

Volume 24, Number 4
ISSN:1521-1398 PRINT,1572-9206 ONLINE

April 2018



Journal of
Computational
Analysis and
Applications

EUDOXUS PRESS,LLC

Journal of Computational Analysis and Applications

ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE

SCOPE OF THE JOURNAL

An international publication of Eudoxus Press, LLC

(sixteen times annually)

Editor in Chief: George Anastassiou

Department of Mathematical Sciences,

University of Memphis, Memphis, TN 38152-3240, U.S.A

ganastss@memphis.edu

<http://www.msci.memphis.edu/~ganastss/jocaaa>

The main purpose of "J.Computational Analysis and Applications" is to publish high quality research articles from all subareas of Computational Mathematical Analysis and its many potential applications and connections to other areas of Mathematical Sciences. Any paper whose approach and proofs are computational, using methods from Mathematical Analysis in the broadest sense is suitable and welcome for consideration in our journal, except from Applied Numerical Analysis articles. Also plain word articles without formulas and proofs are excluded. The list of possibly connected mathematical areas with this publication includes, but is not restricted to: Applied Analysis, Applied Functional Analysis, Approximation Theory, Asymptotic Analysis, Difference Equations, Differential Equations, Partial Differential Equations, Fourier Analysis, Fractals, Fuzzy Sets, Harmonic Analysis, Inequalities, Integral Equations, Measure Theory, Moment Theory, Neural Networks, Numerical Functional Analysis, Potential Theory, Probability Theory, Real and Complex Analysis, Signal Analysis, Special Functions, Splines, Stochastic Analysis, Stochastic Processes, Summability, Tomography, Wavelets, any combination of the above, e.t.c.

"J.Computational Analysis and Applications" is a peer-reviewed Journal. See the instructions for preparation and submission of articles to JoCAAA. Assistant to the Editor:

Dr.Razvan Mezei,mezei_razvan@yahoo.com, Madison,WI,USA.

Journal of Computational Analysis and Applications(JoCAAA) is published by **EUDOXUS PRESS,LLC**,1424 Beaver Trail

Drive,Cordova,TN38016,USA,anastassioug@yahoo.com

<http://www.eudoxuspress.com>. **Annual Subscription Prices:**For USA and

Canada,Institutional:Print \$800, Electronic OPEN ACCESS. Individual:Print \$400. For any other part of the world add \$150 more(handling and postages) to the above prices for Print. No credit card payments.

Copyright©2018 by Eudoxus Press,LLC,all rights reserved.JoCAAA is printed in USA.

JoCAAA is reviewed and abstracted by AMS Mathematical Reviews,MATHSCI,and Zentralblat MATH.

It is strictly prohibited the reproduction and transmission of any part of JoCAAA and in any form and by any means without the written permission of the publisher.It is only allowed to educators to Xerox articles for educational purposes.The publisher assumes no responsibility for the content of published papers.

Editorial Board

Associate Editors of Journal of Computational Analysis and Applications

Francesco Altomare

Dipartimento di Matematica
Universita' di Bari
Via E.Orabona, 4
70125 Bari, ITALY
Tel+39-080-5442690 office
+39-080-3944046 home
+39-080-5963612 Fax
altomare@dm.uniba.it
Approximation Theory, Functional
Analysis, Semigroups and Partial
Differential Equations, Positive
Operators.

Ravi P. Agarwal

Department of Mathematics
Texas A&M University - Kingsville
700 University Blvd.
Kingsville, TX 78363-8202
tel: 361-593-2600
Agarwal@tamuk.edu
Differential Equations, Difference
Equations, Inequalities

George A. Anastassiou

Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152, U.S.A
Tel. 901-678-3144
e-mail: ganastss@memphis.edu
Approximation Theory, Real
Analysis,
Wavelets, Neural Networks,
Probability, Inequalities.

J. Marshall Ash

Department of Mathematics
De Paul University
2219 North Kenmore Ave.
Chicago, IL 60614-3504
773-325-4216
e-mail: mash@math.depaul.edu
Real and Harmonic Analysis

Dumitru Baleanu

Department of Mathematics and
Computer Sciences,
Cankaya University, Faculty of Art
and Sciences,
06530 Balgat, Ankara,
Turkey, dumitru@cankaya.edu.tr

Fractional Differential Equations
Nonlinear Analysis, Fractional
Dynamics

Carlo Bardaro

Dipartimento di Matematica e
Informatica
Universita di Perugia
Via Vanvitelli 1
06123 Perugia, ITALY
TEL+390755853822
+390755855034
FAX+390755855024
E-mail carlo.bardaro@unipg.it
Web site:
<http://www.unipg.it/~bardaro/>
Functional Analysis and
Approximation Theory, Signal
Analysis, Measure Theory, Real
Analysis.

Martin Bohner

Department of Mathematics and
Statistics, Missouri S&T
Rolla, MO 65409-0020, USA
bohner@mst.edu
web.mst.edu/~bohner
Difference equations, differential
equations, dynamic equations on
time scale, applications in
economics, finance, biology.

Jerry L. Bona

Department of Mathematics
The University of Illinois at
Chicago
851 S. Morgan St. CS 249
Chicago, IL 60601
e-mail: bona@math.uic.edu
Partial Differential Equations,
Fluid Dynamics

Luis A. Caffarelli

Department of Mathematics
The University of Texas at Austin
Austin, Texas 78712-1082
512-471-3160
e-mail: caffarel@math.utexas.edu
Partial Differential Equations

George Cybenko

Thayer School of Engineering

Dartmouth College
8000 Cummings Hall,
Hanover, NH 03755-8000
603-646-3843 (X 3546 Secr.)
e-mail:george.cybenko@dartmouth.edu
Approximation Theory and Neural
Networks

Sever S. Dragomir

School of Computer Science and
Mathematics, Victoria University,
PO Box 14428,
Melbourne City,
MC 8001, AUSTRALIA
Tel. +61 3 9688 4437
Fax +61 3 9688 4050
sever.dragomir@vu.edu.au
Inequalities, Functional Analysis,
Numerical Analysis, Approximations,
Information Theory, Stochastics.

Oktay Duman

TOBB University of Economics and
Technology,
Department of Mathematics, TR-
06530,
Ankara, Turkey,
oduman@etu.edu.tr
Classical Approximation Theory,
Summability Theory, Statistical
Convergence and its Applications

Saber N. Elaydi

Department Of Mathematics
Trinity University
715 Stadium Dr.
San Antonio, TX 78212-7200
210-736-8246
e-mail: selaydi@trinity.edu
Ordinary Differential Equations,
Difference Equations

J .A. Goldstein

Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152
901-678-3130
jgoldste@memphis.edu
Partial Differential Equations,
Semigroups of Operators

H. H. Gonska

Department of Mathematics
University of Duisburg
Duisburg, D-47048
Germany

011-49-203-379-3542
e-mail: heiner.gonska@uni-due.de
Approximation Theory, Computer
Aided Geometric Design

John R. Graef

Department of Mathematics
University of Tennessee at
Chattanooga
Chattanooga, TN 37304 USA
John-Graef@utc.edu
Ordinary and functional
differential equations, difference
equations, impulsive systems,
differential inclusions, dynamic
equations on time scales, control
theory and their applications

Weimin Han

Department of Mathematics
University of Iowa
Iowa City, IA 52242-1419
319-335-0770
e-mail: whan@math.uiowa.edu
Numerical analysis, Finite element
method, Numerical PDE, Variational
inequalities, Computational
mechanics

Tian-Xiao He

Department of Mathematics and
Computer Science
P.O. Box 2900, Illinois Wesleyan
University
Bloomington, IL 61702-2900, USA
Tel (309)556-3089
Fax (309)556-3864
the@iwu.edu
Approximations, Wavelet,
Integration Theory, Numerical
Analysis, Analytic Combinatorics

Margareta Heilmann

Faculty of Mathematics and Natural
Sciences, University of Wuppertal
Gaußstraße 20
D-42119 Wuppertal, Germany,
heilmann@math.uni-wuppertal.de
Approximation Theory (Positive
Linear Operators)

Xing-Biao Hu

Institute of Computational
Mathematics
AMSS, Chinese Academy of Sciences
Beijing, 100190, CHINA
hxb@lsec.cc.ac.cn

Computational Mathematics

Jong Kyu Kim

Department of Mathematics
Kyungnam University
Masan Kyungnam, 631-701, Korea
Tel 82-(55)-249-2211
Fax 82-(55)-243-8609
jongkyuk@kyungnam.ac.kr
Nonlinear Functional Analysis,
Variational Inequalities, Nonlinear
Ergodic Theory, ODE, PDE,
Functional Equations.

Robert Kozma

Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152, USA
rkozma@memphis.edu
Neural Networks, Reproducing Kernel
Hilbert Spaces,
Neural Percolation Theory

Mustafa Kulenovic

Department of Mathematics
University of Rhode Island
Kingston, RI 02881, USA
kulenm@math.uri.edu
Differential and Difference
Equations

Irena Lasiecka

Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152
PDE, Control Theory, Functional
Analysis, lasiecka@memphis.edu

Burkhard Lenze

Fachbereich Informatik
Fachhochschule Dortmund
University of Applied Sciences
Postfach 105018
D-44047 Dortmund, Germany
e-mail: lenze@fh-dortmund.de
Real Networks, Fourier Analysis,
Approximation Theory

Hrushikesh N. Mhaskar

Department Of Mathematics
California State University
Los Angeles, CA 90032
626-914-7002
e-mail: hmhaska@gmail.com
Orthogonal Polynomials,
Approximation Theory, Splines,
Wavelets, Neural Networks

Ram N. Mohapatra

Department of Mathematics
University of Central Florida
Orlando, FL 32816-1364
tel.407-823-5080
ram.mohapatra@ucf.edu
Real and Complex Analysis,
Approximation Th., Fourier
Analysis, Fuzzy Sets and Systems

Gaston M. N'Guerekata

Department of Mathematics
Morgan State University
Baltimore, MD 21251, USA
tel: 1-443-885-4373
Fax 1-443-885-8216
Gaston.N'Guerekata@morgan.edu
nguerekata@aol.com
Nonlinear Evolution Equations,
Abstract Harmonic Analysis,
Fractional Differential Equations,
Almost Periodicity & Almost
Automorphy

M.Zuhair Nashed

Department Of Mathematics
University of Central Florida
PO Box 161364
Orlando, FL 32816-1364
e-mail: znashed@mail.ucf.edu
Inverse and Ill-Posed problems,
Numerical Functional Analysis,
Integral Equations, Optimization,
Signal Analysis

Mubenga N. Nkashama

Department OF Mathematics
University of Alabama at Birmingham
Birmingham, AL 35294-1170
205-934-2154
e-mail: nkashama@math.uab.edu
Ordinary Differential Equations,
Partial Differential Equations

Vassilis Papanicolaou

Department of Mathematics
National Technical University of
Athens
Zografou campus, 157 80
Athens, Greece
tel:: +30(210) 772 1722
Fax +30(210) 772 1775
papanico@math.ntua.gr
Partial Differential Equations,
Probability

Choonkil Park

Department of Mathematics
Hanyang University
Seoul 133-791
S. Korea, baak@hanyang.ac.kr
Functional Equations

Svetlozar (Zari) Rachev,

Professor of Finance, College of
Business, and Director of
Quantitative Finance Program,
Department of Applied Mathematics &
Statistics
Stonybrook University
312 Harriman Hall, Stony Brook, NY
11794-3775
tel: +1-631-632-1998,
svetlozar.rachev@stonybrook.edu

Alexander G. Ramm

Mathematics Department
Kansas State University
Manhattan, KS 66506-2602
e-mail: ramm@math.ksu.edu
Inverse and Ill-posed Problems,
Scattering Theory, Operator Theory,
Theoretical Numerical Analysis,
Wave Propagation, Signal Processing
and Tomography

Tomasz Rychlik

Polish Academy of Sciences
Instytut Matematyczny PAN
00-956 Warszawa, skr. poczt. 21
ul. Śniadeckich 8
Poland
trychlik@impan.pl
Mathematical Statistics,
Probabilistic Inequalities

Boris Shekhtman

Department of Mathematics
University of South Florida
Tampa, FL 33620, USA
Tel 813-974-9710
shekhtma@usf.edu
Approximation Theory, Banach
spaces, Classical Analysis

T. E. Simos

Department of Computer
Science and Technology
Faculty of Sciences and Technology
University of Peloponnese
GR-221 00 Tripolis, Greece
Postal Address:
26 Menelaou St.

Anfithea - Paleon Faliron
GR-175 64 Athens, Greece
tsimos@mail.ariadne-t.gr
Numerical Analysis

H. M. Srivastava

Department of Mathematics and
Statistics
University of Victoria
Victoria, British Columbia V8W 3R4
Canada
tel.250-472-5313; office,250-477-
6960 home, fax 250-721-8962
harimsri@math.uvic.ca
Real and Complex Analysis,
Fractional Calculus and Appl.,
Integral Equations and Transforms,
Higher Transcendental Functions and
Appl., q-Series and q-Polynomials,
Analytic Number Th.

I. P. Stavroulakis

Department of Mathematics
University of Ioannina
451-10 Ioannina, Greece
ipstav@cc.uoi.gr
Differential Equations
Phone +3-065-109-8283

Manfred Tasche

Department of Mathematics
University of Rostock
D-18051 Rostock, Germany
manfred.tasche@mathematik.uni-
rostock.de
Numerical Fourier Analysis, Fourier
Analysis, Harmonic Analysis, Signal
Analysis, Spectral Methods,
Wavelets, Splines, Approximation
Theory

Roberto Triggiani

Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152
PDE, Control Theory, Functional
Analysis, rtrggiani@memphis.edu

Juan J. Trujillo

University of La Laguna
Departamento de Analisis Matematico
C/Astr.Fco.Sanchez s/n
38271. LaLaguna. Tenerife.
SPAIN
Tel/Fax 34-922-318209
Juan.Trujillo@ull.es

Fractional: Differential Equations-
Operators-Fourier Transforms,
Special functions, Approximations,
and Applications

Ram Verma

International Publications
1200 Dallas Drive #824 Denton,
TX 76205, USA

Verma99@msn.com

Applied Nonlinear Analysis,
Numerical Analysis, Variational
Inequalities, Optimization Theory,
Computational Mathematics, Operator
Theory

Xiang Ming Yu

Department of Mathematical Sciences
Southwest Missouri State University
Springfield, MO 65804-0094
417-836-5931
xmy944f@missouristate.edu
Classical Approximation Theory,
Wavelets

Lotfi A. Zadeh

Professor in the Graduate School
and Director, Computer Initiative,
Soft Computing (BISC)
Computer Science Division
University of California at
Berkeley
Berkeley, CA 94720
Office: 510-642-4959
Sec: 510-642-8271
Home: 510-526-2569
FAX: 510-642-1712
zadeh@cs.berkeley.edu
Fuzzyness, Artificial Intelligence,
Natural language processing, Fuzzy
logic

Richard A. Zalik

Department of Mathematics
Auburn University
Auburn University, AL 36849-5310
USA.

Tel 334-844-6557 office
678-642-8703 home

Fax 334-844-6555
zalik@auburn.edu

Approximation Theory, Chebychev
Systems, Wavelet Theory

Ahmed I. Zayed

Department of Mathematical Sciences
DePaul University

2320 N. Kenmore Ave.
Chicago, IL 60614-3250
773-325-7808

e-mail: azayed@condor.depaul.edu
Shannon sampling theory, Harmonic
analysis and wavelets, Special
functions and orthogonal
polynomials, Integral transforms

Ding-Xuan Zhou

Department Of Mathematics
City University of Hong Kong
83 Tat Chee Avenue
Kowloon, Hong Kong
852-2788 9708, Fax: 852-2788 8561
e-mail: mazhou@cityu.edu.hk
Approximation Theory, Spline
functions, Wavelets

Xin-long Zhou

Fachbereich Mathematik, Fachgebiet
Informatik
Gerhard-Mercator-Universität
Duisburg
Lotharstr.65, D-47048 Duisburg,
Germany
e-mail: [Xzhou@informatik.uni-
duisburg.de](mailto:Xzhou@informatik.uni-
duisburg.de)
Fourier Analysis, Computer-Aided
Geometric Design, Computational
Complexity, Multivariate
Approximation Theory, Approximation
and Interpolation Theory

Jessada Tariboon

Department of Mathematics,
King Mongkut's University of
Technology N. Bangkok
1518 Pracharat 1 Rd., Wongsawang,
Bangsue, Bangkok, Thailand 10800
jessada.t@sci.kmutnb.ac.th, Time scales,
Differential/Difference Equations,
Fractional Differential Equations

Instructions to Contributors
Journal of Computational Analysis and Applications

An international publication of Eudoxus Press, LLC, of TN.

Editor in Chief: George Anastassiou

Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152-3240, U.S.A.

1. Manuscripts files in Latex and PDF and in English, should be submitted via email to the Editor-in-Chief:

Prof. George A. Anastassiou
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152, USA.
Tel. 901.678.3144
e-mail: ganastss@memphis.edu

Authors may want to recommend an associate editor the most related to the submission to possibly handle it.

Also authors may want to submit a list of six possible referees, to be used in case we cannot find related referees by ourselves.

2. Manuscripts should be typed using any of TEX, LaTeX, AMS-TEX, or AMS-LaTeX and according to EUDOXUS PRESS, LLC. LATEX STYLE FILE. (Click [HERE](#) to save a copy of the style file.) They should be carefully prepared in all respects. Submitted articles should be brightly typed (not dot-matrix), double spaced, in ten point type size and in 8(1/2)x11 inch area per page. Manuscripts should have generous margins on all sides and should not exceed 24 pages.

3. Submission is a representation that the manuscript has not been published previously in this or any other similar form and is not currently under consideration for publication elsewhere. A statement transferring from the authors (or their employers, if they hold the copyright) to Eudoxus Press, LLC, will be required before the manuscript can be accepted for publication. The Editor-in-Chief will supply the necessary forms for this transfer. Such a written transfer of copyright, which previously was assumed to be implicit in the act of submitting a manuscript, is necessary under the U.S. Copyright Law in order for the publisher to carry through the dissemination of research results and reviews as widely and effectively as possible.

4. The paper starts with the title of the article, author's name(s) (no titles or degrees), author's affiliation(s) and e-mail addresses. The affiliation should comprise the department, institution (usually university or company), city, state (and/or nation) and mail code.

The following items, 5 and 6, should be on page no. 1 of the paper.

5. An abstract is to be provided, preferably no longer than 150 words.

6. A list of 5 key words is to be provided directly below the abstract. Key words should express the precise content of the manuscript, as they are used for indexing purposes.

The main body of the paper should begin on page no. 1, if possible.

7. All sections should be numbered with Arabic numerals (such as: 1. INTRODUCTION) .

Subsections should be identified with section and subsection numbers (such as 6.1. Second-Value Subheading).

If applicable, an independent single-number system (one for each category) should be used to label all theorems, lemmas, propositions, corollaries, definitions, remarks, examples, etc. The label (such as Lemma 7) should be typed with paragraph indentation, followed by a period and the lemma itself.

8. Mathematical notation must be typeset. Equations should be numbered consecutively with Arabic numerals in parentheses placed flush right, and should be thusly referred to in the text [such as Eqs.(2) and (5)]. The running title must be placed at the top of even numbered pages and the first author's name, et al., must be placed at the top of the odd numbered pages.

9. Illustrations (photographs, drawings, diagrams, and charts) are to be numbered in one consecutive series of Arabic numerals. The captions for illustrations should be typed double space. All illustrations, charts, tables, etc., must be embedded in the body of the manuscript in proper, final, print position. In particular, manuscript, source, and PDF file version must be at camera ready stage for publication or they cannot be considered.

Tables are to be numbered (with Roman numerals) and referred to by number in the text. Center the title above the table, and type explanatory footnotes (indicated by superscript lowercase letters) below the table.

10. List references alphabetically at the end of the paper and number them consecutively. Each must be cited in the text by the appropriate Arabic numeral in square brackets on the baseline.

**References should include (in the following order):
initials of first and middle name, last name of author(s)
title of article,**

name of publication, volume number, inclusive pages, and year of publication.

Authors should follow these examples:

Journal Article

1. H.H.Gonska, Degree of simultaneous approximation of bivariate functions by Gordon operators, (journal name in italics) *J. Approx. Theory*, 62,170-191(1990).

Book

2. G.G.Lorentz, (title of book in italics) *Bernstein Polynomials* (2nd ed.), Chelsea, New York, 1986.

Contribution to a Book

3. M.K.Khan, Approximation properties of beta operators, in (title of book in italics) *Progress in Approximation Theory* (P.Nevai and A.Pinkus, eds.), Academic Press, New York, 1991, pp.483-495.

11. All acknowledgements (including those for a grant and financial support) should occur in one paragraph that directly precedes the References section.

12. Footnotes should be avoided. When their use is absolutely necessary, footnotes should be numbered consecutively using Arabic numerals and should be typed at the bottom of the page to which they refer. Place a line above the footnote, so that it is set off from the text. Use the appropriate superscript numeral for citation in the text.

13. After each revision is made please again submit via email Latex and PDF files of the revised manuscript, including the final one.

14. Effective 1 Nov. 2009 for current journal page charges, contact the Editor in Chief. Upon acceptance of the paper an invoice will be sent to the contact author. The fee payment will be due one month from the invoice date. The article will proceed to publication only after the fee is paid. The charges are to be sent, by money order or certified check, in US dollars, payable to Eudoxus Press, LLC, to the address shown on the Eudoxus [homepage](#).

No galleys will be sent and the contact author will receive one (1) electronic copy of the journal issue in which the article appears.

15. This journal will consider for publication only papers that contain proofs for their listed results.

Zeros of functions in weighted Dirichlet spaces and Carleson type measure

Ruishen Qian, Songxiao Li* and Yanhua Zhang

Abstract. In this paper, we study the zeros of functions in weighted Dirichlet space and a class of Carleson type measure.

MSC 2000: 30J99, 30H99

Keywords: Zero, weighted Dirichlet space, Carleson measure.

1 Introduction

Let \mathbb{D} denote the open unit disk in the complex plane \mathbb{C} , $\partial\mathbb{D}$ its boundary and $H(\mathbb{D})$ the space of all analytic functions in \mathbb{D} . For $a \in \mathbb{D}$, let σ_a be the automorphism of \mathbb{D} exchanging 0 for a , namely $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$, $z \in \mathbb{D}$. Let H^∞ denote the space of bounded analytic function.

Throughout this paper, we assume that $K : [0, \infty) \rightarrow [0, \infty)$ is a right-continuous and nondecreasing function. An $f \in H(\mathbb{D})$ is said to belong to the weighted Dirichlet space, denoted by D_K , if (see, e.g., [24])

$$\|f\|_{D_K}^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 K(1 - |z|^2) dA(z) < \infty,$$

where $dA(z)$ is the normalized Lebesgue measure on \mathbb{D} . Clearly, D_K is a Hilbert space. When $K(t) = t^s$, $0 \leq s < \infty$, the space D_K gives the usual Dirichlet type space D_s . In particular, if $s = 0$, this gives the classical Dirichlet space \mathcal{D} . We refer to [16, 19, 20] for the space D_s . The space D_K has been extensively studied. For example, under some conditions on K , Kerman and Sawyer [11] characterized Carleson measures and multipliers of D_K in terms of a maximal operator. Aleman [1] proved that each element of the space D_K can be written as a quotient of two bounded functions in the same space. See [2, 3, 8, 14, 18, 24] for more results on weighted Dirichlet spaces.

We say that $Z = \{z_n\} \subset \mathbb{D}$ is a zero set of an analytic function space X defined on \mathbb{D} if there is a $f \in X$ that vanishes on Z and nowhere else. Describing the zero sets for an analytic function space is a difficult problem. See [4, 5, 6, 13, 17, 21] for more information about this topic.

Let μ denote a positive Borel measure on \mathbb{D} . For a subarc $I \subseteq \partial\mathbb{D}$, let $S(I)$ be the Carleson box based on I with

$$S(I) = \{z \in \mathbb{D} : 1 - |I| \leq |z| < 1 \text{ and } \frac{z}{|z|} \in I\}.$$

If $I = \partial\mathbb{D}$, let $S(I) = \mathbb{D}$. Let $0 \leq s < \infty$. We say that μ is a (K, s) -Carleson measure on \mathbb{D} if

$$\sup_{I \subseteq \partial\mathbb{D}} \frac{\mu(S(I))}{K(|I|)|I|^s} < \infty.$$

Here and henceforth $\sup_{I \subseteq \partial\mathbb{D}}$ indicates the supremum taken over all subarcs I of $\partial\mathbb{D}$. We say that μ is a vanishing (K, s) -Carleson measure, if

$$\lim_{|I| \rightarrow 0} \frac{\mu(S(I))}{K(|I|)|I|^s} = 0.$$

If $K(t)t^s = t$, then we get the classical Carleson measure and vanishing Carleson measure, respectively. Carleson measure was firstly introduced in [4] by Carleson and it has many applications, such as in the interpolating sequence, $\bar{\partial}$ -equations, composition operators and integral operators. Hence Carleson measure is a very important tool for the function theory, harmonic analysis and operator theory. For more results on Carleson measure and its' generalization, we refer to [7, 10, 12, 22, 23, 25].

Recently, Pau and Peláez [13, Theorem 1] gave a nice characterization of zero sets of Dirichlet spaces D_s ($0 < s < 1$). Motivated by [13, Theorem 7], in this paper we study the zero sets of the space D_K . Moreover, we will characterize (K, s) -Carleson measure and vanishing (K, s) -Carleson measure. In particular, we will characterize vanishing (K, s) -Carleson measure by functions in the space D_K .

Throughout this paper, we assume that $K(0) = 0$ such that

$$\int_0^1 \frac{\varphi_K(s)}{s} ds < \infty \tag{1}$$

and

$$\int_1^\infty \frac{\varphi_K(s)}{s^2} ds < \infty, \tag{2}$$

where

$$\varphi_K(s) = \sup_{0 \leq t \leq 1} K(st)/K(t), \quad 0 < s < \infty.$$

In this paper, the symbol $f \approx g$ means that $f \lesssim g \lesssim f$. We say that $f \lesssim g$ if there exists a constant C such that $f \leq Cg$.

2 Zero sets

In order to study the zero sets of the D_K space, we define the space S_K , which consists of those $f \in H(\mathbb{D})$ such that

$$\|f\|_{S_K}^2 = |f(0)|^2 + \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 \frac{K(1-|z|^2)}{K\left(\frac{1-|az|^2}{1-|a|^2}\right)} dA(z) < \infty.$$

We say that a positive Borel measure μ on \mathbb{D} is a Carleson measure for D_K if there is a positive constant C such that

$$\int_{\mathbb{D}} |f(z)|^2 d\mu(z) \leq C \|f\|_{D_K}^2$$

for all $f \in D_K$.

We also need the following results.

Lemma 1. [11] *Let K be increasing, concave and $\lim_{x \rightarrow 0} x/K(x) = 0$. Then, μ is a Carleson measure for D_K if and only if there exists a constant $C > 0$ such that*

$$\int_I \sup_{\theta \in J \subset I} \frac{\mu(S(J))^2}{K(|J||J|)} d\theta \leq C\mu(S(I)),$$

for all arcs $I \subset \partial\mathbb{D}$. Here the supremum is taken over all closed arcs $J \subset I$.

Remark 1. From [9, Lemma 2.3], we know that if K satisfy (1) and (2), there exists K_3 , such that K_3 is increasing, concave, $\lim_{x \rightarrow 0} x/K_3(x) = 0$ and $K_3(t) \approx K(t)$, $0 < t < \infty$. As in [9], let $c \in (0, 1)$ be a small constant such that $\frac{K(t)}{t^c}$ and $\frac{t^{1-c}}{K(t)}$ are nondecreasing functions when $0 < t < 1$, moreover

$$\varphi_K(s) \lesssim s^{1-c}, \quad s \geq 1 \tag{3}$$

and

$$\varphi_K(s) \lesssim s^c, \quad s \leq 1. \tag{4}$$

Lemma 2. *Suppose that K satisfy (1) and (2). Then $f \in S_K$ if and only if*

$$\sup_{I \subset \partial\mathbb{D}} \int_{S(I)} |f'(z)|^2 \frac{K(1-|z|^2)}{K(|I|)} dA(z) < \infty. \tag{5}$$

Proof. Assume that $f \in S_K$. For any $I \subset \partial\mathbb{D}$, let $b = (1 - |I|)\eta \in \mathbb{D}$, where η is the center of I . Then

$$1 - |b| \approx |1 - \bar{b}z| \approx |I|, \quad z \in S(I).$$

Thus,

$$K\left(\frac{|1 - \bar{b}z|^2}{1 - |b|^2}\right) \approx K(|I|), \quad z \in S(I).$$

Therefore,

$$\begin{aligned} \int_{S(I)} |f'(z)|^2 \frac{K(1-|z|^2)}{K(|I|)} dA(z) &\lesssim \int_{\mathbb{D}} |f'(z)|^2 \frac{K(1-|z|^2)}{K\left(\frac{|1 - \bar{b}z|^2}{1 - |b|^2}\right)} dA(z) \\ &\lesssim \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 \frac{K(1-|z|^2)}{K\left(\frac{|1 - \bar{a}z|^2}{1 - |a|^2}\right)} dA(z) < \infty, \end{aligned}$$

which implies the desired result.

Conversely, assume that (5) holds. Without loss of generality, we can assume $|a| > 1/2$. Let I be a subarc of $\partial\mathbb{D}$ such that $|I_n| = 2^n|I|$, $n = 0, 1, 2, \dots, N-1$ and $I_N = \partial\mathbb{D}$. Then we have

$$\frac{|1 - \bar{a}\eta|^2}{1 - |a|^2} \approx |I|, \quad \eta \in I,$$

and

$$\frac{|1 - \bar{a}\eta|^2}{1 - |a|^2} \approx 2^{2n}|I|, \quad \eta \in I_{n+1}/I_n, \quad n = 0, 1, 2, \dots, N-1.$$

Since K satisfy (1) and (2), by Remark 1 we obtain

$$\begin{aligned} & \int_{\mathbb{D}} |f'(z)|^2 \frac{K(1 - |z|^2)}{K\left(\frac{|1 - \bar{a}z|^2}{1 - |a|^2}\right)} dA(z) \\ & \lesssim \sum_{n=1}^{\infty} \frac{1}{K(2^{2n}|I|)} \int_{S(2^{n+1}I) \setminus S(2^n I)} |f'(z)|^2 K(1 - |z|^2) dA(z) \\ & \quad + \frac{1}{K(|I|)} \int_{S(2I)} |f'(z)|^2 K(1 - |z|^2) dA(z) \\ & \lesssim \sum_{n=1}^{\infty} \frac{1}{K(2^{2n}|I|)} \int_{S(2^{n+1}I)} |f'(z)|^2 K(1 - |z|^2) dA(z) + \frac{K(2|I|)}{K(|I|)} \\ & \lesssim \sum_{n=1}^{\infty} \frac{K(2^{n+1}|I|)}{K(2^{2n}|I|)} + \varphi_K(2) \\ & \lesssim \sum_{n=1}^{\infty} 2^{(1-n)c} + 2^{1-c} < \infty. \end{aligned}$$

Here c is defined in Remark 1. Hence $f \in S_K$. □

From Lemma 2, we can easily obtain the following corollary.

Corollary 1. *Suppose that K satisfy the conditions (1) and (2). Then $f \in S_K$ if and only if $|f'(z)|^2 K(1 - |z|^2) dA(z)$ is $(K, 0)$ -Carleson measure.*

Let $M(D_K)$ denote the space of multipliers of D_K , that is,

$$M(D_K) = \{g \in H(\mathbb{D}) : gf \in D_K \text{ for all } f \in D_K\}.$$

Theorem 1. *Suppose that K satisfy the conditions (1) and (2). Then $M(D_K)$, $S_K \cap H^\infty$, S_K and D_K have the same zero sets.*

Proof. Since $M(D_K) \subseteq D_K$, we have any zero sets in $M(D_K)$ is zero sets in D_K . Next we prove that any zero set in D_K is zero set in $M(D_K)$.

Suppose that $\|f\|_{D_K} = 1$. Let $\{z_k\}$ be the zeros of f . Fix $z_0 \in \mathbb{D}$ such that $f(z_0) \neq 0$. Set $w_0 = \frac{f(z_0)}{\|R_{z_0}\|_{D_K}}$ and $w_j = 0$ ($j \geq 1$), where

$$R_{z_j}(z) = 1 + \sum_{n=1}^{\infty} \frac{1}{nK(\frac{1}{n})} \bar{z}_j^n z^n, \quad z \in \mathbb{D}, \quad j \geq 0.$$

From [3, Lemma 3.1], we know that $R_{z_j}(z)$ is the reproducing kernel of D_K space at z_j . Then, for each $n \geq 1$ and $a_0, a_1, \dots, a_n \in \mathbb{C}$, we have

$$\begin{aligned} & \sum_{i=0}^n \sum_{j=0}^n a_i \bar{a}_j (1 - \bar{w}_i w_j) \langle R_{z_i}, R_{z_j} \rangle \\ &= \left\| \sum_{j=0}^n a_j R_{z_j} \right\|_{D_K}^2 - |a_0|^2 |f(z_0)|^2 \\ &= \left\| \sum_{j=0}^n a_j R_{z_j} \right\|_{D_K}^2 - \left| \langle f, \sum_{j=0}^n a_j R_{z_j} \rangle \right|^2 \geq 0. \end{aligned}$$

Combine with Lemma 3.2 of [3], we know that D_K has Pick property. Thus, there exists $F_n \in M(D_K)$ with $\|F_n\|_{M(D_K)} \leq 1$ such that

$$F_n(z_0) = \frac{f(z_0)}{\|R_{z_0}\|_{D_K}} \quad \text{and} \quad F_n(z_j) = 0 \quad (j = 1, \dots, n).$$

Then, for all n , we have $\|F_n\|_{H^\infty} \leq \|F_n\|_{M(D_K)} \leq 1$. So $\{F_n\}_{n \geq 1}$ is a normal family, the limit function F is also a multiplier with $\|F\|_{M(D_K)} \leq 1$,

$$F(z_0) = \frac{f(z_0)}{\|R_{z_0}\|_{D_K}} \neq 0 \quad \text{and} \quad F(z_j) = 0 \quad (j = 1, \dots, n).$$

By f -property of D_K space (see [15]), we have every function $f \in D_K$, there exist $F \in M(D_K)$ with the same zero set. That is, D_K and $M(D_K)$ have the same zero sets.

Note that $S_K \cap H^\infty \subseteq S_K \subseteq D_K$. We only need to prove that $M(D_K) \subseteq S_K \cap H^\infty$. Suppose that $f \in M(D_K)$. From [3, Theorem 4.6], we know that $|f'(z)|^2 K(1 - |z|^2) dA(z)$ is a Carleson measure for D_K . Let $|I| = |J|$ in Lemma 1, that is, if μ is a Carleson measure for D_K , we can deduce that $\mu(S(I)) \lesssim K_3(|I|) \approx K(|I|)$. Thus,

$$\int_{S(I)} |f'(z)|^2 K(1 - |z|^2) dA(z) \lesssim K(|I|).$$

Combine with Lemma 2, we deduce that $f \in S_K$. Notice that $M(D_K) \subseteq H^\infty$ (see [3, Theorem 4.6]). That is, $M(D_K) \subseteq S_K \cap H^\infty$. \square

3 Carleson type measure

In this section, we give a characterization for (K, s) -Carleson measure and vanishing (K, s) -Carleson measure.

Theorem 2. *Suppose that K satisfy the conditions (1) and (2). Let μ be a positive Borel measure on \mathbb{D} , $0 \leq s < \infty$ such that $s + c > 1$. Then μ is a (K, s) -Carleson*

measure if and only if

$$\sup_{a \in \mathbb{D}} \frac{1}{K(1 - |a|^2)} \int_{\mathbb{D}} \frac{(1 - |a|^2)^s}{|1 - \bar{a}z|^{2s}} d\mu(z) < \infty. \tag{6}$$

Proof. Suppose that (6) holds. For any $I \subseteq \partial\mathbb{D}$, let $b = (1 - |I|)\zeta \in \mathbb{D}$, where ζ is the center of I . Then, for any $z \in S(I)$, we have

$$1 - |b| \approx |1 - \bar{b}z| \approx |I| \quad \text{and} \quad K(1 - |b|^2) \approx K(|I|).$$

Therefore,

$$\begin{aligned} \frac{\mu(S(I))}{K(|I|)|I|^s} &\lesssim \frac{1}{K(1 - |b|^2)} \int_{S(I)} \frac{(1 - |b|^2)^s}{|1 - \bar{b}z|^{2s}} d\mu(z) \\ &\leq \frac{1}{K(1 - |b|^2)} \int_{\mathbb{D}} \frac{(1 - |b|^2)^s}{|1 - \bar{b}z|^{2s}} d\mu(z) \\ &\leq \sup_{a \in \mathbb{D}} \frac{1}{K(1 - |a|^2)} \int_{\mathbb{D}} \frac{(1 - |a|^2)^s}{|1 - \bar{a}z|^{2s}} d\mu(z) < \infty, \end{aligned}$$

which implies that μ is a (K, s) -Carleson measure by the arbitrary of I .

Conversely, assume that μ is a (K, s) -Carleson measure. Without loss of generality, we assume $|a| > \frac{4}{5}$. Let I_n be the arc on $\partial\mathbb{D}$ such that $\frac{a}{|a|}$ is the center of I_n and $|I_n| = A^{(n-1)}(1 - |a|)$, where $1 < A < 2^{\frac{1}{s}}$, $n = 1, 2, \dots, N$, where N is the smallest integer such that $A^{(N-1)}(1 - |a|) \geq 1$. Since

$$\frac{(1 - |a|^2)^s}{|1 - \bar{a}z|^{2s}} \lesssim \frac{1}{A^{2(n-1)s}(1 - |a|)^s} \lesssim \frac{1}{A^{2ns}(1 - |a|)^s}, \quad z \in S(I_n) \setminus S(I_{n-1}),$$

and

$$\frac{|I_n|^s}{A^{ns}(1 - |a|)^s} = A^{(n-1)s - ns} < 1, \quad \frac{K(A^{(n-1)}(1 - |a|))}{K(1 - |a|^2)} \lesssim A^{(n-1)(1-c)},$$

we obtain

$$\begin{aligned} \frac{1}{K(1 - |a|^2)} \int_{\mathbb{D}} \frac{(1 - |a|^2)^s}{|1 - \bar{a}z|^{2s}} d\mu(z) &\leq \sum_{n=1}^N \frac{|I_n|^s K(|I_n|)}{A^{2ns}(1 - |a|)^s K(1 - |a|^2)} \\ &\lesssim \sum_{n=1}^N \frac{K(A^{(n-1)}(1 - |a|))}{A^{ns} K(1 - |a|^2)} \\ &\lesssim \sum_{n=1}^N \frac{A^{(n-1)(1-c)}}{A^{ns}} < \sum_{n=1}^{\infty} \frac{A^{n(1-c)}}{A^{ns}} \\ &< \infty. \end{aligned}$$

Here we used the fact that $s + c > 1$. □

Theorem 3. *Suppose that K satisfy the conditions (1) and (2). Let μ be a positive Borel measure on \mathbb{D} , $0 \leq s < \infty$ such that $s + c > 1$. Then μ is a vanishing (K, s) -Carleson measure if and only if*

$$\lim_{|a| \rightarrow 1} \frac{1}{K(1 - |a|^2)} \int_{\mathbb{D}} \frac{(1 - |a|^2)^s}{|1 - \bar{a}z|^{2s}} d\mu(z) = 0. \tag{7}$$

Proof. First we assume that μ is a vanishing (K, s) -Carleson measure. For any $\epsilon > 0$, there is a $\eta > 0$ such that for all arcs $I \subseteq \partial\mathbb{D}$ with $|I| \leq \eta$, such that

$$\frac{\mu(S(I))}{K(|I|)|I|^s} < \epsilon.$$

Assume $a = re^{i\theta}$ and $r > 1 - \eta$. Let $I_\eta \subseteq \partial\mathbb{D}$ such that $e^{i\theta}$ is the center of I_η and $|I_\eta| = \eta$. Then

$$\int_{\mathbb{D}} \frac{(1 - |a|^2)^s}{|1 - \bar{a}z|^{2s}} d\mu(z) =: M_1 + M_2,$$

where

$$M_1 =: \int_{\mathbb{D} \setminus S(I_\eta)} \frac{(1 - |a|^2)^s}{|1 - \bar{a}z|^{2s}} d\mu(z)$$

and

$$M_2 =: \int_{S(I_\eta)} \frac{(1 - |a|^2)^s}{|1 - \bar{a}z|^{2s}} d\mu(z).$$

Suppose that $e^{i\theta}$ is also the center of $\{I_n\}$, $|I_n| = A^{n-1}(1 - |a|)$, $A > 1$, $n = 1, 2, \dots, N - 1$ and N is the smallest integer such that $|I_N| > \eta$. Let $I_0 = \phi$. Note that

$$\frac{(1 - |a|^2)^s}{|1 - \bar{a}z|^{2s}} \lesssim \frac{1}{A^{2ns}(1 - |a|)^s}, \quad z \in S(I_n) \setminus S(I_{n-1}).$$

We have

$$\begin{aligned} M_2 &\leq \sum_{n=1}^N \int_{S(I_n) \setminus S(I_{n-1})} \frac{(1 - |a|^2)^s}{|1 - \bar{a}z|^{2s}} d\mu(z) \\ &\lesssim \frac{1}{(1 - |a|)^s} \sum_{n=1}^{N-1} \frac{\mu(S(I_n) \setminus S(I_{n-1}))}{A^{2ns}} + \frac{\mu(S(I_N) \setminus S(I_{N-1}))}{A^{2Ns}(1 - |a|)^s} \\ &\lesssim \frac{1}{(1 - |a|)^s} \sum_{n=1}^N \frac{\mu(S(I_n))}{A^{2ns}}. \end{aligned}$$

Note that $\frac{\mu(S(I))}{K(|I|)|I|^s} < \epsilon$ and $|I_n| = A^{n-1}(1 - |a|)$, using the fact that $\varphi_K(t) \lesssim$

t^{1-c} , $t \geq 1$, we deduce that

$$\begin{aligned}
 M_2 &\lesssim \frac{1}{(1-|a|)^s} \sum_{n=1}^N \frac{\epsilon K(|I_n|)|I_n|^s}{A^{2ns}} \\
 &\lesssim \sum_{n=1}^N \frac{\epsilon K(A^{n-1}(1-|a|))}{K(1-|a|)A^{ns+s}} K(1-|a|) \\
 &\lesssim \sum_{n=1}^N \varphi_K(A^{n-1}) \frac{\epsilon K(1-|a|)}{A^{ns+s}} \\
 &\lesssim \epsilon \sum_{n=1}^N A^{(n-1)(1-c)-ns-s} K(1-|a|) \lesssim \epsilon K(1-|a|). \tag{8}
 \end{aligned}$$

Now, we estimate M_1 . Since $|1-\bar{a}z| \geq \eta$, $z \in \mathbb{D} \setminus S(I_\eta)$, and notice the fact that $\frac{t^{1-c}}{K(t)}$ is a nondecreasing function when $0 < t < 1$, we obtain

$$\begin{aligned}
 M_1 &\lesssim \frac{\mu(\mathbb{D})}{\eta^{2s}} \frac{(1-|a|)^s}{K(1-|a|)} K(1-|a|) \\
 &\lesssim \frac{(1-|a|)^{1-c+s-1+c}}{K(1-|a|)} K(1-|a|) \lesssim (1-|a|)^{s+c-1} K(1-|a|). \tag{9}
 \end{aligned}$$

From (8) and (9) we see that (7) holds.

The proof for another side is similar to Theorem 2. Thus, we omit the details. \square

Finally, we give another characterization of vanishing (K, s) -Carleson measure by using functions in D_K .

