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Editor in Chief: George Anastassiou
Department of Mathematical Sciences,
University of Memphis, Memphis, TN 38152-3240, U.S.A
ganastss@memphis.edu
http://www.msci.memphis.edu/~ganastss/jocaaa

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Editorial Board
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Dartmouth College
8000 Cummings Hall,
Hanover, NH 03755-8000
603-646-3843 (X 3546 Secr.)
e-mail:george.cybenko@dartmouth.edu
Approximation Theory and Neural Networks

Sever S. Dragomir
School of Computer Science and Mathematics, Victoria University,
PO Box 14428,
Melbourne City,
MC 8001, AUSTRALIA
Tel. +61 3 9688 4437
Fax +61 3 9688 4050
sever.dragomir@vu.edu.au
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Oktay Duman
TOBB University of Economics and Technology,
Department of Mathematics, TR-06530,
Ankara, Turkey,
oduman@etu.edu.tr
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Saber N. Elaydi
Department Of Mathematics
Trinity University
715 Stadium Dr.
San Antonio, TX 78212-7200
210-736-8246
e-mail: selaydi@trinity.edu
Ordinary Differential Equations, Difference Equations

Christodoulos A. Floudas
Department of Chemical Engineering
Princeton University
Princeton,NJ 08544-5263
609-258-4595(x4619 assistant)
e-mail: floudas@titan.princeton.edu
Optimization Theory&Applications, Global Optimization

J .A. Goldstein
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152
901-678-3130
jgoldste@memphis.edu
Partial Differential Equations, Semigroups of Operators

H. H. Gonska
Department of Mathematics
University of Duisburg
Duisburg, D-47048
Germany
011-49-203-379-3542
e-mail: heiner.gonska@uni-due.de
Approximation Theory, Computer Aided Geometric Design

John R. Graef
Department of Mathematics
University of Tennessee at Chattanooga
Chattanooga, TN 37304 USA
John-Graef@utc.edu
Ordinary and functional differential equations, difference equations, impulsive systems, differential inclusions, dynamic equations on time scales, control theory and their applications

Weimin Han
Department of Mathematics
University of Iowa
Iowa City, IA 52242-1419
319-335-0770
e-mail: whan@math.uiowa.edu
Numerical analysis, Finite element method, Numerical PDE, Variational inequalities, Computational mechanics

Tian-Xiao He
Department of Mathematics and Computer Science
P.O. Box 2900, Illinois Wesleyan University
Bloomington, IL 61702-2900, USA
Tel (309)556-3089
Fax (309)556-3864
the@iwu.edu
Approximations, Wavelet, Integration Theory, Numerical Analysis, Analytic Combinatorics

Margareta Heilmann
Faculty of Mathematics and Natural Sciences, University of Wuppertal
Gaußstraße 20
D-42119 Wuppertal, Germany,
heilmann@math.uni-wuppertal.de

National Technical University of Athens
Zografou campus, 157 80
Athens, Greece
tel:: +30(210) 772 1722
Fax +30(210) 772 1775
papanico@math.ntua.gr
Partial Differential Equations, Probability

Choonkil Park
Department of Mathematics
Hanyang University
Seoul 133-791
S. Korea, baak@hanyang.ac.kr
Functional Equations

Svetlozar (Zari) Rachev,
Professor of Finance, College of Business, and Director of Quantitative Finance Program, Department of Applied Mathematics & Statistics
Stonybrook University
312 Harriman Hall, Stony Brook, NY 11794-3775
tel: +1-631-632-1998,
svetlozar.rachev@stonybrook.edu

Alexander G. Ramm
Mathematics Department
Kansas State University
Manhattan, KS 66506-2602
e-mail: ramm@math.ksu.edu
Inverse and Ill-posed Problems, Scattering Theory, Operator Theory, Theoretical Numerical Analysis, Wave Propagation, Signal Processing and Tomography

Tomasz Rychlik
Polish Academy of Sciences
Instytut Matematyczny PAN
00-956 Warszawa, skr. poczt. 21
ul. Śniadeckich 8
Poland
trychlik@impan.pl
Mathematical Statistics, Probabilistic Inequalities

Boris Shekhtman
Department of Mathematics
University of South Florida
Tampa, FL 33620, USA
Tel 813-974-9710
shekhtma@usf.edu
Approximation Theory, Banach spaces, Classical Analysis

T. E. Simos
Department of Computer Science and Technology
Faculty of Sciences and Technology
University of Peloponnese
GR-221 00 Tripolis, Greece
Postal Address:
26 Menelaou St.
Anfithea - Paleon Faliron
GR-175 64 Athens, Greece
tsimos@mail.ariadne-t.gr
Numerical Analysis

H. M. Srivastava
Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia V8W 3R4
Canada
tel.250-472-5313; office,250-477-6960 home, fax 250-721-8962
harimsri@math.uvic.ca
Real and Complex Analysis, Fractional Calculus and Appl., Integral Equations and Transforms, Higher Transcendental Functions and Appl.,q-Series and q-Polynomials, Analytic Number Th.

I. P. Stavroulakis
Department of Mathematics
University of Ioannina
451-10 Ioannina, Greece
ipstav@cc.uoi.gr
Differential Equations
Phone +3-065-109-8283

Manfred Tasche
Department of Mathematics
University of Rostock
D-18051 Rostock, Germany
manfred.tasche@mathematik.uni-rostock.de
Numerical Fourier Analysis, Fourier Analysis, Harmonic Analysis, Signal Analysis, Spectral Methods, Wavelets, Splines, Approximation Theory

Roberto Triggiani
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152
PDE, Control Theory, Functional

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Journal of Computational Analysis and Applications

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Editor in Chief: George Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152-3240, U.S.A.

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Prof.George A. Anastassiou
Department of Mathematical Sciences
The University of Memphis
Memphis,TN 38152, USA.
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e-mail: ganastss@memphis.edu

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SOME PERTURBED VERSIONS OF THE GENERALIZED TRAPEZOID INEQUALITY FOR FUNCTIONS OF BOUNDED VARIATION

WENJUN LIU AND JAEKEUN PARK

Abstract. In this paper, we establish some perturbed versions of the generalized Trapezoid inequality for functions of bounded variation in terms of the cumulative variation function.

1. Introduction

In the past few years, many authors have considered various generalizations of some kinds of integral inequalities, which give explicit error bounds for some known and some new quadrature formulae. For example, in [6], Dragomir established the following generalized trapezoidal inequality for functions of bounded variation:

Theorem 1.1. *Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation. Then*

$$\left|\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t - \frac{(x-a)f(a)+(b-x)f(b)}{b-a}\right| \le \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\bigvee_a^b(f), \tag{1.1}$$

where $x \in [a, b]$ and $\bigvee_a^b(f)$ denotes the total variation of f on the interval $[a, b]$. The constant $\frac{1}{2}$ cannot be replaced by a smaller one. The best inequality one can derive from (1.1) *is the trapezoid inequality*

$$\left|\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t - \frac{f(a)+f(b)}{2}\right| \le \frac{1}{2}\bigvee_a^b(f). \tag{1.2}$$

Here the constant $\frac{1}{2}$ is also best possible.

For a function of bounded variation $v : [a, b] \to \mathbb{C}$, the Cumulative Variation Function (CVF) $V : [a, b] \to [0,\infty)$ is defined by

$$V(t) := \bigvee_a^t(v),$$

the total variation of v on the interval $[a, t]$ with $t \in [a, b]$. Recently, Dragomir [7] considered the refinement of (1.1) in terms of the cumulative variation function.

Theorem 1.2. *Let $f : [a, b] \to \mathbb{C}$ be a function of bounded variation on $[a, b]$. Then*

$$\begin{aligned}\left|\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t - \frac{(x-a)f(a)+(b-x)f(b)}{b-a}\right| &\le \frac{1}{b-a}\left[\int_a^x\left(\bigvee_a^t(f)\right)\mathrm{d}t + \int_x^b\left(\bigvee_t^b(f)\right)\mathrm{d}t\right] \\ &\le \frac{1}{b-a}\left[(x-a)\bigvee_a^x(f)+(b-x)\bigvee_x^b(f)\right] \\ &\le \begin{cases}\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\bigvee_a^b(f), \\ \left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right],\end{cases}\end{aligned} \tag{1.3}$$

for any $x \in [a, b]$.

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Key words and phrases. Generalized Trapezoid inequality, Cumulative variation, Function of bounded variation, Lipschitzian function, Monotonic function.

In order to extend the classical Ostrowski's inequality for differentiable functions with bounded derivatives to the larger class of functions of bounded variation, Dragomir obtained the following result in [13]:

Theorem 1.3. *Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation on $[a, b]$. Then, for all $x \in [a, b]$, we have the following inequality*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \right| \le \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \bigvee_a^b (f). \tag{1.4}$$

The constant $\frac{1}{2}$ is the best possible. The best inequality one can obtain from (1.4) *is the midpoint inequality*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \frac{1}{2}\bigvee_a^b (f), \tag{1.5}$$

for which the constant $\frac{1}{2}$ is also sharp.

Recently, Dragomir [8] considered the refinement of (1.4) in terms of the cumulative variation function.

Theorem 1.4. *Let $f : [a, b] \to \mathbb{C}$ be a function of bounded variation on $[a, b]$. Then*

$$\begin{aligned} \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \right| &\le \frac{1}{b-a}\left[\int_a^x \left(\bigvee_a^t (f) \right) \mathrm{d}t + \int_x^b \left(\bigvee_t^b (f) \right) \mathrm{d}t \right] \\ &\le \frac{1}{b-a}\left[(x-a)\bigvee_a^x (f) + (b-x)\bigvee_x^b (f) \right] \\ &\le \begin{cases} \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \bigvee_a^b (f), \\ \left[\frac{1}{2}\bigvee_a^b (f) + \frac{1}{2}\left| \bigvee_a^x (f) - \bigvee_x^b (f) \right| \right], \end{cases} \end{aligned} \tag{1.6}$$

for any $x \in [a, b]$.

Very recently, Dragomir [9] obtained the following perturbed Ostrowski type inequality for functions of bounded variation, in which he denoted $\ell : [a, b] \to [a, b]$ the identity function:

Theorem 1.5. *Let $f : [a, b] \to \mathbb{C}$ be a function of bounded variation on $[a, b]$, and $x \in [a, b]$. Then for any $\lambda_1(x)$ and $\lambda_2(x)$ complex numbers, we have*

$$\begin{aligned} &\left| f(x) + \frac{1}{2(b-a)}\left[(b-x)^2\lambda_2(x) - (x-a)^2\lambda_1(x) \right] - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \right| \\ \le& \frac{1}{b-a}\left[\int_a^x \left(\bigvee_t^x (f - \lambda_1(x)\ell) \right)\mathrm{d}t + \int_x^b \left(\bigvee_x^t (f - \lambda_2(x)\ell) \right)\mathrm{d}t \right] \\ \le& \frac{1}{b-a}\left[(x-a)\bigvee_a^x (f - \lambda_1(x)\ell) + (b-x)\bigvee_x^b (f - \lambda_2(x)\ell) \right] \\ \le& \begin{cases} \max\left\{ \bigvee_a^x (f - \lambda_1(x)\ell), \bigvee_x^b (f - \lambda_2(x)\ell) \right\}, \\ \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \left(\bigvee_a^x (f - \lambda_1(x)\ell) + \bigvee_x^b (f - \lambda_2(x)\ell) \right), \end{cases} \end{aligned} \tag{1.7}$$

where $\bigvee_c^d (g)$ denotes the total variation of g on the interval $[c, d]$.

For related results, see [1]-[5], [11]-[12], [14]-[32].

Motivated by the above works, the purpose of this paper is to establish some perturbed versions of the generalized trapezoid inequality (1.3) for functions of bounded variation in terms of the cumulative variation function.

2. Inequalities for functions of bounded variation

As in [7], it is known that the CVF is monotonic nondecreasing on $[a,b]$ and is continuous at a point $c \in [a,b]$ if and only if the generating function v is continuous at that point. If v is Lipschitzian with the constant $L > 0$, i.e.,

$$|v(t) - v(s)| \le L|t - s| \text{ for any } t, s \in [a,b],$$

then V is also Lipschitzian with the same constant.

The following lemma is of interest in itself as well, see also [10].

Lemma 2.1. *Let $f, u : [a,b] \to \mathbb{C}$. If f is continuous on $[a,b]$ and u is of bounded variation on $[a,b]$, then*

$$\left|\int_a^b f(t)\mathrm{d}u(t)\right| \le \int_a^b |f(t)|\mathrm{d}\left(\bigvee_a^t(u)\right) \le \max_{t\in[a,b]} |f(t)| \bigvee_a^b(u). \tag{2.1}$$

We have the following result:

Theorem 2.1. *Let $f : [a,b] \to \mathbb{C}$ be a function of bounded variation on $[a,b]$ and $x \in [a,b]$. Then for any $\lambda(x)$ complex number, we have the inequalities*

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t - \frac{(x-a)f(a)+(b-x)f(b)}{b-a} - \lambda(x)\left(x - \frac{a+b}{2}\right)\right| \\ \le& \frac{1}{b-a}\left[\int_a^x \left(\bigvee_a^t (f-\lambda(x)\ell)\right)\mathrm{d}t + \int_x^b \left(\bigvee_t^b (f-\lambda(x)\ell)\right)\mathrm{d}t\right] \\ \le& \frac{1}{b-a}\left[(x-a)\bigvee_a^x(f-\lambda(x)\ell) + (b-x)\bigvee_x^b(f-\lambda(x)\ell)\right] \\ \le& \begin{cases} \left[\dfrac{1}{2} + \left|\dfrac{x-\frac{a+b}{2}}{b-a}\right|\right]\bigvee_a^b(f-\lambda(x)\ell) \\ \dfrac{1}{2}\bigvee_a^b(f-\lambda(x)\ell) + \dfrac{1}{2}\left|\bigvee_a^x(f-\lambda(x)\ell) - \bigvee_x^b(f-\lambda(x)\ell)\right|, \end{cases} \end{aligned} \tag{2.2}$$

where $\bigvee_c^d(g)$ denotes the total variation of g on the interval $[c,d]$ and $\ell : [a,b] \to [a,b]$ is the identity function.

Proof. We shall start with the identity obtained in [6]

$$\int_a^b f(t)\mathrm{d}t - [(x-a)f(a) + (b-x)f(b)] = \int_a^b (x-t)\mathrm{d}f(t), \tag{2.3}$$

in which the integrals in the right hand side are taken in the Riemann-Stieltjes sense. If we replace $f(t)$ with $f(t) - \lambda(x)t$ in (2.3), then we can get the following equation:

$$\int_a^b f(t)\mathrm{d}t - [(x-a)f(a)+(b-x)f(b)] - \lambda(x)(b-a)\left(x - \frac{a+b}{2}\right) = \int_a^b (x-t)\mathrm{d}\left[f(t) - \lambda(x)t\right]. \tag{2.4}$$

Taking the modulus in (2.4) and using the property (2.1), we have

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t - \frac{[(x-a)f(a)+(b-x)f(b)]}{b-a} - \lambda(x)\left(x - \frac{a+b}{2}\right)\right| \\ \le& \frac{1}{b-a}\left|\int_a^b (x-t)\mathrm{d}\left[f(t)-\lambda(x)t\right]\right| \\ \le& \frac{1}{b-a}\int_a^b |x-t|\mathrm{d}\left(\bigvee_a^t(f-\lambda(x)\ell)\right) \end{aligned} \tag{2.5}$$

$$=\frac{1}{b-a}\left[\int_a^x (x-t)\mathrm{d}\left(\bigvee_a^t (f-\lambda(x)\ell)\right)+\int_x^b (t-x)\mathrm{d}\left(\bigvee_a^t (f-\lambda(x)\ell)\right)\right].$$

Integrating by parts in the Riemann-Stieltjes integral we have

$$\begin{aligned}(2.6)\qquad \int_a^x (x-t)\mathrm{d}\left(\bigvee_a^t (f-\lambda(x)\ell)\right) &= (x-t)\bigvee_a^t (f-\lambda(x)\ell)\Big|_{t=a}^{x}+\int_a^x \left(\bigvee_a^t (f-\lambda(x)\ell)\right)\mathrm{d}t\\ &= \int_a^x \left(\bigvee_a^t (f-\lambda(x)\ell)\right)\mathrm{d}t\end{aligned}$$

and

$$\begin{aligned}(2.7)\qquad \int_x^b (t-x)\mathrm{d}\left(\bigvee_a^t (f-\lambda(x)\ell)\right) &= (t-x)\bigvee_a^t (f-\lambda(x)\ell)\Big|_{t=x}^{b}-\int_x^b \left(\bigvee_a^t (f-\lambda(x)\ell)\right)\mathrm{d}t\\ &=(b-x)\bigvee_a^b (f-\lambda(x)\ell)-\int_x^b \left(\bigvee_a^t (f-\lambda(x)\ell)\right)\mathrm{d}t\\ &= \int_x^b \left(\bigvee_t^b (f-\lambda(x)\ell)\right)\mathrm{d}t.\end{aligned}$$

Using (2.5)-(2.7), we deduce the first inequality in (2.2).

Since

$$\bigvee_a^t (f-\lambda(x)\ell)\le \bigvee_a^x (f-\lambda(x)\ell)\quad \text{for}\quad t\in[a,x]$$

and

$$\bigvee_t^b (f-\lambda(x)\ell)\le \bigvee_x^b (f-\lambda(x)\ell)\quad \text{for}\quad t\in[x,b],$$

then

$$\int_a^x \left(\bigvee_a^t (f-\lambda(x)\ell)\right)\mathrm{d}t\le (x-a)\bigvee_a^x (f-\lambda(x)\ell)$$

and

$$\int_x^b \left(\bigvee_t^b (f-\lambda(x)\ell)\right)\mathrm{d}t\le (b-x)\bigvee_x^b (f-\lambda(x)\ell),$$

which prove the second inequality in (2.2).

With the max properties we have

$$\begin{aligned}&(x-a)\bigvee_a^x (f-\lambda(x)\ell)+(b-x)\bigvee_x^b (f-\lambda(x)\ell)\\ &\le\begin{cases}\max\{x-a,b-x\}\bigvee_a^b (f-\lambda(x)\ell)\\ \max\left\{\bigvee_a^x (f-\lambda(x)\ell),\bigvee_x^b (f-\lambda(x)\ell)\right\}(b-a)\end{cases}\\ &\le\begin{cases}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b (f-\lambda(x)\ell)\\ \left[\frac{1}{2}\bigvee_a^b (f-\lambda(x)\ell)+\frac{1}{2}\left|\bigvee_a^x (f-\lambda(x)\ell)-\bigvee_x^b (f-\lambda(x)\ell)\right|\right](b-a),\end{cases}\end{aligned}$$

which completes the proof. □

The following trapezoid type inequality holds:

Corollary 2.1. *Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$. Then for any $\lambda\in\mathbb{C}$, we have the inequalities*

$$\left|\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t-\frac{f(a)+f(b)}{2}\right| \le \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}}\left(\bigvee_a^t(f-\lambda\ell)\right)\mathrm{d}t+\int_{\frac{a+b}{2}}^b\left(\bigvee_t^b(f-\lambda\ell)\right)\mathrm{d}t\right] \le \frac{1}{2}\bigvee_a^b(f-\lambda\ell), \tag{2.8}$$

which is equivalent to

$$\left|\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t-\frac{f(a)+f(b)}{2}\right| \le \frac{1}{b-a}\inf_{\lambda\in\mathbb{C}}\left[\int_a^{\frac{a+b}{2}}\left(\bigvee_a^t(f-\lambda\ell)\right)\mathrm{d}t+\int_{\frac{a+b}{2}}^b\left(\bigvee_t^b(f-\lambda\ell)\right)\mathrm{d}t\right] \le \frac{1}{2}\inf_{\lambda\in\mathbb{C}}\left[\bigvee_a^b(f-\lambda\ell)\right]. \tag{2.9}$$

3. Inequalities for Lipschitzian functions

We can state the following result:

Theorem 3.1. *Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$ and $x\in(a,b)$. If $\lambda(x)$ is a complex number and there exists the positive number $L(x)$ such that $f-\lambda(x)\ell$ is Lipschitzian with the constant $L(x)$ on the interval $[a,b]$, then*

$$\left|\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t-\frac{(x-a)f(a)+(b-x)f(b)}{b-a}-\lambda(x)\left(x-\frac{a+b}{2}\right)\right| \le \frac{L(x)}{b-a}\left[\left(x-\frac{a+b}{2}\right)^2+\frac{(b-a)^2}{4}\right]. \tag{3.1}$$

Proof. It's known that, if $g:[c,d]\to\mathbb{C}$ is Riemann integrable and $u:[c,d]\to\mathbb{C}$ is Lipschitzian with the constant $L>0$, then the Riemann-Stieltjes integral $\int_c^d g(t)\mathrm{d}u(t)$ exists and

$$\left|\int_c^d g(t)\mathrm{d}u(t)\right|\le L\int_c^d|g(t)|\mathrm{d}t. \tag{3.2}$$

Taking the modulus in (2.4) and using the property (3.2) we have

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t-\frac{(x-a)f(a)+(b-x)f(b)}{b-a}-\lambda(x)\left(x-\frac{a+b}{2}\right)\right| \\ \le&\frac{1}{b-a}\left|\int_a^b(x-t)\mathrm{d}\left[f(t)-\lambda(x)t\right]\right| \\ \le&\frac{L(x)}{b-a}\left[\int_a^x(x-t)\mathrm{d}t+\int_x^b(t-x)\mathrm{d}t\right] \\ =&\frac{L(x)}{b-a}\left[\left(x-\frac{a+b}{2}\right)^2+\frac{(b-a)^2}{4}\right],\end{aligned} \tag{3.3}$$

which proves the result. □

Corollary 3.1. *Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$. If λ is a complex number and there exists the positive number L such that $f-\lambda\ell$ is Lipschitzian with the constant L on the interval $[a,b]$, then*

$$\left|\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t-\frac{f(a)+f(b)}{2}\right|\le\frac{1}{4}L(b-a). \tag{3.4}$$

4. Inequalities for Monotonic functions

Now, the case of monotonic integrators is as follows:

Theorem 4.1. *Let $f : [a,b] \to \mathbb{C}$ be a function of bounded variation on $[a,b]$ and $x \in (a,b)$. If $\lambda(x)$ is a real number such that $f - \lambda(x)\ell$ is monotonic nondecreasing on the interval $[a,b]$, then*

$$\begin{aligned}
(4.1)\qquad &\left| \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t - \frac{(x-a)f(a)+(b-x)f(b)}{b-a} - \lambda(x)\left(x - \frac{a+b}{2}\right) \right| \\
\le& \frac{1}{b-a}\left[(b-x)f(b) - (x-a)f(a) - \frac{1}{2}\lambda(x)[(b-x)^2+(x-a)^2] - \int_a^b sgn(t-x)f(t)\mathrm{d}t\right] \\
\le& \frac{1}{b-a}\left\{(x-a)[f(x)-f(a)-\lambda(x)(x-a)] + (b-x)[f(b)-f(x)-\lambda(x)(b-x)]\right\}
\end{aligned}$$

Proof. It's known that, if $g : [c,d] \to \mathbb{C}$ is continuous and $u : [c,d] \to \mathbb{C}$ is monotonic nondecreasing, then the Riemann-Stieltjes integral $\int_c^d g(t)\mathrm{d}u(t)$ exists and

$$(4.2)\qquad \left|\int_c^d g(t)\mathrm{d}u(t)\right| \le \int_c^d |g(t)|\mathrm{d}u(t).$$

Taking the modulus in (2.4) and using the property (4.2) we have

$$\begin{aligned}
(4.3)\qquad &\left| \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t - \frac{[(x-a)f(a)+(b-x)f(b)]}{b-a} - \lambda(x)\left(x - \frac{a+b}{2}\right) \right| \\
\le& \frac{1}{b-a}\left|\int_a^b (x-t)\mathrm{d}\left[f(t)-\lambda(x)t\right]\right| \\
\le& \frac{1}{b-a}\left[\int_a^x (x-t)\mathrm{d}[f(t)-\lambda(x)t] + \int_x^b (t-x)\mathrm{d}[f(t)-\lambda(x)t]\right].
\end{aligned}$$

Integrating by parts in the Riemann-Stieltjes integral we have

$$\begin{aligned}
&\int_a^x (x-t)\mathrm{d}[f(t)-\lambda(x)t] \\
=&(x-t)[f(t)-\lambda(x)t]\Big|_{t=a}^{x} + \int_a^x [f(t)-\lambda(x)t]\mathrm{d}t \\
=&-(x-a)[f(a)-\lambda(x)a] + \int_a^x f(t)\mathrm{d}t - \lambda(x)\frac{x^2-a^2}{2} \\
=&-(x-a)f(a) + \lambda(x)a(x-a) + \int_a^x f(t)\mathrm{d}t - \lambda(x)\frac{x^2-a^2}{2} \\
=&-(x-a)f(a) - \frac{1}{2}\lambda(x)(x-a)^2 + \int_a^x f(t)\mathrm{d}t
\end{aligned}$$

and

$$\begin{aligned}
&\int_x^b (t-x)\mathrm{d}[f(t)-\lambda(x)t] \\
=&(t-x)[f(t)-\lambda(x)t]\Big|_{t=x}^{b} - \int_x^b [f(t)-\lambda(x)t]\mathrm{d}t \\
=&(b-x)[f(b)-\lambda(x)b] - \int_x^b f(t)\mathrm{d}t + \lambda(x)\frac{b^2-x^2}{2} \\
=&(b-x)f(b) - \frac{1}{2}\lambda(x)(b-x)^2 - \int_x^b f(t)\mathrm{d}t.
\end{aligned}$$

If we add these equalities, we get

$$\int_a^x (x-t)\mathrm{d}[f(t)-\lambda(x)t] + \int_x^b (t-x)\mathrm{d}[f(t)-\lambda(x)t]$$

$$=(b-x)f(b)-(x-a)f(a)-\frac{1}{2}\lambda(x)[(b-x)^2+(x-a)^2]-\int_a^b sgn(t-x)f(t)\mathrm{d}t$$

and by (4.3) we get the first inequality in (4.1).

Now, since $f-\lambda(x)\ell$ is monotonic nondecreasing on the interval $[a,b]$, then

$$\begin{aligned}&\int_a^x (x-t)\mathrm{d}[f(t)-\lambda(x)t]\\ \le&(x-a)[f(x)-\lambda(x)x-f(a)+\lambda(x)a]\\ =&(x-a)[f(x)-f(a)-\lambda(x)(x-a)]\end{aligned}$$

and

$$\begin{aligned}&\int_x^b (t-x)\mathrm{d}[f(t)-\lambda(x)t]\\ \le&(b-x)[f(b)-\lambda(x)b-f(x)+\lambda(x)x]\\ =&(b-x)[f(b)-f(x)-\lambda(x)(b-x)],\end{aligned}$$

which completes the proof. □

Corollary 4.1. *Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$. If λ is a real number such that $f-\lambda\ell$ is monotonic nondecreasing on the interval $[a,b]$, then*

$$\left|\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t-\frac{f(a)+f(b)}{2}\right|\le\frac{1}{2}[f(b)-f(a)-\lambda(b-a)]. \tag{4.4}$$

5. Conclusions

Some explicit error bounds for known or new quadrature formulae are given recently through various generalizations of some kinds of integral inequalities. In this paper, by using the ideas of Dragomir in [9], we establish some perturbed versions of the generalized trapezoid inequality for functions of bounded variation in terms of the cumulative variation function. These results can be regarded as further generalizations of [6], in which the generalized trapezoidal inequality for functions of bounded variation are established.

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(W. J. Liu) College of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, China
E-mail address: `wjliu@nuist.edu.cn`

(J. K. Park) Department of Mathematics, Hanseo University, Chungnam-do, Seosan-si 356-706, Republic of Korea
E-mail address: `jkpark@hanseo.ac.kr`

A COMPANION OF OSTROWSKI LIKE INEQUALITY AND APPLICATIONS TO COMPOSITE QUADRATURE RULES

WENJUN LIU AND JAEKEUN PARK

Abstract. A companion of Ostrowski like inequality for mappings whose second derivatives belong to L^∞ spaces is established. Applications to composite quadrature rules are also given.

1. Introduction

In 1938, Ostrowski established the following interesting integral inequality (see [24]) for differentiable mappings with bounded derivatives:

Theorem 1.1. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) whose derivative is bounded on (a, b) and denote $\|f'\|_\infty = \sup_{t\in(a,b)} |f'(t)| < \infty$. Then for all $x \in [a, b]$ we have*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \le \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2}\right](b-a)\|f'\|_\infty. \tag{1.1}$$

The constant $\frac{1}{4}$ is sharp in the sense that it can not be replaced by a smaller one.

This inequality has attracted considerable interest over the years, and many authors proved generalizations, modifications and applications of it. For example, the early work of Milovanović and Pečarić [21, 23] extended this inequality for differentiable mappings with bounded derivatives, to functions f that are n times differentiable with $|f^{(n)}| \le M$ and gave an application to quadrature. In [8], motivated by [12], Dragomir proved some companions of Ostrowski's inequality, as follows:

Theorem 1.2. *Let $f : [a, b] \to \mathbb{R}$ be an absolutely continuous function on $[a, b]$. Then the following inequalities*

$$\left|\frac{f(x) + f(a+b-x)}{2} - \frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\le \begin{cases} \left[\frac{1}{8} + 2\left(\frac{x-\frac{3a+b}{4}}{b-a}\right)^2\right](b-a)\|f'\|_\infty, & f' \in L^\infty[a,b], \\ \frac{2^{1/q}}{(q+1)^{1/q}}\left[\left(\frac{x-a}{b-a}\right)^{q+1} + \left(\frac{\frac{a+b}{2}-x}{b-a}\right)^{q+1}\right]^{1/q}(b-a)^{1/q}\|f'\|_p, & \\ \qquad p > 1, \frac{1}{p} + \frac{1}{q} = 1 \quad \text{and} & f' \in L^p[a,b], \\ \left[\frac{1}{4} + \left|\frac{x-\frac{3a+b}{4}}{b-a}\right|\right]\|f'\|_1, & f' \in L^1[a,b] \end{cases} \tag{1.2}$$

hold for all $x \in [a, \frac{a+b}{2}]$.

Recently, Alomari [1] introduced a companion of Dragomir's generalization of Ostrowsk's inequality for absolutely continuous mappings whose first derivatives are belong to $L^\infty([a, b])$.

Theorem 1.3. *Let $f : [a, b] \to \mathbb{R}$ be an absolutely continuous mappings on (a, b) whose derivative is bounded on $[a, b]$. Then the inequality*

$$\left|\left[(1-\lambda)\frac{f(x) + f(a+b-x)}{2} + \lambda\frac{f(a) + f(b)}{2}\right] - \frac{1}{b-a}\int_a^b f(t)dt\right|$$

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$$\leq \left[\frac{1}{8}(2\lambda^2+(1-\lambda)^2)+2\frac{\left(x-\frac{(3-\lambda)a+(1+\lambda)b}{4}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty \tag{1.3}$$

holds for all $\lambda \in [0,1]$ *and* $x \in [a+\lambda\frac{b-a}{2}, \frac{a+b}{2}]$.

In (1.3), choose $\lambda = \frac{1}{2}$, one can get

$$\left|\frac{1}{2}\left[\frac{f(x)+f(a+b-x)}{2}+\frac{f(a)+f(b)}{2}\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\leq \left[\frac{3}{32}+2\frac{(x-\frac{5a+3b}{8})^2}{(b-a)^2}\right](b-a)\|f'\|_\infty. \tag{1.4}$$

And if choose $x = \frac{a+b}{2}$, then one has

$$\left|\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{1}{8}(b-a)\|f'\|_\infty. \tag{1.5}$$

It's shown in [1] that the constant $\frac{1}{8}$ is the best possible.

In related work, Dragomir and Sofo [10] developed the following Ostrowski like integral inequality for twice differentiable mapping.

Theorem 1.4. *Let* $f : [a,b] \to \mathbb{R}$ *be a mapping whose first derivative is absolutely continuous on* $[a,b]$ *and assume that the second derivative* $f'' \in L^\infty([a,b])$*. Then we have the inequality*

$$\left|\frac{1}{2}\left[f(x)+\frac{f(a)+f(b)}{2}\right]-\frac{1}{2}\left(x-\frac{a+b}{2}\right)f'(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\leq \left[\frac{1}{48}+\frac{1}{3}\frac{|x-\frac{a+b}{2}|^3}{(b-a)^3}\right](b-a)^2\|f''\|_\infty, \tag{1.6}$$

for all $x \in [a,b]$.

In (1.6), the authors pointed out that the midpoint $x = \frac{a+b}{2}$ gives the best estimator, i.e.,

$$\left|\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{1}{48}(b-a)^2\|f''\|_\infty. \tag{1.7}$$

In fact, we can choose $f(t) = (t-a)^2$ in (1.7) to prove that the constant $\frac{1}{48}$ in inequality (1.7) is sharp.

For other related results, the reader may be refer to [2, 3, 4, 5, 7, 9, 11, 13, 14, 15, 16, 17, 18, 19, 20, 22, 25, 26, 27, 28, 29, 30] and the references therein. Motivated by previous works [1, 6, 8, 10], we investigate in this paper a companion of the above mentioned Ostrowski like integral inequality for twice differentiable mappings. Our result gives a smaller estimator than (1.7) (see (2.9) below). Some applications to composite quadrature rules are also given.

2. A companion of Ostrowski like inequality

The following companion of Ostrowski like inequality holds:

Theorem 2.1. *Let* $f : [a,b] \to \mathbb{R}$ *be a mapping whose first derivative is absolutely continuous on* $[a,b]$ *and assume that the second derivative* $f'' \in L^\infty([a,b])$*. Then we have the inequality*

$$\left|\frac{1}{2}\left[\frac{f(x)+f(a+b-x)}{2}+\frac{f(a)+f(b)}{2}\right]\right.$$
$$\left.-\frac{1}{2}\left(x-\frac{a+b}{2}\right)\frac{f'(x)-f'(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\leq \left[\frac{1}{3}\frac{(\frac{a+3b}{4}-x)(x-a)^2}{(b-a)^3}+\frac{1}{3}\frac{(\frac{a+b}{2}-x)^3}{(b-a)^3}\right](b-a)^2\|f''\|_\infty \tag{2.1}$$

for all $x \in [a, \frac{a+b}{2}]$*. The first constant* $\frac{1}{3}$ *in the right hand side of* (2.1) *is sharp in the sense that it can not be replaced by a smaller one provided that* $x \neq \frac{a+3b}{4}$ *and* $x \neq a$*.*

Proof. Define the kernel $K(t):[a,b]\to\mathbb{R}$ by

$$K(t):=\begin{cases} t-a, & t\in[a,x],\\ t-\frac{a+b}{2}, & t\in(x,a+b-x],\\ t-b, & t\in(a+b-x,b],\end{cases} \tag{2.2}$$

for all $x\in[a,\frac{a+b}{2}]$. Integrating by parts, we obtain (see [8])

$$\frac{1}{b-a}\int_a^b K(t)g'(t)dt=\frac{g(x)+g(a+b-x)}{2}-\frac{1}{b-a}\int_a^b g(t)dt. \tag{2.3}$$

Now choose in (2.3), $g(x)=(x-\frac{a+b}{2})f'(x)$, to get

$$\begin{aligned}&\frac{1}{b-a}\int_a^b K(t)\left[f'(t)+\left(t-\frac{a+b}{2}\right)f''(t)\right]dt\\ =&\frac{1}{2}\left(x-\frac{a+b}{2}\right)[f'(x)-f'(a+b-x)]-\frac{1}{b-a}\int_a^b\left(t-\frac{a+b}{2}\right)f'(t)dt.\end{aligned} \tag{2.4}$$

Integrating by parts, we have

$$\frac{1}{b-a}\int_a^b\left(t-\frac{a+b}{2}\right)f'(t)dt=\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt. \tag{2.5}$$

Also upon using (2.3), we get

$$\begin{aligned}&\frac{1}{b-a}\int_a^b K(t)\left[f'(t)+\left(t-\frac{a+b}{2}\right)f''(t)\right]dt\\ =&\frac{1}{b-a}\int_a^b K(t)f'(t)dt+\frac{1}{b-a}\int_a^b K(t)\left(t-\frac{a+b}{2}\right)f''(t)dt\\ =&\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)dt+\frac{1}{b-a}\int_a^b K(t)\left(t-\frac{a+b}{2}\right)f''(t)dt.\end{aligned} \tag{2.6}$$

It follows from (2.4)–(2.6) that

$$\begin{aligned}&\frac{1}{2(b-a)}\int_a^b K(t)\left(t-\frac{a+b}{2}\right)f''(t)dt\\ =&\frac{1}{b-a}\int_a^b f(t)dt-\frac{1}{2}\left[\frac{f(x)+f(a+b-x)}{2}+\frac{f(a)+f(b)}{2}\right]\\ &+\frac{1}{2}\left(x-\frac{a+b}{2}\right)\frac{f'(x)-f'(a+b-x)}{2}.\end{aligned} \tag{2.7}$$

Now using (2.7) we obtain

$$\begin{aligned}&\left|\frac{1}{2}\left[\frac{f(x)+f(a+b-x)}{2}+\frac{f(a)+f(b)}{2}\right]\right.\\ &\left.-\frac{1}{2}\left(x-\frac{a+b}{2}\right)\frac{f'(x)-f'(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right|\\ \le&\frac{\|f''\|_\infty}{2(b-a)}\int_a^b|K(t)|\left|t-\frac{a+b}{2}\right|dt.\end{aligned} \tag{2.8}$$

Since $x\in[a,\frac{a+b}{2}]$, we have

$$\begin{aligned}I:=&\int_a^b|K(t)|\left|t-\frac{a+b}{2}\right|dt\\ =&\int_a^x(t-a)\left|t-\frac{a+b}{2}\right|dt+\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^2dt+\int_{a+b-x}^b(b-t)\left|t-\frac{a+b}{2}\right|dt\\ =&\int_a^x(t-a)\left(\frac{a+b}{2}-t\right)dt+\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^2dt+\int_{a+b-x}^b(b-t)\left(t-\frac{a+b}{2}\right)dt\\ =&\frac{(a+3b-4x)(x-a)^2}{12}+\frac{2}{3}\left(\frac{a+b}{2}-x\right)^3+\frac{(a+3b-4x)(x-a)^2}{12}\end{aligned}$$

$$=\frac{(a+3b-4x)(x-a)^2}{6}+\frac{2}{3}\left(\frac{a+b}{2}-x\right)^3,$$

and referring to (2.8), we obtain the result (2.1).

The sharpness of the constant $\frac{1}{3}$ can be proved in a special case for $x=\frac{a+b}{2}$ (see the line behind (1.7)). □

Remark 1. *If we take $x=\frac{a+b}{2}$ in (2.1), we recapture the sharp inequality (1.7). If we take $x=a$ in (2.1), we obtain the perturbed trapezoid type inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{b-a}{8}[f'(b)-f'(a)]-\frac{1}{b-a}\int_a^b f(t)dt\right|\le\frac{(b-a)^2}{24}\|f''\|_\infty,$$

which has a smaller estimator than the sharp trapezoid inequality

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right|\le\frac{(b-a)^2}{8}\|f''\|_\infty$$

stated in [11, Proposition 2.7].

Remark 2. *Consider*

$$F(x)=\left(\frac{a+3b}{4}-x\right)(x-a)^2+\left(\frac{a+b}{2}-x\right)^3$$

for $x\in[a,\frac{a+b}{2}]$. It's easy to know that $F(x)$ obtains its minimal value at $x=\frac{3a+b}{4}$. Therefore, in (2.1), the point $x=\frac{3a+b}{4}$ gives the best estimator, i.e.,

$$\left|\frac{1}{2}\left[\frac{f(\frac{3a+b}{4})+f(\frac{a+3b}{4})}{2}+\frac{f(a)+f(b)}{2}\right]+\frac{b-a}{8}\frac{f'(\frac{3a+b}{4})-f'(\frac{a+3b}{4})}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\le\frac{1}{64}(b-a)^2\|f''\|_\infty, \tag{2.9}$$

the right hand side of which is smaller than that of (1.7).

3. Application to Composite Quadrature Rules

In [10], the authors utilized inequality (1.6) to give estimates of composite quadrature rules which was pointed out have a markedly smaller error than that which may be obtained by the classical results. In this section, we apply our previous inequality (2.1) to give us estimates of new composite quadrature rules which have a further smaller error.

Theorem 3.1. *Let $I_n: a=x_0<x_1<\cdots<x_{n-1}<x_n=b$ be a partition of the interval $[a,b]$, $h_i=x_{i+1}-x_i$, $\nu(h):=\max\{h_i: i=1,\cdots,n\}$, $\xi_i\in[x_i,\frac{x_i+x_{i+1}}{2}]$, and*

$$\begin{aligned}S(f,I_n,\xi)=&\frac{1}{4}\sum_{i=0}^{n-1}\left[f(x_i)+f(\xi_i)+f(x_i+x_{i+1}-\xi_i)+f(x_{i+1})\right]h_i\\&-\frac{1}{4}\sum_{i=0}^{n-1}h_i\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)\left[f'(\xi_i)-f'(x_i+x_{i+1}-\xi_i)\right],\end{aligned}$$

then

$$\int_a^b f(x)dx=S(f,I_n,\xi)+R(f,I_n,\xi)$$

and the remainder $R(f,I_n,\xi)$ satisfies the estimate

$$|R(f,I_n,\xi)|\le\frac{1}{3}\|f''\|_\infty\left[\sum_{i=0}^{n-1}\left(\frac{x_i+3x_{i+1}}{4}-\xi_i\right)(x_i-\xi_i)^2+\sum_{i=0}^{n-1}\left(\frac{x_i+x_{i+1}}{2}-\xi_i\right)^3\right]. \tag{3.1}$$

Proof. Inequality (2.1) can be written as

$$\begin{aligned}&\left|\int_a^b f(t)dt - \frac{1}{4}\left[f(a)+f(x)+f(a+b-x)+f(b)\right](b-a)\right.\\&\left.+\frac{b-a}{4}\left(x-\frac{a+b}{2}\right)\left[f'(x)-f'(a+b-x)\right]\right|\\&\le\frac{1}{3}\left[\left(\frac{a+3b}{4}-x\right)(x-a)^2+\left(\frac{a+b}{2}-x\right)^3\right]\|f''\|_\infty. \qquad (3.2)\end{aligned}$$

Applying (3.2) on $\xi_i \in [x_i, \frac{x_i+x_{i+1}}{2}]$, we have

$$\begin{aligned}&\left|\int_{x_i}^{x_{i+1}} f(t)dt - \frac{1}{4}\left[f(x_i)+f(\xi_i)+f(x_i+x_{i+1}-\xi_i)+f(x_{i+1})\right]h_i\right.\\&\left.+\frac{h_i}{4}\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)\left[f'(\xi_i)-f'(x_i+x_{i+1}-\xi_i)\right]\right|\\&\le\frac{1}{3}\left[\left(\frac{x_i+3x_{i+1}}{4}-\xi_i\right)(x_i-\xi_i)^2+\left(\frac{x_i+x_{i+1}}{2}-\xi_i\right)^3\right]\|f''\|_\infty.\end{aligned}$$

Now summing over i from 0 to $n-1$ and utilizing the triangle inequality, we have

$$\begin{aligned}\left|\int_a^b f(t)dt - S(f,I_n,\xi)\right| =&\left|\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} f(t)dt - \frac{1}{4}\sum_{i=0}^{n-1}\left[f(x_i)+f(\xi_i)+f(x_i+x_{i+1}-\xi_i)+f(x_{i+1})\right]h_i\right.\\&\left.+\frac{1}{4}\sum_{i=0}^{n-1}h_i\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)\left[f'(\xi_i)-f'(x_i+x_{i+1}-\xi_i)\right]\right|\\\le&\frac{1}{3}\|f''\|_\infty\sum_{i=0}^{n-1}\left[\left(\frac{x_i+3x_{i+1}}{4}-\xi_i\right)(x_i-\xi_i)^2+\left(\frac{x_i+x_{i+1}}{2}-\xi_i\right)^3\right]\end{aligned}$$

and therefore (3.1) holds. □

Corollary 3.1. *If we choose* $\xi_i = \frac{3x_i+x_{i+1}}{4}$, *then we have*

$$\begin{aligned}\overline{S}(f,I_n) =&\frac{1}{4}\sum_{i=0}^{n-1}\left[f(x_i)+f\left(\frac{3x_i+x_{i+1}}{4}\right)+f\left(\frac{x_i+3x_{i+1}}{4}\right)+f(x_{i+1})\right]h_i\\&+\frac{1}{16}\sum_{i=0}^{n-1}\left[f'\left(\frac{3x_i+x_{i+1}}{4}\right)-f'\left(\frac{x_i+3x_{i+1}}{4}\right)\right]h_i^2\end{aligned}$$

and

$$|\overline{R}(f,I_n)| \le \frac{1}{64}\|f''\|_\infty\sum_{i=0}^{n-1}h_i^3. \qquad (3.3)$$

Remark 3. *It is obvious that inequality* (3.3) *is better than* [10, inequality (3.1)] *due to a smaller error.*

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(W. J. Liu) College of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, China
E-mail address: wjliu.cn@gmail.com

(J. K. Park) Department of Mathematics, Hanseo University, Chungnam-do, Seosan-si 356-706, Republic of Korea
E-mail address: jkpark@hanseo.ac.kr

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A modified shift-splitting preconditioner for saddle point problems *

Li-Tao Zhang†

Department of Mathematics and Physics, Zhengzhou University of Aeronautics, Zhengzhou, Henan, 450015, P. R. China

Abstract

Recently, Cao, Du and Niu [Shift-splitting preconditioners for saddle point problems, *Journal of Computational and Applied Mathematics*, 272 (2014) 239-250] introduced a shift-splitting preconditioner for saddle point problems. In this paper, we establish a modified shift-splitting preconditioner for solving the large sparse augmented systems of linear equations. Furthermore, the preconditioner is based on a modified shift-splitting of the saddle point matrix, resulting in an unconditional convergent fixed-point iteration, which is a generalization of shift-splitting preconditioners. Finally, numerical examples show the spectrum of the new preconditioned matrix for the different parameters.

Key words: Saddle point problem; Shift-splitting; Krylov subspace methods; Convergence; Preconditioner.

MSC: 65F10; 65F15; 65F50

1 Introduction

For solving the large sparse augmented systems of linear equations

$$\mathcal{A}u = \begin{pmatrix} A & B^T \\ -B & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \equiv b, \tag{1}$$

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†E-mail: litaozhang@163.com.

where $A \in \mathbb{R}^{n\times n}$ is a symmetric and positive definite matrix and $B \in \mathbb{R}^{m\times n}$ is a matrix of full row rank and $m < n, x, f \in \mathbb{R}^n, y, g \in \mathbb{R}^m$. It appears in many different applications of scientific computing, such as constrained optimization [32], the finite element method for solving the Navier-Stokes equation [24, 25, 26], and constrained least squares problems and generalized least squares problems [1, 29, 35, 36]. There have been several recent papers [2-24,25-29,30,31,33,37-40] for solving the augmented system (1). Santos et al. [29] studied preconditioned iterative methods for solving the singular augmented system with $A = I$. Yuan et al. [35, 36] proposed several variants of SOR method and preconditioned conjugate gradient methods for solving general augmented system (1) arising from generalized least squares problems where A can be symmetric and positive semidefinite and B can be rank deficient. The SOR-like method requires less arithmetic work per iteration step than other methods but it requires choosing an optimal iteration parameter in order to achieve a comparable rate of convergence. Golub et al. [27] presented SOR-like algorithms for solving system (1). Darvishi et al. [23] studied SSOR method for solving the augmented systems. Bai et al. [2, 3, 22, 40] presented GSOR method, parameterized Uzawa (PU) and the inexact parameterized Uzawa (PIU) methods for solving systems (1). Zhang and Lu [37] showed the generalized symmetric SOR method for augmented systems. Peng and Li [28] studied unsymmetric block overrelaxation-type methods for saddle point. Bai and Golub [4, 5, 6, 7, 11, 31] presented splitting iteration methods such as Hermitian and skew-Hermitian splitting (HSS) iteration scheme and its preconditioned variants, Krylov subspace methods such as preconditioned conjugate gradient (PCG), preconditioned MINRES (PMINRES) and restrictively preconditioned conjugate gradient (RPCG) iteration schemes, and preconditioning techniques related to Krylov subspace methods such as HSS, block-diagonal, block-triangular and constraint preconditioners and so on. Bai and Wang's 2009 LAA paper [31] and Chen and Jiang's 2008 AMC paper [22] studied some general approaches about the relaxed splitting iteration methods. Wu, Huang and Zhao [33] presented modified SSOR (MSSOR) method for augmented systems. Recently, Cao, Du and Niu [19] introduced a shift-splitting preconditioner and a local shift-splitting preconditioner for saddle point problems (1). Moreover, the authors studied some properties of the local shift-splitting preconditioned matrix and numerical experiments of a model Stokes problem are presented to show the effectiveness of the proposed preconditioners.

For large, sparse or structure matrices, iterative methods are an attractive option. In particular, Krylov subspace methods apply techniques that involve orthogonal projections onto subspaces of the form

$$\mathcal{K}(\mathcal{A}, b) \equiv \text{span}\left\{b, \mathcal{A}b, \mathcal{A}^2 b, ..., \mathcal{A}^{n-1}b, ...\right\}.$$

The conjugate gradient method (CG), minimum residual method (MINRES) and generalized minimal residual method (GMRES) are common Krylov subspace methods. The CG method is used for symmetric, positive definite matrices, MINRES for symmetric and possibly indefinite matrices and GMRES for unsymmetric matrices [30].

In this paper, based on shift-splitting preconditioners presented by Cao, Du and Niu [19], we establish a modified shift-splitting preconditioner for saddle point problems. Furthermore, the preconditioner is based on a modified shift-splitting of the saddle point matrix, resulting in an unconditional convergent fixed-point iteration, which is a generalization of shift-splitting preconditioners. Finally, numerical examples show the effectiveness of the proposed preconditioners. However, the relaxed parameters of the modified shift-splitting methods are not optimal and only lie in the convergence region of the method.

2 Modified shift-splitting preconditioner

Recently, for the coefficient matrix of the augmented system (1), Cao, Du and Niu [19] presented a shift-splitting precnditioner

$$\mathcal{P}_{SS} = \frac{1}{2}(\alpha I + \mathcal{A}),$$

where α is a positive constant and I is an identity matrix. This shift-splitting precoditioner $\mathcal{P}_{SS}$ is constructed by the shift-splitting of the matrix $\mathcal{A}$

$$\mathcal{A} = \mathcal{P}_{SS} - \mathcal{Q}_{SS} = \frac{1}{2}(\alpha I + \mathcal{A}) - \frac{1}{2}(\alpha I - \mathcal{A}),$$

which naturally leads to the shift-splitting scheme

$$u^{k+1} = (\alpha I + \mathcal{A})^{-1}(\alpha I - \mathcal{A})u^k + 2(\alpha I + \mathcal{A})^{-1}b, k = 0, 1, 2, ...$$

Based on shift-splitting preconditioners presented by Cao, Du and Niu [19], we establish a modified shift-splitting preconditione, which is as follows

$$\mathcal{A} = \frac{1}{2}(\Omega + \mathcal{A}) - \frac{1}{2}(\Omega - \mathcal{A}) = \frac{1}{2}\begin{pmatrix} \alpha I_1 + A & B^T \\ -B & \beta I_2 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \alpha I_1 - A & -B^T \\ B & \beta I_2 \end{pmatrix}, \tag{2}$$

where $\alpha \geq 0, \beta > 0$ is a constant, $\Omega = \begin{pmatrix} \alpha I_1 & 0 \\ 0 & \beta I_2 \end{pmatrix}$ and $I_1 \in \mathbb{R}^{n \times n}, I_2 \in \mathbb{R}^{m \times m}$ are the identity matrix. By this special splitting, the following shift-splitting iteration method can be defined for the saddle point problems (1).

The modified shift-splitting iteration method(MSS): Given an initial vector u^0, for $k = 0, 1, 2, ...$, until $\{u^k\}$ converges, compute

$$\frac{1}{2}\begin{pmatrix} \alpha I_1 + A & B^T \\ -B & \beta I_2 \end{pmatrix} u^{k+1} = \frac{1}{2}\begin{pmatrix} \alpha I_1 - A & -B^T \\ B & \beta I_2 \end{pmatrix} u^k + \begin{pmatrix} f \\ g \end{pmatrix}, \tag{3}$$

where $\alpha \geq 0, \beta > 0$ is a constant and $I_1 \in \mathbb{R}^{n \times n}, I_2 \in \mathbb{R}^{m \times m}$ are the identity matrix.

Remark 2.1. When the relaxed parameters $\alpha = \beta$, the modified shift-splitting iteration method (MSS) reduces to the shift-splitting iteration method (SS); When the relaxed parameters $\alpha = 0$, the modified shift-splitting iteration method (MSS) reduces to the local shift-splitting iteration method (LSS). So, MSS iteration method is the generalization of SS iteration method and LSS iteration method. Furthermore, when doing numerical examples, we may choose appropriate parameters to improve the convergence speed.

Obviously, the modified shift-splitting iteration method can naturally induce a splitting preconditioner for the Krylov subspace method. The splitting preconditioner based on iterative scheme (3) is as follows

$$\mathcal{P}_{MSS} = \frac{1}{2}(\Omega + \mathcal{A}) = \frac{1}{2}\begin{pmatrix} \alpha I_1 + A & B^T \\ -B & \beta I_2 \end{pmatrix}. \tag{4}$$

On iterative scheme (3), at each step of applying the modified shift-splitting preconditioner $\mathcal{P}_{MSS}$ within a Krylov subspace method, we need to solve a linear system with $\mathcal{P}_{MSS}$ as the coefficient matrix, which is as follows:

$$\frac{1}{2}\begin{pmatrix} \alpha I_1 + A & B^T \\ -B & \beta I_2 \end{pmatrix} z = r$$

for a given vector r at each step. Moreover, the preconditioner $\mathcal{P}_{MSS}$ can do the following matrix factorization

$$\mathcal{P}_{MSS} = \frac{1}{2}\begin{pmatrix} I_1 & \frac{1}{\beta}B^T \\ 0 & I_2 \end{pmatrix}\begin{pmatrix} \alpha I_1 + A + \frac{1}{\beta}B^T B & 0 \\ 0 & \beta I_2 \end{pmatrix}\begin{pmatrix} I_1 & 0 \\ -\frac{1}{\beta}B & I_2 \end{pmatrix}. \tag{5}$$

Let $r = [r_1^T, r_2^T]$ and $z = [z_1^T, z_2^T]$, where $r_1, z_1 \in \mathbb{R}^n$ and $r_2, z_2 \in \mathbb{R}^m$. So we can obtain

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 2\begin{pmatrix} I_1 & 0 \\ \frac{1}{\beta}B & I_2 \end{pmatrix}\begin{pmatrix} \alpha I_1 + A + \frac{1}{\beta}B^T B & 0 \\ 0 & \beta I_2 \end{pmatrix}^{-1}\begin{pmatrix} I_1 & -\frac{1}{\beta}B^T \\ 0 & I_2 \end{pmatrix}\begin{pmatrix} r_1 \\ r_2 \end{pmatrix}. \tag{6}$$

Hence, the algorithmic on the modified shift-splitting iteration method (MSS) is as follows:

Algorithm 2.1: For a given vector $r = [r_1^T, r_2^T]$, we can compute the vector $z = [z_1^T, z_2^T]$ by (6) from the following steps:

(a) $t_1 = r_1 - \frac{1}{\beta}B^T r_2$;

(b) solve $(\alpha I_1 + A + \frac{1}{\beta}B^T B)z_1 = 2t_1$;

(c) $z_2 = \frac{1}{\beta}(Bz_1 + 2r_2)$.

Remark 2.2. From Algorithm 2.1 in this paper and Algorithm 2.1 in [19], we can see that steps (a) $\sim$ (c) are different because of using different parameter β. In the second step of Algorithm 2.1, we need to solve sub-linear system with the coefficient matrix $\alpha I_1 + A + \frac{1}{\beta}B^T B$. Since the matrix $\alpha I_1 + A + \frac{1}{\beta}B^T B$ is symmetric positive definite, we may employ the CG or preconditioned CG method to solve step (b) in Algorithm 2.1.

3 Convergence of MSS method

Now, we will analyze the unconditional convergence property of the corresponding iterative method for saddle point problems. At first, similar to the proving process in [19], we can obtain the following Lemmas.

Lemma 3.1. *Let A be a symmetric positive definite matrix, and B have full row rank. If λ is an eigenvalue of $\mathcal{T}_{MSS}$, then $\lambda \neq \pm 1$, where $\mathcal{T}_{MSS}$ is the iteration matrix of the modified shift-splitting iteration, which is as follows*

$$\mathcal{T}_{MSS} = \begin{pmatrix} \alpha I_1 + A & B^T \\ -B & \beta I_2 \end{pmatrix}^{-1}\begin{pmatrix} \alpha I_1 - A & -B^T \\ B & \beta I_2 \end{pmatrix}. \tag{7}$$

Lemma 3.2. *Assume that A is symmetric positive definite, B has full row rank. Let λ be an eigenvalue of $\mathcal{T}_{MSS}$ and $[x^*, y^*]$ be the corresponding eigenvector with $x \in \mathbb{C}^n$ and $y \in \mathbb{C}^m$. Moreover, if $y = 0$, then $|\lambda| < 1$.*

4

Lemma 3.3. [34] *Consider the quadratic equation $x^2 - bx + c = 0$, where b and c are real numbers. Both roots of the equation are less than one in modulus if and only if $|c| < 1$ and $|b| < 1 + c$.*

Theorem 3.4. *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric and positive definite matrix, $B \in \mathbb{R}^{m\times m}$ have full row rank and let $\alpha \geq 0, \beta > 0$ be constant numbers. Let $\rho(\mathcal{T}_{MSS})$ be the spectral radius of the modified shift-splitting iteration matrix. Then it holds that*

$$\rho(\mathcal{T}_{MSS}) < 1, \forall \alpha \geq 0, \beta > 0,$$

i.e., the modified shift-splitting iteration converges to the unique solution of the saddle point problems (1).

Proof. Let λ be an eigenvalue of $\rho(\mathcal{T}_{MSS})$ and $\begin{pmatrix} x \\ y \end{pmatrix}$ be the corresponding eigenvector. Then we have

$$\begin{pmatrix} \alpha I_1 - A & -B^T \\ B & \beta I_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} \alpha I_1 + A & B^T \\ -B & \beta I_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{8}$$

Expanding out (8) we obtain

$$\begin{cases} (\lambda - 1)\xi x + (\lambda + 1)Ax + (\lambda + 1)B^T y = 0, \\ (\lambda + 1)Bx + (1 - \lambda)\beta y. \end{cases} \tag{9}$$

By Lemma 3.1, we know that $\lambda \neq 1$. So, we may get from the first equation of (9) that

$$y = \frac{\lambda + 1}{\beta(\lambda - 1)} Bx. \tag{10}$$

Substituting (10) into the first equation of (9) yields

$$\alpha(\lambda - 1)x + (\lambda + 1)Ax + \frac{(\lambda + 1)^2}{\beta(\lambda - 1)} B^T Bx = 0. \tag{11}$$

By Lemma 3.2, we know that $x \neq 0$. Multiplying $\frac{x^*}{x^*x}$ on both sides of the equation (11), we have

$$\alpha\beta(\lambda - 1)^2 + \beta(\lambda^2 - 1)\frac{x^*Ax}{x^*x} + (\lambda + 1)^2 \frac{x^*B^TBx}{x^*x} = 0. \tag{12}$$

Let

$$a = \frac{x^*Ax}{x^*x} > 0, b = \frac{x^*B^TBx}{x^*x} \geq 0.$$

Then, from (12) we know that λ satisfies the following real quadratic equation

$$\lambda^2 + \frac{2b - 2\alpha\beta}{\alpha\beta + \beta a + b}\lambda + \frac{\alpha\beta - \beta a + b}{\alpha\beta + \beta a + b}. \tag{13}$$

By Lemma 3.3, $|\lambda| < 1$ if and only if

$$\left| \frac{\alpha\beta - \beta a + b}{\alpha\beta + \beta a + b} \right| < 1 \tag{14}$$

5

and

$$\left|\frac{2b-2\alpha\beta}{\alpha\beta+\beta a+b}\right| < 1+\frac{\alpha\beta-\beta a+b}{\alpha\beta+\beta a+b}. \tag{15}$$

Obviously, the equations (14) and (15) hold for any $\alpha \geq 0, \beta > 0$. Hence, we have

$$\rho(\mathcal{T}_{MSS}) < 1, \forall \alpha \geq 0, \beta > 0.$$

Remark 3.1. Obviously, from Theorem 3.4, we know that the modified shift-splitting iteration method is convernt unconditionally.

Remark 3.2. In actual operation, when using the Krylov subspace method like GMRES or CG method, we may choose $\mathcal{P}_{MSS}$ as the preconditioner to accelerate the convergence. Actually, the left-preconditioned linear system based on the preconditioner $\mathcal{P}_{MSS}$ is as follows

$$(I-\mathcal{T}_{MSS})u = \mathcal{P}_{MSS}^{-1}\mathcal{A}u = \mathcal{P}_{MSS}^{-1}b.$$

4 Numerical examples

In this section, to further assess the effectiveness of the modified shift-splitting preconditioned matrix $\mathcal{P}_{MSS}^{-1}\mathcal{A}$ combined with Krylov subspace methods, we present a sample of numerical examples which are based on a two-dimensional time-harmonic Maxwell equations in mixed form in a square domain $(-1 \leq x \leq 1, -1 \leq y \leq 1)$. For the simplicity, we take the generic source: $f = 1$ and a finite element subdivision such as Figure 1 based on uniform grids of triangle elements. Three mesh sizes are considered: $h = \frac{\sqrt{2}}{8}, \frac{\sqrt{2}}{12}, \frac{\sqrt{2}}{18}$. The solutions of the preconditioned systems in each iteration are computed exactly. Information on the sparsity of relevant matrices on the different meshes is given in Table 1, where nz(A) denote the nonzero elements of matrix A.

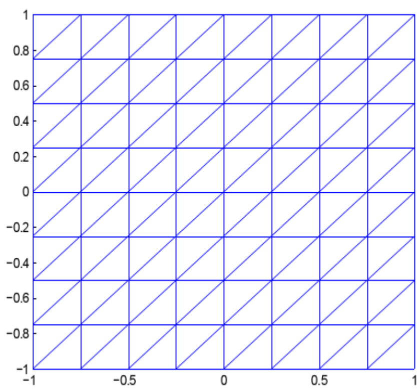

Figure 1: A uniform mesh with $h = \frac{\sqrt{2}}{4}$

Since the modified shift-splitting preconditioners have two parameters, in numerical experiments we will test different values. Numerical experiments show the spectrum of the new preconditioned matrix $\mathcal{P}_{MSS}^{-1}\mathcal{A}$ for the different parameters.

In Figures 2, 3 and 4 we display the eigenvalues of the preconditioned matrix $\mathcal{P}_{MSS}^{-1}\mathcal{A}$ in the case of $h = \frac{\sqrt{2}}{8}$ for different parameters. In Figures 5, 6 and 7 we display the eigenvalues of the preconditioned matrix $\mathcal{P}_{MSS}^{-1}\mathcal{A}$ in the case of $h = \frac{\sqrt{2}}{12}$ for different parameters. In

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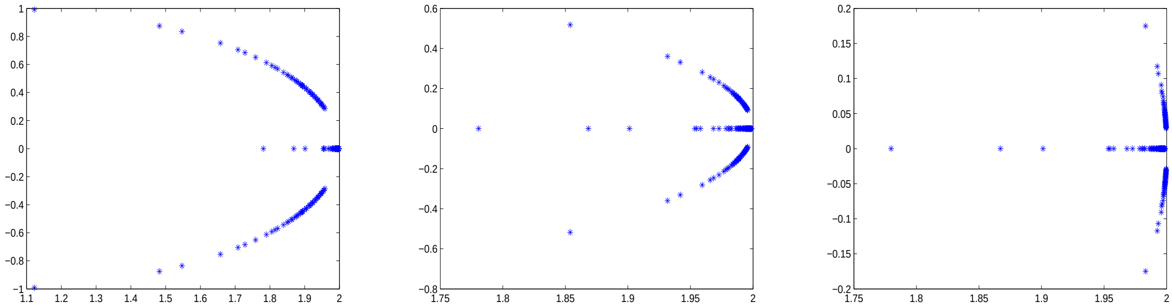

Figure 2: The eigenvalue distribution for the modified shift-splitting preconditioned matrix $\mathcal{P}_{MSS}^{-1}\mathcal{A}$ when $\alpha = 0.01, \beta = 1$(the first), $\alpha = 0.01, \beta = 0.1$(the second) and $\alpha = 0.01, \beta = 0.01$(the third), respectively. Here, $h = \frac{\sqrt{2}}{8}$.

