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Editor in Chief: George Anastassiou
Department of Mathematical Sciences,
University of Memphis, Memphis, TN 38152-3240, U.S.A
ganastss@memphis.edu
http://www.msci.memphis.edu/~ganastss/jocaaa

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8000 Cummings Hall,
Hanover, NH 03755-8000
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e-mail:george.cybenko@dartmouth.edu
Approximation Theory and Neural Networks

Sever S. Dragomir
School of Computer Science and Mathematics, Victoria University,
PO Box 14428,
Melbourne City,
MC 8001, AUSTRALIA
Tel. +61 3 9688 4437
Fax +61 3 9688 4050
sever.dragomir@vu.edu.au
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oduman@etu.edu.tr
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Saber N. Elaydi
Department Of Mathematics
Trinity University
715 Stadium Dr.
San Antonio, TX 78212-7200
210-736-8246
e-mail: selaydi@trinity.edu
Ordinary Differential Equations, Difference Equations

Christodoulos A. Floudas
Department of Chemical Engineering
Princeton University
Princeton,NJ 08544-5263
609-258-4595(x4619 assistant)
e-mail: floudas@titan.princeton.edu
Optimization Theory&Applications, Global Optimization

J .A. Goldstein
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152
901-678-3130
jgoldste@memphis.edu
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H. H. Gonska
Department of Mathematics
University of Duisburg
Duisburg, D-47048
Germany
011-49-203-379-3542
e-mail: heiner.gonska@uni-due.de
Approximation Theory, Computer Aided Geometric Design

John R. Graef
Department of Mathematics
University of Tennessee at Chattanooga
Chattanooga, TN 37304 USA
John-Graef@utc.edu
Ordinary and functional differential equations, difference equations, impulsive systems, differential inclusions, dynamic equations on time scales, control theory and their applications

Weimin Han
Department of Mathematics
University of Iowa
Iowa City, IA 52242-1419
319-335-0770
e-mail: whan@math.uiowa.edu
Numerical analysis, Finite element method, Numerical PDE, Variational inequalities, Computational mechanics

Tian-Xiao He
Department of Mathematics and Computer Science
P.O. Box 2900, Illinois Wesleyan University
Bloomington, IL 61702-2900, USA
Tel (309)556-3089
Fax (309)556-3864
the@iwu.edu
Approximations, Wavelet, Integration Theory, Numerical Analysis, Analytic Combinatorics

Margareta Heilmann
Faculty of Mathematics and Natural Sciences, University of Wuppertal
Gaußstraße 20
D-42119 Wuppertal, Germany,
heilmann@math.uni-wuppertal.de

Analysis, rtrggani@memphis.edu

Juan J. Trujillo
University of La Laguna
Departamento de Analisis Matematico
C/Astr.Fco.Sanchez s/n
38271. LaLaguna. Tenerife.
SPAIN
Tel/Fax 34-922-318209
Juan.Trujillo@ull.es
Fractional: Differential Equations-Operators-Fourier Transforms, Special functions, Approximations, and Applications

Ram Verma
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1200 Dallas Drive #824 Denton, TX 76205, USA
Verma99@msn.com
Applied Nonlinear Analysis, Numerical Analysis, Variational Inequalities, Optimization Theory, Computational Mathematics, Operator Theory

Xiang Ming Yu
Department of Mathematical Sciences
Southwest Missouri State University
Springfield, MO 65804-0094
417-836-5931
xmy944f@missouristate.edu
Classical Approximation Theory, Wavelets

Lotfi A. Zadeh
Professor in the Graduate School and Director, Computer Initiative, Soft Computing (BISC)
Computer Science Division
University of California at Berkeley
Berkeley, CA 94720
Office: 510-642-4959
Sec: 510-642-8271
Home: 510-526-2569
FAX: 510-642-1712
zadeh@cs.berkeley.edu
Fuzzyness, Artificial Intelligence, Natural language processing, Fuzzy logic

Richard A. Zalik
Department of Mathematics
Auburn University
Auburn University, AL 36849-5310
USA.
Tel 334-844-6557 office
678-642-8703 home
Fax 334-844-6555
zalik@auburn.edu
Approximation Theory, Chebychev Systems, Wavelet Theory

Ahmed I. Zayed
Department of Mathematical Sciences
DePaul University
2320 N. Kenmore Ave.
Chicago, IL 60614-3250
773-325-7808
e-mail: azayed@condor.depaul.edu
Shannon sampling theory, Harmonic analysis and wavelets, Special functions and orthogonal polynomials, Integral transforms

Ding-Xuan Zhou
Department Of Mathematics
City University of Hong Kong
83 Tat Chee Avenue
Kowloon, Hong Kong
852-2788 9708,Fax:852-2788 8561
e-mail: mazhou@cityu.edu.hk
Approximation Theory, Spline functions, Wavelets

Xin-long Zhou
Fachbereich Mathematik, Fachgebiet Informatik
Gerhard-Mercator-Universitat Duisburg
Lotharstr.65, D-47048 Duisburg, Germany
e-mail:Xzhou@informatik.uni-duisburg.de
Fourier Analysis, Computer-Aided Geometric Design, Computational Complexity, Multivariate Approximation Theory, Approximation and Interpolation Theory

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Editor in Chief: George Anastassiou

Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152-3240, U.S.A.

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Prof.George A. Anastassiou
Department of Mathematical Sciences
The University of Memphis
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e-mail: ganastss@memphis.edu

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Effect of RTI drug efficacy on the HIV dynamics with two cocirculating target cells

A. M. Elaiw, N. A. Almuallem and Aatef Hobiny
Department of Mathematics, Faculty of Science, King Abdulaziz University,
P.O. Box 80203, Jeddah 21589, Saudi Arabia.
Emails: a_m_elaiw@yahoo.com (A. Elaiw), nalmuallem@gmail.com (N. Almuallem)

Abstract

In this paper, we propose and analyze an HIV dynamics model. The model can be seen as a generalization of many HIV dynamics models presented in the literature since it incorporates (i) two classes of target cells, CD4$^+$ T cells and macrophages, (ii) two types of infected cells, short-lived infected cells and the long-lived chronically infected cells, (iii) intracellular discrete delays, (iv) reverse transcriptase inhibitors (RTIs) drugs with different drug efficacies on CD4$^+$T cells and macrophages. The incidence rate of infection is represented by a general function. A bifurcation parameter, known as the basic reproduction number, R_0 is derived. We established a set of conditions on the general function which are sufficient to determine the global dynamics of the model. Using Lyapunov functionals and LaSalle's invariance principle, the global asymptotic stability of the two equilibria of the model is obtained. An example is presented and some numerical simulations are conducted in order to illustrate the dynamical behavior.

Keywords: Delayed-HIV models; Chronically infected cells; Cocirculating target cells; Immune responses; Lyapunov method.

1 Introduction

Human immunodeficiency virus (HIV) is one of the most dangerous human viruses that destroys the immune system and causes acquired immunodeficiency syndrome (AIDS). During the past decades, several HIV mathematical models have been presented and analyzed (see e.g. [1]-[25]). Global stability of equilibria has become one of the most important features which help us to better understanding of the HIV dynamics. Thus, several researchers have devoted extensive efforts to study the global stability of HIV infection models (see e.g. [7], [8], [9], [11], [25], [14], [15], [16], [17], [22], [23], [19] and [24]). Some of these works assume that HIV infects only the CD4$^+$ T cells ([7], [8], [9], [11], [25], [22], [23], [19] and [24]), while, others assume that HIV infects two types of immune cells, CD4$^+$ T cells and macrophages ([14], [15], [18], [16] and [17]). Callaway and Perelson [3] pointed out that there are two types of infected cells, short-lived infected cells (which produce the most amounts of viruses) and the long-lived chronically infected cells. Moreover, the model presented in [3] incroporates reverse transcriptase inhibitors (RTIs) drugs with different drug efficacies on CD4$^+$T cells and macrophages.

Actually, there exists a time lag between the time the HIV contacts CD4$^+$ T cells or macrophages and the time the production of new infectious HIV particles. Intracellular time delay was first introduced into viral infection model by Herz et al. [5]. Since then, several delayed HIV models have been investigated (see e.g. [6], [7], [8], [9], [11], [25], [14], [17], [18], [22] and [19]). In a very recent work, Elaiw and Almualem [17] have presented the following delayed HIV model:

$$\begin{aligned}
\dot{x}_1(t) &= \lambda_1 - d_1x_1 - (1-\varepsilon)\bar{\beta}_1x_1v, \quad (1)\\
\dot{x}_2(t) &= \lambda_2 - d_2x_2 - (1-\chi\varepsilon)\bar{\beta}_2x_2v,\\
\dot{y}_1(t) &= (1-q_1)(1-\varepsilon)\bar{\beta}_1x_1(t-\tau_1)v(t-\tau_1) - \delta_1y_1,\\
\dot{y}_2(t) &= (1-q_2)(1-\chi\varepsilon)\bar{\beta}_2x_2(t-\tau_2)v(t-\tau_2) - \delta_2y_2,\\
\dot{z}_1(t) &= q_1(1-\varepsilon)\bar{\beta}_1x_1(t-\tau_1)v(t-\tau_1) - a_1z_1,\\
\dot{z}_2(t) &= q_2(1-\chi\varepsilon)\bar{\beta}_2x_1(t-\tau_1)v(t-\tau_1) - a_2z_2,\\
\dot{v}(t) &= \sum_{i=1}^{2}\left(N_i\delta_ie^{-n_i\kappa_i}y_i(t-\kappa_i) + M_ia_ie^{-h_i\omega_i}z_i(t-\omega_i)\right) - uv(t) \quad (2)
\end{aligned}$$

where x_i, y_i, z_i, and v represent the concentrations of uninfected cells, short-lived infected cells, long-lived chronically infected cells and free HIV particles, respectively, where $i = 1$, for the CD4$^+$ T cells and $i = 2$, for the macrophages. The birth and death rates of uninfected cells are given by λ_i and d_ix_i, respectively. Parameter $\bar{\beta}_i$ denotes the infection rate constant. Parameters δ_i and a_i are the death rate constants of the two types of infected cells, and u is the clearance rate of HIV. The uninfected target cells become short-lived infected and long-lived chronically infected cells with fractions $(1-q_i)$ and q_i, respectively, where $q_i.\in(0,1)$. The average number of free viruses produced in the lifetime of the two types of infected cells are given by N_i and M_i, respectively. Parameter τ_i represents for the time between viral contact with an uninfected cell of class i, until it becomes infected but not yet producer cells. The loss of the cells during the delay period $[t-\tau_i, t]$ is given by $e^{-m_i\tau_i}$, where $m_i > 0$. The parameters κ_i and ω_i represent the time necessary for producing new infectious viruses from the short-lived and long-lived chronically infected cells, respectively. The factors $e^{-n_i\kappa_i}$ and $e^{-h_i\omega_i}$ represent the loss of the two types of infected cells during the delay periods $[t-\kappa_i, t]$ and $[t-\omega_i, t]$, where $n_i > 0$ and $h_i > 0$.

The immune system has two main responses to viral infections. The first is based on the Cytotoxic T Lymphocyte (CTL) cells which are responsible to attack and kill the infected cells. The second immune response is based on the antibodies that are produced by the B cells. The function of the antibodies is to attack the viruses [1]. In some infections such as in malaria, the CTL immune response is less effective than the antibody immune response [26]. Several mathematical models have been proposed to consider the antibody immune response into the viral infection models (see [27]-[33])).

All the models presented in [27]-[33] are based on the assumption that, the virus attacks one class of target cells. Moreover, model (1)-(2) did not consider the immune response. Therefore, our aim in this paper is to propose an HIV dynamics model with humoral immunity. Our model generalize model (1)-(2) by taking into account the humoral immune response. We use Lyapunov functionals and LaSalle's invariance principle to prove the global stability of all the equilibria of the models.

2 The model

In this section, we propose and analyze the following HIV model:

$$\dot{x}_i(t) = \lambda_i - d_ix_i(t) - \phi_i(x_i(t), v(t)), \qquad i = 1,2, \tag{3}$$

$$\dot{y}_i(t) = (1-q_i)e^{-m_i\tau_i}\phi_i((t-\tau_i), v(t-\tau_i)) - \delta_iy_i(t), \qquad i = 1,2, \tag{4}$$

$$\dot{z}_i(t) = q_ie^{-m_i\tau_i}\phi_i((t-\tau_i), v(t-\tau_i)) - a_iz_i(t), \qquad i = 1,2, \tag{5}$$

$$\dot{v}(t) = \sum_{i=1}^{2}\left(N_{y_i}\delta_ie^{-n_i\kappa_i}y_i(t-\kappa_i) + M_{z_i}a_ie^{-r_i\omega_i}z_i(t-\omega_i)\right) - uv(t) - bv(t)f(w(t)), \tag{6}$$

$$\dot{w}(t) = cv(t) - pw(t). \tag{7}$$

The incidence rate of infection is given by a general function $\phi_i(x_i, v)$, where $\phi_1(x_1, v) = (1-\varepsilon)\bar{\phi}_1(x_1, v)$, and $\phi_2(x_2, v) = (1-\chi\varepsilon)\bar{\phi}_2(x_2, v)$. In addition, the neutralize rate of viruses is given by a general nonlinear function $f(w)$. Parameter b is the B cells neutralize rate, the antibody response is induced at a rate proportional to the concentration of free viruses. Parameters c and p are the recruited rate and death rate constants of B cells, respectively. All the parameters and variables of the model have the same meanings as given in (1)-(2).

2.1 Initial conditions

The initial conditions for system (3)-(7) take the form

$$\begin{aligned} &x_1(\theta) = \varphi_1(\theta),\ y_1(\theta) = \varphi_3(\theta),\ z_1(\theta) = \varphi_5(\theta),\\ &x_2(\theta) = \varphi_2(\theta),\ y_2(\theta) = \varphi_4(\theta),\ z_2(\theta) = \varphi_6(\theta),\\ &v(\theta) = \varphi_7(\theta),\ w(\theta) = \varphi_8(\theta)\\ &\varphi_j(\theta) \geq 0,\ \theta \in [-\varrho, 0),\ \varphi_j(0) > 0,\ j = 1, ..., 8, \end{aligned} \tag{8}$$

where $\varrho = \max\{\tau_1, \tau_2, \kappa_1, \kappa_2, \omega_1, \omega_2\}$ and $(\varphi_1(\theta), \varphi_2(\theta), ..., \varphi_8(\theta)) \in C([-\varrho, 0], \mathbb{R}^8_{\geq 0})$, where C is the Banach space of continuous functions mapping the interval $[-\varrho, 0]$ into $\mathbb{R}^8_{\geq 0}$. By the fundamental theory of functional differential equations [35], system (3)-(7) has a unique solution satisfying the initial conditions (8).

Assumption A1 Function ϕ_i, is continuously differentiable and satisfies the following:
(i) $\phi_i(x_i, v) > 0$, $\phi_i(x_i, 0) = \phi_i(0, v) = 0$, for all $x_i > 0$, $v > 0$, $i = 1, 2$,
(ii) $\frac{\partial \phi_i(x_i,v)}{\partial v} > 0$, $\frac{\partial \phi_i(x_i,v)}{\partial x_i} > 0$, for any $x_i > 0$, $v > 0$. Furthermore, $\frac{\partial \phi_i(x_i,0)}{\partial v} > 0$ for any $x_i > 0$, $i = 1, 2$.

Assumption A2 The function $f(\theta)$ is locally Lipschitz on $[0, \infty)$, and satisfies $f(\theta) > 0$ for all $\theta > 0$ and $f(0) = 0$, and $f(\theta)$ is strictly increasing in $[0, \infty)$.

2.2 Non-negativity and boundedness of solutions

In the following, we establish the non-negativity and boundedness of solutions of system (3)-(7) with initial conditions (8).

Proposition 1. Let $(x_1(t), x_2(t), y_1(t), y_2(t), z_1(t), z_2(t), v(t), w(t))$ be any solution of (3)-(7) satisfying the initial conditions (8), then $x_i(t), y_i(t), z_i(t), i = 1, 2, v(t)$ and $w(t)$ are all non-negative for $t \geq 0$ and ultimately bounded.

Proof. First, we prove that $x_i(t) > 0$, $i = 1, 2$, for all $t \geq 0$. Assume that $x_i(t)$ lose its positivity on some local existence interval $[0, l]$ for some constant l and let $t_i^* \in [0, l]$ be such that $x_i(t_i^*) = 0$. From Eq. (3) we have $\dot{x}_i(t_i^*) = \lambda_i > 0$. Hence $x_i(t) > 0$ for some $t \in (t_i^*, t_i^* + \epsilon)$, where $\epsilon > 0$ is sufficiently small. This leads to a contradiction and hence $x_i(t) > 0$, for all $t \geq 0$. Furthermore, from Eqs. (4)-(7) we have

$$y_i(t) = y_i(0)e^{-\delta_i t} + (1-q_i)e^{-m_i\tau_i}\int_0^t e^{-\delta_i(t-\theta)}\phi(x_i(\theta-\tau_i), v(\theta-\tau_i))d\theta, \qquad i = 1, 2,$$

$$z_i(t) = z_i(0)e^{-a_i t} + q_i e^{-m_i\tau_i}\int_0^t e^{-a_i(t-\theta)}\phi(x_i(\theta-\tau_i), v(\theta-\tau_i))d\theta, \qquad i = 1, 2,$$

$$v(t) = v(0)e^{-\int_0^t (u+bf(w(\zeta)))d\zeta} + \int_0^t e^{-\int_\theta^t (u+bf(w(\zeta)))d\zeta}\sum_{i=1}^{2}(N_{y_i}\delta_i e^{-n_i\kappa_i}y_i(\theta-\kappa_i) + M_{z_i}a_i e^{-r_i\omega_i}z_i(\theta-\omega_i))d\theta,$$

$$w(t) = w(0)e^{-pt} + c\int_0^t e^{-p(t-\theta)}v(\theta)d\theta,$$

then $y_i(t) \geq 0$, $z_i(t) \geq 0$, $i = 1, 2$, $v(t) \geq 0$ and $w(t) \geq 0$, for all $t \in [0, \varrho]$. By a recursive argument, we obtain $y_i(t) \geq 0$, $z_i(t) \geq 0$, $v(t) \geq 0$ and $w(t) \geq 0$, $i = 1, 2$, for all $t \geq 0$.

Next we show the boundedness of the solutions. From Eq. (3) we have $\dot{x}_i(t) \leq \lambda_i - d_i x_i(t)$, $i = 1, 2$. This implies that $\limsup_{t\to\infty} x_i(t) \leq \frac{\lambda_i}{d_i}$, $i = 1, 2$. Let $T_i(t) = e^{-m_i\tau_i} x_i(t-\tau_i) + y_i(t) + z_i(t)$, $i = 1, 2$ then

$$\begin{aligned}\dot{T}_i(t) &= e^{-m_i\tau_i}\lambda_i - e^{-m_i\tau_i} d_i x_i(t-\tau_i) - \delta_i y_i(t) - a_i z_i(t)\\ &\leq e^{-m_i\tau_i}\lambda_i - \sigma_i\left(e^{-m_i\tau_i} x_i(t-\tau_i) + y_i(t) + z_i(t)\right) \leq \lambda_i - \sigma_i T_i(t),\end{aligned}$$

where $\sigma_i = \min\{d_i, \delta_i, a_i\}$. Hence, $\limsup_{t\to\infty} T_i(t) \leq L_i$, where $L_i = \dfrac{\lambda_i}{\sigma_i}$. Since $x_i(t)$, $y_i(t)$ and $z_i(t)$ are all non-negative, then $\limsup_{t\to\infty} y_i(t) \leq L_i$, and $\limsup_{t\to\infty} z_i(t) \leq L_i$ for all $t \geq 0$. Moreover,

$$\begin{aligned}\dot{v} &= \sum_{i=1}^{2}\left(N_{y_i}\delta_i e^{-n_i\kappa_i} y_i(t-\kappa_i) + M_{z_i} a_i e^{-r_i\omega_i} z_i(t-\omega_i)\right) - uv - bv(t)f(w(t))\\ &\leq \sum_{i=1}^{2}\left(N_{y_i}\delta_i e^{-n_i\kappa_i} + M_{z_i} a_i e^{-r_i\omega_i}\right) L_i - uv.\end{aligned}$$

Then $\limsup_{t\to\infty} v(t) \leq L_3$, for all $t \geq 0$, where $L_3 = \sum_{i=1}^{2}\frac{\left(N_{y_i}\delta_i e^{-n_i\kappa_i} + M_{z_i} a_i e^{-r_i\omega_i}\right)L_i}{u}$. Furthermore, $\dot{w} = cv - pw \leq cL_3 - pw$, then $\limsup_{t\to\infty} w(t) \leq L_4$, for all $t \geq 0$, where $L_4 = \frac{cL_3}{p}$.Therefore, $x_i(t), y_i(t), z_i(t), v(t)$ and $w(t)$ are ultimately bounded.

2.3 Equilibria

Let Assumptions A1 (i) and A2 be satisfied, then system (3)-(7) has a disease-free equilibrium $E_0 = (x_1^0, x_2^0, 0, 0, 0, 0, 0, 0)$, where $x_i^0 = \frac{\lambda_i}{d_i}$, $i = 1, 2$. The system can also has another positive equilibrium $E_1 = (\tilde{x}_1, \tilde{x}_2, \tilde{y}_1, \tilde{y}_2, \tilde{z}_1, \tilde{z}_2, \tilde{v}, \tilde{w})$ which is called endemic equilibrium. The coordinates of the endemic equilibrium, if it exists satisfy the equalities:

$$\begin{aligned}&\lambda_i = d_i\tilde{x}_i + \phi_i(\tilde{x}_i, \tilde{v}), \quad \delta_i\tilde{y}_i = (1-q_i)e^{-m_i\tau_i}\phi_i(\tilde{x}_i, \tilde{v}), \quad a_i\tilde{z}_i = q_i e^{-m_i\tau_i}\phi_i(\tilde{x}_i, \tilde{v}),\\ &u\tilde{v} = \sum_{i=1}^{2}\left(N_{y_i}\delta_i e^{-n_i\kappa_i}\tilde{y}_i + M_{z_i} a_i e^{-r_i\omega_i}\tilde{z}_i\right) - b\tilde{v}f(\tilde{w}), \qquad \tilde{w} = \frac{c}{p}\tilde{v}.\end{aligned}$$

Then the basic infection reproduction number for system (3)-(7) is

$$R_0 = \sum_{i=1}^{2} R_{0i} = \sum_{i=1}^{2}\frac{((1-q_i)N_{y_i}e^{-n_i\kappa_i} + q_i M_{z_i}e^{-r_i\omega_i})e^{-m_i\tau_i}}{u}\frac{\partial\phi_i(x_i^0, 0)}{\partial v}.$$

The term $\partial\phi_i(x_i^0, 0)/\partial v$ represents the maximal average number of target cells of class i that infects by viruses, and R_{01} denotes the basic infection reproduction number of the HIV dynamics with CD4$^+$ T cells (in the absence of macrophages) and R_{02} denotes the basic infection reproduction number of the HIV dynamics with macrophages (in the absence of CD4$^+$T cells), respectively. The parameter R_0 determines whether the infection can be established.

2.4 Global stability analysis

In this subsection, we establish a set of conditions which are sufficient for the global stability of the two equilibria of system (3)-(7) employing Lyapunov method and LaSalle's invariance principle. The following function will be used throughout the paper $H(s) = s - 1 - \ln s$.

Assumption A3 The function ϕ_i, $i = 1, 2$ satisfies:
(i) $\left(\frac{\partial\phi_i(x_i,0)}{\partial v} - \frac{\partial\phi_i(x_i^0,0)}{\partial v}\right)\left(x_i^0 - x_i\right) \leq 0$, for $x_i > 0$,
(ii) $\phi_i(x_i, v) \leq v\frac{\partial\phi_i(x_i,0)}{\partial v}$, for all $x_i, v > 0$.

Theorem 1. Let Assumptions A1-A3 be satisfied and $R_0 \leq 1$, then the disease-free equilibrium E_0 of system (3)-(7) is GAS.

Proof. Define a Lyapunov functional W_0 as follows:

$$W_0 = \sum_{i=1}^{2}\gamma_i \left[x_i - x_i^0 - \int_{x_i^0}^{x_i} \lim_{v\to 0^+} \frac{\phi_i(x_i^0, v)}{\phi_i(s,v)} ds + \frac{N_{y_i} e^{-n_i\kappa_i}}{\gamma_i} y_i + \frac{M_{z_i} e^{-r_i\omega_i}}{\gamma_i} z_i \right.$$
$$\left. + \int_0^{\tau_i} \phi_i(x_i(t-\theta), v(t-\theta))d\theta + \frac{e^{-n_i\kappa_i} N_{y_i}\delta_i}{\gamma_i} \int_0^{\kappa_i} y_i(t-\theta)d\theta + \frac{e^{-r_i\omega_i} M_{z_i} a_i}{\gamma_i} \int_0^{\omega_i} z_i(t-\theta)d\theta \right]$$
$$+ v + \frac{b}{c}\int_0^{w} f(\theta)d\theta,$$

where $\gamma_i = e^{-m_i\tau_i}((1-q_i)e^{-n_i\kappa_i}N_{y_i} + q_i e^{-r_i\omega_i}M_{z_i})$, $i = 1,2$. We calculate $\frac{dW_0}{dt}$ along the trajectories of system (3)-(7) as:

$$\frac{dW_0}{dt} = \sum_{i=1}^{2}\gamma_i \left[\left(1 - \lim_{v\to 0^+} \frac{\phi_i(x_i^0, v)}{\phi_i(x_i,v)}\right)(\lambda_i - d_i x_i - \phi_i(x_i, v)) \right.$$
$$+ \frac{N_{y_i} e^{-n_i\kappa_i}}{\gamma_i}((1-q_i)e^{-m_i\tau_i}\phi_i(x_i(t-\tau_i), v(t-\tau_i)) - \delta_i y)$$
$$+ \frac{M_{z_i} e^{-r_i\omega_i}}{\gamma_i}(q_i e^{-m_i\tau_i}\phi_i(x_i(t-\tau_i), v(t-\tau_i)) - a_i z_i)$$
$$+ \phi_i(x_i, v) - \phi_i(x_i(t-\tau_i), v(t-\tau_i))$$
$$\left. + \frac{e^{-n_i\kappa_i}N_{y_i}\delta_i}{\gamma_i}(y_i - y_i(t-\kappa_i)) + \frac{e^{-r_i\omega_i}M_{z_i}a_i}{\gamma_i}(z_i - z_i(t-\omega_i))\right]$$
$$+ \sum_{i=1}^{2}\left(N_{y_i}\delta_i e^{-n_i\kappa_i} y_i(t-\kappa_i) + M_{z_i}a_i e^{-r_i\omega_i} z_i(t-\omega_i)\right) - uv - bvf(w) + \frac{b}{c}f(w)(cv - pw). \tag{9}$$

Collecting terms of Eq. (9) we get

$$\frac{dW_0}{dt} = \sum_{i=1}^{2}\gamma_i \left[\left(1 - \frac{\partial\phi_i(x_i^0, 0)/\partial v}{\partial\phi_i(x_i, 0)/\partial v}\right)(\lambda_i - d_i x_i) + \phi_i(x_i, v)\frac{\partial\phi_i(x_i^0, 0)/\partial v}{\partial\phi_i(x_i, 0)/\partial v}\right] - uv - \frac{bp}{c}wf(w)$$
$$= \sum_{i=1}^{2}\gamma_i \left[\lambda_i\left(1 - \frac{x_i}{x_i^0}\right)\left(1 - \frac{\partial\phi_i(x_i^0, 0)/\partial v}{\partial\phi_i(x_i, 0)/\partial v}\right) + \phi_i(x_i, v)\frac{\partial\phi_i(x_i^0, 0)/\partial v}{\partial\phi_i(x_i, 0)/\partial v}\right] - uv - \frac{bp}{c}wf(w)$$
$$= \sum_{i=1}^{2}\gamma_i\lambda_i\left(1 - \frac{x_i}{x_i^0}\right)\left(1 - \frac{\partial\phi_i(x_i^0, 0)/\partial v}{\partial\phi_i(x_i, 0)/\partial v}\right) + \sum_{i=1}^{2}\gamma_i\phi_i(x_i, v)\frac{\partial\phi_i(x_i^0, 0)/\partial v}{\partial\phi_i(x_i, 0)/\partial v} - uv - \frac{bp}{c}wf(w). \tag{10}$$

Using A3 we get

$$\frac{dW_0}{dt} \le \sum_{i=1}^{2}\gamma_i\lambda_i\left(1 - \frac{x_i}{x_i^0}\right)\left(1 - \frac{\partial\phi_i(x_i^0, 0)/\partial v}{\partial\phi_i(x_i, 0)/\partial v}\right) + \sum_{i=1}^{2}\gamma_i v\frac{\partial\phi_i(x_i^0, 0)}{\partial v} - uv - \frac{bp}{c}wf(w)$$
$$= \sum_{i=1}^{2}\gamma_i\lambda_i\left(1 - \frac{x_i}{x_i^0}\right)\left(1 - \frac{\partial\phi_i(x_i^0, 0)/\partial v}{\partial\phi_i(x_i, 0)/\partial v}\right) + (R_0 - 1)uv - \frac{bp}{c}wf(w). \tag{11}$$

By using Assumption A2, the last term is less than or equal zero. Therefore, If $R_0 \le 1$ then $\frac{dW_0}{dt} \le 0$ for all $x_1, x_2, v, w > 0$. We note that, the solutions of the system (3)-(7) converge to Γ, the largest invariant subset of $\left\{\frac{dW_0}{dt} = 0\right\}$. From Eq. (11) we have $\frac{dW_0}{dt} = 0$ iff $x_i = x_i^0$, $i = 1,2$, $v = 0$ and $w = 0$. The set Γ is invariant and for any element belongs to Γ satisfies $w = 0$, $v = 0$ then $\dot{v} = 0$. We can see from Eq. (19) that

$$\sum_{i=1}^{2}\left(N_{y_i}\delta_i e^{-n_i\kappa_i} y_i(t-\kappa_i) + M_{z_i}a_i e^{-r_i\omega_i} z_i(t-\omega_i)\right) = 0.$$

Since y_i and z_i are non-negative for $i = 1,2$, then $y_1 = y_2 = 0$ and $z_1 = z_2 = 0$. It follows that, $\frac{dW_0}{dt} = 0$ iff $x_i = x_i^0$, $y_i = z_i = v = w = 0$, i = 1,2. From LaSalle's invariance principle, E_0 is GAS.

To establish the global stability of the endemic equilibrium, we need the following condition.

Assumption A4 Function $\phi_i(x_i, v)$ satisfies the following:

$$\left(\frac{\phi_i(x_i,v)}{\phi_i(x_i,\tilde{v})} - \frac{v}{\tilde{v}}\right)\left(1 - \frac{\phi_i(x_i,\tilde{v})}{\phi_i(x_i,v)}\right) \leq 0, \quad x_i, v > 0$$

Theorem 2. Let Assumptions A1, A2 and A4 hold true and the endemic equilibrium E_1 of system (3)-(7) exists, then E_1 is GAS.

Proof. We consider the Lyapunov functional W_1 as:

$$\begin{aligned} W_1 = \sum_{i=1}^{2}\gamma_i \Bigg[& x_i - \tilde{x}_i - \int_{\tilde{x}_i}^{x_i} \frac{\phi_i(\tilde{x}_i,\tilde{v})}{\phi_i(s,\tilde{v})} ds + \frac{N_{y_i}e^{-n_i\kappa_i}}{\gamma_i}\tilde{y}_i H\left(\frac{y_i}{\tilde{y}_i}\right) \\ & + \frac{M_{z_i}e^{-r_i\omega_i}}{\gamma_i}\tilde{z}_i H\left(\frac{z_i}{\tilde{z}_i}\right) + \phi_i(\tilde{x}_i,\tilde{v}) \int_0^{\tau_i} H\left(\frac{\phi_i(x_i(t-\theta), v(t-\theta))}{\phi_i(\tilde{x}_i,\tilde{v})}\right) d\theta \\ & + \frac{e^{-n_i\kappa_i}N_{y_i}\delta_i\tilde{y}_i}{\gamma_i}\int_0^{\kappa_i} H\left(\frac{y_i(t-\theta)}{\tilde{y}_i}\right) d\theta + \frac{e^{-r_i\omega_i}M_{z_i}a_i\tilde{z}_i}{\gamma_i}\int_0^{\omega_i} H\left(\frac{z_i(t-\theta)}{\tilde{z}_i}\right) d\theta \Bigg] + \tilde{v}H\left(\frac{v}{\tilde{v}}\right) \\ & + \frac{b}{c}\int_{\tilde{w}}^{w} (f(\theta) - f(\tilde{w}))d\theta. \end{aligned}$$

Calculating $\frac{dW_1}{dt}$ along the solutions of system (3)-(7) we obtain

$$\begin{aligned} \frac{dW_1}{dt} = \sum_{i=1}^{2}\gamma_i \Bigg[& \left(1 - \frac{\phi_i(\tilde{x}_i,\tilde{v})}{\phi_i(x_i,\tilde{v})}\right)(\lambda_i - d_i x_i - \phi_i(x_i,v)) \\ & + \frac{N_{y_i}e^{-n_i\kappa_i}}{\gamma_i}\left(1 - \frac{\tilde{y}_i}{y_i}\right)((1-q_i)e^{-m_i\tau_i}\phi_i(x_i(t-\tau_i), v(t-\tau_i)) - \delta_i y_i) \\ & + \frac{M_{z_i}e^{-r_i\omega_i}}{\gamma_i}\left(1 - \frac{\tilde{z}_i}{z_i}\right)(q_i e^{-m_i\tau_i}\phi_i(x_i(t-\tau_i), v(t-\tau_i)) - a_i z_i) \\ & + \phi_i(x_i,v) - \phi_i(x_i(t-\tau_i), v(t-\tau_i)) + \phi_i(\tilde{x}_i,\tilde{v})\ln\left(\frac{\phi_i(x_i(t-\tau_i), v(t-\tau_i))}{\phi_i(x_i,v)}\right) \\ & + \frac{e^{-n_i\kappa_i}N_{y_i}\delta_i\tilde{y}_i}{\gamma_i}\left(\frac{y_i}{\tilde{y}_i} - \frac{y_i(t-\kappa_i)}{\tilde{y}_i} + \ln\left(\frac{y_i(t-\kappa_i)}{y_i}\right)\right) \\ & + \frac{e^{-r_i\omega_i}M_{z_i}a_i\tilde{z}_i}{\gamma_i}\left(\frac{z_i}{\tilde{z}_i} - \frac{z_i(t-\omega_i)}{\tilde{z}_i} + \ln\left(\frac{z_i(t-\omega_i)}{z_i}\right)\right)\Bigg] \\ & + \left(1 - \frac{\tilde{v}}{v}\right)\left(\sum_{i=1}^{2}\left(N_{y_i}\delta_i e^{-n_i\kappa_i}y_i(t-\kappa_i) + M_{z_i}a_i e^{-r_i\omega_i}z_i(t-\omega_i)\right) - uv - bvf(w)\right) \\ & + \frac{b}{c}(f(w) - f(\tilde{w}))(cv - pw). \end{aligned} \tag{12}$$

Collecting terms of Eq. (12) we get

$$\begin{aligned}\frac{dW_1}{dt} &= \sum_{i=1}^{2}\gamma_i\left[\left(1-\frac{\phi_i(\tilde{x}_i,\tilde{v})}{\phi_i(x_i,\tilde{v})}\right)(\lambda_i-d_ix_i)+\phi_i(x_i,v)\frac{\phi_i(\tilde{x}_i,\tilde{v})}{\phi_i(x_i,\tilde{v})}+\frac{N_{y_i}e^{-n_i\kappa_i}\delta_i}{\gamma_i}\tilde{y}_i+\frac{M_{z_i}e^{-r_i\omega_i}a_i}{\gamma_i}\tilde{z}_i\right.\\ &-\frac{(1-q_i)e^{-m_i\tau_i}N_{y_i}e^{-n_i\kappa_i}}{\gamma_i}\frac{\tilde{y}_i\phi_i(x_i(t-\tau_i),v(t-\tau_i))}{y_i}-\frac{q_ie^{-m_i\tau_i}M_{z_i}e^{-r_i\omega_i}}{\gamma_i}\frac{\tilde{z}_i\phi_i(x_i(t-\tau_i),v(t-\tau_i))}{z_i}\\ &+\phi_i(\tilde{x}_i,\tilde{v})\ln\left(\frac{\phi_i(x_i(t-\tau_i),v(t-\tau_i))}{\phi_i(x_i,v)}\right)\\ &\left.+\frac{e^{-n_i\kappa_i}N_{y_i}\delta_i\tilde{y}_i}{\gamma_i}\ln\left(\frac{y_i(t-\kappa_i)}{y_i}\right)+\frac{e^{-r_i\omega_i}M_{z_i}a_i\tilde{z}_i}{\gamma_i}\ln\left(\frac{z_i(t-\omega_i)}{z_i}\right)\right]\\ &-\sum_{i=1}^{2}N_{y_i}\delta_ie^{-n_i\kappa_i}\frac{\tilde{v}y_i(t-\kappa_i)}{v}-\sum_{i=1}^{2}M_{z_i}a_ie^{-r_i\omega_i}\frac{\tilde{v}z_i(t-\omega_i)}{v}-uv+u\tilde{v}\\ &+b\tilde{v}f(w)-\frac{bp}{c}wf(w)-bvf(\tilde{w})+\frac{bp}{c}wf(\tilde{w}).\end{aligned}$$

Using the equilibrium conditions for E_1:

$$\begin{aligned}&\lambda_i=d_i\tilde{x}_i+\phi_i(\tilde{x}_i,\tilde{v}),\quad (1-q_i)e^{-m_i\tau_i}\phi_i(\tilde{x}_i,\tilde{v})=\delta_i\tilde{y}_i,\quad q_ie^{-m_i\tau_i}\phi_i(\tilde{x}_i,\tilde{v})=a_i\tilde{z}_i,\\ &u\tilde{v}=\sum_{i=1}^{2}\left(N_{y_i}\delta_ie^{-n_i\kappa_i}\tilde{y}_i+M_{z_i}a_ie^{-r_i\omega_i}\tilde{z}_i\right)-b\tilde{v}f(\tilde{w}),\quad \tilde{w}=\frac{c}{p}\tilde{v}\end{aligned}$$

and the following equality

$$uv=u\tilde{v}\frac{v}{\tilde{v}}=\frac{v}{\tilde{v}}\left(\sum_{i=1}^{2}(N_{y_i}\delta_ie^{-n_i\kappa_i}\tilde{y}_i+M_{z_i}a_ie^{-r_i\omega_i}\tilde{z}_i)-b\tilde{v}f(\tilde{w})\right)=\frac{v}{\tilde{v}}\sum_{i=1}^{2}\gamma_i\phi_i(\tilde{x}_i,\tilde{v})-bvf(\tilde{w}),$$

we obtain

$$\begin{aligned}\frac{dW_1}{dt} &= \sum_{i=1}^{2}\gamma_i\left[d_i\tilde{x}_i\left(1-\frac{x_i}{\tilde{x}_i}\right)\left(1-\frac{\phi_i(\tilde{x}_i,\tilde{v})}{\phi_i(x_i,\tilde{v})}\right)+\phi_i(\tilde{x}_i,\tilde{v})\left(1-\frac{\phi_i(\tilde{x}_i,\tilde{v})}{\phi_i(x_i,\tilde{v})}\right)\right.\\ &+\phi_i(\tilde{x}_i,\tilde{v})\left(\frac{\phi_i(x_i,v)}{\phi_i(x_i,\tilde{v})}-\frac{v}{\tilde{v}}\right)+\frac{2N_{y_i}e^{-n_i\kappa_i}\delta_i}{\gamma_i}\tilde{y}_i+\frac{2M_{z_i}e^{-r_i\omega_i}a_i}{\gamma_i}\tilde{z}_i\\ &-\frac{N_{y_i}e^{-n_i\kappa_i}\delta_i\tilde{y}_i}{\gamma_i}\left(\frac{\tilde{y}_i\phi_i(x_i(t-\tau_i),v(t-\tau_i))}{y_i\phi_i(\tilde{x}_i,\tilde{v})}+\frac{\tilde{v}y_i(t-\kappa_i)}{v\tilde{y}_i}\right)\\ &-\frac{M_{z_i}e^{-r_i\omega_i}a_i\tilde{z}_i}{\gamma_i}\left(\frac{\tilde{z}_i\phi_i(x_i(t-\tau_i),v(t-\tau_i))}{z_i\phi_i(\tilde{x}_i,\tilde{v})}+\frac{\tilde{v}z_i(t-\omega_i)}{v\tilde{z}_i}\right)\\ &+\frac{N_{y_i}e^{-n_i\kappa_i}\delta_i\tilde{y}_i}{\gamma_i}\left(\ln\left(\frac{\phi_i(x_i(t-\tau_i),v(t-\tau_i))}{\phi_i(x_i,v)}\right)+\ln\left(\frac{y_i(t-\kappa_i)}{y_i}\right)\right)\\ &\left.+\frac{M_{z_i}e^{-r_i\omega_i}a_i\tilde{z}_i}{\gamma_i}\left(\ln\left(\frac{\phi_i(x_i(t-\tau_i),v(t-\tau_i))}{\phi_i(x_i,v)}\right)+\ln\left(\frac{z_i(t-\omega_i)}{z_i}\right)\right)\right]\\ &-b\tilde{v}f(\tilde{w})+b\tilde{v}f(w)-\frac{bp}{c}wf(w)+\frac{bp}{c}wf(\tilde{w}).\end{aligned}\tag{13}$$

Using the following equalities

$$\begin{aligned}
\ln\left(\frac{\phi_i(x_i(t-\tau_i),v(t-\tau_i))}{\phi_i(x_i,v)}\right) &= \ln\left(\frac{\phi_i(\tilde{x}_i,\tilde{v})}{\phi_i(x_i,\tilde{v})}\right)+\ln\left(\frac{\tilde{y}_i\phi_i(x_i(t-\tau_i),v(t-\tau_i))}{y_i\phi_i(\tilde{x}_i,\tilde{v})}\right)\\
&\quad+\ln\left(\frac{v\phi_i(x_i,\tilde{v})}{\tilde{v}\phi_i(x_i,v)}\right)+\ln\left(\frac{\tilde{v}y_i}{v\tilde{y}_i}\right),\\
\ln\left(\frac{y_i(t-\kappa_i)}{y_i}\right) &= \ln\left(\frac{\tilde{v}y_i(t-\kappa_i)}{v\tilde{y}_i}\right)+\ln\left(\frac{v\tilde{y}_i}{\tilde{v}y_i}\right),\\
\ln\left(\frac{\phi_i(x_i(t-\tau_i),v(t-\tau_i))}{\phi_i(x_i,v)}\right) &= \ln\left(\frac{\phi_i(\tilde{x}_i,\tilde{v})}{\phi_i(x_i,\tilde{v})}\right)+\ln\left(\frac{\tilde{z}_i\phi_i(x_i(t-\tau_i),v(t-\tau_i))}{z_i\phi_i(\tilde{x}_i,\tilde{v})}\right)\\
&\quad+\ln\left(\frac{v\phi_i(x_i,\tilde{v})}{\tilde{v}\phi_i(x_i,v)}\right)+\ln\left(\frac{\tilde{v}z_i}{v\tilde{z}_i}\right),\\
\ln\left(\frac{z_i(t-\omega_i)}{z_i}\right) &= \ln\left(\frac{\tilde{v}z_i(t-\omega_i)}{v\tilde{z}_i}\right)+\ln\left(\frac{v\tilde{z}_i}{\tilde{v}z_i}\right).
\end{aligned}$$

Eq. (13) can be rewritten as

$$\begin{aligned}
\frac{dW_1}{dt} &= \sum_{i=1}^{2}\left[\gamma_i d_i\tilde{x}_i\left(1-\frac{x_i}{\tilde{x}_i}\right)\left(1-\frac{\phi_i(\tilde{x}_i,\tilde{v})}{\phi_i(x_i,\tilde{v})}\right)+\gamma_i\phi_i(\tilde{x}_i,\tilde{v})\left(\frac{\phi_i(x_i,v)}{\phi_i(x_i,\tilde{v})}-\frac{v}{\tilde{v}}-1+\frac{v\phi_i(x_i,\tilde{v})}{\tilde{v}\phi_i(x_i,v)}\right)\right.\\
&\quad-\gamma_i\phi_i(\tilde{x}_i,\tilde{v})\left(\frac{\phi_i(\tilde{x}_i,\tilde{v})}{\phi_i(x_i,\tilde{v})}-1-\ln\left(\frac{\phi_i(\tilde{x}_i,\tilde{v})}{\phi_i(x_i,\tilde{v})}\right)\right)-\gamma_i\phi_i(\tilde{x}_i,\tilde{v})\left(\frac{v\phi_i(x_i,\tilde{v})}{\tilde{v}\phi_i(x_i,v)}-1-\ln\left(\frac{v\phi_i(x_i,\tilde{v})}{\tilde{v}\phi_i(x_i,v)}\right)\right)\\
&\quad-N_{y_i}e^{-n_i\kappa_i}\delta_i\tilde{y}_i\left(\frac{\tilde{y}_i\phi_i(x_i(t-\tau_i),v(t-\tau_i))}{y_i\phi_i(\tilde{x}_i,\tilde{v})}-1-\ln\left(\frac{\tilde{y}_i\phi_i(x_i(t-\tau_i),v(t-\tau_i))}{y_i\phi_i(\tilde{x}_i,\tilde{v})}\right)\right)\\
&\quad-N_{y_i}e^{-n_i\kappa_i}\delta_i\tilde{y}_i\left(\frac{\tilde{v}y_i(t-\kappa_i)}{v\tilde{y}_i}-1-\ln\left(\frac{\tilde{v}y_i(t-\kappa_i)}{v\tilde{y}_i}\right)\right)\\
&\quad-M_{z_i}e^{-r_i\omega_i}a_i\tilde{z}_i\left(\frac{\tilde{z}_i\phi_i(x_i(t-\tau_i),v(t-\tau_i))}{z_i\phi_i(\tilde{x}_i,\tilde{v})}-1-\ln\left(\frac{\tilde{z}_i\phi_i(x_i(t-\tau_i),v(t-\tau_i))}{z_i\phi_i(\tilde{x}_i,\tilde{v})}\right)\right)\\
&\quad\left.-M_{z_i}e^{-r_i\omega_i}a_i\tilde{z}_i\left(\frac{\tilde{v}z_i(t-\omega_i)}{v\tilde{z}_i}-1-\ln\left(\frac{\tilde{v}z_i(t-\omega_i)}{v\tilde{z}_i}\right)\right)\right]-\frac{bp}{c}(w-\tilde{w})(f(w)-f(\tilde{w})). \qquad (14)
\end{aligned}$$

Then Eq. (14) becomes,

$$\begin{aligned}
\frac{dW_1}{dt} &= \sum_{i=1}^{2}\left[\gamma_i d_i\tilde{x}_i\left(1-\frac{x_i}{\tilde{x}_i}\right)\left(1-\frac{\phi_i(\tilde{x}_i,\tilde{v})}{\phi_i(x_i,\tilde{v})}\right)+\gamma_i\phi_i(\tilde{x}_i,\tilde{v})\left(\frac{\phi_i(x_i,v)}{\phi_i(x_i,\tilde{v})}-\frac{v}{\tilde{v}}\right)\left(1-\frac{\phi_i(x_i,\tilde{v})}{\phi_i(x_i,v)}\right)\right.\\
&\quad-\gamma_i\phi_i(\tilde{x}_i,\tilde{v})\left\{H\left(\frac{\phi_i(\tilde{x}_i,\tilde{v})}{\phi_i(x_i,\tilde{v})}\right)+H\left(\frac{v\phi_i(x_i,\tilde{v})}{\tilde{v}\phi_i(x_i,v)}\right)\right\}\\
&\quad-N_{y_i}e^{-n_i\kappa_i}\delta_i\tilde{y}_i\left\{H\left(\frac{\tilde{y}_i\phi_i(x_i(t-\tau_i),v(t-\tau_i))}{y_i\phi_i(\tilde{x}_i,\tilde{v})}\right)+H\left(\frac{\tilde{v}y_i(t-\kappa_i)}{v\tilde{y}_i}\right)\right\}\\
&\quad\left.-M_{z_i}e^{-r_i\omega_i}a_i\tilde{z}_i\left\{H\left(\frac{\tilde{z}_i\phi_i(x_i(t-\tau_i),v(t-\tau_i))}{z_i\phi_i(\tilde{x}_i,\tilde{v})}\right)+H\left(\frac{\tilde{v}z_i(t-\omega_i)}{v\tilde{z}_i}\right)\right\}\right]-\frac{bp}{c}(w-\tilde{w})(f(w)-f(\tilde{w})).
\end{aligned}$$

By using Assumption A2, the last term is less than or equal zero. It is easy to see that, if $\tilde{x}_1,\tilde{x}_2,\tilde{y}_1,\tilde{y}_2,\tilde{z}_1,\tilde{z}_2,\tilde{v}$ and $\tilde{w}>0$, then $\frac{dW_1}{dt}\le 0$ for all $x_1,x_2,y_1,y_2,z_1,z_2,v$ and $w>0$. The solutions of the system limit to Γ, the largest invariant subset of $\{\frac{dW_1}{dt}=0\}$. It can be seen that $\frac{dW_1}{dt}=0$ if and only if $x_i=\tilde{x}_i$, $v=\tilde{v}$, $w=\tilde{w}$ and $H=0$ i.e.

$$\frac{\tilde{v}y_i(t-\kappa_i)}{v\tilde{y}_i}=\frac{\tilde{v}z_i(t-\omega_i)}{v\tilde{z}_i}=1 \qquad (15)$$

From Eq. (15), we have $y_i=\tilde{y}_i$ and $z_i=\tilde{z}_i$. It follows that $\frac{dW_1}{dt}$ equal to zero at E_1. LaSalle's invariance principle implies the global stability of E_1.

3 Example and numerical simulations

We introduce the following example:

$$\dot{x}_i(t) = \lambda_i - d_i x_i(t) - \frac{\beta_i x_i^{k_i}(t)v(t)}{(x_i^{k_i}(t)+\rho_i)(v(t)+\varsigma_i)}, \qquad i=1,2, \tag{16}$$

$$\dot{y}_i(t) = (1-q_i)e^{-m_i\tau_i}\frac{\beta_i x_i^{k_i}(t-\tau_i)v(t-\tau_i)}{(x_i^{k_i}(t-\tau_i)+\rho_i)(v(t-\tau_i)+\varsigma_i)} - \delta_i y_i(t), \qquad i=1,2, \tag{17}$$

$$\dot{z}_i(t) = q_i e^{-m_i\tau_i}\frac{\beta_i x_i^{k_i}(t-\tau_i)v(t-\tau_i)}{(x_i^{k_i}(t-\tau_i)+\rho_i)(v(t-\tau_i)+\varsigma_i)} - a_i z_i(t), \qquad i=1,2, \tag{18}$$

$$\dot{v}(t) = \sum_{i=1}^{2}\left(N_{y_i}\delta_i e^{-n_i\kappa_i}y_i(t-\kappa_i) + M_{z_i}a_i e^{-r_i\omega_i}z_i(t-\omega_i)\right) - uv(t) - bv(t)w(t), \tag{19}$$

$$\dot{w}(t) = cv(t) - pw(t). \tag{20}$$

For this example we have

$$\phi_i(x_i,v) = \frac{\beta_i x_i^{k_i}v}{(x_i^{k_i}+\rho_i)(v+\varsigma_i)}, \qquad f(w)=w \tag{21}$$

where $k_i, \rho_i, \varsigma_i > 0$, $i=1,2$. Function ϕ_i satisfies the following:

$$\begin{aligned}
\frac{\partial\phi_i(x_i,v)}{\partial x_i} &= \frac{k_i\rho_i\beta_i x_i^{k_i-1}v}{(x_i^{k_i}+\rho_i)^2(v+\varsigma_i)} > 0, \text{for all } x_i>0,\ v>0,\\
\frac{\partial\phi_i(x_i,v)}{\partial v} &= \frac{\varsigma_i\beta_i x_i^{k_i}}{(x_i^{k_i}+\rho_i)(v+\varsigma_i)^2} > 0, \text{for all } x_i>0,\\
\frac{\partial\phi_i(x_i,0)}{\partial v} &= \frac{\beta_i x_i^{k_i}}{\varsigma_i(x_i^{k_i}+\rho_i)} > 0, \text{for all } x_i>0,\ v>0,\\
\phi_i(x_i,v) &= \frac{\beta_i x_i^{k_i}v}{(x_i^{k_i}+\rho_i)(v+\varsigma_i)} \le \frac{\beta_i x_i^{k_i}v}{\varsigma_i(x_i^{k_i}+\rho_i)} = v\frac{\partial\phi_i(x_i,0)}{\partial v}, \text{for all } x_i>0,\ v>0,\\
\left(\frac{\phi_i(x_i,v)}{\phi_i(x_i,\tilde{v})}-\frac{v}{\tilde{v}}\right)\left(1-\frac{\phi_i(x_i,\tilde{v})}{\phi_i(x_i,v)}\right) &= \frac{-\varsigma_i(v-\tilde{v})^2}{\tilde{v}(\tilde{v}+\varsigma_i)(v+\varsigma_i)} \le 0, \text{for all } x_i,\ v>0.
\end{aligned}$$

Thus Assumption A1-A4 hold true and Theorems 1 and 2 are applicable. The basic reproduction number in this case is given by

$$R_0 = \sum_{i=1}^{2}R_{0i} = \sum_{i=1}^{2}\frac{((1-q_i)e^{-n_i\kappa_i}N_{y_i}+q_i e^{-r_i\omega_i}M_{z_i})e^{-m_i\tau_i}}{u}\frac{\beta_i(x_i^0)^{k_i}}{\varsigma_i((x_i^0)^{k_i}+\rho_i)}.$$

Without loss of generality we let, $\tau_e = \tau_1 = \tau_2 = \kappa_1 = \kappa_2 = \omega_1 = \omega_2$. In Table 1, we present the values of some parameters of model (16)-(20). The effect of the drug efficacy ε and time delay τ_e on the qualitative behavior of the system will be studied below in details. All computations are carried out by MATLAB.

3.1 Evolution of the system state with different initial conditions

We have chosen three different initial conditions as follows:
IC1: $\varphi_1(\theta)=600$, $\varphi_2(\theta)=200$, $\varphi_3(\theta)=1$, $\varphi_4(\theta)=0.5$, $\varphi_5(\theta)=1$, $\varphi_6(\theta)=2$, $\varphi_7(\theta)=1$, $\varphi_8(\theta)=0.02$,
IC2: $\varphi_1(\theta)=700$, $\varphi_2(\theta)=350$, $\varphi_3(\theta)=2$, $\varphi_4(\theta)=2$, $\varphi_5(\theta)=3$, $\varphi_6(\theta)=5$, $\varphi_7(\theta)=6$, $\varphi_8(\theta)=1$
IC3: $\varphi_1(\theta)=800$, $\varphi_2(\theta)=500$, $\varphi_3(\theta)=3.5$, $\varphi_4(\theta)=3.5$, $\varphi_5(\theta)=6$, $\varphi_6(\theta)=8$, $\varphi_7(\theta)=10$, $\varphi_8(\theta)=1.4$,
where $\theta\in[-\varrho,0)$. We will fix the delay parameter $\tau_e = 0.01$ day^{-1}, and using two sets of the parameter ε to get the following two cases.

Case (I): In this case, we choose $\varepsilon=0.8$ then we get $R_0 = 0.79 < 1$. Figure 1 shows that, the state of the system eventually approach to the infection-free equilibrium $E_0 = (1000, 600, 0, 0, 0, 0, 0, 0)$ for the three initial conditions IC1-IC3. This supports the results of Theorem 1 that the infection-free equilibrium E_0 is GAS. In

Table 1: The values of the parameters of model (16)-(20).

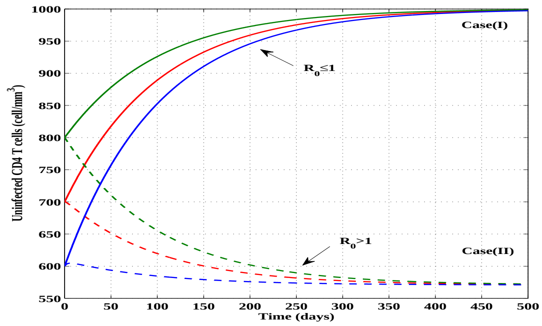
Parameter	Value	Parameter	Value
λ_1	10 cells mm^{-3}day^{-1}	λ_2	6 cells mm^{-3}day^{-1}
$\bar{\beta}_1$	8 cells mm^{-3}day^{-1}	$\bar{\beta}_2$	5 cells mm^{-3}day^{-1}
d_1	0.01 day^{-1}	d_2	0.01 day^{-1}
δ_1	0.5 day^{-1}	δ_2	0.3 day^{-1}
a_1	0.3 day^{-1}	a_2	0.1 day^{-1}
q_1	0.5	q_2	0.5
ς_1	10 virus mm^{-3}	ς_2	10 virus mm^{-3}
k_1	2	k_2	2
N_{y_1}	9 virus cells^{-1}	N_{y_2}	4 virus cells^{-1}
M_{z_1}	4 virus cells^{-1}	M_{z_2}	1 virus cells^{-1}
ρ_1	0.1 cells^{k_1} mm$^{-3k_1}$	ρ_2	0.1 cells^{k_1} mm$^{-3k_1}$
m_1	1 day^{-1}	m_2	1 day^{-1}
n_1	1 day^{-1}	n_2	1 day^{-1}
r_1	1 day^{-1}	r_2	1 day^{-1}
χ	0.5	u	1 day^{-1}
b	1 cells mm^{-3}day^{-1}	p	6 day^{-1}
c	1 day^{-1}	ε	Varied
τ_e	Varied		

this case, the virus particles will be cleared from the body.

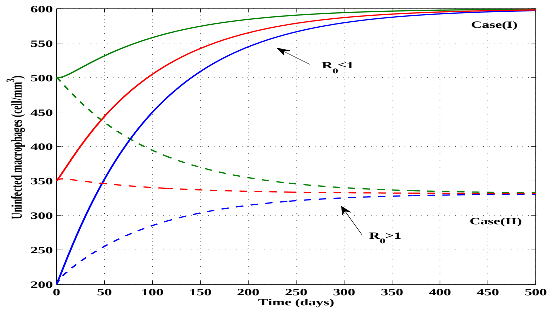
Case (II): In this case, we choose $\varepsilon = 0$ then we calculate $R_0 = 2.13 > 1$. Consequently, the system has two equilibria E_0 and E_1, and based on Theorem 2, E_1 is GAS. From Figure 1 we can see that, our simulation results are consistent with the theoretical results of Theorem 2. We observe that, the state of the system converge the endemic equilibrium $E_1 = (571.06, 332.13, 4.25, 4.43, 7.08, 13.28, 11.58, 1.93)$. for the three initial conditions IC1-IC3. In this case, the infection becomes chronic.

3.2 Effect of the drug efficacy on the dynamical behavior of the system

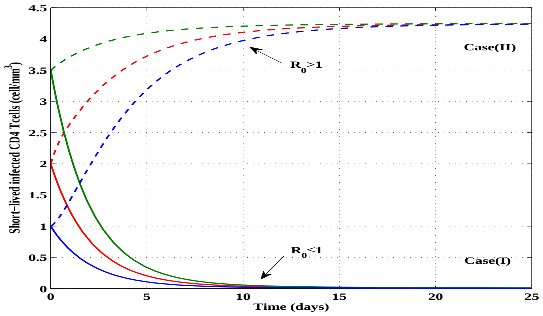
In this case, we will fix the delay parameter $\tau_e = 0.01$ day^{-1}. Figures 2 shows the effect of the parameter ε on the evolution of the uninfected CD4$^+$T cells and macrophages, short-lived infected cells, long-lived chronically infected cells, free virus particles and B cells. When there is no treatment i.e. $\varepsilon = 0$, the trajectory of the system tends to the endemic equilibrium $E_1 = (571.06, 332.13, 4.25, 4.43, 7.08, 13.28, 11.58, 1.93)$. Since E_1 exists, then according to Theorem 2, E_1 is GAS. We can see from the figures that, our simulation results are consistent with the theoretical results of Theorem 2. We observe that, as the drug efficacy is increased from $\varepsilon = 0$ to $\varepsilon = 0.8$, E_1 is still exists and is GAS, moreover, the concentrations of the uninfected CD4$^+$T cells and macrophages are increasing, while the concentrations of the short-lived infected cells, long-lived chronically infected cells, free virus particles and B cells are decreasing. When $\varepsilon = 0.98$, the basic reproduction number is given by $R_0 = 0.73 < 1$, then according to Theorem 1, the disease-free equilibrium E_0 is GAS. We can see that, the concentrations of uninfected CD4$^+$T cells and macrophages are increasing and converge to their normal values $\frac{\lambda_1}{d_1} = 1000$ cells mm^{-3}, $\frac{\lambda_2}{d_2} = 600$ cells mm^{-3}, respectively, while the concentrations of short-lived infected cells, long-lived chronically infected cells, free viruses and B cells are decaying and tend to zero. It means that, the numerical results are also compatible with the results of Theorem 1. In this case, the treatment with such drug



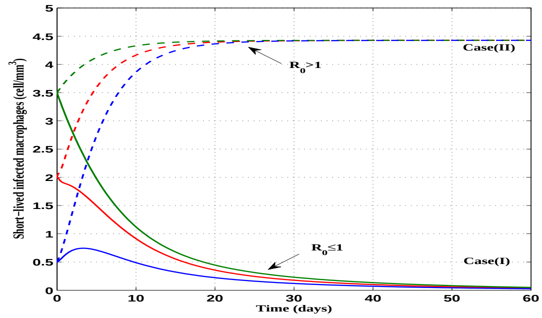
(a) Uninfected CD4$^+$T cells



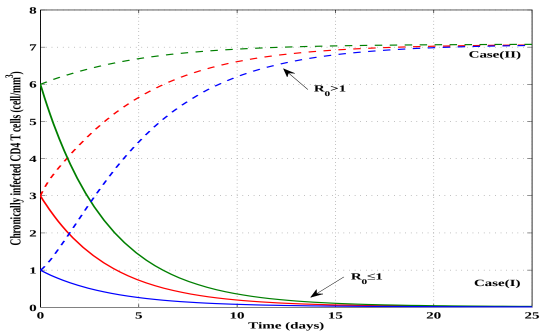
(b) Uninfected macrophages



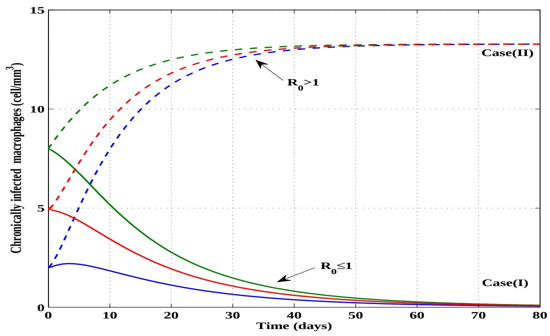
(c) Short-lived infected CD4$^+$T cells



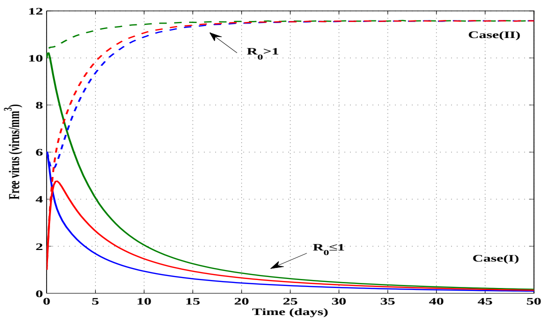
(d) Short-lived infected macrophages



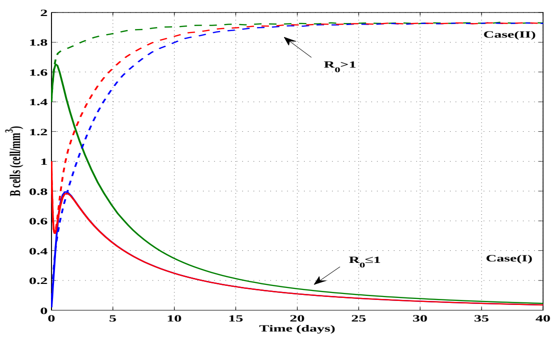
(e) Chronically infected CD4$^+$T cells



(f) Chronically infected macrophages



(g) Free virus



(h) B cells

Figure 1: The evolution of the system state in different initial conditions for model (16) -(20).

efficacy succeeded to eliminate the viruses from the blood.

3.3 Effect of the time delay on the dynamical behavior of the system

In this case, we will fix the drug efficacy $\varepsilon = 0.2$. Figure 3 shows the effect of the parameter τ_e on the evolution of the state variables of the system. When $\tau_e = 0.01$, the trajectory of the system tends to the endemic equilibrium $E_1 = (684.2, 378.23, 3.13, 3.66, 5.2, 10.9, 9.75, 1.62)$. Then E_1 exists and according to Theorem 2 E_1 is GAS. It means that, both the numerical and theoretical results of Theorem 2 are consistent. One can see that, as the time delay is increased from $\tau_e = 0.01$ to $\tau_e = 0.7$, E_1 is still exists and is GAS, in addition, the concentrations of the uninfected CD4$^+$T cells and macrophages are increased, while the concentrations of the short-lived infected cells, long-lived chronically infected cells, free virus particles and B cells are decreased. When $\tau_e = 1$, the basic reproduction number is given by $R_0 = 0.71 < 1$, then according to Theorem 1, E_0 is GAS. We can see that, the concentrations of uninfected CD4$^+$T cells and macrophages are increasing and converge to their normal values $\frac{\lambda_1}{d_1} = 1000$ cells mm^{-3}, $\frac{\lambda_2}{d_2} = 600$ cells mm^{-3}, respectively, while the concentrations of short-lived infected cells, long-lived chronically infected cells, free viruses and B cells are decaying and tend to zero. Figure 3 shows that the numerical results are also compatible with the results of Theorem 1. This shows the effect of time delay on preventing the disease from development.

3.4 Effects of the drug efficacy and the delay on the basic reproduction number:

Figure 4 shows the effect of the parameters ε and τ_e on the basic reproduction number R_0. We note that, $R_0 > 1$ for small values of ε or τ_e, and the endemic equilibrium exists and is GAS, while the disease-free equilibrium is unstable. When $R_0 = 1$ (which is a bifurcation point), both disease-free equilibrium and endemic equilibrium coincide and it is GAS. Moreover, as ε or τ_e is increasing, R_0 is decreasing until it becomes less than one, which makes the endemic equilibrium does not exists and the disease-free equilibrium is GAS. From a biological point of view, the intracellular delay plays a similar role as antiviral treatment in eliminating the virus. We observe that, even if there is no treatment i.e. $\varepsilon = 0$, sufficiently large delay suppress viral replication and clear the virus. This give us some suggestions on new drugs to prolong the increase the intracellular delay period.

3.5 Effects of two types of target cells on the dynamics and controls of HIV infection

In this subsection, we show the effects of two types of target cells on the dynamics and controls of HIV infection. We note that if $R_0 < 1$, then it is sure that $R_{01} < 1$ and $R_{02} < 1$. But if one neglect the presence of the macrophages in the HIV dynamics model, then the HIV model system (16) -(20) will become

$$\dot{x}_1(t) = \lambda_1 - d_1 x_1(t) - \frac{(1-\varepsilon)\bar{\beta}_1 x_1^{k_1}(t) v(t)}{(x_1^{k_1}(t) + \rho_1)(v(t) + \varsigma_1)}, \tag{22}$$

$$\dot{y}_1(t) = (1-q_1) e^{-m_1\tau_1} \frac{(1-\varepsilon)\bar{\beta}_1 x_1^{k_1}(t-\tau_1) v(t-\tau_1)}{(x_1^{k_1}(t-\tau_1) + \rho_1)(v(t-\tau_1) + \varsigma_1)} - \delta_1 y_1(t), \tag{23}$$

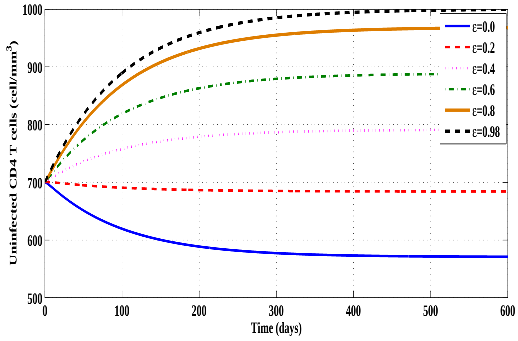
$$\dot{z}_1(t) = q_1 e^{-m_1\tau_1} \frac{(1-\varepsilon)\bar{\beta}_1 x_1^{k_1}(t-\tau_1) v(t-\tau_1)}{(x_1^{k_1}(t-\tau_1) + \rho_1)(v(t-\tau_1) + \varsigma_1)} - a_1 z_1(t), \tag{24}$$

$$\dot{v}(t) = N_{y_1} \delta_1 e^{-n_1\kappa_1} y_1(t-\kappa_1) + M_{z_1} a_1 e^{-r_1\omega_1} z_1(t-\omega_1) - uv(t) - bv(t)w(t), \tag{25}$$

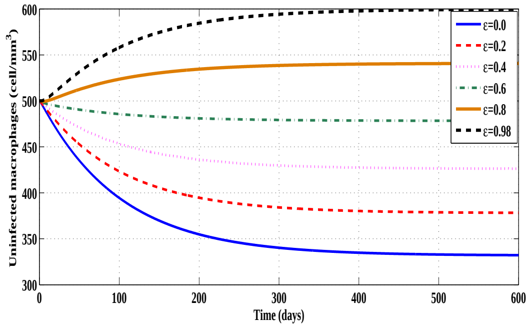
$$\dot{w}(t) = cv(t) - pw(t). \tag{26}$$

The basic reproduction number of model (22)-(26) is given by

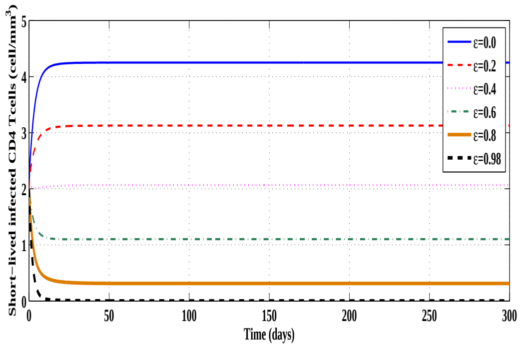
$$R_{01} = \frac{((1-q_1)e^{-n_1\kappa_1} N_{y_1} + q_1 e^{-r_1\omega_1} M_{z_1})}{u} \frac{e^{-m_1\tau_1}(1-\varepsilon)\bar{\beta}_1 (x_1^0)^{k_1}}{\varsigma_1((x_1^0)^{k_1} + \rho_1)}.$$



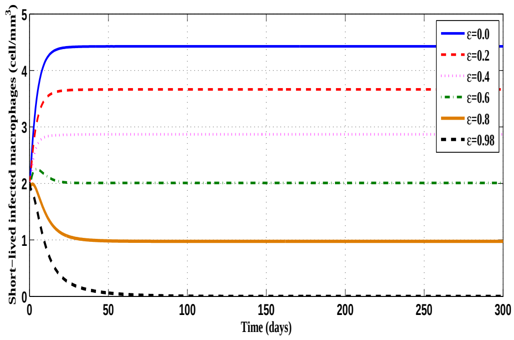
(a) Uninfected CD4$^+$T cells



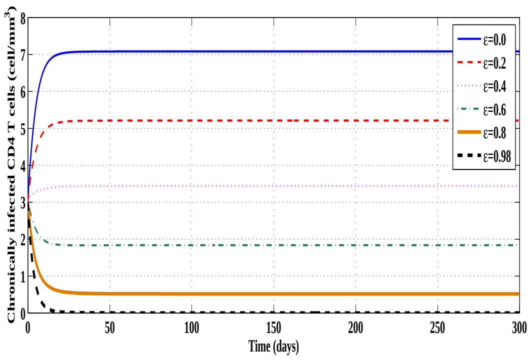
(b) Uninfected macrophages



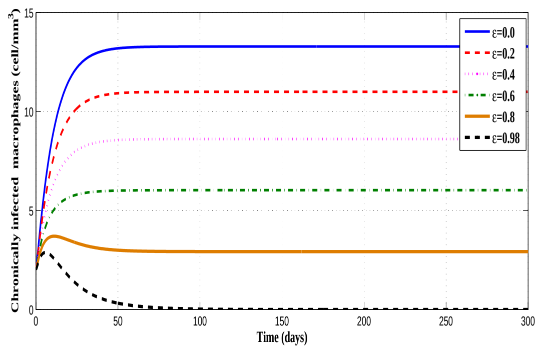
(c) Short-lived infected CD4$^+$T cells



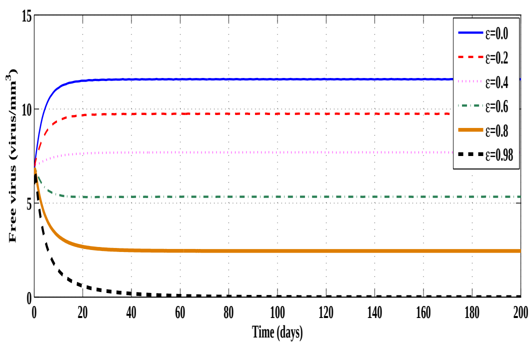
(d) Short-lived infected macrophages



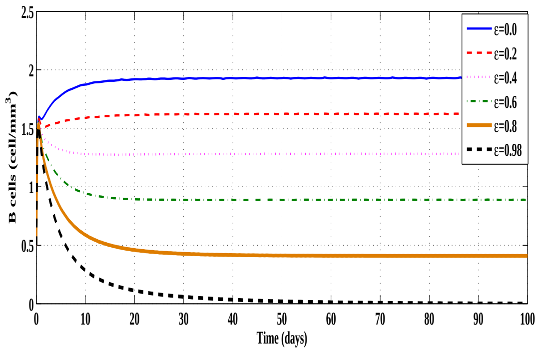
(e) Chronically infected CD4$^+$T cells



(f) Chronically infected macrophages

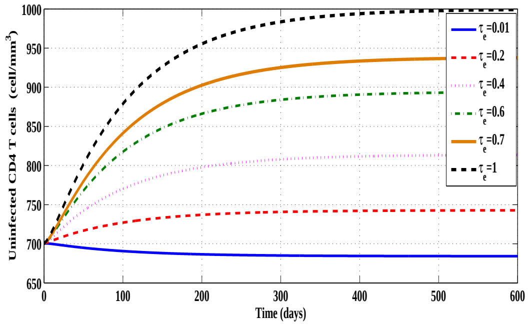


(g) Free virus

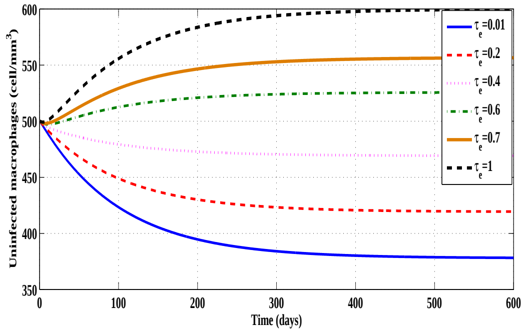


(h) B cells

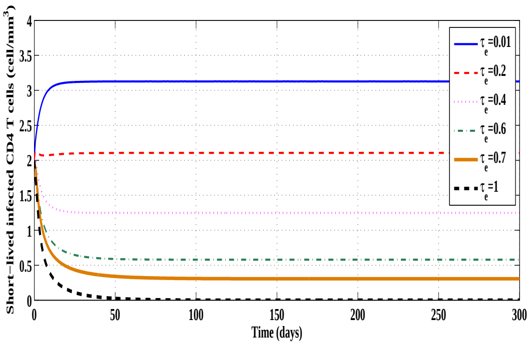
Figure 2: The evolution of the system state with different values of drug efficacy for model (16) -(20).



