow \mathbb{R}$ such that

(1) $|r| \geq 0$ and equality holds if and only if $r = 0$.

(2) $|rs| = |r||s|$, $r, s \in \mathbb{K}$.

(3) $|r + s| \leq \max\{|r|, |s|\}$, $r, s \in \mathbb{K}$.

Any field endowed with a non-Archimedean valuation is said to be a non-Archimedean field. In any such field we have $|\mathbf{1}| = |-\mathbf{1}| = 1$ and $|n \times \mathbf{1}| \leq 1$ for all $n \in \mathbb{N}$, where $\mathbf{1}$ is the neutral element of the semigroup (\mathbb{K}, \cdot) , $1 \times \mathbf{1} = \mathbf{1}$ and $(n + 1) \times \mathbf{1} = (n \times \mathbf{1}) + \mathbf{1}$ for $n \in \mathbb{N}$.

Let X be a linear space over a field \mathbb{K} with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\| : X \rightarrow \mathbb{R}_+$ is a non-Archimedean norm if it satisfies the following conditions:

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†Corresponding author.

E-mail addresses: xutianzhou@bit.edu.cn (T.Z. Xu), dingyaliding@126.com (Y. Ding), jrassias@primedu.uoa.gr (J.M. Rassias).

(1) $\|x\| = 0$ if and only if $x = 0$;

(2) $\|rx\| = |r|\|x\|$ for all $r \in \mathbb{K}$ and $x \in X$;

(3) $\|x + y\| \leq \max\{\|x\|, \|y\|\}$ for all $x, y \in X$. If $\|\cdot\| : X \rightarrow \mathbb{R}_+$ is a non-Archimedean norm in X , then the pair $(X, \|\cdot\|)$ is called a non-Archimedean normed space. Then $(X, \|\cdot\|)$ is called a non-Archimedean normed space.

Let X be a non-Archimedean normed space. Let $\{x_n\}$ be a sequence in X . Then $\{x_n\}$ is said to be convergent if there exists $x \in X$ such that $\lim_{n \rightarrow \infty} \|x_n - x\| = 0$. In that case, x is called the limit of the sequence $\{x_n\}$ and we denote it by $\lim_{n \rightarrow \infty} x_n = x$. A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if $\lim_{n \rightarrow \infty} \|x_{n+p} - x_n\| = 0$ for all $p = 1, 2, \dots$. Due to the fact that

$$\|x_n - x_m\| \leq \max\{\|x_{j+1} - x_j\| : m \leq j \leq n - 1\} \quad (n > m)$$

a sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean normed space.

The most important examples of non-Archimedean spaces are p -adic numbers. The p -adic numbers have gained the interest of physicists because of their connections with some problems coming from quantum physics, p -adic strings and superstrings (see [15]).

In this paper, we first introduce the following new nonic functional equation

$$\begin{aligned} f(x + 5y) - 9f(x + 4y) + 36f(x + 3y) - 84f(x + 2y) + 126f(x + y) - 126f(x) + \\ 84f(x - y) - 36f(x - 2y) + 9f(x - 3y) - f(x - 4y) = 9!f(y). \end{aligned} \tag{1}$$

It is easy to see that the function $f(x) = ax^9$ is a solution of the functional equation (1). Every solution of the functional equation (1) is said to be a nonic mapping.

The study of stability problems for functional equations is related to a question of Ulam [20] concerning the stability of group homomorphisms and affirmatively answered for Banach spaces by Hyers [10]. The result of Hyers was generalized by Aoki [2] for approximate additive mappings and by Rassias [17] for approximate linear mappings by allowing the Cauchy difference operator $CDf(x, y) = f(x + y) - [f(x) + f(y)]$ to be controlled by $\epsilon(\|x\|^p + \|y\|^p)$. In 1994, a further generalization was obtained by Găvruta [7], who replaced $\epsilon(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x, y)$. We refer the reader to (see for instance [1, 3-6, 8, 11-14, 16, 18, 21, 22]) and references therein for more information on Ulam's problem during the last seventy years.

From now on S denotes a nonempty set and X stands for a complete non-Archimedean normed space. Given a set $Z \neq \emptyset$ and functions $\varphi : S \rightarrow S$ and $F : S \times Z \rightarrow Z$, we define an operator $\mathcal{L}_\varphi^F : Z^S \rightarrow Z^S$ (Z^S denotes the family of all functions mapping a set S into a set Z) by

$$\mathcal{L}_\varphi^F(\alpha)(t) := F(t, \alpha(\varphi(t))), \quad \alpha \in Z^S, t \in S.$$

Moreover, if $\Lambda : S \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$, then we write $\Lambda_t := \Lambda(t, \cdot), t \in S$.

For explicitly later use, we recall the following results by Brzdęk and Ciepliński [4].

Theorem 1 Let $\Lambda : S \times \mathbb{R}_+ \rightarrow \mathbb{R}_+, f : S \rightarrow X, \mathcal{T} : X^S \rightarrow X^S, \varphi : S \rightarrow S, \varepsilon : S \rightarrow \mathbb{R}_+$ and

$$\|\mathcal{T}(\alpha)(t) - \mathcal{T}(\beta)(t)\| \leq \Lambda(t, \|\alpha(\varphi(t)) - \beta(\varphi(t))\|), \quad \alpha, \beta \in X^S, t \in S. \tag{2}$$

Assume also that Λ_t is nondecreasing for every $t \in S, \lim_{n \rightarrow \infty} (\mathcal{L}_\varphi^\Lambda)^n(\varepsilon)(t) = 0(t \in S)$ holds and

$$\|\mathcal{T}(f)(t) - f(t)\| \leq \varepsilon(t), \quad t \in S. \tag{3}$$

Then for each $t \in S$ the limit

$$\lim_{n \rightarrow \infty} \mathcal{T}^n(f)(t) =: A(t) \tag{4}$$

exists and the function $A \in X^S$ is the unique fixed point of \mathcal{T} with

$$\|f(t) - A(t)\| \leq \sup_{n \in \mathbb{N}_0} (\mathcal{L}_\varphi^\Lambda)^n(\varepsilon)(t) =: h(t), \quad t \in S. \tag{5}$$

Corollary 1 Let $F : S \times X \rightarrow X$, $\varphi : S \rightarrow S$, $\Lambda : S \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $f : S \rightarrow X$, $\varepsilon : S \rightarrow \mathbb{R}_+$ and

$$\|F(t, x) - F(t, y)\| \leq \Lambda(t, \|x - y\|), \quad t \in S, x, y \in X. \tag{6}$$

Assume also that, for every $t \in S$, Λ_t is nondecreasing, $\lim_{n \rightarrow \infty} (\mathcal{L}_\varphi^\Lambda)^n(\varepsilon)(t) = 0 (t \in S)$ holds and

$$\|f(t) - F(t, f(\varphi(t)))\| \leq \varepsilon(t), \quad t \in S. \tag{7}$$

Then for each $t \in S$ the limit

$$\lim_{n \rightarrow \infty} (\mathcal{L}_\varphi^F)^n(f)(t) =: A(t) \tag{8}$$

exists and the function $A \in X^S$ is the unique solution of the functional equation

$$A(t) = F(t, A(\varphi(t))) \tag{9}$$

such that (5) holds.

We end this section with two corollaries, which are immediate consequences of Corollary 1.

