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On generalized Fibonacci k-sequences and Fibonacci k-numbers

Hee Sik Kim, J. Neggers and Choonkil Park*

ABSTRACT. In this paper analogs of Fibonacci sequences and Fibonacci numbers as well as Fibonacci functions (the case $n = 2$) for cases $n = 3, 4, \cdots$ are introduced. It is shown that these analogs are related to each other in a regular manner and that if $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \varphi_k$ for a Fibonacci k-sequence, then $\varphi_1 = 1, \varphi_2 = \frac{1+\sqrt{5}}{2} < \varphi_3 < \cdots < \varphi_n < \cdots < \lim_{n\to\infty} \varphi_n = 2$. Many identities of types similar to those which hold for the case $n = 2$ (i.e., the Fibonacci case) are also established, indicating the existence of a larger theory of which the Fibonacci case is an integrated part.

1. Introduction

Fibonacci-numbers have been studied in many different forms for centuries and the literature on the subject is consequently incredibly vast (see [1, 3, 4, 9]). Han et al. [5] considered several properties of Fibonacci sequences in arbitrary groupoids. Kim et al. [7] introduced the notion of generalized Fibonacci sequences over a groupoid and discussed these in particular for the case where the groupoid contains idempotents and pre-idempotents.

In [6], Han et al. discussed Fibonacci functions on the real numbers $\mathbf{R}$, i.e., functions $f : \mathbf{R} \to \mathbf{R}$ such that for all $x \in \mathbf{R}$, $f(x + 2) = f(x + 1) + f(x)$, and developed the notion of Fibonacci functions using the concept of f-even and f-odd functions. Moreover, they showed that if f is a Fibonacci function then $\lim_{x\to\infty} \frac{f(x+1)}{f(x)} = \frac{1+\sqrt{5}}{2}$.

Kim et al. [7] discussed Fibonacci functions using the (ultimately) periodicity and they also discussed the exponential Fibonacci functions. Especially, given a non-negative real-valued function, they obtained examples of several classes of exponential Fibonacci functions.

In this paper we introduce the family of Fibonacci k-sequences, where $\{F_n\}_{n=0}^{\omega}$ is a Fibonacci k-sequence provided $F(n + k) = F(n + k - 1) + \cdots + F(n)$ for all $n \in \mathbf{N} = \{1, 2, \cdots\}$. Thus, if $k = 2$, then $F(n + 2) = F(n + 1) + F(n)$ and $\{F(n)\}_{n=0}^{\omega}$ is an ordinary Fibonacci sequence. Similarly, for $k = 3$, we obtain the formula $F(n+3) = F(n+2)+F(n+1)+F(n)$, where $F(0), F(1), F(2)$ may be taken as arbitrary elements of an abelian group A, usually taken to be the group of integers, rationals, real or complex numbers. If $F(0) = F(1) = F(2) = 1$, then we will consider this special case, i.e., $\{1, 1, 1, 3, 5, 9, 17, 31, \cdots\}$ the sequence of Fibonacci 3-numbers. The properties of this sequence can be expected to be analogous to those of the sequence $\{1, 1, 2, 3, 5, 8, \cdots\}$ of Fibonacci (i.e., Fibonacci 2-) numbers. Thus, e.g., $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \varphi_3 (\doteqdot 1.839)$ is an analogue of $\varphi_2 = \frac{1+\sqrt{5}}{2}$, the golden section. What this number φ_3 may mean in other settings is itself a question of interest as is the question of

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whether and where this number can be observed in nature. It turns out that the corresponding sequence $\varphi_1 = 1, \varphi_2 = \frac{1+\sqrt{5}}{2} < \varphi_3 < \cdots < \varphi_n < \cdots < \lim_{n\to\infty} \varphi_n = 2$ is itself a sequence of interest as we hope to show below as well.

From Fibonacci sequences to go to Fibonacci functions in a natural way has proven to be interesting and it has yields a theory of such functions [6, 8]. The second part of this paper is concerned with the introduction of Fibonacci k-functions, where a real-valued function $f : \mathbf{R} \to \mathbf{R}$ is a Fibonacci k-function if $f(x+k) = f(x+k-1) + f(x+k-2) + \cdots + f(x)$. Again, in analogy with the class of Fibonacci 2-functions (i.e., Fibonacci functions), we are able to construct many examples of such functions for $k = 3, 4, 5, \cdots$ etc., indicating that there is a great deal of material which has yet to be uncovered in this area. Nevertheless, it is clear that there is much work to be done, some of which we will discuss below in further detail.

2. Preliminaries

A function f defined on the real numbers is said to be a *Fibonacci function* ([5]) if it satisfies the formula

$$f(x+2) = f(x+1) + f(x) \tag{1}$$

for any $x \in \mathbf{R}$, where $\mathbf{R}$ (as usual) is the set of real numbers.

Example 2.1. ([6]) Let $f(x) := a^x$ be a Fibonacci function on $\mathbf{R}$ where $a > 0$. Then $a^x a^2 = f(x+2) = f(x+1) + f(x) = a^x(a+1)$. Since $a > 0$, we have $a^2 = a+1$ and $a = \frac{1+\sqrt{5}}{2}$. Hence $f(x) = (\frac{1+\sqrt{5}}{2})^x$ is a Fibonacci function, and the unique Fibonacci function of this type on $\mathbf{R}$.

Example 2.2. ([8]) (a). A function $f(x) = (x - \lfloor x \rfloor)(\frac{1+\sqrt{5}}{2})^x$ is a Fibonacci function.
(b). A function $f(x)$ defined by

$$f(x) = \begin{cases} (x - \lfloor x \rfloor)(\frac{1+\sqrt{5}}{2})^x & \text{if } x \in Q \\ -(x - \lfloor x \rfloor)(\frac{1+\sqrt{5}}{2})^x & \text{otherwise} \end{cases}$$

is a Fibonacci function.
(c). A function $f(x) = sin(\pi x)(\frac{\sqrt{5}-1}{2})^x$ is a Fibonacci function.

Proposition 2.3. ([5]) *If $f(x)$ is a Fibonacci function, then*

$$\lim_{x\to\infty} \frac{f(x+1)}{f(x)} = \frac{1+\sqrt{5}}{2}$$

3. Higher Fibonacci k-sequences

Let α, β, γ be non-zero integers. A sequence $\{F_n\}$ is said to be an (α, β)-*Fibonacci 2-sequence* if $F_1 = \alpha, F_2 = \beta$ and $F_{n+2} = F_{n+1} + F_n$ for $n = 1, 2, 3, \cdots$. A sequence $\{F_n\}$ is said to be an (α, β, γ)-*Fibonacci 3-sequence* if $F_1 = \alpha, F_2 = \beta, F_3 = \gamma$ and $F_{n+3} = F_{n+2} + F_{n+1} + F_n$ for $n = 1, 2, 3, \cdots$. Especially, if $\alpha = \beta = 1$ or $\alpha = \beta = \gamma = 1$, then we say $\{F_n\}$ a *Fibonacci 2-sequence* or a *Fibonacci 3-sequence* respectively.

Example 3.1. (a). It is well known that $\{1, 1, 2, 3, 5, 8, 13, \cdots\}$ is the Fibonacci 2-sequence.
(b). $\{1, 1, 1, 3, 5, 9, 17, 31, 57, 105, 193, 355, 653, \cdots\}$ is the Fibonacci 3-sequence.

We give some formulas for the Fibonacci 3-sequence as follows.

Proposition 3.2. *If $\{F_n\}$ is the Fibonacci 3-sequence, then*

$$F_n = \frac{1}{2}[(F_{n+3} - F_{n+1}) - (F_{n+2} - F_n)]. \tag{1}$$

Proof. Since $F_{n+3} = F_{n+2} + F_{n+1} + F_n$, we obtain $F_{n+3} - F_{n+1} = F_{n+2} + F_n$. It follows that

$$\begin{aligned} 2F_n &= (F_{n+2} + F_n) - (F_{n+2} - F_n) \\ &= (F_{n+3} - F_{n+1}) - (F_{n+2} - F_n) \end{aligned}$$

so that we obtain the equality (1). □

Proposition 3.3. *If $\{F_n\}$ is the Fibonacci 3-sequence, then*

$$F_{n+2} = \frac{1}{2}[(F_{n+3} - F_{n+1}) + (F_{n+2} - F_n)]. \tag{2}$$

Proof. Since $F_{n+3} = F_{n+2} + F_{n+1} + F_n$, we obtain $F_{n+3} - F_{n+1} = F_{n+2} + F_n$. It follows that

$$\begin{aligned} (F_{n+3} - F_{n+1}) + (F_{n+2} - F_n) &= (F_{n+2} + F_n) + (F_{n+2} - F_n) \\ &= 2F_{n+2} \end{aligned}$$

so that we obtain the equality (2). □

Proposition 3.4. *If $\{F_n\}$ is the Fibonacci 3-sequence, then*

$$F_n F_{n+2} = \frac{1}{4}[(F_{n+3} - F_{n+1})^2 - (F_{n+2} - F_n)^2]. \tag{3}$$

Proof. It follows that

$$\begin{aligned} 4F_n F_{n+2} &= F_{n+2}^2 + 2F_n F_{n+2} + F_n^2 - F_{n+2}^2 + 2F_n F_{n+2} - F_n^2 \\ &= (F_{n+2} + F_n)^2 - (F_{n+2} - F_n)^2 \\ &= (F_{n+3} - F_{n+1})^2 - (F_{n+2} - F_n)^2 \end{aligned}$$

so that we obtain the equality (3). □

Proposition 3.5. *If $\{F_n\}$ is the Fibonacci 3-sequence, then*

$$F_{n+3} = 2(F_1 + F_2 + \cdots + F_n) + F_{n+1}$$

where $F_0 = -1$.

Proof. Since $F_{n+3} = F_{n+2} + F_{n+1} + F_n$, we obtain $F_3 - F_2 = F_1 + F_0, F_4 - F_3 = F_2 + F_1, \cdots, F_{n+3} - F_{n+2} = F_{n+1} + F_n$, which proves the proposition. □

4. Solutions of a Fibonacci polynomial $\xi_n(x)$.

Given a natural number n, we define a polynomial $\xi_n(x)$ by

$$\xi_n(x) := x^n - x^{n-1} - x^{n-2} - \cdots - x - 1.$$

We call such a polynomial $\xi_n(x)$ a *Fibonacci polynomial.* Let φ_n be the largest real root of the equation $\xi_n(x) = 0$. Then $\xi_n(2) = 2^n - (2^{n-1} + 2^{n-2} + \cdots + 2 + 1) = 1$ and $\xi_n(1) = 1 - n < 0$ when $n > 1$. Let $x \geq 2$. Then $x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \cdots + x + 1) \geq x^{n-1} + x^{n-2} + \cdots + x + 1$. It follows that $\xi_n(x) = x^n - (x^{n-1} + x^{n-2} + \cdots + x + 1) \geq 1 = \xi_n(2)$. Hence we obtain $1 \leq \varphi_n \leq 2$ for all $n \in \mathbf{N}$. If $n = 1$, then $\xi_1(x) = x - 1$ and $\varphi_1 = 1$, and if $n = 2$, then $\xi_2(x) = x^2 - x - 1$ and $\varphi_2 = \frac{1+\sqrt{5}}{2}$.

Proposition 4.1. *Let $\{\varphi_n\}$ be the sequence of the largest roots of $\xi_n(x) = 0$. Then it is increasing, i.e., $\varphi_n < \varphi_{n+1}$ for all $n \in \mathbf{N}$ and $\lim_{n\to\infty} \varphi_n \leq 2$.*

Proof. Since φ_n is the largest real root of $\xi_n(x) = 0$ for all $n \in \mathbf{N}$, $\xi_{n-1}(\varphi_{n-1}) = 0$. It follows that

$$\varphi_{n-1}^{n-1} = \varphi_{n-1}^{n-2} + \varphi_{n-1}^{n-3} + \cdots + \varphi_{n-1} + 1. \tag{4}$$

If we let $x := \varphi_{n-1}$ in $\xi_n(x) = 0$, then by (4) we have

$$\begin{aligned} \xi_n(\varphi_{n-1}) &= \varphi_{n-1}^k - (\varphi_{n-1}^{n-1} + \varphi_{n-1}^{n-2} + \cdots + \varphi_{n-1} + 1 \\ &= \varphi_{n-1}^n - 2\varphi_{n-1}^{n-1} \\ &= \varphi_{n-1}^{n-1}(\varphi_{n-1} - 2) \\ &< 0. \end{aligned}$$

Since φ_n is the largest real root of $\xi_n(x) = 0$, we obtain $\varphi_{n-1} < \varphi_n$ for all $n \in \mathbf{Z}$ with $n \geq 2$. Since $\{\varphi_n\}$ is an increasing sequence and it is bounded above, we have $\lim_{n\to\infty} \varphi_n \leq 2$. □

Given a Fibonacci 3-sequence $\{F_n\}$, we consider the following sequence $\{\frac{F_{n+1}}{F_n}\}$:

$$\{\frac{F_{n+1}}{F_n}\} = \{1, 1, 3, 1.667, 1.8, 1.889, 1.832, 1.839, 1.842, 1.838, 1.839, 1.839, \cdots\}$$

Thus we expect that the limit $\lim_{n\to\infty} \frac{F_{n+1}}{F_n}$ is approximately 1.839.

Note that $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}$ for the Fibonacci 2-sequence.

A quick computation yields $\xi_3(1.839) = (1.839)^3 - (1.839)^2 - 1.839 - 1 = 0.002$, i.e., $F_3(1.839) = 0.002 \doteqdot 0$.

Since φ_3 is the largest real root of $\xi_3(x) = 0$, we obtain $\xi_3(x) = (x - \varphi_3)(x^2 + \alpha x + \beta)$ for some $\alpha, \beta \in \mathbf{R}$. It follows that

$$\xi_3(x) = x^3 + (\alpha - \varphi_3)x^2 + (\beta - \alpha\varphi_3)x - \varphi_3\beta$$

It follows that $\alpha - \varphi_3 = -1, \beta - \alpha\varphi_3 = -1$ and $-\varphi_3\beta = -1$. Hence $\beta = \alpha\varphi_3 - 1 = (\varphi_3 - 1)\varphi_3 - 1 = \varphi_3^2 - \varphi_3 - 1$ and $\alpha^2 - 4\beta = -3\varphi_3^2 + 2\varphi_3 + 5 \doteqdot -1.468 < 0$ if we let $\varphi_3 = 1.839$. This means that $x^2 + \alpha x + \beta = 0$ has imaginary roots. Let ρ_1, ρ_2 be roots of $x^2 + \alpha x + \beta = 0$. Then $\rho_i = -\frac{\alpha}{2} \pm i\sqrt{\beta - \frac{1}{4}\alpha^2}$ and $\|\rho_i\| = \frac{1}{\sqrt{\varphi_3}} < 1$. If we let $\tau := arg(-\frac{\alpha}{2} \pm i\sqrt{\beta - \frac{1}{4}\alpha^2})$, then $\rho_1 = \frac{1}{\sqrt{\varphi_3}} \exp^{i\tau}$ and $\rho_2 = \frac{1}{\sqrt{\varphi_3}} \exp^{-i\tau}$. If we let $\varphi_3 \doteqdot 1.839$, then $\varphi_3^2 = 3.382$ and hence $\beta = \varphi_3^2 - \varphi_3 - 1 \doteqdot 0.543$ and $\alpha \doteqdot 0.839$. Hence $x^3 - x^2 - x - 1 \doteqdot (x - 1.839)(x^2 + 0.839x + 0.543)$ and $\rho_i \doteqdot -0.420 \pm 0.606i$.

Now, we may assume the polynomial $x^3 - x^2 - x - 1 = 0$ is the characteristic equation of the linear operator ξ_n, the Fibonacci 3-sequence (as in a linear differential equation: $y''' - y'' - y' - y = 0$) which means that the roots φ_3, $\frac{1}{\sqrt{\varphi_3}}e^{\tau i}$, $\frac{1}{\sqrt{\varphi_3}}e^{-\tau i}$ provide the expression:

$$F_n = A\varphi_3^n + B(\frac{1}{\sqrt{\varphi_3}})^n e^{n\tau i} + C(\frac{1}{\sqrt{\varphi_3}})^n e^{-n\tau i} \tag{5}$$

for some $A, B, C \in \mathbf{R}$.

Theorem 4.2. *If F_n is of the form (5), then*

$$A = \frac{1}{\triangle}[\ \frac{1}{\varphi_3^2}(e^{2\tau i} - e^{-2\tau i})\ +\ \frac{1}{\varphi_3\sqrt{\varphi_3}}(1 + \frac{1}{\varphi_3})(e^{-\tau i} - e^{\tau i})\ ],$$

where $\triangle = \frac{1}{\varphi_3^2}(e^{2\tau i} - e^{-2\tau i}) + \frac{1}{\sqrt{\varphi_3}}(1 + \frac{1}{\varphi_3^3})(e^{-\tau i} - e^{\tau i})$.

Proof. If we take $n = 1, 2, 3$ in (5) respectively, then we have

$$1 = F_1 = A\varphi_3 + B\frac{1}{\sqrt{\varphi_3}}e^{\tau i} + C\frac{1}{\sqrt{\varphi_3}}e^{-\tau i},$$

$$1 = F_2 = A\varphi_3^2 + B(\frac{1}{\sqrt{\varphi_3}})^2 e^{2\tau i} + C(\frac{1}{\sqrt{\varphi_3}})^2 e^{-2\tau i},$$

$$1 = F_3 = A\varphi_3^3 + B(\frac{1}{\sqrt{\varphi_3}})^3 e^{3\tau i} + C(\frac{1}{\sqrt{\varphi_3}})^3 e^{-3\tau i}.$$

Using Cramer's rule we obtain

$$\triangle = \begin{vmatrix} \varphi_3 & \frac{1}{\sqrt{\varphi_3}}e^{\tau i} & \frac{1}{\sqrt{\varphi_3}}e^{-\tau i} \\ \varphi_3^2 & (\frac{1}{\sqrt{\varphi_3}})^2 e^{2\tau i} & (\frac{1}{\sqrt{\varphi_3}})^2 e^{-2\tau i} \\ \varphi_3^3 & (\frac{1}{\sqrt{\varphi_3}})^3 e^{3\tau i} & (\frac{1}{\sqrt{\varphi_3}})^3 e^{-3\tau i} \end{vmatrix}$$

which becomes

$$\triangle = \frac{1}{\varphi_3^2}(e^{2\tau i} - e^{-2\tau i})\ +\ \frac{1}{\sqrt{\varphi_3}}(1 + \frac{1}{\varphi_3^3})(e^{-\tau i} - e^{\tau i})$$

by simple computations. To obtain A, we change the first column into 1's in $\triangle$.

$$A = \frac{1}{\triangle}\begin{vmatrix} 1 & \frac{1}{\sqrt{\varphi_3}}e^{\tau i} & \frac{1}{\sqrt{\varphi_3}}e^{-\tau i} \\ 1 & (\frac{1}{\sqrt{\varphi_3}})^2 e^{2\tau i} & (\frac{1}{\sqrt{\varphi_3}})^2 e^{-2\tau i} \\ 1 & (\frac{1}{\sqrt{\varphi_3}})^3 e^{3\tau i} & (\frac{1}{\sqrt{\varphi_3}})^3 e^{-3\tau i} \end{vmatrix}$$

which becomes

$$A = \frac{1}{\triangle}[\ \frac{1}{\varphi_3^2}(e^{2\tau i} - e^{-2\tau i})\ +\ \frac{1}{\varphi_3\sqrt{\varphi_3}}(1 + \frac{1}{\varphi_3})(e^{-\tau i} - e^{\tau i}),\ ]$$

by simple computations. This proves the theorem. □

Note that we can obtain the coefficients B, C, but it is not necessary to find those. Since $\frac{1}{\sqrt{\varphi_3}} = \frac{1}{\sqrt{1+\alpha}}$, we obtain that $\lim_{n\to\infty}(\frac{1}{\sqrt{\varphi_3}})^n = 0$ and $\lim_{n\to\infty}(\frac{1}{\sqrt{\varphi_3}})^n[Be^{n\tau i} + Ce^{-n\tau i}] = 0$. Hence $\lim_{n\to\infty}\frac{F_n}{\varphi_3^n} = A$. Thus $\lim_{n\to\infty}\frac{F_{n+1}}{\varphi_3 F_n} = \lim_{n\to\infty}\frac{F_{n+1}/\varphi_3^{n+1}}{F_n/\varphi_3^n} = 1$. It follows that $\lim_{n\to\infty}\frac{F_{n+1}}{F_n} = \varphi_3$.

Example 4.3. To approximate A we may take an expression $A \doteqdot \frac{F_n}{\varphi_3^n}$ for a "sufficiently large n and a sufficiently accurate value of φ_3". Thus, if $n = 5$, then $F_5 = 9$ and $(1.839)^5 = 21.033$ and hence $A \doteqdot 0.428$. If $n = 7$, then $F_7 = 31, \varphi^7 = 71.132$ and hence $A \doteqdot \frac{F_7}{\varphi_3^7} \doteqdot 0.436$. If $n = 12$, then $F_{12} = 653, (1.836)^{12} = 1496.145$ and hence $A \doteqdot \frac{F_{12}}{(1.839)^{12}} = 0.436$.

5. Higher Fibonacci k-functions

Let $k \geq 2$ be an integer. A function f defined on the real numbers is said to be a *Fibonacci k-function* if it satisfies the formula

$$f(x+k) = f(x+k-1) + f(x+k-2) + \cdots + f(x+1) + f(x) \tag{2}$$

for any $x \in \mathbf{R}$, where $\mathbf{R}$ (as usual) is the set of real numbers.

Example 5.1. Let $f(x) := \varphi_3^x$. Then $f(x+3) = \varphi_3^{x+3} = \varphi_3^x\varphi_3^3$ and $f(x+2) + f(x+1) + f(x) = \varphi_3^{x+2} + \varphi_3^{x+1} + \varphi_3^x = \varphi_3^x[\varphi_3^2 + \varphi_3^2 + 1]$. Since $\varphi_3^3 = \varphi_3^2 + \varphi_3^2 + 1$, $f(x) = \varphi_3^x$ is a Fibonacci 3-function.

In a similar manner, we know that $g(x) = \varphi_2^x$ is a Fibonacci 2-function where $\varphi_2 = \frac{1+\sqrt{5}}{2}$. Similarly, $h(x) = \varphi_k^x$ defines a Fibonacci k-function.

Example 5.2. Let $\varphi_k := \omega\theta$ and $f(x) := A\omega^{\lfloor x\rfloor}\theta^x$ where $A \in \mathbf{R}$. Then we have

$$\begin{aligned} f(x+k) &= A\omega^{\lfloor x+k\rfloor}\theta^{x+k} \\ &= A\omega^{\lfloor x\rfloor}\theta^x\omega^k\theta^k \\ &= A\omega^{\lfloor x\rfloor}\theta^x(1 + \omega\theta + \cdots + \omega^{k-1}\theta^{k-1}) \\ &= A\omega^{\lfloor x\rfloor}\theta^x + A\omega^{\lfloor x+1\rfloor}\theta^{x+1} + \cdots + A\omega^{\lfloor x+k-1\rfloor}\theta^{x+k-1} \\ &= f(x) + f(x+1) + \cdots + f(x+k-1). \end{aligned}$$

Hence $f(x) = A\omega^{\lfloor x \rfloor}\theta^x$ is a Fibonacci k-function.

Note that the collection of all Fibonacci k-functions over the real numbers forms a vector space over $\mathbf{R}$.

Proposition 5.3. *Let $f(x)$ be a Fibonacci k-function. If we define $g(x) := f(x+t)$ where $t \in \mathbf{R}$ for any $x \in \mathbf{R}$, then $g(x)$ is also a Fibonacci k-function.*

Proof. Since $f(x)$ is a Fibonacci k-function, we have

$$\begin{aligned} g(x+k) &= f(x+k+t) \\ &= f(x+t+k-1) + f(x+t+k-2) + \cdots + f(x+t) \\ &= g(x+k-1) + g(x+k-2) + \cdots + g(x+1) + g(x), \end{aligned}$$

proving that $g(x)$ is also a Fibonacci k-function. □

For example, a function $g(x) := (\varphi_3)^{x+t}$ is a Fibonacci 3-function.

Theorem 5.4. *Let φ_k be the largest real root of $F_k(x) = 0$ and let $\varphi_k = \omega_1\theta_1 = \omega_2\theta_2 = \cdots = \omega_N\theta_N$, where $\omega_i, \theta_i \in \mathbf{R}$. If we define $F(x) := \sum_{i=1}^N A_i\omega_i^{\lfloor x \rfloor}\theta_i^x$, then $F(x)$ is a Fibonacci k-function, where $A_i \in \mathbf{R}$.*

Proof. Given $x \in \mathbf{R}$, we have

$$\begin{aligned} F(x+k) &= \sum_{i=1}^N A_i\omega_i^{\lfloor x+k \rfloor}\theta_i^{x+k} \\ &= \sum_{i=1}^N A_i\omega_i^{\lfloor x \rfloor}\theta_i^x\omega_i^k\theta_i^k \\ &= \sum_{i=1}^N A_i\omega_i^{\lfloor x \rfloor}\theta_i^x\varphi_k^k \\ &= \sum_{i=1}^N A_i\omega_i^{\lfloor x \rfloor}\theta_i^x(1 + \omega_i\theta_i + \cdots + \omega_i^{k-1}\theta_i^{k-1}) \\ &= \sum_{i=1}^N A_i\omega_i^{\lfloor x \rfloor}\theta_i^x + \sum_{i=1}^N A_i\omega_i^{\lfloor x+1 \rfloor}\theta_i^{x+1} + \cdots + \sum_{i=1}^N A_i\omega_i^{\lfloor x+k-1 \rfloor}\theta_i^{x+k-1} \\ &= F(x) + F(x+1) + \cdots + F(x+k-1), \end{aligned}$$

proving that $F(x)$ is a Fibonacci k-function. □

Example 5.5. In Example 5.2, if we let $A := 1$ and $\omega := n$ a natural number, then $\varphi_k = n\theta$. Assume $f(x) := n^{\lfloor x \rfloor}\theta^x$. Then $f(x) = n^{\lfloor x \rfloor}\theta^x = n^{\lfloor x \rfloor}(\frac{\varphi_k}{n})^x = n^{\lfloor x \rfloor - x}(\varphi_k)^x$. If we let $x := 2.5$, then $\lfloor 2.5 \rfloor = 2$. Hence $f(2.5) = n^{\lfloor 2.5 \rfloor - 2.5}(\varphi_k)^{2.5} = \frac{1}{\sqrt{n}}(\varphi_k)^2\sqrt{\varphi_k}$. If we let $n := 8$, since $\{\varphi_k\}$ goes to 2, we obtain $f(2.5) \doteqdot \frac{1}{\sqrt{8}}2^2\sqrt{2} = 2$.

Proposition 5.6. *Let $f(x)$ be a Fibonacci k-function and differentiable (integrable, resp.). Then its derivative(integration, resp.) is also a Fibonacci k-function.*

Example 5.7. Consider a Fibonacci k-function $f(x) := \varphi_k^x$. Then $f(x) = e^{x\ln\varphi_k}$ and hence $f'(x) = (e^{x\ln\varphi_k})' = \ln\varphi_k e^{x\ln\varphi_k} = \varphi_k^x \ln\varphi_k = f(x)\ln\varphi_k$. Hence $f'(x)$ is a Fibonacci k-function. Similarly, if we define $g(x) := \int_0^x e^{u\ln\varphi_k}du$, then $g(x) = \frac{1}{\ln\varphi_k}[\varphi_k^x - 1]$ and $g'(x) = \frac{1}{\ln\varphi_k}[\varphi_k^x\ln\varphi_k - 0] = \varphi_k^x$. Hence $g(x)$ is also a Fibonacci k-function.

Example 5.8. In Example 5.2, if $\varphi_k = \omega\theta$, then $f(x) := A\omega^{\lfloor x\rfloor}\theta^x$ is a Fibonacci k-function. Since $\frac{f(x+1)}{f(x)} = (A\omega^{\lfloor x+\rfloor}\theta^{x+1})/(A\omega^{\lfloor x\rfloor}\theta^x) = \omega\theta = \varphi_k$, we obtain $\lim_{x\to\infty}\frac{f(x+1)}{f(x)} = \varphi_k$.

Conjecture. *If $f(x)$ is a Fibonacci k-function, then*

$$\lim_{x\to\infty}\frac{f(x+1)}{f(x)} = \varphi_k$$

It is known that if $f(x)$ is a Fibonacci 2-function, then $\lim_{x\to\infty}\frac{f(x+1)}{f(x)} = \varphi_2$ [5].

6. Fibonacci k-sequences and Fibonacci k-numbers

Proposition 6.1. *Let $f(x)$ be a Fibonacci 3-function, then*

$$f(x+n+2) = \alpha_n f(x+2) + \beta_n f(x+1) + \gamma_n f(x)$$

for all natural numbers n, where $\{\alpha_n\}$ is an $(1,2,4)$-Fibonacci 3-sequence, $\{\beta_n\}$ is an $(1,2,3)$-Fibonacci 3-sequence and $\{\gamma_n\}$ is an $(1,1,2)$-Fibonacci 3-sequence.

Proof. Let $f(x)$ be a Fibonacci 3-function. Then $f(x+3) = f(x+2) + f(x+1) + f(x)$, and $f(x+4) = 2f(x+2) + 2f(x+1) + f(x)$ for all $x \in \mathbf{R}$. In this fashion we obtain

$$\begin{aligned} f(x+5) &= 4f(x+2) + 2f(x+1) + 2f(x),\\ f(x+6) &= 7f(x+2) + 6f(x+1) + 4f(x),\\ f(x+7) &= 13f(x+2) + 11f(x+1) + 7f(x),\\ f(x+8) &= 24f(x+2) + 20f(x+1) + 13f(x). \end{aligned}$$

The sequence $\{\alpha_n\}$ of coefficients of $f(x+2)$ is $1,2,4,7,13,24,\cdots$, the sequence $\{\beta_n\}$ of the coefficients of $f(x+1)$ is $1,2,3,6,11,20,\cdots$ and the sequence $\{\gamma_n\}$ of coefficients of $f(x)$ is $1,1,2,4,7,13,\cdots$. This shows that $\{\alpha_n\}$ is the $(1,2,4)$-Fibonacci 3-sequence, $\{\beta_n\}$ is the $(1,2,3)$-Fibonacci 3-sequence and $\{\gamma_n\}$ is the $(1,1,2)$-Fibonacci 3-sequence. This shows that $f(x+n+2) = \alpha_n f(x+2) + \beta_n f(x+1) + \gamma_n f(x)$, proving the theorem. □

Corollary 6.2. *Given a natural number n, we have*

$$\varphi_3^{n+2} = \alpha_n\varphi_3^2 + \beta_n\varphi_3 + \gamma_n$$

where $\{\alpha_n\}$ is the $(1,2,4)$-Fibonacci 3-sequence, $\{\beta_n\}$ is the $(1,2,3)$-Fibonacci 3-sequence and $\{\gamma_n\}$ is the $(1,1,2)$-Fibonacci 3-sequence.

Proof. $f(x) = \varphi_3^x$ is a Fibonacci 3-function. □

Theorem 6.3. *Let a, b, c be non-zero integers. If we define $f(x+3) := af(x+2) + bf(x+1) + cf(x)$ for all $x \in \mathbf{R}$, then*

$$f(x+n+2) = \alpha_n f(x+2) + \beta_n f(x+1) + \gamma_n f(x)$$

for all $x \in \mathbf{R}$ and $n \geq 4$, where $\{\alpha_n\}$ is a $(a, a^2+b, a^3+2ab+c)$-Fibonacci 3-sequence, $\{\beta_n\}$ is a $(b, ab+c, a^2b+ac+b^2)$-Fibonacci 3-sequence and $\{\gamma_n\}$ is a (c, ac, a^2c+bc)-Fibonacci 3-sequence.

Proof. Since $f(x+3) := af(x+2) + bf(x+1) + cf(x)$, we let $(\alpha_1, \beta_1, \gamma_1) := (a, b, c)$. $f(x+4) = af(x+3) + bf(x+2) + cf(x+1) = a[af(x+2) + bf(x+1) + cf(x)] = (a^2+b)f(x+2) + (ab+c)f(x+1) + acf(x)$ leads to $(\alpha_2, \beta_2, \gamma_2) = (a^2+b, ab+c, ac)$. By simple computations, we obtain $f(x+5) = (a^3+2ab+c)f(x+2) + (a^2b+ac+b^2)f(x+1) + (a^2c+bc)f(x)$. Let $(\alpha_3, \beta_3, \gamma_3) := (a^3+2ab+c, a^2b+ac+b^2, a^2c+bc)$. We compute $f(x+6)$ as follows:

$$\begin{aligned} f(x+6) &= af(x+5) + bf(x+4) + cf(x+3) \\ &= a[\alpha_3 f(x+2) + \beta_3 f(x+1) + \gamma_3 f(x)] \\ &+ b[\alpha_2 f(x+2) + \beta_2 f(x+1) + \gamma_2 f(x)] \\ &+ c[\alpha_1 f(x+2) + \beta_1 f(x+1) + \gamma_1 f(x)] \\ &= (a\alpha_3 + b\alpha_2 + c\alpha_1) f(x+2) + (a\beta_3 + b\beta_2 + c\beta_1) f(x+1) \\ &+ (a\gamma_3 + b\gamma_2 + c\gamma_1) f(x), \end{aligned}$$

i.e., $\alpha_4 = a\alpha_3 + b\alpha_2 + c\alpha_1, \beta_4 = a\beta_3 + b\beta_2 + c\beta_1$ and $\gamma_4 = a\gamma_3 + b\gamma_2 + c\gamma_1$. If we let $\alpha_n = a\alpha_{n-1} + b\alpha_{n-2} + c\alpha_{n-3}, \beta_n = a\beta_{n-1} + b\beta_{n-2} + c\beta_{n-3}$ and $\gamma_n = a\gamma_{n-1} + b\gamma_{n-2} + c\gamma_{n-3}$ for $n \geq 4$, then $\{\alpha_n\}$ is an $(a, a^2+b, a^3+2ab+c)$-Fibonacci 3-sequence, $\{\beta_n\}$ is a $(b, ab+c, a^2b+ac+b^2)$-Fibonacci 3-sequence and $\{\gamma_n\}$ is a (c, ac, a^2c+bc)-Fibonacci 3-sequence. Hence $f(x+n+2) = \alpha_n f(x+2) + \beta_n f(x+1) + \gamma_n f(x)$ for all $n \geq 4$. This proves the theorem. □

Note that Proposition 6.1 is a special case of Theorem 6.3 if we let $a = b = c = 1$.

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CONVERGENCE OF SP ITERATIVE SCHEME FOR THREE MULTIVALUED MAPPINGS IN HYPERBOLIC SPACE

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Abstract. The present paper aims to deal with multivalued version of SP iterative scheme to approximate a common fixed point of three multivalued nonexpansive mappings in a uniformly convex hyperbolic space and obtain strong and Δ-convergence theorems for the SP process. Our results extend some existing results in the contemporary literature.

1. Introduction

Fixed point theory is one of the most important area of nonlinear analysis and it has applications in different disciplines of science such as in economics, biology, chemistry, engineering and technology, game theory and physics. It has become attractive to many scientists because it directly affects our daily lives. Iterative methods play an important role in calculating the fixed point of nonlinear mappings (see [16, 17, 18, 19]). The oldest known iterative method is Picard iteration which is the pioneer of iterative approximation of fixed point of different class of nonlinear mappings.

W.R. Mann [1], S. Ishikawa [2], M. A. Noor [3] introduced the Mann, Ishikawa, Noor iteration process respectively for a single valued map T defined on nonempty subset of a normed space. Metric space versions of these iterations are following:

(Mann) $$u_{n+1} = W(u_n, Tu_n, \alpha_n),$$

(Ishikawa) $$\begin{array}{r} u_{n+1} = (u_n, Tv_n, \alpha_n) \\ v_n = (u_n, Tu_n\beta_n) \end{array},$$

(Noor) $$\begin{array}{r} u_{n+1} = (u_n, Tv_n, \alpha_n) \\ v_n = (u_n, Tw_n, \beta_n) \\ w_n = (u_n, Tu_n, \gamma_n) \end{array},$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences of real numbers in $[0,1]$.

In 2008, Thianwan [4] introduced a new two steps iteration. Gunduz et al. [20] modified this iteration process as following and used for computing fixed point of nonexpansive mappings in hyperbolic spaces.

(Thianwan) $$\begin{array}{r} u_{n+1} = (v_n, Tv_n, \alpha_n) \\ v_n = (u_n, Tu_n, \beta_n) \end{array},$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$.

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In 2011, Phuengrattana and Suantai [5] gave the SP iteration which defined in metric spaces as follows:

$$\text{(SP)} \qquad \begin{aligned} u_{n+1} &= (v_n, Tv_n, \alpha_n) \\ v_n &= (w_n, Tw_n, \beta_n) \\ w_n &= (u_n, Tu_n, \gamma_n) \end{aligned} \ ,$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences of real numbers in $[0,1]$.

They also showed that SP-iteration is a generalized version of Mann, Ishikawa, Noor iterations and converges faster than all of others for the class of non-decreasing and continuous functions and they supported their analytical results by numerical examples. Chugh and Kumar [6] showed that the SP iterative scheme converges faster than the new two step iterative schemes of Thianwan [4] for increasing functions and decreasing functions.

On the other hand, it is well known that the theory of multivalued maps is more complex than according to the theory of single valued maps. Now we discourse on multivalued maps.

Let E be a metric space and D be a nonempty subset of E. If there exists an element $k \in D$ such that

$$d(x,k) = \inf\{d(x,y) : y \in D\} = d(x,D)$$

for each $x \in E$, then the set D is called proximinal. We shall denote the compact subsets, proximinal bounded subsets, and closed and bounded subsets of K by $C(D)$, $P(D)$, and $CB(D)$, respectively. A Hausdorff metric H induced by the metric d of E is defined by

$$H(A,B) = \max\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\}$$

for every $A, B \in CB(E)$. A multivalued mapping $T : D \to P(D)$ is said to be a *contraction* if there exists a constant $k \in [0,1)$ such that for any $x, y \in D$,

$$H(Tx,Ty) \le kd(x,y),$$

and T is said to be *nonexpansive* if

$$H(Tx,Ty) \le d(x,y)$$

for all $x, y \in D$. A point $p \in D$ is called a fixed point of T if $p \in Tp$. Denote the set of all fixed points of T by $F(T)$.

By using the Hausdorff metric, Markin [7, 8] studied the fixed points of multivalued nonexpansive mappings and contractions. Moreover, Lim [33] proved the existence of fixed points for multivalued nonexpansive mappings under suitable conditions in uniformly convex Banach spaces. Later on, since the fixed point theory for this kind of mappings has a lot of application areas such as convex optimization problem, control problem and differential inclusion problem (see, [9] and references cited therein), an interesting fixed point theory was developed. From then on different authors have studied on the convergence of fixed points for this class of mappings in convex metric spaces. For instance, Shimizu and Takahashi [25] generalized the results of Lim [33] to the convex metric spaces. The study of multivalued maps is a rapidly growing area of research (see, [10, 11, 12, 13, 14, 15]).

Shahzad and Zegeye [13] studied strong convergenge of Ishikawa iterative process for quasi-nonexpansive multivalued maps in uniformly convex Banach spaces. They defined Ishikawa iteration as follows:

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Let $T : D \to P(D)$ and $P_T(x) = \{y \in Tx : d(x, y) = d(x, Tx)\}$. The sequence of Ishikawa iteration is defined by $x_0 \in D$,

$$\begin{cases} y_n = (1 - b_n)x_n + b_n z_n, \\ x_{n+1} = (1 - a_n)x_n + a_n z_n^{'}, \end{cases}$$

where $z_n \in Tx_n$, $z_n^{'} \in Ty_n$, and $\{a_n\}$, $\{b_n\}$ are sequences of real numbers in $[0, 1]$.

In this paper, we first give multivalued version of the SP iteration scheme in Kohlenbach hyperbolic spaces and then use this scheme to approximate a common fixed point of three multivalued nonexpansive mappings. Our results improve and extend current results in recently literature by via faster, more general iteration and more general space.

2. Hyperbolic Spaces and Lemmas

There are different definitions for hyperbolic space in mathematics. We will study in the hyperbolic space defined by Kohlenbach [22]. We review hyperbolic space and it's features in this section.

Definition 1. [22] *A metric space (E, d) is said to be hyperbolic space if there exists a map $W : E^2 \times [0, 1] \to E$ satisfying:*

W1. $d(u, W(x, y, \alpha)) \le (1 - \alpha) d(u, x) + \alpha d(u, y)$
W2. $d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta| d(x, y)$
W3. $W(x, y, \alpha) = W(y, x, (1 - \alpha))$
W4. $d(W(x, z, \alpha), W(y, w, \alpha)) \le (1 - \alpha) d(x, y) + \alpha d(z, w)$

for all $x, y, z, w \in E$ and $\alpha, \beta \in [0, 1]$.

A metric space (E, d) is called a convex metric space introduced by Takahashi [23] if it satisfies only W1. Every normed space (and Banach space) is a special convex metric space, but the converse of this statement is not true, in general (see [16]). The class of hyperbolic spaces includes normed spaces, the Hilbert ball (see [24] for a book treatment) and CAT (0)-spaces. The readers can found detailed discussion in [21].

Definition 2. [25] *A hyperbolic space (E, d, W) is said to be uniformly convex if for all $u, x, y \in E$, $r > 0$ and $\varepsilon \in (0, 2]$, there exists a $\delta \in (0, 1]$ such that*

$$\left.\begin{array}{r} d(x, u) \le r \\ d(y, u) \le r \\ d(x, y) \ge \varepsilon r \end{array}\right\} \Rightarrow d\left(W\left(x, y, \frac{1}{2}\right), u\right) \le (1 - \delta) r.$$

A map $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ which provides such a $\delta = \eta(r, \varepsilon)$ for given $r > 0$ and $\varepsilon \in (0, 2]$, is called modulus of uniform convexity. We call η monotone if it decreases with r (for a fixed ε). A subset D of a hyperbolic space E is convex if $W(x, y, \alpha) \in D$ for all $x, y \in D$ and $\alpha \in [0, 1]$.

Now, we give definition of Δ-convergence which coined by Lim [26] in general metric spaces. To give the definition of Δ-convergence, we first recall the notions of asymptotic radius and asymptotic center. Let $\{x_n\}$ be a bounded sequence in a metric space E. For $x \in E$, define a continuous functional $r(., \{x_n\}) : E \to [0, \infty)$ by $r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n)$. Then the asymptotic radius $\rho = r(\{x_n\})$

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of $\{x_n\}$ is given by $\rho = \inf\{r(x,\{x_n\}) : x \in E\}$ and the asymptotic center of a bounded sequence $\{x_n\}$ with respect to a subset D of E is defined as follows:

$$A_D(\{x_n\}) = \{x \in E : r(x,\{x_n\}) \le r(y,\{x_n\}) \text{ for any } y \in D\}.$$

The set of all asymptotic centers of $\{x_n\}$ is denoted by $A(\{x_n\})$.

It has been shown in [29] that bounded sequences have unique asymptotic center with respect to closed convex subsets in a complete and uniformly convex hyperbolic space with monotone modulus of uniform convexity.

A sequence $\{x_n\}$ in E is said to Δ-converge to $x \in E$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$ [27]. In this case, we write Δ -$\lim_n x_n = x$.

We want to point out that $\Delta-$convergence coincides with weak convergence in Banach spaces with Opial's property [30].

Kirk and Panyanak [27] specialized this concept to CAT(0) spaces and showed that many Banach space results involving weak convergence have precise analogs in this setting. Dhompongsa and Panyanak [28] continued to work in this direction and studied the Δ-convergence of Picard, Mann and Ishikawa iterates in $CAT(0)$ spaces. Khan et al. [31] was studied this concept in hyperbolic spaces and they gave a couple of helpful lemma as follows.

Lemma 1. [31] *Let (E,d,W) be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x \in E$ and $\{\alpha_n\}$ be a sequence in $[b,c]$ for some $b,c \in (0,1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in E such that $\limsup_{n\to\infty} d(x_n,x) \le r, \limsup_{n\to\infty} d(y_n,x) \le r$ and $\lim_{n\to\infty} d(W(x_n,y_n,\alpha_n),x) = r$ for some $r \ge 0$, then $\lim_{n\to\infty} d(x_n,y_n) = 0$.*

Lemma 2. [31] *Let D be a nonempty closed convex subset of a uniformly convex hyperbolic space and $\{x_n\}$ be a bounded sequence in D such that $A(\{x_n\}) = \{y\}$ and $r(\{x_n\}) = \rho$. If $\{y_m\}$ is another sequence in D such that $\lim_{m\to\infty} r(y_m,\{x_n\}) = \rho$, then $\lim_{m\to\infty} y_m = y$.*

The following useful first lemma can be found in [14] gives some properties of P_T in metric (and hence hyperbolic) spaces and second can be found in [8].

Lemma 3. [14] *Let $T : D \to P(D)$ be a multivalued mapping and $P_T(x) = \{y \in Tx : d(x,y) = d(x,Tx)\}$. Then the following are equivalent.*

(1) $x \in F(T)$, that is, $x \in Tx$,

(2) $P_T(x) = \{x\}$, that is, $x = y$ for each $y \in P_T(x)$,

(3) $x \in F(P_T)$, that is, $x \in P_T(x)$.

Moreover, $F(T) = F(P_T)$.

Lemma 4. *Let $A,B \in CB(E)$ and $a \in A$. If $\eta > 0$, then there exists $b \in B$ such that $d(a,b) \le H(A,B) + \eta$.*

3. Strong and Δ-Convergence Theorems

Before giving our main results, we give the multivalued version of the SP iteration scheme (SP) in hyperbolic spaces. Let E be a hyperbolic space and D be a nonempty convex subset of E. Let $T,S,R : D \to P(D)$ be three multivalued maps and $P_T(x) = \{y \in Tx : d(x, y) = d(x, Tx)\}$. Choose $x_0 \in D$ and define $\{x_n\}$ as

$$\begin{cases} x_{n+1} = W(u_n, y_n, \alpha_n) \\ y_n = W(v_n, z_n, \beta_n) \\ z_n = W(w_n, x_n, \gamma_n) \end{cases}, \tag{3.1}$$

where $u_n \in P_T(y_n)$, $v_n \in P_S(z_n), w_n \in P_R(x_n)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are real sequences such that $0 < a \leq \alpha_n, \beta_n, \gamma_n \leq b < 1$ for all $n \in \mathbb{N}$. The iterative sequence (3.1) is called the *modified SP iterative process* for a multivalued nonexpansive mapping in a Kohlenbach hyperbolic space.

Lemma 5. *Let D be a nonempty closed convex subset of a hyperbolic space X and let $S, T, R : D \to P(D)$ be three multivalued mappings such that P_T, P_S and P_R are nonexpansive mappings with a nonempty common fixed point set F. Then for the modified SP iterative process $\{x_n\}$ in (3.1), $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F$.*

Proof. Let $p \in F$. Then $p \in P_T(p) = \{p\}$, $p \in P_S(p) = \{p\}$ and $p \in P_R(p) = \{p\}$. Using (3.1), we have

$$\begin{aligned} d(x_{n+1}, p) &= d(W(u_n, y_n, \alpha_n), p) \\ &\leq (1-\alpha_n) d(u_n, p) + \alpha_n d(y_n, p) \\ &\leq (1-\alpha_n) d(u_n, P_T(p)) + \alpha_n d(y_n, p) \\ &\leq (1-\alpha_n) H(P_T(y_n), P_T(p)) + \alpha_n d(y_n, p) \\ &\leq (1-\alpha_n) d(y_n, p) + \alpha_n d(y_n, p) \\ &= d(y_n, p) = d(W(v_n, z_n, \beta_n), p) \\ &\leq (1-\beta_n) d(v_n, p) + \beta_n d(z_n, p) \\ &\leq (1-\beta_n) d(v_n, P_S(p)) + \beta_n d(z_n, p) \\ &\leq (1-\beta_n) H(P_S(z_n), P_S(p)) + \alpha_n d(z_n, p) \\ &\leq (1-\beta_n) d(z_n, p) + \alpha_n d(z_n, p) \\ &= d(z_n, p) = d(W(w_n, x_n, \gamma_n), p) \\ &\leq (1-\gamma_n) d(w_n, p) + \gamma_n d(x_n, p) \\ &\leq (1-\gamma_n) d(w_n, P_R(p)) + \gamma_n d(x_n, p) \\ &\leq (1-\gamma_n) H(P_R(x_n), P_R(p)) + \gamma_n d(x_n, p) \\ &\leq (1-\gamma_n) d(x_n, p) + \gamma_n d(x_n, p) \\ &= d(x_n, p) \qquad (3.2) \end{aligned}$$

That is,

$$d(x_{n+1}, p) \leq d(x_n, p).$$

Hence $\lim_{n\to\infty} d(x_n, p)$ exists. □

Lemma 6. *Let D be a nonempty closed convex subset of a uniformly convex hyperbolic space X and let $S, T, R : D \to P(D)$ be three multivalued mappings such that P_T, P_S and P_R are nonexpansive mappings with a nonempty common fixed point set F. Then for modified SP iterative process $\{x_n\}$ in (3.1), we have $\lim_{n\to\infty} d(x_n, P_T(y_n)) = \lim_{n\to\infty} d(x_n, P_S(z_n)) = \lim_{n\to\infty} d(x_n, P_R(x_n)) = 0$.*

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Proof. By Lemma 5, $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F$. Assume that $\lim\limits_{n\to\infty} d(x_n, p) = c$ for some $c \geq 0$. For $c = 0$, the result is trivial. Suppose $c > 0$.

Now $\lim_{n\to\infty} d(x_{n+1}, p) = c$ can be rewritten as

$$\lim_{n\to\infty} d(W(u_n, y_n, \alpha_n), p) = c. \tag{3.3}$$

Since P_T is nonexpansive, we have

$$\begin{aligned} d(u_n, p) &= d(u_n, P_T(p)) \\ &\leq H(P_T(y_n), P_T(p)) \\ &\leq d(y_n, p) = d(W(v_n, z_n, \beta_n), p) \\ &\leq (1-\beta_n) d(v_n, p) + \beta_n d(z_n, p) \\ &\leq (1-\beta_n) H(P_S(z_n), P_S(p)) + \beta_n d(z_n, p) \\ &\leq (1-\beta_n) d(z_n, p) + \beta_n d(z_n, p) \\ &= d(z_n, p) = d(W(w_n, x_n, \gamma_n), p) \\ &\leq (1-\gamma_n) d(w_n, p) + \gamma_n d(x_n, p) \\ &\leq (1-\gamma_n) H(P_R(x_n), P_R(p)) + \gamma_n d(x_n, p) \\ &\leq (1-\gamma_n) d(x_n, p) + \gamma_n d(x_n, p) \\ &\leq d(x_n, p). \end{aligned}$$

Hence

$$\limsup_{n\to\infty} d(u_n, p) \leq c, \limsup_{n\to\infty} d(z_n, p) \leq c, \limsup_{n\to\infty} d(y_n, p) \leq c. \tag{3.4}$$

Next

$$\begin{aligned} d(v_n, p) &= d(v_n, P_S(p)) \\ &\leq H(P_S(z_n), P_S(p)) \\ &\leq d(z_n, p) \end{aligned}$$

and so

$$\limsup_{n\to\infty} d(v_n, p) \leq c. \tag{3.5}$$

Further

$$\begin{aligned} d(w_n, p) &= d(w_n, P_R(p)) \\ &\leq H(P_R(x_n), P_R(p)) \\ &\leq d(x_n, p) \end{aligned}$$

and so

$$\limsup_{n\to\infty} d(w_n, p) \leq c. \tag{3.6}$$

On the other hand, since

$$d(W(w_n, x_n, \gamma_n), p) = d(z_n, p) \leq d(x_n, p),$$

we have

$$\limsup_{n\to\infty} d(W(w_n, x_n, \gamma_n), p) \leq c. \tag{3.7}$$

From (3.2), we have

$$c \leq \liminf_{n\to\infty} d(W(w_n, x_n, \gamma_n), p) \tag{3.8}$$

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So, it follows from (3.7) and (3.8) that

$$\lim_{n\to\infty} d\left(W\left(w_n, x_n, \gamma_n\right), p\right) = \lim_{n\to\infty} d\left(z_n, p\right) = c. \tag{3.9}$$

Since we assume that $\lim_{n\to\infty} d\left(x_n, p\right) = c$, we can write

$$\limsup_{n\to\infty} d\left(x_n, p\right) \leq c. \tag{3.10}$$

By using the inequalities (3.6), (3.9) and (3.10) and Lemma 1, we obtain that

$$\lim_{n\to\infty} d\left(x_n, w_n\right) = 0. \tag{3.11}$$

On the other hand, it follows from (3.3), (3.4) and Lemma 1 that

$$\lim_{n\to\infty} d\left(u_n, y_n\right) = 0.$$

Nothing that

$$\begin{aligned} d\left(x_{n+1}, p\right) &= d\left(W\left(u_n, y_n, \alpha_n\right), p\right) \\ &\leq \left(1-\alpha_n\right) d\left(u_n, p\right) + \alpha_n d\left(y_n, p\right) \\ &\leq \left(1-\alpha_n\right) d\left(y_n, u_n\right) + d\left(u_n, p\right) \end{aligned}$$

which yields that

$$c \leq \liminf_{n\to\infty} d\left(u_n, p\right). \tag{3.12}$$

Then from (3.4) and (3.12), we have

$$c = \lim_{n\to\infty} d\left(u_n, p\right).$$

Since $d\left(x_{n+1}, p\right) \leq d\left(y_n, p\right)$, this implies that

$$\begin{aligned} c &\leq \lim_{n\to\infty} d\left(y_n, p\right) = \lim_{n\to\infty} d\left(W\left(v_n, z_n, \beta_n\right), p\right) \\ &\leq \lim_{n\to\infty} \left[\left(1-\beta_n\right) \limsup_{n\to\infty} d\left(v_n, p\right) + \beta_n \limsup_{n\to\infty} d\left(z_n, p\right)\right] \\ &\leq \lim_{n\to\infty} \left[\left(1-\beta_n\right) c + \beta_n c\right] \\ &= c \end{aligned}$$

Hence, we get

$$c = \lim_{n\to\infty} d\left(y_n, p\right) = \lim_{n\to\infty} d\left(W\left(v_n, z_n, \beta_n\right), p\right). \tag{3.13}$$

So, it follows from Lemma 1, (3.4), (3.5) and (3.13) that

$$\lim_{n\to\infty} d\left(z_n, v_n\right) = 0$$

and

$$\begin{aligned} d\left(y_n, p\right) &= d\left(W\left(v_n, z_n, \beta_n\right), p\right) \\ &\leq \left(1-\beta_n\right) d\left(v_n, p\right) + \beta_n d\left(z_n, p\right) \\ &\leq \beta_n d\left(z_n, v_n\right) + d\left(v_n, p\right) \end{aligned}$$

this yields that

$$c \leq \liminf_{n\to\infty} d\left(v_n, p\right),$$

so (3.5) gives that

$$c = \lim_{n\to\infty} d\left(v_n, p\right).$$

On the other hand, since

$$\begin{aligned}\lim_{n\to\infty} d(z_n,x_n) &= \lim_{n\to\infty} d(W(w_n,x_n,\gamma_n),x_n) \\ &\leq \lim_{n\to\infty}\left[(1-\gamma_n)\limsup_{n\to\infty} d(x_n,x_n)+\gamma_n\limsup_{n\to\infty} d(w_n,x_n)\right],\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty} d(y_n,z_n) &= \lim_{n\to\infty} d(W(v_n,z_n,\beta_n),z_n) \\ &\leq \lim_{n\to\infty}\left[(1-\beta_n)\limsup_{n\to\infty} d(z_n,z_n)+\beta_n\limsup_{n\to\infty} d(v_n,z_n)\right]\end{aligned}$$

we have

$$\lim_{n\to\infty} d(z_n,x_n)=0$$

and

$$\lim_{n\to\infty} d(y_n,z_n)=0.$$

Also,

$$d(u_n,x_n)\leq d(u_n,y_n)+d(y_n,z_n)+d(z_n,x_n),$$

then

$$\lim_{n\to\infty} d(u_n,x_n)=0.$$

Also,

$$d(v_n,x_n)\leq d(v_n,z_n)+d(z_n,x_n),$$

that is,

$$\lim_{n\to\infty} d(v_n,x_n)=0.$$

Since

$$d(x,P_R(x))=\inf_{z\in P_R(x)} d(x,z),$$

therefore

$$d(x_n,P_R(x_n))\leq d(x_n,w_n)\to 0 \text{ as } n\to\infty.$$

Similarly

$$d(x_n,P_T(y_n))\leq d(x_n,u_n)\to 0$$

and

$$d(x_n,P_S(z_n))\leq d(x_n,v_n)\to 0$$

as $n\to\infty$. □

Firstly, we prove that modified SP iterative process defined in (3.1) Δ-converges a common fixed point of S,T and R.

Theorem 1. *Let D be a nonempty closed and convex subset of a complete uniformly convex hyperbolic space E with monotone modulus of uniform convexity η. Let S,T,R,P_S,P_T,P_R and F be as in Lemma 6. Then the modified SP iterative process $\{x_n\}$ Δ-converges to a p in F.*

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Proof. Let $p \in F(T) = F(P_T)$. By the Lemma 5, $\{x_n\}$ is bounded and so $\lim_{n\to\infty} d(x_n, p)$ exists. This gives that $\{x_n\}$ has a unique asymptotic center. In other words, we have $A(\{x_n\}) = \{x\}$. Let $\{v_n\}$ be any subsequence of $\{x_n\}$ such that $A(\{v_n\}) = \{v\}$. From Lemma 6, we have $\lim_{n\to\infty} (x_n, P_S(x_n)) = 0$. We claim that v is a fixed point of P_S.

To prove this, take another sequence $\{z_m\}$ in $P_S(v)$. Then

$$\begin{aligned} r(z_m, \{v_n\}) &= \lim_{n\to\infty} \sup d(z_m, v_n) \\ &\leq \lim_{n\to\infty} \sup \{d(z_m, P_S(v_n)) + d(P_S(v_n), v_n)\} \\ &\leq \lim_{n\to\infty} \sup H(P_S(v), P_S(v_n)) \\ &\leq \lim_{n\to\infty} \sup d(v, v_n) \\ &= r(v, \{v_n\}). \end{aligned}$$

This gives $|r(z_m, \{v_n\} - r(v, \{v_n\})| \to 0$ $(m \to \infty)$. From Lemma 2, we have $\lim_{m\to\infty} z_m = v$. Note that $Sv \in P(K)$ being proximinal is closed, hence $P_S(v)$ is closed. Moreover, $P_S(v)$ is bounded. Consequently $\lim_{m\to\infty} z_m = v \in P_S(v)$. Similarly, $v \in P_T(v)$ and $v \in P_R(v)$. Hence $v \in F$. From the uniqueness of asymptotic center, we have

$$\begin{aligned} \lim_{n\to\infty} \sup d(v_n, v) &< \lim_{n\to\infty} \sup d(v_n, x) \\ &\leq \lim_{n\to\infty} \sup d(x_n, x) \\ &< \lim_{n\to\infty} \sup d(x_n, v) \\ &= \lim_{n\to\infty} \sup d(v_n, v), \end{aligned}$$

and this is a contradiction. Hence $x = u$. Therefore $A(\{v_n\}) = \{v\}$ for every subsequence $\{v_n\}$ of $\{x_n\}$. Hence $\{x_n\}$ Δ-converges to a p in F. □

We now prove a strong convergence theorem. Then we will apply following theorem to obtain new theorem in a complete and uniformly convex hyperbolic space.

Theorem 2. *Let D be a nonempty closed and convex subset of a hyperbolic space E and, S, T, R, P_S, P_T, P_R and F be as in Lemma 6. Let $\{x_n\}$ be the modified SP iterative process defined in* (3.1)*, then $\{x_n\}$ converges strongly to a p in F if and only if* $\liminf_{n\to\infty} d(x_n, F) = 0$.

Proof. The necessity is obvious. Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F) = 0$. By similar methot in Lemma 5, we get

$$d(x_{n+1}, p) \leq d(x_n, p),$$

which implies

$$d(x_{n+1}, F) \leq d(x_n, F).$$

This gives that $\lim_{n\to\infty} d(x_n, F)$ exists and so by the hypothesis of our theorem, $\liminf_{n\to\infty} d(x_n, F) = 0$. Therefore we must have $\lim_{n\to\infty} d(x_n, F) = 0$.

We now prove that $\{x_n\}$ is a Cauchy sequence in D. Let $m, n \in \mathbb{N}$ and assume $m > n$. Then it follows (along the lines similar to Lemma 5) that

$$d(x_m, p) \leq d(x_n, p)$$

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for all $p \in F$. Thus we have

$$d(x_m, x_n) \leq d(x_m, p) + d(p, x_n) \leq 2d(x_n, p).$$

Taking inf on the set F, we have $d(x_m, x_n) \leq d(x_n, F)$. We show that $\{x_n\}$ is a Cauchy sequence in D by taking limit as $m \to \infty$, $n \to \infty$ in the inequality $d(x_m, x_n) \leq d(x_n, F)$. Thus, it converges to a $q \in D$. Now it is left to show that $q \in F(S)$. Indeed, by $d(x_n, F(P_S)) = \inf_{y \in F(P_S)} d(x_n, y)$. So for each $\varepsilon > 0$, there exists $p_n^{(\varepsilon)} \in F(P_S)$ such that,

$$d\left(x_n, p_n^{(\varepsilon)}\right) < d(x_n, F(P_S)) + \frac{\varepsilon}{2}.$$

This implies $\lim_{n\to\infty} d\left(x_n, p_n^{(\varepsilon)}\right) \leq \frac{\varepsilon}{2}$. From $d\left(p_n^{(\varepsilon)}, q\right) \leq d\left(x_n, p_n^{(\varepsilon)}\right) + d(x_n, q)$ it follows that

$$\lim_{n\to\infty} d\left(p_n^{(\varepsilon)}, q\right) \leq \frac{\varepsilon}{2}.$$

Finally,

$$\begin{aligned} d(P_S(q), q) &\leq d\left(q, p_n^{(\varepsilon)}\right) + d\left(p_n^{(\varepsilon)}, P_S(q)\right) \\ &\leq d\left(q, p_n^{(\varepsilon)}\right) + H\left(P_S\left(p_n^{(\varepsilon)}\right), P_S(q)\right) \\ &\leq 2d\left(p_n^{(\varepsilon)}, q\right) \end{aligned}$$

shows $d(P_S(q), q) < \varepsilon$. Therefore $d(P_S(q), q) = 0$. In a similar way, we get $d(P_T(q), q) = 0$ and $d(P_R(q), q) = 0$. Since F is closed, therefore $q \in F$. □

As appropriate our aim, we give definition of multivalued version of condition (I) of Senter and Dotson [32] for three maps and definition of semi-compact map.

Definition 3. *The multivalued nonexpansive mappings $S, T, R : D \to CB(D)$ where D a subset of E, are said to satisfy condition (I) if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that $\frac{1}{3}[d(x, Sx) + d(x, Tx) + d(x, Rx)] \geq f(d(x, F))$ for all $x \in D$.*

Definition 4. *A map $T : D \to P(D)$ is called semi-compact if any bounded sequence $\{x_n\}$ satisfying $d(x_n, Tx_n) \to 0$ as $n \to \infty$ has a convergent subsequence.*

We now give applications of above theorem.

Theorem 3. *Let D be a nonempty closed convex subset of a complete uniformly convex hyperbolic space E with monotone modulus of uniform convexity η and, S, T, R, P_S, P_T, P_R and F be as in Lemma 6. Suppose that P_S, P_T and P_R satisfy condition (I), then the modified SP iterative process $\{x_n\}$ defined in (3.1) converges strongly to $p \in F$.*

Proof. For all $p \in F$, $\lim_{n\to\infty} d(x_n, p)$ exists from Lemma 5. We call it c for some $c \geq 0$.

If $c = 0$, there is nothing to prove. Assume $c > 0$. Now $d(x_{n+1}, p) \leq d(x_n, p)$ gives that

$$\inf_{p \in F(T)} d(x_{n+1}, p) \leq \inf_{p \in F(T)} d(x_n, p),$$

which means that $d(x_{n+1}, F) \leq d(x_n, F)$.Hence $\lim_{n\to\infty} d(x_n, F)$ exists. By using condition (I) and Lemma 6,we get

$$\lim_{n\to\infty} f(d(x_n, F)) \leq \lim_{n\to\infty} \frac{1}{3} [d(x, Sx) + d(x, Tx) + d(x, Rx)] = 0.$$

and so

$$\lim_{n\to\infty} f(d(x_n, F)) = 0.$$

By the properties of f, we obtain that $\lim_{n\to\infty} d(x_n, F) = 0$. Finally applying Theorem 2, we get the result. □

Since the proof of following theorem is similar to proof of theorem proved in Banach spaces by various authors, we omit.

Theorem 4. *Let D be a nonempty closed convex subset of a complete uniformly convex hyperbolic space E with monotone modulus of uniform convexity η and, S, T, R, P_S, P_T, P_R and F be as in Lemma 6. Suppose that one of P_S, P_T, P_R is semi-compact, then the modified SP iterative process $\{x_n\}$ defined in* (3.1) *converges strongly to $p \in F$.*

As a corollary of Theorem 2, we have the following theorem which is new in the literature.

Theorem 5. *Let D be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X and let $T : D \to P(D)$ be a multivalued mapping such that P_T is nonexpansive mapping with a nonempty fixed point set F. Let $\{x_n\}$ defined as*

$$\begin{cases} x_{n+1} = W(u_n, y_n, \alpha_n) \\ y_n = W(v_n, z_n, \beta_n) \\ z_n = W(w_n, x_n, \gamma_n) \end{cases},$$

where $u_n \in P_T(y_n)$, $v_n \in P_T(z_n), w_n \in P_T(x_n)$. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ be real sequences such that $0 < a \leq \alpha_n, \beta_n, \gamma_n \leq b < 1$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ Δ-converges to a fixed point of T.

Remark 1. *(1) Since SP iterative process converges faster than Mann and Ishikawa iterative processes, our theorems are better than results of Fukhar-ud-din et al.* [34].

(2) Since CAT(0)-spaces are uniformly convex hyperbolic spaces with a 'nice' monotone modulus of uniform convexity $\eta(r, \varepsilon) := \frac{\varepsilon^2}{8}$, then our results valid in CAT(0) spaces besides Banach spaces.

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ADDITIVE-QUADRATIC ρ-FUNCTIONAL EQUATIONS IN NON-ARCHIMEDEAN BANACH SPACES

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ABSTRACT. Let

$$\begin{aligned} M_1f(x,y): &= \frac{3}{4}f(x+y) - \frac{1}{4}f(-x-y) \\ &\quad + \frac{1}{4}f(x-y) + \frac{1}{4}f(y-x) - f(x) - f(y), \\ M_2f(x,y): &= 2f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) + f\left(\frac{y-x}{2}\right) - f(x) - f(y). \end{aligned}$$

We solve the additive-quadratic ρ-functional equations

$$M_1f(x,y) = \rho M_2f(x,y), \tag{0.1}$$

where ρ is a fixed non-Archimedean number with $|\rho| < 1$, and

$$M_2f(x,y) = \rho M_1f(x,y), \tag{0.2}$$

where ρ is a fixed non-Archimedean number with $|\rho| < |2|$.

Furthermore, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional equations (0.1) and (0.2) in non-Archimedean Banach spaces.

1. Introduction and preliminaries

A *valuation* is a function $|\cdot|$ from a field K into $[0,\infty)$ such that 0 is the unique element having the 0 valuation, $|rs| = |r|\cdot|s|$ and the triangle inequality holds, i.e.,

$$|r+s| \le |r| + |s|, \qquad \forall r,s \in K.$$

A field K is called a *valued field* if K carries a valuation. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuations.

Let us consider a valuation which satisfies a stronger condition than the triangle inequality. If the triangle inequality is replaced by

$$|r+s| \le \max\{|r|,|s|\}, \qquad \forall r,s \in K,$$

then the function $|\cdot|$ is called a *non-Archimedean valuation*, and the field is called a *non-Archimedean field*. Clearly $|1| = |-1| = 1$ and $|n| \le 1$ for all $n \in \mathbb{N}$. A trivial example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except for 0 into 1 and $|0| = 0$.

Throughout this paper, we assume that the base field is a non-Archimedean field, hence call it simply a field.

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Definition 1.1. ([8]) Let X be a vector space over a field K with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\| : X \to [0, \infty)$ is said to be a *non-Archimedean norm* if it satisfies the following conditions:

(i) $\|x\| = 0$ if and only if $x = 0$;

(ii) $\|rx\| = |r|\|x\| \qquad (r \in K, x \in X)$;

(iii) the strong triangle inequality

$$\|x + y\| \le \max\{\|x\|, \|y\|\}, \qquad \forall x, y \in X$$

holds. Then $(X, \|\cdot\|)$ is called a *non-Archimedean normed space*.

Definition 1.2. (i) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X. Then the sequence $\{x_n\}$ is called *Cauchy* if for a given $\varepsilon > 0$ there is a positive integer N such that

$$\|x_n - x_m\| \le \varepsilon$$

for all $n, m \ge N$.

(ii) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X. Then the sequence $\{x_n\}$ is called *convergent* if for a given $\varepsilon > 0$ there are a positive integer N and an $x \in X$ such that

$$\|x_n - x\| \le \varepsilon$$

for all $n \ge N$. Then we call $x \in X$ a limit of the sequence $\{x_n\}$, and denote by $\lim_{n\to\infty} x_n = x$.

(iii) If every Cauchy sequence in X converges, then the non-Archimedean normed space X is called a *non-Archimedean Banach space*.

The stability problem of functional equations originated from a question of Ulam [19] concerning the stability of group homomorphisms. The functional equation $f(x + y) = f(x) + f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [7] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [12] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [6] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. The functional equation $f\left(\frac{x+y}{2}\right) = \frac{1}{2}f(x) + \frac{1}{2}f(y)$ is called the *Jensen equation*.

The functional equation $f(x + y) + f(x - y) = 2f(x) + 2f(y)$ is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The stability of quadratic functional equation was proved by Skof [18] for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [5] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. The functional equation $2f\left(\frac{x+y}{2}\right)+2\left(\frac{x-y}{2}\right) = f(x)+f(y)$ is called a *Jensen type quadratic equation*. The stability problems of various functional equations have been extensively investigated by a number of authors (see [1, 3, 4, 10, 11, 13, 14, 15, 16, 17, 20, 21]).

In Section 2, we solve the additive-quadratic ρ-functional equation (0.1) and prove the Hyers-Ulam stability of the additive-quadratic ρ-functional equation (0.1) in non-Archimedean Banach spaces.

In Section 3, we solve the additive-quadratic ρ-functional equation (0.2) and prove the Hyers-Ulam stability of the additive-quadratic ρ-functional equation (0.2) in non-Archimedean Banach spaces.

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Throughout this paper, assume that X is a non-Archimedean normed space and that Y is a non-Archimedean Banach space. Let $|2| \neq 1$.

2. Additive-quadratic ρ-functional equation (0.1) in non-Archimedean normed spaces

Throughout this section, assume that ρ is a fixed non-Archimedean number with $|\rho| < 1$.

In this section, we solve the additive-quadratic ρ-functional equation (0.1) in non-Archimedean normed spaces.

Lemma 2.1.
(i) *If an odd mapping $f : X \to Y$ satisfies*

$$M_1 f(x, y) = \rho M_2 f(x, y) \tag{2.1}$$

for all $x, y \in X$, then $f : X \to Y$ is additive.
(ii) *If an even mapping $f : X \to Y$ satisfies* (2.1)*, then $f : X \to Y$ is quadratic.*

Proof. (i) Assume that $f : X \to Y$ satisfies (2.1).

Since f is an odd mapping, $f(0) = 0$.

Letting $y = x$ in (2.1), we get

$$f(2x) - 2f(x) = 0$$

and so $f(2x) = 2f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{2} f(x) \tag{2.2}$$

for all $x \in X$.

It follows from (2.1) and (2.2) that

$$\begin{aligned} f(x+y) - f(x) - f(y) &= \rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right) \\ &= \rho(f(x+y) - f(x) - f(y)) \end{aligned}$$

and so

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$.

(ii) Assume that $f : X \to Y$ satisfies (2.1).

Letting $x = y = 0$ in (2.1), we get

$$-f(0) = 2\rho f(0).$$

So $f(0) = 0$.

Letting $y = x$ in (2.1), we get

$$\frac{1}{2} f(2x) - 2f(x) = 0$$

and so $f(2x) = 4f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{4} f(x) \tag{2.3}$$

for all $x \in X$.

It follows from (2.1) and (2.3) that

$$\begin{aligned}\frac{1}{2}f(x+y)+\frac{1}{2}f(x-y)-f(x)-f(y)\\&=\rho\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right)\\&=\rho\left(\frac{1}{2}f(x+y)+\frac{1}{2}f(x-y)-f(x)-f(y)\right)\end{aligned}$$

and so

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

for all $x, y \in X$. □

We prove the Hyers-Ulam stability of the additive-quadratic ρ-functional equation (2.1) in non-Archimedean Banach spaces for an odd mapping case.

Theorem 2.2. *Let $r < 1$ and θ be nonnegative real numbers and let $f : X \to Y$ be an odd mapping such that*

$$\|M_1f(x,y)-\rho M_2f(x,y)\| \le \theta(\|x\|^r+\|y\|^r) \tag{2.4}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x)-A(x)\| \le \frac{2\theta}{|2|^r}\|x\|^r \tag{2.5}$$

for all $x \in X$.

Proof. Since f is an odd mapping, $f(0) = 0$.

Letting $y = x$ in (2.4), we get

$$\|f(2x)-2f(x)\| \le 2\theta\|x\|^r \tag{2.6}$$

for all $x \in X$. So $\left\|f(x)-2f\left(\frac{x}{2}\right)\right\| \le \frac{2}{|2|^r}\theta\|x\|^r$ for all $x \in X$. Hence

$$\begin{aligned}&\left\|2^lf\left(\frac{x}{2^l}\right)-2^mf\left(\frac{x}{2^m}\right)\right\| \qquad (2.7)\\&\le\max\left\{\left\|2^lf\left(\frac{x}{2^l}\right)-2^{l+1}f\left(\frac{x}{2^{l+1}}\right)\right\|,\cdots,\left\|2^{m-1}f\left(\frac{x}{2^{m-1}}\right)-2^mf\left(\frac{x}{2^m}\right)\right\|\right\}\\&=\max\left\{|2|^l\left\|f\left(\frac{x}{2^l}\right)-2f\left(\frac{x}{2^{l+1}}\right)\right\|,\cdots,|2|^{m-1}\left\|f\left(\frac{x}{2^{m-1}}\right)-2f\left(\frac{x}{2^m}\right)\right\|\right\}\\&\le\max\left\{\frac{|2|^l}{|2|^{rl+r}},\cdots,\frac{|2|^{m-1}}{|2|^{r(m-1)+r}}\right\}2\theta\|x\|^r=\frac{2\theta}{|2|^{(r-1)l+r}}\|x\|^r\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.7) that the sequence $\{2^nf(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{2^nf(\frac{x}{2^n})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} 2^nf(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.7), we get (2.5).

It follows from (2.4) that

$$\begin{aligned} \|M_1A(x,y) - \rho M_2A(x,y)\| &= \lim_{n\to\infty} |2|^n \left\| M_1 f\left(\frac{x}{2^n}, \frac{y}{2^n}\right) - \rho M_2 f\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\right\| \\ &\leq \lim_{n\to\infty} \frac{|2|^n\theta}{|2|^{nr}}(\|x\|^r + \|y\|^r) = 0 \end{aligned}$$

for all $x, y \in X$. So

$$M_1A(x,y) = \rho M_2A(x,y)$$

for all $x, y \in X$. By Lemma 2.1, the mapping $A : X \to Y$ is additive .

