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Journal of Computational Analysis and Applications

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Approximation reduction for multi-granulation dual hesitant fuzzy rough sets

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Abstract

Multi-granulation dual hesitant fuzzy rough set is an extension of intuitionistic fuzzy multi-granulation rough sets and multi-granulation fuzzy rough sets. For further studying the theories and applications of multi-granulation dual hesitant fuzzy rough sets, in this paper, we mainly investigate reduction approaches of the multi-granulation dual hesitant fuzzy rough sets. We develop a reduction approach in multi-granulation dual hesitant fuzzy decision information systems based on multi-granulation dual h-esitant fuzzy rough sets to eliminate redundant dual hesitant fuzzy granulations. And an example is provided to illustrate the validity of this approach.

Key words: Multi-granulation fuzzy rough set; Multi-granulation dual hesitant fuzzy rough set; Reduction approach

1 Introduction

Rough set theory, introduced by Pawlak [19, 20], is a new mathematical approach to cope with imprecision and uncertainty in data analysis, and can be regarded as a valid means of granular computing [21]. In Pawlak's rough set model, a key notion is equivalence

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relation. However, the equivalence relation is a very stringent condition which may limit the application of rough sets. Therefore, by replacing the equivalence relation with other binary relations, such as fuzzy, interval-valued fuzzy, intuitionistic fuzzy, hesitant fuzzy and interval-valued hesitant fuzzy, and so on, lots of researchers have proposed many new rough sets model. For example, Dubois and Prade [4] initiated two rough set models which are called rough fuzzy sets and fuzzy rough sets. Furthermore, Wu et al. [31, 32] studied various generalized fuzzy approximation operators which are characterized by different sets of axioms. According to fuzzy rough sets in the sense of Nanda and Majumda [18], Jena and Ghosh [8] presented the concept of intuitionistic fuzzy rough sets which are not defined by an approximation space. By using a special type of intuitionistic fuzzy triangular norm *min*, Zhou and Wu [48] discussed various relation-based intuitionistic fuzzy rough approximation operators. Meanwhile, they [49] also investigated intuitionistic fuzzy rough approximations on one universe based on intuitionistic fuzzy implicators. In [46], Zhang et al. proposed a generalized interval-valued fuzzy rough set and applied it to decision making. Very recently, rough set theory has been developed into hesitant fuzzy environment and interval-valued hesitant fuzzy environment, and the results are, respectively, called hesitant fuzzy rough sets [41] and interval-valued hesitant fuzzy rough sets [45].

The generalization of Pawlak's rough set model has become a new research hotspot from the perspective of granular computing. Since Qian et al. [22] proposed multi-granulation rough set (MGRS) theory, lots of fruitful results about MGRS theory have been achieved. Qian et al. [23] proposed an incomplete multi-granulation rough set model by using multiple tolerance relations on the universe, and studied decision-theoretic rough sets based on multi-granulations [25]. She et al. [27] investigated topological structures of MGRSs. Yang et al. [42] extended Qian's MGRS model to fuzzy environment and explored a MGRS based on fuzzy relations. Along the lines of Qian's MGRSs, Xu et al. [36] initiated an ordered MGRS model. And they [34, 35] also proposed multi-granulation fuzzy rough sets based on multiple classical equivalence relations and multi-granulation fuzzy rough sets in a fuzzy tolerance approximation space. Through combining MGRSs and intuitionistic fuzzy rough sets, Huang et al. [7] proposed intuitionistic fuzzy MGRSs and gave a reduction approach of this model. Liu et al. [9, 10] presented covering fuzzy rough sets based on MGRSs. To handle data sets in the context of hybrid attributes, Lin et al. [11] introduced the neighborhood-based MGRSs, generalized the covering into multi-granulation environment and proposed the covering based on optimistic and pessimistic MGRSs [12]. More recently, Liang et al. [16] presented an efficient rough feature selection algorithm through a multi-granulation view. Yang et al. [43] proposed a test cost sensitive multi-granulation rough set model to take the test cost into consideration in both data mining and machine learning.

As one of the extensions of Zadeh's fuzzy set [50], hesitant fuzzy (HF) set theory, initiated by Torra [28, 29], permits the membership degree of an element to a set having several possible values, and can express the hesitant information more comprehensively than other extensions of fuzzy set. Since the appearance of hesitant fuzzy set, it has attracted more and more scholars' attention. For example, Xu and Xia [33,37,38] discussed the aggregation operators, correlation measures, distance, and similarity measures for HF sets. Meanwhile, Chen et al. [2] gave correlation coefficients of HF sets and applied them to clustering analysis. Subsequently, Liao et al. [15] proposed novel correlation coefficients between hesitant fuzzy sets and and applied them to decision making. In [5], Farhadinia introduced information measures for HF sets. Rodrguez et al. [26] proposed a HF linguistic term set providing a more powerful form to represent decision makers' preferences in the decision making process. Liao and Xu [13,14] developed a hesitant fuzzy VIKOR method based on some new measures, and proposed some new hybrid weighted aggregation operators under hesitant fuzzy multi-criteria decision making environment. Zhang and Wei [51] proposed an extension of VIKOR method based on hesitant fuzzy set in decision making problem.

Dual hesitant fuzzy (DHF) set, introduced by Zhu et al. [44], is a comprehensive set encompassing fuzzy sets, intuitionistic fuzzy sets [1], hesitant fuzzy sets, and fuzzy multisets [17] as special cases. By several possible values for the membership and nonmembership degrees, dual hesitant fuzzy sets are more objective than hesitant fuzzy sets to describe the vagueness of data or information. In recent years, many authors have investigated multiple attribute decision-making theories and methods under the dual hesitant fuzzy environment [3,6,30,40]. Very recently, the combination of dual hesitant fuzzy sets and other uncertainty theories is becoming a research hotspot. For example, by integrating rough set theory with dual hesitant fuzzy sets, Zhang et al. [47] proposed a single-granulation dual hesitant fuzzy rough set (SGDHFRS). Based on the SGDHFRSs, they presented the concept of multi-granulation dual hesitant fuzzy rough sets (MGDHFRSs) in which two types of this model are proposed: one is called the optimistic MGDHFRS; the other is called the pessimistic MGDHFRS. The relationships among the optimistic MGDHFRS, the pessimistic MGDHFRS and the SGDHFRS are then established. However, reduction approaches of the MGDHFRSs are not still be investigated. In order to develop the application of the MGDHFRSs, topological properties and reduction approaches on MGDHFRSs further need to be studied. The objective of this paper is mainly to focus on the study of reduction approaches of the MGDHFRSs.

The rest of the paper is organized as follows. The next section reviews some basic concepts considered in the study, such as HF sets, DHF sets and MGDHFRSs. In Section 3, we propose a reduction approach of MGDHFRSs to eliminate redundant DHF granulations by a numerical example. Finally, we conclude the paper in Section 4.

2 Preliminaries

2.1 Dual hesitant fuzzy sets

As an extension of hesitant fuzzy sets, dual hesitant fuzzy sets are defined by Zhu et al. [44] as follows:

Definition 2.1 ([44]) *Let U be a fixed set, a dual hesitant fuzzy set $\mathbb{D}$ on U is described as:*

$\mathbb{D} = \{< x, h_{\mathbb{D}}(x), g_{\mathbb{D}}(x) > | x \in U\}$,

in which $h_{\mathbb{D}}(x)$ and $g_{\mathbb{D}}(x)$ are two sets of some values in [0,1], denoting the possible membership degrees and non-membership degrees of the element $x \in U$ to the set $\mathbb{D}$ respectively, with the conditions: $0 \leq \gamma, \eta \leq 1$ and $0 \leq \gamma^+ + \eta^+ \leq 1$, where for all $x \in U$, $\gamma \in h_{\mathbb{D}}(x), \eta \in g_{\mathbb{D}}(x)$, $\gamma^+ \in h^+_{\mathbb{D}}(x) = \cup_{\gamma \in h_{\mathbb{D}}(x)} max\{\gamma\}, \eta^+ \in g^+_{\mathbb{D}}(x) = \cup_{\eta \in g_{\mathbb{D}}(x)} max\{\eta\}$.

For convenience, the pair $d(x) = (h_{\mathbb{D}}(x), g_{\mathbb{D}}(x))$ is called a DHF element denoted by $d = (h, g)$. The set of all DHF sets on U is denoted by $DHF(U)$.

2.2 Multi-granulation dual hesitant fuzzy rough sets

In [47], Zhang et al. proposed a SGDHFRS by integrating rough set theory with dual hesitant fuzzy sets. First, they introduced a DHF relation as follows:

Definition 2.2 ([47]) *Let U, V be two nonempty and finite universes. A DHF subset $\mathbb{R}$ of the universe $U \times V$ is called a DHF relation from U to V, namely, $\mathbb{R}$ is given by*

$\mathbb{R} = \{< (x, y), h_{\mathbb{R}}(x, y), g_{\mathbb{R}}(x, y) > | (x, y) \in U \times V\}$,

where $h_{\mathbb{R}}, g_{\mathbb{R}} : U \times V \to [0, 1]$ are two sets of some values in [0,1], denoting the possible membership degrees and non-membership degrees of the relationships between x and y respectively, with the conditions: $0 \leq \gamma, \eta \leq 1$ and $0 \leq \gamma^+ + \eta^+ \leq 1$, where for all $(x, y) \in U \times V$, $\gamma \in h_{\mathbb{R}}(x, y), \eta \in g_{\mathbb{R}}(x, y)$, $\gamma^+ \in h^+_{\mathbb{R}}(x, y) = \cup_{\gamma \in h_{\mathbb{R}}(x,y)} max\{\gamma\}, \eta^+ \in g^+_{\mathbb{R}}(x, y) = \cup_{\eta \in g_{\mathbb{R}}(x,y)} max\{\eta\}$.

In particular, if $U = V$, we call $\mathbb{R}$ a DHF relation on U. In what follows several special DHF relations are introduced as follows:

Definition 2.3 ([47]) *The DHF relation $\mathbb{R}$ from U to V is said to be serial if for each $x \in U$, there exists a $y \in V$ such that $h_{\mathbb{R}}(x, y) = \{1\}$ and $g_{\mathbb{R}}(x, y) = \{0\}$; $\mathbb{R}$ is said to be reflexive on U if $h_{\mathbb{R}}(x, x) = \{1\}$ and $g_{\mathbb{R}}(x, x) = \{0\}$ for all $x \in U$; $\mathbb{R}$ is referred to as a symmetric DHF relation on U if $h_{\mathbb{R}}(x, y) = h_{\mathbb{R}}(y, x)$ and $g_{\mathbb{R}}(x, y) = g_{\mathbb{R}}(y, x)$ for all $x, y \in U$.*

If a DHF relation $\mathbb{R}$ on U is reflexive and symmetric, it is called a DHF tolerance relation on U.

Based on the above DHF relation, lower and upper DHF approximation operators are defined as follows:

Definition 2.4 ([47]) *Let U be a nonempty and finite universes and $\mathbb{R}$ be a DHF tolerance relation on U. The pair $(U, \mathbb{R})$ is called a DHF tolerance approximation space. For any $\mathbb{A} \in DHF(U)$, the lower and upper approximations of $\mathbb{A}$ with respect to $(U, \mathbb{R})$, denoted by $\underline{\mathbb{R}}(\mathbb{A})$ and $\overline{\mathbb{R}}(\mathbb{A})$, are two DHF sets of U and are, respectively, defined as follows:*

$$\underline{\mathbb{R}}(\mathbb{A}) = \{< x, h_{\underline{\mathbb{R}}(\mathbb{A})}(x), g_{\underline{\mathbb{R}}(\mathbb{A})}(x) > | x \in U\}, \tag{1}$$

$$\overline{\mathbb{R}}(\mathbb{A}) = \{< x, h_{\overline{\mathbb{R}}(\mathbb{A})}(x), g_{\overline{\mathbb{R}}(\mathbb{A})}(x) > | x \in U\}, \tag{2}$$

where

$$h_{\underline{\mathbb{R}}(\mathbb{A})}(x) = \overline{\wedge}_{y_j \in U}\{g_{\mathbb{R}}(x, y_j) \veebar h_{\mathbb{A}}(y_j)\},\ g_{\underline{\mathbb{R}}(\mathbb{A})}(x) = \veebar_{y_j \in U}\{h_{\mathbb{R}}(x, y_j) \overline{\wedge} g_{\mathbb{A}}(y_j)\};$$

$$h_{\overline{\mathbb{R}}(\mathbb{A})}(x) = \veebar_{y_j \in U}\{h_{\mathbb{R}}(x, y_j) \overline{\wedge} h_{\mathbb{A}}(y_j)\},\ g_{\overline{\mathbb{R}}(\mathbb{A})}(x) = \overline{\wedge}_{y_j \in U}\{g_{\mathbb{R}}(x, y_j) \veebar g_{\mathbb{A}}(y_j)\}.$$

$\underline{\mathbb{R}}(\mathbb{A})$ and $\overline{\mathbb{R}}(\mathbb{A})$ are, respectively, called the single-granulation lower and upper approximations of $\mathbb{A}$ with respect to $(U, \mathbb{R})$. The pair $(\underline{\mathbb{R}}(\mathbb{A}), \overline{\mathbb{R}}(\mathbb{A}))$ is called a SGDHFRS of $\mathbb{A}$ with respect to $(U, \mathbb{R})$, and $\underline{\mathbb{R}}, \overline{\mathbb{R}} : DHF(U) \to DHF(U)$ are referred to as single granulation lower and upper DHF rough approximation operators, respectively.

Based on the SGDHFRSs, Zhang et al. [47] presented two MGDHFRS models: one is called the optimistic MGDHFRS; the other is called the pessimistic MGDHFRS.

Definition 2.5 ([47]) *Let U be a nonempty and finite universe of discourse and $\mathbb{R}_i(1 \leq i \leq m)$ be m DHF tolerance relations on U; the pair $(U, \{\mathbb{R}_i | 1 \leq i \leq m\})$ is called the DHF tolerance approximation space. For any $\mathbb{A} \in DHF(U)$, the optimistic multi-granulation dual hesitant fuzzy lower and upper approximations of $\mathbb{A}$ with respect to $(U, \{\mathbb{R}_i | 1 \leq i \leq m\})$, denoted by $\underline{\sum_{i=1}^m \mathbb{R}_i}^O(\mathbb{A})$ and $\overline{\sum_{i=1}^m \mathbb{R}_i}^O(\mathbb{A})$, are two DHF sets and are, respectively, defined as follows:*

$$\underline{\sum_{i=1}^{m} \mathbb{R}_i}^O(\mathbb{A}) = \{< x, h_{\underline{\sum_{i=1}^m \mathbb{R}_i}^O(\mathbb{A})}(x), g_{\underline{\sum_{i=1}^m \mathbb{R}_i}^O(\mathbb{A})}(x) > | x \in U\}, \tag{3}$$

$$\overline{\sum_{i=1}^{m} \mathbb{R}_i}^O(\mathbb{A}) = \{< x, h_{\overline{\sum_{i=1}^m \mathbb{R}_i}^O(\mathbb{A})}(x), g_{\overline{\sum_{i=1}^m \mathbb{R}_i}^O(\mathbb{A})}(x) > | x \in U\}, \tag{4}$$

where

$h_{\underline{\sum_{i=1}^m \mathbb{R}_i}^O(\mathbb{A})}(x) = \veebar_{i=1}^m \barwedge_{y_j \in U} \{g_{\mathbb{R}_i}(x, y_j) \veebar h_{\mathbb{A}}(y_j)\}$,
$g_{\underline{\sum_{i=1}^m \mathbb{R}_i}^O(\mathbb{A})}(x) = \barwedge_{i=1}^m \veebar_{y_j \in U} \{h_{\mathbb{R}_i}(x, y_j) \barwedge g_{\mathbb{A}}(y_j)\}$,
$h_{\overline{\sum_{i=1}^m \mathbb{R}_i}^O(\mathbb{A})}(x) = \barwedge_{i=1}^m \veebar_{y_j \in U} \{h_{\mathbb{R}_i}(x, y_j) \barwedge h_{\mathbb{A}}(y_j)\}$,
$g_{\overline{\sum_{i=1}^m \mathbb{R}_i}^O(\mathbb{A})}(x) = \veebar_{i=1}^m \barwedge_{y_j \in U} \{g_{\mathbb{R}_i}(x, y_j) \veebar g_{\mathbb{A}}(y_j)\}$.

The pair $(\underline{\sum_{i=1}^m \mathbb{R}_i}^O(\mathbb{A}), \overline{\sum_{i=1}^m \mathbb{R}_i}^O(\mathbb{A}))$ *is called an optimistic MGDHFRS of* $\mathbb{A}$ *with respect to* $(U, \{\mathbb{R}_i | 1 \leq i \leq m\})$*. If* $\underline{\sum_{i=1}^m \mathbb{R}_i}^O(\mathbb{A}) = \overline{\sum_{i=1}^m \mathbb{R}_i}^O(\mathbb{A})$*, then* $\mathbb{A}$ *is referred to as optimistic-definable in* $(U, \{\mathbb{R}_i | 1 \leq i \leq m\})$*; otherwise,* $\mathbb{A}$ *is referred to as optimistic-undefinable in* $(U, \{\mathbb{R}_i | 1 \leq i \leq m\})$*.*

Definition 2.6 ([47]) *Let* $(U, \{\mathbb{R}_i | 1 \leq i \leq m\})$ *be the DHF tolerance approximation space. For any* $\mathbb{A} \in DHF(U)$*, the pessimistic multi-granulation dual hesitant fuzzy lower and upper approximations of* $\mathbb{A}$ *with respect to* $(U, \{\mathbb{R}_i | 1 \leq i \leq m\})$*, denoted by* $\underline{\sum_{i=1}^m \mathbb{R}_i}^P(\mathbb{A})$ *and* $\overline{\sum_{i=1}^m \mathbb{R}_i}^P(\mathbb{A})$*, are two DHF sets and are, respectively, defined as follows:*

$$\underline{\sum_{i=1}^m \mathbb{R}_i}^P(\mathbb{A}) = \{< x, h_{\underline{\sum_{i=1}^m \mathbb{R}_i}^P(\mathbb{A})}(x), g_{\underline{\sum_{i=1}^m \mathbb{R}_i}^P(\mathbb{A})}(x) > | x \in U\}, \tag{5}$$

$$\overline{\sum_{i=1}^m \mathbb{R}_i}^P(\mathbb{A}) = \{< x, h_{\overline{\sum_{i=1}^m \mathbb{R}_i}^P(\mathbb{A})}(x), g_{\overline{\sum_{i=1}^m \mathbb{R}_i}^P(\mathbb{A})}(x) > | x \in U\}, \tag{6}$$

where
$h_{\underline{\sum_{i=1}^m \mathbb{R}_i}^P(\mathbb{A})}(x) = \barwedge_{i=1}^m \barwedge_{y_j \in U} \{g_{\mathbb{R}_i}(x, y_j) \veebar h_{\mathbb{A}}(y_j)\}$,
$g_{\underline{\sum_{i=1}^m \mathbb{R}_i}^P(\mathbb{A})}(x) = \veebar_{i=1}^m \veebar_{y_j \in U} \{h_{\mathbb{R}_i}(x, y_j) \barwedge g_{\mathbb{A}}(y_j)\}$,
$h_{\overline{\sum_{i=1}^m \mathbb{R}_i}^P(\mathbb{A})}(x) = \veebar_{i=1}^m \veebar_{y_j \in U} \{h_{\mathbb{R}_i}(x, y_j) \barwedge h_{\mathbb{A}}(y_j)\}$,
$g_{\overline{\sum_{i=1}^m \mathbb{R}_i}^P(\mathbb{A})}(x) = \barwedge_{i=1}^m \barwedge_{y_j \in U} \{g_{\mathbb{R}_i}(x, y_j) \veebar g_{\mathbb{A}}(y_j)\}$.

The pair $(\underline{\sum_{i=1}^m \mathbb{R}_i}^P(\mathbb{A}), \overline{\sum_{i=1}^m \mathbb{R}_i}^P(\mathbb{A}))$ *is called a pessimistic MGDHFRS of* $\mathbb{A}$ *with respect to* $(U, \{\mathbb{R}_i | 1 \leq i \leq m\})$*. If* $\underline{\sum_{i=1}^m \mathbb{R}_i}^P(\mathbb{A}) = \overline{\sum_{i=1}^m \mathbb{R}_i}^P(\mathbb{A})$*, then* $\mathbb{A}$ *is referred to as pessimistic-definable in* $(U, \{\mathbb{R}_i | 1 \leq i \leq m\})$*; otherwise,* $\mathbb{A}$ *is referred to as pessimistic-undefinable in* $(U, \{\mathbb{R}_i | 1 \leq i \leq m\})$*.*

Then, Zhang et al. [47] established the relationships among the optimistic MGDHFRS, the pessimistic MGDHFRS and the SGDHFRS.

Theorem 2.7 ([47]) *Let* U *be a nonempty and finite universe of discourse and* $\mathbb{R}_i (1 \leq i \leq m)$ *be* m *DHF tolerance relations on* U*. For any* $\mathbb{A} \in DHF(U)$*,* $\underline{\sum_{i=1}^m \mathbb{R}_i}^O(\mathbb{A})$ *and* $\overline{\sum_{i=1}^m \mathbb{R}_i}^O(\mathbb{A})$ *are the optimistic multi-granulation dual hesitant fuzzy lower and upper approximations of* $\mathbb{A}$ *with respect to* $(U, \{\mathbb{R}_i | 1 \leq i \leq m\})$*, respectively. Then,* *(1)* $\underline{\sum_{i=1}^{m} \mathbb{R}_i}^{O}(\mathbb{A}) = \Cup_{i=1}^{m} \underline{\mathbb{R}_i}(\mathbb{A})$,

(2) $\overline{\sum_{i=1}^{m} \mathbb{R}_i}^{O}(\mathbb{A}) = \Cap_{i=1}^{m} \overline{\mathbb{R}_i}(\mathbb{A})$.

Theorem 2.8 ([47]) *Let U be a nonempty and finite universe of discourse and $\mathbb{R}_i(1 \leq i \leq m)$ be m DHF tolerance relations on U. For any $\mathbb{A} \in DHF(U)$, $\underline{\sum_{i=1}^{m} \mathbb{R}_i}^{P}(\mathbb{A})$ and $\overline{\sum_{i=1}^{m} \mathbb{R}_i}^{P}(\mathbb{A})$ are the pessimistic multi-granulation DHF lower and upper approximations of $\mathbb{A}$ with respect to $(U, \{\mathbb{R}_i | 1 \leq i \leq m\})$, respectively. Then,*

(1) $\underline{\sum_{i=1}^{m} \mathbb{R}_i}^{P}(\mathbb{A}) = \Cap_{i=1}^{m} \underline{\mathbb{R}_i}(\mathbb{A})$,

(2) $\overline{\sum_{i=1}^{m} \mathbb{R}_i}^{P}(\mathbb{A}) = \Cup_{i=1}^{m} \overline{\mathbb{R}_i}(\mathbb{A})$.

3 Approximation reduction approach in multi-granulation DHF decision information system

In this section, we establish a practical reduction approach in multi-granulation DHF decision information system (MGDHFDIS) based on the MGDHFRS model. The objective of reduction is to obtain a smallest subset of DHF relations that may preserve consistence of MGDHFDIS.

Definition 3.1 *Let apr=$(U, \mathbb{R} = \{\mathbb{R}_i | 1 \leq i \leq m\})$ be the DHF tolerance approximation space, $\mathbb{A}$ be the DHF set and $\underline{\mathbb{R}}^{O}, \overline{\mathbb{R}}^{O}, \underline{\mathbb{R}}^{P}, \overline{\mathbb{R}}^{P} \subseteq \mathbb{R}$.*

(1) If $\underline{\sum_{\mathbb{R}_i \in \underline{\mathbb{R}}^{O}} \mathbb{R}_i}^{O}(\mathbb{A}) = \underline{\sum_{i=1}^{m} \mathbb{R}_i}^{O}(\mathbb{A})$, then $\underline{\mathbb{R}}^{O}$ is referred to as a consistent optimistic lower approximation of apr. If $\underline{\mathbb{R}}^{O}$ is a consistent optimistic lower approximation, and no proper subset of $\underline{\mathbb{R}}^{O}$ is a consistent optimistic lower approximation, then $\underline{\mathbb{R}}^{O}$ is called an optimistic lower approximation reduct of apr.

(2) If $\underline{\sum_{\mathbb{R}_i \in \underline{\mathbb{R}}^{P}} \mathbb{R}_i}^{P}(\mathbb{A}) = \underline{\sum_{i=1}^{m} \mathbb{R}_i}^{P}(\mathbb{A})$, then $\underline{\mathbb{R}}^{P}$ is referred to as a consistent pessimistic lower approximation of apr. If $\underline{\mathbb{R}}^{P}$ is a consistent pessimistic lower approximation, and no proper subset of $\underline{\mathbb{R}}^{P}$ is a consistent pessimistic lower approximation, then $\underline{\mathbb{R}}^{P}$ is called a pessimistic lower approximation reduct of apr.

(3) If $\overline{\sum_{\mathbb{R}_i \in \overline{\mathbb{R}}^{O}} \mathbb{R}_i}^{O}(\mathbb{A}) = \overline{\sum_{i=1}^{m} \mathbb{R}_i}^{O}(\mathbb{A})$, then $\overline{\mathbb{R}}^{O}$ is referred to as a consistent optimistic upper approximation of apr. If $\overline{\mathbb{R}}^{O}$ is a consistent optimistic upper approximation set, and no proper subset of $\overline{\mathbb{R}}^{O}$ is a consistent optimistic upper approximation, then $\overline{\mathbb{R}}^{O}$ is called an optimistic upper approximation reduct of apr.

(4) If $\overline{\sum_{\mathbb{R}_i \in \overline{\mathbb{R}}^{P}} \mathbb{R}_i}^{P}(\mathbb{A}) = \overline{\sum_{i=1}^{m} \mathbb{R}_i}^{P}(\mathbb{A})$, then $\overline{\mathbb{R}}^{P}$ is referred to as a consistent pessimistic upper approximation of apr. If $\overline{\mathbb{R}}^{P}$ is a consistent pessimistic upper approximation, and no proper subset of $\overline{\mathbb{R}}^{P}$ is a consistent pessimistic upper approximation, then $\overline{\mathbb{R}}^{P}$ is called a pessimistic upper approximation reduct of apr.

Table 1: DHF relation $\mathbb{R}_1$ in Example 3.3

$U \times U$	x_1	x_2	x_3	x_4
x_1	({1,1}, {0,0})	({0.4,0.5}, {0.2,0.4})	({0.2,0.3}, {0.5,0.7})	({0.6,0.8}, {0.1,0.2})
x_2	({0.1,0.2}, {0.7,0.8})	({1,1}, {0,0})	({0.2,0.3}, {0.6,0.7})	({0.4,0.5}, {0.5,0.5})
x_3	({0.2,0.2}, {0.6,0.7})	({0.5,0.8}, {0.2,0.2})	({1,1}, {0,0})	({0.3,0.4}, {0.5,0.6})
x_4	({0.3,0.5}, {0.4,0.5})	({0.4,0.5}, {0.2,0.4})	({0.2,0.3}, {0.5,0.7})	({1,1}, {0,0})

Table 2: DHF relation $\mathbb{R}_2$ in Example 3.3

$U \times U$	x_1	x_2	x_3	x_4
x_1	({1,1}, {0,0})	({0.3,0.5}, {0.2,0.5})	({0.2,0.2}, {0.7,0.8})	({0.4,0.5}, {0.3,0.5})
x_2	({0.2,0.2}, {0.6,0.8})	({1,1}, {0,0})	({0.4,0.6}, {0.3,0.4})	({0.2,0.5}, {0.3,0.5})
x_3	({0.1,0.3}, {0.5,0.6})	({0.4,0.5}, {0.3,0.4})	({1,1}, {0,0})	({0.1,0.2}, {0.7,0.8})
x_4	({0.2,0.5}, {0.3,0.5})	({0.1,0.1}, {0.8,0.9})	({0.5,0.6}, {0.3,0.4})	({1,1}, {0,0})

From Definition 3.1, we see that the lower approximation reduct is the smallest subset of $\mathbb{R} = \{\mathbb{R}_i | 1 \leq i \leq m\}$ which preserves the lower approximations of all DHF sets in U. And so is for the upper approximation reduct.

Definition 3.2 *A multi-granulation DHF decision information system is a quads* $S = (U, \{\mathbb{R}_i | 1 \leq i \leq m\}, D, V)$, *where* U *is a nonempty and finite universe;* $\{\mathbb{R}_i | 1 \leq i \leq m\}$ *is a set of* m *DHF relations on* U; D *is a nonempty and finite set of decision attributes;* $V = \{g(x,d) | x \in U, d \in D\}$ *is a set of the relationships between* U *and* D, *and* $g(x,d)$ *is a DHF element denoted as* $g(x,d) = (h_d(x), g_d(x))$. *We call* $g(x,d)$ *the decision DHF value of* x *under decision attribute* d.

Example 3.3 *A MGDHFDIS can be described as follows:* $U = \{x_1, x_2, x_3, x_4\}$; $\mathbb{R}_i (1 \leq i \leq 5)$ *are DHF relations on* U *shown as Tables 1-5;* $D = \{d_1, d_2\}$; $V = \{g(x_i, d_j) | x_i \in U, d_j \in D\}$, *where* $g(x_1, d_1) = (\{0.3, 0.4\}, \{0.2, 0.6\})$, $g(x_2, d_1) = (\{0.3, 0.5\}, \{0.4, 0.5\})$, $g(x_3, d_1) = (\{0.3, 0.7\}, \{0.2, 0.3\})$, $g(x_4, d_1) = (\{0.2, 0.3\}, \{0.6, 0.7\})$, $g(x_1, d_2) = (\{0.5, 0.7\}, \{0.2, 0.3\})$, $g(x_2, d_2) = (\{0.5, 0.6\}, \{0.2, 0.4\})$, $g(x_3, d_2) = (\{0.2, 0.4\}, \{0.3, 0.6\})$ *and* $g(x_4, d_2) = (\{0, 0.1\}, \{0.8, 0.9\})$.

On the basis of Definition 3.1, approximation reducts in MGDHFDIS based on the MGDHFRS model are defined as follows.

Definition 3.4 *Let a MGDHFDIS* $= (U, \{\mathbb{R}_i|1 \leq i \leq m\}, D, V)$, *where* $U = \{x_1, x_2, \cdots, x_n\}$ *and* $D = \{d_1, d_2, \cdots, d_v\}$, $D_j = \{< x_i, g(x_i, d_j) > |x_i \in U, d_j \in D\} \in DHF(U)$.

(1) For all $j(1 \leq j \leq v)$, *if* $\underline{\sum_{\mathbb{R}_i \in \underline{\mathbb{R}}^O} \mathbb{R}_i}^O(D_j) = \underline{\sum_{i=1}^m \mathbb{R}_i}^O(D_j)$, *then* $\underline{\mathbb{R}}^O$ *is referred to as a consistent optimistic lower approximation of MGDHFDIS. If* $\underline{\mathbb{R}}^O$ *is a consistent optimistic lower approximation, and no proper subset of* $\underline{\mathbb{R}}^O$ *is a consistent optimistic lower approximation, then* $\underline{\mathbb{R}}^O$ *is called an optimistic lower approximation reduct of MGDHFDIS.*

(2) For all $j(1 \leq j \leq v)$, *if* $\underline{\sum_{\mathbb{R}_i \in \underline{\mathbb{R}}^P} \mathbb{R}_i}^P(D_j) = \underline{\sum_{i=1}^m \mathbb{R}_i}^P(D_j)$, *then* $\underline{\mathbb{R}}^P$ *is referred to as a consistent pessimistic lower approximation of MGDHFDIS. If* $\underline{\mathbb{R}}^P$ *is a consistent pessimistic lower approximation, and no proper subset of* $\underline{\mathbb{R}}^P$ *is a consistent pessimistic lower approximation, then* $\underline{\mathbb{R}}^P$ *is called a pessimistic lower approximation reduct of MGDHFDIS.*

(3) For all $j(1 \leq j \leq v)$, *if* $\overline{\sum_{\mathbb{R}_i \in \overline{\mathbb{R}}^O} \mathbb{R}_i}^O(D_j) = \overline{\sum_{i=1}^m \mathbb{R}_i}^O(D_j)$, *then* $\overline{\mathbb{R}}^O$ *is referred to as a consistent optimistic upper approximation of MGDHFDIS. If* $\overline{\mathbb{R}}^O$ *is a consistent optimistic upper approximation, and no proper subset of* $\overline{\mathbb{R}}^O$ *is a consistent optimistic upper approximation, then* $\overline{\mathbb{R}}^O$ *is called an optimistic upper approximation reduct of MGDHFDIS.*

(4) For all $j(1 \leq j \leq v)$, *if* $\overline{\sum_{\mathbb{R}_i \in \overline{\mathbb{R}}^P} \mathbb{R}_i}^P(D_j) = \overline{\sum_{i=1}^m \mathbb{R}_i}^P(D_j)$, *then* $\overline{\mathbb{R}}^P$ *is referred to as a consistent pessimistic upper approximation of MGDHFDIS. If* $\overline{\mathbb{R}}^P$ *is a consistent pessimistic upper approximation, and no proper subset of* $\overline{\mathbb{R}}^P$ *is a consistent pessimistic upper approximation, then* $\overline{\mathbb{R}}^P$ *is called a pessimistic upper approximation reduct of MGDHFDIS.*

In order to obtain the optimistic and pessimistic approximation reducts of MGDHFDIS, we introduce the concepts of DHF vectors and DHF matrices. In the text that follows, without loss of generality, we suppose that the first HF elements in all the DHF elements have the same length k, and the second HF elements in all the DHF elements have the same length l.

Definition 3.5 *Let* $n-$*dimensional vector* $\vec{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_n)^T$, *where* $\alpha_i = (h_i, g_i)(1 \leq i \leq n)$ *are* n *DHF elements. Then we call* $\vec{\alpha}$ *a* $n-$*dimensional DHF vector. If* $M_{nm} = (\vec{\alpha}_1, \vec{\alpha}_2, \cdots, \vec{\alpha}_m)$, *where* $\vec{\alpha}_j(1 \leq j \leq m)$ *are* m $n-$*dimensional DHF vectors, then we call* M_{nm} *a* $n \times m$ *DHF matrix. Specially, a* $n-$*dimensional DHF vector can be viewed as a* $n \times 1$ *DHF matrix.*

Based on Definition 3.5, a MGDHFDIS can be described as multiple DHF matrices (DHF relation matrices) and vectors (called decision DHF vectors). For example, by using DHF relation matrices and decision DHF vectors, the MGDHFDIS in Example 3.3 can be described as follows:

$$M_{\mathbb{R}_1} = \begin{pmatrix} (\{1,1\},\{0,0\}) & (\{0.4,0.5\},\{0.2,0.4\}) & (\{0.2,0.3\},\{0.5,0.7\}) & (\{0.6,0.8\},\{0.1,0.2\}) \\ (\{0.1,0.2\},\{0.7,0.8\}) & (\{1,1\},\{0,0\}) & (\{0.2,0.3\},\{0.6,0.7\}) & (\{0.4,0.5\},\{0.5,0.5\}) \\ (\{0.2,0.2\},\{0.6,0.7\}) & (\{0.5,0.8\},\{0.2,0.2\}) & (\{1,1\},\{0,0\}) & (\{0.3,0.4\},\{0.5,0.6\}) \\ (\{0.3,0.5\},\{0.4,0.5\}) & (\{0.4,0.5\},\{0.2,0.4\}) & (\{0.2,0.3\},\{0.5,0.7\}) & (\{1,1\},\{0,0\}) \end{pmatrix}$$

Table 3: DHF relation $\mathbb{R}_3$ in Example 3.3

$U \times U$	x_1	x_2	x_3	x_4
x_1	({1,1}, {0,0})	({0.2,0.4}, {0.5,0.6})	({0.1,0.2}, {0.6,0.8})	({0.3,0.4}, {0.4,0.6})
x_2	({0.1,0.2}, {0.7,0.8})	({1,1}, {0,0})	({0.5,0.6}, {0.2,0.3})	({0.3,0.5}, {0.4,0.4})
x_3	({0.0,0.5}, {0.4,0.4})	({0.1,0.2}, {0.7,0.8})	({1,1}, {0,0})	({0.7,0.9}, {0.1,0.1})
x_4	({0.3,0.4}, {0.5,0.6})	({0.2,0.3}, {0.5,0.7})	({0.3,0.6}, {0.3,0.4})	({1,1}, {0,0})

$$M_{\mathbb{R}_2} = \begin{pmatrix} (\{1,1\},\{0,0\}) & (\{0.3,0.5\},\{0.2,0.5\}) & (\{0.2,0.2\},\{0.7,0.8\}) & (\{0.4,0.5\},\{0.3,0.5\}) \\ (\{0.2,0.2\},\{0.6,0.8\}) & (\{1,1\},\{0,0\}) & (\{0.4,0.6\},\{0.3,0.4\}) & (\{0.2,0.5\},\{0.3,0.5\}) \\ (\{0.1,0.3\},\{0.5,0.6\}) & (\{0.4,0.5\},\{0.3,0.4\}) & (\{1,1\},\{0,0\}) & (\{0.1,0.2\},\{0.7,0.8\}) \\ (\{0.2,0.5\},\{0.3,0.5\}) & (\{0.1,0.1\},\{0.8,0.9\}) & (\{0.5,0.6\},\{0.3,0.4\}) & (\{1,1\},\{0,0\}) \end{pmatrix}$$

$$M_{\mathbb{R}_3} = \begin{pmatrix} (\{1,1\},\{0,0\}) & (\{0.2,0.4\},\{0.5,0.6\}) & (\{0.1,0.2\},\{0.6,0.8\}) & (\{0.3,0.4\},\{0.4,0.6\}) \\ (\{0.1,0.2\},\{0.7,0.8\}) & (\{1,1\},\{0,0\}) & (\{0.5,0.6\},\{0.2,0.3\}) & (\{0.3,0.5\},\{0.4,0.4\}) \\ (\{0.0,0.5\},\{0.4,0.4\}) & (\{0.1,0.2\},\{0.7,0.8\}) & (\{1,1\},\{0,0\}) & (\{0.7,0.9\},\{0.1,0.1\}) \\ (\{0.3,0.4\},\{0.5,0.6\}) & (\{0.2,0.3\},\{0.5,0.7\}) & (\{0.3,0.6\},\{0.3,0.4\}) & (\{1,1\},\{0,0\}) \end{pmatrix}$$

$$M_{\mathbb{R}_4} = \begin{pmatrix} (\{1,1\},\{0,0\}) & (\{0.4,0.6\},\{0.3,0.4\}) & (\{0.6,0.7\},\{0.3,0.3\}) & (\{0.8,0.9\},\{0.1,0.1\}) \\ (\{0.1,0.2\},\{0.7,0.7\}) & (\{1,1\},\{0,0\}) & (\{0.5,0.6\},\{0.2,0.3\}) & (\{0.2,0.3\},\{0.6,0.7\}) \\ (\{0.3,0.4\},\{0.5,0.6\}) & (\{0.5,0.5\},\{0.3,0.4\}) & (\{1,1\},\{0,0\}) & (\{0.3,0.4\},\{0.6,0.6\}) \\ (\{0.4,0.5\},\{0.3,0.5\}) & (\{0.0,0.2\},\{0.7,0.8\}) & (\{0.1,0.4\},\{0.5,0.5\}) & (\{1,1\},\{0,0\}) \end{pmatrix}$$

$$M_{\mathbb{R}_5} = \begin{pmatrix} (\{1,1\},\{0,0\}) & (\{0.5,0.5\},\{0.4,0.5\}) & (\{0.1,0.2\},\{0.6,0.8\}) & (\{0.2,0.3\},\{0.5,0.6\}) \\ (\{0.1,0.1\},\{0.8,0.9\}) & (\{1,1\},\{0,0\}) & (\{0.6,0.7\},\{0.2,0.3\}) & (\{0.2,0.3\},\{0.6,0.7\}) \\ (\{0.0,0.3\},\{0.6,0.7\}) & (\{0.2,0.5\},\{0.4,0.5\}) & (\{1,1\},\{0,0\}) & (\{0.2,0.2\},\{0.6,0.7\}) \\ (\{0.4,0.5\},\{0.3,0.4\}) & (\{0.1,0.2\},\{0.6,0.8\}) & (\{0.1,0.2\},\{0.6,0.8\}) & (\{1,1\},\{0,0\}) \end{pmatrix}$$

and decision DHF vectors:

$\mathbb{D}_1 = ((\{0.3, 0.4\}, \{0.2, 0.6\}), (\{0.3, 0.5\}, \{0.4, 0.5\}), (\{0.3, 0.7\}, \{0.2, 0.3\}), (\{0.2, 0.3\}, \{0.6, 0.7\}))^T$,

$\mathbb{D}_2 = ((\{0.5, 0.7\}, \{0.2, 0.3\}), (\{0.5, 0.6\}, \{0.2, 0.4\}), (\{0.2, 0.4\}, \{0.3, 0.6\}), (\{0, 0.1\}, \{0.8, 0.9\}))^T$.

Now, the union, intersection and complement of two DHF vectors and matrices can be defined as follows:

Definition 3.6 *Let $\vec{\alpha}_1 = (\alpha_{11}, \alpha_{12}, \cdots, \alpha_{1n})^T$ and $\vec{\alpha}_2 = (\alpha_{21}, \alpha_{22}, \cdots, \alpha_{2n})^T$ be two $n-$dimensional DHF vectors, where $\alpha_{1i} = (h_{1i}, g_{1i})$ and $\alpha_{2i} = (h_{2i}, g_{2i})(1 \leq i \leq n)$ are DHF elements. Assume that $M_1 = (\vec{\alpha}_{11}, \vec{\alpha}_{12}, \cdots, \vec{\alpha}_{1m})$ and $M_2 = (\vec{\alpha}_{21}, \vec{\alpha}_{22}, \cdots, \vec{\alpha}_{2m})$ be two $n \times m$ DHF matrices, where $\vec{\alpha}_{1j}$ and $\vec{\alpha}_{2j}(1 \leq j \leq m)$ are $n-$dimensional DHF vectors; then,*

(1) $\vec{\alpha}_1 \Cup \vec{\alpha}_2 = (\alpha_{11} \veebar \alpha_{21}, \alpha_{12} \veebar \alpha_{22}, \cdots, \alpha_{1n} \veebar \alpha_{2n})^T$,
where $\alpha_{1i} \veebar \alpha_{2i} = \{(\{h_{1i}^{\sigma(s)} \vee h_{2i}^{\sigma(s)}\}, \{g_{1i}^{\sigma(t)} \wedge g_{2i}^{\sigma(t)}\}) | 1 \leq s \leq k, 1 \leq t \leq l\}(1 \leq i \leq n)$;

(2) $\vec{\alpha}_1 \Cap \vec{\alpha}_2 = (\alpha_{11} \barwedge \alpha_{21}, \alpha_{12} \barwedge \alpha_{22}, \cdots, \alpha_{1n} \barwedge \alpha_{2n})^T$,
where $\alpha_{1i} \barwedge \alpha_{2i} = \{(\{h_{1i}^{\sigma(s)} \wedge h_{2i}^{\sigma(s)}\}, \{g_{1i}^{\sigma(t)} \vee g_{2i}^{\sigma(t)}\}) | 1 \le s \le k, 1 \le t \le l\} (1 \le i \le n)$;

(3) The complementary vector of $\vec{\alpha}_1$ *is denoted as*
$(\vec{\alpha}_1)^c = (\sim \alpha_{11}, \sim \alpha_{12}, \cdots, \sim \alpha_{1n})^T$,
where $\sim \alpha_{1i} = \{(\{g_{1i}^{\sigma(t)}\}, \{h_{1i}^{\sigma(s)}\}) | 1 \le s \le k, 1 \le t \le l\} (1 \le i \le n)$;

(4) $M_1 \Cup M_2 = (\vec{\alpha}_{11} \Cup \vec{\alpha}_{21}, \vec{\alpha}_{12} \Cup \vec{\alpha}_{22}, \cdots, \vec{\alpha}_{1m} \Cup \vec{\alpha}_{2m})$;

(5) $M_1 \Cap M_2 = (\vec{\alpha}_{11} \Cap \vec{\alpha}_{21}, \vec{\alpha}_{12} \Cap \vec{\alpha}_{22}, \cdots, \vec{\alpha}_{1m} \Cap \vec{\alpha}_{2m})$;

(6) The complementary matrix of M_1 *is denoted as*
${M_1}^c = ((\vec{\alpha}_{11})^c, (\vec{\alpha}_{12})^c, \cdots, (\vec{\alpha}_{1m})^c)^T$.

In the following we introduce the product operation of DHF matrices.

Definition 3.7 *Let* P *and* Q *be two DHF matrices, and*

$$P = \begin{pmatrix} (\overline{p}_{11}, \underline{p}_{11}) & (\overline{p}_{12}, \underline{p}_{12}) & \cdots & (\overline{p}_{1w}, \underline{p}_{1w}) \\ (\overline{p}_{21}, \underline{p}_{21}) & (\overline{p}_{22}, \underline{p}_{22}) & \cdots & (\overline{p}_{2w}, \underline{p}_{2w}) \\ \vdots & \vdots & \ddots & \vdots \\ (\overline{p}_{m1}, \underline{p}_{m1}) & (\overline{p}_{m2}, \underline{p}_{m2}) & \cdots & (\overline{p}_{mw}, \underline{p}_{mw}) \end{pmatrix},$$

$$Q = \begin{pmatrix} (\overline{q}_{11}, \underline{q}_{11}) & (\overline{q}_{12}, \underline{q}_{12}) & \cdots & (\overline{q}_{1n}, \underline{q}_{1n}) \\ (\overline{q}_{21}, \underline{q}_{21}) & (\overline{q}_{22}, \underline{q}_{22}) & \cdots & (\overline{q}_{2n}, \underline{q}_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ (\overline{q}_{w1}, \underline{q}_{w1}) & (\overline{q}_{w2}, \underline{q}_{w2}) & \cdots & (\overline{q}_{wn}, \underline{q}_{wn}) \end{pmatrix}.$$

Then, the product of P *and* Q *is a* $m \times n$ *DHF matrix, denoted as follows:*
$M = P \circ Q = \left((\overline{r}_{ij}, \underline{r}_{ij})\right)_{1 \le i \le m, 1 \le j \le n}$,
where

$$\overline{r}_{ij} = \veebar_{1 \le u \le w} \{\overline{p}_{iu} \barwedge \overline{q}_{uj}\} = \{\bigvee_{1 \le u \le w} (\overline{p}_{iu}^{\sigma(s)} \wedge \overline{q}_{uj}^{\sigma(s)}) | 1 \le s \le k\},$$

$$\underline{r}_{ij} = \barwedge_{1 \le u \le w} \{\underline{p}_{iu} \veebar \underline{q}_{uj}\} = \{\bigwedge_{1 \le u \le w} (\underline{p}_{iu}^{\sigma(t)} \vee \underline{q}_{uj}^{\sigma(t)}) | 1 \le t \le l\}.$$

In the following discussions, for convenience, we don't distinguish between DHF vectors and DHF sets on U.

Theorem 3.8 *Let* $\mathbb{R}$ *be the DHF relation on* U*,* $M_{\mathbb{R}}$ *be DHF matrix of* $\mathbb{R}$ *and* $\mathbb{A} \in DHF(U)$*; then*

(1) $\underline{\mathbb{R}}(\mathbb{A}) = (M_{\mathbb{R}} \circ \mathbb{A}^c)^c$,

(2) $\overline{\mathbb{R}}(\mathbb{A}) = M_{\mathbb{R}} \circ \mathbb{A}$,

where $\underline{\mathbb{R}}(\mathbb{A})$ *and* $\overline{\mathbb{R}}(\mathbb{A})$ *are the single-granulation lower and upper approximations defined in Definition 2.4.*

Proof. It can be easily verified from Definitions 3.7 and 2.4. □

According to Theorems 3.8, 2.7 and 2.8, we conclude that the following theorem holds.

Table 4: DHF relation $\mathbb{R}_4$ in Example 3.3

$U \times U$	x_1	x_2	x_3	x_4
x_1	({1,1}, {0,0})	({0.4,0.6}, {0.3,0.4})	({0.6,0.7}, {0.3,0.3})	({0.8,0.9}, {0.1,0.1})
x_2	({0.1,0.2}, {0.7,0.7})	({1,1}, {0,0})	({0.5,0.6}, {0.2,0.3})	({0.2,0.3}, {0.6,0.7})
x_3	({0.3,0.4}, {0.5,0.6})	({0.5,0.5}, {0.3,0.4})	({1,1}, {0,0})	({0.3,0.4}, {0.6,0.6})
x_4	({0.4,0.5}, {0.3,0.5})	({0.0,0.2}, {0.7,0.8})	({0.1,0.4}, {0.5,0.5})	({1,1}, {0,0})

Table 5: DHF relation $\mathbb{R}_5$ in Example 3.3

$U \times U$	x_1	x_2	x_3	x_4
x_1	({1,1}, {0,0})	({0.5,0.5}, {0.4,0.5})	({0.1,0.2}, {0.6,0.8})	({0.2,0.3}, {0.5,0.6})
x_2	({0.1,0.1}, {0.8,0.9})	({1,1}, {0,0})	({0.6,0.7}, {0.2,0.3})	({0.2,0.3}, {0.6,0.7})
x_3	({0.0,0.3}, {0.6,0.7})	({0.2,0.5}, {0.4,0.5})	({1,1}, {0,0})	({0.2,0.2}, {0.6,0.7})
x_4	({0.4,0.5}, {0.3,0.4})	({0.1,0.2}, {0.6,0.8})	({0.1,0.2}, {0.6,0.8})	({1,1}, {0,0})

Theorem 3.9 *Let $\mathbb{R}_i(1 \le i \le m)$ be m DHF relations on U, $M_{\mathbb{R}_i}$ be the DHF relation matrices of $\mathbb{R}_i(1 \le i \le m)$ and $\mathbb{A} \in DHF(U)$; then*

(1) $\underline{\sum_{i=1}^{m} \mathbb{R}_i}^{O}(\mathbb{A}) = \Cup_{i=1}^{m}(M_{\mathbb{R}_i} \circ \mathbb{A}^c)^c$,

(2) $\overline{\sum_{i=1}^{m} \mathbb{R}_i}^{O}(\mathbb{A}) = \Cap_{i=1}^{m}(M_{\mathbb{R}_i} \circ \mathbb{A})$;

(3) $\underline{\sum_{i=1}^{m} \mathbb{R}_i}^{P}(\mathbb{A}) = \Cap_{i=1}^{m}(M_{\mathbb{R}_i} \circ \mathbb{A}^c)^c$,

(4) $\overline{\sum_{i=1}^{m} \mathbb{R}_i}^{P}(\mathbb{A}) = \Cup_{i=1}^{m}(M_{\mathbb{R}_i} \circ \mathbb{A})$.

Example 3.10 *(Continued from Example 3.3) According to Theorem 3.8(2), we have*

$$\overline{\mathbb{R}_1}(\mathbb{D}_1) = M_{\mathbb{R}_1} \circ \mathbb{D}_1 = ((\{0.3, 0.5\}, \{0.2, 0.5\}), (\{0.3, 0.5\}, \{0.4, 0.5\}), (\{0.3, 0.7\}, \{0.2, 0.3\}), (\{0.3, 0.5\}, \{0.4, 0.5\}))^T,$$

$$\overline{\mathbb{R}_2}(\mathbb{D}_1) = M_{\mathbb{R}_2} \circ \mathbb{D}_1 = ((\{0.3, 0.5\}, \{0.2, 0.5\}), (\{0.3, 0.6\}, \{0.3, 0.4\}), (\{0.3, 0.7\}, \{0.2, 0.3\}), (\{0.3, 0.6\}, \{0.3, 0.4\}))^T,$$

$$\overline{\mathbb{R}_3}(\mathbb{D}_1) = M_{\mathbb{R}_3} \circ \mathbb{D}_1 = ((\{0.3, 0.4\}, \{0.2, 0.6\}), (\{0.3, 0.6\}, \{0.2, 0.3\}), (\{0.3, 0.7\}, \{0.2, 0.3\}), (\{0.3, 0.6\}, \{0.3, 0.4\}))^T,$$

$$\overline{\mathbb{R}_4}(\mathbb{D}_1) = M_{\mathbb{R}_4} \circ \mathbb{D}_1 = ((\{0.3, 0.7\}, \{0.2, 0.3\}), (\{0.3, 0.6\}, \{0.2, 0.3\}),$$
$$(\{0.3, 0.7\}, \{0.2, 0.3\}), (\{0.3, 0.4\}, \{0.3, 0.5\}))^T,$$

$$\overline{\mathbb{R}_5}(\mathbb{D}_1) = M_{\mathbb{R}_4} \circ \mathbb{D}_1 = ((\{0.3, 0.5\}, \{0.2, 0.5\}), (\{0.3, 0.7\}, \{0.2, 0.3\}),$$
$$(\{0.3, 0.7\}, \{0.2, 0.3\}), (\{0.3, 0.4\}, \{0.3, 0.6\}))^T.$$

Then by Theorem 3.9(2) and (4), we obtain

$$\overline{\sum_{i=1}^{5} \mathbb{R}_i}^{O} (\mathbb{D}_1) = ((\{0.3, 0.4\}, \{0.2, 0.6\}), (\{0.3, 0.5\}, \{0.4, 0.5\}),$$
$$(\{0.3, 0.7\}, \{0.2, 0.3\}), (\{0.3, 0.4\}, \{0.4, 0.6\}))^T,$$

and

$$\overline{\sum_{i=1}^{5} \mathbb{R}_i}^{P} (\mathbb{D}_1) = ((\{0.3, 0.7\}, \{0.2, 0.3\}), (\{0.3, 0.7\}, \{0.2, 0.3\}),$$
$$(\{0.3, 0.7\}, \{0.2, 0.3\}), (\{0.3, 0.6\}, \{0.3, 0.4\}))^T.$$

Similarly, we have

$$\overline{\mathbb{R}_1}(\mathbb{D}_2) = M_{\mathbb{R}_1} \circ \mathbb{D}_2 = ((\{0.5, 0.7\}, \{0.2, 0.3\}), (\{0.5, 0.6\}, \{0.2, 0.4\}),$$
$$(\{0.5, 0.6\}, \{0.2, 0.4\}), (\{0.4, 0.5\}, \{0.2, 0.4\}))^T,$$

$$\overline{\mathbb{R}_2}(\mathbb{D}_2) = M_{\mathbb{R}_2} \circ \mathbb{D}_2 = ((\{0.5, 0.7\}, \{0.2, 0.3\}), (\{0.5, 0.6\}, \{0.2, 0.4\}),$$
$$(\{0.4, 0.5\}, \{0.3, 0.4\}), (\{0.2, 0.5\}, \{0.3, 0.5\}))^T,$$

$$\overline{\mathbb{R}_3}(\mathbb{D}_2) = M_{\mathbb{R}_3} \circ \mathbb{D}_2 = ((\{0.5, 0.7\}, \{0.2, 0.3\}), (\{0.5, 0.6\}, \{0.2, 0.4\}),$$
$$(\{0.2, 0.5\}, \{0.3, 0.4\}), (\{0.3, 0.4\}, \{0.3, 0.6\}))^T,$$

$$\overline{\mathbb{R}_4}(\mathbb{D}_2) = M_{\mathbb{R}_4} \circ \mathbb{D}_2 = ((\{0.5, 0.7\}, \{0.2, 0.3\}), (\{0.5, 0.6\}, \{0.2, 0.4\}),$$
$$(\{0.5, 0.5\}, \{0.3, 0.4\}), (\{0.4, 0.5\}, \{0.3, 0.5\}))^T,$$

$$\overline{\mathbb{R}_5}(\mathbb{D}_2) = M_{\mathbb{R}_4} \circ \mathbb{D}_2 = ((\{0.5, 0.7\}, \{0.2, 0.3\}), (\{0.5, 0.6\}, \{0.2, 0.4\}),$$
$$(\{0.2, 0.5\}, \{0.3, 0.5\}), (\{0.4, 0.5\}, \{0.3, 0.4\}))^T.$$

Then

$$\overline{\sum_{i=1}^{5} \mathbb{R}_i}^{O} (\mathbb{D}_2) = ((\{0.5, 0.7\}, \{0.2, 0.3\}), (\{0.5, 0.6\}, \{0.2, 0.4\}),$$
$$(\{0.2, 0.5\}, \{0.3, 0.5\}), (\{0.2, 0.4\}, \{0.3, 0.6\}))^T,$$

and

$$\overline{\sum_{i=1}^{5}\mathbb{R}_i}^{P}(\mathbb{D}_2) =((\{0.5,0.7\},\{0.2,0.3\}),(\{0.5,0.6\},\{0.2,0.4\}),$$
$$(\{0.5,0.6\},\{0.2,0.4\}),(\{0.4,0.5\},\{0.2,0.4\}))^T.$$

According to Theorem 3.8(1), we have

$$\underline{\mathbb{R}_1}(\mathbb{D}_1) = (M_{\mathbb{R}_1} \circ {\mathbb{D}_1}^c)^c =((\{0.2,0.3\},\{0.6,0.7\}),(\{0.3,0.5\},\{0.4,0.5\}),$$
$$(\{0.3,0.5\},\{0.4,0.5\}),(\{0.2,0.3\},\{0.6,0.7\}))^T,$$

$$\underline{\mathbb{R}_2}(\mathbb{D}_1) = (M_{\mathbb{R}_2} \circ {\mathbb{D}_1}^c)^c =((\{0.3,0.4\},\{0.4,0.6\}),(\{0.3,0.5\},\{0.4,0.5\}),$$
$$(\{0.3,0.5\},\{0.4,0.5\}),(\{0.2,0.3\},\{0.6,0.7\}))^T,$$

$$\underline{\mathbb{R}_3}(\mathbb{D}_1) = (M_{\mathbb{R}_3} \circ {\mathbb{D}_1}^c)^c =((\{0.3,0.4\},\{0.3,0.6\}),(\{0.3,0.4\},\{0.4,0.5\}),$$
$$(\{0.2,0.3\},\{0.6,0.7\}),(\{0.2,0.3\},\{0.6,0.7\}))^T,$$

$$\underline{\mathbb{R}_4}(\mathbb{D}_1) = (M_{\mathbb{R}_4} \circ {\mathbb{D}_1}^c)^c =((\{0.2,0.3\},\{0.6,0.7\}),(\{0.3,0.5\},\{0.4,0.5\}),$$
$$(\{0.3,0.5\},\{0.4,0.5\}),(\{0.2,0.3\},\{0.6,0.7\}))^T,$$

$$\underline{\mathbb{R}_5}(\mathbb{D}_1) = (M_{\mathbb{R}_5} \circ {\mathbb{D}_1}^c)^c =((\{0.3,0.4\},\{0.4,0.6\}),(\{0.3,0.5\},\{0.4,0.5\}),$$
$$(\{0.3,0.5\},\{0.2,0.5\}),(\{0.2,0.3\},\{0.6,0.7\}))^T,$$

Then according to Theorem 3.9(1) and (3), we obtain

$$\underline{\sum_{i=1}^{5}\mathbb{R}_i}^{O}(\mathbb{D}_1) =((\{0.3,0.4\},\{0.3,0.6\}),(\{0.3,0.5\},\{0.4,0.5\}),$$
$$(\{0.3,0.5\},\{0.2,0.5\}),(\{0.2,0.3\},\{0.6,0.7\}))^T,$$

and

$$\underline{\sum_{i=1}^{5}\mathbb{R}_i}^{P}(\mathbb{D}_1) =((\{0.2,0.3\},\{0.6,0.7\}),(\{0.3,0.4\},\{0.4,0.5\}),$$
$$(\{0.2,0.3\},\{0.6,0.7\}),(\{0.2,0.3\},\{0.6,0.7\})).$$

Similarly, we have

$$\underline{\mathbb{R}_1}(\mathbb{D}_2) = (M_{\mathbb{R}_1} \circ {\mathbb{D}_2}^c)^c =((\{0.1,0.2\},\{0.6,0.8\}),(\{0.5,0.5\},\{0.4,0.5\}),$$
$$(\{0.2,0.4\},\{0.3,0.6\}),(\{0.0,0.1\},\{0.8,0.9\}))^T,$$

$$\begin{aligned}
\underline{\mathbb{R}_2}(\mathbb{D}_2) = (M_{\mathbb{R}_2} \circ \mathbb{D}_2{}^c)^c =& ((\{0.3, 0.5\}, \{0.4, 0.5\}), (\{0.3, 0.4\}, \{0.3, 0.6\}),\\
& (\{0.2, 0.4\}, \{0.3, 0.6\}), (\{0.0, 0.1\}, \{0.8, 0.9\}))^T,\\
\underline{\mathbb{R}_3}(\mathbb{D}_2) = (M_{\mathbb{R}_3} \circ \mathbb{D}_2{}^c)^c =& ((\{0.4, 0.6\}, \{0.3, 0.4\}), (\{0.2, 0.4\}, \{0.3, 0.6\}),\\
& (\{0.1, 0.1\}, \{0.7, 0.9\}), (\{0.0, 0.1\}, \{0.8, 0.9\}))^T,\\
\underline{\mathbb{R}_4}(\mathbb{D}_2) = (M_{\mathbb{R}_4} \circ \mathbb{D}_2{}^c)^c =& ((\{0.1, 0.1\}, \{0.8, 0.9\}), (\{0.2, 0.4\}, \{0.3, 0.6\}),\\
& (\{0.2, 0.4\}, \{0.3, 0.6\}), (\{0.0, 0.1\}, \{0.8, 0.9\}))^T,\\
\underline{\mathbb{R}_5}(\mathbb{D}_2) = (M_{\mathbb{R}_5} \circ \mathbb{D}_2{}^c)^c =& ((\{0.5, 0.6\}, \{0.2, 0.4\}), (\{0.2, 0.4\}, \{0.3, 0.6\}),\\
& (\{0.2, 0.4\}, \{0.3, 0.6\}), (\{0.0, 0.1\}, \{0.8, 0.9\}))^T.
\end{aligned}$$

Then

$$\begin{aligned}
\underline{\sum_{i=1}^{5} \mathbb{R}_i}^{O} (\mathbb{D}_2) =& ((\{0.5, 0.6\}, \{0.2, 0.4\}), (\{0.5, 0.5\}, \{0.3, 0.5\}),\\
& (\{0.2, 0.4\}, \{0.3, 0.6\}), (\{0.0, 0.1\}, \{0.8, 0.9\}))^T,
\end{aligned}$$

and

$$\begin{aligned}
\underline{\sum_{i=1}^{5} \mathbb{R}_i}^{P} (\mathbb{D}_2) =& ((\{0.1, 0.1\}, \{0.8, 0.9\}), (\{0.2, 0.4\}, \{0.4, 0.6\}),\\
& (\{0.1, 0.1\}, \{0.7, 0.9\}), (\{0.0, 0.1\}, \{0.8, 0.9\})).
\end{aligned}$$

It is well known that a discernibility function is a key notion to reduction algorithms in rough set theory. Therefore, by constructing the discernibility functions, we present a practical method to determine the optimistic and pessimistic approximation reducts of MGDHFDIS.

Definition 3.11 *Let* $MGDHFDIS = (U, \{\mathbb{R}_j | 1 \leq j \leq m\}, D = \{d_i | 1 \leq i \leq v\}, V)$, $|U| = n$ *and* $D_i (1 \leq i \leq v)$ *be decision vectors. Denote*

$$\underline{\sum_{j=1}^{m} \mathbb{R}_j}^{O} (\mathbb{D}_i) = (\underline{o}_{i1}, \underline{o}_{i2}, \cdots, \underline{o}_{in}) (1 \leq i \leq v),$$

where $\underline{o}_{iu} = \{(\{\underline{o}_{iu}^{\sigma(s)}\}, \{\underline{o}_{iu}^{\sigma(t)}\}) | 1 \leq s \leq k, 1 \leq t \leq l\} (1 \leq u \leq n)$;

$$\overline{\sum_{j=1}^{m} \mathbb{R}_j}^{O} (\mathbb{D}_i) = (\overline{o}_{i1}, \overline{o}_{i2}, \cdots, \overline{o}_{in}) (1 \leq i \leq v),$$

where $\overline{o}_{iu} = \{(\{\overline{o}_{iu}^{\sigma(s)}\}, \{\overline{o}_{iu}^{\sigma(t)}\}) | 1 \leq s \leq k, 1 \leq t \leq l\} (1 \leq u \leq n)$;

$$\underline{\sum_{j=1}^{m} \mathbb{R}_j}^{P} (\mathbb{D}_i) = (\underline{p}_{i1}, \underline{p}_{i2}, \cdots, \underline{p}_{in}) (1 \leq i \leq v),$$

where $\underline{p}_{iu} = \{(\{\underline{p}_{iu}^{\sigma(s)}\}, \{\underline{p}_{iu}^{\sigma(t)}\}) | 1 \leq s \leq k, 1 \leq t \leq l\} (1 \leq u \leq n)$;

$$\overline{\sum_{j=1}^{m} \mathbb{R}_j}^{P} (\mathbb{D}_i) = (\overline{p}_{i1}, \overline{p}_{i2}, \cdots, \overline{p}_{in}) (1 \leq i \leq v),$$

where $\overline{p}_{iu} = \{(\{\overline{p}_{iu}^{\sigma(s)}\}, \{\overline{p}_{iu}^{\sigma(t)}\}) | 1 \leq s \leq k, 1 \leq t \leq l\} (1 \leq u \leq n)$;

$$\underline{\mathbb{R}_j}(\mathbb{D}_i) = (\underline{r}_{ij1}, \underline{r}_{ij2}, \cdots, \underline{r}_{ijn}) (1 \leq i \leq v, 1 \leq j \leq m),$$

where $\underline{r}_{iju} = \{(\{\underline{r}_{iju}^{\sigma(s)}\}, \{\underline{r}_{iju}^{\sigma(t)}\}) | 1 \leq s \leq k, 1 \leq t \leq l\} (1 \leq u \leq n)$;

$$\overline{\mathbb{R}_j}(\mathbb{D}_i) = (\overline{r}_{ij1}, \overline{r}_{ij2}, \cdots, \overline{r}_{ijn}) (1 \leq i \leq v, 1 \leq j \leq m),$$

where $\overline{r}_{iju} = \{(\{\overline{r}_{iju}^{\sigma(s)}\}, \{\overline{r}_{iju}^{\sigma(t)}\}) | 1 \leq s \leq k, 1 \leq t \leq l\} (1 \leq u \leq n)$.
Then, the optimistic lower approximation discernibility function of MGDHFDIS is

$$\underline{f}^O = \bigwedge_{i=1}^{v} \bigwedge_{u=1}^{n} \bigwedge_{s=1}^{k} \bigwedge_{t=1}^{l} \Big(\bigvee_{\underline{r}_{iju}^{\sigma(s)} = \underline{o}_{iu}^{\sigma(s)}, 1 \leq j \leq m} \mathbb{R}_j \wedge \bigvee_{\underline{r}_{iju}^{\sigma(t)} = \underline{o}_{iu}^{\sigma(t)}, 1 \leq j \leq m} \mathbb{R}_j \Big);$$

the optimistic upper approximation discernibility function of MGDHFDIS is

$$\overline{f}^O = \bigwedge_{i=1}^{v} \bigwedge_{u=1}^{n} \bigwedge_{s=1}^{k} \bigwedge_{t=1}^{l} \Big(\bigvee_{\overline{r}_{iju}^{\sigma(s)} = \overline{o}_{iu}^{\sigma(s)}, 1 \leq j \leq m} \mathbb{R}_j \wedge \bigvee_{\underline{r}_{iju}^{\sigma(t)} = \underline{o}_{iu}^{\sigma(t)}, 1 \leq j \leq m} \mathbb{R}_j \Big);$$

the pessimistic lower approximation discernibility function of MGDHFDIS is

$$\underline{f}^P = \bigwedge_{i=1}^{v} \bigwedge_{u=1}^{n} \bigwedge_{s=1}^{k} \bigwedge_{t=1}^{l} \Big(\bigvee_{\underline{r}_{iju}^{\sigma(s)} = \underline{p}_{iu}^{\sigma(s)}, 1 \leq j \leq m} \mathbb{R}_j \wedge \bigvee_{\underline{r}_{iju}^{\sigma(t)} = \underline{p}_{iu}^{\sigma(t)}, 1 \leq j \leq m} \mathbb{R}_j \Big);$$

the pessimistic upper approximation discernibility function of MGDHFDIS is

$$\overline{f}^P = \bigwedge_{i=1}^{v} \bigwedge_{u=1}^{n} \bigwedge_{s=1}^{k} \bigwedge_{t=1}^{l} \Big(\bigvee_{\overline{r}_{iju}^{\sigma(s)} = \overline{p}_{iu}^{\sigma(s)}, 1 \leq j \leq m} \mathbb{R}_j \wedge \bigvee_{\overline{r}_{iju}^{\sigma(t)} = \overline{p}_{iu}^{\sigma(t)}, 1 \leq j \leq m} \mathbb{R}_j \Big).$$

According to Definitions 3.11 and 3.4, we can easily obtain the following theorem.

Theorem 3.12 *Let MGDHFDIS* $= (U, \{\mathbb{R}_j | 1 \leq j \leq m\}, D = \{d_i | 1 \leq i \leq v\}, V)$, $|U| = n$. *We can convert the approximation discernibility functions* $\underline{f}^O$, $\overline{f}^O$, $\underline{f}^P$ *and* $\overline{f}^P$ *of MGDHFDIS into their disjunction forms* $\underline{f}^O = \bigvee_{\alpha=1}^{\alpha_1} (\bigwedge_{\beta=1}^{\beta_1} \mathbb{R}_{\alpha\beta 1})$, $\overline{f}^O = \bigvee_{\alpha=1}^{\alpha_2} (\bigwedge_{\beta=1}^{\beta_2} \mathbb{R}_{\alpha\beta 2})$, $\underline{f}^P = \bigvee_{\alpha=1}^{\alpha_3} (\bigwedge_{\beta=1}^{\beta_3} \mathbb{R}_{\alpha\beta 3})$, *and* $\overline{f}^P = \bigvee_{\alpha=1}^{\alpha_4} (\bigwedge_{\beta=1}^{\beta_4} \mathbb{R}_{\alpha\beta 4})$, *respectively. Then,* $\underline{B}_{\alpha}^{O} = \{\mathbb{R}_{\alpha\beta 1} | \beta =$

$1, 2, \cdots, \beta_1\}(\alpha = 1, 2, \cdots, \alpha_1)$, $\overline{B}_\alpha^O = \{\mathbb{R}_{\alpha\beta 2}|\beta = 1, 2, \cdots, \beta_2\}(\alpha = 1, 2, \cdots, \alpha_2)$, $\underline{B}_\alpha^P = \{\mathbb{R}_{\alpha\beta 3}|\beta = 1, 2, \cdots, \beta_3\}(\alpha = 1, 2, \cdots, \alpha_3)$, *and* $\overline{B}_\alpha^P = \{\mathbb{R}_{\alpha\beta 4}|\beta = 1, 2, \cdots, \beta_4\}(\alpha = 1, 2, \cdots, \alpha_4)$ *are the optimistic lower upper, and pessimistic lower and upper approximation reducts of MGDHFDIS, respectively.*

From Theorem 3.12, we see that all the approximation reducts of MGDHFDIS can be obtained through using the discernibility functions defined in Definition 3.11.

Example 3.13 *(Continued from Example 3.10) From Definition 3.11, we obtain*

$\underline{f}^O = ((\mathbb{R}_2 \vee \mathbb{R}_3 \vee \mathbb{R}_5) \wedge \mathbb{R}_3 \wedge \mathbb{R}_5) \wedge (\mathbb{R}_1 \wedge \mathbb{R}_5) = \mathbb{R}_1 \wedge \mathbb{R}_3 \wedge \mathbb{R}_5,$

$\overline{f}^O = (\mathbb{R}_3 \wedge \mathbb{R}_1 \wedge ((\mathbb{R}_4 \vee \mathbb{R}_5) \wedge \mathbb{R}_1 \wedge \mathbb{R}_5)) \wedge (\mathbb{R}_2 \wedge \mathbb{R}_3 \wedge \mathbb{R}_5) = \mathbb{R}_1 \wedge \mathbb{R}_2 \wedge \mathbb{R}_3 \wedge \mathbb{R}_5,$

$\underline{f}^P = (\mathbb{R}_4 \wedge \mathbb{R}_3) \wedge (\mathbb{R}_4 \wedge (\mathbb{R}_3 \vee \mathbb{R}_4 \vee \mathbb{R}_5) \wedge \mathbb{R}_1 \wedge (\mathbb{R}_2 \vee \mathbb{R}_3 \vee \mathbb{R}_4 \vee \mathbb{R}_5) \wedge \mathbb{R}_3) = \mathbb{R}_1 \wedge \mathbb{R}_3 \wedge \mathbb{R}_4,$

and

$\overline{f}^P = (\mathbb{R}_4 \wedge \mathbb{R}_5 \wedge (\mathbb{R}_2 \vee \mathbb{R}_3)) \wedge \mathbb{R}_1 = (\mathbb{R}_1 \wedge \mathbb{R}_2 \wedge \mathbb{R}_4 \wedge \mathbb{R}_5) \vee (\mathbb{R}_1 \wedge \mathbb{R}_3 \wedge \mathbb{R}_4 \wedge \mathbb{R}_5).$

Hence, by virtue of Theorem 3.12, we draw the conclusion that the optimistic lower approximation reducts of MGDHFDIS are $\{\mathbb{R}_1, \mathbb{R}_3, \mathbb{R}_5\}$;

The optimistic upper approximation reducts of MGDHFDIS are $\{\mathbb{R}_1, \mathbb{R}_2, \mathbb{R}_3, \mathbb{R}_5\}$;

The pessimistic lower approximation reducts of MGDHFDIS are $\{\mathbb{R}_1, \mathbb{R}_3, \mathbb{R}_4\}$;

The pessimistic upper approximation reducts of MGDHFDIS are $\{\mathbb{R}_1, \mathbb{R}_2, \mathbb{R}_4, \mathbb{R}_5\}$ *and* $\{\mathbb{R}_1, \mathbb{R}_3, \mathbb{R}_4, \mathbb{R}_5\}$.

