Volume 24, Number 6 June 1st, 2018
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of Computational Analysis and Applications

EUDOXUS PRESS,LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE

SCOPE OF THE JOURNAL

An international publication of Eudoxus Press, LLC
(sixteen times annually)

Editor in Chief: George Anastassiou
Department of Mathematical Sciences,
University of Memphis, Memphis, TN 38152-3240, U.S.A
ganastss@memphis.edu
http://www.msci.memphis.edu/~ganastss/jocaaa

The main purpose of "J.Computational Analysis and Applications" is to publish high quality research articles from all subareas of Computational Mathematical Analysis and its many potential applications and connections to other areas of Mathematical Sciences. Any paper whose approach and proofs are computational,using methods from Mathematical Analysis in the broadest sense is suitable and welcome for consideration in our journal, except from Applied Numerical Analysis articles. Also plain word articles without formulas and proofs are excluded. The list of possibly connected mathematical areas with this publication includes, but is not restricted to: Applied Analysis, Applied Functional Analysis, Approximation Theory, Asymptotic Analysis, Difference Equations, Differential Equations, Partial Differential Equations, Fourier Analysis, Fractals, Fuzzy Sets, Harmonic Analysis, Inequalities, Integral Equations, Measure Theory, Moment Theory, Neural Networks, Numerical Functional Analysis, Potential Theory, Probability Theory, Real and Complex Analysis, Signal Analysis, Special Functions, Splines, Stochastic Analysis, Stochastic Processes, Summability, Tomography, Wavelets, any combination of the above, e.t.c.

"J.Computational Analysis and Applications" is a peer-reviewed Journal. See the instructions for preparation and submission of articles to JoCAAA. Assistant to the Editor: Dr.Razvan Mezei,mezei_razvan@yahoo.com, Madison,WI,USA.

Journal of Computational Analysis and Applications(JoCAAA) is published by **EUDOXUS PRESS,LLC**,1424 Beaver Trail Drive,Cordova,TN38016,USA,anastassioug@yahoo.com http://www.eudoxuspress.com. **Annual Subscription Prices**:For USA and Canada,Institutional:Print $800, Electronic OPEN ACCESS. Individual:Print $400. For any other part of the world add $150 more(handling and postages) to the above prices for Print. No credit card payments.

JoCAAA is reviewed and abstracted by AMS Mathematical Reviews,MATHSCI,and Zentralblaat MATH.

Editorial Board
Associate Editors of Journal of Computational Analysis and Applications

Dartmouth College
8000 Cummings Hall,
Hanover, NH 03755-8000
603-646-3843 (X 3546 Secr.)
e-mail:george.cybenko@dartmouth.edu
Approximation Theory and Neural Networks

Sever S. Dragomir
School of Computer Science and Mathematics, Victoria University,
PO Box 14428,
Melbourne City,
MC 8001, AUSTRALIA
Tel. +61 3 9688 4437
Fax +61 3 9688 4050
sever.dragomir@vu.edu.au
Inequalities, Functional Analysis, Numerical Analysis, Approximations, Information Theory, Stochastics.

Oktay Duman
TOBB University of Economics and Technology,
Department of Mathematics, TR-06530,
Ankara, Turkey,
oduman@etu.edu.tr
Classical Approximation Theory, Summability Theory, Statistical Convergence and its Applications

Saber N. Elaydi
Department Of Mathematics
Trinity University
715 Stadium Dr.
San Antonio, TX 78212-7200
210-736-8246
e-mail: selaydi@trinity.edu
Ordinary Differential Equations, Difference Equations

J .A. Goldstein
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152
901-678-3130
jgoldste@memphis.edu
Partial Differential Equations, Semigroups of Operators

H. H. Gonska
Department of Mathematics
University of Duisburg
Duisburg, D-47048
Germany
011-49-203-379-3542
e-mail: heiner.gonska@uni-due.de
Approximation Theory, Computer Aided Geometric Design

John R. Graef
Department of Mathematics
University of Tennessee at Chattanooga
Chattanooga, TN 37304 USA
John-Graef@utc.edu
Ordinary and functional differential equations, difference equations, impulsive systems, differential inclusions, dynamic equations on time scales, control theory and their applications

Weimin Han
Department of Mathematics
University of Iowa
Iowa City, IA 52242-1419
319-335-0770
e-mail: whan@math.uiowa.edu
Numerical analysis, Finite element method, Numerical PDE, Variational inequalities, Computational mechanics

Tian-Xiao He
Department of Mathematics and Computer Science
P.O. Box 2900, Illinois Wesleyan University
Bloomington, IL 61702-2900, USA
Tel (309)556-3089
Fax (309)556-3864
the@iwu.edu
Approximations, Wavelet, Integration Theory, Numerical Analysis, Analytic Combinatorics

Margareta Heilmann
Faculty of Mathematics and Natural Sciences, University of Wuppertal
Gaußstraße 20
D-42119 Wuppertal, Germany,
heilmann@math.uni-wuppertal.de
Approximation Theory (Positive Linear Operators)

Xing-Biao Hu
Institute of Computational Mathematics
AMSS, Chinese Academy of Sciences
Beijing, 100190, CHINA
hxb@lsec.cc.ac.cn

Computational Mathematics

Instructions to Contributors
Journal of Computational Analysis and Applications

An international publication of Eudoxus Press, LLC, of TN.

Editor in Chief: George Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152-3240, U.S.A.

1. Manuscripts files in Latex and PDF and in English, should be submitted via email to the Editor-in-Chief:

Prof.George A. Anastassiou
Department of Mathematical Sciences
The University of Memphis
Memphis,TN 38152, USA.
Tel. 901.678.3144
e-mail: ganastss@memphis.edu

Authors may want to recommend an associate editor the most related to the submission to possibly handle it.

Also authors may want to submit a list of six possible referees, to be used in case we cannot find related referees by ourselves.

2. Manuscripts should be typed using any of TEX,LaTEX,AMS-TEX,or AMS-LaTEX and according to EUDOXUS PRESS, LLC. LATEX STYLE FILE. (Click HERE to save a copy of the style file.)They should be carefully prepared in all respects. Submitted articles should be brightly typed (not dot-matrix), double spaced, in ten point type size and in 8(1/2)x11 inch area per page. Manuscripts should have generous margins on all sides and should not exceed 24 pages.

3. Submission is a representation that the manuscript has not been published previously in this or any other similar form and is not currently under consideration for publication elsewhere. A statement transferring from the authors(or their employers,if they hold the copyright) to Eudoxus Press, LLC, will be required before the manuscript can be accepted for publication.The Editor-in-Chief will supply the necessary forms for this transfer.Such a written transfer of copyright,which previously was assumed to be implicit in the act of submitting a manuscript,is necessary under the U.S.Copyright Law in order for the publisher to carry through the dissemination of research results and reviews as widely and effective as possible.

4. The paper starts with the title of the article, author's name(s) (no titles or degrees), author's affiliation(s) and e-mail addresses. The affiliation should comprise the department, institution (usually university or company), city, state (and/or nation) and mail code.

The following items, 5 and 6, should be on page no. 1 of the paper.

5. An abstract is to be provided, preferably no longer than 150 words.

6. A list of 5 key words is to be provided directly below the abstract. Key words should express the precise content of the manuscript, as they are used for indexing purposes.

The main body of the paper should begin on page no. 1, if possible.

7. All sections should be numbered with Arabic numerals (such as: 1. INTRODUCTION) .
Subsections should be identified with section and subsection numbers (such as 6.1. Second-Value Subheading).
If applicable, an independent single-number system (one for each category) should be used to label all theorems, lemmas, propositions, corollaries, definitions, remarks, examples, etc. The label (such as Lemma 7) should be typed with paragraph indentation, followed by a period and the lemma itself.

8. Mathematical notation must be typeset. Equations should be numbered consecutively with Arabic numerals in parentheses placed flush right, and should be thusly referred to in the text [such as Eqs.(2) and (5)]. The running title must be placed at the top of even numbered pages and the first author's name, et al., must be placed at the top of the odd numbed pages.

9. Illustrations (photographs, drawings, diagrams, and charts) are to be numbered in one consecutive series of Arabic numerals. The captions for illustrations should be typed double space. All illustrations, charts, tables, etc., must be embedded in the body of the manuscript in proper, final, print position. In particular, manuscript, source, and PDF file version must be at camera ready stage for publication or they cannot be considered.

Tables are to be numbered (with Roman numerals) and referred to by number in the text. Center the title above the table, and type explanatory footnotes (indicated by superscript lowercase letters) below the table.

10. List references alphabetically at the end of the paper and number them consecutively. Each must be cited in the text by the appropriate Arabic numeral in square brackets on the baseline.
References should include (in the following order):
initials of first and middle name, last name of author(s)
title of article,

name of publication, volume number, inclusive pages, and year of publication.

Authors should follow these examples:

<u>Journal Article</u>

1. H.H.Gonska,Degree of simultaneous approximation of bivariate functions by Gordon operators, (journal name in italics) *J. Approx. Theory*, 62,170-191(1990).

<u>Book</u>

2. G.G.Lorentz, (title of book in italics) *Bernstein Polynomials* (2nd ed.), Chelsea,New York,1986.

<u>Contribution to a Book</u>

3. M.K.Khan, Approximation properties of beta operators,in(title of book in italics) *Progress in Approximation Theory* (P.Nevai and A.Pinkus,eds.), Academic Press, New York,1991,pp.483-495.

11. All acknowledgements (including those for a grant and financial support) should occur in one paragraph that directly precedes the References section.

12. Footnotes should be avoided. When their use is absolutely necessary, footnotes should be numbered consecutively using Arabic numerals and should be typed at the bottom of the page to which they refer. Place a line above the footnote, so that it is set off from the text. Use the appropriate superscript numeral for citation in the text.

13. After each revision is made please again submit via email Latex and PDF files of the revised manuscript, including the final one.

14. Effective 1 Nov. 2009 for current journal page charges, contact the Editor in Chief. Upon acceptance of the paper an invoice will be sent to the contact author. The fee payment will be due one month from the invoice date. The article will proceed to publication only after the fee is paid. The charges are to be sent, by money order or certified check, in US dollars, payable to Eudoxus Press, LLC, to the address shown on the Eudoxus <u>homepage</u>.

No galleys will be sent and the contact author will receive one (1) electronic copy of the journal issue in which the article appears.

15. This journal will consider for publication only papers that contain proofs for their listed results.

New types of hesitant fuzzy ideals in $BCK-$algebras

N. O. Alshehri[a] and H. A. Alshehri[b]
[a]Departement of Mathematics, Faculty of Science, King Abdulaziz University, Al-Faisaliah Campus, Jeddah, Saudi Arabia
[b]Departement of Natural and Engineeeing Sciences, Faculty Applied Studies and Community Service, King Saud University, Riyadh, Saudi Arabia

Abstract

The aim of this paper is to intrduoce the new types of ideals of $BCK-$algebras. We define and discuss the hesitant fuzzy (implicative, positive implicative and commutative) ideals in $BCK-$algebras and then we study some of thier proprties.

Key words: $BCK-$algebras, (implicative, positive implicative and commutative) ideals in $BCK-$algebras,fuzzy sets, hesitant fuzzy sets, heaitant fuzzy (subalgebras and ideals), hesitant fuzzy (implicative , positive implicative and commutative) ideals in $BCK-$algebras.

1 Introduction

A BCK-algebra is an important class of logical algebras introduced by Iséki (see [2,3]) and was extensively investigated by several researchers. The concept of fuzzy sets was introduced by Zadeh in [4]. Since then ideals have been applied to other algebraic structures such as semigroups, groups rings, modules, vector spaces and topologies. In 1991, Xi [6] applied the concept of fuzzy sets to BCK-algebra, and he got some results. Further, Jun and Roh [8] , Jun et al. [9] studied fuzzy commutative ideals and fuzzy positive implicative ideals, respectively. Meng [1] introduced the concept of implicative ideals in BCK-algebras and investigated the relationship of it with the concepts of positive implicative ideals and commutative ideals. Torra [7] introduced the notion of hesitant fuzzy sets which ara a very useful to express people hesitancy in daily life. The hesitant fuzzy set is a very uesful tool to deal with uncertainty, which can be accurately and perfectly described in terms of the opinions of decision makers. In 2011, Xi and Xu [5] introduced hesitant fuzzy information aggregation techniques and their applications for decision making. In 2016, Jun and Ahn [10] interoduced the notions of hesitsnt fuzzy subalgebras and hesitant fuzzy ideals of $BCK/BCI-$algebras. In this paper, we introduce the notions of hesitant fuzzy implicative ideals, hesitant fuzzy positive implicative ideals and hesitant fuzzy commutative ideals of BCK-algebras, and discuss some properties of them.

2 Preliminaries

An algebra $(X; *, 0)$ of type $(2, 0)$ is said to be a BCK-algebra if it satisfies the axioms: for all $x, y, z \in X$

$(BCK-1)$ $((x * y) * (x * z)) * (z * y) = 0$,
$(BCK-2)$ $(x * (x * y)) * y = 0$,
$(BCK-3)$ $x * x = 0$,
$(BCK-4)$ $0 * x = 0$,
$(BCK-5)$ $x * y = 0$ and $y * x = 0$ imply $x = y$.

Define a binary relation $\leq$ on X by letting $x \leq y$ if and only if $x * y = 0$. Then $(X; \leq)$ is a partially ordered set with the least element 0. In any BCK-algebra X, the following hold :

(1) $(x * y) * z = (x * z) * y$,
(2) $x * y \leq x$,
(3) $x * 0 = x$,
(4) $(x * z) * (y * z) \leq x * y$,
(5) $x * (x * (x * y)) = x * y$,
(6) $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$, for all $x, y, z \in X$.

A non-empty subset I of X is called an ideal of X if (I_1) $0 \in I$, (I_2) $x * y \in I$

and $y \in I$ imply $x \in I$. A non-empty subset I of is called an implicative ideal if it is satisfies (I_1) and (I_3) $x \in I$ whenever $(x * (y * x)) * z \in I$ and $z \in I$. A commutative ideal if it satisfies (I_1) and (I_4) $x * (y * (y * x)) \in I$ whenever $(x * y) * z \in I$ and $z \in I$; a positive implicative ideal if it satisfies (I_1) and (I_5) $x * z \in I$ whenever $(x * y) * z \in I$ and $y * z \in I$.

A BCK-algebra X is said to be implicative if it satisfies: $\forall x, y \in X$
$x = x * (y * x)$.

A BCK-algebra X is said to be positive implicative if it satisfies:$\forall x, y, z \in X$
$(x * z) * (y * z) = (x * y) * z$.

A BCK-algebra X is said to be commutative if it satisfies:$\forall x, y \in X$
$x * (x * y) = y * (y * x)$

Definition 2.1: [4]
Let S be a set. A fuzzy set in S is a function $\mu : S \to [0, 1]$.

Definition 2.2: [6]
A fuzzy set in X is said to be a fuzzy subalgebra of X if

$$\mu (x * y) \geq \min\{\mu(x), \mu(y)\}$$

for all $x, y \in X$.

Every fuzzy subalgebra μ of X satisfies the inequality $\mu(0) \geq \mu(x)$ for all $x \in X$.

Definition 2.3: [7]
Let E be a reference set. A hesitant fuzzy set on E is defined in terms of a funcation that when applied to E returns a subset of $[0, 1]$ which can be viewed as the following mathematical representation :

$$H_E := \{(e, h_E(e)) \mid e \in E\}$$

where $h_E : E \to p([0, 1])$

Definition 2.4 : [10]
Given a non-empty subset A of X , a hesitant fuzzy set $H_X := \{(x, h_X(x) \mid x \in X\}$ on X satisfying the following condition:

$$h_X(x) = \emptyset \text{ for all } x \notin A$$

is called a hesitant fuzzy set related to A (brielfly , $A-$hesitant fuzzy set) on X,and is represented by $H_A := \{(x, h_A(x)) \mid x \in X\}$, where h_A is a maping from X to $p([0, 1])$ with $h_A = \emptyset$ for all $x \notin A$.

Definition 2.5: [10]
Given a non-empty subset (subalgebra as much as possible) A of X , let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be an $A-$hesitant fuzzy set on X. Then $H_A := \{(x, h_A(x)) \mid x \in X\}$ is called a hesitant fuzzy subalgebra of X related to A (brielfly , $A-$hesitant fuzzy subalgebra of X) if it satisfies the following condition:

$$(\forall x, y \in A)\,(h_A(x * y) \supseteq h_A(x) \cap h_A(y)).$$

An $A-$hesitant fuzzy subalgebra of X with $A = X$ is called a hesitant fuzzy subalgebra of X.

Definition 2.6: [10]
Given a non-empty subset (subalgebra as much as possible) A of X , an $A-$hesitnat fuzzy set $H_A := \{(x, h_A(x)) \mid x \in X\}$ on X is called a hesitant fuzzy ideal of X related to A (brielfly , $A-$hesitant fuzzy ideal of X) if it satisfies:

$$(\forall x, y \in A)\,(h_A(x * y) \cap h_A(y) \subseteq h_A(x) \subseteq h_A(0))$$

An $A-$hesitant fuzzy ideal of X with $A = X$ is called a hesitant fuzzy ideal of X.

Proposition 2.7: [10]
Let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be an $A-$hesitant fuzzy ideal of X where A is a subalgebra of X . Then the following assertions ar valid.
(1) $(\forall x, y \in A)\;\; (x \leq y \;\Rightarrow\; h_A(x) \supseteq h_A(y))$.
(2) $(\forall x, y \in A)\;\; (x * y \leq z \Rightarrow h_A(x) \supseteq h_A(y) \cap h_A(z))$

3

Theorem 2.8: [10]
For a subalgebra A of a BCK-algebra X , every A-hesitant fuzzy ideal is an A-hesitant fuzzy subalgebra.

Proposition 2.9: [1]
In a BCK-algebra X the following hold: for all $x, y, z \in X$
(i) $((x * z) * z) * (y * z) \leq (x * y) * z$
(ii) $(x * z) * (x * (x * z)) = (x * z) * z$
(iii) $(x * (y * (y * x))) * (y * (x * (y * (y * x)))) \leq x * y$.

3 Hesitant fuzzy implicative ideals

Definition 3.1:
Given a non-empty subset (subalgebra as much as possible) A of X, an $A-$hesitnat fuzzy set $H_A := \{(x, h_A(x)) \mid x \in X\}$ on X is called a hesitant fuzzy implicative ideal of X related to A (brielfly, $A-$hesitant fuzzy implicative ideal of X) if it satisfies:
(H_1) $h_A(0) \supseteq h_A(x)$ for all $x \in A$
(H_2) $h_A(x) \supseteq h_A((x * (y * x)) * z) \cap h_A(z)$, for all $x, y, z \in A$
An $A-$hesitant fuzzy implicative ideal of X with $A = X$ is called a hesitant fuzzy implicative ideal of X.

Theorem 3.2:
Any $A-$hesitant fuzzy implicative ideal of X must be $A-$hesitant fuzzy ideal of X.
Proof:
Let $H_A := \{(x, h_A(x)) \mid x \in X\}$ is an $A-$hesitant fuzzy implicative ideal of X. subsitiuting x for y in (H_2) we obtain :

$$\begin{aligned} h_A(x) &\supseteq h_A((x * (x * x)) * z) \cap h_A(z) \\ &= h_A((x * 0) * z) \cap h_A(z) \\ &= h_A(x * z) \cap h_A(z) \end{aligned}$$

Hence $H_A := \{(x, h_A(x)) \mid x \in X\}$ is an $A-$hesitant fuzzy ideal of X . □

Corollary 3.3:
Every $A-$hesitant fuzzy implicative ideal of X must be $A-$hesitant fuzzy subalgebra of X .

Remark 3.4:
The converse of Theorem (3.2)may not be true as shown in the following example.

Example 3.5:
Let $X = \{0, a, b, c\}$ be a BCK-algebra with the following Cayley table:

4

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	0
b	b	b	0	0
c	c	b	a	0

For a subalgebra $A = \{0, a, b\}$ of X, let $H_A = \{(x, h_A(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by

$$H_A = \{(0, [\tfrac{1}{4}, \tfrac{3}{4}]), (a, (\tfrac{1}{4}, \tfrac{1}{2}), (b, [\tfrac{1}{2}, \tfrac{3}{4})), (c, \emptyset)\}$$

Then $H_A = \{(x, h_A(x)) \mid x \in X\}$ is an $A-$hesitant fuzzy ideal of X, but it is not an $A-$hesitant fuzzy implicative ideal of X because $h_A(a) = (\frac{1}{4}, \frac{1}{2}) \nsupseteq h_A((a * (b * a)) * 0) \cap h_A(0) = [\frac{1}{4}, \frac{3}{4}]$

In the following theorem, we can see that the converse of theorem (3.2)also holds if X is an implicative BCK-algebra.

Theorem 3.6:

If X is an implicative BCK-algebra, then every $A-$hesitant fuzzy ideal of X is an $A-$hesitant fuzzy implicative ideal of X.

proof:

Let X is an implicative BCK-algebra, it follows that $x = x*(y*x), \forall x, y \in X$.

Let $H_A = \{(x, h_A(x)) \mid x \in X\}$ be an $A-$hesitant fuzzy ideal of X, then

$$\begin{aligned} h_A(x) &\supseteq h_A(x * z) \cap h_A(z) \\ &\supseteq h_A((x * (y * x)) * z) \cap h_A(z) \end{aligned}$$

for all $x, y, , z \in A$. Hence H_A is an $A-$hesitant fuzzy implicative ideal of X. The proof is complete . □

Theorem 3.7:

Let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be $A-$hesitant fuzzy set on X, where A is non-empty subset (subalgebra as much as possible) of X is a $A-$hesitant fuzzy implicative ideal of X if and only if, for any $\varepsilon \in p([0, 1])$,the set $H_A(\varepsilon) : \{x \in X \mid h_A(x) \supseteq \varepsilon\}$ is either empty or an implicative ideal of X.

The set $H_A(\varepsilon) : \{x \in X \mid h_A(x) \supseteq \varepsilon\}$ is called a $A-$hesitant fuzzy $\varepsilon-$level set of $H_A := \{(x, h_A(x)) \mid x \in X\}$.

proof:

suppose that $H_A := \{(x, h_A(x)) \mid x \in X\}$ is an $A-$hesitant fuzzy implicative ideal of X and $H_A(\varepsilon) : \{x \in X \mid h_A(x) \supseteq \varepsilon\} \neq \emptyset$ for any $\varepsilon \in p([0, 1])$. It is clear that $0 \in H_A(\varepsilon)$ since $h_A(0) \supseteq \varepsilon$. Let $(x * (y * x)) * z \in H_A(\varepsilon)$ and $z \in H_A(\varepsilon)$, then $h_A((x * (y * x)) * z\ ) \supseteq \varepsilon$ and $h_A(z) \supseteq \varepsilon$. It follows from ($H_2$) that :

$h_A(x) \supseteq h_A((x * (y * x)) * z\ ) \cap h_A(z) \supseteq \varepsilon \Rightarrow x \in H_A(x)$. This show that $H_A(\varepsilon)$ is an implicative ideal of X.

conversely, suoppse that for each $\varepsilon \in p([0, 1]), H_A(\varepsilon)$ is either empty or an implicative ideal of X. For any $x \in A$, let $h_A(x) = \varepsilon$. Then $x \in H_A(\varepsilon)$.

Since $H_A(\varepsilon)(\neq \emptyset)$ is an implicative ideal of X, therefore $0 \in H_A(\varepsilon)$ and hence $h_A(0) \supseteq \varepsilon = h_A(x)$ for all $x \in A$.

Now we only need to show that H_A satisfies (H_2). If not , then there exist $x', y', z' \in A$ such that $h_A(x') \subseteq h_A((x' * (y' * x')) * z') \cap h_A(z')$. Taking

$\varepsilon_0 = \frac{1}{2}\{h_A(x') + (h_A(x' * (y' * x')) * z') \cap h_A(z'))\}$; then we have that $h_A(x') \subset \varepsilon_0 \subset \{h_A(x' * (y' * x')) * z') \cap h_A(z')\}$. Hence $x' \notin H_A(\varepsilon_0), ((x' * (y' * x')) * z') \in H_A(\varepsilon_0)$ and $z' \in H_A(\varepsilon_0)$. i.e, $H_A(\varepsilon_0)$ is not an implicative ideal of X, which is contradiction. Therefore, H_A is an $A-$hesitant fuzzy implicative ideal of X, completing the proof of the theorem. □

Now we give characterizations of hesitant fuzzy implicative ideals.

Theorem 3.8 :

suppose that $H_A := \{(x, h_A(x)) \mid x \in X\}$ is an $A-$hesitant fuzzy ideal of X. Then the following are equivalent:

(i) H_A is an $A-$hesitant fuzzy implicative ideal of X.

(ii) $h_A(x) \supseteq h_A(x * (y * x))$ for all $x, y \in A$.

(iii) $h_A(x) = h_A(x * (y * x))$ for all $x, y \in A$.

proof:

(i) $\Rightarrow$ (ii) Let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be $A-$hesitant fuzzy implicative ideal of X. Then, by (H_2) we have

$$\begin{aligned} h_A(x) &\supseteq h_A((x * (y * x)) * 0) \cap h_A(0) \\ &= h_A((x * (y * x)) \cap h_A(0) \\ &= h_A((x * (y * x)) \end{aligned}$$

for all $x, y \in A$. Hence the condition (ii) holds.

(ii) $\Rightarrow$ (iii) Observe that in $BCK-$algebra X, $x * (y * x) \leq x$. Applying proposition 2.7(1) we have $h_A(x) \subseteq h_A(x * (y * x))$

It follows from (ii) that $h_A(x) = h_A(x * (y * x))$. Hence the condition (iii) holds.

(iii) $\Rightarrow$ (i) Suppose the condition (iii) holds. Since H_A is an $A-$hesitant fuzzy ideal , by definition 2.6 we have

$$h_A(x * (y * x)) \supseteq h_A((x * (y * x)) * z) \cap h_A(z)$$

combining (iii) we obtain: $h_A(x) \supseteq h_A((x * (y * x)) * z) \cap h_A(z)$. Thus H_A satisfies (H_2). Obviously, H_A satisfies (H_1).

Therefore, H_A is an $A-$hesitant fuzzy implicative ideal of X. Hence, the condition (i) holds. The proof is complete. □

Now, we give an equivalent condition for which a hesitant fuzzy subalgebra of X is to be a hesitant fuzzy implicative ideal of X.

Theorem 3.9:

A hesitant fuzzy subalgebra H_A of X is a hesitant fuzzy implicative ideal if and only if it satisfies (H_3)$(x * (y * x)) * z \leq u$ implies $h_A(x) \supseteq h_A(z) \cap h_A(u)$ for all $x, y, z, u \in X$.

6

proof:

Assume that H_A is a hesitant fuzzy implicative ideal of X and let $x, y, z, u \in X$ be such that $(x*(y*x))*z \leq u$. Since H_A is also hesitant fuzzy ideal of X by theorem 3.2 it follows from proposition 2.7 that:

$$h_A(x*(y*x)) \supseteq h_A(z) \cap h_A(u)$$

Making use of the theorem 3.8 (iii) we obtain $h_A(x) \supseteq h_A(z) \cap h_A(u)$.

Conversely, suppose that H_A satisfies (H_3). Obviously, H_A satisfies (H_1) since $(x*(y*x))*((x*(y*x))*z) \leq z$. It follows from ($H_3$) that:

$$h_A(x) \supseteq h_A((x*(y*x))*z) \cap h_A(z)$$

which shows that H_A satisfies (H_2) and so H_A is a A-hesitant fuzzy implicative ideal of X. The proof is complete. □

4 Hesitant fuzzy positive implicative ideals

Definition 4.1:

Given a non-empty subset (subalgebra as much as possible) A of X, an $A-$hesitnat fuzzy set $H_A := \{(x, h_A(x)) \mid x \in X\}$ on X is called a hesitant fuzzy positive implicative ideal of X related to A (brielfly, $A-$hesitant fuzzy positive implicative ideal of X) if it satisfies:

(H_1) $h_A(0) \supseteq h_A(x)$ for all $x \in A$

(H_4) $h_A(x*z) \supseteq h_A((x*y)*z) \cap h_A(y*z)$, for all $x, y, z \in A$

An $A-$hesitant fuzzy positive implicative ideal of X with $A = X$ is called a hesitant fuzzy positive implicative ideal of X.

Theorem 4.2:

Every $A-$hesitant fuzzy positive implicative ideal of X is $A-$hesitant fuzzy ideal of X.

Proof:

Assume that $H_A := \{(x, h_A(x)) \mid x \in X\}$ is an $A-$hesitant fuzzy positive implicative ideal of X . Putting $z = 0$ in (H_4) we obtain:

$$\begin{aligned} h_A(x*0) &\supseteq h_A((x*y)*0) \cap h_A(y*0) \\ h_A(x) &\supseteq h_A(x*y) \cap h_A(y). \end{aligned}$$

So, H_A is an $A-$hesitant fuzzy ideal of X. □

Remark 4.3:

The converse of Theorem (4.2) may not be true as shown in the following example.

Example 4.4:

Let X be a BCK-algebra as in Example (3.5) It is easy to check that H_A is an $A-$hesitant fuzzy ideal of X , but it is not an $A-$hesitant fuzzy positive implicative ideal of X because $h_A(a*0) = (\frac{1}{4}, \frac{1}{2}) \not\supseteq [\frac{1}{2}, \frac{3}{4}) = (h_A((a*b)*0) \cap h_A(b*0))$.

Proposition 4.5:
If X is positive implicative, then an $A-$hesitant fuzzy ideal of X is an $A-$hesitant fuzzy positive implicative ideal of X if and only if satisfies:

$$h_A(x*y) = h_A((x*y)*y) \text{ for all } x, y \in A.$$

Proof:
Suppose that the heaitant fuzzy ideal $H_A := \{(x, h_A(x)) \mid x \in X\}$ of X is an $A-$hesitant fuzzy positive implicative ideal of X. So by (H_4)

$$h_A(x*z) \supseteq h_A((x*y)*z) \cap h_A(y*z)$$

subiting $z = y$ in (H_4) we have :

$$\begin{aligned} h_A(x*y) &\supseteq h_A((x*y)*y) \cap h_A(y*y) \\ &= h_A((x*y)*y) \cap h_A(0) \\ &= h_A((x*y)*y) \end{aligned}$$

On other hand, since $(x*y)*y \leq x*y$, it follows from Propsition 2.7(1) that $h_A((x*y)*y) \supseteq h_A(x*y)$. Thus we have $h_A(x*y) = h_A((x*y)*y)$ for all $x, y \in A$.

Conversely, assume that $h_A(x*y) = h_A((x*y)*y)$ for all $x, y \in A$. we want to prove that an $A-$hesitant fuzzy ideal $H_A := \{(x, h_A(x)) \mid x \in X\}$ of X is an $A-$hesitant fuzzy positive implicative ideal of X. It is clear that, $h_A(0) \supseteq h_A(x)$ for all $x \in A$. Since H_A is A-hesitant fuzzy ideal, and X is positive implicative then

$$\begin{aligned} h_A(x*z) &= h_A((x*z)*z) \supseteq h_A((x*z)*(y*z)) \cap h_A(y*z) \\ h_A(x*z) &\supseteq h_A((x*y)*z) \cap h_A(y*z) \end{aligned}$$

Therefore, $H_A := \{(x, h_A(x)) \mid x \in X\}$ of X is an $A-$hesitant fuzzy positive implicative ideal of X. The proof is complete. □

5 Hesitant fuzzy Commutative ideals

Definition 5.1:
Given a non-empty subset (subalgebra as much as possible) A of X, an $A-$hesitnat fuzzy set $H_A := \{(x, h_A(x)) \mid x \in X\}$ on X is called a hesitant fuzzy commutative ideal of X related to A (brielfly, $A-$hesitant fuzzy commutative ideal of X) if it satisfies:

(H_1) $h_A(0) \supseteq h_A(x)$ for all $x \in A$

(H_C) $h_A(x*(y*(y*x))) \supseteq h_A((x*y)*z) \cap h_A(z)$, for all $x, y, z \in A$

An $A-$hesitant fuzzy commutative ideal of X with $A = X$ is called a hesitant fuzzy commutative ideal of X.

Example 5.2:
Let $X = \{0, a, b, c\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by $H_A = \{(0, [0, 1]), (a, [0.2, 0.7], (b, (0.2, 0.3]), (c, [0.6, 0.7))\}$. It is routine to verify that $H_X := \{(x, h_X(x)) \mid x \in X\}$ is hesitant fuzzy commutative ideal of X.

Theorem 5.3:
Any heaitant fuzzy commuatative ideal of BCK-algebra X is hesitant fuzzy ideal of X.

Proof:
Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy commutative ideal of a BCK-algebra X. For any $x, y, z, \in X$, we have

$$\begin{aligned} h(x * z) \cap h(z) &= h((x * 0) * z) \cap h(z) \\ &\subseteq h(x * (0 * (0 * x))) \\ &= h(x) \end{aligned}$$

i.e. H_X is hesitant fuzzy ideal of X. □

Remark 5.4:
A hesitant fuzzy ideal of a BCK- algebra X, may not be hesitant fuzzy commutative ideal. For instance, Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	0	0
c	c	c	c	0	0
d	d	d	d	c	0

Let $S_0, S_1, S_2 \in P([0, 1])$ be such that $S_0 = [0, 1) \supset S_1 = [0.2, 0.7] \supseteq S_2 = [0.4, 0.5]$. Define a mapping $h : X \to P([0, 1])$ by $h(0) = [0, 1), h(a) = [0.2, 0.7]$ and $h(b) = h(c) = h(d) = [0.4, 0.5)$. Routine calculations give that h is a hesitant fuzzy ideal of X. But it is not a hesitant fuzzy commutative ideal of X because $h(b * (c * (c * b))) \not\supseteq h((b * c) * 0) \cap h(0)$.

Theorem 5.5:
Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy ideal of a BCK-algebra X. Then H_X is hesitant fuzzy commutative ideal of X if and only if satisfies the condition

$$h(x * (y * (y * x))) \supseteq h(x * y) \text{ for all } x, y \in X \tag{1.1}$$

proof:
Assume that $H_X := \{(x, h_X(x)) \mid x \in X\}$ is hesitant fuzzy commutative ideal. Taking $z = 0$ in (H$_C$) and using (H$_1$). Also, we use $x * 0 = x$.

$$h(x * (y * (y * x))) \supseteq h((x * y) * 0) \cap h(0) = h(x * y).$$

Conversely, suppose that $H_X := \{(x, h_X(x)) \mid x \in X\}$ satisfies the condition (1.1) as H_X is a hesitant fuzzy ideal. Hence

$$h(x * y) \supseteq h((x * y) * z) \cap h(z) \tag{1.2}$$

for all $x, y, z \in X$. combining (1.2) and(1.1) then we obtain (H$_C$). The proof is complete. □

Observing $x * y \leq x * (y * (y * x))$ and using propoition (2.7) we have $h(x * (y * (y * x))) \subseteq h(x * y)$. Hence, theorem (5.5) can be improved as follows:

Theorem 5.6:
A hesitant fuzzy ideal H_X of a BCK-algebra X is hesitant fuzzy commutative ideal of X if and only if satisfies the identity $h(x*y) = h(x*(y*(y*x)))$ for all $x, y \in X$.

Theorem 5.7:
In a commutative BCK-algebra X. Every hesitant fuzzy ideal is a hesitant fuzzy commutative ideal.
Proof:
Let H_X be a hesitant fuzzy ideal of $BCK-$algebra X. It is sufficient to show that H_X satisfies condition (H_C). Let $x, y, z \in X$.Then

$$\begin{aligned} & ((x * (y * (y * x))) * ((x * y) * z)) * z \\ = \; & ((x * (y * (y * x))) * z) * ((x * y) * z) \\ \leq \; & (x * (y * (y * x))) * (y * x) \\ = \; & (x * (y * y)) * (y * (y * x)) = 0 \end{aligned}$$

that is ,

$$(x * (y * (y * x))) * ((x * y) * z) \leq z$$

It follows from propostion (2.7)

$$h(x * (y * (y * x))) \supseteq h((x * y) * z) \cap h(z)$$

Thus (H_C) holds, and the proof is complete. □

Theorem 5.8:

Let X be a BCK-algebra and let H_A be A-hesitant fuzzy set in X. Then H_A is a A-hesitant fuzzy commutative ideal of X if and only if $H_A(\varepsilon)$ is commutative ideal of X for all $\varepsilon \in P([0,1]), H_A(0) \supseteq \varepsilon$.

Proof:

Suppose H_A is a A-hesitant fuzzy commutative ideal of X. For any fixed $\varepsilon \in P([0,1])$, if $h_A(0) \supseteq \varepsilon$ then $0 \in H_A(\varepsilon)$. Hence $H_A(\varepsilon)$ satisfie (H1). Let $(x*y)*z \in H_A(\varepsilon)$ and $z \in H_A(\varepsilon)$. It follows that $h_A((x*y)*z) \supseteq \varepsilon$ and $h_A(z) \supseteq \varepsilon$. By ($H_C$) we have :

$$h_A(x*(y*(y*x))) \supseteq h_A((x*y)*z) \cap h_A(z) \supseteq \varepsilon$$

namely $(x*(y*(y*x))) \in H_A(\varepsilon)$, so $H_A(\varepsilon)$ satisfies condition (C). Thus, $H_A(\varepsilon)$ is commutative ideal of X.

Conversely, assume that $H_A(\varepsilon)$ is commutative ideal of X for all $\varepsilon \in p([0,1])$, $h_A(0) \supseteq \varepsilon$. Let $h_A((x*y)*z) = \varepsilon_1$ and $h_A(z) = \varepsilon_2$ for $x,y,z \in A$. Then $(x*y)*z \in H_A(\varepsilon_1)$ and $z \in H_A(\varepsilon_2)$. Without loss of generality,we may assume that $\varepsilon_1 \subseteq \varepsilon_2$. Then $H_A(\ \varepsilon_2) \subseteq H_A(\ \varepsilon_1)$ and so $z \in H_A(\varepsilon_1)$. Since $H_A(\varepsilon_1)$ is commutative ideal of X by hypothesis we obtain that $x*(y*(y*x)) \in H_A(\varepsilon_1)$. Thus $h_A(x*(y*(y*x))) \supseteq \varepsilon_1 = h_A((x*y)*z) \cap h_A(z)$. It is clear that $h_A(0) \supseteq h_A(x)$ for all $x \in A$. Therefore H_A is a A-hesitant fuzzy commutative ideal of X. □

Definition 5.9:

Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy commutative ideal of a BCK-algebra X. The commutative ideals $H_X(\ \varepsilon), \varepsilon \in p([0,1])$ and $H_X(\ 0) \supseteq \varepsilon$ are called hesitant $\varepsilon-$level commutative ideals of H_X.

Theorem 5.10:

Any commutative idael of BCK-algebra X can be realized as hesitant $\varepsilon-$level commutative ideal of some hesitant fuzzy commutative ideal of X.

Proof:

Suppose C is a commutative ideal of BCK-algebra X and let H_A be a hesitant fuzzy set in X defined by

$$H_A = \begin{cases} \varepsilon & \text{if } x \in C \\ 0 & \text{if } x \notin C \end{cases}$$

where ε is a fixed interval in $p([0,1])$. Let $x,y,z \in A$. We will divide into the following cases to verify that H_A is $A-$hesitant fuzzy commutative ideal of X.

(i) If $((x*y)*z) \in C$ and $z \in C$, then $(x*(y*(y*x))) \in C$. Thus

$$h_A((x*y)*z) = h_A(z) = h_A(x*(y*(y*x))) = \varepsilon$$

and so

$$h_A(x*(y*(y*x))) \supseteq h_A((x*y)*z) \cap h_A(z)$$

11

(ii) If $((x*y)*z) \notin C$ and $z \notin C$,then $h_A((x*y)*z) = h_A(z) = 0$. Hence

$$h_A(x*(y*(y*x))) \supseteq h_A((x*y)*z) \cap h_A(z)$$

(iii) If exactly one of $(x*y)*z$ and z belongs to C then exactly one of $h_A((x*y)*z)$ and $h_A(z)$ is equal to 0. So

$$h_A(x*(y*(y*x))) \supseteq h_A((x*y)*z) \cap h_A(z)$$

The results above show $h_A(x*(y*(y*x))) \supseteq h_A((x*y)*z) \cap h_A(z)$, for all $x, y, z \in A$. It is clear that $h_A(0) \supseteq h_A(x)$ for all $x \in A$. Hence H_A is $A-$hesitant fuzzy commutative ideal of X and obviously $H_A(\varepsilon) = C$. The proof is completed. □

Theorem 5.11:

Suppose $H_A := \{(x, h_A(x)) \mid x \in X\}$ is a $A-$hesitant fuzzy commutative ideal of BCK-algebra X. Then two hesitant level commutative ideals $H_A(\varepsilon_1), H_A(\varepsilon_2)$ with $(\varepsilon_1 \subset \varepsilon_2)$ of H_A are equal if and only if there is no $x \in X$ such that $\varepsilon_1 \subseteq h_A(x) \subset \varepsilon_2$.

Proof:

Suppose $\varepsilon_1 \subset \varepsilon_2$ and $H_A(\varepsilon_1) = H_A(\varepsilon_2)$. If there exists $x \in X$ such that $\varepsilon_1 \subseteq h_A(x) \subset \varepsilon_2$, then $H_A(\varepsilon_2)$ is proper subset of $H_A(\varepsilon_1)$. This is impossible.

Conversely, suppose that there is no $x \in X$, such that $\varepsilon_1 \subseteq h_A(x) \subset \varepsilon_2$. Note that, $\varepsilon_1 \subset \varepsilon_2$ implies $H_A(\varepsilon_2) \subseteq H_A(\varepsilon_1)$. If $x \in H_A(\varepsilon_1)$, then $h_A(x) \supseteq \varepsilon_1$ and so $h_A(x) \supseteq \varepsilon_2$

because $h_A(x) \nsubseteq \varepsilon_2$. Hence $x \in H_A(\varepsilon_2)$ which says that $H_A(\varepsilon_1) \subseteq H_A(\varepsilon_2)$. Thus, $H_A(\varepsilon_1) = H_A(\varepsilon_2)$. This completes the proof. □

Let h_A be a A-hesitant fuzzy set in X and let $Im(h_A)$ denote the image of h_A.

Theorem 5.12:

Let X be a BCK-algebra and h_A a A-hesitant fuzzy commutative ideal of X. If $Im(h_A) = \{\varepsilon_1, \varepsilon_2, ..., \varepsilon_n\}$ where $\varepsilon_1 \subset \varepsilon_2 \subset ... \subset \varepsilon_n$, then the family of commutative ideals $H_A(\varepsilon_i)(i = 1, 2, ..., n)$ consitiutes all the hesitant level commutative ideals of h_A.

Proof :

Let $\varepsilon \in p([0,1])$ and $\varepsilon \notin \mathrm{Im}(h_A)$. If $\varepsilon \subset \varepsilon_1$, then $h_A(\varepsilon_1) \subseteq h_A(\varepsilon)$. Since $h_A(\varepsilon_1) = X$, we have $h_A(\varepsilon) = X$ and $h_A(\varepsilon) = h_A(\varepsilon_1)$.

If $\varepsilon_i \subset \varepsilon \subset \varepsilon_{i+1}$ $(1 \le i \le n-1)$, then there is no $x \in X$ such that $\varepsilon \subseteq h_A(x) \subset \varepsilon_{i+1.}$ From above theorem(5.10) it follows that $h_A(\varepsilon) = h_A(\varepsilon_{i+1})$. This shows that for any $\varepsilon \in p([0,1])$ with $h_A(0) \supseteq \varepsilon$, the hesitant level commutative ideal $h_A(\varepsilon)$ is in $\{h_A(\varepsilon_i) : 1 \le i \le n\}$.□

Lemma 5.13:

Let X be a finite $BCK-$algebra and H_A a $A-$hesitant fuzzy commutative ideal of X. If α and β belong to $\mathrm{Im}(h_A)$ such that $h_A(\alpha) = h_A(\beta)$, then $\alpha = \beta$.

Proof:

Suppose $\alpha \neq \beta$, say $\alpha < \beta$, then there is $x \in X$ such that $h_A(x) = \alpha < \beta$ and so $x \in h_A(\alpha)$ and $x \notin h_A(\beta)$. Thus $h_A(\alpha) \neq h_A(\beta)$ which is contradiction . The proof is complete. □

Theorem 5.14:

A hesitant fuzzy ideal H_A of X is hestiant fuzzy implicative ideal if and only if H_A is both hestiant fuzzy commutative ideal and hestiant fuzzy positive implicative ideal.

Proof:

Suppose H_A is hestiant fuzzy implicative ideal of X. For all $x, y, z \in A$ we have

$$\begin{aligned} h_A((x*y)*z) \cap h_A(y*z) &\subseteq h_A((x*z)*z), \quad \text{[by proposition(2.9)}(i)\text{and (2.7)]} \\ &= h_A((x*z)*(x*(x*z)), \quad \text{[by proposition(2.9)}(ii)\text{]} \\ &= h_A(x*z), \quad \text{[by theorem(3.8)}(iii)\text{]} \end{aligned}$$

So H_A is hesitant fuzzy positive implicative ideal of X. By proposition (2.7), (2.9)(iii) and theorem (3.8)(iii)

$$h_A(x*y) \subseteq h_A(x*(y*(y*x)))*(y*(x*(y*(y*x))) = h_A(x*(y*(y*x))$$

It follows from proposition (5.5) that H_A is hesitant fuzzy commutative ideal of X.

Conversely, suppose that H_A is both hesitant fuzzy positive implicative and hesitant fuzzy commutative ideal of X. Since $(y*(y*x))*(y*x) \leq x*(y*x)$. It follows from proposition (2.7) that $h_A(x*(y*x)) \subseteq h_A(\ (y*(y*x))*(y*x))$. Using theorm (4.5)we have $h_A(\ (y*(y*x))*(y*x)) = h_A(y*(y*x))$ and so

$$h_A(x*(y*x)) \subseteq h_A(y*(y*x)) \qquad (*)$$

On the other hand, since $x*y \leq x*(y*x)$ implies $h_A(x*(y*x)) \subseteq h_A(\ x*y)$. Since H_A is hesitant fuzzy commutative by theorem (5.6) we have $h_A(x*y) = h_A(x*(y*(y*x)))$, hence $h_A(x*(y*x)) \subseteq h_A(x*(y*(y*x)))$. Combinging $(*)$ we obtain

$$h_A(x*(y*x)) \subseteq h_A(x*(y*(y*x))) \cap h_A(y*(y*x)) \subseteq h_A(x)$$

So H_A is a hesitant fuzzy implicative ideal of X by theorem (3.8) . The proof is complete. □

References:

[1] J. Meng, Y.B. Jun and H.S. Kim, Fuzzy implicative ideals of $BCK-$algebras,Fuzzy Sets and Systems 89 (1997) 243-248.

[2] K. Iséki , An algebraic related with a propositional calclus, Proc. Japan Academy, 42 (1996) 26-29.

[3] K. Iséki and S. Tanaka, An Introduction to the theory of $BCK-$algebras, Math. Japan 23 (1978)1-26.

[4] L. A. Zadah, Fuzzy sets, Inform. and Control 8 (1965) 338-353.

[5] M. M. Xia and Z. S. Xu , Hesitant fuzzy information aggreation in decision making, International Journal of Approximate Reasoning 52 (3) (2011) 395-407.

[6] O. G. Xi, Fuzzy BCK-algebras, Math . Japan 36 (1991) 935-942.

[7] V. Torra, Hesitant fuzzy sets, Int. J. Intell. Syst. 25 (2010) 529-539.

[8] Y. B. Jun and E. H. Roh, Fuzzy commutative ideals of $BCK-$algebras,Fuzzy Sets and Systems 64 (1994) 401-405.

[9] Y. B. Jun, S. M. Hong, J. Meng and X. L. Xin, Characterizations of fuzzy positive implicative ideals in $BCK-$algebras, Math.Japan 40 (1994) 503- 507.

[10] Y. B. Jun and S. S. Ahn, Hesitant fuzzy sets theory applied to in BCK/BCI- algebras, J. Computational Analysis and Appl. Vol. 20 (2016) 635-646.

L-fuzzy (K,E)-soft quasi-uniformities and L-fuzzy (K,E)-soft topologies

A.A.Ramadan, Yong Chan Kim[1]

Mathematics Department, Faculty of Science, Beni-Suef University, Beni-Suef, Egypt

Department of Mathematics, Gangneung-Wonju University, Gangneung, Gangwondo 25457, Korea

---

Abstract The goal of this paper is to focus on the relationships between L-fuzzy (K,E)- soft quasi uniformities and L-fuzzy (K,E)- soft topologies in complete residuated lattices. As main results, we investigate the L-fuzzy (K,E)- soft topologies induced by L-fuzzy (K,E)- soft uniformities. Moreover, we study the L-fuzzy (K,E)- soft quasi uniformities induced by L-fuzzy (K,E)- soft topologies. We give their examples.

Key Words and Phrases: complete residuated lattice, L-fuzzy (K,E)- soft quasi uniformities , L-fuzzy (K,E)- soft topologies.

AMS Subject Classification: 03E72, 06A15, 06F07,54A05

1 Introduction

Molodtsov [18] introduced a completely new concept called soft set theory to model uncertainty, which associates a set with a set of parameters. Pei and Miao [19] showed that soft sets are a class of special information systems. Later, Maji et al. [15] introduced the concept of a fuzzy soft set which combines a fuzzy set and a soft set. Presently, the soft set theory is making progress rapidly [1,2,6,15-19,26,28,31,32]. The topological structures of soft sets have been developed by many researchers [3,5,8,23,27,29,30,33].

Hájek [9] introduced a complete residuated lattice which is an algebraic structure for many valued logic. Bělohlávek [4] investigated information systems and decision rules in complete residuated lattices. Höhle [10] introduced L-fuzzy topologies

[1]Corresponding author

with algebraic structure L(cqm, quantales, MV-algebra). Uniformities in fuzzy sets, have the following approach of Lowen [14] based on powersets of the form $L^{X\times X}$ as a viewpoint of the enourage approach, the uniform covering approach of Kotzé [13], the uniform operator approach of Rodabaugh [25] as a generalization of Hutton [11] based on powersets of the form $(L^X)^{(L^X)}$, the unification approach of Gutiérrez García [7]. Recently, Gutiérrez García introduced L-valued Hutton uniformity where a quadruple $(L,\leq,\otimes,\star)$ is defined by a GL-monoid $(L,\star)$ as an extension of a completely distrbutive lattice L. Kim [12] introduced the notion of L-fuzzy uniformities as an extension of Lowen in a strictly two-sided, commutative quantale. Moreover, he investigated the relations between L- fuzzy topological spaces and L- fuzzy uniform spaces. Ramadan et.al [23] introduced the notion of L-fuzzy (K,E)- soft topogenous orders and L-fuzzy (K,E)- soft quasi uniformities in complete residuated lattices.

The goal of this paper is to focus on the relationships between L-fuzzy (K,E)- soft quasi uniformities and L-fuzzy (K,E)- soft topologies in complete residuated lattices. As main results, we investigate the L-fuzzy (K,E)- soft topologies induced by L-fuzzy (K,E)- soft uniformities. Moreover, we study the L-fuzzy (K,E)- soft quasi uniformities induced by L-fuzzy (K,E)- soft topologies. We give their examples.

2 Preliminaries

Let $L=(L,\leq,\vee,\wedge,0,1)$ be a completely distributive lattice with the least element 0 and the greatest element 1 in L.

Definition 2.1. [4,9,11] An algebra $(L,\wedge,\vee,\odot,\rightarrow,0,1)$ is called a complete residuated lattice if it satisfies the following conditions:

(C1) $L=(L,\leq,\vee,\wedge,1,0)$ is a complete lattice with the greatest element 1 and the least element 0;

(C2) $(L,\odot,1)$ is a commutative monoid;

(C3) $x\odot y\leq z$ iff $x\leq y\rightarrow z$ for $x,y,z\in L$.

Remark 2.2. Every completely distributive lattice $(L,\leq,\wedge,\vee,^*)$ with order reversing involution * is a complete residuated lattice $(L,\leq,\odot,\oplus,^*)$ with a strong negation * where $\odot = \wedge$, $\oplus = \vee$ and

$$x \to y = \begin{cases} 1, & \text{if } x \leq y, \\ y, & \text{otherwise.} \end{cases}$$

In this paper, we assume that $(L, \leq, \odot, \to, \oplus, ^*)$ is a complete residuated lattice with an order reversing involution $x^* = x \to 0$ which is defined by $x \oplus y = (x^* \odot y^*)^*$.

Lemma 2.3. [4,9,11] For each $x, y, z, x_i, y_i, w \in L$, we have the following properties.

(1) $1 \to x = x$, $0 \odot x = 0$,

(2) If $y \leq z$, then $x \odot y \leq x \odot z$, $x \oplus y \leq x \oplus z$, $x \to y \leq x \to z$ and $z \to x \leq y \to x$,

(3) $x \leq y$ iff $x \to y = 1$.

(4) $(\bigwedge_i y_i)^* = \bigvee_i y_i^*$, $(\bigvee_i y_i)^* = \bigwedge_i y_i^*$,

(5) $x \to (\bigwedge_i y_i) = \bigwedge_i (x \to y_i)$,

(6) $(\bigvee_i x_i) \to y = \bigwedge_i (x_i \to y)$,

(7) $x \to (\bigvee_i y_i) \geq \bigvee_i (x \to y_i)$,

(8) $(\bigwedge_i x_i) \to y \geq \bigvee_i (x_i \to y)$,

(9) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$,

(10) $x \odot y = (x \to y^*)^*$ and $x \oplus y = x^* \to y$,

(11) $(x \to y) \odot (z \to w) \leq (x \odot z) \to (y \odot w)$,

(12) $x \to y \leq (x \odot z) \to (y \odot z)$ and $(x \to y) \odot (y \to z) \leq x \to z$,

(13) $(x \to y) \odot (z \to w) \leq (x \oplus z) \to (y \oplus w)$.

(14) $x \to y = y^* \to x^*$.

(15) $(x \vee y) \odot (z \vee w) \leq (x \vee z) \vee (y \odot w) \leq (x \oplus z) \vee (y \odot w)$.

(16) $\bigvee_{i \in \Gamma} x_i \to \bigvee_{i \in \Gamma} y_i \geq \bigwedge_{i \in \Gamma} (x_i \to y_i)$ and $\bigwedge_{i \in \Gamma} x_i \to \bigwedge_{i \in \Gamma} y_i \geq \bigwedge_{i \in \Gamma} (x_i \to y_i)$,

Throughout this paper, X refers to an initial universe, E and K are the sets of all parameters for X, and L^X is the set of all L-fuzzy sets on X.

Definition 2.4. [3,5,23] A map f is called an L- fuzzy soft set on X, where f is a mapping from E into L^X, i.e., $f_e := f(e)$ is an L- fuzzy set on X, for each $e \in E$. The family of all L- fuzzy soft sets on X is denoted by $(L^X)^E$. Let f and g be two L-fuzzy soft sets on X.

(1) f is an L-fuzzy soft subset of g and we write $f \sqsubseteq g$ if $f_e \leq g_e$, for each $e \in E$. f and g are equal if $f \sqsubseteq g$ and $g \sqsubseteq f$.

(2) The intersection of f and g is an L- fuzzy soft set $h = f \sqcap g$, where $h_e = f_e \wedge g_e$, for each $e \in E$.

(3) The union of f and g is an L- fuzzy soft set $h = f \sqcup g$, where $h_e = f_e \vee g_e$, for each $e \in E$.

(4) An L- fuzzy soft set $h = f \odot g$ is defined as $h_e = f_e \odot g_e$, for each $e \in E$.

(5) An L- fuzzy soft set $h = f \oplus g$ is defined as $h_e = f_e \oplus g_e$, for each $e \in E$.

(6) The complement of an L- fuzzy soft sets on X is denoted by f^*, where $f^* : E \to L^X$ is a mapping given by $f^*_e = (f_e)^*$, for each $e \in E$.

(7) f is called a null L- fuzzy soft set and is denoted by 0_X , if $f_e(x) = 0$, for each $e \in E$, $x \in X$.

(8) f is called an absolute L- fuzzy soft set and is denoted by 1_X , if $f_e(x) = 1$, for each $e \in E$, $x \in X$.

Definition 2.5. [3,5] A mapping $\mathcal{T} : K \to L^{(L^X)^E}$ (where $\mathcal{T}_k := \mathcal{T}(k) : (L^X)^E \to L$ is a mapping for each $k \in K$) is called an L-fuzzy (K, E)-soft topology on X if it satisfies the following conditions for each $k \in K$.

(SO1) $\mathcal{T}_k(0_X) = \mathcal{T}_k(1_X) = 1$,

(SO2) $\mathcal{T}_k(f \odot g) \geq \mathcal{T}_k(f) \odot \mathcal{T}_k(g)$ $\forall\, f, g \in (L^X)^E$,

(SO3) $\mathcal{T}_k(\bigsqcup_i f_i) \geq \bigwedge_{i\in I} \mathcal{T}_k(f_i)$ $\forall\, f_i \in (L^X)^E$, $i \in I$.

The pair $(X, \mathcal{T})$ is called an L-fuzzy (K, E)-soft topological space.

Definition 2.6. [3,5] A mapping $\mathcal{F} : K \to L^{(L^X)^E}$ is called an L-fuzzy (K, E)-soft cotopology on X if it satisfies the following conditions for each $e \in E$.

(SF1) $\mathcal{F}_k(0_X) = \mathcal{F}_k(1_X) = 1$,

(SF2) $\mathcal{F}_k(f \oplus g) \geq \mathcal{F}_k(f) \odot \mathcal{F}_k(g)$, $\forall\, f, g \in (L^X)^E$,

(SF3) $\mathcal{F}_k(\sqcap_{i\in I} f_i) \geq \bigwedge_i \mathcal{F}_k(f_i)$, $\forall\, f_i \in (L^X)^E$, $i \in I$.

The pair $(X, \mathcal{F})$ is called an L-fuzzy (K, E)-soft cotopological space.

Definition 2.7. [23] An L- fuzzy (K, E)-soft quasi uniformity is a mapping $\mathcal{U} : K \to L^{(L^{X\times X})^E}$ which satisfies the following conditions .

(SU1) There exists $u \in (L^{X\times X})^E$ such that $\mathcal{U}_k(u) = 1$.

(SU2) If $v \sqsubseteq u$, then $\mathcal{U}_k(v) \leq \mathcal{U}_k(u)$.

(SU3) For every $u, v \in (L^{X\times X})^E, \mathcal{U}_k(u \odot v) \geq \mathcal{U}_k(u) \odot \mathcal{U}_k(v)$.

(SU4) If $\mathcal{U}_k(u) \neq 0$ then $\top_\triangle \sqsubseteq u$ where, for each $e \in E$,

$$(\top_\triangle)_e(x, y) = \begin{cases} 1, & \text{if } x = y, \\ 0, & \text{if } x \neq y. \end{cases}$$

(SU5) $\mathcal{U}_k(u) \leq \bigvee\{\mathcal{U}_k(v) \mid v \circ v \sqsubseteq u\}$, where

$$u_e \circ v_e(x, z) = \bigvee_{y\in X} u_e(x, y) \odot v_e(y, z).$$

The pair $(X,\mathcal{U})$ is called an L-fuzzy (K, E)-soft quasi-uniform space.

An L-fuzzy (K, E)-soft quasi-uniform space $(X,\mathcal{U})$ is said to be an L-fuzzy (K, E)-soft uniform space if

(U) $\mathcal{U}_k(u) \leq \mathcal{U}_k(u^{-1})$, where $(u^{-1})_e(x, y) = u_e(y, x)$ for each $k \in K$ and $u \in (L^{X\times X})^E$.

An L-fuzzy (K, E)-soft quasi-uniformity $\mathcal{U}$ on X is said to be an $(L, \odot)$-fuzzy principle quasi-uniformity if

(P) $\mathcal{U}_k(\bigwedge_{i\in\Gamma} u_i) = \bigwedge_{i\in\Gamma} \mathcal{U}_k(u_i)$ for all $u_i \in (L^{X\times X})^E$.

Remark 2.8. Let $(X,\mathcal{U})$ be an L-fuzzy (K, E)-soft uniform space.

(1) By (SU1) and (SU2), we have $\mathcal{U}_k(1_{X\times X}) = 1$ because $u \sqsubseteq 1_{X\times X}$ for all $u \in (L^{X\times X})^E$.

(2) Since $\mathcal{U}_k(u) \leq \mathcal{U}_k(u^{-1}) \leq \mathcal{U}_k((u^{-1})^{-1}) = \mathcal{U}_k(u)$, then $\mathcal{U}_k(u) = \mathcal{U}_k(u^{-1})$.

3 L-fuzzy (K, E)-soft topologies induced by L-fuzzy (K, E)-soft uniformities

Lemma 3.1. For every $f \in (L^X)^E$, we define $(u_f), (u_f^{-1}) \in (L^{X\times X})^E$ by:

$$(u_f)_e(x, y) = f_e(x) \to f_e(y) \ \ \forall e \in E,$$

$$(u_f^{-1})_e(x, y) = (u_f)_e(y, x) \ ,$$

then $\forall f, g \in (L^X)^E$ we have the following statements

(1) $1_{X\times X} = u_{0_X} = u_{1_X}$,

(2) For every $u_f \in (L^{X\times X})^E$, we have $u_f \circ u_f \sqsubseteq u_f$,

(3) $u_f \odot u_g \sqsubseteq u_{f\odot g}$,

(4) $u_f \odot u_g \sqsubseteq u_{f\oplus g}, u_f \oplus u_g \sqsubseteq u_{f\oplus g}$,

(5)$u_f^{-1} = u_{f^*}$,

(6)$u_{f\odot g}^{-1} = u_{f^*\oplus g^*}$,

(7)$u_{f\oplus g}^{-1} = u_{f^*\odot g^*}$,

Proof. (1) $(1_{X\times X})_e(x,y) = \top = (u_{0_X})_e(x,y) = (0_X)_e(x) \to (0_X)_e(y) = (1_X)_e(x) \to (1_X)_e(y) = (u_{1_X})_e(x,y)$.

(2)

$$\begin{array}{l} ((u_f)_e \circ (u_f)_e)(x,z) = \bigvee_{y\in X}((u_f)_e(x,y) \odot (u_f)_e(y,z)) \\ = \bigvee_{y\in X}(f_e(x) \to f_e(y)) \odot (f_e(y) \to f_e(z))) \\ \le (f_e(x) \to f_e(y)) \odot (f_e(y) \to f_e(z)) \\ \le f_e(x) \to f_e(z). \end{array}$$

Hence $(u_f \circ u_f) \le u_f$.

(3)

$$\begin{array}{l} (u_f \odot u_g)_e(x,y) = (u_f)_e(x,y) \odot (u_g)_e(x,y) \\ \le (f_e(x) \to f_e(y)) \odot (g_e(x) \to g_e(y)) \\ \le f_e(x) \odot g_e(x) \to f_e(y) \odot g_e(y) \\ = (u_{f\odot g})_e(x,y). \end{array}$$

(4)

$$\begin{array}{l} (u_f \odot u_g)(x,y) = u_f(x,y) \odot u_g(x,y) \\ \le (f_e(x) \to f_e(y)) \odot (g_e(x) \to g_e(y)) \\ \le f_e(x) \oplus g_e(x) \to f_e(y) \oplus g_e(y) \\ = u_{f\oplus g}(x,y). \end{array}$$

(5)

$$\begin{array}{l} (u_f^{-1})_e(x,y) = (u_f)_e(y,x) = f_e(y) \to f_e(x) = f_e^*(x) \to f_e^*(y) \\ = (u_{f^*})_e(x,y). \end{array}$$

(6)

$$(u_{f\odot g}^{-1})_e = (u_{(f\odot g)^*})_e = (u_{f^*\oplus g^*})_e.$$

(7)

$$(u_{f\oplus g}^{-1})_e = (u_{(f\oplus g)^*})_e = (u_{f^*\odot g^*})_e.$$

Theorem 3.2. Let $(X,\mathcal{U})$ be an L-fuzzy (K,E)-soft principle quasi-uniform space. Define the mapping $\mathcal{T}^{\mathcal{U}} : K \to L^{(L^X)^E}$ by:

$$\mathcal{T}_k^{\mathcal{U}}(f)=\begin{cases} 1, & \text{if } f=0_X\\ \mathcal{U}_k(u_f), & \text{if } f\neq 0_X.\end{cases}$$

Then, $\mathcal{T}^{\mathcal{U}}$ is an L-fuzzy (K,E)-soft topology on X.

Proof. (SO1) $\mathcal{T}_k^{\mathcal{U}}(0_X)=1$ and $\mathcal{T}_k^{\mathcal{U}}(1_X)=\mathcal{U}_k(u_{1_X})=\mathcal{U}_k(1_{X\times X})=1$.

(SO2) Since $u_f\odot u_g\sqsubseteq u_{f\odot g}$, for each $f,g\in(L^X)^E$, by (SU3), we have $\mathcal{U}_k(u_f\odot u_g)\leq\mathcal{U}_k(u_{f\odot g})$.

Hence, $\mathcal{T}_k^{\mathcal{U}}(f\odot g)=\mathcal{U}_k(u_{f\odot g})\geq\mathcal{U}_k(u_f\odot u_g)\geq\mathcal{U}_k(u_f)\odot\mathcal{U}_k(u_g)=\mathcal{T}_k^{\mathcal{U}}(f)\odot\mathcal{T}_k^{\mathcal{U}}(g)$.

(SO3) Let $\{f_i\}_{i\in\Gamma}$ be a family of fuzzy soft sets in X. Then, by Lemma 2.3(8),we have

$$\begin{aligned}(u_{(\bigvee_{i\in\Gamma}f_i)})_e(x,y) &=(\textstyle\bigvee_{i\in\Gamma}f_i)_e(x)\to(\bigvee_{i\in\Gamma}f_i)_e(y)\\ &\geq\textstyle\bigwedge_i((f_i)_e(x)\to(f_i)_e(y))\\ &=\textstyle\bigwedge_i(u_{f_i})_e(x,y).\end{aligned}$$

Then $\mathcal{T}_k^{\mathcal{U}}(\bigvee_{i\in\Gamma}f_i)=\mathcal{U}_k(u_{(\bigvee_{i\in\Gamma}f_i)})\geq\mathcal{U}_k(\bigwedge_{i\in\Gamma}u_{f_i})=\bigwedge_{i\in\Gamma}\mathcal{U}_k(u_{f_i})=\bigwedge_{i\in\Gamma}\mathcal{T}_k^{\mathcal{U}}(f_i)$, for every $i\in\Gamma$.

Theorem 3.3. Let $(X,\mathcal{U})$ be a L-fuzzy (K,E)-soft principle quasi-uniform space. Define the mapping $\mathcal{F}^{\mathcal{U}}:K\to L^{(L^X)^E}$ by:

$$\mathcal{F}_k^{\mathcal{U}}(f)=\begin{cases} 1, & \text{if } f=1_X\\ \mathcal{U}_k(u_f^{-1}), & \text{if } f\neq 1_X.\end{cases}$$

Then, $\mathcal{F}^{\mathcal{U}}$ is an L- fuzzy (K,E)-soft cotopology on X.

Proof. (SF1) $\mathcal{F}_k^{\mathcal{U}}(1_X)=1$ and $\mathcal{T}_k^{\mathcal{U}}(0_X)=\mathcal{U}_k(u_{0_X}^{-1})=\mathcal{U}(u_{1_X})=\mathcal{U}_k(1_{X\times X})=1$.

(SF2) Since $u_{f^*}\odot u_{g^*}\sqsubseteq u_{f^*\odot g^*}=u_{f\oplus g}^{-1}$, for each $f,g\in(L^X)^E$, by (U3), we have $\mathcal{U}_k(u_f^{-1}\odot u_g^{-1})\leq\mathcal{U}_k(u_{f\odot g}^{-1})$. Hence, $\mathcal{F}_k^{\mathcal{U}}(f\oplus g)=\mathcal{U}_k(u_{f\oplus g}^{-1})=\mathcal{U}_k(u_{f^*\odot g^*})\geq$

$\mathcal{U}_k(u_{f^*}\odot u_{g^*})\geq\mathcal{U}_k(u_{f^*})\odot\mathcal{U}(u_{g^*})=\mathcal{F}_k^{\mathcal{U}}(f)\odot\mathcal{F}_k^{\mathcal{U}}(g)$.

(SF3) Let $\{f_i\}_{i\in\Gamma}$ be a family of fuzzy sets in X. Then, by Lemma 2.3(8),we have

$$\begin{aligned}(u_{(\bigvee_{i\in\Gamma}f_i^*)})_e(x,y) &=(\textstyle\bigvee_{i\in\Gamma}f_i^*)_e(x)\to(\bigvee_{i\in\Gamma}f_i^*)_e(y)\\ &\geq\textstyle\bigwedge_i((f_i^*)_e(x)\to(f_i^*)_e(y))\\ &=\textstyle\bigwedge_i(u_{f_i^*})_e(x,y).\end{aligned}$$

Then $\mathcal{F}_k^{\mathcal{U}}(\bigwedge_{i\in\Gamma} f_i) = \mathcal{U}_k(u^{-1}_{(\bigwedge_{i\in\Gamma} f_i)}) = \mathcal{U}_k(u_{(\bigvee_{i\in\Gamma} f_i^*)}) \geq \mathcal{U}_k(\bigwedge_{i\in\Gamma} u_{f_i^*}) = \bigwedge_{i\in\Gamma} \mathcal{U}_k(u_{f_i^*}) = \bigwedge_{i\in\Gamma} \mathcal{F}_k^{\mathcal{U}}(f_i)$, for every $i \in \Gamma$.