Now, let $T : X \to Y$ be another additive mapping satisfying (2.5). Then we have

$$\begin{aligned} \|A(x) - T(x)\| &= \left\| 2^q A\left(\frac{x}{2^q}\right) - 2^q T\left(\frac{x}{2^q}\right)\right\| \\ &\leq \max\left\{\left\| 2^q A\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\|, \left\| 2^q T\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\|\right\} \leq \frac{2\theta}{|2|^{(r-1)q+r}}\|x\|^r, \end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$. This proves the uniqueness of h. Thus the mapping $A : X \to Y$ is a unique additive mapping satisfying (2.5). □

Theorem 2.3. *Let $r > 1$ and θ be nonnegative real numbers and let $f : X \to Y$ be an odd mapping satisfying* (2.4)*. Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \leq \frac{2\theta}{|2|}\|x\|^r \tag{2.8}$$

for all $x \in X$.

Proof. It follows from (2.6) that

$$\left\| f(x) - \frac{1}{2}f(2x)\right\| \leq \frac{2}{|2|}\theta\|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned} &\left\| \frac{1}{2^l} f\left(2^l x\right) - \frac{1}{2^m} f\left(2^m x\right)\right\| \\ &\leq \max\left\{\left\| \frac{1}{2^l} f\left(2^l x\right) - \frac{1}{2^{l+1}} f\left(2^{l+1} x\right)\right\|, \cdots, \left\| \frac{1}{2^{m-1}} f\left(2^{m-1} x\right) - \frac{1}{2^m} f\left(2^m x\right)\right\|\right\} \\ &= \max\left\{\frac{1}{|2|^l}\left\| f\left(2^l x\right) - \frac{1}{2} f\left(2^{l+1} x\right)\right\|, \cdots, \frac{1}{|2|^{m-1}}\left\| f\left(2^{m-1} x\right) - \frac{1}{2} f\left(2^m x\right)\right\|\right\} \\ &\leq \max\left\{\frac{|2|^{lr}}{|2|^{l+1}}, \cdots, \frac{|2|^{r(m-1)}}{|2|^{(m-1)+1}}\right\} 2\theta\|x\|^r = \frac{2\theta}{|2|^{(1-r)l+1}}\|x\|^r \end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Now, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional equation (2.1) in non-Archimedean Banach spaces for an even mapping case.

Theorem 2.4. *Let $r < 2$ and θ be nonnegative real numbers and let $f : X \to Y$ be an even mapping satisfying* (2.4)*. Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \leq \frac{|2|}{|2|^r}2\theta\|x\|^r \tag{2.9}$$

for all $x \in X$.

Proof. Letting $x = y = 0$ in (2.4), we get $-f(0) = 2\rho f(0)$. So $f(0) = 0$.

Letting $y = x$ in (2.4), we get

$$\left\| \frac{1}{2} f(2x) - 2f(x) \right\| \le 2\theta \|x\|^r \tag{2.10}$$

for all $x \in X$. So $\|f(x) - 4f\left(\frac{x}{2}\right)\| \le \frac{|2|}{|2|^r} 2\theta \|x\|^r$ for all $x \in X$. Hence

$$\begin{aligned} &\left\| 4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right) \right\| \qquad (2.11) \\ &\le \max\left\{ \left\| 4^l f\left(\frac{x}{2^l}\right) - 4^{l+1} f\left(\frac{x}{2^{l+1}}\right) \right\|, \cdots, \left\| 4^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 4^m f\left(\frac{x}{2^m}\right) \right\| \right\} \\ &= \max\left\{ |4|^l \left\| f\left(\frac{x}{2^l}\right) - 4f\left(\frac{x}{2^{l+1}}\right) \right\|, \cdots, |4|^{m-1} \left\| f\left(\frac{x}{2^{m-1}}\right) - 4f\left(\frac{x}{2^m}\right) \right\| \right\} \\ &\le \max\left\{ \frac{|4|^l}{|2|^{rl}}, \cdots, \frac{|4|^{m-1}}{|2|^{r(m-1)}} \right\} \frac{|2|}{|2|^r} 2\theta \|x\|^r = \frac{2\theta}{|2|^{(r-2)l}} \frac{|2|}{|2|^r} \|x\|^r \end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.11) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} 4^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.11), we get (2.9).

It follows from (2.4) that

$$\begin{aligned} \|M_1 Q(x, y) - \rho M_2 Q(x, y)\| &= \lim_{n\to\infty} |4|^n \left\| M_1 f\left(\frac{x}{2^n}, \frac{y}{2^n}\right) - \rho M_2 f\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \right\| \\ &\le \lim_{n\to\infty} \frac{|4|^n \theta}{|2|^{nr}} (\|x\|^r + \|y\|^r) = 0 \end{aligned}$$

for all $x, y \in X$. So

$$M_1 Q(x, y) = \rho M_2 Q(x, y)$$

for all $x, y \in X$. By Lemma 2.1, the mapping $h : X \to Y$ is quadratic.

Now, let $T : X \to Y$ be another quadratic mapping satisfying (2.9). Then we have

$$\begin{aligned} \|Q(x) - T(x)\| &= \left\| 4^q Q\left(\frac{x}{2^q}\right) - 4^q T\left(\frac{x}{2^q}\right) \right\| \\ &\le \max\left\{ \left\| 4^q Q\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right) \right\|, \left\| 4^q T\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right) \right\| \right\} \le \frac{|2|}{|2|^{(r-2)q+r}} 2\theta \|x\|^r, \end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $Q(x) = T(x)$ for all $x \in X$. This proves the uniqueness of Q. Thus the mapping $Q : X \to Y$ is a unique quadratic mapping satisfying (2.9). □

Theorem 2.5. *Let $r > 2$ and θ be positive real numbers, and let $f : X \to Y$ be an even mapping satisfying* (2.4). *Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \le \frac{2\theta}{|2|} \|x\|^r$$

for all $x \in X$.

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Proof. It follows from (2.10) that

$$\left\| f(x) - \frac{1}{4} f(2x) \right\| \leq \frac{2\theta}{|2|} \|x\|^r$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.4. □

3. Additive-quadratic ρ-functional equation (0.2)

Throughout this section, assume that ρ is a fixed non-Archimedean number with $|\rho| < |2|$.

In this section, we solve the additive-quadratic ρ-functional equation (0.2) in non-Archimedean normed spaces.

Lemma 3.1.
(i) *If an odd mapping $f : X \to Y$ satisfies*

$$M_2 f(x, y) = \rho M_1 f(x, y) \tag{3.1}$$

for all $x, y \in X$, then $f : X \to Y$ is additive.
(ii) *If an even mapping $f : X \to Y$ satisfies $f(0) = 0$ and (3.1), then $f : X \to Y$ is quadratic.*

Proof. (i) Assume that $f : X \to Y$ satisfies (3.1).

Letting $y = 0$ in (3.1), we get

$$2f\left(\frac{x}{2}\right) - f(x) = 0 \tag{3.2}$$

and so $f\left(\frac{x}{2}\right) = \frac{1}{2} f(x)$ for all $x \in X$.

It follows from (3.1) and (3.2) that

$$\begin{aligned} f(x + y) - f(x) - f(y) &= 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \\ &= \rho(f(x + y) - f(x) - f(y)) \end{aligned}$$

and so

$$f(x + y) = f(x) + f(y)$$

for all $x, y \in X$.

(ii) Assume that $f : X \to Y$ satisfies (3.1).

Letting $y = 0$ in (3.1), we get

$$4f\left(\frac{x}{2}\right) - f(x) = 0 \tag{3.3}$$

and so $f\left(\frac{x}{2}\right) = \frac{1}{4} f(x)$ for all $x \in X$.

It follows from (3.1) and (3.3) that

$$\begin{aligned} &\frac{1}{2} f(x + y) + \frac{1}{2} f(x - y) - f(x) - f(y) \\ &= 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \\ &= \rho(\frac{1}{2} f(x + y) + \frac{1}{2} f(x - y) - f(x) - f(y)) \end{aligned}$$

and so

$$f(x + y) + f(x - y) = 2f(x) + 2f(y)$$

for all $x, y \in X$. □

We prove the Hyers-Ulam stability of the additive-quadratic ρ-functional equation (3.1) in non-Archimedean Banach spaces for an odd mapping case.

Theorem 3.2. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an odd mapping such that*

$$\|M_2 f(x,y) - \rho M_1 f(x,y)\| \le \theta(\|x\|^r + \|y\|^r) \tag{3.4}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \theta \|x\|^r \tag{3.5}$$

for all $x \in X$.

Proof. Since f is an odd mapping, $f(0) = 0$.

Letting $y = 0$ in (3.4), we get

$$\left\|2f\left(\frac{x}{2}\right) - f(x)\right\| \le \theta \|x\|^r \tag{3.6}$$

for all $x \in X$. So

$$\begin{aligned}
&\left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| \qquad (3.7)\\
&\le \max\left\{\left\|2^l f\left(\frac{x}{2^l}\right) - 2^{l+1} f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, \left\|2^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\|\right\}\\
&= \max\left\{|2|^l \left\|f\left(\frac{x}{2^l}\right) - 2f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, |2|^{m-1}\left\|f\left(\frac{x}{2^{m-1}}\right) - 2f\left(\frac{x}{2^m}\right)\right\|\right\}\\
&\le \max\left\{\frac{|2|^l}{|2|^{rl}}, \cdots, \frac{|2|^{m-1}}{|2|^{r(m-1)}}\right\}\theta\|x\|^r = \frac{\theta}{|2|^{(r-1)l}}\|x\|^r
\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.7) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.7), we get (3.5).

The rest of the proof is similar to the proof of Theorem 2.2. □

Theorem 3.3. *Let $r > 1$ and θ be positive real numbers, and let $f : X \to Y$ be an odd mapping satisfying* (3.4)*. Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \le \frac{|2|^r\theta}{|2|}\|x\|^r \tag{3.8}$$

for all $x \in X$.

Proof. It follows from (3.6) that

$$\left\|f(x) - \frac{1}{2}f(2x)\right\| \le \frac{|2|^r\theta}{|2|}\|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned}
&\left\|\frac{1}{2^l}f(2^lx) - \frac{1}{2^m}f(2^mx)\right\| \qquad (3.9)\\
&\leq \max\left\{\left\|\frac{1}{2^l}f\left(2^lx\right) - \frac{1}{2^{l+1}}f\left(2^{l+1}x\right)\right\|, \cdots, \left\|\frac{1}{2^{m-1}}f\left(2^{m-1}x\right) - \frac{1}{2^m}f\left(2^mx\right)\right\|\right\}\\
&= \max\left\{\frac{1}{|2|^l}\left\|f\left(2^lx\right) - \frac{1}{2}f\left(2^{l+1}x\right)\right\|, \cdots, \frac{1}{|2|^{m-1}}\left\|f\left(2^{m-1}x\right) - \frac{1}{2}f\left(2^mx\right)\right\|\right\}\\
&\leq \max\left\{\frac{|2|^{rl}}{|2|^{l+1}}, \cdots, \frac{|2|^{r(m-1)}}{|2|^{(m-1)+1}}\right\}|2|^r\theta\|x\|^r = \frac{|2|^r\theta}{|2|^{(1-r)l+1}}\|x\|^r
\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.9) that the sequence $\{\frac{1}{2^n}f(2^nx)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n}f(2^nx)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty}\frac{1}{n}f(2^nx)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.9), we get (3.8).

The rest of the proof is similar to the proofs of Theorems 2.2 and 3.2. □

Now, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional equation (3.1) in non-Archimedean Banach spaces for an even mapping case.

Theorem 3.4. *Let $r < 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an even mapping satisfying* (3.4). *Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \leq \theta\|x\|^r \qquad (3.10)$$

for all $x \in X$.

Proof. Letting $x = y = 0$ in (3.4), we get $2f(0) = \rho f(0)$. So $f(0) = 0$.

Letting $y = 0$ in (3.4), we get

$$\left\|4f\left(\frac{x}{2}\right) - f(x)\right\| \leq \theta\|x\|^r \qquad (3.11)$$

for all $x \in X$. So

$$\begin{aligned}
&\left\|4^lf\left(\frac{x}{2^l}\right) - 4^mf\left(\frac{x}{2^m}\right)\right\| \qquad (3.12)\\
&\leq \max\left\{\left\|4^lf\left(\frac{x}{2^l}\right) - 4^{l+1}f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, \left\|4^{m-1}f\left(\frac{x}{2^{m-1}}\right) - 4^mf\left(\frac{x}{2^m}\right)\right\|\right\}\\
&= \max\left\{|4|^l\left\|f\left(\frac{x}{2^l}\right) - 4f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, |4|^{m-1}\left\|f\left(\frac{x}{2^{m-1}}\right) - 4f\left(\frac{x}{2^m}\right)\right\|\right\}\\
&\leq \max\left\{\frac{|4|^l}{|2|^{rl}}, \cdots, \frac{|4|^{m-1}}{|2|^{r(m-1)}}\right\}\theta\|x\|^r = \frac{\theta}{|2|^{(r-2)l}}\|x\|^r
\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.12) that the sequence $\{4^nf(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{4^nf(\frac{x}{2^n})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty}4^nf(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.12), we get (3.10).

The rest of the proof is similar to the proof of Theorem 2.2. □

Theorem 3.5. *Let $r > 2$ and θ be positive real numbers, and let $f : X \to Y$ be an even mapping satisfying* (3.4)*. Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \leq \frac{|2|^r\theta}{|4|}\|x\|^r \tag{3.13}$$

for all $x \in X$.

Proof. It follows from (3.11) that

$$\left\|f(x) - \frac{1}{4}f(2x)\right\| \leq \frac{|2|^r\theta}{|4|}\|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned} &\left\|\frac{1}{4^l}f(2^l x) - \frac{1}{4^m}f(2^m x)\right\| \qquad (3.14)\\ &\leq \max\left\{\left\|\frac{1}{4^l}f\left(2^l x\right) - \frac{1}{4^{l+1}}f\left(2^{l+1}x\right)\right\|, \cdots, \left\|\frac{1}{4^{m-1}}f\left(2^{m-1}x\right) - \frac{1}{4^m}f\left(2^m x\right)\right\|\right\}\\ &= \max\left\{\frac{1}{|4|^l}\left\|f\left(2^l x\right) - \frac{1}{4}f\left(2^{l+1}x\right)\right\|, \cdots, \frac{1}{|4|^{m-1}}\left\|f\left(2^{m-1}x\right) - \frac{1}{4}f\left(2^m x\right)\right\|\right\}\\ &\leq \max\left\{\frac{|2|^{rl}}{|4|^{l+1}}, \cdots, \frac{|2|^{r(m-1)}}{|4|^{(m-1)+1}}\right\}|2|^r\theta\|x\|^r = \frac{|2|^r\theta}{|2|^{(2-r)l+2}}\|x\|^r \end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.14) that the sequence $\{\frac{1}{4^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n}f(2^n x)\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty}\frac{1}{4^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.14), we get (3.13).

The rest of the proof is similar to the proofs of Theorems 2.2 and 3.4. □

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Complete asymptotic expansions for the genuiune Bernstein-Durrmeyer operator [1]

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Abstract In this paper, we discuss properties of asymptotic approximation of the genuiune Bernstein-Durrmeyer operator, for which establish a complete asymptotic expansion formula of approximation and present the saturation theorems as an application.

Key words The genuiune Bernstein-Durrmeyer operator, Jacobi weight, Complete asymptotic expansion, Approximation, Saturation.

1. INTRODUCTION

The Bernstein-Durrmeyer operator with weights, which is one of the objects of interest in approximation theory of operators and play as an important role in learning theory, is defined as follows

$$M_n^{\omega}(f;x) = \sum_{k=0}^{n} \frac{(f, b_{n,k})_{\omega}}{(e_0, b_{n,k})_{\omega}} b_{n,k}(x)$$

where $e_k = e_k(x) = x^k, k = 0, 1, \cdots$, Bernstein basis functions

$$b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k} \qquad x \in [0,1], k = 0, 1, \cdots, n$$

and inner product weighted ω defined by

$$(f, g)_{\omega} = \int_0^1 f(t)g(t)\omega(t)dt.$$

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A popular case is that the weight ω is taken as the Jacobi weight $\omega(t) = t^\alpha(1-t)^\beta$, $\alpha, \beta > -1$, at this time we denote this operator $M_n^\omega(f)$ by $M_n^{(\alpha,\beta)}(f)$. Because when $\alpha = -1, \beta = -1$, the inner products $(e_0, b_{n,k})_\omega$ are no definition at $k = 0$ and n, W. Chen ([1] 1987), T.N.T. Goodman and A. Sharma ([2] 1987) independently modified the operator into

$$M_n^*(f;x) = Lf + (n-1)\sum_{k=1}^{n-1}(f - Lf, b_{n-2,k-1})b_{k,n}(x)$$

where

$$Lf(x) = (1-x)f(0) + xf(1)$$

and

$$(f,g) = \int_0^1 f(x)g(x)dx.$$

The operator M_n^*f is also called as the genuiune Bernstein-Durrmeyer operator(see [3] or [4]) and can be rewritten into the following form

$$M_n^*(f;x) = f(0)(1-x)^n + f(1)x^n + (n-1)\sum_{k=1}^{n-1}(f, b_{n-2,k-1})b_{n,k}(x),$$

which is a particular case of those operators introduced by D. Cárdenas-Morales and V. Gupta in [5]

$$\begin{aligned} M_{n,\alpha,\beta}(f;x) = & \sum_{k\in l_n} b_{n,k}(x)f(\frac{k}{n}) + (n-\alpha+1) \\ & \cdot \sum_{k=1}^{n-\alpha+\beta}\int_0^1 f(t)\ b_{n-\alpha,k-\beta}(t)dt\ b_{n,k}(x) \end{aligned}$$

where $l_n \subset \{0, 1, \cdots, n\}$. Obviously when $\alpha = 2, \beta = 1$ and $l_n = \{0, n\}$, $M_{n,\alpha,\beta}(f)$ is $M_n^*(f)$, for which there has been extensive research (see [3,4] and [6-10]).

In the next sect, we will establish a complete asymptotic expansion formula for the operator $M_n^*(f)$, and in the section 3, we will present the saturation theorems of approximation as a application of the asymptotic expansion formula.

2. Complete Asymptotic expansions

From [5], one can find that

$$M_n^*(e_0;x) = M_n^*(1,x) = 1; \quad M_n^*(e_1;x) = M_n^*(t;x) = x;$$

$$M_n^*(e_2;x) = M_n^*(t^2,x) = x^2 + \frac{2x(1-x)}{n+1}.$$

For higher order moments of the operator $M_n^*(f)$, we have the following result.

Lemma 2.1 For any natural numbers n, m, there holds that

$$M_n^*(e_m;x) = x(1-x)\sum_{k=0}^{m}\binom{m}{k}\frac{1}{n^{\bar{k}}}\Big[x^{m-1}(1-x)^{k-1}\Big]^{(k)},$$

where the up factorial is defined by

$$n^{\bar{k}} = n(n+1)\cdots(n+k-1), \quad n^{\bar{0}} = 1$$

and the fall factorial, which to be used in the following proof, done by

$$n^{\underline{k}} = n(n-1)\dots(n-k+1), \quad n^{\underline{0}} = 1.$$

Proof Since when $n, m \le 1$, the conclusion is true obviously, we only need to consider the case $n, m \ge 2$. At this moment, we have

$$\begin{aligned}
M_n^*(e_m;x) &= (n-1)\sum_{i=1}^{n-1}(t^m, b_{n-2,i-1})b_{n,i}(x) + x^n \\
&= \frac{(n-1)!}{(m+n-1)!}\sum_{i=1}^{n-1}\frac{(m+i-1)!}{(i-1)!}\binom{n}{i}x^i(1-x)^{n-i} + x^n \\
&= \frac{(n-1)!}{(m+n-1)!}\Big[\frac{d^m}{dx^m}\sum_{i=1}^{n-1}\binom{n}{i}x^{m+i-1}y^{n-i}\Big]_{y=1-x} + x^n \\
&= \frac{(n-1)!}{(m+n-1)!}\frac{d^m}{dx^m}\Big[x^{m-1}(x+y)^n\Big]_{y=1-x} \\
&= \frac{(n-1)!}{(m+n-1)!}\Big[\sum_{v=1}^{m}\binom{m}{v}\frac{(m-1)!}{(v-1)!}x^{v-1}\frac{n!}{(n-v)!}(x+y)^{n-v}\Big]_{y=1-x} \\
&= \frac{1}{n^{\overline{m}}}\sum_{v=1}^{m}\binom{m}{v}(m-1)^{\underline{m-v}}n^{\underline{v}}x^v.
\end{aligned}$$

Using the Vandermonde formula we get

$$n^{\underline{v}} = \sum_{k=0}^{v}\binom{v}{k}\Big(n-(1-m)\Big)^{\underline{k}}(1-m)^{\underline{v-k}},$$

and notice that when $v = 0$, $(m-1)^{\underline{m-v}} = 0$, therefore it follows that

$$M_n^*(e_m;x)$$

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$$= \frac{1}{n^{\overline{m}}}\sum_{v=0}^{m}\binom{m}{v}(m-1)^{\underline{m-v}}x^v\sum_{k=0}^{v}\binom{v}{k}\Big(n-(1-m)\Big)^{\underline{k}}(1-m)^{\underline{v-k}}$$

$$= \sum_{k=0}^{m}\frac{1}{(m-k)!}\frac{1}{n^{\overline{k}}}\sum_{v=0}^{k}(-1)^{k-v}\frac{m!}{v!}\frac{(k+m-v-2)^{\underline{k-v}}}{(k-v)!}(m-1)^{\underline{v}}x^{m-v}.$$

Again using the Vandermonde formula we have

$$\frac{(k+m-v-2)^{\underline{k-v}}}{(k-v)!}(m-1)^{\underline{v}} = (m-1)^{\underline{v}}\sum_{i=v}^{k}\binom{k-v}{i-v}(m-1-v)^{\underline{i}}(k-1)^{\underline{k-i}},$$

hence the second sum in the previous equation can be turned into

$$\sum_{v=0}^{k}(-1)^{k-v}\frac{m!}{v!}x^{m-v}\cdot\sum_{i=v}^{k}\frac{(m-1)^{\underline{i}}}{(i-v)!}\frac{(k-1)^{\underline{k-i}}}{(k-i)!}$$

$$= m!\sum_{i=0}^{k}\frac{(m-1)^{\underline{i}}}{i!}\frac{(k-1)^{\underline{k-i}}}{(k-i)!}\cdot\sum_{v=0}^{i}(-1)^{k-v}\binom{i}{v}x^{m-v}$$

$$= m!\sum_{i=0}^{k}\frac{(m-1)^{\underline{i}}}{i!}\frac{(k-1)^{\underline{k-i}}}{(k-i)!}(-1)^{k-i}x^{m-i}(1-x)^i$$

$$= x(1-x)\frac{m!}{k!}\sum_{i=0}^{k}\binom{k}{i}\frac{d^i}{dx^i}\Big[x^{m-1}\Big]\frac{d^{k-i}}{dx^{k-i}}\Big[(1-x)^{k-1}\Big]$$

$$= x(1-x)\frac{m!}{k!}\Big[x^{m-1}(1-x)^{k-1}\Big]^{(k)}.$$

That completes the proof of Lemma 2.1.

If denoting $\psi_x^s(t) = (t-x)^s$, then we have the following assertion.

Lemma 2.2 For arbitrary natural number s and $x \in [0,1]$, there holds

$$M_n^*(\psi_x^s;x) == x(1-x)\sum_{k=[\frac{s+1}{2}]}^{s}\frac{s!}{k!}\frac{1}{n^{\overline{k}}}\binom{k}{s-k}\Big[x^{k-1}(1-x)^{k-1}\Big]^{(2k-s)}$$

Proof On account of

$$M_n^*(\psi_x^s;x) = \sum_{m=0}^{s}\binom{s}{m}(-x)^{s-m}M_n^*(e_m;x),$$

by Lemma 2.1 we see easily

$$M_n^*(\psi_x^s;x) = x(1-x)\sum_{m=0}^{s}\binom{s}{m}(-x)^{s-m}\cdot\sum_{k=0}^{m}\binom{m}{k}\frac{1}{n^{\overline{k}}}\Big[x^{m-1}(1-x)^{k-1}\Big]^{(k)}$$

$$= x(1-x)\sum_{k=0}^{s}\frac{\binom{s}{k}}{n^{\overline{k}}}\frac{d^k}{dy^k}\Big[\sum_{m=0}^{s-k}\binom{s-k}{m}(-x)^{s-k-m}y^{m-1+k}(1-y)^{k-1}\Big]\Big|_{y=x}$$

$$= x(1-x)\sum_{k=0}^{s}\frac{\binom{s}{k}}{n^{\overline{k}}}\frac{d^k}{dy^k}\Big[y^{k-1}(1-y)^{k-1}(y-x)^{s-k}\Big]\Big|_{y=x},$$

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and recall that

$$\frac{d^k}{dy^k}\Big[y^{k-1}(1-y)^{k-1}(y-x)^{s-k}\Big]_{y=x}$$
$$=\begin{cases}(s-k)!\begin{pmatrix}k\\ s-k\end{pmatrix}\dfrac{d^{2k-s}}{dy^{2k-s}}\Big[y^{k-1}(1-y)^{k-1}\Big]_{y=x}, & s\le 2k\\ 0, & s>2k.\end{cases}$$

The proof is completed.

Remark By Lemma 2.2, we immediately obtain

$$M_n^*(\psi_x^s;x)=O(n^{-[\frac{s+1}{2}]}).$$

For arbitrary natural number q and $x\in I$, let $f\in K[q;x]$ denote the class of functions $f\in B(I)$ (space of bounded functions on I) which q times differentiable at x, then we have the following theorems of approximation.

Lemma 2.3 [11] Let q be arbitrary natural number, $x\in I$ and $A_n: B(I)\to C(I)$(space of continuous functions on I) be a sequence of positive linear operators such that

$$A_n(\psi_x^s;x)=O(n^{-[\frac{s+1}{2}]})\quad(n\to\infty)\quad(s=0,1,\cdots,2q+2),$$

then for arbitrary $f\in K[2q;x]$, there holds

$$A_n(f;x)=\sum_{s=0}^{2q}\frac{f^{(s)}(x)}{s!}A_n(\psi_x^s;x)+o(n^{-q})\quad(n\to\infty).$$

In particular, if $f^{(2q+2)}(x)$ exists, then the $o(n^{-q})$ can be replaced by $O(n^{-q-1})$

Theorem 2.1 For arbitrary natural number q, $x\in[0.1]$ and $f\in K[2q;x]$, there holds that

$$M_n^*(f;x)=\sum_{k=0}^{q}\frac{1}{k!n^k}x(1-x)\Big[x^{k-1}(1-x)^{k-1}f^{(k)}(x)\Big]^{(k)}+o(n^{(-q)})$$

Proof By Lemma 2.2 and Lemma 2.3, we have

$$M_n^*(f;x)=\sum_{s=0}^{2q}\frac{f^{(s)}(x)}{s!}x(1-x)$$
$$\cdot\sum_{k=[\frac{s+1}{2}]}^{s}\frac{s!}{k!}\frac{1}{n^k}\begin{pmatrix}k\\ s-k\end{pmatrix}\Big[x^{k-1}(1-x)^{k-1}\Big]^{(2k-s)}+o(n^{-q}).$$

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Interchanging the two sums and substituting $s+k$ into s, it is followed by

$$(M_n^*(f;x) = x(1-x)\sum_{k=0}^{q}\frac{1}{k!n^{\overline{k}}} \cdot\sum_{s=0}^{k}\binom{k}{s}f^{(s+k)}(x)\Big[x^{k-1}(1-x)^{k-1}\Big]^{(k-s)}+o(n^{-q}).$$

With the Leibniz formula, it becomes

$$M_n^*(f;x) = x(1-x)\sum_{k=0}^{q}\frac{1}{k!n^{\overline{k}}}\cdot[x^{k-1}(1-x)^{k-1}f^{(k)}(x)]^{(k)}+o(n^{-q}).$$

That is the proof of Theorem 2.1.

Remark 1 If taking $q=1$, then we can get so-call the Vonorovskaja type asymptotic expansion formula by Theorem 2.1 as below

$$\lim_{n\longrightarrow\infty} n(M_n^*(f;x)-f(x)) = x(1-x)f''(x)$$

Remark 2 Theorem 2.1 shows that the complete asymptotic expansion formula of the operator $M_n^*(f)$ coincides with which of the operator $M_n^{(\alpha,\beta)}(f)$ at $\alpha=-1,\beta=-1$. This seems to be one of reasons why $M_n^*(f)$ is called as the genuiune Bernstein-Durrmeyer operator.

3. Saturations of Approximation

As an application of the asymptotic formula, in this section we will present the saturation theorems of approximation for the operator M_n^*f. Along with those denotations and signs in [12], denote the space of all continuous functions on $[0,1]$ by $C[0,1]$, for $v(x),\omega(x)\in C[0,1]$ and strictly positive, that is $v(x),\omega(x)>0, x\in(0,1)$, let

$$\varphi(x)=\int_0^x v(t)dt,\quad \psi(x)=\int_0^x \omega(t)dt$$

$$\phi(x)=\int_0^x \psi(t)v(t)dt.$$

For a function $f\in C[0,1]$, we define that

$$Df(x)=D_\psi D_\varphi f(x)=\frac{1}{\omega(x)}\Big[\frac{f'(x)}{v(x)}\Big]'$$

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where

$$D_\varphi f(t) = \frac{f'(t)}{v(t)}.$$

Suppose that $L_n : C[0,1] \to C[0,1]$ is a sequence of positive linear operators, $\{\lambda_n\}$ is a sequence of positive numbers such as $\lambda_n \to\propto$ $(n \to\propto)$ and $\rho(x) \in C[0,1]$ is strictly positive. We say that $\{L_n\}$ satisfy with the Voronovskaya condition if and only if for $Df(x), \{\lambda_n\}$ and $\rho(x)$ as above, there holds

$$\lim_{n\longrightarrow\propto} \lambda_n\{L_n(f;x) - f(x)\} = \rho(x)Df(x), \quad x \in [0,1]$$

Lemma 3.1[12] Suppose that $L_n : C[0.1] \to C[0.1]$ is a sequence of positive linear operators and satisfy the Voronovskaya condition and $G \in C[0,1]$. If for all $x \in (0,1)$

$$\lambda_n\Big|L_n(G;x) - G(x)\Big| \leq M\rho(x) + o_x(1),$$

for some positive constant M, than there exists $D_\varphi G(x) \in C[0,1]$ and

$$\Big|D_\varphi G(y) - D_\varphi G(x)\Big| \leq M\Big|\psi(y) - \psi(x)\Big| \quad x, y \in [0,1],$$

and vice versa.

Remark 1 of Theorem 2.1 shows that the operator $M_n^* f$ satisfies the Voronovskaya condition and

$$v(x) = \omega(x) = 1, \rho(x) = x(1-x), \lambda_n = n$$

$$Df(x) = D_\psi D_\varphi f(x) = f''(x),$$

thus from Lemma 3.1 we can get the following pointwise saturation theorem without any difficult.

Theorem 3.1 (1) Let $f \in C[0,1]$, then for all $x \in [0,1]$, $\lim\limits_{n\longrightarrow\propto} n\{M_n^*(f;x) - f(x)\} = 0$ if and only if $f(x) = A + Bx$;

(2) Let $f \in C[0,1]$, than for all $x \in [0,1]$

$$n\Big|M_n^*(f;x) - f(x)\Big| \leq M\rho(x) + o_x(1)$$

if and only if for any $x, y \in [0,1]$,

$$\Big|f'(y) - f'(x)\Big| \leq M\Big|y - x\Big|.$$

For the operator $M_n^*(f)$, we also establish the following saturation theorem in L_P. Since it is the same as the approach taken for the Kantorovich operator in [13] and [14], here the process is omitted.

Let $f \in L_p[0,1], (p \geq 1)$ denote $|f|^p$ Lebesgue integrable on $[0,1]$,

$$U_p = \begin{cases} \{h | h \in L_p[0,1], h(0) = h(1) = 0 \ \& \ h' \in L_p[0,1]\}, & p > 1 \\ \{h | h \in L_p[0,1], h(0) = h(1) = 0 \ \& \ h \in BV[0,1]\}, & p = 1. \end{cases}$$

and

$$S_p = \Big\{ f | \quad f \in L_p[0,1], \exists h \in U_p, \xi \in (0,1) and\ constants\ c, d\ such\ as \\ f(x) = c + dx + \int_\xi^x (\int_\xi^u \frac{h(t)}{t(1-t)} dt) du \Big\}.$$

Theorem 3.2 Suppose $f \in L_p[0,1] (p \geq 1)$, then

(i) $\parallel M_n^*(f) - f \parallel_p = O(\frac{1}{n})$, if and only if $f \in S_p (p \geq 1)$

(ii) $\parallel M_n^*(f) - f \parallel_p = o(\frac{1}{n})$, if and only if $f(x)$ is a linear function.

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IMPULSIVE PERIODIC SOLUTIONS FOR A SINGULAR DAMPED DIFFERENTIAL EQUATION VIA VARIATIONAL METHODS

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Abstract. In this paper, we study impulsive periodic solutions for second order non-autonomous singular damped differential equations. The proof of the main result relies on a variational approach on mountain pass theorem, together with a truncation technique.

1. Introduction

In this work, we are concerned with the existence of periodic solutions for the following second order non-autonomous singular damped problems

$$\begin{cases} u'' + a(t)u' - \frac{b(t)}{u^\alpha} = g(t), \text{ a.e. } t \in (0,T), \\ u(0) = u(T), \quad u'(0) = u'(T), \end{cases} \tag{1.1}$$

under the impulse conditions

$$\Delta u'(t_j) = u'(t_j^+) - u'(t_j^-) = I_j(u(t_j)),\ j = 1, 2, \ldots, p-1, \tag{1.2}$$

where $u'(t_j^\pm) = \lim_{t\to t_j^\pm} u'(t)$, $\alpha > 1$, $a, g \in C(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$ with $\int_0^T a(t)dt = 0$, and $b \in C(\mathbb{R}/T\mathbb{Z}, (0,\infty))$, t_j for $j = 1, 2, \ldots, p-1$, are the instants when the impulses occur and $0 = t_0 < t_1 < t_2 < \ldots < t_{p-1} < t_p = T, I_j : \mathbb{R} \to \mathbb{R} (j = 1, 2, \ldots, p-1)$ are continuous.

Impulsive effects occur widely in many evolution processes in which their states are changed abruptly at certain moments of time. In recent years, second-order differential boundary value problems with impulses have been studied extensively in the literature [1, 3, 11, 12, 14, 15, 16, 17, 18]. In [18], Tian and Ge studied the existence of solutions for impulsive differential equations:

$$\begin{cases} -(\rho(t)\phi_p(u'(t)))' + s(t)\phi_p(u(t)) = f(t, u(t)),\ t \neq t_j \text{ a.e. } t \in [a,b], \\ -\Delta(\rho(t_j)\phi_p(u'(t_j)))' = I_j(u(t_j)),\ j = 1, 2, \ldots, l, \\ \alpha u'(a) + \beta u(a) = A,\ \gamma u(b) + \sigma u'(b) = B, \end{cases}$$

by using a variational method. Later, Nieto and O'Regan [12] further developed the variational framework for impulsive problems and established existence results for the following impulsive differential equations with Dirichlet boundary conditions:

$$\begin{cases} -u'' + \lambda u(t) = f(t, u(t)),\ t \neq t_j \text{ a.e. } t \in (0,T), \\ \Delta(u'(t_j)) = I_j(u(t_j)),\ j = 1, 2, \ldots, l, \\ u(0) = u(T) = 0, \end{cases}$$

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From then on, the variational method has been a powerful tool in the study of impulsive differential equations.

On the other hand, singular problems without impulse effects have been investigated extensively in the literature. Usually, in the literature, the proof is based on variational methods [10], or topological methods[2, 4, 6, 7, 8, 9], which were started with the pioneering paper of Lazer and Solimini [5].

In 1987, Lazer and Solimini [5] considered a the second order singular problem

$$u''(t) - \frac{1}{u^{\alpha}(t)} = e(t), \quad t \in (0, T). \tag{1.3}$$

By using the method of upper and lower solutions, they obtained a famous sufficient and necessary condition on positive T-periodic solution for Problem (1.3) as follows

Theorem 1.1[5] Assume that $e \in L^1([0, T], \mathbb{R})$ is T-periodic. Then Problem (1.3) has a positive T-periodic weak solution if and only if $\int_0^T e(t)dt < 0$.

Motivated by the above fact, in the present paper we shall consider Problem (1.1) with impulsive effects, In general cases, it is impossible to apply variational methods to Eq.(1.1) when $\int_0^T a(t)dt > 0$. In this paper, using a variant of the mountain pass theorem, we consider the case $\int_0^T a(t)dt = 0$, on an appropriate Sobolev space, we establish the corresponding variational framework of periodic solutions to guarantee the existence of at least one nontrivial solution of Eq.(1.1).

In order to state our main result, we need the following assumptions:

(H_1) $a \in C(\mathbb{R}/T\mathbb{Z})$ with $\int_0^T a(t)dt = 0$;
(H_2) $b \in C(\mathbb{R}/T\mathbb{Z}, (0, \infty))$ is $T-$periodic and $b'(t) \geq 0$ for all $t \in [0, T]$;
(H_3) $g \in L^2([0, T], \mathbb{R})$ is $T-$periodic and $\int_0^T g(t)dt < 0$;
(H_4) There exist two constants m, M such that for any $t \in \mathbb{R}$,

$$m \leq I_j(t) \leq M, \ j = 1, 2, \ldots, p-1,$$

where $m < 0$ and $0 \leq M < -\frac{1}{p-1}\int_0^T g(t)dt$;
(H_5) For any $t \in \mathbb{R}$,

$$\int_0^t I_j(s)ds \geq 0, \ j = 1, 2, \ldots, p-1.$$

Theorem 1.1 Assume that $(H_1) - (H_5)$ are satisfied. Then problem (1.1)-(1.2) has at least one solution.

The rest of this paper is organized as follows. In Section 2, some preliminary results will be given. In Section 3, by the use of variational method, we will state and prove the main results.

2. PRELIMINARIES

In this section, we present some results which will be applied in Sections 3.

Let

$$H_T^1 = \{u : [0, T] \to \mathbb{R} \mid u \text{ is absolutely continuous, } u(0) = u(T) \text{ and } u' \in L^2([0, T], \mathbb{R})\}$$

with the inner product

$$(u, v) = \int_0^T u(t)v(t)dt + \int_0^T u'(t)v'(t)dt, \quad \forall u, v \in H_T^1.$$

The corresponding norm is defined by

$$\|u\|_{H_T^1} = \left(\int_0^T |u(t)|^2 dt + \int_0^T |u'(t)|^2 dt) \right)^{\frac{1}{2}}, \ \forall u \in H_T^1.$$

Then H_1^T is a Banach space.

If $u \in H_T^1$, then u is absolutely continuous and $u' \in L^2([0,T],\mathbb{R})$. In this case, $\Delta u'(t) = u'(t^+) - u'(t^-) = 0$ is not necessarily valid for every $t \in (0,T)$ and the derivative u' may exist some discontinuities. It may lead to impulse effects.

From (1.1), we get

$$-\left[e^{A(t)}u'(t)\right]' + e^{A(t)}\left[\frac{b(t)}{u^\alpha} + g(t)\right] = 0, \tag{2.1}$$

where $A(t) = \int_0^t a(s)ds$. Following the ideas of [12], take $v \in H_T^1$ and multiply the two sides of (2.1) by v and integrate from 0 to T, so we have

$$\int_0^T -\left[e^{A(t)}u'(t)\right]' v(t)dt + \int_0^T e^{A(t)}\left[\frac{b(t)}{u^\alpha} + g(t)\right] v(t)dt = 0 \tag{2.2}$$

Note that, since $u'(0) = u'(T)$, one has

$$\begin{aligned}
&\int_0^T \left[e^{A(t)}u'(t)\right]' v(t)dt \\
&= \sum_{j=0}^{p-1} \int_{t_j}^{t_{j+1}} \left[e^{A(t)}u'(t)\right]' v(t)dt \\
&= \sum_{j=0}^{p-1} e^{A(t)}\left[u'(t_{j+1}^-)v(t_{j+1}^-) - u'(t_{j+1}^+)v(t_{j+1}^+)\right] - \sum_{j=0}^{p-1}\int_{t_j}^{t_{j+1}} e^{A(t)}u'(t)v'(t)dt \\
&= e^{A(T)}u'(T)v(T) - e^{A(0)}u'(0)v(0) - \sum_{j=0}^{p-1} e^{A(t)}\Delta u'(t_j)v(t_j) - \int_0^T e^{A(t)}u'(t)v'(t)dt \\
&= -e^{A(t)}\sum_{j=1}^{p-1} I_j(u(t_j))v(t_j) - \int_0^T e^{A(t)}u'(t)v'(t)dt.
\end{aligned}$$

Combining with (2.2), we get

$$\int_0^T e^{A(t)}\left[u'(t)v'(t) + \frac{b(t)}{u^\alpha}v(t)dt + g(t)v(t)\right]dt + e^{A(t)}\sum_{j=1}^{p-1} I_j(u(t_j))v(t_j) = 0.$$

As a result we introduce the following concept of a weak solution for problem (1.1)-(1.2).

Definition 2.1 We say that a function $u \in H_T^1$ is a weak solution of problem (1.1)-(1.2) if

$$\int_0^T e^{A(t)}\left[u'(t)v'(t) + \frac{b(t)}{u^\alpha}v(t)dt + g(t)v(t)\right]dt + e^{A(t)}\sum_{j=1}^{p-1} I_j(u(t_j))v(t_j) = 0$$

holds for any $v \in H_T^1$.

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Define the functional $\Phi : H_T^1 \to \mathbb{R}$ by

$$\Phi(u) := \int_0^T e^{A(t)} \left[\frac{1}{2}|u'(t)|^2 + b(t)\int_1^{u(t)} \frac{1}{s^\alpha} ds + g(t)u(t)\right] dt + e^{A(t)} \sum_{j=1}^{p-1} \int_0^{u(t_j)} I_j(s)ds$$

for every $u \in H_T^1$. Under the conditions of Theorem 1.2, it is easy to verify that Φ is well defined on H_T^1, continuously differentiable and weakly lower semi-continuous. Moreover, the critical points of Φ are the weak solutions of problem (1.1)-(1.2).

In next section, the following version of the mountain pass theorem will be used in our argument.

Theorem 2.2 [13] Let X be a Banach space and let $\varphi \in C(X, \mathbb{R})$. Assume that there exist $x_0, x_1 \in X$ and a bounded open neighborhood Ω of x_0 such that $x_1 \in X/\bar{\Omega}$ and

$$\max\{\varphi(x_0), \varphi(x_1)\} < \inf_{x \in \partial\Omega} \varphi(x).$$

Let

$$\Gamma = \{h \in C([0,1], X) : h(0) = x_0, h(1) = x_1\}$$

and

$$c = \inf_{h \in \Gamma} \max_{s \in [0,1]} \varphi(h(s)).$$

If φ satisfies the (PS)-condition, i.e., a sequence $\{u_n\}$ in X satisfying $\varphi(u_n)$ is bounded and $\varphi'(u_n) \to 0$ as $n \to \infty$ has a convergent subsequence, then c is a critical value of φ and $c > \max\{\varphi(x_0), \varphi(x_1)\}$.

3. Main results

In order to study problem (1.1)-(1.2), for any $\lambda \in (0, 1)$ we consider the following modified problem

$$\text{(3.1)} \qquad \begin{cases} u'' + a(t)u' + b(t)f_\lambda(u(t)) = g(t), \ a.e. \ t \in (0,1), \\ \Delta u'(t_j) = I_j(u(t_j)), \ j = 1, 2, \ldots, p-1, \\ u(0) = u(T), \quad u'(0) = u'(T), \end{cases}$$

where $f_\lambda : [0, T] \times \mathbb{R} \to \mathbb{R}$ is defined by

$$f_\lambda(u) = \begin{cases} -\frac{1}{u^\alpha}, \ u \geq \lambda, \\ -\frac{1}{\lambda^\alpha}, \ u < \lambda. \end{cases}$$

Let $F_\lambda(u) = \int_1^u f_\lambda(s)ds$

$$\Phi_\lambda : H_T^1 \to \mathbb{R}$$

defined by

$$\Phi_\lambda(u) := \int_0^T e^{A(t)} \left[\frac{1}{2}|u'(t)|^2 - b(t)F_\lambda(u(t))dt + g(t)u(t)\right] dt + e^{A(t)} \sum_{j=1}^{p-1} \int_0^{u(t_j)} I_j(s)ds.$$

Clearly, Φ_λ is well defined on H_T^1, continuously differentiable and weakly lower semi-continuous. Moreover, the critical points of Φ_λ are the weak solutions of problem (3.1).

Proof. The proof will be divided into four steps.

Step 1. Φ_λ satisfies the Palais-Smale condition.

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Let a sequence $\{u_n\}$ in H_T^1 satisfy $\Phi_\lambda(u_n)$ is bounded and $\Phi'_\lambda(u_n) \to 0$ as $n \to +\infty$. That is, there exist a constant $c_1 > 0$ and a sequence $\{\epsilon_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ with $\epsilon_n \to 0$ as $n \to +\infty$ such that, for all n,

$$(3.2) \qquad \Big| \int_0^T e^{A(t)} \left[\frac{1}{2}|u_n'(t)|^2 - b(t)F_\lambda(u_n(t))dt + g(t)u_n(t)\right] dt + e^{A(t)} \sum_{j=1}^{p-1} \int_0^{u_n(t_j)} I_j(s)ds \Big| \le c_1,$$

and for every $v \in H_T^1$,

$$(3.3) \qquad \Big| \int_0^T e^{A(t)} \left[u_n'(t)v'(t) - b(t)f_\lambda(u_n(t))v(t) + g(t)v(t)\right] dt + e^{A(t)} \sum_{j=1}^{p-1} I_j(u_n(t_j))v(t_j) \Big| \le \epsilon_n \|v\|_{H_T^1}.$$

Using a standard argument, it is suffices to show that $\{u_n\}$ is bounded when verifying the (PS)-condition.

Taking $v(t) \equiv -1$ in (3.3), one has

$$\left| \int_0^T e^{A(t)} \left[b(t)f_\lambda(u_n(t)) - g(t)\right] dt - e^{A(t)} \sum_{j=1}^{p-1} I_j(u_n(t_j)) \right| \le \epsilon_n \sqrt{T} \text{ for all } n.$$

By (H_3), we have

$$\begin{aligned} \left| \int_0^T e^{A(t)} b(t) f_\lambda(u_n(t)) \right| &\le \epsilon_n \sqrt{T} + \left| \int_0^T e^{A(t)} g(t) dt + \right| + e^{A(t)} \sum_{j=1}^{p-1} |I_j(u_n(t_j))| \\ &\le \epsilon_n \sqrt{T} + e^{\|a\|_{L^1}} \left| \int_0^T g(t) \right| + e^{\|a\|_{L^1}} (p-1)M := c_2. \end{aligned}$$

Note that for any $t \in [0,T], b(t)f_\lambda(u_n(t)) < 0$. Thus

$$\int_0^T \left| e^{A(t)} b(t) f_\lambda(u_n(t)) \right| dt = \left| \int_0^T e^{A(t)} b(t) f_\lambda(u_n(t)) \right| \le c_2.$$

On the other hand, if we take, in (3.3), $v(t) \equiv w_n(t) := u_n(t) - \bar{u}_n$, where $\bar{u}_n$ is the average of u_n over the interval $[0,T]$, we have

$$\begin{aligned} c_3\|w\|_{H_T^1} &\ge \left| \int_0^T e^{A(t)} \left[w_n'(t)^2 - b(t)f_\lambda(u_n(t))w_n(t) + g(t)w_n(t)\right] dt + e^{A(t)} \sum_{j=1}^{p-1} I_j(u_n(t_j))w_n(t_j) \right| \\ &\ge e^{-\|a\|_{L^1}} \|w_n'\|_{L^2} - (c_2 + e^{\|a\|_{L^1}} \|g\|_{L^1}) \|w_n\|_{L^\infty} + e^{-\|a\|_{L^1}} (p-1)m \|w_n\|_{L^\infty} \\ &\ge e^{-\|a\|_{L^1}} \|w_n'\|_{L^2} - (c_2 + e^{\|a\|_{L^1}} \|g\|_{L^1} - e^{-\|a\|_{L^1}} (p-1)m) \|w_n\|_{L^\infty} \\ &\ge e^{-\|a\|_{L^1}} \|w_n'\|_{L^2} - c_4 \|w_n\|_{H_T^1}, \end{aligned}$$

where c_3 and c_4 are two positive constants. Consequently, using the Wirtinger inequality for zero mean functions in the Sobolev space H_T^1, there exists $c_5 > 0$ such that

$$(3.4) \qquad \|u_n'\|_{L^2}^2 \le \|w_n\|_{H_T^1} \le c_5.$$

Now, suppose that

$$\|u_n\|_{H_T^1} \to +\infty \text{ as } n \to +\infty.$$

Since (3.4) holds, we have, passing to subsequence if necessary, that either

$$M_n := \max u_n \to +\infty \text{ as } n \to +\infty, \text{ or}$$
$$m_n := \min u_n \to -\infty \text{ as } n \to +\infty,$$

(i) Assume that the first possibility occurs. By (H_3) and the fact that $f_\lambda < 0$, one has

$$\begin{aligned}
&\int_0^T e^{A(t)}\left[b(t)F_\lambda(u_n(t)) - g(t)u_n(t)\right]dt - e^{A(t)}\sum_{j=1}^{p-1}\int_0^T I_j(s)ds \\
\geq & \int_0^T e^{A(t)}\left[\int_1^{u_n(t)} b(t)f_\lambda(s)ds - g(t)u_n(t)\right]dt - e^{\|a\|_{L^1}}(p-1)MM_n \\
= & \int_0^T e^{A(t)}\left[\int_1^{M_n(t)} b(t)f_\lambda(s)ds - \int_{u_n(t)}^{M_n} b(t)f_\lambda(s)ds - g(t)u_n(t)\right]dt \\
& -e^{\|a\|_{L^1}}(p-1)MM_n \\
= & \int_0^T e^{A(t)}b(t)F_\lambda(M_n)dt - \int_0^T e^{A(t)}M_n g(t)dt - \int_0^T\left[\int_{u_n(t)}^{M_n} e^{A(t)}\left(b(t)f_\lambda(s) - g(t)\right)ds\right]dt \\
& -e^{\|a\|_{L^1}}(p-1)MM_n \\
\geq & \int_0^T e^{A(t)}\left[b(t)F_\lambda(M_n) - M_n g(t)\right]dt + \int_0^T e^{A(t)}(M_n - u_n(t))g(t)dt - e^{\|a\|_{L^1}}(p-1)MM_n \\
\geq & \int_0^T e^{A(t)}\left[b(t)F_\lambda(M_n) - M_n g(t)\right]dt - e^{\|a\|_{L^1}}\|M_n - u_n\|\|g\|_{L^1} - e^{\|a\|_{L^1}}(p-1)MM_n.
\end{aligned}$$

Thus, using Sobolev and Poincare's inequalities, one has

$$\begin{aligned}
& -e^{\|a\|_{L^1}}\left((p-1)M + \int_0^T e^{A(t)}g(t)dt\right)M_n \\
\leq & \int_0^T e^{A(t)}[b(t)F_\lambda(u_n(t)) - g(t)u_n(t)]dt - e^{A(t)}\sum_{j=1}^{p-1}\int_0^{u_n(t_j)} I_j(s)ds \\
& +\sqrt{T}e^{\|a\|_{L^1}}\|g\|_{L^1}\|u_n'\|_{L^2} - F_\lambda(M_n)e^{\|a\|_{L^1}}\int_0^T b(t)dt \\
\leq & \int_0^T e^{A(t)}[b(t)F_\lambda(u_n(t)) - g(t)u_n(t)]dt - e^{A(t)}\sum_{j=1}^{p-1}\int_0^{u_n(t_j)} I_j(s)ds \\
& +\sqrt{T}e^{\|a\|_{L^1}}\|g\|_{L^1}\|u_n'\|_{L^2} - \frac{\int_0^T b(t)dt}{\alpha - 1}\left(\frac{1}{M_n^{\alpha-1}} - 1\right)e^{\|a\|_{L^1}},
\end{aligned}$$

From (3.2),(3.4) and the fact that $\frac{1}{M_n^{\alpha-1}} \to 0$ as $n \to +\infty$, we see that the right hand side of the above inequality is bounded, which is contradiction.

(ii) Assume the second possibility occurs, i.e., $m_n \to -\infty$ as $n \to +\infty$. We replace M_n by m_n in the preceding arguments, and we also get a contradictions.

Therefore, Φ_λ satisfies the Palais-Smale condition. This completes the proof of the claim.

Step 2. Let

$$\Omega = \left\{ u \in H_T^1 | \min_{t\in[0,T]} u(t) > 1 \right\},$$

and

$$\partial\Omega = \left\{ u \in H_T^1 | \min_{t\in[0,T]} u(t) \geq 1 \text{ for all } t \in (0,T), \exists t_u \in (0,T) \text{ such that } u(t_n) = 1 \right\}.$$

We show that there exists $d > 0$ such that $\inf_{u\in\partial\Omega} \Phi_\lambda(u) \geq -d$ whenever $\lambda \in (0,1)$.

For any $u \in \partial\Omega$, there exists some $t_u \in (0,T)$ such that $\min_{t\in[0,T]} u(t) = u(t_n) = 1$. By ($H_4$) and extending the functions by T-periodicity, we obtain that

$$\begin{aligned}
\Phi_\lambda(u) &= \int_{t_u}^{t_u+T} e^{A(t)} \left[\frac{1}{2}|u'(t)|^2 - b(t)F_\lambda(u(t))dt + g(t)u(t)\right] dt + e^{A(t)} \sum_{j=1}^{p-1} \int_0^{u(t_j)} I_j(s)ds \\
&\geq \frac{1}{2}\int_{t_u}^{t_u+T} e^{A(t)}u'(t)^2dt + \frac{1}{\alpha-1}\int_{t_u}^{t_u+T} e^{A(t)}b(t)\left(1 - \frac{1}{u(t)^{\alpha-1}}\right)dt \\
&\quad + \int_{t_u}^{t_u+T} e^{A(t)}g(t)(u(t)-1)dt + \int_{t_u}^{t_u+T} e^{A(t)}g(t)dt \\
&\geq \frac{1}{2}\int_{t_u}^{t_u+T} e^{A(t)}u'(t)^2dt + \int_{t_u}^{t_u+T} e^{A(t)}g(t)(u(t)-1)dt + \int_{t_u}^{t_u+T} e^{A(t)}g(t)dt.
\end{aligned}$$

By the Schwarz inequality and the fact that $u'(t) = (u(\cdot)-1)'(t)$, one has

$$\Phi_\lambda(u) \geq \frac{e^{-\|a\|_{L^1}}}{2}\|(u(\cdot)-1)'\|_{L^2} - e^{\|a\|_{L^1}}\|g\|_{L^2}\|u(\cdot)-1\|_{L^2} - e^{\|a\|_{L^1}}\|g\|_{L^1}.$$

Applying Poincare's inequality to $u(\cdot)-1$, we get

$$\Phi_\lambda(u) \geq \frac{e^{-\|a\|_{L^1}}}{2}\|(u(\cdot)-1)'\|_{L^2} - e^{\|a\|_{L^1}}\gamma\|g\|_{L^2}\|u'\|_{L^2} - e^{\|a\|_{L^1}}\|g\|_{L^1},$$

where $\gamma = \gamma(t_u)$. The above inequality shows that

$$\Phi_\lambda(u) \to +\infty \quad \text{as } \|u'\|_{L^2} \to +\infty.$$

Since $\min_{t\in[0,T]} u(t) = 1$, we have that $\|u(\cdot)-1\|_{H_T^1} \to +\infty$ is equivalent to $\|u'\|_{L^2} \to +\infty$. Hence

$$\Phi_\lambda(u) \to +\infty \quad \text{as } \|u\|_{H_T^1} \to +\infty, \forall u \in \partial\Omega,$$

which shows that Φ_λ is coercive. Thus it has a minimizing sequence. The weak lower semi-continuity of Φ_λ yields

$$\inf_{u\in\partial\Omega} \Phi_\lambda(u) > -\infty.$$

It follows that there exists $d > 0$ such that $\inf\limits_{u\in\partial\Omega} \Phi_\lambda(u) > -d$ for all $\lambda \in (0,1)$.

Step 3. We show that there exists $\lambda_0 \in (0,1)$ with the property that, for every $\lambda \in (0,\lambda_0)$, any solution u of problem (3.1) satisfying $\Phi_\lambda(u) > -d$ such that $\min_{u\in[0,T]} u(t) \geq \lambda_0$, and hence u is a solution of problem (1.1)-(1.2).

Assume on the contrary that there are sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\{u_n\}_{n\in\mathbb{N}}$ such that
(i) $\lambda_n \leq \frac{1}{n}$;
(ii) u_n is a solution of (3.1) with $\lambda = \lambda_n$;

(iii) $\Phi_{\lambda_n}(u_n) \geq -d$;
(iv) $\min_{t\in[0,T]} u_n(t) < \frac{1}{n}$.

Since $f_{\lambda_n} < 0$ and

$$\int_0^T e^{A(t)}[b(t)f_{\lambda_n}(u_n(t)) - g(t)]dt = 0, \tag{3.5}$$

one has

$$\|e^{A(t)}b(\cdot)f_{\lambda_n}(u_n(\cdot))\|_{L^1} \leq c_6, \quad \text{for some constant } c_6 > 0.$$

On the other hand, since $u_n(0) = u_n(T)$, there exists $\tau_n \in (0,T)$ such that

$$u_n'(\tau_n) = 0.$$

Therefore, we obtain that

$$e^{A(t)}u_n'(t) - e^{A(\tau_n)}u_n'(\tau_n) = \int_{\tau_n}^t e^{A(s)}[f_\lambda(u_n(s)) - g(s)]ds,$$

which, from (3.5), yields that

$$\|u_n'\|_{L^\infty} \leq c_7 \quad \text{for some constant } c_7 > 0. \tag{3.6}$$

$$\inf_{u\in\partial\Omega} \Phi_\lambda(u) > -\infty.$$

From $\Phi_{\lambda_n}(u_n) \geq -d$, it follows that there must exist two constants R_1 and R_2, with $0 < R_1 < R_2$ such that

$$\max\{u_n(t); t \in [0,T]\} \subset [R_1, R_2].$$

If not, u_n would tend uniformly to 0 or $+\infty$. In both cases, by $(\mathrm{H}_2) - (\mathrm{H}_3)$ and (3.6), we have

$$\Phi_{\lambda_n}(u_n) \to -\infty \text{ as } n \to +\infty,$$

which contradicts $\Phi_{\lambda_n}(u_n) \geq -d$.

Let τ_n^1, τ_n^2 be such that, for n large enough

$$u_n(\tau_n^1) = \frac{1}{n} < R_1 = u_n(\tau_n^2).$$

Multiplying the differential equation in (3.1) by u_n' and integrating the equation on $[\tau_n^1, \tau_n^2]$, (or $[\tau_n^2, \tau_n^1]$), we get

$$\begin{aligned}\Psi &:= \int_{\tau_n^1}^{\tau_n^2} u_n''(t)u_n'(t)dt + \int_{\tau_n^1}^{\tau_n^2} a(t)u_n'(t)^2(t)dt + \int_{\tau_n^1}^{\tau_n^2} b(t)f_{\lambda_n}(u_n(t))u_n'(t)dt \\ &= \int_{\tau_n^1}^{\tau_n^2} g(t)u_n'(t)dt.\end{aligned}$$

It is easy to verify that

$$\Psi = \Psi_1 + \frac{1}{2}\left[u_n'(\tau_n^2) - u_n'(\tau_n^1)\right] + \int_{\tau_n^2}^{\tau_n^1} a(t)u_n'^2(t)dt,$$

where

$$\Psi_1 = \int_{\tau_n^1}^{\tau_n^2} b(t)f_{\lambda_n}(u_n(t))u_n'(t)dt.$$

From (H_2) and (3.6), it follows that Ψ is bounded, and consequently Ψ_1 is bounded.

On the other hand, it is easy to see that

$$b(t)f_{\lambda_n}(u_n(t))u_n'(t) = \frac{d}{dt}[b(t)F_{\lambda_n}(u_n(t))] - b'(t)F_{\lambda_n}(u_n(t)).$$

Thus, by (H_1) we have

$$\begin{aligned}\Psi_1 &= b(\tau_n^2)F_{\lambda_n}(R_1) - b(\tau_n^1)F_{\lambda_n}\left(\frac{1}{n}\right) - \int_{\tau_n^1}^{\tau_n^2} b'(t)F_{\lambda_n}(u_n(t))dt \\ &\le b(\tau_n^2)F_{\lambda_n}(R_1) - b(\tau_n^1)F_{\lambda_n}\left(\frac{1}{n}\right) - \frac{1}{\alpha-1}\int_{\tau_n^1}^{\tau_n^2} b'(t)\left(\frac{1}{R_2^{\alpha-1}} - 1\right)dt.\end{aligned}$$

From the fact that $F_{\lambda_n}(\frac{1}{n}) \to +\infty$ as $n \to +\infty$, we obtain $\Psi_1 \to -\infty$, i.e., Ψ_1 is unbounded. This is a contradiction.

Step 4. We prove that Φ_λ has a mountain-pass geometry for $\lambda \le \lambda_0$.

Fix $\lambda \in (0, \lambda_0]$, one has

$$\begin{aligned}F_\lambda(0) &= \int_1^0 f_\lambda(s)ds = -\int_0^1 f_\lambda(s)ds \\ &= -\int_0^\lambda f_\lambda(s)ds - \int_\lambda^1 f_\lambda(s)ds \\ &= \frac{1}{\lambda^{\alpha-1}} - \int_\lambda^1 f_\lambda(s)ds.\end{aligned}$$

This implies that

$$F_\lambda(0) > -\int_\lambda^1 f_\lambda(s)ds = \int_1^\lambda f_\lambda(s)ds = F_\lambda(\lambda).$$

Hence

$$\begin{aligned}\Phi_\lambda(0) &= -F_\lambda(0)\int_0^T e^{A(t)}b(t)dt < -F_\lambda(\lambda)\int_0^T e^{A(t)}b(t)dt \\ &\le -\frac{\int_0^T e^{A(t)}b(t)dt}{\alpha-1}\left(\frac{1}{\lambda^{\alpha-1}} - 1\right).\end{aligned} \tag{3.7}$$

Consider $\lambda \in (0, \lambda_0]$ such that

$$\frac{1}{\lambda^{\alpha-1}} > 1 + \frac{d(\alpha-1)}{\int_0^T e^{A(t)}b(t)dt}.$$

Thus it follows from (3.7) that $\Phi_\lambda(0) < -d$.

Also, using (H_3) we can choose $R > 1$ enough large such that

$$-e^{\|a\|_{L^1}}\left(M(p-1) + \int_0^T g(t)dt\right)R - \frac{e^{\|a\|_{L^1}}\int_0^T b(t)dt}{\alpha-1}\left(1 - \frac{1}{R^{\alpha-1}}\right) > d.$$

Thus,

$$\begin{aligned}
\Phi_\lambda(R) &= e^{A(t)}\sum_{j=1}^{p-1}\int_0^R I_j(s)ds - F_\lambda(R)\int_0^T e^{A(t)}b(t)dt + R\int_0^T g(t)dt \\
&\leq e^{\|a\|_{L^1}}M(p-1)R + \frac{1}{\alpha-1}e^{\|a\|_{L^1}}\left(1-\frac{1}{R^{\alpha-1}}\right)\int_0^T b(t)dt \\
&\quad + Re^{\|a\|_{L^1}}\int_0^T g(t)dt \\
&= e^{\|a\|_{L^1}}\left(M(p-1)+\int_0^T g(t)dt\right)R + e^{\|a\|_{L^1}}\frac{\int_0^T b(t)dt}{\alpha-1}\left(1-\frac{1}{R^{\alpha-1}}\right) \\
&< -d.
\end{aligned}$$

Since Ω is a neighborhood of $R, 0 \notin \Omega$ and

$$\max\{\Phi_\lambda(0), \Phi_\lambda(R)\} < \inf_{x\in\partial\Omega}\Phi_\lambda(u).$$

Step 1 and Step 2 imply that Φ_λ has a critical point u_λ such taht

$$\Phi_\lambda(u_\lambda) = \inf_{h\in\Gamma}\max_{s\in[0,1]}\Phi_\lambda(h(s)) \geq \inf_{x\in\partial\Omega}\Phi_\lambda(u),$$

where

$$\Gamma = \{h \in C([0,1], H_T^1) : h(0) = 0, h(1) = R\}.$$

Since $\inf_{u\in\partial\Omega}\Phi_\lambda(u_\lambda) \geq -d$, it follows from Step 3 that u_λ is a solution of problem (1.1)-(1.2). The proof of the main result is complete. □

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Accelerated SNS and accelerated SSS iteration methods for non-Hermitian linear systems

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Abstract. Recently, Bai proposed the skew-normal splitting (SNS) and skew-scaling splitting (SSS) iteration methods for large sparse non-Hermitian positive definite systems. Compared with the Hermitian and skew-Hermitian splitting (HSS) iteration method, both of the SNS and SSS methods are making more use of the skew-Hermitian parts than the HSS method. In this paper, we introduce an accelerated skew-normal splitting (ASNS) iteration method and an accelerated skew-scaling splitting (ASSS) iteration method for solving large sparse non-Hermitian positive definite system of linear equations. We study the convergence properties of the the new iteration methods and the quasi-optimal parameters. Moreover, the inexact forms of the new methods are proposed by employing some subspace methods as the inner iteration processes at each step of the outer iterations. Numerical experiments are given to verify the correctness of the theoretical results and the effectiveness of the new methods.

Mathematics Subject Classification (2010). 65F10; 65F50.

Keywords. HSS iteration method, skew-normal splitting, skew-scaling splitting, quasi-optimal parameters, non-Hermitian positive definite system.

1. Introduction

Consider the numerical solution of the large sparse system of linear equations of the form

$$Ax = b, \quad A \in \mathbb{C}^{n\times n} \quad \text{and} \quad x, b \in \mathbb{C}^n, \tag{1.1}$$

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where A is a non-Hermitian positive definite matrix. The linear system (1.1) arises from many scientific computing areas, such as diffuse optical tomography [1], lattice quantum chromo dynamics [16], structural dynamics [15], eddy current problems [11] and so on. See [2] and references therein for more applications of the linear system of the form (1.1).

Based on the Hermitian and skew-Hermitian (HS) splitting of the coefficient matrix A: $A = H + S$, with

$$H = \frac{1}{2}(A + A^*), \quad S = \frac{1}{2}(A - A^*),$$

Bai, Golub and Ng [2] present and studied an efficient Hermitian and skew-Hermitian splitting (HSS) iteration method. Because of the unconditionally convergent property and effectiveness, the HSS iteration method has captured a lot of researchers' attention. A multitude of researchers focused on the HSS method and proposed varieties of variants based on the Hermitian and skew-Hermitian splitting, such as the HSS-like method [9], the modified HSS method [10], the accelerated HSS method [5] and the preconditioned HSS method [6, 3] and so on, see [12, 20, 22, 17, 4, 18]. As is shown in [7] that the HSS method is more effective when S dominates H than vice versa. Furthermore, when S is very small compared to H, the subsystem

$$(\alpha I + S)x^{(k+1)} = (\alpha I - H)x^{(k+\frac{1}{2})} + b$$

in the HSS method contributes little to convergence and the inner iterations must be designed to terminate properly since

$$(I - \frac{1}{\alpha}S)^{-1} = I + \frac{1}{\alpha}S + \frac{1}{\alpha^2}S^2 + \cdots .$$

As is known that the iterative matrix of HSS

$$(\alpha I + S)^{-1}(\alpha I - H)(\alpha I + H)^{-1}(\alpha I - S)$$

is similar to $W(\alpha)Q(\alpha)$, where

$$W(\alpha) = (I - \frac{1}{\alpha}H)(I + \frac{1}{\alpha}H)^{-1}$$

is Hermitian, so forth $\|W(\alpha)\|_2 < 1$, and

$$Q(\alpha) = (I - \frac{1}{\alpha}S)(I + \frac{1}{\alpha}S)^{-1}$$

is unitary for all $\alpha > 0$. Therefore, according to [7], when S is small compared to H, different iterations combing H and S^*S are more naturally.

Suppose that S is invertible, then we can multiply (1.1) on the left by $-S$ to obtain the following equivalent form:

$$-SHx - S^2x = -Sb, \tag{1.2}$$

i.e.,

$$-SHx = S^2x - Sb \quad \text{or} \quad - S^2x = SHx - Sb.$$

Adding αH to each side will lead to the following two fixed-point equations

$$(\alpha H - SH)x \equiv (\alpha I - S)Hx = (\alpha H + S^2)x - Sb, \tag{1.3}$$

$$(\alpha H - S^2)x = (\alpha H + SH)x - Sb \equiv (\alpha I + S)Hx - Sb. \tag{1.4}$$

Based on the above fixed-point equations, Bai [7] established the skew-normal splitting (SNS) methods as the following algorithm.

Algorithm 1.1. *(SNS method) Given an initial approximate solution $x^{(0)}$, for $k = 0, 1, 2, \cdots$ until convergence, solve*

$$\begin{cases} (\alpha I - S)x^{(k+\frac{1}{2})} = (\alpha H + S^2)x^{(k)} - Sb, \\ (\alpha H - S^2)x^{(k+1)} = (\alpha I + S)x^{(k+\frac{1}{2})} - Sb, \end{cases}$$

where α is a given positive constant.

Another way to use the skew-Hermitian matrix S is to employ it to scale the linear system (1.1). By first adding and then subtracting $\frac{1}{\alpha}S^2x$ to the fixed-point equations

$$-Sx = Hx - b \quad \text{and} \quad Hx = -Sx + b,$$

respectively, it follows,

$$(I - \frac{1}{\alpha}S)(-Sx) = (H + \frac{1}{\alpha}S^2)x - b \quad \text{and} \quad (H - \frac{1}{\alpha}S^2)x = (I + \frac{1}{\alpha}S)(-Sx) + b.$$

In analogy to the SNS method, Bai further present the skew-scaling splitting (SSS) method in [7] as the following algorithm.

Algorithm 1.2. *(SSS method) Given an initial approximate solution $x^{(0)}$, for $k = 0, 1, 2, \cdots$, until convergence, solve*

$$\begin{cases} (\alpha I - S)x^{(k+\frac{1}{2})} = (\alpha H + S^2)x^{(k)} - \alpha b, \\ (\alpha H - S^2)x^{(k+1)} = (\alpha I + S)x^{(k+\frac{1}{2})} + \alpha b, \end{cases}$$

where α is a given positive constant.

For the SNS and SSS methods, it has been shown in [7] that when S is small compared to H, the Corollary 2.3 in [2] makes clear how to choose a good iterative parameter. When S dominates H, it is not clear. However, the SNS and SSS methods give an exact way to choose the optimal iterative parameter no matter S dominates H or not.

In this paper, we will first accelerate the SNS and SSS methods by adding another parameter in the second iterate step of each iterations and then we obtain two new methods, which are named as the ASNS method and the ASSS method. The new methods can be seen as generalized forms of the original SNS and SSS methods. Futher, the iterative parameters can be chosen in a more extensive range.

The outline of this paper is arranged as follows. In Section 2, we present the accelerated SNS (ASNS) and the accelerated SSS (ASSS) methods. Then we analyze the convergence properties of both methods. In Section 3, we determine the quasi-optimal parameters by minimizing the upper bound of the spectral radius of the iteration matrix and then show the case about the new methods superiority to the SNS and SSS methods. Section 4 is devoted to the inexact variant form of the new methods and the asymptotically convergent rate property of the new methods. Numerical experiments are given in Section 5 to illustrate the correctness of the theoretical results obtained in this paper. Section 6 draws some conclusions and remarks to end this paper.

Throughout this paper, the skew-Hermitian part S of the coefficient matrix A is assumed to be invertible and the Hermitian part H to be positive definite. We use $A \sim B$ to denote that the matrix A is similar to the matrix B.

2. The ASNS and ASSS methods

In this section, we will propose the accelerated SNS (ASNS) and the accelerated SSS (ASSS) methods. Firstly, we add αH to each side of the equation $-SHx = S^2x - Sb$ and then add βH to each side of the equation $-S^2x = SHx - Sb$. Then we obtain the fixed-point equations

$$(\alpha H - SH)x \equiv (\alpha I - S)Hx = (\alpha H + S^2)x - Sb \tag{2.1}$$

and

$$(\beta H - S^2)x = (\beta H + SH)x - Sb \equiv (\beta I + S)Hx - Sb. \tag{2.2}$$

Subsequent algorithm is the ASNS iteration method.

Algorithm 2.1. *(ASNS method) Given an initial approximate solution $x^{(0)}$, for $k = 0, 1, 2, \cdots$ until convergence, solve*

$$\begin{cases} (\alpha I - S)x^{(k+\frac{1}{2})} = (\alpha H + S^2)x^{(k)} - Sb, \\ (\beta H - S^2)x^{(k+1)} = (\beta I + S)x^{(k+\frac{1}{2})} - Sb, \end{cases}$$

where α and β are given positive constants.

For the fixed-point equations

$$-Sx = Hx - b \quad \text{and} \quad Hx = -Sx + b,$$

we first add $\frac{1}{\alpha}S^2x$ and then subtract $\frac{1}{\beta}S^2x$ on both sides to obtain

$$(I - \frac{1}{\alpha}S)(-Sx) = (H + \frac{1}{\alpha}S^2)x - b$$

and

$$(H - \frac{1}{\beta}S^2)x = (I + \frac{1}{\beta}S)(-Sx) + b.$$

After rearranging these equations and choosing $x^{(0)}$ wisely, we can straightforwardly get the accelerated skew-scaling splitting (ASSS) method as the next algorithm.

Algorithm 2.2. *(ASSS method) Given an initial approximate solution $x^{(0)}$, for $k = 0, 1, 2, \cdots$, until convergence, solve*

$$\begin{cases} (\alpha I - S)x^{(k+\frac{1}{2})} = (\alpha H + S^2)x^{(k)} - \alpha b, \\ (\beta H - S^2)x^{(k+1)} = (\beta I + S)x^{(k+\frac{1}{2})} + \beta b, \end{cases}$$

where α and β are given positive constants.

Obviously, when $\alpha=\beta$, the ASNS method and the ASSS method reduce to the SNS method and the SSS method, respectively. Comparing the ASSS method with the ASNS method, we find that the coefficient matrices of the ASNS and ASSS method are exactly the same. Therefore, they have the same iteration matrix $M(\alpha,\beta)$, where

$$M(\alpha,\beta):=(\beta H-S^2)^{-1}(\beta I+S)(\alpha I-S)^{-1}(\alpha H+S^2).$$

It can be seen that the ASSS method is much cheaper than the ASNS method, because of the constant vector terms αb, βb instead of Sb.

It is seen that S^{-1} is skew-Hermitian, then $-S^{-1}HS^{-1}=(S^{-1})^*HS^{-1}$ is Hermitian positive definite. Denote

$$\xi_{\max}=\max\{|\xi_j|\big|\mathbf{i}\xi_j\in\sigma(S)\},\xi_{\min}=\min\{|\xi_j|\big|\mathbf{i}\xi_j\in\sigma(S)\},$$

$$\lambda_{\max}=\max\{\lambda_j\big|\lambda_j\in\sigma((S^{-1})^*HS^{-1})\},\lambda_{\min}=\min\{\lambda_j\big|\lambda_j\in\sigma((S^{-1})^*HS^{-1})\},$$

the following theorem concentrates on the convergence property of the ASNS method.

Theorem 2.1. *Given a non-Hermitian matrix $A\in\mathbb{C}^{n\times n}$. Let $H=\frac{1}{2}(A+A^*)$ and $S=\frac{1}{2}(A-A^*)$. If H is positive definite , S is invertible, then the spectral radius $\rho(M(\alpha,\beta))$ of the ASNS and ASSS iteration matrix is bounded by $\delta(\alpha,\beta)$, where*

$$\delta(\alpha,\beta)=\max_{\mathbf{i}\xi_j\in\sigma(S)}\sqrt{\frac{\beta^2+\xi_j^2}{\alpha^2+\xi_j^2}}\cdot\max_{\lambda_k\in\sigma((S^{-1})^*HS^{-1})}\Big|\frac{\alpha\lambda_k-1}{\beta\lambda_k+1}\Big|.$$

Further, if α and β satisfy $\delta(\alpha,\beta)<1$, then the ASNS method and the ASSS method are convergent to the unique solution of the linear system (1.1).