4 Conclusion

As two new mathematical approaches to cope with imprecision and uncertainty in data analysis, DHF sets and MGRS theory have their own advantages. Considering the facts, Zhang et al. [47] proposed a MGDHFRS by combining DHF sets and MGRS theory which includes many existing MGRS models as special types, such as MGRSs [22], MGFRSs in a fuzzy tolerance approximation space [34] and IFMGRSs [7]. Since the MGDHFRS includes both ingredients of DHF sets and MGRSs, it is more effective and flexible than both DHF sets and MGRSs to handle imprecise and imperfect information. In this study, in order to further investigate the applications of MGDHFRSs, we present a reduction method in MGDHFDIS based on MGDHFRSs. An example is also provided to illustrate the validity of this method. Generally, this reduction approach based on discernibility functions can be extended to other various rough set models in the context of defining discernibility functions.

In the future, topological structures of the MGDHFRSs are the main research direction considered by our group. Moreover, it is important and interesting to further investigate the applications of the MGDHFRSs.

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THE FEKETE-SZEGÖ PROBLEM FOR SOME CLASSES OF ANALYTIC FUNCTIONS

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Abstract. Given an analytic standardly normalized function g in $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, by $\mathcal{C}(g)$ will be denoted the class of analytic standardly normalized function f such that

$$\operatorname{Re}\left\{e^{i\delta}\frac{zf'(z)}{g(z)}\right\} > 0, \quad z \in \mathbb{D},$$

for some $\delta \in (-\pi/2, \pi/2)$. For the class $\mathcal{C}(g)$ the Fekete-Szegö problem is examined.

1. Introduction

In [3] Fekete and Szegö found the maximum value of the coefficient functional

$$\Phi_\lambda(f) := \left|a_3 - \lambda a_2^2\right|, \quad \lambda \in [0,1],$$

over the class $\mathcal{S}$ of univalent functions f in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}. \tag{1.1}$$

By applying the Loewner method they proved that

$$\max_{f\in\mathcal{S}} \Phi_\lambda(f) = \begin{cases} 1 + 2\exp\left(-2\lambda/(1-\lambda)\right), & \lambda \in [0,1), \\ 1, & \lambda = 1. \end{cases}$$

The problem of calculating $\max_{f\in\mathcal{F}} \Phi_\lambda(f)$ for various compact subclasses $\mathcal{F}$ of the class $\mathcal{A}$ of all analytic functions f in $\mathbb{D}$ of the form (1.1), as well as for λ being an arbitrary real or complex number, was considered by many authors (see e.g., [8], [12], [23], [14], [10], [20], [13], [2]).

Let $\mathcal{S}^*$ denote the class of *starlike* functions, i.e., $f \in \mathcal{S}^*$ if $f \in \mathcal{A}$ and

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > 0, \quad z \in \mathbb{D}.$$

Let $\mathcal{S}^c$ denote the class of *convex* functions, i.e., $f \in \mathcal{S}^c$ if $f \in \mathcal{A}$ and

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0, \quad z \in \mathbb{D}.$$

Given $\delta \in (-\pi/2, \pi/2)$ and $g \in \mathcal{A}$, let $\mathcal{C}_\delta(g)$ denote the class of all functions $f \in \mathcal{A}$ such that

$$\operatorname{Re}\left\{e^{i\delta}\frac{zf'(z)}{g(z)}\right\} > 0, \quad z \in \mathbb{D}. \tag{1.2}$$

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For $g \in \mathcal{A}$ let

$$\mathcal{C}(g) := \bigcup_{\delta \in (-\pi/2,\pi/2)} \mathcal{C}_\delta(g)$$

and for $\delta \in (-\pi/2, \pi/2)$ let

$$\mathcal{C}_\delta := \bigcup_{g \in \mathcal{A}} \mathcal{C}_\delta(g).$$

Given $\delta \in (-\pi/2, \pi/2)$ and $g \in \mathcal{S}^*$, functions in $\mathcal{C}_\delta(g)$ and in $\mathcal{C}(g)$ are called *close-to-convex with argument δ with respect to g* and *close-to-convex with respect to g*, respectively. For $\delta \in (-\pi/2, \pi/2)$ let

$$\mathcal{C}^*_\delta := \bigcup_{g \in \mathcal{S}^*} \mathcal{C}_\delta(g), \quad \mathcal{C}^c_\delta := \bigcup_{g \in \mathcal{S}^c} \mathcal{C}_\delta(g).$$

Let

$$\mathcal{C}^* := \bigcup_{\delta \in (-\pi/2,\pi/2)} \bigcup_{g \in \mathcal{S}^*} \mathcal{C}_\delta(g)$$

denote the class of *close-to-convex* functions and let

$$\mathcal{C}^c := \bigcup_{\delta \in (-\pi/2,\pi/2)} \bigcup_{g \in \mathcal{S}^c} \mathcal{C}_\delta(g).$$

For details on close-to-convex functions see [22, pp. 184-185], [11], [7] (cf. [6, Vol. II, pp. 1-11]). The class $\mathcal{C}^c_0$ was considered in [1].

For the whole class $\mathcal{C}^*$ of close-to-convex functions, the sharp bound of the Fekete-Szegö functional on $\mathbb{R}$ was calculated by Koepf in [14] who extended the earlier result for the class $\mathcal{C}^*_0$ due to Keogh and Merkes [12], namely, he proved that

$$\max_{f \in \mathcal{C}^*} \Phi_\lambda(f) = \max_{f \in \mathcal{C}^*_0} \Phi_\lambda(f) = \begin{cases} |3 - 4\lambda|, & \lambda \in (-\infty, 1/3] \cup [1, +\infty), \\ 1/3 + 4/(9\lambda), & \lambda \in [1/3, 2/3], \\ 1, & \lambda \in [2/3, 1]. \end{cases} \tag{1.3}$$

For the class $\mathcal{C}^c$ of close-to-convex functions with respect to convex functions, the sharp bound of the Fekete-Szegö functional on the interval $[0, 1]$ was studied by Srivastava, Mishra and Das in [25], who extended the earlier result for the class $\mathcal{C}^c_0$ due to Abdel-Gawad and Thomas [1]. By Theorem 3 of [1], Theorems 1 to 4 of [25] and by observation in Section 2 of the paper [18], the following result holds:

$$\max_{f \in \mathcal{C}^c} \Phi_\lambda(f) = \max_{f \in \mathcal{C}^c_0} \Phi_\lambda(f) = \begin{cases} 5/3 - 9\lambda/4, & \lambda \in [0, 2/9] \\ 2/3 + 1/(9\lambda), & \lambda \in [2/9, 2/3], \end{cases} \tag{1.4}$$

and

$$\max_{f \in \mathcal{C}^c} \Phi_\lambda(f) \leqq 5/6, \quad \lambda \in (2/3, 1]. \tag{1.5}$$

Given $\alpha \in [0, 1]$, let

$$g_\alpha(z) := \frac{z}{(1 - \alpha z)^2}, \quad z \in \mathbb{D},$$

and

$$h_\alpha(z) := \frac{z}{1 - \alpha z}, \quad z \in \mathbb{D}.$$

The corresponding classes $\mathcal{C}(g_\alpha)$ and $\mathcal{C}(h_\alpha)$ are defined, respectively, by the following conditions:

$$\operatorname{Re}\left\{ e^{i\delta}(1 - \alpha z)^2 f'(z) \right\} > 0, \quad z \in \mathbb{D}, \tag{1.6}$$

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and

$$\operatorname{Re}\left\{e^{i\delta}(1-\alpha z)f'(z)\right\} > 0, \quad z \in \mathbb{D}, \tag{1.7}$$

where $\delta \in (-\pi/2, \pi/2)$.

The upper bound on the Fekete-Szegö functional for the class $\mathcal{C}(g_\alpha)$ was obtained in [15], where it was shown that

$$\begin{aligned}&\max_{f\in\mathcal{C}(g_\alpha)} \Phi_\lambda(f)\\ &\le \begin{cases} \left|\dfrac{2}{3}+\dfrac{4}{3}\alpha+\alpha^2-(1+\alpha)^2\lambda\right|, & \lambda \in \mathbb{R}\setminus(\tau_1(\alpha),\tau_2(\alpha)),\\ \dfrac{2}{3}+\alpha^2\left(\dfrac{1}{3}\cdot\dfrac{(2-3\lambda)^2}{2-|2-3\lambda|}+|1-\lambda|\right), & \lambda\in[\tau_1(\alpha),\tau_2(\alpha)],\end{cases}\end{aligned} \tag{1.8}$$

where

$$\tau_1(\alpha) := \frac{2\alpha}{3(1+\alpha)}, \quad \tau_2(\alpha) := \frac{2(2+\alpha)}{3(1+\alpha)}.$$

As it is well known, the Koebe function $k := g_1$ and the function $h := h_1$ are extremal for various computational problems in the class $\mathcal{S}^*$ of starlike and in the class $\mathcal{S}^c$ of convex functions, respectively. The Fekete-Szegö problem was separately considered for the class $\mathcal{C}(k)$ in [16] and for the class $\mathcal{C}(h)$ in [17], i.e., when $\alpha := 1$ in (1.6) and (1.7). Setting $\alpha := 1$ into (1.8) we get the result for the class $\mathcal{C}(k)$.

For $\alpha := 0$ the condition (1.6) as well as (1.7) is of the form

$$\operatorname{Re}\left\{e^{i\delta}f'(z)\right\} > 0, \quad z \in \mathbb{D}. \tag{1.9}$$

Functions f having such a property are called *of bounded turning with argument* δ and form the class $\mathcal{C}_\delta(h)$ denoted usually as $\mathcal{P}'(\delta)$, and further the class $\mathcal{P}'$ of functions called *of bounded turning* (cf. [6, Vol. I, p. 101]). On the other hand, the condition (1.7) is known as a famous criterium of univalence due to Noshiro [21] and Warschawski [27] (cf. [6, p. 88]). By setting $\alpha := 0$ into (1.8) we get the following result published, among other results, in [10, Theorem 2.3]:

$$\max_{f\in\mathcal{P}'} \Phi_\lambda(f) = \frac{2}{3}.$$

In this paper we unify mentioned results proving the Fekete-Szegö inequality for the class $\mathcal{C}(g)$ with $g \in \mathcal{A}$ such that

$$|g''(0)| \le 4.$$

2. Main result

By $\mathcal{P}$ we denote the class of all analytic functions p in $\mathbb{D}$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad z \in \mathbb{D}, \tag{2.1}$$

having a positive real part in $\mathbb{D}$. Let

$$L(z) := \frac{1+z}{1-z}, \quad z \in \mathbb{D}.$$

For each $\varepsilon \in \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ let

$$L_\varepsilon(z) := L(\varepsilon z), \quad z \in \mathbb{D}.$$

Clearly $L_\varepsilon \in \mathcal{P}$ for every $\varepsilon \in \mathbb{T}$.

The inequalities (2.2) and (2.3) below are well known. They can be found in [24, pp. 41 and 166].

Lemma 2.1. *If $p \in \mathcal{P}$ is of the form* (2.1)*, then*

$$|c_n| \le 2, \quad n \in \mathbb{N}, \tag{2.2}$$

and

$$\left| c_2 - \frac{c_1^2}{2} \right| \le 2 - \frac{|c_1|^2}{2}. \tag{2.3}$$

Both inequalities are sharp. The equality in (2.2) *holds for every function $L_\varepsilon \in \mathcal{P}$, $\varepsilon \in \mathbb{T}$. The equality in* (2.3) *holds for every function*

$$\begin{aligned} p_{t,\theta}(z) &:= tL\left(\mathrm{e}^{\mathrm{i}\theta} z\right) + (1-t)L\left(\mathrm{e}^{2\mathrm{i}\theta} z^2\right) \\ &= 1 + 2t\mathrm{e}^{\mathrm{i}\theta} z + 2\mathrm{e}^{2\mathrm{i}\theta} z^2 + \dots, \quad z \in \mathbb{D}, \end{aligned}$$

where $t \in [0,1]$ and $\theta \in \mathbb{R}$.

Now we prove the main theorem of this paper. The idea of the proof is based on the Koepf's method [14] of calculating Φ_λ for close-to-convex functions with λ restricted to the interval $(1/2, 2/3)$. However, we apply it homogenously for the class $\mathcal{C}(g)$ for all real λ in the same manner as in [15] and [16]. Also the Laguerre's rule of counting zeros of polynomials in an interval is the key tool in the proof.

We recall shortly the Laguerre's rule of counting zeros of polynomials in an interval (see [19], [9], [26, pp. 19-20]). Given a real polynomial

$$Q(u) = a_0 u^n + a_1 u^{n-1} + \cdots + a_{n-1} u + a_n \tag{2.4}$$

consider a finite sequence (q_k), $k = 0, 1, \dots, n$, of polynomials of the form

$$q_k(u) = \sum_{j=0}^{k} a_j u^{k-j}. \tag{2.5}$$

For each $u_0 \in \mathbb{R}$ let $N(Q; u_0)$ denote the number of sign changes in the sequence $(q_k(u_0))$, $k = 0, 1, \dots, n$. Given an interval $I \subset \mathbb{R}$, denote by $Z(Q; I)$ the number of zeros of Q in I counted with their orders. Then the following theorem due to Laguerre holds.

Theorem 2.2. *If $a < b$, $Q(a) \neq 0$ and $Q(b) \neq 0$, then $Z(Q; [a,b]) = N(Q; a) - N(Q; b)$ or $N(Q; a) - N(Q; b) - Z(Q; [a,b])$ is an even positive integer.*

Note that $q_k(0) = a_k$ and $q_k(1) = \sum_{j=0}^{k} a_j$. Thus in the case of the interval $[0,1]$ Theorem 2.2 reduces to the following useful corollary.

Corollary 2.3. *If $Q(0) \neq 0$ and $Q(1) \neq 0$, then $Z(Q; [0,1]) = N(Q; 0) - N(Q; 1)$ or $N(Q; 0) - N(Q; 1) - Z(Q; [0,1])$ is an even positive integer, where $N(Q; 0)$ and $N(Q; 1)$ are the numbers of sign changes in the sequence of polynomial coefficients (a_k) and in the sequence of sums $\left(\sum_{j=0}^{k} a_j\right)$, where $k = 0, 1, \dots, n$, respectively.*

The main theorem of the paper is

Theorem 2.4. *If $g \in \mathcal{A}$ is of the form*

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \quad z \in \mathbb{D}, \tag{2.6}$$

with

$$|b_2| \le 2, \tag{2.7}$$

then

$$\max_{f\in\mathcal{C}(g)} \Phi_\lambda(f) \le \begin{cases} \left|\frac{1}{3}b_3 - \frac{1}{4}\lambda b_2^2\right| + (1+|b_2|)\left|\frac{2}{3}-\lambda\right|, & \lambda \in \mathbb{R}\setminus[\tau_1(|b_2|),\tau_2(|b_2|)], \\ \left|\frac{1}{3}b_3 - \frac{1}{4}\lambda b_2^2\right| + \frac{(2-3\lambda)^2|b_2|^2}{12(2-|2-3\lambda|)} + \frac{2}{3}, & \lambda \in [\tau_1(|b_2|),\tau_2(|b_2|)], \end{cases} \tag{2.8}$$

where

$$\tau_1(|b_2|) := \frac{2|b_2|}{3(|b_2|+2)}, \quad \tau_2(|b_2|) := \frac{2(|b_2|+4)}{3(|b_2|+2)}. \tag{2.9}$$

Proof. Let $g \in \mathcal{A}$ be of the form (2.6) and $f \in \mathcal{C}(g)$ be of the form (1.1). Observe that $f \in \mathcal{C}(g)$ if and only if

$$zf'(z) = \mathrm{e}^{-\mathrm{i}\delta} g(z)\left(p(z)\cos\delta + \mathrm{i}\sin\delta\right), \quad z \in \mathbb{D}, \tag{2.10}$$

for some $\delta \in (-\pi/2, \pi/2)$ and $p \in \mathcal{P}$. Setting the series (1.1), (1.3) and (2.1) into (2.10) by comparing coefficients we get

$$\begin{aligned} a_2 &= \frac{1}{2}\left(c_1\mathrm{e}^{-\mathrm{i}\delta}\cos\delta + b_2\right), \\ a_3 &= \frac{1}{3}\left(c_2\mathrm{e}^{-\mathrm{i}\delta}\cos\delta + c_1 b_2\mathrm{e}^{-\mathrm{i}\delta}\cos\delta + b_3\right). \end{aligned} \tag{2.11}$$

Let $\lambda \in \mathbb{R}$. Using (2.3) from the above we have

$$\begin{aligned} \Phi_\lambda(f) &= \left|a_3 - \lambda a_2^2\right| \\ &= \Big|\frac{1}{3}c_2\mathrm{e}^{-\mathrm{i}\delta}\cos\delta + \frac{1}{3}c_1b_2\mathrm{e}^{-\mathrm{i}\delta}\cos\delta + \frac{1}{3}b_3 \\ &\quad -\frac{1}{4}\lambda\left(c_1^2\mathrm{e}^{-2\mathrm{i}\delta}\cos^2\delta + 2c_1b_2\mathrm{e}^{-\mathrm{i}\delta}\cos\delta + b_2^2\right)\Big| \\ &= \Big|\frac{1}{3}b_3 - \frac{1}{4}\lambda b_2^2 + \frac{1}{3}\left(c_2 - \frac{c_1^2}{2}\right)\mathrm{e}^{-\mathrm{i}\delta}\cos\delta \\ &\quad +\frac{1}{6}c_1^2\left(1-\frac{3}{2}\lambda\mathrm{e}^{-\mathrm{i}\delta}\cos\delta\right)\mathrm{e}^{-\mathrm{i}\delta}\cos\delta + \left(\frac{1}{3}-\frac{1}{2}\lambda\right)c_1b_2\mathrm{e}^{-\mathrm{i}\delta}\cos\delta\Big| \\ &\le \left|\frac{1}{3}b_3 - \frac{1}{4}\lambda b_2^2\right| + \frac{1}{3}\left(2-\frac{|c_1|^2}{2}\right)\cos\delta + \frac{|c_1|^2}{6}\left|1-\frac{3}{2}\lambda\mathrm{e}^{-\mathrm{i}\delta}\cos\delta\right|\cos\delta \\ &\quad + \left|\frac{1}{3}-\frac{1}{2}\lambda\right||c_1||b_2|\cos\delta \\ &= \left|\frac{1}{3}b_3 - \frac{1}{4}\lambda b_2^2\right| + \left(\frac{2}{3} + \frac{|c_1|^2}{6}\left(\sqrt{1-\left(3\lambda-\frac{9}{4}\lambda^2\right)\cos^2\delta} - 1\right)\right. \\ &\quad \left. + \left|\frac{1}{3}-\frac{1}{2}\lambda\right||c_1||b_2|\right)\cos\delta. \end{aligned} \tag{2.12}$$

Set $x := |c_1|$ and $y := \cos\delta$. Clearly, $y \in (0,1]$ and, in view of (2.2), $x \in [0,2]$. It is convenient to use in further computation $\gamma := 2 - 3\lambda$ instead of λ. For $\gamma \in \mathbb{R}$ let

$$s_\gamma(y) := \sqrt{1 - \left(1 - \frac{1}{4}\gamma^2\right) y^2}, \quad y \in [0,1].$$

By the assumption (2.7), set $|b_2| := 2\alpha$, where $\alpha \in [0,1]$. Set $R := [0,2] \times [0,1]$. For $\alpha \in [0,1]$ and $\gamma \in \mathbb{R}$ define

$$F_{\alpha,\gamma}(x,y) := \left(4 + x^2\left(s_\gamma(y) - 1\right) + 2\alpha|\gamma|x\right) y, \quad (x,y) \in R.$$

Hence and from (2.12) we have

$$\max_{f\in\mathcal{C}(g)} \Phi_\lambda(f) \le \left|\frac{1}{3}b_3 - \frac{1}{4}\lambda b_2^2\right| + \frac{1}{6}\max_{(x,y)\in R} F_{\alpha,\gamma}(x,y). \tag{2.13}$$

Now for $\alpha \in [0,1]$ and $\gamma \in \mathbb{R}$ we find the maximum value of $F_{\alpha,\gamma}$ on the rectangle R.

1. In the corners of R we have

$$\begin{aligned} F_{\alpha,\gamma}(0,0) &= F_{\alpha,\gamma}(2,0) = 0, \\ F_{\alpha,\gamma}(0,1) &= 4, \\ F_{\alpha,\gamma}(2,1) &= 2(1+2\alpha)|\gamma|. \end{aligned} \tag{2.14}$$

2. For $x = 0$ and $y \in (0,1)$ we have a linear function and for $x \in (0,2)$ and $y = 0$ we have a constant function.

3. For $x \in (0,2)$ and $y = 1$, let

$$G_{\alpha,\gamma}(x) := F_{\alpha,\gamma}(x,1) = \frac{1}{2}\left(|\gamma| - 2\right)x^2 + 2\alpha|\gamma|x + 4.$$

For $|\gamma| = 2$ we get the linear functions, so let $|\gamma| \ne 2$. Then $G'_{\alpha,\gamma}(x) = 0$ if and only if

$$x = \frac{2\alpha|\gamma|}{2 - |\gamma|} =: x_{\alpha,\gamma}.$$

Thus $x_{\alpha,\gamma} \in (0,2)$ if and only if

$$\alpha \ne 0 \wedge 0 < \frac{\alpha|\gamma|}{2 - |\gamma|} < 1. \tag{2.15}$$

The left-hand inequality in (2.15) holds if and only if

$$\alpha \ne 0 \wedge 0 < |\gamma| < 2. \tag{2.16}$$

We can write the right-hand inequality (2.15) as

$$\frac{(1+\alpha)|\gamma| - 2}{2 - |\gamma|} < 0$$

and, in view of (2.16), it holds when $|\gamma| < 2/(1+\alpha)$. But $2/(1+\alpha) < 2$ for $\alpha \in (0,1]$, so this with (2.16) yields that $x_{\alpha,\gamma} \in (0,2)$ if and only if

$$\alpha \ne 0 \wedge 0 < |\gamma| < \frac{2}{1+\alpha}. \tag{2.17}$$

Thus the function $G_{\alpha,\gamma}$ has a critical point in $(0,2)$, namely, $x_{\alpha,\gamma}$ as the unique one, if and only if (2.17) holds. Moreover we have

$$F_{\alpha,\gamma}(x_{\alpha,\gamma},1) = G_{\alpha,\gamma}\left(x_{\alpha,\gamma}\right) = \frac{2\alpha^2\gamma^2}{2 - |\gamma|} + 4. \tag{2.18}$$

4. For $x = 2$ and $y \in (0,1)$, let

$$H_{\alpha,\gamma}(y) := F_{\alpha,\gamma}(2,y) = 4\left(ys_\gamma(y) + \alpha|\gamma|y\right).$$

For $|\gamma| = 2$ we have the linear functions evidently, so let $|\gamma| \neq 2$. Note first that

$$s_\gamma(y) > 0, \quad y \in (0,1), \tag{2.19}$$

since the equation $s_\gamma(y) = 0$, $y \in (0,1)$, equivalently written as

$$(4 - \gamma^2)y^2 = 4, \quad y \in (0,1), \tag{2.20}$$

has no solution. Indeed, as $y^2 > 0$, we have $|\gamma| < 2$. But from (2.20) we obtain

$$y^2 = \frac{4}{4-\gamma^2} > 1,$$

which is a contradiction. Thus (2.20) has no solution, so (2.19) holds. Taking into account (2.19) we have

$$ys'_\gamma(y) = \frac{-\left(1 - \frac{1}{4}\gamma^2\right)y^2}{\sqrt{1 - \left(1 - \frac{1}{4}\gamma^2\right)y^2}} = \frac{s_\gamma^2(y) - 1}{s_\gamma(y)}, \quad y \in (0,1). \tag{2.21}$$

Using (2.21) we get

$$H'_{\alpha,\gamma}(y) = 4\left(s_\gamma(y) + \frac{s_\gamma^2(y) - 1}{s_\gamma(y)} + \alpha|\gamma|\right), \quad y \in (0,1).$$

Hence

$$H'_{\alpha,\gamma}(y) = 0 \tag{2.22}$$

if and only if

$$2s_\gamma^2(y) + \alpha|\gamma|s_\gamma(y) - 1 = 0,$$

i.e., in view of (2.19) if and only if

$$s_\gamma(y) = \frac{-\alpha|\gamma| + \sqrt{8 + \alpha^2\gamma^2}}{4} =: s_{\alpha,\gamma}. \tag{2.23}$$

As $|\gamma| \neq 2$, so from the above we get the equation

$$y^2 = \frac{4 - \alpha^2\gamma^2 + \alpha|\gamma|\sqrt{8 + \alpha^2\gamma^2}}{2(4 - \gamma^2)}. \tag{2.24}$$

Thus the solution of the equation (2.24), and hence of (2.22), exists if and only if

$$0 < \frac{4 - \alpha^2\gamma^2 + \alpha|\gamma|\sqrt{8 + \alpha^2\gamma^2}}{2(4 - \gamma^2)} < 1. \tag{2.25}$$

Let $|\gamma| < 2$. The left-hand inequality in (2.25) is clearly true since $4 - \alpha^2\gamma^2 > 0$. Write the right-hand inequality in (2.25) equivalently as

$$\alpha|\gamma|\sqrt{8 + \alpha^2\gamma^2} < 4 - (2 - \alpha^2)\gamma^2. \tag{2.26}$$

The above inequality can hold only when

$$|\gamma| < \frac{2}{\sqrt{2 - \alpha^2}}. \tag{2.27}$$

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But $2/\sqrt{2-\alpha^2} \le 2$, so squaring (2.26) and reducing we equivalently have

$$(1-\alpha^2)\gamma^4 - 4\gamma^2 + 4 > 0. \tag{2.28}$$

Let $\alpha = 1$. Then, taking into account (2.27), the inequality (2.28) holds if and only if $|\gamma| < 1$. Let $\alpha \in [0,1)$. Then (2.28) holds if and only if

$$|\gamma| > \sqrt{\frac{2}{1-\alpha}} \quad \text{or} \quad |\gamma| < \sqrt{\frac{2}{1+\alpha}}.$$

Hence, from (2.27) and by the fact that for $\alpha \in [0,1)$,

$$\sqrt{\frac{2}{1+\alpha}} \le \frac{2}{\sqrt{2-\alpha^2}} \le \sqrt{\frac{2}{1-\alpha}},$$

we see that (2.28) and, consequently, (2.25) holds if and only if

$$|\gamma| < \sqrt{\frac{2}{1+\alpha}}. \tag{2.29}$$

In this way, we proved that for $\alpha \in [0,1]$, the inequality (2.28), so (2.25) holds if and only if (2.29) holds.

Let $|\gamma| > 2$. Then the left-hand inequality in (2.25) holds if and only if

$$\alpha|\gamma|\sqrt{8+\alpha^2\gamma^2} < \alpha^2\gamma^2 - 4. \tag{2.30}$$

Note that $\alpha^2\gamma^2 - 4 \le 0$ for $|\gamma| \le 2/\alpha$, so then (2.30) is false. Assume that $|\gamma| > 2/\alpha$. Squaring (2.30), after reducing, we get $|\gamma| < 1/\alpha$, which contradicts the assumption.

Thus we proved that the function $H_{\alpha,\gamma}$ has a critical point in $(0,1)$, namely,

$$y = \sqrt{\frac{4-\alpha^2\gamma^2+\alpha|\gamma|\sqrt{\alpha^2\gamma^2+8}}{2(4-\gamma^2)}} =: y_{\alpha,\gamma},$$

as the unique solution of (2.24), if and only if (2.29) holds. Moreover,

$$\begin{aligned} F_{\alpha,\gamma}(2, y_{\alpha,\gamma}) &= H_{\alpha,\gamma}(y_{\alpha,\gamma}) \\ &= \sqrt{\frac{4-\alpha^2\gamma^2+\alpha|\gamma|\sqrt{8+\alpha^2\gamma^2}}{2(4-\gamma^2)}}\left(\sqrt{8+\alpha^2\gamma^2}+3\alpha|\gamma|\right). \end{aligned} \tag{2.31}$$

5. We will prove that for each $\alpha \in [0,1]$ and $\gamma \in \mathbb{R}$ the function $F_{\alpha,\gamma}$ has no critical point in $(0,2)\times(0,1)$.

We have

$$\frac{\partial F_{\alpha,\gamma}}{\partial x} = 0$$

if and only if

$$y\left(x\left(s_\gamma(y)-1\right)+\alpha|\gamma|\right) = 0,$$

and since $y \neq 0$ and $x \neq 0$, if and only if

$$s_\gamma(y) = 1 - \frac{\alpha|\gamma|}{x}, \quad y \in (0,1). \tag{2.32}$$

Observe first that $\gamma \neq 0$ because if $\gamma = 0$, then the equation (2.32) reduces to $s_0(y) = 1,\ y \in (0,1)$, which has no solution in $(0,1)$.

If $\alpha = 0$, then the equation (2.32) reduces to $s_\gamma(y) = 1,\ y \in (0,1)$, which is satisfied if and only if $|\gamma| = 2$ and $y \in (0,1)$ is any. On the other hand, if $|\gamma| = 2$, then the equation (2.32) is satisfied for $\alpha = 0$ only.

Since $x > 0$, by comparing (2.32) and (2.19), we additionally see that $x > \alpha|\gamma|$.

Thus the solution of (2.32) can exist only when

$$(\alpha = 0 \wedge |\gamma| = 2) \vee (\alpha \neq 0 \wedge \gamma \neq 0 \wedge |\gamma| \neq 2 \wedge x > \alpha|\gamma|). \tag{2.33}$$

Squaring then (2.32) we obtain

$$s_\gamma^2(y) - 1 = -\frac{2\alpha|\gamma|}{x} + \frac{\alpha^2\gamma^2}{x^2}. \tag{2.34}$$

Since by (2.19), $s_\gamma(y) \neq 0$ for $y \in (0,1)$, taking into account (2.21) we have

$$\frac{\partial F_{\alpha,\gamma}}{\partial y} = 4 + x^2\left(s_\gamma(y) - 1\right) + 2\alpha|\gamma|x + \frac{(s_\gamma^2(y) - 1)x^2}{s_\gamma(y)}.$$

Thus, by using (2.32) and (2.34), we have

$$\frac{\partial F_{\alpha,\gamma}}{\partial y} = 0$$

if and only if

$$4 + x^2\left(-\frac{\alpha|\gamma|}{x}\right) + 2\alpha|\gamma|x + \frac{\left(-\frac{2\alpha|\gamma|}{x} + \frac{\alpha^2\gamma^2}{x^2}\right)x^2}{1 - \frac{\alpha|\gamma|}{x}} = 0,$$

and after simplifying, if and only if

$$4 + \alpha|\gamma|x + \frac{-2\alpha|\gamma|x^2 + \alpha^2\gamma^2 x}{x - \alpha|\gamma|} = 0.$$

Thus

$$\alpha|\gamma|x^2 - 4x + 4\alpha|\gamma| = 0, \quad x \in (0,2). \tag{2.35}$$

Note first that for $\alpha = 0$ the equation (2.35) has no solution. Let $\alpha \neq 0$. From (2.33), $\gamma \neq 0$. Then the discriminant $\Delta = 16(1 - \alpha^2\gamma^2) \geq 0$ if and only if $0 < |\gamma| \leq 1/\alpha$. Note that $\Delta = 0$ if and only if $|\gamma| = 1/\alpha$, and then the equation (2.35) has no solution. Thus the equation (2.35) has no root when $|\gamma| \geq 1/\alpha$. Consequently, for $\alpha = 0$ and $\gamma \in \mathbb{R}$ as well as for $\alpha \in (0,1]$ and $|\gamma| \geq 1/\alpha$ the function $F_{\alpha,\gamma}$ has no critical point in $(0,2) \times (0,1)$.

Thus by (2.33) we consider

$$\alpha \neq 0 \wedge |\gamma| \neq 2 \wedge 0 < |\gamma| < 1/\alpha \wedge x > \alpha|\gamma|. \tag{2.36}$$

Solving now (2.35) we have

$$x = \frac{2 - 2\sqrt{1 - \alpha^2\gamma^2}}{\alpha|\gamma|} =: x_{1;\alpha,\gamma}, \quad x = \frac{2 + 2\sqrt{1 - \alpha^2\gamma^2}}{\alpha|\gamma|} =: x_{2;\alpha,\gamma}.$$

Since $x_{2;\alpha,\gamma} > 0$ and $x_{1;\alpha,\gamma}x_{2;\alpha,\gamma} = 4$, so we immediately see that $0 < x_{1;\alpha,\gamma} < 2 < x_{2;\alpha,\gamma}$. Thus $x_{2;\alpha,\gamma} \notin (0,2)$ and it remains to consider $x_{1;\alpha,\gamma}$.

Observe that $x_{1;\alpha,\gamma} > \alpha|\gamma|$. Indeed, this follows from the fact that the inequality

$$\frac{2 - 2\sqrt{1 - \alpha^2\gamma^2}}{\alpha|\gamma|} > \alpha|\gamma|$$

is equivalent to

$$2 - \alpha^2\gamma^2 > 2\sqrt{1 - \alpha^2\gamma^2},$$

which is evidently true for $0 < |\gamma| < 1/\alpha$.

Setting $x := x_{1;\alpha,\gamma}$ into (2.34) we have

$$s_\gamma^2(y) - 1 = -\frac{2\alpha|\gamma|}{x_{1;\alpha,\gamma}} + \frac{\alpha^2\gamma^2}{x_{1;\alpha,\gamma}^2}.$$

Hence

$$\begin{aligned} y^2 &= \frac{\dfrac{2\alpha|\gamma|}{x_{1;\alpha,\gamma}} - \dfrac{\alpha^2\gamma^2}{x_{1;\alpha,\gamma}^2}}{1-\dfrac{1}{4}\gamma^2} = \frac{2\alpha|\gamma|x_{1;\alpha,\gamma} - \alpha^2\gamma^2}{x_{1;\alpha,\gamma}^2\left(1-\dfrac{1}{4}\gamma^2\right)} \\ &= \frac{\left(4-\alpha^2\gamma^2 - 4\sqrt{1-\alpha^2\gamma^2}\right)\alpha^2\gamma^2}{\left(1-\sqrt{1-\alpha^2\gamma^2}\right)^2(4-\gamma^2)}. \end{aligned} \tag{2.37}$$

A solution in $(0,1)$ of (2.37) exists if and only if

$$0 < \frac{\left(4-\alpha^2\gamma^2 - 4\sqrt{1-\alpha^2\gamma^2}\right)\alpha^2\gamma^2}{\left(1-\sqrt{1-\alpha^2\gamma^2}\right)^2(4-\gamma^2)} < 1. \tag{2.38}$$

By (2.36) consider

$$\alpha \neq 0 \wedge |\gamma| \neq 2 \wedge 0 < |\gamma| < \frac{1}{\alpha}. \tag{2.39}$$

We will prove that then the condition (2.38) is false.

(A) Suppose that $2 < |\gamma| < 1/\alpha$. Since, as easy to check, the left-hand side of the inequality

$$4-\alpha^2\gamma^2 > 4\sqrt{1-\alpha^2\gamma^2} \tag{2.40}$$

is positive, by squaring and computing, we equivalently get the inequality

$$\alpha^2\gamma^2 + 8 > 0,$$

which is true. Hence and by the fact that $4-\gamma^2 < 0$ we see that the left-hand inequality in (2.38) is false.

(B) By (2.39) it remains to consider

$$\alpha \neq 0 \wedge 0 < |\gamma| < \frac{1}{\alpha} \le 2.$$

(a) As in Part (A), we prove that (2.40) holds. Hence and by the fact that $4-\gamma^2 > 0$ we see that the left-hand inequality in (2.38) holds.

(b) Since $4-\gamma^2 > 0$, write the right-hand inequality in (2.38) as

$$\left(4-\alpha^2\gamma^2 - 4\sqrt{1-\alpha^2\gamma^2}\right)\alpha^2\gamma^2 < \left(1-\sqrt{1-\alpha^2\gamma^2}\right)^2(4-\gamma^2)$$

and, after computing, equivalently as

$$\left(8-2(1+2\alpha^2)\gamma^2\right)\sqrt{1-\alpha^2\gamma^2} < (\alpha^4+\alpha^2)\gamma^4 - 2(1+4\alpha^2)\gamma^2 + 8. \tag{2.41}$$

We will show that (2.41) is false. To verify it, we will prove that the inequality

$$s_\alpha(t) \ge r_\alpha(t), \quad t \in \left(0, 1/\alpha^2\right), \tag{2.42}$$

holds, where

$$s_\alpha(t) := \left(8-2(1+2\alpha^2)t\right)\sqrt{1-\alpha^2 t}, \quad t \in \left[0, 1/\alpha^2\right],$$

and

$$r_\alpha(t) := (\alpha^4+\alpha^2)t^2 - 2(1+4\alpha^2)t + 8, \quad t \in \left[0, 1/\alpha^2\right].$$

Then substituting $t := \gamma^2$ into (2.42), we get the true inequality which shows that (2.41) is false.

Let

$$w_\alpha(t) := s_\alpha^2(t) - r_\alpha^2(t), \quad t \in \left[0, 1/\alpha^2\right].$$

Thus after computing we have

$$w_\alpha(t) = \alpha^4 t^3 \left(4 - (1+\alpha^2)^2 t\right), \quad t \in \left[0, 1/\alpha^2\right]. \tag{2.43}$$

Note that $w_\alpha(t) = 0$ if and only if

$$t = 0 \vee t = \frac{4}{(1+\alpha^2)^2} =: t_\alpha,$$

since, as easy to check, $t_\alpha \in \left[0, 1/\alpha^2\right]$ for $\alpha \in (0,1]$.

Let $\alpha := 1$. Then $t_1 = 1$ and by (2.43),

$$w_1(t) = (s_1(t) - r_1(t))(s_1(t) + r_1(t)) = 4t^3(1-t) > 0, \quad t \in (0,1).$$

Hence and from the fact that

$$s_1(0) + r_1(0) = 16 > 0,$$

it follows that

$$s_1(t) - r_1(t) > 0, \quad t \in (0,1),$$

which confirms (2.42).

Let now $\alpha \in (0,1)$. Then by (2.43),

$$w_\alpha(t) = (s_\alpha(t) - r_\alpha(t))(s_\alpha(t) + r_\alpha(t)) > 0, \quad t \in (0, t_\alpha), \tag{2.44}$$

and

$$w_\alpha(t) = (s_\alpha(t) - r_\alpha(t))(s_\alpha(t) + r_\alpha(t)) < 0, \quad t \in \left(t_\alpha, 1/\alpha^2\right). \tag{2.45}$$

Since

$$s_\alpha(0) + r_\alpha(0) = 16 > 0,$$

from (2.44) it follows that

$$s_\alpha(t) - r_\alpha(t) > 0, \quad t \in (0, t_\alpha). \tag{2.46}$$

Similarly, since

$$s_\alpha\left(\frac{1}{\alpha^2}\right) + r_\alpha\left(\frac{1}{\alpha^2}\right) = 1 - \frac{1}{\alpha^2} < 0,$$

from (2.45) it follows that

$$s_\alpha(t) - r_\alpha(t) > 0, \quad t \in \left(t_\alpha, 1/\alpha^2\right). \tag{2.47}$$

Thus from (2.46), (2.47) and by the continuity of the functions s_α and r_α at $t := t_\alpha$, we have

$$s_\alpha(t) - r_\alpha(t) \geq 0, \quad t \in \left(0, 1/\alpha^2\right),$$

which confirms (2.42).

Thus, taking into account Parts (A) and (B), we proved that (2.41) is false, so the condition (2.38) does not hold and, therefore the equation (2.37) has no solution in $(0,1)$.

In this way, the proof that for $\alpha \in [0,1]$ and $\gamma \in \mathbb{R}$ the function $F_{\alpha,\gamma}$ has no critical point in $(0,2) \times (0,1)$ is finished.

6. Now we calculate the maximum value of $F_{\alpha,\gamma}$ in R, which, as was shown, is attained on the boundary of R. Let $\alpha \in [0,1]$.

(A) $|\gamma| \geq 2/(1+\alpha)$. Taking into account Part 3 with (2.17) and Part 4 with (2.29), we see that the maximum value of $F_{\alpha,\gamma}$ is attained in a corner of R. Thus by (2.14) it suffices to compare the following values:

$$0, \quad 4, \quad 2(1+2\alpha)|\gamma|. \tag{2.48}$$

Since, for $|\gamma| \geq 2/(1+\alpha)$,

$$2(1+2\alpha)|\gamma| \geq \frac{4+8\alpha}{1+\alpha} \geq 4,$$

so from (2.48) we have

$$\max_{(x,y)\in R} F_{\alpha,\gamma}(x,y) = F_{\alpha,\gamma}(2,1) = 2(1+2\alpha)|\gamma|. \tag{2.49}$$

(B) $\sqrt{2/(1+\alpha)} \leq |\gamma| < 2/(\alpha+1)$. Taking into account Part 4 with (2.17) and Part 5 with (2.29), we see that the maximum value of $F_{\alpha,\gamma}$ is attained in a corner of R or in $(x_{\alpha,\gamma},1)$. Thus we compare all values (2.48) and, by (2.18), the value

$$F_{\alpha,\gamma}(x_{\alpha,\gamma},1) = \frac{2\alpha^2\gamma^2}{2-|\gamma|} + 4.$$

Observe that

$$\frac{2\alpha^2\gamma^2}{2-|\gamma|} + 4 \geq 2(1+2\alpha)|\gamma|.$$

Indeed, since $|\gamma| < 2/(\alpha+1) \leq 2$, the above after computing is equivalent to the inequality

$$\left((1+\alpha)^2|\gamma| - 2\right)^2 \geq 0,$$

which clearly holds. Thus

$$\max_{(x,y)\in R} F_{\alpha,\gamma}(x,y) = F_{\alpha,\gamma}(x_{\alpha,\gamma},1) = \frac{2\alpha^2\gamma^2}{2-|\gamma|} + 4. \tag{2.50}$$

(C) $\gamma = 0$. Taking into account Part 3 with (2.17) and Part 4 with (2.29), the maximum value of $F_{\alpha,0}$ is attained in a corner of R or in the point $(2, y_{\alpha,0}) = \left(2, 1/\sqrt{2}\right)$. Thus, by comparing all values (2.48) for $\gamma := 0$ and, by (2.31), the value

$$F_{\alpha,0}(2, y_{\alpha,0}) = 2,$$

we have

$$\max_{(x,y)\in R} F_{\alpha,0}(x,y) = F_{\alpha,0}(0,1) = 4. \tag{2.51}$$

(D) $0 < |\gamma| < \sqrt{2/(\alpha+1)}$. Then we compare all values (2.48) and, by (2.18) and (2.31), $F_{\alpha,\gamma}(x_{\alpha,\gamma},1)$ and $F_{\alpha,\gamma}(2, y_{\alpha,\gamma})$. We will show that the value $F_{\alpha,\gamma}(x_{\alpha,\gamma},1)$ is the largest one.

(D_1) Since $|\gamma| < \sqrt{2/(\alpha+1)} < 2$ for $\alpha \in [0,1]$, so $\alpha^2\gamma^2/(2-|\gamma|) \geq 0$ and therefore

$$F_{\alpha,\gamma}(x_{\alpha,\gamma},1) \geq 4.$$

Moreover, repeating arguments of Part (B), we see that

$$F_{\alpha,\gamma}(x_{\alpha,\gamma},1) \geq 2(1+2\alpha)|\gamma|.$$

(D_2) Thus it remains to prove that

$$F_{\alpha,\gamma}(x_{\alpha,\gamma},1) \geq F_{\alpha,\gamma}(2, y_{\alpha,\gamma}) \tag{2.52}$$

i.e., in view of (2.18) and (2.31), that

$$\frac{2\alpha^2\gamma^2}{2-|\gamma|}+4 \geq \sqrt{\frac{4-\alpha^2\gamma^2+\alpha|\gamma|\sqrt{8+\alpha^2\gamma^2}}{2(4-\gamma^2)}}\left(\sqrt{8+\alpha^2\gamma^2}+3\alpha|\gamma|\right). \tag{2.53}$$

As $|\gamma|<2$, so both sides of (2.53) are positive. Thus squaring (2.53) and computing we equivalently have

$$\alpha^4|\gamma|^5+\left(6\alpha^4-8\alpha^2\right)\gamma^4+\left(20\alpha^2+8\right)|\gamma|^3-\left(8\alpha^2+16\right)\gamma^2-24|\gamma|+48 \geq \alpha|\gamma|(2-|\gamma|)\left(\alpha^2\gamma^2+8\right)^{3/2}. \tag{2.54}$$

To verify that (2.54) holds, we will show that

$$Q_\alpha(u)\geq S_\alpha(u),\quad u\in[0,u_\alpha], \tag{2.55}$$

where $u_\alpha:=\sqrt{2/(\alpha+1)}$,

$$\begin{aligned} Q_\alpha(u) &:= \alpha^4u^5+\left(6\alpha^4-8\alpha^2\right)u^4+\left(20\alpha^2+8\right)u^3 \\ &\quad -\left(8\alpha^2+16\right)u^2-24u+48,\quad u\in[0,u_\alpha], \end{aligned}$$

and

$$S_\alpha(u):=\alpha u(2-u)\left(\alpha^2u^2+8\right)^{3/2},\quad u\in[0,u_\alpha].$$

(1°) $\alpha=0$. Then $u_0=\sqrt{2}$ and the inequality (2.55) reduces to

$$Q_0(u)=u^3-2u^2-3u+6=\left(u^2-3\right)(u-2)>0=S_0(u),\quad u\in\left[0,\sqrt{2}\right],$$

which is true. Thus (2.55) holds, which confirms (2.54).

(2°) $\alpha=1$. Then $u_1=1$ and the inequality (2.55) reduces to

$$u^5-2u^4+28u^3-24u^2-24u+48 \geq u(2-u)\left(u^2+8\right)^{3/2},\quad u\in[0,1]. \tag{2.56}$$

Since

$$\begin{aligned} u^5-2u^4+28u^3-24u^2-24u+48 &\geq u^5-2u^3+28u^3-24-24+48 \\ &= u^5+26u^3\geq 0,\quad u\in[0,1], \end{aligned}$$

so both sides of (2.56) are nonnegative. Thus squaring (2.56) and computing we equivalently get the inequality

$$(u-1)^2\left(2u^6+32u^4+40u^3-92u^2+144u+144\right)\geq 0,\quad u\in[0,1],$$

which clearly holds. To see this, replace u^2 by u. Thus (2.55) holds, which confirms (2.54).

(3°) $\alpha\in(0,1)$. Define

$$V_\alpha(u):=Q_\alpha^2(u)-S_\alpha^2(u),\quad u\in[0,u_\alpha].$$

We will show that

$$V_\alpha(u)>0,\quad u\in[0,u_\alpha], \tag{2.57}$$

i.e., that

$$(Q_\alpha(u)-S_\alpha(u))(Q_\alpha(u)+S_\alpha(u))>0,\quad u\in[0,u_\alpha]. \tag{2.58}$$

Further, taking into account that Q_α and S_α are continuous functions with

$$Q_\alpha(0)-S_\alpha(0)=48>0,$$

from (2.58) we deduce that

$$Q_\alpha(u) - S_\alpha(u) > 0, \quad u \in [0, u_\alpha],$$

i.e., that (2.55) holds.

Now we prove that (2.57) holds, i.e., that the following inequality holds:

$$\begin{aligned} & V_\alpha(u) \\ & = \left(\alpha^4 u^5 + \left(6\alpha^4 - 8\alpha^2\right) u^4 + \left(20\alpha^2 + 8\right) u^3 - \left(8\alpha^2 + 16\right) u^2 - 24u + 48\right)^2 \\ & -\alpha^2 u^2 (2-u)^2 \left(\alpha^2 u^2 + 8\right)^3 > 0, \quad u \in [0, u_\alpha], \end{aligned}$$

which after computation is equivalent to

$$\begin{aligned} V_\alpha(u) &= (\alpha^8 - \alpha^6) u^9 + (2\alpha^8 - 5\alpha^6 + 5\alpha^4) u^8 \\ &+ (20\alpha^6 - 16\alpha^4 - 8\alpha^2) u^7 + (-12\alpha^6 + 6\alpha^4 + 36\alpha^2 + 4) u^6 \\ &+ (16\alpha^4 - 24\alpha^2 - 16) u^5 - (8\alpha^4 + 124\alpha^2 + 8) u^4 + (272\alpha^2 + 96) u^3 \\ &- (176\alpha^2 + 60) u^2 - 144u + 144 =: \sum_{j=0}^{9} a_j u^{9-j} > 0, \quad u \in [0, u_\alpha]. \end{aligned}$$

As in (2.5), let (q_k), $k = 0, 1, \ldots, 9$, be a sequence of polynomials of the form

$$q_k(u) = \sum_{j=0}^{k} a_j u^{k-j}, \quad u \in [0, u_\alpha],$$

corresponding to the polynomial $Q := V_\alpha$ in (2.4) for Laguerre's rule.

(a) Now we check the signs of the elements of the sequence $(q_k(0))$, i.e., of the sequence (a_k) for $k = 0, 1, \ldots, 9$. A simple computing shows that for $\alpha \in (0, 1)$ we have

$$\begin{aligned} q_0(0) &= \alpha^6(\alpha^2 - 1) < 0, \\ q_1(0) &= \alpha^4(2\alpha^4 - 5\alpha^2 + 5) > 0, \\ q_2(0) &= 4\alpha^2(5\alpha^4 - 4\alpha^2 - 2) < 0, \\ q_3(0) &= 2(-6\alpha^6 + 3\alpha^4 + 18\alpha^2 + 2) > 0, \\ q_4(0) &= 8(2\alpha^4 - 3\alpha^2 - 2) < 0, \\ q_5(0) &= -4(2\alpha^4 + 31\alpha^2 + 2) < 0, \\ q_6(0) &= 272\alpha^2 + 96 > 0, \\ q_7(0) &= -176\alpha^2 - 60 < 0, \\ q_8(0) &= -144 < 0, \\ q_9(0) &= 144 > 0. \end{aligned}$$

Hence

$$N(V_\alpha; 0) = 7, \quad \alpha \in (0, 1). \tag{2.59}$$

(b) Now we check the signs of the elements of the sequence $(q_k(u_\alpha))$ for $k = 0, 1, \ldots, 9$.

(i) $k = 0$. We have

$$q_0(u) = \alpha^6(\alpha^2 - 1), \quad u \in [0, u_\alpha].$$

Thus

$$q_0(u_\alpha) < 0, \quad \alpha \in (0, 1). \tag{2.60}$$

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(ii) $k = 1$. We have

$$q_1(u) = \alpha^4 \left((\alpha^4 - \alpha^2)u + 2\alpha^4 - 5\alpha^2 + 5\right), \quad u \in [0, u_\alpha].$$

We will show that

$$q_1(u_\alpha) > 0, \quad \alpha \in (0,1), \tag{2.61}$$

i.e., after computing that

$$-(2\alpha^4 - 5\alpha^2 + 5)\sqrt{\alpha + 1} < (\alpha^4 - \alpha^2)\sqrt{2}, \quad \alpha \in (0,1). \tag{2.62}$$

Observe that since both sides of (2.62) are negative, after squaring and computing we equivalently get

$$4\alpha^8 - 2\alpha^7 - 18\alpha^6 + 2\alpha^5 + 43\alpha^4 - 50\alpha^2 + 25 > 0, \quad \alpha \in (0,1). \tag{2.63}$$

To verify that (2.63) holds, we will show that

$$w(t) > 0, \quad t \in [0,1], \tag{2.64}$$

where

$$\begin{aligned} w(t) &:= 4t^8 - 2t^7 - 18t^6 + 2t^5 + 43t^4 - 50t^2 + 25 \\ &=: \sum_{j=0}^{8} b_j t^{8-j}, \quad t \in [0,1]. \end{aligned}$$

Note that the numbers of sign changes in the sequence of polynomial coefficients (b_k), and in the sequence of sums $\left(\sum_{j=0}^{k} b_j\right)$, where $k = 0, 1, \ldots, 8$, equal 4, i.e., $N(w;0) = N(w;1) = 4$. Applying Corollary 2.3, we see that the polynomial w has no zero in the interval $(0,1)$ and, since $w(0) = 25 > 0$, so (2.64) and, consequently, (2.63) holds. Thus (2.61) is confirmed.

(iii) $k = 2$. We have

$$q_2(u) = (\alpha^8 - \alpha^6)u^2 + (2\alpha^8 - 5\alpha^6 + 5\alpha^4)u + 20\alpha^6 - 16\alpha^4 - 8\alpha^2, \quad u \in [0, u_\alpha].$$

We will show that

$$q_2(u_\alpha) < 0, \quad \alpha \in (0,1), \tag{2.65}$$

i.e., after computing that

$$\begin{aligned} &\sqrt{2}\alpha^2(2\alpha^4 - 5\alpha^2 + 5) \\ &< (-2\alpha^5 - 18\alpha^4 + 16\alpha^2 + 8)\sqrt{\alpha + 1}, \quad \alpha \in (0,1). \end{aligned} \tag{2.66}$$

It is easily seen that the left-hand side of (2.66) is positive and since

$$\begin{aligned} -2\alpha^5 - 18\alpha^4 + 16\alpha^2 + 8 &\geq -2\alpha^2 - 14\alpha^2 - 4\alpha^4 + 16\alpha^2 + 8 \\ &= -4\alpha^4 + 8 > 0, \quad \alpha \in (0,1), \end{aligned} \tag{2.67}$$

so is the right-hand side of (2.66). Thus squaring (2.66) and computing we equivalently get

$$\begin{aligned} &-4\alpha^{12} + 2\alpha^{11} + 58\alpha^{10} + 198\alpha^9 + 85\alpha^8 - 320\alpha^7 - 254\alpha^6 \\ &-32\alpha^5 - 41\alpha^4 + 128\alpha^3 + 128\alpha^2 + 32\alpha + 32 > 0. \end{aligned} \tag{2.68}$$

To verify that (2.68) holds, we will show that

$$w(t) > 0, \quad t \in [0,1], \tag{2.69}$$

where

$$w(t) := -4t^{12} + 2t^{11} + 58t^{10} + 198t^9 + 85t^8 - 320t^7 - 254t^6 -32t^5 - 41t^4 + 128t^3 + 128t^2 + 32t + 32 =: \sum_{j=0}^{12} b_j t^{12-j}, \quad t \in [0,1].$$

Note that the numbers of sign changes in the sequence of polynomial coefficients (b_k) and in the sequence of sums $\left(\sum_{j=0}^{k} b_j\right)$, where $k = 0, 1, \ldots, 12$, equal 3, i.e., $N(w; 0) = N(w; 1) = 3$. Applying Corollary 2.3, we see that the polynomial w has no zero in the interval $(0, 1)$ and, since $w(0) = 32 > 0$, so (2.69) and, consequently, (2.68) holds. Thus (2.65) is confirmed.

(iv) $k = 3$. We have

$$q_3(u) = (\alpha^8 - \alpha^6)u^3 + (2\alpha^8 - 5\alpha^6 + 5\alpha^4)u^2 + (20\alpha^6 - 16\alpha^4 - 8\alpha^2)u -12\alpha^6 + 6\alpha^4 + 36\alpha^2 + 4, \quad u \in [0, u_\alpha]. \tag{2.70}$$

We will show that

$$q_3(u_\alpha) > 0, \quad \alpha \in (0,1), \tag{2.71}$$

i.e., after computing that

$$\alpha^2(\alpha+1)(-\alpha^5 - 9\alpha^4 + 8\alpha^2 + 4)\sqrt{2} < \left(2\alpha^8 - 6\alpha^7 - 11\alpha^6 + 3\alpha^5 + 8\alpha^4 + 18\alpha^3 + 18\alpha^2 +2\alpha + 2\right)\sqrt{\alpha+1}, \quad \alpha \in (0,1). \tag{2.72}$$

Since for $\alpha \in (0,1)$,

$$-\alpha^5 - 9\alpha^4 + 8\alpha^2 + 4 \geq -2\alpha^4 + 4 > 0$$

and

$$-6\alpha^7 - 11\alpha^6 + 18\alpha^3 + 18\alpha^2 \geq -6\alpha^3 - 11\alpha^2 + 18\alpha^3 + 18\alpha^2 = 12\alpha^3 + 7\alpha^2 > 0,$$

so both sides of (2.72) are positive. Thus squaring (2.72) and computing we get equivalently

$$4\alpha^{17} - 22\alpha^{16} - 72\alpha^{15} - 100\alpha^{14} - 67\alpha^{13} + 217\alpha^{12} + 223\alpha^{11} -531\alpha^{10} - 748\alpha^9 - 24\alpha^8 + 652\alpha^7 + 1112\alpha^6 + 1056\alpha^5 +540\alpha^4 + 220\alpha^3 + 84\alpha^2 + 12\alpha + 4 > 0. \tag{2.73}$$

Observe now that the left-hand side of (2.73) is greater or equal to

$$\begin{aligned} &4\alpha^{17} - 22\alpha^{12} - 72\alpha^{12} - 100\alpha^{12} - 67\alpha^{11} + 217\alpha^{12} + 223\alpha^{11} \\ &-531\alpha^7 - 748\alpha^6 - 24\alpha^6 + 652\alpha^7 + 1112\alpha^6 + 1056\alpha^5 \\ &+540\alpha^4 + 220\alpha^3 + 84\alpha^2 + 12\alpha + 4 \\ &= 4\alpha^{17} + 23\alpha^{12} + 156\alpha^{11} + 121\alpha^7 + 340\alpha^6 + 1056\alpha^5 \\ &+540\alpha^4 + 220\alpha^3 + 84\alpha^2 + 12\alpha + 4 \end{aligned}$$

which is clearly positive for $\alpha \in (0,1)$. Thus (2.73) holds, which confirms (2.71).

(v) $k = 4$. We have

$$q_4(u) = (\alpha^8 - \alpha^6)u^4 + (2\alpha^8 - 5\alpha^6 + 5\alpha^4)u^3 + (20\alpha^6 - 16\alpha^4 - 8\alpha^2)u^2 +(-12\alpha^6 + 6\alpha^4 + 36\alpha^2 + 4)u +16\alpha^4 - 24\alpha^2 - 16, \quad u \in [0, u_\alpha]. \tag{2.74}$$

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We will show that there exists a unique $\alpha \in (0,1)$ such that

$$q_4(u_\alpha) = 0, \tag{2.75}$$

i.e., after computing, a unique $\alpha \in (0,1)$ such that

$$\begin{aligned} &\left(2\alpha^8 - 6\alpha^7 - 11\alpha^6 + 3\alpha^5 + 8\alpha^4 + 18\alpha^3 + 18\alpha^2 + 2\alpha + 2\right)\sqrt{2} \\ &= \left(-2\alpha^7 - 18\alpha^6 - 8\alpha^5 + 8\alpha^4 + 12\alpha^3 + 20\alpha^2 + 8\alpha + 8\right)\sqrt{\alpha+1}. \end{aligned} \tag{2.76}$$

To verify that (2.76) holds, we will show that the equation

$$r(t) = s(t) \tag{2.77}$$

has a unique solution in (0,1), where for $t \in [0,1]$,

$$r(t) := \left(2t^8 - 6t^7 - 11t^6 + 3t^5 + 8t^4 + 18t^3 + 18t^2 + 2t + 2\right)\sqrt{2},$$
$$s(t) := \left(-2t^7 - 18t^6 - 8t^5 + 8t^4 + 12t^3 + 20t^2 + 8t + 8\right)\sqrt{t+1}.$$

Define

$$w(t) := s^2(t) - r^2(t), \quad t \in [0,1].$$

Thus

$$\begin{aligned} w(t) &= 4t^{16} - 26t^{15} - 46t^{14} - 70t^{13} - 189t^{12} - 82t^{11} \\ &\quad +145t^{10} + 204t^9 + 424t^8 + 528t^7 + 316t^6 + 92t^5 \\ &\quad -188t^4 - 304t^3 - 180t^2 - 88t - 28 =: \sum_{j=0}^{16} b_j t^{16-j}, \quad t \in [0,1]. \end{aligned}$$

Note that the numbers of sign changes in the sequence of polynomial coefficients (b_k) and in the sequence of sums $\left(\sum_{j=0}^{k} b_j\right)$, where $k = 0, 1, \ldots, 16$, equal 3 and 2, respectively, i.e., $N(w;0) = 3$ and $N(w;1) = 2$. Thus applying Corollary 2.3, we see that the equation

$$w(t) = (s(t) - r(t))\,(s(t) + r(t)) = 0 \tag{2.78}$$

has a unique zero $t =: t_0$. Since $w(0) = -28$ and $w(1) = 512$, so $t_0 \in (0,1)$. Observe that for $t \in [0,1]$ we have

$$\begin{aligned} \frac{r(t)}{\sqrt{2}} &\geq 2t^8 - 6t^4 - 11t^3 + 3t^5 + 8t^4 + 18t^3 + 18t^2 + 2t + 2 \\ &= 2t^8 + 3t^5 + 2t^4 + 7t^3 + 18t^2 + 2t + 2 > 0 \end{aligned}$$

and

$$\begin{aligned} \frac{s(t)}{\sqrt{t+1}} &\geq -2t^4 - 18t^2 - 8t^3 + 8t^4 + 12t^3 + 20t^2 + 8t + 8 \\ &= 6t^4 + 4t^3 + 2t^2 + 8t + 8 > 0. \end{aligned}$$

Hence $r(t) > 0$ and $s(t) > 0$ for $t \in [0,1]$. Thus from (2.78) it follows that $r(t_0) = s(t_0)$. Consequently, the equation (2.77) has a unique solution in $(0,1)$, namely, $t = t_0$. Thus, (2.76) so (2.75) holds with $\alpha := t_0$.

Moreover, since for $\alpha = 1$ we have $u_1 = 1$ and $q_4(u_1) = q_4(1) = 8 > 0$, we deduce that

$$q_4(u_\alpha) < 0, \quad \alpha \in (0, \alpha_0), \tag{2.79}$$

and

$$q_4(u_\alpha) > 0, \quad \alpha \in (\alpha_0, 1). \tag{2.80}$$

(vi) $k = 5$. We have

$$\begin{aligned} q_5(u) &= (\alpha^8-\alpha^6)u^5+(2\alpha^8-5\alpha^6+5\alpha^4)u^4+(20\alpha^6-16\alpha^4-8\alpha^2)u^3 \\ &\quad +(-12\alpha^6+6\alpha^4+36\alpha^2+4)u^2+(16\alpha^4-24\alpha^2-16)u \\ &\quad -8\alpha^4-124\alpha^2-8, \quad u\in[0,u_\alpha]. \end{aligned}$$

We will show that

$$q_5(u_\alpha) < 0, \quad \alpha \in (0,1), \tag{2.81}$$

i.e., after computing that

$$\begin{aligned} &\left(\alpha^8+10\alpha^7+13\alpha^6-10\alpha^4-16\alpha^3-14\alpha^2-8\alpha-4\right)\sqrt{2} \\ &< \alpha(-2\alpha^7+6\alpha^6+13\alpha^5+\alpha^4+25\alpha^3+44\alpha^2 \\ &+15\alpha+2)\sqrt{\alpha+1}, \quad \alpha\in(0,1). \end{aligned} \tag{2.82}$$

Clearly, for $\alpha \in (0,1)$ we have

$$\alpha^8+10\alpha^7+13\alpha^6-10\alpha^4-16\alpha^3-14\alpha^2-8\alpha-4<0,$$

so the left-hand side of (2.82) is negative. But the right-hand side of (2.82) is clearly positive. In this way, (2.82) holds, which confirms (2.81).

(vii) $k = 6$. We have

$$\begin{aligned} q_6(u) &= (\alpha^8-\alpha^6)u^6+(2\alpha^8-5\alpha^6+5\alpha^4)u^5+(20\alpha^6-16\alpha^4-8\alpha^2)u^4 \\ &\quad +(-12\alpha^6+6\alpha^4+36\alpha^2+4)u^3+(16\alpha^4-24\alpha^2-16)u^2 \\ &\quad -(8\alpha^4+124\alpha^2+8)u+272\alpha^2+96, \quad u\in[0,u_\alpha]. \end{aligned}$$

We will show that

$$q_6(u_\alpha) > 0, \quad \alpha \in (0,1), \tag{2.83}$$

i.e., after computing that

$$\begin{aligned} &\alpha(-2\alpha^7+6\alpha^6+13\alpha^5+\alpha^4+25\alpha^3+44\alpha^2+15\alpha+2)\sqrt{2} \\ &< \left(2\alpha^7+18\alpha^6+8\alpha^5+60\alpha^4+124\alpha^3+72\alpha^2\right. \\ &\left.+40\alpha+16\right)\sqrt{\alpha+1}, \quad \alpha\in(0,1). \end{aligned} \tag{2.84}$$

Both sides of (2.84) are positive evidently. Thus squaring (2.84) and computing we get equivalently the inequality

$$\begin{aligned} &4\alpha^{16}-26\alpha^{15}-54\alpha^{14}-62\alpha^{13}-361\alpha^{12}-1474\alpha^{11} \\ &-3097\alpha^{10}-5658\alpha^9-11809\alpha^8-19102\alpha^7-21382\alpha^6 \\ &-18548\alpha^5-12975\alpha^4-6756\alpha^3-2588\alpha^2-768\alpha-128<0, \end{aligned}$$

which is clearly true for $\alpha \in (0,1)$. Thus (2.83) is confirmed.

(viii) $k = 7$. We have

$$\begin{aligned} q_7(u) &= (\alpha^8-\alpha^6)u^7+(2\alpha^8-5\alpha^6+5\alpha^4)u^6+(20\alpha^6-16\alpha^4-8\alpha^2)u^5 \\ &\quad +(-12\alpha^6+6\alpha^4+36\alpha^2+4)u^4+(16\alpha^4-24\alpha^2-16)u^3 \\ &\quad -(8\alpha^4+124\alpha^2+8)u^2+(272\alpha^2+96)u-176\alpha^2-60, \quad u\in[0,u_\alpha]. \end{aligned}$$

We will show that

$$q_7(u_\alpha) > 0, \quad \alpha \in (0,1), \tag{2.85}$$

i.e., after computing that

$$\begin{aligned}&\big(-4\alpha^8+12\alpha^7+26\alpha^6+46\alpha^5+182\alpha^4+235\alpha^3\\&+119\alpha^2+49\alpha+15\big)\sqrt{\alpha+1}\\&<\big(2\alpha^8+20\alpha^7+26\alpha^6+68\alpha^5+184\alpha^4+196\alpha^3\\&+112\alpha^2+56\alpha+16\big)\sqrt{2},\quad \alpha\in(0,1).\end{aligned}\tag{2.86}$$

Both sides of (2.86) are positive evidently. Thus squaring (2.86) and computing we equivalently get

$$\begin{aligned}&(\alpha^2-1)(-16\alpha^{15}+88\alpha^{14}+304\alpha^{13}+904\alpha^{12}+2348\alpha^{11}+3964\alpha^{10}\\&+4560\alpha^9+1228\alpha^8-9016\alpha^7-22876\alpha^6-30417\alpha^5-25691\alpha^4\\&-14838\alpha^3-6286\alpha^2-1889\alpha-287)>0,\quad \alpha\in(0,1).\end{aligned}\tag{2.87}$$

Since for $\alpha\in(0,1)$ we have

$$\begin{aligned}&-16\alpha^{15}+88\alpha^{14}+304\alpha^{13}+904\alpha^{12}+2348\alpha^{11}+3964\alpha^{10}\\&+4560\alpha^9+1228\alpha^8-9016\alpha^7-22876\alpha^6-30417\alpha^5\\&-25691\alpha^4-14838\alpha^3-6286\alpha^2-1889\alpha-287\\&\le-16\alpha^{15}+88\alpha^7+304\alpha^6+904\alpha^5+2348\alpha^4+3964\alpha^3\\&+4560\alpha^2+1228\alpha-9016\alpha^7-22876\alpha^6-30417\alpha^5\\&-25691\alpha^4-14838\alpha^3-6286\alpha^2-1889\alpha-287\\&=-16\alpha^{15}-8928\alpha^7-22572\alpha^6-29513\alpha^5-23343\alpha^4\\&-10874\alpha^3-1726\alpha^2-661\alpha-287<0,\end{aligned}$$

so (2.87) holds. Thus (2.85) is confirmed.

(ix) $k=8$. We have

$$\begin{aligned}q_8(u)&=(\alpha^8-\alpha^6)u^8+(2\alpha^8-5\alpha^6+5\alpha^4)u^7+(20\alpha^6-16\alpha^4-8\alpha^2)u^6\\&\quad+(-12\alpha^6+6\alpha^4+36\alpha^2+4)u^5+(16\alpha^4-24\alpha^2-16)u^4\\&\quad-(8\alpha^4+124\alpha^2+8)u^3+(272\alpha^2+96)u^2\\&\quad-(176\alpha^2+60)u-144,\quad u\in[0,u_\alpha].\end{aligned}$$

We will show that

$$q_8(u_\alpha)<0,\quad \alpha\in(0,1),\tag{2.88}$$

i.e., after computing that

$$r(\alpha)<s(\alpha),\quad \alpha\in(0,1),\tag{2.89}$$

where

$$\begin{aligned}r(\alpha)&:=\big(4\alpha^7+36\alpha^6+16\alpha^5+120\alpha^4+212\alpha^3+36\alpha^2\\&\quad-28\alpha-4\big)\sqrt{\alpha+1},\quad \alpha\in(0,1),\end{aligned}$$

and

$$\begin{aligned}s(\alpha)&:=\big(-4\alpha^8+12\alpha^7+26\alpha^6+46\alpha^5+182\alpha^4+235\alpha^3+119\alpha^2\\&\quad+49\alpha+15)\sqrt{2},\quad \alpha\in(0,1).\end{aligned}$$

We see at once that

$$s(\alpha)>0,\quad \alpha\in(0,1).\tag{2.90}$$

It is easily seen that for $\alpha \in (0,1)$,

$$\begin{aligned} s^2(\alpha) - r^2(\alpha) &= (s(\alpha) - r(\alpha))\,(s(\alpha) + r(\alpha)) \\ &= 16\alpha^{16} - 104\alpha^{15} - 216\alpha^{14} - 600\alpha^{13} - 1444\alpha^{12} - 1472\alpha^{11} \\ &\quad +1276\alpha^{10} + 9956\alpha^9 + 25220\alpha^8 + 48204\alpha^7 + 73421\alpha^6 \\ &\quad +76706\alpha^5 + 50275\alpha^4 + 20320\alpha^3 + 5611\alpha^2 + 1350\alpha + 217 > 0. \end{aligned}$$

Hence either

$$s(\alpha) - r(\alpha) > 0, \quad s(\alpha) + r(\alpha) > 0, \quad \alpha \in (0,1), \tag{2.91}$$

or

$$s(\alpha) - r(\alpha) < 0, \quad s(\alpha) + r(\alpha) < 0, \quad \alpha \in (0,1). \tag{2.92}$$

Supposing that (2.92) holds, we see that then $s(\alpha) < 0$ for $\alpha \in (0,1)$. However this contradicts (2.90). Thus (2.91) holds so (2.88) is confirmed.

(x) $k = 9$. We have

$$q_9(u) = V_\alpha(u), \quad u \in [0, u_\alpha].$$

We will show that

$$q_9(u_\alpha) > 0, \quad \alpha \in (0,1), \tag{2.93}$$

i.e., after computing that

$$\begin{aligned} &\big(-4\alpha^8 + 12\alpha^7 + 26\alpha^6 + 46\alpha^5 + 164\alpha^4 + 163\alpha^3 + 11\alpha^2 \\ &-23\alpha - 3\big)\sqrt{\alpha+1} \\ &< \big(2\alpha^8 + 20\alpha^7 + 26\alpha^6 + 68\alpha^5 + 166\alpha^4 + 124\alpha^3 + 4\alpha^2 \\ &-16\alpha - 2\big)\sqrt{2}, \quad \alpha \in (0,1). \end{aligned} \tag{2.94}$$

To verify that (2.94) holds, we will show that

$$r(t) < s(t), \quad t \in (0,1), \tag{2.95}$$

where

$$\begin{aligned} r(t) &:= \big(-4t^8 + 12t^7 + 26t^6 + 46t^5 + 164t^4 + 163t^3 + 11t^2 \\ &\quad -23t - 3\big)\sqrt{t+1}, \quad t \in [0,1], \end{aligned}$$

and

$$\begin{aligned} s(t) &:= \big(2t^8 + 20t^7 + 26t^6 + 68t^5 + 166t^4 + 124t^3 + 4t^2 \\ &\quad -16t - 2\big)\sqrt{2}, \quad t \in [0,1]. \end{aligned}$$

Let

$$w(t) := s^2(t) - r^2(t), \quad t \in [0,1].$$

Thus after computing we have

$$w(t) = (t-1)^2(t+1)\,(2t+1)\,v(t), \tag{2.96}$$

where

$$\begin{aligned} v(t) &:= -8t^{13} + 40t^{12} + 168t^{11} + 556t^{10} + 1464t^9 + \\ &\quad +2776t^8 + 4148t^7 + 4220t^6 + 2358t^5 + +455t^4 \\ &\quad -176t^3 - 106t^2 - 18t - 1 =: \sum_{j=0}^{13} b_j t^{k-j}, \quad t \in [0,1]. \end{aligned} \tag{2.97}$$

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We use now Corollary 2.3 since $v(0) = -1 \neq 0$ and $v(1) = 15876 \neq 0$. Note that the numbers of sign changes in the sequence of polynomial coefficients (b_k) and in the sequence of sums $\left(\sum_{j=0}^{k} b_j\right)$, where $k = 0, 1, \ldots, 13$, equal 2 and 1, respectively, i.e., $N(v; 0) = 2$ and $N(v; 1) = 1$. Applying Corollary 2.3, we see that the polynomial v has the unique zero $t =: t_0 \in (0, 1)$. Moreover t_0 is the zero of order 1. Hence and from (2.96) it follows that t_0 is the unique zero of order 1 of w in $(0, 1)$, also. Since

$$w(0) = v(0) = -1 < 0,$$

so from (2.96) and by the continuity of the function w we have

$$w(t) = (s(t) - r(t))(s(t) + r(t)) < 0, \quad t \in (0, t_0) \tag{2.98}$$

and

$$w(t) = (s(t) - r(t))(s(t) + r(t)) > 0, \quad t \in (t_0, 1). \tag{2.99}$$

Since

$$s(0) + r(0) = -2\sqrt{2} - 3 < 0,$$

from (2.98) it follows that

$$s(t) - r(t) > 0, \quad t \in (0, t_0), \tag{2.100}$$

and

$$s(t) + r(t) < 0, \quad t \in (0, t_0). \tag{2.101}$$

Similarly, since

$$s(1) + r(1) = 784\sqrt{2} > 0,$$

from (2.99) it follows that

$$s(t) - r(t) > 0, \quad t \in (t_0, 1), \tag{2.102}$$

and

$$s(t) + r(t) > 0, \quad t \in (t_0, 1). \tag{2.103}$$

Thus from (2.100) and (2.102) we have

$$s(t) - r(t) > 0, \quad t \in (0, 1) \setminus \{t_0\}. \tag{2.104}$$

Moreover, from (2.101) and (2.103) is follows that

$$s(t_0) + r(t_0) = 0. \tag{2.105}$$

The continuity of the function $s - r$ and (2.104) yield

$$s(t_0) - r(t_0) \geq 0. \tag{2.106}$$

Suppose now that

$$s(t_0) - r(t_0) = 0. \tag{2.107}$$

Hence and from (2.105) we have

$$s(t_0) = r(t_0) = 0.$$

Thus

$$\frac{r(t)}{\sqrt{t+1}} = (t - t_0)\varrho(t), \quad t \in [0, 1],$$

and

$$\frac{s(t)}{\sqrt{2}} = (t - t_0)\sigma(t), \quad t \in [0, 1],$$

where ϱ and σ are some polynomials in $[0,1]$. Hence

$$w(t) = s^2(t) - r^2(t) = (t-t_0)^2\left(2\sigma^2(t) - (t+1)\varrho^2(t)\right), \quad t \in [0,1],$$

which yields a contradiction since, as was shown, t_0 is the unique zero of order 1 of w in $(0,1)$. Thus the strong inequality in (2.106) holds, which together with (2.104) finishes the proof of (2.95). In this way (2.93) is confirmed.

