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Golub-Kahan-Lanczos based preconditioner for least squares problems in overdetermined and underdetermined cases

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Abstract

We present an effective preconditioner for solving least squares problems in full ranked overdetermined and underdetermined cases. The preconditioner, generated from Golub-Kahan-Lanczos method, can approximately replace a few largest singular values by one without altering the rest. This property accelerates the convergence, thereby improves the efficiency of the algorithm for solving the least squares problems with ill-conditioned system matrix which is caused by large singular values. In this paper we focus on the overdetermined and the underdetermined cases.

Key words: Least squares problems; Preconditioner; Lanczos bidiagonalization process; Krylov subspace method; Golub-Kahan-Lanczos method

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1 Introduction

In this paper, we assume that the least squares problems are in the form as

$$\min \|b - Ax\|_2, \tag{1}$$

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where $A_{m\times n}$ is a full-ranked coefficient matrix which is large and sparse.

In the situation that $m = n$, we can obtain an approximate solution by solving the linear system $Ax = b$ and minimize the residual in the sense of 2-norm. The minimal norm residual method, based on the iterative Krylov methods, is a suitable algorithm to obtain the optimal approximation, and full details can be found in [2]. We have superscript T denoted the transposition of a matrix, and use subscript to indicate the size of matrix. The overdetermined cases

$$\min \|b - Ax\|_2, A \in R_{m\times n}, m > n \tag{2}$$

and the underdetermined cases

$$\min \|b - Ax\|_2, A \in R_{m\times n}, m < n \tag{3}$$

are taken into consideration in the following.

In this paper, we take the preconditioner as a left preconditioner in both overdetermined and underdetermined cases. To the overdetermined system (2) in least squares problems, we generally translate the corresponding linear system

$$Ax = b, A \in R_{m\times n}, m > n, \tag{4}$$

into a normal equation by premultipling A^T on both sides. R is the set of real number here and in the following. Similarly, we translate the underdetermined system (3) into a normal equation in the same way in the corresponding linear system

$$Ax = b, A \in R_{m\times n}, m < n. \tag{5}$$

Thereby we have the normal equation in the following form

$$A^T Ax = A^T b. \tag{6}$$

We notice that the coefficient matrix in (6) is symmetric positive definite, so the normal equation can be solved by the CG method[16]. Thanks to previous researchers, many classic methods, such as CGNE [4] and CGLS[3], can be regarded as an extensions of the CG method and solve least squares problems efficiently. Similarly, the LSQR method[7] is an effective method for solving the least squares problems, so does the LSMR method[15].

For the symmetric positive definition (SPD) matrix, we know the convergence of iterative Krylov methods depends on the condition number κ of the coefficient matrix, in other word, the spectral distribution, where $\kappa(A) = \frac{\lambda_{\max}(A)}{\lambda_{\min}(A)}$ with $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denoting the largest and the smallest eigenvalues of A, respectively. To discuss the spectral distribution of $A^T A$ in (6), we give the singular value decomposition of the original coefficient matrix A as follow. Notice that all the matrixes in this paper are full ranked.

We have the singular value decomposition of A in this form

$$A = \hat{U}_{m\times n} D \hat{V}_{n\times n}^T, D = \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{pmatrix}, \tag{7}$$

where $\hat{U}_{m\times n}$ and $\hat{V}_{n\times n}$ are both unitary matrices, σ_i denotes the singular value that $\sigma_1 > \sigma_2 > \cdots > \sigma_n$. From (7), we have

$$A^T A = \hat{V}_{n\times n} D^2 \hat{V}_{n\times n}^T, \tag{8}$$

which can be regarded as the eigenvalue decomposition of the coefficient matrix in the normal equation (6).

If we denote $\Sigma = diag\{\sigma_1^2, \sigma_2^2, \cdots, \sigma_r^2\}$, where $r = \min(m, n)$, it could be easily concluded that the spectral distribution of the coefficient matrix in (6) is Σ. Therefore, the condition numbers of linear systems can be presented as $\kappa(A^T A) = \frac{\sigma_1^2}{\sigma_r^2}$. To accelerate the convergence, thereby improve the algorithm, we expect the condition number to be as small as possible. Therefore, removing the smallest eigenvalue from the spectrum of the coefficient matrix is purpose of the preconditioner. Also, we leave the rest unchanged. Such kind of preconditioners and relevant applications can be located in [8], [9] and [10].

Also, when the property of ill-condition is caused by a few largest eigenvalues, we expect a preconditioner, from the similar point of view, to eliminate the largest eigenvalues from the spectrum in order to accelerate the convergence. A preconditioner formed by Lanczos bidiagonalization is formulated to change the largest singular values to one approximately without altering the others, so that the preconditioner change the corresponding eigenvalues in normal equations. In the ill-conditioned overdetermined case and the ill-conditioned underdetermined case, we utilize the preconditioner to speed up the convergence. To illustrate the effects of the preconditioners proposed in this paper, we utilize two methods to solve a series of the least squares problems. Of course, we divide every experiments into two parts, using preconditioner and not using it.

In the following sections, the process of Lanczos bidiagonalization will be stated in section 2; the preconditioners for solving overdetermined and underdetermined least squares problems (2) (3) will be defined in section 3; numerical examples are demonstrated in section 4; conclusions are presented in section 5 finally.

2 The process of Lanczos bidiagonalization

2.1 Standard Lanczos bidiagonalization

Lanczos biorthogonalization, which can be located in [6] [4], is an important process in methods like LSQR[7], BiCG[11] and BiCGSTAB[12]. A variation of Lanczos biorthogonalization, formed as

$$AV_n = U_{n+1}B, B = \begin{pmatrix} \alpha_1 & & & \\ \beta_2 & \alpha_2 & & \\ & \ddots & \ddots & \\ & & \beta_n & \alpha_n \\ & & & \beta_{n+1} \end{pmatrix}, \tag{9}$$

is denoted as Golub-Kahan-Lanczos method [5] , where V_n and U_{n+1} are both unitary matrices and we assume A is a matrix of size $n \times n$. One characteristic of decomposition (9) is that the lower bidiagonal matrix B shares the same singular values as A's. Furthermore, we have analyzed and concluded in the previous section that the singular values distribution of A directly reflects the spectral distribution of A^TA in problems (6). Hence we expect a preconditioner based on Lanczos bidiagonalization to optimize spectral distributions of system matrices in least squares problems. Some similar preconditioner based on the Golub-Kahan-Lanczos bidiagonalization for square coefficient matrixes has been proposed and applied. For example, inreference[13], the author optimized the spectral distribution of a ill-posed coefficient matrix by a Lanczos-based preconditioner.

However, limited by the dimension of the coefficient matrix in overdetermined and underdetermined cases, the algorithm will break down when maximal number of iteration is greater than both row dimension and column dimension. Therefore, in order to be applied to overdetermined and underdetermined cases, the standard form of Golub-Kahan-Lanczos method requires modification. To extend applications of the Lanczos-based preconditioner, we define variants of the preconditioner which can be utilized in overdetermined cases and underdetermined cases, thereby it is available for least squares problems. At first, we give the standard algorithm for Golub-Kahan-Lanczos method as stated in [5].

Algorithm 1 Standard Golub-Kahan-Lanczos bidiagonalization

1. $\beta_1 = \|b\|_2$, $u_1 = \frac{b}{\beta_1}$, $v_0 = 0$
2. for $i = 1, 2, ..., n$
3. $\quad p_k = A^T u_k - \beta_k v_{k-1}$
4. $\quad \alpha_k = \|p_k\|_2$
5. $\quad v_k = \frac{p_k}{\alpha_k}$
6. $\quad q_k = Av_k - \alpha k u_k$
7. $\quad \beta_{k+1} = \|q_k\|_2$
8. $\quad u_{k+1} = \frac{q_k}{\beta_{k+1}}$

The $\alpha's$ and $\beta's$ generated in the above algorithm are equal to the ones in (9), also rows of V and U in (9) are obtained through Algorithm 1 as v_k and u_k respectively. Therefore, we could establish the Lanczos bidiagonalization form by a series of iterations performed according to Algorithm 1, when the coefficient matrix A is of size $n \times n$.

To define the Lanczos-based preconditioners in overdetermined cases and underdetermined cases, we have to modify algorithm 1, the standard Lanczos bidiagonalization process, in order to accommodate the situations that the coefficient matrices are m-by-n and $m \neq n$.

2.2 Modified Lanczos bidiagonalization

The main distinction between the overdetermined, or underdetermined, determined and square cases is the dimension of the coefficient matrix A. As stated before, the matrix B, generated by Lanczos bidiagonalization, and A in (9) share the same singular value distribution. We limit the steps of Lanczos bidiagonalizaion process under the minimal number between m and n where A is m-by-n. We utilize iterative Krylov subspace methods to solve the linear systems (6), with symmetric positive definite coefficient matrices. Therefore we conclude easily that the rank of B can not exceed the minimum of m and n. Then, a restrictive condition should be added to the corresponding Lanczos bidiagonalization process to terminate it in appropriate number of steps.

Different from (9), We set a termination rule that the maximal iteration in Golub-Kahan-Lanczos bidiagonalization is less or equal to the minimum between the row dimension and the column dimension toensure that the algorithm will terminate in appropriate number of steps. Following this rule, we have the bidiagonalization decomposition of A in overdetermined situation as

$$AV_{n\times n} = U_{m\times(n+1)}B_n, B_n = \begin{pmatrix} \alpha_1 & & & \\ \beta_2 & \alpha_2 & & \\ & \ddots & \ddots & \\ & & \beta_n & \alpha_n \\ & & & \beta_{n+1} \end{pmatrix}, \tag{10}$$

and the bidiagonalization decomposition of A in underdetermined situation as

$$AV_{n\times m} = U_{m\times(m+1)}B_m, B_m = \begin{pmatrix} \alpha_1 & & & \\ \beta_2 & \alpha_2 & & \\ & \ddots & \ddots & \\ & & \beta_m & \alpha_m \\ & & & \beta_{m+1} \end{pmatrix}. \tag{11}$$

Considering the computational cost of the Lanczos bidiagonalization process, we try to avoid bidiagonalizing A completely. The preconditioner, mentioned in the previous section and defined in the next section, is structured for the purpose of changing the largest singular values to one, in order to optimize the condition numbers of normal equation (6). Hence, we stop the Lanczos dibiagonalization process when the current smallest singular value σ_k, generated in the kth step of Lanczos dibiagonalization process, is much smaller than the largest one σ_1. We set a scalar number δ to be the threshold of termination, i.e, terminates when $\sigma_k < \delta\sigma_1$. If the bidiagonalization process stops at the kth step, the bidiagonalization composition is of the form below

$$AV_{n\times k} = U_{m\times(k+1)}B_k, B_k = \begin{pmatrix} \alpha_1 & & & \\ \beta_2 & \alpha_2 & & \\ & \ddots & \ddots & \\ & & \beta_k & \alpha_k \\ & & & \beta_{k+1} \end{pmatrix}. \tag{12}$$

M Rezghi set the scalar number δ as the square root of machine precision in [13] while applying it in ill-conditioned systems derived from blurring images. Since δ is a scalar to judge whether we should terminate the Lanczos bidiagonalization process and the Lanczos bidiagonalization process aims to remove the largest singular values, the choice of δ has different effects in different numerical examples. We will present the influence caused the change of δ under different numerical examples and iterative methods in the section of experiments. In general ill-conditioned systems, we need not to set δ so small and some cases will be presented in the 4th section. Here we add the above two restrictive conditions to standard Lanczos bidiagonalization, then we have modified Lanczos bidiagonalization as following.

Algorithm 2 Modified Lanczos bidiagonalization

1. $\beta_1 = \|b\|_2$, $u_1 = \frac{b}{\beta_1}$, $v_0 = 0$, $r = min\{m, n\}$, δ
2. for $i = 1, 2, ..., r$
3. $\quad p_k = A^T u_k - \beta_k v_{k-1}$
4. $\quad \alpha_k = \|p_k\|_2$
5. $\quad v_k = \frac{p_k}{\alpha_k}$
6. $\quad q_k = Av_k - \alpha k u_k$
7. $\quad \beta_{k+1} = \|q_k\|_2$
8. $\quad u_{k+1} = \frac{q_k}{\beta_{k+1}}$
9. $\quad$ get singular values of B: $\sigma_1, \sigma_2, \cdots, \sigma_i$
10. $\quad$ if $\sigma_i < \delta\sigma_1$, break down.
11. end

In this section, we introduced the standard Lanczos bidiagonalization process in Algorithm 1, and defined the modified Lanczos bidiagonalization process in

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Algorithm 2, which is adapted to the overdetermined and the underdetermined situations. A preconditioner based on modified Lanczos bidiagonalization process will be introduced and defined in the next section.

3 Lanczos-based preconditioner for least squares problems

To solve the least squares problems formed as (2) and (3), we solve the corresponding linear systems (4) and (5) instead by translating them into normal equations (6) respectively. If we have the singular value decompositions of A which are structured as (7), and the singular value distributions are scattered and wide, that is the largest singular value is much greater than the smallest one, thereby the condition number of the normal equation (6) will be terribly greater according to analysis of (8). For the purpose of speeding up the convergence, we expect to optimize, or reduce, the condition number of A^TA. Since the condition number of normal equations (6) could be presented as $\kappa(A^TA) = \frac{\sigma_1^2}{\sigma_r^2}$ where σ_1 and σ_r denote the largest and the smallest singular value of A, enlargement or elimination of the smallest singular values, and decrease or elimination of the largest singular values are both effective methods to reduce the condition number. Deflation-based preconditioners, like the deflation preconditioner and the balancing preconditioner[8, 9, 10], have such characteristics and properties to eliminate smallest eigenvalues of system matrix. We do not pay much attention to the preconditioners based on deflation, but the preconditioners functioned for decreasing, or eliminating, the largest ones are what we concern. In the following, all the preconditioners based on Lanczos bidiagonalization are defined for the overdetermined cases (2) and the underdetermined cases (3).

First we shall discuss the situation of the underdetermined case. In linear system (5), the coefficient matrix A has the singular value decomposition illustrated as (7). We assume a diagonal matrix

$$D_k = diag\{\sigma_1, \sigma_2, \cdots, \sigma_k\},$$

where σ_i with $i = 1, 2, \cdots, k$, denotes the first k largest singular values of A. The Lanczos bidiagonalization process for underdetermined cases within k steps have been proposed as (12).

On the premise that B, which is structured by Lanczos bidiagonalization, shares the same singular values with A, we have the following conclusion that: the B_m derived from (11) has singular value decomposition form as

$$B_m = \tilde{U}_{(m+1)\times(m+1)} \begin{pmatrix} D \\ 0 \end{pmatrix}_{(m+1)\times m} \tilde{V}_{m\times m}^T,$$

where D in the above equation is equal to the one in (7), with $\tilde{U}_{m+1}$ and $\tilde{V}_{m\times m}$ both unitary matrices. Similarly, the B_k derived from (12) has singular value decomposition form as

$$B_k = \tilde{U}_k \begin{pmatrix} D_k \\ 0 \end{pmatrix} \tilde{V}_k^T, \tag{13}$$

where D_k has been defined at the beginning in this section, with $\tilde{U}_k$ and $\tilde{V}_k$ both unitary matrices.

When we consider the underdetermined case (11), some deductions are stated as follow. We use singular value decomposition of B replacing the one in (11) and we have

$$AV_{n\times m} = U_{m\times(m+1)}\tilde{U}_{(m+1)\times(m+1)} \begin{pmatrix} D \\ 0 \end{pmatrix}_{(m+1)\times m} \tilde{V}^T_{m\times m}.$$

The dimension of matrices are denoted as subscripts in previous sections, and now the subscripts will be omitted for simplification. Then we postmultiply $\tilde{V}$ on both sides and we have

$$AV\tilde{V} = U\tilde{U} \begin{pmatrix} D \\ 0 \end{pmatrix},$$

Here we set $\bar{V} = V\tilde{V} = \{\bar{v}_1, \bar{v}_2, \cdots, \bar{v}_m\}$ and $\bar{U} = U\tilde{U} = \{\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_{m+1}\}$. As for equation

$$A\bar{V} = \bar{U} \begin{pmatrix} D \\ 0 \end{pmatrix},$$

we regard it as a singular value decomposition of A, similar to (7), approximately. If we set $\bar{U}_m = \{\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_m\}$, the first m columns of $U\tilde{U}$, we assume that

$$\begin{aligned} \bar{U}_m &= \hat{U} \\ \bar{V} &= \hat{V} \end{aligned}$$

where $\hat{U}$ and $\hat{V}$ are obtained from (7).

Now we focus on the formulation (8). If a matrix is structured as

$$P = \bar{V}D^{-2}\bar{V}^T,$$

then combining with the previous assumption($\bar{V} = \hat{V}$), it gives that

$$\begin{aligned} PA^TA &= \bar{V}D^{-2}\bar{V}^T\hat{V}D^2\hat{V}^T \\ &= \bar{V}I\bar{V}^T \\ &= I. \end{aligned}$$

It seems that we could have obtained solution directly through the application of such a preconditioner P. In view of computation, however, it is inadvisable for the following reasons: 1. the preconditioner P is based on a complete Lanczos bidiagonalization, so this process has expensive computational cost even no less than direct methods.; 2. the $\bar{V}$ is approximately equal to $\hat{V}$ in practical implement, but we give the above deduction just in theory, without the consideration of computational errors. Although we can not utilize the preconditioner P in practical computation, a variant of P based on incomplete Lanczos bidiagonalization is defined as follow to solve underdetermined least squares problems.

Here we construct a preconditioner P which is similar with the one mentioned above with merely replacing B_m(from (11)) by B_k(from (12)). After simple deduction, we have

$$P = \bar{V} \begin{pmatrix} D_k^{-2} & 0 \\ 0 & I_{m-k} \end{pmatrix} \bar{V}^T,$$

where D_k is from (13). We set $\bar{V}_k = V\tilde{V}_k$ is the first k columns of $\bar{V}$, where $\tilde{V}_k$ is obviously the first k columns of $\tilde{V}$. Hence we set $\bar{V} = [\bar{V}_k, \bar{V}_{m-k}]$. Based on the definition of $\bar{V}$, we have

$$I = \bar{V}\bar{V}^T = \bar{V}_k\bar{V}_k^T + \bar{V}_{m-k}\bar{V}_{m-k}^T.$$

Analyzing the above information, it gives that

$$\begin{aligned} P &= \bar{V}_k D_k^{-2} \bar{V}_k^T + \bar{V}_{m-k}\bar{V}_{m-k}^T \\ &= V\tilde{V}_k D_k^{-2}\tilde{V}_k^T V^T + (I_{m\times m} - \bar{V}_k\bar{V}_k^T) \\ &= V(B_k^T B_k)^{-1}V^T + (I_{m\times m} - VV^T). \end{aligned}$$

where V and B_k can both be obtained through Algorithm 2. If we utilize P as a left preconditioner in normal equation (6) for underdetermined cases (5), we have

$$PA^TA = \hat{V} \begin{pmatrix} I_k & 0 \\ 0 & D_{m-k}^2 \end{pmatrix} \hat{V}^T,$$

where $D_{m-k} = diag\{\sigma_{k+1}, \sigma_{k+2}, \cdots, \sigma_m\}$ with σ_i's denoting the $m-k$ smallest singular values.

According to the statement above, we can conclude that the Laczos-based preconditioner has the property to change k largest singular values of coefficient matrix A, or k largest eigenvalues of the system matrix in normal equation (6) in other word, to one without touching the others. The preconditioner is able to optimize the condition number of normal equation (6) when the ill condition is caused by these large singular values. Since $k \ll m$, the computational cost is greatly reduced, so is the computational error. The conclusion, furthermore, is under the premise that the linear system corresponding to least squares problems is underdetermined, so that

$$P_{under} = V(B_k^T B_k)^{-1}V^T + (I_{n\times n} - VV^T) \tag{14}$$

could be used as a left-preconditioner in underdetermined least squares problems. Next we consider the overdetermined cases.

In the overdetermined cases, we construct a Lanczos-based preconditioner that follows the same strategy as stated in the previous subsection. To solve the overdetermined system (4), we solve the normal equation (6) instead to obtain approximate solution. Considering the decomposition form (8) of A^TA, we expect to construct a preconditioner, similar to the underdetermined cases, presented as

$$P = \hat{V} \begin{pmatrix} D_k^{-2} & 0 \\ 0 & I_{n-k} \end{pmatrix} \hat{V}^T.$$

Through an analogical deduction to underdetermined cases, a preconditioner formed as

$$P_{over} = V(B_k^T B_k)^{-1} V^T + (I_{n\times n} - VV^T) \tag{15}$$

can be used as a left-preconditioner in overdetermined least squares problems. B_k and $V_{n\times k}$ can be obtained from Algorithm 2. Furthermore it is not computationally costly because of $k \ll n$.

From the above discussion, we can see that the forms of the Lanczos-based preconditioners in over- and under- determined cases are the same, although we deduced them in separate ways. Also, such a preconditioner for the linear system with a square coeffcient matrix has the same form. Therefore, we can conclude that we deduce the preconditioners, proposed in this paper, from the point of overdetermined and underdetermined cases and ultimately get a result similar to the one in square problems, which has been proposed in [13]. Of course, the result of this paper can also be regarded as the expansion of the application of the Lanczos-based preconditioner into the overdetermined and underdetermined least squares problems. Now we unify the preconditioner as follow

$$P = V(B_k^T B_k)^{-1} V^T + (I - VV^T), \tag{16}$$

which can be used as a left preconditioner in ordinary linear systems, overdetermined least squares problems and underdetermined least squares problems. The relevant numerical experiments are presented in the following section, from which we can see the effects of Lanczos-based preconditioners.

4 Numerical experiments

In this section, we will take a series of numerical examples to present the effect of the Lanczos-based preconditioner in the least squares problems. At first, we introduce two iterative methods as the basic algorithm for solving these underdetermined and overdetermined problems. Here, we choose an old and classic method as the first one for solving the least squares problems. It is the CGLS method[3]. In this method, we first transform the least squares problems into symmetric positive definite(SPD) problems by the normal equations then solve it by the CG method[16]. Integrating the above ideas, we have the CGLS method. Now we present the preconditioned CGLS method algorithm 3, where we just consider the situation of left precondition.

Algorithm 3 Preconditioned CGLS method

1. select x_0 as the initial guess, $r_0 = b - Ax_0$ and P as the preconditioner
2. initialization: we set $\bar{r}_0 = A^T r_0$, $\hat{r}_0 = P\bar{r}_0, f_0 = z_0$
2. for $i = 0, 1, 2, ...$
3. $\quad g_i = Af_i$
4. $\quad \alpha_i = (\hat{r}_i, \bar{r}_i)/\|g_i\|_2^2$
5. $\quad x_{i+1} = x_i + \alpha_i f_i$
6. $\quad r_{i+1} = r_i - \alpha_i g_i$
8. $\quad \bar{r}_{i+1} = A^T r_{i+1}$
9. $\quad \hat{r}_{i+1} = P\bar{r}_{i+1}$
10. $\quad \beta_i = (\hat{r}_{i+1}, \bar{r}_{i+1})/(\hat{r}_i, \bar{r}_i)$
11. $\quad f_{i+1} = \hat{r}_{i+1} + \beta_i f_i$
12. endfor

The second method to solve the least squares problems is the BAGMRES method[14], a variant of the GMRES method[1]. In this method, the least squares problems will be post-multiplied by a matrix B, an arbitrary nonsingular matrix. Now we give the BAGMRES method as Algorithm4.

Ex.	Group and name	id	#rows	#cols	Nonzeros	Problem kind
1	JGD_Forest/TF10	1944	99	107	622	Combinatorial
2	JGD_Forest/TF11	1945	216	236	1607	Combinatorial
3	HB/wm3	277	207	260	2948	Economic
4	Pajek/Sandi_sandi	1520	314	360	613	Bipartite graph
5	Meszaros/refine	1759	29	62	153	Linear programming
6	JGD_margulies/flower_4_1	2155	121	129	386	Combinatorial

Table 1: The structures of six test underdetermined problems

Algorithm 4 BA-GMRES with k restart

1. select x_0 as the initial guess, $r_0 = B(b - Ax_0)$ and $\nu_1 = r_0/\|r_0\|_2$
2. for $i = 1, 2, ..., m$
3. $\quad \omega_i = BA\nu_i$
4. $\quad$ for $j = 1, 2, ..., i$
5. $\qquad h_{j,i} = (\omega_i, \nu_j)$
6. $\qquad \omega_i = \omega_i - h_{j,i}\nu_j$
7. $\quad$ endfor
8. $\quad h_{i+1,i} = \|\omega_i\|_2$
9. $\quad \nu_{i+1} = \omega_i/h_{i+1,i}$
10. $\quad$ Compute y_m to minimize $\|\hat{r}_i\|_2 = \|\|\hat{r_0}\|_2 e_1 - \overline{H}_i y\|_2$
11. $\quad$ if $\|r_i\|_2 < \tau$
12. $\qquad x_i = x_0 + [\nu_1, ..., \nu_i]y_i$
13. $\qquad$ stop
14. $\quad$ endif
15. endfor
16. set $x_0 = x_k$ and return to line 2 until convergence

In the following numerical experiments, the examples all come from practical applications from [17].

All the required information about the underdetermined and overdetermined cases is contained in Table 1 and Table 2 respectively. They both consist of group, number of rows, columns and nonzero elements and the type of problem of each example.

In the next two subsections, we solve the above 12 problems by the PCGLS method and the BAGMRES method combined with the Lanczos-based preconditioners. Then we change the scalar δ, involving the termination rule of the modified Lanczos bidiagonalization, and show its influence on the iterative process. Because the preconditioner is designed to modify the singular values, the distributions of singular values under different scalar δ's will be presented as well.

Ex.	Group and name	id	#rows	#cols	Nonzeros	Problem kind
7	HB/abb313	5	313	176	1557	Least squares
8	JGD_margulies/cat_ears_3_1	2151	204	181	542	Combinatorial
9	JGD_margulies/cat_ears_4_1	2153	377	313	938	Combinatorial
10	JGD_margulies/flower_5_1	2157	211	201	602	Combinatorial
11	JGD_margulies/flower_7_1	2159	463	393	1178	Combinatorial
12	Pajek/Cities	1457	55	46	1342	Weighted bipartite graph

Table 2: The structures of six test overdetermined problems

4.1 The acceleration of iterative processes

To discuss the acceleration of iterative processes, we refer to the PCGLS method and the BAGMRES method in [14, 4]. For the BAGMRES method, we have the following relation between the initial residual and the one from the kth iteration in underdetermined cases,

$$\|Br_k\|_2 = \|CA^T r_k\| \leq 2(\frac{\sigma_1 - \sigma_m}{\sigma_1 + \sigma_m})^k \|Br_0\|_2, \tag{17}$$

where C is a nonsingular matrix, $\kappa(C)$ is the condition number of matrix C and σ's denote the singular values of BA. And we have the relation between r_0 and r_k as

$$\|Br_k\|_2 = \|CA^T r_k\| \leq 2\sqrt{\kappa(C)}(\frac{\sigma_1 - \sigma_n}{\sigma_1 + \sigma_n})^k \|Br_0\|_2, \tag{18}$$

where C is a nonsingular matrix, $\kappa(C)$ is the condition number of matrix C and σ's denote the singular values of BA. More information of the above conclusion can be found in [14]. Now we give the convergence analysis of the PCGLS method, that is

$$\|e_k\|_A \leq 2(\frac{\sigma_1 - \sigma_r}{\sigma_1 + \sigma_r})^k \|e_0\|_A, \tag{19}$$

where $r = \min(m, n)$ and σ's denoting the singular values of PA^TA.

Based on equation (17), (18) and (19), it is obvious that we can accelerate the convergence if the gap between the largest singular value of normal equations and the smallest one is narrowed. In this paper, the Lanczos-based preconditioner is just for resetting the largest singular values to one, which can be regarded as shrink of the singular value distribution. Now, the effect of the Lanczos-based preconditioner in underdetermined cases is shown from Figure 1 to Figure 6.

In the numerical experiments, we set the tolerance $tol = 10^{-12}$, the maximal number of iteration $max_i t = 1000$ and the restarted number in the BAGMRES method $restart = 600$. Furthermore, the scalar δ upon which to terminates the

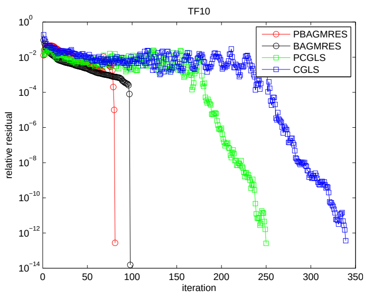

Figure 1: Relative residuals *vs* iterations in `TF10`

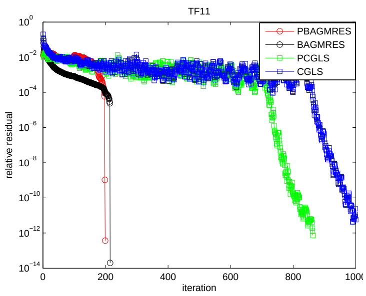

Figure 2: Relative residuals *vs* iterations in `TF11`

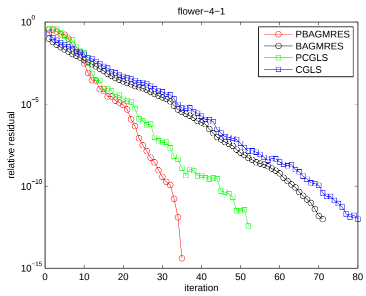

Figure 3: Relative residuals *vs* iterations in `wm3`

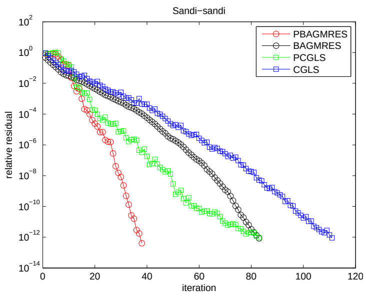

Figure 4: Relative residuals *vs* iterations in `Sandi_sandi`

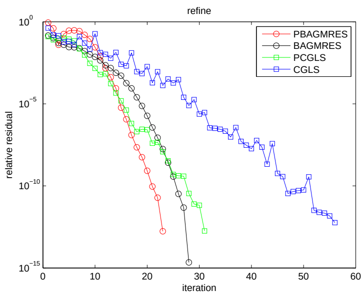

Figure 5: Relative residuals *vs* iterations in `refine`

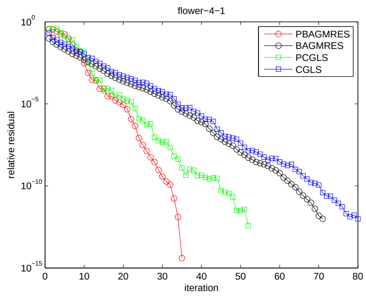

Figure 6: Relative residuals *vs* iterations in `flower_4_1`

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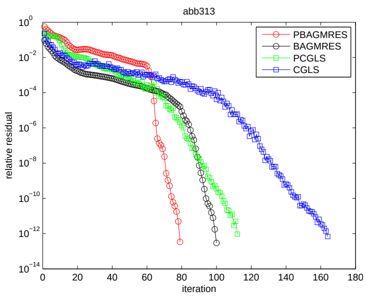

Figure 7: Relative residuals *vs* iterations in `abb313`

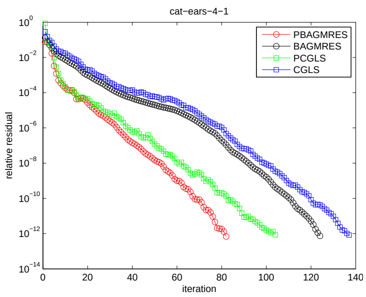

Figure 8: Relative residuals *vs* iterations in `itercat_ears_4_1`

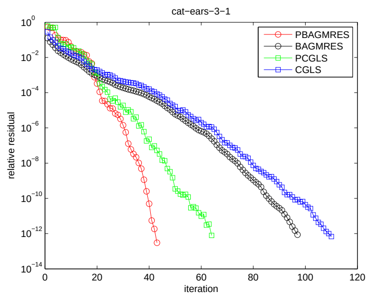

Figure 9: Relative residuals *vs* iterations in `itercat_ears_3_1`

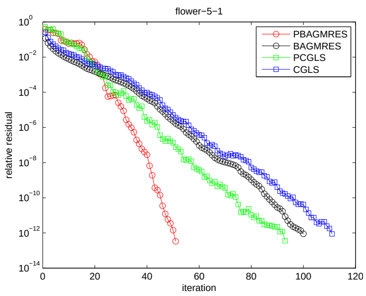

Figure 10: Relative residuals *vs* iterations in `iterflower_5_1`

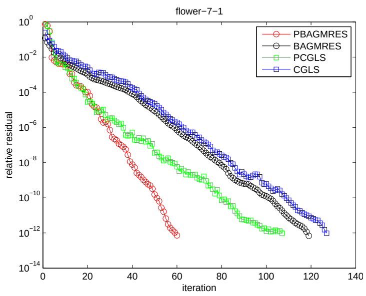

Figure 11: Relative residuals *vs* iterations in `iterflower_7_1`

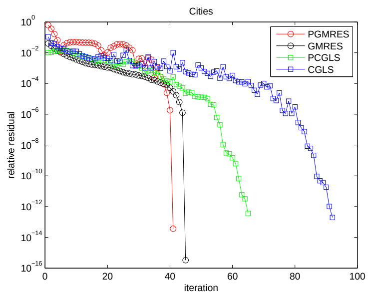

Figure 12: Relative residuals *vs* iterations in `itercities`

Lanczos bidiagonalization process is 0.05 in the test. We set $B = PA^T$ in the preconditioned BAGMRES method and $B = A^T$ in the nonpreconditioned BAGMRES method. From Figure 1 to Figure 6, we can see that both the BAGMRES method and the PCGLS method are accelerated by the Lanczos-based preconditioner as we expected. Next we show the iterative process while solving the overdetermined problems.

Figure 7 to Figure 12 present the results of experiments with the tolerance $tol = 10^{-12}$, the maximal number of iteration $max\_it = 1000$ and the restarted number in the BAGMRES method $restart = 600$. The scalar δ upon which to terminate the Lanczos bidiagonalization process is 0.05 in the test. Similarly, we set $B = PA^T$ in the preconditioned BAGMRES method and $B = A^T$ in the nonpreconditioned BAGMRES method. In Figure 7 to Figure 12, it is obvious that Lanczos-based preconditioners also accelerate the iterative processes in these overdetermined problems, so we think the proconditioner proposed in this paper is helpful to optimize the structure of coefficient matrix thereby accelerate the convergence. Moreover, all the numerical examples here are derived from practical applications. We believe, therefore, the Lanczos preconditioner has the result as

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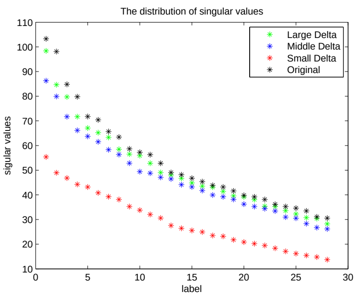

Figure 13: The distribution of singular values in TF10

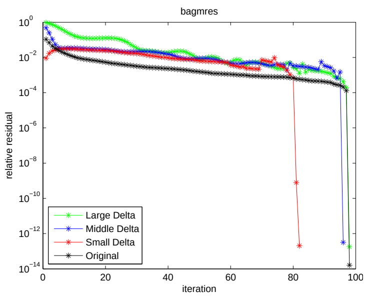

Figure 14: The iterative process of BAGMRE in TF10

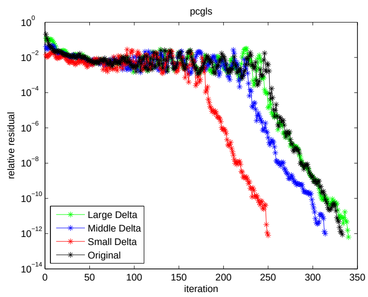

Figure 15: The iterative process of PCGLS in TF10

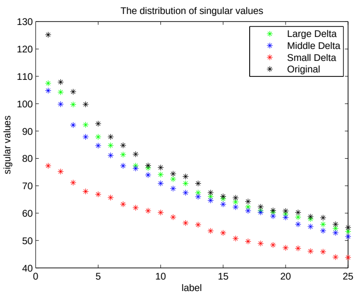

Figure 16: The distribution of singular values in TF11

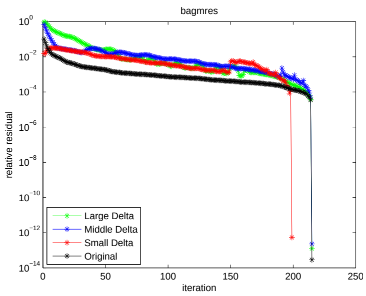

Figure 17: The iterative process of BAGMRE in TF11

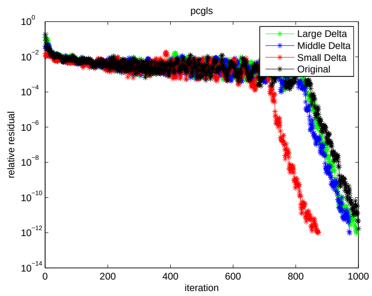

Figure 18: The iterative process of PCGLS in TF11

we expected.

4.2 The influence of the scalar δ

Referring to the illustration above, we have known that the scalar δ is used as a termination rule during the implementation of the Lanczos bidiagonalization process. By the definition of scalar δ, the smaller the δ is, the more large singular values will be replaced by one. It means that we can narrow the distribution of singular values. In the following experiments, we set the scalar δ to three different values and take `TF10` and `TF11` as the underdetermined examples. We test the distributions of the coefficient matrix of corresponding normal equations, the iterative process of the BAGMRES method and the PCGLS method. The results of TF10 and TF11 with varying scalar δ are presented in Figure13-15 and Figure 16-18 respectively.

As for the overdetermined cases, we take `abb313` as the first numerical examples. The singular values distribution and iterative processes of this example are illustrated by Figure 19-21.

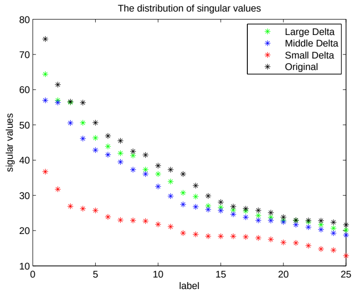

Figure 19: The distribution of singular values in `abb313`

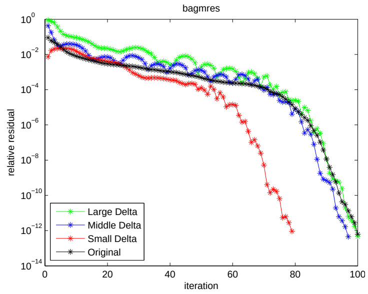

Figure 20: The iterative process of BAGMRE in `abb313`

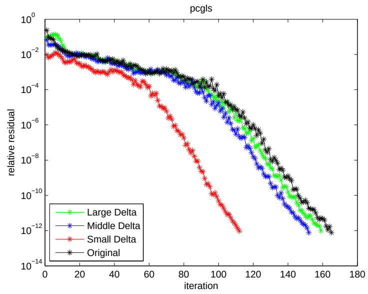

Figure 21: The iterative process of PCGLS in `abb313`

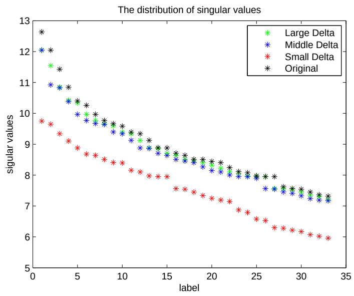

Figure 22: The distribution of singular values in `cat_ears_4_1`

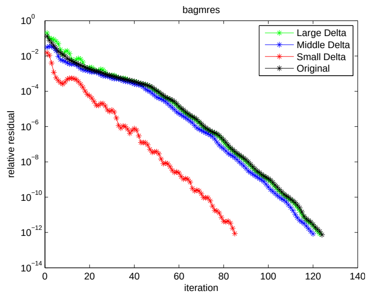

Figure 23: The iterative process of BAGMRE in `cat_ears_4_1`

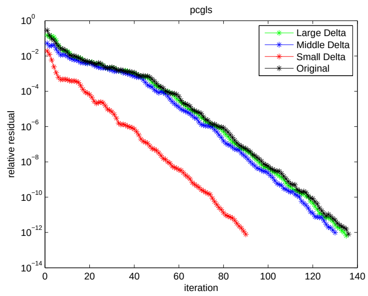

Figure 24: The iterative process of PCGLS in `cat_ears_4_1`

Similarly, the singular value distribution and iterative process regarding to different δ of the example `cat_ears_4_1` are presented in Figure 22-24.

In the above twelve figures, we classify the δ into three classes: the large delta, the middle delta and the small delta. The different δ stand for different preconditioners, upon which we denote the corresponding singular value distribution and iterative process by colorful points and lines. Theoretically, the small delta is able to reset most largest singular values while the large delta reset least largest singular values. Furthermore, required data of the experiments is presented in Table 3 and Table 4, in which k stands for the step of the Lanczos bidiagonalization process, $iter_{BAGMRES}$ and $iter_{PCGLS}$ represent the number of iterations of the BAGMRES method and the PCGLS method, respectively.

From Figure 13, Figure 16, Figure 19 and Figure 22, we can observe that the preconditioner with smaller δ indeed narrows the singular value distribution better than the ones led by larger δ. However, we fail to replace the largest singular values by one, although the improvement has brought us better convergence that is shown in Figure 14-15, Figure 17-18, Figure 20-21 and Figure 23-24. Through Table 3 and Table 4, we can also find that the number of iterations decreases

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Example	TF10		
	k	$iter_{BAGMRES}$	$iter_{PCGLS}$
Nonprec		99	333
$\delta = 0.8$	2	99	340
$\delta = 0.3$	6	97	314
$\delta = 0.05$	22	83	250
Example	TF11		
	k	$iter_{BAGMRES}$	$iter_{PCGLS}$
Nonprec		216	1000
$\delta = 0.8$	2	216	995
$\delta = 0.3$	5	216	972
$\delta = 0.05$	24	200	872

Table 3: The information along with the change of scalar δ in underdetermined cases TF10 and TF11

Example	abb313		
	k	$iter_{BAGMRES}$	$iter_{PCGLS}$
Nonprec		101	165
$\delta = 0.8$	2	101	159
$\delta = 0.3$	5	98	152
$\delta = 0.05$	24	80	112
Example	cat_ears_4_1		
	k	$iter_{BAGMRES}$	$iter_{PCGLS}$
Nonprec		125	136
$\delta = 0.8$	2	124	135
$\delta = 0.3$	5	121	130
$\delta = 0.05$	40	86	90

Table 4: The information along with the change of scalar δ in underdetermined cases abb313 and cat_ears_4_1

obviously while the δ decreasing. In small-scale problem, the Lanczos-based preconditioner can reset the largest singular values closer to one than in large-scale problems, which is easy to testify by a simple numerical deduction. We suppose that the reason why the preconditioner fails to reset the largest singular values to one, just decreasing them instead, is the accumulation of calculation errors and the assumption

$$\bar{U}_m = \hat{U}$$
$$\bar{V} = \hat{V}.$$

From another experiment, the matrix B constructed in Lanczos bidiagonalization

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process has approximately equal singular values with coefficient matrix A. Merely focusing on the numerical value, the gap between the singular values of B and A may be underestimated and even ignored. Nevertheless, the gap will be enlarged when we assume the above equalities without considering the calculation errors. In the above experiments, we can also notice that the different δ influence the iterative process distinctly in different method so the perturbation analysis of the Lanczos-based preconditioner may give us a theoretical explanation of the difference between the theory and the numerical experiment. This supposition is remained to be testified in the future work.

5 Conclusions

To the overdetermined and the underdetermined least squares problems, we choose the BA-GMRES method and the PCGLS method to solve them respectively. Variants of the Lanczos bidiagonalization process are defined in the situation that coefficient matrices are not square, and the algorithm of modified Lanczos bidiagonalization is illustrated as conclusion. When we suffer from the ill-conditioned system matrices, the preconditioners based on modified Lanczos bidiagonalization, P structured for the overdetermined cases and the underdetermined cases respectively, are imposed on iterative Krylov subspace methods to accelerate convergence. Finally we prove our statements with numerical experiments and conclude that the preconditioner defined in this paper is effective to solve least squares problems in overdetermined and underdetermined cases.

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Classical Model of Prandtl's Boundary Layer Theory for Radial Viscous Flow: Application of $(G'/G)-$ Expansion Method

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Abstract

In this paper, the exact closed-form solutions of the Prandtl's boundary layer equation for radial flow models with uniform or vanishing mainstream velocity are derived by using the $(G'/G)-$expansion method. Many new exact solutions are found for the boundary layer equation, which are expressed by the hyperbolic, trigonometric and rational functions. The solutions are valid for all values of the parameter β. It is shown that the $(G'/G)-$expansion method is effective and can be used for many other nonlinear differential equations of mathematical physics.

Keywords: $(G'/G)-$Expansion method; Prandtl's boundary layer equation; Exact solutions

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1 Introduction

Many real world problems in nonlinear science associated with mechanical, structural, aeronautical, ocean, electrical, and control systems can be summarized as solving nonlinear differential equations which arise from mathematically modelling such problems. Therefore, the study of nonlinear differential equations has been an active area of research for the past few years. Investigating integrability and finding exact solutions to such nonlinear differential equations have extensive applications in many scientific fields such as hydrodynamics, fluid dynamics, general relativity, condensed matter physics, solid-state physics, nonlinear optics, neurodynamics, fibre-optic communication and so on. These exact solutions, if reported are helpful for the numerical analyst to verify the complex numerical codes and are also useful in stability analysis for solving special nonlinear models.

In recent years, much attention has been devoted to the development of several powerful and useful methods for finding exact and approximate solutions of nonlinear differential equations. These research methods for solving nonlinear differential equations include the bilinear method and multilinear method [1], classical Lie symmetry method [2], nonclassical Lie group approach [3], Clarkson-Kruskal's direct method [4], deformation mapping method [5], homogenous balance method [6], Weierstrass elliptic function expansion method [7], F-expansion method [8], transformed rational function method [9], auxiliary equation method [10], sine–cosine method [11], tanh-function method [12], Backlund transformation method [13], simplest equation method [14, 15], exponential function rational expansion method [16] and so forth.

Prandtl [17] initiated the concept of a boundary layer in large Reynolds number flows in 1904 and he also showed how the Navier-Stokes equation could be simplified to yield approximate solutions. Prandtl introduced boundary layer theory to understand the flow behavior of a viscous Newtonian fluid near a solid boundary. Prandtl's boundary layer equations arise in various physical models of fluid mechanics. The equations of the boundary layer theory have been the subject of considerable interest, since they represent an important simplification of the original Navier-Stokes equations. These equations arise in the study of steady flows produced by wall jets, free jets and liquid jets, the flow past a stretching plate/surface, flow induced due to a shrinking sheet and so on. These boundary layer equations are usually solved subject to specific boundary conditions depending upon the physical model investigation. Blasius [18] solved the Prandtl's boundary layer equations for a flat moving plate problem and found a power series solution of the model. Falkner and Skan [19] generalized the Blasius problem by considering the boundary layer flow over an wedge inclined at certain angle. Sakiadis [20] studied the boundary layer flow over a continuously moving rigid surface with a constant speed. Crane [21] was the first one who investigated the boundary layer flow due to a stretching surface and developed the exact solutions of boundary layer equations. Gupta and Gupta [22] extended the Crane's work and for the first time introduced the concept of heat transfer with the stretching sheet boundary layer flow. Schlichting [23] was the first to apply the boundary layer theory to the steady flow produced by a free two-dimensional jet emerging into a fluid at rest and solved the resulting ordinary differential equation numerically. Later, Bickley [24] solved the differential equation analytically. The concept of the boundary layer to laminar jets is discussed fully in standard texts on boundary layer theory such as by Schlichting [25] and Rosenhead [26]. More recently, the similarity solution of axisymmetric non-Newtonian wall jet with swirl effects was obtained by Kolar [27]. Naz et al. [28] and Mason [29] studied the general boundary layer equations for two-dimensional and radial flows by using the classical Lie group approach and recently Naz et al. [30] provided the similarity solutions of the Prandtl's boundary layer equations by implementing the non-classical symmetry method.

The $(G'/G)-$expansion method is a powerful mathematical tool for finding exact solutions of certain nonlinear ordinary differential equations. The $(G'/G)-$expansion method was introduced by Wang in [31] for constructing the exact solutions of some nonlinear evolution equations. To express the applicability and effectiveness of the $(G'/G)-$expansion method, further research has been accomplished by a diverse group of researchers (see, for example, papers $[32-34]$). The importance of our present work is to find some new class of exact closed-form solutions of Prandtl's boundary layer equation for radial flow models with constant or uniform main stream velocity by employing the $(G'/G)-$expansion method.

2 Mathematical model

The Prandtl's boundary layer equation, for the stream function $\phi(r,\theta)$, for radial flow with uniform or vanishing mainstream velocity is [26]

$$\frac{1}{r}\frac{\partial\phi}{\partial\theta}\frac{\partial^2\phi}{\partial r\partial\theta}-\frac{1}{r^2}\left(\frac{\partial\phi}{\partial\theta}\right)^2-\frac{1}{r}\frac{\partial\phi}{\partial r}\frac{\partial^2\phi}{\partial\theta^2}-\nu\frac{\partial^3\phi}{\partial\theta^3}=0, \tag{1}$$

where (r,θ) denote the cylindrical polar coordinates and ν is the kinematic viscosity. The velocity components $u(r,\theta)$ and $v(r,\theta)$, in the r and θ directions, are related to stream function $\phi(r,\theta)$ as

$$u(r,\theta)=\frac{1}{r}\frac{\partial\phi}{\partial\theta},\quad v(r,\theta)=-\frac{1}{r}\frac{\partial\phi}{\partial r}. \tag{2}$$

By the use of Lie group theoretic method of infinitesimal transformations [2], the general form of similarity solution for equation (1) is

$$\phi(r,\theta)=r^{2-\beta}H(\xi),\quad \xi=\frac{\theta}{r^\beta}, \tag{3}$$

where β is the constant determined from further conditions and $\xi=\theta/r^\beta$ is the similarity variable. By the substitution of Eq. (3) into Eq. (1), we obtain the third-order nonlinear ordinary differential equation in $H(\xi)$, viz.,

$$\nu\frac{d^3H}{d\xi^3}+(2-\beta)H\frac{d^2H}{d\xi^2}+(2\beta-1)\left(\frac{dH}{d\xi}\right)^2=0. \tag{4}$$

Equation (4) is the general form of Prandtl's boundary layer equation for radial flow of a viscous incompressible fluid. The boundary layer equation is usually solved subject to certain boundary conditions depending upon the particular physical model under investigation. Here, we find the exact closed-form solutions of Eq. (4) using the $(G'/G)-$expansion method. The paper is organised as follows. In Section 3, we provide a brief summary of the $(G'/G)-$expansion method. In Sections 4, we apply this method to solve nonlinear Prandtl's boundary layer equation for radial flow. Finally, some concluding remarks are presented in Section 5.

3 A description of the $(G'/G)-$expansion method

In this section, we present a brief summary of the $(G'/G)-$expansion method for solving nonlinear ordinary differential equations. The essence of the $(G'/G)-$expansion method is given in the following steps:

Step 1: We consider a general form of a nonlinear ordinary differential equation

$$P\left[U(z), \frac{dU}{dz}, \frac{d^2U}{dz^2}, \frac{d^3U}{dz^3}, ...\right] = 0, \tag{5}$$

where U is an unknown function of z and P is a polynomial in U and its various derivatives.

Step 2: According to the $(G'/G)-$expansion method, one assumes that the solution of ODE (5) can be written as a polynomial in (G'/G) as follows:

$$U(z) = \sum_{i=0}^{M} \beta_i \left(\frac{G'}{G}\right)^i, \tag{6}$$

where $G = G(z)$ satisfies the second-order linear ODE with constant coefficients, namely

$$\frac{d^2G}{dz^2} + \lambda\frac{dG}{dz} + \mu G = 0, \tag{7}$$

with β_i $(i = 0, 1, 2, ..., M)$, λ and μ being constants to be determined. The integer M is found by considering the homogenous balance between the highest order derivatives and nonlinear terms appearing in ODE (5).

Step 3: The positive integer M can be accomplished by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in Eq. (5) as follows:

If we define the degree of $U(z)$ as $D[U(z)] = M$, then the degree of other expressions is defined by

$$\begin{aligned} D\left[\frac{d^qU(z)}{dz^q}\right] &= M + q, \\ D\left[U^r\left(\frac{d^qU(z)}{dz^q}\right)^s\right] &= Mr + s(q + M). \end{aligned} \tag{8}$$

Therefore, we can get the value of M in Eq. (6).

Step 4: We substitute Eq. (6) into Eq. (5) and then use ODE (7) to collect all terms with same order of (G'/G) together. The left-hand side of (5) is then converted into polynomial in (G'/G). Now by equating each coefficient of this polynomial to zero, we obtain a system of algebraic equations for β_i, λ and μ.

Step 5: Since the three types of general solutions of Eq. (7) are well known, we substitute the values of β_i and the general solutions of Eq. (7) into Eq. (6) and obtain three types of solutions of the ODE (5).

4 Application of the $(G'/G)-$expansion method

In this section, we employ the $(G'/G)-$expansion method to obtain solutions of Prandtl's boundary layer Eq. (4).

We assume that the solutions of Eq. (4) are of the form

$$H(\xi) = \sum_{i=0}^{M} A_i \left(\frac{G'(\xi)}{G(\xi)} \right)^i, \tag{9}$$

where $G(\xi)$ satisfies the second-order linear ODE with constant coefficients, viz.,

$$\frac{d^2G}{d\xi^2} + \lambda \frac{dG}{d\xi} + \mu G = 0 \tag{10}$$

with λ and μ being constants.

The balancing procedure yields $M = 1$, so the solution of the ODE (4) is of the form

$$H(\xi) = A_0 + A_1 \left(\frac{G'(\xi)}{G(\xi)} \right). \tag{11}$$

Now substituting Eq. (11) into Eq. (4), making use of the ODE (10), collecting all terms with same powers of (G'/G) and equating each coefficient to zero, yields the following system of algebraic equations:

$$\begin{aligned}
2\beta A_1^2\mu^2 - \beta A_0A_1\lambda\mu - A_1\lambda^2\mu\nu + 2A_0A_1\lambda\mu - 2A_1\mu^2\nu - A_1^2\mu^2 &= 0,\\
3\beta A_1^2\lambda\mu - \beta A_0A_1\lambda^2 - 2\beta A_0A_1\mu - A_1\lambda^3\nu + 2A_0A_1\lambda^2 - 8A_1\lambda\mu\nu + 4A_0A_1\mu &= 0,\\
\beta A_1^2\lambda^2 - 3\beta A_0A_1\lambda + 2\beta A_1^2\mu - 7A_1\lambda^2\nu + A_1^2\lambda^2 + 6A_0A_1\lambda - 8A_1\mu\nu + 2A_1^2\mu &= 0,\\
\beta A_1^2\lambda - 2\beta A_0A_1 - 12A_1\lambda\nu + 4A_1^2\lambda + 4A_0A_1 &= 0,\\
3A_1^2 - 6A_1\nu &= 0.
\end{aligned}$$

Solving this system of algebraic equations, with the aid of Mathematica, we obtain

$$\lambda = 2\sqrt{\mu}, \quad A_0 = \lambda\nu, \quad A_1 = 2\nu. \tag{12}$$

Substituting these values of A_0, A_1 and the corresponding solution of ODE (4) into Eq. (11), we obtain the following three types of solutions of Eq. (1):

Case 1: When $\lambda^2 - 4\mu > 0$

For this case we obtain the hyperbolic function solution given by

$$H(\xi) = \lambda\nu + 2\nu\left(-\frac{\lambda}{2} + \delta\frac{C_1\sinh(\delta\xi) + C_2\cosh(\delta\xi)}{C_1\cosh(\delta\xi) + C_2\sinh(\delta\xi)}\right), \tag{13}$$

where $\delta = \frac{1}{2}\sqrt{\lambda^2 - 4\mu}$, C_1 and C_2 are arbitrary constants.

Reverting back to the original variables (r, θ), the corresponding stream function is given by

$$\phi(r,\theta) = r^{2-\beta}\left[\lambda\nu + 2\nu\left(-\frac{\lambda}{2} + \delta\frac{C_1\sinh\left(\delta\frac{\theta}{r^\beta}\right) + C_2\cosh\left(\delta\frac{\theta}{r^\beta}\right)}{C_1\cosh\left(\delta\frac{\theta}{r^\beta}\right) + C_2\sinh\left(\delta\frac{\theta}{r^\beta}\right)}\right)\right]. \tag{14}$$

Case 2: When $\lambda^2 - 4\mu < 0$

Here we obtain the trigonometric function solution

$$H(\xi) = \lambda\nu + 2\nu\left(-\frac{\lambda}{2} + \epsilon\frac{-C_1\sin(\epsilon\xi) + C_2\cos(\delta\xi)}{C_1\cos(\epsilon\xi) + C_2\sin(\epsilon\xi)}\right), \tag{15}$$

where $\epsilon = \frac{1}{2}\sqrt{4\mu - \lambda^2}$, C_1 and C_2 are arbitrary constants. The corresponding stream function is given as

$$\phi(r,\theta) = r^{2-\beta}\left[\lambda\nu + 2\nu\left(-\frac{\lambda}{2} + \epsilon\frac{-C_1\sin\left(\epsilon\frac{\theta}{r^\beta}\right) + C_2\cos\left(\epsilon\frac{\theta}{r^\beta}\right)}{C_1\cos\left(\epsilon\frac{\theta}{r^\beta}\right) + C_2\sin\left(\epsilon\frac{\theta}{r^\beta}\right)}\right)\right]. \tag{16}$$

Case 3: When $\lambda^2 - 4\mu = 0$

For this case we obtain the rational function solution

$$H(\xi) = \lambda\nu + 2\nu\left(-\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2\xi}\right). \tag{17}$$

In the form of stream function, the solution is expressed as

$$\phi(r,\theta) = r^{2-\beta}\left[\lambda\nu + 2\nu\left(-\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2\frac{\theta}{r^\beta}}\right)\right], \tag{18}$$

where C_1 and C_2 are arbitrary constants.

5 Concluding remarks

We have employed the (G'/G)-expansion method for obtaining exact closed-form solutions of the well-known Prandtl's boundary layer equation for radial flow models with uniform main stream velocity. The advantage of this method is that in this method, there is no need to apply the initial and boundary conditions at the outset. This method yields a general solution with free parameters which can be identified by the specific conditions. Also the general solutions obtained by (G'/G)-expansion method are not approximate solutions. Prandtl's boundary layer equations arise in various physical models of fluid dynamics and thus the exact solutions obtained maybe very useful and significant for the explanation of some practical physical models dealing with Prandtl's boundary layer theory.

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On properties of meromorphic solutions for a certain q-difference Painlevé equation

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Abstract

The main purpose of this paper is to investigate some properties on transcendental meromorphic solutions of a certain q-difference Painevé equation

$$f(qz) + f(z) + f(\frac{z}{q}) = \frac{az+b}{f(z)} + c,$$

where a, b and c are complex constants such that $|a| + |b| \neq 0$. We obtain some results on the value distribution of $f(z)$ and $\Delta_q f(z) := f(qz) - f(z)$, and the non-existence of rational solutions, which extend some earlier results by Qi and Yang, Chen et al.

Key words: q-difference equation; solution; zero order.
Mathematical Subject Classification (2010): 39A 50, 30D 35.

1 Introduction and Main Results

In this paper, we shall assume that readers are familiar with the basic theorems and the standard notations of the Nevanlinna value distribution theory of meromorphic functions such as $m(r, f)$, $N(r, f)$, $T(r, f)$, $\cdots$, (see Hayman [12], Yang [19] and Yi and Yang [20]). We also use $S(r, f)$ to denote any quantity satisfying $S(r, f) = o(T(r, f))$ for all r on a set $F \subset [1, +\infty$ of logarithmic density 1, where the logarithmic density of a set F is defined by

$$\limsup_{r\to\infty} \frac{1}{\log r} \int_{[1,r]\cap F} \frac{1}{t} dt.$$

Throughout this paper, the set F of logarithmic density 1 can be not necessarily the same at each occurrence.

A century ago, Painlevé and his colleagues [15] classified all equations of Painlevé type of the form

$$w''(z) = F(z; w; w'),$$

where F is rational in w and w' and (locally) analytic in z. They singled out a list of 50 equations, six of which could not be integrated in terms of known functions. These equations are now known as the differential Painlevé equations. The first two of these equations are P_I and P_{II}:

$$w'' = 6w^2 + z, \quad w'' = 2w^2 + zw + \alpha,$$

where α is a complex constant.

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Differential Painlevé equations have been an important research subject in the field of the Mathematics and the Physics since the beginning of last century. They occur in many physical situations—-plasma physics, statistical mechanics, nonlinear waves, and so on. Therefore, Painlevé equations have attracted much interest as the reduction of solution equations which are solvable by inverse scattering transformations, and so on.

In the past 22 years, the discrete Painlevé equations have become important research problems (see [7]). For example, the discrete P_I equation can be expressed by

$$y_{n+1} + y_{n-1} = \frac{an+b}{y_n} + c,$$

and the discrete P_{II} equation can be expressed by

$$y_{n+1} + y_{n-1} = \frac{(an+b)y_n + c}{1 - y_n^2},$$

where a, b, c are real constants, $n \in \mathbb{N}$.

In 2006-2007, Halburd and Korhonen used the analogues of Nevanlinna value distribution theory to single out the difference Painlevé I and II equations from the following form

$$w(z+1) + w(z-1) = R(z, w), \tag{1}$$

where $R(z, w)$ is rational in w and meromorphic in z (see [9, 10, 11]). They obtained that if (1) has an admissible meromorphic solution of finite order, then either w satisfies a difference Riccati equation, or (1) can be transformed by a linear change in w to some difference equations, which include the difference Painlevé I equation

$$w(z+1) + w(z-1) = \frac{az+b}{w(z)} + c, \tag{2}$$

and the difference Painlevé II equation

$$w(z+1) + w(z-1) = \frac{(az+b)w(z) + c}{1 - w(z)^2}, \tag{3}$$

where a, b, c are complex constants.