Proof. As

$$\begin{aligned}M(\alpha,\beta)&=(\beta H-S^2)^{-1}(\beta I+S)(\alpha I-S)^{-1}(\alpha H+S^2)\\&\sim S^{-1}(\beta I+S)(\alpha I-S)^{-1}(\alpha H+S^2)(\beta H-S^2)^{-1}S\\&=(\beta I+S)S^{-1}(\alpha I-S)^{-1}S(-S^{-1})(\alpha H-S^*S)S^{-1}S(\beta H+S^*S)^{-1}(-S)\\&=(\beta I+S)(\alpha I-S)^{-1}(\alpha(S^{-1})^*HS^{-1}-I)(\beta(S^{-1})^*HS^{-1}+I)^{-1}\\&:=\overline{M}(\alpha,\beta),\end{aligned}\tag{2.3}$$

then it follows

$$\begin{aligned}&\rho(M(\alpha,\beta))\\&\le\|(\beta I+S)(\alpha I-S)^{-1}\|\cdot\|(\alpha(S^{-1})^*HS^{-1}-I)(\beta(S^{-1})^*HS^{-1}+I)^{-1}\|\\&\le\max_{\mathbf{i}\xi_j\in\sigma(S^{-1})}\sqrt{\frac{\beta^2+\xi_j^2}{\alpha^2+\xi_j^2}}\cdot\max_{\lambda_k\in\sigma(((S))^*HS^{-1})}\Big|\frac{\alpha\lambda_k-1}{\beta\lambda_k+1}\Big|\\&=\delta(\alpha,\beta)\end{aligned}$$

If $\delta(\alpha,\beta)<1$, then $\rho(M(\alpha,\beta))\le\delta(\alpha,\beta)<1$, i.e., the ASNS and the ASSS methods are convergent. □

Theorem 2.2. *The ASNS and ASSS iteration methods are convergent if either of the following conditions holds:*
(1) $\alpha \geq \beta$, $(\lambda_{\min}^2\xi_{\max}^2 - 1)\alpha - 2\xi_{\max}^2\lambda_{\min} < \beta(\lambda_{\min}^2\xi_{\max}^2 - 1 + 2\alpha\lambda_{\min})$ *and* $(\lambda_{\max}^2\xi_{\max}^2 - 1)\alpha - 2\xi_{\max}^2\lambda_{\max} < \beta(\lambda_{\max}^2\xi_{\max}^2 - 1 + 2\alpha\lambda_{\max})$*;*
(2) $\alpha < \beta$, $(\lambda_{\min}^2\xi_{\min}^2 - 1)\alpha - 2\xi_{\min}^2\lambda_{\min} < \beta(\lambda_{\min}^2\xi_{\min}^2 - 1 + 2\alpha\lambda_{\min})$ *and* $(\lambda_{\max}^2\xi_{\min}^2 - 1)\alpha - 2\xi_{\min}^2\lambda_{\max} < \beta(\lambda_{\max}^2\xi_{\min}^2 - 1 + 2\alpha\lambda_{\max})$.

Proof. $\delta(\alpha,\beta) < 1$ leads to

$$\max_{\mathrm{i}\xi_j\in\sigma(S)}\sqrt{\frac{\beta^2+\xi_j^2}{\alpha^2+\xi_j^2}}\cdot\max_{\lambda_k\in\sigma((S^{-1})^*HS^{-1})}\left|\frac{\alpha\lambda_k-1}{\beta\lambda_k+1}\right| < 1.$$

Or equivalently,

$$\max_{\lambda_k\in\sigma((S^{-1})^*HS^{-1})}\left|\frac{\alpha\lambda_k-1}{\beta\lambda_k+1}\right| < \min_{\mathrm{i}\xi_j\in\sigma(S)}\sqrt{\frac{\alpha^2+\xi_j^2}{\beta^2+\xi_j^2}}. \tag{2.4}$$

(1) If $\alpha \geq \beta$, then

$$\min_{\mathrm{i}\xi_j\in\sigma(S)}\sqrt{\frac{\alpha^2+\xi_j^2}{\beta^2+\xi_j^2}} = \sqrt{\frac{\alpha^2+\xi_{\max}^2}{\beta^2+\xi_{\max}^2}}.$$

Since

$$\max_{\lambda_k\in\sigma((S^{-1})^*HS^{-1})}\left|\frac{\alpha\lambda_k-1}{\beta\lambda_k+1}\right| = \max\left\{\left|\frac{\alpha\lambda_{\min}-1}{\beta\lambda_{\min}+1}\right|,\left|\frac{\alpha\lambda_{\max}-1}{\beta\lambda_{\max}+1}\right|\right\},$$

then (2.4) is equivalent to

$$\begin{cases}\left(\dfrac{\alpha\lambda_{\min}-1}{\beta\lambda_{\min}+1}\right)^2 < \dfrac{\alpha^2+\xi_{\max}^2}{\beta^2+\xi_{\max}^2},\\[2ex] \left(\dfrac{\alpha\lambda_{\max}-1}{\beta\lambda_{\max}+1}\right)^2 < \dfrac{\alpha^2+\xi_{\max}^2}{\beta^2+\xi_{\max}^2}.\end{cases}$$

After some simple computations, it follows

$$\begin{cases}(\alpha\lambda_{\min}-1)^2(\beta^2+\xi_{\max}^2) < (\alpha^2+\xi_{\max}^2)(\beta\lambda_{\min}+1)^2,\\ (\alpha\lambda_{\max}-1)^2(\beta^2+\xi_{\max}^2) < (\alpha^2+\xi_{\max}^2)(\beta\lambda_{\max}+1)^2.\end{cases} \tag{2.5}$$

Because α and β are positive constants, the first equation of (2.5) leads to

$$(\lambda_{\min}^2\xi_{\max}^2 - 1)\alpha - 2\xi_{\max}^2\lambda_{\min} < \beta(\lambda_{\min}^2\xi_{\max}^2 - 1 + 2\alpha\lambda_{\min}).$$

The second equation of (2.5) leads to

$$(\lambda_{\max}^2\xi_{\max}^2 - 1)\alpha - 2\xi_{\max}^2\lambda_{\max} < \beta(\lambda_{\max}^2\xi_{\max}^2 - 1 + 2\alpha\lambda_{\max}).$$

(2) If $\alpha < \beta$, then

$$\min_{\mathrm{i}\xi_j\in\sigma(S)}\sqrt{\frac{\alpha^2+\xi_j^2}{\beta^2+\xi_j^2}} = \sqrt{\frac{\alpha^2+\xi_{\min}^2}{\beta^2+\xi_{\min}^2}}.$$

Using the same strategy as $\alpha \geq \beta$, we can easily obtain

$$(\lambda_{\min}^2\xi_{\min}^2 - 1)\alpha - 2\xi_{\min}^2\lambda_{\min} < \beta(\lambda_{\min}^2\xi_{\min}^2 - 1 + 2\alpha\lambda_{\min})$$

and

$$(\lambda_{\max}^2\xi_{\min}^2 - 1)\alpha - 2\xi_{\min}^2\lambda_{\max} < \beta(\lambda_{\max}^2\xi_{\min}^2 - 1 + 2\alpha\lambda_{\max}).$$

□

Remark 2.3. *If $\alpha = \beta$, from the result (1) of Theorem 2.2, we obtain that when $\alpha^2+\xi_{\min}^2 > 0$, the ASNS and ASSS iteration methods are convergent. Because $\alpha^2 + \xi_{\min}^2 > 0$ invariably holds, then we know the ASNS and ASSS iteration methods are convergent unconditionally, which agrees with the results in* [7].

3. The quasi-optimal iterative parameters

In this section, we will give the quasi-optimal parameters of the ASNS iteration method and the ASSS iteration method. Then we will analyze the optimal parameters by minimizing the upper bound $\delta(\alpha, \beta)$. According to the analysis in the previous section, we have

$$\delta(\alpha,\beta) = \begin{cases} \sqrt{\dfrac{\beta^2+\xi_{\min}^2}{\alpha^2+\xi_{\min}^2}} \cdot \max\left\{\left|\dfrac{\alpha\lambda_{\min}-1}{\beta\lambda_{\min}+1}\right|, \left|\dfrac{\alpha\lambda_{\max}-1}{\beta\lambda_{\max}+1}\right|\right\}, & \alpha \geq \beta; \\ \sqrt{\dfrac{\beta^2+\xi_{\max}^2}{\alpha^2+\xi_{\max}^2}} \cdot \max\left\{\left|\dfrac{\alpha\lambda_{\min}-1}{\beta\lambda_{\min}+1}\right|, \left|\dfrac{\alpha\lambda_{\max}-1}{\beta\lambda_{\max}+1}\right|\right\}, & \alpha < \beta. \end{cases}$$

We rewrite the matrix $\overline{M}(\alpha, \beta)$ in (2.3) as

$$\begin{aligned} &\overline{M}(\alpha,\beta) \\ &= (\frac{1}{\beta}I + S^{-1})(\frac{1}{\alpha}I - S^{-1})^{-1}(\frac{1}{\alpha}I - (S^{-1})^*HS^{-1})(\frac{1}{\beta}I + (S^{-1})^*HS^{-1})^{-1} \\ &\sim (\frac{1}{\beta}I + (S^{-1})^*HS^{-1})^{-1}(\frac{1}{\alpha}I - (S^{-1})^*HS^{-1})(\frac{1}{\alpha}I - S^{-1})^{-1}(\frac{1}{\beta}I + S^{-1}). \end{aligned}$$

By making use of the same strategy of Theorem 4.2 in [19], we can obtain the optimal iterative parameters and the corresponding upper bound in the following theorem.

Theorem 3.1. *Let $A \in \mathbb{C}^{n\times n}$ be a non-Hermitian matrix. If the ASNS and ASSS iteration methods are convergent, then*
(1) when $\sqrt{\lambda_{\min}\lambda_{\max}} > \frac{1}{\xi_{\min}}$ or $\sqrt{\lambda_{\min}\lambda_{\max}} < \frac{1}{\xi_{\max}}$, the quasi-optimal parameters are given by

$$\alpha_* = \frac{\sqrt{(c^2+\lambda_{\min}^2)(c^2+\lambda_{\max}^2)} + c^2 - \lambda_{\min}\lambda_{\max}}{(\lambda_{\max}+\lambda_{\min})c^2},$$
$$\beta_* = \frac{\sqrt{(c^2+\lambda_{\min}^2)(c^2+\lambda_{\max}^2)} - c^2 + \lambda_{\min}\lambda_{\max}}{(\lambda_{\max}+\lambda_{\min})c^2},$$

and the corresponding optimal upper bound $\delta(\alpha^*, \beta^*)$ of $\rho(M(\alpha, \beta))$ is

$$\delta(\alpha^*,\beta^*) = \frac{\sqrt{(c^2+\lambda_{\min}^2)(c^2+\lambda_{\max}^2)} - (c^2 + \lambda_{\min}\lambda_{\max})}{(\lambda_{\max}-\lambda_{\min})c},$$

where the constant c is given by

$$c = \begin{cases} \dfrac{1}{\xi_{\max}}, & \text{if} \quad \sqrt{\lambda_{\max}\lambda_{\min}} > \dfrac{1}{\xi_{\min}}; \\ \dfrac{1}{\xi_{\min}}, & \text{if} \quad \sqrt{\lambda_{\max}\lambda_{\min}} < \dfrac{1}{\xi_{\max}}. \end{cases}$$

(2) when $\frac{1}{\xi_{\max}} \leq \sqrt{\lambda_{\min}\lambda_{\max}} \leq \frac{1}{\xi_{\min}}$, *the quasi-optimal parameters* α_* *and* β_* *are*

$$\alpha_* = \beta_* = \frac{1}{\sqrt{\lambda_{\min}\lambda_{\max}}}$$

and the corresponding optimal upper bound is

$$\delta(\alpha_*, \beta_*) = \frac{\sqrt{\lambda_{\max}} - \sqrt{\lambda_{\min}}}{\sqrt{\lambda_{\max}} + \sqrt{\lambda_{\min}}}.$$

Remark 3.2. *When* $e_{\min} \leq \sqrt{\lambda_{\min}\lambda_{\max}} \leq e_{\max}$, *the optimal upper bound* $\delta(\alpha, \beta)$ *of the ASNS and GSSS iteration methods reduces to the optimal bound of the SNS and SSS iteration methods, respectively, i.e.,*

$$\alpha_* = \beta_* = \frac{1}{\sqrt{\lambda_{\min}\lambda_{\max}}}$$

and the corresponding optimal bound is

$$\nu_* = \frac{\sqrt{\lambda_{\max}} - \sqrt{\lambda_{\min}}}{\sqrt{\lambda_{\max}} + \sqrt{\lambda_{\min}}}.$$

Theorem 3.3. *When* $\sqrt{\lambda_{\min}\lambda_{\max}} > \frac{1}{\xi_{\min}}$ *or* $\sqrt{\lambda_{\min}\lambda_{\max}} < \frac{1}{\xi_{\max}}$, *then we have* $\delta(\alpha_*, \beta_*) < \nu_*$, *where* $\delta(\alpha_*, \beta_*)$ *and* ν_* *are defined in Theorem* 3.1 *and Remark* 3.2, *respectively.*

Proof. We rewrite ν_* as

$$\nu_* = \frac{(\sqrt{\lambda_{\max}} - \sqrt{\lambda_{\min}})^2}{\lambda_{\max} - \lambda_{\min}}.$$

Then $\delta(\alpha_*, \beta_*) < \nu_*$ if and only if

$$(\lambda_{\min}^2 + c^2)(\lambda_{\max}^2 + c^2) < [c(\sqrt{\lambda_{\min}} - \sqrt{\lambda_{\min}})^2 + (\lambda_{\min}\lambda_{\min} + c^2)]^2.$$

Denote $k_1 = \lambda_{\max} + \lambda_{\min}$ and $k_2 = \lambda_{\max}\lambda_{\min}$, then the above inequality can be simplified as

$$4c^2k_2 - 2c^2k_1\sqrt{k_2} + ck_1k_2 - 2c^2c^3\sqrt{k_2} - 2ck_2\sqrt{k_2} + c^3k_1 > 0.$$

Or equivalently,

$$(k_1 - 2\sqrt{k_2})(\sqrt{k_2} - c)^2 > 0.$$

It follows

$$(\sqrt{\lambda_{\max}} - \sqrt{\lambda_{\min}})^2(\sqrt{\lambda_{\min}\lambda_{\max}} - c)^2 > 0,$$

i.e., $(\sqrt{\lambda_{\min}\lambda_{\max}} - c)^2 > 0$. That is, $\sqrt{\lambda_{\min}\lambda_{\max}} > \frac{1}{\xi_{\min}}$ or $\sqrt{\lambda_{\min}\lambda_{\max}} < \frac{1}{\xi_{\max}}$. □

4. The inexact ASNS and ASSS methods

The two half-steps at each step of the ASNS and ASSS methods require finding solutions with the coefficient matrices $\alpha I - S$ and $\beta H - S^2$. Because $\beta H - S^2$ is Hermitian positive definite, then we may use the CG method to solve the linear system with coefficient matrix $\beta H - S^2$. For solving the linear system with coefficient matrix $\alpha I - S$, we may use some Krylov subspace method [21]. The inexact ASNS (IASNS) method can be described as follows.

Algorithm 4.1. *(IASNS method) Given an initial guess $\bar{x}^{(0)}$, for $k = 0, 1, 2, \cdots$, until $\{\bar{x}^{(k)}\}$ converges, solve $\{\bar{x}^{(k+\frac{1}{2})}\}$ approximately from*

$$(\alpha I - S)\bar{x}^{(k+\frac{1}{2})} = (\alpha H + S^2)\bar{x}^{(k)} - Sb$$

by employing an inner iteration (e.g., some Krylov subspace method) with $\bar{x}^{(k)}$ as the initial guess, then solve $\bar{x}^{(k+1)}$ approximately from

$$(\beta H - S^2)\bar{x}^{(k+1)} = (\beta I + S)\bar{x}^{(k+\frac{1}{2})} - Sb$$

by employing an inner iteration (e.g., the CG method) with $\bar{x}^{(k+\frac{1}{2})}$ as the initial guess, where α and β are given positive constants.

The inexact ASSS (IASSS) method can be described as follows.

Algorithm 4.2. *(IASSS method) Given an initial guess $\bar{x}^{(0)}$, for $k = 0, 1, 2, \cdots$, until $\{\bar{x}^{(k)}\}$ converges, solve $\{\bar{x}^{(k+\frac{1}{2})}\}$ approximately from*

$$(\alpha I - S)\bar{x}^{(k+\frac{1}{2})} = (\alpha H + S^2)\bar{x}^{(k)} - \alpha b$$

by employing an inner iteration (e.g., some Krylov subspace method) with $\bar{x}^{(k)}$ as the initial guess, then solve $\bar{x}^{(k+1)}$ approximately from

$$(\beta H - S^2)\bar{x}^{(k+1)} = (\beta I + S)\bar{x}^{(k+\frac{1}{2})} + \beta b$$

by employing an inner iteration (e.g., the CG method) with $\bar{x}^{(k+\frac{1}{2})}$ as the initial guess, where α and β are given positive constants.

Subsequently, we will concentrate on the IASNS iteration method. The results about the IASSS iteration method can be obtained in the similar way. To simplify numerical implementation and convergence analysis, the IASNS iteration method can be rewritten as the following equivalent scheme.

Given an initial guess $\bar{x}^{(0)}$, for $k = 0, 1, 2, \cdots$, compute Step 1 and Step 2 until $\bar{x}^{(k)}$ converges:

Step 1. approximate the solution of

$$(\alpha I - S)\bar{z}^{(k)} = -S\bar{r}^{(k)}, (\bar{r}^{(k)} = b - A\bar{x}^{(k)})$$

by iterating until $\bar{z}^{(k)}$ is such that the residual

$$\bar{p}^{(k)} = \bar{r}^{(k)} - (\alpha I - S)\bar{z}^{(k)}$$

satisfies

$$\|\bar{p}^{(k)}\| \le \varepsilon_k \|\bar{r}^{(k)}\|,$$

and then compute $\bar{x}^{(k+\frac{1}{2})} = \bar{x}^{(k)} + \bar{z}^{(k)}$;

Step 2. approximate the solution of

$$(\beta H - S^2)\bar{z}^{(k+\frac{1}{2})} = -S\bar{r}^{(k+\frac{1}{2})}, (\bar{r}^{(k+\frac{1}{2})} = b - A\bar{x}^{(k+\frac{1}{2})})$$

by iterating until $\bar{z}^{(k+\frac{1}{2})}$ is such that the residual

$$\bar{q}^{(k+\frac{1}{2})} = \bar{r}^{(k+\frac{1}{2})} - (\beta H - S^2)\bar{z}^{(k+\frac{1}{2})}$$

satisfies

$$\|\bar{p}^{(k+\frac{1}{2})}\| \le \eta_k \|\bar{r}^{(k+\frac{1}{2})}\|,$$

and then compute $\bar{x}^{(k+1)} = \bar{x}^{(k+\frac{1}{2})} + \bar{z}^{(k+\frac{1}{2})}$. Here $\|\cdot\|$ is a norm of a vector.

If the two inner systems are solved inexactly with corresponding quantities $\{\varepsilon_k\}$ and $\{\eta_k\}$. Denote $\varepsilon_{\max} = \max_k\{\varepsilon_k\}$ and $\eta_{\max} = \max_k\{\eta_k\}$. Let $|||\cdot|||_M$ denote

$$|||X|||_M = \|MXM^{-1}\|$$

for all $X \in \mathbb{C}^{n\times n}$. Then the following theorem concentrates on the convergent results of the IASNS and IASSS iteration methods. According to Theorem 3.1 in [8] and by specializing the splitting as

$$\begin{aligned}(-S)A = M_1 - N_1 &:= (\alpha H - SH) - (\alpha H + S^2)\\ = M_2 - N_2 &:= (\beta H - S^2) - (\beta H + SH),\end{aligned}$$

we can immediately obtain the following theorem.

Theorem 4.1. *Let $A \in \mathbb{C}^{n\times n}$ be a non-Hermitian positive definite matrix. $H = \frac{1}{2}(A + A^*)$ and $S = \frac{1}{2}(A - A^*)$ be its Hermitian and skew-Hermitian parts, and let α and β be positive constants. If S is invertible, $\{\bar{x}^{(k)}\}$ is an iterative sequence generated by the IASNS iteration method and $x^* \in \mathbb{C}^n$ is the exact solution of the linear system (1.1), then it holds that*

$$\begin{aligned}&|||\bar{x}^{(k+1)} - x^*|||_{M_2}\\ &\quad \le (\sigma(\alpha,\beta) + \mu(\alpha,\beta)\theta(\beta)\varepsilon_k + \theta(\beta)(\rho(\alpha,\beta) + \theta(\beta)\nu(\alpha,\beta)\varepsilon_k)\eta_k)\\ &\qquad \cdot |||\bar{x}^{(k)} - x^*|||_{M_2},\end{aligned}$$

where

$$\begin{aligned}\sigma(\alpha,\beta) &= \|(\beta I + S)(\alpha I - S)^{-1}(\alpha H + S^2)(\beta H - S^2)^{-1}\|,\\ \rho(\alpha,\beta) &= \|(\beta H - S^2)H^{-1}(\alpha I - S)^{-1}(\alpha H + S^2)(\beta H - S^2)^{-1}\|,\\ \mu(\alpha,\beta) &= \|(\beta I + S)(\alpha I - S)^{-1}\|,\\ \theta(\beta) &= \|(-SA)(\beta H - S^2)^{-1}\|,\\ \nu(\alpha,\beta) &= \|(\beta H - S^2)H^{-1}(\alpha I - S)^{-1}\|.\end{aligned}$$

In particularly, if

$$\sigma(\alpha,\beta) + \mu(\alpha,\beta)\theta(\beta)\varepsilon_{\max} + \theta(\beta)(\rho(\alpha,\beta) + \theta(\beta)\nu(\alpha,\beta)\varepsilon_{\max})\eta_{\max} < 1,$$

then the iterative sequence $\{\bar{x}^{(k)}\}$ converges to $x^* \in \mathbb{C}^n$, where

$$\varepsilon_{\max} = \max_k\{\varepsilon_k\} \quad \textit{and} \quad \eta_{\max} = \max_k\{\eta_k\}.$$

Theorem 4.2. *Let the assumption in Theorem* 4.1 *be satisfied. Suppose that both* $\{\tau_1(k)\}$ *and* $\{\tau_2(k)\}$ *are nondecreasing and positive sequences satisfying* $\tau_1(k) \geq 1$ *and* $\tau_2(k) \geq 1$, *and*

$$\lim_{k\to\infty} \sup \tau_1(k) = \lim_{k\to\infty} \sup \tau_2(k) = +\infty,$$

and that both δ_1 *and* δ_2 *are real constants in the interval* $[0, 1]$ *satisfying*

$$\varepsilon_k \leq c_1 \delta_1^{\tau_1(k)} \quad \text{and} \quad \eta_k \leq c_2 \delta_2^{\tau_2(k)},$$

for $k = 0, 1, 2, \cdots$, *where* c_1 *and* c_2 *are nonnegative constants. Then we have*

$$|||\bar{x}^{(k+1)} - x^*|||_{M_2} \leq (\sqrt{\sigma(\alpha, \beta)} + \omega(\alpha, \beta)\theta(\beta)\delta^{\tau(k)})^2 \cdot |||\bar{x}^{(k)} - x^*|||_{M_2},$$

where $k = 0, 1, 2, \cdots$, $\tau(k) = \min\{\tau_1(k), \tau_2(k)\}, \delta = \max\{\delta_1, \delta_2\}$ *and*

$$\omega = \max\{\sqrt{c_1 c_2 \nu(\alpha, \beta)}, \frac{1}{2\sqrt{\sigma(\alpha, \beta)}}(c_1\mu(\alpha, \beta) + c_2\rho(\alpha, \beta))\}.$$

In particular, we have

$$\lim_{k\to\infty} \sup \frac{|||\bar{x}^{(k+1)} - x^*|||_{M_2}}{|||\bar{x}^{(k)} - x^*|||_{M_2}} = \sigma(\alpha, \beta),$$

i.e., the convergence rate of the IASNS method is asymptotically the same as that of the exact two-step iterative scheme ASNS.

5. Numerical results

In this section, we will consider the three-dimensional convection-diffusion equation

$$-(u_{xx} + u_{yy} + u_{zz}) + q(u_x + u_y + u_z) = f(x, y, z) \tag{5.1}$$

on the unit cube $\Omega = [0, 1] \times [0, 1] \times [0, 1]$, with constant coefficient q and subject to Dirichlet-type boundary conditions. When the seven-point finite difference discretization, for example, the centered differences to diffusive terms, and the centered differences or the first order upwind approximations to the convective terms are applied to the above model convection-diffusion equation, we get the system of linear equations (1.1) with the coefficient matrix

$$A = T_x \otimes I \otimes I + I \otimes T_y \otimes I + I \otimes I \otimes T_z,$$

where the equidistant step-size $h = \frac{1}{n+1}$ is used in the distretization on all of the three directions and the natural lexicographic ordering is employed to the unknowns. In addition, $\otimes$ denotes the Kronecker product, and T_x, T_y, and T_z are tri-diagonal matrices given by

$$T_x = tridiag(t_2, t_1, t_3), \quad T_y = tridiag(t_2, 0, t_3), \quad T_z = tridiag(t_2, 0, t_3),$$

with

$$t_1 = 6, t_2 = -1 - r, t_3 = -1 + r$$

if the first order derivatives are approximated by the centered difference scheme and with

$$t_1 = 6 + 6r, t_2 = -1 - 2r, t_3 = -1$$

if the first order derivatives are approximated by the upwind difference scheme. Here $r = \frac{qh}{2}$ is the mesh Reynolds number. For details, we refer to [13, 14, 2].

Figure 1 plots $\rho(M(\alpha,\beta))$ and $\delta(\alpha,\beta)$ for the centered difference scheme with $n = 8$ and $q = 1, 10$ when α varies in $[0,1]$. When α is fixed, according to [19], we can compute β as

$$\frac{1-\alpha\lambda_{\min}}{\beta\lambda_{\min}+1} = \frac{\alpha\lambda_{\max}-1}{\beta\lambda_{\max}+1}.$$

That is,

$$\beta = \frac{\alpha(\lambda_{\max}+\lambda_{\min})-2}{\lambda_{\max}+\lambda_{\min}-2\alpha\lambda_{\max}\lambda_{\min}}. \tag{5.2}$$

Besides, Figure 2 plots $\rho(M(\alpha,\beta))$ and $\delta(\alpha,\beta)$ for the upwind difference scheme with $n = 8$ and $q = 1, 10$ when α varies in $[0,1]$. β is also computed according to (5.2).

From Figures 1-2, we find that, when β is chosen according to (5.2) and let α vary in $[0,1]$, the point such that the value of $\rho(M(\alpha,\beta))$ reaches the minimum is extremely close to the point such that the value of $\delta(\alpha,\beta)$ reaches the minimum. Therefore, the theoretical optimal parameters in Theorem 3.1 would be intensely close to the real optimal parameters of the methods.

In Figure 3 and Figure 4, we plot the distributions of the eigenvalues of the iterative matrices. Here, we choose the experimental optimal parameters $\alpha = \beta = \frac{qh}{2}$ in the SNS method and we plot the eigenvalues distribution in Figure 3. We replace α in the ASNS method by $\frac{h}{2}$ and we plot the eigenvalues distributions in Figure 4.

It can be seen from Figures 3-4 that the eigenvalues of the ASNS iterative matrix are more cluster around 0 than the eigenvalues of the SNS iterative matrix when the iterative parameters are chosen appropriately. Further, if we can find a cheaper way to choose the optimal parameters, the accelerated methods would be more efficient, which will be our next work.

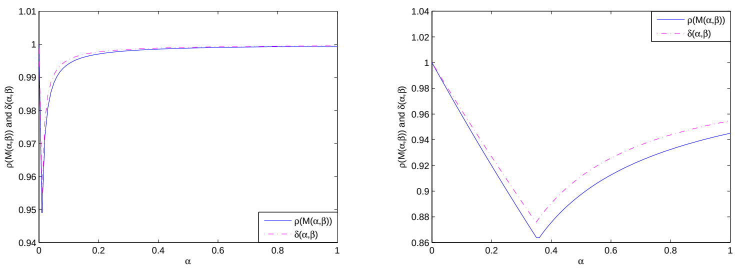

FIGURE 1. The comparison between $\rho(M(\alpha,\beta))$ and $\delta(\alpha,\beta)$ when $q = 1$ (left) and $q = 10$ (right) (centered).

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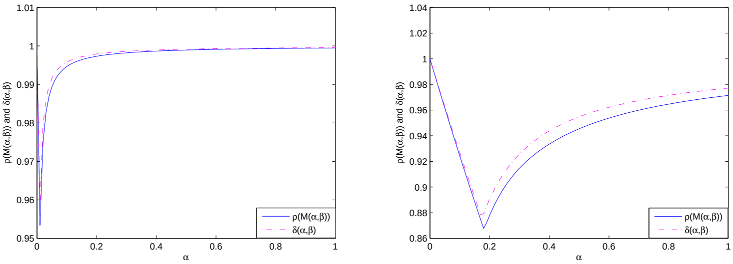

FIGURE 2. The comparison between $\rho(M(\alpha,\beta))$ and $\delta(\alpha,\beta)$ when $q = 1$ (left) and $q = 10$ (right) (upwind).

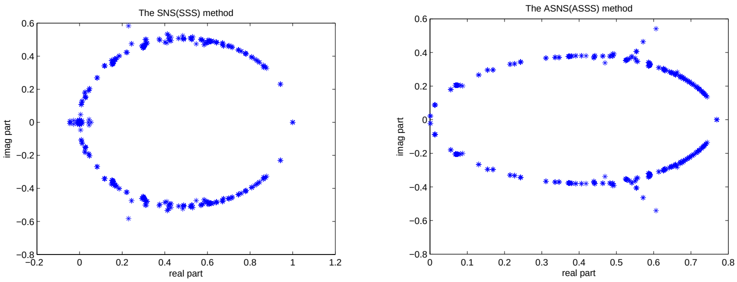

FIGURE 3. The distribution of the eigenvalues of the iterative matrix for the SNS method (left) and the ASNS method (right).

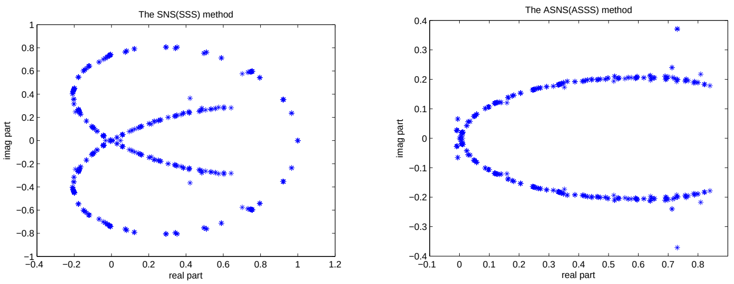

FIGURE 4. The distribution of the eigenvalues of the iterative matrix for the SNS method (left) and the ASNS method (right).

6. Conclusions

For solving the non-Hermitian positive definite with invertible skew-Hermitian parts, we proposed an accelerated SNS and an accelerated SSS iteration method in this paper. Comparied with the HSS iteration method, the new methods not only concentrate more on the balance between the Hermitian parts and the skew-Hermitian parts, but also accelerate the original SNS and SSS methods. The convergence properties and the quasi-optimal parameters are analyzed. In actual implementations, we give the inexact forms of the new methods. Numerical results demonstrate that the point such that $\delta(\alpha,\beta)$ reaches the minimum is exact the same as the point such that $\rho(M(\alpha,\beta))$ reaches the minimum. Therefore, the theoretical results obtained in this paper is correct. Meanwhile, by choosing appropriate parameters, the new methods are more efficient than the SNS and SSS methods.

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Aggregating of Interval-valued Intuitionistic Uncertain Linguistic Variables based on Archimedean t-norm and It Applications in Group Decision Makings†

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Abstract With respect to multiple attribute group decision making (MAGDM) problems in which the attribute weights and the expert weights take the form of real numbers and the attribute values take the form of interval-valued intuitionistic uncertain linguistic variable, we propose group decision making methods of interval-valued intuitionistic uncertain linguistic variable based on Archimedean t-norm and Choquet integral. First, we introduce some concepts of fuzzy measure and interval-valued intuitionistic uncertain lingistic variables based on Archimedean t-norm. Then, interval-valued intuitionistic uncertain linguistic weighted average(geometric) and interval-valued intuitionistic uncertain lingistic ordered weighted average operator based on Archimedean t-norm are developed. Furthermore, some desirable properties of these operators, such as commutativity, idempotency and monotonicity have been studied, and interval-valued intuitionistic uncertain linguistic hybrid average operator based on Archimedean t-norm are developed. Based on these operators, two methods for multiple attribute group decision making problems with intuitionistic uncertain linguistic information have been proposed. Finally, an illustrative example is given to verify the developed approaches and demonstrate their practicality and effectiveness.
Keywords: Interval-valued intuitionistic fuzzy sets; aggregation operators; Archimedean t-norm; group decision making.

1. Introduction

Multiple attribute decision making (MADM) problems are an important research topic in decision theory. Because the objects are fuzzy and uncertain, the attributes involved in decision problems are not always expressed as real numbers, and some better suited to be denoted by fuzzy numbers, such as interval numbers, triangular fuzzy numbers, trapezoidal fuzzy numbers, linguistic numbers on uncertain linguistic variables, and intuitionistic fuzzy numbers. Because Zadeh initially proposed the basic model of fuzzy decision making based on the theory of fuzzy mathematics, fuzzy MADM has been receiving more and more attention. The fuzzy set (FS) theory proposed by Zadeh [1] was a very good tool to research the fuzzy MADM problems, the fuzzy set is used to character the fuzziness just by membership degree. Different from fuzzy set,there is another parameter: non-membership degree in intuitionistic fuzzy set (IFS) which is proposed by Atanassov [2,3]. Clearly, the IFS can describe and character the fuzzy essence of the objective world more accurately [2] than the fuzzy set, and has received more and more attention since its appearance. Later, Atanassov and Gargov [4,5] further introduced the interval-valued intuitionistic fuzzy set (IVIFS), which is a generalization of the IFS. The fundamental characteristic of the IVIFS is that the values of its membership function and non-membership function are interval numbers rather real numbers.

On the other hand, in the real decision-making, there are many qualitative attributes which are difficult to give attribute values by quantitative measurement. While, they are easy to give linguistic assessment values. However, for a linguistic assessment value, it is usually implied that the membership

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degree is one, and the non-membership degree and hesitation degree of decision makers cannot be expressed. On the basis of the intuitionistic fuzzy set and the linguistic assessment set, Wang and Li [6] proposed the concept of intuitionistic linguistic set, the intuitionistic linguistic number, the intuitionistic two-semantic and the Hamming distance between two intuitionistic two-semantics and ranked the alternatives by calculating the comprehensive membership degree to the ideal solution for each alternative.

As an aggregation function, the Choquet integral [7] with respect to fuzzy measures has performed successfully in multicriteria decision making (MCDM). There are many works on the Choquet integral of single-valued functions. It is of interest to combine the Choquet integral and the IFS theory or MCDM under intuitionistic fuzzy environment, because, by doing this, we cannot only deals with the imprecise and uncertain decision information but also efficiently take into account the various interactions among the decision criteria.

Base on Archimedean t-conorm and t-norm [8-11], and the aggregation functions for the classical fuzzy sets (FSs), Beliakov et al. gave some operations about intuitionistic fuzzy sets, proposed two general concepts for constructing other types of aggregation operators for intuitionistic fuzzy sets (IFSs) extending the existing methods and showed that the operators obtained by using the Lukasiewicz t-norm are consistent with the ones on ordinary FSs. We can find above aggregation operators are all based on different relationships of the aggregated arguments, which can provide more choices for the decision makers.

In summary, based on intuitionistic linguistic set proposed by Wang and Li [6] , combining interval-valued uncertain linguistic variables, Archimedean t-norm and Choquet integral, in this paper, we propose the interval-valued uncertain linguistic variables based on Archimedean t-norm and investigate the MAGDM problems. First, we introduced some concepts of fuzzy measure and interval-valued intuitionistic uncertain linguistic variables based on Archimedean t-norm. Then, interval-valued intuitionistic uncertain linguistic weighted average(geometric) operator based on Archimedean t-norm ,interval-valued intuitionistic uncertain linguistic ordered weighted average(geometric) operator based on Archimedean t-norm are developed. Furthermore, some desirable properties of these operators, such as commutativity, idempotency and monotonicity have been studied, and an intuitionistic uncertain linguistic hybrid average(geometric) operator based on Archimedean t-norm was developed. Based on these operators, two methods for multiple attribute group decision making problems with intuitionistic uncertain linguistic information have been proposed.

1. Preliminaries

A function $T : [0,1] \times [0,1] \to [0,1]$ is called a t-norm if it satisfies the following four conditions (ref. to [8,9]):

1) $T(1, x) = x$, for all x.
2) $T(x, y) = T(y, x)$, for all x and y.
3) $T(x, T(y, z)) = T(T(x, y), z)$, for all x, y and z.
4) $x \leqslant x', y \leqslant y'$ implies $T(x, y) \leqslant T(x', y')$, $x, y, x', y' \in [0,1]$.

A function $S : [0,1] \times [0,1] \to [0,1]$ is called a t-conorm if it satisfies the following four conditions (ref. to [8,9]):

1) $S(0, x) = x$, for all x.
2) $S(x, y) = S(y, x)$, for all x and y.
3) $S(x, S(y, z)) = S(S(x, y), z)$, for all x, y and z.
4) $x \leqslant x', y \leqslant y'$ implies $S(x, y) \leqslant S(x', y')$, $x, y, x', y' \in [0,1]$.

A t-norm function $T(x, y)$ is called Archimedean t-norm if it is continuous and $T(x, x) < x$ for all $x \in [0,1]$. An Archimedean t-norm is called strictly Archimedean t-norm if it is strictly increasing in each variable for $x, y \in (0,1)$, (ref. to [8,9]).

A t-conorm function $S(x, y)$ is called Archimedean t-conorm if it is continuous and $S(x, x) > x$ for all $x \in [0,1]$. An Archimedean t-conorm is called strictly Archimedean t-conorm if it is strictly increasing in each variable for $x, y \in (0,1)$, (ref. to [8,9]).

A mapping $N : [0,1] \to [0,1]$ is called negation operator，if N is decreasing and $N(0) = 1$, $N(1) = 0$.

Suppose that $S = (s_0, s_1, \ldots, s_{l-1})$ is a finite and fully ordered discrete term set, where l is an odd number. In real situations, l would be equal to 3,5,7,9,etc. For example, when $l = 7$, a set S can be given as follows:

$$S = (s_0, s_1, s_2, s_3, s_4, s_5, s_6) = \{very\ poor, poor, slightly\ poor, fair, slightly\ good, good, very\ good\}.$$

For any linguistic set $S = (s_0, s_1, \ldots, s_{l-1})$, the relationship between the element s_i and its subscript i is strictly monotonically increasing [12,13,14], so the function can be defined as follows: $f : s_i = f(i)$. Clearly, the function $f(i)$ is a strictly monotonically increasing function about a subscript i. To preserve all of the given information, the discrete linguistic label $S = (s_0, s_1, \ldots, s_{l-1})$ is extended to a continuous linguistic label $\overline{S} = \{s_\alpha \mid \alpha \in R\}$, which satisfies the above characteristics.

Suppose $\widetilde{s} = [s_a, s_b]$, $s_a, s_b \in \overline{S}$ and $a \leqslant b$, s_a and s_b are the lower limit and the upper limit of $\widetilde{s}$, respectively. Then, $\widetilde{s}$ is called an uncertain linguistic variable [15].

For each x, $\mu_A(x)$ and $\nu_A(x)$ are closed intervals and their lower and upper end points are, respectively, denoted by $\mu_A^L(x)$, $\mu_A^U(x)$,$\nu_A^L(x)$,$\nu_A^U(x)$. We can denote by

$$A = \{\langle x[[s_{\theta(x)}, s_{\tau(x)}], [\mu_A^L(x), \mu_A^U(x)], [\nu_A^L(x), \nu_A^U(x)]]\rangle | x \in X\},$$

where $s_{\theta(x)}, s_{\tau(x)} \in \overline{S}$, $0 \leqslant \mu_A^U(x) + \nu_A^U(x) \leqslant 1$, $x \in X$, $\mu_A^L(x) \geqslant 0$ and $\nu_A^L(x) \geqslant 0$.

For each element x, we can compute its hesitation interval of x to uncertain linguistic variable $[s_{\theta(x)}, s_{\tau(x)}]$ as:

$$\pi_A(x) = [\pi_A^L(x), \pi_A^U(x)] = [1 - \nu_A^U(x) - \mu_A^U(x), 1 - \nu_A^L(x) - \mu_A^L(x)].$$

Definition 1.1. Let $A = \{\langle x[[s_{\theta(x)}, s_{\tau(x)}], [\mu_A^L(x), \mu_A^U(x)], [\nu_A^L(x), \nu_A^U(x)]]\rangle | x \in X\}$ be $IVIULS$, 6-Tuple $< [s_{\theta(x)}, s_{\tau(x)}], [\mu_A^L(x), \mu_A^U(x)], [\nu_A^L(x), \nu_A^U(x)] >$ is called an interval-valued intuitionistic uncertain linguistic number ($IVIULN$), and A can also be viewed as a collection of the interval-valued intuitionistic uncertain linguistic variables. So it can also be expressed as $A = \{\langle [s_{\theta(x)}, s_{\tau(x)}], [\mu_A^L(x), \mu_A^U(x)], [\nu_A^L(x), \nu_A^U(x)]\rangle | x \in X\}$.

2. Interval-valued intuitionistic uncertain linguistic variables based on Archimedean t-norm

Definition 2.1. Let $\widetilde{\alpha}_i = \langle [s_{\theta(\alpha_i)}, s_{\tau(\alpha_i)}], [\mu^L(\alpha_i), \mu^U(\alpha_i)], [\nu^L(\alpha_i), \nu^U(\alpha_i)]\rangle$ $(i = 1, 2)$ be two interval-valued intuitionistic uncertain linguistic variables based on Archimedean t-norm and $\lambda \geqslant 0$, we can define the operational rules about $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$ as follows

(1) $\widetilde{\alpha}_1 \oplus \widetilde{\alpha}_2 = \langle [s_{\theta(\alpha_1)+\theta(\alpha_2)}, s_{\tau(\alpha_1)+\tau(\alpha_2)}], [S(\mu^L(\alpha_1), \mu^L(\alpha_2)), S(\mu^U(\alpha_1), \mu^U(\alpha_2))], [T(\nu^L(\alpha_1), \nu^L(\alpha_2)), T(\nu^U(\alpha_1), \nu^U(\alpha_2))]\rangle$
$= \langle [s_{\theta(\alpha_1)+\theta(\alpha_2)}, s_{\tau(\alpha_1)+\tau(\alpha_2)}], [h^{-1}(h(\mu^L(\alpha_1)) + h(\mu^L(\alpha_2))), h^{-1}(h(\mu^U(\alpha_1)) + h(\mu^U(\alpha_2)))], [g^{-1}(g(\nu^L(\alpha_1)) + g(\nu^L(\alpha_2))), g^{-1}(g(\nu^U(\alpha_1)) + g(\nu^U(\alpha_2)))]\rangle$;

(2) $\widetilde{\alpha}_1 \otimes \widetilde{\alpha}_2 = \langle [s_{\theta(\alpha_1)\times\theta(\alpha_2)}, s_{\tau(\alpha_1)\times\tau(\alpha_2)}], [T(\mu^L(\alpha_1), \mu^L(\alpha_2)), T(\mu^U(\alpha_1), \mu^U(\alpha_2))], [S(\nu^L(\alpha_1), \nu^L(\alpha_2)), S(\nu^U(\alpha_1), \nu^U(\alpha_2))]\rangle$
$= \langle [s_{\theta(\alpha_1)\times\theta(\alpha_2)}, s_{\tau(\alpha_1)\times\tau(\alpha_2)}], [g^{-1}(g(\mu^L(\alpha_1)) + g(\mu^L(\alpha_2))), g^{-1}(g(\mu^U(\alpha_1)) + g(\mu^U(\alpha_2)))], [h^{-1}(h(\nu^L(\alpha_1)) + h(\nu^L(\alpha_2))), h^{-1}(h(\nu^U(\alpha_1)) + h(\nu^U(\alpha_2)))]\rangle$;

(3) $\lambda\widetilde{\alpha}_1 = \langle [s_{\lambda\times\theta(\alpha_1)}, s_{\lambda\times\tau(\alpha_1)}], [h^{-1}(\lambda h(\mu^L(\alpha_1))), h^{-1}(\lambda h(\mu^U(\alpha_1)))], [g^{-1}(\lambda g(\nu^L(\alpha_1))), g^{-1}(\lambda g(\nu^U(\alpha_1)))]\rangle$;

(4) $\widetilde{\alpha}_1^\lambda = \langle [s_{(\theta(\alpha_1))^\lambda}, s_{(\tau(\alpha_1))^\lambda}], [g^{-1}(\lambda g(\mu^L(\alpha_1))), g^{-1}(\lambda g(\mu^U(\alpha_1)))], [h^{-1}(\lambda h(\nu^L(\alpha_1))), h^{-1}(\lambda h(\nu^U(\alpha_1)))]\rangle$.

Theorem 2.1. For any two interval-valued intuitionistic uncertain linguistic numbers based on Archimedean t-norm $\widetilde{\alpha}_i = \langle [s_{\theta(\alpha_i)}, s_{\tau(\alpha_i)}], [\mu^L(\alpha_i), \mu^U(\alpha_i)], [\nu^L(\alpha_i), \nu^U(\alpha_i)]\rangle$ $(i = 1, 2)$, it can be proved the calculation rules shown as follows:

(a) $\widetilde{\alpha}_1 \oplus \widetilde{\alpha}_2 = \widetilde{\alpha}_2 \oplus \widetilde{\alpha}_1$;
(b) $\widetilde{\alpha}_1 \otimes \widetilde{\alpha}_2 = \widetilde{\alpha}_2 \otimes \widetilde{\alpha}_1$;
(c) $\lambda(\widetilde{\alpha}_1 \oplus \widetilde{\alpha}_2) = \lambda\widetilde{\alpha}_1 \oplus \lambda\widetilde{\alpha}_2$, $\lambda > 0$;
(d) $\lambda_1\widetilde{\alpha}_1 \oplus \lambda_2\widetilde{\alpha}_1 = (\lambda_1 + \lambda_2)\widetilde{\alpha}_1$, $\lambda_1, \lambda_2 > 0$;
(e) $\widetilde{\alpha}_1^{\lambda_1} \otimes \widetilde{\alpha}_1^{\lambda_2} = (\widetilde{\alpha}_1)^{\lambda_1+\lambda_2}$, $\lambda_1, \lambda_2 > 0$;
(f) $\widetilde{\alpha}_1^\lambda \otimes \widetilde{\alpha}_2^\lambda = (\widetilde{\alpha}_1 \otimes \widetilde{\alpha}_2)^\lambda$, $\lambda > 0$.

Definition 2.2 [16]. Let $\widetilde{\alpha}_1 = \langle [s_{\theta(\alpha_1)}, s_{\tau(\alpha_1)}], [\mu^L(\alpha_1), \mu^U(\alpha_1)], [\nu^L(\alpha_1), \nu^U(\alpha_1)]\rangle$ be an interval-valued intuitionistic uncertain linguistic number based on Archimedean t-norm, an expected value $E(\widetilde{\alpha}_1)$ of $\widetilde{\alpha}_1$ can be represented as follows

$$E(\widetilde{\alpha}_1)=\frac{1}{2}\times(\frac{\mu^L(\alpha_1)+\mu^U(\alpha_1)}{2}+1-\frac{\nu^L(\alpha_1)+\nu^U(\alpha_1)}{2})\times s_{(\theta(\alpha_1)+\tau(\alpha_1))/2}$$
$$=s_{((\theta(\alpha_1)+\tau(\alpha_1))\times(\mu^L(\alpha_1)+\mu^U(\alpha_1)+2-\nu^L(\alpha_1)-\nu^U(\alpha_1))/8}.$$

Definition 2.3 [16]. Let $\widetilde{\alpha}_1=\langle[s_{\theta(\alpha_1)},s_{\tau(\alpha_1)}],[\mu^L(\alpha_1),\mu^U(\alpha_1)],[\nu^L(\alpha_1),\nu^U(\alpha_1)]\rangle$ be an interval-valued intuitionistic uncertain linguistic number based on Archimedean t-norm, an accuracy function $H(\widetilde{\alpha}_1)$ can be represented as follows

$$H(\widetilde{\alpha}_1)=(\frac{\mu^L(\alpha_1)+\mu^U(\alpha_1)}{2}+\frac{\nu^L(\alpha_1)+\nu^U(\alpha_1)}{2})\times s_{(\theta(\alpha_1)+\tau(\alpha_1))/2}$$
$$=s_{((\theta(\alpha_1)+\tau(\alpha_1))\times(\mu^L(\alpha_1)+\mu^U(\alpha_1)+\nu^L(\alpha_1)+\nu^U(\alpha_1))/4}.$$

Definition 2.4 [16]. If $\widetilde{\alpha}_i=\langle[s_{\theta(\alpha_i)},s_{\tau(\alpha_i)}],[\mu^L(\alpha_i),\mu^U(\alpha_i)],[\nu^L(\alpha_i),\nu^U(\alpha_i)]\rangle$ $(i=1,2)$ are any two interval-valued intuitionistic uncertain linguistic numbers based on Archimedean t-norm, then

(1) If $E(\widetilde{\alpha}_1)>E(\widetilde{\alpha}_2)$, then $\widetilde{\alpha}_1\succ\widetilde{\alpha}_2$.
(2) If $E(\widetilde{\alpha}_1)=E(\widetilde{\alpha}_2)$, then:
If $H(\widetilde{\alpha}_1)>H(\widetilde{\alpha}_2)$, then $\widetilde{\alpha}_1\succ\widetilde{\alpha}_2$.
If $H(\widetilde{\alpha}_1)=H(\widetilde{\alpha}_2)$, then $\widetilde{\alpha}_1=\widetilde{\alpha}_2$.

3. Aggregating of the interval-valued intuitionistic uncertain linguistic variables based on Archimedean t-norm aggregating operator and fuzzy measure

A fuzzy measure on X is a set function $\mu:P(X)\rightarrow[0,1]$ such that
(i) $\mu(\emptyset)=0,\mu(X)=1$;
(ii) $A,B\subseteq X$, $A\subseteq B$ implies $\mu(A)\leqslant\mu(B)$.
Let $A,B\in P(X),A\cap B=\varnothing$. If fuzzy measure g satisfies the following conditions:

$$g(A\cup B)=g(A)+g(B)+\lambda g(A)g(B)$$

and $\lambda\in(-1,\infty)$.

If $\lambda=0$, then g is an additive measure, which means there is no interaction between coalitions A and B .

If $\lambda>0$, then g is called a supperadditive measure, which reflects there exists complementary interaction between coalitions A and B.

If $-1<\lambda<0$, then g is said to be a subadditive measure, which shows there exists redundancy interaction between coalitions A and B.

Let $X=\{x_1,x_2,...x_n\}$ be a attribute index set, if $i,j=1,2,...,n$ and $i\neq j,x_i\cap x_j=\varnothing,\bigcup_{i=1}^n x_i=X$, then

$$g(X)=\begin{cases}\frac{1}{\lambda}(\prod_{i=1}^n[1+\lambda g(x_i)]-1) & \lambda\neq 0,\\ \sum_{i=1}^n g(x_i) & \lambda=0,\end{cases}$$

where x_i, $g(x_i)$ is called a fuzzy measure function, and it indicates the importance degree of x_i.

From $g(X)=1$, we know λ is determined by $\lambda+1=\prod_{i=1}^n(1+\lambda g(x_i))$.

Based on the the above operational rules, we propose weighted average (geometric) operator, ordered weighted average (geometric) operator and hybrid average (geometric) operator for interval-valued intuitionistic uncertain linguistic variables based on Archimedean t-norm in this part.

Definition 3.1. Let $\widetilde{\alpha}_i=\langle[s_{\theta(\alpha_i)},s_{\tau(\alpha_i)}],[\mu^L(\alpha_i),\mu^U(\alpha_i)],[\nu^L(\alpha_i),\nu^U(\alpha_i)]\rangle$ $(i=1,2,\ldots,n)$ be a collection of interval-valued intuitionistic uncertain linguistic variables based on Archimedean t-norm, and $ATS-IVIULWA(or\ ATS-IVIULWGA):\Omega^n\rightarrow\Omega$, if

$$ATS-IVIULWA_\mu(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n)=\sum_{j=1}^n\mu_j\widetilde{\alpha}_j,$$

$$(or\quad ATS-IVIULWGA_\mu(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n)=\prod_{j=1}^n(\widetilde{\alpha}_j)^{\mu_j},)$$

where Ω is the set of all interval-valued intuitionistic uncertain linguistic variables based on Archimedean t-norm, and $\mu = (\mu_1, \mu_2, \ldots, \mu_n)^T$ is the weighted vector of $\widetilde{\alpha}_j (j = 1, 2, \ldots, n)$, μ is a fuzzy measure on X with $\mu_j \in [0, 1]$, $\mu_j = \mu(A_{(j)}) - \mu(A_{(j+1)})$, and $\sum_{j=1}^n \mu_j = 1$, $A_{(j)} = (j, \ldots, n)$ with $A_{(n+1)} = \emptyset$, then $ATS - IVIULWA$ ($ATS - IVIULWGA$) is called the interval-valued intuitionistic uncertain linguistic weighted average (weighted geometric average) operator based on Archimedean t-norm.

Specifically, if $\mu = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$, then $ATS - IVIULWA$ operator is interval-valued intuitionistic uncertain linguistic arithmetic average operator based on Archimedean t-norm ($ATS - IVIULAA$):

$$ATS - IVIULAA(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n) = \frac{1}{n}(\widetilde{\alpha}_1 \oplus \widetilde{\alpha}_2 \oplus \ldots \oplus \widetilde{\alpha}_n).$$

Theorem 3.1. Let $\widetilde{\alpha}_i = \langle [s_{\theta(\alpha_i)}, s_{\tau(\alpha_i)}], [\mu^L(\alpha_i), \mu^U(\alpha_i)], [\nu^L(\alpha_i), \nu^U(\alpha_i)] \rangle$ $(i = 1, 2, \ldots, n)$ be a collection of interval-valued intuitionistic uncertain linguistic variables based on Archimedean t-norm, then, the result aggregated by Definition 3.1 is still an interval-valued intuitionistic uncertain linguistic variable based on Archimedean t-norm, and

$$ATS - IVIULWA_\mu(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n) = \langle [s_{\sum_{j=1}^n (\mu_j \times \theta(\alpha_j))}, s_{\sum_{j=1}^n (\mu_j \times \tau(\alpha_j))}],$$
$$[h^{-1}(\sum_{j=1}^n \mu_j h(\mu^L(\alpha_j))), h^{-1}(\sum_{j=1}^n \mu_j h(\mu^U(\alpha_j)))], [g^{-1}(\sum_{j=1}^n \mu_j g(\nu^L(\alpha_j))), g^{-1}(\sum_{j=1}^n \mu_j g(\nu^U(\alpha_j)))] \rangle.$$

$$(or \ \ ATS - IVIULWGA_\mu(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n) = \langle [s_{\prod_{j=1}^n ((\theta(\alpha_j))^{\mu_j})}, s_{\prod_{j=1}^n ((\tau(\alpha_j))^{\mu_j})}],$$
$$[g^{-1}(\sum_{j=1}^n \mu_j g(\mu^L(\alpha_j))), g^{-1}(\sum_{j=1}^n \mu_j g(\mu^U(\alpha_j)))], [h^{-1}(\sum_{j=1}^n \mu_j h(\nu^L(\alpha_j))), h^{-1}(\sum_{j=1}^n \mu_j h(\nu^U(\alpha_j)))] \rangle,)$$

where $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ is a fuzzy measure on X with $\mu_j \in [0, 1]$, $\mu_j = \mu(A_{(j)}) - \mu(A_{(j+1)})$, and $\sum_{j=1}^n \mu_j = 1$, the parentheses used for indices represent a permutation on X such that $\widetilde{\alpha}_1 \leqslant \widetilde{\alpha}_2 \leqslant \cdots \leqslant \widetilde{\alpha}_n$, $A_{(j)} = (j, ..., n), A_{(n+1)} = \varnothing$.

Theorem 3.1 can be proven by mathematical induction. The steps in the proof are shown as follows:

Proof. We only prove the case of $ATS - IVIULWA_\mu(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n)$.

(1) When $n = 1$, obviously, it is right.

(2) When $n = 2$,

$$\mu_1 \widetilde{\alpha}_1 = \langle [s_{\mu_1 \times \theta(\alpha_1)}, s_{\mu_1 \times \tau(\alpha_1)}], [h^{-1}(\mu_1 h(\mu^L(\alpha_1))), h^{-1}(\mu_1 h(\mu^U(\alpha_1)))],$$
$$[g^{-1}(\mu_1 g(\nu^L(\alpha_1))), g^{-1}(\nu_1 g(\nu^U(\alpha_1)))] \rangle.$$
$$\mu_2 \widetilde{\alpha}_2 = \langle [s_{\mu_2 \times \theta(\alpha_2)}, s_{\mu_2 \times \tau(\alpha_2)}], [h^{-1}(\mu_2 h(\mu^L(\alpha_2))), h^{-1}(\mu_2 h(\mu^U(\alpha_2)))],$$
$$[g^{-1}(\mu_2 g(\nu^L(\alpha_2))), g^{-1}(\nu_2 g(\nu^U(\alpha_2)))] \rangle.$$
$$ATS - IVIULWA_\mu(\widetilde{\alpha}_1, \widetilde{\alpha}_2) = \mu_1 \widetilde{\alpha}_1 \oplus \mu_2 \widetilde{\alpha}_2$$
$$= \langle [s_{\mu_1 \times \theta(\alpha_1)}, s_{\mu_1 \times \tau(\alpha_1)}], [h^{-1}(\mu_1 h(\mu^L(\alpha_1))), h^{-1}(\mu_1 h(\mu^U(\alpha_1)))],$$
$$[g^{-1}(\mu_1 g(\nu^L(\alpha_1))), g^{-1}(\nu_1 g(\nu^U(\alpha_1)))] \rangle$$
$$\oplus \langle [s_{\mu_2 \times \theta(\alpha_2)}, s_{\mu_2 \times \tau(\alpha_2)}], [h^{-1}(\mu_2 h(\mu^L(\alpha_2))), h^{-1}(\mu_2 h(\mu^U(\alpha_2)))],$$
$$[g^{-1}(\mu_2 g(\nu^L(\alpha_2))), g^{-1}(\nu_2 g(\nu^U(\alpha_2)))] \rangle$$
$$= \langle [s_{\mu_1 \times \theta(\alpha_1) + \mu_2 \times \theta(\alpha_2)}, s_{\mu_1 \times \tau(\alpha_1) + s_{\mu_2 \times \tau(\alpha_2)}}],$$
$$[h^{-1}(h(h^{-1}(\mu_1 h(\mu^L(\alpha_1)))) + h(h^{-1}(\mu_2 h(\mu^L(\alpha_2))))),$$
$$h^{-1}(h(h^{-1}(\mu_1 h(\mu^U(\alpha_1)))) + h(h^{-1}(\mu_2 h(\mu^U(\alpha_2)))))],$$
$$[g^{-1}(g(g^{-1}(\mu_1 g(\nu^L(\alpha_1)))) + g(g^{-1}(\mu_2 g(\nu^L(\alpha_2))))),$$
$$g^{-1}(g(g^{-1}(\mu_1 g(\nu^U(\alpha_1)))) + g(g^{-1}(\mu_2 g(\nu^U(\alpha_2)))))] \rangle$$
$$= \langle [s_{\sum_{j=1}^2 (\mu_j \times \theta(\alpha_j))}, s_{\sum_{j=1}^2 (\mu_j \times \tau(\alpha_j))}], [h^{-1}(\sum_{j=1}^2 \mu_j h(\mu^L(\alpha_j))), h^{-1}(\sum_{j=1}^2 \mu_j h(\mu^U(\alpha_j)))],$$
$$[g^{-1}(\sum_{j=1}^2 \mu_j g(\nu^L(\alpha_j))), g^{-1}(\sum_{j=1}^2 \mu_j g(\nu^U(\alpha_j)))] \rangle.$$

Therefore, when $n = 2$, the conclusion is right.

(3) Suppose when $n = k$, the conclusion is right, i.e.

$$ATS - IVIULWA_\mu(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_k) = \langle [s_{\sum_{j=1}^k (\mu_j \times \theta(\alpha_j))}, s_{\sum_{j=1}^k (\mu_j \times \tau(\alpha_j))}], [h^{-1}(\sum_{j=1}^k \mu_j h(\mu^L(\alpha_j))),$$
$$h^{-1}(\sum_{j=1}^k \mu_j h(\mu^U(\alpha_j)))], [g^{-1}(\sum_{j=1}^k \mu_j g(\nu^L(\alpha_j))), g^{-1}(\sum_{j=1}^k \mu_j g(\nu^U(\alpha_j)))] \rangle.$$

Then, when $n = k + 1$,

$$ATS - IVIULWA_\mu(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_k, \widetilde{\alpha}_{k+1}) = \langle [s_{\sum_{j=1}^k (\mu_j \times \theta(\alpha_j))}, s_{\sum_{j=1}^k (\mu_j \times \tau(\alpha_j))}], [h^{-1}(\sum_{j=1}^k \mu_j h(\mu^L(\alpha_j))),$$
$$h^{-1}(\sum_{j=1}^k \mu_j h(\mu^U(\alpha_j)))], [g^{-1}(\sum_{j=1}^k \mu_j g(\nu^L(\alpha_j))), g^{-1}(\sum_{j=1}^k \mu_j g(\nu^U(\alpha_j)))] \rangle$$
$$\oplus \langle [s_{\mu_{k+1} \times \theta(\alpha_{k+1})}, s_{\mu_{k+1} \times \tau(\alpha_{k+1})}], [h^{-1}(\mu_{k+1} h(\mu^L(\alpha_{k+1}))), h^{-1}(\mu_{k+1} h(\mu^U(\alpha_{k+1})))],$$
$$[g^{-1}(\mu_{k+1} g(\nu^L(\alpha_{k+1}))), g^{-1}(\nu_{k+1} g(\nu^U(\alpha_{k+1})))] \rangle$$
$$= \langle [s_{\sum_{j=1}^k (\mu_j \times \theta(\alpha_j)) + \mu_{k+1} \times \theta(\alpha_{k+1})}, s_{\sum_{j=1}^k (\mu_j \times \tau(\alpha_j)) + \mu_{k+1} \times \tau(\alpha_{k+1})}],$$

$$[h^{-1}(h(h^{-1}(\sum_{j=1}^{k}\mu_j h(\mu^L(\alpha_j)))) + h(h^{-1}(\mu_{k+1}h(\mu^L(\alpha_{k+1}))))),$$
$$h^{-1}(h(h^{-1}(\sum_{j=1}^{k}\mu_j h(\mu^U(\alpha_j)))) + h(h^{-1}(\mu_{k+1}h(\mu^U(\alpha_{k+1})))))],$$
$$[g^{-1}(g(g^{-1}(\sum_{j=1}^{k}\mu_j g(\nu^L(\alpha_j)))) + g(g^{-1}(\mu_{k+1}g(\nu^L(\alpha_{k+1}))))),$$
$$g^{-1}(g(g^{-1}(\sum_{j=1}^{k}\mu_j g(\nu^U(\alpha_j)))) + g(g^{-1}(\mu_{k+1}g(\nu^U(\alpha_{k+1})))))]\rangle$$
$$= \langle[s_{\sum_{j=1}^{k+1}(\mu_j\times\theta(\alpha_j))}, s_{\sum_{j=1}^{k+1}(\mu_j\times\tau(\alpha_j))}], [h^{-1}(\sum_{j=1}^{k+1}\mu_j h(\mu^L(\alpha_j))), h^{-1}(\sum_{j=1}^{k+1}\mu_j h(\mu^U(\alpha_j)))],$$
$$[g^{-1}(\sum_{j=1}^{k+1}\mu_j g(\nu^L(\alpha_j))), g^{-1}(\sum_{j=1}^{k+1}\mu_j g(\nu^U(\alpha_j)))]\rangle.$$

So, when $n = k+1$, the conclusion is right, too.

According to steps (1), (2) and (3), we can conclude the conclusion is right for all n.

Example 3.1. If $N(x) = 1-x,\ g(t) = -\log t$, and Algebraic t-conorm and t-norm [11] defined by $T^A(x,y) = x\cdot y,\ S^A(x,y) = x+y-xy$, then

$$IVIULWA_\mu(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n) = \langle[s_{\sum_{j=1}^{n}(\mu_j\times\theta(\alpha_j))}, s_{\sum_{j=1}^{n}(\mu_j\times\tau(\alpha_j))}],$$
$$[1-\prod_{j=1}^{n}(1-\mu^L(\alpha_j))^{\mu_j}, 1-\prod_{j=1}^{n}(1-\mu^U(\alpha_j))^{\mu_j}],$$
$$[\prod_{j=1}^{n}(\nu^L(\alpha_j))^{\mu_j}, \prod_{j=1}^{n}(\nu^U(\alpha_j))^{\mu_j}]\rangle,$$

which is the interval-valued intuitionistic uncertain linguistic weighted average(IVIULWA) operator.

Example 3.2. If $N(x) = 1-x,\ g(t) = \log(\frac{2-t}{t})$, and Einstein t-conorm and t-norm [11] defined by $T^E(x,y) = \frac{xy}{1+(1-x)(1-y)}$, $S^E(x,y) = \frac{x+y}{1+xy}$, then

$$EIVIULWA_\mu(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n) = \langle[s_{\sum_{j=1}^{n}(\mu_j\times\theta(\alpha_j))}, s_{\sum_{j=1}^{n}(\mu_j\times\tau(\alpha_j))}],$$
$$[\frac{\prod_{j=1}^{n}(1+\mu^L(\alpha_j))^{\mu_j}-\prod_{j=1}^{n}(1-\mu^L(\alpha_j))^{\mu_j}}{\prod_{j=1}^{n}(1+\mu^L(\alpha_j))^{\mu_j}+\prod_{j=1}^{n}(1-\mu^L(\alpha_j))^{\mu_j}}, \frac{\prod_{j=1}^{n}(1+\mu^U(\alpha_j))^{\mu_j}-\prod_{j=1}^{n}(1-\mu^U(\alpha_j))^{\mu_j}}{\prod_{j=1}^{n}(1+\mu^U(\alpha_j))^{\mu_j}+\prod_{j=1}^{n}(1-\mu^U(\alpha_j))^{\mu_j}}],$$
$$[\frac{2\prod_{j=1}^{n}(\nu^L(\alpha_j))^{\mu_j}}{\prod_{j=1}^{n}(2-\nu^L(\alpha_j))^{\mu_j}+\prod_{j=1}^{n}(\nu^L(\alpha_j))^{\mu_j}}, \frac{2\prod_{j=1}^{n}(\nu^U(\alpha_j))^{\mu_j}}{\prod_{j=1}^{n}(2-\nu^U(\alpha_j))^{\mu_j}+\prod_{j=1}^{n}(\nu^U(\alpha_j))^{\mu_j}}]\rangle,$$

which is called the Einstein interval-valued intuitionistic uncertain linguistic weighted average(EIVIUL WA) operator.

Example 3.3. If $N(x) = 1-x,\ g(t) = \log(\frac{\gamma+(1-\gamma)t}{t})$, $\gamma > 0$, and Hamacher t-conorm and t-norm [11] defined by $T_\gamma^H(x,y) = \frac{xy}{\gamma+(1-\gamma)(x+y-xy)}$, $\gamma > 0,\ S_\gamma^H(x,y) = \frac{x+y-xy-(1-\gamma)xy}{1-(1-\gamma)xy}$, $\gamma > 0$, then

$$HIVIULWA_\mu(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n) = \langle[s_{\sum_{j=1}^{n}(\mu_j\times\theta(\alpha_j))}, s_{\sum_{j=1}^{n}(\mu_j\times\tau(\alpha_j))}],$$
$$[\frac{\prod_{j=1}^{n}(1+(\gamma-1)\mu^L(\alpha_j))^{\mu_j}-\prod_{j=1}^{n}(1-\mu^L(\alpha_j))^{\mu_j}}{\prod_{j=1}^{n}(1+(\gamma-1)\mu^L(\alpha_j))^{\mu_j}+(\gamma-1)\prod_{j=1}^{n}(1-\mu^L(\alpha_j))^{\mu_j}}, \frac{\prod_{j=1}^{n}(1+(\gamma-1)\mu^U(\alpha_j))^{\mu_j}-\prod_{j=1}^{n}(1-\mu^U(\alpha_j))^{\mu_j}}{\prod_{j=1}^{n}(1+(\gamma-1)\mu^U(\alpha_j))^{\mu_j}+(\gamma-1)\prod_{j=1}^{n}(1-\mu^U(\alpha_j))^{\mu_j}}],$$
$$[\frac{\gamma\prod_{j=1}^{n}(\nu^L(\alpha_j))^{\mu_j}}{\prod_{j=1}^{n}(1+(\gamma-1)(1-\nu^L(\alpha_j)))^{\mu_j}+(\gamma-1)\prod_{j=1}^{n}(\nu^L(\alpha_j))^{\mu_j}}, \frac{\gamma\prod_{j=1}^{n}(\nu^U(\alpha_j))^{\mu_j}}{\prod_{j=1}^{n}(1+(\gamma-1)(1-\nu^U(\alpha_j)))^{\mu_j}+(\gamma-1)\prod_{j=1}^{n}(\nu^U(\alpha_j))^{\mu_j}}]\rangle,$$

which is called the Hammer interval-valued intuitionistic uncertain linguistic weighted average(HIVIUL WA) operator.

Especially, if $\gamma = 1$, then the $HIVIULWA$ operator reduces to the $IVIULWA$ operator; if $\gamma = 2$, then the $HIVIULWA$ operator reduces to the $EIVIULWA$ operator.

Example 3.4. If $N(x) = 1-x,\ g(t) = \log(\frac{\gamma-1}{\gamma^t-1})$, $\gamma > 1$, and Frank t-conorm and t-norm [11] defined by $T_\gamma^F(x,y) = \log_\gamma(1+\frac{(\gamma^x-1)(\gamma^y-1)}{\gamma-1})$, $\gamma > 1,\ S_\gamma^F(x,y) = 1-\log_\gamma(1+\frac{(\gamma^{1-x}-1)(\gamma^{1-y}-1)}{\gamma-1})$, $\gamma > 1$, then

$$FIVIULWA_\mu(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n) = \langle[s_{\sum_{j=1}^{n}(\mu_j\times\theta(\alpha_j))}, s_{\sum_{j=1}^{n}(\mu_j\times\tau(\alpha_j))}],$$
$$[1-\log_\gamma(1+\prod_{j=1}^{n}(\gamma^{1-\mu^L(\alpha_j)}-1)^{\mu_j}), 1-\log_\gamma(1+\prod_{j=1}^{n}(\gamma^{1-\mu^U(\alpha_j)}-1)^{\mu_j})],$$
$$[\log_\gamma(1+\prod_{j=1}^{n}(\gamma^{\nu^L(\alpha_j)}-1)^{\mu_j}), \log_\gamma(1+\prod_{j=1}^{n}(\gamma^{\nu^U(\alpha_j)}-1)^{\mu_j})]\rangle,$$

which is called the Frank interval-valued intuitionistic uncertain linguistic weighted average(FIVIULWA) operator.

Especially, if $\gamma \to 1$, then the $FIVIULWA$ operator reduces to the $IVIULWA$ operator.

Example 3.5. If $N(x) = 1-x^2,\ g(t) = -\log t$, and Algebraic t-conorm and t-norm [11] defined by $T_2^A(x,y) = xy$, $S_2^A(x,y) = \sqrt{1-(1-x^2)(1-y^2)}$, then

$$IVIULWA_\mu(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n) = \langle[s_{\sum_{j=1}^{n}(\mu_j\times\theta(\alpha_j))}, s_{\sum_{j=1}^{n}(\mu_j\times\tau(\alpha_j))}],$$
$$[\sqrt{1-\prod_{j=1}^{n}(1-(\mu^L(\alpha_j))^2)^{\mu_j}}, \sqrt{1-\prod_{j=1}^{n}(1-(\mu^U(\alpha_j))^2)^{\mu_j}}],$$
$$[\prod_{j=1}^{n}(\nu^L(\alpha_j))^{\mu_j}, \prod_{j=1}^{n}(\nu^U(\alpha_j))^{\mu_j}]\rangle,$$

which is the interval-valued intuitionistic uncertain linguistic weighted average(IVIULWA) operator.

Example 3.6. If $N(x)=1-x^2,\ g(t)=\log(\frac{2-t}{t})$, and Einstein t-conorm and t-norm [11] defined by $T_2^E(x,y)=\frac{xy}{1+(1-x)(1-y)}$, $S_2^E(x,y)=\sqrt{\frac{x^2+y^2}{1+x^2y^2}}$, then

$$EIVIULWA_\mu(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n)=\langle[s_{\sum_{j=1}^n(\mu_j\times\theta(\alpha_j))},s_{\sum_{j=1}^n(\mu_j\times\tau(\alpha_j))}],$$
$$[\sqrt{\frac{\prod_{j=1}^n(1+(\mu^L(\alpha_j))^2)^{\mu_j}-\prod_{j=1}^n(1-(\mu^L(\alpha_j))^2)^{\mu_j}}{\prod_{j=1}^n(1+(\mu^L(\alpha_j))^2)^{\mu_j}+\prod_{j=1}^n(1-(\mu^L(\alpha_j))^2)^{\mu_j}}},\sqrt{\frac{\prod_{j=1}^n(1+(\mu^U(\alpha_j))^2)^{\mu_j}-\prod_{j=1}^n(1-(\mu^U(\alpha_j))^2)^{\mu_j}}{\prod_{j=1}^n(1+(\mu^U(\alpha_j))^2)^{\mu_j}+\prod_{j=1}^n(1-(\mu^U(\alpha_j))^2)^{\mu_j}}}],$$
$$[\frac{2\prod_{j=1}^n(\nu^L(\alpha_j))^{\mu_j}}{\prod_{j=1}^n(2-\nu^L(\alpha_j))^{\mu_j}+\prod_{j=1}^n(\nu^L(\alpha_j))^{\mu_j}},\frac{2\prod_{j=1}^n(\nu^U(\alpha_j))^{\mu_j}}{\prod_{j=1}^n(2-\nu^U(\alpha_j))^{\mu_j}+\prod_{j=1}^n(\nu^U(\alpha_j))^{\mu_j}}]\rangle,$$

which is called the Einstein interval-valued intuitionistic uncertain linguistic weighted average(EIVIUL WA) operator.