Summarizing, from (2.60), (2.61), (2.65), (2.71), (2.75), (2.79), (2.80), (2.81), (2.83), (2.85), (2.88) and (2.93) it follows that for three cases, namely, for $\alpha \in (0,\alpha_0)$, $\alpha := \alpha_0$ and $\alpha \in (\alpha_0, 1)$, where α_0 is the unique root of the equation (2.75), we have

$$N(V_\alpha; u_\alpha) = 7.$$

Hence, by (2.59) and by Corollary 2.3 we conclude that for each $\alpha \in (0,1)$ the polynomial V_α has no zero in $(0, u_\alpha)$, and since $V_\alpha(0) = 144 > 0$, so (2.57) holds. Thus (2.55) is confirmed, which finishes the proof of the inequality (2.52).

Summarizing, taking into account (2.49)-(2.52), we proved that

$$\max_{(x,y)\in R} F_{\alpha,\gamma}(x,y) = \begin{cases} 2(1+2\alpha)|\gamma|, & |\gamma| \geq \dfrac{2}{1+\alpha}, \\ \dfrac{2\alpha^2\gamma^2}{2-|\gamma|} + 4, & |\gamma| \leq \dfrac{2}{1+\alpha}. \end{cases}$$

Finally, substituting $\gamma = 2 - 3\lambda$ and $\alpha = |b_2|/2$, the above and (2.13) yield (2.8). □

***Remark* 2.5.** Since the condition (2.7), i.e., the inequality $|b_2| \leq 2$ holds for $g \in \mathcal{S}$, Theorem 2.4 is true for the class $\mathcal{C}(g)$, where g is in $\mathcal{S}$.

Now we recall the result for the class $\mathcal{C}(g_\alpha)$ proved in [15].

Theorem 2.6. *Let $\alpha \in [0,1]$. Then*

$$\max_{f\in\mathcal{C}(g_\alpha)} \Phi_\lambda(f)$$

$$\leq \begin{cases} \left|\dfrac{2}{3} + \dfrac{4}{3}\alpha + \alpha^2 - (1+\alpha)^2\lambda\right|, & \lambda \in \mathbb{R} \setminus (\tau'(\alpha), \tau''(\alpha)), \\ \alpha^2\left(|1-\lambda| + \dfrac{1}{3}\cdot\dfrac{(2-3\lambda)^2}{2-|2-3\lambda|}\right) + \dfrac{2}{3}, & \lambda \in [\tau'(\alpha), \tau''(\alpha)], \end{cases} \tag{2.108}$$

where

$$\tau'(\alpha) := \frac{2\alpha}{3(1+\alpha)}, \quad \tau''(\alpha) := \frac{2(2+\alpha)}{3(1+\alpha)}.$$

Proof. Let $\alpha \in [0,1]$. Since

$$g_\alpha(z) = \frac{z}{(1-\alpha z)^2} = \sum_{n=1}^{\infty} n\alpha^{n-1}z^n, \quad z \in \mathbb{D},$$

so

$$b_2 = 2\alpha, \quad b_3 = 3\alpha^2. \tag{2.109}$$

Then in view of (2.9) we have

$$\tau_1(|b_2|) = \frac{2|b_2|}{3(|b_2|+2)} = \frac{2\alpha}{3(1+\alpha)} =: \tau'(\alpha)$$

and

$$\tau_2(|b_2|) = \frac{2(|b_2|+4)}{3(|b_2|+2)} = \frac{2(2+\alpha)}{3(1+\alpha)} =: \tau''(\alpha).$$

Now for $\lambda \in \mathbb{R} \setminus [\tau'(\alpha), \tau''(\alpha)]$ by using (2.109) the inequality (2.8) is of the form

$$\begin{aligned} \max_{f\in\mathcal{C}(g_\alpha)} \Phi_\lambda(f) &\le \left|\frac{1}{3}b_3 - \frac{1}{4}\lambda b_2^2\right| + (1+|b_2|)\left|\frac{2}{3}-\lambda\right| \\ &= \alpha^2|1-\lambda| + (1+2\alpha)\left|\frac{2}{3}-\lambda\right| \\ &= \left|\frac{2}{3}+\frac{4}{3}\alpha+\alpha^2-(1+\alpha)^2\lambda\right| \end{aligned}$$

and for $\lambda \in [\tau'(\alpha), \tau''(\alpha)]$ of the form

$$\begin{aligned} \max_{f\in\mathcal{C}(g_\alpha)} \Phi_\lambda(f) &\le \left|\frac{1}{3}b_3 - \frac{1}{4}\lambda b_2^2\right| + \frac{(2-3\lambda)^2|b_2|^2}{12(2-|2-3\lambda|)} + \frac{2}{3} \\ &= \alpha^2\left(|1-\lambda| + \frac{1}{3}\cdot\frac{(2-3\lambda)^2}{2-|2-3\lambda|}\right) + \frac{2}{3}. \end{aligned}$$

□

For $\alpha := 1$ we get the following result proved in [16].

Corollary 2.7.

$$\max_{f\in\mathcal{C}(k)} \Phi_\lambda(f)$$

$$\le \begin{cases} |3-4\lambda|, & \lambda \in (-\infty, 1/3] \cup [1, +\infty), \\ \dfrac{1}{3}\cdot\dfrac{(2-3\lambda)^2}{2-|2-3\lambda|} + |1-\lambda| + \dfrac{2}{3}, & \lambda \in [1/3, 1]. \end{cases} \tag{2.110}$$

Let now formulate the result for the class $\mathcal{C}(h_\alpha)$.

Theorem 2.8. *Let $\alpha \in [0,1]$. Then*

$$\max_{f\in\mathcal{C}(h_\alpha)} \Phi_\lambda(f)$$

$$\le \begin{cases} \alpha^2\left|\dfrac{1}{3}-\dfrac{1}{4}\lambda\right| + (1+\alpha)\left|\dfrac{2}{3}-\lambda\right|, & \lambda \in \mathbb{R}\setminus[\tau'(\alpha), \tau''(\alpha)], \\ \alpha^2\left(\dfrac{1}{12}\cdot\dfrac{(2-3\lambda)^2}{2-|2-3\lambda|} + \left|\dfrac{1}{3}-\dfrac{1}{4}\lambda\right|\right) + \dfrac{2}{3}, & \lambda \in [\tau'(\alpha), \tau''(\alpha)], \end{cases} \tag{2.111}$$

where

$$\tau'(\alpha) := \frac{2\alpha}{3(2+\alpha)}, \quad \tau''(\alpha) := \frac{2(4+\alpha)}{3(2+\alpha)}.$$

Proof. Let $\alpha \in [0,1]$. Since

$$h_\alpha(z) = \frac{z}{1-\alpha z} = \sum_{n=1}^{\infty} \alpha^{n-1} z^n, \quad z \in \mathbb{D},$$

so

$$b_2 = \alpha, \quad b_3 = \alpha^2. \tag{2.112}$$

Then in view of (2.9) we have

$$\tau_1(|b_2|) = \frac{2|b_2|}{3(|b_2|+2)} = \frac{2\alpha}{3(2+\alpha)} =: \tau'(\alpha)$$

and

$$\tau_2(|b_2|) = \frac{2(|b_2|+4)}{3(|b_2|+2)} = \frac{2(4+\alpha)}{3(2+\alpha)} =: \tau''(\alpha).$$

Now for $\lambda \in \mathbb{R} \setminus [\tau'(\alpha), \tau''(\alpha)]$ by using (2.112) the inequality (2.8) is of the form

$$\begin{aligned} \max_{f\in\mathcal{C}(h_\alpha)} \Phi_\lambda(f) &\le \left|\frac{1}{3}b_3 - \frac{1}{4}\lambda b_2^2\right| + (1+|b_2|)\left|\frac{2}{3}-\lambda\right| \\ &= \alpha^2\left|\frac{1}{3}-\frac{1}{4}\lambda\right| + (1+\alpha)\left|\frac{2}{3}-\lambda\right| \end{aligned}$$

and for $\lambda \in [\tau'(\alpha), \tau''(\alpha)]$ of the form

$$\begin{aligned} \max_{f\in\mathcal{C}(h_\alpha)} \Phi_\lambda(f) &\le \left|\frac{1}{3}b_3 - \frac{1}{4}\lambda b_2^2\right| + \frac{(2-3\lambda)^2|b_2|^2}{12(2-|2-3\lambda|)} + \frac{2}{3} \\ &= \alpha^2\left(\left|\frac{1}{3}-\frac{1}{4}\lambda\right| + \frac{(2-3\lambda)^2}{12(2-|2-3\lambda|)}\right) + \frac{2}{3}. \end{aligned}$$

Thus (2.111) was proved. □

For $\alpha := 1$ we get the following result proved in [17].

Theorem 2.9.

$$\max_{f\in\mathcal{C}(h)} \Phi_\lambda(f)$$

$$\le \begin{cases} \left|\frac{1}{3}-\frac{1}{4}\lambda\right| + \frac{2}{3}|2-3\lambda|, & \lambda \in (-\infty, 2/9] \cup [10/9, +\infty), \\ \frac{1}{12}\cdot\frac{(2-3\lambda)^2}{2-|2-3\lambda|} + \left|\frac{1}{3}-\frac{1}{4}\lambda\right| + \frac{2}{3}, & \lambda \in [2/9, 10/9]. \end{cases} \tag{2.113}$$

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On Mild Solution of Abstract Neutral Fractional Order Impulsive Differential Equations with Infinite Delay

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Abstract

We prove the existence and uniqueness of fractional neutral impulsive differential equations with infinite delay via contraction mapping principle and fixed point technique for condensing map. We use the resolvent operator technique for integral equations to make the mild solution of the problem more appropriate.

Keywords: Fractional differential equations, Fractional order impulsive conditions, Neutral differential equations, Infinite delay, Resolvent operators.

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1 Introduction

In recent years, a significant number of the investigates managed the possibility of the fractional differential equations in different areas of engineering and science disciplines, for example, rheology, viscoelasticity, biomedical, control theory, porous media. Fractional differential equations give an incredible mathematical model for real world phenomena, in which the fractional rate of progress relies on upon the impact of the hereditary effects and describing the long memory of the systems. For detailed investication of the fractional differential equations, we read [4, 13, 19, 23].

The hypothesis of partial neutral integro-differential equations with infinite delay have been utilized for displaying the advancement of physical systems, in which the reaction

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depends on the present and previous history of the system. This sort of equations emerge in the theory of heat conduction in material with fading memory [18]. Since we consider the infinite delay, we use the notion is phase space which acts as an essential part in the study of qualitative theory of delay equations. This idea was presented by Hale and Kato in [7].

Study of impulsive differential equations turn into an essential field of research because of their various applications. The purpose behind this applicability emerges from the way that, numerous real world processes and phenomena which are subjected amid their improvement to short-term external impacts can be demostrated as impulsive differential equations with non-integer order and which cannot be depicted by using classical differential equations [15]. For more subtle elements of fractional impulsive differential equations, see [5, 6, 16, 21].

Similar results for integer order derivative for abstract neutral functional differential equations with impulsive condition was studied by [2, 8, 12]. The work on fractional neutral impulsive differential equations with infinite delay are carried out by [3, 22]. In [14] N. Kosmatov studied the fractional order initial value problems with fractional impulses by the contribution of Caputo and Riemann-Liouville derivatives.

Hernández et al. [9], examined that the concepts of mild solutions utilized as a part of a few late writing on abstract fractional differential equations are not suitable. In [9], he consider the more appropriate mild solution of the abstract fractional differential equations with time by means of resolvent operator for integral equations [20]. The same idea was used by some authors to show the existence of fractional differential equations without impulse, see [1, 10]. But in our best of knowledge this resolvent operator concept was not used in the fractional impulsive differential equations of order lies in $(1,2)$. Note that the order of integration determines the shape of the memory function.

Impulsive fractional differential equations is constructed with either the lower bound as the corresponding impulses or the lower bound as zero at each impulses. Here we construct the solution of fractional order impulsive Cauchy problem involving Caputo derivative with lower bound as zero. That is the different solutions keeping in each impulses the lower bound as zero. This will improve the characterization of the memory property of the factional derivative.

Motivations of the study in [9, 14] and the applications of fractional order derivative give rise in this present article. Here we prove the existence and uniqueness theorems of mild solutions for fractional neutral impulsive differential equations with infinite delay given by

$$ {}^{c}D^{\alpha}_{0^+}(u(t)+q(t,u_t)) = \mathscr{A}u(t)+p(t,u_t),\ t\neq t_{\mathfrak{k}},\ t\in\mathcal{J}:=[0,a], \tag{1.1} $$

$$ {}^{c}D^{\beta}_{0^+}u(t^+_{\mathfrak{k}}) - {}^{c}D^{\beta}_{0^+}u(t^-_{\mathfrak{k}}) = I_{\mathfrak{k}}(u_{t_{\mathfrak{k}}}),\ \mathfrak{k}=1,\cdots,m, \tag{1.2} $$

$$ u(0)=\phi\in\mathscr{B}_A,\ \ u'(0)=z\in\mathscr{E}, \tag{1.3} $$

where $0<\beta<1$ and $\alpha\in(1,2)$. Here $\mathscr{A}$ is the infinitesimal generator of a cosine operator family $\{\mathfrak{C}(t)\}_{t\geq 0}$ on a Banach space $\mathscr{E}$. The memory function $u_t:(-\infty,0]\to\mathscr{E},\ u_t(\sigma)=u(t+\sigma),\ \sigma\leq 0$, associated with some suitable abstract phase space $\mathscr{B}_A$, $0=t_0<t_1<\cdots<t_{m+1}=a$ are pre-fixed values and the appropriate functions $p,q:\mathcal{J}\times\mathscr{B}_A\to\mathscr{E},\ I_{\mathfrak{k}}:\mathscr{B}_A\to\mathscr{E}, \mathfrak{k}=1,\cdots,m$, which are defined later.

We derive the mild solution of (1.1)-(1.3) by resolvent operator technique. The existence results of fractional neutral impulsive differential equations with infinite delay via fixed point technique for condensing map and the uniqueness of the problem is verified by using contraction mapping principle.

2 Preliminaries

Let the space $\mathcal{L}(\mathscr{E}, \mathscr{E}')$ is the set of all bounded linear operators from Banach space $\mathscr{E}$ into Banach space $\mathscr{E}'$ provided with the norm $\|\cdot\|_{\mathcal{L}(\mathscr{E},\mathscr{E}')}$. Here the domain $\mathscr{D}(\mathscr{A})$, takes the norm $\|u\|_{\mathscr{D}(\mathscr{A})} = \|u\| + \|\mathscr{A}u\|$. Further more, $\mathcal{B}_r(u, \mathscr{E})$ symbolizes the closed ball having center at u and distance r in $\mathscr{E}$.

The class of all continuous functions from $\mathcal{J}$ into $\mathscr{E}$ is referred by $C(\mathcal{J}; \mathscr{E})$ with the sup-norm $\|\cdot\|_{C(\mathcal{J};\mathscr{E})}$. Likewise $C^\gamma(\mathcal{J}; \mathscr{E}), 0 < \gamma < 1$ is the set of all $\gamma-$Hölder $\mathscr{E}$-valued continuous functions from $\mathcal{J}$ into $\mathscr{E}$ provided with $\|u\|_{C^\gamma(\mathcal{J};\mathscr{E})} = \|u\|_{C(\mathcal{J};\mathscr{E})} + [|u|]_{C^\gamma(\mathcal{J};\mathscr{E})}$, where $[|u|]_{C^\gamma(\mathcal{J};\mathscr{E})} = \sup_{t\neq s, t,s\in\mathcal{J}} \frac{\|u(t)-u(s)\|_{\mathscr{E}}}{(t-s)^\gamma}$.

Now, we present the piece-wise continuous space $PC(\mathscr{E})$ which is framed by set of all the functions $u : \mathcal{J} \to \mathscr{E}$ such that the function $u(\cdot)$ is continuous at $t \neq t_{\mathfrak{k}}$, $u(t_{\mathfrak{k}}^+)$ and $u(t_{\mathfrak{k}}^-) = u(t_{\mathfrak{k}})$ exists for every $\mathfrak{k} = 1, 2, \cdots, m$. We can easily seen that it is a Banach space concerning the norm $\|u\|_{PC(\mathscr{E})} = \sup_{t\in\mathcal{J}} \|u(t)\|_{\mathscr{E}}$.

We consider the phase space $(\mathscr{B}_A, \|\cdot\|_{\mathscr{B}_A})$, is a linear space of function u_t mapping from $(-\infty, 0]$ into $\mathscr{E}$ with respect to the seminorm $\|\cdot\|_{\mathscr{B}_A}$, which is previously addressed in Hino et al., [11] to examine the infinite delay problem. We assume the space $\mathscr{B}_A$ meets the axioms given below:

(1) If $u : (-\infty, \nu + a] \to \mathscr{E}$, $\nu \in \mathbb{R}, a > 0$ such that $u_\nu \in \mathscr{B}_A$, and $u|_{[\nu,\nu+a]} \in PC([\nu, \nu + a]; \mathscr{E})$, then the subsequent conditions hold for all $t \in [\nu, \nu + a)$

(i) $u_t \in \mathscr{B}_A$.

(ii) $\|u(t)\|_{\mathscr{E}} \leq \mathfrak{H}\|u\|_{\mathscr{B}_A}$

(iii) $\|u_t\|_{\mathscr{B}_A} \leq \mathfrak{K}(t-\nu)\sup\{\|u(s)\|_{\mathscr{E}} : \nu \leq s \leq t\} + \mathfrak{M}(t-\nu)\|u_\nu\|_{\mathscr{B}_A}$, where $\mathfrak{M}, \mathfrak{K} : [0, \infty) \to [1, \infty)$, is locally bounded and continuous respectively; $\mathfrak{H} > 0$ is a constant. $\mathfrak{K}, \mathfrak{H}, \mathfrak{M}$ are independent of $u(\cdot)$.

(2) The phase space $\mathscr{B}_A$ is complete.

We know that the Caputo fractional derivative of a function u of order $\alpha > 0$ defined as follows:

$$^cD^\alpha_{0^+}u(t) = I^{n-\alpha}_{0^+}D^n u(t), \quad n = \lceil\alpha\rceil,$$

where $I^\alpha_{0^+}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds$. Also, in general the Caputo derivative $^cD^\alpha_{0^+}$ is a left inverse of $I^\alpha_{0^+}$ but not a right inverse, i.e., we have $^cD^\alpha_{0^+}I^\alpha_{0^+}u(t) = u(t)$, and $I^\alpha_{0^+}{}^cD^\alpha_{0^+}u(t) = u(t) - u(0) - tu'(0)$, for $0 < \alpha < 2$.

Next, we consider that the Volterra integral equation

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathscr{A} u(s) ds + p(t), \ t \in \mathcal{J}, \tag{2.1}$$

has a corresponding resolvent operator $\{S(t)\}_{t \geq 0}$ on $\mathscr{E}$, see [9] and p in $C(\mathcal{J}; \mathscr{E})$. More detailed explanations about resolvent operator for integral equations one can refer [20]. The definition of mild solution for the integral equation (2.1) by utilizing the concept presented in [20] is given in [9].

Definition 2.1. *[9, Definition 1.2] A function u in the space $C(\mathcal{J}; \mathscr{E})$ is called a mild solution of* (2.1) *on $\mathcal{J}$, if $\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds$ in $C(\mathcal{J}; \mathscr{D}(\mathscr{A}))$ and*

$$u(t) = \frac{\mathscr{A}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds + p(t), \ t \in \mathcal{J}.$$

Definition 2.2. *[20, Definition 1.4] A resolvent operator $S(t)$ for equation (2.1) is said to be differentiable, if $S(\cdot)u \in \mathcal{W}^{1,1}([0, \infty); \mathscr{E})$ for every $u \in \mathscr{D}(\mathscr{A})$ and there is $\varphi \in L^1_{loc}([0, \infty))$ with $\|S'(t)u\| \leq \varphi(t) \|u\|_{\mathscr{D}(\mathscr{A})}$, a.e. on $[0, \infty)$, for every $u \in \mathscr{D}(\mathscr{A})$.*

Lemma 2.1. *[9, Lemma 1.1] Suppose (2.1) admits a differentiable resolvent $S(t)$ and if $p \in C(\mathcal{J}; \mathscr{D}(\mathscr{A}))$, then*

$$u(t) = p(t) + \int_0^t S'(t-s) p(s) ds, \ t \in \mathcal{J},$$

is said to be a mild solution of (2.1).

Now, our point is to present the concept of mild solution for equation (1.1) to (1.3). In this way, we first identify that if $u(\cdot)$ is a solution of (1.1)-(1.3), then one can estimate the corresponding integral equation given by

$$u(t) = \phi(0) + q(0, \phi) + (z + \xi)t - q(t, u_t) + \Gamma(2-\beta) \sum_{0 < t_{\mathfrak{t}} < t} t_{\mathfrak{t}}^{\beta-1} (t - t_{\mathfrak{t}}) I_{\mathfrak{t}}(u_{t_{\mathfrak{t}}})$$
$$+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathscr{A} u(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s, u_s) ds, \ t \in \mathcal{J}, \tag{2.2}$$

where $\frac{d}{dt} q(t, u_t)|_{t=0} = \xi$, ξ is independent of u.

Motivated by Definition 2.1 and the representation (2.2), we introduce the following definition.

Definition 2.3. *A function $u : (-\infty, a] \to \mathscr{E}$ is a mild solution of (1.1)-(1.3), if $u(0) = \phi$, $u'(0) = z$, $u|_{\mathcal{J}} \in PC(\mathscr{E})$, $\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds \in \mathscr{D}(\mathscr{A})$, $\forall \ t \in \mathcal{J}$, and*

$$u(t) = \phi(0) + q(0, \phi) + (z + \xi)t - q(t, u_t) + \Gamma(2-\beta) \sum_{0 < t_{\mathfrak{t}} < t} t_{\mathfrak{t}}^{\beta-1} (t - t_{\mathfrak{t}}) I_{\mathfrak{t}}(u_{t_{\mathfrak{t}}})$$
$$+ \frac{\mathscr{A}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s, u_s) ds, \ t \in \mathcal{J},$$

where $\frac{d}{dt} q(t, u_t)|_{t=0} = \xi$, ξ is independent of u.

3 Existence and Uniqueness Results

Now, we will make the subsequent hypotheses:

(H1) $p : \mathcal{J} \times \mathscr{B}_A \to \mathscr{D}(\mathscr{A})$ is continuous function and let $L_p \in C(\mathcal{J}; \mathbb{R})$ such that $\|p(t, \varpi_1) - p(t, \varpi_2)\|_{\mathscr{D}(\mathscr{A})} \leq L_p(t)\|\varpi_1 - \varpi_2\|_{\mathscr{B}_A},\ t \in \mathcal{J}, \varpi_1, \varpi_2 \in \mathscr{B}_A$.

(H2) The function m_p belongs to $C(\mathcal{J}; \mathbb{R})$ and a non-decreasing function $W : [0, +\infty) \to (0, +\infty)$ such that $\|p(t, \varpi)\|_{\mathscr{D}(\mathscr{A})} \leq m_p(t)W(\|\varpi\|_{\mathscr{B}_A})$, $t \in \mathcal{J}, \varpi \in \mathscr{B}_A$.

(H3) $q : \mathcal{J} \times \mathscr{B}_A \to \mathscr{D}(\mathscr{A})$ is continuous function and $L_q \in C(\mathcal{J}; \mathbb{R})$ with $\|q(t, \varpi_1) - q(t, \varpi_2)\|_{\mathscr{D}(\mathscr{A})} \leq L_q(t)\|\varpi_1 - \varpi_2\|_{\mathscr{B}_A},\ t \in \mathcal{J}, \varpi_1, \varpi_2 \in \mathscr{B}_A$.

(H4) $C_1 > 0$, and $C_2 > 0$ such that $\|q(t, \varpi)\|_{\mathscr{B}_A} \leq C_1\|\varpi\|_{\mathscr{B}_A} + C_2,\ t \in \mathcal{J}, \varpi \in \mathscr{B}_A$.

(H5) $I_{\mathfrak{k}} : \mathscr{B}_A \to \mathscr{D}(\mathscr{A})$ are continuous functions and let positive constants $L_{\mathfrak{k}}$ such that $\|I_{\mathfrak{k}}(\varpi_1) - I_{\mathfrak{k}}(\varpi_2)\|_{\mathscr{D}(\mathscr{A})} \leq L_{\mathfrak{k}}\|\varpi_1 - \varpi_2\|_{\mathscr{B}_A},\ \mathfrak{k} = 1, 2, \cdots, m, \varpi_1, \varpi_2 \in \mathscr{B}_A$.

(H6) Let $d^1_{\mathfrak{k}}, > 0$ and $d^1_{\mathfrak{k}} > 0$ such that $\|I_{\mathfrak{k}}(\varpi)\| \leq d^1_{\mathfrak{k}}\|\varpi\| + d^2_{\mathfrak{k}}$ for all $\mathfrak{k} = 1, 2, \cdots, m,\ \varpi \in \mathscr{B}_A$.

From Lemma 2.1 we note the subsequent Proposition,

Proposition 3.1. *Suppose equation (2.2) admits a differential resolvent operator* $\{S(t)\}_{t\geq 0}$ *and if* $p, q \in C(\mathcal{J} \times \mathscr{B}_A; \mathscr{D}(\mathscr{A})), I_{\mathfrak{k}} \in C(\mathscr{B}_A; \mathscr{D}(\mathscr{A}))$*, then*

$$u(t) = \phi(0) + q(0, \phi) + (z + \xi)t - q(t, u_t) + \Gamma(2 - \beta) \sum_{0<t_{\mathfrak{k}}<t} t_{\mathfrak{k}}^{\beta-1}(t - t_{\mathfrak{k}})I_{\mathfrak{k}}(u_{t_{\mathfrak{k}}})$$
$$+ \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} p(s, u_s)ds + \int_0^t S'(t - s)\,(\phi(0) + q(0, \phi) + (z + \xi)s - q(s, u_s)$$
$$+\Gamma(2 - \beta) \sum_{0<t_{\mathfrak{k}}<s} t_{\mathfrak{k}}^{\beta-1}(s - t_{\mathfrak{k}})I_{\mathfrak{k}}(u_{t_{\mathfrak{k}}}) + \frac{1}{\Gamma(\alpha)} \int_0^s (s - \tau)^{\alpha-1} p(\tau, u_\tau)d\tau\Bigg)ds,\ t \in \mathcal{J},$$

is called a mild solution of the problem (1.1)-(1.3).

Let a function $x : (-\infty, a] \to \mathscr{E}$ be defined by $x_0 = \phi$ and $x(t) = \phi(0) + \int_0^t S'(t - s)\phi(0)ds$ for all $t \in \mathcal{J}$. It is easily say that $\|x_t\| \leq (\mathfrak{K}_a\mathfrak{H}(1 + \|\varphi\|_{L^1(\mathcal{J};\mathbb{R})}) + \mathfrak{M}_a)\|\phi\|_{\mathscr{B}_A}$, where $\mathfrak{M}_a = \sup_{t\in\mathcal{J}} \mathfrak{M}(t),\ \mathfrak{K}_a = \sup_{t\in\mathcal{J}} \mathfrak{K}(t)$.

Theorem 3.1. *Assume that* $(H1), (H3)$ *and* $(H5)$ *are satisfied, and if*

$$\mathfrak{K}_a(\|L_q(t)\| + \Gamma(2 - \beta)a \sum_{0<t_{\mathfrak{k}}<a} t_{\mathfrak{k}}^{\beta-1} L_{\mathfrak{k}} + \frac{a^\alpha}{\alpha\Gamma(\alpha)}\|L_p(t)\|)(1 + \|\varphi\|_{L^1(\mathcal{J};\mathbb{R})}) < 1.$$

Then (1.1)-(1.3) has a unique mild solution.

Proof. Let the space $\mathscr{Z}(a) = \{u : (-\infty, a] \to \mathscr{E} : u_0, u|_{\mathcal{J}} \in PC(\mathscr{E})\}$ endowed with the sup-norm. Now by Proposition 3.1, we consider the operator $\mathfrak{T} : \mathscr{Z}(a) \to \mathscr{Z}(a)$ by

$$\mathfrak{T}u(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ q(0,\phi) + (z+\xi)t - q(t, u_t + x_t) + \Gamma(2-\beta)\sum_{0<t_\mathfrak{k}<t} t_\mathfrak{k}^{\beta-1}(t-t_\mathfrak{k})I_\mathfrak{k}(u_{t_\mathfrak{k}} + x_{t_\mathfrak{k}}) \\ +\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} p(s, u_s + x_s)ds + \int_0^t S'(t-s)\left(q(0,\phi) + (z+\xi)s\right. \\ -q(s, u_s + x_s) + \Gamma(2-\beta)\sum_{0<t_\mathfrak{k}<s} t_\mathfrak{k}^{\beta-1}(s-t_\mathfrak{k})I_\mathfrak{k}(u_{t_\mathfrak{k}} + x_{t_\mathfrak{k}}) \\ \left. +\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1} p(\tau, u_\tau + x_\tau)d\tau\right) ds, t \in \mathcal{J}. \end{cases}$$

It is easily seen that $\|u_t + x_t\|_{\mathscr{B}_A} \le \mathfrak{K}_a\|u\|_t + (\mathfrak{K}_a\mathfrak{H}(1 + \|\varphi\|_{L^1(\mathcal{J};\mathbb{R})}) + \mathfrak{M}_a)\|\phi\|_{\mathscr{B}_A}$, where $\|u\|_t = \sup_{0\le s\le t}\|u(s)\|$.

Let $u \in \mathscr{Z}(a)$ and from the assumption $(H1), (H3)$ and $(H5)$, we get that

$$\begin{aligned} &\int_0^t \Big\| S'(t-s)(q(0,\phi) + (z+\xi)s - q(s, u_s + x_s) + \Gamma(2-\beta) \\ &\quad\times \sum_{0<t_\mathfrak{k}<s} t_\mathfrak{k}^{\beta-1}(s-t_\mathfrak{k})I_\mathfrak{k}(u_{t_\mathfrak{k}} + x_{t_\mathfrak{k}}) + \frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1} p(\tau, u_\tau + x_\tau)d\tau\Big\| ds \\ &\le \Big(\|q(0,\phi)\| + a\|z+\xi\| + \|q(s, u_s + x_s)\| + \Gamma(2-\beta)a \\ &\quad\times \sum_{0<t_\mathfrak{k}<a} t_\mathfrak{k}^{\beta-1}\|I_\mathfrak{k}(u_{t_\mathfrak{k}} + x_{t_\mathfrak{k}})\| + \frac{a^\alpha}{\alpha\Gamma(\alpha)}\|p(\tau, u_\tau + x_\tau)\|\Big)\|\varphi\|_{L^1(\mathcal{J};\mathbb{R})} \end{aligned}$$

which follows that $s \to S'(t-s)(q(0,\phi) + (z+\xi)s - q(s, u_s + x_s) + \Gamma(2-\beta)\sum_{0<t_{k<s}} t_\mathfrak{k}^{\beta-1}(s - t_\mathfrak{k})I_\mathfrak{k}(u_{t_\mathfrak{k}} + x_{t_\mathfrak{k}}) + \frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}p(\tau, u_\tau + x_\tau)d\tau)$ is integrable on $[0, t]$, $\forall t \in \mathcal{J}$. Then, the operator $\mathfrak{T}$ is well defined and $\mathfrak{T}$ have the values in $\mathscr{Z}(a)$.

Now, for u and v in $\mathscr{Z}(a)$ and $t \in \mathcal{J}$, we get

$$\begin{aligned} \|\mathfrak{T}u(t) - \mathfrak{T}v(t)\| \le\ & \|q(t, u_t + x_t) - q(t, v_t + x_t)\| + \Gamma(2-\beta)\sum_{0<t_\mathfrak{k}<t} t_\mathfrak{k}^{\beta-1}(t-t_\mathfrak{k}) \\ & \times\|I_\mathfrak{k}(u_{t_\mathfrak{k}} + x_{t_\mathfrak{k}}) - I_\mathfrak{k}(v_{t_\mathfrak{k}} + x_{t_\mathfrak{k}})\| \\ & + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|p(s, u_s) - p(s, v_s)\|ds \\ & + \int_0^t \varphi(t-s)\,(\|q(s, u_s + x_s) - q(s, v_s + x_s)\| \\ & + \Gamma(2-\beta)\sum_{0<t_\mathfrak{k}<s} t_\mathfrak{k}^{\beta-1}(s-t_\mathfrak{k}))\|I_\mathfrak{k}(u_{t_\mathfrak{k}} + x_{t_\mathfrak{k}}) - I_\mathfrak{k}(v_{t_\mathfrak{k}} + x_{t_\mathfrak{k}})\| \\ & + \frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}\|p(\tau, u_\tau) - p(\tau, v_\tau)\|d\tau\Big) ds \end{aligned}$$

$$\leq \Bigg(\|L_q(t)\|_{C(\mathcal{J};\mathbb{R})} + \Gamma(2-\beta)a \sum_{0<t_\mathfrak{k}<a} t_\mathfrak{k}^{\beta-1} L_\mathfrak{k} + \frac{a^\alpha}{\alpha\Gamma(\alpha)} \|L_p(t)\|_{C(\mathcal{J};\mathbb{R})}\Bigg)(1 + \|\varphi\|_{L^1(\mathcal{J};\mathbb{R})})\|u_t - v_t\|_{\mathscr{B}_A}$$

$$\leq \Bigg[\Re_a\Bigg(\|L_q(t)\|_{C(\mathcal{J};\mathbb{R})} + \Gamma(2-\beta)a \sum_{0<t_\mathfrak{k}<a} t_\mathfrak{k}^{\beta-1} L_\mathfrak{k} + \frac{a^\alpha}{\alpha\Gamma(\alpha)} \|L_p(t)\|_{C(\mathcal{J};\mathbb{R})}\Bigg)(1 + \|\varphi\|_{L^1(\mathcal{J};\mathbb{R})})\Bigg]\|u - v\|_t.$$

Then $\mathfrak{T}$ is a contraction map and has a fixed point $u(\cdot)$ of $\mathfrak{T}$. Thus, we determine that $u(\cdot)$ is a unique mild solution of (1.1)-(1.3). □

Theorem 3.2. *Let $S(t)$ be compact for all $t \geq 0$, $(H2)-(H6)$ are satisfied and if*

$$\Re_a\Bigg(C_1 + \Gamma(2-\beta)a \sum_{0<t_\mathfrak{k}<a} t_\mathfrak{k}^{\beta-1} d_\mathfrak{k}^1 + \frac{a^\alpha}{\alpha\Gamma(\alpha)}\|m_p(t)\| \liminf_{r\to\infty} \frac{W(r)}{r}\Bigg) < 1,$$

$$\Re_a(\|L_q(t)\| + \Gamma(2-\beta)a \sum_{0<t_{k<a}} t_\mathfrak{k}^{\beta-1} L_\mathfrak{k})(1 + \|\varphi\|) < 1.$$

Then (1.1)-(1.3) has a mild solution.

Proof. Take $r > 0$, such that

$$\Bigg(C_1\|\phi\|_{\mathscr{B}_A} + 2C_2 + \Bigg(C_1 + \Gamma(2-\beta)a \sum_{0<t_\mathfrak{k}<a} t_\mathfrak{k}^{\beta-1} d_\mathfrak{k}^1\Bigg)(\Re_a r + (\Re_a \mathfrak{H}(1 + \|\varphi\|_{L^1(\mathcal{J};\mathbb{R})}) + \mathfrak{M}_a)\|\phi\|_{\mathscr{B}_A}) + \Gamma(2-\beta)a \sum_{0<t_\mathfrak{k}<a} t_\mathfrak{k}^{\beta-1} d_\mathfrak{k}^2 + \|z + \xi\|a + \frac{a^\alpha}{\alpha\Gamma(\alpha)}\|m_p(t)\| \times W(\Re_a r + (\Re_a \mathfrak{H}(1 + \|\varphi\|_{L^1(\mathcal{J};\mathbb{R})}) + \mathfrak{M}_a)\|\phi\|_{\mathscr{B}_A})\Bigg)(1 + \|\varphi\|) \leq s,$$

for all $s \geq r$.

Let the operator $\mathfrak{T} : \mathcal{B}_r(0, \mathscr{Z}(a)) \to \mathscr{Z}(a)$ be defined likewise considered in the previous Theorem 3.1, and in a similar manner we can easy to see that $\mathfrak{T}$ is well defined. Now, our aim to show that $\mathfrak{T} : \mathcal{B}_r(0, \mathscr{Z}(a)) \to \mathcal{B}_r(0, \mathscr{Z}(a))$ is a condensing map.

The subsequent steps shows the remaining proof.

Step 1. $\mathfrak{T}$ has values in $\mathcal{B}_r(0, \mathscr{Z}(a))$, i.e., $\mathfrak{T}\mathcal{B}_r(0, \mathscr{Z}(a)) \subset \mathcal{B}_r(0, \mathscr{Z}(a))$.

Let $u \in \mathcal{B}_r(0, \mathscr{Z}(a))$ and $t \in \mathcal{J}$, then

$$
\begin{aligned}
\|\mathfrak{T}u(t)\| &\leq \|q(0,\phi)\| + \|q(t, u_t + x_t)\| + \Gamma(2-\beta) \sum_{0<t_{\mathfrak{k}}<t} t_{\mathfrak{k}}^{\beta-1}(t - t_{\mathfrak{k}})\|I_{\mathfrak{k}}(u_{t_{\mathfrak{k}}} + x_{t_{\mathfrak{k}}})\| \\
&\quad + \|z + \xi\| t + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|p(s, u_s + x_s)\| ds \\
&\quad + \int_0^t \|S'(t-s)\| \Big(\|q(0,\phi)\| + \|q(s, u_s + x_s)\| + \|z + \xi\| s \\
&\quad + \Gamma(2-\beta) \sum_{0<t_{\mathfrak{k}}<s} t_{\mathfrak{k}}^{\beta-1}(s - t_{\mathfrak{k}})\|I_{\mathfrak{k}}(u_{t_{\mathfrak{k}}} + x_{t_{\mathfrak{k}}})\| \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} \|p(\tau, u_\tau + x_\tau)\| d\tau \Big) ds \\
&\leq \Big(C_1 \|\phi\|_{\mathscr{B}_A} + 2C_2 + C_1 \|u_t + x_t\|_{\mathscr{B}_A} + \Gamma(2-\beta) a \sum_{0<t_{\mathfrak{k}}<a} t_{\mathfrak{k}}^{\beta-1} (d_{\mathfrak{k}}^1 \|u_{t_{\mathfrak{k}}} + x_{t_{\mathfrak{k}}}\|_{\mathscr{B}_A} \\
&\quad + d_{\mathfrak{k}}^2) + \|z + \xi\| a + \frac{a^\alpha}{\alpha \Gamma(\alpha)} \|m_p(t)\| W(\|u_t + x_t\|_{\mathscr{B}_A}) \Big) (1 + \|\varphi\|) \\
&\leq \Big(C_1 \|\phi\|_{\mathscr{B}_A} + 2C_2 + \Big(C_1 + \Gamma(2-\beta) a \sum_{0<t_{\mathfrak{k}}<a} t_{\mathfrak{k}}^{\beta-1} d_{\mathfrak{k}}^1 \Big) r^* \\
&\quad + \Gamma(2-\beta) a \sum_{0<t_{\mathfrak{k}}<a} t_{\mathfrak{k}}^{\beta-1} d_{\mathfrak{k}}^2 + \|z + \xi\| a + \frac{a^\alpha}{\alpha \Gamma(\alpha)} \|m_p(t)\| W(r^*) \Big) (1 + \|\varphi\|)
\end{aligned}
$$

where $r^* = \mathfrak{K}_a r + (\mathfrak{K}_a \mathfrak{H}(1 + \|\varphi\|_{L^1(\mathcal{J};\mathbb{R})}) + \mathfrak{M}_a)\|\phi\|_{\mathscr{B}_A}$.

This implies that $\|\mathfrak{T}u(t)\| \leq r$, i.e., $\mathfrak{T}u \in \mathcal{B}_r(0, \mathscr{Z}(a))$ and $\mathfrak{T}\mathcal{B}_r(0, \mathscr{Z}(a)) \subset \mathcal{B}_r(0, \mathscr{Z}(a))$.

The remainder of the proof continuing with the decomposition operator $\mathfrak{T} = \sum_{i=1}^3 \mathfrak{T}_i$, where

$$
\begin{aligned}
\mathfrak{T}_1 u(t) &= q(0,\phi) + (z + \xi)t - q(t, u_t + x_t) + \Gamma(2-\beta) \sum_{0<t_{\mathfrak{k}}<t} t_{\mathfrak{k}}^{\beta-1}(t - t_{\mathfrak{k}}) I_{\mathfrak{k}}(u_{t_{\mathfrak{k}}} + x_{t_{\mathfrak{k}}}) \\
&\quad + \int_0^t S'(t-s) \Big(q(0,\phi) + (z + \xi)s - q(s, u_s + x_s) \\
&\quad + \Gamma(2-\beta) \sum_{0<t_{\mathfrak{k}}<s} t_{\mathfrak{k}}^{\beta-1}(s - t_{\mathfrak{k}}) I_{\mathfrak{k}}(u_{t_{\mathfrak{k}}} + x_{t_{\mathfrak{k}}}) \Big) ds \\
\mathfrak{T}_2 u(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s, u_s + x_s) ds \\
\mathfrak{T}_3 u(t) &= \int_0^t S'(t-s) \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} p(\tau, u_\tau + x_\tau) d\tau ds
\end{aligned}
$$

Step 2. $\mathfrak{T}_1$ is a contraction map on $\mathcal{B}_r(0, \mathscr{Z}(a))$.

Let $u \in \mathcal{B}_r(0, \mathscr{Z}(a))$.

$$\begin{aligned}
\|\mathfrak{T}_1 u(t) - \mathfrak{T}_1 v(t)\| &\leq \|q(t, u_t + x_t) - q(t, v_t + x_t)\| \\
&\quad + \Gamma(2-\beta) \sum_{0<t_\mathfrak{k}<t} t_\mathfrak{k}^{\beta-1}(t - t_\mathfrak{k})\|I_\mathfrak{k}(u_{t_\mathfrak{k}} + x_{t_\mathfrak{k}}) - I_\mathfrak{k}(v_{t_\mathfrak{k}} + x_{t_\mathfrak{k}})\| \\
&\quad + \int_0^t \|S'(t-s)\|(\|q(s, u_s + x_s) - q(s, v_s + x_s)\| \\
&\quad + \Gamma(2-\beta) \sum_{0<t_\mathfrak{k}<s} t_\mathfrak{k}^{\beta-1}(s - t_\mathfrak{k})\|I_\mathfrak{k}(u_{t_\mathfrak{k}} + x_{t_\mathfrak{k}}) - I_\mathfrak{k}(v_{t_\mathfrak{k}} + x_{t_\mathfrak{k}})\|) \\
&\leq (\|L_q(t)\| + \Gamma(2-\beta) a \sum_{0<t_\mathfrak{k}<a} t_\mathfrak{k}^{\beta-1} L_\mathfrak{k})\|u_t - v_t\|_{\mathscr{B}_A}(1 + \|\varphi\|) \\
&\leq \mathfrak{K}_a(\|L_q(t)\| + \Gamma(2-\beta) a \sum_{0<t_\mathfrak{k}<a} t_\mathfrak{k}^{\beta-1} L_\mathfrak{k})(1 + \|\varphi\|)\|u - v\|_t.
\end{aligned}$$

Hence, $\mathfrak{T}_1$ is a contraction map on $\mathcal{B}_r(0, \mathscr{Z}(a))$.

Step 3. $\mathfrak{T}_2$ is a completely continuous map.

It is easy to see that the map $\mathfrak{T}_2$ is continuous, since the function f is continuous.

Next, we only we need to prove that $\mathfrak{T}_2$ is a compact operator.

Let $0 < \epsilon < t \leq a$, $u \in \mathscr{Z}(a)$. From the mean value theorem for the Bochner integral (see [17, Lemma II.1.3]), we have that

$$\begin{aligned}
\mathfrak{T}_2 u(t) &= \frac{1}{\Gamma(\alpha)} \int_0^\epsilon (t-s)^{\alpha-1} p(s, u_s + x_s) ds + \frac{1}{\Gamma(\alpha)} \int_\epsilon^t (t-s)^{\alpha-1} p(s, u_s + x_s) ds \\
&\in \mathcal{B}_{\frac{Q\epsilon^\alpha}{\alpha\Gamma(\alpha)}}(0, \mathscr{E}) + \frac{(t-\epsilon)}{\Gamma(\alpha)} \overline{co(\{(t-s)^{\alpha-1} p(s, u_s + x_s) : s \in [\epsilon, t]\})}
\end{aligned}$$

where $Q = \|m_p(t)\| W(r^*)$, and the notion $co(U)$ refers the convex hull of the set U.

Since from [9, Lemma 2.2], the map i_c is compact and $p \in C(\mathcal{J} \times \mathscr{B}_A; \mathscr{D}(\mathscr{A}))$, from the above inclusion we find that $\mathfrak{T}_2 \mathcal{B}_r(0, \mathscr{Z}(a)) = \{\mathfrak{T}_2 u(t) : u \in \mathcal{B}_r(0, \mathscr{Z}(a))\} \subset C_\epsilon + K_\epsilon$, where K_ϵ is compact and $\text{diam}(C_\epsilon) = \frac{Q\epsilon^\alpha}{\alpha\Gamma(\alpha)} \to 0$ as $\epsilon \to 0$. This proves that the set $\mathfrak{T}_2 \mathcal{B}_r(0, \mathscr{Z}(a))$ is relatively compact in space $\mathscr{E}$ for all t in $\mathcal{J}$.

Consider $l > 0$, $0 \leq t < a$ such that $0 \leq t + l \leq a$, and for $u \in \mathcal{B}_r(0, \mathscr{Z}(a))$,

$$\begin{aligned}
\|\mathfrak{T}_2 u(t+l) - \mathfrak{T}_2 u(t)\| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t ((t-s)^{\alpha-1} - (t+l-s)^{\alpha-1})\|p(s, u_s + x_s)\|_{\mathscr{D}(\mathscr{A})} ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_t^{t+l} (t+l-s)^{\alpha-1}\|p(s, u_s + x_s)\|_{\mathscr{D}(\mathscr{A})} ds \\
&\leq \frac{2l^\alpha}{\alpha\Gamma(\alpha)}\|p(s, u_s + x_s)\|_{\mathscr{D}(\mathscr{A})} \\
&\leq \frac{2Ql^\alpha}{\alpha\Gamma(\alpha)}
\end{aligned}$$

which implies that $\mathfrak{T}_2 \mathcal{B}_r(0, \mathscr{Z}(a))$ is equicontinuous.

Hence from the above results $\mathfrak{T}_2$ is completely continuous.

Step 4. The operator $\mathfrak{T}_3$ is completely continuous.

Let $t \in [0, a)$ and consider $\mathscr{P}(s) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s, u_s + x_s) ds$. For $u \in \mathcal{B}_r(0, \mathscr{Z}(a))$ and there exist $\epsilon > 0$, we take $l \in (0, \epsilon)$ such that $t + l \leq a$, and from [10, Lemma 2.2],

$$\begin{aligned}
&\| \mathfrak{T}_3 u(t+l) - \mathfrak{T}_3 u(t) \| \\
&\leq \int_0^l \| S'(t+l-s)\mathscr{P}(s) \| ds + \int_0^t \| S'(s)(\mathscr{P}(t-s+l) - \mathscr{P}(t-s)) \| ds \\
&\leq \int_0^l \varphi(t+l-s) \| \mathscr{P}(s) \|_{\mathscr{D}(\mathscr{A})} ds + \int_0^t \varphi(s)[|\mathscr{P}|]_{C^\alpha(\mathcal{J};\mathscr{D}(\mathscr{A}))} l^\alpha ds \\
&\leq \frac{2Q}{\alpha\Gamma(\alpha)} \left(a^\alpha \int_0^l \varphi(t+l-s) ds + l^\alpha \| \varphi \|_{L^1(\mathcal{J})} \right)
\end{aligned}$$

which proves that $\mathfrak{T}_3 \mathcal{B}_r(0, \mathscr{Z}(a))$ is right equicontinuous at t in $[0, a)$. The above discussion allow us to show that $\mathfrak{T}_3 \mathcal{B}_r(0, \mathscr{Z}(a))$ is left equicontinuous at t in the interval $(0, a]$. From this argument we say that $\mathfrak{T}_3 \mathcal{B}_r(0, \mathscr{Z}(a))$ is equicontinuous.

In this sequel we finally prove that $\{\mathfrak{T}_3 u(t) : u \in \mathcal{B}_r(0, \mathscr{Z}(a))\}$ is relatively compact in $\mathscr{E}$, $\forall\, t \in (0, a]$.

Take $0 < t \leq a$ and $Q_1 = \|m_p(t)\| W(r^*) \|\varphi\|_{L^1([0,\epsilon])}$. The set $V = \{\mathscr{P}(s) : s \in \mathcal{J}, u \in \mathcal{B}_r(0, \mathscr{Z}(a))\}$ is relatively compact in $\mathscr{E}$, since from the previous Step 2. If u belongs to $\mathcal{B}_r(0, \mathscr{Z}(a))$, by using the concept in [17, Lemma II.1.3], we get

$$\begin{aligned}
\mathfrak{T}_3 u(t) &= \int_0^\epsilon S'(t-s)\mathscr{P}(s) ds + \int_\epsilon^t S'(t-s)\mathscr{P}(s) ds \\
&\in \mathcal{B}_{\frac{Q_1 a^\alpha}{\alpha\Gamma(\alpha)}}(0, \mathscr{E}) + (t-\epsilon)\overline{co(\{S'(s)y : s \in [\epsilon, t], y \in \overline{V}\})}
\end{aligned}$$

and hence, $\{\mathfrak{T}_3 u(t) : u \in \mathcal{B}_r(0, \mathscr{Z}(a))\} \subset \mathcal{B}_{\frac{Q_1 a^\alpha}{\alpha\Gamma(\alpha)}}(0, \mathscr{E}) + K_\epsilon$, where K_ϵ is compact and $\frac{Q_1 a^\alpha}{\alpha\Gamma(\alpha)} \to 0$ as $\epsilon \to 0$. This proves that $\{\mathfrak{T}_3 u(t) : u \in \mathcal{B}_r(0, \mathscr{Z}(a))\}$ is relatively compact in $\mathscr{E}$. Hence we finally conclude that $\mathfrak{T}_3$ is completely continuous.

From the above steps we can say that the operator $\mathfrak{T} : \mathcal{B}_r(0, \mathscr{Z}(a)) \to \mathcal{B}_r(0, \mathscr{Z}(a))$ is a condensing map. Then the existence results follows from [17, Theorem IV.3.2]. □

4 Application

We look at the following partial fractional impulsive neutral differential equations with infinite delay of the form

$$
\begin{aligned}
&{}^cD^{\alpha}_{0^+}\left(v(t,\eta)+\int_{-\infty}^{t}\int_{0}^{\pi}a(t-s,\zeta,\eta)v(s,\zeta)d\zeta ds\right)\\
&\quad=\frac{\partial^2}{\partial\eta^2}v(t,\eta)+\int_{-\infty}^{t}d(t,t-s,\eta,v(s,\eta))ds,\ (t,\eta)\in\mathcal{J}\times[0,\pi], \qquad (4.1)\\
&v(t,0)=v(t,\pi)=0,\ t\in\mathcal{J}, \qquad (4.2)\\
&v(\tau,\eta)=\phi(\tau,\eta),\ 0\le\eta\le\pi,\ \tau\le 0, \qquad (4.3)\\
&{}^cD^{\beta}_{0^+}v(t_{\mathfrak{k}}^{+})(\eta)-{}^cD^{\beta}_{0^+}v(t_{\mathfrak{k}})(\eta)=\int_{-\infty}^{t_{\mathfrak{k}}}e_{\mathfrak{k}}(t_{\mathfrak{k}}-s)v(s,\eta)ds, \qquad (4.4)
\end{aligned}
$$

where $0<\beta<1, 1<\alpha<2$, and $\phi\in\mathscr{B}_A=\mathcal{PC}_0\times L^2(g,\mathscr{E})$. Assume that $d:\mathbb{R}^4\to\mathbb{R},\ e_{\mathfrak{k}}:\mathbb{R}\to\mathbb{R},\ a:\mathbb{R}^3\to\mathbb{R}$ are continuous functions and $\frac{\partial^{\alpha}a(s,\zeta,\eta)}{\partial\eta^{\alpha}}$ exists. $0<t_1<\cdots<t_m<a$ are prefixed numbers.

Let the space $\mathscr{E}=L^2([0,\pi])$. Let $\mathscr{A}:\mathscr{D}(\mathscr{A})\subset\mathscr{E}\to\mathscr{E}$ be defined by $\mathscr{A}u=u''$ with $\mathscr{D}(\mathscr{A})$ consist of set of all u and u'' in $\mathscr{E}$ such that $u(0)=u(\pi)=0$. If $\{\mathfrak{C}(t)\}_{t\ge0}$ is a strongly continuous cosine family on $\mathscr{E}$, then $\mathscr{A}$ is its infinitesimal generator. The well known result that the associated sine operator $\mathfrak{S}(t)$ is compact for every $t\in\mathbb{R}$ and hence $(\lambda-\mathscr{A})^{-1}$ is compact for every λ belongs to $\rho(\mathscr{A})$.

Consider

$$u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\mathscr{A}u(s)ds,\quad s\ge 0,$$

have an analytic resolvent $\{S(t)\}_{t\ge0}$ on $\mathscr{E}$ given by

$$S(t)=\begin{cases}\frac{1}{2\pi i}\int_{\Gamma_{\varrho,\upsilon}}e^{\lambda t}\lambda^{\alpha-1}(\lambda^{\alpha}-\mathscr{A})^{-1}d\lambda, & t>0,\\ I, & t=0,\end{cases}$$

with $\Gamma_{\varrho,\upsilon}$ consisting of the rays $\{\varrho e^{i\upsilon}:\varrho\ge 0\}$ and $\{\varrho e^{-i\upsilon}:\varrho\ge 0\}$. Here $\Gamma_{\varrho,\upsilon},\ \upsilon\in(\pi,\frac{\pi}{2})$, is a contour, [20, Example II.2.1].

To represent the equations (4.1)-(4.4) in the form of (1.1)-(1.3) by

$$q(t,\varsigma)(\eta)=\int_{-\infty}^{0}\int_0^{\pi}a(s,\zeta,\eta)\varsigma(s,\zeta)d\zeta ds,$$

$$p(t,\varsigma)(\eta)=\int_{-\infty}^{0}d(t,s,\eta,\varsigma(s,\eta))ds$$

$$I_{\mathfrak{k}}(\varsigma)(\eta)=\int_{-\infty}^{0}e_{\mathfrak{k}}(s)\varsigma(s,\eta)ds$$

Therefore, under the appropriate conditions on the functions $a, d, e_{\mathfrak{k}}$, the mild solution exists for partial fractional impulsive problem (4.1)-(4.4) in view of Theorem 3.2 and uniqueness results exists from Theorem 3.1.

Conclusion

In this work we consider the fractional neutral infinite delay differential equations with fractional impulsive conditions involving Caputo derivative of order lies in the interval (1,2). To improve the characterization of the memory property of the fractional derivative, we consider the lower bound at each impulse as zero. We use resolvent operator to derive the mild solutions in order to make it as more appropriate.

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The solution to matrix inequality $AXB + (AXB)^* \geqslant C$ and its applications

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Abstract

In this paper, firstly, we study the solution to linear matrix inequality $AXB + (AXB)^* \geqslant C$ for Hermitian matrix C. Furthermore, for the applications, we derive the representations for the common Re-nnd solution to equations $AX = C$ and $XB = D$, and the Re-nnd $\{1, 3, 4\}$-inverse for square matrix.

Keywords: Matrix inequality, Re-nnd solution, Re-nnd generalized inverse

AMS(2000) Subject Classification: 15A09, 15A24

1 Introduction

Let $\mathbb{C}^{m\times n}$ denote the set of all $m \times n$ matrices over the complex field $\mathbb{C}$, $\mathbb{C}_H^m$ denote the set of all $m \times m$ Hermitian matrices, $\mathbb{U}_n$ denote the set of all $n \times n$ unitary matrices. For $A \in \mathbb{C}^{m\times n}$, its range space, rank and conjugate transpose will be denoted by $R(A)$, $r(A)$ and A^* respectively. $i_+(A)$ and $i_-(A)$ denote the numbers of the positive and negative eigenvalues of a Hermitian matrix A counted with multiplicities, respectively. The identity matrix of order n is denoted by I_n.

For a matrix $A \in \mathbb{C}^{m\times n}$, the Moore-Penrose inverse $A^\dagger$ is defined to be the unique solution of the four Penrose equations [1]

$$(1)\ AXA = A,\ (2)\ XAX = X,\ (3)\ (AX)^* = AX,\ (4)\ (XA)^* = XA.$$

For convenience, we denote $E_A = I - AA^\dagger$ and $F_A = I - A^\dagger A$.

The Hermitian part of $A \in \mathbb{C}^{m\times m}$ is defined by $H(A) = \frac{1}{2}(A + A^*)$. We say that A is Re-nnd (Re-nonnegative definite) if $H(A) \geqslant 0$ and A is Re-pd (Re-positive definite) if $H(A) > 0$. Let $A_{re}^{(i,j,\cdots,k)}$ be the

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Re-nnd $\{i, j, \cdots, k\}$-inverse of square matrix A. Recently, some researches on Re-nnd solution and Re-nnd generalized inverse were done by several authors [2-7].

The Löwner partial ordering is one of the most basic concepts for characterizing relations between two Hermitian matrices. A challenging research topic on Hermitian matrices is to solve linear matrix inequalities (LMIs) induced from the Löwner partial ordering, such as

$$AXB + (AXB)^* \geqslant C, \tag{1.1}$$

$$AXB + (AXB)^* \leqslant (>, <)C, \;\; AX + (AX)^* \geqslant (\leqslant, >, <)C, \;\; AXA^* \geqslant (\leqslant, >, <)C.$$

In this article, we consider the matrix inequality (1.1), where $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times m}$ and $C \in \mathbb{C}_H^m$ are given, $X \in \mathbb{C}^{n\times p}$ is variable matrix.

Newly, some special cases of (1.1) were considered by several authors, such as: the case that C is nonnegative definite matrix [8], the case $B = I_m$ [9], the case that block matrix $\begin{pmatrix} A & B^* \end{pmatrix}$ is full row rank [10]. Researches on other linear matrix inequalities can be found in [8, 11]. For the applications, (1.1) can be used to establish the general forms of Re-nnd solution of matrix equation $AXB = C$ [10], and the solution of matrix equation $AXA^* = B$ (or $AX = B$) subject matrix inequality constraint $CXC^* \geqslant D$ [12, 13], and the Re-nnd inverses $A_{re}^{(1,2,i)}$, $A_{re}^{(1,i)}$ $(i = 3, 4)$ of square matrix [3, 4, 10]. In [2, 6], the authors provided some necessary and sufficient conditions for the existence of common Re-nnd and Re-pd solutions to $AX = C$ and $XB = D$, however, the general solutions are still unsolved.

We are, therefore, motivated to focus our research interest on (1.1) without any restrictions on matrices A, B, C.

It is well known that (1.1) can equivalently be written as

$$AXB + (AXB)^* = C + VV^* \tag{1.2}$$

for some V. Tian and Rosen [8] shown that equation (1.2) is solvable for X if and only if VV^* satisfies

$$E_G VV^* = -E_G C, \;\; E_A VV^* E_A = -E_A C E_A, \;\; F_B VV^* F_B = -F_B C F_B, \tag{1.3}$$

where $G = \begin{pmatrix} A & B^* \end{pmatrix}$.

This paper is organized as follows. In section 2, firstly, we establish some necessary and sufficient conditions for the solvability of matrix inequality (1.1), secondly, we derive a general form for VV^*, finally, we present a general solution of X to matrix inequality (1.1). Furthermore, for the applications, we provide the explicit expressions for the common Re-nnd solution to equations $AX = C$ and $XB = D$, and the Re-nnd generalized inverse $A^{(1,3,4)}$ of square matrix A.

Before proceeding to the next section, we list some useful results which will facilitate the proof of our theorems.

Lemma 1.1. ([14]) Let $A \in \mathbb{C}_H^m$, $B \in \mathbb{C}^{m\times n}$ and $C \in \mathbb{C}^{p\times m}$ be given. Then

$$\begin{aligned}\max_{X\in\mathbb{C}^{n\times m}} i_{\pm}[A - BXC - (BXC)^*] &= \min\{i_{\pm}(M_1),\ i_{\pm}(M_2)\},\\ \min_{X\in\mathbb{C}^{n\times m}} i_{\pm}[A - BXC - (BXC)^*] &= r\begin{pmatrix} A & B & C^*\end{pmatrix} + \max\{i_{\pm}(M_1) - r(N_1),\ i_{\pm}(M_2) - r(N_2)\},\end{aligned}$$

where

$$M_1 = \begin{pmatrix} A & B \\ B^* & 0\end{pmatrix},\ M_2 = \begin{pmatrix} A & C^* \\ C & 0\end{pmatrix},\ N_1 = \begin{pmatrix} A & B & C^* \\ B^* & 0 & 0\end{pmatrix},\ N_2 = \begin{pmatrix} A & B & C^* \\ C & 0 & 0\end{pmatrix}.$$

Lemma 1.2. ([14]) Let $A \in \mathbb{C}_H^m$, $B \in \mathbb{C}^{m\times n}$, and denote $M = \begin{pmatrix} A & B \\ B^* & 0\end{pmatrix}$. Then

$$i_{\pm}(M) = r(B) + i_{\pm}(E_B A E_B).$$

Lemma 1.3. ([15]) Let $A, B \in \mathbb{C}^{m\times n}$ be given. Then the matrix equation $AXX^* = B$ has a solution for XX^* if and only if $R(B) \subseteq R(A)$, $AB^* \geqslant 0$ and $r(AB^*) = r(B)$. In this case, the general solution can be written in the following parametric form

$$XX^* = B^*(AB^*)^{\dagger}B + F_A WW^* F_A,$$

where $W \in \mathbb{C}^{n\times n}$ is arbitrary.

Lemma 1.4. ([16]) Given matrices $A, B, C, D \in \mathbb{C}^{p\times n}$. The matrix equations $AXX^*A^* = BB^*$ and $CXX^*C^* = DD^*$ have a common Hermitian nonnegative-definite solution if and only if $R(B) \subseteq R(A)$ (*or* $r\begin{pmatrix} A & B\end{pmatrix} = r(A)$) and there exists $T \in \mathbb{U}_n$ such that

$$E_{CF_A}(DT - CA^{\dagger}B) = 0. \tag{1.4}$$

If a common Hermitian nonnegative-definite solution exists, then a representation of the general common Hermitian nonnegative-definite solution is XX^* with

$$X = A^{\dagger}B + F_A(CF_A)^{\dagger}(DT - CA^{\dagger}B) + F_A F_{CF_A} Z,$$

where $Z \in \mathbb{C}^{n\times n}$ is arbitrary and $T \in \mathbb{U}_n$ is a parameter matrix satisfying (1.4).

Lemma 1.5. ([8]) Let $A \in \mathbb{C}^{m\times p}$ and $B \in \mathbb{C}^{q\times m}$ and $C \in \mathbb{C}_H^m$ are given. Then the matrix equation $AXB + (AXB)^* = C$ has a solution $X \in \mathbb{C}^{p\times q}$ if and only if

$$\begin{pmatrix} A & B^*\end{pmatrix}\begin{pmatrix} A & B^*\end{pmatrix}^{\dagger} C = C,\ \ E_A C E_A = 0,\ \ F_B C F_B = 0.$$

In this case, the general solution can be written as

$$X = \frac{1}{2}(X_1 + X_2^*),$$

where X_1 and X_2 are general solutions of the equation $AX_1B + B^*X_2A^* = C$.

Lemma 1.6. ([17]) Let $A_1 \in \mathbb{C}^{m\times n}$, $B_1 \in \mathbb{C}^{p\times k}$, $A_2 \in \mathbb{C}^{m\times l}$, $B_2 \in \mathbb{C}^{q\times k}$ and $C \in \mathbb{C}^{m\times k}$ be known and $X_1 \in \mathbb{C}^{n\times p}$, $X_2 \in \mathbb{C}^{l\times q}$ unknown; $M = E_{A_1}A_2$, $N = B_2F_{B_1}$, $S = A_2F_M$. Then the following statements are equivalent:

(i) The system $A_1X_1B_1 + A_2X_2B_2 = C$ is solvable;

(ii) The following rank equalities are satisfied,

$$r\begin{pmatrix} A_1 & C \\ 0 & B_2 \end{pmatrix} = r\begin{pmatrix} A_1 & 0 \\ 0 & B_2 \end{pmatrix}, \; r\begin{pmatrix} A_2 & C \\ 0 & B_1 \end{pmatrix} = r\begin{pmatrix} A_2 & 0 \\ 0 & B_1 \end{pmatrix},$$
$$r\begin{pmatrix} C & A_1 & A_2 \end{pmatrix} = r\begin{pmatrix} A_1 & A_2 \end{pmatrix}, \; r\begin{pmatrix} B_1 \\ B_2 \\ C \end{pmatrix} = r\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}.$$

In this case, the general solution can be expressed as

$$\begin{aligned} X_1 &= A_1^{\dagger}CB_1^{\dagger} - A_1^{\dagger}A_2M^{\dagger}E_{A_1}CB_1^{\dagger} - A_1^{\dagger}SA_2^{\dagger}CF_{B_1}N^{\dagger}B_2B_1^{\dagger} - A_1^{\dagger}SVE_NB_2B_1^{\dagger} + F_{A_1}U + ZE_{B_1}, \\ X_2 &= M^{\dagger}E_{A_1}CB_2^{\dagger} + F_MS^{\dagger}SA_2^{\dagger}CF_{B_1}N^{\dagger} + F_M(V - S^{\dagger}SVNN^{\dagger}) + WE_{B_2}, \end{aligned}$$

where U, V, W and Z are arbitrary matrices over complex field with appropriate sizes.

Lemma 1.7. ([8]) Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$ and $C \in \mathbb{C}^{l\times n}$. Then

$$r\begin{pmatrix} A & B \end{pmatrix} = r(A) + r(E_AB), \; r\begin{pmatrix} A \\ C \end{pmatrix} = r(A) + r(CF_A),$$
$$r\begin{pmatrix} A & B \\ C & 0 \end{pmatrix} = r(B) + r(C) + r(E_BAF_C).$$

Lemma 1.8. ([9]) Let $A, C \in \mathbb{C}^{n\times m}$. There exists a Re-nnd solution to equation $AX = C$ if and only if $R(C) \subseteq R(A)$, AC^* is Re-nnd. There exists a Re-pd solution to equation $AX = C$ if and only if $R(C) \subseteq R(A)$, $i_+(AC^* + CA^*) = r(A)$.

2 Main results

In this section, our purpose is to investigate the solution to the linear matrix inequality (1.1), and then apply our result to establish the general expressions for the common Re-nnd solution to $AX = C$ and $XB = D$, and the Re-nnd $\{1, 3, 4\}$-inverse for square matrix A.

First, we come to establish some necessary and sufficient conditions for the solvability of matrix inequality (1.1).

Theorem 2.1. Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times m}$ and $C \in \mathbb{C}_H^m$ be given, $X \in \mathbb{C}^{n\times p}$ be variable matrix, denote $G = \begin{pmatrix} A & B^* \end{pmatrix}$. Then the following statements are equivalent:

(1) Matrix inequality (1.1) is solvable;

(2) $E_ACE_A \leqslant 0$, $F_BCF_B \leqslant 0$, and

$$r\begin{pmatrix} C & A & B^* \end{pmatrix} + r(A) = r\begin{pmatrix} C & A & B^* \\ A^* & 0 & 0 \end{pmatrix},\quad r\begin{pmatrix} C & A & B^* \end{pmatrix} + r(B) = r\begin{pmatrix} C & A & B^* \\ B & 0 & 0 \end{pmatrix};$$

(3) $r(E_GCE_A) = r(E_GCF_B) = r(E_GC)$, $E_ACE_A \leqslant 0$ and $F_BCF_B \leqslant 0$.

Proof. Note that (1.1) can be rewritten as $C - AXB - (AXB)^* \leqslant 0$. So, (1.1) is solvable if and only if

$$\min_X i_+[C - AXB - (AXB)^*] = 0.$$

Applying Lemma 1.1, we get

$$\begin{aligned}
&\min_X i_+[C - AXB - (AXB)^*] \\
= \;& r\begin{pmatrix} C & A & B^* \end{pmatrix} + \max\left\{ i_+\begin{pmatrix} C & A \\ A^* & 0 \end{pmatrix} - r\begin{pmatrix} C & A & B^* \\ A^* & 0 & 0 \end{pmatrix}, i_+\begin{pmatrix} C & B^* \\ B & 0 \end{pmatrix} - r\begin{pmatrix} C & A & B^* \\ B & 0 & 0 \end{pmatrix} \right\} \\
= \;& r\begin{pmatrix} C & A & B^* \end{pmatrix} + \max\left\{ r(A) + i_+(E_ACE_A) - r\begin{pmatrix} C & A & B^* \\ A^* & 0 & 0 \end{pmatrix}, \right. \\
& \left. r(B) + i_+(F_BCF_B) - r\begin{pmatrix} C & A & B^* \\ B & 0 & 0 \end{pmatrix} \right\}. \qquad (2.1)
\end{aligned}$$

Letting the right hand side of (2.1) be zero yields

$$\begin{aligned}
r\begin{pmatrix} C & A & B^* \end{pmatrix} + r(A) + i_+(E_ACE_A) &= r\begin{pmatrix} C & A & B^* \\ A^* & 0 & 0 \end{pmatrix}, \\
r\begin{pmatrix} C & A & B^* \end{pmatrix} + r(B) + i_+(F_BCF_B) &= r\begin{pmatrix} C & A & B^* \\ B & 0 & 0 \end{pmatrix},
\end{aligned}$$

which are equivalent to

$$r\begin{pmatrix} C & A & B^* \end{pmatrix} + r(A) = r\begin{pmatrix} C & A & B^* \\ A^* & 0 & 0 \end{pmatrix}, \qquad (2.2)$$

$$r\begin{pmatrix} C & A & B^* \end{pmatrix} + r(B) = r\begin{pmatrix} C & A & B^* \\ B & 0 & 0 \end{pmatrix}, \qquad (2.3)$$

$$E_ACE_A \leqslant 0 \quad and \quad F_BCF_B \leqslant 0.$$

Applying Lemma 1.7 to (2.2) and (2.3) yields $r(E_GCE_A) = r(E_GC)$ and $r(E_GCF_B) = r(E_GC)$ respectively. Thus, the proof is complete. □

Next, we present some properties for matrices A, B and C which satisfy the conditions in Theorem 2.1.

Corollary 2.1. Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times m}$ and $C \in \mathbb{C}_H^m$ be given, denote $G = \begin{pmatrix} A & B^* \end{pmatrix}$. If the conditions in the statement (2) or (3) of Theorem 2.1 are satisfied, then the following hold,

$$r(E_GCE_G) = r(E_GC) \;\; or \;\; R(E_GCE_G) = R(E_GC), \;\; E_GCE_G \leqslant 0, \tag{2.4}$$

$$R[E_ACE_G(E_GCE_G)^\dagger E_GCE_A - E_ACE_A] \subseteq R(E_AGG^\dagger), \tag{2.5}$$

$$E_ACE_G(E_GCE_G)^\dagger E_GCE_A - E_ACE_A \geqslant 0, \tag{2.6}$$

$$R[F_BCE_G(E_GCE_G)^\dagger E_GCF_B - F_BCF_B] \subseteq R(F_BGG^\dagger), \tag{2.7}$$

$$F_BCE_G(E_GCE_G)^\dagger E_GCF_B - F_BCF_B \geqslant 0. \tag{2.8}$$

Proof. It follows from the two rank equalities of statement (2) in Theorem 2.1 that

$$R\left(\begin{pmatrix} C & A & B^* \end{pmatrix}^*\right) \cap R\left(\begin{pmatrix} A^* & 0 & 0 \end{pmatrix}^*\right) = \emptyset, \;\; R\left(\begin{pmatrix} C & A & B^* \end{pmatrix}^*\right) \cap R\left(\begin{pmatrix} B & 0 & 0 \end{pmatrix}^*\right) = \emptyset.$$

Hence,

$$R\left(\begin{pmatrix} C & A & B^* \end{pmatrix}^*\right) \cap R\left(\begin{pmatrix} A^* & 0 & 0 \\ B & 0 & 0 \end{pmatrix}^*\right) = \emptyset,$$

which means that

$$r\begin{pmatrix} C & A & B^* \\ A^* & 0 & 0 \\ B & 0 & 0 \end{pmatrix} = r\begin{pmatrix} C & A & B^* \end{pmatrix} + r\begin{pmatrix} A & B^* \end{pmatrix}.$$

By Lemma 1.7, we have

$$r(E_GCE_G) + 2r(G) = r\begin{pmatrix} C & A & B^* \\ A^* & 0 & 0 \\ B & 0 & 0 \end{pmatrix} = r\begin{pmatrix} C & A & B^* \end{pmatrix} + r\begin{pmatrix} A & B^* \end{pmatrix} = r(E_GC) + 2r(G),$$

so,

$$r(E_GCE_G) = r(E_GC) \;\; or \;\; R(E_GCE_G) = R(E_GC).$$

On the other hand, it follows from Lemma 1.2 that

$$\begin{aligned} i_+(E_GCE_G) + r(G) &= i_+\begin{pmatrix} C & A & B^* \\ A^* & 0 & 0 \\ B & 0 & 0 \end{pmatrix} = i_+\begin{pmatrix} F_BCF_B & F_BA \\ A^*F_B & 0 \end{pmatrix} + r(B) \\ &= r(B) + r(F_BA) + i_+(E_{F_BA}F_BCF_BE_{F_BA}) \\ &= r(B) + r(F_BA) = r(G), \quad (F_BCF_B \leqslant 0 \; is \; used) \end{aligned}$$

means that $i_+(E_GCE_G) = 0$ or $E_GCE_G \leqslant 0$. Then (2.4) holds.