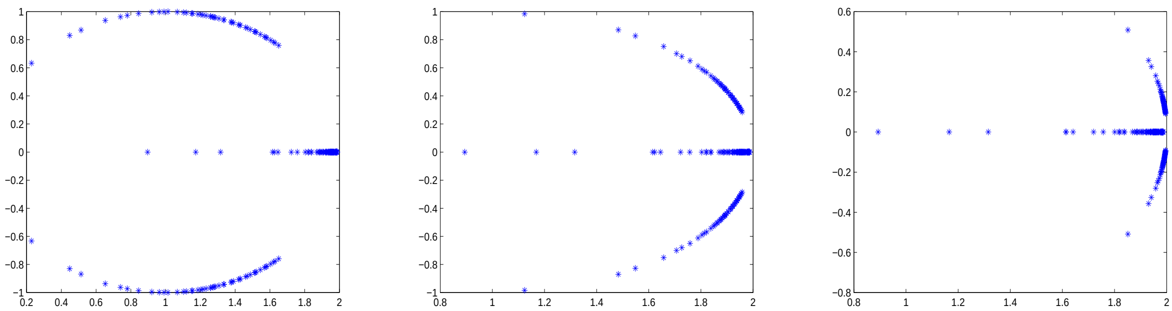

Figure 3: The eigenvalue distribution for the modified shift-splitting preconditioned matrix $\mathcal{P}_{MSS}^{-1}\mathcal{A}$ when $\alpha = 0.1, \beta = 1$(the first), $\alpha = 0.1, \beta = 0.1$(the second) and $\alpha = 0.1, \beta = 0.01$(the third), respectively. Here, $h = \frac{\sqrt{2}}{8}$.

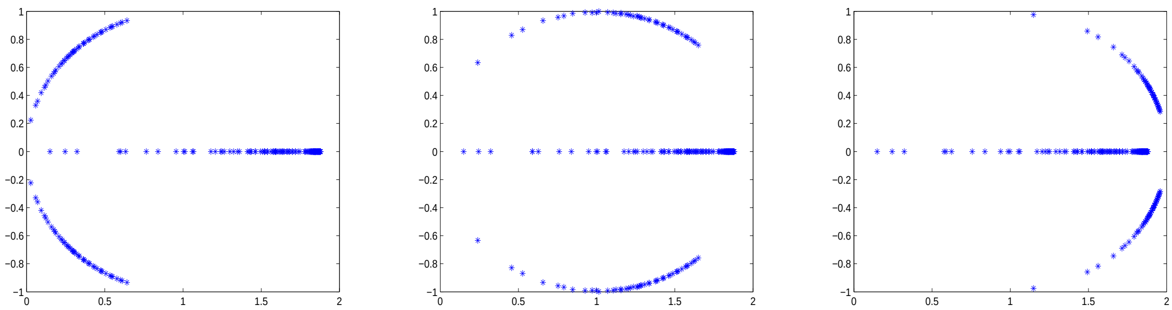

Figure 4: The eigenvalue distribution for the modified shift-splitting preconditioned matrix $\mathcal{P}_{MSS}^{-1}\mathcal{A}$ when $\alpha = 1, \beta = 1$(the first), $\alpha = 1, \beta = 0.1$(the second) and $\alpha = 1, \beta = 0.01$(the third), respectively. Here, $h = \frac{\sqrt{2}}{8}$.

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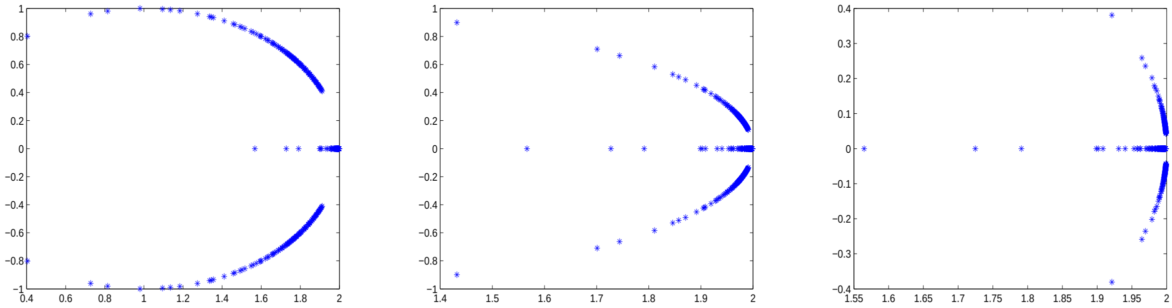

Figure 5: The eigenvalue distribution for the modified shift-splitting preconditioned matrix $\mathcal{P}_{MSS}^{-1}\mathcal{A}$ when $\alpha = 0.01, \beta = 1$(the first), $\alpha = 0.01, \beta = 0.1$(the second) and $\alpha = 0.01, \beta = 0.01$(the third), respectively. Here, $h = \frac{\sqrt{2}}{12}$.

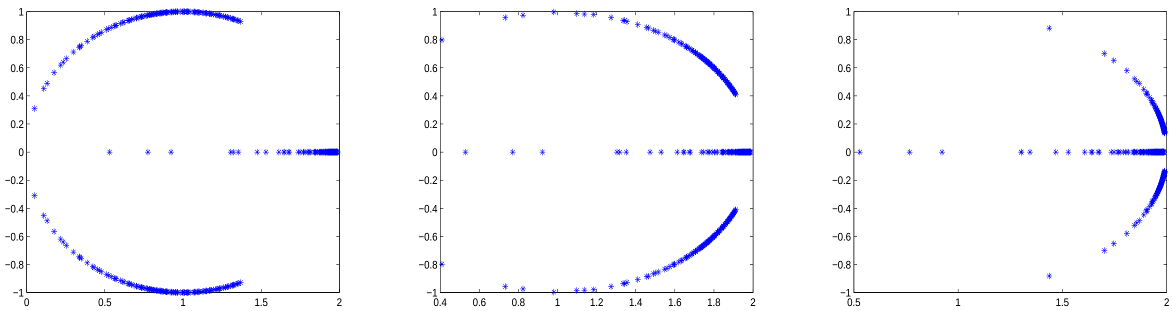

Figure 6: The eigenvalue distribution for the modified shift-splitting preconditioned matrix $\mathcal{P}_{MSS}^{-1}\mathcal{A}$ when $\alpha = 0.1, \beta = 1$(the first), $\alpha = 0.1, \beta = 0.1$(the second) and $\alpha = 0.1, \beta = 0.01$(the third), respectively. Here, $h = \frac{\sqrt{2}}{12}$.

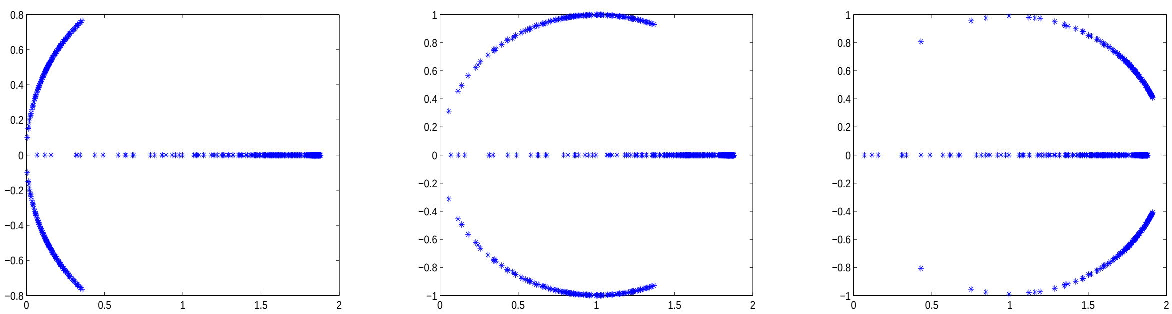

Figure 7: The eigenvalue distribution for the modified shift-splitting preconditioned matrix $\mathcal{P}_{MSS}^{-1}\mathcal{A}$ when $\alpha = 1, \beta = 1$(the first), $\alpha = 1, \beta = 0.1$(the second) and $\alpha = 1, \beta = 0.01$(the third), respectively. Here, $h = \frac{\sqrt{2}}{12}$.

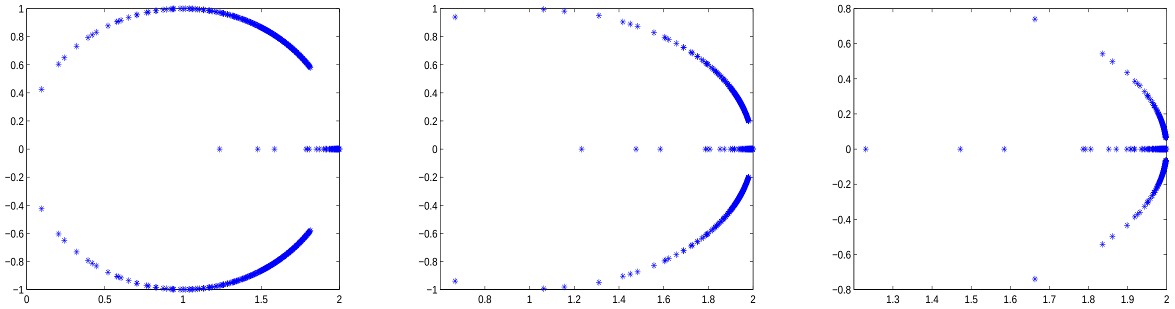

Figure 8: The eigenvalue distribution for the modified shift-splitting preconditioned matrix $\mathcal{P}_{MSS}^{-1}\mathcal{A}$ when $\alpha = 0.01, \beta = 1$(the first), $\alpha = 0.01, \beta = 0.1$(the second) and $\alpha = 0.01, \beta = 0.01$(the third), respectively. Here, $h = \frac{\sqrt{2}}{18}$.

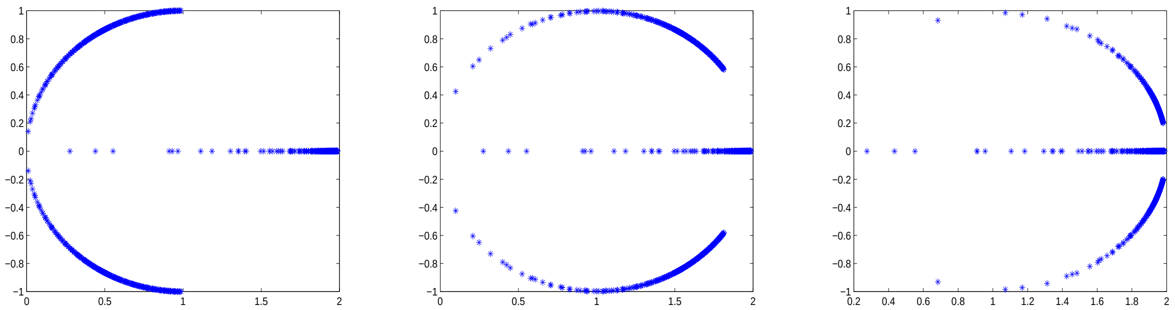

Figure 9: The eigenvalue distribution for the modified shift-splitting preconditioned matrix $\mathcal{P}_{MSS}^{-1}\mathcal{A}$ when $\alpha = 0.1, \beta = 1$(the first), $\alpha = 0.1, \beta = 0.1$(the second) and $\alpha = 0.1, \beta = 0.01$(the third), respectively. Here, $h = \frac{\sqrt{2}}{18}$.

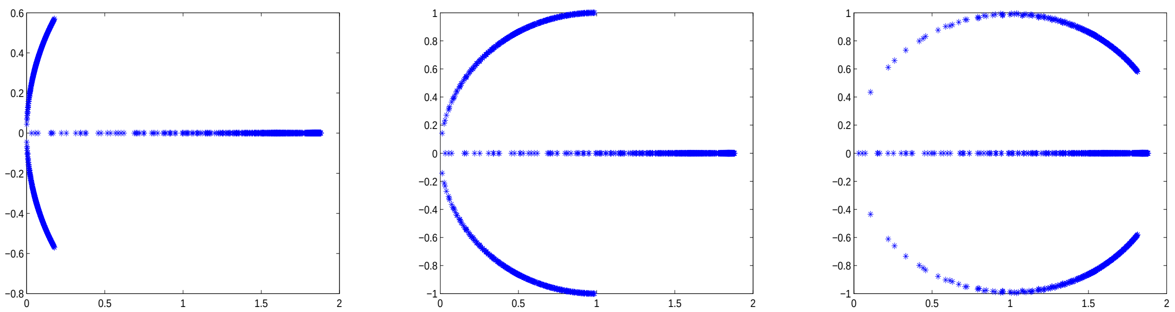

Figure 10: The eigenvalue distribution for the modified shift-splitting preconditioned matrix $\mathcal{P}_{MSS}^{-1}\mathcal{A}$ when $\alpha = 1, \beta = 1$(the first), $\alpha = 1, \beta = 0.1$(the second) and $\alpha = 1, \beta = 0.01$(the third), respectively. Here, $h = \frac{\sqrt{2}}{18}$.

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Table 1: datasheet for different grids

Grid	m	n	nz(A)	nz(B)	nz(W)	order of $\mathcal{A}$
8×8	176	49	820	462	217	225
16×16	736	225	3556	2190	1065	961
32×32	3008	961	14788	9486	4681	3969
64×64	12160	3969	60292	39438	19593	16129

Table 2: Iteration counts and relative residual about the modified shift-splitting preconditioned matrix $\mathcal{P}_{MSS}^{-1}\mathcal{A}$ when choosing different parameters, where the number of iterations and relative residual of unpreconditioned BICGSTAB and GMRES are $-$ and $-$, 171(1) and 7.4545×10^{-7}, respectively. Here, $h = \frac{\sqrt{2}}{8}$ denotes the size of the corresponding grid.

α	β	$It_{BICGSTAB(\mathcal{P}_{MSS}^{-1}\mathcal{A})}$	$Res_{BICGSTAB(\mathcal{P}_{MSS}^{-1}\mathcal{A})}$	$It_{GMRES(\mathcal{P}_{MSS}^{-1}\mathcal{A})}$	$Res_{GMRES(\mathcal{P}_{MSS}^{-1}\mathcal{A})}$
0.01	1	6	7.6716×10^{-7}	10(1)	7.4779×10^{-7}
0.01	0.1	3	5.4416×10^{-7}	6(1)	7.4225×10^{-7}
0.01	0.01	2	8.7718×10^{-7}	5(1)	1.8299×10^{-7}
0.1	1	21.5	5.4960×10^{-7}	24(1)	9.6647×10^{-7}
0.1	0.1	6.5	6.2392×10^{-7}	12(1)	9.3667×10^{-7}
0.1	0.01	5	3.8958×10^{-7}	8(1)	7.3712×10^{-7}
1	1	82.5	4.2920×10^{-7}	65(1)	6.5701×10^{-7}
1	0.1	31	6.0454×10^{-7}	33(1)	8.5683×10^{-7}
1	0.01	13	6.3508×10^{-7}	20(1)	5.1740×10^{-7}

Figures 8, 9 and 10 we display the eigenvalues of the preconditioned matrix $\mathcal{P}_{MSS}^{-1}\mathcal{A}$ in the case of $h = \frac{\sqrt{2}}{18}$ for different parameters. Figures 2 $\sim$ 10 show that the distribution of eigenvalues of the preconditioned matrix confirms our above theoretical analysis. In Tables 2 $\sim$ 4 we show iteration counts and relative residual about preconditioned matrices $\mathcal{P}_{MSS}^{-1}\mathcal{A}$ when choosing different parameters and applying to BICGSTAB and GMRES Krylov subspace iterative methods on three meshes, where $It_{BICGSTAB(\mathcal{P}_{MSS}^{-1}\mathcal{A})}$ and $Res_{BICGSTAB(\mathcal{P}_{MSS}^{-1}\mathcal{A})}$ are the iteration numbers and relative residual of the preconditioned matrices $\mathcal{P}_{MSS}^{-1}\mathcal{A}$ when applying to BICGSTAB Krylov subspace iterative methods, respectively. $It_{GMRES(\mathcal{P}_{MSS}^{-1}\mathcal{A})}$ and $Res_{GMRES(\mathcal{P}_{MSS}^{-1}\mathcal{A})}$ are the iteration numbers and relative residual of the preconditioned matrices $\mathcal{P}_{MSS}^{-1}\mathcal{A}$ when applying to GMRES Krylov subspace iterative methods, respectively.

Remark 4.1. From the above figures and tables, we know that the smaller the parameter β is, the gather the eigenvalues are and the fewer the iteration counts are.

Remark 4.2. From Tables 2, 3 and 4, it is very easy to see that the preconditioner $\mathcal{P}_{MSS}^{-1}\mathcal{A}$ will improve the convergence of BICGSTAB and GMRES iteration efficiently when they are applied to the preconditioned BICGSTAB and GMRES to solove the Stokes equation and two-dimensional time-harmonic Maxwell equations by choosing different parameters.

5 Conclusions

In this paper, we establish the modified shift-splitting preconditioner for solving the large sparse augmented systems of linear equations. Furthermore, the preconditioner is based on a modified shift-splitting of the saddle point matrix, resulting in an unconditional conver-

Table 3: Iteration counts and relative residual about the modified shift-splitting preconditioned matrix $\mathcal{P}_{MSS}^{-1}\mathcal{A}$ when choosing different parameters, where the number of iterations and relative residual of unpreconditioned BICGSTAB and GMRES are − and −, 362(1) and 9.4148×10^{-7}, respectively. Here, $h = \frac{\sqrt{2}}{12}$ denotes the size of the corresponding grid.

α	β	$It_{BICGSTAB(\mathcal{P}_{MSS}^{-1}\mathcal{A})}$	$Res_{BICGSTAB(\mathcal{P}_{MSS}^{-1}\mathcal{A})}$	$It_{GMRES(\mathcal{P}_{MSS}^{-1}\mathcal{A})}$	$Res_{GMRES(\mathcal{P}_{MSS}^{-1}\mathcal{A})}$
0.01	1	14.5	4.1689×10^{-7}	19(1)	5.2459×10^{-7}
0.01	0.1	5.5	9.0310×10^{-7}	9(1)	7.4043×10^{-7}
0.01	0.01	3	5.2030×10^{-7}	6(1)	9.3857×10^{-7}
0.1	1	63.5	5.2347×10^{-7}	50(1)	6.7889×10^{-7}
0.1	0.1	13.5	6.1091×10^{-7}	23(1)	4.9215×10^{-7}
0.1	0.01	7.5	4.5380×10^{-7}	12(1)	8.6233×10^{-7}
1	1	216.5	4.7653×10^{-7}	123(1)	8.0138×10^{-7}
1	0.1	88	9.6032×10^{-7}	65(1)	7.5718×10^{-7}
1	0.01	27.5	1.1257×10^{-7}	34(1)	8.5489×10^{-7}

Table 4: Iteration counts and relative residual about the modified shift-splitting preconditioned matrix $\mathcal{P}_{MSS}^{-1}\mathcal{A}$ when choosing different parameters, where the number of iterations and relative residual of unpreconditioned BICGSTAB and GMRES are 742 and 8.0810×10^{-7}, 1− and −, respectively. Here, $h = \frac{\sqrt{2}}{18}$ denotes the size of the corresponding grid.

α	β	$It_{BICGSTAB(\mathcal{P}_{MSS}^{-1}\mathcal{A})}$	$Res_{BICGSTAB(\mathcal{P}_{MSS}^{-1}\mathcal{A})}$	$It_{GMRES(\mathcal{P}_{MSS}^{-1}\mathcal{A})}$	$Res_{GMRES(\mathcal{P}_{MSS}^{-1}\mathcal{A})}$
0.01	1	58	6.7835×10^{-7}	34(1)	8.5510×10^{-7}
0.01	0.1	7.5	7.7089×10^{-7}	16(1)	3.1469×10^{-7}
0.01	0.01	4	6.1349×10^{-7}	9(1)	2.6837×10^{-7}
0.1	1	2644.5	4.2297×10^{-7}	94(1)	9.9981×10^{-7}
0.1	0.1	34.5	8.1807×10^{-7}	43(1)	7.0956×10^{-7}
0.1	0.01	13	9.4646×10^{-7}	21(1)	5.0204×10^{-7}
1	1	8517.5	9.3710×10^{-7}	229(1)	9.1052×10^{-7}
1	0.1	116	7.8164×10^{-7}	132(1)	9.2308×10^{-7}
1	0.01	93	6.9354×10^{-7}	66(1)	8.8886×10^{-7}

gent fixed-point iteration, which is a generalization of shift-splitting preconditioners. Finally, numerical examples show the preconditioner $\mathcal{P}_{MSS}^{-1}\mathcal{A}$ will improve the convergence of BICGSTAB and GMRES iteration efficiently when they are applied to the preconditioned BICGSTAB and GMRES to solove the Stokes equation and two-dimensional time-harmonic Maxwell equations by choosing different parameters.

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CLOSED-RANGE GENERALIZED COMPOSITION OPERATORS BETWEEN BLOCH-TYPE SPACES

CUI WANG
DEPARTMENT OF MATHEMATICS, TIANJIN UNIVERSITY, TIANJIN 300072 P.R. CHINA.
CUIWANG2016@126.COM

ZE-HUA ZHOU*
DEPARTMENT OF MATHEMATICS, TIANJIN UNIVERSITY, TIANJIN 300072, P.R. CHINA.
ZEHUAZHOUMATH@ALIYUN.COM; ZHZHOU@TJU.EDU.CN

Abstract. Let φ denote a nonconstant analytic self-map of the open unit disk $\mathbb{D}$, g be an analytic function on $\mathbb{D}$. In this paper, we characterize the necessary or sufficient conditions for generalized composition operators

$$C_\varphi^g f(z) = \int_0^z f'(\varphi(\xi))g(\xi)d\xi,$$

on the Bloch-type spaces to have a closed range. Moreover, if $g \in H^\infty$, according to relationship between α and β, we show several conclusions.

1. Introduction

Let $H(\mathbb{D})$ be the class of all holomorphic functions on $\mathbb{D}$, where $\mathbb{D}$ is the open unit disk in the complex plane $\mathbb{C}$. Denote by $H^\infty = H^\infty(\mathbb{D})$ the space of all bounded holomorphic functions on $\mathbb{D}$ with the supremum norm $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$.

For $0 < \alpha < \infty$, a holomorphic function f is said to be in the Bloch-type space $\mathcal{B}^\alpha$ or $\alpha-$Bloch space, if

$$\|f\|_\alpha = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha |f'(z)| < \infty.$$

The little Bloch-type space $\mathcal{B}_0^\alpha$, consists of all $f \in \mathcal{B}^\alpha$, such that

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha |f'(z)| = 0.$$

It is well-known that both $\mathcal{B}^\alpha$ and $\mathcal{B}_0^\alpha$ are Banach spaces under the norm

$$\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha |f'(z)|.$$

Moreover, $\mathcal{B}_0^\alpha$ is the closure of polynomials in $\mathcal{B}^\alpha$. When $0 < \alpha < 1$, $\mathcal{B}^\alpha$ is the analytic Lipschitz space $Lip_{1-\alpha}$, which consists of all $f \in H(\mathbb{D})$ satisfying

$$|f(z) - f(w)| \le C|z-w|^{1-\alpha},$$

for some constant $C > 0$ and all $z, w \in \mathbb{D}$. When $\alpha = 1$, $\mathcal{B}^\alpha$ becomes the classical Bloch space $\mathcal{B}$. When $\alpha > 1$, $\mathcal{B}^\alpha$ is equivalent to the weighted Banach space $H^\infty_{\alpha-1}$. Let H^∞_α be the weighted Banach space of holomorphic functions f on $\mathbb{D}$ satisfying

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha |f(z)| < \infty.$$

We refer the readers to the book [13] by K. Zhu, which is an excellent resource for the development of the theory of function spaces.

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*Corresponding author.
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We say that a subset H of $\mathbb{D}$ is called a *sampling set* for the Bloch-type space $\mathcal{B}^\alpha$, if there is $k > 0$ such that

$$\sup\{(1-|z|^2)^\alpha |f'(z)|, z \in \mathbb{D}\} \leq k \sup\{(1-|z|^2)^\alpha |f'(z)|, z \in H\}.$$

The *pseudo-hyperbolic metric* is given by

$$\rho(z,a) = |\sigma_a(z)|, \text{ where } \sigma_a(z) = \frac{a-z}{1-\bar{a}z}, \ z, \ a \in \mathbb{D}.$$

$\sigma_a(z)$ is the automorphism of $\mathbb{D}$ which changes 0 and a. It is well-known that the pseudo-hyperbolic metric is Möbius-invariant. Moreover, we have that $\sigma_a'(z) = \frac{1-|a|^2}{(1-\bar{a}z)^2}$.

A subset G of $\mathbb{D}$ is an *r-net* for some $r \in (0,1)$, if for every $w \in \mathbb{D}$, there exists a $z \in G$ such that $\rho(z,w) < r$. If we define $\rho(z,E) = \inf\{\rho(z,w) : w \in E\}$ for a set $E \subset D$, then a relatively closed subset E of D is an r-net if and only if $\rho(z,E) \leq r$.

For every analytic self-map φ of $\mathbb{D}$ and $g \in H(\mathbb{D})$, the generalized composition operator C_φ^g is defined by

$$C_\varphi^g f(z) = \int_0^z f'(\varphi(\xi))g(\xi)d\xi, \quad z \in \mathbb{D},$$

which was firstly introduced by Li and Stević [9]. For further references and details about the generalized composition operator, we refer the readers to [10, 11] and their references. S. Li and S. Stević [9] gave the boundedness and compactness of $C_\varphi^g : \mathcal{B}^\alpha \to \mathcal{B}^\beta$, which will play a central roll in our paper, so we use the notation $\tau_{\alpha,\beta}(z)$ to state the results. For $\alpha > 0$ and $\beta > 0$, let

$$\tau_{\alpha,\beta}(z) = \frac{(1-|z|^2)^\beta |g(z)|}{(1-|\varphi(z)|^2)^\alpha}, \ z \in \mathbb{D}.$$

Theorem A. Let $\alpha, \beta > 0$, $g \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. Then $C_\varphi^g : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded if and only if

$$\sup_{z \in \mathbb{D}} \tau_{\alpha,\beta}(z) < \infty.$$

Theorem B. Let $\alpha, \beta > 0$, $g \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. Then $C_\varphi^g : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is compact if and only if $C_\varphi^g : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded and

$$\lim_{|\varphi(z)| \to 1} \tau_{\alpha,\beta}(z) = 0.$$

The composition operator is defined by $C_\varphi(f)(z) = f(\varphi(z))$ on the spaces of analytic functions on $\mathbb{D}$. In 2000, Gathage, Yan and Zheng [7] characterized closed-range composition operators on Bloch spaces firstly. Chen [5] not only added a sufficient condition for [7], but also studied a sufficient and necessary condition of the boundedness from below for C_φ on the Bloch space of the unit ball. Then Gathatage, Zheng and Zorboska [8] introduced the notion of sampling sets for the bloch space and gave a necessary and sufficient condition for C_φ on the Bloch space to have closed-range. This result has been extended by Chen and Gauthire [4] to $\alpha-$Bloch spaces with $\alpha \geq 1$. Soon after Zorboska [14] added new and general results on the closed-range determination of C_φ on Bloch-type spaces. There are also many articles on various other holomorphic function spaces. G. R. Chacón [3] provided a geometric characterization for those composition operators having closed-range on Dirichlet-type spaces. Recently, necessary and sufficient conditions for a closed-range composition operator on Besov spaces and more generally on Besov type spaces were given by M. Tjani [12]. Akeroyd and Fulmer [1, 2] characterized the closed range composition operators on weighted Bergman spaces.

In this paper, we give some results to determine when the generalized composition operator C_φ^g has closed-range. To some extent, our results generalize some existing results. For example, the results obtained in this paper also hold for the classical composition operator $C_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$, which we get by choosing $g = \varphi'$, so some results of [14] can be got easily by this paper. In section 2, we show several necessary and sufficient conditions for the generalized composition operator C_φ^g between Bloch-type spaces to have closed-range; apart from these, we use a set to describe when $C_\varphi^g : \mathcal{B}^\alpha/\mathbb{C} \to \mathcal{B}^\beta$ is bounded below. In section 3, if $g \in H^\infty$, according to relationship between α and β, we show several conclusions.

In order to state our main results conveniently, from now on we note $\Omega_{\varepsilon,\alpha,\beta} = \{z \in \mathbb{D}, \tau_{\alpha,\beta}(z) \geq \varepsilon\}$ and $G_{\varepsilon,\alpha,\beta} = \varphi(\Omega_{\varepsilon,\alpha,\beta})$.

Throughout the remainder of this paper, C will denote a positive constant, the exact value of which will vary from one appearance to the next. The notations $A \asymp B$, $A \preceq B$, $A \succeq B$ mean that there exist different positive constants C such that $B/C \leq A \leq CB$, $A \leq CB$, $CB \leq A$.

2. Sampling sets and r-net

A bounded generalized composition operator $C_\varphi^g : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is said to be bounded below, if there exists a constant $k > 0$ such that $\|C_\varphi^g f\|_{\mathcal{B}^\beta} \geq k\|f\|_{\mathcal{B}^\alpha}$. Meanwhile, we know that C_φ^g maps any constant function to 0 function, so it is only useful to consider spaces of analytic functions modulo the constants. It follows that we can replace the norm $\|f\|_{\mathcal{B}^\alpha}$ with the seminorm $\|f\|_\alpha$ in the definition of boundedness below. Therefore, in this paper, we just show some results on $X/\mathbb{C}$, which means that a Banach space X of analytic functions on $\mathbb{D}$ modulo the constants.

Lemma 1. *Let X be Banach spaces of analytic functions. If φ is a nonconstant analytic self-map of $\mathbb{D}$, then C_φ^g is one-to-one on $X/\mathbb{C}$.*

Proof. If $C_\varphi^g f_1 = C_\varphi^g f_2$, we obtain $f_1'(\varphi(z))g(z) = f_2'(\varphi(z))g(z)$. Excluding the isolated points where g vanishes, since f_1 and f_2 are analytic, φ is a nonconstant analytic self-map of $\mathbb{D}$, the open mapping theorem for analytic functions ensures that $f_1'(z) = f_2'(z)$ for every $z \in \mathbb{D}$, and hence C_φ^g is one-to-one on $X/\mathbb{C}$. □

A basic operator theory result asserts that a one-to-one operator has a closed range if and only if it is bounded below. Therefore, Lemma 1 implies the following theorem. The detailed proof is similar to Proposition 3.30 of [6], and so we omit it.

Theorem 1. *Let $0 < \alpha, \beta < \infty$, φ be a nonconstant analytic self-map of $\mathbb{D}$. Then $C_\varphi^g : \mathcal{B}^\alpha/\mathbb{C} \to \mathcal{B}^\beta$ has a closed range if and only if it is bounded below from $\mathcal{B}^\alpha/\mathbb{C}$ to $\mathcal{B}^\beta$. This is equivalent to the condition that there exists $M > 0$ such that*

$$\|C_\varphi^g f\|_\beta \geq M\|f\|_\alpha, \ \forall f \in \mathcal{B}^\alpha/\mathbb{C}.$$

Remark 1. *Since φ is an open map, a generalized composition operator C_φ^g never has a finite rank. However, the closed subspaces of the range of a compact operator are only the finite dimensional ones, so a compact generalized composition operator can never have a closed range.*

Theorem 2. *Let $0 < \alpha, \beta < \infty$, φ be a nonconstant analytic self-map of $\mathbb{D}$. Suppose that $C_\varphi^g : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded. Then $C_\varphi^g : \mathcal{B}^\alpha/\mathbb{C} \to \mathcal{B}^\beta$ has a closed range if and only if there exists $\varepsilon > 0$ such that the set $G_{\varepsilon,\alpha,\beta}$ is a sampling set on $\mathcal{B}^\alpha/\mathbb{C}$.*

Proof. Suppose that there exists $\varepsilon > 0$ such that the set $G_{\varepsilon,\alpha,\beta}$ is a sampling set on $\mathcal{B}^\alpha/\mathbb{C}$. In this case, we can find a constant $k > 0$ such that

$$\begin{aligned}
\|f\|_\alpha &\leq k\sup\{(1-|\varphi(z)|^2)^\alpha|f'(\varphi(z))|, z \in \Omega_{\varepsilon,\alpha,\beta}\} \\
&\leq k\sup\{\frac{(1-|\varphi(z)|^2)^\alpha}{(1-|z|^2)^\beta|g(z)|}(1-|z|^2)^\beta|f'(\varphi(z))g(z)|, z \in \Omega_{\varepsilon,\alpha,\beta}\} \\
&= k\sup\{\frac{1}{\tau_{\alpha,\beta}(z)}(1-|z|^2)^\beta|f'(\varphi(z))g(z)|, z \in \Omega_{\varepsilon,\alpha,\beta}\} \\
&\leq \frac{k}{\varepsilon}\sup\{(1-|z|^2)^\beta|f'(\varphi(z))g(z)|, z \in \mathbb{D}\} \\
&\leq \frac{k}{\varepsilon}\|C_\varphi^g f\|_\beta
\end{aligned}$$

and because $C_\varphi^g : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded, it is bounded below. By Theorem 1, we obtain that $C_\varphi^g : \mathcal{B}^\alpha/\mathbb{C} \to \mathcal{B}^\beta$ has a closed range.

Conversely, assume that $C_\varphi^g : \mathcal{B}^\alpha/\mathbb{C} \to \mathcal{B}^\beta$ has a closed range. Then there exists $k > 0$, such that for $\forall f \in \mathcal{B}^\alpha/\mathbb{C}$, $\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|f'(\varphi(z))g(z)| \geq k\|f\|_\alpha$. Without loss of generality, we suppose that $\|f\|_\alpha = 1$. Thus, by the definition of supremum, we can choose $\omega \in \mathbb{D}$, such that $(1-|\omega|^2)^\beta|f'(\varphi(\omega))g(\omega)| \geq k/2$, that is to say,

$$\begin{aligned}(1-|\omega|^2)^\beta|f'(\varphi(\omega))g(\omega)| &= \frac{(1-|\omega|^2)^\beta|g(\omega)|}{(1-|\varphi(\omega)|^2)^\alpha}(1-|\varphi(\omega)|^2)^\alpha|f'(\varphi(\omega))| \\ &= \tau_{\alpha,\beta}(w)(1-|\varphi(\omega)|^2)^\alpha|f'(\varphi(\omega))| \\ &\geq \frac{k}{2}. \end{aligned} \tag{2.1}$$

Since $(1-|\varphi(\omega)|^2)^\alpha|f'(\varphi(\omega))| \leq 1$, $\tau_{\alpha,\beta}(w) \geq k/2$. If $\varepsilon = \frac{k}{2}$, then $\Omega_{\varepsilon,\alpha,\beta}$ contains the point ω, and so $\varphi(\omega) \in G_{\varepsilon,\alpha,\beta}$. On the other hand, $C_\varphi^g : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded, Theorem A implies that there exists a constant $M > 0$, such that

$$\tau_{\alpha,\beta}(w) \leq M.$$

Combining the above inequality with (1), we conclude that

$$M(1-|\varphi(\omega)|^2)^\alpha|f'(\varphi(\omega))| \geq \tau_{\alpha,\beta}(w)(1-|\varphi(\omega)|^2)^\alpha|f'(\varphi(\omega))| \geq \frac{k}{2}.$$

Thus

$$(1-|\varphi(\omega)|^2)^\alpha|f'(\varphi(\omega))| \geq \frac{k}{2M}.$$

Since $\varphi(\omega) \in G_{\varepsilon,\alpha,\beta}$,

$$\sup\{(1-|z|^2)^\alpha|f'(z)|, z \in G_{\varepsilon,\alpha,\beta}\} \geq (1-|\varphi(\omega)|^2)^\alpha|f'(\varphi(\omega))| \geq \frac{k}{2M}.$$

Hence $G_{\varepsilon,\alpha,\beta}$ is a sampling set on $\mathcal{B}^\alpha/\mathbb{C}$. □

Theorem 3. *Let $0 < \alpha, \beta < \infty$, and φ be a nonconstant analytic self-map of $\mathbb{D}$. Suppose that $C_\varphi^g : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded. If $C_\varphi^g : \mathcal{B}^\alpha/\mathbb{C} \to \mathcal{B}^\beta$ has a closed range, then there exist $c > 0$ and $0 < r < 1$, such that $G_{c,\alpha,\beta}$ is an r-net for $\mathbb{D}$.*

Proof. We assume that C_φ^g is bounded and has a closed-range. By Theorem A, there exists $K > 0$ such that $\sup \tau_{\alpha,\beta}(z) = K$ for $z \in \mathbb{D}$. Meanwhile, there exists $M > 0$ such that $\|C_\varphi^g f\|_\beta \geq M\|f\|_\alpha$ for all $f \in \mathcal{B}^\alpha/\mathbb{C}$.

Let $\omega \in \mathbb{D}$ and consider the function $\varphi_\omega(z)$ with $\varphi_\omega(0) = 0$ and $\varphi_\omega'(z) = (\sigma_\omega'(z))^\alpha$, where $\sigma_\omega(z) = \frac{\omega - z}{1-\overline{\omega}z}$. We have that $\varphi_\omega(z) \in \mathcal{B}^\alpha/\mathbb{C}$ and

$$\begin{aligned}\|\varphi_\omega\|_\alpha &= \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|\varphi_\omega'(z)| \\ &= \sup_{z\in\mathbb{D}}(1-|\sigma_\omega'(z)|^2)^\alpha \\ &= 1.\end{aligned}$$

In the above equation we use the fact that

$$1-|\sigma_\omega(z)|^2 = \frac{(1-|\omega|^2)(1-|z|^2)}{|1-\overline{\omega}z|^2} = |\sigma_\omega'(z)|(1-|z|^2).$$

Thus,

$$\begin{aligned}\|C_\varphi^g\varphi_\omega\|_\beta &= \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|\varphi_\omega'(\varphi(z))g(z)| \\ &= \sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta|g(z)|}{(1-|\varphi(z)|^2)^\alpha}(1-|\varphi(z)|^2)^\alpha|\sigma_\omega'(\varphi(z))|^\alpha \\ &= \sup_{z\in\mathbb{D}}\tau_{\alpha,\beta}(z)(1-|\sigma_\omega(\varphi(z))|^2)^\alpha.\end{aligned}$$

We shall frequently get that

$$K \geq \sup_{z\in\mathbb{D}} \tau_{\alpha,\beta}(z)(1-|\sigma_\omega(\varphi(z))|^2)^\alpha \geq M(1-|\sigma_\omega(\varphi(z))|^2)^\alpha \geq M,$$

which reveals that there exists $z_0 \in \mathbb{D}$ such that

$$\tau_{\alpha,\beta}(z_0) \geq M/2, \quad (1-|\sigma_\omega(\varphi(z_0))|^2)^\alpha \geq M/2K.$$

Thus let $\varepsilon = M/2$, $r = \sqrt{1-(M/2K)^{1/\alpha}}$, we have for all $\omega \in \mathbb{D}$, there exists $z_0 \in \Omega_{\varepsilon,\alpha,\beta}$ such that $\rho(\omega, \varphi(z_0)) < r$, and so $G_{c,\alpha,\beta}$ is an r-net for $\mathbb{D}$. □

Theorem 4. *Let $0 < \alpha, \beta < \infty$, and φ be a nonconstant analytic self-map of $\mathbb{D}$. Suppose that $C_\varphi^g : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded. If there exist $\varepsilon > 0$ and $0 < r < 1$, such that $G_{\varepsilon,\alpha,\beta}$ contains the annulus $A = \{z : r < |z| < 1\}$, then $C_\varphi^g : \mathcal{B}^\alpha/\mathbb{C} \to \mathcal{B}^\beta$ has a closed range.*

Proof. Suppose that $C_\varphi^g : \mathcal{B}^\alpha/\mathbb{C} \to \mathcal{B}^\beta$ is not bounded below. Then there exists a sequence of functions $\{f_n\}$ with $\|f_n\|_\alpha = 1$ and $\|C_\varphi^g f_n\|_\beta \to 0$. It follows that for $\forall\varepsilon > 0$, there exists N_ε when $n > N_\varepsilon$, we have $\|C_\varphi^g f_n\|_\beta < \varepsilon$. Then

$$\begin{aligned}
\sup_{\omega\in G_{\varepsilon,\alpha,\beta}} (1-|\omega|^2)^\alpha |f_n'(\omega)| &= \sup_{\omega\in\Omega_{\varepsilon,\alpha,\beta}} (1-|\varphi(z)|^2)^\alpha |f_n'(\varphi(z))| \\
&= \sup_{\omega\in\Omega_{\varepsilon,\alpha,\beta}} \frac{(1-|\varphi(z)|^2)^\alpha}{(1-|z|^2)^\beta |g(z)|}(1-|z|^2)^\beta |f_n'(\varphi(z))g(z)| \\
&= \sup_{\omega\in\Omega_{\varepsilon,\alpha,\beta}} \frac{1}{\tau_{\alpha,\beta}(z)}(1-|z|^2)^\beta |f_n'(\varphi(z))g(z)| \\
&\leq \frac{1}{\varepsilon} \sup_{z\in\Omega_{\varepsilon,\alpha,\beta}} (1-|z|^2)^\beta |f_n'(\varphi(z))g(z)| \\
&= \frac{1}{\varepsilon}\|C_\varphi^g f_n\|_\beta \\
&< \varepsilon. \qquad (2.2)
\end{aligned}$$

Since $\|f_n\|_\alpha = 1$, there exists a sequence $\{z_n\}_{n\in\mathbb{N}} \subseteq \mathbb{D}$, such that

$$(1-|z_n|^2)^\alpha |f_n'(z_n)| \geq 1/2 \qquad (2.3)$$

for all $n \geq 1$. If we choose $\varepsilon < 1/2$, by (2) and (3), $z_n \in \mathbb{D}/G_{\varepsilon,\alpha,\beta}$ when $n > N_\varepsilon$. Because $G_{\varepsilon,\alpha,\beta}$ contains the annulus $A = \{z : r < |z| < 1\}$, there exists $r_0 < r$ such that $|z_n| \leq r_0 \leq 1$ and $z_n \to z_0$ with $|z_0| < r_0$.

Since $\|f_n\|_\alpha = 1$, by Montel's theorem, there exists a subsequence $f_{n_k} \to f$ uniformly on every compact subsets of $\mathbb{D}$, where $f \in \mathcal{B}^\alpha/\mathbb{C}$. Cauchy's estimate gives that $f_{n_k}' \to f'$ uniformly on every compact subsets of $\mathbb{D}$. By (2), $\sup_{\omega\in G_{\varepsilon,\alpha,\beta}}(1-|\omega|^2)^\alpha|f_n'(\omega)| \to 0$ as $n \to \infty$. On the other hand, $G_{\varepsilon,\alpha,\beta}$ contains an infinite compact subset of $\mathbb{D}$, we get that $f' \equiv 0$. This contradicts the fact that $|(1-|z_0|^2)^\alpha f_n'(z_0)| \geq 1/2$. Hence, $C_\varphi^g : \mathcal{B}^\alpha/\mathbb{C} \to \mathcal{B}^\beta$ has a closed range. □

3. THE CASE OF $g \in H^\infty$

In this section we will give a special case $g \in H^\infty$. Combine α and β, we get several results.

Theorem 5. *Let φ be a nonconstant analytic self-map of $\mathbb{D}$, $\varphi(0) = 0$, $g \in H^\infty$ and $C_\varphi^g : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded.*

(i) *If $0 < \alpha < \beta < \infty$ then $C_\varphi^g : \mathcal{B}^\alpha/\mathbb{C} \to \mathcal{B}^\beta$ can not have a closed range.*

(ii) *If $\alpha > \beta > 0$ and $\beta < 1$ then $C_\varphi^g : \mathcal{B}^\alpha/\mathbb{C} \to \mathcal{B}^\beta$ can not have a closed range.*

Proof. (i) Since $g \in H^\infty$, there exists a constant $k > 0$, such that $|g(z)| \leq k$, for every $z \in \mathbb{D}$. For $\varphi(0) = 0$, by Schwarz-Pick Theorem in [6], we know

$$\frac{1-|z|^2}{1-|\varphi(z)|^2} \leq 1, z \in \mathbb{D}.$$

So we have

$$\begin{aligned}\tau_{\alpha,\beta}(z) &= \frac{(1-|z|^2)^\beta|g(z)|}{(1-|\varphi(z)|^2)^\alpha}\\ &\le \frac{k(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha}\\ &= \frac{k(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^\alpha}(1-|z|^2)^{\beta-\alpha}\\ &\le k(1-|\varphi(z)|^2)^{\beta-\alpha}.\end{aligned}$$

Since $0<\alpha<\beta<\infty$, as $|\varphi(z)|\to 1$, $\tau_{\alpha,\beta}(z)$ converges to 0. By Theorem B, $C_\varphi^g:\mathcal{B}^\alpha\to\mathcal{B}^\beta$ is compact. Hence $C_\varphi^g:\mathcal{B}^\alpha/\mathbb{C}\to\mathcal{B}^\beta$ can not have a closed range.

(ii) Replacing ϕ by φ, ϕ' by g in the proof of (i) of Theorem 3.6 in [14], we can get this result easily, so we omit the details here. □

Remark 2. *(i) Let φ be a nonconstant analytic self-map of $\mathbb{D}$, $\varphi(0)=0$, $g\in H^\infty$. If $\alpha=\beta$, then*

$$\begin{aligned}\tau_{\alpha,\beta}(z) &= \frac{(1-|z|^2)^\beta|g(z)|}{(1-|\varphi(z)|^2)^\alpha}\\ &\le \frac{k(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha}\\ &\le k.\end{aligned}$$

By Theorem A, we obtain $C_\varphi^g:\mathcal{B}^\alpha\to\mathcal{B}^\beta$ is bounded. While apart from this, we can not get whether $C_\varphi^g:\mathcal{B}^\alpha/\mathbb{C}\to\mathcal{B}^\beta$ has a closed range or not.

(ii) Under the conditions of Theorem 5, if $\alpha>\beta\ge 1$, whether $C_\varphi^g:\mathcal{B}^\alpha/\mathbb{C}\to\mathcal{B}^\beta$ has a closed range or not is uncertain. We just give an example ((ii) of Example 1) showing that this operator sometimes do not have a closed range. While, we fail to give the concrete proof that this operator do not have a closed range always or an example to show this operator has a closed range sometimes. So this can be an open problem.

Example 1. *Let $\varphi(z)=z$, $g(z)=1$.*

(i) If $\alpha=\beta=2$, then

$$\tau_{\alpha,\beta}(z)=\frac{(1-|z|^2)^2|g(z)|}{(1-|\varphi(z)|^2)^2}=1$$

and so $\Omega_{\varepsilon,\alpha,\beta}=\mathbb{D}$ for every $0<\varepsilon<1$. In addition, $\varphi(z)=z$ is a one-to-one analytic map of the disk onto itself, therefore, $G_{\varepsilon,\alpha,\beta}=\varphi(\Omega_{\varepsilon,\alpha,\beta})=\mathbb{D}$. Then $G_{\varepsilon,\alpha,\beta}$ is a sampling set on $\mathcal{B}^\alpha/\mathbb{C}$, and by Theorem 2, $C_\varphi^g:\mathcal{B}^\alpha/\mathbb{C}\to\mathcal{B}^\beta$ has a closed range.

(ii) If $\alpha=3$, $\beta=2$, then

$$\begin{aligned}\tau_{\alpha,\beta}(z) &= \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha}\\ &= (1-|z|^2)^{\beta-\alpha}\to\infty\end{aligned}$$

as $\varphi(z)\to 1$. By Theorem A, $C_\varphi^g:\mathcal{B}^\alpha\to\mathcal{B}^\beta$ is not bounded. Hence $C_\varphi^g:\mathcal{B}^\alpha/\mathbb{C}\to\mathcal{B}^\beta$ can not have a closed range.

Example 2. *Let $g(z)=z+1$, $\varphi(z)=\frac{z-1}{2}$. If $\alpha=\beta$, then*

$$\begin{aligned}\tau_{\alpha,\beta}(z) &= \frac{(1-|z|^2)^\alpha|g(z)|}{(1-|\varphi(z)|^2)^\alpha}\\ &\le \frac{4(1-|z|^2)^\alpha|z+1|}{(1-|z|)^\alpha(3+|z|)^\alpha}\\ &= \frac{4(1+|z|)^\alpha|z+1|}{(3+|z|)^\alpha}\to 0\end{aligned}$$

as $z \to -1$*. By Theorem B,* $C_\varphi^g : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ *is compact. Hence* $C_\varphi^g : \mathcal{B}^\alpha/\mathbb{C} \to \mathcal{B}^\beta$ *can not have a closed range.*

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Approximate ternary Jordan bi-derivations on Banach Lie triple systems

Madjid Eshaghi Gordji[1], Vahid Keshavarz[1], Choonkil Park[2] and Jung Rye Lee[3*]

[1]Department of Mathematics, Semnan University, P. O. Box 35195-363, Semnan, Iran

[2]Research Institute for Naturan Sciences, Hanyang University, Seoul 133-791, Korea

[3]Department of Mathematics, Daejin University, Kyeonggi 487-711, Korea

E-mail: meshaghi@semnan.ac.kr, v.keshavarz68@yahoo.com, baak@hanyang.ac.kr, jrlee@daejin.ac.kr

Abstract. We prove the Hyers-Ulam stability of ternary Jordan bi-derivations on Banach Lie triple systems associated to the Cauchy functional equation.

1. Introduction and preliminaries

We say that a functional equation (Q) is stable if any function g satisfying the equation (Q) approximately is near to true solution of (Q).

Ternary algebraic operations were considered in the 19th century by several mathematicians and physicists. Cayley [8] introduced the notion of cubic matrix which in turn was generalized by Kapranov, Gelfand and Zelevinskii [6]. As an application in physics, the quark model inspired a particular brand of ternary algebraic systems. The so-called Nambu mechanics which has been proposed by Nambu [11], is based on such structures. There are also some applications, although still hypothetical, in the fractional quantum Hall effect, the non-standard statistics (the anyons), supersymmetric theories, Yang-Baxter equation, etc, (cf. [15, 27]).

The comments on physical applications of ternary structures can be found in [1, 5, 10, 14, 17, 23, 24, 29].

A normed (Banach) Lie triple system is a normed (Banach) space $(A, \|\cdot\|)$ with a trilinear mapping $(x, y, z) \mapsto [x, y, z]$ from $A \times A \times A$ to A satisfying the following axioms:

$$\begin{aligned} [x,y,z] &= -[y,x,z]\,, \\ [x,y,z] &= -[y,z,x]-[z,x,y]\,, \\ [u,v,[x,y,z]] &= [[u,v,x]\,,y,z]+[x,[u,v,y]\,,z]+[x,y,[u,v,z]]\,, \\ \|\,[x,y,z]\,\| &\leq \|x\|\|y\|\|z\| \end{aligned}$$

for all $u, v, x, y, z \in A$ (see [12, 16]).

Definition 1.1. Let A be a normed Lie triple system with involution $*$. A $\mathbb{C}$-bilinear mapping $D : A \times A \to A$ is called a ternary Jordan bi-derivation if it satisfies

$$\begin{aligned} D([x,x,x],w) &= [D(x,w),x,x]+[x,D(x,w^*),x]+[x,x,D(x,w)], \\ D(x,[w,w,w]) &= [D(x,w),w,w]+[w,D(x^*,w),w]+[w,w,D(x,w)] \end{aligned}$$

for all $x, w \in A$.

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[0*]Corresponding author.

Approximate ternary Jordan bi-derivations

The stability problem of functional equations originated from a question of Ulam [28] concerning the stability of group homomorphisms. Hyers [13] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [3] for additive mappings and by Th.M. Rassias [21] for linear mappings by considering an unbounded Cauchy difference. J.M. Rassias [20] followed the innovative approach of the Th.M. Rassias theorem in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p\|y\|^p$ for $p, q \in \mathbb{R}$ with $p + q \neq 1$. The stability problems of various functional equations have been extensively investigated by a number of authors (see [2, 7, 9, 10, 18, 19, 22, 23, 24, 25, 26, 30, 31]).

2. Hyers-Ulam stability of ternary Jordan bi-derivations on Banach Lie triple systems

Throughout this section, assume that A is a normed Lie triple system.

For a given mapping $f : A \times A \to A$, we define

$$D_{\lambda,\mu} f(x, y, z, w) = f(\lambda x + \lambda y, \mu z + \mu w) + f(\lambda x + \lambda y, \mu z - \mu w)$$
$$+ f(\lambda x - \lambda y, \mu z + \mu w) + f(\lambda x - \lambda y, \mu z - \mu w) - 4\lambda\mu f(x, z)$$

for all $x, y, z, w \in A$ and all $\lambda, \mu \in \mathbb{T}^1 := \{\nu \in \mathbb{C} : \ |\nu| = 1\}$.

From now on, assume that $f(0, z) = f(x, 0) = 0$ for all $x, z \in A$.

We need the following lemma to obtain the main results.

Lemma 2.1. ([4]) *Let $f : A \times A \to B$ be a mapping satisfying $D_{\lambda,\mu} f(x, y, z, w) = 0$ for all $x, y, z, w \in A$ and all $\lambda, \mu \in \mathbb{T}^1$. Then the mapping $f : A \times A \to A$ is $\mathbb{C}$-bilinear.*

Lemma 2.2. *Let $f : A \times A \to A$ be a bi-additive mapping. Then the following assertions are equivalent:*

$$\begin{aligned} f([a, a, a], [w, w, w]) &= [f(a, w), a, a] + [a, f(a, w^*), a] + [a, a, f(a, w)], \\ f([a, a, a], [w, w, w]) &= [f(a, w), a, a] + [a, f(a^*, w), a] + [a, a, f(a, w)] \end{aligned} \tag{2.1}$$

for all $a, w \in A$, and

$$\begin{aligned} &f([a, b, c] + [b, c, a] + [c, a, b], [w, w, w]) = [f(a, w), b, c] + [a, f(b, w^*), c] + [a, b, f(c, w)] \\ &+ [f(b, w), c, a] + [b, f(c, w^*), a] + [b, c, f(a, w)] + [f(c, w), a, b] + [c, f(a, w^*), b] + [c, a, f(b, w)], \\ &f([a, a, a], [b, c, w] + [c, w, b] + [w, b, c]) = [f(a, b), c, w] + [b, f(a^*, c), w] + [b, c, f(a, w)] \\ &+ [f(a, c), w, b] + [c, f(a^*, w), b] + [c, w, f(a, b)] + [f(a, w), b, c] + [w, f(a^*, b), c] + [w, b, f(a, w)] \end{aligned} \tag{2.2}$$

for all $a, b, c, w \in A$.

Proof. Replacing a by $a + b + c$ in the first equation of (2.1), we have

$$f([a + b + c, a + b + c, a + b + c], [w, w, w]) = [f(a + b + c, w), a + b + c, a + b + c]$$
$$+ [a + b + c, f(a + b + c, w^*), a + b + c] + [a + b + c, a + b + c, f(a + b + c, w)].$$

Then we have

$$\begin{aligned} &f([a + b + c, a + b + c, a + b + c], [w, w, w]) \\ &= f([a, a, a], [w, w, w]) + f([a, b, a], [w, w, w]) + f([a, c, a], [w, w, w]) + f([b, a, a], [w, w, w]) + f([b, b, a], [w, w, w]) \\ &+ f([b, c, a], [w, w, w]) + f([c, a, a], [w, w, w]) + f([c, b, a], [w, w, w]) + f([c, c, a], [w, w, w]) + f([a, a, b], [w, w, w]) \\ &+ f([a, b, b], [w, w, w]) + f([a, c, b], [w, w, w]) + f([b, a, b], [w, w, w]) + f([b, b, b], [w, w, w]) + f([b, c, b], [w, w, w]) \end{aligned}$$

$$
\begin{aligned}
&+ f([c,a,b],[w,w,w]) + f([c,b,b],[w,w,w]) + f([c,c,b],[w,w,w]) + f([a,a,c],[w,w,w]) + f([a,b,c],[w,w,w])\\
&+ f([a,c,c],[w,w,w]) + f([b,a,c],[w,w,w]) + f([b,b,c],[w,w,w]) + f([b,c,c],[w,w,w]) + f([c,a,c],[w,w,w])\\
&+ f([c,b,c],[w,w,w]) + f([c,c,c],[w,w,w])\\
&= [f(a,w),a,a] + [a,f(a,w^*),a] + [a,a,f(a,w)] + [f(a,w),b,a] + [a,f(b,w^*),a] + [a,b,f(a,w)] + [f(a,w),c,a]\\
&+ [a,f(c,w^*),a] + [a,c,f(a,w)] + [f(b,w),a,a] + [b,f(a,w^*),a] + [b,a,f(a,w)] + [f(b,w),b,a] + [b,f(b,w^*),a]\\
&+ [b,b,f(a,w)] + [f(b,w),c,a] + [b,f(c,w^*),a] + [b,c,f(a,w)] + [f(c,w),a,a] + [c,f(a,w^*),a] + [c,a,f(a,w)]\\
&+ [f(c,w),b,a] + [c,f(b,w^*),a] + [c,b,f(a,w)] + [f(c,w),c,a] + [c,f(c,w^*),a] + [c,c,f(a,w)] + [f(a,w),a,b]\\
&+ [a,f(a,w^*),b] + [a,a,f(b,w)] + [f(a,w),b,b] + [a,f(b,w^*),b] + [a,b,f(b,w)] + [f(a,w),c,b] + [a,f(c,w^*),b]\\
&+ [a,c,f(b,w)] + [f(b,w),a,b] + [b,f(a,w^*),b] + [b,a,f(b,w)] + [f(b,w),b,b] + [b,f(b,w^*),b] + [b,b,f(b,w)]\\
&+ [f(b,w),c,b] + [b,f(c,w^*),b] + [b,c,f(b,w)] + [f(c),a,b] + [c,f(a^*),b] + [c,a,f(b)] + [f(c),b,b] + [c,f(b^*),b]\\
&+ [c,b,f(b)] + [f(c,w),c,b] + [c,f(c,w^*),b] + [c,c,f(b,w)] + [f(a,w),a,c] + [a,f(a,w^*),c] + [a,a,f(c,w)]\\
&+ [f(a,w),b,c] + [a,f(b,w^*),c] + [a,b,f(c,w)] + [f(a,w),c,c] + [a,f(c,w^*),c] + [a,c,f(c,w)] + [f(b,w),a,c]\\
&+ [b,f(a,w^*),c] + [b,a,f(c,w)] + [f(b,w),b,c] + [b,f(b,w^*),c] + [b,b,f(c,w)] + [f(b,w),c,c] + [b,f(c,w^*),c]\\
&+ [b,c,f(c,w)] + [f(c,w),a,c] + [c,f(a,w^*),c] + [c,a,f(c,w)] + [f(c,w),b,c] + [c,f(b,w^*),c] + [c,b,f(c,w)]\\
&+ [f(c,w),c,c] + [c,f(c,w^*),c] + [c,c,f(c,w)]
\end{aligned}
$$

for all $a, b, c, w \in A$.

On the other hand, for the right side of equation, we have

$$
\begin{aligned}
&[f(a+b+c,w),a+b+c,a+b+c] + [a+b+c,f(a+b+c,w^*),a+b+c] + [a+b+c,a+b+c,f(a+b+c,w)]\\
&= [f(a,w),a,a] + [f(a,w),a,b] + [f(a,w),a,c] + [f(a,w),b,a] + [f(a,w),b,b] + [f(a,w),b,c] + [f(a,w),c,a]\\
&+ [f(a,w),c,b] + [f(a,w),c,c] + [f(b,w),a,a] + [f(b,w),a,b] + [f(b,w),a,c] + [f(b,w),b,a] + [f(b,w),b,b]\\
&+ [f(b,w),b,c] + [f(b,w),c,a] + [f(b,w),c,b] + [f(b,w),c,c] + [f(c,w),a,a] + [f(c,w),a,b] + [f(c,w),a,c]\\
&+ [f(c,w),b,a] + [f(c,w),b,b] + [f(c,w),b,c] + [f(c,w),c,a] + [f(c,w),c,b] + [f(c,w),c,c] + [a,f(a,w^*),a]\\
&+ [a,f(a,w^*),b] + [a,f(a,w^*),c] + [b,f(a,w^*),a] + [b,f(a,w^*),b] + [b,f(a,w^*),c] + [c,f(a,w^*),a] + [c,f(a,w^*),b]\\
&+ [c,f(a,w^*),c] + [a,f(b,w^*),a] + [a,f(b,w^*),b] + [a,f(b,w^*),c] + [b,f(b,w^*),a] + [b,f(b,w^*),b] + [b,f(b,w^*),c]\\
&+ [c,f(b,w^*),a] + [c,f(b,w^*),b] + [c,f(b,w^*),c] + [a,f(c,w^*),a] + [a,f(c,w^*),b] + [a,f(c,w^*),c] + [b,f(c,w^*),a]\\
&+ [b,f(c,w^*),b] + [b,f(c,w^*),c] + [c,f(c,w^*),a] + [c,f(c,w^*),b] + [c,f(c,w^*),c] + [a,a,f(a,w)] + [a,b,f(a,w)]\\
&+ [a,c,f(a,w)] + [b,a,f(a,w)] + [b,b,f(a,w)] + [b,c,f(,w)] + [c,a,f(a,w)] + [c,b,f(a,w)] + [c,c,f(a,w)]\\
&+ [a,a,f(b,w)] + [a,b,f(b,w)] + [a,c,f(b,w)] + [b,a,f(b,w)] + [b,b,f(b,w)] + [b,c,f(b,w)] + [c,a,f(b,w)]\\
&+ [c,b,f(b,w)] + [c,c,f(b,w)] + [a,a,f(c,w)] + [a,b,f(c,w)] + [a,c,f(c,w)] + [b,a,f(c,w)] + [b,b,f(c,w)]\\
&+ [b,c,f(c,w)] + [c,a,f(c,w)] + [c,b,f(c,w)] + [c,c,f(c,w)]
\end{aligned}
$$

for all $a, b, c, w \in A$. It follows that

$$
\begin{aligned}
f([a,b,c]+[b,c,a]+[c,a,b],[w,w,w]) &= [f(a,w),b,c] + [a,f(b,w^*),c] + [a,b,f(c,w)]\\
&+ [f(b,w),c,a] + [b,f(c,w^*),a] + [b,c,f(a,w)] + [f(c,w),a,b] + [c,f(a,w^*),b] + [c,a,f(b,w)]
\end{aligned}
$$

for all $a, b, c, w \in A$. Hence (2.2) holds.

Similarly, we can show that

$$f([a,a,a],[b,c,w]+[c,w,b]+[w,b,c]) = [f(a,b),c,w]+[b,f(a^*,c),w]+[b,c,f(a,w)]$$
$$+[f(a,c),w,b]+[c,f(a^*,w),b]+[c,w,f(a,b)]+[f(a,w),b,c]+[w,f(a^*,b),c]+[w,b,f(a,w)]$$

for all $a, b, c, w \in A$.

For the converse, replacing b and c by a in the first equation of (2.2), we have

$$f([a,a,a]+[a,a,a]+[a,a,a],[w,w,w]) = [f(a,w),a,a]+[a,f(a,w^*),a]+[a,a,f(a,w)]+[f(a,w),a,a]$$
$$+[a,f(a,w^*),a]+[a,a,f(a,w)]+[f(a,w),a,a]+[a,f(a,w^*),a]+[a,a,f(a,w)],$$

and so

$$f\Big(([a,a,a],[w,w,w])+([a,a,a],[w,w,w])+([a,a,a],[w,w,w])\Big) = 3([f(a,w),a,a]+[a,f(a,w^*),a]+[a,a,f(a,w)]).$$

Thus

$$f\Big(3([a,a,a],[w,w,w])\Big) = 3([f(a,w),a,a]+[a,f(a,w^*),a]+[a,a,f(a,w)])$$

and so

$$f([a,a,a],[w,w,w]) = [f(a,w),a,a]+[a,f(a,w^*),a]+[a,a,f(a,w)]$$

for all $a, w \in A$.

Similarly, we can show that

$$f([a,a,a],[w,w,w]) = [f(a,w),a,a]+[a,f(a^*,w),a]+[a,a,f(a,w)]$$

for all $a, w \in A$. This completes the proof. □

Now we prove the Hyers-Ulam stability of ternary Jordan bi-derivations on Banach Lie triple systems.

Theorem 2.3. *Let p and θ be positive real numbers with $p < 2$, and let $f : A \times A \to A$ be a mapping such that*

$$\|D_{\lambda,\mu}f(x,y,z,w)\| \le \theta(\|x\|^p+\|y\|^p+\|z\|^p+\|w\|^p), \tag{2.3}$$

$$\begin{aligned}
&\|f\Big(([x,y,z]+[y,z,x]+[z,x,y]),w\Big)-[f(x,w),y,z]+[x,f(y,w^*),z]-[x,y,f(z,w)]-[f(y,w),z,x]\\
&-[y,f(z,w^*),x]-[y,z,f(x,w)]-[f(z,w),x,y]-[z,f(x,w^*),y]-[z,x,f(y,w)]\|\\
&+\|f\Big(x,([y,z,w]+[z,w,y]+[w,y,z])\Big)-[f(x,y),z,w]-[y,f(x^*,z),w]-[y,z,f(x^*,w)]-[f(x,z),w,y]\\
&-[z,f(x^*,w),y]-[z,w,f(x,y)]-[f(x,w),y,z]-[w,f(x^*,y),z]-[w,y,f(x,z)]\|\\
&\quad\le\theta(\|x\|^p+\|y\|^p+\|z\|^p+\|w\|^p)
\end{aligned} \tag{2.4}$$

for all $\lambda, \mu \in \mathbb{T}^1$ and all $x, y, z, w \in A$. Then there exists a unique ternary Jordan bi-derivations $D : A \times A \to A$ such that

$$\|f(x,y)-D(x,y)\|_B \le \frac{2\theta}{4-2^p}(\|x\|^p+\|y\|^p) \tag{2.5}$$

for all $x, y \in A$.