Theorem 4. *Suppose that K satisfy the conditions (1) and (2). Let μ be a positive Borel measure on \mathbb{D} , $0 \leq s < \infty$ such that $s + c > 1$. Let $\{g_n\}$ be a bounded sequence in D_K such that $g_n \rightarrow 0$ uniformly on compact subset of \mathbb{D} as $n \rightarrow \infty$. Then μ is a vanishing (K, s) -Carleson measure if and only if*

$$\lim_{n \rightarrow \infty} \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |g_n(a)|^2 \frac{(1-|a|^2)^s}{|1-\bar{a}z|^{2s}} d\mu(z) = 0. \tag{10}$$

Proof. First we assume that μ is a vanishing (K, s) -Carleson measure. Following the proof of Theorem 3, for any given $\epsilon > 0$, we may find $\kappa > 0$ such that

$$\sup_{a \in \mathbb{D} \setminus \mathbb{D}_\kappa} \frac{1}{K(1-|a|)} \int_{\mathbb{D}} \frac{(1-|a|^2)^s}{|1-\bar{a}z|^{2s}} d\mu(z) < \epsilon,$$

where $\mathbb{D}_\kappa = \{z \in \mathbb{D} : |z| < \kappa\}$. Since

$$|g(z)| \lesssim \frac{\|g\|_{D_K}}{\sqrt{K(1-|z|)}}, \quad g \in D_K,$$

we obtain

$$\begin{aligned} & \sup_{a \in \mathbb{D} \setminus \overline{\mathbb{D}_\kappa}} \int_{\mathbb{D}} |g_n(a)|^2 \frac{(1 - |a|^2)^s}{|1 - \bar{a}z|^{2s}} d\mu(z) \\ & \lesssim \sup_{a \in \mathbb{D} \setminus \overline{\mathbb{D}_\kappa}} \frac{1}{K(1 - |a|)} \int_{\mathbb{D}} \frac{(1 - |a|^2)^s}{|1 - \bar{a}z|^{2s}} d\mu(z) < \epsilon. \end{aligned} \tag{11}$$

Also, since $g_n \rightarrow 0$ uniformly on compact subsets of \mathbb{D} , we see that for n sufficiently large,

$$\sup_{a \in \overline{\mathbb{D}_\kappa}} \int_{\mathbb{D}} |g_n(a)|^2 \frac{(1 - |a|^2)^s}{|1 - \bar{a}z|^{2s}} d\mu(z) \leq \epsilon \sup_{a \in \overline{\mathbb{D}_\kappa}} \int_{\mathbb{D}} \frac{(1 - |a|^2)^s}{|1 - \bar{a}z|^{2s}} d\mu(z) \lesssim \epsilon. \tag{12}$$

From (11) and (12) we see that (10) holds.

Conversely, assume that (10) holds. For $a \in \mathbb{D}$, it is easy to check that

$$g_a(z) = \frac{1 - |a|}{(1 - \bar{a}z)\sqrt{K(1 - |a|)}} \in D_K.$$

For any $I_n \subseteq \partial\mathbb{D}$ such that $|I_n| \rightarrow 1$ as $n \rightarrow \infty$, let $a_n = (1 - |I_n|)e^{i\theta_n} \in \mathbb{D}$, where $e^{i\theta_n}$ is the center of I_n . It is easy to check that $\{g_{a_n}\}$ is a bounded sequence in D_K and $g_{a_n} \rightarrow 0$ uniformly on compact subsets of \mathbb{D} as $n \rightarrow \infty$. By (10), we have

$$\lim_{n \rightarrow \infty} \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |g_{a_n}(z)|^2 \frac{(1 - |a|^2)^s}{|1 - \bar{a}z|^{2s}} d\mu(z) = 0.$$

Thus,

$$\lim_{n \rightarrow \infty} \frac{1}{|I_n|^s} \int_{S(I_n)} |g_{a_n}(z)|^2 d\mu(z) = 0.$$

Notice the fact that $|g_{a_n}(z)|^2 \gtrsim \frac{1}{K(|I_n|)}$, we get

$$\frac{\mu(S(I_n))}{K(|I_n|)|I_n|^s} \rightarrow 0, \text{ as } n \rightarrow \infty,$$

as desired. □

References

- [1] A. Aleman, Hilbert spaces of analytic functions between the Hardy space and the Dirichlet space, *Proc. Amer. Math. Soc.* **115** (1992), 97–104.
- [2] N. Arcozzi, R. Rochberg and E. Sawyer, Carleson measures for analytic Besov spaces, *Rev. Mat. Iberoamericana*. **18** (2002), 443–510.
- [3] G. Bao, Z. Lou, R. Qian and H. Wulan, On multipliers of Dirichlet type spaces, *Complex Anal. Oper. Theory* **9** (2015), 1701–1732.
- [4] L. Carleson, An interpolation problem for bounded analytic functions, *Amer. J. Math.* **80** (1985), 921–930.

- [5] L. Carleson, Interpolation by bounded analytic functions and the Corona problem, *Ann. Math.* **76** (1962), 547–559.
- [6] J. Caughran, Two results concerning the zeros of functions with finite Dirichlet integral, *Canad. J. Math.* **21** (1969), 312–316.
- [7] P. Duren, *Theory of H^p Spaces*, Academic Press, New York, 1970.
- [8] O. El-Fallah, K. Kellay, J. Mashreghi and T. Ransford, *A Primer on the Dirichlet Space*, Cambridge Tracts in Mathematics 203, Cambridge University Press, 2014.
- [9] M. Essen, H. Wulan and J. Xiao, Several function-theoretic characterizations of Möbius invariant Q_K spaces, *J. Funct. Anal.* **230** (2006), 78–115.
- [10] J. Garnett, *Bounded Analytic Functions*, Springer, New York, 2007.
- [11] R. Kerman and E. Sawyer, Carleson measures and multipliers of Dirichlet-type spaces, *Trans. Amer. Math. Soc.* **309** (1988), 87–98.
- [12] B. MacCluer and R. Zhao, Vanishing logarithmic Carleson measures, *Illinois J. Math.* **46** (2002), 507–518.
- [13] J. Pau and J. Peláez, On the zeros of functions in Dirichlet-type spaces, *Trans. Amer. Math. Soc.* **363** (2011), 1981–2002.
- [14] R. Qian and Y. Shi, Inner function in Dirichlet type spaces, *J. Math. Anal. Appl.* **421** (2015), 1844–1854.
- [15] M. Rabinathan, Toeplitz operators and division by inner function, *Indiana Univ. J. Math.* **22** (1972), 523–529.
- [16] R. Rochberg and Z. Wu, A new characterization of Dirichlet type spaces and applications, *Illinois J. Math.* **37** (1993), 101–122.
- [17] H. Shapiro and A. Shields, On the zeros of functions with finite Dirichlet integral and some related function spaces, *Math. Z.* **80** (1962), 217–229.
- [18] S. Shimorin, Complete Navanlinna-Pick property of Dirichlet-type spaces, *J. Funct. Anal.* **191** (2002), 272–296.
- [19] D. Stegenga, Multipliers of the Dirichlet space, *Illinois J. Math.* **24** (1980), 113–139.
- [20] G. Taylor, Multipliers on D_α , *Trans. Amer. Math. Soc.* **123** (1966), 229–240.
- [21] B. Taylor and D. Williams, Zeros of Lipschitz functions analytic in the unit disc, *Michigan Math. J.* **18** (1971), 129–139.
- [22] J. Xiao, *Holomorphic Q Classes*, Springer, LNM 1767, Berlin, 2001.
- [23] R. Zhao, On logarithmic Carleson measures, *Acta Sci. Math. (Szeged)* **69** (2003), 605–618.
- [24] J. Zhou and Y. Wu, Decomposition theorems and conjugate pair in D_K spaces, *Acta Math. Sinica* **30** (2014), 1513–1525.
- [25] K. Zhu, *Operator Theory in Function Spaces*, American Mathematical Society, Providence, RI, 2007.

Ruishen Qian: School of Mathematics and Statistics, Lingnan normal University, 524048, Zhanjiang, Guangdong, P. R. China.
 Email: qianruishen@sina.cn

Songxiao Li: Institute of System Engineering, Macau University of Science and Technology, Avenida Wai Long, Taipa, Macau.
 Email: jyulsx@163.com

Yanhua Zhang: Department of Mathematics, Qufu Normal University, 273165, Qufu, ShanDong, P. R. China.
 Email: qfuzhangyanhua@163.com

★ Corresponding author: Songxiao Li

Smarandache fuzzy *BCI*-algebras

Sun Shin Ahn¹ and Young Joo Seo^{*,2}

¹Department of Mathematics Education, Dongguk University, Seoul 04620, Korea

²Research Institute for Natural Sciences, Department of Mathematics, Hanyang University, Seoul, 04763, Korea

Abstract. The notions of a Smarandache fuzzy subalgebra (ideal) of a Smarandache *BCI*-algebra, a Smarandache fuzzy clean(fresh) ideal of a Smarandache *BCI*-algebra are introduced. Examples are given, and several related properties are investigated.

1. Introduction

Generally, in any human field, a Smarandache structure on a set A means a weak structure W on A such that there exists a proper subset B of A with a strong structure S which is embedded in A . In [4], R. Padilla showed that Smarandache semigroups are very important for the study of congruences. Y. B. Jun ([1,2]) introduced the notion of Smarandache *BCI*-algebras, Smarandache fresh and clean ideals of Smarandache *BCI*-algebras, and obtained many interesting results about them.

In this paper, we discuss a Smarandache fuzzy structure on *BCI*-algebras and introduce the notions of a Smarandache fuzzy subalgebra (ideal) of a Smarandache *BCI*-algebra, a Smarandache fuzzy clean (fresh) ideal of a Smarandache *BCI*-algebra are introduced, and we investigate their properties.

2. Preliminaries

An algebra $(X; *, 0)$ of type (2,0) is called a *BCI-algebra* if it satisfies the following conditions:

- (I) $(\forall x, y, z \in X)((x * y) * (x * z)) * (z * y) = 0$,
- (II) $(\forall x, y \in X)((x * (x * (x * y))) * y = 0)$,
- (III) $(\forall x \in X)(x * x = 0)$,
- (IV) $(\forall x, y \in X)(x * y = 0 \text{ and } y * x = 0 \text{ imply } x = y)$.

If a *BCI*-algebra X satisfies the following identity;

- (V) $(\forall x \in X)(0 * x = 0)$,

then X is said to be a *BCK-algebra*. We can define a partial order “ \leq ” on X by $x \leq y$ if and only if $x * y = 0$.

Every *BCI*-algebra X has the following properties:

- (a₁) $(\forall x \in X)(x * 0 = x)$,
- (a₁) $(\forall x, y, z \in X)(x \leq y \text{ implies } x * z \leq y * z, z * y \leq z * x)$.

A non-empty subset I of a *BCI*-algebra X is called an *ideal* of X if it satisfies the following conditions:

- (i) $0 \in I$,
- (ii) $(\forall x \in X)(\forall y \in I)(x * y \in I \text{ implies } x \in I)$.

^{0*} Correspondence: Tel: +82 10 9247 6575 (Y. J. Seo).

Sun Shin Ahn¹ and Young Joo Seo²

Definition 2.1. ([1]) A *Smarandache BCI-algebra* is defined to be a *BCI-algebra* X in which there exists a proper subset Q of X such that

- (i) $0 \in Q$ and $|Q| \geq 2$,
- (ii) Q is a *BCK-algebra* under the same operation of X .

By a *Smarandache positive implicative* (resp. *commutative* and *implicative*) *BCI-algebra*, we mean a *BCI-algebra* X which has a proper subset Q of X such that

- (i) $0 \in Q$ and $|Q| \geq 2$,
- (ii) Q is a positive implicative (resp. commutative and implicative) *BCK-algebra* under the same operation of X .

Let $(X; *, 0)$ be a *Smarandache BCI-algebra* and H be a subset of X such that $0 \in H$ and $|H| \geq 2$. Then H is called a *Smarandache subalgebra* of X if $(H; *, 0)$ is a *Smarandache BCI-algebra*.

A non-empty subset I of X is called a *Smarandache ideal* of X related to Q if it satisfies:

- (i) $0 \in I$,
- (ii) $(\forall x \in Q)(\forall y \in I)(x * y \in I \text{ implies } x \in I)$,

where Q is a *BCK-algebra* contained in X . If I is a *Smarandache ideal* of X related to every *BCK-algebra* contained in X , we simply say that I is a *Smarandache ideal* of X .

In what follows, let X and Q denote a *Smarandache BCI-algebra* and a *BCK-algebra* which is properly contained in X , respectively.

Definition 2.2. ([2]) A non-empty subset I of X is called a *Smarandache ideal* of X related to Q (or briefly, a *Q-Smarandache ideal*) of X if it satisfies:

- (c₁) $0 \in I$,
- (c₂) $(\forall x \in Q)(\forall y \in I)(x * y \in I \text{ implies } x \in I)$.

If I is a *Smarandache ideal* of X related to every *BCK-algebra* contained in X , we simply say that I is a *Smarandache ideal* of X .

Definition 2.3. ([2]) A non-empty subset I of X is called a *Smarandache fresh ideal* of X related to Q (or briefly, a *Q-Smarandache fresh ideal* of X) if it satisfies the conditions (c₁) and

- (c₃) $(\forall x, y, z \in Q)((x * y) * z \in I \text{ and } y * z \in I \text{ imply } x * z \in I)$.

Theorem 2.4. ([2]) Every *Q-Smarandache fresh ideal* which is contained in Q is a *Q-Smarandache ideal*.

The converse of Theorem 2.4 need not be true in general.

Theorem 2.5. ([2]) Let I and J be *Q-Smarandache ideals* of X and $I \subset J$. If I is a *Q-Smarandache fresh ideal* of X , then so is J .

Definition 2.6. ([2]) A non-empty subset I of X is called a *Smarandache clean ideal* of X related to Q (or briefly, a *Q-Smarandache clean ideal* of X) if it satisfies the conditions (c₁) and

- (c₄) $(\forall x, y \in Q)(z \in I)((x * (y * x)) * z \in I \text{ implies } x \in I)$.

Smarandache fuzzy BCI-algebras

Theorem 2.7. ([2]) Every Q -Smarandache clean ideal of X is a Q -Smarandache ideal.

The converse of Theorem 2.7 need not be true in general.

Theorem 2.8. ([2]) Every Q -Smarandache clean ideal of X is a Q -Smarandache fresh ideal.

Theorem 2.9. ([2]) Let I and J be Q -Smarandache ideals of X and $I \subset J$. If I is a Q -Smarandache clean ideal of X , then so is J .

A fuzzy set μ in X is called a *fuzzy subalgebra* of a BCI-algebra X if $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$. A fuzzy set μ in X is called a *fuzzy ideal* of X if

- (F₁) $\mu(0) \geq \mu(x)$ for all $x \in X$,
- (F₂) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$ for all $x, y \in X$.

Let μ be a fuzzy set in a set X . For $t \in [0, 1]$, the set $\mu_t := \{x \in X | \mu(x) \geq t\}$ is called a *level subset* of μ .

3. Smarandache fuzzy ideals

Definition 3.1. Let X be a Smarandache BCI-algebra. A map $\mu : X \rightarrow [0, 1]$ is called a *Smarandache fuzzy subalgebra* of X if it satisfies

- (SF₁) $\mu(0) \geq \mu(x)$ for all $x \in P$,
- (SF₂) $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in P$,

where $P \subsetneq X$, P is a BCK-algebra with $|P| \geq 2$.

A map $\mu : X \rightarrow [0, 1]$ is called a *Smarandache fuzzy ideal* of X if it satisfies (SF₁) and

- (SF₂) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$ for all $x, y \in P$,

where $P \subsetneq X$, P is a BCK-algebra with $|P| \geq 2$. This Smarandache fuzzy subalgebra (ideal) is denoted by μ_P , i.e., $\mu_P : P \rightarrow [0, 1]$ is a fuzzy subalgebra(ideal) of X .

Example 3.2. Let $X := \{0, 1, 2, 3, 4, 5\}$ be a Smarandache BCI-algebra ([1]) with the following Cayley table:

*	0	1	2	3	4	5
0	0	0	0	3	3	3
1	1	0	1	3	3	3
2	2	2	0	3	3	3
3	3	3	3	0	0	0
4	4	3	4	1	0	0
5	5	3	5	1	1	0

Define a map $\mu : X \rightarrow [0, 1]$ by

$$\mu(x) := \begin{cases} 0.5 & \text{if } x \in \{0, 1, 2, 3\}, \\ 0.7 & \text{otherwise} \end{cases}$$

Clearly μ is a Smarandache fuzzy subalgebra of X . It is verified that μ restricted to a subset $\{0, 1, 2, 3\}$ which is a subalgebra of X is a fuzzy subalgebra of X , i.e., $\mu_{\{0,1,2,3\}} : \{0, 1, 2, 3\} \rightarrow [0, 1]$ is a fuzzy subalgebra of X . Thus $\mu : X \rightarrow [0, 1]$ is a Smarandache fuzzy subalgebra of X . Note that $\mu : X \rightarrow [0, 1]$ is not a fuzzy subalgebra of X , since $\mu(5 * 4) = \mu(0) = 0.5 \not\geq \min\{\mu(5), \mu(4)\} = 0.7$.

Sun Shin Ahn¹ and Young Joo Seo²

Example 3.3. Let $X := \{0, 1, 2, 3, 4, 5\}$ be a Smarandache BCI-algebra ([1]) with the following Cayley table:

*	0	1	2	3	4	5
0	0	0	0	0	4	4
1	1	0	0	1	4	4
2	2	2	0	2	4	4
3	3	3	3	0	4	4
4	4	4	4	0	0	
5	5	4	4	5	1	0

Define a map $\mu : X \rightarrow [0, 1]$ by

$$\mu(x) := \begin{cases} 0.5 & \text{if } x \in \{0, 1, 2\} \\ 0.7 & \text{otherwise} \end{cases}$$

Clearly μ is a Smarandache fuzzy ideal of X . It is verified that μ restricted to a subset $\{0, 1, 2\}$ which is an ideal of X is a fuzzy ideal of X , i.e., $\mu_{\{0,1,2\}} : \{0, 1, 2\} \rightarrow [0, 1]$ is a fuzzy ideal of X . Thus $\mu : X \rightarrow [0, 1]$ is a Smarandache fuzzy ideal of X . Note that $\mu : X \rightarrow [0, 1]$ is not a fuzzy ideal of X , since $\mu(2) = 0.5 \not\geq \min\{\mu(2*4) = \mu(4), \mu(4)\} = \mu(4) = 0.7$.

Lemma 3.4. Every Smarandache fuzzy ideal μ_P of a Smarandache BCI-algebra X is order reversing.

Proof. Let P be a BCK-algebra with $P \subsetneq X$ and $|P| \geq 2$. If $x, y \in P$ with $x \leq y$, then $x * y = 0$. Hence we have $\mu(x) \geq \min\{\mu(x * y), \mu(y)\} = \min\{\mu(0), \mu(y)\} = \mu(y)$. \square

Theorem 3.5. Any Smarandache fuzzy ideal μ_P of a Smarandache BCI-algebra X must be a Smarandache fuzzy subalgebra of X .

Proof. Let P be a BCK-algebra with $P \subsetneq X$ and $|X| \geq 2$. Since $x * y \leq x$ for any $x, y \in P$, it follows from Lemma 3.4 that $\mu(x) \leq \mu(x * y)$, so by (SF_2) we obtain $\mu(x * y) \geq \mu(x) \geq \min\{\mu(x * y), \mu(y)\} \geq \min\{\mu(x), \mu(y)\}$. This shows that μ is a Smarandache fuzzy subalgebra of X , proving the theorem. \square

Proposition 3.6. Let μ_P be a Smarandache fuzzy ideal of a Smarandache BCI-algebra X . If the inequality $x * y \leq z$ holds in P , then $\mu(x) \geq \min\{\mu(x), \mu(z)\}$ for all $x, y, z \in P$.

Proof. Let P be a BCK-algebra with $P \subsetneq X$ and $|P| \geq 2$. If $x * y \leq z$ in P , then $(x * y) * z = 0$. Hence we have $\mu(x * y) \geq \min\{\mu((x * y) * z), \mu(z)\} = \min\{\mu(0), \mu(z)\} = \mu(z)$. It follows that $\mu(x) \geq \min\{\mu(x * y), \mu(y)\} \geq \min\{\mu(y), \mu(z)\}$. \square

Theorem 3.7. Let X be a Smarandache BCI-algebra. A Smarandache fuzzy subalgebra μ_P of X is a Smarandache fuzzy ideal of X if and only if for all $x, y \in P$, the inequality $x * y \leq z$ implies $\mu(x) \geq \min\{\mu(y), \mu(z)\}$.

Proof. Suppose that μ_P is a Smarandache fuzzy subalgebra of X satisfying the condition $x * y \leq z$ implies $\mu(x) \geq \min\{\mu(y), \mu(z)\}$. Since $x * (x * y) \leq y$ for all $x, y \in P$, it follows that $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$. Hence μ_P is a Smarandache fuzzy ideal of X . The converse follows from Proposition 3.6. \square

Definition 3.8. Let X be a Smarandache BCI-algebra. A map $\mu : X \rightarrow [0, 1]$ is called a Smarandache fuzzy clean ideal of X if it satisfies (SF_1) and

$$(SF_3) \mu(x) \geq \min\{\mu(x * (y * x)) * z, \mu(z)\} \text{ for all } x, y, z \in P,$$

Smarandache fuzzy BCI-algebras

where $P \subsetneq X$ and P is a BCK-algebra with $|P| \geq 2$. This Smarandache fuzzy clean ideal is denoted by μ_P , i.e., $\mu_P : P \rightarrow [0, 1]$ is a Smarandache fuzzy clean ideal of X .

Example 3.9. Let $X := \{0, 1, 2, 3, 4, 5\}$ be a Smarandache BCI-algebra ([2]) with the following Cayley table:

*	0	1	2	3	4	5
0	0	0	0	0	0	5
1	1	0	0	0	0	5
2	2	1	0	1	0	5
3	3	4	4	4	0	5
4	4	4	4	4	0	5
5	5	5	5	5	5	0

Define a map $\mu : X \rightarrow [0, 1]$ by

$$\mu(x) := \begin{cases} 0.4 & \text{if } x \in \{0, 1, 2, 3\} \\ 0.8 & \text{otherwise} \end{cases}$$

Clearly μ is a Smarandache fuzzy clean ideal of X , but μ is not a fuzzy clean ideal of X , since $\mu(3) = 0.4 \not\geq \min\{\mu((3 * (0 * 3)) * 5), \mu(5)\} = \min\{\mu(5), \mu(5)\} = \mu(5) = 0.8$.

Theorem 3.10. Let X be a Smarandache BCI-algebra. Any Smarandache fuzzy clean ideal μ_P of X must be a Smarandache fuzzy ideal of X .

Proof. Let X be a BCK-algebra with $P \subsetneq X$ and $|P| \geq 2$. Let $\mu_P : P \rightarrow [0, 1]$ be a Smarandache fuzzy clean ideal of X . If we let $y := x$ in (SF_3) , then $\mu(x) \geq \min\{\mu((x * (x * x)) * z), \mu(z)\} = \min\{\mu((x * 0) * z), \mu(z)\} = \min\{\mu(x * z), \mu(z)\}$, for all $x, y, z \in P$. This shows that μ satisfies (SF_2) . Combining (SF_1) , μ_P is a Smarandache fuzzy ideal of X , proving the theorem. □

Corollary 3.11. Every Smarandache fuzzy clean ideal μ_P of a Smarandache BCI-algebra X must be a Smarandache fuzzy subalgebra of X .

Proof. It follows from Theorem 3.5 and Theorem 3.10. □

The converse of Theorem 3.10 may not be true as shown in the following example.

Example 3.12. Let $X := \{0, 1, 2, 3, 4, 5\}$ be a Smarandache BCI-algebra with the following Cayley table:

*	0	1	2	3	4	5
0	0	0	0	0	0	5
1	1	0	1	0	0	5
2	2	2	0	0	0	5
3	3	3	3	0	0	5
4	4	3	4	1	0	5
5	5	5	5	5	5	0

Let μ_P be a fuzzy set in $P = \{0, 1, 2, 3, 4\}$ defined by $\mu(0) = \mu(2) = 0.8$ and $\mu(1) = \mu(3) = \mu(4) = 0.3$. It is easy to check that μ_P is a fuzzy ideal of X . Hence $\mu : X \rightarrow [0, 1]$ is a Smarandache fuzzy ideal of X . But it is not a Smarandache fuzzy clean ideal of X since $\mu(1) = 0.3 \not\geq \min\{\mu((1 * (3 * 1)) * 2), \mu(2)\} = \min\{\mu(0), \mu(2)\} = 0.8$.

Theorem 3.13. Let X be a Smarandache implicative BCI-algebra. Every Smarandache fuzzy ideal μ_P of X is a Smarandache fuzzy clean ideal of X .

Sun Shin Ahn¹ and Young Joo Seo²

Proof. Let P be a BCK -algebra with $P \subsetneq X$ and $|P| \geq 2$. Since X is a Smarandache implicative BCI -algebra, we have $x = x * (y * x)$ for all $x, y \in P$. Let μ_P be a Smarandache fuzzy ideal of X . It follows from (SF_2) that $\mu(x) \geq \min\{\mu(x * z), \mu(z)\} \geq \min\{\mu((x * (y * x)) * z), \mu(z)\}$, for all $x, y, z \in P$. Hence μ_P is a Smarandache clean ideal of X . The proof is complete. \square

In what follows, we give characterizations of fuzzy implicative ideals.

Theorem 3.14. *Let X be a Smarandache BCI -algebra. Suppose that μ_P is a Smarandache fuzzy ideal of X . Then the following equivalent:*

- (i) μ_P is Smarandache fuzzy clean,
- (ii) $\mu(x) \geq \mu(x * (y * x))$ for all $x, y \in P$,
- (iii) $\mu(x) = \mu(x * (y * x))$ for all $x, y \in P$.

Proof. (i) \Rightarrow (ii): Let μ_P be a Smarandache fuzzy clean ideal of X . It follows from (SF_3) that $\mu(x) \geq \min\{\mu((x * (y * x)) * 0), \mu(0)\} = \min\{\mu(x * (y * x)), \mu(0)\} = \mu(x * (y * x))$, $\forall x, y \in P$. Hence the condition (ii) holds.

(ii) \Rightarrow (iii): Since X is a Smarnadache BCI -algebra, we have $x * (y * x) \leq x$ for all $x, y \in P$. It follows from Lemma 3.4 that $\mu(x) \leq \mu(x * (y * x))$. By (ii), $\mu(x) \geq \mu(x * (y * x))$. Thus the condition (iii) holds.

(iii) \Rightarrow (i): Suppose that the condition (iii) holds. Since μ_P is a Smarandache fuzzy ideal, by (SF_2) , we have $\mu(x * (y * x)) \geq \min\{\mu((x * (y * x)) * z), \mu(z)\}$. Combining (iii), we obtain $\mu(x) \geq \min\{\mu((x * (y * x)) * z), \mu(z)\}$. Hence μ satisfies the condition (SF_3) . Obviously, μ satisfies (SF_1) . Therefore μ is a fuzzy clean ideal of X . Hence the condition (i) holds. The proof is complete. \square

For any fuzzy sets μ and ν in X , we write $\mu \leq \nu$ if and only if $\mu(x) \leq \nu(x)$ for any $x \in X$.

Definition 3.15. Let X be a Smarandache BCI -algebra and let $\mu_P : P \rightarrow [0, 1]$ be a Smarandache fuzzy BCI -algebra of X . For $t \leq \mu(0)$, the set $\mu_t := \{x \in P | \mu(x) \geq t\}$ is called a *level subset* of μ_P .

Theorem 3.16. *A fuzzy set μ in P is a Smarandache fuzzy clean ideal of X if and only if, for all $t \in [0, 1]$, μ_t is either empty or a Smarandache clean ideal of X .*

Proof. Suppose that μ_P is a Smarandache fuzzy clean ideal of X and $\mu_t \neq \emptyset$ for any $t \in [0, 1]$. It is clear that $0 \in \mu_t$ since $\mu(0) \geq t$. Let $\mu((x * (y * x)) * z) \geq t$ and $\mu(z) \geq t$. It follows from (SF_3) that $\mu(x) \geq \min\{\mu((x * (y * x)) * z), \mu(z)\} \geq t$, namely, $x \in \mu_t$. This shows that μ_t is a Smarandache clean ideal of X .

Conversely, assume that for each $t \in [0, 1]$, μ_t is either empty or a Smaranadche clean ideal of X . For any $x \in P$, let $\mu(x) = t$. Then $x \in \mu_t$. Since $\mu_t (\neq \emptyset)$ is a Smarandache clean ideal of X , therefore $0 \in \mu_t$ and hence $\mu(0) \geq \mu(x) = t$. Thus $\mu(0) \geq \mu(x)$ for all $x \in P$. Now we show that μ satisfies (SF_3) . If not, then there exist $x', y', z' \in P$ such that $\mu(x') < \min\{\mu((x' * (y' * z')) * z'), \mu(z')\}$. Taking $t_0 := \frac{1}{2}\{\mu(x') + \min\{\mu((x' * (y' * z')) * z'), \mu(z')\}\}$, we have $\mu(x') < t_0 < \min\{\mu((x' * (y' * z')) * z'), \mu(z')\}$. Hence $x' \notin \mu_{t_0}$, $(x' * (y' * z')) * z \in \mu_{t_0}$, and $z' \in \mu_{t_0}$, i.e., μ_{t_0} is not a Smarandache clean of X , which is a contradiction. Therefore, μ_P is a Smarnadche fuzzy clean ideal, completing the proof. \square

Theorem 3.17. ([2]) (Extension Property) *Let X be a Smarandache BCI -algebra. Let I and J be Q -Smarandache ideals of X and $I \subseteq J \subseteq Q$. If I is a Q -Smarandache clean ideal of X , then so is J .*

Next we give the extension theorem of Smarandache fuzzy clean ideals.

Smarandache fuzzy BCI-algebras

Theorem 3.18. *Let X be a Smarandache BCI-algebra. Let μ and ν be Smarandache fuzzy ideals of X such that $\mu \leq \nu$ and $\mu(0) = \nu(0)$. If μ is a Smarandache fuzzy clean ideal of X , then so is ν .*

Proof. It suffices to show that for any $t \in [0, 1]$, ν_t is either empty or a Smarandache clean ideal of X . If the level subset ν_t is non-empty, then $\mu_t \neq \emptyset$ and $\mu_t \subseteq \nu_t$. In fact, if $x \in \mu_t$, then $t \leq \mu(x)$; hence $t \leq \nu(x)$, i.e, $x \in \nu_t$. So $\mu_t \subseteq \nu_t$. By the hypothesis, since μ is a Smarandache fuzzy clean ideal of X , μ_t is a Smarandache clean of X by Theorem 3.16. It follows from Theorem 3.17 that ν_t is a Smarandache clean ideal of X . Hence ν is a Smarandache fuzzy clean of X . The proof is complete. □

Definition 3.19. Let X be a Smarandache BCI-algebra. A map $\mu : X \rightarrow [0, 1]$ is called a *Smarandache fuzzy fresh ideal* of X if it satisfies (SF_1) and

$$(SF_4) \mu(x * z) \geq \min\{\mu((x * y) * z), \mu(y * z)\} \text{ for all } x, y, z \in P,$$

where P is a BCK-algebra with $P \subsetneq X$ and $|P| \geq 2$. This Smarandache fuzzy ideal is denoted by μ_P , i.e., $\mu_P : P \rightarrow [0, 1]$ is a Smarandache fuzzy fresh ideal of X .

Example 3.20. Let $X := \{0, 1, 2, 3, 4, 5\}$ be a Smarandache BCI-algebra ([2]) with the following Cayley table:

*	0	1	2	3	4	5
0	0	0	0	0	0	5
1	1	0	1	0	1	5
2	2	2	0	2	0	5
3	3	1	3	0	3	5
4	4	4	4	4	0	5
5	5	5	5	5	5	0

Define a map $\mu : X \rightarrow [0, 1]$ by

$$\mu(x) := \begin{cases} 0.5 & \text{if } x \in \{0, 1, 3\}, \\ 0.9 & \text{otherwise} \end{cases}$$

Clearly μ is a Smarandache fuzzy fresh ideal of X . But it is not a fuzzy fresh ideal of X , since $\mu(2 * 4) = \mu(0) = 0.5 \not\geq \min\{\mu((2 * 5) * 4), \mu(5 * 4)\} = \mu(5) = 0.9$.

Theorem 3.21. *Any Smarandache fuzzy fresh ideal of a Smarandache BCI-algebra X must be a Smarandache fuzzy ideal of X .*

Proof. Taking $z := 0$ in (SF_4) and $x * 0 = x$, we have $\mu(x * 0) \geq \min\{\mu((x * y) * 0), \mu(y * 0)\}$. Hence $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$. Thus (SF_2) holds. □

The converse of Theorem 3.21 may not be true as show in the following example.

Example 3.22. Let $X := \{0, 1, 2, 3, 4, 5\}$ be a Smarandache BCI-algebra ([2]) with the following Cayley table:

*	0	1	2	3	4	5
0	0	0	0	0	0	5
1	1	0	0	0	1	5
2	2	1	0	1	2	5
3	3	1	1	0	3	5
4	4	4	4	4	0	5
5	5	5	5	5	5	0

Sun Shin Ahn¹ and Young Joo Seo²

Define a map $\mu : X \rightarrow [0, 1]$ by

$$\mu(x) := \begin{cases} 0.5 & \text{if } x \in \{0, 4\}, \\ 0.4 & \text{otherwise} \end{cases}$$

Clearly $\mu(x)$ is a Smarandache fuzzy ideal of X . But $\mu(x)$ is not a Smarandache fuzzy fresh ideal of X , since $\mu(2 * 3) = \mu(1) = 0.4 \not\geq \min\{\mu((2 * 1) * 3), \mu(1 * 3)\} = \min\{\mu(1 * 3), \mu(0)\} = \mu(0) = 0.5$.

Proposition 3.23. *Let X be a Smarandache BCI-algebra. A Smarandache fuzzy ideal μ_P of X is a Smarandache fuzzy fresh ideal of X if and only if it satisfies the condition $\mu(x * y) \geq \mu((x * y) * y)$ for all $x, y \in P$.*

Proof. Assume that μ_P is a Smarandache fuzzy fresh ideal of X . Putting $z := y$ in (SF_4) , we have $\mu(x * y) \geq \min\{\mu((x * y) * y), \mu(y * y)\} = \min\{\mu((x * y) * y), \mu(0)\} = \mu((x * y) * y), \forall x, y \in P$.

Conversely, let μ_P be Smarandache fuzzy ideal of X such that $\mu(x * y) \geq \mu((x * y) * y)$. Since, for all $x, y, z \in P$, $((x * z) * z) * (y * z) \leq (x * z) * y = (x * y) * z$, we have $\mu((x * y) * z) \leq \mu(((x * z) * z) * (y * z))$. Hence $\mu(x * z) \geq \mu((x * z) * z) \geq \min\{\mu(((x * z) * z) * (y * z)), \mu(y * z)\} \geq \min\{\mu((x * y) * z), \mu(y * z)\}$. This completes the proof. \square

Since $(x * y) * y \leq x * y$, it follows from Lemma 3.4 that $\mu(x * y) \leq \mu((x * y) * y)$. Thus we have the following theorem.

Theorem 3.24. *Let X be a Smarandache BCI-algebra. A Smarandache fuzzy ideal μ_P of X is a Smarandache fuzzy fresh if and only if it satisfies the identity*

$$\mu(x * y) = \mu((x * y) * y), \text{ for all } x, y \in X.$$

We give an equivalent condition for which a Smarandache fuzzy subalgebra of a Smarandache BCI-algebra to be a Smarandache fuzzy clean ideal of X .

Theorem 3.25. *A Smarandache fuzzy subalgebra μ_P of X is a Smarandache fuzzy clean ideal of X if and only if it satisfies*

$$(x * (y * x)) * z \leq u \text{ implies } \mu(x) \geq \min\{\mu(z), \mu(u)\} \text{ for all } x, y, z, u \in P. \tag{*}$$

Proof. Assume that μ_P is a Smarandache fuzzy clean ideal of X . Let $x, y, z, u \in P$ be such that $(x * (y * x)) * z \leq u$. Since μ is a Smarandache fuzzy ideal of X , we have $\mu(x * (y * x)) \geq \min\{\mu(z), \mu(u)\}$ by Theorem 3.7. By Theorem 3.14-(iii), we obtain $\mu(x) \geq \min\{\mu(z), \mu(u)\}$.

Conversely, suppose that μ_P satisfies (*). Obviously, μ_P satisfies (SF_1) , since $(x * (y * x)) * ((x * (y * x)) * z) \leq z$, by (*), we obtain $\mu(x) \geq \min\{\mu((x * (y * x)) * z), \mu(z)\}$, which shows that μ_P satisfies (SF_3) . Hence μ_P is a Smarandache fuzzy clean ideal of X . The proof is complete. \square

REFERENCES

[1] Y. B. Jun, *Smarandache BCI-algebras*, Sci. Math. Japo. **62(1)** (2005), 137–142; e2005, 271–276.
 [2] Y. B. Jun, *Smarandache fresh and clean ideals of Smarandache BCI-algebras*, Kyungpook Math. J. **46** (2006), 409–416.
 [3] J. Meng and Y. B. Jun, *BCK-algebras*, Kyung Moon Sa, Seoul, 1994.

Smarandache fuzzy *BCI*-algebras

- [4] R. Padilla, *Smarandache algebraic structures*, Bull. Pure Sci. Delhi, **17E** (1998), no.1, 119–121;<http://www.gallup.unm.edu/smarandache/alg-s-tx.txt>.

On Fibonacci derivative equations

Hee Sik Kim¹, J. Neggers² and Keum Sook So^{3,*}

¹Department of Mathematics, Hanyang University, Seoul, 133-791, Korea

²Department of Mathematics, University of Alabama, Tuscaloosa, AL 35487-0350, U. S. A

^{3,*}Department of Mathematics, Hallym University, Chuncheon 200-702, Korea

Abstract. In this paper, we introduce the notions of Fibonacci (co-)derivative of real-valued functions. We find general solutions of the equations $\Delta(f(x)) = g(x)$ and $(\Delta + I)(f(x)) = g(x)$.

1. Introduction

The theory of Fibonacci-numbers has been studied in many different forms for centuries and the literature on the subject is consequently incredibly vast. The most amazing qualities of these numbers is the variety of mathematical models where they play some sort of role and where their properties are of importance in elucidating the ability of the model under discussion to explain whatever implications are inherent in it. Atanassov et al. [1] and Dunlap [2] provided general and fundamental surveys on the theory of Fibonacci numbers. Hyers-Ulam studied the stability of Fibonacci functional equations [5]. Han et al. [3] discussed Fibonacci sequences in both several groupoids and groups. The present authors [6] introduced the notion of generalized Fibonacci sequences over a groupoid, and investigated these in particular for the case of a groupoid containing idempotents and pre-idempotents.

Han et al. [4] studied Fibonacci functions on the real numbers \mathbf{R} , i.e., functions $f : \mathbf{R} \rightarrow \mathbf{R}$ such that for all $x \in \mathbf{R}$, $f(x + 2) = f(x + 1) + f(x)$, and they developed the notion of Fibonacci functions using the concept of f -even and f -odd functions. The present authors [7] studied Fibonacci functions using the (ultimately) periodicity and also discussed the exponential Fibonacci functions. Especially, given a non-negative real-valued function, the present authors obtained several exponential Fibonacci functions.

In this paper, we introduce the notions of Fibonacci (co-)derivative of real-valued functions. We find general solutions of the equations $\Delta(f(x)) = g(x)$ and $(\Delta + I)(f(x)) = g(x)$.

2. Preliminaries

A function f defined on the real numbers is said to be a *Fibonacci function* ([4]) if it satisfies the formula

$$f(x + 2) = f(x + 1) + f(x)$$

for any $x \in \mathbf{R}$, where \mathbf{R} (as usual) is the set of real numbers.

Example 2.1. ([4]) Let $f(x) := a^x$ be a Fibonacci function on \mathbf{R} where $a > 0$. Then $a^x a^2 = f(x + 2) = f(x + 1) + f(x) = a^x(a + 1)$. Since $a > 0$, we have $a^2 = a + 1$ and $a = \frac{1+\sqrt{5}}{2}$. Hence $f(x) = (\frac{1+\sqrt{5}}{2})^x$ is a Fibonacci function, and the unique Fibonacci function of this type on \mathbf{R} .

If we let $u_0 = 0, u_1 = 1$, then we consider the *full Fibonacci sequence*: $\dots, 5, -3, 2, -1, 1, 0, 1, 1, 2, 3, 5, \dots$, i.e., $u_{-n} = (-1)^n u_n$ for $n > 0$, and $u_n = F_n$, the n th Fibonacci number.

^{0*} Correspondence: Tel.: +82 33 248 2011, Fax: +82 33 256 2011 (K. S. So).

Hee Sik Kim, J. Neggers and Keum Sook So*

Example 2.2. ([4]) Let $\{u_n\}_{n=-\infty}^{\infty}$ and $\{v_n\}_{n=-\infty}^{\infty}$ be full Fibonacci sequences. We define a function $f(x)$ by $f(x) := u_{\lfloor x \rfloor} + v_{\lfloor x \rfloor}t$, where $t = x - \lfloor x \rfloor \in (0, 1)$. Then $f(x + 2) = u_{\lfloor x+2 \rfloor} + v_{\lfloor x+2 \rfloor}t = u_{(\lfloor x \rfloor+2)} + v_{(\lfloor x \rfloor+2)}t = (u_{(\lfloor x \rfloor+1)} + u_{\lfloor x \rfloor}) + (v_{(\lfloor x \rfloor+1)} + v_{\lfloor x \rfloor})t = f(x + 1) + f(x)$ for any $x \in \mathbf{R}$. This proves that f is a Fibonacci function.

Note that if a Fibonacci function is differentiable on \mathbf{R} , then its derivative is also a Fibonacci function.

Proposition 2.3. ([4]) Let f be a Fibonacci function. If we define $g(x) := f(x + t)$ where $t \in \mathbf{R}$ for any $x \in \mathbf{R}$, then g is also a Fibonacci function.

For example, since $f(x) = (\frac{1+\sqrt{5}}{2})^x$ is a Fibonacci function, $g(x) = (\frac{1+\sqrt{5}}{2})^{x+t} = (\frac{1+\sqrt{5}}{2})^t f(x)$ is also a Fibonacci function where $t \in \mathbf{R}$.

3. Fibonacci derivatives

Let $f : \mathbf{R} \rightarrow \mathbf{R}$ be a real-valued function. We shall consider the expression

$$(\Delta f)(x) := f(x + 2) - f(x + 1) - f(x)$$

to be the *Fibonacci derivative* of $f(x)$. For example, if $\Phi := \frac{1+\sqrt{5}}{2}$, then $f(x) = \Phi^x$ yields $(\Delta f)(x) = \Phi^{x+2} - \Phi^{x+1} - \Phi^x = \Phi^x(\Phi^2 - \Phi - 1) = 0$. If f is any Fibonacci function, then $(\Delta f)(x) = 0$ for all $x \in \mathbf{R}$ and conversely. Note that if $\Delta f = \Delta g$, then $f - g$ is a Fibonacci function.

Example 3.1. If $f(x) := ax + b$, then

$$\begin{aligned} \Delta(ax + b) &= [a(x + 2) + b] - [a(x + 1) + b] - [ax + b] \\ &= -ax + (a - b) \end{aligned}$$

and $\Delta(b) = -b$, $\Delta(x) = -x + 1$.

Simultaneously we shall also consider the *Fibonacci co-derivative* of f , denoted $(\Delta + I)(f)$, by the formula

$$(\Delta + I)(f)(x) = \Delta(f) + f(x) = f(x + 2) - f(x + 1)$$

Thus for example, if $f(x) = ax + b$, then $(\Delta + I)(ax + b) = [a(x + 2) + b] - [a(x + 1) + b] - [ax + b] = a$, which coincides with $\frac{d}{dx}(ax + b)$.

We pose a question: what is the “anti-derivative” of a function $f : \mathbf{R} \rightarrow \mathbf{R}$, i.e., given $f : \mathbf{R} \rightarrow \mathbf{R}$, find $g : \mathbf{R} \rightarrow \mathbf{R}$ such that $\Delta g = f$. For example, $\Delta(-x - 1) = [-(x + 2) - 1] - [-(x + 1) - 1] - [-x - 1] = x$. Hence the Fibonacci anti-derivative of x is $-x - 1 + \varphi$ where φ is a Fibonacci function.