Example 3.7. If $N(x)=1-x^2,\ g(t)=\log(\frac{\gamma+(1-\gamma)t}{t})$, $\gamma>0$, and Hamacher t-conorm and t-norm defined by $T_{2\gamma}^H(x,y)=\frac{xy}{\gamma+(1-\gamma)(x+y-xy)}$, $\gamma>0,\ S_{2\gamma}^H(x,y)=\sqrt{\frac{x^2+y^2-x^2y^2-(1-\gamma)x^2y^2}{1-(1-\gamma)x^2y^2}}$, $\gamma>0$, then

$$HIVIULWA_\mu(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n)=\langle[s_{\sum_{j=1}^n(\mu_j\times\theta(\alpha_j))},s_{\sum_{j=1}^n(\mu_j\times\tau(\alpha_j))}],$$
$$[\sqrt{\frac{\prod_{j=1}^n(1+(\gamma-1)(\mu^L(\alpha_j))^2)^{\mu_j}-\prod_{j=1}^n(1-(\mu^L(\alpha_j))^2)^{\mu_j}}{\prod_{j=1}^n(1+(\gamma-1)(\mu^L(\alpha_j))^2)^{\mu_j}+(\gamma-1)\prod_{j=1}^n(1-(\mu^L(\alpha_j))^2)^{\mu_j}}},\sqrt{\frac{\prod_{j=1}^n(1+(\gamma-1)(\mu^U(\alpha_j))^2)^{\mu_j}-\prod_{j=1}^n(1-(\mu^U(\alpha_j))^2)^{\mu_j}}{\prod_{j=1}^n(1+(\gamma-1)(\mu^U(\alpha_j))^2)^{\mu_j}+(\gamma-1)\prod_{j=1}^n(1-(\mu^U(\alpha_j))^2)^{\mu_j}}}],$$
$$[\frac{\gamma\prod_{j=1}^n(\nu^L(\alpha_j))^{\mu_j}}{\prod_{j=1}^n(1+(\gamma-1)(1-\nu^L(\alpha_j)))^{\mu_j}+(\gamma-1)\prod_{j=1}^n(\nu^L(\alpha_j))^{\mu_j}},\frac{\gamma\prod_{j=1}^n(\nu^U(\alpha_j))^{\mu_j}}{\prod_{j=1}^n(1+(\gamma-1)(1-\nu^U(\alpha_j)))^{\mu_j}+(\gamma-1)\prod_{j=1}^n(\nu^U(\alpha_j))^{\mu_j}}]\rangle,$$

which is called the Hammer interval-valued intuitionistic uncertain linguistic weighted average(HIVIUL WA) operator.

Especially, if $\gamma=1$, then the $HIVIULWA$ operator reduces to the $IVIULWA$ operator; if $\gamma=2$, then the $HIVIULWA$ operator reduces to the $EIVIULWA$ operator.

Example 3.8. If $N(x)=1-x^2,\ g(t)=\log(\frac{\gamma-1}{\gamma^t-1})$, $\gamma>1$, and Frank t-conorm and t-norm defined by $T_{2\gamma}^F(x,y)=\log_\gamma(1+\frac{(\gamma^x-1)(\gamma^y-1)}{\gamma-1})$, $\gamma>1,\ S_{2\gamma}^F(x,y)=\sqrt{1-\log_\gamma(1+\frac{(\gamma^{1-x^2}-1)(\gamma^{1-y^2}-1)}{\gamma-1})}$, $\gamma>1$, then

$$FIVIULWA_\mu(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n)=\langle[s_{\sum_{j=1}^n(\mu_j\times\theta(\alpha_j))},s_{\sum_{j=1}^n(\mu_j\times\tau(\alpha_j))}],$$
$$[\sqrt{1-\log_\gamma(1+\prod_{j=1}^n(\gamma^{1-(\mu^L(\alpha_j))^2}-1)^{\mu_j})},\sqrt{1-\log_\gamma(1+\prod_{j=1}^n(\gamma^{1-(\mu^U(\alpha_j))^2}-1)^{\mu_j})}],$$
$$[\log_\gamma(1+\prod_{j=1}^n(\gamma^{\nu^L(\alpha_j)}-1)^{\mu_j}),\log_\gamma(1+\prod_{j=1}^n(\gamma^{\nu^U(\alpha_j)}-1)^{\mu_j})]\rangle,$$

which is called the Frank interval-valued intuitionistic uncertain linguistic weighted average(FIVIULWA) operator.

Especially, if $\gamma\to1$, then the $FIVIULWA$ operator reduces to the $IVIULWA$ operator.

The $ATS-IVIULWA$ operator has the following properties, such as idempotency, monotonicity, bounded, and so on.

In the following section, we propose the interval-valued intuitionistic uncertain linguistic ordered weighted average operator based on Archimedean t-norm.

Definition 3.2. Let $\widetilde{\alpha}_i=\langle[s_{\theta(\alpha_i)},s_{\tau(\alpha_i)}],[\mu^L(\alpha_i),\mu^U(\alpha_i)],[\nu^L(\alpha_i),\nu^U(\alpha_i)]\rangle$ $(i=1,2,\ldots,n)$ be a collection of interval-valued intuitionistic uncertain linguistic variables based on Archimedean t-norm, and $ATS-IVIULOWA$ $(ATS-IVIULOWG):\Omega^n\to\Omega$, if

$$ATS-IVIULOWA_w(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n)=\sum_{j=1}^n w_j\widetilde{\alpha}_{\sigma_j},$$

$$(or\ \ ATS-IVIULOWG_w(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n)=\prod_{j=1}^n\widetilde{\alpha}_{\sigma_j}^{w_j},)$$

where Ω is the set of all interval-valued intuitionistic uncertain linguistic variables based on Archimedean t-norm, and $w=(w_1,w_2,\ldots,w_n)^T$ is an associated weighted vector with $ATS-IVIULOWA$ $(ATS-IVIULOWG)$ and $w_j\in[0,1]$, $\sum_{j=1}^n w_j=1$. If $(\sigma_1,\sigma_2,\ldots,\sigma_n)$ is any permutation of $(1,2,\ldots,n)$, such that $\widetilde{\alpha}_{\sigma_{j-1}}\succeq\widetilde{\alpha}_{\sigma_j}$ for all $j=1,2,\ldots,n$, then $ATS-IVIULOWA$ $(ATS-IVIULOWG)$ operator is called the interval-valued intuitionistic uncertain linguistic ordered weighted average (weighted geometric) operator based on Archimedean t-norm. w_j is decided only by the jth position in the aggregation process. Therefore, w can also be called the position-weighted vector.

The position-weighted vector w can be determined according to actual needs, or it can be determined based on the fuzzy semantic quantitative operator[17] or the combination number[18].

Theorem 3.2. Let $\widetilde{\alpha}_i = \langle [s_{\theta(\alpha_i)}, s_{\tau(\alpha_i)}], [\mu^L(\alpha_i), \mu^U(\alpha_i)], [\nu^L(\alpha_i), \nu^U(\alpha_i)]\rangle$ $(i = 1, 2, \ldots, n)$ be a collection of interval-valued intuitionistic uncertain linguistic variables based on Archimedean t-norm, the result aggregated from the Definition 3.2 is still an interval-valued intuitionistic uncertain linguistic variable based on Archimedean t-norm, and

$ATS-IVIULOWA_w(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n) = \langle [s_{\sum_{j=1}^n (w_j \times \theta(\alpha_{\sigma_j}))}, s_{\sum_{j=1}^n (\mu_j \times \tau(\alpha_{\sigma_j}))}],\ [h^{-1}(\sum_{j=1}^n w_j h(\mu^L(\alpha_{\sigma_j}))), h^{-1}(\sum_{j=1}^n w_j h(\mu^U(\alpha_{\sigma_j})))], [g^{-1}(\sum_{j=1}^n w_j g(\nu^L(\alpha_{\sigma_j}))), g^{-1}(\sum_{j=1}^n w_j g(\nu^U(\alpha_{\sigma_j})))]\rangle$.

$(or\ ATS-IVIULOWG_w(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n) = \langle [s_{\prod_{j=1}^n ((\theta(\alpha_{\sigma_j}))^{w_j})}, s_{\prod_{j=1}^n ((\tau(\alpha_{\sigma_j}))^{w_j})}], [g^{-1}(\sum_{j=1}^n w_j g(\mu^L(\alpha_{\sigma_j}))), g^{-1}(\sum_{j=1}^n w_j g(\mu^U(\alpha_{\sigma_j})))], [h^{-1}(\sum_{j=1}^n w_j h(\nu^L(\alpha_{\sigma_j}))), h^{-1}(\sum_{j=1}^n w_j h(\nu^U(\alpha_{\sigma_j})))]\rangle,)$

where $w = (w_1, w_2, \ldots, w_n)^T$ is an associated weighted vector with $ATS-IVIULOWA$ $(ATS-IVIULOWG)$ and $w_j \in [0, 1]$, $\sum_{j=1}^n w_j = 1$. $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ is any permutation of $(1, 2, \ldots, n)$, such that $\widetilde{\alpha}_{\sigma_{j-1}} \succeq \widetilde{\alpha}_{\sigma_j}$ for all $j = 1, 2, \ldots, n$.

In a similar way to the proof of Theorem 3.1, Theorem 3.2 can be proven by mathematical induction, and the proof steps are therefore omitted.

The $ATS-IVIULOWA$ operator has some similar properties to the $ATS-IVIULWA$ operator, such as idempotency, monotonicity, bounded, commutativity and so on.).

Definition 3.3. Let $\widetilde{\alpha}_i = \langle [s_{\theta(\alpha_i)}, s_{\tau(\alpha_i)}], [\mu^L(\alpha_i), \mu^U(\alpha_i)], [\nu^L(\alpha_i), \nu^U(\alpha_i)]\rangle$ $(i = 1, 2, \ldots, n)$ be a collection of interval-valued intuitionistic uncertain linguistic variables based on Archimedean t-norm, and $ATS-IVIULHA : \Omega^n \to \Omega$, if

$$ATS-IVIULHA_{\mu,w}(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n) = \sum_{j=1}^n w_j \widetilde{\beta}_{\sigma_j},$$

$$(or\quad ATS-IVIULHG_{\mu,w}(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n) = \prod_{j=1}^n (\widetilde{\beta}_{\sigma_j})^{w_j},)$$

where Ω is the set of all interval-valued intuitionistic uncertain linguistic variables based on Archimedean t-norm, and $w = (w_1, w_2, \ldots, w_n)^T$ is an associated weighted vector with $ATS-IVIULHA$ $(ATS-IVIULHG)$ and $w_j \in [0, 1]$, $\sum_{j=1}^n w_j = 1$. If $\widetilde{\beta}_{\sigma_j}$ is the jth the largest of the interval-valued intuitionistic uncertain linguistic weighted argument $\widetilde{\beta}_k(\widetilde{\beta}_k = n\mu_k\widetilde{\alpha}_k, k = 1, 2, \ldots, n)$, $\mu = (\mu_1, \mu_2, \ldots, \mu_n)^T$ is the weighted vector of $\widetilde{\alpha}_j(j = 1, 2, \ldots, n)$, μ is a fuzzy measure on X with $\mu_j \in [0, 1]$, $\mu_j = \mu(A_{(j)}) - \mu(A_{(j+1)})$, $\sum_{j=1}^n \mu_j = 1$, $A_{(j)} = (j, \ldots, n)$ with $A_{(n+1)} = \emptyset$ and n is the balancing coefficient, then $ATS-IVIULHA$ $(ATS-IVIULHG)$ is called the interval-valued intuitionistic uncertain linguistic hybrid average (hybrid geometric) operator based on Archimedean t-norm.

Theorem 3.3. Let $\widetilde{\alpha}_i = \langle [s_{\theta(\alpha_i)}, s_{\tau(\alpha_i)}], [\mu^L(\alpha_i), \mu^U(\alpha_i)], [\nu^L(\alpha_i), \nu^U(\alpha_i)]\rangle$ $(i = 1, 2, \ldots, n)$ be a collection of interval-valued intuitionistic uncertain linguistic variables based on Archimedean t-norm, then, the result aggregated from Definition 3.3 is still an interval-valued intuitionistic uncertain linguistic variable based on Archimedean t-norm, and

$ATS-IVIULHA_{\mu,w}(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n) = \langle [s_{\sum_{j=1}^n (w_j \times \theta(\beta_{\sigma_j}))}, s_{\sum_{j=1}^n (w_j \times \tau(\beta_{\sigma_j}))}], [h^{-1}(\sum_{j=1}^n w_j h(\mu^L(\beta_{\sigma_j}))), h^{-1}(\sum_{j=1}^n w_j h(\mu^U(\beta_{\sigma_j})))], [g^{-1}(\sum_{j=1}^n w_j g(\nu^L(\beta_{\sigma_j}))), g^{-1}(\sum_{j=1}^n w_j g(\nu^U(\beta_{\sigma_j})))]\rangle$.

$(or\ ATS-IVIULHG_{\mu,w}(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n) = \langle [s_{\prod_{j=1}^n ((\theta(\beta_{\sigma_j}))^{w_j})}, s_{\prod_{j=1}^n ((\tau(\beta_{\sigma_j}))^{w_j})}], [g^{-1}(\sum_{j=1}^n w_j g(\mu^L(\beta_{\sigma_j}))), g^{-1}(\sum_{j=1}^n w_j g(\mu^U(\beta_{\sigma_j})))], [h^{-1}(\sum_{j=1}^n w_j h(\nu^L(\beta_{\sigma_j}))), h^{-1}(\sum_{j=1}^n w_j h(\nu^U(\beta_{\sigma_j})))]\rangle)$,

where $w = (w_1, w_2, \ldots, w_n)^T$ is an associated weighted vector with $ATS-IVIULHA$ $(ATS-IVIULHG)$ and $w_j \in [0, 1]$, $\sum_{j=1}^n w_j = 1$. If $\widetilde{\beta}_{\sigma_j}$ is the jth the largest of the interval-valued intuitionistic uncertain linguistic weighted argument $\widetilde{\beta}_k(\widetilde{\beta}_k = n\mu_k\widetilde{\alpha}_k, k = 1, 2, \ldots, n)$, $\mu = (\mu_1, \mu_2, \ldots, \mu_n)^T$ is the weighted vector of $\widetilde{\alpha}_j(j = 1, 2, \ldots, n)$, μ is a fuzzy measure on X with $\mu_j \in [0, 1]$, $\mu_j = \mu(A_{(j)}) - \mu(A_{(j+1)})$, $\sum_{j=1}^n \mu_j = 1$, $A_{(j)} = (j, \ldots, n)$ with $A_{(n+1)} = \emptyset$ and n is the balancing coefficient.

4. Aggregating of the interval-valued intuitionistic uncertain linguistic variables in group decision making based on Archimedean t-norm

Consider a multiple attribute group decision making problem with intuitionistic linguistic information. Let $A = \{A_1, A_2, \ldots, A_m\}$ be a discrete set of alternatives and $C = \{C_1, C_2, \ldots, C_n\}$ be the set of attributes. $\mu = (\mu_1, \mu_2, \ldots, \mu_n)^T$ is the weighting vector of the attribute $C_j (j = 1, 2, \ldots, n)$, where $\mu_j \geqslant 0$ $(j = 1, 2, \ldots .n)$, $\sum_{j=1}^{n} \mu_j = 1$. Let $D = \{D_1, D_2, \ldots, D_p\}$ be the set of decision makers and $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_P)^T$ be the weighted vector of the decision makers, where $\lambda_k \geqslant 0$ $(k = 1, 2, \ldots, p)$, $\sum_{k=1}^{p} \lambda_k = 1$. Suppose that $\widetilde{R}^k = [\widetilde{r}_{ij}^k]_{m\times n}$ is the decision matrix, where $\widetilde{r}_{ij}^k = \langle [\alpha_{ijk}^L, \alpha_{ijk}^U], [\mu_{ijk}^L(\alpha_i), \mu_{ijk}^U(\alpha_i)], [\nu_{ijk}^L(\alpha_i), \nu_{ijk}^U(\alpha_i)]\rangle$ takes the form of the intuitionistic uncertain linguistic variable given by the decision maker D_k for an alternative A_i with respect to an attribute C_j, and $0 \leqslant \mu_{ijk}^L \leqslant 1, 0 \leqslant \nu_{ijk}^L \leqslant 1, \mu_{ijk}^L \leqslant \mu_{ijk}^U, \nu_{ijk}^L \leqslant \nu_{ijk}^U, \mu_{ijk}^U + \nu_{ijk}^U \leqslant 1, \alpha_{ijk}^L, \alpha_{ijk}^U \in S$. Then, the ranking of alternatives is required.

In the following section, we will apply the above operators to solve multiple group decision making problems based on intuitionistic uncertain linguistic information. There are two methods, which are as follows:

4.1 The method of aggregating the attribute values first

Step 1. We can utilize the $ATS - IVIULWA$ operator(or $ATS - IVIULWGA$ operator) to aggregate the attribute values $(\widetilde{r}_{i1}^k, \widetilde{r}_{i2}^k, \ldots, \widetilde{r}_{in}^k)$ given by each decision maker with respect to each alternative into a comprehensive attribute value $\widetilde{r}_i^k$,

$$\widetilde{r}_i^k = ATS - IVIULWA_\mu(\widetilde{r}_{i1}^k, \widetilde{r}_{i2}^k, \ldots, \widetilde{r}_{in}^k) = \sum_{j=1}^{n} \mu_j \widetilde{r}_{ij}^k$$

where $i = 1, 2, \ldots, m$; $k = 1, 2, \ldots, p, \mu = (\mu_1, \mu_2, \ldots, \mu_n)^T$, $\mu_j = \mu(A_{(j)}) - \mu(A_{(j+1)}) \in [0, 1]$ is the attribute weight vector, $A_{(j)} = (j, \ldots, n)$ with $A_{(n+1)} = \emptyset$.

Step 2. We can utilize the $ATS - IVIULHA$ operator(or $ATS - IVIULHG$ operator) operator to aggregate the information $(\widetilde{r}_i^1, \widetilde{r}_i^2, \ldots, \widetilde{r}_i^p)$ of each decision maker into a collective value $\widetilde{r}_i$ for each alternative,

$$\widetilde{r}_i = ATS - IVIULHA_{\lambda,w}(\widetilde{r}_i^1, \widetilde{r}_i^2, \ldots, \widetilde{r}_i^p) = \sum_{k=1}^{p} w_k \widetilde{b}_i^{\sigma_k}$$

$i = 1, 2, \ldots, m$, where $w = (w_1, w_2, \ldots, w_p)^T$ is an associated weighted vector with $ATS - IVIULHA$ (or $ATS - IVIULHG$) and $w_k \in [0, 1]$, $\sum_{k=1}^{p} w_k = 1$. $\widetilde{b}_i^{\sigma_k}$is the kth the largest of the interval-valued intuitionistic uncertain linguistic weighted argument $\widetilde{b}_i^l(\widetilde{b}_i^l = p\lambda_l \widetilde{r}_i^l$ or $\widetilde{b}_i^l = \widetilde{r}_i^{lp\lambda_l}, l = 1, 2, \ldots, p)$, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)^T$ is the weighted vector of the decision makers, $\lambda_k \in [0, 1]$, $\sum_{k=1}^{n} \lambda_k = 1$ and p is the balancing coefficient.

Step 3. By Definition 2.2, we can calculate the expected value $E(\widetilde{r}_i)(i = 1, 2, \ldots, m)$ of the collective value $\widetilde{r}_i(i = 1, 2, \ldots, m)$, rank all of the alternatives $A_i(i = 1, 2, \ldots, m)$ and then select the best one(s). If there is no difference between two expected values $E(\widetilde{r}_i)$ and $E(\widetilde{r}_j)$, then by Definition 2.3, we must calculate the accuracy degrees $H(\widetilde{r}_i)$ and $H(\widetilde{r}_j)$ of the collective overall preference values $\widetilde{r}_i$ and $\widetilde{r}_j$, respectively, and then rank the alternatives A_i and A_j in accordance with the accuracy function values $H(\widetilde{r}_i)$ and $H(\widetilde{r}_j)$.

Step 4. Rank all of the alternatives $A_i(i = 1, 2, \ldots, m)$ and select the best one(s) in accordance with $E(\widetilde{r}_i)$ and $H(\widetilde{r}_i)$ $(i = 1, 2, \ldots, m)$.

4.2 The method of first aggregating the information from different decision makers

Step 1. We can utilize the $ATS - IVIULWA$ operator(or $ATS - IVIULWGA$ operator) to aggregate all of the decision matrices $\widetilde{R}^k(k = 1, 2, \ldots, p)$ into a collective decision matrix $\widetilde{R} = [\widetilde{r}_{ij}]_{m\times n}$,

$$\widetilde{r}_{ij} = ATS - IVIULWA_\lambda(\widetilde{r}_{ij}^1, \widetilde{r}_{ij}^2, \ldots, \widetilde{r}_{ij}^p) = \sum_{k=1}^{p} \lambda_k \widetilde{r}_{ij}^k$$

$i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$ where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)^T$ is the weighted vector of the decision makers.

Step 2. We can utilize the $ATS-IVIULHA$ operator(or $ATS-IVIULHG$ operator) to derive the collective overall preference value $\widetilde{r}_i(i=1,2,\ldots,m)$ of the alternative A_i,

$$\widetilde{r}_i = ATS-IVIULHA_{\mu,w}=(\widetilde{r}_{i1},\widetilde{r}_{i2},\ldots,\widetilde{r}_{in})=\sum_{j=1}^{n} w_j\widetilde{b}_{i\sigma_j}$$

$i=1,2,\ldots,m$, where $w=(w_1,w_2,\ldots,w_n)^T$ is an associated weighted vector with $ATS-IVIULHA$ (or $ATS-IVIULHG$) and $w_j\in[0,1]$, $\sum_{j=1}^{n} w_j=1$. $\widetilde{b}_{i\sigma_j}$ is the jth the largest of the interval-valued intuitionistic uncertain linguistic weighted argument $\widetilde{b}_{ik}$ ($\widetilde{b}_{ik}=n\mu_k\widetilde{r}_{ik}$ or $\widetilde{b}_{ik}=(\widetilde{r}_{ik})^{n\mu_k}, k=1,2,\ldots,n$), $\mu=(\mu_1,\mu_2,\ldots,\mu_n)^T$ is the weight vector of $\widetilde{r}_{ij}(j=1,2,\ldots,n), \mu_j\in[0,1]$, $\sum_{j=1}^{n}\mu_j=1$, and n is the balancing coefficient.

Step 3. By Definition 2.2, we can calculate the expected value $E(\widetilde{r}_i)(i=1,2,\ldots,m)$ of the collective value $\widetilde{r}_i(i=1,2,\ldots,m)$, rank all of the alternatives $A_i(i=1,2,\ldots,m)$ and then select the best one(s). If there is no difference between two expected values $E(\widetilde{r}_i)$ and $E(\widetilde{r}_j)$, then by Definition 2.3, we must calculate the accuracy degrees $H(\widetilde{r}_i)$ and $H(\widetilde{r}_j)$ of the collective overall preference values $\widetilde{r}_i$ and $\widetilde{r}_j$, respectively, and then rank the alternatives A_i and A_j in accordance with the accuracy function values $H(\widetilde{r}_i)$ and $H(\widetilde{r}_j)$.

Step 4. Rank all of the alternatives $A_i(i=1,2,\ldots,m)$ and select the best one(s) in accordance with $E(\widetilde{r}_i)$ and $H(\widetilde{r}_i)$ $(i=1,2,\ldots,m)$.

5. An application example

Let us consider an investment company that wants to invest a sum of money in the best option. There is a panel with four possible alternatives in which to invest the money:

(1) A_1 is a car company.
(2) A_2 is a computer company.
(3) A_3 is a TV company.
(4) A_4 is a food company.

The investment company must make a decision according to the following four attributes:

(1) C_1 is the risk index.
(2) C_2 is the growth index.
(3) C_3 is the social-political impact index.
(4) C_4 is the environment impact index.

The fuzzy measure of each attributes: $g(x_1)=0.4$, $g(x_2)=0.25$, $g(x_3)=0.37$, $g(x_4)=0.2$. Since

$$\lambda+1=\prod_{i=1}^{n}(1+\lambda g(x_i)), i=1,2,3,4,$$

we have $\lambda=-0.44$, then $g(x_1,x_2)=0.60$, $g(x_1,x_3)=0.70$, $g(x_1,x_4)=0.56$, $g(x_2,x_3)=0.68$, $g(x_2,x_4)=0.43$, $g(x_3,x_4)=0.54$, $g(x_1,x_2,x_4)=0.88$, $g(x_2,x_3,x_4)=0.75$, $g(x_1,x_3,x_4)=0.73$, $g(x_1,x_2,x_3,x_4)=1$.

The four possible alternatives A_1,A_2,A_3,A_4 are evaluated by three decision makers $D_k(k=1,2,3)$ (whose weight vector is $\lambda=(0.4,0.32,0.28)^T$) using the linguistic term set $S=(s_0,s_1,s_2,s_3,s_4,s_5,s_6)$ about the interval-valued intuitionistic uncertain linguistic variables based on Archimedean t-norm $\widetilde{r}_{ij}^k=\langle[\alpha_{ijk}^L,\alpha_{ijk}^U],[\mu_{ijk}^L,\mu_{ijk}^U],[\nu_{ijk}^L,\nu_{ijk}^U]\rangle$ under the above four attributes. The decision matrices $\widetilde{R}^k=[\widetilde{r}_{ij}^k]_{4\times4}$ $(k=1,2,3)$) are listed in Tables 1-3.

To obtain the best alternative(s), we can use the two methods introduced to obtain the selection results.

Let $N(x)=1-x$, $g(t)=\log(\frac{\gamma+(1-\gamma)t}{t})$, $\gamma=2$, i.e. $g(t)=\log(\frac{2-t}{t})$.

5.1 The method of aggregating the attribute values first

Step 1. We can utilize the $ATS-IVIULWGA$ operator to aggregate the attribute values $(\widetilde{r}_{i1}^k,\widetilde{r}_{i2}^k,\widetilde{r}_{i3}^k,\widetilde{r}_{i4}^k)$ into a comprehensive attribute value $\widetilde{r}_i^k$, we can obtain the aggregating results shown in Table 4.

Step 2. We can utilize the $ATS - IVIULHG$ operator to aggregate the information $(\widetilde{r}_i^1, \widetilde{r}_i^2, \widetilde{r}_i^3)$ of each decision maker into a collective value $\widetilde{r}_i$ for each alternative.

Suppose the position weight is $w = (0.25, 0.5, 0.25)^T$. We can obtain

$\widetilde{r}_1 = \langle [s_{3.97}, s_{4.85}], [0.73, 0.78], [0.12, 0.16] \rangle$, $\widetilde{r}_2 = \langle [s_{4.12}, s_{5.03}], [0.61, 0.75], [0.17, 0.14] \rangle$,

$\widetilde{r}_3 = \langle [s_{4.28}, s_{4.40}], [0.72, 0.75], [0.10, 0.18] \rangle$, $\widetilde{r}_4 = \langle [s_{3.28}, s_{3.97}], [0.63, 0.70], [0.16, 0.22] \rangle$.

Step 3. We can calculate the expected value $E(\widetilde{r}_i)(i = 1, 2, 3, 4)$ of the collective interval-valued intuitionistic uncertain linguistic variables values based on Archimedean t-norm $\widetilde{r}_i(i = 1, 2, 3, 4)$ as

$E(\widetilde{r}_1) = s_{3.976}$, $E(\widetilde{r}_2) = s_{3.701}$, $E(\widetilde{r}_3) = s_{3.499}$, $E(\widetilde{r}_4) = s_{2.984}$.

Step 4. By definition 4.2, we can rank all of the alternatives (A_1, A_2, A_3, A_4) in accordance with the expected values $(E(\widetilde{r}_1), E(\widetilde{r}_2), E(\widetilde{r}_3), E(\widetilde{r}_4))$ of the collective interval-valued intuitionistic uncertain linguistic variables values based on Archimedean t-norm $(\widetilde{r}_1, \widetilde{r}_2, \widetilde{r}_3, \widetilde{r}_4)$. We can obtain $A_1 \succ A_2 \succ A_3 \succ A_4$, and thus, the most desirable alternatives is A_1.

5.2 The method of first aggregating the information from different decision makers

Step 1. We can utilize the $ATS - IVIULWGA$ operator to aggregate all of the decision matrices $\widetilde{R}^k(k = 1, 2, 3)$ into a collective decision matrix $\widetilde{R} = [\widetilde{r}_{ij}]_{4\times 4}$. We can obtain the aggregation result shown in Table 5.

Step 2. We can utilize

$$\widetilde{r}_i = ATS - IVIULHG_{\mu,w} = (\widetilde{r}_{i1}, \widetilde{r}_{i2}, \ldots, \widetilde{r}_{i4}) = \prod_{j=1}^{4} (\widetilde{b}_{i\sigma_j})^{w_j} \ (i = 1, 2, 3, 4)$$

to derive the collective overall values $\widetilde{r}_i$ of alterative A_i, where $\widetilde{b}_{i\sigma_j}$ is the jth the largest of the interval-valued intuitionistic uncertain linguistic weighted argument $\widetilde{b}_{ik}(\widetilde{b}_{ik} = (\widetilde{r}_{ik})^{4\mu_k}, k = 1, 2, 3, 4)$. Suppose the position weight is $w = (0.15, 022., 0.35, 0.28)^T$. We can obtain

$\widetilde{r}_1 = \langle [s_{4.65}, s_{5.11}], [(0.72, 0.78], [0.10, 0.15] \rangle$, $\widetilde{r}_2 = \langle [s_{4.74}, s_{5.17}], [(0.63, 0.70], [0.15, 0.20] \rangle$,

$\widetilde{r}_3 = \langle [s_{4.34}, s_{4.45}], [(0.70, 0.76], [0.11, 0.17] \rangle$, $\widetilde{r}_4 = \langle [s_{3.71}, s_{4.08}], [(0.66, 0.73], [0.13, 0.23] \rangle$.

Step 3. We can calculate the expected value $E(\widetilde{r}_i)(i = 1, 2, 3, 4)$ of the collective interval-valued intuitionistic uncertain linguistic variables values based on Archimedean t-norm $\widetilde{r}_i(i = 1, 2, 3, 4)$ as

$E(\widetilde{r}_1) = s_{3.971}$, $E(\widetilde{r}_2) = s_{3.694}$, $E(\widetilde{r}_3) = s_{3.494}$, $E(\widetilde{r}_4) = s_{2.952}$.

Step 4. By definition 4.2, we can rank all of the alternatives (A_1, A_2, A_3, A_4) in accordance with the expected values $(E(\widetilde{r}_1), E(\widetilde{r}_2), E(\widetilde{r}_3), E(\widetilde{r}_4))$ of the collective interval-valued intuitionistic uncertain linguistic variables values based on Archimedean t-norm $(\widetilde{r}_1, \widetilde{r}_2, \widetilde{r}_3, \widetilde{r}_4)$. We can obtain $A_1 \succ A_2 \succ A_3 \succ A_4$, and thus, the most desirable alternatives is A_1.

Table 1: Decision matrix $\widetilde{R}^1$

	C_1	C_2	C_3	C_4
A_1	$\langle [s_4, s_5], [0.7, 0.8], [0.1, 0.2] \rangle$	$\langle [s_5, s_5], [0.6, 0.6], [0.1, 0.3] \rangle$	$\langle [s_5, s_6], [0.8, 0.8], [0.1, 0.1] \rangle$	$\langle [s_4, s_4], [0.8, 0.8], [0.1, 0.1] \rangle$
A_2	$\langle [s_5, s_5], [0.6, 0.6], [0.1, 0.2] \rangle$	$\langle [s_5, s_6], [0.7, 0.7], [0.2, 0.2] \rangle$	$\langle [s_4, s_5], [0.5, 0.6], [0.2, 0.3] \rangle$	$\langle [s_4, s_5], [0.5, 0.6], [0.1, 0.3] \rangle$
A_3	$\langle [s_4, s_4], [0.7, 0.7], [0.2, 0.2] \rangle$	$\langle [s_4, s_4], [0.7, 0.8], [0.1, 0.2] \rangle$	$\langle [s_5, s_5], [0.7, 0.7], [0.1, 0.2] \rangle$	$\langle [s_5, s_5], [0.7, 0.8], [0.1, 0.3] \rangle$
A_4	$\langle [s_3, s_4], [0.6, 0.7], [0.2, 0.3] \rangle$	$\langle [s_3, s_3], [0.5, 0.6], [0.2, 0.3] \rangle$	$\langle [s_4, s_4], [0.6, 0.7], [0.2, 0.3] \rangle$	$\langle [s_3, s_4], [0.7, 0.7], [0.2, 0.2] \rangle$

Table 2: Decision matrix $\widetilde{R}^2$

	C_1	C_2	C_3	C_4
A_1	$\langle [s_5, s_6], [0.6, 0.7], [0.1, 0.1] \rangle$	$\langle [s_5, s_5], [0.7, 0.7], [0.1, 0.1] \rangle$	$\langle [s_4, s_5], [0.7, 0.9], [0.2, 0.1] \rangle$	$\langle [s_5, s_5], [0.7, 0.8], [0.1, 0.2] \rangle$
A_2	$\langle [s_5, s_5], [0.5, 0.7], [0.2, 0.2] \rangle$	$\langle [s_4, s_5], [0.6, 0.7], [0.2, 0.2] \rangle$	$\langle [s_5, s_4], [0.7, 0.7], [0.1, 0.2] \rangle$	$\langle [s_6, s_6], [0.6, 0.7], [0.1, 0.1] \rangle$
A_3	$\langle [s_5, s_5], [0.6, 0.7], [0.0, 0.2] \rangle$	$\langle [s_4, s_5], [0.8, 0.9], [0.1, 0.1] \rangle$	$\langle [s_4, s_4], [0.6, 0.6], [0.2, 0.2] \rangle$	$\langle [s_4, s_4], [0.7, 0.7], [0.2, 0.2] \rangle$
A_4	$\langle [s_5, s_5], [0.7, 0.8], [0.1, 0.2] \rangle$	$\langle [s_4, s_4], [0.5, 0.6], [0.2, 0.3] \rangle$	$\langle [s_3, s_3], [0.9, 0.9], [0.0, 0.1] \rangle$	$\langle [s_3, s_4], [0.8, 0.8], [0.1, 0.2] \rangle$

Table 3: Decision matrix $\widetilde{R}^3$

	C_1	C_2	C_3	C_4
A_1	$\langle [s_5, s_5], [0.7, 0.8], [0.1, 0.1] \rangle$	$\langle [s_5, s_5], [0.8, 0.9], [0.1, 0.1] \rangle$	$\langle [s_5, s_5], [0.8, 0.9], [0.1, 0.1] \rangle$	$\langle [s_5, s_6], [0.7, 0.8], [0.2, 0.2] \rangle$
A_2	$\langle [s_5, s_6], [0.6, 0.7], [0.1, 0.2] \rangle$	$\langle [s_5, s_6], [0.7, 0.7], [0.1, 0.2] \rangle$	$\langle [s_5, s_5], [0.8, 0.8], [0.1, 0.1] \rangle$	$\langle [s_5, s_5], [0.9, 0.9], [0.1, 0.1] \rangle$
A_3	$\langle [s_5, s_5], [0.8, 0.8], [0.0, 0.1] \rangle$	$\langle [s_5, s_5], [0.7, 0.8], [0.1, 0.2] \rangle$	$\langle [s_4, s_4], [0.7, 0.8], [0.1, 0.2] \rangle$	$\langle [s_4, s_4], [0.7, 0.8], [0.1, 0.1] \rangle$
A_4	$\langle [s_4, s_5], [0.8, 0.9], [0.1, 0.1] \rangle$	$\langle [s_4, s_4], [0.8, 0.8], [0.0, 0.2] \rangle$	$\langle [s_4, s_5], [0.8, 0.8], [0.0, 0.1] \rangle$	$\langle [s_4, s_5], [0.7, 0.7], [0.1, 0.2] \rangle$

Table 4: The comprehensive attribute value $\widetilde{r}_i^k$

	D_1	D_2	D_3
A_1	$\langle[s_{4.07}, s_{4.89}], [0.58, 0.67], [0.12, 0.16]\rangle$	$\langle[s_{4.21}, s_{4.97}], [0.64, 0.80], [0.11, 0.18]\rangle$	$\langle[s_{5.21}, s_{5.43}], [0.56, 0.73], [0.14, 0.19]\rangle$
A_2	$\langle[s_{4.46}, s_{5.10}], [0.61, 0.75], [0.1, 0.2]\rangle$	$\langle[s_{4.21}, s_{4.86}], [0.67, 0.78], [0.13, 0.17]\rangle$	$\langle[s_{4.60}, s_{4.96}], [0.68, 0.80], [0.13, 0.20]\rangle$
A_3	$\langle[s_{3.82}, s_{4.19}], [0.68, 0.79], [0.07, 0.10]\rangle$	$\langle[s_{4.30}, s_{4.81}], [0.61, 0.74], [0.12, 0.22]\rangle$	$\langle[s_{4.33}, s_{4.62}], [0.71, 0.84], [0.11, 0.23]\rangle$
A_4	$\langle[s_{4.37}, s_{4.94}], [0.72, 0.80], [0.10, 0.16]\rangle$	$\langle[s_{4.79}, s_{5.04}], [0.61, 0.74], [0.08, 0.20]\rangle$	$\langle[s_{4.57}, s_{5.08}], [0.64, 0.76], [0.12, 0.19]\rangle$

Table 5: The collective decision matrix $\widetilde{R} = (C_1, C_2, C_3, C_4)$.

	C_1	C_2
A_1	$\langle[s_{4.57}, s_{5.28}], [0.67, 0.77], [0.10, 0.14]\rangle$	$\langle[s_{5.00}, s_{5.00}], [0.69, 0.71], [0.10, 0.19]\rangle$
A_2	$\langle[s_{5.00}, s_{5.28}], [0.57, 0.66], [0.13, 0.20]\rangle$	$\langle[s_{4.68}, s_{5.68}], [0.67, 0.70], [0.17, 0.20]\rangle$
A_3	$\langle[s_{4.57}, s_{4.57}], [0.70, 0.73], [0.09, 0.17]\rangle$	$\langle[s_{4.28}, s_{4.57}], [0.73, 0.83], [0.10, 0.17]\rangle$
A_4	$\langle[s_{3.81}, s_{4.57}], [0.69, 0.79], [0.14, 0.21]\rangle$	$\langle[s_{3.57}, s_{3.57}], [0.58, 0.65], [0.14, 0.27]\rangle$

	C_3	C_4
A_1	$\langle[s_{4.68}, s_{5.38}], [0.83, 0.86], [0.07, 0.10]\rangle$	$\langle[s_{4.57}, s_{4.83}], [0.74, 0.80], [0.13, 0.16]\rangle$
A_2	$\langle[s_{4.57}, s_{4.68}], [0.64, 0.69], [0.14, 0.21]\rangle$	$\langle[s_{4.83}, s_{5.28}], [0.63, 0.71], [0.19, 0.19]\rangle$
A_3	$\langle[s_{4.37}, s_{4.37}], [0.67, 0.70], [0.13, 0.20]\rangle$	$\langle[s_{4.37}, s_{4.37}], [0.70, 0.77], [0.13, 0.17]\rangle$
A_4	$\langle[s_{3.67}, s_{3.92}], [0.74, 0.79], [0.09, 0.19]\rangle$	$\langle[s_{3.50}, s_{4.28}], [0.73, 0.73], [0.14, 0.20]\rangle$

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Non-integer variable order dynamic equations on time scales involving Caputo-Fabrizio type differential operator

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Abstract

This work deals with the conecept of a Caputo-Fabrizio type non-integer variable order differential opertor on time scales that involves a non-singular kernel. A measure theoretic discussion on the limit cases for the order of differentiation is presented. Then, corresponding to the fractional derivative, we discuss on an integral for constant and variable orders. Beside the obtaining solutions to some dynamic problems on time scales involving the proposed derivative, a fractional folrmulation for the viscoelastic oscillation problem is studied and its conversion into a third order dynamic equation is presented.

keywords: Time scales, Fractional calculus, Caputo-Fabrizio derivative, Non-integer variable order derivative and integral , Dirac delta functional, Viscoelastic Oscillation.

MSC 2010: 34N05, 26A33.

1 Introduction

This work deals with two theories, namely non-integer order calculus (or as what it is called, fractional calculus) and Δ-calculus (calculus on time scales). The first one, originally is as old as the classical calculus in the sense of Leibnitz and the latter which was started by an effort in 80's, was aimed to unify the difference and the differential. For an overview on the trends and achievements in Δ-calculus, see [5].

In the recent years, to propose a non-integer order counterpart of Δ-calculus, a number of efforts have been made [1, 3]. One of the main challenges for such proposals was to overcome the limitations caused by the nature of the time scales, since a typical cluster of points may appear in a variety of scattered or dense patterns.

In view of the real world applications of the non-local fractional calculus, one presumption is to find a suitable kernel for the purpose of the better description to a class of phenomena.

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Recently, a Caputo-like non-integer order derivative with non-singular kernel is proposed by Caputo and Fabrizio [7],

$$\mathfrak{D}_t^{\alpha} f(t) = \frac{M(\alpha)}{1-\alpha}\int_a^t f'(\tau)\exp\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right]d\tau, \tag{1.1}$$

in which, $M(\alpha)$ is called, the normalization function and it satisfies $M(0) = M(1) = 1$. As is is graphically illustrated in [7], this new definition of the non-integer order derivative seems to be more appropriate in describing a process which is affected by its past. Indeed, compare to Caputo derivative, the new derivative with an exponential kernel, shows rapid stabilization with respect to the memory effect.

A substitution of the exponential kernel with Mittag-Leffler type, to define another non-local derivative, is suggested by Atangana and Baleanu with an application to the non-integer order heat transfer model [2].

Fractional integral of variable order with singular kernel first was introduced in 1993 by Samko and Ross in [14] using direct and Fourier based approaches. The direct appraoch formulation for the derivative reads

$$D_{a^+}^{\alpha(t)} = \frac{1}{\Gamma(1-\alpha(t))}.\frac{d}{dt}\int_a^t \frac{f(\tau)}{(t-\tau)^{-\alpha(\tau)}} \tag{1.2}$$

with the assumtion that $\Re(\alpha(t)) \in (0,1)$. The corresponded integral was defined in a similar manner. Having no idea about any prospect application at that time, it has been noticed by the authors in the introduction that, "the work is stimulated by intelectual curiosity". However, later on, the theory found its applicability in viscoelastic dynamics and anomalous diffusion as well. (see [13] for a variable order model of a harmonically forced oscillator with viscoelastic damping).

Lorenzo & Hartley [11], in 2002, proposed the following formulation for the integral

$${}_0d_t^{-q(t)} f(t) = \int_0^t \frac{(t-\tau)^{q(t,\tau)(1-(dq(\tau)/d\tau))-1}}{\gamma(q(t,\tau)(1-\frac{dq(\tau)}{d\tau}))} f(\tau)d\tau, \tag{1.3}$$

then a simplified version of (1.3), that is

$${}_0d_t^{-q(t)} f(t) = \int_0^t \frac{(t-\tau)^{q(t,\tau))-1}}{\Gamma(q(t,\tau))} f(\tau)d\tau, \tag{1.4}$$

is considered. A slightly different approach is introduced by Coimbra non-integer variable order derivative ([6, 13]) which is defined by

$$D^{\alpha(t)} f(t) = \frac{1}{1-\alpha(t)}\left\{\int_a^t \left[\frac{df(\tau)}{d\tau}\right]\frac{1}{(t-\tau)^{\alpha(t)}}d\tau + f(0)t^{-\alpha(t)}.\right\}. \tag{1.5}$$

From the definition, it is clear that, (1.3) and (1.5) are of the Riemann-Liouville and Caputo type. However, according to the other well-known singular non-integer (constant) order integrals and derivatives, the variable order counterparts are reported (see [16], section 1.1.5, and [12] for a comprehensive list of singular and non-singular fractional derivatives).

All the above mentioned works was dealt with the continuous time rather than considering lattices or in its more general form the time scales approach. The appearance of variable order fractional differnce operator of Caputo type, i.e., constant pace time scale of the form $(h\mathbb{N})_{a_0} = \{a_0, a_0 + h, \dots\}$, was in [15], in which, the order $\alpha(t)$ involved a chaos,

$$ {}_{a_0}^{C}\Delta_t^{\alpha(i)} f\Big[t + (1-\alpha(N-1))h\Big] = \frac{1}{\Gamma(1-\alpha(N-1))} \sum_{i=0}^{N-1} \frac{\Gamma(N-i-\alpha(i))}{\Gamma(n-i)h^{\alpha(i)}}, \Delta f(a_0 + ih), \tag{1.6} $$

where

$$ \begin{aligned} \alpha(i) &= \beta + \omega(0.5 - X(i)), \\ X(i) &= 4X(i)(1 - X(i)), \quad i = 1, 2, \dots. \end{aligned} \tag{1.7} $$

In this work, relaxing the order α to be a function of a time domain, we propose the non-integer and non-singular (in the sense of Caputo-Fabrizio) variable order differential operator which is based on the calculus on time scales.

Definition 1.1. *Suppose $\mathbb{T}_0 \subset [0, +\infty)$ is a time scale with $0 \in \mathbb{T}$ and let $\alpha : \mathbb{T}_0 \to [0,1]$ be such that $\mu(t) \neq \frac{1-\alpha(t)}{\alpha(t)}$ ($\frac{1-\alpha}{\alpha} \in \mathcal{R}$). Let $M : [0,1] \to \mathbb{R}$ be a function which satisfies $M(0) = M(1) = 1$. The non-integer derivative of f at t of order $\alpha(s)$, denoted by the operational notation $\mathfrak{D}\{f\}(t,s)$, is defined by*

$$ \mathfrak{D}_{0^+}^{\alpha(s)} f(t) = \frac{M(\alpha(s))}{1-\alpha(s)} \int_0^t f^{\Delta}(\tau) e_{\ominus \frac{\alpha(s)}{1-\alpha(s)}}(t, \sigma(\tau)) \Delta\tau. \quad s \in \mathbb{T}_0 \tag{1.8} $$

An specific case, which is defined in [11] as a variable order derivative is $t = s$, i.e., pertaining to the values of $\mathfrak{D}\{f\}$ on the diagonal line in $\mathbb{T}_0 \times \mathbb{T}_0$.

Remark 1.1. *Interpreting the order α as the rate of anomalousity of a diffusion process, a suitable plausible candidate for α may be regarded below*

$$ \alpha(t) = \sum_{j \in \mathbb{N}} a_j . \chi_{A_j}(t), \tag{1.9} $$

where $\sum_{j\in\mathbb{N}} a_j = 1$ ($a_j > 0$), $\bigcup_{j\in\mathbb{N}} A_j = \mathbb{T}_0$ and A_j can be voided for infinitely many $j \in \mathbb{N}$. Indeed, (1.9) is pertaining to a process for which, the order of proportionality between average of displacement and time is subject to a sudden change caused by some external forces or an unknown mechanism. The special case $t = s$ reflects a synchronization between the process and its anomalousity variable order.

2 An Investigation on the Limit Cases

One of the aims of this section is to clarify that, in the original CF definition, the denominator, i.e., $\frac{1}{1-\alpha}$ will be restrictive (in the case $\alpha \to 1^-$) when we extend the theory (with loyalty) to include the time scales. However, not being confined by the original CF, it will be possible to modify the multiplier $\frac{M(\alpha)}{1-\alpha}$ in such a way that, it complies with the expectations in the sense of limit cases.

It is noticeable that, the limit case $\alpha \to 0^+$ gives $f(t) - f(0)$. The limit is understood in the sense of existing a sequence in $\mathbb{T}_0$ for which, $\alpha(t)$ tends to 0 for a subsequence. In this section, we put aside variable order and only the constant order will be considered.

Investigating the other limit case, namley $\alpha \to 1^-$, in the case t is right dense, demands some time sclae type measure theoretical background (see [9] for measure theory on time scales).

To this end, let t_0 be right-scattered. The (Hilger) Dirac delta function δ_{t_0} is defined by $\frac{1}{\mu(t_0)}$ if $t = t_0$ and 0 if $t \neq t_0$. We define $L_{t_0} \in (C_c^\infty(\mathbb{T}_0, \mathbb{R}))^*$ by ([8], section 3)

$$\langle L_{t_0}, f\rangle = \begin{cases} \int_0^\infty \delta_{t_0}(\tau) f(\tau)\Delta\tau, & \mu(t_0) > 0 \\ \lim_{\epsilon\to 0} \int_0^\infty \frac{1}{\epsilon} h(\frac{\tau}{\epsilon}) f(\sigma(\tau))\Delta\tau. & \mu(t_0) = 0 \end{cases} \tag{2.1}$$

where $h : \mathbb{T}_0 \to \mathbb{R}$ has the property $\int_0^\infty h(\tau)\Delta\tau = 1$. It can be observed that, for a function f with $f^\sigma \in C_c^\infty(\mathbb{T}_0, \mathbb{R})$, we have [8]:

$$\langle L_{t_0}, f\rangle = f(t_0). \tag{2.2}$$

For the time scale version of the exponential funation to be taken into account and be substitited with h, wee need to modify the above mentioned theory in this manner:

Let $t \in \mathbb{T}_0$ be a left scattered then, the corresponding Hilger Dirac delta function is defined by

$$\delta_t(\tau) = \begin{cases} \frac{1}{\mu(\rho(t))}, & \rho(t) = \tau \\ 0, & t \neq \tau \end{cases} \tag{2.3}$$

Now supose $t \in \mathbb{T}_0$ is left dense and define

$$g(\epsilon, t, \sigma(\tau)) = \begin{cases} \frac{1}{\epsilon}, & t - \epsilon \leq \sigma(\tau) \leq t \\ 0, & elsewhere \end{cases} \tag{2.4}$$

we infer that

$$\lim_{\epsilon\to 0^+} g(\epsilon, t, \sigma(\tau)) = \begin{cases} +\infty, & t = \sigma(\tau) \\ 0. & t \neq \sigma(\tau) \end{cases} \tag{2.5}$$

The definition (2.1) with a slight change based on the left density property (instead of right density property) is

$$\langle L_t, f\rangle = \begin{cases} \int_0^\infty \delta_t(\tau) f(\sigma(\tau))\Delta\tau, & \mu(\rho(t)) > 0 \\ \lim_{\epsilon\to 0} \int_0^\infty g(\epsilon, t, \sigma(\tau)) f(\tau)\Delta\tau. & \mu(\rho(t)) = 0 \end{cases} \tag{2.6}$$

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To obtain the expected result, i.e., $\langle L_t, f\rangle = f(t)$, first we assume that $\rho(t) < t$ and we observe

$$\begin{aligned}\langle L_t, f\rangle &= \int_0^{\infty} \delta_t(\tau) f(\sigma(\tau))\Delta\tau \\ &= \int_{\rho(t)}^{t} \delta_t(\tau) f(\sigma(\tau))\Delta\tau \\ &= f(t). \end{aligned} \tag{2.7}$$

For the case $\rho(t) = t$, we have

$$\begin{aligned}\langle L_t, f\rangle &= \lim_{\epsilon\to 0^+} \int_{t-\epsilon}^{t} g(\epsilon, t, \sigma(\tau)) f(\tau)\Delta\tau \\ &= \lim_{\epsilon\to 0^+} \frac{1}{\epsilon}\int_{t-\epsilon}^{t} f(\tau)\Delta\tau \\ &= F^{\Delta}(t) = f(t), \end{aligned} \tag{2.8}$$

where $F(t) = \int_a^t f(\tau)\Delta\tau$ $(a \in \mathbb{T}_0)$.

Now suppose $\{\epsilon_n\}_n$ be such that $\frac{1}{\epsilon_n} \in \mathcal{R}(\mathbb{T}_0)$ (for $n \in \mathbb{N}$) and let $\epsilon_n \to 0$. Since in the case $\mu(s) > 0$, we have $1 \ominus \frac{1}{\epsilon_n}\mu(s) \in [0, 1]$ for $s \in \mathbb{T}$, then,

$$\lim_{n\to+\infty} \frac{1}{\epsilon_n} e_{\ominus\frac{1}{\epsilon_n}}(t, \sigma(\tau)) = \begin{cases} +\infty, & \sigma(\tau) = t \\ 0, & \sigma(\tau) < t \end{cases} \tag{2.9}$$

and in fact

$$\lim_{\epsilon\to 0^+} g(\epsilon, t, \sigma(\tau)) = \lim_{n\to+\infty} \frac{1}{\epsilon_n} e_{\ominus\frac{1}{\epsilon_n}}(t, \sigma(\tau)). \tag{2.10}$$

Suppose $\{\alpha_n\}_n \subset (0, 1)$ be such that $\alpha_n \to 1^-$ and satisfying $\frac{\alpha_n}{1-\alpha_n} \in \mathcal{R}(\mathbb{T}_0)$. Making use of the auxillary variable $\epsilon_n = \frac{1-\alpha_n}{\alpha_n}$, we have $\frac{M(\alpha_n)}{1-\alpha_n} = \frac{N(\epsilon_n)}{\epsilon_n}$, where $N(\epsilon_n) = (1 + \epsilon_n)M(\frac{1}{1+\epsilon_n})$. For ϵ_n small enough, we have

$$\int_0^t \frac{1}{\epsilon_n} e_{\ominus\frac{1}{\epsilon_n}}(t, \sigma(\tau))\Delta\tau \cong \int_{t-\epsilon_n}^{t} \frac{1}{\epsilon_n} e_{\ominus\frac{1}{\epsilon_n}}(t, \sigma(\tau))\Delta\tau \cong 1, \tag{2.11}$$

and we infer that

$$\begin{aligned}\lim_{\alpha\to 1^-} \mathfrak{D}^{\alpha}_{0^+} f(t) &= \lim_{n\to+\infty} N(\epsilon_n) \int_0^t \frac{1}{\epsilon_n} e_{\ominus\frac{1}{\epsilon_n}}(t, \sigma(\tau)) f^{\Delta}(\tau)\Delta\tau \\ &= \lim_{n\to+\infty} N(\epsilon_n) \int_{t-\epsilon_n}^{t} \frac{1}{\epsilon_n} e_{\ominus\frac{1}{\epsilon_n}}(t, \sigma(\tau)) f^{\Delta}(\tau)\Delta\tau \\ &= \lim_{n\to+\infty} N(\epsilon_n) \int_0^t g(\epsilon_n, t, \sigma(\tau)) f^{\Delta}(\tau)\Delta\tau \\ &= \langle L_t, f^{\Delta}\rangle = f^{\Delta}(t). \end{aligned} \tag{2.12}$$

In the right scattered case, i.e., $\rho(t) < t$, the continuation may gives rise to a divergence. Let $\mathbb{T}_0 = \mathbb{N}_0$. Since $e_{\ominus\frac{\alpha}{1-\alpha}}(t, \sigma(\tau)) = (1-\alpha)^{t-\sigma(\tau)}$, then

$$\begin{aligned}
\frac{M(\alpha)}{1-\alpha}\int_0^t f^{\Delta}(\tau)e_{\ominus\frac{\alpha}{1-\alpha}}(t,\sigma(\tau))\Delta\tau &= \frac{M(\alpha)}{1-\alpha}\sum_{\tau=0}^{\rho(t)} f^{\Delta}(\tau)(1-\alpha)^{t-\sigma(\tau)} \\
&= M(\alpha)\sum_{\tau=0}^{\rho(t)} f^{\Delta}(\tau)(1-\alpha)^{t-\sigma(\tau)-1} \\
&= M(\alpha)\sum_{\tau=0}^{\rho^2(t)} f^{\Delta}(\tau)(1-\alpha)^{t-\sigma(\tau)-1} \\
&\quad + M(\alpha)\frac{f^{\Delta}(\rho(t))}{1-\alpha},
\end{aligned} \tag{2.13}$$

and in the case $f^{\Delta}(\rho(t) \neq 0$, (2.13) gives rise to $\lim_{\alpha\to 1^-}\mathfrak{D}^{\alpha}_{0^+}f(t) = \infty$. Assuming $\mathbb{T}_0$ arbitraryily and a left scattered $t \in \mathbb{T}_0$, we obtain

$$\begin{aligned}
\lim_{\alpha\to 1^-}\mathfrak{D}^{\alpha}_{0^+}f(t) &= \lim_{n\to+\infty}\frac{N(\epsilon_n)}{\epsilon_n}\int_0^t e_{\ominus\frac{1}{\epsilon_n}}(t,\sigma(\tau))f^{\Delta}(\tau)\Delta\tau \\
&= \lim_{n\to+\infty}\frac{N(\epsilon_n)}{\epsilon_n}\int_{\rho(t)}^t e_{\ominus\frac{1}{\epsilon_n}}(t,\sigma(\tau))f^{\Delta}(\tau)\Delta\tau \\
&= \Big(f^{\Delta}(\rho(t))\mu(\rho(t))\Big)\lim_{n\to+\infty}\frac{N(\epsilon_n)}{\epsilon_n} \\
&= \begin{cases} \infty, & f^{\Delta}(\rho(t)) \neq 0 \\ 0, & f^{\Delta}(\rho(t)) = 0. \end{cases}
\end{aligned} \tag{2.14}$$

The above discussion is summarized below:

Proposition 2.1. *Let $f : \mathbb{T}_0 \to \mathbb{R}$ be such that $\mathfrak{D}^{\alpha_n}_{0^+}f(t)$ exists for $\alpha_n \in (0,1)$. The*

a) For $t \in \mathbb{T}_0$, we have

$$\lim_{\alpha\to 0^+}\mathfrak{D}^{\alpha}_{0^+}f(t) = f(t). \tag{2.15}$$

b) If $t \in \mathbb{T}_0$ is left dense then

$$\lim_{\alpha\to 1^-}\mathfrak{D}^{\alpha}_{0^+}f(t) = f^{\Delta}(t). \tag{2.16}$$

c) If $t \in \mathbb{T}_0$ is left scattered, then

$$\lim_{\alpha\to 1^-}\mathfrak{D}^{\alpha}_{0^+}f(t) = \begin{cases} \infty, & f^{\Delta}(\rho(t)) \neq 0 \\ 0, & f^{\Delta}(\rho(t)) = 0. \end{cases} \tag{2.17}$$

Remark 2.1. *As it is illustrated, the restriction in the limit case $\alpha \to 1^-$ for a left scattered point, is originated from the definition 1.1. In the special case $\mathbb{T}_0 = [0, +\infty)$, we have the equality below*

$$\lim_{\alpha\to 1^-}\int_0^t \frac{1}{1-\alpha}f(\tau)e_{\ominus\frac{\alpha}{1-\alpha}}(t,\sigma(\tau)) = f(t). \tag{2.18}$$

Unlike (2.18), in the time scale counterpart theory of the Dirac functional, generally, there is no consistency between the formal definition of L_t and the exponential based one, i.e.,

$$\lim_{\alpha\to 1^-}\int_0^t \frac{1}{1-\alpha} f(\tau)e_{\ominus\frac{\alpha}{1-\alpha}}(t,\sigma(\tau)) \neq \langle L_t, f\rangle. \tag{2.19}$$

An alternative multiplier based on the original CF definition, to overcome this pitfall, is proposed below:

Suppose $T:\mathbb{T}_0\times[0,1)\to\mathbb{R}$ satisfies the following asymptotic condintions:

$$\begin{cases} \lim_{\alpha\to 1^-} T(t,\alpha)=1, & \rho(t)<t\\ T(t,\alpha)\sim\frac{1}{1-\alpha}, \quad as\ \alpha\to 1^- & \rho(t)=t\\ \lim_{\alpha\to 0^+} T(t,\alpha)=T(t,0)=1. & \forall t\in\mathbb{T}_0 \end{cases} \tag{2.20}$$

It is clear that, for the CF derivative (with constant order) on time scales introduced by

$$\mathfrak{D}^{\alpha}_{0^+} f(t) = T(t,\alpha)\int_0^t f^{\Delta}(\tau)e_{\ominus\frac{\alpha}{1-\alpha}}(t,\sigma(\tau))\Delta\tau, \qquad s\in\mathbb{T}_0 \tag{2.21}$$

we have

$$\begin{cases} \lim_{\alpha\to 0^+}\mathfrak{D}^{\alpha}_{0^+} f(t) = f(t)-f(0), & \forall t\in\mathbb{T}_0\\ \lim_{\alpha\to 0^+}\mathfrak{D}^{\alpha}_{0^+} f(t) = f^{\Delta}(t), & \rho(t)=t\\ \lim_{\alpha\to 0^+}\mathfrak{D}^{\alpha}_{0^+} f(t) = f^{\Delta}(\rho(t))\mu(\rho(t)). & \rho(t)=t \end{cases} \tag{2.22}$$

For the sake of loyalty to the Caputo and Fabrizio approach, in the rest of this paper, only the Definition 1.1 will be considered.

3 Laplace Transform

Let $\mathbb{T}_0$ be as in the Definition 1.1. we need the following background for the Laplace transform on time scales. For the details and more discussion, see [4].

Definition 3.1. *Assume that $f:\mathbb{T}_0\to\mathbb{R}$ is regulated. Then the Laplace transform of f is defined by*

$$\mathcal{L}\{f\}(z) := \int_0^{\infty} f(t)e^{\sigma}_{\ominus z}(t,0)\Delta t, \tag{3.1}$$

for $z\in\mathcal{D}(f)$, where $\mathcal{D}(f)$ consists of all complex numbers $z\in\mathcal{R}$ for which the improper integral exists.

As in the classical case, the time scale counterpart satisfies almost all expected properties. However, one should be careful about the considering sufficient conditions for the convergence issues. An important one, which will be needed in this sequel, is given below (see [4], Theorems 3.87 & 3.89)

Theorem 3.1. *Supposez $\in\mathbb{C}$ is regressive. 1) Assume $f:\mathbb{T}_0\to\mathbb{R}$ is such that, f^{Δ} is regulated. Then*

$$\mathcal{L}\{f^{\Delta}\}(z) = z\mathcal{L}\{f\}(z) - f(0), \tag{3.2}$$

provided

$$\lim_{t\to\infty}\{f(t)e_{\ominus z}(t,0)\}=0. \tag{3.3}$$

@) Suupose f satisfies the the limit condition

$$\lim_{t\to+\infty}\left\{e_{\ominus z}(t,0)\int_0^t f(\tau)\Delta\tau\right\}=0, \tag{3.4}$$

then

$$\mathcal{L}\Big\{\int_0^t f(\tau)\Delta\tau\Big\}(z)=\frac{\mathcal{L}\{f\}(z)}{z}. \tag{3.5}$$

Remark 3.1. *The limit condition (3.3) is satisfied provided f is of exponentioal type II, i.e.,*

$$|f(t)|\le ce_{c_0}(t,0), \tag{3.6}$$

for some positive constants c, c_0 and $\Re_\mu(z)>\Re_\mu(c_0)$ ([8], Theorem 1.1). Also, this is sufficient to the convergence of (3.1) [8].

Note that, the limit condition (3.3) implies

$$\begin{aligned}\lim_{t\to\infty}\{tf(t)e_{\ominus z}(t,0)\} &= 0,\\ \lim_{t\to\infty}\{\|f\|_{[0,t]}e_{\ominus z}(t,0)\} &= 0.\end{aligned} \tag{3.7}$$

($\|.\|$ is the maximum norm.) The first limit is obvious by the definition of $e_{\ominus z}$ and for the second limit we have

$$\max_{[0,t]}\{|f(\tau)|e_{\ominus z}(t,0)\}\le\max_{[0,t]}\{|f(\tau(t))|e_{\ominus z}(\tau(t),0)\}\to 0, \tag{3.8}$$

as $t\to+\infty$. In view of the estimation $\left|\int_0^t u(\tau)\Delta\tau\right|\le t\|u\|_{[0,t]}$, we conclude that, (3.3) gives rise to (3.4).

An special case of the convolution integral is defined here

Definition 3.2. *For a regulated function $f:\mathbb{T}_0\to\mathbb{R}$, and $\alpha\in\mathcal{R}$, the convolution of f and $e_\alpha(t,\tau)$ is defined by*

$$(e_\alpha * f)(t):=\int_0^t f(\tau)e_\alpha(t,\sigma(\tau))\Delta\tau. \tag{3.9}$$

Likewise the classical case, the convolution theorem (see [4], Theorem 3.106) is stated:

Theorem 3.2. *(Convolution Theorem) For a regulated function $f:\mathbb{T}_0\to\mathbb{R}$, we have*

$$\mathcal{L}\{e_\alpha * f\}(z)=\frac{1}{z-\alpha}.\mathcal{L}\{f\}(z), \tag{3.10}$$

where $\frac{1}{z-\alpha}$ is $\mathcal{L}\{e_\alpha(.,0)\}(z)$.

In view of Remark 2.1, The statement below is immediate.

Corollary 3.1. *Let $\alpha(s) \equiv \alpha$, i.e., equivalent to the case $s \in \mathbb{T}_0$ is fixed. Assume $f : \mathbb{T}_0 \to \mathbb{R}$ be such that, $\mathfrak{D}^{\alpha}_{0^+}(f(t))$ is regulated, continuous and satisfies (3.3) (substituting f with $\mathfrak{D}^{\alpha}_{0^+}$). Then, one can simply observe the following relation*

$$\mathcal{L}\Big\{\mathfrak{D}^{\alpha}_{0^+}(f(t))\Big\}(z) = \frac{M(\alpha)(1-\alpha+\alpha\mu(t))\Big\{z\mathcal{L}\{f\}(z) - f(0)\Big\}}{(1-\alpha)\Big[z(1-\alpha+\alpha\mu(t))+\alpha\Big]}. \tag{3.11}$$

4 Non-integer order integral

In this section, following the same method as in [10] section 2, we tilize the Laplace transform given in Definition 2.1, to obtain an associated integral for the derivative introduced in definition 1.1.

Suppose $\alpha : \mathbb{T}_0 \to \mathbb{R}$ be as in definition 1.1. we assume two cases:

Case 1. Assume that α is constant, i.e., $\alpha(s) \equiv \alpha$ and consider

$$\mathfrak{D}^{\alpha(s)}_{0^+} f(t) = u(t), \tag{4.1}$$

for some $u : \mathbb{T}_0 \to \mathbb{R}$. Applying $\mathcal{L}$ on the both sides of (4.1) and in view of (3.11), we solve the equation for $\mathcal{L}\{f\}(z)$ to obtain

$$\mathcal{L}\{f\}(z) = \frac{1-\alpha}{M(\alpha)}\mathcal{L}\{u\}(z) + \frac{\alpha(1-\alpha)}{zM(\alpha)(1-\alpha+\alpha\mu(t))}\mathcal{L}\{u\}(z) + \frac{f(0)}{z}, \tag{4.2}$$

that is

$$f(t) = \frac{1-\alpha}{M(\alpha)}u(t) + \frac{\alpha(1-\alpha)}{M(\alpha)(1-\alpha+\alpha\mu(t))}\int_0^t u(s)\Delta s + f(0). \tag{4.3}$$

In order to introduce an integral (which is denoted by $\mathfrak{I}^{\alpha(t)}_{0^+} f(t)$), we expect that, it satisfies

$$\mathfrak{D}^{\alpha(t)}_{0^+}\mathfrak{I}^{\alpha}_{0^+} f(t) = f(t). \tag{4.4}$$

In view of (4.3) that is the solution of (4.1), we arrive at the definition below.

Definition 4.1. *The non-singular and constant non-integer order integral of a regulated function $f : \mathbb{T}_0 \to \mathbb{R}$ is defined by*

$$\mathfrak{I}^{\alpha}_{0^+} f(t) = \frac{1-\alpha}{M(\alpha)}f(t) + \frac{\alpha(1-\alpha)}{M(\alpha)(1-\alpha+\alpha\mu(t))}\int_0^t f(s)\Delta s. \tag{4.5}$$

Remark 4.1. *Substituting $\frac{M(\alpha)}{1-\alpha}$ in Definition 1.1 with $\frac{(2-\alpha)M(\alpha)}{2(1-\alpha)}$, as Losada and Nieto did in [10] to define ${}^{CF}\mathfrak{D}^{\alpha}$, the associated integral ${}^{CF}\mathfrak{I}^{\alpha}$ ([10], Definition 1) takes the form*

$${}^{CF}\mathfrak{I}^{\alpha} f(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}f(t) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t f(s)ds. \quad t \geq 0 \tag{4.6}$$

In comparison with (4.5) with the presence of the graininess function $\mu(t)$, the Losada & Nieto's integral introduced by (4.6) is an espcial case of (4.5) up to the multiplier, while assuming $\frac{(2-\alpha)M(\alpha)}{2(1-\alpha)}$ as the multiplier and $\mu(t) = 0$ in (4.5), we obtain (4.6).

Case 2. Suppose $s = t$. Let α be non-constant and Δ-differentiable. Further assume that, $\alpha(\mathbb{T}_0)$, (i.e., the image of α) is closed and $M : \alpha(\mathbb{T}_0) \to \mathbb{R}$ is Δ-differentiable. Making use of the properties of exponential function and considering (4.1) as follows

$$\int_0^t f^\Delta(\tau) e_{\frac{\alpha(t)}{1-\alpha(t)}}(\sigma(\tau), 0)\Delta\tau = \frac{1-\alpha(t)}{M(\alpha(t))} e_{\frac{\alpha(t)}{1-\alpha(t)}}(t, 0).u(t), \tag{4.7}$$

then Δ-differentiating, we obtain

$$\begin{aligned} f^\Delta(t).e_{\frac{\alpha(t)}{1-\alpha(t)}}(\sigma(t), 0) &= \frac{\alpha(t)}{1-\alpha(t)}.e_{\frac{\alpha(t)}{1-\alpha(t)}}(t, 0).\frac{1-\alpha(t)}{M(\alpha(t))}.u(t) \\ &+ e_{\frac{\alpha(t)}{1-\alpha(t)}}(t, 0).\Big(\frac{1-\alpha(t)}{M(\alpha(t))}\Big)^\Delta.u(t) + e_{\frac{\alpha(t)}{1-\alpha(t)}}(t, 0).\frac{1-\alpha(t)}{M(\alpha(t))}.u^\Delta(t). \end{aligned} \tag{4.8}$$

Multiplying the both sides by $e_{\ominus\frac{\alpha(t)}{1-\alpha(t)}}(\sigma(t), 0)$

$$f^\Delta(t) = e_{\ominus\frac{\alpha(t)}{1-\alpha(t)}}(\sigma(t), t).\Big\{\Big(\frac{\alpha(t)}{M(\alpha(t))} + \Big(\frac{1-\alpha(t)}{M(\alpha(t))}\Big)^\Delta\Big).u(t) + \frac{1-\alpha(t)}{M(\alpha(t))}.u^\Delta(t)\Big\}, \tag{4.9}$$

then Δ-Integrating (4.9), we get

$$f(t) = \int_0^t e_{\ominus\frac{\alpha(t)}{1-\alpha(t)}}(\sigma(t), t).\Big(p(\tau)u(\tau) + q(\tau)u^\Delta(\tau)\Big)\Delta\tau + f(0), \tag{4.10}$$

where

$$\begin{cases} p(t) = \frac{\alpha(t)}{M(\alpha(t))} + \frac{\alpha^\Delta(t)}{M(\alpha(t)).M(\alpha^\sigma(t))}.\Big(\alpha(t).M^\Delta(\alpha(t)) - 2M(\alpha(t))\Big), \\ q(t) = \frac{1-\alpha(t)}{M(\alpha(t))}. \end{cases} \tag{4.11}$$

Denoting The corresponded integral to the variable order $\alpha(t)$ by $\mathfrak{I}_{0^+}^{\alpha(t)}$, the discussion above, together with having the expectation similar to (4.4), is summarized in the definition below

Definition 4.2. *The non-singular and non-integer variable order integral of a Δ-differentiable function $f : \mathbb{T}_0 \to \mathbb{R}$ is introduced by*

$$\mathfrak{I}_{0^+}^{\alpha(t)} f(t) = \int_0^t e_{\ominus\frac{\alpha(t)}{1-\alpha(t)}}(\sigma(t), t).\Big(p(\tau)f(\tau) + q(\tau)f^\Delta(\tau)\Big)\Delta\tau, \tag{4.12}$$

where p and q are defined by (4.11).

Example 4.1. *Suppose the dynamic problem below*

$$\begin{cases} \mathfrak{D}_{0^+}^\alpha f(t) = \lambda f(t), \qquad \lambda \in \mathbb{R} \\ \\ f(0) = f_0. \end{cases} \tag{4.13}$$

with constant $\alpha \in [0, 1]$. If we assume $\mathbb{T}_0 = [0, 1]$ then, it will be easily observed that, the exact solution is

$$f(t) = \frac{f_0 M(\alpha)}{M(\alpha)\lambda(1-\alpha)} \exp\Big[\frac{\lambda\alpha t}{M(\alpha) - \lambda(1-\alpha)}\Big]. \tag{4.14}$$

With the same assumption on α and $\mathbb{T}_0$, considering the IVP

$$\begin{cases} \mathfrak{D}^{\alpha}_{0^+} f(t) = a(t)f(t), \qquad \lambda \in \mathbb{R} \\ \\ f(0) = f_0. \end{cases} \tag{4.15}$$

from (4.5), it is obvious that, the differentiability of $a(t)$ as asufficient condition, supposrts the existence issue. Then, (4.15) is converted into the ODE

$$f'(t) + \frac{M(\alpha)a(t)}{a(t)(1-\alpha) - M(\alpha)} f(t) = \frac{M(\alpha)f_0}{M(\alpha) - a(t)(1-\alpha)}, \tag{4.16}$$

which has a unique solution provided the denominator is non-vanishing.

Example 4.2. *Suppose $\mathbb{T}_0 = (h\mathbb{N})_0 \cap [0,1]$. Let $\alpha(t) = t$, then*

$$\begin{aligned} e_{\ominus \frac{\alpha(t)}{1-\alpha(t)}}(\sigma(t), t) &= \exp \int_t^{\sigma(t)} \frac{\log(1 + \mu(\tau). \ominus \frac{\alpha(t)}{1-\alpha(t)})\Delta\tau}{\mu(\tau)} \\ &= \frac{\alpha(t) - 1}{\alpha(t)(1 - \mu(t)) - 1} \\ &= \frac{t-1}{t-1-th}. \end{aligned} \tag{4.17}$$

and by $e_{\frac{\alpha}{1-\alpha}}(\sigma(t), t) = \frac{1}{e_{\ominus \frac{\alpha}{1-\alpha}}(\sigma(t),t)}$, we have $e_{\frac{\alpha}{1-\alpha}}(\sigma(t), t) = \frac{1-t+th}{1-t}$

Assume $M(\alpha) \equiv 1$ and consider the IVP

$$\begin{cases} \mathfrak{D}^{\alpha}_{0^+} f(t) = \lambda f(t), \\ \\ f(0) = f_0. \end{cases} \tag{4.18}$$

In equation (4.10) we have $p(t) = t - 2$, $q(t) = 1 - t$, so

$$\Big(\frac{\lambda(1-t+th)}{1-t}\Big) f(t) = \int_0^t (\tau - 2) f(\tau)\Delta\tau + \int_0^t (1-\tau) f^{\Delta}(\tau)\Delta\tau. \tag{4.19}$$

If we Δ-differentiate of the both sides in (4.19), then we have

$$f^{\Delta}(t) = \beta(t) f(t), \tag{4.20}$$

where

$$\beta(t) = \frac{t(1-t+th) - \lambda(1-t)^2(t-2)}{(1-t)\Big[\lambda(1-t)^2 - th + t + 1\Big]}. \tag{4.21}$$

Therefore, the unique solution to the problem (4.18) is

$$f(t) = f_0 e_{\beta(t)}(t, 0), \tag{4.22}$$

provided $\beta \in \mathcal{R}$. This condition holds true if the roots of the polynomial $Q(t)$ (denoted by $t(\lambda, h)$) defined by

$$Q(t) = \lambda(1+h)t^3 + (h^2 - 4\lambda h - 3\lambda + 1)t^2 + (5\lambda h + 2h + 3\lambda - 2)t + 2\lambda h - \lambda + 1, \tag{4.23}$$

satisfy $t(\lambda, h) \notin \mathbb{T}_0$. Indeed, a sufficient condition is $t(\lambda, h) > 1$.

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5 Dynamic Equations on Viscoelastic Oscillations

The governing equation of motion for the viscoelastic oscillator in a dimensionless form is given by

$$x^{(\alpha)}(t) = \lambda(x(t) + x''(t)), \tag{5.1}$$

in which, $x^{(\alpha)}$ means a fractional order derivative in any of the existing senses and λ simplifies the conastants such as mass, charactristic length and natural frequency. Note that, compare to the classic form, the damping term appears as a fractional order derivative, rather than first order, that is interpreted as a viscoelasticity. The variable order derivative in the sense of Coimbra, (which involves non-singular kernel and continuous time) has been studied in [13].