Chen et al [4, 5, 16] studied some properties of finite order transcendental meromorphic solutions of (2)-(3), and obtained a lot of interesting results.

Recently, there were lots of results about q-difference operators, q-difference equations, and so on (see [2, 6, 8, 18, 21, 22]), by applying the analogue of Logarithmic Derivative Lemma on q-difference operators, which was firstly established by Barnett, Halburd, Korhonen and Morgan [1] in 2007. By comparing these results of differences and q-differences, we find that the usual shift $f(z+c)$ of a meromorphic function are replaced by the q-difference $f(qz)$, and the difference $\Delta_c f = f(z+c) - f(z)$ are replaced by $\Delta_q f(z) = f(qz) - f(z)$, $q \in \mathbb{C} \backslash \{0, 1\}$.

In 2015, Qi and Yang [17] investigated the following equations

$$f(qz) + f(\frac{z}{q}) = \frac{az+b}{f(z)} + c, \tag{4}$$

$$f(qz) + f(\frac{z}{q}) = \frac{(az+b)f(z) + c}{1 - f(z)^2}, \tag{5}$$

which can be seen as q-difference analogues of (2) and (3), and obtained some theorems as follows.

Theorem 1.1 *[17, Theorem 1.1]. Let $f(z)$ be a transcendental meromorphic solution with zero order of equation (4), and a, b, c be three constants such that a, b cannot vanish simultaneously. Then,*

(i) $f(z)$ has infinitely many poles.

(ii) If $a \neq 0$, then $f(z)$ has infinitely many finite values.

(iii) If $a = 0$ and $f(z)$ takes a finite value A finitely often, then A is a solution of $2z^2 - cz - b = 0$.

Theorem 1.2 *[17, Theorem 1.2]. Let a, b, c and $|q| \neq 1$ be four constants, (i) if $a \neq 0$, then equation (4) has no rational solution;*

(ii) if $a = 0$, then the rational solutions of the equation (4) must satisfy $f(z) = B + \frac{P(z)}{Q(z)}$, where $P(z)$ and $Q(z)$ are relatively prime polynomials and satisfy $\deg P < \deg Q$ and $2z^2 - cz - b = 0$.

Theorem 1.3 *[17, Theorem 1.3]. Let a, b, c be constants with $ac \neq 0$, and $f(z)$ be a transcendental meromorphic solution with zero order of equation (5). Then $f(z)$ has infinitely many poles and infinitely many finite values.*

Inspired by the above results, we further investigate some properties of transcendental meromorphic solutions of the q-difference Painlevé equation

$$f(qz) + f(z) + f(\frac{z}{q}) = \frac{az + b}{f(z)} + c, \tag{6}$$

which is different from (4) and (5) to some extent, and obtain the following theorems.

Theorem 1.4 *Let a, b, c be complex constants such that $|a| + |b| \neq 0$, and $f(z)$ be a zero-order transcendental meromorphic solution of the q-difference Painlevé equation (6).*

(i) If $a \neq 0$, $p(z)$ is a polynomial of degree $k (\geq 0)$ and $|q| \neq 1$, then $f(z) - p(z)$ has infinitely many zeros; if $a = 0$, then the Borel exceptional values of $f(z)$ can only come from the set $E = \{z | \ 3z^2 - cz - b = 0\}$;

(ii) $f(z)$ and $\Delta_q f(z)$ have infinitely many poles, where $\Delta_q f(z) = f(qz) - f(z)$.

Theorem 1.5 *Let a, b, c be complex constants such that $|a| + |b| \neq 0$.*

(i) If $a \neq 0$, then (6) has no rational solution.

(ii) If $a = 0$, then (6) has a nonzero constant solution $f(z) = B$, where B satisfies $3B^2 - cB - b = 0$. Furthermore, if $c^2 + 12b = 0$, then (6) has no nonconstant rational solution.

2 Some Lemmas

To prove our results, we require some lemmas as follows.

Lemma 2.1 *[14, Theorem 2.5] Let $f(z)$ be a transcendental meromorphic solution of order zero of a q-difference equation of the form*

$$U_q(z, f) P_q(z, f) = Q_q(z, f),$$

where $U_q(z, f), P_q(z, f)$ and $Q_q(z, f)$ are q-difference polynomials such that the total degree $\deg U_q(z, f) = n$ in $f(z)$ and its q-shifts, whereas $\deg Q_q(z, f) \leq n$. Moreover, we assume that $U_q(z, f)$ contains just one term of maximal total degree in $f(z)$ and its q-shifts. Then

$$m(r, P_q(z, f)) = o(T(r, f)),$$

on a set of logarithmic density 1.

Remark 2.1 *The above lemma can be called see as a type of a q-difference analogue of Clunie lemma, recently proved by Barnett et al.; see [1, Theorem 2.1].*

Remark 2.2 *Here, a q-difference polynomial of $f(z)$ for $q \in \mathbb{C}\backslash\{0,1\}$ is a polynomial in $f(z)$ and finitely many of its q-shifts $f(qz), \ldots, f(q^n z)$ with meromorphic coefficients in the sense that their Nevanlinna characteristic functions are $o(T(r,f))$ on a set of logarithmic density 1.*

Lemma 2.2 *[1, Theorem 2.5] Let $f(z)$ be a nonconstant zero-order meromorphic solution of $P_q(z,f)=0$, where $P_q(z,f)$ is a q-difference polynomial in $f(z)$. If $P_q(z,a) \not\equiv 0$ for slowly moving target $a(z)$, then*

$$m(r, \frac{1}{f-a}) = o(T(r,f))$$

on a set of logarithmic density 1.

Lemma 2.3 *[21, Theorem 1.1 and 1.3] Let $f(z)$ be a nonconstant zero-order meromorphic function and $q \in \mathbb{C} \setminus \{0\}$. Then*

$$T(r, f(qz)) = (1+o(1))T(r,f), \qquad N(r, f(qz)) = (1+o(1))N(r,f),$$

on a set of lower logarithmic density 1.

Lemma 2.4 *(Valiron-Mohon'ko) ([13]). Let $f(z)$ be a meromorphic function. Then for all irreducible rational functions in $f(z)$,*

$$R(z, f(z)) = \frac{\sum_{i=0}^{m} a_i(z) f(z)^i}{\sum_{j=0}^{n} b_j(z) f(z)^j},$$

with meromorphic coefficients $a_i(z), b_j(z)$, the characteristic function of $R(z, f(z))$ satisfies that

$$T(r, R(z, f(z))) = dT(r,f) + O(\Psi(r)),$$

where $d = \max\{m, n\}$ and $\Psi(r) = \max_{i,j}\{T(r, a_i), T(r, b_j)\}$.

3 Proof of Theorem 1.4

Suppose that $f(z)$ is a zero-order transcendental meromorphic solution of (6).

(i) If$a \neq 0$, and $p(z)$ is a polynomial of degree $k (\geq 0)$. Let $p(z) = a_k z^k + \cdots + a_1 z + a_0$. Let $g(z) = f(z) - p(z)$. Substituting $f(z) = g(z) + p(z)$ into equation (6), we have

$$g(qz) + p(qz) + g(z) + p(z) + g(\frac{z}{q}) + p(\frac{z}{q}) = \frac{az+b}{g(z)+p(z)} + c.$$

It follows that

$$\begin{aligned} P_q(z,g) := & \left[g(qz) + p(qz) + g(z) + p(z) + g(\frac{z}{q}) + p(\frac{z}{q}) \right] [g(z) + p(z)] \\ & - (az+b) - c\,[g(z)+p(z)] = 0. \end{aligned} \tag{7}$$

From (7), we have

$$P_q(z,0) = \left[p(qz) + p(z) + p(\frac{z}{q}) \right] p(z) - (az+b) - cp(z). \tag{8}$$

If $p(z) \equiv 0$, then $P_q(z,0) = -(az+b) \not\equiv 0$. If $k = 0$ and $p(z) = a_0 \equiv \alpha \in \mathbb{C} \setminus \{0\}$, then $P_q(z,0) = 3\alpha^2 - (az+b) - c\alpha \not\equiv 0$. If $k \geq 1$ and a_k is a nonzero constant, then, we have from (8) that

$$P_q(z,0) = \left[p(qz) + p(z) + p(\frac{z}{q})\right] p(z) - (az+b) - cp(z) = (q^k + 1 + \frac{1}{q^k})a_k^2 z^{2k} + \cdots. \quad (9)$$

Since $|q| \neq 1$, we have $q^k + 1 + \frac{1}{q^k} \neq 0$, then $P_q(z,0) \not\equiv 0$. Thus, we have by Lemma 2.2 that

$$m(r, \frac{1}{g}) = S(r,g).$$

Then, we get

$$N\left(r, \frac{1}{f-p}\right) = N\left(r, \frac{1}{g}\right) = T(r,g) + S(r,g) = T(r,f) + S(r,f). \quad (10)$$

Since $f(z)$ is transcendental, $f(z) - p(z)$ has infinitely many zeros.

If $a = 0$ and $p(z) = \beta \notin E$, then we have

$$P_q(z,0) = 3\beta^2 - c\beta - b \not\equiv 0.$$

Set $g(z) = f(z) - \beta$, by using the same argument as above, we can obtain $N(r, \frac{1}{f-\beta}) = T(r,f) + S(r,f)$.. Therefore, we can obtain that the Borel exceptional values of $f(z)$ can only come from the set $E = \{z | 3z^2 - cz - b = 0\}$.

(ii) From (6), we have

$$f(z)\left[f(qz) + f(z) + f(\frac{z}{q})\right] = az + b + cf(z). \quad (11)$$

It follows from Lemma 2.1 and (11) that

$$m\left(r, f(qz) + f(z) + f(\frac{z}{q})\right) = S(r,f). \quad (12)$$

By applying Lemma 2.4 for (6), we have

$$T\left(r, f(qz) + f(z) + f(\frac{z}{q})\right) = T(r,f) + S(r,f). \quad (13)$$

And by Lemma 2.3 we get

$$\begin{aligned} N\left(r, f(qz) + f(z) + f(\frac{z}{q})\right) &\leq N(r, f(qz)) + N(r, f(z)) + N\left(r, f(\frac{z}{q})\right) \\ &= 3(1 + o(1))N(r,f) \end{aligned} \quad (14)$$

on a set of lower logarithmic density 1. Thus, by combining (12)-(14), we have

$$T(r,f) \leq 3(1 + o(1))N(r,f) + S(r,f). \quad (15)$$

Since $f(z)$ is transcendental, $f(z)$ has infinitely many poles.

Next, we prove that $\Delta_q f(z)$ has infinitely many poles. Set $z = qw$, then we can rewrite (6) as the form

$$f(q^2 w) + f(qw) + f(w) = \frac{aqw + b}{f(qw)} + c. \quad (16)$$

Then it follows from (16) that

$$f(qw)\left[f(q^2w)+f(qw)+f(w)\right]=aqw+b+cf(qw). \tag{17}$$

Since $\Delta_q f(w)=f(qw)-f(w)$, we have $f(qw)=\Delta_q f(w)+f(w)$ and $f(q^2w)=\Delta_q f(qw)+\Delta_q f(w)+f(w)$. Substituting them into (17), we get

$$[\Delta_q f(w)+f(w)]\,[\Delta_q f(qw)+2\Delta_q f(w)+3f(w)]=(aqw+b)+c\,[\Delta_q f(w)+f(w)]\,,$$

i.e.,

$$\begin{aligned}-3f(w)^2=&\,[\Delta_q f(qw)+5\Delta_q f(w)-c]\,f(w)-(aqw+b)\\&+[\Delta_q f(qw)+2\Delta_q f(w)-c]\,\Delta_q f(w).\end{aligned} \tag{18}$$

Since $f(z)$ is a zero-order transcendental meromorphic function and $z=qw$, by Lemma 2.3, we get that $f(w)$ is of zero order. Thus, by Lemma 2.3 again, we have that $f(w),\Delta_q f(w),\Delta_q f(qw)$ are of zero-order. Then by Lemma 2.3 again, we have

$$N(r,\Delta_q f(qw))\le N(r,\Delta_q f(w))+S(r,f). \tag{19}$$

Thus, from (18) and (19) we have

$$\begin{aligned}2N(r,f(w))=&N\left(r,[\Delta_q f(qw)+3\Delta_q f(w)-c]\,f(w)-(aqw+b)\right.\\&+[\Delta_q f(qw)+\Delta_q f(w)-c]\,\Delta_q f(w)\\ \le&N(r,f(w))+5N(r,\Delta_q f(w))+O(\log r)+S(r,f).\end{aligned}$$

That is,

$$N(r,f(w))\le 5N(r,\Delta_q f(w))+S(r,f). \tag{20}$$

Then, it follows from (15) and (20) that

$$T(r,f(w))\le 15N(r,\Delta_q f(w))+S(r,f). \tag{21}$$

Since $f(z)$ is transcendental, that is, $f(w)$ is transcendental, we have from (21) that $\Delta_q f(w)$ has infinitely many poles, that is, $\Delta_q f(z)$ has infinitely many poles.

Therefore, we complete the proof of Theorem 1.4.

4 Proof of Theorem 1.5

Sppose that $f(z)$ is a nonzero rational solution of (6), and has poles $z_1,z_2,\dots,z_k$. Then, we let

$$\frac{\alpha_{is_i}}{(z-z_i)^{s_i}}+\cdots+\frac{\alpha_{is_1}}{(z-z_i)},\quad i=1,2,\dots,k$$

be the principal parts of $f(z)$ at z_i respectively, where $\alpha_{is_i}\ne 0,\dots,\alpha_{is_1}$ are constants, Thus, we can write $f(z)$ as the following form

$$f(z)=\sum_{i=1}^{k}\left(\frac{\alpha_{is_i}}{(z-z_i)^{s_i}}+\cdots+\frac{\alpha_{is_1}}{(z-z_i)}\right)+\beta_0+\beta_1 z+\cdots+\beta_m z^m, \tag{22}$$

where $\beta_0,\beta_1,\dots,\beta_m$ are constants.

Next, we affirm that $\beta_m = \cdots = \beta_1 = 0$. Suppose that $\beta_m \neq 0 (m \geq 1)$. For sufficiently large z, by (22), we have

$$f(z) = \beta_m z^m (1 + o(1)), \tag{23}$$
$$f(qz) = \beta_m q^m z^m (1 + o(1)), \tag{24}$$
$$f(\frac{z}{q}) = \beta_m q^{-m} z^m (1 + o(1)). \tag{25}$$

By (6), we have

$$\left[f(qz) + f(z) + f(\frac{z}{q}) \right] f(z) = az + b + cf(z). \tag{26}$$

Substituting (23)-(25) into (26), we have

$$(1 + q^m + q^{-m})\beta_m^2 z^{2m}(1 + o(1)) = az + b + c\beta_m z^m (1 + o(1)).$$

Since $|q| \neq 1$, we have $1 + q^m + q^{-m} \neq 0$. And since $\beta_m \neq 0$, we can see the above equation is a contradiction for sufficiently large z. Hence we have $\beta_1 = \cdots = \beta_m = 0$.

(i) Suppose that $a \neq 0$. If $\beta_0 \neq 0$, then for sufficiently large z, by (23)-(25), we have

$$f(qz) = f(z) = f(\frac{z}{q}) = \beta_0 + o(1). \tag{27}$$

Substituting (27) into (26), we conclude that

$$(3\beta_0 + o(1))(\beta_0 + o(1)) = az + b + c(\beta_0 + o(1)),$$

which is a contradiction to the assumption that $a \neq 0$. Thus, $\beta_0 = 0$. Then we have $\beta_0 = \beta_1 = \cdots = \beta_m = 0$. Thus, $f(z)$ can be rewritten by (22) as

$$f(z) = \frac{P(z)}{R(z)}, \tag{28}$$

where

$$P(z) = pz^k + p_{k-1}z^{k-1} + \cdots + p_0, \quad R(z) = rz^t + r_{t-1}z^{t-1} + \cdots + r_0, \tag{29}$$

where $p, p_{k-1}, \ldots, p_0$ and $r, r_{t-1}, \ldots, r_0$ are constants such that $pr \neq 0$ and $k < t$. Then substituting (28) into (6), we have

$$\begin{aligned} &P(qz)P(z)R(z)R(\frac{z}{q}) + P(z)^2 R(qz)R(\frac{z}{q}) + P(\frac{z}{q})P(z)R(qz)R(z) \\ &= (az + b)R(qz)R(z)^2 R(\frac{z}{q}) + cP(z)R(qz)R(z)R(\frac{z}{q}). \end{aligned} \tag{30}$$

Then since $k < t$, we can see that the degree of the left side of (30) does not exceed $2k + 2t$, and the degree of the right side of (30) is equal to $1 + 4t$ by $a \neq 0$. Thus, we can get a contradiction. Therefore, we have that (6) has no nonzero rational solution when $a \neq 0$.

(ii) Suppose that $a = 0$. If $f(z) = B$ is a nonzero constant solution of (6), we can easily get from (6) that B satisfies $3B^2 - cB - b = 0$. Now, we prove that (6) has no rational solution if $a = 0$ and $c^2 + 12b = 0$. Suppose that $f(z)$ is a nonconstant rational solution of (6). Since $\beta_m = 0 (m \geq 1)$, $f(z)$ can be rewritten as the form (28), where

$P(z)$ and $R(z)$ satisfy (29) with $k \leq t$. Suppose that $k < t$. Substituting (28) into (6), we have

$$\begin{aligned} &P(qz)P(z)R(z)R(\frac{z}{q}) + P(z)^2R(qz)R(\frac{z}{q}) + P(\frac{z}{q})P(z)R(qz)R(z) \\ &\qquad = bR(qz)R(z)^2R(\frac{z}{q}) + cP(z)R(qz)R(z)R(\frac{z}{q}). \end{aligned} \tag{31}$$

If $k < t$, then it follows from (31) that there exists only one term $bR(qz)R(z)^2R(\frac{z}{q})$ with maximal degree, which is a contradiction. Thus, we have $k = t$. Then, it follows by (29) and (30) that

$$\begin{aligned} &\frac{pq^kz^k + p_{k-1}q^{k-1}z^{k-1} + \cdots + p_0}{rq^tz^t + r_{t-1}q^{t-1}z^{t-1} + \cdots + r_0} + \frac{pz^z + p_{k-1}z^{k-1} + \cdots + p_0}{rz^t + r_{t-1}z^{t-1} + \cdots + r_0} \\ &\quad + \frac{pq^{-k}z^k + p_{k-1}q^{-(k-1)}z^{k-1} + \cdots + p_0}{rq^{-t}z^t + r_{t-1}q^{-(t-1)}z^{t-1} + \cdots + r_0} \\ &\quad = \frac{b(rz^t + r_{t-1}z^{t-1} + \cdots + r_0)}{pz^k + p_{k-1}z^{k-1} + \cdots + p_0} + c. \end{aligned} \tag{32}$$

Then it follows from (32) that

$$3B^2 - cB - b = 0,$$

as $z \to \infty$, where $B = \frac{p}{r} \neq 0$. Therefore, $f(z)$ can be rewritten as

$$f(z) = B + \frac{G(z)}{H(z)}, \tag{33}$$

where $G(z)$ and $H(z)$ are relatively prime polynomials and satisfy $\deg G(z) = \mu < \deg H(z) = \nu$, B is a constant satisfying $3B^2 - cB - b = 0$. Denote

$$G(z) = \xi z^\mu + \xi_{\mu-1}z^{\mu-1} + \cdots + \xi_0, \quad H(z) = \eta z^\nu + \eta_{\nu-1}z^{\nu-1} + \cdots + \eta_0, \tag{34}$$

where $\xi, \xi_{\mu-1}, \ldots, p_0$ and $\eta, \eta_{\nu-1}, \ldots, \eta_0$ are constants such that $\xi\eta \neq 0$. Substituting (34) into (6) and noting $3B^2 - cB - b = 0$, we have

$$\begin{aligned} &(4B - c)G(z)H(qz)H(z)H(\frac{z}{q}) + BG(qz)H(z)^2H(\frac{z}{q}) + BG(\frac{z}{q})H(z)^2H(qz) \\ &= -G(qz)G(z)H(z)H(\frac{z}{q}) - G(z)^2H(qz)H(\frac{z}{q}) - G(\frac{z}{q})G(z)H(z)H(qz). \end{aligned} \tag{35}$$

By observing the coefficients and degrees of all terms of the above equation, and combining with $\nu > \mu$, we have that the term with maximal degree of (35) is

$$\left[(4B - c) + Bq^{\mu-\nu} + Bq^{\nu-\mu}\right]\xi\eta^3z^{\mu+3\nu}.$$

Since $3B^2 - cB - b = 0$ and $c^2 + 12b = 0$, we have $B = \frac{c}{6}$. And by $|q| \neq 1$, we can get that $(4B - c) + Bq^{\mu-\nu} + Bq^{\nu-\mu} \neq 0$. In fact, if $(4B - c) + Bq^{\mu-\nu} + Bq^{\nu-\mu} = 0$, *i.e.*

$$B = \frac{c}{4 + q^{\mu-\nu} + q^{\nu-\mu}}.$$

Then, we have

$$\frac{c}{4 + q^{\mu-\nu} + q^{\nu-\mu}} = \frac{c}{6}.$$

By solving the above equation, we get $|q| = 1$, a contradiction. Thus, (35) is a contradiction for sufficiently large z. Therefore, if $a = 0$ and $c^2 + 12b = 0$, then (6) has no nonconstant rational solution.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors drafted the manuscript, read and approved the final manuscript.

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New approximation of fixed points of asymptotically demicontractive mappings in arbitrary Banach spaces

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Abstract

We prove necessary and sufficient conditions for the strong convergence of the modified two-step iteration process to the fixed point of asymptotically demicontractive mappings in real Banach spaces.

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1 Introduction

Let K be a nonempty subset of a real Banach space X and X^* be its dual space. We denote by J the normalized duality mapping from X into 2^{X^*} defined by

$$J(x) = \{f^* \in X^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. If X is strictly convex, then J is single-valued. In the sequel, we shall denote the single-valued duality mapping by j.

Let $T : K \to K$ be a mapping.

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Definition 1.1. T is called a *k-strictly asymptotically pseudo-contractive mapping* with sequence $\{k_n\} \subset [1, \infty)$, $\lim_{n\to\infty} k_n = 1$ if for all x, $y \in K$ there exists $j(x-y) \in J(x-y)$ and a constant $k \in [0, 1)$ such that

$$\begin{aligned} &\langle (I - T^n)x - (I - T^n)y, j(x-y)\rangle \\ &\geq \frac{1}{2}(1-k)\left\|(I-T^n)x - (I-T^n)y\right\|^2 - \frac{1}{2}(k_n^2 - 1)\left\|x-y\right\|^2 \end{aligned} \tag{1.1}$$

for all $n \in \mathbb{N}$.

Definition 1.2. T is called an *asymptotically demicontractive mapping* with sequence $\{k_n\} \subset [0, \infty)$, $\lim_{n\to\infty} k_n = 1$ if $F(T) = \{x \in K : Tx = x\} \neq \emptyset$ and for all $x \in K$ and $x^* \in F(T)$, there exists $k \in [0, 1)$ and $j(x - x^*) \in J(x - x^*)$ such that

$$\langle x - T^n x, j(x - x^*)\rangle \geq \frac{1}{2}(1-k)\left\|x - T^n x\right\|^2 - \frac{1}{2}(k_n^2 - 1)\left\|x - x^*\right\|^2 \tag{1.2}$$

for all $n \in \mathbb{N}$.

Definition 1.3. $T : K \to K$ is called *uniformly L-Lipschitizian* if there exists a constant $L > 0$ such that

$$\left\|T^n x - T^n y\right\| \leq L\left\|x - y\right\|, \tag{1.3}$$

for all $x, y \in K$ and $n \in \mathbb{N}$.

The classes of k-strictly asymptotically pseudo-contractive and asymptotically demicontractive mappings are introduced by Liu [3]. It is easy to see that a k-strictly asymptotically pseudo-contrative mapping with a non-empty fixed point set $F(T)$ is asymptotically demicontractive.

In Hilbert spaces, it is shown in [3] that (1.1) and (1.2) are equivalent to the following inequalities:

$$\left\|T^n x - T^n y\right\| \leq k_n^2\left\|x-y\right\|^2 + k\left\|(I-T^n)x - (I-T^n)y\right\|^2$$

and

$$\left\|T^n x - T^n y\right\|^2 \leq k_n^2\left\|x-y\right\|^2 + \left\|x - T^n x\right\|^2,$$

respectively.

By using the modified Mann iteration method [4] introduced by Schu [7], Liu [3] proved a convergence theorem for the iterative approximation of fixed points of k-strictly asymptotically pseudo-contractive mappings and asymptotically demicontractive mappings in Hilbert spaces.

Osilike [6] extended the results of Liu [3] about the iterative approximation of fixed points of k-strictly asymptotically demicontractive mappings from Hilbert spaces to much more general real q-uniformly smooth Banach spaces, $1 < q < \infty$ and specifically proved the following results.

Theorem 1.4. *Let $q > 1$ and X be a real q-uniformly smooth Banach space. Let K be a closed convex and bounded subset of X and $T : K \to K$ a completely continuous uniformly L-Lipschitizian asymptotically demicontractive mapping with a sequence $k_n \subset [1, \infty)$ satisfying $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences satisfying the conditions*

(i) $0 \leq \alpha_n, \beta_n \leq 1,\ n \geq 1$;

(ii) $0 < \epsilon \leq c_q \alpha_n^{q-1}(1 + L\beta_n)^q \leq \frac{1}{2}q(1-k)(1+L)^{-(q-2)} - \epsilon$ *for all* $n \geq 1$ *and for some* $\epsilon > 0$; *and*

(iii) $\sum_{n=1}^{\infty} \beta_n < \infty$.

Then the sequence $\{x_n\}$ generated from an arbitrary $x_1 \in K$ by

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n T^n x_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n, \quad n \geq 1 \end{cases}$$

converges strongly to a fixed point of T.

Remark 1.5. For Hilbert spaces, in Theorem 1.4, if we put $q = 2$, $c_q = 1$ and $\beta_n = 0$, then Theorems 1 and 2 of Liu [3] follow.

Recently Chidume and Măruşter [1] made a comprehensive and very useful survey on the main convergence properties of the modified Mann iteration method for the demicontractive mappings.

The purpose of this work is to prove necessary and sufficient conditions for the strong convergence of the modified two-step iteration process to the fixed point of asymptotically demicontractive mappings in real Banach spaces. Our results extend and improve the results of Igbokwe [2], Liu [3], Moore and Nnoli [5].

2 Main results

The following results are useful:

Lemma 2.1. ([8]) *For all $\varrho, \varsigma \in X$ and $j(\varrho + \varsigma) \in J(\varrho + \varsigma)$,*

$$\|\varrho + \varsigma\|^2 \leq \|\varrho\|^2 + 2\mathrm{Re}\,\langle \varsigma, j(\varrho + \varsigma)\rangle .$$

Lemma 2.2. ([2]) *Let X be a normed space and K be a nonempty convex subset of X. Let $T : K \to K$ be uniformly L-Lipschitzian mapping and let $\{t_n\}$ and $\{\beta_n\}$ be the sequences in $[0, 1]$. For arbitrary $\varrho_1 \in K$, generate the sequence $\{\varrho_n\}$ by*

$$\begin{cases} \varrho_{n+1} = (1-t_n)\varrho_n + t_n T^n \varsigma_n, \\ \varsigma_n = (1-\beta_n)\varrho_n + \beta_n T^n \varrho_n, \quad n \geq 1. \end{cases}$$

Then

$$\|\varrho_n - T\varrho_n\| \leq \|\varrho_n - T^n\varrho_n\| + L(1+L)^2 \left\|\varrho_{n-1} - T^{n-1}\varrho_{n-1}\right\| . \tag{2.1}$$

We now prove our main results.

Lemma 2.3. *Let X be a real Banach space and K be a nonempty convex subset of X. Let $T : K \to K$ be an uniformly $L-$Lipschitzian asymptotically demicontractive mapping with a sequence $\{k_n\} \subset [1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$. For arbitrary $\varrho_1 \in K$, generate the sequence $\{\varrho_n\}$ by*

$$\begin{cases} \varrho_{n+1} = (1-t_n)\varrho_n + t_n T^n \varsigma_n, \\ \varsigma_n = (1-\beta_n)\varrho_n + \beta_n T^n \varrho_n, \quad n \geq 1, \end{cases} \tag{2.2}$$

where $\{t_n\}$ and $\{\beta_n\}$ are the sequences in $[0, 1]$ satisfying

(i) $\sum_{n=1}^{\infty} t_n = \infty$,

(ii) $\lim_{n\to\infty} t_n = 0 = \lim_{n\to\infty} \beta_n$.

Then (a) the sequence $\{\varrho_n\}$ is bounded,

(b) $\liminf_{n\to\infty} \|\varrho_{n+1} - T^n \varrho_{n+1}\| = 0$,

(c) $\liminf_{n\to\infty} \|\varrho_n - T^n \varrho_n\| = 0$,

(d) $\liminf_{n\to\infty} \|\varrho_n - T\varrho_n\| = 0$.

Proof. Since T is asymptotically demicontractive, then

$$\langle \varrho - T^n \varrho, j(\varrho - \varrho^*)\rangle \geq \frac{1}{2}(1-k)\|\varrho - T^n\varrho\|^2 - \frac{1}{2}(k_n^2 - 1)\|\varrho - \varrho^*\|^2$$

and hence

$$\|\varrho - T^n \varrho\| \leq \sqrt{\frac{(2\|\varrho - T^n\varrho\| + (k_n^2 - 1)\|\varrho - \varrho^*\|)\|\varrho - \varrho^*\|}{1-k}}.$$

Therefore, by the triangle inequality,

$$\|\varrho - \varrho^*\| \leqslant \|T^n\varrho - \varrho^*\| + \sqrt{\frac{(2\|\varrho - T^n\varrho\| + (k_n^2 - 1)\|\varrho - \varrho^*\|)\|\varrho - \varrho^*\|}{1-k}}. \tag{2.3}$$

Now we shall prove that

$$\liminf_{n\to\infty} \|\varrho_{n+1} - T^n \varrho_{n+1}\| = 0.$$

If $\varrho_n = T\varrho_n$ for all $n \geqslant m$ for some $m \in \mathbb{N}$, then (2.3) trivially holds, as we have

$$\begin{aligned} \|\varrho_{n+1} - T^n \varrho_{n+1}\| &= \|\varrho_{n+1} - T^n T \varrho_{n+1}\| = \left\|\varrho_{n+1} - T^{n+1}\varrho_{n+1}\right\| \\ &= 0 \end{aligned}$$

for all $n \geq m$.

Suppose now that there exists the smallest positive integer n_0 such that $\varrho_{n_0} \neq T\varrho_{n_0}$. Put

$$\begin{aligned} a_0 := &\|T^{n_0}\varrho_{n_0} - \varrho^*\| \\ &+ \sqrt{\frac{\left(2\|\varrho_{n_0} - T^{n_0}\varrho_{n_0}\| + (k_{n_0}^2 - 1)\|\varrho_{n_0} - \varrho^*\|\right)\|\varrho_{n_0} - \varrho^*\|}{1-k}} + 1. \end{aligned}$$

Then clearly

$$\|\varrho_{n_0} - \varrho^*\| \leq a_0. \tag{2.4}$$

To prove that $\liminf_{n\to\infty} \|\varrho_{n+1} - T^n\varrho_{n+1}\| = 0$, we shall assume, to the contrary, that $\liminf_{n\to\infty} \|\varrho_{n+1} - T^n\varrho_{n+1}\| = 2\delta > 0$. Then there exists $n_0' \in \mathbb{N}$ such that $\|\varrho_{n+1} - T^n\varrho_{n+1}\| \geqslant \delta$ for all $n \geq n_0'$.

Also, by $\lim_{n\to\infty} k_n = 1$ and (ii), we may suppose that

$$\begin{aligned} t_n &\leq \min\left\{\frac{1}{1+2L}, \frac{(1-k)\delta^2}{24(1+L)(1+2L)a_0^2}\right\}, \\ \beta_n &\leq \min\left\{\frac{1}{1+L}, \frac{(1-k)\delta^2}{24L(1+L)a_0^2}\right\}, \\ k_n^2 - 1 &\leq \frac{(1-k)\delta^2}{24a_0^2} \end{aligned} \tag{2.5}$$

for all $n \geq n_0'$.

We now show that the sequence $\{\varrho_n\}$ is bounded. By induction we shall show that

$$\|\varrho_n - \varrho^*\| \leqslant a_0 \tag{2.6}$$

for all $n \geq n_0'$.

It is clear that (2.6) holds for $n = n_0$. Assume it is true for some $n > N := \max\{n_0, n_0'\}$, that is, $\|\varrho_n - \varrho^*\| \leq a_0$ for some $n \geq N$. Then

$$\begin{aligned} \|\varrho_n - T^n\varrho_n\| &\leq \|\varrho_n - \varrho^*\| + \|T^n\varrho_n - \varrho^*\| \\ &\leq (1+L)\|\varrho_n - \varrho^*\| \\ &\leq (1+L)a_0, \end{aligned}$$

$$\begin{aligned} \|\varsigma_n - \varrho^*\| &= \|(1-\beta_n)\varrho_n + \beta_n T^n\varrho_n - \varrho^*\| \\ &= \|\varrho_n - \varrho^* - \beta_n(\varrho_n - T^n\varrho_n)\| \\ &\leq \|\varrho_n - \varrho^*\| + \beta_n\|\varrho_n - T^n\varrho_n\| \\ &\leq a_0 + (1+L)a_0\beta_n \\ &\leq 2a_0, \end{aligned}$$

$$\begin{aligned} \|\varrho_n - T^n\varsigma_n\| &\leq \|\varrho_n - \varrho^*\| + \|T^n\varsigma_n - \varrho^*\| \\ &\leq \|\varrho_n - \varrho^*\| + L\|\varsigma_n - \varrho^*\| \\ &\leq (1+2L)a_0, \end{aligned}$$

and

$$\begin{aligned} \|\varrho_{n+1} - \varrho^*\| &= \|(1-t_n)\varrho_n + t_n T^n\varsigma_n - \varrho^*\| \\ &= \|\varrho_n - \varrho^* - t_n(\varrho_n - T^n\varsigma_n)\| \\ &\leq \|\varrho_n - \varrho^*\| + t_n\|\varrho_n - T^n\varsigma_n\| \\ &\leq a_0 + (1+2L)a_0 t_n \\ &\leq 2a_0. \end{aligned} \tag{2.7}$$

On the other hand, by Lemma 2.1,

$$\begin{aligned}\|\varrho_{n+1}-\varrho^*\|^2 &= \|(1-t_n)\varrho_n + t_n T^n \varsigma_n - \varrho^*\|^2 \\ &= \|\varrho_n - \varrho^* - t_n(\varrho_n - T^n\varsigma_n)\|^2 \\ &\leq \|\varrho_n-\varrho^*\|^2 - 2t_n \left\langle \varrho_n - T^n\varsigma_n, j(\varrho_{n+1}-\varrho^*)\right\rangle \\ &= \|\varrho_n-\varrho^*\|^2 - 2t_n \left\langle \varrho_{n+1} - T^n\varrho_{n+1}, j(\varrho_{n+1}-\varrho^*)\right\rangle \\ &\quad + 2t_n \left\langle T^n\varsigma_n - T^n\varrho_{n+1}, j(\varrho_{n+1}-\varrho^*)\right\rangle + 2t_n \left\langle \varrho_{n+1}-\varrho_n, j(\varrho_{n+1}-\varrho^*)\right\rangle.\end{aligned}$$

Since T is asymptotically demicontractive mapping, we obtain

$$\begin{aligned}\|\varrho_{n+1}-\varrho^*\|^2 \leq& \|\varrho_n-\varrho^*\|^2 - (1-k)t_n\|\varrho_{n+1}-T^n\varrho_{n+1}\|^2 \\ &+ \left(k_n^2-1\right)t_n\|\varrho_{n+1}-\varrho^*\|^2 \\ &+ 2(1+L)t_n\|\varrho_{n+1}-\varrho_n\|\|\varrho_{n+1}-\varrho^*\| \\ &+ 2Lt_n\|\varsigma_n-\varrho_n\|\|\varrho_{n+1}-\varrho^*\|.\end{aligned} \tag{2.8}$$

Consider the following estimates,

$$\begin{aligned}\|\varsigma_n-\varrho_n\| &= \|(1-\beta_n)\varrho_n + \beta_n T^n\varrho_n - \varrho_n\| \\ &= \beta_n\|\varrho_n - T^n\varrho_n\| \\ &\leq (1+L)a_0t_n,\end{aligned}$$

and

$$\begin{aligned}\|\varrho_{n+1}-\varrho_n\| &= \|(1-t_n)\varrho_n + t_nT^n\varsigma_n - \varrho_n\| \\ &= t_n\|\varrho_n - T^n\varsigma_n\| \\ &\leq (1+2L)a_0t_n,\end{aligned}$$

so that (2.8), takes the form

$$\begin{aligned}\|\varrho_{n+1}-\varrho^*\|^2 \leq& \|\varrho_n-\varrho^*\|^2 - (1-k)t_n\|\varrho_{n+1}-T^n\varrho_{n+1}\|^2 \\ &+ \left(k_n^2-1\right)t_n\|\varrho_{n+1}-\varrho^*\|^2 \\ &+ 2(1+L)(1+2L)a_0t_n^2\|\varrho_{n+1}-\varrho^*\| \\ &+ 2L(1+L)a_0t_n\beta_n\|\varrho_{n+1}-\varrho^*\|.\end{aligned}$$

Then, by (2.5),

$$\begin{aligned}\|\varrho_{n+1}-\varrho^*\|^2 \leq& \|\varrho_n-\varrho^*\|^2 - (1-k)\delta^2t_n \\ &+ 4a_0^2\left[\left(k_n^2-1\right) + (1+L)(1+2L)t_n + L(1+L)\beta_n\right]t_n \\ \leq& \|\varrho_n-\varrho^*\|^2 - (1-k)\delta^2t_n + \frac{1}{2}(1-k)\delta^2t_n\end{aligned}$$

and hence

$$\|\varrho_{n+1}-\varrho^*\|^2 \leqslant \|\varrho_n-\varrho^*\|^2 - \frac{1}{2}(1-k)\delta^2t_n. \tag{2.9}$$

Thus $\|\varrho_{n+1}-\varrho^*\| \leq \|\varrho_n-\varrho^*\| \leq a_0$ and so we proved (2.6). Therefore, we proved (a).

From (2.9) we have that for every $r > N$,

$$\begin{aligned}\frac{1}{2}(1-k)\delta^2 \sum_{n=N}^{r} t_n &\leq \sum_{n=N}^{r} (\|\varrho_n - \varrho^*\|^2 - \|\varrho_{n+1} - \varrho^*\|^2) \\ &\leq \|\varrho_N - \varrho^*\|^2 .\end{aligned}$$

Hence we have $\sum_{n=1}^{\infty} t_n < \infty$, a contradiction with the condition (i). Therefore, our assumption $\delta > 0$ was wrong. Thus

$$\liminf_{n\to\infty} \|\varrho_{n+1} - T^n \varrho_{n+1}\| = 0. \tag{2.10}$$

Therefore, we proved (b).

Now according to Lemma 2.1, substituting $\varrho = u + v$ and $\varsigma = -v$, we obtain

$$\|u+v\|^2 \geq \|u\|^2 + 2\langle v, j(u)\rangle ,$$

which is mainly due to Igbokwe [2].

By (2.2) we have

$$\begin{aligned}\|\varrho_{n+1} - T^n \varrho_{n+1}\|^2 &= \|(1-t_n)\varrho_n + t_n T^n \varsigma_n - T^n \varrho_{n+1}\|^2 \\ &= \|\varrho_n - T^n \varrho_n - t_n (\varrho_n - T^n \varsigma_n) - (T^n \varrho_{n+1} - T^n \varrho_n)\|^2 .\end{aligned} \tag{2.11}$$

Then by (2.11) we get

$$\begin{aligned}\|\varrho_{n+1} - T^n \varrho_{n+1}\|^2 \geq{}& \|\varrho_n - T^n \varrho_n\|^2 \\ &- 2\langle t_n (\varrho_n - T^n \varsigma_n) + (T^n \varrho_{n+1} - T^n \varrho_n), j(\varrho_n - T^n \varrho_n)\rangle .\end{aligned}$$

Thus

$$\begin{aligned}\|\varrho_n - T^n \varrho_n\|^2 &\leq \|\varrho_{n+1} - T^n \varrho_{n+1}\|^2 \\ &\quad + 2\langle t_n (\varrho_n - T^n \varsigma_n) + (T^n \varrho_{n+1} - T^n \varrho_n), j(\varrho_n - T^n \varrho_n)\rangle \\ &\leq \|\varrho_{n+1} - T^n \varrho_{n+1}\|^2 \\ &\quad + 2\|t_n (\varrho_n - T^n \varsigma_n) + (T^n \varrho_{n+1} - T^n \varrho_n)\| \|\varrho_n - T^n \varrho_n\| .\end{aligned} \tag{2.12}$$

Further,

$$\begin{aligned}\|t_n (\varrho_n - T^n \varsigma_n) + (T^n \varrho_{n+1} - T^n \varrho_n)\| &\leq t_n \|\varrho_n - T^n \varsigma_n\| + \|T^n \varrho_{n+1} - T^n \varrho_n\| \\ &\leq (1+2L)a_0 t_n + L\|\varrho_{n+1} - \varrho_n\| \\ &\leq (1+2L)a_0 t_n + L(1+2L)a_0 t_n \\ &= (1+L)(1+2L)a_0 t_n.\end{aligned}$$

Therefore, from (2.12), we get

$$\|\varrho_n - T^n \varrho_n\|^2 \leq \|\varrho_{n+1} - T^n \varrho_{n+1}\|^2 + 2(1+L)^2(1+2L)a_0^2 t_n. \tag{2.13}$$

From (2.13), (ii) and (b),

$$\liminf_{n\to\infty} \|\varrho_n - T^n \varrho_n\| = 0. \tag{2.14}$$

Thus we proved (c).

At last, from (2.14) and Lemma 2.2, we obtain (d). This completes the proof. □

Theorem 2.4. *Let X be a real Banach space and K be a nonempty convex subset of X. Let $T : K \to K$ be an uniformly L-Lipschitzian asymptotically demicontractive mapping with a sequence $\{k_n\} \subset [1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$. For arbitrary $\varrho_1 \in K$, generate the sequence $\{\varrho_n\}$ by*

$$\begin{cases} \varrho_{n+1} = (1 - t_n)\varrho_n + t_n T^n \varsigma_n, \\ \varsigma_n = (1 - \beta_n)\varrho_n + \beta_n T^n \varrho_n, \quad n \geq 1, \end{cases}$$

where $\{t_n\}$ and $\{\beta_n\}$ are the sequences in $[0, 1]$ satisfying

(i) $\sum_{n=1}^{\infty} t_n = \infty$,

(ii) $\lim_{n\to\infty} t_n = 0 = \lim_{n\to\infty} \beta_n$.

If T is completely continuos, then $\{\varrho_n\}$ converges strongly to some fixed point of T in K.

Proof. From Lemma 2.3, $\liminf_{n\to\infty} \|\varrho_n - T\varrho_n\| = 0$. Therefore, there exists a subsequence $\{\varrho_{n_j}\}$ of $\{\varrho_n\}$ such that $\lim_{j\to\infty} \left\|\varrho_{n_j} - T\varrho_{n_j}\right\| = 0$. Since $\{\varrho_{n_j}\}$ is bounded and T is completely continuous, then $\{T\varrho_{n_j}\}$ has a subsequence $\{T\varrho_{n_{j_k}}\}$, which converges strongly. Hence $\{\varrho_{n_{j_k}}\}$ converges strongly. Let $\lim_{k\to\infty} \varrho_{n_{j_k}} = p$. Then $\lim_{k\to\infty} T\varrho_{n_{j_k}} = Tp$. Thus we have $\lim_{k\to\infty} \left\|\varrho_{n_{j_k}} - T\varrho_{n_{j_k}}\right\| = \|p - Tp\| = 0$. Hence $p \in F(T)$. From (2.9) and Lemma 2.3 it follows that $\lim_{n\to\infty} \|\varrho_n - p\| = 0$. This completes the proof. □

Remark 2.5. 1. We generalize the results of Liu [3] from Hilbert spaces to more general Banach spaces. Moreover the boundedness assumption on the subset K is removed.

2. One can see that, with $\sum_{n=1}^{\infty} t_n = \infty$, the condition $\sum_{n=1}^{\infty} t_n^2 < \infty$ is not always true. Let us take $t_n = \frac{1}{\sqrt{n}}$. Then obviously $\sum_{n=1}^{\infty} t_n = \infty$, but $\sum_{n=1}^{\infty} t_n^2 = \infty$. Hence the results of Igbokwe [2] are need to be improve.

3. We improve the results of Moore and Nnoli [5] by removing the conditions like $\liminf_{n\to\infty} d(\varrho_n, F(T)) = 0$.

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VISCOSITY APPROXIMATION OF SOLUTIONS OF FIXED POINT AND VARIATIONAL INCLUSION PROBLEMS

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Abstract. In this paper, fixed point and variational inclusion problems are investigated based on a viscosity approximation method. Strong convergence theorems are established without the aid of metric projections in the framework of Hilbert spaces.

Keywords: maximal monotone operator; fixed point; proximal point algorithm; zero point.

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1. Introduction

A very common problem in diverse areas of mathematics and physical sciences consist of finding a solution which satisfies certain constraints. This problem is referred to as the convex feasibility problem. It can be described as follows: Suppose $C_1, C_2, \cdots, C_r$, where r is some positive integer, are finitely many nonempty convex closed subset of a Hilbert space H with $C = \cap_{i=1}^{r} \neq \emptyset$. The convex feasibility problem is to find a point in C. In the real world, many important problems have reformulations which require finding fixed points of some nonlinear operators, for instance, evolution equations, complementarity problems, mini-max problems, variational inequalities and zero point problems; see [1-13] and the references therein.

In this paper, we are concerned with the problem of finding a common solution of fixed point and inclusion problems. Many nonlinear problems arising in applied areas such as image recovery, signal processing, and machine learning are mathematically modeled as this problem. One of the most popular methods for solving inclusion problems goes back to the work of Browder [14]. The basic ideas is to reduce inclusion problems to fixed point problems of nonlinear operators. In this paper, we study a regularization method for two monotone and a nonexpansive mappings. The organization of this paper is as follows. In Section 2, we provide some necessary preliminaries. In Section 3, a viscosity approximation method is introduced. A strong convergence theorem of common solutions is established. In Section 4, applications of the main results are discussed.

2. Preliminaries

In what follows, we always assume that H is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty, convex and closed subset of H. Let $S : C \to C$ be a mapping. $Fix(S)$ stands for the fixed point set of S; that is, $Fix(S) := \{x \in C : x = Sx\}$. Recall that S is said to be κ-*contractive* iff there exists a constant $\kappa \in (0, 1)$ such that

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$\|Sx - Sy\| \leq \kappa\|x - y\|$, $\forall x, y \in C$. It is well known that every contractive mapping has a unique fixed point in metric spaces. The Picard iterative algorithm $x_{n+1} = Sx_n$ converge to the fixed point of S. S is said to be *nonexpansive* iff $\|Sx - Sy\| \leq \|x - y\|$, $\forall x, y \in C$. If C is a bounded, closed, and convex subset of H, then $F(S)$ is not empty; see [15] and the references therein. Since the nonexpansivity of S, the Picard iterative algorithm may not converge to fixed points of S. The Mann iterative algorithm is powerful and efficient to study fixed points of nonexpansive mappings. However, in infinite dimensional spaces, the Mann iterative algorithm is only weak convergence. To obtain strong convergence of the Mann iterative algorithm, different regularization methods have been investigated recently; see [16]-[29] and the references therein.

Let $A : C \to H$ be a mapping. Recall that A is said to be *monotone* iff $\langle Ax-Ay, x-y\rangle \geq 0$, $\forall x, y \in C$. Recall that A is said to be *inverse-strongly monotone* iff there exists a constant $\alpha > 0$ such that $\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2$, $\forall x, y \in C$. For such a case, A is also said to be *α-inverse-strongly monotone*. It is not hard to see that every inverse-strongly monotone mapping is monotone and continuous. Recall that a set-valued mapping $B : H \rightrightarrows H$ is said to be *monotone* iff, for all $x, y \in H$, $f \in Bx$ and $g \in By$ imply $\langle x - y, f - g\rangle \geq 0$. In this paper, we use $B^{-1}(0)$ to stand for the zero point of B. A monotone mapping $B : H \rightrightarrows H$ is *maximal* iff the graph $Graph(B)$ of B is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping B is maximal if and only if, for any $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$, for all $(y, g) \in Graph(B)$ implies $f \in Bx$. For a maximal monotone operator B on H, and $r > 0$, we may define the single-valued resolvent $J_r : H \to Dom(B)$, where $Dom(B)$ denote the domain of B. It is known that J_r is firmly nonexpansive, and $B^{-1}(0) = F(J_r)$.

In this paper, we study fixed points of nonexpansive mappings and zero points of two monotone mappings based on a viscosity approximation method. Strong convergence theorems are established in the framework of Hilbert spaces. The results obtained in this paper mainly improve the corresponding results in [23]-[29]. In order to prove our main results, we also need the following lemmas.

Lemma 2.1 [30] *Let $\{a_n\}$ be a sequence of nonnegative numbers satisfying the condition $a_{n+1} \leq (1 - t_n)a_n + t_nb_n + c_n$, $\forall n \geq 0$, where $\{t_n\}$ is a number sequence in $(0, 1)$ such that $\lim_{n\to\infty} t_n = 0$ and $\sum_{n=0}^{\infty} t_n = \infty$, $\{b_n\}$ is a number sequence such that $\limsup_{n\to\infty} b_n \leq 0$, and $\{c_n\}$ is a positive number sequence such that $\sum_{n=0}^{\infty} c_n < \infty$. Then $\lim_{n\to\infty} a_n = 0$.*

Lemma 2.2. [31] *Let C be a nonempty convex closed subset of a real Hilbert space H. Let $A : C \to H$ be an α-inverse-strongly monotone mapping and let B be a maximal monotone operator on H. Then $(A + B)^{-1}(0) = F(J_r(I - rA))$.*

Lemma 2.3. [32] *Let H be a Hilbert space, and A an maximal monotone operator. For $\lambda > 0$, $\mu > 0$, and $x \in E$, we have $J_\lambda x = J_\mu\left(\left(1 - \frac{\mu}{\lambda}\right)J_\lambda x + \frac{\mu}{\lambda}x\right)$, where $J_\lambda = (I + \lambda A)^{-1}$ and $J_\mu = (I + \mu A)^{-1}$.*

Lemma 2.4. [14] *Let C be a nonempty convex closed subset of a real Hilbert space H. Let T be a nonexpansive mapping on C. Then $I - T$ is demiclosed at origin.*

3. Main results

Theorem 3.1. *Let C be a nonempty convex closed subset of a real Hilbert space H. Let $A : C \to H$ be an α-inverse-strongly monotone mapping and let B be a maximal* *monotone operator on H. Let S be a fixed κ-contraction and let T be a nonexpansive mapping on C. Assume $Dom(B) \subset C$ and $(A+B)^{-1}(0) \cap Fix(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a real number sequence in $(0,1)$ and let $\{r_n\}$ be a positive real number sequence in $(0,2\alpha)$. Let $\{x_n\}$ be a sequence in C in the following process: $x_0 \in C$, $y_n = \alpha_n Sx_n + (1-\alpha_n)Tx_n$, $x_{n+1} \approx (I+r_nB)^{-1}(y_n - r_nAy_n)$, $\forall n \geq 0$. Let the criterion for the approximate computation of x_{n+1} be $\|x_{n+1} - (I+r_nB)^{-1}(y_n - r_nAy_n)\| \leq e_n$, where $\sum_{n=1}^{\infty} e_n < \infty$. Assume that the control sequences $\{\alpha_n\}$ and $\{r_n\}$ satisfy the following restrictions: $\sum_{n=1}^{\infty}|r_n - r_{n-1}| < \infty$, $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=0}^{\infty}\alpha_n = \infty$, $\sum_{n=1}^{\infty}|\alpha_n - \alpha_{n-1}| < \infty$, and $0 < r \leq r_n \leq r' < 2\alpha$, where r and r' are two real numbers. Then $\{x_n\}$ converges strongly to a point $\bar{x} \in (A+B)^{-1}(0) \cap Fix(T)$, where $\bar{x} = Proj_{(A+B)^{-1}(0)\cap Fix(T)} S\bar{x}$.*

Proof. First, we show that $\{x_n\}$ and $\{y_n\}$ are bounded sequences. Using the restrictions imposed on $\{r_n\}$, one see that $I - r_nA$ is nonexpansive. Indeed, we have

$$\begin{aligned}
&\|(I-r_nA)x - (I-r_nA)y\|^2 \\
&\leq \|x-y\|^2 - r_n(2\alpha - r_n)\|Ax - Ay\|^2 \\
&\leq \|x-y\|^2.
\end{aligned}$$

That is, $\|(I-r_nA)x - (I-r_nA)y\| \leq \|x-y\|$. Fixing $p \in (A+B)^{-1}(0) \cap Fix(T)$, we find that

$$\begin{aligned}
\|y_n - p\| &\leq \alpha_n\|Sx_n - p\| + (1-\alpha_n)\|Tx_n - p\| \\
&\leq \alpha_n\|Sx_n - p\| + (1-\alpha_n)\|x_n - p\| \\
&\leq \big(1 - \alpha_n(1-\kappa)\big)\|x_n - p\| + \alpha_n\|Sp - p\|.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\|x_{n+1} - p\| &\leq \|e_n\| + \|(I+r_nB)^{-1}(y_n - r_nAy_n) - p\| \\
&\leq e_n + \|(y_n - r_nAy_n) - (I - r_nA)p\| \\
&\leq e_n + \big(1-\alpha_n(1-\kappa)\big)\|x_n - p\| + \alpha_n(1-\kappa)\frac{\|Sp-p\|}{1-\kappa} \\
&\leq \max\{\|x_n - p\|, \frac{\|Sp-p\|}{1-\kappa}\} + e_n \\
&\;\;\vdots \\
&\leq \max\{\|x_0 - p\|, \frac{\|Sp-p\|}{1-\kappa}\} + \sum_{i=0}^{\infty} e_i < \infty.
\end{aligned}$$

This proves that the sequence $\{x_n\}$ is bounded, so is $\{y_n\}$. Notice that

$$\|y_n - y_{n-1}\| \leq \big(1-\alpha_n(1-\kappa)\big)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|Sx_{n-1} - x_{n-1}\|.$$

Setting $z_n = y_n - r_nAy_n$, one further has

$$\begin{aligned}
\|z_n - z_{n-1}\| &\leq \|y_n - y_{n-1}\| + \|r_n - r_{n-1}\|\|Ay_{n-1}\| \\
&\leq \big(1-\alpha_n(1-\kappa)\big)\|x_n - x_{n-1}\| + |r_n - r_{n-1}|\|Ay_{n-1}\| \\
&\quad + |\alpha_n - \alpha_{n-1}|\|Sx_{n-1} - x_{n-1}\|.
\end{aligned} \tag{3.1}$$

Putting $J_{r_n} = (I + r_n B)^{-1}$, it follows from Lemma 2.3 that

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\leq e_n + e_{n-1} + \|J_{r_{n-1}} z_{n-1} - J_{r_{n-1}}\big(\frac{r_{n-1}}{r_n} z_n + (1 - \frac{r_{n-1}}{r_n}) J_{r_n} z_n\big)\| \\
&\leq e_n + e_{n-1} + \|(1 - \frac{r_{n-1}}{r_n})(J_{r_n} z_n - z_{n-1}) + \frac{r_{n-1}}{r_n}(z_n - z_{n-1})\| \\
&\leq e_n + e_{n-1} + \frac{|r_n - r_{n-1}|}{r_n}\|z_n - J_{r_n} z_n\| + \|z_n - z_{n-1}\|,
\end{aligned}$$

which implies from (3.1) that

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\leq e_n + e_{n-1} + \frac{|r_n - r_{n-1}|}{r_n}\|z_n - J_{r_n} z_n\| + \big(1 - \alpha_n(1-\kappa)\big)\|x_n - x_{n-1}\| \\
&\quad + |r_n - r_{n-1}|\|Ay_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|Sx_{n-1} - x_{n-1}\| \\
&\leq \big(1 - \alpha_n(1-\kappa)\big)\|x_n - x_{n-1}\| + e_n + e_{n-1} \\
&\quad + |r_n - r_{n-1}|\Big(\|Ay_{n-1}\| + \frac{\|J_{r_n} z_n - z_n\|}{r_n}\Big) + |\alpha_n - \alpha_{n-1}|\|Sx_{n-1} - x_{n-1}\|.
\end{aligned}$$

From the restrictions imposed on the control sequences, we have

$$\sum_{n=1}^{\infty}\Big(e_n + e_{n-1} + |r_n - r_{n-1}|\Big(\|Ay_{n-1}\| + \frac{\|J_{r_n} z_n - z_n\|}{r_n}\Big) + |\alpha_n - \alpha_{n-1}|\|Sx_{n-1} - x_{n-1}\|\Big) < \infty.$$

Using Lemma 2.1, we find $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. Since $\|\cdot\|^2$ is convex, we have $\|y_n - p\|^2 \leq \alpha_n\|Sx_n - p\|^2 + (1-\alpha_n)\|x_n - p\|^2$, from which it follows that

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\leq \|(y_n - r_n Ay_n) - (p - r_n Ap)\|^2 + 2e_n\|(I + r_n B)^{-1}(y_n - r_n Ay_n) - p\| + e_n^2 \\
&\leq \|y_n - p\|^2 - r_n(2\alpha - r_n)\|Ay_n - Ap\|^2 + 2e_n\|(I + r_n B)^{-1}(y_n - r_n Ay_n) - p\| + e_n^2 \\
&\leq \alpha_n\|Sx_n - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 - r_n(2\alpha - r_n)\|Ay_n - Ap\|^2 \\
&\quad + 2e_n\|(I + r_n B)^{-1}(y_n - r_n Ay_n) - p\| + e_n^2.
\end{aligned}$$

This implies that

$$\begin{aligned}
r_n(2\alpha - r_n)\|Ay_n - Ap\|^2 \leq{}& \alpha_n\|Sx_n - p\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&+ 2e_n\|(I + r_n B)^{-1}(y_n - r_n Ay_n) - p\| + e_n^2.
\end{aligned}$$

Hence, we have

$$\lim_{n\to\infty}\|Ay_n - Ap\| = 0. \tag{3.2}$$

Put $\lambda_n = (I + r_n B)^{-1}(y_n - r_n Ay_n)$. Since $(I + r_n B)^{-1}$ is firmly nonexpansive, one has

$$\begin{aligned}
\|\lambda_n - p\|^2 &\leq \langle (y_n - r_n Ax_n) - (p - r_n Ap), \lambda_n - p\rangle \\
&\leq \frac{1}{2}\big(\|y_n - p\|^2 + \|\lambda_n - p\|^2 - \|y_n - \lambda_n - r_n(Ay_n - Ap)\|^2\big) \\
&\leq \frac{1}{2}\big(\|y_n - p\|^2 + \|\lambda_n - p\|^2 - \|y_n - \lambda_n\|^2 + 2r_n\|\lambda_n - y_n\|\|Ay_n - Ap\|\big).
\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le e_n^2+\alpha_n\|Sx_n-p\|^2+\|x_n-p\|^2-\|y_n-\lambda_n\|^2\\ &\quad+2r_n\|\lambda_n-y_n\|\|Ay_n-Ap\|+2e_n\|\lambda_n-p\|.\end{aligned}$$

Hence, we have

$$\begin{aligned}\|y_n-\lambda_n\|^2 &\le e_n^2+\alpha_n\|Sx_n-p\|^2+\|x_n-p\|^2-\|x_{n+1}-p\|^2\\ &\quad+2r_n\|\lambda_n-y_n\|\|Ay_n-Ap\|+2e_n\|\lambda_n-p\|.\end{aligned}$$

Using the restrictions imposed on the control sequences and (3.2), we arrive at

$$\lim_{n\to\infty}\|y_n-\lambda_n\|=0. \tag{3.3}$$

Note that $\|x_n-Tx_n\|\le\|x_n-x_{n+1}\|+\|\lambda_n-y_n\|+\|y_n-Tx_n\|+e_n$. This finds from (3.3) $\lim_{n\to\infty}\|x_n-Tx_n\|=0$.