Proof. By the same reasoning as in the proof of [4, Theorem 2.3], there exists a unique $\mathbb{C}$-bilinear mapping $D : A \times A \to A$ satisfying (2.5). The $\mathbb{C}$-bilinear mapping $D : A \times A \to A$ is given by

$$D(x,y) := \lim_{n\to\infty} \frac{1}{4^n} f(2^n x, 2^n y),$$

for all $x, y \in A$. It is easy to show that

$$D(x,y) = \lim_{n\to\infty} \frac{1}{16^n} f(8^n x, 2^n y) = \lim_{n\to\infty} \frac{1}{16^n} f(2^n x, 8^n y)$$

for all $x, y \in A$, since f is bi-additive. It follows from (2.4) that

$$\begin{aligned}
&\|D\Big(([x,y,z]+[y,z,x]+[z,x,y]),w\Big) - [D(x,w),y,z] - [x,D(y,w^*),z] - [x,y,D(z,w)] - [D(y,w),z,x] \\
&-[y,D(z,w^*),x] - [y,z,D(x,w)] - [D(z,w),x,y] - [z,D(x,w^*),y] - [z,x,D(y,w)]\| \\
&+ \|D\Big(x,([y,z,w]+[z,w,y]+[w,y,z])\Big) - [D(x,y),z,w] - [y,D(x^*,z),w] - [y,z,D(x^*,w)] - [D(x,z),w,y] \\
&- [z,f(x^*,w),y] - [z,w,f(x,y)] - [f(x,w),y,z] - [w,f(x^*,y),z] - [w,y,f(x,z)]\| \\
&= \lim_{n\to\infty} \Big(\|\frac{1}{16^n} f\Big(2^{3n}[x,y,z] + 2^{3n}[y,z,x] + 2^{3n}[z,x,y], 2^n w\Big) - [\frac{1}{4^n} f(2^n x, 2^n w), y, z] - [x, \frac{1}{4^n} f(2^n y, 2^n w^*), z] \\
&- [x,y,\frac{1}{4^n} f(2^n z, 2^n w)] - [\frac{1}{4^n} f(2^n y, 2^n w), z, x] - [y, \frac{1}{4^n} f(2^n z, 2^n w^*), x] - [y,z,\frac{1}{4^n} f(2^n x, 2^n w)] \\
&- [\frac{1}{4^n} f(2^n z, 2^n w), x, y] - [z, \frac{1}{4^n} f(2^n x, 2^n w^*), y] - [z,x,\frac{1}{4^n} f(2^n y, 2^n w)]\| \\
&+ \Big(\|\frac{1}{16^n} f\Big(2^n x, 2^{3n}[y,z,w] + 2^{3n}[z,w,y] + 2^{3n}[z,w,y]\Big) - [\frac{1}{4^n} f(2^n x, 2^n y), z, w] + [y, \frac{1}{4^n} f(2^n x^*, 2^n z), w] \\
&- [y,z,\frac{1}{4^n} f(2^n x, 2^n w)] - [\frac{1}{4^n} f(2^n x, 2^n z), w, y] - [z, \frac{1}{4^n} f(2^n x^*, 2^n w), y] - [z,w,\frac{1}{4^n} f(2^n x, 2^n y)] \\
&- [\frac{1}{4^n} f(2^n x, 2^n w), y, z] - [w, \frac{1}{4^n} f(2^n x^*, 2^n y), z] - [w,y,\frac{1}{4^n} f(2^n x, 2^n z)]\| \\
&\le \lim_{n\to\infty} \frac{2^{np}}{16^n} \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p) = 0
\end{aligned}$$

for all $x, y, z, w \in A$. So

$$\begin{aligned}
&\|D\Big(([x,y,z]+[y,z,x]+[z,x,y]),w\Big) - [D(x,w),y,z] - [x,D(y,w^*),z] - [x,y,D(z,w)] - [D(y,w),z,x] \\
&- [y,D(z,w^*),x] - [y,z,D(x,w)] - [D(z,w),x,y] - [z,D(x,w^*),y] - [z,x,D(y,w)]\|
\end{aligned}$$

and

$$\begin{aligned}
&+ \|D\Big(x,([y,z,w]+[z,w,y]+[w,y,z])\Big) - [D(x,y),z,w] - [y,D(x^*,z),w] - [y,z,D(x^*,w)] - [D(x,z),w,y] \\
&- [z,f(x^*,w),y] - [z,w,f(x,y)] - [f(x,w),y,z] - [w,f(x^*,y),z] - [w,y,f(x,z)]\|
\end{aligned}$$

for all $x, y, z, w \in A$. By Lemma 2.2, the mapping D is a unique ternary Jordan bi-derivation satisfying (2.5). □

For the case $p > 4$, one can obtain a similar result.

Theorem 2.4. *Let p and θ be positive real numbers with $p > 4$, and let $f : A \times A \to A$ be a mapping satisfying* (2.3) *and* (2.4)*. Then there exists a unique ternary Jordan bi-derivation $D : A \times A \to A$ such that*

$$\|f(x,y) - D(x,y)\| \le \frac{6\theta}{2^p - 4}(\|x\|^p + \|y\|^p)$$

for all $x, y \in A$.

Proof. The proof is similar to the proof of Theorem 2.3. □

Theorem 2.5. *Let p and θ be positive real numbers with $p < \frac{1}{2}$, and let $f : A \times A \to A$ be a mapping such that*

$$\|D_{\lambda,\mu} f(x,y,z,w)\| \le \theta \cdot \|x\|^p \cdot \|y\|^p \cdot \|z\|^p \cdot \|w\|^p,$$

$$\|f\Big(([x,y,z]+[y,z,x]+[z,x,y]),w\Big)-[f(x,w),y,z]+[x,f(y,w^*),z]-[x,y,f(z,w)]-[f(y,w),z,x]$$
$$-[y,f(z,w^*),x]-[y,z,f(x,w)]-[f(z,w),x,y]-[z,f(x,w^*),y]-[z,x,f(y,w)]\|$$
$$+\|f\Big(x,([y,z,w]+[z,w,y]+[w,y,z])\Big)-[f(x,y),z,w]-[y,f(x^*,z),w]-[y,z,f(x^*,w)]-[f(x,z),w,y]$$
$$-[z,f(x^*,w),y]-[z,w,f(x,y)]-[f(x,w),y,z]-[w,f(x^*,y),z]-[w,y,f(x,z)]\|$$
$$\leq\theta\cdot\|x\|_A^p\cdot\|y\|_A^p\cdot\|z\|_A^p\cdot\|w\|_A^p$$

for all $\lambda,\mu\in\mathbb{T}^1$ and all $x,y,z,w\in A$. Then there exists a unique ternary Jordan bi-derivations $D:A\times A\to A$ such that

$$\|f(x,y)-D(x,y)\|\leq\frac{2\theta}{4-2^{4p}}\|x\|^{2p}\|y\|^{2p} \tag{2.6}$$

for all $x,y\in A$.

Proof. By the same reasoning as in the proof of [4, Theorem 2.6], there exists a unique $\mathbb{C}$-bilinear mapping $D:A\times A\to A$ satisfying (2.6). The $\mathbb{C}$-bilinear mapping $D:A\times A\to A$ is given by

$$D(x,y):=\lim_{n\to\infty}\frac{1}{4^n}f(2^nx,2^ny),$$

for all $x,y\in A$.

The rest of the proof is similar to the proof of Theorem 2.3. □

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SOME GENERALIZED DIFFERENCE SEQUENCE SPACES OF IDEAL CONVERGENCE AND ORLICZ FUNCTIONS

KULDIP RAJ[1], AZIMHAN ABZHAPBAROV[2] AND ASHIRBAYEV KHASSYMKHAN[3]

Abstract. In this paper we shall introduce some generalized difference sequence spaces by using Musielak-Orlicz function, ideal convergence and an infinite matrix defined on n-normed spaces. We shall study these spaces for some linear toplogical structures and algebraic properties. We also prove some inclusion relations between these spaces

1. Introduction and Preliminaries

The notion of statistical convergence was introduced by Fast [5] and Schoenberg [31] independently. Over the years and under different names, statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Later on, it was further investigated from the sequence space point of view and linked with summability theory by Fridy [6], Connor [1], Salat [29], Isik [14], Savaş [30], Malkowsky and Savaş [19], Kolk [16], Tripathy and Sen [32] and many others. In recent years, generalizations of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on locally compact spaces. Statistical convergence and its generalizations are also connected with subsets of the Stone-Cech compactification of natural numbers. Moreover, statistical convergence is closely related to the concept of convergence in probability. The notion depends on the density of subsets of the set $\mathbb{N}$ of natural numbers.

A subset E of $\mathbb{N}$ is said to have the natural density $\delta(E)$ if the following limit exists: $\delta(E) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \chi_E(k)$, where χ_E is the characteristic function of E. It is clear that any finite subset of $\mathbb{N}$ has zero natural density and $\delta(E^c) = 1 - \delta(E)$.

The notion of ideal convergence was first introduced by P.Kostyrko et.al [13] as a generalization of statistical convergence which was further studied in topological spaces by Das, Kostyrko, Wilczynski and Malik (see [2]). More applications of ideals can be seen in ([2], [3]). We continue in this direction and introduce I-convergence of generalized sequences in more general setting.

A family $\mathcal{I} \subset 2^Y$ of subsets of a non empty set Y is said to be an ideal in Yif

(1) $\phi \in \mathcal{I}$;

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(2) $A, B \in \mathcal{I}$ imply $A \cup B \in \mathcal{I}$;
(3) $A \in \mathcal{I}$, $B \subset A$ imply $B \in \mathcal{I}$, while an admissible ideal $\mathcal{I}$ of Y further satisfies $\{x\} \in \mathcal{I}$ for each $x \in Y$ (see [11]).

Given $\mathcal{I} \subset 2^{\mathbb{N}}$ be a non trivial ideal in $\mathbb{N}$. A sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be I-convergent to $x \in X$, if for each $\epsilon > 0$ the set $A(\epsilon) = \Big\{n \in \mathbb{N} : ||x_n - x|| \geq \epsilon\Big\}$ belongs to $\mathcal{I}$ (see [10]).
The notion of difference sequence spaces was introduced by Kızmaz [15], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [4] by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$. Let w be the space of all complex or real sequences $x = (x_k)$ and let m, n be non-negative integers, then for $Z = l_\infty,\ c,\ c_0$ we have sequence spaces

$$Z(\Delta^m_n) = \{x = (x_k) \in w : (\Delta^m_n x_k) \in Z\},$$

where $\Delta^m_n x = (\Delta^m_n x_k) = (\Delta^{m-1}_n x_k - \Delta^{m-1}_n x_{k+1})$ and $\Delta^0_n x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta^m_n x_k = \sum_{v=0}^{m} (-1)^v \begin{pmatrix} m \\ v \end{pmatrix} x_{k+nv}.$$

Taking $n = 1$, we get the spaces which were studied by Et and Çolak [4]. Taking $m = n = 1$, we get the spaces which were introduced and studied by Kızmaz [15].

The concept of 2-normed spaces was initially developed by Gähler [7] in the mid of 1960's, while that of n-normed spaces one can see in Misiak[19]. Since then, many others have studied this concept and obtained various results, see Gunawan ([8], [9]) and Gunawan and Mashadi [10] and many others. Let $n \in \mathbb{N}$ and X be a linear space over the field $\mathbb{K}$, where $\mathbb{K}$ is field of real or complex numbers of dimension d, where $d \geq n \geq 2$. A real valued function $||\cdot, \cdots, \cdot||$ on X^n satisfying the following four conditions:

(1) $||x_1, x_2, \cdots, x_n|| = 0$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent in X;
(2) $||x_1, x_2, \cdots, x_n||$ is invariant under permutation;
(3) $||\alpha x_1, x_2, \cdots, x_n|| = |\alpha|\ \ ||x_1, x_2, \cdots, x_n||$ for any $\alpha \in \mathbb{K}$, and
(4) $||x + x', x_2, \cdots, x_n|| \leq ||x, x_2, \cdots, x_n|| + ||x', x_2, \cdots, x_n||$

is called a n-norm on X, and the pair $(X, ||\cdot, \cdots, \cdot||)$ is called a n-normed space over the field $\mathbb{K}$.
For example, we may take $X = \mathbb{R}^n$ being equipped with the n-norm $||x_1, x_2, \cdots, x_n||_E$ = the volume of the n-dimensional parallelopiped spanned by the vectors $x_1, x_2, \cdots, x_n$ which may be given explicitly by the formula

$$||x_1, x_2, \cdots, x_n||_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots, n$, where script E denotes Euclidean space. Let $(X, ||\cdot, \cdots, \cdot||)$ be an n-normed space of dimension $d \geq n \geq 2$ and

$\{a_1, a_2, \cdots, a_n\}$ be linearly independent set in X. Then the following function $||\cdot, \cdots, \cdot||_\infty$ on X^{n-1} defined by

$$||x_1, x_2, \cdots, x_{n-1}||_\infty = \max\{||x_1, x_2, \cdots, x_{n-1}, a_i|| : i = 1, 2, \cdots, n\}$$

defines an $(n-1)$-norm on X with respect to $\{a_1, a_2, \cdots, a_n\}$.
A sequence (x_k) in a n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to converge to some $L \in X$ if

$$\lim_{k \to \infty} ||x_k - L, z_1, \cdots, z_{n-1}|| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$

A sequence (x_k) in a n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to be Cauchy if

$$\lim_{k,i \to \infty} ||x_k - x_i, z_1, \cdots, z_{n-1}|| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space. An Orlicz function $M : [0, \infty) \to [0, \infty)$ is a continuous, non-decreasing and convex function such that $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.
Lindenstrauss and Tzafriri [17] used the idea of Orlicz function to define the following sequence space,

$$\ell_M = \Big\{(x_k) \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty, \text{ for some } \rho > 0\Big\}$$

which is called as an Orlicz sequence space. Also ℓ_M is a Banach space with the norm

$$||(x_k)|| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1\Big\}.$$

Also, it was shown in [17] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (p \geq 1)$. An Orlicz function M satisfies Δ_2-condition if and only if for any constant $L > 1$ there exists a constant $K(L)$ such that $M(Lu) \leq K(L)M(u)$ for all values of $u \geq 0$. An Orlicz function M can always be represented in the following integral form

$$M(x) = \int_0^x \eta(t)dt$$

where η is known as the kernel of M, is right differentiable for $t \geq 0, \eta(0) = 0, \eta(t) > 0$, η is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$.
A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is called a Musielak-Orlicz function see ([18], [25]). A sequence $\mathcal{N} = (N_k)$ is defined by

$$N_k(v) = \sup\{|v|u - M_k(u) : u \geq 0\}, \ k = 1, 2, \cdots$$

is called the complementary function of a Musielak-Orlicz function $\mathcal{M}$. For a given Musielak-Orlicz function $\mathcal{M}$, the Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows

$$t_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for some } c > 0\Big\},$$

$$h_{\mathcal{M}} = \Big\{ x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for all } c > 0 \Big\},$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k), x = (x_k) \in t_{\mathcal{M}}.$$

We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$||x|| = \inf \Big\{ k > 0 : I_{\mathcal{M}}\Big(\frac{x}{k}\Big) \leq 1 \Big\}$$

or equipped with the Orlicz norm

$$||x||^0 = \inf \Big\{ \frac{1}{k}\Big(1 + I_{\mathcal{M}}(kx)\Big) : k > 0 \Big\}.$$

A Musielak-Orlicz function (M_k) is said to satisfy Δ_2-condition if there exist constants $a, K > 0$ and a sequence $c = (c_k)_{k=1}^{\infty} \in \ell_+^1$ (the positive cone of ℓ^1) such that the inequality

$$M_k(2u) \leq K M_k(u) + c_k$$

holds for all $k \in N$ and $u \in R_+$ whenever $M_k(u) \leq a$.
Let X be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if

(1) $p(x) \geq 0$ for all $x \in X$,
(2) $p(-x) = p(x)$ for all $x \in X$,
(3) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$,
(4) if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [33], Theorem 10.4.2, pp. 183). For more details about sequence spaces (see [21], [22], [23], [24], [26], [27], [28]) and reference therein.
A sequence space E is said to be solid(or normal) if $(x_k) \in E$ implies $(\alpha_k x_k) \in E$ for all sequences of scalars (α_k) with $|\alpha_k| \leq 1$ and for all $k \in \mathbb{N}$.
Let I be an admissible ideal of $\mathbb{N}$, let $p = (p_k)$ be a bounded sequence of positive real numbers for all $k \in \mathbb{N}$ and $A = (a_{nk})$ be an infinite matrix. Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $u = (u_k)$ be a sequence of strictly positive real numbers and $(X, ||., ..., .||)$ be a n-normed space. Further $w(n - x)$ denotes the space of all X-valued sequences. For every $z_1, z_2, ..., z_{n-1} \in X$, for each $\epsilon > 0$ and for some $\rho > 0$ we define the following sequence spaces:
$W^I\big[A, \Delta_n^m, \mathcal{M}, u, p, ||., ..., .||\big] = \Big\{ x = (x_k) \in w(n - x) : \text{ for given } \epsilon > 0, \Big\{ n \in \mathbb{N} :$

$$\sum_{k=1}^{\infty} a_{nk} \Big[M_k\Big(|| \frac{u_k \Delta_n^m x_k - L}{\rho}, z_1, z_2, ..., z_{n-1} || \Big) \Big]^{p_k} \geq \epsilon \Big\} \in I, \text{ for } L \in X \text{ and } n \in \mathbb{N} \Big\},$$

$$W_0^I\big[A,\Delta_n^m,\mathcal{M},u,p,||.,...,.||\big] = \Big\{x=(x_k)\in w(n-x): \text{ for given } \epsilon>0, \Big\{n\in\mathbb{N}: \sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(||\frac{u_k\Delta_n^m x_k}{\rho}, z_1, z_2, ..., z_{n-1}||\Big)\Big]^{p_k} \geq \epsilon\Big\}\in I\Big\}$$

and

$$W_\infty^I\big[A,\Delta_n^m,\mathcal{M},u,p,||.,...,.||\big] = \Big\{x=(x_k)\in w(n-x): \exists\, k>0, \Big\{n\in\mathbb{N}: \sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(||\frac{u_k\Delta_n^m x_k}{\rho}, z_1, z_2, ..., z_{n-1}||\Big)\Big]^{p_k} \geq K\Big\}\in I\Big\}.$$

Some special cases of the above defined sequence spaces are arises:
If $m=n=0$, then we obtain the spaces as follows

$$W^I\big[A,\mathcal{M},u,p,||.,...,.||\big] = \Big\{x=(x_k)\in w(n-x): \text{ for given } \epsilon>0, \Big\{n\in\mathbb{N}: \sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(||\frac{u_k x_k - L}{\rho}, z_1, z_2, ..., z_{n-1}||\Big)\Big]^{p_k} \geq \epsilon\Big\}\in I, \text{ for } L\in X \text{ and } n\in\mathbb{N}\Big\},$$

$$W_0^I\big[A,\mathcal{M},u,p,||.,...,.||\big] = \Big\{x=(x_k)\in w(n-x): \text{ for given } \epsilon>0, \Big\{n\in\mathbb{N}: \sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(||\frac{u_k x_k}{\rho}, z_1, z_2, ..., z_{n-1}||\Big)\Big]^{p_k} \geq \epsilon\Big\}\in I\Big\}$$

and

$$W_\infty^I\big[A,\mathcal{M},u,p,||.,...,.||\big] = \Big\{x=(x_k)\in w(n-x): \exists\, k>0, \Big\{n\in\mathbb{N}: \sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(||\frac{u_k x_k}{\rho}, z_1, z_2, ..., z_{n-1}||\Big)\Big]^{p_k} \geq K\Big\}\in I\Big\}.$$

If $m=n=1$, then the above spaces are as follows

$$W^I\big[A,\Delta,\mathcal{M},u,p,||.,...,.||\big] = \Big\{x=(x_k)\in w(n-x): \text{ for given } \epsilon>0, \Big\{n\in\mathbb{N}: \sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(||\frac{u_k\Delta x_k - L}{\rho}, z_1, z_2, ..., z_{n-1}||\Big)\Big]^{p_k} \geq \epsilon\Big\}\in I, \text{ for } L\in X \text{ and } n\in\mathbb{N}\Big\},$$

$$W_0^I\big[A,\Delta,\mathcal{M},u,p,||.,...,.||\big] = \Big\{x=(x_k)\in w(n-x): \text{ for given } \epsilon>0, \Big\{n\in\mathbb{N}: \sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(||\frac{u_k\Delta x_k}{\rho}, z_1, z_2, ..., z_{n-1}||\Big)\Big]^{p_k} \geq \epsilon\Big\}\in I\Big\}$$

and

$$W_\infty^I\big[A,\Delta,\mathcal{M},u,p,||.,...,.||\big] = \Big\{x=(x_k)\in w(n-x): \exists\, k>0, \Big\{n\in\mathbb{N}: \sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(||\frac{u_k\Delta x_k}{\rho}, z_1, z_2, ..., z_{n-1}||\Big)\Big]^{p_k} \geq K\Big\}\in I\Big\}.$$

If $\mathcal{M}(x) = x$ for all $x \in [0, \infty)$, then we have

$$W^I\big[A, \Delta_n^m, u, p, ||., ..., .||\big] = \Big\{x = (x_k) \in w(n-x): \text{ for given } \epsilon > 0, \Big\{n \in \mathbb{N}: \sum_{k=1}^{\infty} a_{nk}\Big(||\frac{u_k \Delta_n^m x_k - L}{\rho}, z_1, z_2, ..., z_{n-1}||\Big)^{p_k} \geq \epsilon\Big\} \in I, \text{ for } L \in X \text{ and } n \in \mathbb{N}\Big\},$$

$$W_0^I\big[A, \Delta_n^m, u, p, ||., ..., .||\big] = \Big\{x = (x_k) \in w(n-x): \text{ for given } \epsilon > 0, \Big\{n \in \mathbb{N}: \sum_{k=1}^{\infty} a_{nk}\Big(||\frac{u_k \Delta_n^m x_k}{\rho}, z_1, z_2, ..., z_{n-1}||\Big)^{p_k} \geq \epsilon\Big\} \in I\Big\}$$

and

$$W_\infty^I\big[A, \Delta_n^m, u, p, ||., ..., .||\big] = \Big\{x = (x_k) \in w(n-x): \exists\, k > 0, \Big\{n \in \mathbb{N}: \sum_{k=1}^{\infty} a_{nk}\Big(||\frac{u_k \Delta_n^m x_k}{\rho}, z_1, z_2, ..., z_{n-1}||\Big)^{p_k} \geq K\Big\} \in I\Big\}.$$

If $p = (p_k) = 1$ for all k, then the above spaces are as follows

$$W^I\big[A, \Delta_n^m, \mathcal{M}, u, ||., ..., .||\big] = \Big\{x = (x_k) \in w(n-x): \text{ for given } \epsilon > 0, \Big\{n \in \mathbb{N}: \sum_{k=1}^{\infty} a_{nk} M_k\Big(||\frac{u_k \Delta_n^m x_k - L}{\rho}, z_1, z_2, ..., z_{n-1}||\Big) \geq \epsilon\Big\} \in I, \text{ for } L \in X \text{ and } n \in \mathbb{N}\Big\},$$

$$W_0^I\big[A, \Delta_n^m, \mathcal{M}, u, ||., ..., .||\big] = \Big\{x = (x_k) \in w(n-x): \text{ for given } \epsilon > 0, \Big\{n \in \mathbb{N}: \sum_{k=1}^{\infty} a_{nk} M_k\Big(||\frac{u_k \Delta_n^m x_k}{\rho}, z_1, z_2, ..., z_{n-1}||\Big) \geq \epsilon\Big\} \in I\Big\}$$

and

$$W_\infty^I\big[A, \Delta_n^m, \mathcal{M}, u, ||., ..., .||\big] = \Big\{x = (x_k) \in w(n-x): \exists\, k > 0, \Big\{n \in \mathbb{N}: \sum_{k=1}^{\infty} a_{nk} M_k\Big(||\frac{u_k \Delta_n^m x_k}{\rho}, z_1, z_2, ..., z_{n-1}||\Big) \geq K\Big\} \in I\Big\}.$$

If $A = (C, 1)$, the Cesàro matrix, then the above spaces are as follows

$$W^I\big[\Delta_n^m, \mathcal{M}, u, p, ||., ..., .||\big] = \Big\{x = (x_k) \in w(n-x): \text{ for given } \epsilon > 0, \Big\{n \in \mathbb{N}: \sum_{k=1}^{\infty} \Big[M_k\Big(||\frac{u_k \Delta_n^m x_k - L}{\rho}, z_1, z_2, ..., z_{n-1}||\Big)\Big]^{p_k} \geq \epsilon\Big\} \in I, \text{ for } L \in X \text{ and } n \in \mathbb{N}\Big\},$$

$$W_0^I\big[\Delta_n^m, \mathcal{M}, u, p, ||., ..., .||\big] = \Big\{x = (x_k) \in w(n-x): \text{ for given } \epsilon > 0, \Big\{n \in \mathbb{N}: \sum_{k=1}^{\infty} \Big[M_k\Big(||\frac{u_k \Delta_n^m x_k}{\rho}, z_1, z_2, ..., z_{n-1}||\Big)\Big]^{p_k} \geq \epsilon\Big\} \in I\Big\}$$

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and

$$W_\infty^I\big[\Delta_n^m, \mathcal{M}, u, p, ||., ..., .||\big] = \Big\{x = (x_k) \in w(n-x) : \exists\, k > 0, \Big\{n \in \mathbb{N} : \sum_{k=1}^{\infty}\Big[M_k\Big(||\frac{u_k \Delta_n^m x_k}{\rho}, z_1, z_2, ..., z_{n-1}||\Big)\Big]^{p_k} \geq K\Big\} \in I\Big\}.$$

If we take $A = (a_{nk})$ is a de La Valee Poussin mean i.e.

$$a_{nk} = \begin{cases} \frac{1}{\lambda_n}, & \text{if } k \in I_n = [n - \lambda_n + 1, n] \\ 0, & \text{otherwise} \end{cases}$$

where (λ_n) is a non-decreasing sequence of positive numbers tending to ∞ and $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1$, then the above sequence spaces are denoted by $W^I\big[\lambda, \Delta_n^m, \mathcal{M}, u, p, ||., ..., .||\big]$, $W_0^I\big[\lambda, \Delta_n^m, \mathcal{M}, u, p, ||., ..., .||\big]$ and $W_\infty^I\big[\lambda, \Delta_n^m, \mathcal{M}, u, p, ||., ..., .||\big]$.

By a lacunary sequence $\theta = (k_r); r = 0, 1, 2, ...$ where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and $h_r = k_r - k_{r-1}$. We finally arrived, let

$$a_{nk} = \begin{cases} \frac{1}{h_r}, & \text{if } k_{r-1} < k < k_r \\ 0, & \text{otherwise.} \end{cases}$$

Then the above classes of sequences are denoted by $W^I\big[\theta, \Delta_n^m, \mathcal{M}, p, ||., ..., .||\big]$, $W_0^I\big[\theta, \Delta_n^m, \mathcal{M}, p, ||., ..., .||\big]$ and $W_\infty^I\big[\theta, \Delta_n^m, \mathcal{M}, p, ||., ..., .||\big]$.

The following inequality will be used throughout the paper. If $0 \leq p_k \leq \sup p_k = G$, $D = \max(1, 2^{G-1})$ then

$$|a_k + b_k|^{p_k} \leq D\{|a_k|^{p_k} + |b_k|^{p_k}\} \tag{1.1}$$

for all k and $a_k, b_k \in \mathbb{C}$. Also $|a|^{p_k} \leq \max(1, |a|^G)$ for all $a \in \mathbb{C}$.

The main aim of this paper is to introduce some generalized difference sequence spaces defined by ideal convergence, a Musielak-Orlicz function $\mathcal{M} = (M_k)$ and an infinite matrix $A = (a_{nk})$. I have also make an effort to study some inclusion relations and their topological properties.

2. Main Results

Theorem 2.1 *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers. Then $W^I\big[A, \Delta_n^m, \mathcal{M}, u, p, ||., ..., .||\big]$, $W_0^I\big[A, \Delta_n^m, \mathcal{M}, u, p, ||., ..., .||\big]$ and $W_\infty^I\big[A, \Delta_n^m, \mathcal{M}, u, p, ||., ..., .||\big]$ are linear spaces over the field of complex numbers $\mathbb{C}$.*

Proof. We shall prove the result for the space $W_0^I\big[A, \Delta_n^m, \mathcal{M}, u, p, ||., ..., .||\big]$. Let $x = (x_k)$

and $y = (y_k)$ be two elements of $W_0^I\big[A, \Delta_n^m, \mathcal{M}, u, p, ||., ..., .||\big]$. Then there exists $\rho_1 > 0$ and $\rho_2 > 0$ and for $z_1, z_2, ..., z_{n-1} \in X$ such that

$$A_{\frac{\epsilon}{2}} = \Big\{n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(||\frac{u_k\Delta_n^m x_k}{\rho_1}, z_1, z_2, ..., z_{n-1}||\Big)\Big]^{p_k} \geq \frac{\epsilon}{2}\Big\} \in I$$

and

$$B_{\frac{\epsilon}{2}} = \Big\{n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(||\frac{u_k\Delta_n^m y_k}{\rho_2}, z_1, z_2, ..., z_{n-1}||\Big)\Big]^{p_k} \geq \frac{\epsilon}{2}\Big\} \in I.$$

Let $\alpha, \beta \in \mathbb{C}$. Since $||., ..., .||$ is a n-norm, Δ_n^m is linear and the contributing of $\mathcal{M} = (M_k)$, the following inequality holds:

$$\begin{aligned}
&\sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(||\frac{u_k\Delta_n^m(\alpha x_k + \beta y_k)}{|\alpha|\rho_1 + |\beta|\rho_2}, z_1, z_2, ..., z_{n-1}||\Big)\Big]^{p_k} \\
&\leq D\sum_{k=1}^{\infty} a_{nk}\Big[\frac{|\alpha|}{|\alpha|\rho_1 + |\beta|\rho_2}M_k\Big(||\frac{u_k\Delta_n^m x_k}{\rho_1}, z_1, z_2, ..., z_{n-1}||\Big)\Big]^{p_k} \\
&+ D\sum_{k=1}^{\infty} a_{nk}\Big[\frac{|\beta|}{|\alpha|\rho_1 + |\beta|\rho_2}M_k\Big(||\frac{u_k\Delta_n^m y_k}{\rho_2}, z_1, z_2, ..., z_{n-1}||\Big)\Big]^{p_k} \\
&\leq DK\sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(||\frac{u_k\Delta_n^m x_k}{\rho_1}, z_1, z_2, ..., z_{n-1}||\Big)\Big]^{p_k} \\
&+ DK\sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(||\frac{u_k\Delta_n^m y_k}{\rho_2}, z_1, z_2, ..., z_{n-1}||\Big)\Big]^{p_k}
\end{aligned}$$

where $K = \max\Big\{1, \frac{|\alpha|}{|\alpha|\rho_1+|\beta|\rho_2}, \frac{|\beta|}{|\alpha|\rho_1+|\beta|\rho_2}\Big\}$.
From the above relation , we get

$$\begin{aligned}
&\Big\{n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(||\frac{u_k\Delta_n^m(\alpha x_k + \beta y_k)}{|\alpha|\rho_1 + |\beta|\rho_2}, z_1, z_2, ..., z_{n-1}||\Big)\Big]^{p_k} \geq \epsilon\Big\} \\
&\subseteq \Big\{n \in \mathbb{N} : DK\sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(||\frac{u_k\Delta_n^m x_k}{\rho_1}, z_1, z_2, ..., z_{n-1}||\Big)\Big]^{p_k} \geq \frac{\epsilon}{2}\Big\} \\
&\cup \Big\{n \in \mathbb{N} : DK\sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(||\frac{u_k\Delta_n^m y_k}{\rho_2}, z_1, z_2, ..., z_{n-1}||\Big)\Big]^{p_k} \geq \frac{\epsilon}{2}\Big\}.
\end{aligned}$$

Since both the sets on the R.H.S of above relation are belongs to I, so the set on the L.H.S of the inclusion relation belongs to I. Similarly we can prove other cases. This completes the proof of the theorem.

Theorem 2.2 *Let $\mathcal{M}' = (M_k')$ and $\mathcal{M}'' = (M_k'')$ be two Musielak-orlicz functions. Then we have $W_0^I\big[A, \Delta_n^m, \mathcal{M}', u, p, ||., ..., .||\big] \cap W_0^I\big[A, \Delta_n^m, \mathcal{M}'', u, p, ||., ..., .||\big] \subseteq W_0^I\big[A, \Delta_n^m, \mathcal{M}' + \mathcal{M}'', u, p, ||., ..., .||\big]$.*

Proof. Let $x = (x_k) \in W_0^I\big[A, \Delta_n^m, \mathcal{M}', u, p, ||., ..., .||\big] \cap W_0^I\big[A, \Delta_n^m, \mathcal{M}'', u, p, ||., ..., .||\big]$.

Then we get the result by the following inequality:

$$\sum_{k=1}^{\infty} a_{nk}\Big[(M'_k + M''_k)\Big(||\frac{u_k\Delta_n^m x_k}{\rho}, z_1, z_2, ..., z_{n-1}||\Big)\Big]^{p_k}$$

$$\begin{aligned} &\leq D\sum_{k=1}^{\infty} a_{nk}\Big[M'_k\Big(||\frac{u_k\Delta_n^m x_k}{\rho}, z_1, z_2, ..., z_{n-1}||\Big)\Big]^{p_k} \\ &+ D\sum_{k=1}^{\infty} a_{nk}\Big[M''_k\Big(||\frac{u_k\Delta_n^m x_k}{\rho}, z_1, z_2, ..., z_{n-1}||\Big)\Big]^{p_k}. \end{aligned}$$

Hence

$$\Big\{n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk}\Big[(M'_k + M''_k)\Big(||\frac{u_k\Delta_n^m x_k}{\rho}, z_1, z_2, ..., z_{n-1}||\Big)\Big]^{p_k} \geq \epsilon\Big\}$$

$$\begin{aligned} &\subseteq \Big\{n \in \mathbb{N} : D\sum_{k=1}^{\infty} a_{nk}\Big[M'_k\Big(||\frac{u_k\Delta_n^m x_k}{\rho}, z_1, z_2, ..., z_{n-1}||\Big)\Big]^{p_k} \geq \frac{\epsilon}{2}\Big\} \\ &\cup \Big\{n \in \mathbb{N} : D\sum_{k=1}^{\infty} a_{nk}\Big[M''_k\Big(||\frac{u_k\Delta_n^m x_k}{\rho}, z_1, z_2, ..., z_{n-1}||\Big)\Big]^{p_k} \geq \frac{\epsilon}{2}\Big\} \end{aligned}$$

Since both the sets on the R.H.S of above relation are belongs to I, so the set on the L.H.S of the inclusion relation belongs to I. This completes the proof of the theorem.

Theorem 2.3 *The inclusions* $Z\big[\Delta_n^{m-1}, \mathcal{M}, u, p, ||., ..., .||\big] \subseteq Z\big[A, \Delta_n^m, \mathcal{M}, u, p, ||., ..., .||\big]$ *are strict for* $m \geq 1$. *In general* $Z\big[\Delta_n^{m-1}, \mathcal{M}, u, p, ||., ..., .||\big] \subseteq Z\big[A, \Delta_n^m, \mathcal{M}, u, p, ||., ..., .||\big]$, *for* $m = 0, 1, 2, ...$ *where* $Z = W^I, W_0^I, W_\infty^I$.

Proof. We give the proof for $W_0^I\big[A, \Delta_n^{m-1}, \mathcal{M}, u, p, ||., ..., .||\big]$ only. The others can be proved by similar argument. Let $x = (x_k)$ be any element in the space $W_0^I\big[A, \Delta_n^{m-1}, \mathcal{M}, u, p, ||., ..., .||\big]$. Let $\epsilon > 0$ be given. Then there exists $\rho > 0$ such that the set

$$\Big\{n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(||\frac{u_k\Delta_n^{m-1} x_k}{\rho}, z_1, z_2, ..., z_{n-1}||\Big)\Big]^{p_k} \geq \epsilon\Big\} \in I.$$

Since $\mathcal{M}=(M_k)$ is non-decreasing and convex for every k, it follows that

$$\sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(||\frac{u_k\Delta_n^m x_k}{2\rho},z_1,z_2,...,z_{n-1}||\Big)\Big]^{p_k}$$

$$\begin{aligned}
&= \sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(||\frac{u_k\Delta_n^{m-1}x_{k+1}-u_k\Delta_n^{m-1}x_k}{2\rho},z_1,z_2,...,z_{n-1}||\Big)\Big]^{p_k}\\
&\leq D\sum_{k=1}^{\infty} a_{nk}\Big[\frac{1}{2}M_k\Big(||\frac{u_k\Delta_n^{m-1}x_{k+1}}{\rho},z_1,z_2,...,z_{n-1}||\Big)\Big]^{p_k}\\
&+ D\sum_{k=1}^{\infty} a_{nk}\Big[\frac{1}{2}M_k\Big(||\frac{u_k\Delta_n^{m-1}x_k}{\rho},z_1,z_2,...,z_{n-1}||\Big)\Big]^{p_k}\\
&\leq DH\sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(||\frac{u_k\Delta_n^{m-1}x_{k+1}}{\rho},z_1,z_2,...,z_{n-1}||\Big)\Big]^{p_k}\\
&+ DH\sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(||\frac{u_k\Delta_n^{m-1}x_k}{\rho},z_1,z_2,...,z_{n-1}||\Big)\Big]^{p_k},
\end{aligned}$$

where $H=\max\Big\{1,(\frac{1}{2})^G\Big\}$. Thus we have

$$\Big\{n\in\mathbb{N}:\sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(||\frac{u_k\Delta_n^m x_k}{2\rho},z_1,z_2,...,z_{n-1}||\Big)\Big]^{p_k}\geq\epsilon\Big\}$$

$$\begin{aligned}
&\subseteq \Big\{n\in\mathbb{N}:\sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(||\frac{u_k\Delta_n^{m-1}x_{k+1}}{\rho},z_1,z_2,...,z_{n-1}||\Big)\Big]^{p_k}\geq\frac{\epsilon}{2}\Big\}\\
&\cup \Big\{n\in\mathbb{N}:\sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(||\frac{u_k\Delta_n^{m-1}x_k}{\rho},z_1,z_2,...,z_{n-1}||\Big)\Big]^{p_k}\geq\frac{\epsilon}{2}\Big\}
\end{aligned}$$

Since both the sets in right hand side of the above relation belongs to I, therefore we get the set

$$\Big\{n\in\mathbb{N}:\sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(||\frac{u_k\Delta_n^m x_k}{\rho},z_1,z_2,...,z_{n-1}||\Big)\Big]^{p_k}\geq\epsilon\Big\}\in I.$$

This inclusion is strict follows from the following example.

Example. Let $M_k(x)=x$, for all $k\in\mathbb{N}$, $u_k=p_k=1$ for all $k\in\mathbb{N}$ and $A=(C,1)$, the Cesaro matrix. Now consider a sequence $x=(x_k)=(k^s)$. Then for $n=1, x=(x_k)$ belongs to $W_0^I\big[\Delta_n^m,\mathcal{M},u,p,||.,...,.||\big]$ but does not belongs to $W_0^I\big[\Delta_n^{m-1},\mathcal{M},u,p,||.,...,.||\big]$, because $\Delta_n^m x_k=0$ and $\Delta_n^{m-1}x_k=(-1)^{m-1}(m-1)!$.

Theorem 2.4 *For any two sequences $p=(p_k)$ and $q=(q_k)$ of positive real numbers and for any two n-norms $||.,...,.||_1$ and $||.,...,.||_2$ on X, we have the following*

$Z\big[A,\Delta_n^m,\mathcal{M},u,p,||.,...,.||_1\big]\cap Z\big[A,\Delta_n^m,\mathcal{M},u,q,||.,...,.||_2\big]\neq\phi$ *where* $Z=W^I,W_0^I$ *and* W_∞^I.

Proof. Since the zero element belongs to both the classes of sequences, so the intersection is non-empty.

Theorem 2.5 *The sequence spaces* $W_0^I\big[A,\Delta_n^m,\mathcal{M},u,p,||.,...,.||\big]$ *and* $W_\infty^I\big[A,\Delta_n^m,\mathcal{M},u,p,||.,...,.||\big]$ *are normal as well as monotone.*

Proof. We shall prove the theorem for $W_0^I\big[A,\Delta_n^m,\mathcal{M},u,p,||.,...,.||\big]$. Let $x=(x_k)\in W_0^I\big[A,\Delta_n^m,\mathcal{M},u,p,||.,...,.||\big]$ and $\alpha=(\alpha_k)$ be a sequence of scalars such that $|\alpha_k|\leq 1$ for all $k\in\mathbb{N}$. Then for given $\epsilon>0$, we have

$$\Big\{n\in\mathbb{N}:\sum_{k=1}^{\infty}a_{nk}\Big[M_k\Big(||\frac{u_k\Delta_n^m(\alpha_k x_k)}{\rho},z_1,z_2,...,z_{n-1}||\Big)\Big]^{p_k}\geq\epsilon\Big\}$$

$$\subseteq\quad\Big\{n\in\mathbb{N}:\sum_{k=1}^{\infty}a_{nk}\Big[M_k\Big(||\frac{u_k\Delta_n^m(x_k)}{\rho},z_1,z_2,...,z_{n-1}||\Big)\Big]^{p_k}\geq\epsilon\Big\}\in I.$$

Hence $\alpha_k x_k\in W_0^I\big[A,\Delta_n^m,\mathcal{M},u,p,||.,...,.||\big]$. Thus the space $W_0^I\big[A,\Delta_n^m,\mathcal{M},u,p,||.,...,.||\big]$ is normal. Therefore $W_0^I\big[A,\Delta_n^m,\mathcal{M},u,p,||.,...,.||\big]$ is monotone also (see [12]). Similarly we can prove the theorem for other case. This completes the proof of the theorem.

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[1]School of Mathematics Shri Mata Vaishno Devi University, Katra-182320, J & K, India.

[2,3]Science-Pedagogical Faculty, M. Auezov South Kazakhstan State University, Tauke Khan Avenue 5, Shymkent 160012, Kazakhstan

E-mail address: kuldipraj68@gmail.com

E-mail address: azeke_55@mail.ru

E-mail address: ashirbaev54@mail.ru

A general stability theorem for a class of functional equations including quadratic-additive functional equations

Yang-Hi Lee and Soon-Mo Jung

Department of Mathematics Education, Gongju National University of Education, Gongju 314-711, Republic of Korea
E-mail: yanghi2@hanmail.net

Mathematics Section, College of Science and Technology, Hongik University, 339-701 Sejong, Republic of Korea
E-mail: smjung@hongik.ac.kr

Abstract. We prove a general stability theorem of an n-dimensional quadratic-additive type functional equation

$$Df(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{m} c_i f\big(a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n\big) = 0$$

by using the direct method.

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Key Words: generalized Hyers-Ulam stability; functional equation; n-dimensional quadratic-additive type functional equation; quadratic-additive mapping; direct method.

1 Introduction

Let G_1 and G_2 be abelian groups. For any mapping $f : G_1 \to G_2$, let us define

$$Af(x, y) := f(x + y) - f(x) - f(y),$$
$$Qf(x, y) := f(x + y) + f(x - y) - 2f(x) - 2f(y)$$

for all $x, y \in G_1$. A mapping $f : G_1 \to G_2$ is called an additive mapping (or a quadratic mapping) if f satisfies the functional equation $Af(x, y) = 0$ (or $Qf(x, y) = 0$) for all $x, y \in G_1$. We notice that the mappings $g, h : \mathbb{R} \to \mathbb{R}$ given by $g(x) = ax$ and $h(x) = ax^2$ are solutions of $Ag(x, y) = 0$ and $Qh(x, y) = 0$, respectively.

A mapping $f : G_1 \to G_2$ is called a quadratic-additive mapping if and only if f is represented by the sum of an additive mapping and a quadratic mapping. A functional equation is called a quadratic-additive type functional equation if and only if each of its solutions is a quadratic-additive mapping (see [9]). For example, the mapping $f(x) = ax^2 + bx$ is a solution of the quadratic-additive type functional equation.

In the study of stability problems of quadratic-additive type functional equations, we have followed out a routine and monotonous procedure for proving the stability of the quadratic-additive type functional equations under various conditions. We can find in the books [2, 3, 7, 8] a lot of references concerning the Hyers-Ulam stability of functional equations (see also [1, 4, 5, 6, 14, 15]).

Throughout this paper, let V and W be real vector spaces, let X and Y be a real normed space resp. a real Banach space, and let $\mathbb{N}_0$ denote the set of all nonnegative integers.

In this paper, we prove a general stability theorem that can be easily applied to the (generalized) Hyers-Ulam stability of a large class of functional equations of the form $Df(x_1, x_2, \ldots, x_n) = 0$, which includes quadratic-additive type functional equations. In practice, given a mapping $f : V \to W$, $Df : V^n \to W$ is defined by

$$Df(x_1, x_2, \ldots, x_n) := \sum_{i=1}^{m} c_i f\big(a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n\big) \tag{1.1}$$

for all $x_1, x_2, \ldots, x_n \in V$, where m is a positive integer and c_i, a_{ij} are real constants.

Indeed, this stability theorem can save us much trouble of proving the stability of relevant solutions repeatedly appearing in the stability problems for various functional equations (see [11, 12, 13]).

2 Preliminaries

Let V and W be real vector spaces and let X and Y be a real normed space resp. a real Banach space. For a given mapping $f : V \to W$, we use the following abbreviations

$$f_o(x) := \frac{f(x) - f(-x)}{2} \quad \text{and} \quad f_e(x) := \frac{f(x) + f(-x)}{2}$$

for all $x \in V$.

We now introduce a lemma from the paper [10, Corollary 2].

Lemma 2.1 *Let $k > 1$ be a real constant, let $\phi : V\backslash\{0\} \to [0, \infty)$ be a function satisfying either*

$$\Phi(x) := \sum_{i=0}^{\infty} \frac{1}{k^i}\phi(k^i x) < \infty \tag{2.1}$$

for all $x \in V\backslash\{0\}$ or

$$\Phi(x) := \sum_{i=0}^{\infty} k^{2i}\phi\left(\frac{x}{k^i}\right) < \infty \tag{2.2}$$

for all $x \in V\backslash\{0\}$, and let $f : V \to Y$ be an arbitrarily given mapping. If there exists a mapping $F : V \to Y$ satisfying

$$\|f(x) - F(x)\| \le \Phi(x) \tag{2.3}$$

for all $x \in V\backslash\{0\}$ and

$$F_e(kx) = k^2 F_e(x), \quad F_o(kx) = kF_o(x) \tag{2.4}$$

for all $x \in V$, then F is a unique mapping satisfying (2.3) *and* (2.4).

We introduce a lemma that is the same as [10, Corollary 3].

Lemma 2.2 *Let $k > 1$ be a real number, let $\phi, \psi : V\backslash\{0\} \to [0, \infty)$ be functions satisfying each of the following conditions*

$$\sum_{i=0}^{\infty} k^i \psi\left(\frac{x}{k^i}\right) < \infty, \qquad \sum_{i=0}^{\infty} \frac{1}{k^{2i}} \phi(k^i x) < \infty,$$

$$\tilde{\Phi}(x) := \sum_{i=0}^{\infty} k^i \phi\left(\frac{x}{k^i}\right) < \infty, \quad \tilde{\Psi}(x) := \sum_{i=0}^{\infty} \frac{1}{k^{2i}} \psi(k^i x) < \infty$$

for all $x \in V\backslash\{0\}$, and let $f : V \to Y$ be an arbitrarily given mapping. If there exists a mapping $F : V \to Y$ satisfying the inequality

$$\|f(x) - F(x)\| \le \tilde{\Phi}(x) + \tilde{\Psi}(x) \tag{2.5}$$

for all $x \in V\backslash\{0\}$ and the conditions in (2.4) *for all $x \in V$, then F is a unique mapping satisfying* (2.4) *and* (2.5).

3 Main results

In this section, let a be a real constant with $a \notin \{-1, 0, 1\}$. Lemma 2.1 plays an important role in the proofs of the following two main theorems.

Theorem 3.1 *Let n be a fixed integer greater than 1, let $\mu : V\backslash\{0\} \to [0, \infty)$ be a function satisfying the condition*

$$\begin{cases} \displaystyle\sum_{i=0}^{\infty} \frac{\mu(a^i x)}{a^{2i}} < \infty & \text{when } |a| < 1, \\ \displaystyle\sum_{i=0}^{\infty} \frac{\mu(a^i x)}{|a|^i} < \infty & \text{when } |a| > 1 \end{cases} \tag{3.1}$$

for all $x \in V\backslash\{0\}$, and let $\varphi : (V\backslash\{0\})^n \to [0, \infty)$ be a function satisfying the condition

$$\begin{cases} \displaystyle\sum_{i=0}^{\infty} \frac{\varphi(a^i x_1, a^i x_2, \ldots, a^i x_n)}{a^{2i}} < \infty & \text{when } |a| < 1, \\ \displaystyle\sum_{i=0}^{\infty} \frac{\varphi(a^i x_1, a^i x_2, \ldots, a^i x_n)}{|a|^i} < \infty & \text{when } |a| > 1 \end{cases} \tag{3.2}$$

for all $x_1, x_2, \ldots, x_n \in V\backslash\{0\}$. *If a mapping* $f : V \to Y$ *satisfies* $f(0) = 0$,

$$\left\| f(ax) - \frac{a^2+a}{2} f(x) - \frac{a^2-a}{2} f(-x) \right\| \le \mu(x) \tag{3.3}$$

for all $x \in V\backslash\{0\}$, *and*

$$\|Df(x_1, x_2, \ldots, x_n)\| \le \varphi(x_1, x_2, \ldots, x_n) \tag{3.4}$$

for all $x_1, x_2, \ldots, x_n \in V\backslash\{0\}$, *then there exists a unique mapping* $F : V \to Y$ *such that*

$$DF(x_1, x_2, \ldots, x_n) = 0 \tag{3.5}$$

for all $x_1, x_2, \ldots, x_n \in V\backslash\{0\}$,

$$F_e(ax) = a^2 F_e(x) \quad \text{and} \quad F_o(ax) = aF_o(x) \tag{3.6}$$

for all $x \in V$, *and*

$$\|f(x) - F(x)\| \le \sum_{i=0}^{\infty} \left(\frac{\mu(a^i x) + \mu(-a^i x)}{2a^{2i+2}} + \frac{\mu(a^i x) + \mu(-a^i x)}{2|a|^{i+1}} \right) \tag{3.7}$$

for all $x \in V\backslash\{0\}$.

Proof. First, we define $A := \{f : V \to Y \mid f(0) = 0\}$ and a mapping $J_m : A \to A$ by

$$J_m f(x) := \frac{f(a^m x) + f(-a^m x)}{2a^{2m}} + \frac{f(a^m x) - f(-a^m x)}{2a^m}$$

for $x \in V$ and $m \in \mathbb{N}_0$. It follows from (3.3) that

$$\begin{aligned}
&\|J_m f(x) - J_{m+l} f(x)\| \\
&\le \sum_{i=m}^{m+l-1} \|J_i f(x) - J_{i+1} f(x)\| \\
&= \sum_{i=m}^{m+l-1} \left\| \frac{f(a^i x) + f(-a^i x)}{2a^{2i}} + \frac{f(a^i x) - f(-a^i x)}{2a^i} \right. \\
&\qquad \left. - \frac{f(a^{i+1} x) + f(-a^{i+1} x)}{2a^{2i+2}} - \frac{f(a^{i+1} x) - f(-a^{i+1} x)}{2a^{i+1}} \right\| \\
&= \sum_{i=m}^{m+l-1} \left\| -\frac{1}{2a^{i+1}} \left(f(a \cdot a^i x) - \frac{a^2+a}{2} f(a^i x) - \frac{a^2-a}{2} f(-a^i x) \right) \right. \qquad (3.8) \\
&\qquad + \frac{1}{2a^{i+1}} \left(f(-a \cdot a^i x) - \frac{a^2+a}{2} f(-a^i x) - \frac{a^2-a}{2} f(a^i x) \right) \\
&\qquad - \frac{1}{2a^{2i+2}} \left(f(a \cdot a^i x) - \frac{a^2+a}{2} f(a^i x) - \frac{a^2-a}{2} f(-a^i x) \right) \\
&\qquad \left. - \frac{1}{2a^{2i+2}} \left(f(-a \cdot a^i x) - \frac{a^2+a}{2} f(-a^i x) - \frac{a^2-a}{2} f(a^i x) \right) \right\| \\
&\le \sum_{i=m}^{m+l-1} \left(\frac{\mu(a^i x) + \mu(-a^i x)}{2a^{2i+2}} + \frac{\mu(a^i x) + \mu(-a^i x)}{2|a|^{i+1}} \right)
\end{aligned}$$

for all $x \in V\backslash\{0\}$. In view of (3.1) and (3.8), the sequence $\{J_m f(x)\}$ is a Cauchy sequence for all $x \in V\backslash\{0\}$. Since Y is complete and $f(0) = 0$, the sequence $\{J_m f(x)\}$ converges for all $x \in V$. Hence, we can define a mapping $F : V \to Y$ by

$$F(x) := \lim_{m\to\infty} J_m f(x) = \lim_{m\to\infty} \left(\frac{f(a^m x) + f(-a^m x)}{2a^{2m}} + \frac{f(a^m x) - f(-a^m x)}{2a^m} \right)$$

for all $x \in V$.

We easily obtain from the definition of F and (3.4) that

$$\begin{aligned} F_e(ax) &= \frac{F(ax) + F(-ax)}{2} \\ &= \lim_{m\to\infty} \frac{f(a^{m+1}x) + f(-a^{m+1}x)}{2a^{2m}} \\ &= a^2 \lim_{m\to\infty} \frac{f(a^{m+1}x) + f(-a^{m+1}x)}{2a^{2m+2}} \\ &= a^2 F_e(x), \\ F_o(ax) &= \frac{F(ax) - F(-ax)}{2} \\ &= \lim_{m\to\infty} \frac{f(a^{m+1}x) - f(-a^{m+1}x)}{2a^m} \\ &= a \lim_{m\to\infty} \frac{f(a^{m+1}x) - f(-a^{m+1}x)}{2a^{m+1}} \\ &= aF_o(x) \end{aligned}$$

for all $x \in V$, and by (1.1) and (3.2), we get

$$\begin{aligned} &\|DF(x_1, x_2, \dots, x_n)\| \\ &= \lim_{m\to\infty} \left\| \frac{Df\big(a^m x_1, a^m x_2, \dots, a^m x_n\big) + Df\big(-a^m x_1, -a^m x_2, \dots, -a^m x_n\big)}{2a^{2m}} \right. \\ &\qquad \left. + \frac{Df\big(a^m x_1, a^m x_2, \dots, a^m x_n\big) - Df\big(-a^m x_1, -a^m x_2, \dots, -a^m x_n\big)}{2a^m} \right\| \\ &\le \lim_{m\to\infty} \left(\frac{\varphi\big(a^m x_1, a^m x_2, \dots, a^m x_n\big) + \varphi\big(-a^m x_1, -a^m x_2, \dots, -a^m x_n\big)}{2a^{2m}} \right. \\ &\qquad \left. + \frac{\varphi\big(a^m x_1, a^m x_2, \dots, a^m x_n\big) + \varphi\big(-a^m x_1, -a^m x_2, \dots, -a^m x_n\big)}{2|a|^m} \right) \\ &= 0 \end{aligned}$$

for all $x_1, x_2, \dots, x_n \in V\backslash\{0\}$, *i.e.*, $DF(x_1, x_2, \dots, x_n) = 0$ for all $x_1, x_2, \dots, x_n \in V\backslash\{0\}$. Moreover, if we put $m = 0$ and let $l \to \infty$ in (3.8), then we obtain the inequality (3.7).

Notice that the equalities

$$F_e(|a|x) = |a|^2 F_e(x), \quad F_e\left(\frac{x}{|a|}\right) = \frac{F_e(x)}{|a|^2},$$

$$F_o(|a|x) = |a| F_o(x), \quad F_o\left(\frac{x}{|a|}\right) = \frac{F_o(x)}{|a|}$$

are true in view of (3.6).

When $|a| > 1$, in view of Lemma 2.1, there exists a unique mapping $F : V \to Y$ satisfying the equalities in (3.6) and the inequality (3.7), since the inequality

$$\begin{aligned}\|f(x) - F(x)\| &\le \sum_{i=0}^{\infty} \left(\frac{\mu(a^i x) + \mu(-a^i x)}{2a^{2i+2}} + \frac{\mu(a^i x) + \mu(-a^i x)}{2|a|^{i+1}} \right) \\ &\le \sum_{i=0}^{\infty} \frac{\phi(|a|^i x)}{|a|^i} \\ &\le \sum_{i=0}^{\infty} \frac{\phi(k^i x)}{k^i}\end{aligned}$$

holds for all $x \in V\backslash\{0\}$, where we set $k := |a|$ and $\phi(x) := \frac{\mu(x)+\mu(-x)}{2a^2} + \frac{\mu(x)+\mu(-x)}{2|a|}$.

When $|a| < 1$, in view of Lemma 2.1, there exists a unique mapping $F : V \to Y$ satisfying the equalities in (3.6) and the inequality (3.7), since the inequality

$$\begin{aligned}\|f(x) - F(x)\| &\le \sum_{i=0}^{\infty} \left(\frac{\mu(a^i x) + \mu(-a^i x)}{2a^{2i+2}} + \frac{\mu(a^i x) + \mu(-a^i x)}{2|a|^{i+1}} \right) \\ &\le \sum_{i=0}^{\infty} \frac{\phi(|a|^i x)}{|a|^{2i}} \\ &= \sum_{i=0}^{\infty} k^{2i} \phi\left(\frac{x}{k^i}\right)\end{aligned}$$

holds for all $x \in V\backslash\{0\}$, where $k := \frac{1}{|a|}$ and $\phi(x) := \frac{\mu(x)+\mu(-x)}{2a^2} + \frac{\mu(x)+\mu(-x)}{2|a|}$. □

The proof of the following theorem runs analogously to that of the previous theorem.

Theorem 3.2 *Let n be a fixed integer greater than 1, let $\mu : V\backslash\{0\} \to [0, \infty)$ be a function satisfying the condition*

$$\begin{cases} \displaystyle\sum_{i=0}^{\infty} |a|^i \mu\left(\frac{x}{a^i}\right) < \infty & \text{when } |a| < 1, \\ \displaystyle\sum_{i=0}^{\infty} a^{2i} \mu\left(\frac{x}{a^i}\right) < \infty & \text{when } |a| > 1 \end{cases} \tag{3.9}$$

for all $x \in V\backslash\{0\}$, and let $\varphi : (V\backslash\{0\})^n \to [0, \infty)$ be a function satisfying the condition

$$\begin{cases} \displaystyle\sum_{i=0}^{\infty} |a|^i \varphi\left(\frac{x_1}{a^i}, \frac{x_2}{a^i}, \ldots, \frac{x_n}{a^i}\right) < \infty & \text{when } |a| < 1, \\ \displaystyle\sum_{i=0}^{\infty} a^{2i} \varphi\left(\frac{x_1}{a^i}, \frac{x_2}{a^i}, \ldots, \frac{x_n}{a^i}\right) < \infty & \text{when } |a| > 1 \end{cases} \tag{3.10}$$

for all $x_1, x_2, \ldots, x_n \in V\backslash\{0\}$. If a mapping $f : V \to Y$ satisfies $f(0) = 0$, (3.3) for all $x \in V\backslash\{0\}$, and (3.4) for all $x_1, x_2, \ldots, x_n \in V\backslash\{0\}$, then there exists a unique mapping $F : V \to Y$ satisfying (3.5) for all $x_1, x_2, \ldots, x_n \in V\backslash\{0\}$ and the conditions in (3.6) for all $x \in V$, and such that

$$\|f(x) - F(x)\| \le \sum_{i=0}^{\infty} \frac{a^{2i} + |a|^i}{2}\left(\mu\left(\frac{x}{a^{i+1}}\right) + \mu\left(\frac{-x}{a^{i+1}}\right)\right) \tag{3.11}$$

for all $x \in V\backslash\{0\}$.

Proof. First, we define $A := \{f : V \to Y \mid f(0) = 0\}$ and a mapping $J_m : A \to A$ by

$$J_m f(x) := \frac{a^{2m}}{2}\left(f\left(\frac{x}{a^m}\right) + f\left(\frac{-x}{a^m}\right)\right) + \frac{a^m}{2}\left(f\left(\frac{x}{a^m}\right) - f\left(\frac{-x}{a^m}\right)\right)$$

for all $x \in V$ and $m \in \mathbb{N}_0$. It follows from (3.3) that

$$\begin{aligned}
&\|J_m f(x) - J_{m+l} f(x)\| \\
&\le \sum_{i=m}^{m+l-1} \|J_i f(x) - J_{i+1} f(x)\| \\
&= \sum_{i=m}^{m+l-1} \left\| \frac{a^{2i}}{2}\left(f\left(\frac{x}{a^i}\right) + f\left(\frac{-x}{a^i}\right)\right) + \frac{a^i}{2}\left(f\left(\frac{x}{a^i}\right) - f\left(\frac{-x}{a^i}\right)\right) \right. \\
&\qquad \left. - \frac{a^{2i+2}}{2}\left(f\left(\frac{x}{a^{i+1}}\right) + f\left(\frac{-x}{a^{i+1}}\right)\right) - \frac{a^{i+1}}{2}\left(f\left(\frac{x}{a^{i+1}}\right) - f\left(\frac{-x}{a^{i+1}}\right)\right)\right\| \\
&= \sum_{i=m}^{m+l-1} \left\| \frac{a^{2i}}{2}\left(f\left(a\frac{x}{a^{i+1}}\right) - \frac{a^2+a}{2} f\left(\frac{x}{a^{i+1}}\right) - \frac{a^2-a}{2} f\left(\frac{-x}{a^{i+1}}\right)\right) \right. \qquad (3.12) \\
&\qquad + \frac{a^{2i}}{2}\left(f\left(a\frac{-x}{a^{i+1}}\right) - \frac{a^2+a}{2} f\left(\frac{-x}{a^{i+1}}\right) - \frac{a^2-a}{2} f\left(\frac{x}{a^{i+1}}\right)\right) \\
&\qquad + \frac{a^{i}}{2}\left(f\left(a\frac{x}{a^{i+1}}\right) - \frac{a^2+a}{2} f\left(\frac{x}{a^{i+1}}\right) - \frac{a^2-a}{2} f\left(\frac{-x}{a^{i+1}}\right)\right. \\
&\qquad \left. - \frac{a^{i}}{2}\left(f\left(a\frac{-x}{a^{i+1}}\right) - \frac{a^2+a}{2} f\left(\frac{-x}{a^{i+1}}\right) - \frac{a^2-a}{2} f\left(\frac{x}{a^{i+1}}\right)\right)\right\| \\
&\le \sum_{i=m}^{m+l-1} \left[\frac{a^{2i}}{2}\left(\mu\left(\frac{x}{a^{i+1}}\right) + \mu\left(\frac{-x}{a^{i+1}}\right)\right) + \frac{|a|^i}{2}\left(\mu\left(\frac{x}{a^{i+1}}\right) + \mu\left(\frac{-x}{a^{i+1}}\right)\right)\right]
\end{aligned}$$

for all $x \in V\backslash\{0\}$.

On account of (3.9) and (3.12), the sequence $\{J_m f(x)\}$ is a Cauchy sequence for all $x \in V\backslash\{0\}$. Since Y is complete and $f(0) = 0$, the sequence $\{J_m f(x)\}$ converges for all $x \in V$. Hence, we can define a mapping $F : V \to Y$ by

$$F(x) := \lim_{m\to\infty}\left[\frac{a^{2m}}{2}\left(f\left(\frac{x}{a^m}\right) + f\left(\frac{-x}{a^m}\right)\right) + \frac{a^m}{2}\left(f\left(\frac{x}{a^m}\right) - f\left(\frac{-x}{a^m}\right)\right)\right]$$

for all $x \in V$. Moreover, if we put $m = 0$ and let $l \to \infty$ in (3.12), we obtain the inequality (3.11).

In view of the definition of F and (3.4), we get the equalities in (3.6) for all $x \in V$ and

$$\begin{aligned}
&\|DF(x_1, x_2, \dots, x_n)\| \\
&= \lim_{m\to\infty} \bigg\| \frac{a^{2m}}{2}\bigg(Df\Big(\frac{x_1}{a^m}, \frac{x_2}{a^m}, \dots, \frac{x_n}{a^m}\Big) + Df\Big(\frac{-x_1}{a^m}, \frac{-x_2}{a^m}, \dots, \frac{-x_n}{a^m}\Big)\bigg) \\
&\qquad + \frac{a^m}{2}\bigg(Df\Big(\frac{x_1}{a^m}, \frac{x_2}{a^m}, \dots, \frac{x_n}{a^m}\Big) - Df\Big(\frac{-x_1}{a^m}, \frac{-x_2}{a^m}, \dots, \frac{-x_n}{a^m}\Big)\bigg)\bigg\| \\
&\le \lim_{m\to\infty} \bigg[\frac{a^{2m}}{2}\bigg(\varphi\Big(\frac{x_1}{a^m}, \frac{x_2}{a^m}, \dots, \frac{x_n}{a^m}\Big) + \varphi\Big(\frac{-x_1}{a^m}, \frac{-x_2}{a^m}, \dots, \frac{-x_n}{a^m}\Big)\bigg) \\
&\qquad + \frac{|a|^m}{2}\bigg(\varphi\Big(\frac{x_1}{a^m}, \frac{x_2}{a^m}, \dots, \frac{x_n}{a^m}\Big) + \varphi\Big(\frac{-x_1}{a^m}, \frac{-x_2}{a^m}, \dots, \frac{-x_n}{a^m}\Big)\bigg)\bigg] \\
&= 0
\end{aligned}$$

for all $x_1, x_2, \dots, x_n \in V\backslash\{0\}$, *i.e.*, $DF(x_1, x_2, \dots, x_n) = 0$ for all $x_1, x_2, \dots, x_n \in V\backslash\{0\}$. We notice that the equalities

$$\begin{aligned}
F_e(|a|x) &= |a|^2 F_e(x), \quad F_e\Big(\frac{x}{|a|}\Big) = \frac{F_e(x)}{|a|^2}, \\
F_o(|a|x) &= |a| F_o(x), \quad F_o\Big(\frac{x}{|a|}\Big) = \frac{F_o(x)}{|a|}
\end{aligned}$$

hold in view of (3.6).