Proposition 3.2. *Fibonacci functions are fixed points for Fibonacci co-derivative operator $\Delta + I$.*

Proof. Let $f(x)$ be a Fibonacci function. Then $(\Delta f)(x) = 0$ and hence $(\Delta + I)(f)(x) = (\Delta f)(x) + f(x) = f(x)$. □

Proposition 3.3. *If $(\Delta + I)(f)(x) = 0$, then $(\Delta^2 f)(x) = f(x)$.*

Proof. If $(\Delta + I)(f)(x) = 0$, then $(\Delta f)(x) = -f(x)$. It follows that $(\Delta^2 f)(x) = \Delta(-f(x)) = (-f)(x + 2) - (-f)(x + 1) - (-f)(x) = -(\Delta f)(x) = -(-f(x)) = f(x)$. □

On Fibonacci derivative equations

Example 3.4. Suppose that $\Delta f = x^2$ and $f(x) = ax^2 + bx + c$. Since $\Delta(x^2) = -x^2 + 2x + 3$ and $\Delta(x) = -x + 1$, we obtain

$$\begin{aligned} x^2 &= \Delta(ax^2 + bx + c) \\ &= a \Delta(x^2) + b \Delta(x) + \Delta(c) \\ &= -ax^2 + (2a - b)x + (3a + b - c) \end{aligned}$$

It follows that $a = -1, b = -2$ and $c = -5$, i.e., $\Delta(-x^2 - 2x - 5) = x^2$. Thus, the general Fibonacci anti-derivative of x^2 is $-x^2 - 2x - 5 + \varphi$ where φ is a Fibonacci function.

Example 3.5. Suppose that $\Delta f = x^3$ and $f(x) = ax^3 + bx^2 + cx + d$. Since $\Delta(x^3) = -x^3 + 3x^2 + 9x + 7$, we obtain $\Delta(-x^3 - 3x^2 - 15x - 31) = x^3$ as in Example 3.4.

Theorem 3.6. Let $\Delta f_n = x^n$ and let f_0, f_1, \dots, f_{n-1} be determined to yield particular solutions for $\Delta f_k = x^k$ ($k = 0, 1, \dots, n - 1$). Then

$$f_n = -x^n + \sum_{k=0}^{n-1} \binom{n}{k} [2^{n-k} - 1] f_k + \varphi$$

where φ is a Fibonacci function.

Proof. Let $\Delta f_n = x^n$ and let $f_n = -x^n + Q_n(x)$ where $Q_n(x)$ is a polynomial of x of degree $n - 1$. Then

$$\begin{aligned} x^n &= \Delta(f_n) \\ &= \Delta(-x^n + Q_n(x)) \\ &= -\Delta(x^n) + \Delta(Q_n(x)) \\ &= -[(x + 2)^n - (x + 1)^n - x^n] + \Delta(Q_n(x)) \end{aligned}$$

It follows that $\Delta(Q_n(x)) = (x+2)^n - (x+1)^n - x^n = \sum_{k=0}^{n-1} \binom{n}{k} [2^{n-k} - 1] x^k$. Assume f_0, f_1, \dots, f_{n-1} are determined to have a particular solutions for $\Delta f_k = x^k$ ($k = 0, 1, \dots, n - 1$). Then

$$\begin{aligned} \Delta(Q_n(x)) &= \sum_{k=0}^{n-1} \binom{n}{k} [2^{n-k} - 1] \Delta f_k \\ &= \Delta\left(\sum_{k=0}^{n-1} \binom{n}{k} [2^{n-k} - 1] f_k\right) \end{aligned}$$

It follows that $Q_n(x) = \sum_{k=0}^{n-1} \binom{n}{k} [2^{n-k} - 1] f_k + \varphi$ for some Fibonacci function φ . Hence $f_n = -x^n + Q_n(x) = \sum_{k=0}^{n-1} \binom{n}{k} [2^{n-k} - 1] f_k + \varphi$. □

Hee Sik Kim, J. Neggers and Keum Sook So*

Example 3.7. In the above examples it was known that $f_0 = -1 + \varphi$, $f_1 = -x - 1 + \varphi$, $f_2 = -x^2 - 2x - 5 + \varphi$ and $f_3 = -x^3 - 3x^2 - 15x - 31 + \varphi$ where φ is a Fibonacci function. We compute $Q_4(x)$ as follows:

$$\begin{aligned} Q_4(x) &= \sum_{k=0}^3 \binom{4}{k} [2^{4-k} - 1] f_k + \varphi \\ &= 15f_0 + 28f_1 + 18f_2 + 4f_3 + \varphi \\ &= 15(-1) + 28(-x - 1) + 18(-x^2 - 2x - 5) \\ &\quad + 4(-x^3 - 3x^2 - 15x - 31) \\ &= -4x^3 - 30x^2 - 124x - 257 + \varphi \end{aligned}$$

This shows that $f_4 = -x^4 + Q_4(x) + \varphi = -x^4 - 4x^3 - 30x^2 - 124x - 257 + \varphi$ where φ is a Fibonacci function.

Theorem 3.8. Given a polynomial $g(x) := a_0 + a_1x + \dots + a_nx^n$, we have a particular solution for $\Delta(f(x)) = g(x)$ as $f(x) = a_0f_0 + a_1f_1 + \dots + a_nf_n$, where $\Delta f_k = x^k$ ($k = 0, 1, \dots, n$) and a general solution $f(x) + \varphi(x)$ where $\Delta(\varphi(x)) = 0$.

Proof. It follows immediately from Theorem 3.6. □

4. Fibonacci co-derivatives

Let us consider the problem $(\Delta + I)^k(f(x)) = x^n$. We have $(\Delta + I)(1) = \Delta(1) + I(1) = 0$, $(\Delta + I)(x) = \Delta(x) + I(x)(-x + 1) + x = 1$ and $(\Delta + I)(x^2) = \Delta(x^2) + I(x^2) = 2x + 3$. Using Theorem 3.6, we obtain the following proposition.

Proposition 4.1. The Fibonacci co-derivative of x^n is

$$(\Delta + I)(x^n) = \sum_{k=0}^{n-1} \binom{n}{k} [2^{n-k} - 1] x^k$$

Proof. Using Theorem 3.6, we obtain

$$\begin{aligned} (\Delta + I)(x^n) &= \Delta(x^n) + I(x^n) \\ &= (x + 2)^n - (x + 1)^n \\ &= \Delta(Q_n(x)) \\ &= \sum_{k=0}^{n-1} \binom{n}{k} [2^{n-k} - 1] x^k, \end{aligned}$$

proving the proposition. □

On Fibonacci derivative equations

Example 4.2. If we let $n := 4$ in Proposition 4.1, then

$$\begin{aligned} (\Delta + I)(x^4) &= \sum_{k=0}^3 \binom{4}{k} [2^{4-k} - 1]x^k \\ &= \binom{4}{0} (2^4 - 1)x^0 + \binom{4}{1} (2^3 - 1)x^1 + \binom{4}{2} (2^2 - 1)x^2 \\ &\quad + \binom{4}{3} (2^3 - 1)x^3 \\ &= 4x^3 + 18x^2 + 28x + 15 \end{aligned}$$

It follows that $\Delta(x^4) = -x^4 + 4x^3 + 18x^2 + 28x + 15$.

Consider now $(\Delta + I)^2(1) = (\Delta + I)[(\Delta + I)(1)] = (\Delta + I)(0) = 0$ and $(\Delta + I)^2(x) = (\Delta + I)[(\Delta + I)(x^2)] = (\Delta + I)(2x + 3) = 2$. Similarly we obtain $(\Delta + I)^2(x^3) = (\Delta + I)[3x^2 + 9x + 7] = 6x + 18$.

Using Proposition 4.1, we obtain the following formula.

Proposition 4.3. For any natural number n , we have

$$(\Delta + I)^2(x^n) = \sum_{k=0}^{n-1} \sum_{j=0}^{k-1} \binom{n}{k} \binom{k}{j} (2^{n-k} - 1)(2^{k-j} - 1)x^j$$

Proof. Using Proposition 4.1, we obtain the following.

$$\begin{aligned} (\Delta + I)^2(x^n) &= (\Delta + I)\left[\sum_{k=0}^{n-1} \binom{n}{k} (2^{n-k} - 1)x^k\right] \\ &= \sum_{k=0}^{n-1} \binom{n}{k} (2^{n-k} - 1) \sum_{j=0}^{k-1} \binom{k}{j} (2^{k-j} - 1)x^j \\ &= \sum_{k=0}^{n-1} \sum_{j=0}^{k-1} \binom{n}{k} \binom{k}{j} (2^{n-k} - 1)(2^{k-j} - 1)x^j \end{aligned}$$

□

Example 4.4. We compute $(\Delta + I)^2(x^4)$ as follows.

$$\begin{aligned} (\Delta + I)^2(x^4) &= (\Delta + I)[4x^3 + 18x^2 + 28x + 15] \\ &= 4(\Delta + I)(x^3) + 18(\Delta + I)(x^2) + 28(\Delta + I)(x) \\ &\quad + 15(\Delta + I)(1) \\ &= 12x^2 + 72x + 110 \end{aligned}$$

Hee Sik Kim, J. Neggers and Keum Sook So*

Upon checking Proposition 4.3 when $n = 4$, we find that

$$\begin{aligned}
 (\Delta + I)^2(x^4) &= \sum_{k=0}^3 \sum_{j=0}^{k-1} \binom{4}{k} \binom{k}{j} (2^{4-k} - 1)(2^{k-j} - 1)x^j \\
 &= \sum_{j=0}^0 \binom{4}{1} \binom{1}{j} (2^3 - 1)(2^{1-j} - 1)x^j \\
 &+ \sum_{j=0}^1 \binom{4}{2} \binom{2}{j} (2^2 - 1)(2^{2-j} - 1)x^j \\
 &+ \sum_{j=0}^2 \binom{4}{3} \binom{3}{j} (2^1 - 1)(2^{3-j} - 1)x^j \\
 &= 28x^0 + 54x^0 + 36x + 28x^0 + 36x + 12x^2 \\
 &= 12x^2 + 72x + 110
 \end{aligned}$$

Next, we want to obtain an exact analog of Theorem 3.6 for the Fibonacci co-derivative $\Delta + I$.

Example 4.5. Let $f_1(x) := ax^2 + bx + c$ be a polynomial satisfying $(\Delta + I)(f_1(x)) = x$. Then $x = (\Delta + I)(ax^2 + bx + c) = 2ax + 3a + b$. It follows that $2a = 1, 3a + b = 0$, i.e., $a = \frac{1}{2}, b = -\frac{3}{2}$ and c is arbitrary. Hence $(\Delta + I)(\frac{1}{2}x^2 - \frac{3}{2}x + c) = x$ where c is a constant. Similarly, we may find a polynomial $f_2(x)$ satisfying $(\Delta + I)(f_2(x)) = x^2$, i.e., $(\Delta + I)(\frac{1}{3}x^3 - \frac{3}{2}x^2 + \frac{13}{6}x + d) = x^2$ where d is a constant. In this fashion, we obtain a polynomial $f_n(x) = \frac{1}{n+1}x^{n+1} + q_{n+1}(x)$ which can be determined so that $(\Delta + I)(f_n(x)) = x^n$ where $q_{n+1}(x)$ is a polynomial of degree n .

Theorem 4.6. Given a polynomial $g(x) := a_0 + a_1x + \dots + a_nx^n$, a particular solution for $(\Delta + I)(f(x)) = g(x)$ as $f(x) = a_0f_0 + a_1f_1 + \dots + a_nf_n$ is obtained, where $f_n(x) = \frac{1}{n+1}x^{n+1} + q_{n+1}(x)$ where $q_{n+1}(x)$ is a polynomial of degree n .

Proof. The proof is similar to the proof of Theorem 3.8. □

5. Solving the equation $(\Delta + I)^n(f(x)) = q(x)$

Consider $(\Delta + I)(f(x)) = (\Delta + I)(g(x))$. It means that $(\Delta + I)(f(x) - g(x)) = 0$, i.e., $(f - g)(x + 2) - (f - g)(x + 1) = 0$ for all $x \in \mathbf{R}$. This shows that there exists a map $\psi : \mathbf{R} \rightarrow \mathbf{R}$ with $\psi(x + 2) = \psi(x + 1)$ for all $x \in \mathbf{R}$ such that $f = g + \psi$. If we let $\mathbf{B}_1 := \{\psi | (\Delta + I)(\psi(x)) = 0, \forall x \in \mathbf{R}\}$, then \mathbf{B}_1 consists of all functions $\varphi : \mathbf{R} \rightarrow \mathbf{R}$ such that φ is periodic of period 1. This means that

$$\varphi \in \mathbf{B}_1 \iff \varphi(x + 1) - \varphi(x) = 0, \forall x \in \mathbf{R}$$

Hence general solution of $(\Delta + I)^n(f(x)) = q(x)$ is $\{p(x) + \psi(x) | \Delta(p(x)) = q(x), \psi(x) \in \mathbf{B}_1\} = \{p(x) + \psi(x) | \Delta(p(x)) = q(x), \psi(x + 1) = \psi(x), \forall x \in \mathbf{R}\}$. Consider $(\Delta + I)^2(f(x)) = q(x)$. Let $p(x)$ be a polynomial in $\mathbf{R}[x]$ such that $(\Delta + I)^2(p(x)) = q(x)$. Then $(\Delta + I)^2(p(x)) = (\Delta + I)^2(f(x))$. It follows that $(\Delta + I)^2(f(x) - p(x)) = 0$, i.e.,

On Fibonacci derivative equations

there exists a polynomial $\psi(x) \in \mathbf{R}[x]$ such that $f(x) - p(x) = \psi(x)$ and $(\Delta + I)^2(\psi(x)) = 0$. This means that $(\Delta + I)[(\Delta + I)(\psi(x))] = 0$, i.e., $(\Delta + I)(\psi(x)) \in \mathbf{B}_1$. If we let $\mathbf{B}_2 := \{\varphi(x) \in \mathbf{R}[x] \mid (\Delta + I)(\varphi(x)) \in \mathbf{B}_1\}$, then

$$\begin{aligned} \varphi(x) \in \mathbf{B}_2 &\iff (\Delta + I)(\varphi(x)) \in \mathbf{B}_1 \\ &\iff \exists h(x) \in \mathbf{B}_1 \text{ such that } \varphi(x+2) - \varphi(x+1) = h(x) \\ &\iff h(x+1) - h(x) = 0, \varphi(x+2) - \varphi(x+1) = h(x) \\ &\iff \varphi(x+2) - 2\varphi(x+1) + \varphi(x) = 0 \end{aligned}$$

Hence the set of all general solutions of $(\Delta + I)^2(f(x)) = q(x)$ is $\{p(x) + \psi(x) \mid (\Delta + I)^2(p(x)) = q(x), \psi(x) \in \mathbf{B}_2\} = \{p(x) + \psi(x) \mid (\Delta + I)^2(p(x)) = q(x), \psi(x+2) - 2\psi(x+1) + \psi(x) = 0, \forall x \in \mathbf{R}\}$. Similarly, if we let $\mathbf{B}_3 := \{\varphi(x) \mid (\Delta + I)(\varphi(x)) \in \mathbf{B}_2\}$, then $\mathbf{B}_3 = \{\varphi \in \mathbf{R}[x] \mid \varphi(x+3) - 3\varphi(x+2) + 3\varphi(x+1) - \varphi(x) = 0\}$. We generalize this fact as follows:

Lemma 5.1. *If we let $\mathbf{B}_n := \{\varphi(x) \mid (\Delta + I)(\varphi(x)) \in \mathbf{B}_{n-1}\}$, then $\mathbf{B}_n = \{\varphi \in \mathbf{R}[x] \mid \sum_{r=0}^n \binom{n}{r} \varphi(x+n-r) = 0, \forall x \in \mathbf{R}\}$.*

Theorem 5.2. *Given a polynomial $p(x) \in \mathbf{R}[x]$, there exists a polynomial $q_n(x) \in \mathbf{R}[x]$ such that $(\Delta + I)^n(q_n(x)) = p(x)$, and its general solution $f(x)$ is of the form $q_n(x) + \varphi(x)$ where $\varphi(x) \in \mathbf{B}_n$.*

Proof. It follows from Theorem 4.6 and Lemma 5.1. □

6. Concluding remark

Given Theorem 3.6 and the fact the $\varphi(x) \equiv 0$ is a Fibonacci function, a particular solution to the Fibonacci derivative equation $\Delta f_n = x^n$, is given iteratively by the formula:

$$f_n = -x^n + \sum_{k=0}^{n-1} \binom{n}{k} [2^{n-k} - 1] f_k$$

where if we set $f_0 = 1$, we obtain a sequence of polynomials of degree n for $f_n, n = 0, 1, 2, \dots$. From the structure of the formula we may surmise the existence of many combinatorial properties of the sequence. Also upon rewriting:

$$f_n = \sum_{l=0}^n A_{ln} x^l, \quad A_{nn} = -1,$$

the coefficients A_{ln} , thought of as analogs of binomial numbers, should illustrate a great number of combinatorial relations among themselves as well as with other families, including the binomial numbers (coefficients). Since $(\Delta + I)(x^n) = \sum_{k=0}^{n-1} \binom{n}{k} [2^{n-k} - 1] x^k$ exhibits a “similar” form, we expect there to be confirmation of the claim made above in a multitude of ways, above and beyond what has already been illustrated.

Hee Sik Kim, J. Neggers and Keum Sook So*

7. Future works

Given what has been done in this paper, it is clear that very much remains to be done. Thus, a much more detailed study of functions of the $F(x)$ type, as described above, remains to be done. Furthermore, as pointed out in the concluding remarks, there is much to be done still in completing the combinatorial grammar which is associated with the solution of a particular kind to the equation $\Delta^m f_{mn} = x^n$, of which some cases have been looked at above, but for which very significant gaps still remain to be explored. Also, as usual in this type of research, the law of natural growth of problems prevails, i.e., as one problem is successfully resolved, novel gaps noted present themselves for consideration and no finality is in sight (nor expected) for the area of study touched upon in this case as well, to the benefit of those engaged in furthering knowledge of this (as well as any other) subject.

Acknowledgement

This work was supported by Hallym University Research Fund HRF-201604-004.

REFERENCES

- [1] K. Atanassov et al., *New Visual Perspectives on Fibonacci numbers*, World Scientific, New Jersey, 2002.
- [2] R. A. Dunlap, *The Golden Ratio and Fibonacci Numbers*, World Scientific, New Jersey, 1997.
- [3] J. S. Han, H. S. Kim and J. Neggers, J., *Fibonacci sequences in groupoids*, *Advances in Difference Equations* 2012 **2012**:19 (doi:10.1186/1687-1847-2012-19).
- [4] S. M. Jung, *Hyers-Ulam stability of Fibonacci functional equation*, *Bull. Iranian Math. Soc.* **35** (2009), 217-227.
- [5] H. S. Kim and J. Neggers, *Fibonacci Means and Golden Section Mean*, *Computers and Mathematics with Applications* **56** (2008), 228-232.
- [6] H. S. Kim, J. Neggers and K. S. So, *Generalized Fibonacci sequences in groupoids*, *Advances in Difference Equations* 2013 **2013**:26 (doi:10.1186/1687-1847-2013-26).
- [7] H. S. Kim, J. Neggers and K. S. So, *On Fibonacci functions with periodicity*, *Advances in Difference Equations* 2014 **2014**:293 (doi:10.1186/1687-1847-2014-293).

A NOTE ON SYMMETRIC IDENTITIES FOR TWISTED DAEHEE POLYNOMIALS

JONGKYUM KWON¹ AND JIN-WOO PARK^{2,*}

ABSTRACT. In this paper, we consider the twisted Daehee numbers and polynomials. We investigate some new and explicit symmetric identities for the twisted Daehee polynomials arising from p -adic invariant integral on \mathbb{Z}_p .

1. INTRODUCTION

Throughout this paper, \mathbb{Z}_p , \mathbb{Q}_p , and \mathbb{C}_p will respectively denote the ring of p -adic rational integers, the field of p -adic rational numbers and the completions of algebraic closure of \mathbb{Q}_p . The p -adic norm is defined $|p|_p = \frac{1}{p}$.

Let $f(x)$ be a uniformly differentiable function on \mathbb{Z}_p . Then the p -adic invariant integral on \mathbb{Z}_p is defined by

$$\begin{aligned} \int_{\mathbb{Z}_p} f(x) d\mu_0(x) &= \lim_{N \rightarrow \infty} \sum_{x=0}^{p^N-1} f(x) d\mu_0(x + p^N \mathbb{Z}_p) \\ &= \lim_{N \rightarrow \infty} \frac{1}{p^N} \sum_{n=0}^{p^N-1} f(x). \end{aligned} \tag{1.1}$$

Thus, by (1.1), we get

$$\int_{\mathbb{Z}_p} f_1(x) du_0(x) - \int_{\mathbb{Z}_p} f(x) du_0(x) = f'(0), \tag{1.2}$$

where $f_1(x) = f(x + 1)$ (see [1, 4, 9]).

From (1.2), we can derive

$$\int_{\mathbb{Z}_p} f_n(x) du_0(x) - \int_{\mathbb{Z}_p} f(x) du_0(x) = \sum_{l=0}^{n-1} f'(l), \quad (n \in \mathbb{N}), \tag{1.3}$$

where $f_n(x) = f(x + n)$ (see [1, 5, 6]).

As is well known, the Bernoulli polynomials are defined by the generating function to be

$$\left(\frac{t}{e^t - 1} \right) e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad (\text{see [1, 2, 4]}). \tag{1.4}$$

When $x = 0$, $B_n = B_n(0)$ are called the Bernoulli numbers.

2010 *Mathematics Subject Classification.* 11B68, 11S40, 11S80.

Key words and phrases. Daehee polynomials, twisted Daehee polynomials, twisted λ -Daehee polynomials.

* corresponding author.

For $n \in \mathbb{N}$, let T_p be the p -adic locally constant space defined by

$$T_p = \bigcup_{n \geq 1} C_{p^n} = \lim_{n \rightarrow \infty} C_{p^n},$$

where $C_{p^n} = \{\omega | \omega^{p^n} = 1\}$ is the cyclic group of order p^n .

It is well known that for $\xi \in T_p$, the twisted Bernoulli polynomials are defined as

$$\left(\frac{t}{\xi e^t - 1}\right) e^{xt} = \sum_{n=0}^{\infty} B_{n,\xi}(x) \frac{t^n}{n!}, \quad (\text{see [1, 2]}). \tag{1.5}$$

When $x = 0$, $B_{n,\xi} = B_{n,\xi}(0)$ are called the twisted Bernoulli numbers.

For $t \in \mathbb{C}_p$ with $|t|_p < p^{-\frac{1}{p-1}}$, the Daehee polynomials are defined by the generating function to be

$$\frac{\log(1+t)}{t} (1+t)^x = \sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!}, \quad (\text{see [1, 2, 4 - 14, 16,17]}). \tag{1.6}$$

When $x = 0$, $D_n = D_n(0)$ are called the Daehee numbers.

From (1.4) and (1.6), we can derive the following equation:

$$\begin{aligned} \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} &= \sum_{m=0}^{\infty} D_m(x) \frac{1}{m!} (e^t - 1)^m \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n D_m(x) S_2(n, m) \right) \frac{t^n}{n!}, \end{aligned} \tag{1.7}$$

where $S_2(n, m)$ is the Stirling number of the second kind which is given by the generating function to be

$$\frac{1}{m!} (e^t - 1)^m = \sum_{n=m}^{\infty} S_2(n, m) \frac{t^n}{n!}, \quad (\text{see [3, 15]}).$$

By (1.7), we get

$$B_n(x) = \sum_{m=0}^n D_m(x) S_2(n, m), \quad (n \geq 0). \tag{1.8}$$

From (1.4), we have

$$\begin{aligned} \sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!} &= \sum_{m=0}^{\infty} B_m(x) \frac{1}{m!} (\log(1+t))^m \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n B_m(x) S_1(n, m) \right) \frac{t^n}{n!}, \end{aligned} \tag{1.9}$$

where $S_1(n, m)$ is the Stirling number of the first kind which is defined by falling factorials as follows:

$$(x)_0 = 1, \quad (x)_n = x(x-1) \cdots (x-n+1) = \sum_{l=0}^n S_1(n, l) x^l, \quad (n \in \mathbb{N}),$$

Thus, by (1.9), we get

$$D_n(x) = \sum_{m=0}^n B_m(x) S_1(n, m), \quad (n \geq 0). \tag{1.10}$$

From (1.2), we derive Witt's formula for Daehee polynomials as follows:

$$\begin{aligned} \int_{\mathbb{Z}_p} (1+t)^{x+y} du_0(y) &= \frac{\log(1+t)}{t} (1+t)^x \\ &= \sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!}. \end{aligned} \tag{1.11}$$

Thus, by (1.11), we get

$$\int_{\mathbb{Z}_p} (x+y)_n du_0(y) = D_n(x), \quad (n \geq 0), \quad (\text{see [8]}). \tag{1.12}$$

Now, we consider the twisted Daehee polynomials defined by the generating function to be

$$\frac{\log(1+\xi t)}{\xi t} (1+\xi t)^x = \sum_{n=0}^{\infty} D_{n,\xi}(x) \frac{t^n}{n!}, \quad (\text{see [5, 13, 14]}). \tag{1.13}$$

When $x = 0$, $D_{n,\xi} = D_{n,\xi}(0)$ are called the twisted Daehee numbers.

In [5], authors define twisted λ -Daehee polynomials which are given by the p -adic invariant integral on \mathbb{Z}_p to be

$$\begin{aligned} \int_{\mathbb{Z}_p} (1+\xi t)^{\lambda(x+y)} d\mu_0(y) &= \frac{\lambda \log(1+\xi t)}{(1+\xi t)^\lambda - 1} (1+\xi t)^{\lambda x} \\ &= \sum_{n=0}^{\infty} D_{n,\lambda,\xi}(x) \frac{t^n}{n!}. \end{aligned} \tag{1.14}$$

In the special case, $\lambda = 1, \xi = 1$, we note that $D_{n,1,1}(x) = D_n(x)$. When $x = 0$, then $D_{n,\lambda,\xi} = D_{n,\lambda,\xi}(0)$ are called *twisted λ - Daehee numbers*.

Recently, several authors have researched twisted Daehee polynomials in the several areas (see [5, 13, 14]). In this paper, we investigate some explicit and new symmetric identities for the twisted Daehee polynomials which are derived from the p -adic invariant integral on \mathbb{Z}_p .

2. Symmetric identities for the twisted Daehee polynomials

Let $t \in \mathbb{C}_p$ with $|t|_p < p^{-\frac{1}{p-1}}$. Now, we consider the following p -adic integral on \mathbb{Z}_p . From (1.3), we easily get

$$\begin{aligned} & \frac{1}{\log(1 + \xi t)} \left(\int_{\mathbb{Z}_p} (1 + \xi t)^{n+x} du_0(x) - \int_{\mathbb{Z}_p} (1 + \xi t)^x du_0(x) \right) \\ &= \frac{1}{\log(1 + \xi t)} \sum_{l=0}^{n-1} (1 + \xi t)^l \log(1 + \xi t) \\ &= \sum_{l=0}^{n-1} (1 + \xi t)^l \tag{2.1} \\ &= \sum_{i=0}^{n-1} \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n-1} i^m S_1(n, m) \right) \frac{\xi^n t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n S_1(n, m) \sum_{i=0}^{n-1} i^m \right) \frac{\xi^n t^n}{n!}. \end{aligned}$$

Then, by (2.1), we get

$$\begin{aligned} & \frac{1}{\log(1 + \xi t)} \left(\int_{\mathbb{Z}_p} (1 + \xi t)^{n+x} du_0(x) - \int_{\mathbb{Z}_p} (1 + \xi t)^x du_0(x) \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n S_1(n, m) S_m(n-1) \right) \frac{\xi^n t^n}{n!}, \tag{2.2} \end{aligned}$$

where for given positive integer k , $S_k(n) = 0^k + 1^k + 2^k + \dots + n^k$.

From (1.2) and (1.3), we have

$$\begin{aligned} & \frac{1}{\log(1 + \xi t)} \left(\int_{\mathbb{Z}_p} (1 + \xi t)^{n+x} du_0(x) - \int_{\mathbb{Z}_p} (1 + \xi t)^x du_0(x) \right) \\ &= \frac{n \int_{\mathbb{Z}_p} (1 + \xi t)^x du_0(x)}{\int_{\mathbb{Z}_p} (1 + \xi t)^{n+x} du_0(x)} \\ &= \sum_{k=0}^{\infty} \left(\sum_{m=0}^k S_1(k, m) S_k(n-1) \right) \frac{\xi^n t^k}{k!}. \tag{2.3} \end{aligned}$$

We recall that Cauchy numbers are defined by the generating function to be

$$\frac{t}{\log(1 + t)} = \sum_{n=0}^{\infty} C_n \frac{t^n}{n!}. \tag{2.4}$$

By (2.3) and (2.4), we get

$$\begin{aligned} & \sum_{k=0}^{\infty} \left(\sum_{m=0}^k (D_{m,\xi}(k) - D_{m,\xi}) C_{k-m} \binom{k}{m} \right) \frac{t^k}{k!} \\ &= \sum_{k=0}^{\infty} \left(\sum_{m=0}^{k-1} S_1(k-1, m) S_{k-1}(n-1) \right) \frac{t^k}{k!}. \tag{2.5} \end{aligned}$$

From (2.5), we have

$$\begin{aligned} & \frac{1}{k} \left(\sum_{m=0}^k (D_{m,\xi}(k) - D_{m,\xi}) C_{k-m} \binom{k}{m} \right) \\ &= \sum_{m=0}^{k-1} S_1(k-1, m) S_{k-1}(n-1), \quad (k \in \mathbb{N}, n \in \mathbb{N}). \end{aligned} \tag{2.6}$$

Therefore, by (2.6), we obtain the following theorem.

Theorem 2.1. For $k, n \in \mathbb{N}$, we have

$$\frac{1}{k} \left(\sum_{m=0}^k (D_{m,\xi}(k) - D_{m,\xi}) C_{k-m} \binom{k}{m} \right) = \sum_{m=0}^{k-1} S_1(k-1, m) S_{k-1}(n-1). \tag{2.7}$$

Now, we consider symmetric identities for the twisted Daehee polynomials. Let $w_1, w_2 \in \mathbb{N}$. Then, we easily see that

$$\begin{aligned} \frac{\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} (1 + \xi t)^{w_1 x_1 + w_2 x_2} du_0(x_1) du_0(x_2)}{\int_{\mathbb{Z}_p} (1 + \xi t)^{w_1 w_2 x} du_0(x)} &= \frac{\frac{w_1 \log(1 + \xi t)}{((1 + \xi t)^{w_1} - 1)} \frac{w_2 \log(1 + \xi t)}{((1 + \xi t)^{w_2} - 1)}}{\frac{w_1 w_2 \log(1 + \xi t)}{((1 + \xi t)^{w_1 w_2} - 1)}} \\ &= \frac{\log(1 + \xi t) ((1 + \xi t)^{w_1 w_2} - 1)}{((1 + \xi t)^{w_1} - 1) ((1 + \xi t)^{w_2} - 1)}. \end{aligned} \tag{2.8}$$

We consider the following double p -adic invariant integral on \mathbb{Z}_p as follows:

$$\begin{aligned} I &= \frac{\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} (1 + \xi t)^{w_1 x_1 + w_2 x_2 + w_1 w_2 x} d\mu_0(x_1) d\mu_0(x_2)}{\int_{\mathbb{Z}_p} (1 + \xi t)^{w_1 w_2 x} d\mu_0(x)} \\ &= \frac{\log(1 + \xi t) (1 + \xi t)^{w_1 w_2 x} ((1 + \xi t)^{w_1 w_2} - 1)}{((1 + \xi t)^{w_1} - 1) ((1 + \xi t)^{w_2} - 1)}. \end{aligned} \tag{2.9}$$

From (1.2) and (1.3), we have

$$\begin{aligned} \frac{w_1 \int_{\mathbb{Z}_p} (1 + \xi t)^x du_0(x)}{\int_{\mathbb{Z}_p} (1 + \xi t)^{w_1 x} du_0(x)} &= \sum_{k=0}^{w_1-1} (1 + \xi t)^k \\ &= \sum_{l=0}^{\infty} \left(\sum_{m=0}^l S_1(l, m) S_m(w_1 - 1) \right) \frac{\xi^l t^l}{l!} \end{aligned} \tag{2.10}$$

From (2.10), we get

$$\begin{aligned} I &= \left(\frac{1}{w_1} \int_{\mathbb{Z}_p} (1 + \xi t)^{w_1(x_1 + w_2 x)} d\mu_0(x_1) \right) \left(\frac{w_1 \int_{\mathbb{Z}_p} (1 + \xi t)^{w_2 x_2} d\mu_0(x_2)}{\int_{\mathbb{Z}_p} (1 + \xi t)^{w_1 w_2 x} d\mu_0(x)} \right) \\ &= \frac{1}{w_1} \left(\sum_{i=0}^{\infty} D_{i, w_1, \xi}(w_2 x) \frac{t^i}{i!} \right) \left(\sum_{k=0}^{\infty} \left(\sum_{m=0}^k w_2^m S_1(k, m) S_m(w_1 - 1) \right) \frac{\xi^k t^k}{k!} \right) \\ &= \frac{1}{w_1} \sum_{n=0}^{\infty} \left(\sum_{i=0}^n \binom{n}{i} D_{i, w_1, \xi}(w_2 x) \sum_{m=0}^{n-i} w_2^m S_1(n-i, m) S_m(w_1 - 1) \right) \frac{\xi^{n-i} t^n}{n!}. \end{aligned} \tag{2.11}$$

On the other hand, by (2.10), we get

$$\begin{aligned}
 I &= \left(\frac{1}{w_2} \int_{\mathbb{Z}_p} (1 + \xi t)^{w_2(x_2+w_1x)} d\mu_0(x_2) \right) \left(\frac{w_2 \int_{\mathbb{Z}_p} (1 + \xi t)^{w_1x_1} d\mu_0(x_1)}{\int_{\mathbb{Z}_p} (1 + \xi t)^{w_1w_2x} d\mu_0(x)} \right) \\
 &= \frac{1}{w_2} \left(\sum_{i=0}^{\infty} D_{i,w_2,\xi}(w_1x) \frac{t^i}{i!} \right) \left(\sum_{k=0}^{\infty} \left(\sum_{m=0}^k w_1^m S_1(k, m) S_m(w_2 - 1) \right) \frac{\xi^k t^k}{k!} \right) \\
 &= \frac{1}{w_2} \sum_{n=0}^{\infty} \left(\sum_{i=0}^n \binom{n}{i} D_{i,w_2,\xi}(w_1x) \sum_{m=0}^{n-i} w_1^m S_1(n - i, m) S_m(w_2 - 1) \right) \frac{\xi^{n-i} t^n}{n!}.
 \end{aligned} \tag{2.12}$$

Therefore, by (2.9), (2.11) and (2.12), we obtain the following theorem.

Theorem 2.2. For $w_1, w_2 \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned}
 &\frac{1}{w_1} \sum_{i=0}^n \binom{n}{i} D_{i,w_1,\xi}(w_2x) \xi^{n-i} \sum_{m=0}^{n-i} w_2^m S_1(n - i, m) S_m(w_1 - 1) \\
 &= \frac{1}{w_2} \sum_{i=0}^n \binom{n}{i} D_{i,w_2,\xi}(w_1x) \xi^{n-i} \sum_{m=0}^{n-i} w_1^m S_1(n - i, m) S_m(w_2 - 1).
 \end{aligned}$$

Remark. By replacing t by $\frac{1}{\xi}(e^t - 1)$ in (1.14), we get

$$\begin{aligned}
 \sum_{n=0}^{\infty} B_n(x) \lambda^n \frac{t^n}{n!} &= \sum_{n=0}^{\infty} D_{n,\lambda,\xi}(x) \frac{1}{n!} \left(\frac{1}{\xi} (e^t - 1) \right)^n \\
 &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n \xi^{-n} S_2(n, m) D_{m,\lambda,\xi}(x) \right) \frac{t^n}{n!}.
 \end{aligned} \tag{2.13}$$

Thus, by (2.13), we have

$$\lambda^n B_n(x) = \sum_{m=0}^n \xi^{-n} S_2(n, m) D_{m,\lambda,\xi}(x), \quad (n \geq 0). \tag{2.14}$$

By replacing t by $\log\left(t + \frac{1}{\xi}\right)$ in (1.5), we have

$$\begin{aligned}
 \sum_{n=0}^{\infty} D_{n,\xi}(x) \frac{t^n}{n!} &= \sum_{n=0}^{\infty} B_n(x) \frac{1}{n!} \left(\log\left(t + \frac{1}{\xi}\right) \right)^n \\
 &= \sum_{n=0}^{\infty} B_n(x) \sum_{l=n}^{\infty} S_1(l, n) \frac{(\xi t)^n}{n!} \\
 &= \sum_{n=0}^{\infty} \left(\xi^n \sum_{m=0}^n B_m(x) S_1(n, m) \right) \frac{t^n}{n!}.
 \end{aligned} \tag{2.15}$$

Thus, by (2.15), we get

$$\xi^{-n} D_{n,\xi}(x) = \sum_{m=0}^n B_m(x) S_1(n, m), \quad (n \geq 0). \tag{2.16}$$

From (2.9), we note that

$$\begin{aligned}
 I &= \left(\frac{(1 + \xi t)^{w_1 w_2 x}}{w_1} \int_{\mathbb{Z}_p} (1 + \xi t)^{w_1 x_1} d\mu_0(x_1) \right) \left(\frac{w_1 \int_{\mathbb{Z}_p} (1 + \xi t)^{w_2 x_2} d\mu_0(x_2)}{\int_{\mathbb{Z}_p} (1 + \xi t)^{w_1 w_2 x} d\mu_0(x)} \right) \\
 &= \left(\frac{(1 + \xi t)^{w_1 w_2 x}}{w_1} \int_{\mathbb{Z}_p} (1 + \xi t)^{w_1 x} d\mu_0(x_1) \right) \left(\sum_{l=0}^{w_1-1} (1 + \xi t)^{w_2 l} \right) \\
 &= \left(\frac{1}{w_1} \int_{\mathbb{Z}_p} (1 + \xi t)^{w_1 x} d\mu_0(x_1) \right) \left(\sum_{l=0}^{w_1-1} (1 + \xi t)^{w_1(w_2 x + \frac{w_2}{w_1} l)} \right) \\
 &= \frac{1}{w_1} \sum_{l=0}^{w_1-1} \int_{\mathbb{Z}_p} (1 + \xi t)^{w_1(x_1 + w_2 x + \frac{w_2}{w_1} l)} d\mu_0(x_1) \\
 &= \sum_{n=0}^{\infty} \left(\frac{1}{w_1} \sum_{l=0}^{w_1-1} D_{n, w_1, \xi} \left(w_2 x + \frac{w_2}{w_1} l \right) \right) \frac{\xi^n t^n}{n!}.
 \end{aligned} \tag{2.17}$$

On the other hand, we obtain the following equation by the symmetric property of p -adic invariant integral on \mathbb{Z}_p as follows:

$$\begin{aligned}
 I &= \left(\frac{(1 + \xi t)^{w_1 w_2 x}}{w_2} \int_{\mathbb{Z}_p} (1 + \xi t)^{w_2 x_2} d\mu_0(x_2) \right) \left(\frac{w_2 \int_{\mathbb{Z}_p} (1 + \xi t)^{w_1 x_1} d\mu_0(x_1)}{\int_{\mathbb{Z}_p} (1 + \xi t)^{w_1 w_2 x} d\mu_0(x)} \right) \\
 &= \left(\frac{1}{w_2} \int_{\mathbb{Z}_p} (1 + \xi t)^{w_2 x_2} d\mu_0(x_2) \right) \left(\sum_{l=0}^{w_2-1} (1 + \xi t)^{w_1 l} \right) (1 + \xi t)^{w_1 w_2 x} \\
 &= \left(\frac{1}{w_2} \int_{\mathbb{Z}_p} (1 + \xi t)^{w_2 x_2} d\mu_0(x_2) \right) \left(\sum_{l=0}^{w_2-1} (1 + \xi t)^{w_2(w_1 x + \frac{w_1}{w_2} l)} \right) \\
 &= \frac{1}{w_2} \sum_{l=0}^{w_2-1} \int_{\mathbb{Z}_p} (1 + \xi t)^{w_2(x_2 + w_1 x + \frac{w_1}{w_2} l)} d\mu_0(x_2) \\
 &= \sum_{n=0}^{\infty} \left(\frac{1}{w_2} \sum_{l=0}^{w_2-1} D_{n, w_2, \xi} \left(w_1 x + \frac{w_1}{w_2} l \right) \right) \frac{\xi^n t^n}{n!}.
 \end{aligned} \tag{2.18}$$

Therefore, by comparing the coefficients on the both sides of (2.17) and (2.18), we obtain the following theorem.

Theorem 2.3. For $w_1, w_2 \in \mathbb{N}$ and $n \geq 0$, we have

$$\frac{1}{w_1} \sum_{l=0}^{w_1-1} D_{n, w_1, \xi} \left(w_2 x + \frac{w_2}{w_1} l \right) = \frac{1}{w_2} \sum_{l=0}^{w_2-1} D_{n, w_2, \xi} \left(w_1 x + \frac{w_1}{w_2} l \right).$$

Corollary 2.4. For $w_1, w_2 \in \mathbb{N}$ and $n \geq 0$, we have

$$\begin{aligned}
 &\sum_{l=0}^{w_1-1} \sum_{m=0}^n w_1^{m-1} B_m \left(w_2 x + \frac{w_2}{w_1} l \right) S_1(n, m) \\
 &= \sum_{l=0}^{w_2-1} \sum_{m=0}^n w_2^{m-1} B_m \left(w_1 x + \frac{w_1}{w_2} l \right) S_1(n, m).
 \end{aligned}$$

REFERENCES

- [1] Y.-H. Cho, T. Kim, T. Mansour, S.-H. Rim, *Higher-order q -Daehee polynomials*, J. Comput. Anal. Appl. **19** (2015), no. 1, 167-173.
- [2] Y.-H. Cho, T. Kim, T. Mansour, S.-H. Rim, *On a (r,s) -analogue of Changhee and Daehee numbers and polynomials*, Kyungpook Math. J. **55** (2015), 225-232.
- [3] L. Comtet, *Advanced Combinatorics*, Reidel, Dordrecht, 1974.
- [4] D. S. Kim, T. Kim, *Daehee numbers and polynomials*, Appl. Math. Sci. **7(120)**, 5969 - 5976(2013).
- [5] D. S. Kim, T. Kim, S.-H. Lee, J.-J. Seo, *A note on the twisted λ -Daehee polynomials*, Appl. Math. Sci. **7(2013)**.
- [6] D. S. Kim, T. Kim, *Identities arising from higher-order Daehee polynomial bases*, Open Math.**13** 2015, 196-208.
- [7] D. S. Kim, T. Kim, S.-H. Lee, J.-J. Seo, *A note on the λ -Daehee polynomials*, Int. J. Math. Anal.(Ruse) **7** (2013), no. 61-64, 3069-3080.
- [8] D. S. Kim, T. Kim, J.-J. Seo, *Higher-order Daehee polynomials of the first kind with umbral calculus*, Adv. Stud. Contemp. Math.(Kyungshang) **24** (2014), no. 1, 5-18.
- [9] T. Kim, *An invariant p -adic integral associated with Daehee numbers*, Integral Transforms Spec. Funct. **13(1)**, 65-69(2002).
- [10] E.-J. Moon, J.-W. Park, S.-H. Rim, *A note on the generalized q -Daehee numbers of higher order*, Proc. Jangjeon Math. Soc. **17** (2014), no. 4, 557-565.
- [11] H. Ozden, I.N. Cangul, Y. Simsek, *Remarks on the q -Bernoulli numbers associated with Daehee numbers*, Adv. Stud. Contemp. Math.(Kyungshang) **18** (2009), no. 1, 41-48.
- [12] J.-W. Park, *On the q -analogue of λ -Daehee polynomials*, J. Comput. Anal. Appl. **19** (2015), no. 6, 966-974.
- [13] J.-W. Park, S.-H. Rim, J. Kwon, *The twisted Daehee numbers and polynomials*, Adv. Difference Equ. **2014** (2014:1), 9pp.
- [14] J.-W. Park, *On the twisted Daehee polynomials with q -parameter*, Adv. Difference Equ. **2014** (2014:304), 10pp.
- [15] S. Roman, *The umbral calculus*, Dover Publ. Inc. New York, 2005.
- [16] J.-J.Seo, T. Kim, *Some identities of symmetry for Daehee polynomials arising from p -adic invariant integral on \mathbb{Z}_p* , Proc. Jangjeon Math. Soc. **19** (2016), no. 2, 285 - 292.
- [17] J.-J.Seo, S.-H. Rim, T. Kim, S.H. Lee, *Sums products of generalized Daehee numbers*, Proc. Jangjeon Math. Soc. **17** (2014), no. 1, 1-9.