Corollary 2 Let $a : S \rightarrow \mathbb{K} \setminus \{0\}$, $\varphi : S \rightarrow S$, $f : S \rightarrow X$, $\delta : S \rightarrow \mathbb{R}_+$,

$$\|f(\varphi(t)) - a(t)f(t)\| \leq \delta(t), \quad t \in S \tag{10}$$

and

$$\lim_{n \rightarrow \infty} \frac{\delta(\varphi^n(t))}{|\prod_{i=0}^n a(\varphi^i(t))|} = 0, \quad t \in S. \tag{11}$$

Then there exists a unique solution $A \in X^S$ of the functional equation

$$A(\varphi(t)) = a(t)A(t) \tag{12}$$

such that

$$\|f(t) - A(t)\| \leq \sup_{n \in \mathbb{N}_0} \frac{\delta(\varphi^n(t))}{|\prod_{i=0}^n a(\varphi^i(t))|}, \quad t \in S. \tag{13}$$

Corollary 3 Let $b : S \rightarrow \mathbb{K}$, $\psi : S \rightarrow S$, $f : S \rightarrow X$, $\varepsilon : S \rightarrow \mathbb{R}_+$,

$$\|f(t) - b(t)f(\psi(t))\| \leq \varepsilon(t), \quad t \in S \tag{14}$$

and

$$\lim_{n \rightarrow \infty} \left| \prod_{i=0}^n b(\psi^i(t)) \right| \varepsilon(\psi^{n+1}(t)) = 0, \quad t \in S. \tag{15}$$

Then there exists a unique solution $B \in X^S$ of the functional equation

$$B(t) = b(t)B(\psi(t)) \tag{16}$$

such that

$$\|f(t) - B(t)\| \leq \max \left\{ \varepsilon(t), \sup_{n \in \mathbb{N}_0} \left| \prod_{i=0}^n b(\psi^i(t)) \right| \varepsilon(\psi^{n+1}(t)) \right\}, \quad t \in S. \tag{17}$$

2 Solution of the nonic functional equation on commutative groups

In this section, we solve the functional equation (1) on commutative groups with some additional requirements.

A group S is said to be divisible if for every element $b \in S$ and every $n \in \mathbb{N}$, there exists an element $a \in S$ such that $na = b$. If this element a is unique, then S is said to be uniquely divisible. In a uniquely divisible group, this unique element a is denoted by $\frac{b}{n}$. That the equation $na = b$ has a solution is equivalent to saying that the multiplication by n is surjective. Similarly, that the equation $na = b$ has a unique solution is equivalent to saying that the multiplication by n is bijective.

The following two important results due to Székelyhidi (see [19] for the details).

Theorem 2 *Let G be a commutative semigroup with identity, S a commutative group and n a nonnegative integer. Let the multiplication by $n!$ be bijective in S . The function $f : G \rightarrow S$ is a solution of Fréchet functional equation*

$$\Delta_{x_1, \dots, x_{n+1}} f(x_0) = 0 \tag{18}$$

for all $x_0, x_1, \dots, x_{n+1} \in G$ if and only if f is a polynomial of degree at most n , i.e., f is given by

$$f(x) = A^n(x) + \dots + A^1(x) + A^0(x), \quad x \in G, \tag{19}$$

where $A^0(x) = A^0$ is an arbitrary element of S and $A^n(x)$ is the diagonal of an n -additive symmetric function $A_n : G^n \rightarrow S$.

Theorem 3 *Let G and S be commutative groups, n a nonnegative integer, φ_i, ψ_i additive functions from G into G and $\varphi_i(G) \subseteq \psi_i(G) (i = 1, 2, \dots, n + 1)$. If the functions $f, f_i : G \rightarrow S (i = 1, 2, \dots, n + 1)$ satisfy*

$$f(x) + \sum_{i=1}^{n+1} f_i(\varphi_i(x) + \psi_i(y)) = 0, \tag{20}$$

then f satisfies Fréchet functional equation $\Delta_{x_1, \dots, x_{n+1}} f(x_0) = 0$.

Using the results, we have the following theorem.

Theorem 4 *Let S be a commutative group and V be a linear space. Then the function $f : S \rightarrow V$ satisfies the functional equation (1) for all $x, y \in S$, if and only if f is of the form*

$$f(x) = A^9(x), \quad x \in S,$$

where $A^9(x)$ is the diagonal of the 9-additive symmetric map $A_9 : S^9 \rightarrow V$.

Proof. Assume that f satisfies the functional equation (1). We can rewrite the functional equation (1) in the form

$$\begin{aligned} f(x) - \frac{1}{126}f(x + 5y) + \frac{1}{14}f(x + 4y) - \frac{2}{7}f(x + 3y) + \frac{2}{3}f(x + 2y) - f(x + y) \\ - \frac{2}{3}f(x - y) + \frac{2}{7}f(x - 2y) - \frac{1}{14}f(x - 3y) + \frac{1}{126}f(x - 4y) + 2880f(y) = 0. \end{aligned} \tag{21}$$

Thus by Theorems 2 and 3, f is of the form

$$f(x) = \sum_{i=0}^9 A^i(x), \quad x \in S, \tag{22}$$

where $A^0(x) = A^0$ is an arbitrary element of V , and $A^i(x)$ is the diagonal of the i -additive symmetric map $A_i : S^i \rightarrow V$ for $i = 1, 2, \dots, 9$. Replacing $x = 0, y = 0$ in (1), one finds $f(0) = 0$. Hence $A^0(x) = A^0 = 0$.

Replacing $x = 0, y = x$ and $x = x, y = -x$ in (1) and adding the two resulting equations, we get $f(-x) = -f(x)$ for all $x \in S$. So the function f is odd. Thus we have $A^8(x) = A^6(x) = A^4(x) = A^2(x) = 0$ for all $x \in S$. It follows that $f(x) = A^9(x) + A^7(x) + A^5(x) + A^3(x) + A^1(x)$. Replacing (x, y) with $(0, 2x)$ in (1), one obtains

$$f(10x) - 8f(8x) + 27f(6x) - 48f(4x) - (9! - 42)f(2x) = 0. \tag{23}$$

Replacing (x, y) with $(5x, x)$, one gets

$$f(10x) - 9f(9x) + 36f(8x) - 84f(7x) + 126f(6x) - 126f(5x) + 84f(4x) - 36f(3x) + 9f(2x) - (9! + 1)f(x) = 0. \tag{24}$$

Subtracting equations (23) and (24), we find

$$9f(9x) - 44f(8x) + 84f(7x) - 99f(6x) + 126f(5x) - 132f(4x) + 36f(3x) - (9! - 33)f(2x) + (9! + 1)f(x) = 0. \tag{25}$$

Replacing (x, y) with $(4x, x)$, and multiplying the resulting equation by 9, one obtains

$$9f(9x) - 81f(8x) + 324f(7x) - 756f(6x) + 1134f(5x) - 1134f(4x) + 756f(3x) - 324f(2x) - 9(9! - 9)f(x) = 0. \tag{26}$$

Subtracting equations (25) and (26), we get

$$37f(8x) - 240f(7x) + 657f(6x) - 1008f(5x) + 1002f(4x) - 720f(3x) - (9! - 357)f(2x) + (10! - 80)f(x) = 0. \tag{27}$$

Replacing (x, y) with $(3x, x)$, and multiplying the resulting equation by 37, one finds

$$37f(8x) - 333f(7x) + 1332f(6x) - 3108f(5x) + 4662f(4x) - 4662f(3x) + 3108f(2x) - 37(9! + 35)f(x) = 0. \tag{28}$$

Subtracting equations (27) and (28), we arrive at

$$93f(7x) - 675f(8x) + 2100f(5x) - 3660f(4x) + 3942f(3x) - (9! + 2751)f(2x) + (47 \cdot 9! + 1215)f(x) = 0. \tag{29}$$

Replacing (x, y) with $(2x, x)$, and multiplying the resulting equation by 93, one finds

$$93f(7x) - 837f(6x) + 3348f(5x) - 7812f(4x) + 11718f(3x) - 11625f(2x) - 93(9! - 75)f(x) = 0. \tag{30}$$

Subtracting equations (29) and (30) and then dividing by 2, we arrive at

$$81f(6x) - 624f(5x) + 2076f(4x) - 3888f(3x) - \frac{1}{2}(9! - 8874)f(2x) + (70 \cdot 9! - 2880)f(x) = 0. \tag{31}$$