Now consider the IVP

$$\begin{cases} \mathfrak{D}_{0^+}^{\alpha(t)} x(t) = \lambda(x(t) + x^{\Delta\Delta}(t)), \\ \\ f(0) = 0. \end{cases} \tag{5.2}$$

where $\mathfrak{D}_{0^+}^{\alpha(t)}$ is the differential operator introduced by Definition 1.1. Let $\mathbb{T}_0$ be a time scale of the desired type, that is introduced in Definition 1.1 with $\mathbb{T}_0 \subset [0,1]$. Moreover, assume that $\alpha : \mathbb{T}_0 \to \mathbb{T}_0$. By (4.10), we obtain

$$e_{\frac{\alpha(t)}{1-\alpha(t)}}(\sigma(t),t).x(t) = \lambda \int_0^t \Big\{ p(\tau)\Big(x(\tau) + x^{\Delta\Delta}(\tau)\Big) + q(\tau)\Big(x^{\Delta}(\tau) + x^{\Delta\Delta\Delta}(\tau)\Big)\Big\} \Delta\tau. \tag{5.3}$$

Δ-differentiating gives rise to the third order dynamic equation on $\mathbb{T}_0$

$$\begin{aligned} \lambda q(t) x^{\Delta\Delta\Delta}(t) \quad &+ \quad \lambda p(t) x^{\Delta\Delta}(t) + \Big(\lambda q(t) - e_{\frac{\alpha(t)}{1-\alpha(t)}}(\sigma(t),t)\Big) x^{\Delta}(t) \\ &+ \quad \Big(\lambda p(t) - \frac{\alpha(t)}{1-\alpha(t)} e_{\frac{\alpha(t)}{1-\alpha(t)}}(\sigma(t),t)\Big) x(t) = 0. \end{aligned} \tag{5.4}$$

If we further assume that $M : \mathbb{T}_0 \to \mathbb{T}_0$ satisfies

$$\alpha(t) M^{\Delta}(\alpha(t)) = 2M(\alpha(t)), \tag{5.5}$$

then in (4.11), we infer that $p(t) = \frac{\alpha(t)}{1-\alpha(t)}$. Indeed, $M(s) = e_{\frac{2}{s}}(s, s_0)$ ($s_0 \in \mathbb{T}_1 = \alpha(\mathbb{T}_0)$) is well defined and satisfies (5.5), since $\frac{2}{\alpha(t)} \in \mathcal{R}(\mathbb{T}_1)$ for $t \in \mathbb{T}_0$. Thefore, (5.4) takes the form below

$$\frac{\lambda(1-\alpha(t))}{M(\alpha(t))} x^{\Delta^3}(t) + \frac{\lambda \alpha(t)}{M(\alpha(t))} x^{\Delta^2}(t) + \beta(t) x^{\Delta}(t) + \frac{\alpha(t)}{1-\alpha(t)} \beta(t) x(t) = 0. \tag{5.6}$$

in which

$$\beta(t) = \frac{\lambda(1-\alpha(t))^2 - M(\alpha(t))(1-\alpha(t) + \mu(t)\alpha(t))}{M(\alpha(t))(1-\alpha(t))}. \tag{5.7}$$

6 Conclusion

In the present paper, having a look at the anomalous diffusion phenomena, by extending the concept of fractional differential operator of Caputo-Fabrizio type, to include a class of variable orders, a time-scale counterpart of the non-integer order differential operator is introduced. Implementing the measure theory on time scales and introducing Dirac delta functional based on the left density property of a given point, it has been deduced that, the both limit cases, namely $\alpha \to 1^-$or 0^+, give the well-known Δ-derivative and a shifted zeroth derivative, i.e., $f^\Delta(t)$ and $f(t) - f(0)$ respectively provided t is left dense. Moreover, through the discussion, the inconsistency between the formal definition of Dirac delta functional and the exponential based definition with the given multiplier in the original CF definition, is illustrated. By making use of the Laplace transform and direct Δ-calculus based approach, the associated integral for constant and variable orders are discussed. To illustrate the theory, some dynamic equations on time scales are studied and a dynamic problem, which governs a class of viscoelastic oscillation phenomena and involving the new introduced derivative is studied.

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FIXED POINT THEOREMS ON CONE METRIC SPACES WITH $c-$DISTANCE

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ABSTRACT. In this paper, we obtain a new sufficient conditions for existence of a unique coincidence point and a common fixed point for a pair of self mappings satisfying the contractive condition in a cone metric space by using $c-$distance.

1. Introduction

Since Huang and Zhang([5]) introduced the cone metric space which is more general than the concept of a metric space, many fixed point theorems have been proved in normal or non-normal cone metric spaces by some authors ([1], [3], [5], [6], [9] ,[12], [3]). Note that Cho et al.([3]) introduced the $c-$distance in a cone metric space which is a cone version of the $w-$distance of Kada et al.([7]).

In this paper, we obtain a new sufficient conditions for existence of a unique coincidence point and a common fixed point for a pair of self mappings satisfying contractive conditions in a cone metric space by using $c-$distance.

Let E be a real Banach space and θ denote the zero element in E. A *cone* P is a subset of E such that

(i) P is closed, nonempty and $P \neq \{\theta\}$;

(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in P \;\Rightarrow\; ax + by \in P$;

(iii) $P \cap (-P) = \{\theta\}$ i.e, $x \in P$ and $-x \in P$ imply $x = \theta$.

For any cone $P \subseteq E$, the partial ordering $\preceq$ with respect to P is defined by $x \preceq y$ if and only if $y - x \in P$. The notation of $\prec$ stands for $x \preceq y$ but $x \neq y$. Also, we used $x \ll y$ to indicate that $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P. A cone P is called *normal* if there exists a number K such that for all $x, y \in E$,

$$\theta \preceq x \preceq y \quad \text{implies} \quad \|x\| \leq K\|y\|. \tag{1.1.1}$$

Equivalently, the cone P is normal if

$$x_n \preceq y_n \preceq z_n \text{ and } \lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = x \text{ imply } \lim_{n\to\infty} y_n = x \tag{1.1.2}$$

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The least positive number K satisfying condition (1.1.1) is called the *normal constant* of P.

Definition 1.1. Let X be a nonempty set and let E be a real Banach space equipped with the partial ordering $\preceq$ with respect to the cone $P \subseteq E$. Suppose the mapping $d : X \times X \to E$ satisfies the following conditions:

(1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a *cone metric* on X, and (X, d) is called a *cone metric space.*

Definition 1.2. Let (X, d) be a cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$.

(1) If for every $c \in E$ with $\theta \ll c$, there exists a natural number N such that $d(x_n, x) \ll c$ for all $n > N$, then $\{x_n\}$ is said to be *convergent* and $\{x_n\}$ *converges to* x, and the point x is the *limit* of $\{x_n\}$. We denote this by
$$\lim_{n\to\infty} x_n = x \quad \text{or} \quad x_n \to x \quad (n \to \infty).$$
(2) If for all $c \in E$ with $\theta \ll c$, there exists a positive integer N such that $d(x_n, x_m) \ll c$ for all $m, n > N$, then $\{x_n\}$ is called a *Cauchy sequence* in X.
(3) A cone metric space (X, d) is said to be *complete* if every Cauchy sequence in X is convergent.

Lemma 1.3. *([10]) Let E be a real Banach space with a cone P. Then*

(1) *If $a \ll b$ and $b \ll c$, then $a \ll c$.*
(2) *If $a \preceq b$ and $b \ll c$, then $a \ll c$.*

Lemma 1.4. *([10]) Let E be a real Banach space with cone P. Then*

(1) *If $\theta \ll c$, then there exists $\delta > 0$ such that $\|b\| < \delta$ implies $b \ll c$.*
(2) *If $\{a_n\}, \{b_n\}$ are sequences in E such that $a_n \to a, b_n \to b$ and $a_n \preceq b_n$ for all $n \geq 1$, then $a \preceq b$.*

Lemma 1.5. *([5]) Let (X, d) be a cone metric space, P a normal cone, $x \in X$ and $\{x_n\}$ a sequence in X. Then*

(1) *$\{x_n\}$ converges to x if and only if $d(x_n, x) \to \theta$.*
(2) *The limit point of every sequence is unique.*
(3) *Every convergent sequence is a Cauchy sequence.*
(4) *$\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to \theta$ as $n, m \to \infty$.*
(5) *If $x_n \to x$ and $y_n \to y$ then $d(x_n, y_n) \to d(x, y)$ as $n \to \infty$.*

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Definition 1.6. Let (X, d) be a cone metric space. Then a mapping $q : X \times X \to E$ is called a $c-$*distance* on X if the following are satisfied :

(q1) $\theta \preceq q(x, y)$ for all $x, y \in X$.
(q2) $q(x, z) \preceq q(x, y) + q(y, z)$ for all $x, y, z \in X$.
(q3) for all $x \in X$ and all $n \geq 1$, if $q(x, y_n) \preceq u$ for some $u = u_x \in P$, then $q(x, y) \preceq u$ whenever $\{y_n\}$ is a sequence in X converging to a point $y \in X$.
(q4) for all $c \in E$ with $\theta \ll c$, there exists $e \in E$ with $\theta \ll e$ such that

$$q(z, x) \ll e \text{ and } q(z, y) \ll e \text{ imply } d(x, y) \ll c.$$

Example 1.7. ([3]) Let (X, d) be a cone metric space and let P be a normal cone. Put $q(x, y) = d(x, y)$ for all $x, y \in X$. Then q is a $c-$distance.

Example 1.8. ([3]) Let (X, d) be a cone metric space and let P be a normal cone. Put $q(x, y) = d(u, y)$ for all $x, y \in X$, where $u \in X$ is constant. Then q is a $c-$distance.

Example 1.9. ([3]) Let $E = \mathbb{R}$ and $P = \{x \in E : x \geq 0\}$. Let $X = [0, \infty)$ and define a mapping $d : X \times X \to E$ by $d(x, y) = |x - y|$ for all $x, y \in X$. Then (X, d) is a cone metric space. Define a mapping $q : X \times X \to E$ by $q(x, y) = y$ for all $x, y \in X$. Then q is a $c-$distance.

Remark 1.10. (1) $q(x, y) = q(y, x)$ does not necessarily hold for all $x, y \in X$.
(2) $q(x, y) = \theta$ is not necessarily equivalent to $x = y$ for all $x, y \in X$.

Lemma 1.11. *([3]) Let (X, d)be a cone metric space and let q be a c-distance on X. Let $\{x_n\}$ and $\{y_n\}$be sequences in X and $x, y, z \in X$. Suppose that $\{u_n\}$ is a sequence in P converging to θ. Then the following facts hold:*
(1) If $q(x_n, y) \preceq u_n$ and $q(x_n, z) \preceq u_n$, then $y = z$.
(2) If $q(x_n, y_n) \preceq u_n$and $q(x_n, z) \preceq u_n$, then $\{y_n\}$ converges to z.
(3) If $q(x_n, x_m) \preceq u_n$ for $m > n$, then $\{x_n\}$ is a Cauchy sequence in X.
(4) If $q(y, x_n) \preceq u_n$, then $\{x_n\}$is a Cauchy sequence in X.

Definition 1.12. Let T and S be self mappings of a set X.

(1) If $y = Tx = Sx$ for some $x \in X$, then x is called a *coincidence point* of T and S and y is called a *point of coincidence* of T and S.
(2) $T, S : X \to X$ are *weakly compatible* if for every $x \in X$, the following holds:

$$T(Sx) = S(Tx) \quad \text{whenever} \quad Sx = Tx.$$

(2) $T : X \to X$ is *continuous* if $\lim_{n\to\infty} x_n = x$ implies that $\lim_{n\to\infty} Tx_n = Tx$.

2. Main results

In [12], Wang and Guo proved some common fixed point theorem of two mappings (in a normal cone metric space) satisfying the contractive condition (2.2.1). We state Theorem 2.1 in [12] as follows.

Theorem 2.1. *([12]) Let (X,d) be a cone metric space. Let P be a normal cone with normal constant K and let q be a $c-$distance on X. Let $a_i \in [0,1)\,(i=1,2,3,4)$ be constants with $a_1+a_2+a_3+2a_4<1$ and $f,g:X\to X$ be two mappings satisfying the condition*

$$q(fx,fy)\preceq a_1q(gx,gy)+a_2q(gx,fx)+a_3q(gy,fy)+a_4q(gx,fy) \qquad (2.2.1)$$

for all $x,y\in X$. Suppose that $f(X)\subseteq g(X)$ and $g(X)$ is a complete subset of X. If f and g satisfy

$$\inf\{\|q(gx,y)\|+\|q(fx,y)\|+\|q(gx,fx)\|:x\in X\}>0$$

for all $y\in X$ with $y\neq fy$ or $y\neq gy$, then f and g have a common fixed point in X.

In a normal cone metric space we prove a new common fixed point theorem of two mappings satisfying the contractive condition independent of the condition (2.2.1).

Theorem 2.2. *Let (X,d) be a cone metric space, P be a normal cone with normal constant K and q be a $c-$distance on X. Let $a_i\in[0,1)\,(i=1,2,3,4,5)$ be constants with $a_1+2a_2+2a_3+3a_4+a_5<1$ and $f,g:X\to X$ be two mappings satisfying the condition*

$$q(fx,fy)\preceq a_1q(gx,gy)+a_2q(gx,fx)+a_3q(gy,fy)+a_4q(gx,fy)+a_5q(gy,fx)$$

for all $x,y\in X$. Suppose that $f(X)\subseteq g(X)$ and $g(X)$ be a complete subset of X. If f and g satisfy

$$\inf\{\|q(gx,y)\|+\|q(fx,y)\|+\|q(gx,fx)\|\;:\;x\in X\}>0$$

for all $y\in X$ with $y\neq fy$ or $y\neq gy$, then f and g have a unique point of coincidence in X. Moreover if f and g are weakly compatible then f and g have a unique common fixed point in X.

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Proof. Let $x_0, x_1 \in X$. Using the fact that $f(X) \subseteq g(X)$, construct $\{x_{2n}\}, \{x_{2n+1}\}$ such that $gx_{2n} = fx_{2n-2}$ and $gx_{2n+1} = fx_{2n-1} \quad (n \in \mathbb{N})$. Then we have

$$\begin{aligned} q(gx_{2n}, gx_{2n+1}) &= q(fx_{2n-2}, fx_{2n-1}) \\ &\preceq a_1 q(gx_{2n-2}, gx_{2n-1}) + a_2 q(gx_{2n-2}, fx_{2n-2}) + a_3 q(gx_{2n-1}, fx_{2n-1}) \\ &+ a_4 q(gx_{2n-2}, fx_{2n-1}) + a_5 q(gx_{2n-1}, fx_{2n-2}) \\ &= a_1 q(gx_{2n-2}, gx_{2n-1}) + a_2 q(gx_{2n-2}, gx_{2n}) \\ &+ a_3 q(gx_{2n-1}, gx_{2n+1}) + a_4 q(gx_{2n-2}, gx_{2n+1}) + a_5 q(gx_{2n-1}, gx_{2n}) \\ &\preceq a_1 q(gx_{2n-2}, gx_{2n-1}) + a_2 \{q(gx_{2n-2}, gx_{2n-1}) + q(gx_{2n-1}, gx_{2n})\} \\ &+ a_3 \{q(gx_{2n-1}, gx_{2n}) + q(gx_{2n}, gx_{2n+1})\} \\ &+ a_4 \{q(gx_{2n-2}, gx_{2n-1}) + q(gx_{2n-1}, gx_{2n}) + q(gx_{2n}, gx_{2n+1})\} \\ &+ a_5 q(gx_{2n-1}, gx_{2n}). \end{aligned}$$

Hence

$$\begin{aligned} q(gx_{2n}, gx_{2n+1}) &\preceq \frac{a_2 + a_3 + a_4 + a_5}{1 - a_3 - a_4} q(gx_{2n-1}, gx_{2n}) \\ &+ \frac{a_1 + a_2 + a_4}{1 - a_3 - a_4} q(gx_{2n-2}, gx_{2n-1}). \end{aligned} \tag{2.2.2}$$

Similarly,

$$\begin{aligned} q(gx_{2n-1}, gx_{2n}) &\preceq \frac{a_2 + a_3 + a_4 + a_5}{1 - a_3 - a_4} q(gx_{2n-2}, gx_{2n-1}) \\ &+ \frac{a_1 + a_2 + a_4}{1 - a_3 - a_4} q(gx_{2n-3}, gx_{2n-2}). \end{aligned} \tag{2.2.3}$$

Clearly $0 \leq \frac{a_2+a_3+a_4+a_5}{1-a_3-a_4} < 1$ and $0 \leq \frac{a_1+a_2+a_4}{1-a_3-a_4} < 1$. Set

$$b_1 = \alpha = \frac{a_2 + a_3 + a_4 + a_5}{1 - a_3 - a_4} \quad \text{and} \quad c_1 = \beta = \frac{a_1 + a_2 + a_4}{1 - a_3 - a_4}.$$

Applying (2.2.2) and (2.2.3) and putting $b_2 = c_1 + \alpha b_1 = \beta + \alpha b_1, c_2 = \beta b_1$,

$$\begin{aligned} q(gx_{2n}, gx_{2n+1}) &\preceq b_1 q(gx_{2n-1}, gx_{2n}) + c_1 q(gx_{2n-2}, gx_{2n-1}) \\ &\preceq b_2 q(gx_{2n-2}, gx_{2n-1}) + c_2 q(gx_{2n-3}, gx_{2n-2}) \\ &\vdots \\ &\preceq b_{2n-1} q(gx_1, gx_2) + c_{2n-1} q(gx_0, gx_1), \end{aligned} \tag{2.2.4}$$

where $b_{2n-1} = \beta b_{2n-3} + \alpha b_{2n-2}$ and $c_{2n-1} = \beta b_{2n-2}$. Similarly

$$q(gx_{2n-1}, gx_{2n}) \preceq b_{2n-2} q(gx_1, gx_2) + c_{2n-2} q(gx_0, gx_1) \tag{2.2.5}$$

where $b_{2n-2} = \beta b_{2n-4} + \alpha b_{2n-3}$ and $c_{2n-2} = \beta b_{2n-3}$. From (2.2.4) and (2.2.5),

$$q(gx_{n+1}, gx_{n+2}) \preceq b_n q(gx_1, gx_2) + c_n q(gx_0, gx_1)$$

where $b_n = \beta b_{n-2} + \alpha b_{n-1}$ and $c_n = \beta b_{n-1}$.

Consider

$$b_{n+2} = \alpha b_{n+1} + \beta b_n \quad (0 \le \alpha, \beta < 1, b_1, b_2 \ge 0).$$

Then $b_n \ge 0$ for all $n \in \mathbb{N}$. Its characteristic equation is that $t^2 - \alpha t - \beta = 0$. If $1 - \alpha - \beta > 0$ and $1 + \alpha - \beta > 0$ then it has two roots t_1, t_2 such that $-1 < t_1 \le 0 \le t_2 < 1$. Also the hypothesis $a_1 + 2a_2 + 2a_3 + 3a_4 + a_5 < 1$ implies $1 - \alpha - \beta > 0$ and $1 + \alpha - \beta > 0$. For such t_1, t_2, $b_n = k_1(t_1)^n + k_2(t_2)^n$ for some $k_1, k_2 \in \mathbb{R}$.

Let $m > n \ge 1$. It follows that

$$\begin{aligned}
q(gx_n, gx_m) &\preceq q(gx_n, gx_{n+1}) + q(gx_{n+1}, gx_{n+2}) + \cdots + q(gx_{m-1}, gx_m) \\
&\preceq (b_{n-1} + b_n + \cdots + b_{m-2})q(gx_1, gx_2) + (c_{n-1} + c_n + \cdots + c_{m-2})q(gx_0, gx_1) \\
&\preceq \{k_1(t_1^{n-1} + t_1^n + \cdots + t_1^{m-2}) + k_2(t_2^{n-1} + \cdots + t_2^{m-2})\}q(gx_1, gx_2) \\
&+ \beta\{k_1(t_1^{n-2} + \cdots + t_1^{m-3}) + k_2(t_2^{n-2} + \cdots + t_2^{m-3})\}q(gx_0, gx_1) \\
&\preceq (\frac{k_1 t_1^{n-1}}{1-t_1} + \frac{k_2 t_2^{n-1}}{1-t_2})q(gx_1, gx_2) + \beta(\frac{k_1 t_1^{n-2}}{1-t_1} + \frac{k_2 t_2^{n-2}}{1-t_2})q(gx_0, gx_1) \\
&\to \theta
\end{aligned}$$

as $n \to \infty$. Therefore $\{gx_n\}$ is a Cauchy sequence in $g(X)$ by Lemma 1.11 (3). Since $g(X)$ is complete, there exists $x' \in g(X)$ such that $gx_m \to x'$ as $m \to \infty$. By definition 1.6(q3)

$$q(gx_n, x') \preceq (\frac{k_1 t_1^{n-1}}{1-t_1} + \frac{k_2 t_2^{n-1}}{1-t_2})q(gx_1, gx_2) + \beta(\frac{k_1 t_1^{n-2}}{1-t_1} + \frac{k_2 t_2^{n-2}}{1-t_2})q(gx_0, gx_1)$$

Since P is a normal cone with normal constant K, we have

$$\begin{aligned}
\|q(gx_n, gx_m)\| &\le K\|(\frac{k_1 t_1^{n-1}}{1-t_1} + \frac{k_2 t_2^{n-1}}{1-t_2})q(gx_1, gx_2) + \beta(\frac{k_1 t_1^{n-2}}{1-t_1} + \frac{k_2 t_2^{n-2}}{1-t_2})q(gx_0, gx_1)\| \\
&\le K(\frac{k_1 t_1^{n-1}}{1-t_1} + \frac{k_2 t_2^{n-1}}{1-t_2})\|q(gx_1, gx_2)\| + K\beta(\frac{k_1 t_1^{n-2}}{1-t_1} + \frac{k_2 t_2^{n-2}}{1-t_2})\|q(gx_0, gx_1)\| \\
&\to 0
\end{aligned}$$

as $n \to \infty$. Also

$$\begin{aligned}
\|q(gx_n, x')\| &\le K\|(\frac{k_1 t_1^{n-1}}{1-t_1} + \frac{k_2 t_2^{n-1}}{1-t_2})q(gx_1, gx_2) + \beta(\frac{k_1 t_1^{n-2}}{1-t_1} + \frac{k_2 t_2^{n-2}}{1-t_2})q(gx_0, gx_1)\| \\
&\le K(\frac{k_1 t_1^{n-1}}{1-t_1} + \frac{k_2 t_2^{n-1}}{1-t_2})\|q(gx_1, gx_2)\| + K\beta(\frac{k_1 t_1^{n-2}}{1-t_1} + \frac{k_2 t_2^{n-2}}{1-t_2})\|q(gx_0, gx_1)\| \\
&\to 0
\end{aligned}$$

as $n \to \infty$.

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Suppose that x' is not a point of coincidence of f and g. Then by assumption,

$$\begin{aligned} 0 &< \inf\{\|q(gx,x')\| + \|q(fx,x')\| + \|q(gx,fx)\| \,:\, x \in X\} \\ &\leq \inf\{\|q(gx_n,x')\| + \|q(fx_n,x')\| + \|q(gx_n,fx_n)\| \,:\, n \in \mathbb{N}\} \\ &= \inf\{\|q(gx_n,x')\| + \|q(gx_{n+2},x')\| + \|q(gx_n,gx_{n+2})\| \,:\, x \in \mathbb{N}\} \\ &= 0 \end{aligned}$$

which is a contradiction. Therefore x' is a point of coincidence of f and g. So there exists $x \in X$ such that $fx = gx = x'$. If there exists $w \in X$ such that $fy = gy = w$ for some $y \in X$,

$$\begin{aligned} q(x',x') &= q(fx,fx) \\ &\preceq a_1q(gx,gx) + a_2q(gx,fx) + a_3q(gx,fx) + a_4q(gx,fx) + a_5q(gx,fx) \\ &= (a_1+a_2+a_3+a_4+a_5)q(x',x'). \end{aligned}$$

Hence

$$q(x',x') = \theta. \tag{2.2.6}$$

Similarly

$$q(w,w) = \theta. \tag{2.2.7}$$

Now by (2.2.6) and (2.2.7)

$$\begin{aligned} q(x',w) &= q(fx,fy) \\ &\preceq a_1q(gx,gy) + a_2q(gx,fx) + a_3q(gy,fy) + a_4q(gx,fy) + a_5q(gy,fx) \\ &= a_1q(x',w) + a_2q(x',x') + a_3q(w,w) + a_4q(x',w) + a_5q(w,x') \\ &= (a_1+a_4)q(x',w) + a_5q(w,x'). \end{aligned}$$

Similarly $q(w,x') \preceq (a_1+a_4)q(w,x') + a_5q(x',w)$. Thus

$$q(x',w) + q(w,x') \preceq (a_1+a_4+a_5)\{q(x',w) + q(w,x')\}.$$

Therefore $q(x',w) + q(w,x') = \theta$ which implies

$$q(x',w) = q(w,x') = \theta. \tag{2.2.8}$$

By (2.2.7),(2.2.8) and Lemma 1.11(1), $x' = w$. Consequently x' is a unique point of coincidence of f and g.

Moreover if f and g are weakly compatible,

$$gx' = ggx = gfx = fgx = fx'$$

which implies gx' is a point of coincidence of f and g. By uniqueness of the point of coincidence , $fx' = gx' = x'$. In other words, x' is the unique common fixed point of f and g. □

Corollary 2.3. *Let (X, d) be a complete cone metric space and let P be a normal cone with normal constant K and q be a c-distance on X. Let $a_i \in [0,1)$, $i = 1,2,3,4,5$ be constants with $a_1 + 2a_2 + 2a_3 + 3a_4 + a_5 < 1$ and $f : X \to X$ be a mapping satisfying the condition*

$$q(fx, fy) \preceq a_1 q(x, y) + a_2 q(x, fx) + a_3 q(y, fy) + a_4 q(x, fy) + a_5 q(y, fx)$$

for all $x, y \in X$. If f satisfies the condition

$$\inf\{\|q(x, y)\| + \|q(fx, y)\| + \|q(x, fx)\| : x \in X\} > 0$$

if $fy \neq y$, then f has a unique fixed point in X.

Proof. Take $g = I$ in the above theorem. □

Corollary 2.4. *Let (X, d) be a complete cone metric space and let P be a normal cone with normal constant K and q be a c-distance on X. Let $a_i \in [0,1)$, $i = 1,2,3,4$ be constants with $a_1 + 2a_2 + 2a_3 + 3a_4 < 1$. If $f : X \to X$ is a continuous mapping satisfying the condition*

$$q(fx, fy) \preceq a_1 q(x, y) + a_2 q(x, fx) + a_3 q(y, fy) + a_4 q(x, fy),$$

then f has a unique fixed point in X.

Proof. Assume there exists $y \in X$ such that $fy \neq y$ and

$$\inf\{\|q(x, y)\| + \|q(fx, y)\| + \|q(x, fx)\| : x \in X\} = 0.$$

Then we can construct $\{x_n\}$ in X such that

$$\inf\{\|q(x_n, y)\| + \|q(fx_n, y)\| + \|q(x_n, fx_n)\| : n \in \mathbb{N}\} = 0.$$

Hence

$$q(x_n, y) \to \theta, \quad q(fx_n, y) \to \theta, \quad q(x_n, fx_n) \to \theta.$$

By Lemma 1.11(2), $fx_n \to y$. By the contractive condition, we have

$$\begin{aligned} q(fx_n, f^2x_n) &\preceq a_1 q(x_n, fx_n) + a_2 q(x_n, fx_n) + a_3 q(fx_n, f^2x_n) + a_4 q(x_n, f^2x_n) \\ &\preceq a_1 q(x_n, fx_n) + a_2 q(x_n, fx_n) + a_3 q(fx_n, f^2x_n) \\ &+ a_4 q(x_n, fx_n) + a_4 q(fx_n, f^2x_n). \end{aligned}$$

Therefore $q(fx_n, f^2x_n) \preceq \frac{a_1+a_2+a_4}{1-a_3-a_4} q(x_n, fx_n)$. Hence

$$\begin{aligned} q(x_n, f^2x_n) &\preceq q(x_n, fx_n) + q(fx_n, f^2x_n) \\ &\preceq q(x_n, fx_n) + \frac{a_1 + a_2 + a_4}{1 - a_3 - a_4} q(x_n, fx_n) \to \theta \end{aligned}$$

as $n \to \infty$. This implies $q(x_n, f^2x_n) \to \theta$. Consequently, $f^2x_n \to y$ by Lemma 1.11(2) . Since f is continuous, we have

$$fy = f(\lim_{n\to\infty} fx_n) = \lim_{n\to\infty} f^2x_n = y$$

which is a contradiction. Therefore if $fy \neq y$, then

$$\inf\{\|q(x,y)\| + \|q(fx,y)\| + \|q(x,fx)\| \,:\, x \in X\} > 0.$$

By Corollary 2.3, the proof is complete. □

Example 2.5. Let $X = \{0,1,2,3\}$, $E = \mathbb{R}$ and $P = \{x \in \mathbb{R} : x \geq 0\}$. Define $d : X \times X \to E$ by $d(x,y) = |x-y|$. Then (X,d) is a complete cone metric space. Define $q : X \times X \to E$ by the following :

$$\begin{aligned} q(0,0) &= 0, \quad q(0,1) = 1, \quad q(0,2) = 1.1, \quad q(0,3) = 0.5, \\ q(1,0) &= 1, \quad q(1,1) = 0, \quad q(1,2) = 0.1, \quad q(1,3) = 0.5, \\ q(2,0) &= 1, \quad q(2,1) = 1, \quad q(2,2) = 0, \quad q(2,3) = 0.5, \\ q(3,0) &= 1, \quad q(3,1) = 0.5, \quad q(3,2) = 0.6, \quad q(3,3) = 0. \end{aligned}$$

Then q is a $c-$distance. In fact, Definition 1.6 (q1),(q3) are obvious. If we put $e = 0.01$, (q4) is also clear. From the direct calculation, (q2) is satisfied.

Define $f : X \to X$ by $f0 = 1, f1 = 2, f2 = 2, f3 = 2$ and define $g : X \to X$ by $gx = x$. Then $f(X) \subseteq g(X)$ and $g(X)$ is complete. If we take $x = 2, y = 0$, then $q(f2, f0) = q(2,1) = 1$ and

$$\begin{aligned} a_1 q(g2, g0) &+ a_2 q(g2, f2) + a_3 q(g0, f0) + a_4 q(g2, f0) \\ &= a_1 q(2,0) + a_2 q(2,2) + a_3 q(0,1) + a_4 q(2,1) \\ &= a_1 + a_3 + a_4 \leq a_1 + a_3 + 2a_4 < 1 \end{aligned}$$

for any real numbers $a_i \in [0,1)\,(i = 1,2,3,4)$ with $a_1 + a_2 + a_3 + 2a_4 < 1$. Hence the contractive condition (2.2.1) of Theorem 2.1 is not satisfied and so Theorem 2.1 can not be applied to this example.

But Theorem 2.2 can be applied to this example. In fact we take $a_1 = 0.14, a_2 = a_3 = a_4 = 0$ and $a_5 = 0.85$. Then for any $x, y \in X$, the contractive condition of Theorem 2.2 is satisfied. Also

$$\inf\{\|q(gx,y)\| + \|q(fx,y)\| + \|q(gx,fx)\| \,:\, x \in X\} > 0$$

for $y = 0,1,3$. Hence the hypotheses are satisfied and so by Theorem 2.2 f and g have a unique point of coincidence. Since $f2 = 2$ and $g2 = 2$, 2 is a unique point of coincidence. Since $2 = gf2 = fg2$, f and g are weakly compatible. Hence 2 is the unique common fixed point of f and g.

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L-FUZZY CLOSURE OPERATORS, L-FUZZY TOPOLOGIES AND L-FUZZY QUASI-UNIFORMITIES

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Abstract. We investigate L-fuzzy closure operators and L-(fuzzy) topologies in a complete residuated lattice. Also, we study the relationship among L-fuzzy closure operator and L-fuzzy quasi-uniform space. Finally, we give their examples.

Keywords:Complete residuated lattice, L-fuzzy closure space L-fuzzy topological space and L-fuzzy quasi-uniform space.

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1. Introduction

Hájek [6] introduced a complete residuated lattice which is an algebraic structure for many valued logic. Bělohlávek [2] investigated information systems and decision rules in complete residuated lattices. Höhle [7,8]introduced L-fuzzy topological structure with algebraic structure L(cqm, quantales, MV-algebra). Uniformities in fuzzy sets, have the following approach of Lowen [14] based on powersets of the form $L^{X\times X}$ as a viewpoint of the enourage approach, the uniform covering approach of Kotzé [12], the uniform operator approach of Rodabaugh [20] as a generalization of Hutton [9] based on powersets of the form $(L^X)^{(L^X)}$, the unification approach of Gutiérrez García[3]. Many researchers studied the different approach as powerset [12] or the uniform covering [9].

We investigate L-fuzzy closure operators and L-(fuzzy) topologies in a complete residuated lattice. Also, we study the relationship among L-fuzzy closure operator and L-fuzzy quasi-uniform space. Finally, we give their examples.

2. Preliminaries

Definition 2.1. [2,6] An algebra $(L,\wedge,\vee,\odot,\rightarrow,\perp,\top)$ is called a complete residuated lattice if it satisfies the following conditions:

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(L1) $(L, \leq, \vee, \wedge, \perp, \top)$ is a complete lattice with the greatest element $\top$ and the least element $\perp$;

(L2) $(L, \odot, \top)$ is a commutative monoid;

(L3) $x \odot y \leq z$ iff $x \leq y \to z$ for $x, y, z \in L$.

In this paper, we assume that $L = (L, \vee, \wedge, \odot, \to, \perp, \top)$ be a complete residuated lattice with with an order reversing involution * which is defined by $x \oplus y = (x^* \odot y^*)^*$, $x^* = x \to 0$.

Lemma 2.2. [2,6] For each $x, y, z, x_i, y_i, w \in L$, we have the following properties.

(1) $1 \to x = x$, $0 \odot x = 0$ and $x \to 0 = x^*$,

(2) If $y \leq z$, then $x \odot y \leq x \odot z$, $x \oplus y \leq x \oplus z$, $x \to y \leq x \to z$ and $z \to x \leq y \to x$.

(3) $x \odot y \leq x \wedge y \leq x \vee y \leq x \oplus y$,

(4) $(\bigwedge_i y_i)^* = \bigvee_i y_i^*$, $(\bigvee_i y_i)^* = \bigwedge_i y_i^*$,

(5) $x \odot (\bigwedge_i y_i) \leq \bigwedge_i (x \odot y_i)$,

(6) $x \oplus (\bigwedge_i y_i) = \bigwedge_i (x \oplus y_i)$, $x \oplus (\bigvee_i y_i) = \bigvee_i (x \oplus y_i)$,

(7) $x \to (\bigwedge_i y_i) = \bigwedge_i (x \to y_i)$,

(8) $(\bigvee_i x_i) \to y = \bigwedge_i (x_i \to y)$,

(9) $x \to (\bigvee_i y_i) \geq \bigvee_i (x \to y_i)$,

(10) $(\bigwedge_i x_i) \to y \geq \bigvee_i (x_i \to y)$,

(11) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$,

(12) $x \odot (x \to y) \leq y$ and $x \to y \leq (y \to z) \to (x \to z)$,

(13) $x \odot (x^* \oplus y^*) \leq y^*$,

(14) $(x \to y) \odot (z \to w) \leq (x \odot z) \to (y \odot w)$,

(15) $(x \to y) \odot (z \to w) \leq (x \oplus z) \to (y \oplus w)$,

(16) $x \to y \leq (x \odot z) \to (y \odot z)$ and $(x \to y) \odot (y \to z) \leq x \to z$,

(17) $x \odot y \odot (z \oplus w) \leq (x \odot z) \oplus (y \odot w)$.

For $\alpha \in L, \lambda \in L^X$, we denote $(\alpha \to \lambda), (\alpha \odot \lambda), \alpha_X \in L^X$ as $(\alpha \to \lambda)(x) = \alpha \to \lambda(x)$, $(\alpha \odot \lambda)(x) = \alpha \odot \lambda(x)$, $\alpha_X(x) = \alpha$.

Definition 2.3.[2,4] Let X be a set. A mapping $R : X \times X \to L$ is called an *L-partial order* if it satisfies the following conditions:

(E1) reflexive if $R(x, x) = \top$ for all $x \in X$,

(E2) transitive if $R(x, y) \odot R(y, z) \leq R(x, z)$, for all $x, y, z \in X$,

(E3) antisymmetric if $R(x, y) = R(y, x) = \top$, then $x = y$.

Lemma 2.4. [2,4] For a given set X, define a binary mapping $S : L^X \times L^X \to L$ by

$$S(\lambda, \mu) = \bigwedge_{x \in X} (\lambda(x) \to \mu(x)).$$

Then, for each $\lambda, \mu, \rho, \nu \in L^X$, and $\alpha \in L$, the following properties hold.

(1) S is an L-partial order on L^X.

(2) $\lambda \leq \mu$ iff $S(\lambda, \mu) \geq \top$,

(3) If $\lambda \leq \mu$, then $S(\rho, \lambda) \leq S(\rho, \mu)$ and $S(\lambda, \rho) \geq S(\mu, \rho)$,

(4) $S(\lambda, \mu) \odot S(\nu, \rho) \leq S(\lambda \odot \nu, \mu \odot \rho)$ and $S(\lambda, \mu) \odot S(\nu, \rho) \leq S(\lambda \oplus \nu, \mu \oplus \rho)$,

(5) $S(\mu, \rho) \leq S(\lambda, \mu) \to S(\lambda, \rho)$ and $S(\mu, \rho) \leq S(\rho, \lambda) \to S(\mu, \lambda)$,

(6) $\bigvee_{\mu \in L^X} (S(\mu, \rho) \odot S(\lambda, \mu)) = S(\lambda, \rho)$.

(7) If $\phi : X \to Y$ is a map, then for $\lambda, \mu \in L^X$ and $\rho, \nu \in L^Y$,

$$S(\lambda, \mu) \leq S(\phi^{\to}(\lambda), \phi^{\to}(\mu)),$$

$$S(\rho, \nu) \leq S(\phi^{\leftarrow}(\rho), \phi^{\leftarrow}(\nu)),$$

and the equalities hold if ϕ is bijective.

Definition 2.5. [8,22] A map $\mathcal{T} : L^X \to L$ is called an L-fuzzy topology on X if it satisfies the following conditions:

(T1) $\mathcal{T}(\bot_X) = \mathcal{T}(\top_X) = \top$,

(T2) $\mathcal{T}(\lambda \odot \mu) \geq \mathcal{T}(\lambda) \odot \mathcal{T}(\mu), \ \forall \ \lambda, \mu \in L^X$,

(T3) $\mathcal{T}(\bigvee_i \lambda_i) \geq \bigwedge_i \mathcal{T}(\lambda_i), \ \forall \ \{\lambda_i\}_{i \in \Gamma} \subseteq L^X$.

The pair $(X, \mathcal{T})$ is called an L-fuzzy topological space. An L-fuzzy topological space is called enriched if

(R) $\mathcal{T}(\alpha \odot \lambda) \geq \mathcal{T}(\lambda)$ for all $\lambda \in L^X$ and $\alpha \in L$.

Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be two L-fuzzy topological spaces. A mapping $\phi : X \to Y$ is said to be L-fuzzy continuous iff for each $\lambda \in L^Y$, $\mathcal{T}_2(\lambda) \leq \mathcal{T}_1(\phi^{\leftarrow}(\lambda))$.

Remark 2.6. A set $\tau \subset L^X$ is called an L-topology on X if (t1) $\bot_X, \top_X \in \tau$, (t2) $(\lambda \odot \mu) \in \tau$, for each $\lambda, \mu \in \tau$, (t3) $\bigvee_i \lambda_i \in \tau$, for all $\lambda_i \in \tau$. An L-topology τ is called enriched if $\alpha \odot \lambda \in \tau$, for all $\lambda \in \tau$ and $\alpha \in L$.

Definition 2.7. [11] A map $\mathcal{U} : L^{X \times X} \to L$ is called an L-fuzzy quasi-uniformity on X iff the following conditions hold.

(QU1) There exists $u \in L^{X \times X}$ such that $\mathcal{U}(u) = \top$.

(QU2) If $v \leq u$, then $\mathcal{U}(v) \leq \mathcal{U}(u)$.

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(QU3) For every $u, v \in L^{X\times X}$,$\mathcal{U}(u \odot v) \geq \mathcal{U}(u) \odot \mathcal{U}(v)$.

(QU4) If $\mathcal{U}(u) \neq \bot$ then $\top_{\triangle} \leq u$ where

$$\top_{\triangle}(x,y) = \begin{cases} \top, & \text{if x = y} \\ \bot, & \text{if } x \neq y, \end{cases}$$

(QU5) $\mathcal{U}(u) \leq \bigvee\{\mathcal{U}(v) \mid v \circ v \leq u\},\ \forall\ u \in L^{X\times X}$, where

$$v \circ v(x,y) = \bigvee_{z\in X} v(x,z) \odot v(z,y),\ \forall\ x,y \in X.$$

An L-fuzzy quasi-uniformity $\mathcal{U}$ on X is said to be stratified if

(R) $\mathcal{U}(\alpha \odot u) \geq \alpha \odot \mathcal{U}(u),\ \forall\ u \in L^{X\times X}$.

The pair $(X,\mathcal{U})$ is called an L-fuzzy quasi-uniform space.

Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be L-fuzzy quasi-uniform spaces, and $\phi : X \to Y$ be a mapping. Then ϕ is said to be L-fuzzy uniformly continuous if

$\mathcal{V}(v) \leq \mathcal{U}((\phi \times \phi)^{\leftarrow}(v))$, for every $v \in L^{Y\times Y}$.

Remark 2.8. Let $(X,\mathcal{U})$ be an L-fuzzy quasi- uniform space. By (QU1) and (QU2), we have $\mathcal{U}(\top_{X\times X}) = \top$ because $u \leq \top_{X\times X}$ for all $u \in L^{X\times X}$.

3. L-fuzzy closure spaces and L-fuzzy topological spaces

Definition 3.1. A map $\mathcal{C} : L^X \to L^X$ is called an L-fuzzy closure operator if it satisfies the following conditions:

(C1) $\mathcal{C}(\bot_X) = \bot_X$,

(C2) for $\lambda \in L^X$, $\lambda \leq \mathcal{C}(\lambda)$,

(C3) if $\lambda \leq \mu$, $\mathcal{C}(\lambda) \leq \mathcal{C}(\mu)$,

(C4) for all $\lambda, \mu \in L^X$, $\mathcal{C}(\lambda \oplus \mu) \leq \mathcal{C}(\lambda) \oplus \mathcal{C}(\mu)$.

The pair $(X,\mathcal{C})$ is called an L-fuzzy closure space.

An L-fuzzy closure space is called stratified if

(R) $\mathcal{C}(\alpha \to \lambda) \leq \alpha \to \mathcal{C}(\lambda)$ for all $\lambda \in L^X$ and $\alpha \in L$.

Let $(X,\mathcal{C}_1)$ and $(Y,\mathcal{C}_2)$ be two L-fuzzy closure spaces. A mapping $\phi : X \to Y$ is said to be C-map if $\phi^{\to}(\mathcal{C}_1(\lambda)) \leq \mathcal{C}_2(\phi^{\to}(\lambda))$ for each $\lambda \in L^X$.

Lemma 3.2. Let $\mathcal{C} : L^X \to L^X$ a map. The following statement are equivalent.

(1) For all $\lambda, \mu \in L^X$, $S(\lambda,\mu) \leq S(\mathcal{C}(\lambda),\mathcal{C}(\mu))$.

(2) If $\lambda \leq \mu$, then $\mathcal{C}(\lambda) \leq \mathcal{C}(\mu)$ and $\mathcal{C}(\alpha \odot \rho) \geq \alpha \odot \mathcal{C}(\rho)$ for all $\lambda \in L^X$ and $\alpha \in L$.

(3) If $\lambda \leq \mu$, then $\mathcal{C}(\lambda) \leq \mathcal{C}(\mu)$ and $\mathcal{C}(\alpha \to \rho) \leq \alpha \to \mathcal{C}(\rho)$ for all $\lambda \in L^X$ and $\alpha \in L$.

Proof. (1) ($\Rightarrow$) (2). If $\lambda \leq \mu$, then $\top = S(\lambda,\mu) \leq S(\mathcal{C}(\lambda),\mathcal{C}(\mu))$. Hence $\mathcal{C}(\lambda) \leq \mathcal{C}(\mu)$. Put $\mu = \alpha \odot \lambda$. Then $\alpha \leq S(\lambda, \alpha\odot\lambda) \leq S(\mathcal{C}(\lambda),\mathcal{C}(\alpha\odot\lambda))$. Hence $\alpha \odot \mathcal{C}(\lambda) \leq \mathcal{C}(\alpha\odot\lambda)$.

(2) ($\Rightarrow$) (3). Since $\alpha \odot \mathcal{C}(\alpha \to \lambda) \leq \mathcal{C}(\alpha \odot (\alpha\to\lambda)) \leq \mathcal{C}(\lambda)$, $\mathcal{C}(\alpha\to\lambda) \leq \alpha \to \mathcal{C}(\lambda)$.

(3) ($\Rightarrow$) (1). Since $S(\lambda,\mu)\odot\lambda \leq \mu$ iff $\lambda \leq S(\lambda,\mu)\to\mu$, $\mathcal{C}(\lambda) \leq \mathcal{C}(S(\lambda,\mu)\to\mu) \leq S(\lambda,\mu) \to \mathcal{C}(\mu)$. Hence $S(\lambda,\mu) \leq S(\mathcal{C}(\lambda),\mathcal{C}(\mu))$.

Theorem 3.3. Let $(X,\mathcal{T})$ be an L-fuzzy topological space. Define a map $\mathcal{C}_{\mathcal{T}} : L^X \longrightarrow L^X$ as follows:

$$\mathcal{C}_{\mathcal{T}}(\lambda) = \bigwedge_{\mu\in L^X} (\mathcal{T}(\mu) \to (S(\lambda,\mu^*) \to \mu^*))$$

Then $(X,\mathcal{C}_{\mathcal{T}})$ is a stratified L-fuzzy closure space.

Proof. (C1) $\mathcal{C}_{\mathcal{T}}(\bot_X) = \bigwedge_{\mu\in L^X}(\mathcal{T}(\mu) \to (S(\bot_X,\mu^*)\to\mu^*)) = (\bigvee(\mathcal{T}(\mu)\odot\mu))^* = \bot_X$.

(C2), we have $\mathcal{C}_{\mathcal{T}}(\lambda) \geq \lambda$ for each $\lambda \in L^X$ from:

$$\begin{aligned}
S(\lambda,\mathcal{C}_{\mathcal{T}}(\lambda)) &= \bigwedge_{x\in X} (\lambda(x) \to \mathcal{C}_{\mathcal{T}}(\lambda)(x)) \\
&= \bigwedge_{x\in X} \Big(\lambda(x) \to \bigwedge_{\mu\in L^X} (\mathcal{T}(\mu) \to (S(\lambda,\mu^*) \to \mu^*(x)))\Big) \\
&= \bigwedge_{x\in X}\bigwedge_{\mu\in L^X} \Big((\mathcal{T}(\mu)\odot S(\lambda,\mu^*)\odot\lambda(x)) \longrightarrow \mu^*(x))\Big) \\
&= \bigwedge_{x\in X}\bigwedge_{\mu\in L^X} \Big((\mathcal{T}(\mu)\odot S(\lambda,\mu^*) \longrightarrow (\lambda(x) \longrightarrow \mu^*(x))\Big) \\
&= \bigwedge_{\mu\in L^X} \Big((\mathcal{T}(\mu)\odot S(\lambda,\mu^*)) \to \bigwedge_{x\in X} (\lambda(x) \longrightarrow \mu^*(x))\Big) \\
&= \bigwedge_{\mu\in L^X} \Big((\mathcal{T}(\mu)\odot S(\lambda,\mu^*)) \to S(\lambda,\mu^*)\Big) \geq \top.
\end{aligned}$$

(C3) and, by Lemma 3.2, $\mathcal{C}_{\mathcal{T}}$ is stratified from

$$\begin{aligned}
S(\mathcal{C}_{\mathcal{T}}(\lambda), \mathcal{C}_{\mathcal{T}}(\mu)) &= \bigwedge_{x\in X} (\mathcal{C}_{\mathcal{T}}(\lambda)(x) \to \mathcal{C}_{\mathcal{T}}(\mu)(x)) \\
&= \bigwedge_{x\in X} \Big(\bigwedge_{\rho\in L^X} (\mathcal{T}(\rho) \odot S(\lambda,\rho^*) \to \rho^*(x)) \to \bigwedge_{\nu\in L^X} (\mathcal{T}(\nu) \odot S(\mu,\nu^*) \to \nu^*(x))\Big) \\
&\geq \bigwedge_{x\in X} \bigwedge_{\rho\in L^X} \Big((\mathcal{T}(\rho) \odot S(\lambda,\rho^*) \to \rho^*(x)) \to ((\mathcal{T}(\rho) \odot S(\mu,\rho^*) \to \rho^*(x))\Big) \\
&\geq \bigwedge_{x\in X} \bigwedge_{\rho\in L^X} \Big(\mathcal{T}(\rho) \odot S(\mu,\rho^*) \to (\mathcal{T}(\rho) \odot S(\lambda,\rho^*)\Big) \text{ (by Lemma 2.2 (16))} \\
&\geq \bigwedge_{x\in X} \bigwedge_{\rho\in L^X} \Big(S(\mu,\rho^*) \to S(\lambda,\rho^*)\Big) \geq S(\lambda,\mu). \text{(by Lemma 2.4(5))}
\end{aligned}$$

(C4)

$$(a \odot c) \odot (b \oplus d)^* = (a \odot b^*) \odot (c \odot d^*)$$

$$\Leftrightarrow (a \odot c) \to (b \oplus d) = (a \to b) \oplus (c \to d),$$

$$\begin{aligned}
&\mathcal{C}_{\mathcal{T}}(\lambda) \oplus \mathcal{C}_{\mathcal{T}}(\mu) \\
&= \bigwedge_{\rho\in L^X} (\mathcal{T}(\rho) \to (S(\lambda,\rho^*) \to \rho^*)) \oplus \bigwedge_{\nu\in L^X} (\mathcal{T}(\nu) \to (S(\mu,\nu^*) \to \nu^*)) \\
&\text{(by Lemma 2.2(6))} \\
&= \bigwedge_{\rho\in L^X} \bigwedge_{\nu\in L^X} \Big((\mathcal{T}(\rho) \odot S(\lambda,\rho^*) \to \rho^*) \oplus (\mathcal{T}(\nu) \odot S(\mu,\nu^*) \to \nu^*)\Big) \\
&= \bigwedge_{\rho,\nu\in L^X} \Big(\mathcal{T}(\rho) \odot \mathcal{T}(\nu) \odot S(\lambda,\rho^*) \odot S(\mu,\nu^*) \to (\rho^* \oplus \nu^*)\Big) \\
&\geq \bigwedge_{\rho,\nu\in L^X} \Big(\mathcal{T}(\rho \odot \nu) \odot S(\lambda \oplus \mu, (\rho^* \oplus \nu^*)) \to (\rho^* \oplus \nu^*)\Big) \\
&\text{(by Lemma 2.4(4))} \\
&\geq \mathcal{C}_{\mathcal{T}}(\lambda \oplus \mu).
\end{aligned}$$

Remark 3.4. Let (X, τ) be an L-topological space. Define a map $\mathcal{C}_\tau : L^X \to L^X$ as follows:

$$\mathcal{C}_\tau(\lambda) = \bigwedge\{\mu \in L^X \mid \lambda \leq \mu, \mu^* \in \tau\}.$$

Then $(X, \mathcal{C}_\tau)$ is an L-fuzzy closure space. Moreover, if (X, τ) is enriched, $(X, \mathcal{C}_\tau)$ is stratified.

Theorem 3.5. Let $(X,\mathcal{C})$ be an L-fuzzy closure space. Define a map $\mathcal{T}_{\mathcal{C}} : L^X \to L$ by:

$$\mathcal{T}_{\mathcal{C}}(\lambda) = S(\mathcal{C}(\lambda^*), \lambda^*).$$

Then, $\mathcal{T}_{\mathcal{C}}$ is an L-fuzzy topology on X. If $\mathcal{C}$ is stratified, then $\mathcal{T}_{\mathcal{C}}$ is an enriched L-fuzzy topology.

Proof. (T1)

$$\mathcal{T}_{\mathcal{C}}(\top_X) = \bigwedge_{x\in X}(\mathcal{C}(\top_X^*) \to \top_X^*(x)) = \bigwedge_{x\in X}(\bot_X(x) \to \bot_X(x)) = \top,$$

$$\mathcal{T}_{\mathcal{C}}(\bot_X) = \bigwedge_{x\in X}(\mathcal{C}(\bot_X^*) \to \bot_X^*(x)) = \bigwedge_{x\in X}(\top_X(x) \to \top_X(x)) = \top.$$

(T2) By Lemma 2.2(12), we havre

$$\begin{aligned}
\mathcal{T}_{\mathcal{C}}(\lambda\odot\mu) &= \bigwedge_{x\in X}(\mathcal{C}((\lambda\odot\mu)^*)(x) \to (\lambda\odot\mu)^*(x))\\
&\geq \bigwedge_{x\in X}(\mathcal{C}(\lambda^*)(x)\oplus\mathcal{C}(\mu^*)(x) \to \lambda^*(x)\oplus\mu^*(x))\\
&\geq \bigwedge_{x\in X}(\mathcal{C}(\lambda^*)(x) \to \lambda^*(x)) \odot \bigwedge_{x\in X}(\mathcal{C}(\mu^*)(x) \to \mu^*(x))\\
&= \mathcal{T}_{\mathcal{C}}(\lambda)\odot\mathcal{T}_{\mathcal{C}}(\mu).
\end{aligned}$$

(T3) By Lemma 2.2(8),we have

$$\begin{aligned}
\mathcal{T}_{\mathcal{C}}(\bigvee_i \lambda_i) &= \bigwedge_{x\in X}(\mathcal{C}((\bigvee_i \lambda_i)^*)(x) \to (\bigvee_i \lambda_i)^*(x))\\
&\geq \bigwedge_{x\in X}(\mathcal{C}(\bigwedge_i \lambda_i^*)(x) \to \bigwedge_i \lambda_i^*(x))\\
&\geq \bigwedge_{x\in X}(\bigwedge_i \mathcal{C}(\lambda_i^*)(x) \to \bigwedge_i \lambda_i^*(x))\\
&\geq \bigwedge_i \bigwedge_{x\in X}(\mathcal{C}(\lambda_i^*)(x) \to \lambda_i^*(x)) = \bigwedge_i \mathcal{T}_{\mathcal{C}}(\lambda_i).
\end{aligned}$$

(R) By Lemma 2.2 (12), we have

$$\begin{aligned}
\mathcal{T}_{\mathcal{C}}(\alpha \odot \lambda) &= \bigwedge_{x\in X} (\mathcal{C}((\alpha \odot \lambda)^*)(x) \to (\alpha \odot \lambda)^*(x)) \\
&= \bigwedge_{x\in X} (\mathcal{C}(\alpha \to \lambda^*)(x) \to (\alpha \to \lambda^*(x)) \\
&\geq \bigwedge_{x\in X} ((\alpha \to \mathcal{C}(\lambda^*)(x)) \to (\alpha \to \lambda^*(x))) \\
&\geq \bigwedge_{x\in X} (\mathcal{C}(\lambda^*)(x) \to \lambda^*(x)) = \mathcal{T}_{\mathcal{C}}(\lambda).
\end{aligned}$$

Remark 3.6. Let $(X, \mathcal{C})$ be an L-fuzzy closure space. Define a subset $\tau_{\mathcal{C}} \subset L^X$ by:

$$\tau_{\mathcal{C}} = \{\lambda \in L^X \mid \mathcal{C}(\lambda^*) = \lambda^*\}.$$

Then, $\tau_{\mathcal{C}}$ is an L-topology on X. If $\mathcal{C}$ is stratified, then $\tau_{\mathcal{C}}$ is an enriched L-topology.

Theorem 3.7. (1) If $\mathcal{C}$ is an L-fuzzy closure operator on X, then $\mathcal{C}_{\mathcal{T}_{\mathcal{C}}} \geq \mathcal{C}$ and $\mathcal{C}_{\tau_{\mathcal{C}}} \geq \mathcal{C}$.
(2) If $\mathcal{T}$ is an L-fuzzy topology on X, then $\mathcal{T}_{\mathcal{C}_{\mathcal{T}}} \geq \mathcal{T}$ and $\tau_{\mathcal{C}_{\mathcal{T}}} = \{\lambda \in L^X \mid \mathcal{T}_{\mathcal{C}_{\mathcal{T}}}(\lambda^*) = \top\}$.
Proof. (1)

$$\begin{aligned}
\mathcal{C}_{\mathcal{T}_{\mathcal{C}}}(\lambda)(x) &= \bigwedge_{\mu\in L^X} (\mathcal{T}_{\mathcal{C}}(\mu) \to (S(\lambda, \mu^*) \to \mu^*(x))) \\
&= \bigwedge_{\mu\in L^X} ((\mathcal{T}_{\mathcal{C}}(\mu) \odot S(\lambda, \mu^*)) \to \mu^*(x))) \\
&\geq \bigwedge_{\mu\in L^X} (S(\mathcal{C}(\mu^*), \mu^*) \odot S((\mathcal{C}(\lambda), (\mathcal{C}(\mu^*))) \to \mu^*(x))) \\
&= S(\mathcal{C}(\lambda), \mu^*) \to \mu^*(x) \geq (\mathcal{C}(\lambda)(x) \to \mu^*(x)) \to \mu^*(x) \geq \mathcal{C}(\lambda)(x).
\end{aligned}$$

$$\mathcal{C}_{\tau_{\mathcal{C}}}(\lambda) = \bigwedge\{\mu \mid \lambda \leq \mu, \mathcal{C}(\mu) = \mu\} \geq \mathcal{C}(\lambda).$$

(2)

$$\begin{aligned}\mathcal{T}_{\mathcal{C}_{\mathcal{T}}}(\lambda) &= S\Big(\mathcal{C}_{\mathcal{T}}(\lambda^*), \lambda^*\Big)\\ &= \bigwedge_{x\in X}\Big(\bigwedge_{\mu\in L^X}(\mathcal{T}(\mu)\to(S(\lambda^*,\mu^*)\to\mu^*))\to\lambda^*(x)\Big)\\ &\geq \bigwedge_{x\in X}\Big((\mathcal{T}(\lambda)\to(S(\lambda^*,\lambda^*)\to\lambda^*))\to\lambda^*(x)\Big)\\ &= \bigwedge_{x\in X}\Big((\mathcal{T}(\lambda)\to\lambda^*(x))\to\lambda^*(x)\Big)\\ &\geq \mathcal{T}(\lambda).\end{aligned}$$

$$\tau_{\mathcal{C}_{\mathcal{T}}} = \{\lambda\in L^X \mid \mathcal{C}_{\mathcal{T}}(\lambda^*) = \lambda^*\} = \{\lambda\in L^X \mid \mathcal{T}_{\mathcal{C}_{\mathcal{T}}}(\lambda^*) = \top\}.$$

Theorem 3.8 Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be L-fuzzy topological spaces and $\phi : X \to Y$ be a map. Then

(1) For each $\lambda \in L^X$,

$$\bigwedge_{\nu\in L^Y}(\mathcal{T}_Y(\nu)\to(\mathcal{T}_X(\phi^{\leftarrow}(\nu))\leq S(\phi^{\rightarrow}(\mathcal{C}_{\mathcal{T}_X}(\lambda)),\mathcal{C}_{\mathcal{T}_Y}(\phi^{\rightarrow}(\lambda)))$$

(2) If a mapping $\phi : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is continuous, then $\phi : (X, \mathcal{C}_{\mathcal{T}_X}) \to (Y, \mathcal{C}_{\mathcal{T}_Y})$ is a C-map.

Proof. (1)

$$\begin{aligned}&S(\phi^{\rightarrow}(\mathcal{C}_{\mathcal{T}_X}(\lambda)),\mathcal{C}_{\mathcal{T}_Y}(\phi^{\rightarrow}(\lambda)))\\ &= \bigwedge_{y\in Y}(\phi^{\rightarrow}(\mathcal{C}_{\mathcal{T}_X}(\lambda))(y)\to\mathcal{C}_{\mathcal{T}_Y}(\phi^{\rightarrow}(\lambda))(y))\\ &= \bigwedge_{x\in X}(\phi^{\rightarrow}(\mathcal{C}_{\mathcal{T}_X}(\lambda))(\phi(x))\to\mathcal{C}_{\mathcal{T}_Y}(\phi^{\rightarrow}(\lambda))(\phi(x)))\\ &= \bigwedge_{x\in X}(\mathcal{C}_{\mathcal{T}_X}(\lambda)(x)\to\mathcal{C}_{\mathcal{T}_Y}(\phi^{\rightarrow}(\lambda))(\phi(x)))\\ &= \bigwedge_{x\in X}(\bigwedge_{\rho\in L^X}((\mathcal{T}_X(\rho)\odot S(\lambda,\rho^*)\to\rho^*(x))\\ &\to\bigwedge_{\nu\in L^Y}((\mathcal{T}_Y(\nu)\odot S(\phi^{\rightarrow}(\lambda),\nu^*)\to\nu^*(\phi(x)))))\\ &= \bigwedge_{x\in X}(\bigwedge_{\nu\in L^Y}((\mathcal{T}_X(\phi^{\leftarrow}(\nu))\odot S(\lambda,\phi^{\leftarrow}(\nu)^*)\to\phi^{\leftarrow}(\nu)^*(x))\\ &\to\bigwedge_{\nu\in L^Y}((\mathcal{T}_Y(\nu)\odot S(\phi^{\rightarrow}(\lambda),\nu^*)\to\nu^*(\phi(x)))))\end{aligned}$$

$$= \bigwedge_{x\in X}(\bigwedge_{\nu\in L^Y} \Big((\mathcal{T}_X(\phi^{\leftarrow}(\nu)) \odot S(\lambda, \phi^{\leftarrow}(\nu)^*) \to \phi^{\leftarrow}(\nu)^*(x))$$
$$\to ((\mathcal{T}_Y(\nu) \odot S(\phi^{\to}(\lambda), \nu^*) \to \nu^*(\phi(x))))\Big)$$
$$= \bigwedge_{x\in X}(\bigwedge_{\nu\in L^Y} \Big((\mathcal{T}_Y(\nu) \odot S(\phi^{\to}(\lambda), \nu^*) \to (\mathcal{T}_X(\phi^{\leftarrow}(\nu)) \odot S(\lambda, \phi^{\leftarrow}(\nu)^*)\Big)$$
$$= \bigwedge_{\nu\in L^Y} (\mathcal{T}_Y(\nu) \to \mathcal{T}_X(\phi^{\leftarrow}(\nu)))$$

(2) Since $\mathcal{T}_Y(\nu) \leq \mathcal{T}_X(\phi^{\leftarrow}(\nu))$, by (1), $\phi^{\to}(\mathcal{C}_{\mathcal{T}_X}(\lambda)) \leq \mathcal{C}_{\mathcal{T}_Y}(\phi^{\to}(\lambda))$.

Theorem 3.9 Let $(X, \mathcal{C}_X)$ and $(Y, \mathcal{C}_Y)$ be fuzzy closure spaces and $\phi : X \to Y$ be a map. Then
(1) $S(\mathcal{C}_X(\phi^{\leftarrow}(\lambda^*)), \phi^{\leftarrow}(\mathcal{C}_Y(\lambda^*))) \leq \mathcal{T}_{\mathcal{C}_Y}(\lambda) \to \mathcal{T}_{\mathcal{C}_X}(\phi^{\leftarrow}(\lambda))$ for each $\lambda \in L^Y$.
(2) If a mapping $\phi : (X, \mathcal{C}_X) \to (Y, \mathcal{C}_Y)$ is an C-map, then $\phi : (X, \mathcal{T}_{\mathcal{C}_X}) \to (Y, \mathcal{T}_{\mathcal{C}_Y})$ is continuous.

Proof. (1) By Lemma 2.2, we have

$$\mathcal{T}_{\mathcal{C}_Y}(\lambda) \to \mathcal{T}_{\mathcal{C}_X}(\phi^{\leftarrow}(\lambda))$$
$$= \bigwedge_{y\in Y} \Big(\mathcal{C}_Y(\lambda^*)(y) \to \lambda^*(y)\Big) \to \bigwedge_{x\in X} \Big(\mathcal{C}_X(\phi^{\leftarrow}(\lambda^*))(x) \to \phi^{\leftarrow}(\lambda^*)(x)\Big)$$
$$\geq \bigwedge_{x\in X} \Big(\phi^{\leftarrow}(\mathcal{C}_Y(\lambda^*))(x) \to \phi^{\leftarrow}(\lambda^*)(x)\Big) \to \bigwedge_{x\in X} \Big(\mathcal{C}_X(\phi^{\leftarrow}(\lambda^*))(x) \to \phi^{\leftarrow}(\lambda^*)(x)\Big)$$
$$\geq \bigwedge_{x\in X} \Big(\mathcal{C}_X(\phi^{\leftarrow}(\lambda^*))(x) \to \phi^{\leftarrow}(\mathcal{C}_Y(\lambda^*))(x)\Big)$$

(2) Let $\phi^{\to}(\mathcal{C}_X(\lambda)) \leq \mathcal{C}_Y(\phi^{\to}(\lambda))$. Then, put $\lambda = \phi^{\leftarrow}(\mu)$,

$$\mathcal{C}_X(\phi^{\leftarrow}(\mu)) \leq \phi^{\leftarrow}(\phi^{\to}(\mathcal{C}_X(\phi^{\leftarrow}(\mu)))) \leq \phi^{\leftarrow}(\mathcal{C}_Y(\phi^{\to}(\phi^{\leftarrow}(\mu)))) \leq \phi^{\leftarrow}(\mathcal{C}_Y(\mu)).$$

Thus, by (1), if $\mathcal{C}_X(\phi^{\leftarrow}(\lambda)) \leq \phi^{\leftarrow}(\mathcal{C}_Y(\lambda))$, then $\mathcal{T}_{\mathcal{C}_Y}(\lambda) \leq \mathcal{T}_{\mathcal{C}_X}(\phi^{\leftarrow}(\lambda))$.

Example 3.10. Let $(L = [0, 1], \odot, \to)$ be a complete residuated lattice defined by

$$x \odot y = (x + y - 1) \vee 0, \;\; x \to y = (1 - x + y) \wedge 1.$$

$$x \oplus y = (x + y) \wedge 1, \;\; x^* = 1 - x.$$

Let $X = \{x, y, z\}$ be a set and $\rho, \rho \odot \rho \in L^X$ such that

$$\rho(x) = 0.1, \rho(y) = 0.8, \rho(z) = 0.7,$$

$$\rho \odot \rho(x) = 0, \rho \odot \rho(y) = 0.6, \rho \odot \rho(z) = 0.4.$$

(1) We define an L-fuzzy topology $\mathcal{T} : L^X \to L$ as follows

$$\mathcal{T}(\lambda) = \begin{cases} 1, & \text{if } \lambda = 1_X,\ \lambda = 0_X, \\ 0.6, & \text{if } \lambda = \rho, \\ 0.3, & \text{if } \lambda = \rho \odot \rho, \\ 0, & \text{otherwise.} \end{cases}$$

From Theorem 3.3, we obtain an L-fuzzy closure operator $\mathcal{C}_{\mathcal{T}} : L^X \to L^X$ as follows

$$\begin{aligned} &\mathcal{C}_{\mathcal{T}}(\lambda) = \bigwedge_{\mu \in L^X} (\mathcal{T}(\mu) \to (S(\lambda, \mu^*) \to \mu^*)) \\ &= (S(\lambda, 0_X) \to 0_X) \wedge (0.6 \to (S(\lambda, \rho^*) \to \rho^*)) \wedge (0.3 \to (S(\lambda, \rho^* \oplus \rho^*) \to \rho^* \oplus \rho^*)). \end{aligned}$$

For $\lambda_1 = (0.9, 0.4, 0.2)$ and $\lambda_2 = (1, 0, 0)$,

$$\begin{aligned} \mathcal{C}_{\mathcal{T}}(\lambda_1) \quad &= (S(\lambda, 0_X) \to 0_X) \wedge (0.6 \to (S(\lambda, \rho^*) \to \rho^*)) \\ &\wedge (0.3 \to (S(\lambda, \rho^* \oplus \rho^*) \to \rho^* \oplus \rho^*)) = (0.9, 0.8, 0.9) \\ \mathcal{C}_{\mathcal{T}}(\lambda_2) \quad &= 0.6 \to (0.7 \to \rho^*) = (1, 0.7, 0.8), \end{aligned}$$

$$\begin{aligned} \mathcal{T}_{\mathcal{C}_{\mathcal{T}}}(\lambda_1^*) \quad &= S(\mathcal{C}_{\mathcal{T}}(\lambda_1), \lambda_1) = 0.3 \geq \mathcal{T}(\lambda_1^*) = 0 \\ \mathcal{T}_{\mathcal{C}_{\mathcal{T}}}(\lambda_2^*) \quad &= S(\mathcal{C}_{\mathcal{T}}(\lambda_2), \lambda_2) = 0 = \mathcal{T}(\lambda_2^*). \end{aligned}$$

(2) We define an L-fuzzy closure operator $\mathcal{C} : L^X \to L^X$ as follows

$$\mathcal{C}(\lambda) = \begin{cases} 0_X, & \text{if } \lambda = 0_X, \\ \rho, & \text{if } 0_X \neq \lambda \leq \rho, \\ \rho \oplus \rho, & \text{if } \rho \not\geq \lambda \leq \rho \oplus \rho, \\ 1_X, & \text{otherwise.} \end{cases}$$

$\mathcal{C}$ is not stratified because

$$\mathcal{C}(0.9 \to \rho) = \mathcal{C}((0.2, 0.9, 0.8)) = (1, 1, 1) \not\leq 0.9 \to \mathcal{C}(\rho) = (0.2, 0.9, 0.8).$$

From Theorem 3.5, we obtain an L-fuzzy topology $\mathcal{T}_{\mathcal{C}} : L^X \to L$ as follows

$$\mathcal{T}_{\mathcal{C}}(\lambda) = \begin{cases} 1_X, & \text{if } \lambda = 1_X, \\ S(\rho, \lambda^*), & \text{if } \lambda \geq \rho^*, \\ S(\rho \oplus \rho, \lambda^*), & \text{if } \rho^* \not\leq \lambda \geq \rho^* \odot \rho^*, \\ S(1_X, \lambda^*), & \text{otherwise.} \end{cases}$$

Moreover,

$$\begin{aligned} &\mathcal{C}_{\mathcal{T}_{\mathcal{C}}}(\lambda) = (S(\lambda, 0_X) \to 0_X) \wedge (\wedge_{\mu \geq \rho^*} S(\rho, \mu^*) \to (S(\lambda, \mu^*) \to \mu^*) \\ &\wedge (\wedge_{\rho^* \not\leq \mu \geq \rho^* \odot \rho^*} S(\rho \oplus \rho, \mu^*) \to (S(\lambda, \mu^*) \to \mu^*) \\ &\wedge (\wedge_{otherwise} S(1_X, \mu^*) \to (S(\lambda, \mu^*) \to \mu^*) \end{aligned}$$

$\mathcal{C}_{\mathcal{T}_\mathcal{C}}(\lambda_1) = (S(\lambda_1, 0_X) \to 0_X) \wedge (S(\rho,\rho) \to (S(\lambda,\rho) \to \rho)$
$\wedge(S(\rho\oplus\rho, \rho\oplus\rho) \to (S(\lambda, \rho\oplus\rho) \to \rho\oplus\rho)$
$\wedge(S(1_X, \rho\oplus\rho) \to (S(\lambda, \rho\oplus\rho) \to \rho\oplus\rho)$
$= (0.9, 0.9, 0.9) \wedge (0.9, 1, 1) \wedge (0.9, 1, 1) \wedge (1, 1, 1) = (0.9, 0.9, 0.9)$
$\leq \mathcal{C}(\lambda_1) = (1, 1, 1).$
$\mathcal{C}_{\mathcal{T}_\mathcal{C}}(\lambda_2) = (1, 0.9, 0.9) \leq \mathcal{C}(\lambda_2) = (1, 1, 1).$

4. L-fuzzy uniformities induced by L-fuzzy closure spaces

Lemma 4.1. For every $\rho \in L^X$, we define $u_\rho, u_\rho^{-1} : L^{X\times X} \to L$ by

$$u_\rho(x,y) = \rho(x) \to \rho(y)$$

$$u_\rho^{-1}(x,y) = u_\rho(y,x).$$

then we have the following statements

(1) $1_{X\times X} = u_{0_X} = u_{1_X}$,

(2) $1_\triangle \leq u_\rho$,

(3) For every $u_\rho \in L^{X\times X}$, $u_\rho \circ u_\rho = u_\rho$.

(4) $u_{\rho_1} \odot u_{\rho_2} \leq u_{\rho_1\odot\rho_2}$,

(5) $u_{\rho_1} \odot u_{\rho_2} \leq u_{\rho_1\oplus\rho_2}$

(6) $u_\rho^{-1} = u_{\rho^*}$

(7) $u^{-1}_{\rho_1\odot\rho_2} = u_{\rho_1^*\oplus\rho_2^*}$

Proof.

(1) Since $u_{0_X}(x,y) = 0_X(x) \to 0_X(y) = 1$ and $u_{1_X}(x,y) = 1_X(x) \to 1_X(y) = 1$, we have

$$1_{X\times X} = u_{0_X} = u_{1_X}.$$

(2) Since $u\rho(x,x) = \rho(x) \to \rho(x) = 1$, then $1_\triangle \leq u_\rho$.

(3)

$$\begin{aligned} u_\rho \circ u_\rho(x,z) &= \bigvee_{y\in X}(u_\rho(x,y) \odot u_\rho(y,z)) \\ &= \bigvee_{y\in X}((\rho(x) \to \rho(y)) \odot (\rho(y) \to \rho(z))) \\ &= \rho(x) \to \rho(z) = u_\rho(x,z) \end{aligned}$$

(4) Since $(\rho_1(x) \to \rho_1(y)) \odot (\rho_2(x) \to \rho_2(y)) \leq (\rho_1(x) \odot \rho_2(x) \to \rho_1(y) \odot \rho_2(y))$, we have $u_{\rho_1} \odot u_{\rho_2} \leq u_{\rho_1\odot\rho_2}$.

(5) Since $(\rho_1(x) \to \rho_1(y)) \odot (\rho_2(x) \to \rho_2(y)) = (\rho_1^*(y) \to \rho_1^*(x)) \odot (\rho_2^*(y) \to \rho_2^*(x)) \leq (\rho_1^*(y) \odot \rho_2^*(y) \to \rho_1^*(x) \odot \rho_2^*(x)) = (\rho_1^*(x) \odot \rho_2^*(x))^* \to (\rho_1^*(y) \odot \rho_2^*(y))^*$, we have $u_{\rho_1} \odot u_{\rho_2} \leq u_{\rho_1\oplus\rho_2}$.

(6) $u^{-1}_{\rho}(x,y) = \rho(y) \to \rho(x) = \rho^*(x) \to r^*(y) = u_{\rho^*}(x,y)$.

(7) $u^{-1}_{\rho_1 \odot \rho_2}(x,y) = (\rho_1 \odot \rho_2)(y) \to (\rho_1 \odot \rho_2)(x) = (\rho_1 \odot \rho_2)^*(x) \to (\rho_1 \odot \rho_2)^*(y) = u_{\rho_1^* \oplus \rho_2^*}$.

In the following theorem, we obtain an L-fuzzy quasi-uniform structure from an L-fuzzy closure operator.

Theorem 4.2 Let $(X,\mathcal{C})$ be an L-fuzzy closure space. Define $\mathcal{U}_{\mathcal{C}} : L^{X\times X} \to L$ by

$$\mathcal{U}_{\mathcal{C}}(u) = \bigvee \{ \bigvee_{x\in X} \odot_{i=1}^n \mathcal{C}^*(\lambda_i)(x) \mid \odot_{i=1}^n u_{\lambda_i} \le u \},$$

where $\bigvee$ is taken over every finite family $\{u_{\lambda_i} \mid i = 1,2,3,....,n\}$. Then $\mathcal{U}_{\mathcal{C}}$ is an L-fuzzy quasi-uniformity on X.

Proof. (QU1) Since $\mathcal{C}(0_X) = 0_X$, there exists $1_{X\times X} = u_{0_X} \in L^{X\times X}$ such that $\mathcal{U}_{\mathcal{C}}(1_{X\times X}) = 1$.