Theorem 3.4 Let $(X,\mathcal{U})$ be an L-fuzzy (K,E)-soft quasi-uniform space. Define the mapping $\mathcal{T}^{\mathcal{U}} : K \to L^{(L^X)^E}$ by

$$\mathcal{T}_k^{\mathcal{U}}(f) = \bigwedge_{e\in E} \bigwedge_{x\in X} \{f_e^*(x) \vee \bigvee_{u[x]\sqsubseteq f} \mathcal{U}_k(u)\}$$

where $(u[x])_e(y) = u_e(y,x)$. Then, $\mathcal{T}^{\mathcal{U}}$ is an L-fuzzy (K,E)-soft topology on X.

Proof (SO1) It is easily checked.

(SO2) Suppose that

$$(\bigvee_{u[x]\sqsubseteq f} \mathcal{U}_k(u)) \odot (\bigvee_{v[x]\sqsubseteq g} \mathcal{U}_k(v)) \not\leq \bigvee_{w[x]\sqsubseteq f\odot g} \mathcal{U}_k(w).$$

For each $i \in \{1,2\}$, there exists u_i with $u_i[x] \sqsubseteq f_i$ such that

$$\mathcal{U}_k(u_1) \odot \mathcal{U}_k(u_2) \not\leq \bigvee_{w[x]\sqsubseteq f\odot g} \mathcal{U}_k(w).$$

It implies $(u_1 \odot u_2)[x] \sqsubseteq f \odot g$ such that

$$\bigvee_{w[x]\sqsubseteq f_1\odot f_2} \mathcal{U}_k(w) \geq \mathcal{U}_k(u_1 \odot u_2) \geq \mathcal{U}_k(u_1) \odot \mathcal{U}_k(u_2).$$

It is a contradiction.

$$\begin{aligned}
&\mathcal{T}_k^{\mathcal{U}}(f) \odot \mathcal{T}_k^{\mathcal{U}}(g)\\
&= \Big(\bigwedge_{e\in E}\bigwedge_{x\in X}\{f_e^*(x) \vee \bigvee_{u[x]\sqsubseteq f}\mathcal{U}_k(u)\}\Big) \odot \Big(\bigwedge_{e\in E}\bigwedge_{y\in X}\{g_e^*(y) \vee \bigvee_{v[y]\sqsubseteq g}\mathcal{U}_k(v)\}\Big)\\
&\leq \bigwedge_{e\in E}\bigwedge_{x\in X}\Big(\{f_e^*(x) \vee \bigvee_{u[x]\sqsubseteq f}\mathcal{U}_k(u)\} \odot \{g_e^*(x) \vee \bigvee_{v[x]\sqsubseteq g}\mathcal{U}_k(v)\}\Big)\\
&\qquad \text{(by Lemma 2.3)}\\
&\leq \bigwedge_{e\in E}\bigwedge_{x\in X}\Big((f_e^* \oplus g_e^*)(x) \vee (\bigvee_{u[x]\sqsubseteq f}\mathcal{U}_k(u)) \odot (\bigvee_{v[x]\sqsubseteq g}\mathcal{U}_k(v))\Big)\\
&\leq \bigwedge_{e\in E}\bigwedge_{x\in X}\Big((f_e^* \oplus g_e^*)(x) \vee (\bigvee_{u\odot v[x]\sqsubseteq f\odot g}\mathcal{U}_k(u \odot v)\Big)\\
&\leq \mathcal{T}_k^{\mathcal{U}}(f \odot g).
\end{aligned}$$

(SO3)

$$\begin{array}{ll} \mathcal{T}^{\mathcal{U}}_k(\bigvee_{j\in J} f_j) & = \bigwedge_{e\in E}\bigwedge_{x\in X}\{(\bigvee_{j\in J} f_j)^*_e(x) \vee \bigvee_{u[x]\sqsubseteq \vee_j f_j} \mathcal{U}(u)\} \\ & \quad \text{(since } L \text{ is a completely distributive lattice)} \\ & = \bigwedge_{e\in E}\bigwedge_{x\in X}\{\bigwedge_j\{(f_j)^*_e(x) \vee \bigvee_{u[x]\sqsubseteq \vee_j f_j} \mathcal{U}(u)\}\} \\ & = \bigwedge_j\{\bigwedge_{e\in E}\bigwedge_{x\in X}\{(f_j)^*_e(x) \vee \bigvee_{u[x]\sqsubseteq \vee_j f_j} \mathcal{U}(u)\}\} \\ & \geq \bigwedge_j\{\bigwedge_{e\in E}\bigwedge_{x\in X}\{(f_j)^*_e(x) \vee \bigvee_{u[x]\sqsubseteq f_j} \mathcal{U}(u)\}\} \\ & = \bigwedge_j \mathcal{T}^{\mathcal{U}}_k(f_j). \end{array}$$

Corollary 3.5. Let $(X,\mathcal{U})$ be an L-fuzzy (K,E)-soft quasi-uniform space. Define the mapping $\mathcal{F}^{\mathcal{U}} : K \to L^{(L^X)^E}$ by

$$\mathcal{F}^{\mathcal{U}}_k(f) = \bigwedge_{e\in E}\bigwedge_{x\in X}\{f_e(x) \vee \bigvee_{u[x]\sqsubseteq f^*} \mathcal{U}_k(u)\}.$$

Then, $\mathcal{F}^{\mathcal{U}}$ is an L-fuzzy (K,E)-soft cotopology on X.

4 L-fuzzy (K,E)-soft quasi-uniformities induced by L-fuzzy (K,E)-soft topologies

Theorem 4.1. Let $(X,\mathcal{T})$ be an L-fuzzy (K,E)-soft topological space. Define the mapping $\mathcal{U}^{\mathcal{T}} : K \to L^{(L^{X\times X})^E}$ by

$$\mathcal{U}^{\mathcal{T}}_k(u) = \bigvee\{\odot_{i=1}^n \mathcal{T}_k(f_i) \mid \odot_{i=1}^n u_{f_i} \sqsubseteq u, f_i \neq 0_X\},$$

where $\bigvee$ is taken over finite family $\{u_{f_i} \mid i = 1,2,...,n\}$. Then $\mathcal{U}^{\mathcal{T}}$ is an L-fuzzy (K,E)-soft quasi -uniformity on X.

Proof.(1) Put $\mathcal{T}^0 = \{f \in (L^X)^E \mid \mathcal{T}_k(f) \neq 0\}$.

(SU1) Since $u_{1_X} \sqsubseteq 1_{X\times X}$, we have

$$\mathcal{U}(1_{X\times X}) \geq \mathcal{T}_k(1_X) = 1.$$

(SU2) If $u_1 \sqsubseteq u_2,\ u_1,u_2 \in (L^{X\times X})^E$, then

$$\begin{array}{ll} \mathcal{U}_k(u_1) & = \bigvee\{\odot_{i=1}^n \mathcal{T}_k(f_i) \mid \odot_{i=1}^n u_{f_i} \sqsubseteq u_1, f_i \neq 0_X\} \\ & \leq \bigvee\{\odot_{i=1}^n \mathcal{T}_k(f_i) \mid \odot_{i=1}^n u_{f_i} \sqsubseteq u_2, f_i \neq 0_X\} \\ & = \mathcal{U}_k(u_2). \end{array}$$

(SU3) Let $f, g \in (L^X)^E$ and $u \in (L^{X\times X})^E$, then $u_f \odot u_g \sqsubseteq u_{f\odot g}$ and

$$\begin{array}{l} \mathcal{U}_k^{\mathcal{T}}(u) \odot \mathcal{U}_k^{\mathcal{T}}(w) \\ = \bigvee\{\odot_{i=1}^n \mathcal{T}_k(f_i) \mid \odot_{i=1}^n u_{f_i} \sqsubseteq u, f_i \neq 0_X\} \\ \odot \bigvee\{\odot_{j=1}^m \mathcal{T}_k(g_j) \mid \odot_{j=1}^m u_{g_j} \sqsubseteq w, g_j \neq 0_X\} \\ \leq \bigvee\{\odot_{i=1}^n \mathcal{T}_k(f_i) \odot \odot_{j=1}^m \mathcal{T}_k(g_j) \mid \\ \odot_{i=1}^n u_{f_i} \odot \odot_{j=1}^m u_{g_j} \sqsubseteq u \odot w, f_i \neq 0_X, g_j \neq 0_X\} \\ \leq \mathcal{U}_k^{\mathcal{T}}(u \odot w). \end{array}$$

(SU4) If $\mathcal{U}_k(u) \neq 0$, then there exists $f \in (L^X)^E$ with $\mathcal{T}_k(f) \neq 0$ such that $u_f \sqsubseteq u$. Hence, $1_\triangle \sqsubseteq u_f \sqsubseteq u$.

(SU5) Suppose there exists $u \in (L^{X\times X})^E$ such that

$$\bigvee\{\mathcal{U}_k^{\mathcal{T}}(v) \mid v \circ v \sqsubseteq u\} \not\geq \mathcal{U}_k^{\mathcal{T}}(u).$$

There exists a finite family $\{g_i \in \mathcal{T}^0 \mid \odot_{i=1}^m u_{g_i} \leq u\}$. such that

$$\bigvee\{\mathcal{U}_k^{\mathcal{T}}(v) \mid v \circ v \sqsubseteq u\} \not\geq \odot_{i=1}^m \mathcal{T}_k(g_i).$$

On the other hand, since $u_{g_i} \circ u_{g_i} \sqsubseteq u_{g_i}$, for each $i \in \{1, ..., m\}$, we have

$$\begin{array}{l} (\odot_{i=1}^m u_{g_i}) \circ (\odot_{i=1}^m u_{g_i}) = \odot_{i=1}^m (u_{g_i} \circ u_{g_i}) \\ \sqsubseteq \odot_{i=1}^m u_{g_i}. \end{array}$$

Put $v = \odot_{i=1}^m u_{g_i}$. Then $v \circ v \sqsubseteq u$ and

$$\bigvee\{\mathcal{U}_k^{\mathcal{T}}(v) \mid v \circ v \sqsubseteq u\} \geq \mathcal{U}_k^{\mathcal{T}}(v) \geq \odot_{i=1}^m \mathcal{T}_k(g_i).$$

It is a contradiction. Thus $\bigvee\{\mathcal{U}_k^{\mathcal{T}}(v) \mid v \circ v \sqsubseteq u\} \geq \mathcal{U}_k^{\mathcal{T}}(u)$.

Corollary 4.2. Let $(X, \mathcal{F})$ be an L-fuzzy (K, E)-soft cotopological space. Define the mapping $\mathcal{U}^{\mathcal{F}} : K \to L^{(L^{X\times X})^E}$ by

$$\mathcal{U}_k^{\mathcal{F}}(u) = \bigvee\{\odot_{i=1}^n \mathcal{F}_k(g_i^*) \mid \odot_{i=1}^n u_{g_i} \sqsubseteq u, g_i \neq 0_X\},$$

where the first $\bigvee$ is taken over every finite family $\{u_{G_i} \mid i = 1, ..., n\}$. Then $\mathcal{U}^{\mathcal{F}}$ is an L-fuzzy (K, E)-soft quasi-uniformity on X.

Example 4.3. Let $(L = [0, 1], \odot, \to)$ be a complete residuated lattice (ref. [4,9,11]) defined by

$$x \odot y = (x + y - 1) \vee 0, \;\; x \to y = (1 - x + y) \wedge 1,$$

$$x \oplus y = (x + y) \wedge 1, \quad x^* = 1 - x.$$

Let $X = \{x, y\}$ be a set , $K = E = \{e_1, e_2\}$ and $f, g \in (L^X)^E$ such that

$$\begin{array}{c} f_{e_1}(x) = 0.6, f_{e_1}(y) = 0.5, \\ f_{e_2}(x) = 0.3, f_{e_2}(y) = 0.6. \\ g_{e_1}(x) = 0.2, g_{e_1}(y) = 0.5, \\ g_{e_2}(x) = 0.4, g_{e_2}(y) = 0.3. \end{array}$$

We obtain $f \odot f, f^*, g^*, f^* \oplus f^* \in (L^X)^E$ such that

$$\begin{array}{c} (f \odot f)_{e_1}(x) = 0.2, \ \ (f \odot f)_{e_1}(y) = 0, \\ (f \odot f)_{e_2}(x) = 0, \ \ (f \odot f)_{e_2}(y) = 0.2. \end{array}$$

$$\begin{array}{c} f^*_{e_1}(x) = 0.4, \ \ f^*_{e_1}(y) = 0.5, \\ f^*_{e_2}(x) = 0.7, \ \ f^*_{e_2}(y) = 0.4. \\ g^*_{e_1}(x) = 0.8, \ \ g^*_{e_1}(y) = 0.5, \\ g^*_{e_2}(x) = 0.6, \ \ g^*_{e_2}(y) = 0.7. \end{array}$$

$$\begin{array}{c} (f^* \oplus f^*)_{e_1}(x) = 0.8, \ (f^* \oplus f^*)_{e_1}(y) = 1, \\ (f^* \oplus f^*)_{e_2}(x) = 1, \ (f^* \oplus f^*)_{e_2}(y) = 0.8. \end{array}$$

(1) Define $\mathcal{T} : E \to L^{(L^X)^E}$ as follows

$$\mathcal{T}_{e_1}(h) = \begin{cases} 1, & \text{if } h \in \{1_X, 0_X\} \\ 0.6, & \text{if } h = f, \\ 0.3, & \text{if } \ h = f \odot f, \\ 0, & \text{otherwise}, \end{cases}$$

$$\mathcal{T}_{e_2}(h) = \begin{cases} 1, & \text{if } h \in \{1_X, 0_X\} \\ 0.4, & \text{if } h = g, \\ 0, & \text{otherwise}. \end{cases}$$

Since $0.3 = \mathcal{T}_{e_1}(f \odot f) \geq \mathcal{T}_{e_1}(f) \odot \mathcal{T}_{e_1}(f) = 0.2$, $\mathcal{T}$ is an L-fuzzy (E, E)-soft topology on X.

(2) From (1), we obtain an L-fuzzy (E, E)-soft cotopology $\mathcal{F} : E \to L^{(L^X)^E}$ with $\mathcal{F}_{e_i}(f) = \mathcal{T}_{e_i}(f^*)$ as follows

$$\mathcal{F}_{e_1}(h) = \begin{cases} 1, & \text{if } h \in \{1_X, 0_X\} \\ 0.6, & \text{if } h = f^*, \\ 0.3, & \text{if } \ h = f^* \oplus f^*, \\ 0, & \text{otherwise}, \end{cases}$$

$$\mathcal{F}_{e_2}(h) = \begin{cases} 1, & \text{if } h \in \{1_X, 0_X\} \\ 0.4, & \text{if } h = g^*, \\ 0, & \text{otherwise}, \end{cases}$$

(3) We obtain $u_f, u_g, u_{f\odot f}$ as

$$(u_f)_{e_1} = \begin{pmatrix} 1 & 0.9 \\ 1 & 1 \end{pmatrix} \quad (u_f)_{e_2} = \begin{pmatrix} 1 & 1 \\ 0.7 & 1 \end{pmatrix}$$

$$(u_{f\odot f})_{e_1} = \begin{pmatrix} 1 & 0.8 \\ 1 & 1 \end{pmatrix} \quad (u_{f\odot f})_{e_2} = \begin{pmatrix} 1 & 1 \\ 0.8 & 1 \end{pmatrix}$$

$$(u_g)_{e_1} = \begin{pmatrix} 1 & 1 \\ 0.7 & 1 \end{pmatrix} \quad (u_g)_{e_2} = \begin{pmatrix} 1 & 0.9 \\ 1 & 1 \end{pmatrix}$$

From Theorem 4.1, we obtain an L-fuzzy (E,E)-soft quasi-uniformity $\mathcal{U}^{\mathcal{T}} : E \to L^{(L^X \times L^X)^E}$ as follows

$$(\mathcal{U}^{\mathcal{T}})_{e_1}(u) = \begin{cases} 1, & \text{if } u = 1_{X\times X} \\ 0.6, & \text{if } u_f \sqsubseteq u \neq 1_{X\times X}, \\ 0.3, & \text{if } u_{f\odot f} \sqsubseteq u \not\sqsupseteq u_f, \\ 0.2, & \text{if } u_f \odot u_f \sqsubseteq u \not\sqsupseteq u_{f\odot f}, \\ 0, & \text{otherwise,} \end{cases}$$

$$(\mathcal{U}^{\mathcal{T}})_{e_2}(u) = \begin{cases} 1, & \text{if } u = 1_{X\times X} \\ 0.4, & \text{if } u_g \sqsubseteq u \neq 1_{X\times X}, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 4.4. Let $\mathcal{T}$ be an L- fuzzy (K,E)-soft topology on X and $\mathcal{U}^{\mathcal{T}}$ the L-fuzzy (K,E)-soft principle quasi-uniformity. Then

(1) $\mathcal{T}^{\mathcal{U}^{\mathcal{T}}} \geq \mathcal{T}$,

(2) $\mathcal{U}^{\mathcal{T}^{\mathcal{U}}} \leq \mathcal{U}$.

Proof. (1) $\mathcal{T}_k^{\mathcal{U}^{\mathcal{T}}}(f) = \mathcal{U}_k^{\mathcal{T}}(u_f) = \bigvee\{\odot_{i=1}^n \mathcal{T}_k(f_i) \mid \odot_{i=1}^n u_{f_i} \leq u_f, f_i \neq 0_X\} \geq \mathcal{T}_k(f)$.

(2) Suppose that there exists $u \in (L^{X\times X})^E$ such that

$$\mathcal{U}_k^{\mathcal{T}^{\mathcal{U}}}(u) \not\leq \mathcal{U}_k(u).$$

There exists a family $\{u_{f_i} \mid \odot_{i=1}^m u_{f_i} \sqsubseteq u\}$ such that $\odot_{j=1}^m \mathcal{T}_k^{\mathcal{U}}(f_i) \not\leq \mathcal{U}_k(u)$. For each $i = 1, 2, ..., m$, by the definition of $\mathcal{T}^{\mathcal{U}}$, there exists $u_{f_i} \in (L^{X\times X})^E$ such that

$$\odot_{j=1}^m \mathcal{U}_k^{\mathcal{T}}(u_{f_i}) \not\leq \mathcal{U}_k(u).$$

On the other hand,

$$\mathcal{U}_k(u) \geq \mathcal{U}_k(\odot_{i=1}^m u_{f_i}) \geq \odot_{i=1}^m \mathcal{U}_k(u_{f_i}) = \odot_{i=1}^m \mathcal{T}_k(f_i).$$

It is a contradiction.

Example 4.5. Let $X = \{h_i \mid i = \{1,2,3\}\}$ with h_i=house and $E = \{e, b\}$ with e=expensive, b= beautiful. Define a binary operation $\odot$ on $[0,1]$ by

$$x \odot y = \max\{0, x+y-1\},\ x \to y = \min\{1-x+y, 1\}$$

$$x \oplus y = \min\{1, x+y\},\ x^* = 1-x$$

Then $([0,1], \odot, \to, 0, 1)$ is a complete residuated lattice.

(1) Put $v, v \odot v, w \in ([0,1]^{X\times X})^E$ as

$$v_e = \begin{pmatrix} 1 & 0.6 & 0.5 \\ 0.3 & 1 & 0.5 \\ 0.4 & 0.6 & 1 \end{pmatrix} v_b = \begin{pmatrix} 1 & 0.5 & 0.3 \\ 0.7 & 1 & 0.5 \\ 0.6 & 0.6 & 1 \end{pmatrix}$$

$$(v\odot v)_e = \begin{pmatrix} 1 & 0.2 & 0 \\ 0 & 1 & 0 \\ 0 & 0.2 & 1 \end{pmatrix} (v\odot v)_b = \begin{pmatrix} 1 & 0 & 0 \\ 0.4 & 1 & 0 \\ 0.2 & 0.2 & 1 \end{pmatrix}$$

$$w_e = \begin{pmatrix} 1 & 0.4 & 0.5 \\ 0.4 & 1 & 0.5 \\ 0.4 & 0.6 & 1 \end{pmatrix} v_b = \begin{pmatrix} 1 & 0.5 & 0.3 \\ 0.3 & 1 & 0.5 \\ 0.2 & 0.3 & 1 \end{pmatrix}$$

We define $\mathcal{U} : E \to [0,1]^{([0,1]^{X\times X})^E}$ as follows:

$$\mathcal{U}_e(u) = \begin{cases} 1, & \text{if } u = 1_{Y\times Y} \\ 0.6, & \text{if } v \sqsubseteq u \neq 1_{Y\times Y}, \\ 0.3, & \text{if } v\odot v \sqsubseteq u \not\sqsupseteq v, \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathcal{U}_b(u) = \begin{cases} 1, & \text{if } u = 1_{Y\times Y} \\ 0.5, & \text{if } w \sqsubseteq u \neq 1_{Y\times Y}, \\ 0, & \text{otherwise.} \end{cases}$$

Since $v \circ v = v$, $w \circ w = w$ and $(v \odot v) \circ (v \odot v) = (v \odot v)$, $\mathcal{U}$ is a $[0,1]$-fuzzy (E,E)-soft principle quasi-uniformity on X. From Theorem 3.2, we obtain a $[0,1]$-fuzzy soft (E,E)-topology $\mathcal{T}^{\mathcal{U}} : E \to [0,1]^{([0,1]^X)^E}$ such that $\mathcal{T}^{\mathcal{U}}_e(v[f]) = 0.6$ because, for $e \in \{e, b\}$, $f \neq 0_X$,

$$\begin{aligned}
&(u_{v[f]})_e(y,z) = v[f]_e(y) \to v[f]_e(z) \\
&= \bigvee_{x\in X}(v_e(x,y) \odot f_e(x)) \to \bigvee_{x\in X}(v_e(x,z) \odot f_e(x)) \\
&\geq \bigwedge_{x\in X}((v_e(x,y) \odot f_e(x)) \to (v_e(x,z) \odot f_e(x))) \\
&\geq \bigwedge_{x\in X}(v_e(x,y) \to v_e(x,z)) \\
&\geq v_e(y,z).
\end{aligned}$$

Since $(u_{1_{X\times X}[\alpha_X]})_e(y,z) = 1_{X\times X}[\alpha_X]_e(y) \to 1_{X\times X}[\alpha_X]_e(z) = (1_{X\times X})_e(y,z)$, we have $\mathcal{T}^{\mathcal{U}}_e(1_{X\times X}[\alpha_X]) = \mathcal{T}^{\mathcal{U}}_e(\alpha_X) = 1$. Moreover, $\mathcal{T}^{\mathcal{U}}_e(v \odot v[f]) = 0.3$ and $\mathcal{T}^{\mathcal{U}}_d(w[f]) = 0.5$.

Hence

$$\mathcal{T}_e^{\mathcal{U}}(g) = \begin{cases} 1, & \text{if } g = \alpha_X, \\ 0.6, & \text{if } g = v[f], \\ 0.3, & \text{if } \ g = (v \odot v)[f] \\ 0, & \text{otherwise,} \end{cases}$$

$$\mathcal{T}_b^{\mathcal{U}}(g) = \begin{cases} 1, & \text{if } g = \alpha_X, \\ 0.5, & \text{if } g = w[f], \\ 0, & \text{otherwise,} \end{cases}$$

From Theorem 4.1, we obtain $\mathcal{U}^{\mathcal{T}^{\mathcal{U}}} : E \to [0,1]^{([0,1]^{X \times X})^E}$ as follows:

$$\mathcal{U}_e^{\mathcal{T}^{\mathcal{U}}}(u) = \begin{cases} 1, & \text{if } u = 1_{X \times X} \\ 0.6, & \text{if } u_{v[f]} \sqsubseteq u \neq 1_{Y \times Y}, \\ 0.3, & \text{if } u_{(v \odot v)[f]} \sqsubseteq u \not\sqsupseteq u_{v[f]}, \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathcal{U}_b^{\mathcal{T}^{\mathcal{U}}}(u) = \begin{cases} 1, & \text{if } u = 1_{X \times X} \\ 0.5, & \text{if } u_{v[f]} \sqsubseteq u \neq 1_{Y \times Y}, \\ 0, & \text{otherwise.} \end{cases}$$

Since

$$\begin{aligned} & \bigvee_{x \in X}(v_e(x,y) \odot f_e(x)) \to \bigvee_{x \in X}(v_e(x,z) \odot f_e(x)) \\ & \geq \bigwedge_{x \in X}((v_e(x,y) \odot f_e(x)) \to (v_e(x,z) \odot f_e(x)) \\ & \geq \bigwedge_{x \in X}(v_e(x,y) \to v_e(x,z)) \geq v_e(y,z), \end{aligned}$$

$u_{v[f]} \sqsubseteq v$ and $u_{(v \odot v)[f]} \sqsubseteq v$. Hence $\mathcal{U}^{\mathcal{T}^{\mathcal{U}}} \leq \mathcal{U}$.

References

[1] H. Aktas, N. Cağman, Soft sets and soft groups, *Inf. Sci.*, **177**(13)(2007),2726-2735.

[2] A. Aygünoglu, H. Aygün, Introduction to fuzzy soft groups, *Computers and Mathematics with Appl.*, **58**(2009), 1279-1286.

[3] A. Aygünoglu, V. Cetkin, H. Aygün, An introduction to fuzzy soft topological spaces, *Hacettepe Journal of Math. and Stat.*, **43**(2)(2014), 193-204.

[4] R. Bělohlávek, *Fuzzy Relational Systems*, Kluwer Academic Publishers, New York , 2002.

[5] V. Cetkin , H. Aygün , On fuzzy soft topogenous structure, *J. Intell. Fuzzy Syst.* , **27**(2014), 247-255.

[6] F. Feng, C. Li, B. Davvaz, M. Arfan Ali, *Soft sets combined with fuzzy sets and rough sets: a tentative approach* , Soft Comput. **14**(9)(2010), 899-911.

[7] J.Gutiérrez García , M. A. de Prade Vicente, Šostak A. P., *A unified approach to the concept of fuzzy L-uniform spaces*, **Chapter 3** in [15], 81-114.

[8] D.N. Georgiou, A.C. Megaritis, V.I. Petropoulos, On soft topological spaces, *Appl. Math. Inf. Sci.*, **7**(5)(2013), 1889-1901.

[9] P. Hájek, *Metamathematices of Fuzzy Logic*, Kluwer Academic Publishers, Dordrecht (1998).

[10] U. Höhle, S.E. Rodabaugh, *Mathematics of Fuzzy Sets: Logic, Topology, and Measure Theory*, The Handbooks of Fuzzy Sets Series 3, Kluwer Academic Publishers, Boston, 1999.

[11] B. Hutton, Uniformities in fuzzy topological spaces, *J. Math. Anal. Appl.*, **58** (1977), 74-79.

[12] Y.C. Kim, A.A. Ramadan, M. A. Usama, L-fuzzy Uniform Spaces, *The Journal of Fuzzy Mathematics*, **14** (2006), 821-850.

[13] W. Kotzé , *Uniform spaces*, in: Höhle U., Rodabaugh S. E.(Eds.), Mathematics of Fuzzy Sets: Logic, Topology and Measure Theory, Handbook Series, **Chapter 8**, Vol. 3, Kluwer Academic Publishers, Boston, Dordrecht, London, 1999, 553-580.

[14] R. Lowen, Fuzzy uniform spaces, *J. Math. Anal. Appl.*, **82** (1981), 370-385, **doi:** 10.1016/0022-247x(81)90202-x.

[15] P.K. Maji, R. Biswas, A.R. Roy, Fuzzy soft sets, *Journal of Fuzzy Mathematics*, **9**(3)(2001), 589-602.

[16] P.K. Maji, R. Biswas, A.R. Roy, Soft set theory, *Computers Mathematics with Appl.*, **45**(2003), 555-562.

[17] P. Majumdar, S.K. Samanta, Generalized fuzzy soft sets, *Computers Mathematics with Appl.*, 59(2010), 1425-1432.

[18] D. Molodtsov, Soft set theory, *Computers Mathematics with Appl.*, **37**(1999), 19-31.

[19] D. Pei, D. Miao, *From soft sets to information systems*, Granular Computing,2005 IEEE,International Conferences on (2)(2005),617-621.

[20] A.A. Ramadan,Y.C. Kim, M.K. El-Gayyar, On fuzzy uniform spaces, *The Journal of Fuzzy Mathematics*, **11** (2003), 279-299.

[21] A.A. Ramadan, E.H. Elkordy and Yong Chan Kim, Relationships between L-fuzzy quasi-uniform structures and L-fuzzy topologies, *Journal of Intelligent and Fuzzy Systems*, **28** (2015), 2319-2327.

[22] A.A. Ramadan, E.H. Elkordy, Y.C. Kim , Perfect L-fuzzy topogenous space, L-fuzzy quasi-proximities and L-fuzzy quasi-uniform spaces, *Journal of Intelligent and Fuzzy Systems*, **28**(6)(2015), 2591-2604.

[23] A.A. Ramadan, Yong Chan Kim , On L-fuzzy (K, E)- soft topogenous orders and L-fuzzy (K, E)- soft topologies, *Journal of Intelligent and Fuzzy Systems*,(2016) submitted.

[24] S.E. Rodabaugh, *Categorical foundations of variable-basis fuzzy topology*, in: Höhle U., Rodabaugh S. E.(Eds.), Mathematics of Fuzzy Sets: Logic, Topology and Measure Theory, Handbook Series , Chapter 4, Kluwer Academic Publishers, 1999.

[25] S.E. Rodabaugh, E.P. Klement, *Topological and Algebraic Structures In Fuzzy Sets*, The Handbook of Recent Developments in the Mathematics of Fuzzy Sets, Kluwer Academic Publishers, Boston, Dordrecht, London, 2003.

[26] A.R. Roy, P.K. Maji, A fuzzy soft set theoretic approach to decision making problems, *International of Computational and Applied Mathematics*, 203(2007),412-418.

[27] M. Shabir, M. Naz, On soft topological spaces, *Computers and Mathematics with Appl.*, **61**(2011), 1786-1799.

[28] S.A. Solovyov, Lattice-valued soft algebras, *Soft Comput.* **17(10)**(2013),1751-1766.

[29] B. Tanay, M.B. Kandemir, Topological structures of fuzzy soft sets, *Computers and Mathematics with Appl.*, **61**(2011),412-418.

[30] B.P. Varol, H. Aygün , Fuzzy soft topology, *Hacet J. Math. Stat.* **41**(3)(2012), 407-419.

[31] G. Xuechang, L. Yongming, F. Feng, A new order relation on fuzzy soft sets and its application, *Soft Comput.*, **17**(1)(2013),63-70.

[32] J. Zhan, Y.B. Jun, Soft BL-algebras based on fuzzy sets, *Comput Math. Appl.* **59**(6)(2010), 2037-2046.

[33] L. Zhaowen, X. Tusheng, The relationship among soft sets, soft rough sets and topologies, *Soft Comput.* , **18**(4)(2014),717-728.

E-mail: aramadan58@hotmail.com

E-mail: yck@gwnu.ac.kr

A FIXED POINT APPROACH TO STABILITY OF ADDITIVE MAPPINGS IN MODULAR SPACES WITHOUT Δ_2-CONDITIONS

CHOONKIL PARK, ABASALT BODAGHI, AND SANG OG KIM*

ABSTRACT. In this article, we prove the generalized Hyers-Ulam-Rassias stability of the following equivalent functional equations

$$f(x+y) = f(x) + f(y),$$
$$f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) = f(x)$$

via fixed point method. We obtain the result in the framework of modular spaces without Δ_2-conditions.

1. Introduction and preliminaries

We recall some basic facts concerning modular spaces.

Definition 1.1. Let X be a vector space over a field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). A generalized function $\rho : X \to [0, \infty]$ is called a *modular* if it satisfies for all $\alpha, \beta \in \mathbb{K}$, $x, y \in X$

(i) $\rho(x) = 0$ if and only if $x = 0$,
(ii) $\rho(\alpha x) = \rho(x)$ for every scalar α with $|\alpha| = 1$,
(iii) $\rho(\alpha x + \beta y) \le \rho(x) + \rho(y)$ if $\alpha + \beta = 1$ and $\alpha, \beta \ge 0$.
If we replace (iii) by
(iii′) $\rho(\alpha x + \beta y) \le \alpha\rho(x) + \beta\rho(y)$ if $\alpha + \beta = 1$ and $\alpha, \beta \ge 0$

then we say that ρ is a *convex modular*.

A modular ρ defines a corresponding *modular space*, denoted by X_ρ, given by

$$X_\rho = \{x \in X \mid \rho(\lambda x) \to 0 \text{ as } \lambda \to 0\}.$$

X_ρ is a vector subspace of X.

Definition 1.2. Let X_ρ be a modular space and $\{x_n\}$ be a sequence in X_ρ.

(1) $\{x_n\}$ *ρ-converges* to $x \in X_\rho$ if $\rho(x_n - x) \to 0$ as $n \to \infty$. The point x is called the *ρ-limit* of the sequence $\{x_n\}$, which is denoted by $x_n \xrightarrow{\rho} x$.
(2) $\{x_n\}$ is a *ρ-Cauchy sequence* if $\rho(x_n - x_m) \to 0$ as $m, n \to \infty$.
(3) A subset $S \subseteq X$ is called *ρ-complete* if every ρ-Cauchy sequence in S is ρ-convergent to an element of S.

Remark 1.3. Note that for a fixed $x \in X_\rho$, the valuation $\lambda \in \mathbb{K} \mapsto \rho(\lambda x)$ is increasing. In case the modular ρ is convex, one has $\rho(x) \le \delta\rho\left(\frac{1}{\delta}x\right)$ for all $x \in X_\rho$, provided $0 < \delta \le 1$. In

2010 *Mathematics Subject Classification.* Primary 39B52, 39B82, 47H09.
Key words and phrases. generalized Ulam-Hyers-Rassias stability, additive mapping, modular space, fixed point.
* Corresponding author.

particular, $\rho(x) \le \frac{1}{2^n}\rho(2^n x)$ for all $x \in X_\rho$ and $n \in \mathbb{N}$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}$ be sequences in X_ρ that ρ-converge to a, b, c, d, respectively. It is easy to show that

$$\rho\left(\frac{1}{4}(a_n - a) + \frac{1}{4}(b_n - b) + \frac{1}{4}(c_n - c) + \frac{1}{4}(d_n - d)\right) \le \frac{1}{4}\big(\rho(a_n - a) + \rho(b_n - b) + \rho(c_n - c) + \rho(d_n - d)\big).$$

Unlike a norm, a modular need not be continuous. The convergence of a sequence $\{x_n\}$ does not imply that of $\{cx_n\}$ for a scalar c. In order to avoid such difficulties, some additional conditions are imposed on the modular so that the multiple of $\{x_n\}$ converges naturally. One of such conditions is the so-called Δ_2-condition. A modular ρ is said to satisfy the Δ_2-*condition* if there exists $\kappa \ge 0$ such that $\rho(2x) \le \kappa\rho(x)$ for all $x \in X_\rho$.

Example 1.4. A convex function $\phi(t)$ defined on $[0, \infty)$, nondecreasing and continuous for $t \ge 0$ and such that $\phi(0) = 0, \phi(t) > 0$ for $t > 0$, and $\phi(t) \to \infty$ as $t \to \infty$, is called an Orlicz function. Let (Ω, Σ, μ) be a measure space. Let $L^0(\mu)$ be the set of all measurable real-valued (or complex-valued) functions on Ω. Define for $f \in L^0(\mu)$,

$$\rho_\phi(f) = \int_\Omega \phi(|f|)d\mu.$$

Then ρ_ϕ is a modular and the associated modular function space is called an Orlicz space and denoted by

$$L^\phi = \{f \in L^0(\mu) \mid \rho_\phi(\lambda f) \to 0 \text{ as } \lambda \to 0\}.$$

It is known that L^ϕ is ρ_ϕ-complete. Moreover, $(L^\phi, \|\cdot\|_{\rho_\phi})$ is a Banach space, where the Luxemburg norm $\|\cdot\|_{\rho_\phi}$is defined as follows;

$$\|f\|_{\rho_\phi} = \inf\left\{\lambda > 0 \;\middle|\; \int_\Omega \phi\left(\frac{|f|}{\lambda}\right) d\mu \le 1\right\}.$$

Note that if μ is the Lebesgue measure on $\mathbb{R}$ and $\phi(t) = e^t - 1$, then ρ_ϕ does not satisfy the Δ_2-condition.

The notion of modulars on linear spaces and the corresponding theory of modular spaces, as a generalization of metric spaces, were initiated by Nakano [15] in connection with the theory of ordered spaces. Further and the most complete development of the theories are due to Luxemburg, Musielak, Orlicz, Mazur [11, 12, 13, 14] and their collaborators. In the present time, the theory of modular and modular spaces are extensively applied, in particular, in the theory of various Orlicz spaces and interpolation theory, which have broad applications. For a review of Musielak-Orlicz space and modular spaces, the reader is referred to the book of Musielak and Orlicz [13].

The stability problem of functional equations originated from a question of Ulam [23] in 1940, concerning the stability of group homomorphisms. In 1941, Hyers [7] gave the first affirmative answer to the problem of Ulam for Banach spaces. Hyers' result was generalized by Aoki [1] for additive mappings and by Rassias [16] for linear mappings by considering an unbounded Cauchy difference. Generalizations of the Rassias' theorem were obtained by Forti [4] and Găvruta [5] who permitted the Cauchy difference to become arbitrary unbounded. Since then a wide spectrum of stability problems has been investigated for a variety of functional equations and spaces. A large list of references concerning the stability of various functional equations can be found e.g., in [2, 3, 8, 9, 18, 19, 20, 21, 22]. Recently, Sadeghi [17] showed a fixed point method to prove the stabilities of the Cauchy and Jensen functional equations on modular spaces with the Δ_2-conditions. Wongkum et al. [24] obtained a stability result of the quadratic functional equation without the Δ_2-conditions. Also, Eshaghi Gordji et al. [6] proved a generalized Ulam-Hyers-Rassias stability of Cauchy mappings in modular spaces endowed with a partial order without the Δ_2-condition.

Motivated by the ideas and results of [6], [17] and [24], we prove the generalized Ulam-Hyers-Rassias stability of additive functional equations in the framework of modular spaces via fixed point theory. It is very important to note that we dropped the Δ_2-condition. The results are interesting and many results on the stability of additive functional equations in normed spaces can be reformulated. Note that the functional equations we are considering in Theorems 2.2 and 3.1 are equivalent, but in the stability of those functional equations the control functions are different.

2. Stability of the functional equation $\mathbf{f(x+y)=f(x)+f(y)}$

Let X_ρ be a modular space, $C \subset X_\rho$, and $T : C \to C$ be a mapping. The *orbit* of T at $x \in C$ is the set

$$\mathcal{O}(x) = \{x, Tx, T^2x, \ldots\}.$$

The quantity $\delta_\rho(x) = \sup\{\rho(T^n(x) - T^m(x)) \mid n, m \in \mathbb{N}\}$ is called the *ρ-diameter* of T at x.

We start with a known fixed point theorem in modular spaces.

Theorem 2.1. ([10]) *Let X_ρ be a modular space whose induced modular is lower semicontinuous and let $C \subset X_\rho$ be a ρ-complete subset. If $T : C \to C$ is a ρ-contraction, that is, there is a constant $k \in [0, 1)$ such that*

$$\rho(Tx - Ty) \le k\rho(x - y), \quad x, y \in C,$$

and $\delta_\rho(x_0) < \infty$ for $x_0 \in C$, then the sequence $\{T^n(x_0)\}$ is ρ-convergent to a point $w \in C$. If $\rho\big(w - T(w)\big) < \infty$ and $\rho\big(x_0 - T(w)\big) < \infty$, then w is a fixed point of T.

Theorem 2.2. *Let V be a linear space, X_ρ be a ρ-complete modular space where ρ is lower semicontinuous and convex, and $f : V \to X_\rho$ be a mapping with $f(0) = 0$. Let $0 \le L < 1$ be a constant and suppose that*

$$\rho\big(2f(x + y) - 2f(x) - 2f(y)\big) \le \varphi(x, y), \quad x, y \in V, \tag{2.1}$$

where $\varphi : V \times V \to [0, \infty)$ is a nonnegative real-valued function with the following properties;

$$\begin{aligned} &\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y)}{2^n} = 0, \\ &\varphi(2x, 2x) \le 2L\varphi(x, x), \quad x, y \in V. \end{aligned} \tag{2.2}$$

Then there exists a unique additive mapping $w : V \to X_\rho$ such that

$$\rho\big(w(x) - f(x)\big) \le \frac{1}{4(1 - L)}\varphi(x, x) \tag{2.3}$$

for all $x \in V$.

Proof. Let

$$M = \{g \mid g : V \to X_\rho,\ g(0) = 0\}.$$

Then by a standard argument as in [24, Lemma 10] or [17, Theorem 2.1], M is a linear space, and the generalized function $\widetilde{\rho} : M \to [0, \infty]$ defined by

$$\widetilde{\rho}(g) = \inf\{c > 0 \mid \rho\big(g(x)\big) \le c\varphi(x, x),\ x \in V\}$$

is a convex modular on M. Moreover the corresponding modular space $M_{\widetilde{\rho}}$ is the whole space M and is $\widetilde{\rho}$-complete. Also $\widetilde{\rho}$ is lower semicontinuous.

Define $T : M_{\widetilde{\rho}} \to M_{\widetilde{\rho}}$ by

$$T(g)(x) = \frac{1}{2}g(2x), \quad g \in M_{\widetilde{\rho}},\ x \in V.$$

We first show that T is a $\widetilde{\rho}$-contraction. Let $x \in V,\ g, h \in M_{\widetilde{\rho}}$ and c be an arbitrary constant with $\widetilde{\rho}(g-h) \le c$. Then we have

$$\rho\big(g(2x) - h(2x)\big) \le c\varphi(2x, 2x),$$

so that by (2.2)

$$\rho\left(\frac{g(2x)}{2} - \frac{h(2x)}{2}\right) \le \frac{1}{2}\big(g(2x) - h(2x)\big) \le \frac{c}{2}\varphi(2x, 2x) \le Lc\varphi(x, x).$$

Hence we have $\widetilde{\rho}(Tg - Th) \le L\widetilde{\rho}(g-h)$, from which T is a $\widetilde{\rho}$-contraction.

Next, we show that T has a bounded orbit at f. For that, we first show by induction on $n \in \mathbb{N}$

$$\rho\left(\frac{f(2^n x)}{2^{n-1}} - 2f(x)\right) \le \sum_{i=1}^{n} \frac{1}{2^i}\varphi\left(2^{i-1}x, 2^{i-1}x\right), \quad x \in V. \tag{2.4}$$

In fact, for $n = 1$, letting $x = y$ in (2.1), we have

$$\rho\big(f(2x) - 2f(x)\big) \le \frac{1}{2}\rho\big(2f(2x) - 4f(x)\big) \le \frac{1}{2}\varphi(x, x).$$

Assume that (2.4) holds for $n - 1$. Then, for all $x \in V$,

$$\begin{aligned}
&\rho\left(\frac{f(2^n x)}{2^{n-1}} - 2f(x)\right) \\
&\le \frac{1}{2}\rho\left(\frac{f(2^n x)}{2^{n-2}} - 2f(2x)\right) + \frac{1}{2}\rho\big(2f(2x) - 4f(x)\big) \\
&\le \frac{1}{2}\sum_{i=1}^{n-1} \frac{1}{2^i}\varphi\left(2^i x, 2^i x\right) + \frac{1}{2}\varphi(x, x) \\
&= \sum_{i=1}^{n} \frac{1}{2^i}\varphi\left(2^{i-1}x, 2^{i-1}x\right),
\end{aligned}$$

from which it follows that (2.4) holds for every $n \in \mathbb{N}$. Hence we deduce that

$$\begin{aligned}
\rho\left(\frac{f(2^n x)}{2^{n-1}} - 2f(x)\right) &\le \sum_{i=1}^{n} \frac{1}{2^i}\varphi(2^{i-1}x, 2^{i-1}x) \\
&\le \frac{1}{2}\varphi(x, x)\sum_{i=1}^{n} L^{i-1} \\
&\le \frac{1}{2}\varphi(x, x)\sum_{i=1}^{\infty} L^{i-1} \\
&\le \frac{1}{2(1-L)}\varphi(x, x)
\end{aligned} \tag{2.5}$$

for all $x \in V$.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 24, NO.6, 2018, COPYRIGHT 2018 EUDOXUS PRESS, LLC

Now for $m, n \in \mathbb{N}$, we have by (2.5)

$$\begin{aligned} &\rho\left(\frac{f(2^n x)}{2^n} - \frac{f(2^m x)}{2^m}\right) \\ &\leq \frac{1}{2}\rho\left(2\frac{f(2^n x)}{2^n} - 2f(x)\right) + \frac{1}{2}\rho\left(2\frac{f(2^n x)}{2^m} - 2f(x)\right) \\ &\leq \frac{2}{4(1-L)}\varphi(x,x) \\ &= \frac{1}{2(1-L)}\varphi(x,x) \end{aligned} \tag{2.6}$$

for all $x \in V$. We also have

$$\rho\left(\frac{f(2^n x)}{2^n} - f(x)\right) \leq \frac{1}{2}\rho\left(2\frac{f(2^n x)}{2^n} - 2f(x)\right) \leq \frac{1}{4(1-L)}\varphi(x,x) \tag{2.7}$$

for all $x \in V$. Hence it follows by (2.6) that for $m, n \in \mathbb{N}$,

$$\widetilde{\rho}(T^n f - T^m f) \leq \frac{1}{2(1-L)} < \infty,$$

and hence, the $\widetilde{\rho}$-diameter of T at f is finite, i.e., $\delta_{\widetilde{\rho}}(f) < \infty$.

By Theorem 2.1, there exists an element $w \in M_{\widetilde{\rho}}$ such that $T^n f \xrightarrow{\widetilde{\rho}} w$. By the $\widetilde{\rho}$-contractivity of T, one has

$$\widetilde{\rho}(Tw - T^{n+1} f) \leq L\widetilde{\rho}(w - T^n f). \tag{2.8}$$

Letting $n \to \infty$ in (2.8) and applying the lower semicontinuity of $\widetilde{\rho}$, we have

$$\begin{aligned} \widetilde{\rho}(Tw - w) &\leq \liminf_{n\to\infty}\ \widetilde{\rho}(Tw - T^{n+1} f) \\ &\leq \liminf_{n\to\infty}\ L\widetilde{\rho}(w - T^n f) \\ &= 0, \end{aligned}$$

so that w is a fixed point of T, i.e., $w(2x) = 2w(x)$ for all $x \in V$.

Replacing (x, y) by $(2^n x, 2^n y)$ in (2.1), we have

$$\rho\Big(2f\big(2^n(x+y)\big) - 2f(2^n x) - 2f(2^n y)\Big) \leq \varphi(2^n x, 2^n y).$$

Then by using Remark 1.3 and (2.2), we have

$$\begin{aligned} &\rho\left(\frac{f(2^n(x+y))}{2^n} - \frac{f(2^n x)}{2^n} - \frac{f(2^n y)}{2^n}\right) \\ &\leq \rho\left(\frac{2f(2^n(x+y))}{2^n} - \frac{2f(2^n x)}{2^n} - \frac{2f(2^n y)}{2^n}\right) \\ &\leq \frac{1}{2^n}\rho\Big(2f\big(2^n(x+y)\big) - 2f(2^n x) - 2f(2^n y)\Big) \\ &\leq \frac{1}{2^n}\varphi(2^n x, 2^n y) \to 0 \end{aligned} \tag{2.9}$$

as $n \to \infty$.

Applying the lower semicontinuity of ρ, Remark 1.3 and (2.9), we deduce that

$$w(x+y) = w(x) + w(y)$$

for all $x, y \in V$, that is, w is additive. Moreover, by (2.7), it follows that

$$\rho\big(w(x) - f(x)\big) \leq \frac{1}{4(1-L)}\varphi(x,x)$$

for all $x \in V$. Hence the inequality (2.3) is proved.

Finally, we show the uniqueness of the additive mapping w. Assume that w_1, w_2 are additive mappings that satisfy (2.3). Then we deduce that

$$\begin{aligned}
&\rho\left(\frac{w_1(x)}{2} - \frac{w_2(x)}{2}\right) \\
&= \rho\left(\frac{w_1(2^n x)}{2^{n+1}} - \frac{f(2^n x)}{2^{n+1}} + \frac{f(2^n x)}{2^{n+1}} - \frac{w_2(2^n x)}{2^{n+1}}\right) \\
&\leq \frac{1}{2^n}\rho\left(\frac{w_1(2^n x)}{2} - \frac{f(2^n x)}{2} + \frac{f(2^n x)}{2} - \frac{w_2(2^n x)}{2}\right) \\
&\leq \frac{1}{2^n}\left(\frac{1}{2}\rho\big(w_1(2^n x) - f(2^n x)\big) + \frac{1}{2}\rho\big(f(2^n x) - w_2(2^n x)\big)\right) \\
&\leq \frac{1}{2^{n+1}} \cdot \frac{2}{4(1-L)}\varphi(2^n x, 2^n x) \\
&\leq \frac{1}{2^{n+2}} \cdot \frac{1}{1-L} \cdot (2L)^n \varphi(x,x) \\
&\to 0 \text{ as } n \to \infty
\end{aligned}$$

for all $x \in V$, from which it follows that $w_1 = w_2$. This completes the proof. □

If the control function φ is replaced by a constant, we obtain the following result.

Corollary 2.3. *Let V be a linear space, X_ρ be a ρ-complete modular space where ρ is lower semicontinuous and convex, and $f : V \to X_\rho$ be a mapping with $f(0) = 0$. If there exists a constant $\delta > 0$ such that*

$$\rho\big(2f(x+y) - 2f(x) - 2f(y)\big) \leq \delta, \quad x, y \in V,$$

then there exists a unique additive mapping $w : V \to X_\rho$ such that

$$\rho\big(w(x) - f(x)\big) \leq \frac{\delta}{2}$$

for all $x \in V$.

Proof. It is easy to see that we can take $L = \frac{1}{2}$ in Theorem 2.2 if $\varphi(x,y) = \delta$ for all $x, y \in V$. □

It is known that every normed space is a modular space with $\rho(x) = \|x\|$. Applying Theorem 2.2, we have the following result.

Corollary 2.4. *Let V be a linear space, $(X, \|\cdot\|)$ be a Banach space and $f : V \to X$ be a mapping with $f(0) = 0$. Let $0 \leq L < 1$ be a constant and suppose that*

$$\|f(x+y) - f(x) - f(y)\| \leq \varphi(x,y), \quad x, y \in V,$$

where $\varphi : V \times V \to [0, \infty)$ is a nonnegative real-valued function with the following properties;

$$\begin{aligned}
&\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y)}{2^n} = 0, \\
&\varphi(2x, 2x) \leq 2L\varphi(x,x), \quad x, y \in V.
\end{aligned}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 24, NO.6, 2018, COPYRIGHT 2018 EUDOXUS PRESS, LLC

Then there exists a unique additive mapping $w : V \to X$ such that

$$\|w(x) - f(x)\| \leq \frac{1}{2(1-L)}\varphi(x, x)$$

for all $x \in V$.

Example 2.5. Let ϕ be an Orlicz function and L^ϕ be the Orlicz space. Let $f : V \to L^\phi$ be a mapping with $f(0) = 0$. Let $0 \leq L < 1$ be a constant and suppose that

$$\int_\Omega \phi\left(|2f(x+y) - 2f(x) - 2f(y)|\right) d\mu \leq \varphi(x, y), \quad x, y \in V,$$

where $\varphi : V \times V \to [0, \infty)$ is a nonnegative real-valued function with the following properties;

$$\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y)}{2^n} = 0,$$
$$\varphi(2x, 2x) \leq 2L\varphi(x, x), \quad x, y \in V.$$

Then there exists a unique additive mapping $w : V \to L^\phi$ such that

$$\int_\Omega \phi\left(|w(x) - f(x)|\right) d\mu \leq \frac{1}{4(1-L)}\varphi(x, x)$$

for all $x \in V$.

3. Stability of the functional equation $\mathbf{f}\left(\frac{\mathbf{x}+\mathbf{y}}{\mathbf{2}}\right) + \mathbf{f}\left(\frac{\mathbf{x}-\mathbf{y}}{\mathbf{2}}\right) = \mathbf{f}(\mathbf{x})$

Theorem 3.1. *Let V be a linear space, X_ρ be a ρ-complete modular space where ρ is lower semicontinuous and convex, and $f : V \to X_\rho$ be a mapping with $f(0) = 0$. Let $0 \leq L < 1$ be a constant and suppose that*

$$\rho\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - 2f(x)\right) \leq \varphi(x, y), \quad x, y \in V, \tag{3.1}$$

where $\varphi : V \times V \to [0, \infty)$ is a nonnegative real-valued function with the following properties;

$$\begin{aligned} &\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y)}{2^n} = 0, \\ &\varphi(2x, 0) \leq 2L\varphi(x, 0), \quad x, y \in V. \end{aligned} \tag{3.2}$$

Then there exists a unique additive mapping $w : V \to X_\rho$ such that

$$\rho\big(w(x) - f(x)\big) \leq \frac{L}{2(1-L)}\varphi(x, 0) \tag{3.3}$$

for all $x \in V$.

Proof. Let

$$M = \{g \mid g : V \to X_\rho,\ g(0) = 0\}.$$

Then as in the proof of Theorem 2.2, M is a linear space, and the generalized function $\widetilde{\rho} : M \to [0, \infty]$ defined by

$$\widetilde{\rho}(g) = \inf\{c > 0 \mid \rho\big(g(x)\big) \leq c\varphi(x, 0),\ x \in V\}$$

is a convex modular on M. Moreover the corresponding modular space $M_{\widetilde{\rho}}$ is the whole space M and is $\widetilde{\rho}$-complete. Also $\widetilde{\rho}$ is lower semicontinuous. Define $T : M_{\widetilde{\rho}} \to M_{\widetilde{\rho}}$ by

$$T(g)(x) = \frac{1}{2}g(2x), \quad g \in M_{\widetilde{\rho}},\ x \in V.$$

We first show that T is a $\widetilde{\rho}$-contraction. Let $x \in V$, $g, h \in M_{\widetilde{\rho}}$ and c be an arbitrary constant with $\widetilde{\rho}(g-h) \le c$. Then we have

$$\rho\big(g(2x) - h(2x)\big) \le c\varphi(2x, 0),$$

so that by (3.2)

$$\rho\left(\frac{g(2x)}{2} - \frac{h(2x)}{2}\right) \le \frac{1}{2}\rho\big(g(2x) - h(2x)\big) \le \frac{c}{2}\varphi(2x,0) \le Lc\varphi(x,0).$$

Hence we have $\widetilde{\rho}(Tg - Th) \le L\widetilde{\rho}(g-h)$, from which it follows that T is a $\widetilde{\rho}$-contraction.

Next, we show that T has a bounded orbit at f. For that, we first show by induction on $n \in \mathbb{N}$

$$\rho\left(\frac{f(2^n x)}{2^{n-1}} - 2f(x)\right) \le (L^n + \ldots + L^2 + L)\varphi(x,0), \quad x \in V. \tag{3.4}$$

In fact, for $n = 1$, letting $y = 0$ in (3.1), we have

$$\rho\left(4f\left(\frac{x}{2}\right) - 2f(x)\right) \le \varphi(x,0), \quad x \in V.$$

Then we have

$$\rho\big(4f(x) - 2f(2x)\big) \le \varphi(2x,0) \le 2L\varphi(x,0), \quad x \in V,$$

and hence

$$\rho\big(f(2x) - 2f(x)\big) \le \frac{1}{2}\rho\big(4f(x) - 2f(2x)\big) \le L\varphi(x,0), \quad x \in V.$$

Assume that (3.4) holds for $n-1$. Then, for all $x \in V$,

$$\begin{aligned}
&\rho\left(\frac{f(2^n x)}{2^{n-1}} - 2f(x)\right) \\
&\le \frac{1}{2}\rho\left(\frac{f(2^n x)}{2^{n-2}} - 2f(2x)\right) + \frac{1}{2}\rho\big(2f(2x) - 4f(x)\big) \\
&\le \frac{1}{2}(L^{n-1} + \ldots + L^2 + L)\varphi(2x,0) + \frac{1}{2}\cdot 2L\varphi(x,0) \\
&\le (L^n + \ldots + L^2 + L)\varphi(x,0),
\end{aligned} \tag{3.5}$$

from which it follows that (3.4) holds for every $n \in \mathbb{N}$.

Now for $m, n \in \mathbb{N}$, we get by (3.5)

$$\begin{aligned}
&\rho\left(\frac{f(2^n x)}{2^n} - \frac{f(2^m x)}{2^m}\right) \\
&\le \frac{1}{2}\rho\left(2\frac{f(2^n x)}{2^n} - 2f(x)\right) + \frac{1}{2}\rho\left(2\frac{f(2^m x)}{2^m} - 2f(x)\right) \\
&\le \frac{1}{2}\left(\frac{L(1-L^n)}{1-L} + \frac{L(1-L^m)}{1-L}\right)\varphi(x,0) \\
&\le \frac{L}{(1-L)}\varphi(x,0)
\end{aligned} \tag{3.6}$$

for all $x \in V$. We also have by (3.5)

$$\rho\left(\frac{f(2^n x)}{2^n} - f(x)\right) \le \frac{1}{2}\rho\left(2\frac{f(2^n x)}{2^n} - 2f(x)\right) \le \frac{L}{2(1-L)}\varphi(x,0) \tag{3.7}$$

for all $x \in V$. Hence it follows by (3.6) that for $m, n \in \mathbb{N}$,

$$\widetilde{\rho}(T^n f - T^m f) \leq \frac{L}{1-L} < \infty,$$

which implies that the $\widetilde{\rho}$-diameter of T at f is finite, i.e., $\delta_{\widetilde{\rho}}(f) < \infty$. By Theorem 2.1, there exists an element $w \in M_{\widetilde{\rho}}$ such that $T^n f \xrightarrow{\widetilde{\rho}} w$. By the $\widetilde{\rho}$-contractivity of T, one has

$$\widetilde{\rho}(Tw - T^{n+1} f) \leq L\widetilde{\rho}(w - T^n f). \tag{3.8}$$

Letting $n \to \infty$ in (3.8) and applying the lower semicontinuity of $\widetilde{\rho}$, we have

$$\begin{aligned} \widetilde{\rho}(Tw - w) &\leq \liminf_{n\to\infty} \ \widetilde{\rho}(Tw - T^{n+1} f) \\ &\leq \liminf_{n\to\infty} \ L\widetilde{\rho}(w - T^n f) \\ &= 0, \end{aligned}$$

so that w is a fixed point of T, i.e., $w(2x) = 2w(x)$ for all $x \in V$.

Replacing (x, y) by $(2^{n+1}x, 2^{n+1}y)$ in (3.1), we have

$$\rho\big(2f(2^n(x+y)) + 2f(2^n(x-y)) - 2f(2^{n+1}x)\big) \leq \varphi(2^{n+1}x, 2^{n+1}y).$$

Then by using Remark 1.3 and (3.2), we get

$$\begin{aligned} &\rho\left(\frac{2f\big(2^n(x+y)\big)}{2^{n+1}} + \frac{2f\big(2^n(x-y)\big)}{2^{n+1}} - \frac{2f(2^{n+1}x)}{2^{n+1}}\right) \\ &\leq \frac{1}{2^{n+1}}\rho\Big(2f\big(2^n(x+y)\big) + 2f\big(2^n(x-y)\big) - 2f(2^{n+1}x)\Big) \\ &\leq \frac{1}{2^{n+1}}\varphi\left(2^{n+1}x, 2^{n+1}y\right) \\ &\to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.9}$$

Since $w(2x) = 2w(x)$, applying the lower semicontinuity of ρ, Remark 1.3 and (3.9), we deduce that

$$w(x+y) + w(x-y) = 2w(x) \tag{3.10}$$

for all $x, y \in V$. Since $w(0) = 0$, letting $x = 0$ in (3.10), it follows that

$$w(-y) = -w(y)$$

for all $y \in V$. Replacing (x, y) by (y, x) in (3.10), we have

$$w(x+y) + w(y-x) = 2w(y) \tag{3.11}$$

for all $x, y \in V$. From (3.10) and (3.11), we obtain that w is additive, that is,

$$w(x+y) = w(x) + w(y)$$

for all $x, y \in V$. Moreover, by (3.7), it follows that

$$\rho\big(w(x) - f(x)\big) \leq \frac{L}{2(1-L)}\varphi(x, 0)$$

for all $x \in V$. Hence the inequality (3.3) is proved. Finally, we show the uniqueness of the additive mapping w. Assume that w_1, w_2 are additive mappings that satisfy (3.3). Then we deduce that

$$\begin{aligned}
&\rho\left(\frac{w_1(x)}{2}-\frac{w_2(x)}{2}\right)\\
&=\rho\left(\frac{w_1(2^nx)}{2^{n+1}}-\frac{f(2^nx)}{2^{n+1}}+\frac{f(2^nx)}{2^{n+1}}-\frac{w_2(2^nx)}{2^{n+1}}\right)\\
&\le\frac{1}{2^n}\rho\left(\frac{w_1(2^nx)}{2}-\frac{f(2^nx)}{2}+\frac{f(2^nx)}{2}-\frac{w_2(2^nx)}{2}\right)\\
&\le\frac{1}{2^n}\left(\frac{1}{2}\rho\big(w_1(2^nx)-f(2^nx)\big)+\frac{1}{2}\rho\big(f(2^nx)-w_2(2^nx)\big)\right)\\
&\le\frac{1}{2^{n+1}}\cdot\frac{2L}{2(1-L)}\varphi(2^nx,0)\\
&\le\frac{1}{2^{n+1}}\cdot\frac{L}{(1-L)}\cdot(2L)^n\varphi(x,0)\\
&\to 0 \text{ as } n\to\infty
\end{aligned}$$

for all $x \in V$, from which it follows that $w_1 = w_2$. This completes the proof. □

Remark 3.2. It is curious that the multiple of 2 in the inequalities (2.1) and (3.1) appears. It is an interesting question whether the constant 2 in (2.1) and (3.1) can be dropped.

Acknowledgements

This research was supported by Hallym University Research Fund, 2016 (HRF-201606-008).

References

[1] T. Aoki, *On the stability of the linear transformation mappings in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64–66.
[2] A. Bodaghi, S. O. Kim, *Ulam's type stability of a functional equation deriving from quadratic and additive functions*, J. Math. Inequal. **9** (2015), 73–84.
[3] S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific, River Edge, 2002.
[4] G. L. Forti, *An existence and stability theorem for a class of functional equations*, Stochastica **4** (1980), 23-30.
[5] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximate additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.
[6] M. Eshaghi Gordji, F. Sajadian, Y. J. Cho, M. Ramezani, *A fixed point theorem for quasi-contraction mappings in partially order modular spaces with an application*, U. P. B. Sci. Bull., Series A **76** (2014), 135–146.
[7] D. H. Hyers, *On the stability of linear functional equations*, Proc. Nat. Acad. Sci. U. S. A. **27** (1941), 222–224.
[8] D. H. Hyers, G. Isac, Th. M. Rassias, *Stability of Functional Equations in Several Variables*, Basel: Birkhäuser, 1998.
[9] S. Jung, *Hyers-Ulam-Rassis Stability of Functional Equations in Mathematical Analysis*, Springer, New York, 2011.

[10] M. A. Khamsi, *Quasicontraction mappings in modular spaces without Δ_2-condition*, Fixed Point Theory Appl. **2008** Article ID 916187, 6 pages (2008).

[11] W. A. J. Luxemburg, *Banach function spaces*, Ph.D. Thesis, Delft University of Technology, Delft, The Netherlands (1955).

[12] S. Mazur, W. Orlicz, *On some classes of linear spaces*, Studia Math. **17** (1958), 97–119.

[13] J. Musielak, W. Orlicz, *On modular spaces*, Studia Math. **18** (1959), 591–597.

[14] J. Musielak, W. Orlicz, *Some remarks on modular spaces*, Bull. Acad. Polon. Sci. Sr. Math. Astron. Phys. **7** (1959), 661–668.

[15] H. Nakano, *Modulared Semi-Ordered Linear Spaces*, Maruzen, Tokyo, Japan, 1950.

[16] Th. M. Rassias, *On the stability of linear mappings in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.

[17] G. Sadeghi, *A fixed point approach to stability of functional equations in modular spaces*, Bull. Malays. Math. Sci. Soc. Second Ser., **37** (2014), 333–344.

[18] S. Schin, D. Ki, J. Chang, M. Kim, *Random stability of quadratic functional equations: a fixed point approach*, J. Nonlinear Sci. Appl. **4** (2011), 37–49.

[19] S. Shagholi, M. Bavand Savadkouhi, M. Eshaghi Gordji, *Nearly ternary cubic homomorphism in ternary Fréchet algebras,* J. Comput. Anal. Appl. **13** (2011), 1106–1114.

[20] S. Shagholi, M. Eshaghi Gordji, M. Bavand Savadkouhi, *Stability of ternary quadratic derivation on ternary Banach algebras,* J. Comput. Anal. Appl. **13** (2011), 1097–1105.

[21] D. Shin, C. Park, Sh. Farhadabadi, *On the superstability of ternary Jordan C^*-homomorphisms,* J. Comput. Anal. Appl. **16** (2014), 964–973.

[22] D. Shin, C. Park, Sh. Farhadabadi, *Stability and superstability of J^*-homomorphisms and J^*-derivations for a generalized Cauchy-Jensen equation,* J. Comput. Anal. Appl. **17** (2014), 125–134.

[23] S. M. Ulam, *Problems of modern mathematics*, Sciences Editions, John Wiley & Sons Inc., New York, 1964.

[24] K. Wongkum, P. Chaipunya, P. Kumam, *On the generalized Ulam-Hyers-Rassias stability of quadratic mappings in modular spaces without Δ_2-conditions*, J. Funct. Spaces **2015** Article ID 461719, 6 pages (2015).

Department of Mathematics, Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Korea
E-mail address: baak@hanyang.ac.kr

Young Researchers and Elite Club, Islamshahr Branch, Islamic Azad University, Islamshahr, Iran
E-mail address: abasalt.bodaghi@gmail.com

Department of Mathematics, Hallym University, Chuncheon 24252, Korea
E-mail address: sokim@hallym.ac.kr

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 24, NO.6, 2018, COPYRIGHT 2018 EUDOXUS PRESS, LLC

FUZZY STABILITY OF ADDITIVE-QUADRATIC ρ-FUNCTIONAL INEQUALITIES

CHANG IL KIM AND GILJUN HAN*

Abstract. In this paper, we prove the generalized Hyers-Ulam stability of the following additive-quadratic ρ-functional inequalities

$$N\Big(f(x+y)+f(x-y)-2f(x)-f(y)-f(-y), -\rho\Big[4f\Big(\frac{x+y}{2}\Big)+4f\Big(\frac{x-y}{2}\Big)-3f(x) + f(-x)-f(y)-f(-y)\Big], t\Big) \geq \frac{t}{t+\phi(x,y)}, \quad \Big(\rho \neq 1, \frac{1}{2}\ \Big)$$

and

$$N\Big(4f\Big(\frac{x+y}{2}\Big)+4f\Big(\frac{x-y}{2}\Big)-3f(x)+f(-x)-f(y)-f(-y)-\rho[f(x+y)+f(x-y) - 2f(x)-f(y)-f(-y)\Big], t\Big) \geq \frac{t}{t+\phi(x,y)}, \quad (\rho \neq 0, 1, 2)$$

in fuzzy Banach spaces.

1. Introduction and preliminaries

The concept of a fuzzy norm on a linear space was introduced by Katsaras [12] in 1984. Later, Cheng and Mordeson [3] gave a new definition of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [14].

Definition 1.1. Let X be a real vector space. A function $N : X \times \mathbb{R} \longrightarrow [0,1]$ is called *a fuzzy norm on X* if for all $x, y \in X$ and all $c, s, t \in \mathbb{R}$,

(N1) $N(x,\ t) = 0$ for all $t \leq 0$;
(N2) $x = 0$ if and only if $N(x,\ t) = 1$ for all $t > 0$;
(N3) $N(cx,\ t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
(N4) $N(x+y,\ s+t) \geq min\{N(x,s),\ N(y,t)\}$;
(N5) $N(x,\cdot)$ is a nondecreasing function on $\mathbb{R}$ and $\lim_{t\to\infty} N(x,\ t) = 1$;
(N6) for any $x \neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

In this case, the pair (X, N) is called *a fuzzy normed space.*

Let (X, N) be a fuzzy normed space. A sequence $\{x_n\}$ in X is said to be *convergent in* (X, N) if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called *the limit of the sequence* $\{x_n\}$ *in* X and one denotes it by $N - \lim_{n\to\infty} x_n = x$. A sequence $\{x_n\}$ in X is said to be *Cauchy in* (X, N) if for any $\epsilon > 0$, $t > 0$, there is an $m \in N$ such that for any $n \geq m$ and any positive integer p, $N(x_{n+p} - x_n, t) > 1 - \epsilon$ for all $t > 0$. It is well known that every convergent sequence in a fuzzy normed space is Cauchy. A fuzzy normed space

2010 *Mathematics Subject Classification.* 39B62, 39B72, 54A40, 47H10.

Key words and phrases. Hyers-Ulam stability, additive-quadratic ρ-functional inequality, fuzzy normed space, fixed point theorem.

* Corresponding author.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 24, NO.6, 2018, COPYRIGHT 2018 EUDOXUS PRESS, LLC

is said to be *complete* if each Cauchy sequence in it is convergent and a complete fuzzy normed space is called *a fuzzy Banach space*.