Replacing (x, y) with (x, x) , and multiplying the resulting equation by 81, one finds

$$81f(6x) - 729f(5x) + 2916f(4x) - 6723f(3x) + 9477f(2x) - 81(9! + 90)f(x) = 0. \tag{32}$$

Subtracting equations (31) and (32), we arrive at

$$105f(5x) - 840f(4x) + 2835f(3x) - \frac{1}{2}(9! + 10080)f(2x) + (151 \cdot 9! + 4410)f(x) = 0. \tag{33}$$

Replacing (x, y) with $(0, x)$, and multiplying the resulting equation by 105, one finds

$$105(5x) - 840f(4x) + 2835f(3x) - 5040f(2x) - 105(9! - 42)f(x) = 0. \tag{34}$$

Subtracting equations (33) and (34), we arrive at

$$f(2x) = 2^9 f(x). \tag{35}$$

By (35) and $A^n(rx) = r^n A^n(x)$ whenever $x \in S$ and $r \in \mathbb{Q}$, we obtain $2^9(A^9(x) + A^7(x) + A^5(x) + A^3(x) + A^1(x)) = 2^9 A^9(x) + 2^7 A^7(x) + 2^5 A^5(x) + 2^3 A^3(x) + 2 A^1(x)$. It follows that $A^7(x) = A^5(x) = A^3(x) = A^1(x) = 0$ for all $x \in S$. Hence $f(x) = A^9(x)$. The converse is easily verified. \square

3 Stability results

Throughout this section, we assume that S is a commutative group and X is a complete non-Archimedean normed space. For a given mapping $f : S \rightarrow X$, we define the difference operators

$$Df(x, y) := f(x + 5y) - 9f(x + 4y) + 36f(x + 3y) - 84f(x + 2y) + 126f(x + y) - 126f(x) + 84f(x - y) - 36f(x - 2y) + 9f(x - 3y) - f(x - 4y) - 9!f(y)$$

for all $x, y \in S$.

Theorem 5 Let $\varphi : S^2 \rightarrow \mathbb{R}_+$ be a function such that

$$\lim_{n \rightarrow \infty} |2|^{-9n} \varphi(2^n x, 2^n y) = 0, \quad x, y \in S. \tag{36}$$

Assume also that $f : S \rightarrow X$ be a mapping such that

$$\|Df(x, y)\| \leq \varphi(x, y), \quad x, y \in S. \tag{37}$$

Then there exists a unique nonic mapping $T : S \rightarrow X$ such that

$$\|f(x) - T(x)\| \leq \sup_{n \in \mathbb{N}_0} |2|^{-9(n+1)} \delta(2^n x), \quad x \in S, \tag{38}$$

where

$$\begin{aligned} \delta(x) := & \frac{1}{|9!|} \max \left\{ |210|\varphi(0, x), \frac{|210|}{|8!|} \varphi(0, 3x), \frac{|210|}{|8!|} \varphi(3x, -3x), \frac{|15|}{|6!|} \varphi(2x, -2x), \right. \\ & \frac{|35|}{|6!|} \varphi(x, -x), \frac{|2940|}{|8!|} \varphi(0, 0), \frac{|210|}{|9!|} \varphi(0, 4x), \frac{|210|}{|9!|} \varphi(4x, -4x), |162|\varphi(x, x), \\ & \frac{|18|}{|8!|} \varphi(3x, -3x), |93|\varphi(2x, x), |37|\varphi(3x, x), |9|\varphi(4x, x), \varphi(5x, x), \varphi(0, 2x), \\ & \left. \frac{1}{|9!|} \varphi(0, 8x), \frac{1}{|9!|} \varphi(8x, -8x), \frac{1}{|8!|} \varphi(0, 6x), \frac{1}{|8!|} \varphi(6x, -6x) \right\}. \end{aligned}$$

Proof. Replacing $x = y = 0$ in (37), we get

$$\|f(0)\| \leq \frac{1}{|9!|} \varphi(0, 0). \tag{39}$$

Replacing x and y by 0 and x in (37), respectively, we get

$$\begin{aligned} \|f(5x) - 9f(4x) + 36f(3x) - 84f(2x) + 126f(x) - 126f(0) + 84f(-x) \\ - 36f(-2x) + 9f(-3x) - f(-4x) - 9!f(x)\| \leq \varphi(0, x) \end{aligned} \tag{40}$$

for all $x \in S$. Replacing x and y by x and $-x$ in (37), respectively, we have

$$\begin{aligned} \|f(-4x) - 9f(-3x) + 36f(-2x) - 84f(-x) + 126f(0) - 126f(x) + 84f(2x) \\ - 36f(3x) + 9f(4x) - f(5x) - 9!f(-x)\| \leq \varphi(-x, x) \end{aligned} \tag{41}$$

for all $x \in S$. By (40) and (41), we obtain

$$\|f(x) + f(-x)\| \leq \frac{1}{|9!|} \max\{\varphi(0, x), \varphi(x, -x)\} \tag{42}$$

for all $x \in S$. Replacing x and y by 0 and $2x$ in (37), respectively, and using (39) and (42), we find

$$\begin{aligned} \|f(10x) - 8f(8x) + 27f(6x) - 48f(4x) - (9! - 42)f(2x)\| \\ \leq \max \left\{ \varphi(0, 2x), \frac{1}{|9!|} \varphi(0, 8x), \frac{1}{|9!|} \varphi(8x, -8x), \frac{1}{|8!|} \varphi(0, 6x), \frac{1}{|8!|} \varphi(6x, -6x), \right. \\ \left. \frac{|4|}{|8!|} \varphi(0, 4x), \frac{|4|}{|8!|} \varphi(4x, -4x), \frac{|84|}{|9!|} \varphi(2x, -2x) \right\} \end{aligned} \tag{43}$$

for all $x \in S$. Replacing x and y by $5x$ and x in (37), respectively, we get

$$\|f(10x) - 9f(9x) + 36f(8x) - 84f(7x) + 126f(6x) - 126f(5x) + 84f(4x) - 36f(3x) + 9f(2x) - (9! + 1)f(x)\| \leq \varphi(5x, x) \tag{44}$$

for all $x \in S$. By (43) and (44), we obtain

$$\begin{aligned} & \|9f(9x) - 44f(8x) + 84f(7x) - 99f(6x) + 126f(5x) - 132f(4x) \\ & \quad + 36f(3x) - (9! - 33)f(2x) + (9! + 1)f(x)\| \\ & \leq \max \left\{ \varphi(5x, x), \varphi(0, 2x), \frac{1}{|9!|} \varphi(0, 8x), \frac{1}{|9!|} \varphi(8x, -8x), \frac{1}{|8!|} \varphi(0, 6x), \right. \\ & \quad \left. \frac{1}{|8!|} \varphi(6x, -6x), \frac{|4|}{|8!|} \varphi(0, 4x), \frac{|4|}{|8!|} \varphi(4x, -4x), \frac{|84|}{|9!|} \varphi(2x, -2x) \right\} \end{aligned} \tag{45}$$

for all $x \in S$. Replacing x and y by $4x$ and x in (37), respectively, and using (39) we have

$$\|f(9x) - 9f(8x) + 36f(7x) - 84f(6x) + 126f(5x) - 126f(4x) + 84f(3x) - 36f(2x) - (9! - 9)f(x)\| \leq \max \left\{ \varphi(4x, x), \frac{1}{|9!|} \varphi(0, 0) \right\} \tag{46}$$