(QU2) It is trivial from the definition of $\mathcal{U}_{\mathcal{C}}$.

(QU3) Suppose there exist $u, v \in L^{X\times X}$ such that

$$\mathcal{U}_{\mathcal{C}}(u \odot v) \not\ge \mathcal{U}_{\mathcal{C}}(u) \odot \mathcal{U}_{\mathcal{C}}(v).$$

There exist two finite families $\{\lambda_i \in L^X \mid \odot_{i=1}^m u_{\lambda_i} \le u\}$ and $\{\rho_j \in L^X \mid \odot_{j=1}^n u_{\rho_j} \le v\}$ such that

$$\mathcal{U}_{\mathcal{C}}(u \odot v) \not\ge \bigvee_{x\in X} (\odot_{i=1}^m \mathcal{C}^*(\lambda_i)(x)) \odot \bigvee_{x\in X} (\odot_{j=1}^n \mathcal{C}^*(\rho_i)(x)).$$

On the other hand, since $u \odot v \ge (\odot_{i=1}^m u_{\lambda_i}) \odot (\odot_{j=1}^n u_{\rho_j})$, we have

$$\mathcal{U}_{\mathcal{C}}(u \odot v) \ge \bigvee_{x\in X} (\odot_{i=1}^m \mathcal{C}^*(\lambda_i)(x)) \odot \bigvee_{x\in X} (\odot_{j=1}^n \mathcal{C}^*(\rho_i)(x)).$$

It is a contradiction.

(QU4) Let $\mathcal{U}_{\mathcal{C}}(u) \ne 0$. Then there exists a finite family $\{\lambda_i \in L^X \mid \odot_{i=1}^m u_{\lambda_i} \le u\}$ such that

$$\mathcal{U}_{\mathcal{C}}(u) = \bigvee_{x\in X} (\odot_{i=1}^m \mathcal{C}^*(\lambda_i)(x)) \ne 0.$$

Since $u_{\lambda_i} \ge \top_\triangle$ from Lemma 4.1(2),

$$\top_\triangle \le \odot_{i=1}^m u_{\lambda_i} \le u.$$

(QU5) Suppose there exists $u \in L^{X\times X}$ such that

$$\bigvee \{\mathcal{U}_{\mathcal{C}}(v) \mid v \circ v \le u\} \not\ge \mathcal{U}_{\mathcal{C}}(u).$$

There exists a finite family $\{\rho_i \in L^{X\times X} \mid \odot_{i=1}^m u_{\rho_i} \le u\}$ such that

$$\bigvee \{\mathcal{U}_{\mathcal{C}}(v) \mid v \circ v \le u\} \not\ge \bigvee_{x\in X} (\odot_{i=1}^m \mathcal{C}^*(\rho_i)(x)).$$

On the other hand, since $u_{\rho_i} \circ u_{\rho_i} = u_{\rho_i}$ for each $i \in \{1, ..., m\}$ from Lemma 4.1(3), we have $(\odot_{i=1}^m u_{\rho_i}) \circ (\odot_{i=1}^m u_{\rho_i}) \leq \odot_{i=1}^m u_{\rho_i}$ from

$$\begin{aligned}
&\bigvee_{y\in X}((\odot_{i=1}^m u_{\rho_i}(x,y)) \odot (\odot_{i=1}^m u_{\rho_i}(y,z))) \\
&= \bigvee_{y\in X}((\odot_{i=1}^m(\rho_i(x) \to \rho_i(y)) \odot (\odot_{i=1}^m(\rho_i(y) \to \rho_i(z)))) \\
&= \bigvee_{y\in X}((\odot_{i=1}^m(\rho_i(x) \to \rho_i(y)) \odot (\rho_i(y) \to \rho_i(z)))) \\
&\leq \odot_{i=1}^m(\rho_i(x) \to \rho_i(z)).
\end{aligned}$$

Put $v = \odot_{i=1}^m u_{\rho_i}$. Since $\odot_{i=1}^m u_{\rho_i} \leq v$, $v \circ v \leq u$ and

$$\bigvee\{\mathcal{U}_{\mathcal{C}}(w) \mid w \circ w \leq u\} \geq \mathcal{U}_{\mathcal{C}}(v) \geq \bigvee_{x\in X}(\odot_{i=1}^m \mathcal{C}^*(\rho_i)(x)).$$

It is a contradiction. Thus $\bigvee\{\mathcal{U}_{\mathcal{C}}(v) \mid v \circ v \leq u\} \geq \mathcal{U}_{\mathcal{C}}(u)$. Hence $\mathcal{U}_{\mathcal{C}}$ is an L-fuzzy quasi-uniformity on X.

Theorem 4.3. Let $(X, \mathcal{C}_X)$ and $(Y, \mathcal{C}_Y)$ be L-fuzzy closure spaces. Let $\phi : X \to Y$ be a surjective map. Then the following properties hold.

(1) $\mathcal{U}_{\mathcal{C}_Y}(v) \to \mathcal{U}_{\mathcal{C}_X}((\phi\times\phi)^{\leftarrow}(v)) \geq \bigwedge\{\odot_{i=1}^m S(\mathcal{C}_X(\phi^{\leftarrow}(\rho_i)), \phi^{\leftarrow}(\mathcal{C}_Y(\rho_i))) \mid \odot_{i=1}^m v_{\rho_i} \leq v\}$ for each $v \in L^{X\times X}$.

(2) Let $\phi : (X, \mathcal{C}_X) \to (Y, \mathcal{C}_Y)$ be a surjective C-map. Then $\phi : (X, \mathcal{U}_{\mathcal{C}_X}) \to (Y, \mathcal{U}_{\mathcal{C}_Y})$ is fuzzy uniformly continuous.

Proof. Since $v(\phi(x), \phi(y)) = (\phi\times\phi)^{\leftarrow}(v)(x,y)$ and $(\phi\times\phi)^{\leftarrow}(v_\lambda)(x,y) = v_{\phi^{\leftarrow}(\lambda)}(x,y)$, we have

$$\begin{aligned}
&\mathcal{U}_{\mathcal{C}_Y}(v) \to \mathcal{U}_{\mathcal{C}_X}((\phi\times\phi)^{\leftarrow}(v)) \\
&= \bigvee\{\odot_{i=1}^m \bigvee_{y\in Y} \mathcal{C}_Y^*(\rho_i)(y) \mid \odot_{i=1}^m v_{\rho_i} \leq v\} \\
&\to \bigvee\{\odot_{j=1}^n \bigvee_{x\in X} \mathcal{C}_X^*(\mu_i)(x) \mid \odot_{j=1}^n u_{\mu_j} \leq (\phi\times\phi)^{\leftarrow}(v)\} \\
&\geq \bigvee\{\odot_{i=1}^m \bigvee_{x\in X} \mathcal{C}_Y^*(\rho_i)(\phi(x)) \mid \odot_{i=1}^m v_{\rho_i} \leq v\} \\
&\to \bigvee\{\odot_{i=1}^m \bigvee_{x\in X} \mathcal{C}_X^*(\phi^{\leftarrow}(\rho_i))(x) \mid \odot_{j=1}^n (\phi\times\phi)^{\leftarrow}(\rho_i) \leq (\phi\times\phi)^{\leftarrow}(v)\} \\
&\geq \bigvee\{\odot_{i=1}^m \bigvee_{x\in X} \mathcal{C}_Y^*(\rho_i)(\phi(x)) \mid \odot_{i=1}^m v_{\rho_i} \leq v\} \\
&\to \bigvee\{\odot_{i=1}^m \bigvee_{x\in X} \mathcal{C}_X^*(\phi^{\leftarrow}(\rho_i))(x) \mid \odot_{j=1}^n v_{\phi^{\leftarrow}(\rho_i)} \leq (\phi\times\phi)^{\leftarrow}(v)\} \\
&\geq \bigwedge\{\odot_{i=1}^m \bigvee_{x\in X} \mathcal{C}_Y^*(\rho_i)(\phi(x)) \to \odot_{i=1}^m \bigvee_{x\in X} \mathcal{C}_X^*(\phi^{\leftarrow}(\rho_i))(x) \mid \odot_{i=1}^m v_{\rho_i} \leq v\} \\
&\geq \bigwedge\{\odot_{i=1}^m \bigwedge_{x\in X}(\mathcal{C}_Y^*(\rho_i)(\phi(x)) \to \mathcal{C}_X^*(\phi^{\leftarrow}(\rho_i))(x)) \mid \odot_{i=1}^m v_{\rho_i} \leq v\} \\
&= \bigwedge\{\odot_{i=1}^m \bigwedge_{x\in X}(\mathcal{C}_X(\phi^{\leftarrow}(\rho_i))(x) \to \mathcal{C}_Y(\rho_i)(\phi(x)) \mid \odot_{i=1}^m v_{\rho_i} \leq v\}. \\
&= \bigwedge\{\odot_{i=1}^m S(\mathcal{C}_X(\phi^{\leftarrow}(\rho_i)), \phi^{\leftarrow}(\mathcal{C}_Y(\rho_i))) \mid \odot_{i=1}^m v_{\rho_i} \leq v\}
\end{aligned}$$

(2) Since $\mathcal{C}_X(\phi^{\leftarrow}(\rho_i)) \leq \phi^{\leftarrow}(\mathcal{C}_Y(\rho_i))$, by (1), $\mathcal{U}_{\mathcal{C}_Y}(v) \leq \mathcal{U}_{\mathcal{C}_X}((\phi\times\phi)^{\leftarrow}(v))$.

Theorem 4.4 Let $(X,\mathcal{U})$ be an L-fuzzy quasi-uniform space. Define $\mathcal{C}_{\mathcal{U}} : L^X \to L^X$ by

$$\mathcal{C}_{\mathcal{U}}(\lambda)(x) = \bigwedge \Big(\mathcal{U}(u) \to \bigvee_{y \in X} (u(x,y) \odot \lambda(y)) \Big).$$

Then $\mathcal{C}_{\mathcal{U}}$ is a stratified L-fuzzy closure operator on X.

Proof. (C1)

$$\mathcal{C}_{\mathcal{U}}(\perp_X) = \bigwedge \Big(\mathcal{U}(u) \to \bigvee_{y \in X} (u(x,y) \odot \perp_X(y)) \Big) = (\bigvee_u \mathcal{U}(u))^* = \perp_X.$$

(C2)

$$\begin{aligned}
S(\lambda, \mathcal{C}_{\mathcal{U}}(\lambda)) &= \bigwedge_{x \in X} (\lambda(x) \to \mathcal{C}_{\mathcal{U}}(\lambda)(x)) \\
&= \bigwedge_{x \in X} \Big(\lambda(x) \to \bigwedge \Big(\mathcal{U}(u) \to \bigvee_{y \in X} (u(x,y) \odot \lambda(y)) \Big) \Big) \\
&= \bigwedge_{x \in X} \bigwedge \Big((\lambda(x) \odot \mathcal{U}(u)) \to \bigvee_{y \in X} (u(x,y) \odot \lambda(y)) \Big) \\
&\geq \bigwedge_{x \in X} \bigwedge \Big((\lambda(x) \odot \mathcal{U}(u)) \to (u(x,x) \odot \lambda(x)) \Big) \\
&\geq \bigwedge_{x \in X} \bigwedge \Big(\mathcal{U}(u) \to u(x,x) \Big) = \top
\end{aligned}$$

That is, $\mathcal{C}_{\mathcal{U}}(\lambda) \geq \lambda$ for each $\lambda \in L^X$.

(C3) and, by Lemma 3.2, $\mathcal{C}_{\mathcal{U}}$ is stratified from

$$\begin{aligned}
&S(\mathcal{C}_{\mathcal{U}}(\lambda), \mathcal{C}_{\mathcal{U}}(\mu)) = \bigwedge_{x \in X} \mathcal{C}_{\mathcal{U}}(\lambda)(x) \to \mathcal{C}_{\mathcal{U}}(\mu)(x) \\
&= \bigwedge_{x \in X} \Big(\bigwedge \Big(\mathcal{U}(u) \to \bigvee_{y \in X} (u(x,y) \odot \lambda(y)) \Big) \to \bigwedge \Big(\mathcal{U}(u) \to \bigvee_{y \in X} (u(x,y) \odot \mu(y)) \Big) \\
&\geq \bigwedge_{x \in X} \bigwedge \bigwedge_{y \in X} \Big(u(x,y) \odot \lambda(y) \to u(x,y) \odot \mu(y) \Big) \\
&\geq \bigwedge_{y \in X} (\lambda(y) \to \mu(y)) = S(\lambda, \mu)
\end{aligned}$$

(C4), Since

$$(a \odot c) \odot (b \oplus d)^* = (a \odot b^*) \odot (c \odot d^*)$$

$$\Leftrightarrow (a \odot c) \to (b \oplus d) = (a \to b) \oplus (c \to d),$$

$$\begin{aligned}
&\mathcal{C}_{\mathcal{U}}(\lambda)(x) \oplus \mathcal{C}_{\mathcal{U}}(\mu)(x) \\
&= \bigwedge_{u} \Big(\mathcal{U}(u) \to \bigvee_{y\in X} (u(x,y) \odot \lambda(y))\Big) \oplus \bigwedge_{v} \Big(\mathcal{U}(v) \to \bigvee_{y\in X} (v(x,y) \odot \mu(y))\Big) \\
&= \bigwedge_{u,v} \Big(\mathcal{U}(u) \odot \mathcal{U}(v) \to \bigvee_{y\in X} (u(x,y) \odot \lambda(y))\Big) \oplus \bigvee_{y\in X} (v(x,y) \odot \mu(y))\Big) \\
&= \bigwedge_{u,v} \Big(\mathcal{U}(u \odot v) \to \bigvee_{y\in X} ((u \odot v)(x,y) \odot (\lambda \oplus \mu)(y))\Big) \\
&\geq \mathcal{C}_{\mathcal{U}}(\lambda \oplus \mu).
\end{aligned}$$

Theorem 4.5. Let $(X,\mathcal{U}_X)$ and $(Y,\mathcal{U}_Y)$ be L-fuzzy quasi-uniform spaces and t $\phi : X \to Y$ be a surjective map. Then we have the following properties.

(1) $\bigwedge \Big(\mathcal{U}_Y(v) \to \mathcal{U}_X((\phi\times\phi)^{\leftarrow}(v))\Big) \leq S(\mathcal{C}_{\mathcal{U}_X}(\phi^{\leftarrow}(\rho)), \phi^{\leftarrow}(\mathcal{C}_{\mathcal{U}_Y}(\rho)))$.

(2) If $\phi : (X,\mathcal{U}_X) \to (Y,\mathcal{U}_Y)$ is fuzzy uniformly continuous, then $\phi : (X,\mathcal{C}_{\mathcal{U}_X}) \to (Y,\mathcal{C}_{\mathcal{U}_Y})$ is a C-map.

Proof. (1)

$$\begin{aligned}
&S(\mathcal{C}_{\mathcal{U}_X}(\phi^{\leftarrow}(\rho)), \phi^{\leftarrow}(\mathcal{C}_{\mathcal{U}_Y}(\rho))) \\
&= \bigwedge_{x\in X} \Big(\mathcal{C}_{\mathcal{U}_X}(\phi^{\leftarrow}(\rho))(x) \to \phi^{\leftarrow}(\mathcal{C}_{\mathcal{U}_Y}(\rho))(x)\Big) \\
&= \bigwedge_{x\in X} \Big(\bigwedge \Big(\mathcal{U}_X(u) \to \bigvee_{z\in X}(u(x,z) \odot \phi^{\leftarrow}(\rho)(z))\Big) \\
&\to \bigwedge \Big(\mathcal{U}_Y(v) \to \bigvee_{y\in Y}(v(\phi(x),y) \odot \rho(y))\Big)\Big) \\
&\geq \bigwedge_{x\in X} \Big(\bigwedge \Big(\mathcal{U}_X((\phi\times\phi)^{\leftarrow}(v)) \to \bigvee_{z\in X}((\phi\times\phi)^{\leftarrow}(v)(x,z) \odot \phi^{\leftarrow}(\rho)(z))\Big) \\
&\to \bigwedge \Big(\mathcal{U}_Y(v) \to \bigvee_{y\in Y}(v(\phi(x),y) \odot \rho(y))\Big)\Big) \\
&\geq \bigwedge_{x\in X} \bigwedge_{v} \Big(\Big(\mathcal{U}_X((\phi\times\phi)^{\leftarrow}(v)) \to \bigvee_{z\in X}((\phi\times\phi)^{\leftarrow}(v)(x,z) \odot \phi^{\leftarrow}(\rho)(z))\Big) \\
&\to \Big(\mathcal{U}_Y(v) \to \bigvee_{z\in X}(v(\phi(x),\phi(z)) \odot \rho(\phi(z)))\Big)\Big) \\
&= \bigwedge \Big(\mathcal{U}_Y(v) \to \mathcal{U}_X((\phi\times\phi)^{\leftarrow}(v))\Big).
\end{aligned}$$

(2) Let $\mathcal{U}_Y(v) \leq \mathcal{U}_X((\phi\times\phi)^{\leftarrow}(v))$ for $v \in L^{X\times X}$. By (1), $\mathcal{C}_{\mathcal{U}_X}(\phi^{\leftarrow}(\rho)) \leq \phi^{\leftarrow}(\mathcal{C}_{\mathcal{U}_Y}(\rho))$ for all $\rho \in L^Y$.

Example 4.6. Let $(L = [0,1], \odot, \to)$ be a complete residuated lattice as in Example 3.10. Let $X = \{x, y, z\}$ be a set and $\rho, \rho \odot \rho \in L^X$ such that

$$\rho(x) = 0.4, \rho(y) = 0.6, \rho(z) = 0.7,$$

$$\rho \oplus \rho(x) = 0.8, \rho \oplus \rho(y) = 1, \rho \oplus \rho(z) = 1.$$

We define an L-fuzzy closure operator $\mathcal{C} : L^X \to L^X$ as follows

$$\mathcal{C}(\lambda) = \begin{cases} 0_X, & \text{if } \lambda = 0_X, \\ \rho, & \text{if } 0_X \neq \lambda \leq \rho, \\ \rho \oplus \rho, & \text{if } \rho \not\geq \lambda \leq \rho \oplus \rho, \\ 1_X, & \text{otherwise.} \end{cases}$$

From Lemma 4.1, we obtain

$$u_\rho = \begin{pmatrix} 1 & 1 & 1 \\ 0.8 & 1 & 1 \\ 0.7 & 0.9 & 1 \end{pmatrix} u_{\rho\oplus\rho} = \begin{pmatrix} 1 & 1 & 1 \\ 0.8 & 1 & 1 \\ 0.8 & 1 & 1 \end{pmatrix}.$$

$$u_\rho \odot u_\rho = \begin{pmatrix} 1 & 1 & 1 \\ 0.6 & 1 & 1 \\ 0.4 & 0.8 & 1 \end{pmatrix}.$$

From Theorem 4.2, we obtain an L-fuzzy quasi-uniformity $\mathcal{U}_{\mathcal{C}} : L^{X\times X} \to L$ as follows

$$\mathcal{U}_{\mathcal{C}}(u) = \begin{cases} 1, & \text{if } u = 1_{X\times X},\ \lambda = 0_X, \\ 0.6, & \text{if } u_\rho \leq u \neq 1_{X\times X}, \\ 0.2, & \text{if } u_{\rho\odot\rho} \leq u \not\geq u_\rho, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, from Theorem 4.4, we obtain a stratified L-fuzzy closure operator $\mathcal{C}_{\mathcal{U}_{\mathcal{C}}} : L^X \to L^X$ as follows

$$\begin{aligned}
\mathcal{C}_{\mathcal{U}_{\mathcal{C}}}(\lambda)(x) &= (\textstyle\bigvee_{y\in X} \lambda(y)) \wedge (0.6 \to (\lambda(x) \vee \lambda(y) \vee \lambda(z))) \\
&\wedge (0.2 \to (\lambda(x) \vee \lambda(y) \vee \lambda(z))) \\
\mathcal{C}_{\mathcal{U}_{\mathcal{C}}}(\lambda)(y) &= (\textstyle\bigvee_{y\in X} \lambda(y)) \wedge (0.6 \to ((0.8 \odot \lambda(x)) \vee \lambda(y) \vee \lambda(z))) \\
&\wedge (0.2 \to ((0.6 \odot \lambda(x)) \vee \lambda(y) \vee \lambda(z))) \\
\mathcal{C}_{\mathcal{U}_{\mathcal{C}}}(\lambda)(z) &= (\textstyle\bigvee_{y\in X} \lambda(y)) \wedge (0.6 \to ((0.7 \odot \lambda(x)) \vee (0.9 \odot \lambda(y)) \vee \lambda(z))) \\
&\wedge (0.2 \to ((0.4 \odot \lambda(x)) \vee (0.8 \odot \lambda(y)) \vee \lambda(z))).
\end{aligned}$$

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SOFT L-NEIGHBORHOOD SYSTEMS AND SOFT L-FUZZY QUASI-UNIFORM SPACES

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Abstract. In this paper, we introduce soft L-neighborhood spaces and investigate the topological properties of soft L-fuzzy quasi-uniformity in complete residuated lattices. We obtain soft L-fuzzy topologies and soft L-neighborhood spaces induced by soft L-fuzzy quasi-uniformity. Moreover, we study the relations among soft L-fuzzy topology, soft L-neighborhood system and soft L-fuzzy quasi-uniformity.

Keywords: Complete residuated lattice, soft L-neighborhood systems, soft L-fuzzy topologies, soft L-fuzzy quasi-uniformities, (uniformly) continuous soft maps

2000 AMS Subject Classification: 54A40; 03E72; 03G10; 06A15

1. Introduction

Molodtsov [19] introduced the soft set as a mathematical tool for dealing information as the uncertainty of data in engineering, physics, computer sciences and many other diverse field. Presently, the soft set theory is making progress rapidly [1,3,6,11,15,16,28]. Pawlak's rough set [20,21] can be viewed as a special case of soft rough sets [6]. The topological structures of soft sets have been developed by many researchers [3,25,26,30,31].

On the other hand, Hájek [7] introduced a complete residuated lattice which is an algebraic structure for many valued logic. It is an important mathematical tool for algebraic structure of fuzzy contexts [2,8,12-14,23,24,27].

Kim [13,14] introduced a fuzzy soft $F : A \to L^U$ as an extension as the soft $F : A \to P(U)$ where L is a complete residuated lattice. He introduced soft L-fuzzy topologies, soft L-fuzzy quasi-uniformities and soft L-fuzzy topogenous orders in complete residuated lattices.

In this paper, we introduce soft L-neighborhood spaces and investigate the topological properties of soft L-fuzzy quasi-uniformity in complete residuated lattices. We obtain soft L-fuzzy topologies and soft L-neighborhood spaces induced by soft L-fuzzy quasi-uniformity. Moreover, we study

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the relations among soft L-fuzzy topology, soft L-neighborhood system and soft L-fuzzy quasi-uniformity. We give their examples.

2. Preliminaries

Definition 2.1. [2,8,9,30] An algebra $(L, \wedge, \vee, \odot, \to, 0, 1)$ is called a complete residuated lattice if it satisfies the following conditions:

(C1) $L = (L, \leq, \vee, \wedge, 1, 0)$ is a complete lattice with the greatest element 1 and the least element 0;

(C2) $(L, \odot, 1)$ is a commutative monoid;

(C3) $x \odot y \leq z$ iff $x \leq y \to z$ for $x, y, z \in L$.

In this paper, we assume that $(L, \leq, \odot, \to, \oplus, ^*)$ is a complete residuated lattice with an order reversing involution $x^* = x \to 0$ which is defined by $x \oplus y = (x^* \odot y^*)^*$ unless otherwise specified and we denote $L_0 = L - \{0\}$.

Lemma 2.2. [2,8,9,30] For each $x, y, z, x_i, y_i, w \in L$, we have the following properties.

(1) $1 \to x = x$, $0 \odot x = 0$,

(2) If $y \leq z$, then $x \odot y \leq x \odot z$, $x \oplus y \leq x \oplus z$, $x \to y \leq x \to z$ and $z \to x \leq y \to x$,

(3) $x \odot y \leq x \wedge y \leq x \vee y \leq x \oplus y$,

(4) $(\bigwedge_i y_i)^* = \bigvee_i y_i^*$, $(\bigvee_i y_i)^* = \bigwedge_i y_i^*$,

(5) $x \odot (\bigvee_i y_i) = \bigvee_i (x \odot y_i)$,

(6) $x \oplus (\bigwedge_i y_i) = \bigwedge_i (x \oplus y_i)$,

(7) $x \to (\bigwedge_i y_i) = \bigwedge_i (x \to y_i)$,

(8) $(\bigvee_i x_i) \to y = \bigwedge_i (x_i \to y)$,

(9) $x \to (\bigvee_i y_i) \geq \bigvee_i (x \to y_i)$,

(10) $(\bigwedge_i x_i) \to y \geq \bigvee_i (x_i \to y)$,

(11) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$,

(12) $x \odot (x \to y) \leq y$ and $x \to y \leq (y \to z) \to (x \to z)$,

(13) $(x \to y) \odot (z \to w) \leq (x \odot z) \to (y \odot w)$,

(14) $(x \to y) \odot (z \to w) \leq (x \oplus z) \to (y \oplus w)$,

(15) $x \to y \leq (x \odot z) \to (y \odot z)$ and $(x \to y) \odot (y \to z) \leq x \to z$,

(16) $(x \oplus z) \odot y \leq x \oplus (y \odot z)$,

(17) $x \to y = y^* \to x^*$.

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Definition 2.3. [13,14] Let X be an initial universe of objects and E the set of parameters (attributes) in X. A pair (F, A) is called a *fuzzy soft set* over X, where $A \subset E$ and $F : A \to L^X$ is a mapping. We denote $S(X, A)$ as the family of all fuzzy soft sets under the parameter A.

Definition 2.4. [13,14] Let (F, A) and (G, A) be two fuzzy soft sets over a common universe X.

(1) (F, A) is a fuzzy soft subset of (G, A), denoted by $(F, A) \le (G, A)$ if $F(a) \le G(a)$, for each $a \in A$.

(2) $(F, A) \wedge (G, A) = (F \wedge G, A)$ if $(F \wedge G)(a) = F(a) \wedge G(a)$ for each $a \in A$.

(3) $(F, A) \vee (G, A) = (F \vee G, A)$ if $(F \vee G)(a) = F(a) \vee G(a)$ for each $a \in A$.

(4) $(F, A) \odot (G, A) = (F \odot G, A)$ if $(F \odot G)(a) = F(a) \odot G(a)$ for each $a \in A$.

(5) $(F, A)^* = (F^*, A)$ if $F^*(a) = (F(a))^*$ for each $a \in A$.

(6) $(F, A) \oplus (G, A) = (F \oplus G, A)$ if $(F \oplus G)(a) = (F^*(a) \odot G^*(a))^*$ for each $a \in A$.

(7) $\alpha \odot (F, A) = (\alpha \odot F, A)$ for each $\alpha \in L$.

Definition 2.5. [13,14] Let $S(X, A)$ and $S(Y, B)$ be the families of all fuzzy soft sets over X and Y, respectively. The mapping $f_\phi : S(X, A) \to S(Y, B)$ is a soft mapping where $f : X \to Y$ and $\phi : A \to B$ are mappings.

(1) The image of $(F, A) \in S(X, A)$ under the mapping f_ϕ is denoted by $f_\phi((F, A)) = (f_\phi(F), B)$ where

$$f_\phi(F)(b)(y) = \begin{cases} \bigvee_{a \in \phi^{-1}(\{b\})} f^{\to}(F(a))(y), & \text{if } \phi^{-1}(\{b\}) \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

(2) The inverse image of $(G, B) \in S(Y, B)$ under the mapping f_ϕ is denoted by $f_\phi^{-1}((G, B)) = (f_\phi^{-1}(G), A)$ where

$$f_\phi^{-1}(G)(a)(x) = f^{\leftarrow}(G(\phi(a)))(x), \ \forall a \in A, x \in X.$$

(3) The soft mapping $f_\phi : S(X, A) \to S(Y, B)$ is called injective (resp. surjective, bijective) if f and ϕ are both injective (resp. surjective, bijective).

Lemma 2.6. [13,14] Let $f_\phi : S(X, A) \to S(Y, B)$ be a soft mapping. Then we have the following properties. For $(F, A), (F_i, A) \in S(X, A)$ and $(G, B), (G_i, B) \in S(Y, B)$,

(1) $(G, B) \ge f_\phi(f_\phi^{-1}((G, B)))$ with equality if f is surjective,

(2) $(F, A) \le f_\phi^{-1}(f_\phi((F, A)))$ with equality if f is injective,

(3) $f_\phi^{-1}((G, B)^*) = (f_\phi^{-1}((G, B)))^*$,

(4) $f_\phi^{-1}(\bigvee_{i \in I}(G_i, B)) = \bigvee_{i \in I} f_\phi^{-1}((G_i, B))$,

(5) $f_\phi^{-1}(\bigwedge_{i \in I}(G_i, B)) = \bigwedge_{i \in I} f_\phi^{-1}((G_i, B))$,

(6) $f_\phi(\bigvee_{i \in I}(F_i, A)) = \bigvee_{i \in I} f_\phi((F_i, A))$,

(7) $f_\phi(\bigwedge_{i \in I}(F_i, A)) \le \bigwedge_{i \in I} f_\phi((F_i, A))$ with equality if f is injective,

(8) $f_\phi^{-1}((G_1,B)\odot(G_2,B)) = f_\phi^{-1}((G_1,B))\odot f_\phi^{-1}((G_2,B))$,

(9) $f_\phi^{-1}((G_1,B)\oplus(G_2,B)) = f_\phi^{-1}((G_1,B))\oplus f_\phi^{-1}((G_2,B))$,

(10) $f_\phi((F_1,A)\odot(F_2,A)) \leq f_\phi((F_1,A))\odot f_\phi((F_2,A))$ with equality if f is injective,

(11) $f_\phi((F_1,A)\oplus(F_2,A)) \leq f_\phi((F_1,A))\oplus f_\phi((F_2,A))$ with equality if f is injective.

Definition 2.7. [13,14] A map $\mathcal{T} : S(X,A) \to L$ is called a soft L-fuzzy topology on X if it satisfies the following conditions.

(ST1) $\mathcal{T}((0_X,A)) = \mathcal{T}((1_X,A)) = 1$, where $0_X(a)(x) = 0, 1_X(a)(x) = 1$ for all $a \in A, x \in X$,

(ST2) $\mathcal{T}((F,A)\odot(G,A)) \geq \mathcal{T}((F,A))\odot\mathcal{T}((F,A))$,

(ST3) $\mathcal{T}(\bigvee_{i\in I}(F_i,A)) \geq \bigwedge_{i\in I}\mathcal{T}((F_i,A))$.

The triple $(X,A,\mathcal{T})$ is called a soft L-fuzzy topological space.

A soft L-fuzzy topology is called enriched if $\mathcal{T}(\alpha\odot(F,A)) \geq \mathcal{T}((F,A))$ for all $\alpha \in L$ and $(F,A) \in S(X,A)$.

Let $(X,A,\mathcal{T}_1)$ and $(X,A,\mathcal{T}_2)$ be soft L-fuzzy topological spaces. Then $\mathcal{T}_1$ is finer than $\mathcal{T}_2$ if $\mathcal{T}_2((F,A)) \leq \mathcal{T}_1((F,A))$, for all $(F,A) \in S(X,A)$.

Let $(X,A,\mathcal{T}_X)$ and $(Y,B,\mathcal{T}_Y)$ be soft L-fuzzy topological spaces and $f_\phi : S(X,A) \to S(Y,B)$ be a soft map. Then f_ϕ is called a continuous soft map if

$$\mathcal{T}_Y((G,B)) \leq \mathcal{T}_X(f_\phi^{-1}((G,B)),\ \forall (G,B) \in S(Y,B).$$

Definition 2.8. [13,14] A mapping $\mathcal{U} : S(X\times X,A) \to L$ is called a soft L-fuzzy quasi-uniformity on X iff it satisfies the properties.

(SU1) There exists $(U,A) \in S(X\times X,A)$ such that $\mathcal{U}((U,A)) = 1$,

(SU2) If $(V,A) \leq (U,A)$, then $\mathcal{U}((V,A)) \leq \mathcal{U}((U,A))$,

(SU3) For every $(U,A),(V,A) \in S(X\times X,A)$,

$$\mathcal{U}((U,A)\odot(U,A)) \geq \mathcal{U}((U,A))\odot\mathcal{U}((V,A))$$

(SU4) If $\mathcal{U}((U,A)) \neq 0$, then $(1_\triangle,A) \leq (U,A)$, where, for all $a \in A$,

$$1_\Delta(a)(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y, \end{cases}$$

(SU5) $\bigvee\{\mathcal{U}((V,A)) \mid (V,A)\circ(V,A) \leq (U,A)\} \geq \mathcal{U}((U,A))$

$$((V,A)\circ(V,A))(a)(x,y) = (V(a)\circ V(a))(x,y)$$
$$= \bigvee_{z\in X}(V(a)(z,x)\odot V(a)(x,y)),\ \forall\, x,y \in X, a \in A.$$

The triple $(X,A,\mathcal{U})$ is called a soft L-fuzzy quasi-uniform space.

A soft L-fuzzy quasi-uniformity $\mathcal{U}$ is called stratified if $\mathcal{U}(\alpha\odot(U,A)) \geq \alpha\odot\mathcal{U}((U,A))$ for all $\alpha \in L$ and $(U,A) \in S(X\times X,A)$.

Let $(X, A, \mathcal{U}_1)$ and $(Y, B, \mathcal{U}_2)$ be soft L-fuzzy quasi-uniform spaces and $(f \times f)_\phi$ be a soft map. Then $(f \times f)_\phi$ is called an uniformly continuous soft map if, for all $(V, B) \in S(Y \times Y, B)$,

$$\mathcal{U}_2((V, B)) \leq \mathcal{U}_1((f \times f)_\phi^{-1}((V, B))).$$

Definition 2.9. [14] Let X be a set. A function $e_X : X \times X \to L$ is called:

(E1) reflexive if $e_X(x, x) = 1$ for all $x \in X$,

(E2) transitive if $e_X(x, y) \odot e_X(y, z) \leq e_X(x, z)$, for all $x, y, z \in X$,

(E3) if $e_X(x, y) = e_X(y, x) = 1$, then $x = y$.

If e_X satisfies (E1) and (E2), e_X is a fuzzy preorder on X. If e_X satisfies (E1), (E2) and (E3), e_X is a fuzzy partially order on X.

Lemma 2.10. [14] For a given set X, define a binary mapping $e_X : S(X, A) \times S(X, A) \to L$ by

$$e_X((F, A), (G, A)) = \bigwedge_{a \in A} \bigwedge_{x \in X} (F(a)(x) \to G(a)(x)).$$

Then, for each $(F, A), (G, A), (H, A), (K, A) \in S(X, A)$ and $\alpha \in L$ the following properties hold.

(1) $(F, A) \leq (G, A)$ iff $e_X((F, A), (G, A)) = 1$.

(2) e_X is a fuzzy partially order on $S(X, A)$,

(3) If $(F, A) \leq (G, A)$, then

$$e_X((H, A), (F, A)) \leq e_X((H, A), (G, A)),$$
$$e_X((F, A), (H, A)) \geq e_X((G, A), (H, A)).$$

(4) $e_X((F, A), (G, A)) \odot e_X((K, A), (H, A)) \leq e_X((F, A) \odot (K, A), (G, A) \odot (H, A))$.

(5) $e_X((F, A), (G, A)) \odot e_X((K, A), (H, A)) \leq e_X((F, A) \oplus (K, A), (G, A) \oplus (H, A))$.

(6) $e_X((F, A), \alpha \to (G, A)) = e_X(\alpha \odot (F, A), (G, A)) = \alpha \to e_X((F, A), (G, A))$ and $\alpha \odot e_X((F, A), (G, A)) \leq e_X((F, A), \alpha \odot (G, A))$.

(7) $(G, A) \odot e_X((G, A), (F, A)) \leq (F, A)$ and $(G, A) \leq e_X((G, A), (F, A)) \to (F, A)$.

(8) $e_X((G, A), (H, A)) \leq e_X((F, A), (G, A)) \to e_X((F, A), (H, A))$.

(9) $e_X((F, A), (G, A)) \leq e_X((G, A), (H, A)) \to e_X((F, A), (H, A))$.

(10) if $x^* = x \to 0$, then $e_X((F, A), (G, A)) = e_X((G, A)^*, (F, A)^*)$.

(11) Let $f_\phi : (X, A) \to (Y, B)$ be a soft map. Then for $(F, A), (G, A) \in S(X, A)$ and $(H, A), (K, A) \in S(Y, B)$,

$$e_X((F, A), (G, A)) \leq e_Y(f_\phi((F, A)), f_\phi((G, A))),$$

$$e_Y((H, A), (K, A)) \leq e_X(f_\phi^{-1}((H, A)), f_\phi^{-1}((K, A))),$$

and the equalities hold if f_ϕ is bijective.

3. Soft L-neighborhood systems and soft L-fuzzy quasi-uniform spaces

Definition 3.1. [13,14] A map $N : X \to L^{S(X,A)}$ is called a soft L-neighborhood system on X if $N = \{N_x = N(x) \mid x \in X\}$ satisfies the following conditions

(SN1) $N_x((1_X, A)) = 1$ and $N_x((0_X, A)) = 0$,

(SN2) $N_x((F, A) \odot (G, A)) \geq N_x((F, A)) \odot N_x((G, A))$ for each $(F, A), (G, A) \in S(X, A)$,

(SN3) If $(F, A) \leq (G, A)$, then $N_x((F, A)) \leq N_x((G, A))$,

(SN4) $N_x((F, A)) \leq (F, A)(x)$ for all $(F, A) \in S(X, A)$ where $(F, A)(x) = F(-)(x)$.

A soft L-neighborhood system is called stratified if

(S) $N_x(\alpha \odot (F, A)) \geq \alpha \odot N_x((F, A))$ for all $(F, A) \in S(X, A)$ and $\alpha \in L$.

The triple (X, A, N) is called a soft L-neighborhood space.

Let (X, A, N) and (Y, B, M) be soft L-neighborhood spaces. A mapping $f_\phi : (X, A, N) \to (Y, B, M)$ is said to be a continuous soft map iff $M_{f(x)}((G, B)) \leq N_x(f_\phi^{-1}((G, B)))$ for each $x \in X, (G, B) \in S(Y, B)$.

Theorem 3.2. *Let $(X, A, \mathcal{U})$ be a soft L-fuzzy quasi-uniform space. Define a map $rN^{\mathcal{U}} : X \to L^{S(X,A)}$ by, $\forall$ $(F, A) \in S(X, A)$, $x \in X$,*

$$rN_x^{\mathcal{U}}((F, A)) = \bigvee_{(U,A)\in S(X\times X,A)} \mathcal{U}((U, A)) \odot e_X((U[x], A), (F, A)),$$

where $(U[x], A)(y) = U(-)(y, x)$. Then the following properties hold.

(1) *$(X, A, rN^{\mathcal{U}})$ is a soft L-neighborhood space.*

(2) *If $\mathcal{U}$ is stratified, then $rN^{\mathcal{U}}$ is also stratified.*

Proof. (1) (SN1) For $\mathcal{U}((U, A)) \neq 0$, by (SU4), $(1_\triangle, A) \leq (U, A)$. Then

$$\begin{aligned} rN_x^{\mathcal{U}}((0_X, A)) &= \textstyle\bigvee_{(U,A)\in S(X\times X,A)} \mathcal{U}((U, A)) \odot e_X((U[x], A), (0_X, A)) \\ &\leq \textstyle\bigvee_{(U,A)\in S(X\times X,A)} \big(\mathcal{U}((U, A)) \odot \bigwedge_{a\in A, x\in X}(U(a)(x, x) \to 0(a)(x))\big) \\ &= \textstyle\bigvee_{(U,A)\in S(X\times X,A)} \big(\mathcal{U}((U, A)) \odot (1_\triangle(a)(x, x) \to 0)\big) = 0. \end{aligned}$$

Hence $rN_x^{\mathcal{U}}((0_X, A)) = 0$. Also, $rN_x^{\mathcal{U}}((1_X, A)) = 1$, because

$$rN_x^{\mathcal{U}}((1_X, A)) \geq \mathcal{U}((1_{X\times X}, A)) \odot \bigwedge_{a\in A} \bigwedge_{y\in X} (1_{X\times X}(a)(x, y) \to 1_X(a)(y)) = 1.$$

(SN2) By Lemma 2.4 (4) , we have

$$\begin{aligned}
&rN_x^{\mathcal{U}}((F,A)) \odot rN_x^{\mathcal{U}}((G,A)) = \big(\bigvee_{(U,A)\in S(X\times X,A)} \mathcal{U}((U,A)) \odot e_X((U[x],A),(F,A))\big)\\
&\odot\big(\bigvee_{(U,A)\in S(X\times X,A)} \mathcal{U}(V,A) \odot e_X((V[x],A),(G,A))\big)\\
&= \bigvee_{(U,A),(V,A)\in S(X\times X,A)} \mathcal{U}((U,A)) \odot \mathcal{U}(V,A)\\
&\odot e_X((U[x],A),(F,A)) \odot e_X((V[x],A),(G,A))(\text{by Lemma 2.2 (13)})\\
&\leq \bigvee_{(U,A),(V,A)\in S(X\times X,A)} \mathcal{U}((U\odot V),A) \odot e_X(((U\odot V)[x],A),(F,A)\odot(G,A))\\
&\leq \bigvee_{(W,A)\in S(X\times X,A)} \mathcal{U}((W,A)) \odot e_X((W[x],A),(F,A)\odot(G,A))\\
&= rN_x^{\mathcal{U}}((F,A)\odot(G,A)).
\end{aligned}$$

(SN3) It follows from the definition of $rN_x^{\mathcal{U}}$ and Lemma 2.10(3).

(SN4) For $\mathcal{U}((U,A)) \neq 0,\ 1_\triangle \leq U$.

$$\begin{aligned}
&rN_x^{\mathcal{U}}((F,A)) = \bigvee_{(U,A)\in S(X\times X,A)} \mathcal{U}((U,A)) \odot \bigwedge_{a\in A}\bigwedge_{y\in X}(U(a)(y,x) \to F(a)(y))\\
&\leq \bigvee_{(U,A)\in S(X\times X,A)} \{\mathcal{U}((U,A)) \odot (U(-)(x,x) \to (F,A)(x))\} \leq (F,A)(x).
\end{aligned}$$

Hence $(X, A, rN^{\mathcal{U}})$ is a soft L-neighborhood space.

(2)

$$\begin{aligned}
&\alpha \odot rN_x^{\mathcal{U}}((F,A)) = \alpha \odot \bigvee_{(U,A)\in S(X\times X,A)} \mathcal{U}((U,A)) \odot e_X((U[x],A),(F,A))\\
&= \bigvee_{(U,A)\in S(X\times X,A)} \alpha \odot \mathcal{U}((U,A)) \odot e_X(\alpha,\alpha) \odot e_X((U[x],A),(F,A))\\
&\leq \bigvee_{(U,A)\in S(X\times X,A)} \mathcal{U}(\alpha \odot (U,A)) \odot e_X(\alpha \odot (U[x],A), \alpha \odot (F,A))\\
&\leq rN_x^{\mathcal{U}}(\alpha \odot (F,A)).
\end{aligned}$$

Corollary 3.3. *Let $(X, A, \mathcal{U})$ be a soft L-fuzzy quasi-uniform space. Define a map $lN^{\mathcal{U}} : X \to L^{S(X,A)}$ by, $\forall\ (F,A) \in S(X,A),\ x \in X$,*

$$lN_x^{\mathcal{U}}((F,A)) = \bigvee_{(U,A)\in S(X\times X,A)} \mathcal{U}((U,A)) \odot e_X((U[[x]],A),(F,A)),$$

where $(U[[x]],A)(y) = U(-)(x,y)$. Then the following properties hold.

(1) *$(X, A, lN^{\mathcal{U}})$ is a soft L-neighborhood space.*

(2) *If $\mathcal{U}$ is stratified, then $lN^{\mathcal{U}}$ is also stratified.*

Theorem 3.4. (1) *The soft L-neighborhood system $rN^{\mathcal{U}} = \{rN_x^{\mathcal{U}} \mid x \in X\}$ can be constructed from the cuts $\{(U,A) \in S(X \times X, A) \mid \mathcal{U}((U,A)) \geq \alpha\}$ as follows:*

$$rN_x^{\mathcal{U}}((F,A)) = \bigvee_{\alpha\in L} \alpha \odot rN_x^{\mathcal{U}}((F,A),\alpha),$$

where

$$rN_x^{\mathcal{U}}((F,A),\alpha) = \bigvee_v \{e_X((U[x],A),(F,A)) \mid \mathcal{U}((U,A)) \geq \alpha\}.$$

(2)

$$rN_x^{\mathcal{U}}((F,A)) \le \bigvee_{(V,A)\in S(X\times X,A)} \{rN_x^{\mathcal{U}}((G,A)) \mid (G,A)(z) \le rN_z^{\mathcal{U}}((F,A),\mathcal{U}(V,A))\}.$$

Proof. (1) For $F \in L^X$ with $F(y) \ge \alpha$, we have $F(y) \odot G(y) \ge \alpha \odot G(y)$ and

$$\bigvee\{F(x) \odot G(x) \mid F(x) \ge \alpha\} \ge \bigvee\{\alpha \odot G(y) \mid F(x) \ge \alpha\}.$$

Suppose

$$\bigvee\{F(x) \odot G(x) \mid x \in X\} \not\le \bigvee_{\alpha\in L} \bigvee\{\alpha \odot G(x) \mid F(x) \ge \alpha\}.$$

There exists $x_0 \in X$ such that

$$F(x_0) \odot G(x_0) \not\le \bigvee_{\alpha\in L} \bigvee\{\alpha \odot G(x) \mid F(x) \ge \alpha\}.$$

It is a contradiction. Hence

$$\bigvee\{F(x) \odot G(x) \mid x \in X\} = \bigvee_{\alpha\in L} \bigvee\{\alpha \odot G(x) \mid F(x) \ge \alpha\}.$$

By the above equality, we obtain

$$\begin{aligned} rN_x^{\mathcal{U}}((F,A)) &= \bigvee_{(U,A)\in S(X\times X,A)} (\mathcal{U}((U,A)) \odot e_X((U[x],A),(F,A))) \\ &= \bigvee_{\alpha\in L} \bigvee\{\alpha \odot e_X((U[x],A),(F,A)) \mid \mathcal{U}((U,A)) \ge \alpha\} \\ &= \bigvee_{\alpha\in L}\{\alpha \odot \bigvee\{e_X((U[x],A),(F,A)) \mid \mathcal{U}((U,A)) \ge \alpha\}\} \\ &= \bigvee_{\alpha\in L}\{\alpha \odot rN_x^{\mathcal{U}}((F,A),\alpha)\}. \end{aligned}$$

(2) For $(U,A) \in S(X\times X,A)$ and $(F,A) \in S(X,A)$, we have

$$rN_x^{\mathcal{U}}((F,A)) = \bigvee_{(U,A)\in S(X\times X,A)} \mathcal{U}((U,A)) \odot e_X((U[x],A),(F,A))$$
$$= \bigvee_{(U,A)\in S(X\times X,A)} \{\mathcal{U}((U,A)) \odot \bigwedge_{a\in A} \bigwedge_{y\in X} (U(a)(y,x) \to F(a)(y))\} \text{ (by SU(5))}$$
$$\le \bigvee_{(V,A)\in S(X\times X,A)} \{\mathcal{U}((V,A)) \odot \bigwedge_{a\in A} \bigwedge_{y\in X} ((V\circ V)(a)(y,x) \to F(a)(y))\}$$
$$= \bigvee_{(V,A)\in S(X\times X,A)} \{\mathcal{U}((V,A)) \odot \bigwedge_{a\in A} \bigwedge_{y\in X} ((\bigvee_{z\in X} V(a)(z,x) \odot V(a)(y,z)) \to F(a)(y))\}$$
$$= \bigvee_{(V,A)\in S(X\times X,A)} \{\mathcal{U}(V,A) \odot \bigwedge_{a\in A} \bigwedge_{y\in X} \bigwedge_{z\in X} (V(a)(z,x) \odot V(a)(y,z)) \to F(a)(y))\}$$
$$= \bigvee_{(V,A)\in S(X\times X,A)} \{\mathcal{U}(V,A) \odot \bigwedge_{a\in A} \bigwedge_{y\in X} \bigwedge_{z\in X} (V(a)(z,x) \to (V(a)(y,z) \to (F,A)(y))\}$$
(by Lemma 2.2 (12))
$$= \bigvee_{(V,A)\in S(X\times X,A)} \{\mathcal{U}(V,A) \odot \bigwedge_{a\in A} \bigwedge_{z\in X} (V(a)(z,x) \to \bigwedge_{y\in X} (V(a)(y,z) \to (F,A)(y))\}.$$

Put $(G,A)(z) = \bigwedge_{a\in A} \bigwedge_{y\in X} (V(a)(y,z) \to (F,A)(y))$. Then

$$rN_z^{\mathcal{U}}((F,A),\mathcal{U}(V,A)) = \bigvee_{(V,A)\in S(X\times X,A)} \{e_X((U[x],A),(F,A)) \mid \mathcal{U}((U,A)) \ge \mathcal{U}(V,A)\}$$
$$\ge e_X((V[z],A),(F,A)) = \bigwedge_{a\in A} \bigwedge_{y\in X} (V(a)(y,z) \to (F,A)(y)) = (G,A)(z).$$

Thus,

$$\begin{aligned} & rN^{\mathcal{U}}_x((F,A)) \leq \bigvee_{(V,A)\in S(X\times X,A)} \{\mathcal{U}(V,A)\odot \bigwedge_{a\in A}\bigwedge_{z\in X}(V(a)(z,x)\to (G,A)(z)) \\ & \mid (G,A)(z)\leq rN^{\mathcal{U}}_z((F,A),\mathcal{U}(V,A))\} \\ & \leq \bigvee_{(V,A)\in S(X\times X,A)} \{rN^{\mathcal{U}}_x((G,A)) \mid (G,A)(z)\leq rN^{\mathcal{U}}_z((F,A),\mathcal{U}(V,A))\}. \end{aligned}$$

Theorem 3.5. *If $f_\phi : (X,A,\mathcal{U}) \to (Y,B,\mathcal{V})$ is a uniformly continuous soft map, then $f_\phi : (X,A,rN^{\mathcal{U}}) \to (Y,B,rN^{\mathcal{V}})$ and $f_\phi : (X,A,lN^{\mathcal{U}}) \to (Y,B,lN^{\mathcal{V}})$ are continuous soft maps.*

Proof. First we show that $f_\phi^{-1}((V[f(x)],B)) = ((f\times f)_\phi^{-1}(V)[x],A)$ from

$$\begin{aligned} & f_\phi^{-1}((V[f(x)])(a)(z) = f^{\leftarrow}(V[f(x)](\phi(a))(z) = V(\phi(a))[f(x)](f(z)) \\ & = V(\phi(a))(f(z),f(x)) = (f\times f)_\phi^{-1}(V)(a)(z,x) = (f\times f)_\phi^{-1}(V)[x](a)(z). \end{aligned}$$

Thus, by Lemma 2.10(11), we have

$$\begin{aligned} & e_Y((V[f(x)],B),(F,B)) \leq e_X(f_\phi^{-1}((V[f(x)]),A),f_\phi^{-1}((F,B))) \\ & = e_X(((f\times f)_\phi^{-1}(V)[x],A),f_\phi^{-1}((F,B))). \end{aligned}$$

$f_\phi : (X,A,rN^{\mathcal{U}}) \to (Y,B,rN^{\mathcal{V}})$ is a continuous soft map from:

$$\begin{aligned} & rN^{\mathcal{V}}_{f(x)}((F,B)) = \bigvee_{(V,A)\in S(X\times X,A)} \mathcal{V}(V,A)\odot e_Y((V[f(x)],B),(F,B)) \\ & \leq \bigvee_{(V,A)\in S(X\times X,A)} \mathcal{V}(V,A)\odot e_X(((f\times f)_\phi^{-1}(V)[x],A),f_\phi^{-1}((F,B))) \\ & \leq \bigvee_{(U,A)\in S(X\times X,A)} \mathcal{U}((f\times f)_\phi^{-1}(V,A))\odot e_X(((f\times f)_\phi^{-1}(V)[x],A),f_\phi^{-1}((F,B))) \\ & \leq rN^{\mathcal{U}}_x(f_\phi^{-1}((F,B))). \end{aligned}$$

Similarly, $f_\phi : (X,A,lN^{\mathcal{U}}) \to (Y,B,lN^{\mathcal{V}})$ is a continuous soft map.

Theorem 3.6. *Let $f_\phi : (X,A) \to (Y,B)$ be a soft mapping and (Y,B,M) be a soft L-neighborhood space. We define $N : X \to L^{S(X,A)}$ as follows*

$$N_x((F,A)) = \bigvee\{M_{f(x)}((G,B)) \mid f_\phi^{-1}((G,B)) \leq (F,A)\}.$$

Then (X,A,N) is the coarsest soft L-neighborhood space for which $f_\phi : (X,A,N) \to (Y,B,N)$ be a continuous soft map. Moreover, if M is stratified, then N is also stratified.

Proof. (SN1) and (SN3) are clearly true.

(SN2) Let $(F_1,A),(F_2,A)\in S(X,A)$, $(G_1,B),(G_2,B)\in S(Y,B)$ and $x\in X$, then we have

$N_x((F_1,A)) \odot N_x((F_2,A))$
$= \bigvee\{M_{f(x)}((G_1,B)) \mid f_\phi^{-1}((G_1,B)) \le (F_1,A)\} \odot \bigvee\{M_{f(x)}((G_2,B)) \mid f_\phi^{-1}((G_2,B)) \le (F_2,A)\}$
$= \bigvee\{M_{f(x)}((G_1,B)) \odot M_{f(x)}((G_2,B)) \mid f_\phi^{-1}((G_1,B)) \le (F_1,A),\ f_\phi^{-1}((G_2,B)) \le (F_2,A)\}$
$\le \bigvee\{M_{f(x)}((G_1,B) \odot (G_2,B)) \mid f_\phi^{-1}((G_1,B)) \odot f_\phi^{-1}((G_2,B)) \le (F_1,A) \odot (F_2,A)\}$
$= \bigvee\{M_{f(x)}((G,B)) \mid f_\phi^{-1}((G,B)) \le (F_1,A) \odot (F_2,A)\}$
$= N_x((F_1,A) \odot (F_2,A))$.

(SN4) For any $(F,A) \in S(X,A), (G,B) \in S(Y,B)$ and $x \in X$, we have

$N_x((F,A)) = \bigvee\{M_{f(x)}((G,B)) \mid f_\phi^{-1}((G,B)) \le (F,A)\} \le \bigvee\{(G,B)(f(x)) \mid f_\phi^{-1}((G,B)) \le (F,A)\}$
$= \bigvee\{f_\phi^{-1}((G,B))(x) \mid f_\phi^{-1}((G,B)) \le (F,A)\} \le (F,A)(x)$.

The a map $f_\phi : (X,N) \to (Y,M)$ is a continuous soft map because, $x \in X, (G,B) \in S(Y,B)$,

$$N_x(f_\phi^{-1}((G,B))) = \bigvee\{M_{f(x)}((F,B)) \mid f_\phi^{-1}((F,B)) \le f_\phi^{-1}((G,B))\}$$
$$\ge \bigvee\{M_{f(x)}((F,B)) \mid (F,B) \le (G,B)\} \ge M_{f(x)}((G,B)).$$

Let $f_\phi : (X,N') \to (Y,M)$ be a continuous soft map. Suppose there exists $x \in X, (F,A) \in S(X,A)$ such that

$$N_x((F,A)) \not\le N'_x((F,A)).$$

By the definition of N_x, there exists $(G,B) \in S(Y,B)$ with $f_\phi^{-1}((G,B)) \le (F,A)$ such that

$$M_{f(x)}((G,B)) \not\le N'((F,A)).$$

On the other hand,

$$M_{f(x)}((G,B)) \le N'(f_\phi^{-1}((G,B))) \le N'((F,A)).$$

It is a contradiction. Hence $N \le N'$.

Finally, if M is stratified, then N is also stratified. In fact, for any $\alpha \in L$ and $(F,A) \in S(X,A)$, we have

$$\alpha \odot N_x((F,A)) = \alpha \odot \bigvee\{M_{f(x)}((G,B)) \mid f_\phi^{-1}((G,B)) \le (F,A)\}$$
$$\le \bigvee\{\alpha \odot M_{f(x)}((G,B)) \mid \alpha \odot f_\phi^{-1}((G,B)) \le \alpha \odot (F,A)\}$$
$$\le \bigvee\{M_{f(x)}(\alpha \odot (G,B)) \mid f_\phi^{-1}(\alpha \odot (G,B)) \le \alpha \odot (F,A)\} = N_x(\alpha \odot (F,A)).$$

Theorem 3.7. *Let (X,A,N) be a soft L-neighborhood space. Define a map $\mathcal{T}_N : S(X,A) \to L$ by*

$$\mathcal{T}_N((F,A)) = \bigwedge_{a\in A}\bigwedge_{x\in X}(F(a)(x) \to N_x((F,A))).$$

Then (1) $\mathcal{T}_N$ is a soft L-fuzzy topology on X,

(2) If N_x is stratified, then $\mathcal{T}_N$ is an enriched soft L-fuzzy topology.

Proof. (1) (ST1)

$$\mathcal{T}_N((1_X, A)) = \bigwedge_{a\in A}\bigwedge_{x\in X}(1_X(a)(x) \to N_x((1_X, A)) = 1,$$
$$\mathcal{T}_N((0_X, A)) = \bigwedge_{a\in A}\bigwedge_{x\in X}(0_X(a)(x) \to N_x((0_X, A))) = 0.$$

(ST2)

$$\begin{aligned}
&\mathcal{T}_N((F,A)\odot(G,A))\\
&= \bigwedge_{a\in A}\bigwedge_{x\in X}((F(a)(x)\odot G(a)x) \to N_x((F,A)\odot(G,A)))\\
&\geq \bigwedge_{a\in A}\bigwedge_{x\in X}((F(a)(x)\odot G(a)x) \to (N_x((F,A))\odot N_x((G,A))))\\
&\quad \text{(by Lemma 2.2 (12))}\\
&\geq \bigwedge_{a\in A}\bigwedge_{x\in X}(F(a)(x) \to N_x((F,A)))\odot \bigwedge_{a\in A}\bigwedge_{x\in X}(G(a)(x) \to N_x((G,A)))\\
&= \mathcal{T}_N((F,A))\odot\mathcal{T}_N((G,A)).
\end{aligned}$$

(ST3)

$$\begin{aligned}
&\mathcal{T}_N(\bigvee_i(F_i,A) = \bigwedge_{a\in A}\bigwedge_{x\in X}((\bigvee_i F_i(a)(x) \to N_x(\bigvee_i(F_i,A))\\
&\geq \bigwedge_{a\in A}\bigwedge_{x\in X}((\bigvee_i(F_i(a)(x) \to \bigvee_i N_x((F_i,A)) \quad \text{(by Lemma 2.2 (8))}\\
&\geq \bigwedge_i\bigwedge_{a\in A}\bigwedge_{x\in X}((F_i(a)(x) \to N_x((F_i,A)) = \bigwedge_i \mathcal{T}_N((F_i,A).
\end{aligned}$$

(2) By Lemma 2.2 (12), we have

$$\begin{aligned}
&\mathcal{T}_N(\alpha\odot(F,A)) = \bigwedge_{x\in X}((\alpha\odot(F,A))(x) \to N_x(\alpha\odot(F,A)))\\
&\geq \bigwedge_{x\in X}((\alpha\odot(F,A))(x) \to (\alpha\odot N_x((F,A))))\\
&\geq \bigwedge_{x\in X}((F,A)(x) \to N_x((F,A))) = \mathcal{T}_N((F,A)).
\end{aligned}$$

Corollary 3.8. *Let Let $(X,\mathcal{U})$ be a soft L-fuzzy quasi-uniform space, $\{rN_x^{\mathcal{U}} \mid x\in X\}$ and $\{lN_x^{\mathcal{U}} \mid x\in X\}$ be soft L-neighorhood systems on X. Define maps $\mathcal{T}_{rN^{\mathcal{U}}}, \mathcal{T}_{lN^{\mathcal{U}}} : S(X,A)\to L$ by*

$$\mathcal{T}_{rN^{\mathcal{U}}}((F,A)) = \bigwedge_{x\in X}((F,A)(x) \to rN_x^{\mathcal{U}}((F,A))),$$

$$\mathcal{T}_{lN^{\mathcal{U}}}((F,A)) = \bigwedge_{x\in X}((F,A)(x) \to lN_x^{\mathcal{U}}((F,A))).$$

Then,

(1) *$\mathcal{T}_{rN^{\mathcal{U}}}$ and $\mathcal{T}_{lN^{\mathcal{U}}}$ are soft L-fuzzy topologies on X,*

(2) *If $rN^{\mathcal{U}}$ (resp. $rN^{\mathcal{U}}$) is stratified, then $\mathcal{T}_{rN^{\mathcal{U}}}$ (resp. $\mathcal{T}_{lN^{\mathcal{U}}}$) is an enriched soft L-fuzzy topology.*

Theorem 3.9. *Let $(X,A,\mathcal{U})$ and $(Y,B,\mathcal{V})$ be soft L-fuzzy quasi-uniform spaces. If a map f_ϕ : $(X,A,\mathcal{U}) \to (Y,B,\mathcal{V})$ is an uniformly continuous soft map, then maps $f_\phi : (X,A,\mathcal{T}_{rN^{\mathcal{U}}}) \to (Y,B,\mathcal{T}_{rN^{\mathcal{V}}})$ and $f_\phi : (X,A,\mathcal{T}_{lN^{\mathcal{U}}}) \to (Y,B,\mathcal{T}_{lN^{\mathcal{V}}})$ are continuous soft maps.*

Proof.

$\mathcal{T}_{rN^{\mathcal{V}}}((G,B)) \to \mathcal{T}_{rN^{\mathcal{V}}}(f_\phi^{-1}((G,B)))$
$= \bigwedge_{b\in B}\bigwedge_{y\in Y}((G(b)(y) \to rN^{\mathcal{V}}_y((G,B))) \to \bigwedge_{a\in A}\bigwedge_{x\in X}(f_\phi^{-1}(G)(a)(x) \to rN^{\mathcal{U}}_x(f_\phi^{-1}((G,B))))$
$\geq \bigwedge_{a\in A}\bigwedge_{x\in X}((G(\phi(a))(f(x)) \to rN^{\mathcal{V}}_{f(x)}((G,B)))$
$\to \bigwedge_{a\in A}\bigwedge_{x\in X}(G(\phi(a))(f(x)) \to rN^{\mathcal{U}}_x(f_\phi^{-1}((G,B))))$
$\geq \bigwedge_{a\in A}\bigwedge_{x\in X}((G(\phi(a))(f(x)) \to rN^{\mathcal{V}}_{f(x)}((G,B)))$
$\to (G(\phi(a))(f(x)) \to rN^{\mathcal{U}}_x(f_\phi^{-1}((G,B))))$ (by Lemma 2.2 (8))
$\geq \bigwedge_{x\in X}\left(rN^{\mathcal{V}}_{f(x)}((G,B)) \to rN^{\mathcal{U}}_x(f_\phi^{-1}((G,B)))\right)$ (by Lemma 2.2 (10)).

Thus, if $rN^{\mathcal{V}}_{f(x)}((G,B)) \leq rN^{\mathcal{U}}_x(f_\phi^{-1}((G,B)))$, then $\mathcal{T}_{\mathcal{V}}((G,B)) \leq \mathcal{T}_{\mathcal{U}}(f_\phi^{-1}((G,B)))$. So, f_ϕ is a continuous soft map.

Example 3.10. Let $H = \{h_i \mid i = \{1,...,6\}\}$ with h_i=house and $E = \{e,b,w,c,i\}$ with e=expensive,b= beautiful, w=wooden, c= creative, i=in the green surroundings.

Define a binary operation $\odot$ on $[0,1]$ by

$$x \odot y = \max\{0, x+y-1\},\ x \to y = \min\{1-x+y, 1\}$$

$$x \oplus y = \min\{1, x+y\},\ x^* = 1-x$$

Then $([0,1],\odot,\to,0,1)$ is a complete residuated lattice (ref.[2,8,27]). Let $A = \{b,c\} \subset E$ and $X = \{h^1,h^4,h^5\}$. Put a fuzzy soft set (U,A) as follow:

$$(U,\{b\}) = \begin{pmatrix} b & h^1 & h^4 & h^5 \\ h^1 & 1 & 0.6 & 0.7 \\ h^4 & 0.1 & 1 & 0.5 \\ h^5 & 0.4 & 0.6 & 1 \end{pmatrix} (U,\{c\}) = \begin{pmatrix} c & h^1 & h^4 & h^5 \\ h^1 & 1 & 0.3 & 0.6 \\ h^4 & 0.1 & 1 & 0.6 \\ h^5 & 0.7 & 0.5 & 1 \end{pmatrix}$$

Then we obtain $(U\odot U, A)$ as

$$(U\odot U,\{b\}) = \begin{pmatrix} b & h^1 & h^4 & h^5 \\ h^1 & 1 & 0.2 & 0.4 \\ h^4 & 0 & 1 & 0 \\ h^5 & 0 & 0.2 & 1 \end{pmatrix} (U\odot U,\{c\}) = \begin{pmatrix} c & h^1 & h^4 & h^5 \\ h^1 & 1 & 0 & 0.2 \\ h^4 & 0 & 1 & 0.2 \\ h^5 & 0.4 & 0 & 1 \end{pmatrix}$$

Define $\mathcal{U} : S(X\times X, A) \to L$ as follows

$$\mathcal{U}((V,A)) = \begin{cases} 1, & \text{if } (V,A) = (1_{X\times X}, A), \\ 0.6, & \text{if } (U,A) \leq (V,A) \neq (1_{X\times X}, A), \\ 0.3, & \text{if } (U,A)\odot(U,A) \leq (V,A) \not\geq (U,A), \\ 0, & \text{otherwise.} \end{cases}$$

$rN^{\mathcal{U}}_{h_1}((F,A)) = \bigvee_{(U,A)\in S(X\times X,A)} \mathcal{U}((U,A)) \odot e_X((U[x],A),(F,A)),\ \forall\ (F,A)\in S(X,A),\ x\in X,$
(1)

$$\begin{aligned}
&rN^{\mathcal{U}}_{h^1}((F,A)) = \bigvee_{(U,A)\in S(X\times X,A)} \mathcal{U}((U,A)) \odot e_X((U[x],A),(F,A))\\
&= (F(b)(h^1)\wedge F(b)(h^4)\wedge F(b)(h^5)) \vee \Big(0.6\odot (F(b)(h^1)\wedge (0.9+F(b)(h^4))\wedge\\
&(0.6+F(b)(h^5)))\Big) \vee 0.3\odot \Big(F(b)(h^1))\Big) \vee (F(c)(h^1)\wedge F(c)(h^4)\wedge F(c)(h^5))\\
&\vee\Big(0.6\odot (0.7+F(c)(h^1))\wedge F(c)(h^4)\ \ \wedge (0.5+F(c)(h^5)))\Big)\\
&\ \vee 0.3\odot \Big(F(c)(h^1)\wedge (0.6+F(c)(h^5)))\Big)
\end{aligned}$$

$$\begin{aligned}
&rN^{\mathcal{U}}_{h^2}((F,A)) = \bigvee_{(U,A)\in S(X\times X,A)} \mathcal{U}((U,A)) \odot e_X((U[x],A),(F,A))\\
&= (F(b)(h^1)\wedge F(b)(h^4)\wedge F(b)(h^5)) \vee \Big(0.6\odot ((0.4+F(b)(h^1))\wedge F(b)(h^4)\wedge (0.4+F(b)(h^5)))\Big)\\
&\ \vee(0.3\odot (0.8+F(b)(h^1))\wedge F(b)(h^4)\wedge (0.8+F(b)(h^5))) \vee (F(c)(h^1)\wedge F(c)(h^4)\wedge F(c)(h^5))\\
&\vee\Big(0.6\odot ((0.7+F(c)(h^1))\wedge F(c)(h^4)\wedge (0.5+F(c)(h^5)))\Big)\ \ \vee 0.3\odot (F(c)(h^4))
\end{aligned}$$

$$\begin{aligned}
&rN^{\mathcal{U}}_{h^5}((F,A)) = \bigvee_{(U,A)\in S(X\times X,A)} \mathcal{U}((U,A)) \odot e_X((U[x],A),(F,A))\\
&= (F(b)(h^1)\wedge F(b)(h^4)\wedge F(b)(h^5)) \vee \Big(0.6\odot ((0.3+F(b)(h^1))\wedge (0.5+F(b)(h^4))\wedge F(b)(h^5))\Big)\\
&\ \vee 0.3\odot \Big((0.6+F(b)(h^1))\wedge F(b)(h^5))\Big) \vee (F(c)(h^1)\wedge F(c)(h^4)\wedge F(c)(h^5))\\
&\vee\Big(0.6\odot ((0.4+F(c)(h^1))\wedge (0.4+F(c)(h^4))\wedge F(c)(h^5))\Big)\\
&\vee 0.3\odot \Big((0.8+F(c)(h^1))\wedge (0.8+F(c)(h^4))\wedge F(c)(h^5)\Big)
\end{aligned}$$

Put (F,A) be a fuzzy soft set as follow:

(F,A)	h^1	h^4	h^5
b	0.5	0.6	0.2
c	0.9	0.5	0.3

Since $rN^{\mathcal{U}}_{h^1}((F,A)) = 0.2, rN^{\mathcal{U}}_{h^4}((F,A)) = 0.2, rN^{\mathcal{U}}_{h^5}((F,A)) = 0.2,$

$$\begin{aligned}
&\mathcal{T}_N((F,A)) = \bigwedge_{a\in A}\bigwedge_{x\in X}(F(a)(x)\to rN_x((F,A)))\\
&= (F(b)(h^1)\to rN_{h^1}((F,A)))\wedge (F(b)(h^4)\to rN_{h^4}((F,A)))\wedge (F(b)(h^5)\to rN_{h^5}((F,A)))\\
&\wedge(F(c)(h^1)\to rN_{h^1}((F,A)))\wedge (F(c)(h^4)\to rN_{h^4}((F,A)))\wedge (F(c)(h^5)\to rN_{h^5}((F,A))) = 0.3.
\end{aligned}$$

(2)

$$lN^{\mathcal{U}}_{h^1}((F,A)) = \bigvee_{(U,A)\in S(X\times X,A)} \mathcal{U}((U,A)) \odot e_X((U[[x]],A),(F,A))$$
$$= (F(b)(h^1) \wedge F(b)(h^4) \wedge F(b)(h^5)) \vee \Big(0.6 \odot (F(b)(h^1) \wedge (0.4 + F(b)(h^4))$$
$$\wedge (0.3 + F(b)(h^5)))\Big) \vee 0.3 \odot \Big(F(b)(h^1) \wedge (0.8 + F(b)(h^4))$$
$$\wedge (0.6 + F(b)(h^5))\Big) \vee (F(c)(h^1) \wedge F(c)(h^4) \wedge F(c)(h^5))$$
$$\vee \Big(0.6 \odot (F(c)(h^1) \wedge (0.7 + F(c)(h^4)) \wedge (0.4 + F(c)(h^5))\Big)$$
$$\vee 0.3 \odot \Big(F(c)(h^1) \wedge (0.8 + F(c)(h^5))\Big)$$

$$lN^{\mathcal{U}}_{h^2}((F,A)) = (F(b)(h^1) \wedge F(b)(h^4) \wedge F(b)(h^5))$$
$$\vee \Big(0.6 \odot (0.9 + F(b)(h^1)) \wedge F(b)(h^4) \wedge (0.5 + F(b)(h^5))\Big)$$
$$\vee (0.3 \odot F(b)(h^4)) \vee (F(c)(h^1) \wedge F(c)(h^4) \wedge F(c)(h^5))$$
$$\vee \Big(0.6 \odot (0.9 + F(c)(h^1)) \wedge F(c)(h^4) \wedge (0.4 + F(c))(h^5)\Big)$$
$$\vee 0.3 \odot (F(c)(h^4) \wedge (0.8 + F(c)(h^5)))$$

$$lN^{\mathcal{U}}_{h^5}((F,A)) = (F(b)(h^1) \wedge F(b)(h^4) \wedge F(b)(h^5))$$
$$\vee \Big(0.6 \odot (0.6 + F(b)(h^1)) \wedge (0.4 + F(b)(h^4)) \wedge F(b)(h^5)\Big)$$
$$\vee 0.3 \odot \Big((0.8 + F(b)(h^4)) \wedge F(b)(h^5)\Big) \vee (F(c)(h^1) \wedge F(c)(h^4) \wedge F(c)(h^5))$$
$$\vee \Big(0.6 \odot (0.3 + F(c)(h^1)) \wedge (0.5 + F(c)(h^4)) \wedge F(c)(h^5)\Big)$$
$$\vee 0.3 \odot \Big((0.6 + F(c)(h^1)) \wedge F(c)(h^5)\Big)$$

For a fuzzy soft set (F,A) in (1), since

$$lN^{\mathcal{U}}_{h^1}((F,A)) = 0.3, lN^{\mathcal{U}}_{h^4}((F,A)) = 0.2, lN^{\mathcal{U}}_{h^5}((F,A)) = 0.2,$$

we have

$$\mathcal{T}_N((F,A)) = (F(b)(h^1) \to lN_{h^1}((F,A)))$$
$$\wedge (F(b)(h^4) \to lN_{h^4}((F,A))) \wedge (F(b)(h^5) \to lN_{h^5}((F,A)))$$
$$\wedge (F(c)(h^1) \to lN_{h^1}((F,A))) \wedge (F(c)(h^4) \to lN_{h^4}((F,A)))$$
$$\wedge (F(c)(h^5) \to lN_{h^5}((F,A))) = 0.3.$$

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Existence of pseudo almost periodic mild solutions to impulsive partial stochastic differential equations in Hilbert spaces

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Abstract: In this paper, we introduce the concept of p-mean piecewise pseudo almost periodic for a stochastic process. Also it establish a new composition theorem for such processes. With this new composition theorem and by virtue of the theory of operator semigroups, the stochastic analysis techniques and Leray-Schauder nonlinear alternative, we investigate the existence of p-mean piecewise pseudo almost periodic mild solutions for a class of impulsive partial stochastic differential equations. Finally, an example of impulsive stochastic heat equation is also provided to illustrate our results.

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1 Introduction

The study of almost periodic type functions constitutes one of the most attractive topics in qualitative theory of differential equations since their applications. Among them, pseudo almost periodic function was introduced by Zhang as a natural generalization of almost periodic function in [1]. Some contributions on pseudo almost periodic type solutions to abstract differential equations have recently been made [2-7] and the references therein. Recently, there has been an increasing interest in extending certain classical deterministic results to stochastic differential equations. This is due to the fact that most problems in a real life situation to which mathematical models are applicable are basically stochastic rather than deterministic. The existence of almost periodic, asymptotically almost periodic, and pseudo almost periodic solutions to some stochastic differential equations has been considered in many publications such as [8-17] and references therein. In particular, Bezandry and Diagana [18,19] introduced the concepts of p-mean pseudo pseudo almost periodicity, and studied the existence of p-mean pseudo almost periodic mild solutions to partial stochastic differential equations. Diop et al. [20] obtained the existence, uniqueness and global attractiveness of an p-mean pseudo almost periodic solution for stochastic evolution equation driven by a fractional Brownian motion.

The theory of impulsive differential equations is an important branch of differential equations, which has an extensively physical background [21]. Therefore, it seems interesting to study the various types of impulsive differential equations. The asymptotic properties of solutions of impulsive differential equations have been considered by many authors. For example, Henríquez et al. [22], Liu and Zhang [23], Stamov et al. [24-26] discussed the piecewise almost periodic solutions of impulsive differential equations. Liu and Zhang [27], Chérif [28] established the existence and stability of piecewise pseudo almost periodic solutions to abstract impulsive differential equations. Bainov et al. [29] concerned with the asymptotic equivalence of impulsive differential equations. However, besides impulse effects and delays, stochastic effects likewise exist in real systems. In recent years, several interesting results on impulsive partial stochastic systems have been reported in [30-32] and the references therein. Further, Zhang [33] obtained the existence and uniqueness of almost periodic solutions for a class of impulsive stochastic differential equations with delay by mean of the Banach contraction principle. In [34], the authors investigated the existence and stability of square-mean piecewise almost periodic solutions for nonlinear impulsive stochastic differential equations by using Schauder's fixed point theorem.