In 1940, Ulam proposed the following stability problem (cf. [24]):

"Let G_1 be a group and G_2 a metric group with the metric d. Given a constant $\delta > 0$, does there exist a constant $c > 0$ such that if a mapping $f : G_1 \longrightarrow G_2$ satisfies $d(f(xy), f(x)f(y)) < c$ for all $x, y \in G_1$, then there exists a unique homomorphism $h : G_1 \longrightarrow G_2$ with $d(f(x), h(x)) < \delta$ for all $x \in G_1$?"

In the next year, Hyers [11] gave a partial solution of Ulam's problem for the case of approximate additive mappings. Subsequently, his result was generalized by Aoki ([1]) for additive mappings, and by Rassias [23] for linear mappings, to consider the stability problem with unbounded Cauchy differences. A generalization of the Rassias theorem was obtained by Găvruta [9] by replacing the unbounded Cauchy difference by a general control function in the spirit of the Rassias' approach. During the last decades, the stability problems of functional equations have been extensively investigated by a number of mathematicians ([4], [5], [6], [17]).

In 2008, for the first time, Mirmostafaee and Moslehian [15], [16] used the definition of a fuzzy norm in [2] to obtain a fuzzy version of the stability for the Cauchy functional equation

$$f(x + y) = f(x) + f(y) \tag{1.1}$$

and the quadratic functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y). \tag{1.2}$$

Glányi [10] and Fechner [8] proved the Hyers-Ulam stability of the following functional inequality

$$\|2f(x) + 2f(y) - f(x - y)\| \leq \|f(x + y)\|.$$

Park, Cho, and Han [22] proved the generalized Hyers-Ulam stability of the following functional inequalities associated with the following Cauchy additive functional inequality:

$$\|f(x) + f(y) + f(z)\| \leq \|f(x + y + z)\|.$$

and the following Cauchy-Jesen additive functional inequality:

$$\|f(x) + f(y) + f(z)\| \leq \|2f(\frac{x + y}{2} + z)\|.$$

Park [19, 20, 21] defined additive ρ-functional inequalities and proved the Hyers-Ulam stability of the additive ρ-functional inequalities.

Now, we consider the following fixed point theorem on generalized metric spaces.

Theorem 1.2. [7] *Let (X, d) be a complete generalized metric space and let $J : X \longrightarrow X$ be a strictly contractive mapping with some Lipschitz constant L with $0 < L < 1$. Then for each given element $x \in X$, either $d(J^n x, J^{n+1}x) = \infty$ for all nonnegative integers n or there exists a positive integer n_0 such that*
(1) *$d(J^n x, J^{n+1}x) < \infty$ for all $n \geq n_0$;*
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J ;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$* and
(4) *$d(y, y^*) \leq \dfrac{1}{1 - L}\, d(y, Jy)$ for all $y \in Y$.*

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 24, NO.6, 2018, COPYRIGHT 2018 EUDOXUS PRESS, LLC

In this paper, we investigate the solution of following additive-quadratic ρ-functional inequalities

$$\begin{aligned} &N\Big(f(x+y)+f(x-y)-2f(x)-f(y)-f(-y) \\ &-\rho\Big[4f\Big(\frac{x+y}{2}\Big)+4f\Big(\frac{x-y}{2}\Big)-3f(x)+f(-x)-f(y)-f(-y)\Big],t\Big) \\ &\geq \frac{t}{t+\phi(x,y)} \ \Big(\rho\neq 1\frac{1}{2}\ \Big), \end{aligned} \tag{1.3}$$

$$\begin{aligned} &N\Big(4f\Big(\frac{x+y}{2}\Big)+4f\Big(\frac{x-y}{2}\Big)-3f(x)+f(-x)-f(y)-f(-y) \\ &-\rho[f(x+y)+f(x-y)-2f(x)-f(y)-f(-y)\Big],t\Big) \\ &\geq \frac{t}{t+\phi(x,y)} \ (\rho\neq 0,1,2), \end{aligned} \tag{1.4}$$

and prove the generalized Hyers-Ulam stability for them in fuzzy Banach spaces.

Throughout this paper, we assume that $(X,\|\cdot\|)$ is a linear space and (Y,N) is a fuzzy Banach space.

2. Solutions and stability of (1.3) and (1.4)

In this section, we investigate the solution and prove the generalized Hyers-Ulam stability of the ρ-functional inequalities (1.3) and (1.4) in fuzzy Banach spaces. For any mapping $f : X \longrightarrow Y$, let

$$f_o(x)=\frac{f(x)-f(-x)}{2}, \ \ f_e(x)=\frac{f(x)+f(-x)}{2},$$

$$\begin{aligned} D_1f(x,y) &= f(x+y)+f(x-y)-2f(x)-f(y)-f(-y) \\ &-\rho\Big[4f\Big(\frac{x+y}{2}\Big)+4f\Big(\frac{x-y}{2}\Big)-3f(x)+f(-x)-f(y)-f(-y)\Big] \end{aligned}$$

and

$$\begin{aligned} D_2f(x,y) &= 4f\Big(\frac{x+y}{2}\Big)+4f\Big(\frac{x-y}{2}\Big)-3f(x)+f(-x)-f(y)-f(-y) \\ &-\rho[f(x+y)+f(x-y)-2f(x)-f(y)-f(-y)\Big]. \end{aligned}$$

Lemma 2.1. *Let $\rho\neq 1,\frac{1}{2}(\rho\neq 0,1,2,$ resp.) A mapping $f : X \longrightarrow Y$ saisfies $f(0)=0$ and $D_1f(x,y)=0$ $(D_2f(x,y)=0,$ resp.) if and only if f is an additive-quadratic mapping.*

Proof. Suppose that f satisfies $f(0)=0$ and $D_1f(x,y)=0$. Setting $y=x$ in $D_1f(x,y)=0$, we have

$$f(2x)=3f(x)-f(-x)$$

for all $x\in X$ and so we get

$$f\Big(\frac{x}{2}\Big)=\frac{3}{8}f(x)-\frac{1}{8}f(-x)$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 24, NO.6, 2018, COPYRIGHT 2018 EUDOXUS PRESS, LLC

for all $x \in X$. Hence we have

$$\begin{aligned}
&f(x+y)+f(x-y)-2f(x)-f(y)-f(-y)\\
=&\rho\Big[4f\Big(\frac{x+y}{2}\Big)+4f\Big(\frac{x-y}{2}\Big)-3f(x)+f(-x)-f(y)-f(-y)\Big]\\
=&\rho\Big[\frac{3}{2}f(x+y)+\frac{3}{2}f(x-y)-\frac{1}{2}f(-x-y)-\frac{1}{2}f(y-x)-3f(x)+f(-x)-f(y)-f(-y)\Big]
\end{aligned}$$

for all $x \in X$ and thus

$$(1-2\rho)[f_o(x+y)+f_o(x-y)-2f_o(x)]=0,$$

and

$$(1-\rho)[f_e(x+y)+f_e(x-y)-2f_e(x)-2f_e(y)]=0.$$

Since $\rho \neq 1, \frac{1}{2}$, f_o is additive and f_e is quadratic and thus $f = f_o + f_e$ is additive-quadratic. The converse is trivial.

The proof for $D_2 f(x,y) = 0$ is similar to that for $D_1 f(x,y) = 0$. □

Now, we will prove the generalized Hyers-Ulam stability for (1.3) in fuzzy normed spaces.

Theorem 2.2. *Assume that $\phi : X^3 \longrightarrow Z$ is a function such that*

$$\phi(x,y) \leq \frac{L}{4}\phi(2x,2y) \tag{2.1}$$

for all x, y and some L with $0 < L < 1$. Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and

$$N(D_1 f(x,y),t) \geq \frac{t}{t+\phi(x,y)} \tag{2.2}$$

for all $x, y \in X$ and all $t > 0$. Suppose that $\rho \neq 1, \frac{1}{2}$. Then there exists a unique additive-quadratic mapping $F : X \longrightarrow Y$ such that

$$N(f(x)-F(x),t) \geq \min\Big\{\frac{Lt}{Lt+4(1-L)\phi(x,x)}, \frac{Lt}{Lt+4(1-L)\phi(-x,-x)}\Big\} \tag{2.3}$$

for all $x \in X$ and all $t > 0$. Moreover

$$F_o(x) = N-\lim_{n\to\infty} 2^n f_o\Big(\frac{x}{2^n}\Big),\ \ F_e(x) = N-\lim_{n\to\infty} 2^{2n} f_e\Big(\frac{x}{2^n}\Big) \tag{2.4}$$

for all $x \in X$.

Proof. Consider the set $S = \{g \mid g : X \longrightarrow Y\}$ and the generalized metric d on S defined by

$$\begin{aligned}
d(g,h) =& \inf\{c \in [0,\infty) \mid N(g(x)-h(x),ct)\\
&\geq \min\Big\{\frac{t}{t+\phi(x,x)}, \frac{t}{t+\phi(-x,-x)}\Big\},\ \forall x \in X,\ \forall t > 0\}.
\end{aligned}$$

Then (S,d) is a complete metric space(see [18]). Define a mapping $J : S \longrightarrow S$ by

$$Jg(x) = 3g\Big(\frac{x}{2}\Big) + g\Big(-\frac{x}{2}\Big)$$

for all $x \in X$ and all $g \in S$.

Let $g, h \in S$ and $d(g,h) \le c$ for some $c \in [0, \infty)$. Then by (2.1), we have

$$\begin{aligned} N(Jg(x) - Jh(x), cLt) &= N\Big(3g\Big(\frac{x}{2}\Big) + g\Big(-\frac{x}{2}\Big) - 3h\Big(\frac{x}{2}\Big) - h\Big(-\frac{x}{2}\Big), cLt\Big) \\ &\ge \min\Big\{N\Big(g\Big(\frac{x}{2}\Big) - h\Big(\frac{x}{2}\Big), \frac{1}{4}cLt\Big), N\Big(g(-\frac{x}{2}\Big) - h\Big(-\frac{x}{2}\Big), \frac{1}{4}cLt\Big)\Big\} \\ &\ge \min\Big\{\frac{\frac{Lt}{4}}{\frac{Lt}{4} + \phi\Big(\frac{x}{2}, \frac{x}{2}\Big)}, \frac{\frac{Lt}{4}}{\frac{Lt}{4} + \phi\Big(-\frac{x}{2}, -\frac{x}{2}\Big)}\Big\} \\ &\ge \min\Big\{\frac{t}{t + \phi(x,x)}, \frac{t}{t + \phi(-x,-x)}\Big\} \end{aligned}$$

for all $x \in X$. Hence we have $d(Jg, Jh) \le Ld(g,h)$ for any $g, h \in S$ and so J is a strictly contractive mapping.

Putting $y = x$ in (2.2), we get

$$N(f(2x) - 3f(x) - f(-x), t) \ge \frac{t}{t + \phi(x,x)} \tag{2.5}$$

for all $x \in X$, $t > 0$ and hence

$$N\Big(f(x) - 3f\Big(\frac{x}{2}\Big) - f\Big(-\frac{x}{2}\Big), t\Big) \ge \frac{t}{t + \phi\Big(\frac{x}{2}, \frac{x}{2}\Big)} \ge \frac{\frac{4}{L}t}{\frac{4}{L}t + \phi(x,x)}$$

for all $x \in X$ and all $t > 0$. Thus we have

$$N\Big(f(x) - Jf(x), \frac{L}{4}t\Big) \ge \frac{t}{t + \phi(x,x)} \ge \min\Big\{\frac{t}{t + \phi(x,x)}, \frac{t}{t + \phi(-x,-x)}\Big\}$$

for all $x \in X$, $t > 0$ and so we have $d(f, Jf) \le \frac{L}{4} < \infty$. By Theorem 1.2, there exists a mapping $F : X \longrightarrow Y$ which is a fixed point of J such that $d(J^n f, F) \to 0$ as $n \to \infty$. By induction, we have

$$J^n f(x) = \frac{2^n(2^n + 1)}{2} f\Big(\frac{x}{2^n}\Big) + \frac{2^n(2^n - 1)}{2} f\Big(-\frac{x}{2^n}\Big)$$

for all $x \in X$ and all $n \in \mathbb{N}$. Hence we have

$$N - \lim_{n\to\infty} \Big[\frac{2^n(2^n + 1)}{2} f\Big(\frac{x}{2^n}\Big) + \frac{2^n(2^n - 1)}{2} f\Big(-\frac{x}{2^n}\Big)\Big] = F(x) \tag{2.6}$$

for all $x \in X$ and so we get (2.4). Replacing x, y, and t by $\frac{x}{2^n}$, $\frac{y}{2^n}$, and $\frac{t}{2^{2n}}$ in (2.2), respectively, by (2.2), (N3), and (N4), we have

$$\begin{aligned} &N\Big(2^{2n} D_1 f_e\Big(\frac{x}{2^n}, \frac{y}{2^n}\Big), 2^{2n} t\Big) \\ &\ge \min\Big\{N\Big(2^{2n} D_1 f\Big(\frac{x}{2^n}, \frac{y}{2^n},\Big), 2^{2n} t\Big), N\Big(2^{2n} D_1 f\Big(-\frac{x}{2^n}, -\frac{y}{2^n}\Big), 2^{2n} t\Big)\Big\} \\ &\ge \min\Big\{\frac{t}{t + \phi\Big(\frac{x}{2^n}, \frac{y}{2^n}\Big)}, \frac{t}{t + \phi\Big(-\frac{x}{2^n}, -\frac{y}{2^n}\Big)}\Big\} \end{aligned}$$

for all $x, y \in X$ and all $n \in \mathbb{N}$. Hence

$$N\Big(2^{2n} D_1 f_e\Big(\frac{x}{2^n}, \frac{y}{2^n}\Big), t\Big) \ge \min\Big\{\frac{t}{t + L^n \phi(x,y)}, \frac{t}{t + L^n \phi(-x,-y)}\Big\} \tag{2.7}$$

for all $x, y \in X$, all $t > 0$, and all $n \in \mathbb{N}$. Letting $n \to \infty$ in (2.7), by (2.4),

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 24, NO.6, 2018, COPYRIGHT 2018 EUDOXUS PRESS, LLC

$$D_1F_e(x, y) = 0$$

for all $x, y \in X$. Similarly, we have

$$D_1F_o(x, y) = 0$$

for all $x, y \in X$. Hence

$$D_1F(x, y) = 0$$

for all $x, y \in X$ and by Lemma 2.1, F is an additive-quadratic mapping. Since $d(f, Jf) \le \frac{L}{4}$, by Theorem 1.2, we have (2.3).

Now, we show the uniqueness of F. Let G be an additive-quadratic mapping with (2.3). Then clearly, G is a fixed point of J and

$$d(Jf, G) = d(Jf, JG) \le Ld(f, G) \le \frac{L}{4(1-L)} < \infty \tag{2.8}$$

and hence by (3) in Theorem 1.2, $F = G$. □

As examples of $\phi(x, y)$ in Theorem 2.2, we can take $\phi(x, y) = \epsilon(\|x\|^{2p} + \|y\|^{2p} + \|x\|^p\|y\|^p)$. Then we can formulate the following corollary

Corollary 2.3. *Let $\epsilon \ge 0$ and p be a real number with $1 < p$. Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and*

$$N(D_1f(x, y), t) \ge \frac{t}{t + \epsilon(\|x\|^{2p} + \|y\|^{2p} + \|x\|^p\|y\|^p)}$$

for all $x, y \in X$ and all $t > 0$. Suppose that $\rho \ne 1, \frac{1}{2}$. Then there exists a unique additive-quadratic mapping $F : X \longrightarrow Y$ such that

$$N(f(x) - F(x), t) \ge \frac{t}{t + 3\epsilon(2^{2p} - 4)\|x\|^{2p}}$$

for all $x \in X$ and all $t > 0$. Moreover

$$F_o(x) = N - \lim_{n\to\infty} 2^n f_o\Big(\frac{x}{2^n}\Big),\ \ F_e(x) = N - \lim_{n\to\infty} 2^{2n} f_e\Big(\frac{x}{2^n}\Big)$$

for all $x \in X$.

Related with Theorem 2.2, we can also have the following theorem. And the proof is similar to that of Theorem 2.2.

Theorem 2.4. *Assume that $\phi : X^3 \longrightarrow Z$ is a function such that*

$$\phi(x, y) \le 2L\phi\Big(\frac{x}{2}, \frac{y}{2}\Big) \tag{2.9}$$

for all x, y and some L with $0 < L < 1$. Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and

$$N(D_1f(x, y), t) \ge \frac{t}{t + \phi(x, y)} \tag{2.10}$$

for all $x, y \in X$ and all $t > 0$. Suppose that $\rho \ne 1, \frac{1}{2}$. Then there exists a unique additive-quadratic mapping $F : X \longrightarrow Y$ such that

$$N(f(x) - F(x), t) \ge \min\Big\{\frac{t}{t + 2(1-L)\phi(x, x)}, \frac{t}{t + 2(1-L)\phi(-x, -x)}\Big\} \tag{2.11}$$

for all $x \in X$ and all $t > 0$. Moreover

$$F_o(x) = N - \lim_{n\to\infty} \frac{1}{2^n} f_o(2^n x), \quad F_e(x) = N - \lim_{n\to\infty} \frac{1}{2^{2n}} f_e(2^n x)$$

for all $x \in X$.

Proof. Consider the set $S = \{g \mid g : X \longrightarrow Y\}$ and the generalized metric d on S defined by

$$\begin{aligned} d(g,h) = &\inf\{c \in [0,\infty) \mid N(g(x) - h(x), ct) \\ &\geq \min\Big\{\frac{t}{t+\phi(x,x)}, \frac{t}{t+\phi(-x,-x)}\Big\}, \ \forall x \in X, \ \forall t > 0\}. \end{aligned}$$

Then (S,d) is a complete metric space(see [18]). Define a mapping $J : S \longrightarrow S$ by

$$Jg(x) = \frac{3}{8}g(2x) - \frac{1}{8}g(-2x)$$

for all $x \in X$ and all $g \in S$. Let $g, h \in S$ and $d(g,h) \leq c$ for some $c \in [0,\infty)$. Then by (2.9), we have

$$\begin{aligned} N(Jg(x) - Jh(x), cLt) &= N\Big(\frac{3}{8}g(2x) - \frac{1}{8}g(-2x) - \frac{3}{8}h(2x) + \frac{1}{8}h(-2x), cLt\Big) \\ &\geq \min\Big\{N\Big(\frac{3}{8}\Big[g(2x) + h(2x)\Big], \frac{3}{4}cLt\Big), N\Big(\frac{1}{8}\Big[g(-2x) + h(-2x)\Big], \frac{1}{4}cLt\Big)\Big\} \\ &\geq \min\Big\{\frac{2Lt}{2Lt + \phi(2x,2x)}, \frac{2Lt}{2Lt + \phi(-2x,-2x)}\Big\} \\ &\geq \min\Big\{\frac{t}{t+\phi(x,x)}, \frac{t}{t+\phi(-x,-x)}\Big\} \end{aligned}$$

for all $x \in X$. Hence we have $d(Jg, Jh) \leq Ld(g,h)$ for any $g, h \in S$ and so J is a strictly contractive mapping. By (2.5), we have

$$\begin{aligned} &N\Big(f(x) - Jf(x), \frac{1}{2}t\Big) \\ &= N\Big(\frac{3}{8}\Big[f(2x) - 3f(x) - f(-x)\Big] - \frac{1}{8}\Big[f(-2x) - 3f(-x) - f(x)\Big], \frac{1}{2}t\Big) \\ &\geq \min\{N(f(2x) - 3f(x) - f(-x), t), N(f(-2x) - 3f(-x) - f(x), t)\} \\ &\geq \min\Big\{\frac{t}{t+\phi(x,x)}, \frac{t}{t+\phi(-x,-x)}\Big\} \end{aligned}$$

for all $x \in X$, $t > 0$ and so we have $d(f, Jf) \leq \frac{1}{2} < \infty$. By Theorem 1.2, there exists a mapping $F : X \longrightarrow Y$ which is a fixed point of J such that $d(J^n f, F) \to 0$ as $n \to \infty$. The rest of the proof is similar to that of Theorem 2.2. □

As examples of $\phi(x,y)$ in Theorem 2.4, we can take $\phi(x,y) = \epsilon(\|x\|^{2p} + \|y\|^{2p} + \|x\|^p\|y\|^p)$. Then we can formulate the following corollary

Corollary 2.5. *Let $\epsilon \geq 0$ and p be a real number with $0 < p < \frac{1}{2}$. Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and*

$$N(D_1 f(x,y), t) \geq \frac{t}{t + \epsilon(\|x\|^{2p} + \|y\|^{2p} + \|x\|^p\|y\|^p)}$$

for all $x, y \in X$ and all $t > 0$. Suppose that $\rho \neq 1, \frac{1}{2}$. Then there exists a unique additive-quadratic mapping $F : X \longrightarrow Y$ such that

$$N(f(x) - F(x), t) \geq \frac{t}{t + 3\epsilon(2^{2p} - 2)\|x\|^{2p}}$$

for all $x \in X$ and all $t > 0$. Moreover

$$F_o(x) = N - \lim_{n\to\infty} \frac{1}{2^n} f_o(2^n x), \ \ F_e(x) = N - \lim_{n\to\infty} \frac{1}{2^{2n}} f_e(2^n x)$$

for all $x \in X$.

Now, we will prove the stability of the functional inequality (1.4) in fuzzy Banach spaces.

Theorem 2.6. *Assume that $\phi : X^3 \longrightarrow Z$ is a function such that*

$$\phi(x, y) \leq \frac{L}{4}\phi(2x, 2y) \tag{2.12}$$

for all x, y and some L with $0 < L < 1$. Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and

$$N(D_2 f(x, y), t) \geq \frac{t}{t + \phi(x, y)} \tag{2.13}$$

for all $x, y \in X$ and all $t > 0$. Suppose that $\rho \neq 0, 1, 2$. Then there exists a unique additive-quadratic mapping $F : X \longrightarrow Y$ such that

$$N\Big(f(x) - F(x), t\Big) \geq \min\Big\{\frac{Lt}{Lt + 4\rho(1 - L)\phi(x, x)}, \frac{Lt}{Lt + 4\rho(1 - L)\phi(-x, -x)}\Big\} \tag{2.14}$$

for all $x \in X$ and all $t > 0$. Moreover

$$F_o(x) = N - \lim_{n\to\infty} 2^n f_o\Big(\frac{x}{2^n}\Big), \ \ F_e(x) = N - \lim_{n\to\infty} 2^{2n} f_e\Big(\frac{x}{2^n}\Big)$$

for all $x \in X$.

Proof. Consider the set $S = \{g \mid g : X \longrightarrow Y\}$ and the generalized metric d on S defined by

$$\begin{aligned} d(g, h) = &\inf\{c \in [0, \infty) \mid N(g(x) - h(x), ct) \\ &\geq \min\Big\{\frac{t}{t + \phi(x, x)}, \frac{t}{t + \phi(-x, -x)}\Big\}, \ \forall x \in X, \ \forall t > 0\}. \end{aligned}$$

Then (S, d) is a complete metric space(see [18]). Define a mapping $J : S \longrightarrow S$ by

$$Jg(x) = 3g\Big(\frac{x}{2}\Big) + g\Big(-\frac{x}{2}\Big)$$

for all $x \in X$ and all $g \in S$.

Let $g, h \in S$ and $d(g,h) \le c$ for some $c \in [0,\infty)$. Then by (2.12), we have

$$\begin{aligned} N(Jg(x)-Jh(x), cLt) &= N\Big(3g\Big(\frac{x}{2}\Big)+g\Big(-\frac{x}{2}\Big)-3h\Big(\frac{x}{2}\Big)-h\Big(-\frac{x}{2}\Big), cLt\Big) \\ &\ge \min\Big\{N\Big(g\Big(\frac{x}{2}\Big)-h\Big(\frac{x}{2}\Big), \frac{1}{4}cLt\Big), N\Big(g(-\frac{x}{2}\Big)-h\Big(-\frac{x}{2}\Big), \frac{1}{4}cLt\Big)\Big\} \\ &\ge \min\Big\{\frac{\frac{Lt}{4}}{\frac{Lt}{4}+\phi\Big(\frac{x}{2},\frac{x}{2}\Big)}, \frac{\frac{Lt}{4}}{\frac{Lt}{4}+\phi\Big(-\frac{x}{2},-\frac{x}{2}\Big)}\Big\} \\ &\ge \min\Big\{\frac{t}{t+\phi(x,x)}, \frac{t}{t+\phi(-x,-x)}\Big\} \end{aligned}$$

for all $x \in X$. Hence we have $d(Jg, Jh) \le Ld(g,h)$ for any $g, h \in S$ and so J is a strictly contractive mapping.

Putting $y = x$ in (2.13), we get

$$N(\rho[f(2x)-3f(x)-f(-x)], t) \ge \frac{t}{t+\phi(x,x)} \tag{2.15}$$

for all $x \in X$, $t > 0$ and hence

$$N(f(x)-Jf(x), \frac{t}{\rho}) \ge \frac{t}{t+\phi\Big(\frac{x}{2},\frac{x}{2}\Big)} \ge \frac{t}{t+\frac{L}{4}\phi(x,x)}$$

for all $x \in X$ and all $t > 0$. Thus we have

$$N\Big(f(x)-Jf(x), \frac{L}{4\rho}t\Big) \ge \frac{t}{t+\phi(x,x)} \ge \min\Big\{\frac{t}{t+\phi(x,x)}, \frac{t}{t+\phi(-x,-x)}\Big\}$$

for all $x \in X$, $t > 0$ and so we have $d(f, Jf) \le \frac{L}{4\rho} < \infty$. The rest of the proof is similar to that of Theorem 2.2. □

As examples of $\phi(x,y)$ in Theorem 2.6, we can take $\phi(x,y) = \epsilon(\|x\|^{2p}+\|y\|^{2p}+\|x\|^p\|y\|^p)$. Then we can formulate the following corollary

Corollary 2.7. *Let $\epsilon \ge 0$ and p be a real number with $1 < p$. Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and*

$$N(D_2f(x,y), t) \ge \frac{t}{t+\epsilon(\|x\|^{2p}+\|y\|^{2p}+\|x\|^p\|y\|^p)}$$

for all $x, y \in X$ and all $t > 0$. Suppose that $\rho \ne 0, 1, 2$. Then there exists a unique additive-quadratic mapping $F : X \longrightarrow Y$ such that

$$N(f(x)-F(x), t) \ge \frac{t}{t+3\epsilon\rho(2^{2p}-4)\|x\|^{2p}}$$

for all $x \in X$ and all $t > 0$. Moreover

$$F_o(x) = N - \lim_{n\to\infty} 2^n f_o\Big(\frac{x}{2^n}\Big), \ \ F_e(x) = N - \lim_{n\to\infty} 2^{2n} f_e\Big(\frac{x}{2^n}\Big)$$

for all $x \in X$.

Theorem 2.8. *Assume that $\phi : X^3 \longrightarrow Z$ is a function such that*

$$\phi(x,y) \le 2L\phi\Big(\frac{x}{2},\frac{y}{2}\Big) \tag{2.16}$$

for all x, y and some L with $0 < L < 1$. Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and

$$N(D_2 f(x,y), t) \geq \frac{t}{t + \phi(x,y)} \tag{2.17}$$

for all $x, y \in X$ and all $t > 0$. Suppose that $\rho \neq 0, 1, 2$. Then there exists a unique additive-quadratic mapping $F : X \longrightarrow Y$ such that

$$N(f(x) - F(x), t) \geq \min\left\{\frac{t}{t + 2\rho(1-L)\phi(x,x)}, \frac{t}{t + 2\rho(1-L)\phi(-x,-x)}\right\} \tag{2.18}$$

for all $x \in X$ and all $t > 0$. Moreover

$$F_o(x) = N - \lim_{n\to\infty} \frac{1}{2^n} f_o(2^n x), \quad F_e(x) = N - \lim_{n\to\infty} \frac{1}{2^{2n}} f_e(2^n x)$$

for all $x \in X$.

Proof. Consider the set $S = \{g \mid g : X \longrightarrow Y\}$ and the generalized metric d on S defined by

$$\begin{aligned} d(g,h) = &\inf\{c \in [0,\infty) \mid N(g(x) - h(x), ct) \\ &\geq \min\left\{\frac{t}{t + \phi(x,x)}, \frac{t}{t + \phi(-x,-x)}\right\}, \ \forall x \in X, \ \forall t > 0\}. \end{aligned}$$

Then (S, d) is a complete metric space(see [18]). Define a mapping $J : S \longrightarrow S$ by

$$Jg(x) = \frac{3}{8} g(2x) - \frac{1}{8} g(-2x)$$

for all $x \in X$ and all $g \in S$.

Let $g, h \in S$ and $d(g,h) \leq c$ for some $c \in [0,\infty)$. Then by (2.16), we have

$$\begin{aligned} N(Jg(x) - Jh(x), cLt) &= N\Big(\frac{3}{8} g(2x) - \frac{1}{8} g(-2x) - \frac{3}{8} h(2x) + \frac{1}{8} h(-2x), cLt\Big) \\ &\geq \min\Big\{N\Big(\frac{3}{8}\Big[g(2x) + h(2x)\Big], \frac{3}{4} cLt\Big), N\Big(\frac{1}{8}\Big[g(-2x) + h(-2x)\Big], \frac{1}{4} cLt\Big)\Big\} \\ &\geq \min\left\{\frac{2Lt}{2Lt + \phi(2x,2x)}, \frac{2Lt}{2Lt + \phi(-2x,-2x)}\right\} \\ &\geq \min\left\{\frac{t}{t + \phi(x,x)}, \frac{t}{t + \phi(-x,-x)}\right\} \end{aligned}$$

for all $x \in X$. Hence we have $d(Jg, Jh) \leq L d(g,h)$ for any $g, h \in S$ and so J is a strictly contractive mapping. By (2.15), we have

$$\begin{aligned} &N(\rho[f(x) - Jf(x)], \frac{t}{2}) \\ &= N\Big(\frac{3}{8}\rho\Big[f(2x) - 3f(x) - f(-x)\Big] - \frac{1}{8}\rho\Big[f(-2x) - 3f(-x) - f(x)\Big], \frac{t}{2}\Big) \end{aligned}$$

for all $x \in X$, $t > 0$ and so we have $d(f, Jf) \leq \frac{1}{2\rho} < \infty$. By Theorem 1.2, there exists a mapping $F : X \longrightarrow Y$ which is a fixed point of J such that $d(J^n f, F) \to 0$ as $n \to \infty$. The rest of the proof is similar to that of Theorem 2.2. □

As examples of $\phi(x,y)$ in Theorem 2.6, we can take $\phi(x,y) = \epsilon(\|x\|^{2p} + \|y\|^{2p} + \|x\|^p \|y\|^p)$. Then we can formulate the following corollary

Corollary 2.9. *Let $\epsilon \geq 0$ and p be a real number with $0 < p < \frac{1}{2}$. Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and*

$$N(D_2 f(x,y), t) \geq \frac{t}{t + \epsilon(\|x\|^{2p} + \|y\|^{2p} + \|x\|^p \|y\|^p)}$$

for all $x, y \in X$ and all $t > 0$. Suppose that $\rho \neq 0, 1, 2$. Then there exists a unique additive-quadratic mapping $F : X \longrightarrow Y$ such that

$$N(f(x) - F(x), t) \geq \frac{t}{t + 3\epsilon\rho(2 - 2^{2p})\|x\|^{2p}}$$

for all $x \in X$ and all $t > 0$. Moreover

$$F_o(x) = N - \lim_{n\to\infty} \frac{1}{2^n} f_o(2^n x), \quad F_e(x) = N - \lim_{n\to\infty} \frac{1}{2^{2n}} f_e(2^n x)$$

for all $x \in X$.

Acknowledgment

The first author was supported by the research fund of Dankook University in 2017.

References

[1] T. Aoki, *On the stability of the linear transformation in Banach space*, J.Math.Soc.Japan. **2** (1950), 64-66.
[2] T. Bag and S. K. Samanta, Finite dimensional fuzzy normed linear spaces, J. Fuzzy Math. **3**(2003), 687-705.
[3] S. C. Cheng and J. N. Mordeson, Fuzzy linear operator and fuzzy normed linear spaces, Bull. Calcutta Math. Soc. **86**(1994), 429-436.
[4] P. W. Cholewa, Remarkes on the stability of functional equations, Aequationes Math., **27**(1984), 76-86.
[5] K. Cieplinski, Applications of fixed point theorems to the hyers-ulam stability of functional equation-A survey, Ann. Funct. Anal. **3** (2012), 151-164.
[6] S. Czerwik, On the stability of the quadratic mapping in normed spaces, Abh. Math. Sem. Univ. Hamburg **62**(1992), 59-64.
[7] J. B. Diaz and B. Margolis, *A fixed point theorem of the alternative, for contractions on a generalized complete metric space*, Bulletin of the American Mathematical Society, **74**(1968), 305-309.
[8] W. Fechner, *Stability of a functional inequalty associated with the Jordan-Von Neumann functional equation*, Aequationes Mathematicae **71**(2006), 149-161.
[9] P. Găvruta, A generalization of the Hyer-Ulam-Rassias stability of approximately additive mappings, J. Math. Anal. Appl. **184**(1994), 431-436.
[10] A. Gilányi, *On a problem by K. Nikoden, Math. Inequal. and Appl.*, **5**(2002), 701-710.
[11] D. H. Hyers, On the stability of the linear functional equation, Proc. Nat. Acad. Sci. **27**(1941), 222-224.
[12] A. K. Katsaras, Fuzzy topological vector spaces II, Fuzzy Sets Syst, **12**(1984), 143-154.
[13] H. M. Kim, K. W. Jun, and E. Son, Hyers-Ulam stability of Jensen functional inequality in p-Banach spaces, Abstract and Applied Analysis, **2012**(2012), 1-16.
[14] I. Kramosil and J. Michalek, Fuzzy metric and statistical metric spaces, Kybernetica **11**(1975), 326-334.
[15] A. K. Mirmostafaee and M. S. Moslehian, Fuzzy almost quadratic functions, Results Math. **52**(2008), 161-177.
[16] A. K. Mirmostafaee and M. S. Moslehian, Fuzzy versions of Hyers-Ulam-Rassias theorem, Fuzzy Sets Syst. **159**(2008), 720-729.
[17] M. Mirzavaziri and M. S. Moslehian, A fixed point approach to stability of a quadratic equation, Bulletin of the Brazilian Mathematical Society **37**(2006), 361-376.
[18] M. S. Moslehian and T. H. Rassias, *Stability of functional equations in non-Archimedean spaces*, , Applicable Anal. Discrete Math. **1**(2007), 325-334.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 24, NO.6, 2018, COPYRIGHT 2018 EUDOXUS PRESS, LLC

[19] C. Park, *Additive ρ-functional inequalities and equations*, J. Math. Inequal. **9**(2015), 17-26.
[20] C. Park, *Additive ρ-functional inequalities in non-Archimedean normed spaces*, J. Math. Inequal. **9**(2015), 397-407.
[21] C. Park and S. Y. Jang, *Quadratic ρ-functional inequalities in fuzzy normed spaces*, J. Comput. Anal. Appl. **22**(2017), 527-537.
[22] C. Park, Y. S. Cho, and M. H. Han, *Functional inequalities associated with Jordan-von Neumann type additive functional equations*, J. Inequal. Appl. **2007**(2007).
[23] Th. M. Rassias, On the stability of the linear mapping in Banach sapces, Proc. Amer. Math. Sco. **72**(1978), 297-300.
[24] S. M. Ulam, A collection of mathematical problems, Interscience Publisher, New York, 1964.

Department of Mathematics Education, Dankook University, 152, Jukjeon-ro, Suji-gu, Yongin-si, Gyeonggi-do, 448-701, Korea
E-mail address: kci206@hanmail.net

Department of Mathematics Education, Dankook University, 152, Jukjeon-ro, Suji-gu, Yongin-si, Gyeonggi-do, 448-701, Korea
E-mail address: gilhan@dankook.ac.kr

Symmetric identities for Dirichlet-type multiple twisted q-l-function

Cheon Seoung Ryoo

Department of Mathematics, Hannam University, Daejeon 306-791, Korea

Abstract : In this paper we give some interesting symmetric identities for Dirichlet-type multiple twisted q-l-function in complex field.

Key words : Symmetric properties, power sums, Euler numbers and polynomials, twisted q-Euler numbers and polynomials, generalized twisted q-Euler polynomials of higher order, Dirichlet-type multiple twisted q-l-function.

2000 Mathematics Subject Classification : 11B68, 11S40, 11S80.

1. Introduction

Euler numbers and polynomials possess many interesting properties and arising in many areas of mathematics, mathematical physics and statistical physics. Many mathematicians have studied in the area of the q- extension of Euler numbers and polynomials(see [1, 2, 3, 5, 6, 7, 8, 9, 11, 13]). Recently, D. Kim *et al.*[4] derived some identities of symmetry for (h, q)-extension of higher-order Euler numbers and polynomials. D. V. Dolgy *et al.*[2] derived some identities of symmetry for higher-order generalized q-Euler polynomials. Y. He studied several identities of symmetry for Carlitz's q-Bernoulli numbers and polynomials in complex field(see [3]). In this paper, we establish some interesting symmetric identities for generalized twisted q-Euler polynomials of higher order in complex field.

The purpose of this paper is to present a systemic study of the generalized twisted q-Euler numbers and polynomials of higher-order by using the multiple q-Euler zeta function. Throughout this paper, the notations $\mathbb{N}, \mathbb{Z}, \mathbb{R}$, and $\mathbb{C}$ denote the sets of positive integers, integers, real numbers, and complex numbers, respectively, and $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. We assume that $q \in \mathbb{C}$ with $|q| < 1$. Throughout this paper we use the notation:

$$[x]_q = \frac{1 - q^x}{1 - q} \text{ (cf. [1, 2, 3, 5]) .}$$

Note that $\lim_{q \to 1}[x] = x$. Let χ be a Dirichlet character with conductor $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$ and ε be the p^N-th root of unity(see [10, 12, 13]).

In [5], T. Kim introduced the multiple q-Euler zeta function which interpolates higher-order q-Euler polynomials at negative integers as follows:

$$\zeta_{q,r}(s, x) = [2]_q^r \sum_{m_1, \cdots, m_r = 0}^{\infty} \frac{(-1)^{\sum_{j=1}^r m_j} q^{\sum_{j=1}^r m_j}}{[m_1 + \cdots + m_r + x]_q^s}, \tag{1}$$

where $s \in \mathbb{C}$ and $x \in \mathbb{R}$, with $x \neq 0, -1, -2, \ldots$.

Recently, D. V. Dolgy *et al.*[2] considered some symmetric identities for higher-order generalized q-Euler polynomials. The generalized Euler polynomials of order $r \in \mathbb{N}$ attached to χ are also defined by the generating function:

$$\left(2 \sum_{l=0}^{d-1} \frac{\chi(l)(-1)^l e^{(x+l)t}}{e^{dt} + 1} \right)^r = \sum_{m=0}^{\infty} E_{m,\chi}^{(r)}(x) \frac{t^m}{m!}. \tag{2}$$

When $x = 0, E^{(r)}_{n,\chi} = E^{(r)}_{n,\chi}(0)$ are called the generalized Euler numbers $E^{(r)}_{n,\chi}$ attached to χ(see [2, 4]).

For $h \in \mathbb{Z}, \alpha, k \in \mathbb{N}$, and $n \in \mathbb{Z}_+$, we introduced the higher order twisted q-Euler polynomials with weight α as follows(see [7]):

$$\widetilde{E}^{(\alpha)}_{n,q,\varepsilon}(h,k|x) = \frac{[2]^k_q}{(1-q^\alpha)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l \frac{q^{\alpha l x}}{(1+\varepsilon q^{\alpha l+h})\cdots(1+\varepsilon q^{\alpha l+h-k+1})}.$$

In the special case, $x = 0$, $\widetilde{E}^{(\alpha)}_{n,q,w}(h,k|0) = \widetilde{E}^{(\alpha)}_{n,q,w}(h,k)$ are called the higher-order twisted q-Euler numbers with weight α.

We consider the higher order generalized q-Euler polynomials of order r attached to χ twisted by ramified roots of unity as follows(see [10]):

$$\sum_{n=0}^{\infty} E^{(r)}_{n,\chi,\zeta,q}(x)\frac{t^n}{n!} = 2^r \sum_{m_1,\ldots,m_r=0}^{\infty}(-\zeta)^{\sum_{j=0}^{r} m_j}\left(\prod_{i=1}^{r}\chi(m_i)\right)e^{[x+\sum_{j=1}^{r} m_j]_q t}.$$

In the special case $x = 0$, the sequence $E^{(r)}_{n,\chi,\zeta,q}(0) = E^{(r)}_{n,\chi,\zeta,q}$ are called the n-th generalized q-Euler numbers of order r attached to χ twisted by ramified roots of unity.

As is well known, the higher-order generalized twisted q-Euler polynomials $E^{(k)}_{n,\chi,q,\varepsilon}(x)$ attached to χ are defined by the following generating function to be

$$\begin{aligned}\widetilde{F}^{(k)}_{\chi,q,\varepsilon}(t,x) &= [2]^k_q \sum_{m_1,\cdots,m_k=0}^{\infty}(-1)^{m_1+\cdots+m_k}\varepsilon^{m_1+\cdots+m_k}\left(\prod_{j=1}^{k}\chi(m_j)\right)e^{[m_1+\cdots+m_k+x]_q t} \\ &= \sum_{n=0}^{\infty} E^{(k)}_{n,\chi,q,\varepsilon}(x)\frac{t^n}{n!},\end{aligned} \tag{3}$$

where $k \in \mathbb{N}$. When $x = 0, E^{(h,k)}_{n,\chi,q,\varepsilon} = E^{(k)}_{n,\chi,q,\varepsilon}(0)$ are called the higher-order generalized twisted q-Euler numbers $E^{(k)}_{n,\chi,q,\varepsilon}$ attached to χ. Observe that if $q \to 1, \varepsilon \to 1$, then $E^{(k)}_{n,\chi,q,\varepsilon} \to E^{(k)}_{n,\chi}$ and $E^{(k)}_{n,\chi,q,\varepsilon}(x) \to E^{(k)}_{n,\chi}(x)$.

By using (3) and Cauchy product, we have

$$\begin{aligned}E^{(k)}_{n,\chi,q,\varepsilon}(x) &= \sum_{l=0}^{n}\binom{n}{l}q^{lx}E^{(k)}_{l,\chi,q,\varepsilon}[x]^{n-l}_q \\ &= (q^x E^{(k)}_{\chi,q,\varepsilon} + [x]_q)^n,\end{aligned} \tag{4}$$

with the usual convention about replacing $(E^{(k)}_{\chi,q,\varepsilon})^n$ by $E^{(k)}_{n,\chi,q,\varepsilon}$.

By using complex integral and (3), we can also obtain the Dirichlet-type multiple twisted q-l-function as follows:

$$\begin{aligned}l^{(k)}_{\chi,q,\varepsilon}(s,x) &= \frac{1}{\Gamma(s)}\int_0^\infty \widetilde{F}^{(k)}_{\chi,q,\varepsilon}(-t,x)t^{s-1}dt \\ &= [2]^k_q \sum_{m_1,\cdots,m_k=0}^{\infty}\frac{(-1)^{\sum_{j=1}^{k} m_j}\left(\prod_{j=1}^{k}\chi(m_j)\right)\varepsilon^{\sum_{j=1}^{k} m_j}}{[m_1+\cdots+m_k+x]^s_q},\end{aligned} \tag{5}$$

where $s \in \mathbb{C}$ and $x \in \mathbb{R}$, with $x \neq 0, -1, -2, \ldots$

By using Cauchy residue theorem, the value of Dirichlet-type multiple twisted q-l-function at negative integers is given explicitly by the following theorem:

Theorem 1. Let $k \in \mathbb{N}$ and $n \in \mathbb{Z}_+$. We obtain

$$l^{(k)}_{\chi,q,\varepsilon}(-n,x) = E^{(k)}_{n,\chi,q,\varepsilon}(x).$$

The purpose of this paper is to obtain some interesting identities of the power sums and the higher-order generalized twisted q-Euler polynomials $E^{(k)}_{n,\chi,q,\varepsilon}(x)$ attached to χ using the symmetric properties for Dirichlet-type multiple twisted q-l-function. In this paper, if we take $\chi^0 = 1, \varepsilon = 1$, then [4] is the special case of this paper. If we take $\varepsilon = 1$ in all equations of this article, then [2] are the special case of our results.

2. Symmetry identities for multiple twisted q-l-function

In this section, we investigate some symmetric identities for higher-order generalized twisted q-Euler polynomials $E^{(k)}_{n,\chi,q,\varepsilon}(x)$ attached to χ using the symmetric properties for Dirichlet-type multiple twisted q-l-function. We assume that χ is a Dirichlet character with conductor $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$ and ε be the p^N-th root of unity. Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$. For $k \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, we obtain certain symmetry identities for Dirichlet-type multiple twisted q-l-function.

Observe that $[xy]_q = [x]_{q^y}[y]_q$ for any $x, y \in \mathbb{C}$. In (5), we derive next result by substitute $w_2 x + \dfrac{w_2}{w_1}(j_1 + \cdots + j_k)$ for x in and replace q and ε by q^{w_1} and ε^{w_1}, respectively.

$$
\begin{aligned}
&\frac{1}{[2]^k_{q^{w_1}}} l^{(k)}_{\chi,q^{w_1},\varepsilon^{w_1}}(s, w_2x + \frac{w_2}{w_1}(j_1 + \cdots + j_k)) \\
&= \sum_{m_1,\cdots,m_k=0}^{\infty} \frac{(-1)^{\sum_{j=1}^k m_j} \left(\prod_{j=1}^k \chi(m_j)\right) \varepsilon^{w_1(m_1+\cdots+m_k)}}{[m_1 + \cdots + m_k + w_2x + \frac{w_2}{w_1}(j_1 + \cdots + j_k)]^s_{q^{w_1}}} \\
&= \sum_{m_1,\cdots,m_k=0}^{\infty} \frac{(-1)^{\sum_{j=1}^k m_j} \left(\prod_{j=1}^k \chi(m_j)\right) \varepsilon^{w_1(m_1+\cdots+m_k)}}{\left[\frac{w_1(m_1 + \cdots + m_k) + w_1w_2x + w_2(j_1 + \cdots + j_k)}{w_1}\right]^s_{q^{w_1}}} \\
&= [w_1]^s_q \sum_{m_1,\cdots,m_k=0}^{\infty} \sum_{i_1,\cdots,i_k=0}^{dw_2-1} \frac{(-1)^{\sum_{j=1}^k m_j} \left(\prod_{j=1}^k \chi(m_j)\right) \varepsilon^{w_1(m_1+\cdots+m_k)}}{[w_1(m_1 + \cdots + m_k) + w_1w_2x + w_2(j_1 + \cdots + j_k)]^s_q} \\
&= [w_1]^s_q \sum_{m_1,\cdots,m_k=0}^{\infty} \sum_{i_1,\cdots,i_k=0}^{dw_2-1} (-1)^{\sum_{j=1}^k m_j} (-1)^{\sum_{j=1}^k i_j} \left(\prod_{j=1}^k \chi(i_j)\right) \\
&\quad \times \varepsilon^{dw_1w_2\sum_{j=1}^k m_j} \varepsilon^{w_1\sum_{j=1}^k i_j} \\
&\quad \times \left([w_1w_2(x + dm_1 + \cdots + dm_k) + w_1(i_1 + \cdots + i_k) + w_2(j_1 + \cdots + j_k)]^s_q\right)^{-1}
\end{aligned} \tag{6}
$$

Thus, from (6), we can derive the following equation.

$$
\begin{aligned}
&\frac{[w_2]^s_q}{[2]^k_{q^{w_1}}} \sum_{j_1,\cdots,j_k=0}^{dw_1-1} (-1)^{\sum_{l=1}^k j_l} \left(\prod_{l=1}^k \chi(j_l)\right) \varepsilon^{w_2(j_1+\cdots+j_k)} \\
&\qquad \times l^{(k)}_{\chi,q^{w_1},\varepsilon^{w_1}}(s, w_2x + \frac{w_2}{w_1}(j_1 + \cdots + j_k)) \\
&= [w_1]^s_q[w_2]^s_q \sum_{m_1,\cdots,m_k=0}^{\infty} \sum_{i_1,\cdots,i_k=0}^{dw_2-1} \sum_{j_1,\cdots,j_k=0}^{dw_1-1} (-1)^{\sum_{l=1}^k (j_l+i_l+m_l)} \left(\prod_{l=1}^k \chi(j_l)\right) \left(\prod_{l=1}^k \chi(i_l)\right) \\
&\quad \times \varepsilon^{dw_1w_2\sum_{l=1}^k m_l} \varepsilon^{w_1\sum_{l=1}^k i_l} \varepsilon^{w_2\sum_{l=1}^k j_l} \\
&\quad \times \left([w_1w_2(x + dm_1 + \cdots + dm_k) + w_1(i_1 + \cdots + i_k) + w_2(j_1 + \cdots + j_k)]^s_q\right)^{-1}
\end{aligned} \tag{7}
$$

By using the same method as (7), we have

$$\begin{aligned}
&\frac{[w_1]_q^s}{[2]_{q^{w_2}}^k}\sum_{j_1,\cdots,j_k=0}^{dw_2-1}(-1)^{\sum_{l=1}^k j_l}\left(\prod_{l=1}^k\chi(j_l)\right)\varepsilon^{w_2(j_1+\cdots+j_k)}\\
&\quad\times l^{(k)}_{\chi,q^{w_2},\varepsilon^{w_2}}(s,w_1x+\frac{w_1}{w_2}(j_1+\cdots+j_k))\\
&=[w_1]_q^s[w_2]_q^s\sum_{m_1,\cdots,m_k=0}^{\infty}\sum_{j_1,\cdots,j_k=0}^{dw_2-1}\sum_{i_1,\cdots,i_k=0}^{dw_1-1}(-1)^{\sum_{l=1}^k(j_l+i_l+m_l)}\left(\prod_{l=1}^k\chi(j_l)\right)\left(\prod_{l=1}^k\chi(i_l)\right)\\
&\quad\times\varepsilon^{dw_1w_2\sum_{l=1}^k m_l}\varepsilon^{w_2\sum_{l=1}^k i_l}\varepsilon^{w_1\sum_{l=1}^k j_l}\\
&\quad\times\left([w_1w_2(x+dm_1+\cdots+dm_k)+w_1(j_1+\cdots+j_k)+w_2(i_1+\cdots+i_k)]_q^s\right)^{-1}
\end{aligned}\tag{8}$$

Therefore, by (7) and (8), we have the following theorem.

Theorem 2. Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$. For $k \in \mathbb{N}$, we have

$$\begin{aligned}
&[w_2]_q^s[2]_{q^{w_2}}^k\sum_{j_1,\cdots,j_k=0}^{dw_1-1}(-1)^{\sum_{l=1}^k j_l}\left(\prod_{l=1}^k\chi(j_l)\right)\varepsilon^{w_2(j_1+\cdots+j_k)}\\
&\quad\times l^{(k)}_{\chi,q^{w_1},\varepsilon^{w_1}}\left(s,w_2x+\frac{w_2}{w_1}(j_1+\cdots+j_k)\right)\\
&[w_1]_q^s[2]_{q^{w_1}}^k\sum_{j_1,\cdots,j_k=0}^{dw_2-1}(-1)^{\sum_{l=1}^k j_l}\left(\prod_{l=1}^k\chi(j_l)\right)\varepsilon^{w_1(j_1+\cdots+j_k)}\\
&\quad\times l^{(k)}_{\chi,q^{w_2},\varepsilon^{w_2}}\left(s,w_1x+\frac{w_1}{w_2}(j_1+\cdots+j_k)\right)
\end{aligned}\tag{9}$$

By (9) and Theorem 1, we obtain the following theorem.

Theorem 3. Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$. For $k \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, we obtain

$$\begin{aligned}
&[w_2]_q^s[2]_{q^{w_2}}^k\sum_{j_1,\cdots,j_k=0}^{dw_1-1}(-1)^{\sum_{l=1}^k j_l}\left(\prod_{l=1}^k\chi(j_l)\right)\varepsilon^{w_2(j_1+\cdots+j_k)}\\
&\quad\times E^{(k)}_{n,\chi,q^{w_1},\varepsilon^{w_1}}\left(w_2x+\frac{w_2}{w_1}(j_1+\cdots+j_k)\right)\\
&=[w_1]_q^s[2]_{q^{w_1}}^k\sum_{j_1,\cdots,j_k=0}^{dw_2-1}(-1)^{\sum_{l=1}^k j_l}\left(\prod_{l=1}^k\chi(j_l)\right)\varepsilon^{w_1(j_1+\cdots+j_k)}\\
&\quad\times E^{(k)}_{n,\chi,q^{w_2},\varepsilon^{w_2}}\left(w_1x+\frac{w_1}{w_2}(j_1+\cdots+j_k)\right).
\end{aligned}\tag{10}$$

From (4), we note that

$$\begin{aligned}
E^{(k)}_{n,\chi,q,\varepsilon}(x+y)&=(q^{x+y}E^{(k)}_{n,\chi,q,\varepsilon}+[x+y]_q)^n\\
&=\sum_{i=0}^{n}\binom{n}{i}q^{xi}E^{(k)}_{i,\chi,q,\varepsilon}(y)[x]_q^{n-i}.
\end{aligned}\tag{11}$$

with the usual convention about replacing $(E^{(k)}_{\chi,q,\varepsilon})^n$ by $E^{(k)}_{n,\chi,q,\varepsilon}$.

By (11), we have

$$\sum_{j_1,\cdots,j_k=0}^{dw_1-1}(-1)^{\sum_{l=1}^k j_l}\left(\prod_{l=1}^k\chi(j_l)\right)\varepsilon^{w_2(j_1+\cdots+j_k)}E^{(k)}_{n,\chi,q^{w_1},\varepsilon^{w_1}}\left(w_2x+\frac{w_2}{w_1}(j_1+\cdots+j_k)\right)$$
$$=\sum_{j_1,\cdots,j_k=0}^{dw_1-1}(-1)^{\sum_{l=1}^k j_l}\left(\prod_{l=1}^k\chi(j_l)\right)\varepsilon^{w_2(j_1+\cdots+j_k)}\sum_{i=0}^{n}\binom{n}{i}E^{(k)}_{i,\chi,q^{w_1},\varepsilon^{w_1}}(w_2x)\left[\frac{w_2}{w_1}(j_1+\cdots+j_k)\right]^{n-i}_{q^{w_1}}$$
$$=\sum_{j_1,\cdots,j_k=0}^{dw_1-1}(-1)^{\sum_{l=1}^k j_l}\left(\prod_{l=1}^k\chi(j_l)\right)\varepsilon^{w_2(j_1+\cdots+j_k)}\sum_{i=0}^{n}\binom{n}{i}E^{(k)}_{n-i,\chi,q^{w_1},\varepsilon^{w_1}}(w_2x)\left[\frac{w_2}{w_1}(j_1+\cdots+j_k)\right]^{i}_{q^{w_1}}$$

Hence we have the following theorem.

Theorem 4. Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$. For $k \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, we obtain

$$\sum_{j_1,\cdots,j_k=0}^{dw_1-1}(-1)^{\sum_{l=1}^k j_l}\left(\prod_{l=1}^k\chi(j_l)\right)\varepsilon^{w_2(j_1+\cdots+j_k)}E^{(k)}_{n,\chi,q^{w_1},\varepsilon^{w_1}}\left(w_2x+\frac{w_2}{w_1}(j_1+\cdots+j_k)\right)=$$
$$=\sum_{i=0}^{n}\binom{n}{i}[w_2]_q^i[w_1]_q^{-i}E^{(k)}_{n-i,\chi,q^{w_1},\varepsilon^{w_1}}(w_2x)\sum_{j_1,\cdots,j_k=0}^{dw_1-1}(-1)^{\sum_{l=1}^k j_l}\left(\prod_{l=1}^k\chi(j_l)\right)\varepsilon^{w_2(j_1+\cdots+j_k)}[j_1\cdots+j_k]^i_{q^{w_2}}.$$

For each integer $n \geq 0$, let

$$\mathcal{S}^{(k)}_{n,i,\chi,q,\varepsilon}(w)=\sum_{j_1,\cdots,j_k=0}^{w-1}(-1)^{(j_1+\cdots+j_k)}\left(\prod_{l=1}^k\chi(j_l)\right)\varepsilon^{(j_1+\cdots+j_k)}[j_1\cdots+j_k]^i_q.$$

The above sum $\mathcal{S}^{(k)}_{n,i,\chi,q,\varepsilon}(w)$ is called the alternating generalized q-power sums.

By Theorem 4, we have

$$\begin{aligned}&[2]^k_{q^{w_2}}[w_1]^n_q\sum_{j_1,\cdots,j_k=0}^{dw_1-1}(-1)^{\sum_{l=1}^k j_l}\left(\prod_{l=1}^k\chi(j_l)\right)\varepsilon^{w_2(j_1+\cdots+j_k)}\\&\quad\times E^{(k)}_{n,\chi,q^{w_1},\varepsilon^{w_1}}\left(w_2x+\frac{w_2}{w_1}(j_1+\cdots+j_k)\right)\\&=[2]^k_{q^{w_2}}\sum_{i=0}^{n}\binom{n}{i}[w_2]^i_q[w_1]^{n-i}_qE^{(k)}_{n-i,\chi,q^{w_1},\varepsilon^{w_1}}(w_2x)\mathcal{S}^{(k)}_{n,i,\chi,q^{w_2},\varepsilon^{w_2}}(dw_1)\end{aligned}\tag{12}$$

By using the same method as in (12), we have

$$\begin{aligned}&[2]^k_{q^{w_1}}[w_2]^n_q\sum_{j_1,\cdots,j_k=0}^{dw_2-1}(-1)^{\sum_{l=1}^k j_l}\left(\prod_{l=1}^k\chi(j_l)\right)\varepsilon^{w_1(j_1+\cdots+j_k)}\\&\quad\times E^{(k)}_{n,\chi,q^{w_2},\varepsilon^{w_2}}\left(w_1x+\frac{w_1}{w_2}(j_1+\cdots+j_k)\right)\\&=[2]^k_{q^{w_1}}\sum_{i=0}^{n}\binom{n}{i}[w_1]^i_q[w_2]^{n-i}_qE^{(k)}_{n-i,\chi,q^{w_2},\varepsilon^{w_2}}(w_1x)\mathcal{S}^{(k)}_{n,i,\chi,q^{w_1},\varepsilon^{w_1}}(dw_2)\end{aligned}\tag{13}$$

Therefore, by (12), (13), and Theorem 3, we have the following theorem.

Theorem 5. Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$. For $k \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, we obtain

$$\begin{aligned}&[2]^k_{q^{w_2}}\sum_{i=0}^{n}\binom{n}{i}[w_2]^i_q[w_1]^{n-i}_qE^{(k)}_{n-i,\chi,q^{w_1},\varepsilon^{w_1}}(w_2x)\mathcal{S}^{(k)}_{n,i,\chi,q^{w_2},\varepsilon^{w_2}}(dw_1)\\&=[2]^k_{q^{w_1}}\sum_{i=0}^{n}\binom{n}{i}[w_1]^i_q[w_2]^{n-i}_qE^{(k)}_{n-i,\chi,q^{w_2},\varepsilon^{w_2}}(w_1x)\mathcal{S}^{(k)}_{n,i,\chi,q^{w_1},\varepsilon^{w_1}}(dw_2).\end{aligned}$$

By Theorem 5, we obtain the interesting symmetric identity for the higher-order generalized twisted q-Euler numbers $E^{(k)}_{n,\chi,q,\varepsilon}$ in complex field.

Corollary 6. Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$. For $k \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, we obtain

$$\begin{aligned}
&[2]^k_{q^{w_2}} \sum_{i=0}^{n} \binom{n}{i} [w_2]^i_q [w_1]^{n-i}_q \mathcal{S}^{(k)}_{n,i,\chi,q^{w_2},\varepsilon^{w_2}}(dw_1) E^{(k)}_{n-i,\chi,q^{w_1},\varepsilon^{w_1}} \\
&= [2]^k_{q^{w_1}} \sum_{i=0}^{n} \binom{n}{i} [w_1]^i_q [w_2]^{n-i}_q \mathcal{S}^{(k)}_{n,i,\chi,q^{w_1},\varepsilon^{w_1}}(dw_2) E^{(k)}_{n-i,\chi,q^{w_2},\varepsilon^{w_2}}.
\end{aligned}$$

REFERENCES

1. M. Cenkci, The p-adic generalized twisted (h, q)-Euler-l-function and its applications, Adv. Stud. Contemp. Math., 15(2007), 34-47.

2. D. V. Dolgy, D.S. Kim, T.G. Kim, J.J. Seo, Identities of Symmetry for Higher-Order Generalized q-Euler Polynomials, Abstract and Applied Analysis, 2014(2014), Article ID 286239, 6 pages.

3. Yuan He, Symmetric identities for Carlitz's q-Bernoulli numbers and polynomials, Adv. Difference Equ., 246(2013), 10 pages.

4. D. Kim, T. Kim, J.-J. Seo, Identities of symmetric for (h, q)-extension of higher-order Euler polynomials, Applied Mathemtical Sciences 8 (2014), 3799-3808.

5. T. Kim, New approach to q-Euler polynomials of higher order, Russ. J. Math. Phys. 17(2010), 218-225.

6. T. Kim, Barnes type multiple q-zeta function and q-Euler polynomials, J. phys. A : Math. Theor. 43(2010) 255201(11pp).

7. H. Y. Lee, N. S. Jung, J. Y. Kang, C. S. Ryoo, Some identities on the higher-order-twisted q-Euler numbers and polynomials with weight α, Adv. Difference Equ., 2012:21(2012), 10pp.

8. E.-J. Moon, S.-H. Rim, J.-H. Jin, S.-J. Lee, On the symmetric properties of higher-order twisted q-Euler numbers and polynomials, Adv. Difference Equ., 2010, Art ID 765259, 8pp.

9. H. Ozden, Y. Simsek, I. N. Cangul, Euler polynomials associated with p-adic q-Euler measure, Gen. Math., 15(2007), 24-37.

10. C. S. Ryoo, On the generalized Barnes type multiple q-Euler polynomials twisted by ramified roots of unity, Proc. Jangjeon Math. Soc. 13(2010), 255-263.

11. C. S. Ryoo, A note on the weighted q-Euler numbers and polynomials, Adv. Stud. Contemp. Math., 21(2011), 47-54.

12. Y. Simsek, q-analogue of twisted l-series and q-twisted Euler numbers, Journal of Number Theory, 110(2005), 267-278.

13. Y. Simsek, Twisted (h, q)-Bernoulli numbers and polynomials related to twisted (h, q)-zeta function and L-function, J. Math. Anal. Appl., 324(2006), 790-804.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 24, NO.6, 2018, COPYRIGHT 2018 EUDOXUS PRESS, LLC

ON A q-ANALOGUE OF DEGENERATE λ-CHANGHEE POLYNOMIALS

SANG JO YUN[1], SANG-KI LEE[2], AND JIN-WOO PARK[3,*]

ABSTRACT. In this paper, we derive the q-analog of degenerate λ-Changhee polynomials, and found some new and interesting identities and properties of those numbers and polynomials.

1. INTRODUCTION

Let d be a fixed positive integer and let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will respectively denote the ring of p-adic rational integers, the field of p-adic rational numbers and the completions of algebraic closure of $\mathbb{Q}_p$. the p-adic norm is defined $|p|_p = \frac{1}{p}$

When one talks of q-extension, q is various considered as an indeterminate, a complex $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for each $x \in \mathbb{Z}_p$.

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the *p-adic fermionic integral on* $\mathbb{Z}_p$ is defined by Kim as follows :

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x \text{ (see [8, 9, 10]).} \quad (1.1)$$

If we put $f_1(x) = f(x+1)$, then, by (1.1), we can get the following well-known integral identity

$$qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0), \quad (1.2)$$

where $f'(0) = \frac{df(x)}{dx}|_{x=0}$.

It is well known that the *q-Euler polynomials of order k* are defined by the generating function to be

$$\left(\frac{[2]_q}{1+qe^t}\right)^k e^{xt} = \sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{n!}, \text{ (see [1, 5, 8, 9, 15, 18]).}$$

In the special case $x = 0$, $E_{n,q} = E_{n,q}(0)$ are called the *n-th q-Euler numbers.*

Recently, D. S. Kim et. al introduced the *Changhee polynomials* as follows :

$$Ch_n(x) = \int_{\mathbb{Z}_p} (x+y)_n d\mu_{-1}(y), (n \geq 0), \text{ (see [6, 13, 17]).}$$

2010 *Mathematics Subject Classification.* 11B68, 11S40, 11S80.

Key words and phrases. degenerate, q-Changhee polynomials, q-analogue of λ-Changhee polynomials.

* corresponding author.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 24, NO.6, 2018, COPYRIGHT 2018 EUDOXUS PRESS, LLC

When $x = 0$, $Ch_n = Ch_n(0)$ are called the *Changhee numbers*. In [12], authors defined the *q-Changhee polynomials* as follows.

$$\frac{1+q}{q(1+t)+1}(1+t)^x = \sum_{n=0}^{\infty} Ch_{n,q}(x)\frac{t^n}{n!}.$$

The *Stirling number of the first kind* is defined by

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l)x^l,\ (n \geq 0), \tag{1.3}$$

and the *Stirling numbers of the second kind* is defined by

$$(e^t-1)^n = n!\sum_{l=n}^{\infty} S_2(l,n)\frac{t^l}{l!}, \tag{1.4}$$

(see [3, 16]). Note that

$$(\log(x+1))^n = n!\sum_{l=n}^{\infty} S_1(l,n)\frac{x^l}{l!},\ (n \geq 0), \tag{1.5}$$

(see [3, 4, 6, 11-14]).

In [2], L. Carlitz consider the degenerate Bernoulli polynomials which are given by the generating function to be

$$\frac{t}{(1+\lambda t)^{\frac{x}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_n(x|\lambda)\frac{t^n}{n!}. \tag{1.6}$$

When $x = 0$, $\beta_n(\lambda) = \beta_n(0|\lambda)$ are called the *degenerate Bernoulli numbers*. Note that $\lim_{\lambda\to 0}\beta_n(\lambda) = B_n$.

It is well known that

$$e^t = \lim_{u\to 0}(1+ut)^{\frac{1}{u}},\ \text{(see [2, 5, 7])}.$$

The function $(1+ut)^{\frac{1}{u}}$ is called the *degenerate function of* e^t. Thus, for $t = \log e^t$, we have $\log(1+ut)^{\frac{1}{u}}$ as the degenerate function.

Recently, Changhee numbers and polynomials are introduced by Kim et. al., and found interesting identities by many researchers(see [6, 12, 13, 14, 17]).

In this paper, we derive the q-analog of degenerate λ-Changhee polynomials and found some new and interesting identities and properties of those numbers and polynomials.

2. λ-q-Changhee polynomials of the first kind

In this section, we assume that $u, t \in \mathbb{C}_p$ with $|ut|_p < p^{-\frac{1}{p-1}}$ and $\lambda \in \mathbb{Z}_p$.

Now, as a generalization of Changhee polynomials, we consider the *degenerate λ-q-Changhee polynomials* :

$$\frac{1+q}{1+q\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda}}\left(1+\frac{1}{u}\log(1+ut)\right)^x = \sum_{n=0}^{\infty} Ch_{n,\lambda,q}(x|u)\frac{t^n}{n!}. \tag{2.1}$$

When $x = 0$, $Ch_{n,\lambda,q}(u) = Ch_{n,\lambda,q}(0|u)$ are called the *degenerate λ-q-Changhee numbers*. It is easy to check that $\lim_{u\to 0} Ch_{n,\lambda,q}(x|u) = Ch_{n,\lambda,q}(x)$ and $Ch_{n,q} = Ch_{n,1,q}(0)$.

Let us take $f(x)=\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda x}$. From (1.2), we have

$$\int_{\mathbb{Z}_p}\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda x}d\mu_{-q}(x)=\frac{1+q}{1+q\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda}}. \tag{2.2}$$

By (2.1) and (2.2), we have

$$\begin{aligned}\sum_{n=0}^{\infty}Ch_{n,\lambda,q}(x|u)\frac{t^n}{n!}=&\frac{1+q}{1+q\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda}}\left(1+\frac{1}{u}\log(1+ut)\right)^{x}\\ =&\int_{\mathbb{Z}_p}\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda y+x}d\mu_{-q}(y),\end{aligned} \tag{2.3}$$

and, by (1.5),

$$\begin{aligned}&\int_{\mathbb{Z}_p}\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda y+x}d\mu_{-q}(y)\\ =&\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\binom{\lambda y+x}{n}u^{-n}\left(\log(1+ut)\right)^n d\mu_{-q}(y)\\ =&\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\binom{\lambda y+x}{n}u^{-n}n!\sum_{k=n}^{\infty}S_1(k,n)\frac{u^k t^k}{k!}d\mu_{-q}(y)\\ =&\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}u^{n-m}S_1(n,m)\int_{\mathbb{Z}_p}(\lambda y+x)_m d\mu_q(y)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.4}$$

Thus, by (2.3) and (2.4), we obtain the following theorem.