When $|a| > 1$, according to Lemma 2.1, there exists a unique mapping $F : V \to Y$ satisfying the equalities in (3.6) and the inequality (3.11), since the inequality

$$\begin{aligned}
\|f(x) - F(x)\| &\le \sum_{i=0}^{\infty} \bigg[\frac{a^{2i}}{2}\bigg(\mu\Big(\frac{x}{a^{i+1}}\Big) + \mu\Big(\frac{-x}{a^{i+1}}\Big)\bigg) + \frac{|a|^i}{2}\bigg(\mu\Big(\frac{x}{a^{i+1}}\Big) + \mu\Big(\frac{-x}{a^{i+1}}\Big)\bigg)\bigg] \\
&\le \sum_{i=0}^{\infty} |a|^{2i}\phi\Big(\frac{x}{|a|^i}\Big) \\
&= \sum_{i=0}^{\infty} k^{2i}\phi\Big(\frac{x}{k^i}\Big)
\end{aligned}$$

holds for all $x \in V\backslash\{0\}$, where $k := |a|$ and $\phi(x) := \mu\big(\frac{x}{a}\big) + \mu\big(\frac{-x}{a}\big)$.

When $|a| < 1$, according to Lemma 2.1, there exists a unique mapping $F : V \to Y$ satisfying the equalities in (3.6) and the inequality (3.11), since the inequality

$$\begin{aligned}
\|f(x) - F(x)\| &\le \sum_{i=0}^{\infty} \bigg[\frac{a^{2i}}{2}\bigg(\mu\Big(\frac{x}{a^{i+1}}\Big) + \mu\Big(\frac{-x}{a^{i+1}}\Big)\bigg) + \frac{|a|^i}{2}\bigg(\mu\Big(\frac{x}{a^{i+1}}\Big) + \mu\Big(\frac{-x}{a^{i+1}}\Big)\bigg)\bigg] \\
&\le \sum_{i=0}^{\infty} |a|^{i}\phi\Big(\frac{x}{|a|^i}\Big) \\
&\le \sum_{i=0}^{\infty} \frac{\phi(k^i x)}{k^i}
\end{aligned}$$

holds for all $x \in V\backslash\{0\}$, where $k := \frac{1}{|a|}$ and $\phi(x) := \mu\big(\frac{x}{a}\big) + \mu\big(\frac{-x}{a}\big)$. □

Lemma 2.2 is necessary for the proof of the following main theorem.

Theorem 3.3 *Let n be a fixed integer greater than 1, let $\mu : V\backslash\{0\} \to [0,\infty)$ be a function satisfying the condition*

$$\begin{cases} \sum_{i=0}^{\infty} \frac{\mu(a^i x)}{a^{2i}} < \infty \quad \text{and} \quad \sum_{i=0}^{\infty} |a|^i \mu\left(\frac{x}{a^i}\right) < \infty \quad \text{when } |a| > 1, \\ \sum_{i=0}^{\infty} \frac{\mu(a^i x)}{|a|^i} < \infty \quad \text{and} \quad \sum_{i=0}^{\infty} a^{2i} \mu\left(\frac{x}{a^i}\right) < \infty \quad \text{when } |a| < 1 \end{cases} \tag{3.13}$$

for all $x \in V\backslash\{0\}$, and let $\varphi : (V\backslash\{0\})^n \to [0,\infty)$ be a function satisfying the conditions

$$\begin{cases} \sum_{i=0}^{\infty} \frac{\varphi(a^i x_1, a^i x_2, \ldots, a^i x_n)}{a^{2i}} < \infty \quad \text{and} \quad \sum_{i=0}^{\infty} |a|^i \varphi\left(\frac{x_1}{a^i}, \frac{x_2}{a^i}, \ldots, \frac{x_n}{a^i}\right) < \infty \\ \qquad \text{when } |a| > 1, \\ \sum_{i=0}^{\infty} \frac{\varphi(a^i x_1, a^i x_2, \ldots, a^i x_n)}{|a|^i} < \infty \quad \text{and} \quad \sum_{i=0}^{\infty} a^{2i} \varphi\left(\frac{x_1}{a^i}, \frac{x_2}{a^i}, \ldots, \frac{x_n}{a^i}\right) < \infty \\ \qquad \text{when } |a| < 1 \end{cases} \tag{3.14}$$

for all $x_1, x_2, \ldots, x_n \in V\backslash\{0\}$. If a mapping $f : V \to Y$ satisfies $f(0) = 0$ and the inequality (3.3) for all $x \in V\backslash\{0\}$ and (3.4) for all $x_1, x_2, \ldots, x_n \in V\backslash\{0\}$, then there exists a unique mapping $F : V \to Y$ satisfying the equality (3.5) for all $x_1, x_2, \ldots, x_n \in V\backslash\{0\}$, the equalities in (3.6) for all $x \in V$, and

$$\|f(x) - F(x)\| \leq \begin{cases} \sum_{i=0}^{\infty} \left[\frac{\mu(a^i x) + \mu(-a^i x)}{2a^{2i+2}} + \frac{|a|^i}{2}\left(\mu\left(\frac{x}{a^{i+1}}\right) + \mu\left(\frac{-x}{a^{i+1}}\right)\right)\right] \\ \qquad \text{when } |a| > 1, \\ \sum_{i=0}^{\infty} \left[\frac{a^{2i}}{2}\left(\mu\left(\frac{x}{a^{i+1}}\right) + \mu\left(\frac{-x}{a^{i+1}}\right)\right) + \frac{\mu(a^i x) + \mu(-a^i x)}{2|a|^{i+1}}\right] \\ \qquad \text{when } |a| < 1 \end{cases} \tag{3.15}$$

for all $x \in V\backslash\{0\}$.

Proof. We will divide the proof of this theorem into two cases, one is for $|a| > 1$ and the other is for $|a| < 1$.

Case 1. Assume that $|a| > 1$. We define a set $A := \{f : V \to Y \mid f(0) = 0\}$ and a mapping $J_m : A \to A$ by

$$J_m f(x) := \frac{f(a^m x) + f(-a^m x)}{2a^{2m}} + \frac{a^m}{2}\left(f\left(\frac{x}{a^m}\right) - f\left(\frac{-x}{a^m}\right)\right)$$

for all $x \in V$ and $m \in \mathbb{N}_0$. It follows from (3.3) that

$$\begin{aligned}
&\|J_m f(x) - J_{m+l} f(x)\| \\
&\quad \le \sum_{i=m}^{m+l-1} \|J_i f(x) - J_{i+1} f(x)\| \\
&\quad = \sum_{i=m}^{m+l-1} \bigg\| \frac{f(a^i x) + f(-a^i x)}{2a^{2i}} + \frac{a^i}{2}\Big(f\Big(\frac{x}{a^i}\Big) - f\Big(\frac{-x}{a^i}\Big)\Big) \\
&\qquad\qquad - \frac{f(a^{i+1}x) + f(-a^{i+1}x)}{2a^{2i+2}} - \frac{a^{i+1}}{2}\Big(f\Big(\frac{x}{a^{i+1}}\Big) - f\Big(\frac{-x}{a^{i+1}}\Big)\Big)\bigg\| \\
&\quad = \sum_{i=m}^{m+l-1} \bigg\| - \frac{1}{2a^{2i+2}}\Big(f(a\cdot a^i x) - \frac{a^2+a}{2} f(a^i x) - \frac{a^2-a}{2} f(-a^i x)\Big) \qquad (3.16) \\
&\qquad\qquad - \frac{1}{2a^{2i+2}}\Big(f(-a\cdot a^i x) - \frac{a^2+a}{2} f(-a^i x) - \frac{a^2-a}{2} f(a^i x)\Big) \\
&\qquad\qquad + \frac{a^i}{2}\Big(f\Big(a\frac{x}{a^{i+1}}\Big) - \frac{a^2+a}{2} f\Big(\frac{x}{a^{i+1}}\Big) - \frac{a^2-a}{2} f\Big(\frac{-x}{a^{i+1}}\Big) \\
&\qquad\qquad - \frac{a^i}{2}\Big(f\Big(a\frac{-x}{a^{i+1}}\Big) - \frac{a^2+a}{2} f\Big(\frac{-x}{a^{i+1}}\Big) - \frac{a^2-a}{2} f\Big(\frac{x}{a^{i+1}}\Big)\Big)\bigg\| \\
&\quad \le \sum_{i=m}^{m+l-1} \bigg[\frac{\mu(a^i x) + \mu(-a^i x)}{2a^{2i+2}} + \frac{|a|^i}{2}\Big(\mu\Big(\frac{x}{a^{i+1}}\Big) + \mu\Big(\frac{-x}{a^{i+1}}\Big)\Big)\bigg]
\end{aligned}$$

for all $x \in V\backslash\{0\}$.

In view of (3.13) and (3.16), the sequence $\{J_m f(x)\}$ is a Cauchy sequence for all $x \in V\backslash\{0\}$. Since Y is complete and $f(0) = 0$, the sequence $\{J_m f(x)\}$ converges for all $x \in V$. Hence, we can define a mapping $F : V \to Y$ by

$$F(x) := \lim_{m\to\infty} \bigg[\frac{f(a^m x) + f(-a^m x)}{2a^{2m}} + \frac{a^m}{2}\Big(f\Big(\frac{x}{a^m}\Big) - f\Big(\frac{-x}{a^m}\Big)\Big)\bigg]$$

for all $x \in V$. Moreover, if we put $m = 0$ and let $l \to \infty$ in (3.16), we obtain the first inequality of (3.15).

Using the definition of F, (3.4), and (3.14), we get the equalities in (3.6) for all $x \in V$ and

$$\begin{aligned}
&\|DF(x_1, x_2, \ldots, x_n)\| \\
&\quad = \lim_{m\to\infty} \bigg\| \frac{Df\big(a^m x_1, a^m x_2, \ldots, a^m x_n\big) + Df\big(-a^m x_1, -a^m x_2, \ldots, -a^m x_n\big)}{2a^{2m}} \\
&\qquad\qquad + \frac{a^m}{2}\Big(Df\Big(\frac{x_1}{a^m}, \frac{x_2}{a^m}, \ldots, \frac{x_n}{a^m}\Big) - Df\Big(\frac{-x_1}{a^m}, \frac{-x_2}{a^m}, \ldots, \frac{-x_n}{a^m}\Big)\Big)\bigg\| \\
&\quad \le \lim_{m\to\infty} \bigg[\frac{\varphi\big(a^m x_1, a^m x_2, \ldots, a^m x_n\big) + \varphi\big(-a^m x_1, -a^m x_2, \ldots, -a^m x_n\big)}{2a^{2m}} \\
&\qquad\qquad + \frac{|a|^m}{2}\Big(\varphi\Big(\frac{x_1}{a^m}, \frac{x_2}{a^m}, \ldots, \frac{x_n}{a^m}\Big) + \varphi\Big(\frac{-x_1}{a^m}, \frac{-x_2}{a^m}, \ldots, \frac{-x_n}{a^m}\Big)\Big)\bigg] \\
&\quad = 0
\end{aligned}$$

for all $x_1, x_2, \ldots, x_n \in V\backslash\{0\}$, *i.e.*, $DF(x_1, x_2, \ldots, x_n) = 0$ for all $x_1, x_2, \ldots, x_n \in V\backslash\{0\}$. We notice that the equalities

$$F_e(|a|x) = |a|^2 F_e(x) \quad \text{and} \quad F_o(|a|x) = |a|F_o(x)$$

are true in view of (3.6).

Using Lemma 2.2, we conclude that there exists a unique mapping $F : V \to Y$ satisfying the equalities in (3.6) and the first inequality in (3.15), since the inequality

$$\begin{aligned}\|f(x) - F(x)\| &\le \sum_{i=0}^{\infty} \left[\frac{\mu(a^i x) + \mu(-a^i x)}{2a^{2i+2}} + \frac{|a|^i}{2}\left(\mu\left(\frac{x}{a^{i+1}}\right) + \mu\left(\frac{-x}{a^{i+1}}\right)\right)\right] \\ &\le \sum_{i=0}^{\infty} \left(\frac{\psi(k^i x)}{k^{2i}} + k^i \phi\left(\frac{x}{k^i}\right)\right)\end{aligned}$$

holds for all $x \in V\backslash\{0\}$, where $k := |a|$, $\phi(x) := \frac{\mu\left(\frac{x}{a}\right)+\mu\left(\frac{-x}{a}\right)}{2}$, and $\psi(x) := \frac{\mu(x)+\mu(-x)}{2a^2}$.

Case 2. We now consider the case of $|a| < 1$ and define a mapping $J_m : A \to A$ by

$$J_m f(x) := \frac{a^{2m}}{2}\left(f\left(\frac{x}{a^m}\right) + f\left(\frac{-x}{a^m}\right)\right) + \frac{f(a^m x) - f(-a^m x)}{2a^m}$$

for all $x \in V$ and $n \in \mathbb{N}_0$. It follows from (3.3) that

$$\begin{aligned}&\|J_m f(x) - J_{m+l} f(x)\| \\ &\le \sum_{i=m}^{m+l-1} \|J_i f(x) - J_{i+1} f(x)\| \\ &= \sum_{i=m}^{m+l-1} \left\| \frac{a^{2i}}{2}\left(f\left(\frac{x}{a^i}\right) + f\left(\frac{-x}{a^i}\right)\right) + \frac{f(a^i x) - f(-a^i x)}{2a^i} \right. \\ &\qquad \left. - \frac{a^{2i+2}}{2}\left(f\left(\frac{x}{a^{i+1}}\right) + f\left(\frac{-x}{a^{i+1}}\right)\right) - \frac{f(a^{i+1} x) - f(-a^{i+1} x)}{2a^{i+1}} \right\| \\ &= \sum_{i=m}^{m+l-1} \left\| \frac{a^{2i}}{2}\left(f\left(a\frac{x}{a^{i+1}}\right) - \frac{a^2+a}{2} f\left(\frac{x}{a^{i+1}}\right) - \frac{a^2-a}{2} f\left(\frac{-x}{a^{i+1}}\right)\right) \right. \\ &\qquad + \frac{a^{2i}}{2}\left(f\left(a\frac{-x}{a^{i+1}}\right) - \frac{a^2+a}{2} f\left(\frac{-x}{a^{i+1}}\right) - \frac{a^2-a}{2} f\left(\frac{x}{a^{i+1}}\right)\right) \\ &\qquad - \frac{1}{2a^{i+1}}\left(f(a \cdot a^i x) - \frac{a^2+a}{2} f(a^i x) - \frac{a^2-a}{2} f(-a^i x)\right) \\ &\qquad \left. + \frac{1}{2a^{i+1}}\left(f(-a \cdot a^i x) - \frac{a^2+a}{2} f(-a^i x) - \frac{a^2-a}{2} f(a^i x)\right) \right\| \\ &\le \sum_{i=m}^{m+l-1} \left[\frac{a^{2i}}{2}\left(\mu\left(\frac{x}{a^{i+1}}\right) + \mu\left(\frac{-x}{a^{i+1}}\right)\right) + \frac{\mu(a^i x) + \mu(-a^i x)}{2|a|^{i+1}}\right]\end{aligned} \tag{3.17}$$

for all $x \in V\backslash\{0\}$.

On account of (3.13) and (3.17), the sequence $\{J_m f(x)\}$ is a Cauchy sequence for all $x \in V\backslash\{0\}$. Since Y is complete and $f(0) = 0$, the sequence $\{J_m f(x)\}$ converges for all $x \in V$. Hence, we can define a mapping $F : V \to Y$ by

$$F(x) := \lim_{m\to\infty} \left[\frac{a^{2m}}{2}\left(f\left(\frac{x}{a^m}\right) + f\left(\frac{-x}{a^m}\right)\right) + \frac{f(a^m x) - f(-a^m x)}{2a^m}\right]$$

for all $x \in V$. Moreover, if we put $m = 0$ and let $l \to \infty$ in (3.17), we obtain the second inequality in (3.15).

By the definition of F, (3.4), and (3.14), we get the equalities in (3.6) for all $x \in V$ and

$$\begin{aligned}
&\|DF(x_1, x_2, \dots, x_n)\| \\
&= \lim_{m\to\infty} \Bigg\| \frac{a^{2m}}{2}\left(Df\left(\frac{x_1}{a^m}, \frac{x_2}{a^m}, \dots, \frac{x_n}{a^m}\right) + Df\left(\frac{-x_1}{a^m}, \frac{-x_2}{a^m}, \dots, \frac{-x_n}{a^m}\right)\right) \\
&\qquad + \frac{Df\big(a^m x_1, a^m x_2, \dots, a^m x_n\big) - Df\big(-a^m x_1, -a^m x_2, \dots, -a^m x_n\big)}{2a^m} \Bigg\| \\
&\le \lim_{m\to\infty} \Bigg[\frac{a^{2m}}{2}\left(\varphi\left(\frac{x_1}{a^m}, \frac{x_2}{a^m}, \dots, \frac{x_n}{a^m}\right) + \varphi\left(\frac{-x_1}{a^m}, \frac{-x_2}{a^m}, \dots, \frac{-x_n}{a^m}\right)\right) \\
&\qquad + \frac{\varphi\big(a^m x_1, a^m x_2, \dots, a^m x_n\big) + \varphi\big(-a^m x_1, -a^m x_2, \dots, -a^m x_n\big)}{2|a|^m} \Bigg] \\
&= 0
\end{aligned}$$

for all $x_1, x_2, \dots, x_n \in V\backslash\{0\}$, *i.e.*, $DF(x_1, x_2, \dots, x_n) = 0$ for all $x_1, x_2, \dots, x_n \in V\backslash\{0\}$. We remark that the equalities

$$F_e\left(\frac{x}{|a|}\right) = \frac{F_e(x)}{|a|^2} \quad \text{and} \quad F_o\left(\frac{x}{|a|}\right) = \frac{F_o(x)}{|a|}$$

hold by considering (3.6).

Using Lemma 2.2, we conclude that there exists a unique mapping $F : V \to Y$ satisfying the equalities in (3.6) and the second inequality in (3.15), since the inequality

$$\begin{aligned}
\|f(x) - F(x)\| &\le \sum_{i=0}^{\infty} \left[\frac{a^{2i}}{2}\left(\mu\left(\frac{x}{a^{i+1}}\right) + \mu\left(\frac{-x}{a^{i+1}}\right)\right) + \frac{\mu(a^i x) + \mu(-a^i x)}{2|a|^{i+1}}\right] \\
&= \sum_{i=0}^{\infty} \left[\frac{\mu(k^{i+1}x) + \mu(-k^{i+1}x)}{2k^{2i}} + \frac{k^{i+1}}{2}\left(\mu\left(\frac{x}{k^i}\right) + \mu\left(\frac{-x}{k^i}\right)\right)\right] \\
&\le \sum_{i=0}^{\infty} \left(\frac{\psi(k^i x)}{k^{2i}} + k^i \phi\left(\frac{x}{k^i}\right)\right)
\end{aligned}$$

holds for all $x \in V\backslash\{0\}$, where $k := \frac{1}{|a|}$, $\phi(x) := \frac{k}{2}\big(\mu(x) + \mu(-x)\big)$, and $\psi(x) := \frac{\mu(kx)+\mu(-kx)}{2}$. □

In the following corollary, we investigate the Hyers-Ulam-Rassias stability version of Theorems 3.1, 3.2, and 3.3.

Corollary 3.4 *Let X and Y be a real normed space and a real Banach space, respectively. Let p, θ, ξ be real constants such that $p \notin \{1, 2\}$, $a \notin \{-1, 0, 1\}$, $\xi > 0$, and $\theta > 0$. If a mapping $f : X \to Y$ satisfies $f(0) = 0$ and*

$$\left\| f(ax) - \frac{a^2 + a}{2} f(x) - \frac{a^2 - a}{2} f(-x) \right\| \leq \xi \|x\|^p \tag{3.18}$$

for all $x \in X \backslash \{0\}$, as well as if f satisfies the inequality

$$\|Df(x_1, x_2, \ldots, x_n)\| \leq \theta\big(\|x_1\|^p + \cdots + \|x_n\|^p\big) \tag{3.19}$$

for all $x_1, x_2, \ldots, x_n \in X\backslash\{0\}$, then there exists a unique mapping $F : X \to Y$ satisfying (3.5) for all $x_1, x_2, \ldots, x_n \in X\backslash\{0\}$, and the equalities in (3.6) for all $x \in X$, as well as

$$\|f(x) - F(x)\| \leq \frac{\xi\|x\|^p}{|a^2 - |a|^p|} + \frac{\xi\|x\|^p}{||a| - |a|^p|} \tag{3.20}$$

for all $x \in X\backslash\{0\}$.

Proof. If we put $\varphi(x_1, x_2, \ldots, x_n) := \theta\big(\|x_1\|^p + \cdots + \|x_n\|^p\big)$ for all $x_1, x_2, \ldots, x_n \in X\backslash\{0\}$, then φ satisfies (3.2) when either $|a| > 1$ and $p < 1$ or $|a| < 1$ and $p > 2$, and φ satisfies (3.10) when either $|a| > 1$ and $p > 2$ or $|a| < 1$ and $p < 1$. Moreover, φ satisfies (3.14) when $1 < p < 2$. Therefore, by Theorems 3.1, 3.2, and 3.3, there exists a unique mapping $F : X \to Y$ such that (3.5) holds for all $x_1, x_2, \ldots, x_n \in X\backslash\{0\}$, and (3.6) holds for all $x \in X$, and such that (3.20) holds for all $x \in X\backslash\{0\}$. □

4 Quadratic-additive type functional equations

In this section, let a be a rational constant such that $a \notin \{-1, 0, 1\}$. Assume that the functional equation $Df(x_1, x_2, \ldots, x_n) = 0$ is a quadratic-additive type functional equation. Then $F : V \to Y$ is a solution of the functional equation $Df(x_1, x_2, \ldots, x_n) = 0$ if and only if $F : V \to Y$ is a quadratic-additive mapping. If $F : V \to Y$ is a quadratic-additive mapping, then $F_e(x)$ and $F_o(x)$ are a quadratic mapping and an additive mapping, respectively. Hence, $F_e(ax) = a^2F_e(x)$ and $F_o(ax) = aF_o(x)$ for all $x \in V$, *i.e.*, F satisfies the conditions in (3.6).

Therefore, the following theorems are direct consequences of Theorems 3.1, 3.2, and 3.3.

Theorem 4.1 *Let n be a fixed integer greater than 1, let $\mu : V \to [0, \infty)$ be a function satisfying the condition (3.1) for all $x \in V$, and let $\varphi : V^n \to [0, \infty)$ be a function satisfying the condition (3.2) for all $x_1, x_2, \ldots, x_n \in V$. If a mapping $f : V \to Y$ satisfies $f(0) = 0$, (3.3) for all $x \in V$, and (3.4) for all $x_1, x_2, \ldots, x_n \in V$, then there exists a unique quadratic-additive mapping $F : V \to Y$ such that (3.7) holds for all $x \in V$.*

Theorem 4.2 *Let n be a fixed integer greater than 1, let $\mu : V \to [0,\infty)$ be a function satisfying the condition* (3.9) *for all $x \in V$, and let $\varphi : V^n \to [0,\infty)$ be a function satisfying the condition* (3.10) *for all $x_1, x_2, \ldots, x_n \in V$. If a mapping $f : V \to Y$ satisfies $f(0) = 0$,* (3.3) *for all $x \in V$, and* (3.4) *for all $x_1, x_2, \ldots, x_n \in V$, then there exists a unique quadratic-additive mapping $F : V \to Y$ such that* (3.11) *holds for all $x \in V$.*

Theorem 4.3 *Let n be a fixed integer greater than 1, let $\mu : V \to [0,\infty)$ be a function satisfying the condition* (3.13) *for all $x \in V$, and let $\varphi : V^n \to [0,\infty)$ be a function satisfying the condition* (3.14) *for all $x_1, x_2, \ldots, x_n \in V$. If a mapping $f : V \to Y$ satisfies $f(0) = 0$,* (3.3) *for all $x \in V$, and* (3.4) *for all $x_1, x_2, \ldots, x_n \in V$, then there exists a unique quadratic-additive mapping $F : V \to Y$ satisfying the inequality* (3.15) *for all $x \in V$.*

Corollary 4.4 *Let X and Y be a real normed space and a real Banach space, respectively. Let p, θ, ξ be real constants such that $p \notin \{1,2\}$, $a \notin \{-1,0,1\}$, $p > 0$, $\xi > 0$, and $\theta > 0$. If a mapping $f : X \to Y$ satisfies* (3.18) *for all $x \in X$ and the inequality* (3.19) *for all $x_1, x_2, \ldots, x_n \in X$, then there exists a unique quadratic-additive mapping $F : X \to Y$ such that* (3.20) *holds for all $x \in X$.*

Corollary 4.5 *Let X and Y be a real normed space and a real Banach space, respectively. Let θ and ξ be real constants such that $a \notin \{-1,0,1\}$, $\xi > 0$, and $\theta > 0$. If a mapping $f : X \to Y$ satisfies $f(0) = 0$, and*

$$\left\| f(ax) - \frac{a^2+a}{2} f(x) - \frac{a^2-a}{2} f(-x) \right\| \le \xi$$

for all $x \in X$, as well as if f satisfies the inequality

$$\|Df(x_1, x_2, \ldots, x_n)\| \le \theta$$

for all $x_1, x_2, \ldots, x_n \in X$, then there exists a unique quadratic-additive mapping $F : X \to Y$ such that

$$\|f(x) - F(x)\| \le \frac{\xi \|x\|^p}{|a^2 - 1|} + \frac{\xi \|x\|^p}{||a| - 1|}$$

for all $x \in X$.

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A Dynamic Programming Approach to Subsistence Consumption Constraints on Optimal Consumption and Portfolio

Ho-Seok Lee*   Yong Hyun Shin†

We investigate an optimal consumption and portfolio selection problem of an infinitely-lived economic agent with a constant relative risk aversion (CRRA) utility function who faces subsistence consumption constraints. We provide the closed form solutions for the optimal consumption and investment policies by using the dynamic programming method and compare the solutions with those obtained by the martingale method. We show that they coincide with each other. Comparison of optimal policies with and without subsistence consumption constraints shows that the constraints have effect on the optimal consumption and portfolio policies even when the constraints do not bind.

*Department of Mathematics, Kwangwoon University, 20, Kwangwoon-ro, Nowon-gu, Seoul 01897, Republic of Korea, e-mail: kaist.hoseoklee@gmail.com

†Department of Mathematics, Sookmyung Women's University, Seoul 04310, Republic of Korea, Corresponding author. e-mail: yhshin@sookmyung.ac.kr, Tel.: +82-2-2077-7682, FAX: +82-2-2077-7323

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1 Introduction

Following the seminal contributions of Merton [6, 7] on continuous-time optimal consumption and portfolio selection problems, there have been a number of research works on the optimization problems under various economic constraints. One of the most interesting topics is optimal consumption and portfolio selection with subsistence consumption constraints (see [1, 4, 5, 8, 10, 11, 12]). Subsistence consumption constraints mean that there exists a positive minimum consumption level (that can be a constant or a deterministic/stochastic process) such that the agent can live with.

We consider the optimal consumption and investment problem with subsistence consumption constraints and a constant relative risk aversion (CRRA) utility function. We derive the optimal solutions in closed form by using the dynamic programming approach based on Karatzas *et al.* [2]. We also compare the solutions with those of Shin *et al.* [11] by using the martingale duality approach for the same optimization problem. We show that they agree with each other.

Besides the methodological contribution through the dynamic programming method, we quantitatively compare our results to those of the agent without subsistence consumption constraints. The comparison shows that the existence of the subsistence consumption constraints affects the optimal consumption and portfolio policies even when the constraints do not bind.

The prospect that the subsistence consumption constraints become binding later compels the agent to consume less and to invest in the risky asset more conservatively.

The rest of this paper is organized as follows. The financial market is introduced in Section 2. In Section 3 the optimal consumption and investment problem is considered with subsistence consumption constraints. Section 4 demonstrates the impact of the subsistence consumption constraints on the optimal policies. Section 5 summarizes the paper.

2 The Economy

In a financial market, we assume that an economic agent has investment opportunities given by a riskless asset with a constant rate of return $r > 0$ and one risky asset S_t which follows a geometric Brownian motion with a constant mean rate of return μ and a constant volatility σ, $dS_t/S_t = \mu dt + \sigma dB_t$, where B_t is a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\{\mathcal{F}_t\}_{t\geq 0}$ is the $\mathbb{P}$-augmentation of the filtration generated by the standard Brownian motion $\{B_t\}_{t\geq 0}$.

A portfolio process $\boldsymbol{\pi} := \{\pi_t\}_{t\geq 0}$ meaning amounts of money invested in the risky asset at time t is a measurable process adapted to $\{\mathcal{F}_t\}_{t\geq 0}$ and satisfies

$$\int_0^t \pi_s^2 ds < \infty, \text{ for all } t \geq 0 \text{ a.s.} \tag{1}$$

A consumption process $\mathbf{c} := \{c_t\}_{t\geq 0}$ is a measurable nonnegative process adapted to $\{\mathcal{F}_t\}_{t\geq 0}$ and satisfies

$$\int_0^t c_s ds < \infty, \text{ for all } t \geq 0 \text{ a.s.}$$

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Then, with a given initial endowment $X_0 = x > 0$, the agent's wealth process X_t at time t evolves according to

$$dX_t = [rX_t + \pi_t(\mu - r) - c_t]\,dt + \pi_t\sigma dB_t. \tag{2}$$

3 The Optimization Problem

Now we investigate the agent's optimization problem with subsistence consumption constraints. Given a positive subsistence level of consumption $R > 0$, the agent's problem is to maximize the total expected discounted utility from consumption with the constraint

$$c_t \geq R, \text{ for all } t \geq 0. \tag{3}$$

In this paper, we assume that the utility function $u(\cdot)$ is of the CRRA type

$$u(c) := \frac{c^{1-\gamma}}{1-\gamma}, \ \gamma > 0 \ (\gamma \neq 1),$$

where γ is the agent's coefficient of relative risk aversion. A pair $(\mathbf{c}, \boldsymbol{\pi})$ of the optimal consumption/investment processes is called *admissible* at initial capital $x > 0$, if the wealth process X_t in (2) is strictly positive and it satisfies the constraint (3). Let $\mathcal{A}(x)$ denote the set of all admissible consumption/investment pair at $x > 0$.

Then, the agent's optimization problem is given by

$$V(x) := \max_{(\mathbf{c},\boldsymbol{\pi})\in\mathcal{A}(x)} J(x; \mathbf{c}, \boldsymbol{\pi}), \tag{4}$$

where

$$J(x; \mathbf{c}, \boldsymbol{\pi}) := \mathbb{E}\left[\int_0^\infty e^{-\rho t}\frac{c_t^{1-\gamma}}{1-\gamma}dt\right], \tag{5}$$

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subject to the budget constraint (2) and the subsistence consumption constraint (3). Here $\rho > 0$ is the subjective discount factor. In addition, we should impose a lower bound on initial wealth x as follows:

$$x > \frac{R}{r}$$

such that a pair $(\mathbf{c}, \boldsymbol{\pi})$ corresponding to the wealth dynamics (2) should be admissible (see Lemma 3.1 of Gong and Li [1]).

By the dynamic programming principle, the value function $V(x)$ in the optimization problem (4) satisfies the following Bellman equation

$$\max_{c \geq R, \pi} \left[\{rx + \pi(\mu - r) - c\} V'(x) + \frac{1}{2}\sigma^2\pi^2 V''(x) - \rho V(x) + \frac{c^{1-\gamma}}{1-\gamma} \right] = 0. \tag{6}$$

We assume that the wealth process X_t satisfies a transversality condition

$$\lim_{t \to \infty} e^{-\rho t} V(X_t) = 0, \tag{7}$$

if $V(\cdot)$ is the solution to the Bellman equation (6).

The first order conditions (FOCs) of the Bellman equation (6) for the optimal consumption/portfolio (c^*, π^*) imply

$$c^* = \left((V'(x)\right)^{-\frac{1}{\gamma}}$$

and

$$\pi^* = -\frac{\mu - r}{\sigma^2} \frac{V'(x)}{V''(x)}. \tag{8}$$

The subsistence consumption constraint (3) forces us to impose a threshold wealth level $\widetilde{x} > 0$ such that

$$c^* = \begin{cases} R, & \text{for} \quad R/r < x < \widetilde{x}, \\ (V'(x))^{-\frac{1}{\gamma}}, & \text{for} \quad x \geq \widetilde{x}. \end{cases} \tag{9}$$

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Substituting the FOCs (8) and (9) into the equation (6) yields

$$(rx-R)V'(x)-\frac{1}{2}\theta^2\frac{(V'(x))^2}{V''(x)}-\rho V(x)+\frac{R^{1-\gamma}}{1-\gamma}=0,\ \text{for}\ \ R/r<x<\widetilde{x} \quad (10)$$

and

$$rxV'(x)-\frac{1}{2}\theta^2\frac{(V'(x))^2}{V''(x)}-\rho V(x)+\frac{\gamma}{1-\gamma}V'(x)^{-\frac{1-\gamma}{\gamma}}=0,\ \text{for}\ \ x\geq\widetilde{x}, \quad (11)$$

where $\theta := (\mu - r)/\sigma$ is the market price of risk. Moreover, we define a Merton constant K such that

$$K := r+\frac{\rho-r}{\gamma}+\frac{\gamma-1}{2\gamma^2}\theta^2 \quad (12)$$

and assume that $K > 0$ to guarantee the well-definedness of the optimization problem (4).

Lemma 3.1. *The value function V(x) in (4) is strictly concave and strictly increasing for $x > R/r$.*

Proof. The proof follows a similar line to that of Proposition 2.1 in Zariphopoulou [14]. □

Remark 3.1. *For later use, we define two quadratic algebraic equations as follows:*

$$f(m) := rm^2-\left(\rho+r+\frac{1}{2}\theta^2\right)m+\rho=0 \quad (13)$$

and

$$g(n) := \frac{1}{2}\theta^2n^2+\left(\rho-r+\frac{1}{2}\theta^2\right)n-r=0. \quad (14)$$

$f(m) = 0$ *has two real roots* m_1 *and* m_2 *satisfying* $m_1 > 1 > m_2 > 0$ *and* $g(n) = 0$ *has two real roots* n_1 *and* n_2 *satisfying* $n_1 > 0$ *and* $n_2 < -1$. *Also* *we have the following relationships between roots of two quadratic equations (13) and (14):*

$$n_1 = \frac{1}{m_1 - 1}, \; n_2 = \frac{1}{m_2 - 1}. \tag{15}$$

Theorem 3.1. *Assume that a strictly increasing and strictly concave function $v(\cdot)$ such that $v(\cdot) \in C^2(R/r, \infty)$ solves the Bellman equation (6) for $x > R/r$. Then $v(x) \geq J(x; \mathbf{c}, \boldsymbol{\pi})$ for all admissible pair $(\mathbf{c}, \boldsymbol{\pi})$. If (c_t^*, π_t^*) is the maximizer of the Bellman equation (6), then we derive*

$$v(x) = V(x) = \max_{(\mathbf{c}, \boldsymbol{\pi}) \in \mathcal{A}(x)} J(x; \mathbf{c}, \boldsymbol{\pi}) = J(x; \mathbf{c}^*, \boldsymbol{\pi}^*).$$

Proof. Let us define a function $U(\cdot, \cdot)$ as follows:

$$U(t, X_t) := e^{-\rho t} v(X_t). \tag{16}$$

The Itô's formula implies

$$\begin{aligned} dU(t, X_t) &= e^{-\rho t} \left[\{rX_t + \pi_t(\mu - r) - c_t\} v'(X_t) + \frac{1}{2}\sigma^2 \pi_t^2 v''(X_t) - \rho v(X_t) \right] dt + e^{-\rho t} \sigma \pi_t v'(X_t) dB_t \\ &\leq -e^{-\rho t} \frac{c_t^{1-\gamma}}{1-\gamma} dt + e^{-\rho t} \sigma \pi_t v'(X_t) dB_t \end{aligned} \tag{17}$$

for any admissible pair (c_t, π_t) of consumption/portfolio processes. For any $t \geq 0$, we obtain

$$v(X_0) \geq \int_0^t e^{-\rho s} \frac{c_s^{1-\gamma}}{1-\gamma} ds + e^{-\rho t} v(X_t) - \int_0^t e^{-\rho s} \sigma \pi_s v'(X_s) dB_s. \tag{18}$$

From (1), the second integral of the right-hand side of (18) is a bounded local martingale and hence a martingale, so we have

$$v(x) \geq \mathbb{E} \left[\int_0^t e^{-\rho s} \frac{c_s^{1-\gamma}}{1-\gamma} ds + e^{-\rho t} v(X_t) \right]. \tag{19}$$

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Letting $t \uparrow \infty$ and using the monotone convergence theorem, the Lebesgue dominated convergence theorem and the transversality condition in (7), we derive

$$v(x) \geq \mathbb{E}\left[\int_0^\infty e^{-\rho s}\frac{c_s^{1-\gamma}}{1-\gamma}ds\right] = J(x;\mathbf{c},\boldsymbol{\pi}). \tag{20}$$

If (c_t, π_t) is the maximizer of the Bellman equation (6), the inequality in (20) becomes the equality and consequently we obtain $v(x) = V(x)$. □

Theorem 3.2. *The value function $V(x)$ of the optimization problem (4) is given by*

$$V(x) = \begin{cases} C_2\left(x-\frac{R}{r}\right)^{m_2} + \frac{R^{1-\gamma}}{\rho(1-\gamma)}, & \text{for } R/r < x < \widetilde{x}, \\ \frac{r-\frac{1}{2}\theta^2 n_1}{\rho} D_1 \xi^{-\gamma(n_1+1)} + \frac{\xi^{1-\gamma}}{K(1-\gamma)}, & \text{for } x \geq \widetilde{x}, \end{cases} \tag{21}$$

where

$$D_1 = \frac{\left(\frac{m_2-1}{\gamma}+1\right)\frac{1}{K}-\frac{1}{r}}{(m_2-1)n_1-1}R^{\gamma n_1+1}, \ \widetilde{x} = D_1 R^{-\gamma n_1} + \frac{R}{K} \tag{22}$$

and

$$C_2 = \frac{1}{m_2}\left(\widetilde{x}-\frac{R}{r}\right)^{1-m_2} R^{-\gamma}.$$

For $x \geq \widetilde{x}$, ξ is determined from the following algebraic equation

$$x = D_1\xi^{-\gamma n_1} + \frac{\xi}{K}.$$

Proof. For $R/r < x < \widetilde{x}$, trying a homogeneous solution of the form $\left(x-\frac{R}{r}\right)^m$ to the equation (10), then we obtain the algebraic equation $f(m) = 0$ in (13). Thus we can find the homogeneous solution $\widetilde{V}(x)$ to the equation (10) as follows:

$$\widetilde{V}(x) = C_1\left(x-\frac{R}{r}\right)^{m_1} + C_2\left(x-\frac{R}{r}\right)^{m_2},$$

for some constants C_1 and C_2. The particular solution $\frac{R^{1-\gamma}}{\rho(1-\gamma)}$ to the equation (10) can be easily derived. Thus $V(x)$ is given by

$$V(x) = \widetilde{V}(x) + \frac{R^{1-\gamma}}{\rho(1-\gamma)} = C_1\left(x - \frac{R}{r}\right)^{m_1} + C_2\left(x - \frac{R}{r}\right)^{m_2} + \frac{R^{1-\gamma}}{\rho(1-\gamma)}.$$

If $C_1 = 0$ and $C_2 > 0$, then $V(x)$ is a concave function. Thus in order to guarantee the existence of the well-defined value function $V(x)$ we set $C_1 = 0$ and we will prove that $C_2 > 0$ in Proposition 3.1 later. Therefore $V(x)$ is given by

$$V(x) = C_2\left(x - \frac{R}{r}\right)^{m_2} + \frac{R^{1-\gamma}}{\rho(1-\gamma)}. \tag{23}$$

For $x \geq \widetilde{x}$, we set the optimal consumption $c = C(x)$ and $X(\cdot) = C^{-1}(\cdot)$, that is, $X(c) = X(C(x)) = x$. Then, from the FOCs (9), we obtain

$$V'(x) = C(x)^{-\gamma},\ V''(x) = -\gamma\frac{C(x)^{-\gamma-1}}{X'(c)}. \tag{24}$$

Plugging the conditions (24) into the equation (11), we have

$$rc^{-\gamma}X(c) + \frac{1}{2\gamma}\theta^2c^{1-\gamma}X'(c) - \rho V(X(c)) + \frac{\gamma}{1-\gamma}c^{1-\gamma} = 0. \tag{25}$$

Taking the derivative of (25) with respect to c implies

$$\frac{1}{2\gamma}\theta^2c^2X''(c) + \left(r - \rho + \frac{1-\gamma}{2\gamma}\theta^2\right)cX'(c) - r\gamma X(c) + \gamma c = 0. \tag{26}$$

Trying a homogeneous solution of the form $c^{-\gamma n}$ to the equation (26), then we obtain the algebraic equation $g(n) = 0$. Thus the homogeneous solution $\widetilde{X}(c)$ is given by

$$\widetilde{X}(c) = D_1c^{-\gamma n_1} + D_2c^{-\gamma n_2},$$

for some constants D_1 and D_2. The particular solution $\frac{c}{K}$ to the equation (26) can be easily derived. Thus $X(c)$ is given by

$$X(c) = \widetilde{X}(c) + \frac{c}{K} = D_1c^{-\gamma n_1} + D_2c^{-\gamma n_2} + \frac{c}{K}.$$

Now we should discard the rapidly growing term by setting $D_2 = 0$. Therefore $X(c)$ is given by

$$X(c) = D_1 c^{-\gamma n_1} + \frac{c}{K}. \tag{27}$$

We will prove that $X'(c) > 0$ in Proposition 3.1 later. Thus, from (24), we obtain

$$V''(x) = -\gamma \frac{C(x)^{-\gamma-1}}{X'(c)} < 0$$

and hence $V(x)$ is a concave function for $x \geq \widetilde{x}$. From (25), we have

$$V(x) = V(X(\xi)) = \frac{r - \frac{1}{2}\theta^2 n_1}{\rho} D_1 \xi^{-\gamma(n_1+1)} + \frac{\xi^{1-\gamma}}{K(1-\gamma)},$$

where ξ is determined from the algebraic equation

$$x = D_1 \xi^{-\gamma n_1} + \frac{\xi}{K}. \tag{28}$$

From (27), we see that

$$\widetilde{x} = X(R) = D_1 R^{-\gamma n_1} + \frac{R}{K} \tag{29}$$

and

$$X'(R) = -\gamma n_1 D_1 R^{-\gamma n_1 - 1} + \frac{1}{K}. \tag{30}$$

From (23) and (24), we use C^1 and C^2 conditions at $x = \widetilde{x}$ to obtain

$$V'(\widetilde{x}) = m_2 C_2 \left(\widetilde{x} - \frac{R}{r}\right)^{m_2-1} = R^{-\gamma} \tag{31}$$

and

$$V''(\widetilde{x}) = m_2(m_2-1) C_2 \left(\widetilde{x} - \frac{R}{r}\right)^{m_2-2} = -\gamma \frac{R^{-\gamma-1}}{X'(R)}. \tag{32}$$

From (30), (31) and (32) we have

$$\widetilde{x} = -\frac{m_2-1}{\gamma} R X'(R) + \frac{R}{r} = (m_2-1) n_1 D_1 R^{-\gamma n_1} - \frac{m_2-1}{\gamma}\frac{R}{K} + \frac{R}{r}. \tag{33}$$

From (29) and (33), we derive

$$D_1 = \frac{\left(\frac{m_2-1}{\gamma}+1\right)\frac{1}{K}-\frac{1}{r}}{(m_2-1)n_1-1}R^{\gamma n_1+1} \tag{34}$$

and

$$C_2 = \frac{1}{m_2}\left(\widetilde{x}-\frac{R}{r}\right)^{1-m_2}R^{-\gamma}. \tag{35}$$

□

Proposition 3.1. *$\widetilde{x}$ is an increasing function with respect to R, $X'(c) > 0$ and $\widetilde{x} > R/r$. Also $C_2 > 0$ as promised before.*

Proof. From (29) and (34) we have

$$\begin{aligned}\widetilde{x} &= \left[\frac{\left(\frac{m_2-1}{\gamma}+1\right)\frac{1}{K}-\frac{1}{r}}{(m_2-1)n_1-1}+\frac{1}{K}\right]R \\ &= \frac{(m_2-1)\left(\frac{1}{\gamma}+n_1\right)\frac{1}{K}-\frac{1}{r}}{(m_2-1)n_1-1}R.\end{aligned}$$

Thus $\widetilde{x}$ is a linear function of R and is an increasing function with respect to R since

$$\frac{(m_2-1)\left(\frac{1}{\gamma}+n_1\right)\frac{1}{K}-\frac{1}{r}}{(m_2-1)n_1-1} > 0,$$

because of $m_2 - 1 < 0$.

Now we use the Merton constant K in (12) and the quadratic equation (14) to obtain the inequality

$$\begin{aligned}\frac{\gamma n_1}{r}-\frac{\gamma n_1}{K}-\frac{1}{K} &= \frac{\gamma n_1 K - r\gamma n_1 - r}{rK} = \frac{n_1(\rho-r)+n_1\frac{\gamma-1}{2\gamma}\theta^2-r}{rK} \\ &= \frac{\left(\rho-r+\frac{1}{2}\theta^2\right)n_1-\frac{n_1}{2\gamma}\theta^2-r}{rK} = \frac{-\frac{1}{2}\theta^2 n_1^2-\frac{n_1}{2\gamma}\theta^2}{rK} < 0.\end{aligned}$$

Thus we have

$$X'(R) = -\gamma n_1 D_1 R^{-\gamma n_1 - 1} + \frac{1}{K} = -\gamma n_1 \frac{\left(\frac{m_2-1}{\gamma}+1\right)\frac{1}{K} - \frac{1}{r}}{(m_2-1)n_1 - 1} + \frac{1}{K}$$
$$= \frac{\frac{\gamma n_1}{r} - \frac{\gamma n_1}{K} - \frac{1}{K}}{(m_2-1)n_1 - 1} > 0. \quad (36)$$

From the fact $c > R$, we have

$$1 > \left(\frac{R}{c}\right)^{\gamma n_1+1} \quad \text{and} \quad \frac{1}{K} > \frac{1}{K}\left(\frac{R}{c}\right)^{\gamma n_1+1}. \quad (37)$$

Thus we have

$$\begin{aligned} X'(c) &= -\gamma n_1 \frac{\left(\frac{m_2-1}{\gamma}+1\right)\frac{1}{K} - \frac{1}{r}}{(m_2-1)n_1 - 1}\left(\frac{R}{c}\right)^{\gamma n_1+1} + \frac{1}{K} \\ &> -\gamma n_1 \frac{\left(\frac{m_2-1}{\gamma}+1\right)\frac{1}{K} - \frac{1}{r}}{(m_2-1)n_1 - 1}\left(\frac{R}{c}\right)^{\gamma n_1+1} + \frac{1}{K}\left(\frac{R}{c}\right)^{\gamma n_1+1} \\ &= \left(\frac{R}{c}\right)^{\gamma n_1+1} X'(R) \\ &> 0, \end{aligned}$$

where the first inequality is obtained from (37) and the second inequality is obtained from (36). Consequently, from (33), we see that $\widetilde{x} > R/r$ and $C_2 > 0$ from (35). □

Remark 3.2. *For $R/r < x < \widetilde{x}$, $V''(x)$ has a lower bound. From Proposition 3.1 and (24), $V''(x)$ has a lower bound for $\tilde{x} \le x$. From Lemma 3.1, $V'(x)$ is bounded away from zero. Hence, π^* in (8) is bounded away from zero and the Bellman equation (6) is uniformly elliptic. Therefore the solution in Theorem 3.2 is the unique solution to the Bellman equation (6) by Krylov [3]. Vila and Zariphopoulou [13] provided an alternative proof by a similar argument.*

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Now we will describe the related results of Shin *et al.* [11] in the following remark. They also pay their attention to the optimal consumption and portfolio selection problem with a subsistence consumption constraint, but they use the martingale method with Lagrangian duality to derive their solutions.

Remark 3.3. *With the notations in this paper, the value function* $V^S(x)$ *and the threshold wealth level* $\widetilde{x}^S$ *based on Section 4 of Shin et al. [11] are given as follows:*

$$V^S(x) = \begin{cases} d_2\left(\frac{\frac{R}{r}-x}{d_2p_2}\right)^{\frac{p_2}{p_2-1}} + \left(x-\frac{R}{r}\right)\left(\frac{\frac{R}{r}-x}{d_2p_2}\right)^{\frac{1}{p_2-1}} + \frac{R^{1-\gamma}}{\rho(1-\gamma)}, & \text{for } R/r < x < \widetilde{x}^S, \\ c_1\left(\lambda^*\right)^{p_1} + \frac{\gamma}{K(1-\gamma)}\left(\lambda^*\right)^{-\frac{1-\gamma}{\gamma}} + \left(\lambda^*\right)x, & \text{for } x \geq \widetilde{x}^S \end{cases} \tag{38}$$

and

$$\widetilde{x}^S = -c_1p_1R^{-\gamma(p_1-1)} + \frac{R}{K},$$

where

$$c_1 = \frac{\frac{1}{K}\left(\frac{\gamma p_2}{1-\gamma}+1\right) + \frac{p_2-1}{r} - \frac{p_2}{\rho(1-\gamma)}}{p_1-p_2}R^{1-\gamma+\gamma p_1} \tag{39}$$

and

$$d_2 = \frac{\frac{1}{K}\left(\frac{\gamma p_1}{1-\gamma}+1\right) + \frac{p_1-1}{r} - \frac{p_1}{\rho(1-\gamma)}}{p_1-p_2}R^{1-\gamma+\gamma p_2}. \tag{40}$$

$p_1 > 1$ *and* $p_2 < 0$ *are two real roots of the following quadratic algebraic equation*

$$h(p) := \frac{1}{2}\theta^2p^2 + \left(\rho - r - \frac{1}{2}\theta^2\right)p - \rho = 0, \tag{41}$$

and λ^* *is determined by the following algebraic equation*

$$x = -c_1p_1\left(\lambda^*\right)^{p_1-1} + \frac{1}{K}\left(\lambda^*\right)^{-\frac{1}{\gamma}}. \tag{42}$$

Lemma 3.2.

$$m_2 C_2 = (-d_2 p_2)^{\frac{1}{1-p_2}} \quad and \quad D_1 = -c_1 p_1. \tag{43}$$

Proof. From (29), (34) and (15), we have

$$\begin{aligned}\widetilde{x} &= D_1 R^{-\gamma n_1} + \frac{R}{K} = \frac{\left(\frac{m_2-1}{\gamma}+1\right)\frac{1}{K} - \frac{1}{r}}{(m_2-1)n_1 - 1} R + \frac{R}{K} \\ &= \frac{\left(\frac{1}{\gamma}+n_2\right)\frac{1}{K} - \frac{n_2}{r}}{n_1 - n_2} R + \frac{R}{K}.\end{aligned}$$

It can be easily shown that

$$p_1 = n_1 + 1, \quad p_2 = n_2 + 1. \tag{44}$$

Thus we obtain

$$\widetilde{x} = \frac{\left(\frac{1}{\gamma}+p_2-1\right)\frac{1}{K} - \frac{p_2-1}{r}}{p_1 - p_2} R + \frac{R}{K}.$$

From (35), we have

$$\begin{aligned}m_2 C_2 &= \left(\widetilde{x} - \frac{R}{r}\right)^{1-m_2} R^{-\gamma} \\ &= \left(\widetilde{x} R^{\gamma(p_2-1)} - \frac{R^{1+\gamma(p_2-1)}}{r}\right)^{1-m_2} \\ &= \left\{\left(\frac{\left(\frac{1}{\gamma}+p_1-1\right)\frac{1}{K} + \frac{1-p_1}{r}}{p_1-p_2}\right) R^{1+\gamma(p_2-1)}\right\}^{1-m_2} \\ &= (-d_2 p_2)^{1-m_2},\end{aligned}$$

where the last equality is obtained from the following relationships between roots and coefficients of the quadratic equation $h(p) = 0$ in (41)

$$p_1 + p_2 = \frac{\theta^2 - 2\rho + 2r}{\theta^2}, \quad p_1 p_2 = -\frac{2\rho}{\theta^2} \tag{45}$$

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and (40). Therefore we obtain

$$m_2C_2 = (-d_2p_2)^{\frac{1}{1-p_2}}.$$

From (34), we have

$$\begin{aligned} D_1 &= \frac{\left(\frac{m_2-1}{\gamma}+1\right)\frac{1}{K}-\frac{1}{r}}{(m_2-1)n_1-1}R^{\gamma n_1+1} \\ &= \frac{\left(\frac{1}{\gamma}+p_2-1\right)\frac{1}{K}-\frac{p_2-1}{r}}{p_1-p_2}R^{1+\gamma(p_1-1)} \qquad (46)\\ &= -c_1p_1, \end{aligned}$$

where the last equality is also obtained from the relationships (45) and (39). □

Corollary 3.1. *D_1 in (22) is positive.*

Proof. For $p_2 < x < p_1$, we define a decreasing function $F(x)$ as follows:

$$F(x) := -\frac{h(x)}{x-p_2} = -\frac{1}{2}\theta^2(x-p_1) > 0.$$

Since $0 < F(1) < F\left(\frac{\gamma-1}{\gamma}\right)$, we have $\frac{1}{F\left(\frac{\gamma-1}{\gamma}\right)} < \frac{1}{F(1)}$ and

$$\left(\frac{1}{\gamma}+p_2-1\right)\frac{1}{K}-\frac{p_2-1}{r} > 0$$

(see also Shim and Shin [9]). From (46), we have $D_1 > 0$. □

Proposition 3.2. *The value function $V(x)$ and the threshold wealth level $\widetilde{x}$ in our optimization problem coincide with $V^S(x)$ and $\widetilde{x}^S$ of Shin et al. [11], respectively.*

Proof. From (43) and (44), we can easily show that $\widetilde{x} = \widetilde{x}^{\mathrm{S}}$.

For $R/r < x < \widetilde{x}$, we have

$$\begin{aligned}
d_2 \left(\frac{\frac{R}{r} - x}{d_2 p_2} \right)^{\frac{p_2}{p_2 - 1}} + \left(x - \frac{R}{r} \right) \left(\frac{\frac{R}{r} - x}{d_2 p_2} \right)^{\frac{1}{p_2 - 1}} &= \frac{p_2 - 1}{p_2} (-d_2 p_2)^{\frac{1}{1 - p_2}} \left(x - \frac{R}{r} \right)^{\frac{p_2}{p_2 - 1}} \\
&= \frac{p_2 - 1}{p_2} m_2 C_2 \left(x - \frac{R}{r} \right)^{\frac{p_2}{p_2 - 1}} \\
&= C_2 \left(x - \frac{R}{r} \right)^{m_2},
\end{aligned}$$

where the second equality is obtained from (43) and the third equality is obtained from (15) and (44). This equality means $V(x) = V^{\mathrm{S}}(x)$ for $R/r < x < \widetilde{x}$.

For $x \geq \widetilde{x}$, if we set $\xi = (\lambda^*)^{-1/\gamma}$, then the algebraic equation (28) coincides with the algebraic equation (42). From (38) and (42), we obtain

$$\begin{aligned}
V^{\mathrm{S}}(x) &= c_1 (\lambda^*)^{p_1} + \frac{\gamma}{K(1-\gamma)} (\lambda^*)^{-\frac{1-\gamma}{\gamma}} + (\lambda^*) x \\
&= \frac{p_1 - 1}{p_1} (-c_1 p_1) (\lambda^*)^{p_1} + \frac{(\lambda^*)^{\frac{\gamma - 1}{\gamma}}}{K(1-\gamma)} \\
&= \frac{n_1}{n_1 + 1} D_1 \xi^{-\gamma(n_1 + 1)} + \frac{\xi^{1-\gamma}}{K(1-\gamma)} \\
&= \frac{r - \frac{1}{2}\theta^2 n_1}{\rho} D_1 \xi^{-\gamma(n_1 + 1)} + \frac{\xi^{1-\gamma}}{K(1-\gamma)} \\
&= V(x),
\end{aligned}$$

where the third equality is obtained from (43) and (44) and the fourth equality is obtained from (14). $\square$

Finally we use the FOCs (8), (9) and (24) with the derived value function $V(x)$ in (21) to obtain the optimal consumption and investment strategies of this optimization problem.

Theorem 3.3. *The optimal consumption and portfolio pair* (c^*, π^*) *is given by*

$$c_t^* = \begin{cases} R, & \text{for } R/r < X_t < \widetilde{x} \\ \xi_t, & \text{for } X_t \geq \widetilde{x} \end{cases}$$

and

$$\pi_t^* = \begin{cases} \frac{\theta}{\sigma}\frac{1}{1-m_2}\left(X_t - \frac{R}{r}\right), & \text{for } R/r < X_t < \widetilde{x} \\ \frac{\theta}{\sigma\gamma}\left(-\gamma n_1 D_1 \xi_t^{-\gamma n_1} + \frac{\xi_t}{K}\right), & \text{for } X_t \geq \widetilde{x}, \end{cases}$$

where ξ_t *is determined by the following algebraic equation*

$$X_t = D_1 \xi_t^{-\gamma n_1} + \frac{\xi_t}{K}. \tag{47}$$

Proof. The proof directly follows from the FOCs (8) and (9). □

Remark 3.4. *It is easily seen that the optimal consumption and portfolio pair* (c^*, π^*) *in our optimization problem coincides with that of Shin et al. [11].*

4 Implications

In this section, we compare the agent's optimal consumption and portfolio policies with subsistence consumption constraints to those without subsistence consumption constraints. Without subsistence consumption constraints, the optimal consumption and portfolio policies are those of the well-known Merton's problems. Let us denote by $(\mathbf{c}^M, \boldsymbol{\pi}^M)$ the optimal consumption and portfolio pair without subsistence consumption constraints.

Then

$$c_t^M = KX_t, \tag{48}$$

$$\pi_t^M = \frac{\theta}{\sigma\gamma} X_t, \tag{49}$$

for $X_t > 0$. If we let $R \to 0$ to the consumption and portfolio pair (c^*, π^*) in Theorem 3.3, we also arrive at (c_t^M, π_t^M). Due to the subsistence consumption constraints, it is natural to consider the myopic strategies defined by

$$c_t^{myopic} := \max\{R, c_t^M\}.$$

But the myopic strategies are not optimal and the existence of the subsistence consumption constraints affect the consumption and portfolio policies even at the wealth level where the subsistence consumption constraints do not bind. This is because it is possible that the constraints will become binding later. The following proposition demonstrates quantitatively the impact of the subsistence consumption constraints on the consumption and portfolio policies when the constraints are not binding.

Proposition 4.1. *For $X_t \geq \tilde{x}$, $c_t^* < c_t^M$ and $\pi_t^* < \pi_t^M$.*

Proof. From (47) and (48), the optimal wealth process is given by

$$X_t = D_1 c_t^{*-\gamma n_1} + \frac{c_t^*}{K} = \frac{c_t^M}{K}.$$

Since $D_1 > 0$ and $X(c) := D_1 c^{-\gamma n_1} + \dfrac{c}{K}$ is an increasing function from Proposition 3.1, we obtain

$$c_t^* < c_t^M = KX_t. \tag{50}$$

Also we derive

$$\pi_t^* = \frac{\theta}{\sigma\gamma}\left(-\gamma n_1 D_1 c_t^{*-\gamma n_1} + \frac{c_t^*}{K}\right) < \frac{\theta}{\sigma\gamma}\frac{c_t^*}{K} < \frac{\theta}{\sigma\gamma}X_t = \pi_t^M,$$

where the first inequality follows from $D_1 > 0$ and the second one from (50). □

5 Concluding Remarks

In this paper we study an optimal consumption and investment problem with subsistence consumption constraints. We use the dynamic programming method to derive the closed form solutions with a CRRA utility function. We also compare our solutions with those of Shin *et al.* [11] derived by the martingale approach. We show that they coincide with each other. In addition, we point out that the optimal consumption and portfolio policies may alter even when the constraints do not bind. This is attributed to the prospect that the subsistence consumption constraints become binding later. In this case, the agent consume less and invest in the risky asset more conservatively.

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THE STABILITY OF CUBIC FUNCTIONAL EQUATION WITH INVOLUTION IN NON-ARCHIMEDEAN SPACES

CHANG IL KIM AND CHANG HYEOB SHIN*

Abstract. In this paper, using fixed point method, we prove the Hyers-Ulam stability of the following functional equation

$$f(2x+y)+f(2x+\sigma(y))-2f(x+y)-2f(x+\sigma(y))-12f(x)=0$$

with involution.

1. Introduction and Preliminaries

In 1940, Ulam [18] proposed the following problem concerning the stability of group homomorphism: *Let G_1 be a group and let G_2 a meric group with the metric $d(\cdot,\cdot)$. Given $\epsilon>0$, does there exist a $\delta>0$ such that if a mapping $h:G_1\longrightarrow G_2$ satisfies the inequality $d(h(xy),h(x)h(y))<\delta$ for all $x,y\in G_1$, then there exists a homomorphism $H:G_1\longrightarrow G_2$ with $d(h(x),H(x))<\epsilon$ for all $x\in G_1$?*

Hyers [7] solved the Ulam's problem for the case of approximately additive functions in Banach spaces. Since then, the stability of several functional equations have been extensively investigated by several mathematicians [2, 3, 5, 8, 9, 13, 14, 15, 16]. Jun and Kim [11] introduced the following functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x) \tag{1.1}$$

and they established the general solution and generalized Hyers-Ulam-Rassias stability problem for this functional equation. It is easy to see that the function $f(x)=cx^3$ is a solution of the functional equation (1.1). Thus, it is natural that (1.1) is called *a cubic functional equation* and every solution of the cubic functional equation is said to be *a cubic function*.

Let X and Y be real vector spaces. For an additive mapping $\sigma:X\longrightarrow X$ with $\sigma(\sigma(x))=x$ for all $x\in X$, then σ is called *an involution* of X [1, 17]. Stetkær [17] introduced the following quadratic functional equation with involution

$$f(x+y)+f(x+\sigma(y))=2f(x)+2f(\sigma(y)) \tag{1.2}$$

and solved the general solution, Belaid et al. [1] established generalized Hyers-Ulam stability in Banach space for this functional equation. Jung and Lee [12] investigated the Hyers-Ulam-Rassias stability of (1.2) in a complete β-normed space, using fixed point method.

For a given involution $\sigma:X\longrightarrow X$, the functional equation

$$f(2x+y)+f(2x+\sigma(y))=2f(x+y)+2f(x+\sigma(y))+12f(x) \tag{1.3}$$

for all $x,y\in X$ is called *the cubic functional equation with involution* and a solution of (1.3) is called *a cubic mapping with involution*.

In this paper, using fixed point method, we prove the generalized Hyers-Ulam stability of the following functional equation

$$f(2x+y)+f(2x+\sigma(y))-2f(x+y)-2f(x+\sigma(y))-12f(x)=0. \tag{1.4}$$

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*Corresponding Author.

1

A *valuation* is a function $|\cdot|$ from a field K into $[0, \infty)$ such that for any $r, s \in K$, the following conditions hold: (i) $|r| = 0$ if and only if $r = 0$, (ii) $|rs| = |r||s|$, and (iii) $|r + s| \leq |r| + |s|$.

A field K is called *a valued field* if K carries a valuation. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuations. If the triangle inequality is replaced by $|r + s| \leq max\{|r|, |s|\}$ for all $r, s \in K$, then the valuation $|\cdot|$ is called a *non-Archimedean valuation* and the field with a non-Archimedean valuation is called *non-Archimedean field.* If $|\cdot|$ is a non-Archimedean valuation on K, then clearly, $|1| = |-1|$ and $|n| \leq 1$ for all $n \in \mathbb{N}$.

Definition 1.1. Let X be a vector space over a scalar field K with a non-Archimedean nontrivial valuation $|\cdot|$. A function $\|\cdot\| : X \longrightarrow \mathbb{R}$ is called *a non-Archimedean norm* if satisfies the following conditions:
(a) $\|x\| = 0$ if and only if $x = 0$,
(b) $\|rx\| = |r|\|x\|$, and
(c) the strong triangle inequality (ultrametric) holds, that is,

$$||x + y|| \leq max\{\|x\|, \|y\|\}$$

for all $x, y \in X$ and all $r \in K$.

If $\|\cdot\|$ is a non-Archimedean norm, then $(X, \|\cdot\|)$ is called *a non-Archimedean normed space.* Let $(X, \|\cdot\|)$ be a non-Archimedean normed space. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be *convergent* if there exists $x \in X$ such that $\lim_{n\longrightarrow\infty} \|x_n - x\| = 0$. In that case, x is called *the limit of the sequence* $\{x_n\}$, and one denotes it by $\lim_{n\longrightarrow\infty} x_n = x$. A sequence $\{x_n\}$ is said to be *a Cauchy sequence* if $\lim_{n\longrightarrow\infty} \|x_{n+p} - x_n\| = 0$ for all $p \in \mathbb{N}$. Since

$$\|x_n - x_m\| \leq max\{\|x_{j+1} - x_j\| \mid m \leq j \leq n - 1\} \quad (n > m),$$

a sequence $\{x_n\}$ is Cauchy in $(X, \|\cdot\|)$ if and only if $\{x_{n+1} - x_n\}$ converges to zero in $(X, \|\cdot\|)$. By *a complete non-Archimedean space* we mean one in which every Cauchy sequence is convergent.