Furthermore, applying Lemma 1.7, and elementary block matrix operations, we get

$$r(E_AGG^\dagger) = r(E_AG) = r\begin{pmatrix} A & G \end{pmatrix} - r(A) = r(G) - r(A),$$

and

$$\begin{aligned}
& r\begin{pmatrix} E_AGG^{\dagger} & E_ACE_G(E_GCE_G)^{\dagger}E_GCE_A - E_ACE_A \end{pmatrix} \\
= & r\begin{pmatrix} E_AG & E_ACE_G(E_GCE_G)^{\dagger}E_GCE_A - E_ACE_A \end{pmatrix} \\
= & r\begin{pmatrix} A & G & CE_G(E_GCE_G)^{\dagger}E_GCE_A - CE_A \end{pmatrix} - r(A) \\
= & r\begin{pmatrix} G & CE_G(E_GCE_G)^{\dagger}E_GCE_A - CE_A \end{pmatrix} - r(A) \\
= & r[E_GCE_G(E_GCE_G)^{\dagger}E_GCE_A - E_GCE_A] + r(G) - r(A) \\
= & r(G) - r(A) = r(E_AGG^{\dagger}). \quad (R(E_GCE_G) = R(E_GC) \ \textit{is used})
\end{aligned}$$

Thus, (2.5) is evident.

By Lemma 1.2 and (2.4), one can compute that

$$\begin{aligned}
& i_-[E_ACE_G(E_GCE_G)^{\dagger}E_GCE_A - E_ACE_A] \\
= & i_+[E_ACE_A - E_ACE_G(E_GCE_G)^{\dagger}E_GCE_A] = i_+\begin{pmatrix} E_GCE_G & E_GCE_A \\ E_ACE_G & E_ACE_A \end{pmatrix} - i_+(E_GCE_G) \\
= & i_+\left\{\begin{pmatrix} E_G & 0 \\ 0 & E_A \end{pmatrix}\begin{pmatrix} C & C \\ C & C \end{pmatrix}\begin{pmatrix} E_G & 0 \\ 0 & E_A \end{pmatrix}\right\} = i_+\begin{pmatrix} C & C & G & 0 \\ C & C & 0 & A \\ G^* & 0 & 0 & 0 \\ 0 & A^* & 0 & 0 \end{pmatrix} - r(G) - r(A) \\
= & i_+\begin{pmatrix} C & 0 & G & 0 \\ 0 & 0 & -G & A \\ G^* & -G^* & 0 & 0 \\ 0 & A^* & 0 & 0 \end{pmatrix} - r(G) - r(A) = i_+\begin{pmatrix} C & 0 & 0 & A \\ 0 & 0 & -G & A \\ 0 & -G^* & 0 & 0 \\ A^* & A^* & 0 & 0 \end{pmatrix} - r(G) - r(A) \\
= & i_+\begin{pmatrix} C & 0 & 0 & 0 & A \\ 0 & 0 & -A & -B^* & A \\ 0 & -A^* & 0 & 0 & 0 \\ 0 & -B & 0 & 0 & 0 \\ A^* & A^* & 0 & 0 & 0 \end{pmatrix} - r(G) - r(A) \\
= & i_+\begin{pmatrix} C & 0 & 0 & 0 & A \\ 0 & 0 & -A & -B^* & 0 \\ 0 & -A^* & 0 & 0 & 0 \\ 0 & -B & 0 & 0 & 0 \\ A^* & 0 & 0 & 0 & 0 \end{pmatrix} - r(G) - r(A) \\
= & i_+\begin{pmatrix} C & 0 & 0 & A \\ 0 & 0 & -G & 0 \\ 0 & -G^* & 0 & 0 \\ A^* & 0 & 0 & 0 \end{pmatrix} - r(G) - r(A) = i_+(E_ACE_A) = 0,
\end{aligned}$$

which is equivalent to (2.6).

Similarly, (2.7) and (2.8) can be proved. □

When the matrices A, B and C satisfy the conditions in Theorem 2.1, then, by Lemma 1.3 and Lemma 1.4, we get the solution of VV^* to (1.3).

Lemma 2.1. Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times m}$ and $C \in \mathbb{C}_H^m$ be given, $X \in \mathbb{C}^{n\times p}$ be variable matrix. Denote $G = \begin{pmatrix} A & B^* \end{pmatrix}$, $P = E_A GG^\dagger$, $Q = F_B GG^\dagger F_P$, and

$$H_1 = E_A C E_G (E_G C E_G)^\dagger E_G C E_A - E_A C E_A, \ \ H_2 = F_B C E_G (E_G C E_G)^\dagger E_G C F_B - F_B C F_B.$$

Suppose that the conditions in the statement (2) or (3) of Theorem 2.1 are satisfied, then equations in (1.3) have a common solution for VV^*, which can be written as

$$VV^* = -CE_G(E_G C E_G)^\dagger E_G C + GG^\dagger WW^* GG^\dagger, \tag{2.9}$$

where

$$W = P^\dagger H_1^{\frac{1}{2}} + Q^\dagger (H_2^{\frac{1}{2}} T - F_B P^\dagger H_1^{\frac{1}{2}}) + F_P F_Q Z, \tag{2.10}$$

with $T \in \mathbb{U}_m$ and $Z \in \mathbb{C}^{m\times m}$ are arbitrary.

Proof. In view of Lemma 1.3 and Corollary 2.1, we know that $E_G VV^* = -E_G C$ is solvable, and the solution of VV^* can be formed by

$$VV^* = -CE_G(E_G C E_G)^\dagger E_G C + GG^\dagger WW^* GG^\dagger, \tag{2.11}$$

where $W \in \mathbb{C}^{m\times m}$ is arbitrary. Substituting VV^* into the last two equations in (1.3) produces

$$E_A GG^\dagger WW^* GG^\dagger E_A = E_A C E_G (E_G C E_G)^\dagger E_G C E_A - E_A C E_A \triangleq H_1, \tag{2.12}$$

$$F_B GG^\dagger WW^* GG^\dagger F_B = F_B C E_G (E_G C E_G)^\dagger E_G C F_B - F_B C F_B \triangleq H_2. \tag{2.13}$$

Corollary 2.1 shows that both (2.12) and (2.13) are consistent. Next, we come to prove that (2.12) and (2.13) have a common Hermitian nonnegative-definite solution WW^*.

By Lemma 1.4, the matrix equations (2.12) and (2.13) have a common Hermitian nonnegative-definite solution if and only if there exists $T \in \mathbb{U}_m$ such that

$$E_{F_B GG^\dagger F_{E_A GG^\dagger}} (H_2^{\frac{1}{2}} T - F_B GG^\dagger (E_A GG^\dagger)^\dagger H_1^{\frac{1}{2}}) = 0. \tag{2.14}$$

It follows from Lemma 1.7 that

$$\begin{aligned} r\left(F_B GG^\dagger F_{E_A GG^\dagger}\right) &= r\begin{pmatrix} F_B GG^\dagger \\ E_A GG^\dagger \end{pmatrix} - r(E_A GG^\dagger) = r\begin{pmatrix} F_B G \\ E_A G \end{pmatrix} - r(E_A G) \\ &= r\begin{pmatrix} B^* & 0 & G \\ 0 & A & G \end{pmatrix} - r(A) - r(B) - [r\begin{pmatrix} A & G \end{pmatrix} - r(A)] \\ &= r(G) - r(B) = r(F_B GG^\dagger), \end{aligned}$$

i.e., $R\left(F_B GG^\dagger F_{E_A GG^\dagger}\right) = R(F_B GG^\dagger)$, therefore $E_{F_B GG^\dagger F_{E_A GG^\dagger}} = E_{F_B GG^\dagger}$ and

$$E_{F_B GG^\dagger F_{E_A GG^\dagger}} F_B GG^\dagger (E_A GG^\dagger)^\dagger H_1^{\frac{1}{2}} = 0.$$

Applying Lemma 1.7 again and (2.4), we have

$$\begin{aligned} r[E_{F_BGG^\dagger}H_2] &= r\begin{pmatrix} F_BGG^\dagger & H_2 \end{pmatrix} - r(F_BGG^\dagger) = r\begin{pmatrix} F_BG & H_2 \end{pmatrix} - r(F_BG) \\ &= r\begin{pmatrix} B^* & G & CE_G(E_GCE_G)^\dagger E_GCF_B - CF_B \end{pmatrix} - r(B) - r(F_BG) \\ &= r\begin{pmatrix} G & CE_G(E_GCE_G)^\dagger E_GCF_B - CF_B \end{pmatrix} - r(G) \\ &= r[E_GCE_G(E_GCE_G)^\dagger E_GCF_B - E_GCF_B] = 0, \end{aligned}$$

means that $E_{F_BGG^\dagger}H_2 = 0$, i.e., $E_{F_BGG^\dagger F_{E_AGG^\dagger}}H_2^{\frac{1}{2}} = 0$. Hence, (2.14) holds for any $T \in \mathbb{U}_m$, and there exists a common Hermitian nonnegative-definite solution to (2.12) and (2.13). By Lemma 1.4, the common Hermitian nonnegative-definite solution is WW^* with

$$\begin{aligned} W &= P^\dagger H_1^{\frac{1}{2}} + F_PQ^\dagger(H_2^{\frac{1}{2}}T - F_BGG^\dagger P^\dagger H_1^{\frac{1}{2}}) + F_PF_QZ \\ &= P^\dagger H_1^{\frac{1}{2}} + Q^\dagger(H_2^{\frac{1}{2}}T - F_BP^\dagger H_1^{\frac{1}{2}}) + F_PF_QZ, \end{aligned} \tag{2.15}$$

where $T \in \mathbb{U}_m$ and $Z \in \mathbb{C}^{m\times m}$ are arbitrary.

Substituting (2.15) into (2.14) yields (2.9). □

Combining Theorem 2.1 and Lemma 2.1, we can deduce the following result.

Theorem 2.2. Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times m}$, $C \in \mathbb{C}_H^m$ be given, $X \in \mathbb{C}^{n\times p}$ be variable matrix, and suppose that matrix inequality (1.1) is solvable. Then, a general solution to (1.1) can be expressed as

$$X = \frac{1}{2}(X_1 + X_2^*), \tag{2.16}$$

where

$$\begin{aligned} X_1 &= A^\dagger(C+VV^*)B^\dagger - A^\dagger B^*M^\dagger(C+VV^*)B^\dagger - A^\dagger S(B^*)^\dagger(C+VV^*)N^\dagger A^*B^\dagger \\ &\quad - A^\dagger SY_1E_NA^*B^\dagger + F_AY_2 + Y_3E_B, \end{aligned} \tag{2.17}$$

$$X_2 = M^\dagger(C+VV^*)(A^*)^\dagger + S^\dagger S(B^*)^\dagger(C+VV^*)N^\dagger + F_M(Y_1 - S^\dagger SY_1NN^\dagger) + Y_4F_A, \tag{2.18}$$

with VV^* is given by (2.9), $M = E_AB^*$, $N = A^*F_B$, $S = B^*F_M$, and Y_i $(i = 1,2,3,4)$ are arbitrary matrices over complex field with appropriate sizes.

Proof. Since the matrix inequality (1.1) is equivalent to (1.2), where VV^* is given by (2.9). In view of Lemma 1.5, the general solution to (1.2) can be written as

$$X = \frac{1}{2}(X_1 + X_2^*),$$

where X_1 and X_2 are general solutions of the equation

$$AX_1B + B^*X_2A^* = C + VV^*. \tag{2.19}$$

It follows from (1.3) and Lemma 1.6 that (2.19) is solvable, and

$$\begin{aligned} X_1 &= A^\dagger(C+VV^*)B^\dagger - A^\dagger B^* M^\dagger E_A(C+VV^*)B^\dagger - A^\dagger S(B^*)^\dagger(C+VV^*)F_B N^\dagger A^* B^\dagger \\ &\quad - A^\dagger S Y_1 E_N A^* B^\dagger + F_A Y_2 + Y_3 E_B, \\ X_2 &= M^\dagger E_A(C+VV^*)(A^*)^\dagger + F_M S^\dagger S(B^*)^\dagger(C+VV^*)F_B N^\dagger + F_M(Y_1 - S^\dagger S Y_1 N N^\dagger) + Y_4 F_A, \end{aligned}$$

where $M = E_A B^*$, $N = A^* F_B$, $S = B^* F_M$, and $Y_i, (i = 1, 2, 3, 4)$ are arbitrary matrices over complex field with appropriate sizes. Together with $M^\dagger E_A = M^\dagger$, $F_B N^\dagger = N^\dagger$ and $F_M S^\dagger = S^\dagger$, then (2.17) and (2.18) are followed. □

In [2], the author presented some sufficient and necessary conditions for the existence of common Re-nnd solution to $AX = C$ and $XB = D$, however, the general solution has not been established by now. Next, we restudy this problem, and derive its general solution.

Theorem 2.3. Let $A, C \in \mathbb{C}^{n\times m}$, and $B, D \in \mathbb{C}^{m\times n}$, suppose that both $AX = C$ and $XB = D$ have a a Re-nnd solution. If the pair of equations have a common solution (*i.e.*, $AD = CB$), then there exists a common Re-nnd solution if and only if

$$r\begin{pmatrix} A & C \\ B^* & -D^* \end{pmatrix} = r\begin{pmatrix} A & CA^* \\ B^* & -D^*A^* \end{pmatrix} = r\begin{pmatrix} A & CB \\ B^* & -D^*B \end{pmatrix}. \tag{2.20}$$

In this case, a general common Re-nnd solution can be written as

$$X = A^\dagger C + F_A D B^\dagger + \frac{1}{2}(\tilde{Y}_1 + \tilde{Y}_2^*), \tag{2.21}$$

where,

$$\begin{aligned} \tilde{Y}_1 &= F_A(\tilde{C}+VV^*)E_B - F_A M^\dagger(\tilde{C}+VV^*)E_B - E_B F_M(\tilde{C}+VV^*)N^\dagger E_B - E_B F_M Z_1 E_N F_A, \\ \tilde{Y}_2 &= M^\dagger(\tilde{C}+VV^*)F_A + S^\dagger S(\tilde{C}+VV^*)N^\dagger + E_B F_M(Z_1 - S^\dagger S Z_1 N N^\dagger)F_A, \\ VV^* &= -\tilde{C}E_G(E_G\tilde{C}E_G)^\dagger E_G\tilde{C} + GG^\dagger WW^* GG^\dagger, \\ W &= P^\dagger H_1^{\frac{1}{2}} + Q^\dagger(H_2^{\frac{1}{2}}T - BB^\dagger P^\dagger H_1^{\frac{1}{2}}) + F_P F_Q Z, \\ H_1 &= A^\dagger A\tilde{C}E_G(E_G\tilde{C}E_G)^\dagger E_G\tilde{C}A^\dagger A - A^\dagger A\tilde{C}A^\dagger A, \\ H_2 &= BB^\dagger\tilde{C}E_G(E_G\tilde{C}E_G)^\dagger E_G\tilde{C}BB^\dagger - BB^\dagger\tilde{C}BB^\dagger. \end{aligned}$$

with $\tilde{C} = -[(A^\dagger C + F_A D B^\dagger) + (A^\dagger C + F_A D B^\dagger)^*]$, $G = \begin{pmatrix} F_A & E_B \end{pmatrix}$, $M = A^\dagger A E_B$, $N = F_A B B^\dagger$, $S = E_B F_M$, $P = A^\dagger A G G^\dagger$, $Q = BB^\dagger G G^\dagger F_P$, $T \in \mathbb{U}_m$ and $Z, Z_1 \in \mathbb{C}^{m\times m}$ are arbitrary.

Proof. The rank equality (2.20) was obtained by [Theorem 2.1, 2]. Furthermore, by [Lemma 1.1, 2], a general common solution to $AX = C$ and $XB = D$ can be expressed as

$$X = A^{\dagger}C + F_A DB^{\dagger} + F_A Y E_B, \tag{2.22}$$

where $Y \in \mathbb{C}^{m\times m}$ is arbitrary. Therefore, there exists a common Re-nnd solution X if and only if $X + X^* \geqslant 0$ for some Y, i.e.,

$$F_A Y E_B + (F_A Y E_B)^* \geqslant -[(A^{\dagger}C + F_A DB^{\dagger}) + (A^{\dagger}C + F_A DB^{\dagger})^*] \triangleq \tilde{C} \tag{2.23}$$

is solvable. Applying Theorem 2.2 to (2.23) yields

$$Y = \frac{1}{2}(Y_1 + Y_2^*), \tag{2.24}$$

where

$$\begin{aligned} Y_1 &= F_A(\tilde{C} + VV^*)E_B - F_A E_B M^{\dagger}(\tilde{C} + VV^*)E_B - F_A S E_B(\tilde{C} + VV^*)N^{\dagger}F_A E_B \\ &\quad -F_A S Z_1 E_N F_A E_B + A^{\dagger}AZ_2 + Z_3 BB^{\dagger}, \\ Y_2 &= M^{\dagger}(\tilde{C} + VV^*)F_A + S^{\dagger}SE_B(\tilde{C} + VV^*)N^{\dagger} + F_M(Z_1 - S^{\dagger}SZ_1NN^{\dagger}) + Z_4 A^{\dagger}A, \end{aligned}$$

with $M = A^{\dagger}AE_B$, $N = F_A BB^{\dagger}$, $S = E_B F_M$, and $Z_i, (i = 1, 2, 3, 4)$ are arbitrary matrices over complex field with appropriate sizes. Together with $F_A S = E_B F_M$, $F_M E_B = E_B F_M$, we have

$$\begin{aligned} F_A Y_1 E_B &= F_A(\tilde{C} + VV^*)E_B - F_A E_B M^{\dagger}(\tilde{C} + VV^*)E_B - F_A S E_B(\tilde{C} + VV^*)N^{\dagger}F_A E_B - F_A S Z_1 E_N F_A E_B \\ &= F_A(\tilde{C} + VV^*)E_B - F_A M^{\dagger}(\tilde{C} + VV^*)E_B - E_B F_M(\tilde{C} + VV^*)N^{\dagger}E_B - E_B F_M Z_1 E_N F_A, \\ E_B Y_2 F_A &= E_B M^{\dagger}(\tilde{C} + VV^*)F_A + E_B S^{\dagger}SE_B(\tilde{C} + VV^*)N^{\dagger}F_A + E_B F_M(Z_1 - S^{\dagger}SZ_1NN^{\dagger})F_A \\ &= M^{\dagger}(\tilde{C} + VV^*)F_A + S^{\dagger}S(\tilde{C} + VV^*)N^{\dagger} + E_B F_M(Z_1 - S^{\dagger}SZ_1NN^{\dagger})F_A. \end{aligned}$$

Denote $\tilde{Y}_1 = F_A Y_1 E_B$ and $\tilde{Y}_2 = E_B Y_2 F_A$. Combining (2.22) and (2.24) produces (2.21). □

Since the Re-nnd generalized inverse $A^{(1,3,4)}$ can be regarded as the common Re-nnd solution of $A^*AX = A^*$ and $XAA^* = A^*$, where $A \in \mathbb{C}^{m\times m}$, therefore, by Theorem 2.3, we have the following result.

Theorem 2.4. Let $A \in \mathbb{C}^{m\times m}$. Then there exists a Re-nnd generalized inverse $A^{(1,3,4)}$ if and only if A^*A^2, A^2A^* are Re-nnd, and

$$r\begin{pmatrix} A^*A & A^* \\ AA^* & -A \end{pmatrix} = r\begin{pmatrix} A^*A & (A^*)^2 \\ AA^* & -AA^* \end{pmatrix} = r\begin{pmatrix} A^*A & A^*A \\ AA^* & -A^2 \end{pmatrix}. \tag{2.25}$$

In this case, a general Re-nnd generalized inverse $A^{(1,3,4)}$ can be written as

$$A_{re}^{(1,3,4)} = A^{\dagger} + \frac{1}{2}(\tilde{Y}_1 + \tilde{Y}_2^*),$$

where,

$$\begin{aligned}
\tilde{Y}_1 &= F_A(\tilde{C}+VV^*)E_A - F_AM^\dagger(\tilde{C}+VV^*)E_A - E_AF_M(\tilde{C}+VV^*)N^\dagger E_A - E_AF_MZ_1E_NF_A,\\
\tilde{Y}_2 &= M^\dagger(\tilde{C}+VV^*)F_A + S^\dagger S(\tilde{C}+VV^*)N^\dagger + E_AF_M(Z_1 - S^\dagger SZ_1NN^\dagger)F_A,\\
VV^* &= -\tilde{C}E_G(E_G\tilde{C}E_G)^\dagger E_G\tilde{C} + GG^\dagger WW^*GG^\dagger,\\
W &= P^\dagger H_1^{\frac{1}{2}} + Q^\dagger(H_2^{\frac{1}{2}}T - AA^\dagger P^\dagger H_1^{\frac{1}{2}}) + F_PF_QZ,\\
H_1 &= A^\dagger A\tilde{C}E_G(E_G\tilde{C}E_G)^\dagger E_G\tilde{C}A^\dagger A - A^\dagger A\tilde{C}A^\dagger A,\\
H_2 &= AA^\dagger\tilde{C}E_G(E_G\tilde{C}E_G)^\dagger E_G\tilde{C}AA^\dagger - AA^\dagger\tilde{C}AA^\dagger.
\end{aligned}$$

with $\tilde{C} = -[A^\dagger + (A^\dagger)^*]$, $G = \begin{pmatrix} F_A & E_A \end{pmatrix}$, $M = A^\dagger AE_A$, $N = F_AAA^\dagger$, $S = E_AF_M$, $P = A^\dagger AGG^\dagger$, $Q = AA^\dagger GG^\dagger F_P$, $T \in \mathbb{U}_m$ and $Z, Z_1 \in \mathbb{C}^{m\times m}$ are arbitrary.

Proof. In view of Lemma 1.8, $A^*AX = A^*$ and $XAA^* = A^*$ have Re-nnd solution if and only if A^*A^2 and A^2A^* are Re-nnd respectively. Moreover, by Theorem 2.3, these two equations have a common Re-nnd solution if and only if

$$r\begin{pmatrix} A^*A & A^* \\ AA^* & -A \end{pmatrix} = r\begin{pmatrix} A^*A & (A^*)^2A \\ AA^* & -AA^*A \end{pmatrix} = r\begin{pmatrix} A^*A & A^*AA^* \\ AA^* & -A^2A^* \end{pmatrix},$$

which is equivalent to (2.25). The formula of $A_{re}^{(1,3,4)}$ follows directly by (2.21). The proof is complete. □

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Existence and Stability Results for Quaternion Fuzzy Fractional Differential Equations

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Abstract

We consider the initial value problem of quaternion fuzzy fractional differential equations in the generalized regular fuzzy function space. And we propose a notion of the disturbed fuzzy Dirac operator. By using the associate space method and fixed point theorem, a sufficient condition for the existence and stability of the solution of the initial value problem is given.

Keywords: quaternion-valued grades of membership, quaternion fuzzy fractional differential equation, associate space, generalized regular function, Hyers–Ulam stability

1 Introduction

The notion of fuzzy complex number was first proposed by Buckley in [1]. In [2], Tamir et al. pointed out the limitations of the mixed fuzzy and crisp definition of [3] and generalized it by allowing a fuzzy phase term. As illustrated with examples in [2], the advantage of this augmented definition of complex fuzzy sets is its ability to accommodate fuzzy cycles. In order to extent fuzzy complex number, the concept of the fuzzy quaternion number was introduced by Moura et al., who in [4] discuss some concepts such as their arithmetic properties, infimum, supremum, distance, and so on. The quaternion membership function was given by a mapping $u : \mathbb{H} \to [0, 1]$ such that

$$u(a + bi + cj + dk) = \min\{\bar{A}(a), \bar{B}(b), \bar{C}(c), \bar{D}(d)\},$$

where $\bar{A}, \bar{B}, \bar{C}, \bar{D}$ are all real fuzzy numbers. Yang et al. proposed a different definition of quaternion fuzzy sets and discussed entailed results which parallel those of regular fuzzy numbers in [5].

The study on fractional differential equations has been rapidly advancing in recent years. Fractional equations have received increasing attentions [6, 7, 8, 9, 10, 11]. Recently, Agarwal et al. considered a differential equation of fractional order with uncertainty and presented the concept of solution [12]. They considered the Riemann-Liouville differentiability which was a combination of Hukuhara difference and Riemann-Liouville derivative. The shortcomings of applications of Hukuhara difference was discussed in [13] by Bede and Gal. The results on existence and uniqueness of the solution were later established in [14, 15, 16, 17], and in [18, 19]. Salahshour et

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al. applied fuzzy Laplace transforms to solve fuzzy differential equations [20, 21]. The numerical solution of the fuzzy differential equation was obtained in [22, 23, 24]. Furthermore, Malinowski introduced random fuzzy fractional integral equations-theoretical [25].

The study of stability problems for functional equations is related to a question of Ulam [26] concerning the stability of group homomorphisms and affirmatively answered for Banach spaces by Hyers [27]. Some authors then considered the stability of the fuzzy difference and functional equations [28, 29, 30, 31]. In this paper, we consider existence and stability of the solution for quaternion fuzzy fractional differential equations. By the associate space method and fixed point theorem, we given a sufficient condition of the Hyers–Ulam stability for quaternion fuzzy fractional differential equations. Moreover, We provide a way of incorporating such the theory of fuzzy fractional differential equations into quaternionic analysis.

2 Notation and Basic results

Let $P_K(\mathbb{R}^3)$ denote the set of all nonempty convex compact subsets of $\mathbb{R}^3$. The Hausdorff metric for $A,\ B \in P_K(\mathbb{R}^3)$ is defines by

$$d(A,B) = \inf\{\varepsilon \mid A \subset N(B,\varepsilon) \text{ and } B \subset N(A,\varepsilon)\},$$

where $N(A,\varepsilon) = \{x \in \mathbb{R}^3 \mid \|x-y\| < \varepsilon \text{ for some } y \in A\}$.

Throughout this paper, we put $\Lambda := \{0,1,2,3\}$ and denote by $e_0 = 1,\ e_1 = i,\ e_2 = j,\ e_3 = k$, where $i,\ j,\ k$ are units of the real quaternion algebra $\mathbb{H}$.

In [5], Yang et al. considered quaternion fuzzy sets on $\mathbb{R}^3$, i.e., quaternion grades of membership.

Definition 1. [5] *The quaternion membership function f is defined by*

$$f(V,x) = e_0 f_0(V) + e_1 f_1(x) + e_2 f_2(x) + e_3 f_3(x),$$

where V is to be interpreted as a set in a fuzzy set of sets and x as an element of V.

In particluar, for $x \in \mathbb{R}^3$, we have

$$f(x) = f_0(x)e_0 + f_1(x)e_1 + f_2(x)e_2 + f_3(x)e_3,$$

where $f_0, f_1, f_2, f_3 : \mathbb{R}^3 \to [0,1]$. Denote f by (f_0, f_1, f_2, f_3). The $\overline{r} = (r_0, r_1, r_2, r_3)$-level sets for $f = (f_0, f_1, f_2, f_3)$ is defined by

$$[f]^{\overline{r}} = [f_0]^{r_0} \cap [f_1]^{r_1} \cap [f_2]^{r_2} \cap [f_3]^{r_3}. \tag{2.1}$$

Denote $\mathcal{F}^n$ the set of all $\nu : \mathbb{R}^n \to [0,1]$ satisfying all of the following conditions:

(i) ν is normal, i.e., there exists $x_0 \in \mathbb{R}^n$ such that $\nu(x_0) = 1$;

(ii) ν is fuzzy convex, i.e., for all $t_1, t_2 \in \mathbb{R}^n, \lambda \in [0,1]$:

$$\nu\left(\lambda t_1 + (1-\lambda)t_2\right) \geq \min\{\nu(t_1), \nu(t_2)\};$$

(iii) ν is upper semi-continuous;
(iv) $[\nu]^0$ is compact.

Moreover, we define $\hat{\mathcal{F}}^{4n}$ as follows:

$$\hat{\mathcal{F}}^{4n} = \{(\nu_0, \nu_1, \nu_2, \nu_3) \in \mathcal{F}^n \times \mathcal{F}^n \times \mathcal{F}^n \times \mathcal{F}^n | \\ \exists t_0,\ s.t.,\ v_l(t_0) = 1, l \in \Lambda\}.$$

Then, for $\nu = (\nu_0, \nu_1, \nu_2, \nu_3) \in \hat{\mathcal{F}}^{4n}$, $[f]^{\bar{\alpha}} = \cap_{l\in\Lambda}[\nu_l]^{\alpha_l} \in P_K(\mathbb{R}^3)$ for all $\alpha_l \in [0,1], l \in \Lambda$.

For $f, g \in \hat{\mathcal{F}}^{4n}$, where $f = (f_0, f_1, f_2, f_3)$ and $g = (g_0, g_1, g_2, g_3)$, and λ is a scalar, let

$$\begin{aligned} f + g &= (f_0 + g_0, f_1 + g_1, f_2 + g_2, f_3 + g_3), \\ \lambda f &= (\lambda f_0, \lambda f_1, \lambda f_2, \lambda f_3). \end{aligned}$$

Let us define $D : \mathcal{F}^n \times \mathcal{F}^n \to [0, \infty)$ by

$$D(\nu_1, \nu_2) = \sup\{d([\nu_1]^r, [\nu_2]^r) \mid r \in [0,1]\}, \tag{2.2}$$

where d is the Hausdorff metric. $(\mathcal{F}^n, D)$ is a metric space which can be embedded isomorphically as a cone in a Banach space [32]. However, D is not a suitable metric for our space of interest, $\hat{\mathcal{F}}^{4n}$, as we quickly see that linearity is violated. Instead, let us consider the product metric D' on $\mathcal{F}^{4n} = \mathcal{F}^n \times \mathcal{F}^n \times \mathcal{F}^n \times \mathcal{F}^n$. For $f = (f_0, f_1, f_2, f_3) \in \mathcal{F}^{4n}$ and $g = (g_0, g_1, g_2, g_3) \in \mathcal{F}^{4n}$, we define $D' : \mathcal{F}^{4n} \times \mathcal{F}^{4n} \to [0, \infty)$ by the relation

$$\begin{aligned} D'(f, g) &= D'((f_0, f_1, f_2, f_3), (g_0, g_1, g_2, g_3)) \\ &= \max_{l\in\Lambda}\{D(f_l, g_l)\}. \end{aligned} \tag{2.3}$$

Then, D' is a linearity preserving metric for $\mathcal{F}^{4n}$. Since $\hat{\mathcal{F}}^{4n} \subset \mathcal{F}^{4n}$, D' is also a metric for $\hat{\mathcal{F}}^{4n}$. Hence, $(\hat{\mathcal{F}}^{4n}, D')$ is a complete metric space. Now, as $(\hat{\mathcal{F}}^{4n}, D')$ is a metric space and D' preserves linearity, by the Arens-Eells theorem [33] there exists an embedding $\hat{\mathcal{F}}^{4n} \hookrightarrow \mathcal{B}$ where $\mathcal{B}$ is a Banach space. The zero element on $\hat{\mathcal{F}}^{4n}$ then reads $\hat{0}_4(x) = (\hat{0}(x), \hat{0}(x), \hat{0}(x), \hat{0}(x)) \in \mathcal{F}^{4n}$.

We define strongly generalized differentiability as in [13] in terms of the generalize Hukuhara difference. For $x, y \in \hat{\mathcal{F}}^{4n}$, if there exists $z \in \hat{\mathcal{F}}^{4n}$ such that $x = z + y$ or $y = x + (-1)z$, we write $x \ominus y = z$ and call z the difference of x and y.

A fuzzy-valued function f defined in the bounded, simply connected domain $\Omega \subset \mathbb{R}^3$ is a mapping $f : \Omega \to \hat{\mathcal{F}}^{4n}$, and f can be represented in a form $f = \sum_{j=0}^{3} e_j f_j(x)$. Its conjugate $\bar{f}$ is defined by

$$\bar{f} = e_0 f_0(x) \ominus \sum_{j=1}^{3} e_j f_j(x),$$

where $f_j(x)$ are continuous fuzzy-valued functions in $x = (x_1, x_2, x_3) \in \Omega$.

Definition 2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded, simply connected domain. We call a mapping $F : \Omega \to \hat{\mathcal{F}}^{4n}$ strongly generalized partial derivative at $x = (x_1, x_2, x_3) \in \Omega$ if there exists some $\frac{\partial F}{\partial x_i} \in \hat{\mathcal{F}}^{4n}$ such that* *(i) there exists the differences* $F(\cdot, x_i+h, \cdot)\ominus F(\cdot, x_i, \cdot)$, $F(\cdot, x_i, \cdot)\ominus F(\cdot, x_i-h, \cdot)$ *and*

$$\frac{\partial F}{\partial x_i}=\lim_{h\to 0^+}\frac{F(\cdot, x_i+h, \cdot)\ominus F(\cdot, x_i, \cdot)}{h}=\lim_{h\to 0^+}\frac{F(\cdot, x_i, \cdot)\ominus F(\cdot, x_i-h, \cdot)}{h}, \tag{2.4}$$

or

(ii) there exists the differences $F(\cdot, x_i, \cdot)\ominus F(\cdot, x_i+h, \cdot)$, $F(\cdot, x_i-h, \cdot)\ominus F(\cdot, x_i, \cdot)$ *and*

$$\frac{\partial F}{\partial x_i}=\lim_{h\to 0^+}\frac{F(\cdot, x_i, \cdot)\ominus F(\cdot, x_i+h, \cdot)}{-h}=\lim_{h\to 0^+}\frac{F(\cdot, x_i-h, \cdot)\ominus F(\cdot, x_i, \cdot)}{-h}, \tag{2.5}$$

or

(iii) there exists the differences $F(\cdot, x_i+h, \cdot)\ominus F(\cdot, x_i, \cdot)$, $F(\cdot, x_i-h, \cdot)\ominus F(\cdot, x_i, \cdot)$ *and*

$$\frac{\partial F}{\partial x_i}=\lim_{h\to 0^+}\frac{F(\cdot, x_i+h, \cdot)\ominus F(\cdot, x_i, \cdot)}{h}=\lim_{h\to 0^+}\frac{F(\cdot, x_i-h, \cdot)\ominus F(\cdot, x_i, \cdot)}{-h}, \tag{2.6}$$

or

(iv) there exists the differences $F(\cdot, x_i, \cdot)\ominus F(\cdot, x_i+h, \cdot)$, $F(\cdot, x_i, \cdot)\ominus F(\cdot, x_i-h, \cdot)$ *and*

$$\frac{\partial F}{\partial x_i}=\lim_{h\to 0^+}\frac{F(\cdot, x_i, \cdot)\ominus F(\cdot, x_i+h, \cdot)}{-h}=\lim_{h\to 0^+}\frac{F(\cdot, x_i, \cdot)\ominus F(\cdot, x_i-h, \cdot)}{h}. \tag{2.7}$$

In general, we have the following results on the connection between the strongly generalized partial derivative of F and its endpoint function F_l^α and F_r^α.

Let $F:\Omega\to\hat{\mathcal{F}}^{4n}$ be a quaternion fuzzy function. If F is strongly generalized partial derivative at $x\in\Omega$, then we have the following case:

If F is strongly generalized partial derivative at $x\in\Omega$ in (i), then, for each $\alpha_i\in[0,1], F_{il}$ and F_{ir} are strongly generalized partial derivative functions at x and

$$\left[\frac{\partial F}{\partial x_i}\right]^\alpha=\left[\left(\frac{\partial F}{\partial x_i}\right)_l^\alpha,\left(\frac{\partial F}{\partial x_i}\right)_r^\alpha\right],$$

where

$$\left(\frac{\partial F}{\partial x_i}\right)_l^\alpha=\left[\left(\frac{\partial F}{\partial x_i}\right)_{0l}^{\alpha_0},\left(\frac{\partial F}{\partial x_i}\right)_{1l}^{\alpha_1},\left(\frac{\partial F}{\partial x_i}\right)_{2l}^{\alpha_2},\left(\frac{\partial F}{\partial x_i}\right)_{3l}^{\alpha_3}\right] \tag{2.8}$$

and

$$\left(\frac{\partial F}{\partial x_i}\right)_r^\alpha=\left[\left(\frac{\partial F}{\partial x_i}\right)_{0r}^{\alpha_0},\left(\frac{\partial F}{\partial x_i}\right)_{1r}^{\alpha_1},\left(\frac{\partial F}{\partial x_i}\right)_{2r}^{\alpha_2},\left(\frac{\partial F}{\partial x_i}\right)_{3r}^{\alpha_3}\right]. \tag{2.9}$$

Definition 3. *Let* $F:\Omega\to\hat{\mathcal{F}}^{4n}$ *be a continuous mapping. The fuzzy Riemann-Liouville integral of* F *is defined by*

$$(I_{0^+}^\beta F)(x)=\frac{1}{\Gamma(\beta)}\int_0^{x_i}(x_i-\tau)^{\beta-1}F(\cdot,\tau,\cdot)d\tau, \tag{2.10}$$

where $x\in\Omega, x_i>0, 0<\beta<1$.

Then, the Riemann-Liouville integral of a quaternion fuzzy-valued function F can be expressed as follow:

$$(I_{0^+}^\beta F^\alpha)(x)=[(I_{0^+}^\beta F_l^\alpha)(x),(I_{0^+}^\beta F_r^\alpha)(x)],$$

where

$$(I_{0^+}^{\beta}F_l^{\alpha})(x) = \frac{1}{\Gamma(\beta)}\int_0^{x_i}(x_i-\tau)^{\beta-1}F_l^{\alpha}(\cdot,\tau,\cdot)\mathrm{d}\tau$$

and

$$(I_{0^+}^{\beta}F_r^{\alpha})(x) = \frac{1}{\Gamma(\beta)}\int_0^{x_i}(x_i-\tau)^{\beta-1}F_r^{\alpha}(\cdot,\tau,\cdot)\mathrm{d}\tau.$$

Definition 4. *The fuzzy Riemann-Liouville fractional derivatives of order $n-1<\beta<n$ for fuzzy-valued function F is defined by (provided it exists)*

$$({}^{RL}D_{0^+}^{\beta}F)(x) = \frac{1}{\Gamma(n-\beta)}\frac{\partial^n}{\partial x_i^n}\int_0^{x_i}(x_i-\tau)^{n-\beta-1}F(\cdot,\tau,\cdot)\,d\tau. \tag{2.11}$$

Similarly, we have

$$({}^{RL}D_{0^+}^{\beta}F^{\alpha})(x) = [({}^{RL}D_{0^+}^{\beta}F_l^{\alpha})(x), ({}^{RL}D_{0^+}^{\beta}F_r^{\alpha})(x)],$$

where $({}^{RL}D_{0^+}^{\beta}F_l^{\alpha})(x) =$

$$\frac{1}{\Gamma(n-\beta)}\frac{\partial^n}{\partial x_i^n}\int_0^{x_i}(x_i-\tau)^{n-\beta-1}F_l^{\alpha}(\cdot,\tau,\cdot)\mathrm{d}\tau$$

and $({}^{RL}D_{0^+}^{\beta}F_r^{\alpha})(x) =$

$$\frac{1}{\Gamma(n-\beta)}\frac{\partial^n}{\partial x_i^n}\int_0^{x_i}(x_i-\tau)^{n-\beta-1}F_r^{\alpha}(\cdot,\tau,\cdot)\mathrm{d}\tau.$$

Definition 5. *The fuzzy Caputo derivative of F for $n-1<\beta<n$ and $x\in\Omega$ is denoted by $({}^{C}D_{0^+}^{\beta}F)(x)$ (provided it exists) and defined by*

$$({}^{C}D_{0^+}^{\beta}F)(x) = \frac{1}{\Gamma(n-\beta)}\int_0^{x_i}(x_i-\tau)^{n-\beta-1}\frac{\partial^n}{\partial\tau^n}F(\cdot,\tau,\cdot)\,d\tau. \tag{2.12}$$

Then,

$$({}^{C}D_{0^+}^{\beta}F^{\alpha})(x) = [({}^{C}D_{0^+}^{\beta}F_l^{\alpha})(x), ({}^{C}D_{0^+}^{\beta}F_r^{\alpha})(x)],$$

where

$$({}^{C}D_{0^+}^{\beta}F_l^{\alpha})(x) = \frac{1}{\Gamma(n-\beta)}\int_0^{x_i}(x_i-\tau)^{n-\beta-1}\frac{\partial^n}{\partial\tau^n}F_l^{\alpha}(\cdot,\tau,\cdot)\mathrm{d}\tau$$

and

$$({}^{C}D_{0^+}^{\beta}F_r^{\alpha})(x) = \frac{1}{\Gamma(n-\beta)}\int_0^{x_i}(x_i-\tau)^{n-\beta-1}\frac{\partial^n}{\partial\tau^n}F_r^{\alpha}(\cdot,\tau,\cdot)\mathrm{d}\tau.$$

Now let us introduce the fuzzy Dirac operator as

$$D = \sum_{k=1}^{3} e_k\frac{\partial}{\partial x_k}.$$

The fuzzy Dirac operator acts on f as follows

$$Df = \sum_{k=1,j=0}^{3} e_k e_j\frac{\partial f_j}{\partial x_k}.$$

Definition 6. *The disturbed fuzzy Dirac operator is the operator which is defined by*

$$D_\beta u = Du + \beta D,$$

where β is a real number.

Definition 7. *A fuzzy function $u : \Omega \to \hat{\mathcal{F}}^{4n}$ is called a generalized regular fuzzy function if it satisfies $D_\beta u = \hat{0}_4$.*

Definition 8. *Let $L(t, x, u)$ be a first order differential operator depending on t, x, u and the first order derivative $\frac{\partial u}{\partial x_j}$, while $l(t, x, u)$ is a differential operator on the time t. Then L is called "associated" to l if L transforms solutions of $lu = \hat{0}_4$ into solutions of the same equation for fixed t, i.e. $lu = \hat{0}_4$ implies $l[Lu] = \hat{0}_4$.*

If $A : \mathcal{Y} \to \mathcal{X}$ is an operator, let us consider the fixed point equation

$$x = A(x), \quad x \in \mathcal{Y} \tag{2.13}$$

and the inequation

$$d(y, A(y)) \le \epsilon. \tag{2.14}$$

Definition 9. *The equation (2.13) is called generalized Hyers–Ulam stable if there exists $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ increasing, continuous at 0 and $\psi(0) = 0$ such that for each $\epsilon > 0$ and for each solution y^* of (2.14) there exists a solution x^* of the fixed point equation (2.13) such that*

$$d(y^*, x^*) \le \psi(\epsilon).$$

If there exists $c > 0$ such that $\psi(t) := ct$, for each $t \in \mathbb{R}^+$, the equation (2.13) is said to be Hyers–Ulam stable.

3 Main results

In this section, we consider the initial value problem

$$\begin{cases} {}^C D^\alpha_{0^+t} u = \sum_{j=1}^{3} A^{(j)} \dfrac{\partial u}{\partial x_j} + Bu + C := L(u), \\ u(0, x) = \varphi(x), \end{cases} \tag{3.1}$$

where $x = (x_1, x_2, x_3) \in \Omega$ and Ω is a bounded, simply connected domain in $\mathbb{R}^3$; $t \in [0, T]$ is the time variable; ${}^C D^\alpha_{0^+t}$ is the Caputo fractional derivative of t; $u = u(t, x)$ is quaternion fuzzy-valued functions defined in $[0, T] \times \Omega$. $A^{(j)} = A^{(j)}(t, x)$; $B = B(t, x)$ and $C = C(t, x)$ are quaternion-valued functions defined in $[0, T] \times \Omega$. The initial function $\varphi(x)$ is a generalized regular fuzzy function.

It is easy to show that solutions of the initial value problem are fixed points of the operator

$$B(u) := u(t, x) = \varphi(x) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} L(u) \mathrm{d}\tau. \tag{3.2}$$

In order to use fixed points theorem, we have to estimate the integro-differential operator on the right-hand side of (3.2). That is a little bit difficult because the integrand contains derivative with the spacelike variables x_j. But we can estimation it by using the following two properties of the associated function space:

(i) The operator maps the space into itself. Here we use the concept "associated pair" [34, 35, 36].

(ii) For the element of the associated space one has an "interior estimate" [36], that is, the norm (metric) of the derivative with respect to spacelike variables of the element of the associated space can be estimated by the norm of the element.

For our subsequent results, we need the following hypotheses.

(H1) $A_0^{(1)} = A_3^{(2)} = -A_2^{(3)}$,
$A_1^{(1)} = A_2^{(2)} = -A_3^{(3)}$,
$A_2^{(1)} = -A_1^{(2)} = A_0^{(3)}$,
$A_3^{(1)} = -A_0^{(2)} = -A_1^{(3)}$;

(H2) $(DA^{(1)} + \beta A^{(1)} - 2B_1e_0)e_1 = (DA^{(2)} + \beta A^{(2)} - 2B_2e_0)e_2 =$
$(DA^{(3)} + \beta A^{(3)} - 2B_3e_0)e_3$;

(H3) $\beta DA^{(1)}e_1 + 2\beta^2 \sum_{j=1}^{3} A_j^{(1)} e_j e_1 + 2\beta^2 A_1^{(1)} e_0 + DB + 2\beta(B_2e_2 + B_3e_3) = 0$;

(H4) $D_\beta C = DC + \beta C = 0$ for each $t \in [0, T]$.

Theorem 1. *Assume that $A^{(j)}(t, x)(j = 1, 2, 3), B(t, x)$ and $C(t, x)$ are all quaternion-valued function for $t \in [0, T]$. The operator L is associated with the operator D_β if hypotheses* (H1)–(H4) *are satisfied.*

According to Definition 8, we can obtain that the operator L is associated with the operator D_β, if $D_\beta u = \hat{0}_4$ implies $D_\beta(Lu) = \hat{0}_4$. Here, we omit the proof.

To solve the initial value problem (3.1) we need the interior estimate of generalized fuzzy regular functions.

Theorem 2. *Let $\Omega_{s_1} \subset \Omega_{s_2}$ and $\overline{\Omega}_{s_2} \subset \Omega$. Let $m\Omega$ denote the finite measure of $\Omega \subset \mathbb{R}^n$ and u be a generalized fuzzy regular function. We obtain the interior estimate of generalized regular functions*

$$\begin{aligned} D'\left(\frac{\partial u}{\partial x_i}, \hat{0}_4\right) &\leq \frac{\beta^2 (\frac{3m\Omega}{4\pi})^{\frac{1}{3}} [3 + \frac{1}{2}(\frac{3m\Omega}{4\pi})^{\frac{1}{3}}]}{dist(\Omega_{s_1}, \partial\Omega_{s_2})} D'(u, \hat{0}_4) \\ &= \eta D'(u, \hat{0}_4). \end{aligned} \tag{3.3}$$

Proof. Assume that u is a quaternion-valued function. By Theorem 5 in [37], we have

$$\left\| \frac{\partial u}{\partial x_i} \right\|_{s_1} \leq \frac{\beta^2 (\frac{3m\Omega}{4\pi})^{\frac{1}{3}} [3 + \frac{1}{2}(\frac{3m\Omega}{4\pi})^{\frac{1}{3}}]}{dist(\Omega_{s_1}, \partial\Omega_{s_2})} \|u\|_{s_2} = \eta \|u\|_{s_2}. \tag{3.4}$$

Now, for a generalized fuzzy regular function u, we consider its endpoint function u_l^α and u_r^α. It easy to see that u_l^α and u_r^α are also generalized regular functions. Then, we obtain their interior estimate as follows:

$$\left\|\frac{\partial u_l^\alpha}{\partial x_i}\right\|_{s_1} \leq \frac{\beta^2(\frac{3m\Omega}{4\pi})^{\frac{1}{3}}[3+\frac{1}{2}(\frac{3m\Omega}{4\pi})^{\frac{1}{3}}]}{dist(\Omega_{s_1},\partial\Omega_{s_2})}\|u_l^\alpha\|_{s_2} \tag{3.5}$$

and

$$\left\|\frac{\partial u_r^\alpha}{\partial x_i}\right\|_{s_1} \leq \frac{\beta^2(\frac{3m\Omega}{4\pi})^{\frac{1}{3}}[3+\frac{1}{2}(\frac{3m\Omega}{4\pi})^{\frac{1}{3}}]}{dist(\Omega_{s_1},\partial\Omega_{s_2})}\|u_r^\alpha\|_{s_2}. \tag{3.6}$$

Moreover, we can obtain

$$\begin{aligned}
&D'\left(\frac{\partial u}{\partial x_i},\hat{0}_4\right) = \sup_{0\leq\alpha\leq1}\left\{d\left(\left[\frac{\partial u}{\partial x_i}\right]^\alpha,\hat{(0)}_4\right)\right\}\\
&= \sup_{0\leq\alpha\leq1}\left\{d\left(\left[\left(\frac{\partial u}{\partial x_i}\right)_l^\alpha,\left(\frac{\partial u}{\partial x_i}\right)_r^\alpha\right],\hat{(0)}_4\right)\right\}\\
&\leq \frac{\beta^2(\frac{3m\Omega}{4\pi})^{\frac{1}{3}}[3+\frac{1}{2}(\frac{3m\Omega}{4\pi})^{\frac{1}{3}}]}{dist(\Omega_{s_1},\partial\Omega_{s_2})}\\
&\sup_{0\leq\alpha\leq1}\left\{d([u_l^\alpha,u_r^\alpha],\hat{(0)}_4)\right\} = \eta D'(u,\hat{0}_4).
\end{aligned} \tag{3.7}$$

This concludes the proof. □

Theorem 3. *Assume that L satisfies the hypotheses of Theorem 1 and assume that φ is an arbitrary generalized fuzzy regular function. The initial value problem (3.1) is solvable in the conical domain $M_\sigma = \{(t,x) : x \in \Omega, 0 \leq t \leq \sigma \cdot dist(x,\partial\Omega)\}$($\sigma$ is small enough). The solution $u(t,x)$ is also generalized fuzzy regular function for each t. Moreover, the fixed point equation $u = B(u)$ is Hyers–Ulam stable.*

Proof. To prove this, we know that the solution of the differential equation (3.1) must satisfy the Volterra equation

$$B(u) := u(t,x) = \varphi(x) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}L(u)\mathrm{d}\tau. \tag{3.8}$$

We then proof that the operator B has a fixed point. It is easy to see that B maps $C([0,T]\times\Omega,E^*)$ to itself. Moreover, we have

$$\begin{aligned}
&D'(B(u)\ominus B(v),\hat{0}_4) = D'(\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}L(u)\mathrm{d}\tau,\\
&\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}L(v)\mathrm{d}\tau)\\
&= \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}(\sum_{j=1}^3 A^{(j)}\frac{\partial}{\partial x_j}D'(u\ominus v,\hat{0}_4) + BD'(u\ominus v,\hat{0}_4)\mathrm{d}\tau\\
&\leq \frac{1}{\Gamma(\alpha)}(M+3\eta N)D'(u\ominus v,\hat{0}_4)\int_0^t (t-\tau)^{\alpha-1}\mathrm{d}\tau\\
&= \frac{1}{\Gamma(\alpha+1)}(M+3\eta N)t^\alpha D'(u\ominus v,\hat{0}_4)\\
&:= \gamma D'(u\ominus v,\hat{0}_4),
\end{aligned} \tag{3.9}$$

where $M = \|B\|$, $N = \max_{j=1,2,3}\{\|A^{(j)}\|\}$.

We may then choose a number $\tau > 0$ such that

$$\gamma = \frac{1}{\Gamma(\alpha+1)}(M + 3\eta N)\tau^{\alpha} < 1.$$

Then in the domain $M_\sigma = \{(t,x) : x \in \Omega, 0 \le t \le \sigma \cdot dist(x, \partial\Omega) \le \tau\}$, B is a contraction mapping. Thus, by the Banach's fixed point theorem, we obtain the desired uniqueness of the solution of the differential equation. Theorem 2.10 in [38] implies that the operator B is a c-weakly Picard operator with the positive constant $c = \frac{1}{1-\gamma}$ and the fixed point equation $u = B(u)$ is Hyers–Ulam stable.

Moreover, the solution $u(t,x)$ belongs to the associated space for each t. The solution $u(t,x)$ is also generalized regular. □

Competing interests

The author declares to have no competing interests.

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Set-valued quadratic ρ-functional inequalities

Choonkil Park and Jung Rye Lee*

Abstract. In this paper, we introduce set-valued quadratic ρ-functional inequalities and prove the Hyers-Ulam stability of the set-valued quadratic ρ-functional inequalities by using the fixed point method.

1. Introduction and preliminaries

Set-valued functions in Banach spaces have been developed in the last decades. The pioneering paper by Aumann [5] and Debreu [14] were inspired by problems arising in Control Theory and Mathematical Economics. We can refer to the papers by Arrow and Debreu [3], McKenzie [27], the momographs by Hindenbrand [20], Aubin and Frankowska [4], Castaing and Valadier [8], Klein and Thompson [25] and the survey by Hess [19].

The stability problem of functional equations originated from a question of Ulam [53] concerning the stability of group homomorphisms. Hyers [21] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [42] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [18] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. A Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [52] for mappings $f: X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [12] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [13] proved the Hyers-Ulam stability of the quadratic functional equation. The functional equation

$$2f(x+y)+2f(x-y)=f(2x)+f(2y)$$

is called a *Jensen quadratic functional equation*. In particular, every solution of the Jensen quadratic functional equation is said to be a *Jensen quadratic mapping*. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1, 17, 18, 22, 23], [39]–[41], [43]–[51], [54, 55]).

Let X be a set. A function $d: X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;

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(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Let (X, d) be a generalized metric space. An operator $T : X \to X$ satisfies a Lipschitz condition with Lipschitz constant L if there exists a constant $L \geq 0$ such that $d(Tx, Ty) \leq Ld(x, y)$ for all $x, y \in X$. If the Lipschitz constant L is less than 1, then the operator T is called a strictly contractive operator. Note that the distinction between the generalized metric and the usual metric is that the range of the former is permitted to include the infinity. We recall the following theorem by Margolis and Diaz.

Theorem 1.1. [9, 15] *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \geq n_0$;

(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*

(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;*

(4) *$d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.*

In 1996, Isac and Rassias [24] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [10, 11, 29, 34, 35, 38]).

Let Y be a Banach space. We define the following:

2^Y : the set of all subsets of Y;

$C_b(Y)$: the set of all closed bounded subsets of Y;

$C_c(Y)$: the set of all closed convex subsets of Y;

$C_{cb}(Y)$: the set of all closed convex bounded subsets of Y.

On 2^Y we consider the addition and the scalar multiplication as follows:

$$C + C' = \{x + x' : x \in C, x' \in C'\}, \qquad \lambda C = \{\lambda x : x \in C\},$$

where $C, C' \in 2^Y$ and $\lambda \in \mathbb{R}$. Further, if $C, C' \in C_c(Y)$, then we denote by $C \oplus C' = \overline{C + C'}$.

It is easy to check that

$$\lambda C + \lambda C' = \lambda(C + C'), \qquad (\lambda + \mu)C \subseteq \lambda C + \mu C.$$

Furthermore, when C is convex, we obtain $(\lambda + \mu)C = \lambda C + \mu C$ for all $\lambda, \mu \in \mathbb{R}^+$.

For a given set $C \in 2^Y$, the distance function $d(\cdot, C)$ and the support function $s(\cdot, C)$ are respectively defined by

$$\begin{aligned} d(x, C) &= \inf\{\|x - y\| : y \in C\}, \qquad x \in Y, \\ s(x^*, C) &= \sup\{\langle x^*, x\rangle : x \in C\}, \qquad x^* \in Y^*. \end{aligned}$$

For every pair $C, C' \in C_b(Y)$, we define the Hausdorff distance between C and C' by

$$h(C, C') = \inf\{\lambda > 0 : C \subseteq C' + \lambda B_Y, \qquad C' \subseteq C + \lambda B_Y\},$$

where B_Y is the closed unit ball in Y.

The following proposition reveals some properties of the Hausdorff distance.

Proposition 1.2. *For every $C, C', K, K' \in C_{cb}(Y)$ and $\lambda > 0$, the following properties hold*
(a) $h(C \oplus C', K \oplus K') \leq h(C, K) + h(C', K')$;
(b) $h(\lambda C, \lambda K) = \lambda h(C, K)$.

Let $(C_{cb}(Y), \oplus, h)$ be endowed with the Hausdorff distance h. Since Y is a Banach space, $(C_{cb}(Y), \oplus, h)$ is a complete metric semigroup (see [8]). Debreu [14] proved that $(C_{cb}(Y), \oplus, h)$ is isometrically embedded in a Banach space as follows.

Lemma 1.3. [14] *Let $C(B_{Y^*})$ be the Banach space of continuous real-valued functions on B_{Y^*} endowed with the uniform norm $\|\cdot\|_u$. Then the mapping $j : (C_{cb}(Y), \oplus, h) \to C(B_{Y^*})$, given by $j(A) = s(\cdot, A)$, satisfies the following properties:*
(a) $j(A \oplus B) = j(A) + j(B)$;
(b) $j(\lambda A) = \lambda j(A)$;
(c) $h(A, B) = \|j(A) - j(B)\|_u$;
(d) $j(C_{cb}(Y))$ *is closed in* $C(B_{Y^*})$
for all $A, B \in C_{cb}(Y)$ and all $\lambda \geq 0$.

Let $f : \Omega \to (C_{cb}(Y), h)$ be a set-valued function from a complete finite measure space (Ω, Σ, ν) into $C_{cb}(Y)$. Then f is *Debreu integrable* if the composition $j \circ f$ is Bochner integrable (see [7]). In this case, the Debreu integral of f in Ω is the unique element $(D) \int_\Omega f d\nu \in C_{cb}(Y)$ such tha $j((D) \int_\Omega f d\nu)$ is the Bochner integral of $j \circ f$. The set of Debreu integrable functions from Ω to $C_{cb}(Y)$ will be denoted by $D(\Omega, C_{cb}(Y))$. Furthermore, on $D(\Omega, C_{cb}(Y))$, we define $(f + g)(\omega) = f(\omega) \oplus g(\omega)$ for all $f, g \in D(\Omega, C_{cb}(Y))$. Then we obtain that $((\Omega, C_{cb}(Y)), +)$ is an abelian semigroup.

Set-valued functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [6], [30]–[33], [36, 37]).

Using the fixed point method, we prove the Hyers-Ulam stability of the following set-valued quadratic ρ-functional inequalities

$$h(f(x+y) \oplus f(x-y), 2f(x) \oplus 2f(y)) \leq \rho \cdot h\left(2f(x+y) \oplus 2f(x-y), f(2x) \oplus f(2y)\right) \tag{1.1}$$

and

$$h(2f(x+y) \oplus 2f(x-y), f(2x) \oplus f(2y)) \leq \rho \cdot h\left(f(x+y) \oplus f(x-y), 2f(x) \oplus 2f(y)\right). \tag{1.2}$$

Throughout this paper, let X be a real vector space and Y a real Banach space.

2. Stability of the set-valued quadratic ρ-functional inequality (1.1)

Throughout this section, assume that ρ is a positive real number less than $\frac{1}{2}$.

Using the fixed point method, we prove the Hyers-Ulam stability of the set-valued quadratic ρ-functional inequality (1.1).

Definition 2.1. Let $f : X \to C_{cb}(Y)$. The quadratic set-valued functional equation is defined by

$$f(x+y) \oplus f(x-y) = 2f(x) \oplus 2f(y)$$

for all $x, y \in X$. Every solution of the quadratic set-valued functional equation is called a *quadratic set-valued mapping*.

Definition 2.2. [26] Let $f : X \to C_{cb}(Y)$. The Jensen quadratic set-valued functional equation is defined by

$$2f(x+y) \oplus 2f(x-y) = f(2x) \oplus f(2y)$$

for all $x, y \in X$. Every solution of the Jensen quadratic set-valued functional equation is called a *Jensen quadratic set-valued mapping.*

Lemma 2.3. *Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying $f(0) = \{0\}$ and*

$$h(f(x+y) \oplus f(x-y), 2f(x) \oplus 2f(y)) \le \rho \cdot h\left(2f(x+y) \oplus 2f(x-y), f(2x) \oplus f(2y)\right) \tag{2.1}$$

for all $x, y \in X$. Then $f : X \to (C_{cb}(Y), h)$ is a quadratic set-valued mapping.

Proof. Letting $y = x$ in (2.1), we get $h(f(2x), 4f(x)) = 0$ for all $x \in X$. Thus $f(2x) = 4f(x)$ and so

$$\begin{aligned} h(f(x+y) \oplus f(x-y), 2f(x) \oplus 2f(y)) &\le \rho \cdot h\left(2f(x+y) \oplus 2f(x-y), f(2x) \oplus f(2y)\right) \\ &= \rho \cdot h\left(2f(x+y) \oplus 2f(x-y), 4f(x) \oplus 4f(y)\right) \\ &= 2\rho \cdot h(f(x+y) \oplus f(x-y), 2f(x) \oplus 2f(y)) \end{aligned}$$

for all $x, y \in X$. Since $\rho < \frac{1}{2}$, $h(f(x+y) \oplus f(x-y), 2f(x) \oplus 2f(y)) = 0$ and so

$$f(x+y) \oplus f(x-y) = 2f(x) \oplus 2f(y)$$

for all $x, y \in X$. □

Theorem 2.4. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \le \frac{L}{4} \varphi(2x, 2y)$$

for all $x, y \in X$. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying $f(0) = \{0\}$ and

$$h(f(x+y) \oplus f(x-y), 2f(x) \oplus 2f(y)) \le \rho \cdot h\left(2f(x+y) \oplus 2f(x-y), f(2x) \oplus f(2y)\right) + \varphi(x, y) \tag{2.2}$$

for all $x, y \in X$. Then there exists a unique quadratic set-valued mapping $Q : X \to (C_{cb}(Y), h)$ such that

$$h(f(x), Q(x)) \le \frac{L}{4 - 4L} \varphi(x, x) \tag{2.3}$$

for all $x \in X$.

Proof. Let $y = x$ in (2.2). Since $f(x)$ is convex, we get

$$h(f(2x), 4f(x)) \le \varphi(x, x) \tag{2.4}$$

and so

$$h\left(f(x), 4f\left(\frac{x}{2}\right)\right) \le \varphi\left(\frac{x}{2}, \frac{x}{2}\right) \le \frac{L}{4} \varphi(x, x) \tag{2.5}$$

for all $x \in X$.

Consider

$$S := \{g : \ g : X \to C_{cb}(Y), \ g(0) = \{0\}\}$$

and introduce the generalized metric on X,

$$d(g, f) = inf\{\mu \in (0, \infty) : \ h(g(x), f(x)) \le \mu\varphi(x, x), \ x \in X\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [16, Theorem 2.4] and [28, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, f \in S$ be given such that $d(g, f) = \varepsilon$. Then

$$h(g(x), f(x)) \le \varepsilon\varphi(x, x)$$

for all $x \in X$. Hence

$$h(Jg(x), Jf(x)) = h\left(4g\left(\frac{x}{2}\right), 4f\left(\frac{x}{2}\right)\right) = 4h\left(g\left(\frac{x}{2}\right), f\left(\frac{x}{2}\right)\right) \le L\varphi(x, x)$$

for all $x \in X$. So $d(g, f) = \varepsilon$ implies that $d(Jg, Jf) \le L\varepsilon$. This means that

$$d(Jg, Jf) \le Ld(g, f)$$

for all $g, f \in S$.

It follows from (2.5) that $d(f, Jf) \le \frac{L}{4}$.

By Theorem 1.1, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q\left(\frac{x}{2}\right) = \frac{1}{4}Q(x) \tag{2.6}$$

for all $x \in X$. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (2.6) such that there exists a $\mu \in (0, \infty)$ satisfying

$$h(f(x), Q(x)) \le \mu\varphi(x, x)$$

for all $x \in X$;

(2) $d(J^n f, Q) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right) = Q(x)$$

for all $x \in X$;

(3) $d(f, Q) \le \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, Q) \le \frac{L}{4 - 4L}.$$

This implies that the inequality (2.3) holds.

By (2.2),

$$\begin{aligned}
&h\left(4^n f\left(\frac{x+y}{2^n}\right) \oplus 4^n f\left(\frac{x-y}{2^n}\right), 2\cdot 4^n f\left(\frac{x}{2^n}\right) \oplus 2\cdot 4^n f\left(\frac{y}{2^n}\right)\right)\\
&\le \rho\cdot h\left(2\cdot 4^n f\left(\frac{x+y}{2^n}\right) \oplus 2\cdot 4^n f\left(\frac{x-y}{2^n}\right), 4^n f\left(\frac{x}{2^{n-1}}\right) \oplus 4^n f\left(\frac{y}{2^{n-1}}\right)\right) + 4^n\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\\
&\le \rho\cdot h\left(2\cdot 4^n f\left(\frac{x+y}{2^n}\right) \oplus 2\cdot 4^n f\left(\frac{x-y}{2^n}\right), 4^n f\left(\frac{x}{2^{n-1}}\right) \oplus 4^n f\left(\frac{y}{2^{n-1}}\right)\right) + L^n\varphi(x, y)
\end{aligned}$$

and so

$$h(Q(x+y)\oplus Q(x-y), 2Q(x)\oplus 2Q(y)) \le \rho\cdot h\left(2Q(x+y)\oplus 2Q(x-y), Q(2x)\oplus Q(2y)\right)$$

for all $x, y \in X$. By Lemma 2.3, $Q(x+y)\oplus Q(x-y) = 2Q(x)\oplus 2Q(y)$, as desired. □

Corollary 2.5. *Let $p > 2$ and $\theta \ge 0$ be real numbers, and let X be a real normed space. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying $f(0) = \{0\}$ and*

$$\begin{aligned} h(f(x+y)\oplus f(x-y), 2f(x)\oplus 2f(y)) &\le \rho\cdot h\left(2f(x+y)\oplus 2f(x-y), f(2x)\oplus f(2y)\right) \\ &+ \theta(||x||^p + ||y||^p) \end{aligned} \tag{2.7}$$

for all $x, y \in X$. Then there exists a unique quadratic set-valued mapping $Q : X \to Y$ satisfying

$$h(f(x), Q(x)) \le \frac{2\theta}{2^p - 4}||x||^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.4 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then we can choose $L = 2^{2-p}$ and we get the desired result. □

Theorem 2.6. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \le 4L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying $f(0) = \{0\}$ and (2.2). Then there exists a unique quadratic set-valued mapping $Q : X \to (C_{cb}(Y), h)$ such that

$$h(f(x), Q(x)) \le \frac{1}{4-4L}\varphi(x, x)$$

for all $x \in X$.

Proof. It follows from (2.4) that

$$h\left(f(x), \frac{1}{2}f\left(2x\right)\right) \le \frac{1}{4}\varphi\left(x, x\right)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.4. □

Corollary 2.7. *Let $2 > p > 0$ and $\theta \ge 0$ be real numbers, and let X be a real normed space. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying $f(0) = \{0\}$ and (2.7). Then there exists a unique quadratic set-valued mapping $Q : X \to Y$ satisfying*

$$h(f(x), Q(x)) \le \frac{2\theta}{4 - 2^p}||x||^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.6 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then we can choose $L = 2^{p-2}$ and we get the desired result. □

3. Stability of the set-valued quadratic ρ-functional inequality (1.2)

Throughout this section, assume that ρ is a positive real number less than 2.

Using the fixed point method, we prove the Hyers-Ulam stability of the set-valued quadratic ρ-functional inequality (1.2).

Lemma 3.1. *Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying $f(0) = \{0\}$ and*

$$h(2f(x+y) \oplus 2f(x-y), f(2x) \oplus f(2y)) \le \rho \cdot h\left(f(x+y) \oplus f(x-y), 2f(x) \oplus 2f(y)\right) \tag{3.1}$$

for all $x, y \in X$. Then $f : X \to (C_{cb}(Y), h)$ is a Jensen quadratic set-valued mapping.

Proof. Letting $y = 0$ in (3.1), we get $h(4f(x), f(2x)) = 0$ for all $x \in X$. Thus $f(2x) = 4f(x)$ and so

$$\begin{aligned} 2h(f(x+y) \oplus f(x-y), 2f(x) \oplus 2f(y)) &= h(2f(x+y) \oplus 2f(x-y), 4f(x) \oplus 4f(y)) \\ &= h\left(2f(x+y) \oplus 2f(x-y), f(2x) \oplus f(2y)\right) \\ &\le \rho \cdot h\left(f(x+y) \oplus f(x-y), 2f(x) \oplus 2f(y)\right) \end{aligned}$$

for all $x, y \in X$. Since $\rho < 2$, $h(2f(x+y) \oplus 2f(x-y), f(2x) \oplus f(2y)) = 0$ and so

$$2f(x+y) \oplus 2f(x-y) = f(2x) \oplus f(2y)$$

for all $x, y \in X$. □

Theorem 3.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \le \frac{L}{4}\varphi\left(2x, 2y\right)$$

for all $x, y \in X$. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying $f(0) = \{0\}$ and

$$h(2f(x+y) \oplus 2f(x-y), f(2x) \oplus f(2y)) \le \rho \cdot h(f(x+y) \oplus f(x-y), 2f(x) \oplus 2f(y)) + \varphi(x, y) \tag{3.2}$$

for all $x, y \in X$. Then there exists a unique Jensen quadratic set-valued mapping $Q : X \to (C_{cb}(Y), h)$ such that

$$h(f(x), Q(x)) \le \frac{L}{4 - 4L}\varphi(x, 0)$$

for all $x \in X$.

Proof. Let $y = 0$ in (3.2). Since $f(x)$ is convex, we get

$$h(f(2x), 4f(x)) \le \varphi(x, 0) \tag{3.3}$$

and

$$h\left(f(x), 4f\left(\frac{x}{2}\right)\right) \le \varphi\left(\frac{x}{2}, 0\right) \le \frac{L}{4}\varphi(x, 0) \tag{3.4}$$

for all $x \in X$.

Consider

$$S := \{g : \ g : X \to C_{cb}(Y), \ g(0) = \{0\}\}$$

and introduce the generalized metric on X,

$$d(g, f) = inf\{\mu \in (0, \infty) : \ h(g(x), f(x)) \le \mu\varphi(x, 0), \ x \in X\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [16, Theorem 2.4] and [28, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

By the same reasoning as in the proof of Theorem 2.4, one can show that

$$d(Jg, Jf) \le Ld(g, f)$$

for all $g, f \in S$.

It follows from (3.4) that $d(f, Jf) \le \frac{L}{4}$.

The rest of the proof is similar to the proof of Theorem 2.4. □

Corollary 3.3. *Let $p > 2$ and $\theta \ge 0$ be real numbers, and let X be a real normed space. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying $f(0) = \{0\}$ and*

$$\begin{aligned} h(2f(x+y) \oplus 2f(x-y), f(2x) \oplus f(2y)) &\le \rho \cdot h(f(x+y) \oplus f(x-y), 2f(x) \oplus 2f(y)) \\ &+ \theta(||x||^p + ||y||^p) \end{aligned} \tag{3.5}$$

for all $x, y \in X$. Then there exists a unique Jensen quadratic set-valued mapping $Q : X \to Y$ satisfying

$$h(f(x), Q(x)) \le \frac{\theta}{2^p - 4}||x||^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.2 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then we can choose $L = 2^{2-p}$ and we get the desired result. □

Theorem 3.4. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \le 4L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying $f(0) = \{0\}$ and (3.2). Then there exists a unique Jensen quadratic set-valued mapping $Q : X \to (C_{cb}(Y), h)$ such that

$$h(f(x), Q(x)) \le \frac{1}{4 - 4L}\varphi(x, 0)$$

for all $x \in X$.

Proof. It follows from (3.3) that

$$h\left(f(x), \frac{1}{4}f(2x)\right) \le \frac{1}{4}\varphi(x, 0)$$

for all $x \in X$.

The rest of the proof is similar to the proofs of Theorems 2.4 and 3.2. □

Corollary 3.5. *Let $0 < p < 2$ and $\theta \ge 0$ be real numbers, and let X be a real normed space. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying $f(0) = \{0\}$ and (3.5). Then there exists a unique Jensen quadratic set-valued mapping $Q : X \to Y$ satisfying*

$$h(f(x), Q(x)) \le \frac{\theta}{4 - 2^p}||x||^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.4 by taking

$$\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then we can choose $L = 2^{p-2}$ and we get the desired result. □

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APPROXIMATE TERNARY QUADRATIC 3-DERIVATIONS ON TERNARY BANACH ALGEBRAS AND C^*-TERNARY RINGS

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ABSTRACT. In the current article, we use a fixed point alternative theorem to establish the Hyers-Ulam stability and also the superstability of a ternary quadratic 3-derivation on ternary Banach algebras and C^*-ternary rings.

1. **Introduction and preliminaries**

Ternary algebraic operations were considered in the 19th century by several mathematicians and physicists such as Cayley [5] who introduced the notion of cubic matrix which in turn was generalized by Kapranov, Gelfand and Zelevinskii in 1990 [22]. As an application in physics, the quark model inspired a particular brand of ternary algebraic systems. The so-called Nambu mechanics which has been proposed by Nambu [25] in 1973, is based on such structures. There are also some applications, although still hypothetical, in the fractional quantum Hall effect, the non-standard statistics (the anyons), supersymmetric theories, Yang-Baxter equation, etc, (see [1, 35]). The comments on physical applications of ternary structures can be found in ([6, 12, 17, 18, 26, 27, 31]).