Theorem 2.1. *For $n\geq 0$, we have*

$$Ch_{n,\lambda,q}(x|u)=\sum_{m=0}^{n}u^{n-m}S_1(n,m)\int_{\mathbb{Z}_p}(\lambda y+x)_m d\mu_{-q}(y).$$

By replacing t by $\frac{1}{u}\left(e^{ut}-1\right)$ in (2.1), we get

$$\begin{aligned}\frac{1+q}{q(1+t)^{\lambda}+1}(1+t)^x=&\sum_{n=0}^{\infty}Ch_{n,\lambda,q}(x|u)\frac{1}{n!}\left(\frac{1}{u}\left(e^{ut}-1\right)\right)^n\\ =&\sum_{n=0}^{\infty}Ch_{n,\lambda,q}(x|u)\frac{1}{n!}n!\sum_{l=n}^{\infty}S_2(l,n)u^{-n}\frac{(ut)^l}{l!}\\ =&\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}Ch_{m,\lambda,q}(x|u)u^{n-m}S_2(n,m)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.5}$$

On the other hands, by replacing t by $\frac{1}{u}\left(e^{u(e^t-1)}-1\right)$ in (2.1), we have

$$\begin{aligned}\frac{1+q}{1+qe^{\lambda t}}e^{xt} &= \sum_{n=0}^{\infty} Ch_{n,\lambda,q}(x|u)\frac{1}{n!}\left(\frac{1}{u}\left(e^{u(e^t-1)}-1\right)\right)^n \\ &= \sum_{n=0}^{\infty} Ch_{n,\lambda,q}(x|u)\frac{1}{n!}u^{-n}n!\sum_{l=n}^{\infty} S_2(l,n)\frac{u^l}{l!}\left(e^t-1\right)^l \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n} Ch_{m,\lambda,q}(x|u)S_2(n,m)\frac{u^{n-m}}{n!}\left(e^t-1\right)^n \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{l=0}^{m} Ch_{l,\lambda,q}(x|u)u^{m-l}S_2(m,l)S_2(n,m)\right)\frac{t^n}{n!},\end{aligned} \tag{2.6}$$

and

$$\frac{1+q}{1+qe^{\lambda t}}e^{xt} = \frac{1+q}{1+qe^{\lambda t}}e^{\lambda t\left(\frac{x}{\lambda}\right)} = \sum_{n=0}^{\infty}\lambda^n E_{n,q}\left(\frac{x}{\lambda}\right)\frac{t^n}{n!}. \tag{2.7}$$

By (1.3) and Theorem 2.1,

$$\begin{aligned}Ch_{n,\lambda,q}(x|u) &= \sum_{k=0}^{n} u^{n-k}S_1(n,k)\int_{\mathbb{Z}_p}(\lambda y+x)_k d\mu_{-q}(y) \\ &= \sum_{k=0}^{n} u^{n-k}S_1(n,k)\sum_{l=0}^{k} S_1(k,l)\lambda^l\int_{\mathbb{Z}_p}\left(y+\frac{x}{\lambda}\right)d\mu_{-q}(y) \\ &= \sum_{k=0}^{n}\sum_{l=0}^{k} u^{n-k}\lambda^l S_1(n,k)S_1(k,l)E_{l,q}\left(\frac{x}{\lambda}\right).\end{aligned} \tag{2.8}$$

Therefore, by (2.4), (2.5), (2.6) and (2.8), we obtain the following theorem.

Theorem 2.2. *For $n \geq 0$, we have*

$$Ch_{n,\lambda,q}(x) = \sum_{m=0}^{n} Ch_{m,\lambda,q}(x|u)u^{n-m}S_2(n,m),$$

and

$$Ch_{n,\lambda,q}(x|u) = \sum_{k=0}^{n}\sum_{l=0}^{k} u^{n-k}\lambda^l S_1(n,k)S_1(k,l)E_{l,q}\left(\frac{x}{\lambda}\right).$$

In addition,

$$\lambda^n E_{n,q}\left(\frac{x}{\lambda}\right) = \sum_{m=0}^{n}\sum_{l=0}^{m} Ch_{l,\lambda,q}(x|u)u^{m-l}S_2(m,l)S_2(n,m).$$

Let us consider the *degenerate λ-q-Changhee polynomials of the first kind* with order $k(\in\mathbb{N})$ as follows:

$$Ch^{(k)}_{n,\lambda,q}(x|u) = \sum_{l=0}^{n} u^{n-l}S_1(n,l)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(\lambda y_1+\cdots+\lambda y_k+x)_l d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k). \tag{2.9}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 24, NO.6, 2018, COPYRIGHT 2018 EUDOXUS PRESS, LLC

From (2.9), we can derive the generating function of $Ch_{n,\lambda,q}^{(k)}(x)$ as follows:

$$\begin{aligned}
&\sum_{n=0}^{\infty} Ch_{n,\lambda,q}^{(k)}(x|u)\frac{t^n}{n!}\\
=&\sum_{n=0}^{\infty}\sum_{l=0}^{n} u^{n-l}S_1(n,l)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(\lambda y_1+\cdots+\lambda y_k+x)_l d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_k)\\
=&\sum_{l=0}^{\infty} u^{-l}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(\lambda y_1+\cdots+\lambda y_k+x)_l d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_k)\frac{1}{l!}l!\sum_{m=l}^{\infty}S_1(m,l)\frac{(ut)^m}{m!}\\
=&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\sum_{l=0}^{\infty}\binom{\lambda y_1+\cdots+\lambda y_k+x}{l}u^{-l}\left(\log(1+ut)\right)^l d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_k)\\
=&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda y_1+\cdots+\lambda y_k+x} d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_k).
\end{aligned}\tag{2.10}$$

Note that by (1.3),

$$\begin{aligned}
&Ch_{n,\lambda,q}^{(k)}(x|u)\\
=&\sum_{l=0}^{n} u^{n-l}S_1(n,l)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(\lambda y_1+\cdots+\lambda y_k+x)_l d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_k)\\
=&\sum_{l=0}^{n} u^{n-l}S_1(n,l)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\sum_{s=0}^{l}S_1(l,s)(\lambda y_1+\cdots+\lambda y_k+x)^s d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_k)\\
=&\sum_{l=0}^{n}\sum_{s=0}^{l} u^{n-l}S_1(n,l)S_1(l,s)\lambda^s\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left(y_1+\cdots+y_k+\frac{x}{\lambda}\right)^s d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_k)\\
=&\sum_{l=0}^{n}\sum_{s=0}^{l} u^{n-l}S_1(n,l)S_1(l,s)\lambda^s E_{s,q}^{(k)}\left(\frac{x}{\lambda}\right).
\end{aligned}\tag{2.11}$$

From (2.6) and (2.10), we have

$$\begin{aligned}
\left(\frac{1+q}{1+qe^{\lambda t}}\right)^k e^{xt} &= \sum_{n=0}^{\infty} Ch_{n,\lambda,q}^{(k)}(x|u)\frac{1}{n!}\left(\frac{1}{u}\left(e^{u(e^t-1)}-1\right)\right)^n\\
&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{l=0}^{m} Ch_{l,\lambda,q}^{(k)}(x|u)u^{m-l}S_2(m,l)S_2(n,m)\right)\frac{t^n}{n!}
\end{aligned}\tag{2.12}$$

and

$$\left(\frac{1+q}{1+qe^{\lambda t}}\right)^k e^{\lambda t\left(\frac{x}{\lambda}\right)} = \sum_{n=0}^{\infty}\lambda^n E_{n,q}^{(k)}\left(\frac{x}{\lambda}\right)\frac{t^n}{n!}.\tag{2.13}$$

Thus, by (2.11), (2.12) and (2.13), we obtain the following theorem.

Theorem 2.3. *For $n \geq 0$, $k \in \mathbb{N}$, we have*

$$\lambda^n E_{n,q}^{(k)}\left(\frac{x}{\lambda}\right) = \sum_{m=0}^{n}\sum_{l=0}^{m} Ch_{l,\lambda,q}^{(k)}(x|u)u^{m-l}S_2(m,l)S_2(n,m),$$

and

$$\begin{aligned}Ch_{n,\lambda,q}^{(k)}(x|u) =& \sum_{l=0}^{n}\sum_{s=0}^{l} u^{n-l}S_1(n,l)S_1(l,s)\lambda^s E_{s,q}^{(k)}\left(\frac{x}{\lambda}\right)\\ =& \sum_{l=0}^{n}\sum_{s=0}^{l}\sum_{m=0}^{s}\sum_{r=0}^{m} u^{n+m-l-r}\lambda^{-s}Ch_{r,\lambda,q}^{(k)}(x|u)S_1(n,l)S_1(l,s)S_2(s,m)S_2(m,r).\end{aligned}$$

3. λ-q-Changhee polynomials of the second kind

For $n \geq 0$, the *rising factorial sequence* is defined by

$$\begin{aligned}x^{(n)} =& x(x+1)\cdots(x+n-1) = (-1)^n(-x)_n\\ =& \sum_{l=0}^{n}(-1)^{n-l}S_1(n,l)x^l.\end{aligned} \tag{3.1}$$

Let us define the *degenerate λ-q-Changhee polynomials of the second kind* as follows:

$$\sum_{n=0}^{\infty}\widehat{Ch}_{n,\lambda,q}(x|u)\frac{t^n}{n!} = \frac{1+q}{q\left(1+\frac{1}{u}\log(1+ut)\right)^{-\lambda}+1}\left(1+\frac{1}{u}\log(1+ut)\right)^x. \tag{3.2}$$

When $\lambda = 1$, $\widehat{Ch}_{n,1,q}(x|u) = \widehat{Ch}_{n,q}(x|u)$ are called the *degenerate q-Changhee polynomials of the second kind.* In particular, if $x = 0$, then $\widehat{Ch}_{n,\lambda,q}(0|u) = \widehat{Ch}_{n,\lambda,q}(u)$ are called the *degenerate λ-q-Changhee numbers of the second kind*, and $\widehat{Ch}_{n,1,q}(0|u) = \widehat{Ch}_{n,q}(u)$ are called the *degenerate q-Changhee numbers of the second kind.*

Let us take $f(x) = (1+t)^{-\lambda x}$. Then, by (1.2), we have

$$\int_{\mathbb{Z}_p}\left(1+\frac{1}{u}\log(1+ut)\right)^{-\lambda x}d\mu_{-q}(x) = \frac{1+q}{q\left(1+\frac{1}{u}\log(1+ut)\right)^{-\lambda}+1}, \tag{3.3}$$

and so

$$\begin{aligned}&\int_{\mathbb{Z}_p}\left(1+\frac{1}{u}\log(1+ut)\right)^{-\lambda y+x}d\mu_q(y)\\ =& \frac{1+q}{q\left(1+\frac{1}{u}\log(1+ut)\right)^{-\lambda}+1}\left(1+\frac{1}{u}\log(1+ut)\right)^x\\ =& \sum_{n=0}^{\infty}\widehat{Ch}_{n,\lambda,q}(x|u)\frac{t^n}{n!}.\end{aligned} \tag{3.4}$$

By (3.4), we get

$$\sum_{n=0}^{\infty}\widehat{Ch}_{n,\lambda,q}(x|u)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{m=0}^{n}u^{n-m}S_1(n,m)\int_{\mathbb{Z}_p}(-\lambda y+x)_m\,d\mu_{-q}(y)\frac{t^n}{n!}. \tag{3.5}$$

By (3.3) and (3.5), we get

$$\begin{aligned}\widehat{Ch}_{n,\lambda,q}(x|u) &= \sum_{m=0}^{n} u^{n-m} S_1(n,m) \int_{\mathbb{Z}_p} (-\lambda y + x)_m \, d\mu_{-q}(y) \\ &= \sum_{m=0}^{n}\sum_{l=0}^{m} u^{n-m}(-\lambda)^l S_1(n,m) S_1(m,l) \int_{\mathbb{Z}_p} \left(y - \frac{x}{\lambda}\right)^l d\mu_{-q}(y) \\ &= \sum_{m=0}^{n}\sum_{l=0}^{m} u^{n-m}(-\lambda)^l S_1(n,m) S_1(m,l) E_{l,q}\left(-\frac{x}{\lambda}\right).\end{aligned} \tag{3.6}$$

By (3.4), we get

$$\begin{aligned}\frac{q+1}{qe^{-\lambda t}+1}e^{xt} &= \sum_{n=0}^{\infty} \widehat{Ch}_{n,\lambda,q}(x|u)\frac{1}{n!}\left(\frac{1}{u}e^{u(e^t-1)}-1\right)^n \\ &= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m} \widehat{Ch}_{n,\lambda,q}(x|u) u^{m-n} S_2(m,n)\right)\frac{t^m}{m!},\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}\frac{q+1}{qe^{-\lambda t}+1}e^{xt} &= \frac{q+1}{qe^{-\lambda t}+1}e^{\left(-\frac{x}{\lambda}\right)(-\lambda t)} \\ &= \sum_{m=0}^{\infty}(-\lambda)^m E_{m,q}\left(-\frac{x}{\lambda}\right)\frac{t^m}{m!}.\end{aligned} \tag{3.8}$$

Therefore, by (3.6), (3.7) and (3.8), we obtain the following theorem.

Theorem 3.1. *For $m \geq 0$, we have*

$$\widehat{Ch}_{m,\lambda,q}(x|u) = \sum_{n=0}^{m}\sum_{l=0}^{n} u^{m-n}(-\lambda)^l S_1(m,n) S_1(n,l) E_{l,q}\left(-\frac{x}{\lambda}\right),$$

and

$$\begin{aligned}(-\lambda)^m E_{m,q}\left(-\frac{x}{\lambda}\right) &= \sum_{n=0}^{m} \widehat{Ch}_{n,\lambda,q}(x|u) u^{m-n} S_2(m,n) \\ &= \sum_{n=0}^{m}\sum_{k=0}^{n}\sum_{l=0}^{k} u^{m-k}(-\lambda)^l S_1(n,k) S_1(k,l) S_2(m,n) E_{l,q}\left(-\frac{x}{\lambda}\right).\end{aligned}$$

By the Theorem 3.1, we obtain the following corollary.

Corollary 3.2. *For $n \geq 0$, we have*

$$\widehat{Ch}_{n,\lambda,q}(u) = \sum_{n=0}^{m}\sum_{l=0}^{n} u^{m-n}(-\lambda)^l S_1(m,n) S_1(n,l) E_{l,q},$$

and

$$\begin{aligned}E_{m,q} &= (-\lambda)^{-m}\sum_{n=0}^{m} \widehat{Ch}_{n,\lambda,q}(u) u^{m-n} S_2(m,n) \\ &= \sum_{n=0}^{m}\sum_{k=0}^{n}\sum_{l=0}^{k} u^{m-k}(-\lambda)^l S_1(n,k) S_1(k,l) S_2(m,n) E_{l,q}.\end{aligned}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 24, NO.6, 2018, COPYRIGHT 2018 EUDOXUS PRESS, LLC

As the special case of the Corollary 3.2, $\lambda = 1$ and $u = 1$, we have

$$\widehat{Ch}_{n,q} = \sum_{n=0}^{m}\sum_{l=0}^{n}\frac{(-1)^l}{n!}S_1(m,n)S_1(n,l)E_{l,q}$$

and

$$\begin{aligned}E_{m,q} =&(-1)^{-m}\sum_{n=0}^{m}\widehat{Ch}_{n,q}S_2(m,n)\\ =&\sum_{n=0}^{m}\sum_{k=0}^{n}\sum_{l=0}^{k}(-1)^l S_1(n,k)S_1(k,l)S_2(m,n)E_{l,q}.\end{aligned}$$

For $k \in \mathbb{N}$, we define the *degenerate λ-q-Changhee polynomials of the second kind with order k*:

$$\widehat{Ch}^{(k)}_{n,\lambda,q}(x|u) = \sum_{m=0}^{n} u^{n-m}S_1(n,m)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(-\lambda x_1-\cdots-\lambda x_k+x)_m d\mu_q(x_1)\cdots d\mu_q(x_k). \tag{3.9}$$

From (3.9), we can derive the generating function of

$$\begin{aligned}&\sum_{n=0}^{\infty}\widehat{Ch}^{(k)}_{n,\lambda,q}(x|u)\frac{t^n}{n!}\\ =&\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(-\lambda x_1-\cdots-\lambda x_k+x)_n d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k)\frac{t^n}{n!}\\ =&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left(1+\frac{1}{u}\log(1+ut)\right)^{-\lambda x_1-\cdots-\lambda x_k+x} d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k)\\ =&\left(\frac{1+q}{q\left(1+\frac{1}{u}\log(1+ut)\right)^{-\lambda}+1}\right)^k\left(1+\frac{1}{u}\log(1+ut)\right)^x.\end{aligned} \tag{3.10}$$

Replacing t by $\frac{1}{u}\left(e^{ut}-1\right)$ in (3.10), we get

$$\begin{aligned}&\left(\frac{q+1}{q(1+t)^{-\lambda}+1}\right)^k(1+t)^x\\ =&\sum_{n=0}^{\infty}\widehat{Ch}^{(k)}_{n,\lambda,q}(x|u)\frac{\left(\frac{1}{u}(e^{ut}-1)\right)^n}{n!}\\ =&\sum_{n=0}^{\infty}\widehat{Ch}^{(k)}_{n,\lambda,q}(x|u)\frac{1}{n!}u^{-n}n!\sum_{l=n}^{\infty}S_2(l,n)\frac{(ut)^l}{l!}\\ =&\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\widehat{Ch}^{(k)}_{m,\lambda,q}(x|u)u^{n-m}S_2(n,m)\right)\frac{t^n}{n!}.\end{aligned} \tag{3.11}$$

On the other hands, by replacing t by $\frac{1}{u}\left(e^{u(e^t-1)}-1\right)$ in (3.10), we have

$$\begin{aligned}
&\left(\frac{q+1}{qe^{-\lambda t}+1}\right)^k e^{tx}\\
&=\sum_{n=0}^{\infty}\widehat{Ch}_{n,\lambda,q}^{(k)}(x|u)\frac{1}{n!}\left(\frac{1}{u}\left(e^{u(e^t-1)}-1\right)\right)^n\\
&=\sum_{n=0}^{\infty}\widehat{Ch}_{n,\lambda,q}^{(k)}(x|u)\frac{1}{n!}u^{-n}n!\sum_{l=n}^{\infty}S_2(l,n)\frac{u^l}{l!}\left(e^t-1\right)^l\\
&=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\widehat{Ch}_{m,\lambda,q}^{(k)}(x|u)u^{n-m}S_2(n,m)\frac{(e^t-1)^n}{n!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{l=0}^{m}\widehat{Ch}_{l,\lambda,q}^{(k)}(x|u)u^{m-l}S_2(m,l)S_2(n,m)\right)\frac{t^n}{n!},
\end{aligned}\tag{3.12}$$

and

$$\begin{aligned}
\left(\frac{q+1}{qe^{-\lambda t}+1}\right)^k e^{tx}&=\left(\frac{q+1}{qe^{-\lambda t}+1}\right)^k e^{-\lambda t\left(-\frac{\lambda}{x}\right)}\\
&=\sum_{n=0}^{\infty}(-\lambda)^n E_{n,q}^{(k)}\left(-\frac{x}{\lambda}\right)\frac{t^n}{n!}.
\end{aligned}\tag{3.13}$$

By (1.1) and (3.9), we get

$$\begin{aligned}
&\widehat{Ch}_{n,\lambda,q}^{(k)}(x|u)\\
&=\sum_{m=0}^{n}u^{n-m}S_1(n,m)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(-\lambda x_1-\cdots-\lambda x_k+x)_m d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k)\\
&=\sum_{m=0}^{n}\sum_{l=0}^{m}u^{n-m}S_1(n,m)S_1(m,l)(-\lambda)^l E_{l,q}^{(k)}\left(-\frac{x}{\lambda}\right).
\end{aligned}\tag{3.14}$$

Hence, by (3.11), (3.12), (3.13) and (3.14), we obtain the following theorem.

Theorem 3.3. *For $n\geq 0$, we have*

$$\widehat{Ch}_{n,\lambda,q}^{(k)}(x|u)=\sum_{m=0}^{n}\sum_{l=0}^{m}u^{n-m}S_1(n,m)S_1(m,l)(-\lambda)^l E_{l,q}^{(k)}\left(-\frac{x}{\lambda}\right),$$

and

$$Ch_{n,-\lambda,q}^{(k)}(x)=\sum_{m=0}^{n}\widehat{Ch}_{m,\lambda,q}^{(k)}(x|u)u^{n-m}S_2(n,m).$$

In addition,

$$\begin{aligned}
(-\lambda)^n E_{n,q}^{(k)}\left(-\frac{x}{\lambda}\right)&=\sum_{m=0}^{n}\sum_{l=0}^{m}\widehat{Ch}_{l,\lambda,q}^{(k)}(x|u)u^{m-l}S_2(m,l)S_2(n,m)\\
&=\sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{r=0}^{l}\sum_{s=0}^{r}u^{m-r}S_1(l,r)S_1(r,s)S_2(m,l)S_2(n,m)(-\lambda)^s E_{s,q}^{(k)}\left(-\frac{x}{\lambda}\right).
\end{aligned}$$

As the special case of Theorem 3.3, if we put $x = 0$, $\lambda = 1$ and $u = 1$, then

$$\widehat{Ch}_{n,q}^{(k)} = \sum_{m=0}^{n} \sum_{l=0}^{m} S_1(n,m) S_1(m,l)(-\lambda)^l E_{l,q}^{(k)},$$

and

$$\begin{aligned} E_{n,q}^{(k)} =& (-1)^n \sum_{m=0}^{n} \sum_{l=0}^{m} \widehat{Ch}_{l,\lambda,q}^{(k)} S_2(m,l) S_2(n,m) \\ =& \sum_{m=0}^{n} \sum_{l=0}^{m} \sum_{r=0}^{l} \sum_{s=0}^{r} S_1(l,r) S_1(r,s) S_2(m,l) S_2(n,m)(-1)^{n+s} E_{s,q}^{(k)}. \end{aligned}$$

References

[1] S. Araci and M. Açiköz, *A note on the Frobenius-Euler numbers and plynomials associated with Bernstein polynomials*, Adv. Stud. Contemp. Math., **22** (2012), no. 3, 399-406.

[2] L. Carlitz, *Degenerate Stirling, Bernoulli and Eulerian numbers*, Utilitas Math., **15** (1979), 51-88.

[3] L. Comtet, *Advanced Combinatorics*, Reidel, Dordrecht, 1974.

[4] J. H. Jin, T. Mansour, E. J. Moon and J. W. Park, *On the (r,q)-Bernoulli and (r,q)-Euler numbers and polynomials*, J. Comput. Anal. Appl., **19** (2015), no. 2, 250-259.

[5] D. S. Kim, T. Kim and D. V. Dolgy, *Degenerate poly-Cauchy polynomials with a q-parameter*, J. Inequal. Appl., **2015**, 2015:364, 15 pp.

[6] D. S. Kim, T. Kim and J. J. Seo, *A note on Changhee polynomials and numbers*, Adv. Studies Theor. Phys., **7** (2013), no. 20, 993-1003.

[7] T. Kim, D. S. Kim and D. V. Dolgy, *Degenerate q-Euler polynomials*, Adv. Difference Equ., **2015** (2015), 13662.

[8] T. Kim, *On q-analogye of the p-adic log gamma functions and related integral*, J. Number Theory, **76** (1999), no. 2, 320-329.

[9] T. Kim, *An invariant p-adic integral associated with Daehee numbers*, Integral Transforms Spec. Funct., **13** (2002), no. 1, 65-69.

[10] T. Kim, *q-Volkenborn integration*, Russ. J. Math. Phys., **9** (2002), no. 3, 288-299.

[11] T. Kim, *Degenerate Euler zeta function*, Russ. J. Math. Phys., **22** (2015), no. 4, 469-472.

[12] T. Kim, T. Mansour, S. H. Rim and J. J. Seo, *A Note on q-Changhee Polynomials and Numbers*, Adv. Studies Theor. Phys., **8** (2014), no. 1, 35-41.

[13] H. I. Kwon, T. Kim and J. J. Seo, *A note on degenerate Changhee numbers and polynomials*, Proc. Jangjeon Math. Soc., **18** (2015), no. 3, 295-305.

[14] J. W. Park, *On the twisted q-Changhee polynomials of higher-order*, J. Comput. Anal. Appl., **20** (2016), no. 3, 424-431.

[15] S. H. Rim and J. Jeong, *On the modified q-Euler numbers of higher-order with weight*, Adv. Stud. Contemp. Math., **22** (2012), 93-98.

[16] S. Roman, *The umbral calculus*, Dover Publ. Inc. New York, 2005.

[17] S. H. Rim, J. W. Park, S. S. Pyo and J. Kwon, *On the twisted Changhee polynomials and numbers*

[18] E. Sen, *Theorems on Apostol-Euler polynomials of higher-order arising from Euler basis*, Adv. Stud. Contemp. Math., **23** (2013), 337-345.

[1] Department of Mathematics, Dong-A University, Busan 604-714, Republic of Korea.
E-mail address: pitt0202@hanmail.net

[2] Department of Mathematics education, Daegu University, Gyeongsan-si, Gyeongsangbuk-do, 712-714, Republic of Korea.
E-mail address: sanklee@daegu.ac.kr

[3] Department of Mathematics Education, Daegu University, Gyeongsan-si, Gyeongsangbuk-do, 712-714, Republic of Korea.
E-mail address: a0417001@knu.ac.kr

Existence of Generalized Solutions for Fuzzy Impulsive Retarded Differential Equations*

Qiang Ma[1], Yabin Shao [2], † Zizhong Chen[3]

1. Network Information Management Center

Northwest University for Nationalities, Lanzhou 730030, Gansu, P.R. China

2. School of Science

Chongqing University of Posts and Telecommunications

Nan'an 400065, Chongqing, P. R. China

3. School of Computer Science

Chongqing University of Posts and Telecommunications

Nan'an 400065, Chongqing, P. R. China

May 24, 2016

Abstract

In this paper, we study the existence of solution for fuzzy initial valued problems of fuzzy impulsive retarded differential equations in the setting of a generalized Hukuhara derivative and by using the Strong Lusin Condition of fuzzy Henstock integrable functions.

Keywords: Fuzzy number; Strong Lusin Condition; Discontinuous impulsive retarded differential equations; Fuzzy Henstock integrals.

1 INTRODUCTION

It is known that the theory of retarded functional differential equations has been well known when the right side function is continuous, hence Riemann integral. Hale [11] prove that the results still hold true when continuity of right function is weakened to satisfaction of a Carathéodory condition. The further step of generalisation was done in [3] and [22] which applies the Henstock integrals to the study of retarded functional differential equations with finite delays and unbounded delays. By using generalized differential equation theory [16], M. Federson and P. Táboas [6] proved that a local flow can be constructed for a

*The authors would like to thank National Natural Science Foundation of China (No.11161041, 61472056, 61262022)and Basic and Advanced Research Projects in Chongqing (No.cstc2015jcyjA00015) and the PhD Research Startup Foundation of Chongqing University of Posts and Telecommunications (No. A2014-90, A2016-13).

†Corresponding author. E-mail: yb-shao@163.com(Y. Shao)

general class of non-autonomous retarded functional differential equations. On the other hand, it is meaningful to study the fuzzy retarded function differential. In [19], Lupulescu applied a successive approximation method to discuss the fuzzy differential equations with distributed delays. Guo et al. [10] discussed the oscillation properties of a class of fuzzy delay differential equation of second order and provided an oscillation criterion. In [15], Kloeden and Lorenz removed the assumption of fuzzy convexity of fuzzy differential equations and discussed the fuzzy delay differential equations in this perspective. In [14], Khastan et al. provided sufficient conditions for the global existence of a unique (ii)-solution to an initial value problem for fuzzy functional differential equations using generalized derivative and were of broader applicability than those using Hukuhara derivative.

However, there are discontinuous systems in which the right-hand side functions $\tilde{f} : [a, b]\times E^n \to E^n$ are not integrable in the sense of Kaleva [13] on certain intervals and their solutions are not absolute continuous functions. Recently, Wu and Gong [25, 26] have combined the fuzzy set theory [28] and nonabsolute integration theory [12], and discussed the fuzzy Henstock integrals of fuzzy-number-valued functions which extended Kaleva[13] integration. In order to complete the theory of fuzzy calculus and to meet the solving need of transferring a fuzzy differential equation into a fuzzy integral equation, Gong and Shao [7, 8] have defined the strong fuzzy Henstock integrals and discussed some of their properties and the controlled convergence theorem. So, in [20, 23, 24], the authors used the strong fuzzy Henstock integrals [8], and deal with the Cauchy problem of discontinuous fuzzy systems. In this paper, we use the fuzzy Henstock integral to establish the existence of generalized solutions as well as continuous dependence on the interval conditions of fuzzy impulsive retarded differential equations

$$\begin{cases} x'(t) = \tilde{f}(t, x_t), & t \neq t_k, \\ \triangle\, x(t_k) = I_k(x(t_k)), & k = 1, 2, \cdots\cdots, m, \\ x_{t_0} = \phi, \quad x(t_0) = x_0 \end{cases} \tag{1}$$

where $\tilde{f} : I\times FH([-r, 0], \mathbb{R}_{\mathcal{F}}) \to \mathbb{R}_{\mathcal{F}}$, and $I_k : \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ are give fuzzy mapping, $\phi \in FH([-r, 0], \mathbb{R}_{\mathcal{F}})$ and $x_0 \in \mathbb{R}_{\mathcal{F}}$.

To make our analysis possible, in section 2, we will first recall some basic results of fuzzy numbers and given the definition of Strong Lusin Condition of fuzzy-number-valued functions. Under this notion, we give another look at the fundamental theorem of calculus of fuzzy Henstock integrals. In section 3, we deal with the Cauchy problem of discontinuous fuzzy impulsive retarded differential equations. And in section 4, we present some concluding remarks.

2 PRELIMINARIES

Let $P_k(R^n)$ denote the family of all nonempty compact convex subset of R^n and define the addition and scalar multiplication in $P_k(R^n)$ as usual. Let A and

B be two nonempty bounded subset of R^n. The distance between A and B is defined by the Hausdorff metric [5]:

$$d_H(A,B) = \max\{\sup_{a\in A}\inf_{b\in B} \parallel a-b \parallel, \sup_{b\in B}\inf_{a\in A} \parallel b-a \parallel\}.$$

Denote $E^n = \{u : R^n \to [0,1] | u$ satisfies (1)-(4) below$\}$ is a fuzzy number space. where

(1)u is normal, i.e. there exists an $x_0 \in R^n$ such that $u(x_0) = 1$,

(2)u is fuzzy convex, i.e. $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$ for any $x, y \in R^n$ and $0 \leq \lambda \leq 1$,

(3)u is upper semi-continuous,

(4)$[u]^0 = cl\{x \in R^n | u(x) > 0\}$ is compact.

For $0 < \alpha \leq 1$, denote $[u]^\alpha = \{x \in R^n | u(x) \geq \alpha\}$. Then from above (1)-(4), it follows that the α-level set $[u]^\alpha \in P_k(R^n)$ for all $0 \leq \alpha < 1$.

According to Zadeh's extension principle, we have addition and scalar multiplication in fuzzy number space E^n as follows [5]:

$$[u+v]^\alpha = [u]^\alpha + [v]^\alpha, \quad [ku]^\alpha = k[u]^\alpha,$$

where $u, v \in E^n$ and $0 \leq \alpha \leq 1$.

Define $D : E^n \times E^n \to [0, \infty)$

$$D(u,v) = \sup\{d_H([u]^\alpha, [v]^\alpha) : \alpha \in [0,1]\},$$

where d is the Hausdorff metric defined in $P_k(R^n)$. Then it is easy see that D is a metric in E^n. Using the results [4], we know that

(1) (E^n, D) is a complete metric space,

(2) $D(u+w, v+w) = D(u,v)$ for all $u, v, w \in E^n$,

(3)$D(\lambda u, \lambda v) = |\lambda| D(u,v)$ for all $u, v, w \in E^n$ and $\lambda \in R$.

A fuzzy-number-valued function $f : [a,b] \to E^n$ is said to satisfy the condition (H) on $[a,b]$, if for any $x_1 < x_2 \in [a,b]$ there exists $u \in E^n$ such that $f(x_2) = f(x_1) + u$. We call u is the H-difference of $f(x_2)$ and $f(x_1)$, denoted $f(x_2) -_H f(x_1)$ ([13]).

For brevity, we always assume that it satisfies the condition (H) when dealing with the operation of subtraction of fuzzy numbers throughout this paper.

In this paper we consider a more general definition of a derivative for fuzzy-number-valued functions enlarging the class of differentiable fuzzy-number-valued functions, which has been introduced in [1] and [2].

Definition 1 ([1]) *Let $\tilde{f} : (a,b) \to E^n$ and $x_0 \in (a,b)$. We say that $\tilde{f}$ is differentiable at x_0, if there exists an element $\tilde{f}'(t_0) \in E^n$, such that*

(1) for all $h > 0$ sufficiently small, there exists $\tilde{f}(x_0+h) -_H \tilde{f}(x_0)$, $\tilde{f}(x_0) -_H \tilde{f}(x_0 - h)$ and the limits (in the metric D)

$$\lim_{h\to 0} \frac{\tilde{f}(x_0+h) -_H \tilde{f}(x_0)}{h} = \lim_{h\to 0} \frac{\tilde{f}(x_0) -_H \tilde{f}(x_0-h)}{h} = \tilde{f}'(x_0)$$

or

(2) for all $h > 0$ sufficiently small, there exists $\tilde{f}(x_0) -_H \tilde{f}(x_0 + h), \tilde{f}(x_0 - h) -_H \tilde{f}(x_0)$ and the limits

$$\lim_{h\to 0} \frac{\tilde{f}(x_0) -_H \tilde{f}(x_0 + h)}{-h} = \lim_{h\to 0} \frac{\tilde{f}(x_0 - h) -_H \tilde{f}(x_0)}{-h} = \tilde{f}'(x_0)$$

(h and $-h$ at denominators mean $\frac{1}{h}\cdot$ and $-\frac{1}{h}\cdot$, respectively).

Definition 2 ([18]) *Let $\delta(\xi)$ be a positive real function on a closed set $[a, b]$. A division $P = \{(\xi_i, [x_{i-1}, x_i])\}$ is said to be δ-fine, if the following conditions are satisfied:*

(1) $a = x_1 < x_2 < \cdots < x_n = b$;

(2) $\xi_i \in [x_{i-1}, x_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$.

Definition 3 ([25, 26]) *A fuzzy-number-valued function $\tilde{f}$ is said to be Henstock integrable on $[a, b]$ if there exists a fuzzy number $\tilde{A}$ such that for every $\varepsilon > 0$ there is a function $\delta(\xi) > 0$ such that for any δ-fine division $P = \{[u, v], \xi\}$ of $[a, b]$, we have*

$$D\bigg(\sum \tilde{f}(\xi)(v - u), \tilde{A}\bigg) < \varepsilon.$$

We write $(FH) \int_a^b \tilde{f}(x)\mathrm{d}x = \tilde{A}$ and $\tilde{f} \in FH[a, b]$.

Remark 1 *If the fuzzy-number-valued function $\tilde{f}$ and fuzzy number $\tilde{A}$ are replaced by a real valued function and real number, respectively, then the real valued function f is said to be Henstock integrable on $[a, b]$ and we write $f \in H[a, b]$.*

Remark 2 *When the function $\delta : [a, b] \to R^+$ is constant, then we obtain the Riemann integrability for fuzzy-number-valued functions. In this case, $\tilde{A} \in \mathbb{R}_{\mathcal{F}}$ is called the Riemann integral of $\tilde{f}$ on $[a, b]$, beingdenoted by $(FR) \int_a^b \tilde{f}(x)\mathrm{d}x$. Consequently, the fuzzy Riemann integrability is a particular case of the fuzzy Henstock integrability.*

Definition 4 *A fuzzy-number-valued function $\tilde{f} : [a, b] \to \mathbb{R}_{\mathcal{F}}$ is said to satisfy Strong Lusin Condition, that is, $\tilde{f} \in SL([a, b], \mathbb{R}_{\mathcal{F}})$, if for every $\varepsilon > 0$ and $E \subset [a, b]$ with $|E| = 0$, there exists a $\delta-$fine division of $[a, b]$ with $I_i = [t_{i-1}, t_i]$ and $\xi_i \in E$ such that*

$$\sum_i D(\tilde{f}(t_i), \tilde{f}(t_{i-1})) < \varepsilon.$$

Remark 3 *If $AC([a, b], \mathbb{R}_{\mathcal{F}})$ denotes the space of all absolutely continuous functions, the inclusions $AC([a, b], \mathbb{R}_{\mathcal{F}}) \subset SL([a, b], \mathbb{R}_{\mathcal{F}}) \subset C([a, b], \mathbb{R}_{\mathcal{F}})$ hold.*

Theorem 1 ([9]) *Let $\tilde{f} \in AC([a, b], \mathbb{R}_{\mathcal{F}})$ be a fuzzy-number-valued function. Then there exists a absolutely Henstock integrable (i.e. Kaleva integrable) fuzzy-number-valued function $\tilde{f}$ such that*

$$\tilde{F}(x) = (K) \int_a^x \tilde{f}(t)\mathrm{d}t + \tilde{F}(a)$$

for any $x \in [a, b]$.

Now, by Theorem 1 and Definition 4 (Strong Lusin Condition), we give the fundamental theorem of calculus of fuzzy Henstock integrals.

Theorem 2 *For any $\tilde{f} \in SL([a,b],\mathbb{R}_{\mathcal{F}})$ $(GH)-$differentiable a.e., and $\tilde{F}' = \tilde{f}$ a.e. Then $\tilde{f} \in FH([a,b],\mathbb{R}_{\mathcal{F}})$ and*

$$\tilde{F}(x) = (FH)\int_a^x \tilde{f}(t)\mathrm{d}t + \tilde{F}(a)$$

or

$$\tilde{F}(x) = (FH)\int_a^x \tilde{f}(t)\mathrm{d}t \ominus_H (-1)\cdot\tilde{F}(a)$$

for any $x \in [a,b]$.

Conversely, if $\tilde{f} \in FH([a,b],\mathbb{R}_{\mathcal{F}})$, then $\tilde{f} \in SL([a,b],\mathbb{R}_{\mathcal{F}})$ and there exists $\tilde{F}' = \tilde{f}$ for almost every $t \in [a,b]$.

Theorem 3 *If $\tilde{f} \in FH([a,b],\mathbb{R}_{\mathcal{F}})$, then there exists a sequence of closed sets $X_i \subset X_{i+1} \subset [a,b]$, for every $i \in \mathbb{N}$ and $\cup X_i = [a,b]$ such that $\tilde{f} \in K([a,b],\mathbb{R}_{\mathcal{F}})$, and*

$$\lim_{i\to\infty}(K)\int_{X_i\cap[a,x]} \tilde{f}(t)\mathrm{d}t = (FH)\int_a^x \tilde{f}(t)\mathrm{d}t$$

uniformly for $x \in [a,b]$.

By the definition of improper fuzzy Riemann integral in [27], we can also give the proposition of the improper fuzzy Henstock integral.

Proposition 1 *If $\tilde{f} \in FH([a,c),\mathbb{R}_{\mathcal{F}})$ for every $c \in [a,b)$, then $\tilde{f} \in FH([a,b],\mathbb{R}_{\mathcal{F}})$ and*

$$\int_a^b \tilde{f}(t)\mathrm{d}t = \lim_{c\to b}\int_a^c \tilde{f}(t)\mathrm{d}t.$$

3 MAIN RESULTS

We start this section by defining some basic concept. Given a fuzzy-number-valued function $x : [t_0, t_0+a] \to \mathbb{R}_{\mathcal{F}}$, let $\triangle\ x(t_k)$ denote the jump of $x(t)$ at $t = t_k$ for $k = 1, 2, \cdots\cdots, m$, where $t_0 < t_1 < \cdots\cdots < t_k \cdots\cdots < t_m \leq t_0 + a$, that is $\triangle\ x(t_k) = x(t_k+) \ominus_H x(t_k-) = I_k(x(t_k))$ is the $(H)-$difference, and $I_k(x(t_k)) : \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$.

Definition 5 *Denote C_{t_0} is the set of all functions $x : [t_0, t_0+a] \to \mathbb{R}_{\mathcal{F}}$ such that*

(C1) x is continuous at $t \neq t_k, k = 1, 2, \cdots\cdots, m$;

(C2) x is left continuous at $t = t_k, k = 1, 2, \cdots\cdots, m$;

(C3) there exists the right limit $x(t_k+)$ for $k = 1, 2, \cdots\cdots, m$.

The same as above, let SL_{t_0} (respectively AC_{t_0}) be the set of all functions $x : [t_0 - r, t_0 + a] \to \mathbb{R}_{\mathcal{F}}$ such that $x \in SL([I, \mathbb{R}_{\mathcal{F}}])$ (resp. $x \in \mathbb{R}_{\mathcal{F}}$) for every close interval $I \subset (t_k, t_{k+1})$ and such (C1) and (C3) holds.

Let $r > 0, x_0 \in \mathbb{R}_{\mathcal{F}}$ and $\phi \in FH([-r, 0], \mathbb{R}_{\mathcal{F}})$. We consider the set:

$$C_{\phi,x_0} = \{x : [t_0 - r, t_0 + a] \to \mathbb{R}_{\mathcal{F}} | x(t_0) = x_0; x_{t_0} = \phi; x|_{[t_0,t_0+a]} \in C_{t_0}\}$$
$$SL_{\phi,x_0} = \{x \in C_{\phi,x_0} : x|_{[t_0,t_0+a]} \in SL_{t_0}\}$$
$$AC_{\phi,x_0} = \{x \in C_{\phi,x_0} : x|_{[t_0,t_0+a]} \in AC_{t_0}\}$$

Definition 6 *Consider a function $x(t) = x(t, t_0, x_0, \phi)$ and the following conditions:*

(i) $x \in C_{\phi,x_0}$*;*

(ii) $(t, x_t) \in [t_0, t_0 + a] \times FH([-r, 0], \mathbb{R}_{\mathcal{F}})$*, a.e.;*

(iii) $x(t) = x(s) + \int_s^t \tilde{f}(\sigma, x_\sigma)\mathrm{d}\sigma$ *for every* $t_{k-1} < s \le t \le t_k$ *and* $t_m < s \le t \le t_0 + a, k = 1, 2 \cdots, m$*;*

(iv) $x(t) = x(s) \ominus_H (-\int_s^t \tilde{f}(\sigma, x_\sigma)\mathrm{d}\sigma)$ *for every* $t_{k-1} < s \le t \le t_k$ *and* $t_m < s \le t \le t_0 + a, k = 1, 2 \cdots, m$*;*

(v) $I_(k)(x(t_k)) = x(t_k+) \ominus_H x(t_k), k = 1, 2 \cdots, m.$

If x satisfies conditions (i)-(iii) and (v), we say that it is called to be a generalized (i)-solution of problem (5) through (t_0, x_0, ϕ)*. If x satisfies conditions (i),(ii) (iv) and (v), we say that it is called to be a generalized (ii)-solution of problem (5) through* (t_0, x_0, ϕ)*.*

Lemma 1 *If x is a generalized (i)-solution of problem (5), then x satisfies the following integral equation*

$$\begin{cases} x(t) = x_0 + \int_{t_0}^t \tilde{f}(s, x_s)\mathrm{d}s + \sum\limits_{t_0 < t_k < t} [x(t_k+) \ominus_H x(t_k)], \\ x_{t_0} = \phi, \quad x(t_0) = x_0, \end{cases} \tag{2}$$

where $t \in [t_0, t_0 + a]$*.*

Proof. Let $k \in \{0, 1, 2, \cdots, m\}$ and $t_k < s \le u \le t_{k+1}$. By Theorem 53, $x(u) = x(s) + \int_s^u \tilde{f}(t, x_t)\mathrm{d}t$. By Definition 57, we have $x'(t) = \tilde{f}(t, x_t)$ for $t_k < t \le t_{k+1}$. Hence $x(u) = x(s) + \int_s^u x'(t)\mathrm{d}t$. By Proposition 55, for $t_k < t \le t_{k+1}$, there exists $\int_{t_k}^t x'(\sigma)\mathrm{d}\sigma = \lim\limits_{s\to t_k} \int_s^t x'(\sigma)\mathrm{d}\sigma = x(t) \ominus_H x(t_k+)$. Thus, for $t \in (t_0, t_0 + a]$, there exists $k \in \{0, 1, 2, \cdots, m\}$ such that $t_k < t \le t_{k+1}$ and we have

$$\begin{aligned}
\int_{t_0}^{t} x'(\sigma)\mathrm{d}\sigma &= \int_{t_0}^{t_1} x'(\sigma)\mathrm{d}\sigma + \sum_{i=1}^{k-1}\int_{t_i}^{t_{i+1}} x'(\sigma)\mathrm{d}\sigma \\
&\quad + \int_{t_k}^{t} x'(\sigma)\mathrm{d}\sigma \\
&= x(t_1) \ominus_H [x_0 + \sum_{i=1}^{k-1}(x(t_{i+1}) \ominus_H x(t_i+))] + [x(t) \ominus_H x(t_k+)] \\
&= x(t) \ominus_H [x_0 + \sum_{i=1}^{k-1}(x(t_i+) \ominus_H x(t_i))].
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
x(t) &= x_0 + \int_{t_0}^{t} x'(\sigma)\mathrm{d}\sigma + \sum_{t_0<t_k<t}(x(t_k+) \ominus_H x(t_k)) \\
&= x_0 + \int_{t_0}^{t} \tilde{f}(\sigma, x_\sigma)\mathrm{d}\sigma + \sum_{t_0<t_k<t}(x(t_k+) \ominus_H x(t_k)).
\end{aligned}$$

Since $x'(t) = \tilde{f}(t, x_t)$, for $t_k < t \le t_{k+1}$ and $k \in \{0, 1, 2, \cdots, m\}$, and due to Theorem 2.8 in [26]. The proof is complete.

Lemma 2 *If x is a generalized (ii)-solution of problem (5), then x satisfies the following integral equation*

$$\begin{cases} x(t) = x_0 \ominus_H (-\int_{t_0}^{t} \tilde{f}(s, x_s)\mathrm{d}s) + \sum\limits_{t_0<t_k<t}[x(t_k+) \ominus_H x(t_k)], \\ x_{t_0} = \phi, \quad x(t_0) = x_0, \end{cases} \tag{3}$$

where $t \in [t_0, t_0 + a]$.

Proof. The proof of this result is analogous to that of Lemma 1.

Lemma 3 *If x is a generalized (i)-solution of the problem (6), the x is a generalized (i)-solution of the problem (5) and $x \in SL_{\phi,x_0}$.*

Proof. Given $\varepsilon > 0$ then

$$x(t_k + \varepsilon) = x(t_k) + \int_{t_k}^{t_k+\varepsilon} \tilde{f}(s, x_s)\mathrm{d}s + I_k(x(t_k))$$

which tend to $I_k(x(t_k))$ as $\varepsilon \to 0$. By hypothesis, given $x \in C_{\phi,x_0}$, and the fuzzy Henstock integral $\int_{t_0,t} \tilde{f}(\sigma, x_\sigma)\mathrm{d}\sigma$ exists, for every $t \in [t_0, t_0 + a]$. By Theorem 53, we have $\int_{t_0}^{t_0+a} \tilde{f}(t, x_t)\mathrm{d}t \in SL([[t_0, a], \mathbb{R}_{\mathcal{F}})$ and $x'(t) = \mathrm{d}/\mathrm{d}t \int_{t_0}^{t} \tilde{f}(\sigma, x_\sigma)\mathrm{d}\sigma = \tilde{f}(t, x_t)$ a.e. on $[t_0, a]$. That is, $x(t) = x(s) + \int_{s}^{t} \tilde{f}(\sigma, x_\sigma)\mathrm{d}\sigma$.

Now, we shall prove that $x(t_{k+1}) = x(s) + \int_s^{t_{k+1}} \tilde{f}(\sigma, x_\sigma)\mathrm{d}\sigma$ for $s \in (t_k, t_{k+1}]$. By Proposition 55, we have

$$\begin{aligned}\int_s^{t_{k+1}} \tilde{f}(\sigma, x_\sigma)\mathrm{d}\sigma &= \lim_{t\to t_{k+1}} \int_s^t \tilde{f}(\sigma, x_\sigma)\mathrm{d}\sigma = \lim_{t\to t_{k+1}} D(x(s), x(t)) \\ &= x(t_{k+1}-) \ominus_H x(s) = x(t_{k+1}) \ominus_H x(s)\end{aligned}$$

This completed the proof.

Lemma 4 *If x is a generalized (ii)-solution of the problem (7), the x is a generalized (ii)-solution of the problem (5) and $x \in SL_{\phi,x_0}$.*

Theorem 4 *Let T be the operator given by*

$$Tx(t) = x_0 + \int_{t_0}^t \tilde{f}(s, x_s)\mathrm{d}s + \sum_{t_0<t_k<t} [x(t_k+) \ominus_H x(t_k)]$$

for $t \in [t_0.t_0 + a]$. Then T maps C_{ϕ,x_0} into C_{ϕ,x_0}.

Proof. Let $x : [t_0 - r, t_0 + a] \to \mathbb{R}_\mathcal{F}$ be such that $x \in C_{\phi,x_0}$. Then $Tx(t_0) = x_0$. Given $\varepsilon > 0$, we have $Tx(t)$ is continuous for $t \in [t_0-r, t_0+a]\backslash\{t_1, t_2, \cdots\cdots, t_m\}$. On the other hand, we have

$$\begin{aligned}& \lim_{\varepsilon\to 0} Tx(t_k - \varepsilon) \\ = \;& \lim_{\varepsilon\to 0} \left[x_0 + \int_{t_0}^{t_k-\varepsilon} \tilde{f}(s, x_s)\mathrm{d}s + \sum_{t_0<t_i<t_k-\varepsilon} [x(t_i+) \ominus_H x(t_i)]\right] \\ = \;& x_0 + \int_{t_0}^{t_k} \tilde{f}(s, x_s)\mathrm{d}s + \sum_{t_0<t_i<t_k} [x(t_i+) \ominus_H x(t_i)] \\ = \;& Tx(t_k).\end{aligned}$$

Hence, $Tx(t_k)$ is left continuous for $k = 1, 2, \cdots, m$. Finally, we consider

$$\begin{aligned}& \lim_{\varepsilon\to 0} Tx(t_k + \varepsilon) \\ = \;& \lim_{\varepsilon\to 0} \left[x_0 + \int_{t_0}^{t_k+\varepsilon} \tilde{f}(s, x_s)\mathrm{d}s + \sum_{t_0<t_i\le t_k+\varepsilon} [x(t_i+) \ominus_H x(t_i)]\right] \\ = \;& x_0 + \int_{t_0}^{t_k} \tilde{f}(s, x_s)\mathrm{d}s + \sum_{t_0<t_i\le t_k} [x(t_i+) \ominus_H x(t_i)] \\ = \;& Tx(t_k) + [x(t_k+) \ominus_H x(t_k)].\end{aligned}$$

and hence there exists $Tx(t_k+)$ for $k = 1, 2, \cdots, m$.

Theorem 5 *Suppose that $\tilde{f}(t, x_t)$ is fuzzy Henstock integrable for any $x \in C_{\phi,x_0}$. If there exist positive number M and $a_1, a_2, \cdots, a_m$ such that*

8

(i) $D(\tilde{f}(t,\varphi_1),\tilde{f}(t,\varphi_2)) \leq M\cdot D(\varphi_1,\varphi_2)$ *for every* $t\in[t_0,t_0+a]$ *and every* $\varphi_1,\varphi_2\in FH([-r,0],\mathbb{R}_{\mathcal{F}})$*;*

(ii) $D(I_k(u),I_k(v))\leq a_k\cdot D(u,v)$ *for every* $u,v\in\mathbb{R}_{\mathcal{F}}$ *and* $k\in\{1,2,\cdots,m\}$*;*

(iii) $a^2\cdot M+\sum_{k=1}^{m}a_k<1$*;*

then there exist two generalized solutions $x(t)=x(t,x_0,\phi)$ *of problem (1),*$x(\cdot,x_0,\phi)\in SL_{\phi,x_0}$ *depending continously on each variable.*

Proof. Let $r,a>0$ and $F=\{y\in C_0: y(0)=\tilde{0}\}$. For y^1, $y^2\in F$, we define the distence

$$H(y^1_{\phi,x_0},y^2_{\phi,x_0})=\sup_{t\in[0,a]}D(y^1_{\phi,x_0},y^2_{\phi,x_0}).$$

For any $y\in F, x_0\in\mathbb{R}_{\mathcal{F}}$ and $\phi\in FH([-r,0],\mathbb{R}_{\mathcal{F}})$ we define an auxiliary fuzzy-number-valued function $y_{\phi,x_0}:[-r,a]\to\mathbb{R}_{\mathcal{F}}$ by

$$y_{\phi,x_0}(t)=\begin{cases} y(t), & t\in[0,a]\\ \phi(t)\ominus_H x_0, & x\in[-r,0].\end{cases}$$

Then, $(y_{\phi,x_0})_t\in FH([-r,0],\mathbb{R}_{\mathcal{F}})$ for all $t\in[0,a]$ and $(y_{\phi,x_0})_t(\theta)=y_{\phi,x_0}(t+\theta),\theta\in[-r,0]$.

We consider the family

$$\mathcal{U}_F=\{U_{\phi,x_0}:\phi\in FH([-r,0],\mathbb{R}_{\mathcal{F}}),x_0\in\mathbb{R}_{\mathcal{F}}\}$$

of operators from F to C_0 given by

$$U_{\phi,x_0}y(t)=\int_{t_0}^{t+t_0}\tilde{f}(s,(y_{\phi,x_0})_{s-t_0}+x_0)\mathrm{d}s+\sum_{t_0<t_k<t+t_0}I_k(y(t_k-t_0))$$

for all $t\in[0,a]$.

Firstly, we prove that $\mathcal{U}_F$ is a contraction operators. In fact, by using hypothesis (i) and (ii)

$$\begin{aligned} & H(U_{\phi,x_0}y^1(t),U_{\phi,x_0}y^2(t))=\sup_{t\in[0,a]}D(U_{\phi,x_0}y^1(t),U_{\phi,x_0}y^2(t))\\ = & \sup_{t\in[0,a]}D(\int_{t_0}^{t+t_0}\tilde{f}(s,(y^1_{\phi,x_0})_{s-t_0}+x_0)\mathrm{d}s,\int_{t_0}^{t+t_0}\tilde{f}(s,(y^2_{\phi,x_0})_{s-t_0}+x_0)\mathrm{d}s)\\ + & \sum_{t_0<t_k<t+t_0}I_k(y^1(t_k-t_0)\ominus_H I_k(y^2(t_k-t_0))\\ \leq & \sup_{t\in[0,a]}D(\int_{t_0}^{t+t_0}\tilde{f}(s,(y^1_{\phi,x_0})_{s-t_0})\mathrm{d}s,\int_{t_0}^{t+t_0}\tilde{f}(s,(y^2_{\phi,x_0})_{s-t_0})\mathrm{d}s)\\ \leq & M\cdot((L)\int_{t_0}^{a+t_0}D((y^1_{\phi,x_0})_{s-t_0},(y^2_{\phi,x_0})_{s-t_0})\mathrm{d}s)+\sum_{t_0<t_k<t+t_0}a_kD(y^1,y^2).\end{aligned}$$

In addition, since $(y^1_{\phi,x_0})_{s-t_0}, (y^2_{\phi,x_0})_{s-t_0} \in FH([-r,0],\mathbb{R}_{\mathcal{F}})$, we have

$$\begin{aligned} & D(y^1_{\phi,x_0})_{s-t_0}, (y^2_{\phi,x_0})_{s-t_0}) \\ = & \sup_{-v\in[-r,0]} D(\int_{-r}^{-v} (y^1_{\phi,x_0})_{s-t_0}(\theta)\mathrm{d}\theta, \int_{-r}^{-v} (y^2_{\phi,x_0})_{s-t_0}(\theta)\mathrm{d}\theta). \end{aligned}$$

Thus, we have

$$\begin{aligned} & D(\int_{-r}^{-v} (y^1_{\phi,x_0})_{s-t_0}(\theta)\mathrm{d}\theta, \int_{-r}^{-v} (y^2_{\phi,x_0})_{s-t_0}(\theta)\mathrm{d}\theta) \\ = & D(\int_{t_0-s}^{-v} y^1(s-t_0+\theta)\mathrm{d}\theta, \int_{t_0-s}^{-v} y^2(s-t_0+\theta)\mathrm{d}\theta) \\ \leq & (L)\int_0^{-v+s-t_0} D(y^1(u), y^2(u))\mathrm{d}u \\ \leq & (-v+s-t_0)D(y^1,y^2) \leq aD(y^1,y^2). \end{aligned}$$

This implies that $\mathcal{U}_F$ is a contraction operators.

Next, we shall prove $\mathcal{U}_F$ is a continuous at x_0 and ϕ. Given $\varepsilon > 0$, let $\phi_1, \phi_2 \in FH([-r,0],\mathbb{R}_{\mathcal{F}})$ and $0 < \delta \leq \varepsilon/2$ be such that $D(\phi_1,\phi_2) < \delta$. Then, by hypothesis (i), we have

$$\begin{aligned} & H(T_{\phi_1,x_0}y(t), T_{\phi_2,x_0}y(t)) = \sup_{t\in[0,a]} D(T_{\phi_1,x_0}y(t), T_{\phi_2,x_0}y(t)) \\ = & D(\int_{t_0}^{t} +t_0\tilde{f}(s,(y_{\phi_1,x_0})_{s-t_0}+x_0)\mathrm{d}s, \int_{t_0}^{t} +t_0\tilde{f}(s,(y_{\phi_2,x_0})_{s-t_0}+x_0)\mathrm{d}s) \\ \leq & M\cdot(L)\int_{t_0}^{a} +t_0 D((y_{\phi_1,x_0})_{s-t_0}, (y_{\phi_2,x_0})_{s-t_0})\mathrm{d}s. \end{aligned}$$

Therefore, we get

$$\begin{aligned} & D(\int_{-r}^{-v} (y_{\phi_1,x_0})_{s-t_0}(\theta)\mathrm{d}\theta, \int_{-r}^{-v} (y_{\phi_2,x_0})_{s-t_0}(\theta)\mathrm{d}\theta) \\ = & D(\int_{-r}^{t_0-s} \phi_1(s-t_0+\theta)\mathrm{d}\theta, \int_{-r}^{t_0-s} \phi_2(s-t_0+\theta)\mathrm{d}\theta) \\ = & D(\int_{-r-t_0+s}^{0} \phi_1(u)\mathrm{d}u, \int_{-r-t_0+s}^{0} \phi_2(u)\mathrm{d}u) \\ \leq & D(\int_{-r}^{0} \phi_1(u)\mathrm{d}u, \int_{-r}^{0} \phi_2(u)\mathrm{d}u) + D(\int_{-r}^{-r-t_0+s} \phi_1(u)\mathrm{d}u, \int_{-r}^{-r-t_0+s} \phi_2(u)\mathrm{d}u) \\ \leq & 2\cdot \sup_{-v\in[-r,0]} D(\int_{-r}^{-v} \phi_1(u)\mathrm{d}u, \int_{-r}^{-r} \phi_2(u)\mathrm{d}u) < \varepsilon. \end{aligned}$$

At last, since $FH([-r,0],\mathbb{R}_{\mathcal{F}})\times\mathbb{R}_{\mathcal{F}}$ is a complete space and by using Lemma 3, we conclude that $x(\cdot,x_0,\phi) \in SL_{\phi,x_0}$ and it is a generalized solutions $x(t) = x(t,x_0,\phi)$ of problem (1).

4 CONCLUSIONS

In this paper, we study the Cauchy problem of discontinuous fuzzy impulsive retarded differential equations involving the fuzzy Henstock integral in fuzzy number space. The function governing the equations is supposed to be discontinuous with respect to some variables and satisfy nonabsolute fuzzy integrablility. Our result improves the result given in Ref. [14] and [17] (where uniform continuity was required), as well as those referred therein.

References

[1] B. Bede, S. Gal, Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equation, *Fuzzy Sets and Systems* 151, 581-599 (2005).

[2] Y. Chalco-Cano, H. Roman-Flores, On the new solution of fuzzy differential equations, *Chaos, Solition* & *Fractals* 38, 112-119 (2008).

[3] T. S. Chew, W. Van Brunt, G. C. Wake, On retarded functional differential equations and Henstock-Kurzweil integrals, *Diff. Integ. Equa.* 9, 569-580 (1996).

[4] P. Diamond, P. Kloeden, *Metric Space of Fuzzy Sets: Theory and Applications*, World Scientific, Singapore, 1994.

[5] D. Dubois, H. Prade, Towards Fuzzy Differential Calculus, Part 1. Integration of fuzzy mappings, *Fuzzy Sets and Syst.* 8, 1-17 (1982).

[6] M. Federson , P. Táboas, Topological dynamics of retarded functional differential equations, *J. Diff. Eq.* 195(2), 313-331 (2003).

[7] Z. Gong, On the Problem of Characterizing Derivatives for the Fuzzy-valued Functions (II): almost everywhere differentiability and strong Henstock integral, *Fuzzy Sets and Syst.* 145, 381-393 (2004).

[8] Z. Gong, Y. Shao, The Controlled Convergence Theorems for the Strong Henstock Integrals of Fuzzy-Number-Valued Functions, *Fuzzy Sets and Syst.* 160, 1528-1546 (2009).

[9] Z. Gong, C. Wu, Bounded variation, absolute continuity and absolute integrability for fuzzy-number-valued functions, *Fuzzy Sets Syst.* 129, 83-94 (2002).

[10] M. Guo, X. Peng, Y. Xu, Oscillation property for fuzzy delay differential equations, *Fuzzy Sets and Syst.* 200, 25-35 (2012).

[11] J. K. Hale, *Theory of Functional Differential Equations*, Springer, NewYork, 1997.

[12] R. Henstock, *Theory of Integration.* Butterworth, London, (1963)

[13] O. Kaleva, The Cauchy Problem for Fuzzy Differential Equations, *Fuzzy sets and syst.* 35, 389-396 (1990).

[14] A. Khastan, J.J. Nieto, R. R. Rodríguez-López, Fuzzy delay differential equations under generalized differentiability, *Inf. Sci.* 275, 145-167 (2014).

[15] P. E. Kloeden, T. Lorenz, Fuzzy differential equations without fuzzy convexity, *Fuzzy Sets Syst.* 230(1), 65-81 (2013).

[16] J. Kurzweil, Generalized ordinary differential equations and continuous dependence on a Parameter, *Czechoslovak Math. J.* 7, 418-446 (1957).

[17] V. Lakshmikantham, R.N. Mohapatra, *Theory of Fuzzy Differential Equations and Inclusions*, Taylor & Francis, London, 2003.

[18] P. Y. Lee, *Lanzhou Lectures on Henstock Integration*, World Scientific, Singapore, New Jersey, London, Hongkong, 1989.

[19] V. Lupulescu, On a class of fuzzy functional differential equations, *Fuzzy Sets Syst.* 160, 1547-1562 (2009).

[20] Qiang Ma, Ya-bin Shao, and Zi-zhong Chen, A Kind of Generalized Fuzzy Integro-Differential Equations of Mixed Type and Strong Fuzzy Henstock Integrals, *J. Comput. Anal. and Appl.* 23(1), 92-107 (2017).

[21] M. L. Puri, D. A. Ralescu, Differentials of Fuzzy Functions, *J. Math. Anal. Appl.* 91, 552-558 (1983).

[22] T. L. Toh, T. S. Chew, On functional differential equations with unbounded delay and Henstock-Kurzweil integrals, New Zealand J. Math. 28, (1999) 111-123.

[23] Yabin Shao, Huanhuan Zhang, Fuzzy integral equations and strong fuzzy Henstock integrals, *Abstract and Applied Analysis* Volume 2014, Article ID 932696, 8 pages.

[24] Yabin Shao, Huanhuan Zhang, The strong fuzzy Henstock integrals and discontinuous fuzzy differential equations, *Journal of Applied Mathematics* Volume 2013, Article ID 419701, 8 pages.

[25] C. Wu, Z. Gong, On Henstock Intergrals of Interval-Valued and Fuzzy-Number-Valued Functions, *Fuzzy Sets and Systems* 115, 377-391 (2000).

[26] C. Wu, Z. Gong, On Henstock Intergrals of Fuzzy-valued Functions (I), *Fuzzy Sets and Syst.* 120, 523-532 (2001).

[27] Hsien-Chung Wu, The improper fuzzy Riemann integral and its numerical integration, *Inf. Sci.* 111, 109-137 (1998).

[28] L. Zadeh, Fuzzy Sets, *Information and Control* 3, 338-353 (1965).

PRODUCT-TYPE SYSTEM OF DIFFERENCE EQUATIONS WITH MULTIPLICATIVE COEFFICIENTS SOLVABLE IN CLOSED FORM

STEVO STEVIĆ*, BRATISLAV IRIČANIN, AND ZDENĚK ŠMARDA

ABSTRACT. Solvability of the following system of difference equations

$$z_{n+1} = \alpha z_n^a w_{n-1}^b, \quad w_{n+1} = \beta w_n^c z_{n-1}^d, \quad n \in \mathbb{N}_0,$$

where $a, b, c, d \in \mathbb{Z}$, $\alpha, \beta \in \mathbb{C}$, $z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$, is studied.

1. INTRODUCTION

Numerous concrete nonlinear difference equations and systems have attracted some recent attention (see, e.g., [1]-[6], [8], [9], [13]-[49]). There has been also a renewed interest in the problem of their solvability (see, e.g., [1]-[4], [6], [17], [20], [21], [25]-[41], [43]-[49]), especially after the publication of [20] where S. Stević used a nice change of variables for explaining and extending the formula for solutions to the second-order difference equation in [6]. This motivated several authors to develop the idea and apply it for the case of some extensions of the equation, as well as some other equations and systems (see, e.g., [1], [3], [4], [17], [21], [25]-[30], [32]-[36], [39], [41], [43]-[47], [49]). Some classical methods for solving difference equations and systems can be found, e.g., in [7], [10]-[12]. On the other hand, there has been some recent interest in close to symmetric systems of difference equations (see, e.g., [4], [13]-[15], [18], [19], [25], [27]-[29], [31]-[44], [46], [48], [49]). Many of the above mentioned classes of difference equations and systems are obtained from the product-type ones by the translation or max-type operators (see, e.g., [16], [22]-[24], [42]). Some max-type systems are even solvable ([31]).

Product-type equations are solvable if the initial values and coefficients are positive, which is not the case in general. This suggests investigation of product-type equations and systems for the case when

2000 *Mathematics Subject Classification.* Primary 39A10; 39A20.

Key words and phrases. System of difference equations, product-type system, solvable system.

*Corresponding author.

the coefficients and initial values are not positive. The problem was studied for the first time in [40] (an extension of the system in [40] was later studied in [38]), but with methods not so close to those in [20] (or in [26], [30] etc.). They are, in fact, more related to the ones in [22], [23] and [42]. In [48] was investigated the problem of solvability of the system

$$z_{n+1} = \frac{z_n^{\hat{a}}}{w_{n-1}^{\hat{b}}}, \quad w_{n+1} = \frac{w_n^{\hat{c}}}{z_{n-1}^{\hat{d}}}, \quad n \in \mathbb{N}_0, \tag{1}$$

where $\hat{a}, \hat{b}, \hat{c}, \hat{d} \in \mathbb{Z}$ (the condition is posed to avoid multi-valued solutions) and $z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C}\setminus\{0\}$. Soon after that, in [37], was shown that for the case of another product-type system some coefficients can be included to get again a solvable system.

Here we show that the same holds for the next natural extension of system (1)

$$z_{n+1} = \alpha z_n^a w_{n-1}^b, \quad w_{n+1} = \beta w_n^c z_{n-1}^d, \quad n \in \mathbb{N}_0, \tag{2}$$

where $a, b, c, d \in \mathbb{Z}$, $\alpha, \beta \in \mathbb{C}$ and $z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C}$. Since the cases $\alpha = 0$ and $\beta = 0$ are trivial, we will assume that $\alpha, \beta \in \mathbb{C} \setminus \{0\}$.

Note that the domain of undefinable solutions ([32]) to system (2) is a subset of

$$\mathcal{U} = \{(z_{-1}, z_0, w_{-1}, w_0) \in \mathbb{C}^4 : z_{-1} = 0 \ \text{ or } \ z_0 = 0 \ \text{ or } \ w_{-1} = 0 \ \text{ or } \ w_0 = 0\}.$$

This is why we will also assume that $z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$.

2. Main result

The problem of solvability of system (2) is studied in this section.

Theorem 1. *Assume that $a, b, c, d \in \mathbb{Z}$, $\alpha, \beta \in \mathbb{C}\setminus\{0\}$ and $z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. Then system (2) is solvable in closed form.*

Proof. Case $b = 0$. Since $b = 0$ system (2) is

$$z_{n+1} = \alpha z_n^a, \quad w_{n+1} = \beta w_n^c z_{n-1}^d, \quad n \in \mathbb{N}_0. \tag{3}$$

The first equation in (3) yields

$$z_n = \alpha^{\sum_{j=0}^{n-1} a^j} z_0^{a^n}, \quad n \in \mathbb{N}. \tag{4}$$

Hence, if $a \neq 1$ we have

$$z_n = \alpha^{\frac{1-a^n}{1-a}} z_0^{a^n}, \quad n \in \mathbb{N}, \tag{5}$$

while if $a = 1$ we have

$$z_n = \alpha^n z_0, \quad n \in \mathbb{N}. \tag{6}$$

Using (4) in the second equation in (3) we have

$$w_n = \beta\alpha^{d\sum_{j=0}^{n-3} a^j} z_0^{da^{n-2}} w_{n-1}^c, \tag{7}$$

for $n \geq 3$.