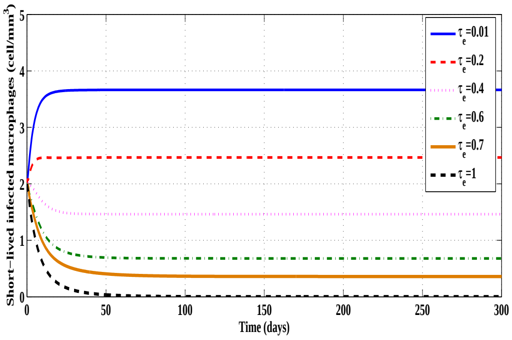
(a) Uninfected CD4$^+$T cells



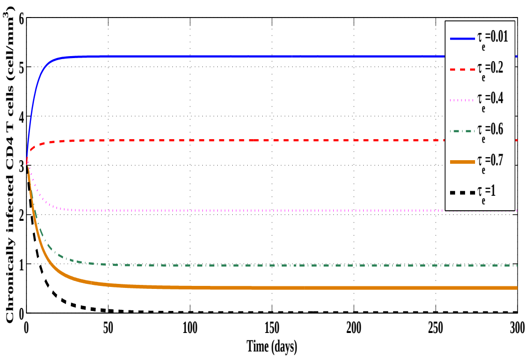
(b) Uninfected macrophages



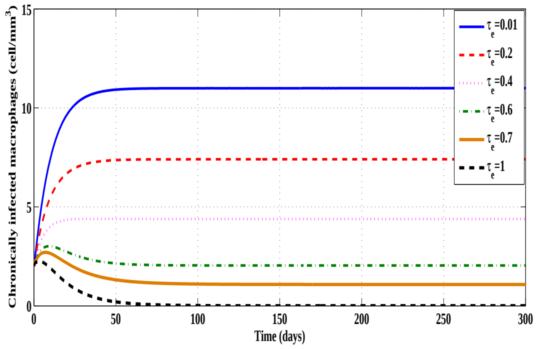
(c) Short-lived infected CD4$^+$T cells



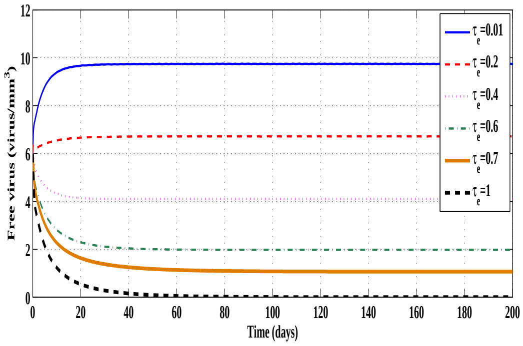
(d) Short-lived infected macrophages



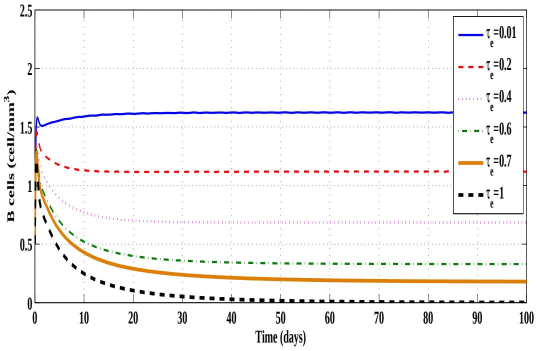
(e) Chronically infected CD4$^+$T cells



(f) Chronically infected macrophages



(g) Free virus



(h) B cells

Figure 3: The evolution of the system state with different values of delayed for model (16) -(20).

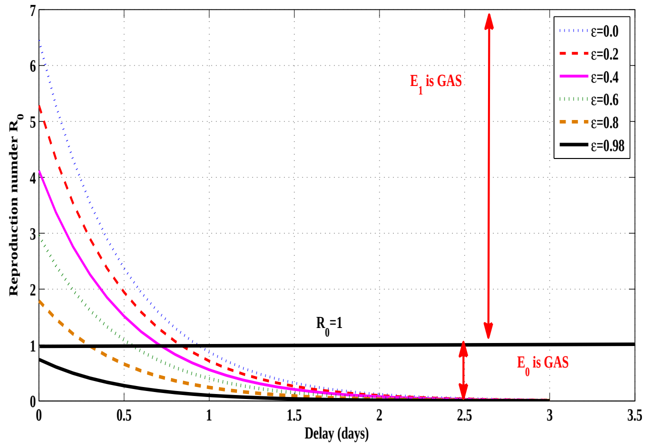


Figure 4: Effects of the drug efficacy and delays on the basis reproduction number of model (3)-(7)

Now we show that there is a number of parameter values for which $R_{01} \leq 1$, but $R_0 > 1$, and in such cases the solutions of system (22)-(26) tend to E_0 (in $\mathbb{R}^5_{\geq 0}$) as $t \to \infty$, while those of (16) -(20) tend to E_1 (in $\mathbb{R}^8_{\geq 0}$) as $t \to \infty$. We calculate the critical drug efficacy for system (16) -(20), E_0 is GAS when $R_0 \leq 1$ i.e.

$$\varepsilon_1^{crit} \leq \varepsilon < 1, \qquad \varepsilon_1^{crit} = \max\left\{0, \frac{\overline{R}_0 - 1}{\overline{R}_{01} + \chi \overline{R}_{02}}\right\},$$

where $\overline{R}_0 = R_0 \mid_{\varepsilon=0}$ and $\overline{R}_{0i} = R_{0i} \mid_{\varepsilon=0}$, $i = 1, 2$.
For system (22)-(26), E_0 is GAS when $R_{01} \leq 1$ i.e.

$$\varepsilon_2^{crit} \leq \varepsilon < 1, \qquad \varepsilon_2^{crit} = \max\left\{0, \frac{\overline{R}_{01} - 1}{\overline{R}_{01}}\right\}.$$

Clearly, $\varepsilon_1^{crit} > \varepsilon_2^{crit}$. Then, if one design treatment with drug efficacy $\varepsilon_2^{crit} \leq \varepsilon \leq \varepsilon_1^{crit}$, then E_0 is GAS for system (22)-(26) but unstable for system (16) -(20). Using the data in Table 1 and $\tau_e = 0.01$, we have $\varepsilon_1^{crit} = 0.93$ and $\varepsilon_2^{crit} = 0.80$. Let us choose $\varepsilon = 0.88$, then $R_{01} \mid_{\varepsilon=0.88} = 0.62 < 1$, but $R_0 \mid_{\varepsilon=0.88} = 1.31 > 1$. Therefore, more accurate treatment can be designed using the model (16) -(20) than those designed using model (22)-(26). Figure 5 shows the effect of two target cells on dynamics and control of HIV infection. We observe that, if we choose $\varepsilon = 0.88$, then the trajectory of model (16) -(20 tends to the infection-free equilibrium $E_0 = (1000, 0, 0, 0, 0, 0)$, while the trajectory of model (16) -(20) tends to the endemic equilibrium $E_1 = (990.54, 573.24, 0.1, 0.4, 0.15, 1.31, 1.04, 0.17)$.

3.6 Effect of long-lived chronically infected cells on the dynamics and controls of HIV infection

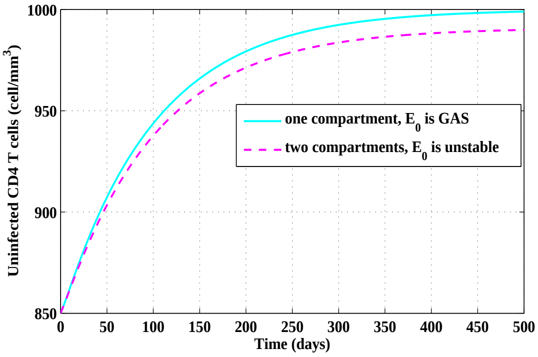
To show the effect of the presence of long-lived chronically infected cells on the dynamics and controls of HIV infection, we write the HIV model without long-lived chronically infected cells as:

$$\dot{x}_i(t) = \lambda_i - d_i x_i(t) - \frac{\beta_i x_i^{k_i}(t) v(t)}{(x_i^{k_i}(t) + \rho_i)(v(t) + \varsigma_i)}, \tag{27}$$

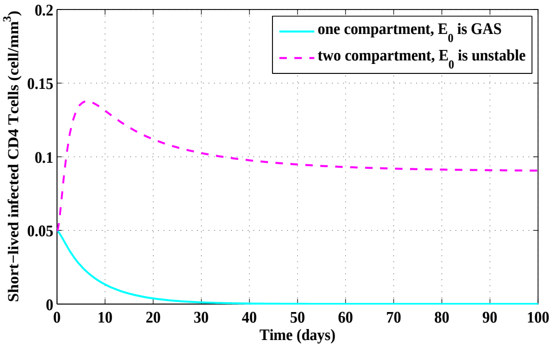
$$\dot{y}_i(t) = \frac{e^{-m_i \tau_i} \beta_i x_i^{k_i}(t - \tau_i) v(t - \tau_i)}{(x_i^{k_i}(t - \tau_i) + \rho_i)(v(t - \tau_i) + \varsigma_i)} - \delta_i y_i(t), \tag{28}$$

$$\dot{v}(t) = \sum_{i=1}^{2} N_{y_i} \delta_i e^{-n_i \kappa_i} y_i(t - \kappa_i) - uv(t) - bv(t)w(t), \tag{29}$$

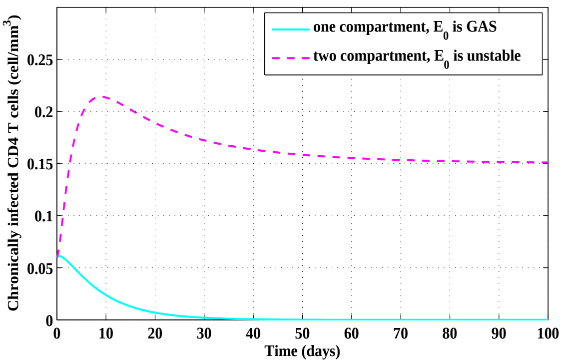
$$\dot{w}(t) = cv(t) - pw(t). \tag{30}$$



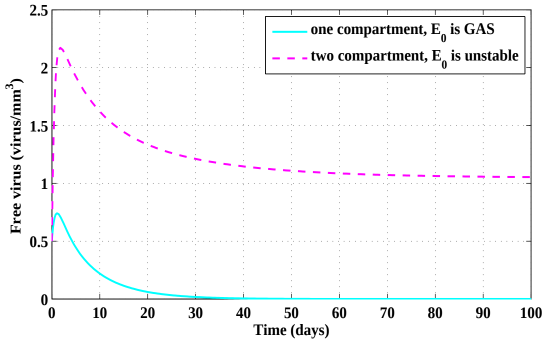
(a) Uninfected CD4$^+$T cells for model (16)-(20) and model (22)-(26).



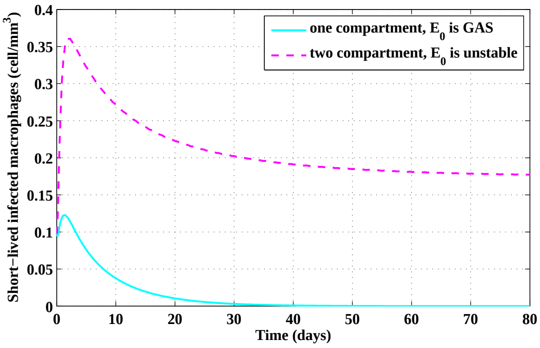
(b) Short-lived CD4$^+$T cells for model (16) -(20) and ((22)-(26).



(c) Chronically infected CD4$^+$T cells for model (16) -(20) and (22)-(26).



(d) Free virus for model (16) -(20)) and (22)-(26).



(e) B cells for model (16) -(20) and (22)-(26).

Figure 5: Effect of two types of target cells on the dynamics and controls of HIV infection

The basic reproduction number for system (27)-(30) is given by

$$\widetilde{R}_0 = \sum_{i=1}^{2} \widetilde{R}_{0i} = \sum_{i=1}^{2} \frac{e^{-n_i\kappa_i} e^{-m_i\tau_i} N_{y_i} \beta_i (x_i^0)^{k_i}}{u\varsigma_i((x_i^0)^{k_i} + \rho_i)},$$

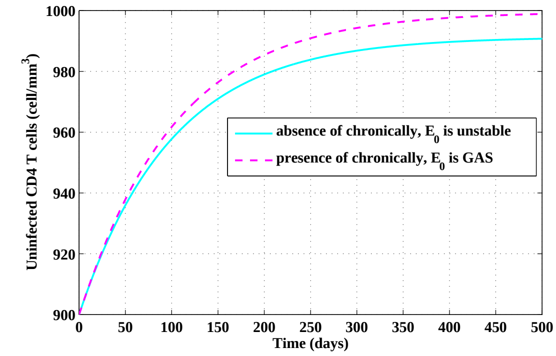
where $\widetilde{R}_0 = R_0 \mid_{q_1=q_2=0}$. Since $e^{-n_i\kappa_i} N_{y_i} > e^{-r_i\omega_i} M_{z_i}$, $i = 1, 2$, then we have

$$\begin{aligned} R_0 &= \sum_{i=1}^{2} \frac{((1-q_i)e^{-n_i\kappa_i} N_{y_i} + q_i e^{-r_i\omega_i} M_{z_i})}{u} \frac{e^{-m_i\tau_i} \beta_i (x_i^0)^{k_i}}{\varsigma_i((x_i^0)^{k_i} + \rho_i)} \\ &= \sum_{i=1}^{2} \frac{e^{-n_i\kappa_i} e^{-m_i\tau_i} N_{y_i} \beta_i (x_i^0)^{k_i}}{u\varsigma_i((x_i^0)^{k_i} + \rho_i)} - \sum_{i=1}^{2} \frac{(e^{-n_i\kappa_i} N_{y_i} - e^{-r_i\omega_i} M_{z_i}) q_i e^{-m_i\tau_i} \beta_i (x_i^0)^{k_i}}{u\varsigma_i((x_i^0)^{k_i} + \rho_i)} \\ &= \widetilde{R}_0 - \sum_{i=1}^{2} \frac{(e^{-n_i\kappa_i} N_{y_i} - e^{-r_i\omega_i} M_{z_i}) q_i e^{-m_i\tau_i} \beta_i (x_i^0)^{k_i}}{u\varsigma_i((x_i^0)^{k_i} + \rho_i)} < \widetilde{R}_0. \end{aligned}$$

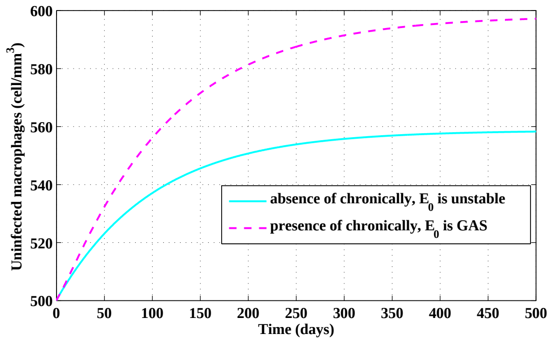
Therefore even without the incorporation of treatment, the long-lived infected cell population decreases the basic reproduction number of the system. Now, we calculate the critical drug efficacy needed in order stabilize the system around the infection-free equilibrium. The critical drug efficacy for systems (16) -(20) and (27)-(30) is given by ε_1^{crit} and ε_3^{crit}, respectively, where,

$$\varepsilon_3^{crit} = \max\left\{0, \frac{\widehat{R}_0 - 1}{\widehat{R}_0}\right\}$$

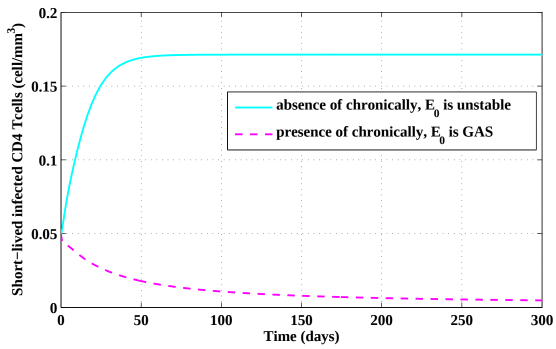
where $\widehat{R}_0 = \widetilde{R}_0 \mid_{\varepsilon=0} = R_0 \mid_{\varepsilon=q_1=q_2=0}$. Using the data given in Table 1 with $\tau_e = 0.01$, we have $\varepsilon_1^{crit} = 0.93$ and $\varepsilon_3^{crit} = 0.99$. Therefore the drug efficacy necessary to drive the system to the infection-free equilibrium is actually less for system (16) -(20) than that for system (27)-(30). Figure 6 shows the effect of chronically infected cells on dynamic and control of HIV infection. We observed that, if we choose $\varepsilon = 0.93$, then the trajectory of model (16)-(20) tends to infection-free equilibrium $E_0 = (1000, 600, 0, 0, 0, 0, 0, 0)$, while in the model (27)-(30), $\widetilde{R}_0 = 1.54 > 1$ and the trajectory tends to the endemic equilibrium with humoral immunity $E_1 = (990.99, 558.53, 0.17, 1.36, 1.83, 0.3)$.



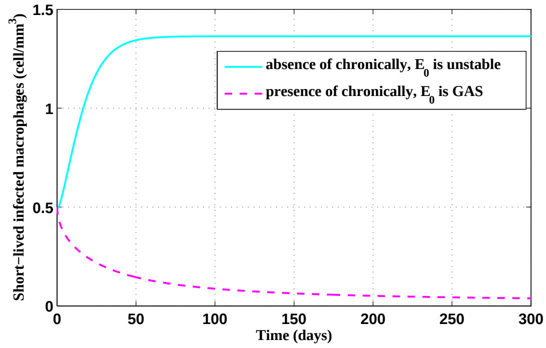
(a) Uninfected CD4$^+$T cells for model (16)-(20) and (27)-(30).



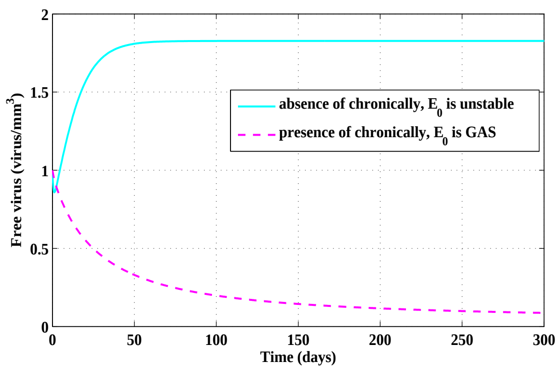
(b) Uninfected macrophages for model (16) -(20) and (27)-(30).



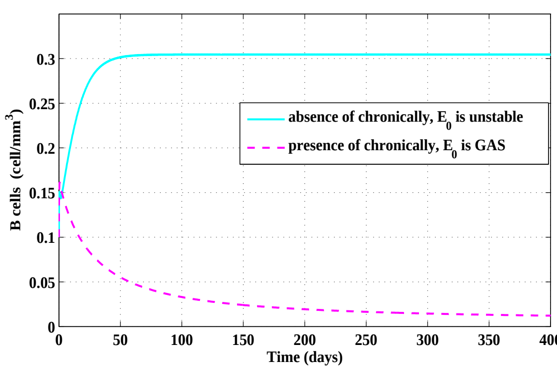
(c) Short-lived CD4$^+$T cells for model (16) -(20) and (27)-(30).



(d) Short-lived macrophages for model (16) -(20) and (27)-(30).



(e) Free virus for model (16) -(20) and (27)-(30).



(f) B cells for model (16) -(20) and (27)-(30).

Figure 6: Effect of long-lived chronically infected cells on the dynamics and controls of HIV infection

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COMPOSITION OPERATORS ON DIRICHLET-TYPE SPACES

LIU YANG, YECHENG SHI*

ABSTRACT. In this note, motivated by [8], under the conditions of weighted function in [10], we characterize bounded and compact composition operator on Dirichlet-type spaces D_K. We also give an equavalent charaterization of composition operator on D_K, if the composition operator on D_K spaces is Hilbert-Schmidt.

Keywords: D_K spaces; composition operators; Hilbert-Schmidt

1. INTRODUCTION

Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the class of functions analytic in $\mathbb{D}$. Let $K : [0,\infty) \to [0,\infty)$ be a right-continuous and nondecreasing function. The Dirichlet-type spaces D_K, consists of those functions $f \in H(\mathbb{D})$, such that

$$\|f\|^2_{D_K} = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 K(1-|z|^2)dA(z) < \infty.$$

When $K(t) = t^\alpha$, $0 < \alpha < 1$, it give the classical Dirichlet-type space D_α. For more informations on D_α and D_K spaces, we refer to [1], [3], [12], [19], [25].

Let φ be a holomorphic self-map of $\mathbb{D}$. The composition operator C_φ on D_K is defined by

$$C_\varphi(f) = f \circ \varphi, \quad f \in D_K.$$

There are many papers study composition operator, we refer to [4], [13], [14], [15], [17], [20], [21], [22], [24], [26]. Recently, Kellay and Lefèvre using Nevanlinna counting function, characterize bounded and compact composition operator on Dirichlet-type space D_K under certain conditions in [13]. Later, Pau and Pèrez studied the essential norm and closed ranged of composition operator on D_α in [17].

2000 *Mathematics Subject Classification.* 30D45, 30D50.

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*Corresponding author.

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In this paper, motivated by [8], we generalize Theorem 2.2 of [8] to D_K spaces. We also give a characterizations of boundedness and compactness of composition operator C_φ on D_K spaces by φ^n. Furthermore, equivalent characterizations of composition operator on D_K spaces belong to Hilbert-Schmidt was gave.

Throughout this paper, suppose that $K : [0,\infty) \to [0,\infty)$ is a right-continuous and nondecreasing function. Satisfying

$$\int_0^1 \frac{\varphi_K(s)}{s} ds < \infty \tag{1.1}$$

and

$$\int_1^\infty \frac{\varphi_K(s)}{s^2} ds < \infty, \tag{1.2}$$

where

$$\varphi_K(s) = \sup_{0\le t\le 1} K(st)/K(t), \quad 0 < s < \infty.$$

To learn more about weight function K, we refer to [2], [3], [9], [10] and [16].

Throughout this paper, for two functions f and g, $f \asymp g$ means that $g \lesssim f \lesssim g$, that is, there are positive constants C_1 and C_2 depend on K and index s, α, such that $C_1 g \le f \le C_2 g$.

2. Auxiliary results

Before to proof, we need to know some results. The following lemma can be found in Lemma 2.1 of [2].

Lemma 1. *Let (1.1) and (1.2) hold for K. If $2 - \frac{\alpha}{2} < s < 1 + c$, then*

$$\int_{\mathbb{D}} \frac{K\left(1-|\sigma_a(w)|^2\right)}{(1-|w|^2)^s |1-\overline{w}z|^\alpha} dA(w) \lesssim \frac{K\left(1-|\sigma_a(z)|^2\right)}{(1-|z|^2)^{s+\alpha-2}}$$

for all $a, z \in \mathbb{D}$, where $\sigma_a(z) = \frac{z-a}{1-\bar{a}z}$.

Lemma 2. *Suppose that K satisfies (1.1) and (1.2). Then*

$$1 + \sum_{n=1}^\infty \frac{n+1}{K(\frac{1}{n+1})} t^n \asymp \frac{1}{(1-t)^2 K(1-t)}$$

for all $0 \le t < 1$.

Proof. Without loss of generality, we can assume $1/3 < t < 1$, otherwise, it obvious. Make change of variables $y = \frac{1}{x}$, an easy computation gives

$$\sum_{n=1}^{\infty} \frac{n+1}{K(\frac{1}{n+1})} t^n \asymp \sum_{n=1}^{\infty} \int_{\frac{1}{n+1}}^{\frac{1}{n}} \frac{t^{\frac{1}{x}}}{x^3 K(x)} dx$$
$$\asymp \int_0^1 \frac{t^{\frac{1}{x}}}{x^3 K(x)} dx \asymp \int_1^{\infty} \frac{y t^y}{K(\frac{1}{y})} dy.$$

Let $y = \frac{\gamma}{-\ln t}$. We can deduce that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{n+1}{K(\frac{1}{n+1})} t^n &\asymp \frac{1}{(\ln \frac{1}{t})^2} \int_{-\ln t}^{\infty} \frac{\gamma e^{-\gamma}}{K(\frac{1}{\gamma} \ln \frac{1}{t})} d\gamma \\ &= \frac{1}{(\ln \frac{1}{t})^2 K(\ln \frac{1}{t})} \int_{-\ln t}^{\infty} \frac{\gamma e^{-\gamma} K(\ln \frac{1}{t})}{K(\frac{1}{\gamma} \ln \frac{1}{t})} d\gamma \\ &\lesssim \frac{1}{(1-t)^2 K(1-t)} \int_{-\ln t}^{\infty} \gamma e^{-\gamma} \varphi_K(\gamma) d\gamma. \end{aligned}$$

By [10], under conditions (1.1) and (1.2), there exists an enough small $c > 0$ only depending on K such that

$$\varphi_K(s) \lesssim s^c, \ 0 < s \leq 1$$

and

$$\varphi_K(s) \lesssim s^{1-c}, \ s \geq 1.$$

Therefore,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{n+1}{K(\frac{1}{n+1})} t^n &\lesssim \frac{1}{(1-t)^2 K(1-t)} \int_{-\ln t}^{\infty} \gamma e^{-\gamma} \varphi_K(\gamma) d\gamma \\ &\lesssim \frac{1}{(1-t)^2 K(1-t)} \left(\int_0^{\infty} e^{-\gamma} \gamma^{2-c} d\gamma + \int_0^{\infty} e^{-\gamma} \gamma^{1+c} d\gamma \right) \\ &\asymp \frac{1}{(1-t)^2 K(1-t)} \left(\Gamma(3-c) + \Gamma(2+c) \right), \end{aligned}$$

where $\Gamma(.)$ is the Gamma function. It follows that

$$1 + \sum_{n=1}^{\infty} \frac{n+1}{K(\frac{1}{n+1})} t^n \lesssim \frac{1}{(1-t)^2 K(1-t)}.$$

Conversely, since K is nondecreasing, we deduce that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{n+1}{K(\frac{1}{n+1})}t^n &\asymp \frac{1}{(\ln\frac{1}{t})^2K(\ln\frac{1}{t})}\int_{-\ln t}^{\infty}\frac{\gamma e^{-\gamma}K(\ln\frac{1}{t})}{K(\frac{1}{\gamma}\ln\frac{1}{t})}d\gamma\\ &\gtrsim \frac{1}{(1-t)^2K(1-t)}\int_{\ln 2}^{\infty}\frac{\gamma e^{-\gamma}K(\ln\frac{1}{t})}{K(\frac{1}{\gamma}\ln\frac{1}{t})}d\gamma\\ &\gtrsim \frac{1}{(1-t)^2K(1-t)}\int_{\ln 2}^{\infty}\gamma e^{-\gamma}d\gamma\\ &\asymp \frac{1}{(1-t)^2K(1-t)}.\end{aligned}$$

The proof is completed. □

The next lemma can be found in Theorem 5 of [23].

Lemma 3. *Let (1.2) hold for K. Then for any $\alpha > 0$ and $0 \le \beta < 1$, we have*

$$\int_0^1 r^{\alpha-1}(\log\frac{1}{r})^{-\beta}K(\log\frac{1}{r})dr \asymp \left(\frac{1-\beta}{\alpha}\right)^{1-\beta}K\left(\frac{1-\beta}{\alpha}\right).$$

3. Boundedness and compactness

In this section, motivated by [8], we discuss the boundedness and compactness of compostion operators by a general computation.

Theorem 1. *Suppose that (1.1) and (1.2) hold for K, $s \ge 0$. Suppose $\varphi(\mathbb{D}) \subset \mathbb{D}$ and $\varphi \in D_K$. Then C_φ is bounded on D_K if and only if*

$$\sup_{a\in\mathbb{D}}\frac{(1-|a|^2)^{2+2s}}{K(1-|a|^2)}\int_{\mathbb{D}}\frac{|\varphi'(z)|^2}{|1-\overline{a}\varphi(z)|^{4+2s}}K(1-|z|^2)dA(z) < \infty.$$

Proof. Let

$$F_a(z) = \frac{(1-|a|^2)^{1+s}}{\sqrt{K(1-|a|^2)}}\frac{1}{(1-\overline{a}z)^{1+s}},\quad s \ge 0.$$

Using Lemma 1, it is easy to check that $F_a \in D_K$. If C_φ is bounded on D_K, then $\|C_\varphi(F_a)\|_{D_K} < \infty$, that is,

$$\sup_{a\in\mathbb{D}}\frac{(1-|a|^2)^{2+2s}}{K(1-|a|^2)}\int_{\mathbb{D}}\frac{|\varphi'(z)|^2}{|1-\overline{a}\varphi(z)|^{4+2s}}K(1-|z|^2)dA(z) < \infty.$$

On the other hand, we know that for any pseudohyperbolic discs $D(z,r)$, we have $1-|w| \asymp 1-|z| \asymp |1-\overline{w}z|$, for any $w \in D(z,r)$ (see [27, page 69]). Let $f \in D_K$. Applying sub-mean-property to $|f'|^2$, we have

$$\begin{aligned} |f'(z)|^2 &\leq \int_{D(z,r)} \frac{|f'(w)|^2}{|1-\overline{w}z|^2} dA(w) \\ &\asymp \int_{D(z,r)} \frac{|f'(w)|^2(1-|w|^2)^{2+2s}}{|1-\overline{w}z|^{4+2s}} dA(w) \\ &\lesssim \int_{\mathbb{D}} \frac{|f'(w)|^2(1-|w|^2)^{2+2s}}{|1-\overline{w}z|^{4+2s}} dA(w). \end{aligned}$$

Therefore, we get

$$\int_{\mathbb{D}} |f'(\varphi(z))|^2 |\varphi'(z)|^2 K(1-|z|^2) dA(z)$$

$$\begin{aligned} &\leq \int_{\mathbb{D}} \left(\int_{\mathbb{D}} \frac{|f'(w)|^2}{|1-\overline{w}\varphi(z)|^{4+2s}} (1-|w|^2)^{2+2s} dA(w) \right) |\varphi'(z)|^2 K(1-|z|^2) dA(z) \\ &\leq \left(\sup_{w \in \mathbb{D}} \frac{(1-|w|^2)^{2+2s}}{K(1-|w|^2)} \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{|1-\overline{w}\varphi(z)|^{4+2s}} K(1-|z|^2) dA(z) \right) \\ &\qquad \times \int_{\mathbb{D}} |f'(w)|^2 K(1-|w|^2) dA(w) < \infty. \end{aligned}$$

The proof is completed. □

Theorem 2. *Suppose that (1.1) and (1.2) hold for K, $s \geq 0$. Suppose $\varphi(\mathbb{D}) \subset \mathbb{D}$ and $\varphi \in D_K$. Then C_φ is compact on D_K if and only if*

$$\lim_{|a| \to 1} \frac{(1-|a|^2)^{2+2s}}{K(1-|a|^2)} \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{|1-\overline{a}\varphi(z)|^{4+2s}} K(1-|z|^2) dA(z) = 0.$$

Proof. Let

$$G(w) = \frac{(1-|w|^2)^{2+2s}}{K(1-|w|^2)} \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{|1-\overline{w}\varphi(z)|^{4+2s}} K(1-|z|^2) dA(z).$$

Let $\{f_k\}_{k=1}^{\infty}$ be a bounded sequence of D_K such that $f_k \to 0$ weakly. Therefore, $f_k' \to 0$ uniformly on compact sets. From the proof of Theorem 1 and

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dominated convergence theorem, when $k \to \infty$, and $r \to 1$, it follows that

$$\begin{aligned}
&\|C_\varphi(f_k)\|^2_{D_K} - |f_k(\varphi(0))|^2 \\
\le &\int_{\mathbb{D}} |f'_k(w)|^2 G(w) K(1-|w|^2) dA(w) \\
\le &\int_{r\mathbb{D}} |f'_k(w)|^2 G(w) K(1-|w|^2) dA(w) \\
&\qquad + \int_{\mathbb{D}\setminus r\mathbb{D}} |f'_k(w)|^2 G(w) K(1-|w|^2) dA(w) \to 0.
\end{aligned}$$

Thus, C_φ is compact.

Conversely, if C_φ is compact, let $\{a_k\}_{k=1}^\infty \subseteq \mathbb{D}$, $|a_k| \to 1$,

$$F_{a_k}(z) = \frac{(1-|a_k|^2)^{1+s}}{\sqrt{K(1-|a_k|^2)}} \frac{1}{(1-\overline{a_k}z)^{1+s}}.$$

Then, it is easy to verify that $F_{a_k} \to 0$ uniformly on compact sets. Thus, $\|C_\varphi(F_{a_k})\|_{D_K} \to 0$ as $k \to \infty$. The proof is completed. □

4. φ^n-TYPE CHARACTERIZATIONS

In [24], Wulan, Zheng and Zhu gave an interesting characterizations of compostion operators C_φ by φ^n. In this section, we are going to give an analogy results on D_K spaces.

Theorem 3. *Let (1.1) and (1.2) hold for K. Suppose $\varphi \in D_K$ satisfies $\varphi(\mathbb{D}) \subset \mathbb{D}$ and $C_\varphi : D_K \to D_K$. Then*

(1) *If*

$$\sup_n \frac{1}{K(\frac{1}{n})} \|\varphi^n\|^2_{D_K} < \infty,$$

then C_φ is bounded;

(2) *If C_φ is bounded, then*

$$\sup_n \frac{1}{nK(\frac{1}{n})} \|\varphi^n\|^2_{D_K} < \infty.$$

Proof. (1). Let $a, z \in \mathbb{D}$ and $s > 0$. Since

$$|1 - \bar{a}\varphi(z)| \ge 1 - |a||\varphi(z)|$$

and

$$\frac{1}{(|1-|a||\varphi(z)|)^{4+2s}} \asymp \frac{1}{(|1-|a|^2|\varphi(z)|^2)^{4+2s}}.$$

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Note that

$$\frac{1}{(|1-|a|^2|\varphi(z)|^2)^{4+2s}} \asymp \sum_{n=0}^{\infty} \frac{\Gamma(n+4+2s)}{n!\Gamma(4+2s)} |a|^{2n}|\varphi(z)|^{2n},$$

it follows that

$$\begin{aligned}
&\frac{(1-|a|^2)^{2+2s}}{K(1-|a|^2)} \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{|1-\overline{a}\varphi(z)|^{4+2s}} K(1-|z|^2)dA(z) \\
\lesssim & \frac{(1-|a|^2)^{2+2s}}{K(1-|a|^2)} \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1-|a||\varphi(z)|)^{4+2s}} K(1-|z|^2)dA(z) \\
\asymp & \frac{(1-|a|^2)^{2+2s}}{K(1-|a|^2)} \int_{\mathbb{D}} \sum_{n=0}^{\infty} \frac{\Gamma(n+4+2s)}{n!\Gamma(4+2s)} |a|^{2n}|\varphi(z)|^{2n}|\varphi'(z)|^2 K(1-|z|^2)dA(z).
\end{aligned}$$

By Stirling formula, we get

$$\frac{\Gamma(n+4+2s)}{n!\Gamma(4+2s)} \sim n^{3+2s},\ n\to\infty.$$

Therefore,

$$\begin{aligned}
&\frac{(1-|a|^2)^{2+2s}}{K(1-|a|^2)} \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{|1-\overline{a}\varphi(z)|^{4+2s}} K(1-|z|^2)dA(z) \\
\lesssim & \frac{(1-|a|^2)^{2+2s}}{K(1-|a|^2)} \sum_{n=0}^{\infty} n^{3+2s}|a|^{2n} \int_{\mathbb{D}} |\varphi(z)|^{2n}|\varphi'(z)|^2 K(1-|z|^2)dA(z) \\
\leq & \frac{(1-|a|^2)^{2+2s}}{K(1-|a|^2)} \sum_{n=0}^{\infty} (n+1)^{3+2s}|a|^{2n} \int_{\mathbb{D}} |\varphi(z)|^{2n}|\varphi'(z)|^2 K(1-|z|^2)dA(z) \\
\leq & \frac{(1-|a|^2)^{2+2s}}{K(1-|a|^2)} \sum_{n=0}^{\infty} (n+1)^{1+2s}|a|^{2n}\|\varphi^n\|_{D_K}^2 \\
\lesssim & \sup_n \frac{1}{K(\frac{1}{n})}\|\varphi^n\|_{D_K}^2 \frac{(1-|a|^2)^{2+2s}}{K(1-|a|^2)} \sum_{n=0}^{\infty} (n+1)^{1+2s} K(\frac{1}{n})|a|^{2n}.
\end{aligned}$$

Following the proof of Lemma 2, we have

$$\sum_{n=0}^{\infty} (n+1)^{1+2s} K(\frac{1}{n})|a|^{2n} \asymp \frac{K(1-|a|^2)}{(1-|a|^2)^{2+2s}}.$$

Thus,

$$\begin{aligned}
&\frac{(1-|a|^2)^{2+2s}}{K(1-|a|^2)} \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{|1-\overline{a}\varphi(z)|^{4+2s}} K(1-|z|^2)dA(z) \\
\lesssim & \sup_n \frac{1}{K(\frac{1}{n})}\|\varphi^n\|_{D_K}^2.
\end{aligned}$$

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Hence, by Theorem 1, we prove (1).

(2). Suppose that C_φ is bounded on D_K. Let $f_n(z) = z^n/\|z^n\|^2_{D_K}$. Then, we have $\|f_n\|^2_{D_K} = 1$. An easy computation gives,

$$\infty > \|C_\varphi f_n\|^2_{D_K} = \frac{\|\varphi^n\|^2_{D_K}}{\|z^n\|^2_{D_K}} \gtrsim \frac{1}{nK(\frac{1}{n})}\|\varphi^n\|^2_{D_K}.$$

The last inequality is deduced by Lemma 3. The proof is completed. □

Theorem 4. *Let (1.1) and (1.2) hold for K. Suppose $\varphi \in D_K$ satisfies $\varphi(\mathbb{D}) \subset \mathbb{D}$ and $C_\varphi : D_K \to D_K$. Then*

(1) *If*

$$\lim_{n\to\infty} \frac{1}{K(\frac{1}{n})}\|\varphi^n\|^2_{D_K} = 0,$$

then C_φ is compact;

(2) *If C_φ is compact, then*

$$\lim_{n\to\infty} \frac{1}{nK(\frac{1}{n})}\|\varphi^n\|^2_{D_K} = 0.$$

Proof. (1). The proof is similar to (1) of Theorem 3.

(2). Let $\{f_n\}$ be a bounded sequence in D_K that convergence to 0 weakly. If C_φ is compact on D_K, then $\|C_\varphi f_n\|_{D_K} \to 0$, as $n \to \infty$. Thus, for any $z \in \mathbb{D}$, we have

$$f_n(\varphi(z)) \to 0, \quad n \to \infty.$$

Since $\{z^n/\|z^n\|_{D_K}, n \geq 1\}$ is bounded in D_K and it converges to 0 pointwise, the compactness of C_φ on D_K implies that

$$\lim_{n\to\infty} \frac{\|\varphi^n\|^2_{D_K}}{\|z^n\|^2_{D_K}} = \frac{1}{nK(\frac{1}{n})}\|\varphi^n\|^2_{D_K} = 0.$$

The proof is completed. □

5. Hilbert-Schmidt class

Let Hilbert-Schmidt class be the space of all compact operators on Hilbert space with its singular value sequence $\{\lambda_n\} \in l^2$, the 2-summable sequence space (see [27, page 18]). The following theorem give an equavalent charaterizations of composition operator on D_K spaces, when it belong to Hilbert-Schmidt class.

Theorem 5. *Let (1.1) and (1.2) hold for K. Suppose $\varphi(\mathbb{D}) \subset \mathbb{D}$, $\varphi \in D_K$ and C_φ is compact. Then C_φ is Hilbert-Schmidt on D_K if and only if*

$$\int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2} \frac{K(1-|z|^2)}{K(1-|\varphi(z)|^2)} dA(z) < \infty.$$

Proof. Without loss of generality, we can assume $\{1\} \cup \{\frac{z^n}{\sqrt{n}\sqrt{K(\frac{1}{n})}}\}_{n=1}^\infty$ is an orthonormal basis in D_K and $\varphi(0) = 0$. From Theorem 1.22 of [27], C_φ is Hilbert-Schmidt on D_K if and only if

$$\sum_{n=1}^\infty \frac{D_K(\varphi^n)}{nK(\frac{1}{n})} < \infty.$$

Applying Lemma 2, we have

$$\begin{aligned}\sum_{n=1}^\infty \frac{D_K(\varphi^n)}{nK(\frac{1}{n})} &= \sum_{n=1}^\infty \frac{n}{K(\frac{1}{n})} \int_{\mathbb{D}} |\varphi^2(z)|^{n-1}|\varphi'(z)|^2 K(1-|z|^2) dA(z) \\ &= \sum_{n=0}^\infty \frac{n+1}{K(\frac{1}{n+1})} \int_{\mathbb{D}} |\varphi^2(z)|^{n}|\varphi'(z)|^2 K(1-|z|^2) dA(z) \\ &\asymp \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2} \frac{K(1-|z|^2)}{K(1-|\varphi(z)|^2)} dA(z).\end{aligned}$$

The proof is completed. □

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DEPARTMENT OF MATHEMATICS, SHAANXI XUEQIAN NORMAL UNIVERSITY, SHAANXI XI'AN 710100, P. R. CHINA
E-mail address: 381900567@qq.com (L. Yang)

DEPARTMENT OF MATHEMATICS AND APPLICATION, SHANWEI VOCATIONAL AND TECHNICAL COLLEGE, GUANGDONG SHANWEI 516600, P. R. CHINA
E-mail address: 09ycshi@sina.cn (Y. Shi)

On left multidimensional Riemann-Liouville fractional integral

George Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

Abstract

Here we study some important properties of left multidimensional Riemann-Liouville fractional integral operator, such as of continuity and boundedness.

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1 Motivation

From [1], p. 388 we have

Theorem 1 *Let* $r > 0$, $F \in L_{\infty}(a, b)$, *and*

$$G(s) = \int_a^s (s-t)^{r-1} F(t)\, dt,$$

all $s \in [a, b]$. *Then* $G \in AC([a, b])$ *(absolutely continuous functions) for* $r \geq 1$, *and* $G \in C([a, b])$, *only for* $r \in (0, 1)$.

2 Main Results

We give

Theorem 2 *Let* $f \in L_{\infty}([a, b] \times [c, d])$, $\alpha_1, \alpha_2 > 0$. *Consider the function*

$$F(x_1, x_2) = \int_{a_1}^{x_1} \int_{a_2}^{x_2} (x_1 - t_1)^{\alpha_1 - 1} (x_2 - t_2)^{\alpha_2 - 1} f(t_1, t_2)\, dt_1 dt_2, \quad (1)$$

where $a_1, x_1 \in [a,b]$, $a_2, x_2 \in [c,d] : a_1 \leq x_1,\ a_2 \leq x_2$.

Then F *is continuous on* $[a_1, b] \times [a_2, d]$.

Proof. (I) Let $a_1, b_1, b_1^* \in [a,b]$ with $b_1 > b_1^* > a_1$, and $a_2, b_2, b_2^* \in [c,d]$ with $b_2 > b_2^* > a_2$.

We observe that

$$F(b_1, b_2) - F(b_1^*, b_2^*) =$$
$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} (b_1 - t_1)^{\alpha_1 - 1} (b_2 - t_2)^{\alpha_2 - 1} f(t_1, t_2)\, dt_1 dt_2 -$$
$$\int_{a_1}^{b_1^*} \int_{a_2}^{b_2^*} (b_1^* - t_1)^{\alpha_1 - 1} (b_2^* - t_2)^{\alpha_2 - 1} f(t_1, t_2)\, dt_1 dt_2 = \tag{2}$$
$$\int_{a_1}^{b_1^*} \int_{a_2}^{b_2^*} (b_1 - t_1)^{\alpha_1 - 1} (b_2 - t_2)^{\alpha_2 - 1} f(t_1, t_2)\, dt_1 dt_2 -$$
$$\int_{a_1}^{b_1^*} \int_{a_2}^{b_2^*} (b_1^* - t_1)^{\alpha_1 - 1} (b_2^* - t_2)^{\alpha_2 - 1} f(t_1, t_2)\, dt_1 dt_2 +$$
$$\int_{b_1^*}^{b_1} \int_{a_2}^{b_2^*} (b_1 - t_1)^{\alpha_1 - 1} (b_2 - t_2)^{\alpha_2 - 1} f(t_1, t_2)\, dt_1 dt_2 +$$
$$\int_{a_1}^{b_1^*} \int_{b_2^*}^{b_2} (b_1 - t_1)^{\alpha_1 - 1} (b_2 - t_2)^{\alpha_2 - 1} f(t_1, t_2)\, dt_1 dt_2 +$$
$$\int_{b_1^*}^{b_1} \int_{b_2^*}^{b_2} (b_1 - t_1)^{\alpha_1 - 1} (b_2 - t_2)^{\alpha_2 - 1} f(t_1, t_2)\, dt_1 dt_2.$$

Call

$$I(b_1^*, b_2^*) = \int_{a_1}^{b_1^*} \int_{a_2}^{b_2^*} \left| (b_1 - t_1)^{\alpha_1 - 1} (b_2 - t_2)^{\alpha_2 - 1} - (b_1^* - t_1)^{\alpha_1 - 1} (b_2^* - t_2)^{\alpha_2 - 1} \right| dt_1 dt_2. \tag{3}$$

Thus

$$|F(b_1, b_2) - F(b_1^*, b_2^*)| \leq$$
$$\left\{ I(b_1^*, b_2^*) + \frac{(b_1 - b_1^*)^{\alpha_1}}{\alpha_1} \left[\frac{(b_2 - a_2)^{\alpha_2} - (b_2 - b_2^*)^{\alpha_2}}{\alpha_2} \right] + \right.$$
$$\left. \left(\frac{(b_1 - a_1)^{\alpha_1} - (b_1 - b_1^*)^{\alpha_1}}{\alpha_1} \right) \frac{(b_2 - b_2^*)^{\alpha_2}}{\alpha_2} + \frac{(b_1 - b_1^*)^{\alpha_1}}{\alpha_1} \frac{(b_2 - b_2^*)^{\alpha_2}}{\alpha_2} \right\} \|f\|_\infty . \tag{4}$$

Hence, by (4), it holds

$$\delta := \lim_{\substack{(b_1^*, b_2^*) \to (b_1, b_2) \\ \text{or} \\ (b_1, b_2) \to (b_1^*, b_2^*)}} |F(b_1, b_2) - F(b_1^*, b_2^*)| \leq \Big(\lim_{\substack{(b_1^*, b_2^*) \to (b_1, b_2) \\ \text{or} \\ (b_1, b_2) \to (b_1^*, b_2^*)}} I(b_1^*, b_2^*) \Big) \|f\|_\infty =: \rho. \tag{5}$$

If $\alpha_1 = \alpha_2 = 1$, then $\rho = 0$, proving $\delta = 0$.

If $\alpha_1 = 1$, $\alpha_2 > 0$ we get

$$I\left(b_1^*, b_2^*\right) = \left(b_1^* - a_1\right)\left(\int_{a_2}^{b_2^*}\left|\left(b_2 - t_2\right)^{\alpha_2 - 1} - \left(b_2^* - t_2\right)^{\alpha_2 - 1}\right| dt_2\right). \tag{6}$$

Assume $\alpha_2 > 1$, then $\alpha_2 - 1 > 0$. Hence by $b_2 > b_2^*$, then $b_2 - t_2 > b_2^* - t_2 \geq 0$, and $\left(b_2 - t_2\right)^{\alpha_2 - 1} > \left(b_2^* - t_2\right)^{\alpha_2 - 1}$ and $\left(b_2 - t_2\right)^{\alpha_2 - 1} - \left(b_2^* - t_2\right)^{\alpha_2 - 1} > 0$.

That is

$$I\left(b_1^*, b_2^*\right) = \left(b_1^* - a_1\right)\left[\frac{\left(b_2 - t_2\right)^{\alpha_2}}{\alpha_2}\bigg|_{b_2^*}^{a_2} - \frac{\left(b_2^* - a_2\right)^{\alpha_2}}{\alpha_2}\right]$$

$$= \left(b_1^* - a_1\right)\left[\frac{\left(b_2 - a_2\right)^{\alpha_2} - \left(b_2 - b_2^*\right)^{\alpha_2} - \left(b_2^* - a_2\right)^{\alpha_2}}{\alpha_2}\right]. \tag{7}$$

Clearly, then

$$\lim_{\substack{b_2^* \to b_2 \\ \text{or} \\ b_2 \to b_2^*}} I\left(b_1^*, b_2^*\right) = 0. \tag{8}$$

Similarly and symmetrically, we obtain that

$$\lim_{\substack{b_1^* \to b_1 \\ \text{or} \\ b_1 \to b_1^*}} I\left(b_1^*, b_2^*\right) = 0, \tag{9}$$

for the case of $\alpha_2 = 1$, $\alpha_1 > 1$.

If $\alpha_1 = 1$, and $0 < \alpha_2 < 1$, then $\alpha_2 - 1 < 0$. Hence

$$I\left(b_1^*, b_2^*\right) = \left(b_1^* - a_1\right)\left(\int_{a_2}^{b_2^*}\left(\left(b_2^* - t_2\right)^{\alpha_2 - 1} - \left(b_2 - t_2\right)^{\alpha_2 - 1}\right) dt_2\right) =$$

$$\left(b_1^* - a_1\right)\left[\frac{\left(b_2^* - a_2\right)^{\alpha_2} - \left(b_2 - a_2\right)^{\alpha_2} + \left(b_2 - b_2^*\right)^{\alpha_2}}{\alpha_2}\right]. \tag{10}$$

Clearly, then

$$\lim_{\substack{b_2^* \to b_2 \\ \text{or} \\ b_2 \to b_2^*}} I\left(b_1^*, b_2^*\right) = 0. \tag{11}$$

Similarly and symmetrically, we derive that

$$\lim_{\substack{b_1^* \to b_1 \\ \text{or} \\ b_1 \to b_1^*}} I\left(b_1^*, b_2^*\right) = 0, \tag{12}$$

for the case of $\alpha_2 = 1$, $0 < \alpha_1 < 1$.

Case now of $\alpha_1, \alpha_2 > 1$, then

$$I\left(b_1^*, b_2^*\right) = \int_{a_1}^{b_1^*}\int_{a_2}^{b_2^*}\left[\left(b_1 - t_1\right)^{\alpha_1 - 1}\left(b_2 - t_2\right)^{\alpha_2 - 1} - \left(b_1^* - t_1\right)^{\alpha_1 - 1}\left(b_2^* - t_2\right)^{\alpha_2 - 1}\right] dt_1 dt_2 = \tag{13}$$

$$\left(\frac{\left(b_1 - a_1\right)^{\alpha_1} - \left(b_1 - b_1^*\right)^{\alpha_1}}{\alpha_1}\right)\left(\frac{\left(b_2 - a_2\right)^{\alpha_2} - \left(b_2 - b_2^*\right)^{\alpha_2}}{\alpha_2}\right) - \frac{\left(b_1^* - a_1\right)^{\alpha_1}}{\alpha_1}\frac{\left(b_2^* - a_2\right)^{\alpha_2}}{\alpha_2}.$$

That is

$$\lim_{\substack{(b_1, b_2) \to \left(b_1^*, b_2^*\right) \\ \text{or} \\ \left(b_1^*, b_2^*\right) \to (b_1, b_2)}} I\left(b_1^*, b_2^*\right) = 0. \tag{14}$$

Case now of $0 < \alpha_1, \alpha_2 < 1$, then

$$I\left(b_1^*, b_2^*\right) = \int_{a_1}^{b_1^*}\int_{a_2}^{b_2^*}\left[\left(b_1^* - t_1\right)^{\alpha_1 - 1}\left(b_2^* - t_2\right)^{\alpha_2 - 1} - \left(b_1 - t_1\right)^{\alpha_1 - 1}\left(b_2 - t_2\right)^{\alpha_2 - 1}\right] dt_1 dt_2 =$$

$$\frac{\left(b_1^* - a_1\right)^{\alpha_1}}{\alpha_1}\frac{\left(b_2^* - a_2\right)^{\alpha_2}}{\alpha_2} - \left(\frac{\left(b_1 - a_1\right)^{\alpha_1} - \left(b_1 - b_1^*\right)^{\alpha_1}}{\alpha_1}\right)\left(\frac{\left(b_2 - a_2\right)^{\alpha_2} - \left(b_2 - b_2^*\right)^{\alpha_2}}{\alpha_2}\right). \tag{15}$$

That is again, when $0 < \alpha_1, \alpha_2 < 1$,

$$\lim_{\substack{(b_1, b_2) \to \left(b_1^*, b_2^*\right) \\ \text{or} \\ \left(b_1^*, b_2^*\right) \to (b_1, b_2)}} I\left(b_1^*, b_2^*\right) = 0. \tag{16}$$

Next we treat the case of $\alpha_1 > 1$, $0 < \alpha_2 < 1$.

We observe that

$$I\left(b_1^*, b_2^*\right) \leq I^*\left(b_1^*, b_2^*\right) := \int_{a_1}^{b_1^*}\int_{a_2}^{b_2^*}\left(b_1 - t_1\right)^{\alpha_1 - 1}\left|\left(b_2 - t_2\right)^{\alpha_2 - 1} - \left(b_2^* - t_2\right)^{\alpha_2 - 1}\right| dt_1 dt_2$$

$$+ \int_{a_1}^{b_1^*}\int_{a_2}^{b_2^*}\left(b_2^* - t_2\right)^{\alpha_2 - 1}\left|\left(b_1 - t_1\right)^{\alpha_1 - 1} - \left(b_1^* - t_1\right)^{\alpha_1 - 1}\right| dt_1 dt_2. \tag{17}$$

Therefore it holds

$$I^*\left(b_1^*, b_2^*\right) = \int_{a_1}^{b_1^*}\int_{a_2}^{b_2^*}\left(b_1 - t_1\right)^{\alpha_1 - 1}\left(\left(b_2^* - t_2\right)^{\alpha_2 - 1} - \left(b_2 - t_2\right)^{\alpha_2 - 1}\right) dt_1 dt_2 \tag{18}$$

$$+ \int_{a_1}^{b_1^*}\int_{a_2}^{b_2^*}\left(b_2^* - t_2\right)^{\alpha_2 - 1}\left(\left(b_1 - t_1\right)^{\alpha_1 - 1} - \left(b_1^* - t_1\right)^{\alpha_1 - 1}\right) dt_1 dt_2 =$$

$$\left(\frac{\left(b_1 - a_1\right)^{\alpha_1} - \left(b_1 - b_1^*\right)^{\alpha_1}}{\alpha_1}\right)\left[\frac{\left(b_2^* - a_2\right)^{\alpha_2} - \left(b_2 - a_2\right)^{\alpha_2} + \left(b_2 - b_2^*\right)^{\alpha_2}}{\alpha_2}\right] +$$

$$\frac{\left(b_2^* - a_2\right)^{\alpha_2}}{\alpha_2}\left[\frac{\left(b_1 - a_1\right)^{\alpha_1} - \left(b_1 - b_1^*\right)^{\alpha_1} - \left(b_1^* - a_1\right)^{\alpha_1}}{\alpha_1}\right]. \tag{19}$$

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So, in case of $\alpha_1 > 1$, $0 < \alpha_2 < 1$, we proved that

$$\lim_{\substack{(b_1,b_2)\to(b_1^*,b_2^*)\\ \text{or}\\ (b_1^*,b_2^*)\to(b_1,b_2)}} I\left(b_1^*, b_2^*\right) = 0. \tag{20}$$

Finally, we prove the case of $\alpha_2 > 1$ and $0 < \alpha_1 < 1$. We have that

$$I^*\left(b_1^*, b_2^*\right) \overset{(17)}{=} \int_{a_1}^{b_1^*} \int_{a_2}^{b_2^*} \left(b_1 - t_1\right)^{\alpha_1 - 1} \left[\left(b_2 - t_2\right)^{\alpha_2 - 1} - \left(b_2^* - t_2\right)^{\alpha_2 - 1}\right] dt_1 dt_2$$

$$+ \int_{a_1}^{b_1^*} \int_{a_2}^{b_2^*} \left(b_2^* - t_2\right)^{\alpha_2 - 1} \left(\left(b_1^* - t_1\right)^{\alpha_1 - 1} - \left(b_1 - t_1\right)^{\alpha_1 - 1}\right) dt_1 dt_2 = \tag{21}$$

$$\left(\frac{\left(b_1 - a_1\right)^{\alpha_1} - \left(b_1 - b_1^*\right)^{\alpha_1}}{\alpha_1}\right) \left[\frac{\left(b_2 - a_2\right)^{\alpha_2} - \left(b_2 - b_2^*\right)^{\alpha_2} - \left(b_2^* - a_2\right)^{\alpha_2}}{\alpha_2}\right] +$$

$$\frac{\left(b_2^* - a_2\right)^{\alpha_2}}{\alpha_2} \left[\frac{\left(b_1^* - a_1\right)^{\alpha_1} - \left(b_1 - a_1\right)^{\alpha_1} + \left(b_1 - b_1^*\right)^{\alpha_1}}{\alpha_1}\right]. \tag{22}$$

Hence again it holds

$$\lim_{\substack{(b_1,b_2)\to(b_1^*,b_2^*)\\ \text{or}\\ (b_1^*,b_2^*)\to(b_1,b_2)}} I\left(b_1^*, b_2^*\right) = 0. \tag{23}$$

We proved $\rho = 0$, and $\delta = 0$ in all cases of this section.

The case of $b_1^* > b_1$ and $b_2^* > b_2$, as symmetric to $b_1 > b_1^*$ and $b_2 > b_2^*$ we treated, it is omitted, a totally similar treatment.

(II) The remaining cases are: let $a_1, b_1, b_1^* \in [a, b]$; $a_2, b_2, b_2^* \in [c, d]$, we can have

(II_1) $b_1 > b_1^*$ and $b_2 < b_2^*$,

or

(II_2) $b_1 < b_1^*$ and $b_2 > b_2^*$.

Notice that (II_1) and (II_2) cases are symmetric, and treated the same way. As such we treat only the case (II_1).

We observe again that

$$F\left(b_1, b_2\right) - F\left(b_1^*, b_2^*\right) =$$

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \left(b_1 - t_1\right)^{\alpha_1 - 1} \left(b_2 - t_2\right)^{\alpha_2 - 1} f\left(t_1, t_2\right) dt_1 dt_2 -$$

$$\int_{a_1}^{b_1^*} \int_{a_2}^{b_2^*} \left(b_1^* - t_1\right)^{\alpha_1 - 1} \left(b_2^* - t_2\right)^{\alpha_2 - 1} f\left(t_1, t_2\right) dt_1 dt_2 = \tag{24}$$

$$\int_{a_1}^{b_1^*} \int_{a_2}^{b_2} \left(b_1 - t_1\right)^{\alpha_1 - 1} \left(b_2 - t_2\right)^{\alpha_2 - 1} f\left(t_1, t_2\right) dt_1 dt_2 +$$

$$\int_{b_1^*}^{b_1}\int_{a_2}^{b_2}(b_1-t_1)^{\alpha_1-1}(b_2-t_2)^{\alpha_2-1}f(t_1,t_2)\,dt_1dt_2-$$
$$\int_{a_1}^{b_1^*}\int_{a_2}^{b_2}(b_1^*-t_1)^{\alpha_1-1}(b_2^*-t_2)^{\alpha_2-1}f(t_1,t_2)\,dt_1dt_2-$$
$$\int_{a_1}^{b_1^*}\int_{b_2}^{b_2^*}(b_1^*-t_1)^{\alpha_1-1}(b_2^*-t_2)^{\alpha_2-1}f(t_1,t_2)\,dt_1dt_2=$$
$$\int_{a_1}^{b_1^*}\int_{a_2}^{b_2}\left((b_1-t_1)^{\alpha_1-1}(b_2-t_2)^{\alpha_2-1}-(b_1^*-t_1)^{\alpha_1-1}(b_2^*-t_2)^{\alpha_2-1}\right)f(t_1,t_2)\,dt_1dt_2$$
$$+\int_{b_1^*}^{b_1}\int_{a_2}^{b_2}(b_1-t_1)^{\alpha_1-1}(b_2-t_2)^{\alpha_2-1}f(t_1,t_2)\,dt_1dt_2-$$
$$\int_{a_1}^{b_1^*}\int_{b_2}^{b_2^*}(b_1^*-t_1)^{\alpha_1-1}(b_2^*-t_2)^{\alpha_2-1}f(t_1,t_2)\,dt_1dt_2. \quad (25)$$

We call

$$I(b_1^*,b_2):=\int_{a_1}^{b_1^*}\int_{a_2}^{b_2}\left|(b_1-t_1)^{\alpha_1-1}(b_2-t_2)^{\alpha_2-1}-(b_1^*-t_1)^{\alpha_1-1}(b_2^*-t_2)^{\alpha_2-1}\right|dt_1dt_2. \quad (26)$$

Hence, we have

$$|F(b_1,b_2)-F(b_1^*,b_2^*)|\le$$
$$\left\{I(b_1^*,b_2)+\frac{(b_1-b_1^*)^{\alpha_1}}{\alpha_1}\frac{(b_2-a_2)^{\alpha_2}}{\alpha_2}+\frac{(b_1^*-a_1)^{\alpha_1}}{\alpha_1}\frac{(b_2^*-b_2)^{\alpha_2}}{\alpha_2}\right\}\|f\|_\infty. \quad (27)$$

Therefore it holds

$$\delta:=\lim_{\substack{|b_1-b_1^*|\to0,\\|b_2-b_2^*|\to0}}|F(b_1,b_2)-F(b_1^*,b_2^*)|\le\Big(\lim_{\substack{|b_1-b_1^*|\to0,\\|b_2-b_2^*|\to0}}I(b_1^*,b_2)\Big)\|f\|_\infty=:\theta. \quad (28)$$

We will prove that $\theta=0$, hence $\delta=0$, in all possible cases.

If $\alpha_1=\alpha_2=1$, then $I(b_1^*,b_2)=0$, hence $\theta=0$.

If $\alpha_1=1$, $\alpha_2>0$ we get

$$I(b_1^*,b_2)=(b_1^*-a_1)\left(\int_{a_2}^{b_2}\left|(b_2-t_2)^{\alpha_2-1}-(b_2^*-t_2)^{\alpha_2-1}\right|dt_2\right). \quad (29)$$

Assume $\alpha_2>1$, then $\alpha_2-1>0$. Hence

$$I(b_1^*,b_2)=(b_1^*-a_1)\left(\int_{a_2}^{b_2}\left((b_2^*-t_2)^{\alpha_2-1}-(b_2-t_2)^{\alpha_2-1}\right)dt_2\right)$$
$$=(b_1^*-a_1)\left[\frac{(b_2^*-t_2)^{\alpha_2}|_{b_2}^{a_2}-(b_2-a_2)^{\alpha_2}}{\alpha_2}\right]$$

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$$= (b_1^* - a_1) \left[\frac{(b_2^* - a_2)^{\alpha_2} - (b_2^* - b_2)^{\alpha_2} - (b_2 - a_2)^{\alpha_2}}{\alpha_2} \right]. \tag{30}$$

Clearly, then

$$\lim_{\substack{|b_1 - b_1^*| \to 0, \\ |b_2 - b_2^*| \to 0}} I(b_1^*, b_2) = 0, \tag{31}$$

hence $\theta = 0$.

Let the case now of $\alpha_2 = 1$, $\alpha_1 > 1$. Then

$$I(b_1^*, b_2) = (b_2 - a_2) \left(\int_{a_1}^{b_1^*} \left| (b_1 - t_1)^{\alpha_1 - 1} - (b_1^* - t_1)^{\alpha_1 - 1} \right| dt_1 \right)$$

$$= (b_2 - a_2) \left[\frac{(b_1 - a_1)^{\alpha_1} - (b_1 - b_1^*)^{\alpha_1} - (b_1^* - a_1)^{\alpha_1}}{\alpha_1} \right]. \tag{32}$$

Then $\theta = 0$.

If $\alpha_1 = 1$, and $0 < \alpha_2 < 1$, then $\alpha_2 - 1 < 0$. Hence

$$I(b_1^*, b_2) = (b_1^* - a_1) \int_{a_2}^{b_2} \left| (b_2 - t_2)^{\alpha_2 - 1} - (b_2^* - t_2)^{\alpha_2 - 1} \right| dt_2 =$$

$$(b_1^* - a_1) \int_{a_2}^{b_2} \left((b_2 - t_2)^{\alpha_2 - 1} - (b_2^* - t_2)^{\alpha_2 - 1} \right) dt_2 =$$

$$(b_1^* - a_1) \left[\frac{(b_2 - a_2)^{\alpha_2} - (b_2^* - a_2)^{\alpha_2} + (b_2^* - b_2)^{\alpha_2}}{\alpha_2} \right]. \tag{33}$$

Hence $\theta = 0$.

Let now $\alpha_2 = 1$, $0 < \alpha_1 < 1$. Then

$$I(b_1^*, b_2) = (b_2 - a_2) \int_{a_1}^{b_1^*} \left| (b_1 - t_1)^{\alpha_1 - 1} - (b_1^* - t_1)^{\alpha_1 - 1} \right| dt_1$$

$$= (b_2 - a_2) \int_{a_1}^{b_1^*} \left((b_1^* - t_1)^{\alpha_1 - 1} - (b_1 - t_1)^{\alpha_1 - 1} \right) dt_1$$

$$= (b_2 - a_2) \left[\frac{(b_1^* - a_1)^{\alpha_1} - (b_1 - a_1)^{\alpha_1} + (b_1 - b_1^*)^{\alpha_1}}{\alpha_1} \right]. \tag{34}$$

Hence $\theta = 0$.

We observe that:

$$I(b_1^*, b_2) \leq \int_{a_1}^{b_1^*} \int_{a_2}^{b_2} \left| (b_1 - t_1)^{\alpha_1 - 1} (b_2 - t_2)^{\alpha_2 - 1} - (b_1 - t_1)^{\alpha_1 - 1} (b_2^* - t_2)^{\alpha_2 - 1} \right| dt_1 dt_2$$

$$+ \int_{a_1}^{b_1^*} \int_{a_2}^{b_2} \left| (b_1 - t_1)^{\alpha_1 - 1} (b_2^* - t_2)^{\alpha_2 - 1} - (b_1^* - t_1)^{\alpha_1 - 1} (b_2^* - t_2)^{\alpha_2 - 1} \right| dt_1 dt_2 =: J(b_1^*, b_2), \tag{35}$$

i.e.

$$I\left(b_1^*, b_2\right) \leq J\left(b_1^*, b_2\right).$$

Hence it holds

$$J\left(b_1^*, b_2\right) = \int_{a_1}^{b_1^*} \int_{a_2}^{b_2} \left(b_1 - t_1\right)^{\alpha_1 - 1} \left| \left(b_2 - t_2\right)^{\alpha_2 - 1} - \left(b_2^* - t_2\right)^{\alpha_2 - 1} \right| dt_1 dt_2 \quad (36)$$

$$+ \int_{a_1}^{b_1^*} \int_{a_2}^{b_2} \left(b_2^* - t_2\right)^{\alpha_2 - 1} \left| \left(b_1 - t_1\right)^{\alpha_1 - 1} - \left(b_1^* - t_1\right)^{\alpha_1 - 1} \right| dt_1 dt_2.$$

Case of $\alpha_1, \alpha_2 > 1$. Then

$$J\left(b_1^*, b_2\right) = \int_{a_1}^{b_1^*} \int_{a_2}^{b_2} \left(b_1 - t_1\right)^{\alpha_1 - 1} \left(\left(b_2^* - t_2\right)^{\alpha_2 - 1} - \left(b_2 - t_2\right)^{\alpha_2 - 1} \right) dt_1 dt_2$$

$$+ \int_{a_1}^{b_1^*} \int_{a_2}^{b_2} \left(b_2^* - t_2\right)^{\alpha_2 - 1} \left(\left(b_1 - t_1\right)^{\alpha_1 - 1} - \left(b_1^* - t_1\right)^{\alpha_1 - 1} \right) dt_1 dt_2 =$$

$$\left(\frac{\left(b_1 - a_1\right)^{\alpha_1} - \left(b_1 - b_1^*\right)^{\alpha_1}}{\alpha_1} \right) \left[\left(\frac{\left(b_2^* - a_2\right)^{\alpha_2} - \left(b_2^* - b_2\right)^{\alpha_2}}{\alpha_2} \right) - \frac{\left(b_2 - a_2\right)^{\alpha_2}}{\alpha_2} \right]$$

$$+ \left(\frac{\left(b_2^* - a_2\right)^{\alpha_2} - \left(b_2^* - b_2\right)^{\alpha_2}}{\alpha_2} \right) \left[\left(\frac{\left(b_1 - a_1\right)^{\alpha_1} - \left(b_1 - b_1^*\right)^{\alpha_1}}{\alpha_1} \right) - \frac{\left(b_1^* - a_1\right)^{\alpha_1}}{\alpha_1} \right]. \quad (37)$$

So that $\theta = 0$.

Case of $0 < \alpha_1, \alpha_2 < 1$, then

$$J\left(b_1^*, b_2\right) = \int_{a_1}^{b_1^*} \int_{a_2}^{b_2} \left(b_1 - t_1\right)^{\alpha_1 - 1} \left(\left(b_2 - t_2\right)^{\alpha_2 - 1} - \left(b_2^* - t_2\right)^{\alpha_2 - 1} \right) dt_1 dt_2$$

$$+ \int_{a_1}^{b_1^*} \int_{a_2}^{b_2} \left(b_2^* - t_2\right)^{\alpha_2 - 1} \left(\left(b_1^* - t_1\right)^{\alpha_1 - 1} - \left(b_1 - t_1\right)^{\alpha_1 - 1} \right) dt_1 dt_2 =$$

$$\left(\frac{\left(b_1 - a_1\right)^{\alpha_1} - \left(b_1 - b_1^*\right)^{\alpha_1}}{\alpha_1} \right) \left[\frac{\left(b_2 - a_2\right)^{\alpha_2}}{\alpha_2} - \left(\frac{\left(b_2^* - a_2\right)^{\alpha_2} - \left(b_2^* - b_2\right)^{\alpha_2}}{\alpha_2} \right) \right] \quad (38)$$

$$+ \left(\frac{\left(b_2^* - a_2\right)^{\alpha_2} - \left(b_2^* - b_2\right)^{\alpha_2}}{\alpha_2} \right) \left[\frac{\left(b_1^* - a_1\right)^{\alpha_1}}{\alpha_1} - \left(\frac{\left(b_1 - a_1\right)^{\alpha_1} - \left(b_1 - b_1^*\right)^{\alpha_1}}{\alpha_1} \right) \right].$$

One more time $\theta = 0$.

Next case of $\alpha_1 > 1$, $0 < \alpha_2 < 1$. We observe that

$$J\left(b_1^*, b_2\right) = \int_{a_1}^{b_1^*} \int_{a_2}^{b_2} \left(b_1 - t_1\right)^{\alpha_1 - 1} \left(\left(b_2 - t_2\right)^{\alpha_2 - 1} - \left(b_2^* - t_2\right)^{\alpha_2 - 1} \right) dt_1 dt_2 \quad (39)$$

$$+ \int_{a_1}^{b_1^*} \int_{a_2}^{b_2} \left(b_2^* - t_2\right)^{\alpha_2 - 1} \left(\left(b_1 - t_1\right)^{\alpha_1 - 1} - \left(b_1^* - t_1\right)^{\alpha_1 - 1} \right) dt_1 dt_2 =$$

$$\left(\frac{(b_1-a_1)^{\alpha_1}-(b_1-b_1^*)^{\alpha_1}}{\alpha_1}\right)\left[\frac{(b_2-a_2)^{\alpha_2}}{\alpha_2}-\left(\frac{(b_2^*-a_2)^{\alpha_2}-(b_2^*-b_2)^{\alpha_2}}{\alpha_2}\right)\right]$$
$$+\left(\frac{(b_2^*-a_2)^{\alpha_2}-(b_2^*-b_2)^{\alpha_2}}{\alpha_2}\right)\left[\left(\frac{(b_1-a_1)^{\alpha_1}-(b_1-b_1^*)^{\alpha_1}}{\alpha_1}\right)-\frac{(b_1^*-a_1)^{\alpha_1}}{\alpha_1}\right]. \tag{40}$$

Hence $\theta=0$.