for all $x \in S$. By (45) and (46), we get

$$\begin{aligned} & \|37f(8x) - 240f(7x) + 657f(6x) - 1008f(5x) + 1002f(4x) \\ & \quad - 720f(3x) - (9! - 357)f(2x) + (10! - 80)f(x)\| \\ & \leq \max \left\{ |9| \varphi(4x, x), \frac{1}{|8!|} \varphi(0, 0), \varphi(5x, x), \varphi(0, 2x), \frac{1}{|9!|} \varphi(0, 8x), \frac{1}{|9!|} \varphi(8x, -8x), \right. \\ & \quad \left. \frac{1}{|8!|} \varphi(0, 6x), \frac{1}{|8!|} \varphi(6x, -6x), \frac{|4|}{|8!|} \varphi(0, 4x), \frac{|4|}{|8!|} \varphi(4x, -4x), \frac{|84|}{|9!|} \varphi(2x, -2x) \right\} \end{aligned} \tag{47}$$

for all $x \in S$. Replacing x and y by $3x$ and x in (37), respectively, then using (39) and (42), we have

$$\|f(8x) - 9f(7x) + 36f(6x) - 84f(5x) + 126f(4x) - 126f(3x) + 84f(2x) - (9! + 35)f(x)\| \leq \max \left\{ \varphi(3x, x), \frac{1}{|8!|} \varphi(0, 0), \frac{1}{|9!|} \varphi(0, x), \frac{1}{|9!|} \varphi(x, -x) \right\} \tag{48}$$

for all $x \in S$. By (47) and (48), we get

$$\begin{aligned} & \|93f(7x) - 675f(6x) + 2100f(5x) - 3660f(4x) \\ & \quad + 3942f(3x) - (9! + 2751)f(2x) + (47 \cdot 9! + 1215)f(x)\| \\ & \leq \max \left\{ |37| \varphi(3x, x), \frac{|37|}{|8!|} \varphi(0, 0), \frac{|37|}{|9!|} \varphi(0, x), \frac{|37|}{|9!|} \varphi(x, -x), |9| \varphi(4x, x), \right. \\ & \quad \varphi(5x, x), \varphi(0, 2x), \frac{1}{|9!|} \varphi(0, 8x), \frac{1}{|9!|} \varphi(8x, -8x), \frac{1}{|8!|} \varphi(0, 6x), \\ & \quad \left. \frac{1}{|8!|} \varphi(6x, -6x), \frac{|4|}{|8!|} \varphi(0, 4x), \frac{|4|}{|8!|} \varphi(4x, -4x), \frac{|84|}{|9!|} \varphi(2x, -2x) \right\} \end{aligned} \tag{49}$$

for all $x \in S$. Replacing x and y by $2x$ and x in (37), respectively, then using (39) and (42), we have

$$\|f(7x) - 9f(6x) + 36f(5x) - 84f(4x) + 126f(3x) - 125f(2x) - (9! - 75)f(x)\| \leq \max \left\{ \varphi(2x, x), \frac{1}{|9!|} \varphi(0, 2x), \frac{1}{|9!|} \varphi(2x, -2x), \frac{1}{|8!|} \varphi(0, x), \frac{1}{|8!|} \varphi(x, -x), \frac{|4|}{|8!|} \varphi(0, 0) \right\} \tag{50}$$

for all $x \in S$. By (49) and (50), we get

$$\begin{aligned} & \|81f(6x) - 624f(5x) + 2076f(4x) - 3888f(3x) - \frac{1}{2}(9! - 8874)f(2x) \\ & \quad + (70 \cdot 9! - 2880)f(x)\| \\ & \leq \frac{1}{|2|} \max \left\{ |93| \varphi(2x, x), \frac{|93|}{|9!|} \varphi(2x, -2x), \frac{|93|}{|8!|} \varphi(0, x), \frac{|93|}{|8!|} \varphi(x, -x), \frac{|372|}{|8!|} \varphi(0, 0), \right. \\ & \quad |37| \varphi(3x, x), |9| \varphi(4x, x), \varphi(5x, x), \varphi(0, 2x), \frac{1}{|9!|} \varphi(0, 8x), \frac{1}{|9!|} \varphi(8x, -8x), \\ & \quad \left. \frac{1}{|8!|} \varphi(0, 6x), \frac{1}{|8!|} \varphi(6x, -6x), \frac{|4|}{|8!|} \varphi(0, 4x), \frac{|4|}{|8!|} \varphi(4x, -4x) \right\} \end{aligned} \tag{51}$$

for all $x \in S$. Replacing x and y by x and x in (37), respectively, then using (39) and (42), we have

$$\begin{aligned} & \|f(6x) - 9f(5x) + 36f(4x) - 83f(3x) + 117f(2x) - (9! + 90)f(x)\| \\ & \leq \max \left\{ \varphi(x, x), \frac{|84|}{|9!|} \varphi(0, 0), \frac{|4|}{|8!|} \varphi(0, x), \frac{|4|}{|8!|} \varphi(x, -x), \frac{1}{|8!|} \varphi(0, 2x), \right. \\ & \quad \left. \frac{1}{|8!|} \varphi(2x, -2x), \frac{1}{|9!|} \varphi(0, 3x), \frac{1}{|9!|} \varphi(3x, -3x) \right\} \end{aligned} \tag{52}$$

for all $x \in S$. By (51) and (52), we get

$$\begin{aligned} & \|105f(5x) - 840f(4x) + 2835f(3x) - \left(\frac{9!}{2} + 5040\right)f(2x) + (151 \cdot 9! + 4410)f(x)\| \\ & \leq \max \left\{ |81| \varphi(x, x), \frac{|756|}{|8!|} \varphi(0, 0), \frac{|324|}{|8!|} \varphi(0, x), \frac{|324|}{|8!|} \varphi(x, -x), \frac{|81|}{|8!|} \varphi(2x, -2x), \right. \\ & \quad \frac{|9|}{|8!|} \varphi(0, 3x), \frac{|9|}{|8!|} \varphi(3x, -3x), \frac{|93|}{|2!|} \varphi(2x, x), \frac{|37|}{|2!|} \varphi(3x, x), \frac{|9|}{|2!|} \varphi(4x, x), \\ & \quad \frac{1}{|2!|} \varphi(5x, x), \frac{1}{|2!|} \varphi(0, 2x), \frac{1}{|2 \cdot 9!|} \varphi(0, 8x), \frac{1}{|2 \cdot 9!|} \varphi(8x, -8x), \\ & \quad \left. \frac{1}{|2 \cdot 8!|} \varphi(0, 6x), \frac{1}{|2 \cdot 8!|} \varphi(6x, -6x), \frac{|2|}{|8!|} \varphi(0, 4x), \frac{|2|}{|8!|} \varphi(4x, -4x) \right\} \end{aligned} \tag{53}$$

for all $x \in S$. Replacing x and y by 0 and x in (37), respectively, then using (39) and (42), we have

$$\begin{aligned} & \|f(5x) - 8f(4x) + 27f(3x) - 48f(2x) - (9! - 42)f(x)\| \\ & \leq \max \left\{ \varphi(0, x), \frac{1}{|8!|} \varphi(0, 3x), \frac{1}{|8!|} \varphi(3x, -3x), \frac{|4|}{|8!|} \varphi(0, 2x), \frac{|4|}{|8!|} \varphi(2x, -2x), \right. \\ & \quad \left. \frac{|84|}{|9!|} \varphi(x, -x), \frac{|14|}{|8!|} \varphi(0, 0), \frac{1}{|9!|} \varphi(0, 4x), \frac{1}{|9!|} \varphi(4x, -4x) \right\} \end{aligned} \tag{54}$$

for all $x \in S$. By (53) and (54), we get

$$\begin{aligned} \|f(2x) - 2^9 f(x)\| & \leq \frac{1}{|9!|} \max \left\{ |210| \varphi(0, x), \frac{|210|}{|8!|} \varphi(0, 3x), \frac{|210|}{|8!|} \varphi(3x, -3x), \right. \\ & \quad \frac{|15|}{|6!|} \varphi(2x, -2x), \frac{|35|}{|6!|} \varphi(x, -x), \frac{|2940|}{|8!|} \varphi(0, 0), \frac{|210|}{|9!|} \varphi(0, 4x), \\ & \quad \frac{|210|}{|9!|} \varphi(4x, -4x), |162| \varphi(x, x), \frac{|18|}{|8!|} \varphi(3x, -3x), |93| \varphi(2x, x), \\ & \quad |37| \varphi(3x, x), |9| \varphi(4x, x), \varphi(5x, x), \varphi(0, 2x), \frac{1}{|9!|} \varphi(0, 8x), \\ & \quad \left. \frac{1}{|9!|} \varphi(8x, -8x), \frac{1}{|8!|} \varphi(0, 6x), \frac{1}{|8!|} \varphi(6x, -6x) \right\} \\ & =: \delta(x), \quad x \in S. \end{aligned} \tag{55}$$