Next, we show that

$$\limsup_{n\to\infty}\langle S\bar{x}-\bar{x},y_n-\bar{x}\rangle\le 0, \tag{3.4}$$

where $\bar{x}$ is the unique fixed point of the mapping $Proj_{(A+B)^{-1}(0)\cap Fix(T)}S$. To show this inequality, we choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $\limsup_{n\to\infty}\langle S\bar{x}-\bar{x},y_n-\bar{x}\rangle=\lim_{i\to\infty}\langle S\bar{x}-\bar{x},y_{n_i}-\bar{x}\rangle\le 0$, Since $\{y_{n_i}\}$ is bounded, there exists a subsequence $\{y_{n_{i_j}}\}$ of $\{y_{n_i}\}$ which converges weakly to $\hat{x}$. Without loss of generality, we assume that $y_{n_i}\rightharpoonup\hat{x}$. Since $\|x_n-y_n\|\le\|x_n-x_{n+1}\|+\|\lambda_n-y_n\|+e_n$, one has $x_{n_i}\rightharpoonup\hat{x}$. Using Lemma 2.4, one has $\hat{x}\in Fix(T)$. Since $y_n-r_nAy_n\in\lambda_n+r_nB\lambda_n$, that is, $\frac{y_n-\lambda_n-r_nAy_n}{r_n}\in B\lambda_n$. Let $\mu\in B\nu$. Since B is monotone, we find that $\langle\frac{y_n-\lambda_n}{r_n}-\mu-Ay_n,\lambda_n-\nu\rangle\ge 0$. Hence, one has $0\le\langle -A\hat{x}-\mu,\hat{x}-\nu\rangle$. This implies that $-A\hat{x}\in B\hat{x}$, that is, $\hat{x}\in(A+B)^{-1}(0)$. This shows (3.4) holds. Notice that

$$\begin{aligned}\|y_n-\bar{x}\|^2 &\le \alpha_n\langle Sx_n-S\bar{x},y_n-\bar{x}\rangle+\alpha_n\langle S\bar{x}-\bar{x},y_n-\bar{x}\rangle+(1-\alpha_n)\|Tx_n-p\|\|y_n-\bar{x}\|\\ &\le \big(1-\alpha_n(1-\kappa)\big)\|x_n-\bar{x}\|\|y_n-\bar{x}\|+\alpha_n\langle S\bar{x}-\bar{x},y_n-\bar{x}\rangle.\end{aligned}$$

It follows that $\|y_n-\bar{x}\|^2\le(1-\alpha_n(1-\kappa))\|x_n-\bar{x}\|^2+2\alpha_n\langle S\bar{x}-\bar{x},y_n-\bar{x}\rangle$.
Hence, we have

$$\begin{aligned}\|x_{n+1}-\bar{x}\|^2 &\le \|(y_n-r_nAy_n)-(I-r_nA)\bar{x}\|^2+2e_n\|\lambda_n-\bar{x}\|+e_n^2\\ &\le (1-\alpha_n(1-\kappa))\|x_n-\bar{x}\|^2+2\alpha_n\langle S\bar{x}-\bar{x},y_n-\bar{x}\rangle+2e_n\|\lambda_n-\bar{x}\|+e_n^2.\end{aligned}$$

An application of Lemma 2.1 to the above inequality yields that $\lim_{n\to\infty}\|x_n-\bar{x}\|=0$. This completes the proof.

4. Applications

Let C be a nonempty closed and convex subset of a Hilbert space H. Let i_C be the indicator function of C, that is, $i_C(x)=\infty, x\notin C$, $i_C(x)=0, x\in C$. Since i_C is a proper lower and semicontinuous convex function on H, the subdifferential ∂i_C of i_C is maximal monotone. So, we can define the resolvent J_r of ∂i_C for $r>0$, i.e., $J_r:=(I+r\partial i_C)^{-1}$. Letting $x=J_ry$, we find that

$$y\in x+r\partial i_Cx\iff y\in x+rN_Cx\iff x=Proj_Cy,$$

where $Proj_C$ is the metric projection from H onto C and $N_Cx:=\{e\in H:\langle e,v-x\rangle,\forall v\in C\}$.

Theorem 4.1. *Let C be a nonempty convex closed subset of a real Hilbert space H. Let $A : C \to H$ be an α-inverse-strongly monotone mapping and let $T : C \to C$ be a nonexpansive mapping. Assume that $VI(C, A) \cap Fix(T)$ is not empty. Let $S : C \to C$ be a fixed κ-contraction. Let $\{x_n\}$ be a sequence in C in the following process: $x_0 \in C$, $y_n = \alpha_n Sx_n + (1-\alpha_n)Tx_n$, $x_{n+1} \approx Proj_C(y_n - r_nAy_n)$, $\forall n \geq 0$. Let the criterion for the approximate computation of x_{n+1} be $\|x_{n+1} - Proj_C(y_n - r_nAy_n)\| \leq e_n$, where $\sum_{n=1}^{\infty} e_n < \infty$. Assume that the control sequences $\{\alpha_n\}$ and $\{r_n\}$ satisfy the following restrictions: $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$, $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$, and $0 < r \leq r_n \leq r' < 2\alpha$, where r and r' are two real numbers. Then $\{x_n\}$ converges strongly to a point $\bar{x} \in VI(C, A) \cap Fix(T)$, where $\bar{x} = Proj_{VI(C,A)\cap Fix(T)} S\bar{x}$.*

Proof. Putting $B = \partial i_C$ in Theorem 3.1, we find that $J_{r_n} = Proj_C$. This finds from Theorem 3.1 the desired conclusion immediately.

Next, we consider the problem of finding a solution of a Ky Fan inequality [7], which is known as an equilibrium problem in the terminology of Blum and Oettli; see [33] and the references therein.

Let B be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. Recall the following equilibrium problem:

$$\text{Find } x \in C \text{ such that } B(x, y) \geq 0, \quad \forall y \in C. \tag{4.1}$$

To study equilibrium problem (4.1), we may assume that B satisfies the following restrictions:

(R-a) $B(y, x) + B(x, y) \leq 0$, $\forall x, y \in C$;
(R-b) $B(x, x) = 0$, $\forall x \in C$;
(R-c) $B(x, y) \geq \limsup_{t\downarrow 0} B(tz + (1-t)x, y)$, $\forall x, y, z \in C$,
(R-d) $y \mapsto B(x, y)$, $\forall x \in C$, is lower semi-continuous and convex.

The following lemmas can be found in [22] and [33].

Lemma 4.2. *Let C be a nonempty convex closed subset of a real Hilbert space H. Let $B : C \times C \to \mathbb{R}$ be a bifunction with (R-a), (R-b), (R-c) and (R-d). Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that $rB(z, y) + \langle y - z, z - x\rangle \geq 0$, $\forall y \in C$. Further, define*

$$T_r x = \Big\{ z \in C : rB(z, y) + \langle y - z, z - x\rangle \geq 0, \ \forall y \in C \Big\} \tag{4.2}$$

for all $r > 0$ and $x \in H$. Then T_r is single-valued and firmly nonexpansive and $F(T_r) = EP(F)$ is closed convex.

Lemma 4.3. *Let C be a nonempty convex closed subset of a real Hilbert space H. Let B be a bifunction from $C \times C$ to $\mathbb{R}$ with (R-a), (R-b), (R-c) and (R-d). Let A_B be a multivalued mapping of H into itself defined by*

$$A_B x = \begin{cases} \{z \in H : \langle y - x, z\rangle \leq B(x, y), \ \forall y \in C\}, & x \in C, \\ \emptyset, & x \notin C. \end{cases} \tag{4.3}$$

Then A_B is a maximal monotone operator with domain $D(A_B) \subset C$, $EP(B) = A_B^{-1}(0)$, where $FP(B)$ stands for the solution set of (4.1), and $T_r x = (I + rA_B)^{-1}x$, $\forall x \in H$, $r > 0$, where T_r is defined as in (4.2).

Theorem 4.4. *Let C be a nonempty convex closed subset of a real Hilbert space H. Let $B : C \times C \to \mathbb{R}$ be a bifunction with (R-a), (R-b), (R-c) and (R-d). Let $T : C \to C$ be a nonexpansive mapping. Assume that $EP(B) \cap Fix(T)$ is not empty. Let $S : C \to C$ be a fixed κ-contraction and let $T_{r_n} = (I+r_nA_B)^{-1}$. Let $\{x_n\}$ be a sequence in C in the following process: $x_0 \in C$ and $x_{n+1} \approx T_{r_n}(\alpha_nSx_n + (1-\alpha_n)Tx_n)$, $\forall n \geq 0$, Let the criterion for the approximate computation of x_{n+1} be $\|x_{n+1} - T_{r_n}(\alpha_nSx_n + (1-\alpha_n)Tx_n)\| \leq e_n$, where $\sum_{n=1}^{\infty} e_n < \infty$. Assume that the control sequences $\{\alpha_n\}$ and $\{r_n\}$ satisfy the following restrictions: $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$, $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$, and $0 < r \leq r_n \leq r' < 2\alpha$, where r and r' are two real numbers. Then $\{x_n\}$ converges strongly to a point $\bar{x} \in EP(B) \cap Fix(T)$, where $\bar{x} = Proj_{EP(B)\cap Fix(T)}S\bar{x}$.*

Proof. Putting $A = 0$ in Theorem 3.1, we find that $J_{r_n} = T_{r_n}$. From Theorem 3.1, we draw the desired conclusion immediately.

Recall that a mapping $T : C \to T$ is said to be α-strictly pseudocontractive iff there exits a constant $\alpha \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \alpha\|(I-T)x - (I-T)y\|^2 + \|x-y\|^2, \quad \forall x, y \in C.$$

The class of strictly pseudocontractive mappings was first introduced by Browder and Petryshyn [28]. It is known if T is α-strictly pseudocontractive, then $I - T$ is $\frac{1-\alpha}{2}$-inverse strongly monotone.

Finally, we consider the problem of common fixed point problems of nonlinear mappings.

Theorem 4.5. *Let C be a nonempty convex closed subset of a real Hilbert space H. Let T_1 be a nonexpansive mapping and let T_2 be a α-strictly pseudocontractive mapping on C. Let S be a fixed κ-contraction on C. Let $\{x_n\}$ be a sequence generated in the following manner: $x_0 \in C$, $y_n = \alpha_nSx_n + (1-\alpha_n)T_1x_n$, $x_{n+1} \approx (1-r_n)y_n + r_nT_2y_n$, $\forall n \geq 0$, Let the criterion for the approximate computation of x_{n+1} be $\|x_{n+1} - (1-r_n)y_n - r_nT_2y_n\| \leq e_n$, where $\sum_{n=1}^{\infty} e_n < \infty$. Assume that the control sequences $\{\alpha_n\}$ and $\{r_n\}$ satisfy the following restrictions: $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$, $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$, and $0 < r \leq r_n \leq r' < 1-\alpha$, where r and r' are two real numbers. Then $\{x_n\}$ converges strongly to a point $\bar{x} \in Fix(T_1) \cap Fix(T_2)$, where $\bar{x} = Proj_{Fix(T_1)\cap Fix(T_2)}S\bar{x}$.*

Proof. Putting $A = I - T_2$, we find A is $\frac{1-\alpha}{2}$-inverse strongly monotone. We also have $VI(C, A) = Fix(T_2)$ and $r_nT_2y_n + (1-r_n)y_n = Proj_C(y_n - r_nAy_n)$. In view of Theorem 3.1, we obtain the desired result immediately.

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ON THE STABILITY OF ADDITIVE ρ-FUNCTIONAL INEQUALITIES IN FUZZY NORMED SPACES

CHOONKIL PARK

Abstract. In this paper, we solve the following additive ρ-functional inequalities

$$N\left(f(x+y)-f(x)-f(y)-\rho\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right),t\right)\geq\frac{t}{t+\varphi(x,y)} \tag{0.1}$$

and

$$N\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y)-\rho\left(f(x+y)-f(x)-f(y)\right),t\right)\geq\frac{t}{t+\varphi(x,y)} \tag{0.2}$$

in fuzzy normed spaces, where ρ is a fixed real number with $\rho \neq 1$.

Using the direct method, we prove the Hyers-Ulam stability of the additive ρ-functional inequalities (0.1) and (0.2) in fuzzy Banach spaces.

1. Introduction and preliminaries

Katsaras [10] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view [6, 12, 27]. In particular, Bag and Samanta [2], following Cheng and Mordeson [5], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [11]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [3].

We use the definition of fuzzy normed spaces given in [2, 16, 17] to investigate the Hyers-Ulam stability of additive ρ-functional inequalities in fuzzy Banach spaces.

Definition 1.1. [2, 16, 17, 18] Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0,1]$ is called a *fuzzy norm* on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

(N_1) $N(x,t) = 0$ for $t \leq 0$;
(N_2) $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$;
(N_3) $N(cx,t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
(N_4) $N(x+y, s+t) \geq \min\{N(x,s), N(y,t)\}$;
(N_5) $N(x,\cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$.
(N_6) for $x \neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

The pair (X, N) is called a *fuzzy normed vector space*.

The properties of fuzzy normed vector spaces and examples of fuzzy norms are given in [15, 16].

Definition 1.2. [2, 16, 17, 18] Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to *be convergent* or *converge* if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the *limit* of the sequence $\{x_n\}$ and we denote it by N-$\lim_{n\to\infty} x_n = x$.

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Definition 1.3. [2, 16, 17, 18] Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called *Cauchy* if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well-known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space*.

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to x_0 in X, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be *continuous* on X (see [3]).

The stability problem of functional equations originated from a question of Ulam [26] concerning the stability of group homomorphisms.

The functional equation $f(x+y) = f(x) + f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [8] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [24] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [7] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach.

The functional equation $f\left(\frac{x+y}{2}\right) = \frac{1}{2}f(x) + \frac{1}{2}f(y)$ is called the *Jensen equation*. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [4, 9, 13, 14, 19, 22, 23, 25]).

Park [20, 21] defined additive ρ-functional inequalities and proved the Hyers-Ulam stability of the additive ρ-functional inequalities in Banach spaces and non-Archimedean Banach spaces.

In Section 2, we solve the additive ρ-functional inequality (0.1) and prove the Hyers-Ulam stability of the additive ρ-functional inequality (0.1) in fuzzy Banach spaces by using the direct method.

In Section 3, we solve the additive ρ-functional inequality (0.2) and prove the Hyers-Ulam stability of the additive ρ-functional inequality (0.2) in fuzzy Banach spaces by using the direct method.

Throughout this paper, assume that X is a real vector space and (Y, N) is a fuzzy Banach space.

2. Additive ρ-functional inequality (0.1)

In this section, we prove the Hyers-Ulam stability of the additive ρ-functional inequality (0.1) in fuzzy Banach spaces. Let ρ be a real number with $\rho \neq 1$. We need the following lemma to prove the main results.

Lemma 2.1. *Let $f : X \to Y$ be a mapping satisfying*

$$f(x+y) - f(x) - f(y) = \rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right) \tag{2.1}$$

for all $x, y \in X$. Then $f : X \to Y$ is additive.

Proof. Letting $x = y = 0$ in (2.1), we get $-f(0) = 0$ and so $f(0) = 0$.

Replacing y by x in (2.1), we get $f(2x) - 2f(x) = 0$ and so $f(2x) = 2f(x)$ for all $x \in X$. Thus

$$f(x+y) - f(x) - f(y) = \rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right) = \rho(f(x+y) - f(x) - f(y))$$

and so $f(x+y) = f(x) + f(y)$ for all $x, y \in X$. $\square$

Theorem 2.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that*

$$\Phi(x, y) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty \tag{2.2}$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying

$$N\left(f(x+y) - f(x) - f(y) - \rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right), t\right) \geq \frac{t}{t + \varphi(x, y)} \tag{2.3}$$

for all $x, y \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N(f(x) - A(x), t) \geq \frac{t}{t + \frac{1}{2}\Phi(x, x)} \tag{2.4}$$

for all $x \in X$ and all $t > 0$.

Proof. Letting $y = x$ in (2.3), we get

$$N(f(2x) - 2f(x), t) \geq \frac{t}{t + \varphi(x, x)} \tag{2.5}$$

and so

$$N\left(f(x) - 2f\left(\frac{x}{2}\right), t\right) \geq \frac{t}{t + \varphi\left(\frac{x}{2}, \frac{x}{2}\right)}$$

for all $x \in X$. Hence

$$\begin{aligned}
&N\left(2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right), t\right) &(2.6)\\
&\geq \min\left\{N\left(2^l f\left(\frac{x}{2^l}\right) - 2^{l+1} f\left(\frac{x}{2^{l+1}}\right), t\right), \cdots, N\left(2^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 2^m f\left(\frac{x}{2^m}\right), t\right)\right\}\\
&= \min\left\{N\left(f\left(\frac{x}{2^l}\right) - 2f\left(\frac{x}{2^{l+1}}\right), \frac{t}{2^l}\right), \cdots, N\left(f\left(\frac{x}{2^{m-1}}\right) - 2f\left(\frac{x}{2^m}\right), \frac{t}{2^{m-1}}\right)\right\}\\
&\geq \min\left\{\frac{\frac{t}{2^l}}{\frac{t}{2^l} + \varphi\left(\frac{x}{2^{l+1}}, \frac{x}{2^{l+1}}\right)}, \cdots, \frac{\frac{t}{2^{m-1}}}{\frac{t}{2^{m-1}} + \varphi\left(\frac{x}{2^m}, \frac{x}{2^m}\right)}\right\}\\
&= \min\left\{\frac{t}{t + 2^l \varphi\left(\frac{x}{2^{l+1}}, \frac{x}{2^{l+1}}\right)}, \cdots, \frac{t}{t + 2^{m-1}\varphi\left(\frac{x}{2^m}, \frac{x}{2^m}\right)}\right\}\\
&\geq \frac{t}{t + \frac{1}{2}\sum_{j=l+1}^{m} 2^j \varphi\left(\frac{x}{2^j}, \frac{x}{2^j}\right)}
\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$ and all $t > 0$. It follows from (2.2) and (2.6) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := N\text{-}\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.6), we get (2.4).

By (2.3),

$$N\left(2^n\left(f\left(\frac{x+y}{2^n}\right)-f\left(\frac{x}{2^n}\right)-f\left(\frac{y}{2^n}\right)\right)\right.$$
$$\left.-\rho\left(2^{n+1}f\left(\frac{x+y}{2^{n+1}}\right)-2^nf\left(\frac{x}{2^n}\right)-2^nf\left(\frac{y}{2^n}\right)\right),2^nt\right)\geq\frac{t}{t+\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)}$$

for all $x,y\in X$, all $t>0$ and all $n\in\mathbb{N}$. So

$$N\left(2^n\left(f\left(\frac{x+y}{2^n}\right)-f\left(\frac{x}{2^n}\right)-f\left(\frac{y}{2^n}\right)\right)\right.$$
$$\left.-\rho\left(2^{n+1}f\left(\frac{x+y}{2^{n+1}}\right)-2^nf\left(\frac{x}{2^n}\right)-2^nf\left(\frac{y}{2^n}\right)\right),t\right)\geq\frac{\frac{t}{2^n}}{\frac{t}{2^n}+\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)}=\frac{t}{t+2^n\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)}$$

for all $x,y\in X$, all $t>0$ and all $n\in\mathbb{N}$. Since $\lim_{n\to\infty}\frac{t}{t+2^n\varphi(\frac{x}{2^n},\frac{y}{2^n})}=1$ for all $x,y\in X$ and all $t>0$,

$$A(x+y)-A(x)-A(y)=\rho\left(2A\left(\frac{x+y}{2}\right)-A(x)-A(y)\right)$$

for all $x,y\in X$. By Lemma 2.1, the mapping $A:X\to Y$ is Cauchy additive, as desired. □

Corollary 2.3. *Let $\theta\geq 0$ and let p be a real number with $p>1$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f:X\to Y$ be a mapping satisfying*

$$N\left(f(x+y)-f(x)-f(y)-\rho\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right),t\right)$$
$$\geq\frac{t}{t+\theta(\|x\|^p+\|y\|^p)}\tag{2.7}$$

for all $x,y\in X$ and all $t>0$. Then $A(x):=N\text{-}\lim_{n\to\infty}2^nf(\frac{x}{2^n})$ exists for each $x\in X$ and defines an additive mapping $A:X\to Y$ such that

$$N\left(f(x)-A(x),t\right)\geq\frac{(2^p-2)t}{(2^p-2)t+2\theta\|x\|^p}$$

for all $x\in X$ and all $t>0$.

Proof. The proof follows from Theorem 2.2 by taking $\varphi(x,y):=\theta(\|x\|^p+\|y\|^p)$ for all $x,y\in X$, as desired. □

Theorem 2.4. *Let $\varphi:X^2\to[0,\infty)$ be a function such that*

$$\Phi(x,y):=\sum_{j=0}^{\infty}\frac{1}{2^j}\varphi\left(2^jx,2^jy\right)<\infty$$

for all $x,y\in X$. Let $f:X\to Y$ be a mapping satisfying (2.3). Then $A(x):=N\text{-}\lim_{n\to\infty}\frac{1}{2^n}f\left(2^nx\right)$ exists for each $x\in X$ and defines an additive mapping $A:X\to Y$ such that

$$N\left(f(x)-A(x),t\right)\geq\frac{1}{t+\frac{1}{2}\Phi(x,x)}$$

for all $x\in X$ and all $t>0$.

Proof. It follows from (2.5) that

$$N\left(f(x)-\frac{1}{2}f(2x),\frac{1}{2}t\right)\geq\frac{t}{t+\varphi(x,x)}$$

and so

$$N\left(f(x) - \frac{1}{2}f(2x), t\right) \geq \frac{2t}{2t + \varphi(x,x)} = \frac{t}{t + \frac{1}{2}\varphi(x,x)}$$

for all $x \in X$ and all $t > 0$.

The rest of the proof is similar to the proof of Theorem 2.2. $\square$

Corollary 2.5. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 1$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be a mapping satisfying (2.7). Then $A(x) := N\text{-}\lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that*

$$N\left(f(x) - A(x), t\right) \geq \frac{(2-2^p)t}{(2-2^p)t + 2\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.4 by taking $\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$, as desired. $\square$

3. Additive ρ-functional inequality (0.2)

In this section, we prove the Hyers-Ulam stability of the additive ρ-functional inequality (0.2) in fuzzy Banach spaces. Let ρ be a fuzzy number with $\rho \neq 1$.

Lemma 3.1. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$2f\left(\frac{x+y}{2}\right) - f(x) - f(y) = \rho\left(f(x+y) - f(x) - f(y)\right) \tag{3.1}$$

for all $x, y \in X$. Then $f : X \to Y$ is additive.

Proof. Letting $y = 0$ in (3.1), we get $2f\left(\frac{x}{2}\right) - f(x) = 0$ and so $f(2x) = 2f(x)$ for all $x \in X$. Thus

$$f(x+y) - f(x) - f(y) = 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) = \rho(f(x+y) - f(x) - f(y))$$

and so $f(x+y) = f(x) + f(y)$ for all $x, y \in X$. $\square$

Theorem 3.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that*

$$\Phi(x,y) := \sum_{j=0}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty \tag{3.2}$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$N\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y) - \rho\left(f(x+y) - f(x) - f(y)\right), t\right) \geq \frac{t}{t + \varphi(x,y)} \tag{3.3}$$

for all $x, y \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\left(f(x) - A(x), t\right) \geq \frac{t}{t + \Phi(x,0)} \tag{3.4}$$

for all $x \in X$ and all $t > 0$.

Proof. Letting $y = 0$ in (3.3), we get

$$N\left(f(x) - 2f\left(\frac{x}{2}\right), t\right) = N\left(2f\left(\frac{x}{2}\right) - f(x), t\right) \geq \frac{t}{t + \varphi(x, 0)} \tag{3.5}$$

for all $x \in X$. Hence

$$\begin{aligned}
&N\left(2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right), t\right) \qquad (3.6)\\
&\geq \min\left\{N\left(2^l f\left(\frac{x}{2^l}\right) - 2^{l+1} f\left(\frac{x}{2^{l+1}}\right), t\right), \cdots, N\left(2^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 2^m f\left(\frac{x}{2^m}\right), t\right)\right\}\\
&= \min\left\{N\left(f\left(\frac{x}{2^l}\right) - 2f\left(\frac{x}{2^{l+1}}\right), \frac{t}{2^l}\right), \cdots, N\left(f\left(\frac{x}{2^{m-1}}\right) - 2f\left(\frac{x}{2^m}\right), \frac{t}{2^{m-1}}\right)\right\}\\
&\geq \min\left\{\frac{\frac{t}{2^l}}{\frac{t}{2^l} + \varphi\left(\frac{x}{2^l}, 0\right)}, \cdots, \frac{\frac{t}{2^{m-1}}}{\frac{t}{2^{m-1}} + \varphi\left(\frac{x}{2^{m-1}}, 0\right)}\right\}\\
&= \min\left\{\frac{t}{t + 2^l \varphi\left(\frac{x}{2^l}, 0\right)}, \cdots, \frac{t}{t + 2^{m-1}\varphi\left(\frac{x}{2^{m-1}}, 0\right)}\right\}\\
&\geq \frac{t}{t + \sum_{j=l}^{m-1} 2^j \varphi\left(\frac{x}{2^j}, 0\right)}
\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$ and all $t > 0$. It follows from (3.2) and (3.6) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := N\text{-}\lim_{n\to\infty} 2^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.6), we get (3.4).

By (3.3),

$$\begin{aligned}
&N\left(2^{n+1} f\left(\frac{x+y}{2^{n+1}}\right) - 2^n f\left(\frac{x}{2^n}\right) - 2^n f\left(\frac{y}{2^n}\right)\right.\\
&\quad \left. - \rho\left(2^n\left(f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right), 2^n t\right) \geq \frac{t}{t + \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)}
\end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. So

$$\begin{aligned}
&N\left(2^{n+1} f\left(\frac{x+y}{2^{n+1}}\right) - 2^n f\left(\frac{x}{2^n}\right) - 2^n f\left(\frac{y}{2^n}\right)\right.\\
&\left. - \rho\left(2^n\left(f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right), t\right) \geq \frac{\frac{t}{2^n}}{\frac{t}{2^n} + \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)} = \frac{t}{t + 2^n\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)}
\end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{t}{t + 2^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$2A\left(\frac{x+y}{2}\right) - A(x) - A(y) = \rho\left(A(x+y) - A(x) - A(y)\right)$$

for all $x, y \in X$. By Lemma 3.1, the mapping $A : X \to Y$ is Cauchy additive, as desired. $\square$

Corollary 3.3. *Let $\theta \geq 0$ and let p be a real number with $p > 1$. Let X be a normed vector space with norm $\| \cdot \|$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$N\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y) - \rho\left(f(x+y) - f(x) - f(y)\right), t\right) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)} \tag{3.7}$$

for all $x, y \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n\to\infty} 2^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\left(f(x) - A(x), t\right) \geq \frac{(2^p - 2)t}{(2^p - 2)t + 2^p\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 3.2 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$, as desired. □

Theorem 3.4. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that*

$$\Phi(x, y) := \sum_{j=1}^{\infty} \frac{1}{2^j} \varphi\left(2^j x, 2^j y\right) < \infty$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (3.3). Then $A(x) := N\text{-}\lim_{n\to\infty} \frac{1}{2^n} f\left(2^n x\right)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\left(f(x) - A(x), t\right) \geq \frac{t}{t + \Phi(x, 0)}$$

for all $x \in X$ and all $t > 0$.

Proof. It follows from (3.5) that

$$N\left(f(x) - \frac{1}{2}f(2x), \frac{t}{2}\right) \geq \frac{t}{t + \varphi(2x, 0)}$$

and so

$$N\left(f(x) - \frac{1}{2}f(2x), t\right) \geq \frac{2t}{2t + \varphi(2x, 0)} = \frac{t}{t + \frac{1}{2}\varphi(2x, 0)}$$

for all $x \in X$ and all $t > 0$.

The rest of the proof is similar to the proof of Theorem 3.2. □

Corollary 3.5. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 1$. Let X be a normed vector space with the norm $\| \cdot \|$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (3.7). Then $A(x) := N\text{-}\lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that*

$$N\left(f(x) - A(x), t\right) \geq \frac{(2 - 2^p)t}{(2 - 2^p)t + 2^p\theta\|x\|^p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.4 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$, as desired. □

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On the Difference equation $x_{n+1} = Ax_n + \frac{B\sum\limits_{i=0}^{k} x_{n-i}}{C+D\prod\limits_{i=0}^{k} x_{n-i}}$

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Abstract

The main objective of this paper is to study the global stability of the positive solutions and the periodic character of the difference equation

$$x_{n+1} = \frac{A\sum\limits_{i=0}^{k} x_{n-i}}{B + C\prod\limits_{i=0}^{k} x_{n-i}}, \qquad n = 0,\ 1,\ ...,$$

where the parameters $A,\ B$ and C are positive real numbers and the initial conditions $x_{-k},\ x_{-k+1}, ...,\ x_{-},\ x_0$ are nonnegative real numbers.

Keywords: difference equations, stability, global stability, periodic solutions.
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1 Introduction

Difference equations have always played an important role in the construction and analysis of mathematical models of biology, ecology, probability theory, genetics, number theory, physics, economic process, and so forth.

The study of nonlinear rational difference equations of higher order is of paramount importance, since we still know so little about such equations.

Ahmed [1] investigated the global asymptotic stability and the periodic character for the rational difference equation,

$$x_{n+1} = \frac{\alpha x_{n-l}}{\beta+\gamma \prod_{i=l}^{k} x_{n-2i}^{p_i}}, \qquad n = 0, 1, ...,$$

where the parameters $\alpha,\ \beta,\ \gamma,\ p_1,\ p_2,\ ...,\ p_k$ are nonnegative real numbers, and $l,\ k$ are nonnegative integers such that $l \leq k$ and the initial conditions x_{-2k}, x_{-2k+1}, ..., x_{-1}, x_0 are arbitrary nonnegative real numbers.

Wang et al. [2] studied the asymptotic behavior of the solutions of the nonlinear difference equation

$$x_{n+1} = \frac{\sum_{i=0}^{l} A_{s_i}\ x_{n-s_i}}{B+C \sum_{j=0}^{k} x_{n-t_j}}, \qquad n = 0, 1, ...,$$

where the initial conditions $x_{-m},\ x_{-m+1},\ ...,\ x_{-1},\ x_0$ are positive real numbers, $m = max \{s_1,\ ...,\ s_l,\ t_1,\ ...,\ t_k\}$, $s_1,\ ...,\ s_l,\ t_1,\ ...,\ t_k$ are nonnegative integers, and A_{s_i}, B, C are arbitrary positive real numbers.

Zayed et al. [3] investigated the boundedness character, the periodic character, the convergence and the global stability of positive solutions of the difference equation

$$x_{n+1} = \frac{A+\sum_{i=0}^{k} \alpha_i\ x_{n-i}}{\sum_{i=0}^{k} \beta_i\ x_{n-i}}, \qquad n = 0, 1, ...,$$

where the coefficients $A,\ \alpha_i,\ ,\beta_i$ and the initial conditions $x_{-k},\ x_{-k+1},\ ...,\ x_{-1},\ x_0$ are positive real numbers, while k is a positive integer number.

In [4] Ibrahim et al. studied the global behavior of the difference equation

$$x_{n+1} = \frac{\alpha x_{n-m}}{\beta+\gamma \prod_{j=0}^{k} x_{n-i_j}}, \qquad n = 0, 1, ...,$$

where the parameters $\alpha,\ \beta,\ \gamma$ and initial conditions are non-negative real numbers, $\{i_0 < i_1 < ... < i_k\}$ is a set of nonnegative even integers and m is an odd positive integer

Hamza et al. [5] studied the global asymptotic stability of the difference equation

$$x_{n+1} = \frac{A \prod_{i=l}^{k} x_{n-2i-1}}{B+C \prod_{j=0}^{k-1} x_{n-2i}}, \qquad n = 0, 1, ...,$$

where $A,\ B,\ C$ are nonnegative parameters and $l,\ k$ are nonnegative integers for $l < k$. They discussed the existence of unbounded solutions under certain conditions for $l = 0$.

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In [6] El-Metwally investigated the global stability character and the oscillatory of the solutions of the following difference equation

$$y_{n+1} = \frac{\alpha y_n \prod\limits_{i=l}^{k} x_{n-2i-1}}{\beta+\gamma \sum\limits_{i=0}^{k} y_{n-2i-1}^{p} \prod\limits_{i=0}^{k} y_{n-2i-1}}, \qquad n = 0, 1, ...,$$

where $\alpha,\ \beta,\ \gamma,\ p \in (0,\ \infty)$ with the initial conditions $y_0,\ y_{-1},\ ...,\ y_{-2k},\ y_{-2k-1} \in (0,\ \infty)$. For more results in the direction of this study, see, for example, [1–27] and the papers therein.

The aim of this paper to study some qualitative behavior of the positive solutions of a higher order difference equation

$$x_{n+1} = Ax_n + \frac{B \sum\limits_{i=0}^{k} x_{n-i}}{C+D \prod\limits_{i=0}^{k} x_{n-i}}, \qquad n = 0,\ 1,\ ..., \tag{1}$$

where the parameters $A,\ B,\ C$ and D are positive real numbers and the initial conditions $x_{-k},\ x_{-k+1}, ...,\ x_{-},\ x_0$ are nonnegative real numbers.

2 Preliminaries

Let I be some interval of real numbers and let

$$F : I^{k+1} \to I,$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, ..., x_0 \in I$, the difference equation

$$x_{n+1} = F(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, ..., \tag{2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 1 *(Equilibrium Point)*
A point $\overline{x} \in I$ is called an equilibrium point of the difference equation (2) if

$$\overline{x} = F(\overline{x}, \overline{x}, ..., \overline{x}).$$

That is, $x_n = \overline{x}$ for $n \geq 0$, is a solution of the difference equation (2), or equivalently, $\overline{x}$ is a fixed point of F.

Definition 2 *(Stability)*
Let $\overline{x} \in (0,\ \infty)$ be an equilibrium point of the difference equation (2). Then, we have

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(i) The equilibrium point $\overline{x}$ of the difference equation (2) is called locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + ... + |x_{-1} - \overline{x}| + |x_0 - \overline{x}| < \delta,$$

we have

$$|x_n - \overline{x}| < \epsilon \quad \text{for all} \quad n \geq -k.$$

(ii) The equilibrium point $\overline{x}$ of the difference equation (2) is called locally asymptotically stable if $\overline{x}$ is locally stable solution of Eq.(2) and there exists $\gamma > 0$, such that for all $x_{-k}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + ... + |x_{-1} - \overline{x}| + |x_0 - \overline{x}| < \gamma,$$

we have

$$\lim_{n \to \infty} x_n = \overline{x}.$$

(iii) The equilibrium point $\overline{x}$ of the difference equation (2) is called global attractor if for all $x_{-k}, ..., x_{-1}, x_0 \in I$, we have

$$\lim_{n \to \infty} x_n = \overline{x}.$$

(iv) The equilibrium point $\overline{x}$ of the difference equation (2) is called globally asymptotically stable if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of the difference equation (2).
(v) The equilibrium point $\overline{x}$ of the difference equation (2) is called unstable if $\overline{x}$ is not locally stable.

Definition 3 *(Periodicity)*
A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$. A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with prime period p if p is the smallest positive integer having this property.

Definition 4 *The linearized equation of the difference equation (2) about the equilibrium $\overline{x}$ is the linear difference equation*

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial F(\overline{x}, \overline{x}, ..., \overline{x})}{\partial x_{n-i}} y_{n-i}. \tag{3}$$

Now, assume that the characteristic equation associated with (3) is

$$p(\lambda) = p_0\lambda^k + p_1\lambda^{k-1} + ... + p_{k-1}\lambda + p_k = 0, \tag{4}$$

where

$$p_i = \frac{\partial F(\overline{x}, \overline{x}, ..., \overline{x})}{\partial x_{n-i}}.$$

Theorem 1 *[1]: Assume that $p_i \in R$, $i = 1, 2, ..., k$ and k is non-negative integer. Then*

$$\sum_{i=1}^{k} |p_i| < 1,$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+k} + p_1 x_{n+k-1} + ... + p_k x_n = 0, \quad n = 0, 1, ... \,.$$

3 Change of variables

By using the change of variables $x_n = \left(\frac{C}{D}\right)^{\frac{1}{k+1}} y_n$, the equation (1) reduces to the following difference equation

$$y_{n+1} = Ay_n + \frac{r \sum\limits_{i=0}^{k} y_{n-i}}{1 + \prod\limits_{i=0}^{k} y_{n-i}}, \qquad n = 0,\ 1,\ ..., \tag{5}$$

where $r = \frac{B}{C}$ and the initial conditions $y_n,\ y_{n-1}, ...,\ y_{n-k+1},\ y_{n-k}$ are positive real numbers.

4 Local Stability of the Equilibrium Point

In this section, we study the local stability character of the equilibrium point of Eq.(5).

Eq.(5) has equilibrium point and is given by

$$\overline{y} = A\overline{y} + \frac{r \sum\limits_{i=0}^{k} \overline{y}_{n-i}}{1 + \prod\limits_{i=0}^{k} \overline{y}_{n-i}},$$

or

$$\overline{y}\,(1 - A)\left(1 + \overline{y}^{k+1}\right) = r(k+1)\overline{y}.$$

Thus $\overline{y}_1 = 0$ is always an equilibrium point of Eq. (5). If $A < 1$ and $\frac{r(k+1)}{1-A} > 1$; then the only positive equilibrium point $\overline{y}_2$ of Eq. (5) is given by

$$\overline{y}_2 = \left(\frac{r(k+1)}{1-A} - 1\right)^{\frac{1}{k+1}}.$$

Theorem 2 *The equilibrium $\overline{y}_1$ of Eq. (5) is locally asymptotically stable if*

$$A + r(k+1) < 1.$$

Proof: Let $f:(0,\ \infty)^{k+1}\longrightarrow(0,\ \infty)$ be a continuous function defined by

$$f(u_n,\ u_{n-1},\ u_{n-2},\ ...,\ u_{n-k})=Au_n+\frac{r\sum\limits_{i=0}^{k}u_{n-i}}{1+\prod\limits_{i=0}^{k}u_{n-i}}. \tag{7}$$

Therefore, it follows that

$$\begin{aligned}
\frac{\partial f(u_n,\ u_{n-1},\ u_{n-2},\ ...,\ u_{n-k})}{\partial u_n} &= A+\frac{r\left(1+\prod\limits_{i=0}^{k}u_{n-i}\right)-r\left(\sum\limits_{i=0}^{k}u_{n-i}\right)\left(\prod\limits_{i=1}^{k}u_{n-i}\right)}{\left(1+\prod\limits_{i=0}^{k}u_{n-i}\right)^2},\\
\frac{\partial f(u_n,\ u_{n-1},\ u_{n-2},\ ...,\ u_{n-k})}{\partial u_{n-1}} &= \frac{r\left(1+\prod\limits_{i=0}^{k}u_{n-i}\right)-ru_n\left(\sum\limits_{i=0}^{k}u_{n-i}\right)\left(\prod\limits_{i=2}^{k}u_{n-i}\right)}{\left(1+\prod\limits_{i=0}^{k}u_{n-i}\right)^2},\\
\frac{\partial f(u_n,\ u_{n-1},\ u_{n-2},\ ...,\ u_{n-k})}{\partial u_{n-2}} &= \frac{r\left(1+\prod\limits_{i=0}^{k}u_{n-i}\right)-ru_nu_{n-1}\left(\sum\limits_{i=0}^{k}u_{n-i}\right)\left(\prod\limits_{i=3}^{k}u_{n-i}\right)}{\left(1+\prod\limits_{i=0}^{k}u_{n-i}\right)^2},\\
&\ \ \vdots\\
\frac{\partial f(u_n,\ u_{n-1},\ u_{n-2},\ ...,\ u_{n-k})}{\partial u_{n-k}} &= \frac{r\left(1+\prod\limits_{i=0}^{k}u_{n-i}\right)-r\left(\sum\limits_{i=0}^{k}u_{n-i}\right)\left(\prod\limits_{i=0}^{k-1}u_{n-i}\right)}{\left(1+\prod\limits_{i=0}^{k}u_{n-i}\right)^2}.
\end{aligned}$$

At $\overline{y}_1=0$, we have

$$\begin{aligned}
\frac{\partial f(u_n,\ u_{n-1},\ u_{n-2},\ ...,\ u_{n-k})}{\partial u_n} &= A+r\\
\frac{\partial f(u_n,\ u_{n-1},\ u_{n-2},\ ...,\ u_{n-k})}{\partial u_{n-1}} &= ...=\frac{\partial f(u_n,\ u_{n-1},\ u_{n-2},\ ...,\ u_{n-k})}{\partial u_{n-k}}=r,
\end{aligned}$$

and the linearized equation of Eq. (5) about $\overline{y}_1=0$, is the equation

$$z_{n+1}-(A+r)\,z_n-rz_{n-l}-...-ry_{n-k}=0,$$

It follows by Theorem 1 that, Eq. (5) is asymptotically stable if and only if

$$|A+r|+|r|+...+|r|<1,$$

and so

$$A+r\,(k+1)<1.$$

The proof is complete.

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Theorem 3 *The equilibrium $\overline{y}_1$ of Eq. (5) is unstable if $A + r(k+1) > 1$.*

Theorem 4 *The equilibrium $\overline{y}_2$ of Eq. (5) is stable if*

$$Ar + (1-A)(1-rk-A) < r.$$

Proof: At $\overline{y}_2 = \left(\frac{r(k+1)}{1-A} - 1\right)^{\frac{1}{k+1}}$, we have

$$\begin{aligned}
\frac{\partial f}{\partial u_n} &= A + \frac{r\left(1+\frac{r(k+1)}{1-A}-1\right) - r(k+1)\left(\frac{r(k+1)}{1-A}-1\right)}{\left(1+\frac{r(k+1)}{1-A}-1\right)^2} \\
&= A + \frac{r\left(\frac{r(k+1)}{1-A}\right) - r(k+1)\left(\frac{r(k+1)-1+A}{1-A}\right)}{\left(\frac{r(k+1)}{1-A}\right)^2} = A + \frac{\left(\frac{r(k+1)}{1-A}\right)(r - r(k+1) + 1 - A)}{\left(\frac{r(k+1)}{1-A}\right)^2} \\
&= A + \frac{(r - rk - r + 1 - A)}{\left(\frac{r(k+1)}{1-A}\right)} = A + \frac{(1-A)(1-rk-A)}{r(k+1)} \\
\frac{\partial f}{\partial u_{n-1}} &= \dots = \frac{\partial f}{\partial u_{n-k}} = \frac{(1-A)(1-rk-A)}{r(k+1)},
\end{aligned}$$

and the linearized equation of Eq. (5) about $\overline{y}_2 = \left(\frac{r(k+1)}{1-A} - 1\right)^{\frac{1}{k+1}}$, is the equation

$$z_{n+1} - \left(A + \frac{(1-A)(1-rk-A)}{r(k+1)}\right) z_n - \frac{(1-A)(1-rk-A)}{r(k+1)} z_{n-l} - \dots - \frac{(1-A)(1-rk-A)}{r(k+1)} y_{n-k} = 0,$$

It follows by Theorem A that, Eq.(5) is stable if and only if

$$\left| A + \frac{(1-A)(1-rk-A)}{r(k+1)} \right| + \left| \frac{(1-A)(1-rk-A)}{r(k+1)} \right| + \dots + \left| \frac{(1-A)(1-rk-A)}{r(k+1)} \right| < 1,$$

for $rk + A < 1$ we get

$$A + \frac{(1-A)(1-rk-A)}{r} < 1.$$

The proof is complete.

5 Existence of Boundedness Solutions

Here we look at the boundedness nature of solutions of Eq.(5).

Theorem 5 *Every solution of Eq.(5) is bounded if $A + r(k+1) < 1$.*

Proof: Let $\{y_n\}_{n=0}^{\infty}$ be a solution of Eq.(5). It follows from Eq.(5) that

$$0 \le y_{n+1} = Ay_n + \frac{r\sum\limits_{i=0}^{k} y_{n-i}}{1+\prod\limits_{i=0}^{k} y_{n-i}} < Ay_n + r\sum_{i=0}^{k} y_{n-i} < (A + r(k+1))\,\overline{y}.$$

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this equation is locally asymptotically stable if $A + r\,(k+1) < 1,$ and converges to the equilibrium point $\overline{y}$. Therefore

$$\limsup_{n\to\infty} y_n \leq (A + r\,(k+1))\,\overline{y}.$$

Hence, the solution is bounded.

Theorem 6 *Every solution of Eq.(5) is unbounded if $A > 1$.*

Proof: Let $\{y_n\}_{n=0}^{\infty}$ be a solution of Eq.(5). Then from Eq.(5) we see that

$$y_{n+1} = Ay_n + \frac{r\sum\limits_{i=0}^{k} y_{n-i}}{1+\prod\limits_{i=0}^{k} y_{n-i}} > Ay_n + r\sum_{i=0}^{k} y_{n-i} > A\overline{y}.$$

This equation is unbounded because $A > 1$, and $\lim\limits_{n\to\infty} y_n = \infty$. Then by using ratio test $\{y_n\}_{n=0}^{\infty}$ is unbounded from above.

6 Global Stability of the Equilibrium Point

In this section we study the global stability of the positive solutions of Equation (1).

Theorem 7 *The following statements are true*

(a) If $A + r(k+1) < 1$ then the equilibrium point $\overline{y}_1 = 0$ is a global attractor of equation (1).

(b) If $rk + A < 1$ then the equilibrium point $\overline{y}_2 = \left(\frac{r(k+1)}{1-A} - 1\right)^{\frac{1}{k+1}}$ is a global attractor of equation (1).

Proof. **(a)** From Eq. (7) we can see that the function is increasing of all arguments. Now, we can see that the function $F(y_n,\ y_{n-1},\ ...,\ y_{n-k})$ increasing in $y_n,\ y_{n-1},\ ...,\ y_{n-k+1}$ and x_{n-k}. Then

$$\begin{aligned} & \left[Ay + \frac{r(k+1)y}{1+y^{k+1}} - y\right](y - \overline{y}_1) \\ \leq\ & [Ay + r(k+1)y - y]\,(y - 0) \\ \leq\ & -(1 - A - r(k+1))\,y^2 < 0 \end{aligned}$$

If $A + r(k+1) < 1$, then $F(y,\ y,\ ...,\ y)$ satisfies the inequality

$$[F(y,\ y,\ ...,\ y) - y]\,(y - \overline{y}_1) < 0, \quad \text{for } \overline{y}_1 = 0.$$

According to Theorem 1.10 page 15 in [1], then $\overline{x}_1$ is a global attractor of Eq. (1). This completes the proof.

(b) If $rk + A < 1$, then we can see that the function $f(u_n,\ u_{n-1},\ u_{n-2},\ ...,\ u_{n-k})$ defined by Eq. (7) increasing of all arguments. Suppose that (m, M) is a solution of the system

$$M = f(M,\ M,\ ...,\ M) \quad \text{and} \quad m = f(m,\ m,\ ...,\ m).$$

Then from Equation (1), we see that

$$M = AM + \frac{r(k+1)M}{1+M^{k+1}}, \text{ and } m = Am + \frac{r(k+1)m}{1+m^{k+1}},$$

then

$$\begin{aligned} (1-A) + (1-A)M^{k+1} &= r(k+1), \\ (1-A) + (1-A)m^{k+1} &= r(k+1), \end{aligned}$$

Subtracting this two equations, we obtain

$$(1-A)\left(M^{k+1} - m^{k+1}\right) = 0$$

under the condition $A \neq 1$,we see that $M = m$. According to Theorem 1.15 page 18 in [1], we see that $\overline{y}_2$ is a global attractor of Equation (1).

7 Existence of Periodic Solutions

In this section we investigate the existence of periodic solutions of Eq.(5).

Theorem 8 *If k is even, then equation (5) has not prime period two solution.*

Proof: Equation (5) can be expressed that

$$y_{n+1} = Ay_n + \frac{r\left(y_n + y_{n-1} + y_{n-2} + ... + y_{n-k}\right)}{1 + y_n y_{n-1} y_{n-2} ... y_{n-k}},$$

For $k = 2m$ is even, then $y_n,\ y_{n-2},\ y_{n-4},\ ...,\ y_{n-k-2},\ y_{n-k}$ are even and $y_{n-1},\ y_{n-3},\ y_{n-5},\ ...,\ y_{n-k-3},\ y_{n-k-1}$ are odd. Suppose that exists there distinct positive solutions

$$...p,\ q,\ p,\ q,\ ...,$$

of Equation (5). Then

$$p = Aq + \frac{r\left(\left(m+1\right)q + mp\right)}{1 + q^{m+1}p^m} \text{ and } q = Ap + \frac{r\left(\left(m+1\right)p + mq\right)}{1 + p^{m+1}q^m}.$$

Therefore,

$$\begin{aligned} p - Aq + q^{m+1}p^{m+1} - Aq^{m+1}p^{m+1} &= r\left(m+1\right)q + rmp, && (7) \\ q - Ap + p^{m+1}q^{m+1} - Ap^{m+1}q^{m+1} &= r\left(m+1\right)p + rmq, && (8) \end{aligned}$$

By subtracting (8) from (7), we have

$$(1 + A + r)(p - q) = 0$$

Since $r + A + 1 \neq 0$, then $p = q$. This is a contradiction. Thus, the proof is completed.

Theorem 9 *If k is odd, then equation (5) has not prime period two solution.*

Proof: When $k = 2m + 1$ is odd, then $y_n,\ y_{n-2},\ y_{n-4},\ ...,\ y_{n-k-3},\ y_{n-k-1}$ are even and $y_{n-1},\ y_{n-3},\ y_{n-5},\ ...,\ y_{n-k-2},\ y_{n-k}$ are odd.

First suppose that there exists distinct positive solutions

$$...p,\ q,\ p,\ q,\ ...,$$

of Equation (5). Then

$$p = Aq + \frac{r\left(\left(m+1\right)q + \left(m+1\right)p\right)}{1 + q^{m+1}p^{m+1}},$$

and

$$q = Ap + \frac{r\left(\left(m+1\right)p + \left(m+1\right)q\right)}{1 + p^{m+1}q^{m+1}}.$$

Therefore,

$$p - Aq + q^{m+1}p^{m+2} - Aq^{m+2}p^{m+1} = r\left(m+1\right)q + r\left(m+1\right)p, \tag{9}$$

$$q - Ap + p^{m+1}q^{m+2} - Ap^{m+2}q^{m+1} = r\left(m+1\right)p + r\left(m+1\right)q, \tag{10}$$

Subtracting (10) from (9), we get

$$(p - q)\left((A+1)p^{m+1}q^{m+1} + 1 + A\right) = 0$$

Since $A+1 \neq 0$, then $p = q$. This is a contradiction. Thus, the proof is completed.

8 Numerical Examples

For confirming the results of this paper, we consider numerical examples which represent different types of solutions to Eq. (5).

Example 1. The zero solution of the difference equation (5) is local stability if $k = 3$, $A = 0.2$, $r = 0.1$ and the initial conditions $x_{-3} = 0.8$, $x_{-2} = 0.2$, $x_{-1} = 0.4$ and $x_0 = 0.7$ (See Fig. 1).

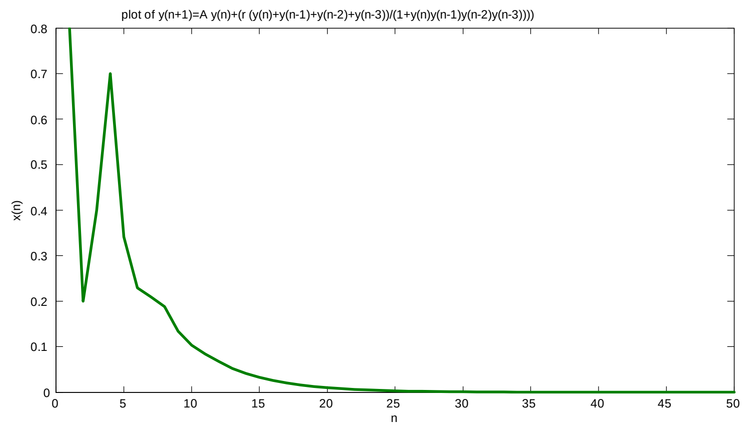

Figure 1. Plot the behavior of the zero solution of equation (5).

Example 2. The positive solution of the difference equation (5) is local stability if $k = 3$, $A = 0.6$, $r = 0.2$ and the initial conditions $x_{-3} = 0.8$, $x_{-2} = 0.2$, $x_{-1} = 0.4$ and $x_0 = 0.7$ (See Fig. 2).

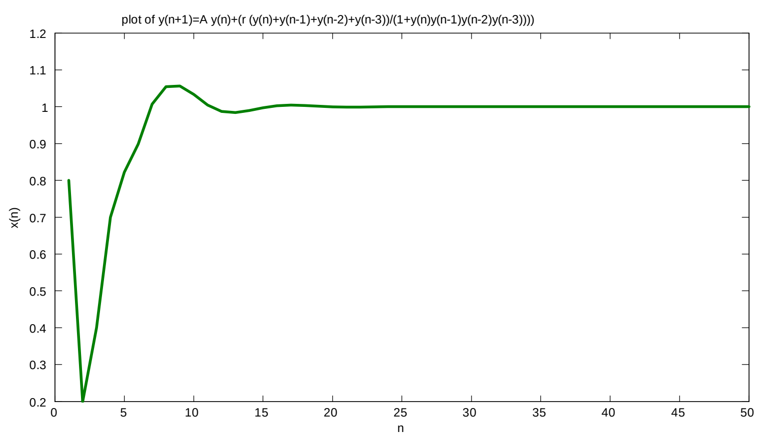

Figure 2. Plot the behavior of the positive solution of equation (5).

Example 3. The solution of the difference equation (5) is global stability if $k = 3$, $A = 0.02$, $r = 0.33$ and the initial conditions $x_{-3} = 0.8$, $x_{-2} = 0.2$, $x_{-1} = 0.4$ and $x_0 = 0.7$ (See Fig. 3).

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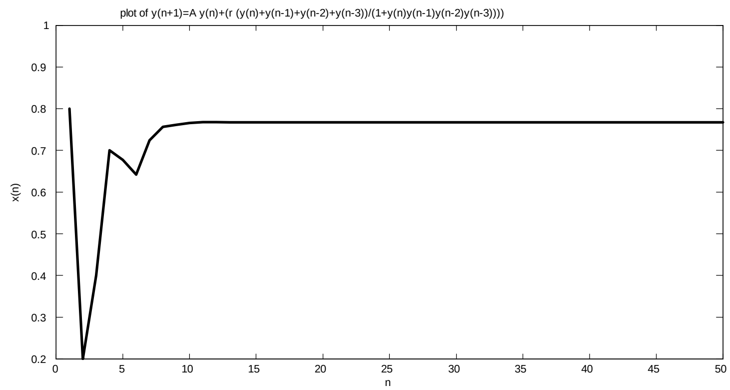

Figure 3. Plot the behavior of the positive solution of equation (5).

Example 4. Figure (4) shows the equation (5) is unbounded when $k = 3$, $A = 1.1$, $r = 0.1$ and the initial conditions $x_{-3} = 0.8$, $x_{-2} = 0.2$, $x_{-1} = 0.4$ and $x_0 = 0.7$.

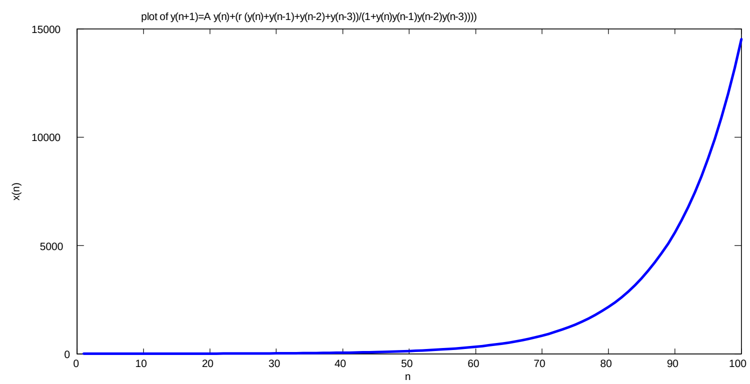

Figure 4. Plot the behavior of the solution of equation (5).

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A Kind of Generalized Fuzzy Integro-differential Equations of Mixed Type and Strong Fuzzy Henstock Integrals*

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Abstract

In this paper, we prove the existence theorem of solutions for a kind of discontinuous fuzzy integro-differential equation of mixed type by using the definition of the $\omega - ACG^*$ for a fuzzy-number-valued function and a generalized controlled convergence theorem of strong fuzzy Henstock integrals.

Keywords: Fuzzy number; $\omega - ACG^*$; Discontinuous fuzzy Integro-differential equation; Controlled convergence theorem; Strong fuzzy Henstock integrals.

1 INTRODUCTION

The Cauchy problems for fuzzy differential equations have been studied by several authors [11, 9, 12, 16, 17, 18] on the metric space (E^n, D) of normal fuzzy convex set with the distance D given by the maximum of the Hausdorff distance between the corresponding level sets. In [16], the author has been proved the Cauchy problem has a uniqueness result if f was continuous and bounded. In [11, 12], the authors presented a uniqueness result when f satisfies a Lipschitz condition. For a general reference to fuzzy differential equations, see a recent

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book by Lakshmikantham and Mohapatra [13] and references therein. In 2002, Xue and Fu [26] established solutions to fuzzy differential equations with right-hand side functions satisfying Caratheodory conditions on a class of Lipschitz fuzzy sets.

However, there are discontinuous systems in which the right-hand side functions $\tilde{f}:[a,b]\times E^n\to E^n$ are not integrable in the sense of Kaleva [11] on certain intervals and their solutions are not absolute continuous functions. Recently, Wu and Gong [24, 25] have combined the fuzzy set theory [27] and nonabsolute integration theory [10], and discussed the fuzzy Henstock integrals of fuzzy-number-valued functions which extended Kaleva[11] integration. In order to complete the theory of fuzzy calculus and to meet the solving need of transferring a fuzzy differential equation into a fuzzy integral equation, Gong and Shao [7, 8] have defined the strong fuzzy Henstock integrals and discussed some of their properties and the controlled convergence theorem. So, in [19, 20, 21, 22, 23], the authors used the strong fuzzy Henstock integrals [8], and deal with the Cauchy problem of discontinuous fuzzy systems. In this paper, according to the idea of [4] and using the concept of generalized differentiability [2], the operator j which is the isometric embedding from (E^n, D) onto its range in the Banach space X and the generalized controlled convergence theorems for the strong fuzzy Henstock integrals, we will deal with the Cauchy problem of discontinuous fuzzy integro-differential equations of mixed type as following:

$$\begin{cases} x'(t)=\tilde{f}(t,x(t),\int_0^t k_1(t,s)\tilde{g}(s,x(s))\mathrm{d}s,\int_0^a k_2(t,s)\tilde{h}(s,x(s))\mathrm{d}s), \\ x(0)=x_0, \quad x_0\in E^n, t\in I_a=[0,a], a\in R^+ \end{cases} \tag{1}$$

where $\tilde{f},\tilde{g},\tilde{h},x$ will be assumed strong fuzzy Henstock integrable and k_1,k_2 are real-valued functions.

To make our analysis possible, in section 2, we will first recall some basic results of fuzzy numbers. In section 3, we give some definitions of $\omega-ACG^*$ of fuzzy-number-valued function. In addition, we present the concept of strong fuzzy Henstock integral and a generalized controlled convergence theorem for the strong fuzzy Henstock integrals. In section 4, we deal with the Cauchy problem of discontinuous fuzzy integro-differential equation of mixed type. And in section 5, we present some concluding remarks.

2 PRELIMINARIES

Let $P_k(R^n)$ denote the family of all nonempty compact convex subset of R^n and define the addition and scalar multiplication in $P_k(R^n)$ as usual. Let A and B be two nonempty bounded subset of R^n. The distance between A and B is defined by the Hausdorff metric [6]:

$$d_H(A,B)=\max\{\sup_{a\in A}\inf_{b\in B}\parallel a-b\parallel,\sup_{b\in B}\inf_{a\in A}\parallel b-a\parallel\}.$$

2

Denote $E^n = \{u : R^n \to [0,1] | u$ satisfies (1)-(4) below$\}$ is a fuzzy number space. where

(1)u is normal, i.e. there exists an $x_0 \in R^n$ such that $u(x_0) = 1$,

(2)u is fuzzy convex, i.e. $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$ for any $x, y \in R^n$ and $0 \leq \lambda \leq 1$,

(3)u is upper semi-continuous,

(4)$[u]^0 = cl\{x \in R^n | u(x) > 0\}$ is compact.

For $0 < \alpha \leq 1$, denote $[u]^\alpha = \{x \in R^n | u(x) \geq \alpha\}$. Then from above (1)-(4), it follows that the α-level set $[u]^\alpha \in P_k(R^n)$ for all $0 \leq \alpha < 1$.

According to Zadeh's extension principle, we have addition and scalar multiplication in fuzzy number space E^n as follows [6]:

$$[u+v]^\alpha = [u]^\alpha + [v]^\alpha, \quad [ku]^\alpha = k[u]^\alpha,$$

where $u, v \in E^n$ and $0 \leq \alpha \leq 1$.

Define $D : E^n \times E^n \to [0, \infty)$

$$D(u,v) = \sup\{d_H([u]^\alpha, [v]^\alpha) : \alpha \in [0,1]\},$$

where d is the Hausdorff metric defined in $P_k(R^n)$. Then it is easy see that D is a metric in E^n. Using the results [5], we know that

(1) (E^n, D) is a complete metric space,

(2) $D(u+w, v+w) = D(u,v)$ for all $u, v, w \in E^n$,

(3)$D(\lambda u, \lambda v) = |\lambda| D(u,v)$ for all $u, v, w \in E^n$ and $\lambda \in R$.

The metric space (E^n, D) has a linear structure, it can be imbedded isomorphically as a cone in a Banach space of function $u^* : I \times S^{n-1} \longrightarrow R$, where S^{n-1} is the unit sphere in R^n, with an imbedding function $u^* = j(u)$ defined by

$$u^*(r,x) = \sup_{\alpha \in [u]^\alpha} < \alpha, x >$$

for all $< r, x > \in I \times S^{n-1}$. (see [5])

Theorem 1 *There exist a real Banach space X such that E^n can be imbedding as a convex cone C with vertex 0 into X. Furthermore the following conclusions hold:*

(1) the imbedding j is isometric,

(2) addition in X induces addition in E^n,

(3) multiplication by nonnegative real number in X induces the corresponding operation in E^n,

(4) $C - C$ is dense in X,

(5) C is closed.

A fuzzy-number-valued function $f : [a,b] \to E^n$ is said to satisfy the condition (H) on $[a,b]$, if for any $x_1 < x_2 \in [a,b]$ there exists $u \in E^n$ such that $f(x_2) = f(x_1) + u$. We call u is the H-difference of $f(x_2)$ and $f(x_1)$, denoted $f(x_2) -_H f(x_1)$ ([11]).

For brevity, we always assume that it satisfies the condition (H) when dealing with the operation of subtraction of fuzzy numbers throughout this paper.