From (7) we get

$$\begin{aligned} w_n &= \beta\alpha^{d\sum_{j=0}^{n-3} a^j} z_0^{da^{n-2}} (\beta\alpha^{d\sum_{j=0}^{n-4} a^j} z_0^{da^{n-3}} w_{n-2}^c)^c \\ &= \beta^{1+c}\alpha^{d\sum_{j=0}^{n-3} a^j + dc\sum_{j=0}^{n-4} a^j} z_0^{da^{n-2}+dca^{n-3}} w_{n-2}^{c^2}, \end{aligned} \tag{8}$$

for $n \geq 4$.

Equalities (7) and (8) suggest that for a $k \in \mathbb{N}$

$$w_n = \beta^{\sum_{i=0}^{k-1} c^i} \alpha^{d\sum_{i=0}^{k-1} c^i \sum_{j=0}^{n-i-3} a^j} z_0^{d\sum_{i=0}^{k-1} c^i a^{n-i-2}} w_{n-k}^{c^k}, \tag{9}$$

for $n \geq k+2$. Assume that (9) holds for some $k \in \mathbb{N}$ and every $n \geq k+2$.

Applying (7) with $n \to n-k$ in (9), it follows that

$$\begin{aligned} w_n &= \beta^{\sum_{i=0}^{k-1} c^i} \alpha^{d\sum_{i=0}^{k-1} c^i \sum_{j=0}^{n-i-3} a^j} z_0^{d\sum_{i=0}^{k-1} c^i a^{n-i-2}} (\beta\alpha^{d\sum_{j=0}^{n-k-3} a^j} z_0^{da^{n-k-2}} w_{n-k-1}^c)^{c^k} \\ &= \beta^{\sum_{i=0}^{k} c^i} \alpha^{d\sum_{i=0}^{k} c^i \sum_{j=0}^{n-i-3} a^j} z_0^{d\sum_{i=0}^{k} c^i a^{n-i-2}} w_{n-k-1}^{c^{k+1}}, \end{aligned} \tag{10}$$

for $n \geq k+3$.

Equalities (7), (10) along with the induction show that (9) holds for all natural numbers k and n such that $1 \leq k \leq n-2$.

For $k = n-2$ equality (9) becomes

$$w_n = \beta^{\sum_{i=0}^{n-3} c^i} \alpha^{d\sum_{i=0}^{n-3} c^i \sum_{j=0}^{n-i-3} a^j} z_0^{d\sum_{i=0}^{n-3} c^i a^{n-i-2}} w_2^{c^{n-2}}, \tag{11}$$

for $n \geq 3$.

From (11) and since

$$w_2 = \beta w_1^c z_0^d = \beta(\beta w_0^c z_{-1}^d)^c z_0^d = \beta^{1+c} z_{-1}^{cd} z_0^d w_0^{c^2},$$

we have

$$\begin{aligned} w_n &= \beta^{\sum_{i=0}^{n-3} c^i} \alpha^{d\sum_{i=0}^{n-3} c^i \sum_{j=0}^{n-i-3} a^j} z_0^{d\sum_{i=0}^{n-3} c^i a^{n-i-2}} (\beta^{1+c} z_{-1}^{cd} z_0^d w_0^{c^2})^{c^{n-2}} \\ &= \beta^{\sum_{i=0}^{n-1} c^i} \alpha^{d\sum_{i=0}^{n-3} c^i \sum_{j=0}^{n-i-3} a^j} z_0^{d\sum_{i=0}^{n-2} c^i a^{n-i-2}} z_{-1}^{dc^{n-1}} w_0^{c^n}, \end{aligned} \tag{12}$$

for $n \geq 3$. A direct verification shows that (12) also holds for $n = 2$.

Case $a \neq c$. In this case from (12) we get

$$w_n = \beta^{\sum_{i=0}^{n-1} c^i} \alpha^{d\sum_{i=0}^{n-3} c^i \sum_{j=0}^{n-i-3} a^j} z_0^{d\frac{a^{n-1}-c^{n-1}}{a-c}} z_{-1}^{dc^{n-1}} w_0^{c^n}, \quad n \geq 2. \tag{13}$$

Subcase $a \neq 1 \neq c$. In this case we have

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 24, NO.6, 2018, COPYRIGHT 2018 EUDOXUS PRESS, LLC

$$\begin{aligned}
w_n =&\beta^{\frac{1-c^n}{1-c}}\alpha^{d\sum_{i=0}^{n-3}c^i\frac{1-a^{n-i-2}}{1-a}}z_0^{d\frac{a^{n-1}-c^{n-1}}{a-c}}z_{-1}^{dc^{n-1}}w_0^{c^n}\\
=&\beta^{\frac{1-c^n}{1-c}}\alpha^{\frac{d}{1-a}\left(\frac{1-c^{n-2}}{1-c}-a\frac{a^{n-2}-c^{n-2}}{a-c}\right)}z_0^{d\frac{a^{n-1}-c^{n-1}}{a-c}}z_{-1}^{dc^{n-1}}w_0^{c^n}\\
=&\beta^{\frac{1-c^n}{1-c}}\alpha^{\frac{d(a-c+(1-a)c^{n-1}+(c-1)a^{n-1})}{(1-a)(1-c)(a-c)}}z_0^{d\frac{a^{n-1}-c^{n-1}}{a-c}}z_{-1}^{dc^{n-1}}w_0^{c^n},\quad n\ge 2. \qquad (14)
\end{aligned}$$

Subcase $a \neq 1 = c$. In this case we have

$$\begin{aligned}
w_n =&\beta^n\alpha^{d\sum_{i=0}^{n-3}\frac{1-a^{n-i-2}}{1-a}}z_0^{d\frac{a^{n-1}-1}{a-1}}z_{-1}^{d}w_0\\
=&\beta^n\alpha^{\frac{d}{1-a}\left(n-2-a\frac{a^{n-2}-1}{a-1}\right)}z_0^{d\frac{a^{n-1}-1}{a-1}}z_{-1}^{d}w_0\\
=&\beta^n\alpha^{\frac{d(a^{n-1}-(n-1)a+n-2)}{(a-1)^2}}z_0^{d\frac{a^{n-1}-1}{a-1}}z_{-1}^{d}w_0,\quad n\ge 2. \qquad (15)
\end{aligned}$$

Subcase $a = 1 \neq c$. In this case we have

$$\begin{aligned}
w_n =&\beta^{\frac{1-c^n}{1-c}}\alpha^{d\sum_{i=0}^{n-3}c^i(n-i-2)}z_0^{d\frac{1-c^{n-1}}{1-c}}z_{-1}^{dc^{n-1}}w_0^{c^n}\\
=&\beta^{\frac{1-c^n}{1-c}}\alpha^{d\left((n-2)\frac{1-c^{n-2}}{1-c}-c\frac{1-(n-2)c^{n-3}+(n-3)c^{n-2}}{(1-c)^2}\right)}z_0^{d\frac{1-c^{n-1}}{1-c}}z_{-1}^{dc^{n-1}}w_0^{c^n}\\
=&\beta^{\frac{1-c^n}{1-c}}\alpha^{d\frac{n-2-(n-1)c+c^{n-1}}{(1-c)^2}}z_0^{d\frac{1-c^{n-1}}{1-c}}z_{-1}^{dc^{n-1}}w_0^{c^n},\quad n\ge 2. \qquad (16)
\end{aligned}$$

Case $a = c$. In this case from (12) we get

$$w_n = \beta^{\sum_{i=0}^{n-1}c^i}\alpha^{d\sum_{i=0}^{n-3}c^i\sum_{j=0}^{n-i-3}c^j}z_0^{(n-1)dc^{n-2}}z_{-1}^{dc^{n-1}}w_0^{c^n}, \qquad (17)$$

for $n \ge 2$.

Subcase $a = c \neq 1$. In this case we have

$$\begin{aligned}
w_n =&\beta^{\frac{1-c^n}{1-c}}\alpha^{d\sum_{i=0}^{n-3}c^i\frac{1-c^{n-i-2}}{1-c}}z_0^{(n-1)dc^{n-2}}z_{-1}^{dc^{n-1}}w_0^{c^n}\\
=&\beta^{\frac{1-c^n}{1-c}}\alpha^{\frac{d}{1-c}\left(\frac{1-c^{n-2}}{1-c}-(n-2)c^{n-2}\right)}z_0^{(n-1)dc^{n-2}}z_{-1}^{dc^{n-1}}w_0^{c^n}\\
=&\beta^{\frac{1-c^n}{1-c}}\alpha^{\frac{d(1-(n-1)c^{n-2}+(n-2)c^{n-1})}{(1-c)^2}}z_0^{(n-1)dc^{n-2}}z_{-1}^{dc^{n-1}}w_0^{c^n},\quad n\ge 2. \qquad (18)
\end{aligned}$$

Subcase $a = c = 1$. In this case we have

$$\begin{aligned}
w_n =&\beta^n\alpha^{d\sum_{i=0}^{n-3}(n-i-2)}z_0^{(n-1)d}z_{-1}^{d}w_0\\
=&\beta^n\alpha^{d\frac{(n-2)(n-1)}{2}}z_0^{(n-1)d}z_{-1}^{d}w_0,\quad n\ge 2. \qquad (19)
\end{aligned}$$

Case $d = 0$. Since $d = 0$ system (2) is

$$z_{n+1} = \alpha z_n^a w_{n-1}^b,\quad w_{n+1} = \beta w_n^c,\quad n \in \mathbb{N}_0. \qquad (20)$$

Second equation in (20) yields

$$w_n = \beta^{\sum_{j=0}^{n-1}c^j}w_0^{c^n},\quad n \in \mathbb{N}, \qquad (21)$$

from which it follows that

$$w_n = \beta^{\frac{1-c^n}{1-c}} w_0^{c^n}, \quad n \in \mathbb{N}, \tag{22}$$

if $c \neq 1$, while if $c = 1$, then

$$w_n = \beta^n w_0, \quad n \in \mathbb{N}. \tag{23}$$

Using (21) in the first equation in (20) we have

$$z_n = \alpha\beta^{b\sum_{j=0}^{n-3} c^j} w_0^{bc^{n-2}} z_{n-1}^a, \tag{24}$$

for $n \geq 3$.

From (24) we get

$$\begin{aligned} z_n &= \alpha\beta^{b\sum_{j=0}^{n-3} c^j} w_0^{bc^{n-2}} (\alpha\beta^{b\sum_{j=0}^{n-4} c^j} w_0^{bc^{n-3}} z_{n-2}^a)^a \\ &= \alpha^{1+a}\beta^{b\sum_{j=0}^{n-3} c^j + ba\sum_{j=0}^{n-4} c^j} w_0^{bc^{n-2}+bac^{n-3}} z_{n-2}^{a^2}, \end{aligned}$$

for $n \geq 4$.

Assume that for a $k \in \mathbb{N}$ we have proved

$$z_n = \alpha^{\sum_{i=0}^{k-1} a^i} \beta^{b\sum_{i=0}^{k-1} a^i \sum_{j=0}^{n-i-3} c^j} w_0^{b\sum_{i=0}^{k-1} a^i c^{n-i-2}} z_{n-k}^{a^k}, \tag{25}$$

for $n \geq k+2$.

Applying (24) with $n \to n-k$ in (25), it follows that

$$\begin{aligned} z_n &= \alpha^{\sum_{i=0}^{k-1} a^i} \beta^{b\sum_{i=0}^{k-1} a^i \sum_{j=0}^{n-i-3} c^j} w_0^{b\sum_{i=0}^{k-1} a^i c^{n-i-2}} (\alpha\beta^{b\sum_{j=0}^{n-k-3} c^j} w_0^{bc^{n-k-2}} z_{n-k-1}^a)^{a^k} \\ &= \alpha^{\sum_{i=0}^{k} a^i} \beta^{b\sum_{i=0}^{k} a^i \sum_{j=0}^{n-i-3} c^j} w_0^{b\sum_{i=0}^{k} a^i c^{n-i-2}} z_{n-k-1}^{a^{k+1}}, \end{aligned} \tag{26}$$

for $n \geq k+3$.

Equalities (24), (26) along with the induction show that (25) holds for all natural numbers k and n such that $1 \leq k \leq n-2$.

For $k = n-2$ equality (25) becomes

$$z_n = \alpha^{\sum_{i=0}^{n-3} a^i} \beta^{b\sum_{i=0}^{n-3} a^i \sum_{j=0}^{n-i-3} c^j} w_0^{b\sum_{i=0}^{n-3} a^i c^{n-i-2}} z_2^{a^{n-2}}, \tag{27}$$

for $n \geq 3$.

From (27) and since

$$z_2 = \alpha z_1^a w_0^b = \alpha(\alpha z_0^a w_{-1}^b)^a w_0^b = \alpha^{1+a} w_{-1}^{ab} w_0^b z_0^{a^2},$$

we have

$$\begin{aligned} z_n &= \alpha^{\sum_{i=0}^{n-3} a^i} \beta^{b\sum_{i=0}^{n-3} a^i \sum_{j=0}^{n-i-3} c^j} w_0^{b\sum_{i=0}^{n-3} a^i c^{n-i-2}} (\alpha^{1+a} w_{-1}^{ab} w_0^b z_0^{a^2})^{a^{n-2}} \\ &= \alpha^{\sum_{i=0}^{n-1} a^i} \beta^{b\sum_{i=0}^{n-3} a^i \sum_{j=0}^{n-i-3} c^j} w_0^{b\sum_{i=0}^{n-2} a^i c^{n-i-2}} w_{-1}^{ba^{n-1}} z_0^{a^n}, \end{aligned} \tag{28}$$

for $n \geq 3$. A direct verification shows that (28) also holds for $n = 2$.

Case $a \neq c$. In this case from (28) we get

$$z_n = \alpha^{\sum_{i=0}^{n-1} a^i} \beta^{b\sum_{i=0}^{n-3} a^i \sum_{j=0}^{n-i-3} c^j} w_0^{b\frac{a^{n-1}-c^{n-1}}{a-c}} w_{-1}^{ba^{n-1}} z_0^{a^n}, \quad n \geq 2. \tag{29}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 24, NO.6, 2018, COPYRIGHT 2018 EUDOXUS PRESS, LLC

Subcase $a \neq 1 \neq c$. In this case we have

$$\begin{aligned} z_n =& \alpha^{\frac{1-a^n}{1-a}} \beta^{b \sum_{i=0}^{n-3} a^i \frac{1-c^{n-i-2}}{1-c}} w_0^{b \frac{a^{n-1}-c^{n-1}}{a-c}} w_{-1}^{ba^{n-1}} z_0^{a^n} \\ =& \alpha^{\frac{1-a^n}{1-a}} \beta^{\frac{b}{1-c}\left(\frac{1-a^{n-2}}{1-a} - c\frac{a^{n-2}-c^{n-2}}{a-c}\right)} w_0^{b \frac{a^{n-1}-c^{n-1}}{a-c}} w_{-1}^{ba^{n-1}} z_0^{a^n} \\ =& \alpha^{\frac{1-a^n}{1-a}} \beta^{\frac{b(a-c+(1-a)c^{n-1}+(c-1)a^{n-1})}{(1-b)(1-c)(a-c)}} w_0^{b \frac{a^{n-1}-c^{n-1}}{a-c}} w_{-1}^{ba^{n-1}} z_0^{a^n}, \quad n \geq 2. \end{aligned} \tag{30}$$

Subcase $a \neq 1 = c$. In this case we have

$$\begin{aligned} z_n =& \alpha^{\frac{1-a^n}{1-a}} \beta^{b \sum_{i=0}^{n-3} a^i (n-i-2)} w_0^{b \frac{a^{n-1}-1}{a-1}} w_{-1}^{ba^{n-1}} z_0^{a^n} \\ =& \alpha^{\frac{1-a^n}{1-a}} \beta^{b\left((n-2)\frac{1-a^{n-2}}{1-a} - a\frac{1-(n-2)a^{n-3}+(n-3)a^{n-2}}{(1-a)^2}\right)} w_0^{b \frac{a^{n-1}-1}{a-1}} w_{-1}^{ba^{n-1}} z_0^{a^n} \\ =& \alpha^{\frac{1-a^n}{1-a}} \beta^{b\frac{n-2-(n-1)a+a^{n-1}}{(1-a)^2}} w_0^{b \frac{a^{n-1}-1}{a-1}} w_{-1}^{ba^{n-1}} z_0^{a^n}, \quad n \geq 2. \end{aligned} \tag{31}$$

Subcase $a = 1 \neq c$. In this case we have

$$\begin{aligned} z_n =& \alpha^n \beta^{b \sum_{i=0}^{n-3} \frac{1-c^{n-i-2}}{1-c}} w_0^{b \frac{1-c^{n-1}}{1-c}} w_{-1}^{b} z_0 \\ =& \alpha^n \beta^{\frac{b}{1-c}\left((n-2) - c\frac{1-c^{n-2}}{1-c}\right)} w_0^{b \frac{1-c^{n-1}}{1-c}} w_{-1}^{b} z_0 \\ =& \alpha^n \beta^{b\frac{n-2-(n-1)c+c^{n-1}}{(1-c)^2}} w_0^{b \frac{1-c^{n-1}}{1-c}} w_{-1}^{b} z_0, \quad n \geq 2. \end{aligned} \tag{32}$$

Case $a = c$. In this case from (28) we get

$$z_n = \alpha^{\sum_{i=0}^{n-1} a^i} \beta^{b \sum_{i=0}^{n-3} a^i \sum_{j=0}^{n-i-3} a^j} w_0^{b(n-1)a^{n-2}} w_{-1}^{ba^{n-1}} z_0^{a^n}, \tag{33}$$

for $n \geq 2$.

Subcase $a = c \neq 1$. In this case we have

$$\begin{aligned} z_n =& \alpha^{\frac{1-a^n}{1-a}} \beta^{b \sum_{i=0}^{n-3} a^i \frac{1-a^{n-i-2}}{1-a}} w_0^{b(n-1)a^{n-2}} w_{-1}^{ba^{n-1}} z_0^{a^n} \\ =& \alpha^{\frac{1-a^n}{1-a}} \beta^{\frac{b}{1-a}\left(\frac{1-a^{n-2}}{1-a} - (n-2)a^{n-2}\right)} w_0^{b(n-1)a^{n-2}} w_{-1}^{ba^{n-1}} z_0^{a^n} \\ =& \alpha^{\frac{1-a^n}{1-a}} \beta^{b\frac{(1-(n-1)a^{n-2}+(n-2)a^{n-1})}{(1-a)^2}} w_0^{b(n-1)a^{n-2}} w_{-1}^{ba^{n-1}} z_0^{a^n}, \end{aligned} \tag{34}$$

for $n \geq 2$.

Subcase $a = c = 1$. In this case we have

$$\begin{aligned} z_n =& \alpha^n \beta^{b \sum_{i=0}^{n-3} (n-i-2)} w_0^{b(n-1)} w_{-1}^{b} z_0 \\ =& \alpha^n \beta^{b\frac{(n-2)(n-1)}{2}} w_0^{b(n-1)} w_{-1}^{b} z_0, \end{aligned} \tag{35}$$

for $n \geq 2$.

Case $bd \neq 0$. Since $z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$, it easily follows that $z_n \neq 0 \neq w_n$ for every $n \geq -1$. Hence, from the first equation in (2), we have

$$w_{n-1}^b = \frac{z_{n+1}}{\alpha z_n^a}, \quad n \in \mathbb{N}_0. \tag{36}$$

Taking both sides of the second equation in (2) to the b-th power we get

$$w_{n+1}^b = \beta^b w_n^{bc} z_{n-1}^{bd}, \quad n \in \mathbb{N}_0. \tag{37}$$

From (36) and (37) it follows that

$$\frac{z_{n+3}}{\alpha z_{n+2}^a} = \beta^b \frac{z_{n+2}^c}{\alpha^c z_{n+1}^{ac}} z_{n-1}^{bd}, \quad n \in \mathbb{N}_0,$$

that is,

$$z_{n+3} = \alpha^{1-c}\beta^b z_{n+2}^{a+c} z_{n+1}^{-ac} z_{n-1}^{bd}, \quad n \in \mathbb{N}_0, \tag{38}$$

which is a fourth order product-type difference equation.

We also have

$$z_1 = \alpha z_0^a w_{-1}^b \quad \text{and} \quad z_2 = \alpha^{1+a} z_0^{a^2} w_{-1}^{ab} w_0^b. \tag{39}$$

Let $\gamma := \alpha^{1-c}\beta^b$,

$$a_1 = a + c, \quad b_1 = -ac, \quad c_1 = 0, \quad d_1 = bd, \quad x_1 = 1. \tag{40}$$

Using (40), equation (38) can be written as

$$z_{n+3} = \gamma^{x_1} z_{n+2}^{a_1} z_{n+1}^{b_1} z_n^{c_1} z_{n-1}^{d_1}, \quad n \in \mathbb{N}_0. \tag{41}$$

Using (41) with $n \to n-1$ into (41) it follows that

$$\begin{aligned} z_{n+3} &= \gamma^{x_1}(\gamma z_{n+1}^{a_1} z_n^{b_1} z_{n-1}^{c_1} z_{n-2}^{d_1})^{a_1} z_{n+1}^{b_1} z_n^{c_1} z_{n-1}^{d_1}, \\ &= \gamma^{x_1+a_1} z_{n+1}^{a_1a_1+b_1} z_n^{b_1a_1+c_1} z_{n-1}^{c_1a_1+d_1} z_{n-2}^{d_1a_1} \\ &= \gamma^{x_2} z_{n+1}^{a_2} z_n^{b_2} z_{n-1}^{c_2} z_{n-2}^{d_2}, \end{aligned} \tag{42}$$

for $n \in \mathbb{N}$, where

$$a_2 := a_1a_1 + b_1, \quad b_2 := b_1a_1 + c_1, \quad c_2 := c_1a_1 + d_1, \quad d_2 := d_1a_1, \tag{43}$$

$$x_2 := x_1 + a_1. \tag{44}$$

Assume that for some $2 \leq k \leq n$, we have proved that

$$z_{n+3} = \gamma^{x_k} z_{n+3-k}^{a_k} z_{n+2-k}^{b_k} z_{n+1-k}^{c_k} z_{n-k}^{d_k}, \tag{45}$$

for $n \geq k-1$, and that

$$\begin{aligned} a_k &= a_1a_{k-1} + b_{k-1}, & b_k &= b_1a_{k-1} + c_{k-1},\\ c_k &= c_1a_{k-1} + d_{k-1}, & d_k &= d_1a_{k-1}, \end{aligned} \tag{46}$$

$$x_k := x_{k-1} + a_{k-1}. \tag{47}$$

Then by using (41) with $n \to n-k$ into (45) we have

$$\begin{aligned} z_{n+3} &= \gamma^{x_k}(\gamma z_{n+2-k}^{a_1} z_{n+1-k}^{b_1} z_{n-k}^{c_1} z_{n-1-k}^{d_1})^{a_k} z_{n+2-k}^{b_k} z_{n+1-k}^{c_k} z_{n-k}^{d_k}\\ &= \gamma^{x_k+a_k} z_{n+2-k}^{a_1a_k+b_k} z_{n+1-k}^{b_1a_k+c_k} z_{n-k}^{c_1a_k+d_k} z_{n-1-k}^{d_1a_k}\\ &= \gamma^{x_{k+1}} z_{n+2-k}^{a_{k+1}} z_{n+1-k}^{b_{k+1}} z_{n-k}^{c_{k+1}} z_{n-1-k}^{d_{k+1}}, \end{aligned} \tag{48}$$

for $n \ge k$, where

$$\begin{aligned} a_{k+1} &:= a_1a_k + b_k, & b_{k+1} &:= b_1a_k + c_k,\\ c_{k+1} &:= c_1a_k + d_k, & d_{k+1} &:= d_1a_k, \end{aligned} \tag{49}$$

$$x_{k+1} := x_k + a_k. \tag{50}$$

Equalities (48)-(50) along with (42)-(44) and the method of induction show that (45)-(47) hold for $2 \le k \le n+1$ ((45) also holds for $k = 1$).

Hence, for $k = n+1$, (45) becomes

$$\begin{aligned} z_{n+3} =& \gamma^{x_{n+1}} z_2^{a_{n+1}} z_1^{b_{n+1}} z_0^{c_{n+1}} z_{-1}^{d_{n+1}}\\ =& (\alpha^{1-c}\beta^b)^{x_{n+1}} (\alpha^{1+a} z_0^{a^2} w_{-1}^{ab} w_0^b)^{a_{n+1}} (\alpha z_0^a w_{-1}^b)^{b_{n+1}} z_0^{c_{n+1}} z_{-1}^{d_{n+1}}\\ =& \alpha^{(1-c)x_{n+1}+(1+a)a_{n+1}+b_{n+1}} \beta^{bx_{n+1}} z_0^{a^2a_{n+1}+ab_{n+1}+c_{n+1}}\\ &\times z_{-1}^{d_{n+1}} w_0^{ba_{n+1}} w_{-1}^{aba_{n+1}+bb_{n+1}}, \qquad n \in \mathbb{N}_0. \end{aligned} \tag{51}$$

From (46) we easily see that $(a_k)_{k\ge4}$ is a solution to the linear difference equation

$$a_k = a_1a_{k-1} + b_1a_{k-2} + c_1a_{k-3} + d_1a_{k-4}. \tag{52}$$

This and the relations $b_{k-1} = a_k - a_1a_{k-1}$, $c_{k-1} = b_k - b_1a_{k-1}$ and $d_k = d_1a_{k-1}$ show that $(b_k)_{k\in\mathbb{N}}$, $(c_k)_{k\in\mathbb{N}}$ and $(d_k)_{k\in\mathbb{N}}$ are also solutions to equation (52).

On the other hand, from (47) and since $x_1 = 1$, we have that

$$x_k = 1 + \sum_{j=1}^{k-1} a_j. \tag{53}$$

From (46) and (47) with $k = 1$ we get

$$\begin{aligned} &a_1 = a_1a_0 + b_0, \quad b_1 = b_1a_0 + c_0, \quad c_1 = c_1a_0 + d_0,\\ &d_1 = d_1a_0, \quad x_1 = x_0 + a_0. \end{aligned} \tag{54}$$

Since $bd = d_1 \neq 0$, from the fourth equation in (54) we get $a_0 = 1$. Using this fact in the other equalities in (54) we get $b_0 = c_0 = d_0 = 0$ and $x_0 = 0$ (since $x_1 = 1$).

From this and by (46) and (47) with $k = 0$ we get

$$\begin{aligned} &1 = a_0 = a_1 a_{-1} + b_{-1}, \quad 0 = b_0 = b_1 a_{-1} + c_{-1}, \\ &0 = c_0 = c_1 a_{-1} + d_{-1}, \quad 0 = d_0 = d_1 a_{-1}, \quad 0 = x_0 = x_{-1} + a_{-1}. \end{aligned} \tag{55}$$

Since $d_1 \neq 0$, from the fourth equation in (55) we get $a_{-1} = 0$. Using this fact in the other equalities in (55) we get $b_{-1} = 1$, $c_{-1} = d_{-1} = 0$ and $x_{-1} = 0$.

From this and by (46) and (47) with $k = -1$ we get

$$\begin{aligned} &0 = a_{-1} = a_1 a_{-2} + b_{-2}, \ \ 1 = b_{-1} = b_1 a_{-2} + c_{-2}, \\ &0 = c_{-1} = c_1 a_{-2} + d_{-2}, \ 0 = d_{-1} = d_1 a_{-2}, \ 0 = x_{-1} = x_{-2} + a_{-2}. \end{aligned} \tag{56}$$

Since $d_1 \neq 0$, from the fourth equation in (56) we get $a_{-2} = 0$. Using this fact in the other equalities in (56) we get $b_{-2} = 0$, $c_{-2} = 1$ and $d_{-2} = 0$ and $x_{-2} = 0$.

From this and by (46) and (47) with $k = -2$ we get

$$\begin{aligned} &0 = a_{-2} = a_1 a_{-3} + b_{-3}, \ 0 = b_{-2} = b_1 a_{-3} + c_{-3}, \\ &1 = c_{-2} = c_1 a_{-3} + d_{-3}, \ 0 = d_{-2} = d_1 a_{-3}, \ 0 = x_{-2} = x_{-3} + a_{-3}. \end{aligned} \tag{57}$$

Since $d_1 \neq 0$, from the fourth equation in (57) we get $a_{-3} = 0$. Using this fact in the other equalities in (57) we get $b_{-3} = 0$, $c_{-3} = 0$ and $d_{-3} = 1$ and $x_{-3} = 0$.

Hence, sequences $(a_k)_{k\geq -3}$, $(b_k)_{k\geq -3}$, $(c_k)_{k\geq -3}$ and $(d_k)_{k\geq -3}$ are solutions to linear difference equation (52) satisfying the next (shifted) initial conditions

$$\begin{aligned} &a_{-3} = 0, \quad a_{-2} = 0, \quad a_{-1} = 0, \quad a_0 = 1; \\ &b_{-3} = 0, \quad b_{-2} = 0, \quad b_{-1} = 1, \quad b_0 = 0; \\ &c_{-3} = 0, \quad c_{-2} = 1, \quad c_{-1} = 0, \quad c_0 = 0; \\ &d_{-3} = 1, \quad d_{-2} = 0, \quad d_{-1} = 0, \quad d_0 = 0, \end{aligned} \tag{58}$$

respectively, while $(x_k)_{k\geq -3}$ is given by (53) and satisfies the conditions

$$x_{-3} = x_{-2} = x_{-1} = x_0 = 0 \quad \text{and} \quad x_1 = 1. \tag{59}$$

Since difference equation (52) is solvable, it follows that closed form formulas for $(a_k)_{k\geq -3}$, $(b_k)_{k\geq -3}$, $(c_k)_{k\geq -3}$ and $(d_k)_{k\geq -3}$, can be found. From this, (53), (59) and by using some known sums, $(x_k)_{k\geq -3}$ can also be calculated. From this fact and (51) we see that equation (38) is solvable too.

The second equation in (2) yields

$$z_{n-1}^{d}=\frac{w_{n+1}}{\beta w_n^c},\quad n\in\mathbb{N}_0. \tag{60}$$

Since $d\neq 0$, from the first equation in (2) we have

$$z_{n+1}^{d}=\alpha^d z_n^{ad}w_{n-1}^{bd},\quad n\in\mathbb{N}_0. \tag{61}$$

From (60) and (61) we obtain

$$\frac{w_{n+3}}{\beta w_{n+2}^c}=\alpha^d\frac{w_{n+2}^a}{\beta^a w_{n+1}^{ac}}w_{n-1}^{bd},\quad n\in\mathbb{N}_0,$$

which can be written as

$$w_{n+3}=\alpha^d\beta^{1-a}w_{n+2}^{a+c}w_{n+1}^{-ac}w_{n-1}^{bd},\quad n\in\mathbb{N}_0, \tag{62}$$

which differs from equation (38) only by the coefficient $\alpha^d\beta^{1-a}$. Beside this, sequence $(w_n)_{n\geq -1}$ satisfies the following initial conditions

$$w_1=\beta w_0^c z_{-1}^d\quad\text{ and }\quad w_2=\beta^{1+c}w_0^{c^2}z_{-1}^{cd}z_0^d. \tag{63}$$

Let $\delta:=\alpha^d\beta^{1-a}$. Then the above procedure can be repeated and obtained that for $1\leq k\leq n+1$

$$w_{n+3}=\delta^{x_k}w_{n+3-k}^{a_k}w_{n+2-k}^{b_k}w_{n+1-k}^{c_k}w_{n-k}^{d_k}, \tag{64}$$

where $(a_k)_{k\in\mathbb{N}}$, $(b_k)_{k\in\mathbb{N}}$, $(c_k)_{k\in\mathbb{N}}$ and $(d_k)_{k\in\mathbb{N}}$ satisfy recurrent relations (46) with initial conditions (40), while $(x_k)_{k\in\mathbb{N}}$ is given by (53) and (59).

From (64) with $k=n+1$ and by using (63) we get

$$\begin{aligned}w_{n+3}=&\delta^{x_{n+1}}w_2^{a_{n+1}}w_1^{b_{n+1}}w_0^{c_{n+1}}w_{-1}^{d_{n+1}}\\=&(\alpha^d\beta^{1-a})^{x_{n+1}}(\beta^{1+c}w_0^{c^2}z_{-1}^{cd}z_0^d)^{a_{n+1}}(\beta w_0^c z_{-1}^d)^{b_{n+1}}w_0^{c_{n+1}}w_{-1}^{d_{n+1}}\\=&\alpha^{dx_{n+1}}\beta^{(1-a)x_{n+1}+(1+c)a_{n+1}+b_{n+1}}w_0^{c^2a_{n+1}+cb_{n+1}+c_{n+1}}\\&\times w_{-1}^{d_{n+1}}z_0^{da_{n+1}}z_{-1}^{cda_{n+1}+db_{n+1}},\end{aligned} \tag{65}$$

for $n\in\mathbb{N}_0$.

As above the solvability of equation (52) shows that closed form formulas for $(a_k)_{k\geq -3}$, $(b_k)_{k\geq -3}$, $(c_k)_{k\geq -3}$ and $(d_k)_{k\geq -3}$, can be found. From this, (53), (59) and by using some known sums, $(x_k)_{k\geq -3}$ can also be calculated. This facts along with (65) imply that equation (62) is solvable too. Direct calculation shows that formulas (51) and (65) are really solutions to system (2), finishing the proof. □

Remark 1. Since linear difference equation (52) is of the fourth order it is not only theoretically solvable, but also practically.

The following corollary is a consequence of Theorem 1.

Corollary 1. *Consider system* (2) *with* $a, b, c, d \in \mathbb{Z}$, $\alpha, \beta \in \mathbb{Z} \setminus \{0\}$. *Assume that* $z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. *Then the following statements are true.*

(a) *If* $b = 0$ *and* $a \neq 1 \neq c$, *then the general solution to system* (2) *is given by* (5) *and* (14).
(b) *If* $b = 0$ *and* $a \neq 1 = c$, *then the general solution to system* (2) *is given by* (5) *and* (15).
(c) *If* $b = 0$ *and* $a = 1 \neq c$, *then the general solution to system* (2) *is given by* (6) *and* (16).
(d) *If* $b = 0$ *and* $a = c \neq 1$, *then the general solution to system* (2) *is given by* (5) *and* (18).
(e) *If* $b = 0$ *and* $a = c = 1$, *then the general solution to system* (2) *is given by* (6) *and* (19).
(f) *If* $d = 0$ *and* $a \neq 1 \neq c$, *then the general solution to system* (2) *is given by* (22) *and* (30).
(g) *If* $d = 0$ *and* $a \neq 1 = c$, *then the general solution to system* (2) *is given by* (23) *and* (31).
(h) *If* $d = 0$ *and* $a = 1 \neq c$, *then the general solution to system* (2) *is given by* (22) *and* (32).
(i) *If* $d = 0$ *and* $a = c \neq 1$, *then the general solution to system* (2) *is given by* (22) *and* (34).
(j) *If* $d = 0$ *and* $a = c = 1$, *then the general solution to system* (2) *is given by* (23) *and* (35).
(k) *If* $bd \neq 0$, *then the general solution to system* (2) *is given by* (51) *and* (65), *where* $(a_k)_{k\geq -3}$, $(b_k)_{k\geq -3}$, $(c_k)_{k\geq -3}$ *and* $(d_k)_{k\geq -3}$ *are solutions to equation* (52) *satisfying the conditions* (58), *while* $(x_k)_{k\geq -3}$ *is given by* (53) *and* (59).

Remark 2. The formulas appearing in the proof of Theorem 1 can be used in studying of the behavior of solutions to system (2). We leave it to the reader as some exercises.

Acknowledgements

The work of Bratislav Iričanin was supported by the Serbian Ministry of Education and Science projects III 41025 and OI 171007. The work of Zdeněk Šmarda was supported by the project FEKT-S-14-2200 of Brno University of Technology. Bratislav Iričanin expresses his sincere gratitude for hospitality during his fruitful stay with Brno University of Technology, FEEC - Faculty of Electrical Engineering and Communication, Department of Mathematics, Brno, Czech Republic, which resulted also with the present article.

References

[1] M. Aloqeili, Dynamics of a kth order rational difference equation, *Appl. Math. Comput.* **181** (2006), 1328-1335.

[2] A. Andruch-Sobilo and M. Migda, Further properties of the rational recursive sequence $x_{n+1} = ax_{n-1}/(b + cx_n x_{n-1})$, *Opuscula Math.* **26** (3) (2006), 387-394.

[3] I. Bajo and E. Liz, Global behaviour of a second-order nonlinear difference equation, *J. Differ. Equations Appl.* **17**

[4] L. Berg and S. Stević, On some systems of difference equations, *Appl. Math. Comput.* **218** (2011), 1713-1718.

[5] L. Berg and S. Stević, On the asymptotics of the difference equation $y_n(1 + y_{n-1} \cdots y_{n-k+1}) = y_{n-k}$, *J. Difference Equ. Appl.* **17** (4) (2011), 577-586.

[6] C. Cinar, On the positive solutions of difference equation, *Appl. Math. Comput.* **150** (1) (2004), 21-24.

[7] C. Jordan, *Calculus of Finite Differences*, Chelsea Publishing Company, New York, 1956.

[8] G. L. Karakostas, Asymptotic 2-periodic difference equations with diagonally self-invertible responces, *J. Differ. Equations Appl.* **6** (2000), 329-335.

[9] C. M. Kent and W. Kosmala, On the nature of solutions of the difference equation $x_{n+1} = x_n x_{n-3} - 1$, *Int. J. Nonlinear Anal. Appl.* **2** (2) (2011), 24-43.

[10] V. A. Krechmar, *A Problem Book in Algebra*, Mir Publishers, Moscow, 1974.

[11] H. Levy and F. Lessman, *Finite Difference Equations*, Dover Publications, Inc., New York, 1992.

[12] D. S. Mitrinović and J. D. Kečkić, *Methods for Calculating Finite Sums*, Naučna Knjiga, Beograd, 1984 (in Serbian).

[13] G. Papaschinopoulos and C. J. Schinas, On a system of two nonlinear difference equations, *J. Math. Anal. Appl.* **219** (2) (1998), 415-426.

[14] G. Papaschinopoulos and C. J. Schinas, On the behavior of the solutions of a system of two nonlinear difference equations, *Comm. Appl. Nonlinear Anal.* **5** (2) (1998), 47-59.

[15] G. Papaschinopoulos and C. J. Schinas, Invariants for systems of two nonlinear difference equations, *Differential Equations Dynam. Systems* **7** (2) (1999), 181-196.

[16] G. Papaschinopoulos, C. J. Schinas and G. Stefanidou, On the nonautonomous difference equation $x_{n+1} = A_n + (x^p_{n-1}/x^q_n)$, *Appl. Math. Comput.* **217** (2011), 5573-5580.

[17] G. Papaschinopoulos and G. Stefanidou, Asymptotic behavior of the solutions of a class of rational difference equations, *Inter. J. Difference Equations* **5** (2) (2010), 233-249.

[18] G. Stefanidou, G. Papaschinopoulos and C. Schinas, On a system of max difference equations, *Dynam. Contin. Discrete Impuls. Systems Ser. A* **14** (6) (2007), 885-903.

[19] G. Stefanidou, G. Papaschinopoulos, and C. J. Schinas, On a system of two exponential type difference equations, *Commun. Appl. Nonlinear Anal.* **17** (2) (2010), 1-13.

[20] S. Stević, More on a rational recurrence relation, *Appl. Math. E-Notes* **4** (2004), 80-85.

[21] S. Stević, A short proof of the Cushing-Henson conjecture, *Discrete Dyn. Nat. Soc.* Vol. 2006, Article ID 37264, (2006), 5 pages.

[22] S. Stević, Boundedness character of a class of difference equations, *Nonlinear Anal. TMA* **70** (2009), 839-848.

[23] S. Stević, On a generalized max-type difference equation from automatic control theory, *Nonlinear Anal. TMA* **72** (2010), 1841-1849.

[24] S. Stević, Periodicity of max difference equations, *Util. Math.* **83** (2010), 69-71.

[25] S. Stević, On a system of difference equations, *Appl. Math. Comput.* **218** (2011), 3372-3378.

[26] S. Stević, On the difference equation $x_n = x_{n-2}/(b_n + c_n x_{n-1} x_{n-2})$, *Appl. Math. Comput.* **218** (2011), 4507-4513.

[27] S. Stević, On a solvable rational system of difference equations, *Appl. Math. Comput.* **219** (2012), 2896-2908.

[28] S. Stević, On a third-order system of difference equations, *Appl. Math. Comput.* **218** (2012), 7649-7654.

[29] S. Stević, On some solvable systems of difference equations, *Appl. Math. Comput.* **218** (2012), 5010-5018.

[30] S. Stević, On the difference equation $x_n = x_{n-k}/(b + c x_{n-1} \cdots x_{n-k})$, *Appl. Math. Comput.* **218** (2012), 6291-6296.

[31] S. Stević, Solutions of a max-type system of difference equations, *Appl. Math. Comput.* **218** (2012), 9825-9830.

[32] S. Stević, Domains of undefinable solutions of some equations and systems of difference equations, *Appl. Math. Comput.* **219** (2013), 11206-11213.

[33] S. Stević, On a system of difference equations which can be solved in closed form, *Appl. Math. Comput.* **219** (2013), 9223-9228.

[34] S. Stević, On a system of difference equations of odd order solvable in closed form, *Appl. Math. Comput.* **219** (2013) 8222-8230.

[35] S. Stević, On the system of difference equations $x_n = c_n y_{n-3}/(a_n + b_n y_{n-1} x_{n-2} y_{n-3})$, $y_n = \gamma_n x_{n-3}/(\alpha_n + \beta_n x_{n-1} y_{n-2} x_{n-3})$, *Appl. Math. Comput.* **219** (2013), 4755-4764.

[36] S. Stević, On the system $x_{n+1} = y_n x_{n-k}/(y_{n-k+1}(a_n + b_n y_n x_{n-k}))$, $y_{n+1} = x_n y_{n-k}/(x_{n-k+1}(c_n + d_n x_n y_{n-k}))$, *Appl. Math. Comput.* **219** (2013), 4526-4534.

[37] S. Stević, First-order product-type systems of difference equations solvable in closed form, *Electron. J. Differential Equations* Vol. 2015, Article No. 308, (2015), 14 pages.

[38] S. Stević, Product-type system of difference equations of second-order solvable in closed form, *Electron. J. Qual. Theory Differ. Equ.* Vol. 2015, Article No. 56, (2015), 16 pages.

[39] S. Stević, Representation of solutions of bilinear difference equations in terms of generalized Fibonacci sequences, *Electron. J. Qual. Theory Differ. Equ.* Vol. 2014, Article no. 67, (2014), 15 pages.

[40] S. Stević, M. A. Alghamdi, A. Alotaibi and E. M. Elsayed, Solvable product-type system of difference equations of second order, *Electron. J. Differential Equations* Vol. 2015, Article No. 169, (2015), 20 pages.

[41] S. Stević, M. A. Alghamdi, A. Alotaibi and N. Shahzad, On a higher-order system of difference equations, *Electron. J. Qual. Theory Differ. Equ.* Vol. 2013, Article No. 47, (2013), 18 pages.

[42] S. Stević, M. A. Alghamdi, A. Alotaibi and N. Shahzad, Boundedness character of a max-type system of difference equations of second order, *Electron. J. Qual. Theory Differ. Equ.* Vol. 2014, Article No. 45, (2014), 12 pages.

[43] S. Stević, J. Diblik, B. Iričanin and Z. Šmarda, On a third-order system of difference equations with variable coefficients, *Abstr. Appl. Anal.* vol. 2012, Article ID 508523, (2012), 22 pages.

[44] S. Stević, J. Diblik, B. Iričanin and Z. Šmarda, On some solvable difference equations and systems of difference equations, *Abstr. Appl. Anal.* Vol. 2012, Article ID 541761, (2012), 11 pages.

[45] S. Stević, J. Diblik, B. Iričanin and Z. Šmarda, On the difference equation $x_n = a_n x_{n-k}/(b_n + c_n x_{n-1} \cdots x_{n-k})$, *Abstr. Appl. Anal.* Vol. 2012, Article ID 409237, (2012), 19 pages.

[46] S. Stević, J. Diblik, B. Iričanin and Z. Šmarda, On a solvable system of rational difference equations, *J. Difference Equ. Appl.* **20** (5-6) (2014), 811-825.

[47] S. Stević, J. Diblik, B. Iričanin and Z. Šmarda, Solvability of nonlinear difference equations of fourth order, *Electron. J. Differential Equations* Vol. 2014, Article No. 264, (2014), 14 pages.

[48] S. Stević, B. Iričanin and Z. Šmarda, On a product-type system of difference equations of second order solvable in closed form, *J. Inequal. Appl.* Vol. 2015, Article No. 327, (2015), 15 pages.

[49] D. T. Tollu, Y. Yazlik and N. Taskara, On fourteen solvable systems of difference equations, *Appl. Math. Comput.* **233** (2014), 310-319.

Stevo Stević, Mathematical Institute of the Serbian Academy of Sciences, Knez Mihailova 36/III, 11000 Beograd, Serbia

Operator Theory and Applications Research Group, Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

E-mail address: sstevic@ptt.rs

Bratislav Iričanin, Faculty of Electrical Engineering, Belgrade University, Bulevar Kralja Aleksandra 73, 11000 Beograd, Serbia

Brno University of Technology, Faculty of Electrical Engineering and Communication, Department of Mathematics, Technicka 3058/10, CZ - 616 00 Brno, Czech Republic

E-mail address: iricanin@etf.rs

Zdeněk Šmarda, Brno University of Technology, Faculty of Electrical Engineering and Communication, Department of Mathematics, Technicka 3058/10, CZ - 616 00 Brno, Czech Republic

E-mail address: smarda@feec.vutbr.cz

ON SOME RESULTS IN METRIC SPACES USING AUXILIARY SIMULATION FUNCTIONS VIA NEW FUNCTIONS

XIAO-LAN LIU*, ARSLAN HOJAT ANSARI, SUMIT CHANDOK, STOJAN RADENOVIĆ

Abstract. In this paper, we investigate the existence and uniqueness of coincidence points for two nonlinear operators by generalizing the notion of simulation function introduced by F. Khojasteh et al. [F. Khojasteh, S. Shukla and S. Radenović, Filomat 29 (6), 1189-1194 (2015)]. Our results improve, extend, complement and generalize several existing results in the literature. Also, some examples are provided to illustrate the usability of our results.

1. Introduction and Preliminaries

There are many disciplines in which various authors are working, but due to the possible applications, two of them, fixed point theory and equilibrium theory have received major attention from the last few decades. Both of them have many ways to face, from a theoretical point of view, to various problems which arise from real-world contexts. Fixed point theory has been applied in various topics, convex minimization and split feasibility, construction and structure of fixed points, as well as for finding zeros of contractive operators (see [1, 2, 3] and [4]). It also has lots of applications; in particular, in image recovery and signal processing, and in transition operators for initial valued problems of differential inclusions. The mixed equilibrium problems include fixed point problems, optimization problems, variational inequality problems and Nash equilibrium problems as special cases. For other important questions in the fixed point theory with various approach see [7]-[25].

Recently, Khojasteh et al. [5] introduced the notion of simulation function in order to express different contractivity conditions in a unified way, and they obtained some fixed point results. In this paper, we generalized the simulation function using a class of C-function and investigate the existence and uniqueness of coincidence points for two operators in the setting of

Date: Received: xxxxxx; Revised: yyyyyy; Accepted: zzzzzz.

* Corresponding author.

1991 *Mathematics Subject Classification.* [2010]Primary 47H10; Secondary 54H25.

Key words and phrases. Fixed point; coincidence point; new function; simulation function,.C-class functions.

1

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 24, NO.6, 2018, COPYRIGHT 2018 EUDOXUS PRESS, LLC

metric spaces. Some examples are also provided to illustrate the importance of the results obtained.

To begin with, we give some basic notions and definitions.

In the sequel, let X be a nonempty set and $T, g : X \to X$ be two mappings. For simplicity, we denote $T(x)$ by Tx. A mapping g is injective on $A \subseteq X$ if we can deduce that $a = b$ for all $a, b \in A$ such that $ga = gb$. We denote the nth iterate of T by T^n , that is, $T^1 = T$ and $T^{n+1} = T \circ T^n$ for all $n \in \mathbb{N}$.

Definition 1. *A point $x \in X$ is called a*
(1) fixed point of the operator T if $Tx = x$.
(2) coincidence point of T and g if $Tx = gx$.
(3) common fixed point of T and g if $Tx = gx = x$.

Definition 2. *Let $T, g : X \to X$ be mappings on a metric space(X, d). We say that T and g are compatible if $\lim_{n\to\infty} d(Tgx_n, gTx_n) = 0$ for all sequence $\{x_n\} \subseteq X$ such that the sequences $\{gx_n\}$ and $\{Tx_n\}$ are convergent and have the same limit.*

Remark 3. *If T and g are commuting (that is, $Tgx = gTx$ for all $x \in X$), then T and g are compatible.*

Definition 4. *Given two self-mappings $T, g : X \to X$ and a sequence $\{x_n\}_{n\geq 0} \subseteq X$, we say that $\{x_n\}$ is a Picard-Jungck sequence of the pair (T, g) (based on x_0) if $gx_{n+1} = Tx_n$ for all $n \geq 0$.*

We say that X verifies the CLR (T, g)-property at a point $x_0 \in X$ if there exists on X a Picard-Jungck sequence of (T, g) based on x_0.

Remark 5. *1. It is well known that if T and g are two self-mappings such that $T(X) \subseteq g(X)$, then there exists a Picard-Jungck sequence of (T, g) based on any point $x_0 \in X$. In other words, if $T(X) \subseteq g(X)$, then X verifies the CLR (T, g)-property at each point $x \in X$.*

2. If $g = I_X$ is the identity mapping on X, then there exists a unique Picard sequence of (T, I_X) based at each $x_0 \in X$, which is given by $x_{n+1} = Tx_n$ for all $n \geq 0$. Therefore, X satisfies the CLR (T, I_X) -property at every point, whatever the mapping T .

Definition 6. *(see [6]) A mapping $F : [0, \infty)^2 \to \mathbb{R}$ is called C-class function if it is continuous and satisfies following conditions:*
(1) $F(s, t) \leq s$;
(2) $F(s, t) = s$ implies that either $s = 0$ or $t = 0$; for all $s, t \in [0, \infty)$.

Note that for some F we have that $F(0, 0) = 0$. We denote C-class functions as $\mathcal{C}$.

Also special case of C-class functions can be found in [7]. For examples of C-class function see [6], [8].

Definition 7. [9] *An altering distance function is a continuous, non-decreasing mapping $\varphi : [0, \infty) \to [0, \infty)$ such that $\varphi^{-1}(\{0\}) = \{0\}$.*

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 24, NO.6, 2018, COPYRIGHT 2018 EUDOXUS PRESS, LLC

2. C-class Simulation Function

In this section, we generalized the simulation function introduced by Khojasteh et al. [5] using the function of C-class as follows:

Definition 8. *A mapping $F : [0,\infty)^2 \to \mathbb{R}$ has property C_F, if there exists an $C_F \geq 0$ such that*
(1) $F(s,t) > C_F \Rightarrow s > t$;
(2) $F(t,t) \leq C_F$, for all $t \in [0,\infty)$.

Example 9. *The following functions $F : [0,\infty)^2 \to \mathbb{R}$ are elements of $\mathcal{C}$ that have property C_F, for all $s,t \in [0,\infty)$:*
(1) $F(s,t) = s - t, C_F = r,\ r \in [0,\infty)$
(2) $F(s,t) = \frac{s}{(1+t)^r}, r \in (0,\infty); C_F = 1$
(3) $F(s,t) = \frac{s}{1+kt}; k \geq 1, C_F = \frac{r}{1+k},\ r \in [2,\infty)$
(4) $F(s,t) = (s+l)^{\frac{1}{1+T}} - l, l > 1, C_F = 0,1$
(5) $F(s,t) = s - (\frac{2+t}{1+t})t; C_F = 0,$
(6) $F(s,t) = \frac{ks}{1+t}; 0 < k < 1, C_F = k, 1$
(7) $F(s,t) = \frac{ks}{1+kt}; 0 < k, C_F = \frac{k+1}{k}, 1$
(8) $F(s,t) = \frac{s}{1+t}; 0 < k, C_F = 1, 2$

Definition 10. *A simulation function is a mapping $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ satisfying the following axioms:*
(ζ_1) $\zeta(0,0) = 0$;
(ζ_2) $\zeta(t,s) < F(s,t)$ for all $t,s > 0$, here function $F : [0,\infty)^2 \to \mathbb{R}$ is element of $\mathcal{C}$;
(ζ_3) if $\{t_n\}$, $\{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, then $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$.

The third condition is symmetric in both arguments of ζ but, in proofs, this property is not necessary. In fact, in practice, the arguments of ζ have different meanings and they play different roles. Then, we slightly modify the previous definition in order to highlight this difference and to enlarge the family of all simulation functions.

Definition 11. *A C_F-simulation function is a mapping $\zeta : [0,\infty)\times[0,\infty) \to \mathbb{R}$ satisfying the following conditions:*
(ζ_a) $\zeta(0,0) = 0$;
(ζ_b) $\zeta(t,s) < F(s,t)$ for all $t,s > 0$; here function $F : [0,\infty)^2 \to \mathbb{R}$ is element of $\mathcal{C}$ which has property C_F
(ζ_c) if $\{t_n\}, \{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, and $t_n < s_n$, then $\limsup_{n\to\infty} \zeta(t_n, s_n) < C_F$.

Let Z_F be the family of all C_F-simulation functions $\zeta : [0,\infty)\times[0,\infty) \to \mathbb{R}$. Every simulation function as in Definition 10 is also a C_F-simulation function as in Definition 11, but the converse is not true, for this see Example 3.3 in [10] using C-class function $F(s,t) = s - t$.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 24, NO.6, 2018, COPYRIGHT 2018 EUDOXUS PRESS, LLC

Example 12. *Let $k \in \mathbb{R}$ be such that $k < 1$ and let $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be the function defined by $\zeta(t,s) = kF(s,t) - t$, here $C_F = 0$. Note that $\zeta(t,s) = kF(s,t) - t \le kF(s,t) < F(s,t) \le s$ and $\zeta(t,t) = kF(t,t) - t \le kt - t < 0$.*

It is easy to verify that ζ verifies (ζ_a), and (ζ_b).

If $\{t_n\}, \{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = \delta > 0$ and $t_n < s_n$ for all $n \in \mathbb{N}$, then $\limsup_{n\to\infty} \zeta(t_n, s_n) = \limsup_{n\to\infty}(kF(s_n,t_n) - t_n) \le (k-1)\delta < 0$.

Also, if $F(s,t) = ms$, $m < 1$, then of $\zeta(t,s) = kF(s,t) - t$, we have $\zeta(t,s) = kms - t$, or $\zeta(t,s) = \lambda s - t, \lambda \in (0,1)$.

Therefore ζ is an C_F-simulation function as in Definition 11.

Example 13. *Let $k \in \mathbb{R}$ be such that $k < 1$ and $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be the function defined by $\zeta(t,s) = kF(s,t)$, here take $C_F = 1$. Here, $\zeta(t,s) = kF(s,t) \le ks < s$ and $\zeta(t,t) = kF(t,t) < 1$.*

it is easy to verify, ζ verifies (ζ_a), and (ζ_b).

If $\{t_n\}, \{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = \delta > 0$ and $t_n < s_n$ for all $n \in \mathbb{N}$, then $\limsup_{n\to\infty} \zeta(t_n, s_n) = \limsup_{n\to\infty}(kF(s_n,t_n)) \le kF(\delta,\delta) < 1$.

Therefore ζ is an C_F-simulation function as in Definition 11.

Also, if take $F(s,t) = \frac{s}{1+t}$, ζ is an C_F-simulation function.

Example 14. *Let $F : [0,\infty)^2 \to \mathbb{R}$ be C-class, $F(\psi(s), \varphi(t)) - t < F(s,t), \psi(t) < t$,and let $\zeta : [0,\infty) \times [0,\infty) \to R$ be the function defined by $\zeta(t,s) = F(\psi(s), \varphi(t)) - t$, here $C_F = 0$, note that $\zeta(0,0) = F(\psi(0), \varphi(0)) - 0 = 0$, and $\zeta(t,t) = F(\psi(t), \varphi(t)) - t < F(t,t) - t \le t - t = 0$*

Clearly, ζ verifies (ζ_1), and (ζ_2) follows from the following:

If $\{t_n\}, \{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = \delta > 0$ and $t_n < s_n$ for all $n \in \mathbb{N}$, then $\limsup_{n\to\infty} \zeta(t_n, s_n) = \limsup_{n\to\infty}(F(\psi(s_n), \varphi(t_n)) - t_n) \le \psi(\delta) - \delta < 0$.

3. A coincidence point theorem for (Z, g)-contractions

In this section we establish some results on the existence and uniqueness of coincidence point by using simulation functions on metric spaces.

Definition 15. *Let (X,d) be a metric space and $T, g : X \to X$ be self-mappings. A mapping T is called a (Z_F, g)-contraction if there exists $\zeta \in Z_F$ such that*

$$\zeta(d(Tx,Ty), d(gx,gy)) \ge C_F \tag{1}$$

for all $x, y \in X$ such that $gx \ne gy$.

For clarity, we use the term $(Z_{F,d}, g)$-contraction when we want to highlight that T is a (Z_F, g)-contraction on a metric space involving the metric d. In such a case, we say that T is a $(Z_{F,d}, g)$-contraction with respect to ζ.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 24, NO.6, 2018, COPYRIGHT 2018 EUDOXUS PRESS, LLC

If g is the identity mapping on X, T is a $Z_{F,d}$-contraction with respect to ζ .

Remark 16. *1. By axiom (ζ_b), it is clear that a simulation function must verify $\zeta(r,r) < C_F$ for all $r > 0$.*

2. Furthermore, if T is a $(Z_{F,d}, g)-$contraction with respect to $\zeta \in Z_F$, then

$$d(Tx, Ty) < d(gx, gy), \tag{2}$$

for all $x, y \in X$ such that $gx \neq gy$.

To prove it, assume that $gx \neq gy$. Then $d(gx, gy) > 0$. If $Tx = Ty$, then $d(Tx, Ty) = 0 < d(gx, gy)$. On the contrary case, if $Tx \neq Ty$, then $d(Tx, Ty) > 0$, and applying (ζ_b) and (1), we have that

$C_F \leq \zeta(d(Tx, Ty), d(gx, gy)) < F(d(gx, gy), d(Tx, Ty))$, so (2) holds.

Next, we observe some useful properties of $(Z_{F,d}, g)$-contractions in the context of metric spaces.

Lemma 17. *If T is a $(Z_{F,d}, g)$-contraction in a metric space (X, d) and $x, y \in X$ are coincidence points of T and g, then $Tx = gx = gy = Ty$. In particular, the following properties hold.*

1. If g (or T) is injective on the set of all coincidence points of T and g (or, simply, injective), then T and g have, at most, a unique coincidence point.

2. If T and g have a common fixed point, then it is unique.

Proof. We will show that $gx = gy$ reasoning by contradiction. Suppose, on the contrary, that $gx \neq gy$. Then $d(gx, gy) > 0$. By (1), we have $C_F \leq \zeta(d(Tx, Ty), d(gx, gy)) = \zeta(d(gx, gy), d(gx, gy))$, which is a contradiction due to item 1 of Remark 16. Therefore, if x and y are coincidence points of T and g, then $Tx = gx = gy = Ty$. Hence the result. □

Theorem 18. *Let T be a $(Z_{F,d}, g)$-contraction in a metric space (X, d) and suppose that there exists a Picard-Jungck sequence $\{x_n\}_{n \geq 0}$ of (T, g). Also assume that, at least, one of the following conditions hold.*

(a) $(g(X), d)$ (or $(T(X), d)$) is complete.

(b) (X, d) is complete and T and g are continuous and compatible.

(c) (X, d) is complete and T and g are continuous and commuting.

Then T and g have, at least, a coincidence point. Furthermore, either the sequence $\{gx_n\}$ contains a coincidence point of T and g or, at least, one of the following properties holds.

In case (a), the sequence $\{gx_n\}$ converges to $u \in g(X)$ and any point $v \in X$ such that $gv = u$ is a coincidence point of T and g.

In cases (b) and (c), the sequence $\{gx_n\}$ converges to a coincidence point of T and g.

In addition to this, if $x, y \in X$ are coincidence points of T and g, then $Tx = gx = gy = Ty$. And if g (or T) is injective on the set of all coincidence points of T and g (or, simply, it is injective), then T and g have a unique coincidence point.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 24, NO.6, 2018, COPYRIGHT 2018 EUDOXUS PRESS, LLC

Proof. If $\{x_n\}$ contains a coincidence point of T and g, the proof is finished. Assume that $\{x_n\}$does not contain any coincidence point of T and g, that is,

$$gx_n \neq Tx_n = gx_{n+1}, \tag{3}$$

for all $n \geq 0$. In such a case,

$$d(gx_n, gx_{n+1}) > 0, \tag{4}$$

for all $n \geq 0$. We divide the proof in three steps.

Step 1. We claim that

$$\lim_{n\to\infty} d(gx_n, gx_{n+1}) = 0. \tag{5}$$

Using (ζ_b) and (1), for all $n \geq 0$,

$$\begin{aligned} C_F &\leq \zeta(d(Tx_n, Tx_{n+1}), d(gx_n, gx_{n+1})) \\ &= \zeta(d(gx_{n+1}, gx_{n+2}), d(gx_n, gx_{n+1})) \\ &< F(d(gx_n, gx_{n+1}), d(gx_{n+1}, gx_{n+2})), \end{aligned} \tag{6}$$

which means that $0 < d(gx_{_{n+1}}, gx_{_{n+2}}) < d(gx_n, gx_{_{n+1}})$ for all $n \geq 0$. Then $\{d(gx_n, gx_{n+1})\}$ is a non-increasing sequence of nonnegative real numbers, so it is convergent.

Let $r = \lim_{n\to\infty} d(gx_n, gx_{n+1})$. We prove that $r = 0$ reasoning by contradiction. Assume that $r > 0$. Applying the axiom (ζ_c) to the sequences $\{t_n = d(gx_{n+1}, gx_{n+2})\}$ and $\{s_n = d(gx_n, gx_{n+1})\}$ (which have the same limit $r > 0$ and verify $t_n < s_n$ for all n), it follows that

$$\lim_{n\to\infty}\sup \zeta(d(gx_{n+1}, gx_{n+2}), d(gx_n, gx_{n+1})) = \lim_{n\to\infty}\sup \zeta(t_n, s_n) < C_F,$$

which contradicts with (6) because $\zeta(d(gx_{n+1}, gx_{n+2}), d(gx_n, gx_{n+1})) \geq C_F$ for all $n \geq 0$. This contradiction prove that $lim_{n\to\infty} d(gx_n, gx_{n+1}) = r = 0$, that is, (5) holds.

Step 2. Now, we claim that the sequence $\{gx_n\}$ is a Cauchy in (X, d). On the contrary, we assume that $\{gx_n\}$ is not a Cauchy sequence in (X, d). In this case, there exists $\varepsilon_0 > 0$ such that, for all $N \in \mathbb{N}$, there exist $m, n \in \mathbb{N}$ verifying $m > n > N$ and $d(gx_m, gx_n) > \varepsilon_0$. Using (5), there exists $n_0 \in N$ such that

$$d(gx_{n+1}, gx_n) < \varepsilon_0, \tag{7}$$

for all $n \geq n_0$.

Taking successive values for n, we can find two partial subsequences $\{gx_{m(k)}\}$ and $\{gx_{n(k)}\}$ of $\{gx_n\}$ such that

$$n_0 \leq n(k) < m(k) < n(k+1) \text{ and } d(gx_{m(k)}, gx_{n(k)}) > \varepsilon_0 \tag{8}$$

for all $k \in N$.

If we choose $m(k)$ as the least natural number $m \in \{n(k), n(k)+1, n(k)+2, ...\}$ such that (8) holds, then we also have that

$$d(gx_{m(k)-1}, gx_{n(k)}) \leq \varepsilon_0, \tag{9}$$

for all $k \in N$.

Notice that $m(k) \geq n(k)+1$ for all $k \in N$. Indeed, joining (7) and (8), we deduce that the case $m(k) = n(k)+1$ is impossible. Therefore, $m(k) \geq n(k)+2$ for all $k \in N$. It follows that

$n(k)+1 < m(k) < m(k)+1$ for all $k \in N$.

Taking into account (8) and (9), we deduce that

$\varepsilon_0 < d(gx_{m(k)}, gx_{n(k)}) \leq d(gx_{m(k)}, gx_{m(k)-1}) + d(gx_{m(k)-1}, gx_{n(k)}) \leq d(gx_{m(k)}, gx_{m(k)-1}) + \varepsilon_0$

for all k. Using (5), we have that

$$\lim_{k\to\infty} d(gx_{m(k)}, gx_{n(k)}) = \varepsilon_0. \tag{10}$$

Moreover, by

$d(gx_{m(k)}, gx_{n(k)}) \leq d(gx_{m(k)}, gx_{m(k)+1}) + d(gx_{m(k)+1}, gx_{n(k)+1}) + d(gx_{n(k)+1}, gx_{n(k)})$

and

$d(gx_{m(k)+1}, gx_{n(k)+1}) \leq d(gx_{m(k)+1}, gx_{m(k)}) + d(gx_{m(k)}, gx_{n(k)}) + d(gx_{n(k)}, gx_{n(k)+1})$

for all k, and also using (5), it follows that

$$\lim_{k\to\infty} d(gx_{m(k)+1}, gx_{n(k)+1}) = \varepsilon_0. \tag{11}$$

In particular, there exists $n_1 \in N$ such that

$d(gx_{m(k)}, gx_{n(k)}) > \frac{\varepsilon_0}{2} > 0$

and

$d(gx_{m(k)+1}, gx_{n(k)+1}) > \frac{\varepsilon_0}{2} > 0$ for all $k \geq n_1$.

Using the fact that T is a $(Z_{F,d}, g)$-contraction with respect to ζ and the axiom (ζ_b), we deduce that

$C_F \leq \zeta(d(Tx_{m(k)}, Tx_{n(k)}), d(gx_{m(k)}, gx_{n(k)})) = \zeta(d(gx_{m(k)+1}, gx_{n(k)+1}), d(gx_{m(k)}, gx_{n(k)})$

$< F(d(gx_{m(k)}, gx_{n(k)}), d(gx_{m(k)+1}, gx_{n(k)+1}))$, for all $k \geq n_1$. In particular,

$0 < d(gx_{m(k)+1}, gx_{n(k)+1}) < d(gx_{m(k)}, gx_{n(k)})$ for all $k \geq n_1$

Employing the sequences $\{t_k = d(gx_{m(k)+1}, gx_{n(k)+1})\}$ and $\{s_k = d(gx_{m(k)}, gx_{n(k)})\}$ (which have the same positive limit by (10) and (11) and verify $t_k < s_k$ for all k) in axiom (ζ_c), we conclude that which is a contradiction. Therefore, we must admit that the sequence $\{gx_n\}$ is Cauchy in (X, d). Hence, Step 2 holds.

Step 3. Now, we prove that T and g have a coincidence point by taking the assumptions (a), or (b), or (c).

Case (a): Assume that $(g(X), d)$ (or $(T(X), d)$) is complete. In this case, notice that $gx_{n+1} = Tx_n \in T(X) \subseteq g(X)$ for all $n \geq 0$,which means that the sequence $\{gx_{n+1}\}$ is included in $T(X) \subseteq g(X)$. Taking into account that $g(X)$ (or $T\ (X)$) is d-complete,then there exists $u \in g(X)$ such that $\{gx_n\} \to u$, that is,

$$\lim_{n\to\infty} d(gx_n, u) = 0. \tag{12}$$

Since $Tx_n = gx_{n+1}$ for all n, we also have that

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 24, NO.6, 2018, COPYRIGHT 2018 EUDOXUS PRESS, LLC

$$\lim_{n\to\infty} d(Tx_n, u) = 0. \tag{13}$$

Let $v \in X$ be any point such that $gv = u$. We will show that v is a coincidence point of T and g. On the contrary, we assume that v is not a coincidence point of T and g, that is, $u = gv \neq Tv$. In such a case, $\delta = d(Tv, gv) > 0$. Using(13), let $n_0 \in N$ be such that $d(gx_n, gv) < \delta$ for all $n \geq n_0$. This means that

$d(gx_n, gv) < \delta = d(Tv, gv)$ for all $n \geq n_0$.

In particular, $gx_n \neq Tv$ for all $n \geq n_0$, that is,

$$d(Tx_n, Tv) = d(gx_{n+1}, Tv) > 0, \tag{14}$$

for all $n \geq n_0$.

On the other hand, by (4), it is impossible the condition

$\exists n_1 \in N$ such that $gx_n = gv$ for all $n \geq n_1$.

Therefore, there exists a partial subsequence $\{gx_{\delta(n)}\}$ of $\{gx_n\}$ such that

$$gx_{\delta(n)} \neq gv \ \forall n \tag{15}$$

Now, let $n_2 \in N$ be such that $\delta(n_2) \geq n_0$. Therefore, by (14) and (15),

$d(gx_{\delta(n)}, gv) > 0$ and $d(Tx_{\delta(n)}, Tv) > 0$ for all $n \geq n2$.

Using (ζ_b),

$C_F \leq \zeta(d(Tx_{\delta(n)}, Tv), d(gx_{\delta(n)}, gv)) < F(d(gx_{\delta(n)}, gv), d(Tx_{\delta(n)}, Tv))$

for all $n \geq n_2$, which means that

$0 \leq d(Tx_{\delta(n)}, Tv) < d(gx_{\delta(n)}, gv) = d(gx_{\delta(n)}, u)$,

for all $n \geq n_2$. In particular, by (12), $\{Tx_{\delta(n)}\} \to Tv$. However, $\{Tx_{\delta(n)}\} = \{gx_{\delta(n)+1}\}$ is a partial subsequence of $\{gx_n\}$, which converges to gv. By the unicity of the limit, we conclude that $gv = Tv$, which is a contradiction with the fact that we have supposed that $gv \neq Tv$. This contradiction yields v is a coincidence point of T and g.