¹ DEPARTMENT OF MATHEMATICS EDUCATION AND RINS, GYEONGSANG NATIONAL UNIVERSITY, JINJU, GYEONGSANGNAMDO, 52828, REPUBLIC OF KOREA
E-mail address: mathkjk26@gnu.ac.kr

^{2,*} DEPARTMENT OF MATHEMATICS EDUCATION, DAEGU UNIVERSITY, GYEONGSAN-SI, KYUNGSANG-
 BUKDO, 38453, REPUBLIC OF KOREA
E-mail address: a0417001@knu.ac.kr

Dynamics and Behavior of $x_{n+1} = ax_n + bx_{n-1} + \frac{\alpha + cx_{n-2}}{\beta + dx_{n-2}}$

M. M. El-Dessoky^{1,2} and Aatef Hobiny^{1,3}

¹King Abdulaziz University, Faculty of Science, Mathematics Department,
P. O. Box 80203, Jeddah 21589, Saudi Arabia.

²Mansoura University, Faculty of Science, Department of Mathematics,
Mansoura 35516, Egypt.

³Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group,
Department of Mathematics, Faculty of Science, King Abdulaziz University,
P.O. Box 80203, Jeddah 21589, Saudi Arabia.

E-mail: dessokym@mans.edu.eg; ahobany@kau.edu.sa

ABSTRACT

The main objective of this paper is to study the local and the global stability of the solutions, the periodic character and the boundedness of the difference equation

$$x_{n+1} = ax_n + bx_{n-1} + \frac{\alpha + cx_{n-2}}{\beta + dx_{n-2}},$$

where the parameters β , α , a , b , c and d are positive real numbers and the initial conditions x_{-2} , x_{-1} and x_0 are positive real numbers. Some numerical examples will be given to illustrate our results.

Keywords: Difference equations, Stability, Boundedness, Periodic solutions.

Mathematics Subject Classification: 39A10

1. INTRODUCTION

Difference equations or discrete dynamical systems are diverse field which impact almost every branch of pure and applied mathematics. Every dynamical system $x_{n+1} = f(x_n)$ determines a difference equation and vice versa. Recently, there has been great interest in studying difference equations systems. One of the reasons for this is a necessity for some techniques which can be used in investigating equations arising in mathematical models describing real life situations in many applied sciences. The theory of discrete dynamical systems and difference equations developed greatly during the last twenty-five years of the twentieth century. Applications of discrete dynamical systems and difference equations have appeared recently in many areas. The theory of difference equations occupies a central position in applicable analysis. There is no doubt that the theory of difference equations will continue to play an important role in mathematics as a whole. Nonlinear difference equations of order greater than one are of paramount importance in applications. Such equations also appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations which model various diverse phenomena in biology, physiology, ecology, engineering, physics, economics, genetics, probability theory, psychology and resource management. It is very interesting to investigate the behavior of solutions of a system of higher-order rational difference equations and to discuss the local asymptotic stability of their equilibrium points. Systems of rational difference equations have been studied by several authors. Especially there has been a great interest in the study of the attractivity of the solutions of such systems [1-33].

Many research have been done to study the global attractivity, boundedness character, periodicity and the solution form of nonlinear difference equations. For example, Agarwal et al. [2] looked at the global stability, periodicity character and found the solution form of some special cases of the difference equation

$$x_{n+1} = a + \frac{dx_{n-l}x_{n-k}}{b-cx_{n-s}}, \quad n = 0, 1, \dots,$$

where a, b, c, d and the initial conditions $x_{-r}, x_{-r+1}, \dots, x_{-1}, x_0$ are positive real numbers, while $x_i \neq b/c$ for $i = -r, -r + 1, \dots, 0$ where $r = \max\{l, k, s\}$.

Hamza and Morsy in [3] investigated the global behavior of the difference equation

$$x_{n+1} = \alpha + \frac{x_{n-1}}{x_n^k}, \quad n = 0, 1, \dots,$$

where the parameters $\alpha, k \in (0, \infty)$ and the initial values x_{-1} and x_0 are arbitrary positive real numbers.

Elsayed et al. [4] studied the global stability character and the periodicity of solutions of the difference equation

$$x_{n+1} = ax_n + \frac{b+cx_{n-1}}{d+ex_{n-1}}, \quad n = 0, 1, \dots,$$

where the parameters a, b, c, d and e are positive real numbers and the initial conditions x_{-1} and x_0 are positive real numbers.

Zayed et al. [5] studied the behavior of the following rational recursive sequence

$$x_{n+1} = \frac{\left(A + \sum_{i=0}^k \alpha_i x_{n-i} \right)}{\sum_{i=0}^k \beta_i x_{n-i}}, \quad n = 0, 1, 2, \dots,$$

where the coefficients A, α_i, β_i and the initial conditions $x_{-k}, x_{-k+1}, \dots, x_{-1}, x_0$ are positive real numbers, while k is a positive integer number.

Also, in [6] Zayed et al. obtained the global behavior of the difference equation

$$x_{n+1} = \frac{\sum_{i=0}^k \alpha_i x_{n-i}}{B + \sum_{i=0}^k \beta_i x_{n-i}}, \quad n = 0, 1, 2, \dots,$$

where the coefficients B, α_i, β_i and the initial conditions $x_{-k}, x_{-k+1}, \dots, x_{-1}, x_0$ are arbitrary positive real numbers, while k is a positive integer number.

In [7] El-Moneam investigated the periodicity, the boundedness and the global stability of the positive solutions of nonlinear difference equation

$$x_{n+1} = Ax_n + Bx_{n-k} + Cx_{n-l} + Dx_{n-\sigma} + \frac{bx_{n-k}}{dx_{n-k} - ex_{n-l}}, \quad n = 0, 1, 2, \dots,$$

where the coefficients $A, B, C, D, b, d, e \in (0, \infty)$, while k, l and s are positive integers and the initial conditions $x_{-s}, \dots, x_{-l}, \dots, x_{-k}, \dots, x_{-1}, x_0$ are arbitrary positive real numbers such that $k < l < s$.

Yalçınkaya [8] investigated the global behaviour of the difference equation

$$x_{n+1} = \alpha + \frac{x_{n-m}}{x_n^k}, \quad n = 0, 1, \dots,$$

where the parameter $\alpha, k \in (0, \infty)$ and the initial values are arbitrary positive real numbers.

Elabbasy et al. [9] studied the dynamics, the global stability, periodicity character and the solution of special case of the recursive sequence

$$x_{n+1} = ax_n - \frac{bx_n}{cx_n - dx_{n-1}}, \quad n = 0, 1, \dots,$$

where the initial conditions x_{-1}, x_0 are arbitrary real numbers and a, b, c, d are positive constants.

El-Owaidy et al. [10] investigated local stability, oscillation and boundedness character of the difference equation

$$x_{n+1} = \alpha + \frac{x_{n-1}^p}{x_n^p}, \quad n = 0, 1, \dots,$$

under specified conditions.

Elsayed [11] studied some qualitative behavior of the solutions of the difference equation

$$x_{n+1} = ax_n + \frac{bx_n}{cx_n - dx_{n-1}}, \quad n = 0, 1, \dots,$$

where the initial conditions x_{-1}, x_0 are arbitrary real numbers and a, b, c, d are positive constants with $cx_0 - dx_{-1} \neq 0$.

Elsayed and El-Dessoky [12] investigated the global convergence, boundedness, and periodicity of solutions of the difference equation

$$x_{n+1} = ax_{n-s} + \frac{bx_{n-l} + cx_{n-k}}{dx_{n-l} + ex_{n-k}}, \quad n = 0, 1, \dots,$$

where the parameters a, b, c, d and e are positive real numbers and the initial conditions $x_{-t}, x_{-t+1}, \dots, x_{-1}, x_0$ are positive real numbers where $t = \max\{s, l, k\}$.

This paper aims to study the global stability character and the periodicity of solutions of the difference equation

$$x_{n+1} = ax_n + bx_{n-1} + \frac{\alpha + cx_{n-2}}{\beta + dx_{n-2}}, \quad n = 0, 1, \dots, \tag{1}$$

where the parameters a, b, c, d, α and β are positive real numbers and the initial conditions x_{-2}, x_{-1} and x_0 are positive real numbers.

2. SOME BASIC PROPERTIES AND DEFINITIONS

In this section, we state some basic definitions and theorems that we need in this paper.

Let I be some interval of real numbers and let

$$F : I^3 \rightarrow I,$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-2}, x_{-1}, x_0 \in I$, the difference equation

$$x_{n+1} = F(x_n, x_{n-1}, x_{n-2}), \tag{2}$$

has a unique solution $\{x_n\}_{n=-2}^\infty$.

Definition 1. (Equilibrium Point)

A point $\bar{x} \in I$ is called an equilibrium point of Eq.(2) if

$$\bar{x} = F(\bar{x}, \bar{x}, \bar{x}).$$

That is, $x_n = \bar{x}$ for $n \geq 0$, is a solution of Eq.(2), or equivalently, \bar{x} is a fixed point of F .

DEFINITION 2.1. (Stability)

(i) The equilibrium point \bar{x} of Eq.(2) is locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-2}, x_{-1}, x_0 \in I$ with

$$|x_{-2} - \bar{x}| + |x_{-1} - \bar{x}| + |x_0 - \bar{x}| < \delta,$$

we have

$$|x_n - \bar{x}| < \epsilon \quad \text{for all } n \geq -k.$$

(ii) The equilibrium point \bar{x} of Eq.(2) is locally asymptotically stable if \bar{x} is locally stable solution of Eq.(2) and there exists $\gamma > 0$, such that for all $x_{-2}, x_{-1}, x_0 \in I$ with

$$|x_{-2} - \bar{x}| + |x_{-1} - \bar{x}| + |x_0 - \bar{x}| < \gamma,$$

we have

$$\lim_{n \rightarrow \infty} x_n = \bar{x}.$$

(iii) The equilibrium point \bar{x} of Eq.(2) is global attractor if for all $x_{-2}, x_{-1}, x_0 \in I$, we have

$$\lim_{n \rightarrow \infty} x_n = \bar{x}.$$

(iv) The equilibrium point \bar{x} of Eq.(2) is globally asymptotically stable if \bar{x} is locally stable, and \bar{x} is also a global attractor of Eq.(2).

(v) The equilibrium point \bar{x} of Eq.(2) is unstable if \bar{x} is not locally stable.

DEFINITION 2.2. (Boundedness)

A sequence $\{x_n\}_{n=-2}^\infty$ is said to be bounded and persists if there exist positive constants m and M such that

$$m \leq x_n \leq M \quad \text{for all } n \geq -2.$$

DEFINITION 2.3. (Periodicity)

A sequence $\{x_n\}_{n=-2}^\infty$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -1$. A sequence $\{x_n\}_{n=-2}^\infty$ is said to be periodic with prime period p if p is the smallest positive integer having this property.

The linearized equation of Eq.(2) about the equilibrium \bar{x} is the linear difference equation

$$y_{n+1} = \frac{\partial F(\bar{x}, \bar{x}, \bar{x})}{\partial x_n} y_n + \frac{\partial F(\bar{x}, \bar{x}, \bar{x})}{\partial x_{n-1}} y_{n-1} + \frac{\partial F(\bar{x}, \bar{x}, \bar{x})}{\partial x_{n-2}} y_{n-2}. \tag{3}$$

Now, assume that the characteristic equation associated with (3) is

$$p(\lambda) = p_0 \lambda^2 + p_1 \lambda + p_2 = 0, \tag{4}$$

where

$$p_0 = \frac{\partial F(\bar{x}, \bar{x}, \bar{x})}{\partial x_n}, \quad p_1 = \frac{\partial F(\bar{x}, \bar{x}, \bar{x})}{\partial x_{n-1}} \quad \text{and} \quad p_2 = \frac{\partial F(\bar{x}, \bar{x}, \bar{x})}{\partial x_{n-2}}.$$

Theorem A [18]: Assume that $p_i \in R, i = 1, 2, 3$. Then

$$|p_1| + |p_2| + |p_3| < 1,$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+3} + p_1 x_{n+2} + p_2 x_{n+1} + p_3 x_n = 0.$$

Theorem B [19]: Let $g : [a, b]^3 \rightarrow [a, b]$, be a continuous function, where 3 is a positive integer, and $[a, b]$ is an interval of real numbers and consider the difference equation

$$x_{n+1} = g(x_n, x_{n-1}, x_{n-2}). \tag{5}$$

Suppose that g satisfies the following conditions:

- (i) For every integer i with $1 \leq i \leq 3$, the function $g(z_1, z_2, z_3)$ is weakly monotonic in z_i , for fixed z_1, z_2, z_3 .
- (ii) If m, M is a solution of the system

$$m = g(m_1, m_2, m_3) \quad \text{and} \quad M = g(M_1, M_2, M_3),$$

then $m = M$, where for each $i = 1, 2, 3$, we set

$$m_i = \begin{cases} m & \text{if } g \text{ is non-decreasing in } z_i \\ M & \text{if } g \text{ is non-increasing in } z_i \end{cases}$$

and

$$M_i = \begin{cases} M & \text{if } g \text{ is non-decreasing in } z_i \\ m & \text{if } g \text{ is non-increasing in } z_i \end{cases}.$$

Then, there exists exactly one equilibrium point \bar{x} of the difference equation (5), and every solution of (5) converges to \bar{x} .

3. LOCAL STABILITY OF THE EQUILIBRIUM POINT OF EQ.(1)

In his section, we study the local stability character of the equilibrium point of Eq.(1).

Eq.(1) has equilibrium point and is given by

$$\bar{x} = a\bar{x} + b\bar{x} + \frac{\alpha + c\bar{x}}{\beta + d\bar{x}},$$

or

$$d(1 - a - b)\bar{x}^2 + (\beta - \beta a - \beta b - c)\bar{x} - \alpha = 0.$$

Then if $a + b < 1$, the only positive equilibrium point of Eq.(1) is given by

$$\bar{x} = \frac{(\beta a + \beta b + c - \beta) + \sqrt{(\beta a + \beta b + c - \beta)^2 + 4\alpha d(1 - a - b)}}{2d(1 - a - b)}.$$

THEOREM 3.1. *The equilibrium \bar{x} of Eq. (1) is locally asymptotically stable if and only if*

$$(\beta + d\bar{x})^2 > \frac{|c\beta - d\alpha|}{(1 - b - a)}. \tag{6}$$

Proof: Let $f : (0, \infty)^3 \rightarrow (0, \infty)$ be a continuous function defined by

$$f(u, v, w) = au + bv + \frac{\alpha + cw}{\beta + dw}. \tag{7}$$

Therefore,

$$\frac{\partial f(u, v, w)}{\partial u} = a, \quad \frac{\partial f(u, v, w)}{\partial v} = b, \quad \frac{\partial f(u, v, w)}{\partial w} = \frac{(c\beta - d\alpha)}{(\beta + dw)^2}.$$

So, we can write

$$\frac{\partial f(\bar{x}, \bar{x}, \bar{x})}{\partial u} = a = p_1, \quad \frac{\partial f(\bar{x}, \bar{x}, \bar{x})}{\partial v} = b = p_2, \quad \frac{\partial f(\bar{x}, \bar{x}, \bar{x})}{\partial w} = \frac{(c\beta - d\alpha)}{(\beta + d\bar{x})^2} = p_3.$$

Then the linearized equation of Eq.(1) about \bar{x} is

$$y_{n+1} - p_1 y_{n-1} - p_2 y_n - p_3 y_{n-2} = 0, \tag{8}$$

It follows by Theorem A that, Eq.(1) is asymptotically stable if and only if

$$|p_1| + |p_2| + |p_3| < 1.$$

Thus,

$$|a| + \left| \frac{(c\beta - d\alpha)}{(\beta + d\bar{x})^2} \right| + |b| < 1,$$

and so

$$\begin{aligned} \left| \frac{(c\beta - d\alpha)}{(\beta + d\bar{x})^2} \right| &< 1 - b - a, \\ |c\beta - d\alpha| &< (\beta + d\bar{x})^2(1 - b - a), \end{aligned}$$

or

$$\frac{|c\beta - d\alpha|}{(1 - b - a)} < (\beta + d\bar{x})^2.$$

The proof is complete.

Example 1. The solution of the difference equation (1) is global stability if $a = 0.55$, $b = 0.3$, $c = 0.8$, $d = 0.5$, $\alpha = 7$ and $\beta = 2$ and the initial conditions $x_{-2} = 4$, $x_{-1} = 9$ and $x_0 = 0.3$ (See Fig. 1).

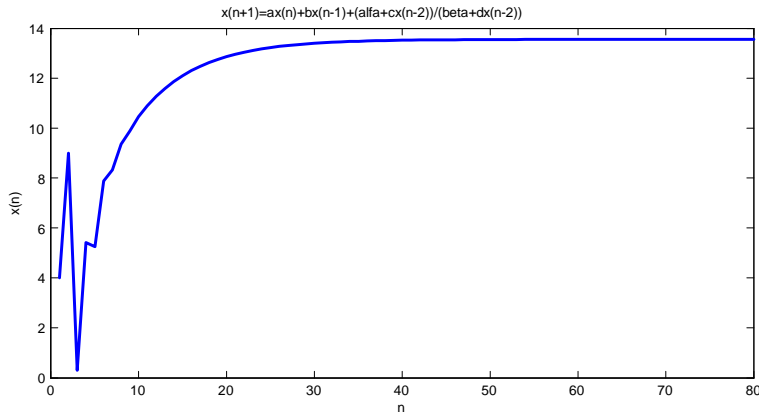


Figure 1. Plot the behavior of the solution of equation (1).

4. EXISTENCE OF BOUNDED AND UNBOUNDED SOLUTIONS OF EQ.(1)

Here we look at the boundedness nature of solutions of Eq.(1).

THEOREM 4.1. *Every solution of Eq.(1) is bounded if $a + b < 1$.*

Proof: Let $\{x_n\}_{n=-2}^\infty$ be a solution of Eq.(1). It follows from Eq.(1) that

$$x_{n+1} = ax_n + bx_{n-1} + \frac{\alpha + cx_{n-2}}{\beta + dx_{n-2}} = ax_n + bx_{n-1} + \frac{\alpha}{\beta + dx_{n-2}} + \frac{cx_{n-2}}{\beta + dx_{n-2}}.$$

Then

$$x_{n+1} < ax_n + bx_{n-1} + \frac{\alpha}{\beta} + \frac{cx_{n-2}}{dx_{n-2}} = ax_n + bx_{n-1} + \frac{\alpha}{\beta} + \frac{c}{d} \quad \text{for all } n \geq 0.$$

By using a comparison, the right hand side can be written as follows

$$y_{n+1} = ay_n + by_{n-1} + \frac{\alpha}{\beta} + \frac{c}{d},$$

and this equation is locally asymptotically stable if $a + b < 1$, and converges to the equilibrium point $\bar{y} = \frac{\alpha d + c\beta}{\beta d(1 - a - b)}$.

Therefore

$$\limsup_{n \rightarrow \infty} x_n \leq \frac{\alpha d + c\beta}{\beta d(1 - a - b)}.$$

Hence, the solution is bounded.

THEOREM 4.2. *Every solution of Eq.(1) is unbounded if $a > 1$ or $b > 1$.*

Proof: Let $\{x_n\}_{n=-2}^\infty$ be a solution of Eq.(1). Then from Eq.(1) we see that

$$x_{n+1} = ax_n + bx_{n-1} + \frac{\alpha + cx_{n-2}}{\beta + dx_{n-2}} > ax_n \quad \text{for all } n \geq 0.$$

The right hand side can be written as follows

$$y_{n+1} = ay_n \quad \Rightarrow \quad y_n = a^n y_0,$$

and this equation is unbounded because $a > 1$, and $\lim_{n \rightarrow \infty} y_n = \infty$. Then by using ratio test $\{x_n\}_{n=-2}^{\infty}$ is unbounded from above.

Similarly from Equation (1) we see that

$$x_{n+1} = ax_n + bx_{n-1} + \frac{\alpha + cx_{n-2}}{\beta + dx_{n-2}} > bx_{n-1} \quad \text{for all } n \geq 0.$$

We see that the right hand side can be written as follows

$$y_{n+1} = by_{n-1} \Rightarrow y_{2n-1} = b^n y_{-1} \text{ and } y_{2n} = b^n y_0,$$

and this equation is unbounded because $b > 1$, and $\lim_{n \rightarrow \infty} y_{2n-1} = \lim_{n \rightarrow \infty} y_{2n} = \infty$. Then by using ratio test $\{x_n\}_{n=-2}^{\infty}$ is unbounded from above.

Example 2. Figure (2) shows that behavior of the solution of the difference equation (1) is boundedness if we take $a = 0.3$, $b = 0.1$, $c = 0.8$, $d = 0.5$, $\alpha = 7$ and $\beta = 2$ and the initial conditions $x_{-2} = 4$, $x_{-1} = 9$ and $x_0 = 0.3$.

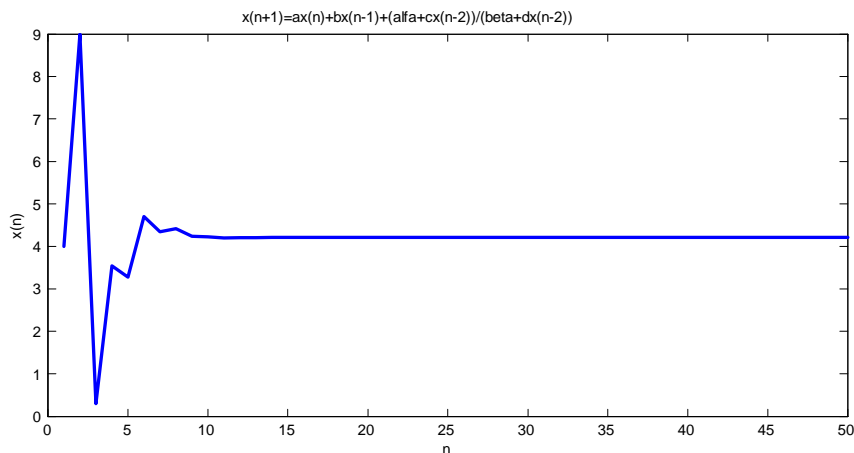


Figure 2. Show the boundedness of the solution of equation (1).

Example 3. Figure (3) shows the behavior of the solution of the difference equation (1) is unbounded when we put $a = 1.5$, $b = 0.8$, $c = 2$, $d = 3$, $\alpha = 6$ and $\beta = 5$ and the initial conditions $x_{-2} = 0.4$, $x_{-1} = 0.9$ and $x_0 = 0.3$.

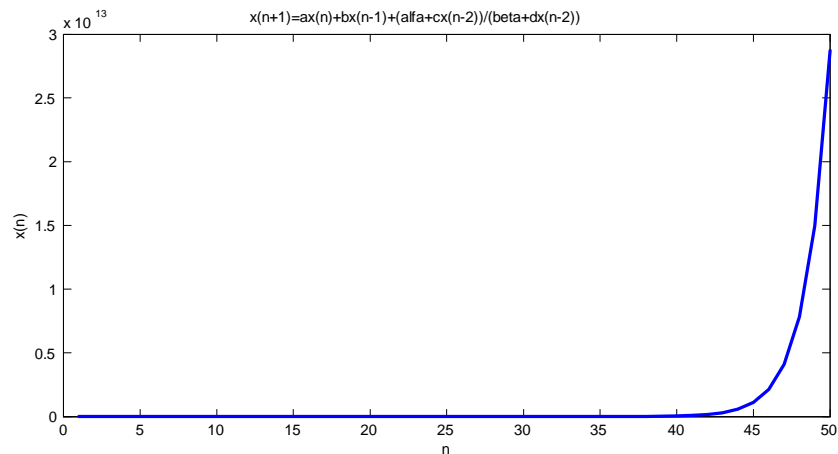


Figure 3. Show the unboundedness of the solution of equation (1).

5. GLOBAL ATTRACTIVITY OF THE EQUILIBRIUM POINT OF EQ.(1)

In this section, the global asymptotic stability of Eq.(1) is studied.

THEOREM 5.1. *The equilibrium point \bar{x} is a global attractor of Eq.(1) if $a + b < 1$.*

Proof: Suppose that ζ and η are real numbers and assume that $g : [\zeta, \eta]^3 \rightarrow [\zeta, \eta]$ is a function defined by

$$g(u, v, w) = au + bv + \frac{\alpha + cw}{\beta + dw}.$$

Then

$$\frac{\partial g(u, v, w)}{\partial u} = a, \quad \frac{\partial g(u, v, w)}{\partial v} = b, \quad \frac{\partial g(u, v, w)}{\partial w} = \frac{c\beta - d\alpha}{(\beta + dw)^2}.$$

Now, two cases must be considered :

Case (1): Let $c\beta - d\alpha < 0$, then we can easily see that the function $g(u, v, w)$ increasing in u, v and decreasing in w .

Let (m, M) be a solution of the system $M = g(M, M, m)$ and $m = g(m, m, M)$. Then from Eq.(1), we see that

$$M = aM + bM + \frac{\alpha + cm}{\beta + dm}, \quad m = am + bm + \frac{\alpha + cM}{\beta + dM},$$

or

$$M(1 - a - b) = \frac{\alpha + cm}{\beta + dm}, \quad m(1 - a - b) = \frac{\alpha + cM}{\beta + dM},$$

then

$$\beta(1 - a - b)M + d(1 - a - b)Mm = \alpha + cm, \quad \beta(1 - a - b)m + d(1 - a - b)mM = \alpha + cM.$$

Subtracting we obtain

$$(M - m)\{\beta(1 - a - b) + c\} = 0,$$

under the condition $a + b < 1$, we see that

$$M = m.$$

It follows by Theorem B that \bar{x} is a global attractor of Eq.(1). This completes the proof of the theorem.

Case (2): Assume that $c\beta - d\alpha > 0$ is true, then we can easily see that the function $g(u, v, w)$ increasing in u, v, w and decreasing in v .

Let (m, M) be a solution of the system $M = g(M, M, M)$ and $m = g(m, m, m)$. Then from Eq.(1), we see that

$$M = aM + bM + \frac{\alpha + cM}{\beta + dM}, \quad m = am + bM + \frac{\alpha + cm}{\beta + dm},$$

or

$$M(1 - a - b) = \frac{\alpha + cM}{\beta + dM}, \quad m(1 - a - b) = \frac{\alpha + cm}{\beta + dm},$$

then

$$\begin{aligned} \beta(1 - a - b)M + d(1 - a - b)M^2 &= \alpha + cM, \\ \beta(1 - a - b)m + d(1 - a - b)m^2 &= \alpha + cm. \end{aligned}$$

Subtracting we obtain

$$\begin{aligned} \beta(1 - a - b)(M - m) + d(1 - a - b)(M^2 - m^2) &= c(M - m) \\ (M - m)\{d(1 - a - b)(M + m) + \beta(1 - a - b) - c\} &= 0, \end{aligned}$$

under the condition $a + b < 1$ and $\beta(1 - a - b) > c$ we see that

$$M = m.$$

It follows by Theorem B that \bar{x} is a global attractor of Eq.(1).

6. EXISTENCE OF PERIODIC SOLUTIONS

In this section we investigate the existence of periodic solutions of Eq.(1). The following theorem states the necessary and sufficient conditions that this equation has periodic solutions of prime period two.

THEOREM 6.1. *Eq.(1) has positive prime period two solutions if and only if*

$$(i) [\beta(b - 1) - (\beta a + c)]^2 (b - a - 1) + 4a\{\beta(1 - b)[\beta(b - 1) - (\beta a + c)] + \alpha ad\} > 0.$$

Proof: Firstly, suppose that there exists a prime period two solution

$$\dots, p, q, p, q, \dots,$$

of Eq.(1). We will show that Condition (i) holds.

From Eq.(1), we get

$$p = aq + bp + \frac{\alpha + cq}{\beta + dq},$$

and

$$q = ap + bq + \frac{\alpha + cp}{\beta + dp}.$$

Therefore,

$$\beta p + dpq = a\beta q + adq^2 + \beta bp + bdpq + \alpha + cq, \tag{9}$$

and

$$\beta q + dpq = a\beta p + adp^2 + \beta bq + bdpq + \alpha + cp. \tag{10}$$

Subtracting (10) from (9) gives

$$\beta(p - q) + ad(p^2 - q^2) = \beta b(p - q) - \beta a(p - q) - c(p - q).$$

Since $p \neq q$, it follows that

$$p + q = \frac{\beta(b - a - 1) - c}{ad}. \tag{11}$$

Again, adding (9) and (10) yields

$$\begin{aligned} \beta(p + q) + 2dpq &= (a\beta + \beta b)(p + q) + ad(p^2 + q^2) + 2bdpq + 2\alpha + c(p + q), \\ ad(p^2 + q^2) &= \beta(p + q) + 2dpq - (a\beta + \beta b)(p + q) - 2bdpq - 2\alpha - c(p + q) \\ ad(p^2 + q^2) &= (\beta - a\beta - \beta b - c)(p + q) + 2dpq - 2bdpq - 2\alpha. \end{aligned} \tag{12}$$

By using (11), (12) and the relation

$$p^2 + q^2 = (p + q)^2 - 2pq \text{ for all } p, q \in R,$$

we obtain

$$\begin{aligned} ad((p + q)^2 - 2pq) &= (\beta - a\beta - \beta b - c)(p + q) + 2dpq - 2bdpq - 2\alpha \\ \frac{(\beta b - \beta a - \beta - c)^2}{ad} - 2adpq &= \frac{(\beta - a\beta - \beta b - c)(\beta b - \beta a - \beta - c)}{ad} + 2dpq - 2bdpq - 2\alpha \end{aligned}$$

Then,

$$pq = -\frac{\beta(b - 1)[\beta(b - 1) - (\beta a + c)] + \alpha ad}{ad^2(b - a - 1)}. \tag{13}$$

Now it is obvious from Eq.(11) and Eq.(13) that p and q are the two distinct roots of the quadratic equation

$$\begin{aligned}
 t^2 - \frac{\beta(b-a-1) - c}{ad}t - \frac{\beta(b-1)[\beta(b-1) - (\beta a + c)] + \alpha ad}{ad^2(b-a-1)} &= 0, \\
 ad t^2 - (\beta b - \beta a - \beta - c)t - \frac{\beta(b-1)[\beta(b-1) - (\beta a + c)] + \alpha ad}{d(b-a-1)} &= 0,
 \end{aligned}
 \tag{14}$$

and so

$$[\beta(b-1) - (\beta a + c)]^2 + \frac{4a\{\beta(b-1)[\beta(b-1) - (\beta a + c)] + \alpha ad\}}{(b-a-1)} > 0,$$

or

$$[\beta(b-1) - (\beta a + c)]^2 (b-a-1) + 4a\{\beta(b-1)[\beta(b-1) - (\beta a + c)] + \alpha ad\} > 0.$$

for $a + 1 < b$ then the inequalities (i) holds.

Conversely, suppose that inequality (i) is true. We will prove that Eq.(1) has a prime period two solution.

Suppose that

$$p = \frac{A + \zeta}{2ad},$$

and

$$q = \frac{A - \zeta}{2ad},$$

where $\zeta = \sqrt{A^2 + \frac{4a[\beta A(b-1) + \alpha ad]}{(b-a-1)}}$ and $A = \beta(b-1) - (\beta a + c)$.

We see from the inequality (i) that

$$A^2(b-a-1) + 4a[\beta A(b-1) + \alpha ad] > 0,$$

which equivalent to

$$A^2 + \frac{4a[\beta A(b-1) + \alpha ad]}{(b-a-1)} > 0.$$

Therefore p and q are distinct real numbers.

Set

$$x_{-2} = p, \quad x_{-1} = q \quad \text{and} \quad x_0 = p.$$

We would like to show that

$$x_1 = x_{-2} = q \quad \text{and} \quad x_2 = x_{-1} = p.$$

It follows from Eq.(1) that

$$x_1 = ap + bq + \frac{\alpha + cp}{\beta + dp} = a \left(\frac{A + \zeta}{2ad} \right) + b \left(\frac{A - \zeta}{2ad} \right) + \frac{\alpha + c \left(\frac{A + \zeta}{2ad} \right)}{\beta + d \left(\frac{A + \zeta}{2ad} \right)}.$$

Dividing the denominator and numerator by $2ad$ we get

$$x_1 = a \left(\frac{A + \zeta}{2ad} \right) + b \left(\frac{A - \zeta}{2ad} \right) + \frac{2ad\alpha + c(A + \zeta)}{2ad\beta + d(A + \zeta)}.$$

Multiplying the denominator and numerator of the right side by $2ad\beta + d(A - \zeta)$ and by computation we obtain

$$x_1 = q.$$

Similarly as before, it is easy to show that

$$x_2 = p.$$

Then by induction we get

$$x_{2n} = p \quad \text{and} \quad x_{2n+1} = q \quad \text{for all} \quad n \geq -2.$$

Thus Eq.(1) has the prime period two solution

$$\dots, p, q, p, q, \dots,$$

where p and q are the distinct roots of the quadratic equation (14) and the proof is complete.

Example 4. Figure (4) shows the period two solution of equation (1) when $a = 0.2$, $b = 5$, $c = 0.4$, $d = 5$, $\alpha = 0.7$ and $\beta = 0.2$ and the initial conditions $x_{-2} = p$, $x_{-1} = q$ and $x_0 = p$ since p and q as in the previous theorem.

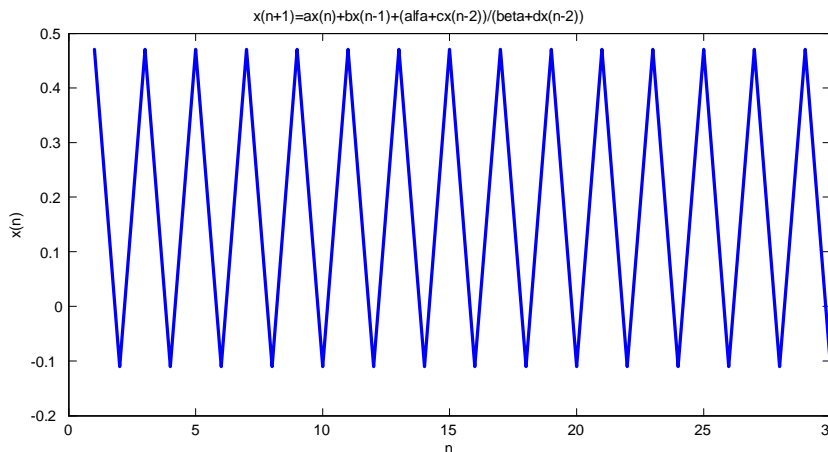


Figure 4. Plot the periodicity of the solution of equation (1).

Acknowledgements

This article was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah. The authors, therefore, acknowledge with thanks DSR technical and financial support.

REFERENCES

1. V. Lakshmikantham and D. Trigiant, Theory of Difference Equations Numerical Methods and Applications, Monographs and textbooks in pure and applied mathematics, Marcel Dekker Inc, second edition, (2002).
2. R. P. Agarwal and E. M. Elsayed, Periodicity and stability of solutions of higher order rational difference equation, Adv. Stud. Contem. Math., 17 (2) (2008), 181-201.
3. A. E. Hamza and A. Morsy, On the recursive sequence $x_{n+1} = \alpha + \frac{x_{n-1}}{x_n^k}$, Appl. Math. Lett. Vol. 22(1), (2009), 91-95.
4. E. M. Elsayed, M. M. El-Dessoky and Asim Asiri, Dynamics and Behavior of a Second Order Rational Difference equation, J. Comput. Anal. Appl., 16 (4), (2014), 794-807.
5. E. M. E. Zayed, M. A. El-Moneam, On the rational recursive sequence $x_{n+1} = \left(A + \sum_{i=0}^k \alpha_i x_{n-i} \right) / \sum_{i=0}^k \beta_i x_{n-i}$, Math. Bohemica, Vol. 133(3), (2008), 225–239.
6. E. M. E. Zayed, M. A. El-Moneam, On the global asymptotic stability for a rational recursive sequence, Iran. J. Sci. & Tech., (2011) A4: 333-339.
7. M. A. El-Moneam, On the Dynamics of the Higher Order Nonlinear Rational Difference Equation, Math. Sci. Lett. 3(2), (2014), 121-129.
8. I. Yalçınkaya, On the difference equation $x_{n+1} = \alpha + \frac{x_{n-m}}{x_n^k}$, Disc. Dyn. Nat. Soc., Vol. 2008, Article ID 805460, 8 pages.

9. E. M. Elabbasy, H. El-Metwally and E. M. Elsayed, On the difference equation $x_{n+1} = ax_n - \frac{bx_n}{cx_n - dx_{n-1}}$, Adv. Differ. Equ., Volume 2006 (2006), Article ID 82579,1–10.
10. H. M.EL-Owaidy, A. M. Ahmed and M. S. Mousa, On asymptotic behavior of the difference equation $x_{n+1} = \alpha + \frac{x_n^p}{x_n}$, J. Appl. Math. Comput. 12 (2003), 31-37.
11. E. M. Elsayed, Dynamics of Recursive Sequence of Order Two, Kyungpook Math. J. 50, (2010), 483-497.
12. E. M. Elsayed and M. M. El-Dessoky, Dynamics and behavior of a higher order rational recursive sequence, Adv. Differ. Equ., 2012, (2012), 69.
13. Taixiang Sun, Xin Wu, Qiuli He, Hongjian Xi, On boundedness of solutions of the difference equation $x_{n+1} = p + \frac{x_{n-1}}{x_n}$ for $p < 1$, J. Appl. Math. Comput. 44(1-2), (2014), 61-68.
14. E. M. Elsayed and M. M. El-Dessoky, Dynamics and global behavior for a fourth-order rational difference equation, Hacettepe J. Math. and Stat., 42 (5) (2013), 479–494.
15. E. M. E. Zayed and M. A. El-Moneam, On the rational recursive sequence $x_{n+1} = Ax_n + Bx_{n-k} + \frac{\beta x_n + \gamma x_{n-k}}{Cx_n + Dx_{n-k}}$, Acta Appl. Math., 111(3), (2010), 287-301.
16. E. M. Elsayed, Behavior and expression of the solutions of some rational difference equations, J. Comp. Anal. Appl., 15 (1) (2013), 73-81.
17. E. M. E. Zayed, Dynamics of the nonlinear rational difference equation $x_{n+1} = Ax_n + Bx_{n-k} + \frac{px_n + x_{n-k}}{q + x_{n-k}}$, Eur. J. Pure Appl. Math., 3 (2) (2010), 254-268.
18. V. L. Kocic and G. Ladas, Global Behavior of Nonlinear Difference Equations of Higher Order with Applications, Kluwer Academic Publishers, Dordrecht, 1993.
19. M. R. S. Kulenovic and G. Ladas, Dynamics of Second Order Rational Difference Equations with Open Problems and Conjectures, Chapman & Hall / CRC Press, 2001.
20. E.A. Grove and G. Ladas, Periodicities in nonlinear difference equations, Vol. 4, Chapman & Hall / CRC, 2005.
21. Kenneth S. Berenhaut, Katherine M. Donadio and John D. Foley, On the rational recursive sequence $y_{n+1} = a + \frac{y_{n-1}}{y_{n-k}}$, Appl. Math. Lett., Vol. 21(9), (2008), 906-909.
22. R. P. Agarwal, Difference Equations and Inequalities, 1st edition, Marcel Dekker, New York, 1992, 2nd edition, 2000.
23. A. Gelişken, C. Cinar and I. Yalçınkaya, On the periodicity of a difference equation with maximum, Disc. Dyn. Nat. Soc., Vol. 2008, Article ID 820629, (2008), 11 pages, doi: 10.1155/2008/820629.
24. C. Wang and S. Wang, Oscillation of partial population model with diffusion and delay, Appl. Math. Lett., 22 (12) (2009), 1793-1797.
25. I. Yalçınkaya, On the global asymptotic behavior of a system of two nonlinear difference equations, ARS Combinatoria, 95 (2010), 151-159.
26. N. Touafek and E. M. Elsayed, On the periodicity of some systems of nonlinear difference equations, Bull. Math. Soc. Sci. Math. Roumanie, 55 (103), No: 2, (2012), 217-224.
27. Y. Yazlik, D. T. Tollu, N. Taskara, On the Behaviour of Solutions for Some Systems of Difference Equations, J. Comp. Anal. Appl., 18 (1), (2015), 166-178.
28. M. R. S. Kulenović and M. Pilling, Global Dynamics of a Certain Two-dimensional Competitive System of Rational Difference Equations with Quadratic Terms, J. Comp. Anal. Appl., 19 (1), (2015), 156-166.
29. Mehmet Gümüş and Özkan Öcalan, Some Notes on the Difference Equation $x_{n+1} = \alpha + \frac{x_{n-1}}{x_n^k}$, Disc. Dyn. Nat. Soc., Vol. 2012, Article ID 258502 ,(2012),12 pages.
30. E. M. E. Zayed and M. A. El-Moneam, On the rational recursive sequence $x_{n+1} = ax_n - \frac{bx_n}{cx_n - dx_{n-k}}$, Comm. Appl. Nonlin. Anal., 15 (2), (2008), 47-57.
31. M. M. El-Dessoky, On the dynamics of a higher Order rational difference equations, Journal of the Egyptian Mathematical Society, 25(1), (2017), 28–36.
32. M. M. El-Dessoky, On the Difference equation $x_{n+1} = ax_{n-l} + bx_{n-k} + \frac{cx_{n-s}}{dx_{n-s} - e}$, Math. Meth. Appl. Sci., 40(3), (2017), 535–545.
33. M. M. El-Dessoky, On the dynamics of higher Order difference equations $x_{n+1} = ax_n + \frac{\alpha x_n x_{n-l}}{\beta x_n + \gamma x_{n-k}}$, J. Comp. Anal. Appl., 22(7), (2017), 1309-1322.

Algebraic and Order Properties of Tracy-Singh Products for Operator Matrices

Arnon Ploymukda¹, Pattrawut Chansangiam^{1*}, Wicharn Lewkeeratiyutkul²

¹Department of Mathematics, Faculty of Science,
King Mongkut's Institute of Technology Ladkrabang,
Bangkok 10520, Thailand.

²Department of Mathematics and Computer Science,
Faculty of Science, Chulalongkorn University,
Bangkok 10330, Thailand.

Abstract

We generalize the tensor product for operators to the Tracy-Singh product for operator matrices acting on the direct sum of Hilbert spaces. This kind of operator product is compatible with algebraic operations and order relations for operators. It follows that this product preserves many structure properties of operators.

Keywords: tensor product, Tracy-Singh product, operator matrix, Moore-Penrose inverse

Mathematics Subject Classifications 2010: 15A69, 47A05, 47A80.

1 Introduction

In scientific computing, we consider a matrix to be a two-dimensional array for stacking data. A processing of such data can be performed using matrix products. One of extremely useful matrix products is the Kronecker product. For any complex matrices $A \in M_{m,n}(\mathbb{C})$ and $B \in M_{p,q}(\mathbb{C})$, the Kronecker product of A and B is given by the block matrix

$$A \hat{\otimes} B = [a_{ij}B]_{ij} \in M_{mp,nq}(\mathbb{C}).$$

Equivalently, $A \hat{\otimes} B$ is the unique complex matrix of order $mp \times nq$ satisfying

$$(A \hat{\otimes} B)(x \hat{\otimes} y) = Ax \hat{\otimes} By \tag{1}$$

for all $x \in \mathbb{C}^n$ and $y \in \mathbb{C}^q$. This matrix product has wide applications in mathematics, computer science, statistics, physics, system theory, signal processing, and related fields. See [2, 5, 6, 12] for more information.