It is well-known that the H-derivative for fuzzy-number-functions was initially introduced by Puri and Ralescu [17] and it is based in the condition (H) of sets. We note that this definition is fairly strong, because the family of fuzzy-number-valued functions H-differentiable is very restrictive. For example, the fuzzy-number-valued function $f : [a, b] \to E^n$ defined by $f(x) = C \cdot g(x)$, where C is a fuzzy number, $\cdot$ is the scalar multiplication (in the fuzzy context) and $g : [a, b] \to R^+$, with $g'(t_0) < 0$, is not H-differentiable in t_0 (see [2]). To avoid the above difficulty, in this paper we consider a more general definition of a derivative for fuzzy-number-valued functions enlarging the class of differentiable fuzzy-number-valued functions, which has been introduced in [2] and [3].

Definition 1 ([2]) *Let $\tilde{f} : (a, b) \to E^n$ and $x_0 \in (a, b)$. We say that $\tilde{f}$ is differentiable at x_0, if there exists an element $\tilde{f}'(t_0) \in E^n$, such that*

(1) for all $h > 0$ sufficiently small, there exists $\tilde{f}(x_0+h) -_H \tilde{f}(x_0), \tilde{f}(x_0) -_H \tilde{f}(x_0 - h)$ and the limits (in the metric D)

$$\lim_{h\to 0} \frac{\tilde{f}(x_0 + h) -_H \tilde{f}(x_0)}{h} = \lim_{h\to 0} \frac{\tilde{f}(x_0) -_H \tilde{f}(x_0 - h)}{h} = \tilde{f}'(x_0)$$

or

(2) for all $h > 0$ sufficiently small, there exists $\tilde{f}(x_0) -_H \tilde{f}(x_0 + h), \tilde{f}(x_0 - h) -_H \tilde{f}(x_0)$ and the limits

$$\lim_{h\to 0} \frac{\tilde{f}(x_0) -_H \tilde{f}(x_0 + h)}{-h} = \lim_{h\to 0} \frac{\tilde{f}(x_0 - h) -_H \tilde{f}(x_0)}{-h} = \tilde{f}'(x_0)$$

or

(3) for all $h > 0$ sufficiently small, there exists $\tilde{f}(x_0 + h) -_H \tilde{f}(x_0), \tilde{f}(x_0 - h) -_H \tilde{f}(x_0)$ and the limits

$$\lim_{h\to 0} \frac{\tilde{f}(x_0 + h) -_H \tilde{f}(x_0)}{h} = \lim_{h\to 0} \frac{\tilde{f}(x_0 - h) -_H \tilde{f}(x_0)}{-h} = \tilde{f}'(x_0)$$

or

(4) for all $h > 0$ sufficiently small, there exists $\tilde{f}(x_0) -_H \tilde{f}(x_0+h), \tilde{f}(x_0) -_H \tilde{f}(x_0 - h)$ and the limits

$$\lim_{h\to 0} \frac{\tilde{f}(x_0) -_H \tilde{f}(x_0 + h)}{-h} = \lim_{h\to 0} \frac{\tilde{f}(x_0) -_H \tilde{f}(x_0 - h)}{h} = \tilde{f}'(x_0)$$

(h and $-h$ at denominators mean $\frac{1}{h}\cdot$ and $-\frac{1}{h}\cdot$, respectively).

4

3 THE STRONG FUZZY HENSTOCK INTEGRAL AND ITS CONTROLLED CONVERGENCE THEOREM

In this section we shall give the definition of the strong Henstock integral for fuzzy-number-valued functions [7, 8] on a finite interval, which is an extension of the usual fuzzy Kaleva integral in [11]. In addition, we define the properties of $\omega - AC^*$ and $\omega - ACG^*$ for fuzzy-number-valued functions. In particular, we shall prove a controlled convergence theorems for the strong fuzzy Henstock integrals.

Definition 2 ([10, 14]) *Let $\delta(x)$ be a positive function defined on the interval $[a, b]$. A division $P = \{[x_{i-1}, x_i] : \xi_i\}$ is said to be $\delta-$fine if the following conditions are satisfied:*
(1) $a - x_0 < x_1 < \cdots < x_n = b$;
(2) $\xi_i \in [x_{i-1}, x_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$.
For brevity, we write $P = \{[u, v]; \xi\}$

Definition 3 ([7, 8]) *A fuzzy-number-valued function $\tilde{f}$ is said to be strong Henstock integrable on $[a, b]$ if there exists a additive fuzzy-number-valued function $\tilde{F}$ on $[a, b]$ such that for every $\varepsilon > 0$ there is a function $\delta(\xi) > 0$ and for any δ-fine division $P = \{([u, v], \xi)\}$ of $[a, b]$, we have*

$$\begin{aligned} & \sum_{i\in K_n} D(\tilde{f}(\xi_i)(v_i - u_i), \tilde{F}([u_i, v_i])) \\ & + \sum_{j\in I_n} D(\tilde{f}(\xi_j)(v_j - u_j), (-1)\cdot \tilde{F}([u_j, v_{j-1}])) \\ < \ & \varepsilon. \end{aligned}$$

where $K_n = \{i \in \{1, 2, \cdot\cdot, n\}$ such that $\tilde{F}([x_{i-1}, x_i])$ is a fuzzy number and $I_n = \{j \in \{1, 2, \cdot\cdot, n\}$ such that $\tilde{F}([x_j, x_{j-1}])$ is a fuzzy number. We write $\tilde{f} \in SFH[a, b]$.

Definition 4 ([10, 14]) *A real-valued function F is strong absolute continuous ($F \in AC^*$) on $[a, b]$ if and only if for every $\varepsilon > 0$ there is a $\eta > 0$ such that for every finite or infinite sequence of non-overlapping interval $\{[a_i, b_i]\}$, satisfying $\sum_i |b_i - a_i| < \eta$, we have $\sum_i \mathcal{O}(F; [a_i, b_i]) < \varepsilon$, where where $\mathcal{O}$ denotes the oscillation of f over $[a_i, b_i]$, i.e.,*

$$\mathcal{O}(f, [a_i, b_i]) = \sup\{|F(x) - F(y)|; x, y \in [a_i, b_i]\}.$$

A real-valued function F is said to be ACG^* on X if X is the union of a sequence of sets $\{X_i\}$ such that on each X_i the function F is $AC^*(X_i)$.

Definition 5 *A fuzzy-number-valued function f defined on $X \subset [a, b]$ is said to be weak generalized absolute continuous ($\tilde{f} \in \omega - ACG^*(X)$) if for every $\lambda \in [0, 1]$, the real-valued function $f^-_\lambda(x)$ and $f^+_\lambda(x)$ are ACG^*.*

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Theorem 2 *If $\tilde{f}$ is strong fuzzy Henstock integrable on $[a,b]$, then its primitive F is $\omega - ACG^*$ on $[a,b]$.*

Proof. For every $\varepsilon > 0$, there is a function $\delta(\xi) > 0$ such that for any δ-fine partial division $P = \{[u,v],\xi\}$ in $[a,b]$, we have

$$\sum D(F([u,v]), f(\xi)(v-u)) < \varepsilon.$$

We assume that $\delta(\xi) \leq 1$. Let

$$X_{n,i} = \{x \in [a,b] : D(f(x),\tilde{0}) \leq n, \frac{1}{n} < \delta(x) \leq \frac{1}{n-1}, x \in [a + \frac{i-1}{n}, a + \frac{i}{n})\}$$

for $n = 2,3,\cdots, i = 1,2,\cdots$. Fixed $X_{n,i}$ and let $\{[a_k,b_k]\}$ be any finite sequence of non-overlapping intervals with $a_k, b_k \in X_{n,i}$ for all k. Then $\{([a_k,b_k],a_k)\}$ is a δ-fine partial division of $[a,b]$. Furthermore, if $a_k \leq u_k \leq v_k \leq b_k$, then $\{([a_k,u_k],a_k)\}, \{([a_k,v_k],a_k)\}$ are δ-fine partial division of $[a,b]$. Thus

$$\begin{aligned}
\sum D(F(u_k),F(v_k)) &\leq \sum D(F(a_k),F(u_k)) + \sum D(F(b_k),F(v_k)) \\
&+ \sum D(F(a_k),F(b_k)) \\
&\leq 3\varepsilon + \sum D(f(a_k)(u_k-a_k),\tilde{0}) + \sum D(f(b_k)(b_k-v_k),\tilde{0}) \\
&+ \sum D(f(a_k)(b_k-a_k),\tilde{0}) \leq 3\varepsilon + 3n\sum(b_k-a_k).
\end{aligned}$$

Choose $\eta \leq \frac{\varepsilon}{3n}$ and $\sum(b_k - a_k) < \eta$. Then

$$\sum \mathcal{O}(F,[a_k,b_k]) \leq 3\varepsilon + \varepsilon.$$

Therefore, F is $\omega - AC^*(X_{n,i})$. Consequently, F is $\omega - ACG^*$ on $[a,b]$.

Theorem 3 *If there exists a fuzzy-number-valued function F is continuous and $\omega - ACG^*$ on $[a,b]$ such that $F'(x) = f(x)$ a.e. in $[a,b]$, then f is strong fuzzy Henstock integrable on $[a,b]$ with primitive F.*

Proof. Let F be the primitive of f and $F'(x) = f(x)$ for $x \in [a,b]\setminus S$ where S is of measure zero. For $\xi \in [a,b]\setminus S$, given $\varepsilon > 0$ there is a $\delta(\xi) > 0$ such that whenever $\xi \in [u,v] \subset (\xi - \delta(\xi), \xi + \delta(\xi))$ we have

$$D(F([u,v]), f(\xi)(v-u)) \leq \varepsilon|v-u|.$$

Since F is continuous and $\omega - ACG^*$ on $[a,b]$, there is a sequence of closed sets $\{X_i\}$ such that $\cup_i X_i = [a,b]$ and F is $\omega - AC^*(X_i)$ for each i. Let $Y_1 = X_1, Y_i = X_i \setminus (X_1 \cup X_2 \cdots \cup X_{i-1})$ for $i = 1,2,\cdots$ and S_{ij} denote the set of points $x \in S \cap Y_i$ such that $j - 1 \leq D(f,\tilde{0}) < j$. Obviously, S_{ij} are pairwise disjointed and their union is the set S. Since F is also $\omega - AC^*(S_{ij})$, there is a $\eta_{ij} < \varepsilon 2^{-i-j} j^{-1}$ such that for any sequence of non-overlapping intervals $\{I_k\}$ with at least one endpoint of I_k belonging to S_{ij} and satisfying $\sum_k |I_k| < \eta_{ij}$

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we have $\sum_k D(F(I_k), \tilde{0}) < \varepsilon 2^{-i-j}$. Again, $F(I)$ denotes $F(v) -_H F(u)$ where $I = [u, v]$. Choose G_{ij} to be the union of a sequence of open intervals such that $|G_{ij}| < \eta_{ij}$ and $G_{ij} \supset S_{ij}$ where $|G_{ij}|$ denotes the total length of G_{ij}. Now for $\xi \in S_{ij}$, put $(\xi - \delta(\xi), \xi + \delta(\xi)) \subset G_{ij}$. Hence we have defined a positive function $\delta(\xi)$.

Take any $\delta-$fine division $P = \{[u, v]; \xi\}$. Split the $\sum$ over P into partial sums $\sum_1$ and $\sum_2$ in which $\xi \bar{\in} S$ and $\xi \in S$ respectively and we obtain

$$\begin{aligned} D(f(\xi)(v-u), F([a,b])) &\le \sum_1 D(f(\xi)(v-u), F([a,b])) \\ &+ \sum_2 D(F([a,b]), \tilde{0}) + \sum_2 D(f(\xi)(v-u), \tilde{0}) \\ &< \varepsilon(b-a) + \sum_{i,j} \varepsilon 2^{-i-j} + \sum_2 j\eta_{ij} \\ &< \varepsilon(b-a) + 2\varepsilon. \end{aligned}$$

That is to say, f is strong fuzzy Henstock integrable to F on $[a, b]$.

Definition 6 *A sequence of fuzzy-number-valued functions $\{G_n(x)\}$ is said to be weak uniformly $ACG^*(U\omega - ACG^*)$ if for every $\lambda \in [0, 1]$, the real-valued functions $\{G_n(x)\}^-_\lambda$ and $\{G_n(x)\}^+_\lambda$ are $UACG^*$.*

Theorem 4 (Controlled Convergence theorem) *If a sequence of strong fuzzy Henstock integrable $\{f_n\}$ satisfies the following conditions:*

(1) $f_n(x) \to f(x)$ almost everywhere in $[a, b]$ as $n \to \infty$;

(2) the primitives $F_n(x) = (SFH) \int_a^x f_n(s)\mathrm{d}x$ of f_n are $\omega - ACG^*$ uniformly in n;

(3) the primitives $F_n(x)$ are equicontinuous on $[a, b]$,
then $f(x)$ is strong fuzzy Henstock integrable on $[a, b]$ and we have

$$\lim_{n\to\infty} (SFH) \int_a^b f_n(x)\mathrm{d}x = (SFH) \int_a^b f(x)\mathrm{d}x.$$

If condition (1) and (2) are replaced by condition (4):

(4) $g(x) \le f(x) \le h(x)$ almost everywhere on $[a, b]$, where $g(x)$ and $h(x)$ are steong fuzzy Henstock integrable.

Proof. In view of condition (3), $F(x)$ exist as the limit of $F_n(x)$ and is continuous. In fact, for $\forall\lambda \in [0, 1]$, $(F_n(x))^-_\lambda$ and $(F_n(x))^+_\lambda$ is uniformly ACG^* on $[a, b]$. By the Controlled Convergence theorem of real valued strong Henstock integral([14] Theorem 7.6), $F(x)$ is continuous. Because $F^-_\lambda(x)$ and $F^+_\lambda(x)$ is Henstock integrable on $[a, b]$, it follows condition (2) that F is $\omega - ACG^*$. From theorem 3.2, it remains to show that $F'(x) = f(x)$ almost everywhere. Hence we obtain $f(x)$ is strong fuzzy Henstock integrable on $[a, b]$.

Next, we put $G(x) = (SFH) \int_a^x F(t)\mathrm{d}t$, in view of condition (3), for $\forall\lambda \in [0, 1]$, we have

$$\lim_{n\to\infty} (F_n(x))^-_\lambda = G^-_\lambda(x) = F^-_\lambda(x)$$

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and

$$\lim_{n\to\infty}(F_n(x))^+_\lambda = G^+_\lambda(x) = F^+_\lambda(x).$$

So, let $x = b$, we have

$$\lim_{n\to\infty}(SFH)\int_a^b f_n(x)\mathrm{d}x = (SFH)\int_a^b f(x)\mathrm{d}x.$$

This completes the proof.

4 AN EXISTENCE RESULT OF GENERALIZED FUZZY INTEGRO-DIFFERENTIAL EQUATIONS

By using the Controlled Convergence theorem of strong fuzzy Henstock integral, in this section, we prove a theorem for the existence of solution to the Cauchy problem (1). For any bounded subset A of the Banach space X we denote $\alpha(A)$ the Kuratowski measure of non-compactness of A, i.e the infimum of all $\varepsilon > 0$ such that there exist a finite covering of A by sets of diameter less than ε. For the properties of α we refer to [1] for example.

Lemma 1 ([1]) *Let $H \subset C(I_\gamma, X)$ be a family of strong equicontinuous functions. Then*

$$\alpha(H) = \sup_{t\in I_\gamma}\alpha(H(t)) = \alpha(H(I_\gamma))$$

where $\alpha(H)$ denote the Kuratowski measure of non-compactness in $C(I_\gamma, X)$ and the function $t \to \alpha(H(t))$ is continuous.

Theorem 5 ([1]) *Let D be a closed convex subset of X, and let F be a continuous function from D into itself. If for $x \in D$ the implication*

$$\bar{V} = c\bar{o}n(\{x\} \cup F(V)) \Rightarrow V$$

is relatively compact, then F has a fixed point.

Theorem 6 *If the fuzzy-number-valued function $\tilde{f} : I_a \longrightarrow E^n$ is (SFH) integrable, then*

$$\int_I \tilde{f}(t)\mathrm{d}t \in |I| \cdot \overline{conv}\tilde{f}(I),$$

where $\overline{conv}\tilde{f}(I)$ is the closure of the convex of $\tilde{f}(I)$, I is an arbitrary subinterval of I_a, and $|I|$ is the length of I..

Proof. Because of $j \circ \tilde{f}$ is abstract (SH) integrable in a Banach Space, by using the mean valued theorem of (SH) integrals, we have

$$(SH)\int_I j\circ\tilde{f}(t)\mathrm{d}t \in |I|\cdot\overline{conv}j\circ\tilde{f}(I) = |I|\cdot j\circ\overline{conv}\tilde{f}(t).$$

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In additional, there exists $(SH)\int_I j\circ\tilde{f}(t)\mathrm{d}t = j\circ\int_I \tilde{f}(t)\mathrm{d}t$.

So, we have $j\circ\int_I \tilde{f}(t)\mathrm{d}t \in |I|\cdot\overline{conv} j\circ\tilde{f}(I)$. And the set $\{|I|\cdot\overline{conv}\tilde{f}(I)\}$ is a closed convex set, we have

$$\int_I \tilde{f}(t)\mathrm{d}t \in |I|\cdot\overline{conv}\tilde{f}(I).$$

Definition 7 *A fuzzy-number-valued function $\tilde{f}: I_a\times E^n \longrightarrow E^n$ is L^1-Carathéodory if the following conditions hold:*

(1) the fuzzy mapping $(x,y)\in E^n\times E^n$ is measurable for all $t\longrightarrow \tilde{f}(t,x,y)$;

(2) the fuzzy mapping $t\in I_a$ is continuous for all $(x,y)\longrightarrow \tilde{f}(t,x,y)$.

We observer that the problem (1) is equivalent to the integral eqution:

$$x(t) = x_0 + \int_0^t \tilde{f}(z, x(z), \int_0^z k_1(z,s)\tilde{g}(s,x(s))\mathrm{d}s, \int_0^a k_2(z,s)\tilde{h}(s,x(s))\mathrm{d}s)\mathrm{d}z$$

or

$$x(t) = x_0 + (-1)\cdot\int_0^t \tilde{f}(z, x(z), \int_0^z k_1(z,s)\tilde{g}(s,x(s))\mathrm{d}s, \int_0^a k_2(z,s)\tilde{h}(s,x(s))\mathrm{d}s)\mathrm{d}z. \tag{2}$$

Now, we define a notion of a solution.

Definition 8 *A $\omega - ACG^*$ function $x: I_a\to E^n$ is said to be the generalized solutions of the problem (1) if it satisfies the following conditions:*

(1) $x(0)=x_0$;

(2)

$$x'(t) = \tilde{f}(t, x(t), \int_0^t k_1(t,s)\tilde{g}(s,x(s))\mathrm{d}s, \int_0^a k_2(t,s)\tilde{h}(s,x(s))\mathrm{d}s).$$

for a. e. $t\in I_a$.

Definition 9 *A continuous function $x: I_a\to E^n$ is said to be the solutions of problem (2) if*

$$x(t) = x_0 + \int_0^t \tilde{f}(z, x(z), \int_0^z k_1(z,s)\tilde{g}(s,x(s))\mathrm{d}s, \int_0^a k_2(z,s)\tilde{h}(s,x(s))\mathrm{d}s)\mathrm{d}z$$

or

$$x(t) = x_0 + (-1)\cdot\int_0^t \tilde{f}(z, x(z), \int_0^z k_1(z,s)\tilde{g}(s,x(s))\mathrm{d}s, \int_0^a k_2(z,s)\tilde{h}(s,x(s))\mathrm{d}s)\mathrm{d}z.$$

for every $t\in I_a$

For every fuzzy number $x\in C(I_a, E^n)$, we define the norm of x by:

$$H(x,\tilde{0}) = \sup_{t\in I_a} D(x,\tilde{0}).$$

Let

$$B(p) = \{x \in C(I_a, E^n) | H(x, \tilde{0}) \leq H(x, \tilde{0}) + p, p > 0\}.$$

Obviously, $B(p)$ is closed and convex in E^n. Define the operator $F : C(I_a, E^n) \to C(I_a, E^n)$ by:

$$F(x)(t) = x_0 + \int_0^t \tilde{f}(z, x(z), \int_0^z k_1(z,s)\tilde{g}(s, x(s))\mathrm{d}s, \int_0^a k_2(z,s)\tilde{h}(s, x(s))\mathrm{d}s)\mathrm{d}z$$

where integrals are in the sense of strong fuzzy Henstock integral.

Let

$$\Gamma(p) = \{F(x) \in C(I_a, E^n) | x \in B(p)\}$$

for each $p > 0$. Let $r(K)$ be the spectral radius of the integral operator K defined by

$$K(u)(t) = \int_0^c k(t,s)u(s)\mathrm{d}s,$$

where the kernel $k \in C(I_a \times I_a, R), u \in C(I_a, E^n)$ and c denotes any fixed valued in I_a.

Next, we give the main result in this section.

Theorem 7 *Suppose that for each $\omega - ACG^*$ function $x : I_a \to E^n$, the functions $\tilde{g}(\cdot, x(\cdot)), \tilde{f}(\cdot, x(\cdot)), \int_0^{(\cdot)} k_1(\cdot, s)\tilde{g}(s, x(s))\mathrm{d}s$, and $\int_0^a k_2(z,s)\tilde{h}(s, x(s))\mathrm{d}s$ are (SFH) integrable, $\tilde{g}, \tilde{f}$, and $\tilde{h}$ are fuzzy L^1-Caratheodory functions. Let $k_1, k_2 : I_a \times I_a \to R^+$ be measurable functions such that $k_1(t, \cdot), k_2(t, \cdot)$ are continuous.*

Assume that there exists $p_0 > 0$ and positive constants L, L_1 and d_1, such that

$$\alpha(j \circ \tilde{g}(I, X)) \leq L\alpha(j \circ X), \quad I \subset I_a, X \subset B(p_0),$$
$$\alpha(j \circ \tilde{h}(I, X)) \leq L_1\alpha(j \circ X), \quad I \subset I_a, X \subset B(p_0),$$
$$\alpha(j \circ \tilde{f}(t, A, C, D)) \leq d_1 \cdot \max\{\alpha(j \circ A), \alpha(j \circ C), \alpha(j \circ D)\} \quad A, C, D \subset B(p_0),$$

where $\tilde{g}(I, X) = \{\tilde{g}(t, x(t)) | t \in I, x \in X\}, \tilde{h}(I, X) = \{\tilde{h}(t, x(t)) | t \in I, x \in X\}$ and

$$\tilde{f}(t, A, C, D) = \{\tilde{f}(t, x_1, x_2, x_3) | (x_1, x_2, x_3) \in A \times C \times D\}$$

where α denotes the Kuratowski measure of non-compactness.

Moreover, let $\Gamma(p_0)$ be equicontinuous, equibounded, and uniformly $\omega-ACG^*$ on I_a. Then, there exists at least on solution of problem (1) on I_c, for some $0 < c \leq a$, such that $d_1 \cdot c < 1$ and $d_1 \cdot c \cdot L \cdot r(K)$.

Proof. By equicontinuity and equiboundedness of $\Gamma(p_0)$ there exists a number $c, 0 < c \leq a$ such that

$$\begin{aligned}
&H(\int_0^t \tilde{f}(z, x(z), \int_0^z k_1(z,s)\tilde{g}(s, x(s))\mathrm{d}s, \int_0^a k_2(z,s)\tilde{h}(s, x(s))\mathrm{d}s)\mathrm{d}z, \tilde{0}) \\
=&\sup_{t \in I_c} D(\int_0^t \tilde{f}(z, x(z), \int_0^z k_1(z,s)\tilde{g}(s, x(s))\mathrm{d}s, \int_0^a k_2(z,s)\tilde{h}(s, x(s))\mathrm{d}s)\mathrm{d}z, \tilde{0}) \\
\leq& p_0,
\end{aligned}$$

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where $p_0 > 0, x \in B(p_0)$. By the definition of F, we have

$$
\begin{aligned}
&H(F(x)(t), \tilde{0}) \\
=&H(x_0 + \int_0^t \tilde{f}(z, x(z), \int_0^z k_1(z,s)\tilde{g}(s, x(s))\mathrm{d}s, \int_0^a k_2(z,s)\tilde{h}(s, x(s))\mathrm{d}s)\mathrm{d}z, \tilde{0}) \\
\leq&H(x_0, \tilde{0}) + H(\int_0^t \tilde{f}(z, x(z), \int_0^z k_1(z,s)\tilde{g}(s, x(s))\mathrm{d}s, \int_0^a k_2(z,s)\tilde{h}(s, x(s))\mathrm{d}s)\mathrm{d}z, \tilde{0}) \\
\leq&H(x_0, \tilde{0}) + p_0, \quad t \in I_c, x_0 \in E^n.
\end{aligned}
$$

Using Theorem 4, we deduce that the fuzzy-number-valued function F is continuous.

Obviously, there exists $V \subset B$ such that $\overline{V} = \overline{conv}(\{x\} \cup F(V))$ for every $x \in B(p_0)$. Next, we will prove that V is relatively compact.

In fact, let $V(t) = \{v(t) \in E^n | v \in V\}$ for $t \in I_c$. Since $V \subset B(p_0)$ and $F(V) \subset \Gamma(p_0)$, then $V \subset \overline{V}$ is equicontinuous. By Lemma 1, we get that $t \to v(t) = \alpha(j \circ V(t))$ is continuous on I_c. For fixed $t \in I_c$, we divide the interval $[0,t]$ into m parts: $0 = t_0 < t_1 < \cdots < t_m = t$, where $t_i = it/m, i = 0, 1, 2 \cdots, m$. Let $V([t_i, t_{i+1}]) = \{u(s) : u \in V, t_i \leq s \leq t_{i+1}, i = 1, 2, \cdots, m-1\}$ By Lemma 1 and the continuity of v, there exists $s_i \in I_i = [t_i, t_{i+1}]$ such that

$$\alpha(j \circ V([t_i, t_{i+1}])) = \sup_{t \in I_c}\{\alpha(j \circ V(s)) | t_i \leq s \leq t_{i+1}\} := v(s_i).$$

For fixed $z \in [0,t]$, we divide the interval $[0,z]$ into m parts: $0 = z_0 < z_1 < \cdots < z_m = z$, where $z_j = jz/m, j = 0, 1, 2 \cdots, m$. Let $V([z_j, z_{j+1}]) = \{u(s) | u \in V, z_j \leq s \leq z_{j+1}\}, j = 0, 1, 2, \cdots, m-1$. By Lemma 1 and the continuity of v, there exists $s_j \in I_j = [z_j, z_{j+1}]$ such that

$$\alpha(j \circ V([z_j, z_{j+1}])) = \sup_{t \in I_c}\{\alpha(j \circ V(s)) | z_j \leq s \leq z_{j+1}\} := v(s_j).$$

Furthermore, we divide the interval $[0,c]$ into m parts: $0 = r_0 < r_1 < \cdots < r_m = c$, where $r_k = kc/m, k = 0, 1, 2 \cdots, m$. Let $V([r_k, r_{k+1}]) = \{u(s) | u \in V, r_k \leq s \leq r_{k+1}\}, j = 0, 1, 2, \cdots, m-1$. By Lemma 1 and the continuity of v, there exists $s_k \in I_k = [r_k, r_{k+1}]$ such that

$$\alpha(j \circ V([r_k, r_{k+1}])) = \sup_{t \in I_c}\{\alpha(j \circ V(s)) | r_k \leq s \leq r_{k+1}\} := v(s_k).$$

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By Theorem 3 and Theorem 4, we have

$$F(x)(t) = x_0 + \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \tilde{f}(z, x(z), \sum_{j=0}^{m-1} \int_{z_j}^{z_{j+1}} k_1(z,s)\tilde{g}(s, x(s))\mathrm{d}s,$$

$$\sum_{k=0}^{m-1} \int_{r_k}^{r_{k+1}} k_2(z,s)\tilde{h}(s, x(s))\mathrm{d}s)\mathrm{d}z \in x_0$$

$$+ \sum_{i=0}^{m-1}(t_{i+1} - t_i)\overline{conv}\tilde{f}(I_i, V(I_i), \sum_{j=0}^{m-1}(z_{j+1} - z_j)\overline{conv}(k_1(I_i, I_j)\tilde{g}(I_j, V(I_j))),$$

$$\sum_{k=0}^{m-1}(r_{j+1} - r_j)\overline{conv}(k_2(I_i, I_j)\tilde{h}(I_k, V(I_k))),$$

where $k(I, J) = \{k(t,s)|t \in I, s \in J\}$ and $\tilde{g}(I, V(I)) = \{\tilde{g}(t, x(t))|t \in I, x \in V\}$.

Using the condition in assumption and the properties of noncompactness α ([1]), we have

$$\alpha(j \circ F(V)(t))$$

$$\leq \sum_{i=0}^{m-1}(t_{i+1} - t_i)\overline{conv}\alpha(j \circ \tilde{f}(I_i, V(I_i), \sum_{j=0}^{m-1}(z_{j+1} - z_j)\overline{conv}(k_1(I_i, I_j)\tilde{g}(I_j, V(I_j))),$$

$$\sum_{k=0}^{m-1}(r_{j+1} - r_j)\overline{conv}(k_2(I_i, I_j)\tilde{h}(I_k, V(I_k))))$$

$$\leq \sum_{i=0}^{m-1}(t_{i+1} - t_i)d_1 \max\{(\alpha(j \circ V(I_i)), \alpha j \circ (\sum_{j=0}^{m-1}(z_{j+1} - z_j)\overline{conv}(k_1(I_i, I_j)\tilde{g}(I_j, V(I_j)))),$$

$$\alpha j \circ (\sum_{k=0}^{m-1}(r_{j+1} - r_j)\overline{conv}(k_2(I_i, I_j)\tilde{h}(I_k, V(I_k)))).$$

We observe that if

$$\alpha(j \circ V(I_i)) = \max\{(\alpha(j \circ V(I_i)), \alpha j \circ (\sum_{j=0}^{m-1}(z_{j+1} - z_j)\overline{conv}(k_1(I_i, I_j)\tilde{g}(I_j, V(I_j)))),$$

$$\alpha(j \circ (\sum_{k=0}^{m-1}(r_{j+1} - r_j)\overline{conv}(k_2(I_i, I_j)\tilde{h}(I_k, V(I_k))))),$$

then

$$\alpha(j \circ V(t)) = \alpha j \circ (\overline{conv}(\{x(t)\} \cup F(V(t))))\alpha(j \circ F(V(t))) \leq d_1 \cdot c \cdot \alpha(j \circ V(t))$$

for every $t \in I_c$. Because $d_1 \cdot c < 1$, we have $\alpha(j \circ V) < \alpha(j \circ V)$. This is a contradiction.

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If

$$\alpha(j\circ(\sum_{j=0}^{m-1}(z_{j+1}-z_j)\overline{conv}(k_1(I_i,I_j)\tilde{g}(I_j,V(I_j))))))$$
$$=\max\{\alpha(j\circ V(I_i)),\alpha j\circ(\sum_{j=0}^{m-1}(z_{j+1}-z_j)\overline{conv}(k_1(I_i,I_j)\tilde{g}(I_j,V(I_j)))),$$
$$\alpha(j\circ(\sum_{k=0}^{m-1}(r_{j+1}-r_j)\overline{conv}(k_2(I_i,I_j)\tilde{h}(I_k,V(I_k))))))\},$$

we have

$$\begin{aligned}&\alpha(j\circ F(V)(t))\\&\leq\sum_{i=0}^{m-1}(t_{i+1}-t_i)\cdot d_1\cdot\sum_{j=0}^{m-1}(z_{j+1}-z_j)k_1(I_i,I_j)\alpha(j\circ\tilde{g}(I_j,V(I_j)))\\&\leq\sum_{i=0}^{m-1}(t_{i+1}-t_i)\cdot d_1\cdot L\cdot\sum_{j=0}^{m-1}(z_{j+1}-z_j)k_1(I_i,I_j)\alpha(j\circ V(I_j))\\&\leq\frac{c}{m}\cdot\sum_{j=0}^{m-1}(z_{j+1}-z_j)\alpha(j\circ V(I_j))\sum_{i=0}^{m-1}k_1(I_i,I_j).\end{aligned}$$

For $j=0,1,2,\ldots,m-1$, there exists $q_j=0,1,2,\ldots,m-1$ such that $k_1(I_i,I_j)\leq k_1(I_{q_j},I_j)$. So,

$$\alpha(j\circ F(V)(t))\leq d_1\cdot c\cdot L\cdot\sum_{j=0}^{m-1}(z_{j+1}-z_j)k_1(I_{q_j},I_j)v(s_j),\quad s_j\in I_j.$$

Hence

$$\begin{aligned}&\alpha(j\circ F(V)(t))\\&\leq d_1\cdot c\cdot L\cdot\sum_{j=0}^{m-1}(z_{j+1}-z_j)k_1(I_{q_j},I_j)(v(s_j)-v(p_j))\\&+d_1\cdot c\cdot L\cdot\sum_{j=0}^{m-1}(z_{j+1}-z_j)k_1(I_{q_j},I_j)v(p_j).\end{aligned}$$

By the continuity of v, we have $j\circ v(s_j)-j\circ v(p_j)<\varepsilon$. Therefore, we have

$$\alpha(j\circ F(V)(t))\leq d_1\cdot c\cdot L\cdot\int_0^c k_1(t,s)v(s)\mathrm{d}s$$

for $t\in I_c$. Since $V=\overline{conv}(\{x\}\cup F(V))$, we have $\alpha(j\circ V(t))\leq\alpha(j\circ F(V)(t))$, so, $v(t)\leq d_1\cdot c\cdot L\cdot\int_0^c k_1(t,s)v(s)\mathrm{d}s$. By Gronwalls inequality, we have $\alpha(j\circ V(t))=0$ for $t\in I_c$. By Arzelá−Ascoli's theorem, we have V is relatively. Consequently,

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by Theorem 5, F has a fixed point. That is to say that problem (1) have at least solutions.

Similary, if

$$\alpha(j\circ(\sum_{k=0}^{m-1}(r_{j+1}-r_j)\overline{conv}(k_2(I_i,I_j)\tilde{h}(I_k,V(I_k)))))$$

$$=\max\{\alpha(j\circ V(I_i)),\alpha j\circ(\sum_{j=0}^{m-1}(z_{j+1}-z_j)\overline{conv}(k_1(I_i,I_j)\tilde{g}(I_j,V(I_j)))),$$

$$\alpha(j\circ(\sum_{k=0}^{m-1}(r_{j+1}-r_j)\overline{conv}(k_2(I_i,I_j)\tilde{h}(I_k,V(I_k)))))\},$$

then we have $\alpha(j\circ V(t))\leq\alpha(j\circ F(V)(t))$. By Arzel$\acute{a}-$Ascoli's theorem, the set V is relatively. By Theorem 5, F has a fixed point which is a solution of the problem (1).

5 CONCLUSIONS

In this paper, we give the definition of the $\omega-ACG^*$ for a fuzzy-number-valued function and a generalized controlled convergence theorem. In addition, we deal with the Cauchy problem of discontinuous fuzzy integro-differential equations of mixed type involving the strong fuzzy Henstock integral in fuzzy number space. The function governing the equations is supposed to be discontinuous with respect to some variables and satisfy nonabsolute fuzzy integrablility. Our result improves the result given in Ref. [11, 2] and [26] (where uniform continuity was required), as well as those referred therein.

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On the Generalized Stieltjes Transform of Fox's Kernel Function and its Properties in the Space of Generalized Functions

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Abstract

In this paper, a Stieltjes transform enfolding some Fox's H-function has been investigated on certain class of generalized functions named as Boehmians. By developing two spaces of Boehmians, the extended transform has been inspected and some general properties are also obtained. An inverse problem is also discussed in some detail.

Keywords: Fox's H-function; Stieltjes transform; Laplace transform; Boehmian space; Distribution space.

1 Introduction

The Fox's H-function is a generalization of the Meijer G-function introduced by Charles Fox [15]. It is defined by the compact notation adopted for

$$H_{p,q}^{m,n}(\omega) = H_{p,q}^{m,n}\left[\omega \left| \begin{array}{l} (a_j,\alpha_j)_{j=1,2,\ldots,p} \\ (b_j,\beta_j)_{j=1,2,\ldots,q} \end{array} \right. \right]$$

and has an exemplification in terms of the Barnes-type integral [2]

$$H_{p,q}^{m,n}(\omega) = \frac{1}{2\pi i}\int_{\mathcal{L}} \jmath_{p,q}^{m,n}(\varsigma)\,\omega^{\varsigma}d\varsigma,$$

where $\mathcal{L}$ is a path in the complex plane, $\omega^{\varsigma} = \exp\{\varsigma(\log|\omega| + i\arg\omega)\}$, and

$$\jmath_{p,q}^{m,n}(\varsigma) = \frac{\boldsymbol{a}(\varsigma)\,\boldsymbol{b}(\varsigma)}{\boldsymbol{c}(\varsigma)\,\boldsymbol{d}(\varsigma)},$$

where

$$\begin{aligned} \boldsymbol{a}(\varsigma) &:= \prod_1^m \Gamma\left(b_j - \beta_j\varsigma\right), \boldsymbol{b}(\varsigma) := \prod_1^n \Gamma\left(1 - a_j + \alpha_j\varsigma\right) \\ \boldsymbol{c}(\varsigma) &:= \prod_{m+1}^q \Gamma\left(1 - b_j - \beta_j\varsigma\right) \text{ and } \boldsymbol{d}(\varsigma) := \prod_{n+1}^p \Gamma\left(a_j + \alpha_j\varsigma\right), \end{aligned}$$

with $m, p, q \in \mathbb{N}$, $a_j, b_j \in \mathbb{C}, \alpha_j, \beta_j \in \mathbb{R}^+, n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ satisfying $0 < n < p$ and $0 < m < q$, and $\mathbb{C}, \mathbb{R}^+$ and $\mathbb{N}$ denote, respectively, the sets of complex numbers, positive real numbers and positive integers.

We refer to the survey article by Braaksma [2] and the book of Charles Fox [15] for asymptotic behaviour of Fox's H-functions.

Fox's H-functions being an extreme generalization of the generalized hypergeometric functions ${}_pF_q$ are utilized for applications in a large variety of problems connected with statistical distribution theory, structures of random variables, generalized distributions, Mathai's pathway models, versatile integrals, reaction, diffusion, reaction diffusion, engineering, communication, fractional differential and integral equations and many areas of theoretical physics and statistical distribution theory as well.

Recently, utility and importance of H-functions are realized due to their occurrence as kernels of certain integral transforms.

The generalized Stieltjes transform of a function $\varphi(t)$ of one variable with kernel involving Fox's H-function is defined by [5, (1.3)]

$$\chi_g^s(\varphi)(\omega) = \int_0^\infty \omega^{\square 1} H_{2,2}^{1,2}\left[\left(\frac{\xi}{\omega}\right)^\lambda \middle| \begin{array}{l} (a_1, \alpha_1), (1 - b_1 - \beta_1, \lambda\beta_1) \\ (e_1, \gamma_1), (e_2, \gamma_2) \end{array}\right] \varphi(\xi)\, \mathtt{d}\xi, \tag{1}$$

where $H_{p,q}^{m,n}[\omega]$ is the usual notation of the Fox H-function.

An interesting fact that we find it worthwhile to be mentioned here is that the transform under consideration is a modulation of the Laplace transform

$$\chi_\ell(\varphi)(\omega) = \int_0^\infty H_{2,2}^{1,2}\left[(\xi\omega)^\lambda \middle| \begin{array}{l} (a_1, \alpha_1) \\ (e_1, \gamma_1), (e_2, \gamma_2) \end{array}\right] \varphi(\xi)\, \mathtt{d}\xi \tag{2}$$

that rectified after some iterations and an appropriate choice on its parameter.

Denote by $\mathcal{J}_{c,d}$ the Fréchet space of smooth functions φ defined for all ξ $(0 < \xi < \infty)$ by the set $\{\boldsymbol{\delta}_{c,d,k}\}$ of seminorms where

$$\boldsymbol{\delta}_{c,d,k}(\varphi) = \sup_{0<\xi<\infty} \left| \varrho_{c,d}(\log\xi)(\xi D_\xi)^k \sqrt{\xi}\varphi(\xi) \right| < \infty \tag{3}$$

for every choice of $k\,(k \in \mathbb{N}_0)$,

$$\varrho_{c,d}(\log\xi) = \begin{cases} \xi^c, & 1 \le \xi < \infty \\ \xi^d, & 0 < \xi < 1 \end{cases},$$

c and d are being real numbers.

The strong dual of continuous linear forms on $\mathcal{J}_{c,d}$ is denoted by $\mathcal{J}_{c,d}'$.

Let p_1 and q_1 be real numbers defined by $p_1 = \min\left(\operatorname{Re}\frac{b_j}{\beta_j}\right)$ $(j = 1, 2, ..., m)$, $q_1 = \max\left(\operatorname{Re}\frac{a_j \square 1}{\alpha_j}\right)$ $(j = 1, 2, ..., n)$ and related by the pair of inequalities $c + \dfrac{1}{2} + \lambda q_1 < 0$ and $d + \dfrac{1}{2} + \lambda p_1 > 0$.

Then, the extended transform of a distribution $f \in \mathcal{J}_{c,d}'$ is defined as the application of $f(t) \in \mathcal{J}_{c,d}'$ to its kernel (see [5, Theorem 3.1])

$$\omega^{\square 1} H_{2,2}^{1,2}\left[\left(\frac{\xi}{\omega}\right)^\lambda \middle| \begin{array}{l} (a_1, \alpha_1), (1 - b_1 - \beta_1, \lambda\beta_1) \\ (e_1, \gamma_1), (e_2, \gamma_2) \end{array}\right]$$

giving, by kernel method,

$$\chi_g^s(f)(\omega) = \left\langle f(\xi), \omega^{\square 1} H_{2,2}^{1,2}\left[\left(\frac{\xi}{\omega}\right)^\lambda \middle| \begin{array}{l} (a_1, \alpha_1), (1 - b_1 - \beta_1, \lambda\beta_1) \\ (e_1, \gamma_1), (e_2, \gamma_2) \end{array}\right]\right\rangle, \tag{4}$$

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where ω is a complex number not lying on the negative real axis.

For our consecutive investigation, we denote by $\mathcal{I}_{c,d}$ the subset of those integrable functions of $\mathcal{J}_{c,d}$ assigned by the set $\{\boldsymbol{\delta}_{c,d,k}\}$ and its strong dual $\mathcal{I}_{c,d}^{'}$ of distributions. Then, indeed, $\mathcal{I}_{c,d} \subseteq \mathcal{J}_{c,d}$ and, hence, $\mathcal{J}_{c,d}^{'} \subseteq \mathcal{I}_{c,d}^{'}$. Denote by $\mathcal{D}$ the standard notation of the space of smooth functions whose supports are compact subset of $(0, \infty)$. Then, it is easy to check that $\mathcal{D} \subset \mathcal{I}_{c,d}$ and that topology of $\mathcal{D}$ is stronger than the topology induced on it by $\mathcal{I}_{c,d}$. Hence, the restriction to any $f \in \mathcal{I}_{c,d}^{'}$ to $\mathcal{D}$ is in $\mathcal{D}^{'}$, where $\mathcal{D}^{'}$ is the space of disributions.

We need to establish the following theorem.

Theorem 1 Given $\varphi \in \mathcal{I}_{c,d}$. Then, $\chi_g^s(\varphi) \in \mathcal{I}_{c,d}$.
Proof Let $\varphi \in \mathcal{I}_{c,d}$ be given. For the convenience of the reader, we write

$$H_{2,2}^{1,2}\left[\left(\frac{y}{\xi}\right)^{\lambda}\right] = H_{2,2}^{1,2}\left[\left(\frac{y}{\xi}\right)^{\lambda} \middle| \begin{array}{l} (a_1,\alpha_1),(1-b_1-\beta_1,\lambda\beta_1) \\ (e_1,\gamma_1),(e_2,\gamma_2) \end{array}\right].$$

By aid of (3) and (1) and simple computation we write

$$\left|\varrho_{c,d}(\log\xi)(\xi\mathcal{D}_\xi)^k\sqrt{\xi}\left(\chi_g^s\right)(\varphi)(\xi)\right| \leq \int_0^\infty \left|\varrho_{c,d}(\log\xi)(\xi\mathcal{D}_\xi)^k\sqrt{\xi}\xi^{-1}H_{2,2}^{1,2}\left[\left(\frac{y}{\xi}\right)^{\lambda}\right]\right| \times|\varphi(y)|\,\mathtt{d}y.$$

This can also be revised to give

$$\left|\varrho_{c,d}(\log\xi)(\xi\mathcal{D}_\xi)^k\sqrt{\xi}\left(\chi_g^s\right)(\varphi)(\xi)\right| \leq \int_0^\infty \left|\varrho_{c,d}(\log\xi)(\xi\mathcal{D}_\xi)^k\left(\xi^{-1}\right)^{\frac{1}{2}}H_{2,2}^{1,2}\left[\left(\frac{y}{\xi}\right)^{\lambda}\right]\right| \times|\varphi(y)|\,\mathtt{d}y.$$

By utilizing the Property 2.8

$$\mathcal{D}_z^k\left\{z^w H_{p,q}^{m,n}\left[cz^\sigma \middle| \begin{array}{l} (a_i,\alpha_i)_{1,p} \\ (b_j,\beta_j)_{1,q} \end{array}\right]\right\} = z^{w-k}H_{p+1,q+1}^{m,n+1}\left[cz^\sigma \middle| \begin{array}{l} (-w,\sigma),(a_i,\alpha_i)_{1,p} \\ (b_j,\beta_j)_{1,q},(k-w,\sigma) \end{array}\right]$$

of Kilbas and Saigo [1, p.33] we get

$$\left|\varrho_{c,d}(\log\xi)(\xi D_\xi)^k\sqrt{\xi}\left(\chi_g^s\right)(\varphi)(\xi)\right| \leq \int_0^\infty \left|\varrho_{c,d}(\log\xi)\,\xi^{-\frac{1}{2}}\,\widehat{H_{3,3}^{1,3}}\left[\left(\frac{y}{\xi}\right)^{\lambda}\right]\right||\varphi(y)|\,\mathtt{d}y,$$

where

$$\widehat{H_{3,3}^{1,3}}\left[\left(\frac{y}{\xi}\right)^{\lambda}\right] = H_{3,3}^{1,3}\left[\left(\frac{y}{\xi}\right)^{\lambda} \middle| \begin{array}{l} \left(\frac{1}{2},\lambda\right),(a_1,\alpha_1),(1-b_1-\beta_1,\lambda\beta_1) \\ (e_1,\gamma_1),(e_2,\gamma_2),\left(k-\frac{1}{2},\lambda\right) \end{array}\right].$$

Therefore, the asymptotic properties of H-functions, for large ξ, imply

$$\sup_{0<\xi<\infty}\left|\varrho_{c,d}(\log\xi)\,\xi^{-\frac{1}{2}}\,\widehat{H_{3,3}^{1,3}}\left[\left(\frac{y}{\xi}\right)^{\lambda}\right]\right| = \sup_{0<\xi<\infty}\left|\xi^c\xi^{-\frac{1}{2}}\,\widehat{H_{3,3}^{1,3}}\left[\left(\frac{y}{\xi}\right)^{\lambda}\right]\right| < M_1,$$

where M_1 is some positive constant. Similarly, for small ξ, it implies

$$\sup_{0<\xi<\infty}\left|\varrho_{c,d}(\log\xi)\,\xi^{-\frac{1}{2}}\,\widehat{H_{3,3}^{1,3}}\left[\left(\frac{y}{\xi}\right)^{\lambda}\right]\right| = \sup_{0<\xi<\infty}\left|\xi^d\xi^{-\frac{1}{2}}\,\widehat{H_{3,3}^{1,3}}\left[\left(\frac{y}{\xi}\right)^{\lambda}\right]\right| < M_2,$$

where M_2 is a postive constant.
Let $M = \max\{M_1, M_2\}$. Then, by the preceding two formulas, we have

$$\sup_{0<\xi<\infty} \left| \varrho_{c,d}(\log \xi)(\xi \mathcal{D}_\xi)^k \sqrt{\xi} \left(\chi_g^s\right)(\varphi)(\xi) \right| \leq M \int_0^\infty |\varphi(y)| \, \mathtt{d}y < \infty,$$

since φ is integrable.

The proof of this theorem is finished.

Definition 2 Let $f \in \acute{\mathcal{I}}_{c,d}$. Then, the Stieltjes transform χ_g^s of $f \in \acute{\mathcal{I}}_{c,d}$ is defined by the inner product

$$\left\langle \chi_g^s(f)(\omega), \varphi(\omega) \right\rangle = \left\langle f(\omega), \chi_g^s(\varphi)(\omega) \right\rangle, \tag{6}$$

where $\varphi \in \mathcal{I}_{c,d}$ is aritrary.

The inner product on the left hand side of (6) is well-defined by Theorem 1. Hence, it may be noted from Equation 6 that the Stieltjes transform of $f \in \acute{\mathcal{I}}_{c,d}$ is a distribution in $\acute{\mathcal{I}}_{c,d}$.

2 Generalized Distributions; Boehmian Spaces

We always assume that readers are aquainted with the concept of Boehmian spaces, if it were otherwise, we would refer to [4], [6 − 14] and [16, 17].

Let us now prove the following Theorems that legitimate the existence of our Boehmian spaces.

The following definition is important for our next investigation.

Definition 3 Given $\varphi, \psi \in \mathcal{I}_{c,d}$, then, for φ and ψ, the product $\otimes$ is defined by

$$(\varphi \otimes \psi)(\omega) = \int_0^\infty \varphi\left(\xi^{\square 1}\omega\right) \frac{\psi(\xi)}{\xi} \mathtt{d}\xi, \tag{7}$$

provided the integral exists.

Theorem 4 Given $\varphi \in \mathcal{I}_{c,d}$, then $\varphi \otimes \psi \in \mathcal{I}_{c,d}$, for every $\psi \in \mathcal{D}$.

Proof On account of (3), we write

$$\begin{aligned} \left| \varrho_{c,d}(\log \xi)(\xi \mathcal{D}_\xi)^k \sqrt{\xi}(\varphi \otimes \psi)(\xi) \right| &\leq \int_0^\infty \left| y^{\square 1}\psi(y) \right| \\ &\quad \times \left| \varrho_{c,d}(\log \xi)(\xi \mathcal{D}_\xi)^k \sqrt{\xi}\varphi\left(y^{\square 1}\xi\right) \right| \mathtt{d}y \\ &\leq A^* \int_0^\infty \left| y^{\square 1}\psi(y) \right| \mathtt{d}y. \end{aligned}$$

Let $[a_1, a_2]$ be a closed interval containing the support of ψ. Since $\varphi \in \mathcal{I}_{c,d}$, it by considering supremum over all $\xi\,(0 < \xi < \infty)$ follows that

$$\boldsymbol{\delta}_{c,d,k}(\varphi \otimes \psi) \leq A^* \int_{a_1}^{a_2} \left| y^{\square 1}\psi(y) \right| \mathtt{d}y < \infty,$$

for some constant A^*.

Hence, the proof of this theorem is finished.

Let $\curlyvee$ be the product of Mellin type given by

$$(\varphi \curlyvee \psi)(y) = \int_0^\infty \xi^{\square 1}\varphi\left(\xi^{\square 1}y\right)\psi(\xi)\,\mathtt{d}\xi. \tag{8}$$

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We generate the space $\mathcal{B}\left(\left(\mathcal{I}_{c,d},\otimes\right);\left(\mathcal{D},\curlyvee\right)\right)$ where Δ is the subset of $\mathcal{D}$ of sequences (δ_n) such that

$$\left.\begin{array}{l} \text{(i)}\ \displaystyle\int_0^\infty \delta_n(\xi)\,\mathtt{d}\xi = 1; \\ \text{(ii)}\ \left|\delta_n(\xi)\right| < A,\ A \in \mathbb{R},\ A > 0; \\ \text{(iii)}\ \operatorname{supp}\delta_n(\xi) \subseteq (a_n, b_n)\,,\ a_n, b_n \to 0 \text{ as } n \to \infty, \end{array}\right\} \tag{9}$$

$n \in \mathbb{N},\ \xi \in (0,\infty)$.
In what follows we shall make a free use of the properties of the product $\curlyvee$ that we briefly describe them as follows :

(i) $\varphi_1 \curlyvee \varphi_2 = \varphi_2 \curlyvee \varphi_1$;
(ii) $(\varphi_1 \curlyvee \varphi_2) \curlyvee \varphi_3 = \varphi_1 \curlyvee (\varphi_2 \curlyvee \varphi_3)\,$;
(iii) $(\alpha\varphi_1) \curlyvee \varphi_2 = \alpha\,(\varphi_1 \curlyvee \varphi_2)\,$;
(iv) $\varphi_1 \curlyvee (\varphi_2 + \varphi_3) = \varphi_1 \curlyvee \varphi_2 + \varphi_1 \curlyvee \varphi_3$.

Following theorem follows from elementary rules of integral calculus. Hence, its proof is deleted.

Theorem 5 Given $\varphi_n, \varphi, \varphi_1, \varphi_2 \in \mathcal{I}_{c,d}$, $\alpha \in \mathbb{C}$, and $\psi, \psi_1, \psi_2 \in \mathcal{D}$ such that $\varphi_n \to \varphi$ as $n \to \infty$, then the following are true :

(i) $\varphi_n \otimes \psi \to \varphi \otimes \psi$ as $n \to \infty$.
(ii) $\varphi_1 \otimes (\psi_1 + \psi_2) = \varphi_1 \otimes \psi_1 + \varphi_2 \otimes \psi_2$.
(iii) $\alpha\,(\varphi \otimes \psi) = \alpha\varphi \otimes \psi = \varphi \otimes (\alpha\psi)\,$.

Theorem 6 Given $\varphi \in \mathcal{I}_{c,d}$ and $\psi_1, \psi_2 \in \mathcal{D}$, then $\varphi \otimes (\psi_1 \curlyvee \psi_2) = (\varphi \otimes \psi_1) \otimes \psi_2$.
Proof Let $\varphi \in \mathcal{I}_{c,d}$ and $\psi_1, \psi_2 \in \mathcal{D}$. Then, by aid of the integrals (7) and (8) , we write

$$\begin{aligned} (\varphi \otimes (\psi_1 \curlyvee \psi_2))(\omega) &= \int_0^\infty \varphi\left(\xi^{\square 1}\omega\right)\frac{(\psi_1 \curlyvee \psi_2)(\xi)}{\xi}\mathtt{d}\xi \\ &= \int_0^\infty \psi_2(y)\,y^{\square 1}\int_0^\infty \varphi\left(\xi^{\square 1}\omega\right)\frac{\psi_1\left(\xi y^{\square 1}\right)}{\xi}\mathtt{d}y\mathtt{d}\xi \\ &= \int_0^\infty \psi_2(y)\,\frac{\int_0^\infty \varphi\left(y^{\square 1}z^{\square 1}\omega\right)z^{\square 1}\psi(z)\,\mathtt{d}z}{y}\mathtt{d}y \\ &= \int_0^\infty \psi_2(y)\,\frac{(\varphi \otimes \psi_1)\left(y^{\square 1}\omega\right)}{y}\mathtt{d}y. \end{aligned}$$

The proof of this theorem is finished.

Theorem 7 Given $(\delta_n) \in \Delta$ and $\varphi \in \mathcal{I}_{c,d}$, then $\varphi \otimes \delta_n \in \mathcal{I}_{c,d}$.
Proof Let $\varphi \in \mathcal{I}_{c,d}$ and $(\delta_n) \in \Delta$ be given. Then, by (3) and the Identity (i) of (9) we have

$$\begin{aligned} \left|\varrho_{c,d}(\log\xi)\,(\xi\mathcal{D}_\xi)^k\sqrt{\xi}\,(\varphi \otimes \delta_n - \varphi)(\xi)\right| &= \int_0^\infty \left|\varrho_{c,d}(\log\xi)\,(\xi\mathcal{D}_\xi)^k\sqrt{\xi}\varphi_y(\xi)\right| \\ &\quad \times \left|\delta_n(y)\right|\mathtt{d}y, \end{aligned} \tag{10}$$

where $\varphi_y(\xi) = \varphi\left(\xi y^{\square 1}\right)y^{\square 1} - \varphi(\xi)\,$. Since $\varphi_y(\xi) \in \mathcal{I}_{c,d}$, we from (10) , get that

$$\left|\varrho_{c,d}(\log\xi)\,(\xi\mathcal{D}_\xi)^k\sqrt{\xi}\,(\varphi \otimes \delta_n - \varphi)(\xi)\right| \leq A\int_0^\infty \left|\delta_n(y)\right|\mathtt{d}y, \tag{11}$$

where A is some positive constant.
Hence, by the identities (ii) and (iii) of (9) , Equation (11) can be expressed as

$$\left|\varrho_{c,d}(\log\xi)\,(\xi\mathcal{D}_\xi)^k\sqrt{\xi}\,(\varphi \otimes \delta_n - \varphi)(\xi)\right| \leq AA_1(b_n - a_n) \to 0 \text{ as } n \to \infty. \tag{12}$$

Hence, the proof of this theorem is finished.

THEOREM 8 Given $\varphi \in \mathcal{I}_{c,d}$, then, for every $(\delta_n) \in \Delta$, we have $\varphi \otimes \delta_n \to \varphi$ in $\mathcal{I}_{c,d}$ as $n \to \infty$.
PROOF Let F_n be a compact subset of $(0,\infty)$ containing $\operatorname{supp} \delta_n$, for all n. Then, on account of (i) of (9), we get

$$\begin{aligned} \left| \varrho_{c,d} (\log \xi) (\xi \mathcal{D}_\xi)^k \sqrt{\xi} (\varphi \otimes \delta_n - \varphi)(\xi) \right| \quad \leq \quad & \int_{F_n} \left| \varrho_{c,d} (\log \xi) (\xi \mathcal{D}_\xi)^k \sqrt{\xi} \varphi \left(x^{\square 1} \xi \right) \right| \\ & \times \frac{|\delta_n (x)|}{x} \mathrm{d}x \\ & + \int_{F_n} \left| \varrho_{c,d} (\log \xi) (\xi \mathcal{D}_\xi)^k \sqrt{\xi} \varphi (\xi) \right| \\ & \times |\delta_n (x)| \,\mathrm{d}x. \end{aligned} \tag{13}$$

Therefore, (13) gives

$$\left| \varrho_{c,d} (\log \xi) (\xi \mathcal{D}_\xi)^k \sqrt{\xi} (\varphi \otimes \delta_n - \varphi)(\xi) \right| \leq A_1 \int_{F_n} \frac{|\delta_n (x)|}{x} \mathrm{d}x + A_2 \int_{F_n} |\delta_n (x)| \,\mathrm{d}x.$$

Considering the supremum over all ξ, $0 < \xi < \infty$, implies

$$\boldsymbol{\delta}_{c,d,k} (\varphi \otimes \delta_n - \varphi) < \infty,$$

for any choice of the real numbers c, d and $k \in \mathbb{N}_0$. Thus, we find that

$$\varphi \otimes \delta_n \to \varphi \text{ in } \mathcal{I}_{c,d} \text{ as } n \to \infty.$$

The proof has been completed .

COROLLARY 9 Given $(\delta_n) \in \Delta$ and $\varphi_1 \otimes \delta_n = \varphi_2 \otimes \delta_n$, then $\varphi_1 = \varphi_2$ for all $\varphi_1, \varphi_2 \in \mathcal{I}_{c,d}$.

The space $\mathcal{B}((\mathcal{I}_{c,d}, \otimes); (\mathcal{D}, \curlyvee))$ is constructed.

Addition and multiplication by a scalar in $\mathcal{B}((\mathcal{I}_{c,d}, \otimes); (\mathcal{D}, \curlyvee))$ are defined by

$$\left[\frac{\varphi_n}{\delta_n}\right] + \left[\frac{\psi_n}{\varepsilon_n}\right] =: \left[\frac{\varphi_n \otimes \delta_n + \psi_n \otimes \delta_n}{\delta_n \curlyvee \varepsilon_n}\right] \text{ and } \mu \left[\frac{\varphi_n}{\delta_n}\right] =: \left[\frac{\mu \varphi_n}{\delta_n}\right] \quad (\mu \in \mathbb{C}).$$

An extension of $\otimes$ and differentiation to $\mathcal{B}((\mathcal{I}_{c,d}, \otimes); (\mathcal{D}, \curlyvee))$ is given as follows

$$\left[\frac{\varphi_n}{\delta_n}\right] \otimes \left[\frac{\psi_n}{\varepsilon_n}\right] =: \left[\frac{\varphi_n \otimes \psi_n}{\delta_n \curlyvee \varepsilon_n}\right] \text{ and } \mathcal{D}^\alpha \left[\frac{\varphi_n}{\delta_n}\right] =: \left[\frac{\mathcal{D}^\alpha \varphi_n}{\delta_n}\right] \quad (\alpha \in \mathbb{R}).$$

Given $\left[\frac{\varphi_n}{\delta_n}\right] \in \mathcal{B}((\mathcal{I}_{c,d}, \otimes); (\mathcal{D}, \curlyvee))$ and $\varpi \in \mathcal{I}_{c,d}$. Then, $\otimes$ can be extended to $\mathcal{B}((\mathcal{I}_{c,d}, \otimes); (\mathcal{D}, \curlyvee)) \times \mathcal{I}_{c,d}$ by

$$\left[\frac{\varphi_n}{\delta_n}\right] \otimes \varpi =: \left[\frac{\varphi_n \otimes \varpi}{\delta_n}\right].$$

$\beta_n \xrightarrow{\delta} \beta$ in $\mathcal{B}((\mathcal{I}_{c,d}, \otimes); (\mathcal{D}, \curlyvee))$ if there can be (δ_n) in Δ satisfying $(\beta_n \otimes \delta_k), (\beta \otimes \delta_k) \in \mathcal{I}_{c,d}$ $(k, n \in \mathbb{N})$ and $(\beta_n \otimes \delta_k) \to (\beta \otimes \delta_k)$ in $\mathcal{I}_{c,d}$ as $n \to \infty$ $(k \in \mathbb{N})$. This can be expressed to mean :

$\beta_n \xrightarrow{\delta} \beta$ $(n \to \infty)$ in $\mathcal{B}((\mathcal{I}_{c,d}, \otimes); (\mathcal{D}, \curlyvee))$ if there are $\varphi_{n,k}$ and $\varphi_k \in \mathcal{I}_{c,d}$, and $(\delta_k) \in \Delta$ where $\beta_n = \left[\frac{\varphi_{n,k}}{\delta_k}\right], \beta = \left[\frac{\varphi_k}{\delta_k}\right]$ and for each $k \in \mathbb{N}$ we have $f_{n,k} \to f_k$ as $n \to \infty$ in $\mathcal{I}_{c,d}$.