We say that a functional equation (Q) is stable if any function g satisfying the equation (Q) approximately is near to true solution of (Q). A basic question in the theory of functional equations is as follows: when is it true that a function, which approximately satisfies a functional equation, must be close to an exact solution of the equation? If the problem accepts a unique solution, we say the equation is stable. Also, if every approximately solution is an exact solution of it, we say the functional equation is superstable (see [3]). The first stability problem concerning group homomorphisms was raised by Ulam [34] and partially solved by Hyers [20]. In [29], Rassias [16] generalized the result of Hyers for approximately linear mappings. Gajda [15] answered the question for another case of linear mapping, which was rased by Rassias. The stability problems of various functional equations have been extensively investigated by a number of authors (see [13, 14, 21]).

The functional equation $f(x+y)+f(x-y)=2f(x)+2f(y)$ is called quadratic functional equation. In addition, every solution of the above equation is said to be a quadratic mapping. A Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [33] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Later, Czerwik [7] proved the Cauchy-Rassias stability of the quadratic functional equation. Since then, the stability problems of various functional equation have been extensively investigated by a number of authors (see [4, 9, 11, 23, 28]). As it is extensively discussed in [30], the full description of a physical system S implies the knowledge of three basic ingredients: the set of

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the observables, the set of the states and the dynamics that describes the time evolution of the system by means of the time dependence of the expectation value of a given observable on a given statue. Originally the set of the observables were considered to be a C^*-algebra [19]. In many applications, however, this was shown not to be the most convenient choice, and so the C^*-algebra was replaced by a von Neumann algebra. This is because the role of the representation turns out to be crucial, mainly when long range interactions are involved. Here we used a different algebraic structure.

A ternary algebra is a complex Banach space, equipped with a ternary product $(x, y, z) \to [x, y, z]$ of A^3 into A, which is linear in the outer variables, conjugate linear in the middle variable, and associative in the sense that $\Big[x, y, [z, u, v]\Big] = \Big[x, [y, z, u]v\Big] = \Big[[x, y, z], u, v\Big]$, and satisfies $\|[x, y, z]\| \le \|x\| \cdot \|y\| \cdot \|z\|$. A C^*-ternary ring is a complex Banach space, A equipped with a ternary product which is associative and linear in the outer variables, conjugate linear in the middle variable, and $\|[x, x, x]\| = \|x\|^3$ (see [37]).

If a C^*-ternary algebra $(A, [\cdot, \cdot, \cdot])$ has an identity, that is, an element $e \in A$ such that $x = [x, e, e] = [e, e, x]$ for all $x \in A$, then it is routine to verify that A, endowed with the operation $xoy := [x, e, y]$, $\quad x^* := [e, x, e]$ is a unital C^*-algebra. Conversely, if (A, o) is a unital C^*-algebra, then $[x, y, z] := xoy^*oz$ makes A into a C^*-ternary ring.

Recently, Shagholi et al. [32] proved the stability of ternary quadratic derivations on ternary Banach algebras. Moslehian investigated the stability and the superstability of ternary derivations on C^*-ternary rings [24]. Xu et al. [36] used the fixed point alternative (Theorem 4.2 of current article) to establish the Hyers-Ulam stability of the general mixed additive-cubic functional equation, where functions map a linear space into a complete quasi fuzzy p-normed space. The Hyers-Ulam stability of an additive-cubic-quartic functional equation in NAN -spaces was also proved by using the mentioned theorem in [2].

In this article, we prove the Hyers-Ulam stability and the superstability of ternary quadratic 3-derivations on ternary Banach algebras and C^*-ternary rings associated with the quadratic functional equation $f(x + y) + f(x - y) = 2f(x) + 2f(y)$ using the fixed point theorem.

2. Stability of ternary quadratic 3-derivations

Throughout this article, for a ternary Banach algebra (or C^*-ternary ring) A , we denote

$$\overbrace{A \times A \times \cdots \times A}^{n-times}$$

by A^n.

Definition 2.1. *Let A be a ternary Banach algebra or C^*-ternary ring. Then a mapping $D : A \to A$ is called a ternary quadratic 3-derivation if it is a quadratic mapping that satisfies*

$$\begin{aligned}
&D\left(\Big[[x_1, x_2, x_3], [y_1, y_2, y_3], [z_1, z_2, z_3]\Big]\right) \\
&= \Big[D([x_1, x_2, x_3]), [y_1, y_2, y_3], \Big[[y_1^*, y_2^*, y_3^*], [z_1^*, z_2^*, z_3^*], [z_1, z_2, z_3]\Big]\Big] \\
&+ \Big[[x_1, x_2, x_3], [x_1^*, x_2^*, x_3^*], \Big[D([y_1^*, y_2^*, y_3^*]), [z_1^*, z_2^*, z_3^*], [z_1, z_2, z_3]\Big]\Big] \\
&+ \Big[[x_1, x_2, x_3], [x_1^*, x_2^*, x_3^*], \Big[[y_1^*, y_2^*, y_3^*], [y_1, y_2, y_3], D([z_1, z_2, z_3])\Big]\Big]
\end{aligned}$$

for all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$.

It was proved in [10] that for the vector spaces X and Y and a fixed positive integer k, the mapping $f : X \to Y$ is quadratic if and only if the following equality holds:

$$2f\left(\frac{kx+ky}{2}\right)+2f\left(\frac{kx-ky}{2}\right)=k^2f(x)+k^2f(y)$$

for all $x, y \in X$. Also, we can show that f is quadratic if and only if for a fixed positive integer k, we have

$$f(kx+ky)+f(kx-ky)=2k^2f(x)+2k^2f(y)$$

for all $x, y \in X$. Before proceeding to the main results, to achieve our aim, we need the following known fixed point theorem which has been proved in [8].

Theorem 2.2. *Suppose that (Ω, d) is a complete generalized metric space and $J : \Omega \to \Omega$ is a strictly contractive mapping with the Lipschitz constant L. Then, for any $x \in \Omega$, either*

$$d(J^nx, J^{n+1}x)=\infty, \quad \forall n \geq 0,$$

or there exists a positive integer n_0 such that

(1) $d(J^nx, J^{n+1}x) < \infty$ *for all* $n \geq n_0$;
(2) *the sequence* $\{J^nx\}$ *is convergent to a fixed point* y^* *of* J;
(3) y^* *is the unique fixed point of* J *in* $\Lambda = \{y \in \Omega : d(J^{n_0}x, y) < \infty\}$;
(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ *for all* $y \in \Lambda$.

In the following theorem, we prove the Hyers-Ulam stability of ternary quadratic 3-derivation on C^*-ternary rings.

Theorem 2.3. *Let A be a C^*-ternary ring, $f : A \to A$ be a mapping with $f(0) = 0$, and also let $\varphi : A^{11} \to [0,\infty)$ be a function such that*

$$\left\|2f\left(\frac{\mu a+\mu b}{2}\right)+2f\left(\frac{\mu a-\mu b}{2}\right)-\mu^2(f(a)+f(b))\right\| \leq \varphi(a,b,0,0,0,0,0,0,0,0,0), \tag{1}$$

$$\begin{aligned}\Big\|f\Big(\big[[x_1,x_2,x_3],[y_1,y_2,y_3],[z_1,z_2,z_3]\big]\Big)\\ &-\Big[f([x_1,x_2,x_3]),[y_1,y_2,y_3],\big[[y_1^*,y_2^*,y_3^*],[z_1^*,z_2^*,z_3^*],[z_1,z_2,z_3]\big]\Big]\\ &-\Big[[x_1,x_2,x_3],[x_1^*,x_2^*,x_3^*],\big[f([y_1^*,y_2^*,y_3^*]),[z_1^*,z_2^*,z_3^*],[z_1,z_2,z_3]\big]\Big]\\ &-\Big[[x_1,x_2,x_3],[x_1^*,x_2^*,x_3^*],\big[[y_1^*,y_2^*,y_3^*],[y_1,y_2,y_3],f([z_1,z_2,z_3])\big]\Big]\Big\|\\ &\leq \varphi(0,0,x_1,x_2,x_3,y_1,y_2,y_3,z_1,z_2,z_3)\end{aligned} \tag{2}$$

for all $\mu \in T = \{\lambda \in C : |\lambda| = 1\}$ and all $a, b, x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$. If there exists a constant $M \in (0, 1)$ such that

$$\begin{aligned}&\varphi(2a,2b,2x_1,2x_2,2x_3,2y_1,2y_2,2y_3,2z_1,2z_2,2z_3)\\ &\leq 4M\varphi(a,b,x_1,x_2,x_3,y_1,y_2,y_3,z_1,z_2,z_3)\end{aligned} \tag{3}$$

for all $a, b, x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$, then there exists a unique ternary quadratic 3-derivation $D : A \to A$ such that

$$\|f(a)-D(a)\| \leq \frac{M}{1-M}\psi(a) \tag{4}$$

for all $a \in A$, where $\psi(a) = \varphi(a, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$.

Proof. It follows from (3) that

$$\lim_{j\to\infty}\frac{\varphi(2^j a, 2^j b, 2^j x_1, 2^j x_2, 2^j x_3, 2^j y_1, 2^j y_2, 2^j y_3, 2^j z_1, 2^j z_2, 2^j z_3)}{4^j}=0 \tag{5}$$

for all $a, b, x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$. Putting $\mu = 1, b = 0$ and replacing a by $2a$ in (1), we have

$$\left\|4f(a)-f(2a)\right\| \le \psi(2a) \le \psi(2a) \le 4M\psi(a)$$

and so

$$\left\|f(a)-\frac{1}{4}f(2a)\right\| \le M\psi(a) \tag{6}$$

for all $a \in A$. We consider the set $\Omega := \{h : A \to A\ |h(0) = 0\}$ and introduce the generalized metric on X as follows:

$$d(h_1, h_2) := inf\{K \in (0,\infty) : \|h_1(a) - h_2(a)\| \le K\psi(a), \forall a \in A\},$$

if there exists such a constant K, and $d(h_1, h_2) = \infty$, otherwise. One can show that (Ω, d) is a complete metric space. We now show that $J : \Omega \to \Omega$ by

$$J(h)(a) = \frac{1}{4}h(2a) \tag{7}$$

for all $a \in A$. Given $h_1, h_2 \in \Omega$, let $K \in R^+$ an arbitrary constant with $d(h_1, h_2) \le K$, that is,

$$d(h_1(a), h_2(a)) \le K\psi(a) \tag{8}$$

for all $a \in A$. Substituting a by $2a$ in (8) and using (3) and (7), we have

$$\|(Jh_1)(a) - (Jh_2)(a)\| = \frac{1}{4}\|h_1(2a) - h_2(2a)\| \le \frac{1}{4}K\psi(2a) \le KM\psi(a)$$

for all $a \in A$ and thus $d(Jh_1, Jh_2) \le KM$. Therefore, we conclude that $d(Jh_1, Jh_2) \le Md(h_1, h_2)$ for all $h_1, h_2 \in \Omega$. It follows from (6) that

$$d(Jf, f) \le M. \tag{9}$$

By Theorem 2.2, the sequence $\{J^n f\}$ converges to a unique fixed point $D : A \to A$ in the set $\Omega_1 = \{h \in \Omega, d(f, h) < \infty\}$, i.e.,

$$\lim_{n\to\infty}\frac{2^n a}{4^n} = D(a), \tag{10}$$

for all $a \in A$. By Theorem 2.2 and (9), we have

$$d(f, D) \le \frac{d(Jf, f)}{1-M} \le \frac{M}{1-M}.$$

The last inequality shows that (4) holds for all $a \in A$. Replace $2^n a$ and $2^n b$ by a and b, respectively. Now, dividing both sides of the resulting inequality by 2^n, and letting n goes to infinity, we obtain

$$2D\left(\frac{\mu a+\mu b}{2}\right)+2D\left(\frac{\mu a-\mu b}{2}\right)=\mu^2(D(a)+D(b)) \tag{11}$$

for all $a, b \in A$ and $\mu \in T$. Putting $\mu = 1$ in (11), we have

$$2D\left(\frac{a+b}{2}\right) + 2D\left(\frac{a-b}{2}\right) = D(a) + D(b)$$

for all $a, b \in A$. Hence D is a quadratic mapping by [33, Proposition 1]. So it follows from the definition of D, (2), (5) and (10) that

$$\begin{aligned}
&D\left(\Big[[x_1,x_2,x_3],[y_1,y_2,y_3],[z_1,z_2,z_3]\Big]\right)\\
&\quad - \Big[D([x_1,x_2,x_3]),[y_1,y_2,y_3],\Big[[y_1^*,y_2^*,y_3^*],[z_1^*,z_2^*,z_3^*],[z_1,z_2,z_3]\Big]\Big]\\
&\quad - \Big[[x_1,x_2,x_3],[x_1^*,x_2^*,x_3^*],\Big[D([y_1^*,y_2^*,y_3^*]),[z_1^*,z_2^*,z_3^*],[z_1,z_2,z_3]\Big]\Big]\\
&\quad - \Big[[x_1,x_2,x_3],[x_1^*,x_2^*,x_3^*],\Big[[y_1^*,y_2^*,y_3^*],[y_1,y_2,y_3],D([z_1,z_2,z_3])\Big]\Big]\\
&= \lim_{n\to\infty}\left(\frac{1}{4^{9n}}f\left(\Big[[2^n x_1,2^n x_2,2^n x_3],[2^n y_1,2^n y_2,2^n y_3],[2^n z_1,2^n z_2,2^n z_3]\Big]\right)\right.\\
&\quad - \Big[\frac{1}{4^{3n}}f\left([2^n x_1,2^n x_2,2^n x_3]\right),[y_1,y_2,y_3],\Big[[y_1^*,y_2^*,y_3^*],[z_1^*,z_2^*,z_3^*],[z_1,z_2,z_3]\Big]\Big]\\
&\quad - \Big[[x_1,x_2,x_3],[x_1^*,x_2^*,x_3^*],\Big[\frac{1}{4^{3n}}f([2^n y_1^*,2^n y_2^*,2^n y_3^*]),[z_1^*,z_2^*,z_3^*],[z_1,z_2,z_3]\Big]\Big]\\
&\quad \left.-\Big[[x_1,x_2,x_3],[x_1^*,x_2^*,x_3^*],\Big[[y_1^*,y_2^*,y_3^*],[y_1,y_2,y_3],\frac{1}{4^{3n}}f([2^n z_1,2^n z_2,2^n z_3])\Big]\Big]\right)\\
&\le \lim_{n\to\infty}\frac{1}{4^{9n}}\varphi(0,0,2^n x_1,2^n x_2,2^n x_3,2^n y_1,2^n y_2,2^n y_3,2^n z_1,2^n z_2,2^n z_3) = 0
\end{aligned}$$

for all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$ and so

$$\begin{aligned}
&D\left(\Big[[x_1,x_2,x_3],[y_1,y_2,y_3],[z_1,z_2,z_3]\Big]\right)\\
&= \Big[D([x_1,x_2,x_3]),[y_1,y_2,y_3],\Big[[y_1^*,y_2^*,y_3^*],[z_1^*,z_2^*,z_3^*],[z_1,z_2,z_3]\Big]\Big]\\
&+ \Big[[x_1,x_2,x_3],[x_1^*,x_2^*,x_3^*],\Big[D([y_1^*,y_2^*,y_3^*]),[z_1^*,z_2^*,z_3^*],[z_1,z_2,z_3]\Big]\Big]\\
&+ \Big[[x_1,x_2,x_3],[x_1^*,x_2^*,x_3^*],\Big[[y_1^*,y_2^*,y_3^*],[y_1,y_2,y_3],D([z_1,z_2,z_3])\Big]\Big],
\end{aligned}$$

which means that D is a ternary quadratic 3-derivation. □

Corollary 2.4. *Let p, θ be nonnegative real numbers such that $p < 2$ and let f be a mapping on a C^*-ternary ring A with $f(0) = 0$ and*

$$\left\|2f\left(\frac{\mu a+\mu b}{2}\right) + 2f\left(\frac{\mu a-\mu b}{2}\right) - \mu^2(f(a)+f(b))\right\| \le \theta(\|a\|^p + \|b\|\|^p),$$

$$\begin{aligned}&\Big\|f\Big(\Big[[x_1,x_2,x_3],[y_1,y_2,y_3],[z_1,z_2,z_3]\Big]\Big)\\&\quad-\Big[f([x_1,x_2,x_3]),[y_1,y_2,y_3],\Big[[y_1^*,y_2^*,y_3^*],[z_1^*,z_2^*,z_3^*],[z_1,z_2,z_3]\Big]\Big]\\&\quad-\Big[[x_1,x_2,x_3],[x_1^*,x_2^*,x_3^*],\Big[f([y_1^*,y_2^*,y_3^*]),[z_1^*,z_2^*,z_3^*],[z_1,z_2,z_3]\Big]\Big]\\&\quad-\Big[[x_1,x_2,x_3],[x_1^*,x_2^*,x_3^*],\Big[[y_1^*,y_2^*,y_3^*],[y_1,y_2,y_3],f([z_1,z_2,z_3])\Big]\Big]\Big\|\\&\quad\le\theta(\|x_1\|^p+\|x_2\|^p+\|x_3\|^p+\|y_1\|^p+\|y_2\|^p+\|y_3\|^p+\|z_1\|^p+\|z_2\|^p+\|z_3\|^p)\end{aligned}$$

for all $\mu\in T$ and all $a,b,x_1,x_2,x_3,y_1,y_2,y_3,z_1,z_2,z_3\in A$. Then there exists a unique ternary quadratic 3-derivation $D:A\to A$ satisfying

$$\|f(a)-D(a)\|\le\frac{2^p\theta}{4-2^p}\|a\|^p$$

for all $a\in A$.

Proof. The result follows from Theorem 2.3 by putting

$$\begin{aligned}\varphi(a,b,x_1,x_2,x_3,y_1,y_2,y_3,z_1,z_2,z_3):=\theta(\|a\|^p+\|b\|\|^p+\|x_1\|^p+\|x_2\|^p+\|x_3\|^p\\+\|y_1\|^p+\|y_2\|^p+\|y_3\|^p+\|z_1\|^p+\|z_2\|^p+\|z_3\|^p)\end{aligned}$$

for all $a,b,x_1,x_2,x_3,y_1,y_2,y_3,z_1,z_2,z_3\in A$. □

Now, we establish the superstability of ternary quadratic 3-derivations on C^*-ternary rings as follows:

Corollary 2.5. *Let p,θ be nonnegative real numbers such that $11p<2$ and let f be a mapping on a C^*-ternary ring A with $f(0)=0$ and*

$$\left\|2f\left(\frac{\mu a+\mu b}{2}\right)+2f\left(\frac{\mu a-\mu b}{2}\right)-\mu^2(f(a)+f(b))\right\|\le\theta\cdot\|a\|^p\cdot\|b\|^p,\tag{12}$$

$$\begin{aligned}&\Big\|f\Big(\Big[[x_1,x_2,x_3],[y_1,y_2,y_3],[z_1,z_2,z_3]\Big]\Big)\\&\quad-\Big[f([x_1,x_2,x_3]),[y_1,y_2,y_3],\Big[[y_1^*,y_2^*,y_3^*],[z_1^*,z_2^*,z_3^*],[z_1,z_2,z_3]\Big]\Big]\\&\quad-\Big[[x_1,x_2,x_3],[x_1^*,x_2^*,x_3^*],\Big[f([y_1^*,y_2^*,y_3^*]),[z_1^*,z_2^*,z_3^*],[z_1,z_2,z_3]\Big]\Big]\\&\quad-\Big[[x_1,x_2,x_3],[x_1^*,x_2^*,x_3^*],\Big[[y_1^*,y_2^*,y_3^*],[y_1,y_2,y_3],f([z_1,z_2,z_3])\Big]\Big]\Big\|\\&\quad\le\theta\cdot\|x_1\|^p\cdot\|x_2\|^p\cdot\|x_3\|^p\cdot\|y_1\|^p\cdot\|y_2\|^p\cdot\|y_3\|^p\cdot\|z_1\|^p\cdot\|z_2\|^p\cdot\|z_3\|^p\end{aligned}$$

for all $\mu\in T$ and all $a,b,x_1,x_2,x_3,y_1,y_2,y_3,z_1,z_2,z_3\in A$. Then f is a ternary quadratic 3-derivation on A.

Proof. Putting $a=b=0$ in (12), we get $f(0)=0$. Now, if we put $b=0$, $\mu=1$ and replace a by $2a$ in (12), then we have $f(2a)=4f(a)$ for all $a\in A$. It is easy to see by induction that

$f(2^n a) = 4^n f(a)$, and so $f(a) = \frac{f(2^n a)}{4^n}$ for all $a \in A$ and $n \in N$. It follows from Theorem 2.3 that f is a quadratic mapping. Putting

$$\varphi(a, b, x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3) := \theta \cdot \|a\|^p \cdot \|b\|^p \cdot \|x_1\|^p \cdot \|x_2\|^p \cdot \|x_3\|^p \cdot \|y_1\|^p \cdot \|y_2\|^p \cdot \|y_3\|^p \cdot \|z_1\|^p \cdot \|z_2\|^p \cdot \|z_3\|^p$$

in Theorem 2.3, we can obtain the desired result. □

Theorem 2.6. *Let A be a ternary Banach algebra, and let $f : A \to A$ be a mapping with $f(0) = 0$, and also let $\varphi : A^5 \to [0, \infty)$ be a function such that*

$$\Big\|f(\mu a + \mu b) + f(\mu a - \mu b) - 2\mu^2(f(a) + f(b))\Big\| \leq \varphi(a, b, 0, 0, 0, 0, 0, 0, 0, 0, 0) \tag{13}$$

$$\begin{aligned}
&\Big\|f\Big(\Big[[x_1, x_2, x_3], [y_1, y_2, y_3], [z_1, z_2, z_3]\Big]\Big) \\
&- \Big[f([x_1, x_2, x_3]), [y_1, y_2, y_3], \Big[[y_1^*, y_2^*, y_3^*], [z_1^*, z_2^*, z_3^*], [z_1, z_2, z_3]\Big]\Big] \\
&- \Big[[x_1, x_2, x_3], [x_1^*, x_2^*, x_3^*], \Big[f([y_1^*, y_2^*, y_3^*]), [z_1^*, z_2^*, z_3^*], [z_1, z_2, z_3]\Big]\Big] \\
&- \Big[[x_1, x_2, x_3], [x_1^*, x_2^*, x_3^*], \Big[[y_1^*, y_2^*, y_3^*], [y_1, y_2, y_3], f([z_1, z_2, z_3])\Big]\Big]\Big\| \\
&\leq \varphi(0, 0, x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3)
\end{aligned} \tag{14}$$

for all $\mu \in T$ and all $a, b, x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$. If there exists a constant $m \in (0, 1)$ such that

$$\begin{aligned}
&\varphi(2a, 2b, 2x_1, 2x_2, 2x_3, 2y_1, 2y_2, 2y_3, 2z_1, 2z_2, 2z_3) \\
&\leq 4m\varphi(a, b, x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3)
\end{aligned} \tag{15}$$

for all $a, b, x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$, then there exists a unique ternary quadratic 3-derivation $D : A \to A$ satisfying

$$\|f(a) - D(a)\| \leq \frac{4m}{1 - m}\psi(a) \tag{16}$$

for all $a \in A$, where $\psi(a) = \varphi(a, a, 0, 0, 0, 0, 0, 0, 0, 0, 0)$.

Proof. Using (15), we obtain

$$\lim_{n\to\infty} \frac{\varphi(2^n a, 2^n b, 2^n x_1, 2^n x_2, 2^n x_3, 2^n y_1, 2^n y_2, 2^n y_3, 2^n z_1, 2^n z_2, 2^n z_3)}{4^n} = 0 \tag{17}$$

for all $a, b, x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$. Putting $\mu = 1, a = b$ and replacing a by $2a$ in (13), we get

$$\Big\|f(2a) - 4f(a)\Big\| \leq \psi(a)$$

for all $a \in A$. By the last inequality, we have

$$\Big\|\frac{1}{4}f(2a) - f(a)\Big\| \leq \frac{1}{4}\psi(a) \tag{18}$$

for all $a \in A$. Similar to the proof of Theorem 2.3, we consider the set $\Omega := \{h : A \to A | h(0) = 0\}$ and introduce a generalized metric on Ω by

$$d(g, h) := inf\{C \in (0, \infty) : \|g(a) - h(a)\| \leq C\psi(a), \forall a \in A\},$$

if there exists such constant C, and $d(g,h)=\infty$, otherwise. Again, it is easy to check the fact that (Ω,d) is a complete metric space. We now define the linear mapping $T:\Omega\to\Omega$ by

$$T(h)(a)=\frac{1}{4}h(2a) \tag{19}$$

for all $a\in A$. For arbitrary elements $g,h\in\Omega$ and $C\in(0,\infty)$ with $d(g,h)\le C$, we have

$$\|g(a)-h(a)\|\le C\psi(a) \tag{20}$$

for all $a\in A$. Replacing a by 2a in the inequality (20) and using (15) and (19), we we have

$$\|(Tg)(a)-(Th)(a)\|=\frac{1}{4}\|G(2a)-h(2a)\|\le\frac{1}{4}C\psi(2a)\le Cm\psi(a)$$

for all $a\in A$. Thus $d(Tg,Th)\le Cm$. Therefore, we conclude that $d(Tg,Th)\le md(g,h)$ for all $g,h\in\Omega$. It follows from (18) that

$$d(Tf,f)\le\frac{1}{4}. \tag{21}$$

Hence T is a strictly contractive mapping on Ω. Now, Theorem 2.2 shows that T has a unique fixed point $D:A\to A$ in the set $\Omega_1=\{h\in\Omega,d(f,h)<\infty\}$. On the other hand,

$$\lim_{n\to\infty}\frac{2^na}{4^n}=D(a) \tag{22}$$

for all $a\in A$. By Theorem 2.2 and (21), we obtain

$$d(f,D)\le\frac{d(Tf,f)}{1-m}\le\frac{m}{4(1-m)},$$

i.e., the inequality (16) is true for all $a\in A$. Let us replace a and b in (13) by 2^na and 2^nb respectively, and then divide both sides by 2^n. Passing to the limit as $n\to\infty$, we get

$$D(\mu a+\mu b)+D(\mu a-\mu b)=2\mu^2D(a)+2\mu^2D(b) \tag{23}$$

for all $a,b\in A$ and $\lambda\in T$. Putting $\mu=1$ in (23), we have

$$D(a+b)+D(a-b)=2D(a)+2D(b) \tag{24}$$

for all $a,b\in A$. Hence D is a quadratic mapping.

It follows from (14) that

$$\begin{aligned}\Bigg\|\frac{1}{4^{9n}}f\Big(\Big[[2^nx_1,2^nx_2,2^nx_3],[2^ny_1,2^ny_2,2^ny_3],[2^nz_1,2^nz_2,2^nz_3]\Big]\Big)\\ -\Big[\frac{1}{4^{3n}}f([2^nx_1,2^nx_2,2^nx_3]),[y_1,y_2,y_3],\Big[[y_1^*,y_2^*,y_3^*],[z_1^*,z_2^*,z_3^*],[z_1,z_2,z_3]\Big]\Big]\\ -\Big[[x_1,x_2,x_3],[x_1^*,x_2^*,x_3^*],\Big[\frac{1}{4^{3n}}f([2^ny_1^*,2^ny_2^*,2^ny_3^*]),[z_1^*,z_2^*,z_3^*],[z_1,z_2,z_3]\Big]\Big]\\ -\Big[[x_1,x_2,x_3],[x_1^*,x_2^*,x_3^*],\Big[[y_1^*,y_2^*,y_3^*],[y_1,y_2,y_3],\frac{1}{4^{3n}}f([2^nz_1,2^nz_2,2^nz_3])\Big]\Big]\Bigg\|\\ \le\frac{1}{4^{9n}}\varphi(0,0,2^nx_1,2^nx_2,2^nx_3,2^ny_1,2^ny_2,2^ny_3,2^nz_1,2^nz_2,2^nz_3)\end{aligned} \tag{25}$$

for all $x_1,x_2,x_3,y_1,y_2,y_3,z_1,z_2,z_3\in A$.

Taking the limit in the equality (25) and using (17), one obtains that

$$\begin{aligned}
&D\left(\Big[[x_1,x_2,x_3],[y_1,y_2,y_3],[z_1,z_2,z_3]\Big]\right)\\
&=\Big[D([x_1,x_2,x_3]),[y_1,y_2,y_3],\Big[[y_1^*,y_2^*,y_3^*],[z_1^*,z_2^*,z_3^*],[z_1,z_2,z_3]\Big]\Big]\\
&+\Big[[x_1,x_2,x_3],[x_1^*,x_2^*,x_3^*],\Big[D([y_1^*,y_2^*,y_3^*]),[z_1^*,z_2^*,z_3^*],[z_1,z_2,z_3]\Big]\Big]\\
&+\Big[[x_1,x_2,x_3],[x_1^*,x_2^*,x_3^*],\Big[[y_1^*,y_2^*,y_3^*],[y_1,y_2,y_3],D([z_1,z_2,z_3])\Big]\Big]
\end{aligned}$$

for all $a,b,x_1,x_2,x_3,y_1,y_2,y_3,z_1,z_2,z_3\in A$. Therefore, D is a ternary quadratic 3-derivation. This completes the proof. □

The following corollaries are some applications to show the stability and superstability of ternary quadratic 3-derivations under some conditions.

Corollary 2.7. *Let A be a ternary Banach algebra. Let p,θ be nonnegative real numbers such that $p<2$ and let f be a mapping on a C^*-ternary ring A with $f(0)=0$ and*

$$\Big\|f(\mu a+\mu b)+f(\mu a-\mu b)-2\mu^2(f(a)+f(b))\Big\|\le\theta(\|a\|^p+\|b\|\|^p),$$

$$\begin{aligned}
&\Big\|f\left(\Big[[x_1,x_2,x_3],[y_1,y_2,y_3],[z_1,z_2,z_3]\Big]\right)\\
&-\Big[f([x_1,x_2,x_3]),[y_1,y_2,y_3],\Big[[y_1^*,y_2^*,y_3^*],[z_1^*,z_2^*,z_3^*],[z_1,z_2,z_3]\Big]\Big]\\
&-\Big[[x_1,x_2,x_3],[x_1^*,x_2^*,x_3^*],\Big[f([y_1^*,y_2^*,y_3^*]),[z_1^*,z_2^*,z_3^*],[z_1,z_2,z_3]\Big]\Big]\\
&-\Big[[x_1,x_2,x_3],[x_1^*,x_2^*,x_3^*],\Big[[y_1^*,y_2^*,y_3^*],[y_1,y_2,y_3],f([z_1,z_2,z_3])\Big]\Big]\Big\|\\
&\le\theta(\|x_1\|^p+\|x_2\|^p+\|x_3\|^p+\|y_1\|^p+\|y_2\|^p+\|y_3\|^p+\|z_1\|^p+\|z_2\|^p+\|z_3\|^p)
\end{aligned}$$

for all $\mu\in T$ and all $a,b,x_1,x_2,x_3,y_1,y_2,y_3,z_1,z_2,z_3\in A$. Then there exists a unique ternary quadratic 3-derivation $D:A\to A$ satisfying

$$\|f(a)-D(a)\|\le\frac{2^p\theta}{4-2^p}\|a\|^p$$

for all $a\in A$.

Proof. The result follows from Theorem 2.6 by putting

$$\begin{aligned}
\varphi(a,b,x_1,x_2,x_3,y_1,y_2,y_3,z_1,z_2,z_3):=\theta(&\|a\|^p+\|b\|\|^p+\|x_1\|^p+\|x_2\|^p+\|x_3\|^p\\
&+\|y_1\|^p+\|y_2\|^p+\|y_3\|^p+\|z_1\|^p+\|z_2\|^p+\|z_3\|^p)
\end{aligned}$$

for all $a,b,x_1,x_2,x_3,y_1,y_2,y_3,z_1,z_2,z_3\in A$. □

Corollary 2.8. *Let p,θ be nonnegative real numbers such that $11p<2$ and let f be a mapping on a C^*-ternary ring A with $f(0)=0$ and*

$$\Big\|f(\mu a+\mu b)+f(\mu a-\mu b)-2\mu^2(f(a)+f(b))\Big\|\le\theta(\|a\|^p\|b\|^p),\tag{26}$$

$$\Big\| f\left(\Big[[x_1,x_2,x_3],[y_1,y_2,y_3],[z_1,z_2,z_3]\Big]\right) - \Big[f([x_1,x_2,x_3]),[y_1,y_2,y_3],\Big[[y_1^*,y_2^*,y_3^*],[z_1^*,z_2^*,z_3^*],[z_1,z_2,z_3]\Big]\Big] - \Big[[x_1,x_2,x_3],[x_1^*,x_2^*,x_3^*],\Big[f([y_1^*,y_2^*,y_3^*]),[z_1^*,z_2^*,z_3^*],[z_1,z_2,z_3]\Big]\Big] - \Big[[x_1,x_2,x_3],[x_1^*,x_2^*,x_3^*],\Big[[y_1^*,y_2^*,y_3^*],[y_1,y_2,y_3],f([z_1,z_2,z_3])\Big]\Big]\Big\| \leq \theta\cdot\|x_1\|^p\cdot\|x_2\|^p\cdot\|x_3\|^p\cdot\|y_1\|^p\cdot\|y_2\|^p\cdot\|y_3\|^p\cdot\|z_1\|^p\cdot\|z_2\|^p\cdot\|z_3\|^p$$

for all $\mu \in T$ and all $a, b, x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$. Then f is a ternary quadratic 3-derivation on A.

Proof. If we put $a = b = 0$ in (26), then we have $f(0) = 0$. Moreover, letting $b = 0$, $\mu = 1$ and replacing a by $2a$ in (26), we obtain $f(2a) = 4f(a)$ for all $a \in A$. Similar to the proof of Corollary 2.5, we can show that f is a quadratic mapping. Putting

$$\varphi(a,b,x_1,x_2,x_3,y_1,y_2,y_3,z_1,z_2,z_3) := \theta\cdot\|a\|^p\cdot\|b\|^p\cdot\|x_1\|^p\cdot\|x_2\|^p\cdot\|x_3\|^p\cdot\|y_1\|^p\cdot\|y_2\|^p\cdot\|y_3\|^p\cdot\|z_1\|^p\cdot\|z_2\|^p\cdot\|z_3\|^p$$

in Theorem 2.6, we can obtain the desired result. □

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Existence results for a coupled system of fractional differential equations with multiple orders of fractional derivatives and integrals

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Abstract

In this paper, we introduce a new class of coupled systems of boundary value problems for fractional differential equations which contains multiple orders of fractional derivatives and integrals, and discuss the existence and uniqueness of solutions. We apply Leray-Schauder's alternative and Banach's contraction mapping principle to obtain the desired results. Illustrative examples is also included.

Key words and phrases: Fractional differential systems; nonlocal boundary conditions; integral boundary conditions; fixed point theorem.
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1 Introduction

Differential equations of fractional order have played a significant role in engineering, science, and pure and applied mathematics in recent years. Fractional differential equations arise in the mathematical modeling of systems and processes occurring in many engineering and scientific disciplines such as physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, etc. [1]-[4]. Fractional-order boundary value problems involving a variety of classical, nonlocal and integral boundary conditions have been addressed by many authors, for instance, see [5]-[13] and the references cited therein.

Coupled systems of fractional-order differential equations also constitute an interesting and important field of research in view of their applications in many real world problems such as anomalous diffusion [14], disease models [15]-[18], ecological models [19], synchronization of chaotic systems [20]-[22], etc. For some theoretical works on coupled systems of fractional-order differential equations, we refer the reader to a series of papers [23]-[28].

Recently in [29] a new class of fractional boundary valued problems was introduced, which contains four orders of Riemann-Liouville fractional derivatives, two in fractional differential equation and two

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in boundary conditions of the form

$$\begin{cases} \left(\lambda D^{\alpha} + (1-\lambda)D^{\beta}\right) x(t) = f(t, x(t)), \quad t \in (0,T), \\ x(0) = 0, \quad \mu D^{\gamma_1} x(T) + (1-\mu) D^{\gamma_2} x(T) = \gamma_3, \end{cases} \tag{1}$$

where D^{ϕ} is the Riemann-Liouville fractional derivative of order $\phi \in \{\alpha, \beta, \gamma_1, \gamma_2\}$ such that $1 < \alpha, \beta \le 2$ and $0 < \gamma_1, \gamma_2 < \alpha - \beta$, $\gamma_3 \in \mathbb{R}$, the given constants $0 < \lambda \le 1$, $0 \le \mu \le 1$ and $f \in C([0,T] \times \mathbb{R}, \mathbb{R})$ is a continuous function. Existence and uniqueness results were obtained by means of Banach's contraction mapping principle, Krasnoselskii's fixed point theorem and Leray-Schauder's nonlinear alternative.

In this paper, we study a coupled system of fractional differential equations

$$\begin{cases} \left(\lambda D^{\alpha} + (1-\lambda)D^{\beta}\right) x(t) = f(t, x(t), y(t)), \quad t \in (0,T), \quad 1 < \alpha, \beta \le 2 \\ \left(\lambda_1 D^{\alpha_1} + (1-\lambda_1)D^{\beta_1}\right) y(t) = g(t, x(t), y(t)), \quad t \in (0,T), \quad 1 < \alpha_1, \beta_1 \le 2, \end{cases} \tag{2}$$

subject to the following type of boundary conditions

$$\begin{cases} x(0) = 0, \quad \mu D^{\gamma_1} x(T) + (1-\mu) D^{\gamma_2} x(T) = \gamma_3, \\ y(0) = 0, \quad \mu_1 I^{\delta_1} y(T) + (1-\mu_1) I^{\delta_2} y(T) = \delta_3, \end{cases} \tag{3}$$

where D^{ϕ} denotes the Caputo fractional derivatives of order $\phi \in \{\alpha, \beta, \alpha_1, \beta_1, \gamma_1, \gamma_2\}$, I^{χ} denotes the Riemann-Liouville fractional integral of order $\chi \in \{\delta_1, \delta_2\}$, $\gamma_3, \delta_3 \in \mathbb{R}$, $0 < \lambda, \lambda_1 \le 1$, $0 \le \mu, \mu_1 \le 1$ and $f, g : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ are appropriately chosen functions.

The paper is organized as follows. In Section 2, we recall some basic definitions of fractional calculus and present two auxiliary lemmas. The main results are presented in Section 3. We give two results: the first one derives the existence of solutions via Leray-Schauder's alternative, whereas the second one concerning existence and uniqueness of solutions is established by Banach's contraction principle. We also discuss two examples for illustration of the existence-uniqueness results.

2 Preliminaries

Before presenting two auxiliary lemmas, we recall some basic definitions of fractional calculus [1, 2].

Definition 2.1 *For $(n-1)-$times absolutely continuous function $y : [0, \infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as*

$${}^{c}D^{q} y(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} y^{(n)}(s) ds, \quad n-1 < q < n, n = [q]+1,$$

where $[q]$ denotes the integer part of the real number q.

Definition 2.2 *The Riemann-Liouville fractional integral of order q is defined as*

$$I^{q} y(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{y(s)}{(t-s)^{1-q}} ds, \quad q > 0,$$

provided the integral exists.

Lemma 2.3 *The boundary value problem*

$$\begin{cases} \left(\lambda D^{\alpha} + (1-\lambda)D^{\beta}\right) x(t) = \omega(t), \quad t \in (0,T), \\ x(0) = 0, \quad \mu D^{\gamma_1} x(T) + (1-\mu) D^{\gamma_2} x(T) = \gamma_3, \end{cases} \tag{4}$$

is equivalent to the following integral equation

$$x(t) \quad = \quad \frac{\lambda - 1}{\lambda \Gamma(\alpha - \beta)} \int_0^t (t-s)^{\alpha-\beta-1} x(s) ds + \frac{1}{\lambda \Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \omega(s) ds$$

$$\begin{aligned}
&+\frac{t}{\Lambda_1}\Bigg(\gamma_3-\frac{\mu(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma_1)}\int_0^T(T-s)^{\alpha-\beta-\gamma_1-1}x(s)ds\\
&-\frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)}\int_0^T(T-s)^{\alpha-\gamma_1-1}\omega(s)ds\\
&-\frac{(1-\mu)(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma_2)}\int_0^T(T-s)^{\alpha-\beta-\gamma_2-1}x(s)ds\\
&-\frac{1-\mu}{\lambda\Gamma(\alpha-\gamma_2)}\int_0^T(T-s)^{\alpha-\gamma_2-1}\omega(s)ds\Bigg),\quad t\in J:=[0,T], \qquad (5)
\end{aligned}$$

where the non zero constant Λ_1 *is defined by*

$$\Lambda_1=\frac{\mu T^{1-\gamma_1}}{\Gamma(2-\gamma_1)}+\frac{(1-\mu)T^{1-\gamma_2}}{\Gamma(2-\gamma_2)}. \qquad (6)$$

Proof. The first equation of (4) can be rewritten as

$$D^\alpha x(t)=\frac{\lambda-1}{\lambda}D^\beta x(t)+\frac{1}{\lambda}\omega(t),\quad t\in J. \qquad (7)$$

Applying the Riemann-Liouville fractional integral of order α to both sides of (7), we obtain

$$x(t)\;=\;\frac{\lambda-1}{\lambda\Gamma(\alpha-\beta)}\int_0^t(t-s)^{\alpha-\beta-1}x(s)ds+\frac{1}{\lambda\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\omega(s)ds+C_1+C_2t,$$

where constants C_1, $C_2\in\mathbb{R}$. The first boundary condition of (4) implies that $C_1=0$. Hence

$$x(t)=\frac{\lambda-1}{\lambda\Gamma(\alpha-\beta)}\int_0^t(t-s)^{\alpha-\beta-1}x(s)ds+\frac{1}{\lambda\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\omega(s)ds+C_2t. \qquad (8)$$

Taking the Caputo fractional derivative of order $\psi\in\{\gamma_1,\gamma_2\}$ such that $0<\psi<\alpha-\beta$ to (8), we deduce that

$$\begin{aligned}
D^\psi x(t)\;=\;&\frac{\lambda-1}{\lambda\Gamma(\alpha-\beta-\psi)}\int_0^t(t-s)^{\alpha-\beta-\psi-1}x(s)ds\\
&+\frac{1}{\lambda\Gamma(\alpha-\psi)}\int_0^t(t-s)^{\alpha-\psi-1}\omega(s)ds+C_2\frac{1}{\Gamma(2-\psi)}t^{1-\psi}.
\end{aligned}$$

Substituting the values $\psi=\gamma_1$ and $\psi=\gamma_2$ to the above relation and using the second condition of (4), we obtain a constant γ_3 as

$$\begin{aligned}
\gamma_3\;=\;&\frac{\mu(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma_1)}\int_0^T(T-s)^{\alpha-\beta-\gamma_1-1}x(s)ds\\
&+\frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)}\int_0^T(T-s)^{\alpha-\gamma_1-1}\omega(s)ds+\frac{\mu T^{1-\gamma_1}}{\Gamma(2-\gamma_1)}C_2\\
&+\frac{(1-\mu)(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma_2)}\int_0^T(T-s)^{\alpha-\beta-\gamma_2-1}x(s)ds\\
&+\frac{1-\mu}{\lambda\Gamma(\alpha-\gamma_2)}\int_0^T(T-s)^{\alpha-\gamma_2-1}\omega(s)ds+\frac{(1-\mu)T^{1-\gamma_2}}{\Gamma(2-\gamma_2)}C_2,
\end{aligned}$$

which yields

$$C_2\;=\;\frac{1}{\Lambda_1}\Bigg[\gamma_3-\frac{\mu(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma_1)}\int_0^T(T-s)^{\alpha-\beta-\gamma_1-1}x(s)ds-\frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)}\int_0^T(T-s)^{\alpha-\gamma_1-1}\omega(s)ds$$

$$- \frac{(1-\mu)(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma_2)} \int_0^T (T-s)^{\alpha-\beta-\gamma_2-1} x(s)ds - \frac{1-\mu}{\lambda\Gamma(\alpha-\gamma_2)} \int_0^T (T-s)^{\alpha-\gamma_2-1} \omega(s)ds \Bigg].$$

Substituting the value of the constant C_2 into (8), we deduce the integral equation (5). The converse follows by direct computation. This completes the proof. □

In the same way, we obtain the following result with the Riemann-Liouville fractional integal boundary conditions.

Lemma 2.4 *The boundary value problem*

$$\begin{cases} \left(\lambda_1 D^{\alpha_1} + (1-\lambda_1) D^{\beta_1}\right) y(t) = \omega_1(t), \quad t \in (0,T), \\ y(0) = 0, \quad \mu_1 I^{\delta_1} y(T) + (1-\mu_1) I^{\delta_2} y(T) = \delta_3, \end{cases} \tag{9}$$

is equivalent to the following integral equation

$$\begin{aligned} y(t) \quad = \quad & \frac{\lambda_1 - 1}{\lambda_1\Gamma(\alpha_1-\beta_1)} \int_0^t (t-s)^{\alpha_1-\beta_1-1} y(s)ds + \frac{1}{\lambda_1\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1-1} \omega_1(s)ds \\ & + \frac{t}{\Lambda_2}\Bigg(\delta_3 - \frac{\mu_1(\lambda_1-1)}{\lambda_1\Gamma(\delta_1+\alpha_1-\beta_1)} \int_0^T (T-s)^{\delta_1+\alpha_1-\beta_1-1} y(s)ds \\ & - \frac{\mu_1}{\lambda_1\Gamma(\delta_1+\alpha_1)} \int_0^T (T-s)^{\delta_1+\alpha_1-1} \omega_1(s)ds \\ & - \frac{(1-\mu_1)(\lambda_1-1)}{\lambda_1\Gamma(\delta_2+\alpha_1-\beta_1)} \int_0^T (T-s)^{\delta_2+\alpha_1-\beta_1-1} y(s)ds \\ & - \frac{1-\mu_1}{\lambda_1\Gamma(\delta_2+\alpha_1)} \int_0^T (T-s)^{\delta_2+\alpha_1-1} \omega_1(s)ds \Bigg), \quad t \in J, \end{aligned} \tag{10}$$

where the non zero constant Λ_2 is defined by

$$\Lambda_2 = \frac{\mu_1 T^{1+\delta_1}}{\Gamma(2+\delta_1)} + \frac{(1-\mu_1)T^{1+\delta_2}}{\Gamma(2+\delta_2)}. \tag{11}$$

3 Main Results

Let us introduce the space $X = \{u(t) \,|\, u(t) \in C(J,\mathbb{R})\}$ endowed with the norm $\|u\| = \sup\{|u(t)|, t \in J\}$. Obviously $(X, \|\cdot\|)$ is a Banach space. Also $Y = \{v(t)|v(t) \in C(J,\mathbb{R})\}$ endowed with the norm $\|v\| = \sup\{|v(t)|, t \in J\}$ is a Banach space. Then the product space $(X \times Y, \|(u,v)\|)$ is also a Banach space equipped with norm $\|(u,v)\| = \|u\| + \|v\|$.

In view of Lemmas 2.3 and 2.4, we define the operator $\mathcal{T} : X \times Y \to X \times Y$ by

$$\mathcal{T}(u,v)(t) = \begin{pmatrix} \mathcal{T}_1(u,v)(t) \\ \mathcal{T}_2(u,v)(t) \end{pmatrix},$$

where

$$\begin{aligned} \mathcal{T}_1(u,v)(t) \quad = \quad & \frac{\lambda-1}{\lambda\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} u(s)ds + \frac{1}{\lambda\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,u(s),v(s))ds \\ & + \frac{t}{\Lambda_1}\Bigg(\gamma_3 - \frac{\mu(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma_1)} \int_0^T (T-s)^{\alpha-\beta-\gamma_1-1} u(s)ds \\ & - \frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)} \int_0^T (T-s)^{\alpha-\gamma_1-1} f(s,u(s),v(s))ds \end{aligned}$$

$$
\begin{aligned}
&- \frac{(1-\mu)(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma_2)} \int_0^T (T-s)^{\alpha-\beta-\gamma_2-1} u(s)ds \\
&- \frac{1-\mu}{\lambda\Gamma(\alpha-\gamma_2)} \int_0^T (T-s)^{\alpha-\gamma_2-1} f(s,u(s),v(s))ds \Bigg),
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{T}_2(u,v)(t) &= \frac{\lambda_1-1}{\lambda_1\Gamma(\alpha_1-\beta_1)} \int_0^t (t-s)^{\alpha_1-\beta_1-1} v(s)ds + \frac{1}{\lambda_1\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1-1} g(s,u(s),v(s))ds \\
&+ \frac{t}{\Lambda_2}\Bigg(\delta_3 - \frac{\mu_1(\lambda_1-1)}{\lambda_1\Gamma(\delta_1+\alpha_1-\beta_1)} \int_0^T (T-s)^{\delta_1+\alpha_1-\beta_1-1} v(s)ds \\
&- \frac{\mu_1}{\lambda_1\Gamma(\delta_1+\alpha_1)} \int_0^T (T-s)^{\delta_1+\alpha_1-1} g(s,u(s),v(s))ds \\
&- \frac{(1-\mu_1)(\lambda_1-1)}{\lambda_1\Gamma(\delta_2+\alpha_1-\beta_1)} \int_0^T (T-s)^{\delta_2+\alpha_1-\beta_1-1} v(s)ds \\
&- \frac{1-\mu_1}{\lambda_1\Gamma(\delta_2+\alpha_1)} \int_0^T (T-s)^{\delta_2+\alpha_1-1} g(s,u(s),v(s))ds \Bigg).
\end{aligned}
$$

Let us introduce the following hypotheses which are used hereafter.

(H_1) Assume that there exist real constants $k_i,\ \nu_i \geq 0\ (i=1,2)$ and $k_0>0, \nu_0>0$ such that $\forall x_i \in \mathbb{R}$, $(i=1,2)$ we have

$$|f(t,x_1,x_2)| \leq k_0 + k_1|x_1| + k_2|x_2|,$$

$$|g(t,x_1,x_2)| \leq \nu_0 + \nu_1|x_1| + \nu_2|x_2|.$$

(H_2) Assume that $f,h : J\times\mathbb{R}^2 \to \mathbb{R}$ are continuous functions and there exist constants $m_i, n_i,\ i=1,2$ such that for all $t\in J$ and $u_i, v_i \in \mathbb{R},\ i=1,2,$

$$|f(t,u_1,u_2) - f(t,v_1,v_2)| \leq m_1|u_1-v_1| + m_2|u_2-v_2|$$

and

$$|g(t,u_1,u_2) - g(t,v_1,v_2)| \leq n_1|u_1-v_1| + n_2|u_2-v_2|.$$

For the sake of convenience, we set constants

$$M_1 = \frac{T^\alpha}{\lambda\Gamma(\alpha+1)} + \frac{T^{\alpha-\gamma_1+1}\mu}{\lambda\Lambda_1\Gamma(\alpha-\gamma_1+1)} + \frac{T^{\alpha-\gamma_2+1}(1-\mu)}{\lambda\Lambda_1\Gamma(\alpha-\gamma_2+1)}, \tag{12}$$

$$N_1 = \frac{T^{\alpha-\beta}|\lambda-1|}{\lambda\Gamma(\alpha-\beta+1)} + \frac{T^{\alpha-\beta-\gamma_1+1}\mu|\lambda-1|}{\lambda\Lambda_1\Gamma(\alpha-\beta-\gamma_1+1)} + \frac{T^{\alpha-\beta-\gamma_2+1}(1-\mu)|\lambda-1|}{\lambda\Lambda_1\Gamma(\alpha-\beta-\gamma_2+1)}, \tag{13}$$

$$M_2 = \frac{T^{\alpha_1}}{\lambda_1\Gamma(\alpha_1+1)} + \frac{T^{\delta_1+\alpha_1+1}\mu_1}{\lambda_1\Lambda_2\Gamma(\delta_1+\alpha_1+1)} + \frac{T^{\delta_2+\alpha_1+1}(1-\mu_1)}{\lambda_1\Lambda_2\Gamma(\delta_2+\alpha_1+1)}, \tag{14}$$

$$N_2 = \frac{T^{\alpha_1-\beta_1}|\lambda_1-1|}{\lambda_1\Gamma(\alpha_1-\beta_1+1)} + \frac{T^{\delta_1+\alpha_1-\beta_1+1}\mu_1|\lambda_1-1|}{\lambda_1\Lambda_2\Gamma(\delta_1+\alpha_1-\beta_1+1)} + \frac{T^{\delta_2+\alpha_1-\beta_1+1}(1-\mu_1)|\lambda_1-1|}{\lambda_1\Lambda_2\Gamma(\delta_2+\alpha_1-\beta_1+1)} \tag{15}$$

and

$$M_0 = \min\{1-(M_1k_1+N_1+M_2\nu_1),\ 1-(M_1k_2+M_2\nu_2+N_2)\},\quad k_i,\ \nu_i \geq 0\ (i=1,2). \tag{16}$$

The first result is based on Leray-Schauder alternative.

Lemma 3.1 *(Leray-Schauder alternative) ([30] p. 4.) Let $F : E \to E$ be a completely continuous operator (i.e., a map that restricted to any bounded set in E is compact). Let*

$$\mathcal{E}(F) = \{x \in E : x = \lambda F(x) \text{ for some } 0 < \lambda < 1\}.$$

Then either the set $\mathcal{E}(F)$ is unbounded, or F has at least one fixed point.

Theorem 3.2 *Assume that (H_1) holds. In addition, it is assumed that*

$$M_1k_1 + N_1 + M_2\nu_1 < 1 \quad \text{and} \quad M_1k_2 + M_2\nu_2 + N_2 < 1,$$

where M_1 and M_2 are given by (13) and (15) respectively. Then the system (2)-(3) has at least one solution.

Proof. First we show that the operator $\mathcal{T} : X \times Y \to X \times Y$ is completely continuous. By continuity of functions f and g, the operator $\mathcal{T}$ is continuous.

Let $\Omega = \{(u, v) \in X \times Y : \|(u, v)\| \le r\} \subset X \times Y$ be a bounded set. Then there exist positive constants L_1 and L_2 such that

$$|f(t, u(t), v(t))| \le L_1, \quad |g(t, u(t), v(t))| \le L_2, \quad \forall (u, v) \in \Omega.$$

Then for any $(u, v) \in \Omega$, we have

$$\begin{aligned}
|\mathcal{T}_1(u, v)(t)| &\le \frac{|\lambda - 1|}{\lambda\Gamma(\alpha - \beta)} \int_0^t (t - s)^{\alpha-\beta-1}|u(s)|ds + \frac{1}{\lambda\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1}|f(s, u(s), v(s))|ds \\
&\quad + \frac{t}{\Lambda_1}\left(|\gamma_3| + \frac{\mu(\lambda - 1)}{\lambda\Gamma(\alpha - \beta - \gamma_1)} \int_0^T (T - s)^{\alpha-\beta-\gamma_1-1}|u(s)|ds\right. \\
&\quad + \frac{\mu}{\lambda\Gamma(\alpha - \gamma_1)} \int_0^T (T - s)^{\alpha-\gamma_1-1}|f(s, u(s), v(s))|ds \\
&\quad + \frac{(1 - \mu)|\lambda - 1|}{\lambda\Gamma(\alpha - \beta - \gamma_2)} \int_0^T (T - s)^{\alpha-\beta-\gamma_2-1}|u(s)|ds \\
&\quad \left. + \frac{1 - \mu}{\lambda\Gamma(\alpha - \gamma_2)} \int_0^T (T - s)^{\alpha-\gamma_2-1}|f(s, u(s), v(s))|ds\right) \\
&\le L_1\left[\frac{T^\alpha}{\lambda\Gamma(\alpha + 1)} + \frac{T^{\alpha-\gamma_1+1}\mu}{\lambda\Lambda_1\Gamma(\alpha - \gamma_1 + 1)} + \frac{T^{\alpha-\gamma_2+1}(1 - \mu)}{\lambda\Lambda_1\Gamma(\alpha - \gamma_2 + 1)}\right] \\
&\quad + \|u\|\left[\frac{T^{\alpha-\beta}|\lambda - 1|}{\lambda\Gamma(\alpha - \beta + 1)} + \frac{T^{\alpha-\beta-\gamma_1+1}\mu|\lambda - 1|}{\lambda\Lambda_1\Gamma(\alpha - \beta - \gamma_1 + 1)}\right. \\
&\quad \left. + \frac{T^{\alpha-\beta-\gamma_2+1}(1 - \mu)|\lambda - 1|}{\lambda\Lambda_1\Gamma(\alpha - \beta - \gamma_2 + 1)}\right] + \frac{|\gamma_3|T}{\Lambda_1} \\
&= L_1M_1 + N_1r + |\gamma_3|T/\Lambda_1
\end{aligned}$$

and consequently,

$$\|\mathcal{T}_1(u, v)\| \le L_1M_1 + N_1r + |\gamma_3|T/\Lambda_1.$$

Similarly, we get

$$\|\mathcal{T}_2(u, v)\| \le L_2M_2 + N_2r + |\delta_3|T/\Lambda_2.$$

Thus, it follows from the above inequalities that the operator $\mathcal{T}$ is uniformly bounded.

Next, we show that $\mathcal{T}$ is equicontinuous. Let $t_1, t_2 \in [0, T]$ with $t_1 < t_2$. Then we have

$$|\mathcal{T}_1(u(t_2), v(t_2)) - \mathcal{T}_1(u(t_1), v(t_1))|$$

$$\begin{aligned}
\leq\ & \frac{|\lambda-1|}{\lambda\Gamma(\alpha-\beta)}\left[\int_0^{t_2}(t_2-s)^{\alpha-\beta-1}u(s)ds-\int_0^{t_1}(t_1-s)^{\alpha-\beta-1}u(s)ds\right]\\
&+\frac{1}{\lambda\Gamma(\alpha)}\left[\int_0^{t_2}(t_2-s)^{\alpha-1}f(s,u(s),v(s))ds-\int_0^{t_1}(t_1-s)^{\alpha-1}f(s,u(s),v(s))ds\right]\\
&+\frac{|t_2-t_1|}{\Lambda_1}\Bigg(|\gamma_3|+\frac{\mu|\lambda-1|}{\lambda\Gamma(\alpha-\beta-\gamma_1)}\int_0^T(T-s)^{\alpha-\beta-\gamma_1-1}u(s)ds\\
&+\frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)}\int_0^T(T-s)^{\alpha-\gamma_1-1}f(s,u(s),v(s))ds\\
&+\frac{(1-\mu)|\lambda-1|}{\lambda\Gamma(\alpha-\beta-\gamma_2)}\int_0^T(T-s)^{\alpha-\beta-\gamma_2-1}u(s)ds\\
&+\frac{1-\mu}{\lambda\Gamma(\alpha-\gamma_2)}\int_0^T(T-s)^{\alpha-\gamma_2-1}f(s,u(s),v(s))ds\Bigg)\\
\leq\ & r\Bigg[\frac{[2(t_2-t_1)^{\alpha-\beta}+|t_2^{\alpha-\beta}-t_1^{\alpha-\beta}|]|\lambda-1|}{\lambda\Gamma(\alpha-\beta+1)}+\frac{|t_2-t_1|}{\Lambda_1}\Bigg(\frac{|\lambda-1|\mu T^{\alpha-\beta-\gamma_1}}{\lambda\Gamma(\alpha-\beta-\gamma_1+1)}\\
&+\frac{|\lambda-1|(1-\mu)T^{\alpha-\beta-\gamma_2}}{\lambda\Gamma(\alpha-\beta-\gamma_2+1)}\Bigg)\Bigg]+\frac{|t_2-t_1||\gamma_3|}{\Lambda_1}\\
&+L_1\left[\frac{2(t_2-t_1)^{\alpha-1}+|t_2^{\alpha-1}-t_1^{\alpha-1}|}{\lambda\Gamma(\alpha+1)}+\frac{|t_2-t_1|}{\Lambda_1}\left(\frac{\mu T^{\alpha-\gamma_1}}{\lambda\Gamma(\alpha-\gamma_1+1)}+\frac{(1-\mu)T^{\alpha-\gamma_2}}{\lambda\Gamma(\alpha-\gamma_2+1)}\right)\right].
\end{aligned}$$

Analogously, we can obtain

$$\begin{aligned}
&|\mathcal{T}_2(u(t_2),v(t_2))-\mathcal{T}_2(u(t_1),v(t_1))|\\
\leq\ & r\Bigg[\frac{[2(t_2-t_1)^{\alpha_1-\beta_1}+|t_2^{\alpha_1-\beta_1}-t_1^{\alpha_1-\beta_1}|]|\lambda_1-1|}{\lambda_1\Gamma(\alpha_1-\beta_1+1)}+\frac{|t_2-t_1|}{\Lambda_2}\Bigg(\frac{|\lambda_1-1|\mu_1T^{\delta_1+\alpha_1-\beta_1}}{\lambda_1\Gamma(\delta_1+\alpha_1-\beta_1+1)}\\
&+\frac{|\lambda_1-1|(1-\mu)T^{\delta_2+\alpha_1-\beta_1}}{\lambda_1\Gamma(\delta_2+\alpha_1-\beta_1+1)}\Bigg)\Bigg]+\frac{|t_2-t_1||\delta_3|}{\Lambda_2}\\
&+L_2\left[\frac{2(t_2-t_1)^{\alpha_1-1}+|t_2^{\alpha_1-1}-t_1^{\alpha_1-1}|}{\lambda_1\Gamma(\alpha_1+1)}+\frac{|t_2-t_1|}{\Lambda_2}\left(\frac{\mu_1T^{\delta_1+\alpha_1}}{\lambda_1\Gamma(\delta_1+\alpha_1+1)}+\frac{(1-\mu_1)T^{\delta_2+\alpha_1}}{\lambda_1\Gamma(\delta_2+\alpha_1+1)}\right)\right].
\end{aligned}$$

As $t_2-t_1\to 0$, the right-hand sides of the above inequalities tends to zero independently of $(u,v)\in\Omega$. Therefore, by the Arzelá-Ascoli theorem, the operator $\mathcal{T}(u,v)$ is equicontinuous, and thus the operator $\mathcal{T}(u,v)$ is completely continuous.

Finally, it will be verified that the set $\mathcal{E}=\{(u,v)\in X\times Y|(u,v)=\theta\mathcal{T}(u,v),0\leq\theta\leq 1\}$ is bounded. Let $(u,v)\in\mathcal{E}$, with $(u,v)=\theta T(u,v)$. For any $t\in[0,T]$, we have

$$u(t)=\theta\mathcal{T}_1(u,v)(t),\quad v(t)=\theta\mathcal{T}_2(u,v)(t).$$

Then

$$\begin{aligned}
|u(t)|\ \leq\ & (k_0+k_1\|u\|+k_2\|v\|)\left[\frac{T^\alpha}{\lambda\Gamma(\alpha+1)}+\frac{T^{\alpha-\gamma_1+1}\mu}{\lambda\Lambda_1\Gamma(\alpha-\gamma_1+1)}+\frac{T^{\alpha-\gamma_2+1}(1-\mu)}{\lambda\Lambda_1\Gamma(\alpha-\gamma_2+1)}\right]\\
&+\|u\|\left[\frac{T^{\alpha-\beta}|\lambda-1|}{\lambda\Gamma(\alpha-\beta+1)}+\frac{T^{\alpha-\beta-\gamma_1+1}\mu|\lambda-1|}{\lambda\Lambda_1\Gamma(\alpha-\beta-\gamma_1+1)}\right.\\
&\left.+\frac{T^{\alpha-\beta-\gamma_2+1}(1-\mu)|\lambda-1|}{\lambda\Lambda_1\Gamma(\alpha-\beta-\gamma_2+1)}\right]+\frac{|\gamma_3|T}{\Lambda_1}
\end{aligned}$$

and

$$\begin{aligned}|v(t)| \quad &\leq \quad (\nu_0+\nu_1\|u\|+\nu_2\|v\|)\left[\frac{T^{\alpha_1}}{\lambda_1\Gamma(\alpha_1+1)}+\frac{T^{\delta_1+\alpha_1+1}\mu_1}{\lambda_1\Lambda_2\Gamma(\delta_1+\alpha_1+1)}\right.\\ &\left.+\frac{T^{\delta_2+\alpha_1+1}(1-\mu_1)}{\lambda_1\Lambda_2\Gamma(\delta_2+\alpha_1+1)}\right]+\|v\|\left[\frac{T^{\alpha_1-\beta_1}|\lambda_1-1|}{\lambda_1\Gamma(\alpha_1-\beta_1+1)}\right.\\ &\left.+\frac{T^{\delta_1+\alpha_1-\beta_1+1}\mu_1|\lambda_1-1|}{\lambda_1\Lambda_2\Gamma(\delta_1+\alpha_1-\beta_1+1)}+\frac{T^{\delta_2+\alpha_1-\beta_1+1}(1-\mu_1)|\lambda_1-1|}{\lambda_1\Lambda_2\Gamma(\delta_2+\alpha_1-\beta_1+1)}\right]+\frac{|\delta_3|T}{\Lambda_2}.\end{aligned}$$

Hence we have

$$\|u\|\leq M_1(k_0+k_1\|u\|+k_2\|v\|)+N_1\|u\|+|\gamma_3|T/\Lambda_1$$

and

$$\|v\|\leq M_2(\nu_0+\nu_1\|u\|+\nu_2\|v\|)+N_2\|v\|+|\delta_3|T/\Lambda_2,$$

which imply that

$$\begin{aligned}\|u\|+\|v\| \quad &= \quad \Big(M_1k_0+M_2\nu_0+|\gamma_3|T/\Lambda_1+|\delta_3|T/\Lambda_2\Big)\\ &+(M_1k_1+N_1+M_2\nu_1)\|u\|+(M_1k_2+M_2\nu_2+N_2)\|v\|.\end{aligned}$$

Consequently,

$$\|(u,v)\|\leq\frac{M_1k_0+M_2\nu_0+|\gamma_3|T/\Lambda_1+|\delta_3|T/\Lambda_2}{M_0},$$

for any $t\in[0,T]$, where M_0 is defined by (16), which proves that $\mathcal{E}$ is bounded. Thus, by Lemma 3.1, the operator $\mathcal{T}$ has at least one fixed point. Hence the boundary value problem (2)-(3) has at least one solution. The proof is complete. □

In the second result, we prove existence and uniqueness of solutions of the boundary value problem (2)-(3) via Banach's contraction principle.

Theorem 3.3 *Assume that (H_2) holds. In addition, assume that*

$$M_1(m_1+m_2)+N_1+M_2(n_1+n_2)+N_2<1,$$

where M_1, N_1, M_2 and N_2 are given by (12) and (15), respectively. Then the system (2)-(3) has a unique solution on J.

Proof. Define $\sup_{t\in J}f(t,0,0)=F_0<\infty$ and $\sup_{t\in J}g(t,0,0)=G_0<\infty$ such that

$$r\geq\frac{N_1M_1+N_2M_2+|\gamma_3|T/\Lambda_1+|\delta_3|T/\Lambda_2}{1-M_1(m_1+m_2)-M_2(n_1+n_2)-(N_1+N_2)}.$$

We show that $\mathcal{T}B_r\subset B_r$, where $B_r=\{(u,v)\in X\times Y:\|(u,v)\|\leq r\}$.

For $(u,v)\in B_r$, we have

$$\begin{aligned}|\mathcal{T}_1(u,v)(t)| \quad &\leq \quad \frac{|\lambda-1|}{\lambda\Gamma(\alpha-\beta)}\int_0^T(T-s)^{\alpha-\beta-1}|u(s)|ds\\ &+\frac{1}{\lambda\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}[|f(s,u(s),v(s))-f(s,0,0)|+|f(s,0,0)|]ds\\ &+\frac{T}{\Lambda_1}\Bigg(|\gamma_3|+\frac{\mu|\lambda-1|}{\lambda\Gamma(\alpha-\beta-\gamma_1)}\int_0^T(T-s)^{\alpha-\beta-\gamma_1-1}|u(s)|ds\\ &+\frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)}\int_0^T(T-s)^{\alpha-\gamma_1-1}[|f(s,u(s),v(s))-f(s,0,0)|+|f(s,0,0)|]ds\end{aligned}$$

$$
\begin{aligned}
&+\frac{(1-\mu)|\lambda-1|}{\lambda\Gamma(\alpha-\beta-\gamma_2)}\int_0^T(T-s)^{\alpha-\beta-\gamma_2-1}|u(s)|ds\\
&+\frac{1-\mu}{\lambda\Gamma(\alpha-\gamma_2)}\int_0^T(T-s)^{\alpha-\gamma_2-1}[|f(s,u(s),v(s))-f(s,0)|+|f(s,0,0)|]ds\Bigg)\\
\leq\ & (m_1\|u\|+m_2\|v\|+F_0)\left[\frac{T^\alpha}{\lambda\Gamma(\alpha+1)}+\frac{T^{\alpha-\gamma_1+1}\mu}{\lambda\Lambda_1\Gamma(\alpha-\gamma_1+1)}+\frac{T^{\alpha-\gamma_2+1}(1-\mu)}{\lambda\Lambda_1\Gamma(\alpha-\gamma_2+1)}\right]\\
&+\|x\|\left[\frac{T^{\alpha-\beta}|\lambda-1|}{\lambda\Gamma(\alpha-\beta+1)}+\frac{T^{\alpha-\beta-\gamma_1+1}\mu|\lambda-1|}{\lambda\Lambda_1\Gamma(\alpha-\beta-\gamma_1+1)}\right.\\
&\left.+\frac{T^{\alpha-\beta-\gamma_2+1}(1-\mu)|\lambda-1|}{\lambda\Lambda_1\Gamma(\alpha-\beta-\gamma_2+1)}\right]+\frac{|\gamma_3|T^{\alpha-1}}{\Lambda_1}\\
\leq\ & M_1[(m_1+m_2)r+F_0]+N_1r+|\gamma_3|T/\Lambda_1.
\end{aligned}
$$

Hence

$$\|\mathcal{T}_1(u,v)\|\leq M_1[(m_1+m_2)r+F_0]+N_1r+|\gamma_3|T/\Lambda_1.$$

In the same way, we can obtain that

$$\|\mathcal{T}_2(u,v)\|\leq M_2[(n_1+n_2)r+G_0]+N_2r+|\delta_3|T/\Lambda_2.$$

Consequently, $\|T(u,v)\|\leq r$.

Now for $(u_2,v_2),(u_1,v_1)\in X\times Y$, and for any $t\in[0,e]$, we get

$$
\begin{aligned}
&|\mathcal{T}_1(u_2,v_2)(t)-\mathcal{T}_1(u_1,v_1)(t)|\\
\leq\ & \frac{|\lambda-1|}{\lambda\Gamma(\alpha-\beta)}\int_0^T(T-s)^{\alpha-\beta-1}|u_2(s)-u_1(s)|ds\\
&+\frac{1}{\lambda\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}|f(s,u_2(s),v_2(s))-f(s,u_1(s),v_1(s))|ds\\
&+\frac{T^{\alpha-1}}{\Lambda_1}\Bigg(\frac{\mu|\lambda-1|}{\lambda\Gamma(\alpha-\beta-\gamma_1)}\int_0^T(T-s)^{\alpha-\beta-\gamma_1-1}|u_2(s)-u_1(s)|ds\\
&+\frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)}\int_0^T(T-s)^{\alpha-\gamma_1-1}|f(s,u_2(s),v_2(s))-f(s,u_1(s),v_1(s))|ds\\
&+\frac{(1-\mu)|\lambda-1|}{\lambda\Gamma(\alpha-\beta-\gamma_2)}\int_0^T(T-s)^{\alpha-\beta-\gamma_2-1}|u_2(s)-u_1(s)|ds\\
&+\frac{1-\mu}{\lambda\Gamma(\alpha-\gamma_2)}\int_0^T(T-s)^{\alpha-\gamma_2-1}|f(s,u_2(s),v_2(s))-f(s,u_1(s),v_1(s))|ds\Bigg)\\
\leq\ & (m_1\|u_2-u_1\|+m_2\|v_2-v_1\|)\left[\frac{T^\alpha}{\lambda\Gamma(\alpha+1)}+\frac{T^{\alpha-\gamma_1+1}\mu}{\lambda\Lambda_1\Gamma(\alpha-\gamma_1+1)}\right.\\
&\left.+\frac{T^{\alpha-\gamma_2+1}(1-\mu)}{\lambda\Lambda_1\Gamma(\alpha-\gamma_2+1)}\right]+\|u_2-u_1\|\left[\frac{T^{\alpha-\beta}|\lambda-1|}{\lambda\Gamma(\alpha-\beta+1)}+\frac{T^{\alpha-\beta-\gamma_1+1}\mu|\lambda-1|}{\lambda\Lambda_1\Gamma(\alpha-\beta-\gamma_1+1)}\right.\\
&\left.+\frac{T^{\alpha-\beta-\gamma_2+1}(1-\mu)|\lambda-1|}{\lambda\Lambda_1\Gamma(\alpha-\beta-\gamma_2+1)}\right]\\
\leq\ & M_1[(m_1\|u_2-u_1\|+m_2\|v_2-v_1\|)+N_1\|u_2-u_1\|\\
\leq\ & M_1(m_1+m_2)[\|u_2-u_1\|+\|v_2-v_1\|)]+N_1[\|u_2-u_1\|+\|v_2-v_1\|]\\
\leq\ & [M_1(m_1+m_2)+N_1][\|u_2-u_1\|+\|v_2-v_1\|],
\end{aligned}
$$

and consequently we obtain

$$\|\mathcal{T}_1(u_2,v_2)-\mathcal{T}_1(u_1,v_1)\|\leq[M_1(m_1+m_2)+N_1][\|u_2-u_1\|+\|v_2-v_1\|]. \tag{17}$$

Similarly,

$$\|\mathcal{T}_2(u_2,v_2)-\mathcal{T}_2(u_1,v_1)\| \leq [M_2(n_1+n_2)+N_2][\|u_2-u_1\|+\|v_2-v_1\|]. \tag{18}$$

It follows from (17) and (18) that

$$\|\mathcal{T}(u_2,v_2)-\mathcal{T}(u_1,v_1)\| \leq [M_1(m_1+m_2)+N_1+M_2(n_1+n_2)+N_2](\|u_2-u_1\|+\|v_2-v_1\|).$$

Since $M_1(m_1+m_2)+N_1+M_2(n_1+n_2)+N_2 < 1$, therefore, $\mathcal{T}$ is a contraction operator. So, By Banach's fixed point theorem, the operator $\mathcal{T}$ has a unique fixed point, which is the unique solution of problem (2)-(3). This completes the proof. □

Example 3.4 *Consider the following coupled system of fractional differential equations with multiple orders of fractional derivatives and integrals boundary conditions of the form*

$$\begin{cases} \left(\frac{33}{38}D^{29/15}+\frac{5}{38}D^{16/15}\right)x(t)=\frac{t}{t+1}+\frac{1}{3}\sin\left(\frac{|x(t)|}{4}\right)+\frac{y^2(t)}{10(1+|y(t)|)}, \ t\in[0,3/2], \\ \left(\frac{24}{27}D^{19/13}+\frac{3}{27}D^{15/13}\right)y(t)=\sqrt{t+3}+\frac{x(t)}{10}e^{-|x(t)|}+\frac{t}{3}\tan^{-1}\left(\frac{|y(t)|}{6}\right), \quad t\in[0,3/2], \\ x(0)=0, \qquad \frac{9}{16}D^{8/15}x\left(\frac{3}{2}\right)+\frac{7}{16}D^{11/15}x\left(\frac{3}{2}\right)=\frac{1}{3}, \\ y(0)=0, \qquad \frac{2}{5}I^{1/2}y\left(\frac{3}{2}\right)+\frac{3}{5}I^{3/2}y\left(\frac{3}{2}\right)=\frac{2}{7}. \end{cases} \tag{19}$$

Here $\lambda=33/38$, $\alpha=29/15$, $\beta=16/15$, $T=3/2$, $\lambda_1=24/27$, $\alpha_1=19/13$, $\beta_1=15/13$, $\mu=9/16$, $\gamma_1=8/15$, $\gamma_2=11/15$, $\gamma_3=1/3$, $\mu_1=2/5$, $\delta_1=1/2$, $\delta_2=3/2$, $\delta_3=2/7$. From all constants, we can compute that $\Lambda_1=1.307202573$, $\Lambda_2=1.050302214$. $M_1=3.248792650$, $N_1=0.4373542422$, $M_2=2.869543745$ and $N_2=0.3962406719$. Clearly,

$$\begin{aligned} |f(t,x,y)| &= \left|\frac{t}{t+1}+\frac{1}{3}\sin\left(\frac{|x|}{4}\right)+\frac{y^2}{10(1+|y|)}\right| \\ &\leq \frac{3}{5}+\frac{1}{12}|x|+\frac{1}{10}|y|, \end{aligned}$$

and

$$\begin{aligned} |g(t,x,y)| &= \left|\sqrt{t+3}+\frac{x}{10}e^{-|x|}+\frac{t}{3}\tan^{-1}\left(\frac{|y|}{6}\right)\right| \\ &\leq \frac{3}{\sqrt{2}}+\frac{1}{10}|x|+\frac{1}{12}|y|. \end{aligned}$$

Setting $k_0=3/5$, $k_1=1/12$, $k_2=1/10$, $\nu_0=3/\sqrt{2}$, $\nu_1=1/10$ and $\nu_2=1/12$, we have

$$M_1k_1+M_2\nu_1+N_1=0.9950413375<1 \quad \text{and} \quad M_1k_2+M_2\nu_2+N_2=0.9602485823<1.$$

Therefore, by applying Theorem 3.2, the boundary value problem (19) has at least one solution on $[0,3/2]$.