In 1897, Hensel [6] discovered the p-adic numbers as a number theoretical analogue of power series in complex analysis. Fix a prime number p. For any nonzero rational number x, there exists a unique integer $n_x \in \mathbb{Z}$ such that $x = \frac{a}{b}p^{n_x}$, where a and b are integers not divisible by p. Then $|x|_p := p^{-n_x}$ defines a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric $d(x, y) = |x - y|_p$ is denoted by $\mathbb{Q}_p$, which is called the *p-adic number field.* In fact, $\mathbb{Q}_p$ is the set of all formal series $x = \sum_{k\geq n_x}^{\infty} a_k p^k$, where $|a_k| \leq p - 1$ are integers. The addition and multiplication between any two elements of $\mathbb{Q}_p$ are defined naturally. The norm $\left|\sum_{k\geq n_x}^{\infty} a_k p^k\right|_p = p^{-n_x}$ is a non-Archimedean norm on $\mathbb{Q}_p$ and it makes $\mathbb{Q}_p$ a locally compact field.

Let (X, d) be a generalized metric space. An operator $T : X \longrightarrow X$ satisfies a Lipschitz condition with Lipschitz constant L if there exists a constant $L \geq 0$ such that $d(Tx, Ty) \leq Ld(x, y)$ for all $x, y \in X$. If the Lipschitz constant L is less than 1, then the operator T is called a *strictly contractive operator.* Note that the distinction between the generalized metric and the usual metric is that the range of the former is permitted to include the infinity.

Theorem 1.2. [4] *Let (X, d) be a complete generalized metric space and let $J : X \longrightarrow X$ be a strictly contractive mapping with some Lipschitz constant L with $0 < L < 1$. Then for each given element $x \in X$, either $d(J^n x, J^{n+1}x) = \infty$ for all nonnegative integers n or there exists a positive integer n_0 such that*
(1) $d(J^n x, J^{n+1}x) < \infty$ *for all* $n \geq n_0$;
(2) *the sequence* $\{J^n x\}$ *converges to a fixed point* x^* *of* J ;
(3) x^* *is the unique fixed point of* J *in the set* $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$ and
(4) $d(y, y^*) \leq \dfrac{1}{1 - L}\, d(y, Jy)$ *for all* $y \in Y$.

In 1996, Issac and Rassias [10] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorem with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors.

Throughout this paper, we assume that X is a non-Archimedean normed space and Y is a complete non-Archimedean normed space.

2. The generalized Hyers-Ulam stability for (1.4)

Using the fixed point methods, we will prove the generalized Hyers-Ulam stability of the cubic functional equation (1.4) with involution σ in non-Archimedean normed spaces. For a given mapping $f : X \longrightarrow Y$, we define the difference operator $Df : X^2 \longrightarrow Y$ by

$$Df(x,y) = f(2x+y) + f(2x+\sigma(y)) - 2f(x+y) - 2f(x+\sigma(y)) - 12f(x)$$

for all $x, y \in X$.

Theorem 2.1. *Assume that $\phi : X^2 \longrightarrow [0,\infty)$ is a mapping and there exists a real number L with $0 < L < 1$ such that*

$$\phi(2x,2y) \le |8|L\phi(x,y), \ \phi(x+\sigma(x), y+\sigma(y)) \le |8|L\phi(x,y) \tag{2.1}$$

for all $x, y \in X$. Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and

$$\|Df(x,y)\| \le \phi(x,y) \tag{2.2}$$

for all $x, y \in X$. Then there exists a unique cubic mapping $C : X \longrightarrow Y$ with involution such that

$$\|f(x) - C(x)\| \le \frac{1}{|2|^4(1-L)}\Phi(x) \tag{2.3}$$

for all $x \in X$, where $\Phi(x) = max\{\phi(x,0), \phi(0,x)\}$.

Proof. Consider the set $S = \{g \mid g : X \longrightarrow Y\}$ and the generalized metric d in S defined by $d(g,h) = inf\{c \in [0,\infty) |\ \|g(x) - h(x)\| \le c\ \Phi(x)$ for all $x \in X\}$. Then (S,d) is a complete metric space(See [12]). Define a mapping $J : S \longrightarrow S$ by

$$Jg(x) = \frac{1}{8}\{g(2x) + g(x+\sigma(x))\}$$

for all $x \in X$ and all $g \in S$. Let $g, h \in S$ and $d(g,h) \le c$ for some non-negative real number c. Then by (2.1), we have

$$\begin{aligned} \|Jg(x) - Jh(x)\| &= \frac{1}{|8|}\ \|g(2x) + g(x+\sigma(x)) - h(2x) - h(x+\sigma(x))\| \\ &\le \frac{1}{|8|} max\{\|g(2x) - h(2x)\|, \|g(x+\sigma(x)) - h(x+\sigma(x))\|\} \\ &\le cL\Phi(x) \end{aligned}$$

for all $x \in X$. Hence we have $d(Jg, Jh) \le Ld(g,h)$ for any $g, h \in S$ and so J is a strictly contractive mapping.

Next, we claim that $d(Jf, f) < \infty$. Putting $y = 0$ in (2.2), we get

$$\|f(2x) - 8f(x)\| \le \frac{1}{|2|}\phi(x,0) \tag{2.4}$$

for all $x \in X$ and putting $x = 0$ in (2.2), we get

$$\|f(y) + f(\sigma(y))\| \le \phi(0,y) \tag{2.5}$$

for all $y \in X$ and putting $y = x + \sigma(x)$ in (2.5), we get

$$\|f(x+\sigma(x))\| \le \frac{1}{|2|}\phi(0, x+\sigma(x)) \tag{2.6}$$

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for all $x \in X$. By (2.4) and (2.6), we have

$$\begin{aligned}\|Jf(x) - f(x)\| &= \frac{1}{|8|}\Big\|f(2x) - 8f(x) + f(x+\sigma(x))\Big\| \\ &\leq \frac{1}{|8|}max\Big\{\|f(2x) - 8f(x)\|, \|f(x+\sigma(x))\|\Big\} \\ &\leq \frac{1}{|2|^4}\Phi(x)\end{aligned}$$

for all $x \in X$. Hence

$$d(Jf, f) \leq \frac{1}{|2|^4} < \infty. \tag{2.7}$$

By Theorem 1.2, there exists a mapping $C : X \longrightarrow Y$ which is a fixed point of J such that $d(J^n f, C) \to 0$ as $n \to \infty$. By induction, we can easily show that

$$(J^n f)(x) = \frac{1}{2^{3n}}\Big\{f(2^n x) + (2^n - 1)f\Big(2^{n-1}\big(x + \sigma(x)\big)\Big)\Big\}$$

for all $x \in X$ and $n \in \mathbb{N}$. Since $d(J^n f, C) \to 0$ as $n \to \infty$, there exists a sequence $\{c_n\}$ in $\mathbb{R}$ such that $c_n \to 0$ as $n \to \infty$ and $d(J^n f, C) \leq c_n$ for every $n \in \mathbb{N}$. Hence, it follows from the definition of d that

$$\|(J^n f)(x) - C(x)\| \leq c_n \Phi(x)$$

for all $x \in X$. Thus for each fixed $x \in X$, we have

$$\lim_{n\longrightarrow\infty} ||(J^n f)(x) - C(x)|| = 0$$

and so

$$C(x) = \lim_{n\longrightarrow\infty} \frac{1}{2^{3n}}\Big\{f(2^n x) + (2^n - 1)f\Big(2^{n-1}\big(x + \sigma(x)\big)\Big)\Big\}. \tag{2.8}$$

It follows from (2.2) and (2.8) that

$$\begin{aligned}&\|C(2x+y) + C(2x+\sigma(y)) - 2C(x+y) - 2C(x+\sigma(y)) - 12C(x)\| \\ &\leq \lim_{n\longrightarrow\infty} \frac{1}{|8|^n}\ max\{\phi(2^n x, 2^n y), |2^n - 1|\phi(2^{n-1}(x+\sigma(x)), 2^{n-1}(y+\sigma(y)))\} \\ &\leq \lim_{n\longrightarrow\infty} L^n\ max\{\phi(x,y), |2^n - 1|\phi(x,y)\} = \lim_{n\longrightarrow\infty} L^n\phi(x,y) = 0\end{aligned}$$

for all $x, y \in X$, because $|2^n - 1| \leq 1$ for all $n \in \mathbb{N}$. Hence C satisfies (1.4), C is a cubic mapping with involution. By (4) in Theorem 1.2 and (2.4), f satisfies (2.3).

Assume that $C_1 : X \longrightarrow Y$ is another solution of (1.4) satisfying (2.3). We know that C_1 is a fixed point of J. Due to (3) in Theorem 1.2, we get $C = C_1$. This proves the uniqueness of C. □

Theorem 2.2. *Assume that $\phi : X^2 \longrightarrow [0,\infty)$ is a mapping and there exists a real number L with $0 < L < 1$ such that*

$$\phi(x,y) \leq \frac{L}{|8|}\phi(2x,2y),\ \ \phi(x+\sigma(x), y+\sigma(y)) \leq \phi(2x,2y) \tag{2.9}$$

for all $x, y \in X$. Let $f : X \longrightarrow Y$ be a mapping satisfying (2.2) and $f(0) = 0$. Then there exists a unique cubic mapping $C : X \longrightarrow Y$ with involution such that

$$\|f(x) - C(x)\| \leq \frac{L}{|2|^4(1-L)}\Phi(x) \tag{2.10}$$

for all $x \in X$, where $\Phi(x) = max\{\phi(x,0), \phi(0,x)\}$.

Proof. Consider the set $S = \{g \mid g : X \longrightarrow Y\}$ and the generalized metric d in S defined by $d(g,h) = inf\{c \in [0,\infty)| \ \|g(x) - h(x)\| \leq c\ \Phi(x)$ for all $x \in X\}$. Then (S,d) is a complete metric space. Define a mapping $J : S \longrightarrow S$ by

$$Jg(x) = 8\Big\{g\Big(\frac{x}{2}\Big) - g\Big(\frac{x+\sigma(x)}{4}\Big)\Big\}$$

for all $x \in X$ and all $g \in S$. Let $g,h \in S$ and $d(g,h) \leq c$ for some non-negative real number c. Then by (2.9), we have

$$\begin{aligned}\|Jg(x) - Jh(x)\| &= |8|\Big\|g\Big(\frac{x}{2}\Big) - g\Big(\frac{x+\sigma(x)}{4}\Big) - h\Big(\frac{x}{2}\Big) + h\Big(\frac{x+\sigma(x)}{4}\Big)\Big\| \\ &\leq |8|max\Big\{\Big\|g\Big(\frac{x}{2}\Big) - h\Big(\frac{x}{2}\Big)\Big\|, \Big\|g\Big(\frac{x+\sigma(x)}{4}\Big) - h\Big(\frac{x+\sigma(x)}{4}\Big)\Big\|\Big\} \\ &\leq cL\Phi(x)\end{aligned}$$

for all $x \in X$. Hence $d(Jg, Jh) \leq Ld(g,h)$ for any $g,h \in S$ and so J is a strictly contractive mapping.

Next, we claim that $d(Jf, f) < \infty$. By (2.4), (2.5) and (2.6), we have

$$\|Jf(x) - f(x)\| = \Big\|8f\Big(\frac{x}{2}\Big) - 8f\Big(\frac{x+\sigma(x)}{4}\Big) - f(x)\Big\| \leq \frac{L}{|2|^4}\Phi(x)$$

for all $x \in X$ and hence

$$d(Jf, f) \leq \frac{L}{|2|^4} < \infty.$$

By Theorem 1.2, there exists a mapping $C : X \longrightarrow Y$ which is a fixed point of J such that $d(J^n f, C) \to 0$ as $n \to \infty$. By induction, we can easily show that

$$(J^n f)(x) = 2^{3n}\Big\{f\Big(\frac{x}{2^n}\Big) - f\Big(\frac{x+\sigma(x)}{2^{n+1}}\Big)\Big\}$$

for each $n \in \mathbb{N}$. Since $d(J^n f, C) \to 0$ as $n \to \infty$, there exists a sequence $\{c_n\}$ in $\mathbb{R}$ such that $c_n \to 0$ as $n \to \infty$ and $d(J^n f, C) \leq c_n$ for every $n \in \mathbb{N}$. Hence, it follows from the definition of d that

$$\|(J^n f)(x) - C(x)\| \leq c_n\Phi(x)$$

for all $x \in X$. Thus for each fixed $x \in X$, we have

$$\lim_{n\longrightarrow\infty} \|(J^n f)(x) - C(x)\| = 0$$

and

$$C(x) = 2^{3n}\Big\{f\Big(\frac{x}{2^n}\Big) - f\Big(\frac{x+\sigma(x)}{2^{n+1}}\Big)\Big\}.$$

Analogously to the proof of Theorem 2.2, we can show that C is a unique cubic mapping with involution satisfying (2.10)

□

We can use Theorem 2.1 and Theorem 2.2 to get a classical result in the framework of non-Archimedean normed spaces. Taking $\phi(x,y) = \theta(\|x\|^p + \|y\|^p)$ or $\phi(x,y) = \theta(\|x\|^p\|y\|^p + \|x\|^{2p} + \|y\|^{2p})$, we have the following examples.

Example 2.3. Let $\theta \geq 0$ and p be a positive real number with $p \neq 3$. Let $f : X \longrightarrow Y$ be a mapping satisfying

$$\|Df(x,y)\| \leq \theta(\|x\|^p + \|y\|^p) \tag{2.11}$$

for all $x, y \in X$. Suppose that $\|x + \sigma(x)\| \leq |2|\|x\|$ for all $x \in X$. Then there exists a unique mapping $C : X \longrightarrow Y$ with involution such that the inequality

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$$||f(x) - C(x)|| \le \begin{cases} \dfrac{\theta||x||^p}{|2|(|2|^3 - |2|^p)}, & \text{if } p > 3, \\ \dfrac{\theta||x||^p}{|2|(|2|^p - |2|^3)}, & \text{if } 0 < p < 3 \end{cases}$$

holds for all $x \in X$.

Proof. Let $\phi(x, y) = \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$ and $L = |2|^{p-3}$. Then $\phi(2x, 2y) = |8||2|^{p-3}\phi(x, y)$ for all $x, y \in X$. Since $\|x + \sigma(x)\| \le |2|\|x\|$ for all $x \in X$, $\phi(x + \sigma(x), y + \sigma(y)) \le |8||2|^{p-3}\phi(x, y)$ for all $x, y \in X$. Hence if $p > 3$, then we have the results of Theorem 2.1.

Suppose that $L = |2|^{3-p}$. Then $\phi(x, y) = \frac{|2|^{3-p}}{|8|}\phi(2x, 2y)$ for all $x, y \in X$ and $\phi(x + \sigma(x), y + \sigma(y)) \le |2|^p\phi(x, y) = \frac{|2|^{3-p}}{|8|}\phi(x, y)$ for all $x, y \in X$. Hence if $0 < p < 3$, then we have the results of Theorem 2.2. Thus the proof is complete. □

Example 2.4. Let $\theta \ge 0$ and p be a positive real number with $p \ne \frac{3}{2}$. Let $f : X \longrightarrow Y$ be a mapping satisfying

(2.12) $$\|Df(x, y)\| \le \theta(\|x\|^p\|y\|^p + \|x\|^{2p} + \|y\|^{2p})$$

for all $x, y \in X$. Suppose that $\|x + \sigma(x)\| \le |2|\|x\|$ for all $x \in X$. Then there exists a unique mapping $C : X \longrightarrow Y$ with involution such that C is a solution of the functional equation (1.4) and the inequality

$$||f(x) - C(x)|| \le \begin{cases} \dfrac{\theta||x||^p}{|2|(|2|^3 - |2|^{2p})}, & \text{if } p > \frac{3}{2}, \\ \dfrac{\theta||x||^p}{|2|(|2|^{2p} - |2|^3)}, & \text{if } 0 < p < \frac{3}{2} \end{cases}$$

holds for all $x \in X$.

Using Theorem 2.1 and Theorem 2.2, we obtain the following corollary concerning the stability of (1.4).

Corollary 2.5. *Let $\alpha_i : [0, \infty) \longrightarrow [0, \infty)$ $(i = 1, 2, 3)$ be increasing mappings satisfying*
(i) $0 < \alpha_i(|2|) < 1$ and $\alpha_i(0) = 0$,
(ii) $\alpha_i(|2|t) \le \alpha_i(|2|)\alpha_i(t)$ for all $t \ge 0$.
Let $f : X \longrightarrow Y$ be a mapping such that for some $\delta \ge 0$

(2.13) $$\|Df(x, y)\| \le \delta[\alpha_1(\|x\|)\alpha_1(\|y\|) + \alpha_2(\|x\|) + \alpha_3(\|y\|)]$$

for all $x, y \in X$. Suppose that $\|x + \sigma(x)\| \le |2|\|x\|$ for all $x \in X$. Then there exists a unique cubic mapping $C : X \longrightarrow Y$ with involution such that

$$||f(x) - C(x)|| \le \begin{cases} \dfrac{1}{|2|(|2|^3 - M)}\widetilde{\Phi}(x), & \text{if } 0 < M < |2|^3, \\ \dfrac{1}{|2|(N - |2|^3)}\widetilde{\Phi}(x), & \text{if } N > |2|^3 \end{cases}$$

holds for all $x \in X$, where $M = \max\{(\alpha_1(|2|))^2, \alpha_2(|2|), \alpha_3(|2|)\}$, $N = \min\{(\alpha_1(|2|))^2, \alpha_2(|2|), \alpha_3(|2|)\}$ and $\widetilde{\Phi}(x) = \delta \ \max\{\alpha_2(\|x\|), \alpha_3(\|x\|)\}$.

As examle of Corollary 2.5, we can take $\alpha_1(t) = \alpha_2(t) = \alpha_3(t) = t^p$ for all $t \ge 0$. Then we have the following example.

Example 2.6. Let $\delta \ge 0$ and p be a positive real number with $p \ne \frac{3}{2}$. Let $f : X \longrightarrow Y$ be a mapping satisfying

(2.14) $$\|Df(x, y)\| \le \delta(\|x\|^p\|y\|^p + \|x\|^p + \|y\|^p)$$

and $\|x + \sigma(x)\| \le |2|\|x\|$ for all $x, y \in X$. Then there exists a unique mapping $C : X \longrightarrow Y$ with involution such that the inequality

$$||f(x)-C(x)|| \leq \begin{cases} \dfrac{\delta||x||^p}{|2|(|2|^3-|2|^p)}, & \text{if } p>3, \\ \dfrac{\delta||x||^p}{|2|(|2|^{2p}-|2|^3)}, & \text{if } 0<p<\frac{3}{2} \end{cases}$$

holds for all $x \in X$.

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Department of Mathematics Education, Dankook University, Yongin 448-701, Republic of Korea
E-mail address: kci206@hanmail.net

Department of Mathematics, Soongsil University, Seoul 156-743, Republic of Korea
E-mail address: seashin@hanmail.net

VALUE SHARING RESULTS FOR MEROMORPHIC FUNCTIONS WITH THEIR Q-SHIFTS

XIAOGUANG QI, JIA DOU AND LIANZHONG YANG

ABSTRACT. This research is a continuation of a recent paper [16, 17]. Shared value problems related to a meromorphic function $f(z)$ and its q-shift $f(qz+c)$ are studied. Moreover, we also consider uniqueness problems on meromorphic functions $f(z)$ share sets with $f(qz+c)$.

1. INTRODUCTION

We assume that the reader is familiar with the elementary Nevanlinna Theory, see, e.g. [8, 18]. Meromorphic functions are always non-constant, unless otherwise specified. As usual, by $S(r,f)$ we denote any quantity satisfying $S(r,f)=o(T(r,f))$ for all r outside of a possible exceptional set of finite linear measure. In particular, we denote by $S_1(r,f)$ any quality satisfying $S_1(r,f)=o(T(r,f))$ for all r on a set of logarithmic density 1.

For a meromorphic function f and a set S of complex numbers, we define the set $E(S,f)=\bigcup_{a\in S}\{z|f(z)-a=0\}$, where a zero of $f-a$ with multiplicity m counts m times in $E(S,f)$. As a special case, when $S=\{a\}$ contains only one element a, if $E(a,f)=E(a,g)$, then we say $f(z)$ and $g(z)$ share a CM; if $\overline{E}(a,f)=\overline{E}(a,g)$, then we say $f(z)$ and $g(z)$ share a IM, see [18].

The classical results due to Nevanlinna [14] in the uniqueness theory of meromorphic functions are the five-point, resp. four-point, theorems:

Theorem A. *If two meromorphic functions $f(z)$ and $g(z)$ share five distinct values a_1, a_2, a_3, a_4, $a_5\in\mathbb{C}\cup\{\infty\}$ IM, then $f(z)\equiv g(z)$.*

Theorem B. *If two meromorphic functions $f(z)$ and $g(z)$ share four distinct values a_1, a_2, a_3, $a_4\in\mathbb{C}\cup\{\infty\}$ CM, then $f(z)\equiv g(z)$ or $f(z)\equiv T\circ g(z)$, where T is a Möbius transformation.*

It is well-known that 4 CM can not be improved to 4 IM, see [6]. Further, Gundersen [7, Theorem 1] has improved the assumption 4 CM to 2 CM+2 IM, while 1 CM+3 IM is still an open problem.

Heittokangas et al. [9, 10] considered the uniqueness of a finite order meromorphic function sharing values with its shift. They proved the following theorem:

Theorem C. *Let $f(z)$ be a meromorphic function of finite order, let $c\in\mathbb{C}$, and let $a_1,a_2,a_3\in S(f)\cup\{\infty\}$ be three distinct periodic functions with period*

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c. If $f(z)$ and $f(z+c)$ share a_1, a_2 CM and a_3 IM, then $f(z) = f(z+c)$ for all $z \in \mathbb{C}$.

Here, denote by $S(f)$ the family of all meromorphic functions $a(z)$ that satisfy $T(r,a) = o(T(r,f))$, for $r \to \infty$ outside a possible exceptional set of finite logarithmic measure.

Some improvements of Theorem C can be found in [1, 11, 12, 15]. A natural question is: what is the uniqueness result in the case when $f(z)$ shares values with $f(qz+c)$ for a zero-order meromorphic function $f(z)$. Corresponding to this question, we got the following result in [16]:

Theorem D. *Let $f(z)$ be a zero-order meromorphic function, and $q \in \mathbb{C} \setminus \{0\}$, $c \in \mathbb{C}$, and let $a_1, a_2, a_3 \in \mathbb{C} \cup \{\infty\}$ be three distinct values. If $f(z)$ and $f(qz+c)$ share a_1, a_2 CM and a_3 IM, then $f(z) = f(qz+c)$ and $|q| = 1$.*

Theorem E. *Let $f(z)$ be a zero-order entire function, $q \in \mathbb{C} \setminus \{0\}$, $c \in \mathbb{C}$, and let $a_1, a_2 \in \mathbb{C}$ be two distinct values. If $f(z)$ and $f(qz+c)$ share a_1 and a_2 IM, then $f(z) = f(qz+c)$ and $|q| = 1$.*

It seems natural to ask whether the assumption "constants a_i" can be replaced by "small functions a_i" in Theorem E. We will give a positive answer in this paper. The reminder of this paper is organized as follows: Firstly, Section 2 contains some auxiliary results. We consider the value sharing problem for $f(z)$ and $f(qz+c)$ in Section 3. Section 4 is devoted to proving some uniqueness results for meromorphic functions $f(z)$ share sets with $f(qz+c)$.

2. Some Lemmas

Lemma 2.1. [13, Theorem 2.1] *Let $f(z)$ be a zero-order meromorphic function, and $q \in \mathbb{C} \setminus \{0\}$, $c \in \mathbb{C}$. Then*

$$m\left(r, \frac{f(qz+c)}{f(z)}\right) = S_1(r,f).$$

Lemma 2.2. [16, Theorem 3.2] *Let $f(z)$ be a zero-order meromorphic function, and $q \in \mathbb{C} \setminus \{0\}$, $c \in \mathbb{C}$. Then*

$$m\left(r, \frac{f(z)}{f(qz+c)}\right) = S_1(r,f) \tag{2.1}$$

and

$$T(r, f(qz+c)) = T(r, f(z)) + S_1(r,f). \tag{2.2}$$

Lemma 2.3. [13, Theorem 2.4] *Let $f(z)$ be a zero-order meromorphic solution of*

$$f(z)^n P(z,f) = Q(z,f),$$

where $P(z,f)$ and $Q(z,f)$ are q-shift difference polynomials in $f(z)$. If the degree of $Q(z,f)$ as a polynomial in $f(z)$ and its q-shifts is at most n, then

$$m(r, P(z,f)) = S_1(r,f).$$

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3. Improvement of Theorem E

Next we show that "constants a_i" in Theorems E can be replaced by "small functions a_i".

Theorem 3.1. *Let $f(z)$ be a zero-order entire function, $q \in \mathbb{C} \setminus \{0\}$, $c \in \mathbb{C}$, and let $a_1, a_2 \in S(f)$. If $f(z)$ and $f(qz+c)$ share a_1 and a_2 IM, then $f(z) = f(qz+c)$ and $|q| = 1$.*

Remarks. (1). Theorem E and Theorem 3.1 seem to be so similar. However, our proof is different to the one in Theorem E.

(2). We tried to improve Theorem D, unfortunately, we cannot get any improvement in this paper.

Proof of Theorem 3.1. From the fact that a non-constant meromorphic function of zero-order can have at most one Picard exceptional value (see, e. g., [3, p. 114]), it can be concluded that $N(r, \frac{1}{f-a_1}) \neq 0$ and $N(r, \frac{1}{f-a_2}) \neq 0$. Define

$$H(z) = \frac{H_1(z)(f(z) - f(qz+c))}{(f(z) - a(z))(f(z) - b(z))}, \tag{3.1}$$

where

$$H_1(z) = (f(z) - a(z))(f'(z) - b'(z)) - (f'(z) - a'(z))(f(z) - b(z)).$$

And

$$G(z) = \frac{G_1(z)(f(z) - f(qz+c))}{(f(qz+c) - a(z))(f(qz+c) - b(z))}, \tag{3.2}$$

where

$$G_1(z) = (f(qz+c) - a(z))(f'(qz+c) - b'(z)) - (f'(qz+c) - a'(z))(f(qz+c) - b(z)).$$

Equation (3.1) can be rewritten as

$$\begin{aligned} H(z) &= \left(\frac{f'(z) - b'(z)}{f(z) - b(z)} - \frac{f'(z) - a'(z)}{f(z) - a(z)} \right) (f(z) - f(qz+c)) \\ &= \frac{H_1(z)(f(z) - a(z) + a(z))}{(f(z) - a(z))(f(z) - b(z))} \left(1 - \frac{f(qz+c)}{f(z)} \right). \end{aligned} \tag{3.3}$$

Note

$$\begin{aligned} H_1(z) &= (f(z) - a(z))(f'(z) - b'(z)) - (f'(z) - a'(z))(f(z) - b(z)) \\ &= (f(z) - b(z))(a'(z) - b'(z)) - (f'(z) - b'(z))(a(z) - b(z)), \end{aligned}$$

hence equation (3.3) can be expressed as

$$\begin{aligned} H(z) &= \left(1 - \frac{f(qz+c)}{f(z)} \right) \left(\frac{H_1(z)}{f(z) - b(z)} + a(z) \frac{H_1(z)}{(f(z) - a(z))(f(z) - b(z))} \right) \\ &= \left(1 - \frac{f(qz+c)}{f(z)} \right) \left(\frac{(f(z) - b(z))(a'(z) - b'(z)) - (f'(z) - b'(z))(a(z) - b(z))}{f(z) - b(z)} \right. \\ &\quad \left. + a(z) \frac{(f(z) - a(z))(f'(z) - b'(z)) - (f'(z) - a'(z))(f(z) - b(z))}{(f(z) - a(z))(f(z) - b(z))} \right). \end{aligned} \tag{3.4}$$

By the assumption $f(z)$ and $f(qz+c)$ share $a(z)$, $b(z)$ IM and equation (3.3), we get

$$N(r,H(z)) \leq N(r,a(z)) + N(r,b(z)) = S(r,f). \tag{3.5}$$

From equation (3.4), Lemma 2.1 and the lemma of logarithmic derivative, we know

$$m(r,H(z)) = S_1(r,f).$$

Hence,

$$T(r,H(z)) = S_1(r,f). \tag{3.6}$$

Similarly as above, we know

$$G(z) = \left(\frac{f'(qz+c)-b'(z)}{f(qz+c)-b(z)} - \frac{f'(qz+c)-a'(z)}{f(qz+c)-a(z)}\right)(f(z)-f(qz+c)). \tag{3.7}$$

Using a similar way, we obtain that

$$T(r,G(z)) = S_1(r,f). \tag{3.8}$$

Denote

$$U(z) = mH(z) - nG(z). \tag{3.9}$$

Next, suppose on the contrary that $f(z) \not\equiv f(qz+c)$, and head for a contradiction.

Case 1. Assume that there exists two integers m,n such that $U(z) = 0$. Then from (3.3) and (3.7), we deduce that

$$m\left(\frac{f'(z)-b'(z)}{f(z)-b(z)} - \frac{f'(z)-a'(z)}{f(z)-a(z)}\right) = n\left(\frac{f'(qz+c)-b'(z)}{f(qz+c)-b(z)} - \frac{f'(qz+c)-a'(z)}{f(qz+c)-a(z)}\right),$$

which implies that

$$\left(\frac{f(z)-b(z)}{f(z)-a(z)}\right)^m = A\left(\frac{f(qz+c)-b(z)}{f(qz+c)-a(z)}\right)^n,$$

where A is a non-zero constant. If $m \neq n$, then we get from above equality and (2.2) that

$$mT(r,f(z)) = nT(r,f(qz+c)) + S_1(r,f) = nT(r,f(z)) + S_1(r,f),$$

which is a contradiction. If $m = n$, then we get

$$\frac{f(z)-b(z)}{f(z)-a(z)} = B\frac{f(qz+c)-b(z)}{f(qz+c)-a(z)}, \tag{3.10}$$

where B satisfies $B^m = A$.

If $B = 1$, then we obtain $f(z) = f(qz+c)$, which contradicts the assumption $f(z) \not\equiv f(qz+c)$. It remains to consider the case that $B \neq 1$. The equation (3.10) gives

$$f(z)((B-1)f(qz+c)+a(z)-Bb(z)) = (Ba(z)-b(z))f(qz+c)+(1-B)a(z)b(z).$$

Apply Lemma 2.3 to the above equation, resulting in

$$m(r,((B-1)f(qz+c)+a(z)-Bb(z))) = S_1(r,f).$$

Consequently,

$$T(r,f(qz+c)) = T(r,f) + S_1(r,f) = S_1(r,f),$$

which is impossible.

Case 2. There does not exist two positive integers m, n such that $U(z) = 0$. In what follows, we denote $S_{f\sim g(n,m)}(a)$ for the set of those points $z \in \mathbb{C}$ such that z is an a-point of f with multiplicity n and an a-point of g with multiplicity m such that $a(z) \neq \infty, b(z) \neq \infty, a(z) - b(z) \neq 0$. Let $N_{(n,m)}(r, \frac{1}{f-a})$ and $\overline{N}_{(n,m)}(r, \frac{1}{f-a})$ denote the counting function and reduced counting function of $f(z)$ with respect to the set $S_{f\sim g(n,m)}(a)$, respectively.

Take z_0 such that $z_0 \in S_{f(z)\sim f(qz+c)(n,m)}(a(z))$, we have $mn \neq 0$, since $a(z)$ is not a Picard exceptional value of $f(z)$ as we discuss above. Combining (3.3), (3.7) with (3.9), by calculating carefully, it follows that $U(z_0) = 0$. From (3.6), (3.8) and (3.9), we have

$$\overline{N}_{(n,m)}\left(r, \frac{1}{f(z) - a(z)}\right) \leq N\left(r, \frac{1}{U(z)}\right) = N\left(r, \frac{1}{mH(z) - nG(z)}\right) = S_1(r, f).$$

Using the same reason, we get

$$\overline{N}_{(n,m)}\left(r, \frac{1}{f(z) - b(z)}\right) \leq N\left(r, \frac{1}{U(z)}\right) = N\left(r, \frac{1}{nH(z) - mG(z)}\right) = S_1(r, f).$$

Consequently,

$$\overline{N}_{(n,m)}\left(r, \frac{1}{f(z) - a(z)}\right) + \overline{N}_{(n,m)}\left(r, \frac{1}{f(z) - b(z)}\right) = S_1(r, f). \quad (3.11)$$

Combining (2.2) with (3.11), it follows that

$$\begin{aligned}
T(r, f(z)) &\leq \overline{N}\Big(r, \frac{1}{f(z) - a(z)}\Big) + \overline{N}\Big(r, \frac{1}{f(z) - b(z)}\Big) + S_1(r, f) \\
&= \sum_{n,n} \left(\overline{N}_{(n,m)}(r, \frac{1}{f(z) - a(z)}) + \overline{N}_{(n,m)}(r, \frac{1}{f(z) - b(z)})\right) + S_1(r, f) \\
&= \sum_{m+n\geq 5} \left(\overline{N}_{(n,n)}(r, \frac{1}{f(z) - a(z)}) + \overline{N}_{(n,m)}(r, \frac{1}{f(z) - b(z)})\right) + S_1(r, f) \\
&\leq \frac{1}{5} \sum_{m+n\geq 5} \Big(N_{(n,m)}(r, \frac{1}{f(z) - a(z)}) + N_{(n,m)}(r, \frac{1}{f(z) - b(z)}) \\
&+ N_{(n,m)}(r, \frac{1}{f(qz+c) - a(z)}) + N_{(n,m)}(r, \frac{1}{f(qz+c) - b(z)})\Big) + S_1(r, f) \\
&\leq \frac{2}{5}T(r, f) + \frac{2}{5}T(r, f(qz+c)) + S_1(r, f) \\
&= \frac{4}{5}T(r, f) + S_1(r, f),
\end{aligned}$$

which is a contradiction. Therefore, we get $f(z) = f(qz + c)$.

The rest of proof consists of the conclusion that $|q| = 1$. The proof is similar as [10, Theorem 1.5]. In fact, we have given the proof in [16]. The proof is stated explicitly for the convenience of the reader. If $f(z)$ is transcendental and suppose first $|q| < 1$. It can be assumed that there exists one point z_0 such that $f(z_0) = a_1$ and that z_0 is not a fixed point of $qz + c$. By the sharing assumptions of Theorem 3.1, we get $f(qz_0 + c) = a_1$ as well. By calculation, we know $f(q^n z_0 + c(1 + \cdots + q^{n-1})) = a_1$ for all $n \in \mathbb{N}$. Letting $n \to \infty$, it is concluded that a_1-points of f accumulate to $z = \frac{c}{1-q}$, which is a contradiction. If $|q| > 1$, then set $g(z) = f(qz + c)$. Assume that g has at least one a_1 point, say at z_0. From the sharing assumptions, we get $g(\frac{1}{q^n}z - c(\frac{1}{q} + \cdots + \frac{1}{q^n})) = a_1$ for all all $n \in \mathbb{N}$. Using the same way above, we get a_1-point of g accumulate to $z = \frac{c}{1-q}$, which is a contradiction. Hence $|q| = 1$.

If f is a rational function, then set $f(z) = \frac{\sum_{i=1}^{m} a_i z^i}{\sum_{j=1}^{n} b_i z^j}$ and $f(qz + c) = \frac{\sum_{i=1}^{m} a_i (qz+c)^i}{\sum_{j=1}^{n} b_i (qz+c)^j}$. By simply calculations, it follow that $|q| = 1$. This completes the proof of Theorem 3.1.

4. Sharing sets results

Gross [4, Question 6] asked the following question:

Question. Can one find (even one set) finite sets S_j $(j = 1, 2)$ such that any two entire functions $f(z)$ and $g(z)$ satisfying $E(S_j, f) = E(S_j, g)$ $(j = 1, 2)$ must be identical?

Since then, many results have been obtained for this and related topics (see [2, 19, 20, 21]). Here, we just recall the following two results only.

Theorem F [5]. *Let $S_1 = \{1, -1\}$, $S_2 = \{0\}$. If $f(z)$ and $g(z)$ are entire functions of finite order such that $E(S_j, f) = E(S_j, g)$ for $j = 1, 2$, then $f(z) = \pm g(z)$ or $f(z)g(z) = 1$.*

Theorem G [22]. *Let $S_1 = \{1, \omega, \ldots \omega^{n-1}\}$ and $S_2 = \{\infty\}$, where $\omega = \cos(2\pi/n) + i\sin(2\pi/n)$ and $n \geq 6$ be a positive integer. Suppose that $f(z)$ and $g(z)$ are meromorphic functions such that $E(S_j, f) = E(S_j, g)$ for $j = 1, 2$, then $f(z) = tg(z)$ or $f(z)g(z) = t$, where $t^n = 1$.*

It is natural to ask what will happen if $g(z)$ is replaced by q-shift of $f(z)$ in Theorems F and G. In the following, we answer this problem, and get shared sets results for $f(z)$ and its q-shift $f(qz + c)$.

Theorem 4.1. *Let S_1, S_2 be given as in Theorem G, and let $f(z)$ be a zero-order meromorphic function satisfying $E(S_j, f(z)) = E(S_j, f(qz + c))$ for $j = 1, 2$, $c \in \mathbb{C}$ and $q \in \mathbb{C} \setminus \{0\}$. If $n \geq 4$, then $f(z) = tf(qz + c)$, $t^n = 1$ and $|q| = 1$.*

By the same reasoning as in the proof of Theorem 4.1, we obtain the following result. We omit the proof here.

Corollary 4.2. *Theorem 4.1 still holds if f is a zero-order entire function and $n \geq 3$.*

In the following, we give a partial answer as to what may happen if $n = 2$ in Corollary 4.2, which can be seen an analogue for q-shift of Theorem F.

Theorem 4.3. *Suppose $f(z)$ is a zero-order entire function and $q \in \mathbb{C} \setminus \{0\}$, $c \in \mathbb{C}$. If $f(z)$ and $f(qz + c)$ share the set $\{a(z), -a(z)\}$ CM, where $a(z)$ is a non-vanishing small function of $f(z)$, then one of the following statements hold:*
(1). $C^2 f(z) = f(q^2 z + qc + c)$, where C is a constant such that $C^2 \neq 1$;
(2). $f(z) = \pm f(qz + c)$, and $|q| = 1$.

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Corollary 4.4. *Suppose a is a non-zero constant in Theorem 4.3, then we get $f(z) = \pm f(qz + c)$, where $|q| = 1$.*

Corollary 4.5. *Under the assumptions of Theorem 4.3, if $f(z)$ and $f(qz+c)$ share sets $\{a(z), -a(z)\}$, $\{0\}$ CM, then $f(z) = \pm f(qz + c)$, where $|q| = 1$.*

Proof of Theorem 4.1. By the sharing assumption, we get $f(z)^n$ and $f(qz + c)^n$ share 1 and ∞ CM. This implies,

$$\frac{f(qz + c)^n - 1}{f(z)^n - 1} = \gamma, \tag{4.1}$$

where γ is a non-zero constant. This gives

$$f(qz + c)^n = \gamma(f(z)^n - 1 + \frac{1}{\gamma}). \tag{4.2}$$

Denote

$$G(z) = \frac{f(z)^n}{1 - \frac{1}{\gamma}}.$$

Suppose $\gamma \not\equiv 1$, then by the second main theorem and Lemma 2.2 to $G(z)$, it follows that

$$\begin{aligned} nT(r, f) + S(r, f) = T(r, G) &\leq \overline{N}\left(r, \frac{1}{G}\right) + \overline{N}(r, G) + \overline{N}\left(r, \frac{1}{G - 1}\right) + S(r, G) \\ &\leq \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f(z)^n - 1 + \frac{1}{\gamma}}\right) + S(r, f) \\ &\leq \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f(qz + c)}\right) + S(r, f) \\ &\leq 2T(r, f) + T(r, f(qz + c)) + S(r, f) \leq 3T(r, f) + S_1(r, f). \end{aligned}$$

This together with the assumption $n \geq 4$ results in a contradiction. Hence, $\gamma \equiv 1$, that is, $f(z)^n = f(qz + c)^n$. This yields $f(z) = tf(qz + c)$ for a constant t with $t^n = 1$. Let $F(z) = f(z)^n$ and $F(qz + c) = f(qz + c)^n$, then we get $F(z) = F(qz + c)$. Similarly as Theorem 3.1, we have $|q| = 1$. The conclusion follows.

Proof of Theorem 4.3. It follows by the assumptions that

$$(f(qz + c) - a(z))(f(qz + c) + a(z)) = C^2(f(z) - a(z))(f(z) + a(z)), \tag{4.3}$$

where C is a non-zero constant.

Case 1. Suppose first that $C^2 \not\equiv 1$. Denote

$$h_1(z) = f(z) - \frac{1}{C}f(qz + c), \quad h_2(z) = f(z) + \frac{1}{C}f(qz + c).$$

Then

$$f(z) = \frac{1}{2}(h_1(z) + h_2(z)), \quad f(qz + c) = \frac{C}{2}(h_2(z) - h_1(z)). \tag{4.4}$$

Moreover, we have

$$h_1(z)h_2(z) = (1 - \frac{1}{C^2})a^2(z). \tag{4.5}$$

From above equation, we get

$$N\left(r,\frac{1}{h_1}\right)=S(r,f),\quad N\left(r,\frac{1}{h_2}\right)=S(r,f). \tag{4.6}$$

By definitions of $h_1(z)$ and $h_2(z)$, Lemma 2.2 yields

$$T(r,h_i)\leq 2T(r,f)+S_1(r,f),$$

which means $S_1(r,h_i)=o(T(r,f))$ for all r on a set of logarithmic density 1, $i=1,2$.
Denote

$$\alpha(z)=\frac{h_1(qz+c)}{h_1(z)},\quad \beta(z)=\frac{h_2(qz+c)}{h_2(z)}.$$

From (4.6) and Lemma 2.1, we obtain that

$$\begin{aligned} T(r,\alpha)&=m(r,\alpha)+N\left(r,\frac{1}{h_1}\right)=S_1(r,f),\\ T(r,\beta)&=m(r,\beta)+N\left(r,\frac{1}{h_2}\right)=S_1(r,f). \end{aligned} \tag{4.7}$$

From definitions of $h_1(z)$, $h_2(z)$ and equation (4.4), we conclude that

$$Ch_2(z)-Ch_1(z)=h_1(qz+c)+h_2(qz+c).$$

Dividing above equation with $h_1(z)h_2(z)$, we obtain

$$(\alpha+C)h_1(z)=(C-\beta)h_2(z). \tag{4.8}$$

By combining (4.5) and (4.8), it follows that

$$(\alpha+C)h_1^2(z)-(C-\beta)(1-\frac{1}{C^2})a^2(z)=0. \tag{4.9}$$

From (4.7) and (4.9), we get $\alpha=-C$ and $\beta=C$. Otherwise, we know $T(r,h_1)=S_1(r,f)$, which means $T(r,f)=S_1(r,f)$ from (4.4) and (4.5), a contradiction. Hence, we have

$$h_1(qz+c)=-Ch_1(z),\quad h_2(qz+c)=Ch_2(z),$$

from definitions of $\alpha(z)$ and $\beta(z)$, that is

$$\begin{cases} -C(f(z)-\frac{1}{C}f(qz))=f(qz)-\frac{1}{C}f(q(qz+c)+c),\\ C(f(z)+\frac{1}{C}f(qz))=f(qz)+\frac{1}{C}f(q(qz+c)+c). \end{cases}$$

The above equations give $C^2f(z)=f(q^2z+qc+c)$.

Case 2. $C^2\equiv 1$. The equation (4.3) implies that $f(z)=\pm f(qz+c)$. Using a similar way as Theorem 3.1, we get $|q|=1$ in Case 2.

Proof of Corollary 4.4. Similarly as Theorem 4.3, we obtain equations (4.4) and (4.5) hold as well. Equation (4.5) and the assumption that a is non-zero constant give

$$N\left(r,\frac{1}{h_1}\right)=0,\quad N\left(r,\frac{1}{h_2}\right)=0. \tag{4.10}$$

Combining (4.10) with the definitions of $h_1(z)$ and $h_2(z)$, we conclude that $h_1(z)$ and $h_2(z)$ are constants. From (4.4), we get $f(z)$ is a constant, which contradicts the assumption. Hence, only Case 2 of Theorem 4.3 holds, the conclusion follows.

Proof of Corollary 4.5. It suffices to prove the case $C^2 f(z) = f(q^2 z + qc + c)$ in Theorem 4.3 cannot hold. Suppose that $f(z_0) = 0$, then by the sharing assumption and (4.4), it follows that

$$h_1(z_0) + h_2(z_0) = 0, \quad h_1(qz_0 + c) + h_2(qz_0 + c) = 0.$$

Hence,

$$\frac{h_1(qz_0 + c)}{h_1(z_0)} \frac{h_2(z_0)}{h_2(qz_0 + c)} = 1.$$

From the proof of Theorem 4.3, we know

$$\alpha = \frac{h_1(qz_0 + c)}{h_1(z_0)} = -C, \quad \beta = \frac{h_2(qz_0 + c)}{h_2(z_0)} = C,$$

which means that

$$\frac{h_1(qz_0 + c)}{h_1(z_0)} \frac{h_2(z_0)}{h_2(qz_0 + c)} = -1.$$

which is impossible. This contradiction is only avoided when 0 is the Picard exceptional value of $f(z)$ and $f(qz + c)$. Since $f(z)$ is a zero-order entire function, we conclude that $f(z)$ must be a constant, which contradicts the assumption. Hence, $f(z) = \pm f(qz + c)$, where $|q| = 1$.

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Xiaoguang Qi
University of Jinan, School of Mathematics,
Jinan, Shandong, 250022, P. R. China
E-mail address: xiaoguang.202@163.com or xiaogqi@mail.sdu.edu.cn

Jia Dou
School of Mathematics, Shandong Normal University,
Jinan, Shandong, 250358, P. R. China
E-mail address: doujia.1983@163.com

Lianzhong Yang
Shandong University, School of Mathematics,
Jinan, Shandong, 250100, P. R. China
E-mail address: lzyang@sdu.edu.cn

RANDOM NORMED SPACE AND MIXED TYPE AQ–FUNCTIONAL EQUATION

ICK-SOON CHANG* AND YANG-HI LEE

ABSTRACT. We investigate the stability problems for the following functional equation

$$f(x+ay)+f(x-ay)-2f(x)+\frac{a-a^2}{2}f(y)-\frac{a+a^2}{2}f(-y)-f(ay)=0$$

in random normed spaces.

1. Introduction and Preliminaries

We first demonstrate the usual terminology, notations and conventions of the theory of random normed spaces [7, 8]. The space of all probability distribution functions is denoted by

$$\begin{aligned}\Delta^+ := \{F:\mathbb{R}\cup\{-\infty,\infty\}\to[0,1]\,\big|\,&F \text{ is left-continuous and nondecreasing on } \mathbb{R},\\ &\text{where } F(0)=0 \text{ and } F(+\infty)=1\}.\end{aligned}$$

And let $D^+ := \{F\in\Delta^+\,|\,l^-F(+\infty)=1\}$, where $l^-f(x)$ denotes the left limit of the function f at the point x. The space Δ^+ is partially ordered by the usual pointwise ordering of functions, i.e., $F\le G$ if and only if $F(t)\le G(t)$ for all $t\in\mathbb{R}$. The maximal element for Δ^+ in this order is the distribution function $\varepsilon_0:\mathbb{R}\cup\{0\}\to[0,\infty)$ given by

$$\varepsilon_0(t)=\begin{cases}0, & \text{if } t\le 0,\\ 1, & \text{if } t>0.\end{cases}$$

Definition 1.1. ([7]) A mapping $\tau:[0,1]\times[0,1]\to[0,1]$ is called a *continuous triangular norm* (briefly, a *continuous t-norm*) if τ satisfies the following conditions:

(TN1) τ is commutative and associative;
(TN2) τ is continuous;
(TN3) $\tau(a,1)=a$ for all $a\in[0,1]$;
(TN4) $\tau(a,b)\le\tau(c,d)$ whenever $a\le c$ and $b\le d$ for all $a,b,c,d\in[0,1]$.

Typical examples of continuous t-norms are $\tau_P(a,b)=ab$, $\tau_M(a,b)=\min(a,b)$ and $\tau_L(a,b)=\max(a+b-1,0)$.

Definition 1.2. ([8]) A *random normed space* (briefly, *RN-space*) is a triple (X,μ,τ), where X is a vector space, τ is a continuous t-norm and μ is a mapping from X into D^+ such that the following conditions hold:

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*Corresponding author

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(RN1) $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$,
(RN2) $\mu_{\alpha x}(t) = \mu_x(t/|\alpha|)$ for all $x \in X$, $\alpha \neq 0$ and all $t \geq 0$,
(RN3) $\mu_{x+y}(t+s) \geq \tau(\mu_x(t), \mu_y(s))$ for all $x, y \in X$ and all $t, s \geq 0$.

If $(X, \|\cdot\|)$ is a normed space, we can define a mapping $\mu : X \to D^+$ by $\mu_x(t) = \frac{t}{t+\|x\|}$ for all $x \in X$ and all $t > 0$. Then (X, μ, τ_M) is a random normed space, which is called the *induced random normed space*.

Definition 1.3. Let (X, μ, τ) be an *RN*-space.

(A_1) A sequence $\{x_n\}$ in X is said to be *convergent* to a point $x \in X$ if for every $t > 0$ and $\varepsilon > 0$, there exists a positive integer N such that $\mu_{x_n - x}(t) > 1 - \varepsilon$ whenever $n \geq N$.

(A_2) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if for every $t > 0$ and $\varepsilon > 0$, there exists a positive integer N such that $\mu_{x_n - x_m}(t) > 1-\varepsilon$ whenever $n \geq m \geq N$.

(A_3) An RN-space (X, μ, τ) is said to be *complete* if and only if every Cauchy sequence in X is convergent to a point in X.

Theorem 1.4. ([7]) *If (X, μ, τ) is an RN-space and $\{x_n\}$ is a sequence such that $x_n \to x$, then $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$.*

The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. The stability problem for functional equations originated from questions of Ulam [9] concerning the stability of group homomorphisms. Hyers [2] had answered affirmatively the question of Ulam for Banach spaces. A generalized version of the theorem of Hyers for additive mappings was given by Aoki [1] and for linear mappings was presented by Rassias [6]. Since then, many interesting results of the stability of various functional equation have been extensively investigated.

Now we take into account the following *mixed type additive-quadratic functional equation* (briefly, *AQ-functional equation*)

$$f(x+ay) + f(x-ay) - 2f(x) + \frac{a-a^2}{2} f(y) - \frac{a+a^2}{2} f(-y) - f(ay) = 0. \tag{1.1}$$

Here we promise that each solution of equation (1.1) is said to be an *additive-quadratic mapping.* Quite recently, the stability of functional equation (1.1) in the case when $a = 1$ was investigated in [3, 4, 5].

The main aim of this work is to establish the stability for the functional equation (1.1) in random normed spaces.

2. Main results

Let E_1 and E_2 be vector spaces. For convenience, we use the following abbreviations for a given mapping $f : E_1 \to E_2$,

$$Af(x, y) := f(x + y) - f(x) - f(y),$$
$$Qf(x, y) := f(x + y) + f(x - y) - 2f(x) - 2f(y),$$
$$Df(x, y) := f(x+ay) + f(x-ay) - 2f(x) + \frac{a-a^2}{2} f(y) - \frac{a+a^2}{2} f(-y) - f(ay)$$

for all $x, y \in E_1$, where $a > \frac{1}{2}$ is a rational number.

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A solution of $Af = 0$ is said to be an *additive mapping* and a solution of $Qf = 0$ is called a *quadratic mapping*. If a mapping f is represented by sum of additive mapping and quadratic mapping, we say that f is an *additive-quadratic mapping*.

Lemma 2.1. *A mapping $f : E_1 \to E_2$ satisfies the functional equation $Df(x, y) = 0$ for all $x, y \in E_1$ if and only if there exist a quadratic mapping $g : E_1 \to E_2$ and an additive mapping $h : E_1 \to E_2$ such that $f(x) = g(x) + h(x)$ for all $x \in E_1$.*

Proof. (Necessity) We decompose f into the even part and the odd part by considering

$$g(x) = \frac{f(x) + f(-x)}{2},\ h(x) = \frac{f(x) - f(-x)}{2}$$

for all $x \in E_1$. It is note that $f(0) = \frac{-Df(0,0)}{a^2+1} = 0$. The following functional equalities

$$\begin{aligned} Qg(x, y) =& Dg(x, y/a) - Dg(0, y/a) = 0, \\ Ah(x, y) =& - Dh\Big(\frac{x+y}{2}, \frac{x-y}{2a}\Big) + Dh\Big(\frac{x+y}{2}, \frac{x+y}{2a}\Big) + Dh\Big(0, \frac{x-y}{2a}\Big) - Dh\Big(0, \frac{x+y}{2a}\Big) \\ =& 0 \end{aligned}$$

give that g is a quadratic mapping and h is an additive mapping.

(Sufficiency) Assume that there exist a quadratic mapping $g : E_1 \to E_2$ and an additive mapping $h : E_1 \to E_2$ such that $f(x) = g(x) + h(x)$ for all $x \in E_1$. Then we see that

$$\begin{aligned} Df(x, y) &= Dg(x, y) + Dh(x, y) \\ &= Qg(x, ay) + g(ay) - a^2 g(y) - Ah(x + ay, x - ay) + Ah(x, x) + ah(y) - h(ay) \\ &= 0 \end{aligned}$$

for all $x, y \in E_1$. Therefore we arrive at the desired conclusion. □

In the following theorem, we establish the stability of the functional equation (1.1) in random normed spaces.

Theorem 2.2. *Let (Y, μ, τ_M) and (Z, μ', τ_M) be a complete RN-space and an RN-space, respectively. Suppose that V is a vector space and $f : V \to Y$ is a mapping with $f(0) = 0$ for which there exists a mapping $\varphi : V^2 \to Z$ such that*

$$\mu_{Df(x,y)}(t) \geq \mu'_{\varphi(x,y)}(t) \tag{2.1}$$

for all $x, y \in V$ and all $t > 0$. If a mapping φ satisfies one of the following conditions:

(i) $\mu'_{\alpha\varphi(x,y)}(t) \leq \mu'_{\varphi(2ax,2ay)}(t)$ *for some* $0 < \alpha < 2a$,

(ii) $\mu'_{\varphi(2ax,2ay)}(t) \leq \mu'_{\alpha\varphi(x,y)}(t) \leq \mu'_{\varphi((2a)^2x,(2a)^2y)}(t)$ *for some* $2a < \alpha < (2a)^2$,

(iii) $\mu'_{\varphi((2a)^2x,(2a)^2y)}(t) \leq \mu'_{\alpha\varphi(x,y)}(t)$ *for some* $(2a)^2 < \alpha$

for all $x, y \in V$ and all $t > 0$, then there exists a unique additive-quadratic mapping $F : V \to Y$ such that

$$\mu_{f(x)-F(x)}(t) \geq \begin{cases} \sup_{t'<t}\{M(x, (2a - \alpha)t')\} & \text{if } \varphi \text{ satisfies (i)}, \\ \sup_{t'<t}\{M(x, \frac{((2a)^2-\alpha)(2a-\alpha)t'}{4((2a)^2-2a)})\} & \text{if } \varphi \text{ satisfies (ii)}, \\ \sup_{t'<t}\{M(x, (\alpha - (2a)^2)t')\} & \text{if } \varphi \text{ satisfies (iii)} \end{cases} \tag{2.2}$$

for all $x \in V$ and all $t > 0$, where

$$M(x, t) := \tau_M\big\{\mu'_{\varphi(ax,x)}(t), \mu'_{\varphi(-ax,-x)}(t), \mu'_{\varphi(0,x)}(t), \mu'_{\varphi(0,-x)}(t)\big\}.$$

Proof. We will take into account three different cases for the assumption of φ.

Case 1. Let φ satisfy the condition (i) for some α with $0 < \alpha < 2a$ and let $J_nf : V \to Y$ be a mapping defined by

$$J_nf(x) := \frac{f((2a)^n x) - f(-(2a)^n x)}{2(2a)^n} + \frac{f((2a)^n x) + f(-(2a)^n x)}{2(2a)^{2n}}$$

for all $x \in V$ and all $n \in \mathbb{N}$. Then $J_0f(x) = f(x)$, $J_jf(0) = f(0)$ and

$$\begin{aligned} J_jf(x) - J_{j+1}f(x) =& \frac{(2a)^{j+1} - 1}{2(2a)^{2j+2}} \left[Df(-(2a)^j ax, -(2a)^j x) - 3Df(0, (2a)^j x)\right] \\ &- \frac{(2a)^{j+1} + 1}{2(2a)^{2j+2}} \left[Df((2a)^j ax, (2a)^j x) - 3Df(0, -(2a)^j x)\right] \end{aligned} \tag{2.3}$$

for all $x \in V$ and all $j \geq 0$. It implies that if $n + m > n \geq 0$, then we get by (RN2), (RN3), (2.1) and (2.2)

$$\begin{aligned} &\mu_{J_nf(x)-J_{n+m}f(x)}\Big(\sum_{j=n}^{n+m-1} \frac{4\alpha^j t}{(2a)^{j+1}}\Big) \\ &\geq \mu_{\sum_{j=n}^{n+m-1}(J_jf(x)-J_{j+1}f(x))}\Big(\sum_{j=n}^{n+m-1} \frac{4\alpha^j t}{(2a)^{j+1}}\Big) \\ &\geq \tau M_{j=n}^{n+m-1}\Big\{\mu_{J_jf(x)-J_{j+1}f(x)}\Big(\frac{4\alpha^j t}{(2a)^{j+1}}\Big)\Big\} \\ &\geq \tau M_{j=n}^{n+m-1}\Big\{\tau\Big\{\mu_{-\frac{((2a)^{j+1}+1)Df((2a)^j\cdot ax,(2a)^j x)}{2(2a)^{2j+2}}}\Big(\frac{((2a)^{j+1}+1)\alpha^j t}{2(2a)^{2j+2}}\Big), \\ &\quad \mu_{\frac{((2a)^{j+1}-1)Df(-(2a)^j\cdot ax,-(2a)^j x)}{2(2a)^{2j+2}}}\Big(\frac{((2a)^{j+1}-1)\alpha^j t}{2(2a)^{2j+2}}\Big), \\ &\quad \mu_{\frac{3((2a)^{j+1}+1)Df(0,-(2a)^j x)}{2(2a)^{2j+2}}}\Big(\frac{3((2a)^{j+1}+1)\alpha^j t}{2(2a)^{2j+2}}\Big), \\ &\quad \mu_{-\frac{3((2a)^{j+1}-1)Df(0,(2a)^j x)}{2(2a)^{2j+2}}}\Big(\frac{3((2a)^{j+1}-1)\alpha^j t}{2(2a)^{2j+2}}\Big)\Big\} \\ &\geq M(x,t) \end{aligned} \tag{2.4}$$

for all $x \in V$ and all $t > 0$. Let $c > 0$ and $\varepsilon > 0$ be given. Since $\lim_{t\to\infty} \mu'_z(t) = 1$ for all $z \in Z$, there is some $t_0 > 0$ such that $M(x, t_0) \geq 1 - \varepsilon$. Fix some $t > t_0$. Since $\alpha < 2a$, we know that the series $\sum_{j=0}^{\infty} \frac{4\alpha^j t}{(2a)^{j+1}}$ converges. It guarantees that there exists some $n_0 \geq 0$ such that $\sum_{j=n}^{n+m-1} \frac{4\alpha^j t}{(2a)^{j+1}} < c$ for all $n \geq n_0$ and all $m > 0$. Together with (RN3) and (2.4), this implies that

$$\begin{aligned} \mu_{J_nf(x)-J_{n+m}f(x)}(c) &\geq \mu_{J_nf(x)-J_{n+m}f(x)}\Big(\sum_{j=n}^{n+m-1} \frac{4\alpha^j t}{(2a)^{j+1}}\Big) \\ &\geq M(x,t) \geq M(x,t_0) \geq 1 - \varepsilon \end{aligned}$$

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for all $x \in V$. Hence $\{J_n f(x)\}$ is a Cauchy sequence in the complete RN-space (Y, μ, τ_M) and so we can define a mapping $F : X \to Y$ by $F(x) := \lim_{n\to\infty} J_n f(x)$. Moreover, if we put $m = 0$ in (2.4), we have

$$\mu_{f(x)-J_nf(x)}(t) \geq M\left(x, \frac{t}{\sum_{j=0}^{n-1}\frac{4\alpha^j t}{(2a)^{j+1}}}\right) \tag{2.5}$$

for all $x \in V$.

Next we are in the position to show that F is an additive-quadratic mapping. In view of (RN3), we figure out the relation

$$\begin{aligned}\mu_{DF(x,y)}(t) \geq \tau_M\Big\{&\mu_{(F-J_nf)(x+ay)}\Big(\frac{t}{12}\Big), \mu_{(F-J_nf)(x-ay)}\Big(\frac{t}{12}\Big), \mu_{2(J_n-Ff)(x)}\Big(\frac{t}{12}\Big),\\ &\mu_{\frac{a-a^2}{2}(F-J_nf)(y)}\Big(\frac{t}{12}\Big), \mu_{-\frac{a+a^2}{2}(F-J_nf)(-y)}\Big(\frac{t}{12}\Big), \mu_{-(F-J_nf)(ay)}\Big(\frac{t}{12}\Big),\\ &\mu_{DJ_nf(x,y)}\Big(\frac{t}{2}\Big)\Big\}\end{aligned} \tag{2.6}$$

for all $x, y \in V$ and all $n \in \mathbb{N}$. The first six terms on the right hand side of the previous inequality tend to 1 as $n \to \infty$ by the definition of F. Also we consider that

$$\begin{aligned}\mu_{DJ_nf(x,y)}\Big(\frac{t}{2}\Big) &\geq \tau_M\Big\{\mu_{\frac{Df((2a)^nx,(2a)^ny)}{2\cdot(2a)^{2n}}}\Big(\frac{t}{8}\Big), \mu_{\frac{Df(-(2a)^nx,-(2a)^ny)}{2\cdot(2a)^{2n}}}\Big(\frac{t}{8}\Big),\\ &\qquad \mu_{\frac{Df((2a)^nx,(2a)^ny)}{2\cdot(2a)^{n}}}\Big(\frac{t}{8}\Big), \mu_{\frac{Df(-(2a)^nx,-(2a)^ny)}{2\cdot(2a)^{n}}}\Big(\frac{t}{8}\Big)\Big\}\\ &\geq \tau_M\Big\{\mu_{\frac{\varphi((2a)^nx,(2a)^ny)}{2\cdot(2a)^{2n}}}\Big(\frac{t}{8}\Big), \mu_{\frac{\varphi(-(2a)^nx,-(2a)^ny)}{2\cdot(2a)^{2n}}}\Big(\frac{t}{8}\Big),\\ &\qquad \mu_{\frac{\varphi((2a)^nx,(2a)^ny)}{2\cdot(2a)^{n}}}\Big(\frac{t}{8}\Big), \mu_{\frac{\varphi(-(2a)^nx,-(2a)^ny)}{2\cdot(2a)^{n}}}\Big(\frac{t}{8}\Big)\Big\}\\ &\geq \tau_M\Big\{\mu_{\varphi(x,y)}\Big(\frac{(2a)^{2n}t}{4\alpha^n}\Big), \mu_{\varphi(-x,-y)}\Big(\frac{(2a)^{2n}t}{4\alpha^n}\Big),\\ &\qquad \mu_{\varphi(x,y)}\Big(\frac{(2a)^{n}t}{4\alpha^n}\Big), \mu_{\varphi(-x,-y)}\Big(\frac{(2a)^{n}t}{4\alpha^n}\Big)\Big\},\end{aligned}$$

which tends to 1 as $n \to \infty$ by (RN3). It follows from (2.6) that $\mu_{DF(x,y)}(t) = 1$ for all $x, y \in V$ and all $t > 0$. By (RN1), this means that $DF(x, y) = 0$ for all $x, y \in V$.

We now approximate the difference between f and F. Fix $x \in V$, $t > 0$ and choose $t' < t$. For arbitrary $\varepsilon > 0$, by $F(x) := \lim_{n\to\infty} J_n f(x)$, there is a $n \in \mathbb{N}$ such that

$$\mu_{F(x)-J_nf(x)}(t - t') \geq 1 - \varepsilon.$$

It follows by (2.5) that

$$\begin{aligned}\mu_{F(x)-f(x)}(t) &\geq \tau_M\{\mu_{F(x)-J_nf(x)}(t-t'), \mu_{J_nf(x)-f(x)}(t')\}\\ &\geq \tau_M\Big\{1-\varepsilon, M\Big(x, \frac{t'}{\sum_{j=0}^{n-1}\frac{4\alpha^j t}{(2a)^{j+1}}}\Big)\Big\}\\ &\geq \tau_M\Big\{1-\varepsilon, M\Big(x, \frac{(2a-\alpha)t'}{4}\Big)\Big\}.\end{aligned}$$

Because $\varepsilon > 0$ is arbitrary, we find that

$$\mu_{F(x)-f(x)}(t) \geq M(x, (2a-\alpha)t')$$

for all $x \in V$ and $t' < t$. The first inequality in (2.2) follows from the previous inequality.