By Corollary 2, there exists a unique mapping $T : S \rightarrow X$ such that $T(2x) = 2^9 T(x)$ and (38) holds. By (8) in Corollary 1,

$$T(x) := \lim_{n \rightarrow \infty} (\mathcal{L}_\varphi^F)^n(f)(x) = \lim_{n \rightarrow \infty} 2^{-9n} f(2^n x), \quad x \in S. \tag{56}$$

It remains to show that T is a nonic map. By (37), we have

$$\|Df(2^n x, 2^n y)/2^{9n}\| \leq |2|^{-9n} \varphi(2^n x, 2^n y) \tag{57}$$

for all $x, y \in S$ and $n \in \mathbb{N}$. So, by (36), (54) and (55), $\|DT(x, y)\| = 0$ for all $x, y \in S$. Thus the mapping $T : S \rightarrow X$ is nonic. \square

Similar to Theorem 5, one can prove the following result.

Theorem 6 Assume that the multiplication by 2^n ($n \in \mathbb{N}$) be bijective in S . Let $\varphi : S^2 \rightarrow \mathbb{R}_+$ be a function such that

$$\lim_{n \rightarrow \infty} |2|^{9n} \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0, \quad x, y \in S. \tag{58}$$

Assume also that $f : S \rightarrow X$ be a mapping such that

$$\|Df(x, y)\| \leq \varphi(x, y), \quad x, y \in S. \tag{59}$$

Then there exists a unique nonic mapping $T : S \rightarrow X$ such that

$$\|f(x) - T(x)\| \leq \max \left\{ \delta \left(\frac{x}{2} \right), \sup_{n \in \mathbb{N}_0} |2|^{9(n+1)} \delta \left(\frac{x}{2^{n+2}} \right) \right\}, \quad x \in S, \tag{60}$$

where $\delta(x)$ is defined as in Theorem 5.

Proof. From (55), we have

$$\left\| f(x) - 2^9 f \left(\frac{x}{2} \right) \right\| \leq \delta \left(\frac{x}{2} \right), \quad x \in S. \tag{61}$$

By Corollary 3, there exists a unique mapping $T : S \rightarrow X$ such that $T(x) = 2^9 T(\frac{x}{2})$ and (60) holds. By (8) in Corollary 1,

$$T(x) := \lim_{n \rightarrow \infty} (\mathcal{L}_\varphi^F)^n(f)(t) = \lim_{n \rightarrow \infty} 2^{9n} f \left(\frac{t}{2^n} \right), \quad x \in S. \tag{62}$$

The rest of the proof is similar to the proof of Theorem 5. □

Corollary 4 Let S be a non-Archimedean normed space and X be a complete non-Archimedean normed space with $|2| < 1$. Let ϵ, λ be positive numbers with $\lambda \neq 9$, and $f : S \rightarrow X$ be a mapping satisfying

$$\|Df(x, y)\| \leq \epsilon(\|x\|^\lambda + \|y\|^\lambda), \quad x, y \in S.$$

Then there exists a unique nonic mapping $T : S \rightarrow X$ such that

$$\|f(x) - T(x)\| \leq \begin{cases} \frac{2\epsilon\|x\|^\lambda}{|9!|^2 \cdot |2|^9}, & \lambda > 9, x \in S; \\ \frac{2\epsilon\|x\|^\lambda}{|9!|^2 \cdot |2|^\lambda}, & \lambda < 9, x \in S. \end{cases}$$

Proof. Let $\varphi : S^2 \rightarrow \mathbb{R}_+$ be defined by $\varphi(x, y) = \epsilon(\|x\|^\lambda + \|y\|^\lambda)$ for all $x, y \in S$. Then the corollary is followed from Theorems 5 and 6. □

Similar to Corollary 4, one can obtain the following corollary.

Corollary 5 Let S be a non-Archimedean normed space and X be a complete non-Archimedean normed space with $|2| < 1$. Let ϵ, λ, μ be positive numbers with $\lambda + \mu \neq 9$, and $f : S \rightarrow X$ be a mapping satisfying

$$\|Df(x, y)\| \leq \epsilon\|x\|^\lambda \cdot \|y\|^\mu, \quad x, y \in S.$$

Then there exists a unique nonic mapping $T : S \rightarrow X$ such that

$$\|f(x) - T(x)\| \leq \begin{cases} \frac{\epsilon\|x\|^{\lambda+\mu}}{|9!|^2 \cdot |2|^9}, & \lambda + \mu > 9, x \in S; \\ \frac{\epsilon\|x\|^{\lambda+\mu}}{|9!|^2 \cdot |2|^{\lambda+\mu}}, & \lambda + \mu < 9, x \in S. \end{cases}$$

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Global Attractivity and Periodicity Behavior of a Recursive Sequence

E. M. Elsayed^{1,2}, and Abdul Khaliq¹

¹Mathematics Department, Faculty of Science,
King Abdulaziz University, P. O. Box 80203, Jeddah 21589, Saudi Arabia.

²Department of Mathematics, Faculty of Science,
Mansoura University, Mansoura 35516, Egypt.

E-mails: emmelsayed@yahoo.com, khaliqsyed@gmail.com

ABSTRACT

Our aim in this paper is to study the global stability character and the periodic nature of the solutions of the difference equation

$$x_{n+1} = ax_{n-l} + \frac{bx_{n-k} + cx_{n-s}}{d + ex_{n-t}}, \quad n = 0, 1, \dots,$$

where the initial conditions $x_{-r}, x_{-r+1}, x_{-r+2}, \dots, x_0$ are arbitrary positive real numbers, $r = \max\{l, k, s, t\}$ is nonnegative integer and a, b, c, d, e are positive constants.

Keywords: stability, periodic solutions, global attractor, difference equations.

Mathematics Subject Classification: 39A10

1. INTRODUCTION

Our goal in this paper is to investigate the global stability character and the periodicity of the solutions of the difference equation

$$x_{n+1} = ax_{n-l} + \frac{bx_{n-k} + cx_{n-s}}{d + ex_{n-t}}, \quad n = 0, 1, \dots, \tag{1}$$

where the initial conditions $x_{-r}, x_{-r+1}, x_{-r+2}, \dots, x_0$ are arbitrary positive real numbers, $r = \max\{l, k, s, t\}$ is nonnegative integer and a, b, c, d, e are positive constants.

Recently there has been a lot of interest in studying the global attractivity, the boundedness character and the periodicity nature of nonlinear difference equations see for example [1-20].

The study of the nonlinear rational difference equations is interesting and attractive to many researchers working in this field It is quite challenging and rewarding, many real life phenomena are modelling using these equations. Examples from economy, biology,etc. may be obtained in [3,7,11,12] The study of some properties of these equations via the global attractivity, the boundedness and the periodicity of these equations is of great interest. For examples in the articles [11,12,15]. Recently, many researchers have investigated the behavior of the solution of difference equations for example: In [1] Ahmed investigated the behavior of the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-2k+1}}{\pm 1 \pm \prod_{i=1}^k x_{n-2i+1}}.$$

Elabbasy et al. [8] studied the boundedness, global stability, periodicity character and gave the solution of some special cases of the difference equation

$$x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^k x_{n-i}}.$$

Elabbasy et al. [9] investigated the global stability, periodicity character and gave the solution of some special cases of the difference equation

$$x_{n+1} = a + \frac{dx_{n-l}x_{n-k}}{cx_{n-s} - b}.$$

Yalçınkaya [32] has studied the following difference equation

$$x_{n+1} = \alpha + \frac{x_{n-m}}{x_n^k}.$$

For some related work see [21–35].