Finally, we prove the case of $\alpha_2>1$ and $0<\alpha_1<1$. In that case it holds

$$J(b_1^*,b_2)=\int_{a_1}^{b_1^*}\int_{a_2}^{b_2}(b_1-t_1)^{\alpha_1-1}\left((b_2^*-t_2)^{\alpha_2-1}-(b_2-t_2)^{\alpha_2-1}\right)dt_1dt_2 \tag{41}$$

$$+\int_{a_1}^{b_1^*}\int_{a_2}^{b_2}(b_2^*-t_2)^{\alpha_2-1}\left((b_1^*-t_1)^{\alpha_1-1}-(b_1-t_1)^{\alpha_1-1}\right)dt_1dt_2=$$

$$\left(\frac{(b_1-a_1)^{\alpha_1}-(b_1-b_1^*)^{\alpha_1}}{\alpha_1}\right)\left[-\frac{(b_2-a_2)^{\alpha_2}}{\alpha_2}+\left(\frac{(b_2^*-a_2)^{\alpha_2}-(b_2^*-b_2)^{\alpha_2}}{\alpha_2}\right)\right]$$
$$+\left(\frac{(b_2^*-a_2)^{\alpha_2}-(b_2^*-b_2)^{\alpha_2}}{\alpha_2}\right)\left[-\left(\frac{(b_1-a_1)^{\alpha_1}-(b_1-b_1^*)^{\alpha_1}}{\alpha_1}\right)+\frac{(b_1^*-a_1)^{\alpha_1}}{\alpha_1}\right]. \tag{42}$$

Hence again $\theta=0$.

We have proved that $\delta=0$, in all possible subcases of (II_1).

We have proved that F is a continuous function over $[a_1,b]\times[a_2,d]$. ■

Now we can state:

Theorem 3 *Let $f\in L_\infty\left(\prod_{i=1}^k[a_i,b_i]\right)$, $\alpha_i>0$, $i=1,...,k\in\mathbb{N}$. Consider the function*

$$F(x_1,...,x_k)=\int_{a_1^*}^{x_1}...\int_{a_k^*}^{x_k}\prod_{i=1}^k(x_i-t_i)^{\alpha_i-1}f(t_1,...,t_k)\,dt_1...dt_k, \tag{43}$$

where $a_i^*,x_i\in[a_i,b_i]$, $a_i^*\le x_i$, $i=1,...,k$.

Then F is continuous on $\prod_{i=1}^k[a_i^*,b_i]$.

Remark 4 *In the setting of Theorem 3: Consider the left multidimensional Riemann-Liouville fractional integral of order $\alpha=(\alpha_1,...,\alpha_k)$:*

$$\left(I_{a_+^*}^\alpha f\right)(x)=\frac{1}{\prod_{i=1}^k\Gamma(\alpha_i)}\int_{a_1^*}^{x_1}...\int_{a_k^*}^{x_k}\prod_{i=1}^k(x_i-t_i)^{\alpha_i-1}f(t_1,...,t_k)\,dt_1...dt_k, \tag{44}$$

where $a^*=(a_1^*,...,a_k^*)$, $x=(x_1,...,x_k)$, $a_i^*\le x_i$, $i=1,...,k$. Here Γ denotes the gamma function.

By Theorem 3 we get that $\left(I_{a_+^*}^\alpha f\right)(x)$ is a continuous function for every $x\in\prod_{i=1}^k[a_i^*,b_i]$.

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We notice that

$$\left|\left(I_{a_+^*}^{\alpha} f\right)(x)\right| \leq \frac{1}{\prod_{i=1}^{k} \Gamma(\alpha_i)} \left(\int_{a_1^*}^{x_1} \ldots \int_{a_k^*}^{x_k} \prod_{i=1}^{k} (x_i - t_i)^{\alpha_i - 1} dt_1 \ldots dt_k\right) \|f\|_\infty$$

$$= \frac{\|f\|_\infty}{\prod_{i=1}^{k} \Gamma(\alpha_i)} \prod_{i=1}^{k} \left(\int_{a_i^*}^{x_i} (x_i - t_i)^{\alpha_i - 1} dt_i\right) = \frac{\|f\|_\infty}{\prod_{i=1}^{k} \Gamma(\alpha_i)} \prod_{i=1}^{k} \frac{(x_i - a_i^*)^{\alpha_i}}{\alpha_i} \quad (45)$$

$$= \|f\|_\infty \left(\prod_{i=1}^{k} \frac{(x_i - a_i^*)^{\alpha_i}}{\Gamma(\alpha_i + 1)}\right).$$

That is

$$\left|\left(I_{a_+^*}^{\alpha} f\right)(x)\right| \leq \left(\prod_{i=1}^{k} \frac{(x_i - a_i^*)^{\alpha_i}}{\Gamma(\alpha_i + 1)}\right) \|f\|_\infty . \quad (46)$$

In particular we get that

$$\left(I_{a_+^*}^{\alpha} f\right)(a^*) = 0, \quad (47)$$

and

$$\left\|I_{a_+^*}^{\alpha} f\right\|_\infty \leq \left(\prod_{i=1}^{k} \frac{(b_i - a_i^*)^{\alpha_i}}{\Gamma(\alpha_i + 1)}\right) \|f\|_\infty . \quad (48)$$

That is $I_{a_+^*}^{\alpha} f$ *is a bounded linear operator, which here is also a positive operator.*

References

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Weak closure operations on ideals of BCK-algebras

Hashem Bordbar[1], **Mohammad Mehdi Zahedi**[2], **Sun Shin Ahn**[3,*] **and Young Bae Jun**[4]

[1]*Faculty of Mathematics, Statistics and Computer Science, Shahid Bahonar University, Kerman, Iran*
[2]Department of Mathematics, Graduate University of Advanced Technology, Mahan-Kerman, Iran
[3]*Department of Mathematics Education, Dongguk University, Seoul 04620, Korea*
[4] *The Research Institute of Nature Sciences, Department of Mathematics Education, Gyeongsang National University, Jinju 52828, Korea*

Abstract. Weak closure operation, which is more general form than closure operation, on ideals of BCK-algebras is introduced, and related properties are investigated. Regarding weak closure operation, finite type and (strong) quasi-primeness are considered. Also positive implicative (resp., commutative and implicative) weak closure operations are discussed.

1. Introduction

Semi-prime closure operations on ideals of BCK-algebras are introduced in the paper [1], and a finite type of closure operations on ideals of BCK-algebras are discussed in [2].

In this paper, we consider more general form than closure operations on ideals of BCK-algebras. We introduce the notion of weak closure operations on ideals of BCK-algebras. Regarding weak closure operation, we define finite type and (strong) quasi-primeness, and investigate related properties. We also discuss positive implicative (resp., commutative and implicative) weak closure operations, and provide several examples to illustrate notions and properties.

2. Preliminaries

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers.

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a *BCI-algebra* if it satisfies the following conditions:

(I) $(\forall x, y, z \in X)$ $(((x * y) * (x * z)) * (z * y) = 0)$,

(II) $(\forall x, y \in X)$ $((x * (x * y)) * y = 0)$,

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* The corresponding author.

[0]**E-mail**: `bordbar.amirh@gmail.com` (H. Bordbar); `zahedi_mm@kgut.ac.ir` (M. M. Zahedi); `sunshine@dongguk.edu` (S. S. Ahn); `skywine@gmail.com` (Y. B. Jun)

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(III) $(\forall x \in X)\ (x * x = 0)$,
(IV) $(\forall x, y \in X)\ (x * y = 0,\ y * x = 0\ \Rightarrow\ x = y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\ (0 * x = 0)$,

then X is called a *BCK-algebra.* Any BCK/BCI-algebra X satisfies the following axioms:

(a1) $(\forall x \in X)\ (x * 0 = x)$,
(a2) $(\forall x, y, z \in X)\ (x \leq y\ \Rightarrow\ x * z \leq y * z,\ z * y \leq z * x)$,
(a3) $(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y)$,
(a4) $(\forall x, y, z \in X)\ ((x * z) * (y * z) \leq x * y)$

where $x \leq y$ if and only if $x * y = 0$.

A subset A of a BCK/BCI-algebra X is called an *ideal* of X (see [4]) if it satisfies:

$$0 \in A, \tag{2.1}$$

$$(\forall x \in X)\,(\forall y \in A)\,(x * y \in A\ \Rightarrow\ x \in A)\,. \tag{2.2}$$

For any subset A of X, the ideal generated by A is defined to be the intersection of all ideals of X containing A, and it is denoted by $\langle A \rangle$. If A is finite, then we say that $\langle A \rangle$ is *finitely generated ideal* of X (see [4]).

A subset A of a BCK-algebra X is called a *commutative ideal* of X (see [4]) if it satisfies (2.1) and

$$(\forall x, y \in X)(\forall z \in A)\,((x * y) * z \in A\ \Rightarrow\ x * (y * (y * x)) \in A)\,. \tag{2.3}$$

A subset A of a BCK-algebra X is called a *positive implicative ideal* of X (see [4]) if it satisfies (2.1) and

$$(\forall x, y, z \in X)\,((x * y) * z \in A,\, y * z \in A\ \Rightarrow\ x * z \in A)\,. \tag{2.4}$$

A subset A of a BCK-algebra X is called an *implicative ideal* of X (see [4]) if it satisfies (2.1) and

$$(\forall x, y \in X)(\forall z \in A)\,((x * (y * y)) * z \in A\ \Rightarrow\ x \in A)\,. \tag{2.5}$$

Denote by $\mathcal{I}_{pi}(X)$ (resp., $\mathcal{I}_c(X)$ and $\mathcal{I}_m(X)$) the set of all positive implicative (resp., commutative and implicative) ideals of X.

We refer the reader to the books [3, 4] for further information regarding BCK/BCI-algebras.

3. **Weak Closure operations**

In what follows, let X and $\mathcal{I}(X)$ be a BCK-algebra and a set of all ideals of X, respectively, unless otherwise specified .

Definition 3.1. A mapping $c : \mathcal{I}(X) \to \mathcal{I}(X)$ is called a *weak closure operation* on $\mathcal{I}(X)$ if the following conditions are valid.

$$(\forall A \in \mathcal{I}(X))\,(A \subseteq c(A))\,, \tag{3.1}$$

$$(\forall A, B \in \mathcal{I}(X))\,(A \subseteq B \;\Rightarrow\; c(A) \subseteq c(B))\,. \tag{3.2}$$

If a weak closure operation $c : \mathcal{I}(X) \to \mathcal{I}(X)$ satisfies the condition

$$(\forall A \in \mathcal{I}(X))\,(c(c(A)) = c(A))\,, \tag{3.3}$$

then we say that c is a closure operation on $\mathcal{I}(X)$ (see [2]). In what follows, we use A^{cl} instead of $c(A)$.

Example 3.2. Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	0	2
3	3	2	1	0	3
4	4	4	4	4	0

We have 8 ideals of X, and they are $A_0 = \{0\}$, $A_1 = \{0, 1\}$, $A_2 = \{0, 2\}$, $A_3 = \{0, 4\}$, $A_4 = \{0, 1, 4\}$, $A_5 = \{0, 1, 2, 3\}$, $A_6 = \{0, 2, 4\}$, and $A_7 = X$. Define a mapping $c : \mathcal{I}(X) \to \mathcal{I}(X)$ by $A_0^{cl} = A_0$, $A_1^{cl} = A_4$, $A_2^{cl} = A_5$, $c(A_3) = A_6$, and $c(A_4) = c(A_5) = c(A_6) = c(A_7) = A_7$. Then c is a weak closure operation on $\mathcal{I}(X)$. But it is not a closure operation on $\mathcal{I}(X)$ since $c(A_2^{cl}) = c(A_5) = A_7$.

In a BCK-algebra X, let $x \wedge y$ denote the greatest lower bound of x and y. Note that $0 \wedge x = 0$ for all $x \in X$. For any element x of X, consider the following condition

$$(\exists\, y \in X \setminus \{0\})\,(x \wedge y = 0)\,. \tag{3.4}$$

In the following example, we know that there are two kinds of element. One is an element x satisfying the condition (3.4). The other is an element x which does not satisfy the condition (3.4).

Example 3.3. Let $X = \{0, 1, 2, 3, 4\}$ be a set with the following Cayley table.

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	2	1	0	0
4	4	4	4	4	0

Then X is a BCK-algebra. We know that 1 and 2 satisfy the condition (3.4), but 3 and 4 do not satisfy the condition (3.4).

On the basis of this consideration, we define the zeromeet element in a BCK-algebra.

Definition 3.4. An element x of X is called a *zeromeet element* of X if the condition (3.4) is valid. Otherwise, x is called a non-zeromeet element of X.

Denote by $Z(X)$ the set of all zeromeet elements of X, that is,

$$Z(X) = \{x \in X \mid x \wedge y = 0 \text{ for some nonzero element } y \in X\}.$$

Obviously, $0 \in Z(X)$. We know that $0, 1, 2 \in Z(X)$ and $3, 4 \notin Z(X)$ in Example 3.3.

Lemma 3.5. *For any $x, y \in X$, if $x, y \notin Z(X)$, then $x \wedge y \notin Z(X)$, that is, the set $X \setminus Z(X)$ is closed under the operation $\wedge$.*

Proof. Let $x, y \in X \setminus Z(X)$ and assume that $x \wedge y \in Z(X)$. Then $x \wedge (y \wedge a) = (x \wedge y) \wedge a = 0$ for some nonzero element $a \in X$. Since $x \notin Z(X)$, it follows that $y \wedge a = 0$ and so that $a = 0$ since $y \notin Z(X)$. This is a contradiction, and thus $x \wedge y \notin Z(X)$. □

For any subsets A and B of X, we define

$$A \wedge B := \langle \{a \wedge b \mid a \in A,\ b \in B\rangle.$$

We use $x \wedge A$ instead of $\{x\} \wedge A$, that is, $x \wedge A := \langle \{x \wedge a \mid a \in A\}\rangle$.

Definition 3.6. A weak closure operation $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ is said to be *quasi-prime* if it satisfies:

$$(\forall a \in X \setminus Z(X))\, (\forall A \in \mathcal{I}(X))\, \left(a \wedge A^{cl} \subseteq (a \wedge A)^{cl}\right). \tag{3.5}$$

Example 3.7. Consider a BCK-algebra $X = \{0, 1, 2, 3\}$ with the following Cayley table.

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	2	0	0
3	3	3	3	0

We know that $Z(X) = \{0\}$ and there are four ideals in X, that is, $A_0 = \{0\}$, $A_1 = \{0, 1\}$, $A_2 = \{0, 1, 2\}$ and $A_3 = X$. Define a mapping $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ by $A_0^{cl} = A_0, A_1^{cl} = A_2, A_2^{cl} = A_3$ *and* $A_3^{cl} = A_3$. Then "cl" is a weak closure operation on $\mathcal{I}(X)$. For $1, 2, 3 \in X \setminus Z(X)$, we have

$1 \wedge A_0^{cl} = 1 \wedge A_0 = \langle \{0\}\rangle = A_0 = A_0^{cl} = (1 \wedge A_0)^{cl}$,
$1 \wedge A_1^{cl} = 1 \wedge A_2 = \langle \{0, 1\}\rangle = A_1 \subseteq A_2 = A_1^{cl} = (1 \wedge A_1)^{cl}$,
$1 \wedge A_2^{cl} = 1 \wedge A_3 = \langle \{0, 1\}\rangle = A_1 \subseteq A_2 = A_1^{cl} = (1 \wedge A_2)^{cl}$,
$1 \wedge A_3^{cl} = 1 \wedge A_3 = \langle \{0, 1\}\rangle = A_1 \subseteq A_2 = A_1^{cl} = (1 \wedge A_3)^{cl}$,

$2 \wedge A_0^{cl} = 2 \wedge A_0 = \langle\{0\}\rangle = A_0 = A_0^{cl} = (2 \wedge A_0)^{cl}$,
$2 \wedge A_1^{cl} = 2 \wedge A_2 = \langle\{0,1,2\}\rangle = A_2 = A_1^{cl} = (2 \wedge A_1)^{cl}$,
$2 \wedge A_2^{cl} = 2 \wedge A_3 = \langle\{0,1,2\}\rangle = A_2 \subseteq A_3 = A_2^{cl} = (2 \wedge A_2)^{cl}$,
$2 \wedge A_3^{cl} = 2 \wedge A_3 = \langle\{0,1,2\}\rangle = A_2 \subseteq A_3 = A_2^{cl} = (2 \wedge A_3)^{cl}$,
$3 \wedge A_0^{cl} = 3 \wedge A_0 = \langle\{0\}\rangle = A_0 = A_0^{cl} = (3 \wedge A_0)^{cl}$,
$3 \wedge A_1^{cl} = 3 \wedge A_2 = \langle\{0,1,2\}\rangle = A_2 = A_1^{cl} = (3 \wedge A_1)^{cl}$,
$3 \wedge A_2^{cl} = 3 \wedge A_3 = \langle\{0,1,2,3\}\rangle = A_3 = A_2^{cl} = (3 \wedge A_2)^{cl}$,
$3 \wedge A_3^{cl} = 3 \wedge A_3 = \langle\{0,1,2,3\}\rangle = A_3 = A_3^{cl} = (3 \wedge A_3)^{cl}$,
Therefore "cl" is a quasi-prime weak closure operation on $\mathcal{I}(X)$.

Definition 3.8. A weak closure operation $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ is said to be *strong quasi-prime* if it satisfies:

$$(\forall a \in X \setminus Z(X))\,(\forall A \in \mathcal{I}(X))\,\left(a \wedge A^{cl} = (a \wedge A)^{cl}\right). \tag{3.6}$$

Example 3.9. Consider a BCK-algebra $X = \{0,1,2,3,4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	4	4	4	0

We know that $Z(X) = \{0,1,2\}$ and there are six ideals in X, that is, $A_0 = \{0\}$, $A_1 = \{0,1\}$, $A_2 = \{0,2\}$, $A_3 = \{0,1,2\}$, $A_4 = \{0,1,2,3\}$ and $A_5 = X$. Define a mapping $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ as follows: $A_0^{cl} = A_1$, $A_1^{cl} = A_2^{cl} = A_3$, $A_3^{cl} = A_4^{cl} = A_4$ and $A_5^{cl} = A_5$. Then "cl" is a weak closure operation on $\mathcal{I}(X)$. For $3, 4 \in X \setminus Z(X)$, we have
$3 \wedge A_0^{cl} = 3 \wedge A_1 = \langle\{0,1\}\rangle = A_1 = A_0^{cl} = (3 \wedge A_0)^{cl}$,
$3 \wedge A_1^{cl} = 3 \wedge A_3 = \langle\{0,1,2\}\rangle = A_3 = A_1^{cl} = (3 \wedge A_1)^{cl}$,
$3 \wedge A_2^{cl} = 3 \wedge A_3 = \langle\{0,1,2\}\rangle = A_3 = A_2^{cl} = (3 \wedge A_2)^{cl}$,
$3 \wedge A_3^{cl} = 3 \wedge A_4 = \langle\{0,1,2,3\}\rangle = A_4 = A_3^{cl} = (3 \wedge A_3)^{cl}$,
$3 \wedge A_4^{cl} = 3 \wedge A_4 = \langle\{0,1,2,3\}\rangle = A_4 = A_4^{cl} = (3 \wedge A_4)^{cl}$,
$3 \wedge A_5^{cl} = 3 \wedge A_5 = \langle\{0,1,2,3\}\rangle = A_4 = A_4^{cl} = (3 \wedge A_5)^{cl}$,
$4 \wedge A_0^{cl} = 4 \wedge A_1 = \langle\{0,1\}\rangle = A_1 = A_0^{cl} = (4 \wedge A_0)^{cl}$,
$4 \wedge A_1^{cl} = 4 \wedge A_3 = \langle\{0,1,2\}\rangle = A_3 = A_1^{cl} = (4 \wedge A_1)^{cl}$,
$4 \wedge A_2^{cl} = 4 \wedge A_3 = \langle\{0,1,2\}\rangle = A_3 = A_2^{cl} = (4 \wedge A_2)^{cl}$,
$4 \wedge A_3^{cl} = 4 \wedge A_4 = \langle\{0,1,2,3\}\rangle = A_4 = A_3^{cl} = (4 \wedge A_3)^{cl}$,
$4 \wedge A_4^{cl} = 4 \wedge A_4 = \langle\{0,1,2,3\}\rangle = A_4 = A_4^{cl} = (4 \wedge A_4)^{cl}$,
$4 \wedge A_5^{cl} = 4 \wedge A_5 = \langle\{0,1,2,3\}\rangle = A_4 = A_4^{cl} = (4 \wedge A_5)^{cl}$.
Therefore "cl" is a strong quasi-prime weak closure operation on $\mathcal{I}(X)$.

Given an ideal A of X and an operation $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ on $\mathcal{I}(X)$, we consider the following set:

$$K := \cup\{B^{cl} \mid B \subseteq A,\ B \in \mathcal{I}_f(X)\} \tag{3.7}$$

where $\mathcal{I}_f(X)$ is the set of all finitely generated ideals of X. The following example shows that the set K in (3.7) may not be an ideal of X in general.

Example 3.10. Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	2	0
3	3	3	3	0	0
4	4	4	3	2	0

There are five ideals in X, that is, $A_0 = \{0\}$, $A_1 = \{0, 1\}$, $A_2 = \{0, 1, 2\}$, $A_3 = \{0, 1, 3\}$ and $A_4 = X$. Define a mapping $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ as follows: $A_0^{cl} = A_3$, $A_1^{cl} = A_2$, $A_2^{cl} = A_0$, $A_3^{cl} = A_4$ and $A_4^{cl} = A_3$. For the ideal A_2 of X, we have

$$\cup\{B^{cl} \mid B \subseteq A,\ B \in \mathcal{I}_f(X)\} = A_0^{cl} \cup A_1^{cl} \cup A_2^{cl} = \{0, 1, 2, 3\}$$

which is not an ideal of X.

We provide a condition for the set K in (3.7) to be an ideal of X.

Theorem 3.11. *If $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ is a weak closure operation on $\mathcal{I}(X)$, then the set K in* (3.7) *is an ideal of X for any ideal A of X.*

Proof. Obviously, $0 \in K$. Let $x, y \in X$ such that $x * y \in K$ and $y \in K$. Then there exist B_x, $B_y \in \mathcal{I}_f(X)$ such that $B_x \subseteq A$, $B_y \subseteq A$, $x * y \in B_x^{cl}$ and $y \in B_y^{cl}$. Since $B_x, B_y \subseteq B_x + B_y = \langle B_x \cup B_y\rangle$, we have $x * y \in B_x^{cl} \subseteq (B_x + B_y)^{cl}$ and $y \in B_y^{cl} \subseteq (B_x + B_y)^{cl}$, which imply that $x \in (B_x + B_y)^{cl}$. Since B_x, $B_y \in \mathcal{I}_f(X)$, we get $B_x + B_y \in \mathcal{I}_f(X)$ and $B_x + B_y \subseteq A$. Therefore $x \in K$, and K is an ideal of X. □

Corollary 3.12. *If $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ is a closure operation on $\mathcal{I}(X)$, then the set K in* (3.7) *is an ideal of X for any ideal A of X.*

Lemma 3.13 ([4])**.** (Extension property) *Let A and B be ideals of X such that $A \subseteq B$. If A is a positive implicative (resp., commutative and implicative) ideal, then so is B.*

Using Lemma 3.13 and (3.1), we have the following theorem.

Theorem 3.14. *Let "cl" be a weak closure operation on $\mathcal{I}(X)$. If A is a positive implicative (resp., commutative and implicative) ideal of X, then so is A^{cl}.*

The following example shows that the converse of Theorem 3.14 is not true in general.

Example 3.15. Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	3	4	1	0

There are five ideals in X, that is, $A_0 = \{0\}$, $A_1 = \{0, 1\}$, $A_2 = \{0, 2\}$, $A_3 = \{0, 1, 2\}$ and $A_4 = X$. Define a mapping $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ as follows: $(A_0)^{cl} = A_0$, $(A_1)^{cl} = (A_2)^{cl} = A_3$, and $(A_3)^{cl} = (A_4)^{cl} = A_4$. Then "$cl$"" is a weak closure operation on $\mathcal{I}(X)$. The ideal $A_2 = \{0, 2\}$ is not positive implicative (resp., commutative and implicative) ideal, but $(A_2)^{cl} = A_3 = \{0, 1, 2\}$ is a positive implicative (resp., commutative and implicative) ideal of X.

Theorem 3.16. *An operation $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ on $\mathcal{I}(X)$ defined by*

$$(\forall A \in \mathcal{I}(X)) \left(A^{cl} = \cap\{I_\lambda \mid I_\lambda \in \mathcal{I}_\Gamma(X),\, A \subseteq I_\lambda, \lambda \in \Lambda\} \right) \tag{3.8}$$

is a weak closure operation on $\mathcal{I}(X)$ where $\mathcal{I}_\Gamma(X) \in \{\mathcal{I}_{pi}(X), \mathcal{I}_c(X), \mathcal{I}_m(X)\}$ and Λ is any index set.

Proof. Obviously, $A \subseteq A^{cl}$ for every $A \in \mathcal{I}(X)$. Let $A, B \in \mathcal{I}(X)$ be such that $A \subseteq B$. Then

$$\begin{aligned} B^{cl} &= \cap\{I_\lambda \mid I_\lambda \in \mathcal{I}_\Gamma(X),\, B \subseteq I_\lambda, \lambda \in \Lambda\} \\ &\supseteq \cap\{I_\lambda \mid I_\lambda \in \mathcal{I}_\Gamma(X),\, A \subseteq I_\lambda, \lambda \in \Lambda\} \\ &= A^{cl}, \end{aligned}$$

and so "cl" is a weak closure operation on $\mathcal{I}(X)$. □

The following example illustrates Theorem 3.16.

Example 3.17. Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	2	0
3	3	1	3	0	3
4	4	4	4	4	0

There are six ideals in X, that is, $A_0 = \{0\}$, $A_1 = \{0, 1, 3\}$, $A_2 = \{0, 2\}$, $A_3 = \{0, 1, 2, 3\}$, $A_4 = \{0, 2, 4\}$ and $A_5 = X$.

(1) Define a mapping $cl_1 : \mathcal{I}(X) \to \mathcal{I}(X)$ by

$$A^{cl_1} = \cap\{B \mid A \subseteq B \text{ and } B \in \mathcal{I}_{pi}(X)\}.$$

Then we have
$A_0^{cl_1} = A_1 \cap A_3 \cap A_5 = A_1$, $A_1^{cl_1} = A_1 \cap A_3 \cap A_5 = A_1$,
$A_2^{cl_1} = A_3 \cap A_5 = A_3$, $A_3^{cl_1} = A_3 \cap A_5 = A_3$, $A_4^{cl_1} = A_5 = A_5^{cl_1}$.
We can check that "cl_1" is a weak closure operation on $\mathcal{I}(X)$.

(2) We define an operation "cl_2" on $\mathcal{I}(X)$ by

$$A^{cl_2} = \cap\{B \mid A \subseteq B \text{ and } B \in \mathcal{I}_c(X)\}.$$

Then we have
$A_0^{cl_2} = A_2 \cap A_3 \cap A_4 \cap A_5 = A_2$, $A_1^{cl_2} = A_3 \cap A_5 = A_3$,
$A_2^{cl_2} = A_2 \cap A_3 \cap A_4 \cap A_5 = A_2$, $A_3^{cl_2} = A_3 \cap A_5 = A_3$,
$A_4^{cl_2} = A_4 \cap A_5 = A_4$, $A_5^{cl_2} = A_5$.
It is routine to verify that "cl_2" is a weak closure operation on $\mathcal{I}(X)$.

(3) We define an operation "cl_3" on $\mathcal{I}(X)$ by

$$A^{cl_3} = \cap\{B \mid A \subseteq B \text{ and } B \in \mathcal{I}_m(X)\}.$$

Then we have
$A_0^{cl_3} = A_3 \cap A_5 = A_3$, $A_1^{cl_3} = A_3 \cap A_5 = A_3$,
$A_2^{cl_3} = A_3 \cap A_5 = A_3$, $A_3^{cl_3} = A_3 \cap A_5 = A_3$,
$A_4^{cl_3} = A_5$, $A_5^{cl_3} = A_5$.
It is easy to show that "cl_3" is weak closure operation on $\mathcal{I}(X)$.

Let $\{cl_\lambda \mid \lambda \in \Lambda\}$ be a collection of operations on $\mathcal{I}(X)$. We define the intersection of cl_λ's, denoted by $\underset{\lambda\in\Lambda}{\cap} cl_\lambda$, as follows:

$$\underset{\lambda\in\Lambda}{\cap} cl_\lambda : \mathcal{I}(X) \to \mathcal{I}(X),\ A \mapsto \underset{\lambda\in\Lambda}{\cap} A^{cl_\lambda}.$$

Note that if cl_λ is a weak closure operation on $\mathcal{I}(X)$ for all $\lambda \in \Lambda$, then $\underset{\lambda\in\Lambda}{\cap} cl_\lambda$ is a weak closure operation on $\mathcal{I}(X)$ (see [2]). But the following example shows that the union of weak closure operations may not be a weak closure operation.

Example 3.18. Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	2
3	3	2	1	0	2
4	4	1	4	1	0

There are four ideals in X, that is, $A_0 = \{0\}$, $A_1 = \{0, 1, 4\}$, $A_2 = \{0, 2\}$ and $A_3 = X$. Define a mapping $cl_1 : \mathcal{I}(X) \to \mathcal{I}(X)$ as follows: $A_0^{cl_1} = A_1$, $A_1^{cl_1} = A_3$, $A_2^{cl_1} = A_3$, $A_3^{cl_1} = A_3$. Then "cl_1" is a weak closure operation on $\mathcal{I}(X)$. Also, define a mapping $cl_2 : \mathcal{I}(X) \to \mathcal{I}(X)$ as follows: $A_0^{cl_2} = A_2$, $A_1^{cl_2} = A_3$, $A_2^{cl_2} = A_3$, $A_3^{cl_2} = A_3$. Then "cl_2" is a weak closure operation on $\mathcal{I}(X)$.

Now if we define "cl_3" by $A^{cl_3} = A^{cl_1} \cup A^{cl_2}$, then "$cl_3$" is not a weak closure operation on $\mathcal{I}(X)$ because for an ideal A_0 of X, we have

$$A_0^{cl_3} = A_0^{cl_1} \cup A_0^{cl_2} = A_1 \cup A_2 = \{0, 1, 2, 4\}$$

which is not an ideal of X. Thus "cl_3" is not a weak closure operation on $\mathcal{I}(X)$.

Definition 3.19. Given a (weak) closure operation $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ on $\mathcal{I}(X)$, we define a new operation $cl_f : \mathcal{I}(X) \to \mathcal{I}(X)$ by

$$(\forall A \in \mathcal{I}(X)) \left(A^{cl_f} = \cup\{B^{cl} \mid B \subseteq A,\ B \in \mathcal{I}_f(X)\} \right), \tag{3.9}$$

where $\mathcal{I}_f(X)$ is the set of all finitely generated ideals of X.

Definition 3.20. A (weak) closure operation cl on $\mathcal{I}(X)$ is said to be of *finite type* if the following assertion is valid.

$$(\forall A \in \mathcal{I}(X)) \left(A^{cl} = A^{cl_f} \right). \tag{3.10}$$

Note that every weak closure operation on a finite BCK-algebra is of finite type.

Example 3.21. Let X be a BCK-algebra of infinite order. Define an operation "cl'' on $\mathcal{I}(X)$ as follows:

$$A^{cl} = \begin{cases} X & \text{if } A \text{ is a maximal ideal or } A = X, \\ M & \text{otherwise,} \end{cases} \tag{3.11}$$

where M is a maximal ideal of X containing A. We can easily check that "cl" is a weak closure operation. Now let A be a maximal ideal of X which is not finitely generated. Then

$$A^{cl_f} = \cup\{B^{cl} \mid B \subseteq A \text{ and } B \in \mathcal{I}_f(X)\} \subseteq M \subsetneq X = A^{cl},$$

and thus "cl" is a weak closure operation which is not of finite type.

For two operations "cl_1" and "cl_2" on $\mathcal{I}(X)$, we say that "cl_1" is *weaker* than "cl_2", denoted by $cl_1 \le cl_2$, if $A^{cl_1} \subseteq A^{cl_2}$ for every $A \in \mathcal{I}(X)$.

Theorem 3.22. *Given an operation "cl" on $\mathcal{I}(X)$, we have*

(i) *If "cl" is a weak closure operation of finite type, then so is "cl_f", and it is largest in the set of weak closure operations which are weaker than "cl".*
(ii) *If "cl" is a (strong) quasi-prime weak closure operation, then so is "cl_f".*

Proof. (i) Let "cl" be a weak closure operation of finite type. Then "cl_f" is a weak closure operation on $\mathcal{I}(X)$ (see [2]). To prove that "cl_f" is of finite type, we should prove that $A^{cl_f} = A^{(cl_f)_f}$ for every ideal A of X. Clearly, we have $A^{cl_f} \subseteq A^{(cl_f)_f}$. Suppose that $x \in A^{(cl_f)_f}$. Then there exists a finitely generated ideal B such that $B \subseteq A$ and $x \in B^{cl_f}$. Since "cl" is a weak closure operation of finite type, we have $B^{cl} = B^{cl_f}$. Thus $x \in B^{cl}$, $B \subseteq A$ and B is finitely generated ideal. Therefore $x \in A^{cl_f}$ and $A^{cl_f} = A^{(cl_f)_f}$ which means that "cl_f" is a weak closure operation on $\mathcal{I}(X)$ of finite type. Now let c be a weak closure operation on $\mathcal{I}(X)$ of finite type which is weaker than "*cl*". Let A be an ideal of X and $a \in A^c$. Then there exists a finitely generated ideal B of X such that $B \subseteq A$ and $a \in B^c$. It follows from $c \leq cl$ that $a \in B^{cl}$. Therefore $a \in A^{cl_f}$, and so $c \leq cl_f$.

(ii) Suppose that "*cl*" be a quasi prime weak closure operation on $\mathcal{I}(X)$. To prove that "cl_f" is a quasi prime weak closure operation, it is enough to show that $a \wedge A^{cl_f} \subseteq (a \wedge A)^{cl_f}$. Now let $x \in a \wedge A^{cl_f} = \langle \{a \wedge \alpha \mid \alpha \in A^{cl_f}\} \rangle$. Then there exist $\alpha_1, \alpha_2, \cdots, \alpha_n \in A^{cl_f}$ such that

$$(\cdots((x \wedge (a \wedge \alpha_1)) * (a \wedge \alpha_2)) * \cdots) * (a \wedge \alpha_n) = 0.$$

Since $\alpha_i \in A^{cl_f} = \cup\{B^{cl} \mid B \subseteq A \text{ and } B \in \mathcal{I}_f(X)\}$ for each $1 \leq i \leq n$, we have $\alpha_i \in A^{cl_f} = \cup\{B^{cl} \mid B \subseteq A \text{ and } B \in \mathcal{I}_f(X)\}$, and so there exists a finitely generated ideal B such that $\alpha_i \in B^{cl}$ and $B \subseteq A$. Since $\alpha_i \in B^{cl}$, we have

$$a \wedge \alpha_i \in \{a \wedge \beta \mid \beta \in B\} \subseteq \langle \{a \wedge \beta \mid \beta \in B\} \rangle = a \wedge B^{cl},$$

which implies that $a \wedge \alpha_i \in a \wedge B$ and

$$(\cdots((x \wedge (a \wedge \alpha_1)) * (a \wedge \alpha_2)) * \cdots) * (a \wedge \alpha_n) = 0.$$

This means that $x \in a \wedge B^{cl}$. Since "*cl*" is a quasi prime weak closure operation on $\mathcal{I}(X)$, it follows that

$$x \in a \wedge B^{cl} \subseteq (a \wedge B)^{cl} \subseteq (a \wedge A)^{cl} \subseteq (a \wedge A)^{cl_f}.$$

Therefore $x \in (a \wedge A)^{cl_f}$ and "cl_f" is a quasi-prime weak closure operation on $\mathcal{I}(X)$. Similarly, we can check that if "*cl*" is a strong quasi-prime weak closure operation on $\mathcal{I}(X)$, then "cl_f" is a strong quasi-prime weak closure operation on $\mathcal{I}(X)$. □

Definition 3.23. An operation $\alpha : \mathcal{I}(X) \to \mathcal{I}(X)$ is called a *positive implicative* (resp. *commutative* and *implicative*) *weak closure operation* if the following conditions are valid.

(i) For any $A, B \in \mathcal{I}_{pi}(X)$ (resp. $\mathcal{I}_c(X)$ and $\mathcal{I}_m(X)$),

$$A \subseteq A^\alpha, \tag{3.12}$$

$$A \subseteq B \ \Rightarrow \ A^\alpha \subseteq B^\alpha. \tag{3.13}$$

(ii) $(\forall A \notin \mathcal{I}_{pi}(X)(\text{resp., } \mathcal{I}_c(X) \text{ and } \mathcal{I}_m(X)))\,(A^\alpha = A)$.

Obviously, every positive implicative (resp., commutative and implicative) weak closure operation is a weak closure operation, but the converse is not true in general as seen in the following example.

Example 3.24. Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	2	0
3	3	1	3	0	3
4	4	4	4	4	0

There are six ideals in X, that is, $A_0 = \{0\}$, $A_1 = \{0,1,3\}$, $A_2 = \{0,2\}$, $A_3 = \{0,1,2,3\}$, $A_4 = \{0,2,4\}$ and $A_5 = X$. Note that A_1, A_3 and A_5 are positive implicative ideals and A_0, A_2 and A_4 are not positive implicative ideals. Define a mapping $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ as follows: $A_0^{cl} = A_0$ $A_1^{cl} = A_3$, $A_2^{cl} = A_2$, $A_3^{cl} = A_5$, $A_4^{cl} = A_4$ and $A_5^{cl} = X$. Then "cl" is a positive implicative weak closure operation on $\mathcal{I}(X)$. Now we define an operation "cl_1" on $\mathcal{I}(X)$ as follows:

$$A_0^{cl_1} = A_1,\ A_1^{cl_1} = A_3,\ A_2^{cl_1} = A_4,\ A_3^{cl_1} = A_5,\ A_4^{cl_1} = A_5 \text{ and } A_5^{cl_1} = X.$$

Then "cl_1" is a weak closure operation on $\mathcal{I}(X)$, but it is not positive implicative because the ideal A_2 is not a positive implicative ideal and $A_2^{cl_1} = A_4 \neq A_2$.

Example 3.25. Consider a BCK-algebra $X = \{0,1,2,3,4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	4	4	4	0

There are five ideals in X, that is, $A_0 = \{0\}$, $A_1 = \{0,1\}$, $A_2 = \{0,1,2\}$, $A_3 = \{0,1,2,3\}$ and $A_4 = X$ where A_3 and A_4 are commutative ideals and A_0, A_1 and A_2 are not commutative ideals. Now define "cl" as follows:

$$A_0^{cl} = A_1,\ A_1^{cl} = A_2,\ A_2^{cl} = A_3,\ A_3^{cl} = A_4 \text{ and } A_4^{cl} = X$$

Then "cl" is a weak closure operation on $\mathcal{I}(X)$, but it is not commutative since the ideal A_2 is not a commutative ideal and $A_2^{cl} = A_3 \neq A_2$.

Example 3.26. Let $X = \{0,1,2,3,4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	4	4	4	0

Then X is a BCK-algebra with seven ideals $A_0 = \{0\}$, $A_1 = \{0,1\}$, $A_2 = \{0,1,2\}$, $A_3 = \{0,1,4\}$ $A_4 = \{0,1,2,3\}$, $A_5 = \{0,1,2,4\}$ and $A_6 = X$. Note that A_2, A_4, A_5 and A_6 are implicative ideals and A_0, A_1 and A_3 are not implicative ideals. Now we define an operation define "*cl*" on $\mathcal{I}(X)$ by

$$A_0^{cl} = A_1,\ A_1^{cl} = A_2,\ A_2^{cl} = A_5,\ A_3^{cl} = A_5,\ A_4^{cl} = A_6,\ A_5^{cl} = A_6 \text{ and } A_6^{cl} = X.$$

Then "*cl*" is a weak closure operation on $\mathcal{I}(X)$, but it is not implicative since the ideal A_3 is not an implicative ideal and $A_3^{cl} = A_5 \neq A_3$.

Given a weak closure operation, we kame a positive implicative weak closure operation.

Theorem 3.27. *Given $A \in \mathcal{I}(X)$, let "cl" be a weak closure operation on $\mathcal{I}(X)$ and "cl_{pi}" be an operation on $\mathcal{I}(X)$ such that $cl \leq cl_{pi}$ and*

(i) $(\forall C \in \mathcal{I}(X))\left(A \subseteq C \ \Rightarrow\ C^{cl_{pi}} = C^{cl}\right)$.

(ii) $(\forall C \in \mathcal{I}(X))\left(C \subsetneq A \ \Rightarrow\ C^{cl_{pi}} = C\right)$.

(iii) *For any $C \in \mathcal{I}(X)$, if A and C have no inclusion relation, then $C^{cl_{pi}} = C$.*

If A is positive implicative (resp., commutative and implicative) ideals of X, then "cl_{pi}" is a positive implicative (resp., commutative and implicative) weak closure operation on $\mathcal{I}(X)$.

Proof. Let A and C be ideals of X such that $A \subseteq C$. Suppose that A is a positive implicative (resp., commutative and implicative) ideal of X. Then C is a positive implicative (resp., commutative and implicative) ideal of X by Lemma 3.13. Let A and C be ideals of X such that $C \subseteq A$. If A is not a positive implicative (resp., commutative and implicative) ideal of X, then C is not a positive implicative (resp., commutative and implicative) ideal of X. Therefore "*cl*" is a positive implicative (resp., commutative and implicative) weak closure operation on $\mathcal{I}(X)$. □

The following examples illustrate Theorem 3.27.

Example 3.28. Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table,

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	1
2	2	2	0	0	2
3	3	2	1	0	3
4	4	4	4	4	0

There are six ideals in X, that is, $A_0 = \{0\}$, $A_1 = \{0, 1\}$, $A_2 = \{0, 4\}$, $A_3 = \{0, 1, 2, 3\}$, $A_4 = \{0, 1, 4\}$ and $A_5 = X$ in which A_1, A_3, A_4 and A_5 are positive implicative ideals and A_0 and A_2 are not positive implicative ideals. Now define "*cl*" as follows:

$$A_0^{cl} = A_0, A_1^{cl} = A_3, A_2^{cl} = A_4, A_3^{cl} = A_3, A_4^{cl} = A_5 \text{ and } A_5^{cl} = X.$$

Then "*cl*" is a weak closure operation. Now let $A = \{0, 4\} = A_2$ which is not a positive implicative ideal. By using Theorem 3.27 we have "cl_{pi} as follows:

$$A_0^{cl_{pi}} = A_0,\ A_1^{cl_{pi}} = A_1,\ A_2^{cl_{pi}} = A_4,\ A_3^{cl_{pi}} = A_3,\ A_4^{cl_{pi}} = A_5 \text{ and } A_5^{cl_{pi}} = X.$$

Clearly, $cl \leq cl_{pi}$. But, "cl_{pi}" is not a positive implicative weak closure operation because $A_2^{cl_{pi}} = A_4 \neq A_2$. Now let $A = \{0, 1\} = A_1$ which is a positive implicative ideal. By using Theorem 3.27 we have "cl_{pi}" as follows:

$$A_0^{cl_{pi}} = A_0,\ A_1^{cl_{pi}} = A_3,\ A_2^{cl_{pi}} = A_2,\ A_3^{cl_{pi}} = A_3,\ A_4^{cl_{pi}} = A_5 \text{ and } A_5^{cl_{pi}} = X.$$

Clearly, $cl \leq cl_{pi}$ and "cl_{pi}" is a positive implicative weak closure operation.

Example 3.29. Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	2	1	0	0
4	4	4	4	4	0

There are five ideals in X, that is, $A_0 = \{0\}$, $A_1 = \{0, 1\}$, $A_2 = \{0, 2\}$, $A_3 = \{0, 1, 2, 3\}$, and $A_4 = X$ in which A_3 and A_4 are commutative ideals and A_0, A_1 and A_2 are not commutative ideals. Now define "cl" as follows:

$$A_0^{cl} = A_1, A_1^{cl} = A_3, A_2^{cl} = A_3, A_3^{cl} = A_4 \text{ and } A_4^{cl} = X.$$

Then "cl" is a weak closure operation. Now let $A = \{0, 1\} = A_1$ which is not a commutative ideal. By using Theorem 3.27 we have "cl_c as follows:

$$A_0^{cl_c} = A_0, A_1^{cl_c} = A_3, A_2^{cl_c} = A_2, A_3^{cl_c} = A_4 \text{ and } A_4^{cl_c} = X.$$

Clearly, $cl \leq cl_c$. But, "cl_c" is not a commutative weak closure operation because $A_1^{cl_c} = A_3 \neq A_1$. Now let $A = \{0, 1, 2, 3\} = A_3$ which is a commutative ideal. By using Theorem 3.27 we have "cl_c" as follows:

$$A_0^{cl_c} = A_0, A_1^{cl_c} = A_1, A_2^{cl_c} = A_2, A_3^{cl_c} = A_4 \text{ and } A_4^{cl_c} = X.$$

Clearly, $cl \leq cl_c$ and "cl_c" is a commutative weak closure operation.

Example 3.30. Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	2
3	3	2	1	0	2
4	4	4	4	4	0

There are six ideals in X, that is, $A_0 = \{0\}$, $A_1 = \{0, 2\}$, $A_2 = \{0, 1\}$, $A_3 = \{0, 1, 2, 3\}$, $A_4 = \{0, 1, 4\}$ and $A_5 = X$ in which A_2, A_3, A_4 and A_5 are implicative ideals and A_0 and A_1 are not implicative ideals. Now define "cl" as follows:

$$A_0^{cl} = A_1, A_1^{cl} = A_3, A_2^{cl} = A_4, A_3^{cl} = A_5, A_4^{cl} = A_4 \text{ and } A_5^{cl} = X.$$

Then "cl" is a weak closure operation. Now let $A = \{0, 2\} = A_1$ which is not an implicative ideal. By using Theorem 3.27 we have "cl_m" as follows:

$$A_0^{cl_m} = A_0, A_1^{cl_m} = A_3, A_2^{cl_m} = A_2, A_3^{cl_m} = A_5, A_4^{cl_m} = A_4 \text{ and } A_5^{cl_m} = X.$$

Clearly, $cl \leq cl_m$. But, "cl_m" is not an implicative weak closure operation because $A_1^{cl_m} = A_3 \neq A_1$. Now let $A = \{0, 1\} = A_2$ which is an implicative ideal. By using Theorem 3.27 we have "cl_m" as follows:

$$A_0^{cl_m} = A_0, A_1^{cl_m} = A_1, A_2^{cl_m} = A_4, A_3^{cl_m} = A_5, A_4^{cl_m} = A_4 \text{ and } A_5^{cl_m} = X.$$

Clearly, $cl \leq cl_m$ and "cl_m" is an implicative weak closure operation.

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Communication between relation information systems*

Funing Lin† Shenggang Li‡

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Abstract: Communication between information systems is considered as an important issue in granular computing. A relation information system is the generalization of an information system. This paper investigates communication between relation information systems and obtain some invariant characterizations of relation information systems under homomorphism.

Keywords: Relation information system; Reduction; Consistent function; Relation mapping; Homomorphism.

1 Introduction

Rough set theory, proposed by Pawlak [17], is an important tool for dealing with fuzzyness and uncertainty of knowledge and has become an active branch of information science. With more than thirty years development, rough set theory has been successfully applied to machine learning, intelligent systems, inductive reasoning, pattern recognition, mereology, image processing, signal analysis, knowledge discovery, decision analysis, expert systems and many other fields [13, 14, 15, 16].

Communication between information systems is a very important topic in the field of artificial intelligence. In mathematics, it can be explained as a mapping between information systems. The approximations and reductions in the original system can be regarded as encoding while the image system is seen as an interpretive system. The concept of homomorphisms as a kind of tool to study relationships between information systems with rough sets was introduced by Grzymala-Busse [1, 2]. A homomorphism can be viewed as a special communication between two information systems. As explained in [23], homomorphisms allow one to translate the information contained in one granular world into the

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†Corresponding Author, College of Mathematics and Information Science, Shaanxi Normal University, Xi'an, Shaanxi 710062, P.R.China; linfuning518@126.com.

‡College of Mathematics and Information Science, Shaanxi Normal University, Xi'an, Shaanxi 710062, P.R.China; shengganglinew@126.com.

granularity of another granular world and thus provide a communication mechanism for exchanging information with other granular worlds. Li et al. [5] studied invariant characters of information systems under some homomorphism. Wang et al. [20, 21] introduced the notions of consistent functions, relation mappings and relation information systems which are the generalization of information systems. By using these notions, they proposed the homomorphisms as a mechanism for communicating between relation information systems. Zhu et al. [26] obtained some improved results on communication between relation information systems. Li et al. [12] investigated communication between knowledge bases. It should be pointed out that some other related works investigating information systems through homomorphisms [1, 2, 3, 5, 25] are based on equivalence relations or other particular relations and are quite different from [20, 21, 26].

The purpose of this paper is to investigate some invariant characterizations of relation information systems under homomorphisms.

2 Preliminaries

In this section, we recall some basic concepts on consistent functions, relation mappings and relation information systems.

Throughout this paper, U denotes a non-empty finite set called the universe, 2^U denotes the family of all subsets of U, $2^{U\times U}$ denotes the family of all binary relations on U, All mappings are assumed to be surjective.

For $R \in 2^{U\times U}$, the successor neighborhood of $x \in U$ with respect to R will be denoted by $R_s(x)$, that is, $R_s(x) = \{y \in U : xRy\}$ ([22]). Denote

$$U/R = \{R_s(x) : x \in U\}.$$

If R is an equivalence relation on U, then $\forall\ x \in U$, $R_s(x) = [x]_R$.

For $\mathcal{R} \subseteq 2^{U\times U}$, denote $ind(\mathcal{R}) = \bigcap_{R\in\mathcal{R}} R$.

2.1 Consistent functions

Definition 2.1 ([20, 21]). *Let U and V be two finite nonempty universes, $f\colon U \to V$ a mapping and $R \in 2^{U\times U}$. Define*

$$[x]_f = \{u \in U : f(u) = f(x)\},$$

$$(x)_R = \{u \in U : R_s(u) = R_s(x)\}.$$

Then $\{[x]_f : x \in U\}$ and $\{(x)_R : x \in U\}$ are two partitions on U. If $[x]_f \subseteq R_s(u)$ or $[x]_f \cap R_s(u) \neq \emptyset$ for any $x, u \in U$, then f is called a type-1 consistent function with respect to R on U. If $[x]_f \subseteq (x)_R$ for any $x \in U$, then f is called a type-2 consistent function with respect to R on U.

Remark 2.2. (1) $\forall\ x \in U$, $[x]_f = f^{-1}(f(x))$.

(2) *If R is an equivalence relation on U, then $\forall\ x \in U$, $(x)_R = [x]_R$.*

(3) *If f is type-2 consistent with respect to R on U and $f(u) = f(x)$, then $R_s(u) = R_s(x)$.*

Obviously,

f is type-1 $\Longleftrightarrow$ If $[x]_f \cap R_s(y) \neq \emptyset$, then $[x]_f \subseteq R_s(y)$

$\Longleftrightarrow$ If $[x]_f \not\subseteq R_s(y)$, then $[x]_f \cap R_s(y) = \emptyset$,

f is type-2 $\Longleftrightarrow$ If $f(x_1) = f(x_2)$, then $R_s(x_1) = R_s(x_2)$.

2.2 Relation mappings

Definition 2.3 ([20, 21]). *Let $f : U \to V$ be a mapping. Define*

$$\hat{f} : 2^{U\times U} \to 2^{V\times V},\ R| \to \hat{f}(R) = \bigcup_{x\in U}(\{f(x)\} \times f(R_s(x)));$$

$$\hat{f}^{-1} : 2^{V\times V} \to 2^{U\times U},\ T| \to \hat{f}^{-1}(T) = \bigcup_{y\in V}(\{f^{-1}(y)\} \times f^{-1}(T_s(y))).$$

Then $\hat{f}$ and $\hat{f}^{-1}$ are called the relation mapping and inverse relation mapping induced by f, respectively.

Obviously,

$$y_1\hat{f}(R)y_2 \Longleftrightarrow\ \exists\ x_1, x_2 \in U,\ y_1 = f(x_1),\ y_2 = f(x_1) \text{ and } x_1Rx_2,$$

$$x_1\hat{f}^{-1}(T)x_2 \Longleftrightarrow \exists\ y_1, y_2 \in V,\ y_1 = f(x_1), y_2 = f(x_1) \text{ and } y_1Ty_2.$$

For $\mathcal{R} \subseteq 2^{U\times U}$, denote

$$\hat{f}(\mathcal{R}) = \{\hat{f}(R) \mid R \in \mathcal{R}\}.$$

Proposition 2.4 ([20]). *If $f : U \to V$ is both type-1 and type-2 consistent with respect to $R \in 2^{U\times U}$, then*

$$\hat{f}^{-1}(\hat{f}(R)) = R.$$

2.3 Relation information systems

Definition 2.5 ([13]). *An information system is a pair (U, A) of non-empty finite sets U and A, where U is a set of objects and A is a set of attributes; each attribute $a \in A$ is a function $a : U \to V_a$, where V_a is the set of values (called domain) of attribute a.*

If (U, A) is an information system and $B \subseteq A$, then an equivalence relation (or indiscernibility relation) R_B can be defined by

$$(x, y) \in R_B \iff a(x) = a(y),\ \forall\ a \in B.$$

Definition 2.6 ([20]). *A pair $(U, \mathcal{R})$ is called a relation information system, if $\mathcal{R} \subseteq 2^{U\times U}$.*

Definition 2.7. *Let (U, A) be an information system. Put*

$$\mathcal{R} = \{R_{\{a\}} :\ a \in A\}.$$

Then the pair $(U, \mathcal{R})$ is called the relation information system induced by (U, A).

3

Definition 2.8 ([20])**.** *Let $f\colon U \to V$ be a mapping and $\mathcal{R} \subseteq 2^{U\times U}$. If f is type-1 (resp. type-2) consistent with respect to R on U for every $R \in \mathcal{R}$, then f is called type-1 (resp. type-2) consistent with respect to $\mathcal{R}$ on U.*

Proposition 2.9 ([20])**.** *Let $f : U \to V$ be a mapping and $\mathcal{R} \subseteq 2^{U\times U}$. If f is both type-1 and type-2 consistent with respect to $\mathcal{R}$, then $\hat{f}(ind(\mathcal{R})) = ind(\hat{f}(\mathcal{R}))$.*

Proposition 2.10 ([20])**.** *Let $f : U \to V$ be a mapping and $\mathcal{R} \subseteq 2^{U\times U}$. If f is both type-1 and type-2 consistent with respect to $\mathcal{R}$, then $\hat{f}^{-1}(\hat{f}(ind(\mathcal{R})) = ind(\mathcal{R})$.*

Definition 2.11 ([20])**.** *Let $f : U \to V$ be a mapping and $\mathcal{R} \subseteq 2^{U\times U}$. Then the pair $(V, \hat{f}(\mathcal{R}))$ is called an f-induced relation information system of $(U, \mathcal{R})$.*

Definition 2.12 ([20])**.** *Let $(U, \mathcal{R})$ be a relation information system and $(V, \hat{f}(\mathcal{R}))$ an f-induced relation information system of $(U, \mathcal{R})$. If f is both type-1 and type-2 consistent with respect to $\mathcal{R}$ on U, then f is called a homomorphism from $(U, \mathcal{R})$ to $(V, \hat{f}(\mathcal{R}))$. We write*

$$(U, \mathcal{R}) \ \sim_f \ (V, \hat{f}(\mathcal{R})).$$

We often consider reductions in a relation information system by deleting unrelated or unimportant elements with the requirement of keeping the ability of classification.

Definition 2.13 ([20])**.** *Let $(U, \mathcal{R})$ be a relation information system and $\mathcal{P} \subseteq \mathcal{R}$.*

(1) *$\mathcal{P}$ is called a coordination subfamily of $\mathcal{R}$, if $ind(\mathcal{P}) = ind(\mathcal{R})$.*

(2) *$R \in \mathcal{P}$ is called independent in $\mathcal{P}$, if $ind(\mathcal{P} - \{R\}) \neq ind(\mathcal{P})$; $\mathcal{P}$ is called a independent subfamily of $\mathcal{R}$, if $\forall\ R \in \mathcal{P}$, R is independent in $\mathcal{P}$.*

(3) *$\mathcal{P}$ is called a reductions of $\mathcal{R}$, if $\mathcal{P}$ is both coordination and independent.*

In this paper, the set of all coordination subfamilies (resp., all reductions) of $\mathcal{R}$ is denoted by $co(\mathcal{R})$ (resp., $red(\mathcal{R})$).

Obviously,

$\mathcal{P} \in red(\mathcal{R}) \ \Leftrightarrow \mathcal{P} \in co(\mathcal{R})$ and $\forall\ \mathcal{Q} \subset \mathcal{P}$, $\mathcal{Q} \notin co(\mathcal{R})$.

3 Some results on reductions in relation information systems

Proposition 3.1. *Let $(U, \mathcal{R})$ be a relation information system. Then $red(\mathcal{R}) \neq \emptyset$.*

Proof. Suppose $\forall\ R \in \mathcal{R}$, $\mathcal{R} - \{R\} \notin co(\mathcal{R})$. Then $\mathcal{R} \in red(\mathcal{R})$.

Suppose $\exists\ R_1 \in \mathcal{R}$, $\mathcal{R} - \{R_1\} \in co(\mathcal{R})$. Then, we consider $\mathcal{R} - \{R_1\}$. Again suppose $\forall\ R \in \mathcal{R} - \{R_1\}$, $(\mathcal{R} - \{R_1\}) - \{R\} \notin co(\mathcal{R})$. Then $\mathcal{R} - \{R_1\} \in red(\mathcal{R})$. Again suppose $\exists\ R_2 \in \mathcal{R} - \{R_1\}$, $(\mathcal{R} - \{R_1\}) - \{R_2\} \in co(\mathcal{R})$. Then, we consider

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$\mathcal{R}-\{R_1,R_2\}$. Repeat this process. Since $\mathcal{R}$ is finite, we can find a reductions of $\mathcal{R}$.

Thus $red(\mathcal{R})\neq\emptyset$. □

Definition 3.2. *Let $(U,\mathcal{R})$ be a relation information system. Put*

$$\mathcal{D}(x,y)=\{R\in\mathcal{R}|(x,y)\notin R\}\ (x,y\in U).$$

Then

(1) $\mathcal{D}(x,y)$ is called is called the discernibility subfamily of $\mathcal{R}$ on x and y.

(2) $\mathfrak{D}(\mathcal{R})=(d_{ij})_{n\times n}$ is called the discernibility matrix of $\mathcal{R}$ where $U=\{x_1,x_2,\cdots,x_n\}$ and $d_{ij}=\mathcal{D}(x_i,x_j)$ $(1\leq i,j\leq n)$.

Example 3.3. *Let $U=\{x_1,x_2,x_3,x_4,x_5,x_6\}$. We consider the relation information system $(U,\mathcal{R})$ where $\mathcal{R}=\{R_1,R_2,R_3,R_4\}$ and*

$U/R_1=\{\{x_1,x_2,x_5\},\{x_3,x_4,x_6\}\}$,
$U/R_2=\{\{x_1,x_6\},\{x_2,x_3,x_4,x_5\}\}$,
$U/R_3=\{\{x_1,x_2,x_5,x_6\},\{x_3,x_4\}\}$,
$U/R_4=\{\{x_1,x_2,x_5\},\{x_3,x_4,x_6\}\}$.

We can obtain the discernibility matrix $\mathfrak{D}(\mathcal{R})$ as follows:

$$\begin{pmatrix} \emptyset & \{R_2\} & \mathcal{R} & \mathcal{R} & \{R_2\} & \{R_1,R_4\} \\ \{R_2\} & \emptyset & \{R_1,R_3,R_4\} & \{R_1,R_3,R_4\} & \emptyset & \{R_1,R_2,R_4\} \\ \mathcal{R} & \{R_1,R_3,R_4\} & \emptyset & \emptyset & \{R_1,R_3,R_4\} & \{R_2,R_3\} \\ \mathcal{R} & \{R_1,R_3,R_4\} & \emptyset & \emptyset & \{R_1,R_3,R_4\} & \{R_2,R_3\} \\ \{R_2\} & \emptyset & \{R_1,R_3,R_4\} & \{R_1,R_3,R_4\} & \emptyset & \{R_1,R_2,R_4\} \\ \{R_1,R_4\} & \{R_1,R_2,R_4\} & \{R_2,R_3\} & \{R_2,R_3\} & \{R_1,R_2,R_4\} & \emptyset \end{pmatrix}$$

Discernibility family can expediently judge coordination families and reductions.

Proposition 3.4. *Let $(U,\mathcal{R})$ be a relation information system. Then*

$$\mathcal{P}\in co(\mathcal{R})\iff If\ (x,y)\notin ind(\mathcal{R}),\ then\ \mathcal{P}\cap\mathcal{D}(x,y)\neq\emptyset.$$

Proof. (1) "$\Longrightarrow$". Let $(x,y)\notin ind(\mathcal{R})$. Since $\mathcal{P}\in co(\mathcal{R})$, we have $ind(\mathcal{P})=ind(\mathcal{R})$. Then $(x,y)\notin ind(\mathcal{P})$. It follows $(x,y)\notin P\ for\ some\ P\in\mathcal{P}$.

$(x,y)\notin P$ implies $P\in\mathcal{D}(x,y)$. Then $P\in\mathcal{P}\cap\mathcal{D}(x,y)$.

Thus $\mathcal{P}\cap\mathcal{D}(x,y)\neq\emptyset$.

"$\Longleftarrow$". Suppose $\mathcal{P}\notin co(\mathcal{R})$. Then $ind(\mathcal{P})\neq ind(\mathcal{R})$. It follows $ind(\mathcal{P})-ind(\mathcal{R})\neq\emptyset$. Pick

$$(x,y)\in ind(\mathcal{P})-ind(\mathcal{R}).$$

Since $(x,y)\notin ind(\mathcal{R})$, we have $\mathcal{P}\cap\mathcal{D}(x,y)\neq\emptyset$.

Note that $(x,y)\in ind(\mathcal{P})$. Then $\forall\ P\in\mathcal{P}$, $(x,y)\in P$. So $P\notin\mathcal{D}(x,y)$. Thus $\mathcal{P}\cap\mathcal{D}(x,y)=\emptyset$. This is a contradiction.

Thus $\mathcal{P}\in co(\mathcal{R})$. □

Theorem 3.5. *Let $(U,\mathcal{R})$ be a relation information system. Then $\mathcal{P} \in red(\mathcal{R})$ $\Longleftrightarrow$ (1) If $(x,y) \notin ind(\mathcal{R})$, then $\mathcal{P} \cap \mathcal{D}(x,y) \neq \emptyset$;*
(2) $\forall\ R \in \mathcal{P}$, $\exists\ (x_R, y_R) \in ind(\mathcal{R})$, $(\mathcal{P} - \{R\}) \cap \mathcal{D}(x_R, y_R) = \emptyset$.

Proof. This holds by Proposition 3.4. □

Definition 3.6. *Let $(U,\mathcal{R})$ be a relation information system. Put*

$$core(\mathcal{R}) = \bigcap_{\mathcal{P} \in red(\mathcal{R})} \mathcal{P}.$$

Then $core(\mathcal{R})$ is called the core of $\mathcal{R}$. Moreover,
(1) $R \in \mathcal{R}$ is called necessary, if $R \in core(\mathcal{R})$.
(2) $R \in \mathcal{R}$ is called relatively necessary, if $R \in \bigcup_{\mathcal{P} \in red(\mathcal{R})} \mathcal{P} - core(\mathcal{R})$.
(3) $R \in \mathcal{R}$ is called unnecessary, if $R \in \mathcal{R} - \bigcup_{\mathcal{P} \in red(\mathcal{R})} \mathcal{P}$.

Discernibility family can easily determine the core.

Proposition 3.7. *Let $(U,\mathcal{R})$ be a relation information system. The following are equivalent:*
(1) R is necessary;
(2) R is independent in $\mathcal{R}$;
(3) $\exists\ x, y \in U$, $\mathcal{D}(x,y) = \{R\}$.

Proof. (1) $\Longrightarrow$ (2). Suppose that R is not independent in $\mathcal{R}$. Then

$$ind(\mathcal{R} - \{R\}) = ind(\mathcal{R}).$$

It follows $\mathcal{R} - \{R\} \in co(\mathcal{R})$. Consider $\mathcal{R} - \{R\}$. By Proposition 3.1, $\exists\ \mathcal{P} \subseteq \mathcal{R} - \{R\}$, $\mathcal{P} \in red(\mathcal{R})$.

$\mathcal{P} \subseteq \mathcal{R} - \{R\}$ implies $R \notin \mathcal{P}$. Then R is not necessary. This is a contradiction.

(2) $\Longrightarrow$ (1). Suppose that R is not necessary. Then $\exists\ \mathcal{P} \in red(\mathcal{R})$, $R \notin \mathcal{P}$. So $\mathcal{P} \subseteq \mathcal{R} - \{R\} \subseteq \mathcal{R}$. It follows

$$ind(\mathcal{P}) \supseteq ind(\mathcal{R} - \{R\}) \supseteq ind(\mathcal{R}).$$

By $\mathcal{P} \in red(\mathcal{R})$, $ind(\mathcal{P}) = ind(\mathcal{R})$. Then $ind(\mathcal{R} - \{R\}) = ind(\mathcal{R})$. So R is not independent in $\mathcal{R}$. This is a contradiction.

(2) $\Longrightarrow$ (3). Since R is independent in $\mathcal{R}$, we have $ind(\mathcal{R} - \{R\}) \neq ind(\mathcal{R})$. Then $ind(\mathcal{R} - \{R\}) - ind(\mathcal{R}) \neq \emptyset$. Pick

$$(x,y) \in ind(\mathcal{R} - \{R\}) - ind(\mathcal{R}).$$

Denote $\mathcal{R} = \{R_1, R_2, \ldots, R_n\}$. Then $R = R_j$ for some $j \leq n$. So

$$(x,y) \in \bigcap_{1 \leq i \leq n, i \neq j} R_i - \bigcap_{1 \leq i \leq n} R_i.$$

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It follows $(x,y) \notin R_j$ and $(x,y) \in R_i$ $(i \neq j)$.

Thus $\mathcal{D}(x,y) = \{R\}$.

(3) $\Longrightarrow$ (2). Since $\exists\ x, y \in U$, $\mathcal{D}(x,y) = \{R\}$, we have

$$(x,y) \notin R,\ (x,y) \in R^{'}\ (R^{'} \neq R).$$

Then $(x,y) \in ind(\mathcal{R} - \{R\})$. But $(x,y) \notin ind(\mathcal{R})$.

Thus $ind(\mathcal{R} - \{R\}) \neq ind(\mathcal{R})$.

Hence R is independent in $\mathcal{R}$. □

Proposition 3.8. *Let $(U, \mathcal{R})$ be a relation information system. Denote*

$$R^{\star} = \bigcup_{\mathcal{P} \in co(\mathcal{R})} ind(\mathcal{P} - \{R\}).$$

Then the following are equivalent.

(1) *R is unnecessary;*

(2) $\forall\ \mathcal{P} \in co(\mathcal{R})$, $\mathcal{P} - \{R\} \in co(\mathcal{R})$;

(3) *$R^{\star} = ind(\mathcal{R})$;*

(4) *$R^{\star} \subseteq R$.*

Proof. (1) $\Longrightarrow$ (2). By Proposition 3.1, $\exists\ \mathcal{Q} \subseteq \mathcal{P}$, $\mathcal{Q} \in red(\mathcal{R})$. Since R is unnecessary, we have $R \notin \mathcal{Q}$. It follows $\mathcal{Q} \subseteq \mathcal{R} - \{R\}$. Then

$$\mathcal{Q} \subseteq \mathcal{P} \cap (\mathcal{R} - \{R\}) = \mathcal{P} - \{R\} \subseteq \mathcal{P}.$$

We have

$$ind(\mathcal{Q}) \supseteq ind(\mathcal{R} - \{R\}) \supseteq ind(\mathcal{P}).$$

Note that $\mathcal{P} \in co(\mathcal{R})$ and $\mathcal{Q} \in red(\mathcal{R})$. Then $ind(\mathcal{P}) = ind(\mathcal{R}) = ind(\mathcal{Q})$.

Thus $ind(\mathcal{P} - \{R\}) = ind(\mathcal{R})$. This shows $\mathcal{P} - \{R\} \in co(\mathcal{R})$.

(2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) are obvious.

(4) $\Longrightarrow$ (1). Suppose $\exists\ \mathcal{P} \in red(\mathcal{R})$, $R \in \mathcal{P}$. Then $\mathcal{P} - \{R\} \subset \mathcal{P}$. Since $\mathcal{P} \in red(\mathcal{R})$, we have $\mathcal{P} - \{R\} \notin co(\mathcal{R})$. Then $ind(\mathcal{P} - \{R\}) - ind(\mathcal{R}) \neq \emptyset$. $\mathcal{P} \in red(\mathcal{R})$ implies $ind(\mathcal{P}) = ind(\mathcal{R})$. Then

$$ind(\mathcal{P} - \{R\}) - ind(\mathcal{P}) \neq \emptyset.$$

Pick $(x,y) \in ind(\mathcal{P} - \{R\}) - ind(\mathcal{P})$. Note that $ind(\mathcal{P}) = ind(\mathcal{P} - \{R\}) \cap R$. Then $(x,y) \notin R$.

Since $\mathcal{P} \in co(\mathcal{R})$ and $R^{\star} \subseteq R$, we have $ind(\mathcal{P} - \{R\}) \subseteq R$. Then $(x,y) \in R$. This is a contradiction.

Thus R is unnecessary. □

Theorem 3.9. *Let $(U, \mathcal{R})$ be a relation information system. Then*

(1) *R is necessary $\Leftrightarrow \mathcal{R} - \{R\} \notin co(\mathcal{R})$.*

(2) *R is relatively necessary $\Leftrightarrow \mathcal{R} - \{R\} \in co(\mathcal{R})$ and $R^{\star} \nsubseteq R$.*

(3) *R is unnecessary $\Leftrightarrow R^{\star} \subseteq R$.*

Proof. This holds by Proposition 3.7 and Proposition 3.8. □

Example 3.10. *In Example 3.3, we have*
(1) R_2 *is necessary.*
(2) R_1 *and* R_4 *are relatively necessary.*
(3) R_3 *is unnecessary.*
(4) $red(\mathcal{R}) = \{\{R_1, R_2\}, \{R_2, R_4\}\}$, $core(\mathcal{R}) = \{R_2\}$.

4 Communication between relation information systems

Proposition 4.1. *Let* $(U, \mathcal{R}) \sim_f (V, \hat{f}(\mathcal{R}))$. *Then*
(1) $\mathcal{P} \in co(\mathcal{R}) \iff \hat{f}(\mathcal{P}) \in co(\hat{f}(\mathcal{R}))$.
(2) $co(\hat{f}(\mathcal{R})) = \hat{f}(co(\mathcal{R}))$.

Proof. (1) "$\Longrightarrow$". Since $\mathcal{P} \in co(\mathcal{R})$, we have $ind(\mathcal{P}) = ind(\mathcal{R})$. Then

$$\hat{f}(ind(\mathcal{P})) = \hat{f}(ind(\mathcal{R})).$$

By Proposition 2.6,

$$ind(\hat{f}(\mathcal{P})) = ind(\hat{f}(\mathcal{R})).$$

Thus $\hat{f}(\mathcal{P}) \in co(\hat{f}(\mathcal{R}))$.
"$\Longleftarrow$". Since $\hat{f}(\mathcal{P}) \in co(\hat{f}(\mathcal{R}))$, we have

$$ind(\hat{f}(\mathcal{P})) = ind(\hat{f}(\mathcal{R})).$$

By Proposition 2.6,

$$\hat{f}(ind(\mathcal{P})) = \hat{f}(ind(\mathcal{R})).$$

Then

$$\hat{f}^{-1}(\hat{f}(ind(\mathcal{P}))) = \hat{f}^{-1}(\hat{f}(ind(\mathcal{R}))).$$

By Proposition 2.7, $ind(\mathcal{P}) = ind(\mathcal{R})$.
Thus $\mathcal{P} \in co(\mathcal{R})$.
(2) By (1),

$$\begin{aligned} \hat{f}(co(\mathcal{R})) &= \{\hat{f}(\mathcal{P}) | \mathcal{P} \in co(\mathcal{R})\} \\ &= \{\hat{f}(\mathcal{P}) | \hat{f}(\mathcal{P}) \in co(\hat{f}(\mathcal{R}))\} \\ &= co(\hat{f}(\mathcal{R})). \end{aligned}$$

□

Theorem 4.2. *Let* $(U, \mathcal{R}) \sim_f (V, \hat{f}(\mathcal{R}))$. *Then*
(1) $\mathcal{P} \in red(\mathcal{R}) \iff \hat{f}(\mathcal{P}) \in red(\hat{f}(\mathcal{R}))$.
(2) $red(\hat{f}(\mathcal{R})) = \hat{f}(red(\mathcal{R}))$.

Proof. (1) "$\Longrightarrow$". Since $\mathcal{P} \in red(\mathcal{R})$, we have $\mathcal{P} \in co(\mathcal{R})$. By Proposition 4.1, $\hat{f}(\mathcal{P}) \in co(\hat{f}(\mathcal{R}))$.

$\forall\ \mathcal{T} \subset \hat{f}(\mathcal{P})$. Pick $\mathcal{Q} \subseteq \mathcal{R}$, $\mathcal{T} = \hat{f}(\mathcal{Q})$. Then $\hat{f}(\mathcal{Q}) \subset \hat{f}(\mathcal{P})$. By Proposition 2.4,

$$\mathcal{Q} = \hat{f}^{-1}(\hat{f}(\mathcal{Q})) \subseteq \hat{f}^{-1}(\hat{f}(\mathcal{P})) = \mathcal{P}.$$

Suppose $\mathcal{Q} = \mathcal{P}$. Then $\mathcal{T} = \hat{f}(\mathcal{Q}) = \hat{f}(\mathcal{P})$. This is a contradiction.

Thus $\mathcal{Q} \subset \mathcal{P}$.