In order to prove the uniqueness of F, we assume that F' is another additive-quadratic mapping from V to Y satisfying the first inequality in (2.2) with $F'(0) = f(0)$. Note that if F' is an additive-quadratic mapping, then we have by (2.3)

$$F'(x) - J_nF'(x) = \sum_{j=0}^{n-1}(J_jF'(x) - J_{j+1}F'(x)) = 0$$

for all $x \in V$ and all $n \in \mathbb{N}$. With the help of (RN3) and the first inequality in (2.2), this result yields that for all $x \in V$ and all $n \in \mathbb{N}$,

$$\begin{aligned}\mu_{F'(x)-J_nf(x)}(t) &= \mu_{J_nF'(x)-J_nf(x)}(t) \\ &\geq \tau_M\Big\{\mu_{\frac{(F'-f)((2a)^nx)}{2\cdot(2a)^{2n}}}\Big(\frac{t}{4}\Big), \mu_{\frac{(F'-f)(-(2a)^nx)}{2\cdot(2a)^{2n}}}\Big(\frac{t}{4}\Big), \mu_{\frac{(F'-f)((2a)^nx)}{2\cdot(2a)^{n}}}\Big(\frac{t}{4}\Big), \\ &\qquad \mu_{\frac{(F'-f)(-(2a)^nx)}{2\cdot(2a)^{n}}}\Big(\frac{t}{4}\Big)\Big\} \\ &\geq \tau_M\Big\{\sup_{t'<t}\Big\{M\Big(x, \Big(\frac{2a}{\alpha}\Big)^n\frac{(2a-\alpha)t'}{4}\Big), \sup_{t'<t}\Big\{M\Big(x, \Big(\frac{4a^2}{\alpha}\Big)^n\frac{(2a-\alpha)t'}{4}\Big)\Big\}.\end{aligned}$$

Observe that

$$\lim_{n\to\infty}\Big(\frac{2a}{\alpha}\Big)^n\frac{(2a-\alpha)t'}{4} = \infty,$$

which gives that

$$\lim_{n\to\infty}\mu_{F'(x)-J_nf(x)}(t) = 1$$

and then we have by (RN1)

$$F'(x) = \lim_{n\to\infty} J_nf(x) = F(x)$$

for all $x \in V$.

Case 2. Assume that φ satisfies the condition (ii) for some α with $2a < \alpha < 4a^2$ and $J_nf : V \to Y$ is a mapping defined by

$$J_nf(x) := \frac{f((2a)^nx) + f(-(2a)^nx)}{2\cdot(2a)^{2n}} + \frac{(2a)^n}{2}\Big[f\Big(\frac{x}{(2a)^n}\Big) - f\Big(\frac{-x}{(2a)^n}\Big)\Big]$$

for all $x \in V$. Then we have $J_0f(x) = f(x)$, $J_jf(0) = f(0)$ and

$$\begin{aligned}J_jf(x) - J_{j+1}f(x) = &-\frac{Df(-(2a)^jax, -(2a)^jx) - 3Df(0, (2a)^jx)}{2(2a)^{2j+2}} \\ &-\frac{Df((2a)^jax, (2a)^jx) - 3Df(0, -(2a)^jx)}{2(2a)^{2j+2}} \\ &+\frac{(2a)^j}{2}\Big[Df\Big(\frac{x}{2(2a)^j}, \frac{x}{(2a)^{j+1}}\Big) - 3Df\Big(0, \frac{-x}{(2a)^{j+1}}\Big)\Big] \\ &-\frac{(2a)^j}{2}\Big[Df\Big(\frac{-x}{2(2a)^j}, \frac{-x}{(2a)^{j+1}}\Big) - 3Df\Big(0, \frac{x}{(2a)^{j+1}}\Big)\Big]\end{aligned} \tag{2.7}$$

for all $x \in V$ and all $j \geq 0$. If $n+m > n \geq 0$, then we deduce that

$$\begin{aligned}
&\mu_{J_n f(x) - J_{n+m} f(x)}\left(\sum_{j=n}^{n+m-1}\Big(\frac{4}{(2a)^2}\Big(\frac{\alpha}{(2a)^2}\Big)^j + \frac{4}{\alpha}\Big(\frac{(2a)}{\alpha}\Big)^j\Big)t\right)\\
&= \mu_{\sum_{j=m}^{l+m-1}(J_j f(x) - J_{j+1} f(x))}\left(\sum_{j=n}^{n+m-1}\Big(\frac{4}{(2a)^2}\Big(\frac{\alpha}{(2a)^2}\Big)^j + \frac{4}{\alpha}\Big(\frac{(2a)}{\alpha}\Big)^j\Big)t\right)\\
&\geq \tau_M{}_{j=n}^{n+m-1}\Big\{\mu_{J_j f(x) - J_{j+1} f(x)}\Big(\Big(\frac{4}{(2a)^2}\Big(\frac{\alpha}{(2a)^2}\Big)^j + \frac{4}{\alpha}\Big(\frac{(2a)}{\alpha}\Big)^j\Big)t\Big)\Big\} \qquad (2.8)\\
&\geq \tau_M{}_{j=n}^{n+m-1}\Big\{\tau_M\Big\{\mu_{-\frac{Df((2a)^j ax,(2a)^j x)}{2(2a)^{2j+2}}}\Big(\frac{\alpha^j t}{2(2a)^{2j+2}}\Big), \mu_{\frac{-Df(-(2a)^j ax,-(2a)^j x)}{2(2a)^{2j+2}}}\Big(\frac{\alpha^j t}{2(2a)^{2j+2}}\Big),\\
&\quad \mu_{\frac{3Df(0,-(2a)^j x)}{2(2a)^{2j+2}}}\Big(\frac{3\alpha^j t}{2(2a)^{2j+2}}\Big), \mu_{\frac{3Df(0,(2a)^j x)}{2(2a)^{2j+2}}}\Big(\frac{3\alpha^j t}{2(2a)^{2j+2}}\Big), \mu_{\frac{(2a)^j}{2}Df\left(\frac{x}{2(2a)^j},\frac{x}{(2a)^{j+1}}\right)}\Big(\frac{(2a)^j t}{2\alpha^{j+1}}\Big),\\
&\quad \mu_{\frac{-3(2a)^j}{2}Df\left(0,\frac{-x}{(2a)^{j+1}}\right)}\Big(\frac{3(2a)^j t}{2\alpha^{j+1}}\Big), \mu_{\frac{-(2a)^j}{2}Df\left(\frac{-x}{2(2a)^j},\frac{-x}{(2a)^{j+1}}\right)}\Big(\frac{(2a)^j t}{2\alpha^{j+1}}\Big),\\
&\quad \mu_{\frac{3(2a)^j}{2}Df\left(0,\frac{x}{(2a)^{j+1}}\right)}\Big(\frac{3(2a)^j t}{2\alpha^{j+1}}\Big)\Big\}\Big\}\\
&\geq M(x,t)
\end{aligned}$$

for all $x \in V$ and all $t > 0$. Therefore the Cauchy sequence $\{J_n f(x)\}$ has the limit $F(x) := \lim_{n\to\infty} J_n f(x)$ for all $x \in V$ and

$$\mu_{f(x)-J_n f(x)}(t) \geq M\left(x, \frac{t}{\sum_{j=0}^{n-1}\left(\frac{4}{(2a)^2}\left(\frac{\alpha}{(2a)^2}\right)^j + \frac{4}{\alpha}\left(\frac{(2a)}{\alpha}\right)^j\right)}\right) \tag{2.9}$$

for all $x \in V$.

Now, to prove that $DF(x,y) = 0$ for all $x, y \in V$, we consider (2.6) in case 1. By virtue of (RN3) and (2.1), we see that

$$\begin{aligned}
\mu_{DJ_n f(x,y)}\Big(\frac{t}{2}\Big) &\geq \tau_M\Big\{\mu_{\frac{Df((2a)^n x,(2a)^n y)}{2\cdot(2a)^{2n}}}\Big(\frac{t}{8}\Big), \mu_{\frac{Df(-(2a)^n x,-(2a)^n y)}{2\cdot(2a)^{2n}}}\Big(\frac{t}{8}\Big),\\
&\quad \mu_{\frac{(2a)^n}{2}Df\left(\frac{x}{(2a)^n},\frac{y}{(2a)^n}\right)}\Big(\frac{t}{8}\Big), \mu_{-\frac{(2a)^n}{2}Df\left(\frac{-x}{(2a)^n},\frac{-y}{(2a)^n}\right)}\Big(\frac{t}{8}\Big)\Big\}\\
&\geq \tau_M\Big\{\mu_{\varphi(x,y)}\Big(\frac{(2a)^{2n}t}{4\alpha^n}\Big), \mu_{\varphi(-x,-y)}\Big(\frac{(2a)^{2n}t}{4\alpha^n}\Big),\\
&\quad \mu_{\varphi(x,y)}\Big(\frac{\alpha^n t}{4(2a)^n}\Big), \mu_{\varphi(-x,-y)}\Big(\frac{\alpha^n t}{4(2a)^n}\Big)\Big\}
\end{aligned}$$

for all $x, y \in V$ and all $t > 0$, which tends to 1 as $n \to \infty$. It implies that all the terms of (2.6) are equal to 1 as $n \to \infty$ and then we know that F is an additive-quadratic mapping.

Employing the same argument as in the proof of case 1, the second inequality in (2.2) follows from (2.9).

Finally, it remains to prove the uniqueness of F. Let us assume that $F' : V \to Y$ is another additive-quadratic mapping satisfying (2.2). Note that if F' is an additive-quadratic mapping then by (2.7)

$$F'(x) - J_nF'(x) = \sum_{j=0}^{n-1} \big(J_jF'(x) - J_{j+1}F'(x)\big) = 0$$

for all $x \in V$ and all $n \in \mathbb{N}$. This relation with (RN3) and (2.2) imply that

$$\begin{aligned}\mu_{F'(x)-J_nf(x)}(t) &= \mu_{J_nF'(x)-J_nf(x)}(t)\\ &\geq \tau_M\Big\{\mu_{\frac{(F'-f)((2a)^n x)}{2\cdot(2a)^{2n}}}\Big(\frac{t}{4}\Big), \mu_{\frac{(F'-f)(-(2a)^n x)}{2\cdot(2a)^{2n}}}\Big(\frac{t}{4}\Big),\\ &\quad \mu_{\frac{(2a)^n}{2}(F'-f)\left(\frac{x}{(2a)^n}\right)}\Big(\frac{t}{4}\Big), \mu_{\frac{(2a)^n}{2}(F'-f)\left(\frac{-x}{(2a)^n}\right)}\Big(\frac{t}{4}\Big)\Big\}\\ &\geq \tau_M\Big\{\sup_{t'<t}\Big\{M\Big(x, \frac{((2a)^2-\alpha)(2a-\alpha)t'}{2((2a)^2-2a)}\Big(\frac{\alpha}{2a}\Big)^n\Big),\\ &\quad \sup_{t'<t}\Big\{M\Big(x, \frac{((2a)^2-\alpha)(2a-\alpha)t'}{2((2a)^2-2a)}\Big)^n\Big)\Big\}\end{aligned}$$

for all $x \in V$ and all $n \in \mathbb{N}$. Due to the fact that

$$\lim_{n\to\infty} \frac{((2a)^2-\alpha)(2a-\alpha)t'}{2((2a)^2-2a)}\Big(\frac{(2a)^2}{\alpha}\Big)^n = \infty, \ \lim_{n\to\infty} \frac{((2a)^2-\alpha)(2a-\alpha)t'}{2((2a)^2-2a)}\Big(\frac{\alpha}{2a}\Big)^n = \infty$$

for $2a < \alpha < 4a^2$, we have

$$\lim_{n\to\infty} \mu_{F'(x)-J_nf(x)}(t) = 1.$$

Of course, by virtue of (RN1), we see that

$$F'(x) = \lim_{n\to\infty} J_nf(x) = F(x)$$

for all $x \in V$.

Case 3. Suppose that φ satisfies the condition (iii) for some α with $\alpha > (2a)^2$ and and $J_nf : V \to Y$ is a mapping defined by

$$J_nf(x) = \frac{(2a)^{2n} + (2a)^n}{2} f\Big(\frac{x}{(2a)^n}\Big) + \frac{(2a)^{2n} - (2a)^n}{2} f\Big(\frac{-x}{(2a)^n}\Big)$$

for all $x \in V$. Then we have $J_0f(x) = f(x)$ and

$$\begin{aligned}J_jf(x) - J_{j+1}f(x) = \frac{(2a)^{2j} + (2a)^j}{2}\Big[Df\Big(\frac{x}{2\cdot(2a)^j}, \frac{x}{(2a)^{j+1}}\Big) - 3Df\Big(0, \frac{-x}{(2a)^{j+1}}\Big)\Big]\\ + \frac{(2a)^{2j} - (2a)^j}{2}\Big[Df\Big(\frac{-x}{2\cdot(2a)^j}, \frac{-x}{(2a)^{j+1}}\Big) - 3Df\Big(0, \frac{x}{(2a)^{j+1}}\Big)\Big]\end{aligned} \tag{2.10}$$

for all $x \in V$ and all $j \geq 0$. Moreover, if $n+m > n \geq 0$, then we get the inequality

$$\begin{aligned}
&\mu_{J_n f(x) - J_{n+m} f(x)}\Big(\sum_{j=n}^{n+m-1} \Big(\Big(\frac{(2a)^2}{\alpha}\Big)^j \frac{4t}{\alpha} \Big) \Big) \\
&\geq \mu_{\sum_{j=n}^{n+m-1} (J_j f(x) - J_{j+1} f(x))}\Big(\sum_{j=n}^{n+m-1} \Big(\Big(\frac{(2a)^2}{\alpha}\Big)^j \frac{4t}{\alpha} \Big) \Big) \\
&\geq \tau_M{}_{j=n}^{n+m-1} \Big\{ \mu_{J_j f(x) - J_{j+1} f(x)} \Big(\Big(\frac{(2a)^2}{\alpha}\Big)^j \frac{4t}{\alpha} \Big) \Big\} \\
&\geq \tau_M{}_{j=n}^{n+m-1} \Big\{ \tau_M \Big\{ \mu_{\frac{(2a)^j((2a)^j+1)}{2} Df\left(\frac{x}{2(2a)^j}, \frac{x}{(2a)^{j+1}}\right)} \Big(\frac{(2a)^j((2a)^j+1)t}{2\alpha^{j+1}} \Big), \\
&\quad \mu_{\frac{-3(2a)^j((2a)^j+1)}{2} Df\left(0, \frac{-x}{(2a)^{j+1}}\right)} \Big(\frac{3(2a)^j((2a)^j+1)t}{2\alpha^{j+1}} \Big), \\
&\quad \mu_{\frac{(2a)^j((2a)^j-1)}{2} Df\left(\frac{-x}{2(2a)^j}, \frac{-x}{(2a)^{j+1}}\right)} \Big(\frac{(2a)^j((2a)^j-1)t}{2\alpha^{j+1}} \Big) \\
&\quad \mu_{\frac{3(2a)^j((2a)^j-1)}{2} Df\left(0, \frac{x}{(2a)^{j+1}}\right)} \Big(\frac{3(2a)^j((2a)^j-1)t}{2\alpha^{j+1}} \Big) \Big\} \Big\} \\
&\geq M(x,t)
\end{aligned}$$

for all $x \in V$ and all $t > 0$. And so we can define a mapping $F : V \to Y$ by $F(x) := \lim_{n\to\infty} J_n f(x)$ for all $x \in V$ and

$$\mu_{f(x) - J_n f(x)}(t) \geq M\left(x, \frac{t}{\sum_{j=0}^{n-1} \left(\frac{(2a)^2}{\alpha}\right)^j \frac{4}{\alpha}} \right) \tag{2.11}$$

for all $x \in V$. Note that for all $x, y \in V$ and all $t > 0$,

$$\begin{aligned}
\mu_{DJ_n f(x,y)}\Big(\frac{t}{2}\Big) &\geq \tau_M \Big\{ \mu_{\frac{(2a)^{2n}}{2} Df\left(\frac{x}{(2a)^n}, \frac{y}{(2a)^n}\right)} \Big(\frac{t}{8}\Big), \mu_{\frac{(2a)^{2n}}{2} Df\left(\frac{-x}{(2a)^n}, \frac{-y}{(2a)^n}\right)} \Big(\frac{t}{8}\Big), \\
&\quad \mu_{\frac{(2a)^n}{2} Df\left(\frac{x}{(2a)^n}, \frac{y}{(2a)^n}\right)} \Big(\frac{t}{8}\Big), \mu_{-\frac{(2a)^n}{2} Df\left(\frac{-x}{(2a)^n}, \frac{-y}{(2a)^n}\right)} \Big(\frac{t}{8}\Big) \Big\} \\
&\geq \tau_M \Big\{ \mu_{\varphi(x,y)} \Big(\frac{\alpha^n t}{4(2a)^{2n}} \Big), \mu_{\varphi(-x,-y)} \Big(\frac{\alpha^n t}{4(2a)^{2n}} \Big), \\
&\quad \mu_{\varphi(x,y)} \Big(\frac{\alpha^n t}{4(2a)^n} \Big), \mu_{\varphi(-x,-y)} \Big(\frac{\alpha^n t}{4(2a)^n} \Big) \Big\},
\end{aligned}$$

which tends to 1 as $n \to \infty$. Therefore we can show that F is an additive-quadratic mapping by using the similar fashion after (2.6).

By the same reasoning as in the proof of case 1, the relation (2.2) yields the third inequality in (2.11).

To complete the proof of the theorem, we are enough to show the uniqueness of F. Suppose that $F' : V \to Y$ is another mapping satisfying the third inequality in (2.2). If g is an additive-quadratic mapping, then, by (2.9), we have $g(x) = J_n g(x)$ for all $x \in V$

and all $n \in \mathbb{N}$. Observe that

$$\begin{aligned}
\mu_{F(x)-F'(x)}(t) &= \mu_{J_nF(x)-J_nF'(x)}(t) \\
&\geq \tau_M\Big\{\mu_{J_nF(x)-J_nf(x)}\Big(\frac{t}{2}\Big), \mu_{J_nf(x)-J_nF'(x)}\Big(\frac{t}{2}\Big)\Big\} \\
&\geq \tau_M\Big\{\mu_{\frac{(2a)^{2n}}{2}(F-f)\left(\frac{x}{(2a)^n}\right)}\Big(\frac{t}{8}\Big), \mu_{\frac{(2a)^{2n}}{2}(f-F')\left(\frac{x}{(2a)^n}\right)}\Big(\frac{t}{8}\Big), \\
&\quad \mu_{\frac{(2a)^{2n}}{2}(F-f)\left(\frac{-x}{(2a)^n}\right)}\Big(\frac{t}{8}\Big), \mu_{\frac{(2a)^{2n}}{2}(f-F')\left(\frac{-x}{(2a)^n}\right)}\Big(\frac{t}{8}\Big), \\
&\quad \mu_{\frac{(2a)^{n}}{2}(F-f)\left(\frac{x}{(2a)^n}\right)}\Big(\frac{t}{8}\Big), \mu_{\frac{(2a)^{n}}{2}(f-F')\left(\frac{x}{(2a)^n}\right)}\Big(\frac{t}{8}\Big), \\
&\quad \mu_{\frac{(2a)^{n}}{2}(F-f)\left(\frac{-x}{(2a)^n}\right)}\Big(\frac{t}{8}\Big), \mu_{\frac{(2a)^{n}}{2}(f-F')\left(\frac{-x}{(2a)^n}\right)}\Big(\frac{t}{8}\Big)\Big\} \\
&\geq \tau_M\Big\{\sup_{t'<t}\Big\{M\Big(x, \frac{(\alpha-n^2)t'}{4}\Big(\frac{\alpha}{n}\Big)^m\Big), \sup_{t'<t}\Big\{M\Big(x, \frac{(\alpha-n^2)t'}{4}\Big(\frac{\alpha}{(2a)^2}\Big)^n\Big)\Big\}\Big\}
\end{aligned}$$

for all $x \in V$ and all $n \in \mathbb{N}$. Since $\alpha > (2a)^2$, the last term in (2.6) tends to 1 as $n \to \infty$ by (RN3) and $F(0) = F'(0)$. Therefore $F = F'$. □

Corollary 2.3. *Let X and Y be a vector space and a complete normed space, respectively. Suppose that $f : X \to Y$ is a mapping with $f(0) = 0$ for which there is $\varphi : X^2 \to \mathbb{R}$ such that*

$$\|Df(x, y)\| \leq \varphi(x, y) \tag{2.12}$$

for all $x, y \in X$. If φ satisfies one of the following conditions:

(i) $\alpha\varphi(x, y) \geq \varphi(2ax, 2ay)$ *for some* $0 < \alpha < 2a$,
(ii) $\varphi(2ax, 2ay) \geq \alpha\varphi(x, y) \geq \varphi(4a^2x, 4a^2y)$ *for some* $2a < \alpha < 4a^2$,
(iii) $\varphi(4a^2x, 4a^2y) \geq \alpha\varphi(x, y)$ *for some* $4a^2 < \alpha$

for all $x, y \in X$, then there exists a unique additive-quadratic mapping $F : X \to Y$ such that

$$\|f(x) - F(x)\| \leq \begin{cases} \frac{\Phi(x)}{2a-\alpha} & \text{if } \varphi \text{ satisfies (i)}, \\ \frac{(4a^2-2a)\Phi(x)}{(4a^2-\alpha)(\alpha-2a)} & \text{if } \varphi \text{ satisfies (ii)}, \\ \frac{\Phi(x)}{\alpha-4a^2} & \text{if } \varphi \text{ satisfies (iii)} \end{cases} \tag{2.13}$$

for all $x, y \in X$, where

$$\Phi(x) = \max\{\varphi(ax, x), \varphi(-ax, -x), \varphi(0, x), \varphi(0, -x)\}.$$

Proof. Let (Y, μ, τ_M) and $(\mathbb{R}, \mu', \tau_M)$ be the induced random normed RN-spaces. Then the inequality

$$\mu_{Df(x,y)}(t) \geq \mu'_{\varphi(x,y)}(t)$$

follows from the inequality (2.12) and φ satisfies one of the conditions in Theorem 2.2. So there exists a unique additive-quadratic mapping $F : X \to Y$ satisfying (2.13). □

From Corollary 2.3, we can obtain the following result.

Corollary 2.4. *Let X be a normed space and let $p \neq 1, 2$ be a positive real number. If a mapping $f : X \to Y$ satisfies the inequality*

$$\|Df(x,y)\| \leq \theta\big(\|x\|^p + \|y\|^p\big)$$

for all $x, y \in X$ and for some $\theta \geq 0$, then there exists a unique additive-quadratic mapping $F : X \to Y$ such that

$$\|f(x) - F(x)\| \leq \begin{cases} \frac{2\theta\|x\|^p}{2a-(2a)^p} & \text{if } p < 1, \\ \frac{2\theta\|x\|^p}{(2a)^p-2a} + \frac{2\theta\|x\|^p}{4a^2-(2a)^p} & \text{if } 1 < p < 2, \\ \frac{2\theta\|x\|^p}{(2a)^p-4a^2} & \text{if } p > 2 \end{cases}$$

for all $x \in X$.

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Ick-Soon Chang*, Department of Mathematics, Chungnam National University, 79 Daehangno, Yuseong-gu, Daejeon 305-764, Republic of Korea
E-mail address: ischang@cnu.ac.kr

Yang-Hi Lee, Department of Mathematics Education,, Gongju National University of Education, Gongju 314-711, Republic of Korea
E-mail address: yanghi2@hanmail.net

Blow-up of solutions for a vibrating riser equation with dissipative term

Junping Zhao
College of Science, Xi'an University of Architecture & Technology, Xi'an 710055, China.
e-mail: junpingzhao@yeah.net

Abstract: In this paper we consider a vibrating riser equation with dissipative term and the homogeneous Dirichlet boundary condition. By developing the method in [9] and [16], we establish a blow-up result for certain solutions with non-positive initial energy as well as positive initial energy. Estimates of the lifespan of solutions are also given.

Keywords: Blow-up of solution, quasilinear riser problem, positive initial energy

AMS Subject Classification (2000): 35L70, 35L15

1 Introduction and main result

In this paper we consider the problem

$$\begin{cases} u_{tt}+pu_t+2qu_{xxxx}-2[(ax+b)u_x]_x+\dfrac{q}{3}(u_x^3)_{xxx} \\ \qquad -[(ax+b)u_x^3]_x-q(u_{xx}^2u_x)_x=f(u), \quad (x,t)\in[0,1]\times(0,T), \\ u(0,t)=u(1,t)=u_{xx}(0,t)=u_{xx}(1,t)=0, \quad t\in(0,T), \\ u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), \qquad x\in[0,1], \end{cases} \tag{1.1}$$

where a,b,p,q are nonnegative constant, $f(u)$ is a $C(R)$ function satisfying some conditions to be special later.

Problem (1.1) models the behavior of a riser vibrating due to effects of waves and current [14]. In 1997, Bayrack and Can [1] studied problem (1.1) and proved that, under suitable conditions on f and the initial data, all solutions of (1.1) blow up in finite time in the L^2 space. To establish their result, the authors used the standard concavity method due to [7]. Gmira and Guedda [4] extended the result of [1] to the multi-dimensional version of the problem (1.1) by using the modified concavity method introduced in [6].

More recently, Hao et al. [5] discussed (1.1) and showed that, under suitable conditions, the solution blows up in finite time *with a negative initial energy* while exists globally with a nonnegative initial energy for the case $p=0$. Precisely, the following blow-up result was established.

Theorem 1 *Let $u(x,t)$ be a classical solution of the system (1.1). Assume that there exists a positive constant A such that the function $f(s)$ satisfies*

$$sf(s)\geq(4+A)\int_0^s f(\upsilon)\mathrm{d}\upsilon \quad for\ s\in R, \tag{1.2}$$

and the initial values satisfy

$$E(0) = \frac{1}{2}\|u_1\|_2^2 + q\|u_{0xx}\|_2^2 + \int_0^1 (ax+b)u_{0x}^2 \mathrm{d}x + \frac{q}{2}\|u_{0x}u_{0xx}\|_2^2 + \frac{1}{4}\int_0^1 (ax+b)u_{0x}^4 \mathrm{d}x - \int_0^1 \int_0^{u_0} f(\upsilon)\mathrm{d}\upsilon \mathrm{d}x < 0 \tag{1.3}$$

and

$$\int_0^1 u_0 u_1 \mathrm{d}x > 0. \tag{1.4}$$

Then the solution $u(x,t)$ of the system (1.1) blows up in a finite time.

In the present paper, we shall improve the results of [5] and derive the blow-up properties of solutions of problem (1.1) *with non-positive initial energy as well as positive initial energy* by developing the method in [9] and [16] (see Remark 2). Estimates of the lifespan of solutions will also be given. For the convenience of our computation, we set $p = q = 1$ and $f(s) = |s|^{r-1}s$. Then the condition (1.2) holds when $r > 4$.

We define the energy function for the solution u of (1.1) by

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + \|u_{xx}\|_2^2 + \int_0^1 (ax+b)u_x^2 \mathrm{d}x + \frac{1}{2}\|u_x u_{xx}\|_2^2 + \frac{1}{4}\int_0^1 (ax+b)u_x^4 \mathrm{d}x - \frac{1}{r}\|u\|_r^r. \tag{1.5}$$

Then

$$E'(t) = -\|u_t\|_2^2 \le 0, \quad \text{for } t \ge 0, \tag{1.6}$$

and

$$E(t) = E(0) - \int_0^t \|u_\tau(\tau)\|_2^2 \mathrm{d}\tau, \quad t \ge 0. \tag{1.7}$$

We also set

$$\alpha_1 = \left(\frac{2}{B^r}\right)^{\frac{1}{r-2}}, \quad E_1 = \frac{r-2}{r}\alpha_1^2 = \left(\frac{1}{2} - \frac{1}{r}\right) B^r \alpha_1^r. \tag{1.8}$$

where B is the optimal constant of the embedding inequality

$$\|u\|_r \le B\|u_{xx}\|_2, \quad u \in H^2([0,1]) \cap H_0^1([0,1]), \tag{1.9}$$

for $2 < r < +\infty$, that is

$$B^{-1} = \inf_{u \in H^2([0,1]) \cap H_0^1([0,1]), u \ne 0} \frac{\|u_{xx}\|_2}{\|u\|_r}.$$

We introduce the functionals

$$a(t) = \int_0^1 u^2 \mathrm{d}x + \int_0^t \int_0^1 u^2 \mathrm{d}x\mathrm{d}t, \ t \ge 0 \tag{1.10}$$

and

$$G(t) = [a(t) + (T_1 - t)\|u_0\|_2^2]^{-\delta}, \quad t \in [0, T_1], \tag{1.11}$$

where $\delta = \frac{r-2}{4}$ and $T_1 > 0$ is a certain constant to be specified later.

Our main result reads as follows.

Theorem 2 *Let $u(x,t)$ be a classical solution of the system (1.1). Assume that $r > 4$ and either one of the following four conditions is satisfied:*

1. $E(0) < 0$,
2. $E(0) = 0$ *and* $\int_0^1 u_0u_1\mathrm{d}x > 0$,
3. $0 < E(0) < E_1$ *and* $\|u_{0xx}\|_2 > \alpha_1$,
4. $E_1 \le E(0) < \min\left\{\frac{r+2}{2r}\left[\left(1+\sqrt{\frac{r-2}{r+2}}\right)\int_0^1 u_0u_1\mathrm{d}x - 2\|u_0\|_2^2\right],\ \frac{\left(\int_0^1 u_0u_1\mathrm{d}x\right)^2}{2(1+T_1)\|u_0\|_2^2}\right\}$.

Then the solution u of the problem (1.1) blows up in a finite time T^* in the sense of (2.25). Moreover, the upper bounds for T^* can be estimated according to the sign of $E(0)$:

For the case 1,

$$T^* \le t_0 - \frac{G(t_0)}{G'(t_0)}.$$

Furthermore, if $G(t_0) < \min\{1, \sqrt{\frac{\alpha}{-\beta}}\}$, then

$$T^* \le t_0 + \frac{1}{\sqrt{-\beta}}\ln\frac{\sqrt{\frac{\alpha}{-\beta}}}{\sqrt{\frac{\alpha}{-\beta}} - G(t_0)}.$$

For the case 2,

$$T^* \le -\frac{G(0)}{G'(0)} = \frac{2(T_1 - t + 1)\|u_0\|_2^2}{(r-2)\int_0^1 u_0u_1\mathrm{d}x} \quad or \quad T^* \le \frac{G(0)}{\sqrt{\alpha}}.$$

For the case 3,

$$T^* \le t_0 - \frac{G(t_0)}{G'(t_0)}.$$

Furthermore, if $G(t_0) < \min\{1, \sqrt{\frac{\alpha'}{-\beta'}}\}$, then

$$T^* \le t_0 + \frac{1}{\sqrt{-\beta'}}\ln\frac{\sqrt{\frac{\alpha'}{-\beta'}}}{\sqrt{\frac{\alpha'}{-\beta'}} - G(t_0)}.$$

For the case 4,

$$T^* \le 2^{(3\delta+1)/2\delta}\frac{\delta c}{\sqrt{\alpha}}\{1 - [1 + cG(0)]^{-1/2\delta}\}.$$

where $c = (\alpha/\beta)^{2+1/\delta}$. Here α, β, α' and β' are given in (2.23), (2.24), (2.27) and (2.28), respectively. And $t_0 = t^*$ is given by (2.12) for the case 1 and $t_0 = t_1^*$ is given by (2.13) for the case 3.

Remark 1 *Compared with Theorem 1, we have no the restriction $\int_0^1 u_0u_1\mathrm{d}x > 0$ in Theorem 2 when $E(0) < 0$.*

Remark 2 *E_1 defined in (1.8) is exactly the potential well depth obtained by Payne and Sattinger (see [13]). In [16], a global nonexistence theorem for abstract evolution equations with* *nonlinear damping terms was proved by combining the arguments in [3] and [8], where positive initial energy less than E_1 was demanded while we allow here a larger positive initial energy (see the case 4). In this work, we divide the case $E(0) > 0$ into two cases: the case 3 and 4. Unlike [9], we discuss cautiously the case 3 by combining the method of [16] (see Lemma 7). We also note that the case 4 is allowed here since the damping term involved in problem (1.1) is linear.*

There are many related works on the existence and non-existence of global solutions to the hyperbolic equations with dissipative terms and damping terms, please see [2, 11, 12, 15] and the references therein.

2 Blow-up of the solutions

In this section, we shall prove Theorem 2. We start with a series of Lemmas.

Lemma 3 *Suppose $u(x,t)$ is a classical solution of the system (1.1). Assume that $E(0) < E_1$ and $\|u_{0xx}\|_2 > \alpha_1$. Then there exists a positive constant $\alpha_2 > \alpha_1$, such that*

$$\|u_{xx}(\cdot,t)\|_2 \geq \alpha_2, \quad \forall\ t \geq 0, \tag{2.1}$$

and

$$\|u(\cdot,t)\|_r \geq B\alpha_2, \quad \forall\ t \geq 0. \tag{2.2}$$

Proof. The idea follows from [16] where different type of equations were discussed. We first note that, by (1.5) and (1.9),

$$E(t) \geq \|u_{xx}\|_2^2 - \frac{1}{r}\|u\|_r^r \geq \|u_{xx}\|_2^2 - \frac{1}{r}B^r\|u_{xx}\|_2^r = \alpha^2 - \frac{1}{r}B^r\alpha^r := g(\alpha), \tag{2.3}$$

where $\alpha = \|u_{xx}\|_2$. It is easy to verify that g is increasing for $0 < \alpha < \alpha_1$, decreasing for $\alpha > \alpha_1$; $g(\alpha) \to -\infty$ as $\alpha \to +\infty$ and $g(\alpha_1) = E_1$, where α_1 is given in (1.8). Since $E(0) < E_1$, there exists $\alpha_2 > \alpha_1$ such that $g(\alpha_2) = E(0)$. Let $\alpha_0 = \|u_{0xx}\|_2$, then by (2.3) we have $g(\alpha_0) \leq E(0) = g(\alpha_2)$, which implies that $\alpha_0 \geq \alpha_2$.

To establish (2.1), we suppose by contradiction that $\|u_{xx}(t_0)\|_2 < \alpha_2$ for some $t_0 > 0$. By the continuity of $\|u_{xx}(\cdot,t)\|_2$ we can choose t_0 such that $\|u_{xx}(t_0)\|_2 > \alpha_1$. It follows from (2.3) that

$$E(t_0) \geq g(\|u_{xx}(t_0)\|_2) > g(\alpha_2) = E(0).$$

This is impossible since $E(t) \leq E(0)$ for all $t \geq 0$. Hence (2.1) is established.

To prove (2.2), we exploit (1.5) to see that

$$\|u_{xx}\|_2^2 \leq E(0) + \frac{1}{r}\|u\|_r^r.$$

Consequently,

$$\frac{1}{r}\|u\|_r^r \geq \|u_{xx}\|_2^2 - E(0) \geq \alpha_2^2 - E(0) \geq \alpha_2^2 - g(\alpha_2) = \frac{1}{r}B^r\alpha_2^r. \tag{2.4}$$

Therefore (2.2) is concluded.

Lemma 4 *[9] Let $\delta > 0$ and $B(t) \in C^2(0,\infty)$ be a nonnegative function satisfying*

$$B''(t) - 4(\delta+1)B'(t) + 4(\delta+1)B(t) \geq 0. \tag{2.5}$$

If

$$B'(0) > r_2 B(0) + k_0, \tag{2.6}$$

then $B'(t) > k_0$ for $t > 0$, where $r_2 = 2(\delta+1) - 2\sqrt{(\delta+1)\delta}$ is the smallest root of the equation

$$r^2 - 4(\delta+1)r + 4(\delta+1) = 0.$$

Lemma 5 *[9] If $G(t)$ is a non-increasing function on $[t_0, +\infty), t_0 \geq 0$ and satisfies the differential inequality*

$$G'(t)^2 \geq a + bG(t)^{2+\frac{1}{\delta}}, \quad \text{for } t \geq 0, \tag{2.7}$$

where $a > 0, \delta > 0$ and $b \in R$, then there exists a finite time T^* such that

$$\lim_{t\to T^{*-}} G(t) = 0$$

and the upper bound of T^* is estimated respectively by the following cases:

(i) If $b < 0$ and $G(t_0) < \min\{1, \sqrt{\frac{a}{-b}}\}$, then

$$T^* \leq t_0 + \frac{1}{\sqrt{-b}} \ln \frac{\sqrt{\frac{a}{-b}}}{\sqrt{\frac{a}{-b}} - G(t_0)}.$$

(ii) If $b = 0$, then

$$T^* \leq t_0 + \frac{G(t_0)}{\sqrt{a}}.$$

(iii) If $b > 0$, then

$$T^* \leq t_0 + 2^{(3\delta+1)/2\delta} \frac{\delta c}{\sqrt{a}} \{1 - [1 + cG(t_0)]^{-1/2\delta}\},$$

where $c = (a/b)^{2+1/\delta}$.

Lemma 6 *Assume that $r > 4$, $a(t)$ is defined by (1.10) and let u be a solution of (1.1), then we have*

$$a''(t) - 4(\delta+1)\|u_t\|_2^2 \geq Q_1(t), \tag{2.8}$$

where

$$Q_1(t) = (-4-8\delta)E(0) + 2r\int_0^t \|u_\tau\|_2^2 \mathrm{d}\tau + 2(r-2)\|u_{xx}\|_2^2.$$

Proof. By the definition of $a(t)$, we have

$$a'(t) = 2\int_0^1 uu_t \mathrm{d}x + \int_0^1 u^2 \mathrm{d}x, \tag{2.9}$$

and by (1.1) and the divergence theorem we get

$$\begin{aligned} a''(t) &= 2\int_0^1 u_t^2 \mathrm{d}x + 2\int_0^1 uu_{tt}\mathrm{d}x + 2\int_0^1 uu_t \mathrm{d}x\mathrm{d}x \\ &= 2\int_0^1 u_t^2 \mathrm{d}x + 2\int_0^1 u\left(|u|^{r-2}u + 2[(ax+b)u_x]_x + [(ax+b)u_x^3]_x \right. \\ &\qquad \left. +(u_{xx}^2 u_x)_x - 2u_{xxxx} - \frac{1}{3}(u_x^3)_{xxx}\right)\mathrm{d}x \\ &= 2\|u_t\|_2^2 + 2\|u\|_r^r - 4\int_0^1 (ax+b)u_x^2 \mathrm{d}x - 2\int_0^1 (ax+b)u_x^4 \mathrm{d}x - 4\|u_x u_{xx}\|_2^2 - 4\|u_{xx}\|_2^2 \end{aligned} \tag{2.10}$$

Using (1.5) and (1.7) we get

$$\begin{aligned} & a''(t) - 4(\delta+1)\|u_t\|_2^2 \\ =\ & a''(t) - 2\|u_t\|_2^2 - \frac{1}{2}(8\delta+4)\|u_t\|_2^2 \\ =\ & 2\|u\|_r^r - 4\int_0^1 (ax+b)u_x^2 \mathrm{d}x - 2\int_0^1 (ax+b)u_x^4 \mathrm{d}x - 4\|u_x u_{xx}\|_2^2 - 4\|u_{xx}\|_2^2 \\ & -2r\left(E(0) - \int_0^t \|u_\tau\|_2^2 \mathrm{d}\tau - \|u_{xx}\|_2^2 - \int_0^1 (ax+b)u_x^2 \mathrm{d}x - \frac{1}{2}\|u_x u_{xx}\|_2^2 \right. \\ & \left. -\frac{1}{4}\int_0^1 (ax+b)u_x^4 \mathrm{d}x + \frac{1}{r}\|u\|_r^r\right) \\ \geq\ & (-4-8\delta)E(0) + 2r\int_0^t \|u_\tau\|_2^2 \mathrm{d}\tau + 2(r-2)\|u_{xx}\|_2^2 + 2(r-2)\int_0^1 (ax+b)u_x^2 \mathrm{d}x \\ & +\frac{1}{2}(r-4)\int_0^1 (ax+b)u_x^4 \mathrm{d}x + (r-4)\|u_x u_{xx}\|_2^2 \\ \geq\ & (-4-8\delta)E(0) + 2r\int_0^t \|u_\tau\|_2^2 \mathrm{d}\tau + 2(r-2)\|u_{xx}\|_2^2 \end{aligned} \tag{2.11}$$

since $r > 4$.

Lemma 7 *Assume that $r > 4$ and that either one of the following is satisfied:*

1. $E(0) < 0$,
2. $E(0) = 0$ *and* $\int_0^1 u_0 u_1 \mathrm{d}x > 0$,
3. $0 < E(0) < E_1$ *and* $\|u_{0xx}\|_2 > \alpha_1$,
4. $E_1 \leq E(0) < \frac{r+2}{2r}\left[\left(1+\sqrt{\frac{r-2}{r+2}}\right)\int_0^1 u_0 u_1 \mathrm{d}x - 2\|u_0\|_2^2\right]$.

Then $a'(t) > \|u_0\|_2^2$ for $t > t_0$, where $t_0 = t^*$ is given by (2.12) for the case 1, $t_0 = 0$ for the cases 2 and 4, and $t_0 = t_1^*$ is given by (2.14) for the case 3.

Proof. We consider different cases on the sign of the initial energy $E(0)$.

1. If $E(0) < 0$, then from (2.8), we have

$$a'(t) \geq a'(0) - 4(1+2\delta)E(0)t, \quad t \geq 0.$$

Thus $a'(t) > \|u_0\|_2^2$ for $t > t^*$, where

$$t^* = \max\left\{\frac{a'(0) - \|u_0\|_2^2}{4(1+2\delta)E(0)},\ 0\right\} = \max\left\{\frac{\int_0^1 u_0 u_1 \mathrm{d}x}{2(1+2\delta)E(0)},\ 0\right\}. \tag{2.12}$$

2. If $E(0) = 0$, then $a''(t) \geq 0$ for $t \geq 0$. Furthermore, if $a'(0) > \|u_0\|_2^2$ (i.e., $\int_0^1 u_0 u_1 \mathrm{d}x > 0$), then $a'(t) > \|u_0\|_2^2$, $t \geq 0$.

3. If $0 < E(0) < E_1$, then using Lemma 3 and (1.8) we see that

$$\begin{aligned} Q_1(t) &\geq -(4+8\delta)E(0) + 2(r-2)\alpha_2^2 \\ &> (4+8\delta)(-E(0)+E_1) := C_1 > 0, \quad t > 0. \end{aligned} \tag{2.13}$$

Thus, from (2.8), we have

$$a''(t) \geq Q_1(t) > C_1 > 0, \quad t > 0.$$

Hence $a'(t) > \|u_0\|_2^2$ for $t > t_1^*$, where

$$t_1^* = \max\left\{\frac{\|u_0\|_2^2 - a'(0)}{C_1},\ 0\right\} = \max\left\{\frac{-2\int_0^1 u_0 u_1 \mathrm{d}x}{C_1},\ 0\right\}. \tag{2.14}$$

4. If $E(0) \geq E_1$, we first note

$$\int_0^1 u^2 \mathrm{d}x - \int_0^1 u_0^2 \mathrm{d}x = 2\int_0^t \int_0^1 u u_t \mathrm{d}x\mathrm{d}t. \tag{2.15}$$

By the Hölder inequality and Young inequality, we have

$$\int_0^1 u^2 \mathrm{d}x \leq \int_0^1 u_0^2 \mathrm{d}x + \int_0^t \|u\|_2^2 \mathrm{d}t + \int_0^t \|u_\tau\|_2^2 \mathrm{d}\tau.$$

By the Hölder inequality, Young inequality again, and (2.15), it follows from (2.9) that

$$a'(t) \leq a(t) + \int_0^1 u_0^2 \mathrm{d}x + \int_0^1 u_t^2 \mathrm{d}x + \int_0^t \|u_\tau\|_2^2 \mathrm{d}\tau. \tag{2.16}$$

In view of (2.8) and (2.16), we obtain

$$\begin{aligned} &a''(t) - 4(\delta+1)a'(t) + 4(\delta+1)a(t) + K_1 \\ &\geq a''(t) + 4(\delta+1)\left(-\|u_0\|_2^2 - \|u_t\|_2^2 - \int_0^t \|u_\tau\|_2^2 \mathrm{d}\tau\right) + K_1 \\ &\geq (-4-8\delta)E(0) + 2r\int_0^t \|u_\tau\|_2^2 \mathrm{d}\tau + 2(r-2)\|u_{xx}\|_2^2 - 4(\delta+1)\|u_0\|_2^2 - 4(\delta+1)\int_0^t \|u_\tau\|_2^2 \mathrm{d}\tau + K_1 \\ &\geq 4\delta\int_0^t \|u_\tau\|_2^2 \mathrm{d}\tau + 2(r-2)\|u_{xx}\|_2^2 \geq 0, \end{aligned}$$

where

$$K_1 = (4+8\delta)E(0) + 4(\delta+1)\|u_0\|_2^2.$$

Let

$$b(t) = a(t) + \frac{K_1}{4(1+\delta)}, \quad t > 0.$$

Then $b(t)$ satisfies (2.5). By (2.6), we see that if

$$a'(0) > r_2\left(a(0) + \frac{K_1}{4(1+\delta)}\right) + \|u_0\|_2^2, \tag{2.17}$$

i.e.,

$$E(0) < \frac{r+2}{2r}\left[\left(1+\sqrt{\frac{r-2}{r+2}}\right)\int_0^1 u_0u_1\mathrm{d}x - 2\|u_0\|_2^2\right],$$

then $a'(t) > \|u_0\|_2^2,\ t > 0$. The proof is completed.

Hereafter, we will find an estimate for the life span of $a(t)$ and prove Theorem 2.

Proof of Theorem 2. By the definition of $G(t)$, we have

$$G'(t) = -\delta G(t)^{1+1/\delta}(a'(t) - \|u_0\|_2^2)$$

$$G''(t) = -\delta G^{1+2/\delta}(t)V(t), \tag{2.18}$$

where

$$V(t) = a''(t)[a(t) + (T_1 - t)\|u_0\|_2^2] - (1+\delta)(a'(t) - \|u_0\|_2^2)^2. \tag{2.19}$$

For simplicity of calculation, we denote

$$P = \|u\|_2^2, \quad Q = \int_0^t \|u\|_2^2\mathrm{d}t, \quad R = \|u_t\|_2^2, \quad S = \int_0^t \|u_\tau\|_2^2\mathrm{d}\tau.$$

From (2.9), (2.15) and the Hölder inequality, we get

$$a'(t) \le 2\left(\sqrt{PR} + \sqrt{QS}\right) + \int_0^1 u_0^2\mathrm{d}x. \tag{2.20}$$

For the case 1 and 2, it follows from (2.8) that

$$a''(t) \ge (-4 - 8\delta)E(0) + 4(1+\delta)(R+S). \tag{2.21}$$

Applying (2.20) and (2.21), it yields

$$V(t) \ge [(-4-8\delta)E(0) + 4(1+\delta)(R+S)][a(t) + (T_1 - t)\|u_0\|_2^2] - 4(1+\delta)\left(\sqrt{PR} + \sqrt{QS}\right)^2.$$

Applying (1.11) and (1.10), it follows

$$\begin{aligned} V(t) &\ge (-4-8\delta)E(0)G^{-1/\delta}(t) + 4(1+\delta)(R+S)(T_1 - t)\|u_0\|_2^2 \\ &\quad + 4(1+\delta)\left[(R+S)(P+Q) - \left(\sqrt{PR} + \sqrt{QS}\right)^2\right] \\ &\ge (-4-8\delta)E(0)G^{-1/\delta}(t). \end{aligned}$$

In view of (2.18) we have

$$G''(t) \le \delta(4+8\delta)E(0)G^{1+1/\delta}(t), \quad t \ge t_0. \tag{2.22}$$

Note that by Lemma 7, $G'(t) < 0$ for $t > t_0$. Multiplying (2.22) by $G'(t)$ and integrating it from t_0 to t, we obtain

$$G'(t)^2 \ge \alpha + \beta G^{2+1/\delta}(t), \quad \text{for} \quad t \ge t_0,$$

where

$$\alpha = \delta^2 G(t_0)^{2+2/\delta}\left[(a'(t_0) - \|u_0\|_2^2)^2 - 8E(0)G^{-1/\delta}(t_0)\right] > 0 \tag{2.23}$$

and

$$\beta = 8\delta^2 E(0). \tag{2.24}$$

Then by Lemma 5, there exists a finite time T^* such that $\lim_{t\nearrow T^*-} G(t) = 0$. Therefore

$$\lim_{t\nearrow T^*-}\left(\int_0^1 u^2 \mathrm{d}x + \int_0^t\int_0^1 u^2 \mathrm{d}x\mathrm{d}t\right) = \infty. \tag{2.25}$$

For the case 3: $0 < E(0) < E_1$, it follows from (2.8) and (2.13) that

$$a''(t) \geq (-4-8\delta)E(0) + 2(r-2)\|u_{xx}\|_2^2 + 4(1+\delta)(R+S) > C_1 + 4(1+\delta)(R+S). \tag{2.26}$$

Then using the same arguments as in (1), we have

$$G''(t) \leq -\delta C_1 G^{1+1/\delta}(t),\ G'(t)^2 \geq \alpha' + \beta' G^{2+1/\delta}(t), \quad t \geq t_0,$$

where

$$\alpha' = \delta^2 G^{2+2/\delta}(t_0)\left[(a'(t_0) - \|u_0\|_2^2)^2 + \frac{2C_1}{1+2\delta}G^{-1/\delta}(t_0)\right] > 0 \tag{2.27}$$

and

$$\beta' = -\frac{2C_1\delta^2}{1+2\delta}. \tag{2.28}$$

Then by Lemma 5, there exists a finite time T^* such that (2.25) holds.

For the case 4: $E(0) \geq E_1$, applying the same discussion as in the case 1, we may get the equalities (2.23) and (2.24) under the condition

$$E(0) < \frac{(a'(t_0) - \|u_0\|_2^2)^2}{8a(t_0) + 8(T_1 - t_0)\|u_0\|_2^2} = \frac{\left(\int_0^1 u_0 u_1 \mathrm{d}x\right)^2}{2(1+T_1)\|u_0\|_2^2}.$$

Then by Lemma 5, there exists a finite time T^* such that (2.25) holds.

Remark 3 *The choice of T_1 in (1.11) is possible provided that $T_1 \geq T^*$.*

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Existence, uniqueness and asymptotic behavior of solutions for a fourth-order degenerate pseudo-parabolic equation with $p(x)$-growth conditions

Junping Zhao
College of Science, Xi'an University of Architecture & Technology, Xi'an 710055, China.
e-mail: junpingzhao@yeah.net

Abstract: In this paper, we consider an initial-boundary value problem for a fourth order degenerate pseudo-parabolic equation with $p(x)$-growth conditions. Under some assumptions on the initial value, we establish the existence of weak solutions by the time-discrete method. The uniqueness and asymptotic behavior of solutions are also discussed.

Keywords: Existence, asymptotic behavior, pseudo-parabolic equation

AMS Subject Classification (2000): 35G25, 35Q99, 35K55, 35K70.

1 INTRODUCTION

This paper is concerned with a fourth order degenerate pseudo-parabolic equation with $p(x)$-growth conditions

$$\frac{\partial u}{\partial t} - k\frac{\partial \Delta u}{\partial t} + \triangle(|\triangle u|^{p(x)-2}\triangle u) = 0, \quad x \in \Omega,\ t > 0, \tag{1.1}$$

with boundary condition

$$u = \triangle u = 0, \quad x \in \partial\Omega,\ t > 0, \tag{1.2}$$

and initial condition

$$u(x,0) = u_0(x), \quad x \in \Omega. \tag{1.3}$$

Here $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $p(x)$ is a function defined on $\bar{\Omega}$ and $k > 0$ is the viscosity coefficient. The term $k\frac{\partial \Delta u}{\partial t}$ in (1.1) is interpreted as due to viscous relaxation effects, or viscosity.

Equation (1.1) arises as a regularization of the pseudo-parabolic equation

$$\frac{\partial u}{\partial t} - k\frac{\partial \Delta u}{\partial t} = \Delta u, \tag{1.4}$$

which arises in various physical phenomena. (1.4) can be assumed as a model for diffusion of fluids in fractured porous media [1, 5, 6], or as a model for heat conduction involving a thermodynamic temperature $\theta = u - k\Delta u$ and a conductive temperature u [4, 13]. In [2], Bernis investigates a class of higher order parabolic with degeneracy depending on both the unknown functions and its derivatives, the fourth order case of which is the equation

$$\frac{\partial}{\partial t}(|u|^{q-1} sgnu) + D^2(|D^2u|^{p-1} sgnD^2u) = f \tag{1.5}$$

where $p > 1$, $q > 1$ are constants. Some existence result of energy solutions was proved by energy method (see also [12, 17]).

Motivated by (1.4) and (1.5), we study the problem (1.1)-(1.3) in this paper. Under some assumptions on the initial value, we will establish the existence, uniqueness and asymptotic behavior of weak solutions by the time-discrete method as used in [10, 11].

Equation (1.1) is something like the p-Laplacian equation, but many methods which are useful for the p-Laplacian equation are no longer valid for this equation. Because of the degeneracy, problem (1.1)-(1.3) does not admit classical solutions in general. So, we study weak solutions in the sense of following

Definition A function u is said to be a weak solution of (1.1)-(1.3), if the following conditions are satisfied:

1. $u \in L^\infty(0,T;W_0^{2,p(x)}(\Omega)) \cap C(0,T;H^1(\Omega))$, $\frac{\partial u}{\partial t} \in L^\infty(0,T;(W^{2,p(x)})'(\Omega))$, where $(W^{2,p(x)})'(\Omega)$ is the conjugate space of $W^{2,p(x)}(\Omega)$.
2. For any $\varphi \in C_0^\infty(Q_T)$ and $Q_T = \Omega \times (0,T)$, the following integral equality holds
$$\iint_{Q_T} u\frac{\partial \varphi}{\partial t}dx\,dt + k\iint_{Q_T} \nabla u\frac{\partial \nabla\varphi}{\partial t}dx\,dt - \iint_{Q_T} |\triangle u|^{p(x)-2}\triangle u \triangle\varphi dx\,dt = 0\,.$$
3. $u(x,0) = u_0(x)$.

We need some theories on spaces $W^{m,p(x)}$ which we call generalized Lebesgue-Sobolev spaces. We refer the reader to [8] (see also [7, 9]) for some basic properties of spaces $W^{m,p(x)}$ which will be used later. For simplicity we set $k = 1$ in this paper.

This paper is arranged as following. We first discuss the existence of weak solutions in Section 2. Our method for investigating the existence of weak solutions is based on the time discrete method to construct an approximate solutions. By means of the uniform estimates on solutions of the time difference equations, we prove the existence of weak solutions of the problem (1.1)-(1.3). We also prove the uniqueness and asymptotic behavior in Section 3 and Section 4 subsequently.

2 EXISTENCE OF WEAK SOLUTIONS

In this section, we are going to prove the existence of weak solutions.

Theorem 1 *If $u_0 \in W_0^{2,p(x)}(\Omega)$, $p(x) \in C(\bar\Omega)$, $p(x)$ satisfies for some constant L*
$$-|p(x) - p(y)|\ln|x-y| \le L, \quad \textit{for any } x, y \in \bar\Omega$$
and $p_- = \min_{\bar\Omega} p(x) > 2$. Then the problem (1.1)-(1.3) has at least one solution.

We use the a discrete method for constructing an approximate solution. First, divide the interval $(0,T)$ in N equal segments and set $h = \frac{T}{N}$. Then consider the problem
$$\frac{1}{h}(u_{k+1} - u_k) - \frac{1}{h}(\Delta u_{k+1} - \Delta u_k) + \triangle(|\triangle u_{k+1}|^{p(x)-2}\triangle u_{k+1}) = 0, \tag{2.1}$$
$$u_{k+1}|_{\partial\Omega} = \triangle u_{k+1}|_{\partial\Omega} = 0, \quad k = 0, 1, \ldots, N-1, \tag{2.2}$$
where u_0 is the initial value.

Lemma 2 *For a fixed k, if $u_k \in H_0^1(\Omega)$, problem (2.1)-(2.2) admits a weak solution $u_{k+1} \in W_0^{2,p(x)}(\Omega)$, such that for any $\varphi \in C_0^\infty(\Omega)$, have*

$$\frac{1}{h}\int_\Omega (u_{k+1}-u_k)\varphi dx + \frac{1}{h}\int_\Omega (\nabla u_{k+1}-\nabla u_k)\nabla\varphi dx + \int_\Omega |\triangle u_{k+1}|^{p(x)-2}\triangle u_{k+1}\triangle\varphi dx = 0. \quad (2.3)$$

Proof. Let us consider the following functionals on the space $W_0^{2,p(x)}(\Omega)$

$$F_1[u] = \int_\Omega \frac{1}{p(x)}|\triangle u|^{p(x)}dx, \quad F_2[u] = \frac{1}{2}\int_\Omega |u|^2 dx, \quad F_3[u] = \frac{1}{2}\int_\Omega |\nabla u|^2 dx,$$

$$H[u] = F_1[u] + \frac{1}{h}F_2[u] + \frac{1}{h}F_3[u] - \int_\Omega fudx,$$

where $f \in H^{-1}(\Omega)$ is a known function. Using Young's inequality, there exist constants $C_1 > 0$, such that

$$\begin{aligned} H[u] &= \int_\Omega \frac{1}{p(x)}|\triangle u|^{p(x)}dx + \frac{1}{2h}\int_\Omega |u|^2dx + \frac{1}{2h}\int_\Omega |\nabla u|^2 dx - \int_\Omega fu\,dx \\ &\geq \frac{1}{p_+}\int_\Omega |\triangle u|^{p(x)}dx - C_1\|f\|_{-1}. \end{aligned}$$

We need to check that $H[u]$ satisfies the coercive condition. For this purpose, we notice that by $u|_{\partial\Omega} = \triangle u|_{\partial\Omega} = 0$ and using the L^p theory for elliptic equation ([4]),

$$\|u\|_{W^{2,p(x)}} \leq C|\triangle u|_{p(x)}.$$

Therefore, we have $H[u] \to \infty$, as $\|u\|_{W^{2,p(x)}} \to +\infty$.

Since the norm is lower semi-continuous and $\int_\Omega fudx$ is a continuous functional, $H[u]$ is weakly lower semi-continuous on $W_0^{2,p(x)}(\Omega)$ and satisfying the coercive condition. From [3] we conclude that there exists $u_* \in W_0^{2,p(x)}(\Omega)$, such that

$$H[u_*] = \inf H[u],$$

and u_* is the weak solutions of the Euler equation corresponding to $H[u]$,

$$\frac{1}{h}u - \frac{1}{h}\Delta u + \triangle(|\triangle u|^{p-2}\triangle u) = f.$$

Taking $f = (u_k - \Delta u_k)/h$, we obtain a weak solutions u_{k+1} of (2.1)–(2.2). The proof is complete.

Now, we construct an approximate solution u^h of the problem (1.1)-(1.3) by defining

$$u^h(x,t) = u_k(x), \quad kh < t \leq (k+1)h, \ k = 0,1,\ldots,N-1,$$
$$u^h(x,0) = u_0(x).$$

The desired solution of the problem (1.1)-(1.3) will be obtained as the limit of some subsequence of $\{u^h\}$. To this purpose, we need some uniform estimates on u^h.

Lemma 3 *The weak solutions u_k of (2.1)–(2.2) satisfy*

$$h\sum_{k=1}^N \int_\Omega |\triangle u_k|^{p(x)}dx \leq C, \quad (2.4)$$

$$\sup_{0<t<T}\int_\Omega |\triangle u^h(x,t)|^{p(x)}dx \leq C, \quad (2.5)$$

where C is a constant independent of h and k.

Proof. i) We take $\varphi = u_{k+1}$ in the integral equality (2.3) (we can easily prove that for $\varphi \in W_0^{2,p(x)}(\Omega)$, (2.3) also holds) and obtain

$$\begin{aligned}&\frac{1}{h}\int_\Omega |u_{k+1}|^2dx + \frac{1}{h}\int_\Omega |\nabla u_{k+1}|^2dx + \int_\Omega |\triangle u_{k+1}|^{p(x)}dx\\ &= \frac{1}{h}\int_\Omega u_k u_{k+1}dx + \frac{1}{h}\int_\Omega \nabla u_{k+1}\nabla u_k dx.\end{aligned}$$

By Young's inequality,

$$\begin{aligned}&\frac{1}{h}\int_\Omega |u_{k+1}|^2dx + \frac{1}{h}\int_\Omega |\nabla u_{k+1}|^2dx + \int_\Omega |\triangle u_{k+1}|^{p(x)}dx\\ &\le \frac{1}{2h}\int_\Omega |u_k|^2dx + \frac{1}{2h}\int_\Omega |u_{k+1}|^2dx + \frac{1}{2h}\int_\Omega |\nabla u_k|^2dx + \frac{1}{2h}\int_\Omega |\nabla u_{k+1}|^2dx;\end{aligned}$$

that is,

$$\begin{aligned}&\frac{1}{2}\int_\Omega |u_{k+1}|^2dx + \frac{1}{2}\int_\Omega |\nabla u_{k+1}|^2dx + h\int_\Omega |\triangle u_{k+1}|^{p(x)}dx\\ &\le \frac{1}{2}\int_\Omega |u_k|^2dx + \frac{1}{2}\int_\Omega |\nabla u_k|^2dx.\end{aligned} \tag{2.6}$$

Adding these inequalities for k from 0 to $N-1$, we have

$$h\sum_{k=1}^{N}\int_\Omega |\triangle u_k|^{p(x)}dx \le \frac{1}{2}\int_\Omega |u_0|^2dx + \frac{1}{2}\int_\Omega |\nabla u_0|^2dx.$$

Therefore, (2.4) holds.

ii) We take $\varphi = u_{k+1} - u_k$ in the integral equality (2.3) and integrating by parts, we have

$$\begin{aligned}&\frac{1}{h}\int_\Omega |u_{k+1} - u_k|^2dx + \frac{1}{h}\int_\Omega |\nabla u_{k+1} - \nabla u_k|^2dx\\ &\quad + \int_\Omega |\triangle u_{k+1}|^{p(x)-2}\triangle u_{k+1}\triangle(u_{k+1} - u_k)dx = 0\,.\end{aligned}$$

Since the first term and the second term of the left hand side of the above equality are nonnegative, it follows that

$$\begin{aligned}\int_\Omega |\triangle u_{k+1}|^{p(x)}dx &\le \int_\Omega |\triangle u_{k+1}|^{p(x)-2}\triangle u_{k+1}\triangle u_k\,dx\\ &\le \int_\Omega \frac{p(x)-1}{p(x)}|\triangle u_{k+1}|^{p(x)}dx + \int_\Omega \frac{1}{p(x)}|\triangle u_k|^{p(x)}dx;\end{aligned}$$

thus,

$$\int_\Omega \frac{1}{p(x)}|\triangle u_{k+1}|^{p(x)}dx \le \int_\Omega \frac{1}{p(x)}|\triangle u_k|^{p(x)}dx.$$

For any m, with $1 \le m \le N-1$, adding the above inequality for k from 0 to $m-1$, we have

$$\int_\Omega \frac{1}{p(x)}|\triangle u_m|^{p(x)}dx \le \int_\Omega \frac{1}{p(x)}|\triangle u_0|^{p(x)}dx,$$

that is

$$\frac{1}{p_+}\int_\Omega |\triangle u_m|^{p(x)}dx \le \frac{1}{p_-}\int_\Omega |\triangle u_0|^{p(x)}dx.$$

Therefore, (2.5) holds.

Lemma 4 *For a weak solutions u_{k+1} of (2.1)–(2.2), we have*

$$-Ch \leq \int_\Omega |u_{k+1}|^2 dx + \int_\Omega |\nabla u_{k+1}|^2 dx - \int_\Omega |u_k|^2 dx - \int_\Omega |\nabla u_k|^2 dx \leq 0, \tag{2.7}$$

where C is a constant independently of h.

Proof. The second inequality in (2.7) is an immediate consequence of (2.6). To prove the first inequality, we choose $\varphi = u_k$ in (2.3) and obtain

$$\begin{aligned}
&\int_\Omega |u_k|^2 dx + \int_\Omega |\nabla u_k|^2 dx - \int_\Omega u_{k+1} u_k dx - \int_\Omega \nabla u_{k+1} \nabla u_k dx \\
&= h \int_\Omega |\triangle u_{k+1}|^{p(x)-2} \triangle u_{k+1} \triangle u_k dx \\
&\leq h \int_\Omega \frac{p(x)-1}{p(x)} |\triangle u_{k+1}|^{p(x)} dx + h \int_\Omega \frac{1}{p(x)} |\triangle u_k|^{p(x)} dx.
\end{aligned}$$

Here we have used Hölder inequality. By (2.5) again, we obtain

$$\int_\Omega |u_k|^2 dx + \int_\Omega |\nabla u_k|^2 dx - \int_\Omega u_{k+1} u_k dx - \int_\Omega \nabla u_{k+1} \nabla u_k dx \leq Ch.$$

Therefore,

$$\int_\Omega |u_k|^2 dx + \int_\Omega |\nabla u_k|^2 dx - \int_\Omega |u_{k+1}|^2 dx - \int_\Omega |\nabla u_{k+1}|^2 dx \leq Ch,$$

which completes the proof.