*Corresponding author. Email: pattrawut.ch@kmitl.ac.th

Tracy-Singh Products for Operator Matrices

Kronecker product was generalized to the Tracy-Singh product of partitioned matrices by Tracy and Singh [10]. Let $A = [A_{ij}] \in M_{m,n}(\mathbb{C})$ be a partitioned matrix with A_{ij} of order $m_i \times n_j$ as the (i, j) th submatrix where $\sum_i m_i = m$ and $\sum_j n_j = n$. Let $B = [B_{kl}] \in M_{p,q}(\mathbb{C})$ be a partitioned matrix with B_{kl} of order $p_k \times q_l$ as the (k, l) th submatrix where $\sum_k p_k = p$ and $\sum_l q_l = q$. The Tracy-Singh product of A and B is defined by

$$A \hat{\otimes} B = [[A_{ij} \hat{\otimes} B_{kl}]_{kl}]_{ij} \in M_{mp,nq}(\mathbb{C}),$$

where each block $A_{ij} \hat{\otimes} B_{kl}$ is of order $m_i p_k \times n_j q_l$. This kind of matrix product has several attractive properties in algebraic, order, and analytic points of views; see, *e.g.*, [3, 8, 9, 10]. The Tracy-Singh product can be applied widely in statistics, econometrics and related fields; see, *e.g.*, [9, 10].

As a natural generalization of a complex matrix, we consider a bounded linear operator between complex Hilbert spaces. The tensor product of Hilbert space operators can be viewed as an extension of the Kronecker product of complex matrices. Using the universal mapping property in the monoidal category of Hilbert spaces, the tensor product of $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ and $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$ is the unique bounded linear operator from $\mathcal{H} \otimes \mathcal{K}$ into $\mathcal{H}' \otimes \mathcal{K}'$ such that for all $x \in \mathcal{H}$ and $y \in \mathcal{K}$,

$$(A \otimes B)(x \otimes y) = Ax \otimes By. \tag{2}$$

A fundamental property of tensor product is the mixed product property:

$$(A \otimes B)(C \otimes D) = AC \otimes BD. \tag{3}$$

The theory of tensor product of operators has been continuously developed in the literature; see, *e.g.*, [4, 11].

From the previous discussion, it is natural to extend the notion of tensor product for operators to the ‘‘Tracy-Singh product’’ of operators. We shall propose a natural definition of such operator product. It turns out that this product is compatible with algebraic operations and order relations for operators. One of the most attractive properties, the mixed product property, also holds for Tracy-Singh products. It follows that this product preserves attractive properties of operators, such as being invertible, Hermitian, unitary, positive, and normal. Our results generalize the results known so far in the literature for both Tracy-Singh products of matrices and tensor products of operators.

This paper is organized as follows. In section 2, we introduce the Tracy-Singh product for operator matrices and deduce its algebraic properties. In section 3, we show that the Tracy-Singh product is compatible with various kinds of operator inverses. We investigate the relationship between Tracy-Singh products and operator orderings in Section 4.

2 Tracy-Singh products and algebraic operations for operators

In this section, we introduce the Tracy-Singh product of operators on a Hilbert space. Then we will show that this product is compatible with addition, scalar multiplication, adjoint operation, usual multiplication, power, and direct sum of operator inverses.

Throughout this paper, let \mathcal{H} , \mathcal{H}' , \mathcal{K} and \mathcal{K}' be complex Hilbert spaces. When \mathcal{X} and \mathcal{Y} are Hilbert spaces, denote by $\mathbb{B}(\mathcal{X}, \mathcal{Y})$ the Banach space of bounded linear operators from \mathcal{X} into \mathcal{Y} , and abbreviate $\mathbb{B}(\mathcal{X}, \mathcal{X})$ to $\mathbb{B}(\mathcal{X})$.

The projection theorem for Hilbert spaces allows us to decompose

$$\mathcal{H} = \bigoplus_{j=1}^n \mathcal{H}_j, \quad \mathcal{H}' = \bigoplus_{i=1}^m \mathcal{H}'_i, \quad \mathcal{K} = \bigoplus_{l=1}^q \mathcal{K}_l, \quad \mathcal{K}' = \bigoplus_{k=1}^p \mathcal{K}'_k$$

where each $\mathcal{H}_j, \mathcal{H}'_i, \mathcal{K}_l, \mathcal{K}'_k$ are Hilbert spaces. Such decompositions are fixed throughout the paper. For each $j = 1, \dots, n$, let E_j be the canonical embedding from \mathcal{H}_j into \mathcal{H} , defined by

$$x_j \mapsto (0, \dots, 0, x_j, 0, \dots, 0).$$

Similarly, let F_l be the canonical embedding from \mathcal{K}_l into \mathcal{K} for each $l = 1, \dots, q$. For each $i = 1, \dots, m$ and $k = 1, \dots, p$, let $P'_i : \mathcal{H}' \rightarrow \mathcal{H}'_i$ and $Q'_k : \mathcal{K}' \rightarrow \mathcal{K}'_k$ be the orthogonal projections. Thus, each operator $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ and $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$ can be expressed uniquely as operator matrices

$$A = [A_{ij}]_{i,j=1}^{m,n} \quad \text{and} \quad B = [B_{kl}]_{k,l=1}^{p,q}$$

where $A_{ij} = P'_i A E_j$ and $B_{kl} = Q'_k B F_l$ for each i, j, k, l .

Definition 1. Let $A = [A_{ij}]_{i,j=1}^{m,n} \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ and $B = [B_{kl}]_{k,l=1}^{p,q} \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$ be operator matrices defined as above. We define the Tracy-Singh product of A and B to be the operator matrix

$$A \boxtimes B = [[A_{ij} \otimes B_{kl}]_{kl}]_{ij} \tag{4}$$

which is a bounded linear operator from $\bigoplus_{j=1}^n \bigoplus_{l=1}^q \mathcal{H}_j \otimes \mathcal{K}_l$ to $\bigoplus_{i=1}^m \bigoplus_{k=1}^p \mathcal{H}'_i \otimes \mathcal{K}'_k$.

Note that if both A and B are 1×1 block operator matrices i.e. $m = n = p = q = 1$, then their Tracy-Singh product $A \boxtimes B$ is just the tensor product $A \otimes B$.

Next, we shall show that the Tracy-Singh product of two linear maps induced by two matrices is just the linear map induced by the Tracy-Singh product of these matrices. Recall that for each $A \in M_{m,n}(\mathbb{C})$ and $B \in M_{p,q}(\mathbb{C})$, the induced maps

$$L_A : \mathbb{C}^n \rightarrow \mathbb{C}^m, \quad x \mapsto Ax \quad \text{and} \quad L_B : \mathbb{C}^q \rightarrow \mathbb{C}^p, \quad y \mapsto By$$

Tracy-Singh Products for Operator Matrices

are bounded linear operators. Using the universal mapping property, we identify $\mathbb{C}^n \otimes \mathbb{C}^q$ with $\mathbb{C}^{nq} \cong M_{n,q}(\mathbb{C})$ together with the canonical bilinear map $(x, y) \mapsto x \hat{\otimes} y$ for each $(x, y) \in \mathbb{C}^n \times \mathbb{C}^q$. It is similar for $\mathbb{C}^m \otimes \mathbb{C}^p$.

Lemma 2. For each $A \in M_{m,n}(\mathbb{C})$ and $B \in M_{p,q}(\mathbb{C})$, we have

$$L_A \otimes L_B = L_{A \hat{\otimes} B}. \tag{5}$$

Proof. For any $x \otimes y \in \mathbb{C}^n \otimes \mathbb{C}^q$, we obtain from the mixed product property of the Kronecker product (1) that

$$\begin{aligned} (L_A \otimes L_B)(x \otimes y) &= L_A(x) \otimes L_B(y) = L_A(x) \hat{\otimes} L_B(y) \\ &= Ax \hat{\otimes} By = (A \hat{\otimes} B)(x \hat{\otimes} y) \\ &= (A \hat{\otimes} B)(x \otimes y) = L_{A \hat{\otimes} B}(x \otimes y). \end{aligned}$$

Thus, by the uniqueness of tensor product, $L_A \otimes L_B = L_{A \hat{\otimes} B}$. □

Proposition 3. For any complex matrices $A = [A_{ij}]$ and $B = [B_{kl}]$ partitioned in block-matrix forms, we have

$$L_A \boxtimes L_B = L_{A \boxtimes B}. \tag{6}$$

Proof. Recall that the (i, j) th block of the matrix representation of L_A is the matrix A_{ij} . It follows from Lemma 2 that

$$L_A \boxtimes L_B = \left[[L_{A_{ij}} \otimes L_{B_{kl}}]_{kl} \right]_{ij} = \left[[L_{A_{ij} \hat{\otimes} B_{kl}}]_{kl} \right]_{ij} = L_{A \boxtimes B}.$$

□

The next proposition shows that the Tracy-Singh product is compatible with the addition, the scalar multiplication and the adjoint operation of operators.

Proposition 4. Let $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ and $B, C \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$ be operator matrices, and let $\alpha \in \mathbb{C}$. Then

$$(\alpha A) \boxtimes B = \alpha(A \boxtimes B) = A \boxtimes (\alpha B), \tag{7}$$

$$(A \boxtimes B)^* = A^* \boxtimes B^*, \tag{8}$$

$$A \boxtimes (B + C) = A \boxtimes B + A \boxtimes C, \tag{9}$$

$$(B + C) \boxtimes A = B \boxtimes A + C \boxtimes A. \tag{10}$$

Proof. Since each (i, j) th block of αA is given by $(\alpha A)_{ij} = \alpha A_{ij}$, we get

$$(\alpha A) \boxtimes B = \left[[(\alpha A_{ij}) \otimes B_{kl}]_{kl} \right]_{ij} = \left[[\alpha(A_{ij} \otimes B_{kl})]_{kl} \right]_{ij} = \alpha(A \boxtimes B).$$

Similarly, $A \boxtimes (\alpha B) = \alpha(A \boxtimes B)$. Since $A^* = [A_{ji}^*]_{ij}$ and $B^* = [B_{lk}^*]_{kl}$ for all i, j, k, l , we obtain

$$(A \boxtimes B)^* = \left[[A_{ji}^* \otimes B_{kl}^*]_{kl} \right]_{ij} = \left[[A_{ji}^* \otimes B_{lk}^*]_{kl} \right]_{ij} = A^* \boxtimes B^*.$$

The proofs of (9) and (10) are done by using the fact that $(B + C)_{kl} = B_{kl} + C_{kl}$ for all k, l together with the left/right distributivity of the tensor product over the addition. □

A. Ploymukda, P. Chansangiam, W. Lewkeeratiyutkul

Properties (7), (9) and (10) say that the map $(A, B) \mapsto A \boxtimes B$ is bilinear.

Proposition 5. *Let $A = [A_{ij}] \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ and let $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$ be operator matrices. Then*

$$A \boxtimes B = [A_{ij} \boxtimes B]_{ij} = \begin{bmatrix} A_{11} \boxtimes B & \cdots & A_{1n} \boxtimes B \\ \vdots & \ddots & \vdots \\ A_{m1} \boxtimes B & \cdots & A_{mn} \boxtimes B \end{bmatrix}.$$

That is, the (i, j) th block of $A \boxtimes B$ is just $A_{ij} \boxtimes B$, regardless of how to partition B .

Proof. It follows directly from the definition of the Tracy-Singh product. \square

Remark 6. *It is not true in general that the (k, l) th block of $A \boxtimes B$ is $A \boxtimes B_{kl}$.*

When $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$, the direct sum of $A_1 \in \mathbb{B}(\mathcal{H}_1, \mathcal{K}_1)$ and $A_2 \in \mathbb{B}(\mathcal{H}_2, \mathcal{K}_2)$ is defined to be the operator

$$A_1 \oplus A_2 = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \in \mathbb{B}(\mathcal{H}, \mathcal{K}).$$

The next result gives a relation between the direct sum and the Tracy-Singh product.

Proposition 7. *The Tracy-Singh product is right distributive over the direct sum of operators. That is, for any operator matrices A, B and C , we have*

$$(A \oplus B) \boxtimes C = (A \boxtimes C) \oplus (B \boxtimes C). \tag{11}$$

Proof. It follows from Proposition 5 that

$$\begin{aligned} (A \oplus B) \boxtimes C &= \begin{bmatrix} A \boxtimes C & 0 \boxtimes C \\ 0 \boxtimes C & B \boxtimes C \end{bmatrix} = \begin{bmatrix} A \boxtimes C & 0 \\ 0 & B \boxtimes C \end{bmatrix} \\ &= (A \boxtimes C) \oplus (B \boxtimes C). \end{aligned} \quad \square$$

It is not true in general that the Tracy-Singh product is left distributive over the direct sum of operators.

The next theorem shows that the Tracy-Singh product is compatible with the ordinary product of operators. This fundamental property, called the *mixed product property*, will be used many times in later discussions.

Theorem 8. *Let $\mathcal{H}, \mathcal{H}', \mathcal{H}'', \mathcal{K}, \mathcal{K}'$ and \mathcal{K}'' be complex Hilbert spaces. Let $A = [A_{ij}]_{i,j=1}^{m,n} \in \mathbb{B}(\mathcal{H}', \mathcal{H}'')$, $C = [C_{ij}]_{i,j=1}^{n,r} \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$, $B = [B_{kl}]_{k,l=1}^{p,q} \in \mathbb{B}(\mathcal{K}', \mathcal{K}'')$ and $D = [D_{kl}]_{k,l=1}^{q,s} \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$ be operator matrices partitioned so that they are compatible with the decompositions of the corresponding Hilbert spaces. Then*

$$(A \boxtimes B)(C \boxtimes D) = AC \boxtimes BD. \tag{12}$$

Tracy-Singh Products for Operator Matrices

Proof. Using block multiplication of operators and the mixed product property of the tensor product (3), we have

$$\begin{aligned}
 (A \boxtimes B)(C \boxtimes D) &= [[A_{ij} \otimes B_{kl}]_{kl}]_{ij} [[C_{ij} \otimes D_{kl}]_{kl}]_{ij} \\
 &= \left[\left[\sum_{\alpha=1}^n \sum_{\beta=1}^q (A_{i\alpha} \otimes B_{k\beta})(C_{\alpha j} \otimes D_{\beta l}) \right]_{kl} \right]_{ij} \\
 &= \left[\left[\sum_{\alpha=1}^n \sum_{\beta=1}^q (A_{i\alpha} C_{\alpha j} \otimes B_{k\beta} D_{\beta l}) \right]_{kl} \right]_{ij} \\
 &= \left[\sum_{\alpha=1}^n A_{i\alpha} C_{\alpha j} \right]_{ij} \boxtimes \left[\sum_{\beta=1}^q B_{k\beta} D_{\beta l} \right]_{kl} \\
 &= AC \boxtimes BD. \quad \square
 \end{aligned}$$

Corollary 9. For any operator matrices $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$, we have

$$(A \boxtimes B)^r = A^r \boxtimes B^r \tag{13}$$

for any $r \in \mathbb{N}$.

In the rest of section, we investigate structure properties of operators under taking Tracy-Singh products. Recall that an operator $T \in \mathbb{B}(\mathcal{H})$ is said to be *involutory* if $T^2 = I$, *idempotent* if $T^2 = T$, an *isometry* if $T^*T = I$, a *partial isometry* if the restriction of T to a closed subspace is an isometry, or equivalently, $TT^*T = T$.

Corollary 10. Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$. If both A and B satisfy one of the following properties, then the same property holds for $A \boxtimes B$: Hermitian, unitary, isometry, co-isometry, partial isometry, idempotent, involutory, projection.

Proof. Applying Theorem 8 and Proposition 4, we get the results. □

If A and B are skew-Hermitian operators, then $A \boxtimes B$ is Hermitian. Recall that an operator $T \in \mathbb{B}(\mathcal{H})$ is said to be *nilpotent* if there is a positive integer k such that $T^k = 0$. The smallest such integer k is called the degree of nilpotency of T . If $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ are nilpotent operators with degrees of nilpotency r and s , respectively, then $A \boxtimes B$ is also nilpotent with degree of nilpotency not exceed $\min\{r, s\}$.

3 Tracy-Singh products and operator inverses

Next, we discuss the invertibility of the Tracy-Singh product of operators. Recall that an operator $A \in \mathbb{B}(\mathcal{H}, \mathcal{K})$ is said to be *regular* if there is an operator $A^- \in \mathbb{B}(\mathcal{K}, \mathcal{H})$ such that $AA^-A = A$. The operator A^- is called an *inner inverse* of A . An operator $X \in \mathbb{B}(\mathcal{K}, \mathcal{H})$ is said to be an *outer inverse* of A if $XAX = X$.

Proposition 11. *Let $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ and $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$.*

- (i) *If A and B are left invertible with left inverses \hat{A} and \hat{B} respectively, then $A \boxtimes B$ is left invertible and $\hat{A} \boxtimes \hat{B}$ is its left inverse.*
- (ii) *If A and B are right invertible with right inverses \hat{A} and \hat{B} respectively, then $A \boxtimes B$ is right invertible and $\hat{A} \boxtimes \hat{B}$ is its right inverse.*
- (iii) *If A and B are regular with inner inverses A^- and B^- respectively, then $A \boxtimes B$ is regular with $A^- \boxtimes B^-$ as its inner inverse.*
- (iv) *If A and B have A^- and B^- as their outer inverses respectively, then $A \boxtimes B$ has $A^- \boxtimes B^-$ as its outer inverse.*

Proof. It follows from Theorem 8 and the facts that $I_{\mathcal{X}} \boxtimes I_{\mathcal{Y}} = I_{\mathcal{X} \otimes \mathcal{Y}}$ for any Hilbert spaces \mathcal{X} and \mathcal{Y} . □

As a consequence of (i) and (ii) in Proposition 11, we obtain the following result.

Corollary 12. *Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$. If A and B are invertible, then $A \boxtimes B$ is invertible and*

$$(A \boxtimes B)^{-1} = A^{-1} \boxtimes B^{-1}. \tag{14}$$

Next, we consider a kind of operator inverse, called Moore-Penrose inverse. Recall that a *Moore-Penrose inverse* of $A \in \mathbb{B}(\mathcal{H}, \mathcal{K})$ is an operator $A^\dagger \in \mathbb{B}(\mathcal{K}, \mathcal{H})$ satisfying the following Penrose conditions ([7])

- (i) A^\dagger is an inner inverse of A ;
- (ii) A^\dagger is an outer inverse of A ;
- (iii) AA^\dagger is Hermitian ;
- (iv) $A^\dagger A$ is Hermitian.

It is well known that the following statements are equivalent for $A \in \mathbb{B}(\mathcal{H}, \mathcal{K})$ (see *e.g.* [1]):

- (i) a Moore-Penrose inverse of A exists ;
- (ii) a Moore-Penrose inverse of A is unique ;
- (iii) the range of A is closed.

Theorem 13. *Let $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ and $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$. If A and B have closed ranges, then*

1. *the range of $A \boxtimes B$ is closed ;*
2. *$(A \boxtimes B)^\dagger = A^\dagger \boxtimes B^\dagger$.*

Tracy-Singh Products for Operator Matrices

Proof. Since the ranges of A and B are closed, the Moore-Penrose inverses A^\dagger and B^\dagger exist and are unique. Making use of Theorem 8 and Proposition 4, we can verify that $A^\dagger \boxtimes B^\dagger$ satisfies the following Penrose equations:

- (i) $(A \boxtimes B)(A^\dagger \boxtimes B^\dagger)(A \boxtimes B) = A \boxtimes B$
- (ii) $(A^\dagger \boxtimes B^\dagger)(A \boxtimes B)(A^\dagger \boxtimes B^\dagger) = A^\dagger \boxtimes B^\dagger$
- (iii) $((A \boxtimes B)(A^\dagger \boxtimes B^\dagger))^* = (A \boxtimes B)(A^\dagger \boxtimes B^\dagger)$
- (iv) $((A^\dagger \boxtimes B^\dagger)(A \boxtimes B))^* = (A^\dagger \boxtimes B^\dagger)(A \boxtimes B)$.

Hence, a Moore-Penrose inverse of $A \boxtimes B$ exists and it is uniquely determined by $A^\dagger \boxtimes B^\dagger$. It follows that $A \boxtimes B$ has a closed range. □

The results in this section indicate that the Tracy-Singh product is compatible with various kinds of operator inverses.

4 Tracy-Singh products and operator orderings

Now, we focus on order properties of Tracy-Singh products related to algebraic properties.

Theorem 14. *Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$.*

- (i) *If $A, B \geq 0$, then $A \boxtimes B \geq 0$.*
- (ii) *If $A, B > 0$, then $A \boxtimes B > 0$.*

Proof. Assume $A, B \geq 0$. Using Theorem 8 and property (8), we obtain

$$\begin{aligned} A \boxtimes B &= A^{\frac{1}{2}} A^{\frac{1}{2}} \boxtimes B^{\frac{1}{2}} B^{\frac{1}{2}} = \left(A^{\frac{1}{2}} \boxtimes B^{\frac{1}{2}} \right) \left(A^{\frac{1}{2}} \boxtimes B^{\frac{1}{2}} \right) \\ &= \left(A^{\frac{1}{2}} \boxtimes B^{\frac{1}{2}} \right)^* \left(A^{\frac{1}{2}} \boxtimes B^{\frac{1}{2}} \right) \geq 0. \end{aligned}$$

Consider the case $A, B > 0$. We have immediately by (i) that $A \boxtimes B \geq 0$. By Corollary 12, $A \boxtimes B$ is invertible. This implies that $A \boxtimes B > 0$. □

The next result provides the monotonicity of Tracy-Singh product.

Corollary 15. *Let $A_1, A_2 \in \mathbb{B}(\mathcal{H})$ and $B_1, B_2 \in \mathbb{B}(\mathcal{K})$.*

- (i) *If $A_1 \geq A_2 \geq 0$ and $B_1 \geq B_2 \geq 0$, then $A_1 \boxtimes B_1 \geq A_2 \boxtimes B_2$.*
- (ii) *If $A_1 > A_2 > 0$ and $B_1 > B_2 > 0$, then $A_1 \boxtimes B_1 > A_2 \boxtimes B_2$.*

Proof. Suppose that $A_1 \geq A_2 \geq 0$ and $B_1 \geq B_2 \geq 0$. Applying Proposition 4 and Theorem 14 yields

$$\begin{aligned} A_1 \boxtimes B_1 - A_2 \boxtimes B_2 &= A_1 \boxtimes B_1 - A_2 \boxtimes B_1 + A_2 \boxtimes B_1 - A_2 \boxtimes B_2 \\ &= (A_1 - A_2) \boxtimes B_1 + A_2 \boxtimes (B_1 - B_2) \\ &\geq 0. \end{aligned}$$

The proof of (ii) is similar to that of (i). □

A. Ploymukda, P. Chansangiam, W. Lewkeeratiyutkul

Acknowledgement. This research was supported by Thailand Research Fund grant no. MRG6080102.

References

- [1] S. R. Caradus, *Generalized Inverses and Operator Theory, Queen's Papers in Pure and Applied Mathematics no. 50*, Queen's University, Kingston, Ontario, 1978.
- [2] R. A. Horn, C. R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, 1999.
- [3] R. H. Koning, H. Neudecker, T. Wansbeek, Block Kronecker products and the vecb operator. *Linear Algebra Appl.*, 149, 165-184 (1991).
- [4] C. S. Kubrusly, P. C.M. Vieira, Convergence and decomposition for tensor products of Hilbert space operators. *Oper. Matrices*, 2, 407-416 (2008).
- [5] J. R. Magnus, H. Neudecker, Matrix differential calculus with applications to simple, Hadamard, and Kronecker products., *J. Math. Psych.*, 29, 474-492 (1985).
- [6] H. Neudecker, A. Satorra, G. Trenkler, S. Liu, A Kronecker matrix inequality with a statistical application. *Econometric Theory*, 11, 654-655 (1985).
- [7] R. Penrose, A generalized inverse for matrices. *Proc. Cambridge Philos. Soc.*, 51, 406-413 (1955).
- [8] L. Shuangzhe, Matrix results on the Khatri-Rao and Tracy-Singh products. *Linear Algebra Appl.*, 289, 267-277 (1999).
- [9] D. S. Tracy, K. G. Jinadasa, Partitioned Kronecker products of matrices and applications. *Canad. J. Statist.*, 17, 107-120 (1989).
- [10] D. S. Tracy, R. P. Singh, A new matrix product and its applications in partitioned matrix differentiation. *Stat. Neerl.*, 26, 143-157 (1972).
- [11] J. Zanni, C. S. Kubrusly, A note on compactness of tensor products., *Acta Math. Univ. Comenian. (N.S.)*, 84, 59-62 (2015).
- [12] H. Zhang, F. Ding, On the Kronecker products and their applications. *J. Appl. Math.*, 2013, 8 pages (2013).

Analytic Properties of Tracy-Singh Products for Operator Matrices

Arnon Ploymukda¹, Pattrawut Chansangiam^{1*}, Wicharn Lewkeeratiyutkul²

¹Department of Mathematics, Faculty of Science,
King Mongkut's Institute of Technology Ladkrabang,
Bangkok 10520, Thailand.

²Department of Mathematics and Computer Science,
Faculty of Science, Chulalongkorn University,
Bangkok 10330, Thailand.

Abstract

We show that the Tracy-Singh product of Hilbert space operators is continuous with respect to the operator-norm topology. The Tracy-Singh product of two nonzero operators is compact if and only if both factors are compact. We provide upper and lower bounds for certain Schatten p -norms of the Tracy-Singh product of operators. It turns out that this product is continuous with respect to the topologies on norm ideals of compact operators, trace class operators, and Hilbert-Schmidt class operators. Thus the Tracy-Singh product preserves such classes of operators.

Keywords: tensor product, Tracy-Singh product, operator matrix, compact operator, Schatten p -class operator

Mathematics Subject Classifications 2010: 47A80, 47A30, 47B10.

1 Introduction

In matrix theory, one of useful matrix products is the Kronecker product. Recall that the Kronecker product of two complex matrices $A \in M_{m,n}(\mathbb{C})$ and $B \in M_{p,q}(\mathbb{C})$ is given by the block matrix

$$A \hat{\otimes} B = [a_{ij}B]_{ij} \in M_{mp,nq}(\mathbb{C}).$$

This matrix product was generalized to the Tracy-Singh product by Tracy and Singh [3]. Let $A = [A_{ij}] \in M_{m,n}(\mathbb{C})$ be a partitioned matrix with A_{ij} as the (i, j) th submatrix. Let $B = [B_{kl}] \in M_{p,q}(\mathbb{C})$ be a partitioned matrix with B_{kl} as the (k, l) th submatrix. The Tracy-Singh product of A and B is defined by

$$A \hat{\boxtimes} B = [[A_{ij} \hat{\otimes} B_{kl}]_{kl}]_{ij} \in M_{mp,nq}(\mathbb{C}).$$

*Corresponding author. Email: pattrawut.ch@kmitl.ac.th

Analytic Properties of Tracy-Singh Products for Operator Matrices

This kind of matrix product has several attractive properties and can be applied widely in statistics, econometrics and related fields; see *e.g.*, [3, 5, 7, 8, 9].

The tensor product of Hilbert space operators is a natural extension of the Kronecker product to infinite-dimensional setting. Theory of Hilbert tensor product has been continuously investigated in the literature; see, *e.g.*, [2, 4, 10]. It is well known that the tensor product is continuous with respect to the operator-norm topology. Moreover, on the norm ideals of compact operators generated by Schatten p -norm for $p = 1, 2, \infty$, the tensor product are also continuous. Recently, the tensor product for operators was generalized to the Tracy-Singh product for operator matrices acting on the direct sum of Hilbert spaces in [6]. This kind of operator product satisfies certain pleasing algebraic and order properties.

In this paper, we discuss continuity, convergence, and compactness of the Tracy-Singh product for operators in the operator-norm topology. Then we obtain relations between Tracy-Singh product and certain analytic functions. We also investigate the Tracy-Singh product on norm ideals of compact operators generated by certain Schatten p -norms. In fact, this product is continuous with respect to the Schatten p -norm for $p = 1, 2, \infty$. Estimations by such norms for Tracy-Singh products are provided. It follows that trace class operators and Hilbert-Schmidt class operators are preserved under this product.

This paper is organized as follows. In section 2, we give preliminaries on Tracy-Singh products for operators on a Hilbert space. In section 3, we establish analytic properties of the Tracy-Singh product in the operator-norm topology. We investigate the Tracy-Singh product on the norm ideals of compact operators generated by certain Schatten p -norms in Section 4.

2 Preliminaries on Tracy-Singh products for operator matrices

Throughout, let $\mathcal{H}, \mathcal{H}', \mathcal{K}$ and \mathcal{K}' be complex Hilbert spaces. When X and Y are Hilbert spaces, denote by $\mathbb{B}(X, Y)$ the Banach space of bounded linear operators from X into Y , and abbreviate $\mathbb{B}(X, X)$ to $\mathbb{B}(X)$.

In order to define the Tracy-Singh product, we have to fix the decompositions of Hilbert spaces, namely,

$$\mathcal{H} = \bigoplus_{j=1}^n \mathcal{H}_j, \quad \mathcal{H}' = \bigoplus_{i=1}^m \mathcal{H}'_i, \quad \mathcal{K} = \bigoplus_{l=1}^q \mathcal{K}_l, \quad \mathcal{K}' = \bigoplus_{k=1}^p \mathcal{K}'_k$$

where each $\mathcal{H}_j, \mathcal{H}'_i, \mathcal{K}_l, \mathcal{K}'_k$ are Hilbert spaces. For each $j = 1, \dots, n$ and $l = 1, \dots, q$, let $E_j : \mathcal{H}_j \rightarrow \mathcal{H}$ and $F_l : \mathcal{K}_l \rightarrow \mathcal{K}$ be the canonical embeddings. For each $i = 1, \dots, m$ and $k = 1, \dots, p$, let P'_i and Q'_k be the orthogonal projections. Thus, each operator $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ and B in $\mathbb{B}(\mathcal{K}, \mathcal{K}')$ can be expressed uniquely as operator matrices

$$A = [A_{ij}]_{i,j=1}^{m,n} \quad \text{and} \quad B = [B_{kl}]_{k,l=1}^{p,q}$$

A. Ploymukda, P. Chansangiam, Wicharn Lewkeeratiyutkul

where $A_{ij} = P'_i A E_j : \mathcal{H}_j \rightarrow \mathcal{H}'_i$ and $B_{kl} = Q'_k B F_l : \mathcal{K}_l \rightarrow \mathcal{K}'_k$ for each i, j, k, l . We define the *Tracy-Singh product* of A and B to be a bounded linear operator from $\bigoplus_{j,l=1}^{n,q} \mathcal{H}_j \otimes \mathcal{K}_l$ to $\bigoplus_{i,k=1}^{m,p} \mathcal{H}'_i \otimes \mathcal{K}'_k$ represented in the block-matrix form as follows:

$$A \boxtimes B = \left[[A_{ij} \otimes B_{kl}]_{kl} \right]_{ij}.$$

When $m = n = p = q = 1$, the Tracy-Singh product $A \boxtimes B$ becomes the tensor product $A \otimes B$.

Lemma 1 ([6]). *Fundamental properties of the Tracy-Singh product for operators are listed below (provided that each term is well-defined):*

1. *The map $(A, B) \mapsto A \boxtimes B$ is bilinear.*
2. *Compatibility with adjoints: $(A \boxtimes B)^* = A^* \boxtimes B^*$.*
3. *Mixed-product property: $(A \boxtimes B)(C \boxtimes D) = AC \boxtimes BD$.*
4. *Compatibility with powers: $(A \boxtimes B)^r = A^r \boxtimes B^r$ for any $r \in \mathbb{N}$.*
5. *Compatibility with inverses: if A and B are invertible, then $A \boxtimes B$ is invertible with $(A \boxtimes B)^{-1} = A^{-1} \boxtimes B^{-1}$.*
6. *Positivity: if $A \geq 0$ and $B \geq 0$, then $A \boxtimes B \geq 0$.*
7. *Strictly positivity: if $A > 0$ and $B > 0$, then $A \boxtimes B > 0$.*
8. *If A and B are partial isometries, then so is $A \boxtimes B$. Recall that an operator T is a partial isometry if and only if the restriction of T to a closed subspace is an isometry.*

3 Analytic properties of the Tracy-Singh product

In this section, we establish some analytic properties of the Tracy-Singh product involving operator norms. These properties involve continuity, convergence, norm estimates, and certain analytic functions. We denote the operator norm by $\|\cdot\|_\infty$.

In order to discuss the continuity of the Tracy-Singh product, recall the following bounds for the operator norm of operator matrices.

Lemma 2 ([1]). *Let $A = [A_{ij}]_{i,j=1}^{n,n} \in \mathbb{B}(\mathcal{H})$ be an operator matrix. Then*

$$n^{-2} \sum_{i,j=1}^n \|A_{ij}\|_\infty^2 \leq \|A\|_\infty^2 \leq \sum_{i,j=1}^n \|A_{ij}\|_\infty^2. \tag{1}$$

Lemma 3. *Let $A = [A_{ij}]_{i,j=1}^{n,n} \in \mathbb{B}(\mathcal{H})$ be an operator matrix and let $(A_r)_{r=1}^\infty$ be a sequence in $\mathbb{B}(\mathcal{H})$ where $A_r = [A_{ij}^{(r)}]_{i,j=1}^{n,n}$ for each $r \in \mathbb{N}$. Then $A_r \rightarrow A$ if and only if $A_{ij}^{(r)} \rightarrow A_{ij}$ for all $i, j = 1, \dots, n$.*

Analytic Properties of Tracy-Singh Products for Operator Matrices

Proof. It is a direct consequence of Lemma 2. □

The next theorem explains that the Tracy-Singh product is (jointly) continuous with respect to the topology induced by the operator norm.

Theorem 4. *Let $A = [A_{ij}] \in \mathbb{B}(\mathcal{H})$ and $B = [B_{kl}] \in \mathbb{B}(\mathcal{K})$ be operator matrices, and let $(A_r)_{r=1}^\infty$ and $(B_r)_{r=1}^\infty$ be sequences in $\mathbb{B}(\mathcal{H})$ and $\mathbb{B}(\mathcal{K})$, respectively. If $A_r \rightarrow A$ and $B_r \rightarrow B$, then $A_r \boxtimes B_r \rightarrow A \boxtimes B$.*

Proof. Suppose that $A_r \rightarrow A$ and $B_r \rightarrow B$. By Lemma 3, we have $A_{ij}^{(r)} \rightarrow A_{ij}$ and $B_{kl}^{(r)} \rightarrow B_{kl}$ for each i, j, k, l . Since the tensor product is continuous, we have

$$A_{ij}^{(r)} \otimes B_{kl}^{(r)} \rightarrow A_{ij} \otimes B_{kl}$$

for each i, j, k, l . It follows that $A_r \boxtimes B_r \rightarrow A \boxtimes B$ by Lemma 3. □

The next theorem provides upper/lower bounds for the operator norm of the Tracy-Singh product.

Theorem 5. *For any operator matrices $A = [A_{ij}]_{i,j=1}^{n,n} \in \mathbb{B}(\mathcal{H})$ and $B = [A_{kl}]_{k,l=1}^{q,q} \in \mathbb{B}(\mathcal{K})$, we have*

$$\frac{1}{nq} \|A\|_\infty \|B\|_\infty \leq \|A \boxtimes B\|_\infty \leq nq \|A\|_\infty \|B\|_\infty. \tag{2}$$

Proof. It follows from Lemma 2 that

$$\begin{aligned} \|A \boxtimes B\|_\infty^2 &\leq \sum_{k,l} \sum_{i,j} \|A_{ij} \otimes B_{kl}\|_\infty^2 = \sum_{k,l} \sum_{i,j} \|A_{ij}\|_\infty^2 \|B_{kl}\|_\infty^2 \\ &= \left(\sum_{i,j} \|A_{ij}\|_\infty^2 \right) \left(\sum_{k,l} \|B_{kl}\|_\infty^2 \right) \leq (nq)^2 \|A\|_\infty^2 \|B\|_\infty^2. \end{aligned}$$

We also have

$$\begin{aligned} \|A \boxtimes B\|_\infty^2 &\geq (nq)^{-2} \sum_{k,l} \sum_{i,j} \|A_{ij} \otimes B_{kl}\|_\infty^2 = (nq)^{-2} \sum_{k,l} \sum_{i,j} \|A_{ij}\|_\infty^2 \|B_{kl}\|_\infty^2 \\ &= (nq)^{-2} \left(\sum_{i,j} \|A_{ij}\|_\infty^2 \right) \left(\sum_{k,l} \|B_{kl}\|_\infty^2 \right) \geq (nq)^{-2} \|A\|_\infty^2 \|B\|_\infty^2. \end{aligned}$$

Hence, we obtain the bound (2). □

Theorem 6. *Let $A \in \mathbb{B}(\mathcal{H})$.*

(i) *If f is an analytic function on a region containing the spectra of A and $I \boxtimes A$, then*

$$f(I \boxtimes A) = I \boxtimes f(A). \tag{3}$$

(ii) *If f is an analytic function on a region containing the spectra of A and $A \boxtimes I$, then*

$$f(A \boxtimes I) = f(A) \boxtimes I. \tag{4}$$

Proof. (i) Since f is analytic on spectra of A and $I \boxtimes A$, we have the Taylor series expansion

$$f(z) = \sum_{r=0}^{\infty} \alpha_r z^r.$$

It follows that

$$f(A) = \sum_{r=0}^{\infty} \alpha_r A^r \quad \text{and} \quad f(I \boxtimes A) = \sum_{r=0}^{\infty} \alpha_r (I \boxtimes A)^r.$$

Making use of the bilinearity of Tracy-Singh product and Theorem 4 yields

$$\begin{aligned} f(I \boxtimes A) &= \sum_{r=0}^{\infty} \alpha_r (I \boxtimes A^r) = \sum_{r=0}^{\infty} (I \boxtimes \alpha_r A^r) \\ &= I \boxtimes \sum_{r=0}^{\infty} \alpha_r A^r = I \boxtimes f(A). \end{aligned}$$

Similarly, we obtain the assertion (ii). □

Theorem 7. *Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ be positive operators. For any $\alpha > 0$, we have*

$$(A \boxtimes B)^\alpha = A^\alpha \boxtimes B^\alpha. \tag{5}$$

Proof. First, note that $A \boxtimes B$ is positive by property (6) of Lemma 1. It follows from the property (4) in Lemma 1 that for any $r, s \in \mathbb{N}$,

$$(A^{\frac{r}{s}} \boxtimes B^{\frac{r}{s}})^s = A^r \boxtimes B^r = (A \boxtimes B)^r,$$

and thus $(A \boxtimes B)^{\frac{r}{s}} = A^{\frac{r}{s}} \boxtimes B^{\frac{r}{s}}$. Now, for $\alpha > 0$, there is a sequence (q_n) of positive rational numbers such that $q_n \rightarrow \alpha$. It follows from the previous claim and the continuity of Tracy-Singh product (Theorem 4) that

$$\begin{aligned} (A \boxtimes B)^\alpha &= \lim_{n \rightarrow \infty} (A \boxtimes B)^{q_n} = \lim_{n \rightarrow \infty} A^{q_n} \boxtimes B^{q_n} \\ &= \lim_{n \rightarrow \infty} A^{q_n} \boxtimes \lim_{n \rightarrow \infty} B^{q_n} = A^\alpha \boxtimes B^\alpha. \end{aligned} \tag{□}$$

Corollary 8. *Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ be strictly positive operators. For any real number α , we have*

$$(A \boxtimes B)^\alpha = A^\alpha \boxtimes B^\alpha. \tag{6}$$

Proof. Note that $A \boxtimes B$ is strictly positive by property (7) of Lemma 1. For $\alpha < 0$, it follows from Theorem 7 and the property (5) in Lemma 1 that

$$\begin{aligned} (A \boxtimes B)^\alpha &= [(A \boxtimes B)^{-1}]^{-\alpha} = (A^{-1} \boxtimes B^{-1})^{-\alpha} \\ &= (A^{-1})^{-\alpha} \boxtimes (B^{-1})^{-\alpha} = A^\alpha \boxtimes B^\alpha. \end{aligned} \tag{□}$$

Corollary 9. *Let $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ and $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$. Then*

$$|A \boxtimes B| = |A| \boxtimes |B|. \tag{7}$$

Proof. Applying Lemma 1 and property (5), we get

$$\begin{aligned} |A \boxtimes B| &= [(A \boxtimes B)^*(A \boxtimes B)]^{\frac{1}{2}} = [(A^* \boxtimes B^*)(A \boxtimes B)]^{\frac{1}{2}} \\ &= (A^*A \boxtimes B^*B)^{\frac{1}{2}} = (A^*A)^{\frac{1}{2}} \boxtimes (B^*B)^{\frac{1}{2}} = |A| \boxtimes |B|. \quad \square \end{aligned}$$

Recall the polar decomposition theorem: for any $A \in \mathbb{B}(\mathcal{H}, \mathcal{K})$, there exists a partial isometry U such that $A = U|A|$. The next result is a polar decomposition for the Tracy-Singh product of operators.

Corollary 10. *Let $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ and $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$. If $A = U|A|$ and $B = V|B|$ are polar decompositions of A and B , respectively, then a polar decomposition of $A \boxtimes B$ is given by*

$$A \boxtimes B = (U \boxtimes V)|A \boxtimes B|. \tag{8}$$

Proof. Let U and V be partial isometries such that $A = U|A|$ and $B = V|B|$. It follows from Lemma 1(3) and Corollary 9 that

$$A \boxtimes B = U|A| \boxtimes V|B| = (U \boxtimes V)(|A| \boxtimes |B|) = (U \boxtimes V)|A \boxtimes B|.$$

Note that $U \boxtimes V$ is also a partial isometry, according to property (8) in Lemma 1. Hence, the decomposition (8) is a polar one. \square

4 Tracy-Singh products on norm ideals of compact operators

In this section, we investigate the Tracy-Singh product on norm ideals of $\mathbb{B}(\mathcal{H})$. Recall that any proper ideal of $\mathbb{B}(\mathcal{H})$ is contained in the ideal \mathcal{S}_∞ of compact operators. For any compact operator $A \in \mathbb{B}(\mathcal{H})$, let $(s_i(A))_{i=1}^\infty$ be the sequence of decreasingly-ordered singular values of A (i.e. eigenvalues of $|A|$). For each $1 \leq p < \infty$, the Schatten p -norm of A is defined by

$$\|A\|_p = \left(\sum_{i=1}^\infty s_i^p(A) \right)^{1/p}.$$

If $\|A\|_p$ is finite, we say that A is a Schatten p -class operator. The Schatten ∞ -norm is just the operator norm. For each $1 \leq p \leq \infty$, let \mathcal{S}_p be the Schatten p -class operators. In particular, \mathcal{S}_1 and \mathcal{S}_2 are the trace class and the Hilbert-Schmidt class, respectively. Each Schatten p -norm induces a norm ideal of $\mathbb{B}(\mathcal{H})$ and this ideal is closed under the topology generated by this norm.