Since $\mathcal{P} \in red(\mathcal{R})$, we have $\mathcal{Q} \notin co(\mathcal{R})$. By Proposition 4.1, $\mathcal{T} = \hat{f}(\mathcal{Q}) \notin co(\hat{f}(\mathcal{R}))$.

Hence $\hat{f}(\mathcal{P}) \in red(\hat{f}(\mathcal{R}))$.

"$\Longleftarrow$". Since $\hat{f}(\mathcal{P}) \in red(\hat{f}(\mathcal{R}))$, we have $\hat{f}(\mathcal{P}) \in co(\hat{f}(\mathcal{R}))$. By Proposition 4.1, $\mathcal{P} \in co(\mathcal{R})$.

$\forall\ \mathcal{Q} \subset \mathcal{P}$, $\hat{f}(\mathcal{Q}) \subseteq \hat{f}(\mathcal{P})$. Suppose $\hat{f}(\mathcal{Q}) = \hat{f}(\mathcal{P})$. By Proposition 2.4,

$$\mathcal{Q} = \hat{f}^{-1}(\hat{f}(\mathcal{Q})) = \hat{f}^{-1}(\hat{f}(\mathcal{P})) = \mathcal{P}.$$

This is a contradiction. Thus $\hat{f}(\mathcal{Q}) \subset \hat{f}(\mathcal{P})$.

Since $\hat{f}(\mathcal{P}) \in red(\hat{f}(\mathcal{R}))$, we have $\hat{f}(\mathcal{Q}) \notin co(\hat{f}(\mathcal{R}))$. By Proposition 4.1, $\mathcal{Q} \notin co(\mathcal{R})$.

Hence $\mathcal{P} \in red(\mathcal{R})$.

(2) By (1),

$$\begin{aligned}\hat{f}(red(\mathcal{R})) &= \{\hat{f}(\mathcal{P}) | \mathcal{P} \in red(\mathcal{R})\}\\ &= \{\hat{f}(\mathcal{P}) | \hat{f}(\mathcal{P}) \in red(\hat{f}(\mathcal{R}))\}\\ &= red(\hat{f}(\mathcal{R})).\end{aligned}$$

□

Remark 4.3. *Theorem 3.20(1) is Theorem 4.4 in [20]. We just prove this result from another angle.*

Lemma 4.4. *Let* $(U, \mathcal{R}) \ \sim_f \ (V, \hat{f}(\mathcal{R}))$. *Then*

$$\hat{f}(\mathcal{R} - \{R\}) = \hat{f}(\mathcal{R}) - \{\hat{f}(R)\}.$$

Proof. $\forall\ S \in \mathcal{R} - \{R\}$, $S \neq R$. By Proposition 2.4, $\hat{f}(S) \neq \hat{f}(R)$. It follows $\hat{f}(S) \in \hat{f}(\mathcal{R}) - \{\hat{f}(R)\}$. Thus

$$\hat{f}(\mathcal{R} - \{R\}) \subseteq \hat{f}(\mathcal{R}) - \{\hat{f}(R)\}.$$

On the other hand, $\forall\ T \in \hat{f}(\mathcal{R}) - \{\hat{f}(R)\}$, $T = \hat{f}(S)$ for some $S \in \mathcal{R}$. $T \notin \{\hat{f}(R)\}$ implies $\hat{f}(S) \neq \hat{f}(R)$. Then $S \neq R$. So $S \in \mathcal{R} - \{R\}$. It follows $T \in \hat{f}(\mathcal{R} - \{R\})$. Thus

$$\hat{f}(\mathcal{R} - \{R\}) \supseteq \hat{f}(\mathcal{R}) - \{\hat{f}(R)\}.$$

Hence $\hat{f}(\mathcal{R} - \{R\}) = \hat{f}(\mathcal{R}) - \{\hat{f}(R)\}$. □

Theorem 4.5. *Let* $(U,\mathcal{R}) \sim_f (V,\hat{f}(\mathcal{R}))$. *Then*

$$R \in core(\mathcal{R}) \iff \hat{f}(R) \in core(\hat{f}(\mathcal{R})).$$

Proof. This holds by Theorem 3.9(1), Proposition 4.1(1) and Lemma 4.4. □

Theorem 4.6. *Let* $(U,\mathcal{R}) \sim_f (V,\hat{f}(\mathcal{R}))$. *Then*

$$\hat{f}(core(\mathcal{R})) = core(\hat{f}(\mathcal{R})).$$

Proof. By Theorem 3.23,

$$\begin{aligned}\hat{f}(core(\mathcal{R})) &= \{\hat{f}(R)|R \in core(\mathcal{R})\}\\ &= \{\hat{f}(R)|\hat{f}(R) \in core(\hat{f}(\mathcal{R}))\}\\ &= core(\hat{f}(\mathcal{R})).\end{aligned}$$

□

Theorem 4.7. *Let* $(U,\mathcal{R}) \sim_f (V,\hat{f}(\mathcal{R}))$. *Then*

$$R \textit{ is unnecessary} \Leftrightarrow \hat{f}(R) \textit{ is unnecessary.}$$

Proof. "$\Longrightarrow$". $\forall\, \mathcal{T} \in co(\hat{f}(\mathcal{R}))$, pick $\mathcal{P} \subseteq \mathcal{R}$, $\mathcal{T} = \hat{f}(\mathcal{P})$. Then $\hat{f}(\mathcal{P}) \in co(\hat{f}(\mathcal{R}))$. By Proposition 3.19(1), $\mathcal{P} \in co(\mathcal{R})$.

Since R is unnecessary, by Proposition 3.8, we have $\mathcal{P} - \{R\} \in co(\mathcal{R})$. Then $ind(\mathcal{P} - \{R\}) = ind(\mathcal{R})$. By Proposition 2.6 and Lemma 4.4,

$$ind(\hat{f}(\mathcal{P}) - \{\hat{f}(R)\}) = ind(\hat{f}(\mathcal{P} - \{R\}) = \hat{f}(ind(\mathcal{P} - \{R\})),$$

$$ind(\hat{f}(\mathcal{R})) = \hat{f}(ind(\mathcal{R})).$$

Then $ind(\mathcal{T} - \{\hat{f}(R)\}) = ind(\hat{f}(\mathcal{R}))$. This implies $\mathcal{T} - \{\hat{f}(R)\} \in co(\hat{f}(\mathcal{R}))$.

By Proposition 3.8, $\hat{f}(R)$ is unnecessary.

"$\Longleftarrow$". $\forall\, \mathcal{P} \in co(\mathcal{R})$, by Proposition 4.1(1), $\hat{f}(\mathcal{P}) \in co(\hat{f}(\mathcal{R}))$.

Since $\hat{f}(R)$ is unnecessary, by Proposition 3.8, we have

$$\hat{f}(\mathcal{P}) - \{\hat{f}(R)\} \in co(\hat{f}(\mathcal{R})).$$

Then

$$ind(\hat{f}(\mathcal{P}) - \{\hat{f}(R)\}) = ind(\hat{f}(\mathcal{R})).$$

By Proposition 2.6 and Lemma 4.4,

$$\hat{f}(ind(\mathcal{P} - \{R\})) = ind(\hat{f}(\mathcal{P} - \{R\}) = ind(\hat{f}(\mathcal{P}) - \{\hat{f}(R)\}),$$

$$\hat{f}(ind(\mathcal{R})) = ind(\hat{f}(\mathcal{R})).$$

Then $\hat{f}(ind(\mathcal{P} - \{R\})) = \hat{f}(ind(\mathcal{R}))$.

By Proposition 2.7,

$$ind(\mathcal{P} - \{R\}) = \hat{f}^{-1}(\hat{f}(ind(\mathcal{P} - \{R\}))) = \hat{f}^{-1}(\hat{f}(ind(\mathcal{R}))) = ind(\mathcal{R}).$$

Then $\mathcal{P} - \{R\} \in co(\mathcal{R})$.

By Proposition 3.8, R is unnecessary. □

Corollary 4.8. *Let* $(U, \mathcal{R}) \sim_f (V, \hat{f}(\mathcal{R}))$. *Then*

$$R \text{ is relatively necessary} \iff \hat{f}(R) \text{ is relatively necessary.}$$

Proof. This hods by Theorem 4.5 and Theorem 4.7. □

Example 4.9. *Let* $U = \{x_i | 1 \leq i \leq 15\}$. *We consider the relation information system* $(U, \mathcal{R})$ *where* $\mathcal{R} = \{R_1, R_2, R_3, R_4\}$,

$U/R_1 = \{\{x_1, x_2, x_4, x_7, x_8, x_9, x_{10}, x_{11}\}, \{x_3, x_5, x_6, x_{12}, x_{13}, x_{14}, x_{15}\}\}$,
$U/R_2 = \{\{x_1, x_4, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}\}, \{x_2, x_3, x_5, x_6, x_7, x_8, x_9, x_{10}\}\}$,
$U/R_3 = \{\{x_1, x_2, x_4, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}\}, \{x_3, x_5, x_6\}\}$,
$U/R_4 = \{\{x_1, x_2, x_4, x_7, x_8, x_9, x_{10}, x_{11}\}, \{x_3, x_5, x_6, x_{12}, x_{13}, x_{14}, x_{15}\}\}$.

Let $V = \{y_1, y_2, y_3, y_4, y_5, y_6\}$. *Define a mapping as follows:*

x_1, x_4, x_{11}	x_2, x_8	x_3, x_6	x_5	x_7, x_9, x_{10}	$x_{12}, x_{13}, x_{14}, x_{15}$
y_1	y_2	y_3	y_4	y_5	y_6

.

Let $(V, \hat{f}(\mathcal{R}))$ *be the* f*-induced relation information system of* (U, R). *It is very easy to verify that* f *is a homomorphism from* (U, R) *to* $(V, \hat{f}(\mathcal{R}))$.

We have $\hat{f}(\mathcal{R}) = \{\hat{f}(R_1), \hat{f}(R_3), \hat{f}(R_3), \hat{f}(R_4)\}$ *where*

$V/\hat{f}(R_1) = \{\{y_1, y_2, y_5\}, \{y_3, y_4, y_6\}\}$,
$V/\hat{f}(R_2) = \{\{y_1, y_6\}, \{y_2, y_3, y_4, y_5\}\}$,
$V/\hat{f}(R_3) = \{\{y_1, y_2, y_5, y_6\}, \{y_3, y_4\}\}$,
$V/\hat{f}(R_4) = \{\{y_1, y_2, y_5\}, \{y_3, y_4, y_6\}\}$.

By Example 3.10,

$$red(\hat{f}(\mathcal{R})) = \{\{\hat{f}(R_1), \hat{f}(R_2)\}, \{\hat{f}(R_2), \hat{f}(R_4)\}\}, \quad core(\hat{f}(\mathcal{R})) = \{\hat{f}(R_2)\}.$$

By Proposition 2.4, Theorem 4.2(2) and Theorem 4.6,

$$red(\mathcal{R}) = \{\{R_1, R_2\}, \{R_2, R_4\}\}, \quad core(\mathcal{R}) = \{R_2\}.$$

5 Conclusions

In this paper, we have investigated the original relation information system and image relation information system, and obtained some invariant characterizations of relation information systems under homomorphism. These results will be significant for establishing a framework of granular computing in knowledge bases and may have potential applications to knowledge discovery, decision making and reasoning about data. In the future, we will consider concrete applications of our results.

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Global stability in a discrete Lotka-Volterra competition model

Sangmok Choo[a], Young-Hee Kim[*,b]

[a]Department of Mathematics, University of Ulsan, Ulsan 44610, Korea.
[b] Division of General Education-Mathematics, Kwangwoon University, Seoul 01897, Korea

Abstract

We consider the Euler difference scheme for two-dimensional Lotka-Volterra competition equations and show that the difference scheme has positive and bounded solutions. In addition, we present sufficient conditions that the solutions of the scheme converge to the equilibrium points of the scheme. The convergence is shown based on the two approaches: first, partition of the domain used for the boundedness of the solutions and second, calculation of the movement of the species started in each partitioned region. Numerical examples are presented to verify the results.

Key words: Euler difference scheme, positivity, global stability, competition model

1. Introduction

The competition model in the two-dimensional case represents two species which are competing for a common resource; an additional term is included within the logistic prey growth Lotka-Volterra model to incorporate this interspecific competition for some limiting resource. This limiting resource can be anything for which supply is smaller than demand. The classic two-dimensional competition model is given by

$$\frac{dx}{dt} = x(t)(r_1 - a_{11}x(t) - a_{12}y(t)), \ \frac{dy}{dt} = y(t)(r_2 - a_{21}x(t) - a_{22}y(t)), \tag{1}$$

where $r_i > 0$ and $a_{ij} > 0$. Here $x(t)$ and $y(t)$ denote the population sizes or population density in the species x and y at time t; the parameters r_i's are the intrinsic growth rates for the two species x and y; a_{ii}'s measure the inhibiting effect on the two species; a_{12} and a_{21} are the interspecific acting coefficients.

The species x in the model (1) acts on y with functional response of type $a_{12}x(t)y(t)$. However other types of functional responses including Holling types [1–5], Beddington-DeAngelis type [6–8], Crowley-Martin type [9–11], and Ivlev-type of functional responses [12–14] have been applied to many population models

The dynamics of the model (1) is well-known [15–17] ; the solutions of (1) are positive and bounded, and the stability of the system (1) has been studied. There are a number of works on investigating continuous time Lokta-Volterra models, but relatively few theoretical papers are published on their discretized models [18–21]. The author in [22] has

*Corresponding author
Email addresses: smchoo@ulsan.ac.kr (Sangmok Choo), yhkim@kw.ac.kr (Young-Hee Kim)

introduced a method to present global stability in discrete Lokta-Volterra predator-prey models for the case that all species coexist at a unique equilibrium. In [23], the authors have shown the global stability of the Euler difference scheme for a three-dimensional predator-prey model using a new approach.

As far as we know, there is no theoretical research on the global stability of the discrete-time competition model of (1), so that we consider the Euler difference scheme

$$x_{n+1} = F_{y_n}(x_n),\ y_{n+1} = G_{x_n}(y_n),\ n \geq 0 \tag{2}$$

with

$$F_y(x) = x\{1 + (r_1 - a_{11}x - a_{12}y)\Delta t\}, \tag{3}$$
$$G_x(y) = y\{1 + (r_2 - a_{21}x - a_{22}y)\Delta t\}, \tag{4}$$

where Δt is a time step size, $x_n = x_0 + n\Delta t$ and $y_n = y_0 + n\Delta t$ with $(x_0, y_0) = (x(0), y(0))$.

The paper is organized as follows. Section 2 gives the positivity and boundedness of solutions of (2). In Section 3, we partition the domain used for the boundedness of the discrete solutions and find the geometric properties of the movement of the solutions starting in the partitioned regions. Using the properties, we present sufficient conditions that the solutions converge to equilibrium points of (2). In Section 4, some numerical examples are presented to verify our results.

2. Positivity of the discrete solutions

In this section, we consider the positivity and boundedness of the solutions of (2). Note that if τ_1 and τ_2 are positive constants satisfying

$$U_1(\tau_2) = \frac{1 + r_1\Delta t - a_{12}\tau_2\Delta t}{2a_{11}\Delta t} > 0,\quad U_2(\tau_1) = \frac{1 + r_2\Delta t - a_{21}\tau_1\Delta t}{2a_{22}\Delta t} > 0, \tag{5}$$

then

$$F_{\tau_2}(x), G_{\tau_1}(y) \text{ are increasing on } 0 \leq x \leq U_1(\tau_2),\ 0 \leq y \leq U_2(\tau_1). \tag{6}$$

For the positivity and boundedness of the solutions (x_n, y_n) we assume

$$\max\{r_1, r_2\} < 1/\Delta t \tag{7}$$

and consider constants x^* and y^* such that

$$r_1 a_{11}^{-1} \leq x^* \leq U_1(y^*),\ r_2 a_{22}^{-1} \leq y^* \leq U_2(x^*). \tag{8}$$

Remark 1. For every point (x^*, y^*) satisfying

$$\frac{r_1}{a_{11}} \leq x^* \leq \min\left\{\frac{1 + r_1\Delta t}{4a_{11}\Delta t}, \frac{1 + r_2\Delta t}{2a_{21}\Delta t}\right\},\quad \frac{r_2}{a_{22}} \leq y^* \leq \min\left\{\frac{1 + r_1\Delta t}{2a_{12}\Delta t}, \frac{1 + r_2\Delta t}{4a_{22}\Delta t}\right\}, \tag{9}$$

the two conditions in (8) hold since

$$\begin{aligned} U_1(y^*) &= \frac{1+r_1\Delta t - a_{12}y^*\Delta t}{2a_{11}\Delta t} \geq \frac{1+r_1\Delta t - a_{12}\min\left\{\frac{1+r_1\Delta t}{2a_{12}\Delta t},\ \frac{1+r_2\Delta t}{4a_{22}\Delta t}\right\}\Delta t}{2a_{11}\Delta t} \\ &= \frac{1+r_1\Delta t}{2a_{11}\Delta t} - \frac{a_{12}}{2a_{11}}\min\left\{\frac{1+r_1\Delta t}{2a_{12}\Delta t},\ \frac{1+r_2\Delta t}{4a_{22}\Delta t}\right\} \\ &= \max\left\{\frac{1+r_1\Delta t}{4a_{11}\Delta t},\ \frac{1+r_1\Delta t}{2a_{11}\Delta t} - \frac{a_{12}}{2a_{11}}\frac{1+r_2\Delta t}{4a_{22}\Delta t}\right\} \geq \frac{1+r_1\Delta t}{4a_{11}\Delta t} \\ &\geq \min\left\{\frac{1+r_1\Delta t}{4a_{11}\Delta t},\ \frac{1+r_2\Delta t}{2a_{21}\Delta t}\right\} \geq x^* \end{aligned}$$

and

$$\begin{aligned} U_2(x^*) &= \frac{1+r_2\Delta t - a_{21}x^*\Delta t}{2a_{22}\Delta t} \geq \frac{1+r_2\Delta t - a_{21}\min\left\{\frac{1+r_1\Delta t}{4a_{11}\Delta t},\ \frac{1+r_2\Delta t}{2a_{21}\Delta t}\right\}\Delta t}{2a_{22}\Delta t} \\ &= \frac{1+r_2\Delta t}{2a_{22}\Delta t} - \frac{a_{21}}{2a_{22}}\min\left\{\frac{1+r_1\Delta t}{4a_{11}\Delta t},\ \frac{1+r_2\Delta t}{2a_{21}\Delta t}\right\} \\ &= \max\left\{\frac{1+r_2\Delta t}{2a_{22}\Delta t} - \frac{a_{21}}{2a_{22}}\frac{1+r_1\Delta t}{4a_{11}\Delta t},\ \frac{1+r_2\Delta t}{4a_{22}\Delta t}\right\} \geq \frac{1+r_2\Delta t}{4a_{22}\Delta t} \\ &\geq \min\left\{\frac{1+r_1\Delta t}{2a_{12}\Delta t},\ \frac{1+r_2\Delta t}{4a_{22}\Delta t}\right\} \geq y^*. \end{aligned}$$

Using x^* and y^* in (8), we can obtain the positivity and boundedness of (x_n, y_n).

Theorem 1. *Let (x_n, y_n) be the solution of (2). Assume that (7) and (8) hold.*

If $(x_0, y_0) \in (0, x^*) \times (0, y^*)$, then $(x_n, y_n) \in (0, x^*) \times (0, y^*)$ for all n.

Proof. Using the condition in this theorem and (5), we have

$$0 < x_0 < x^* \leq U_1(y^*) < U_1(y_0),\ 0 < y_0 < y^* \leq U_2(x^*) < U_2(x_0), \tag{10}$$

and then the increasing property (6) gives the positivity of x_1 and y_1:

$$x_1 = F_{y_0}(x_0) > F_{y_0}(0) = 0, \quad y_1 = G_{x_0}(y_0) > G_{x_0}(0) = 0. \tag{11}$$

Now, we claim that $x_1 < x^*$ and $y_1 < y^*$. If $r_1 - a_{11}x_0 - a_{12}y_0 \leq 0$, then

$$x_1 = F_{y_0}(x_0) \leq x_0 < x^*.$$

Otherwise, we get

$$0 < x_0 < (r_1 - a_{12}y_0)a_{11}^{-1} < (1 + r_1\Delta t - a_{12}y_0\Delta t)(2a_{11}\Delta t)^{-1} = U_1(y_0),$$

where the last inequality is obtained from $r_1\Delta t < 1$ in (7). Hence (6) and (8) imply the boundedness of x_1:

$$x_1 = F_{y_0}(x_0) < F_{y_0}\left((r_1 - a_{12}y_0)a_{11}^{-1}\right) = (r_1 - a_{12}y_0)a_{11}^{-1} < r_1 a_{11}^{-1} \leq x^*. \tag{12}$$

Similarly if $r_2 - a_{21}x_0 - a_{22}y_0 \le 0$, then $y_1 = G_{x_0}(y_0) \le y_0 < y^*$. Otherwise, we have

$$0 < y_0 < (r_2 - a_{21}x_0)a_{22}^{-1} < (1 + r_2\Delta t - a_{21}x_0\Delta t)(2a_{22}\Delta t)^{-1} = U_2(x_0),$$

where the last inequality is obtained from $r_2\Delta t < 1$ in (7). Thus (6) and (8) imply the boundedness of y_1 that

$$y_1 = G_{x_0}(y_0) < G_{x_0}\left((r_2 - a_{21}x_0)a_{22}^{-1}\right) = (r_2 - a_{21}x_0)a_{22}^{-1} < r_2a_{22}^{-1} \le y^*. \tag{13}$$

Hence using (11), (12) and (13), we have that

$$\text{if } (x_0, y_0) \in (0, x^*) \times (0, y^*), \text{ then } (x_1, y_1) \in (0, x^*) \times (0, y^*).$$

Therefore, using the mathematical induction, we can obtain the desired result. □

Remark 2. Due to (9), we can choose sufficiently large values of x^* and y^* when letting Δt be sufficiently small, so that the area of $(0, x^*) \times (0, y^*)$ for the initial state (x_0, y_0) in Theorem 1 can be taken large.

3. Stability of the discrete solutions

Let $\mathcal{D} = (0, x^*) \times (0, y^*)$ for x^* and y^* defined in (8). In order to discuss the stability of the Euler scheme (2) for each initial position (x_0, y_0) contained in $\mathcal{D}$, we partition $\mathcal{D}$ into the four regions

$$\begin{aligned} \text{I} &= \{\mathbf{x} \in \mathcal{D} \mid f(\mathbf{x}) \ge 0,\ g(\mathbf{x}) > 0\}, \quad \text{II} = \{\mathbf{x} \in \mathcal{D} \mid f(\mathbf{x}) < 0,\ g(\mathbf{x}) \ge 0\}, \\ \text{III} &= \{\mathbf{x} \in \mathcal{D} \mid f(\mathbf{x}) \le 0,\ g(\mathbf{x}) < 0\}, \quad \text{IV} = \{\mathbf{x} \in \mathcal{D} \mid f(\mathbf{x}) > 0,\ g(\mathbf{x}) \le 0\}, \end{aligned} \tag{14}$$

where $\mathbf{x} = (x, y)$ and

$$f(x, y) = r_1 - a_{11}x - a_{12}y,\ g(x, y) = r_2 - a_{21}x - a_{22}y. \tag{15}$$

Since the location of the regions depends on the x and y-intercepts of the two lines $f(x, y) = 0$ and $g(x, y) = 0$, we partition $\mathcal{D}$ by using the four categories $\mathcal{C}_i (1 \le i \le 4)$ as in Figure 1; we use the symbol $\mathcal{C}_1$ for the two conditions $r_1a_{11}^{-1} < r_2a_{21}^{-1}$ and $r_1a_{12}^{-1} < r_2a_{22}^{-1}$, the symbol $\mathcal{C}_2$ for $r_1a_{11}^{-1} > r_2a_{21}^{-1}$ and $r_1a_{12}^{-1} > r_2a_{22}^{-1}$, the symbol $\mathcal{C}_3$ for $r_1a_{11}^{-1} < r_2a_{21}^{-1}$ and $r_1a_{12}^{-1} > r_2a_{22}^{-1}$, and finally the symbol $\mathcal{C}_4$ for $r_1a_{11}^{-1} > r_2a_{21}^{-1}$ and $r_1a_{12}^{-1} < r_2a_{22}^{-1}$. The magenta circles in Figure 1 denote the stable points of the difference model (2) in the categories, which will be proved.

Remark 3. In the case of $\mathcal{C}_1$

$$r_1a_{11}^{-1} < r_2a_{21}^{-1},\ r_1a_{12}^{-1} < r_2a_{22}^{-1}, \tag{16}$$

the region IV is empty. In order to prove this emptiness, suppose, on the contrary, that there exists $(x, y) \in \text{IV}$, which means, from (14), that

$$r_1 - a_{11}x - a_{12}y > 0,\ r_2 - a_{21}x - a_{22}y \le 0. \tag{17}$$

Eliminating x and y from (17), we have the two inequalities, respectively:

$$-r_1a_{21} + r_2a_{11} < (a_{11}a_{22} - a_{12}a_{21})y, \tag{18}$$

$$-r_1a_{22} + r_2a_{12} < (a_{12}a_{21} - a_{11}a_{22})x. \tag{19}$$

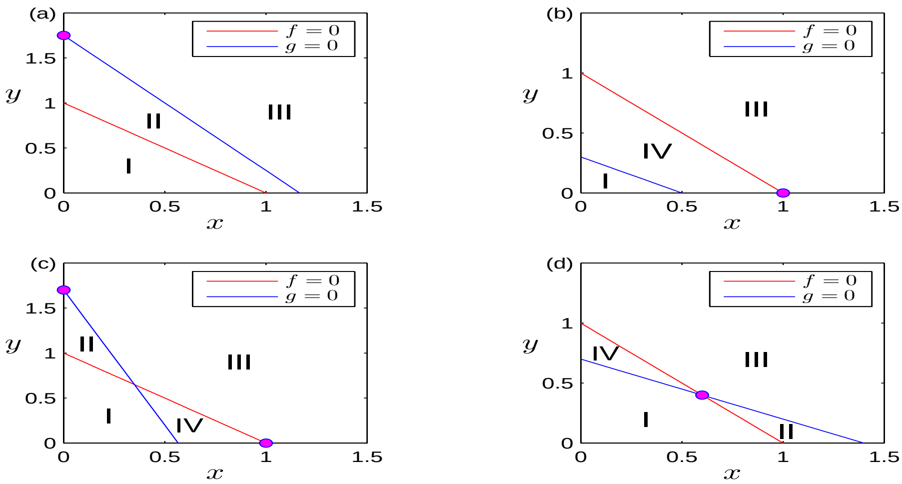

Figure 1: Two lines $f = 0$ and $g = 0$ and regions with stable points. (a) $r_2 = 3.5, a_{21} = 3.0, a_{22} = 2$ (b) $r_2 = 1.5, a_{21} = 3, a_{22} = 5$ (c) $r_2 = 1.7, a_{21} = 3, a_{22} = 1$ (d) $r_2 = 3.5, a_{21} = 2.5, a_{22} = 5$

We find a contradiction by using the following three cases:
Case 1. Let $a_{11}a_{22} - a_{12}a_{21} = 0$.
In this case, (18) becomes $-r_1a_{21} + r_2a_{11} < 0$, which contradicts (16).
Case 2. Let $a_{11}a_{22} - a_{12}a_{21} < 0$.
Using the positivity of y, (18) becomes $-r_1a_{21} + r_2a_{11} < 0$, which contradicts (16).
Case 3. Let $a_{11}a_{22} - a_{12}a_{21} > 0$.
Using the positivity of x, (19) becomes $-r_1a_{22} + r_2a_{12} < 0$, which contradicts (16).
Therefore it follows from Cases 1, 2 and 3 that the region IV is empty and then

$$\mathcal{D} = \mathrm{I} \cup \mathrm{II} \cup \mathrm{III} \text{ for } \mathcal{C}_1 \tag{20}$$

as in Figure 1-(a). Similarly we can obtain

$$\mathcal{D} = \mathrm{I} \cup \mathrm{III} \cup \mathrm{IV} \text{ for } \mathcal{C}_2 \tag{21}$$

as in Figure 1-(b).

For convenience, we use the difference equations

$$x_{n+1} = x_n\{1 + f(x_n, y_n)\Delta t\}, \tag{22}$$
$$y_{n+1} = y_n\{1 + g(x_n, y_n)\Delta t\} \tag{23}$$

as well as (2), where $f(x, y)$ and $g(x, y)$ are defined in (15).
For the stability we need to assume

$$1 > \Delta t\,(a_{11}x^* + a_{22}y^* + x^*y^*|a_{12}a_{21} - a_{11}a_{22}|\Delta t)\,. \tag{24}$$

Lemma 1. *Let (x_n, y_n) be the solution of (2). Assume that (7), (8) and (24) hold. If $(x_k, y_k) \in \mathrm{I}$ for some k, then (x_{k+1}, y_{k+1}) is not contained in* III.

Proof. The condition $(x_k, y_k) \in \mathrm{I}$ gives

$$g(x_k, y_k) > 0. \tag{25}$$

Suppose, on the contrary, that (x_{k+1}, y_{k+1}) is contained in III, which means

$$f(x_{k+1}, y_{k+1}) \leq 0 \text{ and } g(x_{k+1}, y_{k+1}) < 0.$$

Then (22) and (23) give

$$\begin{aligned} 0 \geq f(x_{k+1}, y_{k+1}) &= f\left(x_k + x_k f(x_k, y_k)\Delta t,\ y_k + y_k g(x_k, y_k)\Delta t\right) \\ &= f(x_k, y_k) + (-a_{11})x_k f(x_k, y_k)\Delta t + (-a_{12})y_k g(x_k, y_k)\Delta t \end{aligned} \tag{26}$$

and

$$\begin{aligned} 0 > g(x_{k+1}, y_{k+1}) &= g(x_k + x_k f(x_k, y_k)\Delta t,\ y_k + y_k g(x_k, y_k)\Delta t) \\ &= g(x_k, y_k) + (-a_{21})x_k f(x_k, y_k)\Delta t + (-a_{22})y_k g(x_k, y_k)\Delta t. \end{aligned} \tag{27}$$

We write (26) and (27) as

$$\begin{aligned} f(x_k, y_k)(1 - a_{11}x_k\Delta t) &\leq a_{12}y_k g(x_k, y_k)\Delta t, \\ g(x_k, y_k)(1 - a_{22}y_k\Delta t) &< a_{21}x_k f(x_k, y_k)\Delta t. \end{aligned} \tag{28}$$

Combining (24) and Theorem 1 gives

$$0 < 1 - a_{11}x^*\Delta t < 1 - a_{11}x_k\Delta t$$

and so (28) implies

$$g(x_k, y_k)(1 - a_{22}y_k\Delta t) < a_{21}x_k\Delta t\frac{a_{12}y_k g(x_k, y_k)\Delta t}{(1 - a_{11}x_k\Delta t)}. \tag{29}$$

Using (24) and (25), we can simplify (29) as follows.

$$\begin{aligned} 1 &< \Delta t\{a_{11}x_k(1 - a_{22}y_k\Delta t) + a_{22}y_k + a_{12}y_k a_{21}x_k\Delta t\} \\ &\leq \Delta t\{a_{11}x_k + a_{22}y_k + x_k y_k|a_{12}a_{21} - a_{11}a_{22}|\Delta t\}, \end{aligned} \tag{30}$$

where the last inequality contradicts (24). Hence (x_{k+1}, y_{k+1}) is not contained in III. □

Remark 4. Similarly to Lemma 1 under the same assumption, we can obtain that

$$\text{if } (x_k, y_k) \in \text{ III for some } k, \text{ then } (x_{k+1}, y_{k+1}) \text{ is not contained in I} \tag{31}$$

as follows. The condition $(x_k, y_k) \in \mathrm{III}$ gives

$$g(x_k, y_k) < 0. \tag{32}$$

Suppose, on the contrary, that

$$f(x_{k+1}, y_{k+1}) \geq 0 \text{ and } g(x_{k+1}, y_{k+1}) > 0. \tag{33}$$

Using (33) instead of $f(x_{k+1}, y_{k+1}) \leq 0$ and $g(x_{k+1}, y_{k+1}) < 0$ in the proof of Lemma 1 and following the proof of Lemma 1 with (32), we have

$$g(x_k, y_k)(1 - a_{22}y_k\Delta t) > a_{21}x_k\Delta t\frac{a_{12}y_k g(x_k, y_k)\Delta t}{(1 - a_{11}x_k\Delta t)}$$

and then obtain the contradiction (30) due to (32). Therefore we obtain (31).

Lemma 2. *Let (x_n, y_n) be the solution of (2). Assume that (7), (8) and (24) hold.*

$$\text{If } (x_k, y_k) \in \text{II} \text{ for some } k, \text{ then } (x_n, y_n) \in \text{II} \text{ for all } n \geq k.$$

Proof. Let $(x_k, y_k) \in \text{II}$, which implies $f(x_k, y_k) < 0 \leq g(x_k, y_k)$ and then

$$x_{k+1} < x_k, \quad y_{k+1} \geq y_k. \tag{34}$$

It follows from Theorem 1, (34) and (10) that

$$0 < x_{k+1} < x_k < U_1(y_k), \ 0 < y_k \leq y_{k+1} < y^* < U_2(x_k). \tag{35}$$

Using the decreasing function $F_y(x)$ of y and combining (6) with (35), we have

$$x_{k+2} = F_{y_{k+1}}(x_{k+1}) \leq F_{y_k}(x_{k+1}) < F_{y_k}(x_k) = x_{k+1} \tag{36}$$

and then (22) gives

$$f(x_{k+1}, y_{k+1}) < 0. \tag{37}$$

Similarly, the strictly decreasing function $G_x(y)$ of x with (6) and (35) gives

$$y_{k+2} = G_{x_{k+1}}(y_{k+1}) > G_{x_k}(y_{k+1}) \geq G_{x_k}(y_k) = y_{k+1}. \tag{38}$$

Substituting (23) into (38) yields

$$g(x_{k+1}, y_{k+1}) > 0,$$

with which (37) gives

$$f(x_{k+1}, y_{k+1}) < 0 < g(x_{k+1}, y_{k+1}).$$

This implies

$$(x_{k+1}, y_{k+1}) \in \text{II}.$$

Hence

$$\text{if } (x_k, y_k) \in \text{II}, \text{ then } (x_{k+1}, y_{k+1}) \in \text{II}.$$

Therefore using mathematical induction, we can obtain the desired result. □

Remark 5. Similarly to Lemma 2 under the same assumption, we can obtain that

$$\text{if } (x_k, y_k) \in \text{IV for some } k, \text{ then } (x_n, y_n) \in \text{IV for all } n \geq k \tag{39}$$

as follows. Let $(x_k, y_k) \in \text{IV}$, which implies

$$f(x_k, y_k) > 0 \geq g(x_k, y_k). \tag{40}$$

Then replacing $f(x_k, y_k) < 0 \leq g(x_k, y_k)$ in the proof of Lemma 2 with (40) and following the proof of Lemma 2, we have

$$f(x_{k+1}, y_{k+1}) > 0 > g(x_{k+1}, y_{k+1}),$$

which implies

$$(x_{k+1}, y_{k+1}) \in \text{IV}.$$

Hence mathematical induction gives (39).

In the following theorem, we show the global stability of the solutions of (2) for the category $\mathcal{C}_1$ as in Figure 1-(a); we present the condition that the species y always outcompetes the species x.

Theorem 2. *Assume that (7), (8) and (24) hold.*

$$\text{If } r_1 a_{11}^{-1} < r_2 a_{21}^{-1} \text{ and } r_1 a_{12}^{-1} < r_2 a_{22}^{-1}, \text{ then } \left(0, r_2 a_{22}^{-1}\right) \text{ is globally stable.}$$

Proof. The condition in this theorem is corresponding to $\mathcal{C}_1$, so that $\mathcal{D}$ is partitioned into the three regions I, II and III due to (20). We claim the global stability for $(x_0, y_0) \in$ $\mathrm{I} \cup \mathrm{II} \cup \mathrm{III}$ by using mathematical induction as follows.

Case 2-1. Let $(x_0, y_0) \in \mathrm{II}$.

Using Lemma 2 and Theorem 1, we have that

$$0 < x_{n+1} < x_n, \; 0 < y_n \leq y_{n+1} < y^*, \tag{41}$$

which give the convergence of $\{x_n\}$ and $\{y_n\}$ with limits ω_1 and ω_2, respectively.

Note that the increasing property of $\{y_n\}$ gives $\omega_2 > 0$.

In addition, the limit ω_1 is zero, which can be obtained by indirect proof. Suppose, on the contrary, that ω_1 is nonzero. Taking the limit of (2) and using $\omega_i > 0$ $(i = 1, 2)$, we have

$$(a_{11}a_{22} - a_{12}a_{21})(\omega_1, \omega_2) = (r_1 a_{22} - r_2 a_{12}, -r_1 a_{21} + r_2 a_{11}). \tag{42}$$

Since $r_1 a_{22} - r_2 a_{12} < 0$ and $-r_1 a_{21} + r_2 a_{11} > 0$ from the conditions in this theorem, the equality (42) with $\omega_i > 0$ gives

$$0 > a_{11}a_{22} - a_{12}a_{21} > 0, \tag{43}$$

which is a contradiction. Consequently, ω_1 is zero.

Taking the limit of the second equation in (2) with $\omega_1 = 0$ and $\omega_2 > 0$, we have $\omega_2 = r_2 a_{22}^{-1}$, which completes the proof for Case 2-1.

Case 2-2. Let $(x_0, y_0) \in \mathrm{I}$.

This case implies that $f(x_0, y_0) \geq 0$ and $g(x_0, y_0) > 0$. We use the following three steps to prove this theorem in this case.

Step 1. There exists a positive integer m_1 such that $(x_{m_1}, y_{m_1}) \notin \mathrm{I}$.

Suppose, on the contrary, that $(x_n, y_n) \in \mathrm{I}$ for all n, which means $f(x_n, y_n) \geq 0$ and $g(x_n, y_n) > 0$ for all n. Then

$$x_{n+1} = x_n\{1 + f(x_n, y_n)\Delta t\} \geq x_n > 0, \quad y_{n+1} = y_n\{1 + g(x_n, y_n)\Delta t\} > y_n > 0$$

and hence the boundedness of (x_n, y_n) in Theorem 1 gives the convergence of the increasing sequences $\{x_n\}$ and $\{y_n\}$, which have positive limits ω_1 and ω_2, respectively. Therefore we have a contradiction by using (42)–(43).

Step 2. There exists a positive integer m such that $(x_m, y_m) \in \mathrm{II}$.

Using $(x_0, y_0) \in \mathrm{I}$ and Step 1, there exists a positive integer m_1 such that $(x_{m_1-1}, y_{m_1-1}) \in$ I and $(x_{m_1}, y_{m_1}) \in \mathcal{D}-\mathrm{I}$. Since $\mathcal{D}-\mathrm{I} = \mathrm{II} \cup \mathrm{III}$, we have

$$(x_{m_1}, y_{m_1}) \in \mathrm{II} \text{ or } (x_{m_1}, y_{m_1}) \in \mathrm{III}. \tag{44}$$

Applying Lemma 1 with $(x_{m_1-1}, y_{m_1-1}) \in \mathrm{I}$, it is not true that $(x_{m_1}, y_{m_1}) \in \mathrm{III}$ and then $(x_{m_1}, y_{m_1}) \in \mathrm{II}$. Taking $m = m_1$ gives the desired result.

Step 3. If $(x_0, y_0) \in \mathrm{I}$, then $(x_m, y_m) \in \mathrm{II}$ for some positive integer m due to Step 2. Therefore the proof for Case 2-1 completes the proof for Case 2-2.

Case 2-3. Let $(x_0, y_0) \in \mathrm{III}$.

This case implies that $f(x_0, y_0) \leq 0$ and $g(x_0, y_0) < 0$. We use the following two steps to prove this theorem in this case.

Step 1. If $(x_n, y_n) \in \mathrm{III}$ for all n, then $\lim_{n\to\infty}(x_n, y_n) = (0, r_2 a_{22}^{-1})$.
Assume that $(x_n, y_n) \in \mathrm{III}$ for all n, which implies

$$f(x_n, y_n) \leq 0, \;\; g(x_n, y_n) < 0 \tag{45}$$

for all n. The assumption gives the decreasing property

$$0 < x_{n+1} = x_n\{1 + f(x_n, y_n)\Delta t\} \leq x_n, \;\; 0 < y_{n+1} = y_n\{1 + g(x_n, y_n)\Delta t\} < y_n$$

and then Theorem 1 gives the convergence of $\{x_n\}$ and $\{y_n\}$ with the nonnegative limits ω_1 and ω_2, respectively. It is only possible that $\omega_1 = 0$ and $\omega_2 > 0$ as follows.
If $\omega_1 > 0$ and $\omega_2 > 0$, then (42)–(43) give a contradiction.
If $\omega_1 > 0$ and $\omega_2 = 0$, then $\omega_1 = r_1 a_{11}^{-1}$. This is impossible due to the unstability of $(r_1 a_{11}^{-1}, 0)$ since the linearized system of (2) at $(r_1 a_{11}^{-1}, 0)$ has the eigenvalue

$$1 + \Delta t a_{11}^{-1}\left(r_2 a_{11} - r_1 a_{21}\right) > 1$$

under the condition $a_{21} a_{11}^{-1} < r_2 {r_1}^{-1}$. Therefore $\{(x_n, y_n)\}$ cannot have the limit $(r_1 a_{11}^{-1}, 0)$.
If $\omega_1 = 0$ and $\omega_2 = 0$, then

$$\lim_{n\to\infty} f\left(x_n, y_n\right) = r_1 > 0, \quad \lim_{n\to\infty} g\left(x_n, y_n\right) = r_2 > 0,$$

which are contradictory to (45).
Therefore it remains that $\omega_1 = 0$ and $\omega_2 > 0$, which gives $(\omega_1, \omega_2) = (0, r_2 a_{22}^{-1})$.

Step 2. If $(x_m, y_m) \notin \mathrm{III}$ for some m, then $\lim_{n\to\infty}(x_n, y_n) = (0, r_2 a_{22}^{-1})$.
Since $(x_m, y_m) \in \mathcal{D}-\mathrm{III}$ and $\mathcal{D}-\mathrm{III} = \mathrm{I} \cup \mathrm{II}$, we have

$$(x_m, y_m) \in \mathrm{I} \text{ or } (x_m, y_m) \in \mathrm{II}.$$

However it is not true that $(x_m, y_m) \in \mathrm{I}$ due to Remark 4 and so we have $(x_m, y_m) \in \mathrm{II}$. Therefore, following the proof for Case 2-1, we obtain $\lim_{n\to\infty}(x_n, y_n) = (0, r_2 a_{22}^{-1})$.

Finally, we obtain the desired result from the proofs for Cases 2-1, 2-2 and 2-3. □

In the following theorem, we show the global stability of (2) for $\mathcal{C}_2$ as in Figure 1-(b) and present the condition that the species x always outcompetes the species y.

Theorem 3. *Assume that (7), (8) and (24) hold.*

If $r_1 a_{11}^{-1} > r_2 a_{21}^{-1}$ *and* $r_1 a_{12}^{-1} > r_2 a_{22}^{-1}$*, then* $\left(r_1 a_{11}^{-1}, 0\right)$ *is globally stable.*

Proof. The condition in this theorem is corresponding to $\mathcal{C}_1$ and so $\mathcal{D}$ is partitioned into the three regions I, III and IV due to (21). We claim the global stability for $(x_0, y_0) \in \mathrm{I} \cup \mathrm{III} \cup \mathrm{IV}$ by using mathematical induction as follows.

Case 3-1. Let $(x_0, y_0) \in \mathrm{IV}$.

In this case, (39) gives $(x_n, y_n) \in \mathrm{IV}$ for all n, with which (22) and (23) give $x_n < x_{n+1}$ and $y_{n+1} \leq y_n$. Then Theorem 1 gives

$$0 < x_n < x_{n+1} < x^*, \;\; 0 < y_{n+1} \leq y_n, \tag{46}$$

which imply the convergence of $\{x_n\}$ and $\{y_n\}$ with limits ω_1 and ω_2, respectively. The increasing property of $\{x_n\}$ gives $\omega_1 > 0$.
In addition, the limit ω_2 is zero, which can be obtained by indirect proof as in Case 2-1. Suppose, on the contrary, that ω_2 is nonzero. Taking the limit of (2) and using the positivity of ω_1 and ω_2, we have (42). Applying the conditions $r_1a_{11}^{-1} > r_2a_{21}^{-1}$ and $r_1a_{12}^{-1} > r_2a_{22}^{-1}$ to (42) yields the contradiction (43) Consequently, ω_2 is zero.
Taking the limit of the first equation in (2) with $\omega_1 > 0$ and $\omega_2 = 0$, we have $\omega_1 = r_1a_{11}^{-1}$, which completes the proof for Case 3-1.
Case 3-2. Let $(x_0, y_0) \in \text{I}$.
In this case we have $f(x_0, y_0) \geq 0$ and $g(x_0, y_0) > 0$, and use the following three steps.

Step 1. There exists a positive integer m_1 such that $(x_{m_1}, y_{m_1}) \notin \text{I}$.
Suppose, on the contrary, that $(x_n, y_n) \in \text{I}$ for all n, which means $f(x_n, y_n) \geq 0$ and $g(x_n, y_n) > 0$ for all n. Then

$$x_{n+1} = x_n\{1 + f(x_n, y_n)\Delta t\} \geq x_n > 0, \quad y_{n+1} = y_n\{1 + g(x_n, y_n)\Delta t\} > y_n > 0,$$

and hence the boundedness of (x_n, y_n) in Theorem 1 gives the convergence of the increasing sequences $\{x_n\}$ and $\{y_n\}$, which have positive limits ω_1 and ω_2, respectively. Therefore we have the contradiction (43) as in Case 3-1.

Step 2. There exists a positive integer m such that $(x_m, y_m) \in \text{IV}$.
Using $(x_0, y_0) \in \text{I}$ and Step 1, there exists a positive integer m_1 such that $(x_{m_1-1}, y_{m_1-1}) \in \text{I}$ and $(x_{m_1}, y_{m_1}) \in \mathcal{D}-\text{I}$ for some m_1. Since $\mathcal{D}-\text{I} = \text{III} \cup \text{IV}$, we have

$$(x_{m_1}, y_{m_1}) \in \text{III or } (x_{m_1}, y_{m_1}) \in \text{IV}.$$

Applying Lemma 1 with $(x_{m_1-1}, y_{m_1-1}) \in \text{I}$, it is not true that $(x_{m_1}, y_{m_1}) \in \text{III}$ and then $(x_{m_1}, y_{m_1}) \in \text{IV}$. Taking $m = m_1$ gives $(x_m, y_m) \in \text{IV}$.

Step 3. If $(x_0, y_0) \in \text{I}$, then $(x_m, y_m) \in \text{IV}$ for some positive integer m due to Step 2. Therefore the proof used in Case 3-1 completes the proof for Case 3-2.
Case 3-3. Let $(x_0, y_0) \in \text{III}$.
In this case we have $f(x_0, y_0) \leq 0$ and $g(x_0, y_0) < 0$,and use the following two steps.

Step 1. If $(x_n, y_n) \in \text{III}$ for all n, then $\lim_{n\to\infty}(x_n, y_n) = (r_1a_{11}^{-1}, 0)$.
As in Step 1 of Case 2-3 in Theorem 2, $\{(x_n, y_n)\}$ has the limit (ω_1, ω_2). It is only possible that $\omega_1 > 0$ and $\omega_2 = 0$ as follows.
If $\omega_1 > 0$ and $\omega_2 > 0$, then (46)–(**??**) give a contradiction.
If $\omega_1 = 0$ and $\omega_2 > 0$, then $\omega_2 = r_2a_{22}^{-1}$. This is impossible due to the unstability of $(0, r_2a_{22}^{-1})$ since the linearized system of (2) at $(0, r_2a_{22}^{-1})$ has the eigenvalue

$$1 + \Delta t a_{22}^{-1}(r_1a_{22} - r_2a_{12}) > 1$$

under the condition $a_{22}a_{12}^{-1} > r_2{r_1}^{-1}$. Therefore $\{(x_n, y_n)\}$ cannot have the limit $(r_1a_{11}^{-1}, 0)$.
If $\omega_1 = 0$ and $\omega_2 = 0$, then

$$\lim_{n\to\infty} f(x_n, y_n) = r_1 > 0, \quad \lim_{n\to\infty} g(x_n, y_n) = r_2 > 0,$$

which are contradictory to (45).
It remains that $\omega_1 > 0$ and $\omega_2 = 0$, which yields the desired result $(\omega_1, \omega_2) = (r_1a_{11}^{-1}, 0)$.

Step 2. If $(x_m, y_m) \notin \text{III}$ for some m, then $\lim_{n\to\infty}(x_n, y_n) = (r_1a_{11}^{-1}, 0)$.
Since $(x_m, y_m) \in \mathcal{D}-\text{III} = \text{I} \cup \text{IV}$, we have

$$(x_m, y_m) \in \text{I or } (x_m, y_m) \in \text{IV}.$$

However it is not true that $(x_m, y_m) \in \text{I}$ due to Remark 4. Therefore, we have $(x_m, y_m) \in \text{IV}$ and then $\lim_{n\to\infty}(x_n, y_n) = (r_1 a_{11}^{-1}, 0)$ by following the proof for Case 3-1.

Finally, we obtain the desired result from the proofs for Cases 3-1 and 3-2. □

In the following theorem, we show the convergence of the solutions of (2) for the category $\mathcal{C}_3$ as in Figure 1-(c) and the dependence of the limit on the region in which the initial state is located.

From now on, in the case that $a_{11}a_{22} - a_{12}a_{21} \neq 0$, we use the symbol (θ_1, θ_2) to mean

$$(\theta_1, \theta_2) = (a_{11}a_{22} - a_{12}a_{21})^{-1} (r_1 a_{22} - r_2 a_{12}, -r_1 a_{21} + r_2 a_{11}), \tag{47}$$

where (θ_1, θ_2) satisfies

$$f(\theta_1, \theta_2) = g(\theta_1, \theta_2) = 0. \tag{48}$$

Theorem 4. *Let the conditions (7), (8) and (24) hold. Assume that*

$$r_1 a_{11}^{-1} > r_2 a_{21}^{-1} \text{ and } r_1 a_{12}^{-1} < r_2 a_{22}^{-1}.$$

(a) *If* $(x_0, y_0) \in \text{II}$, *then* $\lim_{n\to\infty}(x_n, y_n) = (0, r_2 a_{22}^{-1})$.
(b) *If* $(x_0, y_0) \in \text{IV}$, *then* $\lim_{n\to\infty}(x_n, y_n) = (r_1 a_{11}^{-1}, 0)$.
(c) *If* $(x_0, y_0) \in \text{I} \cup \text{III}$, *then* $\{(x_n, y_n)\}$ *converges with the limit* $(r_1 a_{11}^{-1}, 0)$ *or* $(0, r_2 a_{22}^{-1})$.

Proof. For the proof of (a), let $(x_0, y_0) \in \text{II}$. We have from Lemma 2 and Theorem 1 that

$$0 < x_{n+1} < x_n, \ 0 < y_n \leq y_{n+1} < y^*, \tag{49}$$

which gives the convergence of $\{x_n\}$ and $\{y_n\}$ with limits ω_1 and ω_2, respectively. The increasing property of $\{y_n\}$ gives $\omega_2 > 0$.

In addition, the limit ω_1 is zero, which can be obtained by indirect proof. Suppose, on the contrary, that ω_1 is nonzero. Taking the limit of (2) and using the positivity of ω_1 and ω_2, we have

$$(a_{11}a_{22} - a_{12}a_{21})\omega_1 = r_1 a_{22} - r_2 a_{12}. \tag{50}$$

Since $(x_0, y_0) \in \text{II}$, the definition of the region II gives

$$f(x_0, y_0) < 0 \leq g(x_0, y_0). \tag{51}$$

Solving (51) for x_0, we obtain

$$(r_1 a_{22} - r_2 a_{12}) - (a_{11}a_{22} - a_{12}a_{21})x_0 < 0. \tag{52}$$

The conditions $a_{21}a_{11}^{-1} > r_2 {r_1}^{-1} > a_{22}a_{12}^{-1}$ in this theorem give

$$a_{11}a_{22} - a_{12}a_{21} < 0. \tag{53}$$

Applying (53) into both (52) and (50) yields

$$\omega_1 > x_0. \tag{54}$$

Combining (54) with (49), we have that for all n

$$\omega_1 > x_0 > x_n,$$

which is contradictory to $\lim_{n\to\infty} x_n = \omega_1$. Consequently, ω_1 is zero.
Taking the limit of the second equation in (2) with $\omega_1 = 0$ and $\omega_2 > 0$, we have $\omega_2 = r_2 a_{22}^{-1}$, which completes the proof of (a).

For the proof of (b), let $(x_0, y_0) \in$ IV. Using (46), we have the convergence of $\{x_n\}$ and $\{y_n\}$ with limits ω_1 and ω_2, respectively. The increasing property of $\{x_n\}$ gives $\omega_1 > 0$. In addition, the limit ω_2 is zero, which can be obtained by indirect proof. Suppose, on the contrary, that ω_2 is nonzero. Taking the limit of (2) and using the positivity of ω_1 and ω_2, we have

$$(a_{11}a_{22} - a_{12}a_{21})\omega_2 = -r_1 a_{21} + r_2 a_{11}. \tag{55}$$

Since $(x_0, y_0) \in$ IV, the definition of the region IV gives

$$f(x_0, y_0) > 0 \geq g(x_0, y_0). \tag{56}$$

Solving (56) for y_0, we obtain

$$(r_1 a_{21} - r_2 a_{11}) + (a_{11}a_{22} - a_{12}a_{21})y_0 > 0. \tag{57}$$

Applying (53) into (57) yields

$$\omega_2 > y_0. \tag{58}$$

Combining (58) with (46), we have that for all n

$$\omega_2 > y_0 > y_n,$$

which is contradictory to $\lim_{n\to\infty} y_n = \omega_2$. Consequently, ω_2 is zero.
Taking the limit of the first equation in (2) with $\omega_1 > 0$ and $\omega_2 = 0$, we have $\omega_1 = r_1 a_{11}^{-1}$, which completes the proof of (b).

For the proof of (c), we consider the following two cases.
Case 4-1. Let $(x_0, y_0) \in$ I.
We use the following three steps to obtain the desired result in this case.

Step 1. There exists a positive constant m such that $(x_m, y_m) \notin$ I.
Suppose, on the contrary, that $(x_n, y_n) \in$ I for all n. Then $\{x_n\}$ and $\{y_n\}$ have the positive limits (θ_1, θ_2) defined in (47) by applying (53) and the approach used in Step1 of Case 2-2 in Theorem 2. However the system (2) under the condition $r_1 a_{11}^{-1} < r_2 a_{21}^{-1}$ is unstable at the point (θ_1, θ_2) since the linearized system at (θ_1, θ_2) has the eigenvalue $1 + \Delta t a_{11}^{-1}(r_2 a_{11} - r_1 a_{21})$ greater than 1. Therefore $\{x_n\}$ and $\{y_n\}$ cannot have the positive limits θ_1 and θ_2, respectively, which is contradictory.

Step 2. There exists a positive constant m such that $(x_m, y_m) \in \text{II} \cup \text{IV}$.
Since $(x_0, y_0) \in$ I, Step 1 gives the existence of a positive integer m such that

$$(x_{m-1}, y_{m-1}) \in \text{I and } (x_m, y_m) \notin \text{I},$$

which implies $(x_m, y_m) \in \text{II} \cup \text{IV}$ due to Lemma 1 and $\mathcal{D} = \text{I} \cup \text{II} \cup \text{III} \cup \text{IV}$.

Step 3. It follows from (a), (b) and Step 2 in this theorem that (x_n, y_n) converges and has the limit $(r_1 a_{11}^{-1}, 0)$ or $(0, r_2 a_{22}^{-1})$.
Case 4-2. Let $(x_0, y_0) \in$ III.
We use the following two steps to obtain the desired result in this case.

Step 1. If $(x_n, y_n) \in \mathrm{III}$ for all n, then $\{(x_n, y_n)\}$ converges with the limit $(r_1 a_{11}^{-1}, 0)$ or $(0, r_2 a_{22}^{-1})$. To prove this, note that we have the convergence of $\{(x_n, y_n)\}$ with the limit (ω_1, ω_2) by following the proof of Step 1 of Case 2-3 in Theorem 2.
If $\omega_1 > 0$ and $\omega_2 > 0$, then $(\omega_1, \omega_2) = (\theta_1, \theta_2)$. This is impossible due to the unstability of (θ_1, θ_2) since the linearized system of (2) at (θ_1, θ_2) has the eigenvalue greater than 1:

$$1 + 0.5\Delta t \left\{ -(a_{11}\theta_1 + a_{22}\theta_2) + \sqrt{(a_{11}\theta_1 + a_{22}\theta_2)^2 + \alpha} \right\} > 1$$

since $\alpha = 4(a_{12}a_{21} - a_{11}a_{22})\theta_1\theta_2 > 0$ under the condition $a_{21}a_{11}^{-1} > r_2 {r_1}^{-1} > a_{22}a_{12}^{-1}$. Therefore it is not possible that $\omega_1 > 0$ and $\omega_2 > 0$.
If $\omega_1 = 0$ and $\omega_2 = 0$, then we have the contradictions to to (45):

$$\lim_{n\to\infty} f(x_n, y_n) = r_1 > 0, \quad \lim_{n\to\infty} g(x_n, y_n) = r_2 > 0.$$

Therefore the remaining signs of ω_1 and ω_2 are

$$(+, 0) \text{ and } (0, +),$$

which give the desired result

$$(\omega_1, \omega_2) = (r_1 a_{11}^{-1}, 0) \text{ and } (0, r_2 a_{22}^{-1}),$$

respectively, by taking the limit of (2) and using the signs of ω_1 and ω_2.

Step 2. If $(x_m, y_m) \notin \mathrm{III}$ for some m, then $\{(x_n, y_n)\}$ converges with the limit $(r_1 a_{11}^{-1}, 0)$ or $(0, r_2 a_{22}^{-1})$. To prove this, we follow the proof used in Step 2 of Case 4-1.
Since $(x_0, y_0) \in \mathrm{III}$, using the condition $(x_m, y_m) \notin \mathrm{III}$ for some m, we can assume that there exists a positive constant m_1 such that

$$(x_{m_1-1}, y_{m_1-1}) \in \mathrm{III} \text{ and } (x_{m_1}, y_{m_1}) \notin \mathrm{III},$$

which implies

$$(x_{m_1}, y_{m_1}) \in \mathrm{II} \cup \mathrm{IV} \tag{59}$$

due to $\mathcal{D} = \mathrm{I} \cup \mathrm{II} \cup \mathrm{III} \cup \mathrm{IV}$ and Lemma 1. Therefore, using (59) and (a) and (b) in this theorem, we have that (x_n, y_n) converges and has the limit $(r_1 a_{11}^{-1}, 0)$ or $(0, r_2 a_{22}^{-1})$.

Finally, we obtain the desired result from the proofs for Cases 4-1 and 4-2. □

In the following theorem, we show the global stability of the solutions of (2) for the category $\mathcal{C}_4$ as in Figure 1-(d) where each component of the equilibrium point is positive.

Theorem 5. *Let the conditions (7), (8) and (24) hold. Assume that*

$$r_1 a_{11}^{-1} < r_2 a_{21}^{-1} \text{ and } r_1 a_{12}^{-1} > r_2 a_{22}^{-1}.$$

Then for (θ_1, θ_2) defined in (47)

$$(\theta_1, \theta_2) \text{ is globally stable.}$$

Proof. Note that the conditions $r_1 a_{11}^{-1} < r_2 a_{21}^{-1}$ and $r_1 a_{12}^{-1} > r_2 a_{22}^{-1}$ in this theorem give

$$a_{11}a_{22} - a_{12}a_{21} > 0. \tag{60}$$

We prove this theorem by using the four cases and mathematical induction.
Case 5-1. Let $(x_0, y_0) \in \mathrm{II}$.
Lemma 2 and Theorem 1 give (49). Then we have

$$\lim_{n\to\infty}(x_n, y_n) = (\omega_1, \omega_2),\ \omega_2 > 0$$

and

$$f(x_n, y_n) < 0 \leq g(x_n, y_n). \tag{61}$$

Solving (61) for x_n as in (51) and (52) and using (60), we have that for all n

$$0 < \theta_1 < x_n$$

and then $\omega_1 \geq \theta_1 > 0$. Since ω_1 and ω_2 are positive, we have

$$(\omega_1, \omega_2) = (\theta_1, \theta_2).$$

Case 5-2. Let $(x_0, y_0) \in$ IV.
Using Remark 5 and Theorem 1, we have

$$0 < x_n < x_{n+1} < x^*,\ 0 < y_{n+1} \leq y_n \tag{62}$$

and

$$\lim_{n\to\infty}(x_n, y_n) = (\omega_1, \omega_2),\ \omega_1 > 0.$$

The inequalities (62) implies

$$f(x_n, y_n) > 0 \geq g(x_n, y_n). \tag{63}$$

Solving (63) for y_n as in (56) and (57), we have that for all n

$$0 < \theta_2 < y_n$$

and then $\omega_2 \geq \theta_2 > 0$. Since ω_1 and ω_2 are positive, we have

$$(\omega_1, \omega_2) = (\theta_1, \theta_2).$$

Case 5-3. Let $(x_0, y_0) \in$ I.
If $(x_m, y_m) \notin$ I for some m, then

$$(x_m, y_m) \in \mathcal{D} - \text{I} = \text{II} \cup \text{III} \cup \text{IV}$$

and further

$$(x_m, y_m) \in \text{II} \cup \text{IV}$$

due to Lemma 1. By Case 5-1 and 5-2, we have

$$\lim_{n\to\infty}(x_n, y_n) = (\theta_1, \theta_2).$$

On the other hand, if $(x_n, y_n) \in$ I for all n, then we have the positive limits ω_1 and ω_2 of $\{x_n\}$ and $\{y_n\}$, respectively, due to the definition of I and Theorem 1. Taking the limit of (2) and using ω_i $(i = 1, 2)$, we have

$$(\omega_1, \omega_2) = (\theta_1, \theta_2).$$

Case 5-4. Let $(x_0, y_0) \in$ III.
Replacing I in the proof of Case 5-3 with III, we can obtain

$$(\omega_1, \omega_2) = (\theta_1, \theta_2).$$

Finally, we obtain the desired result from the proofs for Cases 5-1 to 5-4. □

4. Numerical examples

In this section, we provide simulations that illustrate our results in Theorem 2 to Theorem 5 for the difference scheme (2) with $\Delta t = 0.001$ and $(x^*, y^*) = (r_1 a_{11}^{-1}+50, r_2 a_{22}^{-1}+50)$. The values of parameters used in the following four examples satisfy the three conditions (7), (8) and (24).

Example 1. Let $(r_1, a_{11}, a_{12}, r_2, a_{21}, a_{22}) = (1, 1, 2, 3.5, 3, 2)$, which satisfies the two conditions $r_1 a_{11}^{-1} < r_2 a_{21}^{-1}$ and $r_1 a_{12}^{-1} < r_2 a_{22}^{-1}$ in Theorem 2. Then the solutions (x_n, y_n) of (2) converge to $(0, r_2 a_{22}^{-1} = 1.75)$ as displayed in Figure 2-(a).

Example 2. Let $(r_1, a_{11}, a_{12}, r_2, a_{21}, a_{22}) = (1, 1, 1, 1.5, 3, 5)$, which satisfies the two conditions $r_1 a_{11}^{-1} > r_2 a_{21}^{-1}$ and $r_1 a_{12}^{-1} > r_2 a_{22}^{-1}$ in Theorem 3. Then the solutions (x_n, y_n) of (2) converge to $(r_1 a_{11}^{-1} = 1, 0)$ as displayed in Figure 2-(b).

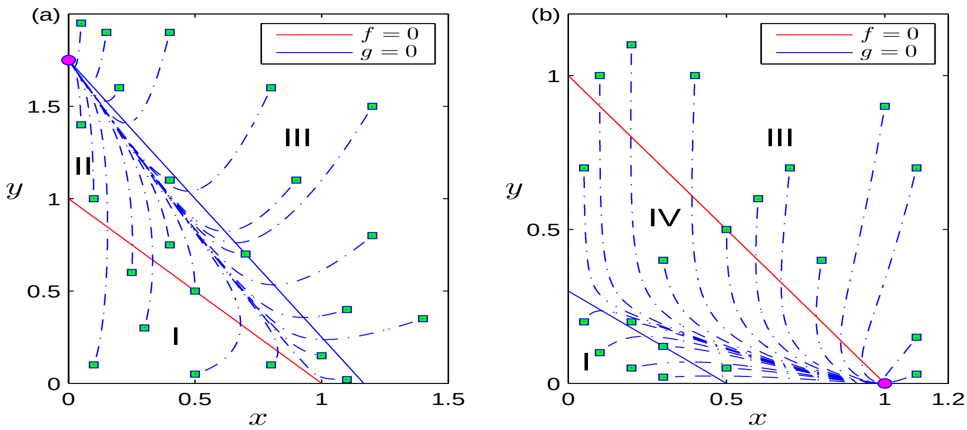

Figure 2: (a) Trajectories for different initial points in the regions I, II, III with $r_1 = 1, a_{11} = 1, a_{12} = 2, r_2 = 3.5, a_{21} = 3, a_{22} = 2$ in the category $\mathcal{C}_1$. (b) Trajectories for different initial points in the regions I, III, IV with $r_1 = 1, a_{11} = 1, a_{12} = 1, r_2 = 1.5, a_{21} = 3, a_{22} = 5$ in the category $\mathcal{C}_2$. The box and circle symbols denote initial and equilibrium points, respectively.

Example 3. Let $(r_1, a_{11}, a_{12}, r_2, a_{21}, a_{22}) = (1, 1, 1, 1.7, 3, 1)$, which satisfies the two conditions $r_1 a_{11}^{-1} > r_2 a_{21}^{-1}$ and $r_1 a_{12}^{-1} < r_2 a_{22}^{-1}$ in Theorem 4. Then as displayed in Figure 3-(a), we obtain the results in Theorem 4. If $(x_0, y_0) \in$ II, then the solutions (x_n, y_n) of (2) converge to $(0, r_2 a_{22}^{-1}) = (0, 1.7)$. If $(x_0, y_0) \in$ IV, then $\lim_{n\to\infty}(x_n, y_n) = (r_1 a_{11}^{-1}, 0) = (1, 0)$. If $(x_0, y_0) \in \text{I} \cup \text{III}$, then $\{(x_n, y_n)\}$ converges with the limit $(r_1 a_{11}^{-1}, 0) = (1, 0)$ or $(0, r_2 a_{22}^{-1}) = (0, 1.7)$. Especially, Figure 3-(a) shows that there exist at least two initial points contained in I converging to $(r_1 a_{11}^{-1}, 0) = (1, 0)$ and $(0, r_2 a_{22}^{-1}) = (0, 1.7)$, respectively. In the region III, the same phenomenon happens. The outcome depends on the initial abundances of the two species.

Example 4. Let $(r_1, a_{11}, a_{12}, r_2, a_{21}, a_{22}) = (1, 1, 1, 3.5, 2.5, 5)$, which satisfies the two conditions $r_1 a_{11}^{-1} < r_2 a_{21}^{-1}$ and $r_1 a_{12}^{-1} > r_2 a_{22}^{-1}$ in Theorem 5. Then the solutions x_n and y_n of (2) converge to

$$(r_1 a_{22} - r_2 a_{12})(a_{11} a_{22} - a_{12} a_{21})^{-1} = 0.6$$

and

$$(-r_1 a_{21} + r_2 a_{11})(a_{11}a_{22} - a_{12}a_{21})^{-1} = 0.4,$$

respectively, as displayed in Figure 3-(b). Although the outcome in Example 3 depends on the initial abundances of the two species, the outcome in Example 4 is independent of the initial abundances.

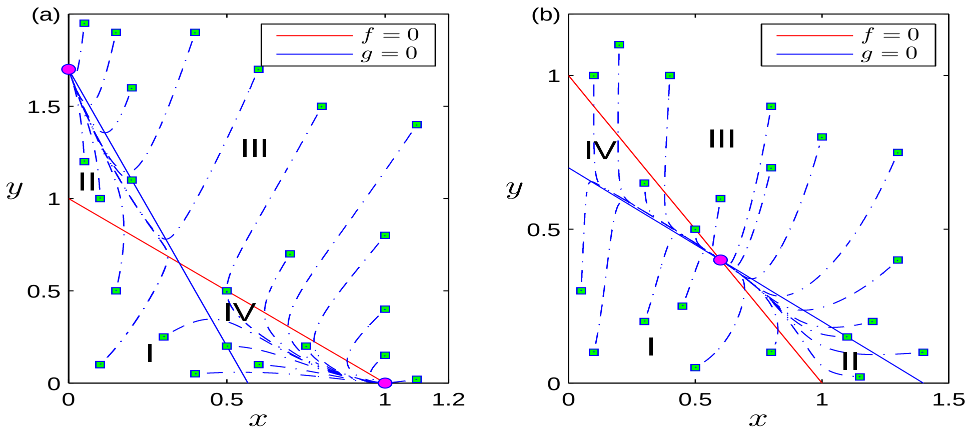

Figure 3: Trajectories for different initial points in the regions I, II, III and IV. The values of the parameters are (a) $r_1 = 1, a_{11} = 1, a_{12} = 1, r_2 = 1.7, a_{21} = 3, a_{22} = 1$ in the category $\mathcal{C}_3$. (b) $r_1 = 1, a_{11} = 1, a_{12} = 1, r_2 = 3.5, a_{21} = 2.5, a_{22} = 5$ in the category $\mathcal{C}_4$. The box and circle symbols denote initial and equilibrium points, respectively.

5. Conclusions and future work

In this paper, we have studied the Euler difference scheme for a two-dimensional Lotka-Volterra competition model and presented sufficient conditions for the global stability of the fixed points of a discrete competition model with two species. The main idea of our approach is to divide the domain used for the boundedness of solutions of the discrete model and to describe how to trace the trajectories with respect to each partition. Although we have applied our method for the two-dimensional discrete model, this method can be utilized to two-dimensional and other higher dimensional discrete models.

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Weighted Composition Operators from Bloch spaces into Zygmund spaces*

Shanli Ye†

(School of Sciences, Zhejiang University of Science and Technology, Hangzhou 310023, China)

(Department of Mathematics, Fujian Normal University, Fuzhou 350007, China)

Abstract

In this paper we characterize the boundedness and compactness of the weighted composition operator from the classical Bloch space β to the Zygmund space $\mathcal{Z}$, and from the little Bloch space β_0 to the little Zygmund space $\mathcal{Z}_0$, respectively.

Keywords Bloch space, Zygmund space; Weighted composition operator; Boundedness; Compactness

2010 MR Subject Classification 47B38, 30D99, 30H05

1 Introduction

Let $D = \{z : |z| < 1\}$ be the open unit disk in the complex plane and $H(D)$ denote the set of all analytic functions on D. Let $u, \varphi \in H(D)$, where φ is an analytic self-map of D. Then the well-known *weighted composition operator* uC_φ on $H(D)$ is defined by $uC_\varphi(f)(z) = u(z)\cdot(f\circ\varphi(z))$ for $f \in H(D)$ and $z \in D$. Weighted composition operators can be regarded as a generalization of multiplication operators and composition operators. In 2001, Ohno and Zhao studied the weighted composition operators on the classical Bloch space β in [14], which has led many researchers to study this operator on other Banach spaces of analytic functions. The boundedness and compactness of it have been studied on various Banach spaces of analytic functions, such as Hardy, Bergman, BMOA, Bloch-type spaces, see, e.g. [2, 4, 8, 18, 27].

In 2006, the boundedness of composition operators on the Zygmund space $\mathcal{Z}$ was first studied by Choe, Koo, and Smith in [1]. Later, many researchers have studied composition operators and weighted composition operators acting on the Zygmund space $\mathcal{Z}$. Li and Stević in [9] studied the boundedness and compactness of the generalized composition operators on Zygmund spaces and Bloch type spaces. They in [11] considered the boundedness and compactness of the weighted composition operators from Zygmund spaces to Bloch spaces. Ye and Hu in [22] characterized boundedness and compactness of weighted composition operators on the Zygmund space $\mathcal{Z}$. Esmaeili and Lindström in [7] studied weighted composition operators from Zygmund type spaces to Bloch type spaces and their essential norms. Sanatpour and Hassanlou in [17] gave the essential norms of this operators between Zygmund-type spaces and Bloch-type spaces. See also

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†E-mail: slye@zust.edu.cn

[5, 15, 16, 19, 20, 21, 23, 24, 25, 26] for corresponding results for weighted composition operators from one Banach space of analytic functions to another. It is well-known that $\mathcal{Z} \subset \beta$. It is more interesting to characterize u, φ such that this operator uC_φ has the pull-back properly, that is, $uC_\varphi f \in \mathcal{Z}$ whenever $f \in \beta$. In this paper we consider this question.

Now we give a detailed definition of these spaces. A function f analytic on the unit disk is said to belong to the *Bloch* space β if

$$b(f) = \sup_{z\in\mathbb{D}}\{(1-|z|^2)|f'(z)|\} < \infty,$$

and to the little *Bloch* space β_0 if $f \in \beta$ and

$$\lim_{|z|\to 1^-}(1-|z|^2)|f'(z)| = 0.$$

It is well known that β is a Banach space under the norm

$$\|f\|_\beta = |f(0)| + b(f),$$

and β_0 is a closed subspace of β.

The Zygmund space $\mathcal{Z}$ consists of all analytic functions f defined on D such that

$$z(f) = \sup\{(1-|z|^2)|f''(z)| : z \in D\}, \quad 0 < \alpha < +\infty.$$

From a theorem of Zygmund (see [29, vol. I, p. 263] or [6, Theorem 5.3]), we see that $f \in \mathcal{Z}$ if and only if f is continuous in the close unit disk $\overline{D} = \{z : |z| \le 1\}$ and the boundary function $f(e^{i\theta})$ such that

$$\sup_{h>0,\theta}\frac{|f(e^{i(\theta+h)}) + f(e^{i(\theta-h)}) - 2f(e^{i\theta})|}{h} < \infty.$$

An analytic function $f \in H(D)$ is said to belong to the little Zymund space $\mathcal{Z}_0$ consists of all $f \in \mathcal{Z}$ satisfying $\lim_{|z|\to 1^-}(1-|z|^2)|f''(z)| = 0$. It can easily proved that $\mathcal{Z}$ is a Banach space under the norm

$$\|f\|_{\mathcal{Z}} = |f(0)| + |f'(0)| + z(f)$$

and the polynomials are norm-dense in closed subspace $\mathcal{Z}_0$ of $\mathcal{Z}$. For some other information on this space and some operators on it, see, for example, [9, 10, 11].