Here, we recall some basic definitions and some theorems that we need in the sequel.

Let I be some interval of real numbers and let

$$F : I^{k+1} \rightarrow I,$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, \dots, x_0 \in I$, the difference equation

$$x_{n+1} = F(x_n, x_{n-1}, \dots, x_{n-k}), \quad n = 0, 1, \dots, \tag{2}$$

has a unique solution $\{x_n\}_{n=-k}^\infty$.

A point $\bar{x} \in I$ is called an equilibrium point of Eq.(2) if

$$\bar{x} = f(\bar{x}, \bar{x}, \dots, \bar{x}).$$

That is, $x_n = \bar{x}$ for $n \geq 0$, is a solution of Eq.(2), or equivalently, \bar{x} is a fixed point of f .

DEFINITION 1.1. (*Periodicity*)

A sequence $\{x_n\}_{n=-k}^\infty$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.

DEFINITION 1.2. (*Stability*)

(i) The equilibrium point \bar{x} of Eq.(2) is locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, \dots, x_{-1}, x_0 \in I$ with

$$|x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + \dots + |x_0 - \bar{x}| < \delta,$$

we have

$$|x_n - \bar{x}| < \epsilon \quad \text{for all } n \geq -k.$$

(ii) The equilibrium point \bar{x} of Eq.(2) is locally asymptotically stable if \bar{x} is locally stable solution of Eq.(2) and there exists $\gamma > 0$, such that for all $x_{-k}, x_{-k+1}, \dots, x_{-1}, x_0 \in I$ with

$$|x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + \dots + |x_0 - \bar{x}| < \gamma,$$

we have

$$\lim_{n \rightarrow \infty} x_n = \bar{x}.$$

(iii) The equilibrium point \bar{x} of Eq.(2) is global attractor if for all $x_{-k}, x_{-k+1}, \dots, x_{-1}, x_0 \in I$, we have

$$\lim_{n \rightarrow \infty} x_n = \bar{x}.$$

(iv) The equilibrium point \bar{x} of Eq.(2) is globally asymptotically stable if \bar{x} is locally stable, and \bar{x} is also a global attractor of Eq.(2).

(v) The equilibrium point \bar{x} of Eq.(2) is unstable if \bar{x} is not locally stable.

The linearized equation of Eq.(2) about the equilibrium \bar{x} is the linear difference equation

$$y_{n+1} = \sum_{i=0}^k \frac{\partial F(\bar{x}, \bar{x}, \dots, \bar{x})}{\partial x_{n-i}} y_{n-i}. \tag{3}$$

Theorem A [26] Assume that $p, q \in R$ and $k \in \{0, 1, 2, \dots\}$. Then

$$|p| + |q| < 1,$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+1} + px_n + qx_{n-k} = 0, \quad n = 0, 1, \dots .$$

REMARK 1. *Theorem A can be easily extended to a general linear equations of the form*

$$x_{n+k} + p_1x_{n+k-1} + \dots + p_kx_n = 0, \quad n = 0, 1, \dots, \tag{4}$$

where $p_1, p_2, \dots, p_k \in R$ and $k \in \{1, 2, \dots\}$. Then Eq.(4) is asymptotically stable provided that

$$\sum_{i=1}^k |p_i| < 1.$$

Consider the following equation

$$x_{n+1} = g(x_n, x_{n-1}, \dots, x_{n-K}) \quad n = 0, 1, 2, \dots \tag{5}$$

The following theorem will be useful for the proof of our results in this paper.

Theorem B [27]: Let $[\alpha, \beta]$ be an interval of real numbers and assume that

$$g : [\alpha, \beta]^{k+1} \rightarrow [\alpha, \beta],$$

is a continuous function satisfying the following properties :

(a) $g(x_1, x_2, \dots, x_{k+1})$ is non-increasing in one component (for example x_σ) for each $x_r (r \neq \sigma)$ in $[\alpha, \beta]$, and is non-increasing in the remaining components for each $x_\sigma \in [\alpha, \beta]$;

(b) If $(m, M) \in [\alpha, \beta] \times [\alpha, \beta]$ is a solution of the system

$$M = g(m, m, \dots, m, M, m, \dots, m, m) \quad \text{and} \quad m = g(M, M, \dots, M, m, M, \dots, M, M),$$

then

$$m = M.$$

Then Eq.(5) has a unique equilibrium $\bar{x} \in [\alpha, \beta]$ and every solution of Eq.(5) converges to \bar{x} .

2. LOCAL STABILITY OF THE EQUILIBRIUM POINT OF EQ.(1)

In this section we study the local stability character of the solutions of Eq.(1). The equilibrium points of Eq.(1) are given by the relation

$$\bar{x} = a\bar{x} + \frac{b\bar{x} + c\bar{x}}{cd + d\bar{x}}.$$

If $a \neq 1$, then the equilibrium points of Eq.(1) is given by

$$\bar{x} = 0 \quad \text{and} \quad \bar{x} = \frac{b + c + d(a - 1)}{e(1 - a)}.$$

Let $f : (0, \infty)^4 \rightarrow (0, \infty)$ be a function defined by

$$f(u_0, u_1, u_2, u_3) = au_0 + \frac{bu_1 + cu_2}{d + eu_3}.$$

Therefore at $\bar{x} = \frac{b+c+d(a-1)}{e(1-a)}$

$$\begin{aligned} \frac{\partial f(\bar{x}, \bar{x}, \bar{x}, \bar{x})}{\partial u_0} &= a = -c_0, & \frac{\partial f(\bar{x}, \bar{x}, \bar{x}, \bar{x})}{\partial u_1} &= -\frac{b(a-1)}{(b+c)} = -c_1, \\ \frac{\partial f(\bar{x}, \bar{x}, \bar{x}, \bar{x})}{\partial u_2} &= -\frac{c(a-1)}{(b+c)} = -c_2, & \frac{\partial f(\bar{x}, \bar{x}, \bar{x}, \bar{x})}{\partial u_3} &= \frac{(a-1)(b+c-d+ad)}{(b+c)} = -c_3 \end{aligned}$$

Then we see that at $\bar{x} = 0$

$$\begin{aligned} \frac{\partial f(\bar{x}, \bar{x}, \bar{x}, \bar{x})}{\partial u_0} &= a = -c_0, & \frac{\partial f(\bar{x}, \bar{x}, \bar{x}, \bar{x})}{\partial u_1} &= \frac{b}{d} = -c_1 \\ \frac{\partial f(\bar{x}, \bar{x}, \bar{x}, \bar{x})}{\partial u_2} &= \frac{c}{d} = -c_2, & \frac{\partial f(\bar{x}, \bar{x}, \bar{x}, \bar{x})}{\partial u_3} &= 0 = -c_3 \end{aligned}$$

Then the linearized equation of Eq.(1) about \bar{x} is

$$y_{n+1} + c_0y_{n-l} + c_1y_{n-k} + c_2y_{n-s} + c_3y_{n-t} = 0. \tag{6}$$

THEOREM 2.1. *Assume that*

$$1 < \frac{d(1-a)}{(b+c)}.$$

Then the positive equilibrium point $\bar{x} = 0$ of Eq.(1) is locally asymptotically stable.

Proof: It follows by Theorem A that, Eq.(6) is asymptotically stable if

$$|c_3| + |c_2| + |c_1| + |c_0| < 1.$$

$$|0| + \left| \frac{c}{d} \right| + \left| \frac{b}{d} \right| + |a| < 1,$$

and so

$$1 < \frac{d(1-a)}{(b+c)}.$$

This completes the proof.