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$\beta_n \xrightarrow{\Delta} \beta$ in $\mathcal{B}((\mathcal{I}_{c,d}, \otimes);(\mathcal{D}, \curlyvee))$, in a sense of Δ, if there can be $(\delta_n) \in \Delta$ where $(\beta_n - \beta) \otimes \delta_n \in \mathcal{I}_{c,d}$ $(\forall n \in \mathbb{N})$ and that $(\beta_n - \beta) \otimes \delta_n \to 0$ as $n \to \infty$ in $\mathcal{I}_{c,d}$.

By techniques similar to above, the space $\mathcal{B}((\mathcal{I}_{c,d}, \curlyvee);(\mathcal{D}, \curlyvee))$ can similarly be generated. In $\mathcal{B}((\mathcal{I}_{c,d}, \curlyvee);(\mathcal{D}, \curlyvee))$, addition and multiplication by a scalar has the following meanings

$$\left[\frac{\varphi_n}{\delta_n}\right] + \left[\frac{\psi_n}{\varepsilon_n}\right] =: \left[\frac{\varphi_n \curlyvee \delta_n + \psi_n \curlyvee \delta_n}{\delta_n \curlyvee \varepsilon_n}\right] \text{ and } \rho \left[\frac{\varphi_n}{\delta_n}\right] =: \left[\frac{\alpha\varphi_n}{\delta_n}\right] \quad (\rho \in \mathbb{C}).$$

We extend $\curlyvee$ and the differentiation to $\mathcal{B}((\mathcal{I}_{c,d}, \curlyvee);(\mathcal{D}, \curlyvee))$ as

$$\left[\frac{\varphi_n}{\delta_n}\right] \curlyvee \left[\frac{\psi_n}{\varepsilon_n}\right] = \left[\frac{\varphi_n \curlyvee \psi_n}{\delta_n \curlyvee \varepsilon_n}\right], \ \mathcal{D}^{\alpha} \left[\frac{\varphi_n}{\delta_n}\right] = \left[\frac{\mathcal{D}^{\alpha}\varphi_n}{\delta_n}\right],$$

α being real number.

Given $\left[\frac{\varphi_n}{\delta_n}\right] \in \mathcal{B}((\mathcal{I}_{c,d}, \curlyvee);(\mathcal{D}, \curlyvee))$ and $\varpi \in \mathcal{I}_{c,d}$. We define $\curlyvee$ for $\mathcal{B}((\mathcal{I}_{c,d}, \curlyvee);(\mathcal{D}, \curlyvee)) \times \mathcal{I}_{c,d}$ as

$$\left[\frac{\varphi_n}{\delta_n}\right] \curlyvee \varpi =: \left[\frac{\varphi_n \curlyvee \varpi}{\delta_n}\right].$$

Convergence in $\mathcal{B}((\mathcal{I}_{c,d}, \curlyvee);(\mathcal{D}, \curlyvee))$ is as follows :

$\beta_n \xrightarrow{\delta} \beta$ $(n \to \infty)$ in $\mathcal{B}((\mathcal{I}_{c,d}, \curlyvee);(\mathcal{D}, \curlyvee))$ if and only if there can be (δ_n) in Δ such that $(\beta_n \curlyvee \delta_k), (\beta \curlyvee \delta_k) \in \mathcal{I}_{c,d}$ $(\forall k, n \in \mathbb{N})$ and $(\beta_n \curlyvee \delta_k) \to (\beta \curlyvee \delta_k)$ in $\mathcal{I}_{c,d}$ as $n \to \infty$ $(\forall k \in \mathbb{N})$. Or, if there can be found $\varphi_{n,k}, \varphi_k \in \mathcal{I}_{c,d}$, $(\delta_k) \in \Delta$, $\beta_n = \left[\frac{\varphi_{n,k}}{\delta_k}\right]$, $\beta = \left[\frac{\varphi_k}{\delta_k}\right]$ and $f_{n,k} \to f_k$ as $n \to \infty$ in $\mathcal{I}_{c,d}$ $(k \in \mathbb{N})$.

$\beta_n \xrightarrow{\Delta} \beta$ $(n \to \infty)$, in $\mathcal{B}((\mathcal{I}_{c,d}, \curlyvee);(\mathcal{D}, \curlyvee))$, if there can be $(\delta_n) \in \Delta$ satisfying $(\beta_n - \beta) \curlyvee \delta_n \in \mathcal{I}_{c,d}$ and $(\beta_n - \beta) \curlyvee \delta_n \to 0$ as $n \to \infty$ in $\mathcal{I}_{c,d}$.

3 The Generalized χ_g^s Transform of $\mathcal{B}((\mathcal{I}_{c,d}, \curlyvee);(\mathcal{D}, \curlyvee))$

We devote this section to the definition of the generalized Stieltjes transform and to derive some desired properties. The following theorem specifies the relation between $\curlyvee$ and $\otimes$.
THEOREM 10 Given $\varphi \in \mathcal{I}_{c,d}$, then $\chi_g^s(\varphi \curlyvee \psi)(\omega) = (\chi_g^s(\varphi)\psi)(\omega)$ for every $\psi \in \mathcal{D}$.
PROOF Let $\varphi \in \mathcal{I}_{c,d}$ and $\psi \in \mathcal{D}$ be given. Then, by (1), we have

$$\begin{aligned} \chi_g^s(\varphi \curlyvee \psi)(\omega) &= \int_0^{\infty} \omega^{\square 1} H_{2,2}^{1,2}\left[\left(\frac{\xi}{\omega}\right)^{\lambda} \middle| \begin{array}{l} (a_1, \alpha_1), (1 - b_1 - \beta_1, \lambda\beta_1) \\ (e_1, \gamma_1), (e_2, \gamma_2) \end{array}\right] \\ &\quad \times (\varphi \curlyvee \psi)(\omega)\, \mathrm{d}\xi, \end{aligned}$$

which can be expressed after setting the variables and using Fubini's theorem as

$$\begin{aligned} \chi_g^s(\varphi \curlyvee \psi)(\omega) &= \int_0^{\infty} \psi(y) \int_0^{\infty} \omega^{\square 1} \\ &\quad \times H_{2,2}^{1,2}\left[\left(\frac{z}{y^{\square 1}\omega}\right)^{\lambda} \middle| \begin{array}{l} (a_1, \alpha_1), (1 - b_1 - \beta_1, \lambda\beta_1) \\ (e_1, \gamma_1), (e_2, \gamma_2) \end{array}\right] \varphi(z)\, \mathrm{d}z\mathrm{d}y. \end{aligned} \tag{14}$$

Simple motivation on (14) yields

$$\begin{aligned}\chi_g^s(\varphi \curlyvee \psi)(\omega) &= \int_0^\infty \psi(y) \int_0^\infty \left(y\omega^{\square 1}\right) \\ &\times H_{2,2}^{1,2}\left[\left(\frac{z}{y^{\square 1}\omega}\right)^\lambda \middle| \begin{array}{l}(a_1,\alpha_1),(1-b_1-\beta_1,\lambda\beta_1) \\ (e_1,\gamma_1),(e_2,\gamma_2)\end{array}\right]\varphi(z)\,\mathtt{d}z\mathtt{d}y.\end{aligned}$$

Hence, the above equation is interpreted to mean

$$\chi_g^s(\varphi \curlyvee \psi)(\omega) = \int_0^\infty \psi(y)\, y^{\square 1}\left(\chi_g^s\right)(\varphi)\left(y\omega^{\square 1}\right)\mathtt{d}y.$$

Hence, the proof of this theorem is finished.

In view of the preceeding result we give the definition of χ_g^s transform of $\left[\frac{\varphi_n}{\delta_n}\right]$ in the space $\mathcal{B}((\mathcal{I}_{c,d},\curlyvee),(\mathcal{D},\curlyvee))$ as

$$\widehat{\chi_g^s}\left(\left[\frac{\varphi_n}{\delta_n}\right]\right) =: \left[\frac{\chi_g^s\varphi_n}{\delta_n}\right] \tag{15}$$

which belongs to $\mathcal{B}((\mathcal{I}_{c,d},\otimes),(\mathcal{D},\curlyvee))$ by means of Theorem 10.

Theorem 11 The operator $\widehat{\chi_g^s}$ is well - defined and linear, mapping from $\mathcal{B}((\mathcal{I}_{c,d},\curlyvee);(\mathcal{D},\curlyvee))$ into $\mathcal{B}((\mathcal{I}_{c,d},\otimes);(\mathcal{D},\curlyvee))$.

Proof Let $\left[\frac{\varphi_n}{\delta_n}\right] = \left[\frac{\psi_n}{\varepsilon_n}\right]$ in the sense of $\mathcal{B}((\mathcal{I}_{c,d},\curlyvee);(\mathcal{D},\curlyvee))$. Then, by the concept of equivalent classes, $\frac{\varphi_n}{\delta_n}$ and $\frac{\psi_n}{\varepsilon_n}$ are equivalent in $\mathcal{B}((\mathcal{I}_{c,d},\curlyvee);(\mathcal{D},\curlyvee))$. Thus, it has been obtained $\varphi_n \curlyvee \varepsilon_m = \psi_n \curlyvee \delta_m$.

Applying χ_g^s to the sides of the above equation and employing Theorem 10 imply

$$\chi_g^s\varphi_n \otimes \varepsilon_m = \chi_g^s\psi_n \otimes \delta_m \ \ (\forall n,m \in \mathbb{N}).$$

That is,

$$\left[\frac{\chi_g^s\varphi_n}{\delta_n}\right] = \left[\frac{\chi_g^s\psi_n}{\varepsilon_n}\right].$$

To show that the $\widehat{\chi_g^s} : \mathcal{B}((\mathcal{I}_{c,d},\curlyvee);(\mathcal{D},\curlyvee)) \to \mathcal{B}((\mathcal{I}_{c,d},\otimes);(\mathcal{D},\curlyvee))$ is linear, let $\rho_1 = \left[\frac{\varphi_n}{\delta_n}\right], \rho_2 = \left[\frac{\psi_n}{\varepsilon_n}\right] \in \mathcal{B}((\mathcal{I}_{c,d},\curlyvee);(\mathcal{D},\curlyvee))$. Then, addition of Boehmians of $\mathcal{B}((\mathcal{I}_{c,d},\curlyvee);(\mathcal{D},\curlyvee))$ and Equation 15, suggest to write

$$\widehat{\chi_g^s}(\rho_1+\rho_2) = \left[\frac{\chi_g^s(\varphi_n \curlyvee \varepsilon_n) + \chi_g^s(\psi_n \curlyvee \delta_n)}{\delta_n \curlyvee \varepsilon_n}\right].$$

By aid of Theorem 10, we obtain

$$\widehat{\chi_g^s}(\rho_1+\rho_2) = \left[\frac{\chi_g^s\varphi_n \otimes \varepsilon_n + \chi_g^s\psi_n \otimes \delta_n}{r_n \curlyvee \varepsilon_n}\right].$$

Employing the product $\otimes$ that assigned to the $\mathcal{B}((\mathcal{I}_{c,d},\otimes);(\mathcal{D},\curlyvee))$ gives

$$\widehat{\chi_g^s}(\rho_1+\rho_2) = \left[\frac{\chi_g^s\varphi_n}{\delta_n}\right] + \left[\frac{\chi_g^s\psi_n}{\varepsilon_n}\right].$$

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Hence, we have obtained that

$$\widehat{\chi_g^s}\left(\rho_1+\rho_2\right)=\widehat{\chi_g^s}\left(\left[\frac{\varphi_n}{\delta_n}\right]\right)+\widehat{\chi_g^s}\left(\left[\frac{\psi_n}{\varepsilon_n}\right]\right).$$

Also, it is easy for readers to check that

$$\lambda\widehat{\chi_g^s}\left(\rho_1\right)=\widehat{\chi_g^s}\left(\lambda\rho_1\right)\ \ (\lambda\in\mathbb{C}).$$

Hence, the proof of this theorem is finished.

Theorem 12 The mapping $\widehat{\chi_g^s}:\mathcal{B}\left(\left(\mathcal{I}_{c,d},\curlyvee\right);\left(\mathcal{D},\curlyvee\right)\right)\to\mathcal{B}\left(\left(\mathcal{I}_{c,d},\otimes\right);\left(\mathcal{D},\curlyvee\right)\right)$ is an isomorphism.

Proof Given $\left[\frac{\chi_g^s\varphi_n}{\delta_n}\right]=\left[\frac{\chi_g^s\psi_n}{\varepsilon_n}\right]\in\mathcal{B}\left(\left(\mathcal{I}_{c,d},\otimes\right);\left(\mathcal{D},\curlyvee\right)\right)$. Then, by virtue of Theorem 10, we get

$$\chi_g^s\varphi_n\otimes\varepsilon_m=\chi_g^s\psi_m\otimes\delta_n\ \ (m,n\in\mathbb{N}).$$

Once again, Theorem 10 implies

$$\chi_g^s\left(\varphi_n\otimes\varepsilon_m\right)=\chi_g^s\left(\psi_m\otimes\delta_n\right).$$

Hence $\varphi_n\otimes\varepsilon_m=\psi_m\otimes\delta_n$. Therefore,

$$\left[\frac{\varphi_n}{\delta_n}\right]=\left[\frac{\psi_n}{\varepsilon_n}\right]\in\mathcal{B}\left(\left(\mathcal{I}_{c,d},\curlyvee\right);\left(\mathcal{D},\curlyvee\right)\right).$$

This proves that the above mapping is an injection. surjectivity of $\widehat{\chi_g^s}$ is obvious.
The proof is finished.

Definition 13 Let $\rho^*\in\mathcal{B}\left(\left(\mathcal{I}_{c,d},\otimes\right);\left(\mathcal{D},\curlyvee\right)\right)$, $\rho^*=\left[\frac{\chi_g^s\varphi_n}{\delta_n}\right]$. Then, we the inverse mapping $\widehat{\chi_g^s}$ is defined as

$$\left(\widehat{\chi_g^s}\right)^{\square 1}\left(\rho^*\right)=\left[\frac{\left(\chi_g^s\right)^{\square 1}\left(\chi_g^s\varphi_n\right)}{\delta_n}\right]=\left[\frac{\varphi_n}{\delta_n}\right],$$

for each $(\delta_n)\in\Delta$.

Theorem 14 Let $\rho^*=\left[\frac{\chi_g^s\varphi_n}{\delta_n}\right]\in\mathcal{B}\left(\left(\mathcal{I}_{c,d},\otimes\right);\left(\mathcal{D},\curlyvee\right)\right)$ for some $\left[\frac{\varphi_n}{\delta_n}\right]\in\mathcal{B}\left(\left(\mathcal{I}_{c,d},\otimes\right);\left(\mathcal{D},\curlyvee\right)\right)$ and $\phi,\psi\in\mathcal{D}$. Then we have

(i) $\left(\widehat{\chi_g^s}\right)^{\square 1}\left(\rho^*\otimes\phi\right)=\left[\frac{\varphi_n}{\delta_n}\right]\curlyvee\phi$,

(ii) $\widehat{\chi_g^s}\left(\left[\frac{\varphi_n}{\delta_n}\right]\curlyvee\psi\right)=\rho^*\otimes\psi$.

Proof Assume $\rho^*=\left[\frac{\chi_g^s\varphi_n}{\delta_n}\right]\in\mathcal{B}\left(\left(\mathcal{I}_{c,d},\otimes\right);\left(\mathcal{D},\curlyvee\right)\right)$ be given. Then, by Theorem 10, we write

$$\left(\widehat{\chi_g^s}\right)^{\square 1}\left(\rho^*\otimes\phi\right)=\left(\widehat{\chi_g^s}\right)^{\square 1}\left(\left[\frac{\chi_g^s\varphi_n\otimes\phi}{\delta_n}\right]\right)=\left[\frac{\left(\chi_g^s\right)^{\square 1}\left(\chi_g^s\varphi_n\otimes\phi\right)}{\delta_n}\right].$$

Hence,

$$\left(\widehat{\chi_g^s}\right)^{\square 1}\left(\rho^*\otimes\phi\right)=\left[\frac{\varphi_n\curlyvee\phi}{\delta_n}\right].$$

Therefore,

$$\left(\widehat{\chi_g^s}\right)^{\square 1}(\rho^* \otimes \phi) = \left[\frac{\varphi_n}{\delta_n}\right] \curlyvee \phi.$$

To prove the second identity, we apply Theorem 10 to get

$$\widehat{\chi_g^s}\left(\left[\frac{\varphi_n}{\delta_n}\right] \curlyvee \psi\right) = \widehat{\chi_g^s}\left(\left[\frac{\varphi_n \curlyvee \psi}{\delta_n}\right]\right) = \rho^* \otimes \psi.$$

This finishes the proof of the theorem.

Conclusion : This paper provides some integral products which were implemented to extend a new type of Stieltjes transforms enfolding Fox's H-functions as kernels to generalized functions. The generalized Sitieltjes transform was formed to satisfy the desired properties of the classical transform. It may be concluded here that the employed Stieltjes transform method is a very efficient technique in extending integral transforms to generalized functions and could lead to a promising approach for many integrals of special functions kernels .

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Decision making based on interval-valued intuitionistic fuzzy soft sets and its algorithm *

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Abstract: This paper investigates an approach to interval-valued intuitionistic fuzzy soft sets in decision making by means of grey relational analysis and D-S theory of evidence. An algorithm based on this approach in decision making is presented.

Keywords: Interval-valued intuitionistic fuzzy soft set; Decision making; Grey relational analysis; D-S theory of evidence.

1 Introduction

In 1999, Molodtsov [18] initiated soft sets as a mathematical tool for dealing with vagueness and uncertainties. Compared with some traditional tools for dealing with uncertainties, such as probability theory, fuzzy set theory [32], rough set theory [23], soft set theory has the advantage of freeing from the inadequacy of the parametrization tools of those theories.

Recently, many efforts have been devoted to further generalizations and extensions of Molodtsov's soft sets. Maji et al. [19, 20] defined fuzzy soft sets and intuitionistic fuzzy soft sets by combining soft sets with fuzzy sets and and intuitionistic fuzzy sets, Yang et al. [31] defined the interval-valued fuzzy soft sets. Jiang et al. [7] proposed a more general soft set model called interval-valued intuitionistic fuzzy soft set, which is a substantial and important combination of the soft set and the interval-valued intuitionistic fuzzy set. The intuitionistic fuzzy soft set theory makes descriptions of the objective world more realistic, practical and accurate in some cases, making it very promising.

With the rapid development of soft set theory, there has been some progress on the practical applications, especially the use of soft sets in decision making. Roy et al. [25] discussed score value as the evaluation basis to find an optimal choice object in fuzzy soft sets. But Kong et al. [10] argued that the Roy's method was incorrect by using a counter example to discuss two evaluation bases

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of choice value and score value, and they proposed a revised algorithm. Later Feng et al. [5] applied level soft sets to discuss fuzzy soft sets based decision making and subsequently extended the approach to interval-valued fuzzy soft set based decision making [6]. Jiang et al. [8] generalize the approach to solve intuitionistic fuzzy soft sets. Based on Feng' works, Basu et al. [2] further investigated the previous methods to fuzzy soft sets in decision making and introduced the mean potentiality approach, which was showed more efficient and more accurate than the previous methods. Zhang [36] proposed a rough set approach to intuitionistic fuzzy soft set based decision making. Li et al. [15] investigated decision making based on intuitionistic fuzzy soft sets. Li et al. [16] considered fuzzy soft set based decision making for applications in medical diagnosis. Ma et al. [22] presented the algorithm to solve decision making problems based on interval-valued intuitionistic fuzzy soft sets. Qin et al. [24] present an adjustable approach to interval-valued intuitionistic fuzzy soft set based decision making by using reduct intuitionistic fuzzy soft sets and level soft sets of intuitionistic fuzzy soft sets.

All of the above methods for soft sets in decision making are mainly based on the level soft set to obtain useful information such as choice values and score values. However, the existing methods have their limitations. For example, it is very difficult for decision maker to select a suitable level soft set to reduce subjectivity and uncertainty (see [36]). Moreover, there has been rather little work completed for interval-valued intuitionistic fuzzy soft set based decision making. Then it is necessary to pay attention to this issue.

Grey relational analysis, initiated by Deng [4], is an important method to reflect uncertainty in grey system theory, which is utilized for generalizing estimates under small samples and uncertain conditions. It has been successfully applied in solving decision-making problems [9, 27, 28, 35]. D-S theory of evidence, proposed by Dempster [3] and Shafer [26], is a powerful method for combining accumulative evidence of changing prior opinions in the light of new evidences [26]. Compared to probability theory, this theory captures more information to support decision making by identifying the uncertain and unknown evidence. It provides a mechanism to derive solutions from various vague evidences without knowing much prior information. Therefore, combining both theories enables the decision makers to take advantage of both methods' merits and make evaluation experts to deal with uncertainty and risk confidently. The hybrid model is effective and practical under uncertainty [27, 29]. It is very meaningful to extend the hybrid model to interval-valued fuzzy soft set based decision making Thus, this not only allows us to avoid selecting a suitable level soft set, but also helps reducing humanistic and subjective in nature to raise the choices decision level.

The purpose of this paper is to investigate decision making based on the interval-valued intuitionistic fuzzy soft sets.

2 Preliminaries

Throughout this paper, U denotes an initial universe, E denotes the set of all possible parameters, 2^U denotes the family of all subsets of U. We only consider the case where U and E are both nonempty finite sets. Int$[0,1]$ denotes a set of all closed subintervals of $[0,1]$.

2.1 Interval-valued intuitionistic fuzzy soft sets

Definition 2.1 ([1]). *An interval-valued intuitionistic fuzzy set $\widetilde{X}$ over U is an object having the form $\widetilde{X} = \{(x, \mu_{\widetilde{X}}(x), \nu_{\widetilde{X}}(x)) \mid x \in U\}$ $(e \in A)$, where $\mu_{\widetilde{X}} : U \to Int[0,1]$ and $\nu_{\widetilde{X}} : U \to Int\ [0,1]$ satisfy $0 \leqslant sup\ \mu_{\widetilde{X}}(x) + sup\ \nu_{\widetilde{X}}(x) \leqslant 1$ for all $x \in U$.*

$\mu_{\widetilde{X}}(x)$ and $\nu_{\widetilde{X}}(x)$ are called the membership degree and non-membership degree of the element $x \in U$ to $\widetilde{X}$.

The set of all interval-valued intuitionistic fuzzy subsets of U is denoted by $IVIF(U)$.

Definition 2.2 ([18]). *Let $A \subseteq E$. A pair (F, A) is called a soft set over U, where F is a mapping given by $F : A \to 2^U$.*

Definition 2.3 ([7]). *Let $A \subseteq E$. A pair (F, A) is called an interval-valued intuitionistic fuzzy soft set over U, where F is a mapping given by $F : A \to IVIF(U)$.*

In other words, an interval-valued intuitionistic fuzzy soft set over U is a parameterized family of interval-valued intuitionistic fuzzy subsets of U. For any $e \in A$, $F(e)$ is referred as the set of e-approximate elements of (F, A) and can be written as:

$$F(e) = \{(x, \mu_{F(e)}(x), \nu_{F(e)}(x)) \mid x \in U\}\ (e \in A),$$

where $\mu_{F(e)} : U \to Int[0,1]$ and $\nu_{F(e)} : U \to Int[0,1]$ satisfy $0 \leqslant sup\ \mu_{F(e)}(x) + sup\ \nu_{F(e)}(x) \leqslant 1$. $\mu_{F(e)}(x)$ and $\nu_{F(e)}(x)$ are called the membership degree and non-membership degree that x holds e, respectively. $\pi_{F(e)}(x) = 1 - \mu_{F(e)}(x) - \nu_{F(e)}(x)$ is called the hesitating degree of x holds e.

The set of all interval-valued intuitionistic fuzzy soft subsets of U is denoted by $IVIFS(U)$.

Example 2.4. *Let $U = \{h_1, h_2, h_3, h_4, h_5\}$ be a set of houses and let $A = \{e_1, e_2, e_3, e_4, \} \subseteq E$ be a set of status of houses where e_j $(j = 1, 2, 3, 4)$ stand for the parameters "beautiful", "modern", "cheap" and "in the green surroundings", respectively.*

Now, we consider an interval-valued intuitionistic fuzzy soft set (F, A) over U, which describes "the attractiveness of the houses" to this decision maker and its tabular representation is shown in Table 1.

Obviously, we can see that the precise evaluation for each object on each parameter is unknown while the lower and upper limits of such an evaluation are given. For example, we cannot present the precise membership degree and non-membership degree of how beautiful house h_1 is, however, house h_1 is at least beautiful on the membership degree of 0.6 *and it is at most beautiful on the membership degree of* 0.8*; house h_1 is not at least beautiful on the non-membership degree of 0.1 and it is not at most beautiful on the non-membership degree of 0.2.*

Table 1: Tabular representation of the interval-valued intuitionistic soft set (F, A)

	e_1	e_2	e_3	e_4
h_1	[0.6, 0.8],[0.1,0.2]	[0.7, 0.8],[0.15,0.2]	[0.75, 0.85],[0.1,0.15]	[0.8, 0.9],[0.01,0.1]
h_2	[0.8, 0.9],[0.05,0.1]	[0.6, 0.7],[0.15,0.21]	[0.5, 0.6],[0.2,0.35]	[0.65, 0.75],[0.2,0.25]
h_3	[0.6, 0.7],[0.2,0.25]	[0.5, 0.7],[0.2,0.3]	[0.6, 0.8],[0.1,0.18]	[0.66, 0.77],[0.2,0.22]
h_4	[0.65, 0.78],[0.15,0.21]	[0.7, 0.75],[0.15,0.25]	[0.68, 0.75],[0.1,0.2]	[0.69, 0.78],[0.1,0.2]

2.2 Basic concepts of D-S theory of evidence

D-S theory of evidence is a new important reasoning method under uncertainty. It has an advantage to deal with subjective judgments and to synthesize the uncertainty knowledge [34].

A frame of discernment, denoted Θ, is a finite nonempty set of mutually exclusive and exhaustive hypotheses, denoted $\{A_1, A_2, \cdots, A_n\}$ and $A_i \cap A_j = \emptyset$. 2^Θ denotes the set of all subsets of Θ.

Definition 2.5 ([26])**.** *Let Θ be a frame of discernment. A basic probability assignment function (or Mass function) on Θ is defined a mapping $m : 2^\Theta \to [0, 1]$, m satisfies*

$$m(\emptyset) = 0 \ , \ \sum_{A \subseteq \Theta} m(A) = 1 \ for \ A \in 2^\Theta.$$

For any $A \subseteq \Theta$, A is called as focal elements if $m(A) > 0$, $m(A)$ represents the belief measuser that one is willing to commit exactly to A, given a certain piece of evidence.

Definition 2.6 ([26])**.** *Let Θ be the frame of discernment and $m : 2^\Theta \to [0, 1]$ be a Mass function. Then a belief function on Θ is defined a mapping $Bel : 2^\Theta \to [0, 1]$, Bel satisfies*

$$Bel(\emptyset) = 0, \ Bel(\Theta) = 1, \ Bel(A) = \sum_{B \subseteq A} m(B) \ for \ A \subseteq \Theta.$$

$Bel(A)$ can be interpreted as a global belief measure that the hypothesis A is true, and represents the imprecision and uncertainty in the decision-making process. In the case of single hypothesis, $Bel(A) = m(A)$.

4

Definition 2.7 ([26]). *Let Θ be the frame of discernment. Suppose there are two Mass functions are m_1 and m_2 over Θ, induced by two independent items of evidences $A_1, A_2, \cdots, A_s$ and $B_1, B_2, \cdots, B_t$, respectively. D-S rule of evidence combination is defined and denoted as follows:*

$$m(A) = m_1 \oplus m_2(A) = \begin{cases} \frac{1}{1-K} \sum\limits_{A_i \cap B_j = A} m_1(A_i) m_2(B_j), \ \forall\ A \subseteq \Theta, A \neq \emptyset, \\ 0, \ A = \emptyset, \end{cases}$$

where $K = \sum\limits_{A_i \cap B_j = \emptyset} m_1(A_i) m_2(B_j) < 1.$

K is called the conflict probability and reflects the extent of the conflict between the evidences. Coefficient $\frac{1}{1-K}$ is called normalized factor, its role is to avoid the probability of assigning non-0 to empty set $\emptyset$ in the combination.

D-S rule of evidence combination can be generalized to multiple Mass functions, the belief measure resulting from the combination of multiply evidences A_i is as follows:

$$m_1 \oplus m_2 \cdots \oplus m_n(A) = \frac{1}{1-K} \sum_{\bigcap_{i=1}^{n} A_i = A, A_i \subset \Theta} m_1(A_1) m_2(A_2) \cdots m_n(A_n),$$

where $K = \sum\limits_{\bigcap_{i=1}^{n} A_i = \emptyset, A_i \subset \Theta} m_1(A_1) m_2(A_2) \cdots m_n(A_n) < 1.$

D-S rule of evidence combination can increase belief measure of hypotheses and reduce the uncertain degree to improve reliability.

Example 2.8. *Let $\Theta = \{A_1, A_2\}$ be the frame of discernment. Suppose there are two Mass functions m_1 and m_2 over Θ, induced by the independent items of evidences A_1, A_2, given by*

$m_1(A_1) = 0.3, \quad m_1(A_2) = 0.4, \quad m_1(\Theta) = 0.3,$
$m_2(A_1) = 0.4, \quad m_2(A_2) = 0.3, \quad m_2(\Theta) = 0.3.$

Combining the two evidences by D-S rule of evidence combination leads to:

$m(A_1) = m_1 \oplus m_2(A_1) = \frac{m_1(A_1)m_2(A_1) + m_1(A_1)m_2(\Theta) + m_1(\Theta)m_2(A_1)}{1-K} = 0.44,$
$m(A_2) = m_1 \oplus m_2(A_2) = \frac{m_1(A_2)m_2(A_2) + m_1(A_2)m_2(\Theta) + m_1(\Theta)m_2(A_2)}{1-K} = 0.44,$
$m(\Theta) = m_1 \oplus m_2(\Theta) = \frac{m_1(\Theta)m_2(\Theta)}{1-K} = 0.12,$
where $K = m_1(A_1)m_2(A_2) + m_1(A_2)m_2(A_1) = 0.25.$

3 An approach to interval-valued intuitionistic fuzzy soft sets in decision making

Recently, research on soft sets based decision making has attracted more and more attention. The works of Roy et al. [10, 25, 5, 2, 11] are fundamental and significant. Later other authors like Qin et al. further studied and proposed an adjustable approach to interval-valued intuitionistic fuzzy soft set based decision making using the level soft sets and reductions . Generally, there does not exist any unique or uniform criterion for the evaluation of decision alternatives under uncertain condition. However, it is very difficult for decision makers to select suitable level soft sets and discuss reduct intuitionistic fuzzy soft sets.

Now we investigate interval-valued intuitionistic fuzzy soft sets based decision making by means of grey relational analysis and D-S theory of evidence. It is divided three phases: First, grey relational analysis is applied to calculate the grey mean relational degree and the uncertain degree of each parameter is obtained. Second, the corresponding Mass function with respect to each parameter is constructed by the uncertain degree of each parameter. Third, we apply D-S rule of evidence combination to aggregate individual alternatives into a collective alternative, by which the candidate alternatives are ranked and the best alternative is obtained.

In the following, we consider the decision making problem with m mutually exclusive alternatives x_i and n evaluation parameters (or indexes) e_j. d_{ij} denotes the degree that the alternative x_i satisfies the parameter e_j. Put

$$\Theta = \{x_1, x_2, \cdots, x_m\} \ and \ A = \{e_1, e_2, \cdots, e_n\}.$$

Define $F : A \to IVIF(\Theta)$ by $F(e_j) = \{(x_i, \mu_{F(e_j)}(x_i), \nu_{F(e_j)}(x_i)) \mid x_i \in \Theta\}$ $(e_j \in A)$ where $\mu_{F(e_j)} : U \to Int[0,1]$ and $\nu_{F(e_j)} : U \to Int[0,1]$ satisfy $0 \leqslant sup\ \mu_{F(e_j)}(x_i) + sup\ \nu_{F(e_j)}(x_i) \leqslant 1$. Then (F, A) is an interval-valued intuitionistic fuzzy soft set over Θ. Denote $\mu_{F(e_j)}(x_i) = [\mu_{ij}^-, \mu_{ij}^+]$, $\nu_{F(e_j)}(x_i) = [\nu_{ij}^-, \nu_{ij}^+]$, $a_{ij} = (\mu_{F(e_j)}(x_i), \nu_{F(e_j)}(x_i))$. $D = (a_{ij})_{m\times n}$ is called an interval-valued intuitionistic fuzzy soft decision matrix induced by (F, A). Here, we see the set of parameters as a item of evidences information.

The key to solve decision problems by using D-S theory of evidence is how to obtain the uncertain degree of evidences (or parameters).

First, inspired by Xu [12], we define the score function of as follows.

Definition 3.1. *Suppose that (F, A) is an interval-valued intuitionistic fuzzy soft over Θ. Suppose that $D = (a_{ij})_{m\times n}$ is an interval-valued intuitionistic fuzzy soft decision matrix induced by (F, A). Denote $\mu_{F(e_j)}(x_i) = [\mu_{ij}^-, \mu_{ij}^+]$, $\nu_{F(e_j)}(x_i) = [\nu_{ij}^-, \nu_{ij}^+]$, $a_{ij} = (\mu_{F(e_j)}(x_i), \nu_{F(e_j)}(x_i))$. Then score function of d_{ij} is defined and denoted as*

$$s(a_{ij}) = (\mu_{ij}^- + \mu_{ij}^+ - \nu_{ij}^- - \nu_{ij}^+)/2 + \alpha(\mu_{ij}^+ + \nu_{ij}^+ - \mu_{ij}^- - \nu_{ij}^-)/2.$$

By Definition 4.1, we can convert d_{ij} into real numbers. $s(a_{ij})$ presents the global degree that the alternative x_i holds the parameter e_j. Obviously, $0 \leqslant s(a_{ij}) \leqslant 1$. α is called a risk factor. For $\alpha = 0, > 0, < 0$, they imply the attitude of decision makers for risk is neutral, positive, oppose, respectively. Decision makers can select a α value according to their risk preference. In this paper, we pick $\alpha = 0$.

To obtain Mass functions of each alternative with respect to each parameter, we consider score function values may be negative, so we should normalize the score function values by the following formula:

$$d_{ij} = \frac{s(a_{ij}) - min_{1\leqslant i\leqslant m} s(a_{ij})}{max_{1\leqslant i\leqslant m} s(a_{ij}) - min_{1\leqslant i\leqslant m} S(a_{ij})}, \quad 1 \leqslant i \leqslant m, \ 1 \leqslant j \leqslant n.$$

Hence, we can get normalized matrix of score function values $D = (d_{ij})_{m\times n}$.

Next, inspired by the paper [12], we define the grey mean relational degree and the uncertain degree of the parameter as follows.

Definition 3.2. *Let $\Theta = \{x_1, x_2, \cdots, x_m\}$, $A = \{e_1, e_2, \cdots, e_n\}$ and let (F, A) be an intuitionistic fuzzy soft set on Θ. Suppose that $D = (d_{ij})_{m\times n}$ is normalized matrix of score function values. For any i, j, denote*

$$\widetilde{d_i} = \frac{1}{n}\sum_{j=1}^{n} d_{ij}, \quad \triangle d_{ij} = |d_{ij} - \widetilde{d_i}|,$$

$$r_{ij} = \frac{min_{1\leqslant j\leqslant n} min_{1\leqslant i\leqslant m} \ \triangle d_{ij} + \rho \ max_{1\leqslant j\leqslant n} max_{1\leqslant i\leqslant m} \ \ \triangle d_{ij}}{\triangle d_{ij} + \rho \ max_{1\leqslant j\leqslant n} max_{1\leqslant i\leqslant m} \ \triangle d_{ij}}, \ \rho \in (0, 1),$$

$$DOI(e_j) = \frac{1}{m}(\sum_{i=1}^{m}(r_{ij})^q)^{\frac{1}{q}} \ (j = 1, 2, \cdots, n).$$

r_{ij} is called the grey mean relational degree between d_{ij} and $\widetilde{d_i}$. $DOI(e_j)$ is called q order uncertain degree of the parameter e_j.

ρ aims to expand or compress the range of the grey relational coefficient. Decision makers can select q, ρ values according to different circumstance. To obtain strong distinguishing effectiveness, we pick $q = 2$, $\rho = 0.5$ in this paper. We call $DOI(e_j)$ the uncertain degree of e_j for short.

It is worthy to notice that the method to obtain the uncertain degree varies from different situation in Definition 4.2. General speaking, since a index (or parameter) is specially more matching the mean of the index set than other indexes, it contains more satisfying information for decision making and the uncertain degree of the index information is lower. Then, in this paper we just consider grey mean relational degree between d_{ij} and $\widetilde{d_i}$.

Definition 3.3 ([36])**.** *Let $X = (x_1, x_2, \cdots, x_m)$ be a finite difference information sequence, where there exists $x_{i_k} \neq 0$ for $k = 1, 2, \cdots, m$ and $1 \leqslant i_k \leqslant m$. Then the information structure image sequence $Y = (y_1, y_2, \cdots, y_m)$ is given by $y_i = \frac{x_i}{\sum_{i=1}^{m} x_i}$.*

In the normalized matrix of score function values $D = (d_{ij})_{m\times n}$, the information structure image sequence with respect to a parameter e_j is denoted by $d_j = \{\widetilde{d_{1j}}, \widetilde{d_{2j}}, \widetilde{d_{3j}}, \cdots, \widetilde{d_{mj}}\}$, where $\widetilde{d_{ij}} = \frac{d_{ij}}{\sum_{i=1}^{m} d_{ij}}$. Then we obtain an information structure image matric $\widetilde{D} = (\widetilde{d_{ij}})_{m\times n}$ induced by d_j $(j = 1, 2, \cdots, n)$.

D-S theory of evidence is a powerful method for combining accumulative evidence of changing prior opinions in the light of new evidences [26]. The primary procedure of combining the known evidences or information with other evidences is to construct suitable Mass functions of evidences.

Now, by the uncertain degree of each parameter, we can obtain Mass function of each alternative with respect to each parameter.

Theorem 3.4. *Let* $\Theta = \{x_1, x_2, \cdots, x_m\}$*,* $A = \{e_1, e_2, \cdots, e_n\}$ *and let* (F, A) *be an intuitionistic fuzzy soft set on* Θ*. Suppose that* $D = (d_{ij})_{m\times n}$ *is the normalized matrix of score function values and* $DOI(e_j)$ *is the uncertain degree of* e_j*. Denote* $\widetilde{d_{ij}} = \frac{d_{ij}}{\sum\limits_{i=1}^{m} d_{ij}}$*. For any* i, j*, we define functions* $m_{e_j}(j = 1, 2, \cdots, n)$ *with respect to the parameter* e_j*, it satisfies:*

$$m_{e_j}(x_i) = \widetilde{d_{ij}}\ (1 - DOI(e_j)),\quad m_{e_j}(\Theta) = 1 - \sum_{i=1}^{m} m_j(i).$$

Then $m_{e_j}(j = 1, 2, \cdots, n)$ *are Mass functions.*

In a normalized matrix of score function values $D = (d_{ij})_{m\times n}$, denote $m_{e_j}(x_i)$, $m_{e_j}(\Theta)$ by $m_j(i)$ and $m_j(m+1)$, respectively. $m_j(i)$ implies the belief measure that holds the alternative x_i with the parameter e_j and $m_j(m+1)$ implies the belief measure of the whole uncertainty with parameter e_j.

Next, using D-S rule of evidence combination to compose m_j $(j = 1, 2, \cdots, n)$, we get the belief measure of each alternative with all the parameters, by which the candidate alternatives are ranked and thus the best alternative is obtained.

4 Algorithm

4.1 Algorithm

Based on the above analysis, the detailed step-wise procedure as an algorithm is given as follows:

Input: An interval-value intuitionistic fuzzy soft set (F, A).

Output: The optimal decision-making results.

Step 1. Input an interval-value intuitionistic fuzzy soft set (F, A) and construct an interval-value intuitionistic fuzzy soft decision matrix induced by (F, A).

Step 2. Compute the normalized matrix of score function values $(D = (d_{ij})_{m\times n})$.

Step 3. Compute the mean of all the score function values $(\widetilde{d_i})$ with respect to each alternative.

Step 4. Compute the difference information between d_{ij} and $\widetilde{d_i}$.

Step 5. Compute the gray mean relational degree between d_{ij} and $\widetilde{d_i}$.

Step 6. Compute the uncertain degree $DOI(e_j)$ of each parameter e_j.

Step 7. Compute the information structure image sequence $\widetilde{d_{ij}}$ with respect to each parameter e_j by Definition 3.3.

Step 8. Compute Mass function values of the alternative x_i and Θ with respect to the parameter e_j by Theorem 3.4.

Step 9. Compute belief measure of each alternative x_i by combining these Mass functions $m_{e_j}(j = 1, 2, \cdots, n)$ respectively by Definition 2.8.

Step 10. The optimal decision is to select x_k if $c_k = max_i \{Bel(x_i)\}$. k has more than one value then any one of x_k may be optimal choices .

4.2 An illustrative example

Suppose that a fund manager in a wealth management wants to invest a company. Suppose that the set of four potential investment companies $U = \{x_1, x_2, x_3, x_4\}$ which are characterized by a set of parameters $A = \{e_1, e_2, e_3, e_4\}$. For $i = 1, 2, 3, 4$, the parameters e_i stand for "risk", "growth ,"socio-political issues" ,and "environmental impacts", respectively. The fund manager provide his/her assessment of each investment company on each parameter as an interval-valued intuitionistic fuzzy soft set (F, A). Its tabular representation is shown in Table 2.

Table 2: Tabular representation of the interval-valued intuitionistic soft set (F, A)

	e_1	e_2	e_3	e_4
x_1	[0.4, 0.5],[0.3,0.4]	[0.4, 0.6],[0.2,0.4]	[0.1, 0.3],[0.5,0.6]	[0.5, 0.7],[0.2,0.3]
x_2	[0.4, 0.5],[0.4,0.5]	[0.5, 0.8],[0.1,0.2]	[0.3, 0.6],[0.3,0.4]	[0.6, 0.7],[0.1,0.3]
x_3	[0.3, 0.5],[0.4,0.5]	[0.1, 0.3],[0.2,0.4]	[0.7, 0.8],[0.1,0.2]	[0.5, 0.7],[0.1,0.2]
x_4	[0.2, 0.4],[0.4,0.5]	[0.6, 0.7],[0.2,0.3]	[0.5, 0.6],[0.2,0.3]	[0.7, 0.8],[0.1,0.2]

Now, we suppose that the four mutually exclusive and exhaustive investment companies consist a frame of discernment, denoted $\Theta = \{x_1, x_2, x_3, x_4\}$. And we consider the set of parameters $A = \{e_1, e_2, e_3, e_4\}$ as a set of evidences.

Step 1. Construct an interval-valued intuitionistic fuzzy soft decision matrix induced by (F, A) as follows:

$$\begin{pmatrix} ([0.4, 0.5], [0.3, 0.4]) & ([0.4, 0.6], [0.2, 0.4]) & ([0.1, 0.3], [0.5, 0.6]) & ([0.5, 0.7], [0.2, 0.3]) \\ ([0.4, 0.5], [0.4, 0.5]) & ([0.5, 0.8], [0.1, 0.2]) & ([0.3, 0.6], [0.3, 0.4]) & ([0.6, 0.7], [0.1, 0.3]) \\ ([0.3, 0.5], [0.4, 0.5]) & ([0.1, 0.3], [0.2, 0.4]) & ([0.7, 0.8], [0.1, 0.2]) & ([0.5, 0.7], [0.1, 0.2]) \\ ([0.2, 0.4], [0.4, 0.5]) & ([0.6, 0.7], [0.2, 0.3]) & ([0.5, 0.6], [0.2, 0.3]) & ([0.7, 0.8], [0.1, 0.2]) \end{pmatrix}$$

Step 2. Compute the normalized matrix of score function values as follows:

$$D = (d_{ij})_{4\times4} = \begin{pmatrix} 1.0000 & 0.5000 & 0 & 0 \\ 0.6000 & 1.0000 & 0.4737 & 0.4000 \\ 0.4000 & 0 & 1.0000 & 0.4000 \\ 0 & 0.8333 & 0.6842 & 1.0000 \end{pmatrix}$$

Step 3. Compute the mean of all parameters with respect to each investment company x_i as follows:

$$\widetilde{d_1} = 0.3750, \ \widetilde{d_2} = 0.6184, \ \widetilde{d_3} = 0.4500, \widetilde{d_4} = 0.6294$$

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Step 4. Compute the difference information between d_{ij} and $\widetilde{d_i}$, and construct the difference matrix as follows:

$$\triangle D = \begin{pmatrix} 0.6250 & 0.1250 & 0.3750 & 0.3750 \\ 0.0184 & 0.3816 & 0.1447 & 0.2184 \\ 0.0500 & 0.4500 & 0.5500 & 0.0500 \\ 0.6294 & 0.2039 & 0.0548 & 0.3706 \end{pmatrix}$$

Step 5. Compute the gray mean relational degree between d_{ij} and $\widetilde{d_i}$ based on $\triangle D$ as follows:

$$(r_{ij})_{4\times 4} = \begin{pmatrix} 0.3545 & 0.7576 & 0.4830 & 0.4830 \\ 1.0000 & 0.4784 & 0.7251 & 0.6248 \\ 0.9134 & 0.4356 & 0.3852 & 0.9134 \\ 0.3528 & 0.6423 & 0.9015 & 0.4861 \end{pmatrix}$$

Step 6. Compute the uncertain degree of each parameter e_j by Definition 3.2 as follows:

$$DOI(e_1) = 0.3609, \ \ DOI(e_2) = 0.2963, \ \ DOI(e_3) = 0.3279, \ \ DOI(e_4) = 0.3254.$$

Step 7. Compute the information structure image sequence with respect to each parameter and construct the matrix as follows:

$$\widetilde{D} = (\widetilde{d_{ij}})_{4\times 4} = \begin{pmatrix} 0.5000 & 0.2143 & 0 & 0 \\ 0.3000 & 0.4286 & 0.2195 & 0.2222 \\ 0.2000 & 0 & 0.4634 & 0.2222 \\ 0 & 0.3571 & 0.3171 & 0.5556 \end{pmatrix}$$

Step 8. Let $2^\Theta = \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \Theta\}$. Compute Mass function values of x_i and Θ with respect to the parameter e_j by Theorem 3.4:

$$(m_j(i))_{4\times 4} = \begin{pmatrix} 0.3195 & 0.1508 & 0 & 0 \\ 0.1917 & 0.3016 & 0.1475 & 0.1499 \\ 0.1278 & 0 & 0.3115 & 0.1499 \\ 0 & 0.2513 & 0.2131 & 0.3748 \end{pmatrix}$$

and

$$m_1(5) = 0.3609, \quad m_2(5) = 0.2963, \quad m_3(5) = 0.3279, \quad m_4(5) = 0.3254,$$

$$\frac{1}{4}\sum_{j=1}^{4} m_j(5) = 0.3276.$$

Step 9. We combine these Mass functions and compute each belief measure of each candidate x_i respectively as follows:

$$Bel(\{x_1\}) = m_1 \oplus m_2 \oplus m_3 \oplus m_4(\{x_1\}) = 0.1098,$$
$$Bel(\{x_2\}) = m_1 \oplus m_2 \oplus m_3 \oplus m_4(\{x_2\}) = 0.3298,$$
$$Bel(\{x_3\}) = m_1 \oplus m_2 \oplus m_3 \oplus m_4(\{x_3\}) = 0.1700,$$
$$Bel(\{x_4\}) = m_1 \oplus m_2 \oplus m_3 \oplus m_4(\{x_4\}) = 0.3309,$$
$$Bel(\{x_5\}) = m_1 \oplus m_2 \oplus m_3 \oplus m_4(\Theta) = 0.0595.$$

Then the final rang order is $x_4 \succ x_2 \succ x_3 \succ x_1$.

Step 10. x_4 is the optimal investment company for $max_i \{Bel(x_i)\} = 0.3309$.

From the above results, the belief measure of the uncertainty with respect to the whole candidates Θ is declined from 0.3276 to 0.0595, after applying grey relational analysis to construct the corresponding Mass functions for different evidences and then using the rule of evidence combination to compose these information. This implies the above algorithm is effective and practical under uncertainties. It not only allows us to avoid selecting the suitable level soft set, but also helps reducing humanistic and subjective in nature to raise the choices decision level. Moreover, it broadens the application field of the grey system theory and D-S theory of evidence.

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PRODUCT-TYPE OPERATORS FROM WEIGHTED ZYGMUND SPACES TO BLOCH-ORLICZ SPACES

YONG YANG AND ZHI-JIE JIANG

Abstract. Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the class of all analytic functions on $\mathbb{D}$. Let φ be an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$. The boundedness and compactness of the product-type operators $D^n M_u C_\varphi$, $D^n C_\varphi M_u$, $M_u D^n C_\varphi$, $C_\varphi D^n M_u$, $M_u C_\varphi D^n$ and $C_\varphi M_u D^n$ from weighted Zygmund spaces to Bloch-Orlicz spaces are characterized by constructing some test functions in weighted Zygmund spaces.

1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the class of all analytic functions on $\mathbb{D}$. For $\alpha > 0$, the weighted Zygmund space $\mathcal{Z}^\alpha$ consists of all $f \in H(\mathbb{D})$ such that

$$b_{\mathcal{Z}^\alpha}(f) = \sup_{z\in\mathbb{D}} (1 - |z|^2)^\alpha |f''(z)| < \infty.$$

It is a Banach space with the norm

$$\|f\|_{\mathcal{Z}^\alpha} = |f(0)| + |f'(0)| + b_{\mathcal{Z}^\alpha}(f).$$

If $\alpha = 1$, then it becomes the famous Zygmund space, usually denoted by $\mathcal{Z}$. For some results of weighted Zygmund spaces and some concrete operators on them, see, for example, [9, 22, 24, 43, 56] and the references therein.

Next we introduce the Bloch-Orlicz space which was defined by Ramos Fernández in [32]. Let Ψ be a Young's function, i.e., Ψ is a strictly increasing convex function on $[0, +\infty)$ such that $\Psi(0) = 0$ and $\lim_{t\to+\infty} \Psi(t) = +\infty$. The Bloch-Orlicz space $\mathcal{B}^\Psi$ consists of all $f \in H(\mathbb{D})$ such that

$$\sup_{z\in\mathbb{D}} (1 - |z|^2)\Psi(\lambda|f'(z)|) < \infty$$

for some $\lambda > 0$ depending on f. The Minkowski's functional

$$\|f\|_\Psi = \inf\left\{k > 0 : S_\Psi\left(\frac{f'}{k}\right) \le 1\right\}$$

defines a seminorm for $\mathcal{B}^\Psi$, where

$$S_\Psi(f) = \sup_{z\in\mathbb{D}} (1 - |z|^2)\Psi(|f(z)|).$$

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$\mathcal{B}^\Psi$ becomes a Banach space with the norm $\|f\|_{\mathcal{B}^\Psi} = |f(0)| + \|f\|_\Psi$. Ramos Fernández in [32] proved that it is isometrically equal to a special μ_Ψ-Bloch space, where

$$\mu_\Psi(z) = \frac{1}{\Psi^{-1}(\frac{1}{1-|z|^2})}, \ \ z \in \mathbb{D}.$$

Consequently, a equivalent norm on $\mathcal{B}^\Psi$ is given by $\|f\|_{\mathcal{B}^\Psi} = |f(0)| + b_{\mathcal{B}^\Psi}(f)$, where

$$b_{\mathcal{B}^\Psi}(f) = \sup_{z\in\mathbb{D}} \mu_\Psi(z)|f'(z)|.$$

Clearly, the quantity $b_{\mathcal{B}^\Psi}(f)$ is a seminorm on the space $\mathcal{B}^\Psi$ and a norm on the quotient space $\mathcal{B}^\Psi/\mathbb{P}_0$, where $\mathbb{P}_0$ is the set of all constant functions. The Bloch-Orlicz space generalizes some spaces. For example, if $\Psi(t) = t^p$ with $p > 0$, then $\mathcal{B}^\Psi$ coincides with the weighted Bloch space $\mathcal{B}^\alpha$, where $\alpha = 1/p$; if $\Psi(t) = t\log(1 + t)$, then $\mathcal{B}^\Psi$ coincides with the Log-Bloch space (see [2]).

Let φ be an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$. The weighted composition operator $W_{\varphi,u}$ on $H(\mathbb{D})$ is defined by

$$W_{\varphi,u}f(z) = u(z)f(\varphi(z)), \ z \in \mathbb{D}.$$

If $u \equiv 1$, it becomes the composition operator, usually denoted by C_φ. If $\varphi(z) = z$, it becomes the multiplication operator, usually denoted by M_u. Since $W_{\varphi,u} = M_uC_\varphi$, it is a product-type operator. For some studies on weighted composition operators, see, for example, [1, 4, 7, 10, 19, 22, 29, 42, 49, 50] and the references therein.

Let $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The nth differentiation operator D^n on $H(\mathbb{D})$ is defined by

$$D^nf(z) = f^{(n)}(z), \ \ z \in \mathbb{D},$$

where $f^{(0)} = f$. If $n = 1$, it is the well-known differentiation operator D. Zhu in [57] introduced the following, so-called, generalized weighted composition operator:

$$D^n_{\varphi,u}f(z) = u(z)f^{(n)}(\varphi(z)), \ z \in \mathbb{D}.$$

If $n = 0$, it becomes the weighted composition operator. Since $D^n_{\varphi,u} = M_uC_\varphi D^n$, it is also a product-type operator. For generalized weighted composition operators, see, for example, [3, 28, 47, 53, 54, 59, 60] and the references therein. Before the operator $D^n_{\varphi,u}$ some other product-type operators were introduced and studied. For example, the next product-type operators

$$M_uC_\varphi D, \ C_\varphi M_uD, \ M_uDC_\varphi, \ C_\varphi DM_u, \ DC_\varphi M_u, \ DM_uC_\varphi$$

were studied by Sharma in [34]. They were also studied on weighted Bergman spaces by Stević et al. in [51] and [52]. However, a normally systematic study of product-type operators started by Stević et al. since the publication of papers [21] and [25]. Before that there were a few papers in the topic, e.g., [8]. The publication of paper [21] first attracted some attention in product-type operators DC_φ and $C_\varphi D$ (see, e.g., [23, 30, 39, 41] and the references therein). The publication of paper [25] attracted some attention in product-type operators involving integral-type ones (see, e.g., [16, 26, 37, 43, 48] and the references therein). Recently there is a great interest in various product-type operators between two given spaces of holomorphic functions (see, e.g., [11, 12, 17, 31, 33, 36, 38, 40, 45, 57] and the references therein).

Before this paper some product-type operators from Zygmund spaces or weighted Zygmund spaces to some other spaces were studied, for example, in [3, 13, 14, 18, 27]. In this paper we consider the following product-type operators:

$$D^nM_uC_\varphi, \ D^nC_\varphi M_u, \ M_uD^nC_\varphi, \ C_\varphi D^nM_u, \ M_uC_\varphi D^n, \ C_\varphi M_uD^n. \quad (1)$$

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The boundedness and compactness of operators in (1) from Zygmund spaces to Bloch-Orlicz spaces were characterized in [14]. As a continuation and completeness of our work, we consider the same problems for operators in (1) from weighted Zygmund spaces with $\alpha \neq 1$ to Bloch-Orlicz spaces. Because these operators are more complicated than those above mentioned, we need seek some other test functions in weighted Zygmund spaces to achieve our objective.

Let X and Y be Banach spaces. A linear operator $L : X \to Y$ is bounded if there exists a positive constant K such that $\|Lf\|_Y \leq K\|f\|_X$ for all $f \in X$. The operator $L : X \to Y$ is compact if it maps bounded sets into relatively compact sets. The norm of the operator $L : X \to Y$ is defined by

$$\|L\|_{X\to Y} = \sup_{\|f\|_X \leq 1} \|Lf\|_Y.$$

In this paper, the letter C denotes a positive constant which may differ from one occurrence to the other. The notation $a \lesssim b$ means that there exists a positive constant C such that $a \leq Cb$. When $a \lesssim b$ and $b \lesssim a$, we write $a \asymp b$.

2. Preliminaries and test functions

We first state the following result which was essentially proved in [35] and [46].

Lemma 2.1. *For $\alpha > 0$ and $f \in \mathcal{Z}^\alpha$. Then*

(a) For $0 < \alpha < 1$, $|f(z)| \leq \frac{2}{1-\alpha}\|f\|_{\mathcal{Z}}$ and $|f'(z)| \leq \frac{2}{1-\alpha}\|f\|_{\mathcal{Z}}$.
(b) For $\alpha = 1$, $|f(z)| \leq \|f\|_{\mathcal{Z}}$ and $|f'(z)| \leq \|f\|_{\mathcal{Z}} \log \frac{e}{1-|z|^2}$.
(c) For $1 < \alpha < 2$, $|f(z)| \leq \frac{1}{(\alpha-1)(2-\alpha)}\|f\|_{\mathcal{Z}^\alpha}$ and $|f'(z)| \leq \frac{2}{\alpha-1}\frac{\|f\|_{\mathcal{Z}^\alpha}}{(1-|z|^2)^{\alpha-1}}$.
(d) For $\alpha = 2$, $|f(z)| \leq 2\|f\|_{\mathcal{Z}^2} \log \frac{e}{1-|z|^2}$ and $|f'(z)| \leq \frac{e}{1-|z|^2}\|f\|_{\mathcal{Z}^2}$.
(e) For $\alpha > 2$, $|f(z)| \leq \frac{1}{(\alpha-1)(\alpha-2)}\frac{\|f\|_{\mathcal{Z}^\alpha}}{(1-|z|^2)^{\alpha-2}}$ and $|f'(z)| \leq \frac{2}{\alpha-1}\frac{\|f\|_{\mathcal{Z}^\alpha}}{(1-|z|^2)^{\alpha-1}}$.

The following result directly follows from the corresponding result for the Bloch type spaces when a function f is replaced by f' (see, e.g., [55]).

Lemma 2.2. *For each $k \in \mathbb{N}$ and $k \geq 2$, there exists a positive constant C_k independent of $f \in \mathcal{Z}^\alpha$ and $z \in \mathbb{D}$ such that*

$$|f^{(k)}(z)| \leq \frac{C_k\|f\|_{\mathcal{Z}^\alpha}}{(1-|z|^2)^{\alpha+k-2}}.$$

Let $w \in \mathbb{D}$ and $i \in \mathbb{N}_0$. It is easily shown that the next function is in the space $\mathcal{Z}^\alpha$

$$r_{w,i}(z) = \frac{(1-|w|^2)^{2+i}}{(1-\overline{w}z)^{\alpha+i}}, \ z \in \mathbb{D}.$$

The following result provides the needed test functions for the cases $0 < \alpha < 1$, $1 < \alpha < 2$, $\alpha = 2$ and $\alpha > 2$.

Lemma 2.3. *(a) If $0 < \alpha < 1$, then for each fixed $k \in \{2, 3, \ldots, n+1\}$, there exist constants $a_{0,k}$, $a_{1,k}$, $\ldots$, $a_{n+1,k}$ such that the function*

$$f_{w,k}(z) = \sum_{i=0}^{n+1} a_{i,k} r_{w,i}(z)$$

satisfies

$$f_{w,k}^{(k)}(w) = \frac{\overline{w}^k}{(1-|w|^2)^{\alpha+k-2}} \quad \textit{and} \quad f_{w,k}^{(j)}(w) = 0 \tag{2}$$

for each $j \in \{0, 1, \dots, n+1\} \setminus \{k\}$.

(b) If $1 < \alpha \le 2$*, then for each fixed* $k \in \{1, 2, \dots, n+1\}$*, there exist constants* $b_{0,k}$*,* $b_{1,k}$*, …,* $b_{n+1,k}$ *such that the function*

$$g_{w,k}(z) = \sum_{i=0}^{n+1} b_{i,k} r_{w,i}(z)$$

satisfies

$$g_{w,k}^{(k)}(w) = \frac{\overline{w}^k}{(1-|w|^2)^{\alpha+k-2}} \quad \textit{and} \quad g_{w,k}^{(j)}(w) = 0 \tag{3}$$

for each $j \in \{0, 1, \dots, n+1\} \setminus \{k\}$.

(c) If $\alpha > 2$*, then for each fixed* $k \in \{0, 1, \dots, n+1\}$*, there exist constants* $c_{0,k}$*,* $c_{1,k}$*, …,* $c_{n+1,k}$ *such that the function*

$$h_{w,k}(z) = \sum_{i=0}^{n+1} c_{i,k} r_{w,i}(z)$$

satisfies

$$h_{w,k}^{(k)}(w) = \frac{\overline{w}^k}{(1-|w|^2)^{\alpha+k-2}} \quad \textit{and} \quad h_{w,k}^{(j)}(w) = 0 \tag{4}$$

for each $j \in \{0, 1, \dots, n+1\} \setminus \{k\}$.