Example 3.5 *Consider the following coupled system of fractional differential equations with multiple orders of fractional derivatives and integrals boundary conditions of the form*

$$\begin{cases} \left(\frac{49}{53}D^{17/9}+\frac{4}{53}D^{10/9}\right)x(t)=\frac{t+1}{2}+\frac{|x(t)|e^{-t^2}}{2(1+|x(t)|)}+\frac{1}{3}\sin|y(t)|\cos 2\pi t, \ t\in[0,1/2], \\ \left(\frac{41}{46}D^{13/7}+\frac{5}{46}D^{8/7}\right)y(t)=\frac{t}{4}+\tan^{-1}\left(\frac{|x(t)|}{3}\right)+\frac{1}{8}\left(\frac{y^2(t)+2|y(t)|}{1+|y(t)|}\right), \quad t\in[0,1/2], \\ x(0)=0, \qquad \frac{13}{31}D^{5/9}x\left(\frac{1}{2}\right)+\frac{18}{31}D^{4/9}x\left(\frac{1}{2}\right)=\frac{3}{4}, \\ y(0)=0, \qquad \frac{6}{11}I^{5/2}y\left(\frac{1}{2}\right)+\frac{5}{11}I^{7/2}y\left(\frac{1}{2}\right)=\frac{2}{3}. \end{cases} \tag{20}$$

Here $\lambda = 49/53$, $\alpha = 17/9$, $\beta = 10/9$, $T = 1/2$, $\lambda_1 = 41/46$, $\alpha_1 = 13/7$, $\beta_1 = 8/7$, $\mu = 13/31$, $\gamma_1 = 5/9$, $\gamma_2 = 4/9$, $\gamma_3 = 3/4$, $\mu_1 = 6/11$, $\delta_1 = 5/2$, $\delta_2 = 7/2$, $\delta_3 = 2/3$, $f(t,x,y) = ((t+1)/2) + ((|x|e^{-t^2})/(2(1+|x|))) + ((\sin|y|\cos 2\pi t)/(3))$ and $g(t,x,y) = (t/4) + \tan^{-1}(|x|/3) + ((y^2+2|y|)/(8(1+|y|)))$. From above information, we can calculate that $\Lambda_1 = 0.7921804090$, $\Lambda_2 = 0.004528637717$. $M_1 = 0.3706636539$, $N_1 = 0.09832444532$, $M_2 = 0.4209829845$ and $N_2 = 0.1927580748$. It is easy to see that

$$|f(t,x,y) - f(t,u,v)| \le \frac{1}{2}|x-u| + \frac{1}{3}|y-v|,$$

and

$$|g(t,x,y) - g(t,u,v)| \le \frac{1}{3}|x-u| + \frac{1}{4}|y-v|.$$

Putting $m_1 = 1/2$, $m_2 = 1/3$, $n_1 = 1/3$ and $n_2 = 1/4$, we deduce that

$$M_1(m_1+m_2) + N_1 + M_2(n_1+n_2) + N_2 = 0.8455423059 < 1.$$

Hence, by using Theorem 3.3, the boundary value problem (20) has a unique solution on $[0, 1/2]$.

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On solvability of a coupled system of fractional differential equations supplemented with a new kind of flux type integral boundary conditions

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Abstract

In this paper, we introduce a new kind of nonlocal nonlinear flux type integral boundary conditions and discuss the existence and uniqueness of solutions for a coupled system of fractional differential equations supplemented with these conditions. We apply Leray-Schauder's alternative and Banach's contraction mapping principle to obtain the desired results. An illustrative example is also included. Our results are new and enrich the existing material on coupled systems of fractional differential equations equipped with integral boundary conditions.

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1 Introduction

Fractional differential equations appear in the mathematical modeling of several systems and processes occurring in many branches of applied sciences such as blood flow phenomena, control theory, signal and image processing, reaction-diffusion models, aerodynamics, electrodynamics of complex medium, polymer rheology, etc. [1]-[4]. Fractional order differential equations are also found to be of great support in describing the hereditary properties of various materials and processes. With this advantage, fractional-order models have become more realistic and practical than the corresponding classical integer-order models. Fractional-order boundary value problems involving a variety of classical, nonlocal and integral boundary conditions have been addressed by many authors, for instance, see [5]-[10] and the references cited therein.

Coupled systems of fractional-order differential equations also constitute an interesting and important field of research in view of their applications in many real world problems such as anomalous diffusion [11], disease models [12]-[15], ecological models [16], synchronization of chaotic systems [17]-[19], etc. For some theoretical works on coupled systems of fractional-order differential equations, we refer the reader to a series of papers [20]-[24].

The integral boundary conditions provide a descent approach to relax the limitation of circular cross-section of blood vessels with an arbitrary shaped cross-section of such vessels in the study of blood flow problems [25] and model the problem of bacterial self-organization [26]. Recently, in [27, 28], the authors investigated fractional-order differential inclusions and equations with nonlocal nonlinear flux type integral boundary conditions.

In this paper, we consider a more generalized version of flux type integral boundary conditions and develop the existence criteria for a coupled system of Caputo type fractional differential equations equipped with these new conditions. Precisely, we investigate the following coupled system of Caputo type fractional differential equations

$$\begin{cases} {}^cD^q x(t) = f(t, x(t), y(t)), & t \in [0,1], \quad 1 < q \le 2, \\ {}^cD^p y(t) = h(t, x(t), y(t)), & t \in [0,1], \quad 1 < p \le 2, \end{cases} \tag{1}$$

supplemented with the nonlocal nonlinear flux type integral boundary conditions:

$$\begin{cases} x'(0) = \alpha \int_0^\xi x'(s)ds, \quad x(1) = \beta \int_0^1 g(x'(s))ds, \quad 0 \le \xi \le 1, \\ y'(0) = \alpha_1 \int_0^\theta y'(s)ds, \quad y(1) = \beta_1 \int_0^1 g(y'(s))ds, \quad 0 \le \theta \le 1, \end{cases} \tag{2}$$

where ${}^cD^q, {}^cD^p$ denote the Caputo fractional derivatives of order q and p respectively, $f, h : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ are appropriately chosen functions, and $\alpha, \beta, \alpha_1, \beta_1$ are real constants.

The objective of the present paper is to enhance the theoretical treatment of coupled systems further by considering a new boundary value problem of coupled fractional-order differential equations supplemented with nonlocal nonlinear flux type integral boundary conditions. The paper is organized as follows. In Section 2, we recall some basic definitions of fractional calculus and present an auxiliary lemma. The main results are presented in Section 3. We give two results: the first one derives the existence of solutions via Leray-Schauder's alternative, whereas the second one concerning existence and uniqueness of solutions is established by Banach's contraction principle. We also discuss an example for illustration of the existence-uniqueness result.

2 Preliminaries

Before presenting an auxiliary lemma, we recall some basic definitions of fractional calculus [3, 2].

Definition 2.1 *For $(n-1)-$times absolutely continuous function $y : [0,\infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as*

$${}^cD^q y(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} y^{(n)}(s)ds, \quad n-1 < q < n, n = [q]+1,$$

where $[q]$ denotes the integer part of the real number q.

Definition 2.2 *The Riemann-Liouville fractional integral of order q is defined as*

$$I^q y(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{y(s)}{(t-s)^{1-q}} ds, \quad q > 0,$$

provided the integral exists.

To define the solution for the problem (1)-(2), we use the following lemma.

Lemma 2.3 *Let $\alpha\xi \ne 1$. For $\phi \in C([0,1], \mathbb{R})$, the linear problem consisting by the equation*

$${}^cD^q x(t) = \phi(t), \qquad t \in [0,1], \quad 1 < q \le 2, \tag{3}$$

supplemented with the boundary conditions

$$x'(0) = \alpha \int_0^\xi x'(s)ds, \quad x(1) = \beta \int_0^1 g(x'(s))ds, \quad 0 \le \xi \le 1, \tag{4}$$

is equivalent to the integral equation

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \phi(s)ds + \frac{\alpha(t-1)}{1-\alpha\xi} \int_0^\xi \int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)} \phi(\tau)d\tau ds - \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} \phi(s)ds \\ &\quad + \beta \int_0^1 g\left(\int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)} \phi(\tau)d\tau + \frac{\alpha}{1-\alpha\xi} \int_0^\xi \int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)} \phi(\tau)d\tau ds \right) ds. \end{aligned} \tag{5}$$

Proof. It is well known that the general solution of the fractional differential equation (3) can be written as

$$x(t) = c_0 + c_1 t + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\phi(s)ds, \tag{6}$$

where $c_0, c_1 \in \mathbb{R}$ are arbitrary constants. Using the boundary conditions (4) in (6), we find that

$$\begin{aligned} c_0 &= \beta \int_0^1 g\bigg(\int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)}\phi(\tau)d\tau + \frac{\alpha}{1-\alpha\xi}\int_0^\xi \int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)}\phi(\tau)d\tau ds\bigg) ds \\ &\quad - \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}\phi(s)ds - \frac{\alpha}{1-\alpha\xi}\int_0^\xi \int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)}\phi(\tau)d\tau ds \end{aligned}$$

and

$$c_1 = \frac{\alpha}{1-\alpha\xi}\int_0^\xi \int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)}\phi(\tau)d\tau ds.$$

Substituting the values of c_0, c_1 in (6), we get (5). The converse follows by direct computation. This completes the proof. □

3 Main Results

Let us introduce the space $X = \{u(t)|u(t) \in C([0,1],\mathbb{R})\}$ endowed with the norm $\|u\| = \sup\{|u(t)|, t \in [0,1]\}$. Obviously $(X, \|\cdot\|)$ is a Banach space. Also $Y = \{v(t)|v(t) \in C([0,1],\mathbb{R})\}$ endowed with the norm $\|v\| = \sup\{|v(t)|, t \in [0,1]\}$ is a Banach space. Then the product space $(X \times Y, \|(u,v)\|)$ is also a Banach space equipped with norm $\|(u,v)\| = \|u\| + \|v\|$.

In view of Lemma 2.3, we define the operator $T : X \times Y \to X \times Y$ by $T(u,v)(t) = \begin{pmatrix} T_1(u,v)(t) \\ T_2(u,v)(t) \end{pmatrix}$, where

$$\begin{aligned} T_1(u,v)(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,u(s),v(s))ds + \frac{\alpha(t-1)}{1-\alpha\xi}\int_0^\xi \int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)} f(\tau,u(\tau),v(\tau))d\tau ds \\ &\quad - \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s,u(s),v(s))ds + \beta \int_0^1 g\bigg(\int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)} f(\tau,u(\tau),v(\tau))d\tau \\ &\quad + \frac{\alpha}{1-\alpha\xi}\int_0^\xi \int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)} f(\tau,u(\tau),v(\tau))d\tau ds\bigg) ds, \end{aligned}$$

and

$$\begin{aligned} T_2(u,v)(t) &= \int_0^t \frac{(t-s)^{p-1}}{\Gamma(p)} h(s,u(s),v(s))ds + \frac{\alpha_1(t-1)}{1-\alpha_1\theta}\int_0^\theta \int_0^s \frac{(s-\tau)^{p-2}}{\Gamma(p-1)} h(\tau,u(\tau),v(\tau))d\tau ds \\ &\quad - \int_0^1 \frac{(1-s)^{p-1}}{\Gamma(p)} h(s,u(s),v(s))ds + \beta_1 \int_0^1 g\bigg(\int_0^s \frac{(s-\tau)^{p-2}}{\Gamma(p-1)} h(\tau,u(\tau),v(\tau))d\tau \\ &\quad + \frac{\alpha_1}{1-\alpha_1\theta}\int_0^\theta \int_0^s \frac{(s-\tau)^{p-2}}{\Gamma(p-1)} h(\tau,u(\tau),v(\tau))d\tau ds\bigg) ds, \ \alpha_1\theta \neq 1. \end{aligned}$$

Let us introduce the following hypotheses which are used hereafter.

(H_1) Assume that there exist real constants $k_i, \ \lambda_i \geq 0 \ (i = 1,2)$ and $k_0 > 0, \lambda_0 > 0$ such that $\forall x_i \in \mathbb{R}, \ (i = 1,2)$ we have

$$|f(t,x_1,x_2)| \leq k_0 + k_1|x_1| + k_2|x_2|, \ |h(t,x_1,x_2)| \leq \lambda_0 + \lambda_1|x_1| + \lambda_2|x_2|.$$

(H_2) $|g(v)| \leq |v|, \quad \forall v \in \mathbb{R}.$

(H$_3$) Assume that $f, h : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ are continuous functions and there exist constants $m_i, n_i, i = 1,2$ such that for all $t \in [0,1]$ and $u_i, v_i \in \mathbb{R}, i = 1,2,$

$$|f(t,u_1,u_2) - f(t,v_1,v_2)| \le m_1|u_1 - v_1| + m_2|u_2 - v_2|$$

and

$$|h(t,u_1,u_2) - h(t,v_1,v_2)| \le n_1|u_1 - v_1| + n_2|u_2 - v_2|.$$

For the sake of convenience, we set

$$M_1 = \frac{1}{\Gamma(q+1)}\left(2 + \frac{|\alpha|\xi^q}{|1-\alpha\xi|}\right) + \frac{|\beta|}{\Gamma(q+2)}\left(q+1+\frac{|\alpha|}{|1-\alpha\xi|}\right), \tag{7}$$

$$M_2 = \frac{1}{\Gamma(p+1)}\left(2 + \frac{|\alpha_1|\theta^p}{|1-\alpha_1\theta|}\right) + \frac{|\beta_1|}{\Gamma(p+2)}\left(p+1+\frac{|\alpha_1|}{|1-\alpha_1\theta|}\right), \tag{8}$$

and

$$M_0 = \min\{1-(M_1k_1+M_2\lambda_1),\ 1-(M_1k_2+M_2\lambda_2)\},\quad k_i,\ \lambda_i \ge 0\ (i=1,2). \tag{9}$$

The first result is based on Leray-Schauder alternative.

Lemma 3.1 *(Leray-Schauder alternative) ([29] p. 4.) Let $F : E \to E$ be a completely continuous operator (i.e., a map that restricted to any bounded set in E is compact). Let*

$$\mathcal{E}(F) = \{x \in E : x = \lambda F(x) \text{ for some } 0 < \lambda < 1\}.$$

Then either the set $\mathcal{E}(F)$ is unbounded, or F has at least one fixed point.

Theorem 3.2 *Assume that (H$_1$), (H$_2$) hold. In addition it is assumed that*

$$M_1k_1 + M_2\lambda_1 < 1 \quad \text{and} \quad M_1k_2 + M_2\lambda_2 < 1,$$

where M_1 and M_2 are given by (7) and (8) respectively. Then the system (1)-(2) has at least one solution.

Proof. First we show that the operator $T : X \times Y \to X \times Y$ is completely continuous. By continuity of functions f, h and g, the operator T is continuous.

Let $\Omega \subset X \times Y$ be bounded. Then there exist positive constants L_1 and L_2 such that

$$|f(t,u(t),v(t)| \le L_1, \quad |h(t,u(t),v(t)| \le L_2, \quad \forall (u,v) \in \Omega.$$

Then for any $(u,v) \in \Omega$, we have

$$\begin{aligned} |T_1(u,v)(t)| \le\ & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}|f(s,u(s),v(s))|ds + \frac{|\alpha(t-1)|}{|1-\alpha\xi|}\int_0^\xi\int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)}|f(\tau,u(\tau),v(\tau))|d\tau ds \\ & + \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}|f(s,u(s),v(s))|ds + \beta\int_0^1 g\bigg(\int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)}|f(\tau,u(\tau),v(\tau))|d\tau \\ & + \frac{|\alpha|}{|1-\alpha\xi|}\int_0^\xi\int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)}|f(\tau,u(\tau),v(\tau))|d\tau ds\bigg)ds \\ \le\ & L_1\left\{\frac{1}{\Gamma(q+1)}\left(2+\frac{|\alpha|\xi^q}{|1-\alpha\xi|}\right) + \frac{|\beta|}{\Gamma(q+2)}\left(q+1+\frac{|\alpha|}{|1-\alpha\xi|}\right)\right\}, \end{aligned}$$

which implies that

$$\|T_1(u,v)\| \le L_1\left\{\frac{1}{\Gamma(q+1)}\left(2+\frac{|\alpha|\xi^q}{|1-\alpha\xi|}\right) + \frac{|\beta|}{\Gamma(q+2)}\left(q+1+\frac{|\alpha|}{|1-\alpha\xi|}\right)\right\} = L_1M_1.$$

Similarly, we get

$$\|T_2(u,v)\| \leq L_2\left\{\frac{1}{\Gamma(p+1)}\left(2+\frac{|\alpha_1|\theta^p}{|1-\alpha_1\theta|}\right)+\frac{|\beta_1|}{\Gamma(p+2)}\left(p+1+\frac{|\alpha_1|}{|1-\alpha_1\theta|}\right)\right\} = L_2M_2.$$

Thus, it follows from the above inequalities that the operator T is uniformly bounded.

Next, we show that T is equicontinuous. Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$. Then we have

$$\begin{aligned}
& |T_1(u(t_2),v(t_2)) - T_1(u(t_1),v(t_1))| \\
\leq\ & L_1\left\{\left|\frac{1}{\Gamma(q)}\int_0^{t_2}(t_2-s)^{q-1}ds - \frac{1}{\Gamma(q)}\int_0^{t_1}(t_1-s)^{q-1}vds\right|\right. \\
& \left. + \frac{|\alpha||t_2-t_1|}{|1-\alpha\xi|}\int_0^{\xi}\int_0^{s}\frac{(s-\tau)^{q-2}}{\Gamma(q-1)}d\tau ds\right\} \\
\leq\ & L_1\left\{\frac{1}{\Gamma(q)}\int_0^{t_1}[(t_2-s)^{q-1}-(t_1-s)^{q-1}]ds + \frac{1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}ds + \frac{|\alpha||t_2-t_1|}{|1-\alpha\xi|}\frac{\xi^q}{\Gamma(q+1)}\right\} \\
\leq\ & \frac{L_1}{\Gamma(q+1)}[2(t_2-t_1)^q + |t_2^q - t_1^q|] + L_1\frac{|\alpha||t_2-t_1|}{|1-\alpha\xi|}\frac{\xi^q}{\Gamma(q+1)}.
\end{aligned}$$

Analogously, we can obtain

$$|T_2(u(t_2),v(t_2)) - T_2(u(t_1),v(t_1))| \leq \frac{L_2}{\Gamma(p+1)}[2(t_2-t_1)^p + |t_2^p - t_1^p|] + L_2\frac{|\alpha_1||t_2-t_1|}{|1-\alpha_1\theta|}\frac{\theta^p}{\Gamma(p+1)}.$$

Therefore, the operator $T(u,v)$ is equicontinuous, and thus the operator $T(u,v)$ is completely continuous.

Finally, it will be verified that the set $\mathcal{E} = \{(u,v) \in X \times Y | (u,v) = \lambda T(u,v), 0 \leq \lambda \leq 1\}$ is bounded. Let $(u,v) \in \mathcal{E}$, with $(u,v) = \lambda T(u,v)$. For any $t \in [0,1]$, we have

$$u(t) = \lambda T_1(u,v)(t), \quad v(t) = \lambda T_2(u,v)(t).$$

Then

$$|u(t)| \leq \left\{\frac{1}{\Gamma(q+1)}\left(2+\frac{|\alpha|\xi^q}{|1-\alpha\xi|}\right)+\frac{|\beta|}{\Gamma(q+2)}\left(q+1+\frac{|\alpha|}{|1-\alpha\xi|}\right)\right\}(k_0 + k_1\|u\| + k_2\|v\|),$$

and

$$|v(t)| \leq \left\{\frac{1}{\Gamma(p+1)}\left(2+\frac{|\alpha_1|\theta^p}{|1-\alpha_1\theta|}\right)+\frac{|\beta_1|}{\Gamma(p+2)}\left(p+1+\frac{|\alpha_1|}{|1-\alpha_1\theta|}\right)\right\}(\lambda_0 + \lambda_1\|u\| + \lambda_2\|v\|).$$

Hence we have

$$\|u\| \leq M_1(k_0 + k_1\|u\| + k_2\|v\|),\ \|v\| \leq M_2(\lambda_0 + \lambda_1\|u\| + \lambda_2\|v\|),$$

which imply that

$$\|u\| + \|v\| = (M_1k_0 + M_2\lambda_0) + (M_1k_1 + M_2\lambda_1)\|u\| + (M_1k_2 + M_2\lambda_2)\|v\|.$$

Consequently,

$$\|(u,v)\| \leq \frac{M_1k_0 + M_2\lambda_0}{M_0},$$

for any $t \in [0,1]$, where M_0 is defined by (9), which proves that $\mathcal{E}$ is bounded. Thus, by Lemma 3.1, the operator T has at least one fixed point. Hence the boundary value problem (1)-(2) has at least one solution. The proof is complete. □

In the second result, we prove existence and uniqueness of solutions of the boundary value problem (1)-(2) via Banach's contraction principle.

Theorem 3.3 *Assume that (H_2), (H_3) hold. In addition, assume that*

$$M_1(m_1+m_2)+M_2(n_1+n_2)<1,$$

where M_1 and M_2 are given by (7) and (8) respectively. Then the system (1)-(2) has a unique solution.

Proof. Define $\sup_{t\in[0,1]} f(t,0,0)=N_1<\infty$ and $\sup_{t\in[0,1]} g(t,0,0)=N_2<\infty$ such that

$$r\geq \frac{N_1M_1+N_2M_2}{1-M_1(m_1+m_2)-M_2(n_1+n_2)}.$$

We show that $TB_r\subset B_r$, where $B_r=\{(u,v)\in X\times Y:\|(u,v)\|\leq r\}$. For $(u,v)\in B_r$, we have

$$\begin{aligned}
|T_1(u,v)(t)| &\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}(|f(s,u(s),v(s))-f(t,0,0)|+|f(t,0,0)|)ds\\
&\quad+\frac{|\alpha|}{|1-\alpha\xi|}\int_0^{\xi}\int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)}(|f(\tau,u(\tau),v(\tau))-f(t,0,0)|+|f(t,0,0)|)d\tau ds\\
&\quad+\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}(|f(s,u(s),v(s))-f(t,0,0)|+|f(t,0,0)|)ds\\
&\quad+|\beta|\int_0^1 g\bigg(\int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)}(|f(\tau,u(\tau),v(\tau))-f(t,0,0)|+|f(t,0,0)|)d\tau\\
&\quad+\frac{|\alpha|}{|1-\alpha\xi|}\int_0^{\xi}\int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)}(|f(\tau,u(\tau),v(\tau))-f(t,0,0)|+|f(t,0,0)|)d\tau ds\bigg)ds\\
&\leq \bigg\{\frac{1}{\Gamma(q+1)}+\frac{|\alpha|\xi^q}{|1-\alpha\xi|\Gamma(q+1)}+\frac{1}{\Gamma(q+1)}\\
&\quad+|\beta|\bigg(\frac{1}{\Gamma(q+1)}+\frac{|\alpha|}{|1-\alpha\xi|\Gamma(q+2)}\bigg)\bigg\}(m_1\|u\|+m_2\|v\|+N_1)\\
&= M_1[(m_1+m_2)r+N_1].
\end{aligned}$$

Hence

$$\|T_1(u,v)(t)\|\leq M_1[(m_1+m_2)r+N_1].$$

In the same way, we can obtain that

$$\|T_2(u,v)(t)\|\leq M_2[(n_1+n_2)r+N_2].$$

Consequently, $\|T(u,v)(t)\|\leq r$.

Now for $(u_2,v_2),(u_1,v_1)\in X\times Y$, and for any $t\in[0,e]$, we get

$$\begin{aligned}
&|T_1(u_2,v_2)(t)-T_1(u_1,v_1)(t)|\\
&\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}|f(s,u_2(s),v_2(s))-f(s,u_1(s),v_1(s))|ds\\
&\quad+\frac{|\alpha|}{|1-\alpha\xi|}\int_0^{\xi}\int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)}|f(\tau,u_2(\tau),v_2(\tau))-|f(\tau,u_1(\tau),v_1(\tau))|d\tau ds\\
&\quad+\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}|f(s,u_2(s),v_2(s))-f(s,u_1(s),v_1(s))|ds\\
&\quad+|\beta|\int_0^1 g\bigg(\int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)}|f(s,u_2(s),v_2(s))-f(s,u_1(s),v_1(s))|d\tau\\
&\quad+\frac{|\alpha|}{|1-\alpha\xi|}\int_0^{\xi}\int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)}|f(s,u_2(s),v_2(s))-f(s,u_1(s),v_1(s))|d\tau ds\bigg)ds
\end{aligned}$$

$$
\begin{aligned}
&\le \Bigg\{\frac{1}{\Gamma(q+1)}+\frac{|\alpha|\xi^q}{|1-\alpha\xi|\Gamma(q+1)}+\frac{1}{\Gamma(q+1)} \\
&\quad +|\beta|\left(\frac{1}{\Gamma(q+1)}+\frac{|\alpha|}{|1-\alpha\xi|\Gamma(q+2)}\right)\Bigg\}(m_1\|u_2-u_1\|+m_2\|v_2-v_1\|) \\
&= M_1(m_1\|u_2-u_1\|+m_2\|v_2-v_1\|) \\
&\le M_1(m_1+m_2)(\|u_2-u_1\|+\|v_2-v_1\|),
\end{aligned}
$$

and consequently we obtain

$$\|T_1(u_2,v_2)(t)-T_1(u_1,v_1)\| \le M_1(m_1+m_2)(\|u_2-u_1\|+\|v_2-v_1\|). \tag{10}$$

Similarly,

$$\|T_2(u_2,v_2)(t)-T_2(u_1,v_1)\| \le M_2(n_1+n_2)(\|u_2-u_1\|+\|v_2-v_1\|). \tag{11}$$

It follows from (10) and (11) that

$$\|T(u_2,v_2)(t)-T(u_1,v_1)(t)\| \le [M_1(m_1+m_2)+M_2(n_1+n_2)](\|u_2-u_1\|+\|v_2-v_1\|).$$

Since $M_1(m_1+m_2)+M_2(n_1+n_2)<1$, therefore, T is a contraction operator. So, By Banach's fixed point theorem, the operator T has a unique fixed point, which is the unique solution of problem (1)-(2). This completes the proof. □

Example. Consider the following system of fractional boundary value problem

$$
\begin{cases}
{}^cD^{3/2}x(t)=\dfrac{1}{4(t+2)^2}\dfrac{|x(t)|}{1+|x(t)|}+1+\dfrac{1}{32}\sin^2 y(t), \quad t\in[0,1], \\
{}^cD^{3/2}y(t)=\dfrac{1}{32\pi}\sin(2\pi x(t))+\dfrac{|y(t)|}{16(1+|y(t)|)}+\dfrac{1}{2}, \quad t\in[0,1], \\
x'(0)=\dfrac{1}{2}\displaystyle\int_0^{1/3}x'(s)ds, \quad x(1)=\dfrac{1}{3}\int_0^1 g(x'(s))ds, \\
y'(0)=\dfrac{4}{5}\displaystyle\int_0^{1/4}y'(s)ds, \quad y(1)=\dfrac{3}{4}\int_0^1 g(y'(s))ds.
\end{cases} \tag{12}
$$

Here $q=p=3/2,\ \alpha=1/2,\ \alpha_1=4/5,\ \xi=1/3,\ \theta=1/4,\ \beta=1/3,\ \beta_1=3/4,\ g(v)=\begin{cases}\sqrt{v}, & |v|\ge 1,\\ v^2, & |v|<1.\end{cases}$ $f(t,u,v)=\dfrac{1}{4(t+2)^2}\dfrac{|u|}{1+|u|}+1+\dfrac{1}{32}\sin^2 v$, and $h(t,u,v)=\dfrac{1}{32\pi}\sin(2\pi u)+\dfrac{|v|}{16(1+|v|)}+\dfrac{1}{2}$. With the given data, we find that $M_1\approx 1.9027815$, $M_2\approx 1.6365646$. Note that $|f(t,u_1,u_2)-f(t,v_1,v_2)|\le \dfrac{1}{16}|u_1-u_2|+\dfrac{1}{16}|v_1-v_2|$, $|g(t,u_1,u_2)-g(t,v_1,v_2)|\le\dfrac{1}{16}|u_1-u_2|+\dfrac{1}{16}|v_1-v_2|$, and $M_1(m_1+m_2)+M_2(n_1+n_2)\approx 0.4424181<1$. Thus all the conditions of Theorem 3.3 are satisfied and consequently, its conclusion applies to the problem (12).

4 Conclusions

We have obtained the existence criteria for the solutions of a coupled system of nonlinear Caputo type fractional differential equations equipped with a new kind of nonlocal nonlinear flux type integral boundary conditions. Our results are new in the sense of introduced integral boundary conditions (2) and contribute to the theory of coupled systems of fractional differential equations.

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Strong Convergence Theorems of Non-convex Hybrid Algorithm for Quasi-Lipschitz Mappings

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Abstract

The aim of this paper is to introduce a new non-convex hybrid algorithm for a family of countable quasi-Lipschitz mappings. We establish strong convergence theorems of common fixed points for a uniformly closed asymptotically family of countable quasi-Lipschitz mappings in a Hilbert space.

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1 Introduction

Fixed point theory of special mappings like nonexpansive, asymptotically nonexpansive, contractive and other mappings is an active area of interest and finds applications in many related fields like image recovery, signal processing and geometry of objects [13]. From time to time, some versions of theorems relating to fixed points of functions of special nature keep on appearing in almost in all branches of mathematics. Consequently, we apply them in industry, toy making, finance, aircrafts and manufacturing of new model cars. For example, a fixed-point iteration scheme has been applied in intensity modulated radiation therapy optimization to pre-compute dose-deposition coefficient matrix, see [12]. Because of its vast range of applications almost in all directions, the research in it is moving rapidly

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and an immense literature is present currently. The construction of fixed point theorems (for example, Banach fixed point theorem) which not only claim the existence of a fixed point but yield an algorithm, too (in the Banach case fixed point iteration $x_{n+1} = f(x_n)$). Any equation that can be written as $x = f(x)$ for some mapping f that is contracting with respect to some (complete) metric will provide such a fixed point iteration. Mann's iteration method was the stepping stone in this regard and is invariably used in most of the occasions, see [4]. But it only ensures weak convergence, see [2] but more often then not, we require strong convergence in many real world problems relating to Hilbert spaces, see [1]. So mathematician are in search for the modifications of the Mann's process to control and ensure the strong convergence (see [2, 3, 5–9, 11] and references therein).

Most probably the first noticeable modification of Mann's Iteration process was proposed by Nakajo and Takahashi [9] in 2003. They introduced this modification for only one nonexpansive mapping in a Hilbert space, where Kim and Xu [5] introduced a modification for asymptotically nonexpansive mappings in the Hilbert space in 2006. In the same year Martinez-Yanes ad Xu [7] introduced a modification of the Ishikawa iteration process for a nonexpansive mapping for a Hilbert space. They also gave modification of the Halpern iteration method in a Hilbert space. Su and Qin [11] gave a monotone hybrid iteration process for nonexpansive mappings in a Hilbert space. Liu et al. [6] gave a novel iteration method for a finite family of quasi-asymptotically pseudo-contractive mappings in a Hilbert space.

Let H be a Hilbert space and C be a nonempty closed and convex subset of H. Let $P_c(\cdot)$ be the metric projection onto C. A mapping $T : C \to C$ is said to be *nonexpensive* if $\|Tx-Ty\| \leq \|x-y\|$ for all $x, y \in C$. Denote by $F(T)$ the set of fixed points of T. It is well known that $F(T)$ is closed and convex. A mapping $T : C \to C$ is said to be *quasi-Lipschitz* if $F(T) \neq \emptyset$ and $\|Tx - p\| \leq L\|x - p\|$ for all $x \in C,\ p \in F(T)$, where $1 \leq L < \infty$ is a constant. If $L = 1$, then T is known as *quasi-nonexpansive*. It is well-known that T is said to be *closed* if $x_n \to x$ and $\|Tx_n - x_n\| \to 0$ as $n \to \infty$ implies $Tx = x$. T is said to be *weak closed* if $x_n \rightharpoonup x$ and $\|Tx_n - x_n\| \to 0$ as for $n \to \infty$ implies $Tx = x$. It is admitted fact that a mapping which is weak closed should be closed but converse is no longer true.

Let $\{T_n\}$ be a sequence of mappings from C into itself with a nonempty common fixed points set F. Then $\{T_n\}$ is said to be *uniformly closed* if for any convergent sequences $\{z_n\} \subset C$ with conditions $\|T_n z_n - z_n\| \to 0$ as $n \to \infty$, the limit of $\{z_n\}$ belongs to F.

In 1953 Mann [4] proposed an iterative scheme given as

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T(x_n), \quad n = 0, 1, 2, \ldots.$$

Guan et al. [3] established the following non-convex hybrid iteration algorithm corresponding to Mann iterative scheme:

$$\begin{cases} x_0 \in C = Q_0, \quad \text{choosen arbitrarily,} \\ y_n = (1 - \alpha_n)x_n + \alpha_n T_n x_n, \quad n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq (1 + (L_n - 1)\alpha_n)\|x_n - z\| \cap A, \quad n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0. \end{cases}$$

They also established non-convex hybrid iteration algorithms and proved some strong convergence results relating to common fixed points for a uniformly closed asymptotically family of countable quasi-Lipschitz mappings in a Hilbert space. They applied their results for the finite case to obtain fixed points. In this article we established a kind of non-convex hybrid iteration algorithm concerning SP-iterative process [10] and proves strong convergence theorems of common fixed points for a uniformly closed asymptotically family of countable quasi-Lipschitz mappings in a Hilbert space. We also present an application of our algorithm.

2 Main results

In this section we formulate our main results.

Definition 2.1. Let C be a closed convex subset of a Hilbert space H, and Let $\{T_n\}$ be a family of countable quasi-L_n-Lipschitz mappings from C into itself. $\{T_n\}$ is said to be *asymptotically* if $\lim_{n\to\infty} L_n = 1$.

The following lemmas is well known.

Proposition 2.2. *Let C be a closed convex subset of a Hilbert space H. For $x \in H$ and $z \in C$, $z = P_C x$ if and only if we have $\langle x - z, z - y\rangle \geq 0$ for all $y \in C$.*

Proposition 2.3. *Let C be a closed convex subset of a Hilbert space H. For any given $x_0 \in H$, we have $p = P_C x_0$ if and only if $\langle p - z, x_0 - p\rangle \geq 0$ for all $z \in C$.*

Proposition 2.4. ([3]) *Let C be a closed convex subset of a Hilbert space H and let $\{T_n\}$ be a uniformly closed asymptotically family of countable quasi-L_n-Lipschitz mappings from C into itself. Then the common fixed point set F is closed and convex.*

Theorem 2.5. *Let C be a closed convex subset of a Hilbert space H, and let $\{T_n\} : C \to C$ be a uniformly closed asymptotically family of countable quasi-L_n-Lipschitz mappings from C into itself. Assume that $\alpha_n \in (0, 1]$ and $\beta_n, \gamma_n \in [0, 1]$ for all $n \in N$. Then $\{x_n\}$ generated by*

$$\begin{cases} x_0 \in C = Q_0, \quad \text{choosen arbitrarily,} \\ y_n = (1-\alpha_n)z_n + \alpha_n T_n z_n, \quad n \geq 0, \\ z_n = (1-\beta_n)t_n + \beta_n T_n t_n, \quad n \geq 0, \\ t_n = (1-\gamma_n)x_n + \gamma_n T_n x_n, \quad n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq [1 + L_n(\alpha_n + \beta_n + \gamma_n - 2\alpha_n\beta_n \\ \qquad -2\alpha_n\gamma_n - 2\beta_n\gamma_n + 3\alpha_n\beta_n\gamma_n) + L_n^2(\alpha_n\beta_n + \alpha_n\gamma_n \\ \qquad +\beta_n\gamma_n - 3\alpha_n\beta_n\gamma_n) + \alpha_n\beta_n\gamma_n L_n^3 - \alpha_n - \beta_n - \gamma_n \\ \qquad +\alpha_n\beta_n + \alpha_n\gamma_n + \beta_n\gamma_n - \alpha_n\beta_n\gamma_n]\|x_n - z\|\} \cap A, \quad n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0, \end{cases}$$

converges strongly to $P_F x_0$, where $\overline{co}C_n$ denotes the closed convex closure of C_n for all $n \geq 1$ and $A = \{z \in H : \|z - P_F x_0\| \leq 1\}$.

Proof. We give our proof in following steps.

STEP 1. We know that $\overline{co}C_n$ and Q_n are closed and convex for all $n \geq 0$. Next, we show that $F \cap A \subset \overline{co}C_n$ for all $n \geq 0$. Indeed, for each $p \in F \cap A$, we have

$$\begin{aligned}
\|y_n - p\| &= \|(1-\alpha_n)z_n + \alpha_n T_n z_n - p\| \\
&= \|(1-\alpha_n)((1-\beta_n)t_n + \beta_n T_n t_n) + \alpha_n T_n((1-\beta_n)t_n + \beta_n T_n t_n) - p\| \\
&= \|(1-\alpha_n)((1-\beta_n)[(1-\gamma_n)x_n + \gamma_n T_n x_n] + \beta_n T_n[(1-\gamma_n)x_n + \gamma_n T_n x_n]) \\
&\quad + \alpha_n T_n((1-\beta_n)[(1-\gamma_n)x_n + \gamma_n T_n x_n] + \beta_n T_n[(1-\gamma_n)x_n + \gamma_n T_n x_n]) - p\| \\
&= \|(1-\alpha_n-\beta_n-\gamma_n+\alpha_n\beta_n+\alpha_n\gamma_n+\beta_n\gamma_n-\alpha_n\beta_n\gamma_n)(x_n-p) \\
&\quad + (\alpha_n+\beta_n+\gamma_n-2\alpha_n\beta_n-2\alpha_n\gamma_n-2\beta_n\gamma_n+3\alpha_n\beta_n\gamma_n)(T_n x_n - p) \\
&\quad + (\alpha_n\beta_n+\alpha_n\gamma_n+\beta_n\gamma_n-3\alpha_n\beta_n\gamma_n)(T_n^2 x_n - p) + \alpha_n\beta_n\gamma_n(T_n^3 x_n - p)\| \\
&\leq (1-\alpha_n-\beta_n-\gamma_n+\alpha_n\beta_n+\alpha_n\gamma_n+\beta_n\gamma_n-\alpha_n\beta_n\gamma_n)\|x_n-p\| \\
&\quad + (\alpha_n+\beta_n+\gamma_n-2\alpha_n\beta_n-2\alpha_n\gamma_n-2\beta_n\gamma_n+3\alpha_n\beta_n\gamma_n)L_n\|x_n-p\| \\
&\quad + (\alpha_n\beta_n+\alpha_n\gamma_n+\beta_n\gamma_n-3\alpha_n\beta_n\gamma_n)L_n^2\|x_n-p\| + \alpha_n\beta_n\gamma_n L_n^3\|x_n-p\| \\
&= [1 + L_n(\alpha_n+\beta_n+\gamma_n-2\alpha_n\beta_n-2\alpha_n\gamma_n-2\beta_n\gamma_n+3\alpha_n\beta_n\gamma_n) \\
&\quad + L_n^2(\alpha_n\beta_n+\alpha_n\gamma_n+\beta_n\gamma_n-3\alpha_n\beta_n\gamma_n) + \alpha_n\beta_n\gamma_n L_n^3 \\
&\quad - \alpha_n-\beta_n-\gamma_n+\alpha_n\beta_n+\alpha_n\gamma_n+\beta_n\gamma_n-\alpha_n\beta_n\gamma_n]\|x_n-p\|
\end{aligned}$$

and $p \in A$, so $p \in C_n$ which implies that $F \cap A \subset C_n$ for all $n \geq 0$. therefore, $F \cap A \subset \overline{co}C_n$ for all $n \geq 0$.

STEP 2. We show that $F \cap A \subset \overline{co}C_n \cap Q_n$ for all $n \geq 0$. it suffices to show that $F \cap A \subset Q_n$ for all $n \geq 0$. We prove this by mathematical induction. For $n = 0$ we have $F \cap A \subset C = Q_0$. Assume that $F \cap A \subset Q_n$. Since x_{n+1} is the projection of x_0 onto $\overline{co}C_n \cap Q_n$, from Proposition 2.2, we have

$$\langle x_{n+1} - z, x_{n+1} - x_0 \rangle \leq 0, \quad \forall z \in \overline{co}C_n \cap Q_n$$

as $F \cap A \subset \overline{co}C_n \cap Q_n$, the last inequality holds, in particular, for all $z \in F \cap A$. This together with the definition of Q_{n+1} implies that $F \cap A \subset Q_{n+1}$. Hence the $F \cap A \subset \overline{co}C_n \cap Q_n$ holds for all $n \geq 0$.

STEP3. We prove $\{x_n\}$ is bounded. Since F is a nonempty closed and convex subset of C, there exists a unique element $z_0 \in F$ such that $z_0 = P_F x_0$. From $x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0$, we have

$$\|x_{n+1} - x_0\| \leq \|z - x_0\|$$

for every $z \in \overline{co}C_n \cap Q_n$. As $z_0 \in F \cap A \subset \overline{co}C_n \cap Q_n$, we get

$$\|x_{n+1} - x_0\| \leq \|z_0 - x_0\|$$

for each $n \geq 0$. This implies that $\{x_n\}$ is bounded.

STEP 4. We show that $\{x_n\}$ converges strongly to a point of C (we show that $\{x_n\}$ is a Cauchy sequence). As $x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0 \subset Q_n$ and $x_n = P_{Q_n} x_0$ (Proposition 2.3), we have

$$\|x_{n+1} - x_0\| \geq \|x_n - x_0\|$$

for every $n \geq 0$, which together with the boundedness of $\|x_n - x_0\|$ implies that there exists the limit of $\|x_n - x_0\|$. On the other hand, from $x_{n+m} \in Q_n$, we have $\langle x_n - x_{n+m}, x_n - x_0 \rangle \leq 0$ and hence

$$\begin{aligned}\|x_{n+m} - x_n\|^2 &= \|(x_{n+m} - x_0) - (x_n - x_0)\|^2 \\ &\leq \|x_{n+m} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+m} - x_n, x_n - x_0 \rangle \\ &\leq \|x_{n+m} - x_0\|^2 - \|x_n - x_0\|^2 \\ &\to 0, \quad n \to \infty\end{aligned}$$

for any $m \geq 1$. Therefore $\{x_n\}$ is a cauchy sequence in C, then there exists a point $q \in C$ such that $\lim_{n\to\infty} x_n = q$.

STEP 5. We show that $y_n \to q$ as $n \to \infty$. Let

$$D_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + (L_n^3 - 1)(L_n^3 + 1)\}.$$

From the definition of D_n, we have

$$\begin{aligned}D_n &= \{z \in C : \langle y_n - z, y_n - z \rangle \leq \langle x_n - z, x_n - z \rangle + (L_n^3 - 1)(L_n^3 + 1)\} \\ &= \{z \in C : \|y_n\|^2 - 2\langle y_n, z \rangle + \|z\|^2 \leq \|x_n\|^2 - 2\langle x_n, z \rangle + \|z\|^2 + (L_n^3 - 1)(L_n^3 + 1)\} \\ &= \{z \in C : 2\langle x_n - y_n, z \rangle \leq \|x_n\|^2 - \|y_n\|^2 + (L_n^3 - 1)(L_n^3 + 1)\}\end{aligned}$$

This shows that D_n is convex and closed, $n \in Z^+ \cup \{0\}$. Next, we want to prove that $C_n \subset D_n$, $n \geq 0$.

In fact, for any $z \in C_n$, we have

$$\begin{aligned}\|y_n - z\|^2 &\leq [1 + L_n(\alpha_n + \beta_n + \gamma_n - 2\alpha_n\beta_n - 2\alpha_n\gamma_n - 2\beta_n\gamma_n + 3\alpha_n\beta_n\gamma_n) \\ &\quad + L_n^2(\alpha_n\beta_n + \alpha_n\gamma_n + \beta_n\gamma_n - 3\alpha_n\beta_n\gamma_n) + \alpha_n\beta_n\gamma_n L_n^3 - \alpha_n - \beta_n - \gamma_n \\ &\quad + \alpha_n\beta_n + \alpha_n\gamma_n + \beta_n\gamma_n - \alpha_n\beta_n\gamma_n]^2\|x_n - z\|^2 \\ &= \|x_n - z\|^2 + [2(L_n(\alpha_n + \beta_n + \gamma_n - 2\alpha_n\beta_n - 2\alpha_n\gamma_n - 2\beta_n\gamma_n + 3\alpha_n\beta_n\gamma_n) \\ &\quad + L_n^2(\alpha_n\beta_n + \alpha_n\gamma_n + \beta_n\gamma_n - 3\alpha_n\beta_n\gamma_n) + \alpha_n\beta_n\gamma_n L_n^3 - \alpha_n - \beta_n - \gamma_n \\ &\quad + \alpha_n\beta_n + \alpha_n\gamma_n + \beta_n\gamma_n - \alpha_n\beta_n\gamma_n)\alpha_n + (L_n(\alpha_n + \beta_n + \gamma_n - 2\alpha_n\beta_n \\ &\quad - 2\alpha_n\gamma_n - 2\beta_n\gamma_n + 3\alpha_n\beta_n\gamma_n) + L_n^2(\alpha_n\beta_n + \alpha_n\gamma_n + \beta_n\gamma_n - 3\alpha_n\beta_n\gamma_n) \\ &\quad + \alpha_n\beta_n\gamma_n L_n^3 - \alpha_n - \beta_n - \gamma_n + \alpha_n\beta_n + \alpha_n\gamma_n + \beta_n\gamma_n - \alpha_n\beta_n\gamma_n)^2]\|x_n - z\|^2 \\ &\leq \|x_n - z\|^2 + [2(L_n^3 - 1) + (L_n^3 - 1)^2]\|x_n - z\|^2 \\ &= \|x_n - z\|^2 + (L_n^3 - 1)(L_n^3 + 1)\|x_n - z\|^2.\end{aligned}$$

From

$$\begin{aligned}C_n = \{z \in C : \|y_n - z\| &\leq [1 + L_n(\alpha_n + \beta_n + \gamma_n - 2\alpha_n\beta_n - 2\alpha_n\gamma_n - 2\beta_n\gamma_n + 3\alpha_n\beta_n\gamma_n) \\ &+ L_n^2(\alpha_n\beta_n + \alpha_n\gamma_n + \beta_n\gamma_n - 3\alpha_n\beta_n\gamma_n) + \alpha_n\beta_n\gamma_n L_n^3 - \alpha_n - \beta_n - \gamma_n + \alpha_n\beta_n \\ &+ \alpha_n\gamma_n + \beta_n\gamma_n - \alpha_n\beta_n\gamma_n]\|x_n - z\|\} \cap A, \quad n \geq 0,\end{aligned}$$

We have $C_n \subset A$, $n \geq 0$. Since A is convex, we also have $\overline{co}C_n \subset A$, $n \geq 0$. Consider $x_n \in \overline{co}C_{n-1}$, we know that

$$\begin{aligned}\|y_n - z\| &\leq \|x_n - z\|^2 + (L_n^3 - 1)(L_n^3 + 1)\|x_n - z\|^2 \\ &\leq \|x_n - z\|^2 + (L_n^3 - 1)(L_n^3 + 1).\end{aligned}$$

This implies that $z \in D_n$ and hence $C_n \subset D_n$, $n \geq 0$. Sinnce D_n is convex, we have $\overline{co}(C_n) \subset D_n$, $n \geq 0$. Therefore

$$\|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + (L_n^3 - 1)(L_n^3 + 1) \to 0$$

as $n \to \infty$. That is, $y_n \to q$ as $n \to \infty$.

STEP 6. We show that $q \in F$. From the definition of y_n, we have

$$\begin{aligned}
&(\alpha_n + \beta_n + \gamma_n - \alpha_n\gamma_n - \beta_n\gamma_n - \alpha_n\beta_n + \alpha_n\beta_n\gamma_n) \\
&+ (\alpha_n\gamma_n + \beta_n\gamma_n + \alpha_n\beta_n - 2\alpha_n\beta_n\gamma_n)T_n + \alpha_n\beta_n\gamma_n T_n^2 \|T_n x_n - x_n\| \\
&= \|y_n - x_n\| \to 0
\end{aligned}$$

as $n \to \infty$. Since $\alpha_n \in (a, 1] \subset [0, 1]$, from the above limit we have

$$\lim_n \to \infty \|T_n x_n - x_n\| = 0.$$

Since $\{T_n\}$ is uniformly closed and $x_n \to q$, we have $q \in F$.

STEP 7. We claim that $q = z_0 = P_F x_0$, if not, we have that $\|x_0 - p\| > \|x_0 - z_0\|$. There must exist a positive integer N, if $n > N$, then $\|x_0 - x_n\| > \|x_0 - z_0\|$, which leads to

$$\begin{aligned}
\|z_0 - x_n\|^2 &= \|z_0 - x_n + x_n - x_0\|^2 \\
&= \|z_0 - x_n\|^2 + \|x_n - x_0\|^2 + 2\langle z_0 - x_n, x_n - x_0\rangle.
\end{aligned}$$

It follows that $\langle z_0 - x_n, x_n - x_0\rangle < 0$ which implies that $z_0 \overline{\in} Q_n$, so that $z_0 \overline{\in} F$, this is a contradiction. This completes the proof. ☐

In [3], we show an example of C_n which does not involve a convex subset.

Corollary 2.6. *Let C be a closed convex subset of a Hilbert space H, and let T be a closed quasi-nonexpansive mapping from C into itself. Assume that $\alpha_n \in (0, 1]$ and $\beta_n, \gamma_n \in [0, 1]$ for all $n \in N$. Then $\{x_n\}$ generated by*

$$\begin{cases}
x_0 \in C = Q_0, \quad \text{choosen arbitrarily,} \\
y_n = (1 - \alpha_n)z_n + \alpha_n T z_n, \quad n \geq 0, \\
z_n = (1 - \beta_n)t_n + \beta_n T t_n, \quad n \geq 0, \\
t_n = (1 - \gamma_n)x_n + \gamma_n T x_n, \quad n \geq 0, \\
C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\} \cap A, \quad n \geq 0, \\
Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \quad n \geq 1, \\
x_{n+1} = P_{C_n \cap Q_n} x_0,
\end{cases}$$

converges strongly to $P_{F(T)} x_0$, where $A = \{z \in H : \|z - P_F x_0\| \leq 1\}$.

Proof. Take $T_n \equiv T$, $L_n \equiv 1$ in Theorem 2.5, in this case, C_n is convex and closed and , for all $n \geq 0$, by using Theorem 2.5, we obtain Corollary 2.6. ☐

Corollary 2.7. *Let C be a closed convex subset of a Hilbert space H, and let T be a nonexpansive mapping from C into itself. Assume that $\alpha_n \in (0,1]$ and $\beta_n, \gamma_n \in [0,1]$ for all $n \in N$. Then $\{x_n\}$ generated by*

$$\begin{cases} x_0 \in C = Q_0, \quad \text{choosen arbitrarily,} \\ y_n = (1-\alpha_n)z_n + \alpha_n T z_n, \quad n \geq 0, \\ z_n = (1-\beta_n)t_n + \beta_n T t_n, \quad n \geq 0, \\ t_n = (1-\gamma_n)x_n + \gamma_n T x_n, \quad n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\} \cap A, \quad n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases}$$

converges strongly to $P_{F(T)}x_0$, where $A = \{z \in H : \|z - P_F x_0\| \leq 1\}$.

3 Applications

Here, we give an application of our result for the following case of finite family of asymptotically quasi-nonexpansive mappings $\{T_n\}_{n=0}^{N-1}$. Let

$$\|T_i^j x - p\| \leq k_{i,j}\|x - p\|, \quad \forall x \in C, \, p \in F,$$

where F is the common fixed point set of $\{T_n\}_{n=0}^{N-1}$ and $\lim_{j\to\infty} k_{i,j} = 1$ for all $0 \leq i \leq N-1$. The finite family of asymptotically quasi-nonexpansive mappings $\{T_n\}_{n=0}^{N-1}$ is said to be *uniformly L-Lipschitz* if

$$\|T_i^j x - T_i^j y\| \leq L_{i,j}\|x - y\|, \quad \forall x, y \in C$$

for all $i \in \{0, 1, 2, ..., N-1\}$, $j \geq 1$, where $L \geq 1$.

Theorem 3.1. *Let C be a closed convex subset of a Hilbert space H, and let $\{T_n\}_{n=0}^{N-1} : C \to C$ be a uniformly L-Lipschitz finite family of asymptotically quasi-nonexpansive mappings with a nonempty common fixed point set F. Assume that $\alpha_n \in (0,1]$ and $\beta_n, \gamma_n \in [0,1]$ for all $n \in N$. Then $\{x_n\}$ generated by*

$$\begin{cases} x_0 \in C = Q_0, \quad \text{choosen arbitrarily,} \\ y_n = (1-\alpha_n)z_n + \alpha_n T_{i(n)}^{j(n)} z_n, \quad n \geq 0, \\ z_n = (1-\beta_n)t_n + \beta_n T_{i(n)}^{j(n)} t_n, \quad n \geq 0, \\ t_n = (1-\gamma_n)x_n + \gamma_n T_{i(n)}^{j(n)} x_n, \quad n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq [1 + k_{i(n),j(n)}(\alpha_n + \beta_n + \gamma_n - 2\alpha_n\beta_n \\ \qquad -2\alpha_n\gamma_n - 2\beta_n\gamma_n + 3\alpha_n\beta_n\gamma_n) + k_{i(n),j(n)}^2(\alpha_n\beta_n + \alpha_n\gamma_n \\ \qquad +\beta_n\gamma_n - 3\alpha_n\beta_n\gamma_n) + \alpha_n\beta_n\gamma_n k_{i(n),j(n)}^3 - \alpha_n - \beta_n - \gamma_n \\ \qquad +\alpha_n\beta_n + \alpha_n\gamma_n + \beta_n\gamma_n - \alpha_n\beta_n\gamma_n]\|x_n - z\|\} \cap A, \quad n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0, \end{cases}$$

converges strongly to $P_F x_0$, *where* $\overline{co}C_n$ *denotes the closed convex closure of* C_n *for all* $n \geq 1$, $n = (j(n)-1)N + i(n)$ *for all* $n \geq 0$ *and* $A = \{z \in H : \|z - P_F x_0\| \leq 1\}$.

Proof. We can drive the prove from the following two conclusions.
Conclusion 1. $\{T_{n=0}^{N-1}\}_{n=0}^{\infty}$ is a uniformly closed asymptotically family of countable quasi-L_n-Lipschitz mappings from C into itself.
Conclusion 2. $F = \bigcap_{n=0}^{N} F(T_n) = \bigcap_{n=0}^{\infty} F(T_{i(n)}^{j(n)})$, where $F(T_n)$ denotes the fixed point set of the mappings T_n. □

Corollary 3.2. *Let* C *be a closed convex subset of a Hilbert space* H*, and let* $T : C \to C$ *be a L-Lipschitz asymptotically quasi-nonexpansive mappings with nonempty common fixed point set* F*. Assume that* $\alpha_n \in (0,1]$ *and* $\beta_n, \gamma_n \in [0,1]$ *for all* $n \in N$*. Then* $\{x_n\}$ *generated by*

$$\begin{cases} x_0 \in C = Q_0, \quad \text{choosen arbitrarily,} \\ y_n = (1-\alpha_n)z_n + \alpha_n T^n z_n, \quad n \geq 0, \\ z_n = (1-\beta_n)t_n + \beta_n T^n t_n, \quad n \geq 0, \\ t_n = (1-\gamma_n)x_n + \gamma_n T^n x_n, \quad n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq [1 + k_n(\alpha_n + \beta_n + \gamma_n - 2\alpha_n\beta_n \\ \qquad -2\alpha_n\gamma_n - 2\beta_n\gamma_n + 3\alpha_n\beta_n\gamma_n) + k_n^2(\alpha_n\beta_n + \alpha_n\gamma_n \\ \qquad +\beta_n\gamma_n - 3\alpha_n\beta_n\gamma_n) + \alpha_n\beta_n\gamma_n k_n^3 - \alpha_n - \beta_n - \gamma_n \\ \qquad +\alpha_n\beta_n + \alpha_n\gamma_n + \beta_n\gamma_n - \alpha_n\beta_n\gamma_n]\|x_n - z\|\} \cap A, \quad n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0, \end{cases}$$

converges strongly to $P_F x_0$, *where* $\overline{co}C_n$ *denotes the closed convex closure of* C_n *for all* $n \geq 1$ *and* $A = \{z \in H : \|z - P_F x_0\| \leq 1\}$.

Proof. Take $T_n \equiv T$ in Theorem 3.1, we get the desired result. □

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An Intermixed Algorithm for Three Strict Pseudo-contractions in Hilbert Spaces

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Abstract

We generalize an intermixed algorithm to three and m-strict pseudo-contractions in Hilbert spaces and show that this algorithm converges strongly to the fixed points of three and m-strict pseudo-contractions in Hilbert spaces, independently. Consequently, we can find the common fixed points of these mappings.

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1 Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H with its inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. A mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$. We use $Fix(T)$ to denote the set of fixed points of T. A mapping $T : C \to C$ is said to be *strictly pseudo-contractive* if there exists a constant $0 \leq \lambda < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \lambda \|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C.$$

It is well known that every strictly pseudo-contractive mapping is also nonexpansive mapping but a nonexpansive mapping may not be pseudo-contractive mapping. For the

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rest of this article, we reserve C to be a nonempty closed convex subset of a Hilbert space H.

Iterative construction of fixed points is a celebrated idea in these days in the realm of nonlinear mappings. $T : C \to C$ be a nonlinear mapping and $\{\alpha_n\}$ be a real number sequence in $(0, 1)$. For fixed $x_0 \in C$ arbitrarily, define a sequence $\{x_n\}$ by the following manner

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \geq 0. \tag{1.1}$$

which is the Mann's iteration scheme ([11]). If T is a nonexpansive mapping with $Fix(T) \neq \emptyset$ and $\{\alpha_n\}$ satisfies the condition $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by Mann's algorithm converges weakly to a fixed point of T ([14]). Now, it is a common fact that, in infinite-dimensional Hilbert spaces, Mann's algorithm fails to converge strongly . An active area of research today is to develop Iterative methods for nonexpansive mappings; see [1–4, 7–10, 14–19] . But for strict pseudo-contraction mappings, iterative methods are far less developed though Browder and Petryshyn [1] started this work in 1967. Because of some powerful applications, (see Scherzer [15]), we desired to create algorithms for computation of the fixed points of strict pseudo-contraction mappings. As Mann's algorithm is too strong enough to approximate fixed points of pseudo-contractions, we need to find other type of iterative algorithms, see [6, 12, 21]. The first attempt was made by Ishikawa [9] with the following Ishikawa algorithm which can be viewed as a double-step Mann's algorithm.

$$\begin{cases} y_n = (1 - \beta_n)x_n + \beta_n Tx_n, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n, \quad n \geq 0, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in the interval $[0, 1]$, T is a (nonlinear) self-mapping of C, where $x_0 \in C$ arbitrarily. Ishikawa proved that his algorithm converges in norm to a fixed point of a Lipschitz pseudo-contraction T if $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy certain conditions and if T is compact.

In 2000, Noor [13] gave following three step Noor iterative scheme

$$\begin{cases} z_n = (1 - \gamma_n)x_n + \gamma_n Tx_n, \\ y_n = (1 - \beta_n)x_n + \beta_n Tz_n, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n, \quad n \geq 0, \end{cases}$$

In [20], the following algorithm for two strict pseudo-contraction mappings S and T is given which converges strongly.

Algorithm 1.1. *For given $x_0 \in C$, $y_0 \in C$ arbitrarily, let the sequences $\{x_n\}$ and $\{y_n\}$ be generated iteratively by*

$$\begin{cases} x_{n+1} = (1 - \beta_n)x_n + \beta_n P_C[\alpha_n f(y_n) + (1 - k - \alpha_n)x_n + kTx_n], \quad n \geq 0, \\ y_{n+1} = (1 - \beta_n)y_n + \beta_n P_C[\alpha_n g(x_n) + (1 - k - \alpha_n)y_n + kSy_n], \quad n \geq 0, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two real number sequences in $(0, 1)$.

The quest for the answer of the question, can we develop an iterative algorithm which strongly converges to fixed points of finite many strict pseudo-contractions? However the answer of this problem is still not known. In this paper, Our main purpose is to give a redundant intermixed algorithms for three and m-strict pseudo-contractions. It is shown that the above said algorithm converges strongly to the fixed points of three and m-strict pseudo-contractions, independently. As applications, we can find these common fixed points in the settings of Hilbert spaces.

2 Preliminaries

The metric projection from H onto C is defined as: for each point $x \in H$, $P_C x$ is the unique point in C with the property:

$$\|x - P_C x\| \leq \|x - y\|, \quad y \in C,$$

where P_C is given by

$$P_C x \in C, \quad \langle x - P_C x, y - P_C x\rangle \leq 0, \quad y \in C.$$

Consequently, P_C is nonexpansive. Following well-known lemmas will be important for our results.

Lemma 2.1. ([12]) *Let $T : C \to C$ be a λ-strictly pseudo-contractive mapping. Then $I - T$ is demi-closed at 0, that is, if $x_n \rightharpoonup x \in C$ and $x_n - Tx_n \to 0$, then $x = Tx$.*

Lemma 2.2. ([10]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} = (1 - \beta_n)x_n + \beta_n z_n$ for all $n \geq 0$ and $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n\to\infty} \|z_n - x_n\| = 0$.*

Lemma 2.3. ([16]) *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n\delta_n$, $n \geq 0$, where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in R such that*

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=0}^{\infty} |\delta_n\gamma_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3 Main results

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a λ-strict pseudo-contraction. Let $f : C \to H$ be a ρ_1-contraction, $g : C \to H$ be a ρ_2-contraction and $h : C \to H$ be a ρ_3-contraction. Let $k \in (0, 1 - \lambda)$ be a constant.

Now we give the following redundant intermixed algorithm for three strict pseudo-contractions T_1, T_2 and T_3.

Algorithm 3.1. *For given $x_0 \in C$, $y_0 \in C$ and $z_0 \in C$ arbitrarily, let the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be generated iteratively by*

$$\begin{cases} x_{n+1} = (1-\beta_n)x_n + \beta_n P_C[\alpha_n f(y_n) + (1-k-\alpha_n)x_n + kT_1x_n], & n \geq 0, \\ y_{n+1} = (1-\beta_n)y_n + \beta_n P_C[\alpha_n g(z_n) + (1-k-\alpha_n)y_n + kT_2y_n], & n \geq 0, \\ z_{n+1} = (1-\beta_n)z_n + \beta_n P_C[\alpha_n h(x_n) + (1-k-\alpha_n)z_n + kT_3z_n], & n \geq 0, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two real number sequences in $(0, 1)$.

Remark 3.2. Note that this algorithm is said to be the redundant intermixed algorithm as $\{x_n\}$ in $\{z_n\}$ and $\{z_n\}$ is in $\{y_n\}$ and $\{y_n\}$ is in $\{x_n\}$. So we can use this algorithm to find the fixed points of T_1, T_2 and T_3, independently.

Theorem 3.3. *Suppose that $Fix(T_1) \neq \emptyset$, $Fix(T_2) \neq \emptyset$ and $Fix(T_3) \neq \emptyset$. Assume the following conditions are satisfied:*

$(C1)$ $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

$(C2)$ $\beta_n \in [\xi_1, \xi_2] \subset (0, 1)$ *for all* $n \geq 0$.

Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ generated by (3.1) converge strongly to the fixed points $P_{Fix(T_1)}f(y^*)$, $P_{Fix(T_2)}g(x^*)$ and $P_{Fix(T_3)}h(x^*)$ of T_1, T_2 and T_3, respectively, where $x^* \in Fix(T_1)$, $y^* \in Fix(T_2)$ and $z^* \in Fix(T_3)$.

Note that, $P_C[\alpha f + (1-k-\alpha)I + kT]$ is contractive for small enough α, see [20].

First, we give the following propositions.

Proposition 3.4. *The sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are bounded.*

Proof. Since $Fix(T_1) \neq \emptyset$, $Fix(T_2) \neq \emptyset$ and $Fix(T_3) \neq \emptyset$, we can choose $x^* \in Fix(T_1)$, $y^* \in Fix(T_2)$ and $z^* \in Fix(T_3)$. From (3.1), we have

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|(1-\beta_n)x_n + \beta_n P_C[\alpha_n f(y_n) + (1-k-\alpha_n)x_n + kT_1x_n] - x^*\| \\ &\leq \beta_n \|P_C[\alpha_n f(y_n) + (1-k-\alpha_n)x_n + kT_1x_n] - x^*\| \\ &\quad + (1-\beta_n)\|x_n - x^*\| \\ &\leq \beta_n\alpha_n\|f(y_n) - x^*\| + \beta_n\|(1-k-\alpha_n)(x_n - x^*) + k(T_1x_n - T_1x^*)\| \\ &\quad + (1-\beta_n)\|x_n - x^*\| \\ &\leq \beta_n\alpha_n\|f(y_n) - f(y^*)\| + \beta_n\alpha_n\|f(y^*) - x^*\| + (1-\beta_n)\|x_n - x^*\| \\ &\quad + \beta_n(1-\alpha_n)\|x_n - x^*\| \\ &\leq \rho_1\beta_n\alpha_n\|y_n - y^*\| + \beta_n\alpha_n\|f(y^*) - x^*\| + (1-\alpha_n\beta_n)\|x_n - x^*\| \\ &\leq \rho\beta_n\alpha_n\|y_n - y^*\| + \beta_n\alpha_n\|f(y^*) - x^*\| + (1-\alpha_n\beta_n)\|x_n - x^*\|, \end{aligned} \tag{3.2}$$

where $\rho = \max\{\rho_1, \rho_2, \rho_3\}$.

Similarly, we have

$$\begin{aligned} \|y_{n+1} - y^*\| &\leq \rho_2\beta_n\alpha_n\|z_n - z^*\| + \beta_n\alpha_n\|g(z^*) - y^*\| + (1-\alpha_n\beta_n)\|y_n - y^*\| \\ &\leq \rho\beta_n\alpha_n\|z_n - z^*\| + \beta_n\alpha_n\|g(z^*) - y^*\| + (1-\alpha_n\beta_n)\|y_n - y^*\| \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \|z_{n+1} - z^*\| &\leq \rho_3\beta_n\alpha_n\|x_n - x^*\| + \beta_n\alpha_n\|h(x^*) - z^*\| + (1-\alpha_n\beta_n)\|z_n - z^*\| \\ &\leq \rho\beta_n\alpha_n\|x_n - x^*\| + \beta_n\alpha_n\|h(x^*) - z^*\| + (1-\alpha_n\beta_n)\|z_n - z^*\|. \end{aligned} \tag{3.4}$$

By adding (3.2), (3.3) and (3.4), we obtain

$$\begin{aligned}
&\|x_{n+1} - x^*\| + \|y_{n+1} - y^*\| + \|z_{n+1} - z^*\| \\
&\le [1 - (1-\rho)\alpha_n\beta_n](\|x_n - x^*\| + \|y_n - y^*\| + \|z_n - z^*\|) + \alpha_n\beta_n(\|f(y^*) - x^*\| \\
&\quad + \|g(x^*) - y^*\| + \|h(z^*) - z^*\|) \\
&\le \max\Bigg\{\|x_n - x^*\| + \|y_n - y^*\| + \|z_n - z^*\|, \\
&\qquad \frac{\|f(y^*) - x^*\| + \|g(x^*) - y^*\| + \|g(x^*) - z^*\|}{1-\rho}\Bigg\}.
\end{aligned}$$

By induction, we have

$$\begin{aligned}
&\|x_n - x^*\| + \|y_n - y^*\| + \|z_n - z^*\| \\
&\le \max\Bigg\{\|x_0 - x^*\| + \|y_0 - y^*\| + \|z_0 - z^*\|, \\
&\qquad \frac{\|f(y^*) - x^*\| + \|g(z^*) - y^*\| + \|h(x^*) - z^*\|}{1-\alpha}\Bigg\}.
\end{aligned}$$

So, $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are bounded. This completes the proof. □

Proposition 3.5. $\|x_n - T_1x_n\| \to 0$, $\|y_n - T_2y_n\| \to 0$ *and* $\|z_n - T_3z_n\| \to 0$.

Proof. We will prove it for $\{x_n\}$ and $\{z_n\}$, for $\{y_n\}$ it is similar. We first estimate $\|x_{n+1} - x_n\|$. Set $u_n = P_C[\alpha_n f(y_n) + (1-k-\alpha_n)x_n + kT_1x_n], n \ge 0$. It follows that

$$\begin{aligned}
\|u_{n+1} - u_n\| &\le \|\alpha_{n+1}f(y_{n+1}) + (1-k-\alpha_{n+1})x_{n+1} + kT_1x_{n+1} \\
&\quad - \alpha_n f(y_n) - (1-k-\alpha_n)x_n + kT_1x_n\| \\
&\le \|(1-k-\alpha_{n+1})(x_{n+1} - x_n) + k(T_1x_{n+1} - T_1x_n)\| \\
&\quad + \alpha_{n+1}(\|f(y_{n+1})\| + \|x_n\|) + \alpha_n(\|f(y_n)\| + \|x_n\|) \\
&\le (1-\alpha_{n+1})\|x_{n+1} - x_n\| + \alpha_{n+1}(\|f(y_{n+1})\| + \|x_n\|) \\
&\quad + \alpha_n(\|f(y_n)\| + \|x_n\|).
\end{aligned}$$

Since $\alpha_n \to 0$, we deduce that

$$\limsup_{n\to\infty}(\|u_{n+1} - u_n\| - \|x_{n+1} - x_n\|) \le 0.$$

From Lemma 2.2, we get

$$\lim_{n\to\infty}\|u_n - x_n\| = 0 \text{ and } \lim_{n\to\infty}\|x_{n+1} - x_n\| = 0.$$

From (3.1), we derive

$$\begin{aligned}
\|x_{n+1} - T_1x_n\| &\le (1-\beta_n)\|x_n - T_1x_n\| + \beta_n\alpha_n\|f(y_n) - T_1x_n\| \\
&\quad + \beta_n(1-k-\alpha_n)\|x_n - T_1x_n\| \\
&= [1 - (k+\alpha_n)\beta_n]\|x_n - T_1x_n\| + \beta_n\alpha_n\|f(y_n) - T_1x_n\|.
\end{aligned}$$

Thus

$$\begin{aligned}\|x_n - T_1x_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_1x_n\| \\ &\leq [1-(k+\alpha_n)\beta_n]\|x_n - T_1x_n\| + \beta_n\alpha_n\|f(y_n) - T_1x_n\| \\ &\quad + \|x_n - x_{n+1}\|.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_n - T_1x_n\| &\leq \frac{1}{(k+\alpha_n)\beta_n}(\|x_n - x_{n+1}\| + \beta_n\alpha_n\|f(y_n) - T_1x_n\|) \\ &\to 0.\end{aligned}$$

Similarly, we can obtain

$$\lim_{n\to\infty}\|y_n - T_2y_n\| = 0.$$

Now, we will prove

$$\lim_{n\to\infty}\|z_n - T_3z_n\| = 0.$$

Set $w_n = P_C[\alpha_n h(z_n) + (1-k-\alpha_n)z_n + kT_3z_n]$, $n \geq 0$. It follows that

$$\begin{aligned}\|w_{n+1} - w_n\| &\leq \|\alpha_{n+1}h(x_{n+1}) + (1-k-\alpha_{n+1})z_{n+1} + kT_3z_{n+1} \\ &\quad - \alpha_n h(x_n) - (1-k-\alpha_n)z_n + kT_3z_n\| \\ &\leq \|(1-k-\alpha_{n+1})(z_{n+1} - z_n) + k(T_3z_{n+1} - T_3z_n)\| \\ &\quad + \alpha_{n+1}(\|h(x_{n+1})\| + \|z_n\|) + \alpha_n(\|h(x_n)\| + \|z_n\|) \\ &\leq (1-\alpha_{n+1})\|z_{n+1} - z_n\| + \alpha_{n+1}(\|h(x_{n+1})\| + \|z_n\|) \\ &\quad + \alpha_n(\|h(x_n)\| + \|z_n\|).\end{aligned}$$

Since $\alpha_n \to 0$, we deduce that

$$\limsup_{n\to\infty}(\|w_{n+1} - w_n\| - \|z_{n+1} - z_n\|) \leq 0.$$

From Lemma 2.2, we get

$$\lim_{n\to\infty}\|w_n - z_n\| = 0 \text{ and } \lim_{n\to\infty}\|z_{n+1} - z_n\| = 0.$$

From (3.1), we derive

$$\begin{aligned}\|z_{n+1} - T_3z_n\| &\leq (1-\beta_n)\|z_n - T_3z_n\| + \beta_n\alpha_n\|h(x_n) - T_3z_n\| \\ &\quad + \beta_n(1-k-\alpha_n)\|z_n - T_3z_n\| \\ &= [1-(k+\alpha_n)\beta_n]\|z_n - T_3z_n\| + \beta_n\alpha_n\|h(x_n) - T_3z_n\|.\end{aligned}$$

Thus

$$\begin{aligned}\|z_n - T_3z_n\| &\leq \|z_n - z_{n+1}\| + \|z_{n+1} - T_3z_n\| \\ &\leq [1-(k+\alpha_n)\beta_n]\|z_n - T_3z_n\| + \beta_n\alpha_n\|h(x_n) - T_3z_n\| \\ &\quad + \|z_n - z_{n+1}\|.\end{aligned}$$

It follows that

$$\begin{aligned}\|z_n - T_3z_n\| &\leq \frac{1}{(k+\alpha_n)\beta_n}(\|z_n - z_{n+1}\| + \beta_n\alpha_n\|h(x_n) - T_3z_n\|) \\ &\to 0.\end{aligned}$$

This completes the proof. □

Note that the mapping $P_C[\alpha f + (1-k-\alpha)I + kT_1]$ is contractive for small enough α. Thus, the equation $x = P_C[tf(x) + (1-k-t)x + kT_1x]$ has a unique fixed point, denoted by x_t, that is,

$$x_t = P_C[tf(x_t) + (1-k-t)x_t + kT_1x_t] \tag{3.5}$$

for small enough t.