Case (b): Assume that (X, d) is complete and T and g are continuous and compatible. In this case, the sequence $\{gx_n\}$ is a Cauchy sequence in the complete metric space (X, d) , so there exists $u \in X$such that $\{gx_n\} \to u$. Since T and g are continuous,it follows that $\{ggx_n\} \to gu$ and $\{Tgx_n\} \to Tu$. Moreover, as T and g are compatible and the sequences$\{Tx_n = gx_{n+1}\}$and $\{gx_n\}$ have the same limit, we deduce that

$lim_{n\to\infty} d(Tgx_n, gTx_n) = 0$. It follows that

$d(Tu, gu) = lim_{n\to\infty} d(Tgx_n, ggx_{n+1}) = lim_{n\to\infty} d(Tgx_n, gTx_n) = 0$.

Therefore, $Tu = gu$ and we conclude that u is a coincidence point of T and g.

Case (c): Assume that (X, d) is complete and T and g are continuous and commuting. It follows from case (b) taking into account Remark 16. Last statement follows from Lemma 17. □

Now if let $F(s, t) = s - t$, we have the following result of [10].

Corollary 19. *Let T be a $(Z_{d,}g)$-contraction in a metric space (X, d) and suppose that there exists a Picard-Jungck sequence $\{x_n\}_{n\geq 0}$ of (T, g). Also assume that, at least, one of the following conditions holds.*

(a) $(g(X), d)$ (or $(T(X), d)$) is complete.

(b) (X, d) is complete and T and g are continuous and compatible.

(c) (X, d) is complete and T and g are continuous and commuting.

Then T and g have, at least, a coincidence point. Furthermore, either the sequence $\{gx_n\}$ contains a coincidence point of T and g or, at least, one of the following properties holds.

In case (a), the sequence $\{gx_n\}$ converges to $u \in g(X)$ and any point $v \in X$ such that $gv = u$ is a coincidence point of T and g.

In cases (b) and (c), the sequence $\{gx_n\}$ converges to a coincidence point of T and g.

In addition to this, if $x, y \in X$ are coincidence points of T and g, then $Tx = gx = gy = Ty$. And if g (or T) is injective on the set of all coincidence points of T and g (or, simply, it is injective), then T and g have a unique coincidence point.

In the following example, we describe how to use our main result in order to guarantee existence and uniqueness of a solution for nonlinear equations.

Example 20. *Let $X = [0, \infty)$ provided with the Euclidean metric $d(x, y) = |x - y|$ for all $x, y \in X$, and consider the operators $T, g : X \to X$ given, for all $x \in X$, by*

$$Tx = x + 2,\ \ gx = 4x + e^{2x}$$

It is clear that T is not a contraction in the classical Banach sense (in fact, it is an isometry). In order to solve the nonlinear equation

$$x + 2 = 4x + e^{2x}, \tag{16}$$

*Theorem 18 can be applied using the simulation function $\zeta(t, s) = \frac{9}{10}[\psi(s)-(\frac{2+\varphi(t)}{1+\varphi(t)})\varphi(t)]$,
where ψ and φ are the altering distance functions given by $\psi(t) = t$ and $\varphi(t) = t$ for all $t \geq 0$ and $C_F = 0, F(s, t) = s - (\frac{2+t}{1+t})t$, now we have that*

$$\begin{aligned}
& \zeta(d(Tx, Ty), d(gx, gy)) \\
= & \frac{9}{10}[\psi(d(gx, gy)) - (\frac{2 + \varphi(d(Tx, Ty))}{1 + \varphi(d(Tx, Ty))})\varphi(d(Tx, Ty))] \\
= & \frac{9}{10}[d(gx, gy) - (\frac{2 + d(Tx, Ty)}{1 + d(Tx, Ty)})d(Tx, Ty)] \\
= & \frac{9}{10}[\left|4(x - y) + (e^{2x} - e^{2y})\right| - (\frac{2 + |x - y|}{1 + |x - y|})\,|x - y|] \\
\geq & \ 0.
\end{aligned}$$

Therefore T is a (Z_d, g)-contraction with respect to ζ_1. As all conditions of Theorem 4.8 are satisfied (for instance, $T(X) = [2,\infty) \subset [1,\infty) = g(X)$ and $g(X)$ is complete), it is ensured that T and g have a unique coincidence point, which is the only solution of equation (16). □

4. Competing interests

The authors declare that they have no competing interests.

5. Authors contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

6. Acknowledgements

This work is partially supported by the National Natural Science Foundation of China (Grant No.61573010), Natural Science Foundation of HaiNan Province (Grant No.114014), Scientific Research Fund of Science and Technology Department of Sichuan Province (2017JY0125), Opening Project of Sichuan Province University Key Laboratory of Bridge Non-destruction Detecting and Engineering Computing (2015QZJ01), Artificial Intelligence of Key Laboratory of Sichuan Province(2015RZJ01), Scientific Research Fund of Sichuan Provincial Education Department (16ZA0256), Scientific Research Fund of Sichuan University of Science and Engineering (2014RC01, 2014RC03)

References

[1] H. Iiduka, *Fixed point optimization algorithm and its application to network bandwidth allocation,* J. Comput. Appl. Anal. 236 (2012) 1733–1742.
[2] Y. Lu, L.-P. Pang, X.-J. Liang, Z.-Q. Xia, *An approximate decomposition algorithm for convex minimization,* J. Comput. Appl. Anal. 234 (2010) 658–666.
[3] Y. Yao, Y.-C. Liou, N. Shahzad, *A strongly convergent method for the split feasibility problem,* Abstr. Appl. Anal. 2012 (2012) 15. Article ID 125046.
[4] F. Khojasteh, E. Karapinar, and S. Radenović, *$\theta-$Metric Spaces: A Generalization*, Math. Problems Engin. Volume 2013, Article ID 504-609, 7 pages
[5] F. Khojasteh, S. Shukla, S. Radenović, *A new approach to the study of fixed point theorems via simulation functions,* Filomat 29 (6), 1189-1194 (2015)
[6] A. H. Ansari, *Note on" φ -ψ-contractive type mappings and related fixed point"*, The 2nd Regional Conference on Math. Appl. PNU, Sept. 2014, pages 377-380
[7] A. H. Ansari, S. Chandok and C. Ionescu, *Fixed point theorems on b-metric spaces for weak contractions with auxiliary functions,* J. Inequalities Appl. 2014, 2014:429,17 pages
[8] A.H. Ansari, M. Berzig and S. Chandok, *Some fixed point theorems for (CAB)-contractive mappings and related results,* Math. Moravica 19 (2) (2015), 97-112.
[9] M. S. Khan, M. Swaleh, S. Sessa, *Fixed point theorems by altering distances between the points,* Bull. Aust. Math. Soc. 30 (1) (1984) 1–9.
[10] Antonio-Francisco Roldán-López-de-Hierro , E. Karapınar, C. Roldán-López-de-Hierro, J. Martínez-Morenoa, *Coincidence point theorems on metric spaces via simulation functions,* J. Comput. Appl. Math. 275 (2015) 345–355.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 24, NO.6, 2018, COPYRIGHT 2018 EUDOXUS PRESS, LLC

[11] Du, W-S, Khojasteh, F: *New results and generalizations for approximate fixed point property and their applications,* Abstr. Appl. Anal. 2014, Article ID 581-267 (2014)

[12] R. Agarwal, E. Karapınar, A. Roldán, *Fixed point theorems in metric spaces and applications to coupled /tripled fixed points on G^*-metric spaces,* J. Nonlinear Convex Anal. (2014) in press.

[13] S.A. Al-Mezel, H. H. Alsulami, E. Karapınar, A. Roldán, *Discussion on "Multidimensional coincidence points" via recent publications,* Abstr. Appl. Anal. (2014) Article ID 287492, 13 pages.

[14] Antonio Francisco Roldán López de Hierro and Naseer Shahzad, *New fixed point theorem under R-contractions,* Fixed Point Theory Appl. (2015) 2015:98

[15] E. Karapınar, A. Roldán, N. Shahzad, W. Sintunaravat, *Discussion on coupled and tripled coincidence point theorems for φ-contractive mappings without the mixed g-monotone property,* Fixed Point Theory Appl. 2014 (2014) 92.

[16] F. Khojasteh, V. Rakočević, *Some new common fixed point results for generalized contractive multi-valued non-self-mappings,* Appl. Math. Lett. **25** (2012) 287–293.

[17] A. Roldán, J. Martínez-Moreno, C. Roldán, Y.J. Cho, *Multidimensional coincidence point results for compatible mappings in partially ordered fuzzy metric spaces,* Fuzzy Sets and Systems 251 (2014) 71–82.

[18] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales,* Fund. Math. 3 (1922) 133–181.

[19] B.E. Rhoades, *Some theorems on weakly contractive maps,* Nonlinear Anal. **47** (2001) 2683–2693.

[20] Antonio-Francisco Roldán-López-de-Hierro and Naseer Shahzad, *Common fixed point theorems under (R,S)-contractivity conditions,* Fixed Point Theory Appl. (2016) 2016:55

[21] V. Ozturk and S. Radenović, *Some remarks on b-(E.A)-property in b-metric spaces,* Springer Plus (2016) 5:544

[22] A. Nastasi and P. Vetro, *Existence and Uniqueness for a First-Order Periodic Differential Problem Via Fixed Point Results,* Results. Math. Online First, DOI 10.1007/s00025-016-0551-x

[23] H. Argoubi, B. Samet, C. Vetro, *Nonlinear contractions involving simulation functions in a metric space with a partial order,* J. Nonlinear Sci. Appl. 8, 1082-1094 (2015)

[24] A. Nastasi, P. Vetro, *Fixed point results on metric and partial metric spaces via simulation functions,* J. Nonlinear Sci. Appl. 8, 1059-1069 (2015)

[25] M. Turinici, *Abstract comparison principles and multivariable Gronwall-Bellman inequalities,* J. Math. Anal. Appl. 117, 100-127 (1986)

X.Liu School of Science, Sichuan University of Science and Engineering, Zigong, Sichuan 643000, China, &, Sichuan Province University Key Laboratory of Bridge Non-destruction Detecting and Engineering Computing, 643000, China

E-mail address: stellalwp@163.com

A.H. Ansari Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran.

E-mail address: mathanalsisamir4@gmail.com

S. Chandok School of Mathematics, Thapar University, Patiala 147-004, Punjab, India.

E-mail address: chansok.s@gmail.com, sumit.chandok@thapar.edu

S. Radenović Faculty of Mechanical Engineering, University of Belgrade, Kraljice Marije 16, 11120 Beograd, Serbia, &, State University of Novi Pazar, Novi Pazar, Serbia

E-mail address: stojan.radenovic@tdt.edu.vn

The uniqueness of meromorphic functions and weight-shared *

Jin Tu[a], Hong Yan Xu[b]†, Hong Zhang[c]

[a]Institute of Mathematics and informatics, Jiangxi Normal University,
Nanchang, Jiangxi, 330022, China
<e-mail: tujin2008@sina.com>

[a]Department of Informatics and Engineering, Jingdezhen Ceramic Institute,
Jingdezhen, Jiangxi, 333403, China
<e-mail: xhyhhh@126.com>

[c] School of the Tourism and Urban Management,Jiangxi University of Finance and Economics
Nanchang, Jiangxi, 330032, China
<e-mail: zhanghongjxcd@163.com>

Abstract

The purpose of this paper is to study some uniqueness problem of meromorphic functions sharing two sets and three sets with finite weight. Our results are improvement and complement of some results given by Zhang-Xu, Fang-Xu, Lahiri-Banerjee.

Key words: Meromorphic function; Weighted sharing; Uniqueness.
Mathematical Subject Classification (2000): 30D 30, 30D 35.

1 Introduction and main results

This purpose of this paper is to study some uniqueness problem of meromorphic functions sharing two sets and three sets with finite weight. The fundamental theorems and the standard notations of the Nevanlinna value distribution theory of meromorphic functions will be used (see Hayman [6] and Yi and Yang [12]). Let f be a nonconstant meromorphic function and $a \in \mathbb{C} \cup \{\infty\}$ and S be a subset of $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Define

$$E(S,f) = \bigcup_{a \in S} \{z : f(z) - a = 0, \ \textbf{counting} \quad \textbf{multiplicity}\},$$

$$\overline{E}(S,f) = \bigcup_{a \in S} \{z : f(z) - a = 0, \ \textbf{ignoring} \quad \textbf{multiplicity}\}.$$

If $E(S,f) = E(S,g)$ we say that f and g share the set S CM. On the other hand, if $\overline{E}(S,f) = \overline{E}(S,g)$, we say that f and g share the set S IM. Especially, let $S = \{a\}$, we say f and g share the value a CM. If $E(S,f) = E(S,g)$, and we say that f and g share the value a IM if $\overline{E}(S,f) = \overline{E}(S,g)$ (see [5]).

Let m be a nonnegative integer, we denote by $E_{m)}(a;f)$ the set of all a-points of f with multiplicities not exceeding m, where an a-point is counted according to its multiplicity. Also we denote by $\overline{E}_{m)}(a;f)$ the set of distinct a-points of f with multiplicities not greater than m. If for

*The authors were supported by the NSF of China(11561031, 11561033) and the Natural Science Foundation of Jiangxi Province in China (20151BAB201008,20161BAB201020).

†Corresponding author.

some $a \in \mathbb{C} \cup \{\infty\}$, $E_{\infty)}(a; f) = E_{\infty)}(a; g)$ we say that f, g share the value a CM. For any positive integer m, we define

$$E_{m)}(S, f) = \bigcup_{a \in S} E_{m)}(a; f), \quad \text{and} \quad \overline{E}_{m)}(S, f) = \bigcup_{a \in S} \overline{E}_{m)}(a; f).$$

In 1977, Gross [4] proved that there exist three finite sets $S_j (j = 1, 2, 3)$, such that any two entire functions f and g satisfying $E(S_j, f) = E(S_j, g)$ for $j = 1, 2, 3$ must be identical, and posed the following question:

Question A *Does there exist a finite set S such that, for any pair of nonconstant entire functions f and g, $E(S, f) = E(S, g)$ implies $f \equiv g$?*

If the answer of this question is affirmative, a natural question is the following:

Question B *What is the smallest cardinal of S?*

In 1995, Yi [11] first proved that such that a set exist. In fact, Yi proved the following theorem

Theorem A [11]. *There exists a set S with 7 elements such that $E(S, f) = E(S, g)$ implies $f \equiv g$, for any pair of nonconstant entire functions f and g.*

For meromorphic functions, the present best answer to Question B was obtained by Frank and Reinders [3].

Theorem B [3]. *There exists a set S with 11 elements such that $E(S, f) = E(S, g)$ implies $f \equiv g$ for any pair of nonconstant meromorphic functions f and g.*

A natural problem arises: *What can we say if nonconstant meromorphic functions f and g have "few" poles?*

Lahiri and Banerjee [8] investigated the situation for $\Theta(\infty; f) \leq \frac{1}{2}$ and $\Theta(\infty; g) \leq \frac{1}{2}$ in Theorem C and proved the following result.

Theorem C [8]. *Let $S_1 = \{z : z^n + az^{n-1} + b = 0\}, S_2 = \{0\}$ and $S_3 = \{\infty\}$, where a, b are nonzero constants such that $z^n + az^{n-1} + b = 0$ has no repeated root and $n(\geq 4)$ is an integer. Suppose that f, g are two nonconstant meromorphic functions satisfying $\Theta(\infty; f) + \Theta(\infty; g) > 0$. If $E(S_j, f) = E(S_j, g)$ for $j = 1, 2, 3$, then $f \equiv g$.*

Recently, Zhang and Xu [13] proved the following result:

Theorem D [13]. *Let $S = \{z : z^7 - z^6 = 1\}$. Suppose that f, g are two nonconstant meromorphic functions satisfying $\Theta(\infty; f) + \Theta(\infty; g) > 1$. If $E(S, f) = E(S, g)$ and $E(\infty, f) = E(\infty, g)$, then $f \equiv g$.*

Now considering all the above theorems it is natural to ask the following question:

Question 1.1 *Is it possible in any way to further relax the nature of sharing the set S_1, S_2, S_3 in Theorem C?*

In the present paper we shall investigate this problem and obtain two results which will improve all the previous theorems mentioned earlier. Also we shall provide an answer to the question of Gross in a more compact and convenient way than the previous authors have given.

Now we state the following theorems which are the main results of this paper.

Theorem 1.1 *Let S_1, S_3 be as in Theorem C. Suppose that f, g are nonconstant meromorphic functions satisfying $E_{1)}(S_1, f) = E_{1)}(S_1, g)$, $\overline{E}_{m)}(S_1, f) = \overline{E}_{m)}(S_1, g)$, $m \geq 3$, $E(S_3, f) = E(S_3, g)$ and satisfy one of the following conditions*
(i) $n = 8$ and $\Theta(\infty; g) + \Theta(\infty; g) > \frac{3}{2}$;
(ii) $n = 9$ and $\Theta(\infty; f) + \Theta(\infty; g) > \frac{3}{4}$;

(iii) $n \geq 10$ *and* $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n-1}$, *n is a positive integer.*
Then $f \equiv g$.

Theorem 1.2 *Let* S_1, S_3 *be as in Theorem C. Suppose that* f, g *are nonconstant meromorphic functions satisfying* $E_{2)}(S_1, f) = E_{2)}(S_1, g)$, $\overline{E}_{m)}(S_1, f) = \overline{E}_{m)}(S_1, g)$, $m \geq 4$, $E(S_3, f) = E(S_3, g)$ *and satisfy one of two conditions:*
(i) $n = 7$ *and* $\Theta(\infty; f) + \Theta(\infty; g) > 1$;
(ii) $n \geq 8$ *and* $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n-1}$, *n is a positive integer.*
Then $f \equiv g$.

Remark 1.1 *When* $n = 7$, *it obviously that Theorem 1.2 is an improvement of Theorem D.*

Theorem 1.3 *Let* S_1, S_3 *be as in Theorem C. Suppose that* f, g *are nonconstant meromorphic functions satisfying* $E_{1)}(S_1, f) = E_{1)}(S_1, g)$, $\overline{E}_{m)}(S_1, f) = \overline{E}_{m)}(S_1, g)$, $m \geq 3$, $E(S_2, f) = E(S_2, g), E(S_3, f) = E(S_3, g)$ *and satisfy one of the conditions:*
(i) $n = 6$ *and* $\Theta(\infty; g) + \Theta(\infty; g) > \frac{3}{2}$;
(ii) $n = 7$ *and* $\Theta(\infty; f) + \Theta(\infty; g) > \frac{3}{4}$;
(iii) $n \geq 8$ *and* $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n-1}$, *n is a positive integer.*
Then $f \equiv g$.

Theorem 1.4 *Let* S_1, S_2, S_3 *be as in Theorem C. Suppose that* f, g *are nonconstant meromorphic functions satisfying* $E_{2)}(S_1, f) = E_{2)}(S_1, g)$, $\overline{E}_{m)}(S_1, f) = \overline{E}_{m)}(S_1, g)$, $m \geq 4$, $E(S_2, f) = E(S_2, g), E(S_3, f) = E(S_3, g)$ *and satisfy one of the conditions:*
(i) $n = 5$ *and* $\Theta(\infty; f) + \Theta(\infty; g) > 1$;
(ii) $n \geq 6$ *and* $\Theta(\infty; f) + \Theta(\infty; g) > 0$.
Then $f \equiv g$.

Though the standard definitions and notations of the value distribution theory are available in [6], we explain some definitions and notations which are used in the paper.

Definition 1.1 *(see [1]). Let* k *and* r *be two positive integers such that* $1 \leq r < k - 1$ *and for* $a \in \mathbf{C}$, $\overline{E}_{k)}(a, f) = \overline{E}_{k)}(a, g), E_{r)}(a, f) = E_{r)}(a, g)$. *Let* z_0 *be a zero of* $f - a$ *of multiplicity* p *and a zero of* $g - a$ *of multiplicity* q. *We denote by* $\overline{N}_L(r, a; f)(\overline{N}_L(r, a; g))$ *the reduced counting function of those a-points of* f *and* g *for which* $p > q \geq r + 1(q > p \geq r + 1)$, *by* $\overline{N}_E^{(r+1}(r, a; f)$ *the reduced counting function of those a-points of* f *and* g *for which* $p = q \geq r + 1$, *by* $\overline{N}_{f \geq k+1}(r, a; f|g \neq a)(\overline{N}_{g \geq k+1}(r, a; g|f \neq a))$ *the reduced counting functions of those a-points of* f *and* g *for which* $p \geq k + 1$ *and* $q = 0(q \geq k + 1$ *and* $p = 0)$.

Definition 1.2 *(see [1]). If* $r = 0$ *in definition 1.1 then we use the same notations as in definition 1.1 except by* $\overline{N}_E^{1)}(r, a; f)$ *we mean the common simple a-points of* f *and* g *and by* $\overline{N}_E^{(2}(r, a; f)$ *we mean the reduced counting functions of those a-points of* f *and* g *for which* $p = q \geq 2$.

Definition 1.3 *(see [9]). Let* $a, b \in \mathbb{C} \cup \{\infty\}$, *We denote by* $N(r, a; f|g = b)$ *the counting function of those* $a-$*points of* f, *counted according to multiplicity, which are b-points of* g*; by* $N(r, a; f|g \neq b)$ *the counting function of those a-points of* f, *counted according to multiplicity, which are not the b-points of* g.

2 Some Lemmas

In this section we shall denote by H and V the following two functions

$$H = (\frac{F''}{F'} - \frac{2F'}{F-1}) - (\frac{G''}{G'} - \frac{2G'}{G-1}),$$

where F, G are two nonconstant meromorphic functions.

Lemma 2.1 *(see [12]). Let f be a nonconstant meromorphic function and let*

$$R(f) = \sum_{k=0}^{n} a_k f^k / \sum_{j=0}^{m} b_j f^j$$

be an irreducible rational function in f with coefficients $\{a_k\}$ and $\{b_j\}$, where $a_n \neq 0$ and $b_m \neq 0$. Then

$$T(r, R(f)) = dT(r, f) + S(r, f),$$

where $d = max\{n, m\}$.

Lemma 2.2 *(see [14]). Let F, G be two nonconstant meromorphic functions such that $E_{1)}(1; F) = E_{1)}(1; F)$ and $H \not\equiv 0$. Then*

$$N_E^{1)}(r, 1; F) \leq N(r, \infty; H) + S(r, F) + S(r, G).$$

Lemma 2.3 *(see [8, Lemma 9]). Let f, g be two nonconstant meromorphic functions. Let*

$$F = \frac{f^{n-1}(f + a)}{-b}, \quad and \quad G = \frac{g^{n-1}(g + a)}{-b}. \tag{1}$$

If $F \equiv G$ and $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n-1}$, where $n \geq 4$, then $f \equiv g$.

Lemma 2.4 *(see [8, Lemma 3]). Let f, g be two nonconstant meromorphic functions sharing $(0, \infty)$ and $\Theta(\infty; f) + \Theta(\infty; g) > 0$. Then $f^{n-1}(f + a) \equiv g^{n-1}(g + a)$ implies $f \equiv g$, where $n(\geq 2)$ is an integer and a is a nonzero finite constant.*

Lemma 2.5 *(see [7, Lemma 5]). If two nonconstant meromorphic functions f, g share $(\infty, 0)$ then for $n \geq 2$,*

$$f^{n-1}(f + a)g^{n-1}(g + a) \not\equiv b^2,$$

where a, b are finite nonzero constants.

Lemma 2.6 *(see [1, Lemma 2.2]). Let $\overline{E}_{m)}(1, F) = \overline{E}_{m)}(1, G), E_{1)}(1, F) = E_{1)}(1, G)$ and $H \not\equiv 0$, where $m \geq 3$. Then*

$$\begin{aligned} & N(r, \infty; H) \\ \leq \; & \overline{N}(r, 0; F| \geq 2) + \overline{N}(r, 0; G| \geq 2) + \overline{N}_L(r, \infty; F) + \overline{N}_L(r, \infty; G) \\ & + \overline{N}_L(r, 1; F) + \overline{N}_L(r, 1; G) + \overline{N}_{F \geq m+1}(r, 1; F|G \neq 1) \\ & + \overline{N}_{G \geq m+1}(r, 1; G|F \neq 1) + \overline{N}_0(r, 0; F') + \overline{N}_0(r, 0; G'), \end{aligned}$$

where $\overline{N}_0(r, 0; F')$ is the reduced counting function of those zeros of F' which are not the zeros of $F(F - 1)$ and $\overline{N}_0(r, 0; G')$ is similarly defined.

From Lemma 2.6, we can get the following lemmas easily.

Lemma 2.7 *Let $\overline{E}_{m)}(1, F) = \overline{E}_{m)}(1, G), E_{1)}(1, F) = E_{1)}(1, G)$, $E(\infty, F) = E(\infty, G)$ and $H \not\equiv 0$, where $m \geq 3$. Then*

$$\begin{aligned} N(r, \infty; H) \leq & \overline{N}(r, 0; F| \geq 2) + \overline{N}(r, 0; G| \geq 2) + \overline{N}_L(r, 1; F) + \overline{N}_L(r, 1; G) \\ & + \overline{N}_{F \geq m+1}(r, 1; F|G \neq 1) + \overline{N}_{G \geq m+1}(r, 1; G|F \neq 1) \\ & + \overline{N}_0(r, 0; F') + \overline{N}_0(r, 0; G'), \end{aligned}$$

where $\overline{N}_0(r, 0; F')$ and $\overline{N}_0(r, 0; G')$ is stated as in Lemma 2.6.

Lemma 2.8 *Let $\overline{E}_{m)}(1,F) = \overline{E}_{m)}(1,G), E_{1)}(1,F) = E_{1)}(1,G)$, $E(\infty;F) = E(\infty;G)$, $E(0,F) = E(0,G)$ and $H \not\equiv 0$, where $m \geq 3$. Then*

$$\begin{aligned} N(r,\infty;H) \leq & \overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + \overline{N}_{F\geq m+1}(r,1;F|G \neq 1) \\ & + \overline{N}_{G\geq m+1}(r,1;G|F \neq 1) + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G'), \end{aligned}$$

where $\overline{N}_0(r,0;F')$ and $\overline{N}_0(r,0;G')$ is stated as in Lemma 2.6.

Lemma 2.9 *(see [1, Lemma 2.6]). Let $\overline{E}_{m)}(1,F) = \overline{E}_{m)}(1,G), E_{1)}(1,F) = E_{1)}(1,G)$ and $H \not\equiv 0$, where $m \geq 3$. Then*

$$\begin{aligned} & 2\overline{N}_L(r,1;F) + 2\overline{N}_L(r,1;G) + \overline{N}_E^{(2}(r,1;F) \\ & + m\overline{N}_{G\geq m+1}(r,1;G|F \neq 1) - \overline{N}_{F>2}(r,1;G) \\ \leq & N(r,1;G) - \overline{N}(r,1;G). \end{aligned}$$

Lemma 2.10 *(see [10]). If $N(r,0;f^{(k)}|f \neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity then*

$$N(r,0;f^{(k)}|f \neq 0) \leq k\overline{N}(r,\infty;f) + N(r,0;f| < k) + k\overline{N}(r,0;f| \geq k) + S(r,f).$$

Lemma 2.11 *(see [1, Lemma 2.9]). Let $\overline{E}_{m)}(1,F) = \overline{E}_{m)}(1,G), E_{1)}(1,F) = E_{1)}(1,G)$, where $m \geq 3$. Then*

$$\begin{aligned} & 2\overline{N}_{F>2}(r,1;G) + 2\overline{N}_{F\geq m+1}(r,1;F|G \neq 1) \\ \leq & \frac{2}{3}\overline{N}(r,0;F) + \frac{2}{3}\overline{N}(r,\infty;F) - \frac{2}{3}N_0(r,0;F') + S(r,F). \end{aligned}$$

Lemma 2.12 *Let $\overline{E}_{m)}(1,F) = \overline{E}_{m)}(1,G), E_{1)}(1,F) = E_{1)}(1,G)$ and $E(\infty,f) = E(\infty,g)$, where $m \geq 3$ and $H \not\equiv 0$, then*

$$T(r,F) \leq N_2(r,0;F) + \frac{8}{3}\overline{N}(r,\infty;F) + N_2(r,0;G) + \frac{2}{3}\overline{N}(r,0;F) + S(r,F) + S(r,G). \quad (2)$$

Proof: From the condition of Lemma 2.12, and by Lemma 2.7, we have

$$\begin{aligned} & \overline{N}(r,1;F) + \overline{N}(r,1;G) \\ \leq & N(r,1;F| = 1) + \overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + \overline{N}_E^{(2}(r,1;F) + \overline{N}_{F\geq m+1}(r,1;F|G \neq 1) + \overline{N}(r,1;G) \\ \leq & \overline{N}(r,0;F| \geq 2) + \overline{N}(r,0;G| \geq 2) + \overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) \\ & + \overline{N}_{F\geq m+1}(r,1;F|G \neq 1) + \overline{N}_{G\geq m+1}(r,1;G|F \neq 1) + \overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) \\ & + \overline{N}_E^{(2}(r,1;F) + \overline{N}_{F\geq m+1}(r,1;F|G \neq 1) + T(r,G) - m(r,1;G) \\ & + O(1) - 2\overline{N}_L(r,1;F) - 2\overline{N}_L(r,1;G) - \overline{N}_E^{(2}(r,1;F) \\ & - m\overline{N}_{G\geq m+1}(r,1;G|F \neq 1) + \overline{N}_{F>2}(r,1;G) + \overline{N}_0(r,0;F') \\ & + \overline{N}_0(r,0;G') + S(r,F) + S(r,G) \\ \leq & \overline{N}(r,0;F| \geq 2) + \overline{N}(r,0;G| \geq 2) + T(r,G) - m(r,1;G) + 2\overline{N}_{F\geq m+1}(r,1;F|G \neq 1) \\ & + \overline{N}_{F>2}(r,1;G) - (m-1)\overline{N}_{G\geq m+1}(r,1;G|F \neq 1) + \overline{N}_0(r,0;F') \\ & + \overline{N}_0(r,0;G') + S(r,F) + S(r,G). \end{aligned}$$

By Lemma 2.11, it follows that

$$\begin{aligned} \overline{N}(r,1;F) + \overline{N}(r,1;G) \leq & \overline{N}(r,0;F| \geq 2) + \overline{N}(r,0;G| \geq 2) + T(r,G) - m(r,1;G) \quad (3) \\ & + \frac{2}{3}\overline{N}(r,0;F) + \frac{2}{3}\overline{N}(r,\infty;F) - (m-1)\overline{N}_{G\geq m+1}(r,1;G|F \neq 1) \\ & + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') + S(r,F) + S(r,G). \end{aligned}$$

By the second fundamental theorem, we have

$$T(r,F) \leq \overline{N}(r,\infty;F) + \overline{N}(r,0;F) + \overline{N}(r,1;F) - \overline{N}_0(r,0;F') + S(r,F), \tag{4}$$

$$T(r,G) \leq \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,1;G) - \overline{N}_0(r,0;G') + S(r,G). \tag{5}$$

Adding (4) and (5) and from (3), it follows

$$\begin{aligned} T(r,F)+T(r,G) \leq & \overline{N}(r,\infty;F) + \overline{N}(r,0;F) + \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,1;F) \\ & + \overline{N}(r,1;G) - \overline{N}_0(r,0;F') - \overline{N}_0(r,0;G') + S(r,F) + S(r,G) \\ \leq & N_2(r,0;F) + \frac{8}{3}\overline{N}(r,\infty;F) + N_2(r,0;G) + T(r,G) - m(r,1;G) \\ & + \frac{2}{3}\overline{N}(r,0;F) - (m-1)\overline{N}_{G\geq m+1}(r,1;G|F\neq 1) + S(r,F) + S(r,G). \end{aligned} \tag{6}$$

Thus, we can get (2) from (6) easily. □

Similar to the above argument and by Lemma 2.8, we can get the following lemma.

Lemma 2.13 *Let* $\overline{E}_{m)}(1,F) = \overline{E}_{m)}(1,G), E_{1)}(1,F) = E_{1)}(1,G)$, $E(\infty,f) = E(\infty,g)$ *and* $E(0,F) = E(0,G)$, *where* $m \geq 3$ *and* $H \not\equiv 0$, *then*

$$T(r,F) \leq \frac{8}{3}\overline{N}(r,0;F) + \frac{8}{3}\overline{N}(r,\infty;F) + S(r,F) + S(r,G). \tag{7}$$

Lemma 2.14 *If* $\overline{E}_{m)}(1,F) = \overline{E}_{m)}(1,G), E_{2)}(1,F) = E_{2)}(1,G)$ *and* $E(\infty,F) = E(\infty,G)$, *where* $m \geq 4$ *and* $H \not\equiv 0$, *then*

$$T(r,F) + T(r,G) \leq 2N_2(r,0;F) + 2N_2(r,0;G) + 4\overline{N}(r,\infty;F) + S(r,F) + S(r,G). \tag{8}$$

Proof: From (4), (5) and by Lemma 2.7, we have

$$\begin{aligned} T(r,F)+T(r,G) \leq & \overline{N}(r,\infty;F) + \overline{N}(r,0;F) + \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,1;F) \\ & + \overline{N}(r,1;G) - \overline{N}_0(r,0;F') - \overline{N}_0(r,0;G') + S(r,F) + S(r,G) \\ \leq & \overline{N}(r,\infty;F) + \overline{N}(r,0;F) + \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + N(r,1;F|=1) \\ & + \overline{N}(r,1;F|\geq 2) + \overline{N}(r,1;G) - \overline{N}_0(r,0;F') - \overline{N}_0(r,0;G') \\ & + S(r,F) + S(r,G) \\ \leq & \overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + N_2(r,0;F) + N_2(r,0;G) + \overline{N}(r,1;G) \\ & + \overline{N}(r,1;F|\geq 2) + \overline{N}_L(r,1;G) + \overline{N}_L(r,1;F) + \overline{N}_{F\geq m+1}(r,1;F|G\neq 1) \\ & + \overline{N}_{G\geq m+1}(r,1;G|F\neq 1) + S(r,F) + S(r,G). \end{aligned} \tag{9}$$

Since

$$\overline{N}(r,1;F|=m;G|=m-1) + \cdots + \overline{N}(r,1;F|=m;G|=3) \leq \overline{N}(r,1;F|=m);$$

and

$$\overline{N}(r,1;G|=m;F|=m-1) + \cdots + \overline{N}(r,1;G|=m;F|=3) \leq \overline{N}(r,1;G|=m),$$

we see that

$$\begin{aligned}
& \overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_{F\geq m+1}(r,1;F|G\neq 1) \\
& +\overline{N}_{G\geq m+1}(r,1;G|F\neq 1)+\overline{N}(r,1;F|\geq 2)+\overline{N}(r,1;G) \\
\leq\ & \overline{N}(r,1;F|=m;G|=m-1)+\cdots+\overline{N}(r,1;F|=m;G|=3) \\
& +\overline{N}(r,1;F|\geq m+2)+\overline{N}(r,1;G|=m;F|=m-1)+\cdots \\
& +\overline{N}(r,1;G|=m;F|=3)+\overline{N}(r,1;G|\geq m+2) \\
& +\overline{N}(r,1;G|\geq m+2)+\overline{N}(r,1;F|\geq m+1) \\
& +\overline{N}(r,1;G|\geq m+1)+\overline{N}(r,1;F|=2)+\cdots \\
& +\overline{N}(r,1;F|=m)+\overline{N}(r,1;F|\geq m+1)+\overline{N}(r,1;G|=1) \qquad (10)\\
& +\cdots+\overline{N}(r,1;G|=m)+\overline{N}(r,1;G|\geq m+1) \\
\leq\ & \tfrac{1}{2}N(r,1;F|=1)+\overline{N}(r,1;F|=2)+\cdots+2\overline{N}(r,1;F|=m) \\
& +2\overline{N}(r,1;F|\geq m+1)+\overline{N}(r,1;F|\geq m+2)+\tfrac{1}{2}N(r,1;G|=1) \\
& +\overline{N}(r,1;G|=2)+\cdots+2\overline{N}(r,1;G|=m)+2\overline{N}(r,1;G|\geq m+1) \\
& +\overline{N}(r,1;G|\geq m+2) \\
\leq\ & \tfrac{1}{2}[N(r,1;F)+N(r,1;G)] \\
\leq\ & \tfrac{1}{2}[T(r,F)+T(r,G)].
\end{aligned}$$

From (9) and (10), we can get (8) easily.

Thus, this completes the proof of Lemma 2.14. □

Similar to the argument as in Lemma 2.14, and by Lemma 2.8, we can get the following lemma

Lemma 2.15 *If $\overline{E}_{m)}(1,F)=\overline{E}_{m)}(1,G), E_{2)}(1,F)=E_{2)}(1,G)$, $E(0,F)=E(0,G)$ and $E(\infty,F)$ $=E(\infty,G)$, where $m\geq 4$ and $H\not\equiv 0$, then*

$$T(r,F)+T(r,G)\leq 2\overline{N}(r,\infty;F)+2\overline{N}(r,\infty;G)+2\overline{N}(r,0;F)+2\overline{N}(r,0;G)+S(r,F)+S(r,G).$$

3 Proofs of the Theorems

3.1 Proof of Theorem 1.1

Proof: Let F,G be stated as in (1). Suppose that $H\not\equiv 0$. From the condition of Theorem 1.1, we have $\overline{E}_{m)}(1,F)=\overline{E}_{m)}(1,G), E_{1)}(1,F)=E_{1)}(1,G)$ and $E(\infty,f)=E(\infty,g)$, where $m\geq 3$.

From the definitions of F,G, it follows that

$$\begin{aligned}
& N_2(r,0;F)\leq 2\overline{N}(r,0;f)+N(r,0;f+a)+S(r,f); \qquad (11)\\
& N_2(r,0;G)\leq 2\overline{N}(r,0;g)+N(r,0;g+a)+S(r,g);\\
& \overline{N}(r,0;F)=\overline{N}(r,0;f)+\overline{N}(r,0;f+a);\\
& \overline{N}(r,0;G)=\overline{N}(r,0;g)+\overline{N}(r,0;g+a);\\
& \overline{N}(r,\infty;F)=\overline{N}(r,\infty;f),\quad \overline{N}(r,\infty;G)=\overline{N}(r,\infty;g);\\
& T(r,F)=nT(r,f)+S(r,f),\quad T(r,G)=nT(r,g)+S(r,g).
\end{aligned}$$

It follows from (11) and Lemma 2.12 that

$$(n-\frac{13}{3})T(r,f)\leq 3T(r,g)+\frac{8}{3}\overline{N}(r,\infty;f)+S(r,f)+S(r,g). \qquad (12)$$

Similarly, we have

$$(n-\frac{13}{3})T(r,g)\leq 3T(r,f)+\frac{8}{3}\overline{N}(r,\infty;g)+S(r,f)+S(r,g). \qquad (13)$$

Combining (12) and (13), we have

$$\left\{n-10+\frac{4}{3}[\Theta(\infty,f)+\Theta(\infty,g)-2\varepsilon]\right\}T(r)\le S(r), \tag{14}$$

where $T(r)=\max\{T(r,f),T(r,g)\}$ and $S(r)=o(T(r))$ as $r\to\infty$.

If $n=8$ and $\Theta(\infty,f)+\Theta(\infty,g)>\frac{3}{2}$, we can take any real number $\varepsilon(0<2\varepsilon<\Theta(\infty,f)+\Theta(\infty,g)-\frac{3}{2})$. Thus, we can get a contradiction from (14).

If $n=9$ and $\Theta(\infty,f)+\Theta(\infty,g)>\frac{3}{4}$, we can take any real number $\varepsilon(0<2\varepsilon<\Theta(\infty,f)+\Theta(\infty,g)-\frac{3}{4})$. Thus, we can get a contradiction from (14).

If $n\ge 10$ and $\Theta(\infty,f)+\Theta(\infty,g)>\frac{4}{n-1}$, we can take any real number $\varepsilon(>0)$, we also get a contradiction from (14).

Thus, it follows $H\equiv 0$, that is

$$\frac{F''}{F'}-\frac{2F'}{F-1}\equiv\frac{G''}{G'}-\frac{2G'}{G-1}.$$

By a simple calculate, we have either $F\cdot G\equiv 1$ or $F\equiv G$.

If $F\cdot G\equiv 1$, that is $f^{n-1}(f+a)g^{n-1}(g+a)\not\equiv b^2$. By Lemma 2.6, we can get a contradiction.

If $F\equiv G$, that is $f^{n-1}(f+a)\equiv g^{n-1}(g+a)$. From the condition of Theorem 1.1 and Lemma 2.3, we can get $f\equiv g$.

Thus, we complete the proof of Theorem 1.1. □

3.2 The proof of Theorem 1.2

Proof: From the condition of Theorem 1.2, we have $\overline{E}_{m)}(1,F)=\overline{E}_{m)}(1,G)$, $E_{2)}(1,F)=E_{2)}(1,G)$ and $E(\infty,F)=E(\infty,G)$,where $m\ge 4$. Let $H\equiv 0$. Then it follows from (11) and Lemma 2.14 that

$$(n-6)[T(r,f)+T(r,g)]\le 2\overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)+S(r,f)+S(r,g),$$

that is,

$$\{n-8+\Theta(\infty;f)+\Theta(\infty;g)-2\varepsilon\}\,T(r)\le S(r), \tag{15}$$

where $T(r)=\max\{T(r,f),T(r,g)\}$ and $S(r)=o(T(r))$ as $r\to\infty$.

If $n=7$ and $\Theta(\infty,f)+\Theta(\infty,g)>1$, we can take any real number $\varepsilon(0<2\varepsilon<\Theta(\infty,f)+\Theta(\infty,g)-1)$. Thus, we can get a contradiction from (15).

If $n\ge 8$ and $\Theta(\infty,f)+\Theta(\infty,g)>\frac{4}{n-1}>0$, we can take any real number $\varepsilon(0<2\varepsilon<\Theta(\infty,f)+\Theta(\infty,g))$. Thus, we can get a contradiction from (15).

Suppose that $H\equiv 0$, by using the same argument as in Theorem 1.1, we can get the conclusion of Theorem 1.2 easily.

Thus, we complete the proof of Theorem 1.2. □

3.3 The proof of Theorem 1.3

Proof: From the condition of Theorem 1.3, we have $\overline{E}_{m)}(1,F)=\overline{E}_{m)}(1,G), E_{1)}(1,F)=E_{1)}(1,G)$, $E(\infty,F)=E(\infty,G)$ and $E(0,F)=E(0,G)$, where $m\ge 3$.

Suppose that $H\not\equiv 0$. From (11) and Lemma 2.13, it follows

$$(n-\frac{16}{3})T(r,f)\le\frac{8}{3}\overline{N}(r,\infty;f)+S(r,f)+S(r,g),$$

that is,

$$\left(n-8+\frac{4}{3}(\Theta(\infty;f)+\Theta(\infty;g)-2\varepsilon)\right)T(r)\le S(r). \tag{16}$$

If $n = 6$ and $\Theta(\infty, f) + \Theta(\infty, g) > \frac{3}{2}$, we can take any real number $\varepsilon(0 < 2\varepsilon < \Theta(\infty, f) + \Theta(\infty, g) - \frac{3}{2})$. Thus, we can get a contradiction from (16).

If $n = 7$ and $\Theta(\infty, f) + \Theta(\infty, g) > \frac{3}{4}$, we can take any real number $\varepsilon(0 < 2\varepsilon < \Theta(\infty, f) + \Theta(\infty, g) - \frac{3}{4})$. Thus, we can get a contradiction from (16).

If $n \geq 8$ and $\Theta(\infty, f) + \Theta(\infty, g) > 0$, we can take any real number $\varepsilon(> 0)$, we also get a contradiction from (16).

Let $H \equiv 0$,that is

$$\frac{F''}{F'} - \frac{2F'}{F-1} \equiv \frac{G''}{G'} - \frac{2G'}{G-1}.$$

By a simple calculate, we have either $F \cdot G \equiv 1$ or $F \equiv G$.

If $F \cdot G \equiv 1$, that is $f^{n-1}(f + a)g^{n-1}(g + a) \not\equiv b^2$. By Lemma 2.5, we can get a contradiction.

If $F \equiv G$, that is $f^{n-1}(f + a) \equiv g^{n-1}(g + a)$. Since $E(0; f) = E(0, g)$, from the condition of Theorem 1.3 and Lemma 2.4, we can get $f \equiv g$.

Thus, we complete the proof of Theorem 1.3. □

3.4 The proof of Theorem 1.4

Proof: From the condition of Theorem 1.4, we have $\overline{E}_{m)}(1, F) = \overline{E}_{m)}(1, G), E_{2)}(1, F) = E_{2)}(1, G)$, $E(\infty, F) = E(\infty, G)$ and $E(0, F) = E(0, G)$, where $m \geq 4$.

Suppose that $H \not\equiv 0$. From (11) and Lemma 2.15, it follows

$$(n - 4)[T(r, f) + T(r, g)] \leq 2\overline{N}(r, \infty; f) + 2\overline{N}(r, \infty; g) + S(r, f) + S(r, g),$$

that is,

$$(n - 6 + \Theta(\infty; f) + \Theta(\infty; g) - 2\varepsilon)\, T(r) \leq S(r). \tag{17}$$

If $n = 5$ and $\Theta(\infty, f) + \Theta(\infty, g) > 1$, we can take any real number $\varepsilon(0 < 2\varepsilon < \Theta(\infty, f) + \Theta(\infty, g) - 1)$. Thus, we can get a contradiction from (17).

If $n \geq 6$ and $\Theta(\infty, f) + \Theta(\infty, g) > 0$, we can take any real number $\varepsilon(0 < 2\varepsilon < \Theta(\infty, f) + \Theta(\infty, g))$. Thus, we can get a contradiction from (17).

Suppose that $H \equiv 0$, by using the same argument as in Theorem 1.3, we can get the conclusion of Theorem 1.4 easily.

Thus, we complete the proof of Theorem 1.4. □

References

[1] A. Banerjee, On uniqueness of meromorphic functions when two differential monomials share one value, Bull. Korean Math. Soc., 44(4) (2007), 607-622.

[2] M. L Fang, W. S. Xu, A note on a problem of Gross, Chin. Ann. Math., 18A:5(1997), 563-568.

[3] G. Frank, M. Reinders, Aunique range set for meromorphic functions with 11 elements, Complex Var. Elliptic Equ. 37 (1998), 185-193.

[4] F. Gross, Factorization of meromorphic functions and some open problem, in:Proc. Conf. Univ. Kentucky. Lexington. KY, 1976, in: Lecture Notes in Math., vol.599, Springer, Berlin, 1977, pp.51-69.

[5] G. G. Gundersen, Meromorphic functions that share three or four values, J. London Math. Soc., 20 (1979), 457-466.

[6] W. K. Hayman, Meromorphic Functions, The Clarendon Press, Oxford,1964.

[7] I. Lahiri, On a question of Hong Xun Yi, Arch. Math. (Brno), 38 (2002), 119-128.

[8] I. Lahiri, A.Banerjee, Uniqueness of meromorphic functions with deficient poles, Kyungpook Math. J. 44 (2004), 575-584.

[9] I. Lahiri, A.Banerjee, Weighted sharing of two sets, Kyungpook Math. J., 46 (1) (2006), 79-87.

[10] I. Lahiri, S. Dewan, Value distribution of the product of a meromorphic function and its derivative, Kodai Math.J., 26(2003)(1), 95-100.

[11] H. X. Yi, On a question of Gross, Sci. China (Ser. A), 38(1995), 8-16.

[12] H. X. Yi, C. C.Yang, Uniqueness theory of meromorphic functions, Science Press, Beijing, (1995).

[13] J. Zhang, Y. Xu, Meromorphic functions sharing two sets, Applied Mathematics Letters, 21 (2008),471-476.

[14] H. X. Yi, Meromorphic functions that share one or two values II. Kodai Math. J., 22 (1999), 264-272.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 24, NO.6, 2018, COPYRIGHT 2018 EUDOXUS PRESS, LLC

Global Dynamics of Nonlinear Difference Equation $x_{n+1} = A + \frac{x_n}{x_{n-1}\,x_{n-2}}$

Keying Liu[1,2], Peng Li[2], Fei Han[3], Weizhou Zhong[1,4]

1. School of Economics and Finance, Xi'an Jiaotong University, Xi'an 710061, China
2. School of Mathematics and Information Sciences, North China University of Water Resources and Electric Power, Zhengzhou 450045, China
3. The Hilltiop Institute, University of Maryland Baltimore County, Baltimore, Maryland 21250, USA
4. College of Business Administration, Huaqiao University, Quanzhou 362021, China

Correspondence should be addressed to Zhong Weizhou, weizhou@mail.xjtu.edu.cn.

Abstract

In this paper, global dynamics of a kind of nonlinear difference equation was investigated, which had only one positive equilibrium. Every positive solution of the equation either converges to its positive equilibrium if it is locally asymptotically stable or nonhyperbolic or oscillates about its positive equilibrium if it is a saddle point. For the latter case, the length of positive or negative semicycles is no more than four.

Key words: Difference equations; Stable; Semicycles

AMS 2000 Subject Classification: 39A10, 39A11

1 Introduction

In this paper, we focus on the nonlinear difference equation of third order

$$x_{n+1} = A + \frac{x_n}{x_{n-1}x_{n-2}}, \qquad n = 0, 1, \cdots, \tag{1.1}$$

with the parameter A and initial conditions $x_{-2},\ x_{-1},\ x_0$ being positive. We will investigate its global dynamics and present the following results:

If $A \geq 1/\sqrt{2}$, then every positive solution of (1.1) converges to $\bar{x} = (A + \sqrt{A^2+4})/2$.

If $0 < A < 1/\sqrt{2}$, then every positive solution of (1.1) oscillates about $\bar{x}$.

Equation (1.1) can be regarded as a variation of the following equation

$$x_{n+1} = A + \frac{x_n^p}{x_{n-1}^q\, x_{n-2}^r}, \qquad n = 0, 1, 2, \ldots \tag{1.2}$$

In 2009, Iričanin and Stević [1] studied the boundedness character of positive solutions of (1.2) when all parameters A, p, q and r are positive.

It can be also regarded as a variation of the following systems of difference equations.

In 2015, Zhang et al.[2] investigated the following system

$$x_{n+1} = A + \frac{x_n}{y_{n-1}\, y_{n-2}}, \qquad y_{n+1} = A + \frac{y_n}{x_{n-1}\, x_{n-2}}, \qquad n = 0, 1, 2, \ldots, \tag{1.3}$$

where A is a positive constant and initial conditions $x_{-i},\ y_{-i} \in (0, \infty)$ for $i = 0, -1, -2$.

The equilibria of system (1.3) satisfy the following system of equations:

$$\bar{x} = A + \frac{\bar{x}}{\bar{y}^2}, \qquad \bar{y} = A + \frac{\bar{y}}{\bar{x}^2}, \tag{1.4}$$

By eliminating y from (1.4), we obtain

$$(A^2 - 1)\bar{x}^4 - A^3\bar{x}^3 + 2\bar{x}^2 - 1 = 0. \tag{1.5}$$

Denote $c = (A + \sqrt{A^2 + 4})/2$, $a = 2/(A - \sqrt{4 - 3A^2})$ and $b = 2/(A + \sqrt{4 - 3A^2})$.

If $A > 0$, then system (1.3) always has the positive equilibrium $(c,\ c)$.

If $1 < A < 2/\sqrt{3}$, then system (1.3) has two additional positive equilibrium $(a,\ b),\ (b,\ a)$.

The main results[2] are listed in the following:

(A.1) If $A > 1$, every positive solution of system (1.3) is bounded.

(A.2) If $A > 2/\sqrt{3}$, $(c,\ c)$ is locally asymptotically stable.

(A.3) If $A > \sqrt{3}$, every positive solution of system (1.3) approaches $(c,\ c)$.

(A.4) If $1 < A < 2/\sqrt{3}$, $(a,\ b)$ and $(b,\ a)$ are locally asymptotically stable.

In 2005, Yang[3] studied the global behavior of the following system

$$x_n = A + \frac{y_{n-1}}{x_{n-p}\, y_{n-q}}, \qquad y_n = A + \frac{x_{n-1}}{x_{n-r}\, y_{n-s}}, \qquad n = 1, 2, \ldots, \tag{1.6}$$

where $p, q, r, s \geq 2$, A is a positive constant and initial conditions are positive real numbers. System (1.6) has the unique positive equilibrium $(c,\ c)$. He obtained the following results:

(B.1) If $A > 1$, every positive solution of system (1.6) is bounded.

(B.2) If $A > 2/\sqrt{3}$, $(c,\ c)$ is locally asymptotically stable.

(B.3) If $A > \sqrt{2}$, every positive solution of system (1.6) approaches $(c,\ c)$.

For these two systems, they gave the same condition $A > 2/\sqrt{3}$ to obtain the local asymptotical stability of $(c,\ c)$ by the same method. Other related articles see[4]-[8]. Other methods include Routh-Hurwitz criterion[9] and Rouche's Theorem[10]. Other types of equation and boundedness nature and so on see [11]-[18].

2 Local Stability

First of all, we obtain the local stability of $\bar{x}$ of (1.1).

As for the definition of stability and the method of linearized stability, see [17],[18] and related papers[2–10].

As is known, $\bar{x}$ of (1.1) satisfies $\bar{x} = A + 1/\bar{x}$ and $\bar{x} = c$ is positive, which is unique.

The linearized equation of (1.1) about $\bar{x} = c$ is

$$x_{n+1} = c^{-2}x_n - c^{-2}x_{n-1} - c^{-2}x_{n-2} \tag{2.2}$$

and its characteristic polynomial is

$$f(\lambda) = \lambda^3 - c^{-2}\lambda^2 + c^{-2}\lambda + c^{-2}, \tag{2.3}$$

from which we have

$$f(0) = c^{-2} > 0, \tag{2.4}$$

$$f(1) = c^{-2} + 1 > 0, \tag{2.5}$$

$$f(-0.5) = \frac{1}{8c^2}(2 - c^2), \tag{2.6}$$

$$f(-1) = -c^{-2} - 1 < 0, \tag{2.7}$$

which lead to the existence of a solution λ_0 of $f(\lambda) = 0$ in the interval $(-1, 0)$.

On the other hand,

$$f'(\lambda) = 3\lambda^2 - 2c^{-2}\lambda + c^{-2} > 0 \tag{2.8}$$

because $\Delta = 4c^{-4} - 12c^{-2} = 4c^{-4}(1 - 3c^2) < 0$ for $A > 0$ and $c > 1$. Hence, we claim that $f(\lambda) = 0$ has a pair of conjugate complex roots $\lambda_{1,2}$.

Let $f(\lambda)$ be factorize as the following

$$\begin{aligned} f(\lambda) &= \lambda^3 - c^{-2}\lambda^2 + c^{-2}\lambda + c^{-2} \\ &= (\lambda - \lambda_0)(\lambda^2 + a\lambda + b) \\ &= \lambda^3 + (a - \lambda_0)\lambda^2 + (b - a\lambda_0)\lambda - b\lambda_0, \end{aligned} \tag{2.9}$$

from which we have

$$a - \lambda_0 = -c^{-2}, \qquad b - a\lambda_0 = c^{-2}, \qquad -b\lambda_0 = c^{-2}.$$

Thus, we have the modulus of the complex roots $\lambda_{1,2}$

$$|\lambda_{1,2}|^2 = b = \lambda_0^2 - c^{-2}\lambda_0 + c^{-2}. \tag{2.10}$$

We claim that $|\lambda_{1,2}| < 1$ if $A > 1/\sqrt{2}$.

In fact, we have that

$$|\lambda_{1,2}| < 1 \Longleftrightarrow b < 1 \Longleftrightarrow h(\lambda_0) < 0 \tag{2.11}$$

with $h(x) = x^2 - c^{-2}x + c^{-2} - 1$ for $\lambda_0 \in (-1, 0)$. As is known, $h(x) = 0$ has two distinct roots: $x_1 = 1$ and $x_2 = 2c^{-2} - 2$. Further, $h(-0.5) = 3(2 - c^2)/(4c^2)$.

If $A > 1/\sqrt{2}$, then we have that $c^2 > 2$ and $\lambda_0 \in (-0.5, 0)$, from which $|\lambda_{1,2}| < 1$.

If $A = 1/\sqrt{2}$, then $c = \sqrt{2}$ and $\lambda_0 = -0.5$, thus $|\lambda_{1,2}| = 1$.

If $A < 1/\sqrt{2}$, then $c < \sqrt{2}$ and $\lambda_0 \in (-1, -0.5)$, thus $|\lambda_{1,2}| > 1$.

A nonhyperbolic equilibrium point of difference equations or system of difference equations is called non-hyperbolic point of stable type (resp. of unstable type) if the other characteristic value of the Jacobian matrix about it is in interval (-1, 1) (resp. outside of interval [-1, 1]).

We generalize the above into the following theorem.

Theorem 2.1. *Assume that $\bar{x} = c$ is the positive equilibrium of* (1.1).

1. *If $A > 1/\sqrt{2}$, then $\bar{x} = c$ is locally asymptotically stable.*
2. *If $A = 1/\sqrt{2}$, then $\bar{x} = c$ is nonhyperbolic of stable type.*
3. *If $0 < A < 1/\sqrt{2}$, then $\bar{x} = c$ is a saddle point .*

3 Global Dynamics

In this section, we present the main results by investigating the global dynamics of (1.1), which has only one positive equilibrium – being locally asymptotically stable or nonhyperbolic of stable type or a saddle point depending on the parameter A .

First, we show the boundedness of positive solutions of (1.1).

By Theorem 3[1], if $p^2 < 3q$, then every positive solution of (1.2) is bounded. Thus, every positive solution of (1.1) is bounded and $x_{n+1} \le A + 1/A + 1/A^3$ for $n > 4$.

Second, we show the semicycles of solutions of (1.1).

From (1.1), we obtain

$$x_{n+1} - \bar{x} = \frac{x_n}{x_{n-1}\, x_{n-2}} - \frac{1}{\bar{x}} = \frac{\bar{x}}{x_{n-1}\, x_{n-2}} \left(\frac{x_n}{\bar{x}} - \frac{x_{n-1}}{\bar{x}} \frac{x_{n-2}}{\bar{x}} \right) \tag{3.1}$$

and

$$x_{n+2} \;=\; A + \frac{x_{n+1}}{x_n\, x_{n-1}} = A + \frac{A}{x_n\, x_{n-1}} + \frac{1}{x_{n-1}^2\, x_{n-2}} \tag{3.2}$$

Then the following statements are true:

1. If for some $N \ge 0,\ x_{N-2},\ x_{N-1} \le \bar{x}$ and $x_N \ge \bar{x}$, then $x_{N+1} \ge \bar{x}$.
2. If for some $N \ge 0,\ x_{N-2},\ x_{N-1} \ge \bar{x}$ and $x_N \le \bar{x}$, then $x_{N+1} \le \bar{x}$.
3. If for some $N \ge 0,\ x_{N-2},\ x_{N-1},\ x_N \ge \bar{x}$, then $x_{N+2} \le \bar{x}$.
4. If for some $N \ge 0,\ x_{N-2},\ x_{N-1},\ x_N \le \bar{x}$, then $x_{N+2} \ge \bar{x}$.

From the above, we obtain that every positive solution of (1.1) is bounded and converges to $\bar{x}$ or oscillates about $\bar{x}$.

Now, we try to analyze the global dynamics of (1.1) for three cases.

3.1 Case $A > 1/\sqrt{2}$

In this case, the positive equilibrium $\bar{x} = c$ of (1.1) is locally asymptotically stable. We will show that it is an attractor and is globally asymptotically stable.

Theorem 3.1. *Assume that $\bar{x} = c$ is the positive equilibrium of* (1.1).
If $A > 1/\sqrt{2}$, then $\bar{x} = c$ is globally asymptotically stable.

Proof.
Let $(m,\, M) \in [A,\, A + 1/A + 1/A^3\,]^2$ be a solution of the system

$$m = A + \frac{m}{M^2}, \qquad M = A + \frac{M}{m^2}, \tag{3.3}$$

From (3.3), we could derive

$$\frac{m^3}{m^2-1} = \frac{M^3}{M^2-1}. \tag{3.4}$$

As is known, the function $k(x) = x^3/(x^2-1)$ is strictly increasing for $x > \sqrt{3}$ and is strictly decreasing for $0 < x < 1$ and $1 < x < \sqrt{3}$. Therefore, (3.4) holds only for $A \le m = M$.

By Theorem A.0.5 [17], every positive solution of (1.1) converges to $\bar{x} = c$ for $A > 1/\sqrt{2}$, that is, $\bar{x}$ is a global attractor. Hence, we obtain the result. □

It is worth pointing out that under the condition $A > 1/\sqrt{2}$, $\bar{x} = c$ of (1.1) is locally asymptotically stable which implies that it is globally asymptotically stable, that is, locally asymptotically stable $\Longrightarrow$ globally asymptotically stable.

3.2 Case $A = 1/\sqrt{2}$

In this case, the positive equilibrium $\bar{x} = \sqrt{2}$ of (1.1) is nonhyperbolic of stable type.

Theorem 3.2. *If $A = 1/\sqrt{2}$, then every positive solution of* (1.1) *converges to $\bar{x}$.*

Proof.
Let $\{x_n\}$ be a positive solution of (1.1). We assume that $m = \liminf_{n\to\infty} x_n$ and $M = \limsup_{n\to\infty} x_n$, then $A \le m \le M < \infty$. From (1.1), we obtain

$$m \ge A + \frac{m}{M^2}, \qquad M \le A + \frac{M}{m^2}, \tag{3.5}$$

from which we get

$$m \ge \frac{AM^2}{M^2-1}, \qquad m^2 \le \frac{M}{M-A}. \tag{3.6}$$

It follows that

$$(A^2-1)M^4 - A^3M^3 + 2M^2 - 1 \le 0. \tag{3.7}$$

Similarly, we obtain

$$(A^2-1)m^4 - A^3m^3 + 2m^2 - 1 \ge 0, \tag{3.8}$$

As is known, for $A = 1/\sqrt{2}$, the equation $(A^2-1)x^4 - A^3x^3 + 2x^2 - 1 = 0$, which is (1.5), has exactly one root $c = \sqrt{2}$ greater than A, which implies that $\lim\limits_{n\to\infty} x_n = c = \sqrt{2}$

□

3.3 Case $0 < A < 1/\sqrt{2}$

In this case, the positive equilibrium $\bar{x} = c$ of (1.1) is a saddle point.

Theorem 3.3. *Suppose that $0 < A < 1/\sqrt{2}$ and let $\{x_n\}$ be a nontrivial solution of* (1.1)*. Then the solution oscillates about $\bar{x}$ with semicycles of length no more than four.*

4 Numerical Results

Some numerical simulations are given to support our theoretical analysis with Matlab7.0.

Exam1 For $A = 0.8 > 1/\sqrt{2}$ and $(x_{-2},\ x_{-1},\ x_0) = (2.2,\ 2.3,\ 2)$, the solution of (1.1) converges to $\bar{x} = 1.4770$ by Theorem 3.1. See Figure 1.

Exam2 For $A = 1/\sqrt{2}$ and $(x_{-2},\ x_{-1},\ x_0) = (0.2,\ 1.8,\ 0.5)$, the solution of (1.1) converges to $\bar{x} = 1.4142$ by Theorem 3.2. See Figure 2.

Exam3 For $A = 0.5 < 1/\sqrt{2}$ and $(x_{-2},\ x_{-1},\ x_0) = (1.5,\ 1.6,\ 2.5)$, the solution of (1.1) oscillates about $\bar{x} = 1.2808$ by Theorem 3.3. See Figure 3. The length of positive or negative semicycles is no more than four.

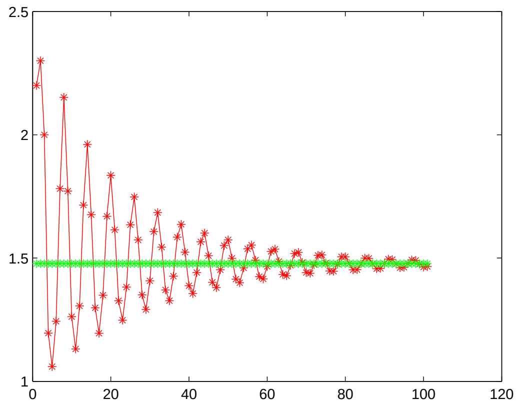

Figure 1: The solution of (1.1) converges to $\bar{x} = 1.4770$.

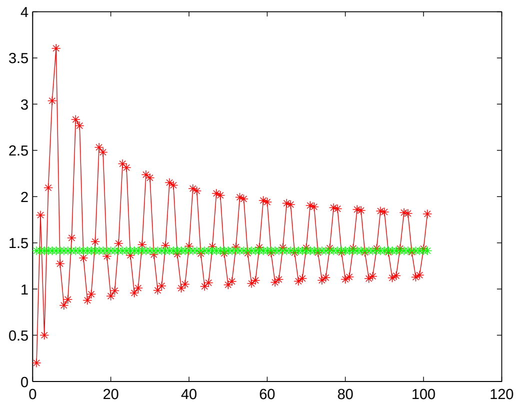

Figure 2: The solution of (1.1) converges to $\bar{x} = 1.4142$.

5 Conclusion

Difference equation appeals more and more attention in recent years. It is of great interest to

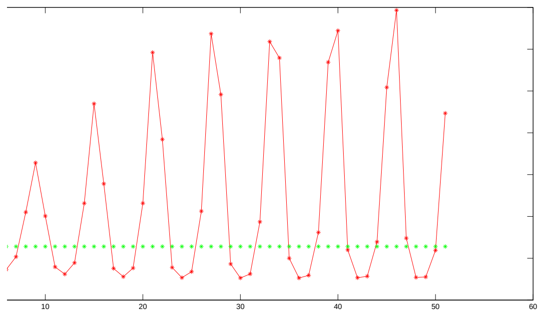

Figure 3: The solution of (1.1) oscillates about $\bar{x} = 1.2808$.

investigate of the monotonicity, periodicity, boundedness nature and global dynamics of kinds of difference equations and system of difference equations.

It is known that the techniques in the investigation of the behavior of difference equations can be used in investigating equations arising in mathematical models describing real life situations in biology, economics, physics, sociology, control theory and vice versa.

Here, we consider the local stability of the positive equilibrium of such a system by investigating the distribution of roots of the corresponding characteristic polynomial with order three. For the global dynamics, we analyze three cases according to the positive equilibrium, especially we consider the the distribution of roots of an auxiliary equation of order four which arises naturally in the proof. In one word, we consider the distribution of roots of two particular polynomial of higher order. The popular methods including the Schur-Cohn criterion, Rouché Theorem, Routh-Hurwitz criterion and Jury criterion and so on.

The complex dynamics of such a equation we considered should play an important role on the dynamics of systems such as (1.3) and (1.6) and so on. We believe that they should be paid more attention on the global dynamics.

Conflict of Interests

The authors declare that they have no competing interests.

Acknowledgement

This work is financially supported by the Natural Science Foundation of China (No. 71271086, 71172184) and the Education Department of Henan Province (No. $12A110014$).

References

[1] B. Iričanin, S. Stević, *On a class of third-order nonlinear difference equations*, Applied Mathematics and Computation, 213 (2009): 479-483.

[2] Qianhong Zhang, Jingzhong Liu, Zhenguo Luo, *Dynamical behavior of a system of third-order rational difference equation*, Discrete Dynamics in Nature and Society, 2015, Article ID 530453.

[3] Xiaofang Yang, *On system of rational difference equations* $x_n = A + y_{n-1}/\ x_{n-p}\, y_{n-q}$, $y_n = A + x_{n-1}/\ x_{n-r}\, y_{n-s}$, Journal of Mathematica Analysis and Applications, 307 (2005): 305-311.

[4] Yu Zhang, Xiaofan Yang, G.M. Megson, D.J. Evans, *On system of rational difference equations* $x_n = A + 1/\ y_{n-p}$, $y_n = A + y_{n-1}/\ x_{n-r}\, y_{n-s}$, Applied Mathematics and Computation, 176 (2006): 403-408.

[5] A.Q. Khan, M.N. Qureshi, Q. Din, *Global dynamics of some systems of higher-order rational difference equations*, Advances in Difference Equations, doi:10.1186/1687-1847-2013-354.

[6] Qianhong Zhang, Wenzhuan Zhang, *On a system of two high-order nonlinear difference equations*, Advances in Mathematical Physics, 2014, Article ID 729273.

[7] Yu Zhang, Xiaofan Yang, D.J. Evans, Ce Zhu, *On the nonlinear difference equation system* $x_{n+1} = A + y_{n-m}/x_n,\ y_{n+1} = A + x_{n-m}/y_n$, Computers and Mathematics with Applications, 53 (2007): 1561-1566.

[8] B. Sroysang, *Dynamics of a system of rational higher-order difference equation*, Discrete Dynamics in Nature and Society, 2013, Article ID 179401.

[9] Q. Din, M.N. Qureshi, A.Q. Khan, *Dynamics of a fourth-order system of rational difference equations*, Advances in Difference Equations, doi:10.1186/1687-1847-2012-215.

[10] A.Q. Khan, *Global dynamics of two systems of exponential difference equations by Lyapunov function*, Advances in Difference Equations, doi:10.1186/1687-1847-2014-297.