THEOREM 2.2. *Assume that*

$$3 < \frac{d(1-a)}{(b+c)}, \quad a < 1$$

Then the positive equilibrium point $\bar{x} = \frac{b+c+d(a-1)}{e(1-a)}$ of Eq.(1) is locally asymptotically stable.

Proof: It follows by Theorem A that, Eq.(6) is asymptotically stable if

$$\left| \frac{(a-1)(b+c-d+ad)}{(b+c)} \right| + \left| -\frac{c(a-1)}{(b+c)} \right| + \left| -\frac{b(a-1)}{(b+c)} \right| + |a| < 1.$$

$$3a - \frac{d(a-1)^2}{(b+c)} < 3,$$

and so

$$1 < \frac{d(1-a)}{(b+c)}.$$

This completes the proof.

3. EXISTENCE OF PERIODIC SOLUTIONS

In this section we study the existence of periodic solutions of Eq.(1.1).

THEOREM 3.1. *Eq.(1) has a prime period two solutions if and only if*

$$e^2(b + c + d + ad)^2(a + 1)^2 - 4ade^2(a + 1)(b + c + d + ad) > 0, \quad , \quad k, l, s, t - \text{even}. \tag{7}$$

Proof: First suppose that there exists a prime period two solution

$$\dots, p, q, p, q, \dots,$$

of Eq.(1). We will prove that Condition (7) holds.

We see from Eq.(1) (when k, l, s, t -even) that

$$p = aq + \frac{bq + cq}{d + eq}, \quad q = ap + \frac{bp + cp}{d + ep}.$$

Then

$$dp + epq = adq + aeq^2 + bq + cq, \tag{8}$$

and

$$dq + epq = adp + aep^2 + bp + cp. \tag{9}$$

Subtracting (8) from (9) gives

$$d(p - q) = ad(q - p) + ae(q^2 - p^2) + b(q - p) + c(q - p).$$

Since $p \neq q$, it follows that

$$p + q = -\frac{(b + c + d + ad)}{ae}. \tag{10}$$

Again, adding (8) and (9) yields

$$2epq + d(p + q) = ad(p + q) + ad(p + q)^2 + 2aepq + b(p + q) + c(p + q). \tag{11}$$

It follows by (10), (11) and the relation $p^2 + q^2 = (p + q)^2 - 2pq$ for all $p, q \in R$, that

$$pq = \frac{d(a + b + c + d + ad)}{ae^2(a + 1)}. \tag{12}$$

Now it is clear from Eq.(10) and Eq.(12) that p and q are the two positive distinct roots of the quadratic equation

$$t^2 + \left(\frac{(b+c+d+ad)}{ae}\right)t + \left(\frac{d(a+b+c+d+ad)}{ae^2(a+1)}\right) = 0, \tag{13}$$

$$ae^2(a + 1)t^2 + e(a + 1)(b + c + d + ad)t + d(a + b + c + d + ad) = 0,$$

and so

$$((a + 1)(b + c + d + ad))^2 > 4ad(a + 1)(b + c + d + ad),$$

thus

$$(a + 1)(b + c + d + ad) > 4ad.$$

Therefore Inequality (7) holds.

Second suppose that Inequality (7) is true. We will show that Eq.(1) has a prime period two solution. Assume that

$$p = \frac{-e(a+1)(b+c+d+ad)+\sqrt{\xi}}{2ae^2(a+1)} = \frac{-eAB + \sqrt{\xi}}{2ae^2A},$$

and

$$q = \frac{-eAB - \sqrt{\xi}}{2ae^2A}, \text{ where } A = (a + 1), B = (a + b + c + d + ad)$$

where $\xi = e^2(a + 1)^2(b + c + d + ad)^2 - 4ade^2(a + 1)(b + c + d + ad)$.

We see from Inequality (7) that

$$e^2(a + 1)^2(b + c + d + ad)^2 - 4ade^2(a + 1)(b + c + d + ad) > 0,$$

then after dividing by $e^2(a + 1)(b + c + d + ad)$ we see that

$$\Rightarrow (a + 1)(b + c + d + ad) > 4ad,$$

Therefore p and q are distinct real numbers.

Set

$$\begin{aligned} x_{-l} &= p, x_{-l+1} = q, \dots, x_{-k} = q, x_{-k+1} = p, \\ x_{-s} &= p, x_{-s+1} = q, x_{-t} = p, x_{-t+1} = q, \text{ and } x_0 = q. \end{aligned}$$

We wish to show that

$$x_1 = x_{-1} = p \text{ and } x_2 = x_0 = q.$$

It follows from Eq.(1.1) that

$$\begin{aligned} x_1 &= ax_{-l} + \frac{bx_{-k} + cx_{-s}}{d + ex_{-t}} = ap + \frac{bp + cp}{d + ep} = ap + \frac{(b + c)p}{d + ep} \\ &= ap + \frac{(b + c) \left(\frac{-eAB + \sqrt{\xi}}{2ae^2A} \right)}{d + e \left(\frac{-eAB + \sqrt{\xi}}{2ae^2A} \right)}. \end{aligned}$$

Multiplying the denominator and numerator of the right side by $2ae^2A$ gives

$$x_1 = ap + \frac{(b+c)(-eAB + \sqrt{\xi})}{2ae^2Ad - e^2AB + e\sqrt{\xi}},$$

Multiplying the denominator and numerator of the right side by $\{2ae^2Ad - e^2AB - e\sqrt{\xi}\}$

$$\begin{aligned} x_1 &= ap + \frac{(b+c)(-eAB + \sqrt{\xi})(2ae^2Ad - e^2AB - e\sqrt{\xi})}{(2ae^2Ad - e^2AB + e\sqrt{\xi})(2ae^2Ad - e^2AB - e\sqrt{\xi})}, \\ &= ap + \frac{(b+c)[-2ade^3A^2B + e^3A^2B^2 - e(e^2A^2B^2 - 4ade^2AB) + 2ade^2A\sqrt{\xi}]}{(2ade^2A - e^2AB)^2 - (e\sqrt{\xi})^2}, \\ &= ap + \frac{(b+c)[2ade^3AB(2-A) + 2ade^2A\sqrt{\xi}]}{e^4A^2(4a^2d^2 + B^2 - 4adB) - e^4A^2B^2 + 4ade^4AB}, \end{aligned}$$

Replacing $A = (a + 1)$ and $B = (b + c + d + ad)$ in denominator of above equation gives

$$\begin{aligned} x_1 &= ap + \frac{(b+c)[2ade^3AB(1-a) + 2ade^2A\sqrt{\xi}]}{4a^2d^2e^4(a+1)^2 - 4ade^4(a+1)^2(b+c+d+ad) + 4ade^4(1+a)(b+c+d+ad)} \\ &= ap + \frac{(b+c)[2ade^3AB(1-a) + 2ade^2A\sqrt{\xi}]}{4a^2d^2e^4(a+1)^2 - 4ade^4(a+1)^2(b+c+d+ad) + 4ade^4(1+a)(b+c+d+ad)} \\ &= ap - \frac{(b+c)[2ade^3AB(1-a) + 2ade^2A\sqrt{\xi}]}{4a^2de^2(a+1)(b+c)} \end{aligned}$$

Dividing numerator and denominator by $(b + c)$ we get

$$x_1 = ap - \frac{2ade^3AB(1-a)+2ade^2A\sqrt{\xi}}{4a^2de^2(a+1)} = ap - \frac{eB(1-a) + \sqrt{\xi}}{2ae^2}$$

Now inserting the value of p we get

$$\begin{aligned} x_1 &= \frac{-eAB + \sqrt{\xi}}{2e^2A} - \frac{eB(1-a) + \sqrt{\xi}}{2ae^2} \\ &= \frac{1}{2e^2} \left(\frac{-eAB + \sqrt{\xi}}{A} - \frac{eB(1-a) + \sqrt{\xi}}{a} \right) = \frac{-eAB + a\sqrt{\xi} - eB(1-a)(1+a) - (a+1)\sqrt{\xi}}{2ae^2(a+1)} \\ &= \frac{-eaB(a+1) - eB + eBa^2 - \sqrt{\xi}}{2ae^2(a+1)} \end{aligned}$$

putting the value of $B = (b + c + d + ad)$ we get

$$x_1 = \frac{-e(b+c+d+ad)(a+1) - \sqrt{\xi}}{2ae^2(a+1)} = q$$

Similarly as before one can easily show that

$$x_2 = p.$$

Then it follows by induction that

$$x_{2n} = p \quad \text{and} \quad x_{2n+1} = q \quad \text{for all} \quad n \geq -1.$$

Thus Eq.(1) has the positive prime period two solution

$$\dots, p, q, p, q, \dots,$$

where p and q are the distinct roots of the quadratic equation (13) and the proof is complete.