In order to prove Theorem 3.3, we need the following lemma.

Lemma 3.6. *Suppose $Fix(T_i) \neq \emptyset$, $i = 1, 2, 3$. Then as $t \to 0$, the net $\{x_t\}$ defined by* (3.5) *converges strongly to a fixed point of T_i.*

Proof. Let $z^* \in Fix(T_3)$. From (3.5), we have

$$\begin{aligned}\|z_t - z^*\| &= \|P_C[th(z_t) + (1-k-t)z_t + kT_3z_t] - z^*\| \\ &\leq t\|h(z_t) - z^*\| + \|(1-k-t)(z_t - z^*) + k(T_3z_t - z^*)\| \\ &\leq t\rho_1\|z_t - z^*\| + t\|h(z^*) - z^*\| + (1-t)\|z_t - z^*\|,\end{aligned}$$

hence

$$\|z_t - z^*\| \leq \frac{1}{1-\rho_1}\|h(z^*) - z^*\|.$$

Thus, $\{z_t\}$ is bounded. Again, from (3.5), we get

$$\|z_t - T_3z_t\| \leq t\|h(z_t) - T_3z_t\| + (1-k-t)\|z_t - T_3z_t\|.$$

It follows that

$$\|z_t - T_3z_t\| \leq \frac{t}{k+t}\|h(z_t) - T_3z_t\| \to 0.$$

Let $\{t_n\} \subset (0, 1)$. Assume that $t_n \to 0$ as $n \to \infty$. Put $z_n := z_{t_n}$. We have $\lim_{n\to\infty} \|z_n - T_3z_n\| = 0$. Set $m_t = th(z_t) + (1-k-t)z_t + kT_3z_t$, for all t. Then, we have $z_t = P_Cm_t$, and for any $z^* \in Fix(T_3)$,

$$\begin{aligned}z_t - z^* &= z_t - m_t + m_t - z^* \\ &= z_t - m_t + t(h(z_t) - z^*) + (1-k-t)(z_t - z^*) + k(T_3z_t - z^*).\end{aligned}$$

From the property of the metric projection, we deduce

$$\langle z_t - m_t, z_t - z^*\rangle \leq 0.$$

So

$$\begin{aligned}\|z_t - z^*\|^2 &= \langle z_t - m_t, z_t - z^*\rangle + \langle(1-k-t)(z_t - z^*) + k(T_3z_t - z^*), z_t - z^*\rangle \\ &\quad + t\langle h(z_t) - z^*, z_t - z^*\rangle \\ &\leq \|(1-k-t)(z_t - z^*) + k(T_3z_t - z^*)\|\|z_t - z^*\| \\ &\quad + t\langle h(z_t) - h(z^*), z_t - z^*\rangle + t\langle h(z^*) - z^*, z_t - z^*\rangle \\ &\leq [1 - (1-\rho_1)t]\|z_t - z^*\|^2 + t\langle h(z^*) - z^*, z_t - z^*\rangle.\end{aligned}$$

Hence

$$\|z_t - z^*\|^2 \leq \frac{1}{(1-\rho_1)}\langle h(z^*) - z^*, z_t - z^*\rangle, \quad \forall z^* \in Fix(T).$$

By the similar arguments as that in [12], we can obtain that the net $\{z_t\}$ converges strongly to $z^* \in Fix(T_3)$. This completes the proof. □

From Lemma 3.6, we know that the net $\{z_t\}$ defined by $z_t = P_C[tu+(1-k-t)z_t+kT_3z_t]$, where $u \in H$ converges to $P_{Fix(T_3)}u$. Let $z^* \in Fix(T_3)$ and $y^* \in Fix(T_2)$ and $x^* \in Fix(T_1)$. If we take $u = h(z^*)$, then the net $\{z_t\}$ defined by $z_t = P_C[th(z^*) + (1-k-t)z_t + kT_3z_t]$ converges to $P_{Fix(T_3)}h(y^*)$.

Finally, we prove Theorem 3.3.

Proof. Now, we prove that $x_n \to P_{Fix(T_1)}f(y^*)$, $y_n \to P_{Fix(T_2)}g(z^*)$ and $z_n \to P_{Fix(T_3)}h(x^*)$, where $x^* \in Fix(T_1)$, $y^* \in Fix(T_2)$ and $z^* \in Fix(T_3)$. First we observe that, if the sequence $\{w_n\}$ is bounded and $\|w_n - Tw_n\| \to 0$, we easily deduce that

$$\limsup_{n\to\infty}\langle f(P_{Fix(T_2)}g(z^*)) - P_{Fix(T_1)}f(y^*), w_n - P_{Fix(T_1)}f(y^*)\rangle \le 0,$$

$$\limsup_{n\to\infty}\langle g(P_{Fix(T_3)}h(x^*)) - P_{Fix(T_2)}g(z^*), w_n - P_{Fix(T_2)}g(z^*)\rangle \le 0$$

and

$$\limsup_{n\to\infty}\langle h(P_{Fix(T_1)}f(y^*)) - P_{Fix(T_3)}h(x^*), w_n - P_{Fix(T_3)}h(x^*)\rangle \le 0.$$

We set

$$\begin{cases} u_n = P_C[\alpha_n f(y_n) + (1-k-\alpha_n)x_n + kT_1x_n], \quad n \ge 0, \\ v_n = P_C[\alpha_n g(z_n) + (1-k-\alpha_n)y_n + kT_2y_n], \quad n \ge 0, \\ m_n = P_C[\alpha_n h(x_n) + (1-k-\alpha_n)z_n + kT_3z_n], \quad n \ge 0. \end{cases}$$

Thus, we deduce that the sequences $\{u_n\}$, $\{v_n\}$ and $\{m_n\}$ are bounded; and $\|u_n - T_1u_n\| \to 0$, $\|v_n - T_2v_n\| \to 0$ and $\|m_n - T_3m_n\| \to 0$. Therefore,

$$\limsup_{n\to\infty}\langle f(P_{Fix(T_2)}g(z^*)) - P_{Fix(T_1)}f(y^*), u_n - P_{Fix(T_1)}f(y^*)\rangle \le 0,$$

$$\limsup_{n\to\infty}\langle g(P_{Fix(T_3)}h(x^*)) - P_{Fix(T_2)}g(z^*), v_n - P_{Fix(T_2)}g(z^*)\rangle \le 0$$

and

$$\limsup_{n\to\infty}\langle h(P_{Fix(T_1)}f(y^*)) - P_{Fix(T_3)}h(x^*), m_n - P_{Fix(T_3)}h(x^*)\rangle \le 0.$$

Next, we estimate $\|u_n - P_{Fix(T_1)}f(y^*)\|$. Set $\tilde{u}_n = \alpha_n f(y_n) + (1-k-\alpha_n)x_n + kT_1x_n$, $\tilde{v}_n = \alpha_n g(z_n) + (1-k-\alpha_n)y_n + kT_2y_n$ and $\tilde{m}_n = \alpha_n h(x_n) + (1-k-\alpha_n)z_n + kT_3z_n$ for all n.

$$\begin{aligned} &\|U_n - P_{Fix(T_1)}f(y^*)\|^2 \\ &= \|P_C[\tilde{U}_n] - P_{Fix(T_1)}f(y^*)\|^2 \\ &\le \langle \tilde{U}_n - P_{Fix(T_1)}f(y^*), U_n - P_{Fix(T_1)}f(y^*)\rangle \\ &= \langle \alpha_n f(y_n) + (1-k-\alpha_n)x_n + kT_1x_n - P_{Fix(T_1)}f(y^*), U_n - P_{Fix(T_1)}f(y^*)\rangle \\ &\le \alpha_n\langle f(y_n) - P_{Fix(T_1)}f(y^*), U_n - P_{Fix(T_1)}f(y^*)\rangle \\ &\quad + (1-\alpha_n)\|x_n - P_{Fix(T_1)}f(y^*)\|\|U_n - P_{Fix(T_1)}f(y^*)\| \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1-\alpha_n}{2}\|x_n - P_{Fix(T_1)}f(y^*)\|^2 + \frac{1}{2}\|U_n - P_{Fix(T_1)}f(y^*)\|^2 \\
&\quad + \alpha_n\langle f(y_n) - f(P_{Fix(T_2)}g(z^*)), U_n - P_{Fix(T_1)}f(y^*)\rangle \\
&\quad + \alpha_n\langle f(P_{Fix(T_2)}g(z^*)) - P_{Fix(T_1)}f(y^*), U_n - P_{Fix(T_1)}f(y^*)\rangle \\
&\leq \frac{1-\alpha_n}{2}\|x_n - P_{Fix(T_1)}f(y^*)\|^2 + \frac{1}{2}\|U_n - P_{Fix(T_1)}f(y^*)\|^2 \\
&\quad + \alpha_n\rho\|y_n - P_{Fix(T_2)}g(z^*)\|\|U_n - P_{Fix(T_1)}f(y^*)\| \\
&\quad + \alpha_n\langle f(P_{Fix(T_2)}g(z^*)) - P_{Fix(T_1)}f(y^*), U_n - P_{Fix(T_1)}f(y^*)\rangle \\
&\leq \frac{1-\alpha_n}{2}\|x_n - P_{Fix(T_1)}f(y^*)\|^2 + \frac{1}{2}\|U_n - P_{Fix(T_1)}f(y^*)\|^2 \\
&\quad + \frac{\alpha_n\rho}{2}(\|y_n - P_{Fix(T_2)}g(z^*)\|^2 + \|U_n - P_{Fix(T_1)}f(y^*)\|^2) \\
&\quad + \alpha_n\langle f(P_{Fix(T_2)}g(z^*)) - P_{Fix(T_1)}f(y^*), U_n - P_{Fix(T_1)}f(y^*)\rangle,
\end{aligned}$$

so, we have

$$\begin{aligned}
&\|U_n - P_{Fix(T_1)}f(y^*)\|^2 \\
&\leq \frac{1-\alpha_n}{1-\alpha_n\rho}\|x_n - P_{Fix(T_1)}f(y^*)\|^2 + \frac{\alpha_n\rho}{1-\alpha_n\rho}\|y_n - P_{Fix(T_2)}g(z^*)\|^2 \\
&\quad + \frac{2\alpha_n}{1-\alpha_n\rho}\langle f(P_{Fix(T_2)}g(z^*)) - P_{Fix(T_1)}f(y^*), U_n - P_{Fix(T_1)}f(y^*)\rangle.
\end{aligned}$$

Thus

$$\begin{aligned}
&\|x_{n+1} - P_{Fix(T_1)}f(y^*)\|^2 \\
&\leq (1-\beta_n)\|x_n - P_{Fix(T_1)}f(y^*)\|^2 + \beta_n\|U_n - P_{Fix(T_1)}f(y^*)\|^2 \\
&\leq (1 - \frac{1-\rho}{1-\alpha_n\rho}\alpha_n\beta_n)\|x_n - P_{Fix(T_1)}f(y^*)\|^2 + \frac{\alpha_n\beta_n\rho}{1-\alpha_n\rho}\|y_n - P_{Fix(T_2)}g(z^*)\|^2 \\
&\quad + \frac{2\alpha_n\beta_n}{1-\alpha_n\rho}\langle f(P_{Fix(T_2)}g(z^*)) - P_{Fix(T_1)}f(y^*), U_n - P_{Fix(T_1)}f(y^*)\rangle.
\end{aligned}$$

Similarly

$$\begin{aligned}
&\|y_{n+1} - P_{Fix(T_2)}g(z^*)\|^2 \\
&\leq \left(1 - \frac{1-\rho}{1-\alpha_n\rho}\alpha_n\beta_n\right)\|y_n - P_{Fix(T_2)}g(z^*)\|^2 + \frac{\alpha_n\beta_n\rho}{1-\alpha_n\rho}\|z_n - P_{Fix(T_3)}h(y^*)\|^2 \\
&\quad + \frac{2\alpha_n\beta_n}{1-\alpha_n\rho}\langle g(P_{Fix(T_3)}h(y^*)) - P_{Fix(T_2)}g(z^*), V_n - P_{Fix(T_2)}g(z^*)\rangle
\end{aligned}$$

and

$$\begin{aligned}
&\|z_{n+1} - P_{Fix(T_3)}h(x^*)\|^2 \\
&\leq \left(1 - \frac{1-\rho}{1-\alpha_n\rho}\alpha_n\beta_n\right)\|z_n - P_{Fix(T_3)}h(x^*)\|^2 + \frac{\alpha_n\beta_n\rho}{1-\alpha_n\rho}\|x_n - P_{Fix(T_1)}f(y^*)\|^2 \\
&\quad + \frac{2\alpha_n\beta_n}{1-\alpha_n\rho}\langle h(P_{Fix(T_1)}f(y^*)) - P_{Fix(T_3)}h(x^*), M_n - P_{Fix(T_3)}h(x^*)\rangle.
\end{aligned}$$

Combing all above, we have

$$\begin{aligned}
&\|x_{n+1} - P_{Fix(T_1)} f(y^*)\|^2 + \|y_{n+1} - P_{Fix(T_2)} g(z^*)\|^2 + \|z_{n+1} - P_{Fix(T_3)} h(x^*)\|^2 \\
&\leq \left(1 - \frac{1-\rho}{1-\alpha_n\rho}\alpha_n\beta_n\right)(\|x_n - P_{Fix(T_1)} f(y^*)\|^2 + \|y_n - P_{Fix(T_2)} g(z^*)\|^2) \\
&\quad + \|z_n - P_{Fix(T_3)} h(x^*)\|^2) \\
&\quad + \frac{2\alpha_n\beta_n}{1-\alpha_n\rho}\langle f(P_{Fix(T_2)} g(z^*)) - P_{Fix(T_1)} f(y^*), U_n - P_{Fix(T_1)} f(y^*)\rangle \\
&\quad + \frac{2\alpha_n\beta_n}{1-\alpha_n\rho}\langle g(P_{Fix(T_3)} h(x^*)) - P_{Fix(T_2)} g(z^*), V_n - P_{Fix(T_2)} g(z^*)\rangle \\
&\quad + \frac{2\alpha_n\beta_n}{1-\alpha_n\rho}\langle h(P_{Fix(T_1)} f(y^*)) - P_{Fix(T_3)} h(x^*), M_n - P_{Fix(T_3)} h(x^*)\rangle.
\end{aligned}$$

Therefore, $x_n \to P_{Fix(T_1)} f(y^*)$, $y_n \to P_{Fix(T_2)} g(z^*)$ and $z_n \to P_{Fix(T_3)} h(x^*)$. This completes the proof. □

4 An redundant intermixed algorithm for m-strict pseudo-contractions

Let $T_i : C \to C$ be λ-strict pseudo-contractions, $f_i : C \to H$ be ρ_i-contractions for $i = 1, 2, 3, ..., m$ and $k \in (0, 1-\lambda)$ be a constant.

We propose the following redundant intermixed algorithm for m-strict pseudo-contraction mappings T_i for $i = 1, 2, 3, ..., m$.

Algorithm 4.1.

$$\begin{cases}
x^1_{n+1} = (1-\beta_n)x^1_n + \beta_n P_C[\alpha_n f_1(x^2_n) + (1-k-\alpha_n)x^1_n + kT_1 x^1_n], & n \geq 0, \\
x^2_{n+1} = (1-\beta_n)x^2_n + \beta_n P_C[\alpha_n f_2(x^3_n) + (1-k-\alpha_n)x^2_n + kT_2 x^2_n], & n \geq 0, \\
x^3_{n+1} = (1-\beta_n)x^3_n + \beta_n P_C[\alpha_n f_3(x^4_n) + (1-k-\alpha_n)x^3_n + kT_3 x^3_n], & n \geq 0, \\
\qquad \vdots \\
x^m_{n+1} = (1-\beta_n)x^m_n + \beta_n P_C[\alpha_n f_4(x^1_n) + (1-k-\alpha_n)x^m_n + kT_m x^m_n], & n \geq 0,
\end{cases} \tag{4.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two real number sequences in $(0, 1)$.

Theorem 4.2. *Suppose that $Fix(T_i) \neq \emptyset$. Assume the following conditions are satisfied:*
$(C1)$ $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$;
$(C2)$ $\beta_n \in [\xi_1, \xi_2] \subset (0, 1)$ for all $n \geq 0$.
Then the sequences $\{x^i_n\}$ generated by (4.1) converge strongly to the fixed points $P_{Fix(T_i)} f_i(x^*)$ of T_i, where $x^{i*} \in Fix(T_i)$ for all $i = 1, 2, 3, ..., m$.

5 Conclusions

In this article, we presented an intermixed algorithm for three and m-strict pseudo-contractions in Hilbert spaces which are extensions of the results in [20]. We also proved that, the above algorithm converges strongly to the fixed points for three and m-strict pseudo-contractions in Hilbert spaces, independently. Consequently, we can find the common fixed points of three and m-strict pseudo-contractions in Hilbert spaces.

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ON FIXED POINT THEOREMS IN DUALISTIC PARTIAL METRIC SPACES

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Abstract. In this paper, we introduce dualistic contractive mappings and use such mappings to prove some fixed point theorems. The results extend various comparable results existing in the literature. Moreover, we give examples that show the superiority and effectiveness of our results among corresponding fixed point theorems in partial metric spaces.

Keywords: Fixed point, dualistic partial metric, monotone mapping.
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1. Introduction and preliminaries

In [6], Matthews introduced the concept of partial metric space as a suitable mathematical tool for program verification and proved an analogue of Banach fixed point theorem in complete partial metric spaces. O'Neill [7] introduced the concept of dualistic partial metric, which is more general than partial metric and established a robust relationship between dualistic partial metric and quasi metric. In [10], Oltra and Valero presented a Banach fixed point theorem on complete dualistic partial metric spaces. They also showed that the contractive condition in Banach fixed point theorem in complete dualistic partial metric spaces cannot be replaced by the contractive condition of Banach fixed point theorem for complete partial metric spaces. Later, Valero [10] generalized the main theorem of [9] using nonlinear contractive condition instead of Banach contractive condition.

Alghamdi et. al.[1], presented the following theorems in partial metric spaces, which are stated below:

Theorem 1. *Let (X, p) be a complete partial metric space and let $T : X \to X$ be a weakly contractive mapping. Then T has a unique fixed point $x^* \in X$ and the Picard iterative sequence $\{T^n(x)\}_{n\in\mathbb{N}}$ converges to x^* with respect to $\tau(p^s)$, for every $x \in X$. Moreover, $p(x^*, x^*) = 0$.*

Theorem 2. *Let (X, p) be a complete partial metric space and let $T : X \to X$ be a Kannan mapping. Then T has a unique fixed point $x \in X$ and the Picard iterative sequence $\{T^n(x)\}_{n\in\mathbb{N}}$ converges to x^* with respect to $\tau(p^s)$, for every $x \in X$. Moreover, $p(x^*, x^*) = 0$.*

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We shall prove new fixed point theorems that generalize fixed point theorems provided by Alghamdi, Shahzad and Valero in [1]. We will show, with the help of examples, that the new results allow us to find fixed points of mappings in some cases in which the results in partial metric spaces cannot be applied. The key feature in these fixed point theorems is that the contractivity condition on the nonlinear map is only assumed to hold on elements that are comparable in the partial order. However, the map is assumed to be monotone.

Throughout, in this paper, the letters $\mathbb{R}^+$, $\mathbb{R}$ and $\mathbb{N}$ will represent the set of nonnegative real numbers, real numbers and positive integers, respectively.
Let us recall some mathematical basics of dualistic partial metric space to make this paper self-sufficient.

Definition 1. [7] A dualistic partial metric on a nonempty set X is a function $D : X\times X \longrightarrow \mathbb{R}$ satisfying the following properties, for all $x, y, z, \in X$:

(D_1) $x = y \Leftrightarrow D(x, x) = D(y, y) = D(x, y)$.
(D_2) $D(x, x) \leq D(x, y)$.
(D_3) $D(x, y) = D(y, x)$.
(D_4) $D(x, z) \leq D(x, y) + D(y, z) - D(y, y)$.
And the pair (X, D) represents a dualistic partial metric space.

If (X, D) is a dualistic partial metric space, then the function $d_D : X \times X \to \mathbb{R}^+$ defined by

$$d_D(x, y) = D(x, y) - D(x, x)$$

is a quasi metric on X such that $\tau(D) = \tau(d_D)$ for all $x, y \in X$.

Remark 1. It is obvious that every partial metric is a dualistic partial metric but the converse is not true. To support this comment, define $D_\vee : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$D_\vee(x, y) = x \vee y = \sup\{x, y\}$$

for all $x, y \in \mathbb{R}$. It is clear that $D_\vee$ is a dualistic partial metric. Note that $D_\vee$ is not a partial metric, since $D_\vee(-1, -2) = -1 \notin \mathbb{R}^+$. However, the restriction of $D_\vee$ to $\mathbb{R}^+$, $D_\vee|_{\mathbb{R}^+}$, is a partial metric.

Example 1. If (X, d) is a metric space and $c \in \mathbb{R}$ is arbitrary constant, then

$$D(x, y) = d(x, y) + c.$$

defines a dualistic partial metric on X.

Example 2. Let $X = \mathbb{R}$ and define the function $D : X \times X \to \mathbb{R}$ by

$$D(x, y) = x + y - xy$$

for all $x \leq y \wedge 1$. Then (X, D) is a dualistic partial metric space.

Following [7], each dualistic partial metric D on X generates a T_0 topology $\tau(D)$ on X which has, as a base, the family of D-balls $\{B_D(x, \epsilon) : x \in X, \epsilon > 0\}$ and $B_D(x, \epsilon) = \{y \in X : D(x, y) < \epsilon + D(x, x)\}$.

Definition 2. [7] Let (X, D) be a dualistic partial metric space.

(1) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in (X, D) converges to a point $x \in X$ if and only if $D(x, x) = \lim_{n\to\infty} D(x, x_n)$.
(2) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in (X, D) is called a Cauchy sequence if $\lim_{n,m\to\infty} D(x_n, x_m)$ exists and is finite.
(3) A dualistic partial metric space (X, D) is said to be complete if every Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converges, with respect to $\tau(D)$, to a point $x \in X$ such that $D(x, x) = \lim_{n,m\to\infty} D(x_n, x_m)$.

Following lemma will be helpful in the sequel.

Lemma 1. [7, 10]

(1) A dualistic partial metric (X, D) is complete if and only if the metric space (X, d^s_D) is complete.
(2) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converges to a point $x \in X$, with respect to $\tau(d^s_D)$ if and only if $\lim_{n\to\infty} D(x, x_n) = D(x, x) = \lim_{n,m\to\infty} D(x_n, x_m)$.
(3) If $\lim_{n\to\infty} x_n = \upsilon$ such that $D(\upsilon, \upsilon) = 0$ then $\lim_{n\to\infty} D(x_n, y) = D(\upsilon, y)$ for every $y \in X$.

Later on, Oltra and Valero [9] established a Banach fixed point theorem for dualistic partial metric spaces in such a way that the Matthews fixed point theorem is obtained as a particular case. The aforesaid result can be stated as follows:

Theorem 3. *Let (X, D) be a complete dualistic partial metric space and let $T : X \to X$ be a mapping such that there exists $\alpha \in [0, 1[$ satisfying*

$$|D(T(x), T(y))| \le \alpha |D(x, y)|,$$

for all $x, y \in X$. Then T has a unique fixed point $x^* \in X$. Moreover, $D(x^*, x^*) = 0$ and the Picard iterative sequence $\{T^n(x_0)\}_{n\in\mathbb{N}}$ converges to x^* with respect to $\tau(d^s_D)$, for every $x \in X$.

2. Main results

In this section, we shall prove the dualistic partial metric versions of Theorems 1 and 2.

Definition 3. *Let $(X, \preceq, D)$ be an ordered dualistic partial metric space. A self map T defined on X is said to be a Kannan type dualistic contractive mapping if there exists $k \in [0, 1[$ such that*

$$|D(T(x), T(y))| \le \frac{k}{2}[|D(x, T(x))| + |D(y, T(y))|] \tag{2.1}$$

for all comparable $x, y \in X$.

Our first main result is given below.

Theorem 4. *Let $(X, \preceq)$ be a partially ordered set and (X, D) be a complete dualistic partial metric space. Let $T : X \to X$ be a nondecreasing mapping. If T satisfies following conditions;*

(1) *T is a Kannan type dualistic contractive mapping.*

(2) *there exists* $x_0 \in X$ *such that* $x_0 \preceq T(x_0)$.
(3) *if* $\{x_n\}$ *is a nondecreasing sequence in* X *such that* $\{x_n\} \to x \in X$, *then* $x_n \preceq x$.

Then T *has a fixed point* x^* *such that* $D(x^*, x^*) = 0$.

Proof. Let us consider the Picard iterative sequence $\{x_n\}_{n\in\mathbb{N}}$ with initial point $x_0 \in X$ (i.e., $x_n = T(x_{n-1})$ for all $n \in \mathbb{N}$). Of course, if there exists $n \in \mathbb{N}$ such that $x_n = x_{n+1} = T(x_n)$, then x_n is a fixed point of T. On the other hand, if $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$, then $x_n \preceq x_{n+1}$. Indeed by $x_0 \preceq T(x_0)$, we obtain $x_0 \preceq x_1$. Since T is nondecreasing, $x_0 \preceq x_1$ implies $T(x_0) \preceq T(x_1)$ and so $x_1 \preceq x_2$. Continuing in this way, we get

$$x_0 \preceq x_1 \preceq x_2 \preceq x_3 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots$$

Since $x_n \preceq x_{n+1}$ for each $n \in \mathbb{N}$, using contractive condition (2.1), we have

$$\begin{aligned} |D(x_1, x_2)| &= |D(T(x_0), T(x_1))| \\ &\leq \frac{k}{2}[|D(x_0, T(x_0))| + |D(x_1, T(x_1))|] \\ &= \frac{k}{2}[|D(x_0, x_1)| + |D(x_1, x_2)|], \end{aligned}$$

which implies

$$(1 - \frac{k}{2})|D(x_1, x_2)| \leq \frac{k}{2}|D(x_0, x_1)|$$

and so

$$|D(x_1, x_2)| \leq \lambda |D(x_0, x_1)|,$$

where $\lambda = \frac{k}{2-k}$ and $0 < \lambda < 1$.

Similarly,

$$\begin{aligned} |D(x_2, x_3)| &= |D(T(x_1), T(x_2))| \\ &\leq \frac{k}{2}[|D(x_1, T(x_1))| + |D(x_2, T(x_2))|]. \end{aligned}$$

Thus,

$$|D(x_2, x_3)| \leq \lambda |D(x_1, T(x_1))| \leq \lambda^2 |D(x_0, x_1)|.$$

Continuing in this way, we have

$$|D(x_n, x_{n+1})| \leq \lambda^n |D(x_0, x_1)|. \tag{2.2}$$

Since $x_n \preceq x_n$, from the contractive condition (2.1), we get

$$|D(x_n, x_n)| \leq k\lambda^{n-1} |D(x_0, x_1)|. \tag{2.3}$$

In order to prove that $\{x_n\}$ is a Cauchy sequence in (X, D), we shall prove that $\{x_n\}$ is a Cauchy sequence in (X, d_D^s). Clearly,

$$\begin{aligned} D(x_n, x_{n+1}) - D(x_n, x_n) &\leq |D(x_n, x_{n+1})| + |D(x_n, x_n)| \\ &\leq \lambda^n |D(x_0, x_1)| + k\lambda^{n-1}|D(x_0, x_1)| \\ &\leq \lambda^n (3-k)|D(x_0, x_1)| \end{aligned}$$

for all $n \in \mathbb{N}$. Thus for a fixed $p \in \mathbb{N}$,

$$D(x_{n+p-1}, x_{n+p}) - D(x_{n+p-1}, x_{n+p-1}) \leq \lambda^{n+p-1}(3-k)|D(x_0, x_1)| \tag{2.4}$$

for all $n \in \mathbb{N}$.

Now using (D_4) and (2.4), we have

$$\begin{aligned} D(x_n, x_{n+p}) - D(x_n, x_n) &\leq D(x_n, x_{n+1}) + D(x_{n+1}x_{n+2}) + \dots \\ &+ D(x_{n+p-1}, x_{n+p}) - \sum_{i=0}^{\eta-1} D(x_{n+i}, x_{n+i}) \\ &\leq (\lambda^n + \lambda^{n+1} + \dots + \lambda^{n+p-1})(3-k)|D(x_0, x_1)| \\ &\leq \frac{\lambda^n}{1-\lambda}(3-k)|D(x_0, x_1)|. \end{aligned}$$

Similarly,

$$D(x_{n+p}, x_n) - D(x_{n+p}, x_{n+p}) \leq \frac{\lambda^n}{1-\lambda}(1+k)|D(x_0, x_1)|.$$

Consequently,

$$d_D^s(x_n, x_m) \leq 4\frac{\lambda^n}{1-\lambda}|D(x_0, x_1)|$$

for all $n + p = m > n \in \mathbb{N}$

This leads to $\lim_{n,m\to\infty} d_D^s(x_n, x_m) = 0$. Thus, $\{x_n\}$ is a Cauchy sequence in (X, d_D^s). Since (X, D) is a complete dualistic partial metric space, by Lemma 1, (X, d_D^s) is also complete and there exists $x^* \in (X, d_D^s)$ such that $x_n \to x^*$ as $n \to \infty$, i.e.,

$$\lim_{n\to\infty} d_D^s(x_n, x^*) = 0.$$

By Lemma 1, we have

$$\lim_{n\to\infty} D(x^*, x_n) = D(x^*, x^*) = \lim_{n,m\to\infty} D(x_n, x_m). \tag{2.5}$$

Since $\lim_{n,m\to\infty} d_D(x_n, x_m) = 0$, the inequality (2.3) implies that $\lim_{n,m\to\infty} D(x_n, x_m) = 0$, which shows that $\{x_n\}$ is a Cauchy sequence in (X, D). From (2.5), we get

$$D(x^*, x^*) = \lim_{n\to\infty} D(x_n, x^*) = 0. \tag{2.6}$$

Now, it follows from the hypotheses (3), (2.1) and (D_4) that

$$\begin{aligned} D(x^*, T(x^*)) &\leq D(x^*, x_n) + D(x_n, T(x^*)) - D(x_n, x_n), \\ &\leq D(x^*, x_n) + |D(x_n, T(x^*))| + |D(x_n, x_n)|, \\ &\leq D(x^*, x_n) + \frac{k}{2}[|D(x_{n-1}, x_n)| + D(x^*, T(x^*))] + |D(x_n, x_n)|. \end{aligned}$$

Hence we obtain

$$(1 - \frac{k}{2})D(x^*, T(x^*)) \leq D(x^*, x_n) + \frac{k}{2}|D(x_{n-1}, x_n)| + |D(x_n, x_n)|.$$

Letting $n \to \infty$ and using (2.3) and (2.2), we obtain

$$(1 - \frac{k}{2})D(x^*, T(x^*)) \leq 0.$$

and so $D(x^*, T(x^*)) \leq 0$, but also $0 = D(x^*, x^*) \leq D(x^*, T(x^*))$. We deduce that

$$D(x^*, Tx^*) = D(x^*, x^*) = D(T(x^*), T(x^*)) = 0.$$

This implies that $x^* = T(x^*)$. Hence x^* is a fixed point of T with $D(x^*, x^*) = 0$ and $\{T^n(x)\}_{n\in\mathbb{N}}$ converges to x^* with respect to $\tau(d_D^s)$ for any $x \in X$. □

Remark 2. *In case when $D(x, y) \in \mathbb{R}^+$ for all $x, y \in X$, Theorem 4 reduces to Theorem 2.*

A natural question that can be raised is whether the contractive condition in the statement of Theorem 4 can be replaced by the contractive condition in the statement of Theorem 2. The following easy example provides a negative answer to this question.

Example 3. *Consider the complete ordered dualistic partial metric $(\mathbb{R}, \leq, D_\vee)$. Define the self-mapping $T_0 : \mathbb{R} \to \mathbb{R}$ by*

$$T_0(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ -1 & \text{if } x = 0 \end{cases}.$$

It is easy to check that T_0 is nondecreasing with respect to usual order on $\mathbb{R}$ and for all comparable $x, y \in \mathbb{R}$, contractive condition

$$D_\vee(T_0(x), T_0(y)) \leq \frac{1}{2}[D_\vee(x, T_0(x)) + D_\vee(y, T_0(y))]$$

holds. However, T_0 does not have a fixed point. Observe that T_0 does not satisfy the contractive condition in the statement of Theorem 4. Indeed, note that for all $k \in [0, 1[$, we have

$$\begin{aligned} 1 = |D_\vee(-1, -1)| = |D_\vee(T_0(0), T_0(0))| &> \frac{k}{2}[|D_\vee(0, T_0(0))| + |D_\vee(0, T_0(0))|] \\ &= k|(0 \vee (-1)| = 0. \end{aligned}$$

For next result, we begin with following definition.

Definition 4. *Let $(X, \preceq, D)$ be an ordered dualistic partial metric space. A mapping $T : X \to X$ is said to be a weakly dualistic contractive if there exists $\alpha : X \times X \to [0,1[$ such that for all $0 \leq a \leq b$*

$$\theta(a,b) = \sup\{\alpha(x,y) : a \leq |D(x,y)| \leq b\} < 1,$$

and for all comparable $x, y \in X$

$$|D(T(x), T(y))| \leq \alpha(x,y)|D(x,y)|. \tag{2.7}$$

Example 4. *Consider $([-1,1], \leq, D_\vee)$ an ordered dualistic partial metric space. Define the mapping $T_3 : X \to X$ by*

$$T_3(x) = \frac{x^3}{x^2+1}$$

for all $x \in X$. We define $\alpha : [-1,1] \times [-1,1] \to [0,1]$ by

$$\alpha(x,y) = \begin{cases} \dfrac{D_\vee(T_3x, T_3y)}{D_\vee(x,y)} & \text{if } D_\vee(x,y) \neq 0 \\ \\ 0 & \text{if } D_\vee(x,y) = 0 \end{cases}.$$

Observe that $\dfrac{D_\vee(T_3x, T_3y)}{D_\vee(x,y)} > 0$ provided that $D_\vee(x,y) \neq 0$. It is easy to check that $\alpha(x,y) \leq \frac{1}{2}$ for all comparable $x, y \in [-1,1]$ and that $\theta(a,b) < 1$ for all $a, b \in \mathbb{R}$ with $0 \leq a \leq b$. Moreover,

$$|D_\vee(T_3x, T_3y)| \leq \alpha(x,y)|D_\vee(x,y)|$$

for all comparable $x, y \in [-1,1]$.

Theorem 5. *Let $(X, \preceq)$ be a partially ordered set and (X, D) be a complete dualistic partial metric space. Let $T : X \to X$ be a nondecreasing mapping. Assume that T satisfies following conditions;*

1. *T is a weakly dualistic contractive mapping.*
2. *there exists $x_0 \in X$ such that $x_0 \preceq T(x_0)$.*
3. *either T is continuous or if $\{x_n\}$ is a nondecreasing sequence in X such that $\{x_n\} \to x \in X$, then $x_n \preceq x$.*

Then T has a fixed point x^* with $D(x^*, x^*) = 0$.

Proof. Consider the Picard iterative sequence $\{x_n\}_{n\in\mathbb{N}}$ with an initial point $x_0 \in X$ (i.e., $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$). It is clear that if there exists $n \in \mathbb{N}$ such that $x_n = x_{n+1}$, then x_n is a fixed point of T. On the other hand, if $x_n \neq x_{n+1}$ for each $n \in \mathbb{N}$, then $x_n \preceq x_{n+1}$. Indeed by $x_0 \preceq T(x_0)$, we obtain $x_0 \preceq x_1$. Since T is nondecreasing, $x_0 \preceq x_1$ implies $T(x_0) \preceq T(x_1)$, and so $x_1 \preceq x_2$. Continuing in this way, we get

$$x_0 \preceq x_1 \preceq x_2 \preceq x_3 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots$$

Since $x_n \preceq x_{n+1}$ for each $n \in \mathbb{N}$, using contractive condition (2.7), we have

$$\begin{aligned}|D(x_n, x_{n+1})| &= |D(T(x_{n-1}), T(x_n))| \\ &\leq \alpha(x_{n-1}, x_n)|D(x_{n-1}, x_n)| \\ &\leq |D(x_{n-1}, x_n)|.\end{aligned}$$

This implies that the sequence $\{|D(x_n, x_{n+1})|\}_{n\in\mathbb{N}}$ is decreasing and bounded below. So it converges to $r \in \mathbb{R}$ with

$$r = \inf_{n\in\mathbb{N}} |D(x_{n-1}, x_n)| \geq 0.$$

We claim that $r = 0$. For the purpose of contradiction, assume $r > 0$.

$$0 < r \leq |D(x_n, x_{n+1})| \leq |D(x_{n-1}, x_n)| \leq \cdots \leq |D(x_0, x_1)|.$$

It implies $0 < r \leq |D(x_0, x_1)|$ and so we deduce that

$$\theta = \theta(r, |D(x_0, x_1)|) = \sup\{\alpha(x, y) : \quad r \leq |D(x, y)| \leq |D(x_0, x_1)|\} < 1.$$

Now from contractive conition (2.7), we get

$$\begin{aligned} r &\leq |D(x_n, x_{n+1})| \\ &\leq \alpha(x_{n-1}, x_n)|D(x_{n-1}, x_n)| \\ &\leq \theta(r, |D(x_0, x_1)|)|D(x_{n-1}, x_n)| \\ &\leq \theta^2(r, |D(x_0, x_1)|)|D(x_{n-2}, x_{n-1})| \leq \ldots \\ &\leq \theta^n(r, |D(x_0, x_1)|)|D(x_0, x_1)|.\end{aligned}$$

Therefore,

$$r \leq \lim_{n\to\infty} \theta^n(r, |D(x_0, x_1)|)|D(x_0, x_1)|.$$

This implies that $r \leq 0$, which is a contradiction. Consequently, $r = 0$ and hence

$$\lim_{n\to\infty} |D(x_n, x_{n+1})| = 0 = \lim_{n\to\infty} D(x_n, x_{n+1}) = 0. \tag{2.8}$$

Now since $x_n \preceq x_n$, by arguing like above, we can show that $\lim_{n\to\infty} |D(x_n, x_n)| = \lim_{n\to\infty} D(x_n, x_n) = 0$, since

$$|D(x_n, x_n)| \leq \alpha(x_{n-1}, x_{n-1})|D(x_{n-1}, x_{n-1})|$$

for all $n \in \mathbb{N}$ and thus the sequence $\{|D(x_n, x_n)|\}_{n\in\mathbb{N}}$ is decreasing and bounded below.

Next we show that $\{x_n\}$ is a Cauchy sequence in the metric space (X, d^s_D). It is clear that

$$\begin{aligned} D(x_n, x_{n+1}) - D(x_n, x_n) &\leq \theta^n(0, |D(x_0, x_1)|)|D(x_0, x_1)| + \theta^n(0, |D(x_0, x_0)|)|D(x_0, x_0)| \\ &\leq \theta^n \left[|D(x_0, x_1)| + |D(x_0, x_0)|\right]\end{aligned}$$

for all $n \in \mathbb{N}$, where $\theta^n = (\theta^n(0, |D(x_0, x_1)|) \vee \theta^n(0, |D(x_0, x_0)|))$ for all $n \in \mathbb{N}$. This implies that, for a fixed $p \in \mathbb{N}$, we have

$$\begin{aligned} D(x_n, x_{n+p}) - D(x_n, x_n) &\leq D(x_n, x_{n+1}) + D(x_{n+1}x_{n+2}) + \ldots \\ &+ D(x_{n+p-1}, x_{n+p}) - \sum_{i=0}^{p-1} D(x_{n+i}, x_{n+i}) \\ &\leq (\theta^n + \theta^{n+1} + \ldots + \theta^{n+p-1})[|D(x_0, x_1)| + |D(x_0, x_0)|] \\ &\leq \frac{\theta^n}{1-\theta}[|D(x_0, x_1)| + |D(x_0, x_0)|] \end{aligned}$$

for all $n \in \mathbb{N}$. Similarly, we can calculate that

$$D(x_{n+p}, x_n) - D(x_{n+p}, x_{n+p}) \leq \frac{\theta^n}{1-\theta}[|D(x_0, x_1)| + |D(x_0, x_0)|],$$

which implies that $\lim_{n\to\infty} d^s_D(x_n, x_{n+p}) = 0$. Hence $\{x_n\}$ is a Cauchy sequence in (X, d^s_D). Since (X, D) is a complete dualistic partial metric space, by Lemma 1, (X, d^s_D) is also complete and there exists $x^* \in (X, d^s_D)$ such that $x_n \to x^*$ as $n \to \infty$, i.e., $\lim_{n\to\infty} d^s_D(x_n, x^*) = 0$. Now again from Lemma 1, we get

$$D(x^*, x^*) = \lim_{n\to\infty} D(x_n, x^*) = \lim_{n,m\to\infty} D(x_n, x_m); \qquad m = n + p. \tag{2.9}$$

Now since $\lim_{n,m\to\infty} d_D(x_n, x_m) = 0$, $\lim_{n,m\to\infty}[D(x_n, x_m) - D(x_n, x_n)] = 0$ and

$$\lim_{n,m\to\infty} D(x_n, x_m) = \lim_{n\to\infty} D(x_n, x_n)$$

but (2.8) implies that

$$\lim_{n\to\infty} D(x_n, x_n) = 0.$$

It follows directly that

$$D(x^*, x^*) = \lim_{n\to\infty} D(x_n, x^*) = 0. \tag{2.10}$$

Now if T is continuous, then

$$x^* = \lim_{n\to\infty} x_n = \lim_{n\to\infty} T^n(x_0) = \lim_{n\to\infty} T^{n+1}(x_0) = T(\lim_{n\to\infty} T^n(x_0)) = T(x^*).$$

Now if T is discontinuous, then by the hypotheses (3), we have

$$\begin{aligned} D(x^*, T(x^*)) &\leq D(x^*, x_n) + D(x_n, T(x^*)) - D(x_n, x_n) \\ &\leq D(x^*, x_n) + |D(x_n, T(x^*))| + |D(x_n, x_n)| \\ &\leq D(x^*, x_n) + \alpha(x_{n-1}, x^*)|D(x_{n-1}, x^*)| + |D(x_n, x_n)| \\ &\leq D(x^*, x_n) + |D(x_{n-1}, x^*)| + |D(x_n, x_n)|. \end{aligned}$$

Since $\lim_{n\to\infty} D(x_n, x_n) = \lim_{n\to\infty} D(x_n, x^*) = 0$, $D(x^*, T(x^*)) \leq 0$, but also

$$0 = D(x^*, x^*) \leq D(x^*, T(x^*)).$$

We deduce that

$$D(x^*, Tx^*) = D(x^*, x^*) = D(T(x^*), T(x^*)) = 0.$$

This implies that $x^* = T(x^*)$. Hence x^* is a fixed point of T with $D(x^*, x^*)$. □

Remark 3. Since every dualistic partial metric is an extension of partial metric, Theorem 5 is an extension of Theorem 1.

There arises the following natural question:

Whether the contractive condition in the statement of Theorem 5 can be replaced by the contractive condition in Theorem 1?

The following example provides a negative answer to the above question.

Example 5. *Consider the complete ordered dualistic partial metric* $(\mathbb{R}, \leq, D_\vee)$ *and the self-mapping* T_0 *defined as in Example 3. Then, for fixed* $k \in [0, 1[$*, it is easy to verify that for all comparable* $x, y \in \mathbb{R}$*, the contractive condition*

$$D_\vee(T_0(x), T_0(y)) \leq \alpha(x, y) D_\vee(x, y)$$

holds with $\alpha(x, y) = k$*. However,* T_0 *does not have a fixed point. Observe that* T_0 *does not satisfy the contractive condition of Theorem 5. Indeed, there is no mapping* $\alpha : \mathbb{R} \times \mathbb{R} \to [0, 1[$ *such that*

$$1 = |D_\vee(-1, -1)| = |D_\vee(T_0(0), T_0(0))| > \alpha(0, 0)|D_\vee(0, 0)| = 0.$$

Remark 4. Significance of the above results lies in the fact that these results are true for all real numbers whereas such results proved in partial metric spaces are only true for positive real numbers.

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Approximation of a kind of new Stancu-Bézier type operators

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Abstract. In this paper, a kind of new Stancu-Bézier type operators is introduced. The Korovkin type approximation theorem of these operators is investigated. The rates of convergence of these operators are studied by means of modulus of continuity. Then, by using the Ditzian-Totik modulus of smoothness, a direct theorem concerned with an approximation for these operators is also obtained.

Keywords: Stancu-Bézier type operators; Korovich type approximation theorem; Rate of convergence; Modulus of continuity; Modulus of smoothness

Mathematical subject classification: 41A10, 41A25, 41A36

1. Introduction

In 2012, Ren [6] introduced Bernstein type operators as follows:

$$L_n(f;x) = f(0)P_{n,0}(x) + \sum_{k=1}^{n-1} P_{n,k}(x)B_{n,k}(f) + f(1)P_{n,n}(x), \tag{1}$$

where $f \in C[0,1]$, $x \in [0,1]$, $P_{n,k}(x) = \begin{pmatrix} n \\ k \end{pmatrix} x^k(1-x)^{n-k}$ $(k = 0,1,...,n)$, and $B_{n,k}(f) = \frac{1}{B(nk,n(n-k))}\int_0^1 t^{nk-1}(1-t)^{n(n-k)-1}f(t)dt$ $(k = 1,...,n-1)$, $B(.,.)$ is the beta function.

The moments of the operators $L_n(f;x)$ were obtained as follows (see [6]):

Remark 1. For $L_n(t^m;x)$, $m = 0,1,2$, we have

(i) $L_n(1;x) = 1$;

(ii) $L_n(t;x) = x$;

(iii) $L_n(t^2;x) = \frac{n(n-1)}{n^2+1}x^2 + \frac{n+1}{n^2+1}x$.

In 2015, Inspired by [1], Ren and Zeng [7] introduced new type Bézier operators, which is the Bézier variant of the Bernstein type operators $Ln(f;x)$, as follows:

$$L_{n,\alpha}(f;x) = f(0)Q_{n,0}^{(\alpha)}(x) + \sum_{k=1}^{n-1} Q_{n,k}^{(\alpha)}(x)B_{n,k}(f) + f(1)Q_{n,n}^{(\alpha)}(x), \tag{2}$$

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where $f \in C[0,1]$, $x \in [0,1]$, $\alpha > 0$, $Q_{n,k}^{(\alpha)}(x) = J_{n,k}^{\alpha}(x) - J_{n,k+1}^{\alpha}(x)$, $J_{n,n+1}(x) = 0$, $J_{n,k}(x) = \sum\limits_{i=k}^{n} P_{n,i}(x)$, $P_{n,k}(x)$ $(k = 0,1,...n)$, $B_{n,k}(f)$ $(k = 1,...,n-1)$ and $B(.,.)$ are as stated in (1).

In the present paper, we will study the Stancu variant of the new type Bézier operators $L_{n,\alpha}(f;x)$, which have been given by (2). We introduce new Stancu-Bézier type operators as follows:

$$L_{n,\alpha}^{(\beta,\gamma)}(f;x) = f(\frac{\beta}{n+\gamma})Q_{n,0}^{(\alpha)}(x) + \sum_{k=1}^{n-1} Q_{n,k}^{(\alpha)}(x)B_{n,k}^{(\beta,\gamma)}(f) + f(\frac{n+\beta}{n+\gamma})Q_{n,n}^{(\alpha)}(x), \quad (3)$$

where f, x, α, $Q_{n,k}^{(\alpha)}(x)(k = 0,1,...,n)$ are as stated in (2), β, γ are two given real parameters satisfying the condition $0 \le \beta \le \gamma$, $B(.,.)$ is the beta function, and $B_{n,k}^{(\beta,\gamma)}(f) = \frac{1}{B(nk,n(n-k))}\int_0^1 t^{nk-1}(1-t)^{n(n-k)-1} f(\frac{nt+\beta}{n+\gamma})dt$, $k = 1,...,n-1$, .

It is clear that $L_{n,\alpha}^{(\beta,\gamma)}(f;x)$ are bounded and positive on C[0,1]. When $\beta = \gamma = 0$, $L_{n,\alpha}^{(\beta,\gamma)}(f;x)$ become the operators $L_{n,\alpha}(f;x)$.

The goal of this paper is to study the approximation properties of these operators with the help of the Korovkin type approximation theorem. We also estimate the rates of convergence of these operators by using a modulus of continuity. Then, we obtain the direct theorem concerned with an approximation for these operators by means of the Ditzian-Totik modulus of smoothness.

In the paper, for $f \in C[0,1]$, we denote $\|f\| = max\{|f(x)| : x \in [0,1]\}$. $\omega(f,\delta)$ $(\delta > 0)$ denotes the usual modulus of continuity of $f \in C[0,1]$.

2. Auxiliary results

Now, we give some lemmas, which are necessary to prove our results.

Lemma 1. Let $\alpha > 0, x \in [0,1]$, $0 \le \beta \le \gamma$. We have

(i) $L_{n,\alpha}^{(\beta,\gamma)}(1;x) = 1$;

(ii) $\lim\limits_{n\to\infty} L_{n,\alpha}^{(\beta,\gamma)}(t;x) = x$ *uniformly on* $[0,1]$;

(iii) $\lim\limits_{n\to\infty} L_{n,\alpha}^{(\beta,\gamma)}(t^2;x) = x^2$ *uniformly on* $[0,1]$.

Proof (i) Since $\sum_{k=0}^{n} Q_{n,k}^{(\alpha)}(x) = 1$, so, by (3), we get $L_{n,\alpha}^{(\beta,\gamma)}(1;x) = 1$.

(ii) by (2) and (3), we have

$$L_{n,\alpha}^{(\beta,\gamma)}(t;x) = \frac{n}{n+\gamma}L_{n,\alpha}(t;x) + \frac{\beta}{n+\gamma},$$

thus, by Lemma 2 (ii) in [7], we have $\lim\limits_{n\to\infty} L_{n,\alpha}^{(\beta,\gamma)}(t;x) = x$ uniformly on [0,1].

(iii) by (2) and (3), we have

$$L_{n,\alpha}^{(\beta,\gamma)}(t^2;x) = (\frac{n}{n+\gamma})^2 L_{n,\alpha}(t^2;x) + \frac{2n\beta}{(n+\gamma)^2}L_{n,\alpha}(t;x) + (\frac{\beta}{n+\gamma})^2,$$

thus, by Lemma 2 (iii) in [7], we have $\lim\limits_{n\to\infty} L_{n,\alpha}^{(\beta,\gamma)}(t^2;x) = x^2$ uniformly on [0,1].

Lemma 2.(see [4]) For $x \in [0,1]$, $k = 0,1,...,n$, we have

$$0 \le Q_{n,k}^{(\alpha)}(x) \le \begin{cases} \alpha P_{n,k}(x), & \alpha \ge 1; \\ P_{n,k}^{\alpha}(x), & 0 < \alpha < 1. \end{cases}$$

Lemma 3.(see [5]) For $0 < \alpha < 1$, $\nu > 0$, we have

$$\sum_{k=0}^{n} |k - nx|^{\nu} P_{n,k}^{\alpha}(x) \le (n+1)^{1-\alpha}(A_{\frac{\nu}{\alpha}})^{\alpha} n^{\frac{\nu}{2}},$$

where the constant A_s only depends on s.

Lemma 4. Let $\alpha > 0$, $0 \le \beta \le \gamma$, We have

$$\text{(i) } B_{n,k}^{(\beta,\gamma)}(1) = 1;$$

$$\text{(ii) } B_{n,k}^{(\beta,\gamma)}(t) = \frac{k+\beta}{n+\gamma};$$

$$\text{(iii) } B_{n,k}^{(\beta,\gamma)}(t^2) = \frac{n^2}{(n+\gamma)^2(n^2+1)}(k^2+\frac{k}{n}) + \frac{2k\beta}{(n+\gamma)^2} + \frac{\beta^2}{(n+\gamma)^2}.$$

Proof By [7], we have $B_{n,k}(1) = 1$, $B_{n,k}(t) = \frac{k}{n}$, $B_{n,k}(t^2) = \frac{1}{n^2+1}(k^2 + \frac{k}{n})$, so, by simple calculation, we obtain

$$\text{(i)} B_{n,k}^{(\beta,\gamma)}(1) = 1;$$

$$\text{(ii)} B_{n,k}^{(\beta,\gamma)}(t) = \frac{n}{n+\gamma} B_{n,k}(t) + \frac{\beta}{n+\gamma} B_{n,k}(1) = \frac{k+\beta}{n+\gamma};$$

$$\begin{aligned}\text{(iii)} B_{n,k}^{(\beta,\gamma)}(t^2) &= \frac{n^2}{(n+\gamma)^2} B_{n,k}(t^2) + \frac{2n\beta}{(n+\gamma)^2} B_{n,k}(t) + \frac{\beta^2}{(n+\gamma)^2} B_{n,k}(1) \\ &= \frac{n^2}{(n+\gamma)^2(n^2+1)}(k^2+\frac{k}{n}) + \frac{2k\beta}{(n+\gamma)^2} + \frac{\beta^2}{(n+\gamma)^2}.\end{aligned}$$

Lemma 5. For $\alpha \ge 1, x \in [0,1]$, $0 \le \beta \le \gamma$, we have

$$\text{(i)} L_{n,\alpha}^{(\beta,\gamma)}((t-x)^2; x) \le \frac{2\alpha(1+\gamma)}{n+\gamma};$$

$$\text{(ii)} L_{n,\alpha}^{(\beta,\gamma)}(|t-x|; x) \le \sqrt{\frac{2\alpha(1+\gamma)}{n+\gamma}}.$$

Proof (i) For $\alpha \ge 1, x \in [0,1]$, $0 \le \beta \le \gamma$, by (3), Lemma 2, (1) and Remark 1, we obtain

$$\begin{aligned}
&L_{n,\alpha}^{(\beta,\gamma)}((t-x)^2; x) \\
&= (\frac{\beta}{n+\gamma} - x)^2 Q_{n,0}^{(\alpha)}(x) + \sum_{k=1}^{n-1} Q_{n,k}^{(\alpha)}(x) B_{n,k}^{(\beta,\gamma)}((t-x)^2) + (\frac{n+\beta}{n+\gamma} - x)^2 Q_{n,n}^{(\alpha)}(x) \\
&\le \alpha[(\frac{\beta}{n+\gamma} - x)^2 P_{n,0}(x) + \sum_{k=1}^{n-1} P_{n,k}(x) B_{n,k}^{(\beta,\gamma)}((t-x)^2) + (\frac{n+\beta}{n+\gamma} - x)^2 P_{n,n}(x)] \\
&= \alpha[\frac{n^2}{(n+\gamma)^2} L_n(t^2; x) + \frac{2n\beta}{(n+\gamma)^2} L_n(t; x) + \frac{\beta^2}{(n+\gamma)^2} L_n(1; x)] \\
&\quad -2\alpha x(\frac{n}{n+\gamma} L_n(t; x) + \frac{\beta}{n+\gamma} L_n(1; x)) + \alpha x^2 L_n(1; x) \\
&= \alpha\{\frac{n^2}{(n+\gamma)^2}[\frac{n(n-1)}{n^2+1} x^2 + \frac{n+1}{n^2+1} x] + \frac{2n\beta x}{(n+\gamma)^2} + \frac{\beta^2}{(n+\gamma)^2}\} - 2\alpha x(\frac{nx+\beta}{n+\gamma}) \\
&\quad +\alpha x^2 \\
&= \alpha[\frac{n^3+n^2-2\beta\gamma n^2-2\beta\gamma}{(n+\gamma)^2(n^2+1)} x(1-x) + \frac{-2\beta\gamma n^2 - 2\beta\gamma + \gamma^2 n^2 + \gamma^2}{(n+\gamma)^2(n^2+1)} x^2 + \frac{\beta^2}{(n+\gamma)^2}] \\
&\le \alpha[\frac{n+1}{(n+\gamma)^2} x(1-x) + \frac{\beta^2+\gamma^2}{(n+\gamma)^2}] \\
&\le \frac{\alpha(n+1+\beta^2+\gamma^2)}{(n+\gamma)^2} \\
&\le \frac{2\alpha(1+\gamma)}{n+\gamma}.
\end{aligned}$$

(ii) In view of $L_{n,\alpha}^{(\beta,\gamma)}(1;x)=1$, by the Cauchy-Schwarz inequality, we have

$$L_{n,\alpha}^{(\beta,\gamma)}(|t-x|;x)\leq\sqrt{L_{n,\alpha}^{(\beta,\gamma)}(1;x)}\sqrt{L_{n,\alpha}^{(\beta,\gamma)}((t-x)^2;x)},$$

thus, we get

$$L_{n,\alpha}^{(\beta,\gamma)}(|t-x|;x)\leq\sqrt{\frac{2\alpha(1+\gamma)}{n+\gamma}}.$$

Lemma 6. For $0<\alpha<1, x\in[0,1]$, $0\leq\beta\leq\gamma$, we have

$$\text{(i) } L_{n,\alpha}^{(\beta,\gamma)}((t-x)^2;x)\leq M_\alpha^{(\beta,\gamma)}n^{-\alpha};$$

$$\text{(ii) } L_{n,\alpha}^{(\beta,\gamma)}(|t-x|;x)\leq\sqrt{M_\alpha^{(\beta,\gamma)}}\cdot n^{-\frac{\alpha}{2}}.$$

where the constant $M_\alpha^{(\beta,\gamma)}$ only depends on α,β,γ.

Proof (i) For $0<\alpha<1, x\in[0,1]$, $0\leq\beta\leq\gamma$, by (3), Lemma 2 and Lemma 4, we obtain

$$\begin{aligned}
&L_{n,\alpha}^{(\beta,\gamma)}((t-x)^2;x)\\
&=(\frac{\beta}{n+\gamma}-x)^2Q_{n,0}^{(\alpha)}(x)+\sum_{k=1}^{n-1}Q_{n,k}^{(\alpha)}(x)B_{n,k}^{(\beta,\gamma)}((t-x)^2)+(\frac{n+\beta}{n+\gamma}-x)^2Q_{n,n}^{(\alpha)}(x)\\
&\leq(\frac{\beta}{n+\gamma}-x)^2P_{n,0}^{\alpha}(x)+\sum_{k=1}^{n-1}P_{n,k}^{\alpha}(x)B_{n,k}^{(\beta,\gamma)}((t-x)^2)+(\frac{n+\beta}{n+\gamma}-x)^2P_{n,n}^{\alpha}(x)\\
&\leq\sum_{k=0}^{n}P_{n,k}^{\alpha}(x)[\frac{1}{(n+\gamma)^2}(k^2+\frac{k}{n})+\frac{2k\beta}{(n+\gamma)^2}+\frac{\beta^2}{(n+\gamma)^2}-2x\frac{k+\beta}{n+\gamma}+x^2]\\
&=\frac{1}{(n+\gamma)^2}\sum_{k=0}^{n}(k-nx)^2P_{n,k}^{\alpha}(x)+\frac{2(\beta-\gamma x)}{(n+\gamma)^2}\sum_{k=0}^{n}(k-nx)P_{n,k}^{\alpha}(x)\\
&\quad+\frac{1}{(n+\gamma)^2}\sum_{k=0}^{n}(\frac{k}{n}+\beta^2-2\beta\gamma x+\gamma^2x^2)P_{n,k}^{\alpha}(x)\\
&=:I_1+I_2+I_3.
\end{aligned}$$

By Lemma 3, we get $I_1\leq\frac{n(n+1)}{(n+\gamma)^2}(n+1)^{-\alpha}(A_{\frac{2}{\alpha}})^\alpha\leq2(A_{\frac{2}{\alpha}})^\alpha n^{-\alpha}$, $I_2\leq\frac{2(\beta+\gamma)}{(n+\gamma)^2}\sum\limits_{k=0}^{n}|k-nx|P_{n,k}^{\alpha}(x)\leq\frac{2(\beta+\gamma)\sqrt{n}(n+1)}{(n+\gamma)^2}(n+1)^{-\alpha}(A_{\frac{1}{\alpha}})^\alpha\leq4(\beta+\gamma)(A_{\frac{1}{\alpha}})^\alpha n^{-\alpha}$, here the constant $A_{\frac{i}{\alpha}}(i=1,2)$ only depends on α.

Using the Hölder inequality, we have $\sum\limits_{k=0}^{n}P_{n,k}^{\alpha}(x)\leq(n+1)^{1-\alpha}[\sum\limits_{k=0}^{n}P_{n,k}(x)]^\alpha$, and $|\frac{k}{n}+\beta^2-2\beta\gamma x+\gamma^2x^2|\leq1+(\beta+\gamma)^2$, so, we have

$$I_3\leq\frac{1+(\beta+\gamma)^2}{(n+\gamma)^2}(n+1)^{1-\alpha}[\sum_{k=0}^{n}P_{n,k}(x)]^\alpha\leq2[1+(\beta+\gamma)^2]n^{-\alpha}.$$

Denote $M_\alpha^{(\beta,\gamma)}=2(A_{\frac{2}{\alpha}})^\alpha+4(\beta+\gamma)(A_{\frac{1}{\alpha}})^\alpha+2[1+(\beta+\gamma)^2]$, then we can get

$$L_{n,\alpha}^{(\beta,\gamma)}((t-x)^2;x)\leq M_\alpha^{(\beta,\gamma)}n^{-\alpha}.$$

(ii) Since

$$L_{n,\alpha}^{(\beta,\gamma)}(|t-x|;x)\leq\sqrt{L_{n,\alpha}^{(\beta,\gamma)}(1;x)}\sqrt{L_{n,\alpha}^{(\beta,\gamma)}((t-x)^2;x)},$$

thus, we get

$$L_{n,\alpha}^{(\beta,\gamma)}(|t-x|;x) \leq \sqrt{M_{\alpha}^{(\beta,\gamma)}} \cdot n^{-\frac{\alpha}{2}}.$$

Lemma 7. *For* $f \in C[0,1],\ x \in [0,1],\ \alpha \geq 0,$ *and* $0 \leq \beta \leq \gamma,$ *we have*

$$| L_{n,\alpha}^{(\beta,\gamma)}(f;x) | \leq \| f \| .$$

Proof By (3) and Lemma 1 (i), we have

$$| L_{n,\alpha}^{(\beta,\gamma)}(f;x) | \leq \|f\| L_{n,\alpha}^{(\beta,\gamma)}(1;x) = \|f\|.$$

3. Main results

First of all we give the following convergence theorem for the sequence $\{L_{n,\alpha}^{(\beta,\gamma)}(f;x)\}$.

Theorem 1. Let $\alpha > 0, x \in [0,1],\ 0 \leq \beta \leq \gamma$. *Then the sequence* $\{L_{n,\alpha}^{(\beta,\gamma)}(f;x)\}$ *converges to* f *uniformly on* $[0,1]$ *for any* $f \in C[0,1]$.

Proof Since $L_{n,\alpha}^{(\beta,\gamma)}(f;x)$ is bounded and positive on $C[0,1]$, and by Lemma 1, we have $\lim\limits_{n\to\infty} \|L_{n,\alpha}^{(\beta,\gamma)}(e_m;\cdot) - e_m\| = 0$ for $e_m(t) = t^m$, $m = 0,1,2$. So, according to the well-known Bohman-korovkin theorem ([2, P.40, Theorem 1.9]), we see that the sequence $\{L_{n,\alpha}^{(\beta,\gamma)}(f;x)\}$ converges to f uniformly on [0,1] for any $f \in C[0,1]$.

Next we estimate the rates of convergence of the sequence $\{L_{n,\alpha}^{(\beta,\gamma)}\}$ by means of modulus of continuity.

Theorem 2. *Let* $f \in C[0,1]$, $x \in [0,1]$, $0 \leq \beta \leq \gamma$. Then

(i) when $\alpha \geq 1$, we have

$$|L_{n,\alpha}^{(\beta,\gamma)}(f;\cdot) - f\| \leq [1 + \sqrt{2\alpha(1+\gamma)}]\omega(f, \frac{1}{\sqrt{n+\gamma}});$$

(ii) when $0 < \alpha < 1$, we have

$$|L_{n,\alpha}^{(\beta,\gamma)}(f;x) - f(x)| \leq (1 + \sqrt{M_{\alpha}^{(\beta,\gamma)}})\omega(f, n^{-\frac{\alpha}{2}}).$$

Here the constant $M_{\alpha}^{(\beta,\gamma)}$ only depends on α, β, γ.

Proof (i) When $\alpha \geq 1$, by Lemma 1 (i), we have

$$\begin{aligned}
&|L_{n,\alpha}^{(\beta,\gamma)}(f;x) - f(x)| \\
&\leq |f(\frac{\beta}{n+\gamma}) - f(x)|Q_{n,0}^{(\alpha)}(x) + \sum_{k=1}^{n-1} Q_{n,k}^{(\alpha)}(x) B_{n,k}^{(\beta,\gamma)}(|f(t) - f(x)|) \\
&\quad + |f(\frac{n+\beta}{n+\gamma}) - f(x)|Q_{n,n}^{(\alpha)}(x) \\
&\leq \omega(f, |\frac{\beta}{n+\gamma} - x|)Q_{n,0}^{(\alpha)}(x) + \sum_{k=1}^{n-1} Q_{n,k}^{(\alpha)}(x) B_{n,k}^{(\beta,\gamma)}(\omega(f,|t-x|)) \\
&\quad + \omega(f, |\frac{n+\beta}{n+\gamma} - x|)Q_{n,n}^{(\alpha)}(x) \\
&\leq (1 + \sqrt{n+\gamma}|\frac{\beta}{n+\gamma} - x|)\omega(f, \frac{1}{\sqrt{n+\gamma}})Q_{n,0}^{(\alpha)}(x)
\end{aligned}$$

$$+\sum_{k=1}^{n-1} Q_{n,k}^{(\alpha)}(x) B_{n,k}^{(\beta,\gamma)}((1+\sqrt{n+\gamma}|t-x|)\omega(f,\frac{1}{\sqrt{n+\gamma}}))$$
$$+(1+\sqrt{n+\gamma}|\frac{n+\beta}{n+\gamma}-x|)\omega(f,\frac{1}{\sqrt{n+\gamma}})Q_{n,n}^{(\alpha)}(x)$$
$$\leq \omega(f,\frac{1}{\sqrt{n+\gamma}})+\sqrt{n+\gamma}\omega(f,\frac{1}{\sqrt{n+\gamma}})L_{n,\alpha}^{(\beta,\gamma)}(|t-x|;x),$$

so, by Lemma 5 (ii), we obtain

$$|L_{n,\alpha}^{(\beta,\gamma)}(f;x)-f(x)| \leq [1+\sqrt{2\alpha(1+\gamma)}]\omega(f,\frac{1}{\sqrt{n+\gamma}}).$$

The desired result follows immediately.

(ii) When $0<\alpha<1$, by Lemma 1 (i), we have

$$|L_{n,\alpha}^{(\beta,\gamma)}(f;x)-f(x)|$$
$$\leq \omega(f,|\frac{\beta}{n+\gamma}-x|)Q_{n,0}^{(\alpha)}(x)+\sum_{k=1}^{n-1} Q_{n,k}^{(\alpha)}(x) B_{n,k}^{(\beta,\gamma)}(\omega(f,|t-x|))$$
$$+\omega(f,|\frac{n+\beta}{n+\gamma}-x|)Q_{n,n}^{(\alpha)}(x)$$
$$\leq (1+n^{\frac{\alpha}{2}}|\frac{\beta}{n+\gamma}-x|)\omega(f,n^{-\frac{\alpha}{2}})Q_{n,0}^{(\alpha)}(x)$$
$$+\sum_{k=1}^{n-1} Q_{n,k}^{(\alpha)}(x) B_{n,k}^{(\beta,\gamma)}((1+n^{\frac{\alpha}{2}}|t-x|)\omega(f,n^{-\frac{\alpha}{2}}))$$
$$+(1+n^{\frac{\alpha}{2}}|\frac{n+\beta}{n+\gamma}-x|)\omega(f,n^{-\frac{\alpha}{2}})Q_{n,n}^{(\alpha)}(x)$$
$$\leq \omega(f,n^{-\frac{\alpha}{2}})+n^{\frac{\alpha}{2}}\omega(f,n^{-\frac{\alpha}{2}})L_{n,\alpha}^{(\beta,\gamma)}(|t-x|;x),$$

so, by Lemma 6 (ii), we obtain $|L_{n,\alpha}^{(\beta,\gamma)}(f;x)-f(x)| \leq (1+\sqrt{M_{\alpha}^{(\beta,\gamma)}})\omega(f,n^{-\frac{\alpha}{2}})$. The desired result follows immediately.

Theorem 3. *Let* $f\in C^1[0,1]$, $x\in[0,1]$, $0\leq\beta\leq\gamma$. Then

(i) when $\alpha\geq 1$, we have

$$|L_{n,\alpha}^{(\beta,\gamma)}(f;x)-f(x)|$$
$$\leq [\|f'\|+\omega(f',\frac{1}{\sqrt{n+\gamma}})(1+\sqrt{2\alpha(1+\gamma)})]\sqrt{\frac{2\alpha(1+\gamma)}{n+\gamma}};$$

(ii) when $0<\alpha<1$, we have

$$|L_{n,\alpha}^{(\beta,\gamma)}(f;x)-f(x)| \leq [\|f'\|+\omega(f',n^{-\frac{\alpha}{2}})(1+\sqrt{M_{\alpha}^{(\beta,\gamma)}})]\sqrt{M_{\alpha}^{(\beta,\gamma)}n^{-\alpha}}.$$

Here the constant $M_{\alpha}^{(\beta,\gamma)}$ only depends on α,β,γ.

Proof Let $f\in C^1[0,1]$. For any $t,x\in[0,1]$, $\delta>0$, we have

$$\begin{aligned} |f(t)-f(x)-f'(x)(t-x)| &\leq |\int_x^t |f'(u)-f'(x)|du| \\ &\leq \omega(f',|t-x|)|t-x| \\ &\leq \omega(f',\delta)(|t-x|+\delta^{-1}(t-x)^2), \end{aligned}$$

hence, by the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&|L_{n,\alpha}^{(\beta,\gamma)}(f(t)-f(x)-f'(x)(t-x);x)|\\
&\le \omega(f',\delta)\left(L_{n,\alpha}^{(\beta,\gamma)}(|t-x|;x)+\delta^{-1}L_{n,\alpha}^{(\beta,\gamma)}((t-x)^2;x)\right)\\
&\le \omega(f',\delta)(\sqrt{L_{n,\alpha}^{(\beta,\gamma)}(1;x)}+\delta^{-1}\sqrt{L_{n,\alpha}^{(\beta,\gamma)}((t-x)^2;x)})\sqrt{L_{n,\alpha}^{(\beta,\gamma)}((t-x)^2;x)}.
\end{aligned}$$

So, we get

$$\begin{aligned}
&|L_{n,\alpha}^{(\beta,\gamma)}(f;x)-f(x)|\\
&\le \|f'\|L_{n,\alpha}^{(\beta,\gamma)}(|t-x|;x)\\
&+\omega(f',\delta)(1+\delta^{-1}\sqrt{L_{n,\alpha}^{(\beta,\gamma)}((t-x)^2;x)})\sqrt{L_{n,\alpha}((t-x)^2;x)}. \qquad (4)
\end{aligned}$$

(i) When $\alpha \ge 1$, taking $\delta = \frac{1}{\sqrt{n+\gamma}}$ in (4), by Lemma 5 and inequality (4), we obtain the desired result.

(ii)When $0 < \alpha < 1$, taking $\delta = n^{-\frac{\alpha}{2}}$ in (4), by Lemma 6 and inequality (4), we obtain the desired result.

Finally we study the direct theorem concerned with an approximation for the sequence $\{L_{n,\alpha}^{(\beta,\gamma)}\}$ by means of the Ditzian-Totik modulus of smoothness. For the next theorem we shall use some notations.