Throughout this paper, constants are denoted by C, they are positive and only depending on p, and may differ from one occurrence to the other.

2 Auxiliary results

In order to prove the main results of this paper. we need some auxiliary results. The first part of the following lemma is a well known.

Lemma 2.1 *Suppose that* $f \in \beta$, *then*

(i) $|f(z)| \le \log\dfrac{e}{(1-|z|^2)}\|f\|_\beta$ *for every* $z \in D$;

(ii) $|f''(z)| \le \dfrac{8}{(1-|z|^2)^2}b(f)$ *for every* $z \in D$.

Proof For any $f \in \beta$. Fix $z \in D$ and let $\rho = \dfrac{1+|z|}{2}$, by the Cauchy integral formula, we obtain that

$$|f''(z)| = |\frac{1}{2\pi i}\int_{|\xi|=\rho}\frac{f'(\xi)}{(\xi-z)^2}\,d\xi| \le \frac{b(f)}{1-\rho^2}\frac{1}{2\pi}\int_0^{2\pi}\frac{\rho\, d\theta}{|\rho e^{i\theta}-z|^2} = \frac{\|f\|_\infty}{1-\rho^2}\frac{\rho}{\rho^2-|z|^2} \le \frac{8}{(1-|z|^2)^2}.$$

Hence (ii) holds.

Lemma 2.2 *[28] Suppose that* $f \in \beta_0$, *then*

(i) $\lim\limits_{|z|\to 1^-}\dfrac{|f(z)|}{\log(e/(1-|z|^2))} = 0$;
(ii) $\lim\limits_{|z|\to 1^-}(1-|z|^2)^2|f''(z)| = 0$.

Lemma 2.3 *Suppose* $uC_\varphi : \beta_0 \to \mathcal{Z}_0$ *is a bounded operator, then* $uC_\varphi : \beta \to \mathcal{Z}$ *is a bounded operator.*

The proof is similar to that of Lemma 2.3 in [21]. The details are omitted.

Lemma 2.4 *Suppose that* uC_φ *be a bounded operator from* β *to* $\mathcal{Z}$, *then* uC_φ *is compact if and only if for any bounded sequence* $\{f_n\}$ *in* β *which converges to* 0 *uniformly on compact subsets of* D. *We have* $\|uC_\varphi(f_n)\|_{\mathcal{Z}} \to 0$, *as* $n \to \infty$.

The proof is similar to that of Proposition 3.11 in [3] . The details are omitted.

Lemma 2.5 *Let* $U \subset \mathcal{Z}_0$. *Then* U *is compact if and only if it is closed, bounded and satisfies*

$$\lim_{|z|\to 1}\sup_{f\in U}(1-|z|^2)|f''(z)| = 0.$$

The proof is similar to that of Lemma 1 in [12], we omit it.

3 Main results

Theorem 3.1 *Let* u *be an analytic function on the unit disc* D, *and* φ *an analytic self-map of* D. *Then* uC_φ *is a bounded operator from the classical space* β *to the Zygmund space* $\mathcal{Z}$ *if and only if the following are satisfied:*

$$\sup_{z\in D}(1-|z|^2)|u''(z)|\log\frac{e}{1-|\varphi(z)|^2} < \infty; \tag{3.1}$$

$$\sup_{z\in D}\frac{(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)|}{1-|\varphi(z)|^2} < \infty; \tag{3.2}$$

$$\sup_{z\in D}\frac{(1-|z|^2)|u(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^2} < \infty. \tag{3.3}$$

Proof Suppose uC_φ is bounded from the Bloch space β to the Zygmund space $\mathcal{Z}$. Then we can easily obtain the following results by taking $f(z) = 1$ and $f(z) = z$ in β respectively:

$$u \in \mathcal{Z}; \tag{3.4}$$

$$\sup_{z\in D}(1-|z|^2)|2\varphi'(z)u'(z) + \varphi''(z)u(z) + \varphi(z)u''(z)| < +\infty. \tag{3.5}$$

By (3.4), (3.5) and the boundedness of the function $\varphi(z)$, we get

$$K_1 = \sup_{z\in D}(1-|z|^2)|2\varphi'(z)u'(z) + \varphi''(z)u(z)| < +\infty. \tag{3.6}$$

Let $f(z) = z^2$ in β again, in the same way we have

$$\sup_{z\in D}(1-|z|^2)|4\varphi(z)\varphi'(z)u'(z) + \varphi^2(z)u''(z) + 2u(z)(\varphi(z)\varphi''(z) + (\varphi'(z))^2)| < \infty.$$

Using these facts and the boundedness of the function $\varphi(z)$ again, we get

$$K_2 = \sup_{z\in D}(1-|z|^2)|(\varphi'(z))^2u(z)| < +\infty. \tag{3.7}$$

Fix $a \in D$, we take the test functions

$$f_a(z) = 3\log\frac{e}{1-\bar{a}z} + \frac{3}{\log\frac{e}{1-|a|^2}}(\log\frac{e}{1-\bar{a}z})^2 - \frac{1}{\log^2\frac{e}{1-|a|^2}}(\log\frac{e}{1-\bar{a}z})^3 \tag{3.8}$$

for $z \in D$. By a directly calculation we obtain that $f_a \in \beta$ and $\sup_a \|f_a\|_\beta \le C < \infty$, where C is not depended on a. Since $f_a(a) = 5\log\frac{e}{1-|a|^2}$, $f_a'(a) = 0$, $f_a''(a) = 0$, we have

$$\begin{aligned} C\|f_a\|_\beta &\ge \|uC_\varphi f_a\|_{\mathcal{Z}} \ge \sup_{z\in D}(1-|z|^2)|(uC_\varphi f_a)''(z)| \\ &= \sup_{z\in D}(1-|z|^2)|\big(2\varphi'(z)u'(z) + \varphi''(z)u(z)\big)f_a'(\varphi(z)) \\ &+ f_a''(\varphi(z))(\varphi'(z))^2u(z) + u''(z)f_a(\varphi(z))|. \end{aligned}$$

Let $a = \varphi(\lambda)$, it follows that

$$\begin{aligned} C\|f_a\|_\beta &\ge (1-|\lambda|^2)^\alpha|\big(2\varphi'(\lambda)u'(\lambda) + \varphi''(\lambda)u(\lambda)\big)f_{\varphi(\lambda)}'(\varphi(\lambda)) \\ &+ f_{\varphi(\lambda)}''(\varphi(\lambda))(\varphi'(\lambda))^2u(\lambda) + u''(\lambda)f_{\varphi(\lambda)}(\varphi(\lambda))| \\ &= 5(1-|\lambda|^2)^\alpha|u''(\lambda)\log\frac{e}{1-|\varphi(\lambda)|^2}|. \end{aligned}$$

Hence (3.1) holds.

Next, we will show that (3.2) holds. Fix $a \in D$ with $|a| > \frac{1}{2}$, we take another test functions:

$$g_a(z) = \frac{8(1-|a|^2)^2}{(1-\bar{a}z)^2} - \frac{14(1-|a|^2)^3}{(1-\bar{a}z)^3} + \frac{6(1-|a|^2)^4}{(1-\bar{a}z)^4} \tag{3.9}$$

for $z \in D$. By a directly calculation we obtain that $g_a \in \beta$ and $\sup_a \|g_a\|_\beta \le C < \infty$, where C is not depended on a. Since $g_a(a) = 0$, $g_a'(a) = \dfrac{-2\bar{a}}{1-|a|^2}$, $g_a''(a) = 0$, it follows that for all $\lambda \in D$ with $|\varphi(\lambda)| > \frac{1}{2}$, we have

$$\begin{aligned} C\|g_a\|_\beta &\ge \|uC_\varphi g_a\|_{\mathcal{Z}} \ge \sup_{z\in D}(1-|z|^2)|(uC_\varphi g_a)''(z)| \\ &= \sup_{z\in D}(1-|z|^2)|\big(2\varphi'(z)u'(z) + \varphi''(z)u(z)\big)g_a'(\varphi(z)) \\ &+ g_a''(\varphi(z))(\varphi'(z))^2u(z) + u''(z)g_a(\varphi(z))|. \end{aligned}$$

Let $a = \varphi(\lambda)$, it follows that

$$\begin{aligned} C\|g_a\|_\beta &\geq (1-|\lambda|^2)|\big(2\varphi'(\lambda)u'(\lambda)+\varphi''(\lambda)u(\lambda)\big)g'_{\varphi(\lambda)}(\varphi(\lambda)) \\ &+ g''_{\varphi(\lambda)}(\varphi(\lambda))(\varphi'(\lambda))^2u(\lambda)+u''(\lambda)g_{\varphi(\lambda)}(\varphi(\lambda))| \\ &= (1-|\lambda|^2)|\big(2\varphi'(\lambda)u'(\lambda)+\varphi''(\lambda)u(\lambda)\big)\frac{-2\overline{\varphi(\lambda)^2}}{1-|\varphi(\lambda)|^2}| \\ &\geq \frac{1}{2}\frac{(1-|\lambda|^2)|\big(2\varphi'(\lambda)u'(\lambda)+\varphi''(\lambda)u(\lambda)\big)|}{1-|\varphi(\lambda)|^2}. \end{aligned}$$

For $\forall\lambda \in D$ with $|\varphi(\lambda)| \leq \frac{1}{2}$, by (3.6), we have

$$\sup_{\lambda\in D}\frac{(1-|\lambda|^2)|\big(2\varphi'(\lambda)u'(\lambda)+\varphi''(\lambda)u(\lambda)\big)|}{1-|\varphi(\lambda)|^2} \leq \frac{4}{3}\sup_{\lambda\in D}(1-|\lambda|^2)|\big(2\varphi'(\lambda)u'(\lambda)+\varphi''(\lambda)u(\lambda)\big)| < +\infty.$$

Hence (3.2) holds.

Finally we will show (3.3) holds. Fix $a \in D$ with $|a| > \frac{1}{2}$, we take the test functions:

$$h_a(z) = -\frac{3(1-|a|^2)^2}{(1-\bar{a}z)^2} + \frac{6(1-|a|^2)^3}{(1-\bar{a}z)^3} - \frac{3(1-|a|^2)^4}{(1-\bar{a}z)^4} \tag{3.10}$$

for $z \in D$. It is easily proved that $\sup_{\frac{1}{2}<|a|<1}\|h_a\|_\beta \leq C < \infty$, where C is not depended on a. For $w \in D$, let $a = \varphi(w)$, since

$$h_{\varphi(w)}(\varphi(w)) = 0, \quad h'_{\varphi(w)}(\varphi(w)) = 0, \quad h''_{\varphi(w)}(\varphi(w)) = \frac{-6(\overline{\varphi(w)})^2}{(1-|\varphi(w)|^2)^2},$$

then, for all $w \in D$ with $|\varphi(w)| > \frac{1}{2}$, we obtain that

$$C\|h_a\|_\beta \geq \|uC_\varphi g_a\|_{\mathcal{Z}} \geq (1-|w|^2)\frac{|6u(w)(\varphi'(w))^2(\overline{\varphi(w)})^2|}{(1-|\varphi(w)|^2)^2}.$$

Then, by (3.7), we have

$$\begin{aligned} &\sup_{w\in D}\frac{(1-|w|^2)|u(w)(\varphi'(w))^2|}{(1-|\varphi(w)|^2)^2} \leq \sup_{|\varphi(w)|>\frac{1}{2}}\frac{(1-|w|^2)|u(w)(\varphi'(w))^2|}{(1-|\varphi(w)|^2)^2} \\ &+ \sup_{|\varphi(w)|\leq\frac{1}{2}}\frac{(1-|w|^2)|u(w)(\varphi'(w))^2|}{(1-|\varphi(w)|^2)^2} \\ &\leq 4\sup_{|\varphi(w)|>\frac{1}{2}}(1-|w|^2)|\frac{|u(w)(\varphi'(w))^2(\overline{\varphi(w)})^2|}{(1-|\varphi(w)|^2)^2} + \frac{16}{9}\sup_{|\varphi(w)|\leq\frac{1}{2}}(1-|w|^2)|u(w)(\varphi'(w))^2| \\ &< \infty. \end{aligned}$$

Hence (3.3) holds.

Conversely, suppose that (3.1), (3.2), and (3.2) hold. For $f \in \beta$, by Lemma 2.1, we have the following inequality:

$$\begin{aligned}
(1-|z|^2)|(uC_\varphi f)''(z)| &= (1-|z|^2)|\big(2\varphi'(z)u'(z)+\varphi''(z)u(z)\big)f'(\varphi(z)) \\
&+ f''(\varphi(z))(\varphi'(z))^2u(z)+u''(z)f(\varphi(z))| \\
&\leq (1-|z|^2)|\big(2\varphi'(z)u'(z)+\varphi''(z)u(z)\big)f'(\varphi(z))| \\
&+ (1-|z|^2)|f''(\varphi(z))(\varphi'(z))^2u(z)|+(1-|z|^2)|u''(z)f(\varphi(z))| \\
&\leq \frac{(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)|}{1-|\varphi(z)|^2}b(f) \\
&+ 8\frac{(1-|z|^2)|(\varphi'(z))^2u(z)|}{(1-|\varphi(z)|^2)^2}b(f)+(1-|z|^2)|u''(z)|\log(\frac{e}{1-|\varphi(z)|^2})\|f\|_\beta \\
&\leq C\|f\|_\beta,
\end{aligned}$$

and

$$\begin{aligned}
&|u(0)f(\varphi(0))|+|u'(0)f(\varphi(0))|+|u(0)f'(\varphi(0))\varphi'(0)| \\
&\leq \big((|u(0)|+|u'(0)|)\log(\frac{e}{1-|\varphi(0)|^2})+\frac{|u(0)\varphi'(0)|}{1-|\varphi(0)|^2}\big)\|f\|_\beta.
\end{aligned}$$

This shows that uC_φ is bounded. This completes the proof of Theorem 3.1.

Corollary 3.1 *Let φ be an analytic self-map of D. Then C_φ is a bounded operator from the Bloch space β to the Zygmund space $\mathcal{Z}$ if and only if*

$$\sup_{z\in D}\frac{(1-|z|^2)|(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^2}<\infty \quad\text{and}\quad \sup_{z\in D}\frac{(1-|z|^2)|\varphi''(z)|}{1-|\varphi(z)|^2}<\infty.$$

In the formulation of lemma, we use the notation M_u on $H(D)$ defined by $M_uf=uf$ for $f\in H(D)$.

Corollary 3.2 *The pointwise multiplier $M_u:\beta\to\mathcal{Z}$ is a bounded operator if and only if $u=0$.*

Theorem 3.2 *Let u be an analytic function on the unit disc D and φ an analytic self-map of D. Then uC_φ is a compact operator from β to $\mathcal{Z}$ if and only if uC_φ is a bounded operator and the following are satisfied:*

$$\lim_{|\varphi(z)|\to1^-}(1-|z|^2)|u''(z)|\log\frac{e}{1-|\varphi(z)|^2}=0; \tag{3.11}$$

$$\lim_{|\varphi(z)|\to1^-}\frac{(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)|}{1-|\varphi(z)|^2}=0; \tag{3.12}$$

$$\lim_{|\varphi(z)|\to1^-}\frac{(1-|z|^2)|u(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^2}=0. \tag{3.13}$$

Proof Suppose that uC_φ is compact from β to the Zygmund space $\mathcal{Z}$. Let $\{z_n\}$ be a sequence in D such that $|\varphi(z_n)| \to 1$ as $n \to \infty$. If such a sequence does not exist then (3.11), (3.12) and (3.13) are automatically satisfied. Without loss of generality we may suppose that $|\varphi(z_n)| > \frac{1}{2}$ for all n. We take the test functions

$$f_n(z) = \frac{6}{a_n} \log^2 \frac{e}{1-\overline{\varphi(z_n)}z} - \frac{8}{a_n^2} \log^3 \frac{e}{1-\overline{\varphi(z_n)}z} + \frac{3}{a_n^3} \log^4 \frac{e}{1-|\varphi(z_n)|^2}. \tag{3.14}$$

where $a_n = \log \dfrac{e}{1-|\varphi(z_n)|^2}$. By a directly calculation, we may easily prove that $\{f_n\}$ converges to 0 uniformly on compact subsets of D and $\sup_n \|f_n\|_\beta \le C < \infty$. Then $\{f_n\}$ is a bounded sequence in β which converges to 0 uniformly on compact subsets of D. Then $\lim_{n\to\infty} \|uC_\varphi(f_n)\|_{\mathcal{Z}} = 0$ by Lemma 2.4. Note that

$$f_n(\varphi(z_n)) = a_n, \quad f_n'(\varphi(z_n)) = 0, \quad f_n''(\varphi(z_n)) = 0.$$

It follows that

$$\|uC_\varphi f_n\|_{\mathcal{Z}} \ge (1-|z_n|^2)|(2u'(z_n)\varphi'(z_n) + \varphi''(z_n)u(z_n))f_n'(\varphi(z_n))$$

$$+u(z_n)f_n''(\varphi(z_n))(\varphi'(z_n))^2 + u''(z_n)f_n(\varphi(z_n))|$$

$$= (1-|z_n|^2)|u''(z_n)| \log \frac{e}{1-|\varphi(z_n)|^2}.$$

Then

$$\lim_{n\to\infty} (1-|z_n|^2)|u''(z_n)| \log \frac{e}{1-|\varphi(z_n)|^2} = 0.$$

Next, let

$$g_n(z) = \frac{8(1-|\varphi(z_n)|^2)^2}{(1-\overline{\varphi(z_n)}z)^2} - \frac{14(1-|\varphi(z_n)|^2)^3}{(1-\overline{\varphi(z_n)}z)^3} + \frac{6(1-|\varphi(z_n)|^2)^4}{(1-\overline{\varphi(z_n)}z)^4}.$$

By a directly calculation we obtain that $g_n \rightrightarrows 0$ $(n \to \infty)$ on compact subsets of D and $\sup_n \|g_n\|_\beta \le C < \infty$. Consequently, $\{g_n\}$ is a bounded sequence in β which converges to 0 uniformly on compact subsets of D. Then $\lim_{n\to\infty} \|uC_\varphi(g_n)\|_{\mathcal{Z}} = 0$ by Lemma 2.4. Note that $g_n(\varphi(z_n)) \equiv 0$, $g_n''(\varphi(z_n)) \equiv 0$ and $g_n'(\varphi(z_n)) = \dfrac{-2\overline{\varphi(z_n)}}{1-|\varphi(z_n)|^2}$.

It follows that

$$\|uC_\varphi g_n\|_{\mathcal{Z}} \ge (1-|z_n|^2)|(2u'(z_n)\varphi'(z_n) + \varphi''(z_n)u(z_n))g_n'(\varphi(z_n))$$

$$+u(z_n)g_n''(\varphi(z_n))(\varphi'(z_n))^2 + u''(z_n)g_n(\varphi(z_n))|$$

$$= 2(1-|z_n|^2)|(2u'(z_n)\varphi'(z_n) + \varphi''(z_n)u(z_n))\frac{\overline{\varphi(z_n)}}{1-|\overline{\varphi(z_n)}|^2}|.$$

Then $\displaystyle\lim_{n\to\infty} \frac{(1-|z_n|^2)|2u'(z_n)\varphi'(z_n) + \varphi''(z_n)u(z_n)|}{1-|\varphi(z_n)|^2} = 0.$

Finally, let

$$h_n(z) = -\frac{3(1-|\varphi(z_n)|^2)^2}{(1-\overline{\varphi(z_n)}z)^2} + \frac{6(1-|\varphi(z_n)|^2)^3}{(1-\overline{\varphi(z_n)}z)^3} - \frac{3(1-|\varphi(z_n)|^2)^4}{(1-\overline{\varphi(z_n)}z)^4}.$$

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By a directly calculation we obtain that $h_n \rightrightarrows 0$ $(n \to \infty)$ on compact subsets of D and $\sup_n \|h_n\|_{\mathcal{Z}} \le C < \infty$. Consequently, $\{h_n\}$ is a bounded sequence in $\mathcal{Z}$ which converges to 0 uniformly on compact subsets of D. Then $\lim_{n\to\infty} \|uC_\varphi(h_n)\|_{\mathcal{Z}} = 0$ by Lemma 2.4. Note that $h_n(\varphi(z_n)) \equiv 0$, $h_n'(\varphi(z_n)) \equiv 0$ and $h_n''(\varphi(z_n)) = \dfrac{-6(\overline{\varphi(z_n)})^2}{(1-|\varphi(z_n)|^2)^2}$. It follows that

$$\|uC_\varphi h_n\|_{\mathcal{Z}} \ge (1-|z_n|^2)|(2u'(z_n)\varphi'(z_n) + \varphi''(z_n)u(z_n))h_n'(\varphi(z_n))$$

$$+u(z_n)h_n''(\varphi(z_n))(\varphi'(z_n))^2 + u''(z_n)h_n(\varphi(z_n))|$$

$$= 6(1-|z_n|^2)|u(z_n)(\varphi'(z_n))^2|\frac{|\overline{\varphi(z_n)}|^2}{(1-|\varphi(z_n)|^2)^2}.$$

Then $\lim\limits_{n\to\infty}(1-|z_n|^2)\dfrac{|u(z_n)(\varphi'(z_n))^2|}{(1-|\varphi(z_n)|^2)^2} = 0$. The proof of the necessary is completed.

Conversely, suppose that (3.11), (3.12), and (3.13) hold. Since uC_φ is a bounded operator, by Theorem 3.1, we have

$$M_1 \triangleq \sup_{z\in D}(1-|z|^2)|u''(z)|\log\frac{1}{1-|\varphi(z)|^2} < \infty, \qquad M_3 \triangleq \sup_{z\in D}\frac{(1-|z|^2)|u(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^2} < \infty,$$

and

$$M_2 \triangleq \sup_{z\in D}\frac{(1-|z|^2)|2\varphi'(z)u'(z) + \varphi''(z)u(z)|}{1-|\varphi(z)|^2} < \infty.$$

Let $\{f_n\}$ be a bounded sequence in β with $\|f_n\|_\beta \le 1$ and $f_n \to 0$ uniformly on compact subsets of D. We only prove $\lim\limits_{n\to\infty}\|uC_\varphi(f_n)\|_{\mathcal{Z}} = 0$ by Lemma 2.4. By the assumption, for any $\epsilon > 0$, there is a constant δ, $0 < \delta < 1$, such that $\delta < |\varphi(z)| < 1$ implies

$$\frac{(1-|z|^2)|u(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^2} < \epsilon, \qquad (1-|z|^2)|u''(z)|\log\frac{e}{1-|\varphi(z)|^2} < \epsilon,$$

and

$$\frac{(1-|z|^2)|2\varphi'(z)u'(z) + \varphi''(z)u(z)|}{1-|\varphi(z)|^2} < \epsilon.$$

Let $K = \{w \in D : |w| \le \delta\}$. Noting that K is a compact subset of D, we get that

$$\begin{aligned}
z(uC_\varphi f_n) &= \sup_{z\in D}(1-|z|^2)|(uC_\varphi f_n)''(z)| \\
&\le \sup_{z\in D}(1-|z|^2)|\big(2\varphi'(z)u'(z) + \varphi''(z)u(z)\big)f_n'(\varphi(z))| \\
&+ \sup_{z\in D}(1-|z|^2)|f_n''(\varphi(z))(\varphi'(z))^2u(z)| + \sup_{z\in D}(1-|z|^2)|u''(z)f_n(\varphi(z))| \\
&\le 10\epsilon + \sup_{|\varphi(z)|\le\delta}(1-|z|^2)|\big(2\varphi'(z)u'(z) + \varphi''(z)u(z)\big)f_n'(\varphi(z))| \\
&+ \sup_{|\varphi(z)|\le\delta}(1-|z|^2)|f_n''(\varphi(z))(\varphi'(z))^2u(z)| + \sup_{|\varphi(z)|\le\delta}(1-|z|^2)|u''(z)f_n(\varphi(z))| \\
&\le 10\epsilon + M_2\sup_{w\in K}|f_n'(w)| + M_3\sup_{w\in K}|f_n''(w)| + M_1\sup_{w\in K}|f_n(w)|.
\end{aligned}$$

As $n \to \infty$, $\|uC_\varphi f_n\|_{\mathcal{Z}} \to 0$. Hence uC_φ is compact. This completes the proof of Theorem 3.2.

Corollary 3.3 *Let φ be an analytic self-map of D. Then C_φ is a compact operator from the Bloch space β to the Zygmund space $\mathcal{Z}$ if and only if C_φ is bounded,*

$$\lim_{|\varphi(z)|\to 1^-} \frac{(1-|z|^2)|(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^2} = 0 \quad \text{and} \quad \lim_{|\varphi(z)|\to 1^-} \frac{(1-|z|^2)|\varphi''(z)|}{1-|\varphi(z)|^2} = 0.$$

Theorem 3.3 *Let u be an analytic function on the unit disc D, and φ an analytic self-map of D. Then $uC_\varphi : \beta_0 \to \mathcal{Z}_0$ is a bounded operator if and only if $u \in \mathcal{Z}_0$, (3.1), (3.2), and (3.3) hold, and the following are satisfied:*

$$\lim_{|z|\to 1^-} (1-|z|^2)|2\varphi'(z)u'(z) + \varphi''(z)u(z)| = 0. \tag{3.15}$$

$$\lim_{|z|\to 1^-} (1-|z|^2)|u(z)(\varphi'(z))^2| = 0; \tag{3.16}$$

Proof Suppose that uC_φ is bounded from the little Bloch space β_0 to the little Zygmund type spaces $\mathcal{Z}_0$. Then $u = uC_\varphi 1 \in \mathcal{Z}_0$. Also $u\varphi = uC_\varphi z \in \mathcal{Z}_0$, thus

$$(1-|z|^2)|2\varphi'(z)u'(z) + \varphi''(z)u(z) + \varphi(z)u''(z)| \longrightarrow 0 \quad (|z| \to 1^-).$$

Since $|\varphi| \le 1$ and $u \in \mathcal{Z}_0$, we have $\lim_{|z|\to 1^-} (1-|z|^2)|2\varphi'(z)u'(z) + \varphi''(z)u(z)| = 0$. Hence (3.15) holds.

Similarly, $uC_\varphi z^2 \in \mathcal{Z}_0$, then

$$(1-|z|^2)|4\varphi(z)\varphi'(z)u'(z) + \varphi^2(z)u''(z) + 2u(z)(\varphi(z)\varphi''(z) + (\varphi'(z))^2)| \longrightarrow 0 \quad (|z| \to 1^-).$$

By (3.15), $|\varphi| \le 1$ and $u \in \mathcal{Z}_0$, we get that $\lim_{|z|\to 1^-} (1-|z|^2)|u(z)(\varphi'(z))^2| = 0$, i. e. that (3.16) holds. On the other hand, from Lemma 2.3 and Theorem 3.1, we obtain that (3.1), (3.2), and (3.3) hold.

Conversely, for $\forall f \in \beta_0$, we have both $(1-|z|^2)^2|f''(z)| \to 0$ and $\frac{|f(z)|}{\ln \frac{e}{1-|z|^2}} \to 0$ as $|z| \to 1^-$ by Lemma 2.2. Given $\epsilon > 0$ there is a $0 < \delta < 1$ such that $(1-|z|^2)|f'(z)| < \frac{\epsilon}{3M_2}$, $(1-|z|^2)^2|f''(z)| < \frac{\epsilon}{3M_3}$ and $\frac{|f(z)|}{\log \frac{e}{1-|z|^2}} < \frac{\epsilon}{3M_1}$ for all z with $\delta < |z| < 1$, where $M_1,\ M_2,\ M_3$ are defined in above.

If $|\varphi(z)| > \delta$, it follows that

$$\begin{aligned}
(1-|z|^2)|(uC_\varphi f)''(z)| &= (1-|z|^2)|[2\varphi'(z)u'(z) + \varphi''(z)u(z)]f'(\varphi(z)) \\
&+ f''(\varphi(z))(\varphi'(z))^2u(z) + u''(z)f(\varphi(z))| \\
&\le (1-|z|^2)|[2\varphi'(z)u'(z) + \varphi''(z)u(z)]f'(\varphi(z))| \\
&+ (1-|z|^2)|f''(\varphi(z))(\varphi'(z))^2u(z)| + (1-|z|^2)|u''(z)f(\varphi(z))| \\
&\le M_2(1-|\varphi(z)|^2)|f'(\varphi(z))| + M_3(1-|\varphi(z)|^2)^2|f''(\varphi(z))| + M_1\frac{|f(\varphi(z))|}{\log \frac{e}{1-|\varphi(z)|^2}} \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

We know that there exists a constant M_4 such that $|f(z)| \le M_3$, $|f'(z)| \le M_4$ and $|f''(z)| \le M_4$ for all $|z| \le \delta$.

If $|\varphi(z)| \le \delta$, it follows that

$$\begin{aligned}(1-|z|^2)|(uC_\varphi f)''(z)| &= (1-|z|^2)|[2\varphi'(z)u'(z)+\varphi''(z)u(z)]f'(\varphi(z)) \\ &+ f''(\varphi(z))(\varphi'(z))^2u(z)+u''(z)f(\varphi(z))| \\ &\le M_4(1-|z|^2)||2\varphi'(z)u'(z)+\varphi''(z)u(z)| \\ &+ M_4(1-|z|^2)|(\varphi'(z))^2u(z)|+M_4(1-|z|^2)|u''(z)|.\end{aligned}$$

Thus we conclude that $(1-|z|^2)|(uC_\varphi(f))''(z)| \to 0$ as $|z| \to 1^-$. Hence $uC_\varphi f \in \mathcal{Z}_0$ for all $f \in \beta_0$. On the other hand, uC_φ is a bounded operator from β to $\mathcal{Z}$ by Theorem 3.1. Hence uC_φ is a bounded operator from the little Bloch space β_0 to the little Zygmund space $\mathcal{Z}_0$.

Corollary 3.4 *Let φ be an analytic self-map of D. Then C_φ is a bounded operator from β_0 to $\mathcal{Z}_0$ if and only if C_φ is a bounded operator from β to $\mathcal{Z}$ and $\varphi \in \mathcal{Z}_0$.*

Proof By Theorem 3.3 we have that C_φ is a bounded operator from β_0 to $\mathcal{Z}_0$ if and only $C_\varphi : \beta \to \mathcal{Z}$ is bounded, $\varphi \in \mathcal{Z}_0$, and

$$\lim_{|z|\to 1^-}(1-|z|^2)|(\varphi'(z))^2| = 0.$$

However, That $\varphi \in \mathcal{Z}_0$ means $\varphi' \in \beta_0$. Then we have that $|\varphi'(z)| \le \log \frac{e}{1-|z|^2}\|\varphi'\|_\beta$ by Lemma 2.1. It follows that

$$(1-|z|^2)|(\varphi'(z))^2| \le (1-|z|^2)\log^2\frac{e}{1-|z|^2}\|\varphi'\|_\beta^2 \to 0,$$

as $|z| \to 1^-$.

Theorem 3.4 *Let u be an analytic function on the unit disc D, and φ an analytic self-map of D. Then uC_φ is compact from β_0 to $\mathcal{Z}_0$ if and only if the following are satisfied:*

$$\lim_{|z|\to 1^-}(1-|z|^2)|u''(z)|\log\frac{e}{1-|\varphi(z)|^2} = 0; \tag{3.17}$$

$$\lim_{|z|\to 1^-}\frac{(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)|}{1-|\varphi(z)|^2} = 0; \tag{3.18}$$

$$\lim_{|z|\to 1^-}\frac{(1-|z|^2)|u(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^2} = 0. \tag{3.19}$$

Proof Assume (3.17), (3.18), and (3.19) hold. From Theorem 3.3, we know that uC_φ is bounded from β_0 to $\mathcal{Z}_0$. Suppose that $f \in \beta_0$ with $\|f\|_\beta \le 1$. We obtain that

$$\begin{aligned}&(1-|z|^2)|(uC_\varphi f)''(z)| \le (1-|z|^2)|\big(2\varphi'(z)u'(z)+\varphi''(z)u(z)\big)f'(\varphi(z))| \\ &+(1-|z|^2)|f''(\varphi(z))(\varphi'(z))^2u(z)|+(1-|z|^2)|u''(z)f(\varphi(z))| \\ &\le (1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)|\frac{1}{1-|\varphi(z)|^2}b(f) \\ &+8\frac{(1-|z|^2)|(\varphi'(z))^2u(z)|}{(1-|\varphi(z)|^2)^2}b(f)+(1-|z|^2)|u''(z)|\log\frac{e}{1-|\varphi(z)|^2}\|f\|_\beta,\end{aligned}$$

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thus

$$\sup\{|(1-|z|^2)(uC_\varphi f)''(z)| : f \in \beta_0, \|f\|_\beta \le 1\}$$

$$\le (1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)|\frac{1}{1-|\varphi(z)|^2}$$

$$+\frac{8(1-|z|^2)|(\varphi'(z))^2u(z)|}{(1-|\varphi(z)|^2)^2}+(1-|z|^2)|u''(z)|\log\frac{e}{1-|\varphi(z)|^2},$$

and it follows that

$$\lim_{|z|\to 1^-}\sup\{|(1-|z|^2)(uC_\varphi f)''(z)| : f \in \beta_0, \|f\|_\beta \le 1\}=0,$$

hence $uC_\varphi : \beta_0 \to \mathcal{Z}_0$ is compact by Lemma 2.5.

Conversely, suppose that $uC_\varphi : \beta_0 \to \mathcal{Z}_0$ is compact.

First, it is obvious $uC_\varphi : \beta_0 \to \mathcal{Z}_0$ is bounded, then by Theorem 3.3, we have $u \in \mathcal{Z}_0$ and that (3.15) and (3.16) hold. On the other hand, by Lemma 2.5 we have

$$\lim_{|z|\to 1^-}\sup\{|(1-|z|^2)(uC_\varphi f)''(z)| : f \in \beta_0, \|f\|_\beta \le M\}=0,$$

for some $M > 0$.

Next, noting that the proof of Theorem 3.1 and the fact that the functions given in (3.8) are in β_0 and have norms bounded independently of a, we obtain that

$$\lim_{|z|\to 1^-}(1-|z|^2)|u''(z)|\log\frac{e}{1-|\varphi(z)|^2}=0.$$

Similarly, noting that the functions given in (3.9) are in β_0 and have norms bounded independently of a, we obtain that

$$\lim_{|z|\to 1^-}\frac{(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)|}{1-|\varphi(z)|^2}=0 \tag{3.20}$$

for $|\varphi(z)| > \frac{1}{2}$. However, if $|\varphi(z)| \le \frac{1}{2}$, by (3.15), we easily have

$$\lim_{|z|\to 1^-}\frac{(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)|}{1-|\varphi(z)|^2}$$

$$\le \frac{4}{3}\lim_{|z|\to 1^-}(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)|=0.$$

Thus (3.18) holds.

Also, the third statement, that (3.19), is proved similarly. We omitted it here. This completes the proof of Theorem 4.2.

Corollary 3.5 *Let φ be an analytic self-map of D. Then C_φ is a compact operator from β_0 to $\mathcal{Z}_0$ if and only if*

$$\lim_{|z|\to 1^-}\frac{(1-|z|^2)|(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^2}=0$$

and

$$\lim_{|z|\to 1^-}\frac{(1-|z|^2)|\varphi''(z)|}{1-|\varphi(z)|^2}=0.$$

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Approximate homomorphisms and derivations on non-Archimedean Lie JC^*-algebras

Javad Shokri[1] and Dong Yun Shin[2*]

[1]Department of Mathematics, Urmia University, P. O. Box 165, Urmia, Iran

[2]Department of Mathematics, University of Seoul, Seoul 02504, Republic of Korea

Abstract. In this paper, by using the fixed point method, we prove the Hyers-Ulam stability of homomorphisms in non-Archimedean Lie JC^*-algebras and derivations on non-Archimedean Lie JC^*-algebras associated with the following additive mapping:

$$\sum_{k=2}^{n}\Big(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{n-k+1}=i_{n-k}+1}^{n}\Big)f\Big(\sum_{i=1,i\neq i_1,\cdots,i_{n-k+1}}^{n}x_i-\sum_{r=1}^{n-k+1}x_{i_r}\Big)+f\Big(\sum_{i=1}^{n}x_i\Big)=2^{n-1}f(x_1)$$

for a fixed positive integer n with $n\geqslant 2$.

1. Introduction

In 1896, Hensel [4] introduced a field with a valuation in which does not have the Archimedean property. Let $\mathcal{K}$ be a field. A non-Archimedean absolute value on$\mathcal{K}$ is a function $|\cdot|:\mathcal{K}\to[0,+\infty)$ such that, for any $a,b\in\mathcal{K}$, the following conditions are satisfying

(i) $|a|\geqslant 0$ and equality holds if and only if $a=0$,

(ii) $|ab|=|a||b|$,

(iii) $|a+b|\leqslant\max\{|a|,|b|\}$ (the strict triangle inequality).

Note that $|1|=|-1|=1$ and $|n|\leqslant 1$ for each integer n. We always assume, in addition, that $|\cdot|$ is non-trival, i.e., there exists an $a_0\neq 0,1$. A function $\|.\|:X\to[0,\infty)$ is called a non-Archimedean norm if it satisfies the following conditions:

(i) $\|x\|=0$ if and only if $x=0$;

(ii) for any $r\in K, x\in X$, $\|rx\|=|r|\|x\|$;

(iii) the strong triangle inequality holds, namely,

$$\|x+y\|\leqslant\max\{\|x\|,\|y\|\}\quad(x,y\in X).$$

Then $(X,\|\cdot\|)$ is called a non-Archimedean normed space. From the fact that

$$\|x_n-x_m\|\leqslant\max\{\|n_n-x_m\|:m\leqslant j\leqslant n-1\}\quad(n>m),$$

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*Corresponding author.

[0]**E-mail**: [1]j.shokri@urmia.ac.ir, [2]dyshin@uos.ac.kr

holds, a sequence $\{x_n\}$ is Cauchy if and only if $\{x_n - x_m\}$ converges to zero in a non-Archimedean normed space. By a complete non-Archimedean normed space we mean one in which every Cauchy sequence is convergent.

A non-Archimedean Banach algebra is a complete non-Archimedean algebra $\mathcal{A}$ which satisfies $\|ab\| \leqslant \|a\| \cdot \|b\|$ for all $a, b \in \mathcal{A}$. For more detailed definitions of non-Archimedean Banach algebras, we refer the reader to [15].

If $\mathcal{U}$ is a non-Archimedean Banach algebra, then an involution on $\mathcal{U}$ is mapping $t \to t^*$ from $\mathcal{U}$ into $\mathcal{U}$ which satisfies

$(i)\, t^{**} = t$ for $t \in \mathcal{U}$;

$(ii)\, (\alpha s + \beta t)^* = \bar{\alpha} s^* + \bar{\beta} t^*$;

$(iii)\, (st)^* = t^* s^*$ for all $s, t \in \mathcal{U}$.

If, in addition, $\|t^* t\| = \|t\|^2$ for $t \in \mathcal{U}$, then $\mathcal{U}$ is a non-Archimedean C^*-algebra.

The stability problem of functional equations originated from a question of Ulam [16] concerning the stability of group homomorphisms: Let $(G_1, *)$ be a group and let $(G_2, \diamond)$ be a metric group (a metric is defined on a set with group property) with the metric $d(.,.)$. Given $\epsilon > 0$, does there exist a $\delta(\epsilon) > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(x * y), h(x) * h(y)) < \delta$ for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$? If the answer is affirmative, we would say that the equation of homomorphism $h(x * y) = h(x) * h(y)$ is stable (see also [3, 5, 9, 10, 12, 13, 14]).

For explicitly later use, we recall a fundamental result in fixed point theory.

Theorem 1.1. *(The fixed point alternative theorem* [2]*) Let* (Ω, d) *be a complete generalized metric space and* $J : \Omega \to \Omega$ *be a strictly contractive mapping with Lipschitz constant* $0 < L < 1$*, that is,*

$$d(Jx, Jy) \leqslant L d(x, y), \quad x, y \in \Omega.$$

Then, for each given $x \in \Omega$*, either*

$$d(J^n x, J^{n+1} x) = \infty, \quad \forall n \geqslant 0,$$

or there exists a positive integer n_0 *such that*

(1) $d(J^n x, J^{n+1} x) < \infty, \quad \forall n \geqslant n_0$;

(2) *the sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of* J;

(3) y^* *is the unique fixed point of* J *in the set* $\Delta = \{y \in \Omega : d(J^{n_0} x, y) < \infty\}$;

(4) $d(y, y^*) \leqslant \frac{1}{1-L} d(y, Jy)$ *for all* $y \in \Delta$.

A non-Archimedean C^*-algebra $\mathcal{C}$, endowed with the Lie product $[x, y] := \frac{xy - yx}{2}$ and endowed with *anticommutator product* (Jordan product) $x \circ y := \frac{xy + yx}{2}$ on $\mathcal{C}$, is called a non-Archimedean Lie JC^*-algebra (see [6, 7, 8]).

Jordan algebras as coordinates for Lie algebras were created to illuminate a particular aspect of physics, quantum-mechanical observables, but turned out to have illuminating connections with many areas of mathematics.

In this paper, we prove the Hyers-Ulam stability of homomorphisms and derivations in non-Archimedean Lie JC^*-algebras associated with the following additive functional equation:

$$\sum_{k=2}^{n}\Big(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{n-k+1}=i_{n-k+1}}^{n}\Big)f\Big(\sum_{i=1,i\neq i_1,\cdots,i_{n-k+1}}^{n}x_i-\sum_{r=1}^{n-k+1}x_{i_r}\Big)$$
$$+f\Big(\sum_{i=1}^{n}x_i\Big)=2^{n-1}f(x_1) \tag{1.1}$$

for a fixed positive integer n with $n\geqslant 2$.

2. Stability of homomorphisms in non-Archimedean Lie JC^*-algebras

Definition 2.1. [7] *Let $\mathcal{A}$ and $\mathcal{B}$ be non-Archimedean Lie JC^*-algebras. A $\mathbb{C}$-linear mapping $H:\mathcal{A}\to\mathcal{B}$ is called a (non-Archimedean Lie JC^*-algebra) homomorphism if H satisfies*

$$H([x,y])=[H(x),H(y)],$$
$$H(x\circ y)=H(x)\circ h(y),$$
$$H(x^*)=H(x)^*$$

for all $x,y\in\mathcal{A}$.

Throughout this section, assume that A and B are two non-Archimedean Lie JC^*-algebras, respectively, with norm $\|\cdot\|_{\mathcal{A}}$ and $\|\cdot\|_{\mathcal{B}}$.

For a given mapping $f:\mathcal{A}\to\mathcal{B}$, we define

$$D_\mu f(x_1,\cdots,x_n):=\sum_{k=2}^{n}\Big(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{n-k+1}=i_{n-k+1}}^{n}\Big)f\Big(\sum_{i=1,i\neq i_1,\cdots,i_{n-k+1}}^{n}\mu x_i-\sum_{r=1}^{n-k+1}\mu x_{i_r}\Big)$$
$$+f\Big(\sum_{i=1}^{n}\mu x_i\Big)-2^{n-1}f(\mu x_1)$$

for all $\mu\in\mathbb{T}^1:=\{\lambda\in\mathbb{C}:|\lambda|=1\}$ and all $x_1,\cdots,x_n\in\mathcal{A}$.

We recall the following needed lemmas in this paper.

Lemma 2.2. [11] *Let $\mathcal{V}$ and $\mathcal{W}$ be linear spaces and $f:\mathcal{V}\to\mathcal{W}$ be an additive mapping such that $f(\mu x)=\mu f(x)$ for all $x\in\mathcal{V}$ and $\mu\in\mathbb{T}^1$. Then the mapping f is $\mathbb{C}$-linear.*

Lemma 2.3. [7] *A mapping $f:\to\mathcal{A}\to\mathcal{B}$ with $f(0)=0$ satisfies the functional equation (1.1) if and only if $f:\mathcal{A}\to\mathcal{B}$ is additive.*

We prove the Hyers-Ulam stability of homomorphisms in non-Archimedean Lie JC^*-algebras for the functional equation $D_\mu f(x_1,\cdots,x_n)=0$.

Theorem 2.4. *Let $f:\mathcal{A}\to\mathcal{B}$ be a mapping for which there are functions $\varphi:\mathcal{A}^n\to[0,\infty),\psi:\mathcal{A}^2\to[0,\infty)$, and $\eta:\mathcal{A}\to[0,\infty)$ such that $|2|<1$ is far from zero and*

$$\lim_{m\to\infty}\frac{1}{|2|^m}\varphi(2^mx_1,2^mx_2,\cdots,2^mx_n)=0, \tag{2.1}$$

$$\lim_{m\to\infty}\frac{1}{|2|^{2m}}\psi(2^mx,2^my)=0, \tag{2.2}$$

$$\lim_{m\to\infty}\frac{1}{|2|^m}\eta(2^m x)=0, \tag{2.3}$$

$$\|D_\mu f(x_1,\cdots,x_n)\|_{\mathcal{B}}\leqslant\varphi(x_1,\cdots,x_n), \tag{2.4}$$

$$\|f([x,y])-[f(x),f(y)]\|_{\mathcal{B}}\leqslant\psi(x,y), \tag{2.5}$$

$$\|f(x\circ y)-f(x)\circ f(y)\|_{\mathcal{B}}\leqslant\psi(x,y), \tag{2.6}$$

$$\|f(x^*)-f(x)^*\|_{\mathcal{B}}\leqslant\eta(x), \tag{2.7}$$

for all $x,y,x_1,\cdots,x_n\in\mathcal{A}$ and $\mu\in\mathbb{T}^1$. If there exists a constant $0<L<1$ such that $\varphi(x_1,x_2,\cdots,x_n)\leqslant\alpha L\varphi(\frac{x_1}{2},\frac{x_2}{2},\cdots,\frac{x_n}{2})$ for all $x_1,x_2,\cdots,x_n\in\mathcal{A}$, where $\alpha=|2|^{n-1}$, then there exists a unique homomorphism $H:\mathcal{A}\to\mathcal{B}$ such that

$$\|f(x)-H(x)\|\leqslant\frac{L}{1-L}\varphi(\frac{x}{2},\frac{x}{2},0,\cdots,0) \tag{2.8}$$

for all $x\in\mathcal{A}$.

Proof. Let $\mu=1$. Using the following relation

$$1+\sum_{k=1}^{n-k}\binom{n-k}{k}=\sum_{k=0}^{n-k}\binom{n-k}{k}=2^{n-k} \tag{2.9}$$

for all $n>k$ and putting $x_1=x_2=x$ and $x_3=x_4=\cdots=x_n=0$ in (2.4), we obtain

$$\|\frac{\alpha}{2}f(2x)-\alpha f(x)\|_{\mathcal{B}}\leqslant\varphi(x,x,0,\cdots,0)$$

for all $x\in\mathcal{A}$. So

$$\|\frac{1}{2}f(2x)-f(x)\|_{\mathcal{B}}\leqslant\frac{1}{\alpha}\varphi(x,x,0,\cdots,0)\leqslant L\varphi\Big(\frac{x}{2},\frac{x}{2},0,\cdots,0\Big) \tag{2.10}$$

for all $x\in\mathcal{A}$. Let define $\Omega:=\{g:\mathcal{A}\to\mathcal{B}\}$ and introduce a generalized metric on Ω as follows

$$d(g,h)=\inf\{k\in(0,\infty):\|g(x)-h(x)\|_{\mathcal{B}}<k\varphi\Big(\frac{x}{2},\frac{x}{2},0,\cdots,0\Big),\forall x\in\mathcal{A}\}.$$

It is easy to show that (Ω,d) is a generalized complete metric space (see [1]).

Now we consider the function $J:\Omega\to\Omega$ define by $Jg(x)=\frac{1}{|2|}g(2x)$ for all $x\in\mathcal{A}$ and $g\in\Omega$. Let for all $g,h\in\Omega$ and an arbitrary constant $k\in[0,\infty)$ with $d(x,y)\leqslant k$, we have

$$\|g(x)-h(x)\|_{\mathcal{B}}\leqslant k\varphi\Big(\frac{x}{2},\frac{x}{2},0,\cdots,0\Big)$$

for all $x\in\mathcal{A}$. Then we can write

$$\|Jg(x)-Jh(x)\|_{\mathcal{B}}=\frac{1}{|2|}\|g(2x)-h(2x)\|_{\mathcal{B}}\leqslant\frac{k}{|2|}\varphi(x,x,0,\cdots,0)\leqslant\frac{\alpha kL}{|2|}\varphi\Big(\frac{x}{2},\frac{x}{2},0,\cdots,0\Big)$$

for all $x\in\mathcal{A}$. So we conclude that $d(Jg,Jh)\leqslant\frac{\alpha}{|2|}Ld(g,h)$ for all $g,h\in\Omega$. It follows from (2.9) that $d(Jf,f)\leqslant L$, that is, J is a self-function of Ω with the Lipchitz constant L. Therefore, from Theorem 1.1, there eists a fixed point H of J set $\Omega_1=\{h\in X:d(f,h)<\infty\}$ such that

$$H(x)=\lim_{m\to\infty}\frac{1}{|2|^m}f(2^m x) \tag{2.11}$$

for all $x \in \mathcal{A}$, since $\lim_{m\to\infty} d(J^n f, H) = 0$. Also $2H(\frac{x}{2}) = H(x)$ for all $x \in \mathcal{A}$. Thus $H : \mathcal{A} \to \mathcal{B}$ is the unique fixed point of J in Ω_1 such that

$$d(H, f) \leqslant \frac{1}{1-L} d(Jf, f) \leqslant \frac{L}{1-L},$$

i.e., H satisfies (2.8) for all $x \in \mathcal{A}$. It follows from the definition of H, (2.1) and (2.4) that

$$\sum_{k=2}^{n} \Big(\sum_{i_1=2}^{k} \sum_{i_2=i_1+1}^{k+1} \cdots \sum_{i_{n-k+1}=i_{n-k+1}}^{n} \Big) H\Big(\sum_{i=1, i\neq i_1,\cdots,i_{n-k+1}}^{n} x_i - \sum_{r=1}^{n-k+1} x_{i_r} \Big) + H\Big(\sum_{i=1}^{n} x_i \Big) = 2^{n-1} H(x_1)$$

for all $x_1, x_2, \cdots, x_n \in \mathcal{A}$. Since $H(0) = 0$, by Lemma 2.3, the mapping H is additive.

Put $x_1 = x$ and $x_2 = x_3 = \cdots = 0$ in (2.4). It follows from (2.9) that

$$\|f(\mu x) - \mu f(x)\| \leqslant \frac{1}{\alpha} \varphi(x, 0, \cdots, 0) \tag{2.12}$$

for all $x \in \mathcal{A}$ and all $\mu \in \mathbb{T}^1$. Also we conclude

$$\|\frac{1}{2^m}(f(\mu 2^m x) - \mu f(2^m x))\|_{\mathcal{B}} \leqslant \frac{1}{\alpha |2|^m} \varphi(2^m x, 0, \cdots, 0)$$

for all $x \in \mathcal{A}$ and all $\mu \in \mathbb{T}^1$. The right hand side of the above inequality tends to zero as $m \to \infty$, and so we obtain

$$H(\mu x) = \lim_{m\to\infty} \frac{1}{|2|^m} f(\mu 2^m x) = \lim_{m\to\infty} \frac{1}{|2|^m} \mu f(2^m x) = \mu H(x)$$

for all $x \in \mathcal{A}$ and all $\mu \in \mathbb{T}^1$. Hence by Lemma 2.2, the mapping $H : \mathcal{A} \to \mathcal{B}$ is $\mathbb{C}$-linear.

It follows from (2.2), (2.5), (2.6) and (2.11) that

$$\begin{aligned} \|H([x, y]) - [H(x), H(y)]\|_{\mathcal{B}} &= \lim_{m\to\infty} \frac{1}{|2|^{2m}} \|f([2^m x, 2^m y]) - [f(2^m x), f(2^m y)]\|_{\mathcal{B}} \\ &\leqslant \lim_{m\to\infty} \frac{1}{|2|^{2m}} \psi(2^m x, 2^m y) = 0 \end{aligned}$$

and

$$\begin{aligned} \|H(x \circ y) - H(x) \circ H(y)\|_{\mathcal{B}} &= \lim_{m\to\infty} \frac{1}{|2|^{2m}} \|f(2^m x \circ 2^m y) - f(2^m x) \circ f(2^m y)\|_{\mathcal{B}} \\ &\leqslant \lim_{m\to\infty} \frac{1}{|2|^{2m}} \psi(2^m x, 2^m y) = 0 \end{aligned}$$

for all $x, y \in \mathcal{A}$. So

$$H([x, y]) = [H(x), H(y) \quad \text{and} \quad H(x \circ y) = H(x) \circ H(y)$$

for all $x, y \in \mathcal{A}$.

Similarly, by (2.3), (2.7) and (2.11), we have

$$\|H(x^*) - H(x)^*\|_{\mathcal{B}} = \lim_{m\to\infty} \frac{1}{|2|^m} \|f(2^m x^*) - f(2^m x)^*\|_{\mathcal{B}} \leqslant \lim_{m\to\infty} \frac{1}{|2|^m} \eta(2^m x) = 0$$

and so $H(x^*) = H(x)^*$ for all $x, y \in \mathcal{A}$. Thus $H : \mathcal{A} \to \mathcal{B}$ is the desired homomorphism satisfying (2.8). □

Corollary 2.5. *Let $r > 1$ and θ be nonnegative real number, and let $f : \mathcal{A} \to \mathcal{B}$ be a mapping such that*

$$\begin{aligned}
\|D_\mu f(x_1, x_2, \cdots, x_n)\|_{\mathcal{B}} &\leqslant \theta(\||x_1\|^r_{\mathcal{A}} + \||x_2\|^r_{\mathcal{A}} + \cdots + \||x_n\|^r_{\mathcal{A}}),\\
\|f([x, y]) - [f(x), f(y)]\|_{\mathcal{B}} &\leqslant \theta \cdot \||x\|^r_{\mathcal{A}} \cdot \||y\|^r_{\mathcal{A}},\\
\|f(x \circ y) - f(x) \circ f(y)\|_{\mathcal{B}} &\leqslant \theta \cdot \||x\|^r_{\mathcal{A}} \cdot \||y\|^r_{\mathcal{A}},\\
\|f(x^*) - f(x)^*\|_{\mathcal{B}} &\leqslant \theta \cdot \||x\|^r_{\mathcal{A}},
\end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and $x, y, x_1, \cdots, x_n \in \mathcal{A}$. Then there exists a unique homomorphism $H : \mathcal{A} \to \mathcal{B}$ such that

$$\|f(x) - H(x)\|_{\mathcal{B}} \leqslant \frac{|2|\theta}{|2| - |2|^r}\|x\|^r_{\mathcal{A}}$$

for all $x \in \mathcal{A}$.

Proof. The proof follows from Theorem 2.4 by taking

$$\begin{aligned}
\varphi(x_1, x_2, \cdots, x_n) &:= \theta(\||x_1\|^r_{\mathcal{A}} + \||x_2\|^r_{\mathcal{A}} + \cdots + \||x_n\|^r_{\mathcal{A}}),\\
\psi(x, y) &:= \theta \cdot \||x\|^r_{\mathcal{A}} \cdot \||y\|^r_{\mathcal{A}},\\
\eta(x) &:= \theta \cdot \||x\|^r_{\mathcal{A}}
\end{aligned}$$

for all $x, y, x_1, \cdots, x_n \in \mathcal{A}$ and $L = |2|^{r-1}$. □

3. Stability of derivations on non-Archimedean Lie JC^*-algebras

Definition 3.1. [7] *Let $\mathcal{A}$ be a non-Archimedean Lie JC^*-algebra. A $\mathbb{C}$-linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ is called a (non-Archimedean Lie JC^*-algebra) derivation if δ satisfies*

$$\begin{aligned}
\delta([x, y]) &= [\delta(x), y] + [x, \delta(y)],\\
\delta(x \circ y) &= \delta(x) \circ y + x \circ \delta(y),\\
\delta(x^*) &= \delta(x)^*
\end{aligned}$$

for all $x \in \mathcal{A}$.

Throughout this section, assume that $\mathcal{A}$ is a non-Archimedean Lie JC^*-algebra with norm $\| \cdot \|_{\mathcal{A}}$.

We prove the Hyers-Ulam stability of derivation on non-Archimedean Lie JC^*-algebras for the functional equation $D_\mu f(x_1, \cdots, x_n) = 0$.

Theorem 3.2. *Let $f : \mathcal{A} \to \mathcal{A}$ be a mapping for which there are function $\varphi : \mathcal{A}^n \to [0, \infty)$, $\psi : \mathcal{A}^2 \to [0, \infty)$ and $\eta : \mathcal{A} \to [0, \infty)$ such that (2.1), (2.2), (2.3). (2.4) and (2.7) hold and*

$$\|f([x, y]) - [f(x), y]0[x, f(y)]\|_{\mathcal{A}} \leqslant \psi(x, y), \tag{3.1}$$

$$\|f(x \circ y) - f(x) \circ y - x \circ f(y)\|_{\mathcal{A}} \leqslant \psi(x, y) \tag{3.2}$$

for all $x, y \in \mathcal{A}$. If there exists a constant $0 < L < 1$ such that $\varphi(x_1, x_2, \cdots, x_n) \leqslant \alpha L \varphi(\frac{x_1}{2}, \frac{x_2}{2}, \cdots, \frac{x_n}{2})$ for all $x_1, x_2, \cdots, x_n \in \mathcal{A}$, where $\alpha = |2|^{n-1}$, then there exists a unique derivation $\delta : \mathcal{A} \to \mathcal{A}$ such that

$$\|f(x) - \delta(x)\| \leqslant \frac{L}{1-L} \varphi\Big(\frac{x}{2}, \frac{x}{2}, 0, \cdots, 0\Big) \tag{3.3}$$

for all $x \in \mathcal{A}$.

Proof. By the same reasoning as in the proof of Theorem 2.4, there exists a unique $\mathbb{C}$-linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ satisfying in the desired inequality (3.3) and the mapping $\delta : \mathcal{A} \to \mathcal{A}$ is defined by

$$\delta(x) = \lim_{m\to\infty} \frac{1}{|2|^m} f(2^m x) \tag{3.4}$$

for all $x \in \mathcal{A}$.

It follows from (2.2), (3.1), (3.3) and (3.4) that

$$\begin{aligned} &\|\delta([x,y]) - [\delta(x), y] - [x, \delta(y)]\|_{\mathcal{A}} \\ &\quad = \lim_{m\to\infty} \frac{1}{|2|^{2m}} \|f([2^m x, 2^m y]) - [f(2^m x), 2^m y] - [2^m x, f(2^m y)]\|_{\mathcal{A}} \\ &\quad \leqslant \lim_{m\to\infty} \frac{1}{|2|^{2m}} \psi(2^m x, 2^m y) = 0 \end{aligned}$$

and

$$\begin{aligned} &\|\delta(x \circ y) - \delta(x) \circ y - x \circ \delta(y)\|_{\mathcal{A}} \\ &\quad = \lim_{m\to\infty} \frac{1}{|2|^{2m}} \|f(2^m x \circ 2^m y) - f(2^m x) \circ 2^m y - 2^m x \circ f(2^m y)\|_{\mathcal{A}} \\ &\quad \leqslant \lim_{m\to\infty} \frac{1}{|2|^{2m}} \psi(2^m x, 2^m y) = 0 \end{aligned}$$

for all $x, y \in \mathcal{A}$. So

$$\begin{aligned} \delta([x,y]) &= [\delta(x), y] + [x, \delta(y)], \\ \delta(x \circ y) &= \delta(x) \circ y + x \circ \delta(y) \end{aligned}$$

for all $x, y \in \mathcal{A}$.

Similarly, as in the proof of Theorem 2.4, one can show $\delta(x^*) = \delta(x)^*$ for all $x \in \mathcal{A}$. Therefore, $\delta : \mathcal{A} \to \mathcal{A}$ is a non-Archimedean Lie JC^*-algebra derivation satisfying (3.4). □

Corollary 3.3. *Let $r > 1$ and θ be nonnegative and real number, and let $f : \mathcal{A} \to \mathcal{A}$ be a mapping such that*

$$\begin{aligned} \|D_\mu f(x_1, x_2, \cdots, x_n)\|_{\mathcal{B}} &\leqslant \theta(\|x_1\|_{\mathcal{A}}^r + \|x_2\|_{\mathcal{A}}^r + \cdots, \|x_n\|_{\mathcal{A}}^r), \\ \|f([x,y]) - [f(x), y] - [x, f(y)]\|_{\mathcal{B}} &\leqslant \theta \cdot \|x\|_{\mathcal{A}}^r \cdot \|y\|_{\mathcal{A}}^r, \\ \|f(x \circ y) - f(x) \circ y - x \circ f(y)\|_{\mathcal{B}} &\leqslant \theta \cdot \|x\|_{\mathcal{A}}^r \cdot \|y\|_{\mathcal{A}}^r, \\ \|f(x^*) - f(x)^*\|_{\mathcal{B}} &\leqslant \theta \cdot \|x\|_{\mathcal{A}}^r \end{aligned}$$

for all $\mu \in \mathbb{T}^1$ *and* $x, y, x_1, \cdots, x_n \in \mathcal{A}$*. Then there exists a unique homomorphism* $H : \mathcal{A} \to \mathcal{A}$ *such that*

$$\|f(x) - \delta(x)\|_{\mathcal{B}} \leqslant \frac{|2|\theta}{|2| - |2|^r} \|x\|_{\mathcal{A}}^r$$

for all $x \in \mathcal{A}$*.*

Proof. The proof follows from Theorem 3.2 by taking

$$\begin{aligned} \varphi(x_1, x_2, \cdots, x_n) &:= \theta.(\|x_1\|_{\mathcal{A}}^r + \|x_2\|_{\mathcal{A}}^r + \cdots + \|x_n\|_{\mathcal{A}}^r), \\ \psi(x, y) &:= \theta.(\|x\|_{\mathcal{A}}^r.\|y\|_{\mathcal{A}}^r), \\ \eta(x) &:= \theta.\|x\|_{\mathcal{A}}^r \end{aligned}$$

for all $x, y, x_1, \cdots, x_n \in \mathcal{A}$ and $L = |2|^{r-1}$. □

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ON DISTRIBUTION AND PROBABILITY DENSITY FUNCTIONS OF ORDER STATISTICS ARISING FROM INDEPENDENT BUT NOT NECESSARILY IDENTICALLY DISTRIBUTED RANDOM VECTORS

M. GÜNGÖR[1] and Y. BULUT[2]
[1,2] Department of Econometrics, Inonu University, 44280, Malatya, TURKEY.
[1]mgungor44@gmail.com and [2]ybulut79@gmail.com

ABSTRACT

In this study, joint probability density and distribution functions of any ***d*** order statistics of *innid* continuous random vectors are expressed. Then, some results connecting distributions of order statistics of *innid* random vectors to that of order statistics of *iid* random vectors are given.

Keywords: Order Statistics, Distribution Function, Probability Density Function, Continuous Random Variable.

MSC 2010: 62G30, 62E15.

1. Introduction

Several identities and recurrence relations for probability density function (*pdf*) and distribution function (*df*) of order statistics of independent and identically distributed (*iid*) random variables were established by numerous authors including (Arnold et al., 1992; Balasubramanian, Beg, 2003; David, 1981; Reiss, 1989). Furthermore, (Arnold et al., 1992; David, 1981; Gan, Bain, 1995; Khatri, 1962) obtained the probability function (*pf*) and *df* of order statistics of *iid* random variables from a discrete parent. (Corley, 1984) defined a multivariate generalization of classical order statistics for random samples from a continuous multivariate distribution. (Goldie, Maller, 1999) derived expressions for generalized joint densities of order statistics of *iid* random variables in terms of Radon-Nikodym derivatives with respect to product measures based on *df*. (Guilbaud, 1982) expressed the probability of the functions of independent but not necessarily identically distributed (*innid*) random vectors as a linear combination of probabilities of the functions of *iid* random vectors and thus also for order statistics of random variables.

(Cao, West, 1997) obtained recurrence relationships among the distribution functions of order statistics arising from *innid* random variables. (Vaughan, Venables, 1972) derived the joint *pdf* and marginal *pdf* of order statistics of *innid* random variables by means of permanents. (Balakrishnan, 2007; Bapat, Beg, 1989) obtained the joint *pdf* and *df* of order statistics of *innid* random variables by means of permanents. (Childs, Balakrishnan, 2006) obtained, using multinomial arguments, the *pdf* of $X_{r:n+1}$ $(1 \le r \le n+1)$ by adding another independent random variable to the original n variables $X_1, X_2, ..., X_n$. Also, (Balasubramanian et al.,1994) established the identities satisfied by distributions of order statistics from non-independent non-identical variables through operator methods based on the difference and differential operators.

In this paper, joint *df* and *pdf* of order statistics from *innid* continuous random vectors are obtained.

As far as we know, these approaches have not been considered in the framework of order statistics from *innid* continuous random vectors.

From now on, subscripts and superscripts are defined in first place in which they are used and these definitions will be valid unless they are redefined.

Consider $\mathrm{x}=(x^{(1)},x^{(2)},...,x^{(b)})$ and $\mathrm{y}=(y^{(1)},y^{(2)},...,y^{(b)})$, then it can be written as;

$\mathrm{x}\leq\mathrm{y}$ if $x^{(v)}\leq y^{(v)}$ (v=1, 2, …, b) and $\mathrm{x}+\mathrm{y}=(x^{(1)}+y^{(1)},x^{(2)}+y^{(2)},...,x^{(b)}+y^{(b)})$.

Let $\xi_i=(\xi_i^{(1)},\xi_i^{(2)},...,\xi_i^{(b)})$ (i=1,2,…,n) be n *innid* continuous random vectors which components of ξ_i are independent.

$$X_{r:n}^{(v)}=Z_{r:n}(\xi_1^{(v)},\xi_2^{(v)},...,\xi_n^{(v)}) \tag{1.1}$$

is stated as rth order statistic of vth components of $\xi_1,\xi_2,\ldots,\xi_n$.

From (1.1), ordered values of vth components of $\xi_1,\xi_2,\ldots,\xi_n$ are expressed as

$$X_{1:n}^{(v)}\leq X_{2:n}^{(v)}\leq...\leq X_{n:n}^{(v)}. \tag{1.2}$$

From (1.2), we can write $\mathrm{X}_{r:n}=(X_{r:n}^{(1)},X_{r:n}^{(2)},...,X_{r:n}^{(b)})$ $(1\leq r\leq n)$.

Also, $\mathrm{x}_w=(x_w^{(1)},x_w^{(2)},\ldots,x_w^{(b)})$, $x_w^{(v)}\in R$ ($w=1,2,...,\mathrm{d}$; $d=1,2,...,\mathrm{n}$).

Let F_i and f_i be *df* and *pdf* of $\xi_i^{(v)}$, respectively.

Moreover, $X_{1:n}^{(v),s},X_{2:n}^{(v),s},...,X_{n:n}^{(v),s}$ are order statistics of *iid* continuous random variables with *df* F^s and *pdf* f^s, respectively, defined by

$$F^s=\frac{1}{n_s}\sum_{i\in s}F_i \tag{1.3}$$

and

$$f^s=\frac{1}{n_s}\sum_{i\in s}f_i\,. \tag{1.4}$$

Here, s is a subset of integers {1, 2,…, n} with $n_s\geq 1$ elements.

In follows, *df* and *pdf* of $\mathrm{X}_{r_1:n}, \mathrm{X}_{r_2:n}, ..., \mathrm{X}_{r_d:n}$ ($1 \le r_1 < r_2 < ... < r_d \le n$) are given. Let $\mathrm{X}^{(v)} = (X_{r_1:n}^{(v)}, X_{r_2:n}^{(v)}, ..., X_{r_d:n}^{(v)})$ and $\mathrm{x}^{(v)} = (x_1^{(v)}, x_2^{(v)}, ..., x_d^{(v)})$. For notational convenience we write $\sum\sum$ and $\sum_{m_d,...,m_2,m_1}^{n,...,m_3,m_2}$ instead of $\sum_{\kappa=1}^{n}(-1)^{n-\kappa}\frac{\kappa^n}{n!}\sum_{n_s=\kappa}$ and $\sum_{m_d=r_d}^{n}...\sum_{m_2=r_2}^{m_3}\sum_{m_1=r_1}^{m_2}$ in the expressions below, respectively.

2. Distribution function of order statistics from *innid* random vectors

In this section, *df* of $\mathrm{X}_{r_1:n}, \mathrm{X}_{r_2:n}, ..., \mathrm{X}_{r_d:n}$ and its results are given. The results connect *df* of order statistics of *innid* random vectors to that of order statistics of *iid* random vectors using (1.3).

Now, we give the following theorem for establish joint *df* of *d* order statistics of *innid* continuous random vectors.

Theorem 2.1.

$$F_{r_1,r_2,...,r_d:n}(\mathrm{x}_1, \mathrm{x}_2, ..., \mathrm{x}_d) = \prod_{v=1}^{b}\{\sum_{m_d,...,m_2,m_1}^{n,...,m_3,m_2} C\sum_{P}\prod_{w=1}^{d+1}\prod_{l=m_{w-1}+1}^{m_w}[F_{j_l}(x_w^{(v)}) - F_{j_l}(x_{w-1}^{(v)})]\}, \quad (2.1)$$

$\mathrm{x}_1 < \mathrm{x}_2 < ... < \mathrm{x}_d$, where $C = [\prod_{w=1}^{d+1}(m_w - m_{w-1})!]^{-1}$, $m_0 = 0$, $m_{d+1} = n$, $\sum_P$ denotes sum over all n! permutations $(j_1, j_2, ..., j_n)$ of (1,2,…,n), $F_{j_l}(x_0^{(v)}) = 0$ and $F_{j_l}(x_{d+1}^{(v)}) = 1$.

Proof. It can be written

$$\begin{aligned} F_{r_1,r_2,...,r_d:n}(\mathrm{x}_1, \mathrm{x}_2, ..., \mathrm{x}_d) &= P\{\mathrm{X}_{r_1:n} \le \mathrm{x}_1, \mathrm{X}_{r_2:n} \le \mathrm{x}_2, ..., \mathrm{X}_{r_d:n} \le \mathrm{x}_d\} \\ &= P\{\mathrm{X}^{(1)} \le \mathrm{x}^{(1)}, \mathrm{X}^{(2)} \le \mathrm{x}^{(2)}, ..., \mathrm{X}^{(b)} \le \mathrm{x}^{(b)}\} \\ &= \prod_{v=1}^{b} P\{\mathrm{X}^{(v)} \le \mathrm{x}^{(v)}\} \\ &= \prod_{v=1}^{b} P\{X_{r_1:n}^{(v)} \le x_1^{(v)}, X_{r_2:n}^{(v)} \le x_2^{(v)}, ..., X_{r_d:n}^{(v)} \le x_d^{(v)}\} \quad (2.2) \end{aligned}$$

(2.2) can be expressed as

$$F_{r_1,r_2,...,r_d:n}(\mathrm{x}_1, \mathrm{x}_2, ..., \mathrm{x}_d) =$$
$$\prod_{v=1}^{b}\{\sum_{m_d,...,m_2,m_1}^{n,...,m_3,m_2} C\sum_{P}\left(\prod_{l=1}^{m_1}F_{j_l}(x_1^{(v)})\right)\left(\prod_{l=m_1+1}^{m_2}[F_{j_l}(x_2^{(v)}) - F_{j_l}(x_1^{(v)})]\right)...\prod_{l=m_d+1}^{n}[1 - F_{j_l}(x_d^{(v)})]\}.$$

Thus, (2.1) is obtained.

The approach in Theorem 2.1 can also be adapted to Theorem 2.2 for *iid* case.

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Theorem 2.2.

$$F_{r_1,r_2,...,r_d:n}(\mathrm{x}_1,\mathrm{x}_2,...,\mathrm{x}_d)=\prod_{v=1}^{b}\{\sum\sum\sum_{m_d,...,m_2,m_1}^{n,...,m_3,m_2} n!C\prod_{w=1}^{d+1}[F^s(x_w^{(v)})-F^s(x_{w-1}^{(v)})]^{m_w-m_{w-1}}\}. \quad (2.3)$$

Proof. (2.2) can be expressed as

$$F_{r_1,r_2,...,r_d:n}(\mathrm{x}_1,\mathrm{x}_2,...,\mathrm{x}_d)=\prod_{v=1}^{b}[\sum\sum P\{X_{r_1:n}^{(v),s}\le x_1^{(v)},X_{r_2:n}^{(v),s}\le x_2^{(v)},...,X_{r_d:n}^{(v),s}\le x_d^{(v)}\}]. \quad (2.4)$$

(2.3) is obtained from (2.1) and (2.4).

We now obtain the following three results for ***df*** of order statistics of ***innid*** continuous random vectors from the above theorems.

Result 2.1.

$$F_{r_1:n}(x_1^{(1)})=\sum_{m_1=r_1}^{n}\frac{1}{m_1!(n-m_1)!}\sum_P\left(\prod_{l=1}^{m_1}\left(F_{j_l}(x_1^{(1)})\right)\right)\prod_{l=m_1+1}^{n}[1-F_{j_l}(x_1^{(1)})]$$

$$=\sum\sum\sum_{m_1=r_1}^{n}\binom{n}{m_1}[F^s(x_1^{(1)})]^{m_1}[1-F^s(x_1^{(1)})]^{n-m_1}. \quad (2.5)$$

Proof. In (2.1) and (2.3), if $b=1$, $d=1$, (2.5) is obtained.