Proof. (a). From a calculation, it follows that (2) is equivalent to the following system

$$\begin{cases} \sum\limits_{i=0}^{n+1} (\alpha+i) a_{i,k} = 0 \\ \sum\limits_{i=0}^{n+1} (\alpha+i)(\alpha+i+1) a_{i,k} = 0 \\ \qquad \cdots\cdots\cdots \\ \sum\limits_{i=0}^{n+1} \prod\limits_{j=0}^{k-1} (\alpha+i+j) a_{i,k} = 1 \\ \qquad \cdots\cdots\cdots \\ \sum\limits_{i=0}^{n+1} \prod\limits_{j=0}^{n} (\alpha+i+j) a_{i,k} = 0. \end{cases} \tag{5}$$

Hence, we only need to prove that there exist constants $a_{0,k}$, $a_{1,k}$, …, $a_{n+1,k}$ such that the system (5) holds. By Lemma 3 in [47], the determinant of the system (5) equals to $\prod_{j=1}^{n+1} j!$, which is different from zero. So there exist constants $a_{0,k}$, $a_{1,k}$, …, $a_{n+1,k}$ such that the system (5) holds. Results (b) and (c) can be proved similarly, so we omit. □

Let $w \in \mathbb{D}$ and

$$q_w(z) = \Big(1 + \log^2 \frac{e}{1-\overline{w}z}\Big) \log^{-1} \frac{e}{1-|w|^2}.$$

Lemma 2.4. *For the function q_w, it follows that*

$$q_w^{(k)}(w) = c_k \frac{\overline{w}^k}{(1-|w|^2)^k} + d_k \frac{\overline{w}^k}{(1-|w|^2)^k} \log^{-1} \frac{e}{1-|w|^2}, \tag{6}$$

where $c_k > 0$ for each $k \geq 1$, $d_1 = 0$ and $d_k > 0$ for each $k \geq 2$.

Proof. By a direct computation, we have

$$q'_w(z) = 2\frac{\overline{w}}{1-\overline{w}z} \log \frac{e}{1-\overline{w}z} \log^{-1} \frac{e}{1-|w|^2}, \tag{7}$$

and

$$q''_w(z) = 2\frac{\overline{w}^2}{(1-\overline{w}z)^2} \log \frac{e}{1-\overline{w}z} \log^{-1} \frac{e}{1-|w|^2} + 2\frac{\overline{w}^2}{(1-\overline{w}z)^2} \log^{-1} \frac{e}{1-|w|^2}. \tag{8}$$

Also, from a direct computation, we see that for $k \geq 2$

$$\begin{aligned} q_w^{(k)}(z) = {} & 2(k-1)! \frac{\overline{w}^k}{(1-\overline{w}z)^k} \log \frac{e}{1-\overline{w}z} \log^{-1} \frac{e}{1-|w|^2} \\ & + \big[k-1+2(k-1)!\big] \frac{\overline{w}^k}{(1-\overline{w}z)^k} \log^{-1} \frac{e}{1-|w|^2}. \end{aligned} \tag{9}$$

Set $c_k = 2(k-1)!$, $d_1 = 0$ and $d_k = k-1+2(k-1)!$ for $k \geq 2$. Then (6) follows from (7)-(9). □

Remark 2.1. *Let X_w be the functions in Lemmas 2.3 and 2.4. Then*

$$\sup_{w\in\mathbb{D}} \|X_w\|_{\mathcal{Z}^\alpha} \lesssim 1, \tag{10}$$

and $X_w \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $|w| \to 1$. In fact, if X_w are the functions in Lemma 2.3, then this remark follows from the facts that $\sup_{w\in\mathbb{D}} \|r_{w,i}\|_{\mathcal{Z}^\alpha} \lesssim 1$ and $r_{w,i} \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $|w| \to 1$; if X_w is the function in Lemma 2.4, then it follows from [44].

Stević in [47] used Faà di Bruno's formula of the following version

$$(f\circ\varphi)^{(n)}(z) = \sum_{k=0}^{n} f^{(k)}(\varphi(z)) B_{n,k}(\varphi'(z), \ldots, \varphi^{(n-k+1)}(z)), \tag{11}$$

where $B_{n,k}(x_1, ..., x_{n-k+1})$ is the Bell polynomial. See [15] for the Faà di Bruno's formula. For $n \in \mathbb{N}$ the sum can go from $k = 1$ since $B_{n,0}(\varphi'(z), ..., \varphi^{(n-k+1)}(z)) = 0$, however we will keep the summation since for $n = 0$ the only existing term $B_{0,0}$ is equal to 1. From (11) and the Leibniz formula the next lemma follows.

Lemma 2.5. *Let f, $u \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. Then*

$$\big(u(z)f(\varphi(z))\big)^{(n+1)} = \sum_{k=0}^{n+1} f^{(k)}(\varphi(z)) \sum_{j=k}^{n+1} C_{n+1}^{j} u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z), \ldots, \varphi^{(j-k+1)}(z)\big).$$

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3. Boundedness the product-type operators

We first characterize the boundedness of the operator $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$.

Theorem 3.1. *Let φ be an analytic self-map of $\mathbb{D}$, $u \in H(\mathbb{D})$, C_{n+1}^j the binomial coefficient and $0 < \alpha < 1$. Then the following statements are equivalent.*

(a) The operator $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is bounded.

(b) The functions u and φ satisfy the following conditions:

$$I_0 := \sup_{z\in\mathbb{D}} \mu_\Psi(z)\Big|\sum_{j=0}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,0}\big(\varphi'(z), \dots, \varphi^{(j+1)}(z)\big)\Big| < \infty,$$

$$I_1 := \sup_{z\in\mathbb{D}} \mu_\Psi(z)\Big|\sum_{j=1}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,1}\big(\varphi'(z), \dots, \varphi^{(j)}(z)\big)\Big| < \infty,$$

and

$$I_k := \sup_{z\in\mathbb{D}} \frac{\mu_\Psi(z)\Big|\sum\limits_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z), \dots, \varphi^{(j-k+1)}(z)\big)\Big|}{(1-|\varphi(z)|^2)^{\alpha+k-2}} < \infty$$

for each $k \in \{2, 3, \dots, n+1\}$.

Moreover, if the operator $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is bounded, then

$$\big\|D^n M_u C_\varphi\big\|_{\mathcal{Z}^\alpha \to \mathcal{B}^\Psi/\mathbb{P}_0} \asymp \sum_{k=0}^{n+1} I_k.$$

Proof. $(a) \Rightarrow (b)$. Let $h_k(z) = z^k \in \mathcal{Z}$, $k = 0, 1, \dots, n+1$. Then applying the operator $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ to the function h_0, we have

$$I_0 = \sup_{z\in\mathbb{D}} \mu_\Psi(z)\Big|\sum_{j=0}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,0}(\varphi'(z), \dots, \varphi^{(j+1)}(z))\Big| \le C\big\|D^n M_u C_\varphi\big\|. \quad (12)$$

By the fact $\|\varphi\|_\infty \le 1$, the boundedness of $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$, the triangle inequality and (12), we have

$$I_1 \le I_0 + C\|D^n M_u C_\varphi\|. \quad (13)$$

Assume now that we have proved the following inequalities

$$\sup_{z\in\mathbb{D}} \mu_\Psi(z)\Big|\sum_{j=l}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,l}(\varphi'(z), \dots, \varphi^{(j-l+1)}(z))\Big| \le C\|D^n M_u C_\varphi\| \quad (14)$$

for each $l \in \{0, 1, ..., k-1\}$ and a $k \le n+1$. Applying Lemma 2.5 to the function h_k, and noticing that $h_k^{(s)}(z) \equiv 0$ for $s > k$, we get

$$\begin{aligned}\big(D^n M_u C_\varphi h_k\big)'(z) &= \sum_{j=0}^{k} h_k^{(j)}(\varphi(z)) \sum_{i=j}^{n+1} C_{n+1}^i u^{(n+1-i)}(z) B_{i,j}(\varphi'(z), \dots, \varphi^{(i-j+1)}(z)) \\ &= \sum_{j=0}^{k} k \cdots (k-j+1)(\varphi(z))^{k-j} \sum_{i=j}^{n+1} C_{n+1}^i u^{(n+1-i)}(z) B_{i,j}(\varphi'(z), \dots, \varphi^{(i-j+1)}(z)). \quad (15)\end{aligned}$$

From (15), the boundedness of function φ and the triangle inequality, by noticing that the coefficient at

$$\sum_{j=k}^{n+1} C_{n+1}^{j} u^{(n+1-j)}(z) B_{j,k}(\varphi'(z), \dots, \varphi^{(j-k+1)}(z))$$

is independent of z and finally using hypothesis (14) we easily obtain

$$L_k := \sup_{z\in\mathbb{D}} \mu_\Psi(z) \Big| \sum_{j=k}^{n+1} C_{n+1}^{j} u^{(n+1-j)}(z) B_{j,k}(\varphi'(z), \dots, \varphi^{(j-k+1)}(z)) \Big| \le C\|D^n M_u C_\varphi\|. \tag{16}$$

By induction we see that (16) holds for each $k \in \{0, 1, \dots, n+1\}$.

For a fixed $w \in \mathbb{D}$ and a fixed $k \in \{2, 3, \dots, n+1\}$, by Lemma 2.3 (a) there exists a function

$$f_{w,k}(z) = \sum_{i=0}^{n+1} a_{i,k} r_{\varphi(w),i}(z)$$

such that

$$f_{w,k}^{(k)}(\varphi(w)) = \frac{\overline{\varphi(w)}^k}{(1-|\varphi(w)|^2)^{\alpha+k-2}} \quad \text{and} \quad f_{w,k}^{(j)}(\varphi(w)) = 0 \tag{17}$$

for each $j \in \{0, 1, \dots, n+1\} \setminus \{k\}$, and

$$\sup_{w\in\mathbb{D}} \|f_{w,k}\|_{\mathcal{Z}^\alpha} \le C. \tag{18}$$

Then by (17), (18) and the boundedness of $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$, we have

$$\begin{aligned} I_k(w) &:= \frac{\mu_\Psi(w)|\varphi(w)|^k \Big| \sum_{j=k}^{n+1} C_{n+1}^{j} u^{(n+1-j)}(w) B_{j,k}\big(\varphi'(w), \dots, \varphi^{(j-k+1)}(w)\big) \Big|}{(1-|\varphi(w)|^2)^{\alpha+k-2}} \\ &\le \|D^n M_u C_\varphi f_{w,k}\|_{\mathcal{B}^\Psi} \le C\|D^n M_u C_\varphi\|. \end{aligned} \tag{19}$$

From (19) we see that

$$\sup_{z\in\mathbb{D}} I_k(z) \le C\big\|D^n M_u C_\varphi\big\|,$$

which leads to

$$\sup_{|\varphi(z)|>1/2} \frac{\mu_\Psi(z) \Big| \sum_{j=k}^{n+1} C_{n+1}^{j} u^{(n+1-j)}(z) B_{j,k}(\varphi'(z), \dots, \varphi^{(j-k+1)}(z)) \Big|}{(1-|\varphi(z)|^2)^{\alpha+k-2}} \le C\|D^n M_u C_\varphi\|. \tag{20}$$

On the other hand, by (16) we have

$$\sup_{|\varphi(z)|\le 1/2} \frac{\mu_\Psi(z) \Big| \sum_{j=k}^{n+1} C_{n+1}^{j} u^{(n+1-j)}(z) B_{j,k}(\varphi'(z), \dots, \varphi^{(j-k+1)}(z)) \Big|}{(1-|\varphi(z)|^2)^{\alpha+k-2}} \le C\|D^n M_u C_\varphi\|. \tag{21}$$

Hence from (20) and (21) we obtain

$$I_k \le C\|D^n M_u C_\varphi\| < \infty. \tag{22}$$

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$(b) \Rightarrow (a)$. By Lemmas 2.1, 2.2 and 2.5, for all $f \in \mathcal{Z}^\alpha$ we have

$$\begin{aligned}
&\sup_{z\in\mathbb{D}} \mu_\Psi(z)\big|\big(D^n M_u C_\varphi f\big)'(z)\big| \\
&= \sup_{z\in\mathbb{D}} \mu_\Psi(z)\Big|\sum_{k=0}^{n+1} f^{(k)}(\varphi(z))\sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z)B_{j,k}\big(\varphi'(z),\ldots,\varphi^{(j-k+1)}(z)\big)\Big| \\
&\le \sup_{z\in\mathbb{D}} \mu_\Psi(z)\sum_{k=0}^{n+1} \big|f^{(k)}(\varphi(z))\big|\Big|\sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z)B_{j,k}\big(\varphi'(z),\ldots,\varphi^{(j-k+1)}(z)\big)\Big| \\
&\le \Big(\frac{1}{1-\alpha}(I_0+I_1)+\sum_{k=2}^{n+1} C_k I_k\Big)\|f\|_{\mathcal{Z}^\alpha}. \qquad (23)
\end{aligned}$$

It is clear that

$$|(D^n M_u C_\varphi f)(0)| \le C\|f\|_{\mathcal{Z}^\alpha}. \qquad (24)$$

Hence, from (23) and (24) it follows that the operator $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is bounded.

Clearly, if the operator $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is bounded, then the operator $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi/\mathbb{P}_0$ is also bounded. By the definition of the norm in the quotient spaces, and using the same functions in the proofs of (12), (13) and (22), we obtain

$$I_k \le C\|D^n M_u C_\varphi\|_{\mathcal{Z}^\alpha\to\mathcal{B}^\Psi/\mathbb{P}_0},$$

for each $k \in \{0,1,2,\ldots,n+1\}$, and then

$$\sum_{k=0}^{n+1} I_k \le C\|D^n M_u C_\varphi\|_{\mathcal{Z}^\alpha\to\mathcal{B}^\Psi/\mathbb{P}_0}. \qquad (25)$$

By (23) we have

$$\|D^n M_u C_\varphi\|_{\mathcal{Z}^\alpha\to\mathcal{B}^\Psi/\mathbb{P}_0} \le C\sum_{k=0}^{n+1} I_k. \qquad (26)$$

The asymptotic expression of $\|D^n M_u C_\varphi\|_{\mathcal{Z}^\alpha\to\mathcal{B}^\Psi/\mathbb{P}_0}$ follows from (25) and (26). □

Remark 3.1. *In fact, from the fact $z^k \in \mathcal{Z}^\alpha$, in the proof of Theorem 3.1 we have seen that if the operator $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is bounded, then $L_k < \infty$ for all $\alpha > 0$.*

Theorem 3.2. *Let φ be an analytic self-map of $\mathbb{D}$, $u \in H(\mathbb{D})$, C_{n+1}^j the binomial coefficient and $1 < \alpha < 2$. Then the following statements are equivalent.*

(a) The operator $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is bounded.
(b) The functions u and φ are such that $I_0 < \infty$ and for each $k \in \{1,2,\ldots,n+1\}$

$$M_k := \sup_{z\in\mathbb{D}} \frac{\mu_\Psi(z)\Big|\sum\limits_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z)B_{j,k}\big(\varphi'(z),\ldots,\varphi^{(j-k+1)}(z)\big)\Big|}{(1-|\varphi(z)|^2)^{\alpha+k-2}} < \infty.$$

Moreover, if the operator $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is bounded, then

$$\big\|D^n M_u C_\varphi\big\|_{\mathcal{Z}^\alpha\to\mathcal{B}^\Psi/\mathbb{P}_0} \asymp I_0 + \sum_{k=1}^{n+1} M_k.$$

Proof. $(a) \Rightarrow (b)$. Let $h_0(z) \equiv 1 \in \mathcal{Z}^\alpha$. Then $I_0 < \infty$. For a fixed $w \in \mathbb{D}$ and each fixed $k \in \{1, 2, \ldots, n+1\}$, by Lemma 2.3 (b) there exists a function

$$g_{w,k}(z) = \sum_{i=0}^{n+1} b_{i,k} r_{\varphi(w),i}(z)$$

such that

$$g_{w,k}^{(k)}(\varphi(w)) = \frac{\overline{\varphi(w)}^k}{(1 - |\varphi(w)|^2)^{\alpha+k-2}} \quad \text{and} \quad g_{w,k}^{(j)}(\varphi(w)) = 0 \tag{27}$$

for each $j \in \{0, 1, \ldots, n+1\} \setminus \{k\}$. Moreover,

$$\sup_{w\in\mathbb{D}} \|g_{w,k}\|_{\mathcal{Z}^\alpha} \le C. \tag{28}$$

Then from (27), (28) and the boundedness of $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$, we have

$$\begin{aligned} M_k(w) :=& \frac{\mu_\Psi(w)|\varphi(w)|^k \Big| \sum\limits_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(w) B_{j,k}\big(\varphi'(w), \ldots, \varphi^{(j-k+1)}(w)\big)\Big|}{(1 - |\varphi(w)|^2)^{\alpha+k-2}} \\ &\le \|D^n M_u C_\varphi g_{\varphi(w),k}\|_{\mathcal{B}^\Psi} \le C\|D^n M_u C_\varphi\|. \end{aligned} \tag{29}$$

From (29) we see

$$\sup_{z\in\mathbb{D}} M_k(z) \le C\big\|D^n M_u C_\varphi\big\|, \tag{30}$$

and then

$$\sup_{|\varphi(z)|>1/2} \frac{\mu_\Psi(z)\Big| \sum\limits_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}(\varphi'(z), \ldots, \varphi^{(j-k+1)}(z))\Big|}{(1 - |\varphi(z)|^2)^{\alpha+k-2}} \le C\|D^n M_u C_\varphi\|. \tag{31}$$

On the other hand, by using the fact $L_k < \infty$ for each $k \in \{0, 1, \ldots, n+1\}$, we get

$$\sup_{|\varphi(z)|\le 1/2} \frac{\mu_\Psi(z)\Big| \sum\limits_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}(\varphi'(z), \ldots, \varphi^{(j-k+1)}(z))\Big|}{(1 - |\varphi(z)|^2)^{\alpha+k-2}} \le C\|D^n M_u C_\varphi\|. \tag{32}$$

Hence from (31) and (32) we see that $M_k < \infty$ for each $k \in \{1, 2, ..., n+1\}$.

$(b) \Rightarrow (a)$. By Lemmas 2.1, 2.2 and 2.5, for all $f \in \mathcal{Z}^\alpha$ we have

$$\begin{aligned} &\sup_{z\in\mathbb{D}} \mu_\Psi(z)\big|(D^n M_u C_\varphi f)'(z)\big| \\ &= \sup_{z\in\mathbb{D}} \mu_\Psi(z)\Big| \sum_{k=0}^{n+1} f^{(k)}(\varphi(z)) \sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z), \ldots, \varphi^{(j-k+1)}(z)\big)\Big| \\ &\le \sup_{z\in\mathbb{D}} \mu_\Psi(z) \sum_{k=0}^{n+1} \big|f^{(k)}(\varphi(z))\big| \Big| \sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z), \ldots, \varphi^{(j-k+1)}(z)\big)\Big| \\ &\le \Big(\frac{I_0}{(\alpha-1)(2-\alpha)} + \frac{2M_1}{\alpha-1} + \sum_{k=2}^{n+1} C_k M_k\Big)\|f\|_{\mathcal{Z}^\alpha}. \end{aligned} \tag{33}$$

It is clear that

$$|(D^n M_u C_\varphi f)(0)| \le C\|f\|_{\mathcal{Z}^\alpha}. \tag{34}$$

Hence from (33) and (34) it follows that the operator $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is bounded. Similarly is obtained the asymptotic formula of $\|D^n M_u C_\varphi\|_{\mathcal{Z}^\alpha \to \mathcal{B}^\Psi/\mathbb{P}_0}$, hence we omit. □

Theorem 3.3. *Let φ be an analytic self-map of $\mathbb{D}$, $u \in H(\mathbb{D})$, C_{n+1}^j the binomial coefficient and $\alpha = 2$. Then the following statements are equivalent.*

(a) The operator $D^n M_u C_\varphi : \mathcal{Z}^2 \to \mathcal{B}^\Psi$ is bounded.
(b) The functions u and φ satisfy the following conditions:

$$R_0 := \sup_{z\in\mathbb{D}} \mu_\Psi(z)\Big|\sum_{j=0}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,0}\big(\varphi'(z), \dots, \varphi^{(j+1)}(z)\big)\Big| \log\frac{e}{1-|\varphi(z)|^2} < \infty.$$

and for each $k \in \{1, 2, \dots, n+1\}$

$$R_k := \sup_{z\in\mathbb{D}} \frac{\mu_\Psi(z)\Big|\sum\limits_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z), \dots, \varphi^{(j-k+1)}(z)\big)\Big|}{(1-|\varphi(z)|^2)^k} < \infty.$$

Moreover, if the operator $D^n M_u C_\varphi : \mathcal{Z}^2 \to \mathcal{B}^\Psi$ is bounded, then

$$\big\|D^n M_u C_\varphi\big\|_{\mathcal{Z}^\alpha \to \mathcal{B}^\Psi/\mathbb{P}_0} \asymp \sum_{k=0}^{n+1} R_k.$$

Proof. $(a) \Rightarrow (b)$. By using Lemma 2.3 (b), we can prove that $R_k < \infty$ for each $k \in \{1, 2, \dots, n+1\}$, so we do not give the proof again. For a fixed $w \in \mathbb{D}$, by Lemma 2.4 there exists a function

$$s_{\varphi(w)}(z) = p_{\varphi(w)}(z) + \sum_{i=0}^{n+1} d_i r_{\varphi(w),i}(z)$$

such that

$$s_{\varphi(w)}(\varphi(w)) = \log\frac{e}{1-|\varphi(w)|^2} \quad \text{and} \quad s_{\varphi(w)}^{(j)}(\varphi(w)) = 0 \tag{35}$$

for each $j \in \{1, 2, \dots, n+2\}$, moreover, $\sup_{w\in\mathbb{D}} \|s_{\varphi(w)}\|_{\mathcal{Z}^2} \le C$. Then from these and the boundedness of $D^n M_u C_\varphi : \mathcal{Z}^2 \to \mathcal{B}^\Psi$, we have

$$\begin{aligned} R_0(w) &:= \mu_\Psi(w)\Big|\sum_{j=0}^{n+1} C_{n+1}^j u^{(n+1-j)}(w) B_{j,0}\big(\varphi'(w), \dots, \varphi^{(j+1)}(w)\big)\Big| \log\frac{e}{1-|\varphi(w)|^2} \\ &\le \|D^n M_u C_\varphi s_{\varphi(w)}\|_{\mathcal{B}^\Psi} \le C\big\|D^n M_u C_\varphi\big\|. \end{aligned} \tag{36}$$

Then from (36) it follows that $R_0 < \infty$.

$(b) \Rightarrow (a)$. From Lemmas 2.1, 2.2 and 2.5, for all $f \in \mathcal{Z}^2$ we have

$$\begin{aligned} &\sup_{z\in\mathbb{D}} \mu_\Psi(z)\big|(D^n M_u C_\varphi f)'(z)\big| \\ &= \sup_{z\in\mathbb{D}} \mu_\Psi(z)\Big|\sum_{k=0}^{n+1} f^{(k)}(\varphi(z)) \sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z), \dots, \varphi^{(j-k+1)}(z)\big)\Big| \end{aligned}$$

$$\leq \sup_{z\in\mathbb{D}} \mu_\Psi(z) \sum_{k=0}^{n+1} \left|f^{(k)}(\varphi(z))\right| \Big| \sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z), \dots, \varphi^{(j-k+1)}(z)\big)\Big|$$

$$\leq \Big(2R_0 + eR_1 + \sum_{k=2}^{n+1} C_k R_k\Big)\|f\|_{\mathcal{Z}^2}. \tag{37}$$

It is clear that

$$|(D^n M_u C_\varphi f)(0)| \leq C\|f\|_{\mathcal{Z}^2}. \tag{38}$$

Hence from (37) and (38) it follows that the operator $D^n M_u C_\varphi : \mathcal{Z}^2 \to \mathcal{B}^\Psi$ is bounded. The asymptotic expression of $\|D^n M_u C_\varphi\|_{\mathcal{Z}^\alpha \to \mathcal{B}^\Psi/\mathbb{P}_0}$ can be similarly obtained. □

Theorem 3.4. *Let φ be an analytic self-map of $\mathbb{D}$, $u \in H(\mathbb{D})$ and $\alpha > 2$. Then the following statements are equivalent.*

(a) The operator $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is bounded.

(b) The functions u and φ satisfy

$$S_k := \sup_{z\in\mathbb{D}} \frac{\mu_\Psi(z)\Big|\sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z), \dots, \varphi^{(j-k+1)}(z)\big)\Big|}{(1-|\varphi(z)|^2)^{\alpha+k-2}} < \infty,\ k = 0, \dots, n+1.$$

Moreover, if the operator $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is bounded, then

$$\big\|D^n M_u C_\varphi\big\|_{\mathcal{Z}^\alpha \to \mathcal{B}^\Psi/\mathbb{P}_0} \asymp \sum_{k=0}^{n+1} S_k.$$

Proof. Similarly to the proofs of Theorems 3.1-3.3, this result can be proved. □

Remark 3.2. *By using the similar methods and techniques, the boundedness of the operators $D^n C_\varphi M_u$, $C_\varphi D^n M_u$, $M_u D^n C_\varphi$, $M_u C_\varphi D^n$ and $C_\varphi M_u D^n$ from weighted Zygmund spaces to Bloch-Orlicz spaces can be characterized, so we omit.*

4. Compactness of the product-type operators

The first result is an alternative to Proposition 3.11 in [5], which characterizes the compactness in terms of sequential convergence. So the proof is omitted.

Lemma 4.1. *Let $T \in \{D^n M_u C_\varphi, D^n C_\varphi M_u, M_u D^n C_\varphi, C_\varphi D^n M_u, M_u C_\varphi D^n, C_\varphi M_u D^n\}$. Then the bounded operator $T : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is compact if and only if for every bounded sequence $\{f_j\}$ in $\mathcal{Z}^\alpha$ such that $f_j \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $j \to \infty$, it follows that* $\lim_{j\to\infty} \|Tf_j\|_{\mathcal{B}^\Psi} = 0$.

The following lemma was proved in [46].

Lemma 4.2. *(a) If $0 < \alpha < 2$ and $\{f_j\}$ is a bounded sequence in $\mathcal{Z}^\alpha$ which uniformly converges to zero on compact subsets of $\mathbb{D}$ as $j \to \infty$, then*

$$\lim_{j\to\infty} \sup_{z\in\mathbb{D}} |f_j(z)| = 0.$$

(b) If $0 < \alpha < 1$ and $\{f_j\}$ is a bounded sequence in $\mathcal{Z}^\alpha$ which uniformly converges to zero on compact subsets of $\mathbb{D}$ as $j \to \infty$, then

$$\lim_{j\to\infty} \sup_{z\in\mathbb{D}} |f_j'(z)| = 0.$$

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Now we characterize the compactness of the operator $D^nM_uC_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$.

Theorem 4.1. *Let φ be an analytic self-map of $\mathbb{D}$, $u \in H(\mathbb{D})$ and $0 < \alpha < 1$. Then the following statements are equivalent.*

(a) The operator $D^nM_uC_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is compact.

(b) The functions u and φ satisfy $L_k < \infty$ for each $k \in \{0, 1, \dots, n+1\}$, and for each $k \in \{2, 3, \dots, n+1\}$

$$\lim_{|\varphi(z)|\to 1} \frac{\mu_\Psi(z)\Big|\sum\limits_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z)B_{j,k}\big(\varphi'(z), \dots, \varphi^{(j-k+1)}(z)\big)\Big|}{(1 - |\varphi(z)|^2)^{\alpha+k-2}} = 0.$$

Proof. $(a) \Rightarrow (b)$. Suppose that the operator $D^nM_uC_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is compact. Clearly the operator $D^nM_uC_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is bounded. By Remark 2.1, $L_k < \infty$ for each $k \in \{0, 1, \dots, n+1\}$. Consider a sequence $\{\varphi(z_i)\}$ in $\mathbb{D}$ such that $|\varphi(z_i)| \to 1$ as $i \to \infty$. If such a sequence does not exist, then the last condition in (b) obviously holds. Without loss of generality, we may suppose that $|\varphi(z_i)| > 1/2$ for all $i \in \mathbb{N}$. For each fixed $k \in \{2, 3, \dots, n+1\}$, using this sequence we define the function sequence $f_{i,k}(z) = f_{\varphi(z_i),k}(z),\ i \in \mathbb{N}$. Then by Lemma 2.3 (a) we have that $\sup_{i\in\mathbb{N}} \|f_{i,k}\|_{\mathcal{Z}^\alpha} \le C$ and $f_{i,k} \to 0$ uniformly on every compact subset of $\mathbb{D}$ as $i \to \infty$, moreover

$$f_{i,k}^{(k)}(\varphi(z_i)) = \frac{\overline{\varphi(z_i)}^k}{(1 - |\varphi(z_i)|^2)^{\alpha+k-2}} \quad\text{and}\quad f_{i,k}^{(j)}(\varphi(z_i)) = 0 \tag{39}$$

for each $j \in \{0, 1, \dots, n+1\} \setminus \{k\}$. By Lemma 4.1 and (39), we have

$$\lim_{i\to\infty} \frac{\mu_\Psi(z_i)\big|\sum\limits_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z_i)B_{j,k}(\varphi'(z_i), \dots, \varphi^{(j-k+1)}(z_i))\big|}{(1 - |\varphi(z_i)|^2)^{\alpha+k-2}} = 0. \tag{40}$$

$(b) \Rightarrow (a)$. We first check that $D^nM_uC_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is bounded. We observe that the last condition in (b) implies that for every $\varepsilon > 0$, there is an $\eta \in (0, 1)$ such that for all $z \in K = \{z \in \mathbb{D} : |\varphi(z)| > \eta\}$ and for each $k \in \{2, 3, \dots, n+1\}$

$$\frac{\mu_\Psi(z)\big|\sum\limits_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z)B_{j,k}(\varphi'(z), \dots, \varphi^{(j-k+1)}(z))\big|}{(1 - |\varphi(z)|^2)^{\alpha+k-2}} < \varepsilon. \tag{41}$$

From the fact $L_k < \infty$ for each $k \in \{2, 3, \dots, n+1\}$, and (41), we have

$$I_k \le \varepsilon + \frac{L_k}{(1 - \eta^2)^{\alpha+k-2}}. \tag{42}$$

From (42) and the fact $L_k < \infty$, it follows that $D^nM_uC_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is bounded.

To prove that $D^nM_uC_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is compact, by Lemma 4.1 we just need to prove that, if $\{f_i\}$ is a sequence in $\mathcal{Z}^\alpha$ such that $\sup_{i\in\mathbb{N}} \|f_i\|_{\mathcal{Z}^\alpha} \le M$ and $f_i \to 0$ uniformly on any compact subset of $\mathbb{D}$ as $i \to \infty$, then

$$\lim_{i\to\infty} \|D^nM_uC_\varphi f_i\|_{\mathcal{B}^\Psi} = 0.$$

For such chosen ε and η, by using (39), Lemma 2.1 and Lemma 2.2, we have

$$
\begin{aligned}
&\sup_{z\in\mathbb{D}}\mu_\Psi(z)\big|(D^nM_uC_\varphi f_i)'(z)\big| \\
&=\sup_{z\in\mathbb{D}}\mu_\Psi(z)\Big|\sum_{k=0}^{n+1} f_i^{(k)}(\varphi(z))\sum_{j=k}^{n+1}C_{n+1}^j u^{(n+1-j)}(z)B_{j,k}\big(\varphi'(z),\dots,\varphi^{(j-k+1)}(z)\big)\Big| \\
&\le \sup_{z\in\mathbb{D}}\mu_\Psi(z)\sum_{k=0}^{n+1}\big|f_i^{(k)}(\varphi(z))\big|\Big|\sum_{j=k}^{n+1}C_{n+1}^j u^{(n+1-j)}(z)B_{j,k}\big(\varphi'(z),\dots,\varphi^{(j-k+1)}(z)\big)\Big| \\
&\le \sup_{z\in\mathbb{D}}\mu_\Psi(z)\Big|\sum_{j=0}^{n+1}C_{n+1}^j u^{(n+1-j)}(z)B_{j,0}\big(\varphi'(z),\dots,\varphi^{(j+1)}(z)\big)\Big|\big|f_i(\varphi(z))\big| \\
&\quad+\sup_{z\in\mathbb{D}}\mu_\Psi(z)\Big|\sum_{j=1}^{n+1}C_{n+1}^j u^{(n+1-j)}(z)B_{j,1}\big(\varphi'(z),\dots,\varphi^{(j)}(z)\big)\Big|\big|f_i'(\varphi(z))\big| \\
&\quad+\big(\sup_{z\in K}+\sup_{z\in\mathbb{D}\setminus K}\big)\mu_\Psi(z)\sum_{k=2}^{n+1}\big|f_i^{(k)}(\varphi(z))\big|\Big|\sum_{j=k}^{n+1}C_{n+1}^j u^{(n+1-j)}(z)B_{j,k}\big(\varphi'(z),\dots,\varphi^{(j-k+1)}(z)\big)\Big| \\
&\le L_0\sup_{z\in\mathbb{D}}\big|f_i(\varphi(z))\big|+L_1\sup_{z\in\mathbb{D}}\big|f_i'(\varphi(z))\big|+\sum_{k=2}^{n+1}L_k\sup_{|z|\le\eta}\big|f_i^{(k)}(z)\big|+C\varepsilon. \qquad (43)
\end{aligned}
$$

From (43), Lemma 4.2 and the fact $f_i \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $i\to\infty$ implies that for each $k\in\mathbb{N}$, $f_i^{(k)}\to 0$ uniformly on compact subsets of $\mathbb{D}$ as $i\to\infty$, we finally get

$$\lim_{i\to\infty}\sup_{z\in\mathbb{D}}\mu_\Psi(z)\big|(D^nM_uC_\varphi f_i)'(z)\big|=0. \qquad (44)$$

It is clear that

$$\lim_{i\to\infty}\big|(D^nM_uC_\varphi f_i)(0)\big|=0. \qquad (45)$$

From (44) and (45) we obtain

$$\lim_{i\to\infty}\|D^nM_uC_\varphi f_i\|_{\mathcal{B}^\Psi}=0.$$

This shows that the operator $D^nM_uC_\varphi:\mathcal{Z}^\alpha\to\mathcal{B}^\Psi$ is compact. □

Theorem 4.2. *Let φ be an analytic self-map of $\mathbb{D}$, $u\in H(\mathbb{D})$ and $1<\alpha<2$. Then the following statements are equivalent.*

(a) The operator $D^nM_uC_\varphi:\mathcal{Z}^\alpha\to\mathcal{B}^\Psi$ is compact.

(b) The functions u and φ are such that $L_k<\infty$ for each $k\in\{0,1,\dots,n+1\}$, and for each $k\in\{1,2,\dots,n+1\}$

$$\lim_{|\varphi(z)|\to1}\frac{\mu_\Psi(z)\big|\sum_{j=k}^{n+1}C_{n+1}^j u^{(n+1-j)}(z)B_{j,k}\big(\varphi'(z),\dots,\varphi^{(j-k+1)}(z)\big)\big|}{(1-|\varphi(z)|^2)^{\alpha+k-2}}=0.$$

Proof. $(a)\Rightarrow(b)$. Suppose that the operator $D^nM_uC_\varphi:\mathcal{Z}^\alpha\to\mathcal{B}^\Psi$ is compact. Obviously the operator $D^nM_uC_\varphi:\mathcal{Z}^\alpha\to\mathcal{B}^\Psi$ is bounded. Then $L_k<\infty$ for each $k\in\{0,1,\dots,n+1\}$. Consider a sequence $\{\varphi(z_i)\}_{i\in\mathbb{N}}$ in $\mathbb{D}$ such that $|\varphi(z_i)|\to1$ as $i\to\infty$. If such a sequence does not exist, then the last condition in (b) obviously holds. Without loss of generality,

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we may suppose that $|\varphi(z_i)| > 1/2$ for all $i \in \mathbb{N}$. For each fixed $k \in \{1, 2, \ldots, n+1\}$, by using this sequence we define the function sequence $g_{i,k}(z) = g_{\varphi(z_i),k}(z)$, $i \in \mathbb{N}$. Then from Lemma 2.3 (b) we see that $\sup_{i\in\mathbb{N}} \|g_{i,k}\|_{\mathcal{Z}^\alpha} \le C$ and $g_{i,k} \to 0$ uniformly on every compact subset of $\mathbb{D}$ as $i \to \infty$, moreover

$$g_{i,k}^{(k)}(\varphi(z_i)) = \frac{\overline{\varphi(z_i)}^k}{(1-|\varphi(z_i)|^2)^{\alpha+k-2}} \quad \text{and} \quad g_{i,k}^{(j)}(\varphi(z_i)) = 0 \tag{46}$$

for each $j \in \{0, 1, \ldots, n+1\} \setminus \{k\}$. From Lemma 4.1 and (46), for each fixed $k \in \{1, 2, \ldots, n+1\}$ we have

$$\lim_{i\to\infty} \frac{\mu_\Psi(z_i)\big|\sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z_i) B_{j,k}(\varphi'(z_i), \ldots, \varphi^{(j-k+1)}(z_i))\big|}{(1-|\varphi(z_i)|^2)^{\alpha+k-2}} = 0. \tag{47}$$

$(b) \Rightarrow (a)$. We first check that $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is bounded. We observe that the last condition in (b) implies that for every $\varepsilon > 0$, there is an $\eta \in (0,1)$ such that for all $z \in K = \{z \in \mathbb{D} : |\varphi(z)| > \eta\}$ and for each $k \in \{1, 2, \ldots, n+1\}$

$$\frac{\mu_\Psi(z)\big|\sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}(\varphi'(z), \ldots, \varphi^{(j-k+1)}(z))\big|}{(1-|\varphi(z)|^2)^{\alpha+k-2}} < \varepsilon. \tag{48}$$

From the fact $L_k < \infty$ for each $k \in \{0, 1, \ldots, n+1\}$, and (48), we have

$$M_k \le \varepsilon + \frac{L_k}{(1-\eta^2)^{\alpha+k-2}}. \tag{49}$$

From (49) and the fact $I_0 = L_0 < \infty$, it follows that $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is bounded.

In order to prove that $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is compact, by Lemma 4.1 we just need to prove that, if $\{f_i\}$ is a sequence in $\mathcal{Z}^\alpha$ such that $\sup_{i\in\mathbb{N}} \|f_i\|_{\mathcal{Z}^\alpha} \le M$ and $f_i \to 0$ uniformly on any compact subset of $\mathbb{D}$ as $i \to \infty$, then $\lim_{i\to\infty} \|D^n M_u C_\varphi f_i\|_{\mathcal{B}^\Psi} = 0$. For such chosen ε and η, by using (46), Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned}
&\sup_{z\in\mathbb{D}} \mu_\Psi(z)\big|(D^n M_u C_\varphi f_i)'(z)\big| \\
&= \sup_{z\in\mathbb{D}} \mu_\Psi(z)\Big|\sum_{k=0}^{n+1} f_i^{(k)}(\varphi(z)) \sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z), \ldots, \varphi^{(j-k+1)}(z)\big)\Big| \\
&\le \sup_{z\in\mathbb{D}} \mu_\Psi(z) \sum_{k=0}^{n+1} \big|f_i^{(k)}(\varphi(z))\big|\Big|\sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z), \ldots, \varphi^{(j-k+1)}(z)\big)\Big| \\
&\le \sup_{z\in\mathbb{D}} \mu_\Psi(z)\Big|\sum_{j=0}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,0}\big(\varphi'(z), \ldots, \varphi^{(j+1)}(z)\big)\Big|\big|f_i(\varphi(z))\big| \\
&\quad + \big(\sup_{z\in K} + \sup_{z\in\mathbb{D}\setminus K}\big)\mu_\Psi(z) \sum_{k=1}^{n+1} \big|f_i^{(k)}(\varphi(z))\big|\Big|\sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z), \ldots, \varphi^{(j-k+1)}(z)\big)\Big| \\
&\le L_0 \sup_{z\in\mathbb{D}} \big|f_i(\varphi(z))\big| + \sum_{k=1}^{n+1} L_k \sup_{|z|\le\eta} \big|f_i^{(k)}(z)\big| + C\varepsilon.
\end{aligned} \tag{50}$$

From (50), Lemma 4.2 and the fact $f_i \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $i \to \infty$ implies that for each $k \in \mathbb{N}$, $f_i^{(k)} \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $i \to \infty$, we

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get

$$\lim_{i\to\infty}\sup_{z\in\mathbb{D}}\mu_\Psi(z)\big|(D^nM_uC_\varphi f_i)'(z)\big|=0. \tag{51}$$

It is clear that

$$\lim_{i\to\infty}\big|(D^nM_uC_\varphi f_i)(0)\big|=0. \tag{52}$$

From (51) and (52) we obtain

$$\lim_{i\to\infty}\|D^nM_uC_\varphi f_i\|_{\mathcal{B}^\Psi}=0.$$

This shows that the operator $D^nM_uC_\varphi:\mathcal{Z}^\alpha\to\mathcal{B}^\Psi$ is compact. □

Theorem 4.3. *Let φ be an analytic self-map of $\mathbb{D}$, $u\in H(\mathbb{D})$ and $\alpha=2$. Then the following statements are equivalent.*

(a) The operator $D^nM_uC_\varphi:\mathcal{Z}^2\to\mathcal{B}^\Psi$ is compact.
(b) The functions u and φ are such that $L_k<\infty$ for each $k\in\{0,1,\dots,n+1\}$,

$$\lim_{|\varphi(z)|\to1}\mu_\Psi(z)\Big|\sum_{j=0}^{n+1}C_{n+1}^ju^{(n+1-j)}(z)B_{j,0}\big(\varphi'(z),\dots,\varphi^{(j+1)}(z)\big)\Big|\log\frac{e}{1-|\varphi(z)|^2}=0,$$

and for each $k\in\{1,2,\dots,n+1\}$

$$\lim_{|\varphi(z)|\to1}\frac{\mu_\Psi(z)\Big|\sum_{j=k}^{n+1}C_{n+1}^ju^{(n+1-j)}(z)B_{j,k}\big(\varphi'(z),\dots,\varphi^{(j-k+1)}(z)\big)\Big|}{(1-|\varphi(z)|^2)^k}=0.$$

Proof. $(a)\Rightarrow(b)$. Suppose that the operator $D^nM_uC_\varphi:\mathcal{Z}^2\to\mathcal{B}^\Psi$ is compact. Clearly the operator $D^nM_uC_\varphi:\mathcal{Z}^2\to\mathcal{B}^\Psi$ is bounded. Then $L_k<\infty$ for each $k\in\{0,1,\dots,n+1\}$. Consider a sequence $\{\varphi(z_i)\}_{i\in\mathbb{N}}$ in $\mathbb{D}$ such that $|\varphi(z_i)|\to1$ as $i\to\infty$. If such a sequence does not exist, then the last two conditions in (b) obviously hold. Without loss of generality, we may suppose that $|\varphi(z_i)|>1/2$ for all $i\in\mathbb{N}$. For each fixed $k\in\{1,2,\dots,n+1\}$, by using this sequence we define the function sequence $g_{i,k}(z)=g_{\varphi(z_i),k}(z)$, $i\in\mathbb{N}$. Then from Lemma 2.3 (b) we see that $\sup_{i\in\mathbb{N}}\|g_{i,k}\|_{\mathcal{Z}^2}\le C$ and $g_{i,k}\to0$ uniformly on every compact subset of $\mathbb{D}$ as $i\to\infty$, moreover

$$g_{i,k}^{(k)}(\varphi(z_i))=\frac{\overline{\varphi(z_i)}^k}{(1-|\varphi(z_i)|^2)^k}\ \text{ and }\ g_{i,k}^{(j)}(\varphi(z_i))=0 \tag{53}$$

for each $j\in\{0,1,\dots,n+1\}\setminus\{k\}$. From Lemma 4.1 and (53), for each fixed $k\in\{1,2,\dots,n+1\}$ we have

$$\lim_{i\to\infty}\frac{\mu_\Psi(z_i)\Big|\sum_{j=k}^{n+1}C_{n+1}^ju^{(n+1-j)}(z_i)B_{j,k}(\varphi'(z_i),\dots,\varphi^{(j-k+1)}(z_i))\Big|}{(1-|\varphi(z_i)|^2)^k}=0. \tag{54}$$

Now consider another function sequence $q_i(z)=q_{\varphi(z_i)}(z)$. Then by Lemma 2.4 we have

$$q_i^{(k)}(\varphi(z_i))=c_k\frac{\overline{\varphi(z_i)}^k}{(1-|\varphi(z_i)|^2)^k}+d_k\frac{\overline{\varphi(z_i)}^k}{(1-|\varphi(z_i)|^2)^k}\log^{-1}\frac{e}{1-|\varphi(z_i)|^2}, \tag{55}$$

where $c_k > 0$ for each $k \geq 1$, $d_1 = 0$ and $d_k > 0$ for each $k \geq 2$. Moreover, $\sup_{i\in\mathbb{N}} \|q_i\|_{\mathcal{Z}^2} \leq C$, and $q_i \to 0$ uniformly on every compact subset of $\mathbb{D}$ as $i \to \infty$. From Lemma 4.1, we get

$$\lim_{i\to\infty} \|D^n M_u C_\varphi q_i\|_{\mathcal{B}^\Psi} = 0. \tag{56}$$

By (55) and the triangle inequality, we have

$$\begin{aligned}
&\mu_\Psi(z_i)\Big|\sum_{j=0}^{n+1} C_{n+1}^j u^{(n+1-j)}(z_i) B_{j,0}\big(\varphi'(z_i),\dots,\varphi^{(j+1)}(z_i)\big)\Big|\Big(\log\frac{e}{1-|\varphi(z_i)|^2} + \log^{-1}\frac{e}{1-|\varphi(z_i)|^2}\Big)\\
&\leq \|D^n M_u C_\varphi q_i\|_{\mathcal{B}^\Psi} + \sum_{k=1}^{n+1} \frac{c_k \mu_\Psi(z_i)|\varphi(z_i)|^k \Big|\sum\limits_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z_i) B_{j,k}(\varphi'(z_i),\dots,\varphi^{(j-k+1)}(z_i))\Big|}{(1-|\varphi(z_i)|^2)^k}\\
&+ \sum_{k=1}^{n+1} \frac{d_k \mu_\Psi(z_i)|\varphi(z_i)|^k \Big|\sum\limits_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z_i) B_{j,k}(\varphi'(z_i),\dots,\varphi^{(j-k+1)}(z_i))\Big|}{(1-|\varphi(z_i)|^2)^k} \log^{-1}\frac{e}{1-|\varphi(z_i)|^2}.
\end{aligned} \tag{57}$$

Therefore, taking the limit in (57) as $i \to \infty$, from (54), (56) and the fact

$$\log^{-1}\frac{e}{1-|\varphi(z_i)|^2} \to 0 \ \text{ as } \ i \to \infty,$$

we get

$$\lim_{i\to\infty} \mu_\Psi(z_i)\Big|\sum_{j=0}^{n+1} C_{n+1}^j u^{(n+1-j)}(z_i) B_{j,0}\big(\varphi'(z),\dots,\varphi^{(j+1)}(z_i)\big)\Big| \log\frac{e}{1-|\varphi(z_i)|^2} = 0.$$

$(b) \Rightarrow (a)$. We first check that $D^n M_u C_\varphi : \mathcal{Z}^2 \to \mathcal{B}^\Psi$ is bounded. We observe that the conditions in (b) imply that for every $\varepsilon > 0$, there is an $\eta \in (0,1)$, such that for any $z \in K = \{z \in \mathbb{D} : |\varphi(z)| > \eta\}$

$$\mu_\Psi(z)\Big|\sum_{j=0}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,0}\big(\varphi'(z),\dots,\varphi^{(j+1)}(z)\big)\Big| \log\frac{e}{1-|\varphi(z)|^2} < \varepsilon \tag{58}$$

and

$$\frac{\mu_\Psi(z)\Big|\sum\limits_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}(\varphi'(z),\dots,\varphi^{(j-k+1)}(z))\Big|}{(1-|\varphi(z)|^2)^k} < \varepsilon \tag{59}$$

for each $k \in \{1,2,\dots,n+1\}$. From the fact $L_0 < \infty$ and (58), we see

$$R_0 \leq \varepsilon + L_0 \log\frac{e}{1-\eta^2}. \tag{60}$$

From (59) and the fact $L_k < \infty$ for each $k \in \{1,2,\dots,n+1\}$, we see

$$R_k \leq \varepsilon + \frac{L_k}{(1-\eta^2)^k}. \tag{61}$$

Then from (60), (61) and Theorem 3.3, it follows that $D^n M_u C_\varphi : \mathcal{Z}^2 \to \mathcal{B}^\Psi$ is bounded.

In order to prove that $D^n M_u C_\varphi : \mathcal{Z}^2 \to \mathcal{B}^\Psi$ is compact, by Lemma 4.1 we just need to prove that, if $\{f_i\}$ is a sequence in $\mathcal{Z}^2$ such that $\sup_{i\in\mathbb{N}} \|f_i\|_{\mathcal{Z}^2} \leq M$ and $f_i \to 0$ uniformly on any compact subset of $\mathbb{D}$ as $i \to \infty$, then $\lim_{i\to\infty} \|D^n M_u C_\varphi f_i\|_{\mathcal{B}^\Psi} = 0$. For such chosen ε and η, by using (58), (59), Lemma 2.1 and Lemma 2.2, we have

$$
\begin{aligned}
&\sup_{z\in\mathbb{D}} \mu_\Psi(z)\big|(D^n M_u C_\varphi f_i)'(z)\big| \\
&= \sup_{z\in\mathbb{D}} \mu_\Psi(z)\Big|\sum_{k=0}^{n+1} f_i^{(k)}(\varphi(z)) \sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z),\ldots,\varphi^{(j-k+1)}(z)\big)\Big| \\
&\le \sup_{z\in\mathbb{D}} \mu_\Psi(z)\sum_{k=0}^{n+1} \big|f_i^{(k)}(\varphi(z))\big|\Big|\sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z),\ldots,\varphi^{(j-k+1)}(z)\big)\Big| \\
&\le \big(\sup_{z\in K} + \sup_{z\in\mathbb{D}\setminus K}\big)\mu_\Psi(z)\Big|\sum_{j=0}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,0}\big(\varphi'(z),\ldots,\varphi^{(j+1)}(z)\big)\Big|\big|f_i(\varphi(z))\big| \\
&\quad + \big(\sup_{z\in K} + \sup_{z\in\mathbb{D}\setminus K}\big)\mu_\Psi(z)\sum_{k=1}^{n+1}\big|f_i^{(k)}(\varphi(z))\big|\Big|\sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z),\ldots,\varphi^{(j-k+1)}(z)\big)\Big| \\
&\le \sum_{k=0}^{n+1} L_k \sup_{|z|\le\eta}\big|f_i^{(k)}(z)\big| + C\varepsilon. \qquad (62)
\end{aligned}
$$

From (62) and the fact $f_i \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $i \to \infty$ implies that for each $k \in \mathbb{N}$, $f_i^{(k)} \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $i \to \infty$, we get

$$\lim_{i\to\infty} \sup_{z\in\mathbb{D}} \mu_\Psi(z)\big|(D^n M_u C_\varphi f_i)'(z)\big| = 0. \qquad (63)$$

It is clear that

$$\lim_{i\to\infty} \big|(D^n M_u C_\varphi f_i)(0)\big| = 0. \qquad (64)$$

From (63) and (64) we obtain

$$\lim_{i\to\infty} \|D^n M_u C_\varphi f_i\|_{\mathcal{B}^\Psi} = 0.$$

Hence this shows that the operator $D^n M_u C_\varphi : \mathcal{Z}^2 \to \mathcal{B}^\Psi$ is compact. □

Theorem 4.4. *Let φ be an analytic self-map of $\mathbb{D}$, $u \in H(\mathbb{D})$ and $\alpha > 2$. Then the following statements are equivalent.*

(a) The operator $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is compact.

(b) The functions u and φ are such that $L_k < \infty$ and for each $k \in \{0, 1, \ldots, n+1\}$

$$\lim_{|\varphi(z)|\to1} \frac{\mu_\Psi(z)\Big|\sum\limits_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z),\ldots,\varphi^{(j-k+1)}(z)\big)\Big|}{(1-|\varphi(z)|^2)^{\alpha+k-2}} = 0.$$

Proof. Similarly to the proofs of Theorems 4.1-4.3, this result can be proved. □

Remark 4.1. *By using the similar methods and techniques, the compactness of the operators $D^n C_\varphi M_u$, $C_\varphi D^n M_u$, $M_u D^n C_\varphi$, $M_u C_\varphi D^n$ and $C_\varphi M_u D^n$ from weighted Zygmund spaces to Bloch-Orlicz spaces can be characterized, so we omit.*

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Union soft p-ideals and union soft sub-implicative ideals in BCI-algebras

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Abstract. The aim of this article is to lay a foundation for providing a soft algebraic tool in considering many problems that contain uncertainties. In order to provide these soft algebraic structures, the notion of union soft p-ideals(sub-implicative ideals) are introduced, and related properties are investigated. Conditions for a union soft ideal to be a union soft p-ideal(sub-implicative ideal) are established. Characterizations of a union soft p-ideal(sub-implicative ideal) are considered, and a new union soft p-ideal(sub-implicative ideal) from an old one is constructed.

1. Introduction

The real world is inherently uncertain, imprecise and vague. Various problems in system identification involve characteristics which are essentially non-probabilistic in nature [16]. In response to this situation Zadeh [17] introduced *fuzzy set theory* as an alternative to probability theory. Uncertainty is an attribute of information. In order to suggest a more general framework, the approach to uncertainty is outlined by Zadeh [18]. To solve complicated problem in economics, engineering, and environment, we can't successfully use classical methods because of various uncertainties typical for those problems. There are three theories: theory of probability, theory of fuzzy sets, and the interval mathematics which we can consider as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties. Uncertainties can't be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as probability theory, theory of (intuitionistic) fuzzy sets, theory of vague sets, theory of interval mathematics, and theory of rough sets. However, all of these theories have their own difficulties which are pointed out in [14]. Maji et al. [13] and Molodtsov [14] suggested that one reason for these difficulties may be due to the inadequacy of the parametrization tool of the theory. To overcome these difficulties, Molodtsov [14] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. Worldwide, there has been a rapid growth in interest in soft set theory and its applications in recent years. Evidence of this can be found in the increasing number of high-quality articles on soft sets and related topics that have been published in a variety of international journals, symposia, workshops, and international conferences in recent years. Maji et al. [13] described the application of soft set theory to a decision making problem. Maji et

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al. [12] also studied several operations on the theory of soft sets. Aktaş and Çağman [2] studied the basic concepts of soft set theory, and compared soft sets to fuzzy and rough sets, providing examples to clarify their differences. They also discussed the notion of soft groups. Jun [9] discussed the union soft sets with applications in BCK/BCI-algebras. We refer the reader to the papers [1, 3, 6, 8, 10] for further information regarding algebraic structures/properties of soft set theory.

In this paper, we discuss applications of the union soft sets in p-ideals of BCI-algebras. We introduce the notion of union soft p-ideals, and investigated related properties. We provide conditions for a union soft ideal to be a union soft p-ideal, and establish characterizations of a union soft p-ideal. We construct a new union soft p-ideal from an old one.

Secondly, we define the notion of union soft sub-implicative ideals, and investigated related properties. We provide conditions for a union soft ideal to be a union soft sub-implicative ideal, and study characterizations of a union soft sub-implicative ideal. We find a new union soft sub-implicative ideal from an old one.

2. Preliminaries

We review some definitions and properties that will be useful in our results.

By a BCI-*algebra* we mean an algebra $(X, *, 0)$ of type (2,0) satisfying the following conditions:

(a1) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,
(a2) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,
(a3) $(\forall x \in X)\ (x * x = 0)$,
(a4) $(\forall x, y \in X)\ (x * y = 0,\ y * x = 0\ \Rightarrow\ x = y)$.

If a BCI-algebra X satisfies the following identity:

(a5) $(\forall x \in X)(0 * x = 0)$,

then X is called a BCK-algebra. A BCI-algebra X is said to be p-*semisimple* if $0 * (0 * x) = x$ for all $x \in X$. A BCI-algebra X is said to be *implicative* if $(x * (x * y)) * (y * x) = y * (y * x)$ for all $x, y \in X$.

In any BCI-algebra X one can define a partial order "$\leq$" by putting $x \leq y$ if and only if $x * y = 0$.

A BCI-algebra X has the following properties:

(b1) $(\forall x \in X)\ (x * 0 = x)$,
(b2) $(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y)$,
(b3) $(\forall x, y \in X)\ (0 * (x * y) = (0 * x) * (0 * y))$,
(b4) $(\forall x, y \in X)\ (x * (x * (x * y)) = x * y)$.
(b5) $(\forall x, y, z \in X)\ (x \leq y\ \Rightarrow\ x * z \leq y * z,\ z * y \leq z * x)$,
(b6) $(\forall x, y, z \in X)\ ((x * z) * (y * z) \leq x * y)$,
(b7) $(\forall x, y, z \in X)\ (0 * (0 * ((x * z) * (y * z))) = (0 * y) * (0 * x))$,

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(b8) $(\forall x, y \in X)$ $(0 * (0 * (x * y)) = (0 * y) * (0 * x))$.

A non-empty subset S of a BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ whenever $x, y \in S$. A non-empty subset A of a BCI-algebra X is called an *ideal* of X if it satisfies:

(c1) $0 \in A$,
(c2) $(\forall x \in X)$ $(\forall y \in A)$ $(x * y \in A \Rightarrow x \in A)$.

Note that every ideal A of a BCI-algebra X satisfies:

$$(\forall x \in X)\,(\forall y \in A)\,(x \leq y \Rightarrow x \in A).$$

A non-empty subset A of a BCI-algebra X is called a *p-ideal* ([15]) of X if it satisfies (c1) and

(c3) $(\forall x, y, z \in X)((x * z) * (y * z) \in A$ and $y \in A \Rightarrow x \in A)$.

Note that any p-ideal is an ideal, but the converse is not true in general.

Theorem 2.1. ([15]) *An ideal I of a BCI-algebra X is a p-ideal if and only if $0 * (0 * x) \in I$ implies $x \in I$ for any $x \in X$.*

For any elements x and y of a BCI-algebra X, $x^n * y$ denotes $x * (x * \cdots * (x * (x * y \cdots)$ in which x occurs n times. A non-empty subset A of a BCI-algebra X is called a *sub-implicative ideal* ([11]) of X if it satisfies (c1) and

(c4) $(\forall x, y, z \in X)(((x^2 * y) * (y * x)) * z \in A$ and $z \in A \Rightarrow y^2 * x \in A)$.

Note that any sub-implicative ideal is an ideal, but the converse is not true in general.

Theorem 2.2. ([11]) *An ideal I of a BCI-algebra X is a sub-implicative ideal if and only if $(x^2 * y) * (y * x) \in I$ implies $y^2 * x \in I$ for any $x, y \in X$.*

We refer the reader to the book [7] for further information regarding BCI-algebras. A soft set theory is introduced by Molodtsov [14].

In what follows, let U be an initial universe set and E be a set of parameters. We say that the pair (U, E) is a *soft universe.* Let $\mathscr{P}(U)$ denotes the power set of U and $A, B, C, \cdots \subseteq E$.

Definition 2.3. ([14]) A soft set $\mathscr{F}_A$ over U is defined to be the set of ordered pairs

$$\mathscr{F}_A := \{(x, f_A(x)) : x \in E,\ f_A(x) \in \mathscr{P}(U)\},$$

where $f_A : E \to \mathscr{P}(U)$ such that $f_A(x) = \emptyset$ if $x \notin A$.

The function f_A is called the approximate function of the soft set $\mathscr{F}_A$. The subscript A in the notation f_A indicates that f_A is the approximate function of $\mathscr{F}_A$.

In what follows, denote by $S(U)$ the set of all soft sets over U.

Definition 2.4. ([12]) For two soft sets $\mathscr{F}_A$ and $\mathscr{G}_B$ over a common universe U, we say that $\mathscr{F}_A$ is a *soft subset* of $\mathscr{G}_B$, denoted by $\mathscr{F}_A \tilde{\subset} \mathscr{G}_B$, if it satisfies:

(i) $A \subset B$,
(ii) For every $\epsilon \in A$, $\mathscr{F}(\epsilon)$ and $\mathscr{G}(\epsilon)$ are identical approximations.

Let $\mathscr{F}_A \in S(U)$ and let $\tau \subseteq U$. Then the *τ-exclusive set* of $\mathscr{F}_A$ is defined to be the set

$$e(\mathscr{F}_A;\tau) := \{x \in A \mid f_A(x) \subseteq \tau\}.$$

Obviously, we have the following properties:

(1) $e(\mathscr{F}_A;U) = A$,
(2) $f_A(x) = \cap\{\tau \subseteq U \mid x \in e(\mathscr{F}_A;\tau)\}$,
(3) $(\forall \tau_1, \tau_2 \subseteq U)\,(\tau_1 \subseteq \tau_2 \;\Rightarrow\; e(\mathscr{F}_A;\tau_1) \subseteq e(\mathscr{F}_A;\tau_2))$.

3. Union soft p-ideals

Definition 3.1. ([9]) Let $(U,E) = (U,X)$ where X is a BCI-algebra. Given a subalgebra A of E, we let $\mathscr{F}_A \in S(U)$. Then $\mathscr{F}_A$ is called a *union soft deal* over U (briefly, *U-soft ideal*) if the approximate function f_A of $\mathscr{F}_A$ satisfies:

$$(\forall x \in A)\,(f_A(0) \subseteq f_A(x)), \tag{3.1}$$

$$(\forall x, y \in A)\,(f_A(x) \subseteq f_A(x * y) \cup f_A(y)). \tag{3.2}$$

Definition 3.2. Let $(U,E) = (U,X)$ where X is a BCI-algebra. Given a subalgebra A of E, let $\mathscr{F}_A \in S(U)$. Then $\mathscr{F}_A$ is called a *union soft p-ideal* over U (briefly, *U-soft p-ideal*) if the approximate function f_A of $\mathscr{F}_A$ satisfies (3.1) and

$$(\forall x, y, z \in A)\,(f_A(x) \subseteq f_A((x * z) * (y * z)) \cup f_A(y)). \tag{3.3}$$

Example 3.3. Let $(U,E) = (U,X)$ where $X = \{0, 1, a, b, c\}$ is a BCI-algebra ([10]) with the following Cayley table:

$*$	0	1	a	b	c
0	0	0	a	b	c
1	1	0	a	b	c
a	a	a	0	c	b
b	b	b	c	0	a
c	c	c	b	a	0

Let τ_1, τ_2 and τ_3 be subsets of U such that $\tau_1 \subsetneq \tau_2 \subsetneq \tau_3$. Define a soft set $\mathscr{F}_E$ over U as follows:

$$\mathscr{F}_E = \{(0,\tau_1), (1,\tau_2), (a,\tau_3), (b,\tau_3), (c,\tau_2)\}.$$

Routine calculations show that $\mathscr{F}_E$ is a U-soft p-ideal over U.