[11] Changyou Wang, Min Hu, *On the solutions of a rational recursive sequence*, Journal of Mathematics and Informatics, 1 (2013): 25-33.

[12] R. Abo-Zeid, *On the oscillation of a third order rational difference equation*, Journal of the Egyptian Mathematical Society, 23 (2015): 62-66.

[13] S. Stević, B. Iričanin, Z. Šmarda, *Boundedness character of a fourth-order system of difference equations*, Advances in Difference Equations, doi 10.1186/s13662-015-0644-y.

[14] H. Shojaei, S. Parvandeh, T. Mohammadi, Z. Mohammadi, N, Mohammadi, *Stability and convergence of a higher order rational difference equation*, Australian Journal of Basic and Applied Sciences, 5(11 (2011): 72-77.

[15] M. Dehghan, N. Rastegar, *Stability and periodic character of a third order difference equation*, Mathematical and Computer Modelling,(2011) doi:10.1016/j.mcm.2011.06.025.

[16] S. Jašarević Hrustić, M.R.S. Kulenović, M. Nurkanović, *Local dynamics and global stability of certain second-order rational difference equation with quadratic terms*, Discrete Dynamics in Nature and Society, 2016, Article ID 3716042.

[17] M.R.S. Kulenović, G. Ladas. *Dynamics of Second Order Rational Difference Equations with Open Problems and Conjectures*, Chapman & Hall/CRC Press, Boca Raton, Fla, USA, 2002.

[18] E. Camouzis, G. Ladas. *Dynamics of Third-Order Rational Difference Equations with Open Problems and Conjectures*, Chapman & Hall/CRC Press, Boca Raton, Fla, USA, 2008.

Regular hesitant fuzzy filters and MV-hesitant fuzzy filters of residuated lattices

G. Muhiuddin[a,*] **and Abdullah M. Al-roqi**[b]

[a] *Department of Mathematics, University of Tabuk, Tabuk 71491, Saudi Arabia*

[b] *Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia*

Abstract. The notion of regular hesitant fuzzy filters, Boolean hesitant fuzzy filters and MV-hesitant fuzzy filters are introduced, and related properties are investigated. Characterizations of regular hesitant fuzzy filters, Boolean hesitant fuzzy filters and MV-hesitant fuzzy filters are considered, relations between regular hesitant fuzzy filters and MV-hesitant fuzzy filters are discussed. Extension property for Boolean hesitant fuzzy filters (resp. MV-hesitant fuzzy filters) is established.

1. Introduction

Since the original definition of fuzzy sets by Zadeh in 1965, several extensions have been proposed for fuzzy sets, for example, type 2 fuzzy sets, intuitionistic fuzzy sets, interval-valued fuzzy sets and fuzzy multisets etc. Another extension of fuzzy sets, so called hesitant fuzzy sets, has been proposed in [7]. The motivation for introducing hesitant fuzzy sets is that it is sometimes difficult to determine the membership of an element into a set and in some circumstances this difficulty is caused by a doubt between a few different values. As a non-classical logic system, residuated lattices are a formal and useful tool for computer science to deal with uncertain and fuzzy information. Using the notion of hesitant fuzzy sets, Jun and Song [4] have studied filter theory in MTL-algebras. Also, Muhiuddin [5] have discussed filter theory in residuated lattices.

In this paper, we introduce the notion of regular hesitant fuzzy filters, Boolean hesitant fuzzy filters and MV-hesitant fuzzy filters, and investigate related properties. We consider

2010 Mathematics Subject Classification: 03G25, 06D72 06E99.

Keywords: (Regular, Boolean) hesitant fuzzy filter, MV-hesitant fuzzy filter.

*Corresponding author.

E-mail: chishtygm@gmail.com (G. Muhiuddin), aalroqi@kau.edu.sa (Abdullah M. Al-roqi)

characterizations of regular hesitant fuzzy filters, Boolean hesitant fuzzy filters and MV-hesitant fuzzy filters. We discuss relations between regular hesitant fuzzy filters and MV-hesitant fuzzy filters. We establish extension property for Boolean hesitant fuzzy filters (resp. MV-hesitant fuzzy filters).

2. Preliminaries

Definition 2.1 ([1, 2, 3])**.** A residuated lattice is an algebra $\mathcal{L} := (L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ such that

(1) $(L, \vee, \wedge, 0, 1)$ is a bounded lattice.
(2) $(L, \odot, 1)$ is a commutative monoid.
(3) $\odot$ and $\rightarrow$ form an adjoint pair, that is,

$$(\forall x, y, z \in L)\,(x \leq y \rightarrow z \;\Leftrightarrow\; x \odot y \leq z)\,.$$

A regular residuated lattice is a residuated lattice $\mathcal{L}$ satisfying the following regularity equation:

$$(\forall x \in L)\,((x \rightarrow 0) \rightarrow 0 = x)\,. \tag{2.1}$$

A residuated lattice L is called an MTL-algebra if it satisfies:

$$(\forall x, y \in L)\,((x \rightarrow y) \vee (y \rightarrow x) = 1)\,. \tag{2.2}$$

A residuated lattice L is called a BL-algebra if it satisfies the condition (2.2) and

$$(\forall x, y \in L)\,(x \wedge y = x \odot (x \rightarrow y))\,. \tag{2.3}$$

In a residuated lattice L, the ordering $\leq$ is defined as follows:

$$(\forall x, y \in L)\,(x \leq y \;\Leftrightarrow\; x \wedge y = x \;\Leftrightarrow\; x \vee y = y \;\Leftrightarrow\; x \rightarrow y = 1)$$

and x' will be reserved for $x \rightarrow 0$, and $x'' = (x')'$, etc. for all $x \in L$.

Proposition 2.2 ([1, 2, 3, 8, 9])**.** *In a residuated lattice $\mathcal{L}$, the following properties are valid.*

(2.4) $1 \to x = x,\ x \to 1 = 1,\ x \to x = 1,\ 0 \to x = 1,\ x \to (y \to x) = 1.$

(2.5) $x \to (y \to z) = (x \odot y) \to z = y \to (x \to z).$

(2.6) $x \leq y \ \Rightarrow\ z \to x \leq z \to y,\ y \to z \leq x \to z.$

(2.7) $z \to y \leq (x \to z) \to (x \to y),\ z \to y \leq (y \to x) \to (z \to x).$

(2.8) $x' = x''',\ x \leq x'',\ 1' = 0,\ 0' = 1.$

(2.9) $x' \wedge y' = (x \vee y)'.$

(2.10) $x \vee x' = 1 \ \Rightarrow\ x \wedge x' = 0.$

Definition 2.3 ([6])**.** A nonempty subset F of a residuated lattice $\mathcal{L}$ is called a filter of $\mathcal{L}$ if it satisfies the conditions:

(2.11) $$(\forall x, y \in L)\,(x, y \in F \ \Rightarrow\ x \odot y \in F)\,.$$

(2.12) $$(\forall x, y \in L)\,(x \in F,\ x \leq y \ \Rightarrow\ y \in F)\,.$$

Proposition 2.4 ([6])**.** *A nonempty subset F of a residuated lattice $\mathcal{L}$ is a filter of $\mathcal{L}$ if and only if it satisfies:*

(2.13) $$1 \in F.$$

(2.14) $$(\forall x \in F)\,(\forall y \in L)\,(x \to y \in F \ \Rightarrow\ y \in F)\,.$$

Definition 2.5 ([6])**.** A nonempty subset F of $\mathcal{L}$ is called a *Boolean filter* of a residuated lattice $\mathcal{L}$ if it is a filter of $\mathcal{L}$ that satisfies the following condition:

(2.15) $$(\forall x \in L)\,(x \vee x' \in F)\,.$$

Zhu and Xu [10] introduced the notion of a regular filter in a residuated lattice.

Definition 2.6 ([10])**.** A filter F of $\mathcal{L}$ is said to be regular if it satisfies the following condition:

(2.16) $$(\forall x \in L)\,(x'' \to x \in F)\,.$$

Lemma 2.7 ([10])**.** *Let F be a filter of $\mathcal{L}$. Then the following assertions are equivalent:*

(1) *F is regular.*
(2) $(\forall x, y \in L)\,(x' \to y \in F \ \Rightarrow\ y' \to x \in F)\,.$

Definition 2.8 ([10])**.** A subset F of $\mathcal{L}$ is called an MV-filter of $\mathcal{L}$ if it is a filter of $\mathcal{L}$ that satisfies:

$$(\forall x, y \in L)\,(y \to x \in F \;\Rightarrow\; ((x \to y) \to y) \to x \in F)\,. \tag{2.17}$$

Lemma 2.9 ([10])**.** *A filter F of $\mathcal{L}$ is an MV-filter of $\mathcal{L}$ if and only if it satisfies the condition:*

$$(\forall x, y \in L)\,(((x \to y) \to y) \to ((y \to x) \to x) \in F)\,. \tag{2.18}$$

3. Regular hesitant fuzzy filters

Let E be a reference set. A *hesitant fuzzy set* on E (see [7]) is defined in terms of a function $\mathcal{H}$ that when applied to E returns a subset of $[0, 1]$, that is, $\mathcal{H} : E \to \mathscr{P}([0, 1])$.

In what follows, we take a residuated lattice $\mathcal{L}$ as a reference set. For a hesitant fuzzy set $\mathcal{H}$ on $\mathcal{L}$ and $\tau \in \mathscr{P}([0, 1])$, we consider a set

$$\mathcal{H}_\tau := \{x \in L \mid \tau \subseteq \mathcal{H}(x)\}$$

which is called the τ*-hesitant level set* on $\mathcal{L}$.

Definition 3.1 ([5])**.** A hesitant fuzzy set $\mathcal{H}$ on $\mathcal{L}$ is called a hesitant fuzzy filter of $\mathcal{L}$ if the τ-hesitant level set $\mathcal{H}_\tau$ on $\mathcal{L}$ is a filter of $\mathcal{L}$ for all $\tau \in \mathscr{P}([0, 1])$ with $\mathcal{H}_\tau \neq \emptyset$.

Lemma 3.2 ([5])**.** *A hesitant fuzzy set $\mathcal{H}$ on $\mathcal{L}$ is a hesitant fuzzy filter of $\mathcal{L}$ if and only if the following assertions are valid.*

$$(\forall x, y \in L)\,(x \leq y \;\Rightarrow\; \mathcal{H}(x) \subseteq \mathcal{H}(y))\,, \tag{3.1}$$

$$(\forall x, y \in L)\,(\mathcal{H}(x) \cap \mathcal{H}(y) \subseteq \mathcal{H}(x \odot y))\,. \tag{3.2}$$

Lemma 3.3 ([5])**.** *A hesitant fuzzy set $\mathcal{H}$ on $\mathcal{L}$ is a hesitant fuzzy filter of $\mathcal{L}$ if and only if it satisfies*

$$(\forall x \in L)\,(\mathcal{H}(x) \subseteq \mathcal{H}(1))\,. \tag{3.3}$$

$$(\forall x, y \in L)\,(\mathcal{H}(x) \cap \mathcal{H}(x \to y) \subseteq \mathcal{H}(y))\,. \tag{3.4}$$

Definition 3.4. A hesitant fuzzy filter $\mathcal{H}$ of $\mathcal{L}$ is said to be regular if it satisfies:

$$(\forall x \in L)\,(\mathcal{H}(x'' \to x) = \mathcal{H}(1))\,. \tag{3.5}$$

Note that the notion of regular hesitant fuzzy filters coincides with the notion of hesitant fuzzy filters in a regular residuated lattice.

Theorem 3.5. *For a hesitant fuzzy filter $\mathcal{H}$ of $\mathcal{L}$, the following assertions are equivalent:*

(1) *$\mathcal{H}$ is regular.*

(2) $(\forall x, y \in L)\,(\mathcal{H}(x' \to y') \subseteq \mathcal{H}(y \to x))$.
(3) $(\forall x, y \in L)\,(\mathcal{H}(x' \to y) \subseteq \mathcal{H}(y' \to x))$.

Proof. Assume that $\mathcal{H}$ is a regular hesitant fuzzy filter of $\mathcal{L}$ and let $x, y \in L$. Using (2.6) and (2.8), we have

$$x' \to y' \leq y'' \to x'' \leq y \to x''.$$

It follows from (2.7) and (2.6) that

$$\begin{aligned} x'' \to x &\leq (y \to x'') \to (y \to x) \\ &\leq (x' \to y') \to (y \to x). \end{aligned}$$

Hence, by (3.3), (3.5) and (3.4), we have

$$\begin{aligned} \mathcal{H}(x' \to y') &= \mathcal{H}(x' \to y') \cap \mathcal{H}(1) \\ &= \mathcal{H}(x' \to y') \cap \mathcal{H}(x'' \to x) \\ &\subseteq \mathcal{H}(x' \to y') \cap \mathcal{H}((x' \to y') \to (y \to x)) \\ &\subseteq \mathcal{H}(y \to x), \end{aligned}$$

and so the second assertion holds. Since $x' \to y \leq y' \to x''$, we have

$$x'' \to x \leq (y' \to x'') \to (y' \to x) \leq (x' \to y) \to (y' \to x)$$

by (2.7) and (2.6). It follows from (3.3), (3.5) and (3.4) that

$$\begin{aligned} \mathcal{H}(x' \to y) &= \mathcal{H}(x' \to y) \cap \mathcal{H}(1) \\ &= \mathcal{H}(x' \to y) \cap \mathcal{H}(x'' \to x) \\ &\subseteq \mathcal{H}(x' \to y) \cap \mathcal{H}((x' \to y) \to (y' \to x)) \\ &\subseteq \mathcal{H}(y' \to x). \end{aligned}$$

Hence the third condition is valid. Next, suppose that the second condition holds. The second condition together with the condition (2.8) induces

$$\mathcal{H}(1) = \mathcal{H}(x' \to x''') \subseteq \mathcal{H}(x'' \to x)$$

for all $x \in L$, and so $\mathcal{H}(x'' \to x) = \mathcal{H}(1)$. Hence $\mathcal{H}$ is regular. Finally, assume that the third condition is valid. Since $x' \to x' = 1$ for all $x \in L$, it follows from the third condition that $\mathcal{H}(1) = \mathcal{H}(x' \to x') \subseteq \mathcal{H}(x'' \to x)$, and that $\mathcal{H}(x'' \to x) = \mathcal{H}(1)$ by (3.3). Therefore $\mathcal{H}$ is regular. □

Theorem 3.6. *A hesitant fuzzy set $\mathcal{H}$ on $\mathcal{L}$ is a regular hesitant fuzzy filter of $\mathcal{L}$ if and only if it satisfies the condition* (3.3) *and*

$$(\forall x, y, z \in L)\,(\mathcal{H}(z) \cap \mathcal{H}(z \to (x' \to y)) \subseteq \mathcal{H}(y' \to x))\,. \tag{3.6}$$

Proof. Assume that $\mathcal{H}$ is a regular hesitant fuzzy filter of $\mathcal{L}$. Clearly the condition (3.3) is true. If we use the condition (3.4) and Theorem 3.5(3), then get

$$\mathcal{H}(z) \cap \mathcal{H}(z \to (x' \to y)) \subseteq \mathcal{H}(x' \to y) \subseteq \mathcal{H}(y' \to x)$$

for all $x, y, z \in L$.

Conversely, suppose that $\mathcal{H}$ satisfies two conditions (3.3) and (3.6). Let $x, y \in L$. Since

$$x \to y = x \to (1 \to y) = x \to (0' \to y)$$

and

$$y'' = 1 \to y'' = 1 \to (y' \to 0),$$

it follows from (2.4), (3.3) and (3.6) that

$$\begin{aligned}
\mathcal{H}(x) \cap \mathcal{H}(x \to y) &= \mathcal{H}(x) \cap \mathcal{H}(x \to (0' \to y)) \\
&\subseteq \mathcal{H}(y' \to 0) \\
&= \mathcal{H}(y'') \\
&= \mathcal{H}(1) \cap \mathcal{H}(1 \to (y' \to 0)) \\
&\subseteq \mathcal{H}(0' \to y) \\
&= \mathcal{H}(1 \to y) = \mathcal{H}(y).
\end{aligned}$$

Therefore $\mathcal{H}$ is a hesitant fuzzy filter of $\mathcal{L}$ by Lemma 3.3. If we take $z := 1$ in (3.6) and use (2.4) and (3.3), then

$$\begin{aligned}
\mathcal{H}(y' \to x) &\supseteq \mathcal{H}(1) \cap \mathcal{H}(1 \to (x' \to y)) \\
&= \mathcal{H}(1 \to (x' \to y)) = \mathcal{H}(x' \to y).
\end{aligned}$$

Hence $\mathcal{H}$ is regular by Theorem 3.5. □

Theorem 3.7. *A hesitant fuzzy set $\mathcal{H}$ on $\mathcal{L}$ is a regular hesitant fuzzy filter of $\mathcal{L}$ if and only if it satisfies the condition* (3.3) *and*

$$(\forall x, y, z \in L)\,(\mathcal{H}(z) \cap \mathcal{H}(z \to (x' \to y')) \subseteq \mathcal{H}(y \to x))\,. \tag{3.7}$$

Proof. The proof is similar to the proof of Theorem 3.6. □

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 24, NO.6, 2018, COPYRIGHT 2018 EUDOXUS PRESS, LLC

Theorem 3.8. *A hesitant fuzzy set $\mathcal{H}$ on $\mathcal{L}$ is a regular hesitant fuzzy filter of $\mathcal{L}$ if and only if the τ-hesitant level set $\mathcal{H}_\tau$ on $\mathcal{L}$ is a regular filter of $\mathcal{L}$ for all $\tau \in \mathscr{P}([0,1])$ with $\mathcal{H}_\tau \neq \emptyset$.*

Proof. Assume that $\mathcal{H}$ is a regular hesitant fuzzy filter of $\mathcal{L}$. Let $\tau \in \mathscr{P}([0,1])$ be such that $\mathcal{H}_\tau \neq \emptyset$. Then $\mathcal{H}_\tau$ is a filter of $\mathcal{L}$. Let $x, y \in L$ be such that $x' \to y \in \mathcal{H}_\tau$. Then $\tau \subseteq \mathcal{H}(x' \to y) \subseteq \mathcal{H}(y' \to x)$ by Theorem 3.5, and so $y' \to x \in \mathcal{H}_\tau$. Hence $\mathcal{H}_\tau$ is regular by Lemma 2.7.

Conversely, suppose that $\mathcal{H}_\tau$ is a regular filter of $\mathcal{L}$ for all $\tau \in \mathscr{P}([0,1])$ with $\mathcal{H}_\tau \neq \emptyset$. Then $\mathcal{H}_\tau$ is a filter of $\mathcal{L}$, and thus $\mathcal{H}$ is a hesitant fuzzy filter of $\mathcal{L}$. For any $x, y \in L$, let $\mathcal{H}(x' \to y) = \delta$. Then $x' \to y \in \mathcal{H}_\delta$ which implies from Lemma 2.7 that $y' \to x \in \mathcal{H}_\delta$. Hence $\mathcal{H}(x' \to y) = \delta \subseteq \mathcal{H}(y' \to x)$, and so $\mathcal{H}$ is regular by Theorem 3.5. □

Theorem 3.9. *Every regular filter of $\mathcal{L}$ can be represented as a τ-hesitant level set $\mathcal{H}_\tau$ on $\mathcal{L}$ for some $\tau \in \mathscr{P}([0,1]) \setminus \{\emptyset\}$ and a regular hesitant fuzzy filter $\mathcal{H}$ of $\mathcal{L}$.*

Proof. Let F be a regular filter of $\mathcal{L}$ and let $\mathcal{H}$ be a hesitant fuzzy set on $\mathcal{L}$ defined by

$$\mathcal{H} : L \to \mathscr{P}([0,1]),\ x \mapsto \begin{cases} \tau & \text{if } x \in F, \\ \emptyset & \text{otherwise,} \end{cases}$$

where $\tau \in \mathscr{P}([0,1]) \setminus \{\emptyset\}$. Since $1 \in F$, we have $\mathcal{H}(x) \subseteq \tau = \mathcal{H}(1)$ for all $x \in L$. Let $x, y, z \in L$. If $z \in F$ and $z \to (x' \to y) \in F$, then $y' \to x \in F$ by Proposition 2.4 and Lemma 2.7. Hence $\mathcal{H}(z) \cap \mathcal{H}(z \to (x' \to y)) = \tau = \mathcal{H}(y' \to x)$. Suppose that $z \notin F$ or $z \to (x' \to y) \notin F$. Then $\mathcal{H}(z) = \emptyset$ or $\mathcal{H}(x' \to y) = \emptyset$, and so $\mathcal{H}(z) \cap \mathcal{H}(z \to (x' \to y)) = \emptyset \subseteq \mathcal{H}(y' \to x)$. It follows from by Theorem 3.6 that $\mathcal{H}$ is a regular hesitant fuzzy filter of $\mathcal{L}$, and it is obvious that $F = \mathcal{H}_\tau$. This completes the proof. □

4. MV-hesitant fuzzy filters

Definition 4.1. A hesitant fuzzy set $\mathcal{H}$ on $\mathcal{L}$ is called an MV-hesitant fuzzy filter of $\mathcal{L}$ if the τ-hesitant level set $\mathcal{H}_\tau$ on $\mathcal{L}$ is an MV-filter of $\mathcal{L}$ for all $\tau \in \mathscr{P}([0,1])$ with $\mathcal{H}_\tau \neq \emptyset$.

Theorem 4.2. *A hesitant fuzzy set $\mathcal{H}$ on $\mathcal{L}$ is an MV-hesitant fuzzy filter of $\mathcal{L}$ if it is a hesitant fuzzy filter of $\mathcal{L}$ with the following additional condition:*

$$(\forall x, y \in L)\,(\mathcal{H}(y \to x) \subseteq \mathcal{H}(((x \to y) \to y) \to x))\,. \tag{4.1}$$

Proof. If $\mathcal{H}$ is an MV-hesitant fuzzy filter of $\mathcal{L}$, then $\mathcal{H}_\tau$ is an MV-filter of $\mathcal{L}$ for all $\tau \in \mathscr{P}([0,1])$ with $\mathcal{H}_\tau \neq \emptyset$. Hence it is filter of $\mathcal{L}$, and so $\mathcal{H}$ is a hesitant fuzzy filter of $\mathcal{L}$. For any $x, y \in L$, let $\mathcal{H}(y \to x) = \delta$. Then $y \to x \in \mathcal{H}_\delta$, and so $((x \to y) \to y) \to x \in \mathcal{H}_\delta$ since $\mathcal{H}_\delta$ is an MV-filter of $\mathcal{L}$. Thus $\mathcal{H}(y \to x) = \delta \subseteq \mathcal{H}(((x \to y) \to y) \to x)$.

Conversely, suppose that $\mathcal{H}$ is a hesitant fuzzy filter of $\mathcal{L}$ which satisfies the condition (4.1). Let $\tau \in \mathscr{P}([0,1])$ be such that $\mathcal{H}_\tau \neq \emptyset$. Then $\mathcal{H}_\tau$ is a filter of $\mathcal{L}$. Let $x, y \in L$ be such that $y \to x \in \mathcal{H}_\tau$. Using the condition (4.1), we have

$$\tau \subseteq \mathcal{H}(y \to x) \subseteq \mathcal{H}(((x \to y) \to y) \to x)$$

which implies that $((x \to y) \to y) \to x \in \mathcal{H}_\tau$. Therefore $\mathcal{H}_\tau$ is an MV-filter of $\mathcal{L}$, and thus $\mathcal{H}$ is an MV-hesitant fuzzy filter of $\mathcal{L}$. □

Theorem 4.3. *A hesitant fuzzy set $\mathcal{H}$ on $\mathcal{L}$ is an MV-hesitant fuzzy filter of $\mathcal{L}$ if and only if it satisfies the condition* (3.3) *and*

$$(\forall x, y, z \in L)\,(\mathcal{H}(z) \cap \mathcal{H}(z \to (y \to x)) \subseteq \mathcal{H}(((x \to y) \to y) \to x))\,. \tag{4.2}$$

Proof. Assume that $\mathcal{H}$ is an MV-hesitant fuzzy filter of $\mathcal{L}$. Using (3.4) and (4.1), we have

$$\mathcal{H}(z) \cap \mathcal{H}(z \to (y \to x)) \subseteq \mathcal{H}(y \to x) \subseteq \mathcal{H}(((x \to y) \to y) \to x)$$

for all $x, y \in L$.

Conversely, let $\mathcal{H}$ be a hesitant fuzzy set on $\mathcal{L}$ which satisfies two conditions (3.3) and (4.2). Taking $y := 1$ in (4.2) and using (2.4) induces the condition (3.4). Hence $\mathcal{H}$ is a hesitant fuzzy filter of $\mathcal{L}$ by Lemma 3.3. If we take $z := 1$ in (4.2) and use (2.4) and (3.3), then we know that $\mathcal{H}$ satisfies the condition (4.1). Therefore $\mathcal{H}$ is an MV-hesitant fuzzy filter of $\mathcal{L}$. □

Theorem 4.4. *Let $\mathcal{H}$ be a hesitant fuzzy filter of $\mathcal{L}$. Then $\mathcal{H}$ is an MV-hesitant fuzzy filter of $\mathcal{L}$ if and only if the following assertion is valid:*

$$(\forall x, y \in L)\,(\mathcal{H}(((x \to y) \to y) \to ((y \to x) \to x)) = \mathcal{H}(1))\,. \tag{4.3}$$

Proof. Assume that $\mathcal{H}$ is an MV-hesitant fuzzy filter of $\mathcal{L}$. Then $\mathcal{H}$ is a hesitant fuzzy filter of $\mathcal{L}$, and so $\mathcal{H}_\tau$ is a filter of $\mathcal{L}$ for all $\tau \in \mathscr{P}([0,1])$ with $\mathcal{H}_\tau \neq \emptyset$. In particular, $\mathcal{H}_{\mathcal{H}(1)}$ is a filter of $\mathcal{L}$. Let $x, y \in L$ be such that $y \to x \in \mathcal{H}_{\mathcal{H}(1)}$. Then

$$\mathcal{H}(1) \subseteq \mathcal{H}(y \to x) \subseteq \mathcal{H}(((x \to y) \to y) \to x),$$

and so $((x \to y) \to y) \to x \in \mathcal{H}_{\mathcal{H}(1)}$. Therefore $\mathcal{H}_{\mathcal{H}(1)}$ is an MV-filter of $\mathcal{L}$, and thus

$$((x \to y) \to y) \to ((y \to x) \to x) \in \mathcal{H}_{\mathcal{H}(1)}$$

by Lemma 2.9. Hence $\mathcal{H}(1) \subseteq \mathcal{H}(((x \to y) \to y) \to ((y \to x) \to x))$, which implies from (3.3) that $\mathcal{H}(((x \to y) \to y) \to ((y \to x) \to x)) = \mathcal{H}(1)$.

Conversely, let $\mathcal{H}$ be a hesitant fuzzy filter of $\mathcal{L}$ that satisfies the condition (4.3). Using (3.3), (4.3), (2.5) and (3.4), we obtain

$$\begin{aligned}\mathcal{H}(y \to x) &= \mathcal{H}(y \to x) \cap \mathcal{H}(1)\\ &= \mathcal{H}(y \to x) \cap \mathcal{H}(((x \to y) \to y) \to ((y \to x) \to x))\\ &= \mathcal{H}(y \to x) \cap \mathcal{H}((y \to x) \to (((x \to y) \to y) \to x))\\ &\subseteq \mathcal{H}(((x \to y) \to y) \to x).\end{aligned}$$

Therefore $\mathcal{H}$ is an MV-hesitant fuzzy filter of $\mathcal{L}$. □

Theorem 4.5. *Every MV-hesitant fuzzy filter is a regular hesitant fuzzy filter.*

Proof. Let $\mathcal{H}$ be an MV-hesitant fuzzy filter of $\mathcal{L}$. If we take $y := 0$ in (4.1) and use (2.4), then

$$\mathcal{H}(1) = \mathcal{H}(0 \to x) \subseteq \mathcal{H}(((x \to 0) \to 0) \to x) = \mathcal{H}(x'' \to x)$$

and so $\mathcal{H}(x'' \to x) = \mathcal{H}(1)$ by (3.1). Therefore $\mathcal{H}$ is a regular hesitant fuzzy filter of $\mathcal{L}$. □

The converse of Theorem 4.5 is not true in general as seen in the following example.

Example 4.6. Let $\mathcal{L} := [0, 1]$ (unit interval). For any $a, b \in L$, define

$$a \vee b = \max\{a, b\},\ a \wedge b = \min\{a, b\},$$

$$a \to b = \begin{cases} 1 & \text{if } a \leq b,\\ (1-a) \vee b & \text{otherwise,}\end{cases} \quad \text{and } a \odot b = \begin{cases} 0 & \text{if } a + b \leq 1,\\ a \wedge b & \text{otherwise.}\end{cases}$$

Then $\mathcal{L} := (L, \vee, \wedge, \odot, \to, 0, 1)$ is a residuated lattice (see [10]). Let $\mathcal{H}$ be a hesitant fuzzy set on $\mathcal{L}$ defined by

$$\mathcal{H} : L \to \mathscr{P}([0, 1]),\ x \mapsto \begin{cases} (x, 1] & \text{if } x \in [0.5, 1],\\ \emptyset & \text{otherwise,}\end{cases}$$

Then $\mathcal{H}$ is a regular hesitant fuzzy filter of $\mathcal{L}$. Let $F := (c, 1]$ for any $c \in L$. Note that if $c \in [0.5, 1]$ then F is a regular filter of $\mathcal{L}$. But, if $c \in (0.7, 1]$ then F is not an MV-filter of $\mathcal{L}$ since $0.4 \to 0.7 = 1 \in F$, but $((0.7 \to 0.4) \to 0.4) \to 0.7 = 0.7 \notin F$. Hence the hesitant fuzzy set $\mathcal{H}$ on $\mathcal{L}$ which is given as follows:

$$\mathcal{H} : L \to \mathscr{P}([0, 1]),\ x \mapsto \begin{cases} U & \text{if } x \in F,\\ \emptyset & \text{otherwise,}\end{cases}$$

is a hesitant fuzzy filter of $\mathcal{L}$ which is regular. But, since $\mathcal{H}(0.4 \to 0.7) = \mathcal{H}(1) = U$ and $\mathcal{H}(((0.7 \to 0.4) \to 0.4) \to 0.7) = \mathcal{H}(0.7) = \emptyset$. Therefore $\mathcal{H}$ is not an MV-hesitant fuzzy filter of $\mathcal{L}$.

The following theorem shows that the converse of Theorem 4.5 is true in BL-algebras.

Theorem 4.7. *In a BL-algebra $\mathcal{L}$, the notion of an MV-hesitant fuzzy filter coincides with the notion of a regular hesitant fuzzy filter.*

Proof. Based on Theorem 4.5, it is sufficient to show that every regular hesitant fuzzy filter is an MV-hesitant fuzzy filter. Let $\mathcal{H}$ be a regular hesitant fuzzy filter of a BL-algebra $\mathcal{L}$ and let $x, y \in L$. Then $\mathcal{H}(x' \to y') \subseteq \mathcal{H}(y \to x)$ by Theorem 3.5. Since $y \to x \leq x' \to y'$, we have $\mathcal{H}(y \to x) \subseteq \mathcal{H}(x' \to y')$ by (3.1). Hence

$$\begin{aligned}
\mathcal{H}(y \to x) = \mathcal{H}(x' \to y') &= \mathcal{H}(x' \to (x' \to y')) \\
&= \mathcal{H}(x' \to (y' \odot (y' \to x'))) \\
&= \mathcal{H}(x' \to (y' \odot (x \to y''))) \\
&= \mathcal{H}((y' \odot (x \to y''))' \to x) \\
&= \mathcal{H}(((x \to y'') \to (y' \to 0)) \to x) \\
&= \mathcal{H}(((x \to y'') \to y'') \to x)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{H}(1) = \mathcal{H}(y' \to y') &= \mathcal{H}(y'' \to y) \\
&\subseteq \mathcal{H}((x \to y'') \to (x \to y)) \\
&\subseteq \mathcal{H}(((x \to y) \to y'') \to ((x \to y'') \to y'')) \\
&\subseteq \mathcal{H}((((x \to y'') \to y'') \to x) \to (((x \to y) \to y'') \to x)).
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathcal{H}(y \to x) &= \mathcal{H}(y \to x) \cap \mathcal{H}(1) \\
&\subseteq \mathcal{H}(((x \to y'') \to y'') \to x) \\
&\quad \cap \mathcal{H}((((x \to y'') \to y'') \to x) \to (((x \to y) \to y'') \to x)) \\
&\subseteq \mathcal{H}((((x \to y) \to y'') \to x)) \\
&\subseteq \mathcal{H}(((x \to y) \to y) \to x).
\end{aligned}$$

Therefore $\mathcal{H}$ is an MV-hesitant fuzzy filter of $\mathcal{L}$. □

Definition 4.8. A hesitant fuzzy set $\mathcal{H}$ on $\mathcal{L}$ is called a Boolean hesitant fuzzy filter of $\mathcal{L}$ if it is a hesitant fuzzy filter of $\mathcal{L}$ that satisfies the following condition:

$$(\forall x \in L)\,(\mathcal{H}(x \vee x') = \mathcal{H}(1)). \tag{4.4}$$

Theorem 4.9. *A hesitant fuzzy set $\mathcal{H}$ on $\mathcal{L}$ is a Boolean hesitant fuzzy filter of $\mathcal{L}$ if and only if the τ-hesitant level set $\mathcal{H}_\tau$ on $\mathcal{L}$ is a Boolean filter of $\mathcal{L}$ for all $\tau \in \mathscr{P}([0,1])$ with $\mathcal{H}_\tau \neq \emptyset$.*

Proof. Suppose that $\mathcal{H}$ is a Boolean hesitant fuzzy filter of $\mathcal{L}$ and let $\tau \in \mathscr{P}([0,1])$ with $\mathcal{H}_\tau \neq \emptyset$. Then $\mathcal{H}_\tau$ is a filter of $\mathcal{L}$, and so $1 \in \mathcal{H}_\tau$, that is, $\tau \subseteq \mathcal{H}(1)$. It follows from (4.4) that $\tau \subseteq \mathcal{H}(1) = \mathcal{H}(x \vee x')$ for all $x \in L$. Hence $x \vee x' \in \mathcal{H}_\tau$ for all $x \in L$, and therefore $\mathcal{H}_\tau$ is a Boolean filter of $\mathcal{L}$.

Conversely assume that $\mathcal{H}_\tau$ is a Boolean filter of $\mathcal{L}$ for all $\tau \in \mathscr{P}([0,1])$ with $\mathcal{H}_\tau \neq \emptyset$. Then $\mathcal{H}_\tau$ is a filter of $\mathcal{L}$, and so $\mathcal{H}$ is a hesitant fuzzy filter of $\mathcal{L}$. Note that $1 \in \mathcal{H}_{\mathcal{H}(1)}$. Since $\mathcal{H}_{\mathcal{H}(1)}$ is a Boolean filter of $\mathcal{L}$, we have $x \vee x' \in \mathcal{H}_{\mathcal{H}(1)}$ for all $x \in L$. Hence $\mathcal{H}(x \vee x') = \mathcal{H}(1)$, and therefore $\mathcal{H}$ is a Boolean hesitant fuzzy filter of $\mathcal{L}$. □

Theorem 4.10. (Extension property) *Let $\mathcal{H}$ and $\mathcal{G}$ be hesitant fuzzy filters of $\mathcal{L}$ satisfying two conditions:*

(1) $\mathcal{H}(1) = \mathcal{G}(1)$,
(2) $(\forall x \in L)\,(\mathcal{H}(x) \subseteq \mathcal{G}(x))$.

If $\mathcal{H}$ is an MV-hesitant fuzzy filter (resp., a Boolean hesitant fuzzy filter) of $\mathcal{L}$, then so is $\mathcal{G}$.

Proof. Assume that $\mathcal{H}$ is a Boolean hesitant fuzzy filter of $\mathcal{L}$. Then $\mathcal{H}(x \vee x') = \mathcal{H}(1)$ for all $x \in L$. Using two conditions, we have

$$\mathcal{G}(x \vee x') \supseteq \mathcal{H}(x \vee x') = \mathcal{H}(1) = \mathcal{G}(1) \tag{4.5}$$

for all $x \in L$. Combining (3.3) and (4.5) implies that $\mathcal{G}(x \vee x') = \mathcal{G}(1)$. Therefore $\mathcal{G}$ is a Boolean hesitant fuzzy filter of $\mathcal{L}$.

Now suppose that $\mathcal{H}$ is an MV-hesitant fuzzy filter of $\mathcal{L}$. Using Theorem 4.4, we have

$$\begin{aligned}\mathcal{G}(1) &= \mathcal{H}(1)\\ &= \mathcal{H}(((x \to y) \to y) \to ((y \to x) \to x))\\ &\subseteq \mathcal{G}(((x \to y) \to y) \to ((y \to x) \to x)),\end{aligned}$$

and so $\mathcal{G}(((x \to y) \to y) \to ((y \to x) \to x)) = \mathcal{G}(1)$ for all $x, y \in L$. It follows from Theorem 4.4 that $\mathcal{G}$ is an MV-hesitant fuzzy filter of $\mathcal{L}$. □

References

[1] R. Belohlavek, Some properties of residuated lattices, Czechoslovak Math. J. 53(123) (2003) 161–171.

[2] F. Esteva and L. Godo, Monoidal t-norm based logic: towards a logic for left-continuous t-norms, Fuzzy Sets and Systems 124 (2001) 271–288.

[3] P. Hájek, Metamathematics of Fuzzy Logic, Kluwer Academic Press, Dordrecht, 1998.

[4] Y. B. Jun and S. Z. Song, Hesitant fuzzy set theory applied to filters in MTL-algebras, Honam Math. J. 36, (2014), no. 4, 813-830.

[5] G. Muhiuddin, Hesitant fuzzy filters and hesitant fuzzy G-filters in residuated lattices, Journal of Computational Analysis and Applications 21, No. 2 (2016) 394-404.

[6] J. G. Shen and X. H. Zhang, Filters of residuated lattices, Chin. Quart. J. Math. 21 (2006) 443–447.

[7] V. Torra, Hesitant fuzzy sets, Int. J. Intell. Syst. 25 (2010), 529–539.

[8] E. Turunen, BL-algebras of basic fuzzy logic, Mathware & Soft Computing 6 (1999), 49–61.

[9] E. Turunen, Boolean deductive systems of BL-algebras, Arch. Math. Logic 40 (2001) 467–473.

[10] Y. Q. Zhu and Y. Xu, On filter theory of residuated lattices, Inform. Sci. 180 (2010) 3614–3632.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 24, NO.6, 2018, COPYRIGHT 2018 EUDOXUS PRESS, LLC

Hyers-Ulam Stability of Set-Valued AQ-Functional Equations in Non-Archimedean Metric Spaces

Xian Sun, Zhanpeng Yang*, Hongqi Wang, Wenjuan Ren, Guangluan Xu and Kun Fu

Key Laboratory of Technology in Geo-spatial Information Processing and Application System, Institute of Electronics, Chinese Academy of Sciences, Beijing 100190, PR China

Abstract

In the paper, we introduce a notion of a non-Archimedean metric space endowed with the non-Archimedean Pompeiu-Hausdorff metric. Using the direct and fixed point methods, we study the Hyers-Ulam stability of set-valued AQ-functional equations in the framework of complete non-Archimedean metric spaces. We indeed present an interdisciplinary relations between the theory of set-valued mappings, the theory of non-Archimedean spaces and the stability theory of functional equations.

Keywords: Hyers-Ulam stability, non-Archimedean Pompeiu-Hausdorff metric, AQ-functional equation, non-Archimedean metric space, fixed point theorem

1 Introduction

The stability of functional equations was originally raised by Ulam in 1940 in a talk given at Wisconsin University [1]. The first answer to Ulam's question was given by Hyers in 1941 in the case of Banach spaces in [2]. Thereafter, this type of stability is called the Hyers–Ulam stability. In 1978, Rassias [3] provided a remarkable generalization of the Hyers–Ulam stability of mappings by considering variables. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Considerable attention has been given to the study of the Ulam-Hyers and Ulam-Hyers-Rassias stability of all kinds of functional equations [4, 5, 6]. Bota-Boriceanu and Petrusel [7], Shen [8], Popa [9], Xu [10, 11], and Rus [12, 13] discussed the Hyers–Ulam stability for operatorial equations and inclusions. Castro and Ramos [14], and Jung [15] considered the Hyers-Ulam-Rassias stability for a class of Volterra integral equations.

In 1897, Hensel discovered the p-adic numbers as a number theoretical analogue of power series in complex analysis. The most important examples of non-Archimedean spaces are p-adic numbers. A key property of p-adic numbers is that they do not satisfy the Archimedean axiom:

*Corresponding author. Email: yzhp1980@163.com(Z. Yang), wiecas@sina.com(H. Wang), sunxian@mail.ie.ac.cn(X. Sun), wjren2011@mail.ie.ac.cn(W. Ren), gluanxu@mail.ie.ac.cn(G. Xu), fukun@mail.ie.ac.cn(K. Fu).

for all $x, y > 0$, there exists an integer n such that $x < ny$. It turned out that non-Archimedean spaces have many nice applications [16]. During the last three decades, theory of non-Archimedean spaces has gained the interest of physicists for their research in particular in problems coming from quantum physics, p-adic strings and superstrings [17]. In [18], Huy obtained some Hyers-Ulam stability results concerning fixed point equations in non-Archimedean cone metric spaces. Using a fixed point approach, Brzdęk proved the Hyers-Ulam stability of a quite wide class of functional equations in a single variable [19]. Mirmostafaee gave some stability results of the Cauchy equation in the context of non-Archimedean fuzzy spaces [20].

Set-valued functions in Banach spaces have been developed in the last decades. The pioneering papers by Aumann [21] and Debreu [22] were inspired by problems arising in control theory and mathematical economics. We can refer to the papers by Arrow [23] and McKenzie [24]. In particular, the stability of set-valued functional equations has been considered by many scholars. Chu proved the Hyers-Ulam stability of the generalized cubic set-valued functional equations in Banach spaces [25]. Stability of two types of cubic fuzzy set-valued functional equations was considered in [26]. Lee studied additive set-valued and quadratic set-valued functional equations in Banach spaces [27]. Notice that, the spaces discussed above are all Banach spaces. So far as we know, the work related with the stability of set-valued functions equation in non-Archimedean spaces needs to study.

We introduce a notion of a complete non-Archimedean metric space endowed with the non-Archimedean Pompeiu-Hausdorff distance, which is following [19] and modifying the definition of a complete metric space endowed with the Hausdorff distance [27]. Although many results in the classic normed space theory have corresponding non-Archimedean counterparts, our proofs of the results on the complete non-Archimedean metric space are different and require a new kind of intuition. In Section 3 and Section 4, we prove the stability of the set-valued AQ-functional equations in the framework of complete non-Archimedean metric spaces.

2 Preliminaries

We will give some definitions which will be used in this paper. Let $\mathbb{N}$ denote the set of positive integers and we put $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{R}_+ := [0, \infty)$. In a non-Archimedean metric space $(\mathcal{X}, d)$ the triangle inequality holds in the stronger form as follows

$$d(\mu, \nu) \leq \max\{d(\mu, w), d(w, \nu)\}, \ \mu, \nu, w \in \mathcal{X}. \tag{2.1}$$

The non-Archimedean space theory has many applications in the filed of superstrings, p-adic strings and quantum physics. A typical example of non-Archimedean metric spaces is the non-Archimedean normed space that can be described below.

Definition 1. *A mapping $|\cdot| : \mathbb{K} \to \mathbb{R}_+$ in the field $\mathbb{K}$ is called a non-Archimedean valuation if it satisfies the following conditions*
(i) $|\mu| = 0$ *if and only if* $\mu = 0$*;*
(ii) $|\mu\nu| = |\mu||\nu|, \ \mu, \nu \in \mathbb{K}$*;*

(iii) $|\mu+\nu| \le \max\{|\mu|, |\nu|\},\ \mu, \nu \in \mathbb{K}$.

The condition (iii) is called the strong triangle inequality. By (ii), it is easy to verify that $|\cdot|$ is a homomorphism of groups. In addition, we will always suppose that $|\cdot|$ is nontrivial, i.e., there exists an $\mu_0 \in \mathbb{K}$ such that $|\mu_0| \neq 0, 1$.

It is easy to show that $|n \cdot 1| \le 1$ for $n \in \mathbb{N}$. If $|\mu| \neq |\nu|$ for some $\mu, \nu \in \mathbb{K}$, then (iii) can be sharpened into the equality $|\mu+\nu| = \max\{|\mu|, |\nu|\}$. Set $d(\mu, \nu) := |\mu - \nu|$. And the space $(\mathbb{K}, d)$ is a metric space. Denote the closed ball in $\mathbb{K}$ by $N_\epsilon(\mu) := \{\nu \in \mathbb{K} : |\nu - \mu| \le \epsilon\}$ for $\mu \in \mathbb{K}$ and $\epsilon > 0$. Then they form a fundamental system of neighbourhoods of μ.

Definition 2. *A space $(\mathcal{X}, |\cdot|)$ is non-Archimedean if it is equipped with a non-Archimedean valuation $|\cdot|$ such that the corresponding metric space $\mathcal{X}$ is complete (i.e., every Cauchy sequence in $\mathcal{X}$ converges).*

Let $\mathcal{X}$ be a non-Archimedean space. We need the following definitions:
$2^{\mathcal{X}}$: the set of subsets of $\mathcal{X}$;
$C_b(\mathcal{X})$: the set of closed bounded subsets of $\mathcal{X}$;
$C_c(\mathcal{X})$: the set of closed convex subsets of $\mathcal{X}$;
$C_{cb}(\mathcal{X})$: the set of closed convex bounded subsets of $\mathcal{X}$.

Definition 3. *For any two nonempty subsets $A, B \in \mathcal{X}$, the (Minkowski) addition is defined as $A + B = \{w \in \mathcal{X} \mid w = \mu + \nu, \mu \in A, \nu \in B\}$ and the scalar multiplication as $\lambda A = \{w \in \mathcal{X} \mid w = \lambda\mu, \mu \in A\}$ for $\lambda \in \mathbb{R}$. Moreover, for $A, B \in C_c(\mathcal{X})$, $A \oplus B := \overline{A + B}$.*

By Definition 3, we can obtain the following two properties:
$(1)\lambda A + \lambda B = \lambda(A + B)$, $(2)(\lambda + \mu)A \subseteq \lambda A + \mu A$;
if A is convex, then $(\lambda + \mu)A = \lambda A + \mu A$ for all $\lambda\mu \ge 0$.

Definition 4. *Let $(\mathcal{X}, d)$ $(d(\mu, \nu) := |\mu - \nu|)$ be a non-Archimedean metric space. A gap function in $2^{\mathcal{X}}$ is defined as*

$$D_d : 2^{\mathcal{X}} \times 2^{\mathcal{X}} \to \mathbb{R}_+,\ D_d(A, B) = \inf\{d(\mu, \nu) \mid \mu \in A, \nu \in B\}.$$

In particular, $D_d(\mu, B) := D_d(\{\mu\}, B)$ for $\mu \in \mathcal{X}$.

Definition 5. *The non-Archimedean Pompeiu-Hausdorff distance H_d on 2^X is defined as*

$$H_d : 2^{\mathcal{X}} \times 2^{\mathcal{X}} \to \mathbb{R}_+ \cup \{+\infty\},\ H_d(A, B) = \max\{\sup_{\mu \in A} D_d(\mu, B), \sup_{\nu \in B} D_d(\nu, A)\}.$$

By using Definition 5, we have the following properties of the non-Archimedean Pompeiu-Hausdorff distance.

Proposition 1. *For $A, A_1, B, B_1 \in C_{cb}(\mathcal{X})$ and $\lambda > 0$, the following properties hold*
(i) $H_d(A \oplus A_1, B \oplus B_1) \le \max\{H_d(A, B), H_d(A_1, B_1)\}$;
(ii) $H_d(\lambda A, \lambda B) = \lambda H_d(A, B)$;
(iii) $H_d(A, A_1) = H_d(A \oplus B, A_1 \oplus B)$.

Then, $(C_{cb}(\mathcal{X}), H_d)$ is a complete non-Archimedean metric semigroup. Using the method given by Debreu [22], one can prove that $(C_{cb}(\mathcal{X}), H_d)$ can be isometrically embedded in a non-Archimedean normed space.

Now we consider that a set-valued function $f: \mathcal{X} \to C_{cb}(Y)$ satisfies the following additive-quadratic (AQ) set-valued functional equation,

$$f(kx+y) \oplus f(kx-y) = f(x+y) \oplus f(x-y) \oplus (k-1)[(k+2)f(x) \oplus kf(-x)], \tag{2.2}$$

for a fixed integer with $k \geq 2$. Eskandani et al. [32] and Xu et al. [31] have established the general solution and investigated the generalized Hyers-Ulam stability of (2.2) in quasi-β-normed spaces and non-Archimedean normed spaces.

The main purpose in our paper is to prove the generalized Hyers-Ulam stability of the set-valued AQ-functional equation (2.2) in complete non-Archimedean normed spaces using the direct and fixed point methods.

3 Stability of the Set-Valued Functional Equation: a Direct Method

Throughout this paper, we always suppose that $\mathcal{X}$ is a linear space over $\mathbb{Q}$ or a non-Archimedean field of characteristic different from 2 and k (i.e. $|2| \neq 0, |k| \neq 0$), and that $\mathcal{Y}$ is a complete non-Archimedean normed space over a non-Archimedean field of characteristic different from $2, 3$ and k (a fixed integer $k \geq 2$).

For convenience, we use the following abbreviation for a given function $f : \mathcal{X} \to C_{cb}(\mathcal{Y})$:

$$\mathcal{L}f(x,y) := H_d(f(kx+y) \oplus f(kx-y), f(x+y) \oplus f(x-y) \oplus (k-1)[(k+2)f(x) \oplus kf(-x)]). \tag{3.1}$$

Theorem 1. *Assume that $\varphi: \mathcal{X} \times \mathcal{X} \to [0, \infty)$ is a mapping satisfying*

$$\lim_{n\to\infty} \frac{\varphi(k^n x, k^n y)}{|k|^n} = 0, \ x, y \in \mathcal{X}. \tag{3.2}$$

And suppose that for each $x \in \mathcal{X}$ the limit

$$\lim_{n\to\infty} \max\left\{\frac{\varphi(k^j x, 0)}{|k|^j} : 0 \leq j < n\right\}, \tag{3.3}$$

denoted by $\tilde{\varphi}_a(x)$, exists. If $f: \mathcal{X} \to C_{cb}(\mathcal{Y})$ is an odd mapping such that

$$\mathcal{L}f(x,y) \leq \varphi(x,y), \ x, y \in \mathcal{X}, \tag{3.4}$$

then there exists an additive mapping $A: X \to C_{cb}(\mathcal{Y})$ such that

$$H_d(f(x), A(x)) \leq \frac{1}{|2k|}\tilde{\varphi}_a(x), \ x \in \mathcal{X}. \tag{3.5}$$

Moreover, if

$$\lim_{i\to\infty}\lim_{n\to\infty} \max\left\{\frac{\varphi(k^j x, 0)}{|k|^j} : i \leq j < n+i\right\} = 0, \tag{3.6}$$

then A is the unique additive mapping satisfying (3.5).

Proof. Setting $y = 0$ in (3.4), we get

$$H_d(f(kx), kf(x)) \leq \frac{1}{|2|}\varphi(x, 0), x \in \mathcal{X}. \tag{3.7}$$

Replacing x by $k^{n-1}x$ in (3.7), we get

$$H_d\left(\frac{f(k^n x)}{k^n}, \frac{f(k^{n-1}x)}{k^{n-1}}\right) \leq \frac{1}{|2 \cdot k^n|}\varphi(k^{n-1}x, 0), x \in \mathcal{X}. \tag{3.8}$$

(3.8) and (3.2) imply that $\left\{\frac{f(k^n x)}{k^n}\right\}$ is a Cauchy sequence. Since $\mathcal{Y}$ is complete, we conclude that the sequence $\left\{\frac{f(k^n x)}{k^n}\right\}$ is convergent. Let $A(x) = \lim_{n\to\infty} \frac{1}{k^n} f(k^n x)$. Using induction one can show that

$$H_d\left(\frac{f(k^n x)}{k^n}, f(x)\right) \leq \frac{1}{|2k|} \max\left\{\frac{\varphi(k^j x, 0)}{|k|^j} : 0 \leq j < n\right\}, x \in \mathcal{X}, n \in \mathbb{N}. \tag{3.9}$$

Letting $n \to \infty$ and using (3.3) we can get (3.5). (3.2) and (3.4) imply that

$$\mathcal{L}A(x, y) = \lim_{n\to\infty} \frac{1}{|k|^n}\mathcal{L}f(k^n x, k^n y) \leq \lim_{n\to\infty} \frac{1}{|k|^n}\varphi(k^n x, k^n y) = 0, x, y \in \mathcal{X}.$$

Therefore, the mapping $A : \mathcal{X} \to C_{cb}(\mathcal{Y})$ satisfies (2.2). By Lemma 2.2 in [32], we get that the mapping A is additive. To prove the uniqueness property of A, let A' be another additive mapping satisfying (3.5). Then

$$\begin{aligned} H_d(A(x), A'(x)) &\leq \lim_{i\to\infty} \frac{1}{|k|^i} H_d(A(k^i x), A'(k^i x)) \\ &\leq \lim_{i\to\infty} \frac{1}{|k|^i} \max\{H_d(A(k^i x), f(k^i x)), H_d(A'(k^i x), f(k^i x))\} \\ &\leq \frac{1}{|2k|} \lim_{i\to\infty} \lim_{n\to\infty} \max\left\{\frac{\varphi(k^j x, 0)}{|k|^j} : i \leq j < n + i\right\} \end{aligned} \tag{3.10}$$

for all $x \in \mathcal{X}$. If

$$\lim_{i\to\infty} \lim_{n\to\infty} \max\left\{\frac{\varphi(k^j x, 0)}{|k|^j} : i \leq j < n + i\right\} = 0, \tag{3.11}$$

then $A = A'$. □

Corollary 1. *Assume that the function $\beta : [0, \infty) \to [0, \infty)$ satisfies*

(i) $\beta(|k|t) \leq \beta(|k|)\beta(t)$ *for all* $t > 0$;

(ii) $\beta(|k|) < |k|$.

Let $\delta > 0$. Assume that $\mathcal{X}$ is a normed space and that $f : \mathcal{X} \to C_{cb}(\mathcal{Y})$ is an odd mapping such that

$$\mathcal{L}f(x, y) \leq \delta[\beta(|x|) + \beta(|y|)]$$

for all $x, y \in \mathcal{X}$. Then there exists a unique additive mapping $A : \mathcal{X} \to C_{cb}(\mathcal{Y})$ such that

$$H_d(f(x), A(x)) \leq \frac{1}{|2k|}\delta\beta(|x|), x \in \mathcal{X}.$$

Proof. If we set $\psi(x,y)=\delta[\beta(|x|)+\beta(|y|)]$ for all $x,y\in\mathcal{X}$, then we have

$$\lim_{n\to\infty}\frac{\psi(k^nx,k^ny)}{|k|^n}\le\lim_{n\to\infty}\left(\frac{\beta(|k|)}{|k|}\right)^n\psi(x,y)=0\ (x,y\in\mathcal{X}),$$

$$\widetilde{\psi}_a(x)=\lim_{n\to\infty}\max\left\{\frac{\psi(k^jx,0)}{|k|^j}:0\le j<n\right\}=\psi(x,0),$$

$$\lim_{i\to\infty}\lim_{n\to\infty}\max\left\{\frac{\psi(k^jx,0)}{|k|^j}:i\le j<n+i\right\}=\lim_{i\to\infty}\frac{\psi(k^ix,k^iy)}{|k|^i}=0.$$

By Theorem 1, we can get the corollary. □

Remark 1. The classical example of the function β is the mapping $\beta(t)=t^r$ for all $t\ge 0$, where $r>1$ with the further assumption that $|k|<1$. The assumption $|k|<1$ cannot be omitted.

Remark 2. A similar result can be formulated as Theorem 1 if we define the sequence $A(x):=\lim_{n\to\infty}k^nf(\frac{x}{k^n})$ under suitable conditions on the function $\psi(x)$.

Theorem 2. *Assume that a mapping $\varphi:X\times X\to[0,\infty)$ satisfies*

$$\lim_{n\to\infty}\frac{\varphi(k^nx,k^ny)}{|k|^{2n}}=0,x,y\in\mathcal{X}. \tag{3.12}$$

And suppose that for $x\in\mathcal{X}$ the limit

$$\lim_{n\to\infty}\max\left\{\frac{\varphi(k^jx,0)}{|k|^{2j}}:0\le j<n\right\}, \tag{3.13}$$

denoted by $\tilde{\varphi}_q(x)$, exists. If an even mapping $f:X\to C_{cb}(\mathcal{Y})$ satisfies the following condition

$$\mathcal{L}f(x,y)\le\varphi(x,y) \tag{3.14}$$

for all $x,y\in\mathcal{X}$, then there exists a quadratic mapping $Q:\mathcal{X}\to C_{cb}(\mathcal{Y})$ such that

$$H_d(f(x),Q(x))\le\frac{1}{|2k^2|}\tilde{\varphi}_q(x),x\in\mathcal{X}. \tag{3.15}$$

Moreover, if

$$\lim_{i\to\infty}\lim_{n\to\infty}\max\left\{\frac{\varphi(k^jx,0)}{|k|^{2j}}:i\le j<n+i\right\}=0, \tag{3.16}$$

then the quadratic mapping Q is unique.

Proof. Setting $y=0$ in (3.14), we have

$$H_d(f(kx),k^2f(x))\le\frac{1}{|2|}\varphi(x,0) \tag{3.17}$$

for all $x\in\mathcal{X}$. Replacing x by $k^{n-1}x$ in (3.17), we get

$$H_d\left(\frac{f(k^nx)}{k^{2n}},\frac{f(k^{n-1}x)}{k^{2(n-1)}}\right)\le\frac{1}{|2\cdot k^{2n}|}\varphi(k^{n-1}x,0) \tag{3.18}$$

for all $x \in \mathcal{X}$. (3.18) and (3.12) imply that $\left\{\frac{f(k^n x)}{k^{2n}}\right\}$ is a Cauchy sequence. Since $\mathcal{Y}$ is complete, we can obtain that $\left\{\frac{f(k^n x)}{k^{2n}}\right\}$ is convergent. Let $Q(x) = \lim_{n\to\infty} \frac{1}{k^{2n}} f(k^n x)$. Using induction one can show that

$$H_d\left(\frac{f(k^n x)}{k^{2n}}, f(x)\right) \leq \frac{1}{|2k^2|} \max\left\{\frac{\varphi(k^j x, 0)}{|k|^{2j}} : 0 \leq j < n\right\} \tag{3.19}$$

for $x \in \mathcal{X}$ and $n \in \mathbb{N}$. By taking n to approach infinity in (3.19) and using (3.13) one can obtain (3.15). By (3.12) and (3.14), we get

$$\mathcal{L}Q(x,y) = \lim_{n\to\infty} \frac{1}{|k|^{2n}} \mathcal{L}f(k^n x, k^n y) \leq \lim_{n\to\infty} \frac{1}{|k|^{2n}} \varphi(k^n x, k^n y) = 0$$

for all $x, y \in \mathcal{X}$. Therefore, the mapping Q satisfies (2.2). By [[32], Lemma 2.1] we get that the mapping Q is quadratic. Let now

$$\lim_{i\to\infty} \lim_{n\to\infty} \max\left\{\frac{\varphi(k^j x, 0)}{|k|^{2j}} : i \leq j < n+i\right\} = 0.$$

Assume that Q' is another quadratic mapping satisfying (3.15). Then

$$\begin{aligned} H_d(Q(x), Q'(x)) &\leq \lim_{i\to\infty} \frac{1}{|k|^{2i}} H_d(Q(k^i x), Q'(k^i x)) \\ &\leq \lim_{i\to\infty} \frac{1}{|k|^{2i}} \max\{H_d(Q(k^i x), f(k^i x)), H_d(Q'(k^i x), f(k^i x))\} \\ &\leq \frac{1}{|2k^2|} \lim_{i\to\infty} \lim_{n\to\infty} \max\left\{\frac{\varphi(k^j x, 0)}{|k|^{2j}} : i \leq j < n+i\right\} = 0 \end{aligned} \tag{3.20}$$

for all $x \in \mathcal{X}$. Hence $Q = Q'$. □

Corollary 2. *Suppose that the function $\beta : [0,\infty) \to [0,\infty)$ satisfies the following conditions*

(i) $\beta(|k|t) \leq \beta(|k|)\beta(t)$ *for any* $t > 0$*;*

(ii) $\beta(|k|) < |k|^2$.

Let $\mathcal{X}$ be a normed space and $\delta > 0$. If the even mapping $f : \mathcal{X} \to C_{cb}(\mathcal{Y})$ satisfies

$$|\mathcal{L}f(x,y)| \leq \delta[\beta(|x|) + \beta(|y|)]$$

for $x, y \in \mathcal{X}$, then there exists a unique quadratic mapping $Q : X \to C_{cb}(Y)$ such that

$$H_d(f(x), Q(x)) \leq \frac{1}{|2k^2|} \delta\beta(|x|)$$

for $x \in \mathcal{X}$.

Proof. Setting $\varphi(x,y) = \delta[\beta(|x|) + \beta(|y|)]$ for all $x, y \in \mathcal{X}$, we can obtain

$$\lim_{n\to\infty} \frac{\varphi(k^n x, k^n y)}{|k|^{2n}} \leq \lim_{n\to\infty} \left(\frac{\beta(|k|)}{|k|^2}\right)^n \varphi(x,y) = 0 \ (x, y \in \mathcal{X}),$$

$$\widetilde{\varphi}_q(x) = \lim_{n\to\infty} \max\left\{\frac{\varphi(k^j x, 0)}{|k|^{2j}} : 0 \leq j < n\right\} = \varphi(x, 0),$$

$$\lim_{i\to\infty} \lim_{n\to\infty} \max\left\{\frac{\varphi(k^j x, 0)}{|k|^{2j}} : i \leq j < n+i\right\} = \lim_{i\to\infty} \frac{\varphi(k^i x, k^i y)}{|k|^{2i}} = 0.$$

The corollary can be deduced by Theorem 2. □

Theorem 3. *Assume that the function* $\varphi : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ *satisfies*

$$\lim_{n\to\infty} \frac{\varphi(k^n x, k^n y)}{|k|^{2n}} = 0, x, y \in \mathcal{X}.$$

Suppose that for each $x \in \mathcal{X}$ *the limit*

$$\lim_{n\to\infty} \max\left\{\frac{\varphi(k^j x, 0)}{|k|^j} : 0 \le j < n\right\},$$

denoted by $\tilde{\varphi}_a(x)$*, and*

$$\lim_{n\to\infty} \max\left\{\frac{\varphi(k^j x, 0)}{|k|^{2j}} : 0 \le j < n\right\},$$

denoted by $\tilde{\varphi}_q(x)$*, both exist. Let* $f : \mathcal{X} \to C_{cb}(Y)$ *be a mapping satisfying* $\mathcal{L}f(x, y) \le \varphi(x, y)$ *for all* $x, y \in \mathcal{X}$*. If* f *can be decomposed into an even and odd part, then there exist an additive mapping* $A : \mathcal{X} \to C_{cb}(\mathcal{Y})$ *and a quadratic mapping* $Q : \mathcal{X} \to C_{cb}(\mathcal{Y})$ *such that*

$$H_d(f(x), A(x) \oplus Q(x)) \le \frac{1}{|4k|} \max\left\{\max\{\tilde{\varphi}_a(x), \tilde{\varphi}_a(-x)\}, \frac{1}{|k|}\max\{\tilde{\varphi}_q(x), \tilde{\varphi}_q(-x)\}\right\} \tag{3.21}$$

for all $x \in \mathcal{X}$*.*

Moreover, if

$$\begin{aligned} &\lim_{i\to\infty}\lim_{n\to\infty} \max\left\{\frac{\varphi(k^j x, 0)}{|k|^j} : i \le j < n+i\right\} \\ &= \lim_{i\to\infty}\lim_{n\to\infty} \max\left\{\frac{\varphi(k^j x, 0)}{|k|^{2j}} : i \le j < n+i\right\} = 0, \end{aligned} \tag{3.22}$$

then A *and* Q *are both unique.*

Proof. Assume that $f(x) = f_e(x) \oplus f_o(x)$. We have for all $x, y \in \mathcal{X}$

$$\mathcal{L}f_o(x, y) \le \frac{1}{|2|}\max\{\varphi(x, y), \varphi(-x, -y)\},$$

$$\mathcal{L}f_e(x, y) \le \frac{1}{|2|}\max\{\varphi(x, y), \varphi(-x, -y)\}.$$

By Theorems 1 and 2, there exist an additive mapping $A : \mathcal{X} \to C_{cb}(\mathcal{Y})$ and a quadratic mapping $Q : \mathcal{X} \to C_{cb}(\mathcal{Y})$ satisfying

$$H_d(f_o(x), A(x)) \le \frac{1}{|4k|}\max\{\widetilde{\varphi}_a(x), \widetilde{\varphi}_a(-x)\},$$

$$H_d(f_e(x), Q(x)) \le \frac{1}{|4k^2|}\max\{\widetilde{\varphi}_q(x), \widetilde{\varphi}_q(-x)\}$$

for all $x \in \mathcal{X}$. Therefore

$$\begin{aligned} H_d(f(x), A(x) \oplus Q(x)) &\le \max\{H_d(f_o(x), A(x)), H_d(f_e(x), Q(x))\} \\ &\le \frac{1}{|4k|}\max\left\{\max\{\widetilde{\varphi}_a(x), \widetilde{\varphi}_a(-x)\}, \frac{1}{|k|}\max\{\widetilde{\varphi}_q(x), \widetilde{\varphi}_q(-x)\}\right\} \end{aligned} \tag{3.23}$$

for all $x \in \mathcal{X}$.

4 Stability of the Set-Valued Functional Equation: a Fixed Point Method

In this section, we establish the generalized Hyers-Ulam stability results for the mixed set-valued AQ-functional equation (2.2) in non-Archimedean Banach spaces by using the fixed point method introduced by Radu in [28].

Let Ω be a set. A mapping $\mathcal{D} : \Omega \times \Omega \to [0, \infty]$ is called a generalized metric on Ω if $\mathcal{D}$ satisfies the following three conditions:

(i) $\mathcal{D}(\mu, \nu) = 0$ if only if $\mu = \nu$;

(ii) $\mathcal{D}(\mu, \nu) = \mathcal{D}(\nu, \mu)$, $\nu, \mu \in \Omega$;

(iii) $\mathcal{D}(\nu, \mu) \leq \mathcal{D}(\nu, w) + \mathcal{D}(w, \mu)$, $\nu, \mu, w \in \Omega$.

For explicitly later use, we recall the following lemma proposed by Diaz and Margolis in [30].

Lemma 1 ([30]). *Assume that $(\Omega, \mathcal{D})$ is a complete generalized metric space, and that $\mathcal{T} : \Omega \to \Omega$ is a strictly contractive mapping with Lipschitz constant $L < 1$, that is*

$$\mathcal{D}(\mathcal{T}x, \mathcal{T}y) \leq L\mathcal{D}(x, y), x, y \in \Omega.$$

Then, for each given $x \in \Omega$, either

$$\mathcal{D}(\mathcal{T}^n x, \mathcal{T}^{n+1} x) = \infty, n \geq 0,$$

or there exists a non-negative integer n_0 such that

(1) *$\mathcal{D}(\mathcal{T}^n x, \mathcal{T}^{n+1} x) < \infty$ for all $n \geq n_0$;*

(2) *the sequence $\{\mathcal{T}^n(x)\}$ is converges to a fixed point y^*;*

(3) *y^* is the unique fixed point of $\mathcal{T}$ in the set $\Omega^* = \{y \in \Omega \mid \mathcal{D}(\mathcal{T}^{n_0} x, y) < \infty\}$;*

(4) *$\mathcal{D}(y, y^*) \leq \frac{1}{1-L}\mathcal{D}(y, \mathcal{T}y)$ for all $y \in \Omega^*$.*

Theorem 4. *Let $\varphi : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ be a mapping such that there exists an $0 < L < 1$ with*

$$\varphi(kx, ky) \leq |k|L\varphi(x, y) \tag{4.1}$$

for all $x, y \in \mathcal{X}$. If the odd mapping $f : \mathcal{X} \to C_{cb}(\mathcal{Y})$ satisfies

$$\mathcal{L}f(x, y) \leq \varphi(x, y), \quad x, y \in \mathcal{X}, \tag{4.2}$$

then there exists a unique additive mapping $A : \mathcal{X} \to C_{cb}(\mathcal{Y})$ such that

$$H_d(f(x), A(x)) \leq \frac{1}{|2k|(1-L)}\varphi(x, 0) \tag{4.3}$$

for all $x \in \mathcal{X}$.