In addition,

$$F_{r_1:n}(x_1^{(1)})=\sum_{m_1=r_1}^{n}\frac{1}{m_1!(n-m_1)!}\sum_P\left(\prod_{l=1}^{m_1}F_{j_l}(x_1^{(1)})\right)\prod_{l=m_1+1}^{n}[1-F_{j_l}(x_1^{(1)})]$$

$$=\sum_{m_1=r_1}^{n}\frac{1}{m_1!(n-m_1)!}\sum_P\left(\prod_{l=1}^{m_1}F_{j_l}(x_1^{(1)})\right)\sum_{t=m_1}^{n}(-1)^{n-t}\sum_{n_\tau=n-t}\prod_{l=1}^{n-t}F_{\tau_l}(x_1^{(1)}),$$

where $\sum_{n_\tau=n-t}$ denotes sum over all $\binom{n-m_1}{n-t}$ subsets $\tau=\{\tau_1,\tau_2,...,\tau_{n-t}\}$ of $\{j_{m_1+1},j_{m_1+2},...,j_n\}$.

Result 2.2.

$$F_{1:n}(x_1^{(1)})=1-\frac{1}{n!}\sum_P\prod_{l=1}^{n}[1-F_{j_l}(x_1^{(1)})]$$

$$=\sum\sum[1-(1-F^s(x_1^{(1)}))^n]. \quad (2.6)$$

Proof. In (2.5), if $r_1=1$, (2.6) is obtained.

Result 2.3.

$$F_{n:n}(x_1^{(1)}) = \frac{1}{n!}\sum_P \prod_{l=1}^{n} F_{j_l}(x_1^{(1)})$$

$$= \sum\sum [F^s(x_1^{(1)})]^n . \tag{2.7}$$

Proof. In (2.5), if $r_1 = n$, (2.7) is obtained.

3. Probability density function of order statistics from *innid* random vectors

In this section, *pdf* of $\mathrm{X}_{r_1:n}, \mathrm{X}_{r_2:n}, ..., \mathrm{X}_{r_d:n}$ and its results are given. The results connect *pdf* of order statistics of *innid* random vectors to that of order statistics of *iid* random vectors using (1.3) and (1.4).

Joint *pdf* of *d* order statistics of *innid* continuous random vectors is expressed in the following theorem.

Theorem 3.1.

$$f_{r_1,r_2,...,r_d:n}(\mathrm{x}_1, \mathrm{x}_2, ..., \mathrm{x}_d) = \prod_{v=1}^{b}\{D\sum_P\left(\prod_{w=1}^{d+1}\prod_{l=r_{w-1}+1}^{r_w-1}[F_{j_l}(x_w^{(v)}) - F_{j_l}(x_{w-1}^{(v)})]\right)\prod_{w=1}^{d} f_{j_{r_w}}(x_w^{(v)})\}, \tag{3.1}$$

$\mathrm{x}_1 < \mathrm{x}_2 < ... < \mathrm{x}_d$, where $D = [\prod_{w=1}^{d+1}(r_w - r_{w-1} - 1)!]^{-1}$, $r_0 = 0$ and $r_{d+1} = n+1$.

Proof. Let $\delta\mathrm{x}_w = (\delta x_w^{(1)}, \delta x_w^{(2)}, ..., \delta x_w^{(b)})$ and $\delta\,\mathrm{x}^{(v)} = (\delta x_1^{(v)}, \delta x_2^{(v)}, ..., \delta x_d^{(v)})$.

Consider

$\{\mathrm{x}_1 < \mathrm{X}_{r_1:n} \le \mathrm{x}_1 + \delta\mathrm{x}_1, \mathrm{x}_2 < \mathrm{X}_{r_2:n} \le \mathrm{x}_2 + \delta\mathrm{x}_2, ..., \mathrm{x}_d < \mathrm{X}_{r_d:n} \le \mathrm{x}_d + \delta\mathrm{x}_d\}$.

It can be written

$$P\{\mathrm{x}_1 < \mathrm{X}_{r_1:n} \le \mathrm{x}_1 + \delta\,\mathrm{x}_1, \mathrm{x}_2 < \mathrm{X}_{r_2:n} \le \mathrm{x}_2 + \delta\,\mathrm{x}_2, ..., \mathrm{x}_d < \mathrm{X}_{r_d:n} \le \mathrm{x}_d + \delta\,\mathrm{x}_d\}$$

$$= P\{\mathrm{x}^{(1)} < \mathrm{X}^{(1)} \le \mathrm{x}^{(1)} + \delta\,\mathrm{x}^{(1)}, \mathrm{x}^{(2)} < \mathrm{X}^{(2)} \le \mathrm{x}^{(2)} + \delta\,\mathrm{x}^{(2)}, ..., \mathrm{x}^{(b)} < \mathrm{X}^{(b)} \le \mathrm{x}^{(b)} + \delta\,\mathrm{x}^{(b)}\}$$

$$= \prod_{v=1}^{b} P\{\mathrm{x}^{(v)} < \mathrm{X}^{(v)} \le \mathrm{x}^{(v)} + \delta\,\mathrm{x}^{(v)}\}$$

$$= \prod_{v=1}^{b} P\{x_1^{(v)} < X_{r_1:n}^{(v)} \le x_1^{(v)} + \delta\,x_1^{(v)}, x_2^{(v)} < X_{r_2:n}^{(v)} \le x_2^{(v)} + \delta x_2^{(v)}, ..., x_d^{(v)} < X_{r_d:n}^{(v)} \le x_d^{(v)} + \delta x_d^{(v)}\}. \tag{3.2}$$

Dividing (3.2) by $\prod_{v=1}^{b}\prod_{w=1}^{d}\delta x_w^{(v)}$ and then letting $\delta x_1^{(v)}, \delta x_2^{(v)}, ..., \delta x_d^{(v)}$ tend to zero, we obtain

$$f_{r_1,r_2,...,r_d:n}(\mathrm{x}_1, \mathrm{x}_2, ..., \mathrm{x}_d) = \prod_{v=1}^{b}\{D\sum_P F_{j_1}(x_1^{(v)})...F_{j_{r_1-1}}(x_1^{(v)}) f_{j_{r_1}}(x_1^{(v)})[F_{j_{r_1+1}}(x_2^{(v)}) - F_{j_{r_1+1}}(x_1^{(v)})]$$

$$...[F_{j_{r_2}-1}(x_2^{(v)}) - F_{j_{r_2}-1}(x_1^{(v)})] f_{j_{r_2}}(x_2^{(v)}) ... f_{j_{r_d}}(x_d^{(v)})[1 - F_{j_{r_d+1}}(x_d^{(v)}]...[1 - F_{j_n}(x_d^{(v)})]\}. \qquad (3.3)$$

From (3.3), we can write

$$f_{r_1, r_2, ..., r_d : n}(\mathrm{x}_1, \mathrm{x}_2, ..., \mathrm{x}_d) = \prod_{v=1}^{b} \{ D \sum_{p} \left(\prod_{l=1}^{r_1 - 1} [F_{j_l}(x_1^{(v)})] \right) f_{j_{r_1}}(x_1^{(v)})$$

$$\cdot \left(\prod_{l=r_1+1}^{r_2 - 1} [F_{j_l}(x_2^{(v)}) - F_{j_l}(x_1^{(v)})] \right) f_{j_{r_2}}(x_2^{(v)}) ... f_{j_{r_d}}(x_d^{(v)}) \prod_{l=r_d+1}^{n} [1 - F_{j_l}(x_d^{(v)})] \}. \qquad (3.4)$$

Thus, (3.1) is obtained.

Next theorem shows that *pdf* of *d* order statistics of *innid* continuous random vectors can be expressed in terms of *pdf* of *d* order statistics of *iid* continuous random vectors.

Theorem 3.2.

$$f_{r_1, r_2, ..., r_d : n}(\mathrm{x}_1, \mathrm{x}_2, ..., \mathrm{x}_d) = \prod_{v=1}^{b} \{ \sum \sum n! D \left(\prod_{w=1}^{d+1} [F^s(x_w^{(v)}) - F^s(x_{w-1}^{(v)})]^{r_w - r_{w-1} - 1} \right) \prod_{w=1}^{d} f^s(x_w^{(v)}) \}. \qquad (3.5)$$

Proof. (3.2) can be expressed as

$$\prod_{v=1}^{b} [\sum \sum P\{x_1^{(v)} < X_{r_1:n}^{(v),s} \le x_1^{(v)} + \delta x_1^{(v)}, x_2^{(v)} < X_{r_2:n}^{(v),s} \le x_2^{(v)} + \delta x_2^{(v)}, ..., x_d^{(v)} < X_{r_d:n}^{(v),s} \le x_d^{(v)} + \delta x_d^{(v)}\}]. \qquad (3.6)$$

Dividing (3.6) by $\prod_{v=1}^{b} \prod_{w=1}^{d} \delta x_w^{(v)}$ and then letting $\delta x_1^{(v)}, \delta x_2^{(v)}, ..., \delta x_d^{(v)}$ tend to zero, (3.5) is obtained.

The following five results of which first three are belong to *pdf* of single order statistic and last two are belong to joint *pdf* of *d* order statistics of *innid* continuous random vectors can be written from last two theorems.

Result 3.1.

$$f_{r_1:n}(x_1^{(1)}) = \frac{1}{(r_1 - 1)!(n - r_1)!} \sum_{P} \left(\prod_{l=1}^{r_1 - 1} F_{j_l}(x_1^{(1)}) \right) \left(\prod_{l=r_1+1}^{n} [1 - F_{j_l}(x_1^{(1)})] \right) f_{j_{r_1}}(x_1^{(1)})$$

$$= \sum \sum r_1 \binom{n}{r_1} [F^s(x_1^{(1)})]^{r_1 - 1} [1 - F^s(x_1^{(1)})]^{n - r_1} f^s(x_1^{(1)}). \qquad (3.7)$$

Proof. In (3.1) and (3.5), if $b = 1$, $d = 1$, (3.7) is obtained.

Result 3.2.

$$f_{1:n}(x_1^{(1)}) = \frac{1}{(n-1)!} \sum_{P} \left(\prod_{l=2}^{n} [1 - F_{j_l}(x_1^{(1)})] \right) f_{j_1}(x_1^{(1)})$$

$$= \sum \sum n [1 - F^s(x_1^{(1)})]^{n-1} f^s(x_1^{(1)}). \qquad (3.8)$$

Proof. In (3.7), if $r_1 = 1$, (3.8) is obtained.

Result 3.3.

$$f_{n:n}(x_1^{(1)}) = \frac{1}{(n-1)!}\sum_P \left(\prod_{l=1}^{n-1} F_{j_l}(x_1^{(1)}) \right) f_{j_n}(x_1^{(1)})$$

$$= \sum\sum n[F^s(x_1^{(1)})]^{n-1} f^s(x_1^{(1)}) . \tag{3.9}$$

Proof. In (3.7), if $r_1 = n$, (3.9) is obtained.

Result 3.4.

$$f_{1,n:n}(x_1^{(1)}, x_2^{(1)}) = \frac{1}{(n-2)!}\sum_P \left(\prod_{l=2}^{n-1} [F_{j_l}(x_2^{(1)}) - F_{j_l}(x_1^{(1)})] \right) f_{j_1}(x_1^{(1)}) f_{j_n}(x_2^{(1)})$$

$$= \sum\sum n(n-1)[F^s(x_2^{(1)}) - F^s(x_1^{(1)})]^{n-2} f^s(x_1^{(1)}) f^s(x_2^{(1)}) . \tag{3.10}$$

Proof. In (3.1) and (3.5), if $b = 1$, $d = 2$ and $r_1 = 1$, $r_2 = n$, (3.10) is obtained.

Result 3.5.

$$f_{1,2,\ldots,k:n}(\mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_\mathrm{k}) = \prod_{v=1}^{b} \{ \frac{1}{(n-k)!}\sum_P \left(\prod_{l=k+1}^{n} [1 - F_{j_l}(x_k^{(v)})] \right) f_{j_1}(x_1^{(v)}) f_{j_2}(x_2^{(v)}) \ldots f_{j_k}(x_k^{(v)}) \}$$

$$= \prod_{v=1}^{b} \{ \sum\sum \frac{n!}{(n-k)!} [1 - F^s(x_k^{(v)})]^{n-k} f^s(x_1^{(v)}) f^s(x_2^{(v)}) \ldots f^s(x_k^{(v)}) \} . \tag{3.11}$$

Proof. In (3.1) and (3.5), if $d = k$ and $r_1 = 1$, $r_2 = 2, \ldots, r_k = k$, (3.11) is obtained.

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Stability of homomorphisms and derivations in non-Archimedean random C^*-algebras via fixed point method

Javad Shokri[1] and Jung Rye Lee[2*]

[1]Department of Mathematics, Urmia University, P. O. Box 165, Urmia, Iran

[2]Department of Mathematics, Daejin University, Kyunggi 11159, Republic of Korea

Abstract. In this paper, using the fixed point method, we investigate the Hyers-Ulam stability of homomorphisms in non-Archimedean random C^*-algebras and non-Archimedean random Lie JC^*-algebras and of derivations on non-Archimedean random C^*-algebras and non-Archimedean random Lie JC^*-algebras related to the generalized Cauchy-Jensen additive functional equation.

1. Introduction

A non-Archimedean field is a field like $\mathcal{K}$ equipped is a function $|\cdot| : \mathcal{K} \to [0, +\infty)$ such that $|a| = 0$ if and only if $a = 0$, $|ab| = |a||b|$ and $|a + b| \leqslant \max\{|a|, |b|\}$ for all $a, b \in \mathcal{K}$. Note that $|1| = |-1| = 1$ and $|n| \leqslant 1$ for each integer n. By the trivial valuation we mean the mapping $|\cdot|$ taking everything but 0 into 1 and $|0| = 0$. We always assume, in addition, that $|\cdot|$ is non-trivial, i.e., there exists an $a_0 \neq 0, 1$. A function $\|.\| : X \to [0, \infty)$ is called a *non-Archimedean norm* if it satisfies the following conditions:

(i) $\|x\| = 0$ if and only if $x = 0$;
(ii) for any $r \in K, x \in X$, $\|rx\| = |r|\|x\|$;
(iii) the strong triangle inequality holds; namely,

$$\|x + y\| \leqslant \max\{\|x\|, \|y\|\} \quad (x, y \in X).$$

Then $(X, \|.\|)$ is called a non-Archimedean normed space. From the fact that

$$\|x_n - x_m\| \leqslant \max\{\|n_n - x_m\| : m \leqslant j \leqslant n - 1\} \quad (n > m)$$

holds, a sequence $\{x_n\}$ is Cauchy if and only if $\{x_n - x_m\}$ converges to zero in a non-Archimedean normed space. By a complete non-Archimedean normed space we mean one in which every Cauchy sequence is convergent.

For any nonzero rational number x, there exists a unique integer $n_x \in \mathbb{Z}$ such that $x = \frac{a}{b} p^{n_x}$, where a and b are integers not divisible by p. Then $|x|_p := p^{-n_x}$ defines a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric $d(x, y) = |x - y|_p$ is denoted by $\mathbb{Q}_p$, which is called the p-adic number field.

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*Corresponding author.

[0]**E-mail**: [1]`j.shokri@urmia.ac.ir`, [2]`jrlee@daejin.ac.kr`

A non-Archimedean Banach algebra is a complete non-Archimedean algebra $\mathcal{A}$ which satisfies $\|ab\| \leqslant \|a\| \cdot \|b\|$ for all $a, b \in \mathcal{A}$. For more detailed definitions of non-Archimedean Banach algebras, we refer the reader to [25].

If $\mathcal{U}$ is a non-Archimedean Banach algebra, then an involution on $\mathcal{U}$ is mapping $t \to t^*$ from $\mathcal{U}$ into $\mathcal{U}$ which satisfies

(i) $t^{**} = t$ for $t \in \mathcal{U}$;

(ii) $(\alpha s + \beta t)^* = \bar{\alpha} s^* + \bar{\beta} t^*$;

(iii) $(st)^* = t^* s^*$ for all $s, t \in \mathcal{U}$.

If, in addition, $\|t^* t\| = \|t\|^2$ for $t \in \mathcal{U}$, then $\mathcal{U}$ is a non-Archimedean C^*-algebra.

The stability problem of functional equations originated from a question of Ulam [26] concerning the stability of group homomorphisms: Let $(G_1, *)$ be a group and let $(G_2, \diamond)$ be a metric group (a metric is defined on a set with group property) with the metric $d(.,.)$. Given $\epsilon > 0$, does there exist a $\delta(\epsilon) > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(x * y), h(x) * h(y)) < \delta$ for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$? If the answer is affirmative, we we would say that the equation of homomorphism $h(x * y) = h(x) * h(y)$ is stable (see also [10, 11, 14, 18, 19, 20, 21, 22]).

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, y) \leqslant d(x, z) + d(z, y)$ for all $x, y, z \in X$.

For explicitly later use, we recall a fundamental result in fixed point theory.

Theorem 1.1. [9] *Let (Ω, d) be a complete generalized metric space and $J : \Omega \to \Omega$ be a strictly contractive mapping with Lipschitz constant $0 < L < 1$. Then for each given $x \in \Omega$, either $d(J^n x, J^{n+1} x) = \infty$ for all nonnegative n or there exists a positive integer n_0 such that*

(1) *$d(J^n x, J^{n+1} x) < \infty, \quad \forall n \geqslant n_0$;*

(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*

(3) *y^* is the unique fixed point of J in the set $\Delta = \{y \in \Omega : d(J^{n_0} x, y) < \infty\}$;*

(4) *$d(y, y^*) \leqslant \frac{1}{1-L} d(y, Jy)$ for all $y \in \Delta$.*

A C^*-algebra $\mathcal{C}$, endowed with the Lie product $[x, y] := \frac{xy - yx}{2}$ and endowed with *anticommutator product* (Jordan product) $x \circ y := \frac{xy + yx}{2}$ on $\mathcal{C}$, is called a Lie JC^*-algebra (see [15, 16, 17]).

Jordan algebras as coordinates for Lie algebras were created to illuminate a particular aspect of physics, quantum-mechanical observables, but turned out to have illuminating connections with many areas of mathematics.

In this paper, using the fixed point method, we prove the Hyers-Ulam stability of homomorphisms and derivations in non-Archimedean random C^*-algebras and non-Archimedean random Lie JC^*-algebras associated with $f : X \to Y$ satisfying the following functional equation (see [1])

$$\sum_{\substack{1 \leqslant i_1 < \cdots < i_m \leqslant n \\ 1 \leqslant k_l (\neq i_j, \forall j \in \{1, \cdots, m\}) \leqslant n}} f\left(\frac{\sum_{j=1}^{m} x_{i_j}}{m} + \sum_{l=1}^{n-m} x_{k_l}\right) = \frac{(n-m+1)}{n} \binom{n}{m} \sum_{i=1}^{n} f(x_i) \quad (1.1)$$

for all $x_1, \cdots, x_n \in X$, where $m, n \in \mathbb{N}$ are fixed integer with $n \geqslant 2$, $1 \leqslant m \leqslant n$. In particular, it is shown that in the case $m = 1$, (1.1) yields the Cauchy additive equation $f(\sum_{l=1}^{n} x_{k_l}) = \sum_{l=1}^{n} f(x_i)$ and also in the case $m = n$, (1.1) yields the Jensen additive equation $f(\frac{\sum_{j=1}^{n} x_j}{n}) = \frac{1}{n}\sum_{l=1}^{n} f(x_i)$. Then (1.1) is a generalized form of the Cauchy-Jensen additive equation, and thus every solution of the equation (1.1) may be analogously called general (m, n)-Cauchy-Jensen additive. For each m with $1 \leqslant m \leqslant n$, a mapping $f : X \to Y$ satisfies (1.1) for all $n \geqslant 2$ if and only if $f(x) - f(0) = A(x)$ is Cauchy additive, where $f(0) = 0$ if $m < n$. In particular, we have $f((n - m + 1)x) = (n - m + 1)f(x)$ and $f(mx) = mf(x)$ for all $x \in X$.

2. Random spaces

In this section, we adopt the usual terminology, notations, and conventions of the theory of random normed spaces as in [2, 3, 6, 7, 8]. Throughout this paper, Δ^+ is the space of distribution functions, that is the space of all mapping $F : \mathbb{R} \cup \{-\infty, \infty\} \to [0, 1]$ such that F is left-continuous and non-decreasing on $\mathbb{R}$, $F(0) = 0$ and $F(+\infty) = 1$. And D^+ is a subset of Δ^+ consisting of all functions $F \in \Delta^+$ for which $l^- F(+\infty) = 1$, where $l^- f(x)$ denotes the left limit of the function f at the point x, that is, $l^- f(x) = \lim_{t \to x^-} f(t)$. The space Δ^+ is partially ordered by the usual point-wise ordering of functions, i.e., $F \leqslant G$ if and only if $F(t) \leqslant G(t)$ for all t in $\mathbb{R}$. The maximal element for Δ^+ in this order is distribution function ε_0 given by

$$\varepsilon_0(t) = \begin{cases} 0 & \text{if } t \leqslant 0, \\ 1 & \text{if } t > 0. \end{cases}$$

Definition 2.1. [23] *A mapping $T : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous triangular norm norm (briefly, a continuous t-norm) if T satisfies the following conditions:*
(a) T is commutative and associative;
(b) T is continuous;
(c) $T(a, 1) = a$ for all $a \in [0, 1]$;
(d) $T(a, b) \leqslant T(c, d)$ whenever $a \leqslant c$ and $b \leqslant d$ for all $a, b, c, d \in [0, 1]$.

Typical examples of continuous t-norms are $T_P(a, b) = ab$, $T_M(a, b) = \min(a, b)$ and $T_L(a, b) = \max(a + b - 1, 0)$ (the Lukasiewicz t-norm).

Definition 2.2. [24] *A non-Archimedean random normed space (briefly, NA-RN-space) is a triple (X, μ, T), where X is a vector space, T is a continuous t-norm, and μ is a mapping from X into D^+ such that the following conditions hold:*
(RN1) $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;
(RN2) $\mu_{\alpha x}(t) = \mu_x(\frac{t}{|\alpha|})$ for all $x \in X, \alpha \neq 0$.
(RN3) $\mu_{x+y}(t) \geqslant T(\mu_x(t), \mu_y(t))$ for all $x, y \in X$ and all $t \geqslant 0$.

Every normed space $(X, \|\cdot\|)$ defines a non-Archimedean random normed space (X, μ, T_M), where

$$\mu_x(t) = \frac{t}{t + \|x\|}$$

for all $t > 0$, and T_M is the minimum t-norm. This space is called the induced random normed space.

Definition 2.3. [12] *A non-Archimedean random normed algebra* (X, μ, T, T') *is a non-Archimedean random normed space* (X, μ, T) *with an algebraic structure such that*

$(RN4)$ $\mu_{xy}(t) \geqslant T'(\mu_x(t), \mu_y(t))$ *for all* $x, y \in X$ *and all* $t > 0$, *in which* T' *is a continuous* t*-norm.*

Every non-Archimedean normed algebra $(X, \|\cdot\|)$ defines a non-Archimedean random normed algebra (X, μ, T_M), where

$$\mu_x(t) = \frac{t}{t + \|x\|}$$

for all $t > 0$ if and only if

$$\|xy\| \leqslant \|x\|\,\|y\| + t\|x\| + t\|y\| \ \ (x, y \in X; t > 0).$$

This space is called an induced non-Archimedean random normed algebra.

Definition 2.4. *Let* (X, μ, T_M) *and* (Y, μ, T_M) *be non-Archimedean random normed algebras.*

(1) *An* $\mathbb{R}$*-linear mapping* $f : X \to Y$ *is called a* homomorphism *if* $f(xy) = f(x)f(y)$ *for all* $x, y \in X$.
(2) *An* $\mathbb{R}$*-linear mapping* $f : X \to Y$ *is called a* derivation *if* $f(xy) = f(x)y + xf(y)$ *for all* $x, y \in X$.

Definition 2.5. *Let* $(\mathcal{U}, \mu, T)$ *be a non-Archimedean random Banach algebra. Then an involution on* $\mathcal{U}$ *is mapping* $u \to u^*$ *from* $\mathcal{U}$ *into* $\mathcal{U}$ *which satisfies*

(i) $u^{**} = u$ *for* $u \in \mathcal{U}$;
(ii) $(\alpha u + \beta v)^* = \bar{\alpha}u^* + \bar{\beta}v^*$;
(iii) $(uv)^* = v^*u^*$ *for all* $u, v \in \mathcal{U}$.

If, in addition, $\mu_{u^*u}(t) = T'(\mu_u(t), \mu_u(t))$ *for* $u \in \mathcal{U}$, *then* $\mathcal{U}$ *is a non-Archimedean random* C^**-algebra.*

Definition 2.6. *Let* (X, μ, T) *be an* NA*-*RN*-space.*

(1) *A sequence* $\{x_n\}$ *in* X *is said be* convergent *to* x *in* X *if, for every* $\epsilon > 0$ *and* $\lambda > 0$, *there exists a positive integer* N *such that* $\mu_{x_n - x}(\epsilon) > 1 - \lambda$ *whenever* $n \geqslant N$.
(2) *A sequence* $\{x_n\}$ *in* X *is called a* Cauchy sequence *if, for every* $\epsilon > 0$ *and* $\lambda > 0$, *there exists a positive integer* N *such that* $\mu_{x_n - x_{n+1}}(\epsilon) > 1 - \lambda$ *whenever* $n \geqslant m \geqslant N$.
(3) *An* RN*-space* (X, μ, T) *is said to be* complete *if and only if every Cauchy sequence in* X *is convergent to a point in* X.

3. Stability of homomorphisms and derivations in non-Archimedean random C^*-algebras

Throughout this section, we suppose that $\mathcal{A}$ and $\mathcal{B}$ are non-Archimedean random C^*-algebras, respectively, with norms $\mu^{\mathcal{A}}_{.}$ and $\mu^{\mathcal{B}}_{.}$.

We use the following abbreviation for a given mapping $f : \mathcal{A} \to \mathcal{B}$:

$$D_\lambda f(x_1, \cdots, x_n) := \sum_{\substack{1 \leqslant i_1 < \cdots < i_m \leqslant n \\ 1 \leqslant k_l (\neq i_j, \forall j \in \{1, \cdots, m\}) \leqslant n}} f\left(\frac{\sum_{j=1}^m \lambda x_{i_j}}{m} + \sum_{l=1}^{n-m} \lambda x_{k_l}\right) - \frac{(n-m+1)\binom{n}{m} \sum_{i=1}^n \lambda f(x_i)}{n}$$

for all $\lambda \in \mathbb{T}^1 := \{\mu \in \mathbb{C} : |\mu| = 1\}$ and all $x_1, \cdots, x_n \in \mathcal{A}$.

It is well-known that a $\mathbb{C}$-linear mapping $H : \mathcal{A} \to \mathcal{B}$ is called a *random homomorphism* in non-Archimedean random C^*-algebras if H satisfies $H(xy) = H(x)H(y)$ and $H(x^*) = H(x)^*$ for all $x, y \in \mathcal{A}$.

We prove the Hyers-Ulam stability oh homomorphisms in non-Archimedean random C^*-algebras for the functional equation $D_\lambda f(x_1, \cdots, x_n) = 0$.

Theorem 3.1. *Let $f : \mathcal{A} \to \mathcal{B}$ be a mapping for which there are functions $\varphi : \mathcal{A}^n \to D^+, \psi : \mathcal{A}^2 \to D^+$, and $\eta : \mathcal{A} \to D^+$ such that $|\mathcal{M}| = |n - m + 1| < 1$ and $|\mathcal{N}| = |(n-m+1)\binom{n}{m}| < 1$ are far from zero and*

$$\mu^{\mathcal{B}}_{D_\lambda f(x_1, \cdots, x_n)}(t) \geqslant \varphi_{x_1, \cdots, x_n}(t), \tag{3.1}$$

$$\mu^{\mathcal{B}}_{f(xy) - f(x)f(y)}(t) \geqslant \psi_{x,y}(t), \tag{3.2}$$

$$\mu^{\mathcal{B}}_{f(x^*) - f(x)^*}(t) \geqslant \eta_x(t), \tag{3.3}$$

for all $\lambda \in \mathbb{T}^1 := \{\mu \in \mathbb{C} : |\mu| = 1\}$ and all $x_1, \cdots, x_n, x, y \in \mathcal{A}$ and $t > 0$. If there exists an $L < 1$ such that

$$\varphi_{\mathcal{M}x_1, \cdots, \mathcal{M}x_n}(|\mathcal{M}|Lt) \geqslant \varphi_{x_1, \cdots, x_n}(t), \tag{3.4}$$

$$\psi_{\mathcal{M}x, \mathcal{M}y}(|\mathcal{M}|^2 Lt) \geqslant \psi_{x,y}(t), \tag{3.5}$$

$$\eta_{\mathcal{M}x}(|\mathcal{M}|Lt) \geqslant \eta_x(t), \tag{3.6}$$

for all $x_1, \cdots, x_n, x, y \in \mathcal{A}$ and $t > 0$, then there exists a unique random homomorphism $H : \mathcal{A} \to \mathcal{B}$ such that

$$\mu^{\mathcal{B}}_{f(x) - H(x)}(t) \geqslant \varphi_{x, \cdots, x}((|\mathcal{N}| - |\mathcal{N}|L)t) \tag{3.7}$$

for all $x \in \mathcal{A}$ and $t > 0$.

Proof. It follows from (3.4), (3.5), (3.6), and $L < 1$ that

$$\lim_{m \to \infty} \varphi_{\mathcal{M}^m x_1, \cdots, \mathcal{M}^m x_n}(|\mathcal{M}|^m t) = 1, \tag{3.8}$$

$$\lim_{m \to \infty} \psi_{\mathcal{M}^m x, \mathcal{M}^m y}(|\mathcal{M}|^{2m} t) = 1, \tag{3.9}$$

$$\lim_{m \to \infty} \eta_{\mathcal{M}^m x}(|\mathcal{M}|^m t) = 1, \tag{3.10}$$

for all $x_1, \cdots, x_n, x, y \in \mathcal{A}$ and $t > 0$.

Now we define $\Omega := \{g : \mathcal{A} \to \mathcal{B}; g(0) = 0\}$ and introduce a generalized metric on Ω as following:

$$d(g, h) = \inf\{k \in (0, \infty) : \mu^{\mathcal{B}}_{g(x) - h(x)}(kt) > \varphi_{x,x,\cdots,x}(t), \forall x \in \mathcal{A},\ t > 0\}$$

where $\inf \emptyset = +\infty$. By the same technique as in the proof of [13, Theorem 3.2], we can show that (Ω, d) is a complete generalized metric space. We define $J : \Omega \to \Omega$ by $Jg(x) = \frac{1}{\mathcal{M}} g(\mathcal{M}x)$ for all $x \in \mathcal{A}$ and $g \in \Omega$. Note that for all $g, h \in \Omega$, from (3.4), we have

$$\begin{aligned} d(g,h) \leqslant k &\Rightarrow \mu^{\mathcal{B}}_{g(x)-h(x)}(kt) > \varphi_{x,\cdots,x}(t) \\ &\Rightarrow \mu^{\mathcal{B}}_{\frac{1}{\mathcal{M}}g(\mathcal{M}x)-\frac{1}{\mathcal{M}}h(\mathcal{M}x)}(kt) > \varphi_{\mathcal{M}x,\cdots,\mathcal{M}x}(|\mathcal{M}|t) \\ &\Rightarrow \mu^{\mathcal{B}}_{\frac{1}{\mathcal{M}}g(\mathcal{M}x)-\frac{1}{\mathcal{M}}h(\mathcal{M}x)}(kLt) > \varphi_{x,\cdots,x}(t) \\ &\Rightarrow d(Jg, Jh) < kL. \end{aligned}$$

Then one can show that $d(Jg, Jh) \leqslant Ld(g, h)$ for all $g, h \in \Omega$ and so J is self-function of Ω with the the Lipschitz constant L.

Letting $\lambda = 1$ and putting $x_1 = x_2 = \cdots = x_n = x$ in (3.1), we obtain

$$\mu^{\mathcal{B}}_{\binom{n}{m}f((n-m+1)x)-\binom{n}{m}(n-m+1)f(x)}(t) \geqslant \varphi_{x,x,\cdots,x}(t)$$

for all $x \in \mathcal{A}$ and $t > 0$. Then

$$\mu^{\mathcal{B}}_{f(x)-\frac{1}{\mathcal{M}}f(\mathcal{M}x)}(t) \geqslant \varphi_{x,x,\cdots,x}(|\mathcal{N}|t)$$

for all $x \in \mathcal{A}$ and $t > 0$. This implies that $d(Jf, f) \leqslant \frac{1}{|\mathcal{N}|} < \infty$. By The fixed point alternative theorem, Theorem 1.1, J has a unique fixed point $H : \mathcal{A} \to \mathcal{B}$ in $\Omega_0 := \{h \in \Omega : d(h, f) < \infty\}$ such that

$$H(x) = \lim_{m\to\infty} \frac{1}{|\mathcal{M}|^m} f(\mathcal{M}^m x) \tag{3.11}$$

for all $x \in \mathcal{A}$, since $\lim_{m\to\infty} d(J^m f, H) = 0$.

On the other hand, it follows from (3.1), (3.8) and (3.11) that

$$\begin{aligned} \mu^{\mathcal{B}}_{D_\lambda H(x_1,\cdots,x_n)}(t) &= \lim_{m\to\infty} \mu_{\frac{1}{\mathcal{M}^m}D_\lambda f(\mathcal{M}^m x_1,\cdots,\mathcal{M}^m x_n)}(t) \\ &\geqslant \lim_{m\to\infty} \varphi_{\mathcal{M}^m x_1,\cdots,\mathcal{M}^m x_n}(|\mathcal{M}|^m t) = 1. \end{aligned}$$

By a similar method to the above, we can get $\lambda H(\mathcal{M}x) = H(\lambda\mathcal{M}x)$ for all $\lambda \in \mathbb{T}$ and all $x \in \mathcal{A}$. Then by using the same technique as in the proof of [10, Theorem 2.1], we can show that H is $\mathbb{C}$-linear.

It follows from (3.2), (3.9) and (3.11) that

$$\begin{aligned} \mu^{\mathcal{B}}_{H(xy)-H(x)H(y)}(t) &= \lim_{m\to\infty} \mu^{\mathcal{B}}_{f(\mathcal{M}^{2m}xy)-f(\mathcal{M}^m x)f(\mathcal{M}^m y)}\left(|\mathcal{M}|^{2m}t\right) \\ &\geqslant \lim_{m\to\infty} \psi_{\mathcal{M}^m x,\mathcal{M}^m y}\left(|\mathcal{M}|^{2m}t\right) = 1 \end{aligned}$$

for all $x, y \in \mathcal{A}$. Therefore, we conclude that $H(xy) = H(x)H(y)$ for all $x, y \in \mathcal{A}$. Thus $H : \mathcal{A} \to \mathcal{B}$ is a homomorphism satisfying (3.7).

By same method as above, from (3.3),(3.10) and (3.11), we can write

$$\begin{aligned} \mu^{\mathcal{B}}_{H(x^*)-H(x)^*}(t) &= \lim_{m\to\infty} \mu^{\mathcal{B}}_{\frac{1}{\mathcal{M}^m}(f(\mathcal{M}^m x^*)-f(\mathcal{M}^m x)^*)}(t) \\ &\geqslant \lim_{m\to\infty} \eta_{\mathcal{M}^m x}\left(|\mathcal{M}|^m t\right) = 1 \end{aligned}$$

for all $x \in \mathcal{A}$ and all $t > 0$. Then we conclude that $H(x^*) = H(x)^*$ and the proof is complete, as desired. □

Corollary 3.2. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : \mathcal{A} \to \mathcal{B}$ be a mapping such that*

$$\mu^{\mathcal{B}}_{D_\lambda f(x_1,\cdots,x_n)}(t) \geqslant \frac{t}{t+\theta(\|x_1\|^r_{\mathcal{A}} + \|x_2\|^r_{\mathcal{A}} + \cdots + \|x_n\|^r_{\mathcal{A}})},$$
$$\mu^{\mathcal{B}}_{f(xy)-f(x)f(y)}(t) \geqslant \frac{t}{t+\theta(\|x\|^r_{\mathcal{A}}.\|y\|^r_{\mathcal{A}})},$$
$$\mu^{\mathcal{B}}_{f(x^*)-f(x)^*}(t) \geqslant \frac{t}{t+\theta\|x\|^r_{\mathcal{A}}}$$

for all $\lambda \in \mathbb{T}^1$, all $x_1, \cdots, x_n, x, y \in \mathcal{A}$ and $t > 0$. Then there exists a unique random homomorphism $H : \mathcal{A} \to \mathcal{B}$ such that

$$\mu^{\mathcal{B}}_{f(x)-H(x)}(t) \geqslant \frac{(|\mathcal{N}| - |\mathcal{N}|^r)t}{(|\mathcal{N}| - |\mathcal{N}|^r)t + n\theta\|x\|^r_{\mathcal{A}}}$$

for all $x \in \mathcal{A}$ and $t > 0$.

Proof. Letting

$$\varphi_{x_1,\cdots,x_n}(t)\frac{t}{t+\theta(\|x_1\|^r_{\mathcal{A}} + \|x_2\|^r_{\mathcal{A}} + \cdots + \|x_n\|^r_{\mathcal{A}})},$$
$$\psi_{x,y}(t) = \frac{t}{t+\theta(\|x\|^r_{\mathcal{A}}.\|y\|^r_{\mathcal{A}})},$$
$$\eta_x(t) = \frac{t}{t+\theta\|x\|^r_{\mathcal{A}}}$$

for all $x_1, \cdots, x_n, x, y \in \mathcal{A}$, $L = |\mathcal{N}|^{r-1}$ and $t > 0$ in Theorem 3.1, we get the desired result. □

In the following theorem, we investigate the Hyers-Ulam stability of derivations on non-Archimedean random C^*-algebras for the functional equation $D_\lambda f(x_1, \cdots, x_n) = 0$.

Theorem 3.3. *Let $f : \mathcal{A} \to \mathcal{A}$ be a mapping for which there are functions $\varphi : \mathcal{A}^n \to D^+$, $\psi : \mathcal{A}^2 \to D^+$, satisfying (3.1), (3.3), and $\eta : \mathcal{A} \to D^+$ such that $|\mathcal{M}| < 1$ and $|\mathcal{N}| < 1$ are far from zero and*

$$\mu^{\mathcal{A}}_{f(xy)-f(x)y-xf(y)}(t) \geqslant \psi_{x,y}(t), \tag{3.12}$$

for all $\lambda \in \mathbb{T}^1$ and all $x_1, \cdots, x_n, x, y \in \mathcal{A}$ and $t > 0$. If there exists an $L < 1$ such that (3.4), (3.5) and (3.6) hold, then there exists a unique random derivation $\delta : \mathcal{A} \to \mathcal{A}$ such that

$$\mu^{\mathcal{A}}_{f(x)-\delta(x)}(t) \geqslant \varphi_{x,\cdots,x}\left((|\mathcal{N}| - |\mathcal{N}|L)\right) \tag{3.13}$$

for all $x \in^{\mathcal{A}}$ and $t > 0$.

Proof. By the same argument as in the proof of Theorem 3.1, there exists a unique $\mathbb{C}$-linear mapping $\delta : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ satisfying (3.13). The mapping δ is given by

$$\delta(x) = \lim_{m\to\infty} \frac{1}{|\mathcal{M}|^m} f(\mathcal{M}^m x) \tag{3.14}$$

for all $x \in \mathcal{A}$.

It follows from (3.12), (3.9) and (3.14) that

$$\mu^{\mathcal{B}}_{\delta(xy)-\delta(x)y-x\delta(y)}(t) = \lim_{m\to\infty} \mu^{\mathcal{B}}_{f(\mathcal{M}^{2m}xy)-f(\mathcal{M}^m x)\mathcal{M}^m y-\mathcal{M}^m x f(\mathcal{M}^m y)}\left(|\mathcal{M}|^{2m}t\right)$$
$$\geqslant \lim_{m\to\infty} \psi_{\mathcal{M}^m x,\mathcal{M}^m y}\left(|\mathcal{M}|^{2m}t\right) = 1$$

for all $x, y \in \mathcal{A}$. Therefore, we conclude that $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in \mathcal{A}$. The remainder of the proof is similar to the proof of Theorem 3.1. □

4. Stability of homomorphisms and derivations in non-Archimedean random Lie JC^*-algebras

A non-Archimedean random C^*-algebra $\mathcal{C}$, endowed with the Lie product $[x, y] := \frac{xy-yx}{2}$and endowed with *anticommutator product* (Jordan product) $x \circ y := \frac{xy+yx}{2}$ on $\mathcal{C}$, is called a non-Archimedean random Lie JC^*-algebra.

Definition 4.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be non-Archimedean random Lie JC^*-algebras. A $\mathbb{C}$-linear mapping $H : \mathcal{A} \to \mathcal{B}$ is called a random Lie JC^*-algebra homomorphism if H satisfies*

$$H([x, y]) = [H(x), H(y)],$$
$$H(x \circ y) = H(x) \circ H(y),$$
$$H(x^*) = H(x)^*$$

for all $x, y \in \mathcal{A}$.

Throughout this section, assume that $\mathcal{A}$ and $\mathcal{B}$ are two non-Archimedean random Lie JC^*-algebras respectively with norm $\mu^{\mathcal{A}}_{.}$ and $\mu^{\mathcal{B}}_{.}$.

In the following theorem, we prove the Hyers-Ulam stability of homomorphisms in non-Archimedean random Lie JC^*-algebra for the functional equation $D_\lambda f(x_1, \cdots, x_n) = 0$.

Theorem 4.2. *Let $f : \mathcal{A} \to \mathcal{B}$ be a mapping for which there are functions $\varphi : \mathcal{A}^n \to D^+$ and $\psi : \mathcal{A}^2 \to D^+$ satisfying (3.1), (3.3) and*

$$\mu^{\mathcal{B}}_{f([x,y])-[f(x),f(y)]}(t) \geqslant \psi_{x,y}(t), \tag{4.1}$$

$$\mu^{\mathcal{B}}_{H(x\circ y)-H(x)\circ H(y)}(t) \geqslant \phi_{x,y}(t) \tag{4.2}$$

for all $\lambda \in \mathbb{T}^1$, all $x, y \in \mathcal{A}$ and $t > 0$. If there exists an $L < 1$ such that (3.4), (3.5) and (3.6) hold, and also

$$\phi_{\mathcal{M}x,\mathcal{M}y}(|\mathcal{M}|^2 Lt) \geqslant \phi_{x,y}(t), \tag{4.3}$$

for all $x, y \in \mathcal{A}$ and $t > 0$, then there exists a unique random Lie JC^*-algebra homomorphism $H : \mathcal{A} \to \mathcal{B}$ satisfying (3.7).

Proof. It follows from (4.3) and $L < 1$ that

$$\lim_{m\to\infty} \phi_{\mathcal{M}^m x,\mathcal{M}^m y}(|\mathcal{M}|^{2m}t) = 1, \tag{4.4}$$

for all $x, y \in \mathcal{A}$ and $t > 0$.

By the same argument as in the proof of Theorem 3.1, there exists a unique $\mathbb{C}$-linear mapping $H:\mathcal{A}\to\mathcal{B}$ satisfying (3.7). The mapping H is given by

$$H(x)=\lim_{m\to\infty}\frac{f(\mathcal{M}^m x)}{|\mathcal{M}|^m} \tag{4.5}$$

for all $x\in\mathcal{A}$. It follows from (3.9), (4.4) and (4.5) that

$$\begin{aligned}\mu^{\mathcal{B}}_{H([x,y])-[H(x),H(y)]}(t)&=\lim_{m\to\infty}\mu^{\mathcal{B}}_{f(\mathcal{M}^{2m}[x,y])-[f(\mathcal{M}^m x),f(\mathcal{M}^m y)]}\left(|\mathcal{M}|^{2m}t\right)\\&\geqslant\lim_{m\to\infty}\psi_{\mathcal{M}^m x,\mathcal{M}^m y}\left(|\mathcal{M}|^{2m}t\right)=1\end{aligned}$$

and

$$\begin{aligned}\mu^{\mathcal{B}}_{H(x\circ y)-H(x)\circ H(y)}&=\lim_{m\to\infty}\mu^{\mathcal{B}}_{f(\mathcal{M}^{2m}(x\circ y))-f(\mathcal{M}x)\circ f(\mathcal{M}y)}\left(|\mathcal{M}|^{2m}t\right)\\&\geqslant\lim_{m\to\infty}\phi_{\mathcal{M}^m x,\mathcal{M}^m y}\left(|\mathcal{M}|^{2m}t\right)=1\end{aligned}$$

for all $x,y\in\mathcal{A}$ and $t>0$, then it is concluded that

$$H([x,y])=[H(x),H(y)]\quad;\quad H(x\circ y)=H(x)\circ H(y)$$

for all $x,y\in\mathcal{A}$. Therefore, $H:\mathcal{A}\to\mathcal{B}$ is the unique random Lie JC^*-algebra homomorphism satisfying (3.7). □

Corollary 4.3. *Let $r>1$ and θ be nonnegative real numbers, and $f:\mathcal{A}\to\mathcal{B}$ be a mapping such that*

$$\begin{aligned}&\mu^{\mathcal{B}}_{D_\lambda f(x_1,\cdots,x_n)}(t)\geqslant\frac{t}{t+\theta(\|x_1\|^r_{\mathcal{A}}+\cdots+\|x_n\|^r_{\mathcal{A}})},\\&\mu^{\mathcal{B}}_{f([x,y])-[f(x),f(y)]}\geqslant\frac{t}{t+\theta(\|x\|^r_{\mathcal{A}}.\|y\|^r_{\mathcal{A}})},\\&\mu^{\mathcal{B}}_{f(x^8)-f(x)^*}(t)\geqslant\frac{t}{t+\theta.\|x\|^r_{\mathcal{A}}}\end{aligned}$$

for all $\lambda\in\mathbb{T}^1$, all $x_1,\cdots,x_n,x,y\in\mathcal{A}$ and $t>0$. Then there exists a unique random Lie JC^*-algebra homomorphism $H:\mathcal{A}\to\mathcal{B}$ such that

$$\mu^{\mathcal{B}}_{f(x)-H(x)}\geqslant\frac{(|\mathcal{N}|-|\mathcal{N}|^r)t}{(|\mathcal{N}|-|\mathcal{N}|^r)t+n\theta\|x\|^r_{\mathcal{A}}}$$

for all $x\in\mathcal{A}$ and $t>0$.

Proof. By the same reasoning as in the proof of Theorem 4.2 and a technique similar to Corollary 3.2, by putting $L=|\mathcal{N}|^{r-1}$, the proof will be completed. □

Definition 4.4. *Let $\mathcal{A}$ be a non-Archimedean random Lie JC^*-algebra. A $\mathbb{C}$-linear mapping $\delta:\mathcal{A}\to\mathcal{A}$ is called a random Lie JC^*-algebra derivation if δ satisfies*

$$\begin{aligned}&\delta([x,y])=[\delta(x),y]+[x,\delta(y)],\\&\delta(x\circ y)=\delta(x)\circ y+x\circ\delta(y),\\&\delta(x^*)=\delta(x)^*\end{aligned}$$

for all $x,y\in\mathcal{A}$.

In the following theorem, we prove the Hyers-Ulam stability of derivation on non-Archimedean random Lie JC^*-algebras for the functional equation $D_\lambda f(x_1, \cdots, x_n) = 0$.

Theorem 4.5. *Let $f : \mathcal{A} \to \mathcal{A}$ be a mapping for which there are functions $\varphi : \mathcal{A}^n \to D^+$ and $\psi : \mathcal{A}^2 \to D^+$ such that (3.1) and (3.3) hold and*

$$\mu^{\mathcal{A}}_{f([x,y])-[f(x),y]-[x,f(y)]}(t) \geqslant \psi_{x,y}(t), \tag{4.6}$$

$$\mu^{\mathcal{A}}_{f(x\circ y)-f(x)\circ y-x\circ f(y)}(t) \geqslant \phi_{x,y}(t) \tag{4.7}$$

for all $x, y \in \mathcal{A}$. If there exists an $L < 1$ and (3.4), (3.5), (3.6) and (4.3) hold, then there exists a unique random Lie JC^*-algebra derivation $\delta : \mathcal{A} \to \mathcal{A}$ such that (3.13) holds.

Proof. By the same argument as in the proof of Theorem 4.2, there exists a unique $\mathbb{C}$-linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ satisfying (3.13), and is given by

$$\delta(x) = \lim_{m\to\infty} \frac{f(\mathcal{M}^m x)}{|\mathcal{M}|^m} \tag{4.8}$$

for all $x \in \mathcal{A}$.

It follows from (3.9), (4.4) and (4.8) that

$$\begin{aligned}\mu_{\delta([x,y])-[\delta(x),y]-[x,\delta(y)]}(t) &= \lim_{m\to\infty} \mu^{\mathcal{A}}_{f(\mathcal{M}^{2m}[x,y])-[f(\mathcal{M}^m x),\mathcal{M}^m y]-[\mathcal{M}^m x,f(\mathcal{M}^m y)]}\left(|\mathcal{M}|^{2m}t\right)\\ &\geqslant \lim_{m\to\infty} \psi_{\mathcal{M}^m x,\mathcal{M}^m y}\left(|\mathcal{M}|^{2m}t\right) = 1\end{aligned}$$

and

$$\begin{aligned}\mu^{\mathcal{A}}_{\delta(x\circ y)-\delta(x)\circ y-x\circ\delta(y)}(t) &= \lim_{m\to\infty} \mu^{\mathcal{A}}_{f(\mathcal{M}^{2m}(x\circ y))-f(\mathcal{M}^m x)\circ y-x\circ f(\mathcal{M}^m y)}\left(|\mathcal{M}|^{2m}t\right)\\ &\geqslant \lim_{m\to\infty} \phi_{\mathcal{M}^m x,\mathcal{M}^m y}\left(|\mathcal{M}|^{2m}t\right) = 1\end{aligned}$$

for all $x, y \in \mathcal{A}$ and $t > 0$, and so we conclude that

$$\delta([x, y]) = [\delta(x), y] + [x, \delta(y)], \quad \delta(x \circ y) = \delta(x) \circ y + x \circ \delta(y)$$

for all $x, y \in \mathcal{A}$. Therefore, $\delta : \mathcal{A} \to \mathcal{A}$ is the unique desired random Lie JC^*-algebra derivation satisfying (3.13). □

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ON THE FUZZY STABILITY PROBLEMS OF GENERALIZED SEXTIC MAPPINGS

HEEJEONG KOH AND DONGSEUNG KANG*

ABSTRACT. We introduce a fuzzy anti-β-norm and generalized sextic mapping and then investigate the Hyers-Ulam-Rassias stability in quasi β-Banach space and the fuzzy stability by using a fixed point in fuzzy anti-β Banach space for the generalized sextic function.

1. INTRODUCTION

The concept of stability problem of a functional equation was first posed by Ulam [33] concerning the stability of group homomorphisms. In the next year, Hyers [14] gave a partial answer to the question of Ulam. Hyers' theorem was generalized in various directions. The very first author who generalized Hyers' theorem to the case of unbounded control functions was Aoki [1]. Rassias [28] succeeded in extending the result of Hyers' theorem by weakening the condition for the Cauchy difference operator $CDf(x, y) = f(x + y) - [f(x) + f(y)]$ to be controlled by $\varepsilon(||x||^p + ||y||^p)$. Rassias' paper [28] has provided a lot of influence in the development of Hyers-Ulam stability or Hyers-Ulam-Rassias stability of functional equations. In 1996, Isac and Rassias [16] were first to provide applications of new fixed point theorems for the proof of stability theory of functional equations. By using fixed point methods the stability problems of several functional equations have been extensively investigated by a number of authors; see [6], [7], [25] and [26]. Recently, the stability problem of functional equations was investigated by using shadowing properties; see [20] and [31].

During the last three decades, several stability problems of a large variety of functional equations have been extensively studied and generalized by a number of authors [9], [12], [15], [28], and [2]. In particular, Xu and et al. [37] introduced the sextic functional equation

$$f(x + 3y) + f(x - 3y) - 6[f(x + 2y) + f(x - 2y)] + 15[f(x + y) + f(x - y)] = 20f(x) + 720f(y)\,. \tag{1.1}$$

In fact, Xu and et al. [37] and Gordji and et al. [13] introduced a quintic mapping and sextic mapping.

In this paper, we deal with the following functional equation

$$f(ax + y) + f(ax - y) + f(x + ay) + f(x - ay) = a^2(a^2 + 1)[f(x + y) + f(x - y)] + 2(a^2 - 1)(a^4 - 1)[f(x) + f(y)] \tag{1.2}$$

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* Corresponding author.

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holds for all $x,\, y \in X$ and all $a \in \mathbb{Z}\,(a \neq 0, \pm 1)$.

We will use the following definition to prove Hyers-Ulam-Rassias stability for the generalized sextic functional equation in the quasi β-normed space. Let β be a real number with $0 < \beta \leq 1$ and $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$.

Definition 1.1. *Let X be a linear space over a field $\mathbb{K}$. A quasi β-norm $||\cdot||$ is a real-valued function on X satisfying the following statements:*

(1) $||x|| \geq 0$ for all $x \in X$ and $||x|| = 0$ if and only if $x = 0$.

(2) $||\lambda x|| = |\lambda|^\beta \cdot ||x||$ for all $\lambda \in \mathbb{K}$ and all $x \in X$.

(3) There is a constant $K \geq 1$ such that $||x+y|| \leq K(||x||+||y||)$ for all $x,\, y \in X$.

The pair $(X, ||\cdot||)$ is called a *quasi β-normed space* if $||\cdot||$ is a quasi β-norm on X. The smallest possible K is called the *modulus of concavity* of $||\cdot||$. A *quasi β-Banach space* is a complete quasi-β-normed space.

A quasi β-norm $||\cdot||$ is called a $(\beta,\, p)$-norm $(0 < p \leq 1)$ if (3) takes the form $||x+y||^p \leq ||x||^p + ||y||^p$ for all $x,\, y \in X$. In this case, a quasi β-Banach space is called a (β, p)-Banach space; see [5], [29] and [27].

In 1984, Katsaras [18] and Wu and Fang [35] independently introduced a notion of a fuzzy norm. Since then some mathematicians have defined fuzzy metrics and norms on a linear space from various points of view; see [3], [11], [19], [36] and [23]. In 2003, Bag and Samanta [3] modified the definition of Cheng and Mordeson [8]. Bag and Samanta [3] introduced the following definition of fuzzy normed spaces. The notion of fuzzy stability of functional equations was given in the paper [24]. Jebril and Samanta [17] introduced a fuzzy anti-norm linear space depending on the idea of fuzzy anti-norm was introduced by Bag and Samanta [4] and investigated their important properties.

We will use the definition of fuzzy anti-normed spaces to investigate a fuzzy version of Hyers-Ulam-Rassias stability in the fuzzy anti-normed algebra setting.

Definition 1.2. [17] *Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0,\, 1]$ is called a fuzzy anti-norm on X if for all $x,\, y \in X$ and all $s,\, t \in \mathbb{R}$,*

(aN1) $N(x,t) = 1$ *for* $t \leq 0$

(aN2) $N(x,t) = 0$ *if and only if* $x = 0$ *for all* $t > 0$

(aN3) $N(cx,t) = N(x, \frac{t}{|c|})$ *for* $c \neq 0$

(aN4) $N(x+y, s+t) \leq max\{N(x,s)\,, N(y,t)\}$

(aN5) $N(x,t)$ *is a non-increasing function of* $t \in \mathbb{R}$ *and* $\lim_{t\to\infty} N(x,t) = 0$,

(aN6) *for* $x \neq 0$, $N(x,\,\cdot)$ *is continuous on* $\mathbb{R}$.

The pair (X, N) is called a fuzzy anti-normed space.

The property (aN3) implies that $N(-x,t) = N(x,t)$ for all $x \in X$ and $t > 0$. It is easy to show that (aN4) is equivalent the following condition:

$$N(x+y,t) \leq \max\{N(x,t)\,, N(y,t)\}\,, \text{ for all } x, y \in X \text{ and } t \in \mathbb{R}.$$

Definition 1.3. *Let X be a real vector space. A fuzzy anti-norm $N : X \times \mathbb{R} \to [0,\, 1]$ is called a fuzzy anti-β-norm on X if (aN_3) in Definition 1.2 takes the form*

$$(aN_3') \quad N(cx,\, t) = N(x,\, \frac{t}{|c|^\beta}) \quad (c \neq 0,\, 0 < \beta \leq 1).$$

Example 1.4. *Let $(X,\, ||\cdot||)$ be a β-normed space. Define*

$$N(x,t) = \begin{cases} \frac{||x||}{t+||x||} & when\ t > 0,\, t \in \mathbb{R} \\ 1 & when\ t \leq 0\,, \end{cases}$$

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where $x \in X$ *. We note that*

$$N(cx,\, t) = \frac{||cx||}{t + ||cx||} = \frac{||x||}{\frac{t}{|c|^\beta} + ||x||} = N(x,\, \frac{t}{|c|^\beta})\,,$$

for all $x \in X$ *and* $c \in \mathbb{R}\,(c \neq 0\,, 0 < \beta \leq 1)$ *. Then* (X, N) *is a fuzzy anti-*β*-normed space induced by the* β*-norm* $||\cdot||$ *.*

Definition 1.5. *Let* $(X,\, N)$ *be a fuzzy anti-*β*-normed vector space. A sequence* $\{x_n\}$ *in* X *is said to be convergent or converge if there exists an* $x \in X$ *such that* $\lim_{n\to\infty} N(x_n - x,\, t) = 0$ *for all* $t > 0$ *. In this case,* x *is called the limit of the sequence* $\{x_n\}$ *and we denote it by* $N\text{-}\lim_{n\to\infty} x_n = x$ *.*

Definition 1.6. *Let* $(X,\, N)$ *be a fuzzy anti-*β*-normed vector space. A sequence* $\{x_n\}$ *in* X *is called Cauchy if for each* $\varepsilon > 0$ *and each* $t > 0$ *there exists an* $n_0 \in \mathbb{N}$ *such that for all* $n \geq n_0$ *and all integer* $d > 0$ *, we have* $N(x_{n+d} - x_n,\, t) < \varepsilon$ *.*

It is well-known that every convergent sequence in a fuzzy anti-β-normed vector space is Cauchy. If each Cauchy sequaence is convergent, then the fuzzy anti-β-normed space is said to be *fuzzy anti-*β *complete* and the fuzzy anti-β-normed vector space is called a *fuzzy anti-*β *Banach space.*

Now, we will state the theorem, the alternative of fixed point in a generalized metric space.

Definition 1.7. *Let* X *be a set. A function* $d : X \times X \to [0,\, \infty]$ *is called a generalized metric on* X *if* d *satisfies*

(1) $d(x,\, y) = 0$ *if and only if* $x = y$ *;*
(2) $d(x,\, y) = d(y,\, x)$ *for all* $x,\, y \in X$ *;*
(3) $d(x,\, z) \leq d(x,\, y) + d(y,\, z)$ *for all* $x,\, y,\, z \in X$ *.*

Theorem 1.8 (The alternative of fixed point [21], [30])**.** *Suppose that we are given a complete generalized metric space* (X, d) *and a strictly contractive mapping* $J : X \to X$ *with Lipschitz constant* $0 < L < 1$ *. Then for each given* $x \in X$ *, either*

$$d(J^n x,\, J^{n+1}x) = \infty \text{ for all } n \geq 0\,,$$

or there exists a natural number n_0 *such that*

(1) $d(J^n x,\, J^{n+1}x) < \infty$ *for all* $n \geq n_0$ *;*
(2) *The sequence* $\{J^n x\}$ *is convergent to a fixed point* y^* *of* J *;*
(3) y^* *is the unique fixed point of* J *in the set*

$$Y = \{y \in X | d(J^{n_0}x, y) < \infty\}\,;$$

(4) $d(y, y^*) \leq \frac{1}{1-L}\, d(y, Jy)$ *for all* $y \in Y$ *.*

In this paper, we investigate the Hyers-Ulam-Rassias stability in quasi β-normed space and then the fuzzy stability by using a fixed point in fuzzy anti-β Banach space for the generalized sextic function $f : X \to Y$ satisfying the equation (1.2). Let us fix some notations which will be used throughout this paper. Let $a \in \mathbb{Z}\,(a \neq 0\,, \pm 1)$.

2. A sextic functional equation

In this section let X and Y be real vector spaces and we investigate the general solution of the functional equation (1.2). Before we proceed, we would like to introduce some basic definitions concerning n-additive symmetric mappings and key concepts which are found in [32] and [34]. A function $A : X \to Y$ is said to be

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additive if $A(x+y) = A(x) + A(y)$ for all $x, y \in X$. Let n be a positive integer. A function $A_n : X^n \to Y$ is called *n-additive* if it is additive in each of its variables. A function A_n is said to be *symmetric* if $A_n(x_1, \cdots, x_n) = A_n(x_{\sigma(1)}, \cdots, x_{\sigma(n)})$ for every permutation $\{\sigma(1), \cdots, \sigma(n)\}$ of $\{1, 2, \cdots, n\}$. If $A_n(x_1, x_2, \cdots, x_n)$ is an n-additive symmetric map, then $A^n(x)$ will denote the diagonal $A_n(x, x, \cdots, x)$ and $A^n(rx) = r^n A^n(x)$ for all $x \in X$ and all $r \in \mathbb{Q}$. such a function $A^n(x)$ will be called a *monomial function* of degree n (assuming $A^n \not\equiv 0$). Furthermore the resulting function after substitution $x_1 = x_2 = \cdots = x_s = x$ and $x_{s+1} = x_{s+2} = \cdots = x_n = y$ in $A_n(x_1, x_2, \cdots, x_n)$ will be denoted by $A^{s,n-s}(x, y)$.

Theorem 2.1. *A function $f : X \to Y$ is a solution of the functional equation (1.2) if and only if f is of the form $f(x) = A^6(x)$ for all $x \in X$, where $A^6(x)$ is the diagonal of the 6-additive symmetric mapping $A_6 : X^6 \to Y$.*

Proof. Assume that f satisfies the functional equation (1.2). Letting $x = y = 0$ in the equation (1.2), we have

$$2a^2(2a^2+1)(a^2-1)f(0) = 0,$$

that is, $f(0) = 0$. Let $y = 0$ in the equation (1.2). Then we get

$$f(ax) = a^6 f(x) \tag{2.1}$$

for all $x \in X$. Putting $x = 0$ in the equation (1.2), we get

$$(a^4-1)(a^2-1)\Big(f(y) - f(-y)\Big) = 0 \tag{2.2}$$

for all $y \in X$. Hence we have $f(y) = f(-y)$, for all $y \in X$. That is, f is even. We can rewrite the functional equation (1.2) in the form

$$\begin{aligned} & f(x) - \frac{1}{2(a^2-1)(a^4-1)} f(ax+y) - \frac{1}{2(a^2-1)(a^4-1)} f(ax-y) \\ & - \frac{1}{2(a^2-1)(a^4-1)} f(x+ay) - \frac{1}{2(a^2-1)(a^4-1)} f(x-ay) \\ & + \frac{a^2(a^2+1)}{2(a^2-1)(a^4-1)} f(x+y) + \frac{a^2(a^2+1)}{2(a^2-1)(a^4-1)} f(x-y) + f(y) = 0 \end{aligned}$$

for all $x, y \in X$ and an integer $a (a \neq 0, \pm 1)$. By Theorem 3.5 and 3.6 in [34], f is a generalized polynomial function of degree at most 6, that is, f is of the form

$$f(x) = A^6(x) + A^5(x) + A^4(x) + A^3(x) + A^2(x) + A^1(x) + A^0(x) \tag{2.3}$$

for all $x \in X$, where $A^0(x) = A^0$ is an arbitrary element of Y, and $A^i(x)$ is the diagonal of the i-additive symmetric mapping $A_i : X^i \to Y$ for $i = 1, 2, 3, 4, 5, 6$. By $f(0) = 0$ and $f(-x) = f(x)$ for all $x \in X$, we get $A^0(x) = A^0 = 0$, $A^5(x) = 0$, $A^3(x) = 0$ and $A^1(x) = 0$. It follows that

$$f(x) = A^6(x) + A^4(x) + A^2(x)$$

for all $x \in X$. By (2.1) and $A^n(rx) = r^n A^n(x)$ for all $x \in X$ and $r \in \mathbb{Q}$, we obtain that $A^2(x) = -\frac{a^2}{a^2+1} A^4(x)$ for all $x \in X$ and an integer $a\,(a \neq 0, \pm 1)$. Hence we get $A^4(x) = A^2(x) = 0$, for all $x \in X$. Thus we have $f(x) = A^6(x)$ for all $x \in X$.

Conversely, assume that $f(x) = A^6(x)$ for all $x \in X$, where $A^6(x)$ is the diagonal of a 6-additive symmetric mapping $A_6 : X^6 \to Y$. Note that

$$\begin{aligned} A^6(qx+ry) &= q^6 A^6(x) + 6q^5 r A^{5,1}(x,y) + 15q^4 r^2 A^{4,2}(x,y) + 20q^3 r^3 A^{3,3}(x,y) \\ &+ 15q^2 r^4 A^{2,4}(x,y) + 6qr^5 A^{1,5}(x,y) + r^6 A^6(y) \end{aligned}$$

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$$c^s A^{s,t}(x,y) = A^{s,t}(cx,y)\,, \quad c^t A^{s,t}(x,y) = A^{s,t}(x,cy)$$

where $1 \le s,t \le 5$ and $c \in \mathbb{Q}$. Thus we may conclude that f satisfies the equation (1.2). □

We note that a mapping $f : X \to Y$ is called *generalized sextic* if f satisfies the functional equation (1.2).

3. Hyers-Ulam-Rassias stability over a quasi β-Banach space

Throughout this section, let X be a real linear space and let Y be a quasi β-Banach space with a quasi β-norm $||\cdot||_Y$. Let K be the modulus of concavity of $||\cdot||_Y$. We will investigate the Hyers-Ulam-Rassias stability for the functional equation (1.2); see also the paper [10].

For a given mapping $f : X \to Y$ and all fixed integer a ($a \ne 0, \pm 1$), let

$$D_a f(x,y) := f(ax+y) + f(ax-y) + f(x+ay) + f(x-ay) -a^2(a^2+1)\Big(f(x+y)+f(x-y)\Big)-2(a^2-1)(a^4-1)\Big(f(x)+f(y)\Big) \tag{3.1}$$

for all $x,\ y \in X$.

Theorem 3.1. *Suppose that there exists a mapping $\phi : X^2 \to [0,\infty)$ for which a mapping $f : X \to Y$ satisfies $f(0) = 0$,*

$$||D_a f(x,y)||_Y \le \phi(x,y) \tag{3.2}$$

and the series $\sum_{j=0}^{\infty} \left(\frac{K}{|a|^{6\beta}}\right)^j \phi(a^j x, a^j y)$ converges for all $x,\ y \in X$. Then there exists a unique generalized sextic mapping $S : X \to Y$ satisfying the equation (1.2) and the inequality

$$||f(x) - S(x)||_Y \le \frac{K}{2^\beta |a|^{6\beta}} \sum_{j=0}^{\infty} \Big(\frac{K}{|a|^{6\beta}}\Big)^j \phi(a^j x, 0)\,, \tag{3.3}$$

for all $x \in X$.

Proof. By letting $y = 0$ in inequality (3.2), since $f(0) = 0$ we have

$$\begin{aligned} ||D_a f(x,0)||_Y &= ||2f(ax) + 2f(x) - 2a^2(a^2+1)f(x) - 2(a^2-1)(a^4-1)f(x)||_Y \\ &= 2^\beta |a|^{6\beta} ||f(x) - \frac{1}{a^6} f(ax)||_Y \le \phi(x,0)\,, \end{aligned}$$

that is,

$$||f(x) - \frac{1}{a^6} f(ax)||_Y \le \frac{1}{2^\beta |a|^{6\beta}}\, \phi(x,0)\,, \tag{3.4}$$

for all $x \in X$.

We note that putting $x = ax$ and multiplying $\frac{1}{|a|^{6\beta}}$ in the inequality (3.4), we get

$$\frac{1}{|a|^{6\beta}} ||f(ax) - \frac{1}{a^6} f(a^2 x)||_Y \le \frac{1}{2^\beta |a|^{6\beta}}\, \frac{1}{|a|^{6\beta}} \phi(ax,0)\,, \tag{3.5}$$

for all $x \in X$.

Combining two inequalities (3.4) and (3.5), we have

$$||f(x) - \Big(\frac{1}{a^6}\Big)^2 f(a^2 x)||_Y \le \frac{K}{2^\beta |a|^{6\beta}} \Big(\phi(x,0) + \frac{1}{|a|^{6\beta}} \phi(ax,0)\Big)\,, \tag{3.6}$$

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for all $x \in X$.

Since $K \geq 1$, inductively using the previous note we have the following inequalities

$$||f(x) - \Big(\frac{1}{a^6}\Big)^k f(a^k x)||_Y \leq \frac{K}{2^\beta |a|^{6\beta}} \sum_{j=0}^{k-1} \Big(\frac{K}{|a|^{6\beta}}\Big)^j \phi(a^j x, 0), \tag{3.7}$$

for all $x \in X$, $k \in \mathbb{N}$ and also

$$||\Big(\frac{1}{a^6}\Big)^k f(a^k x) - \Big(\frac{1}{a^6}\Big)^t f(a^t x)||_Y \leq \frac{K}{2^\beta |a|^{6\beta}} \sum_{j=k}^{t-1} \Big(\frac{K}{|a|^{6\beta}}\Big)^j \phi(a^j x, 0), \tag{3.8}$$

for all $x \in X$ and k, $t \in \mathbb{N}\,(k < t)$.

Since the right-hand side of the previous inequality (3.8) tends to 0 as $t \to \infty$, hence $\{\big(\frac{1}{a^6}\big)^n f(a^n x)\}$ is a Cauchy sequence in the quasi β-Banach space Y. Thus we may define

$$S(x) = \lim_{n\to\infty} \Big(\frac{1}{a^6}\Big)^n f(a^n x),$$

for all $x \in X$. Since $K \geq 1$, replacing x and y by $a^n x$ and $a^n y$ respectively and dividing by $|a|^{6\beta n}$ in the inequality (3.2), we have

$$\begin{aligned}
&\Big(\frac{1}{|a|^{6\beta}}\Big)^n ||D_a f(a^n x, a^n y)||_Y \\
&= \Big(\frac{1}{|a|^{6\beta}}\Big)^n ||f(a^n(ax+y)) + f(a^n(ax-y)) + f(a^n(x+ay)) + f(a^n(x-ay)) \\
&-a^2(a^2+1)\Big(f(a^n(x+y)) + f(a^n(x-y))\Big) \\
&-2(a^2-1)(a^4-1)\Big(f(a^n x) + f(a^n y)\Big)||_Y \\
&\leq \Big(\frac{K}{|a|^{6\beta}}\Big)^n \phi(a^n x, a^n y)
\end{aligned}$$

for all x, $y \in X$.

By taking $n \to \infty$, the definition of S implies that S satisfies (1.2) for all x, $y \in X$, that is, S is the generalized sextic mapping. Also, the inequality (3.7) implies the inequality (3.3).

Now, it remains to show the uniqueness. Assume that there exists $T : X \to Y$ satisfying (1.2) and (3.3). Then

$$\begin{aligned}
||T(x) - S(x)||_Y &= \Big(\frac{1}{|a|^{6\beta}}\Big)^n ||T(a^n x) - S(a^n x)||_Y \\
&\leq \Big(\frac{1}{|a|^{6\beta}}\Big)^n K\Big(||T(a^n x) - f(a^n x)||_Y + ||f(a^n x) - S(a^n x)||_Y\Big) \\
&\leq \frac{2K^2}{2^\beta |a|^{6\beta} K^n} \sum_{j=n}^{\infty} \Big(\frac{K}{|a|^{6\beta}}\Big)^j \phi(a^j x, 0)
\end{aligned}$$

for all $x \in X$. By letting $n \to \infty$, we immediately have the uniqueness of S. □

Corollary 3.2. *Let $\theta \geq 0$, $p < 6$ be a real number and X be a normed linear space with norm $||\cdot||$. Suppose $f : X \to Y$ is a mapping satisfying $f(0) = 0$ and*

$$||D_a f(x,y)||_Y \leq \theta(||x||^p + ||y||^p) \tag{3.9}$$

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for all $x,\, y \in X$ *and all* $t > 0$. *Then* $S(x) := N\text{-}\lim_{n\to\infty} \frac{1}{a^{6n}} f(a^n\, x)$ *exists for each* $x \in X$ *and defines a generalized sextic mapping* $S : X \to Y$ *such that*

$$||f(x) - S(x)||_Y \leq \frac{\theta\, K\, ||x||^p}{2^\beta(|a|^{6\beta} - K|a|^{p\beta})}$$

for all $x \in X$ *and all* $t > 0$.

Proof. The proof follows from Theorem 3.1 by taking $\phi(x,\, y) = \theta(||x||^p + ||y||^p)$ for all $x\,, y \in X$. □

4. Fuzzy fixed point stability over a Fuzzy Banach space

Let us fix some notations which will be used throughout this section. We assume X is a vector space and $(Y,\, N)$ is a fuzzy anti-β Banach space. Using fixed point method, we will prove the Hyers-Ulam stability of the functional equation satisfying equation (1.2) in fuzzy anti-β Banach space.

Theorem 4.1. *Let* $\phi : X^2 \to [0, \infty)$ *be a function such that there exists an* $0 < L < 1$ *with*

$$\phi(x, y) \leq \frac{L}{|a|^{6\beta}}\, \phi(ax, ay) \tag{4.1}$$

for all $x,\, y \in X$. *Let* $f : X \to Y$ *be a mapping satisfying* $f(0) = 0$ *and*

$$N(D_a f(x, y),\, t) \leq \frac{\phi(x, y)}{t + \phi(x, y)} \tag{4.2}$$

for all $x,\, y \in X$ *and all* $t > 0$. *Then* $S(x) := N\text{-}\lim_{n\to\infty} a^{6n} f\left(\frac{x}{a^n}\right)$ *exists for each* $x \in X$ *and defines a generalized sextic mapping* $S : X \to Y$ *such that*

$$N(f(x) - S(x),\, t) \leq \frac{L\, \phi(x, 0)}{2^\beta |a|^{6\beta}(1 - L)\, t + L\, \phi(x, 0)} \tag{4.3}$$

for all $x \in X$ *and all* $t > 0$.

Proof. By letting $y = 0$ in the inequality (4.2), we have

$$N\Big(2f(ax) - 2a^6 f(x),\, t\Big) \;\leq\; \frac{\phi(x,\, 0)}{t + \phi(x,\, 0)} \tag{4.4}$$

for all $x \in X$ and all $t > 0$.