Theorem 3.4. *Let $(U,E) = (U,X)$ where X is a BCI-algebra. Then every U-soft p-ideal is a U-soft ideal.*

Proof. Let $\mathscr{F}_A$ be a U-soft p-ideal over U where A is a subalgebra of E. Taking $z := 0$ in (3.3) and using (b1) we obtain

$$\begin{aligned} f_A(x) \subseteq & f_A((x * 0) * (y * 0)) \cup f_A(y) \\ = & f_A(x * y)) \cup f_A(y) \end{aligned}$$

for all $x, y \in A$. Therefore $\mathscr{F}_A$ is a U-soft ideal over U. □

The following example shows that the converse of Theorem 3.4 is not true.

Example 3.5. Let $(U, E) = (U, X)$ where $X = \{0, 1, 2, 3, 4\}$ is a BCI-algebra ([9]) with the following Cayley table:

$*$	0	1	2	a	b
0	0	0	0	a	b
1	1	0	1	b	a
2	2	2	0	a	a
a	a	a	a	0	0
b	b	a	b	1	0

Let $\tau_1, \tau_2, \tau_3, \tau_4$ and τ_5 be subsets of U such that $\tau_1 \subsetneq \tau_3 \subsetneq \tau_4 \subsetneq \tau_5$ and $\tau_1 \subsetneq \tau_2 \subsetneq \tau_5$. Define a soft set $\mathscr{F}_E$ over U as follows:

$$\mathscr{F}_E = \{(0, \tau_1), (1, \tau_2), (2, \tau_3), (a, \tau_4), (b, \tau_5)\}.$$

Routine calculations show that $\mathscr{F}_E$ is a U-soft ideal over U. But it is not a U-soft p-ideal over U, since

$$f_E(b) = \tau_5 \nsubseteq \tau_4 = \tau_1 \cup \tau_4 = f_E((b * b) * (a * b)) \cup f_E(a).$$

We provide some conditions for a U-soft ideal to be a U-soft p-ideal over U.

Theorem 3.6. *Let $(U, E) = (U, X)$ where X is a BCI-algebra. For a subalgebra A of E, let $\mathscr{F}_A \in S(U)$. Then the following are equivalent:*

(1) *$\mathscr{F}_A$ is a U-soft p-ideal over U,*
(2) *$\mathscr{F}_A$ is a U-soft ideal over U and its approximate function f_A satisfies*

$$(\forall x, y, z \in A)\,(f_A(x * y) \subseteq f_A((x * z) * (y * z))). \tag{3.4}$$

Proof. Assume that $\mathscr{F}_A$ is a U-soft p-ideal over U. By Theorem 3.4, $\mathscr{F}_A$ is a U-soft deal over U. Using (a1) and (b2), we have $0 = ((x * z) * (x * y)) * (y * z) = ((x * z) * (y * z)) * (x * y)$ for any $x, y, z \in A$. Hence $((x * y) * (x * y)) * [((x * z) * (y * z)) * (x * y)] = 0 * 0 = 0$. It follows from (3.3) and (3.1) that

$$\begin{aligned} f_A(x * y) \subseteq & f_A((x * y) * (x * y)) * [((x * z) * (y * z)) * (x * y)]) \cup f_A((x * z) * (y * z)) \\ = & f_A(0) \cup f_A((x * z) * (y * z)) \\ = & f_A((x * z) * (y * z)). \end{aligned}$$

Hence (3.4) holds.

Conversely, suppose that $\mathscr{F}_A$ is a U-soft ideal over U satisfying (3.4). Using (3.2) and (3.4), we have $f_A(x) \subseteq f_A(x*y) \cup f_A(y) \subseteq f_A((x*z)*(y*z)) \cup f_A(y)$ for any $x, y, z \in A$. Hence $\mathscr{F}_A$ is a U-soft p-ideal over U. This completes the proof. □

Lemma 3.7. *Let $(U, E) = (U, X)$ where X is a BCI-algebra. For a subalgebra A of E, let $\mathscr{F}_A \in S(U)$. If $\mathscr{F}_A$ is a U-soft ideal over U, then the approximate function f_A of $\mathscr{F}_A$ satisfies the following condition:*

$$(\forall x \in A)(f_A(0*(0*x)) \subseteq f_A(x)).$$

Proof. Assume that $\mathscr{F}_A$ is a U-soft ideal over U. Note that $0 = (0*x)*(0*x) = (0*(0*x))*x$. Using (3.2) and (3.1), we have

$$\begin{aligned} f_A(0*(0*x)) \subseteq & f_A((0*(0*x)*x)) \cup f_A(x) \\ = & f_A(0) \cup f_A(x) \\ = & f_A(x) \end{aligned}$$

for any $x \in A$. This completes the proof. □

Theorem 3.8. *Let $(U, E) = (U, X)$ where X is a BCI-algebra. For a subalgebra A of E, let $\mathscr{F}_A \in S(U)$. Then the following are equivalent:*

(i) *$\mathscr{F}_A$ is a U-soft p-ideal over U,*
(ii) *$\mathscr{F}_A$ is a U-soft ideal over U and its approximate function f_A satisfies*

$$(\forall x \in A)\left(f_A(x) \subseteq f_A(0*(0*x))\right). \tag{3.5}$$

Proof. Assume that $\mathscr{F}_A$ is a U-soft p-ideal over U. By Theorem 3.4, $\mathscr{F}_A$ is a U-soft deal over U. It follows from (3.3) and (3.1) that

$$\begin{aligned} f_A(x) \subseteq & f_A((x*x)*(0*x)) \cup f_A(0) \\ = & f_A(0*(0*x)) \end{aligned}$$

for any $x \in A$. Hence (3.5) holds.

Conversely, suppose that $\mathscr{F}_A$ is a U-soft ideal over U satisfying (3.5). By Lemma 3.7, we obtain $f_A(0*(0*((x*z)*(y*z)))) \subseteq f_A((x*z)*(y*z))$. It follows from (b7) and (b8) that $0*(0*(x*y)) = (0*y)*(0*x) = 0*(0*(x*z)*(y*z)))$. Using (3.5), we have $f_A(x*y) \subseteq f_A(0*(0*(x*y))) \subseteq f_A((x*z)*(y*z))$. By Theorem 3.6, $\mathscr{F}_A$ is a U-soft p-ideal over U. □

Lemma 3.9. ([9]) *Let $(U, E) = (U, X)$ where X is a BCI-algebra, Given a subalgebra A of E, let $\mathscr{F}_A \in S(U)$. Then the following are equivalent:*

(i) *$\mathscr{F}_A$ is an U-soft ideal over U,*
(ii) *The nonempty τ-exclusive set of $\mathscr{F}_A$ is a ideal of A for any $\tau \subseteq U$.*

Theorem 3.10. *Let $(U, E) = (U, X)$ where X is a BCI-algebra, Given a subalgebra A of E. let $\mathscr{F}_A \in S(U)$. Then the following are equivalent:*

(i) *$\mathscr{F}_A$ is a U-soft p-ideal over U,*
(ii) *The nonempty τ-exclusive set of $\mathscr{F}_A$ is a p-ideal of A for any $\tau \subseteq U$.*

Proof. Assume that $\mathscr{F}_A$ is a U-soft p-ideal over U. Then $\mathscr{F}_A$ is a U-soft ideal over U by Theorem 3.4. Hence $e(\mathscr{F}_A; \tau)$ is an ideal of A for all $\tau \subseteq U$ by Lemma 3.9. Let $\tau \subseteq U$ and let $x, y, z \in A$ be such that $(x * z) * (y * z) \in e(\mathscr{F}_A; \tau)$ and $y \in e(\mathscr{F}_A; \tau)$. Then $f_A((x * z) * (y * z)) \subseteq \tau$, $f_A(y) \subseteq \tau$, and so

$$f_A(x) \subseteq f_A((x * z) * (y * z)) \cup f_A(y) \subseteq \tau.$$

Hence $x \in e(\mathscr{F}_A; \tau)$. Thus $e(\mathscr{F}_A; \tau)$ is a p-ideal of A.

Conversely, suppose that the nonempty τ-exclusive set of $\mathscr{F}_A$ is a p-ideal of A for any $\tau \subseteq U$. Then $e(\mathscr{F}_A; \tau)$ is an ideal of A for all $\tau \subseteq U$. Hence $\mathscr{F}_A$ is a U-soft ideal over U by Lemma 3.9. Let $x \in A$ be such that $f_A(0 * (0 * x)) = \tau$. Then $0 * (0 * x) \in e(\mathscr{F}_A; \tau)$, and so $x \in e(\mathscr{F}_A; \tau)$ by Theorem 2.1. Hence $f_A(x) \subseteq f_A(0 * (0 * x))$. It follows from Theorem 3.8 that $\mathscr{F}_A$ is a U-soft p-ideal over U. □

The p-ideals $e(\mathscr{F}_A; \tau)$ in Theorem 3.10 are called the *exclusive p-ideals* of $\mathscr{F}_A$.

Theorem 3.11. *Let $(U, E) = (U, X)$ and $\mathscr{F}_A \in S(U)$ where X is a BCI-algebra and A is a subalgebra of E. For a subset τ of U, define a soft set $\mathscr{F}^*_A$ over U by*

$$f^*_A : E \to \mathscr{P}(U),\ x \mapsto \begin{cases} f_A(x) & \text{if } x \in e(\mathscr{F}_A; \tau), \\ U & \text{otherwise.} \end{cases}$$

If $\mathscr{F}_A$ is a U-soft p-ideal over U, then so is $\mathscr{F}^*_A$.

Proof. If $\mathscr{F}_A$ is a U-soft p-ideal over U, then $e(\mathscr{F}_A; \tau)$ is a p-ideal of A for any $\tau \subseteq U$. Hence $0 \in e(\mathscr{F}_A; \tau)$, and so $f^*_A(0) = f_A(0) \subseteq f_A(x) \subseteq f^*_A(x)$ for all $x \in A$. Let $x, y, z \in A$. If $(x * z) * (y * z) \in e(\mathscr{F}_A; \tau)$ and $y \in e(\mathscr{F}_A; \tau)$, then $x \in e(\mathscr{F}_A; \tau)$ and so

$$\begin{aligned} f^*_A(x) =& f_A(x) \\ \subseteq& f_A((x * z) * (y * z)) \cup f_A(y) \\ =& f^*_A((x * z) * (y * z)) \cup f^*_A(y). \end{aligned}$$

If $(x * y) * (y * z) \notin e(\mathscr{F}_A; \tau)$ or $y \notin e(\mathscr{F}_A; \tau)$, then $f^*_A((x * z) * (y * z)) = U$ or $f^*_A(y) = U$. Hence

$$f^*_A(x) \subseteq U = f^*_A((x * z) * (y * z)) \cup f^*_A(y).$$

This shows that $\mathscr{F}^*_A$ is a U-soft p-ideal over U. □

Theorem 3.12. *Let $(U, E) = (U, X)$ where X is a BCI-algebra. Then any p-ideal of E can be realized as an exclusive p-ideal of some U-soft p-ideal over U.*

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Proof. Let A be a p-ideal of E. For any subset $\tau \subsetneq U$, let $\mathscr{F}_A$ be a soft set over U defined by

$$f_A : E \to \mathscr{P}(U),\ x \mapsto \begin{cases} \tau & \text{if } x \in A, \\ U & \text{if } x \notin A. \end{cases}$$

Obviously, $f_A(0) \subseteq f_A(x)$ for all $x \in E$. For any $x, y, z \in E$, if $(x * z) * (y * z) \in A$ and $y \in A$ then $x \in A$. Hence

$$f_A((x * z) * (y * z)) \cup f_A(y) = \tau = f_A(x).$$

If $(x * z) * (y * z) \notin A$ or $y \notin A$ then $f_A((x * z) * (y * z)) = U$ or $f_A(y) = U$. It follows from (3.3) that

$$f_A(x) \subseteq U = f_A((x * z) * (y * z)) \cup f_A(y).$$

Therefore $\mathscr{F}_A$ is a U-soft p-ideal over U, and clearly $e(\mathscr{F}_A; \tau) = A$. This completes the proof. □

Example 3.13. Let $(U, E) = (U, X)$ where X is a BCI-algebra.
(1) $B(X) := \{x \in X | 0 * x = 0\}$. Then $B(X)$ is a p-ideal ([15]) of X. For any subset $\tau \subsetneq U$, let $\mathscr{F}_{B(X)}$ be a soft set over U defined by

$$f_{B(X)} : E \to \mathscr{P}(U),\ x \mapsto \begin{cases} \tau & \text{if } x \in B(X), \\ U & \text{if } x \notin B(X). \end{cases}$$

Then $\mathscr{F}_{B(X)}$ is a U-soft p-ideal over U.
(2) $T_n(X) := \{x \in X | 0 * x^n = 0\}$, where $0 * x^n = (\cdots (0 * x) * \cdots) * x$ in which x appears n-times. Then $T_n(X)$ is a p-ideal ([15]) of X. For any subset $\tau \subsetneq U$, let $\mathscr{G}_{T_n(X)}$ be a soft set over U defined by

$$g_{T_n(X)} : E \to \mathscr{P}(U),\ x \mapsto \begin{cases} \tau & \text{if } x \in T_n(X), \\ U & \text{if } x \notin T_n(X). \end{cases}$$

Then $\mathscr{G}_{T_n(X)}$ is a U-soft p-ideal over U.

Theorem 3.14. [Extension property] *Let $(U, E) = (U, X)$ where X is a p-semisimple BCI-algebra. Given subalgebras A and B of E, let $\mathscr{F}_A, \mathscr{F}_B \in S(U)$ such that*

(i) $\mathscr{F}_A \tilde{\subset} \mathscr{F}_B$,
(ii) $\mathscr{F}_B$ *a U-soft ideal over U.*

If $\mathscr{F}_A$ is a U-soft p-ideal over U, then so is $\mathscr{F}_B$.

Proof. Let $\tau \subseteq U$ be such that $e(\mathscr{F}_B; \tau) \neq \emptyset$. It follows from the condition (ii) and Lemma 3.9 that $e(\mathscr{F}_B; \tau)$ is an ideal. Assume that $\mathscr{F}_A$ is a U-soft p-ideal over U. Then $e(\mathscr{F}_A; \tau)$ is a p-ideal for every $\tau \subseteq U$ by Theorem 3.10. Let $x \in E$ and $\tau \subseteq U$ be such that $0 * (0 * x) \in e(\mathscr{F}_B; \tau)$. Since X is a p-semisimple BCI-algebra, $0 * (0 * x) = x$. Hence $x \in e(\mathscr{F}_B; \tau)$. Thus $e(\mathscr{F}_B; \tau)$ is a p-ideal by Theorem 2.1. By Theorem 3.10, $\mathscr{F}_B$ is a U-soft p-ideal over U. □

4. Union soft sub-implicative ideals

Definition 4.1. Let $(U, E) = (U, X)$ where X is a BCI-algebra. Given a subalgebra A of E, let $\mathscr{F}_A \in S(U)$. Then $\mathscr{F}_A$ is called a *union soft sub-implicative ideal* over U (briefly, *U-soft sub-implicative ideal*) if the approximate function f_A of $\mathscr{F}_A$ satisfies (3.1) and

$$(\forall x, y, z \in A)\left(f_A(y^2 * x) \subseteq f_A(((x^2 * y) * (y * x)) * z) \cup f_A(z)\right). \tag{4.1}$$

Example 4.2. Let $(U, E) = (U, X)$ where $X = \{0, 1, 2\}$ is a BCI-algebra ([11]) with the following Cayley table:

$*$	0	1	2
0	0	0	2
1	1	0	2
2	2	2	0

Let τ_1 and τ_2 be subsets of U such that $\tau_1 \subsetneq \tau_2$. Define a soft set $\mathscr{F}_E$ over U as follows:

$$\mathscr{F}_E = \{(0, \tau_1), (1, \tau_1), (2, \tau_2)\}.$$

Routine calculations show that $\mathscr{F}_E$ is a U-soft sub-implicative ideal over U.

Theorem 4.3. *Let $(U, E) = (U, X)$ where X is a BCI-algebra. Then every U-soft sub-implicative ideal is a U-soft ideal.*

Proof. Let $\mathscr{F}_A$ be a U-soft sub-implicative ideal over U where A is a subalgebra of E. Taking $y := x$ in (4.1) we obtain

$$\begin{aligned} f_A(x) &= f_A(x^2 * x) \\ &\subseteq f_A(((x^2 * x) * (x * x)) * z) \cup f_A(z) \\ &= f_A(x * z)) \cup f_A(z) \end{aligned}$$

for all $x, z \in A$. Therefore $\mathscr{F}_A$ is a U-soft ideal over U. □

The following example shows that the converse of Theorem 4.3 is not true.

Example 4.4. Let $(U, E) = (U, X)$ where $X = \{0, 1, 2, 3, 4\}$ is a BCI-algebra ([11]) with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	c
a	a	0	0	c
b	b	b	0	c
c	c	c	c	0

Let τ_1 and τ_2 be subsets of U such that $\tau_1 \subsetneq \tau_2$. Define a soft set $\mathscr{F}_E$ over U as follows:

$$\mathscr{F}_E = \{(0, \tau_1), (a, \tau_2), (b, \tau_2), (c, \tau_2)\}.$$

Routine calculations show that $\mathscr{F}_E$ is a U-soft ideal over U. But it is not a U-soft sub-implicative ideal over U, since

$$f_E(a^2 * b) = f_E(a) = \tau_2 \nsubseteq \tau_1 = f_E(((b^2 * a) * (a * b)) * 0) \cup f_E(0).$$

Proposition 4.5. *Let $(U, E) = (U, X)$ where X is a BCI-algebra. For a subalgebra A of E, let $\mathscr{F}_A \in S(U)$. If $\mathscr{F}_A$ is a U-soft sub-implicative ideal over U, then the approximate function f_A of $\mathscr{F}_A$ satisfies the following condition:*

$$(\forall x, y \in A) \left(f_A(y^2 * x) \subseteq f_A((x^2 * y) * (y * x)) \right). \tag{4.2}$$

.

Proof. Assume that $\mathscr{F}_A$ is a U-soft sub-implicative ideal over U. For any $x, y \in A$, we have

$$\begin{aligned} f_A(y^2 * x) \subseteq & f_A(((x^2 * y) * (y * x)) * 0) \cup f_A(0) \\ = & f_A((x^2 * y) * (y * x)). \end{aligned}$$

This completes the proof. □

We provide conditions for a U-soft BCI-ideal to be a U-soft sub-implicative ideal over U.

Theorem 4.6. *Let $(U, E) = (U, X)$ where X is a BCI-algebra. For a subalgebra A of E, let $\mathscr{F}_A \in S(U)$. If $\mathscr{F}_A$ is a U-soft ideal over U satisfying the condition (4.2), then $\mathscr{F}_A$ is a U-soft sub-implicative ideal over U.*

Proof. Assume that $\mathscr{F}_A$ is a U-soft ideal over U satisfying the condition (4.2). For any $x, y \in A$, we have

$$\begin{aligned} f_A(y^2 * x) \subseteq & f_A((x^2 * y) * (y * x)) \\ \subseteq & f_A((x^2 * y) * (y * x)) * z) \cup f_A(z) \end{aligned}$$

which proves the condition (4.1). This completes the proof. □

Corollary 4.7. *Let $(U, E) = (U, X)$ where X is an implicative BCI-algebra. Then every U-soft sub-implicative ideal is a U-soft ideal.*

Theorem 4.8. *Let $(U, E) = (U, X)$ where X is a p-semisimple BCI-algebra. For a subalgebra A of E, let $\mathscr{F}_A \in S(U)$. The notions of a U-soft ideal over U and a U-soft sub-implicative ideal over U coincide.*

Proof. Note that $x^2 * y = y$ for all $x, y \in X$, since X is a p-semsimple BCI-algebra. Assume that $\mathscr{F}_A$ is a U-soft ideal over U. For any $x, y, z \in A$, we have

$$\begin{aligned} f_A(y^2 * x) = & f_A(x) \\ \subseteq & f_A(x * z) \cup f_A(z) \\ = & f_A((y^2 * x) * z)) \cup f_A(z) \\ = & f_A(((x^2 * y) * (y * x)) * z) \cup f_A(z). \end{aligned}$$

Therefore $\mathscr{F}_A$ ia a U-soft sub-implicative ideal over U. □

Theorem 4.9. *Let $(U, E) = (U, X)$ where X is a BCI-algebra. Then every U-soft p-ideal is a U-soft sub-implicative ideal.*

Proof. Let $\mathscr{F}_A$ be a U-soft p-ideal over U, where A is a subalgebra of E. Then $\mathscr{F}_A$ is a U-soft ideal over U. Then $\mathscr{F}_A$ is a U-soft ideal over U by Theorem 3.4. Note that

$$\begin{aligned}(0^2 * (y^2 * x)) * ((x^2 * y) * (y * x)) &= 0 * ((x^2 * y) * (y * x)) * (0 * (y^2 * x)) \\ &= [(0 * (x^2 * y)) * (0 * (y * x))] * (0 * (y^2 * x)) \\ &= [((0 * x) * (0 * (x * y))) * (0 * (y * x))] * [(0 * y) * (0 * (y * x))] \\ &\leq ((0 * x) * (0 * (x * y))) * (0 * y) \\ &= ((0 * x) * (0 * y)) * (0 * (x * y)) \\ &= 0.\end{aligned}$$

For any $x, y \in A$, we have

$$\begin{aligned}f_A(y^2 * x) &\subseteq f_A(0^2 * (y^2 * x)) \\ &\subseteq f_A((0^2 * (y^2 * x)) * ((x^2 * y) * (y * x))) \cup f_A((x^2 * y) * (y * x)) \\ &\subseteq f_A(0) \cup ((x^2 * y) * (y * x)) \\ &= f_A((x^2 * y) * (y * x)).\end{aligned}$$

It follows from Theorem 4.6 that $\mathscr{F}_A$ is a U-soft sub-implicaticve ideal over U. □

The converse of Theorem 4.9 may not be true in general as seen in the following example.

Example 4.10. Let $(U, E) = (U, X)$ where $X = \{0, a, 1, 2, 3\}$ is a BCI-algebra with the following Cayley table:

$*$	0	a	1	2	3
0	0	0	3	2	1
a	a	0	3	2	1
1	1	1	0	3	2
2	2	2	1	0	3
3	3	3	2	1	0

Let τ_1, τ_2 and τ_3 be subsets of U such that $\tau_1 \subsetneq \tau_2 \subsetneq \tau_3$. Define a soft set $\mathscr{F}_E$ over U as follows:

$$\mathscr{F}_E = \{(0, \tau_1), (a, \tau_2), (1, \tau_3), (2, \tau_3), (3, \tau_3)\}.$$

Routine calculations show that $\mathscr{F}_E$ is a U-soft sub-implicative ideal over U. But it is not a U-soft p-ideal over U, since

$$f_E(a) = \tau_2 \nsubseteq \tau_1 = f_E((a * 1) * (0 * 1)) \cup f_E(0).$$

Theorem 4.11. *Let $(U, E) = (U, X)$ where X is a BCI-algebra, Given a subalgebra A of E. let $\mathscr{F}_A \in S(U)$. Then the following are equivalent:*

(i) *$\mathscr{F}_A$ is a U-soft sub-implicative ideal over U,*
(ii) *The nonempty τ-exclusive set of $\mathscr{F}_A$ is a sub-implicative ideal of A for any $\tau \subseteq U$.*

Proof. Assume that $\mathscr{F}_A$ is a U-soft sub-implicative ideal over U. Then $\mathscr{F}_A$ is a U-soft ideal over U by Theorem 4.3. Hence $e(\mathscr{F}_A;\tau)$ is an ideal of A for all $\tau \subseteq U$ by Lemma 3.9. Let $\tau \subseteq U$ and let $x, y, z \in A$ be such that $((x^2 * y) * (y * x)) * z \in e(\mathscr{F}_A;\tau)$ and $z \in e(\mathscr{F}_A;\tau)$. Then $f_A(((x^2 * y) * (y * x)) * z) \subseteq \tau$, $f_A(z) \subseteq \tau$, and so

$$f_A(y^2 * x) \subseteq f_A(((x^2 * y) * (y * x)) * z) \cup f_A(z) \subseteq \tau.$$

Hence $y^2 * x \in e(\mathscr{F}_A;\tau)$. Thus $e(\mathscr{F}_A;\tau)$ is a sub-implicative ideal of A.

Conversely, suppose that the nonempty τ-exclusive set of $\mathscr{F}_A$ is a sub-implicative ideal of A for any $\tau \subseteq U$. Then $e(\mathscr{F}_A;\tau)$ is an ideal of A for all $\tau \subseteq U$. Hence $\mathscr{F}_A$ is a U-soft ideal over U by Lemma 3.9. Let $x, y \in A$ be such that $f_A((x^2 * y) * (y * x)) = \tau$. Then $(x^2 * y) * (y * x) \in e(\mathscr{F}_A;\tau)$, and so $y^2 * x \in e(\mathscr{F}_A;\tau)$ by Theorem 2.2. Hence $f_A(y^2 * x) \subseteq f_A((x^2 * y) * (y * x))$. It follows from Theorem 4.6 that $\mathscr{F}_A$ is a U-soft sub-implicative ideal over U. $\square$

The sub-implicative ideals $e(\mathscr{F}_A;\tau)$ in Theorem 4.11 are called the *exclusive sub-implicative ideals* of $\mathscr{F}_A$.

Theorem 4.12. *Let $(U, E) = (U, X)$ and $\mathscr{F}_A \in S(U)$ where X is a BCI-algebra and A is a subalgebra of E. For a subset τ of U, define a soft set $\mathscr{F}_A^*$ over U by*

$$f_A^* : E \to \mathscr{P}(U),\ x \mapsto \begin{cases} f_A(x) & \text{if } x \in e(\mathscr{F}_A;\tau), \\ U & \text{otherwise.} \end{cases}$$

If $\mathscr{F}_A$ is a U-soft sub-implicative ideal over U, then so is $\mathscr{F}_A^*$.

Proof. If $\mathscr{F}_A$ is a U-soft sub-implicative ideal over U, then $e(\mathscr{F}_A;\tau)$ is a sub-implicative ideal of A for any $\tau \subseteq U$. Hence $0 \in e(\mathscr{F}_A;\tau)$, and so $f_A^*(0) = f_A(0) \subseteq f_A(x) \subseteq f_A^*(x)$ for all $x \in A$. Let $x, y, z \in A$. If $((x^2 * y) * (y * x) * z) \in e(\mathscr{F}_A;\tau)$ and $z \in e(\mathscr{F}_A;\tau)$, then $y^2 * x \in e(\mathscr{F}_A;\tau)$ and so

$$\begin{aligned} f_A^*(y^2 * x) =& f_A(y^2 * x) \\ \subseteq& f_A(((x^2 * y) * (y * x)) * z) \cup f_A(z) \\ =& f_A^*(((x^2 * y) * (y * x)) * z) \cup f_A^*(z). \end{aligned}$$

If $((x^2 * y) * (y * x)) * z \notin e(\mathscr{F}_A;\tau)$ or $z \notin e(\mathscr{F}_A;\tau)$, then $f_A^*((x^2 * y) * (y * x)) * z)) = U$ or $f_A^*(z) = U$. Hence

$$f_A^*(x) \subseteq U = f_A^*(((x^2 * y) * (y * x)) * z) \cup f_A^*(z).$$

This shows that $\mathscr{F}_A^*$ is a U-soft sub-implicative ideal over U. $\square$

Theorem 4.13. *Let $(U, E) = (U, X)$ where X is a BCI-algebra. Then any sub-implicative ideal of E can be realized as an exclusive sub-implicative ideal of some U-soft sub-implicative ideal over U.*

Proof. Let A be a sub-implicative ideal of E. For any subset $\tau \subsetneq U$, let $\mathscr{F}_A$ be a soft set over U defined by

$$f_A : E \to \mathscr{P}(U),\ x \mapsto \begin{cases} \tau & \text{if } x \in A, \\ U & \text{if } x \notin A. \end{cases}$$

Obviously, $f_A(0) \subseteq f_A(x)$ for all $x \in E$. For any $x, y, z \in E$, if $((x^2*y)*(y*x))*z \in A$ and $z \in A$, then $y^2*x \in A$. Hence $f_A(y^2*x) = \tau = f_A(((x^2*y)*(y*x))*z) \cup f_A(z)$. If $((x^2*y)*(y*x))*z \notin A$ or $z \notin A$, then $f_A(((x^2*y)*(y*x))*z) = U$ or $f_A(z) = U$. It follows from (4.1) that

$$f_A(y^2 * x) \subseteq U = f_A(((x^2 * y) * (y * x)) * z) \cup f_A(z).$$

Therefore $\mathscr{F}_A$ is a U-soft sub-implicative ideal over U, and clearly $e(\mathscr{F}_A; \tau) = A$. This completes the proof. $\square$

Example 4.14. Let $(U, E) = (U, X)$ where X is a BCI-algebra and let $B(X) := \{x \in X | 0 * x = 0\}$. Then $B(X)$ is a sub-implicative ideal ([11]) of X. For any subset $\tau \subsetneq U$, let $\mathscr{F}_{B(X)}$ be a soft set over U defined by

$$f_{B(X)} : E \to \mathscr{P}(U),\ x \mapsto \begin{cases} \tau & \text{if } x \in B(X), \\ U & \text{if } x \notin B(X). \end{cases}$$

Then it is easy to see that $\mathscr{F}_{B(X)}$ is a U-soft sub-implicative ideal over U.

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On interval-valued fuzzy rough approximation operators *

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Abstract: Rough approximation operators based on approximation spaces are a key concept of rough set theory. This paper investigates rough approximation operators in interval-valued fuzzy (for short, IVF) environment by using constructive and axiomatic approaches. Moreover, IVF pseudo-closure operators are considered.

Keywords: IVF set; IVF relation; IVF approximate space; IVF rough set; IVF rough approximation operators.

1 Introduction

Rough set theory was proposed by Pawlak [16] as a mathematical tool for data reasoning. It may be seen as an extension of classical set theory, has been proved to be an effective approach to deal with intelligent systems characterized by insufficient and incomplete information, and has been successfully applied to machine learning, intelligent systems, inductive reasoning, pattern recognition, mereology, image processing, signal analysis, knowledge discovery, decision analysis, expert systems and many other fields [17, 18, 19, 20]. The foundation of its object classification is an equivalence relation. The upper and lower approximation operations are two core notions of this theory. They can also be seen as the closure operator and the interior operator of the topology induced by an equivalence relation on the universe, respectively. In the real world, the equivalence

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relation is, however, too restrictive for many practical applications. To address this issue, many interesting and meaningful extensions of Pawlak's rough sets have been presented in the literature. Equivalence relations can be replaced by tolerance relations [23], similarity relations [24], binary relations [7, 27].

Various fuzzy generalizations of rough approximations have been proposed in the literature [1, 2, 6, 10, 11, 15, 21, 26, 29]. The most common fuzzy rough set is obtained by replacing the crisp binary relations with fuzzy relations on the universe and the crisp subsets with the fuzzy sets.

There are mainly two approaches to the development of rough set theory. One is the constructive approach in which rough approximation operators are constructed by means of relations, partitions, coverings, neighborhood systems and so on. The constructive approach is suitable for practical applications of rough sets. The other one is the axiomatic approach. In this approach, a set of axioms is used to characterize rough approximation operators that guarantee the existence of certain types of relations which produce the same operators. This approach is appropriate for studying algebra structures of rough sets. Under this point of view, rough set theory may be interpreted as an extension of set theory with two additional unary operators.

As a generalization of Zadeh's fuzzy set, interval-valued fuzzy (IVF, for short) sets were introduced by Gorzalczany [4] and Turksen [25], and they were applied to the fields of approximate inference, signal transmission and controller, etc. Mondal et al. [14] defined topology of IVF sets and studied their properties.

By integrating Pawlak rough set theory with IVF set theory, Sun et al. [22] introduced IVF rough sets based on an IVF approximation space, defined IVF information systems and discussed their attribute reduction. Gong et al. [5] studied the knowledge discovery in IVF information systems. Zhang et al. [30] discussed $(\mathcal{I},\mathcal{T})$-IVF rough sets based on an IVF approximation space on two universes of discourse.

The purpose of this paper is to investigate IVF rough approximation operators by using constructive and axiomatic approaches.

2 Preliminaries

Throughout this paper, " interval-valued fuzzy " denote briefly by " IVF ". U denotes a nonempty finite set called the universe of discourse. I denotes $[0,1]$ and $[I]$ denotes $\{[a,b] : a,b \in I \ and \ a \leq b\}$. $\mathscr{P}(U)$ denotes the family of all subsets of U. $F^{(i)}(U)$ denotes the family of all IVF sets in U. $\bar{a}$ denotes $[a,a]$ for each $a \in [0,1]$.

2.1 IVF sets

For any $[a_j,b_j] \in [I](\ j=1,2)$, we define

$$[a_1,b_1]=[a_2,b_2] \iff a_1=a_2, b_1=b_2;$$

$$[a_1,b_1] \leq [a_2,b_2] \iff a_1 \leq a_2, b_1 \leq b_2;$$

$$[a_1, b_1] < [a_2, b_2] \iff [a_1, b_1] \le [a_2, b_2] \text{ and } [a_1, b_1] \ne [a_2, b_2];$$

$$\bar{1} - [a_1, b_1] \text{ or } [a_1, b_1]^c = [1 - b_1, 1 - a_1].$$

Obviously, $([a, b]^c)^c = [a, b]$ for each $[a, b] \in [I]$.

Definition 2.1 ([4, 25]). *For each $\{[a_j, b_j] : j \in J\} \subseteq [I]$, we define*

$$\bigvee_{j\in J} [a_j, b_j] = [\bigvee_{j\in J} a_j, \bigvee_{j\in J} b_j] \text{ and } \bigwedge_{j\in J} [a_j, b_j] = [\bigwedge_{j\in J} a_j, \bigwedge_{j\in J} b_j],$$

where $\bigvee_{j\in J} a_j = sup\ \{a_j : j \in J\}$ and $\bigwedge_{j\in J} a_j = inf\ \{a_j : j \in J\}$.

Definition 2.2 ([4, 25]). *An IVF set A in U is defined by a mapping $A : U \to [I]$. Denote*

$$A(x) = [A^-(x), A^+(x)] \quad (x \in U).$$

Then $A^-(x)$ (resp. $A^+(x)$) is called the lower (resp. upper) degree to which x belongs to A. A^- (resp. A^+) is called the lower (resp. upper) IVF set of A.

The set of all IVF sets in U is denoted by $F^{(i)}(U)$.

Let $a, b \in I$. $\widetilde{[a, b]}$ represents the IVF set which satisfies $\widetilde{[a, b]}(x) = [a, b]$ for each $x \in U$. We denoted $\widetilde{[a, a]}$ by $\tilde{a}$.

We recall some basic operations on $F^{(i)}(U)$ as follows ([4, 25]): for any $A, B \in F^{(i)}(U)$ and $[a, b] \in [I]$,

(1) $A = B \iff A(x) = B(x)$ for each $x \in U$.
(2) $A \subseteq B \iff A(x) \le B(x)$ for each $x \in U$.
(3) $A = B^c \iff A(x) = B(x)^c$ for each $x \in U$.
(4) $(A \cap B)(x)$=$A(x) \wedge B(x)$ for each $x \in U$.
(5) $(A \cup B)(x)$=$A(x) \vee B(x)$ for each $x \in U$.

Moreover,

$$(\bigcup_{j\in J} A)(x) = \bigvee_{j\in J} A(x) \text{ and } (\bigcap_{j\in J} A)(x) = \bigwedge_{j\in J} A(x),$$

where $\{A_j : j \in J\} \subseteq F^{(i)}(U)$.

(6) $([a, b]A)(x) = [a, b] \wedge [A^-(x), A^+(x)]$ for each $x \in U$.

Obviously,

$$A = B \iff A^- = B^- \text{ and } A^+ = B^+ \ ; \ (\widetilde{[a, b]})^c = \widetilde{[a, b]^c} \quad ([a, b] \in [I]).$$

Definition 2.3 ([14]). *$A \in F^{(i)}(U)$ is called an IVF point in U, if there exist $[a, b] \in [I] - \{\bar{0}\}$ and $x \in U$ such that*

$$A(y) = \begin{cases} [a, b], & y = x, \\ \bar{0}, & y \ne x. \end{cases}$$

We denote A by $x_{[a,b]}$.

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If $[a, b] = \bar{1}$, then

$$x_{\bar{1}}(y) = \begin{cases} \bar{1}, & y = x, \\ \bar{0}, & y \neq x. \end{cases}$$

Remark 2.4. $A = \bigcup_{x \in U} (A(x)x_{\bar{1}})$.

2.2 Definition of IVF rough approximation operators

Recall that R is called an IVF relation on U if $R \in F^{(i)}(U \times U)$.

Definition 2.5 ([7, 22])**.** *Let R be an IVF relation on U. Then R is called*
(1) *reflexive, if $R(x, x) = \bar{1}$ for each $x \in U$.*
(2) *transitive, if $R(x, z) \geq R(x, y) \wedge R(y, z)$ for any $x, y, z \in U$.*
(3) *preorder, if R is reflexive and transitive.*

Definition 2.6 ([22])**.** *Let R be an IVF relation on U. The pair (U, R) is called an IVF approximation space. For each $A \in F^{(i)}(U)$, the IVF lower and the IVF upper approximation of A with respect to (U, R), denoted by $\underline{R}(A)$ and $\overline{R}(A)$, are two IVF sets and are respectively defined as follows:*

$$\underline{R}(A)(x) = \bigwedge_{y \in U} (A(y) \vee (\bar{1} - R(x, y))) \ (x \in U)$$

and

$$\overline{R}(A)(x) = \bigvee_{y \in U} (A(y) \wedge R(x, y)) \ (x \in U).$$

The the pair $(\underline{R}(A), \overline{R}(A))$ is called the IVF rough set of A with respect to (U, R).

$\underline{R} : F^{(i)}(U) \to F^{(i)}(U)$ and $\overline{R} : F^{(i)}(U) \to F^{(i)}(U)$ are called the IVF lower approximation operator and the IVF upper approximation operator, respectively. In general, we refer to $\underline{R}$ and $\overline{R}$ as the IVF rough approximation operators.

Remark 2.7. *Let (U, R) be an IVF approximation space. Then*
(1) *for each $x, y \in U$,*

$$\overline{R}(x_{\bar{1}})(y) = R(y, x) \quad and \quad \underline{R}((x_{\bar{1}})^c)(y) = \bar{1} - R(y, x).$$

(2) *for each $[a, b] \in [I]$, $\underline{R}(\widetilde{[a, b]}) \supseteq \widetilde{[a, b]} \supseteq \overline{R}(\widetilde{[a, b]})$.*

Proposition 2.8 ([22])**.** *Let (U, R) be an IVF approximation space. Then for each $A \in F^{(i)}(U)$,*

$$(\underline{R}(A))^- = \underline{R^+}(A^-), \ (\underline{R}(A))^+ = \underline{R^-}(A^+),$$

$$(\overline{R}(A))^- = \overline{R^-}(A^-) \quad and \quad (\overline{R}(A))^+ = \overline{R^+}(A^+).$$

3 IVF rough approximation operators

In this section, we deeply investigate IVF rough approximation operators.

3.1 Construction of IVF rough approximation operators

Theorem 3.1 ([28]). *Let (U,R) be an IVF approximation space. Then for any $A,B\in F^{(i)}(U)$, $\{A_j : j\in J\}\subseteq F^{(i)}(U)$ and $[a,b]\in[I]$,*
(1) $\underline{R}(\tilde{1})=\tilde{1}$, $\overline{R}(\tilde{0})=\tilde{0}$.
(2) $A\subseteq B \Longrightarrow \underline{R}(A)\subseteq \underline{R}(B)$, $\overline{R}(A)\subseteq\overline{R}(B)$.
(3) $\underline{R}(A^c)=(\overline{R}(A))^c$, $\overline{R}(A^c)=(\underline{R}(A))^c$.
(4) $\underline{R}(\bigcap\limits_{j\in J}A_j)=\bigcap\limits_{j\in J}\underline{R}(A_j)$, $\overline{R}(\bigcup\limits_{j\in J}A_j)=\bigcup\limits_{j\in J}\overline{R}(A_j)$.
(5) $\underline{R}(\widetilde{[a,b]}\cup A)=\widetilde{[a,b]}\cup\underline{R}(A)$, $\overline{R}([a,b]A)=[a,b]\overline{R}(A)$.

Theorem 3.2 ([28]). *Let (U,R) be an IVF approximation space. Then*

$$\begin{array}{lll}(1)\ R\ is\ reflexive & \Longleftrightarrow & (ALR)\ \forall A\in F^{(i)}(U),\underline{R}(A)\subseteq A.\\ & \Longleftrightarrow & (AUR)\ \forall A\in F^{(i)}(U), A\subseteq\overline{R}(A).\\ (2)\ R\ is\ transitive & \Longleftrightarrow & (ALT)\ \forall A\in F^{(i)}(U),\underline{R}(A)\subseteq\underline{R}(\underline{R}(A)).\\ & \Longleftrightarrow & (AUT)\ \forall A\in F^{(i)}(U),\overline{R}(\overline{R}(A))\subseteq\overline{R}(A).\end{array}$$

Corollary 3.3 ([28]). *Let (U,R) be an IVF approximation space. If R is pre-order, then*

$$\underline{R}(\underline{R}(A))=\underline{R}(A)\ \ and\ \ \overline{R}(\overline{R}(A))=\overline{R}(A)\ \ (A\in F^{(i)}(U)).$$

Let $A\in F^{(i)}(U)$. Denote

$$\begin{array}{rcl}A_\lambda &=& \{(x)\in U: A^-(x)\geq\lambda\}\ \ (\lambda\in I),\\ A^\lambda &=& \{(x)\in U: A^+(x)\geq\lambda]\}\ \ (\lambda\in I),\\ A_{\lambda^+} &=& \{(x)\in U: A^-(x)>\lambda\}\ \ (\lambda\in[0,1)),\\ A^{\lambda^+} &=& \{(x)\in U: A^+(x)>\lambda]\}\ \ (\lambda\in[0,1)).\end{array}$$

Definition 3.4 ([4, 25]). *Let $A\in F^{(i)}(U)$ and $[\alpha,\beta]\in[I]$. Denote*

$$A_{[\alpha,\beta]}=\{x\in U: A^-(x)\geq\alpha, A^+(x)\geq\beta]\},$$

$$A_{[\alpha,\beta]^+}=\{x\in U: A(x)>[\alpha,\beta]\},$$

$$A_{(\alpha,\beta)}=\{x\in U: A^-(x)>\alpha, A^+(x)>\beta]\}.$$

Then $A_{[\alpha,\beta]}$ (resp.$A_{[\alpha,\beta]^+}, A_{(\alpha,\beta)}$) is called the $[\alpha,\beta]$-level (resp. strong $[\alpha,\beta]$-level, (α,β)-level) set of A.

Obviously, $A_{(\alpha,\beta)}\subseteq A_{[\alpha,\beta]^+}\subseteq A_{[\alpha,\beta]}$.

Proposition 3.5 ([4, 25]). *Let $A,B\in F^{(i)}(U)$ and $[\alpha,\beta]\in[I]$. Then*
(1) $A\subseteq B\Longrightarrow A_{[\alpha,\beta]^+}\subseteq B_{[\alpha,\beta]^+}$;
(2) $(A\cup B)_{[\alpha,\beta]^+}\supseteq A_{[\alpha,\beta]^+}\cup B_{[\alpha,\beta]^+}$;
(2) $(A\cap B)_{[\alpha,\beta]^+}=A_{[\alpha,\beta]^+}\cap B_{[\alpha,\beta]^+}$.

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Let $R \in F^{(i)}(U \times U)$. Denote

$$\begin{array}{rcl} R_\lambda & = & \{(x,y) \in U \times U : R^-(x,y) \geq \lambda\} \quad (\lambda \in I), \\ R^\lambda & = & \{(x,y) \in U \times U : R^+(x,y) \geq \lambda]\} \quad (\lambda \in I), \\ R_{\lambda^+} & = & \{(x,y) \in U \times U : R^-(x,y) > \lambda\} \quad (\lambda \in [0,1)), \\ R^{\lambda^+} & = & \{(x,y) \in U \times U : R^+(x,y) > \lambda]\} \quad (\lambda \in [0,1)), \\ R_{[\alpha,\beta]} & = & \{(x,y) \in U \times U : R(x,y) \geq [\alpha,\beta]\} \quad ([\alpha,\beta] \in [I]), \\ R_{[\alpha,\beta]^+} & = & \{(x,y) \in U \times U : R(x,y) > [\alpha,\beta]\} \quad (\alpha < 1, [\alpha,\beta] \in [I]). \end{array}$$

Proposition 3.6. *Let R be an IVF relation on U.*

(1) *If R is reflexive, then R_λ, R^λ, R_{λ^+}, R^{λ^+} and $R_{[\alpha,\beta]^+}$ are reflexive.*

(2) *If R is transitive, then R_λ, R^λ, R_{λ^+}, R^{λ^+} and $R_{[\alpha,\beta]^+}$ are transitive.*

Proof. (1) are obvious.

(2) For any $x, y, z \in U$, if $(x,y), (y,z) \in R_\lambda$, we have $R^-(x,y) \geq \lambda$ and $R^-(y,z) \geq \lambda$. Note that R is transitive. Then $R(x,z) \geq R(x,y) \wedge R(y,z)$ and so

$$R^-(x,z) \geq R^-(x,y) \wedge R^-(y,z) \geq \lambda.$$

Thus $(x,z) \in R_\lambda$. Hence R_λ is transitive.

Similarly, We can prove that R^λ, R_{λ^+} and R^{λ^+} are transitive.

For any $x, y, z \in U$, if $(x,y), (y,z) \in R_{[\alpha,\beta]^+}$, we have $R(x,y) > [\alpha,\beta]$ and $R(y,z) > [\alpha,\beta]$. Note that R is transitive. Then

$$R(x,z) \geq R(x,y) \wedge R(y,z) > [\alpha,\beta].$$

and so $(x,z) \in R_{[\alpha,\beta]^+}$. Hence $R_{[\alpha,\beta]^+}$ is transitive. $\square$

Theorem 3.7. *Let (U,R) be an IVF approximation space. Then IVF rough approximation operator can be represented as follows: for each $A \in F^{(i)}(U)$,*

(1) $(\underline{R}(A))^- = \bigcup\limits_{\lambda \in I} \lambda \underline{R^{1-\lambda}}(A_\lambda) = \bigcup\limits_{\lambda \in I} \lambda \underline{R^{1-\lambda}}(A_{\lambda^+}),$
$= \bigcup\limits_{\lambda \in I} \lambda \underline{R^{(1-\lambda)^+}}(A_\lambda) = \bigcup\limits_{\lambda \in I} \lambda \underline{R^{(1-\lambda)^+}}(A_{\lambda^+});$

(2) $(\underline{R}(A))^+ = \bigcup\limits_{\lambda \in I} \lambda \underline{R_{1-\lambda}}(A^\lambda) = \bigcup\limits_{\lambda \in I)} \lambda \underline{R_{1-\lambda}}(A^{\lambda^+}),$
$= \bigcup\limits_{\lambda \in I} \lambda \underline{R_{(1-\lambda)^+}}(A^\lambda) = \bigcup\limits_{\lambda \in I} \lambda \underline{R_{(1-\lambda)^+}}(A^{\lambda^+});$

(3) $(\overline{R}(A))^- = \bigcup\limits_{\lambda \in I} \lambda \overline{R_\lambda}(A_\lambda) = \bigcup\limits_{\lambda \in I} \lambda \overline{R_{\lambda^+}}(A_\lambda),$
$= \bigcup\limits_{\lambda \in I} \lambda \overline{R_\lambda}(A_{\lambda^+}) = \bigcup\limits_{\lambda \in I} \lambda \overline{R_{\lambda^+}}(A_{\lambda^+});$

(4) $(\overline{R}(A))^+ = \bigcup\limits_{\lambda \in I} \lambda \overline{R^\lambda}(A^\lambda) = \bigcup\limits_{\lambda \in I} \lambda \overline{R^{\lambda^+}}(A^\lambda),$
$= \bigcup\limits_{\lambda \in I} \lambda \overline{R^\lambda}(A^{\lambda^+}) = \bigcup\limits_{\lambda \in I} \lambda \overline{R^{\lambda^+}}(A^{\lambda^+});$

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Proof. (1) For each $x \in U$, by Proposition 2.10,

$$\begin{aligned}
(\bigcup_{\lambda \in I} \lambda \underline{R^{1-\lambda}}(A_\lambda))(x) &= \bigvee\{\lambda \in I : x \in \underline{R^{1-\lambda}}(A_\lambda)\} \\
&= \bigvee\{\lambda \in I : (R^{1-\lambda})_s(x) \subseteq A_\lambda\} \\
&= \bigvee\{\lambda \in I : R^+(x,y) \geq 1-\lambda \textit{ implies } A^-(y) \geq \lambda\} \\
&= \bigvee\{\lambda \in I : 1 - R^+(x,y) \leq \lambda \textit{ implies } A^-(y) \geq \lambda\} \\
&= \bigvee\{\lambda \in I : \bigwedge_{y \in U} (A^-(x) \vee (1 - R^+(x,y))) \geq \lambda\} \\
&= \bigvee\{\lambda \in I : (\underline{R}(A))^-(x) \geq \lambda\} \\
&= (\underline{R}(A))^-(x).
\end{aligned}$$

Then $(\underline{R}(A))^- = \bigcup_{\lambda \in I} \lambda \underline{R^{1-\lambda}}(A_\lambda)$.

Similarly, we can prove that

$$(\underline{R}(A))^- = \bigcup_{\lambda \in [0,1)} \lambda \underline{R^{1-\lambda}}(A_{\lambda^+}) = \bigcup_{\lambda \in (0,1]} \lambda \underline{R^{(1-\lambda)^+}}(A_\lambda) = \bigcup_{\lambda \in (0,1)} \lambda \underline{R^{(1-\lambda)^+}}(A_{\lambda^+}).$$

(2) The proof is similar to (1).

(3) For each $x \in U$, by Proposition 2.10,

$$\begin{aligned}
(\bigcup_{\lambda \in I} \lambda(\overline{R_\lambda}(A_\lambda)))(x) &= \bigvee\{\lambda \in I : x \in \overline{R_\lambda}(A_\lambda)\} \\
&= \bigvee\{\lambda \in I : (R_\lambda)_s(x) \cap A_\lambda \neq \emptyset\} \\
&= \bigvee\{\lambda \in I : \exists y \in U, y \in A_\lambda \cap (R_\lambda)_s(x)\} \\
&= \bigvee\{\lambda \in I : \exists y \in U, A^-(y) \wedge R^-(x,y) \geq \lambda\} \\
&= \bigvee\{\lambda \in I : \bigvee_{y \in U} (A^-(y) \wedge R^-(x,y)) \geq \lambda\} \\
&= \bigvee\{\lambda \in I : (\overline{R}(A))^-(x) \geq \lambda\} \\
&= (\overline{R}(A))^-(x).
\end{aligned}$$

Then $\bigcup_{\lambda \in I} \lambda(\overline{R_\lambda}(A_\lambda)) = (\overline{R}(A))^-$.

Similarly, we can prove that

$$(\overline{R}(A))^- = \bigcup_{\lambda \in [0,1)} \lambda(\overline{R_{\lambda^+}}(A_\lambda)) = \bigcup_{\lambda \in [0,1)} \lambda(\overline{R_\lambda}(A_{\lambda^+})) = \bigcup_{\lambda \in [0,1)} \lambda(\overline{R_{\lambda^+}}(A_{\lambda^+})).$$

(4) The proof is similar to (3). □

Theorem 3.8. *Let (U, R) be an IVF approximation space. Then IVF rough approximation operator can be represented as follows: for each $A \in F^{(i)}(U)$,*

$$\begin{aligned}\overline{R}(A) &= \bigcup_{[\alpha,\beta]\in[I]} ([\alpha,\beta]\overline{R_{[\alpha,\beta]^+}}(A_{[\alpha,\beta]^+})) = \bigcup_{[\alpha,\beta]\in[I]} ([\alpha,\beta]\overline{R_{[\alpha,\beta]^+}}(A_{[\alpha,\beta]}))\\ &= \bigcup_{[\alpha,\beta]\in[I]} ([\alpha,\beta]\overline{R_{[\alpha,\beta]}}(A_{[\alpha,\beta]^+})).\end{aligned}$$

Proof. Denote $B = \bigcup\limits_{[\alpha,\beta]\in[I]} ([\alpha,\beta]\overline{R_{[\alpha,\beta]^+}}(A_{[\alpha,\beta]^+}))$. By Proposition 2.10,

$$\begin{aligned}B^-(x) &= \bigvee_{\alpha\in I} (\alpha \wedge \overline{R_{[\alpha,\beta]^+}}(A_{[\alpha,\beta]^+})(x))\\ &= \bigvee\{\alpha \in I : x \in (\overline{R_{[\alpha,\beta]^+}}(A_{[\alpha,\beta]^+}))\}\\ &= \bigvee\{\alpha \in I : (R_{[\alpha,\beta]^+})_s(x) \cap A_{[\alpha,\beta]^+} \neq \emptyset\}\\ &= \bigvee\{\alpha \in I : \exists y \in U, R(x,y) > [\alpha,\beta] \ and \ A(y) > [\alpha,\beta]\}\\ &= \bigvee\{\alpha \in I : \exists y \in U, A^-(y) \wedge R^-(x,y) > \alpha \ and \ A^+(y) \wedge R^+(x,y)\\ &\quad \geq \beta \ or \ A^-(y) \wedge R^-(x,y) \geq \alpha \ and \ A^+(y) \wedge R^+(x,y) > \beta\}\\ &= \bigvee_{y\in U} (A^-(y) \wedge R^-(x,y)) = (\overline{R}(A))^-(x).\end{aligned}$$

Then $(\overline{R}(A))^- = B^-$. Similarly, we can prove that $(\overline{R}(A))^+ = B^+$. Hence

$$\overline{R}(A) = B = \bigcup_{[\alpha,\beta]\in[I]} ([\alpha,\beta]\overline{R_{[\alpha,\beta]^+}}(A_{[\alpha,\beta]^+})).$$

Similarly, we can prove that

$$\overline{R}(A) = \bigcup_{[\alpha,\beta]\in[I]} ([\alpha,\beta]\overline{R_{[\alpha,\beta]^+}}(A_{[\alpha,\beta]})) = \bigcup_{[\alpha,\beta]\in[I]} ([\alpha,\beta]\overline{R_{[\alpha,\beta]}}(A_{[\alpha,\beta]^+})).$$

□

3.2 Axiomatic characterizations of IVF rough approximation operators

In an axiomatic approach, rough sets are axiomatized by abstract operators. For the case of IVF rough sets, the primitive notion is the system $(F^{(i)}(U), \bigcap, \bigcup, c, L, H)$, where $L, H : F^{(i)}(U) \to F^{(i)}(U)$ be two IVF set operators. In this subsection, rough approximation operators in the IVF environment are characterized by some axioms.

Definition 3.9. *Let $L, H : F^{(i)}(U) \to F^{(i)}(U)$ be two IVF set operators. If*

$$(L(A))^c = H(A^c) \ (A \in F^{(i)}(U)),$$

then L, H are called two dual operators.

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Remark 3.10. *$L, H : F^{(i)}(U) \to F^{(i)}(U)$ are two dual operators iff $(H(A))^c = L(A^c)$ for each $A \in F^{(i)}(U)$.*

Theorem 3.11. *Let $L, H : F^{(i)}(U) \to F^{(i)}(U)$ be two dual operators. Then there exists an IVF relation R on U such that $L = \underline{R}$ and $H = \overline{R}$ iff L satisfies axioms $(AL1)$ and $(AL2)$, or equivalently, H satisfies axioms $(AU1)$ and $(AU2)$:*

$$(AL1) \quad L(\widetilde{[a,b]} \cup A) = \widetilde{[a,b]} \cup L(A) \ (A \in F^{(i)}(U), [a,b] \in [I]),$$

$$(AL2) \quad L(A \cap B) = \underline{R}(A) \cap L(B) \ (A, B \in F^{(i)}(U);$$

$$(AU1) \quad H([a,b]A) = [a,b]H(A) \ (A \in F^{(i)}(U), [a,b] \in [I]),$$

$$(AU2) \quad H(A \cup B) = H(A) \cup H(B) \ (A, B \in F^{(i)}(U)).$$

Proof. Note that $L, H : F^{(i)}(U) \to F^{(i)}(U)$ are two dual operators. Then $(AL1)$ and $(AL2)$ are equivalent to $(AU1)$ and $(AU2)$. We only need to prove that $L = \underline{R}$ and $H = \overline{R}$ iff H satisfies axioms $(AU1)$ and $(AU2)$.

Necessity. This is obvious.

Sufficiency. Assume that the operator H satisfies axioms $(AU1)$ and $(AU2)$. Define an IVF relation R on U by

$$R(x,y) = H(y_{\bar{1}})(x) \quad (x, y \in U).$$

Let $A \in F^{(i)}(U)$. Note that

$$\begin{aligned} H(A)(x) &= H(\bigcup_{y \in U}(A(y)y_{\bar{1}}))(x) = (\bigcup_{y \in U} H(A(y)y_{\bar{1}}))(x) = (\bigcup_{y \in U}(A(y)H(y_{\bar{1}})))(x) \\ &= \bigvee_{y \in U}(A(y) \wedge H(y_{\bar{1}})(x)) = \bigvee_{y \in U}(A(y) \wedge R(x,y)) = \overline{R}(A)(x) \end{aligned}$$

for each $x \in U$. Then $H(A) = \overline{R}(A)$. By Theorem 3.1(3),

$$L(A) = (H(A^c))^c = (\overline{R}(A^c))^c = \underline{R}(A).$$

Thus $L = \underline{R}$, $H = \overline{R}$. □

Theorem 3.12. *Let $L, H : F^{(i)}(U) \to F^{(i)}(U)$ be two dual operators. Then there exists a reflexive IVF relation R on U such that $L = \underline{R}$ and $H = \overline{R}$ iff L satisfies axiom $(AL1)$, $(AL2)$ and (ALR), or equivalently, H satisfies axiom $(AU1)$, $(AU2)$ and (AUR):*

$$(ALR) \quad L(A) \subseteq A \quad (A \in F^{(i)}(U));$$

$$(AUR) \quad A \subseteq H(A) \quad (A \in F^{(i)}(U)).$$

Proof. This holds by Theorems 3.2(1) and 3.11. □

Theorem 3.13. *Let $L, H : F^{(i)}(U) \to F^{(i)}(U)$ be two dual operators. Then there exists a symmetric IVF relation R on U such that $L = \underline{R}$ and $H = \overline{R}$ iff L satisfies axiom $(AL1)$, $(AL2)$ and (ALS), or equivalently, H satisfies axiom $(AU1)$, $(AU2)$ and (AUS):*

$$(ALS) \quad L((x_{\bar{1}})^c)(y) = L((y_{\bar{1}})^c)(x) \ (x, y \in U);$$
$$(ALS) \qquad H(x_{\bar{1}})(y) = H(y_{\bar{1}})(x) \ (x, y \in U).$$

Proof. This hold by Remark 2.9(1) and Theorem 3.11. □

Theorem 3.14. *Let $L, H : F^{(i)}(U) \to F^{(i)}(U)$ be two dual operators. Then there exists a transitive IVF relation R on U such that $L = \underline{R}$ and $H = \overline{R}$ iff L satisfies axiom $(AL1)$, $(AL2)$ and (ALT), or equivalently, H satisfies axiom $(AU1)$, $(AU2)$ and (AUT):*

$$(ALT) \quad L(A) \subseteq L(L(A)) \quad (A \in F^{(i)}(U));$$
$$(AUT) \quad H(H(A)) \subseteq H(A) \quad (A \in F^{(i)}(U)).$$

Proof. This holds by Theorems 3.2(2) and 3.11. □

4 IVF pseudo-closure operators in IVF approximation spaces

In this section, we investigate IVF pseudo-closure operators in IVF approximation spaces.

For each $[a, b] \in [I], X \in \mathscr{P}(U)$, we define

$$([a, b]X)(x) = \begin{cases} [a, b], & x \in X, \\ \bar{0}, & x \in U - X. \end{cases}$$

Denote

$$\mathscr{E}(U) = \{[a, b]X : [a, b] \in [I], X \in \mathscr{P}(U)\}.$$

Then $\mathscr{E}(U) \subseteq F^{(i)}(U)$.

Definition 4.1. *Let τ be an IVF topology on U. Define*

$$S_\tau(A) = \bigcup_{[\alpha,\beta]\in[I]} cl_\tau([\alpha, \beta]A_{[\alpha,\beta]}\} \quad (A \in F^{(i)}(U)).$$

Then $S_\tau : F^{(i)}(U) \to F^{(i)}(U)$ is called the IVF pseudo-closure operator induced by τ on U.

Theorem 4.2 ([25])**.** *Let $A \in F^{(i)}(U)$. Then*

$$A = \bigcup_{[\alpha,\beta]\in[I]} [\alpha, \beta]A_{[\alpha,\beta]} = \bigcup_{[\alpha,\beta]\in[I]} [\alpha, \beta]A_{(\alpha,\beta)}.$$

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Theorems 4.3(5) and 4.4 below illustrate the meaning on IVF pseudo-closure operators.

Theorem 4.3. *Let τ be an IVF topology on U and let S_τ be the IVF pseudo-closure operator induced by τ on U. Then for any $A, B \in F^{(i)}(U)$,*
(1) $S_\tau(\tilde{0}) = \tilde{0}$.
(2) $A \subseteq S_\tau(A) \subseteq cl_\tau(A)$.
(3) $S_\tau(A \cup B) \supseteq S_\tau(A) \cup S_\tau(B)$. $S_\tau(A \cap B) \subseteq S_\tau(A) \cap S_\tau(B)$.
(4) $A \in \tau^c \Longrightarrow S_\tau(A) = A$.
(5) S_τ coincides with cl_τ as operators from $\mathscr{E}(U)$ to $F^{(i)}(U)$.