Lemma 11. *Let $A = [A_{ij}] \in \mathbb{B}(\mathcal{H})$ be an operator matrix. Then A is compact if and only if A_{ij} is compact for all i, j .*

A. Ploymukda, P. Chansangiam, Wicharn Lewkeeratiyutkul

Proof. If A is compact, then $A_{ij} = P'_i A E_j$ is also compact for each i, j due to the fact that \mathcal{S}_∞ is an ideal of $\mathbb{B}(\mathcal{H})$. Conversely, suppose that A_{ij} is compact for all i, j . Recall that a bounded linear operator is compact if and only if it maps a bounded sequence into a sequence having a convergent subsequence. Let $(x_r)_{r=1}^\infty$ be a bounded sequence in $\mathcal{H} = \bigoplus_{i=1}^n \mathcal{H}_i$. Write $x_r = [x_r^{(1)} \ x_r^{(2)} \ \dots \ x_r^{(n)}]^T \in \bigoplus_{i=1}^n \mathcal{H}_i$ for each $r \in \mathbb{N}$. Consider

$$Ax_r = \begin{bmatrix} A_{11} & \dots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \dots & A_{nn} \end{bmatrix} \begin{bmatrix} x_r^{(1)} \\ \vdots \\ x_r^{(n)} \end{bmatrix} = \begin{bmatrix} A_{11}x_r^{(1)} + \dots + A_{1n}x_r^{(n)} \\ \vdots \\ A_{n1}x_r^{(1)} + \dots + A_{nn}x_r^{(n)} \end{bmatrix}.$$

For each $l = 1, 2, \dots, n$, since $(x_r^{(l)})_{r=1}^\infty$ is bounded, the sequence $(A_{ij}x_r^{(l)})_{r=1}^\infty$ has a convergent subsequence, namely, $(A_{ij}x_{r_k}^{(l)})_{k=1}^\infty$. Hence,

$$\begin{bmatrix} A_{11}x_{r_k}^{(1)} + \dots + A_{1n}x_{r_k}^{(n)} \\ \vdots \\ A_{n1}x_{r_k}^{(1)} + \dots + A_{nn}x_{r_k}^{(n)} \end{bmatrix}$$

is a desired convergent subsequence of $(Ax_r)_{r=1}^\infty$. □

Lemma 12 ([1]). *Let $A = [A_{ij}]_{i,j=1}^{n,n}$ be an operator matrix in the Schatten p -class.*

(i) *For $1 \leq p \leq 2$, we have*

$$\sum_{i,j=1}^n \|A_{ij}\|_p^2 \leq \|A\|_p^2 \leq n^{4/p-2} \sum_{i,j=1}^n \|A_{ij}\|_p^2. \tag{9}$$

(ii) *For $2 \leq p < \infty$, we have*

$$n^{4/p-2} \sum_{i,j=1}^n \|A_{ij}\|_p^2 \leq \|A\|_p^2 \leq \sum_{i,j=1}^n \|A_{ij}\|_p^2. \tag{10}$$

Lemma 13. *Let $1 \leq p < \infty$. An operator matrix $A = [A_{ij}] \in \mathbb{B}(\mathcal{H})$ is a Schatten p -class operator if and only if A_{ij} is a Schatten p -class operator for all i, j .*

Proof. This is a direct consequence of the norm estimations in Lemma 12. □

Lemma 14. *Let $1 \leq p \leq \infty$. Let $A = [A_{ij}]_{i,j=1}^{n,n}$ be an operator matrix in the class \mathcal{S}_p and let $(A_r)_{r=1}^\infty$ be a sequence in \mathcal{S}_p where $A_r = [A_{ij}^{(r)}]_{i,j=1}^{n,n}$ for each $r \in \mathbb{N}$. Then $A_r \rightarrow A$ in \mathcal{S}_p if and only if $A_{ij}^{(r)} \rightarrow A_{ij}$ in \mathcal{S}_p for all $i, j = 1, \dots, n$.*

Analytic Properties of Tracy-Singh Products for Operator Matrices

Proof. Lemma 13 assures that A_{ij} and $A_{ij}^{(r)}$ belong to \mathcal{S}_p for any $i, j = 1, \dots, n$ and $r \in \mathbb{N}$. Consider the case $1 \leq p \leq 2$. Suppose that $A_r \rightarrow A$ in \mathcal{S}_p . For any fixed $i, j \in \{1, \dots, n\}$, we have from the estimation (9) that

$$\|A_{ij}^{(r)} - A_{ij}\|_p^2 \leq \sum_{i,j=1}^n \|A_{ij}^{(r)} - A_{ij}\|_p^2 \leq \|A_r - A\|_p^2.$$

Hence, $A_{ij}^{(r)} \rightarrow A_{ij}$ in \mathcal{S}_p . Conversely, suppose $A_{ij}^{(r)} \rightarrow A_{ij}$ in \mathcal{S}_p for each i, j . Lemma 12 implies that

$$\|A_r - A\|_p^2 \leq n^{4/p-2} \sum_{i,j=1}^n \|A_{ij}^{(r)} - A_{ij}\|_p^2.$$

Hence, $A_r \rightarrow A$ in \mathcal{S}_p . The case $2 < p < \infty$ and the case $p = \infty$ are done by using the norm estimations (10) and (1), respectively. \square

Next, we discuss compactness of Tracy-Singh product of operators.

Lemma 15 ([10]). *Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ be nonzero operators. Then $A \otimes B$ is compact if and only if both A and B are compact.*

Theorem 16. *Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ be nonzero operator matrices. Then $A \boxtimes B$ is compact if and only if both A and B are compact.*

Proof. Write $A = [A_{ij}]$ and $B = [B_{kl}]$. For sufficiency, suppose that A and B are compact. By Lemma 11, we deduce that A_{ij} and B_{kl} are compact for all i, j, k, l . It follows from Lemma 15 that $A_{ij} \otimes B_{kl}$ is compact for all i, j, k, l . Lemma 11 ensures the compactness of $A \boxtimes B$. For necessity part, reverse the previous procedure. \square

The following theorem supplies bounds for Schatten 1-norm of the Tracy-Singh product of operators.

Theorem 17. *For any nonzero compact operator $A = [A_{ij}]_{i,j=1}^{n,n} \in \mathbb{B}(\mathcal{H})$ and $B = [A_{kl}]_{k,l=1}^{q,q} \in \mathbb{B}(\mathcal{K})$, we have*

$$\frac{1}{nq} \|A\|_1 \|B\|_1 \leq \|A \boxtimes B\|_1 \leq nq \|A\|_1 \|B\|_1. \tag{11}$$

Hence, $A \boxtimes B$ is trace-class if and only if both A and B are trace-class.

Proof. Suppose that both A and B are nonzero and compact. Then the operator $A \boxtimes B$ is compact by Theorem 16. It follows from the norm bound (9) that

$$\begin{aligned} \|A \boxtimes B\|_1^2 &\leq (nq)^2 \sum_{k,l} \sum_{i,j} \|A_{ij} \otimes B_{kl}\|_1^2 = (nq)^2 \sum_{k,l} \sum_{i,j} \|A_{ij}\|_1^2 \|B_{kl}\|_1^2 \\ &= (nq)^2 \left(\sum_{i,j} \|A_{ij}\|_1^2 \right) \left(\sum_{k,l} \|B_{kl}\|_1^2 \right) \leq (nq)^2 \|A\|_1^2 \|B\|_1^2. \end{aligned}$$

A. Ploymukda, P. Chansangiam, Wicharn Lewkeeratiyutkul

We also have

$$\begin{aligned} \|A \boxtimes B\|_1^2 &\geq \sum_{k,l} \sum_{i,j} \|A_{ij} \otimes B_{kl}\|_1^2 = \sum_{k,l} \sum_{i,j} \|A_{ij}\|_1^2 \|B_{kl}\|_1^2 \\ &= \left(\sum_{i,j} \|A_{ij}\|_1^2 \right) \left(\sum_{k,l} \|B_{kl}\|_1^2 \right) \geq (nq)^{-2} \|A\|_1^2 \|B\|_1^2. \end{aligned}$$

Hence, we obtain the bound (11). □

Theorem 18. *For any nonzero compact operator matrices $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$, we have*

$$\|A \boxtimes B\|_2 = \|A\|_2 \|B\|_2. \tag{12}$$

Hence, $A \boxtimes B$ is a Hilbert-Schmidt operator if and only if both A and B are Hilbert-Schmidt operators.

Proof. Since both A and B are nonzero and compact, the operator $A \boxtimes B$ is compact by Theorem 16. Write $A = [A_{ij}]$ and $B = [B_{kl}]$. Then by Lemma 12(ii), we have

$$\begin{aligned} \|A \boxtimes B\|_2^2 &= \sum_{k,l} \sum_{i,j} \|A_{ij} \otimes B_{kl}\|_2^2 = \sum_{k,l} \sum_{i,j} \|A_{ij}\|_2^2 \|B_{kl}\|_2^2 \\ &= \left(\sum_{i,j} \|A_{ij}\|_2^2 \right) \left(\sum_{k,l} \|B_{kl}\|_2^2 \right) = \|A\|_2^2 \|B\|_2^2. \end{aligned}$$

Hence, we get the multiplicative property (12). □

The final result asserts that the Tracy-Singh product is continuous with respect to the topology induced by the Schatten p -norm for each $p \in \{1, 2, \infty\}$.

Theorem 19. *Let $p \in \{1, 2, \infty\}$. If a sequence $(A_r)_{r=1}^\infty$ converges to A and a sequence $(B_r)_{r=1}^\infty$ converges to B in the norm ideal \mathcal{S}_p , then $A_r \boxtimes B_r$ converges to $A \boxtimes B$ in \mathcal{S}_p .*

Proof. Write $A = [A_{ij}]$ and $B = [B_{kl}]$. In the viewpoint of Lemma 14, it suffices to show that $A_{ij}^{(r)} \otimes B_{kl}^{(r)} \rightarrow A_{ij} \otimes B_{kl}$ in \mathcal{S}_p for all i, j, k, l . Since $A_r \rightarrow A$ and $B_r \rightarrow B$ in \mathcal{S}_p , we have by Lemma 14 that $A_{ij}^{(r)} \rightarrow A_{ij}$ and $B_{kl}^{(r)} \rightarrow B_{kl}$ for all i, j, k, l . It follows that

$$\begin{aligned} \|A_{ij}^{(r)} \otimes B_{kl}^{(r)} - A_{ij} \otimes B_{kl}\|_p &= \|A_{ij}^{(r)} \otimes B_{kl}^{(r)} - A_{ij}^{(r)} \otimes B_{kl} + A_{ij}^{(r)} \otimes B_{kl} - A_{ij} \otimes B_{kl}\|_p \\ &\leq \|A_{ij}^{(r)} \otimes (B_{kl}^{(r)} - B_{kl})\|_p + \|(A_{ij}^{(r)} - A_{ij}) \otimes B_{kl}\|_p \\ &= \|A_{ij}^{(r)}\|_p \|B_{kl}^{(r)} - B_{kl}\|_p + \|A_{ij}^{(r)} - A_{ij}\|_p \|B_{kl}\|_p \\ &\rightarrow \|A_{ij}\|_p \cdot 0 + 0 \cdot \|B_{kl}\|_p = 0. \end{aligned}$$

Hence, $A_{ij}^{(r)} \otimes B_{kl}^{(r)} \rightarrow A_{ij} \otimes B_{kl}$ in \mathcal{S}_p for all i, j, k, l . □

Acknowledgement. This research was supported by Thailand Research Fund (grant no. MRG6080102).

References

- [1] R. Bhatia, F. Kittaneh, Norm inequalities for partitioned operators and an application. *Math. Ann.*, 287, 719-726 (1990).
- [2] A. Brown, C. Pearcy, Spectra of tensor products of operators. *Proc. Amer. Math. Soc.*, 17, 162-166 (1966).
- [3] R. H. Koning, H. Neudecker, T. Wansbeek, Block Kronecker products and the vecb operator. *Linear Algebra Appl.*, 149, 165-184 (1991).
- [4] C. S. Kubrusly, P. C.M. Vieira, Convergence and decomposition for tensor products of Hilbert space operators. *Oper. Matrices*, 2, 407-416 (2008).
- [5] J.R. Magnus, H. Neudecker, Matrix differential calculus with applications to simple, Hadamard, and Kronecker products. *J. Math. Psych.*, 29, 474-492 (1985).
- [6] A. Ploymukda, P. Chansangiam, W. Lewkeeratiyutkul, Algebraic and order properties of Tracy-Singh products for operator matrices. *J. Comput. Anal. Appl.*, 9 pages, to appear in 2018.
- [7] L. Shuangzhe, Matrix results on the Khatri-Rao and Tracy-Singh products. *Linear Algebra Appl.* 289, 267-277 (1999).
- [8] D. S. Tracy, K. G. Jinadasa, Partitioned Kronecker products of matrices and applications. *Canad. J. Statist.*, 17, 107-120 (1989).
- [9] D. S. Tracy, R. P. Singh, A new matrix product and its applications in partitioned matrix differentiation. *Stat. Neerl.*, 26, 143-157 (1972).
- [10] J. Zanni, C. S. Kubrusly, A note on compactness of tensor products. *Acta Math. Univ. Comenian. (N.S.)*, 84, 59-62 (2015).

ON THE RADIAL DISTRIBUTION OF JULIA SET OF SOLUTIONS OF $f'' + Af' + Bf = 0$

JIANREN LONG

ABSTRACT. The paper is devoted to study the dynamical properties of solutions of $f'' + A(z)f' + B(z)f = 0$, where $A(z)$ is nontrivial solution of $w'' + P(z)w = 0$, $P(z)$ is a polynomial, $B(z)$ is a transcendental entire function of lower order less than $\frac{1}{2}$. The lower bound of the size of the radial distribution of Julia sets of infinite order solutions of the equation are obtained. Another proof of the result in [9] is discussed in which the modified Phragmén-Lindelöf principle is needed.

1. INTRODUCTION AND MAIN RESULTS

For a function meromorphic f in the complex plane \mathbb{C} , the order of growth, lower order of growth and the convergence exponent of zero-sequence of f are given respectively by

$$\rho(f) = \limsup_{r \rightarrow \infty} \frac{\log^+ T(r, f)}{\log r},$$

$$\mu(f) = \liminf_{r \rightarrow \infty} \frac{\log^+ T(r, f)}{\log r}$$

and

$$\lambda(f) = \limsup_{r \rightarrow \infty} \frac{\log^+ N(r, \frac{1}{f})}{\log r}.$$

In what follows, we assume that the reader is familiar with standard notation and basic results in Nevanlinna theory of meromorphic functions, such as $T(r, f)$, $m(r, f)$ and $N(r, f)$, see [10, 12, 23] for more details.

We define the n th iterate of meromorphic function f as follows:

$$f^0(z) = z, f^1(z) = f(z), \dots, f^n(z) = f(f^{n-1}(z)), \quad n \in \mathbb{N}, \quad n \geq 2.$$

The Fatou set $F(f)$ of transcendental meromorphic function f is the subset of \mathbb{C} where the iterates $\{f^n(z)\}_{n=1}^\infty$ of f form a normal family, and its complement $J(f) = \mathbb{C} \setminus F(f)$ is called the Julia set of f . It is well known that $F(f)$ is open and completely invariant under f , and $J(f)$ is closed and

2010 *Mathematics Subject Classification.* 34M10; 37F10; 30D35.

Key words and phrases. Complex differential equation, Entire function, Infinite order, Radial distribution, Julia set.

non-empty, see [4]. We also need the following notations and definitions. For $\alpha < \beta$ and $r \in (0, \infty)$, set

$$\begin{aligned} S(\alpha, \beta) &= \{z : \alpha < \arg z < \beta\}, \\ S(\alpha, \beta, r) &= \{z : |z| < r, \alpha < \arg z < \beta\}, \\ S(r, \alpha, \beta) &= \{z : |z| > r, \alpha < \arg z < \beta\}. \end{aligned}$$

Let \bar{F} denotes the closure of $F \subset \mathbb{C}$. Given $\theta \in [0, 2\pi)$, if $S(\theta-\varepsilon, \theta+\varepsilon) \cap J(f)$ is unbounded for any small $\varepsilon > 0$, then the ray $\arg z = \theta$ from the origin is called the radial distribution of $J(f)$. Define

$$\Delta(f) = \{\theta \in [0, 2\pi) : \arg z = \theta \text{ is the radial distribution of } J(f)\}.$$

Obviously, $\Delta(f)$ is closed and so measurable. Let $m(\Delta(f))$ denotes the linear measure of $\Delta(f)$. What can we say according to $m(\Delta(f))$ of any meromorphic function f in \mathbb{C} ? It is interesting topic, many results have been obtained by several authors. Baker [1] considered the radial distribution of the Julia set and constructed an entire function with infinite lower order whose Julia set lies in a horizontal trip. Qiao [16] proved that if f is a transcendental entire function of finite lower order, then

$$m(\Delta(f)) \begin{cases} = 2\pi, & \mu(f) < 1/2, \\ \geq \pi/\mu(f), & \mu(f) \geq 1/2. \end{cases}$$

Later, some observations on radial distribution of the Julia sets of transcendental meromorphic functions with finite lower order were made; see, for example, [25] and [17]. It seems that there are few work done on the case of meromorphic functions of infinite order. Recently, Huang-Wang [8, 9] studied the radial distribution of the Julia sets of entire functions of infinite lower order by using the tool of differential equations, i.e., for any nontrivial solutions f of (1.2) below, a lower bound of $m(\Delta(f))$ is obtained. Zhang-Wang-Yang [24] also studied the radial distribution of the Julia sets of entire solutions f of (1.2), a lower bound of $m(\Delta(f))$ is obtained when the coefficient $A(z)$ and $B(z)$ satisfy different conditions with [8, 9]. Our idea of this paper comes from [24], a new lower bound of $m(\Delta(f))$ is found when $A(z)$ and $B(z)$ satisfy new conditions which are different with the conditions of [24].

Our starting point is a result which is related to the growth of solutions of (1.2).

Theorem 1.1 ([14]). *Let $A(z)$ be a nontrivial solution of the equation*

$$(1.1) \quad w'' + P(z)w = 0,$$

where $P(z) = a_n z^n + \dots + a_0, a_n \neq 0$, and let $\lambda(A) < \rho(A)$. Let $B(z)$ be a transcendental entire function with $\mu(B) < \frac{1}{2}$. Then every nontrivial solution of the equation

$$(1.2) \quad f'' + A(z)f' + B(z)f = 0$$

is of infinite order.

The following result shows that $m(\Delta(f))$ has a lower bound when $A(z)$ and $B(z)$ satisfy the conditions of Theorem 1.1.

Theorem 1.2. *Let $A(z)$ and $B(z)$ be given as in Theorem 1.1. Then every nontrivial solution f of (1.2) satisfies $m(\Delta(f)) \geq \frac{2\pi}{n+2}$.*

To state the following results, the definition of accumulation lines of zero-sequence is needed, which can be found in [13, 18, 21, 22].

Definition 1.3. Let f be a meromorphic function in \mathbb{C} , and let $\arg z = \theta \in [0, 2\pi)$ be a ray from the origin. We denote, for each $\varepsilon > 0$, the convergence exponent of zero-sequence of f in the region $S(\theta - \varepsilon, \theta + \varepsilon, r)$ by $\lambda_{\theta, \varepsilon}(f)$ and by $\lambda_{\theta}(f) = \lim_{\varepsilon \rightarrow 0^+} \lambda_{\theta, \varepsilon}(f)$. That is,

$$\lambda_{\theta}(f) = \lim_{\varepsilon \rightarrow 0^+} \limsup_{r \rightarrow \infty} \frac{\log^+ n_{\theta - \varepsilon, \theta + \varepsilon}(r, 0, f)}{\log r},$$

where $n_{\theta - \varepsilon, \theta + \varepsilon}(r, 0, f)$ is the number of zeros of f , counting multiplicity in $S(\theta - \varepsilon, \theta + \varepsilon, r)$.

The ray $\arg z = \theta$ is called an accumulation line of the zero-sequence of f if $\lambda_{\theta}(f) = \rho(f)$. By Lemma 2.1 below, we know that the number of accumulation lines of zero-sequence of nontrivial solutions of (1.1) less than or equal to $n + 2$ and the set of the accumulation lines of zero-sequence of nontrivial solutions of (1.1) is the subset of $\{\arg z = \theta_j, 0 \leq j \leq n + 1\}$, where $\theta_j = \frac{2j\pi - \arg(a_n)}{n+2}$. Let w be a nontrivial solution of (1.1), where $P(z) = a_n z^n + \dots + a_0$ is a polynomial of degree $n \geq 1$, let $p(w)$ denotes the number of the rays $\arg z = \theta_j, j = 0, 1, \dots, n + 1$, which are not accumulation lines of zero-sequence of w .

Remark 1.4. It follows from Lemma 2.1 that $p(w)$ must be an even number for every nontrivial solution w of (1.1).

Theorem 1.5. *Let $A(z)$ be a nontrivial solution of (1.1), and the number of accumulation lines of zero-sequence of $A(z)$ strictly less than $n + 2$. Let $B(z)$ be a transcendental entire function with $\mu(B) < \frac{1}{2}$. Then every nontrivial solution f of (1.2) satisfies $\rho(f) = \infty$ and $m(\Delta(f)) \geq \frac{2\pi}{n+2}$.*

Furthermore, we study the radial distribution of Julia set of the derivatives of nontrivial solutions of (1.2).

Theorem 1.6. *Let $A(z)$ and $B(z)$ be given as in Theorem 1.1. Then every nontrivial solution f of (1.2) satisfies $m(\Delta(f) \cap \Delta(f^{(k)})) \geq \frac{2\pi}{n+2}$, where $k \geq 1$ is an integer.*

By using similar reasoning in proving Theorems 1.5 and 1.6, we have the following result.

Theorem 1.7. *Let $A(z)$ and $B(z)$ be given as in Theorem 1.5. Then every nontrivial solution f of (1.2) satisfies $m(\Delta(f) \cap \Delta(f^{(k)})) \geq \frac{2\pi}{n+2}$, where $k \geq 1$ is an integer.*

Applying Theorems 1.6 and 1.7, we immediately obtain the following corollaries.

Corollary 1.8. *Let $A(z)$ and $B(z)$ be given as in Theorem 1.6. Then $m(\Delta(f^{(k)})) \geq \frac{2\pi}{n+2}$ for every nontrivial solution f of (1.2), where $k \geq 1$ is an integer.*

Corollary 1.9. *Let $A(z)$ and $B(z)$ be given as in Theorem 1.7. Then $m(\Delta(f^{(k)})) \geq \frac{2\pi}{n+2}$ for every nontrivial solution f of (1.2), where $k \geq 1$ is an integer.*

Obviously, we can obtain Theorems 1.2 and 1.5 from Theorems 1.6 and 1.7, however, we need the results of Theorems 1.2 and 1.5 in proving Theorems 1.6 and 1.7. So we will give the proofs of Theorems 1.2 and 1.5 in Sections 3 and 4, respectively.

2. AUXILIARY RESULTS

In this section, we will give some auxiliary results for proving our theorems. To this end, we introduce following notations. Let f be an entire function of order $\rho(f) \in (0, \infty)$. For simplicity, set $\rho(f) = \rho$ and $S = S(\alpha, \beta)$. If for any $\theta \in (\alpha, \beta)$,

$$\lim_{r \rightarrow \infty} \frac{\log \log |f(re^{i\theta})|}{\log r} = \rho,$$

then we say that f blows up exponentially in \overline{S} . If for any $\theta \in (\alpha, \beta)$,

$$\lim_{r \rightarrow \infty} \frac{\log \log |f(re^{i\theta})|^{-1}}{\log r} = \rho,$$

then we say that f decays to zero exponentially in \overline{S} .

The following lemma, originally due to Hille [11, Chapter 7.4], which also be found in [6, 19], plays an important role in proving our results.

Lemma 2.1. *Let w be a nontrivial solution of (1.1), where $P(z) = a_n z^n + \dots + a_0$, $a_n \neq 0$. Set $\theta_j = \frac{2j\pi - \arg(a_n)}{n+2}$ and $S_j = S(\theta_j, \theta_{j+1})$, where $j = 0, 1, 2, \dots, n+1$ and $\theta_{n+2} = \theta_0 + 2\pi$. Then w has the following properties.*

- (i) *In each sector S_j , w either blows up or decays to zero exponentially.*
- (ii) *If, for some j , w decays to zero in S_j , then it must blow up in S_{j-1} and S_{j+1} . However, it is possible for w to blow up in many adjacent sectors.*
- (iii) *If w decays to zero in S_j , then w has at most finitely many zeros in any closed subsector within $S_{j-1} \cup \overline{S_j} \cup S_{j+1}$.*
- (iv) *If w blows up in S_{j-1} and S_j , then for each $\varepsilon > 0$, w has infinitely many zeros in each sector $\overline{S}(\theta_j - \varepsilon, \theta_j + \varepsilon)$, and furthermore, as $r \rightarrow \infty$,*

$$n(\overline{S}(\theta_j - \varepsilon, \theta_j + \varepsilon, r), 0, w) = (1 + o(1)) \frac{2\sqrt{|a_n|}}{\pi(n+2)} r^{\frac{n+2}{2}},$$

where $n(\overline{S}(\theta_j - \varepsilon, \theta_j + \varepsilon, r), 0, w)$ is the number of zeros of w , counting multiplicity in $\overline{S}(\theta_j - \varepsilon, \theta_j + \varepsilon, r)$.

Before stating the next lemma, for $E \subset [0, \infty)$, we define the Lebesgue linear measure of E by $m(E) = \int_E dt$, and the logarithmic measure of $F \subset [1, \infty)$ is $m_1(F) = \int_F \frac{dt}{t}$. The upper and lower logarithmic density of $F \subset [1, \infty)$ are given by

$$\overline{\log \text{dens}}(F) = \limsup_{r \rightarrow \infty} \frac{m_1(F \cap [1, r])}{\log r}$$

and

$$\underline{\log \text{dens}}(F) = \liminf_{r \rightarrow \infty} \frac{m_1(F \cap [1, r])}{\log r},$$

respectively.

The following result is due to Barry [3].

Lemma 2.2. *Let f be an entire function with $0 \leq \mu(f) < 1$, and denote $m(r) = \inf_{|z|=r} \log |f(z)|$ and $M(r) = \sup_{|z|=r} \log |f(z)|$. Then, for every $\alpha \in (\mu(f), 1)$,*

$$\overline{\log \text{dens}}(\{r \in [1, \infty) : m(r) > M(r) \cos \pi\alpha\}) \geq 1 - \frac{\mu(f)}{\alpha}.$$

We say that an open set is hyperbolic if it has at least three boundary points in $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Let W be a hyperbolic open set in \mathbb{C} . For an $a \in \overline{\mathbb{C}} \setminus W$, define

$$C_W(a) = \inf\{\lambda_W(z)|z - a| : z \in W\},$$

where $\lambda_W(z)$ is the hyperbolic density on W . We know that if every component of W is simply connected, then $C_W(a) \geq \frac{1}{2}$. The following result was proved in [25, Lemma 2.2].

Lemma 2.3. *Let f be an analytic in $S(r_0, \theta_1, \theta_2)$, let U be a hyperbolic domain and $f : S(r_0, \theta_1, \theta_2) \rightarrow U$. If there exists a point $a \in \partial U \setminus \{\infty\}$ such that $C_U(a) > 0$, then there exists a constant $l > 0$ such that, for sufficiently small $\varepsilon > 0$, one has*

$$|f(z)| = O(|z|^l), \quad z \in S(r_0, \theta_1 + \varepsilon, \theta_2 - \varepsilon), \quad |z| \rightarrow \infty.$$

The following lemma is related to the Nevanlinna theory in an angular domain. To the end, we recall some notations and properties of Nevanlinna theory in an angular domain, see [5] for more details. Let f be meromorphic in $\overline{S}(\alpha, \beta)$, where $0 < \alpha < \beta \leq 2\pi$. Then we have

$$\begin{aligned} A_{\alpha,\beta}(r, f) &= \frac{\omega}{\pi} \int_1^r \left(\frac{1}{t^\omega} - \frac{t^\omega}{r^{2\omega}}\right) \{ \log^+ |f(te^{i\alpha})| + \log^+ |f(te^{i\beta})| \} \frac{dt}{t}, \\ B_{\alpha,\beta}(r, f) &= \frac{2\omega}{\pi r^\omega} \int_\alpha^\beta \log^+ |f(te^{i\varphi})| \sin \omega(\varphi - \alpha) d\varphi, \\ C_{\alpha,\beta}(r, f) &= 2 \sum_{1 < |b_n| < r} \left(\frac{1}{|b_n|^\omega} - \frac{|b_n|^\omega}{r^{2\omega}} \right) \sin \omega(\theta_n - \alpha), \end{aligned}$$

where $\omega = \frac{\pi}{\beta - \alpha}$ and $b_n = |b_n|e^{i\theta_n}$ are poles of f in $\overline{S}(\alpha, \beta)$ appearing according to their multiplicities. The sectorial Nevanlinna characteristic is defined by

$$S_{\alpha,\beta}(r, f) = A_{\alpha,\beta}(r, f) + B_{\alpha,\beta}(r, f) + C_{\alpha,\beta}(r, f).$$

We denote the order of f in an angular domain $S(\alpha, \beta)$ by

$$\sigma_{\alpha,\beta}(f) = \limsup_{r \rightarrow \infty} \frac{\log^+ S_{\alpha,\beta}(r, f)}{\log r}.$$

The definition of an R -set is needed, which can be found in [12]. Set $B(z_n, r_n) = \{z : |z - z_n| < r_n\}$. If $\sum_{n=1}^\infty r_n < \infty$ and $z_n \rightarrow \infty$, then $\bigcup_{n=1}^\infty B(z_n, r_n)$ is called an R -set. Clearly, the $\{|z| : z \in \bigcup_{n=1}^\infty B(z_n, r_n)\}$ is of finite linear measure.

The next lemma shows an estimation for the logarithmic derivative of analytic functions in an angular domain. It is a combination of results in [15, 21] and [8, Lemma 7], which can be found in [9, Lemma 2.2].

Lemma 2.4. *Let $z = re^{i\psi}$, $r_0 + 1 < r$, and $\alpha \leq \psi \leq \beta$, where $0 < \beta - \alpha \leq 2\pi$. Suppose that $n \geq 2$ is an integer and that f is analytic in $S(r_0, \alpha, \beta)$*

with $\sigma_{\alpha,\beta}(f) < \infty$. Choose $\alpha < \alpha_1 < \beta_1 < \beta$. Then, for every $\varepsilon_j \in (0, \frac{\beta_j - \alpha_j}{2})$, $j = 1, 2, \dots, n - 1$, outside a set of linear measure zero with

$$\alpha_j = \alpha + \sum_{s=1}^{j-1} \varepsilon_s, \quad \beta_j = \beta - \sum_{s=1}^{j-1} \varepsilon_s, \quad j = 2, 3, \dots, n - 1,$$

there exist $K > 0$ and $M > 0$ only depending on $f, \varepsilon_1, \dots, \varepsilon_{n-1}$ and $S(\alpha_{n-1}, \beta_{n-1})$, and not depending on z , such that

$$\left| \frac{f'(z)}{f(z)} \right| \leq Kr^M (\sin k(\psi - \alpha))^{-2}$$

and

$$\left| \frac{f^{(n)}(z)}{f(z)} \right| \leq Kr^M \left(\sin k(\psi - \alpha) \prod_{j=1}^{n-1} \sin k_{\varepsilon_j}(\psi - \alpha_j) \right)^{-2}$$

for all $z \in S(\alpha_{n-1}, \beta_{n-1})$ outside an R -set H , where $k = \frac{\pi}{\beta - \alpha}$ and $k_{\varepsilon_j} = \frac{\pi}{\beta_j - \alpha_j}$, $j = 1, 2, \dots, n - 1$.

3. PROOF OF THEOREM 1.2

Set $d = \frac{2\pi}{n+2}$. Suppose on the contrary to the assertion that there exists a nontrivial solution f of (1.2) with $m(\Delta(f)) < d$. We aim for a contradiction. Set $\eta = d - m(\Delta(f))$. Since $\Delta(f)$ is closed, then $S = [0, 2\pi) \setminus \Delta(f)$ consists of at most countable many open intervals. Therefore, we choose finite many open intervals $I_i = (\alpha_i, \beta_i)$, $i = 1, 2, \dots, m$, which satisfy $[\alpha_i, \beta_i] \subset S$ and $m(S \setminus \cup_{i=1}^m I_i) < \frac{\eta}{4}$. For the sector domain $S(\alpha_i, \beta_i)$, and for sufficiently large r_i , we have

$$(\alpha_i, \beta_i) \cap \Delta(f) = \emptyset, \quad S(r_i, \alpha_i, \beta_i) \cap J(f) = \emptyset.$$

This shows that, for each $i = 1, 2, \dots, m$, there exist the corresponding r_i and unbounded Fatou component U_i of $F(f)$, such that $S(r_i, \alpha_i, \beta_i) \subset U_i$ (see [2]). In boundary of U_i , we take an unbounded and connected section $\gamma_i \subset \partial U_i$, then the mapping $f : S(r_i, \alpha_i, \beta_i) \rightarrow \mathbb{C} \setminus \gamma_i$ is analytic. According to the choose of γ_i , we know that $\mathbb{C} \setminus \gamma_i$ is simply connected, thus for any $a \in \gamma_i \setminus \{\infty\}$, $C_{\mathbb{C} \setminus \gamma_i}(a) \geq \frac{1}{2}$. In every $S(r_i, \alpha_i, \beta_i)$, applying Lemma 2.3 to f , there exists a positive constant l_1 such that

$$|f(z)| = O(|z|^{l_1}), \quad z \in \cup_{i=1}^m S(r_i, \alpha_i + \varepsilon, \beta_i - \varepsilon), \quad |z| \rightarrow \infty,$$

where $0 < \varepsilon < \min\{\frac{\eta}{16m}, \frac{\beta_i - \alpha_i}{8}\}$, $i = 1, 2, \dots, m$. Hence we immediately get

$$S_{\alpha_i + \varepsilon, \beta_i - \varepsilon}(r, f) = O(1), \quad i = 1, 2, \dots, m,$$

and then $\sigma_{\alpha_i+\varepsilon, \beta_i-\varepsilon}(f)$ is finite. Applying Lemma 2.4, there exist two constants $M > 0$ and $K > 0$ such that

$$(3.1) \quad \left| \frac{f^{(k)}(z)}{f(z)} \right| \leq Kr^M, \quad k = 1, 2,$$

for all $z \in \cup_{i=1}^m S(r_i, \alpha_i + 2\varepsilon, \beta_i - 2\varepsilon)$, outside a R -set H .

Set $\theta_j = \frac{2j\pi - \arg(a_n)}{n+2}$ and $S_j = \{z : \theta_j < \arg z < \theta_{j+1}\}$, $j = 0, 1, 2, \dots, n + 1$, if $j = n + 1$, set $\theta_{j+1} = \theta_0 + 2\pi$. Since $\lambda(A) < \rho(A)$, by Lemma 2.1, there exists at least one sector of the $n + 2$ sectors, such that $A(z)$ decays to zero exponentially, say $S_{j_0} = \{z : \theta_{j_0} < \arg z < \theta_{j_0+1}\}$, $0 \leq j_0 \leq n + 1$. This implies that for any $\theta \in (\theta_{j_0} + \varepsilon, \theta_{j_0+1} - \varepsilon)$,

$$(3.2) \quad \lim_{r \rightarrow \infty} \frac{\log \log \frac{1}{|A(re^{i\theta})|}}{\log r} = \frac{n + 2}{2}.$$

Set $S'_{j_0} = \{\theta \in [0, 2\pi) : re^{i\theta} \in S_{j_0}(\varepsilon)\}$, where $S_{j_0}(\varepsilon) = \{z : \theta_{j_0} + \varepsilon < \arg z < \theta_{j_0+1} - \varepsilon\}$, then $m(S'_{j_0}) = \theta_{j_0+1} - \varepsilon - (\theta_{j_0} + \varepsilon) \geq d - \frac{\eta}{4}$. Thus

$$m(S'_{j_0} \cap S) = m(S'_{j_0} \setminus (S'_{j_0} \cap \Delta(f))) \geq m(S'_{j_0}) - m(\Delta(f)) > \frac{3\eta}{4} > 0.$$

So,

$$\begin{aligned} m(S'_{j_0} \cap (\cup_{i=1}^m I_i)) &= m(S'_{j_0} \cap S) - m(S'_{j_0} \cap (S \setminus \cup_{i=1}^m I_i)) \\ &> \frac{3\eta}{4} - m(S \setminus \cup_{i=1}^m I_i) \\ &> \frac{\eta}{2} > 0. \end{aligned}$$

Thus, there exists an open interval $I_{i_0} = (\alpha, \beta) \subset \bigcup_{i=1}^m I_i \subset S$, such that

$$(3.3) \quad m(S'_{j_0} \cap I_{i_0}) > \frac{\eta}{2m} > 0.$$

By equation (1.2), we get

$$(3.4) \quad |B(z)| \leq \left| \frac{f''(z)}{f(z)} \right| + |A(z)| \left| \frac{f'(z)}{f(z)} \right|.$$

We divide into two cases to $B(z)$ for finishing the proof.

Case 1. $0 < \mu(B) < \frac{1}{2}$. By Lemma 2.2, there exists a set $E_1^* \subset [1, \infty)$ with $\overline{\log \text{dens}}(E_1^*) \geq 1 - \frac{\mu(B)}{\alpha_0}$, where $\alpha_0 = \frac{\mu(B)+\frac{1}{2}}{2}$, $E_1^* = \{r \in [1, \infty) : m(r) > M(r) \cos \pi \alpha_0\}$, $m(r) = \inf_{|z|=r} \log |B(z)|$, $M(r) = \sup_{|z|=r} \log |B(z)|$. Hence there exists a constant $R_0 > 1$, such that

$$(3.5) \quad |B(z)| > \exp(r^{\mu(B)-\varepsilon})$$

for all $|z| = r \in E_1 = E_1^* \setminus [0, R_0)$. Then there exists a sequence of points $\{r_s e^{i\theta}\}$ outside H , $\theta \in I_{i_0}$, $r_s \in E_1$ satisfying $r_s \rightarrow \infty$ as $s \rightarrow \infty$, such that (3.1), (3.5) hold for $z = r_s e^{i\theta}$, and

$$(3.6) \quad \lim_{s \rightarrow \infty} \frac{\log \log \frac{1}{|A(r_s e^{i\theta})|}}{\log r_s} = \frac{n+2}{2}.$$

It follows from (3.1), (3.4), (3.5) and (3.6) that

$$\exp(r_s^{\mu(B)-\epsilon}) < K r_s^M (1 + o(1))$$

for sufficiently large s . Obviously, this is a contradiction.

Case 2. $\mu(B) = 0$. By Lemma 2.2, there exists a set $E_2 \subset [1, \infty)$ with $\overline{\log \text{dens}}(E_2) = 1$, such that for all z satisfying $|z| = r \in E_2$, we have

$$(3.7) \quad \log |B(z)| > \frac{\sqrt{2}}{2} \log M(r, B),$$

where $M(r, B) = \max_{|z|=r} |B(z)|$. It follows from (3.3) and (3.7), there exists a sequence of points $\{r_s e^{i\theta}\}$ outside H , $\theta \in I_{i_0}$, $r_s \in E_2$ satisfying $r_s \rightarrow \infty$ as $s \rightarrow \infty$, such that (3.1), (3.6) and (3.7) hold for $z = r_s e^{i\theta}$. We deduce from (3.1), (3.4), (3.6) and (3.7) that

$$(3.8) \quad M(r_s, B)^{\frac{\sqrt{2}}{2}} \leq K r_s^M (1 + o(1))$$

for sufficiently large s . However $B(z)$ is a transcendental entire function, we have

$$(3.9) \quad \liminf_{r \rightarrow \infty} \frac{\log M(r, B)}{\log r} = \infty.$$

We get a contradiction from (3.8) and (3.9). This completes the proof.

4. PROOF OF THEOREM 1.5

We begin by recalling a lemma on logarithmic derivative due to Gundersen plays an important role in proving Theorem 1.5, it can be found in [7].

Lemma 4.1. *Let f be a transcendental meromorphic function of finite order $\rho(f)$. Let $\epsilon > 0$ be a given real constant, and let k and j be integers such that $k > j \geq 0$. Then there exists a set $E_1 \subset [0, 2\pi)$ of linear measure zero, such that if $\psi_0 \in [0, 2\pi) \setminus E_1$, then there is a constant $R_0 = R_0(\psi_0) > 1$ such that for all z satisfying $\arg z = \psi_0$ and $|z| \geq R_0$, we have*

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq |z|^{(k-j)(\rho(f)-1+\epsilon)}.$$

Firstly, we prove that every nontrivial solution of (1.2) is of infinite order. To the end, suppose on the contrary to the assertion that there exists a nontrivial solution f of (1.2) with $\rho(f) < \infty$. We aim for a contradiction. Set $\theta_j = \frac{2j\pi - \arg(a_n)}{n+2}$ and $S_j = \{z : \theta_j < \arg z < \theta_{j+1}\}$, $j = 0, 1, 2, \dots, n + 1$, if $j = n + 1$, set $\theta_{j+1} = \theta_0 + 2\pi$. Since $p(A) \geq 2$, by Lemma 2.1, there exists at least one sector of the $n + 2$ sectors, such that $A(z)$ decays to zero exponentially, say $S_{j_0} = \{z : \theta_{j_0} < \arg z < \theta_{j_0+1}\}$, $0 \leq j_0 \leq n + 1$. That is, for any $\theta \in (\theta_{j_0}, \theta_{j_0+1})$, we have (3.2) holds.

In the following, we divide into two cases to $B(z)$.

Case 1. $0 < \mu(B) < \frac{1}{2}$. By using the similar reasoning as in the case 1 of proof of Theorem 1.2, we have (3.5) holds for all $|z| = r \in E_1$, where $\overline{\log \text{dens}}(E_1) \geq 1 - \frac{\mu(B)}{\alpha_0}$, $\alpha_0 = \frac{\mu(B) + \frac{1}{2}}{2}$.

Applying Lemma 4.1, there exists a set $E_2 \subset [0, 2\pi)$ of linear measure zero, such that if $\psi_0 \in [0, 2\pi) \setminus E_2$, then there is a constant $R_0 = R_0(\psi_0) > 1$ such that for all z satisfying $\arg z = \psi_0$ and $|z| \geq R_0$,

$$(4.1) \quad \left| \frac{f^{(k)}(z)}{f(z)} \right| \leq |z|^{2\rho(f)}, \quad k = 1, 2.$$

Thus, there exists a sequence of points $z_s = r_s e^{i\theta}$ with $r_s \rightarrow \infty$ as $s \rightarrow \infty$, $r_s \in E_1$ and $\theta \in (\theta_{j_0}, \theta_{j_0+1}) \setminus E_2$, such that (3.2), (3.5) and (4.1) hold. It follows from (1.2), (3.2), (3.5) and (4.1) that

$$(4.2) \quad \begin{aligned} \exp(r_s^{\mu(B)-\varepsilon}) &\leq |B(r_s e^{i\theta})| \\ &\leq \left| \frac{f''(r_s e^{i\theta})}{f(r_s e^{i\theta})} \right| + |A(r_s e^{i\theta})| \left| \frac{f'(r_s e^{i\theta})}{f(r_s e^{i\theta})} \right| \\ &\leq r_s^{2\rho(f)}(1 + o(1)) \end{aligned}$$

for sufficiently large s . Obviously, this is a contradiction for arbitrary small ε . Hence we have $\rho(f) = \infty$ for every nontrivial solutions f of (1.2).

Case 2. $\mu(B) = 0$. By Lemma 2.2, there exists a set $E_3 \subset [0, \infty)$ with $\overline{\log \text{dens}}(E_3) = 1$, such that for all z satisfying $|z| = r \in E_3$, we have (3.7) holds. Thus, there exists a sequence of points $z_s = r_s e^{i\theta}$ with $r_s \rightarrow \infty$ as $s \rightarrow \infty$, $r_s \in E_3$ and $\theta \in (\theta_{j_0}, \theta_{j_0+1}) \setminus E_2$, such that (3.2), (3.7) and (4.1) hold. Therefore, we deduce from (1.2), (3.2), (3.7) and (4.1) that

$$(4.3) \quad M(r_s, B)^{\frac{\sqrt{2}}{2}} \leq r_s^{2\rho(f)}(1 + o(1))$$

for sufficiently large s . But $B(z)$ is a transcendental entire function, we have (3.9) holds. We obtain a contradiction from (3.9) and (4.3). So, $\rho(f) = \infty$ for every nontrivial solutions f of (1.2).

Secondly, we can prove $m(\Delta(f)) \geq \frac{2\pi}{n+2}$ by using the similar reasoning in proving Theorem 1.2, we omit the details. This completes the proof.