In this paper, we study the existence of p-mean piecewise pseudo almost periodic mild solutions to the following impulsive partial stochastic differential equations:

$$dx(t) = [Ax(t) + g(t, x(t))]dt + f(t, x(t))dW(t), \quad t \in R, t \neq t_i, i \in Z, \tag{1}$$

$$\Delta x(t_i) = x(t_i^+) - x(t_i^-) = I_i(x(t_i)), \quad i \in Z, \tag{2}$$

where A is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ on $L^p(P, H)$ and $W(t)$ is a two-sided standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$, where $\mathcal{F}_t = \sigma\{W(u) - W(v); u, v \leq t\}$. g, f, I_i, t_i satisfy suitable conditions which will be established later. The notations $x(t_i^+), x(t_i^-)$ represent the right-hand side and the left-hand side limits of $x(\cdot)$ at t_i, respectively.

To the best of our knowledge, the existence of p-mean piecewise pseudo almost periodic mild solutions for for nonlinear impulsive stochastic system (1)-(2) is an untreated original topic, which in fact is the main motivation of the present paper. In the paper, we will introduce the notion of p-mean piecewise pseudo almost periodic for stochastic processes, which, in turn generalizes all the above-mentioned concepts, in particular, the notion of piecewise almost periodic. Then we will establish a new composition theorem for p-mean pseudo almost periodic functions under non-Lipschitz conditions. As an application, we study and obtain the existence of p-mean piecewise pseudo almost periodic mild solutions to system (1)-(2) by using Leray-Schauder nonlinear alternative. Such a result generalizes most of known results on the existence of almost periodic solutions of type system (1)-(2). It includes some results of almost periodic and pseudo almost periodic solutions to stochastic differential equations without impulse. Moreover, the results are also new for deterministic systems with impulse.

The paper is organized as follows. In Section 2, we introduce some notations and necessary preliminaries. In Section 3, we give the existence of p-mean piecewise pseudo almost periodic solutions for linear and nonlinear impulsive stochastic differential equations, respectively. In Section 4, an example is given to illustrate our results.

2 Preliminaries

Throughout the paper, N, Z, R and R^+ stand for the set of natural numbers, integers, real numbers, positive real numbers, respectively. We assume that $(H, \|\cdot\|), (K, \|\cdot\|_K)$ are real separable Hilbert spaces and $(\Omega, \mathcal{F}, P)$ is supposed to be a filtered complete probability space. Define $L^p(P, H)$, for $p \geq 1$ to be the space of all H-valued random variables V such that $E \| V \|^p = \int_\Omega \| V \|^p \, dP < \infty$. Then $L^p(P, H)$ is a Banach space when it is equipped with its natural norm $\|\cdot\|_p$ defined by $\| V \|_p = (\int_\Omega E \| V \|^p \, dP)^{1/p} < \infty$ for each $V \in L^p(P, H)$. Let $C(R, L^p(P, H)), BC(R, L^p(P, H))$ stand for the collection of all continuous functions from R into $L^p(P, H)$, the Banach space of all bounded continuous functions from R into $L^p(P, H)$, equipped with the sup norm, respectively. We let $L(K, H)$ be the space of all linear bounded operators from K into H, equipped with the usual operator norm $\|\cdot\|_{L(K,H)}$; in particular, this is simply denoted by $L(H)$ when $K = H$. Furthermore, $L_2^0(K, H)$ denotes the space of all Q-Hilbert-Schmidt operators from K to H with the norm $\| \psi \|_{L_2^0}^2 = \mathrm{Tr}(\psi Q \psi^*) < \infty$ for any $\psi \in L(K, H)$.

Definition 2.1 ([18]). A stochastic process $x : R \to L^p(P, H)$ is said to be continuous provided that for any $s \in R$,

$$\lim_{t \to s} E \| x(t) - x(s) \|^p = 0.$$

Definition 2.2 ([18]). A stochastic process $x : R \to L^p(P, H)$ is said to be stochastically bounded provided that

$$\lim_{n \to \infty} \limsup_{t \in R} P\{\| x(t) \| > n\} = 0.$$

Let T be the set consisting of all real sequences $\{t_i\}_{i \in Z}$ such that $\alpha = \inf_{i \in Z}(t_{i+1} - t_i) > 0$, $\lim_{i \to \infty} t_i = \infty$, and $\lim_{i \to -\infty} t_i = -\infty$. For $\{t_i\}_{i \in Z} \in T$, let $PC(R, L^p(P, H))$ be the space consisting of all stochastically bounded piecewise continuous functions $f : R \to L^p(P, H)$ such that $f(\cdot)$ is stochastically continuous at t for any $t \notin \{t_i\}_{i \in Z}$ and $f(t_i) = f(t_i^-)$ for all $i \in Z$; let $PC(R \times L^p(P, K), L^p(P, H))$ be the space formed by all stochastically piecewise continuous functions $f : R \times L^p(P, K) \to L^p(P, H)$ such that for any $x \in L^p(P, K)$, $f(\cdot, x) \in PC(R, L^p(P, H))$ and for any $t \in R, f(t, \cdot)$ is stochastically continuous at $x \in L^p(P, K)$.

3

Definition 2.3 ([18]). A function $f \in C(R, L^p(P, H))$ is said to be p-mean almost periodic if for each $\varepsilon > 0$, there exists an $l(\varepsilon) > 0$, such that every interval J of length $l(\varepsilon)$ contains a number τ with the property that $E \parallel f(t+\tau) - f(t) \parallel^p < \varepsilon$ for all $t \in R$. Denote by $AP(R, L^p(P, H))$ the set of such functions.

Definition 2.4 (Compare with [21]). A sequence $\{x_n\}$ is called p-mean almost periodic if for any $\varepsilon > 0$, there exists a relatively dense set of its ε-periods, i.e., there exists a natural number $l = l(\varepsilon)$, such that for $k \in Z$, there is at least one number q in $[k, k+l]$, for which inequality $E \parallel x_{n+q} - x_n \parallel^p < \varepsilon$ holds for all $n \in N$. Denote by $AP(Z, L^p(P, H)))$ the set of such sequences.

Define $l^\infty(Z, L^p(P, H)) = \{x : Z \to L^p(P, H) : \parallel x \parallel = \sup_{n \in Z}(E \parallel x(n) \parallel^p)^{1/p} < \infty\}$, and

$$PAP_0(Z, L^p(P, H)) = \left\{ x \in l^\infty(Z, L^p(P, H)) : \lim_{n\to\infty} \frac{1}{2n} \sum_{j=-n}^{n} E \parallel x(n) \parallel^p dt = 0 \right\}.$$

Definition 2.5. A sequence $\{x_n\}_{n\in Z} \in l^\infty(Z, H)$ is called p-mean pseudo almost periodic if $x_n = x_n^1 + x_n^2$, where $x_n^1 \in AP(Z, L^p(P, H)), x_n^2 \in PAP_0(Z, L^p(P, H))$. Denote by $PAP(Z, L^p(P, H))$ the set of such sequences.

Definition 2.6 (Compare with [21]). For $\{t_i\}_{i\in Z} \in T$, the function $f \in PC(R, L^p(P, H))$ is said to be p-mean piecewise almost periodic if the following conditions are fulfilled:

(i) $\{t_i^j = t_{i+j} - t_i\}, j \in Z$, is equipotentially almost periodic, that is, for any $\varepsilon > 0$, there exists a relatively dense set Q_ε of R such that for each $\tau \in Q_\varepsilon$ there is an integer $q \in Z$ such that $|t_{i+q} - t_i - \tau| < \varepsilon$ for all $i \in Z$.

(ii) For any $\varepsilon > 0$, there exists a positive number $\tilde{\delta} = \tilde{\delta}(\varepsilon)$ such that if the points t' and t'' belong to a same interval of continuity of φ and $|t' - t''| < \tilde{\delta}$, then $E \parallel f(t') - f(t'') \parallel^p < \varepsilon$.

(iii) For every $\varepsilon > 0$, there exists a relatively dense set $\tilde{\Omega}(\varepsilon)$ in R such that if $\tau \in \tilde{\Omega}(\varepsilon)$, then
$$E \parallel f(t+\tau) - f(t) \parallel^p < \varepsilon$$
for all $t \in R$ satisfying the condition $|t - t_i| > \varepsilon, i \in Z$. The number τ is called ε-translation number of f.

We denote by $AP_T(R, L^p(P, H))$ the collection of all the p-mean piecewise almost periodic functions. Obviously, the space $AP_T(R, L^p(P, H))$ endowed with the sup norm defined by $\parallel f \parallel_\infty = \sup_{t\in R}(E \parallel f(t) \parallel^p)^{1/p}$ for any $f \in AP_T(R, L^p(P, H))$ is a Banach space. Let $UPC(R, L^p(P, H))$ be the space of all stochastic functions $f \in PC(R, L^p(P, H))$ such that f satisfies the condition (ii) in Definition 2.6.

Definition 2.7. The function $f \in PC(R \times L^p(P, K), L^p(P, H))$ is said to be p-mean piecewise almost periodic in $t \in R$ uniform in $x \in L^p(P, K)$ if for every compact subset $\tilde{K} \subseteq L^p(P,K)$, $\{f(\cdot,x) : x \in \tilde{K}\}$ is uniformly bounded, and given $\varepsilon > 0$, there exists a relatively dense subset Ω_ε such that

$$E \parallel f(t+\tau,x) - f(t,x) \parallel^p < \varepsilon$$

for all $x \in \tilde{K}, \tau \in \Omega_\varepsilon$, and $t \in R$ satisfying $|t - t_i| > \varepsilon$. Denote by $AP_T(R \times L^p(P,K), L^p(P,H))$ the set of all such functions.

Similarly as the proof of [21, Lemma 35], one has

Lemma 2.1. Assume that $f \in AP_T(R, L^p(P,H))$, the sequence $\{x_i\}_{i\in Z} \in AP(Z, L^p(P,H))$, and $\{t_i^j\}, j \in Z$ are equipotentially almost periodic. Then, for each $\varepsilon > 0$, there exist relatively dense sets Ω_ε of R and Ω_ε of Z such that

(i) $E \parallel f(t+\tau) - f(t) \parallel^p < \varepsilon$ for all $t \in R, |t - t_i| > \varepsilon, \tau \in \Omega_\varepsilon$ and $i \in Z$.

(ii) $E \parallel x_{i+q} - x_i \parallel^p < \varepsilon$ for all $q \in \Omega_\varepsilon$ and $i \in Z$.

(iii) $E \parallel x_i^q - \tau \parallel^p < \varepsilon$ for all $q, \tau \in \Omega_\varepsilon$ and $i \in Z$.

Denote

$$PC_T^0(R, L^p(P,H)) = \left\{ f \in PC(R, L^p(P,H)) : \lim_{t\to\infty} E \parallel f(t) \parallel^p = 0 \right\},$$

$$\begin{aligned} &PAP_T^0(R, L^p(P,H)) \\ &\quad = \left\{ f \in PC(R, L^p(P,H)) : \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} E \parallel f(t) \parallel^p dt = 0 \right\}, \end{aligned}$$

$$\begin{aligned} &PAP_T^0(R \times L^p(P,K), L^p(P,H)) \\ &\quad = \Big\{ f \in PC(R \times L^p(P,K), L^p(P,H)) : \\ &\quad\quad \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} E \parallel f(t,x) \parallel^p dt = 0 \text{ uniformly with respect to} \\ &\quad\quad x \in \tilde{K}, \text{where } \tilde{K} \text{ is an arbitrary compact subset of } L^p(P,K) \Big\}. \end{aligned}$$

Definition 2.8. A function $f \in PC(R, L^p(P,H))$ is said to be p-mean piecewise pseudo almost periodic if it can be decomposed as $f = h + \varphi$, where $h \in AP_T(R, L^p(P,H))$ and $\varphi \in PAP_T^0(R, L^p(P,H))$.

Denoted by $PAP_T(R, L^p(P,H))$ the set of all such functions. $PAP_T(R, L^p(P,H))$ is a Banach space with the sup norm $\parallel \cdot \parallel_\infty$.

Similar to [1,27], one has

Remark 2.1. (i) $PAP_T^0(R, L^p(P,H))$ is a translation invariant set of $PC(R, L^p(P,H)))$. (ii) $PC_T^0(R, L^p(P,H)) \subset PAP_T^0(R, L^p(P,H))$.

Lemma 2.2. Let $\{f_n\}_{n\in N} \subset PAP_T^0(R, L^p(P,H))$ be a sequence of functions. If f_n converges uniformly to f, then $f \in PAP_T^0(R, L^p(P,H))$.

One can refer to Lemma 2.5 in [5] for the proof of Lemma 2.2.

Definition 2.9. A function $f \in PC(R \times L^p(P,K), L^p(P,H))$ is said to be p-mean piecewise pseudo almost periodic if it can be decomposed as $f = h + \varphi$, where $h \in AP_T(R\times L^p(P,K), L^p(P,H))$ and $\varphi \in PAP_T^0(R\times L^p(P,K), L^p(P,H))$.

Denoted by $PAP_T(R \times L^p(P,K), L^p(P,H))$ the set of all such functions.

We need the following composition of p-mean pseudo almost periodic processes.

Lemma 2.3. Assume $f \in PAP_T(R\times L^p(P,K), L^p(P,H))$. Suppose that $f(t,x)$ satisfies

$$E \parallel f(t,x) - f(t,y) \parallel^p \leq \Lambda(E \parallel x - y \parallel^p) \tag{3}$$

for all $t \in R, x, y \in L^p(P,K)$, where Λ is a concave and continuous nondecreasing function from R^+ to R^+ such that $\Lambda(0) = 0, \Lambda(s) > 0$ for $s > 0$ and $\int_{0+} \frac{ds}{\Lambda(s)} = +\infty$. Here, the symbol $\int_{0+}$ stands for $\lim_{\varepsilon\to 0+} \int_{\varepsilon}^{+\infty}$. If $\phi(\cdot) \in PAP_T(R, L^p(P,K))$ then $f(\cdot,\phi(\cdot)) \in PAP_T(R, L^p(P,H))$.

Proof. Assume that $f = f_1 + f_2, \phi = \phi_1 + \phi_2$, where $f_1 \in AP_T(R \times L^p(P,K), L^p(P,H))$, $f_2 \in PAP_T^0(R \times L^p(P,K), L^p(P,H))$, $\phi_1 \in AP_T(R, L^p(P,H))$, and $\phi_2 \in PAP_T^0(R, L^p(P,H))$. Consider the decomposition

$$f(t,\phi(t)) = f_1(t,\phi_1(t)) + [f(t,\phi(t)) - f(t,\phi_1(t))] + f_2(t,\phi_1(t)).$$

Since $f_1(\cdot,\phi_1(\cdot)) \in AP_T(R, L^p(P,H))$, it remains to prove that both $[f(\cdot,\phi(\cdot)) - f(\cdot,\phi_1(\cdot))]$ and $f_2(\cdot,\phi_1(\cdot))$ belong to $PAP_T^0(R, L^p(P,H))$. Indeed, using (3), it follows that

$$\begin{aligned}
&\frac{1}{2r}\int_{-r}^{r} E \parallel f(t,\phi(t)) - f(t,\phi_1(t)) \parallel^p dt \\
&\quad \leq \frac{1}{2r}\int_{-r}^{r} \Lambda(E \parallel \phi(t) - \phi_1(t) \parallel^p) dt \\
&\quad = \frac{1}{2r}\int_{-r}^{r} \Lambda(E \parallel \phi_2(t) \parallel^p) dt,
\end{aligned}$$

noting that Λ is concave, continuous and $\Lambda(0) = 0$, we deduce that

$$\begin{aligned}
&\frac{1}{2r}\int_{-r}^{r} \Lambda(E \parallel \phi_2(t) \parallel^p) dt \\
&\quad \leq \Lambda\biggl(\frac{1}{2r}\int_{-r}^{r} E \parallel \phi_2(t) \parallel^p dt\biggr) \to 0 \quad \text{as } r \to \infty,
\end{aligned}$$

which implies that $[f(\cdot,\phi(\cdot)) - f(\cdot,\phi_1(\cdot))] \in PAP_T^0(R, L^p(P,H))$.

Since $\phi_1(R)$ is relatively compact in $L^p(P,K)$ and f_1 is uniformly continuous on sets of the form $R \times \tilde{K}$ where $\tilde{K} \subset L^p(P,K)$ is compact subset, for $\varepsilon > 0$ there exists $\xi \in (0,\varepsilon)$ such that

$$E \parallel f_1(t,z) - f_1(t,\tilde{z}) \parallel^p \leq \varepsilon, \quad z, \tilde{z} \in \phi_1(R)$$

with $|z-\tilde{z}| < \xi$. Now, fix $z_1, ..., z_n \in \phi_1(R)$ such that $\phi_1(R) \subset \bigcup_{j=1}^{n} B_\xi(z_j, L^p(P,K))$. Obviously, the sets $D_j = \phi_1^{-1}(B_\xi(z_j))$ form an open covering of R, and therefore using the sets $B_1 = D_1, B_2 = D_2 \backslash D_1$ and $B_j = D_j \backslash \bigcup_{k=1}^{j-1} D_k$ one obtains a covering of R by disjoint open sets. For $t \in B_j, \phi_1(t) \in B_\xi(z_j)$,

$$\begin{aligned} & E \parallel f_2(t,\phi_1(t)) \parallel^p \\ & \quad \leq 3^{p-1} E \parallel f(t,\phi_1(t)) - f(t,z_j) \parallel^p \\ & \quad\quad +3^{p-1} E \parallel -f_1(t,\phi_1(t)) + f_1(t,z_j) \parallel^p +3^{p-1} E \parallel f_2(t,z_j) \parallel \\ & \quad \leq 3^{p-1}\Lambda(E \parallel \phi_1(t) - z_j \parallel^p) + 3^{p-1}\varepsilon + 3^{p-1} E \parallel f_2(t,z_j) \parallel \\ & \quad \leq 3^{p-1}\Lambda(\varepsilon) + 3^{p-1}\varepsilon + 3^{p-1} E \parallel f_2(t,z_j) \parallel . \end{aligned}$$

Now using the previous inequality it follows that

$$\begin{aligned} & \frac{1}{2r}\int_{-r}^{r} E \parallel f_1(t,\phi_1(t)) \parallel^p dt \\ & \quad = \frac{1}{2r}\sum_{j=1}^{n}\int_{B_j\cap[-r,r]} E \parallel f_1(t,\phi_1(t)) \parallel^p dt \\ & \quad \leq 3^{p-1}\frac{1}{2r}\sum_{j=1}^{n}\int_{B_j\cap[-r,r]} E \parallel f(t,\phi_1(t)) - f(t,z_j) \parallel^p dt \\ & \quad\quad +3^{p-1}\frac{1}{2r}\sum_{j=1}^{n}\int_{B_j\cap[-r,r]} E \parallel f_1(t,\phi_1(t)) - f_1(t,z_j) \parallel^p dt \\ & \quad\quad +3^{p-1}\frac{1}{2r}\sum_{j=1}^{n}\int_{B_j\cap[-r,r]} E \parallel f_2(t,z_j) \parallel^p dt \\ & \quad \leq 3^{p-1}\frac{1}{2r}\int_{-r}^{r}[\Lambda(\varepsilon)+\varepsilon]dt + 3^{p-1}\sum_{j=1}^{n}\frac{1}{2r}\int_{-r}^{r} E \parallel f_2(t,z_j) \parallel^p dt. \end{aligned}$$

In view of the above it is clear that $f_2(\cdot,\phi_1(\cdot))$ belongs to $PAP_T^0(R, L^p(P,H))$. This completes the proof.

Lemma 2.4. Assume the sequence of vector-valued functions $\{I_i\}_{i\in Z}$ is pseudo almost periodic, and there is a concave nondecreasing function from R^+ to R^+ such that $\Lambda_i(0) = 0, \Lambda_i(s) > 0$ for > 0 and $\int_{0^+} \frac{ds}{\Lambda_i(s)} = +\infty$,

$$E \parallel I_i(x) - I_i(y) \parallel^p \leq \Lambda_i(E \parallel x - y \parallel^p)$$

for all $x, y \in L^p(P,K)$, $i \in Z$. If $\phi \in PAP_T(R, L^p(P,H)) \cap UPC(R, L^p(P,H))$ such that $R(\phi) \subset L^p(P,K)$, then $I_i(\phi(t_i))$ is pseudo almost periodic.

Proof. Assume that $\phi = \phi_1 + \phi_2$, where $\phi_1 \in AP_T(R, L^p(P,H))$, $\phi_2 \in PAP_T^0(R, L^p(P,H))$. Fix $\phi \in PAP_T(R, L^p(P,H)) \cap UPC(R, L^p(P,H))$, first we show $\phi(t_i)$ is pseudo almost periodic. One can refer to Lemma 37 in [21] that the sequence $\phi(t_i)$ is almost periodic. Next we need to show that $\phi(t_i) \in PAP_0(Z, L^p(P,H))$. By the hypothesis, $\phi, \phi_1 \in UPC(R, L^p(P,H))$, so

$\phi_2 \in UPC(R, L^p(P,H))$. Let $0<\varepsilon<1$, there exists $0<\xi<\min\{1,\gamma\}$ such that for $t\in(t_i-\xi, t_i), i\in Z$, we have

$$E \parallel \phi_2(t) \parallel^p \leq (1-\varepsilon)E \parallel \phi_2(t_i) \parallel^p, \quad i\in Z.$$

Since $t_i^j, \quad i\in Z, j=0,1,...$ are equipotentially almost periodic, $\{t_i^1\}$ is an almost periodic sequence. Here we assume a bound of $\{t_i^1\}$ is M_t and $|t_i|\geq|t_{-i}|$; therefore,

$$\begin{aligned}
&\frac{1}{2t_i}\int_{-t_i}^{t_i} E \parallel \phi_2(t) \parallel^p dt \\
&\quad \geq \frac{1}{2t_i}\sum_{j=-i+1}^{i}\int_{t_j-\xi}^{t_j} E \parallel \phi_2(t) \parallel^p dt \\
&\quad \geq \frac{1}{2t_i}\sum_{j=-i+1}^{i}\xi(1-\varepsilon)E \parallel \phi_2(t_j) \parallel^p \\
&\quad \geq \frac{\xi(1-\varepsilon)}{M_t}\frac{1}{2t_i}\sum_{j=-i+1}^{i} E \parallel \phi_2(t_j) \parallel^p .
\end{aligned}$$

Since $\phi_2\in PAP_T^0(R, L^p(P,H))$, it follows from the inequality above that $\phi_2(t_i)\in PAP_0(Z, L^p(P,H))$. Hence, $\phi(t_i)$ is pseudo almost periodic.

Now, we show $I_i(\phi(t_i))$ is pseudo almost periodic. Let

$$I(t,x)=(t-n)I_n(x), \quad n\leq t<n+1, n\in Z,$$

$$\vartheta(t)=(t-n)\phi_n(t_n), \quad n\leq t<n+1, n\in Z.$$

Since $I_n, \phi(t_n)$ are two pseudo almost periodic sequences, Refer to Lemma 1.7.12. in [36], we get that $I\in PAP(R\times L^p(P,K), L^p(P,H))$, $\vartheta\in PAP(R, L^p(P,K))$. For every $t\in R$, there exists a number $n\in Z$ such that $|t-n|\leq 1$, we have for $x_1,x_2\in L^p(P,K)$,

$$\begin{aligned}
&E \parallel I(t,x_1)-I(t,x_2) \parallel^p \\
&\quad \leq E \parallel I_n(x_1)-I_n(x_2) \parallel^p \\
&\quad \leq \Lambda_n(E \parallel x_1-x_2 \parallel^p).
\end{aligned}$$

Similar to the proof of Lemma 2.4, $I(\cdot,\vartheta(\cdot))\in PAP(R, L^p(P,H))$. Again, similarly as the proof of Lemma 1.7.12 in [36], we have that $I(i,\vartheta(i))$ is a pseudo almost periodic sequence, that is, $I_i(\phi(t_i))$ is pseudo almost periodic. This completes the proof.

Next, we introduce a useful compactness criterion on $PC(R, L^p(P,H))$.

Let $h: R\to R^+$ be a continuous function such that $h(t)\geq 1$ for all $t\in R$ and $h(t)\to\infty$ as $|t|\to\infty$. Define

$$PC_h^0(R, L^p(P,H))=\left\{f\in PC(R, L^p(P,H)): \lim_{|t|\to\infty}\frac{E \parallel f(t) \parallel^p}{h(t)}=0\right\}$$

endowed with the norm $\| f \|_h = \sup_{t\in R} \frac{E\|f(t)\|^p}{h(t)}$, it is a Banach space.
Lemma 2.5. A set $B \subseteq PC_h^0(R, L^p(P,H))$ is relatively compact if and only if it verifies the following conditions:

(i) $\lim\limits_{|t|\to\infty} \frac{E\|f(t)\|^p}{h(t)} = 0$ uniformly for $f \in B$.

(ii) $B(t) = \{f(t) : f \in B\}$ is relatively compact in $L^p(P,H)$ for every $t \in R$.

(iii) The set B is equicontinuous on each interval $(t_i, t_{i+1})(i \in Z)$.

One can refer to Lemma 4.1 in [27] for the proof of Lemma
Lemma 2.6 (Leray-Schauder nonlinear alternative [35]). Let X be a Banach space with $D \subset X$ closed and convex. Assume U is a relatively open subset of D with $0 \in U$ and $\Psi : \overline{U} \to D$ is a compact map, then either

(i) Ψ has a fixed point in $\overline{U}$, or

(ii) there is a point $x \in \partial U$ and $\lambda \in (0,1)$ with $x \in \lambda\Psi(x)$.

3 Main results

In this section, we investigate the existence of p-mean piecewise pseudo almost periodic mild solution for system (1)-(2). To do this, we first consider the existence of p-mean piecewise pseudo almost periodic mild solutions to the following linear stochastic differential equation

$$dx(t) = [Ax(t) + g(t)]dt + f(t)dW(t), \quad t \in R, t \neq t_k, i \in Z, \tag{4}$$

$$\Delta x(t_i) = x(t_i^+) - x(t_i^-) = \gamma_i, \quad i \in Z, \tag{5}$$

where A is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ on $L^p(P,H)$ such that for all $t \geq 0, \| T(t) \| \leq Me^{-\delta t}$ with $M, \delta > 0$. $W(t)$ is a two-sided standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$, where $\mathcal{F}_t = \sigma\{W(u) - W(v); u, v \leq t\}$. Furthermore, $g : R \to L^p(P,H)$, $f : R \to L^p(P, L_2^0)$ are two stochastic processes, γ_i is an p-mean pseudo almost periodic sequence.
Definition 3.1. An $\mathcal{F}_t$ -progressively measurable process $\{x(t)\}_{t\in R}$ is called a mild solution of system (4)-(5) if for any $t \in R, t > \sigma, \sigma \neq t_i, i \in Z$,

$$x(t) = T(t-\sigma)x(\sigma) + \int_\sigma^t T(t-s)g(s)ds + \int_\sigma^t T(t-s)f(s)dW(s) + \sum_{\sigma<t_i<t} T(t-t_i)\gamma_i. \tag{6}$$

Theorem 3.1. *Assume $g \in PAP_T(R, L^p(P,H)), f \in PAP_T(R, L^p(P, L_2^0)), \{\gamma_i, i \in Z\}$ is an p-mean pseudo almost periodic sequence, then system (4)-(5) has a mild solution $x \in PAP_T(R, L^p(P,H))$.*

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Proof. Consider for each $i \in Z$, the integrals

$$x(t) = \int_{-\infty}^{t} T(t-s)g(s)ds + \int_{-\infty}^{t} T(t-s)f(s)dW(s) + \sum_{t_i<t} T(t-t_i)\gamma_i \tag{7}$$

for each $t \in R$. Next we aim to prove (7) is an p-mean piecewise pseudo almost periodic mild solution of system (4)-(5).

Since $g \in PAP_T(R, L^p(P,H))$, $f \in PAP_T(R, L^p(P, L_2^0))$, $\gamma_i \in PAP(Z, L^p(P,H))$, there exist $g_1 \in AP_T(R, L^p(P,H))$, $f_1 \in AP_T(R, L^p(P, L_2^0))$, $\gamma_{1,i} \in AP(Z, L^p(P,H))$ and $g_2 \in PAP_T^0(R, L^p(P,H))$, $f_2 \in PAP_T^0(R, L^p(P, L_2^0))$, $\gamma_{2,i} \in PAP_0(Z, L^p(P,H))$, such that $g = g_1 + g_2, f = f_1 + f_2, \gamma_i = \gamma_{1,i} + \gamma_{2,i}$. Hence,

$$\begin{aligned} x(t) &= \int_{-\infty}^{t} T(t-s)[g_1(s)+g_2(s)]ds + \int_{-\infty}^{t} T(t-s)[f_1(s)+f_2(s)]dW(s) \\ &\quad + \sum_{t_i<t} T(t-t_i)[\gamma_{1,i}+\gamma_{2,i}] \\ &= \Big[\int_{-\infty}^{t} T(t-s)g_1(s)ds + \int_{-\infty}^{t} T(t-s)f_1(s)dW(s) \\ &\quad + \sum_{t_i<t} T(t-t_i)\gamma_{1,i}\Big] \\ &\quad + \Big[\int_{-\infty}^{t} T(t-s)g_2(s)ds + \int_{-\infty}^{t} T(t-s)f_2(s)dW(s) \\ &\quad + \sum_{t_i<t} T(t-t_i)\gamma_{2,i}\Big] \\ &=: F(t) + \Phi(t). \end{aligned}$$

In order to prove (7) is an p-mean pseudo almost periodic mild solution, we only need to verify $F(t) \in AP_T(R, L^p(P,H))$ and $\Phi(t) \in PAP_T^0(R, L^p(P,H))$. Thus, the following verification procedure is divided into three steps.

Step 1. $F \in UPC(R, L^p(P,H))$.

Let $t', t'' \in (t_i, t_{i+1}), i \in Z, t'' < t'$. By $\{T(t)\}_{t\geq 0}$ is a C_0-semigroup and $\| T(t) \| \leq Me^{-\delta t}, t \geq 0$, for any $\varepsilon > 0$, there exists $0 < \xi < \min\{(\frac{\varepsilon}{5\tilde{g}_1})^{1/p}, (\frac{\varepsilon}{5\tilde{f}_1})^{2(p-1)/p}\}$ such that $0 < t' - t'' < \xi$, we have for $p > 2$,

$$\begin{aligned} &\| T(t'-t'') - I \|^p \\ &\quad \leq \min\left\{\frac{\delta^p \varepsilon}{5\tilde{g}_1}, \frac{(\frac{p\delta}{p-2})^{(p-2)/2}\frac{p\delta}{2}\varepsilon}{5\tilde{f}_1}, \frac{(1-e^{-\delta\alpha})^p\varepsilon}{5\tilde{\gamma}_1}\right\}, \end{aligned}$$

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where $\tilde{g}_1 = 6^{p-1}M^p \parallel g_1 \parallel_\infty^p$, $\tilde{f}_1 = 6^{p-1}M^pC_p \parallel f_1 \parallel_\infty^p$, $\tilde{\gamma}_1 = 3^{p-1}M^p \parallel \gamma_{1,i} \parallel_\infty^p$.
Using Hölder's inequality and the Ito integral [37], we have

$$
\begin{aligned}
&E \parallel F(t') - F(t'') \parallel^p \\
&\quad \leq 3^{p-1}E\left\| \int_{-\infty}^{t'} T(t'-s)g_1(s)ds - \int_{-\infty}^{t''} T(t''-s)g_1(s)ds \right\|^p \\
&\quad\quad +3^{p-1}E\left\| \int_{-\infty}^{t'} T(t'-s)f_1(s)dW(s) - \int_{-\infty}^{t''} T(t''-s)f_1(s)dW(s) \right\|^p \\
&\quad\quad +3^{p-1}E\left\| \sum_{t_i<t'} T(t'-t_i)\gamma_{1,i} - \sum_{t_i<t''} T(t''-t_i)\gamma_{1,i} \right\|^p \\
&\quad \leq 6^{p-1}E\left\| \int_{-\infty}^{t''} T(t''-s)[T(t'-t'')-I]g_1(s)ds \right\|^p \\
&\quad\quad +6^{p-1}E\left\| \int_{t''}^{t'} T(t'-s)g_1(s)ds \right\|^p \\
&\quad\quad +6^{p-1}E\left\| \int_{-\infty}^{t''} T(t''-s)[T(t'-t'')-I]f_1(s)dW(s) \right\|^p \\
&\quad\quad +6^{p-1}E\left\| \int_{t''}^{t'} T(t'-s)f_1(s)dW(s) \right\|^p \\
&\quad\quad +3^{p-1}E\left\| \sum_{t_i<t''} T(t''-t_i)[T(t'-t'')-I]\gamma_{1,i} \right\|^p \\
&\quad \leq 6^{p-1}M^p \parallel T(t'-t'')-I \parallel^p \left(\int_{-\infty}^{t''} e^{-\delta(t''-s)}ds \right)^{p-1} \\
&\quad\quad \times\left(\int_{-\infty}^{t''} e^{-\delta(t''-s)}E \parallel g_1(s) \parallel^p ds \right) \\
&\quad\quad +6^{p-1}M^p\left(\int_{t''}^{t'} e^{-\delta(t'-s)}ds \right)^{p-1}\left(\int_{t''}^{t'} e^{-\delta(t'-s)}E \parallel g_1(s) \parallel^p ds \right) \\
&\quad\quad +6^{p-1}M^pC_pE\left[\int_{-\infty}^{t''} e^{-2\delta(t''-s)} \parallel T(t'-t'')-I \parallel^2 \parallel f_1(s) \parallel_{L_2^0}^2 ds \right]^{p/2} \\
&\quad\quad +6^{p-1}M^pC_pE\left[\int_{t''}^{t'} e^{-2\delta(t'-s)} \parallel f_1(s) \parallel_{L_2^0}^2 ds \right]^{p/2} \\
&\quad\quad +3^{p-1}M^p \parallel T(t'-t'')-I \parallel^p \left(\sum_{t_i<t''} e^{-\delta(t''-t_i)} \right)^{p-1} \\
&\quad\quad \times\left(\sum_{t_i<t''} e^{-\delta(t''-t_i)}E \parallel \gamma_{1,i} \parallel^p \right) \\
&\quad \leq 6^{p-1}M^p \parallel T(t'-t'')-I \parallel^p \left(\int_{-\infty}^{t''} e^{-\delta(t''-s)}ds \right)^p \sup_{s\in R} E \parallel g_1(s) \parallel^p
\end{aligned}
$$

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$$
\begin{aligned}
&+6^{p-1}M^p\bigg(\int_{t''}^{t'} e^{-\delta(t'-s)}ds\bigg)^p \sup_{s\in R} E \parallel g_1(s) \parallel^p \\
&+6^{p-1}M^pC_p \parallel T(t'-t'')-I \parallel^p \bigg(\int_{-\infty}^{t''} e^{-\frac{p}{p-2}\delta(t''-s)}ds\bigg)^{\frac{p-2}{p}} \\
&\times\bigg(\int_{-\infty}^{t''} e^{-\frac{p}{2}\delta(t''-s)}ds\bigg) \sup_{s\in R} \parallel f_1(s) \parallel_{L_2^0}^p \\
&+6^{p-1}M^pC_p\bigg(\int_{t''}^{t'} e^{-\frac{p}{p-2}\delta(t'-s)}ds\bigg)^{\frac{p-2}{p}} \\
&\times\bigg(\int_{t''}^{t'} e^{-\frac{p}{2}\delta(t'-s)}ds\bigg) \sup_{s\in R} \parallel f_1(s) \parallel_{L_2^0}^p \\
&+3^{p-1}M^p \parallel T(t'-t'')-I \parallel^p \bigg(\sum_{t_i<t''} e^{-\delta(t''-t_i)}\bigg)^p \sup_{i\in Z} E \parallel \gamma_{1,i} \parallel^p \\
\le\ & 6^{p-1}M^p \parallel g_1 \parallel_\infty^p \frac{\delta^p\varepsilon}{5\tilde{g}_1}\bigg(\int_{-\infty}^{t''} e^{-\delta(t''-s)}ds\bigg)^p \\
&+6^{p-1}M^p \parallel g_1 \parallel_\infty^p \bigg[\bigg(\frac{\varepsilon}{5\tilde{g}_1}\bigg)^{1/p}\bigg]^p \\
&+6^{p-1}M^pC_p \parallel f_1 \parallel_\infty^p \frac{(\frac{p\delta}{p-2})^{(p-2)/p}\frac{p\delta}{2}\varepsilon}{5\tilde{f}_1}\bigg(\int_{-\infty}^{t''} e^{-\frac{p}{p-2}\delta(t''-s)}ds\bigg)^{\frac{p-2}{p}} \\
&\times\bigg(\int_{-\infty}^{t''} e^{-\frac{p}{2}\delta(t''-s)}ds\bigg) \\
&+6^{p-1}M^pC_p \parallel f_1 \parallel_\infty^p \bigg[\bigg(\frac{\varepsilon}{5\tilde{f}_1}\bigg)^{p/2(p-1)}\bigg]^{2(p-2)/p} \\
&+3^{p-1}M^p\frac{(1-e^{-\delta\alpha})^p\varepsilon}{5\tilde{\gamma}_1}\bigg(\sum_{t_i<t''} e^{-\delta(t''-t_i)}\bigg)^p \parallel \gamma_{1,i} \parallel_\infty^p \\
<\ & \frac{\varepsilon}{5}+\frac{\varepsilon}{5}+\frac{\varepsilon}{5}+\frac{\varepsilon}{5}+\frac{\varepsilon}{5}=\varepsilon.
\end{aligned}
$$

Consequently, $F \in UPC(R, L^p(P, H))$.

Step 2. $F \in AP_T(R, L^p(P, H))$.

Let $t_i < t \le t_{i+1}$. For $\varepsilon > 0$, let Ω_ε be a relatively dense set of R formed by ε-periods of F. For $\tau \in \Omega_\varepsilon$ and $0 < \eta < \min\{\varepsilon, \alpha/2\}$, we have

$$
\begin{aligned}
&E \parallel F(t+\tau)-F(t) \parallel^p \\
&\quad \le 3^{p-1}E\bigg\| \int_{-\infty}^{t} T(t-s)[g_1(s+\tau)-g_1(s)]ds\bigg\|^p \\
&\quad\quad +3^{p-1}E\bigg\| \int_{-\infty}^{t} T(t-s)[f_1(s+\tau)-f_1(s)]dW(s)\bigg\|^p
\end{aligned}
$$

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$$+3^{p-1}E\Big\| \sum_{t_i<t+\tau} T(t+\tau-t_i)\gamma_{1,i} - \sum_{t_i<t} T(t-t_i)\gamma_{1,i}\Big\|^p$$
$$= \sum_{k=1}^{3} J_k.$$

Using Höder's inequality, it follows that

$$\begin{aligned}
J_1 & \\
&\leq 3^{p-1}M^p\Big(\int_{-\infty}^{t} e^{-\delta(t-s)}ds\Big)^{p-1} \\
&\quad\times\Big(\int_{-\infty}^{t} e^{-\delta(t-s)}E \parallel g_1(s+\tau)-g_1(s)\parallel^p ds\Big) \\
&\leq 3^{p-1}M^p\Big(\int_{-\infty}^{t} e^{-\delta(t-s)}ds\Big)^{p-1} \\
&\quad\times\Big[\sum_{j=-\infty}^{i-1}\int_{t_j+\eta}^{t_{j+1}-\eta} e^{-\delta(t-s)}E \parallel g_1(s+\tau)-g_1(s)\parallel^p ds \\
&\quad+\sum_{j=-\infty}^{i-1}\int_{t_j}^{t_j+\eta} e^{-\delta(t-s)}E \parallel g_1(s+\tau)-g_1(s)\parallel^p ds \\
&\quad+\sum_{j=-\infty}^{i-1}\int_{t_{j+1}-\eta}^{t_{j+1}} e^{-\delta(t-s)}E \parallel g_1(s+\tau)-g_1(s)\parallel^p ds \\
&\quad+\int_{t_i}^{t} e^{-\delta(t-s)}E \parallel g_1(s+\tau)-g_1(s)\parallel^p ds\Big].
\end{aligned}$$

Since $g_1 \in AP_T(R, L^p(P,H))$, one has

$$E \parallel g_1(t+\tau)-g_1(t)\parallel^p < \varepsilon$$

for all $t \in [t_j+\eta, t_{j+1}-\eta]$, $j \in Z, j \leq i$, and $t-s \geq t-t_i+t_i-(t_{j+1}-\eta) \geq t-t_i+\alpha(i-1-j)+\eta$. Then,

$$\begin{aligned}
&\sum_{j=-\infty}^{i-1}\int_{t_j+\eta}^{t_{j+1}-\eta} e^{-\delta(t-s)}E \parallel g_1(s+\tau)-g_1(s)\parallel^p \\
&\quad\leq \varepsilon\sum_{j=-\infty}^{i-1}\int_{t_j+\eta}^{t_{j+1}-\eta} e^{-\delta(t-s)}ds \\
&\quad\leq \frac{\varepsilon}{\delta}\sum_{j=-\infty}^{i-1} e^{-\delta(t-t_{j+1}+\eta)} \\
&\quad\leq \frac{\varepsilon}{\delta}\sum_{j=-\infty}^{i-1} e^{-\delta\alpha(i-j-1)}
\end{aligned}$$

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$$\leq \frac{\varepsilon}{\delta(1-e^{-\delta\alpha})},$$

$$\begin{aligned}
&\sum_{j=-\infty}^{i-1}\int_{t_j}^{t_j+\eta} e^{-\delta(t-s)}E\parallel g_1(s+\tau)-g_1(s)\parallel^p ds\\
&\quad\leq 2^{p-1}\sup_{s\in R}E\parallel g_1(s)\parallel^p\sum_{j=-\infty}^{i-1}\int_{t_j}^{t_j+\eta}e^{-\delta(t-s)}ds\\
&\quad\leq 2^{p-1}\parallel g_1\parallel_\infty^p\varepsilon e^{\delta\eta}\sum_{j=-\infty}^{i-1}e^{-\delta(t-t_j)}\\
&\quad\leq 2^{p-1}\parallel g_1\parallel_\infty^p\varepsilon e^{\delta\eta}e^{-\delta(t-t_i)}\sum_{j=-\infty}^{i-1}e^{-\delta\alpha(i-j)}\\
&\quad\leq \frac{2^{p-1}\parallel g_1\parallel_\infty^p e^{\delta\alpha/2}\varepsilon}{1-e^{-\delta\alpha}}.
\end{aligned}$$

Similarly, one has

$$\sum_{j=-\infty}^{i-1}\int_{t_{j+1}-\eta}^{t_{j+1}} e^{-\delta(t-s)}E\parallel g_1(s+\tau)-g_1(s)\parallel^p ds\leq \tilde{M}_1\varepsilon,$$

$$\int_{t_i}^{t} e^{-\delta(t-s)}E\parallel g_1(s+\tau)-g_1(s)\parallel^p ds\leq \tilde{M}_2\varepsilon,$$

where $\tilde{M}_1, \tilde{M}_2$ are some positive constants. Therefore, we get that $J_1\leq \bar{N}_1\varepsilon$ for a positive constant $\bar{N}_1$. Using Hölder's inequality and the Ito integral, we have for $p>2$,

$$\begin{aligned}
&J_2\\
&\leq 3^{p-1}C_pE\Big[\int_{-\infty}^{t}\parallel T(t-s)\parallel^2\parallel f_1(s+\tau)-f_1(s)\parallel_{L_2^0}^2 ds\Big]^{p/2}\\
&\leq 3^{p-1}C_pM^pE\Big[\int_{-\infty}^{t}e^{-2\delta(t-s)}\parallel f_1(s+\tau)-f_1(s)\parallel_{L_2^0}^2 ds\Big]^{p/2}\\
&\leq 3^{p-1}C_pM^p\Big(\int_{-\infty}^{t}e^{-\frac{p}{p-2}\delta(t-s)}ds\Big)^{\frac{p-2}{p}}\\
&\quad\times\Big[\sum_{j=-\infty}^{i-1}\int_{t_j+\eta}^{t_{j+1}-\eta}e^{-\frac{p}{2}\delta(t-s)}E\parallel f_1(s+\tau)-f_1(s)\parallel_{L_2^0}^p ds\\
&\quad+\sum_{j=-\infty}^{i-1}\int_{t_j}^{t_j+\eta}e^{-\frac{p}{2}\delta(t-s)}E\parallel f_1(s+\tau)-f_1(s)\parallel_{L_2^0}^p ds\\
&\quad+\sum_{j=-\infty}^{i-1}\int_{t_{j+1}-\eta}^{t_{j+1}}e^{-\frac{p}{2}\delta(t-s)}E\parallel f_1(s+\tau)-f_1(s)\parallel_{L_2^0}^p ds
\end{aligned}$$

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$$+\int_{t_i}^{t} e^{-\frac{p}{2}\delta(t-s)} E \parallel f_1(s+\tau)-f_1(s) \parallel_{L_2^0}^p ds\Big].$$

Since $f_1 \in AP_T(R, L^p(P, L_2^0))$, one has

$$E \parallel f_1(t+\tau)-f_1(t) \parallel_{L_2^0}^p < \varepsilon$$

for all $t \in [t_j+\eta, t_{j+1}-\eta]$ and $j \in Z, j \leq i$. Then,

$$\begin{aligned}
&\sum_{j=-\infty}^{i-1} \int_{t_j+\eta}^{t_{j+1}-\eta} e^{-\frac{p}{2}\delta(t-s)} E \parallel f_1(s+\tau)-f_1(s) \parallel_{L_2^0}^p ds \\
&\leq \varepsilon \sum_{j=-\infty}^{i-1} \int_{t_j+\eta}^{t_{j+1}-\eta} e^{-\frac{p}{2}\delta(t-s)} ds \\
&\leq \frac{2}{\delta p} \sum_{j=-\infty}^{i-1} e^{-\frac{p}{2}\delta(t-t_{j+1}+\eta)} \\
&\leq \frac{2\varepsilon}{\delta p} \sum_{j=-\infty}^{i-1} e^{-\frac{p}{2}\delta\alpha(i-j-1)} \\
&\leq \frac{2\varepsilon}{\delta p(1-e^{-\delta\alpha})},
\end{aligned}$$

$$\begin{aligned}
&\sum_{j=-\infty}^{i-1} \int_{t_j}^{t_j+\eta} e^{-\frac{p}{2}\delta(t-s)} E \parallel f_1(s+\tau)-f_1(s) \parallel_{L_2^0}^p ds \\
&\leq 2^{p-1} \sup_{s\in R} E \parallel f_1(s) \parallel_{L_2^0}^p \sum_{j=-\infty}^{i-1} \int_{t_j}^{t_{j+1}+\eta} e^{-\frac{p}{2}\delta(t-s)} ds \\
&\leq 2^{p-1} \sup_{s\in R} E \parallel f_1(s) \parallel_{L_2^0}^p \varepsilon e^{\frac{p}{2}\delta\eta} e^{-\frac{p}{2}\delta(t-t_i)} \sum_{j=-\infty}^{i-1} e^{-\frac{p}{2}\delta\alpha(i-j)} \\
&\leq 2^{p-1} \sup_{s\in R} E \parallel f_1(s) \parallel_{L_2^0}^p \varepsilon e^{\frac{p}{2}\delta\eta} e^{-\frac{p}{2}\delta(t-t_i)} \sum_{j=-\infty}^{i-1} e^{-\frac{p}{2}\delta\alpha(i-j)} \\
&\leq \frac{2^{p-1} \parallel f_1 \parallel_\infty^p e^{\delta\alpha/4}\varepsilon}{1-e^{-\frac{p}{2}\delta\alpha}}.
\end{aligned}$$

Similarly, one has

$$\sum_{j=-\infty}^{i-1} \int_{t_{j+1}-\eta}^{t_{j+1}} e^{-\frac{p}{2}\delta(t-s)} E \parallel f_1(s+\tau)-f_1(s) \parallel_{L_2^0}^p ds \leq \tilde{M}_3\varepsilon,$$

$$\int_{t_i}^{t} e^{-\frac{p}{2}\delta(t-s)} E \parallel f_1(s+\tau)-f_1(s) \parallel_{L_2^0}^p ds \leq \tilde{M}_4\varepsilon,$$

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where $\tilde{M}_3, \tilde{M}_4$ are some positive constants. Therefore, we get that $J_2 \leq \bar{N}_2\varepsilon$ for a positive constant $\bar{N}_2$. For $p = 2$, we have

$$\begin{aligned}
&J_2 \\
&\leq 3M^2 E\int_{-\infty}^{t} e^{-2\delta(t-s)} \parallel f_1(s+\tau) - f_1(s) \parallel_{L_2^0}^2 ds \\
&\leq 3M^2\Big[\sum_{j=-\infty}^{i-1}\int_{t_j+\eta}^{t_{j+1}-\eta} e^{-2\delta(t-s)}E \parallel f_1(s+\tau) - f_1(s) \parallel_{L_2^0}^2 ds \\
&\quad + \sum_{j=-\infty}^{i-1}\int_{t_j}^{t_j+\eta} e^{-2\delta(t-s)}E \parallel f_1(s+\tau) - f_1(s) \parallel_{L_2^0}^2 ds \\
&\quad + \sum_{j=-\infty}^{i-1}\int_{t_{j+1}-\eta}^{t_{j+1}} e^{-2\delta(t-s)}E \parallel f_1(s+\tau) - f_1(s) \parallel_{L_2^0}^2 ds \\
&\quad + \int_{t_i}^{t} e^{-2\delta(t-s)}E \parallel f_1(s+\tau) - f_1(s) \parallel_{L_2^0}^2 ds\Big].
\end{aligned}$$

Similarly, we get that $J_2 \leq \bar{N}_3\varepsilon$ for a positive constant $\bar{N}_3$. For any $\varepsilon > 0$, by Lemma 2.1, there exists relative dense sets of real numbers Ω_ε and integers Q_ε, for every $\tau \in \Omega_\varepsilon$, there exists at least one number $q \in Q_\varepsilon$ such that $|t^q - \tau| < \varepsilon, i \in Z$ and $E \parallel \gamma_{1,i+q} - \gamma_{1,i} \parallel^p < \varepsilon, q \in Q_\varepsilon, i \in Z$. Then,

$$\begin{aligned}
&J_3 \\
&\leq 3^{p-1}E\Big[\sum_{t_i<t} \parallel T(t-t_i) \parallel \parallel \gamma_{1,i+q} - \gamma_{1,i} \parallel\Big]^p \\
&\leq 3^{p-1}M^p E\Big[\Big(\sum_{t_i<t} e^{-\delta(t-t_i)}\Big)^{p-1}\Big(\sum_{t_i<t} e^{-\delta(t-t_i)} \parallel \gamma_{1,i+q} - \gamma_{1,i} \parallel^p\Big)\Big] \\
&\leq 3^{p-1}M^p\Big(\sum_{t_i<t} e^{-\delta(t-t_i)}\Big)^p E \parallel \gamma_{1,i+q} - \gamma_{1,i} \parallel^p \\
&\leq \frac{3^{p-1}M^p\varepsilon}{(1-e^{-\delta\alpha})^p}.
\end{aligned}$$

Hence, $F \in AP_T(R, L^p(P, H))$.

Step 3. $\Phi \in PAP_T^0(R, L^p(P, H))$.

In fact, for $r > 0$, one has

$$\begin{aligned}
&\frac{1}{2r}\int_{-r}^{r} E \parallel \Phi(t) \parallel^p dt \\
&\quad \leq 3^{p-1}\frac{1}{2r}\int_{-r}^{r} E\Big\| \int_{-\infty}^{t} T(t-s)g_2(s)ds\Big\|^p dt \\
&\quad\quad +3^{p-1}\frac{1}{2r}\int_{-r}^{r} E\Big\| \int_{-\infty}^{t} T(t-s)f_2(s)dW(s)\Big\|^p dt
\end{aligned}$$

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$$+3^{p-1}\frac{1}{2r}\int_{-r}^{r}E\bigg\|\sum_{t_i<t}T(t-t_i)\gamma_{2,i}\bigg\|^{p}dt.$$

Then, by Hölder's inequality, we obtains that

$$\begin{aligned}
&3^{p-1}\frac{1}{2r}\int_{-r}^{r}E\bigg\|\int_{-\infty}^{t}T(t-s)g_2(s)ds\bigg\|^{p}dt\\
&=3^{p-1}\frac{1}{2r}\int_{-r}^{r}E\bigg\|\int_{0}^{\infty}T(s)g_2(t-s)ds\bigg\|^{p}dt\\
&\le 3^{p-1}M^{p}\frac{1}{2r}\int_{-r}^{r}\left(\int_{0}^{\infty}e^{-\delta s}ds\right)^{p-1}\int_{0}^{\infty}e^{-\delta s}E\parallel g_2(t-s)\parallel^{p}dsdt\\
&=3^{p-1}M^{p}\left(\int_{0}^{\infty}e^{-\delta s}ds\right)^{p-1}\int_{0}^{\infty}e^{-\delta s}ds\frac{1}{2r}\int_{-r}^{r}E\parallel g_2(t-s)\parallel^{p}dt.
\end{aligned}$$

By Hölder's inequality and the Ito integral, we have for $p>2$,

$$\begin{aligned}
&3^{p-1}\frac{1}{2r}\int_{-r}^{r}E\bigg\|\int_{-\infty}^{t}T(t-s)f_2(s)dW(s)\bigg\|^{p}dt\\
&=3^{p-1}\frac{1}{2r}\int_{-r}^{r}E\bigg\|\int_{0}^{\infty}T(s)f_2(t-s)dW(s)\bigg\|^{p}dt\\
&\le 3^{p-1}C_p\frac{1}{2r}\int_{-r}^{r}E\bigg[\int_{0}^{\infty}e^{-2s}\parallel f_2(t-s)\parallel_{L_2^0}^{2}ds\bigg]^{p/2}dt\\
&\le 3^{p-1}M^{p}C_p\frac{1}{2r}\int_{-r}^{r}\left(\int_{0}^{\infty}e^{-\frac{p}{p-2}\delta s}ds\right)^{\frac{p-2}{p}}\\
&\quad\times\int_{0}^{\infty}e^{-\frac{p}{2}\delta s}E\parallel f_2(t-s)\parallel_{L_2^0}^{p}dsdt\\
&=3^{p-1}M^{p}C_p\left(\int_{0}^{\infty}e^{-\frac{p-2}{p}\delta s}ds\right)^{\frac{p-2}{p}}\int_{0}^{\infty}e^{-\frac{p}{2}\delta s}ds\\
&\quad\times\frac{1}{2r}\int_{-r}^{r}E\parallel f_2(t-s)\parallel_{L_2^0}^{p}dt.
\end{aligned}$$

For $p=2$, we have

$$\begin{aligned}
&\frac{3}{2r}\int_{-r}^{r}E\bigg\|\int_{-\infty}^{t}T(t-s)f_2(s)dW(s)\bigg\|^{2}dt\\
&\le 3M^{2}\frac{1}{2r}\int_{-r}^{r}\int_{0}^{\infty}e^{-2s}E\parallel f_2(t-s)\parallel_{L_2^0}^{2}dsdt\\
&=3M^{2}\left(\int_{0}^{\infty}e^{-2\delta s}ds\right)\frac{1}{2r}\int_{-r}^{r}E\parallel f_2(t-s)\parallel_{L_2^0}^{p}dt.
\end{aligned}$$

Since $g_2\in PAP_T^0(R,L^p(P,H))$, $f_2\in PAP_T^0(R,L^p(P,L_2^0))$, it follows that $g_2(\cdot-s)\in PAP_T^0(R,\ L^p(P,H))$, $f_2(\cdot-s)\in PAP_T^0(R,L^p(P,L_2^0))$ for each $s\in R$ by

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Remark 2.1, hence

$$3^{p-1}\frac{1}{2r}\int_{-r}^{r} E\left\| \int_{-\infty}^{t} T(t-s)g_2(s)ds \right\|^p dt \to 0 \quad \text{as } r\to\infty,$$

$$3^{p-1}\frac{1}{2r}\int_{-r}^{r} E\left\| \int_{-\infty}^{t} T(t-s)f_2(s)dW(s) \right\|^p dt \to 0 \quad \text{as } r\to\infty$$

for all $s\in R$.

For a given $i\in Z$, define the function $v(t)$ by $v(t)=T(t-t_i)\gamma_{2,i}, t_i<t\le t_{i+1}$, then

$$\begin{aligned}&\lim_{t\to\infty} E\parallel v(t)\parallel^p\\ &\quad=\lim_{t\to\infty} E\parallel T(t-t_i)\gamma_{2,i}\parallel^p\le \lim_{t\to\infty} M^p e^{-p\delta(t-t_i)}\sup_{i\in Z} E\parallel \gamma_{2,i}\parallel^p=0.\end{aligned}$$

Thus $v\in PC_T^0(R,L^p(P,H))\subset PAP_T^0(R,L^p(P,H))$. Define $v_j: R\to L^p(P,H)$ by

$$v_j(t)=T(t-t_{i-j})\gamma_{2,i-j},\quad t_i<t\le t_{i+1}, j\in N.$$

So $v_j\in PAP_T^0(R,L^p(P,H))$. Moreover,

$$\begin{aligned}&E\parallel v_j(t)\parallel^p\\ &\quad=E\parallel T(t-t_{i-j})\gamma_{2,i-j}\parallel^p\\ &\quad\le M^p e^{-p\delta(t-t_{i-j})}\sup_{i\in Z} E\parallel \gamma_{2,i}\parallel^p\\ &\quad\le M^p e^{-p\delta(t-t_i)}e^{-p\delta\alpha j}\sup_{i\in Z} E\parallel \gamma_{2,i}\parallel^p .\end{aligned}$$

Therefore, the series $\sum_{j=0}^{\infty} v_j$ is uniformly convergent on R. By Lemma 2.2, one has

$$\sum_{t_i<t} T(t-t_i)\gamma_{2,i}=\sum_{j=0}^{\infty} v_j(t)\in PAP_T^0(R,L^p(P,H)),$$

that is

$$3^{p-1}\frac{1}{2r}\int_{-r}^{r} E\left\| \sum_{t_i<t} T(t-t_i)\gamma_{2,i} \right\|^p dt \to 0 \quad \text{as } r\to\infty.$$

Using Lebesgue's dominated convergence theorem, we have $\Phi\in PAP_T^0(R, L^p(P,H))$.

Finally, to prove that x satisfies (6) for all $t\ge s$, all $s\in R$. Fix $\sigma, \sigma\ne t_i, i\in Z$, we have

$$\begin{aligned}x(\sigma)=&\int_{-\infty}^{\sigma} T(\sigma-s)g(s)ds\\ &+\int_{-\infty}^{\sigma} T(\sigma-s)f(s)dW(s)+\sum_{t_i<\sigma} T(\sigma-t_i)\gamma_i.\end{aligned}$$

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Since $\{T(t) : t \geq 0\}$ is a C_0-semigroup, we have for all $t \in R$

$$\begin{aligned} x(t) & \\ &= \int_{-\infty}^{t} T(t-s)g(s)ds + \int_{-\infty}^{t} T(t-s)f(s)dW(s) + \sum_{t_i<t} T(t-t_i)\gamma_i \\ &= \int_{-\infty}^{\sigma} T(t-s)g(s)ds + \int_{-\infty}^{\sigma} T(t-s)f(s)dW(s) + \sum_{t_i<\sigma} T(t-t_i)\gamma_i \\ &\quad + \int_{\sigma}^{t} T(t-s)g(s)ds + \int_{\sigma}^{t} T(t-s)f(s)dW(s) + \sum_{\sigma<t_i<t} T(t-t_i)\gamma_i \\ &= T(t-\sigma)x(\sigma) + \int_{\sigma}^{t} T(t-s)g(s)ds + \int_{\sigma}^{t} T(t-s)f(s)dW(s) \\ &\quad + \sum_{\sigma<t_i<t} T(t-t_i)\gamma_i. \end{aligned}$$

Hence $x \in PAP_T(R, L^p(P,H))$ is an p-mean piecewise pseudo almost periodic mild solution to system (4)-(5). This completes the proof.

Now, we establish the existence theorem of p-mean piecewise pseudo almost periodic mild solutions to partial impulsive stochastic differential equation (1)-(2). For that, we make the following hypotheses:

(H1) A is the infinitesimal generator of an exponentially stable C_0-semigroup $\{T(t) : t \geq 0\}$. Moreover, $T(t)$ is compact for $t > 0$.

(H2) The functions $g \in PAP_T(R \times L^p(P,K), L^p(P,H))$, $f \in PAP_T(R \times L^p(P, K), L^p(P, L_2^0))$, and for each $t \in R$, $\psi_1, \psi_2 \in L^p(P,K)$,

$$E \parallel g(t,\psi_1) - g(t,\psi_2) \parallel^p + E \parallel f(t,\psi_1) - f(t,\psi_2) \parallel^p_{L_2^0} \leq \Lambda(E \parallel \psi_1 - \psi_2 \parallel^p),$$

where Λ is a concave and continuous nondecreasing function from R^+ to R^+ such that $\Lambda(0) = 0, \Lambda(s) > 0$ for $s > 0$ and $\int_{0^+} \frac{ds}{\Lambda(s)} = +\infty$.

(H3) For any $\beta > 0$, there exist a constant $\mu > 0$ and nondecreasing continuous function $\Theta : R^+ \to R^+$ such that, for all $t \in R$, and $x \in L^p(P,K)$ with $E \parallel x \parallel^p > \mu$,

$$E \parallel g(t,x) \parallel^p + E \parallel f(t,x) \parallel^p_{L_2^0} \leq \beta\Theta(E \parallel x \parallel^p).$$

(H4) The functions $I_i \in PAP(Z, L^p(P,H))$, and for each $t \in R$, $\psi_1, \psi_2 \in L^p(P,H), i \in Z$,

$$E \parallel I_i(\psi_1) - I_i(\psi_2) \parallel^p \leq \tilde{\Lambda}_i(E \parallel \psi_1 - \psi_2 \parallel^p),$$

where $\tilde{\Lambda}_i$ are concave and continuous nondecreasing functions from R^+ to R^+ such that $\tilde{\Lambda}_i(0) = 0, \tilde{\Lambda}_i(s) > 0$ for $s > 0$ and $\int_{0^+} \frac{ds}{\tilde{\Lambda}_i(s)} = +\infty$.

(H5) For any $\beta > 0$, there exist a constant $\mu > 0$ and nondecreasing continuous function $\tilde{\Theta}_i : R^+ \to R^+, i \in Z$, such that, for all $t \in R$, and $x \in L^p(P, H)$ with $E \parallel x \parallel^p > \mu$,

$$E \parallel I_i(x) \parallel^p \leq \beta \tilde{\Theta}_i(E \parallel x \parallel^p).$$

Definition 3.2. An $\mathcal{F}_t$ -progressively measurable process $\{x(t)\}_{t \in R}$ is called a mild solution of system (1)-(2) if for any $t \in R, t > \sigma, \sigma \neq t_i, i \in Z$,

$$x(t) = T(t-\sigma)x(\sigma) + \int_\sigma^t T(t-s)g(s, x(s))ds + \int_\sigma^t T(t-s)f(s, x(s))dW(s) + \sum_{\sigma < t_i < t} T(t-t_i)I_i(x(t_i)). \quad (8)$$

Theorem 3.2. *Assume that assumptions* (H1)-(H5) *are satisfied. Then system* (1)-(2) *has a mild solution* $x \in PAP_T(R, L^p(P, H))$.

Proof. Consider the operator $\Psi : PAP_T(R, L^p(P, H)) \cap UPC(R, L^p(P, H)) \to PC(R, L^p(P, H))$ defined by

$$(\Psi x)(t) = \int_{-\infty}^t T(t-s)g(s, x(s))ds + \int_{-\infty}^t T(t-s)f(s, x(s))dW(s) + \sum_{t_i < t} T(t-t_i)I_i(x(t_i)), \quad t \in R.$$

We next show that Ψ has a fixed point in $PAP_T(R, L^p(P, H)) \cap UPC(R, L^p(P, H))$ and divide the proof into several steps.

Step 1. For every $x \in PAP_T(R, L^p(P, H)) \cap UPC(R, L^p(P, H))$, $\Psi x \in PAP_T(R, L^p(P, H)) \cap UPC(R, L^p(P, H))$.

Let $x(\cdot) \in PAP_T(R, L^p(P, H)) \cap UPC(R, L^p(P, H))$, by (H2), (H4) and Lemmas 2.3, 2.4, we deduce that $g(\cdot, x(\cdot)), f(\cdot, x(\cdot)) \in PAP_T(R, L^p(P, H))$ and $I_i(x(t_i)) \in PAP(Z, L^p(P, H))$ Similarly as the proof of Theorem 3.1, one has $\Psi x \in PAP_T(R, L^p(P, H)) \cap UPC(R, L^p(P, H))$.

Step 2. Ψ maps bounded sets into bounded sets in $PAP_T(L^p(P, H)) \cap UPC(R, L^p(P, H))$.

Indeed, let $r^* > 0$ and $x \in B_{r^*} = \{x \in PAP_T(R, L^p(P, H)) \cap UPC(R, L^p(P, H)) : E \parallel x \parallel^p \leq r^*\}$. It is enough to show that there exists a positive constant $\mathcal{L}$ such that for each $x \in B_{r^*}$ one has $E \parallel \Phi x \parallel^p \leq \mathcal{L}$. Let $\beta > 0$ be fixed. By (H3) and (H5) it follows that there exist a positive constant μ such that, for all $t \in R$ and $x \in L^p(P, H)$ with $E \parallel x \parallel^p > \mu$,

$$E \parallel g(t, x) \parallel^p + E \parallel f(t, x) \parallel^p_{L_2^0} \leq \beta \Theta(E \parallel x \parallel^p),$$

$$E \parallel I_i(x) \parallel^p \leq \beta \tilde{\Theta}_i(E \parallel x \parallel^p), i \in Z.$$

Let

$$\nu = \sup_{t \in R}\{E \parallel g(t, x) \parallel^p, E \parallel f(t, x) \parallel^p_{L_2^0} : E \parallel x \parallel^p \leq \mu\},$$

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$$\nu_1 = \sup_{t\in R, i\in Z}\{E \parallel I_i(x) \parallel^p : E \parallel x \parallel^p \leq \mu\}.$$

Thus, for all $t \in R$ and $x \in L^p(P, H)$,

$$E \parallel g(t,x) \parallel^p + E \parallel f(t,x) \parallel^p_{L_2^0} \leq \beta\Theta(E \parallel x \parallel^p) + \nu, \tag{9}$$

$$E \parallel I_i(x) \parallel^p \leq \beta\tilde{\Theta}_i(E \parallel x \parallel^p) + \nu_1, i \in Z. \tag{10}$$

Note that, for β sufficiently small, we can choose $M^* > 0$ such that for $p > 2$,

$$\frac{M^*}{N_1[\beta\Theta(M^*) + \nu] + N_2[\beta \sup_{i\in Z} \tilde{\Theta}_i(M^*) + \nu_1]} > 1, \tag{11}$$

where $N_1 = 3^{p-1}M^p[\frac{1}{\delta^p} + C_p(\frac{p-2}{p\delta})^{\frac{p-2}{p}}\frac{2}{p\delta}]$, $N_2 = 3^{p-1}M^p\frac{1}{(1-e^{-\delta\alpha})^p}$. For the case of $p = 2$, take $N_1 = 3M^2[\frac{1}{\delta^2} + \frac{1}{2\delta}]$, $N_2 = 3M^2\frac{1}{(1-e^{-\delta\alpha})^2}$.

Let $x \in B_{r^*}$, and $t \in R$. By (H1), (9), (10), Hölder's inequality and the Ito integral, we have for $p > 2$,

$$\begin{aligned}
&E \parallel (\Psi x)(t) \parallel^p \\
&\leq 3^{p-1}E\left\| \int_{-\infty}^t T(t-s)g(s,x(s))ds\right\|^p \\
&\quad +3^{p-1}E\left\| \int_{-\infty}^t T(t-s)f(s,x(s))dW(s)\right\|^p \\
&\quad +3^{p-1}E\left\| \sum_{t_i<t} T(t-t_i)I_i(x(t_i))\right\|^p \\
&\leq 3^{p-1}M^p\Big(\int_{-\infty}^t e^{-\delta(t-s)}ds\Big)^{p-1}\Big(\int_{-\infty}^t e^{-\delta(t-s)}E \parallel g(s,x(s)) \parallel^p ds\Big) \\
&\quad +3^{p-1}C_pM^pE\Big(\int_{-\infty}^t e^{-2\delta(t-s)} \parallel f(s,x(s)) \parallel^2_{L_2^0} ds\Big)^{p/2} \\
&\quad +3^{p-1}M^pE\Big[\Big(\sum_{t_i<t} e^{-\delta(t-t_i)}\Big)^{p-1}\Big(\sum_{t_i<t} e^{-\delta(t-t_i)} \parallel I_i(x(t_i)) \parallel^p\Big)\Big] \\
&\leq 3^{p-1}M^p\frac{1}{\delta^{p-1}}\Big(\int_{-\infty}^t e^{-\delta(t-s)}[\beta\Theta(E \parallel x(s) \parallel^p) + \nu]ds\Big) \\
&\quad +3^{p-1}M^pC_p\Big(\int_{-\infty}^t e^{-\frac{p}{p-2}\delta(t-s)}ds\Big)^{\frac{p-2}{p}} \\
&\quad \times\Big(\int_{-\infty}^t e^{-\frac{p}{2}\delta(t-s)}[\beta\Theta(E \parallel x(s) \parallel^p) + \nu]ds\Big) \\
&\quad +3^{p-1}M^p\frac{1}{(1-e^{-\delta\alpha})^{p-1}}\Big(\sum_{t_i<t} e^{-\delta(t-t_i)}[\beta\tilde{\Theta}_i(E \parallel x(t_i) \parallel^p) + \nu_1]\Big) \\
&\leq 3^{p-1}M^p\frac{1}{\delta^p}[\beta\Theta(r^*) + \nu] + 3^{p-1}M^pC_p\Big(\frac{p-2}{p\delta}\Big)^{\frac{p-2}{p}}\frac{2}{p\delta}[\beta\Theta(r^*) + \nu]
\end{aligned}$$

$$+3^{p-1}M^p\frac{1}{(1-e^{-\delta\alpha})^p}[\beta\sup_{i\in Z}\tilde{\Theta}_i(r^*)+\nu_1]:=\mathcal{L}_1.$$

For $p=2$, we have

$$\begin{aligned}&E\parallel(\Psi x)(t)\parallel^2\\&\quad\le 3M^2\frac{1}{\delta^2}[\beta\Theta(r^*)+\nu]+3M^2\frac{1}{2\delta}[\beta\Theta(r^*)+\nu]\\&\qquad+3M^2\frac{1}{(1-e^{-\delta\alpha})^2}[\beta\sup_{i\in Z}\tilde{\Theta}_i(r^*)+\nu_1]:=\mathcal{L}_2.\end{aligned}$$

Take $\mathcal{L}=\max\{\mathcal{L}_1,\mathcal{L}_2\}$. Then for each $x\in B_{r^*}$, we have $E\parallel\Psi x\parallel^p\le\mathcal{L}$.

Step 3. $\Psi: PAP_T(L^p(P,H))\cap UPC(R,L^p(P,H))\to PAP_T(L^p(P,H))\cap UPC(R,L^p(P,H))$ is continuous.

Let $\{x^{(n)}\}\subseteq B_{r^*}$ with $x^{(n)}\to x(n\to\infty)$ in $PAP_T(L^p(P,H))\cap UPC(R,L^p(P,H))$, then there exists a bounded subset $\tilde{K}\subseteq L^p(P,K)$ such that $R(x)\subseteq\tilde{K}$, $R(x^n)\subseteq\tilde{K}, n\in N$. By the assumption (H2) and (H4), for any $\varepsilon>0$, there exists $\xi>0$ such that $x,y\in K$ and $E\parallel x-y\parallel^p<\xi$ implies that

$$E\parallel g(s,x(s))-g(s,y(s))\parallel^p<\varepsilon\quad\text{for all } t\in R,$$

$$E\parallel f(s,x(s))-f(s,y(s))\parallel^p_{L_2^0}<\varepsilon\quad\text{for all } t\in R,$$

and

$$E\parallel I_i(x)-I_i(y)\parallel^p<\varepsilon\quad\text{for all } i\in Z,$$

For the above ξ there exists n_0 such that $E\parallel x^{(n)}(t)-x(t)\parallel^p<\varepsilon$ for $n>n_0$ and $t\in R$, then for $n>n_0$, we have

$$E\parallel g(s,x^{(n)}(s))-g(s,x(s))\parallel^p<\varepsilon\quad\text{for all } t\in R,$$

$$E\parallel f(s,x^{(n)}(s))-f(s,x(s))\parallel^p_{L_2^0}<\varepsilon\quad\text{for all } t\in R,$$

and

$$E\parallel I_i(x^{(n)})-I_i(x)\parallel^p<\varepsilon\quad\text{for all } i\in Z.$$

Then, by (H2), (H4) and Hölder's inequality, we have that for $p>2$,

$$\begin{aligned}&E\parallel(\Psi x^{(n)})(t)-(\Psi x)(t)\parallel^p\\&\quad\le 3^{p-1}E\left\|\int_{-\infty}^t T(t-s)[g(s,x^{(n)}(s))-g(s,x(s))]ds\right\|^p\\&\qquad+3^{p-1}E\left\|\int_{-\infty}^t T(t-s)[f(s,x^{(n)}(s))-f(s,x(s))]dW(s)\right\|^p\\&\qquad+3^{p-1}E\left\|\sum_{t_i<t}T(t-t_i)[I_i(x^{(n)}(t_i))-I_i(x(t_i))]\right\|^p\\&\quad\le 3^{p-1}M^p\left(\int_{-\infty}^t e^{-\delta(t-s)}ds\right)^{p-1}\end{aligned}$$

$$
\begin{aligned}
&\times\Big(\int_{-\infty}^{t} e^{-\delta(t-s)} E \parallel g(s,x^{(n)}(s)) - g(s,x(s)) \parallel^p ds\Big)\\
&+3^{p-1}C_pM^p\Big(\int_{-\infty}^{t} e^{-2\delta(t-s)} E \parallel f(s,x^{(n)}(s)) - f(s,x(s)) \parallel^2_{L_2^0} ds\Big)^{p/2}\\
&+3^{p-1}M^pE\bigg[\Big(\sum_{t_i<t} e^{-\delta(t-t_i)}\Big)^{p-1}\\
&\times\Big(\sum_{t_i<t} e^{-\delta(t-t_i)} \parallel I_i(x^{(n)}(t_i)) - I_i(x(t_i)) \parallel^p\Big)\bigg]\\
\leq\ & 3^{p-1}M^p\Big(\int_{-\infty}^{t} e^{-\delta(t-s)}ds\Big)^p\varepsilon\\
&+3^{p-1}C_pM^p\Big(\int_{-\infty}^{t} e^{-\frac{p}{p-2}\delta(t-s)}ds\Big)^{\frac{p-2}{p}}\Big(\int_{-\infty}^{t} e^{-\frac{p}{2}\delta(t-s)}ds\Big)\varepsilon\\
&+3^{p-1}M^p\frac{1}{(1-e^{-\delta\alpha})^{p-1}}\Big(\sum_{t_i<t} e^{-\delta(t-t_i)}\Big)\varepsilon\\
\leq\ & 3^{p-1}M^p\bigg[\frac{1}{\delta^p} + C_p\Big(\frac{p-2}{p\delta}\Big)^{\frac{p-2}{2}}\frac{2}{p\delta} + \frac{1}{(1-e^{-\delta\alpha})^p}\bigg]\varepsilon.
\end{aligned}
$$

For $p = 2$, we have

$$
\begin{aligned}
E &\parallel (\Psi x^{(n)})(t) - (\Psi x)(t) \parallel^2\\
&\leq 3M^2\bigg[\frac{1}{\delta^2} + \frac{1}{2\delta} + \frac{1}{(1-e^{-\delta\alpha})^2}\bigg]\varepsilon.
\end{aligned}
$$

Thus Ψ is continuous.