Proof of Theorem 2.1. First, we define the operator A^t, $A^t(\triangle u^h) = |\triangle u_k|^{p(x)-2} \triangle u_k$, $\Delta^h u^h = u_{k+1} - u_k$, where $kh < t \leq (k+1)h, k = 0, 1, \dots, N-1$. By the discrete equation (2.1) and the (2.4) in Lemma 2.2, we know that

$$\frac{1}{h} \Delta^h u^h \quad \text{in } L^\infty(0,T; (W^{2,p(x)}(\Omega))') \quad \text{is bounded.} \tag{2.9}$$

By (2.5), (2.7), (2.9) and (2.4) we known that exists a subsequence of $\{u^h\}$ (which we denote as the original sequence) such that

$$\begin{aligned}
u^h \to u \quad &\text{in } L^\infty(0,T; W^{2,p(x)}(\Omega)) \quad \text{weak-}\star, \\
\nabla u^h \to \nabla u \quad &\text{in } L^\infty(0,T; L^2(\Omega)) \quad \text{weak-}\star, \\
\frac{1}{h}(u_{k+1} - u_k) \to \frac{\partial u}{\partial t} \quad &\text{in } L^\infty(0,T; (W^{2,p(x)}(\Omega))') \quad \text{weak-}\star, \\
A^t(\triangle u^h) \to w \quad &\text{in } L^\infty(0,T; L^{p'(x)}(\Omega)) \quad \text{weak-}\star,
\end{aligned}$$

where $p'(x)$ is conjugate exponent of $p(x)$. From (2.3), we known, for any $\varphi \in C_0^\infty(Q_T)$,

$$\iint_{Q_T} \left(\frac{1}{h} \Delta^h u^h \varphi - \frac{1}{h} \Delta^h u^h \triangle \varphi + A^t(\triangle u^h) \triangle \varphi \right) dx\, dt = 0.$$

Letting $h \to 0$, we obtain, in the sense of distributions,

$$\frac{\partial u}{\partial t} - \frac{\partial \Delta u}{\partial t} + \triangle w = 0. \tag{2.10}$$

Similar as in [10], we can easily prove $w = |\triangle u|^{p(x)-2} \triangle u$ a.e. in Q_T. The strong convergence of u^h in $C(0,T; H^1(\Omega))$ and the fact that $u^h(x,0) = u_0(x)$ completes the proof.

3 UNIQUENESS OF SOLUTIONS

In this section, we prove that the weak solution is unique. To this end we need the following lemma.

Lemma 5 *For $\varphi \in L^\infty(t_1, t_2; W_0^{2,p(x)}(\Omega))$ with $\varphi_t \in L^2(t_1, t_2; H^1(\Omega))$, the weak solutions u of the problem (1.1)-(1.3) on Q_T satisfies*

$$\begin{aligned}&\int_\Omega u(x,t_1)\varphi(x,t_1)dx + \int_\Omega \nabla u(x,t_1)\nabla\varphi(x,t_1)dx\\&+\int_{t_1}^{t_2}\int_\Omega\left(u\frac{\partial\varphi}{\partial t}+\nabla u\frac{\partial\nabla\varphi}{\partial t}+|\triangle u|^{p(x)-2}\triangle u\triangle\varphi\right)dx\,dt\\&=\int_\Omega u(x,t_2)\varphi(x,t_2)dx+\int_\Omega\nabla u(x,t_2)\nabla\varphi(x,t_2)dx.\end{aligned}$$

In particular, for $\varphi \in W_0^{2,p(x)}(\Omega)$, we have

$$\begin{aligned}\int_\Omega (u(x,t_1)-u(x,t_2))\varphi dx + \int_\Omega \nabla(u(x,t_1)-u(x,t_2))\nabla\varphi dx\\ -\int_{t_1}^{t_2}\int_\Omega |\triangle u|^{p(x)-2}\triangle u\triangle\varphi dx\,dt = 0\,.\end{aligned}\tag{3.1}$$

Proof. From $\varphi \in L^\infty(t_1, t_2; W_0^{2,p(x)}(\Omega))$ and $\varphi_t \in L^2(t_1, t_2; H^1(\Omega))$, it follows that there exists a sequence of functions $\{\varphi_k\}$, for fixed $t \in (t_1, t_2)$, $\varphi_k(\cdot, t) \in C_0^\infty(\Omega)$, and as $k \to \infty$

$$\|\varphi_{kt}-\varphi_t\|_{L^2(t_1,t_2;H^1(\Omega))}\to 0,\quad \|\varphi_k-\varphi\|_{L^\infty(t_1,t_2;W_0^{2,p(x)}(\Omega))}\to 0.$$

Choose a function $j(s) \in C_0^\infty(R)$ such that $j(s) \ge 0$, for $s \in R$; $j(s) = 0$, for $\forall |s| > 1$; $\int_R j(s)ds = 1$. For $h > 0$, define $j_h(s) = \frac{1}{h}j(\frac{s}{h})$ and

$$\eta_h(t)=\int_{t-t_2+2h}^{t-t_1-2h} j_h(s)ds.$$

Clearly $\eta_h(t) \in C_0^\infty(t_1, t_2)$, $\lim_{h\to 0^+}\eta_h(t) = 1$, for all $t \in (t_1, t_2)$. In the definition of weak solutions, choose $\varphi = \varphi_k(x, t)\eta_h(t)$, we have

$$\begin{aligned}&\int_{t_1}^{t_2}\int_\Omega u\varphi_k j_h(t-t_1-2h)dx\,dt-\int_{t_1}^{t_2}\int_\Omega u\varphi_k j_h(t-t_2+2h)dx\,dt\\&+\int_{t_1}^{t_2}\int_\Omega\nabla u\nabla\varphi_k j_h(t-t_1-2h)dx\,dt-\int_{t_1}^{t_2}\int_\Omega\nabla u\nabla\varphi_k j_h(t-t_2+2h)\,dx\,dt\\&+\int_{t_1}^{t_2}\int_\Omega u\varphi_{kt}\eta_h dx\,dt+\int_{t_1}^{t_2}\int_\Omega\nabla u\nabla\varphi_{kt}\eta_h\,dx\,dt\\&+\int_{t_1}^{t_2}\int_\Omega|\triangle u|^{p(x)-2}\triangle u\triangle\varphi_k\eta_h\,dx\,dt=0.\end{aligned}$$

Observe that

$$\begin{aligned}
&\Big|\int_{t_1}^{t_2}\int_\Omega u\varphi_k j_h(t-t_1-2h)dx\,dt-\int_\Omega (u\varphi_k)|_{t=t_1}dx\Big|\\
&=\Big|\int_{t_1+h}^{t_1+3h}\int_\Omega u\varphi_k j_h(t-t_1-2h)dx\,dt-\int_{t_1+h}^{t_1+3h}\int_\Omega (u\varphi_k)|_{t=t_2}j_h(t-t_1-2h)dx\,dt\Big|\\
&\le \sup_{t_1+h<t<t_1+3h}\int_\Omega \big|(u\varphi_k)|_t-(u\varphi_k)|_{t_1}\big|dx,
\end{aligned}$$

and $u\in C(0,T;L^2(\Omega))$. We see that the right hand side tends to zero as $h\to 0$. Similarly,

$$\Big|\int_{t_1}^{t_2}\int_\Omega u\varphi_k j_h(t-t_2+2h)dx\,dt-\int_\Omega (u\varphi_k)|_{t=t_2}dx\Big|\to 0,\quad \text{as } h\to 0,$$

$$\Big|\int_{t_1}^{t_2}\int_\Omega \nabla u\nabla\varphi_k j_h(t-t_1-2h)dx\,dt-\int_\Omega (\nabla u\nabla\varphi_k)|_{t=t_1}dx\Big|\to 0,\quad \text{as } h\to 0,$$

$$\Big|\int_{t_1}^{t_2}\int_\Omega \nabla u\nabla\varphi_k j_h(t-t_2+2h)dx\,dt-\int_\Omega (\nabla u\nabla\varphi_k)|_{t=t_2}dx\Big|\to 0,\quad \text{as } h\to 0.$$

Letting $h\to 0$ and $k\to\infty$, we obtain

$$\begin{aligned}
&\int_\Omega u(x,t_1)\varphi(x,t_1)dx+\int_\Omega \nabla u(x,t_1)\nabla\varphi(x,t_1)dx\\
&+\int_{t_1}^{t_2}\int_\Omega\Big(u\frac{\partial\varphi}{\partial t}+\nabla u\frac{\partial\nabla\varphi}{\partial t}+|\triangle u|^{p(x)-2}\triangle u\triangle\varphi\Big)dx\,dt\\
&=\int_\Omega u(x,t_2)\varphi(x,t_2)dx+\int_\Omega \nabla u(x,t_2)\nabla\varphi(x,t_2)dx.
\end{aligned}$$

In particular for $\varphi\in W_0^{2,p(x)}(\Omega)$, we have

$$\begin{aligned}
\int_\Omega (u(x,t_1)-u(x,t_2))\varphi dx+\int_\Omega (\nabla u(x,t_1)-\nabla u(x,t_2))\nabla\varphi dx&\\
-\int_{t_1}^{t_2}\int_\Omega |\triangle u|^{p(x)-2}\triangle u\triangle\varphi\,dx\,dt=0&
\end{aligned}$$

which completes the proof.

For a fixed $\tau\in(0,T)$, set h satisfying $0<\tau<\tau+h<T$. Letting $t_1=\tau$, $t_2=\tau+h$, then multiply (3.1) by $\frac{1}{h}$, for $\varphi\in W_0^{2,p(x)}(\Omega)$, we obtain

$$\int_\Omega (u_h(x,\tau))_\tau\varphi(x)dx+\int_\Omega ((\nabla u)_h(x,\tau))_\tau\varphi(x)dx+\int_\Omega (|\triangle u|^{p(x)-2}\triangle u)_h(x,\tau)\triangle\varphi dx=0, \tag{3.2}$$

where

$$u_h(x,t)=\begin{cases}\frac{1}{h}\int_t^{t+h}u(\cdot,\tau)d\tau, & t\in(0,T-h),\\ 0, & t>T-h.\end{cases}$$

Theorem 6 *Problem (1.1)-(1.3) admits only one weak solution.*

Proof. Suppose u_1,u_2 are two solutions of (1.1)-(1.3), then

$$\begin{aligned}
\int_\Omega (u_1(x,\tau)-u_2(x,\tau))_{h\tau}\varphi(x)dx+\int_\Omega ((\nabla u_1-\nabla u_2)_h(x,\tau))_\tau\varphi(x)dx&\\
-\int_\Omega (|\triangle u_1|^{p(x)-2}\triangle u_1-|\triangle u_2|^{p(x)-2}\triangle u_2)_h(x,\tau)\triangle\varphi dx=0.&
\end{aligned}$$

For a fixed τ, we take $\varphi(x) = [u_1 - u_2]_h \in W_0^{2,p(x)}(\Omega)$, and hence

$$\int_\Omega (u_1(x,\tau) - u_2(x,\tau))_{h\tau}(u_1 - u_2)_h dx + \int_\Omega \nabla(u_1(x,\tau) - u_2(x,\tau))_{h\tau}\nabla(u_1 - u_2)_h dx$$
$$= -\int_\Omega [(|\triangle u_1|^{p(x)-2}\triangle u_1 - |\triangle u_2|^{p(x)-2}\triangle u_2)_h](x,\tau)\triangle(u_1 - u_2)_h dx.$$

Integrating the above equality with respect to τ over $(0,t)$,

$$\int_\Omega |(u_1 - u_2)_h|^2(x,t)dx + \int_\Omega |\nabla(u_1 - u_2)_h|^2(x,t)dx \le 0,$$

we have $\int_\Omega |(u_1 - u_2)_h|^2 dx = 0$; therefore, $u_1 = u_2$.

4 ASYMPTOTIC BEHAVIOR

This section is devoted to the asymptotic behavior of solutions. To this purpose, we first show that:

Theorem 7 *The weak solution u obtained in Theorem 3.1, satisfies*

$$\frac{1}{2}\int_\Omega |\nabla u(x,t)|^2 dx + \frac{1}{2}\int_\Omega |u(x,t)|^2 dx - \frac{1}{2}\int_\Omega |\nabla u_0(x)|^2 dx - \frac{1}{2}\int_\Omega |u_0(x)|^2$$
$$= -\iint_{Q_t} |\triangle u|^{p(x)} dx\, d\tau, \tag{4.1}$$

where $Q_t = \Omega \times (0,t)$.

Proof. In the proof of Theorem 2.1, we have

$$f(t) = \frac{1}{2}\int_\Omega |\nabla u(x,t)|^2 dx + \frac{1}{2}\int_\Omega |u(x,t)|^2 dx \in C([0,T]). \tag{4.2}$$

Consider the functional

$$K[v] = \frac{1}{2}\int_\Omega |\nabla v(x)|^2 dx + \frac{1}{2}\int_\Omega |v(x)|^2 dx.$$

It is easy to see that $K[v]$ is a convex functional on $H_0^1(\Omega)$.

For any $\tau \in (0,T)$ and $h > 0$, we have

$$K[u(\tau+h)] - K[u(\tau)] \ge \langle u(\tau+h) - u(\tau), u(x,\tau) - \triangle u(x,\tau)\rangle.$$

By $\frac{\delta K[v]}{\delta v} = v - \triangle v$, for any fixed $t_1, t_2 \in [0,T]$, $t_1 < t_2$, integrating the above inequality with respect to τ over (t_1,t_2), we have

$$\int_{t_2}^{t_2+h} K[u(\tau)]d\tau - \int_{t_1}^{t_1+h} K[u(\tau)]d\tau \ge \int_{t_1}^{t_2} \langle u(\tau+h) - u(\tau), u - \triangle u\rangle d\tau.$$

Multiplying the both side of the above inequality by $1/h$, and letting $h \to 0$, we obtain

$$K[u(t_2)] - K[u(t_1)] \ge \int_{t_1}^{t_2} \left\langle \frac{\partial u}{\partial t}, u - \triangle u \right\rangle d\tau.$$

Similarly, we have

$$K[u(\tau)] - K[u(\tau - h)] \leq \langle u(\tau) - u(\tau - h), u - \triangle u \rangle.$$

Thus

$$K[u(t_2)] - K[u(t_1)] \leq \int_{t_1}^{t_2} \left\langle \frac{\partial u}{\partial t}, u - \triangle u \right\rangle d\tau,$$

and hence

$$K[u(t_2)] - K[u(t_1)] = \int_{t_1}^{t_2} \left\langle \frac{\partial u}{\partial t}, u - \triangle u \right\rangle d\tau.$$

Taking $t_1 = 0, t_2 = t$, we get from the definition of solutions that

$$\begin{aligned} K[u(t)] - K[u(0)] &= \int_0^t \left\langle \frac{\partial u}{\partial t} - \frac{\partial \Delta u}{\partial t}, u(\tau) \right\rangle d\tau. \\ &= -\int_0^t \left\langle \triangle(|\triangle u|^{p(x)-2} \triangle u), u(\tau) \right\rangle d\tau \\ &= -\iint_{Q_t} |\triangle u|^{p(x)} dx\, d\tau. \end{aligned}$$

Theorem 8 *Let u be the weak solution of the problem (1.1)-(1.3), $p_- > 2$. Then*

$$\int_\Omega |\nabla u(x,t)|^2 dx + \int_\Omega |u(x,t)|^2 dx \leq \frac{C_3}{(C_1 t + C_2)^\alpha}, \quad C_i > 0 \ (i = 1, 2, 3), \quad \alpha = \frac{2}{p_- - 2}.$$

Proof. By (4.2), we have

$$f'(t) = -\int_\Omega |\triangle u|^{p(x)} dx \leq 0.$$

By $u \in W_0^{2,p(x)}(\Omega)$, we see that

$$\int_\Omega |\nabla u(x,t)|^2 dx + \int_\Omega |u(x,t)|^2 dx \leq C \int_\Omega |\triangle u|^2 dx \leq C \left(\int_\Omega |\triangle u|^{p(x)} dx \right)^{2/p_-},$$

that is $f(t) \leq C|f'(t)|^{2/p_-}$. Again by $f'(t) \leq 0$, we have $f'(t) \leq -Cf(t)^{p_-/2}$, and hence we complete the proof.

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Generalizations on some meromorphic function spaces in the unit disc

A. El-Sayed Ahmed and M. Al Bogami

Sohag University Faculty of Science, Department of Mathematics, 82524 Sohag, Egypt

Current Address: Taif University, Faculty of Science, Mathematics Department

Box 888 El-Hawiyah, El-Taif 5700, Saudi Arabia

e-mail: ahsayed80@hotmail.com

Abstract

In this paper, we define a general spherical derivative. Making use of this general derivative, we introduce some new classes of meromorphic functions in the unit disk. Also, we introduce some new classes of meromorphic functions which are defined by means of a general chordal distance.

1 Introduction

Let Δ be the unit disk in the complex plane $\mathbb{C}$, and let $dA(z)$ be the Euclidean area element on Δ. Let $H(\Delta)$ (resp. $M(\Delta)$) denote the class of functions that are analytic (resp.meromorphic) in Δ. The Green's function in Δ with singularity at $a \in \Delta$ is given by $g(z,a) = \log \frac{1}{|\varphi_a(z)|}$, where $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$ is the $M\ddot{o}bius$ transformation of Δ. For $0 < r < 1$, let $\Delta(a,r) = \{z \in \Delta : |\varphi_a(z)| < r\}$ be the pseudohyperbolic disk with center $a \in \Delta$ and radius r.
For $0 < p < \infty$, the spaces Q_p and M_p are defined by (see [1]):

$$Q_p = \{f \in H(\Delta) : \sup_{a\in\Delta} \int\int_{\Delta} |f'(z)|^2 (g(z,a))^p dA(z) < \infty\},$$

$$M_p = \{f \in H(\Delta) : \sup_{a\in\Delta} \int\int_{\Delta} |f'(z)|^2 (1 - |\varphi_a(z)|)^p dA(z) < \infty\}.$$

The Bloch space $\mathcal{B}$ (cf. [1] and [16]), is the space of all analytic functions belonging to $H(\Delta)$, for which

$$\mathcal{B} = \{f \in H(\Delta) : \|f\|_{\mathcal{B}} = \sup_{z\in\Delta} (1 - |z|^2)|f'(z)| < \infty\}.$$

When we study meromorphic functions in Δ, it is natural to replace $|f'(z)|$ in these expressions by the spherical derivative $f^{\#}(z) = |f'(z)|/(1 + |f(z)|^2)$ and obtain the classes $Q_p^{\#}, M_p^{\#}$ and $\mathcal{N}$, the class of normal function in Δ, respectively (see, for example, Aulaskari, Xiao and Zhao [4] and Wulan [19]).

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The meromorphic counterpart of BMOA is the set UBC of meromorphic functions of uniformly bounded characteristic introduced by Yamashita [21]. It turns out that we have $Q_p = M_p$ ([3]), $Q_p^\# \subsetneqq M_p^\#$ ([5] and [19]).
Now, let $K : [0, \infty) \to [0, \infty)$ be a right-continuous and nondecreasing function, then the spaces Q_K and $Q_K^\#$ are defined as follows (see [10, 20]):

Definition 1.1 *$f \in H(\Delta)$ belongs to the space Q_K if*

$$\|f\|_K^2 = \|f\|_{Q_K}^2 = \sup_{a\in\Delta} \int\int_\Delta |f'(z)|^2 K(g(z,a))\, dA(z) < \infty. \tag{1}$$

Definition 1.2 *$f \in M(\Delta)$ belongs to the class $Q_K^\#$ if*

$$\sup_{a\in\Delta} \int\int_\Delta (f^\#(z))^2 K(g(z,a))\, dA(z) < \infty. \tag{2}$$

Remark 1.1 *It should be remarked that the space $Q_K^\#$ is not a linear space. It is clear that Q_K and $Q_K^\#$ are Möbius invariant.*

Remark 1.2 *For $0 < p < \infty$, $K(t) = t^p$ gives the space Q_p and the class $Q_p^\#$. Choosing $K(t) = (1 - e^{-2t})^p$, we obtain M_p and $M_p^\#$.*

Remark 1.3 *Choosing $K(t) = 1$, we get the Dirichlet space $\mathcal{D}$ and the spherical Dirichlet class $\mathcal{D}^\#$. For a fixed r, $0 < r < 1$, we choose*

$$K_0(t) = \begin{cases} 1, & t \geq \log(1/r), \\ 0, & 0 < t < \log(1/r). \end{cases}$$

Then, we obtain

$$\int\int_\Delta |f'(z)|^2 K_0(g(z,a))\, dA(z) = \int\int_{\Delta(a,r)} |f'(z)|^2\, dA(z)$$

and

$$\int\int_\Delta (f^\#(z))^2 K_0(g(z,a))\, dA(z) = \int\int_{\Delta(a,r)} (f^\#(z))^2\, dA(z).$$

We conclude that $Q_{K_0} = \mathcal{B}$ (cf. Axler [6]) and $Q_{K_0}^\# = \mathcal{B}^\#$, where $\mathcal{B}^\#$ is the class of spherical Bloch functions (cf. Section 3 in [10]). It is easy to see that $\mathcal{N} \subset \mathcal{B}^\#$ (cf. Lappan [14] and the discussion after Definition 2.1 in Wulan [19]).
Now, let us introduce the following notation general spherical derivative

$$f_n^\#(z) = \frac{|f^{(n)}(z)|}{1 + |f(z)|^{n+1}};\ n \in \mathbb{N}.$$

This general derivative gives a plethora of new results on the meromorphic function spaces.
Note that if $n = 1$, we obtain the usual spherical derivative as defined above.
let $\omega : (0, 1] \to (0, \infty)$ be a nondecreasing function. Let $\mathcal{N}_{n,\omega}^\alpha$ be the class of all normal functions in Δ. We recall that a function f meromorphic in Δ is said to be $\omega-$normal if and only if

$$\sup_{z\in\Delta} \frac{(1 - |z|^2)^\alpha}{\omega(1 - |\varphi_a(z)|)} f_n^\#(z) < \infty.$$

Now, we define some general meromorphic classes as follows:

Definition 1.3 *Let $K : [0, \infty) \to [0, \infty)$ be a nondecreasing function. For $n \in \mathbb{N}$, a function f meromorphic in Δ is said to belong to the class $Q^{\#}_{K,n,\omega}$ if*

$$\sup_{a\in\Delta} \int_{\Delta} (f^{\#}_n(z))^2 \frac{K(g(z,a))}{\omega(1-|\varphi_a(z)|)}\, dA(z) < \infty. \tag{3}$$

Definition 1.4 *A function f meromorphic in Δ is said to be a general spherical Bloch function, denoted by $f \in \mathcal{B}^{\#}_{n,\omega}$, if there exists an r, $0 < r < 1$, such that*

$$\sup_{a\in\Delta} \int_{\Delta} \frac{(f^{\#}_n(z))^2}{\omega(1-|\varphi_a(z)|)}\, dA(z) < \infty. \tag{4}$$

It is easy to see that a normal function is a spherical Bloch function, that is, $\mathcal{N}_{n,\omega} \subset \mathcal{B}^{\#}_{n,\omega}$, but the converse is not true .
For more information of some related meromorphic function spaces, we refer to [1, 2, 7, 8, 9, 10, 11, 18] and others.
For a nondecreasing function $K : [0, \infty) \to [0, \infty)$, we say that the space Q_K is trivial if Q_K contains only constant functions. Whether our space Q_K is trivial or not depends on the integral

$$\int_0^{1/e} K(log(1/\rho))\rho\, d\rho = \int_1^{\infty} K(t)e^{-2t}\, dt. \tag{5}$$

The notation $A \lesssim B$ means that there exists a positive constant C such that $A \leq CB$. The symbol $\gtrsim$ is understood in a similar fashion.

2 General meromorphic classes

It is necessary to know for which functions K the classes $Q^{\#}_{K,n}$ will be trivial. Here, the square of the general spherical derivative $(f^{\#}_n(z))^2$ is not necessarily subharmonic, where $f^{\#}_n(z) = \frac{|f^{(n)}(z)|}{1+|f(z)|^{n+1}}; n \in \mathbb{N}$.

Theorem 2.1 *If the integral*

$$\int_0^r \frac{K(\log(1/R))}{\omega(1-R)} R\, dR$$

is divergent, then the space $Q^{\#}_{K,n,\omega}$ contains only constant functions.

$$\begin{aligned}
\int\int_{\Delta} (f^{\#}_n(z))^2 \frac{K(g(z,a))}{\omega(1-|\varphi_a(z)|)}\, dA(z) &\geq \int\int_{\Delta(a,r)} (f^{\#}_n(z))^2 \frac{K(g(z,a))}{\omega(1-|\varphi_a(z)|)}\, dA(z) \\
&= \int\int_{\Delta(a,r)} \left(\frac{|f^{(n)}(z)|}{1+|f(z)|^{n+1}}\right)^2 \frac{K(g(z,a))}{\omega(1-|\varphi_a(z)|)}\, dA(z) \\
&= \int\int_{|\varphi_a(z)|<r} \left(\frac{|f^{(n)}\varphi_a(z)|}{1+|f(\varphi_a(z))|^{n+1}}\right)^2 |\varphi'_a(z)|^2 \frac{K(\log(1/|z|))}{\omega(1-|z|)}\, dA(z) \\
&\geq \frac{\pi}{2}\left(\frac{(1-|a|^2)|f^{(n)}(a)|}{1+|f(a)|^{n+1}}\right)^2 \int_0^r R\, \frac{K(\log(1/R))}{\omega(1-R)} dR = \infty.
\end{aligned}$$

This is a contradiction, and the proof is complete.

Again, we assume from now on that the functions K and ω are right-continuous and nondecreasing, and that the integral (5) is convergent.
As in [21], we can give the following result.

Theorem 2.2 *For some $r \in (0,1)$, a meromorphic function f belongs to $\mathcal{N}_{n,\omega}$ if and only if*

$$\sup_{a\in\Delta}\int\int_{\Delta(a,r)}\frac{(f_n^{\#}(z))^2}{\omega(1-|\varphi_a(z)|)}dA(z)<\pi.$$

Proof: The proof is very similar to the corresponding result in [21] with simple modifications, so it will be omitted.

Now, we consider the following question:

Question 1

Is the condition that there exists $r \in (0,1)$ such that:

$$\sup_{a\in\Delta}\int\int_{\Delta(a,r)}(f_n^{\#}(z))^2\frac{K(g(z,a))}{\omega(1-|\varphi_a(z)|)}\,dA(z)<\infty \tag{6}$$

necessary and sufficient for $f \in \mathcal{B}^{\#}_{n,\omega}$?

Answer

If (6) holds, we can conclude that, $f \in B^{\#}_{n,\omega}$. In particular, it follows that $Q^{\#}_{K,n,\omega} \subset \mathcal{B}^{\#}_{n,\omega}$. Conversely, if we assume that $f \in \mathcal{B}^{\#}_{n,\omega}$ and that K is bounded, it is easy to see that (6) will hold. If K is unbounded and $f \in \mathcal{B}^{\#}_{n,\omega} \setminus \mathcal{N}_{n,\omega}$, we claim that the supremum in (6) will be infinite for all $r \in (0,1)$. To prove the claim , we note that it follows from Theorem 2.1 that if $f \in \mathcal{B}^{\#}_{n,\omega} \setminus \mathcal{N}_{n,\omega}$, then

$$\sup_{a\in\Delta}\int\int_{\Delta(a,r)}\frac{(f_n^{\#}(z))^2dA(z)}{\omega(1-|\varphi_a(z)|)}\geq\pi \quad for\ \ all\ \ r\in(0,1).$$

if $0<\rho<r$, we see that

$$\int\int_{\Delta(a,r)}(f_n^{\#}(z))^2\frac{K(g(z,a))}{\omega(1-|\varphi_a(z)|)}\,dA(z)\geq K(log(1/\rho))\int\int_{\Delta(a,\rho)}\frac{(f_n^{\#}(z))^2}{\omega(1-|\varphi_a(z)|)}\,dA(z).$$

Using the observation above, we deduce that

$$\sup_{a\in\Delta}\int\int_{\Delta(a,r)}(f_n^{\#}(z))^2\frac{K(g(z,a))}{\omega(1-|\varphi_a(z)|)}\,dA(z)\geq\pi\ K(log(1/\rho)),\ \ 0<\rho<r$$

Letting $\rho \to 0$, we conclude that (6) cannot hold for any $r \in (0,1)$ which completes the proof.
We conclude that (6) is a sufficient condition for $f \in \mathcal{B}^{\#}_{n,\omega}$. It is also a necessary condition when K is bounded, but not when K is unbounded. Finally, if we assume that $f \in \mathcal{N}_{n,\omega}$, it is easy to prove that (7) will hold (see the proof of Theorem 2.3(ii) below).
For the weights, there are some questions, which can be stated as follows:

Question 2

Which additional conditions on K are required for the inclusion $Q^{\#}_{K,n,\omega} \subset \mathcal{N}_{n,\omega}$?
When are the classes $Q^{\#}_{K_1,n}$ and $Q^{\#}_{K_2,n,\omega}$ identical for $K_1 \neq K_2$?
Answers of the above questions can be given by the next results. First, as in [17, 18, 19], we can give the following proposition.

Proposition 2.1 *Assume that $K(r) \to \infty$ as $r \to \infty$. Then $Q^{\#}_{K,n,\omega} \subset \mathcal{N}_{n,\omega}$.*

Next, we prove the following result:

Theorem 2.3 *Assume that $K(\infty) = 1$. Then $f \in \mathcal{N}_{n,\omega}$ if and only if*

$$\sup_{a\in\Delta}\int\int_{\Delta(a,r)}(f_n^{\#}(z))^2\frac{K(g(z,a))}{\omega^2(1-|\varphi_a(z)|)}dA(z)<\pi \tag{7}$$

for some $r \in (0,1)$.

Proof: Suppose that f is a general normal function. Then for $0 < r < 1$,

$$\begin{aligned} \int\int_{\Delta(a,r)} (f_n^{\#}(z))^2 \frac{K(g(z,a))}{\omega^2(1-|\varphi_a(z)|)}\, dA(z) &\leq \|f\|^2_{\mathcal{N}_{n,\omega}} \int\int_{\Delta(a,r)} (1-|z|^2)^{-2} K(g(a,z))\, dA(z) \\ &\leq 2\pi \|f\|^2_{\mathcal{N}_{n,\omega}} (1-r^2)^{-2} \int_0^r K(log1/\rho)\rho\, d\rho. \end{aligned} \tag{8}$$

Since

$$\int_0^r K(log1/\rho)\rho\, d\rho \to 0, r \to 0,$$

we may choose r small enough such that the left hand member in the first inequality in (8) is less than $\pi/2$. Thus (7) holds.
Conversely, let $\lambda(<\pi)$ be the supremum in (7) assumed for some $r_0 \in (0,1)$. Now consider $r \in (0, r_0)$. Since $\Delta(a,r) = \{z \in \Delta : g(z,a) > log(1/r)\}$,

$$\begin{aligned} &\int\int_{\Delta(a,r)} (f_n^{\#}(z))^2 dA(z) \\ &\leq \frac{\omega^2(1-r)}{K(log(1/r))} \int\int_{\Delta(a,r_0)} (f_n^{\#}(z))^2 \frac{K(g(z,a))}{\omega^2(1-|\varphi_a(z)|)}\, dA(z) \leq \lambda \frac{\omega^2(1-r)}{K(log(1/r))} < \pi \end{aligned}$$

here λ is a constant. Hereafter, λ stands for absolute constants, which may indicate different constants from one occurrence to the next. If r is small enough. Hence $f \in \mathcal{N}_{n,\omega}$ according to Theorem 2.1, the proof is established.

Corollary 2.1 *Assume that* $K(\infty) = 1$. *if* $f \in Q^{\#}_{K,n,\omega}$ *and*

$$\sup_{a\in\Delta} \int\int_{\Delta} (f_n^{\#}(z))^2 \frac{K(g(z,a))dA(z)}{\omega(1-|\varphi_a(z)|} < \pi,$$

then $f \in \mathcal{N}_{n,\omega}$.

Another important result on the weights of some meromorphic functions can be given by the following result:

Theorem 2.4 *Assume that* $K(1) > 0$ *and set* $K_1(r) = \inf(K(r), K(1))$.
(i) If K *is bounded, then* $Q^{\#}_{K,n,\omega} = Q^{\#}_{K_1,n,\omega}$.
(ii) If K *is unbounded, then* $Q^{\#}_{K,n,\omega} = \mathcal{N}_{n,\omega} \cap Q^{\#}_{K_1,n,\omega}$.

Proof: (i) If K is bounded , we have

$$K_1(r) \leq K(r) \leq \frac{K(\infty)}{K(1)} K_1(r)$$

and it is clear that $Q^{\#}_{K,n,\omega} = Q^{\#}_{K_1,n,\omega}$.
(ii) By Proposition 2.1, we have $Q^{\#}_{K,n,\omega} \subset \mathcal{N}_{n,\omega} \cap Q^{\#}_{K_1,n,\omega}$. Now assume that $f \in \mathcal{N}_{n,\omega} \cap Q^{\#}_{K_1,n,\omega}$. We note that $K(g(z,a)) = K_1(g(z,a))$ in $\Delta/\Delta(a,1/e)$. (In this domain, we have $g(z,a) \leq 1$) . To compare the two suprema in the integrals defining $Q^{\#}_{K,n,\omega}$ and $Q^{\#}_{K_1,n,\omega}$, it suffices to deal with integrals over $\Delta(a,1/e)$. Using our assumption that $f \in \mathcal{N}_{n,\omega}$, we see that

$$\begin{aligned} \int\int_{\Delta(a,1/e)} (f_n^{\#}(z))^2 \frac{K(g(z,a))}{\omega^2(1-|\varphi_a(z)|)}\, dA(z) &\leq \|f\|^2_{\mathcal{N}_{n,\omega}} \int\int_{\Delta(a,1/e)} (1-|z|^2)^{-2} K(g(z,a))\, dA(z) \\ &= \|f\|^2_{\mathcal{N}_{n,\omega}} \int\int_{\Delta(0,1/e)<r} (1-|z_1|^2)^{-2} K(\log\frac{1}{r})\, dA(z_1) \\ &= 2\pi \|f\|^2_{\mathcal{N}_{n,\omega}} \int_0^{1/e} r(1-|r|^2)^{-2} K(log(1/r))\, dr. \end{aligned}$$

the right hand member gives a bound for the supremum over $a \in \Delta$ of the first term in this chain of inequalities. Hence $f \in Q^{\#}_{K,n,\omega}$ and Theorem 2.3 is proved.

Next, we state conditions on K_1 and K_2 which imply that $Q^{\#}_{K_1,n,\omega} = Q^{\#}_{K_2,n,\omega}$.

Theorem 2.5 *Assume that K_1 and K_2 are either both bounded or both unbounded and that $K_1(r) \approx K_2(r)$ as $r \to 0$. Then $Q^{\#}_{K_1,n,\omega} = Q^{\#}_{K_2,n,\omega}$.*

Proof: We define $K_{i,1}(r) = inf(K_i(r), K_i(1))$, $i = 1, 2$. If K_1 and K_2 are bounded, it follows from our assumptions that $0 < c \leq K_1(r)/K_2(r) \leq c' < \infty, 0 < r < \infty$ and it is clear that we have $Q^{\#}_{K_1,n,\omega} = Q^{\#}_{K_2,n,\omega}$. If K_1 and K_2 are unbounded, we use Theorem 2.4 to deduce that

$$Q^{\#}_{K_1,n,\omega} = \mathcal{N}_{n,\omega} \cap Q^{\#}_{K_{1,1},n,\omega} = \mathcal{N}_{n,\omega} \cap Q^{\#}_{K_{2,1},n,\omega} = Q^{\#}_{K_2,n,\omega}.$$

This completes the proof of Theorem 2.5.

Theorem 2.6 *(i) If K is unbounded and (5) holds, then $Q^{\#}_{K,n,\omega} = \mathcal{N}_{n,\omega}$.*
(ii) If K is bounded and (5) holds, then $Q^{\#}_{K,n,\omega} = \mathcal{B}^{\#}_{n,\omega}$.
(iii) In (i) (resp. (ii)), (5) is a necessary condition for $Q^{\#}_{K,n,\omega} = \mathcal{N}_{n,\omega}$ (resp. $Q^{\#}_{K,n,\omega} = \mathcal{B}^{\#}_{n,\omega}$) .

Proof: (i) By Proposition 2.1 we have $Q^{\#}_{K,n,\omega} \subset \mathcal{N}_{n,\omega}$. Conversely, if $f \in N_{n,\omega}$, we know that $f^{\#}_n(z) \leq \lambda(1-|z|^2)^{-1}$ and we can use the argument in the proof of (Theorem 2.3 in [10]) to prove that $f \in Q^{\#}_{K,n,\omega}$.
(ii) By question 1, we have $Q^{\#}_{K,n,\omega} \subset \mathcal{B}^{\#}_{n,,\omega}$. It suffices to prove that $\mathcal{B}^{\#}_{n,,\omega} \subset Q^{\#}_{K,n,\omega}$. If $f \in B^{\#}_{n,\omega}$, there exists $r \in (0,1)$ such that

$$\int\int_{\Delta(a,r)} \frac{(f^{\#}_n(z))^2}{\omega^2(1-|\varphi_a(z))} \, dA(z) \leq \lambda < \infty \quad \text{for all } \ a \in \Delta. \tag{9}$$

Let us first prove that there exists a constant C_1 depending on r and K (see below) such that

$$\int\int_{\Delta} (f^{\#}_n(z))^2 \frac{K(log(1/|z|))}{\omega^2(1-|z|)} \, dA(z) \leq \lambda \|K\|_\infty + C_1. \tag{10}$$

Our first observation in the proof of this estimate is that

$$\int\int_{|z|<r} (f^{\#}_n(z))^2 \frac{K(log(1/|z|))}{\omega^2(1-|z|)} \, dA(z) \leq B\|K\|_\infty.$$

Let $\Omega_k = \{z - (1-r)^k \leq |z| \leq 1-(1-r)^{k+1}\}$. We wish to cover Ω_k with disks $\Delta(a,r)$ with $|a| = 1-(1-r)^{k+1}$, it suffices to use roughly $C(r(1-r)^{k+1})^{-1}$ such disks, where C is an absolute constant, $k = 1, 2$. Hence .

$$\begin{aligned}
\int\int_{\Omega_k} (f^{\#}_n(z))^2 \frac{K(log(1/|z|))}{\omega^2(1-|z|)} \, dA(z) &\leq K(log\frac{1}{1-(1-r)^k})^{-1} BC(r(1-r)^{k+1})^{-1}, \\
&\leq K((1-r)^k \gamma(r)) BC(r(1-r)^{k+1})^{-1},
\end{aligned}$$

where $\gamma(r) = (1-r)^{-1}\frac{\log(\frac{1}{r})}{\omega(1-r)}$. It follows that

$$\begin{aligned}
\int\int_{r<|z|<1} (f^{\#}_n(z))^2 \frac{K(log(1/|z|))}{\omega^2(1-|z|)} \, dA(z) &\leq \lambda\, r^{-1} \sum_1^\infty (1-r)^{-k-1} K((1-r)^k \gamma(r)) \\
&\leq \lambda\, r^{-2}(1-r)^{-2} \int_0^1 t^{-2} K(t\gamma(r)) \, dt. \\
&= \lambda\gamma(r) r^{-2}(1-r)^{-2} \int_0^{\gamma(r)} s^{-2} K(s) \, ds = C_1 < \infty.
\end{aligned}$$

The convergence of the integral follows from (5). We have proved that (10) holds for all $f \in B^{\#}_{n,\omega}$ satisfying (10). Since for all $b \in \Delta$,

$$\sup_{a\in\Delta}\int\int_{\Delta(a,r)} \frac{((f\circ\varphi_b)^{\#}_n(z))^2}{\omega^2(1-|\varphi_a(z)|)}dA(z) = \sup_{a\in\Delta}\int\int_{\Delta(a,r)} \frac{(f^{\#}_n(z))^2}{\omega^2(1-|\varphi_a(z)|)}\,dA(z) = \lambda.$$

It follows from (9) and (10) with $f^{\#}_n$ replaced by $(f\circ\varphi_b)^{\#}_n$ that

$$\sup_{b\in\Delta}\int\int_{\Delta}(f^{\#}_n(z))^2\frac{K(log\frac{1}{|\varphi_b(z)|})}{\omega^2(1-|\varphi_b(z)|)}dA(z) = \sup_{b\in\Delta}\int\int_{\Delta}((f\circ\varphi_b)^{\#}_n(z))^2\frac{K(log(1/|z|))}{\omega^2(1-|z|)}dA(z) \le C_1 + \lambda\|K\|_\infty$$

this proves Theorem 2.5(ii).
(iii) As given by Lappan and Xiao [15], there exist functions f_1 and f_2 in $\mathcal{N}_{n,\omega}$ such that

$$c_0 = \inf_{z\in\Delta}(1-|z|^2)(f^{\#}_{n,1}(z)+f^{\#}_{n,2}(z)) > 0 \tag{11}$$

If $Q^{\#}_{K,n,\omega} = \mathcal{N}_{n,\omega}$ or $Q^{\#}_{K,n,\omega} = \mathcal{B}^{\#}_{n,\omega} \supset \mathcal{N}_{n,\omega}$, we have

$$\begin{aligned}
\infty &> \sup_{a\in\Delta}\int\int_{\Delta}(f^{\#}_{n,1}(z))^2+(f^{\#}_{n,2}(z))^2\frac{K(g(z,a))}{\omega^2(1-|\varphi_a(z)|)}\,dA(z).\\
&\ge \frac{1}{2}\int\int_{\Delta}(f^{\#}_{n,1}(z)+f^{\#}_{n,2}(z))^2\frac{K(g(z,0))}{\omega^2(1-|\varphi_0(z)|)}\,dA(z).\\
&\ge (c_0^2/2)\int\int_{\Delta}(1-|z|^2)^{-2}\frac{K(g(z,0))}{\omega^2(1-|\varphi_0(z)|)}\,dA(z).\\
&= \pi c_0^2\int_0^1(1-r^2)^{-2}\frac{K(log(1/r))}{\omega^2(1-r)}r\,dr.
\end{aligned}$$

Hence (5) holds which finishes the proof of Theorem 2.5(iii).

Remark 2.1 *There is an analogue of (11) for Bloch functions with the general spherical derivatives* $f^{\#}_{n,1}$ *and* $f^{\#}_{n,2}$ *replaced by* $|f_1^{(n)}|$ *and* $|f_2^{(n)}|$.

Finally we consider the classes

$$\mathcal{B}^{\#}_{n,\omega,0} = \{f\in M(\Delta): \lim_{|a|\to 1}\int\int_{\Delta(a,r)}(f^{\#}_n(z))^2\,dA(z) = 0 \text{ for some } r\in(0,1)\},$$

$$Q^{\#}_{K,n,\omega,0} = \{f\in M(\Delta): \lim_{|a|\to 1}\int\int_{\Delta}(f^{\#}_n(z))^2\frac{K(g(z,a))}{\omega^2(1-|\varphi_a(z)|)}\,dA(z) = 0\},$$

$$\mathcal{N}_{n,\omega,0} = \{f\in M(\Delta): \frac{(1-|z|^2)}{\omega^2(1-|\varphi_a(z)|)}f^{\#}_n(z)\to 0\,,\ |z|\to\ 1\}\,.$$

and the weighted general spherical Dirichlet class can be defined by

$$\mathcal{D}^{\#}_{n,\omega} = \{f\in M(\Delta): \int\int_{\Delta}\frac{(f^{\#}_n(z))^2}{\omega^2(1-|\varphi_a(z)|)}\,dA(z) < \infty\}$$

Arguing as in the proof of (Theorem 2.4 in [10]), we deduce:

Theorem 2.7 $Q^{\#}_{K,n,\omega,0} \subset \mathcal{B}^{\#}_{n,\omega,0} = \mathcal{N}_{n,\omega,0}$.

Theorem 2.8 *If (5) holds, then* $Q^{\#}_{K,n,\omega,0} = \mathcal{N}_{n,\omega,0}$.

Remark 2.2 *It suffices to prove that $\mathcal{N}_{n,\omega,0} \subset Q^{\#}_{K,n,\omega,0}$. We deduce this using the same argument as in the first part of the proof of (Theorem 2.5 in [10]). We note that in this argument, the growth of K at infinity is unimportant since we have $\mathcal{N}_{n,\omega,0} = \mathcal{B}^{\#}_{n,\omega,0}$.*

Theorem 2.9 .
(i) If $K(0) > 0$, then $\mathcal{D}^{\#}_{n,\omega} = Q^{\#}_{K,n,\omega}$.
(ii) $\mathcal{D}^{\#}_{n,\omega} \subset Q^{\#}_{K,n,\omega,0}$ if and only if $K(0) = 0$.
(iii) Assume that $Q^{\#}_{K,n,\omega} \neq Q^{\#}_{K,n,\omega,0}$. If $\mathcal{D}^{\#}_{n,\omega} = Q^{\#}_{K,n,\omega}$, then $K(0) > 0$.
(iv) If $\mathcal{D}^{\#}_{n,\omega} = Q^{\#}_{K,n,\omega} = Q^{\#}_{K,n,\omega,0}$, then $K(0) = 0$.

Proof:
To prove (i), we assume that $K(0) > 0$ and note that $\mathcal{D}^{\#}_{n} \subset \mathcal{B}^{\#}_{n,\omega,o} = \mathcal{N}_{n,\omega,0} \subset \mathcal{N}_{n,\omega}$. If K is bounded, it is clear that $Q^{\#}_{K,n,\omega} = \mathcal{D}^{\#}_{n,\omega}$. If K is unbounded, we use Theorem 2.3 and the fact that $Q^{\#}_{K_1,n,\omega} = \mathcal{D}^{\#}_{n,\omega}$ (we use the notation of Theorem 2.3) to obtain that $Q^{\#}_{K,n,\omega} = \mathcal{N}_{n,\omega} \cap Q^{\#}_{K_1,n,\omega} = \mathcal{N}_{n,\omega} \cap \mathcal{D}^{\#}_{n,\omega} = \mathcal{D}^{\#}_{n,\omega}$
the proof of (i) is completely established.
The proof of (ii) uses the same argument as the proof of Theorem 2.7 in [10] with some simple modifications except that we again use the fact that $\mathcal{D}^{\#}_{n,\omega} \subset \mathcal{B}^{\#}_{\omega,0} = \mathcal{N}_{n,\omega,0}$.
To prove (iii), we remark that assumptions imply that $\mathcal{D}^{\#}_{n,\omega} \not\subseteq Q^{\#}_{K,n,\omega,0}$ and use (ii).
If the assumptions of (iv) hold, we have $\mathcal{D}^{\#}_{n,\omega} \subset Q^{\#}_{K,n,\omega,0}$ and the conclusion follows from (ii).

Corollary 2.2 *$\mathcal{D}^{\#}_{n,\omega} \subset Q^{\#}_{p,n,\omega,0}$ for all p ,$0 < p < \infty$.*

3 General chordal distance

In this section, we introduce and study some certain new scales of meromorphic functions in the unit disk and solve some problems connected with a general Chordal distance in these scales of spaces.
The chordal distance between the points z and w in the extended complex plane $\widehat{C} = C \cup \{\infty\}$ is

$$\chi_n(z,w) = \begin{cases} \frac{|z-w|^n}{(1+|z|^2)^{\frac{1}{n+1}}(1+|w|^2)^{\frac{1}{n+1}}} & if \;\; z,w \neq \infty;\; n \in \mathbb{N}. \\ \\ \frac{1}{(1+|z|^2)^{\frac{1}{n+1}}} & if \;\; w = \infty. \end{cases}$$

Remark 3.1 *If, we put $n = 1$ in the general chordal distance, we obtain the usual chordal distance see [2].*

The meromorphic Bergman class M^P_α is defined as the set of those $f \in M(\Delta)$ for which

$$\|f\|^p_{M^p_{\alpha,\omega}} = \int_\Delta \chi_n(f(z),0)^p \frac{(1-|z|^2)^\alpha}{\omega(1-|z|)}\, dA(z) < \infty.$$

Now, we give the following result:

Theorem 3.1 *Let $1 \leq p < \infty$, and $-1 < \alpha < \infty$ and let $f \in M(\Delta)$. Suppose that*

$$\int_{|w|}^1 \frac{\left(1-\frac{|w|}{t}\right)^\alpha}{\omega\left(1-\frac{|w|}{t}\right)} \frac{dt}{t^3} < \infty.$$

Then there exists a positive constant C, depending only on p and α, such that

$$\int_\Delta \chi_n(f(z),f(0))^p \frac{(1-|z|^2)^\alpha}{\omega(1-|z|)}\, dA(z) \leq C \int_\Delta (f^{\#}_n(z))^p \frac{(1-|z|^2)^{p+\alpha}}{\omega(1-|z|)} \frac{dA(z)}{|z|}.$$

Proof: First let $p = 1$ and let $0 < t < 1$. Since

$$\chi_n(f(z), f(0)) \leq \int_0^1 f_n^{\#}(tz)|z|\, dt,$$

Fubini's theorem and integration by parts yield

$$\begin{aligned}
\int_\Delta \chi_n(f(z), f(0)) \frac{(1-|z|^2)^\alpha}{\omega(1-|z|)}\, dA(z) &\lesssim \int_\Delta \int_0^1 f_n^{\#}(tz)\, dt|z| \frac{(1-|z|^2)^\alpha}{\omega(1-|z|)}\, dA(z) \\
&= \int_0^1 \int_{D(0,t)} f_n^{\#}(w)|w| \frac{\big(1-\frac{|w|}{t}\big)^\alpha}{\omega\big(1-\frac{|w|}{t}\big)} \frac{dt}{t^3}\, dA(w) \\
&= \int_\Delta f_n^{\#}(w)|w| \int_{|w|}^1 \frac{\big(1-\frac{|w|}{t}\big)^\alpha}{\omega\big(1-\frac{|w|}{t}\big)} \frac{dt}{t^3}\, dA(w) \\
&\lesssim \int_\Delta (f_n^{\#}(w))|w| dA(w),
\end{aligned}$$

which is the desired asymptotic inequality for $p = 1$. If $p > 1$, choose $q > ((p-1)/p)$ such that $\alpha - pq + p > 0$. By Hölder's inequality, we obtain

$$\begin{aligned}
\chi_n(f(z), f(0)) &\leq \int_0^1 f_n^{\#}(tz)|z|\, dt = \int_0^1 f_n^{\#}(1-t|z|)^q \frac{|z|\, dt}{(1-t|z|)^q} \\
&\leq \Big(\int_0^1 f_n^{\#}(tz)^p \frac{(1-t|z|)^{pq}}{\omega^p(1-t|z|)}\, dt\Big)^{1/p} \Big(\int_0^1 \frac{|z|^{(p-1)/p}\, dt}{\omega^{\frac{-p}{p-1}}(1-t|z|)(1-t|z|)^{pq/(p-1)}}\Big)^{(p-1)/p} \\
&\lesssim \Big(\int_0^1 f_n^{\#}(tz)^p(1-t|z|)^{pq}\, dt|z|(1-|z|)^{p-1-pq}\Big)^{1/p}
\end{aligned}$$

from which Fubinis theorem yields

$$\begin{aligned}
\int_\Delta \chi_n(f(z), f(0))^p \frac{(1-|z|^2)^\alpha}{\omega(1-t|z|)}\, dA(z) &\lesssim \int_\Delta \int_0^1 \big(f_n^{\#}(tz)\big)^p (1-t|z|)^{pq}\, dt|z| \frac{(1-|z|)^{\alpha+p-1-pq}}{\omega(1-t|z|)}\, dA(z) \\
&= \int_0^1 \int_{D(0,t)} \big(f_n^{\#}(w)\big)^p (1-|w|)^{pq}|w| \frac{\big(1-\frac{|w|}{t}\big)^{\alpha-pq+p-1}}{\omega(1-\frac{|w|}{t})} \frac{dt}{t^3}\, dA(w) \\
&= \int_\Delta \big(f_n^{\#}(w)\big)^p |w| \int_{|w|}^1 \frac{\big(1-\frac{|w|}{t}\big)^{\alpha+p-1}}{\omega(1-\frac{|w|}{t})} \frac{dt}{t^3}\, dA(w) \\
&\lesssim \int_\Delta f_n^{\#}(w)^p |w| dA(w).
\end{aligned}$$

Theorem 3.2 *Let $1 \leq p < \infty$ and $-1 < \alpha < \infty$, and let $f \in M(\Delta)$. Suppose that*

$$\int_\Delta |\varphi_w'(z)|^{\alpha+2} \frac{dA(w)}{\omega(1-|\varphi_w(z)|)|\varphi_w(z)|(1-|w|^2)^2} < C$$

where C is a positive constant. Then,

$$\int\int_\Delta \frac{(\chi_n(f(z), f(w))^p}{|1-\overline{w}z|^4} \frac{(1-|\varphi_w(z)|^2)^\alpha}{\omega(1-|\varphi_w(z)|)}\, dA(w) \leq \lambda \int_\Delta |\varphi_w'(z)|^{\alpha+2} \frac{dA(w)}{\omega(1-|\varphi_w(z)|)|\varphi_w(z)|(1-|w|^2)^2}.$$

Proof: By the change of variable $z = \varphi_w(u)$, Theorem 3.1 and Fubini's theorem,

$$I(f) = \int\int_\Delta \frac{(\chi_n(f(z), f(w)))^p}{|1-\overline{w}z|^4} \frac{(1-|\varphi_w(z)|^2)^\alpha}{\omega(1-|\varphi_w(z)|)}\, dA(z)\, dA(w)$$

$$\begin{aligned}
&= \int\int_{\Delta} (\chi_n((f\circ\varphi_w)(u),(f\circ\varphi_w)(0)))^p \frac{(1-|u|^2)^{\alpha}}{\omega(1-|u|)}\, dA(u) \frac{dA(w)}{(1-|w|^2)^2}\\
&\lesssim \int\int_{\Delta} ((f\circ\varphi_w)^{\#}_n(u))^p \frac{(1-|u|^2)^{p+\alpha}}{\omega(1-|u|)} \frac{dA(u)}{|u|} \frac{dA(w)}{(1-|w|^2)^2}\\
&= \int\int_{\Delta} (f^{\#}_n(\varphi_w(u)))^p (1-|\varphi_w(u)|^2)^p \frac{(1-|u|^2)^{\alpha}}{\omega(1-|u|)} \frac{dA(u)}{|u|} \frac{dA(w)}{(1-|w|^2)^2}\\
&= \int_{\Delta} (f^{\#}_n(z))^p (1-|z|^2)^{p+\alpha} \int_{\Delta} |\varphi'_w(z)|^{\alpha+2} \frac{dA(w)}{\omega(1-|\varphi_w(z)|)|\varphi_w(z)|(1-|w|^2)^2}\, dA(z).
\end{aligned}$$

But since

$$\int_{\Delta} |\varphi'_w(z)|^{\alpha+2} \frac{dA(w)}{\omega(1-|\varphi_w(z)|)|\varphi_w(z)|(1-|w|^2)^2} < C.$$

Then,

$$I(f) \leq \lambda \int_{\Delta} (f^{\#}_n(z))^p (1-|z|^2)^{p+\alpha} dA(z).$$

Remark 3.2 *In Theorem 3.2, if we put $n = 1$, we obtain theorem 1.2 in [2].*

Corollary 3.1 *Let $2 < p < \infty$ and $f \in M(\Delta)$. Then there exists a positive constant C, depending only on p, such that*

$$\int\int_{\Delta} \frac{\chi_n(f(z)-f(w))^p}{|1-\overline{w}z|} \Big(\frac{(1-|z|^2)^{(p/2)-2}}{\omega(1-|z|)}\Big)\Big(\frac{(1-|w|^2)^{(p/2)-2}}{\omega(1-|w|)}\Big)\, dA(z)\, dA(w) \;\leq C\|f\|^p_{B^{\#}_{p,n}}.$$

An application of Theorem 3.1 with $\alpha = 0$ to the function $.(fo\varphi_w)(rz)$ yields

$$\int_{\Delta(w,r)} \chi_n(f(z),f(w))^p\, dA(z) \lesssim \int_{\Delta(w,r)} (f^{\#}_n(z))^p \Big(\frac{(1-|z|^2)^p}{\omega(1-|z|)}\Big) \frac{dA(z)}{|\varphi_w(z)|}, \tag{12}$$

where $\Delta(w,r) = \{z : |\varphi_w(z)| < r\}$ is the pseudohyperbolic disc of (pseudohyperbolic) center $w \in \Delta$ and radius $r \in (0,1)$, and the constant of comparison depends only on r . This fact can be used to prove Theorem 3.3. The class $M^{\#}_{n,\omega}(p,q,s)$ consists of those $f \in M(\Delta)$ for which

$$\|f\|^p_{M^{\#}_{n,\omega}(p,q,s)} = \sup_{a\in\Delta} \int_{\Delta} (f^{\#}_n(z))^p \Big(\frac{(1-|z|^2)^q}{\omega(1-|z|)}\Big)\Big(\frac{(1-|\varphi_a(z)|^2)^s}{\omega(1-|\varphi_a(z)|)}\Big)\, dA(z) < \infty.$$

For the next result, let $|D(z,r)|$ denote the Euclidean area of $D(z,r)$, so by [[12], p. 3], we have that

$$|D(z,r)| = \pi r \frac{(1-|a|^2)^2}{(1-|a|^2r^2)^2} \tag{13}$$

Theorem 3.3 *Let $1 \leq p < \infty$, $-2 < q < \infty$, $0 \leq s < \infty$ and $o < r < 1$. Let $\alpha,\beta,\gamma,\delta \in \mathbb{R}$ such that $\alpha+\beta = q-p$, and $\gamma+\delta = s$, and let $f \in M(\Delta)$. Then*

$$\sup_{a\in\Delta} \int_{\Delta} \Big(\frac{1}{|D(z,r)|} \int_{D(z,r)} \chi_n(f(z),f(w))^p \Big(\frac{(1-|z|^2)^{\alpha}}{\omega(1-|z|)}\Big)\Big(\frac{(1-|w|^2)^{\beta}}{\omega(1-|w|)}\Big)$$

$$.\Big(\frac{(1-|\varphi_a(z)|^2)^{\gamma}}{\omega(1-|\varphi_a(z)|)}\Big)\Big(\frac{(1-|\varphi_a(w)|^2)^{\delta}}{\omega(1-|\varphi_a(w)|)}\Big)\, dA(w)\Big)\, dA(z) \leq \|f\|^p_{M^{\#}_{n,\omega}(p,q,s)}.$$

Proof: Routine calculations and (15) show that for $w \in D(z,r)$ and $a \in \Delta$,

$$1-|z|^2 \simeq 1-|w|^2 \simeq 1-|\overline{w}z|^2 \simeq |D(z,r)|^{1/2}, \tag{14}$$

and

$$1-|\varphi_a(z)|^2 \simeq 1-|\varphi_a(w)|^2, \tag{15}$$

where the constants of comparison depend only on r . By (16), (17) and (14),

$$\begin{aligned} I &= \sup_{a\in\Delta}\int_\Delta \Big(\frac{1}{|D(z,r)|}\int_{D(z,r)} \big(\chi_n(f(z),f(w))\big)^p \Big(\frac{(1-|z|^2)^\alpha}{\omega(1-|z|)}\Big)\Big(\frac{(1-|w|^2)^\beta}{\omega(1-|w|)}\Big) \\ &\quad . \Big(\frac{(1-|\varphi_a(z)|^2)^\gamma}{\omega(1-|\varphi_a(z)|)}\Big)\Big(\frac{(1-|\varphi_a(w)|^2)^\delta}{\omega(1-|\varphi_a(w)|)}\Big)\, dA(w)\Big)\, dA(z) \\ &\lesssim \sup_{a\in\Delta}\int_\Delta \Big(\int_{D(z,r)} (f_n^{\#}(w))^p \Big(\frac{(1-|w|^2)^p}{\omega(1-|w|)}\Big)\frac{dA(w)}{|\varphi_z(w)|}\Big)\Big(\frac{(1-|z|^2)^{q-p-2}}{\omega(1-|z|)}\Big).\Big(\frac{(1-|\varphi_a(z)|^2)^s}{\omega(1-|\varphi_a(z)|)}\Big)\, dA(z) \end{aligned}$$

from which (16), (17) and Fubini's theorem yield

$$\begin{aligned} I &\lesssim \sup_{a\in\Delta}\int_\Delta \Big(\int_{D(z,r)} (f_n^{\#}(w))^p \Big(\frac{(1-|w|^2)^{q-2}}{\omega(1-|w|)}\Big)\Big(\frac{(1-|\varphi_a(w)|^2)^s}{\omega(1-|\varphi_a(w)|)}\Big)\frac{dA(w)}{|\varphi_z(w)|}\Big)\, dA(z) \\ &= \sup_{a\in\Delta}\int_\Delta \Big(\int_{D(z,r)} \frac{dA(z)}{|\varphi_z(w)|}\Big)(f_n^{\#}(w))^p \Big(\frac{(1-|w|^2)^{q-2}}{\omega(1-|w|)}\Big)\Big(\frac{(1-|\varphi_a(w)|^2)^s}{\omega(1-|\varphi_a(w)|)}\Big)\, dA(w) \\ &\simeq \sup_{a\in\Delta}\int_\Delta (f_n^{\#}(w))^p \Big(\frac{(1-|w|^2)^q}{\omega(1-|w|)}\Big)\Big(\frac{(1-|\varphi_a(w)|^2)^s}{\omega(1-|\varphi_a(w)|)}\Big)\, dA(w). \end{aligned}$$

The class $\mathcal{N}$ of normal functions consists of those $f \in M(\Delta)$ for which the family $\{fo\varphi\}$, where φ is a *Möbius* transformation of Δ, is normal in Δ in the sense of Montel. It is known that $f \in M(\Delta)$ is all normal if and only if

$$\|f\|_{\mathcal{N}_n,\omega} = \sup_{z\in\Delta} f_n^{\#}(z)\frac{(1-|z|^2)}{\omega(1-|z|)} < \infty.$$

The following result establishes a sufficient condition for the general normal meromorphic functions to belong to $M^{\#}_{n,\omega}(p,q,s)$.

Theorem 3.4 *Let $1 \le p < \infty$, $-2 < q < \infty$, $0 \le s < \infty$ and $0 < r < 1$, and let $f \in \mathcal{N}_{n,\omega}$. Let $\alpha,\beta,\gamma,\delta \in \mathbb{R}$ such that $\alpha+\beta = q-p$, and $\gamma+\delta = s$. Then*

$$\begin{aligned} \|f\|^P_{M^{\#}_{n,\omega}(p,q,s)} &\lesssim \sup_{a\in\Delta}\int_\Delta \Big(\frac{1}{|D(z,r)|}\int_{D(z,r)} \chi_n(f(z),f(w))\Big(\frac{(1-|w|^2)^{\alpha/p}}{\omega(1-|w|)}\Big)\Big(\frac{(1-|z|^2)^{\beta/p}}{\omega(1-|z|)}\Big) \\ &\quad .\Big(\frac{(1-|\varphi_a(w)|^2)^{\gamma/p}}{\omega(1-|\varphi_a(w)|)}\Big)\Big(\frac{(1-|\varphi_a(z)|^2)^{\delta/p}}{\omega(1-|\varphi_a(z)|)}\Big)\, dA(w)\Big)^p\, dA(z). \end{aligned}$$

Proof: Let $z, w \in \widehat{\mathbb{C}}$, and define

$$F_n(z,w) = \begin{cases} \frac{w-z}{1+\overline{w}z} & if \;\; w \in \mathbb{C}. \\ \frac{1}{z} & if \;\; w = \infty. \end{cases}$$

A direct calculation shows that $|F_n(z,w)|^2 = \chi_n^2(z,w)/(1-\chi_n^2(z,w))$ for all $z,w \in \widehat{\mathbb{C}}$. Denote the pseudohyperbolic distance between the points z and w in Δ by $\rho(z,w) = |\varphi_z(w)|$. By the uniform (ρ,χ)-continuity of f, there is an $r_1 \in (0,1)$ such that $\chi_n(f(z), f(w)) < C$, for $\rho(z,w) < r_1$ [13], where C is a positive constant. Then, it follows that

$$|F_n(f(z), f(w))| = \frac{\chi_n(f(z), f(w))}{\sqrt{1-\chi_n^2(f(z), f(w))}} < C\chi_n(f(z), f(w)) \tag{16}$$

for $\rho(z,w) < r_1$. Since $f \in M(\Delta)$, there is an $r_2 \in (0,1)$ such that the function $g_z(w) = F_n((fo\varphi_z)(w), f(z))$is analytic in $D(0,r_2) = \{w : \rho(0,w) = |w| < r_2\}$ for all $z \in \Delta$, and hence its Maclaurin series is of the form $\sum_{k=1}^{\infty} a_k(z)w^k$ in $D(0,r_2)$.Therefore

$$\begin{aligned} f_n^{\#}(z)(1-|z|^2) &= |a_1| = \frac{2}{r^4}\Big|\int_{D(0,r)} \overline{w} g_z(w)\, dA(w)\Big| \\ &\leq \frac{2}{r^3}\int_{D(0,r)} \big|F_n((fo\varphi_z)(w), f(z))\big|\, dA(w) \end{aligned} \tag{17}$$

for any $r \in (0, r_2)$. Now let $r < \min\{r_1, r_2\}$. Then, we obtain that

$$\begin{aligned} I(f) &= \int_\Delta (f_n^{\#}(z)\big(\frac{(1-|z|^2)^p}{\omega(1-|z|)}\big)\big(\frac{(1-|z|)^{q-p}}{\omega(1-|z|)}\big)\big(\frac{(1-|\varphi_a(z)|^2)^s}{\omega(1-|\varphi_a(z)|)}\big)\, dA(z) \\ &\leq \int_\Delta \big(\frac{2}{r^3}\int_{D(0,r)} \big|F_n((fo\varphi_z)(w), f(z))\big|\, dA(u)\big)^p \big(\frac{(1-|z|)^{q-p}}{\omega(1-|z|)}\big)\big(\frac{(1-|\varphi_a(z)|^2)^s}{\omega(1-|\varphi_a(z)|)}\big)\, dA(z) \\ &= \int_\Delta \big(\frac{2}{r^3}\int_{D(z,r)} \big|F_n((f(u), f(z))\big||\varphi_z'(u)|^2\, dA(u)\big)^p \big(\frac{(1-|z|)^{q-p}}{\omega(1-|z|)}\big)\big(\frac{(1-|\varphi_a(z)|^2)^s}{\omega(1-|\varphi_a(z)|)}\big)\, dA(z) \\ &\leq \int_\Delta \big(\frac{C}{r^3}\int_{D(z,r)} \chi_n(f(u), f(z))|\varphi_z'(u)|^2\, dA(u)\big)^p \big(\frac{(1-|z|)^{q-p}}{\omega(1-|z|)}\big)\big(\frac{(1-|\varphi_a(z)|^2)^s}{\omega(1-|\varphi_a(z)|)}\big)\, dA(z). \end{aligned} \tag{18}$$

from which the assertion for $r < \min\{r_1, r_2\}$ follows by (16) and (17). If $r \geq \min\{r_1, r_2\}$, choose $c > 1$ such that $r^* = r/c < \min\{r_1, r_2\}$. Then, we easily obtain the assertion for r^*. To obtain the assertion for r , it remains to make the set of integration larger by replacing $D(z, r^*)$ by $D(z,r)$ and note that there is a constant C, depending only on c, such that $|D(z,r^*)| \geq C|D(z,r)|$ for all $z \in \Delta$.