We note that by letting $x = \frac{x}{a}$ in the inequality (4.4) we have

$$N\Big(2f(x) - 2a^6 f\Big(\frac{x}{a}\Big),\, t\Big) \;\leq\; \frac{\phi(\frac{x}{a},\, 0)}{t + \phi(\frac{x}{a},\, 0)}\,.$$

The inequality (4.1) implies that

$$N\Big(f(x) - a^6 f\Big(\frac{x}{a}\Big),\, \frac{t}{2^\beta}\Big) \;\leq\; \frac{\frac{L}{|a|^{6\beta}}\, \phi(x,\, 0)}{t + \frac{L}{|a|^{6\beta}}\, \phi(x,\, 0)}\,.$$

By putting $t = \frac{L}{|a|^{6\beta}}\, t$, we have

$$N\Big(f(x) - a^6 f\Big(\frac{x}{a}\Big),\, \frac{L}{2^\beta |a|^{6\beta}}\, t\Big) \;\leq\; \frac{\frac{L}{|a|^{6\beta}}\, \phi(x,\, 0)}{\frac{L}{|a|^{6\beta}}\, t + \frac{L}{|a|^{6\beta}}\, \phi(x,\, 0)}\,,$$

that is,

$$N\Big(f(x) - a^6 f\Big(\frac{x}{a}\Big),\, \frac{L}{2^{\beta}|a|^{6\beta}}\, t\Big) \;\leq\; \frac{\phi(x,\,0)}{t+\phi(x,\,0)}\,, \tag{4.5}$$

for all $x \in X$ and all $t > 0$.

We consider the set

$$F := \{g : X \to X\}$$

and the mapping d defined on $F \times F$ by

$$d(g,\,h) = \inf\{\mu \in \mathbb{R}^+ \mid N\Big(g(x) - h(x),\, \mu t\Big) \leq \frac{\phi(x,\,0)}{t+\phi(x,\,0)}\,, \forall x \in X \text{ and } t > 0\}$$

where $\inf \emptyset = +\infty$, as usual. Then $(F,\,d)$ is a complete generalized metric space; see [22, Lemma 2.1]. Now let's consider the linear mapping $J : F \to F$ such that

$$Jg(x) := a^6 g\Big(\frac{x}{a}\Big)$$

for all $x \in X$. Let $g\,, h \in F$ be given such that $d(g\,, h) = \varepsilon$. Then

$$N\Big(g(x) - h(x),\, \varepsilon t\Big) \leq \frac{\phi(x,\,0)}{t+\phi(x,\,0)}$$

for all $x \in X$ and all $t > 0$.

$$\begin{aligned} N\Big(Jg(x) - Jh(x),\, L\varepsilon t\Big) &= N\Big(a^6 g\Big(\frac{x}{a}\Big) - a^6 h\Big(\frac{x}{a}\Big),\, L\varepsilon t\Big) \\ = \quad N\Big(g\Big(\frac{x}{a}\Big) - h\Big(\frac{x}{a}\Big),\, \frac{L}{|a|^{6\beta}}\varepsilon t\Big) &\leq \frac{\phi(\frac{x}{a},\,0)}{\frac{L}{|a|^{6\beta}}t + \phi(\frac{x}{a},\,0)} \\ \leq \quad \frac{\frac{L}{|a|^{6\beta}}\phi(x,\,0)}{\frac{L}{|a|^{6\beta}}t + \frac{L}{|a|^{6\beta}}\phi(x,\,0)} &= \frac{\phi(x,\,0)}{t+\phi(x,\,0)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. $d(g,\,h) = \varepsilon$ implies that $d(Jg,\,Jh) \leq L\varepsilon$. Hence we get

$$d(Jg,\,Jh) \leq L\, d(g,\,h)$$

for all $g,\,h \in F$. The inequality (4.5) implies that $d(f,\,Jf) \leq \frac{L}{2^{\beta}|a|^{6\beta}}$. By Theorem 1.8, there exists a mapping $S : X \to Y$ such that

(1) S is a fixed point of J, that is,

$$S\Big(\frac{x}{a}\Big) = \frac{1}{a^6}\, S(x) \tag{4.6}$$

for all $x \in X$. The mapping S is a unique fixed point of J in the set $M = \{g \in F \,|\, d(f,\,g) < \infty\}$. This means that S is a unique mapping satisfying the equation (4.6) such that there exists a $\mu \in (0,\,\infty)$ satisfying

$$N\Big(f(x) - S(x),\, \mu t\Big) \leq \frac{\phi(x,\,0)}{t+\phi(x,\,0)}$$

for all $x \in X$ and all $t > 0$;

(2) $d(J^n f,\, S) \to 0$ as $n \to \infty$. This implies the following equality

$$\text{N-}\lim_{n\to\infty} a^{6n} f\Big(\frac{x}{a^n}\Big) = S(x)$$

for all $x \in X$;

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(3) $d(f,\, S) \le \frac{1}{1-L}\, d(f,\, Jf)$, which implies the inequality

$$d(f,\, S) \le \frac{1}{1-L} \cdot \frac{L}{2^\beta |a|^{6\beta}} = \frac{L}{2^\beta |a|^{6\beta}(1-L)}\,.$$

It implies that

$$N\Big(f(x) - S(x),\, \frac{L}{2^\beta |a|^{6\beta}(1-L)} t\Big) \le \frac{\phi(x,\, 0)}{t + \phi(x,\, 0)}$$

for all $x \in X$ and all $t > 0$. By replacing t by $\frac{2^\beta |a|^{6\beta}(1-L)}{L}\, t$, we have

$$N\Big(f(x) - S(x),\, t\Big) \le \frac{L\phi(x,\, 0)}{2^\beta |a|^{6\beta}(1-L)\, t + L\phi(x,\, 0)}$$

for all $x \in X$ and all $t > 0$. That is, the inequality (4.3) holds. By letting $x = \frac{x}{a^n}$ and $y = \frac{y}{a^n}$ in the inequality (4.2), we have

$$N\Big(a^{6n} D_a f\Big(\frac{x}{a^n}, \frac{y}{a^n}\Big),\, |a|^{6\beta n}\, t\Big) \le \frac{\phi(\frac{x}{a^n},\, \frac{y}{a^n})}{t + \phi(\frac{x}{a^n},\, \frac{y}{a^n})}$$

for all $x,\, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Replacing t by $\frac{t}{|a|^{6\beta n}}$,

$$N\Big(a^{6n} D_a f\Big(\frac{x}{a^n}, \frac{y}{a^n}\Big),\, t\Big) \le \frac{\phi(\frac{x}{a^n},\, \frac{y}{a^n})}{\frac{t}{|a|^{6\beta n}} + \phi(\frac{x}{a^n},\, \frac{y}{a^n})} \le \frac{L^n\, \phi(x,\, y)}{t + L^n\, \phi(x,\, y)}$$

for all $x,\, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{L^n\, \phi(x,\, y)}{t + L^n\, \phi(x,\, y)} = 0$ for all $x,\, y \in X$ and all $t > 0$, we may conclude that

$$N\Big(D_a S(x,\, y),\, t\Big) = 0$$

for all $x,\, y \in X$ and all $t > 0$. Thus the mapping $S : X \to Y$ is the generalized sextic mapping. □

Corollary 4.2. *Let $\theta \ge 0$, $p > 6$ be a real number and X be a normed linear space with norm $||\cdot||$. Suppose $f : X \to Y$ is a mapping satisfying $f(0) = 0$ and*

$$N(D_a f(x,y),\, t) \le \frac{\theta(||x||^p + ||y||^p)}{t + \theta(||x||^p + ||y||^p)} \tag{4.7}$$

for all $x,\, y \in X$ and all $t > 0$. Then $S(x) := N\text{-}\lim_{n\to\infty} a^{6n} f\Big(\frac{x}{a^n}\Big)$ exists for each $x \in X$ and defines a generalized sextic mapping $S : X \to Y$ such that

$$N(f(x) - S(x),\, t) \le \frac{\theta\, ||x||^p}{2^\beta(|a|^{p\beta} - |a|^{6\beta})\, t + \theta\, ||x||^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 4.1 by taking $\phi(x,\, y) = \theta(||x||^p + ||y||^p)$ for all $x\,, y \in X$ and $L = |a|^{(6-p)\beta}$. □

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Department of Mathematical Education, Dankook University, 126, Jukjeon, Suji, Yongin, Gyeonggi, South Korea 448-701 , Korea

E-mail address: khjmath@dankook.ac.kr (H. Koh)

E-mail address: dskang@dankook.ac.kr (D. Kang)

Existence and uniqueness of solutions to SFDEs driven by G-Brownian motion with non-Lipschitz conditions

Faiz Faizullah*

Department of BS and H, College of E and ME, National University of Sciences and Technology (NUST) Pakistan.

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Abstract

The main aim of this paper is to study the existence, uniqueness and stability of solution for stochastic functional differential equations driven by G-Brownian motion (in short G-SFDEs). The existence-and-uniqueness theorem is established for G-SFDEs under non-Lipschitz condition and weakened linear growth condition. We have used the Picard approximation scheme, Gronwall's inequality, Bihari's inequality and Burkholder-Davis-Gundy (in short BDG) inequalities to develop the existence theory for the above mentioned stochastic dynamical systems. In addition, the mean square stability of solutions for these systems has been obtained.

Key words: Existence, uniqueness, stability, G-Brownian motion, stochastic functional differential equations.

1 Introduction

Responding to the contemporary developments in the fields of physics, control engineering, economics, and social sciences, a growing concern has recently been witnessed in both stochastic differential and deterministic models. The applications of functional differential equations have been applied in a number of cases in physical phenomena, such as in the relocation of soil moisture, where the fluid flows through the crack of rocks, and the problem of conduction of heat as well as its share in order fluids is investigated. The idea of G-Brownian motion as well as the associated stochastic differential equations were introduced by Peng [8, 10]. These equations were extended to stochastic functional differential equations, which are driven by G-Brownian motion (in short G-SFDEs) by Ren, Bi and Sakthivel [12]. While Faizullah, developed the existence-and-uniqueness theorem for G-SFDEs with Cauchy-Maruyama approximation scheme [3], they used the strong Lipschitz and linear growth conditions to develop the mentioned theory. In this article, we have generalized the existence theory for functional stochastic dynamical systems, driven by G-Brownian motion. We have used non-Lipschitz condition and weak linear growth condition to study the existence, uniqueness and stability theory for G-SFDEs. We have considered the following stochastic dynamical system that

*Corresponding author, E-mail: faiz_math@yahoo.com

is driven by G-Brownian motion. Let $0 \leq t \leq T < \infty$. Suppose $g : [0,T] \times BC([-\theta,0];\mathbb{R}^n) \to \mathbb{R}^n$, $h : [0,T] \times BC([-\theta,0];\mathbb{R}^n) \to \mathbb{R}^n$ and $w : [0,T] \times BC([-\theta,0];\mathbb{R}^n) \to \mathbb{R}^n$ are Borel measurable. Consider stochastic functional differential equation driven by G-Brownian motion of the type

$$dX(t) = g(t,X_t)dt + h(t,X_t)d\langle B,B\rangle(t) + w(t,X_t)dB(t), \tag{1.1}$$

where $X(t)$ is the value of stochastic process at time t and $X_t = \{X(t+\delta) : -\theta \leq \delta \leq 0, \theta > 0\}$ is a $BC([-\theta,0];\mathbb{R}^n)$-valued stochastic process, which presents the family of bounded continuous $\mathbb{R}^n$-valued functions φ defined on $[-\theta,0]$ having norm $\|\varphi\| = \sup_{-\theta\leq\delta\leq 0} \mid \varphi(\delta) \mid$. $\{\langle B,B\rangle(t), t \geq 0\}$ is the quadratic variation process of G-Brownian motion $\{B(t), t \geq 0\}$ and $g,h,w \in M_G^2([-\tau,T];\mathbb{R}^n)$. Denote the space of all $\mathcal{F}_t$-adapted process $X(t), 0 \leq t \leq T$, such that $\| X \|_{L^2} = \sup_{-\theta\leq t\leq T} |X(t)| < \infty$ by L^2. The initial data of equation (1.1) is given as follows

$$X_{t_0} = \zeta = \{\zeta(\delta) : -\theta < \delta \leq 0\} \; is \; \mathcal{F}_0-measurable, \; BC([-\theta,0];\mathbb{R}^n)- \; valued \\ random \; variable \; such \; that \; \zeta \in M_G^2([-\theta,0];\mathbb{R}^n). \tag{1.2}$$

The integral form of G-SFDE (1.1) with initial data (1.2) is given by

$$X(t) = \zeta(0) + \int_0^t g(s,X_s)ds + \int_0^t h(s,X_s)d\langle B,B\rangle(s) + \int_0^t w(s,X_s)dB(s).$$

The solution of G-SFDE (1.1) with initial data (1.2) is an $\mathbb{R}^n$ valued stochastic processes $X(t)$, $t \in [-\theta,T]$ such that

- **(i)** $X(t)$ is $\mathcal{F}_t$-adapted and continuous for all $t \in [0,T]$;
- **(ii)** $g(t,X_t) \in \mathcal{L}^1([o,T];\mathbb{R}^n)$ and $h(t,X_t), w(t,X_t) \in \mathcal{L}^2([0,T];\mathbb{R}^n)$;
- **(iii)** $X_0 = \zeta$ and for each $t \in [0,T]$, $dX(t) = g(t,X_t)dt + h(t,X_t)d\langle B,B\rangle(t) + w(t,X_t)dB(t)$ q.s.

$X(t)$ is called a unique solution if it is indistinguishable from any other solution $Y(t)$, that is,

$$E[\sup_{-\theta\leq q\leq t} |X(q) - Y(q)|^2] = 0.$$

Throughout this paper we assume the following two conditions, known as non-uniform Lipschitz condition and weakened linear growth condition respectively.

($\mathbf{A_i}$) For all $\varphi,\psi \in BC([-\theta,0];\mathbb{R}^d)$ and $t \in [0,T]$,

$$|g(t,\varphi) - g(t,\psi)|^2 + |h(t,\varphi) - |h(t,\psi)|^2 + |w(t,\varphi) - w(t,\psi)|^2 \leq \lambda(|\varphi - \psi|^2), \tag{1.3}$$

where $\lambda(.) : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-decreasing and concave function such that $\lambda(0) = 0$, $\lambda(v) > 0$ for $v > 0$ and

$$\int_{0+} \frac{dv}{\lambda(v)} = \infty. \tag{1.4}$$

As λ is concave and $\lambda(0) = 0$, there exists two positive constants c and d such that

$$\lambda(v) \leq c + dv, \tag{1.5}$$

for all $v \geq 0$.

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($\mathbf{A_{ii}}$) For all $t \in [0,T]$, $g(t,0), h(t,0), w(t,0) \in L^2$ and

$$|g(t,0)|^2 + |h(t,0)|^2 + |w(t,0)|^2 \leq K, \tag{1.6}$$

where K is a positive constant.

We have organized the rest of the paper as follows. In section 2, some well-known basic notions and results are included. In section 3, several important lemmas are developed. In section 4, the existence-and-uniqueness theorem is proved. In section 5, the mean square stability for the solution of G-SFDEs is given.

2 Preliminaries

The main purpose of this section is to give some basic concepts and results, which are used in the subsequent sections of this paper. For more detailed literature of G-expectation, we refer the readers to book [9] and papers [1, 2, 4, 5, 13].

Definition 2.1. Let $\mathcal{H}$ be a linear space of real valued functions defined on a nonempty basic space Ω. Then a sub-linear expectation E is a real valued functional on $\mathcal{H}$ with the following properties:

(i) For all $X, Y \in \mathcal{H}$, if $X \leq Y$ then $E[X] \leq E[Y]$.

(ii) For any real constant α, $E[\alpha] = \alpha$.

(iii) For all $X, Y \in \mathcal{H}$, $E[X+Y] \leq E[X] + E[Y]$.

(iv) For any $\theta > 0$ $E[\theta X] = \theta E[X]$.

Let $C_{b.Lip}(\mathbb{R}^{l\times d})$ denotes the set of bounded Lipschitz functions on $\mathbb{R}^{l\times d}$ and

$$L_G^p(\Omega_T) = \{\phi(B_{t_1}, B_{t_2}, ..., B_{t_l}/l \geq 1, t_1, t_2, ..., t_l \in [0,T], \phi \in C_{b.Lip}(\mathbb{R}^{l\times d}))\}.$$

Let $\xi_i \in L_G^p(\Omega_{t_i})$, $i = 0, 1, ..., N-1$ then $M_G^0(0,T)$ denotes the collection of processes of the following type: For a given partition $\pi_T = \{t_0, t_1, ..., t_N\}$ of $[0,T]$,

$$\eta_t(w) = \sum_{i=0}^{N-1} \xi_i(w) I_{[t_i, t_{i+1}]}(t).$$

Under the norm $\|\eta\| = \{\int_0^T E[|\eta_u|^p]du\}^{1/p}$, $M_G^p(0,T)$, $p \geq 1$, is the completion of $M_G^0(0,T)$. For every $\eta_t \in M_G^{2,0}(0,T)$, the G-Itô's integral $I(\eta)$ and G-quadratic variation process $\{\langle B\rangle_t\}_{t\geq 0}$ are respectively given by

$$I(\eta) = \int_0^T \eta_u dB_u = \sum_{i=0}^{N-1} \xi_i (B_{t_{i+1}} - B_{t_i}),$$

$$\langle B\rangle_t = B_t^2 - 2\int_0^t B_u dB_u.$$

The following definition and lemmas are borrowed from [7, 11].

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Definition 2.2. A solution $X(t)$ of dynamical system (1.1) with initial data (1.2) is said to be stable in mean square if for all $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that $E|\zeta - \xi|^2 \leq \delta(\epsilon)$ follows that $E|X(t) - Y(t)|^2 < \epsilon$ for all $t \geq 0$, where $Y(t)$ is an other solution of system (1.1) having initial data $\xi \in M^2([-\theta, 0] : \mathbb{R}^l)$.

Lemma 2.3. *(Hölder's inequality) If* $\frac{1}{q} + \frac{1}{r} = 1$ *for any* $q, r > 1$, $g \in L^2$ *and* $h \in L^2$ *then* $gh \in L^1$ *and*

$$\int_c^d gh \leq (\int_c^d |g|^q)^{\frac{1}{q}} (\int_c^d |h|^r)^{\frac{1}{r}}.$$

Lemma 2.4. *(Gronwall's inequality) Let* $C \geq 0$, $h(t) \geq 0$ *and* $w(t)$ *be a real valued continuous function on* $[c, d]$. *If for all* $c \leq t \leq d$, $w(t) \leq C + \int_c^d h(s)w(s)ds$, *then*

$$w(t) \leq Ce^{\int_c^t h(s)ds},$$

for all $c \leq t \leq d$.

Lemma 2.5. *(Bihari's inequality) Suppose* $T \geq 0$ *and* $h_0 \geq 0$. *Assume* $h(t)$ *and* $w(t)$ *be continuous functions on* $[0, T]$. *Let* $\lambda(.) : \mathbb{R}^+ \to \mathbb{R}^+$ *be non-decreasing and concave continuous function such that* $\lambda(v) > 0$ *for* $v > 0$. *If for all* $0 \leq t \leq T$, $h(t) \leq h(0) + \int_0^T w(s)\lambda(h(s))ds$, *then for all* $0 \leq t \leq T$,

$$h(t) \leq H^{-1}(H(h_0) + \int_t^T w(s)ds),$$

such that $H(h_0) + \int_t^T w(s)ds \in Dom(H^{-1})$ *where* $H(q) = \int_t^q \frac{1}{\lambda(s)}ds$, $q \geq 0$ *and* H^{-1} *is the inverse function of* H.

Lemma 2.6. *Assume the assumptions of lemma 2.5 are satisfied and for* $0 \leq t \leq T$, $w(t) \geq 0$. *If for all* $\epsilon > 0$, *there exists* $t_1 \geq 0$ *such that for* $0 \leq h_0 \leq \epsilon$, $\int_{t_1}^T w(s)ds \leq \int_{h_0}^T \frac{1}{\lambda(s)}ds$ *holds, then for each* $t_1 \leq t \leq T$

$$h(t) \leq \epsilon,$$

holds.

3 Important results

In this section, we show some important lemmas. They will be used in the forth coming existence-and-uniqueness theorem. Let $X^0(t) = \zeta(0)$ for $t \in [0, T]$. Set $X^l(0) = \zeta$ for each $l = 1, 2, ...,$ and define the following Picard iterations sequence,

$$\begin{aligned} X^l(t) = \zeta(0) + \int_0^t g(s, X_s^{l-1})ds + \int_0^t h(s, X_s^{l-1})d\langle B, B\rangle(s) \\ + \int_0^t w(s, X_s^{l-1})dB(s), \quad t \in [0, T]. \end{aligned} \tag{3.1}$$

First, we show that $X^l(.) \in M_G^2([-\theta, T]; \mathbb{R}^n)$.

Lemma 3.1. *Let assumptions A_i and A_{ii} hold. Then for all $l \geq 1$,*

$$\sup_{-\theta \leq t \leq T} E|X^l(t)|^2 \leq C,$$

where C is a positive constant.

Proof. Obviously, $X^0(.) \in M_G^2([-\theta, T]; \mathbb{R}^n)$. Using the basic inequality $|a+b+c+d|^2 \leq 4|a|^2 + 4|b|^2 + 4|c|^2 + 4|d|^2$, equation (3.1) yields

$$|X^l(t)|^2 \leq 4|\zeta(0)|^2 + 4|\int_0^t g(s, X_s^{l-1})ds|^2 + 4|\int_0^t h(s, X_s^{l-1})d\langle B, B\rangle(s)|^2$$
$$+ 4|\int_0^t w(s, X_s^{l-1})dB(s)|^2.$$

Taking G-expectation on both sides, using the Burkholder-Davis-Gundy (BDG) inequalities [6] and Hölder inequality (lemma 2.3) we have

$$\begin{aligned}
E|X^l(t)|^2 &\leq 4E|\zeta(0)|^2 + 4C_1 E\int_0^t |g(s, X_s^{l-1})|^2 ds \\
&+ 4C_2 E\int_0^t |h(s, X_s^{l-1})|^2 ds + 4C_3 \int_0^t |w(s, X_s^{l-1})|^2 ds \\
&\leq 4E|\zeta(0)|^2 + 8C_1 E\int_0^t (|g(s, X_s^{l-1}) - g(s,0)|^2 + |g(s,0)|^2) ds \\
&+ 8C_2 E\int_0^t (|h(s, X_s^{l-1}) - h(s,0)|^2 + |h(s,0)|^2) ds \\
&+ 8C_3 \int_0^t (|w(s, X_s^{l-1}) - w(s,0)|^2 + |w(s,0)|^2) d(s) \\
&\leq 4E|\zeta(0)|^2 + 8C_1 E\int_0^t |g(s,0)|^2 ds + 8C_1 E\int_0^t |g(s, X_s^{l-1}) - g(s,0)|^2 ds \\
&+ 8C_2 E\int_0^t |h(s,0)|^2 d(s) + 8C_2 E\int_0^t |h(s, X_s^{l-1}) - h(s,0)|^2 ds \\
&+ 8C_3 \int_0^t |w(s,0)|^2 ds + 8C_3 \int_0^t |w(s, X_s^{l-1}) - w(s,0)|^2 ds
\end{aligned}$$

By assumptions A_i and A_{ii}, the above inequality yields

$$\begin{aligned}
E|X^l(t)|^2 &\leq 4E|\zeta(0)|^2 + 8C_1 KT + 8C_2 KT + 8C_3 KT \\
&+ 8C_1 E\int_0^t \lambda(|X_s^{l-1})|^2) ds + 8C_2 E\int_0^t \lambda(|X_s^{l-1})|^2) d(s) + 8C_3 \int_0^t \lambda(|X_s^{l-1})|^2) d(s) \\
&= 4E|\zeta(0)|^2 + 8KT(C_1 + C_2 + C_3) + 8(C_1 + C_2 + C_3) E\int_0^t \lambda(|X_s^{l-1})|^2) ds \\
&\leq 4E|\zeta(0)|^2 + 8KT(C_1 + C_2 + C_3) + 8a(C_1 + C_2 + C_3)T \\
&+ 8b(C_1 + C_2 + C_3) E\int_0^t |X_s^{l-1})|^2 ds \\
&= K_1 + 8b(C_1 + C_2 + C_3) E\int_0^t |X_s^{l-1})|^2 ds,
\end{aligned}$$

where $K_1 = 4E|\zeta(0)|^2 + 8C_0KT + 8aC_0T$. and $C_0 = C_1 + C_2 + C_3$. Noting that

$$\sup_{0\le s\le t} |X^l_s|^2 \le \sup_{0\le s\le t}\sup_{-\theta\le u\le 0} |X^l(s+u)|^2 \le \sup_{-\theta\le q\le t} |X^l(q)|^2 \le |\zeta|^2 + \sup_{0\le q\le t} |X^l(q)|^2,$$

we have

$$\sup_{-\theta\le q\le t} E|X^l(q)|^2 \le E|\zeta|^2 + K_1 + 8b(C_1+C_2+C_3)E\int_0^t \sup_{-\theta\le q\le t} |X^{l-1}(q)|^2 ds.$$

Again noting that for any $j \ge 1$

$$\max_{1\le l\le j} E|X^{l-1}_s|^2 \le E|\zeta|^2 + \max_{1\le l\le j} E|X^l(q)|^2,$$

we obtain

$$\begin{aligned}\max_{1\le l\le j}\sup_{-\theta\le q\le t} E|X^l(q)|^2 &\le E|\zeta|^2 + K_1 + 8b(C_1+C_2+C_3)\int_0^t [E|\zeta|^2 + \max_{1\le l\le j}\sup_{-\theta\le q\le t} E|X^l(q)|^2]ds \\ &\le E|\zeta|^2 + K_1 + 8b(C_1+C_2+C_3)TE|\zeta|^2 + \int_0^t \max_{1\le l\le j}\sup_{-\theta\le q\le t} E|X^l(q)|^2 ds \\ &= K_2 + 8b(C_1+C_2+C_3)\int_0^t \max_{1\le l\le j}\sup_{-\theta\le q\le t} E|X^l(q)|^2 ds,\end{aligned}$$

where $K_2 = K_1 + (1+8bC_0T)E|\zeta|^2$. Now the Gronwall inequality (lemma 2.4) yields

$$\max_{1\le l\le j}\sup_{-\theta\le q\le t} E|X^l(t)|^2 \le C,$$

where $C = K_2e^{8bC_0T}$, but j is arbitrary, so

$$\sup_{-\theta\le t\le T} E|X^l(t)|^2 \le C.$$

The proof is complete. □

Lemma 3.2. *Under the assumptions A_i and A_{ii} there exists a positive constant C^* such that for all $l, d \ge 1$,*

$$\begin{aligned}E\sup_{-\theta\le s\le t} |X^{l+d}(s) - X^l(s)|^2 &\le \hat{C}\int_0^t \lambda(E\sup_{-\theta\le q\le s} |X^{l+d-1}(q) - X^{l-1}(q)|^2)ds \\ &\le C^*t.\end{aligned}$$

Proof. Using the basic inequality $|a+b+c|^2 \le 3|a|^2 + 3|b|^2 + 3|c|^2$, equation (3.1) yields

$$\begin{aligned}|X^{l+d}(t) - X^l(t)|^2 \le 3|\int_0^t [g(s, X^{l+d-1}_s) - g(s, X^{l-1}_s)]ds|^2 + 3|\int_0^t [h(s, X^{l+d-1}_s) - h(s, X^{l-1}_s)]d\langle B, B\rangle(s)|^2 \\ + 3|\int_{t_0}^t [w(s, X^{l+d-1}_s) - w(s, X^{l-1}_s)]dB(s)|^2\end{aligned}$$

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Taking G-expectation on both sides, using the BDG inequalities [6], Jensen inequality $E(\lambda(x)) \leq \lambda(E(x))$, Holder inequality and assumptions A_i, A_i it gives

$$\begin{aligned} E[\sup_{-\theta\leq s\leq t}|X^{l+d}(s)-X^l(s)|^2] &\leq 3C_1\int_0^t \lambda(E[\sup_{-\theta\leq q\leq s}|X^{l+d-1}(q)-X^{l-1}(q)|^2])ds \\ &+ 3C_2\int_0^t \lambda(E[\sup_{-\theta\leq q\leq s}|X^{l+d-1}(q)-X^{l-1}(q)|^2])ds \\ &+ 3C_3\int_0^t \lambda(E[\sup_{-\theta\leq q\leq s}|X^{l+d-1}(q)-X^{l-1}(q)|^2])ds \\ &\leq 3(C_1+C_2+C_3)\int_0^t \lambda(E[\sup_{-\theta\leq q\leq s}|X^{l+d-1}(q)-X^{l-1}(q)|^2])ds. \end{aligned}$$

$$E[\sup_{-\theta\leq s\leq t}|X^{l+d}(s)-X^l(s)|^2] \leq \hat{C}\int_0^t \lambda(E[\sup_{-\theta\leq q\leq s}|X^{l+d-1}(q)-X^{l-1}(q)|^2])ds,$$

where $\hat{C}=3C_0$. Finally, using lemma 3.1 it yields

$$E[\sup_{-\theta\leq s\leq t}|X^{l+d}(s)-X^l(s)|^2] \leq \hat{C}\lambda(4C)t = C^*t,$$

where $C^*=\hat{C}\lambda(4C)$. The proof is complete. □

4 Existence and uniqueness results for G-SFDEs

We introduce the following new notations to prepare a key lemma. Choose $T_1 \in [0,T]$ such that for all $t\in[0,T_1]$

$$\hat{C}\lambda(C^*t) \leq C^*. \tag{4.1}$$

For all $l,d\geq 1$, define the following recursive function

$$\phi_1(t) = C^*t. \tag{4.2}$$

$$\begin{aligned} \phi_{l+1}(t) &= \hat{C}\int_0^t \lambda(\phi_l(s))ds, \\ \phi_{l,d}(t) &= E[\sup_{-\theta\leq q\leq t}|X^{l+d}(q)-X^l(q)|^2]. \end{aligned} \tag{4.3}$$

Lemma 4.1. *Under the hypothesis A_i and A_{ii} for any $d\geq 1$ and all $l\geq 1$ there exists a positive $T_1\in[0,T]$ such that*

$$0 \leq \phi_{l,d}(t) \leq \phi_l(t) \leq \phi_{l-1}(t) \leq ... \leq \phi_1(t), \tag{4.4}$$

for all $t\in[0,T_1]$.

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Proof. We use mathematical induction to prove the inequality (4.4). Using the definition of function $\phi(.)$ and lemma 3.2, we have

$$\phi_{1,d}(t) = E[\sup_{-\theta \le q \le t} |X^{1+d}(q) - X^1(q)|^2] \le C^* t = \phi_1(t).$$

$$\begin{aligned}\phi_{2,d}(t) &= E[\sup_{-\theta \le q \le t} |X^{2+d}(q) - X^2(q)|^2] \\ &\le \hat{C} \int_0^t \lambda(E[\sup_{-\theta \le q \le t} |X^{1+d}(q) - X^1(q)|^2]) ds \\ &\le \hat{C} \int_0^t \lambda(\phi_1(s)) ds = \phi_2(t).\end{aligned}$$

Using (4.1), we have

$$\phi_2(t) = \hat{C} \int_0^t \lambda(\phi_1(s)) ds = \int_0^t \hat{C}\lambda(C^* t) ds \le C^* t = \phi_1(t).$$

Hence for all $t \in [0, T_1]$, we derive that $\phi_{2,d}(t) \le \phi_2(t) \le \phi_1(t)$. Next, suppose that the inequality (4.4) holds for some $l \ge 1$. We now show that lemma 4.1 is valid for $l + 1$, as follows

$$\begin{aligned}\phi_{l+1,d}(t) &= E[\sup_{-\theta \le q \le t} |X^{l+d+1}(q) - X^{l+1}(q)|^2] \\ &\le \hat{C} \int_0^t \lambda(E[\sup_{-\theta \le q \le s} |X^{l+d}(q) - X^l(q)|^2]) ds \\ &= \hat{C} \int_0^t \lambda(\phi_{l,d}(s)) ds \\ &\le \hat{C} \int_0^t \lambda(\phi_l(s)) ds \\ &= \phi_{l+1}(t).\end{aligned}$$

Also

$$\phi_{l+1}(t) = \hat{C} \int_0^t \lambda(\phi_l(s)) ds \le \hat{C} \int_0^t \lambda(\phi_{l-1}(s)) ds = \phi_l(s).$$

Hence for all $t \in [0, T_1]$, we derive that $\phi_{l+1,d}(t) \le \phi_{l+1}(t) \le \phi_l(s)$, that is, lemma 4.1 holds for $l + 1$. The proof is complete. □

Theorem 4.2. *Let assumptions A_i and A_{ii} hold. Then the stochastic system* (1.1) *with initial data* (1.2) *has a unique solution.*

Proof. We split the whole proof in two steps. First, we show uniqueness and then existence. Let system (1.1) with initial data (1.2) has two solutions $X(t)$ and $Y(t)$. Then we have

$$\begin{aligned}|X(t) - Y(t)| \le &\int_0^t |g(s, X_s) - g(s, Y_s)| ds + \int_0^t |h(s, X_s) - h(s, Y_s)| d\langle B, B\rangle(s) \\ &+ \int_0^t |w(s, X_s) - w(s, Y_s)| dB(s).\end{aligned}$$

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Taking G-expectation on both sides and using the basic inequality $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$, Hölder inequality and BDG inequalities [6], it follows

$$E|X(t)-Y(t)|^2 \leq 3C_1 \int_0^t E|g(s,X_s)-g(s,Y_s)|^2 ds + 3C_2 \int_0^t E|h(s,X_s)-h(s,Y_s)|^2 ds$$
$$+ 3C_3 \int_0^t E|w(s,X_s)-w(s,Y_s)|ds.$$

Using assumptions A_i and A_{ii} we have

$$E[\sup_{-\theta \leq q \leq t} |X(q)-Y(q)|^2] \leq 3(C_1+C_2+C_3) \int_0^t \lambda(E[\sup_{-\theta \leq q \leq s} |X(q)-Y(q)|^2])ds,$$

Then lemma 2.5 and lemma 2.6 gives $E[\sup_{-\theta \leq q \leq t} |X(q)-Y(q)|^2] = 0,\ t \in [0,T]$. The proof of uniqueness is complete.

Next we show existence. We note that on $t \in [0,T_1]$, $\phi_l(t)$ is continuous. For $l \geq 1$, it is decreasing on $t \in [0,T_1]$. By dominated convergence theorem, we define the function $\phi(t)$ as follows

$$\phi(t) = \lim_{l \to \infty} \phi_l(t) = \lim_{l \to \infty} \hat{C} \int_0^t \lambda(\phi_{l-1}(s))ds = \hat{C} \int_0^t \lambda(\phi(s))ds, \quad 0 \leq t \leq T_1.$$

So,

$$\phi(t) \leq \phi(0) + \hat{C} \int_0^t \lambda(\phi(s))ds.$$

Thus for all $0 \leq t \leq T_1$, lemma 2.5 and lemma 2.6 follow that $\phi(t) = 0$. From lemma 4.1 for all $t \in [0,T_1]$ we get $\phi_{l,d}(s) \leq \phi_l(s) \to 0$ as $l \to \infty$, which yields $E|X^{l+d}(t) - X^l(t)|^2 \to 0$ as $l \to \infty$. By the property of function $\lambda(.)$, assumptions A_i, A_{ii} and completeness of L^2, it follows that for all $t \in [0,T_1]$,

$$g(t,X_t^l) \to g(t,X_t), h(t,X_t^l) \to h(t,X_t), w(t,X_t^l) \to w(t,X_t) \ \ in\ L^2 \ \ as\ \ l \to \infty.$$

Hence for all $t \in [0,T_1]$,

$$\lim_{l \to \infty} X^l(t) = \zeta(0) + \lim_{l \to \infty} \int_0^t g(s,X_s^{l-1})ds$$
$$+ \lim_{l \to \infty} \int_0^t h(s,X_s^{l-1})d\langle B,B\rangle(s) + \lim_{l \to \infty} \int_0^t w(s,X_s^{l-1})dB(s),$$

that is,

$$X(t) = \zeta(0) + \int_0^t g(s,X_s)ds + \int_0^t h(s,X_s)d\langle B,B\rangle(s) + \int_0^t w(s,X_s)dB(s).$$

Thus $X(t)$ is a unique solution of stochastic system (1.1) with initial data (1.2) on $t \in [0,T_1]$. Thus by iteration, one can obtain that the system (1.1) has a unique solution on $t \in [0,T]$. The proof is complete. □

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5 Dependence of solutions

In this section, we use lemma 2.5 and lemma 2.6 to give continuous dependence of solutions for stochastic system (1.1) with initial data (1.2).

Theorem 5.1. *Let assumptions A_i and A_{ii} hold. Assume $X(t)$ and $Y(t)$ be two solutions of dynamical system (1.1) with initial data ζ and ξ respectively. If for all $\epsilon > 0$ and $t \in [0,T]$ there exists $\delta(\epsilon) > 0$ such that $E|\zeta - \xi|^2 < \delta(\epsilon)$, then*

$$E|X(t) - Y(t)|^2 \leq \epsilon.$$

Proof. Since $X(t)$ and $Y(t)$ are any two solutions of system (1.1). It follows that for any $t \in [0,T]$,

$$X(t) = \zeta(0) + \int_0^t g(s, X_s)ds + \int_0^t h(s, X_s)d\langle B, B\rangle(s) + \int_0^t w(s, X_s)dB(s) \;\; q.s.$$

$$Y(t) = \xi(0) + \int_0^t g(s, Y_s)ds + \int_0^t h(s, Y_s)d\langle B, B\rangle(s) + \int_0^t w(s, Y_s)dB(s) \;\; q.s.$$

Then

$$X(t) - Y(t) = \zeta(0) - \xi(0) + \int_0^t [g(s, X_s) - g(s, Y_s)]ds + \int_0^t [h(s, X_s) - h(s, Y_s)]d\langle B, B\rangle(s) + \int_0^t [w(s, X_s) - w(s, Y_s)]dB(s) \;\; q.s.$$

Taking G-expectation on both sides, using the fundamental inequality $(a + b + c + d)^2 \leq 4(a^2 + b^2 + c^2 + d^2)$, BDG inequalities [6] and Hölder inequality, it follows

$$E[\sup_{-\theta \leq r \leq t} |X(r) - Y(r)|^2] \leq 4E|\zeta(0) - \xi(0)|^2 + 4(C_1 + C_2 + C_3) \int_0^t \lambda(E[\sup_{-\theta \leq r \leq t} |X(r) - Y(r)|^2])ds.$$

Thus from lemma 2.5 and 2.6 we have

$$E[|X(t) - Y(t)|^2] \leq \epsilon,$$

for $t \in [0,T]$. The proof is complete. □

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Approximation of a kind of new Bernstein-Bézier type operators

Mei-Ying Ren[1*], Xiao-Ming Zeng[2*], Wen-Hui Zhang[2]

[1]School of Mathematics and Computer Science, Wuyi University, Wuyishan 354300, China

[2]School of Mathematical Sciences, Xiamen University, Xiamen 361005, Chnia

E-mail: npmeiyingr@163.com, xmzeng@xmu.edu.cn, 213859971@qq.com

Abstract. In this paper, a kind of new Bernstein-Bézier type operators is introduced. The Korovkin type approximation theorem of these operators is investigated. The rates of convergence of these operators are studied by means of modulus of continuity. Then, by using the Ditzian-Totik modulus of smoothness, a direct theorem concerned with an approximation for these operators is also obtained.

Keywords: Bernstein-Bézier type operators; Korovich type approximation theorem; rate of convergence; direct theorem; modulus of smoothness

Mathematical subject classification: 41A10, 41A25, 41A36

1. Introduction

In view of the Bézier basis function, which was introduced by Bézier [1], in 1983, Chang [2] defined the generalized Bernstein-Bézier polynomials for any $\alpha > 0$, and a function f defined on $[0, 1]$ as follows:

$$B_{n,\alpha}(f;x) = \sum_{k=0}^{n} f(\frac{k}{n})[J_{n,k}^{\alpha}(x) - J_{n,k+1}^{\alpha}(x)], \tag{1}$$

where $J_{n,n+1}(x) = 0$, and $J_{n,k}(x) = \sum\limits_{i=k}^{n} P_{n,i}(x)$, $k = 0, 1, ..., n$, $P_{n,i}(x) = \left(\begin{array}{c} n \\ i \end{array}\right) x^i(1-x)^{n-i}$. $J_{n,k}(x)$ is the Bézier basis function of degree n.

Obviously, when $\alpha = 1$, $B_{n,\alpha}(f;x)$ become the well-known Bernstein polynomials $B_n(f;x)$, and for any $x \in [0, 1]$, we have $1 = J_{n,0}(x) > J_{n,1}(x) > ... > J_{n,n}(x) = x^n$, $J_{n,k}(x) - J_{n,k+1}(x) = P_{n,k}(x)$.

During the last ten years, the Bézier basis function was extensively used for constructing various generalizations of many classical approximation processes. Some Bézier type operators, which are based on the Bézier basis function, have been introduced and studied (e.g., see [3-9]).

In 2013, Ren [10] introduced generalized Bernstein operators as follows:

$$E_{n,\beta}(f;x) = f(0)P_{n,0}(x) + \sum_{k=1}^{n-1} P_{n,k}(x)F_{n,k}^{(\beta)}(f) + f(1)P_{n,n}(x), \tag{2}$$

*Corresponding authors: Mei-Ying Ren and Xiao-Ming Zeng

where $f \in C[0,1]$, $x \in [0,1]$, $P_{n,k}(x) = \begin{pmatrix} n \\ k \end{pmatrix} x^k(1-x)^{n-k}$, $k = 0,1,...,n$, and

$$F_{n,k}^{(\beta)}(f) = \frac{1}{B(nk, n(n-k))} \int_0^1 t^{nk-1}(1-t)^{n(n-k)-1} f(\beta t + (1-\beta)\frac{k}{n})dt, \quad (3)$$

where $k = 1,...,n-1$, $\beta \in [0,1]$, $B(.,.)$ is the beta function.

The moments of the operators $E_{n,\beta}(f;x)$ were obtained as follows (see [10]).

Remark For $E_{n,\beta}(t^j;x)$, $j = 0,1,2$, we have

(i) $E_{n,\beta}(1;x) = 1$;
(ii) $E_{n,\beta}(t;x) = x$;
(iii) $E_{n,\beta}(t^2;x) = x^2 + \left[\frac{1}{n} + \frac{(n-1)\beta^2}{(n^2+1)n}\right] x(1-x)$.

In the present paper, we will study the Bézier variant of the generalized Bernstein operators $E_{n,\beta}(f;x)$ given by (2). We introduce Bernstein-Bézier type operators as follows:

$$E_{n,\beta}^{(\alpha)}(f;x) = f(0)Q_{n,0}^{(\alpha)}(x) + \sum_{k=1}^{n-1} Q_{n,k}^{(\alpha)}(x)F_{n,k}^{(\beta)}(f) + f(1)Q_{n,n}^{(\alpha)}(x), \quad (4)$$

where $f \in C[0,1]$, $x \in [0,1]$, $\beta \in [0,1]$, $\alpha > 0$, $Q_{n,k}^{(\alpha)}(x) = J_{n,k}^{\alpha}(x) - J_{n,k+1}^{\alpha}(x)$, $J_{n,n+1}(x) = 0$, $J_{n,k}(x) = \sum_{i=k}^{n} P_{n,i}(x)$, $k = 0,1,...,n$, $P_{n,i}(x) = \begin{pmatrix} n \\ i \end{pmatrix} x^i(1-x)^{n-i}$, and $F_{n,k}^{(\beta)}(f)$ is defined as above (3).

It is clear that $E_{n,\beta}^{(\alpha)}(f;x)$ are bounded and positive on C[0,1]. When $\alpha = 1$, $E_{n,\beta}^{(\alpha)}(f;x)$ become the operators $E_{n,\beta}(f;x)$. When $\beta = 0$, $E_{n,\beta}^{(\alpha)}(f;x)$ become the generalized Bernstein-Bézier operators $B_{n,\alpha}(f;x)$.

The goal of this paper is to study the approximation properties of these operators with the help of the Korovkin type approximation theorem. We also estimate the rates of convergence of these operators by using a modulus of continuity. Then we obtain the direct theorem concerned with an approximation for these operators by means of the Ditzian-Totik modulus of smoothness.

In the paper, for $f \in C[0,1]$, we denote $\|f\| = max\{|f(x)| : x \in [0,1]\}$. $\omega(f,\delta)$ $(\delta > 0)$ denotes the usual modulus of continuity of $f \in C[0,1]$.

2. Auxiliary results

In the sequel, we shall need the following auxiliary results.

Lemma 1 (see [2]) *Let $\alpha > 0$. We have*

(i) $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} J_{n,k}^{\alpha}(x) = x$ *uniformly on* $[0,1]$;

(ii) $\lim_{n\to\infty} \frac{1}{n^2} \sum_{k=1}^{n} kJ_{n,k}^{\alpha}(x) = \frac{x^2}{2}$ *uniformly on* $[0,1]$.

Lemma 2 *Let $\alpha > 0$. We have*

$$\text{(i) } E_{n,\beta}^{(\alpha)}(1;x) = 1;$$
$$\text{(ii) } \lim_{n\to\infty} E_{n,\beta}^{(\alpha)}(t;x) = x \textit{ uniformly on } [0,1];$$
$$\text{(iii) } \lim_{n\to\infty} E_{n,\beta}^{(\alpha)}(t^2;x) = x^2 \textit{ uniformly on } [0,1].$$

Proof By simple calculation, we obtain $F_{n,k}^{(\beta)}(1) = 1,\ F_{n,k}^{(\beta)}(t) = \frac{k}{n},\ F_{n,k}^{(\beta)}(t^2) = \frac{\beta^2}{n^2+1}\cdot\frac{k}{n} + (1-\frac{\beta^2}{n^2+1})\frac{k^2}{n^2}$.

(i) Since $\sum_{k=0}^{n} Q_{n,k}^{(\alpha)}(x) = 1$, by (4) we can get $E_{n,\beta}^{(\alpha)}(1;x) = 1$.

(ii) By (4), we have

$$\begin{aligned} &E_{n,\beta}^{(\alpha)}(t;x) \\ &= \sum_{k=1}^{n-1} Q_{n,k}^{(\alpha)}(x)\frac{k}{n} + Q_{n,n}^{(\alpha)}(x) \\ &= [J_{n,1}^{\alpha}(x) - J_{n,2}^{\alpha}(x)]\frac{1}{n} + \dots + [J_{n,n-1}^{\alpha}(x) - J_{n,n}^{\alpha}(x)]\frac{n-1}{n} + J_{n,n}^{\alpha}(x)\frac{n}{n} \\ &= \frac{1}{n}\sum_{k=1}^{n} J_{n,k}^{\alpha}(x), \end{aligned}$$

thus, by Lemma 1 (i), we have $\lim_{n\to\infty} E_{n,\beta}^{(\alpha)}(t;x) = x$ uniformly on $[0,1]$.

(iii) By (4), we have

$$\begin{aligned} &E_{n,\beta}^{(\alpha)}(t^2;x) \\ &= \sum_{k=1}^{n-1} Q_{n,k}^{(\alpha)}(x)\left[\frac{\beta^2}{n^2+1}\cdot\frac{k}{n} + (1-\frac{\beta^2}{n^2+1})\frac{k^2}{n^2}\right] + Q_{n,n}^{(\alpha)}(x) \\ &= \frac{\beta^2}{n^2+1}\cdot\frac{1}{n}\sum_{k=1}^{n} kQ_{n,k}^{(\alpha)}(x) + (1-\frac{\beta^2}{n^2+1})\cdot\frac{1}{n^2}\sum_{k=1}^{n} k^2 Q_{n,k}^{(\alpha)}(x) \\ &= \frac{\beta^2}{n^2+1}\cdot\frac{1}{n}\sum_{k=1}^{n} J_{n,k}^{\alpha}(x) + (1-\frac{\beta^2}{n^2+1})\cdot\frac{1}{n^2}\sum_{k=1}^{n}(2k-1)J_{n,k}^{\alpha}(x), \end{aligned}$$

thus, by Lemma 1, we have $\lim_{n\to\infty} E_{n,\beta}^{(\alpha)}(t^2;x) = x^2$ uniformly on $[0,1]$.

Lemma 3 (see [11]) *For $x \in [0,1]$, $k = 0,1,...,n$, we have*

$$0 \le Q_{n,k}^{(\alpha)}(x) \le \begin{cases} \alpha P_{n,k}(x), & \alpha \ge 1; \\ P_{n,k}^{\alpha}(x), & 0 < \alpha < 1. \end{cases}$$

Lemma 4 (see [12]) *For $0 < \alpha < 1$, $\gamma > 0$, we have*

$$\sum_{k=0}^{n} |k-nx|^{\gamma} P_{n,k}^{\alpha}(x) \le (n+1)^{1-\alpha}(A_{\frac{\gamma}{\alpha}})^{\alpha} n^{\frac{\gamma}{2}},$$

where the constant A_s only depends on s.

Lemma 5 *For $\alpha \geq 1$, we have*

$$\text{(i) } E_{n,\beta}^{(\alpha)}((t-x)^2;x) \leq \frac{\alpha}{4}(1+\frac{\beta^2}{5}) \cdot \frac{1}{n};$$
$$\text{(ii) } E_{n,\beta}^{(\alpha)}(|t-x|;x) \leq \sqrt{\frac{\alpha}{4}(1+\frac{\beta^2}{5})} \cdot \sqrt{\frac{1}{n}}.$$

Proof Let $\alpha \geq 1$.

(i) By (4), Lemma 3 and Remark 1, we obtain

$$\begin{aligned}
&E_{n,\beta}^{(\alpha)}((t-x)^2;x) \\
&= x^2 Q_{n,0}^{(\alpha)}(x) + \sum_{k=1}^{n-1} Q_{n,k}^{(\alpha)}(x) F_{n,k}^{(\beta)}((t-x)^2) + (1-x)^2 Q_{n,n}^{(\alpha)}(x) \\
&\leq \alpha[x^2 P_{n,0}(x) + \sum_{k=1}^{n-1} P_{n,k}(x) F_{n,k}^{(\beta)}((t-x)^2) + (1-x)^2 P_{n,n}(x)] \\
&= \alpha E_{n,\beta}((t-x)^2;x) \\
&= \frac{\alpha}{n}\left(1+\frac{n-1}{n^2+1}\beta^2\right) x(1-x). \qquad (5)
\end{aligned}$$

Since $\max\limits_{0\leq x\leq 1} x(1-x) = \frac{1}{4}$, and for any $n \in N$, one can get $\frac{n-1}{n^2+1} \leq \frac{1}{5}$, so we have

$$E_{n,\beta}^{(\alpha)}((t-x)^2;x) \leq \frac{\alpha}{4}(1+\frac{\beta^2}{5}) \cdot \frac{1}{n}.$$

(ii) In view of $E_{n,\beta}^{(\alpha)}(1;x) = 1$, by the Cauchy-Schwarz inequality, we have

$$E_{n,\beta}^{(\alpha)}(|t-x|;x) \leq \sqrt{E_{n,\beta}^{(\alpha)}(1;x)}\sqrt{E_{n,\beta}^{(\alpha)}((t-x)^2;x)},$$

thus, we get $E_{n,\beta}^{(\alpha)}(|t-x|;x) \leq \sqrt{\frac{\alpha}{4}(1+\frac{\beta^2}{5})} \cdot \sqrt{\frac{1}{n}}$.

Lemma 6 *For $0 < \alpha < 1$, we have*

$$\text{(i) } E_{n,\beta}^{(\alpha)}((t-x)^2;x) \leq M_\alpha^{(\beta)} n^{-\alpha};$$
$$\text{(ii) } E_{n,\beta}^{(\alpha)}(|t-x|;x) \leq \sqrt{M_\alpha^{(\beta)}} \cdot n^{-\frac{\alpha}{2}}.$$

Where the constant $M_\alpha^{(\beta)}$ only depends on α, β.

Proof Let $0 < \alpha < 1$.

(i) In view of (4), Lemma 3 and $F_{n,k}^{(\beta)}((t-x)^2) = \frac{(k-nx)^2}{n^2} + \frac{\beta^2}{n^2+1}(\frac{k}{n} - \frac{k^2}{n^2})$, we obtain

$$\begin{aligned}
&E_{n,\beta}^{(\alpha)}((t-x)^2;x)\\
&= x^2 Q_{n,0}^{(\alpha)}(x) + \sum_{k=1}^{n-1} Q_{n,k}^{(\alpha)}(x) F_{n,k}^{(\beta)}((t-x)^2) + (1-x)^2 Q_{n,n}^{(\alpha)}(x)\\
&\le x^2 P_{n,0}^{\alpha}(x) + \sum_{k=1}^{n-1} P_{n,k}^{\alpha}(x) F_{n,k}^{(\beta)}((t-x)^2) + (1-x)^2 P_{n,n}^{\alpha}(x)\\
&= \sum_{k=0}^{n} P_{n,k}^{\alpha}(x) \left[\frac{(k-nx)^2}{n^2} + \frac{\beta^2}{n^2+1}(\frac{k}{n} - \frac{k^2}{n^2})\right]\\
&= \frac{1}{n^2}\sum_{k=0}^{n}(k-nx)^2 P_{n,k}^{\alpha}(x) + \frac{\beta^2}{n^2+1}\sum_{k=0}^{n} P_{n,k}^{\alpha}(x)(\frac{k}{n} - \frac{k^2}{n^2})\\
&:= I_1 + I_2.
\end{aligned}$$

By Lemma 4, we have $I_1 \le \frac{n+1}{n}(n+1)^{-\alpha}(A_{\frac{2}{\alpha}})^{\alpha} \le 2(A_{\frac{2}{\alpha}})^{\alpha} n^{-\alpha}$, where the constant $A_{\frac{2}{\alpha}}$ only depends on α.

Using the Hölder inequality, we have $\sum\limits_{k=0}^{n} P_{n,k}^{\alpha}(x) \le (n+1)^{1-\alpha}[\sum\limits_{k=0}^{n} P_{n,k}(x)]^{\alpha}$, and $(\frac{k}{n} - \frac{k^2}{n^2}) \le 1$, so we have

$$I_2 \le \frac{\beta^2}{n^2+1}(n+1)^{1-\alpha}[\sum_{k=0}^{n} P_{n,k}(x)]^{\alpha} = \frac{\beta^2}{n^2+1}(n+1)^{1-\alpha} \le \beta^2 n^{-\alpha}.$$

Denote $M_{\alpha}^{(\beta)} = 2(A_{\frac{2}{\alpha}})^{\alpha} + \beta^2$, then we can get $E_{n,\beta}^{(\alpha)}((t-x)^2;x) \le M_{\alpha}^{(\beta)} n^{-\alpha}$.

(ii) Since

$$E_{n,\beta}^{(\alpha)}(|t-x|;x) \le \sqrt{E_{n,\beta}^{(\alpha)}(1;x)}\sqrt{E_{n,\beta}^{(\alpha)}((t-x)^2;x)},$$

thus, we get

$$E_{n,\beta}^{(\alpha)}(|t-x|;x) \le \sqrt{M_{\alpha}^{(\beta)}} \cdot n^{-\frac{\alpha}{2}}.$$

Lemma 7 *For* $f \in C[0,1],\ x \in [0,1]$ *and* $\alpha > 0,$ *we have*

$$| E_{n,\beta}^{(\alpha)}(f;x) | \le \| f \| .$$

Proof By (4) and Lemma 2 (i), we have

$$| E_{n,\beta}^{(\alpha)}(f;x) | \le \|f\| E_{n,\beta}^{(\alpha)}(1;x) = \|f\|.$$

3. Main results

First of all we give the following convergence theorem for the sequence $\{E_{n,\beta}^{(\alpha)}(f;x)\}$.

Theorem 1 *Let* $\alpha > 0$. *Then the sequence* $\{E_{n,\beta}^{(\alpha)}(f;x)\}$ *converges to* f *uniformly on* $[0,1]$ *for any* $f \in C[0,1]$.

Proof Since $E_{n,\beta}^{(\alpha)}(f;x)$ is bounded and positive on $C[0,1]$, and by Lemma 2, we have $\lim\limits_{n\to\infty}\|E_{n,\beta}^{(\alpha)}(e_j;\cdot)-e_j\|=0$ for $e_j(t)=t^j$, $j=0,1,2$. So, according to the well-known Bohman-korovkin theorem ([13, P.40, Theorem 1.9]), we see that the sequence $\{E_{n,\beta}^{(\alpha)}(f;x)\}$ converges to f uniformly on [0,1] for any $f\in C[0,1]$.

Next we estimate the rates of convergence of the sequence $\{E_{n,\beta}^{(\alpha)}\}$ by means of the modulus of continuity.

Theorem 2 *Let $f\in C[0,1]$, $x\in[0,1]$. Then*

(i) *when* $\alpha\geq 1,$ *we have* $\|E_{n,\beta}^{(\alpha)}(f;\cdot)-f\|\leq\left[1+\sqrt{\frac{\alpha}{4}(1+\frac{\beta^2}{5})}\right]\omega(f,\frac{1}{\sqrt{n}});$

(ii) *when* $0<\alpha<1,$ *we have* $\|E_{n,\beta}^{(\alpha)}(f;\cdot)-f\|\leq(1+\sqrt{M_\alpha^{(\beta)}})\omega(f,n^{-\frac{\alpha}{2}}).$

Where the constant $M_\alpha^{(\beta)}$ only depends on α,β.

Proof (i) When $\alpha\geq 1$, by Lemma 2 (i), we have

$$\begin{aligned}
&|E_{n,\beta}^{(\alpha)}(f;x)-f(x)|\\
&\leq|f(0)-f(x)|Q_{n,0}^{(\alpha)}(x)+\sum_{k=1}^{n-1}Q_{n,k}^{(\alpha)}(x)F_{n,k}^{(\beta)}(|f(t)-f(x)|)+|f(1)-f(x)|Q_{n,n}^{(\alpha)}(x)\\
&\leq\omega(f,|0-x|)Q_{n,0}^{(\alpha)}(x)+\sum_{k=1}^{n-1}Q_{n,k}^{(\alpha)}(x)F_{n,k}^{(\beta)}(\omega(f,|t-x|))+\omega(f,|1-x|)Q_{n,n}^{(\alpha)}(x)\\
&\leq(1+\sqrt{n}|0-x|)\omega(f,\frac{1}{\sqrt{n}})Q_{n,0}^{(\alpha)}(x)+\sum_{k=1}^{n-1}Q_{n,k}^{(\alpha)}(x)F_{n,k}^{(\beta)}((1+\sqrt{n}|t-x|)\omega(f,\frac{1}{\sqrt{n}}))\\
&\quad+(1+\sqrt{n}|1-x|)\omega(f,\frac{1}{\sqrt{n}})Q_{n,n}^{(\alpha)}(x)\\
&\leq\omega(f,\frac{1}{\sqrt{n}})+\sqrt{n}\omega(f,\frac{1}{\sqrt{n}})E_{n,\beta}^{(\alpha)}(|t-x|;x),
\end{aligned}$$

so, by Lemma 5 (ii), we obtain

$$|E_{n,\beta}^{(\alpha)}(f;x)-f(x)|\leq\left[1+\sqrt{\frac{\alpha}{4}(1+\frac{\beta^2}{5})}\right]\omega(f,\frac{1}{\sqrt{n}}).$$

The desired result follows immediately.

(ii) When $0<\alpha<1$, by Lemma 2 (i), we have

$$\begin{aligned}
&|E_{n,\beta}^{(\alpha)}(f;x)-f(x)|\\
&\leq\omega(f,|0-x|)Q_{n,0}^{(\alpha)}(x)+\sum_{k=1}^{n-1}Q_{n,k}^{(\alpha)}(x)F_{n,k}^{(\beta)}(\omega(f,|t-x|))+\omega(f,|1-x|)Q_{n,n}^{(\alpha)}(x)\\
&\leq(1+n^{\frac{\alpha}{2}}|0-x|)\omega(f,n^{-\frac{\alpha}{2}})Q_{n,0}^{(\alpha)}(x)+\sum_{k=1}^{n-1}Q_{n,k}^{(\alpha)}(x)F_{n,k}^{(\beta)}(1+n^{\frac{\alpha}{2}}|t-x|)\omega(f,n^{-\frac{\alpha}{2}})\\
&\quad+(1+n^{\frac{\alpha}{2}}|1-x|)\omega(f,n^{-\frac{\alpha}{2}})Q_{n,n}^{(\alpha)}(x)\\
&=\omega(f,n^{-\frac{\alpha}{2}})+n^{\frac{\alpha}{2}}\omega(f,n^{-\frac{\alpha}{2}})E_{n,\beta}^{(\alpha)}(|t-x|;x),
\end{aligned}$$

so, by Lemma 6 (ii), we obtain

$$|E_{n,\beta}^{(\alpha)}(f;x)-f(x)|\le(1+\sqrt{M_\alpha^{(\beta)}})\omega(f,n^{-\frac{\alpha}{2}}).$$

The desired result follows immediately.

Theorem 3 *Let $f\in C^1[0,1]$, $x\in[0,1]$. Then*
(i) *when* $\alpha\ge 1$, *we have*

$$\begin{aligned}|E_{n,\beta}^{(\alpha)}(f;x)-f(x)| &\le \|f'\|\sqrt{\frac{\alpha}{4}(1+\frac{\beta^2}{5})}\cdot\sqrt{\frac{1}{n}}\\ &+ \omega(f',\frac{1}{\sqrt{n}})\left[1+\sqrt{\frac{\alpha}{4}(1+\frac{\beta^2}{5})}\right]\cdot\sqrt{\frac{\alpha}{4}(1+\frac{\beta^2}{5})}\cdot\sqrt{\frac{1}{n}};\end{aligned}$$

(ii) *when* $0<\alpha<1$, *we have*

$$|E_{n,\beta}^{(\alpha)}(f;x)-f(x)|\le\|f'\|\sqrt{M_\alpha^{(\beta)}n^{-\alpha}}+\omega(f',n^{-\frac{\alpha}{2}})(1+\sqrt{M_\alpha^{(\beta)}})\sqrt{M_\alpha^{(\beta)}n^{-\alpha}}.$$

Where the constant $M_\alpha^{(\beta)}$ only depends on α,β.

Proof Let $f\in C^1[0,1]$. For any $t,x\in[0,1]$, $\delta>0$, we have

$$\begin{aligned}|f(t)-f(x)-f'(x)(t-x)| &\le |\int_x^t|f'(u)-f'(x)|du|\\ &\le \omega(f',|t-x|)|t-x|\\ &\le \omega(f',\delta)(|t-x|+\delta^{-1}(t-x)^2),\end{aligned}$$

hence, by the Cauchy-Schwarz inequality, we have

$$\begin{aligned}&|E_{n,\beta}^{(\alpha)}(f(t)-f(x)-f'(x)(t-x);x)|\\ &\le\omega(f',\delta)\left(E_{n,\beta}^{(\alpha)}(|t-x|;x)+\delta^{-1}E_{n,\beta}^{(\alpha)}((t-x)^2;x)\right)\\ &\le\omega(f',\delta)\left[\sqrt{E_{n,\beta}^{(\alpha)}(1;x)}\right.\\ &\left.+\delta^{-1}\sqrt{E_{n,\beta}^{(\alpha)}((t-x)^2;x)}\right]\sqrt{E_{n,\beta}^{(\alpha)}((t-x)^2;x)}.\end{aligned}$$

So, we get

$$\begin{aligned}&|E_{n,\beta}^{(\alpha)}(f;x)-f(x)|\\ &\le\|f'\|E_{n,\beta}^{(\alpha)}(|t-x|;x)\\ &+\omega(f',\delta)\left[1+\delta^{-1}\sqrt{E_{n,\beta}^{(\alpha)}((t-x)^2;x)}\right]\sqrt{E_{n,\beta}^{(\alpha)}((t-x)^2;x)}.\end{aligned}\tag{6}$$

(i) When $\alpha\ge1$, taking $\delta=\frac{1}{\sqrt{n}}$ in (6), by Lemma 5 and inequality (6), we obtain the desired result.
(ii)When $0<\alpha<1$, taking $\delta=n^{-\frac{\alpha}{2}}$ in (6), by Lemma 6 and inequality (6), we obtain the desired result.

Finally we study the direct theorem concerned with an approximation for the sequence $\{E_{n,\beta}^{(\alpha)}\}$ by means of the Ditzian-Totik modulus of smoothness. For the following theorem we shall use some notations.

For $f \in C[0,1]$, $\varphi(x) = \sqrt{x(1-x)}$, $0 \le \lambda \le 1$, $x \in [0,1]$, let

$$\omega_{\varphi^\lambda}(f,t) = \sup_{0<h\le t} \sup_{x\pm\frac{h\varphi^\lambda(x)}{2}\in[0,1]} |f(x+\frac{h\varphi^\lambda(x)}{2}) - f(x-\frac{h\varphi^\lambda(x)}{2})|$$

be the Ditzian-Totik modulus of first order, and let

$$K_{\varphi^\lambda}(f,t) = \inf_{g\in W_\lambda}\{\| f-g \| + t\|\varphi^\lambda g'\|\} \tag{7}$$

be the corresponding K-functional, where $W_\lambda = \{f | f \in AC_{loc}[0,1], \|\varphi^\lambda f'\| < \infty, \|f'\| < \infty\}$.

It is well known that (see [14])

$$K_{\varphi^\lambda}(f,t) \le C\omega_{\varphi^\lambda}(f,t), \tag{8}$$

for some absolute constant $C > 0$.

Now we state our following main result.

Theorem 4 *Let $f \in C[0,1]$, $\alpha \ge 1$, $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$, $0 \le \beta, \lambda \le 1$. Then there exists an absolute constant $C > 0$ such that*

$$|E_{n,\beta}^{(\alpha)}(f;x) - f(x)| \le C\omega_{\varphi^\lambda}(f, \frac{\varphi^{1-\lambda}(x)}{\sqrt{n}}).$$

Proof Let $g \in W_\lambda$, by Lemma 2 (i) and Lemma 7, we have

$$\begin{aligned} &|E_{n,\beta}^{(\alpha)}(f;x) - f(x)| \\ &\le |E_{n,\beta}^{(\alpha)}(f-g;x)| + |f(x)-g(x)| + |E_{n,\beta}^{(\alpha)}(g;x) - g(x)| \\ &\le 2\|f-g\| + |E_{n,\beta}^{(\alpha)}(g;x) - g(x)|. \end{aligned} \tag{9}$$

Since $g(t) = \int_x^t g'(u)du + g(x)$, $E_{n,\beta}^{(\alpha)}(1;x) = 1$, so, we have

$$\begin{aligned} |E_{n,\beta}^{(\alpha)}(g;x) - g(x)| &\le |E_{n,\beta}^{(\alpha)}(\int_x^t |g'(u)|du;x)| \\ &\le \|\varphi^\lambda g'\| E_{n,\beta}^{(\alpha)}(|\int_x^t \varphi^{-\lambda}(u)du|;x). \end{aligned} \tag{10}$$

By the Hölder inequality, we get

$$|\int_x^t \varphi^{-\lambda}(u)du| \le |\int_x^t \frac{1}{\sqrt{u(1-u)}}du|^\lambda |t-x|^{1-\lambda}, \tag{11}$$

also, in view of $1 \le \sqrt{u} + \sqrt{1-u} < 2$, $0 \le u \le 1$, we have

$$\begin{aligned} |\int_x^t \frac{1}{\sqrt{u(1-u)}}du| &\le |\int_x^t (\frac{1}{\sqrt{u}} + \frac{1}{\sqrt{1-u}})du| \\ &\le 2(|\sqrt{t}-\sqrt{x}| + |\sqrt{1-x}-\sqrt{1-t}|) \\ &\le 2(\frac{|t-x|}{\sqrt{t}+\sqrt{x}} + \frac{|t-x|}{\sqrt{1-t}+\sqrt{1-x}}) \\ &\le 2|t-x|(\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{1-x}}) \\ &\le 4|t-x|\varphi^{-1}(x), \end{aligned} \tag{12}$$

thus, by (11) and (12), we obtain

$$|\int_x^t \varphi^{-\lambda}(u)du| \le C\varphi^{-\lambda}(x)|t-x|. \tag{13}$$

Also, by (10) and (13), we have

$$\begin{aligned} |E_{n,\beta}^{(\alpha)}(g;x) - g(x)| &\le C\|\varphi^\lambda g'\| E_{n,\beta}^{(\alpha)}(\varphi^{-\lambda}(x)|t-x|;x) \\ &= C\|\varphi^\lambda g'\|\varphi^{-\lambda}(x) E_{n,\beta}^{(\alpha)}(|t-x|;x). \end{aligned} \tag{14}$$

In view of (5) and Lemma 2 (i), by the Cauchy-Schwarz inequality, we have

$$\begin{aligned} E_{n,\beta}^{(\alpha)}(|t-x|;x) &\le \sqrt{E_{n,\beta}^{(\alpha)}(1;x)}\sqrt{E_{n,\beta}^{(\alpha)}((t-x)^2;x)} \\ &\le \sqrt{\frac{\alpha}{n}\left(1+\frac{n-1}{n^2+1}\beta^2\right)x(1-x)} \\ &\le C\frac{\varphi(x)}{\sqrt{n}}, \end{aligned} \tag{15}$$

so, by (14) and (15), we obtain

$$|E_{n,\beta}^{(\alpha)}(g;x) - g(x)| \le C\|\varphi^\lambda g'\|\frac{\varphi^{1-\lambda}(x)}{\sqrt{n}}, \tag{16}$$

thus, by (9) and (16), we have

$$|E_{n,\beta}^{(\alpha)}(f;x) - f(x)| \le 2\|f-g\| + C\|\varphi^\lambda g'\|\frac{\varphi^{1-\lambda}(x)}{\sqrt{n}}.$$

Then, in view of (17), (7) and (8), we obtain

$$|E_{n,\beta}^{(\alpha)}(f;x) - f(x)| \le CK_{\varphi^\lambda}(f, \frac{\varphi^{1-\lambda}(x)}{\sqrt{n}}) \le C\omega_{\varphi^\lambda}(f, \frac{\varphi^{1-\lambda}(x)}{\sqrt{n}}),$$

where C is a positive constant, in different places, the value of C may be different.

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Approximation by complex Stancu type summation-integral operators in compact disks

Mei-Ying Ren[1*], Xiao-Ming Zeng[2*], Wen-Hui Zhang[2]
[1]School of Mathematics and Computer Science, Wuyi University,
Wuyishan 354300, China
[2]School of Mathematical Sciences, Xiamen University, Xiamen 361005, Chnia
E-mail: npmeiyingr@163.com, xmzeng@xmu.edu.cn, 213859971@qq.com

Abstract. In this paper we introduce a class of complex Stancu type summation-integral operators and study the approximation properties of these operators. We obtain a Voronovskaja-type result with quantitative estimate for these operators attached to analytic functions on compact disks. We also study the exact order of approximation. More important, our results show the overconvergence phenomenon for these complex operators.

Keywords: complex Stancu type summation-integral operators; Voronovskaja-type result; Exact order of approximation; Simultaneous approximation; Overconvergence

Mathematical subject classification: 30E10, 41A25 , 41A36

1. Introduction

In 1986, some approximation properties of complex Bernstein polynomials in compact disks were initially studied by Lorentz [11]. Very recently, the problem of the approximation of complex operators has been causing great concern, which is becoming a hot topic of research. A Voronovskaja-type result with quantitative estimate for complex Bernstein polynomials in compact disks was obtained by Gal [3]. Also, in [1-2, 4-10, 12-15] similar results for complex Bernstein-Kantorovich polynomials, Bernstein-Stancu polynomials, Kantorovich-Schurer polynomials, Kantorovich-Stancu polynomials, complex Favard-Szász-Mirakjan operators, complex Beta operators of first kind, complex Baskajov-Stancu operators, complex Bernstein-Durrmeyer operators based on Jacobi weights, complex genuine Durrmeyer Stancu polynomials, complex Schurer-Stancu operators, complex q-Szász-Mirakjan operators, complex q-Gamma operators, and complex q-Durrmeyer type operators were obtained.

The aim of the present article is to obtain approximation results for complex Stancu type summation-integral operators which are defined for $f:[0,1]\to \mathbf{C}$ continuous on $[0,1]$ by

$$M_n^{(\alpha,\beta)}(f;z):=p_{n,0}(z)f(\frac{\alpha}{n+\beta})+\sum_{k=1}^{n-1}p_{n,k}(z)L_{n,k}^{(\alpha,\beta)}(f)+p_{n,n}(z)f(\frac{n+\alpha}{n+\beta}),\quad (1)$$

*Corresponding authors: Mei-Ying Ren and Xiao-Ming Zeng where α, β are two given real parameters satisfying the condition $0 \le \alpha \le \beta$, $z \in \mathbf{C}, n \in \mathbf{N}$, $L_{n,k}^{(\alpha,\beta)}(f) = \frac{1}{B(n(n-k),nk)} \int_0^1 t^{nk-1}(1-t)^{n(n-k)-1} f(\frac{nt+\alpha}{n+\beta})dt$ with $B(x,y)$ is Beta function, and $p_{n,k}(z) = \left(\begin{array}{c} n \\ k \end{array}\right) z^k(1-z)^{n-k}$.

Note that, for $\alpha = \beta = 0$, these operators become the complex summation-integral type operators $M_n(f;z) = M_n^{(0,0)}(f;z)$, this case has been investigated in [16].

2. Auxiliary results

In the sequel, we shall need the following auxiliary results.

Lemma 1 *Let $e_m(z) = z^m$, $m \in \mathbf{N} \cup \{0\}$, $z \in \mathbf{C}$, $n \in \mathbf{N}$, $0 \le \alpha \le \beta$, we have $M_n^{(\alpha,\beta)}(e_m;z)$ is a polynomial of degree less than or equal to min (m,n) and*

$$M_n^{(\alpha,\beta)}(e_m;z) = \sum_{j=0}^{m} \left(\begin{array}{c} m \\ j \end{array}\right) \frac{n^j \alpha^{m-j}}{(n+\beta)^m} M_n(e_j;z).$$

Proof By the definition given by (1) , the proof is easy, here the proof is omitted.