The following Theorems can be proved similarly.

THEOREM 3.2. *Eq.(1) has a prime period two solutions if and only if*

$$e^2(a+1)^2(d+ad+b+c)^2 - 4e^2(ad+b+c)(a+1)(d+ad+b+c) > 0, \quad t - \text{odd}, l, k, s - \text{even}.$$

THEOREM 3.3. *Eq.(1) has a prime period two solutions if and only if*

$$e^2(a+1)^2(d+ad-b-c)^2 - 4e^2ad(a+1)(d+ad-b-c) > 0, \quad l - \text{even}, s, k, t - \text{odd}.$$

THEOREM 3.4. *Eq.(1) has a prime period two solutions if and only if*

$$e^2(d-ad+b+c)^2(a-1)^2 - 4e^2(a-1)^2(b+c)(d-ad+b+c) > 0, \quad l, t - \text{odd}, s, k - \text{even}.$$

THEOREM 3.5. *Eq.(1) has a prime period two solutions if and only if*

$$e^2(a+1)^2(b+c-d-ad)^2 + 4ae^2(a+1)(b+c-d-ad)(d-b-c) > 0, \quad l, t - \text{even}, s, k - \text{odd}.$$

4. GLOBAL ATTRACTIVITY OF THE EQUILIBRIUM POINT OF EQ.(1)

In this section we investigate the global attractivity character of solutions of Eq.(1).

THEOREM 4.1. *The equilibrium point \bar{x} of Eq. (1) is global attractor.*

Proof: Let p, q are a real numbers and assume that $f : [p, q]^4 \rightarrow [p, q]$ be a function defined by

$$f(u_0, u_1, u_2, u_3) = au_0 + \frac{bu_1 + cu_2}{d + eu_3}.$$

We can easily see that the function $f(u_0, u_1, u_2, u_3)$ increasing in u_0, u_1, u_2 and decreasing in u_3 .

Suppose that (m, M) is a solution of the system

$$m = f(m, m, m, M) \quad \text{and} \quad M = f(M, M, M, m).$$

Then from Eq.(1), we see that

$$m = am + \frac{(b+c)m}{d+eM}, \quad M = aM + \frac{(b+c)M}{d+em},$$

That is

$$1 - a = \frac{b+c}{d+eM}, \quad 1 - a = \frac{b+c}{d+em},$$

or,

$$\frac{b+c}{d+eM} = \frac{b+c}{d+em},$$

then $d + eM = d + em$. Thus $M = m$. It follows by the Theorem B that \bar{x} is a global attractor of Eq.(1) and then the proof is complete.

5. NUMERICAL EXAMPLES

For confirming the results of this paper, we consider numerical examples which represent different types of solutions to Eq. (1).

Example 1. We assume $l = 3, k = 2, s = 3, t = 2, x_{-3} = 7, x_{-2} = 2, x_{-1} = 1, x_0 = 9, a = 0.1, b = 0.2, c = 0.9, d = 0.6, e = 0.3$. See Fig. 1.

Example 2. See Fig. 2, since $l = 4, k = 3, x_{-4} = 12, x_{-3} = 7, x_{-2} = 9, x_{-1} = 10, x_0 = 5, a = 0.9, b = 2, c = 7, d = 3$.

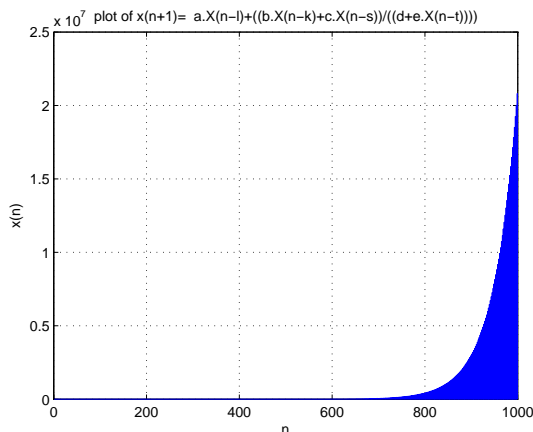


Figure 1.

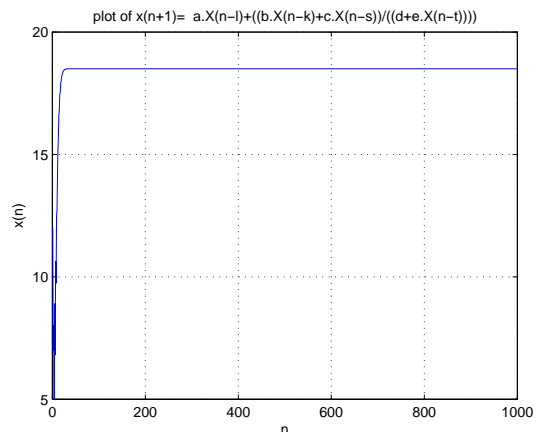


Figure 2.

Example 3. We consider $l = 3, k = 2, x_{-3} = 12, x_{-2} = 7, x_{-1} = 9, x_0 = 10, a = 0.3, b = 1.5, c = 11, d = 8$. See Fig. 3.

Example 4. See Fig. 4, since $l = 3$, $k = 4$, $x_{-4} = 12$, $x_{-3} = 7$, $x_{-2} = 9$, $x_{-1} = 10$, $x_0 = 5$, $a = 0.6$, $b = 2$, $c = 7$, $d = 4$.

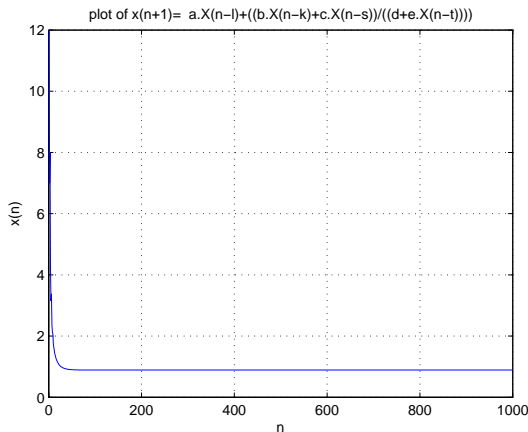


Figure 3.

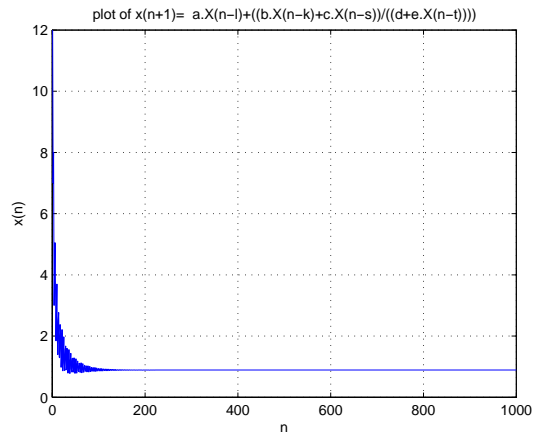


Figure 4.

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