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Maximum Norm Superconvergence of the Trilinear Block Finite Element

Jinghong Liu*and Yinsuo Jia†

In this article we discuss a pointwise superconvergence post-processing technique for the gradient of the trilinear block finite element for the Poisson equation with homogeneous Dirichlet boundary conditions over a fully uniform mesh of the three-dimensional domain Ω. First, the supercloseness of the gradients between the piecewise trilinear finite element solution u_h and the trilinear interpolant Πu is given. Secondly, we analyze a superconvergence post-processing scheme for the gradient of the finite element solution by using the Z-Z recovery technique, which shows that the recovered gradient of u_h is superconvergent to the gradient of the true solution u in the pointwise sense of the L^∞-norm. Finally, a numerical example is given.

1 Introduction

Superconvergence of the gradient for the finite element approximation is a phenomenon whereby the convergent order of the derivatives of the finite element solutions exceeds the optimal global rate. Up to now, superconvergence is still an active research topic; see, for example, Babuška and Strouboulis [1], Chen [2], Chen and Huang [3], Lin and Yan [4], Wahlbin [5] and Zhu and Lin [6] for overviews of this field. Nevertheless, how to obtain the superconvergent numerical solution is an issue to researchers. In general, it needs to use post-processing techniques to get recovered gradients with high order accuracy from the finite element solution. Usual post-processing techniques include interpolation technique, projection technique, average technique, extrapolation technique, superconvergence patch recovery (SPR) technique introduced by Zienkiewicz and Zhu [7–9] and polynomial patch recovery (PPR) technique raised by Zhang and Naga [10]. In previous works, for the linear tetrahedral element, Chen and Wang [11] obtained the recovered gradient with $\mathcal{O}(h^2)$ order accuracy in the average sense of the L^2-norm by using the SPR technique. Using the L^2-projection technique, in the average sense of the L^2-norm, Chen [12] got the recovered gradient with $\mathcal{O}(h^{1+min(\sigma,\frac{1}{2})})$ order accuracy. Goodsell [13] derived by using the average

*Department of Fundamental Courses, Ningbo Institute of Technology, Zhejiang University, Ningbo 315100, China, email: jhliu1129@sina.com

†School of Mathematics and Computer Science, Shangrao Normal University, Shangrao 334001, China, email: 550897472@qq.com

technique the pointwise superconvergence estimate of the recovered gradient with $\mathcal{O}(h^{2-\varepsilon})$ order accuracy. Brandts and Křížek [14] obtained by using the interpolation technique the recovered gradient with $\mathcal{O}(h^2)$ order accuracy in the average sense of the L^2-norm. Zhang [15, 16] gave the theoretical analysis for the SPR technique for the one-dimensional two points boundary value problem and two-dimensional Laplacian equations, which proved two orders higher than the optimal convergence rate of the finite element solution at the internal nodal points over uniform meshes. Zhang and Victory [17] presented the theoretical justification for superconvergence of the SPR technique for a general second-order elliptic equation over the quadrilateral meshes. Zhang and Zhu [18, 19] also analyzed the SPR technique in details as well as its applications to a posteriori error estimation. In this article, we consider a SPR recovery scheme by using the Z-Z technique, by which the pointwise superconvergence recovered gradient from the trilinear finite element approximation can be obtained. We shall use the letter C to denote a generic constant which may not be the same in each occurrence and also use the standard notations for the Sobolev spaces and their norms.

2 Maximum Norm Supercloseness

Suppose $\Omega \subset R^3$ is a rectangular block with boundary, $\partial\Omega$, consisting of faces parallel to the x-, y-, and z-axes. Moreover, Ω is partitioned into a uniform rectangulation $\mathcal{T}^h$ with mesh size $h \in (0, 1)$ such that $\bar{\Omega} = \bigcup_{e\in\mathcal{T}^h} \bar{e}$. We consider the following Poisson equation with homogeneous Dirichlet boundary value conditions

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

The corresponding weak form is

$$a(u\,,\,v) = (f\,,\,v),\,\forall\, v \in H_0^1(\Omega), \tag{2.2}$$

where

$$a(u\,,\,v) \equiv (\nabla u\,,\,\nabla v) = \int_\Omega \nabla u \cdot \nabla v\, dxdydz.$$

We introduce a trilinear polynomial space Q_1, namely

$$q(x, y, z) = \sum_{(i,j,k)\in I} a_{ijk} x^i y^j z^k,\ q \in Q_1,$$

where the indexing set I is as follows:

$$I = \{(i, j, k)|0 \le i, j, k \le 1\}.$$

Denote the trilinear finite element space by

$$S_0^h(\Omega) = \left\{ v \in C(\bar{\Omega}) \bigcap H_0^1(\Omega) : v|_e \in Q_1(e),\, \forall\, e \in \mathcal{T}^h \right\}. \tag{2.3}$$

Thus the finite element method is to find $u_h \in S_0^h(\Omega)$ such that

$$a(u_h\,,\,v) = (f\,,\,v),\ \forall\, v \in S_0^h(\Omega).$$

Obviously, there is the following Galerkin orthogonality relation

$$a(u - u_h\,,\,v) = 0,\ \forall\, v \in S_0^h(\Omega). \tag{2.4}$$

Let the element

$$e = (x_e - h_e, x_e + h_e) \times (y_e - k_e, y_e + k_e) \times (z_e - d_e, z_e + d_e) \equiv I_1 \times I_2 \times I_3,$$

and let $\{l_j(x)\}_{j=0}^{\infty}$, $\{\tilde{l}_j(y)\}_{j=0}^{\infty}$, $\{\bar{l}_j(z)\}_{j=0}^{\infty}$ be the normalized orthogonal Legendre polynomial systems on $L^2(I_1)$, $L^2(I_2)$, and $L^2(I_3)$, respectively. It is easy to see that $\{l_i(x)\tilde{l}_j(y)\bar{l}_k(z)\}_{i,j,k=0}^{\infty}$ is the normalized orthogonal polynomial system on $L^2(e)$. Set

$$\omega_0(x) = \tilde{\omega}_0(y) = \bar{\omega}_0(z) = 1,\ \omega_{j+1}(x) = \int_{x_e-h_e}^{x} l_j(\xi)\,d\xi,$$

$$\tilde{\omega}_{j+1}(y) = \int_{y_e-k_e}^{y} \tilde{l}_j(\xi)\,d\xi,\ \bar{\omega}_{j+1}(z) = \int_{z_e-d_e}^{z} \bar{l}_j(\xi)\,d\xi,\ \ j \geq 0.$$

Define the trilinear interpolation operator of projection type by Π^e: $H^3(e) \to Q_1(e)$ such that

$$\Pi^e u(x,y,z) = \sum_{(i,j,k)\in I} \beta_{ijk}\omega_i(x)\tilde{\omega}_j(y)\bar{\omega}_k(z). \tag{2.5}$$

where $\beta_{000} = u(x_e{-}h_e, y_e{-}k_e, z_e{-}d_e)$, $\beta_{i00} = \int_{I_1} \partial_x u(x, y_e{-}k_e, z_e{-}d_e)l_{i-1}(x)\,dx$, $\beta_{0j0} = \int_{I_2} \partial_y u(x_e{-}h_e, y, z_e{-}d_e)\tilde{l}_{j-1}(y)\,dy$, $\beta_{00k} = \int_{I_3} \partial_z u(x_e{-}h_e, y_e{-}k_e, z)\bar{l}_{k-1}(z)\,dz$, $\beta_{ij0} = \int_{I_1\times I_2} \partial_x\partial_y u(x,y,z_e-d_e)l_{i-1}(x)\tilde{l}_{j-1}(y)\,dxdy$, $\beta_{0jk} = \int_{I_2\times I_3} \partial_y\partial_z u(x_e - h_e, y, z)\tilde{l}_{j-1}(y)\bar{l}_{k-1}(z)\,dydz$, $\beta_{i0k} = \int_{I_1\times I_3} \partial_x\partial_z u(x, y_e{-}k_e, z)l_{i-1}(x)\bar{l}_{k-1}(z)\,dxdz$, $\beta_{ijk} = \int_e \partial_x\partial_y\partial_z u\, l_{i-1}(x)\tilde{l}_{j-1}(y)\bar{l}_{k-1}(z)\,dxdydz,\ i,j,k \geq 1$.
In addition, we define $(\Pi u)|_e = \Pi^e u$. Thus we have the global interpolation operator of projection type Π: $H^3(\Omega) \to S_0^h(\Omega)$. In [20], we obtained the following supercloseness estimate

Lemma 2.1. Let $\{\mathcal{T}^h\}$ be a regular family of rectangular partitions of Ω, and $u \in W^{3,\infty}(\Omega)\bigcap H_0^1(\Omega)$. For u_h and Πu, the trilinear block finite element approximation and the corresponding interpolant of projection type to u, respectively. Then we have the following supercloseness estimate

$$|u_h - \Pi u|_{1,\,\infty,\,\Omega} \leq Ch^2\,|\ln h|^{\frac{4}{3}}\,\|u\|_{3,\,\infty,\,\Omega}. \tag{2.6}$$

3 Maximum Norm Superconvergence

SPR is a gradient recovery method introduced by Zienkiewicz and Zhu. This method is now widely used in engineering practices for its robustness in a posterior error estimation and its efficiency in computer implementation.

For $v \in S_0^h(\Omega)$, we denote by R_x the SPR-recovery operator (or Z-Z recovery operator) with respect to the x-derivative, and begin by defining the point values of $R_x v$ at the element nodes. After the recovered derivative values at all nodes are obtained, we construct a piecewise trilinear interpolant by using these values to obtain a global recovered derivative, namely SPR-recovery derivative $R_x v$. Obviously $R_x v \in S_0^h(\Omega)$. Similarly, we can define by R_y and R_z the recovered derivatives with respect to the y-derivative and the z-derivative, respectively. Consequently, we get a recovered gradient operator $R_h = (R_x, R_y, R_z)$. In the following, we mainly discuss the recovery operator R_x and its superconvergence properties. The superconvergence properties of R_y and R_z can be similarly derived.

Let us first assume N is an interior node of the partition $\mathcal{T}^h$, and denote by ω the element patch around N containing eight elements (see Fig.1).

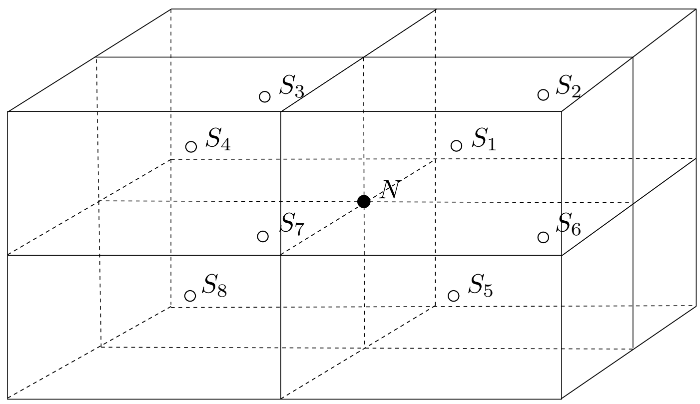

FIG. 1. Element Patch Containing Eight Elements

Under the local coordinate system centered N, we let S_j be the barycenter of an element $e_j \subset \omega$, $j = 1, 2, \cdots, 8$. SPR uses the discrete least-squares fitting to seek linear function $p \in P_1(\omega)$, such that

$$|||p - \partial_x v||| = \min_{g \in P_1(\omega)} |||g - \partial_x v|||, \tag{3.1}$$

where $|||w||| = (\sum_{j=1}^{8} |w(S_j)|^2)^{\frac{1}{2}}$. Obviously, for $w \in P_1(\omega)$, we have

$$|||w||| = 0 \Longleftrightarrow w = 0$$

. It is easy to verify that the problem (3.1) is equivalent to the following problem

$$\sum_{j=1}^{8}[p(S_j)-\partial_x v(S_j)]g(S_j)=0,\ \forall g\in P_1(\omega). \tag{3.2}$$

Then we define $R_x v(N)=p(0,0,0)$. If N is a node on the boundary, $\partial\Omega$, of Ω, we can calculate $R_x v(N)$ by the linear extrapolation from the values of $R_x v$ already obtained at two neighboring interior nodes, N_1 and N_2, namely

$$R_x v(N)=2R_x v(N_1)-R_x v(N_2). \tag{3.3}$$

Lemma 3.1. Let ω be the element patch around an interior node N, S_j the barycenter of the element $e_j\subset\omega,\ j=1.\cdots.8$, and Π the trilinear interpolation operator of projection type. For every $u\in P_2(\omega)$, we have

$$\partial_x(u-\Pi u)(S_j)=0. \tag{3.4}$$

Proof. Obviously, S_j is a Gauss point of the element $e_j\subset\omega$. From the definition of the operator Π,

$$u-\Pi u=\left(\sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=2}^{\infty}+\sum_{i=0}^{1}\sum_{j=2}^{\infty}\sum_{k=0}^{\infty}+\sum_{i=2}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\right)\beta_{ijk}\omega_i(x)\tilde{\omega}_j(y)\bar{\omega}_k(z).$$

By the representation of the coefficient β_{ijk} and the orthogonality of the *Legendre* polynomial system, we obtain for $u\in P_2(\omega)$,

$$\partial_x(u-\Pi u)(S_j)=0,$$

which is the desired result (3.4).

Lemma 3.2. Let ω be the element patch around an interior node N and Π the trilinear interpolation operator of projection type. For every $u\in P_2(\omega)$, we have

$$\partial_x u-R_x\Pi u=0\ \text{ in }\omega. \tag{3.5}$$

Proof. From (3.4) and the definition (3.1) of the recovery operator R_x, we have for $u\in P_2(\omega)$,

$$R_x u=R_x\Pi u. \tag{3.6}$$

Since $u\in P_2(\omega)$, thus $\partial_x u\in P_1(\omega)$. So we obtain

$$R_x u=\partial_x u. \tag{3.7}$$

Combining (3.6) and (3.7) yields the desired result (3.5).

Lemma 3.3. For $\Pi u\in S_0^h(\Omega)$ the trilinear interpolant of projection type to u, the solution of (2.2), and R_x the x-derivative recovered operator by SPR, we have the superconvergent estimate

$$|\partial_x u-R_x\Pi u|_{0,\infty,\Omega}\le Ch^2\|u\|_{3,\infty,\Omega}. \tag{3.8}$$

Proof. By the triangle inequality, the norms equivalence of the finite-dimensional space, and the inverse property, we have

$$\begin{aligned} & |\partial_x u - R_x\Pi u|_{0,\infty,\Omega} = |\partial_x u - R_x\Pi u|_{0,\infty,e} \le |\partial_x u|_{0,\infty,e} + |R_x\Pi u|_{0,\infty,e} \\ \le\ & C\left(|\partial_x u|_{0,\infty,e} + |||R_x\Pi u|||\right) \le C\left(|\partial_x u|_{0,\infty,e} + |||\partial_x\Pi u|||\right) \\ \le\ & C\left(|\partial_x u|_{0,\infty,\omega} + |\partial_x\Pi u|_{0,\infty,\omega}\right) \le C\left(|\partial_x u|_{0,\infty,\omega} + h^{-1}\,|u|_{0,\infty,\omega}\right), \end{aligned} \tag{3.9}$$

where ω is an element patch containing the element e. Let $u_I \in P_2(\omega)$ be a quadratic interpolant to u. From (3.5) and (3.9), we obtain by using the interpolation error estimate,

$$\begin{aligned} |\partial_x u - R_x\Pi u|_{0,\infty,\Omega} &= |\partial_x(u-u_I) - R_x\Pi(u-u_I)|_{0,\infty,e} \\ &\le C\left(|\partial_x(u-u_I)|_{0,\infty,\omega} + h^{-1}\,|u-u_I|_{0,\infty,\omega}\right), \\ &\le Ch^2\,\|u\|_{3,\infty,\Omega}\,. \end{aligned}$$

This proves the statement.

As for the y-derivative recovery operator R_y and the z-derivative recovery operator R_z, we have the following results similar to (3.8).

$$|\partial_y u - R_y\Pi u|_{0,\infty,\Omega} \le Ch^2\|u\|_{3,\infty,\Omega}. \tag{3.10}$$

$$|\partial_z u - R_z\Pi u|_{0,\infty,\Omega} \le Ch^2\|u\|_{3,\infty,\Omega}. \tag{3.11}$$

Set $R_h = (R_x, R_y, R_z)$. Combining (3.8), (3.10) and (3.11) yields

$$|\nabla u - R_h\Pi u|_{0,\infty,\Omega} \le Ch^2\|u\|_{3,\infty,\Omega}. \tag{3.12}$$

In the following, we give the main result of this article.

Theorem 3.1. For $u_h \in S_0^h(\Omega)$ the trilinear block finite element approximation to u, the solution of (2.2), and R_h the gradient recovered operator by SPR, we have the superconvergent estimate

$$|\nabla u - R_h u_h|_{0,\infty,\Omega} \le Ch^2|\ln h|^{\frac{4}{3}}\|u\|_{3,\infty,\Omega}.$$

Proof. Using the triangle inequality and the norms equivalence of the finite-dimensional space, we have

$$\begin{aligned} |\nabla u - R_h u_h|_{0,\infty,\Omega} &\le |R_h(u_h - \Pi u)|_{0,\infty,\Omega} + |\nabla u - R_h\Pi u|_{0,\infty,\Omega} \\ &= |R_h(u_h - \Pi u)|_{0,\infty,e} + |\nabla u - R_h\Pi u|_{0,\infty,\Omega} \\ &\le C\left(|||R_h(u_h - \Pi u)||| + |\nabla u - R_h\Pi u|_{0,\infty,\Omega}\right) \\ &\le C\left(|||\nabla(u_h - \Pi u)||| + |\nabla u - R_h\Pi u|_{0,\infty,\Omega}\right) \\ &\le C\left(|u_h - \Pi u|_{1,\infty,\Omega} + |\nabla u - R_h\Pi u|_{0,\infty,\Omega}\right). \end{aligned} \tag{3.13}$$

Combining (2.6), (3.12) and (3.13) yields

$$|\nabla u - R_h u_h|_{0,\infty,\Omega} \le Ch^2|\ln h|^{\frac{4}{3}}\|u\|_{3,\infty,\Omega}.$$

This proves the statement.

4 A Numerical Example

Example 1. Consider the following Poisson's equation:

$$\begin{cases} -\Delta u = f & \text{in } \Omega = [0,1]\times[0,1]\times[0,1], \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$\begin{aligned} f &= (-e^x(e^y-(e-1)y-1) - e^y(e^x-(e-1)x-1) \\ &\quad +\pi^2(e^x-(e-1)x-1)(e^y-(e-1)y-1))\sin(\pi z). \end{aligned}$$

The exact solution is

$$u = (e^x-(e-1)x-1)(e^y-(e-1)y-1)\sin(\pi z).$$

Let u_h be the trilinear block finite element approximation to the exact solution u and $N_0 = (0.5, 0.5, 0.5)$. We solve Example 1 and obtain the following numerical results:

Table 4.1 Results of the derivatives post-processing at the interior vertex N_0

h	$\lvert\partial_x u(N_0) - R_x u_h(N_0)\rvert$
0.25	1.8364e-003
0.125	4.0003e-004
0.0625	9.6873e-005

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HYERS-ULAM STABILITY OF AN ADDITIVE FUNCTIONAL INEQUALITY

MING FANG AND DONGHE PEI*

Abstract. In this paper, we prove that the generalized Hyers-Ulam stability of the additive functional inequality

$$\|f(2x + y + 2z) + f(2x + 3y + 3z) + f(4x + 4y + 3z)\| \leq \|8f(x + y + z)\|$$

in β-homogeneous F-spaces.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [32] concerning the stability of group homomorphisms. Hyers [11] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [22] for linear mappings by considering an unbounded Cauchy difference. The paper of Rassias [22] has provided a lot of influence in the development of what we call *generalized Hyers-Ulam stability* of functional equations. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [9] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach. The stability problems for several functional equations or inequations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [2]–[8],[10], [12]–[15], [21]–[24],[25]-[30],[34]).

We recall a fundamental result in fixed point theory.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) d(x,y)=0 if and only if x=y;
(2) d(x,y)=d(y,x) for all $x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 1.1 (see[6],[7]). *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for*

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*Corresponding author:peidh340@nenu.edu.cn (D.Pei).

each given element $x \in X$, either

$$d(J^n x, J^{n+1}x) = \infty \tag{1.1}$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1}x) < \infty$, *for all $n \geq n_0$;*
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X | d(J^{n_0}x, y) < \infty\}$;*
(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ *for all $y \in Y$.*

By the using fixed point method, the stability problems of several functional inequations have been extensively investigated by a number of authors(see[5][6][14][17]-[18]).

We recall some basic facts concerning β-homogeneous F-spaces.

Definition 1.2. Let X be a linear space. A nonnegative valued function $\|\cdot\|$ is an F-norm if it satisfies the following conditions:

(FN_1) $\|x\| = 0$ if and only if $x = 0$;
(FN_2) $\|\lambda x\| = \|x\|$ for all $x \in X$ and all λ with $|\lambda| = 1$;
(FN_3) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$;
(FN_4) $\|\lambda_n x\| \to 0$ provided $\lambda_n \to 0$;
(FN_5) $\|\lambda x_n\| \to 0$ provided $\|x_n\| \to 0$.

Then $(X, \|\cdot\|)$ is called an F^*-space. An F-space is a complete F^*-space.

A F-norm is called β-homogeneous ($\beta > 0$) if $\|tx\| = |t|^\beta \|x\|$ for all $x \in X$ and all $t \in \mathbb{R}$ (see [31]).

2. HYers-Ulam Stability In β-homogeneous F-spaces

From now on , Let $\mathcal{X}$ be a normed linear space and $\mathcal{Y}$ a β-homogeneous F-spaces.

This paper,we prove that the generalized Hyers-Ulam stability of the additive functional inequality

$$\|f(2x + y + 2z) + f(2x + 3y + 3z) + f(4x + 4y + 3z)\| \leq \|8f(x + y + z)\|$$

in β-homogeneous F-spaces.

Lemma 2.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping with $f(0) = 0$. Then it is additive if and only if it satisfies*

$$\|f(2x + y + 2z) + f(2x + 3y + 3z) + f(4x + 4y + 3z)\| \leq \|8f(x + y + z)\| \tag{2.1}$$

for all $x, y, z \in \mathcal{X}$.

Proof. If f is additive, then clearly

$$\|f(2x + y + 2z) + f(2x + 3y + 3z) + f(4x + 4y + 3z)\| = \|8f(x + y + z)\|$$

for all $x, y, z \in \mathcal{X}$.

Assume that f satisfies (2.1). Suppose that $f(0) = 0$. putting $z = 0$ and replacing y by $-x$ in (2.1), we get

$$\|f(x) + f(-x)\| \leq \|8f(0)\| = 8^\beta \|f(0)\| = 0$$

and so $f(-x) = -f(x)$ for all $x \in \mathcal{X}$. Replacing y by $-x - z$ in (2.1), we have

$$\|f(-y) + f(-x) + f(x+y)\| \leq 0$$

for all $x, y \in \mathcal{X}$. We obtain

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in \mathcal{X}$. □

Theorem 2.2. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping with $f(0) = 0$. If there is a function $\varphi : X^3 \to [0, \infty)$ such that*

$$\begin{aligned} &\|f(2x + y + 2z) + f(2x + 3y + 3z) + f(4x + 4y + 3z)\| \\ &\leq \|8f(x + y + z)\| + \varphi(x, y, z) \end{aligned} \tag{2.2}$$

and

$$\widetilde{\varphi}(x, y, z) := \sum_{i=0}^{\infty} \frac{1}{2^{\beta j}} \varphi\left((-2)^j x, (-2)^j y, (-2)^j z\right) < \infty \tag{2.3}$$

for all $x, y, z \in \mathcal{X}$, then there exists a unique additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - A(x)\| \leq \widetilde{\varphi}(-x, -x, 2x) \tag{2.4}$$

for all $x \in \mathcal{X}$.

Proof. Letting $y = x$ and $z = -2x$ in (2.2), we get

$$\|2f(-x) + f(2x)\| \leq \varphi(x, x, -2x)$$

for all $x \in \mathcal{X}$. Thus

$$\left\| f(x) - \frac{f(-2x)}{-2} \right\| \leq \frac{1}{2^\beta} \varphi(-x, -x, 2x)$$

for all $x \in \mathcal{X}$.

Hence one may have the following formula for positive integers m, l with $m > l$,

$$\begin{aligned} &\left\| \frac{1}{(-2)^l} f\left((-2)^l x\right) - \frac{1}{(-2)^m} f\left((-2)^m x\right) \right\| \\ &\leq \sum_{i=l}^{m-1} \frac{1}{2^{\beta i}} \varphi\left(-(-2)^i x, -(-2)^i x, (-2)^i 2x\right) \end{aligned} \tag{2.5}$$

for all $x \in \mathcal{X}$. It follows from (2.5) that the sequence $\left\{\frac{f((-2)^k x)}{(-2)^k}\right\}$ is a Cauchy sequence for all $x \in \mathcal{X}$. Since $\mathcal{Y}$ is an F-space, the sequence $\left\{\frac{f((-2)^k x)}{(-2)^k}\right\}$ converges. So one may define the mapping $A : \mathcal{X} \to \mathcal{Y}$ by

$$A(x) := \lim_{k\to\infty} \left\{\frac{f((-2)^k x)}{(-2)^k}\right\}, \quad \forall x \in \mathcal{X}.$$

Taking $m = 0$ and letting l tend to ∞ in (2.5), we have the inequality (2.4).

It follows from (2.2) that

$$\begin{aligned}
&\|A(2x+y+2z) + A(2x+3y+3z) + A(4x+4y+3z)\| \\
&= \lim_{k\to\infty} \left|\frac{1}{(-2)^{k\beta}}\right| \left\|f((-2)^k(2x+y+2z)) + f((-2)^k(2x+3y+3z))\right. \\
&\left.+f((-2)^k(4x+4y+3z))\right\| \qquad (2.6)\\
&\le \lim_{k\to\infty} \left|\frac{1}{(-2)^{k\beta}}\right| \left\|8f((-2)^k(x+y+z))\right\| + \lim_{k\to\infty} \left|\frac{1}{(-2)^{k\beta}}\right| \varphi((-2)^k x, (-2)^k y, (-2)^k z) \\
&\le \|8A(x+y+z)\|
\end{aligned}$$

for all $x, y, z \in \mathcal{X}$. One see that A satisfies the inequality (2.1) and so it is additive by Lemma (2.1).

Now, we show that the uniqueness of A. Let $T : X \to Y$ be another additive mapping satisfying (2.4). Then one has

$$\begin{aligned}
\|A(x) - T(x)\| &= \left\|\frac{1}{(-2)^k} A\left((-2)^k x\right) - \frac{1}{(-2)^k} T\left((-2)^k x\right)\right\| \\
&\le \frac{1}{2^{k\beta}} \left(\left\|A\left((-2)^k x\right) - f\left((-2)^k x\right)\right\|\right. \\
&\left.+ \left\|T\left((-2)^k x\right) - f\left((-2)^k x\right)\right\|\right) \\
&\le 2\frac{1}{2^{k\beta}} \widetilde{\varphi}\left(-(-2)^k x, -(-2)^k x, (-2)^k 2x\right)
\end{aligned}$$

which tends to zero as $k \to \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$. □

Theorem 2.3. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping with $f(0) = 0$. If there is a function $\varphi : X^3 \to [0, \infty)$ satisfying (2.2) such that*

$$\widetilde{\varphi}(x, y, z) := \sum_{j=1}^{\infty} 2^{\beta j} \varphi\left(\frac{x}{(-2)^j}, \frac{y}{(-2)^j}, \frac{z}{(-2)^j}\right) < \infty \qquad (2.7)$$

for all $x, y, z \in \mathcal{X}$, then there exists a unique additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - A(x)\| \le \widetilde{\varphi}(x, x, -2x) \qquad (2.8)$$

for all $x \in \mathcal{X}$.

Proof. Letting $y = x$ and $z = -2x$ in (2.2), we get

$$\|2f(-x) + f(2x)\| \le \varphi\left(\frac{x}{2}, \frac{x}{2}, -x\right)$$

for all $x \in \mathcal{X}$. Thus

$$\left\|f(x) - (-2)f\left(\frac{x}{-2}\right)\right\| \le \varphi\left(\frac{x}{2}, \frac{x}{2}, -x\right)$$

for all $x \in \mathcal{X}$.

Next, we can prove that the sequence $\{(-2)^n f\left(\frac{x}{(-2)^n}\right)$ is a Cauchy sequence for all $x \in \mathcal{X}$,and define a mapping $A : \mathcal{X} \to \mathcal{Y}$ by

$$A(x) := \lim_{n\to\infty} (-2)^n f\left(\frac{x}{(-2)^n}\right)$$

for all $x \in \mathcal{X}$ that is similar to the corresponding part of the proof of Theorem (2.2). □

3. Hyers-Ulam Stability for Fixed Point Methods

Now, using fixed point theorem, we investigate the Hyers-Ulam stability of the functional inequality (2.1) in β-homogeneous F-spaces.

Theorem 3.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping for which there exists a function $\varphi : \mathcal{X}^3 \to [0, \infty)$ such that*

$$\begin{aligned}&\|f(2x + y + 2z) + f(2x + 3y + 3z) + f(4x + 4y + 3z)\| \\ &\le \|8f(x + y + z)\| + \varphi(x, y, z)\end{aligned} \tag{3.1}$$

for all $x, y, z \in X$. If there exists $L \in (0, 1)$ such that

$$\varphi(x, y, z) \le 2L\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \tag{3.2}$$

for all $x, y, z \in \mathcal{X}$. Then there exists a unique additive mapping $H : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - H(x)\| \le \frac{1}{2^\beta(1 - L)}\varphi(-x, -x, 2x) \tag{3.3}$$

for all $x \in X$.

Proof. It follows from $\varphi(x, y, z) \le 2L\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right)$ that

$$\lim_{j\to\infty} \frac{1}{2^j}\varphi(2^j x, 2^j y, 2^j z) = 0$$

for all $x, y, z \in \mathcal{X}$.

Consider the set

$$A := \{g : \mathcal{X} \to \mathcal{Y}\}$$

and introduce the *generalized metric* on A:

$$d(g, h) = \inf\{C \in \mathbb{R}_+ : \|g(x) - h(x)\| \le C\varphi(-x, -x, 2x), \forall x \in \mathcal{X}\}.$$

It is easy to show that (A, d) is complete.

Now we consider the linear mapping $J : A \to A$ such that

$$Jg(x) := \frac{1}{-2}g(-2x)$$

for all $x \in \mathcal{X}$.

By [6, Theorem 3.1]

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in A$.

Letting $y = x$ and $z = -2x$ in (3.1), we get

$$\left\| f(x) - \frac{1}{-2}f(-2x) \right\| \leq \frac{1}{2^\beta}\varphi(-x, -x, 2x)$$

for all $x \in \mathcal{X}$.

Hence $d(f, Jf) \leq \frac{1}{2^\beta}$.

By the Theorem (1.1), there exists a mapping $H : \mathcal{X} \to \mathcal{Y}$ such that

(1) H is a fixed point of J, that is

$$\frac{1}{-2}H(-2x) = H(x) \tag{3.4}$$

for all $x \in \mathcal{X}$. The mapping H is a unique fixed point of J in the set

$$B = \{g \in A : d(f, g) < \infty\}.$$

This implies that H is a unique mapping satisfying (3.4) such that there exists $C \in (0, \infty)$ satisfying

$$\|H(x) - f(x)\| \leq C\varphi(-x, -x, 2x)$$

for all $x \in \mathcal{X}$.

(2) $d(J^n f, H) \to 0$ as $n \to \infty$. This implies the inequality

$$\lim_{n\to\infty} \frac{1}{(-2)^n} f((-2)^n x) = H(x)$$

for all $x \in \mathcal{X}$.

(3) $d(f, H) \leq \frac{1}{1-L}d(f, Jf)$, which implies the inequality

$$d(f, H) \leq \frac{1}{2^\beta(1 - L)}.$$

This implies that the inequality (3.3)holds.

Next, we show that $H(x)$ is an additive mapping.

$$\begin{aligned}
&\|H(2x+y+2z)+H(2x+3y+3z)+H(4x+4y+3z)\| \\
&= \lim_{k\to\infty}\left|\frac{1}{(-2)^{k\beta}}\right| \|f((-2)^k(2x+y+2z))+f((-2)^k(2x+3y+3z)) \\
&+f((-2)^k(4x+4y+3z))\| \qquad (3.5)\\
&\le \lim_{k\to\infty}\left|\frac{1}{(-2)^{k\beta}}\right| \|8f((-2)^k(x+y+z))\| + \lim_{k\to\infty}\left|\frac{1}{(-2)^{k\beta}}\right| \varphi((-2)^k x,(-2)^k y,(-2)^k z) \\
&\le \|8H(x+y+z)\|
\end{aligned}$$

for all $x, y, z \in \mathcal{X}$. □

Theorem 3.2. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping for which there exists a function $\varphi : \mathcal{X}^3 \to [0,\infty)$ satisfying (3.1) If there exists an $L \in (0,1)$ such that*

$$\varphi(x,y,z) \le \frac{1}{2}L\varphi\left(2x,2y,2z\right) \qquad (3.6)$$

for all $x, y, z \in \mathcal{X}$. Then there exists a unique additive mapping $H : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x)-H(x)\| \le \frac{L}{2(1-L)}\varphi(-x,-x,2x) \qquad (3.7)$$

for all $x \in X$.

Proof. It follows from $\varphi(x,y,z) \le 2L\varphi\left(\frac{x}{2},\frac{y}{2},\frac{z}{2}\right)$ that

$$\lim_{j\to\infty} 2^j \varphi\left(\frac{1}{2^j}x,\frac{1}{2^j}y,\frac{1}{2^j}z\right) = 0$$

for all $x, y, z \in \mathcal{X}$.

Consider the set

$$A := \{g : \mathcal{X} \to \mathcal{Y}\}$$

and introduce the *generalized metric* on A:

$$d(g,h) = \inf\{C \in \mathbb{R}_+ : \|g(x)-h(x)\| \le C\varphi\left(x,x,-2x\right), \forall x \in \mathcal{X}\}.$$

It is easy to show that (A, d) is complete.

Now we consider the linear mapping $J : A \to A$ such that

$$Jg(x) := -2g\left(-\frac{x}{2}\right)$$

for all $x \in \mathcal{X}$.

By [6, Theorem 3.1]

$$d(Jg, Jh) \le Ld(g,h)$$

for all $g, h \in A$.

Letting $y = x$ and $z = x + y$ in (3.1), we get

$$\left\|f(x)-(-2)f\left(-\frac{1}{2}x\right)\right\| \le \varphi\left(\frac{x}{2},\frac{x}{2},-x\right) \le \frac{L}{2}\varphi\left(x,x,-2x\right)$$

for all $x \in \mathcal{X}$.

Hence $d(f, Jf) \leq \frac{L}{2}$. The rest of the proof is similar to the corresponding part of the proof of Theorem 3.1.

□

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Ming Fang
1.School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, P.R.China
2.Department of Mathematics, Yanbian University, Yanji 133000, P.R. China
E-mail address: fangming@ybu.edu.cn

DongHe Pei
School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, P.R.China
E-mail address: peidh340@nenu.edu.cn

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Characterization of a Class of Differential Equations

Mohammad Fuad Mohammad Naser[a], Omar M. Bdair[a], Fayçal Ikhouane[b] [1]

[a]Al Balqa' Applied University, Faculty of Engineering Technology, Department of Physics and Basic Sciences,
P.O. Box: (15008) Amman (11134) Jordan.
[b]Universitat Politècnica de Catalunya, Escola Universitària d´Enginyeria, Tècnica Industrial de Barcelona. Departament de Matemàtiques, Comte d´Urgell, 187, 08036, Barcelona, Spain.

Abstract

This paper deals with a characterization of nonlinear systems of the form $\dot{x}_\gamma(t) = f\left(x_\gamma(t), u(t/\gamma)\right)$ when the parameter $\gamma \to \infty$. In particular, we are interested in the uniform convergence of the sequence of functions $x_\gamma\left(\gamma t\right)$. Necessary conditions and sufficient ones are derived for this uniform convergence to happen.

Keywords: nonlinear systems, consistent operator, uniform convergence

1 Introduction

Hysteresis is a nonlinear behavior encountered in a wide variety of processes including biology, optics, electronics, ferroelectricity, magnetism, mechanics, structures, among other areas. The detailed modeling of hysteresis systems using the laws of Physics is an arduous task, and the obtained models are often too complex to be used in applications. For this reason, alternative models of these complex systems have been proposed [15, 1, 8, 6, 9]. These models do not come, in general, from the detailed analysis of the physical behavior of the systems with hysteresis. Instead, they combine some physical understanding of the system along with some kind of black-box modeling.

This way of describing hysteresis systems led to the proliferation of hysteresis models in the last two decades. A search in the Web of Knowledge database gives more than 2000 publications. The question that arises naturally is: do these research works describe really hysteresis phenomena? In other words, does the researcher who proposes a new hysteresis model have a mathematical rule to decide whether the model they propose is indeed a hysteresis one?

Surprisingly enough, such a rule exists only for a limited number of hysteresis processes: those that possess the so-called rate-independence property. This property states that, under a time-scale change, the relationship output versus input is unchanged. Hysteresis systems that are rate-independent are listed in the survey paper [10]. However, in the last two decades, researchers have acknowledged the importance of rate-dependent processes in applications [4, 3, 2]. For this reason, a recent effort [5] proposed a mathematical framework that

[1] *E-mail addresses: mohammad.naser@bau.edu.jo* (Mohammad Fuad Mohammad Naser), *bdairmb@yahoo.com* (Omar M. Bdair), *faycal.ikhouane@upc.edu* (Fayçal Ikhouane).

proposes a rule to decide whether or not a system may be hysteretic. The rule proposed in [5] shows that, for an input/output system with input $u(t/\gamma)$ and output $x_\gamma(t)$, the convergence of the sequence of functions $t \to x_\gamma(\gamma t)$ as $\gamma \to \infty$ is a necessary condition for the hysteresis. The previous formulation is used to study the hysteresis behavior of the generalized Duhem model [11] and the LuGre friction model [12].

In the present paper, we consider the differential equation $\dot{x} = f(x,u)$. Our objective is to derive necessary conditions and also sufficient ones for the uniform convergence of the sequence of functions $t \to x_\gamma(\gamma t)$.

This paper is organized as follows. Section 2 presents the system of study and the assumptions under which the study is performed. Sections 3 and 4 present; respectively, necessary conditions and sufficient ones for the uniform convergence of the sequence of functions $x_\gamma(\gamma t)$ as $\gamma \to \infty$. Conclusions are given in Section 5.

2 Problem Statement

The class of systems under study is

$$\dot{x}(t) = f\big(x(t), u(t)\big),\ t \geq 0, \tag{1}$$

$$x(0) = x_0, \tag{2}$$

where initial condition x_0 and state $x(t)$ take value in $\mathbb{R}^m$, and input $u \in L^\infty(\mathbb{R}_+, \mathbb{R}^n)$ for some strictly positive integers n and m. The mapping $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ is a well-defined continuous function. Because of the continuity of the right-hand side of (1), the system (1)-(2) has a maximal solution which is defined on an interval of the form $[0,\omega)$, $\omega > 0$ [14, p. 67–70]. In this paper, we assume that the system (1)-(2) has a unique Carathéodory solution for all $(u, x_0) \in L^\infty(\mathbb{R}_+, \mathbb{R}^n) \times \mathbb{R}^m$.

Consider the time scale change $s_\gamma(t) = t/\gamma, \forall\gamma > 0, \forall t \geq 0$. When the input $u \circ s_\gamma$ is used instead of u, system (1)-(2) becomes

$$\dot{x}_\gamma(t) = f\big(x_\gamma(t), u \circ s_\gamma(t)\big),\ t \geq 0, \tag{3}$$

$$x_\gamma(0) = x_0, \tag{4}$$

which can be written for all $\gamma > 0$ as

$$\sigma_\gamma(t) = x_0 + \gamma \int_0^t f\big(\sigma_\gamma(\tau), u(\tau)\big) d\tau,\ \forall t \in [0, \omega_\gamma), \tag{5}$$

where $\sigma_\gamma = x_\gamma \circ s_{1/\gamma}$ and $[0, \omega_\gamma)$ is the maximal interval for the existence of solutions σ_γ.

We seek necessary conditions and also sufficient conditions for the uniform convergence of the sequence of functions σ_γ.

3 Necessary Conditions

Our aim in this section is to derive necessary conditions for the uniform covergence of the sequence of functions σ_γ.

Lemma 3.1. *Assume that the maximal solution of system (1)-(2) is defined on* $\mathbb{R}_+$ *for all* $(u, x_0) \in L^\infty(\mathbb{R}_+, \mathbb{R}^n) \times \mathbb{R}^m$*. Suppose that there exists a function* $h : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$|x(t)| \le h\big(|x_0|, \|u\|_\infty\big), \forall t \ge 0, \tag{6}$$

for each initial state $x_0 \in \mathbb{R}^n$ *and each input* $u \in L^\infty(\mathbb{R}_+, \mathbb{R}^n)$*. Assume that there exists a function* $q_u \in L^\infty(\mathbb{R}_+, \mathbb{R}^m) \cap C^0(\mathbb{R}_+, \mathbb{R}^m)$ *such that* $\lim_{\gamma\to\infty} \|\sigma_\gamma - q_u\|_\infty = 0$*. Then, we have* $f\big(x_0, u(0)\big) = 0$*,* $q_u(0) = x_0$*, and* $f\big(q_u(t), u(t)\big) = 0$*,* $\forall t \ge 0$*.*

Proof. From the fact that $\|u\|_\infty = \|u \circ s_\gamma\|, \forall\gamma > 0$ and Inequality (6) it comes that

$$\|x_\gamma\|_\infty \le h(|x_0|, \|u\|_\infty) = a. \forall\gamma > 0,$$

Thus, we get from the continuity of σ_γ that

$$|\sigma_\gamma(t)| \le a, \forall t \ge 0, \ \forall\gamma > 0. \tag{7}$$

Inequality (7) along with the continuity of function f and the boundedness of the input u imply that there exists a constant $r > 0$ independent of γ, such that $|f(\sigma_\gamma(\tau), u(\tau))| \le r, \ \forall\tau \ge 0, \ \forall\gamma > 0$. This means that we can apply the Dominated Lebesgue Theorem in Equation (5) and get

$$\lim_{\gamma\to\infty} \int_0^t f(\sigma_\gamma(\tau), u(\tau))\, d\tau = \int_0^t f(q_u(\tau), u(\tau))\, d\tau, \ \forall t \ge 0, \tag{8}$$

where the continuity of f and the fact that $\lim_{\gamma\to\infty} \|\sigma_\gamma - q_u\|_\infty = 0$ are used. By Equation (7) we have $\|\sigma_\gamma - x_0\|_\infty / \gamma \to 0$ as $\gamma \to \infty$. Thus, we obtain from (5) and (8) that

$$\int_0^t f(q_u(\tau), u(\tau))\, d\tau = 0, \ \forall t \ge 0,$$

which gives $f\big(q_u(t), u(t)\big) = 0$ for almost all $t \ge 0$. From the continuity of functions f, q_u, and u it comes that

$$f\big(q_u(t), u(t)\big) = 0, \quad \text{for all } t \ge 0. \tag{9}$$

Since $\sigma_\gamma(0) = x_0$, $\forall\gamma > 0$ it comes that

$$q_u(0) = x_0. \tag{10}$$

Finally, taking $t = 0$ in (9) and using (10) provides the necessary condition

$$f\big(x_0, u(0)\big) = 0, \tag{11}$$

which completes the proof. □

Remark 1. Once chosen an input u, the term $u(0)$ is given so that any initial condition x_0 for which we have $\lim_{\gamma\to\infty} \|\sigma_\gamma - q_u\|_\infty = 0$ should satisfy (11).

4 Sufficient Conditions

In this section, we derive sufficient conditions to ensure that the sequence of functions σ_γ converges uniformly as $\gamma \to \infty$.

Definition 4.1. *[7] A continuous function $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to class $\mathcal{K}_\infty$ if it is strictly increasing, satisfies $\beta(0) = 0$, and $\lim_{t\to\infty} \beta(t) = \infty$.*

Lemma 4.1. *[11] Consider a function $z : [0, \omega) \subset \mathbb{R}_+ \to \mathbb{R}_+$, where ω may be infinite. Assume the following*

(i) *The function z is absolutely continuous on each compact subset of $[0, \omega)$.*

(ii) *There exist $z_1, z_2 \geq$ such that $z_1, z(0) < z_2$ and $\dot{z}(t) \leq 0$ for almost all $t \in [0, \omega)$ that satisfy $z_1 < z(t) < z_2$.*

Then, $z(t) \leq \max\big(z(0), z_1\big)$, $\forall t \in [0, \omega)$.

Corollary 4.1. *Consider a function $z : [0, \omega) \subset \mathbb{R}_+ \to \mathbb{R}_+$, where ω may be infinite. Assume the following*

(i) *The function z is absolutely continuous on each compact subset of $[0, \omega)$.*

(ii) *There exist a class $\mathcal{K}_\infty$ function $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ and $z_1, z_2, z_3 \geq 0$ such that $\max\big(\beta^{-1}(z_3), z_1, z(0)\big) < z_2$, and $\dot{z}(t) \leq -\beta(z(t)) + z_3$ for almost all $t \in [0, \omega)$ that satisfy $z_1 < z(t) < z_2$.*

Then, $z(t) \leq \max\big(z(0), z_1, \beta^{-1}(z_3)\big)$, $\forall t \in [0, \omega)$.

Proof. We have $\dot{z}(t) \leq 0$ for almost all $t \in [0, \omega)$ that satisfy $\max\big(\beta^{-1}(z_3), z_1\big) < z(t) < z_2$, and hence the result follows directly from Lemma 4.1. □

Lemma 4.2. *Assume that there exists $q_u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$ such that*

$$f\big(q_u(t), u(t)\big) = 0, \ \forall t \geq 0, \tag{12}$$

$$q_u(0) = x_0. \tag{13}$$

Define $y_\gamma : \mathbb{R}_+ \to \mathbb{R}^m$ as

$$y_\gamma(t) = \sigma_\gamma(t) - q_u(t) = x_\gamma(\gamma t) - q_u(t), \forall \gamma > 0, \tag{14}$$

for all $t \in [0, \omega_\gamma)$. Suppose that we can find a continuously differentiable function $V : \mathbb{R}^m \to \mathbb{R}_+$ that satisfies the following:

(i) *V is positive definite, that is $V(0) = 0$ and $V(\alpha) > 0, \forall\, 0 \neq \alpha \in \mathbb{R}^m$.*

(ii) *V is proper, that is $V(\alpha) \to \infty$ as $|\alpha| \to 0$.*

(iii) *There exist $\delta > 0$ and $\beta \in \mathcal{K}_\infty$ satisfying:*

$$\begin{cases} \left.\frac{dV(\alpha)}{d\alpha}\right|_{\alpha = y_\gamma(t)} \cdot f\big(y_\gamma(t) + q_u(t), u(t)\big) \leq -\beta\big(|y_\gamma(t)|\big), \\ \text{for all } t \in [0, \omega_\gamma) \text{ and } \forall \gamma > 0 \text{ that satisfy } |y_\gamma(t)| < \delta. \end{cases} \tag{15}$$

Then,

- *$\omega_\gamma = +\infty, \forall \gamma > 0$. Furthermore, there exist $E, \gamma^* > 0$ such that $\|x_\gamma\|_\infty \leq E, \forall \gamma > \gamma^*$, for any solution x_γ of the system (3)-(4).*

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- $\lim_{\gamma\to\infty} \|\sigma_\gamma - q_u\|_\infty = 0$.

Proof. Since V is positive definite and proper, there exists $\beta_1, \beta_2 \in \mathcal{K}_\infty$ such that (see [7, p. 145])

$$\beta_1(|\alpha|) \le V(\alpha) \le \beta_2(|\alpha|), \forall \alpha \in \mathbb{R}^m. \tag{16}$$

From (5), we get for almost all $t \in [0, \omega_\gamma)$, $\forall \gamma > 0$ that

$$\dot{y}_\gamma(t) = \gamma f\big(y_\gamma(t) + q_u(t), u(t)\big) - \dot{q}_u(t), \tag{17}$$

$$y_\gamma(0) = 0. \tag{18}$$

For any $\gamma > 0$, define $V_\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ as $V_\gamma(t) = V\big(y_\gamma(t)\big), \forall t \in [0, \omega_\gamma)$. Note that the function V_γ is absolutely continuous on each compact subset of $[0, \omega_\gamma)$ as a composition of a continuously differentiable function V and an absolutely continuous function y_γ. Then, we get for almost all $t \in [0, \omega_\gamma)$ and all $\gamma > 0$ that

$$\dot{V}_\gamma(t) = \frac{dV(\alpha)}{d\alpha}\bigg|_{\alpha = y_\gamma(t)} \cdot \dot{y}_\gamma(t) = \frac{dV(\alpha)}{d\alpha}\bigg|_{\alpha = y_\gamma(t)} \cdot \Big[\gamma f\big(y_\gamma(t) + q_u(t), u(t)\big) - \dot{q}_u(t)\Big]. \tag{19}$$

Let $\Omega = \big(0, \beta_1(\delta)\big)$. By (16) we have for any $\gamma > 0$, and for almost all $t \in [0, \omega_\gamma)$ that

$$V_\gamma(t) \in \Omega \Rightarrow |y_\gamma(t)| < \delta. \tag{20}$$

We conclude from (15), (19), and (20) that

$$\dot{V}_\gamma(t) \le -\gamma\,\beta(|y_\gamma(t)|) + \|\dot{q}_u\|_\infty \left|\frac{dV(\alpha)}{d\alpha}\bigg|_{\alpha = y_\gamma(t)}\right|, \text{ for almost all } t \in [0, \omega_\gamma), \forall \gamma > 0 \text{ that satisfy } V_\gamma(t) \in \Omega.$$

Thus, we deduce from the continuity of $\frac{dV(\alpha)}{d\alpha}$, the boundedness of $\dot{q}_u$, and (20) there exists some $b > 0$ independent of γ such that

$$\dot{V}_\gamma(t) \le -\gamma\,\beta(|y_\gamma(t)|) + b, \text{ for almost all } t \in [0, \omega_\gamma), \forall \gamma > 0 \text{ that satisfy } V_\gamma(t) \in \Omega.$$

Hence, (16) implies

$$\dot{V}_\gamma(t) \le -\gamma\,\beta \circ \beta_2^{-1}\big(V_\gamma(t)\big) + b, \text{ for almost all } t \in [0, \omega_\gamma), \forall \gamma > 0 \text{ that satisfy } V_\gamma(t) \in \Omega.$$

Thus, Corollary 4.1 and the fact that $V_\gamma(0) = 0, \forall \gamma > 0$, imply that $V_\gamma(t) \le \beta_2 \circ \beta^{-1}\left(\frac{b}{\gamma}\right)$, $\forall \gamma > \gamma_0, \forall t \in [0, \omega_\gamma)$ where $\gamma_0 = \frac{b}{\beta \circ \beta_2^{-1} \circ \beta_1(\delta)}$. Therefore, (16) implies that

$$|y_\gamma(t)| \le \beta_1 \circ \beta_2 \circ \beta^{-1}\left(\frac{b}{\gamma}\right), \forall \gamma > \gamma_0, \forall t \in [0, \omega_\gamma). \tag{21}$$

Thus, $\omega_\gamma = +\infty, \forall \gamma > \gamma_1$ for some $\gamma_1 > 0$, and $\lim_{\gamma\to\infty} \|y_\gamma\|_\infty = 0$, which is equivalent to $\lim_{\gamma\to\infty} \|\sigma_\gamma - q_u\|_\infty = 0$. On the other hand, (21) and the fact that $\sigma_\gamma = y_\gamma + q_u$ imply that there exists some $E, \gamma^* > 0$ such that $\|\sigma_\gamma\|_\infty \le E, \forall \gamma > \gamma^*$, and hence $\|x_\gamma\|_\infty \le E, \forall \gamma > \gamma^*$. □

Lemma 4.3. *Consider the nonlinear system [13]*

$$\dot{x} = f(x, u) = Ax + \Phi(x) + R(u), \tag{22}$$

$$x(0) = x_0, \tag{23}$$

$$y = Dx, \tag{24}$$

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where $x_0 \in \mathbb{R}^m$*, A is an $m \times m$ Hurwitz matrix*[2]*, D is an $m \times m$ matrix, input $u \in L^\infty(\mathbb{R}_+, \mathbb{R}^n)$, state x, output y take values in $\mathbb{R}^m$, function $R \in C^0(\mathbb{R}^n, \mathbb{R}^m)$, and a locally Lipshitz function $\Phi \in C^0(\mathbb{R}^m, \mathbb{R}^m)$. Assume the following:*

(i) *The exists $q_u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^m)$ such that $q_u(0) = x_0$ and*

$$Aq_u(t) + \Phi\big(q_u(t)\big) + R\big(u(t)\big) = 0, \forall t \geq 0.$$

(ii) *There exist $c_1 > 0$, $c_2 > 0$, $\xi > 0$ and $r > 2$ such that*

$$\left|\alpha \cdot \left[\Phi\big(\alpha + q_u(t)\big) - \Phi\big(q_u(t)\big)\right]\right| \leq c_1 |\alpha|^2 + c_2 |\alpha|^r, \text{ for almost all } t \geq 0, \forall \alpha \in \mathbb{R}^m \text{ that satisfy } |\alpha| < \xi.$$

(iii) *One has $c_1 < \frac{1}{2\lambda_{\max}}$, where $\lambda_{\max}$ is the largest eigenvalue for the $m \times m$ positive-definite symmetric matrix P that satisfies*[3]

$$PA + A^T P = -I_{m \times m}. \tag{25}$$

Let x_γ, y_γ be respectively the state and the output of (22)-(24) when we use the input $u \circ s_\gamma$ instead of u.

Then,

- *All solutions of (22)-(24) are bounded. Furthermore, there exist $E, \gamma^* > 0$ such that $\|x_\gamma\|_\infty \leq E, \forall \gamma > \gamma^*$, for any solution x_γ of the system (3)-(4).*
- $\lim_{\gamma \to \infty} \|F_\gamma - Dq_u\|_\infty = 0$*, where $F_\gamma : \mathbb{R}_+ \to \mathbb{R}^m$ is defined as $F_\gamma(t) = y_\gamma(\gamma t), \forall t \geq 0, \forall \gamma > 0$.*

Proof. Since Φ is locally Lipschitz, the right-hand side of (22) is locally Lipschitz relative to x and hence the system (22) has a unique solution. The function q_u satisfies (12)-(13) in Lemma 4.2 because of (i).

Consider the continuously differentiable quadratic Lyapunov function candidate $V : \mathbb{R}^m \to \mathbb{R}$ such that $V(\alpha) = \alpha^T P \alpha$, $\forall \alpha \in \mathbb{R}^m$. Since P is symmetric, we have $\forall \alpha \in \mathbb{R}^m$ that

$$\lambda_{\min} |\alpha|^2 \leq V(\alpha) = \alpha^T P \alpha \leq \lambda_{\max} |\alpha|^2,$$

where $\lambda_{\min}$ is the smallest eigenvalue of the matrix P. Thus V is positive definite and proper. Since P is symmetric we have

$$\left|\frac{dV(\alpha)}{d\alpha}\right| = 2|P\alpha| \leq 2\lambda_{\max}|\alpha|, \forall \alpha \in \mathbb{R}^m. \tag{26}$$

We have by (25) that

$$\frac{dV(\alpha)}{d\alpha} \cdot A\alpha = 2P\alpha \cdot A\alpha = \alpha^T\left(PA + A^T P\right)\alpha = -|\alpha|^2, \forall \alpha \in \mathbb{R}^m. \tag{27}$$

From Condition (i) we get for all $\gamma > 0$ that

$$\begin{aligned} \frac{dV(\alpha)}{d\alpha}\Big|_{\alpha = y_\gamma} \cdot f(y_\gamma + q_u, u) &= \frac{dV(\alpha)}{d\alpha}\Big|_{\alpha = y_\gamma} \cdot \Big[Ay_\gamma + Aq_u + \Phi(y_\gamma + q_u) + R(u)\Big] \\ &= \frac{dV(\alpha)}{d\alpha}\Big|_{\alpha = y_\gamma} \cdot \Big[Ay_\gamma + \Phi(y_\gamma + q_u) - \Phi(q_u)\Big]. \end{aligned} \tag{28}$$

[2] that is each eigenvalue of A has a strictly negative real part.

[3] the existence of the matrix P in (25) is guaranteed because A is Hurwitz [7, p.136].

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where y_γ is defined in (14).

We get from (28), (27), (26) and Condition (ii) that

$$\left.\frac{dV(\alpha)}{d\alpha}\right|_{\alpha=y_\gamma(t)} \cdot f\big(y_\gamma(t)+q_u(t),u(t)\big) \le (-1+2c_1\,\lambda_{\max})\,|y_\gamma(t)|^2 + 2c_2\,\lambda_{\max}\,|y_\gamma(t)|^r,$$
$$\forall\gamma>0 \text{ for almost all } t\in[0,\omega_\gamma) \text{ that satisfy } |y_\gamma(t)|<\xi, \tag{29}$$

where $[0,\omega_\gamma)$ is the maximal interval of existence of σ_γ and y_γ. This leads to

$$\left.\frac{dV(\alpha)}{d\alpha}\right|_{\alpha=y_\gamma(t)} \cdot f\big(y_\gamma(t)+q_u(t),u(t)\big) \le -\frac{1-2c_1\,\lambda_{\max}}{2}\,|y_\gamma(t)|^2,$$
$$\forall\gamma>0, \text{ for almost all } t\in[0,\omega_\gamma) \text{ that satisfy } |y_\gamma(t)|<\min\left(\sqrt[r-2]{\frac{1-2c_1\,\lambda_{\max}}{4c_2\,\lambda_{\max}}},\ \xi\right). \tag{30}$$

Thus, (15) in is satisfied with $\beta(v)=\frac{1-2c_1\,\lambda_{\max}}{2}v^2, \forall v\ge 0$ and $\delta=\min\left(\sqrt[r-2]{\frac{1-2c_1\,\lambda_{\max}}{4c_2\,\lambda_{\max}}},\ \xi\right)$. Hence all conditions of Lemma 4.2 are satisfied so that the solution of (22) is bounded. Morover, there exist $E,\gamma^*>0$ such that $\|x_\gamma\|_\infty\le E, \forall\gamma>\gamma^*$. Futhermore, we have $\lim_{\gamma\to\infty}\|\sigma_\gamma-q_u\|_\infty=0$. Thus, we deduce from (24) that $\lim_{\gamma\to\infty}\|F_\gamma-Dq_u\|_\infty=0$. □

Example. Consider the system

$$\dot{x} = -x+x^3-u, \tag{31}$$
$$x(0) = 0. \tag{32}$$

where state x takes values in $\mathbb{R}$ and input $u\in W^{1,\infty}(\mathbb{R}_+,\mathbb{R})$ is defined as $u(t)=0.1\sin(t), \forall t\ge 0$. The system (31)-(32) has the form (22)-(24), with $x=y$, $m=n=1$, $A=-1$, $\Phi(\alpha)=\alpha^3$, $R(\alpha)=-\alpha$, $\forall\alpha\in\mathbb{R}$, and $D=1$. Observe that P in (25) equals $1/2$ which mean that $\lambda_{\min}=\lambda_{\max}=1/2$. We have $u(0)=0$ and u is bounded with

$$u(\cdot)\in[u_{\min},u_{\max}]=[-0.1,0.1]. \tag{33}$$

Define the function $\chi:\left[-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right]\to\left[-\frac{2}{3\sqrt{3}},\frac{2}{3\sqrt{3}}\right]$ as $\chi(v)=-v+v^3, \forall v\in\left[-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right]$. The function χ is strictly decreasing, bijective and its inverse function is continuous. Hence, there exists a function $q_u\in C^0(\mathbb{R}_+,\mathbb{R})\cap L^\infty(\mathbb{R}_+,\mathbb{R})$ such that $q_u(\cdot)\in\left[-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right]$, $q_u(0)=0$ and

$$\chi\big(q_u(t)\big)=-q_u(t)+q_u^3(t)=u(t), \forall t\ge 0. \tag{34}$$

It can be checked using (33) that $\|q_u\|_\infty<0.11$ (see Figure (1b)). Thus $q_u(\cdot)\neq\frac{1}{\sqrt{3}}$. This fact and (34) implies that the function $\dot{q}_u=\dot{u}/\left(1-3q_u^2\right)$ is bounded so that $q_u\in W^{1,\infty}(\mathbb{R}_+,\mathbb{R})$. Hence Condition (i) of Lemma 4.3 is satisfied.

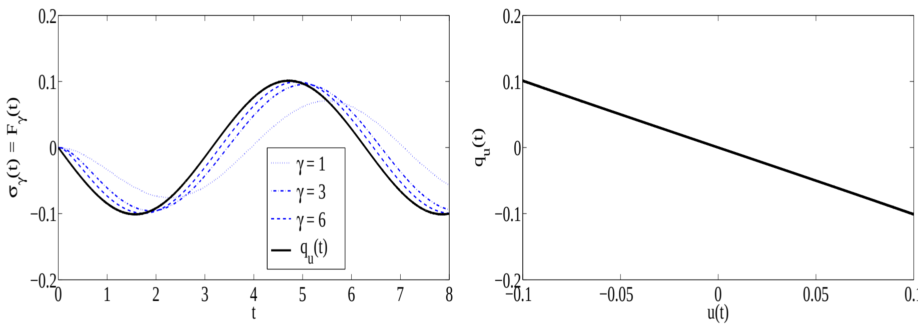
On the other hand, we have for all $\alpha\in\mathbb{R}$ that

$$\alpha\left(\Phi(\alpha+q_u)-\Phi(q_u)\right)=3q_u^2\alpha^2+3q_u\alpha^3+\alpha^4. \tag{35}$$

Since $\|q_u\|_\infty < 0.11$, one has $\left\|3q_u^2\right\|_\infty < 0.0363 = c_1$. Hence it follows from (35) that for any $\xi > 0$ we have

$$\alpha\left[\Phi\big(\alpha + q_u\,(t)\big) - \Phi\big(q_u\,(t)\big)\right] \leq c_1\alpha^2 + \left(3\,\|q_u\|_\infty + \xi\right)\alpha^3 \qquad \forall \alpha \in \mathbb{R}^m \text{ that satisfy } |\alpha| < \xi, \text{ for almost all } t \geq 0. \tag{36}$$

Thus, Condition (ii) in Lemma 4.3 is satisfied with $c_2 = 3\,\|q_u\|_\infty + \xi$. Moreover, we have $c_1 < 1 = \frac{1}{2\,\lambda_{\max}}$ which implies that Condition (ii) in Lemma 4.3 is also satisfied. Therefore, the solution of (31)-(32) is bounded, that there exist $E, \gamma^* > 0$ such that $\|x_\gamma\|_\infty \leq E, \forall \gamma > \gamma^*$, and that $\lim_{\gamma\to\infty} \|\sigma_\gamma - q_u\|_\infty = \lim_{\gamma\to\infty} \|F_\gamma - q_u\|_\infty = 0$ (observe that $\sigma_\gamma(\cdot) = F_\gamma(\cdot)$ because $x(\cdot) = y(\cdot)$). This is illustrated in Figure 1a.



(a) $F_\gamma\,(t)$ versus t for system (31). (b) $q_u(t)$ versus $u(t)$ for system (31).

Figure 1: Simulations.

5 Conclusion

In [5] a rule for deciding whether a process may or may not be a hysteresis is proposed for causal operators such that a constant input leads to a constant output. That rule involves checking whether the so-called consistency and strong consistency properties hold. In this paper we derived necessary conditions and sufficient ones for the uniform convergence of the shifted solutions $\sigma_\gamma : t \to x_\gamma\,(\gamma t)$ of the system $\dot{x} = f(x, u \circ s_\gamma)$. This uniform convergence is related to consistency. Does this mean that the concept of consistency can be extended to study operators for which the property that a constant input leads to a constant output, that property does not hold?

This paper explores this issue for systems of the form $\dot{x} = f(x, u)$, however, no clear cut answer may be drawn for the obtained results.

Indeed, the necessary conditions alone cannot guarantee whether the uniform convergence of σ_γ when $\gamma \to \infty$ happens or not. The sufficient conditions do imply that convergence but do not guarantee that the hysteresis loop of the operator is not trivial. In the example, we have seen that q_u is a function of u so that the hysteresis loop is a curve and we cannot acertain from this whether system (31) is a hysteresis or not. This is a future research line.

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