5. PROOF OF THEOREM 1.6

Set $d = \frac{2\pi}{n+2}$. Suppose on the contrary to the assertion that there exists a nontrivial solution f of (1.2) with $m(\Delta(f) \cap \Delta(f^{(k)})) < d$. We aim for a contradiction. Set $\eta = d - m(\Delta(f) \cap \Delta(f^{(k)}))$. By using the idea as in proving Theorem 1.2, in order to finish the proof, we need find an open interval $I = (\alpha, \beta) \subset \Delta(f^{(k)})^c$, $0 < \beta - \alpha < d$, such that

$$(5.1) \quad m(\Delta(f) \cap S'_{j_0} \cap I) > 0,$$

where $\Delta(f^{(k)})^c = [0, 2\pi) \setminus \Delta(f^{(k)})$, S'_{j_0} is defined as in the proof of Theorem 1.2. First, we claim that

$$(5.2) \quad m(S'_{j_0} \setminus \Delta(f)) = 0.$$

If it is not true, then there exist $\phi_0 \in \Delta(f)^c$ and $\zeta > 0$ satisfying

$$(5.3) \quad m((\phi_0 - \zeta, \phi_0 + \zeta) \cap (S'_{j_0} \setminus \Delta(f))) > 0.$$

Since $\arg z = \phi_0$ is not the radial distribution of $J(f)$, there exists a constant $r_0 > 0$ such that

$$S(r_0, \phi_0 - \zeta, \phi_0 + \zeta) \cap J(f) = \emptyset.$$

It follows that there exists an unbounded component U of Fatou set $F(f)$, such that $S(r_0, \phi_0 - \zeta, \phi_0 + \zeta) \subset U$. In boundary of U , we take an unbounded and connected set $\gamma \subset \partial U$, then the mapping $f : S(r_0, \phi_0 - \zeta, \phi_0 + \zeta) \rightarrow \mathbb{C} \setminus \gamma$ is analytic. Since $\mathbb{C} \setminus \gamma$ is simply connected, then, for any $a \in \gamma \setminus \{\infty\}$, we get $C_{\mathbb{C} \setminus \gamma}(a) \geq \frac{1}{2}$. For any $0 < \xi < \frac{\zeta}{4}$, applying Lemma 2.3 to f in $S(r_0, \phi_0 - \zeta, \phi_0 + \zeta)$, we get

$$(5.4) \quad |f(z)| = O(|z|^{l_1}), \quad z \in S(r_0, \phi_0 - \zeta + \xi, \phi_0 + \zeta - \xi), \quad |z| \rightarrow \infty,$$

where l_1 is a positive constant. Hence we get

$$S_{\phi_0 - \zeta + \xi, \phi_0 + \zeta - \xi}(r, f) = O(1),$$

and then $\sigma_{\phi_0 - \zeta + \xi, \phi_0 + \zeta - \xi}(f)$ is finite. Applying Lemma 2.4, there exist two constants $M > 0$ and $K > 0$ such that (3.1) holds for all $z \in S(r_0, \phi_0 - \zeta + 2\xi, \phi_0 + \zeta - 2\xi)$, outside a R -set H . Since ξ is arbitrary small, from (5.3), we have

$$m((\phi_0 - \zeta + 2\xi, \phi_0 + \zeta - 2\xi) \cap S'_{j_0}) > 0.$$

By the similar reasoning as in the cases 1 and 2 of the proof of Theorem 1.2, then there exists a sequence of points $\{r_s e^{i\phi}\}$, where $\phi \in (\phi_0 - \zeta + 2\xi, \phi_0 + \zeta - 2\xi)$ and $r_s \rightarrow \infty$ as $s \rightarrow \infty$, such that (3.5), (3.6) and (3.7) hold for $z = r_s e^{i\phi}$. Combining (1.2), (3.1), (3.5), (3.6) and (3.7), we get a contradiction. Therefore, (5.2) is valid.

We know that $m(\Delta(f)) \geq d$ from Theorem 1.2. It follows from the definition of S'_{j_0} and Lemma 2.1 that $m(S'_{j_0}) \geq d - 2\varepsilon$ for any small $\varepsilon > 0$. From this and (5.2), we have

$$(5.5) \quad m(\Delta(f) \cap S'_{j_0}) \geq d - \frac{\eta}{4}.$$

Since $\Delta(f^{(k)})$ is closed, then $\Delta(f^{(k)})^c$ consists of at most countable many open intervals. We can choose finite many open intervals I_i such that

$$I_i \subset \Delta(f^{(k)})^c, \quad m\left(\Delta(f^{(k)})^c \setminus \bigcup_{i=1}^m I_i\right) < \frac{\eta}{4}, \quad i = 1, 2, \dots, m.$$

Since

$$\begin{aligned} & m(\Delta(f) \cap S'_{j_0} \cap (\bigcup_{i=1}^m I_i)) + m(\Delta(f) \cap S'_{j_0} \cap \Delta(f^{(k)})) \\ &= m\left(\Delta(f) \cap S'_{j_0} \cap (\Delta(f^{(k)}) \cup (\bigcup_{i=1}^m I_i))\right) \geq d - \frac{\eta}{2}, \end{aligned}$$

then

$$\begin{aligned} & m\left(\Delta(f) \cap S'_{j_0} \cap (\bigcup_{i=1}^m I_i)\right) \\ & \geq d - \frac{\eta}{2} - m(\Delta(f) \cap S'_{j_0} \cap \Delta(f^{(k)})) \\ & \geq d - \frac{\eta}{2} - m(\Delta(f) \cap \Delta(f^{(k)})) = \frac{\eta}{2} > 0. \end{aligned}$$

Thus, there exists an open interval $I_{j_0} = (\alpha, \beta) \subset \bigcup_{i=1}^m I_i \subset \Delta(f^{(k)})^c$, such that

$$m(\Delta(f) \cap S'_{j_0} \cap I_{j_0}) \geq \frac{\eta}{2m} > 0.$$

Thus (5.1) is valid. From (5.1), we know that there are $\tilde{\phi}_0$ and $\tilde{\zeta} > 0$, such that $(\tilde{\phi}_0 - \tilde{\zeta}, \tilde{\phi}_0 + \tilde{\zeta}) \subset I_{j_0}$, and $m(\Delta(f) \cap S'_{j_0} \cap (\tilde{\phi}_0 - \tilde{\zeta}, \tilde{\phi}_0 + \tilde{\zeta})) > 0$. Then there exists $\tilde{r}_0 > 0$, such that $S(\tilde{r}_0, \tilde{\phi}_0 - \tilde{\zeta}, \tilde{\phi}_0 + \tilde{\zeta}) \cap J(f^{(k)}) = \emptyset$. By using similar reasoning as in proving (5.4), for any $0 < \tilde{\xi} < \frac{\tilde{\zeta}}{6}$, we have

$$(5.6) \quad |f^{(k)}(z)| = O(|z|^{l_2}), \quad z \in S(\tilde{r}_0, \tilde{\phi}_0 - \tilde{\zeta} + \tilde{\xi}, \tilde{\phi}_0 + \tilde{\zeta} - \tilde{\xi}), \quad |z| \rightarrow \infty,$$

where l_2 is a positive constant.

Fix $r_* e^{i\phi_*}$, where $r_* > \tilde{r}_0$ and $\phi_* \in (\tilde{\phi}_0 - \tilde{\zeta} + \tilde{\xi}, \tilde{\phi}_0 + \tilde{\zeta} - \tilde{\xi})$, and for any $z = r e^{i\phi} \in S(\tilde{r}_0, \tilde{\phi}_0 - \tilde{\zeta} + \tilde{\xi}, \tilde{\phi}_0 + \tilde{\zeta} - \tilde{\xi})$. Take a simple Jordan arc γ_z in $S(\tilde{r}_0, \tilde{\phi}_0 - \tilde{\zeta} + \tilde{\xi}, \tilde{\phi}_0 + \tilde{\zeta} - \tilde{\xi})$ which connects $r_* e^{i\phi_*}$ to $r_* e^{i\phi}$ along $|z| = r_*$ and connects $r_* e^{i\phi}$ to $r e^{i\phi}$ along $\arg z = \phi$. It follows from (5.6) and Cauchy

integral formula that

$$|f^{(k-1)}(z)| \leq \int_{\gamma_z} |f^{(k)}(z)||dz| + c_k \leq O(|z|^{l_2+1}), \quad |z| \rightarrow \infty.$$

Similarly,

$$|f^{(k-2)}(z)| \leq \int_{\gamma_z} |f^{(k-1)}(z)||dz| + c_{k-1} \leq O(|z|^{l_2+2}), \quad |z| \rightarrow \infty.$$

By induction, we have

$$|f(z)| \leq \int_{\gamma_z} |f'(z)||dz| + c_1 \leq O(|z|^{l_2+k}), \quad |z| \rightarrow \infty,$$

where $c_i, i = 1, 2, \dots, k$, are positive constants. Therefore,

$$S_{\tilde{\phi}_0-\tilde{\zeta}+\tilde{\xi}, \tilde{\phi}_0+\tilde{\zeta}-\tilde{\xi}}(r, f) = O(1),$$

and then $\sigma_{\tilde{\phi}_0-\tilde{\zeta}+\tilde{\xi}, \tilde{\phi}_0+\tilde{\zeta}-\tilde{\xi}}(f) < \infty$. Applying Lemma 2.3, we know that (3.1) holds for all $z \in S(\tilde{r}_0, \tilde{\phi}_0-\tilde{\zeta}+2\tilde{\xi}, \tilde{\phi}_0+\tilde{\zeta}-2\tilde{\xi})$, outside a R -set H . By applying similar reasoning as in the cases 1 and 2 of the proof of Theorem 1.2, we can get a contradiction. Therefore, we have

$$m(\Delta(f) \cap \Delta(f^{(k)})) \geq d.$$

This completes the proof.

6. ANNEX REMARKS

In [9], Huang and Wang proved the following result by using the spread relation and Pólya peaks of meromorphic functions.

Theorem 6.1. *Suppose that $A_i(z), i = 0, 1, \dots, k - 1$, are entire functions satisfying $\rho(A_j) < \mu(A_0) < \infty, j = 1, 2, \dots, k - 1$. Then every nontrivial solution f of the equation*

$$(6.1) \quad f^{(k)} + A_{k-1}(z)f^{(k-1)} + \dots + A_0(z)f = 0$$

satisfies $m(\Delta(f)) \geq \min\{2\pi, \frac{\pi}{\mu(A_0)}\}$.

In this section, we will give a simple proof of Theorem 6.1 than original way which is given in [9], which rely heavily on the following modified Phragmén-Lindelöf principle.

Lemma 6.2 ([20]). *Let f be an entire function of lower order $\mu(f) \in [\frac{1}{2}, \infty)$. Then there exists a sector domain $S(\alpha, \beta) = \{z : \alpha < \arg z < \beta\}$ with $\beta - \alpha \geq \frac{\pi}{\mu(f)}$, where $0 \leq \alpha < \beta \leq 2\pi$, such that*

$$\limsup_{r \rightarrow \infty} \frac{\log \log |f(re^{i\theta})|}{\log r} \geq \mu(f)$$

for all the ray $\arg z = \theta \in (\alpha, \beta)$.

Another proof of Theorem 6.1. We divide into two cases to finish our proof.

Case 1. $\mu(A_0) \geq \frac{1}{2}$. We need prove that $m(\Delta(f)) \geq \frac{\pi}{\mu(A_0)}$ for every nontrivial solution f of (6.1). To the end, suppose on the contrary to the assertion that there exists a nontrivial solution f of (6.1) with $m(\Delta(f)) < \frac{\pi}{\mu(A_0)}$. Set $S = (0, 2\pi) \setminus \Delta(f)$. By Lemma 6.2 to $A_0(z)$, there exists a sector domain $S(\alpha, \beta)$ with $\beta - \alpha \geq \frac{\pi}{\mu(A_0)}$, such that

$$(6.2) \quad \limsup_{r \rightarrow \infty} \frac{\log \log |A_0(re^{i\theta})|}{\log r} \geq \mu(A_0)$$

for any $\theta \in (\alpha, \beta)$.

Since $m(\Delta(f)) < \frac{\pi}{\mu(A_0)}$ and $\beta - \alpha \geq \frac{\pi}{\mu(A_0)}$, then there exists a sector domain $S(\alpha', \beta')$, such that $\alpha < \alpha' < \beta' < \beta$ and $(\alpha', \beta') \subset S$. Then for any $\alpha' < \theta < \beta'$, we have (6.2) holds. For the sector domain $S(\alpha', \beta')$, it is easy to see that

$$(\alpha', \beta') \cap \Delta(f) = \emptyset, \quad S(r, \alpha', \beta') \cap J(f) = \emptyset$$

for sufficiently large r . This implies that there exists $r_0 > 0$ and unbounded Fatou component U of $F(f)$ such that $S(r_0, \alpha', \beta') \subset U$. We take a unbounded and connected section γ of ∂U , then the mapping $f : S(r_0, \alpha', \beta') \rightarrow \mathbb{C} \setminus \gamma$ is analytic. Since we have chosen γ such that $\mathbb{C} \setminus \gamma$ is simply connected, for any $a \in \gamma \setminus \{\infty\}$, we have $C_{\mathbb{C} \setminus \gamma}(a) \geq \frac{1}{2}$. Applying Lemma 2.3 to f , there exists a positive constant l such that

$$(6.3) \quad |f(z)| = O(|z|^l), \quad z \in S(r_0, \alpha' + \varepsilon, \beta' - \varepsilon), \quad |z| \rightarrow \infty,$$

where $0 < \varepsilon < \frac{\beta' - \alpha'}{8}$. Thus we immediately obtain $S_{\alpha' + \varepsilon, \beta' - \varepsilon}(r, f) = O(1)$, and then $\sigma_{\alpha' + \varepsilon, \beta' - \varepsilon}(f) < \infty$. Applying Lemma 2.4, there exist two constants $M > 0$ and $K > 0$, such that

$$(6.4) \quad \left| \frac{f^{(j)}(z)}{f(z)} \right| \leq Kr^M, \quad j = 1, 2, \dots, k,$$

for all $z \in S(\alpha' + 2\varepsilon, \beta' - 2\varepsilon)$, outside a R -set H .

Let $\max_{1 \leq j \leq k-1} \{\rho(A_j)\} = \eta$, and $\delta \in (\eta + \varepsilon, \mu(A_0) - 2\varepsilon)$ be a constant. Since $\rho(A_j) < \mu(A_0)$, $j = 1, 2, \dots, k-1$, then there exists a constant $r_1 > r_0$, such that for any $|z| = r > r_1$, we have

$$(6.5) \quad |A_j(z)| \leq \exp(r^\delta), \quad j = 1, 2, \dots, k-1.$$

Thus there exists a sequence of points $z_n = r_n e^{i\theta}$ outside H , $r_n \rightarrow \infty$ as $n \rightarrow \infty$, $\alpha' < \theta < \beta'$, such that (6.2), (6.4) and (6.5) hold. It follows from

(6.1), (6.2), (6.4) and (6.5) that

$$\begin{aligned}
 \exp(r_n^{\mu(A_0)-\varepsilon}) &\leq |A_0(r_n e^{i\theta})| \\
 &\leq \left| \frac{f^{(k)}(r_n e^{i\theta})}{f(r_n e^{i\theta})} \right| + \dots + |A_1(r_n e^{i\theta})| \left| \frac{f'(r_n e^{i\theta})}{f(r_n e^{i\theta})} \right| \\
 (6.6) \qquad \qquad &\leq K r_n^M (k-1) \exp(r_n^\delta).
 \end{aligned}$$

Obviously, this is a contradiction for sufficiently large n .

Case 2. $\mu(A_0) < \frac{1}{2}$. We need prove that $m(\Delta(f)) = 2\pi$ for every nontrivial solution f of (6.1). Suppose on the contrary to the assertion that there exists a nontrivial solution f of (6.1) with $m(\Delta(f)) < 2\pi$. Then, by similar reasoning as in case 1 above, there exist $(\alpha, \beta) \subset [0, 2\pi] \setminus \Delta(f)$ and constant $r_0 > 1$ such that $S(r_0, \alpha, \beta) \subset F(f)$, and (6.3) holds for $z \in S(r, \alpha + \varepsilon, \beta - \varepsilon)$. Hence we have $S_{\alpha+\varepsilon, \beta-\varepsilon}(r, f) = O(1)$, and then $\sigma_{\alpha+\varepsilon, \beta-\varepsilon}(f) < \infty$. Applying Lemma 2.4, we see that (6.4) holds for all $z \in S(r, \alpha + 2\varepsilon, \beta - 2\varepsilon)$ outside a R -set H .

Since $\mu(A_0) < \frac{1}{2}$, applying Lemma 2.2 to $A_0(z)$, there exists a set $E_1^* \subset [1, \infty)$ with $\overline{\log \text{dens}}(E_1^*) \geq 1 - \frac{\mu(A_0)}{\alpha_0}$, where $\alpha_0 = \frac{\mu(A_0) + \frac{1}{2}}{2}$, $E_1^* = \{r \in [1, \infty) : m(r) > M(r) \cos \pi \alpha_0\}$, $m(r) = \inf_{|z|=r} \log |A_0(z)|$, and $M(r) = \sup_{|z|=r} \log |A_0(z)|$.

Thus, there exists a constant $R_0 > 1$ such that

$$(6.7) \qquad |A_0(z)| > \exp(r^{\mu(A_0)-\varepsilon})$$

for all $|z| = r \in E_1^* \setminus [0, R_0]$. Thus, there exists a sequence of points $z_n = r_n e^{i\theta}$ outside H , $r_n \rightarrow \infty$ as $n \rightarrow \infty$, $\alpha < \theta < \beta$, such that (6.4), (6.5) and (6.7) hold. It follows from (6.1), (6.4), (6.5) and (6.7) that

$$\exp(r_n^{\mu(A_0)-\varepsilon}) \leq K r_n^M (k-1) \exp(r_n^\delta).$$

Obviously, this is a contradiction for sufficiently large n . This completes the proof.

Acknowledgements. This research is supported by the National Natural Science Foundation of China (Grant No. 11501142), and the Foundation of Science and Technology of Guizhou Province of China (Grant No. [2015] 2112).

REFERENCES

[1] I. N. Baker, *Sets of non-normality in iteration theory*, J. London Math. Soc. 40 (1965), 499-502.
 [2] I. N. Baker, *The domains of normality of an entire function*, Ann. Acad. Sci. Fenn. Math. 1 (1975), no. 2, 277-283.

- [3] P. D. Barry, *Some theorems related to the $\cos \pi \rho$ theorem*, Proc. London. Math. Soc. 21 (1970), no. 3, 334-360.
- [4] W. Bergweiler, *Iteration of meromorphic functions*, Bull. Amer. Math. Soc. 29 (1993), 151-188.
- [5] A. A. Gol'dberg and I. V. Ostrovskii, *Value Distribution of Meromorphic Functions*, Translations of Mathematical Monographs, vol. 236, American Mathematical Society, Providence, RI, 2008.
- [6] G. G. Gundersen, *On the real zeros of solutions of $f'' + A(z)f = 0$ where $A(z)$ is entire*, Ann. Acad. Sci. Fenn. Math. 11 (1986), 275-294.
- [7] G. G. Gundersen, *Estimates for the logarithmic derivative of a meromorphic function, plus similar estimates*, J. London Math. Soc. 37 (1988), 88-104.
- [8] Z. G. Huang and J. Wang, *On the radial distribution of Julia sets of entire solutions of $f^{(n)} + A(z)f = 0$* , J. Math. Anal. Appl. 409 (2012), no. 2, 1106-1113.
- [9] Z. G. Huang and J. Wang, *On limit directions of Julia sets of entire solutions of linear differential equations*, J. Math. Anal. Appl. 387 (2014), no. 1, 478-484.
- [10] W. Hayman, *Meromorphic Functions*, Clarendon Press, Oxford, 1964.
- [11] E. Hille, *Lectures on Ordinary Differential Equations*, Addison-Wesley Publishing Company, Reading, Massachusetts-Menlo Park, California-London-Don Mills, Ontario, 1969.
- [12] I. Laine, *Nevanlinna Theory and Complex Differential Equations*, Walter de Gruyter Berlin, New York, 1993.
- [13] J. R. Long, P. C. Wu and X. B. Wu, *On the zero distribution of solutions of second order linear differential equations in the complex plane*, New Zealand J. Math. 42 (2012), 9-16.
- [14] J. R. Long and K. E. Qiu, *Growth of solutions to a second-order complex linear differential equation*, Math. Pract. Theory 45 (2015), no. 2, 243-247.
- [15] A. Z. Mokhon'ko, *An estimate of the modules of the logarithmic derivative of a function which is meromorphic in an angular region, and its application*, Ukrain Mat. Zh. 41 (1989), no. 6, 839-843.
- [16] J. Y. Qiao, *Stable sets for iterations of entire functions*, Acta Math. Sinica 37 (1994), no. 5, 702-708.
- [17] L. Qiu and S. J. Wu, *Radial distributions of Julia sets of meromorphic functions*, J. Austra. Math. Soc. 81 (2006), no. 3, 363-368.

- [18] J. Rossi and S. P. Wang, *The radial oscillation of solutions to ODE's in the complex domain*, Proc. Edinburgh Math. Soc. 39 (1996), 473-483.
- [19] K. C. Shin, *New polynomials P for which $f'' + P(z)f = 0$ has a solution with almost all real zeros*, Ann. Acad. Sci. Fenn. Math. 27 (2002), 491-498.
- [20] X. B. Wu, J. R. Long, J. Heittokangas and K. E. Qiu, *Second-order complex linear differential equations with special functions or extremal functions as coefficients*, Electron. J. Differential Equations 2015 (2015), No. 143, 1-15.
- [21] S. J. Wu, *On the location of zeros of solution of $f'' + Af = 0$ where $A(z)$ is entire*, Math. Scand. 74 (1994), 293-312.
- [22] S. J. Wu, *Angular distribution in the complex oscillation theory*, Sci. China Ser. A 48 (2005), no. 1, 107-114.
- [23] L. Yang, *Value Distribution Theory*, Springer-Verlag, Berlin, 1993.
- [24] G. W. Zhang, J. Wang and L.Z. Yang, *On radial distribution of Julia sets of solutions to certain second order complex linear differential equations*, Abstr. Appl. Anal. 2014 (2014), Article ID 842693, 1-6.
- [25] J. H. Zheng, S. Wang and Z. G. Huang, *Some properties of Fatou and Julia sets of transcendental meromorphic functions*, Bull. Austral. Math. Soc. 66 (2002), no. 1, 1-8.

JIANREN LONG
 SCHOOL OF MATHEMATICAL SCIENCES, GUIZHOU NORMAL UNIVERSITY, GUIYANG,
 550001, P.R. CHINA.

E-mail address: longjianren2004@163.com, jrlong@gznu.edu.cn

Common Fixed Point Results for the Family of Multivalued Mappings Satisfying Contractions on a Sequence in Hausdorff Fuzzy Metric Space

Abdullah Shoaib¹, Akbar Azam², and Aqeel Shahzad³

Abstract: The aim of this paper is to establish common fixed point results on a sequence contained in a closed ball for family of multivalued mapping in complete fuzzy metric space. Simple and different technique has been used. Example has been constructed to demonstrate the novelty of our results. Our results unify, extend and generalize several results in the existing literature.

2010 Mathematics Subject Classification: 46S40; 47H10; 54H25.

Keywords and Phrases: common fixed point; complete fuzzy metric space; closed ball; family of multivalued mappings; Hausdorff fuzzy metric space.

1 Introduction and Preliminaries

The notion of fuzzy sets was first introduced by Zadeh [5]. Kramosil et al. [10] introduced the concept of fuzzy metric space and obtained many fixed point results. Later on many authors [7, 8, 9, 11] used this concept and prove many fixed point results using the different contractive conditions. Lopez et al. [11] discuss the method for constructing a Hausdorff fuzzy metric on nonempty compact subsets of a given fuzzy metric space.

Sometimes, it happens that the fixed point of a mapping exists, but the contraction does not hold. Recently, Shoaib et al. [1, 2, 3, 4, 6, 13] obtained the necessary and sufficient conditions for the existence of a fixed point of such self mapping. In this paper, we prove the existence of a common fixed point of a family of such multivalued mappings which are contractive on a sequence contained in a closed ball instead of the whole space, by using the concept of Hausdorff fuzzy metric space. We also present an example to support our results.

Definition 1.1 [7] A binary operation $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is said to be a continuous t-norm if it satisfies the following conditions:

- i) $*$ is associative and commutative;
- ii) $*$ is continuous;
- iii) $a * 1 = a$ for all $a \in [0, 1]$;
- iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0, 1]$.

Definition 1.2 [10] The 3-tuple $(X, M, *)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm, and F is a fuzzy set on $X^2 \times [0, \infty)$, satisfying the following conditions for all $x, y, z \in X$ and $t, s > 0$:

- F1) $F(x, y, 0) = 0$;
- F2) $F(x, y, t) = 1$ if and only if $x = y$;
- F3) $F(x, y, t) = F(y, x, t)$;
- F4) $F(x, z, t + s) \geq F(x, y, t) * F(y, z, s)$;
- F5) $F(x, y, \cdot) : (0, \infty) \rightarrow [0, 1]$ is left-continuous.

Example 1.3 [7] Let (X, d) be a metric space. Define $a * b = ab$ and

$$F(x, y, t) = \frac{kt^n}{kt^n + md(x, y)},$$

for all $x, y \in X$ and $k, m, n \in \mathbb{R}^+$. Then $(X, F, *)$ is a fuzzy metric space.

Definition 1.4 [9] Let $(X, F, *)$ be a fuzzy metric space. Then, we have

- i) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ denoted $x_n \rightarrow x$, if $\lim_{n \rightarrow \infty} F(x_n, x, t) = 1$ for each $t > 0$.
- ii) A sequence $\{x_n\}$ in X is said to be a Cauchy sequence, if $\lim_{n \rightarrow \infty} F(x_n, x_{n+p}, t) = 1$ for each $t > 0, p > 0$.
- iii) A fuzzy metric space $(X, F, *)$ in which every Cauchy sequence is convergent is called a complete fuzzy metric space.

Definition 1.5 [11] Let $(X, F, *)$ be a fuzzy metric space. Define a function H_{FM} on $\hat{C}_0(X) \times \hat{C}_0(X) \times (0, \infty)$ by

$$H_{FM}(A, B, t) = \min \left\{ \inf_{a \in A} F(a, B, t), \inf_{b \in B} F(A, b, t) \right\},$$

for all $A, B \in \hat{C}_0(X)$ and $t > 0$, where $\hat{C}_0(X)$ is the collection of all nonempty compact subsets of X .

Definition 1.6 [7] Let $(X, F, *)$ be a fuzzy metric space. Then,

$$B_F(x, r, t) = \{y \in X : F(x, y, t) > 1 - r\}$$

and

$$\overline{B_F(x, r, t)} = \{y \in X : F(x, y, t) \geq 1 - r\}$$

are called open and closed balls respectively, with centre $x \in X$ and radius r for $0 < r < 1, t > 0$.

Lemma 1.7 [11] Let $(X, F, *)$ be a complete fuzzy metric space. Then, for each $a \in X, B \in \hat{C}_0(X)$ and for all $t > 0$ there is $b_o \in B$ such that

$$F(a, b_o, t) = F(a, B, t).$$

Lemma 1.8 [12] Let $(X, F, *)$ be a complete fuzzy metric space. $(\hat{C}_0(X), H_{FM}, *)$ is a hausdorff fuzzy metric space on $\hat{C}_0(X)$. Then, for all $A, B \in \hat{C}_0(X)$, for each $a \in A$ and for all $t > 0$ there exists $b_a \in B$, satisfies $F(a, B, t) = F(a, b_a, t)$, then

$$H_{FM}(A, B, t) \leq F(a, b_a, t).$$

2 Main Results

Let $(X, F, *)$ be a fuzzy metric space, $x_0 \in X$ and let $\{S_\beta : \beta \in \Omega\}$ be a family of multivalued mappings from X to $\hat{C}_0(X)$. Then, there exists $x_1 \in S_a x_0$ for some $a \in \Omega$, such that $F(x_0, S_a x_0, t) = F(x_0, x_1, t)$, for all $t > 0$. Let $x_2 \in S_b x_1$ be such that $F(x_1, S_b x_1, t) = F(x_1, x_2, t)$. Continuing this process, we construct a sequence x_n of points in X such that $x_{n+1} \in S_\gamma x_n$, $F(x_n, S_\gamma x_n, t) = F(x_n, x_{n+1}, t)$, for all $t > 0$. We denote this iterative sequence $\{XS_\beta(x_n) : \beta \in \Omega\}$ and say that $\{XS_\beta(x_n) : \beta \in \Omega\}$ is a sequence in X generated by x_0 .

Theorem 2.1 Let $(X, F, *)$ be a complete fuzzy metric space, where $*$ be a continuous t -norm, defined as $a * a \geq a$ or $a * b = \min\{a, b\}$. Let $(\hat{C}_0(X), H_{FM}, *)$ be a Hausdorff fuzzy metric space on $\hat{C}_0(X)$, $\{S_\beta : \beta \in \Omega\}$ be a family of multivalued mappings from X to $\hat{C}_0(X)$ and $\{XS_\beta(x_n) : \beta \in \Omega\}$ be a sequence in X generated by x_0 . Assume that, for some $0 < \alpha_{i,j} \leq k < 1$, for all $t > 0$, $x_0 \in X$, for all $x, y \in \overline{B_F(x_0, r, t)} \cap \{XS_\beta(x_n) : \beta \in \Omega\}$, with $x \neq y$ and for all $i, j \in \Omega$ with $i \neq j$, we have

$$H_{FM}(S_i x, S_j y, \alpha_{i,j} t) \geq F(x, y, t) \tag{2.1}$$

and, for some $t > 0$

$$F(x_0, x_1, (1 - k)t) \geq 1 - r. \tag{2.2}$$

Then, $\{XS_\beta(x_n) : \beta \in \Omega\}$ is a sequence in $\overline{B_F(x_0, r, t)}$ and $\{XS_\beta(x_n) : \beta \in \Omega\} \rightarrow z \in \overline{B_F(x_0, r, t)}$. Also, if (2.1) holds for z , then there exists a common fixed point for the family of multivalued mappings $\{S_\beta : \beta \in \Omega\}$ in $\overline{B_F(x_0, r, t)}$.

Proof: Let $\{XS_\beta(x_n) : \beta \in \Omega\}$ be a sequence in X generated by x_0 . If $x_0 = x_1$, then x_0 is a common fixed point of S_a for all $a \in \Omega$. Let $x_0 \neq x_1$ and by Lemma 1.8, we have

$$F(x_1, x_2, t) \geq H_{FM}(S_a x_0, S_b x_1, t).$$

By induction, we have by Lemma 1.8, we have

$$F(x_n, x_{n+1}, t) \geq H_{FM}(S_i x_{n-1}, S_\gamma x_n, t). \tag{2.3}$$

First, we will show that $x_n \in \overline{B_F(x_0, r, t)}$. By (2.2), we get

$$\begin{aligned} F(x_0, x_1, t) &= F(x_0, S_a x_0, t) > F(x_0, x_1, (1 - k)t) \geq 1 - r \\ F(x_0, x_1, t) &> 1 - r. \end{aligned}$$

This shows that $x_1 \in \overline{B_F(x_0, r, t)}$. Let $x_2, \dots, x_j \in \overline{B_F(x_0, r, t)}$. Now, we have

$$\begin{aligned}
 F(x_j, x_{j+1}, t) &\geq H_{FM}(S_\delta x_{j-1}, S_\eta x_j, t) \geq F(x_{j-1}, x_j, \frac{t}{\alpha_{\delta, \eta}}) \\
 &\geq H_{FM}(S_\rho x_{j-2}, S_\delta x_{j-1}, \frac{t}{\alpha_{\delta, \eta}}) \quad (\text{by Lemma 1.8}) \\
 &\geq F(x_{j-2}, x_{j-1}, \frac{t}{\alpha_{\rho, m}, \alpha_{\delta, \eta}}) \\
 &\geq F(x_{j-2}, x_{j-1}, \frac{t}{k^2}) \geq \dots \geq F(x_0, x_1, \frac{t}{k^j}) \\
 F(x_j, x_{j+1}, t) &\geq F(x_0, x_1, \frac{t}{k^j}) \tag{2.4}
 \end{aligned}$$

Now,

$$\begin{aligned}
 F(x_0, x_{j+1}, t) &\geq F(x_0, x_{j+1}, (1 - k^{j+1})t) \\
 &\geq F(x_0, x_1, (1 - k)t) * F(x_1, x_2, (1 - k)kt) * \dots \\
 &\quad * F(x_j, x_{j+1}, (1 - k)k^j t) \\
 &\geq F(x_0, x_1, (1 - k)t) * F(x_0, x_1, (1 - k)t) * \dots \\
 &\quad * F(x_0, x_1, (1 - k)t) \quad (\text{by (2.4)}) \\
 &\geq 1 - r * 1 - r * \dots * 1 - r = 1 - r \\
 F(x_0, x_{j+1}, t) &\geq 1 - r.
 \end{aligned}$$

This implies that $x_{j+1} \in \overline{B_F(x_0, r, t)}$. Now, inequality (2.4) can be written as

$$F(x_n, x_{n+1}, t) \geq F(x_0, x_1, \frac{t}{k^n}). \tag{2.5}$$

Let $n, m \in \mathbb{N}$ with $m > n$. Assume that $m = n + p$, we have

$$\begin{aligned}
 F(x_n, x_{n+p}, t) &\geq F(x_n, x_{n+1}, (1 - k)t) * F(x_{n+1}, x_{n+p}, kt) \\
 &\geq F(x_n, x_{n+1}, (1 - k)t) * H_{FM}(S_j x_n, S_k x_{n+p-1}, kt) \\
 &\geq F(x_n, x_{n+1}, (1 - k)t) * F(x_n, x_{n+p-1}, \frac{kt}{\alpha_{j, k}}) \\
 &\geq F(x_n, x_{n+1}, (1 - k)t) * F(x_n, x_{n+p-1}, t) \\
 &\geq F(x_n, x_{n+1}, (1 - k)t) * F(x_n, x_{n+1}, (1 - k)t) \\
 &\quad * F(x_{n+1}, x_{n+p-1}, (1 - k)t) \\
 &\geq F(x_n, x_{n+1}, (1 - k)t) * F(x_n, x_{n+1}, (1 - k)t) \\
 &\quad * H_{FM}(S_j x_n, S_l x_{n+p-2}, kt) \\
 &\geq F(x_n, x_{n+1}, (1 - k)t) * F(x_n, x_{n+1}, (1 - k)t) \\
 &\quad * F(x_n, x_{n+p-2}, \frac{kt}{\alpha_{j, l}}) \\
 F(x_n, x_{n+p}, t) &\geq F(x_n, x_{n+1}, (1 - k)t) * F(x_n, x_{n+1}, (1 - k)t) \\
 &\quad * F(x_n, x_{n+p-2}, t).
 \end{aligned}$$

Using the above, we have

$$\begin{aligned}
 F(x_n, x_{n+p}, t) &\geq F(x_n, x_{n+1}, (1-k)t) * F(x_n, x_{n+1}, (1-k)t) * \dots * \\
 &\quad F(x_n, x_{n+1}, t) \\
 F(x_n, x_{n+p}, t) &\geq F(x_n, x_{n+1}, (1-k)t) * F(x_n, x_{n+1}, (1-k)t) * \dots * \\
 &\quad F(x_n, x_{n+1}, (1-k)t) \\
 &\geq F(x_0, x_1, \frac{(1-k)t}{k^n}) * F(x_0, x_1, \frac{(1-k)t}{k^n}) * \dots * \\
 &\quad F(x_0, x_1, \frac{(1-k)t}{k^n}) \quad (\text{by (2.5)}) \\
 F(x_n, x_{n+p}, t) &\geq F(x_0, x_1, \frac{(1-k)t}{k^n}).
 \end{aligned}$$

As, we have

$$\lim_{t \rightarrow \infty} F(x, y, t) = 1 \text{ for all } x, y \in X.$$

In particular

$$F(x_0, x_1, \frac{(1-k)t}{k^n}) = 1 \text{ as } n \rightarrow \infty.$$

By using above, we get

$$F(x_n, x_m, t) = 1 \text{ as } n \rightarrow \infty.$$

Hence, $\{XS_\beta(x_n)\}$ is a Cauchy sequence in $\overline{B_F(x_0, r, t)}$. As every closed ball in a complete fuzzy metric space is complete. So, $\overline{B_F(x_0, r, t)}$ is complete. Then, there exists $z \in B_F(x_0, r, t)$, such that $x_n \rightarrow z$ as $n \rightarrow \infty$. Now, for some $q \in \Omega$, we have

$$F(z, S_q z, t) \geq F(z, x_n, (1-k)t) * F(x_n, S_q z, kt).$$

By Lemma 1.8, we have

$$\begin{aligned}
 F(z, S_q z, t) &\geq F(z, x_n, (1-k)t) * H_{FM}(S_r x_{n-1}, S_q z, kt) \\
 &\geq F(z, x_n, (1-k)t) * F(x_{n-1}, z, \frac{kt}{\alpha_{r,q}}) \\
 &\geq F(z, x_n, (1-k)t) * F(x_{n-1}, z, t).
 \end{aligned}$$

Letting $n \rightarrow \infty$, we have

$$F(z, S_q z, t) \geq 1 * 1 = 1.$$

This implies that $z \in S_q z$. Hence, $z \in \bigcap_{q \in \Omega} S_q z$. This completes the proof.

Let $(X, F, *)$ be a fuzzy metric space, $x_0 \in X$ and let S be a multivalued mapping from X to $\tilde{C}_0(X)$. Then, there exists $x_1 \in Sx_0$, such that $F(x_0, Sx_0, t) = F(x_0, x_1, t)$, for all $t > 0$. Let $x_2 \in Sx_1$ be such that $F(x_1, Sx_1, t) = F(x_1, x_2, t)$.

Continuing this process, we construct a sequence x_n of points in X such that $x_{n+1} \in Sx_n$, $F(x_n, Sx_n, t) = F(x_n, x_{n+1}, t)$, for all $t > 0$. We denote this iterative sequence $\{XS(x_n)\}$ and say that $\{XS(x_n)\}$ is a sequence in X generated by x_0 .

Corollary 2.2 Let $(X, F, *)$ be a complete fuzzy metric space, where $*$ be a continuous t -norm, defined as $a * a \geq a$ or $a * a = \min\{a, b\}$. Let $(\hat{C}_0(X), H_{FM}, *)$ is Hausdorff fuzzy metric space on $\hat{C}_0(X)$, $x_0 \in X$, $S : X \rightarrow \hat{C}_0(X)$ be a multivalued mapping and $\{XS(x_n)\}$ be a sequence in X generated by x_0 . Assume that for some $k \in (0, 1)$ $t > 0$, and $x_o \in X$, we have

$$H_{FM}(Sx, Sy, kt) \geq F(x, y, t) \text{ for all } x, y \in \overline{B_F(x_0, r, t)} \cap \{XS(x_n)\} \quad (2.6)$$

and

$$F(x_0, Sx_0, (1 - k)t) \geq 1 - r$$

Then, $\{XS(x_n)\}$ is a sequence in $\overline{B_F(x_0, r, t)}$ and $\{XS(x_n)\} \rightarrow z \in \overline{B_F(x_0, r, t)}$. Also, if (2.6) holds for z , then there exists a fixed point for S in $\overline{B_F(x_0, r, t)}$.

Proof: By using the similar steps as we have used in Theorem 2.1, it can be proved easily.

Corollary 2.3 Let $(X, F, *)$ be a complete fuzzy metric space, where $*$ be a continuous t -norm, defined as $a * a \geq a$ or $a * a = \min\{a, b\}$. Let $x_0 \in X$ and $S : X \rightarrow X$ be a self mapping. Assume that for some $k \in (0, 1)$, $t > 0$ and $x_o \in X$, we have

$$F(Sx, Sy, kt) \geq F(x, y, t) \text{ for all } x, y \in \overline{B_F(x_0, r, t)}$$

and

$$F(x_0, Sx_0, (1 - k)t) \geq 1 - r.$$

Then S has a fixed point in $\overline{B_F(x_0, r, t)}$.

Example 2.4 Let $X = [0, 5]$ and $d : X \times X \rightarrow \mathbb{R}$ be a complete metric space defined by,

$$d(x, y) = |x - y| \text{ for all } x, y \in X$$

Denote $a * b = ab$ or $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$ and $F(x, y, t) = \frac{t}{t + d(x, y)}$ for all $x, y \in X$ and $t > 0$. Then, we can find that $(X, F, *)$ is a complete fuzzy metric space. Consider the multivalued mappings $S_\beta : X \rightarrow \hat{C}_0(X)$ where $\beta = a, 1, 2, 3, \dots$ defined as,

$$S_n x = \begin{cases} [\frac{x}{3^n}, \frac{x}{2^n}] & \text{if } x \in [0, \frac{7}{2}] \\ [2nx, 3nx] & \text{if } x \in (\frac{7}{2}, 5] \end{cases}, \text{ where } n = 1, 2, \dots,$$

and

$$S_a x = \begin{cases} [\frac{x}{3}, \frac{5x}{12}] & \text{if } x \in [0, \frac{7}{2}] \\ [2x, 3x] & \text{if } x \in (\frac{7}{2}, 5] \end{cases}.$$

Considering, $x_0 = \frac{1}{2}$ and $r = \frac{3}{4}$. then, $\overline{B_F(x_0, r, t)} = [0, \frac{7}{2}]$. Now,

$$\begin{aligned}
 F(x_0, S_a x_0, t) &= F\left(\frac{1}{2}, S_a \frac{1}{2}, t\right) = F\left(\frac{1}{2}, \frac{5}{24}, t\right) \\
 F(x_1, S_1 x_1, t) &= F\left(\frac{5}{24}, S_1 \frac{5}{24}, t\right) = F\left(\frac{5}{24}, \frac{5}{48}, t\right)
 \end{aligned}$$

So, we obtain a sequence $\{XS_\beta(x_n)\} = \{\frac{1}{2}, \frac{5}{24}, \frac{5}{48}, \frac{5}{192}, \dots\}$ in X generated by x_0 . Now, for $x = 4, y = 5, k = \alpha_{1,a} = \frac{5}{6}$ and $t = 1$, we have

$$\begin{aligned}
 H_{FM}(S_1 4, S_a 5, \frac{5}{6}) &= \min \left\{ \inf_{a \in S_1 4} F(a, S_a 5, \frac{5}{6}), \inf_{b \in S_a 5} F(S_1 4, b, \frac{5}{6}) \right\} = 0.22 \\
 F(4, 5, 1) &= \frac{1}{1 + |4 - 5|} = \frac{1}{2} = 0.5
 \end{aligned}$$

So, we have

$$H_{FM}(S_1 4, S_a 5, \frac{5}{6}) \not\geq F(4, 5, 1)$$

So, the contractive condition does not hold on X . Now, for all $x, y \in \overline{B_F(x_0, r, t)} \cap \{XS_\beta(x_n)\}$, we have

$$\begin{aligned}
 H_{FM}(S_n x, S_a y, kt) &= \min \left\{ \inf_{a \in S_n x} F(a, S_a y, kt), \inf_{b \in S_a y} F(S_n x, b, kt) \right\} \\
 &= \min \left\{ \inf_{a \in S_n x} F(a, [\frac{y}{3}, \frac{5y}{12}], \frac{5}{6}t), \inf_{b \in S_a y} F([\frac{x}{3n}, \frac{x}{2n}], b, \frac{5}{6}t) \right\} \\
 &= \min \left\{ F\left(\frac{x}{2n}, \frac{5y}{12}, \frac{5}{6}t\right), F\left(\frac{x}{3n}, \frac{y}{3}, \frac{5}{6}t\right) \right\} \\
 &= \min \left\{ \frac{(5/6)t}{(5/6)t + |x/3 - y/3|}, \frac{(5/6)t}{(5/6)t + |y/3 - x/3|} \right\} \\
 H_{FM}(Sx, Sy, kt) &= \frac{(5/6)t}{(5/6)t + |x/3 - y/3|} \geq \frac{t}{t + |x - y|} = F(x, y, t)
 \end{aligned}$$

So, the contractive condition holds on $\overline{B_F(x_0, r, t)} \cap \{XS_\beta(x_n)\}$. Also, for $t = 1$, we have

$$\begin{aligned}
 F(x_0, x_1, (1 - k)t) &= F\left(\frac{1}{2}, \frac{5}{24}, \frac{1}{6}\right) \\
 &= \frac{4}{11} > \frac{1}{4} = 1 - r
 \end{aligned}$$

Hence, all the conditions of above theorem are satisfied. Now, we have $\{XS_\beta(x_n)\}$ is a sequence in $\overline{B_F(x_0, r, t)}$, and $\{XS_\beta(x_n)\} \rightarrow 0 \in \overline{B_F(x_0, r, t)}$. Moreover, $\{S_\beta : \beta = a, 1, 2 \dots\}$ has a common fixed point 0.