Proof. In (4.2) setting $y = 0$, we have

$$H_d(f(kx), kf(x)) \leq \frac{1}{|2|}\varphi(x, 0) \tag{4.4}$$

for all $x \in \mathcal{X}$. Set $\Omega := \{g \mid g : \mathcal{X} \to C_{cb}(\mathcal{Y}),\ g(0) = \{0\}\}$, and we then introduce the generalized metric on Ω:

$$\mathcal{D}(g,h) = \inf\{C \in (0,\infty) \mid H_d(g(x),h(x)) \le C\varphi(x,0), \forall x \in \mathcal{X}\}. \tag{4.5}$$

It is easy to show that $(\Omega, \mathcal{D})$ is a complete generalized metric space. We now define a mapping $\mathcal{T} : \Omega \to \Omega$ as

$$(\mathcal{T}g)(x) = \frac{1}{k}g(kx), \quad g \in \Omega, x \in \mathcal{X}. \tag{4.6}$$

Let $g, h \in \Omega$ and $C \in [0,\infty]$ be an arbitrary constant with $\mathcal{D}(g,h) < C$. The definition of $\mathcal{D}$ implies

$$H_d(g(x),h(x)) \le C\varphi(x,0), \quad x \in \mathcal{X}. \tag{4.7}$$

By the last inequality, we can obtain

$$H_d\left(\frac{1}{k}g(kx), \frac{1}{k}h(kx)\right) \le CL\varphi(x,0), \quad \forall x \in \mathcal{X}. \tag{4.8}$$

Hence, $\mathcal{D}(\mathcal{T}g, \mathcal{T}h) \le L\mathcal{D}(g,h)$. It follows from (4.4) that $\mathcal{D}(\mathcal{T}f, f) \le \frac{1}{|2k|} < \infty$. Therefore, by Lemma 1, the operator $\mathcal{T}$ has a unique fixed point $A : \mathcal{X} \to C_{cb}(\mathcal{Y})$ in the set $\Omega^* = \{g \in \Omega \mid \mathcal{D}(f,g) < \infty\}$ such that

$$A(x) := \lim_{n\to\infty}(\mathcal{T}^n f)(x) = \lim_{n\to\infty}\frac{1}{k^n}f(k^n x) \tag{4.9}$$

and $A(kx) = kA(x)$ for all $x \in X$. Also,

$$\mathcal{D}(A,f) \le \frac{1}{1-L}\mathcal{D}(\mathcal{T}f,f) \le \frac{1}{|2k|(1-L)}. \tag{4.10}$$

Then (4.3) holds for all $x \in \mathcal{X}$.

Now we will prove that A is additive. By (4.1), (4.2) and (4.9), we have

$$\mathcal{L}A(x,y) = \lim_{n\to\infty}\frac{1}{|k|^n}\mathcal{L}f(k^n x, k^n y) \le \lim_{n\to\infty}\frac{1}{|k|^n}\varphi(k^n x, k^n y) \le \lim_{n\to\infty}L^n\varphi(x,y) = 0$$

for all $x, y \in \mathcal{X}$. Therefore, it follows form [[32], Lemma 2.2] that A is an additive mapping. □

Corollary 3. *Let $\delta > 0,\ r > 1,\ |k| < 1$. If $f : \mathcal{X} \to C_{cb}(\mathcal{Y})$ is an odd mapping and satisfies the following condition*

$$\mathcal{L}f(x,y) \le \delta(|x|^r + |y|^r), \quad x, y \in \mathcal{X},$$

then there exists a unique additive mapping $A : \mathcal{X} \to C_{cb}(\mathcal{Y})$ such that

$$H_d(f(x), A(x)) \le \frac{1}{|2|(|k| - |k|^r)}\delta|x|^r, \quad x \in \mathcal{X}.$$

Proof. Set $\psi(x,y) = \delta(|x|^r + |y|^r)$ for all $x, y \in \mathcal{X}$. Then we can get the corollary by Theorem 4 by $L = |k|^{r-1} < 1$. □

The following example is a modification of the example of [31] and it shows that the assumption $|k| < 1$ cannot be omitted in Corollary 3.

Example 1. For a prime number $p > 2$ if the set-valued mapping $f : \mathbb{Q}_p \to C_{cb}(\mathbb{Q}_p)$ satisfies $f(x) = [0, x^3]$, then for $\delta = 1$ and $r = 3$

$$\mathcal{L}f(x, y) \leq \max\{|x|_p^3, |y|_p^3\} \leq |x|_p^3 + |y|_p^3, \quad x, y \in \mathbb{Q}_p.$$

However, for $k = 2$ we have $|k|_p = |2|_p = 1$ and

$$H_d\left(\frac{1}{2^n}f(2^n x), \frac{1}{2^{n+1}}f(2^{n+1}x)\right) = |2^{2n}|_p|3|_p|x|_p^3 = |3|_p|x|_p^3$$

for all $x \in \mathbb{Q}_p$ and $n \in \mathbb{N}$. Hence $\left\{\frac{1}{2^n}f(2^n x)\right\}$ is not a Cauchy sequence for each nonzero $x \in \mathbb{Q}_p$.

Similar to Theorem 4, we have the following theorem.

Theorem 5. *Assume that* $\varphi : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ *is a function such that there exists an* $0 < L < 1$ *with*

$$\varphi(x, y) \leq \frac{L}{|k|}\varphi(kx, ky) \tag{4.11}$$

for $x, y \in \mathcal{X}$. *If the odd mapping* $f : \mathcal{X} \to C_{cb}(\mathcal{Y})$ *satisfies*

$$\mathcal{L}f(x, y) \leq \varphi(x, y), \quad x, y \in \mathcal{X}, \tag{4.12}$$

then there exists a unique additive mapping $A : \mathcal{X} \to C_{cb}(\mathcal{Y})$ *such that*

$$H_d(f(x), A(x)) \leq \frac{L}{|2k|(1 - L)}\varphi(x, 0), \quad x \in \mathcal{X}. \tag{4.13}$$

Corollary 4. *Let* $\delta > 0$, $0 \leq r < 1$, *and* $|k| < 1$. *If the odd set-valued mapping* $f : \mathcal{X} \to C_{cb}(\mathcal{Y})$ *satisfies*

$$\mathcal{L}f(x, y) \leq \delta(|x|^r + |y|^r),$$

for all $x, y \in \mathcal{X}$, *then there exists a unique additive mapping* $A : \mathcal{X} \to C_{cb}(\mathcal{Y})$ *such that*

$$H_d(f(x), A(x)) \leq \frac{1}{|2|(|k|^r - |k|)}\delta|x|^r, x \in \mathcal{X}$$

Proof. Set $\varphi(x, y) = \delta(|x|^r + |y|^r)$ for $x, y \in \mathcal{X}$. The corollary can be obtained by using Theorem 5 with $L = |k|^{1-r} < 1$. □

We then give an example to show that the assumption $|k| < 1$ cannot be omitted in Corollary 4.

Example 2. For a prime number $p > 2$ if the set-valued function $f : \mathbb{Q}_p \to C_{cb}(\mathbb{Q}_p)$ satisfies $f(x) = [0, x^{\frac{1}{3}}]$, then for $\delta = 1$ and $r = \frac{1}{3}$,

$$\mathcal{L}f(x, y) \leq \max\{|x|_p^r, |y|_p^r\} \leq |x|_p^r + |y|_p^r, \quad x, y \in \mathbb{Q}_p.$$

However, for $k = 2$ we have $|k|_p = |2|_p = 1$ and

$$H_d\left(2^n f(\frac{1}{2^n}x), 2^{n+1}f(\frac{1}{2^{n+1}}x)\right) = |2^{\frac{2n}{3}}|_p|1 - 2^{\frac{2}{3}}|_p|x|_p^{\frac{1}{3}} = |1 - 2^{\frac{2}{3}}|_p|x|_p^{\frac{1}{3}}.$$

for all $x \in \mathbb{Q}_p$ and $n \in \mathbb{N}$. Hence $\{2^n f(\frac{1}{2^n}x)\}$ is not a Cauchy sequence for each nonzero $x \in \mathbb{Q}_p$.

Theorem 6. *Assume that $\varphi : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ is a function such that there exists an $0 < L < 1$ with*

$$\varphi(kx, ky) \le |k|^2 L\varphi(x, y). \tag{4.14}$$

for all $x, y \in \mathcal{X}$. If the even set-valued function $f : \mathcal{X} \to C_{cb}(\mathcal{Y})$ satisfies

$$\mathcal{L}f(x, y) \le \varphi(x, y), \quad x, y \in \mathcal{X}, \tag{4.15}$$

then there exists a unique quadratic mapping $Q : \mathcal{X} \to C_{cb}(\mathcal{Y})$ such that

$$H_d(f(x), Q(x)) \le \frac{1}{|2k^2|(1 - L)}\varphi(x, 0), \quad x \in \mathcal{X}. \tag{4.16}$$

Proof. Setting $y = 0$ in (4.15), we get

$$H_d(f(kx), k^2 f(x)) \le \frac{1}{|2|}\varphi(x, 0), \quad x \in \mathcal{X}. \tag{4.17}$$

Let $\Omega := \{g \mid g : \mathcal{X} \to C_{cb}(\mathcal{Y})\}$, and we then introduce a generalized metric on Ω:

$$\mathcal{D}(g, h) = \inf\{C \in (0, \infty) \mid H_d(g(x), h(x)) \le C\varphi(x, 0), \ \forall x \in \mathcal{X}\}. \tag{4.18}$$

It is easy to show that $\Omega := \{g \mid g : \mathcal{X} \to C_{cb}(\mathcal{Y})\}$ is a complete generalized metric space. Define an operator $\mathcal{T} : \Omega \to \Omega$ as

$$(\mathcal{T}g)(x) = \frac{1}{k^2}g(kx), \quad \forall g \in \Omega, x \in \mathcal{X}. \tag{4.19}$$

Let $g, h \in \Omega$ and $C \in [0, \infty]$ be an arbitrary constant with $\mathcal{D}(g, h) < C$. By the definition of the metric $\mathcal{D}$, we can get

$$H_d(g(x), h(x)) \le C\varphi(x, 0), \quad x \in \mathcal{X}. \tag{4.20}$$

By the given hypothesis and the last inequality, we get

$$H_d\left(\frac{1}{k^2}g(kx), \frac{1}{k^2}h(kx)\right) \le CL\varphi(x, 0), \ \forall x \in \mathcal{X}. \tag{4.21}$$

Hence, $\mathcal{D}(\mathcal{T}g, \mathcal{T}h) \le L\mathcal{D}(g, h)$. (4.17) implies that $\mathcal{D}(\mathcal{T}f, f) \le \frac{1}{|2k^2|} < \infty$. Therefore, by Lemma 1, $\mathcal{T}$ has a unique fixed point $Q : \mathcal{X} \to C_{cb}(\mathcal{Y})$ in the set $\Omega^* = \{g \in \Omega \mid \mathcal{D}(f, g) < \infty\}$ such that

$$Q(x) := \lim_{n\to\infty}(\mathcal{T}^n f)(x) = \lim_{n\to\infty}\frac{1}{k^{2n}}f(k^n x), \tag{4.22}$$

and $Q(kx) = k^2 Q(x)$ for all $x \in \mathcal{X}$. Also,

$$\mathcal{D}(Q, f) \le \frac{1}{1 - L}\mathcal{D}(\mathcal{T}f, f) \le \frac{1}{|2k^2|(1 - L)}. \tag{4.23}$$

Then, (4.16) holds. By (4.14), (4.15) and (4.22) we obtain

$$\mathcal{L}Q(x, y) = \lim_{n\to\infty}\frac{1}{|k|^{2n}}\mathcal{L}f(k^n x, k^n y) \le \lim_{n\to\infty}\frac{1}{|k|^{2n}}\varphi(k^n x, k^n y) \le \lim_{n\to\infty}|k|^n\varphi(x, y) = 0$$

for all $x, y \in \mathcal{X}$. So by Lemma 2.1 in [32], we get that the mapping Q is quadratic. □

Corollary 5. *Let $\delta > 0$, $r > 2$, $|k| < 1$. If $f : \mathcal{X} \to C_{cb}(\mathcal{Y})$ is an even set-valued mapping satisfying*

$$\mathcal{L}f(x,y) \leq \delta(|x|^r + |y|^r), \quad x, y \in \mathcal{X},$$

then there exists a unique quadratic mapping $Q : \mathcal{X} \to C_{cb}(\mathcal{Y})$ such that

$$H_d(f(x), Q(x)) \leq \frac{1}{|2|(|k|^2 - |k|^r)}\delta|x|^r, \quad x \in \mathcal{X}.$$

Proof. Set $\varphi(x,y) = \delta(|x|^r + |y|^r)$ for all $x, y \in \mathcal{X}$. By Theorem 6 with $L = |k|^{r-2} < 1$, we can get the corollary. □

Example 3. For a prime number $p > 2$ if a set-valued mapping $f : \mathbb{Q}_p \to C_{cb}(\mathbb{Q}_p)$ satisfies $f(x) = [0, x^4]$, then for $\delta = 1$ and $r = 4$,

$$\mathcal{L}f(x,y) \leq \max\{|x|_p^4, |y|_p^4\} \leq |x|_p^4 + |y|_p^4, \quad x, y \in \mathbb{Q}_p.$$

However, for $k = 2$ we have $|k|_p = |2|_p = 1$ and

$$H_d\left(\frac{1}{2^{2n}}f(2^n x), \frac{1}{2^{2(n+1)}}f(2^{n+1}x)\right) = |2^{2n}|_p|3|_p|x|_p^4 = |3|_p|x|_p^4.$$

for all $x \in \mathbb{Q}_p$ and $n \in \mathbb{N}$. Hence $\{\frac{1}{2^{2n}}f(2^n x)\}$ is not a Cauchy sequence for each nonzero $x \in \mathbb{Q}_p$.

Similarly, we can obtain the following theorem.

Theorem 7. *Let $\varphi : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ be a mapping such that there exists an $0 < L < 1$ with*

$$\varphi(x,y) \leq \frac{L}{|k|^2}\varphi\left(\frac{x}{k}, \frac{y}{k}\right), x, y \in \mathcal{X}. \tag{4.24}$$

If the even set-valued mapping $f : \mathcal{X} \to C_{cb}(\mathcal{Y})$ satisfies

$$\mathcal{L}f(x,y) \leq \varphi(x,y), \quad x, y \in \mathcal{X}, \tag{4.25}$$

then there exists a unique quadratic set-valued mapping $Q : \mathcal{X} \to C_{cb}(\mathcal{Y})$ such that

$$H_d(f(x), Q(x)) \leq \frac{1}{|2k^2|(1-L)}\varphi(x,0), \quad x \in \mathcal{X}. \tag{4.26}$$

Corollary 6. *Let $\delta > 0$, $0 \leq r < 2$ and $|k| < 1$. If the even mapping $f : X \to C_{cb}(Y)$ satisfies*

$$\mathcal{L}f(x,y) \leq \delta(|x|^r + |y|^r), \quad x, y \in \mathcal{X},$$

then there exists a unique quadratic mapping $Q : \mathcal{X} \to C_{cb}(\mathcal{Y})$ such that

$$H_d(f(x), Q(x)) \leq \frac{1}{|2|(|k|^2 - |k|^r)}\delta|x|^r, \quad x \in \mathcal{X}.$$

In Corollary 6 the assumption $|k| < 1$ cannot be omitted .

Example 4. For a prime number $p > 2$ if $f : \mathbb{Q}_p \to C_{cb}(\mathbb{Q}_p)$ is defined as $f(x) = [0, |2|_p]$, then for $\delta = 1, k = 2$ and $r = 0$,

$$\mathcal{L}f(x, y) = |12|_p \leq 1 = \delta \ (x, y \in \mathbb{Q}_p).$$

Note that if $p > 2$, then $|2^n|_p = 1$ for each integer n, we have

$$H_d\left(2^n f(2^{-n}x), 2^{n+1} f(2^{-(n+1)}x)\right) = |2^{n+1}|_p = 1.$$

for all $x \in \mathbb{Q}_p$ and $n \in \mathbb{N}$. Hence $\{2^n f(2^{-n}x)\}$ is not a Cauchy sequence for $x \in \mathbb{Q}_p$.

Theorem 8. *Assume that $\varphi : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ is a function such that there exists an $0 < L < 1$ with*

$$\varphi(kx, ky) \leq |k|^2 L\varphi(x, y), x, y \in \mathcal{X}. \tag{4.27}$$

Suppose that $f : \mathcal{X} \to C_{cb}(\mathcal{Y})$ is a set-valued mapping such that

$$\mathcal{L}f(x, y) \leq \varphi(x, y), \quad x, y \in \mathcal{X}. \tag{4.28}$$

If f can be decomposed into an even and odd part, then there exist a unique additive mapping $A : \mathcal{X} \to C_{cb}(\mathcal{Y})$ and a unique quadratic mapping $Q : \mathcal{X} \to C_{cb}(\mathcal{Y})$ such that

$$H_d(f(x), A(x) \oplus Q(x)) \leq \frac{1}{|4k^2|(1-L)} \max\{\varphi(x, 0), \varphi(-x, 0)\}, \quad x \in \mathcal{X}. \tag{4.29}$$

Proof. Assume that $f(x) = f_e(x) \oplus f_o(x)$. Let $\psi(x, y) = \frac{1}{|2|} \max\{\varphi(x, y), \varphi(-x, -y)\}$, then by (4.27) and (4.28), we have

$$\psi(kx, ky) \leq |k|^2 L\psi(x, y) \leq |k|L\psi(x, y),$$

$$\mathcal{L}f_o(x, y) \leq \psi(x, y), \mathcal{L}f_e(x, y) \leq \psi(x, y).$$

Hence by Theorems 4 and 6, there exist a unique additive mapping $A : \mathcal{X} \to C_{cb}(\mathcal{Y})$ and a unique quadratic mapping $Q : \mathcal{X} \to C_{cb}(\mathcal{Y})$ such that

$$H_d(f_o(x), A(x)) \leq \frac{1}{|2k|(1-L)}\psi(x, 0),$$

$$H_d(f_e(x), Q(x)) \leq \frac{1}{|2k^2|(1-L)}\psi(x, 0)$$

for all $x \in \mathcal{X}$. Therefore

$$\begin{aligned} H_d(f(x), A(x) \oplus Q(x)) &\leq \max\{H_d(f_o(x), A(x)), H_d(f_e(x), Q(x))\} \\ &\leq \max\left\{\frac{1}{|2k|(1-L)}\psi(x, 0), \frac{1}{|2k^2|(1-L)}\psi(x, 0)\right\} \\ &\leq \frac{1}{|4k^2|(1-L)} \max\{\varphi(x, 0), \varphi(-x, 0)\} \end{aligned} \tag{4.30}$$

for all $x \in \mathcal{X}$. □

Corollary 7. *Let $\delta > 0,\ 2 < r,\ |k| < 1$ and $f : \mathcal{X} \to C_{cb}(\mathcal{Y})$ be a set-valued mapping for which*

$$\mathcal{L}f(x, y) \leq \delta(|x|^r + |y|^r), \quad x, y \in \mathcal{X}.$$

If f can be decomposed into an even and odd part, then there exist a unique additive mapping $A : \mathcal{X} \to C_{cb}(\mathcal{Y})$ and a unique quadratic mapping $Q : \mathcal{X} \to C_{cb}(\mathcal{Y})$ such that

$$H_d(f(x), Q(x) \oplus A(x)) \leq \frac{1}{|4|(|k|^2 - |k|^r)}\delta|x|^r, \quad x \in \mathcal{X}.$$

Similar to Theorem 8, one can obtain the following theorem.

Theorem 9. *Let $\varphi : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ be a function such that there exists an $0 < L < 1$ with*

$$\varphi(x, y) \leq \frac{L}{|k|}\varphi(kx, ky) \tag{4.31}$$

for all $x, y \in \mathcal{X}$. Let $f : \mathcal{X} \to C_{cb}(\mathcal{Y})$ be a mapping such that

$$\mathcal{L}f(x, y) \leq \varphi(x, y), \quad x, y \in \mathcal{X}. \tag{4.32}$$

If f can be decomposed into an even and odd part, then there exist a unique additive mapping $A : \mathcal{X} \to C_{cb}(\mathcal{Y})$ and a unique quadratic mapping $Q : \mathcal{X} \to C_{cb}(\mathcal{Y})$ such that

$$H_d(f(x), A(x) \oplus Q(x)) \leq \frac{1}{|4k^2|(1 - L)} \max\{\varphi(x, 0), \varphi(-x, 0)\}, \quad x \in \mathcal{X}. \tag{4.33}$$

Corollary 8. *Let $\delta > 0,\ 0 \leq r < 1,\ |k| < 1$ and $f : \mathcal{X} \to C_{cb}(\mathcal{Y})$*

$$\mathcal{L}f(x, y) \leq \delta(|x|^r + |y|^r), \quad x, y \in \mathcal{X}.$$

If f can be decomposed into an even and odd part, then there exist a unique additive mapping $A : \mathcal{X} \to C_{cb}(\mathcal{Y})$ and a unique quadratic mapping $Q : \mathcal{X} \to C_{cb}(\mathcal{Y})$ such that

$$H_d(f(x), Q(x) \oplus A(x)) \leq \frac{1}{|4|(|k|^r - |k|)}\delta|x|^r, \quad x \in \mathcal{X}.$$

Acknowledgements

This work is supported by the National Natural Science Foundation of China (Grant No. 41301493).

References

[1] S.M. Ulam. A Collection of Mathematical Problems. Interscience Publishers, New York (1968).

[2] D.H. Hyers. On the stability of the linear functional equation. Proceedings of the National Academy of Sciences 27 (1941) 222-224.

[3] T.M. Rassias. On the stability of linear mappings in Banach spaces. Proceedings of the American Mathematical Society 72 (1978) 297-300.

[4] D.H. Hyers, G. Isac , T.M. Rassias. Stability of Functional Equations in Several Variables. Birkhauser, Boston (1998).

[5] S.M. Jung. Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis. Hadronic Press, Palm Harbor (2001).

[6] C. Mortici, T.M. Rassias, S.M. Jung. The inhomogeneous Euler equation and its Hyers-Ulam stability. Applied Mathematics Letters 40 (2015) 23-28.

[7] M.F. Bota-Boriceanu, A. Petrusel. Ulam-Hyers stability for operatorial equations and inclusions. Analele Univ. I. Cuza Iasi 57 (2011) 65-74.

[8] Y. H. Shen. Hyers-Ulam-Rassias stability of first order linear partial fuzzy differential equations under generalized differentiability. Advances in Difference Equations (2015) 2015:351.

[9] D. Popa, I. Roşa. On the best constant in Hyers-Ulam stability of some positive linear operators. Journal of Mathematical Analysis and Applications 412 (2014) 103-108.

[10] T.Z. Xu, Z.P. Yang. A fixed point approach to the stability of functional equations on noncommutative spaces. Results in Mathematics DOI 10.1007/s00025-015-0448-0.

[11] Z.P. Yang, T.Z. Xu, M. Qi. Ulam-Hyers stability for fractional differential equations in Quaternionic analysis. Advances in Applied Clifford Algebras 26 (2016) 469-478.

[12] I.A. Rus. Ulam stability of ordinary differential equations. Studia Univ. Babes-Bolyai, Math. LIV(4)(2009) 125-133.

[13] I.A. Rus. Remarks on Ulam stability of the operatorial equations. Fixed Point Theory 10 (2009) 305-320.

[14] L.P. Castro, A. Ramos. Hyers-Ulam-Rassias stability for a class of Volterra integral equations. Banach Journal of Mathematical Analysis 3 (2009) 36-43.

[15] S.M. Jung. A fixed point approach to the stability of a Volterra integral equation. Fixed Point Theory and Applications 2007, Article ID 57064, 1-9.

[16] A. Khrennikov. Non-Archimedean Analysis: Quantum Paradoxes, Dynamical Systems and Biological Models. Kluwer Academic Publishers, 1997.

[17] V.S. Vladimirov, I.V. Volovich, E.I. Zelenov. p-adic Analysis and Mathematical Physics. World Scientific, 1994.

[18] N.B. Huy, T.D. Thanh. Fixed point theorems and the Ulam-Hyers stability in non-Archimedean cone metric spaces. Journal of Mathematical Analysis and Applications 414 (2011) 10-20.

[19] J. Brzdęk, K. Ciepliński. A fixed point appraoch to the sability of functional equations in non-Archimedean metric spaces. Nonlinear Analysis 74 (2011) 6861-6867.

[20] A.K. Mirnostafaee, M.S. moslehian. Stability of additive mappings in non-Archimedean fuzzy normed spaces. Fuzzy Sets and Systems 160 (2009) 1643-1652.

[21] R.J. Aumann. Integrals of set-valued functions. Journal of Mathematical Analysis and Applications 12 (1965) 1-12.

[22] G. Debreu. Integration of correspondences. Proceedings of Fifth Berkeley Symposium on Mathematical Statistics and Probability 2 (1966) 351-372.

[23] K.J. Arrow, G. Debreu. Existence of an equilibrium for a competitive economy. Econometrica 22 (1954) 265-290.

[24] L.W. McKenzie. On the existence of general equilibrium for a competitive market. Econometrica 27 (1959) 54-71.

[25] H.Y. Chu, A. Kim, S.Y. Yoo. On the stability of the generalized cubic set-valued functional equation. Applied Mathematics Letters 37 (2014) 7-14.

[26] Z.H. Wang. Stability of two types of cubic fuzzy set-valued functional equations. Results in Mathematics Doi. 10.1007/s00025-015-0457-z.

[27] J.R. Lee, C. Park, T.M. Rassias. Hyers-Ulam stability of set-vaulued mappings. Mathematics Without Boundaries (2014) 323-336.

[28] V. Radu. The fixed point alternative and the stability of functional equations. Fixed Point Theory 4 (2003) 91-96.

[29] S.M. Jung. A fixed point approach to the stability of the equation $f(x+y) = \frac{f(x)f(y)}{f(x)+f(y)}$. Australian Journal of Mathematical Analysis and Applications 6 (2009) 1-6.

[30] J.B. Diaz, B. Margolis. A fixed point theorem of the alternative for the contractions on generalized complete metric space. Bulletin of the American Mathematical Society 74 (1968) 305-309.

[31] T. Z. Xu, Z. P. Yang, J. M. Rassias. Direct and fixed point approaches to the stability of an AQ-functional equation in non-Archimedean normed spaces, Journal of Computational Analysis and Applications 17 (2014) 697-706.

[32] G. Z. Eskandani, P. Găvruţa, J.M. Rassias. Generalized Hyers-Ulam stability for a general mixed functional equation in quasi-β-normed spaces[J]. Mediterranean Journal of Mathematics 8 (2011) 331–348.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 24, NO.6, 2018, COPYRIGHT 2018 EUDOXUS PRESS, LLC

Solution and stability of nonic functional equations in non-Archimedean normed spaces

John Michael Rassias[1], Seong Sik Kim[2] and Soo Hwan Kim[2]

[1] Pedagogical Department E.E., Section of Mathematics and Informatics,
National and Capodistrian University of Athens,
4, Agamemnonos St., Aghia Paraskevi, Athens 15342, Greece
E-mail: jrassias@primedu.uoa.gr; jrass@otenet.gr

[2] Department of Mathematics, Dongeui University,
Busan 614-714, Korea
E-mail: sskim@deu.ac.kr
E-mail: sh-kim@deu.ac.kr

Abstract: In this paper, we establish the general solution and stability of the following nonic functional equation:

$$\begin{aligned} &f(x+5y)-9f(x+4y)+36f(x+3y)-84f(x+2y)+126f(x+y)-126f(x)\\ &+84f(x-y)-36f(x-2y)+9f(x-3y)-f(x-4y)=9!f(y), \end{aligned}$$

where $9! = 362880$ in non-Archimedean normed spaces.

1. **Introduction and Preliminaries**

The stability problem of functional equations originated from a question of Ulam [26] concerning the stability of group homomorphisms. The first affirmative partial answer to Ulam's question was given by Hyers [11]. The result of Hyers by obtaining a unique linear mapping near an approximate additive mapping was generalized by Rassias [22]. The paper of Rassias has provided a lot of influence in the development of what we called the *generalized Hyers-Ulam-Rassias stability* of functional equations. In 1994, Găvruta then generalized the Rassias' result in [8] for unbounded Cauchy difference by a general control function. In 2003, Radu [20] proposed a new method for obtaining the existence of exact solutions and error estimations, based on the fixed point alternative. Cădariu and Radu [5] proposed a novel method for studying the stability of the Cauchy functional equation based on a fixed point result in generalized metric spaces. A great number of papers (see [2, 6, 9, 12, 13, 21, 24, 25, 28] and references therein for more detailed information) on the subject has been published, generalizing Ulam's problem and Hyer's theorem in various functional equations.

In 1897, Hensel [10] discovered the p-adic numbers as a number theoretical analogue of power series in complex analysis and introduced a normed space which does not have the Archimedean property. Let p be a prime number. For any nonzero rational number $x \in \mathbb{Q}$, there exists a unique integer $n_x \in \mathbb{Z}$ such that $x = \frac{a}{b}p^{n_x}$, where a, b are integers divisible by p. Then the p-adic absolute value $|x|_p = p^{-n_x}$ is a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the

[0]**2010 Mathematics Subject Classification**: 39B52, 39B82, 46S50, 47B48
[0]**Keywords**: Generalized Hyers-Ulam stability; Nonic mappings; Non-Archimedean normed spaces

metric $d(x, y) = |x - y|_p$ is denoted by $\mathbb{Q}_p$ which is called the p-adic number field. Note that if $p > 2$, then $|2^n|_p = 1$ for each integer $n \in \mathbb{Z}$; however, $|2|_2 < 1$.

Let us recall some basic definitions and facts concerning non-Archimedean normed spaces [14]. By a *non-Archimedean field* we mean a field $\mathbb{K}$ equipped with a function (valuation) $|\cdot|$ from $\mathbb{K}$ into $[0, \infty)$ such that $|r| = 0$ if and only if $r = 0, |rs| = |r||s|$, and $|r + s| \le \max\{|r|, |s|\}$ for all $r, s \in \mathbb{K}$. Clearly, $|1| = |-1| = 1$ and $|n| \le 1$ for all $n \in \mathbb{N}$. In the rest of the paper, let $|2| \neq 1$.

In any field $\mathbb{K}$ the function $|\cdot| : \mathbb{K} \to [0, \infty)$ defined by

$$|x| = \begin{cases} 0, & x \in X; \\ 1, & \text{otherwise} \end{cases}$$

is a valuation which is called trivial, but the most important examples of non-Archimedean fields are p-adic numbers which have gained the interest of physicists for their research in some problems coming from quantum physics, p-adic strings and superstrings [27].

Let X be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean nontrivial valuation $|\cdot|$. A function $\|\cdot\| : X \to \mathbb{R}$ is a *non-Archimedean norm* (valuation) if it satisfies the following conditions:

(NA1) $\|x\| = 0$ if and only if $x = 0$;

(NA2) $\|rx\| = |r|\|x\|$ for all $r \in \mathbb{K}, x \in X$;

(NA3) $\|x + y\| \le \max\{\|x\|, \|y\|\}$ for all $x, y \in X$ (: the strong triangle inequality).

Then $(X, \|\cdot\|)$ is called a *non-Archimedean normed space*. It follows from (NA3) that

$$\|x_n - x_m\| \le \max\{\|x_{j+1} - x_j\| : m \le j \le n - 1\} \qquad \text{for} \quad n > m$$

and so a sequence $\{x_n\}$ is Cauchy in X if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean normed space. If every Cauchy sequence is convergent, the non-Archimedean normed space is said to be complete and called a *non-Archimedean Banach space*.

In 2005, Arriola and Beyer [1] investigated stability of approximate additive mappings $f : \mathbb{Q}_p \to \mathbb{R}$. They showed that if $f : \mathbb{Q}_p \to \mathbb{R}$ is a continuous mapping for which there exists a fixed ϵ such that $|f(x + y) - f(x) - f(y)| \le \epsilon$ for all $x, y \in \mathbb{Q}_p$, then there exists a unique additive mapping $T : Q_p \to \mathbb{R}$ such that $|f(x) - T(x)| \le \epsilon$ for all $x \in \mathbb{Q}_p$. In 2007, Moslehian and Rassias [18] proved the Hyers-Ulam stability of functional equations in complete non-Archimedean normed spaces. Then a lot of papers on the stability of other many equations have been published in non-Archimedean normed spaces ([3, 4, 16, 17, 19, 23]).

Now, we consider the following nonic functional equation

$$\begin{aligned} f(x + 5y) - 9f(x + 4y) + 36f(x + 3y) - 84f(x + 2y) + 126f(x + y) - 126f(x) \\ + 84f(x - y) - 36f(x - 2y) + 9f(x - 3y) - f(x - 4y) = 9!f(y). \end{aligned} \tag{1.1}$$

Then the function $f(x) = x^9$ is a solution of the functional equation (1.1). Thus, the functional equation (1.1) is called the *nonic functional equation* and every solution of the nonic functional equation is said to be a *nonic mapping*.

In this paper, using the direct and fixed point methods, we prove the generalized Hyers-Ulam stability of (1.1) in non-Archimedean normed spaces.

2. Properties of the nonic functional equation (1.1)

In this section, we investigate the properties and the general solution of the nonic functional equation (1.1).

Theorem 2.1. *Let X and Y be linear spaces. The nonic functional equation (1.1) has the following properties:*
(i) $f(0) = 0$.
(ii) $f(-x) = -f(x)$.
(iii) $f(2x) = 2^9 f(x)$.

Proof. (i) Replacing $x = 0, y = 0$ in (1.1), we get $f(0) = 0$.

(ii) Replacing $x = 0, y = x$ and $x = x, y = -x$ in (1.1) and adding the two resulting equations, we get $f(-x) = -f(x)$.

(iii) Replacing (x, y) with $(0, 2x)$ and $(5x, x)$ in (1.1), respectively, we get

$$f(10x) - 8f(8x) + 27f(6x) - 48f(4x) - 362838f(2x) = 0. \tag{2.1}$$

$$\begin{aligned} &f(10x) - 9f(9x) + 36f(8x) - 84f(7x) + 126f(6x) - 126f(5x) \\ &+ 84f(4x) - 36f(3x) + 9f(2x) - 362881f(x) = 0. \end{aligned} \tag{2.2}$$

Subtracting the equations (2.1) and (2.2), we find

$$\begin{aligned} &9f(9x) - 44f(8x) + 84f(7x) - 99f(6x) + 126f(5x) - 132f(4x) \\ &+ 36f(3x) - 362847f(2x) + 362881f(x) = 0. \end{aligned} \tag{2.3}$$

Replacing (x, y) with $(4x, x)$ in (1.1) and multiplying the resulting equation by 9, we get

$$\begin{aligned} &9f(9x) - 81f(8x) + 324f(7x) - 756f(6x) + 1134f(5x) - 1134f(4x) \\ &+ 756f(3x) - 324f(2x) - 3265839f(x) = 0. \end{aligned} \tag{2.4}$$

Subtracting the equations (2.3) and (2.4), we get

$$\begin{aligned} &37f(8x) - 240f(7x) + 657f(6x) - 1008f(5x) + 1002f(4x) \\ &- 720f(3x) - 362523f(2x) + 3628720f(x) = 0. \end{aligned} \tag{2.5}$$

Replacing (x, y) with $(3x, x)$ in (1.1) and multiplying the resulting equation by 37, we find

$$\begin{aligned} &37f(8x) - 333f(7x) + 1332f(6x) - 3108f(5x) + 4662f(4x) \\ &- 4662f(3x) + 3108f(2x) - 13427855f(x) = 0. \end{aligned} \tag{2.6}$$

Subtracting the equations (2.5) and (2.6), we arrive at

$$\begin{aligned} &93f(7x) - 675f(6x) + 2100f(5x) - 3660f(4x) \\ &+ 3942f(3x) - 365631f(2x) + 17056575f(x) = 0. \end{aligned} \tag{2.7}$$

Replacing (x, y) with $(2x, x)$ in (1.1) and multiplying the resulting equation by 93, we get

$$\begin{aligned} &93f(7x) - 837f(6x) + 3348f(5x) - 7812f(4x) + 11718f(3x) \\ &- 11625f(2x) - 33740865f(x) = 0. \end{aligned} \tag{2.8}$$

Subtracting the equations (2.7) and (2.8) and then dividing by 2, we obtain

$$\begin{aligned} &81f(6x) - 624f(5x) + 2076f(4x) - 3888f(3x) \\ &- 177003f(2x) + 25398720f(x) = 0. \end{aligned} \tag{2.9}$$

Replacing (x, y) with (x, x) in (1.1) and multiplying the resulting equation by 81, we get

$$\begin{aligned} &81f(6x) - 729f(5x) + 2916f(4x) - 6723f(3x) \\ &+ 9477f(2x) - 29400570f(x) = 0. \end{aligned} \tag{2.10}$$

Subtracting the equations (2.9) and (2.10), we arrive at

$$105f(5x) - 840f(4x) + 2835f(3x) - 186480f(2x) + 54799290f(x) = 0. \tag{2.11}$$

Replacing (x, y) with $(0, x)$ in (1.1) and multiplying the resulting equation by 105, we get

$$105f(5x) - 840f(4x) + 2835f(3x) - 5040f(2x) - 38097990f(x) = 0. \tag{2.12}$$

Subtracting the equations (2.11) and (2.12), we can obtain

$$-181440f(2x) + 92897280f(x) = 0,$$

which gives

$$f(2x) = 2^9 f(x). \tag{2.13}$$

This completes the proof. □

A function $A : X \to X$ is said to be *additive* if $A(x + y) = A(x) + A(y)$ for all $x, y \in X$. Let $n \in \mathbb{Z}^+$. A function $A_n : X^n \to Y$ is called *n-additive* if it is additive in each of its variables. A function A_n is called *symmetric* if $A_n(x_1, ..., x_n) = A_n(x_{\pi(1)}, ..., x_{\pi(n)})$ for every permutation $\{\pi(1), ..., \pi(n)\}$ of $1, ..., n$. If $A_n(x_1, ..., x_n)$ is an n-additive symmetric map, then $A_n(x_1, ...x_n)$, its diagonal is the function $A_n(x, ..., x)$ for $x \in X$ and denoted by $A^n(x)$. Evidently, $A^n(rx) = r^n A^n(x)$ whenever $x \in X$ and $r \in \mathbb{Q}$. Further the resulting function after substitution $x_1 = \cdots = x_l = x$ and $x_{l+1} = \cdots = x_n = y$ in $A_n(x_1, ..., x_n)$ will be denoted by $A^{l,n-l}(x, y)$.

A function $p : X \to Y$ is called a *generalized polynomial function of degree n* provided that there exist $A^n(x) = A^0 \in Y$ and i-additive symmetric function $A_i : X \to Y$ $(1 \le i \le n)$ such that

$$p(x) = \sum_{i=0} A^i(x)$$

for all $x \in X$ and $A^n \neq 0$. Let $f : X \to Y$. The difference operator Δ_h defined as follows:

$$\Delta_h f(x) = f(x + h) - f(x)$$

for $\in X$. In fact, a difference operator can be extended to an n-order difference operator in the usual composition way by induction. For each $h \in X$ and $n \in \mathbb{Z}^+ \cup \{0\}$, define

$$\Delta_h^{n+1} f(x) = \Delta_h \circ \Delta_h^n f(x)$$

with the convention $\Delta_h^0 f(x) = f(x)$ and $\Delta_h^1 f(x) = \Delta_h f(x)$. Furthermore, a more general difference operator which was used in the Fréchet functional equation, can be defined as

$$\Delta_{h_1,...,h_{n+1}} f(x) = \Delta_{n+1} \circ \cdots \circ \Delta_{h_1} f(x),$$

where $x, h_1, ..., h_{n+1} \in X$.

Lemma 2.2 (see [29]). *Let G be a commutative semigroup with identity, S be a symmetric group and n be a non-negative integer. Assume that the multiplication by $n!$ is bijective on S, that is, for every $b \in G$, the equation $n!a = b$ has a unique solution in S. Then the function $f : G \to S$ is a solution of Fréchet functional equation*

$$\Delta_{x_1,...,x_{n+1}} f(x_0) = 0$$

for all $x_0, x_1, ..., x_{n+1} \in G$ if and only if f is a polynomial of degree at most n.

Theorem 2.3. *A function $f : X \to Y$ is a solution of the functional equation (1.1) if and only if f is of the form $f(x) = A^9(x)$ for all $x \in X$, where $A^9(x)$ is the diagonal of the 9-additive symmetric map $A^9 : X^9 \to Y$.*

Proof. We can rewrite the functional equation (1.1) in the form

$$\begin{aligned} f(x) = &\frac{1}{126}\big[f(x+5y) - f(x-4y)\big] - \frac{1}{14}\big[f(x+4y) - f(x-3y)\big] \\ &+ \frac{2}{7}\big[f(x+3y) - f(x-2y)\big] - \frac{2}{3}\big[f(x+2y) - f(x-y)\big] + f(x+y) - 2880f(y). \end{aligned} \tag{2.14}$$

It follows from Lemma 2.2 that f is a generalized polynomial function of degree at most 9 and so f is of the form

$$f(x) = \sum_{n=0}^{9} A^n(x) \tag{2.15}$$

for all $x \in X$, where $A^0(x) = A^0$ is an arbitrary element of Y and $A^n(x)$ is the diagonal of the n-additive symmetric map $A^n : X^n \to Y$ for $n = 1, 2, ..., 9$. Since $f(0) = 0$ and f is odd, we have $A^0(x) = 0$ and $A^2(x) = A^4(x) = A^6(x) = A^8(x) = 0$. Thus, the expression (2.15) can be

$$f(x) = A^9(x) + A^7(x) + A^5(x) + A^3(x) + A^1(x)$$

for all $x \in X$. Since $f(2x) = 2^9 f(x)$ and $A^n(rx) = r^n A^n(x)$ whenever $x \in X$ and $r \in \mathbb{Q}$, we obtain

$$2^9 \left(A^7(x) + A^5(x) + A^3(x) + A^1(x)\right) = 2^7 A^7(x) + 2^5 A^5(x) + 2^3 A^3(x) + 2A^1(x).$$

Hence, we get $A^1(x) = -\frac{1}{255}(192A^7(x) + 240A^5(x) + 252A^3(x))$ and so $A^7(x) = A^5(x) = A^3(x) = A^1(x) = 0$ for all $x \in X$. Therefore, we obtain that $f(x) = A^9(x)$ for all $x \in X$.

Conversely, assume that $f(x) = A^9(x)$ for all $x \in X$, where $A^9(x)$ is the diagonal of the 9-additive symmetric map $A^9 : X^9 \to Y$. According to the definition of additive function, we get

$$A^n(x+y) = A^n(x) + A^n(y) + nA^{n-1,1}(x,y) + \frac{n(n-1)}{2}A^{n-2,2}(x,y) + \cdots + nA^{1,n-1}(x,y),$$

$A^n(rx) = r^n A^n(x)$ for all $n = 1, 2, ..., 9$ and $A^{s,t}(x, ry) = r^t A^{s,t}(x,y)$ $(s,t = 1, 2, ..., 8, s+t = 9)$ whenever $x, y \in X$ and $r \in \mathbb{Q}$. Letting the above equalities into (2.14), f satisfies the functional equation (1.1). This completes the proof of Theorem 2.3. □

3. Stability of nonic functional equations

In this section, we prove the generalized Hyers-Ulam stability of the nonic functional equation (1.1) in non-Archimedean normed spaces using the direct and fixed point methods. Throughout this section, we define the difference operator

$$\begin{aligned} \Delta f(x,y) =& f(x+5y) - 9f(x+4y) + 36f(x+3y) - 84f(x+2y) + 126f(x+y) - 126f(x) \\ &+ 84f(x-y) - 36f(x-2y) + 9f(x-3y) - f(x-4y) - 9!f(y). \end{aligned}$$

3.1. Direct method

Theorem 3.1. *Let X be an additive semigroup and Y be a non-Archimedean Banach space. Let $\varphi : X \times X \to [0, \infty)$ be a mapping such that*

$$\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y)}{|2^9|^n} = 0 \tag{3.1}$$

for all $x, y \in X$, and the limit

$$\widetilde{\varphi}_d(x) \overset{\text{def}}{\equiv} \lim_{n\to\infty} \max\left\{ \frac{\widetilde{\varphi}(2^j x)}{|2^9|^j} : 0 \le j < n \right\} \tag{3.2}$$

exists for all $x \in X$, where

$$\widetilde{\varphi}(x) = \frac{1}{|9!|} \max\{\varphi(0, 2x), \varphi(5x, x), |9|\varphi(4x, x), |37|\varphi(3x, x), |93|\varphi(2x, x), |162|\varphi(x, x), |210|\varphi(0, x)\}$$

for all $x \in X$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies

$$\|\Delta f(x, y)\| \le \varphi(x, y) \tag{3.3}$$

for all $x, y \in G$. Then there exists a nonic mapping $\mathcal{N} : X \to Y$ such that

$$\|f(x) - \mathcal{N}(x)\| \le \frac{1}{|2^9|} \widetilde{\varphi}_d(x) \tag{3.4}$$

for all $x \in X$. Moreover, if

$$\lim_{j\to\infty} \lim_{n\to\infty} \max\left\{ \frac{1}{|2^9|^k} \widetilde{\varphi}(2^k x) : j \le k < n + j \right\} = 0 \tag{3.5}$$

for all $x \in X$, then $\mathcal{N}$ is the unique nonic mapping satisfying (3.4).

Proof. Replacing (x, y) with $(0, 2x)$ in (3.3), we get

$$\|f(10x) - 8f(8x) + 27f(6x) - 48f(4x) - 362838f(2x)\| \le \varphi(0, 2x) \tag{3.6}$$

Replacing (x, y) with $(5x, x)$ in (3.3), we have

$$\begin{aligned} &\|f(10x) - 9f(9x) + 36f(8x) - 84f(7x) + 126f(6x) - 126f(5x) \\ &\quad + 84f(4x) - 36f(3x) + 9f(2x) - 362881f(x)\| \le \varphi(5x, x) \end{aligned} \tag{3.7}$$

for all $x \in X$. Subtracting (3.6) and (3.7), then

$$\begin{aligned} &\|9f(9x) - 44f(8x) + 84f(7x) - 99f(6x) + 126f(5x) - 132f(4x) \\ &\quad + 36f(3x) - 362847f(2x) + 362881f(x)\| \\ &\le \max\{\varphi(0, 2x), \varphi(5x, x)\} \end{aligned} \tag{3.8}$$

for all $x \in X$. Replacing (x, y) with $(4x, x)$ in (3.3), we get

$$\begin{aligned} &\|9f(9x) - 81f(8x) + 324f(7x) - 756f(6x) + 1134f(5x) - 1134f(4x) \\ &\quad + 756f(3x) - 324f(2x) - 3265839f(x)\| \le |9|\varphi(4x, x) \end{aligned} \tag{3.9}$$

for all $x \in X$. Subtracting (3.8) and (3.9), we get

$$\begin{aligned} &\|37f(8x) - 240f(7x) + 657f(6x) - 1008f(5x) + 1002f(4x) \\ &- 720f(3x) - 362523f(2x) + 3628720f(x)\| \\ &\le \max\{\varphi(0, 2x), \varphi(5x, x), |9|\varphi(4x, x)\} \end{aligned} \tag{3.10}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 24, NO.6, 2018, COPYRIGHT 2018 EUDOXUS PRESS, LLC

for all $x \in X$. Replacing (x, y) with $(3x, x)$ in (3.3), we have

$$\begin{aligned} &\|37f(8x) - 333f(7x) + 1332f(6x) - 3108f(5x) + 4662f(4x) \\ &\quad - 4662f(3x) + 3108f(2x) - 13427855f(x)\| \le |37|\varphi(3x, x) \end{aligned} \tag{3.11}$$

for all $x \in X$. Subtracting (3.10) and (3.11), we find

$$\begin{aligned} &\|93f(7x) - 675f(6x) + 2100f(5x) - 3660f(4x) + 3942f(3x) - 365631f(2x) + 17056575f(x)\| \\ &\le \max\{\varphi(0, 2x), \varphi(5x, x)), |9|\varphi(4x, x), |37|\varphi(3x, x)\} \end{aligned} \tag{3.12}$$

for all $x \in X$. Replacing (x, y) with $(2x, x)$ in (3.3), we get

$$\begin{aligned} &\|93f(7x) - 837f(6x) + 3348f(5x) - 7812f(4x) + 11718f(3x) \\ &\quad - 11625f(2x) - 33740865f(x)\| \le |93|\varphi(2x, x) \end{aligned} \tag{3.13}$$

for all $x \in X$. Subtracting (3.12) and (3.13), we obtain

$$\begin{aligned} &\|81f(6x) - 624f(5x) + 2076f(4x) - 3888f(3x) - 177003f(2x) + 25398720f(x)\| \\ &\le \frac{1}{|2|} \max\Big\{\varphi(0, 2x), \varphi(5x, x), |9|\varphi(4x, x), |37|\varphi(3x, x), |93|\varphi(2x, x)\Big\} \end{aligned} \tag{3.14}$$

for all $x \in X$. Replacing (x, y) with (x, x) in (3.3), we get

$$\begin{aligned} &\|81f(6x) - 729f(5x) + 2916f(4x) - 6723f(3x) \\ &\quad + 9477f(2x) - 29400570f(x)\| \le |81|\varphi(x, x) \end{aligned} \tag{3.15}$$

for all $x \in X$. Subtracting (3.14) and (3.15), we find

$$\begin{aligned} &\|105f(5x) - 840f(4x) + 2835f(3x) - 186480f(2x) + 54799290f(x)\| \\ &\le \frac{1}{|2|} \max\Big\{\varphi(0, 2x), \varphi(5x, x), |9|\varphi(4x, x), |37|\varphi(3x, x), |93|\varphi(2x, x), |162|\varphi(x, x)\Big\} \end{aligned} \tag{3.16}$$

for all $x \in X$. Replacing (x, y) with $(0, x)$ in (3.16), we get

$$\|105f(5x) - 840f(4x) + 2835f(3x) - 5040f(2x) - 38097990f(x)\| \le |105|\varphi(0, x) \tag{3.17}$$

for all $x \in X$. Subtracting (3.16) and (3.17), we get

$$\begin{aligned} &\|181440f(2x) - 9289280f(x)\| \\ &\le \frac{1}{|2|} \max\Big\{\varphi(0, 2x), \varphi(5x, x)), |9|\varphi(4x, x), |37|\varphi(3x, x), |93|\varphi(2x, x), |162|\varphi(x, x), |210|\varphi(0, x)\Big\} \end{aligned}$$

for all $x \in X$. Thus, we deduce that

$$\begin{aligned} &\|f(2x) - 2^9 f(x)\| \\ &\le \frac{1}{|9!|} \max\Big\{\varphi(0, 2x), \varphi(5x, x), |9|\varphi(4x, x), |37|\varphi(3x, x), |93|\varphi(2x, x), \\ &\qquad |162|\varphi(x, x), |210|\varphi(0, x)\Big\} \\ &\stackrel{\text{def}}{\equiv} \widetilde{\varphi}(x) \end{aligned} \tag{3.18}$$

for all $x \in X$. It follows from (3.18), we have

$$\Big\|\frac{f(2x)}{2^9} - f(x)\Big\| \le \frac{1}{|2^9|}\widetilde{\varphi}(x) \tag{3.19}$$

for all $x \in X$. Replacing x by $2^n x$ in (3.19), then we get

$$\Big\| \frac{f(2^{n+1}x)}{2^{9(n+1)}} - \frac{f(2^n x)}{2^{9n}} \Big\| \le \frac{1}{|2^9|^{n+1}} \widetilde{\varphi}(2^n x) \tag{3.20}$$

for all $x \in X$. From (3.20) and (3.1) the sequence $\{\frac{f(2^n x)}{2^{9n}}\}$ is a Cauchy sequence. Since X is a non-Archimedean Banach space, $\{\frac{f(2^n x)}{2^{9n}}\}$ is convergent. So, we can define the mapping $\mathcal{N} : X \to Y$ by $\mathcal{N}(x) = \lim_{n\to\infty}\{\frac{f(2^n x)}{2^{9n}}\}$ for all $x \in X$. By induction, we can show that

$$\begin{aligned} \Big\| \frac{f(2^n x)}{2^{9n}} - f(x) \Big\| &= \Big\| \sum_{j=0}^{n-1} \frac{f(2^{j+1}x)}{2^{9(j+1)}} - \frac{f(2^j x)}{2^{9j}} \Big\| \\ &\le \frac{1}{|2^9|} \max\Big\{ \frac{\widetilde{\varphi}(2^j x)}{|2^9|^j} : 0 \le j < n \Big\} \end{aligned} \tag{3.21}$$

for all $x \in X$ and $n \in \mathbb{N}$. By taking $n \to \infty$ in (3.21) and using (3.2), we obtain the desired inequality (3.4).

It follows from (3.1) and (3.3) that

$$\|\Delta\mathcal{N}(x,y)\| \le \lim_{n\to\infty} \frac{1}{|2^9|^n} \|\Delta f(2^n x, 2^n y)\| \le \lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y)}{|2^9|^n} = 0$$

for all $x, y \in X$. Then the mapping $\mathcal{N} : X \to Y$ satisfies (1.1), that is, $\mathcal{N}$ is the nonic mapping.

To prove the uniqueness property of $\mathcal{N}$, let $\mathcal{N}'$ be another nonic mapping satisfying (3.4). Then

$$\begin{aligned} \|\mathcal{N}(x) - \mathcal{N}'(x)\| &= \lim_{j\to\infty} \frac{1}{|2^9|^j} \|\mathcal{N}(2^j x) - \mathcal{N}'(2^j x)\| \\ &\le \lim_{j\to\infty} \frac{1}{|2^9|^j} \max\{\|\mathcal{N}(2^j x) - f(2^j x)\|, \|f(2^j x) - \mathcal{N}'(2^j x)\|\} \\ &\le \frac{1}{|2^9|} \lim_{j\to\infty} \lim_{n\to\infty} \max\Big\{ \frac{\widetilde{\varphi}(2^k x)}{|2^9|^k} : j \le k < n+j \Big\} \end{aligned}$$

for all $x \in X$. It follows from

$$\lim_{j\to\infty} \lim_{n\to\infty} \max\Big\{ \frac{\widetilde{\varphi}(2^k x)}{|2^9|^k} : j \le k < n+j \Big\} = 0$$

for all $x \in X$ that $\mathcal{N} = \mathcal{N}'$. This completes the proof.

Corollary 3.2. *Let $\rho : [0,\infty) \to [0,\infty)$ be a mapping satisfying*

$$\rho(|2|t) \le \rho(|2|)\rho(t), \qquad \rho(|2|) < |2|^9$$

for all $t \ge 0$. Let $\theta > 0$ be real number and $f : X \to Y$ be a mapping with $f(0) = 0$ satisfying

$$\|\Delta f(x,y)\| \le \theta(\rho(|x|) + \rho(|y|))$$

for all $x, y \in X$. Then there exists a unique nonic mapping $\mathcal{N} : X \to Y$ such that

$$\|f(x) - \mathcal{N}(x)\| \le \frac{\theta}{|2^9|} \widetilde{\varphi}_d(x) \tag{3.22}$$

for all $x \in X$, where

$$\begin{aligned} \widetilde{\varphi}_d(x) = \frac{\rho(|x|)}{|9!|} \max\Big\{ &\rho(|2|), (\rho(|5|)+1), |9|(\rho(|4|)+1), \\ &|37|(\rho(|3|)+1), |93|(\rho(|2|)+1), |324|, |210| \Big\}. \end{aligned}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 24, NO.6, 2018, COPYRIGHT 2018 EUDOXUS PRESS, LLC

Proof. If a mapping $\varphi : X \times X \to [0, \infty)$ is defined by $\varphi(x, y) = \theta(\rho(|x|) + \rho(|y|))$, then we would have

$$\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y)}{|2^9|^n} = \lim_{n\to\infty} \left(\frac{\rho(|2|)}{|2^9|}\right)^n \varphi(x, y) = 0$$

for all $x, y \in X$ since $\rho(|2|) < |2|^9$, and

$$\widetilde{\varphi}_d(x) = \lim_{n\to\infty} \max\left\{\frac{\widetilde{\varphi}(2^j x)}{|2^9|^j} : 0 \le j < n\right\} = \widetilde{\varphi}(x)$$

exists for all $x \in X$, where

$$\begin{aligned}\widetilde{\varphi}(x) =& \frac{\rho(|x|)}{|9!|} \max\Big\{\rho(|2|), (\rho(|5|) + 1), |9|(\rho(|4|) + 1),\\ & |37|(\rho(|3|) + 1), |93|(\rho(|2|) + 1), |324|, |210|\Big\}\end{aligned}$$

for all $x \in X$. Also, we get

$$\lim_{j\to\infty} \lim_{n\to\infty} \max\left\{\frac{1}{|2|^{9k}} \widetilde{\varphi}(2^k x) : j \le k < n + j\right\} = \lim_{j\to\infty} \frac{1}{|2|^{9j}} \widetilde{\varphi}(2^j x) = 0$$

for all $x \in X$. By applying Theorem 3.1, we obtain the desired result. This completes the proof. □

3.2. Fixed point method

Next, we investigate the generalized Hyers-Ulam stability of the nonic functional equation (1.1) in non-Archimedean normed spaces using fixed point method. We recall the following fixed point theorem which was proved by Diaz and Margolis.

Theorem 3.3. [7] *Let (Ω, d) be a complete generalized metric space and $T : \Omega \to \Omega$ be a strictly contractive mapping with Lipshitz constant L. Then, for any $x \in \Omega$, either $d(T^n x, T^{n+1} x) = \infty$ for all nonnegative integers $n \ge 0$ or other exists a natural number n_o such that*

(i) $d(T^n x, T^{n+1} x) < \infty$ for all $n \ge n_o$;
(ii) the sequence $\{T^n x\}$ is convergent to a fixed point $y*$ of T;
(iii) $y*$ is the unique fixed point of T in the set $\Omega_1 = \{y \in \Omega : d(T^{n_o} x, y) < \infty\}$;
(iv) $d(y, y*) \le \frac{1}{1-L} d(y, Ty)$ for all $y \in \Omega_1$.

Theorem 3.4. *Assume that X is a non-Archimedean normed space and Y is a a non-Archimedean Banach space. Let $\varphi : X \times X \to [0, \infty)$ be a mapping such that there exists a constant $0 < L < 1$ with*

$$\varphi(2x, 2y) \le |2^9| L \varphi(x, y) \tag{3.23}$$

and $\lim_{n\to\infty} \frac{1}{2^{9n}} \varphi(2^n x, 2^n y) = 0$ for all $x, y \in X$. If $f : X \to Y$ is a mapping with $f(0) = 0$ satisfying

$$\|\Delta f(x, y)\| \le \varphi(x, y) \tag{3.24}$$

for all $x, y \in X$, then there exists a unique nonic mapping $\mathcal{N} : X \to Y$ such that

$$\|f(x) - \mathcal{N}(x)\| \le \frac{1}{|2^9|(1 - L)} \widetilde{\varphi}(x) \tag{3.25}$$

for all $x \in X$, where

$$\widetilde{\varphi}(x) = \frac{1}{|9!|} \max\Big\{\varphi(0,2x), \varphi(5x,x), |9|\varphi(4x,x), |37|\varphi(3x,x), |93|\varphi(2x,x), |162|\varphi(x,x), |210|\varphi(0,x)\Big\}.$$

Proof. From (3.18) we obtain

$$\Big\|f(x) - \frac{f(2x)}{2^9}\Big\| \le \frac{1}{|2^9|}\widetilde{\varphi}(x) \tag{3.26}$$

for all $x \in X$. Let Ω be a set of all mappings from X into Y and a generalized metric d on Ω as follows:

$$d(g,h) = \inf\{\alpha \in [0,\infty) : \|g(x) - h(x)\| \le \alpha\widetilde{\varphi}(x),\ \forall\, x \in X\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (Ω, d) is a generalized complete metric space.

Now, we consider the linear mapping $T : \Omega \to \Omega$ defined by

$$Tg(x) = \frac{g(2x)}{2^9}$$

for all $x \in X$. Let $f, g \in \Omega$ and $\alpha \in [0,\infty)$ be an arbitrary constant with $d(f,g) \le \alpha$. Then $\|f(x) - g(x)\| \le \alpha\widetilde{\varphi}(x)$ for all $x \in X$ and so

$$\|Tf(x) - Tg(x)\| \le \frac{1}{|2^9|}\|f(2x) - g(2x)\| \le \frac{\alpha}{|2^9|}\widetilde{\varphi}(2x) \le \alpha L\widetilde{\varphi}(x)$$

for all $x \in X$. Thus, $d(f,g) \le \alpha$ implies that $d(Tf,Tg) \le L\alpha$. This means that

$$d(Tf,Tg) \le Ld(f,g)$$

for all $f, g \in \Omega$. It follows from (3.18) that $d(Tf,f) \le \frac{1}{|2^9|}$ for all $x \in X$. From the conditions (2) and (3) of Theorem 3.3, there exists a mapping $\mathcal{N} : X \to Y$, which is a unique fixed point of T in the set $\Omega_1 = \{g \in \Omega : d(f,g) < \infty\}$, such that

$$\mathcal{N}(x) = \lim_{n\to\infty} \frac{1}{2^{9n}} f(2^n x)$$

for all $x \in X$ since $\lim_{n\to\infty} d(T^n f, \mathcal{N}) = 0$. Again, from the condition (4) of Theorem 3.3, we obtain

$$d(f,\mathcal{N}) \le \frac{1}{1-L} d(Tf,f) \le \frac{1}{|2^9|(1-L)},$$

which implies the inequality (3.25).

If we replace x by $2^n x$ and y by $2^n y$ in (3.3), then we obtain

$$\Big\|\frac{\Delta f(2^n x, 2^n y)}{2^{9n}}\Big\| \le \frac{1}{|2^9|^n}\|\Delta f(2^n x, 2^n y)\| \le \frac{1}{|2^9|^n}\varphi(2^n x, 2^n y) \tag{3.27}$$

for all $x, y \in X$. Taking as the limit $n \to \infty$ in (3.27), we deduce that $\Delta\mathcal{N}(x,y) = 0$ for all $x, y \in X$. That is, the function $\mathcal{N} : X \to Y$ is nonic, as desired. This completes the proof. □

Corollary 3.5. *Let X be a non-Archimedean normed space over $\mathbb{K}$ and Y be a non-Archimedean Banach space. Let $\theta > 0$ be real number and $f : X \to Y$ be a mapping with $f(0) = 0$ satisfying*

$$\|\Delta f(x,y)\| \le \theta$$

for all $x, y \in X$. Then there exists a unique nonic mapping $\mathcal{N} : X \to Y$ such that

$$\|f(x) - \mathcal{N}(x)\| \le \frac{\theta}{|2240|(|2^9| - |2^p|)}. \tag{3.28}$$

Corollary 3.6. *Let $\theta, p > 0$ be real numbers with $0 \le p < 9$ and $f : X \to Y$ be a mapping with $f(0) = 0$ satisfying*

$$\|\Delta f(x,y)\| \le \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then there exists a unique nonic mapping $\mathcal{N} : X \to Y$ such that

$$\|f(x) - \mathcal{N}(x)\| \le \frac{\theta M_p}{|2^9| - |2^p|}\|x\|^p \tag{3.29}$$

for all $x \in X$, where

$$M_p = \frac{1}{|9!|}\max\Big\{|2|^p, |5|^p + 1, |9|(|4|^p + 1), |37|(|3|^p + 1), |93|(|2|^p + 1), |324|, |210|\Big\}.$$

Corollary 3.7. *Let $\theta, p, q > 0$ be real numbers with $p + q < 9$ and $f : X \to Y$ be a mapping with $f(0) = 0$ satisfying*

$$\|\Delta f(x,y)\| \le \theta\|x\|^p\|y\|^q$$

for all $x, y \in X$. Then there exists a unique nonic mapping $\mathcal{N} : X \to Y$ such that

$$\|f(x) - N(x)\| \le \frac{\theta M_{p,q}}{|2^9 - 2^{p+q}|}\|x\|^{p+q} \tag{3.30}$$

for all $x \in X$, where

$$M_{p,q} = \frac{1}{|9!|}\max\Big\{|5|^p, |9||4|^p, |37||3|^p, |93||2|^p, |162|\Big\}$$

Remark. Let X be a normed space and Y be a Banach space in Theorem 3.5. Let $\varphi : X \times X \to [0, \infty)$ be a mapping such that there exists a constant $0 < L < 1$ with

$$\varphi(2x, 2y) \le 2^9 L\varphi(x, y)$$

for all $x, y \in X$. Suppose that $f : X \to Y$ be a mapping with $f(0) = 0$ satisfying $\|\Delta f(x, y)\| \le \varphi(x, y)$ for all $x, y \in X$. Using the fixed point method, we can show that there exists a unique nonic mapping $\mathcal{N} : X \to Y$ satisfying (3.24) and

$$\|f(x) - \mathcal{N}(x)\| \le \frac{1}{2^9(1 - L)}\widetilde{\varphi}(x) \tag{3.31}$$

where

$$\widetilde{\varphi}(x) = \frac{1}{9!}\Big(\varphi(0, 2x) + \varphi(5x, x) + 9\varphi(4x, x) + 37\varphi(3x, x) + 93\varphi(2x, x) + 162\varphi(x, x) + 210\varphi(0, x)\Big)$$

for all $x \in X$ (see [15]).

It is easy to show that the approximation in non-Archimedean normed space (see (3.25)) is better than the approximation in (Archimedean) normed space (see (3.31)).

References

[1] L.M. Arriola and W.A. Beyer, Stability of the Cauchy functional equation over p-adic fields, Real Analysis Exchange, 31 (2005/2006), 125-132.

[2] J. Brzdęk, J. Chudziak and Z. Páles, A fixed point approach to stability of functional equations, Nonlinear Analysis: Theory, Methods and Applications 74 (2011), 6728-6732.

[3] Y.J. Cho, C. Park, R. Saadati, Functional inequalities in non-Archimedean in Banach spaces, Appl. Math. Lett. 60 (2010), 1994-2002.

[4] K. Ciepliński, Stability of multi-additive mappings in non-Archimedean normed spaces, J. Math. Anal. Appl. 373 (2011), 376-383.

[5] L. Cădariu and V. Radu, Fixed points and the stability of Jensen's functional equation, J. Inequal. Pure Appl. Math. 4 (2003), 1-7.

[6] S. Czerwik, On the stability of the quadratic mapping in normed spaces, Abh. Math. Sem. Univ. Hamburg 62 (1992), 59-64.

[7] J.B. Dias and B. Margolis, A fixed point theorem of the alternative for contractions on a generalized complete metric space, Bull. Amer. Math. Soc. 74 (1968), 305-309.

[8] P. Găvruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, J. Math. Anal. Appl. 184 (1994), 431-436.

[9] M.E. Gordji, and M.B. Savadkouhi, Stability of cubic and quartic functional equations in non-Archimedean spaces, Acta Appl. Math. 110 (2010), 1321-1329.

[10] K. Hensel, Über eine neue Begründung der Theorie der algebraischen Zählen. Jahresber. Deutsch. Math. Verein, 6 (1897), 83-88.

[11] D. H. Hyers, On the stability of the linear functional equation, Proc. Nat. Acad. Sci. 27 (1941), 222-224.

[12] S.M. Jung, M.S. Moslehian and P.K. Sahoo, Stability of a generalized Jensen equation on restricted domains, J. Math. Inequal. 4 (2010), 191-206.

[13] H.A. Kenary, S.Y. Jang and C. Park, A fixed point approach to the Hyers-Ulam stability of a functional equation in various normed spaces, Fixed Point Th. Appl. 2011, 2011:67.

[14] A. Khrennikov, Non-Archimedean analysis: quantum paradoxes, dynamical systems and biological models, Math. Appl. 427 (1997), Zbl 0920.11087.

[15] S.S. Kim and J.M. Rassias, A fixed point approach to the stability of nonic functional equations, preprint.

[16] A.K. Mirmostafaee and M.S. Moslehian, Stability of additive mappings in non-Archimedean fuzzy normed spaces, Fuzzy Sets and Systems, 160 (2009), 1643-1652.

[17] M.S. Moslehian and G.H. Sadeghi, A Mazur-Ulam theorem in non-Archimedean normed spaces. Nonlinear Anal. 69 (2008), 3405-3408.

[18] M.S. Moslehian and Th.M. Rassias, Stability of functional equations in non-Archimedean spaces. Appl. Anal. Discrete Math. 1 (2007), 325-334.

[19] A. Najati and F. Moradlou, Hyers-Ulam-Rassias stability of the Apollonius type quadratic mapping in non-Archimedean spaces, Tamsui Oxf. J. Math. Sci. 24(2008), 367-380.

[20] V. Radu, The fixed point alternative and the stability of functional equations, Fixed Point Theory, 4 (2003), 91-96.

[21] J.M. Rassias, Solution of the Ulam stability problem for quartic mapping, Glasnik Math. 34 (1999), 243-252.

[22] Th. M. Rassias, On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. 72 (1978), 297-300.

[23] R. Saadati, Y.J. Cho and J. Vahidi, The stability of the quartic functional equation in various spaces, Comput. Math. Appl. 60 (2010), 1994-2002.

[24] Y. Shen and W. Chen, On the stability of septic and octic functional equations, J. Comp. Anal. Appl. 18(2015), 277-290.

[25] T. Trif, Hyers-Ulam-Rassias stability of a Jensen type functional equation, J. Math. Anal. Appl. 250 (2000), 579-588.

[26] S.M. Ulam, Problems in Modern Mathematics, Science Editions, John Wiley & Sons, New York, USA, 1960.

[27] V.S. Vladimirov, I.V. Volovich and E.I. Zelenov, p-adic Analysis and Mathematical Physics, World Scientific, 1994.

[28] A. Wiwatwanich and P. Nakmahachalasint, On the stability of a cubic functional equation, Thai J. Math. 6 (2008), 69-76.

[29] T.Z. Xu, J.M. Rassias, M.J. Rassias and W.X. Xu, A fixed point approach to the stability of quintic and sextic functional equations in quasi-β-normed spaces, J. Inequal. Appl. 2010, Article ID 423231, 23pp.

TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 24, NO. 6, 2018