Proof. (1) For any $[\alpha, \beta] \in [I]$ and $x \in U$, since

$$([\alpha,\beta]\tilde{0}_{[\alpha,\beta]})(x) = [\alpha,\beta] \wedge \tilde{0}_{[\alpha,\beta]}(x) = \begin{cases} [0,0] \wedge \bar{1} = \bar{0}, & [\alpha,\beta] = \bar{0}, \\ [\alpha,\beta] \wedge \bar{0} = \bar{0}, & [\alpha,\beta] \in [I] - \{\bar{0}\}. \end{cases}$$

we have $[\alpha,\beta]\tilde{0}_{[\alpha,\beta]} = \tilde{0}$. Thus

$$S_\tau(\tilde{0}) = \bigcup_{[\alpha,\beta]\in[I]} cl_\tau([\alpha,\beta]\tilde{0}_{[\alpha,\beta]}) = \bigcup_{[\alpha,\beta]\in[I]} cl_\tau(\tilde{0}) = \tilde{0}.$$

(2) By Theorem 4.2,

$$A = \bigcup_{[\alpha,\beta]\in[I]} [\alpha,\beta]A_{[\alpha,\beta]} \subseteq \bigcup_{[\alpha,\beta]\in[I]} cl_\tau([\alpha,\beta]A_{[\alpha,\beta]}) = S_\tau(A) \quad and$$

$$S_\tau(A) = \bigcup_{[\alpha,\beta]\in[I]} cl_\tau([\alpha,\beta]A_{[\alpha,\beta]}) \subseteq cl_\tau(\bigcup_{[\alpha,\beta]\in[I]} [\alpha,\beta]A_{[\alpha,\beta]}) = cl_\tau(A).$$

(3) For any $A, B \in F^{(i)}(U)$, $[\alpha,\beta] \in [I]$ and $x \in U$ put

$$C(x) = \begin{cases} \bar{1}, & x \in A_{[\alpha,\beta]}, \\ \bar{0}, & x \in U - A_{[\alpha,\beta]} \end{cases}, \quad D(x) = \begin{cases} \bar{1}, & x \in B_{[\alpha,\beta]}, \\ \bar{0}, & x \in U - B_{[\alpha,\beta]}. \end{cases}$$

Obviously,

$$[\alpha,\beta]A_{[\alpha,\beta]} = \widetilde{[\alpha,\beta]} \cap C, \quad [\alpha,\beta]B_{[\alpha,\beta]} = \widetilde{[\alpha,\beta]} \cap D,$$

$$[\alpha,\beta](A_{[\alpha,\beta]} \cup B_{[\alpha,\beta]}) = \widetilde{[\alpha,\beta]} \cap (C \cup D)$$

and

$$[\alpha,\beta](A_{[\alpha,\beta]} \cap B_{[\alpha,\beta]}) = \widetilde{[\alpha,\beta]} \cap (C \cap D).$$

We can easily prove that

$$(A \cup B)_{[\alpha,\beta]} \supseteq A_{[\alpha,\beta]} \cup B_{[\alpha,\beta]} \quad and \quad (A \cap B)_{[\alpha,\beta]} = A_{[\alpha,\beta]} \cap B_{[\alpha,\beta]}.$$

By Proposition 2.6(5),

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$$\begin{aligned}
& S_\tau(A \cup B) \\
= & \bigcup_{[\alpha,\beta]\in[I]} cl_\tau([\alpha,\beta](A\cup B)_{[\alpha,\beta]}) \supseteq \bigcup_{[\alpha,\beta]\in[I]} cl_\tau([\alpha,\beta](A_{[\alpha,\beta]} \cup B_{[\alpha,\beta]})) \\
= & \bigcup_{[\alpha,\beta]\in[I]} cl_\tau(\widetilde{[\alpha,\beta]} \cap (C \cup D)) = \bigcup_{[\alpha,\beta]\in[I]} cl_\tau((\widetilde{[\alpha,\beta]} \cap C) \cup (\widetilde{[\alpha,\beta]} \cap D)) \\
= & \bigcup_{[\alpha,\beta]\in[I]} (cl_\tau(\widetilde{[\alpha,\beta]} \cap C) \cup cl_\tau(\widetilde{[\alpha,\beta]} \cap D)) \\
= & \left(\bigcup_{[\alpha,\beta]\in[I]} cl_\tau(\widetilde{[\alpha,\beta]} \cap C) \right) \cup \left(\bigcup_{[\alpha,\beta]\in[I]} cl_\tau(\widetilde{[\alpha,\beta]} \cap D) \right) \\
= & \left(\bigcup_{[\alpha,\beta]\in[I]} cl_\tau([\alpha,\beta]A_{[\alpha,\beta]}) \right) \cup \left(\bigcup_{[\alpha,\beta]\in[I]} cl_\tau([\alpha,\beta]B_{[\alpha,\beta]}) \right) \\
= & S_\tau(A) \cup S_\tau(B).
\end{aligned}$$

By Proposition 2.6(3),

$$\begin{aligned}
& S_\tau(A \cap B) \\
= & \bigcap_{[\alpha,\beta]\in[I]} cl_\tau([\alpha,\beta](A\cap B)_{[\alpha,\beta]}) = \bigcup_{[\alpha,\beta]\in[I]} cl_\tau([\alpha,\beta](A_{[\alpha,\beta]} \cap B_{[\alpha,\beta]})) \\
= & \bigcup_{[\alpha,\beta]\in[I]} cl_\tau(\widetilde{[\alpha,\beta]} \cap (C \cap D)) = \bigcup_{[\alpha,\beta]\in[I]} cl_\tau((\widetilde{[\alpha,\beta]} \cap C) \cap (\widetilde{[\alpha,\beta]} \cap D)) \\
\subseteq & \bigcup_{[\alpha,\beta]\in[I]} (cl_\tau(\widetilde{[\alpha,\beta]} \cap C) \cap cl_\tau(\widetilde{[\alpha,\beta]} \cap D)) \\
\subseteq & \left(\bigcup_{[\alpha,\beta]\in[I]} cl_\tau(\widetilde{[\alpha,\beta]} \cap C) \right) \cap \left(\bigcup_{[\alpha,\beta]\in[I]} cl_\tau(\widetilde{[\alpha,\beta]} \cap D) \right) \\
= & \left(\bigcup_{[\alpha,\beta]\in[I]} cl_\tau([\alpha,\beta]A_{[\alpha,\beta]}) \right) \cap \left(\bigcup_{[\alpha,\beta]\in[I]} cl_\tau([\alpha,\beta]B_{[\alpha,\beta]}) \right) \\
= & S_\tau(A) \cap S_\tau(B).
\end{aligned}$$

(4) By (2) and Proposition 2.6(6),

$$cl_\tau(A) \subseteq S(cl_\tau(A)) \subseteq cl_\tau(cl_\tau(A)) = cl_\tau(A),$$

Note that $A \in \tau^c$. Then

$$S_\tau(A) = S_\tau(cl_\tau(A)) = cl_\tau(A) = A.$$

(5) Let $A \in \mathscr{E}(U)$. Then there exist $[a,b] \in [I]$ and $X \in \mathscr{P}(U)$ such that $A = [a,b]X$.

(i) If $[a,b] \neq \bar{0}$, then for each $x \in U$,

$$A_{[a,b]}(x) = ([a,b]X)_{[a,b]}(x) = \begin{cases} \bar{1}, & ([a,b]X)(x) \geq [a,b] \\ \bar{0}, & ([a,b]X)(x) \ngeq [a,b] \end{cases} = \begin{cases} \bar{1}, & x \in X, \\ \bar{0}, & x \in U - X. \end{cases}$$

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Thus $A_{[a,b]} = X$. So

$$\begin{aligned} S_\tau(A) &= \bigcup_{[\alpha,\beta]\in[I]} cl_\tau([\alpha,\beta]A_{[\alpha,\beta]}) \\ &\supseteq cl_\tau([a,b]A_{[a,b]}) = cl_\tau([a,b]X) = cl_\tau(A). \end{aligned}$$

By (2), $S_\tau(A) \subseteq cl(A)$. Thus $S_\tau(A) = cl_\tau(A)$.
(ii) If $[a,b] = \bar{0}$, then $A = \tilde{0}$. By (1), $S_\tau(\tilde{0}) = \tilde{0}$. Thus $S_\tau(A) = cl_\tau(A)$.
By (i) and (ii),

$$S_\tau \text{ coincides with } cl_\tau \text{ as operators from } \mathscr{E}(U) \text{ to } F^{(i)}(U).$$

□

Theorem 4.4. *Let (U,R) be an IVF approximation space. If R is preorder, then*

$$\overline{R}(A) = S_{\tau_R}(A) \quad (A \in \mathscr{E}(U)).$$

Proof. For each $A \in \mathscr{E}(U)$, by Theorems 3.11(3) and 4.3(5),

$$\overline{R}(A) = cl_{\tau_R}(A) = S_{\tau_R}(A).$$

□

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SOME WEIGHTED HERMITE-HADAMARD TYPE INEQUALITIES FOR GEOMETRICALLY–ARITHMETICALLY CONVEX FUNCTIONS ON THE CO-ORDINATES

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Abstract. In this paper, the concept of GA-convex functions on the co-ordinates is introduced. By using a concept of GA-convex functions on the co-ordinates, Hermite-Hadamard type inequalities for this class of functions are settled.

1. Introduction

A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ forenamed as convex in the classical touch [24], if the inequality

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

holds for all $x,\ y \in I$ and $\lambda \in [0,1]$.

Indeed, a vast literature has been written on inequalities using classical convexity but one of the most celebrated is the Hermite-Hadamard inequelity. This double inequality is stated as follows:

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a function and $a,\ b \in I$ with $a < b$. Then f is convex on $[a,b]$ iff

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

This also reveals that (1.1) can be compulsary as a adequate and sufficient condition to function f to be convex on $[a,b]$.

Hermite-Hadmard inequality (1.1) has recieved considerable attention of many reserchers because of its various applications and usefulness in the field of mathematical inequalities itself as well as in other areas of mathematics. The inequality (1.1) has been extended to various forms by using various generalizations of the definition of classical convex functions and it has also been refined under different hypotheses, see for instance [6, 9, 10, 11, 15, 24, 32] and the references therein.

As stated above the classical convexity has been generalized to different forms and we mention below one of the generalizations of the classical convexity which is known as GA-convexity.

Definition 1. [18, 19] *A function $f : I \subseteq \mathbb{R}_0 = [0,\infty) \to \mathbb{R}$ is said to be GA-convex function on I if*

$$f\left(x^{\lambda}y^{1-\lambda}\right) \leq \lambda f(x) + (1-\lambda)f(y)$$

holds for all $x,\ y \in I$ and $\lambda \in [0,1]$, where $x^{\lambda}y^{1-\lambda}$ and $\lambda f(x) + (1-\lambda)f(y)$ are respectively the weighted geometric mean of two positive numbers x and y and the weighted arithmetic mean of $f(x)$ and $f(y)$.

For results on Hermite-Hadamard type inequalities on GA-convex functions and their applications we refere to a recent articles of Latif [15] and Zhang et al. [32].

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The definition of classical convexity for functions of of one variables was extended to functions two variables as follows.

Definition 2. [5, 6] *Let* $\Delta =: [a,b] \times [c,d] \subseteq \mathbb{R}^2$ *with* $a < b$ *and* $c < d$ *be a bidimensional interval. A mapping* $f : \Delta \to \mathbb{R}$ *is said to be convex on* Δ *if the inequality*

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \lambda f(x,y) + (1-\lambda)f(z,w)$$

holds for all $(x,y),(z,w) \in \Delta$ *and* $\lambda \in [0,1]$.

The Definition 2 of convex functions on Δ was modified as co-ordinated convex functions by Dragomir in [5].

Definition 3. [5] *A function* $f : \Delta \to \mathbb{R}$ *is said to be convex on the co-ordinates on* Δ *if the partial mappings* $f_y : [a,b] \to \mathbb{R}$, $f_y(u) = f(u,y)$ *and* $f_x : [c,d] \to \mathbb{R}$, $f_x(v) = f(x,v)$ *are convex where defined for all* $x \in [a,b]$, $y \in [c,d]$.

Remark 1. [12] *It is clear that if a function* $f : \Delta \to \mathbb{R}$ *is convex on the co-ordinates on* Δ. *Then*

$$\begin{aligned} & f(tx + (1-t)z, sy + (1-s)w) \\ & \leq tsf(x,y) + t(1-s)f(x,w) + s(1-t)f(z,y) + (1-t)(1-s)f(z,w), \end{aligned}$$

holds for all $(t,s) \in [0,1] \times [0,1]$ *and* $x,z \in [a,b], y,w \in [c,d]$.

It is well-known that every convex mapping $f : \Delta \to \mathbb{R}$ is convex on the co-ordinates but converse may not be true (see [5]).

The following inequalities of Hermite-Hadamard type for co-ordinated convex functions on the rectangle from the plane $\mathbb{R}^2$ were established in [5, Theorem 1, page 778]:

Most recently, the notion of co-ordinated convexity has also been generalized in a diverse manner and as a result, the author [14] extended the defintion of GA-convex functions of one variable to GA-convex functions of two variables.

Definition 4. [14]*A function* $f : \Delta \subseteq (0,\infty) \times (0,\infty) \to \mathbb{R}$ *is GA-convex on* Δ *if*

$$f\left(x^\lambda z^{1-\lambda}, y^\lambda w^{1-\lambda}\right) \leq \lambda f(x,y) + (1-\lambda) f(z,w)$$

holds for all $(x,y),(z,w) \in \Delta$ *and* $\lambda \in [0,1]$.

A modification in Definition 4 resulted in the notion of GA-convex functions on the co-ordinates on Δ.

Definition 5. [14] *A function* $f : \Delta \subseteq (0,\infty) \times (0,\infty) \to \mathbb{R}$ *is said to be GA-convex on the co-ordinates on* Δ *if the partial mappings* $f_y : [a,b] \subseteq (0,\infty) \to \mathbb{R}$, $f_y(u) = f(u,y)$ *and* $f_x : [c,d] \subseteq (0,\infty) \to \mathbb{R}$, $f_x(v) = f(x,v)$ *are GA-convex where defined for all* $x \in [a,b]$, $y \in [c,d]$.

The following result holds as a consequence of the defintion of GA-convex fuctions on the co-ordinates on Δ.

Remark 2. *If a function* $f : \Delta \subseteq (0,\infty) \times (0,\infty) \to \mathbb{R}$ *is GA-convex on the co-ordinates on* Δ. *Then*

$$\begin{aligned} & f\left(x^t z^{1-t}, y^s w^{1-s}\right) \\ & \leq tf\left(x, y^s w^{1-s}\right) + (1-t) f\left(z, y^s w^{1-s}\right) \\ & \leq t\left[sf(x,y) + (1-s) f(x,w)\right] + (1-t)\left[sf(z,y) + (1-s) f(z,w)\right] \\ & \leq tsf(x,y) + t(1-s)f(x,w) + s(1-t)f(z,y) + (1-t)(1-s)f(z,w) \end{aligned}$$

holds for all $(t,s) \in [0,1] \times [0,1]$ *and* $x,z \in [a,b], y,w \in [c,d]$.

In [13], some H-H type inequalities for GA-convex functions on the co-ordinates on Δ were also proved for GA-convex functions on the co-ordinates on Δ. For more results on H-H type inequalities for different generilazations of the defintion of of co-ordinated convex functions we refer the reader to [1], [2], [7]-[12], [16], [20]-[23], [27], [28] and closely related articles mentioned therein.

The main objective of the present paper is to establish some new weighted H-H type inequalities for the class of GA-convex functions on the co-ordinates on a rectangle from the plane in Section 2.

2. Weighted Inequalities for co-ordinated GA-convex functions

For the sake of convenience to the reader, we will use the following notations

$$L_1(t)=a^{\frac{1+t}{2}}b^{\frac{1-t}{2}},L_2(s)=c^{\frac{1+s}{2}}d^{\frac{1-s}{2}},U_1(t)=a^{\frac{1-t}{2}}b^{\frac{1+t}{2}},U_2(s)=c^{\frac{1-s}{2}}d^{\frac{1+s}{2}}.$$

To obtain our main results, we first establish the following weighted identity.

Lemma 1. *Suppose that $f:\Delta\subseteq(0,\infty)\times(0,\infty)\to\mathbb{R}$ has second order partial derivatives on Δ° and $[a,b]\times[c,d]\subseteq\Delta^\circ$ with $a<b$ and $c<d$. If $h:[a,b]\times[c,d]\to[0,\infty)$ is twice partially differentiable mapping and $f_{ts}\in L([a,b]\times[c,d])$, then we have*

$$\begin{aligned}
&\Phi(a,b,c,d;f,h)\\
&\quad=h(a,c)f(a,c)-h(a,d)f(a,d)-h(b,c)f(b,c)+h(b,d)f(b,d)\\
&\quad+\int_c^d h_y(a,y)f(a,y)\,dy-\int_c^d h_y(b,y)f(b,y)\,dy-\int_a^b h_x(x,d)f(x,d)\,dx\\
&\quad+\int_a^b h_x(x,c)f(x,c)\,dx+\int_a^b\int_c^d h_{xy}(x,y)f(x,y)\,dydx\\
&=\frac{(\ln b-\ln a)(\ln d-\ln c)}{4}\left[\int_0^1\int_0^1 L_1(t)L_2(s)h(L_1(t),L_2(s))f_{ts}(L_1(t),L_2(s))\,dsdt\right.\\
&\quad+\int_0^1\int_0^1 U_1(t)L_2(s)h(U_1(t),L_2(s))f_{ts}(U_1(t),L_2(s))\,dsdt\\
&\quad+\int_0^1\int_0^1 L_1(t)U_2(s)h(L_1(t),U_2(s))f_{ts}(L_1(t),U_2(s))\,dsdt\\
&\quad\left.+\int_0^1\int_0^1 U_1(t)U_2(s)h(U_1(t),U_2(s))f_{ts}(U_1(t),U_2(s))\,dsdt\right].\qquad(2.1)
\end{aligned}$$

Proof. By letting $x=a^{\frac{1+t}{2}}b^{\frac{1-t}{2}}$, $y=c^{\frac{1+s}{2}}d^{\frac{1-s}{2}}$ and by integration by parts with respect to y and then with respect to x, we have

$$\begin{aligned}
&\frac{(\ln b-\ln a)(\ln d-\ln c)}{4}\int_0^1\int_0^1 L_1(t)L_2(s)h(L_1(t),L_2(s))f_{ts}(L_1(t),L_2(s))\,dsdt\\
&\quad=\int_a^{\sqrt{ab}}\int_c^{\sqrt{cd}}h(x,y)f_{xy}(x,y)\,dydx=h\left(\sqrt{ab},\sqrt{cd}\right)f\left(\sqrt{ab},\sqrt{cd}\right)\\
&-h\left(a,\sqrt{cd}\right)f\left(a,\sqrt{cd}\right)-h\left(\sqrt{ab},c\right)f\left(\sqrt{ab},c\right)+h(a,c)f(a,c)+\int_c^{\sqrt{cd}}h_y(a,y)f(a,y)\,dy\\
&\quad-\int_c^{\sqrt{cd}}h_y\left(\sqrt{ab},y\right)f\left(\sqrt{ab},y\right)dy-\int_a^{\sqrt{ab}}h_x\left(x,\sqrt{cd}\right)f\left(x,\sqrt{cd}\right)dx\\
&\quad+\int_a^{\sqrt{ab}}h_x(x,c)f(x,c)\,dx+\int_a^{\sqrt{ab}}\int_c^{\sqrt{cd}}h_{xy}(x,y)f(x,y)\,dydx.\qquad(2.2)
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
&\frac{(\ln b-\ln a)(\ln d-\ln c)}{4}\int_0^1\int_0^1 U_1(t)\,L_2(s)\,h(U_1(t),L_2(s))\,f_{ts}(U_1(t),L_2(s))\,dsdt\\
&=h\left(b,\sqrt{cd}\right)f\left(b,\sqrt{cd}\right)-h(b,c)\,f(b,c)-h\left(\sqrt{ab},\sqrt{cd}\right)f\left(\sqrt{ab},\sqrt{cd}\right)\\
&\quad+h\left(\sqrt{ab},c\right)f\left(\sqrt{ab},c\right)-\int_c^{\sqrt{cd}}h_y(b,y)\,f(b,y)\,dy\\
&\quad+\int_c^{\sqrt{cd}}h_y\left(\sqrt{ab},y\right)f\left(\sqrt{ab},y\right)dy-\int_{\sqrt{ab}}^b h_x\left(x,\sqrt{cd}\right)f\left(x,\sqrt{cd}\right)dx\\
&\quad+\int_{\sqrt{ab}}^b h_x(x,c)\,f(x,c)\,dx+\int_{\sqrt{ab}}^b\int_c^{\sqrt{cd}}h_{xy}(x,y)\,f(x,y)\,dydx, \qquad (2.3)
\end{aligned}$$

$$\begin{aligned}
&\frac{(\ln b-\ln a)(\ln d-\ln c)}{4}\int_0^1\int_0^1 L_1(t)\,U_2(s)\,h(L_1(t),U_2(s))\,f_{ts}(L_1(t),U_2(s))\,dsdt\\
&=h\left(\sqrt{ab},d\right)f\left(\sqrt{ab},d\right)-h(a,d)\,f(a,d)-h\left(\sqrt{ab},\sqrt{cd}\right)f\left(\sqrt{ab},\sqrt{cd}\right)\\
&\quad+h\left(a,\sqrt{cd}\right)f\left(a,\sqrt{cd}\right)-\int_c^{\sqrt{cd}}h_y\left(\sqrt{ab},y\right)f\left(\sqrt{ab},y\right)dy\\
&\quad+\int_c^{\sqrt{cd}}h_y(a,y)\,f(a,y)\,dy-\int_a^{\sqrt{ab}}h_x(x,d)\,f(x,d)\,dx\\
&\quad+\int_a^{\sqrt{ab}}h_x\left(x,\sqrt{cd}\right)f\left(x,\sqrt{cd}\right)dx+\int_a^{\sqrt{ab}}\int_{\sqrt{cd}}^d h_{xy}(x,y)\,f(x,y)\,dydx \qquad (2.4)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{(\ln b-\ln a)(\ln d-\ln c)}{4}\int_0^1\int_0^1 U_1(t)\,U_2(s)\,h(U_1(t),U_2(s))\,f_{ts}(U_1(t),U_2(s))\,dsdt\\
&=h(b,d)\,f(b,d)-h\left(b,\sqrt{cd}\right)f\left(b,\sqrt{cd}\right)-h\left(\sqrt{ab},d\right)f\left(\sqrt{ab},d\right)\\
&\quad+h\left(\sqrt{ab},\sqrt{cd}\right)f\left(\sqrt{ab},\sqrt{cd}\right)-\int_{\sqrt{cd}}^d h_y(b,y)\,f(b,y)\,dy\\
&\quad+\int_{\sqrt{cd}}^d h_y\left(\sqrt{ab},y\right)f\left(\sqrt{ab},y\right)dy-\int_{\sqrt{ab}}^b h_x(x,d)\,f(x,d)\,dx\\
&\quad+\int_{\sqrt{ab}}^b h_x\left(x,\sqrt{cd}\right)f\left(x,\sqrt{cd}\right)dx+\int_{\sqrt{ab}}^b\int_{\sqrt{cd}}^d h_{xy}(x,y)\,f(x,y)\,dydx. \qquad (2.5)
\end{aligned}$$

Adding (2.2)-(2.5), we get the desired identity. This completes the proof of the lemma. □

Lemma 2. *Let* $u,\ v>0$, $\eta,k\in\mathbb{R}$ *and* $\eta\neq 0$. *Then*

$$\begin{aligned}
\zeta(u,v;k,\eta)&=\int_0^1(1-kt)\,u^{\frac{1}{2}+\eta t}v^{\frac{1}{2}-\eta t}dt\\
&=\begin{cases}\frac{kv^{\frac{1}{2}-\eta}u^{\frac{1}{2}}[L(u^\eta,v^\eta)-u^\eta]}{\eta(\ln u-\ln v)}+v^{\frac{1}{2}-\eta}u^{\frac{1}{2}}L(u^\eta,v^\eta), & u\neq v,\\ \frac{u\left[1-(1-k)^2\right]}{2k}, & u=v,\end{cases}
\end{aligned}$$

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where $L(u,v)$ is the logarithmic mean

$$L(u,v)=\begin{cases}\frac{v-u}{\ln v-\ln u}, & u\neq v,\\ u, & u=v.\end{cases}$$

Proof. The proof follows by integration by parts. □

Now we present some new weighted H-H type inequality for GA-convex functions on a rectangle from $\mathbb{R}^2$.

In what follows, we will use the following notation to make our presentation compact.

$$\begin{aligned}\sigma_1(u,v,z,w;q) &= \left[\zeta\left(u,v;-1,\frac{1}{2}\right)\zeta\left(z,w;-1,\frac{1}{2}\right)|f_{ts}(a,c)|^q\right.\\ &\quad+\zeta\left(u,v;-1,\frac{1}{2}\right)\zeta\left(z,w;1,\frac{1}{2}\right)|f_{ts}(a,d)|^q+\zeta\left(u,v;1,\frac{1}{2}\right)\\ &\quad\left.\times\zeta\left(z,w;-1,\frac{1}{2}\right)|f_{ts}(b,c)|^q+\zeta\left(u,v;1,\frac{1}{2}\right)\zeta\left(z,w;1,\frac{1}{2}\right)|f_{ts}(b,d)|^q\right]^{\frac{1}{q}},\end{aligned}$$

$$\begin{aligned}\sigma_2(u,v,z,w;q) &= \left[\zeta\left(u,v;1,-\frac{1}{2}\right)\zeta\left(z,w;-1,\frac{1}{2}\right)|f_{ts}(a,c)|^q\right.\\ &\quad+\zeta\left(u,v;1,-\frac{1}{2}\right)\zeta\left(z,w;1,\frac{1}{2}\right)|f_{ts}(a,d)|^q+\zeta\left(u,v;-1,-\frac{1}{2}\right)\\ &\quad\left.\times\zeta\left(z,w;-1,\frac{1}{2}\right)|f_{ts}(b,c)|^q+\zeta\left(u,v;-1,-\frac{1}{2}\right)\zeta\left(z,w;1,\frac{1}{2}\right)|f_{ts}(b,d)|^q\right]^{\frac{1}{q}},\end{aligned}$$

$$\begin{aligned}\sigma_3(u,v,z,w;q) &= \left[\zeta\left(u,v;-1,\frac{1}{2}\right)\zeta\left(z,w;1,-\frac{1}{2}\right)|f_{ts}(a,c)|^q\right.\\ &\quad+\zeta\left(u,v;-1,\frac{1}{2}\right)\zeta\left(z,w;-1,-\frac{1}{2}\right)|f_{ts}(a,d)|^q+\zeta\left(u,v;1,\frac{1}{2}\right)\\ &\quad\left.\times\zeta\left(z,w;1,-\frac{1}{2}\right)|f_{ts}(b,c)|^q+\zeta\left(u,v;1,\frac{1}{2}\right)\zeta\left(z,w;-1,-\frac{1}{2}\right)|f_{ts}(b,d)|^q\right]^{\frac{1}{q}}\end{aligned}$$

and

$$\begin{aligned}\sigma_4(u,v,z,w;q) &= \left[\zeta\left(u,v;-1,\frac{1}{2}\right)\zeta\left(z,w;-1,\frac{1}{2}\right)|f_{ts}(a,c)|^q\right.\\ &\quad+\zeta\left(u,v;-1,\frac{1}{2}\right)\zeta\left(z,w;1,\frac{1}{2}\right)|f_{ts}(a,d)|^q+\zeta(u,v;-1)\\ &\quad\left.\times\zeta(z,w;1)|f_{ts}(b,c)|^q+\zeta\left(u,v;-1,-\frac{1}{2}\right)\zeta\left(z,w;-1,-\frac{1}{2}\right)|f_{ts}(b,d)|^q\right]^{\frac{1}{q}}.\end{aligned}$$

It is easy to observe that when $u=v=z=w=1$, then

$$\sigma_1(1,1,1,1;q)=\left[\frac{9}{4}|f_{ts}(a,c)|^q+\frac{3}{4}|f_{ts}(a,d)|^q+\frac{3}{4}|f_{ts}(b,c)|^q+\frac{1}{4}|f_{ts}(b,d)|^q\right]^{\frac{1}{q}},$$

$$\sigma_2(1,1,1,1;q)=\left[\frac{3}{4}|f_{ts}(a,c)|^q+\frac{1}{4}|f_{ts}(a,d)|^q\left|+\frac{9}{4}f_{ts}(b,c)\right|^q+\frac{3}{4}|f_{ts}(b,d)|^q\right]^{\frac{1}{q}},$$

$$\sigma_3(1,1,1,1;q)=\left[\frac{3}{4}|f_{ts}(a,c)|^q+\frac{9}{4}|f_{ts}(a,d)|^q+\frac{1}{4}|f_{ts}(b,c)|^q+\frac{3}{4}|f_{ts}(b,d)|^q\right]^{\frac{1}{q}}$$

and

$$\sigma_4(1,1,1,1;q)=\left[\frac{1}{4}|f_{ts}(a,c)|^q+\frac{3}{4}|f_{ts}(a,d)|^q+\frac{3}{4}|f_{ts}(b,c)|^q+\frac{9}{4}|f_{ts}(b,d)|^q\right]^{\frac{1}{q}}.$$

Theorem 1. *Let $f:\Delta\subseteq(0,\infty)\times(0,\infty)\to\mathbb{R}$ be a twice partially differentiable mapping on Δ° and $[a,b]\times[c,d]\subseteq\Delta^\circ$ with $a<b$ and $c<d$. If $h:[a,b]\times[c,d]\to[0,\infty)$ is a twice partially differentiable mapping such that $f_{ts}\in L([a,b]\times[c,d])$ and $|f_{ts}|^q$ is GA-convex on the co-ordinates on $[a,b]\times[c,d]$ for $q\geq1$, then we get hands on:*

$$\begin{aligned}|\Phi(a,b,c,d;f,h)|&\leq\left(\frac{1}{4}\right)^{\frac{1}{q}+1}(\ln b-\ln a)(\ln d-\ln c)\|h\|_\infty\\&\times\left\{\left[\zeta\left(a,b;0,\frac{1}{2}\right)\zeta\left(c,d;0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_1(a,b,c,d;q)\right.\\&+\left[\zeta\left(a,b;0,-\frac{1}{2}\right)\zeta\left(c,d;0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_2(a,b,c,d;q)\\&+\left[\zeta\left(a,b;0,\frac{1}{2}\right)\zeta\left(c,d;0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_3(a,b,c,d;q)\\&\left.+\left[\zeta\left(a,b;0,-\frac{1}{2}\right)\zeta\left(c,d;0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_4(a,b,c,d;q)\right\},\qquad(2.6)\end{aligned}$$

where $\|h\|_\infty=\sup_{(x,y)\in[a,b]\times[c,d]}h(x,y)$ and $\zeta(u,v;k,\eta)$ is defined in Lemma 2.

Proof. By virtue of Lemma 1, we have

$$\begin{aligned}&|\Phi(a,b,c,d;f,h)|\\&\leq\frac{(\ln b-\ln a)(\ln d-\ln c)\|h\|_\infty}{4}\left[\int_0^1\int_0^1L_1(t)L_2(s)|f_{ts}(L_1(t),L_2(s))|\,dsdt\right.\\&\quad+\int_0^1\int_0^1U_1(t)L_2(s)|f_{ts}(U_1(t),L_2(s))|\,dsdt\\&\quad+\int_0^1\int_0^1L_1(t)U_2(s)|f_{ts}(L_1(t),U_2(s))|\,dsdt\\&\quad\left.+\int_0^1\int_0^1U_1(t)U_2(s)|f_{ts}(U_1(t),U_2(s))|\,dsdt\right].\qquad(2.7)\end{aligned}$$

Now by using Hölder's inequality for double integrals and by the GA-convexity of $|f_{ts}|^q$ on the co-ordinates on $[a,b]\times[c,d]$ for $q\geq1$, we acquire

$$\begin{aligned}&\int_0^1\int_0^1L_1(t)L_2(s)|f_{ts}(L_1(t),L_2(s))|\,dsdt\\&\leq\left(\int_0^1\int_0^1L_1(t)L_2(s)\,dsdt\right)^{1-\frac{1}{q}}\left(\int_0^1\int_0^1L_1(t)L_2(s)|f_{ts}(L_1(t),L_2(s))|^q\,dsdt\right)^{\frac{1}{q}}\\&\leq\left(\frac{1}{4}\right)^{\frac{1}{q}}\left[\zeta\left(a,b;0,\frac{1}{2}\right)\zeta\left(c,d;0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\left[\zeta\left(a,b;-1,\frac{1}{2}\right)\zeta\left(c,d;-1,\frac{1}{2}\right)\right.\\&\quad\times|f_{ts}(a,c)|^q+\zeta\left(a,b;-1,\frac{1}{2}\right)\zeta\left(c,d;1,\frac{1}{2}\right)|f_{ts}(a,d)|^q+\zeta\left(a,b;1,\frac{1}{2}\right)\\&\quad\left.\times\zeta\left(c,d;-1,\frac{1}{2}\right)|f_{ts}(b,c)|^q+\zeta\left(a,b;1,\frac{1}{2}\right)\zeta\left(c,d;1,\frac{1}{2}\right)|f_{ts}(b,d)|^q\right]^{\frac{1}{q}}.\end{aligned}$$

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Correspondingly

$$\begin{aligned}
&\int_0^1\int_0^1 U_1(t)\,L_2(s)\,|f_{ts}(U_1(t),L_2(s))|\,dsdt \\
&\leq \left(\frac{1}{4}\right)^{\frac{1}{q}}\left[\zeta\left(a,b;0,-\frac{1}{2}\right)\zeta\left(c,d;0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\left[\zeta\left(a,b;1,-\frac{1}{2}\right)\zeta\left(c,d;-1,\frac{1}{2}\right)\right. \\
&\quad\times|f_{ts}(a,c)|^q+\zeta\left(a,b;1,-\frac{1}{2}\right)\zeta\left(c,d;1,\frac{1}{2}\right)|f_{ts}(a,d)|^q+\zeta\left(a,b;-1,-\frac{1}{2}\right) \\
&\quad\left.\times\zeta\left(c,d;-1,\frac{1}{2}\right)|f_{ts}(b,c)|^q+\zeta\left(a,b;-1,-\frac{1}{2}\right)\zeta\left(c,d;1,\frac{1}{2}\right)|f_{ts}(b,d)|^q\right]^{\frac{1}{q}},
\end{aligned}$$

$$\begin{aligned}
&\int_0^1\int_0^1 L_1(t)\,U_2(s)\,|f_{ts}(L_1(t),U_2(s))|\,dsdt \\
&\leq \left(\frac{1}{4}\right)^{\frac{1}{q}}\left[\zeta\left(a,b;0,\frac{1}{2}\right)\zeta\left(c,d;0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\left[\zeta\left(a,b;-1,\frac{1}{2}\right)\zeta\left(c,d;1,-\frac{1}{2}\right)\right. \\
&\quad\times|f_{ts}(a,c)|^q+\zeta\left(a,b;-1,\frac{1}{2}\right)\zeta\left(c,d;-1,-\frac{1}{2}\right)|f_{ts}(a,d)|^q+\zeta\left(a,b;1,\frac{1}{2}\right) \\
&\quad\left.\times\zeta\left(c,d;1,-\frac{1}{2}\right)|f_{ts}(b,c)|^q+\zeta\left(a,b;1,\frac{1}{2}\right)\zeta\left(c,d;-1,-\frac{1}{2}\right)|f_{ts}(b,d)|^q\right]^{\frac{1}{q}},
\end{aligned}$$

by similar argument

$$\begin{aligned}
&\int_0^1\int_0^1 U_1(t)\,U_2(s)\,|f_{ts}(U_1(t),U_2(s))|\,dsdt \\
&\leq \left(\frac{1}{4}\right)^{\frac{1}{q}}\left[\zeta\left(a,b;0,-\frac{1}{2}\right)\zeta\left(c,d;0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\left[\zeta\left(a,b;1,-\frac{1}{2}\right)\zeta\left(c,d;1,-\frac{1}{2}\right)\right. \\
&\quad\times|f_{ts}(a,c)|^q+\zeta\left(a,b;1,-\frac{1}{2}\right)\zeta\left(c,d;-1,-\frac{1}{2}\right)|f_{ts}(a,d)|^q+\zeta\left(a,b;-1,-\frac{1}{2}\right) \\
&\quad\left.\times\zeta\left(c,d;1,-\frac{1}{2}\right)|f_{ts}(b,c)|^q+\zeta\left(a,b;-1,-\frac{1}{2}\right)\zeta\left(c,d;-1,-\frac{1}{2}\right)|f_{ts}(b,d)|^q\right]^{\frac{1}{q}}.
\end{aligned}$$

Using the above four inequalities in (2.7) and by resolution, it reveals (2.6) and proof is completed. □

Corollary 1. *Suppose the assumptions of Theorem 1 are met and if $q=1$,then*

$$\begin{aligned}
|\Phi(a,b,c,d;f,h)| &\leq \frac{(\ln b-\ln a)(\ln d-\ln c)}{16}\|h\|_\infty \\
&\quad\times\{\sigma_1(a,b,c,d;1)+\sigma_2(a,b,c,d;1)+\sigma_3(a,b,c,d;1)+\sigma_4(a,b,c,d;1)\}. \qquad (2.8)
\end{aligned}$$

Corollary 2. *If we consider* $h(x,y)=\frac{1}{(\ln b-\ln a)(\ln d-\ln c)}$, $(x,y)\in[a,b]\times[c,d]$ *in Theorem 1, then*

$$\begin{aligned}
&\left|\Phi\left(a,b,c,d;f,\frac{1}{(\ln b-\ln a)(\ln d-\ln c)}\right)\right|\\
&\le\left(\frac{1}{4}\right)^{\frac{1}{q}+1}\left\{\left[\zeta\left(a,b;0,\frac{1}{2}\right)\zeta\left(c,d;0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_1(a,b,c,d;q)\right.\\
&\quad+\left[\zeta\left(a,b;0,-\frac{1}{2}\right)\zeta\left(c,d;0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_2(a,b,c,d;q)\\
&\quad+\left[\zeta\left(a,b;0,\frac{1}{2}\right)\zeta\left(c,d;0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_3(a,b,c,d;q)\\
&\quad\left.+\left[\zeta\left(a,b;0,-\frac{1}{2}\right)\zeta\left(c,d;0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_4(a,b,c,d;q)\right\}. \qquad (2.9)
\end{aligned}$$

Theorem 2. *Suppose* $f:\Delta\subseteq(0,\infty)\times(0,\infty)\to\mathbb{R}$ *be a twice partially differentiable mapping on* Δ° *and* $[a,b]\times[c,d]\subseteq\Delta^\circ$ *with* $a<b$ *and* $c<d$. *Further let* $h:[a,b]\times[c,d]\to[0,\infty)$ *be a twice partially differentiable mapping. If* $f_{ts}\in L([a,b]\times[c,d])$ *and* $|f_{ts}|^q$ *is GA-convex on the co-ordinates on* $[a,b]\times[c,d]$ *for* $q>1$, *then we have inequality of the form:*

$$\begin{aligned}
&|\Phi(a,b,c,d;f,h)|\le\left(\frac{1}{4}\right)^{1+\frac{1}{q}}(\ln b-\ln a)(\ln d-\ln c)\|h\|_\infty\\
&\times\left\{\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_1(1,1,1,1;q)\right.\\
&+\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_2(1,1,1,1;q)\\
&+\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_3(1,1,1,1;q)\\
&\left.+\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_4(1,1,1,1;q)\right\}, \qquad (2.10)
\end{aligned}$$

where $\|h\|_\infty=\sup_{(x,y)\in[a,b]\times[c,d]}h(x,y)$ *and* $\zeta(u,v;k,\eta)$ *is defined in Lemma 2.*

Proof. From Lemma 1, we may write

$$\begin{aligned}
&|\Phi(a,b,c,d;f,h)|\\
&\le\frac{(\ln b-\ln a)(\ln d-\ln c)\|h\|_\infty}{4}\left[\int_0^1\int_0^1 L_1(t)L_2(s)|f_{ts}(L_1(t),L_2(s))|\,dsdt\right.\\
&\quad+\int_0^1\int_0^1 U_1(t)L_2(s)|f_{ts}(U_1(t),L_2(s))|\,dsdt\\
&\quad+\int_0^1\int_0^1 L_1(t)U_2(s)|f_{ts}(L_1(t),U_2(s))|\,dsdt\\
&\quad\left.+\int_0^1\int_0^1 U_1(t)U_2(s)|f_{ts}(U_1(t),U_2(s))|\,dsdt\right]. \qquad (2.11)
\end{aligned}$$

Now by using Hölder's inequality for double integrals, Lemma 2 and by the GA-convexity of $|f_{ts}|^q$ on the co-ordinates on $[a,b]\times[c,d]$ for $q>1$, consequently we

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have

$$\begin{aligned}
&\int_0^1\int_0^1 L_1(t)\,L_2(s)\,|f_{ts}(L_1(t),L_2(s))|\,dsdt\\
&\le\left[\int_0^1\int_0^1 (L_1(t)\,L_2(s))^{\frac{q}{q-1}}\,dsdt\right]^{1-\frac{1}{q}}\left[\int_0^1\int_0^1 |f_{ts}(L_1(t),L_2(s))|^q\,dsdt\right]^{\frac{1}{q}}\\
&\qquad\le\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\\
&\qquad\times\left[\frac{9}{16}|f_{ts}(a,c)|^q+\frac{3}{16}|f_{ts}(a,d)|^q+\frac{3}{16}|f_{ts}(b,c)|^q+\frac{1}{16}|f_{ts}(b,d)|^q\right]^{\frac{1}{q}}.
\end{aligned}$$

In addition

$$\begin{aligned}
&\int_0^1\int_0^1 U_1(t)\,L_2(s)\,|f_{ts}(U_1(t),L_2(s))|\,dsdt\\
&\le\left[\int_0^1\int_0^1 (U_1(t)\,L_2(s))^{\frac{q}{q-1}}\,dsdt\right]^{1-\frac{1}{q}}\left[\int_0^1\int_0^1 |f_{ts}(U_1(t),L_2(s))|^q\,dsdt\right]^{\frac{1}{q}}\\
&\qquad\le\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\\
&\qquad\times\left[\frac{3}{16}|f_{ts}(a,c)|^q+\frac{1}{16}|f_{ts}(a,d)|^q+\frac{9}{16}|f_{ts}(b,c)|^q+\frac{3}{16}|f_{ts}(b,d)|^q\right]^{\frac{1}{q}},
\end{aligned}$$

$$\begin{aligned}
&\int_0^1\int_0^1 L_1(t)\,U_2(s)\,|f_{ts}(L_1(t),U_2(s))|\,dsdt\\
&\le\left[\int_0^1\int_0^1 (L_1(t)\,U_2(s))^{\frac{q}{q-1}}\,dsdt\right]^{1-\frac{1}{q}}\left[\int_0^1\int_0^1 |f_{ts}(L_1(t),U_2(s))|^q\,dsdt\right]^{\frac{1}{q}}\\
&\qquad\le\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\\
&\qquad\times\left[\frac{3}{16}|f_{ts}(a,c)|^q+\frac{9}{16}|f_{ts}(a,d)|^q+\frac{1}{16}|f_{ts}(b,c)|^q+\frac{3}{16}|f_{ts}(b,d)|^q\right]^{\frac{1}{q}},
\end{aligned}$$

equivalently

$$\begin{aligned}
&\int_0^1\int_0^1 U_1(t)\,U_2(s)\,|f_{ts}(U_1(t),U_2(s))|\,dsdt\\
&\le\left[\int_0^1\int_0^1 (U_1(t)\,U_2(s))^{\frac{q}{q-1}}\,dsdt\right]^{1-\frac{1}{q}}\left[\int_0^1\int_0^1 |f_{ts}(U_1(t),U_2(s))|^q\,dsdt\right]^{\frac{1}{q}}\\
&\qquad\le\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\\
&\qquad\times\left[\frac{1}{16}|f_{ts}(a,c)|^q+\frac{3}{16}|f_{ts}(a,d)|^q+\frac{3}{16}|f_{ts}(b,c)|^q+\frac{9}{16}|f_{ts}(b,d)|^q\right]^{\frac{1}{q}}.
\end{aligned}$$

Using the above four inequalities in (2.11) and simplifying, we get the required inequality (2.10). □

Corollary 3. *If we take* $h(x,y) = \frac{1}{(\ln b - \ln a)(\ln d - \ln c)}$, $(x,y) \in [a,b] \times [c,d]$ *in Theorem 2, then*

$$\left|\Phi\left(a,b,c,d;f,\frac{1}{(\ln b-\ln a)(\ln d-\ln c)}\right)\right|$$
$$\leq \left(\frac{1}{4}\right)^{1+\frac{1}{q}}\left\{\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_1(1,1,1,1;q)\right.$$
$$+\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_2(1,1,1,1;q)$$
$$+\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_3(1,1,1,1;q)$$
$$\left.+\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_4(1,1,1,1;q)\right\}. \quad (2.12)$$

We shall use the following notation for the next theorem and its related corollary.

$$\Delta_1(a,b,c,d;q) = (\theta(q))^{\frac{2}{q}}|f_{ts}(a,c)|^q + (\theta(q))^{\frac{1}{q}}|f_{ts}(a,d)|^q + (\theta(q))^{\frac{1}{q}}|f_{ts}(b,c)|^q + |f_{ts}(b,d)|^q,$$

$$\Delta_2(a,b,c,d;q) = (\theta(q))^{\frac{1}{q}}|f_{ts}(a,c)|^q + |f_{ts}(a,d)|^q + (\theta(q))^{\frac{2}{q}}|f_{ts}(b,c)|^q + (\theta(q))^{\frac{1}{q}}|f_{ts}(b,d)|^q,$$

$$\Delta_3(a,b,c,d;q) = (\theta(q))^{\frac{1}{q}}|f_{ts}(a,c)|^q + (\theta(q))^{\frac{2}{q}}|f_{ts}(a,d)|^q + |f_{ts}(b,c)|^q + (\theta(q))^{\frac{1}{q}}|f_{ts}(b,d)|^q$$

and

$$\Delta_4(a,b,c,d;q) = |f_{ts}(a,c)|^q + (\theta(q))^{\frac{1}{q}}|f_{ts}(a,d)|^q + (\theta(q))^{\frac{1}{q}}|f_{ts}(b,c)|^q + (\theta(q))^{\frac{2}{q}}|f_{ts}(b,d)|^q,$$

where $\theta(q) = 2^{q+1} - 1$.

Theorem 3. *Let* $f : \Delta \subseteq (0,\infty) \times (0,\infty) \to \mathbb{R}$ *be a twice partially differentiable mapping on* Δ° *and* $[a,b] \times [c,d] \subseteq \Delta^\circ$ *with* $a < b$ *and* $c < d$. *Further let* $h : [a,b] \times [c,d] \to [0,\infty)$ *is a twice partially differentiable mapping. If* $f_{ts} \in L([a,b] \times [c,d])$ *and* $|f_{ts}|^q$ *is GA-convex on the co-ordinates on* $[a,b] \times [c,d]$ *for* $q > 1$, *then the following inequality holds:*

$$|\Phi(a,b,c,d;f,h)| \leq \frac{(\ln b-\ln a)(\ln d-\ln c)\|h\|_\infty}{16}\left(\frac{1}{q+1}\right)^{2/q}$$
$$\times\left\{\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\Delta_1(a,b,c,d;q)\right.$$
$$+\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\Delta_2(a,b,c,d;q)$$
$$+\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\Delta_3(a,b,c,d;q)$$
$$\left.+\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\Delta_4(a,b,c,d;q)\right\}, \quad (2.13)$$

where $\|h\|_{\infty}=\sup_{(x,y)\in[a,b]\times[c,d]} h(x,y)$, $\zeta(u,v;k,\eta)$ *is defined in Lemma 2.*

Proof. From Lemma 1, we have

$$\begin{aligned}|\Phi(a,b,c,d;f,h)| \leq{}& \frac{(\ln b-\ln a)(\ln d-\ln c)\|h\|_{\infty}}{4}\\ &\left[\int_0^1\int_0^1 L_1(t)L_2(s)|f_{ts}(L_1(t),L_2(s))|\,dsdt\right.\\ &+\int_0^1\int_0^1 U_1(t)L_2(s)|f_{ts}(U_1(t),L_2(s))|\,dsdt\\ &+\int_0^1\int_0^1 L_1(t)U_2(s)|f_{ts}(L_1(t),U_2(s))|\,dsdt\\ &\left.+\int_0^1\int_0^1 U_1(t)U_2(s)|f_{ts}(U_1(t),U_2(s))|\,dsdt\right]. \quad (2.14)\end{aligned}$$

Now by using the GA-convexity of $|f_{ts}|^q$ on the co-ordinates on $[a,b]\times[c,d]$ for $q>1$, Lemma 2 together with the Hölder's inequality for double integrals, we have

$$\begin{aligned}&\int_0^1\int_0^1 L_1(t)L_2(s)|f_{ts}(L_1(t),L_2(s))|\,dsdt\\ &\leq \int_0^1\int_0^1 (L_1(t)L_2(s))\left[\left(\frac{1+t}{2}\right)\left(\frac{1+s}{2}\right)|f_{ts}(a,c)|+\left(\frac{1+t}{2}\right)\right.\\ &\left.\left(\frac{1-s}{2}\right)|f_{ts}(a,d)|+\left(\frac{1-t}{2}\right)\left(\frac{1+s}{2}\right)|f_{ts}(b,c)|+\left(\frac{1-t}{2}\right)\left(\frac{1-s}{2}\right)|f_{ts}(b,d)|\right]\\ &\leq\left[\int_0^1\int_0^1 (L_1(t)L_2(s))^{\frac{q}{q-1}}\,dsdt\right]^{1-\frac{1}{q}}\left\{\left[\int_0^1\int_0^1\left(\frac{1+t}{2}\right)^q\left(\frac{1+s}{2}\right)^q dsdt\right]^{\frac{1}{q}}|f_{ts}(a,c)|\right.\\ &+\left[\int_0^1\int_0^1\left(\frac{1+t}{2}\right)^q\left(\frac{1-s}{2}\right)^q dsdt\right]^{\frac{1}{q}}|f_{ts}(a,d)|+\left[\int_0^1\int_0^1\left(\frac{1-t}{2}\right)^q\left(\frac{1+s}{2}\right)^q dsdt\right]^{\frac{1}{q}}\\ &\left.\times|f_{ts}(b,c)|+\left[\int_0^1\int_0^1\left(\frac{1-t}{2}\right)^q\left(\frac{1-s}{2}\right)^q dsdt\right]^{\frac{1}{q}}|f_{ts}(b,d)|\right\}\\ &=\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\left[\frac{1}{2^q(q+1)}\right]^{2/q}\left[\left(2^{q+1}-1\right)^{2/q}\right.\\ &\left.\times|f_{ts}(a,c)|^q+\left(2^{q+1}-1\right)^{1/q}|f_{ts}(a,d)|^q+\left(2^{q+1}-1\right)^{1/q}|f_{ts}(b,c)|^q+|f_{ts}(b,d)|^q\right].\end{aligned}$$

Likewise, we have

$$\begin{aligned}&\int_0^1\int_0^1 U_1(t)L_2(s)|f_{ts}(U_1(t),L_2(s))|\,dsdt\\ &\leq\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\\ &\times\left[\frac{1}{2^q(q+1)}\right]^{2/q}\left[\left(2^{q+1}-1\right)^{1/q}|f_{ts}(a,c)|^q+|f_{ts}(a,d)|^q\right.\\ &\left.+\left(2^{q+1}-1\right)^{2/q}|f_{ts}(b,c)|^q+\left(2^{q+1}-1\right)^{1/q}|f_{ts}(b,d)|^q\right],\end{aligned}$$

$$\int_0^1\int_0^1 L_1(t)\,U_2(s)\,|f_{ts}(L_1(t),U_2(s))|\,dsdt$$
$$\leq\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}$$
$$\times\left[\frac{1}{2^q(q+1)}\right]^{2/q}\left[\left(2^{q+1}-1\right)^{1/q}|f_{ts}(a,c)|^q+\left(2^{q+1}-1\right)^{1/q}|f_{ts}(a,d)|^q\right.$$
$$\left.+|f_{ts}(b,c)|^q+\left(2^{q+1}-1\right)^{1/q}|f_{ts}(b,d)|^q\right],$$

and

$$\int_0^1\int_0^1 U_1(t)\,U_2(s)\,|f_{ts}(U_1(t),U_2(s))|\,dsdt$$
$$\leq\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}$$
$$\times\left[\frac{1}{2^q(q+1)}\right]^{2/q}\left[\left(2^{q+1}-1\right)^{1/q}|f_{ts}(a,c)|^q+\left(2^{q+1}-1\right)^{1/q}|f_{ts}(a,d)|^q\right.$$
$$\left.+|f_{ts}(b,c)|^q+\left(2^{q+1}-1\right)^{1/q}|f_{ts}(b,d)|^q\right].$$

Further employing the above four inequalities in (2.14) and after simplification, we built up the required inequality (2.13). □

Corollary 4. *If we take $h(x,y)=\frac{1}{(\ln b-\ln a)(\ln d-\ln c)}$, $(x,y)\in[a,b]\times[c,d]$ in Theorem 3,then*

$$\left|\Phi\left(a,b,c,d;f,\frac{1}{(\ln b-\ln a)(\ln d-\ln c)}\right)\right|\leq\frac{1}{16}\left(\frac{1}{q+1}\right)^{2/q}$$
$$\left\{\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\Delta_1(a,b,c,d;q)\right.$$
$$+\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\Delta_2(a,b,c,d;q)$$
$$+\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\Delta_3(a,b,c,d;q)$$
$$\left.+\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\Delta_3(a,b,c,d;q)\right\},\quad(2.15)$$

where $\zeta(u,v;k,\eta)$ is defined in Lemma 2 and $\theta(q)=2^{q+1}-1$.

Theorem 4. *Let $f:\Delta\subseteq(0,\infty)\times(0,\infty)\to\mathbb{R}$ be a twice partially differentiable mapping on Δ° and $[a,b]\times[c,d]\subseteq\Delta^\circ$ with $a<b$ and $c<d$. Further let $h:[a,b]\times[c,d]\to[0,\infty)$ is a twice partially differentiable mapping. If $f_{ts}\in L([a,b]\times[c,d])$ and $|f_{ts}|^q$ is GA-convex on the co-ordinates on $[a,b]\times[c,d]$ for $q>1$ and $q\geq r\geq 0$,* *then we attain the following inequality:*

$$\begin{aligned}
&\left|\Phi\left(a,b,c,d;f,h\right)\right| \leq \left(\frac{1}{4}\right)^{\frac{1}{q}+1}\left(\ln b-\ln a\right)\left(\ln d-\ln c\right)\left\|h\right\|_{\infty} \\
&\quad\times\left\{\left[\zeta\left(a^{\frac{q-r}{q-1}},b^{\frac{q-r}{q-1}};0,\frac{1}{2}\right)\zeta\left(c^{\frac{q-r}{q-1}},d^{\frac{q-r}{q-1}};0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_1\left(a^r,b^r,c^r,d^r;q\right)\right. \\
&\quad+\left[\zeta\left(a^{\frac{q-r}{q-1}},b^{\frac{q-r}{q-1}};0,-\frac{1}{2}\right)\zeta\left(c^{\frac{q-r}{q-1}},d^{\frac{q-r}{q-1}};0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_2\left(a^r,b^r,c^r,d^r;q\right) \\
&\quad+\left[\zeta\left(a^{\frac{q-r}{q-1}},b^{\frac{q-r}{q-1}};0,\frac{1}{2}\right)\zeta\left(c^{\frac{q-r}{q-1}},d^{\frac{q-r}{q-1}};0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_3\left(a^r,b^r,c^r,d^r;q\right) \\
&\quad\left.+\left[\zeta\left(a^{\frac{q-r}{q-1}},b^{\frac{q-r}{q-1}};0,-\frac{1}{2}\right)\zeta\left(c^{\frac{q-r}{q-1}},d^{\frac{q-r}{q-1}};0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_4\left(a^r,b^r,c^r,d^r;q\right)\right\},
\end{aligned} \tag{2.16}$$

where $\left\|h\right\|_{\infty}=\sup_{(x,y)\in[a,b]\times[c,d]}h\left(x,y\right)$ *and* $\zeta\left(u,v;k,\eta\right)$ *is defined in Lemma 2.*

Proof. From Lemma 1, it follows that

$$\begin{aligned}
&\left|\Phi\left(a,b,c,d;f,h\right)\right| \\
&\leq \frac{\left(\ln b-\ln a\right)\left(\ln d-\ln c\right)\left\|h\right\|_{\infty}}{4}\left[\int_0^1\int_0^1 L_1\left(t\right)L_2\left(s\right)\left|f_{ts}\left(L_1\left(t\right),L_2\left(s\right)\right)\right|dsdt\right. \\
&\qquad+\int_0^1\int_0^1 U_1\left(t\right)L_2\left(s\right)\left|f_{ts}\left(U_1\left(t\right),L_2\left(s\right)\right)\right|dsdt \\
&\qquad+\int_0^1\int_0^1 L_1\left(t\right)U_2\left(s\right)\left|f_{ts}\left(L_1\left(t\right),U_2\left(s\right)\right)\right|dsdt \\
&\qquad\left.+\int_0^1\int_0^1 U_1\left(t\right)U_2\left(s\right)\left|f_{ts}\left(U_1\left(t\right),U_2\left(s\right)\right)\right|dsdt\right].
\end{aligned} \tag{2.17}$$

Now by virtue of GA-convexity of $\left|f_{ts}\right|^q$ on the co-ordinates on $[a,b]\times[c,d]$ for $q>1$, Lemma 2 and by the Hölder's inequality for double integrals, we have in hand

$$\begin{aligned}
&\int_0^1\int_0^1 L_1\left(t\right)L_2\left(s\right)\left|f_{ts}\left(L_1\left(t\right),L_2\left(s\right)\right)\right|dsdt \leq \left(\int_0^1\int_0^1\left(L_1\left(t\right)L_2\left(s\right)\right)^{\frac{q-r}{q-1}}dsdt\right)^{1-\frac{1}{q}} \\
&\qquad\times\left(\int_0^1\int_0^1\left(L_1\left(t\right)L_2\left(s\right)\right)^r\left|f_{ts}\left(L_1\left(t\right),L_2\left(s\right)\right)\right|^q dsdt\right)^{\frac{1}{q}} \\
&\leq\left(\frac{1}{4}\right)^{\frac{1}{q}}\left[\zeta\left(a^{\frac{q-r}{q-1}},b^{\frac{q-r}{q-1}};0,\frac{1}{2}\right)\zeta\left(c^{\frac{q-r}{q-1}},d^{\frac{q-r}{q-1}};0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_1\left(a^r,b^r,c^r,d^r;q\right)
\end{aligned}$$

Similarly

$$\begin{aligned}
&\int_0^1\int_0^1 U_1\left(t\right)L_2\left(s\right)\left|f_{ts}\left(U_1\left(t\right),L_2\left(s\right)\right)\right|dsdt \leq \left(\int_0^1\int_0^1\left(U_1\left(t\right)L_2\left(s\right)\right)^{\frac{q-r}{q-1}}dsdt\right)^{1-\frac{1}{q}} \\
&\qquad\times\left(\int_0^1\int_0^1\left(U_1\left(t\right)L_2\left(s\right)\right)^r\left|f_{ts}\left(U_1\left(t\right),L_2\left(s\right)\right)\right|^q dsdt\right)^{\frac{1}{q}} \\
&\leq\left(\frac{1}{4}\right)^{\frac{1}{q}}\left[\zeta\left(a^{\frac{q-r}{q-1}},b^{\frac{q-r}{q-1}};0,-\frac{1}{2}\right)\zeta\left(c^{\frac{q-r}{q-1}},d^{\frac{q-r}{q-1}};0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_2\left(a^r,b^r,c^r,d^r;q\right)
\end{aligned}$$

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$$\int_0^1\int_0^1 L_1(t)\,U_2(s)\left|f_{ts}\left(L_1(t),U_2(s)\right)\right|dsdt \leq \left(\int_0^1\int_0^1 \left(L_1(t)\,U_2(s)\right)^{\frac{q-r}{q-q}}dsdt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\int_0^1 \left(L_1(t)\,U_2(s)\right)^r\left|f_{ts}\left(L_1(t),U_2(s)\right)\right|^q dsdt\right)^{\frac{1}{q}}$$
$$\leq\left(\frac{1}{4}\right)^{\frac{1}{q}}\left[\zeta\left(a^{\frac{q-r}{q-1}},b^{\frac{q-r}{q-1}};0,\frac{1}{2}\right)\zeta\left(c^{\frac{q-r}{q-1}},d^{\frac{q-r}{q-1}};0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_3\left(a^r,b^r,c^r,d^r;q\right)$$

and

$$\int_0^1\int_0^1 U_1(t)\,U_2(s)\left|f_{ts}\left(U_1(t),U_2(s)\right)\right|dsdt \leq \left(\int_0^1\int_0^1 \left(U_1(t)\,U_2(s)\right)^{\frac{q-r}{q-1}}dsdt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\int_0^1 \left(U_1(t)\,U_2(s)\right)^r\left|f_{ts}\left(U_1(t),U_2(s)\right)\right|^q dsdt\right)^{\frac{1}{q}}$$
$$\leq\left(\frac{1}{4}\right)^{\frac{1}{q}}\left[\zeta\left(a^{\frac{q-r}{q-1}},b^{\frac{q-r}{q-1}};0,-\frac{1}{2}\right)\zeta\left(c^{\frac{q-r}{q-1}},d^{\frac{q-r}{q-1}};0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_4\left(a^r,b^r,c^r,d^r;q\right)$$

Using the above four inequalities in (2.17) and simplifying, we obtained the required inequality (2.16). □

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