Step 4. Ψ maps bounded sets into equicontinuous sets of $PAP_T(L^p(P,H))\cap UPC(R, L^p(P,H))$.

Let $\tau_1,\tau_2 \in (t_i,t_{i+1}), i \in Z, \tau_1 < \tau_2$, and $x \in B_{r^*}$. Then, by (H1), (9), (10), Hölder's inequality and the Ito integral, we have for $p > 2$,

$$
\begin{aligned}
E &\parallel (\Psi x)(\tau_2) - (\Psi x)(\tau_1) \parallel^p\\
&\leq 3^{p-1}E\bigg\| \int_{-\infty}^{\tau_2} T(\tau_2-s)g(s,x(s))ds - \int_{-\infty}^{\tau_1} T(\tau_1-s)g(s,x(s))ds\bigg\|^p\\
&\quad+3^{p-1}E\bigg\| \int_{-\infty}^{\tau_2} T(\tau_2-s)f(s,x(s))dW(s)\\
&\quad-\int_{-\infty}^{\tau_1} T(\tau_1-s)f(s,x(s))dW(s)\bigg\|^p\\
&\quad+3^{p-1}E\bigg\| \sum_{t_i<\tau_2} T(\tau_2-t_i)I_i(x(t_i)) - \sum_{t_i<\tau_1} T(\tau_2-t_i)I_i(x(t_i))\bigg\|^p\\
&\leq 6^{p-1}E\bigg\| \int_{-\infty}^{\tau_1} T(\tau_1-s)[T(\tau_2-\tau_1)-I]g(s,x(s))ds\bigg\|^p
\end{aligned}
$$

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$$
\begin{aligned}
&+6^{p-1}E\bigg\| \int_{\tau_1}^{\tau_2} T(\tau_2-s)g(s,x(s))ds\bigg\|^p \\
&+6^{p-1}E\bigg\| \int_{-\infty}^{\tau_1} T(\tau_1-s)[T(\tau_2-\tau_1)-I]f(s,x(s))dW(s)\bigg\|^p \\
&+6^{p-1}E\bigg\| \int_{\tau_1}^{\tau_2} T(\tau_2-s)f(s,x(s))dW(s)\bigg\|^p \\
&+3^{p-1}E\bigg\| \sum_{t_i<\tau_1} T(\tau_1-t_i)[T(\tau_2-\tau_1)-I]I_i(x(t_i))\bigg\|^p \\
\le\ & 6^{p-1}M^p \parallel T(\tau_2-\tau_1)-I\parallel^p \left(\int_{-\infty}^{\tau_1} e^{-\delta(\tau_1-s)}ds\right)^{p-1} \\
&\times\left(\int_{-\infty}^{\tau_1} e^{-\delta(\tau_1-s)}E\parallel g(s,x(s))\parallel^p ds\right) \\
&+6^{p-1}M^p\left(\int_{\tau_1}^{\tau_2} e^{-\delta(\tau_2-s)}ds\right)^{p-1} \\
&\times\left(\int_{\tau_1}^{\tau_2} e^{-\delta(\tau_2-s)}E\parallel g(s,x(s))\parallel^p ds\right) \\
&+6^{p-1}M^pC_pE\bigg[\int_{-\infty}^{\tau_1} e^{-2\delta(\tau_1-s)}\parallel T(\tau_2-\tau_1)-I\parallel^2 \\
&\times\parallel f(s,x(s))\parallel^2_{L_2^0} ds\bigg]^{p/2} \\
&+6^{p-1}M^pC_pE\bigg[\int_{\tau_1}^{\tau_2} e^{-2\delta(\tau_2-s)}\parallel f(s,x(s))\parallel^2_{L_2^0} ds\bigg]^{p/2} \\
&+3^{p-1}M^p\parallel T(\tau_2-\tau_1)-I\parallel^p\left(\sum_{t_i<\tau_1} e^{-\delta(\tau_1-t_i)}\right)^{p-1} \\
&\times\left(\sum_{t_i<\tau_1} e^{-\delta(\tau_1-t_i)}E\parallel I_i(x(t_i))\parallel^p\right) \\
\le\ & 6^{p-1}M^p \parallel T(\tau_2-\tau_1)-I\parallel^p \left(\int_{-\infty}^{\tau_1} e^{-\delta(\tau_1-s)}ds\right)^{p-1} \\
&\times\left(\int_{-\infty}^{\tau_1} e^{-\delta(\tau_1-s)}[\beta\Theta(E\parallel x(s)\parallel^p)+\nu]ds\right) \\
&+6^{p-1}M^p\left(\int_{\tau_1}^{\tau_2} e^{-\delta(\tau_2-s)}ds\right)^{p-1} \\
&\times\left(\int_{\tau_1}^{\tau_2} e^{-\delta(\tau_2-s)}[\beta\Theta(E\parallel x(s)\parallel^p)+\nu]ds\right) \\
&+6^{p-1}M^pC_p\parallel T(\tau_2-\tau_1)-I\parallel^p\left(\int_{-\infty}^{\tau_1} e^{-\frac{p}{p-2}\delta(\tau_1-s)}ds\right)^{\frac{p-2}{p}}
\end{aligned}
$$

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$$
\begin{aligned}
&\times\bigg(\int_{-\infty}^{\tau_1} e^{-\frac{p}{2}\delta(\tau_1-s)}[\beta\Theta(E\parallel x(s)\parallel^p)+\nu]ds\bigg)\\
&+6^{p-1}M^pC_p\bigg(\int_{\tau_1}^{\tau_2} e^{-\frac{p}{p-2}\delta(\tau_2-s)}ds\bigg)^{\frac{p-2}{p}}\\
&\times\bigg(\int_{\tau_1}^{\tau_2} e^{-\frac{p}{2}\delta(\tau_2-s)}[\beta\Theta(E\parallel x(s)\parallel^p)+\nu]ds\bigg)\\
&+3^{p-1}M^p\parallel T(\tau_2-\tau_1)-I\parallel^p\left(\sum_{t_i<\tau_1} e^{-\delta(\tau_1-t_i)}\right)^{p-1}\\
&\times\bigg(\sum_{t_i<\tau_1} e^{-\delta(\tau_1-t_i)}[\beta\tilde{\Theta}_i(E\parallel x(t_i)\parallel^p)+\nu_1]\bigg)\\
\le\;& 6^{p-1}M^p\parallel T(\tau_2-\tau_1)-I\parallel^p\frac{1}{\delta^p}[\beta\Theta(r^*)+\nu]\\
&+6^{p-1}M^p\bigg(\int_{\tau_1}^{\tau_2} e^{-\delta(\tau_2-s)}ds\bigg)^p[\beta\Theta(r^*)+\nu]\\
&+6^{p-1}M^pC_p\parallel T(\tau_2-\tau_1)-I\parallel^p\left(\frac{p-2}{p\delta}\right)^{\frac{p-2}{p}}\frac{2}{p\delta}[\beta\Theta(r^*)+\nu]\\
&+6^{p-1}M^pC_p\bigg(\int_{\tau_1}^{\tau_2} e^{-\frac{p}{p-2}\delta(\tau_2-s)}ds\bigg)^{\frac{p-2}{p}}\\
&\times\bigg(\int_{\tau_1}^{\tau_2} e^{-\frac{p}{2}\delta(\tau_2-s)}ds\bigg)[\beta\Theta(r^*)+\nu]\\
&+3^{p-1}M^p\parallel T(\tau_2-\tau_1)-I\parallel^p\frac{1}{(1-e^{-\delta\alpha})^p}[\beta\tilde{\Theta}_i(r^*)+\nu_1].
\end{aligned}
$$

For $p=2$, we have

$$
\begin{aligned}
&E\parallel(\Psi x)(\tau_2)-(\Psi x)(\tau_1))\parallel^2\\
&\quad\le 6M^2\parallel T(\tau_2-\tau_1)-I\parallel^2\frac{1}{\delta^2}[\beta\Theta(r^*)+\nu]\\
&\quad\quad+6M^2\bigg(\int_{\tau_1}^{\tau_2} e^{-\delta(\tau_2-s)}ds\bigg)^2[\beta\Theta(r^*)+\nu]\\
&\quad\quad+6M^2\parallel T(\tau_2-\tau_1)-I\parallel^2\frac{2}{\delta}[\beta\Theta(r^*)+\nu]\\
&\quad\quad+6M^2\bigg(\int_{\tau_1}^{\tau_2} e^{-2\delta(\tau_2-s)}ds\bigg)[\beta\Theta(r^*)+\nu]\\
&\quad\quad+3M^2\parallel T(\tau_2-\tau_1)-I\parallel^2\frac{1}{(1-e^{-\delta\alpha})^2}[\beta\tilde{\Theta}_i(r^*)+\nu_1].
\end{aligned}
$$

The right-hand side of the above inequality is independent of $x\in B_{r^*}$ and tends to zero as $\tau_2\to\tau_1$, since the compactness of $T(t)$ for $t>0$ implies imply the continuity in the uniform operator topology. Thus, Ψ maps B_{r^*} into an equicontinuous family of functions.

Step 5. $V(t)=\{(\Psi x)(t): x\in B_{r^*}\}$ is relatively compact in $L^p(P,H)$ for each $t\in R$.

For each $t\in R$, and let ε be a real number satisfying $0<\varepsilon<1$. For $x\in B_{r^*}$, we define

$$\begin{aligned}
&(\Psi_\varepsilon x)(t)\\
&=T(\varepsilon)\bigg[\int_{-\infty}^{t-\varepsilon}T(t-\varepsilon-s)g(s,x(s)ds\\
&\quad+\int_{-\infty}^{t-\varepsilon}T(t-\varepsilon-s)f(s,x(s)dW(s)+\sum_{t_i<t-\varepsilon}T(t-\varepsilon-t_i)I_i(x(t_i))\bigg]\\
&=T(\varepsilon)[(\Psi x)(t-\varepsilon)].
\end{aligned}$$

Since $T(t)(t>0)$ is compact, then the set $V_\varepsilon(t)=\{(\Psi_\varepsilon x)(t): x\in B_{r^*}\}$ is relatively compact in $L^p(P,H)$ for each $t\in R$. Moreover, for every $x\in B_{r^*}$, we have for $p>2$,

$$\begin{aligned}
&E\parallel(\Psi x)(t)-(\Psi_\varepsilon x)(t)\parallel^p\\
&\le 3^{p-1}E\bigg\|\int_{t-\varepsilon}^{t}T(t-s)g(s,x(s))ds\bigg\|^p\\
&\quad+3^{p-1}E\bigg\|\int_{t-\varepsilon}^{t}T(t-s)f(s,x(s))dW(s)\bigg\|^p\\
&\quad+3^{p-1}E\bigg\|\sum_{t-\varepsilon<t_i<t}T(t-t_i)I_i(x(t_i))\bigg\|^p\\
&\le 3^{p-1}M^p\bigg(\int_{t-\varepsilon}^{t}e^{-\delta(t-s)}ds\bigg)^{p-1}\bigg(\int_{t-\varepsilon}^{t}e^{-\delta(t-s)}E\parallel g(s,x(s))\parallel^p ds\bigg)\\
&\quad+3^{p-1}C_pM^pE\bigg(\int_{t-\varepsilon}^{t}e^{-2\delta(t-s)}\parallel f(s,x(s))\parallel_{L_2^0}^2 ds\bigg)^{p/2}\\
&\quad+3^{p-1}M^pE\bigg[\bigg(\sum_{t-\varepsilon<t_i<t}e^{-\delta(t-t_i)}\bigg)^{p-1}\\
&\quad\times\bigg(\sum_{t-\epsilon<t_i<t}e^{-\delta(t-t_i)}\parallel I_i(x(t_i))\parallel^p\bigg)\bigg]\\
&\le 3^{p-1}M^p\bigg(\int_{t-\varepsilon}^{t}e^{-\delta(t-s)}ds\bigg)^{p-1}\\
&\quad\times\bigg(\int_{t-\varepsilon}^{t}e^{-\delta(t-s)}[\beta\Theta(E\parallel x(s)\parallel^p)+\nu]ds\bigg)\\
&\quad+3^{p-1}C_pM^p\bigg(\int_{t-\varepsilon}^{t}e^{-\frac{p}{p-2}\delta(t-s)}ds\bigg)^{\frac{p-2}{p}}\\
&\quad\times\bigg(\int_{t-\varepsilon}^{t}e^{-\frac{p}{2}\delta(t-s)}[\beta\Theta(E\parallel x(s)\parallel^p)+\nu]ds\bigg)
\end{aligned}$$

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$$
\begin{aligned}
&+3^{p-1}M^p\Big(\sum_{t-\varepsilon<t_i<t} e^{-\delta(t-t_i)}\Big)^{p-1}\\
&\times\Big(\sum_{t-\varepsilon<t_i<t} e^{-\delta(t-t_i)}[\beta\tilde{\Theta}_i(E\parallel x_i(t_i)\parallel^p)+\nu_1]\Big)\\
\le\ & 3^{p-1}M^p\Big(\int_{t-\varepsilon}^t e^{-\delta(t-s)}ds\Big)^p[\beta\Theta(r^*)+\nu]\\
&+3^{p-1}C_pM^p\Big(\int_{t-\varepsilon}^t e^{-\frac{p}{p-2}\delta(t-s)}ds\Big)^{\frac{p-2}{p}}\\
&\times\Big(\int_{t-\varepsilon}^t e^{-\frac{p}{2}\delta(t-s)}ds\Big)[\beta\Theta(r^*)+\nu]\\
&+3^{p-1}M^p\Big(\sum_{t-\varepsilon<t_i<t} e^{-\delta(t-t_i)}\Big)^p[\beta\sup_{i\in Z}\tilde{\Theta}_i(r^*)+\nu_1].
\end{aligned}
$$

For $p=2$, we have

$$
\begin{aligned}
&E\parallel(\Psi x)(t)-(\Psi_\varepsilon x)(t)\parallel^2\\
&\le 3M^2\Big(\int_{t-\varepsilon}^t e^{-\delta(t-s)}ds\Big)^2[\beta\Theta(r^*)+\nu]\\
&\quad+3M^2\Big(\int_{t-\varepsilon}^t e^{-2\delta(t-s)}ds\Big)[\beta\Theta(r^*)+\nu]\\
&\quad+3M^2\Big(\sum_{t-\varepsilon<t_i<t} e^{-\delta(t-t_i)}\Big)^2[\beta\sup_{i\in Z}\tilde{\Theta}_i(r^*)+\nu_1].
\end{aligned}
$$

Therefore, letting $\varepsilon\to 0$, it follows that there are relatively compact sets $V_\varepsilon(t)$ arbitrarily close to $V(t)$ and hence $V(t)$ is also relatively compact in $L^p(P,H)$ for each $t\in R$. Since $\{\Psi x: x\in B_{r^*}\}\subset PC_h^0(R,L^p(P,H))$, then $\{\Psi x: x\in B_{r^*}\}$ is a relatively compact set by Lemma 2.5, then Ψ is a compact operator.

Step 6. We now show that there exists an open set $U\subseteq PAP_T(L^p(P,H))\cap UPC(R,L^p(P,H))$ with $x\notin\Psi x$ for $\lambda\in(0,1)$ and $x\in\partial U$.

Let $\lambda\in(0,1)$ and let $x\in L^p(P,H)$ be a possible solution of $x=\lambda\Psi(x)$ for some $0<\lambda<1$. Thus, for each $t\in R$,

$$
\begin{aligned}
x(t)&=\lambda(\Psi x)(t)\\
&=\lambda\int_{-\infty}^t T(t-s)g(s,x(s)ds+\lambda\int_{-\infty}^t T(t-s)f(s,x(s)dW(s)\\
&\quad+\lambda\sum_{t_i<t}T(t-t_i)I_i(x(t_i)).
\end{aligned}
$$

Then, by (H1), (9), (10), Hölder's inequality and the Ito integral, we have for $p>2$,

$$E\parallel x(t)\parallel^p$$

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$$\begin{aligned}&\leq 3^{p-1}M^p\frac{1}{\delta^p}[\beta\Theta(\sup_{s\in R}E\parallel x(s)\parallel^p)+\nu]\\&+3^{p-1}C_pM^p\left(\frac{p-2}{p\delta}\right)^{\frac{p-2}{p}}\frac{2}{p\delta}[\beta\Theta(\sup_{s\in R}E\parallel x(s)\parallel^p)+\nu]\\&+3^{p-1}M^p\frac{1}{(1-e^{-\delta\alpha})^p}[\beta\tilde{\Theta}_i(E\parallel x(t_i)\parallel^p)+\nu_1].\end{aligned}$$

For $p=2$, we have

$$\begin{aligned}&E\parallel x(t)\parallel^2\\&\quad\leq 3M^2\frac{1}{\delta^2}[\beta\Theta(\sup_{s\in R}E\parallel x(s)\parallel^2)+\nu]\\&\quad+3M^2\frac{1}{2\delta}[\beta\Theta(\sup_{s\in R}E\parallel x(s)\parallel^2)+\nu]\\&\quad+3M^2\frac{1}{(1-e^{-\delta\alpha})^2}[\beta\tilde{\Theta}_i(E\parallel x(t_i)\parallel^2)+\nu_1].\end{aligned}$$

Taking the supremum over t, we have for $p>2$,

$$\begin{aligned}&\sup_{t\in R}E\parallel x(t)\parallel^p\\&\quad\leq 3^{p-1}M^p\frac{1}{\delta^p}[\beta\Theta(\sup_{s\in R}E\parallel x(s)\parallel^p)+\nu]\\&\quad+3^{p-1}C_pM^p\left(\frac{p-2}{p\delta}\right)^{\frac{p-2}{p}}\frac{2}{p\delta}[\beta\Theta(\sup_{s\in R}E\parallel x(s)\parallel^p)+\nu]\\&\quad+3^{p-1}M^p\frac{1}{(1-e^{-\delta\alpha})^p}[\beta\tilde{\Theta}_i(\sup_{s\in R}E\parallel x(s)\parallel^p)+\nu_1].\end{aligned}$$

For $p=2$, we have

$$\begin{aligned}&\sup_{t\in R}E\parallel x(t)\parallel^2\\&\quad\leq 3M^2\frac{1}{\delta^2}[\beta\Theta(\sup_{s\in R}E\parallel x(s)\parallel^2)+\nu]\\&\quad+3M^2\frac{1}{2\delta}[\beta\Theta(\sup_{s\in R}E\parallel x(s)\parallel^2)+\nu]\\&\quad+3M^2\frac{1}{(1-e^{-\delta\alpha})^2}[\beta\tilde{\Theta}_i(\sup_{s\in R}E\parallel x(s)\parallel^2)+\nu_1].\end{aligned}$$

Therefore, we have for $p>2$,

$$\frac{\parallel x\parallel_\infty^p}{N_1[\beta\Theta(\parallel x\parallel_\infty^p)+\nu]+N_2[\beta\sup_{i\in Z}\tilde{\Theta}_i(\parallel x\parallel_\infty^p)+\nu_1]}\leq 1,$$

where $N_1=3^{p-1}M^p[\frac{1}{\delta^p}+C_p(\frac{p-2}{p\delta})^{\frac{p-2}{p}}\frac{2}{p\delta}]$, $N_2=3^{p-1}M^p\frac{1}{(1-e^{-\delta\alpha})^p}$. For the case of $p=2$, take $N_1=3M^2[\frac{1}{\delta^2}+\frac{1}{2\delta}]$, $N_2=3M^2\frac{1}{(1-e^{-\delta\alpha})^2}$. Then, by (11), there exists M^* such that $\| x \|_\infty^p \neq M^*$. Set

$$U = \left\{ x \in PAP_T(R, L^p(P,H)) \cap UPC(R, L^p(P,H)) : \| x \|_\infty^p) < M^* \right\}.$$

As a consequence of Steps 1-6, it suffices to show that $\Psi : \overline{U} \to PAP_T(R, L^p(P, H)) \cap UPC(R, L^p(P,H))$ is a compact map. From the choice of U, there is no $x \in \partial U$ such that $x \in \lambda \Psi x$ for $\lambda \in (0,1)$. By Lemma 2.6, we deduce that Ψ has a fixed point $x \in \overline{U}$. The proof is complete.

4 An example

Consider following partial stochastic differential equations of the form

$$\begin{aligned} dz(t,x) = & \frac{\partial^2}{\partial x^2} z(t,x)dt + \varpi_1(t, z(t,x))dt \\ & + \varpi_2(t, z(t,x))dW(t), t \in R, t \neq t_i, \quad i \in Z, x \in [0,\pi], \end{aligned} \tag{12}$$

$$\Delta z(t_i, x) = \beta_i z(t_i, x), \quad i \in Z, x \in [0,\pi], \tag{13}$$

$$z(t,0) = z(t,\pi) = 0, \quad t \in R, \tag{14}$$

where $W(t)$ is a two-sided standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$. In this system, $\beta_i \in PAP(Z,R)$, $t_i = i + \frac{1}{4}|\sin i + \sin \sqrt{2} i|$, $\{t_i^j\}, i \in Z, j \in Z$ are equipotentially almost periodic and $\alpha = \inf_{i \in Z}(t_{i+1} - t_i) > 0$, one can see [21] for more details.

Let $H = L^2([0,\pi])$ with the norm $\| \cdot \|$ and define the operators $A : A(D) \subset H \to H$ by $Av = v''$ with the domain $D(A) := \{v \in H : v'' \in H, v(0) = v(\pi) = 0\}$. It is well known that A is the infinitesimal generator of an analytic compact semigroup $T(t)$ on H and $\| T(t) \| \leq e^{-t}$ for $t \geq 0$ with $M = \delta = 1$. Furthermore, A has a discrete spectrum with eigenvalues of the form $-n^2, n \in N$ and normalized eigenfunctions given by $v_n(\xi) := (\frac{2}{\pi})^{\frac{1}{2}} \sin(n\xi)$. In addition, the following properties hold:

(a) The set of functions $\{v_n : n \in N\}$ is an orthonormal basis for H;

(b) For $v \in H$, $T(t)v = \sum_{n=1}^\infty \exp(-n^2 t)\langle v, v_n\rangle v_n$, and $Av = \sum_{n=1}^\infty n^2 \langle v, v_n\rangle$ $v_n, \ v \in D(A)$.

Taking

$$g(t,\psi)(\cdot) = \varpi_1(t, \psi(t,\cdot)),$$

$$f(t,\psi)(\cdot) = \varpi_2(t, \psi(t,\cdot)),$$

and

$$I_i(\psi)(\cdot) = \beta_i \psi(t_i, \cdot), \quad i \in Z.$$

Then, the above equation (12)-(14) can be written in the abstract form as the system (1)-(2).

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From Theorem 3.2, it follows that the following proposition holds.
Proposition 4.1. Let ϖ_1, ϖ_2 satisfy (H2)-(H5), then system (12)-(14) has an p-mean piecewise pseudo almost periodic mild solution on R.

In the above example, we can take

$$\varpi_1(t, z(t,\cdot)) = \tilde{k}_1[\sin t + \sin\sqrt{2}t + l(t)]z(t,\cdot),$$

$$\varpi_1(t, z(t,\cdot)) = \tilde{k}_2[\sin t + \sin\sqrt{2}t + l(t)]z(t,\cdot),$$

and

$$\beta_i z(t_i,\cdot) = \tilde{c}_i[\sin i + \sin\sqrt{2}i + l(i)]z(t_i,\cdot), \quad i \in Z,$$

where $\tilde{k}_j > 0, j = 1,2,3$, and $\tilde{c}_i > 0, i \in Z$, $l \in UPC(R,R)$ defined by

$$l(t) = \begin{cases} 0, & \text{for } t \le 0, \\ e^{-t}, & \text{for } t \ge 0. \end{cases}$$

From [3], $\sin t + \sin\sqrt{2}t$ is almost periodic. On the other hand,

$$\frac{1}{2r}\int_{-r}^{r} |l(t)|^p dt = \frac{1}{2r}\int_{0}^{r} |l(t)|^p dt = \frac{1}{2r}\int_{0}^{r} e^{-pt} dt = \frac{1}{2r}\frac{1-e^{-pr}}{p}.$$

Consequently

$$\lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r} |l(t)|^p dt = 0.$$

Then, all conditions in Theorem 3.2 are satisfied. Hence, the system (12)-(14) has an p-mean piecewise pseudo almost periodic mild solution.

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Eventual periodicity of a max-type difference equation system

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Abstract In this paper, we investigate the eventual periodicity of the following max-type system of difference equations

$$\begin{cases} x_n = \max\left\{\frac{1}{y_{n-m}}, \min\left\{1, \frac{\alpha}{x_{n-r}}\right\}\right\}, \\ y_n = \max\left\{\frac{1}{x_{n-s}}, \ \min\left\{1, \frac{\beta}{y_{n-t}}\right\}\right\}, \end{cases} \quad n = 0, 1, 2, \cdots,$$

where $\alpha, \beta \in (0, +\infty)$, $m, s, r, t \in \{1, 2, \cdots\}$ with $r \neq m$ and $t \neq s$. We show that every solution of this system with the initial values $x_{-d}, y_{-d}, x_{-d+1}, y_{-d+1}, \cdots, x_{-1}, y_{-1} \in (0, +\infty)$ is eventually periodic with period $m + s$, where $d = \max\{m, s, r, t\}$.

Keywords: System of difference equations; Solution; Eventual periodicity
Mathematics Subject Classification: 39A10; 39A11.

1. Introduction

Recently, there has been a great interest in studying difference equations and systems. A class of difference equations that has attracted recent attention is the class of, so called, max-type difference equations and and systems(see, e.g., [1-14,17,21-24]). On the other hand, some concrete classes of nonlinear systems of difference equations have also attracted some recent attention (see, e.g., [15,16,18-20, 25]).

In 2012, Stević [15] obtained the general solution of the following max-type system of difference equations

$$\begin{cases} x_{n+1} = \max\left\{\frac{\alpha}{x_n}, \frac{y_n}{x_n}\right\}, \\ y_{n+1} = \max\left\{\frac{\alpha}{y_n}, \frac{x_n}{y_n}\right\}, \end{cases} \quad n \in \mathbb{N}_0 \equiv \{0, 1, 2, \cdots\}, \tag{1.1}$$

where $\alpha \in \mathbb{R}_+ \equiv (0, +\infty)$ and the initial values $x_0, y_0 \in [\alpha, +\infty)$ and $y_0/x_0 \geq \max\{\alpha, 1/\alpha\}$.

In 2015, Yazlik et al. [25] studied the following max-type system of difference equations

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$$\begin{cases} x_{n+1} = \max\left\{\frac{1}{x_n}, \min\left\{1, \frac{\alpha}{y_n}\right\}\right\}, \\ y_{n+1} = \max\left\{\frac{1}{y_n}, \min\left\{1, \frac{\alpha}{x_n}\right\}\right\}, \end{cases} \quad n \in \mathbb{N}_0, \tag{1.2}$$

where $\alpha \in \mathbb{R}_+$ and the initial values $x_0, y_0 \in \mathbb{R}_+$, and obtained in an elegant way the general solution of (1.2).

Motivated by aforementioned papers, in this paper, we investigate the eventual periodicity of the following system of difference equations

$$\begin{cases} x_n = \max\left\{\frac{1}{y_{n-m}}, \min\left\{1, \frac{\alpha}{x_{n-r}}\right\}\right\}, \\ y_n = \max\left\{\frac{1}{x_{n-s}}, \ \min\left\{1, \frac{\beta}{y_{n-t}}\right\}\right\}, \end{cases} \quad n \in \mathbb{N}_0 \tag{1.3}$$

where $\alpha, \beta \in \mathbb{R}_+$, $m, s, r, t \in \mathbb{N}$ with $r \neq m$ and $t \neq s$ and the initial values $x_{-d}, y_{-d}, x_{-d+1}, y_{-d+1}, \cdots, x_{-1}, y_{-1} \in \mathbb{R}_+$ with $d = \max\{m, s, r, t\}$. The main result of this paper is the following theorem.

Theorem 1.1 Let $\alpha > 0$ and $\beta > 0$. Then every solution $\{(x_n, y_n)\}_{n \geq -d}$ with the initial values $x_{-d}, y_{-d}, x_{-d+1}, y_{-d+1}, \cdots, x_{-1}, y_{-1} \in \mathbb{R}_+$ of system (1.3) is eventually periodic with period $m + s$.

2. Proof of Theorem 1.1

In this section, we will discuss the periodicity of solutions of system (1.3). Let $\{(x_n, y_n)\}_{n \geq -d}$ be a solution of (1.3) with the initial values $x_{-d}, y_{-d}, x_{-d+1}, y_{-d+1}, \cdots, x_{-1}, y_{-1} \in \mathbb{R}_+$. Write

$$A_n = \min\left\{1, \frac{\alpha}{x_{n-r}}\right\}, \qquad B_n = \min\left\{1, \frac{\beta}{y_{n-t}}\right\}.$$

It is easy to see that $A_n \leq 1$ and $B_n \leq 1$ for any $n \in \mathbb{N}_0$. The main result of this paper is established through the following lemmas.

Lemma 2.1 The following statements are true.

(1) $x_n y_{n-m} \geq 1$ (resp. $y_n x_{n-s} \geq 1$) for all $n \in \mathbb{N}_0$.

(2) $x_n \leq \max\{x_{n-m-s}, A_n\}$ (resp. $y_n \leq \max\{y_{n-m-s}, B_n\}$) for all $n \geq d$.

(3) If $x_n = 1/y_{n-m}$ (resp. $y_n = 1/x_{n-s}$) for some $n \geq d$, then $x_n \leq x_{n-m-s}$(resp. $y_n \leq y_{n-m-s}$). If $x_n = A_n > 1/y_{n-m}$ (resp. $y_n = B_n > 1/x_{n-s}$) for some $n \geq d$, then $x_n > x_{n-m-s}$(resp. $y_n > y_{n-m-s}$).

Proof (1) It follows from $x_n \geq 1/y_{n-m}$ (resp. $y_n \geq 1/x_{n-s}$) that $x_n y_{n-m} \geq 1$ (resp. $y_n x_{n-s} \geq 1$) for all $n \in \mathbb{N}_0$.

(2) Since $y_{n-m} x_{n-m-s} \geq 1$ $(n \geq m)$, one know that for all $n \geq d$,

$$x_n = \max\left\{\frac{x_{n-m-s}}{y_{n-m} x_{n-m-s}}, A_n\right\} \leq \max\{x_{n-m-s}, A_n\}.$$

The other case is treated similarly, which detail is omitted.

(3) If $x_n = A_n > 1/y_{n-m}$ for some $n \geq d$, then combining (1.3) with $A_n \leq 1$ and $B_{n-m} \leq 1$

one know that

$$\begin{aligned} 1 < x_n y_{n-m} &= \max\left\{\frac{x_n}{x_{n-m-s}}, x_n B_{n-m}\right\} \\ &= \max\left\{\frac{x_n}{x_{n-m-s}}, A_n B_{n-m}\right\} \\ &= \frac{x_n}{x_{n-m-s}}, \end{aligned}$$

which implies $x_n > x_{n-m-s}$. If $x_n = 1/y_{n-m}$ for some $n \geq d$, then by $y_{n-m}x_{n-m-s} \geq 1$ $(n \geq m)$ one know that

$$x_n = \frac{x_{n-m-s}}{y_{n-m}x_{n-m-s}} \leq x_{n-m-s}.$$

The other case is treated similarly, which detail is omitted also. This completes the proof of Lemma 2.1.

Lemma 2.2 Let $N, k, l \in \mathbb{N}$ $(l \geq 2)$ such that the following statements hold.

(1) $x_{N+(m+s)\mu} = A_{N+(m+s)\mu} > 1/y_{N+(m+s)\mu-m}$ for every $\mu \in \{k, k+l\}$.

(2) $x_{N+(m+s)\mu} = 1/y_{N+(m+s)\mu-m}$ for every $\mu \in \{k+1, \cdots, k+l-1\}$.

Then $x_{N+(m+s)k} = x_{N+(m+s)(k+1)} = \cdots = x_{N+(m+s)(k+l-1)} < x_{N+(m+s)(k+l)}$.

Proof Combining the conditions (1) and (2) with Lemma 2.1 (3), one know that $x_{N+(m+s)(\mu+1)} \leq x_{N+(m+s)\mu}$ for every $\mu \in \{k, \cdots, k+l-2\}$ and $x_{N+(m+s)\mu} > x_{N+(m+s)(\mu-1)}$ for every $\mu \in \{k, k+l\}$.

Assume on the contrary that $x_{N+(m+s)(\mu+1)} < x_{N+(m+s)\mu}$ for some $k \leq \mu \leq k+l-2$. Then by $A_{N+(m+s)k}B_{N+(m+s)(\mu+1)-m} \leq 1$ one know that

$$\begin{aligned} x_{N+(m+s)k} = A_{N+(m+s)k} &\geq x_{N+(m+s)\mu} \\ &> x_{N+(m+s)(\mu+1)} \\ &= \frac{1}{y_{N+(m+s)(\mu+1)-m}} \\ &= \frac{1}{\max\left\{\frac{1}{x_{N+(m+s)\mu}}, B_{N+(m+s)(\mu+1)-m}\right\}} \\ &= \frac{1}{\frac{1}{x_{N+(m+s)\mu}}} \\ &= x_{N+(m+s)\mu}. \end{aligned}$$

A contradiction. This completes the proof of Lemma 2.2.

In a similar fashion as in the proof of Lemma 2.2, we can obtain the following Lemma 2.3.

Lemma 2.3 Let $N, k, l \in \mathbb{N}$ $(l \geq 2)$ such that the following statements hold.

(1) $y_{N+(m+s)\mu} = B_{N+(m+s)\mu} > 1/x_{N+(m+s)\mu-s}$ for every $\mu \in \{k, k+l\}$.

(2) $y_{N+(m+s)\mu} = 1/x_{N+(m+s)\mu-s}$ for every $\mu \in \{k+1, \cdots, k+l-1\}$.

Then $y_{N+(m+s)k} = y_{N+(m+s)(k+1)} = \cdots = y_{N+(m+s)(k+l-1)} < y_{N+(m+s)(k+l)}$.

By Lemma 2.2 and Lemma 2.3 and Lemma 2.1 (3) one obtain the following Lemma 2.4.

Lemma 2.4 Let $K, L \in \mathbb{N}$.

(1) If $x_{K+(m+s)\mu} = 1/y_{K+(m+s)\mu-m}$ (resp. $y_{K+(m+s)\mu} = 1/x_{N+(m+s)\mu-s}$) for all $\mu \geq L$, then $x_{N+(m+s)\mu}$ (resp. $y_{N+(m+s)\mu}$) is nonincreasing eventually. If $(1 \geq)$ $x_{N+(m+s)\mu} = A_{N+(m+s)\mu} > 1/y_{N+(m+s)\mu-m}$ (resp. $(1 \geq)$ $y_{n+(m+s)\mu} = B_{N+(m+s)\mu} > 1/x_{N+(m+s)\mu-s}$) for all $\mu \geq L$, then $x_{N+(m+s)\mu}$ (resp. $y_{N+(m+s)\mu}$) is increasing eventually .

(2) If $x_{N+(m+s)\mu}$ (resp. $y_{N+(m+s)\mu}$) is nonincreasing eventually and $x_{N+(m+s)\mu} < 1$ (resp. $y_{N+(m+s)\mu} < 1$) for all $\mu \geq L$, then $x_{N+(m+s)\mu}$ (resp. $y_{N+(m+s)\mu}$) is a constant sequence eventually.

(3) If there are two sequences of positive integers $\{p_n\}$ and $\{q_n\}$ with $1 < p_1 < q_1 < p_2 < q_2 < \cdots < p_n < q_n < \cdots$ such that for every $p_n \leq k < q_n$,

$$x_{N+(m+s)k} = A_{N+(m+s)k} > \frac{1}{y_{N+(m+s)k-m}} \text{ (resp. } y_{N+(m+s)k} = B_{N+(m+s)k} > \frac{1}{x_{N+(m+s)k-s}})$$

and for every $p_n \leq k < q_{n+1}$,

$$x_{N+(m+s)k} = \frac{1}{y_{N+(m+s)k-m}} \text{(resp. } y_{N+(m+s)k} = \frac{1}{x_{N+(m+s)k-s}}) ,$$

then $x_{N+(m+s)\mu}$ (resp. $y_{N+(m+s)\mu}$) is nondecreasing eventually and $x_{N+(m+s)\mu} < 1$ (resp. $y_{N+(m+s)\mu} < 1$) eventually.

Proof (1) It follows from Lemma 2.1 (3).

(2) Noting that $1/x_{N+(m+s)\mu} > 1 \geq B_{N+(m+s)\mu+s}$ eventually, we have

$$y_{N+(m+s)\mu+s} = \max\{\frac{1}{x_{N+(m+s)\mu}}, B_{N+(m+s)\mu+s}\} = \frac{1}{x_{N+(m+s)\mu}}$$

eventually. Thus $y_{N+(m+s)\mu+s}$ is nondecreasing eventually. On the other hand, we know from (1) that $y_{N+(m+s)\mu+s}$ is nonincreasing eventually. Thus $y_{N+(m+s)\mu+s}$ and $x_{N+(m+s)\mu}$ are constant sequences eventually. The other case is treated similarly.

(3) Combining Lemma 2.1 (3) with Lemma 2.2 one know that $x_{N+(m+s)(k-1)} < x_{N+(m+s)k}$ for every $p_n \leq k < q_n$ and $x_{N+(m+s)(k-1)} = x_{N+(m+s)k}$ for every $q_n \leq k < p_{n+1}$. Thus $\{x_{N+(m+s)\mu}\}_{\mu \geq p_1}$ is nondecreasing and $x_{N+(m+s)\mu} < 1$ eventually. The other case is treated similarly, which detail is omitted also. This completes the proof of Lemma 2.4.

Now we give the proof of Theorem 1.1.

Proof of Theorem 1.1 It need only to show that $\{x_{(m+s)k+i}\}_{k\geq 1}$ and $\{y_{(m+s)k+i}\}_{k\geq 1}$ are constant sequences eventually for every $i \in \{0, 1, \cdots, m+s-1\}$.

Assume on the contrary that there is some $\theta \in \{0, 1, \cdots, m+s-1\}$ such that $\{x_{(m+s)k+\theta}\}_{k\geq 1}$ (resp. $\{y_{(m+s)k+\theta}\}_{k\geq 1}$) is not a constant sequence eventually. Then by Lemma 2.4 we know that $x_{(m+s)k+\theta}$ (resp. $y_{(m+s)k+\theta}$) is monotone eventually.

We claim that $x_{(m+s)k+\theta}$ is nondecreasing eventually. Indeed, if $x_{(m+s)k+\theta}$ is nonincreasing eventually, then $x_{(m+s)k+\theta} > 1$ eventually and

$$x_{(m+s)k+\theta} = \max\left\{\frac{1}{y_{(m+s)k+\theta-m}}, A_{(m+s)k+\theta}\right\} = \frac{1}{y_{(m+s)k+\theta-m}} \quad \text{eventually.}$$

Thus $y_{(m+s)k+\theta-m}$ is nondecreasing eventually and $y_{(m+s)k+\theta-m} < 1$ eventually. According to Lemma 2.4, there is a sequence of positive integers $k_{11} < k_{12} < \cdots < k_{1n} < \cdots$ such that

$$\begin{aligned} y_{(m+s)k_{1n}+\theta-m} &= \max\left\{\frac{1}{x_{(m+s)k_{1n}+\theta-m-s}}, B_{(m+s)k_{1n}+\theta-m}\right\} \\ &= B_{(m+s)k_{1n}+\theta-m} \\ &= \frac{\beta}{y_{(m+s)k_{1n}+\theta-m-t}} \\ &> \frac{1}{x_{(m+s)k_{1n}+\theta-m-s}}. \end{aligned}$$

From which we see that $y_{(m+s)k+\theta-m-t}$ is nonincreasing eventually and $y_{(m+s)k+\theta-m-t} > 1$ eventually and

$$y_{(m+s)k+\theta-m-t} = \max\left\{\frac{1}{x_{(m+s)k+\theta-m-t-s}}, B_{(m+s)k+\theta-m-t}\right\} = \frac{1}{x_{(m+s)k+\theta-m-t-s}} \quad \text{eventually.}$$

Thus $x_{(m+s)k+\theta-m-t-s}$ is nondecreasing eventually and $x_{(m+s)k+\theta-m-t-s} < 1$ eventually. Again according to Lemma 2.4, there is a sequence of positive integers $\lambda_{11} < \lambda_{12} < \cdots < \lambda_{1n} < \cdots$ such that

$$\begin{aligned} x_{(m+s)\lambda_{1n}+\theta-m-t-s} &= \max\left\{\frac{1}{y_{(m+s)\lambda_{1n}+\theta-m-t-s-m}}, A_{(m+s)\lambda_{1n}+\theta-m-t-s}\right\} \\ &= A_{(m+s)\lambda_{1n}+\theta-m-t-s} \\ &= \frac{\alpha}{x_{(m+s)\lambda_{1n}+\theta-m-t-s-r}} \\ &> \frac{1}{y_{(m+s)\lambda_{1n}+\theta-m-t-s-m}}. \end{aligned}$$

Therefore $x_{(m+s)\lambda_{1n}+\theta-m-t-s-r}$ is nonincreasing eventually and $x_{(m+s)\lambda_{1n}+\theta-m-t-s-r} > 1$ eventually. Write $\omega = m+s+r+t$. By induction, we may obtain that

(1) For every $0 \le \mu \le m+s-1$, $x_{(m+s)k+\theta-\mu\omega} = 1/y_{(m+s)k+\theta-\mu\omega-m}$ is nonincreasing eventually and $x_{(m+s)k+\theta-\mu\omega} > 1$ eventually, and $y_{(m+s)k+\theta-\mu\omega-m-t} = 1/x_{(m+s)k+\theta-\mu\omega-m-t-s}$ is nonincreasing eventually and $y_{(m+s)k+\theta-\mu\omega-m-t} > 1$ eventually

(2) For every $1 \le \mu \le m+s$, $y_{(m+s)k+\theta-(\mu-1)\omega-m}$ is nondecreasing eventually and $y_{(m+s)k+\theta-(\mu-1)\omega-m} < 1$ eventually, and there is a sequence of positive integers $k_{\mu1} < k_{\mu2} < \cdots < k_{\mu n} < \cdots$ such that

$$\begin{aligned} & y_{(m+s)k_{\mu n}+\theta-(\mu-1)\omega-m} \\ =\ & \max\left\{\frac{1}{x_{(m+s)k_{\mu n}+\theta-(\mu-1)\omega-m-s}}, B_{(m+s)k_{\mu n}+\theta-(\mu-1)\omega-m}\right\} \\ =\ & B_{(m+s)k_{\mu n}+\theta-(\mu-1)\omega-m} \\ =\ & \frac{\beta}{y_{(m+s)k_{\mu n}+\theta-(\mu-1)\omega-m-t}} \\ >\ & \frac{1}{x_{(m+s)k_{\mu n}+\theta-(\mu-1)\omega-m-s}}. \end{aligned}$$

(3) For every $1 \le \mu \le m+s$, $x_{(m+s)k+\theta-(\mu-1)\omega-m-t-s}$ is nondecreasing eventually and $x_{(m+s)k+\theta-(\mu-1)\omega-m-t-s} < 1$ eventually, and there is a sequence of positive integers $\lambda_{\mu1} < \lambda_{\mu2} <$

$\cdots < \lambda_{\mu n} < \cdots$ such that

$$\begin{aligned} & x_{(m+s)\lambda_{\mu n}+\theta-(\mu-1)\omega-m-t-s} \\ = & \max\left\{\frac{1}{y_{(m+s)\lambda_{\mu n}+\theta-(\mu-1)\omega-m-t-s-m}}, A_{(m+s)\lambda_{\mu n}+\theta-(\mu-1)\omega-m-t-s}\right\} \\ = & A_{(m+s)\lambda_{\mu n}+\theta-(\mu-1)\omega-m-t-s} \\ = & \frac{\alpha}{x_{(m+s)\lambda_{\mu n}+\theta-(\mu-1)\omega-\omega}} \\ > & \frac{1}{y_{(m+s)\lambda_{\mu n}+\theta-(\mu-1)\omega-m-t-s-m}}. \end{aligned}$$

We may assume without loss of generality that $k_{\mu n} > \lambda_{\mu n} > k_{\mu+1,n} > \lambda_{\mu+1,n}$ for any $1 \leq \mu \leq m+s-1$. Furthermore, we can assume that

$$\lambda_{1n} = \max\{k \leq k_{1n} : x_{(m+s)k+\theta-m-t-s} = A_{(m+s)k+\theta-m-t-s} > \frac{1}{y_{(m+s)k+\theta-m-t-s-m}}\}$$

and for $2 \leq \mu \leq m+s$,

$$\begin{aligned} k_{\mu n} = & \max\{k \leq \lambda_{\mu-1,n} : y_{(m+s)k+\theta-(\mu-1)\omega-m} \\ & = B_{(m+s)k+\theta-(\mu-1)\omega-m} > \frac{1}{x_{(m+s)k+\theta-(\mu-1)\omega-m-s}}\} \end{aligned}$$

and

$$\begin{aligned} \lambda_{\mu n} = & \max\{k \leq k_{\mu n} : x_{(m+s)k+\theta-(\mu-1)\omega-m-t-s} \\ & = A_{(m+s)k+\theta-(\mu-1)\omega-m-t-s} > \frac{1}{y_{(m+s)k+\theta-(\mu-1)\omega-m-t-s-m}}\}. \end{aligned}$$

Then it follows from Lemma 2.2 and Lemma 2.3 that

$$\begin{aligned} & x_{(m+s)k_{1n}+\theta} \\ = & \frac{1}{y_{(m+s)k_{1n}+\theta-m}} = \frac{1}{B_{(m+s)k_{1n}+\theta-m}} = \frac{y_{(m+s)k_{1n}+\theta-m-t}}{\beta} \\ = & \frac{1}{\beta x_{(m+s)k_{1n}+\theta-m-t-s}} = \frac{1}{\beta x_{(m+s)\lambda_{1n}+\theta-m-t-s}} = \frac{x_{(m+s)\lambda_{1n}+\theta-\omega}}{\alpha\beta} \\ = & \frac{1}{\alpha\beta y_{(m+s)\lambda_{1n}+\theta-\omega-m}} = \frac{1}{\alpha\beta y_{(m+s)k_{2n}+\theta-\omega-m}} = \frac{y_{(m+s)k_{2n}+\theta-\omega-m-t}}{\beta^2\alpha} \\ = & \frac{1}{\beta^2\alpha x_{(m+s)k_{2n}+\theta-\omega-m-t-s}} = \frac{1}{\beta^2\alpha x_{(m+s)\lambda_{2n}+\theta-\omega-m-t-s}} = \frac{x_{(m+s)\lambda_{2n}+\theta-2\omega}}{\alpha^2\beta^2} \\ = & \cdots\cdots\cdots\cdots \\ = & \frac{x_{(m+s)\lambda_{m+s,n}+\theta-(m+s)\omega}}{(\alpha\beta)^{m+s}}. \end{aligned}$$

Let $\lim_{k\longrightarrow\infty} x_{(m+s)k+\theta} = \Phi$. Then $\Phi \geq 1$. By taking the limit in the following relationship.

$$x_{(m+s)k_{1n}+\theta} = \frac{x_{(m+s)\lambda_{m+s,n}+\theta-(m+s)\omega}}{(\alpha\beta)^{m+s}}$$

as $n \longrightarrow \infty$, we have $\Phi = \Phi/(\alpha\beta)^{m+s}$. Therefore $\alpha\beta = 1$ and

$$x_{(m+s)k_{1n}+\theta} = x_{(m+s)\lambda_{m+s,n}+\theta-(m+s)\omega} \geq x_{(m+s)(k_{1n}-1)+\theta} > x_{(m+s)k_{1n}+\theta}.$$

A contradiction.

By the above claim we see that $x_{(m+s)k+\theta}$ is nondecreasing eventually. Then $x_{(m+s)k+\theta} < 1$ eventually and there is a sequence of positive integers $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$ such that

$$\begin{aligned} x_{(m+s)\lambda_n+\theta} &= \max\left\{\frac{1}{y_{(m+s)\lambda_n+\theta-m}}, A_{(m+s)\lambda_n+\theta}\right\} \\ = \quad A_{(m+s)\lambda_n+\theta} &= \frac{\alpha}{x_{(m+s)\lambda_n+\theta-r}} > \frac{1}{y_{(m+s)\lambda_n+\theta-m}}. \end{aligned}$$

Thus $x_{(m+s)\lambda_n+\theta-r}$ is nonincreasing eventually and $x_{(m+s)\lambda_n+\theta-r} > 1$ eventually. This contradicts to the above claim.

Therefore, for every $i \in \{0, 1, \cdots, m+s-1\}$, $\{x_{(m+s)k+i}\}_{k\geq 1}$ is a constant sequence eventually. In a similar fashion, also we can show that for every $i \in \{0, 1, \cdots, m+s-1\}$, $\{y_{(m+s)k+i}\}_{k\geq 1}$ is a constant sequence eventually. This completes the proof of Theorem 2.5.

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On uni-soft implicative filters of BE-algebras

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Abstract. The notion of uni-soft implicative filters of a BE-algebra is introduced, and related properties are investigated. The problem of classifying uni-soft implicative by their γ-exclusive filter is solved. Also, as a generalization of uni-soft implicative filters, the foldness of uni-soft implicative filters are considered. Characterizations of uni-soft (n-fold) implicative filters are discussed.

1. **Introduction**

Kim and Kim [6] introduced the notion of a BE-algebra, and investigated several properties. In [2], Ahn and So introduced the notion of ideals in BE-algebras. They gave several descriptions of ideals in BE-algebras.

Various problems in system identification involve characteristics which are essentially non-probabilistic in nature [11]. In response to this situation Zadeh [12] introduced *fuzzy set theory* as an alternative to probability theory. Uncertainty is an attribute of information. In order to suggest a more general framework, the approach to uncertainty is outlined by Zadeh [13]. To solve complicated problem in economics, engineering, and environment, we can't successfully use classical methods because of various uncertainties typical for those problems. There are three theories: theory of probability, theory of fuzzy sets, and the interval mathematics which we can consider as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties. Uncertainties can't be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as probability theory, theory of (intuitionistic) fuzzy sets, theory of vague sets, theory of interval mathematics, and theory of rough sets. However, all of these theories have their own difficulties which are pointed out in [9]. Maji et al. [8] and Molodtsov [9] suggested that one reason for these difficulties may be due to the inadequacy of the parametrization tool of the theory. To overcome these difficulties, Molodtsov [9] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. At present, works on the soft set

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theory are progressing rapidly. Maji et al. [8] described the application of soft set theory to a decision making problem. Maji et al. [7] also studied several operations on the theory of soft sets.

Ahn et al. [1] introduced the notion of an implicative vague filter in BE-algebras, and investigated some properties of it. Jun et al. [4] introduced the notion of int-soft implicative filter of a BE-algebra, and studied their properties.

In this paper, we introduce the notion of a uni-soft implicative filter of a BE-algebra, and investigate their properties. We solve the problem of classifying uni-soft implicative by their γ-exclusive filter. Also, as a generalization of uni-soft implicative filters, the foldness of uni-soft implicative filters are considered and discuss characterizations of uni-soft (n-fold) implicative filters.

2. **Preliminaries**

We recall some definitions and results discussed in [6].

An algebra $(X;*,1)$ of type (2, 0) is called a *BE-algebra* if

(BE1) $x*x=1$ for all $x\in X$;
(BE2) $x*1=1$ for all $x\in X$;
(BE3) $1*x=x$ for all $x\in X$;
(BE4) $x*(y*z)=y*(x*z)$ for all $x,y,z\in X$ (*exchange*)

We introduce a relation "$\leq$" on a BE-algebra X by $x\leq y$ if and only if $x*y=1$. A non-empty subset S of a BE-algebra X is said to be a *subalgebra* of X if it is closed under the operation "$*$". Noticing that $x*x=1$ for all $x\in X$, it is clear that $1\in S$. A BE-algebra $(X;*,1)$ is said to be *self distributive* if $x*(y*z)=(x*y)*(x*z)$ for all $x,y,z\in X$.

Definition 2.1. Let $(X;*,1)$ be a BE-algebra and let F be a non-empty subset of X. Then F is called a *filter* of X if

(F1) $1\in F$;
(F2) $x*y\in F$ and $x\in F$ imply $y\in F$ for all $x,y\in X$.

Definition 2.2. Let $(X;*,1)$ be a BE-algebra and let F be a non-empty subset of X. Then F is called an *implicative filter* of X if

(F1) $1\in F$;
(F2) $x*(y*z)\in F$ and $x*y\in F$ imply $x*z\in F$ for all $x,y,z\in X$.

Proposition 2.3. *Let $(X;*,1)$ be a BE-algebra and let F be a filter of X. If $x\leq y$ and $x\in F$ for any $y\in X$, then $y\in F$.*

Proposition 2.4. *Let $(X;*,1)$ be a self distributive BE-algebra. Then following hold: for any $x,y,z \in X$,*

(i) *if $x \leq y$, then $z*x \leq z*y$ and $y*z \leq x*z$.*
(ii) $y*z \leq (z*x)*(y*z)$.
(iii) $y*z \leq (x*y)*(x*z)$.

A BE-algebra $(X;*,1)$ is said to be *transitive* if it satisfies Proposition 2.4(iii).

A soft set theory is introduced by Molodtsov [9]. In what follows, let U be an initial universe set and X be a set of parameters. Let $\mathscr{P}(U)$ denotes the power set of U and $A,B,C,\cdots \subseteq X$.

Definition 2.5. A *soft set* (f,A) of X over U is defined to be the set of ordered pairs

$$(f,A) := \{(x,f(x)) : x \in X,\ f(x) \in \mathscr{P}(U)\},$$

where $f : X \to \mathscr{P}(U)$ such that $f(x) = \emptyset$ if $x \notin A$.

For a soft set (f,A) of X and a subset γ of U, the *γ-exclusive set* of (f,A), denoted by $e_A(f;\gamma)$, is defined to be the set $e_A(f;\gamma) := \{x \in A \mid f(x) \subseteq \gamma\}$.

For any soft sets (f,X) and (g,X) of X, we call (f,X) a *soft subset* of (g,X), denoted by $(f,X) \tilde{\subseteq} (g,X)$, if $f(x) \subseteq g(x)$ for all $x \in X$. The *soft union* of (f,X) and (g,X), denoted by $(f,X) \tilde{\cup} (g,X)$, is defined to be the soft set $(f \tilde{\cup} g, X)$ of X over U in which $f \tilde{\cup} g$ is defined by

$$(f \tilde{\cup} g)(x) = f(x) \cup g(x) \text{ for all } x \in X.$$

The *soft intersection* of (f,X) and (g,X), denoted by $(f,X) \tilde{\cap} (g,X)$, is defined to be the soft set $(f \tilde{\cap} g, M)$ of X over U in which $f \tilde{\cap} g$ is defined by

$$(f \tilde{\cap} g)(x) = f(x) \cap g(x) \text{ for all } x \in M.$$

3. **Uni-soft implicative filters**

In what follows, we take a BE-algebra X, as a set of parameters unless specified.

Definition 3.1.([5]) A soft set (f,X) of X over U is called a *union-soft filter* (briefly, *uni-soft filter*) of X if it satisfies:

(US1) $(\forall x \in X)(f(1) \subseteq f(x))$,
(US2) $(\forall x,y \in X)(f(y) \subseteq f(x*y) \cup f(x))$.

Proposition 3.2.([5]) Every uni-soft filter (f,X) of X over U satisfies the following properties:

$$(\forall x,y \in X)(x \leq y \Rightarrow f(y) \subseteq f(x)).$$

Definition 3.3. A soft set (f,X) of X over U is called a *union-soft implicative filter* (briefly, *uni-soft implicative filter*) of X if it satisfies (US1) and

(US3) $(\forall x, y, z \in X)\,(f(x * z) \subseteq f(x * (y * z)) \cup f(x * y))$.

Example 3.4. Let $E = X$ be the set of parameters where $X := \{1, a, b, c, d, 0\}$ is a BE-algebra ([6]) with the following Cayley table:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

Let (f, X) be a soft set of X over U defined as follows:

$$f : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \mathbb{N} & \text{if } x \in \{1, a, b\} \\ \mathbb{Z} & \text{if } x \in \{c, d, 0\}. \end{cases}$$

It is easy to check that (f, X) is a uni-soft implicative filter of X.

Proposition 3.5. *Every uni-soft implicative filter of a BE-algebra X is a uni-soft filter of X.*

Proof. Let (f, X) be a uni-soft implicative filter of X. Using (US3) and (BE4), we have

$$\begin{aligned} f(x * z) \subseteq & f(x * (y * z)) \cup f(x * y) \\ = & f(y * (x * z)) \cup f(x * y) \end{aligned} \tag{3.1}$$

for any $x, y, z \in X$. Putting $x := 1$ in (3.1), we get $f(z) = f(1 * z) \subseteq f(y * (1 * z)) \cup f(1 * y) = f(y * z) \cup f(y)$. Hence (US2) holds. Therefore (f, X) is a uni-soft filter of X. □

The converse of Proposition 3.5 is not true in general as seen in the following example.

Example 3.6. Let $E = X$ be the set of parameters where $X := \{1, a, b, c\}$ is a BE-algebra ([6]) with the following Cayley table:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	a
b	1	1	1	a
c	1	a	a	1

Let (f, X) be a soft set of X over $U := \mathbb{Z}$ defined as follows:

$$f : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} 3\mathbb{N} & \text{if } x \in \{1, c\} \\ 3\mathbb{Z} & \text{if } x \in \{a, b\}. \end{cases}$$

It is easy to check that (f, X) is a uni-soft filter of X. But it is not a uni-soft implicative filter of X, since $f(b * c) = f(a) = 3\mathbb{Z} \nsubseteq f(b * (a * c)) \cup f(b * a) = f(1) = 3\mathbb{N}$.

We provide conditions for a uni-soft filter to be a uni-soft implicative filter.

Proposition 3.7. *Let X be a self distributive BE-algebra. Let (f, X) be a soft filter over U satisfying*

$$(\forall x, y, z \in X)(f(y * z) \subseteq f(x * (y * (y * z))) \cup f(y * x)). \tag{3.2}$$

Then (f, X) is a uni-soft implicative filter over U.

Proof. Since $x * (y * z) = y * (x * z) \leq (x * y) * (x * (x * z)) = x * (y * (x * z)) = y * (x * (x * z))$ for all $x, y \in X$, we have $f(x * (y * z)) \supseteq f(y * (x * (x * z)))$ by Proposition 3.2. Using (3.2), we have $f(x * z) \subseteq f(y * (x * (x * z)) \cup f(x * y) \subseteq f(x * (y * z)) \cup f(x * y)$. Thus (f, X) is a uni-soft implicative filter over U. □

Theorem 3.8. *Let X be a self distributive BE-algebra. Then the soft set (f, X) over U is a uni-soft implicative filter of X over U if and only if it is a uni-soft filter of X over U.*

Proof. By Proposition 3.5, every uni-soft implicative filter over U is a uni-soft filter over U.

Conversely, assume that (f, X) is a uni-soft filter of X. For any $x, y, z \in X$, using (US2) we have $f(x * z) \subseteq f((x * y) * (x * z)) \cup f(x * y) = f(x * (y * z)) \cup f(x * y)$. Hence (f, X) is a uni-soft implicative filter of X.

For any element x and y of a BE-algebra X and positive integer n, let $x^n * y$ denote $x * (\cdots * (x * (x * y)) \cdots)$ in which x occurs n times, and $x^0 * y = 1$.

Definition 3.9. A soft set (f, X) of a BE-algebra X over U is called a *uni-soft n-fold implicative filter* of X if it satisfies (US1) and

(US4) $(\forall x, y, z \in X) \left(f(x^n * z) \subseteq f(x^n * (y * z)) \cup f(x^n * y) \right)$.

Note that a uni-soft 1-fold implicative filter of a BE-algebra X is a uni-soft implicative filter of X.

Example 3.10. Let $E = X$ be the set of parameters where $X := \{1, a, b, c, d, 0\}$ is a transitive BE-algebra ([10]) with the following Cayley table:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	b	c	b	c
b	1	a	1	b	a	d
c	1	a	1	1	a	a
d	1	1	1	b	1	b
0	1	1	1	1	1	1

Let (f, X) be a soft set of X over U defined as follows:

$$f : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \gamma_1 & \text{if } x \in \{1, b, c\} \\ \gamma_2 & \text{if } x \in \{a, d, 0\}, \end{cases}$$

where γ_1 and γ_2 are subsets of U with $\gamma_1 \subsetneq \gamma_2$. It is easy to check that (f, X) is a uni-soft n-fold implicative filter over U.

Theorem 3.11. *Every uni-soft n-fold implicative filter of a BE-algebra X is a uni-soft filter of X.*

Proof. Taking $x := 1$ in (US4) and using (BE3), we have $f(z) \subseteq f(y * z) \cup f(y)$. Hence (f, X) is a uni-soft filter over U. □

The converse of Theorem 3.10 is not true in general as seen the following example.

Example 3.12. Let $E = X$ be the set of parameters where $X := \{1, a, b, c, d, 0\}$ is a BE-algebra as in Example 3.10. Let (f, X) be a soft set of X over U defined as follows:

$$f : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \gamma_1 & \text{if } x = 1 \\ \gamma_2 & \text{if } x \in \{a, b, c, d, 0\}, \end{cases}$$

where γ_1 and γ_2 are subsets of U with $\gamma_1 \subsetneq \gamma_2$. It is easy to check that (f, X) is a uni-soft filter of X. But it is not a uni-soft 1-fold implicative filter over U, since $f(d * c) = f(b) = \gamma_2 \nsubseteq \gamma_1 = f(1) = f(d * (b * c)) \cup f(d * b)$.

Theorem 3.13. *Let X be a transitive BE-algebra. For any uni-soft filter (f, X) of a BE-algebra X, the following are equivalent:*

(i) *(f, X) is a uni-soft n-fold implicative filter,*
(ii) $(\forall x, y \in X)\,(f(x^n * y) \subseteq f(x^{n+1} * y))$,
(iii) $(\forall x, y, z \in X)\,(f((x^n * y) * (x^n * z)) \subseteq f(x^n * (y * z)))$.

Proof. (i)⇒(ii) Assume that (f, X) is a uni-soft n-fold implicative filter of X. Putting $z := y, y := x$ in (US4), we have $f(x^n * y) \subseteq f(x^n * (x * y)) \cup f(x^n * x) = f(x^{n+1} * y) \cup f(1) = f(x^{n+1} * y)$. Hence (ii) holds.
(ii)⇒(iii) Suppose that (ii) holds. Since $x^n * (y * z) \leq x^n * ((x^n * y) * (x^n * z))$, we have $f(x^n * ((x^n * y) * (x^n * z))) \subseteq f(x^n * (y * z))$. Since $x^{n+1} * (x^{n-1} * ((x^n * y) * z)) = x^n * (x^n * ((x^n * y) * z))) = x^n * ((x^n * y) * (x^n * z))$ and using (ii), we have

$$\begin{aligned} f(x^{n+1} * (x^{n-2} * ((x^n * y) * z)) &= f(x^n * (x^{n-1} * ((x^n * y) * z)) \\ &\subseteq f(x^{n+1} * (x^{n-1} * ((x^n * y) * z))) \\ &= f(x^n * ((x^n * y) * (x^n * z))) \\ &\subseteq f(x^n * (y * z)). \end{aligned} \tag{3.3}$$

It follows form (ii) and (3.3) that

$$\begin{aligned} f(x^{n+1} * (x^{n-3} * ((x^n * y) * z))) &= f(x^n * (x^{n-2} * ((x^n * y) * z))) \\ &\subseteq f(x^{n+1} * (x^{n-2} * ((x^n * y) * z))) \\ &\subseteq f(x^n * (y * z)). \end{aligned}$$

Repeating this process, we conclude that

$$\begin{aligned} f((x^n * y) * (x^n * z)) =& f(x^n * ((x^n * y) * z)) \\ \subseteq& f(x^n * (y * z)). \end{aligned}$$

(iii)⇒(i) Let $x, y, z \in X$. Using (iii), we have

$$\begin{aligned} f(x^n * z) \subseteq& f((x^n * y) * (x^n * z)) \cap f(x^n * y) \\ \subseteq& f((x^n * (y * z)) \cap f(x^n * y). \end{aligned}$$

Hence (f, X) is a uni-soft n-fold implicative filter □

Definition 3.14. Let n be a positive integer. A BE-algebra X is said to be *n-fold implicative* if it satisfies the equality $x^{n+1} * y = x^n * y$ for all $x, y \in X$.

Corollary 3.15. *In an n-fold implicative BE-algebra, the notion of uni-soft filters and uni-soft n-fold implicative filters coincide.*

Proof. Straightforward. □

Theorem 3.16. *Let X be a BE-algebra. A soft set (f, X) over U is a uni-soft n-fold implicative filter of a BE-algebra X if and only if it satisfies (US1) and*

(US5) $(\forall x, y, z \in X) \left(f(y^n * z) \subseteq f(x * (y^{n+1} * z)) \cup f(x) \right).$

Proof. Suppose that a soft set (f, X) over U is a uni-soft n-fold implicative filter. By Theorems 3.13, 3.11 and (US2), we have $f(y^n * z) \subseteq f(y^{n+1} * z) \subseteq f(x * (y^{n+1} * z)) \cup f(x)$ for any $x, y, z \in X$. Hence (US5) holds.

Conversely, assume that (f, X) satisfies (US1) and (US5). Using (BE3), we obtain $f(y) = f(1^n * y) \subseteq f(x * (1^{n+1} * y)) \cup f(x) = f(x * y) \cup f(x)$. Hence (US2) holds and so (f, X) is a uni-soft filter over U. By (US5), (US1) and (BE3), we get $f(x^n * y) \subseteq f(1 * (x^{n+1} * y)) \cup f(1) = f(x^{n+1} * y)$. By Theorem 3.13, (f, X) is a uni-soft n-fold implicative filter of X. □

Theorem 3.17. *Every implicative filter of a BE-algebra can be represented as a γ-exclusive set of a uni-soft implicative filter, i.e., given an implicative filter F a BE-algebra X, there exists a uni-soft implicative filter (f, X) of X over U such that F is the γ-exclusive set of (f, X) for a non-empty subset γ of U.*

Proof. Let F be an implicative filter of a BE-algebra X. For a subset γ of U, define a soft set (f, X) over U by

$$f : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \gamma & \text{if } x \in F, \\ U & \text{if } x \notin F. \end{cases}$$

Obviously, $F = e_X(f; \gamma)$. We now prove that (f, X) is a uni-soft implicative filter of X. Since $1 \in F = e_X(f; \gamma)$, we have $f(1) = \gamma \subseteq f(x)$ for all $x \in X$. Let $x, y, z \in X$. If $x * (y * z), x * y \in F$, then $x * z \in F$ because F is an implicative filter of X. Hence $f(x * (y * z)) = f(x * y) = f(x * z) = \gamma$, and so $f(x*(y*z)) \cup f(x*y) \supseteq f(x*z)$. If $x*(y*z) \in F$ and $x*y \notin F$, then $f(x*(y*z)) = \gamma$ and $f(x*y) = U$ which imply that $f(x*(y*z)) \cup f(x*y) = \gamma \cup U = U \supseteq f(x*z)$. Similarly, if $x*(y*z) \notin F$ and $x*y \in F$, then $f(x*(y*z)) \cup f(x*y) \supseteq f(x*z)$. Obviously, if $x*(y*z) \notin F$ and $x*y \notin F$, then $f(x*(y*z)) \cup f(x*y) \supseteq f(x*z)$. Therefore (f, X) is a uni-soft implicative filter of X. □

For any elements a and b of X, consider a soft set $(f_{X(a,b)}, X)$ over U where

$$f_{X(a,b)} : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \gamma_1 & \text{if } x \in X(a,b), \\ \gamma_2 & \text{otherwise}, \end{cases}$$

where γ_1 and γ_2 are subsets of U with $\gamma_1 \subsetneq \gamma_2$ and $X(a,b) := \{x \in X | a \leq b*x\}$. In the following example, we know that there exist $a, b \in X$ such that $(f_{X(a,b)}, X)$ is not a uni-soft implicative filter of X.

Example 3.18. Consider the BE-algebra $X = \{1, a, b, c\}$ which is given in Example 3.6. Then $(f_{X(c,c)}, X)$ is not a uni-soft implicative filter of X over U since $f_{X(c,c)}(b*(a*c))) \cup f_{X(c,c)}(b*a) = \gamma_1 \not\supseteq f_{X(c,c)}(b*c) = \gamma_2$.

Now we provide a condition for the soft set $(f_{X(a,b)}, X)$ to be a uni-soft implicative filter of X over U for all $a, b \in X$.

Theorem 3.19. *If X is a self distributive BE-algebra, then the soft set $(f_{X(a,b)}, X)$ is a uni-soft implicative filter of X over U for all $a, b \in X$.*

Proof. Let $a, b \in X$. Obviously, $f_{X(a,b)}(1) \subseteq f_{X(a,b)}(x)$ for all $x \in X$. Let $x, y, z \in X$ be such that $a*(b*(x*(y*z))) \neq 1$ or $a*(b*(x*y)) \neq 1$. Then $f_{X(a,b)}(x*(y*z)) = \gamma_2$ or $f_{X(a,b)}(x*y) = \gamma_2$. Hence $f_{X(a,b)}(x*(y*z)) \cup f_{X(a,b)}(x*y) = \gamma_2 \supseteq f_{X(a,b)}(x*z)$. Assume that $a*(b*(x*(y*z))) = 1$ and $a*(b*(x*y)) = 1$. Then

$$\begin{aligned} 1 &= a*(b*(x*(y*z))) \\ &= a*(b*((x*y)*(x*z))) \\ &= a*((b*(x*y))*(b*(x*z))) \\ &= (a*(b*(x*y)))*(a*(b*(x*z))) \\ &= 1*(a*(b*(x*z))) \\ &= a*(b*(x*z)), \end{aligned}$$

and so $f_{X(a,b)}(x*(y*z)) \cup f_{X(a,b)}(x*y) = \gamma_1 = f_{X(a,b)}(x*z)$. Therefore $(f_{X(a,b)}, X)$ is a uni-soft implicative filter of X over U for all $a, b \in X$. □

Theorem 3.20. *If (f, X) and (g, X) are uni-soft implicative filters of a BE-algebra X, then the soft union $(f, X) \tilde{\cup} (g, X)$ of (f, X) and (g, X) is a uni-soft implicative filter of X.*

Proof. For any $x \in X$, we have $(f \tilde{\cup} g)(1) = f(1) \cup g(1) \subseteq f(x) \cup g(x) = (f \tilde{\cup} g)(x)$. Let $x, y, z \in X$. Then

$$\begin{aligned}(f \tilde{\cup} g)(x * z) &= f(x * z) \cup g(x * z) \\ &\subseteq (f(x * (y * z)) \cup f(x * y)) \cup (g(x * (y * z)) \cup g(x * y)) \\ &= (f(x * (y * z)) \cup g(x * (y * z))) \cup (f(x * y) \cup g(x * y)) \\ &= (f \tilde{\cup} g)(x * (y * z)) \cup (f \tilde{\cup} g)(x * y).\end{aligned}$$

Hence $(f, X) \tilde{\cup} (g, X)$ is an int-soft implicative filter of X. □

The following example shows that the soft intersection of uni-soft implicative filters of X may not be a uni-soft implicative filter of X.

Example 3.21. Let $E = X$ be the set of parameters and $U = X$ be the initial universe set, where $X = \{1, a, b, c, d\}$ is a BE-algebra with the following Cayley table ([6]):

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

Let (f, X) and (g, X) be soft sets of X over U defined, respectively, as follows:

$$f : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \gamma_1 & \text{if } x \in \{1, b\} \\ \gamma_3 & \text{if } x \in \{a, c, d\} \end{cases}$$

and

$$g : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \gamma_2 & \text{if } x \in \{1, a, c\} \\ \gamma_4 & \text{if } x \in \{b, d\} \end{cases}$$

where γ_1, γ_2, γ_3, and γ_4 are subsets of U with $\gamma_1 \subsetneq \gamma_2 \subsetneq \gamma_3 \subsetneq \gamma_4$. It is easy to check that (f, X) and (g, X) are uni-soft implicative filters of X over U. But $(f, X) \tilde{\cap} (g, X) = (f \tilde{\cap} g, X)$ is not a uni-soft implicative filter of X over U, since

$$\begin{aligned}(f \tilde{\cap} g)(1 * (c * d)) \cup (f \tilde{\cap} g)(1 * c) &= (f \tilde{\cap} g)(b) \cup (f \tilde{\cap} g)(c) \\ &= (f(b) \cap g(b)) \cup (f(c) \cap g(c)) \\ &= \gamma_1 \cup \gamma_2 = \gamma_2 \not\supseteq \gamma_3 = \gamma_3 \cap \gamma_4 \\ &= f(1 * d) \cap g(1 * d) \\ &= (f \tilde{\cap} g)(1 * d).\end{aligned}$$

Let (f,X) be a soft set of X. For any $a,b\in X$ and $k\in\mathbb{N}$, consider the set

$$f[a^k;b]:=\left\{x\in X\mid f\left(a^k*(b*x)\right)=f(1)\right\}$$

where $f(a^k*x)=f(a*(a*(\cdots*(a*(a*x))\cdots)))$ in which a appears k-times. Note that $a,b,1\in f[a^k;b]$ for all $a,b\in X$ and $k\in\mathbb{N}$.

There exists a soft set (f,X) of X, $a,b\in X$ and $k\in\mathbb{N}$ such that $f[a^k;b]$ is not an implicative filter of X (see Example 3.22).

Example 3.22. Let $E=X$ be the set of parameters and $U=X$ be the initial universe set where $X=\{1,a,b,c\}$ is a BE-algebra as in Example 3.6. Let (f,X) be a soft set of X over U defined as follows:

$$f:X\to\mathscr{P}(U),\ x\mapsto\begin{cases}\gamma_1 & \text{if } x=1\\ \gamma_2 & \text{if } x\in\{a,b,c\},\end{cases}$$

where γ_1 and γ_2 are subsets of U with $\gamma_1\subsetneq\gamma_2$. Then it is a soft set of X over U. But $f[1;c]=\{x\in X|f(1*(c*x))=f(1)\}=\{1,c\}$ is not an implicative filter, since $b*(a*c)=1\in f[1;c]$, $b*a=1\in f[1;c]$ and $b*c=a\notin f[1;c]$.

We provide conditions for a set $f[a^k;b]$ to be an implicative filter.

Theorem 3.23. *Let X be a self distributive BE-algebra. Let (f,X) be a soft set of X over U satisfying the condition (US1) and*

$$(\forall x,y\in X)\left(f(x*y)=f(x)\cup f(y)\right). \tag{3.4}$$

Then $f[a^k;b]$ is an implicative filter of X for all $a,b\in X$ and $k\in\mathbb{N}$.

Proof. Let $a,b\in X$ and $k\in\mathbb{N}$. Obviously, $1\in f[a^k;b]$. Let $x,y,z\in X$ be such that $x*(y*z)\in f[a^k;b]$ and $x*y\in f[a^k;b]$. Then $f\left(a^k*(b*(x*y))\right)=f(1)$, which implies from (3.4) and (US1) that

$$\begin{aligned}f(1)&=f(a^k*(b*(x*(y*z))))\\&=f(a^{k-1}*(a*(b*(x*(y*z)))))\\&=f(a^{k-1}*(a*((b*(x*y))*(b*(x*z)))))\\&=\cdots\\&=f((a^k*(b*(x*y)))*(a^k*(b*(x*z))))\\&=f(a^k*(b*(x*y)))\cup f(a^k*(b*(x*z)))\\&=f(1)\cup f(a^k*(b*(x*z)))\\&=f(a^k*(b*(x*z))).\end{aligned}$$

Hence $x * z \in f[a^k; b]$ and therefore $f[a^k; b]$ is an implicative filter of X for all $a, b \in X$ and $k \in \mathbb{N}$. □

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