For $f \in C[0,1]$, $\varphi(x) = \sqrt{x(1-x)}$, $0 \le \lambda \le 1$, $x \in [0,1]$, let

$$\omega_{\varphi^\lambda}(f,t) = \sup_{0<h\le t} \sup_{x\pm\frac{h\varphi^\lambda(x)}{2}\in[0,1]} |f(x+\frac{h\varphi^\lambda(x)}{2}) - f(x-\frac{h\varphi^\lambda(x)}{2})|$$

be the Ditzian-Totik modulus of first order, and let

$$K_{\varphi^\lambda}(f,t) = \inf_{g\in W_\lambda}\{\| f-g \| + t\|\varphi^\lambda g'\| + t^{\frac{1}{1-\frac{\lambda}{2}}}\|g'\|\} \qquad (5)$$

be the corresponding K-functional, where $W_\lambda = \{f | f \in AC_{loc}[0,1], \|\varphi^\lambda f'\| < \infty, \|f'\| < \infty\}$.

It is well known that (see [3])

$$K_{\varphi^\lambda}(f,t) \le C\omega_{\varphi^\lambda}(f,t), \qquad (6)$$

for some absolute constant $C > 0$.

Now we state our next main result.

Theorem 4. Let $f \in C[0,1]$, $\alpha \ge 1$, $x \in [0,1]$, $0 \le \beta \le \gamma$, $\varphi(x) = \sqrt{x(1-x)}$, $\delta_n(x) = \varphi(x) + \frac{1}{\sqrt{n+\gamma}}$, $0 \le \lambda \le 1$. Then there exists an absolute constant $C > 0$ such that

$$|L_{n,\alpha}^{(\beta,\gamma)}(f;x) - f(x)| \le C\omega_{\varphi^\lambda}(f, \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n+\gamma}}).$$

Proof Let $g \in W_\lambda$, by Lemma 1 (i) and Lemma 7, we have

$$\begin{aligned}
&|L_{n,\alpha}^{(\beta,\gamma)}(f;x)-f(x)|\\
&\le |L_{n,\alpha}^{(\beta,\gamma)}(f-g;x)| + |f(x)-g(x)| + |L_{n,\alpha}^{(\beta,\gamma)}(g;x)-g(x)|\\
&\le 2\|f-g\| + |L_{n,\alpha}^{(\beta,\gamma)}(g;x)-g(x)|. \qquad (7)
\end{aligned}$$

Since $g(t)=\int_x^t g'(u)du+g(x)$, $L_{n,\alpha}^{(\beta,\gamma)}(1;x)=1$, so, we have

$$\begin{aligned}|L_{n,\alpha}^{(\beta,\gamma)}(g;x)-g(x)| &\leq |L_{n,\alpha}^{(\beta,\gamma)}(\int_x^t |g'(u)|du;x)| \\ &\leq \|\delta_n^\lambda g'\| L_{n,\alpha}^{(\beta,\gamma)}(|\int_x^t \delta_n^{-\lambda}(u)du|;x). \qquad (8)\end{aligned}$$

By the Hölder inequality, we get

$$|\int_x^t \delta_n^{-\lambda}(u)du| \leq |\int_x^t \delta_n^{-1}(u)du|^\lambda |t-x|^{1-\lambda}. \qquad (9)$$

Since

$$\delta_n^{-1}(x) \sim min(\varphi^{-1}(x), \sqrt{n+\gamma}), \qquad (10)$$

here $a \sim b$ means that there exists some constant $C>0$, such that $C^{-1}b \leq a \leq Cb$.

Also, by (11) in [7] , we have

$$|\int_x^t \varphi^{-1}(u)du| \leq 4|t-x|\varphi^{-1}(x), \qquad (11)$$

thus, by (9), (10) and (11), we obtain

$$|\int_x^t \delta_n^{-\lambda}(u)du| \leq C\delta_n^{-\lambda}(x)|t-x|, \qquad (12)$$

also, by (8) and (12), we have

$$\begin{aligned}|L_{n,\alpha}^{(\beta,\gamma)}(g;x)-g(x)| &\leq C\|\delta_n^\lambda g'\| L_{n,\alpha}^{(\beta,\gamma)}(\delta_n^{-\lambda}(x)|t-x|;x) \\ &= C\|\delta_n^\lambda g'\|\delta_n^{-\lambda}(x) L_{n,\alpha}^{(\beta,\gamma)}(|t-x|;x). \qquad (13)\end{aligned}$$

In view of the proof of Lemma 5 (i), we have

$$L_{n,\alpha}^{(\beta,\gamma)}((t-x)^2;x) \leq \alpha[\frac{n+1}{(n+\gamma)^2}x(1-x)+\frac{\beta^2+\gamma^2}{(n+\gamma)^2}],$$

so, by the Cauchy-Schwarz inequality and Lemma 1 (i), we have

$$\begin{aligned}L_{n,\alpha}^{(\beta,\gamma)}(|t-x|;x) &\leq \sqrt{L_{n,\alpha}^{(\beta,\gamma)}(1;x)}\sqrt{L_{n,\alpha}^{(\beta,\gamma)}((t-x)^2;x)} \\ &\leq \sqrt{\alpha[\frac{n+1}{(n+\gamma)^2}x(1-x)+\frac{\beta^2+\gamma^2}{(n+\gamma)^2}]} \\ &\leq C\frac{\delta_n(x)}{\sqrt{n+\gamma}}, \qquad (14)\end{aligned}$$

so, by (13) and (14), we obtain

$$|L_{n,\alpha}^{(\beta,\gamma)}(g;x)-g(x)| \leq C\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n+\gamma}}\|\delta_n^\lambda g'\|, \qquad (15)$$

thus, by (7), (15) and $\delta_n(x)=\varphi(x)+\frac{1}{\sqrt{n+\gamma}}$, we have

$$\begin{aligned}&|L_{n,\alpha}^{(\beta,\gamma)}(f;x)-f(x)| \\ &\leq C[\|f-g\|+\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n+\gamma}}\|\delta_n^\lambda g'\|] \\ &\leq C[\|f-g\|+\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n+\gamma}}\|\varphi^\lambda g'\|+\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n+\gamma}}(\frac{1}{\sqrt{n+\gamma}})^\lambda\|g'\|] \\ &\leq C[\|f-g\|+\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n+\gamma}}\|\varphi^\lambda g'\|+(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n+\gamma}})^{\frac{1}{1-\frac{\lambda}{2}}}\|g'\|]. \qquad (16)\end{aligned}$$

Then, in view of (16), (5) and (6), we obtain

$$|L_{n,\alpha}^{(\beta,\gamma)}(f;x)-f(x)|\leq CK_{\varphi^\lambda}(f,\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n+\gamma}})\leq C\omega_{\varphi^\lambda}(f,\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n+\gamma}}),$$

where C is a positive constant, in different places, the value of C may be different.

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Dynamics of a Higher Order Difference Equations $x_{n+1} = \alpha x_n + \beta x_{n-l} + \gamma x_{n-k} + \frac{ax_{n-l}+bx_{n-k}}{cx_{n-l}+dx_{n-k}}$

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ABSTRACT

The main objective of this paper is to study the global stability of the positive solutions and the periodic character of the difference equation

$$x_{n+1} = \alpha x_n + \beta x_{n-l} + \gamma x_{n-k} + \frac{ax_{n-l} + bx_{n-k}}{cx_{n-l} + dx_{n-k}}, \qquad n = 0,\ 1,\ ...,$$

where the parameters $\alpha,\ \beta,\ \gamma,\ a,\ b,\ c,\ d \in (0,\ \infty)$ and the initial conditions $x_{-s},\ x_{-s+1}\ ...,\ x_{-1}$ and x_0 are positive real numbers where $s = max\{l,\ k\}$. Examples to illustrate the importance of our results.

Keywords: difference equations, stability, global stability, boundedness, periodic solutions.

Mathematics Subject Classification: 39A10

1. INTRODUCTION

Difference equations have always played an important role in the construction and analysis of mathematical models of economic process, biology, ecology, physics and so forth. The study of nonlinear rational difference equations of higher order is of paramount importance, since we still know so little about such equations.

In [1] Papaschinopoulos et al. studied the asymptotic behavior and the periodicity of the positive solutions of the nonautonomous difference equation

$$x_{n+1} = A_n + \frac{x_{n-1}^p}{x_n^q}$$

Kalabušić et al. [2] investigated the global character of the solution of the nonlinear rational difference equation

$$x_{n+1} = \frac{\beta x_{n-l}+\delta x_{n-k}}{Bx_{n-l}+Dx_{n-k}}.$$

Elsayed et al. [3] studied the global stability character and the periodicity of solutions of the recursive sequence

$$x_{n+1} = ax_{n-s} + \frac{bx_{n-l}+cx_{n-k}}{dx_{n-l}+ex_{n-k}}.$$

Zayed et al. [4] investigated the behavior of the following rational recursive sequences

$$x_{n+1} = ax_n - \frac{bx_n}{cx_n-dx_{n-k}}.$$

El-Moneam et al. [5] obtained the boundedness, the periodicity and the global stability of the positive solution of the difference equation,

$$x_{n+1} = ax_n + \frac{bx_{n-1}+cx_{n-2}+fx_{n-3}+rx_{n-4}}{dx_{n-1}+ex_{n-2}+gx_{n-3}+sx_{n-4}}.$$

El-Dessoky [6] studied the global stability, the boundedness and the periodicity of the nonlinear difference equation

$$x_{n+1} = ax_n + bx_{n-k} + cx_{n-l} - \frac{dx_{n-s}}{ex_{n-s}-\alpha x_{n-t}}, \qquad n = 0,\ 1,\ ....$$

For other related results, see [1 - 30].

Our aim in this paper is to obtain some qualitative behavior of the positive solutions of the difference equation

$$x_{n+1} = \alpha x_n + \beta x_{n-l} + \gamma x_{n-k} + \frac{ax_{n-l}+bx_{n-k}}{cx_{n-l}+dx_{n-k}}, \quad n = 0,\ 1,\ ..., \tag{1}$$

where the parameters $\alpha,\ \beta,\ \gamma,\ a,\ b,\ c,\ d\ \in (0,\ \infty)$ and the initial conditions $x_{-s},\ x_{-s+1}\ ...,\ x_{-1}$ and x_0 are positive real numbers where $s = max\{l,\ k\}$.

2. LOCAL STABILITY OF THE EQUILIBRIUM POINT

In this section, we study the local stability character of the equilibrium point of Eq. (1).

Eq. (1) has equilibrium point and is given by

$$x^* = \alpha x^* + \beta x^* + \gamma x^* + \frac{ax^*+bx^*}{cx^*+dx^*},$$

If $\alpha+\beta+\gamma < 1$, then the only positive equilibrium point x^* of Eq. (1) is given by $x^* = \frac{a+b}{[1-\alpha-\beta-\gamma](c+d)}$.

THEOREM 2.1. *(i) Let $ad > cb$, $\alpha+\beta+\gamma < 1$ and $\gamma > \frac{(ad-bc)(1-\alpha-\beta-\gamma))}{(c+d)(a+b)}$, then equilibrium x^* of Eq. (1) is locally asymptotically stable.*

(ii) Let $cb > ad$, $\alpha+\beta+\gamma < 1$ and $\beta > \frac{(ad-cb)(1-\alpha-\beta-\gamma))}{(c+d)(a+b)}$, then equilibrium x^* of Eq. (1) is locally asymptotically stable.

Proof: Suppose that $G : (0,\ \infty)^3 \longrightarrow (0,\ \infty)$ be a continuous function defined by

$$G(u_0,\ u_1,\ u_2) = \alpha u_0 + \beta u_1 + \gamma u_2 + \frac{au_1+bu_2}{cu_1+du_2}. \tag{2}$$

Therefore, it follows that

$$\frac{\partial G(u_0,\ u_1,\ u_2)}{\partial u_0} = \alpha,\ \frac{\partial G(u_0,\ u_1,\ u_2)}{\partial u_1} = \beta + \frac{(ad-cb)u_2}{(cu_1+du_2)^2},\ \frac{\partial G(u_0,\ u_1,\ u_2)}{\partial u_2} = \gamma - \frac{(ad-bc)u_0}{(cu_1+du_2)^2}.$$

Then, we see that

$$\frac{\partial G(x^*,\ x^*,\ x^*)}{\partial u_0} = \alpha,\ \frac{\partial G(x^*,\ x^*,\ x^*)}{\partial u_1} = \beta + \frac{(ad-cb)(1-\alpha-\beta-\gamma)}{(c+d)(a+b)},\ \frac{\partial G(x^*,\ x^*,\ x^*)}{\partial u_2} = \gamma - \frac{(ad-bc)(1-\alpha-\beta-\gamma)}{(c+d)(a+b)}.$$

Under the conditions of part (i), we get

$$\begin{aligned} |\alpha| + \left|\beta + \frac{(ad-cb)(1-\alpha-\beta-\gamma)}{(c+d)(a+b)}\right| + \left|\gamma - \frac{(ad-bc)(1-\alpha-\beta-\gamma)}{(c+d)(a+b)}\right| &< 1, \\ \alpha + \beta + \frac{(ad-cb)(1-\alpha-\beta-\gamma)}{(c+d)(a+b)} + \gamma - \frac{(ad-bc)(1-\alpha-\beta-\gamma)}{(c+d)(a+b)} &< 1, \end{aligned}$$

and so

$$\alpha+\beta+\gamma < 1.$$

Then the equilibrium x^* of Eq. (1) is locally asymptotically stable, the proof of part (i) is complete.

Under the conditions of part (ii), we get

$$\begin{aligned}
|\alpha| + \left|\beta - \tfrac{(bc-ad)(1-\alpha-\beta-\gamma)}{(c+d)(a+b)}\right| + \left|\gamma + \tfrac{(bc-ad)(1-\alpha-\beta-\gamma)}{(c+d)(a+b)}\right| &< 1, \\
\alpha + \beta - \tfrac{(bc-ad)(1-\alpha-\beta-\gamma)}{(c+d)(a+b)} + \gamma + \tfrac{(bc-ad)(1-\alpha-\beta-\gamma)}{(c+d)(a+b)} &< 1,
\end{aligned}$$

and so

$$\alpha + \beta + \gamma < 1.$$

Then the equilibrium x^* of Eq. (1) is locally asymptotically stable, the proof of part (ii) is complete.

Example 1. See Figure (1) when we take the Eq. (1) with $l = 4$, $k = 3, \alpha = 0.2$, $\beta = 0.1$, $\gamma = 0.5$, $a = 0.4$, $b = 0.3$, $c = 0.6$ and $d = 1$ and the initial conditions $x_{-4} = 0.6$, $x_{-3} = 7$, $x_{-2} = 2$, $x_{-1} = 3$ and $x_0 = 5$.

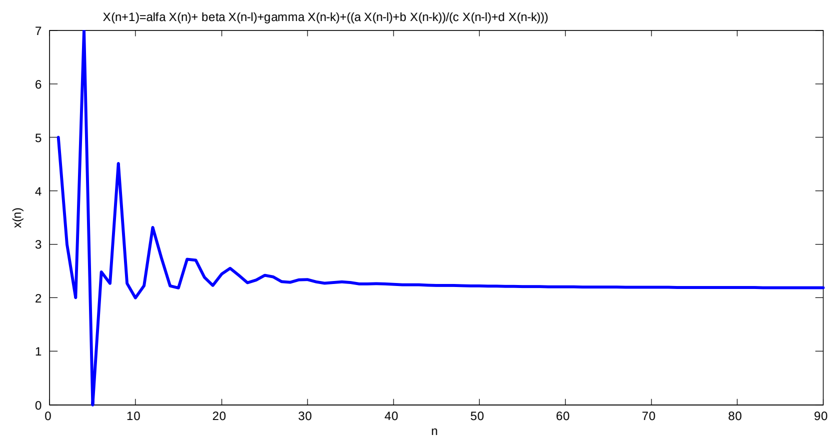

Fig. 1. sketch the behavior of the solution of Eq. (1).

Example 2. The solution of Eq. (1) is local stability if $l = 4$, $k = 3$, $\alpha = 0.2$, $\beta = 0.1$, $\gamma = 0.2$, $a = 0.4$, $b = 0.3$, $c = 0.6$ and $d = 1$ and the initial conditions $x_{-4} = 6$, $x_{-3} = 1.1$, $x_{-2} = 0.8$, $x_{-1} = 2$ and $x_0 = 0.2$ (See Fig. 2).

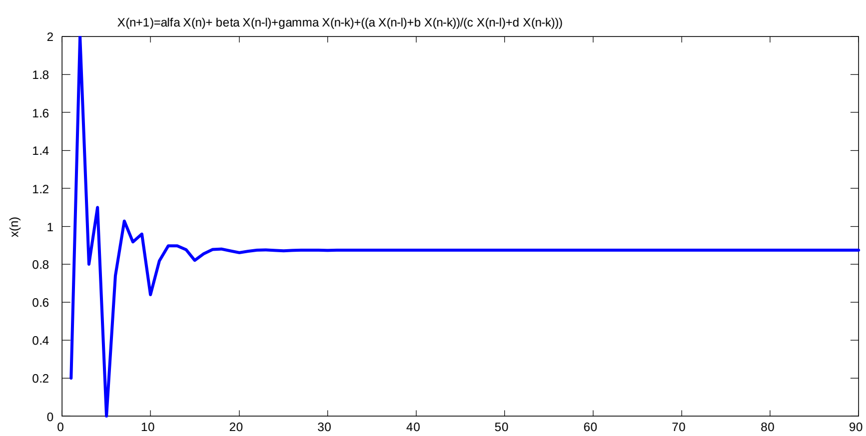

Fig. 2. Plot the behavior of the solution of equation (1) under the conditions (i).

Example 3. Figure (3) shows that if $l = 4$, $k = 3$, $\alpha = 0.2$, $\beta = 0.3$, $\gamma = 0.2$, $a = 0.4$, $b = 2$, $c = 1.6$ and $d = 1$, then the solution of Eq. (1) is local stability with the initial conditions $x_{-4} = 6$, $x_{-3} = 1.1$, $x_{-2} = 0.8$, $x_{-1} = 2$ and $x_0 = 0.2$.

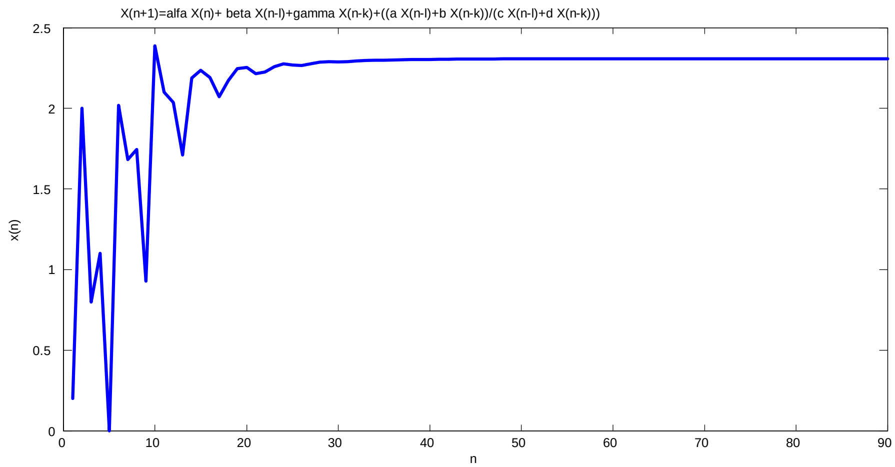

Fig. 3. Plot the behavior of the solution of equation (1) under the conditions (ii).

Example 4. See Figure (4) when we take Eq. (1) with $l = 4$, $k = 3$, $\alpha = 0.2$, $\beta = 0.28$, $\gamma = 0.82$, $a = 0.4$, $b = 0.3$, $c = 0.6$ and $d = 1$ and the initial conditions $x_{-4} = 6$, $x_{-3} = 1.1$, $x_{-2} = 0.8$, $x_{-1} = 2$ and $x_0 = 0.2$.

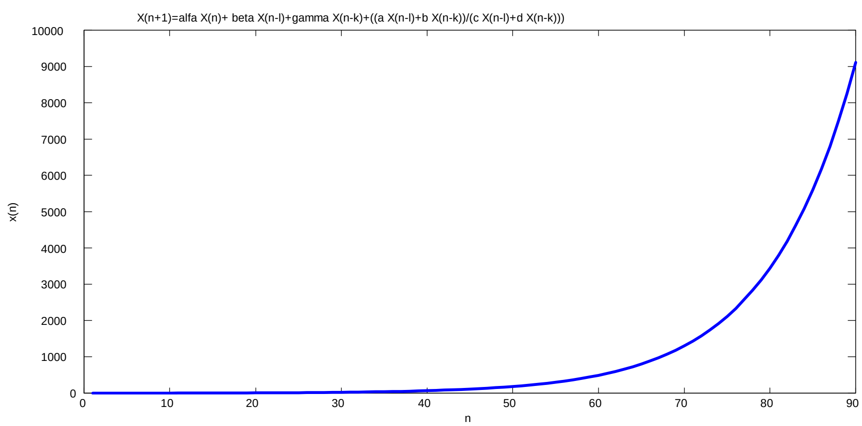

Fig. 4. Draw the behavior of the solution of Eq. (1).

3. GLOBAL STABILITY OF THE EQUILIBRIUM POINT

Theorem 3.1. *The equilibrium point x^* is a global attractor of Eq. (1) if one of the following conditions holds:*

$$(i)\ ad - cb \geqslant 0,\ b \geqslant a.$$
$$(ii)\ cb - ad \geqslant 0,\ a \geqslant b.$$

Proof. Let $r,\ s$ be nonnegative real numbers and assume that $H : [r, s]^3 \to [r, s]$ be a function defined by

$$H(u_0,\ u_1,\ u_1) = \alpha u_0 + \beta u_1 + \gamma u_2 + \frac{au_1+bu_2}{cu_1+du_2}.$$

Then

$$\frac{\partial H(u_0,\ u_1,\ u_1)}{\partial u_0}=a,\ \frac{\partial H(u_0,\ u_1,\ u_1)}{\partial u_1}=\beta+\frac{(ad-cb)u_1}{(cu_0+du_1)^2}\text{ and }\frac{\partial H(u_0,\ u_1,\ u_1)}{\partial u_2}=\gamma-\frac{(ad-bc)u_0}{(cu_0+du_1)^2}.$$

We consider two cases:

Case1: Assume that $ad-cb>0,\ \alpha+\beta+\gamma<1$ and $\gamma>\frac{(ad-bc)(1-\alpha-\beta-\gamma))}{(c+d)(a+b)}$ is true, then we can easily see that the function $H(u_0,\ u_1,\ u_2)$ is increasing in u_0, u_1 and decreasing in u_2. Suppose that (m,M) is a solution of the system

$$M=H(M,\ M,\ m)\quad\text{ and }\quad m=H(m,\ m,\ M).$$

Then from Eq. (1), we see that

$$M=\alpha M+\beta M+\gamma m+\frac{aM+bm}{cM+dm}\quad\text{ and }\quad m=\alpha m+\beta m+\gamma M+\frac{am+bM}{cm+dM},$$

then

$$\begin{aligned}c(1-\alpha-\beta)M^2+d(1-\alpha-\beta)mM-c\gamma mM-d\gamma m^2&=aM+bm,\\ c(1-\alpha-\beta)m^2+d(1-\alpha-\beta)mM-c\gamma mM-d\gamma M^2&=am+bM,\end{aligned}$$

Subtracting this two equations, we obtain

$$(M-m)\left\{(c(1-\alpha-\beta)+d\gamma)\,(M+m)]+(b-a)\right\}=0,$$

under the condition $\alpha+\beta<1,\ b\geqslant a$, we see that $M=m$. Then x^* is a global attractor of Eq. (1).

Case 2: Assume that $cb>ad,\ \beta+\gamma<1$ and $\beta>\frac{(ad-cb)(1-\alpha-\beta-\gamma)}{(c+d)(a+b)}$ is true, then we can easily see that the function $H(u_0,\ u_1,\ u_2)$ is decreasing in $u_0,\ u_1$ and increasing in u_2. Suppose that (m,M) is a solution of the system

$$M=H(m,\ m,\ M)\quad\textit{ and }\quad m=H(M,\ M,\ m).$$

Then from Eq. (1), we see that

$$M=\alpha m+\beta m+\gamma M+\frac{am+bM}{cm+dM},,\quad\text{ and }\quad m=\alpha M+\beta M+\gamma m+\frac{aM+bm}{cM+dm},$$

then

$$\begin{aligned}d(1-\gamma)M^2+c(1-\gamma)mM-c\,(\alpha+\beta)\,m^2-d\,(\alpha+\beta)\,mM&=am+bM,\\ d(1-\gamma)m^2+c(1-\gamma)mM-c\,(\alpha+\beta)\,M^2-d\,(\alpha+\beta)\,mM&=aM+bm,\end{aligned}$$

Subtracting this two equations, we obtain

$$(M-m)\left\{(d(1-\gamma)+c\,(\alpha+\beta))\,(M+m)]+(a-b)\right\}=0,$$

under the condition $\gamma\neq 1,\ a\neq b$, we see that $M=m$. Then x^* is a global attractor of Eq. (1).

Example 5. The solution of Eq. (1) is global stability if $l=4,\ k=3,\ \alpha=0.02,\ \beta=0.01,\ \gamma=0.03,\ a=0.4,$ $b=1$, $c=0.2$ and $d=1$ and the initial conditions $x_{-4}=6,\ x_{-3}=1.1,\ x_{-2}=0.8,\ x_{-1}=2$ and $x_0=0.2$ (See Fig. 5).

Example 6. Figure (6) shows the global stability of the solution of Eq. (1) when $l=4,\ k=3,\ \alpha=0.02,$ $\beta=0.2,\ \gamma=0.1,\ a=1.1,\ b=0.3,\ c=1$ and $d=0.3$ and the initial conditions $x_{-4}=6,\ x_{-3}=1.1,$ $x_{-2}=0.8,\ x_{-1}=2$ and $x_0=0.2$.

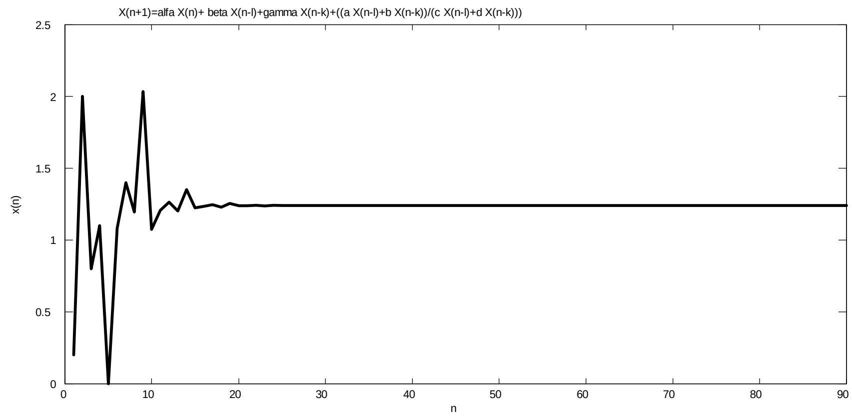

Fig. 5. sketch the behavior of the solution of Eq. (1) when $ad \geqslant cb$ and $b \geqslant a$.

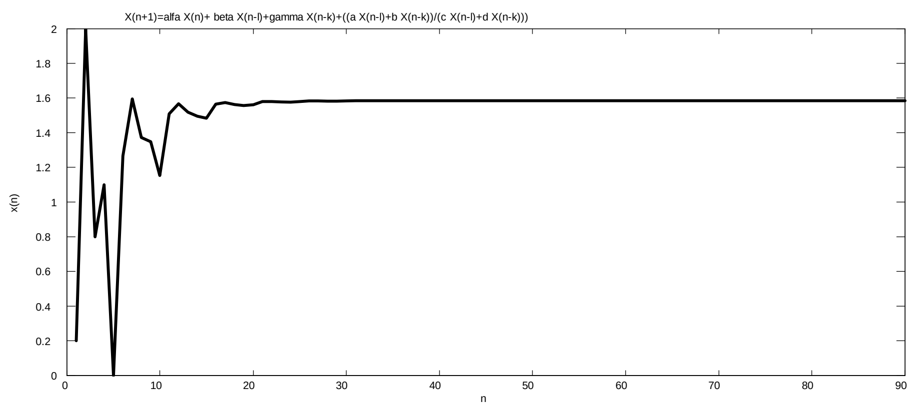

Fig. 6. Shows the behavior of the solution of Eq. (1) when $cb \geqslant ad,\ a \geqslant b$.

4. BOUNDEDNESS OF THE SOLUTIONS

THEOREM 4.1. *Every solution of Eq. (1) is bounded if* $\beta + \gamma < 1$.

Proof. Let $\{x_n\}_{n=-s}^{\infty}$ be a solution of Eq. (1). It follows from Eq. (1) that

$$\begin{aligned} x_{n+1} &= \alpha x_n + \beta x_{n-l} + \gamma x_{n-k} + \tfrac{a x_{n-l} + b x_{n-k}}{c x_{n-l} + d x_{n-k}} \\ &= \alpha x_n + \beta x_{n-l} + \gamma x_{n-k} + \tfrac{a x_{n-l}}{c x_{n-l} + d x_{n-k}} + \tfrac{b x_{n-k}}{c x_{n-l} + d x_{n-k}}. \end{aligned}$$

Then

$$x_{n+1} \leqslant \alpha x_n + \beta x_{n-l} + \gamma x_{n-k} + \tfrac{a x_{n-l}}{c x_{n-l}} + \tfrac{b x_{n-k}}{d x_{n-k}} = \alpha x_n + \beta x_{n-l} + \gamma x_{n-k} + \tfrac{a}{c} + \tfrac{b}{d} \text{ for all } n \geq 0.$$

By using a comparison, we can right hand side as follows

$$y_{n+1} = \alpha y_n + \beta y_{n-l} + \gamma y_{n-k} + \frac{a}{c} + \frac{b}{d}.$$

and this equation is locally asymptotically stable if $\alpha + \beta + \gamma < 1$, and converges to the equilibrium point $y^* = \frac{ad+bc}{cd(1-\alpha-\beta-\gamma)}$. Therefore

$$\lim_{n\to\infty} \sup y_n \leqslant \tfrac{ad+bc}{cd(1-\alpha-\beta-\gamma)}.$$

Thus the solution is bounded.

THEOREM 4.2. *Every solution of Eq. (1) is unbounded if* $\alpha > 1$ *or* $\beta > 1$*or* $\gamma > 1$.

Proof. Let $\{x_n\}_{n=-s}^{\infty}$ be a solution of Eq. (1).Then from Eq. (1) we see that

$$x_{n+1} = \alpha x_n + \beta x_{n-l} + \gamma x_{n-k} + \frac{ax_{n-l}+bx_{n-k}}{cx_{n-l}+dx_{n-k}} > \alpha x_n \quad \text{for all} \quad n \geq 0.$$

We see that the right hand side can be written as follows $y_{n+1} = \alpha y_n$. Then

$$y_{n+1} = \alpha^n y_n + const.,$$

and this equation is unstable because $\alpha > 1$, and $\lim\limits_{n\to\infty} y_n = \infty$.Then by using ratio test $\{x_n\}_{n=-s}^{\infty}$ is unbounded. Using the same technique, we can prove the other cases.

Example 7. When $l = 4$, $k = 3$, $\alpha = 1.3$, $\beta = 0.5$, $\gamma = 0.2$, $a = 0.4$, $b = 0.3$, $c = 0.6$ and $d = 1$, the solution of Eq. (1) with initial conditions $x_{-4} = 6$, $x_{-3} = 1.1$, $x_{-2} = 0.8$, $x_{-1} = 2$ and $x_0 = 0.2$, the solution of the difference equation is unbounded (See Fig. 7).

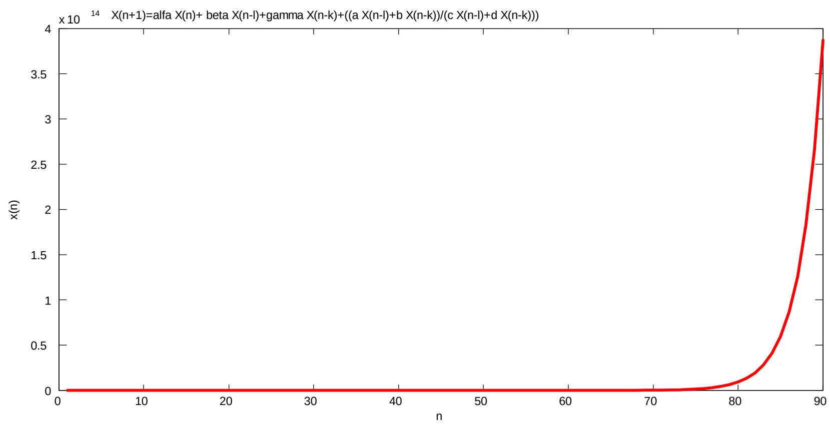

Fig. 7. Plot the behavior of the solution of equation (1) when $\alpha > 1$.

Example 8. Figure (8) shows that $l = 4$, $k = 3$, $\alpha = 0.2$, $\beta = 1.5$, $\gamma = 0.5$, $a = 0.4$, $b = 0.3$, $c = 0.6$ and $d = 1$, the solution of Eq. (1) with initial conditions $x_{-4} = 6$, $x_{-3} = 1.1$, $x_{-2} = 0.8$, $x_{-1} = 2$ and $x_0 = 0.2$ is unbounded.

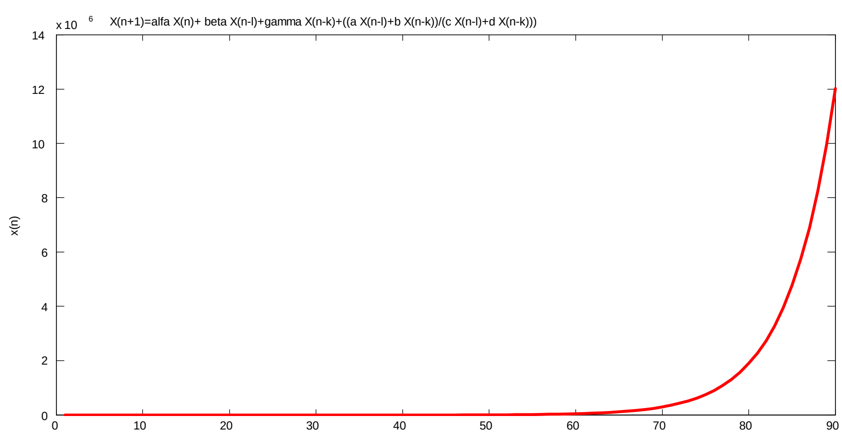

Fig. 8. Draw the behavior of the solution of equation (1) when $\beta > 1$.

Example 9. Figure (9) shows the solution of Eq. (1) is unbounded if $l = 4$, $k = 3$, $\alpha = 0.2$, $\beta = 0.4$, $\gamma = 1.2$, $a = 0.4$, $b = 0.3$, $c = 0.6$ and $d = 1$ and the initial conditions $x_{-4} = 6$, $x_{-3} = 1.1$, $x_{-2} = 0.8$, $x_{-1} = 2$ and $x_0 = 0.2$.

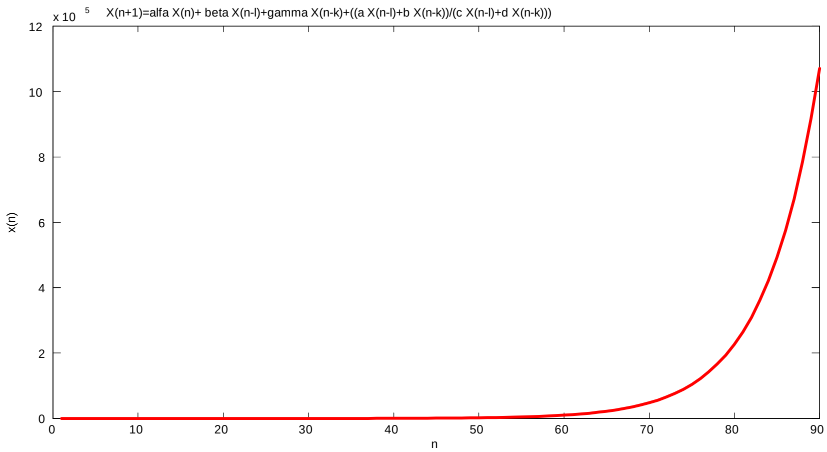

Fig. 9. Shows the behavior of the solution of equation (1) when $\gamma > 1$.

5. EXISTENCE OF PERIODIC SOLUTIONS

THEOREM 5.1. *If l is an even and k is an odd, then Eq. (1) has a prime period two solutions if*

$$(b-a)(c-d)(1+\alpha+\beta-\gamma) > 4(bc(\alpha+\beta)+ad(1-\gamma)), \tag{3}$$

where $a < b$, $c < d$, $\gamma < 1$ and $\gamma < 1+\alpha+\beta$.

Proof. Suppose that there exists a prime period two solution $...P, Q, P, Q, ...$, of Eq. (1). We see from Eq. (1) when l is even, and k is odd that

$$P = \alpha Q + \beta Q + \gamma P + \frac{aQ+bP}{cQ+dP} \quad \text{and} \quad Q = \alpha P + \beta P + \gamma Q + \frac{aP+bQ}{cP+dQ}.$$

Then

$$c(1-\gamma)PQ + d(1-\gamma)P^2 = c(\alpha+\beta)Q^2 + d(\alpha+\beta)PQ + aQ + bP, \tag{4}$$

$$c(1-\gamma)cPQ + d(1-\gamma)Q^2 = c(\alpha+\beta)P^2 + d(\alpha+\beta)PQ + aP + bQ. \tag{5}$$

Subtracting (4) from (5) gives

$$d(1-\gamma)(P^2-Q^2) = c(\alpha+\beta)(Q^2-P^2) - a(P-Q) + b(P-Q),$$

$$(d(1-\gamma)+c(\alpha+\beta))(P-Q)(P+Q) = (b-a)(P-Q),$$

Since $P \neq Q$, it follows that

$$(d(1-\gamma)+c(\alpha+\beta))(P+Q) = b-a,$$

$$P+Q = \frac{b-a}{d(1-\gamma)+c(\alpha+\beta)}. \tag{6}$$

Again, adding (4) and (5) yields

$$2PQ(c(1-\gamma)-d(\alpha+\beta)) = (c(\alpha+\beta)-d(1-\gamma))(P^2+Q^2) + (a+b)(P+Q), \tag{7}$$

It follows by (6), (7) and the relation

$$P^2 + Q^2 = (P+Q)^2 - 2PQ \quad for\ all \quad P, Q \in R,$$

that

$$2PQ\left(c\left(1-\gamma\right)-d\left(\alpha+\beta\right)\right) = \left(c\left(\alpha+\beta\right)-d\left(1-\gamma\right)\right)\left((P+Q)^2-2PQ\right)+(a+b)(P+Q),$$

$$\begin{aligned} 2\left(c-d\right)\left(1+\alpha+\beta-\gamma\right)PQ &= \left(P+Q\right)\left[\left(c\left(\alpha+\beta\right)-d\left(1-\gamma\right)\right)\left(P+Q\right)+\left(a+b\right)\right], \\ 2\left(c-d\right)\left(1+\alpha+\beta-\gamma\right)PQ &= \left(\tfrac{b-a}{d(1-\gamma)+c(\alpha+\beta)}\right)\left(\tfrac{(c(\alpha+\beta)-d(1-\gamma))(b-a)+(a+b)(d(1-\gamma)+c(\alpha+\beta))}{d(1-\gamma)+c(\alpha+\beta)}\right), \\ 2\left(c-d\right)\left(1+\alpha+\beta-\gamma\right)PQ &= 2\left(\tfrac{b-a}{d(1-\gamma)+c(\alpha+\beta)}\right)\left(\tfrac{bc(\alpha+\beta)+ad(1-\gamma)}{d(1-\gamma)+c(\alpha+\beta)}\right) \end{aligned}$$

$$PQ = \tfrac{(b-a)(bc(\alpha+\beta)+ad(1-\gamma))}{(c-d)(1+\alpha+\beta-\gamma)(d(1-\gamma)+c(\alpha+\beta))^2}. \tag{8}$$

Now it is clear from equations (6) and (8) that P and Q are the two distinct roots of the quadratic equation

$$\left(d\left(1-\gamma\right)+c\left(\alpha+\beta\right)\right)t^2-\left(b-a\right)t+\tfrac{(b-a)(da-\gamma ad+\beta cb)}{(c-d)(1+\alpha+\beta-\gamma)(d(1-\gamma)+c(\alpha+\beta))} = 0. \tag{9}$$

and so

$$\begin{aligned} \left(\tfrac{b-a}{d(1-\gamma)+c(\alpha+\beta)}\right)^2 &> \tfrac{4(b-a)(bc(\alpha+\beta)+ad(1-\gamma))}{(c-d)(1+\alpha+\beta-\gamma)(d(1-\gamma)+c(\alpha+\beta))^2}, \\ (b-a) &> \tfrac{4(bc(\alpha+\beta)+ad(1-\gamma))}{(c-d)(1+\alpha+\beta-\gamma)(d(1-\gamma)+c(\alpha+\beta))^2}. \end{aligned}$$

For $b > a,\ c > d$ and $\gamma < 1$, $\gamma < 1+\alpha+\beta$, then

$$\left(b-a\right)\left(c-d\right)\left(1+\alpha+\beta-\gamma\right) > 4\left(bc\left(\alpha+\beta\right)+ad\left(1-\gamma\right)\right).$$

Therefore Inequality (3) holds and the proof is complete.

Example 10. Figure (10) shows the Eq. (1) has a prime period two solution when $l = 4$, $k = 3$, $\alpha = 0.001$, $\beta = 0.03$, $\gamma = 0.06$, $a = 0.1$, $b = 0.9$, $c = 0.8$ and $d = 0.06$ and the initial conditions $x_{-4} = Q$, $x_{-3} = P$, $x_{-2} = Q$, $x_{-1} = P$ and $x_0 = Q$ such that $P = \frac{b-a+\xi}{2(d(1-\gamma)+c(\alpha+\beta))}$ and $Q = \frac{b-a-\xi}{2(d(1-\gamma)+c(\alpha+\beta))}$ where $\xi = \sqrt{(b-a)^2 - \frac{4(b-a)(bc(\alpha+\beta)+ad(1-\gamma))}{(c-d)(1+\alpha+\beta-\gamma)}}$.

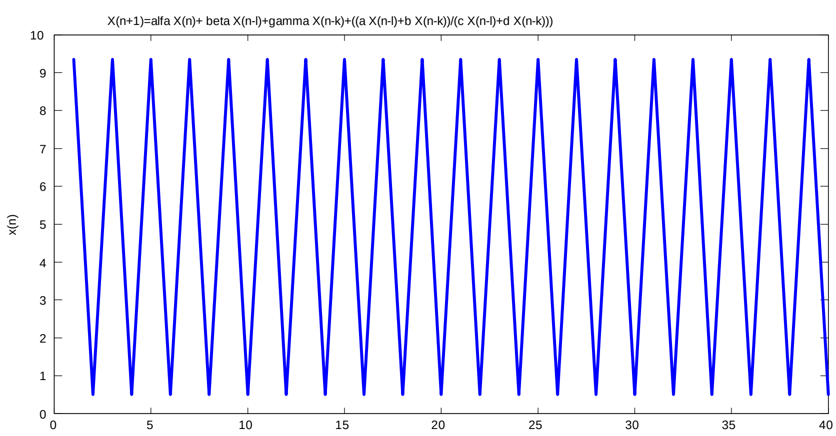

Fig. 9. Plot the solution of Eq. (1) has a periodic solution.

THEOREM 5.2. *If l is an odd and k is an even, then Eq. (1) has a prime period two solutions if*

$$(a-b)(d-c)(d-c)(1+\alpha+\gamma-\beta) > 4(ad(\alpha+\gamma)+cb(1-\beta)), \tag{10}$$

where $b<a,\ c<d$ and $\beta<\alpha+\gamma+1$.

Proof. Suppose that there exists a prime period two solution $...P,\ Q,\ P,\ Q,\ ...,$ of Eq. (1). We see from Eq. (1) when l is odd, and k is even that

$$P=\alpha Q+\beta P+\gamma Q+\tfrac{aP+bQ}{cP+dQ} \quad \text{and} \quad Q=\alpha P+\beta Q+\gamma P+\tfrac{aQ+bP}{cQ+dP}.$$

Then

$$c(1-\beta)P^2+d(1-\beta)PQ = c(\alpha+\gamma)PQ+d(\alpha+\gamma)Q^2+aP+bQ, \tag{11}$$

$$c(1-\beta)Q^2+d(1-\beta)PQ = c(\alpha+\gamma)PQ+d(\alpha+\gamma)P^2+aQ+bP. \tag{12}$$

Subtracting (11) from (12) gives

$$c(1-\beta)(P^2-Q^2)=-d(\alpha+\gamma)(P^2-Q^2)+a(P-Q)-b(P-Q),$$

$$(c(1-\beta)+d(\alpha+\gamma))(P-Q)(P+Q)=(a-b)(P-Q),$$

Since $P\neq Q$, it follows that

$$(c(1-\beta)+d(\alpha+\gamma))(P+Q)=a-b,$$

$$P+Q=\tfrac{a-b}{c(1-\beta)+d(\alpha+\gamma)}. \tag{13}$$

Again, adding (11) and (12) yields

$$2d(1-\beta)PQ+c(1-\beta)(P^2+Q^2)=2c(\alpha+\gamma)PQ+d(\alpha+\gamma)(P^2+Q^2)+(a+b)(P+Q),$$

$$2PQ(d(1-\beta)-c(\alpha+\gamma))=(d(\alpha+\gamma)-c(1-\beta))(P^2+Q^2)+(a+b)(P+Q), \tag{14}$$

It follows by (13), (14) and the relation

$$P^2+Q^2=(P+Q)^2-2PQ \quad \textit{for all} \quad P,Q\in R,$$

that

$$2PQ(d(1-\beta)-c(\alpha+\gamma))=(d(\alpha+\gamma)-c(1-\beta))\left((P+Q)^2-2PQ\right)+(a+b)(P+Q),$$

$$\begin{aligned}2(d-c)(1+\alpha+\gamma-\beta)PQ &= (P+Q)\left[(d(\alpha+\gamma)-c(1-\beta))(P+Q)+(a+b)\right],\\ &= \left(\tfrac{a-b}{c(1-\beta)+d(\alpha+\gamma)}\right)\left[\tfrac{(d(\alpha+\gamma)-c(1-\beta))(a-b)+(c(1-\beta)+d(\alpha+\gamma))(a+b)}{c(1-\beta)+d(\alpha+\gamma)}\right]\\ &= 2\left(\tfrac{a-b}{c(1-\beta)+d(\alpha+\gamma)}\right)\left(\tfrac{ad(\alpha+\gamma)+cb(1-\beta)}{c(1-\beta)+d(\alpha+\gamma)}\right)\end{aligned}$$

$$PQ=\tfrac{(a-b)(ad(\alpha+\gamma)+cb(1-\beta))}{(d-c)(1+\alpha+\gamma-\beta)(c(1-\beta)+d(\alpha+\gamma))^2}. \tag{15}$$

Now it is clear from equations (13) and (15) that P and Q are the two distinct roots of the quadratic equation

$$(c(1-\beta)+d(\alpha+\gamma))t^2-(a-b)t+\tfrac{(a-b)(ad(\alpha+\gamma)+cb(1-\beta))}{(d-c)(1+\alpha+\gamma-\beta)(c(1-\beta)+d(\alpha+\gamma))}=0. \tag{16}$$

and so

$$\begin{aligned}\left(\tfrac{a-b}{c(1-\beta)+d(\alpha+\gamma)}\right)^2 &> \tfrac{4(a-b)(ad(\alpha+\gamma)+cb(1-\beta))}{(d-c)(1+\alpha+\gamma-\beta)(c(1-\beta)+d(\alpha+\gamma))^2},\\ (a-b) &> \tfrac{4(ad(\alpha+\gamma)+cb(1-\beta))}{(d-c)(1+\alpha+\gamma-\beta)}.\end{aligned}$$

For $b < a,\ c < d$ and $\beta < 1 + \alpha + \gamma$, then

$$(a-b)(d-c)(d-c)(1+\alpha+\gamma-\beta) > 4(ad(\alpha+\gamma)+cb(1-\beta)).$$

Therefore Inequality (10) holds and the proof is complete.

Example 11. Figure (11) shows the Eq. (1) has a prime two solution when $l = 1$, $k = 4$, $\alpha = 0.001$, $\beta = 0.6$, $\gamma = 0.02$, $a = 0.9$, $b = 0.2$, $c = 0.05$ and $d = 0.55$ and the initial conditions $x_{-4} = Q$, $x_{-3} = P$, $x_{-2} = Q$, $x_{-1} = P$ and $x_0 = Q$, such that $P = \frac{a-b+\xi}{2(c(1-\beta)+d(\alpha+\gamma))}$ and $Q = \frac{a-b-\xi}{2(c(1-\beta)+d(\alpha+\gamma))}$ where $\xi = \sqrt{(a-b)^2 - \frac{4(a-b)(ad(\alpha+\gamma)+cb(1-\beta))}{(d-c)(1+\alpha+\gamma-\beta)}}$

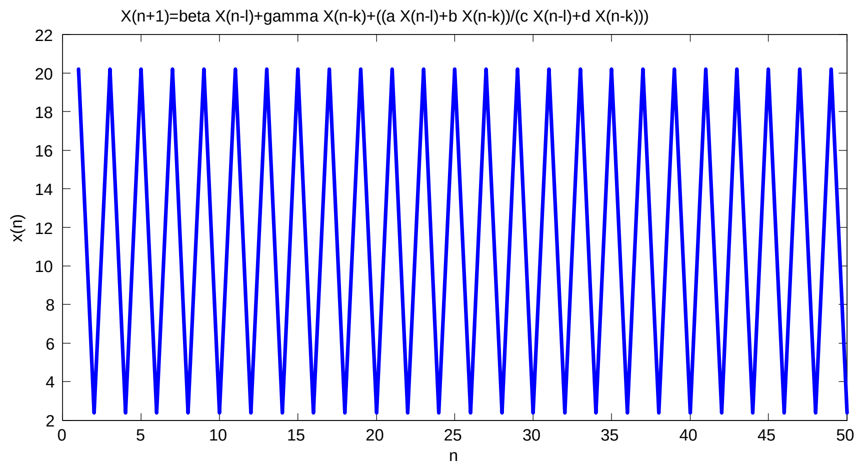

Fig. 11. sketch the solution of Eq. (1) has a periodic solution.

Theorem 5.3. *Equation (1) has no prime period two solutions if l and k are even and $\beta + \gamma + 1 \neq 0$.*

Proof. Suppose that there exists a prime period two solution $...P,\ Q,\ P,\ Q,\ ...,$ of Equation (1). We see from Equation (1) when l and k are even that

$$P = \alpha Q + \beta Q + \gamma Q + \frac{aQ+bQ}{cQ+dQ}, \quad (17)$$

$$Q = \alpha P + \beta P + \gamma P + \frac{aP+bP}{cP+dP}. \quad (18)$$

Subtracting (17) from (18) gives

$$(\alpha+\beta+\gamma+1)(P-Q) = 0,$$

Since $\alpha+\beta+\gamma+1 \neq 0$, then $P = Q$. This is a contradiction. Thus, the proof is completed.

Example 12. Figure (12) shows the Eq. (1) has no period two solution when $l = 4$, $k = 4$, $\alpha = 0.2$, $\beta = 0.7$, $\gamma = 0.4$, $a = 0.8$, $b = 0.3$, $c = 0.6$ and $d = 0.9$ and the initial conditions $x_{-4} = 6$, $x_{-3} = 7$, $x_{-2} = 2$, $x_{-1} = 3$ and $x_0 = 5$.

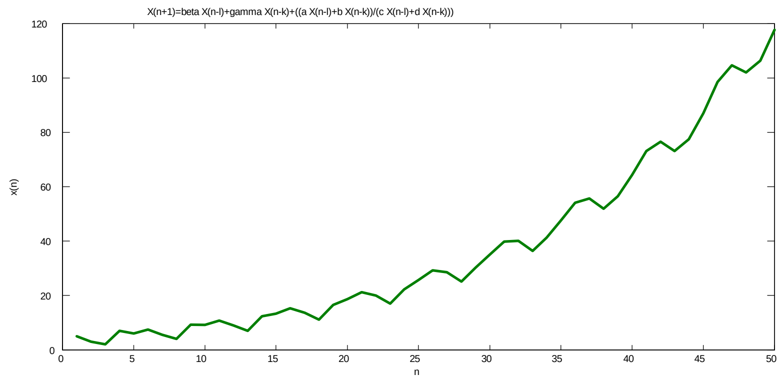

Fig. 12. Draw the solution of Eq. (1) has no periodic when l and k are even.

THEOREM 5.4. *Equation (1) has no prime period two solutions if l and k are odd and $1-\beta-\gamma \neq 0$.*

Proof. Suppose that there exists a prime period two solution $...P,\ Q,\ P,\ Q,\ ...,$ of Eq. (1). We see from Eq. (1) when l and k are odd that

$$P = \alpha Q + \beta P + \gamma P + \tfrac{aP+bP}{cP+dP}, \quad (10)$$

$$Q = \alpha P + \beta Q + \gamma Q + \tfrac{aQ+bQ}{cQ+dQ}. \quad (20)$$

Subtracting (19) from (20) gives

$$(1-\alpha-\beta-\gamma)(P-Q) = 0,$$

Since $1-\alpha-\beta-\gamma \neq 0$, then $P = Q$. This is a contradiction. Thus, the proof is completed.

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OSCILLATION OF SOLUTIONS OF CERTAIN LINEAR DIFFERENTIAL EQUATIONS

YONG LIU AND XIAOGUANG QI

ABSTRACT. In this article, we mainly investigate the growth of solutions of certain higher order linear differential equations. The results we obtain generalize some previous results of P. C. Wu and J. Zhu.

1 INTRODUCTION

In this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna theory(e.g.see [8, 14, 15]). In addition, we will use the notations $\sigma(f), \mu(f), \lambda(f), \lambda(\frac{1}{f})$ to denote the order, the lower order, the exponents of the convergence of the zero-sequence and the exponents of convergence of pole-sequence of a meromorphic function $f(z)$, respectively.

For a set $E \subset R^{+}$, let $m(H)$, respectively $m_l(H)$, denote the linear measure, respectively the logarithmic measure, of H. By $\chi H(t)$, we denote the characteristic function of H. Moreover, the upper logarithmic density and the lower logarithmic density of H are defined by

$$\overline{\log dens}H = \limsup_{r\to\infty}(\int_1^r (\chi H(t)/t)dt)/\log r$$

$$\underline{\log dens}H = \liminf_{r\to\infty}(\int_1^r (\chi H(t)/t)dt)/\log r,$$

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Key words:deficient functions, complex differential equations, value distribution, infinite order.

where $\chi H(t)$ is the characteristic function of the set H.

For the second order linear differential equation

$$f'' + A(z)f' + B(z)f = 0, \tag{1.1}$$

many authors have investigated the growth of solutions of (1.1), where $A(z)$ and $B(z)$ are entire functions. It is well known that if $B(z)$ is transcendental and f_1, f_2 are two linearly independent solutions of equation (1.1), then at least one of f_1, f_2 must have infinite order. On the other hand, there exist some equations of the form (1.1) that possess a solution $f \not\equiv 0$ of finite order; for example, $f(z) = e^{2z}$ satisfies $f'' + e^{-2z}f' - (2e^{-2z} + 4)f = 0$. Thus a natural question is: what conditions on $A(z)$ and $B(z)$ can guarantee that every solution $f \not\equiv 0$ of (1.1) has infinite order? Many authors have focused on this subject, such that ([1-3, 9-13])

Recently, P. C. Wu and J. Zhu [12] proved the following result:

Theorem A. [12] Let $A(z)$ be a meromorphic function with finite order having a finite deficient value. Suppose that $B(z)$ is a meromorphic function satisfying the following condition:

$$\lambda(\frac{1}{B}) < \mu(B) < \frac{1}{2}.$$

Then every solution $f \not\equiv 0$ of equation (1.1) is of infinite order.

Thus a natural question arises: whether does the conclusion hold when $\mu(B) = \frac{1}{2}$? We give an affirmative answer, and get the following interesting result:

Theorem 1.1. Let $A_0(z)$ be a meromorphic function with $\lambda(\frac{1}{A_0}) < \mu(A_0) \leq \frac{1}{2}$. And let $A_j(z)(j = 1, 2, \cdots, k-1)$ be meromorphic functions with finite order having a finite deficient value. Then every solution $f \not\equiv 0$ of

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f'(z) + A_0(z)f = 0 \tag{1.2}$$

satisfies $\sigma(f) = \infty$.

2 SOME LEMMAS

Lemma 2.1. [7] *Let $f(z)$ be a transcendental meromorphic function of finite-order σ, and let $\varepsilon > 0$ be a given constant. Then there exists a set $H \subset (1, \infty)$ that has finite logarithmic measure, such that for all z satisfying $|z| \notin H \cup [0, 1]$ and for all $k, j, 0 \leq j < k$, one has*

$$\Big|\frac{f^{(k)}(z)}{f^{(j)}(z)}\Big| \leq |z|^{(k-j)(\sigma-1+\varepsilon)} \tag{2.1}$$

Similarly, there exists a set $E \subset [0, 2\pi)$ of linear measure zero such that for all $z = re^{i\theta}$ with $|z|$ sufficiently large and $\theta \in [0, 2\pi) \setminus E$, and for all $k, j, 0 \leq j < k$, the inequality (2.1) holds.

Lemma 2.2. [5] *Let $f(z)$ be a meromorphic function of finite order σ. Given $\zeta > 0$ and $l, 0 < l < \frac{1}{2}$, there exist a constant $K(\sigma,\zeta)$ and a set $E_\zeta \subset [0,\infty)$ of lower logarithmic density greater than $1-\zeta$ such that for all $r \in E_\zeta$ and for every interval J of length l*

$$r\int_J \Big|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\Big| d\theta < K(\sigma,\zeta)\Big(l\log\frac{1}{l}\Big)T(r,f).$$

By the remarks following Theorem 8.1 in [4] and ref [9], we can obtain that

Lemma 2.3. *Suppose that $g(z)$ is an entire function of $\mu(f) = \frac{1}{2}$, and satisfies*

$$\log L(r,g) = o(\log M(r,g))$$

where $L(r,g) = \min_{|z|=r}|g(z)|, M(r,g) = \max_{|z|=r}|g(z)|$. There exists a set G of logarithmic density 1, a set H of density 0, a real-valued function $\varphi(r)$, and a positive function $\Psi(r)$ varying slowing in the sense that

$$\lim_{r\to\infty}\frac{\Psi(\sigma(r))}{\Psi(r)} = 1, r \in G \tag{2.1}$$

for all $\sigma > 0$, such that for $r \in G - H$

$$\log|g(re^{i(\varphi+\varphi(r))})| = (\cos(\frac{\varphi}{2}) + o(1))r^{\frac{1}{2}}\Psi(r), r \to \infty, \tag{2.2}$$

uniformly for $\varphi \in [-\pi,\pi]$.

Lemma 2.4. [6] *Suppose that $g(z)$ is transcendental and meromorphic in the plane, of lower order $\mu < \alpha < 1$, and define $L(r,g) = \min\{|g(z)| : |z| = r\}$ and*

$$Y_1 = \{r > 1 : \log L(r,g) > \gamma(\cos\pi\alpha + \delta(\infty,g) - 1)T(r,g)\},.$$

where $\gamma = \frac{\pi\alpha}{\sin\pi\alpha}$. Then Y_1 has upper logarithmic density at least $1 - \frac{\mu}{\alpha}$.

Lemma 2.5. *Suppose $f(z)$ is meromorphic and $\lambda\Big(\frac{1}{f}\Big) < \mu(f) \le \frac{1}{2}$. Then either, for every $\delta < \mu(f)$, there exists $r_m \to \infty$ such that*

$$\log|f(z)| > r_m^\delta \tag{2.3}$$

for all z satisfying $|z| = r_m$. Or, for every $\delta < \mu(f)$, if

$$k_r = \{\theta \in [0,2\pi) : \log|f(re^{i\theta})| < r^\delta\}$$

there exists a set $E_1 \subset [1,\infty)$ of upper logarithmic density 1 such that for $r \in E_1$,

$$m(K_r) \to 0,\ r \to \infty.$$

Proof. Let $f(z) = \frac{g(z)}{l(z)}$, where $l(z)$ is canonical products(or polynomial) formed by the poles of $f(z)$, and $g(z)$ is entire. From $\lambda(\frac{1}{f}) < \mu(f)$, we have

$$\lambda\Big(\frac{1}{f}\Big) = \lambda(l) = \sigma(l) < \mu(f),\ \mu(f) = \mu(g).$$

We divide our proof into two cases:

Case 1: $\mu(f) < \frac{1}{2}$. Since $\lambda(\frac{1}{f}) < \mu(f)$, we have $\delta(\infty, f) = 1$. Let $\lambda\left(\frac{1}{f}\right) < \delta < \alpha_1 < \mu(f) < \alpha < \frac{1}{2}$. By Lemma 2.4, then there exists a set E_1 of $(1, \infty)$, having lower logarithmic density $1 - \frac{\mu(f)}{\alpha}$, such that for all $r \in E_1$ we have

$$\log L(r, f) > \gamma \cos \pi\alpha T(r, f) \geq r^{\alpha_1} > r^{\delta},$$

where $\gamma = \frac{\pi\alpha}{\sin \pi\alpha}$. Hence, for every $\delta < \mu(f)$, there exists $r_m \to \infty$ such that

$$\log |f(z)| > r_m^{\delta}$$

for all z satisfying $|z| = r_m$. So (i) holds.

Case 2: $\mu(f) = \frac{1}{2}$. There exist the following two subcases:

Subcase 2.1. If there exists $r_n \to \infty$ with

$$\log L(r_n, g) > \alpha \log M(r_n, g) \ \ as \ \ r_n \to \infty \tag{2.4}$$

for some $\alpha > 0$. Hence for given $0 < \varepsilon < \min\{\frac{\delta - \sigma(l)}{2}, \frac{\mu(g) - \delta}{2}\}$, by (2.4) we have

$$\begin{aligned} & \log L(r_n, f) \geq \log L(r_n, g) - \log M(r_n, l) \\ \geq \ & \alpha r_n^{\delta+\varepsilon} - r_n^{\sigma(l)+\varepsilon} > r_n^{\delta}. \end{aligned} \tag{2.5}$$

So (i) also hold.

Subcase 2.2. Otherwise

$$\log L(r, g) = o(\log M(r, g)). \tag{2.6}$$

We choose $\max\{\delta, \lambda(\frac{1}{f})\} < \xi < \alpha < \mu(g)$. We note that $E^* = G - H$ has logarithmic density 1, where G, H are defined as in Lemma 2.3. By Lemma 2.3, (2.1), (2.2), (2.6) and the fact that E^* has logarithmic density 1, we obtain

$$\Psi(r) r^{\frac{1}{2} - \xi} \to \infty, \tag{2.7}$$

as $r \to \infty$. Defining

$$K*_r = \{\theta \in [0, 2\pi] : \log |g(re^{i\theta})| < r^{\xi}\}.$$

By (2.2) and (2.7), for all $r \in E^*$ we have that

$$m(K*_r) \to 0.$$

Set $F = \{z | f(z) = \infty\}$, since $\lambda(\frac{1}{f}) < \frac{1}{2}$, we have $m_l(F) < \infty$. Obviously, $E^{**} = E^* - F$ has logarithmic density 1. If $\theta \in K_r$ for all $r \in E^{**}$, we have

$$\log |g(re^{i\theta})| < \log |f(z)| + \log M(r, d) < r^{\delta} + r^{\xi} < r^{\alpha}.$$

So, $K_r \subset K_r^*$. Thus Lemma 2.5 holds. □

Lemma 2.6. *Suppose $f(z)$ is a nonconstant meromorphic function of order $\sigma < \sigma_1 < \infty$. For a positive number α, there exists a set $E(\alpha) \subset [1, \infty)$ with finite linear measure such that*

$$m(E(\alpha) \cap [\frac{r}{e}, er]) < \exp(-r^{\alpha}), r > r_0(f),$$

and that, for $|z| = r \notin E(\alpha)$, we have

$$\Big|\frac{f^{(j)}(z)}{f(z)}\Big| < \exp(r^{3\alpha}),\ r > r_0(f),\ j = 1, 2, \cdots, k.$$

Proof. Let $\Delta(a, \delta) = \{z : |z - a| < \delta\}$ and let $\{l_\mu\}, \{m_\nu\}$ denote all the zeros and poles of $f(z)$, respectively. Let $A = A_1 \cup A_2$, where $A_1 = \cup_\mu \Delta(l_\mu, \frac{1}{k}\exp(-3|l_\mu|^{2\alpha}))$, $A_2 = \cup_\nu \Delta(m_\nu, \frac{1}{k}\exp(-3|m_\nu|^{2\alpha}))$.
Suppose $E_1 = \{t \geq 1 : A \cap \{|z| = t\} \neq \emptyset\}$. Obviously

$$\begin{aligned} & m(E_1(\alpha) \cap [\frac{r}{e}, er]) \\ < \ & \frac{1}{k}\{n(3r, f) + n(3r, \frac{1}{f})\}\exp(-r^{2\alpha}) \\ < \ & \frac{2}{k}(3r)^{\sigma_1}\exp(-r^{2\alpha}) < \frac{1}{k}\exp(-r^{\alpha}), r > r_1(\alpha) \end{aligned}$$

for $|z| = r \notin E_1(\alpha)$. We consider the differentiated Poisson-Jensen formula, for $|z| = r$ and $R = 3r$, we have

$$\begin{aligned} \frac{zf'(z)}{f(z)} &= \frac{1}{2\pi}\int_0^{2\pi}\log|f(3re^{i\theta})|\frac{2z(3re^{i\theta})}{(3re^{i\theta} - z)^2}d\theta \\ &+ \sum_{|l_\mu|<3r}(\frac{z}{z - l_\mu} + \frac{\overline{l_\mu}z}{(3r)^2 - \overline{l_\mu}z}) \\ &+ \sum_{|m_\nu|<3r}(\frac{z}{z - m_\nu} + \frac{\overline{m_\nu}z}{(3r)^2 - \overline{m_\nu}z}). \end{aligned}$$

We use the method of ref [9], we also obtain

$$\begin{aligned} \Big|\frac{zf'(z)}{f(z)}\Big| \quad & \leq \frac{3}{2}\{m(3r, f) + m(3r, \frac{1}{f})\} \\ & +4r(n(3r, f) + n(3r, \frac{1}{f}))e^{(9r)^{2\alpha}} + O(1) \\ & \leq (8r + 3)r^{\sigma_1}e^{(9r)^{2\alpha}}. \end{aligned}$$

Hence

$$\Big|\frac{f'(z)}{f(z)}\Big| < (8r + 3)r^{\sigma_1 - 1}e^{(9r)^{2\alpha}} < e^{r^{3\alpha}}.$$

We use the same method to each of the functions $f', \cdots, f^{(k)}$, we get there exists a set $E(\alpha)$ such that

$$m(E(\alpha) \cap [\frac{r}{e}, er]) < \exp(-r^{\alpha}),$$

and if $|z| = r \notin E(\alpha)$, we obtain

$$\Big|\frac{f^{(j)}(z)}{f(z)}\Big| < e^{r^{3\alpha}}, 1 \le j \le k$$

Evidently

$$m_l(E(\alpha) \cap [\frac{r}{e}, er]) \le \frac{em(E \cap [\frac{r}{e}, er])}{r} = o(1).$$

Since $m_l([\frac{r}{e}, er]) = 2$, we obtain that the logarithmic density of E is 0. □

By the remarks following Theorem 8.1 in [4] and ref [9], we can easily get that

Lemma 2.7. *Suppose $f(z)$ is meromorphic of $\lambda(\frac{1}{f}) < \mu(f) < 1$ and $0 < \varepsilon < \min(\frac{\mu(f)-\lambda(\frac{1}{f})}{2}, 1-\mu(f))$. Suppose there exists an unbounded set of r-valued such that*

$$\log|f(re^{i\theta})| > r^{\mu(f)-\varepsilon}$$

for all $\theta \in [0, 2\pi]$. Suppose also that $E_3 \subset [1, \infty)$ satisfies

$$m(E_3 \cap [\frac{r}{e}, er]) < \exp(-r^{6\varepsilon}), r > R_0.$$

Then there is an unbounded set of s-values with $s \notin E_3$ such that

$$\log|f(se^{i\theta})| > s^{\mu(f)-2\varepsilon}$$

for all $\theta \in [0, 2\pi]$.

3 Proof of Theorem 1.1

Suppose that $A_j(z)$ $(j = 1, 2, \cdots, k-1)$ has a finite deficiency $\delta(a_j, f) = 2\alpha_j > 0$ at $a_j \in C$. By the definition of deficiency, for all sufficiently r, we get

$$m(r, \frac{1}{A_j - a_j}) \ge \alpha_j T(r, A_j).$$

Hence, for all sufficiently r, there exists a point z_r satisfying $|z_r| = r$ and

$$\log|A_j(z_r) - a_j| \le -\alpha_j T(r, A_j). \tag{3.1}$$

By Lemma 2.2, we choose $l > 0$ so small that

$$K(\rho, \varphi)(l, \log\frac{1}{l}) < \frac{\alpha_j}{2}.$$

Then for all $r \in E_\varphi$ and for every interval J, we obtain that

$$r\int_J \Big|\frac{A_j'(re^{i\theta})}{A_j(re^{i\theta})}\Big| d\theta < \frac{\alpha_j}{2} T(r, A_j),$$

where E_φ is a set with lower logarithmic density greater than $1-\varphi$. Suppose $z_r = re^{i\theta_r}$ and $\varphi > 0$ be a sufficiently small number, we choose a number $\theta_0 > 0, |\theta_r - \theta_0| \le l$, and a set $E_\varphi \subset [0, \infty)$ with lower logarithmic density greater than $1-\varphi$. For all given $r \in E_\varphi$ and for all $\theta \in [\theta_r - \beta, \theta_r + \beta]$, we have

$$\begin{aligned}
&\log|A_j(re^{i\theta}) - a_j| \\
&= \log|A_j(re^{i\theta_r}) - a_j| + \int_{\theta_r}^{\theta} \frac{d}{dt} \log|A_j(e^{it}) - a_j| dt \\
&\le -\alpha T(r, A_j) + r \int_{\theta_r}^{\theta} \Big|\frac{A_j'(re^{it})}{A_j(re^{it})}\Big| |dt| \\
&\le -\frac{\alpha}{2} T(r, A_j) \le 0.
\end{aligned}$$

Thus for $|z_r| = r \in E(\varphi) \setminus [0, r_1]$ and $\theta \in [\theta_r - \theta_0, \theta_r + \theta_0]$, we obtain

$$|A_j(re^{i\theta})| \le |a_j| + 1. \tag{3.2}$$

Let transcendental $f \not\equiv 0$ be a finite order solution of (1.2), and suppose $\lambda(\frac{1}{A_0}) < \mu(A_0)$. We divide the proof into two cases depending on the growth property of $A_0(z)$ by Lemma 2.5.

Case 1. For given $\varepsilon, 0 < \varepsilon < \min\{\frac{\mu(f) - \lambda(\frac{1}{f})}{2}, 1 - \mu(f), \frac{\mu(A_0)}{20}\}$. there exists a sequence $r_m \to \infty$ such that

$$\log|A_0(z)| > r_m^{\mu(A_0) - \varepsilon}. \tag{3.3}$$

From (1.2), we get

$$|A_0(z)| \le \Big|\frac{f^{(k)}(z)}{f(z)}\Big| + |A_{k-1}|\Big|\frac{f^{(k-1)}(z)}{f(z)}\Big| + \cdots + |A_1(z)|\Big|\frac{f'(z)}{f(z)}\Big|. \tag{3.4}$$

By Lemma 2.6, set $\alpha = 7\varepsilon$, there exists a set $E_\alpha \subset [1, \infty)$ with finite linear measure satisfying

$$m(E_\alpha \cap [\frac{r}{e}, er]) < e^{-r^{6\varepsilon}},\ r > R_0 \tag{3.5}$$

and if $|z| = r \notin E_\alpha$, we get

$$\Big|\frac{f^{(j)}(z)}{f(z)}\Big| < e^{r^{15\varepsilon}},\ 1 \le j \le k,\ r > R_0 \tag{3.6}$$

By Lemma 2.7, there exists a sequence $s_m \to \infty, s_m \notin E_\alpha$ such that for all $\theta \in [0, 2\pi]$,

$$\log|A_0(s_m e^{i\theta})| > s_m^{\mu(A_0) - 2\varepsilon}. \tag{3.7}$$

With (3.1), (3.4), (3.6), (3.7), as $s_m \to \infty$, we have

$$\exp(s_m^{\mu(A_0) - 2\varepsilon}) \le (|a_1| + \cdots + |a_{k-1}| + k) \exp(s_m^{16\varepsilon}). \tag{3.8}$$

Thus, (3.8) is impossible.

Case 2. For given $\varepsilon, 0 < \varepsilon < \frac{\mu(A_0)}{2}$, If

$$K_r = \{\theta \in [0, 2\pi) : \log |g(r^{i\theta})| < r^{\mu(A_0)-\varepsilon}\},$$

there exists a set $E_2 \subset [0, \infty)$ having logarithmic density 1 such that $m(K_r) \to 0$, as $r \to \infty$ in E_2.
By Lemma 2.1, there exists a set $E_3 \subset [0, \infty)$ with linear measure zero such that for all $|z| = r \notin E_3$, we get

$$\Big|\frac{f^{(k)}(z)}{f(z)}\Big| \leq |z|^{k\sigma(f)-1+\varepsilon}. \tag{3.9}$$

Note that $E_4 = E(\varphi) \cap E_2 \backslash E_3$ has a positive lower logarithmic density, and for all sufficiently large $r \in E_4$, we have $[\theta_r - \phi, \theta_r + \phi] - K_r \neq \emptyset$. Hence, there exist unbounded points $z = re^{i\theta}$ such that (3.2), (3.9) and $\log |A_0(re^{i\theta})| \geq r^{\mu(A_0)-\varepsilon}$, we obtain that

$$\exp\{r^{\mu(A_0)-\varepsilon}\} \leq (k + |a|)r^{k\sigma(f)+1}. \tag{3.10}$$

Obviously, (3.10) is impossible. By case 1, and case 2, we obtain that $\sigma(f) = \infty$.

If rational function $f \not\equiv 0$ be a solution of (1.2), using the above similar method, we can get a contradiction. Hence, Theorem 1.1 hold.

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