Let $m = 0, 1, 2$, according to [16, Lemma 1] , by simple computation, we have

$$\begin{aligned} M_n^{(\alpha,\beta)}(e_0;z) &= 1; \\ M_n^{(\alpha,\beta)}(e_1;z) &= \frac{nz}{n+\beta} + \frac{\alpha}{n+\beta}; \\ M_n^{(\alpha,\beta)}(e_2;z) &= \frac{n^2}{(n+\beta)^2}\left[\frac{n(n-1)}{n^2+1}z^2 + \frac{n+1}{n^2+1}z\right] \\ &\quad + \frac{2n\alpha z}{(n+\beta)^2} + \frac{\alpha^2}{(n+\beta)^2}. \end{aligned}$$

Lemma 2 *Let $e_m(z) = z^m$, $m \in \mathbf{N} \cup \{0\}$, $z \in \mathbf{C}$, $n \in \mathbf{N}$, $0 \le \alpha \le \beta$, for all $|z| \le r$, $r \ge 1$, we have $|M_n^{(\alpha,\beta)}(e_m;z)| \le r^m$.*

Proof The proof follows directly Lemma 1 and [16, Corollary 1].

Lemma 3 *Let $e_m(z) = z^m$, $m, n \in$ N, $z \in \mathbf{C}$ and $0 \le \alpha \le \beta$, we have*

$$\begin{aligned} M_n^{(\alpha,\beta)}(e_{m+1};z) &= \frac{z(1-z)n^2}{(n+\beta)(n^2+m)}(M_n^{(\alpha,\beta)}(e_m;z))' \\ &\quad + \frac{(m+n^2z)n + \alpha(n^2+2m)}{(n+\beta)(n^2+m)} M_n^{(\alpha,\beta)}(e_m;z) \\ &\quad - \frac{\alpha m(n+\alpha)}{(n+\beta)^2(n^2+m)} M_n^{(\alpha,\beta)}(e_{m-1};z). \end{aligned} \tag{2}$$

Proof Let

$$\widetilde{L}_{n,k}^{(\alpha,\beta)}(f) := \frac{1}{B(n(n-k),nk)} \int_0^1 t^{nk-1}(1-t)^{n(n-k)-1} t f(\frac{nt+\alpha}{n+\beta})dt,$$

$$\widehat{L}_{n,k}^{(\alpha,\beta)}(f) := \frac{1}{B(n(n-k), nk)} \int_0^1 t^{nk-1}(1-t)^{n(n-k)-1} t^2 f(\frac{nt+\alpha}{n+\beta})dt,$$

$$E_n^{(\alpha,\beta)}(f;z) := \sum_{k=1}^{n-1} p_{n,k}(z) L_{n,k}^{(\alpha,\beta)}(f),$$

we have

$$M_n^{(\alpha,\beta)}(f;z) = p_{n,0}(z) f(\frac{\alpha}{n+\beta}) + E_n^{(\alpha,\beta)}(f;z) + p_{n,n}(z) f(\frac{n+\alpha}{n+\beta}),$$

$$\widetilde{L}_{n,k}^{(\alpha,\beta)}(e_m) = \frac{n+\beta}{n} L_{n,k}^{(\alpha,\beta)}(e_{m+1}) - \frac{\alpha}{n} L_{n,k}^{(\alpha,\beta)}(e_m),$$

$$\widehat{L}_{n,k}^{(\alpha,\beta)}(e_m) = (\frac{n+\beta}{n})^2 L_{n,k}^{(\alpha,\beta)}(e_{m+2}) - \frac{2\alpha(n+\beta)}{n^2} L_{n,k}^{(\alpha,\beta)}(e_{m+1}) + (\frac{\alpha}{n})^2 L_{n,k}^{(\alpha,\beta)}(e_m).$$

By simple calculation, we obtain

$$z(1-z)p'_{n,k}(z) = (k-nz)p_{n,k}(z),$$

$$t(1-t)[t^{nk-1}(1-t)^{n(n-k)-1}]' = [nk-1-(n^2-2)t] t^{nk-1}(1-t)^{n(n-k)-1},$$

it follows that

$$\begin{aligned}
&z(1-z)(E_n^{(\alpha,\beta)}(e_m;z))' \\
&= \sum_{k=1}^{n-1} (k-nz)\, p_{n,k}(z) L_{n,k}^{(\alpha,\beta)}(e_m) \\
&= \sum_{k=1}^{n-1} k p_{n,k}(z) \frac{1}{B(n(n-k), nk)} \int_0^1 t^{nk-1}(1-t)^{n(n-k)-1} (\frac{nt+\alpha}{n+\beta})^m dt - nz E_n^{(\alpha,\beta)}(e_m;z) \\
&= \frac{1}{n} \sum_{k=1}^{n-1} p_{n,k}(z) \frac{1}{B(n(n-k), nk)} \int_0^1 [nk-1-(n^2-2)t] t^{nk-1}(1-t)^{n(n-k)-1} (\frac{nt+\alpha}{n+\beta})^m dt \\
&\quad + \frac{1}{n} E_n^{(\alpha,\beta)}(e_m;z) + \frac{n^2-2}{n} \sum_{k=1}^{n-1} p_{n,k}(z) \widetilde{L}_{n,k}^{(\alpha,\beta)}(e_m) - nz E_n^{(\alpha,\beta)}(e_m;z),
\end{aligned}$$

where

$$\begin{aligned}
&\frac{1}{n} \sum_{k=1}^{n-1} p_{n,k}(z) \frac{1}{B(n(n-k), nk)} \int_0^1 [nk-1-(n^2-2)t] t^{nk-1}(1-t)^{n(n-k)-1} (\frac{nt+\alpha}{n+\beta})^m dt \\
&= \frac{1}{n} \sum_{k=1}^{n-1} p_{n,k}(z) \frac{1}{B(n(n-k), nk)} \int_0^1 (t-t^2)[t^{nk-1}(1-t)^{n(n-k)-1}]' (\frac{nt+\alpha}{n+\beta})^m dt \\
&= -\frac{1}{n} E_n^{(\alpha,\beta)}(e_m;z) + \frac{2}{n} \sum_{k=1}^{n-1} p_{n,k}(z) \widetilde{L}_{n,k}^{(\alpha,\beta)}(e_m) - \frac{m}{n+\beta} \sum_{k=1}^{n-1} p_{n,k}(z) \widetilde{L}_{n,k}^{(\alpha,\beta)}(e_{m-1}) \\
&\quad + \frac{m}{n+\beta} \sum_{k=1}^{n-1} p_{n,k}(z) \widehat{L}_{n,k}^{(\alpha,\beta)}(e_{m-1}).
\end{aligned}$$

So, in conclusion, we have

$$\begin{aligned}
z(1-z)(E_n^{(\alpha,\beta)}(e_m;z))' = &\frac{(n+\beta)(n^2+m)}{n^2} E_n^{(\alpha,\beta)}(e_{m+1};z) \\
&- (\frac{\alpha n^2 + mn + 2\alpha m}{n^2} + nz) E_n^{(\alpha,\beta)}(e_m;z) \\
&+ \frac{\alpha mn + \alpha^2 m}{n^2(n+\beta)} E_n^{(\alpha,\beta)}(e_{m-1};z),
\end{aligned}$$

which implies the recurrence in the statement.

Lemma 4 *Let $n \in \mathbf{N}$, $m = 2, 3, \cdots$, $e_m(z) = z^m$, $S_{n,m}^{(\alpha,\beta)}(z) := M_n^{(\alpha,\beta)}(e_m; z) - z^m$, $z \in \mathbf{C}$ and $0 \leq \alpha \leq \beta$, we have*

$$\begin{aligned} S_{n,m}^{(\alpha,\beta)}(z) = & \frac{z(1-z)n^2}{(n+\beta)(n^2+m-1)}(M_n^{(\alpha,\beta)}(e_{m-1}; z))' \\ & + \frac{(m-1+n^2z)n + \alpha(n^2+m-1)}{(n+\beta)(n^2+m-1)} S_{n,m-1}^{(\alpha,\beta)}(z) \\ & + \frac{\alpha(m-1)}{(n+\beta)(n^2+m-1)} M_n^{(\alpha,\beta)}(e_{m-1}; z) \\ & - \frac{\alpha(m-1)(n+\alpha)}{(n+\beta)^2(n^2+m-1)} M_n^{(\alpha,\beta)}(e_{m-2}; z) \\ & + \frac{(m-1+n^2z)n + \alpha(n^2+m-1)}{(n+\beta)(n^2+m-1)} z^{m-1} - z^m. \end{aligned} \quad (3)$$

Proof Using the recurrence formula (2), by simple calculation, we can easily get the recurrence (3), the proof is omitted.

3. Main results

The first main result is expressed by the following upper estimates.

Theorem 1 *Let $0 \leq \alpha \leq \beta$, $1 \leq r \leq R$, $D_R = \{z \in \mathbf{C} : |z| < R\}$. Suppose that $f : D_R \to \mathbf{C}$ is analytic in D_R, i.e. $f(z) = \sum_{m=0}^{\infty} c_m z^m$ for all $z \in D_R$.*

(i) for *all $|z| \leq r$ and $n \in \mathbf{N}$, we have*

$$|M_n^{(\alpha,\beta)}(f; z) - f(z)| \leq \frac{K_r^{(\alpha,\beta)}(f)}{n+\beta},$$

where $K_r^{(\alpha,\beta)}(f) = (1+r) \sum_{m=1}^{\infty} |c_m| m(m+1+\alpha+\beta) r^{m-1} < +\infty$.

(ii) (Simultaneous approximation) *If $1 \leq r < r_1 < R$ are arbitrary fired, then for all $|z| \leq r$ and $n, p \in \mathbf{N}$ we have*

$$|(M_n^{(\alpha,\beta)}(f; z))^{(p)} - f^{(p)}(z)| \leq \frac{K_{r_1}^{(\alpha,\beta)}(f) p! r_1}{(n+\beta)(r_1-r)^{p+1}},$$

where $K_{r_1}^{(\alpha,\beta)}(f)$ is defined as at the above point (i).

Proof Taking $e_m(z) = z^m$, by hypothesis that $f(z)$ is analytic in D_R, i.e. $f(z) = \sum_{m=0}^{\infty} c_m z^m$ for all $z \in D_R$, it is easy for us to obtain

$$M_n^{(\alpha,\beta)}(f; z) = \sum_{m=0}^{\infty} c_m M_n^{(\alpha,\beta)}(e_m; z),$$

therefore, we get

$$
\begin{aligned}
|M_n^{(\alpha,\beta)}(f;z)-f(z)| &\le \sum_{m=0}^{\infty}|c_m|\cdot|M_n^{(\alpha,\beta)}(e_m;z)-e_m(z)| \\
&= \sum_{m=1}^{\infty}|c_m|\cdot|M_n^{(\alpha,\beta)}(e_m;z)-e_m(z)|,
\end{aligned}
$$

as $M_n^{(\alpha,\beta)}(e_0;z)=e_0(z)=1$.

(i) For $m\in\mathbf{N}$, taking into account that $M_n^{(\alpha,\beta)}(e_{m-1};z)$ is a polynomial degree $\le min(m-1,n)$, by the well-known Bernstein inequality and Lemma 2 we get

$$
|(M_n^{(\alpha,\beta)}(e_{m-1};z))'| \le \frac{m-1}{r}max\{|M_n^{(\alpha,\beta)}(e_{m-1};z)|:|z|\le r\}\le (m-1)r^{m-2}.
$$

On the one hand, when $m=1$, for $|z|\le r$, by Lemma 1, we have

$$
|M_n^{(\alpha,\beta)}(e_1;z)-e_1(z)| = |\frac{nz}{n+\beta}+\frac{\alpha}{n+\beta}-z| \le \frac{1+r}{n+\beta}(2+\alpha+\beta).
$$

When $m\ge 2$, for $n\in\mathbf{N}$, $|z|\le r$, $0\le\alpha\le\beta$, in view of $|(m-1+n^2z)n+\alpha(n^2+m-1)|\le(n+\beta)(n^2+m-1)r$, using the recurrence formula (3) and the above inequality, we have

$$
\begin{aligned}
|M_n^{(\alpha,\beta)}(e_m;z)-e_m(z)| &= |S_{n,m}^{(\alpha,\beta)}(z)| \\
&\le \frac{r(1+r)}{n+\beta}\cdot(m-1)r^{m-2}+r|S_{n,m-1}^{(\alpha,\beta)}(z)| \\
&\quad +\frac{\alpha}{n+\beta}r^{m-1}+\frac{\alpha}{n+\beta}r^{m-2}+\frac{m+1+\beta}{n+\beta}(1+r)r^{m-1} \\
&\le \frac{m-1}{n+\beta}(1+r)r^{m-1}+r|S_{n,m-1}^{(\alpha,\beta)}(z)| \\
&\quad +\frac{\alpha}{n+\beta}(1+r)r^{m-1}+\frac{m+1+\beta}{n+\beta}(1+r)r^{m-1} \\
&= r|S_{n,m-1}^{(\alpha,\beta)}(z)|+\frac{2m+\alpha+\beta}{n+\beta}(1+r)r^{m-1}.
\end{aligned}
$$

By writing the last inequality, for $m=2,\cdots$, we easily obtain step by step the following

$$
\begin{aligned}
|M_n^{(\alpha,\beta)}(e_m;z)-e_m(z)| &\le r\left(r|S_{n,m-2}^{(\alpha,\beta)}(z)|+\frac{2(m-1)+\alpha+\beta}{n+\beta}(1+r)r^{m-2}\right) \\
&\quad +\frac{2m+\alpha+\beta}{n+\beta}(1+r)r^{m-1} \\
&= r^2|S_{n,m-2}^{(\alpha,\beta)}(z))|+\frac{2(m-1+m)+2(\alpha+\beta)}{n+\beta}(1+r)r^{m-1} \\
&\le \ldots \le \frac{1+r}{n+\beta}m(m+1+\alpha+\beta)r^{m-1}.
\end{aligned}
$$

In conclusion, for any $m,n\in\mathbf{N}$, $|z|\le r$, $0\le\alpha\le\beta$, we have

$$
|M_{n+\beta}^{(\alpha,\beta)}(e_m;z)-e_m(z)| \le \frac{1+r}{n+\beta}m(m+1+\alpha+\beta)r^{m-1},
$$

it follows that

$$|M_n^{(\alpha,\beta)}(f;z)-f(z)|\leq \frac{1+r}{n+\beta}\sum_{m=1}^{\infty}|c_m|m(m+1+\alpha+\beta)r^{m-1}.$$

By assuming that $f(z)$ is analytic in D_R, we have $f^{(2)}(z)=\sum\limits_{m=2}^{\infty}c_m m(m-1)z^{m-2}$ and the series is absolutely convergent in $|z|\leq r$, so we get $\sum\limits_{m=2}^{\infty}|c_m|m(m-1)r^{m-2}<+\infty$, which implies $K_r^{(\alpha,\beta)}(f)=(1+r)\sum\limits_{m=1}^{\infty}|c_m|m(m+1+\alpha+\beta)r^{m-1}<+\infty$.

(ii) For the simultaneous approximation, denoting by Γ the circle of radius $r_1>r$ and center 0, since for any $|z|\leq r$ and $\upsilon\in\Gamma$, we have $|\upsilon-z|\geq r_1-r$, by Cauchy's formulas it follows that for all $|z|\leq r$ and $n\in\mathbf{N}$, we have

$$\begin{aligned}|(M_n^{(\alpha,\beta)}(f;z))^{(p)}-f^{(p)}(z)| &= \frac{p!}{2\pi}\left|\int_\Gamma \frac{M_n^{(\alpha,\beta)}(f;\upsilon)-f(\upsilon)}{(\upsilon-z)^{p+1}}d\upsilon\right| \\ &\leq \frac{K_{r_1}^{(\alpha,\beta)}(f)}{n+\beta}\frac{p!}{2\pi}\frac{2\pi r_1}{(r_1-r)^{p+1}} \\ &= \frac{K_{r_1}^{(\alpha,\beta)}(f)}{n+\beta}\cdot\frac{p!r_1}{(r_1-r)^{p+1}},\end{aligned}$$

which proves the theorem.

Theorem 2 *Let $0\leq\alpha\leq\beta$, $R>1$, $D_R=\{z\in\mathbf{C}:|z|<R\}$. Suppose that $f:D_R\to\mathbf{C}$ is analytic in D_R, i.e. $f(z)=\sum\limits_{k=0}^{\infty}c_kz^k$ for all $z\in D_R$. For any fixed $r\in[1,R]$ and all $n\in\mathbf{N}$, $|z|\leq r$, we have*

$$\begin{aligned}&\left|M_n^{(\alpha,\beta)}(f;z)-f(z)-\frac{\alpha-\beta z}{n+\beta}f'(z)-\frac{z(1-z)}{2(n+\beta)}f''(z)\right| \\ &\leq \frac{M_{r,1}^{(\alpha,\beta)}(f)}{n(n+\beta)}+\frac{M_{r,2}^{(\alpha,\beta)}(f)}{(n+\beta)^2}+\frac{M_{r,2}(f)}{n^2},\end{aligned}\tag{4}$$

where $M_{r,2}(f)=M_r(f)+M_{r,1}(f)$, $M_r(f)=\sum\limits_{k=2}^{\infty}|c_k|(k-1)F_{k,r}r^k$ with $F_{k,r}=10k^3-30k^2+39k-16+4(k-2)(k-1)^2(1+r)$, $M_{r,1}(f)=\sum\limits_{k=2}^{\infty}|c_k|(\beta+1)k(k-1)(1+r)r^{k-1}$, $M_{r,1}^{(\alpha,\beta)}(f)=\sum\limits_{k=2}^{\infty}|c_k|[2k(k-1)^2\alpha+2k^3\beta r]r^{k-1}$, $M_{r,2}^{(\alpha,\beta)}(f)=\sum\limits_{k=2}^{\infty}|c_k|[\frac{k(k-1)(\alpha^2+\beta^2r^2)}{2}+k^2\alpha\beta r+k^2\beta^2r^2]r^{k-2}$.

Proof For all $z\in D_R$, we have

$$\begin{aligned}&M_n^{(\alpha,\beta)}(f;z)-f(z)-\frac{\alpha-\beta z}{n+\beta}f'(z)-\frac{z(1-z)}{2(n+\beta)}f''(z) \\ &=\left[M_n(f;z)-f(z)-\frac{(n+1)z(1-z)}{2(n^2+1)}f''(z)\right] \\ &\quad+\left[M_n^{(\alpha,\beta)}(f;z)-M_n(f;z)-\frac{\alpha-\beta z}{n+\beta}f'(z)+\frac{(\beta+1)n+(\beta-1)}{2(n+\beta)(n^2+1)}z(1-z)f''(z)\right] \\ &:=I_1+I_2.\end{aligned}$$

By [16, Theorem 2], we have $|I_1| \leq \frac{M_r(f)}{n^2}$, where $M_r(f) = \sum_{k=2}^{\infty} |c_k|(k-1)F_{k,r}r^k$ and $F_{k,r} = 10k^3 - 30k^2 + 39k - 16 + 4(k-2)(k-1)^2(1+r)$.

Next let us to estimate $|I_2|$.

Denote $Q_{n,k}^{(\beta)}(z) = \frac{k(k-1)((\beta+1)n+(\beta-1))}{2(n+\beta)(n^2+1)} z^{k-1}(1-z)$. By f is analytic in D_R, i.e. $f(z) = \sum_{k=0}^{\infty} c_k z^k$ for all $z \in D_R$, and $M_n^{(\alpha,\beta)}(e_1;z) = M_n(e_1;z) + \frac{\alpha-\beta z}{n+\beta}$, we have

$$\begin{aligned} |I_2| &= \left| \sum_{k=2}^{\infty} c_k \left[M_n^{(\alpha,\beta)}(e_k;z) - M_n(e_k;z) - \frac{\alpha-\beta z}{n+\beta} k z^{k-1} + Q_{n,k}^{(\beta)}(z) \right] \right| \\ &\leq \sum_{k=2}^{\infty} |c_k| \left| M_n^{(\alpha,\beta)}(e_k;z) - M_n(e_k;z) - \frac{\alpha-\beta z}{n+\beta} k z^{k-1} + Q_{n,k}^{(\beta)}(z) \right|. \end{aligned}$$

When $k \geq 2$, since $\frac{n^k}{(n+\beta)^k} - 1 = -\sum_{j=0}^{k-1} \binom{k}{j} \frac{n^j \beta^{k-j}}{(n+\beta)^k}$, by Lemma 1, we obtain

$$\begin{aligned} & M_n^{(\alpha,\beta)}(e_k;z) - M_n(e_k;z) - \frac{\alpha-\beta z}{n+\beta} k z^{k-1} + Q_{n,k}^{(\beta)}(z) \\ &= \sum_{j=0}^{k-1} \binom{k}{j} \frac{n^j \alpha^{k-j}}{(n+\beta)^k} M_n(e_j;z) + \left[\frac{n^k}{(n+\beta)^k} - 1 \right] M_n(e_k;z) - \frac{\alpha-\beta z}{n+\beta} k z^{k-1} \\ &\quad + Q_{n,k}^{(\beta)}(z) \\ &= \sum_{j=0}^{k-2} \binom{k}{j} \frac{n^j \alpha^{k-j}}{(n+\beta)^k} M_n(e_j;z) + \frac{k n^{k-1} \alpha}{(n+\beta)^k} M_n(e_{k-1};z) \\ &\quad - \sum_{j=0}^{k-1} \binom{k}{j} \frac{n^j \beta^{k-j}}{(n+\beta)^k} M_n(e_k;z) - \frac{\alpha-\beta z}{n+\beta} k z^{k-1} + Q_{n,k}^{(\beta)}(z) \\ &= \sum_{j=0}^{k-2} \binom{k}{j} \frac{n^j \alpha^{k-j}}{(n+\beta)^k} M_n(e_j;z) + \frac{k n^{k-1} \alpha}{(n+\beta)^k} \left[M_n(e_{k-1};z) - e_{k-1}(z) \right] \\ &\quad + \frac{k n^{k-1} \alpha}{(n+\beta)^k} z^{k-1} - \sum_{j=0}^{k-2} \binom{k}{j} \frac{n^j \beta^{k-j}}{(n+\beta)^k} M_n(e_k;z) \\ &\quad - \frac{k n^{k-1} \beta}{(n+\beta)^k} \left[M_n(e_k;z) - e_k(z) \right] - \frac{k n^{k-1} \beta}{(n+\beta)^k} z^k - \frac{\alpha-\beta z}{n+\beta} k z^{k-1} + Q_{n,k}^{(\beta)}(z) \\ &= \sum_{j=0}^{k-2} \binom{k}{j} \frac{n^j \alpha^{k-j}}{(n+\beta)^k} M_n(e_j;z) + \frac{k n^{k-1} \alpha}{(n+\beta)^k} \left[M_n(e_{k-1};z) - e_{k-1}(z) \right] \\ &\quad - \sum_{j=0}^{k-2} \binom{k}{j} \frac{n^j \beta^{k-j}}{(n+\beta)^k} M_n(e_k;z) - \frac{k n^{k-1} \beta}{(n+\beta)^k} \left[M_n(e_k;z) - e_k(z) \right] \\ &\quad - \left[\frac{1}{n+\beta} - \frac{n^{k-1}}{(n+\beta)^k} \right] k \alpha z^{k-1} + \left[\frac{1}{n+\beta} - \frac{n^{k-1}}{(n+\beta)^k} \right] k \beta z^k + Q_{n,k}^{(\beta)}(z). \end{aligned}$$

By the proof of the [16, Theorem 1], for any $k \in \mathbf{N}$, $|z| \leq r$, $r \geq 1$, we have

$$|M_n(e_k;z)| \leq r^k, \ |M_n(e_k;z) - e_k| \leq \frac{2k^2}{n} r^k,$$

hence, for any $k \geq 2$, $|z| \leq r$, $r \geq 1$, we can get

$$\begin{aligned}
&\left|\sum_{j=0}^{k-2} \binom{k}{j} \frac{n^j \alpha^{k-j}}{(n+\beta)^k} M_n(e_j; z)\right| \\
&\leq \sum_{j=0}^{k-2} \binom{k}{j} \frac{n^j \alpha^{k-j}}{(n+\beta)^k} r^{k-2} \\
&= \sum_{j=0}^{k-2} \frac{k(k-1)}{(k-j)(k-j-1)} \binom{k-2}{j} \frac{n^j \alpha^{k-2-j}}{(n+\beta)^{k-2}} \cdot \frac{\alpha^2}{(n+\beta)^2} r^{k-2} \\
&\leq \frac{k(k-1)}{2} \cdot \frac{\alpha^2}{(n+\beta)^2} \sum_{j=0}^{k-2} \binom{k-2}{j} \frac{n^j \alpha^{k-2-j}}{(n+\beta)^{k-2}} r^{k-2} \\
&\leq \frac{k(k-1)}{2} \cdot \frac{\alpha^2}{(n+\beta)^2} r^{k-2}
\end{aligned}$$

and

$$\left|\frac{kn^{k-1}\alpha}{(n+\beta)^k}\left[M_n(e_{k-1}; z) - e_{k-1}(z)\right]\right| \leq \frac{2k(k-1)^2\alpha}{n(n+\beta)} r^{k-1}.$$

Also, using

$$\frac{1}{n+\beta} - \frac{n^{k-1}}{(n+\beta)^k} = \frac{\sum_{j=0}^{k-2} \binom{k-1}{j} n^j \beta^{k-1-j}}{(n+\beta)^k} \leq \frac{(k-1)\beta}{(n+\beta)^2},$$

thus, for any $k \geq 2$, $|z| \leq r$, $r \geq 1$, we get

$$\begin{aligned}
&|M_n^{(\alpha,\beta)}(e_k; z) - M_n(e_k; z) - \frac{\alpha - \beta z}{n+\beta} k z^{k-1} + Q_{n,k}^{(\beta)}(z)| \\
&\leq \frac{k(k-1)}{2} \cdot \frac{\alpha^2}{(n+\beta)^2} r^{k-2} + \frac{2k(k-1)^2\alpha}{n(n+\beta)} r^{k-1} + \frac{k(k-1)}{2} \cdot \frac{\beta^2}{(n+\beta)^2} r^k \\
&\quad + \frac{2k^3\beta}{n(n+\beta)} r^k + \frac{k^2\alpha\beta}{(n+\beta)^2} r^{k-1} + \frac{k^2\beta^2}{(n+\beta)^2} r^k + \frac{(\beta+1)k(k-1)(1+r)r^{k-1}}{n^2} \\
&= \frac{r^{k-1}}{n(n+\beta)} \left[2k(k-1)^2\alpha + 2k^3\beta r\right] + \frac{(\beta+1)k(k-1)(1+r)r^{k-1}}{n^2} \\
&\quad + \frac{r^{k-2}}{(n+\beta)^2} [\frac{k(k-1)(\alpha^2 + \beta^2 r^2)}{2} + k^2\alpha\beta r + k^2\beta^2 r^2].
\end{aligned}$$

Hence, we have

$$|I_2| \leq \frac{M_{r,1}^{(\alpha,\beta)}(f)}{n(n+\beta)} + \frac{M_{r,2}^{(\alpha,\beta)}(f)}{(n+\beta)^2} + \frac{M_{r,1}(f)}{n^2},$$

where

$$\begin{aligned}
M_{r,1}(f) &= \sum_{k=2}^{\infty} |c_k| (\beta+1)k(k-1)(1+r)r^{k-1}, \\
M_{r,1}^{(\alpha,\beta)}(f) &= \sum_{k=2}^{\infty} |c_k| [2k(k-1)^2\alpha + 2k^3\beta r] r^{k-1}, \\
M_{r,2}^{(\alpha,\beta)}(f) &= \sum_{k=2}^{\infty} |c_k| [\frac{k(k-1)(\alpha^2 + \beta^2 r^2)}{2} + k^2\alpha\beta r + k^2\beta^2 r^2] r^{k-2}.
\end{aligned}$$

In conclusion, we obtain

$$\left|M_n^{(\alpha,\beta)}(f;z) - f(z) - \frac{\alpha-\beta z}{n+\beta} f'(z) - \frac{z(1-z)}{2(n+\beta)} f''(z)\right|$$

$$\leq |I_1| + |I_2| \leq \frac{M_{r,1}^{(\alpha,\beta)}(f)}{n(n+\beta)} + \frac{M_{r,2}^{(\alpha,\beta)}(f)}{(n+\beta)^2} + \frac{M_{r,2}(f)}{n^2},$$

where $M_{r,2}(f) = M_r(f) + M_{r,1}(f)$.

In the following theorem, we will obtain the exact order in approximation.

Theorem 3 *Let $0 < \alpha \leq \beta$, $R > 1$, $D_R = \{z \in \mathbf{C} : |z| < R\}$. Suppose that $f : D_R \to \mathbf{C}$ is analytic in D_R. If f is not a polynomial of degree 0 , then for any $r \in [1, R)$ we have*

$$\|M_n^{(\alpha,\beta)}(f;\cdot) - f\|_r \geq \frac{C_r^{(\alpha,\beta)}(f)}{n+\beta},\ n \in \mathbf{N},$$

where $\|f\|_r = max\{|f(z)|; |z| \leq r\}$ and the constant $C_r^{(\alpha,\beta)}(f) > 0$ depends on f, r and α, β but it is independent of n.

Proof Denote $e_1(z) = z$ and

$$H_n^{(\alpha,\beta)}(f;z) = M_n^{(\alpha,\beta)}(f;z) - f(z) - \frac{\alpha-\beta z}{n+\beta} f'(z) - \frac{z(1-z)}{2(n+\beta)} f''(z).$$

For all $z \in D_R$ and $n \in \mathbf{N}$ we have

$$M_n^{(\alpha,\beta)}(f;z) - f(z)$$

$$= \frac{1}{n+\beta}\left\{(\alpha-\beta z)f'(z) + \frac{z(1-z)}{2} f''(z) + (n+\beta)H_n^{(\alpha,\beta)}(f;z)\right\}.$$

In view of the property: $\|F+G\|_r \geq |\|F\|_r - \|G\|_r| \geq \|F\|_r - \|G\|_r$, it follows

$$\|M_n^{(\alpha,\beta)}(f;\cdot) - f\|_r$$

$$\geq \frac{1}{n+\beta}\left\{\|(\alpha-\beta e_1)f' + \frac{e_1(1-e_1)}{2} f''\|_r - (n+\beta)||H_n^{(\alpha,\beta)}(f;\cdot)||_r\right\}.$$

Considering the hypothesis that f is not a polynomial of degree 0 in D_R, we have $\|(\alpha-\beta e_1)f' + \frac{e_1(1-e_1)}{2} f''\|_r > 0$.

Indeed, supposing the contrary, it follows that

$$(\alpha-\beta z)f'(z) + \frac{z(1-z)}{2} f''(z) = 0,\ \text{for all } z \in \overline{D_r}.$$

Denoting $y(z) = f'(z)$ and looking for the analytic function $y(z)$ under the form $y(z) = \sum_{k=0}^{\infty} a_k z^k$, after replacement in the differential equation, the identification of the coefficients method immediately leads to $a_k = 0$, for all $k \in \mathbb{N}\bigcup\{0\}$. This implies that $y(z) = 0$ for all $z \in \overline{D_r}$ and therefore f is constant on $\overline{D_r}$, a contradiction with the hypothesis.

Using the inequality (4), we get

$$\lim_{n\to\infty}(n+\beta)\|H_n^{(\alpha,\beta)}(f;\cdot)\|_r = 0, \tag{5}$$

therefore, there exists an index n_0 depending only on f, r and α, β, such that for all $n \geq n_0$, we have

$$\|(\alpha-\beta e_1)f' + \frac{e_1(1-e_1)}{2}f''\|_r - (n+\beta)||H_n^{(\alpha,\beta)}(f;\cdot)||_r$$
$$\geq \frac{1}{2}\left\|(\alpha-\beta e_1)f' + \frac{e_1(1-e_1)}{2}f''\right\|_r,$$

which implies

$$\|M_n^{(\alpha,\beta)}(f;\cdot) - f\|_r \geq \frac{1}{2n}\left\|(\alpha-\beta e_1)f' + \frac{e_1(1-e_1)}{2}f''\right\|_r, \text{ for all } n \geq n_0.$$

For $n \in \{1, 2, \cdots, n_0 - 1\}$, we have $\|M_n^{(\alpha,\beta)}(f;\cdot) - f\|_r \geq \frac{W_{r,n}^{(\alpha,\beta)}(f)}{n+\beta}$, where $W_{r,n}^{(\alpha,\beta)}(f) = (n+\beta)\|M_n^{(\alpha,\beta)}(f;\cdot) - f\|_r > 0$.

As a conclusion, we have $\|M_n^{(\alpha,\beta)}(f;\cdot) - f\|_r \geq \frac{C_r^{(\alpha,\beta)}(f)}{n+\beta}$, for all $n \in \mathbf{N}$, where

$$C_r^{(\alpha,\beta)}(f) = min\left\{W_{r,1}^{(\alpha,\beta)}(f), W_{r,2}^{(\alpha,\beta)}(f), \ldots, W_{r,n_0-1}^{(\alpha,\beta)}(f),\right.$$
$$\left.\frac{1}{2}\|(\alpha-\beta e_1)f' + \frac{e_1(1-e_1)}{2}f''\|_r\right\},$$

this complete the proof.

Combining Theorem 3 with Theorem 1, we get the following result.

Corollary *Let $0 \leq \alpha \leq \beta$, $R > 1$, $D_R = \{z \in \mathbf{C} : |z| < R\}$. Suppose that $f : D_R \to \mathbf{C}$ is analytic in D_R. If f is not a polynomial of degree 0 , then for any $r \in [1, R)$ we have*

$$\|M_n^{(\alpha,\beta)}(f;\cdot) - f\|_r \asymp \frac{1}{n+\beta}, \quad n \in \mathbf{N},$$

where $\|f\|_r = max\{|f(z)|; |z| \leq r\}$ and the constants in the equivalence depend on f, r and α, β but it is independent of n.

Theorem 4 *Let $0 \leq \alpha \leq \beta$, $R > 1$, $D_R = \{z \in \mathbf{C} : |z| < R\}$. Suppose that $f : D_R \to \mathbf{C}$ is analytic in D_R. Also, let $1 \leq r < r_1 < R$ and $p \in \mathbf{N}$ be fixed. If f is not a polynomial of degree $\leq p-1$, then we have*

$$\|(M_n^{(\alpha,\beta)}(f;\cdot))^{(p)} - f^{(p)}\|_r \asymp \frac{1}{n+\beta}, \; n \in \mathbf{N},$$

where $\|f\|_r = max\{|f(z)|; |z| \leq r\}$ and the constants in the equivalence depend on f, r, r_1, p, α and β, but it is independent of n.

Proof Taking into account that the upper estimate in Theorem 1 , it remains to prove the lower estimate only. Denoting by Γ the circle of radius $r_1 > r$ and center 0 , by the Cauchy's formula, it follows that for all $|z| \leq r$ and $n \in \mathbf{N}$, we have

$$(M_n^{(\alpha,\beta)}(f;z))^{(p)} - f^{(p)}(z) = \frac{p!}{2\pi i}\int_\Gamma \frac{M_n^{(\alpha,\beta)}(f;v) - f(v)}{(v-z)^{p+1}}dv.$$

Keeping the notation there for $H_n^{(\alpha,\beta)}(f;z)$, for all $n \in \mathbf{N}$, we have

$$M_n^{(\alpha,\beta)}(f;z) - f(z)$$
$$= \frac{1}{n+\beta}\left\{(\alpha-\beta z)f'(z) + \frac{z(1-z)}{2}f''(z) + (n+\beta)H_n^{(\alpha,\beta)}(f;z)\right\}.$$

by using Cauchy's formula, for all $v \in \Gamma$ we get

$$(M_n^{(\alpha,\beta)}(f;z))^{(p)} - f^{(p)}(z) = \frac{1}{n+\beta}\left\{\left[(\alpha-\beta z)f'(z) + \frac{z(1-z)}{2}f''(z)\right]^{(p)} + \frac{p!}{2\pi i}\int_\Gamma \frac{(n+\beta)H_n^{(\alpha,\beta)}(f;v)}{(v-z)^{p+1}}dv\right\},$$

passing now to $\|\cdot\|_r$ and denoting $e_1(z) = z$, it follows

$$\left\|(M_n^{(\alpha,\beta)}(f;\cdot))^{(p)} - f^{(p)}\right\|_r \geq \frac{1}{n+\beta}\left[\left\|\left[(\alpha-\beta e_1)f' + \frac{e_1(1-e_1)}{2}f''\right]^{(p)}\right\|_r - \left\|\frac{p!}{2\pi i}\int_\Gamma \frac{(n+\beta)H_n^{(\alpha,\beta)}(f;v)}{(v-\cdot)^{p+1}}dv\right\|_r\right].$$

Since for any $|z| \leq r$ and $v \in \Gamma$, we have $|v-z| \geq r_1 - r$, so,

$$\left\|\frac{p!}{2\pi i}\int_\Gamma \frac{(n+\beta)H_n^{(\alpha,\beta)}(f;v)}{(v-\cdot)^{p+1}}dv\right\|_r \leq \frac{p!}{2\pi}\cdot\frac{2\pi r_1(n+\beta)\|H_n^{(\alpha,\beta)}(f;\cdot)\|_{r_1}}{(r_1-r)^{p+1}},$$

thus, by the inequality (5), we can get $\lim_{n\to\infty}\left\|\frac{p!}{2\pi i}\int_\Gamma \frac{(n+\beta)H_n^{(\alpha,\beta)}(f;v)}{(v-\cdot)^{p+1}}dv\right\|_r = 0.$

Taking into account the function f is analytic in D_R, by following exactly the lines in Gal [5], seeing also the book Gal [6, pp. 77-78], we have $\left\|[(\alpha-\beta e_1)f' + \frac{e_1(1-e_1)}{2}f'']^{(p)}\right\|_r > 0$,

In continuation, reasoning exactly as in the proof of Theorem 3, we can get the desired conclusion.

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On right multidimensional Riemann-Liouville fractional integral

George Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

Abstract

Here we study some important properties of right multidimensional Riemann-Liouville fractional integral operator, such as of continuity and boundedness.

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Key Words and Phrases: Riemann-Liouville fractional integral, continuity, boundedness.

1 Motivation

From [1] we have

Theorem 1 *Let* $r > 0$*,* $F \in L_\infty(a,b)$*, and*

$$G(s) = \int_s^b (t-s)^{r-1} F(t)\, dt,$$

all $s \in [a,b]$*. Then* $G \in AC([a,b])$ *(absolutely continuous functions) for* $r \geq 1$*, and* $G \in C([a,b])$*, only for* $r \in (0,1)$*.*

2 Main Results

We give

Theorem 2 *Let* $f \in L_\infty([a,b] \times [c,d])$*,* $\alpha_1, \alpha_2 > 0$*. Consider the function*

$$F(x_1,x_2) = \int_{x_1}^{b_1} \int_{x_2}^{b_2} (t_1 - x_1)^{\alpha_1 - 1} (t_2 - x_2)^{\alpha_2 - 1} f(t_1,t_2)\, dt_1 dt_2, \quad (1)$$

where $x_1, b_1 \in [a, b],\ x_2, b_2 \in [c, d] : x_1 \leq b_1,\ x_2 \leq b_2$.

Then F *is continuous on* $[a, b_1] \times [c, b_2]$.

Proof. (I) Let $a_1, a_1^*, b_1 \in [a, b]$ with $a_1 < a_1^* < b_1$, and $a_2, a_2^*, b_2 \in [c, d]$ with $a_2 < a_2^* < b_2$.

We observe that

$$F(a_1, a_2) - F(a_1^*, a_2^*) =$$

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} (t_1 - a_1)^{\alpha_1 - 1} (t_2 - a_2)^{\alpha_2 - 1} f(t_1, t_2)\, dt_1 dt_2 -$$

$$\int_{a_1^*}^{b_1} \int_{a_2^*}^{b_2} (t_1 - a_1^*)^{\alpha_1 - 1} (t_2 - a_2^*)^{\alpha_2 - 1} f(t_1, t_2)\, dt_1 dt_2 = \tag{2}$$

$$\int_{a_1^*}^{b_1} \int_{a_2^*}^{b_2} (t_1 - a_1)^{\alpha_1 - 1} (t_2 - a_2)^{\alpha_2 - 1} f(t_1, t_2)\, dt_1 dt_2 +$$

$$\int_{a_1^*}^{b_1} \int_{a_2}^{a_2^*} (t_1 - a_1)^{\alpha_1 - 1} (t_2 - a_2)^{\alpha_2 - 1} f(t_1, t_2)\, dt_1 dt_2 +$$

$$\int_{a_1}^{a_1^*} \int_{a_2}^{a_2^*} (t_1 - a_1)^{\alpha_1 - 1} (t_2 - a_2)^{\alpha_2 - 1} f(t_1, t_2)\, dt_1 dt_2 +$$

$$\int_{a_1}^{a_1^*} \int_{a_2^*}^{b_2} (t_1 - a_1)^{\alpha_1 - 1} (t_2 - a_2)^{\alpha_2 - 1} f(t_1, t_2)\, dt_1 dt_2 - \tag{3}$$

$$\int_{a_1^*}^{b_1} \int_{a_2^*}^{b_2} (t_1 - a_1^*)^{\alpha_1 - 1} (t_2 - a_2^*)^{\alpha_2 - 1} f(t_1, t_2)\, dt_1 dt_2 =$$

$$\int_{a_1^*}^{b_1} \int_{a_2^*}^{b_2} \left[(t_1 - a_1)^{\alpha_1 - 1} (t_2 - a_2)^{\alpha_2 - 1} - (t_1 - a_1^*)^{\alpha_1 - 1} (t_2 - a_2^*)^{\alpha_2 - 1} \right] f(t_1, t_2)\, dt_1 dt_2$$

$$+ \int_{a_1^*}^{b_1} \int_{a_2}^{a_2^*} (t_1 - a_1)^{\alpha_1 - 1} (t_2 - a_2)^{\alpha_2 - 1} f(t_1, t_2)\, dt_1 dt_2 + \tag{4}$$

$$\int_{a_1}^{a_1^*} \int_{a_2}^{a_2^*} (t_1 - a_1)^{\alpha_1 - 1} (t_2 - a_2)^{\alpha_2 - 1} f(t_1, t_2)\, dt_1 dt_2 +$$

$$\int_{a_1}^{a_1^*} \int_{a_2^*}^{b_2} (t_1 - a_1)^{\alpha_1 - 1} (t_2 - a_2)^{\alpha_2 - 1} f(t_1, t_2)\, dt_1 dt_2.$$

Call

$$I(a_1^*, a_2^*) =$$

$$\int_{a_1^*}^{b_1} \int_{a_2^*}^{b_2} \left| (t_1 - a_1)^{\alpha_1 - 1} (t_2 - a_2)^{\alpha_2 - 1} - (t_1 - a_1^*)^{\alpha_1 - 1} (t_2 - a_2^*)^{\alpha_2 - 1} \right| dt_1 dt_2. \tag{5}$$

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Thus

$$|F(a_1,a_2) - F(a_1^*,a_2^*)| \leq$$

$$\left\{ I(a_1^*,a_2^*) + \left(\frac{(b_1-a_1)^{\alpha_1} - (a_1^*-a_1)^{\alpha_1}}{\alpha_1} \right) \frac{(a_2^*-a_2)^{\alpha_2}}{\alpha_2} + \right.$$

$$\left. \frac{(a_1^*-a_1)^{\alpha_1}}{\alpha_1} \frac{(a_2^*-a_2)^{\alpha_2}}{\alpha_2} + \frac{(a_1^*-a_1)^{\alpha_1}}{\alpha_1} \left(\frac{(b_2-a_2)^{\alpha_2} - (a_2^*-a_2)^{\alpha_2}}{\alpha_2} \right) \right\} \|f\|_\infty .$$

Hence, by the last inequality, it holds

$$\delta := \lim_{\substack{(a_1^*,a_2^*)\to(a_1,a_2) \\ \text{or} \\ (a_1,a_2)\to(a_1^*,a_2^*)}} |F(a_1,a_2) - F(a_1^*,a_2^*)| \leq$$

$$\left(\lim_{\substack{(a_1^*,a_2^*)\to(a_1,a_2) \\ \text{or} \\ (a_1,a_2)\to(a_1^*,a_2^*)}} I(a_1^*,a_2^*) \right) \|f\|_\infty =: \rho. \tag{6}$$

If $\alpha_1 = \alpha_2 = 1$, then $\rho = 0$, proving $\delta = 0$.

If $\alpha_1 = 1$, $\alpha_2 > 0$ we get

$$I(a_1^*,a_2^*) = (b_1 - a_1^*) \int_{a_2^*}^{b_2} \left| (t_2-a_2)^{\alpha_2-1} - (t_2-a_2^*)^{\alpha_2-1} \right| dt_2. \tag{7}$$

Assume $\alpha_2 > 1$, then $\alpha_2 - 1 > 0$. Hence

$$I(a_1^*,a_2^*) = (b_1 - a_1^*) \int_{a_2^*}^{b_2} \left((t_2-a_2)^{\alpha_2-1} - (t_2-a_2^*)^{\alpha_2-1} \right) dt_2$$

$$= (b_1 - a_1^*) \left\{ \left(\frac{(b_2-a_2)^{\alpha_2} - (a_2^*-a_2)^{\alpha_2}}{\alpha_2} \right) - \frac{(b_2-a_2^*)^{\alpha_2}}{\alpha_2} \right\}. \tag{8}$$

Clearly, then

$$\lim_{\substack{(a_1,a_2)\to(a_1^*,a_2^*) \\ \text{or} \\ (a_1^*,a_2^*)\to(a_1,a_2)}} I(a_1^*,a_2^*) = 0. \tag{9}$$

Similarly and symmetrically, we obtain that

$$\lim_{\substack{(a_1,a_2)\to(a_1^*,a_2^*) \\ \text{or} \\ (a_1^*,a_2^*)\to(a_1,a_2)}} I(a_1^*,a_2^*) = 0 \tag{10}$$

for the case of $\alpha_2 = 1$, $\alpha_1 > 1$.

If $\alpha_1 = 1$, and $0 < \alpha_2 < 1$, then $\alpha_2 - 1 < 0$. Hence

$$I\left(a_1^*, a_2^*\right) = \left(b_1 - a_1^*\right) \int_{a_2^*}^{b_2} \left[\left(t_2 - a_2^*\right)^{\alpha_2 - 1} - \left(t_2 - a_2\right)^{\alpha_2 - 1}\right] dt_2 =$$

$$\left(b_1 - a_1^*\right) \left\{ \frac{\left(b_2 - a_2^*\right)^{\alpha_2}}{\alpha_2} - \left(\frac{\left(b_2 - a_2\right)^{\alpha_2} - \left(a_2^* - a_2\right)^{\alpha_2}}{\alpha_2} \right) \right\}. \tag{11}$$

Clearly, then

$$\lim_{\substack{a_2^* \to a_2 \\ \text{or} \\ a_2 \to a_2^*}} I\left(a_1^*, a_2^*\right) = 0. \tag{12}$$

Similarly and symmetrically, we derive that

$$\lim_{\substack{a_1^* \to a_1 \\ \text{or} \\ a_1 \to a_1^*}} I\left(a_1^*, a_2^*\right) = 0, \tag{13}$$

for the case of $\alpha_2 = 1$, $0 < \alpha_1 < 1$.

Case now of $\alpha_1, \alpha_2 > 1$, then

$$I\left(a_1^*, a_2^*\right) =$$

$$\int_{a_1^*}^{b_1} \int_{a_2^*}^{b_2} \left(\left(t_1 - a_1\right)^{\alpha_1 - 1} \left(t_2 - a_2\right)^{\alpha_2 - 1} - \left(t_1 - a_1^*\right)^{\alpha_1 - 1} \left(t_2 - a_2^*\right)^{\alpha_2 - 1} \right) dt_1 dt_2 =$$

$$\left(\frac{\left(b_1 - a_1\right)^{\alpha_1} - \left(a_1^* - a_1\right)^{\alpha_1}}{\alpha_1} \right) \left(\frac{\left(b_2 - a_2\right)^{\alpha_2} - \left(a_2^* - a_2\right)^{\alpha_2}}{\alpha_2} \right)$$

$$- \frac{\left(b_1 - a_1^*\right)^{\alpha_1}}{\alpha_1} \frac{\left(b_2 - a_2^*\right)^{\alpha_2}}{\alpha_2}. \tag{14}$$

That is

$$\lim_{\substack{\left(a_1^*, a_2^*\right) \to \left(a_1, a_2\right) \\ \text{or} \\ \left(a_1, a_2\right) \to \left(a_1^*, a_2^*\right)}} I\left(a_1^*, a_2^*\right) = 0. \tag{15}$$

Case now of $0 < \alpha_1, \alpha_2 < 1$, then

$$I\left(a_1^*, a_2^*\right) =$$

$$\int_{a_1^*}^{b_1} \int_{a_2^*}^{b_2} \left(\left(t_1 - a_1^*\right)^{\alpha_1 - 1} \left(t_2 - a_2^*\right)^{\alpha_2 - 1} - \left(t_1 - a_1\right)^{\alpha_1 - 1} \left(t_2 - a_2\right)^{\alpha_2 - 1} \right) dt_1 dt_2 =$$

$$\frac{\left(b_1 - a_1^*\right)^{\alpha_1}}{\alpha_1} \frac{\left(b_2 - a_2^*\right)^{\alpha_2}}{\alpha_2} -$$

$$\left(\frac{\left(b_1 - a_1\right)^{\alpha_1} - \left(a_1^* - a_1\right)^{\alpha_1}}{\alpha_1} \right) \left(\frac{\left(b_2 - a_2\right)^{\alpha_2} - \left(a_2^* - a_2\right)^{\alpha_2}}{\alpha_2} \right). \tag{16}$$

Hence, when $0 < \alpha_1, \alpha_2 < 1$, we get

$$\lim_{\substack{(a_1^*, a_2^*) \to (a_1, a_2) \\ (a_1, a_2) \to (a_1^*, a_2^*)}} I(a_1^*, a_2^*) = 0. \tag{17}$$

We observe that

$$I(a_1^*, a_2^*) \le I^*(a_1^*, a_2^*) :=$$

$$\int_{a_1^*}^{b_1} \int_{a_2^*}^{b_2} (t_1 - a_1)^{\alpha_1 - 1} \left| (t_2 - a_2)^{\alpha_2 - 1} - (t_2 - a_2^*)^{\alpha_2 - 1} \right| dt_1 dt_2$$

$$+ \int_{a_1^*}^{b_1} \int_{a_2^*}^{b_2} (t_2 - a_2^*)^{\alpha_2 - 1} \left| (t_1 - a_1)^{\alpha_1 - 1} - (t_1 - a_1^*)^{\alpha_1 - 1} \right| dt_1 dt_2. \tag{18}$$

Next we treat the case of $\alpha_1 > 1$, $0 < \alpha_2 < 1$.

Therefore it holds

$$I^*(a_1^*, a_2^*) = \int_{a_1^*}^{b_1} \int_{a_2^*}^{b_2} (t_1 - a_1)^{\alpha_1 - 1} \left((t_2 - a_2^*)^{\alpha_2 - 1} - (t_2 - a_2)^{\alpha_2 - 1} \right) dt_1 dt_2$$

$$+ \int_{a_1^*}^{b_1} \int_{a_2^*}^{b_2} (t_2 - a_2^*)^{\alpha_2 - 1} \left((t_1 - a_1)^{\alpha_1 - 1} - (t_1 - a_1^*)^{\alpha_1 - 1} \right) dt_1 dt_2 = \tag{19}$$

$$\left(\frac{(b_1 - a_1)^{\alpha_1} - (a_1^* - a_1)^{\alpha_1}}{\alpha_1} \right) \left(\frac{(b_2 - a_2^*)^{\alpha_2}}{\alpha_2} - \frac{(b_2 - a_2)^{\alpha_2}}{\alpha_2} + \frac{(a_2^* - a_2)^{\alpha_2}}{\alpha_2} \right) +$$

$$\frac{(b_2 - a_2^*)^{\alpha_2}}{\alpha_2} \left(\frac{(b_1 - a_1)^{\alpha_1}}{\alpha_1} - \frac{(a_1^* - a_1)^{\alpha_1}}{\alpha_1} - \frac{(b_1 - a_1^*)^{\alpha_1}}{\alpha_1} \right).$$

Clearly then ($\alpha_1 > 1$, $0 < \alpha_2 < 1$)

$$\lim_{\substack{(a_1, a_2) \to (a_1^*, a_2^*) \\ \text{or} \\ (a_1^*, a_2^*) \to (a_1, a_2)}} I(a_1^*, a_2^*) = 0. \tag{20}$$

Finally, we prove the case of $\alpha_2 > 1$ and $0 < \alpha_1 < 1$. We have that

$$I^*(a_1^*, a_2^*) = \int_{a_1^*}^{b_1} \int_{a_2^*}^{b_2} (t_1 - a_1)^{\alpha_1 - 1} \left((t_2 - a_2)^{\alpha_2 - 1} - (t_2 - a_2^*)^{\alpha_2 - 1} \right) dt_1 dt_2$$

$$+ \int_{a_1^*}^{b_1} \int_{a_2^*}^{b_2} (t_2 - a_2^*)^{\alpha_2 - 1} \left((t_1 - a_1^*)^{\alpha_1 - 1} - (t_1 - a_1)^{\alpha_1 - 1} \right) dt_1 dt_2 = \tag{21}$$

$$\left(\frac{(b_1 - a_1)^{\alpha_1} - (a_1^* - a_1)^{\alpha_1}}{\alpha_1} \right) \left(-\frac{(b_2 - a_2^*)^{\alpha_2}}{\alpha_2} + \frac{(b_2 - a_2)^{\alpha_2}}{\alpha_2} - \frac{(a_2^* - a_2)^{\alpha_2}}{\alpha_2} \right) +$$

$$\frac{(b_2 - a_2^*)^{\alpha_2}}{\alpha_2} \left(-\frac{(b_1 - a_1)^{\alpha_1}}{\alpha_1} + \frac{(a_1^* - a_1)^{\alpha_1}}{\alpha_1} + \frac{(b_1 - a_1^*)^{\alpha_1}}{\alpha_1} \right).$$

Clearly then ($\alpha_2 > 1$, $0 < \alpha_1 < 1$)

$$\lim_{\substack{(a_1,a_2)\to(a_1^*,a_2^*) \\ \text{or} \\ (a_1^*,a_2^*)\to(a_1,a_2)}} I(a_1^*,a_2^*) = 0. \tag{22}$$

We proved $\rho = 0$, and $\delta = 0$ in all cases of this section.

The case of $a_1 > a_1^*$ and $a_2 > a_2^*$, as symmetric to the already treated one of $a_1 < a_1^*$ and $a_2 < a_2^*$, is omitted.

(II) The remaining cases are: let $a_1, a_1^*, b_1 \in [a,b]$; $a_2, a_2^*, b_2 \in [c,d]$, we can have

$$\begin{array}{l} (\mathrm{II}_1)\ a_1 > a_1^* \ \text{ and } \ a_2 < a_2^*, \\ \text{or} \\ (\mathrm{II}_2)\ a_1 < a_1^* \ \text{ and } \ a_2 > a_2^*. \end{array} \tag{23}$$

Notice that the subcases (II_1) and (II_2) are symmetric, and treated the same way. As such we treat only the case (II_2).

We observe again that

$$F(a_1,a_2) - F(a_1^*,a_2^*) = \tag{24}$$

$$\begin{aligned} &\int_{a_1}^{b_1}\int_{a_2}^{b_2} (t_1-a_1)^{\alpha_1-1}(t_2-a_2)^{\alpha_2-1} f(t_1,t_2)\,dt_1dt_2 - \\ &\int_{a_1^*}^{b_1}\int_{a_2^*}^{b_2} (t_1-a_1^*)^{\alpha_1-1}(t_2-a_2^*)^{\alpha_2-1} f(t_1,t_2)\,dt_1dt_2 = \\ &\int_{a_1}^{a_1^*}\int_{a_2}^{b_2} (t_1-a_1)^{\alpha_1-1}(t_2-a_2)^{\alpha_2-1} f(t_1,t_2)\,dt_1dt_2 + \\ &\int_{a_1^*}^{b_1}\int_{a_2}^{b_2} (t_1-a_1)^{\alpha_1-1}(t_2-a_2)^{\alpha_2-1} f(t_1,t_2)\,dt_1dt_2 - \\ &\int_{a_1^*}^{b_1}\int_{a_2^*}^{a_2} (t_1-a_1^*)^{\alpha_1-1}(t_2-a_2^*)^{\alpha_2-1} f(t_1,t_2)\,dt_1dt_2 - \\ &\int_{a_1^*}^{b_1}\int_{a_2}^{b_2} (t_1-a_1^*)^{\alpha_1-1}(t_2-a_2^*)^{\alpha_2-1} f(t_1,t_2)\,dt_1dt_2 = \end{aligned} \tag{25}$$

$$\begin{aligned} &\int_{a_1^*}^{b_1}\int_{a_2}^{b_2} \left((t_1-a_1)^{\alpha_1-1}(t_2-a_2)^{\alpha_2-1} - (t_1-a_1^*)^{\alpha_1-1}(t_2-a_2^*)^{\alpha_2-1}\right) f(t_1,t_2)\,dt_1dt_2 \\ &+\int_{a_1}^{a_1^*}\int_{a_2}^{b_2} (t_1-a_1)^{\alpha_1-1}(t_2-a_2)^{\alpha_2-1} f(t_1,t_2)\,dt_1dt_2 - \\ &\int_{a_1^*}^{b_1}\int_{a_2^*}^{a_2} (t_1-a_1^*)^{\alpha_1-1}(t_2-a_2^*)^{\alpha_2-1} f(t_1,t_2)\,dt_1dt_2. \end{aligned} \tag{26}$$

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Call

$$I\left(a_1^*, a_2\right) :=$$

$$\int_{a_1^*}^{b_1} \int_{a_2}^{b_2} \left| \left(t_1 - a_1\right)^{\alpha_1 - 1} \left(t_2 - a_2\right)^{\alpha_2 - 1} - \left(t_1 - a_1^*\right)^{\alpha_1 - 1} \left(t_2 - a_2^*\right)^{\alpha_2 - 1} \right| dt_1 dt_2. \tag{27}$$

Hence, we have

$$\left|F\left(a_1, a_2\right) - F\left(a_1^*, a_2^*\right)\right| \leq$$

$$\left\{ I\left(a_1^*, a_2\right) + \frac{\left(a_1^* - a_1\right)^{\alpha_1}}{\alpha_1} \frac{\left(b_2 - a_2\right)^{\alpha_2}}{\alpha_2} + \frac{\left(b_1 - a_1^*\right)^{\alpha_1}}{\alpha_1} \frac{\left(a_2 - a_2^*\right)^{\alpha_2}}{\alpha_2} \right\} \|f\|_\infty . \tag{28}$$

Therefore it holds

$$\delta := \lim_{\substack{\left|a_1 - a_1^*\right| \to 0, \\ \left|a_2 - a_2^*\right| \to 0}} \left|F\left(a_1, a_2\right) - F\left(a_1^*, a_2^*\right)\right| \leq \left(\lim_{\substack{\left|a_1 - a_1^*\right| \to 0, \\ \left|a_2 - a_2^*\right| \to 0}} I\left(a_1^*, a_2\right) \right) \|f\|_\infty =: \theta. \tag{29}$$

We will prove that $\theta = 0$, hence $\delta = 0$, in all possible cases.

If $\alpha_1 = \alpha_2 = 1$, then $I\left(a_1^*, a_2\right) = 0$, hence $\theta = 0$.

If $\alpha_1 = 1$, $\alpha_2 > 0$ we get

$$I\left(a_1^*, a_2\right) = \left(b_1 - a_1^*\right) \int_{a_2}^{b_2} \left| \left(t_2 - a_2\right)^{\alpha_2 - 1} - \left(t_2 - a_2^*\right)^{\alpha_2 - 1} \right| dt_2. \tag{30}$$

Assume $\alpha_2 > 1$, then $\alpha_2 - 1 > 0$. Hence

$$I\left(a_1^*, a_2\right) = \left(b_1 - a_1^*\right) \int_{a_2}^{b_2} \left(\left(t_2 - a_2^*\right)^{\alpha_2 - 1} - \left(t_2 - a_2\right)^{\alpha_2 - 1} \right) dt_2$$

$$= \left(b_1 - a_1^*\right) \left\{ \frac{\left(b_2 - a_2^*\right)^{\alpha_2}}{\alpha_2} - \frac{\left(a_2 - a_2^*\right)^{\alpha_2}}{\alpha_2} - \frac{\left(b_2 - a_2\right)^{\alpha_2}}{\alpha_2} \right\}. \tag{31}$$

Clearly, then

$$\lim_{\left|a_2 - a_2^*\right| \to 0,} I\left(a_1^*, a_2\right) = 0, \tag{32}$$

hence $\theta = 0$.

Let the case now of $\alpha_2 = 1$, $\alpha_1 > 1$. Then

$$\begin{aligned} I\left(a_1^*, a_2\right) &= \left(b_2 - a_2\right) \int_{a_1^*}^{b_1} \left| \left(t_1 - a_1\right)^{\alpha_1 - 1} - \left(t_1 - a_1^*\right)^{\alpha_1 - 1} \right| dt_1 \\ &= \left(b_2 - a_2\right) \int_{a_1^*}^{b_1} \left(\left(t_1 - a_1\right)^{\alpha_1 - 1} - \left(t_1 - a_1^*\right)^{\alpha_1 - 1} \right) dt_1 \\ &= \left(b_2 - a_2\right) \left(\frac{\left(b_1 - a_1\right)^{\alpha_1}}{\alpha_1} - \frac{\left(a_1^* - a_1\right)^{\alpha_1}}{\alpha_1} - \frac{\left(b_1 - a_1^*\right)^{\alpha_1}}{\alpha_1} \right). \end{aligned} \tag{33}$$

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Then $\theta = 0$.

If $\alpha_1 = 1$, and $0 < \alpha_2 < 1$, then $\alpha_2 - 1 < 0$. Hence

$$I(a_1^*, a_2) = (b_1 - a_1^*) \int_{a_2}^{b_2} \left((t_2 - a_2)^{\alpha_2 - 1} - (t_2 - a_2^*)^{\alpha_2 - 1} \right) dt_2 =$$

$$(b_1 - a_1^*) \left\{ \frac{(b_2 - a_2)^{\alpha_2}}{\alpha_2} - \frac{(b_2 - a_2^*)^{\alpha_2}}{\alpha_2} + \frac{(a_2 - a_2^*)^{\alpha_2}}{\alpha_2} \right\}, \tag{34}$$

hence $\theta = 0$.

Let now $\alpha_2 = 1$, $0 < \alpha_1 < 1$. Then

$$I(a_1^*, a_2) = (b_2 - a_2) \int_{a_1^*}^{b_1} \left((t_1 - a_1^*)^{\alpha_1 - 1} - (t_1 - a_1)^{\alpha_1 - 1} \right) dt_1$$

$$= (b_2 - a_2) \left\{ \frac{(b_1 - a_1^*)^{\alpha_1}}{\alpha_1} - \frac{(b_1 - a_1)^{\alpha_1}}{\alpha_1} + \frac{(a_1^* - a_1)^{\alpha_1}}{\alpha_1} \right\}, \tag{35}$$

hence $\theta = 0$.

We observe that:

$$I(a_1^*, a_2) \leq \int_{a_1^*}^{b_1} \int_{a_2}^{b_2} (t_1 - a_1)^{\alpha_1 - 1} \left| (t_2 - a_2)^{\alpha_2 - 1} - (t_2 - a_2^*)^{\alpha_2 - 1} \right| dt_1 dt_2$$

$$+ \int_{a_1^*}^{b_1} \int_{a_2}^{b_2} (t_2 - a_2^*)^{\alpha_2 - 1} \left| (t_1 - a_1)^{\alpha_1 - 1} - (t_1 - a_1^*)^{\alpha_1 - 1} \right| dt_1 dt_2 =: J(a_1^*, a_2). \tag{36}$$

I.e.

$$I(a_1^*, a_2) \leq J(a_1^*, a_2). \tag{37}$$

Case of $\alpha_1, \alpha_2 > 1$. Then

$$J(a_1^*, a_2) = \int_{a_1^*}^{b_1} \int_{a_2}^{b_2} (t_1 - a_1)^{\alpha_1 - 1} \left((t_2 - a_2^*)^{\alpha_2 - 1} - (t_2 - a_2)^{\alpha_2 - 1} \right) dt_1 dt_2$$

$$+ \int_{a_1^*}^{b_1} \int_{a_2}^{b_2} (t_2 - a_2^*)^{\alpha_2 - 1} \left((t_1 - a_1)^{\alpha_1 - 1} - (t_1 - a_1^*)^{\alpha_1 - 1} \right) dt_1 dt_2 = \tag{38}$$

$$\left(\frac{(b_1 - a_1)^{\alpha_1}}{\alpha_1} - \frac{(a_1^* - a_1)^{\alpha_1}}{\alpha_1} \right) \left\{ \frac{(b_2 - a_2^*)^{\alpha_2}}{\alpha_2} - \frac{(a_2 - a_2^*)^{\alpha_2}}{\alpha_2} - \frac{(b_2 - a_2)^{\alpha_2}}{\alpha_2} \right\}$$

$$+ \left(\frac{(b_2 - a_2^*)^{\alpha_2}}{\alpha_2} - \frac{(a_2 - a_2^*)^{\alpha_2}}{\alpha_2} \right) \left\{ \frac{(b_1 - a_1)^{\alpha_1}}{\alpha_1} - \frac{(a_1^* - a_1)^{\alpha_1}}{\alpha_1} - \frac{(b_1 - a_1^*)^{\alpha_1}}{\alpha_1} \right\}, \tag{39}$$

hence $\theta = 0$.

Case of $0 < \alpha_1, \alpha_2 < 1$, then

$$J(a_1^*, a_2) = \int_{a_1^*}^{b_1} \int_{a_2}^{b_2} (t_1 - a_1)^{\alpha_1 - 1} \left((t_2 - a_2)^{\alpha_2 - 1} - (t_2 - a_2^*)^{\alpha_2 - 1} \right) dt_1 dt_2$$

$$+\int_{a_1^*}^{b_1}\int_{a_2}^{b_2}(t_2-a_2^*)^{\alpha_2-1}\left((t_1-a_1^*)^{\alpha_1-1}-(t_1-a_1)^{\alpha_1-1}\right)dt_1dt_2= \tag{40}$$

$$\left(\frac{(b_1-a_1)^{\alpha_1}}{\alpha_1}-\frac{(a_1^*-a_1)^{\alpha_1}}{\alpha_1}\right)\left\{\frac{(b_2-a_2)^{\alpha_2}}{\alpha_2}-\frac{(b_2-a_2^*)^{\alpha_2}}{\alpha_2}+\frac{(a_2-a_2^*)^{\alpha_2}}{\alpha_2}\right\}$$
$$+\left(\frac{(b_2-a_2^*)^{\alpha_2}}{\alpha_2}-\frac{(a_2-a_2^*)^{\alpha_2}}{\alpha_2}\right)\left\{\frac{(b_1-a_1^*)^{\alpha_1}}{\alpha_1}-\frac{(b_1-a_1)^{\alpha_1}}{\alpha_1}+\frac{(a_1^*-a_1)^{\alpha_1}}{\alpha_1}\right\}, \tag{41}$$

hence $\theta=0$.

Next case of $\alpha_1>1$, $0<\alpha_2<1$. We observe that

$$J(a_1^*,a_2)=\int_{a_1^*}^{b_1}\int_{a_2}^{b_2}(t_1-a_1)^{\alpha_1-1}\left((t_2-a_2)^{\alpha_2-1}-(t_2-a_2^*)^{\alpha_2-1}\right)dt_1dt_2 \tag{42}$$

$$+\int_{a_1^*}^{b_1}\int_{a_2}^{b_2}(t_2-a_2^*)^{\alpha_2-1}\left((t_1-a_1)^{\alpha_1-1}-(t_1-a_1^*)^{\alpha_1-1}\right)dt_1dt_2=$$

$$\left(\frac{(b_1-a_1)^{\alpha_1}}{\alpha_1}-\frac{(a_1^*-a_1)^{\alpha_1}}{\alpha_1}\right)\left\{\frac{(b_2-a_2)^{\alpha_2}}{\alpha_2}-\frac{(b_2-a_2^*)^{\alpha_2}}{\alpha_2}+\frac{(a_2-a_2^*)^{\alpha_2}}{\alpha_2}\right\} \tag{43}$$
$$+\left(\frac{(b_2-a_2^*)^{\alpha_2}}{\alpha_2}-\frac{(a_2-a_2^*)^{\alpha_2}}{\alpha_2}\right)\left\{\frac{(b_1-a_1)^{\alpha_1}}{\alpha_1}-\frac{(a_1^*-a_1)^{\alpha_1}}{\alpha_1}-\frac{(b_1-a_1^*)^{\alpha_1}}{\alpha_1}\right\},$$

hence $\theta=0$.

Finally, we prove the case of $\alpha_2>1$ and $0<\alpha_1<1$. In that case it holds

$$J(a_1^*,a_2)=\int_{a_1^*}^{b_1}\int_{a_2}^{b_2}(t_1-a_1)^{\alpha_1-1}\left((t_2-a_2^*)^{\alpha_2-1}-(t_2-a_2)^{\alpha_2-1}\right)dt_1dt_2$$

$$+\int_{a_1^*}^{b_1}\int_{a_2}^{b_2}(t_2-a_2^*)^{\alpha_2-1}\left((t_1-a_1^*)^{\alpha_1-1}-(t_1-a_1)^{\alpha_1-1}\right)dt_1dt_2= \tag{44}$$

$$\left(\frac{(b_1-a_1)^{\alpha_1}-(a_1^*-a_1)^{\alpha_1}}{\alpha_1}\right)\left\{\frac{(b_2-a_2^*)^{\alpha_2}}{\alpha_2}-\frac{(a_2-a_2^*)^{\alpha_2}}{\alpha_2}-\frac{(b_2-a_2)^{\alpha_2}}{\alpha_2}\right\}$$
$$+\left(\frac{(b_2-a_2^*)^{\alpha_2}-(a_2-a_2^*)^{\alpha_2}}{\alpha_2}\right)\left\{\frac{(b_1-a_1^*)^{\alpha_1}}{\alpha_1}-\frac{(b_1-a_1)^{\alpha_1}}{\alpha_1}+\frac{(a_1^*-a_1)^{\alpha_1}}{\alpha_1}\right\}, \tag{45}$$

hence $\theta=0$.

We have proved that $\delta=0$, in all possible subcases of (II_2).

We have proved that F is a continuous function over $[a,b_1]\times[c,b_2]$. ■

Now we can state:

Theorem 3 *Let $f \in L_{\infty}\left(\prod_{i=1}^{k}[a_i,b_i]\right)$, $\alpha_i>0$, $i=1,...,k \in \mathbb{N}$. Consider the function*

$$F(x_1,...,x_k)=\int_{x_1}^{b_1^*}...\int_{x_k}^{b_k^*}\prod_{i=1}^{k}(t_i-x_i)^{\alpha_i-1}f(t_1,...,t_k)\,dt_1...dt_k, \tag{46}$$

where $a_i \leq x_i \leq b_i^* \leq b_i$, $i=1,...,k$.

Then F is continuous on $\prod_{i=1}^{k}[a_i,b_i^*]$.

Remark 4 *In the setting of Theorem 3: Consider the right multidimensional Riemann-Liouville fractional integral of order $\alpha=(\alpha_1,...,\alpha_k)$, $\alpha_i>0$, $i=1,...,k$:*

$$\left(I_{b^*-}^{\alpha}f\right)(x)=\frac{1}{\prod_{i=1}^{k}\Gamma(\alpha_i)}\int_{x_1}^{b_1^*}...\int_{x_k}^{b_k^*}\prod_{i=1}^{k}(t_i-x_i)^{\alpha_i-1}f(t_1,...,t_k)\,dt_1...dt_k, \tag{47}$$

where $a_i \leq x_i \leq b_i^* \leq b_i$, $i=1,...,k$, where $b^*=(b_1^*,...,b_k^*)$, $x=(x_1,...,x_k)$, Γ is the gamma function.

By Theorem 3 we get that $\left(I_{b^*-}^{\alpha}f\right)$ is a continuous function for every $x \in \prod_{i=1}^{k}[a_i,b_i^*]$.

We notice that

$$\left|\left(I_{b^*-}^{\alpha}f\right)(x)\right| \leq \frac{1}{\prod_{i=1}^{k}\Gamma(\alpha_i)}\left(\int_{x_1}^{b_1^*}...\int_{x_k}^{b_k^*}\prod_{i=1}^{k}(t_i-x_i)^{\alpha_i-1}dt_1...dt_k\right)\|f\|_{\infty} \tag{48}$$

$$=\frac{1}{\prod_{i=1}^{k}\Gamma(\alpha_i)}\prod_{i=1}^{k}\left(\int_{x_i}^{b_i^*}(t_i-x_i)^{\alpha_i-1}dt_i\right)\|f\|_{\infty}=$$

$$\frac{1}{\prod_{i=1}^{k}\Gamma(\alpha_i)}\prod_{i=1}^{k}\frac{(b_i^*-x_i)^{\alpha_i}}{\alpha_i}\|f\|_{\infty}=\left(\prod_{i=1}^{k}\frac{(b_i^*-x_i)^{\alpha_i}}{\Gamma(\alpha_i+1)}\right)\|f\|_{\infty}. \tag{49}$$

That is

$$\left|\left(I_{b^*-}^{\alpha}f\right)(x)\right| \leq \left(\prod_{i=1}^{k}\frac{(b_i^*-x_i)^{\alpha_i}}{\Gamma(\alpha_i+1)}\right)\|f\|_{\infty}. \tag{50}$$

In particular we get

$$\left(I_{b^*-}^{\alpha}f\right)(b^*)=0, \tag{51}$$

and

$$\left\|I_{b^*-}^{\alpha}f\right\|_{\infty,\prod_{i=1}^{k}[a_i,b_i^*]} \leq \left(\prod_{i=1}^{k}\frac{(b_i^*-a_i)^{\alpha_i}}{\Gamma(\alpha_i+1)}\right)\|f\|_{\infty}. \tag{52}$$

That is $I_{b^*-}^{\alpha}f$ is a bounded linear operator, which here is also a positive operator.

References

[1] G. A. Anastassiou, *Fractional representation formulae and right fractional inequalities*, Mathematical and Computer Modelling, 54 (2011), 3098-3115.

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