Competing interests

The authors declare that they have no competing interests.

References

- [1] M. Arshad, A. Shoaib, I. Beg, Fixed point of a pair of contractive dominated mappings on a closed ball in an ordered complete dislocated metric space, *Fixed Point Theory and Appl.* (2013), 2013:115.
- [2] M. Arshad, A. Shoaib, and P. Vetro, Common Fixed Points of a Pair of Hardy Rogers Type Mappings on a Closed Ball in Ordered Dislocated Metric Spaces, *Journal of Function Spaces*, 2013 (2013), Article ID 63818.
- [3] M. Arshad, A. Shoaib, M. Abbas and A. Azam, Fixed Points of a pair of Kannan Type Mappings on a Closed Ball in Ordered Partial Metric Spaces, *Miskolc Mathematical Notes*, 14(3), 2013.
- [4] M. Arshad, A. Azam, M. Abbas and A. Shoaib, Fixed point results of dominated mappings on a closed ball in ordered partial metric spaces without continuity, *U.P.B. Sci. Bull., Series A*, 76(2), 2014.
- [5] L. A. Zadeh, *Fuzzy Sets. Information and Control*, 8(3), 1965.
- [6] I. Beg, M. Arshad, A. Shoaib, Fixed Point on a Closed Ball in ordered dislocated Metric Space, *Fixed Point Theory*, 16(2), 2015.
- [7] A. George and P. Veeramani, On some results in fuzzy metric spaces, *Fuzzy Sets and Systems* 64(3), 1994.
- [8] A. George and P. Veeramani, On some results of analysis for fuzzy metric spaces, *Fuzzy Sets and Systems*, 90(3), 1997.
- [9] D. Gopal, C. Vetro, Some New Fixed Point Theorems In Fuzzy Metric Spaces, *Iranian Journal of Fuzzy Systems*, 11(3), 2014.
- [10] I. Kramosil, J. Michalek, Fuzzy metric and statistical metric spaces, *Kybernetika* 11(5) (1975).
- [11] J. Rodriguez-Lopez, S. Romaguera, The Hausdorff fuzzy metric on compact sets, *Fuzzy Sets and Systems* 147(2), 2004.
- [12] A. Shoaib, Ph.D thesis, Fixed points theorems for locally and globally contractive mappings in ordered spaces, 2016.
- [13] A. Shoaib, M. Arshad and M. A. Kutbi, Common fixed points of a pair of Hardy Rogers Type Mappings on a Closed Ball in Ordered Partial Metric Spaces, *J. Comput. Anal. Appl.*, 17(2014), 255-264.

^{1,3}Department of Mathematics and Statistics, Riphah International University, Islamabad - 44000, Pakistan. ¹E-mail address: abdullahshoaib15@yahoo.com.
³E-mail address: aqeel4all84@gmail.com.

²Department of Mathematics, COMSATS Institute of Information Technology, Chack Shahzad, Islamabad - 44000, Pakistan. E-mail address: akbarazam@yahoo.com.

ADDITIVE-QUADRATIC ρ -FUNCTIONAL INEQUALITIES IN NON-ARCHIMEDEAN BANACH SPACES

CHOONKIL PARK¹, JUNG RYE LEE², AND DONG YUN SHIN^{3*}

ABSTRACT. Let

$$\begin{aligned} M_1 f(x, y) : &= \frac{3}{4}f(x+y) - \frac{1}{4}f(-x-y) \\ &\quad + \frac{1}{4}f(x-y) + \frac{1}{4}f(y-x) - f(x) - f(y), \\ M_2 f(x, y) : &= 2f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) + f\left(\frac{y-x}{2}\right) - f(x) - f(y). \end{aligned}$$

We solve the additive-quadratic ρ -functional inequalities

$$\|M_1 f(x, y)\| \leq \|M_2 f(x, y)\|, \tag{0.1}$$

where ρ is a fixed non-Archimedean number with $|\rho| < 1$, and

$$\|M_2 f(x, y)\| \leq \|\rho M_1 f(x, y)\|, \tag{0.2}$$

where ρ is a fixed non-Archimedean number with $|\rho| < |2|$.

Furthermore, we prove the Hyers-Ulam stability of the additive-quadratic ρ -functional inequalities (0.1) and (0.2) in non-Archimedean Banach spaces.

1. INTRODUCTION AND PRELIMINARIES

A *valuation* is a function $|\cdot|$ from a field K into $[0, \infty)$ such that 0 is the unique element having the 0 valuation, $|rs| = |r| \cdot |s|$ and the triangle inequality holds, i.e.,

$$|r + s| \leq |r| + |s|, \quad \forall r, s \in K.$$

A field K is called a *valued field* if K carries a valuation. The usual absolute values of \mathbb{R} and \mathbb{C} are examples of valuations.

Let us consider a valuation which satisfies a stronger condition than the triangle inequality. If the triangle inequality is replaced by

$$|r + s| \leq \max\{|r|, |s|\}, \quad \forall r, s \in K,$$

then the function $|\cdot|$ is called a *non-Archimedean valuation*, and the field is called a *non-Archimedean field*. Clearly $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$. A trivial example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except for 0 into 1 and $|0| = 0$.

Throughout this paper, we assume that the base field is a non-Archimedean field, hence call it simply a field.

Definition 1.1. ([8]) Let X be a vector space over a field K with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\| : X \rightarrow [0, \infty)$ is said to be a *non-Archimedean norm* if it satisfies the following conditions:

2010 Mathematics Subject Classification. Primary 46S10, 39B62, 39B52, 47S10, 12J25.

Key words and phrases. Hyers-Ulam stability; non-Archimedean normed space; additive-quadratic ρ -functional inequality.

*Corresponding author: Dong Yun Shin (email: dyshin@uos.ac.kr).

C. PARK, J. LEE, AND D. SHIN

- (i) $\|x\| = 0$ if and only if $x = 0$;
- (ii) $\|rx\| = |r|\|x\|$ ($r \in K, x \in X$);
- (iii) the strong triangle inequality

$$\|x + y\| \leq \max\{\|x\|, \|y\|\}, \quad \forall x, y \in X$$

holds. Then $(X, \|\cdot\|)$ is called a *non-Archimedean normed space*.

Definition 1.2. (i) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X . Then the sequence $\{x_n\}$ is called *Cauchy* if for a given $\varepsilon > 0$ there is a positive integer N such that

$$\|x_n - x_m\| \leq \varepsilon$$

for all $n, m \geq N$.

(ii) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X . Then the sequence $\{x_n\}$ is called *convergent* if for a given $\varepsilon > 0$ there are a positive integer N and an $x \in X$ such that

$$\|x_n - x\| \leq \varepsilon$$

for all $n \geq N$. Then we call $x \in X$ a limit of the sequence $\{x_n\}$, and denote by $\lim_{n \rightarrow \infty} x_n = x$.

(iii) If every Cauchy sequence in X converges, then the non-Archimedean normed space X is called a *non-Archimedean Banach space*.

The stability problem of functional equations originated from a question of Ulam [19] concerning the stability of group homomorphisms. The functional equation $f(x + y) = f(x) + f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [7] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [12] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [6] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. The functional equation $f\left(\frac{x+y}{2}\right) = \frac{1}{2}f(x) + \frac{1}{2}f(y)$ is called the *Jensen equation*.

The functional equation $f(x + y) + f(x - y) = 2f(x) + 2f(y)$ is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The stability of quadratic functional equation was proved by Skof [18] for mappings $f : E_1 \rightarrow E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [5] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. The functional equation $2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) = f(x) + f(y)$ is called a *Jensen type quadratic equation*. The stability problems of various functional equations have been extensively investigated by a number of authors (see [1, 3, 4, 10, 11, 13, 14, 15, 16, 17, 20, 21]).

In Section 2, we solve the additive-quadratic ρ -functional inequality (0.1) and prove the Hyers-Ulam stability of the additive-quadratic ρ -functional inequality (0.1) in non-Archimedean Banach spaces.

In Section 3, we solve the additive-quadratic ρ -functional inequality (0.2) and prove the Hyers-Ulam stability of the additive-quadratic ρ -functional inequality (0.2) in non-Archimedean Banach spaces.

Throughout this paper, assume that X is a non-Archimedean normed space and that Y is a non-Archimedean Banach space. Let $|2| \neq 1$.

ADDITIVE-QUADRATIC ρ -FUNCTIONAL INEQUALITIES

2. ADDITIVE-QUADRATIC ρ -FUNCTIONAL INEQUALITY (0.1) IN NON-ARCHIMEDEAN NORMED SPACES

Throughout this section, assume that ρ is a fixed non-Archimedean number with $|\rho| < 1$.

In this section, we solve the additive-quadratic ρ -functional inequality (0.1) in non-Archimedean normed spaces.

Lemma 2.1.

(i) If an odd mapping $f : X \rightarrow Y$ satisfies

$$\|M_1 f(x, y)\| \leq \|\rho M_2 f(x, y)\| \tag{2.1}$$

for all $x, y \in X$, then $f : X \rightarrow Y$ is additive.

(ii) If an even mapping $f : X \rightarrow Y$ satisfies (2.1), then $f : X \rightarrow Y$ is quadratic.

Proof. (i) Assume that $f : X \rightarrow Y$ satisfies (2.1).

Since f is an odd mapping, $f(0) = 0$.

Letting $y = x$ in (2.1), we get

$$\|f(2x) - 2f(x)\| \leq 0$$

and so $f(2x) = 2f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{2}f(x) \tag{2.2}$$

for all $x \in X$.

It follows from (2.1) and (2.2) that

$$\begin{aligned} \|f(x+y) - f(x) - f(y)\| &\leq \left\| \rho \left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right) \right\| \\ &= |\rho| \|f(x+y) - f(x) - f(y)\| \end{aligned}$$

and so

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$.

(ii) Assume that $f : X \rightarrow Y$ satisfies (2.1).

Letting $x = y = 0$ in (2.1), we get

$$\|f(0)\| \leq \|2\rho f(0)\| = |2| \cdot |\rho| \cdot \|f(0)\|.$$

So $f(0) = 0$.

Letting $y = x$ in (2.1), we get

$$\left\| \frac{1}{2}f(2x) - 2f(x) \right\| \leq 0$$

and so $f(2x) = 4f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{4}f(x) \tag{2.3}$$

for all $x \in X$.

C. PARK, J. LEE, AND D. SHIN

It follows from (2.1) and (2.3) that

$$\begin{aligned} & \left\| \frac{1}{2}f(x+y) + \frac{1}{2}f(x-y) - f(x) - f(y) \right\| \\ & \leq \left\| \rho \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \right) \right\| \\ & = |\rho| \left\| \frac{1}{2}f(x+y) + \frac{1}{2}f(x-y) - f(x) - f(y) \right\| \end{aligned}$$

and so

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in X$. □

We prove the Hyers-Ulam stability of the additive-quadratic ρ -functional inequality (2.1) in non-Archimedean Banach spaces for an odd mapping case.

Theorem 2.2. *Let $r < 1$ and θ be nonnegative real numbers and let $f : X \rightarrow Y$ be an odd mapping such that*

$$\|M_1 f(x, y)\| \leq \|\rho M_2 f(x, y)\| + \theta(\|x\|^r + \|y\|^r) \tag{2.4}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \rightarrow Y$ such that

$$\|f(x) - A(x)\| \leq \frac{2\theta}{|2|^r} \|x\|^r \tag{2.5}$$

for all $x \in X$.

Proof. Since f is an odd mapping, $f(0) = 0$.

Letting $y = x$ in (2.4), we get

$$\|f(2x) - 2f(x)\| \leq 2\theta\|x\|^r \tag{2.6}$$

for all $x \in X$. So $\|f(x) - 2f\left(\frac{x}{2}\right)\| \leq \frac{2}{|2|^r}\theta\|x\|^r$ for all $x \in X$. Hence

$$\begin{aligned} & \left\| 2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right) \right\| \tag{2.7} \\ & \leq \max \left\{ \left\| 2^l f\left(\frac{x}{2^l}\right) - 2^{l+1} f\left(\frac{x}{2^{l+1}}\right) \right\|, \dots, \left\| 2^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 2^m f\left(\frac{x}{2^m}\right) \right\| \right\} \\ & = \max \left\{ |2|^l \left\| f\left(\frac{x}{2^l}\right) - 2f\left(\frac{x}{2^{l+1}}\right) \right\|, \dots, |2|^{m-1} \left\| f\left(\frac{x}{2^{m-1}}\right) - 2f\left(\frac{x}{2^m}\right) \right\| \right\} \\ & \leq \max \left\{ \frac{|2|^l}{|2|^{rl+r}}, \dots, \frac{|2|^{m-1}}{|2|^{r(m-1)+r}} \right\} 2\theta\|x\|^r = \frac{2\theta}{|2|^{(r-1)l+r}} \|x\|^r \end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.7) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $A : X \rightarrow Y$ by

$$A(x) := \lim_{n \rightarrow \infty} 2^n f\left(\frac{x}{2^n}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \rightarrow \infty$ in (2.7), we get (2.5).

ADDITIVE-QUADRATIC ρ -FUNCTIONAL INEQUALITIES

It follows from (2.4) that

$$\begin{aligned} \|M_1A(x, y)\| &= \lim_{n \rightarrow \infty} |2|^n \left\| M_1f \left(\frac{x}{2^n}, \frac{y}{2^n} \right) \right\| \\ &\leq \lim_{n \rightarrow \infty} |2|^n |\rho| \left\| M_2f \left(\frac{x}{2^n}, \frac{y}{2^n} \right) \right\| + \lim_{n \rightarrow \infty} \frac{|2|^{n\theta}}{|2|^{nr}} (\|x\|^r + \|y\|^r) \\ &= |\rho| \|M_2A(x, y)\| = \|\rho M_2A(x, y)\| \end{aligned}$$

for all $x, y \in X$. So

$$\|M_1A(x, y)\| \leq \|\rho M_2A(x, y)\|$$

for all $x, y \in X$. By Lemma 2.1, the mapping $A : X \rightarrow Y$ is additive .

Now, let $T : X \rightarrow Y$ be another additive mapping satisfying (2.5). Then we have

$$\begin{aligned} \|A(x) - T(x)\| &= \left\| 2^q A \left(\frac{x}{2^q} \right) - 2^q T \left(\frac{x}{2^q} \right) \right\| \\ &\leq \max \left\{ \left\| 2^q A \left(\frac{x}{2^q} \right) - 2^q f \left(\frac{x}{2^q} \right) \right\|, \left\| 2^q T \left(\frac{x}{2^q} \right) - 2^q f \left(\frac{x}{2^q} \right) \right\| \right\} \leq \frac{2\theta}{|2|^{(r-1)q+r}} \|x\|^r, \end{aligned}$$

which tends to zero as $q \rightarrow \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$. This proves the uniqueness of h . Thus the mapping $A : X \rightarrow Y$ is a unique additive mapping satisfying (2.5). □

Theorem 2.3. *Let $r > 1$ and θ be nonnegative real numbers and let $f : X \rightarrow Y$ be an odd mapping satisfying (2.4). Then there exists a unique additive mapping $A : X \rightarrow Y$ such that*

$$\|f(x) - A(x)\| \leq \frac{2\theta}{|2|} \|x\|^r \tag{2.8}$$

for all $x \in X$.

Proof. It follows from (2.6) that

$$\left\| f(x) - \frac{1}{2} f(2x) \right\| \leq \frac{2}{|2|} \theta \|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned} &\left\| \frac{1}{2^l} f(2^l x) - \frac{1}{2^m} f(2^m x) \right\| \\ &\leq \max \left\{ \left\| \frac{1}{2^l} f(2^l x) - \frac{1}{2^{l+1}} f(2^{l+1} x) \right\|, \dots, \left\| \frac{1}{2^{m-1}} f(2^{m-1} x) - \frac{1}{2^m} f(2^m x) \right\| \right\} \\ &= \max \left\{ \frac{1}{|2|^l} \left\| f(2^l x) - \frac{1}{2} f(2^{l+1} x) \right\|, \dots, \frac{1}{|2|^{m-1}} \left\| f(2^{m-1} x) - \frac{1}{2} f(2^m x) \right\| \right\} \\ &\leq \max \left\{ \frac{|2|^{lr}}{|2|^{l+1}}, \dots, \frac{|2|^{r(m-1)}}{|2|^{(m-1)+1}} \right\} 2\theta \|x\|^r = \frac{2\theta}{|2|^{(1-r)l+1}} \|x\|^r \end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Now, we prove the Hyers-Ulam stability of the additive-quadratic ρ -functional inequality (2.1) in non-Archimedean Banach spaces for an even mapping case.

C. PARK, J. LEE, AND D. SHIN

Theorem 2.4. *Let $r < 2$ and θ be nonnegative real numbers and let $f : X \rightarrow Y$ be an even mapping satisfying (2.4). Then there exists a unique quadratic mapping $Q : X \rightarrow Y$ such that*

$$\|f(x) - Q(x)\| \leq \frac{|2|}{|2|^r} 2\theta \|x\|^r \tag{2.9}$$

for all $x \in X$.

Proof. Letting $x = y = 0$ in (2.4), we get $\|f(0)\| \leq |\rho| \|2f(0)\|$. So $f(0) = 0$.

Letting $y = x$ in (2.4), we get

$$\left\| \frac{1}{2} f(2x) - 2f(x) \right\| \leq 2\theta \|x\|^r \tag{2.10}$$

for all $x \in X$. So $\|f(x) - 4f\left(\frac{x}{2}\right)\| \leq \frac{|2|}{|2|^r} 2\theta \|x\|^r$ for all $x \in X$. Hence

$$\begin{aligned} & \left\| 4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right) \right\| \tag{2.11} \\ & \leq \max \left\{ \left\| 4^l f\left(\frac{x}{2^l}\right) - 4^{l+1} f\left(\frac{x}{2^{l+1}}\right) \right\|, \dots, \left\| 4^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 4^m f\left(\frac{x}{2^m}\right) \right\| \right\} \\ & = \max \left\{ |4|^l \left\| f\left(\frac{x}{2^l}\right) - 4f\left(\frac{x}{2^{l+1}}\right) \right\|, \dots, |4|^{m-1} \left\| f\left(\frac{x}{2^{m-1}}\right) - 4f\left(\frac{x}{2^m}\right) \right\| \right\} \\ & \leq \max \left\{ \frac{|4|^l}{|2|^{rl}}, \dots, \frac{|4|^{m-1}}{|2|^{r(m-1)}} \right\} \frac{|2|}{|2|^r} 2\theta \|x\|^r = \frac{2\theta}{|2|^{(r-2)l}} \frac{|2|}{|2|^r} \|x\|^r \end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.11) that the sequence $\{4^n f\left(\frac{x}{2^n}\right)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{4^n f\left(\frac{x}{2^n}\right)\}$ converges. So one can define the mapping $Q : X \rightarrow Y$ by

$$Q(x) := \lim_{n \rightarrow \infty} 4^n f\left(\frac{x}{2^n}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \rightarrow \infty$ in (2.11), we get (2.9).

It follows from (2.4) that

$$\begin{aligned} \|M_1 Q(x, y)\| &= \lim_{n \rightarrow \infty} |4|^n \left\| M_1 f\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \right\| \\ &\leq \lim_{n \rightarrow \infty} |4|^n |\rho| \left\| M_2 f\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \right\| + \lim_{n \rightarrow \infty} \frac{|4|^n \theta}{|2|^{nr}} (\|x\|^r + \|y\|^r) \\ &= |\rho| \|M_2 Q(x, y)\| \end{aligned}$$

for all $x, y \in X$. So

$$\|M_1 Q(x, y)\| \leq \|\rho M_2 Q(x, y)\|$$

for all $x, y \in X$. By Lemma 2.1, the mapping $h : X \rightarrow Y$ is quadratic.

Now, let $T : X \rightarrow Y$ be another quadratic mapping satisfying (2.9). Then we have

$$\begin{aligned} \|Q(x) - T(x)\| &= \left\| 4^q Q\left(\frac{x}{2^q}\right) - 4^q T\left(\frac{x}{2^q}\right) \right\| \\ &\leq \max \left\{ \left\| 4^q Q\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right) \right\|, \left\| 4^q T\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right) \right\| \right\} \leq \frac{|2|}{|2|^{(r-2)q+r}} 2\theta \|x\|^r, \end{aligned}$$

which tends to zero as $q \rightarrow \infty$ for all $x \in X$. So we can conclude that $Q(x) = T(x)$ for all $x \in X$. This proves the uniqueness of Q . Thus the mapping $Q : X \rightarrow Y$ is a unique quadratic mapping satisfying (2.9). \square

ADDITIVE-QUADRATIC ρ -FUNCTIONAL INEQUALITIES

Theorem 2.5. *Let $r > 2$ and θ be positive real numbers, and let $f : X \rightarrow Y$ be an even mapping satisfying (2.4). Then there exists a unique quadratic mapping $Q : X \rightarrow Y$ such that*

$$\|f(x) - Q(x)\| \leq \frac{2\theta}{|2|} \|x\|^r$$

for all $x \in X$.

Proof. It follows from (2.10) that

$$\left\| f(x) - \frac{1}{4}f(2x) \right\| \leq \frac{2\theta}{|2|} \|x\|^r$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.4. □

3. ADDITIVE-QUADRATIC ρ -FUNCTIONAL INEQUALITY (0.2)

Throughout this section, assume that ρ is a fixed non-Archimedean number with $|\rho| < |2|$.

In this section, we solve the additive-quadratic ρ -functional inequality (0.2) in non-Archimedean normed spaces.

Lemma 3.1.

(i) *If an odd mapping $f : X \rightarrow Y$ satisfies*

$$\|M_2f(x, y)\| \leq \|\rho M_1f(x, y)\| \tag{3.1}$$

for all $x, y \in X$, then $f : X \rightarrow Y$ is additive.

(ii) *If an even mapping $f : X \rightarrow Y$ satisfies $f(0) = 0$ and (3.1), then $f : X \rightarrow Y$ is quadratic.*

Proof. (i) Assume that $f : X \rightarrow Y$ satisfies (3.1).

Letting $y = 0$ in (3.1), we get

$$\left\| 2f\left(\frac{x}{2}\right) - f(x) \right\| \leq 0 \tag{3.2}$$

and so $f\left(\frac{x}{2}\right) = \frac{1}{2}f(x)$ for all $x \in X$.

It follows from (3.1) and (3.2) that

$$\begin{aligned} \|f(x+y) - f(x) - f(y)\| &= \left\| 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right\| \\ &\leq |\rho| \|f(x+y) - f(x) - f(y)\| \end{aligned}$$

and so

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$.

(ii) Assume that $f : X \rightarrow Y$ satisfies (3.1).

Letting $y = 0$ in (3.1), we get

$$\left\| 4f\left(\frac{x}{2}\right) - f(x) \right\| \leq 0 \tag{3.3}$$

and so $f\left(\frac{x}{2}\right) = \frac{1}{4}f(x)$ for all $x \in X$.

C. PARK, J. LEE, AND D. SHIN

It follows from (3.1) and (3.3) that

$$\begin{aligned} & \left\| \frac{1}{2}f(x+y) + \frac{1}{2}f(x-y) - f(x) - f(y) \right\| \\ &= \left\| 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \right\| \\ &\leq |\rho| \left\| \frac{1}{2}f(x+y) + \frac{1}{2}f(x-y) - f(x) - f(y) \right\| \end{aligned}$$

and so

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in X$. □

We prove the Hyers-Ulam stability of the additive-quadratic ρ -functional inequality (3.1) in non-Archimedean Banach spaces for an odd mapping case.

Theorem 3.2. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : X \rightarrow Y$ be an odd mapping such that*

$$\|M_2f(x, y)\| \leq \|\rho M_1f(x, y)\| + \theta(\|x\|^r + \|y\|^r) \tag{3.4}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \rightarrow Y$ such that

$$\|f(x) - A(x)\| \leq \theta\|x\|^r \tag{3.5}$$

for all $x \in X$.

Proof. Since f is an odd mapping, $f(0) = 0$.

Letting $y = 0$ in (3.4), we get

$$\left\| 2f\left(\frac{x}{2}\right) - f(x) \right\| \leq \theta\|x\|^r \tag{3.6}$$

for all $x \in X$. So

$$\begin{aligned} & \left\| 2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right) \right\| \tag{3.7} \\ &\leq \max \left\{ \left\| 2^l f\left(\frac{x}{2^l}\right) - 2^{l+1} f\left(\frac{x}{2^{l+1}}\right) \right\|, \dots, \left\| 2^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 2^m f\left(\frac{x}{2^m}\right) \right\| \right\} \\ &= \max \left\{ |2|^l \left\| f\left(\frac{x}{2^l}\right) - 2f\left(\frac{x}{2^{l+1}}\right) \right\|, \dots, |2|^{m-1} \left\| f\left(\frac{x}{2^{m-1}}\right) - 2f\left(\frac{x}{2^m}\right) \right\| \right\} \\ &\leq \max \left\{ \frac{|2|^l}{|2|^{rl}}, \dots, \frac{|2|^{m-1}}{|2|^{r(m-1)}} \right\} \theta\|x\|^r = \frac{\theta}{|2|^{(r-1)l}} \|x\|^r \end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.7) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $A : X \rightarrow Y$ by

$$A(x) := \lim_{n \rightarrow \infty} 2^n f\left(\frac{x}{2^n}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \rightarrow \infty$ in (3.7), we get (3.5).

The rest of the proof is similar to the proof of Theorem 2.2. □

ADDITIVE-QUADRATIC ρ -FUNCTIONAL INEQUALITIES

Theorem 3.3. *Let $r > 1$ and θ be positive real numbers, and let $f : X \rightarrow Y$ be an odd mapping satisfying (3.4). Then there exists a unique additive mapping $A : X \rightarrow Y$ such that*

$$\|f(x) - A(x)\| \leq \frac{|2|^{r\theta}}{|2|} \|x\|^r \tag{3.8}$$

for all $x \in X$.

Proof. It follows from (3.6) that

$$\left\| f(x) - \frac{1}{2}f(2x) \right\| \leq \frac{|2|^{r\theta}}{|2|} \|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned} & \left\| \frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x) \right\| \tag{3.9} \\ & \leq \max \left\{ \left\| \frac{1}{2^l}f(2^l x) - \frac{1}{2^{l+1}}f(2^{l+1} x) \right\|, \dots, \left\| \frac{1}{2^{m-1}}f(2^{m-1} x) - \frac{1}{2^m}f(2^m x) \right\| \right\} \\ & = \max \left\{ \frac{1}{|2|^l} \left\| f(2^l x) - \frac{1}{2}f(2^{l+1} x) \right\|, \dots, \frac{1}{|2|^{m-1}} \left\| f(2^{m-1} x) - \frac{1}{2}f(2^m x) \right\| \right\} \\ & \leq \max \left\{ \frac{|2|^{rl}}{|2|^{l+1}}, \dots, \frac{|2|^{r(m-1)}}{|2|^{(m-1)+1}} \right\} |2|^{r\theta} \|x\|^r = \frac{|2|^{r\theta}}{|2|^{(1-r)l+1}} \|x\|^r \end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.9) that the sequence $\{\frac{1}{2^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n}f(2^n x)\}$ converges. So one can define the mapping $A : X \rightarrow Y$ by

$$A(x) := \lim_{n \rightarrow \infty} \frac{1}{2^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \rightarrow \infty$ in (3.9), we get (3.8).

The rest of the proof is similar to the proofs of Theorems 2.2 and 3.2. □

Now, we prove the Hyers-Ulam stability of the additive-quadratic ρ -functional inequality (3.1) in non-Archimedean Banach spaces for an even mapping case.

Theorem 3.4. *Let $r < 2$ and θ be nonnegative real numbers, and let $f : X \rightarrow Y$ be an even mapping satisfying (3.4). Then there exists a unique quadratic mapping $Q : X \rightarrow Y$ such that*

$$\|f(x) - Q(x)\| \leq \theta \|x\|^r \tag{3.10}$$

for all $x \in X$.

Proof. Letting $x = y = 0$ in (3.4), we get $\|2f(0)\| \leq |\rho| \|f(0)\|$. So $f(0) = 0$.

Letting $y = 0$ in (3.4), we get

$$\left\| 4f\left(\frac{x}{2}\right) - f(x) \right\| \leq \theta \|x\|^r \tag{3.11}$$

for all $x \in X$. So

$$\begin{aligned} & \left\| 4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right) \right\| & (3.12) \\ & \leq \max \left\{ \left\| 4^l f\left(\frac{x}{2^l}\right) - 4^{l+1} f\left(\frac{x}{2^{l+1}}\right) \right\|, \dots, \left\| 4^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 4^m f\left(\frac{x}{2^m}\right) \right\| \right\} \\ & = \max \left\{ |4|^l \left\| f\left(\frac{x}{2^l}\right) - 4f\left(\frac{x}{2^{l+1}}\right) \right\|, \dots, |4|^{m-1} \left\| f\left(\frac{x}{2^{m-1}}\right) - 4f\left(\frac{x}{2^m}\right) \right\| \right\} \\ & \leq \max \left\{ \frac{|4|^l}{|2|^{rl}}, \dots, \frac{|4|^{m-1}}{|2|^{r(m-1)}} \right\} \theta \|x\|^r = \frac{\theta}{|2|^{(r-2)l}} \|x\|^r \end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.12) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $Q : X \rightarrow Y$ by

$$Q(x) := \lim_{n \rightarrow \infty} 4^n f\left(\frac{x}{2^n}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \rightarrow \infty$ in (3.12), we get (3.10).

The rest of the proof is similar to the proof of Theorem 2.2. □

Theorem 3.5. *Let $r > 2$ and θ be positive real numbers, and let $f : X \rightarrow Y$ be an even mapping satisfying (3.4). Then there exists a unique quadratic mapping $Q : X \rightarrow Y$ such that*

$$\|f(x) - Q(x)\| \leq \frac{|2|^r \theta}{|4|} \|x\|^r \tag{3.13}$$

for all $x \in X$.

Proof. It follows from (3.11) that

$$\left\| f(x) - \frac{1}{4} f(2x) \right\| \leq \frac{|2|^r \theta}{|4|} \|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned} & \left\| \frac{1}{4^l} f(2^l x) - \frac{1}{4^m} f(2^m x) \right\| & (3.14) \\ & \leq \max \left\{ \left\| \frac{1}{4^l} f(2^l x) - \frac{1}{4^{l+1}} f(2^{l+1} x) \right\|, \dots, \left\| \frac{1}{4^{m-1}} f(2^{m-1} x) - \frac{1}{4^m} f(2^m x) \right\| \right\} \\ & = \max \left\{ \frac{1}{|4|^l} \left\| f(2^l x) - \frac{1}{4} f(2^{l+1} x) \right\|, \dots, \frac{1}{|4|^{m-1}} \left\| f(2^{m-1} x) - \frac{1}{4} f(2^m x) \right\| \right\} \\ & \leq \max \left\{ \frac{|2|^{rl}}{|4|^{l+1}}, \dots, \frac{|2|^{r(m-1)}}{|4|^{(m-1)+1}} \right\} |2|^r \theta \|x\|^r = \frac{|2|^r \theta}{|2|^{(2-r)l+2}} \|x\|^r \end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.14) that the sequence $\{\frac{1}{4^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n} f(2^n x)\}$ converges. So one can define the mapping $Q : X \rightarrow Y$ by

$$Q(x) := \lim_{n \rightarrow \infty} \frac{1}{4^n} f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \rightarrow \infty$ in (3.14), we get (3.13).

The rest of the proof is similar to the proofs of Theorems 2.2 and 3.4. □

ADDITIVE-QUADRATIC ρ -FUNCTIONAL INEQUALITIES

REFERENCES

- [1] M. Adam, *On the stability of some quadratic functional equation*, J. Nonlinear Sci. Appl. **4** (2011), 50–59.
- [2] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64–66.
- [3] L. Cădariu, L. Găvruta and P. Găvruta, *On the stability of an affine functional equation*, J. Nonlinear Sci. Appl. **6** (2013), 60–67.
- [4] A. Chahbi and N. Bounader, *On the generalized stability of d'Alembert functional equation*, J. Nonlinear Sci. Appl. **6** (2013), 198–204.
- [5] P. W. Cholewa, *Remarks on the stability of functional equations*, Aequationes Math. **27** (1984), 76–86.
- [6] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–43.
- [7] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. U.S.A. **27** (1941), 222–224.
- [8] M.S. Moslehian and Gh. Sadeghi, *A Mazur-Ulam theorem in non-Archimedean normed spaces*, Nonlinear Anal.–TMA **69** (2008), 3405–3408.
- [9] C. Park, Y. Cho and M. Han, *Functional inequalities associated with Jordan-von Neumann-type additive functional equations*, J. Inequal. Appl. **2007** (2007), Article ID 41820, 13 pages.
- [10] C. Park, K. Ghasemi, S. G. Ghaleh and S. Jang, *Approximate n -Jordan $*$ -homomorphisms in C^* -algebras*, J. Comput. Anal. Appl. **15** (2013), 365–368.
- [11] C. Park, A. Najati and S. Jang, *Fixed points and fuzzy stability of an additive-quadratic functional equation*, J. Comput. Anal. Appl. **15** (2013), 452–462.
- [12] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.
- [13] S. Schin, D. Ki, J. Chang and M. Kim, *Random stability of quadratic functional equations: a fixed point approach*, J. Nonlinear Sci. Appl. **4** (2011), 37–49.
- [14] S. Shagholi, M. Bavand Savadkouhi and M. Eshaghi Gordji, *Nearly ternary cubic homomorphism in ternary Fréchet algebras*, J. Comput. Anal. Appl. **13** (2011), 1106–1114.
- [15] S. Shagholi, M. Eshaghi Gordji and M. Bavand Savadkouhi, *Stability of ternary quadratic derivation on ternary Banach algebras*, J. Comput. Anal. Appl. **13** (2011), 1097–1105.
- [16] D. Shin, C. Park and Sh. Farhadabadi, *On the superstability of ternary Jordan C^* -homomorphisms*, J. Comput. Anal. Appl. **16** (2014), 964–973.
- [17] D. Shin, C. Park and Sh. Farhadabadi, *Stability and superstability of J^* -homomorphisms and J^* -derivations for a generalized Cauchy-Jensen equation*, J. Comput. Anal. Appl. **17** (2014), 125–134.
- [18] F. Skof, *Proprietà locali e approssimazione di operatori*, Rend. Sem. Mat. Fis. Milano **53** (1983), 113–129.
- [19] S. M. Ulam, *A Collection of the Mathematical Problems*, Interscience Publ. New York, 1960.
- [20] C. Zaharia, *On the probabilistic stability of the monomial functional equation*, J. Nonlinear Sci. Appl. **6** (2013), 51–59.
- [21] S. Zolfaghari, *Approximation of mixed type functional equations in p -Banach spaces*, J. Nonlinear Sci. Appl. **3** (2010), 110–122.

¹RESEARCH INSTITUTE FOR NATURAL SCIENCES,
HANYANG UNIVERSITY, SEOUL 04763,
REPUBLIC OF KOREA
E-mail address: baak@hanyang.ac.kr

²DEPARTMENT OF MATHEMATICS,
DAEJIN UNIVERSITY, KYUNGGI 11159,
REPUBLIC OF KOREA
E-mail address: jrlee@hdaejin.ac.kr

³DEPARTMENT OF MATHEMATICS,
UNIVERSITY OF SEOUL, SEOUL 02504,
REPUBLIC OF KOREA
E-mail address: dyshin@uos.ac.kr

Differential equations associated with the generalized Euler polynomials of the second kind

C. S. Ryoo

Department of Mathematics, Hannam University, Daejeon 306-791, Korea

Abstract : In this paper, we study linear differential equations arising from the generating functions of the generalized Euler polynomials of the second kind. We give explicit identities for the second kind Euler polynomials.

Key words : linear differential equations, the second kind Euler numbers, generalized Euler polynomials of the second kind.

2000 Mathematics Subject Classification : 11B68, 11S40, 11S80.

1. Introduction

Recently, many mathematicians have studied in the area of the Bernoulli numbers, Euler numbers, Genocchi numbers, and the second kind Euler numbers(see [1, 2, 3, 4, 6, 7]). The generalized Euler polynomials $\mathcal{E}_n(x)(n \geq 0)$ of the second kind, were introduced by Ryoo(see [5, 6]). The generalized Euler polynomials $\mathcal{E}_n(x)$ of the second kind are defined by the generating function:

$$F = F(t, x) = \left(\frac{2e^t}{e^{2t} + 1} \right)^x = \sum_{n=0}^{\infty} \mathcal{E}_n(x) \frac{t^n}{n!}. \tag{1.1}$$

We recall that the classical Stirling numbers of the first kind $S_1(n, k)$ and $S_2(n, k)$ are defined by the relations(see [8])

$$(x)_n = \sum_{k=0}^n S_1(n, k)x^k \text{ and } x^n = \sum_{k=0}^n S_2(n, k)(x)_k, \tag{1.2}$$

respectively. Here $(x)_n = x(x-1) \cdots (x-n+1)$ denotes the falling factorial polynomial of order n . The numbers $S_2(n, m)$ also admit a representation in terms of a generating function

$$\sum_{n=m}^{\infty} S_2(n, m) \frac{t^n}{n!} = \frac{(e^t - 1)^m}{m!}. \tag{1.3}$$

We also have

$$\sum_{n=m}^{\infty} S_1(n, m) \frac{t^n}{n!} = \frac{(\log(1+t))^m}{m!}. \tag{1.4}$$

If x is a variable, we use the following notation:

$$\langle x \rangle_k = x(x+1) \cdots (x+k-1), \quad \binom{x}{k} = \frac{(x)_k}{k!}, \quad (1+t)^x = \sum_{k=0}^{\infty} \binom{x}{k} t^k. \tag{1.5}$$

Nonlinear differential equations arising from the generating functions of special polynomials are studied by many authors in order to give explicit identities for special polynomials(see [3, 4]). In this paper, we study linear differential equations arising from the generating functions of the generalized Euler polynomials of the second kind. We give explicit identities for the generalized Euler polynomials of the second kind.

2. Differential equations associated with the generalized Euler polynomials of the second kind

In this section, we study linear differential equations arising from the generating functions of the generalized Euler polynomials $\mathcal{E}_n(x)$ of the second kind.

Let

$$F = F(t, x) = \left(\frac{2e^t}{e^{2t} + 1} \right)^x. \tag{2.1}$$

Then, by (2.1), we have

$$\begin{aligned} F^{(1)} &= \frac{d}{dt} F(t, x) = \frac{d}{dt} \left(\frac{2e^t}{e^{2t} + 1} \right)^x = x \left(\frac{2e^t}{e^{2t} + 1} \right)^{x-1} \left(\frac{2e^t}{e^{2t} + 1} - \left(\frac{2e^t}{e^{2t} + 1} \right)^2 e^t \right) \\ &= xF(t, x) - xF(t, x + 1)e^t, \\ F^{(2)} &= \frac{d}{dt} F^{(1)} = xF(t, x)^{(1)} - xF^{(1)}(t, x + 1)e^t - xF(t, x + 1)e^t \\ &= x(xF(t, x) - xF(t, x + 1)e^t) \\ &\quad - x((x + 1)F(t, x + 1) - (x + 1)F(t, x + 2)e^t)e^t - xF(t, x + 1)e^t, \\ &= x^2F(t, x) - (2x^2 + 2x)F(t, x + 1)e^t + x(x + 1)F(t, x + 2)e^{2t}, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} F^{(3)} &= \frac{d}{dt} F^{(2)} = x^2F^{(1)}(t, x) - (2x^2 + 2x)F^{(1)}(t, x + 1)e^t - (2x^2 + 2x)(t, x + 1)e^t \\ &\quad + x(x + 1)F^{(1)}(t, x + 2)e^{2t} + 2x(x + 1)F(t, x + 2)e^{2t} \\ &= x^3F(t, x) - (3x^3 + 6x^2 + 4x)F(t, x + 1)e^t \\ &\quad - (3x^3 + 9x^2 + 6x)F(t, x + 2)e^{2t} - (x^3 + 3x^2 + 2x)F(t, x + 3)e^{3t}. \end{aligned} \tag{2.3}$$

Continuing this process, we can guess that

$$\begin{aligned} F^{(N)} &= \left(\frac{d}{dt} \right)^N F(t, x) \\ &= \sum_{i=0}^N a_i(N, x)F(t, x + i)e^{it}, \quad (N = 0, 1, 2, \dots). \end{aligned} \tag{2.4}$$

Taking the derivative with respect to t in (2.4), we have

$$\begin{aligned} F^{(N+1)} &= \frac{dF^{(N)}}{dt} \\ &= \sum_{i=0}^N a_i(N, x)ie^{it}F(t, x + i) + \sum_{i=0}^N a_i(N, x)e^{it}F^{(1)}(t, x + i) \\ &= \sum_{i=0}^N a_i(N, x)(x + 2i)e^{it}F(t, x + i) - \sum_{i=1}^{N+1} a_{i-1}(N, x)(x + i - 1)e^{it}F(t, x + i). \end{aligned} \tag{2.5}$$

On the other hand, by replacing N by $N + 1$ in (2.4), we get

$$F^{(N+1)} = \sum_{i=0}^{N+1} a_i(N + 1, x)e^{it}F(t, x + i) \tag{2.6}$$

By (2.5) and (2.6), we have

$$\begin{aligned} &\sum_{i=0}^N a_i(N, x)(x + 2i)e^{it}F(t, x + i) - \sum_{i=1}^{N+1} a_{i-1}(N, x)(x + i - 1)e^{it}F(t, x + i) \\ &= \sum_{i=0}^{N+1} a_i(N + 1, x)e^{it}F(t, x + i). \end{aligned} \tag{2.7}$$

Comparing the coefficients on both sides of (2.7), we obtain

$$a_0(N + 1, x) = xa_0(N, x), \quad a_{N+1}(N + 1, x) = -(x + N)a_N(N, x), \tag{2.8}$$

and

$$a_i(N + 1, x) = (x + 2i)a_i(N, x) - (x + i - 1)a_{i-1}(N, x), \quad (1 \leq i \leq N). \tag{2.9}$$

In addition, by (2.2) and (2.4), we get

$$F^{(0)} = F^{(0)}(t, x) = a_0(0, x) = F(t, x). \tag{2.10}$$

Thus, by (2.10), we obtain

$$a_0(0, x) = 1. \tag{2.11}$$

It is not difficult to show that

$$\begin{aligned} xF(t, x) - xF(t, x + 1)e^t &= \sum_{i=0}^1 a_i(1, x)e^{it}F(t, x + i) \\ &= a_0(1, x)F(t, x) + a_1(1, x)F(t, x + 1)e^t. \end{aligned} \tag{2.12}$$

Thus, by (2.12), we also get

$$a_0(1, x) = x, \quad a_1(1, x) = -x. \tag{2.13}$$

From (2.8), we note that

$$a_0(N + 1, x) = xa_0(N, x) = \dots = x^{N+1}a_0(0, x) = x^{N+1},$$

and

$$\begin{aligned} a_{N+1}(N + 1, x) &= -(x + N)a_N(N, x) = \dots \\ &= (-1)^{N+1}(x + N)(x + N - 1) \dots (x + 1)x. \end{aligned} \tag{2.14}$$

For $i = 1, 2, 3$ in (2.9), we get

$$a_1(N + 1, x) = -x \sum_{k=0}^N (x + 2 \cdot 1)^k a_0(N - k, x),$$

$$a_2(N + 1, x) = -(x + 1) \sum_{k=0}^{N-1} (x + 2 \cdot 2)^k a_1(N - k, x),$$

and

$$a_3(N + 1, x) = -(x + 2) \sum_{k=0}^{N-2} (x + 2 \cdot 3)^k a_2(N - k, x).$$

Continuing this process, we can deduce that, for $1 \leq i \leq N$,

$$a_i(N + 1, x) = -(x + i - 1) \sum_{k=0}^{N+1-i} (x + 2 \cdot i)^k a_{i-1}(N - k, x). \tag{2.15}$$

Here, we note that the matrix $a_i(j, x)_{0 \leq i, j \leq N}$ is given by

$$\begin{pmatrix} 1 & x & x^2 & x^3 & \dots & x^N \\ 0 & (-1) \langle x \rangle_1 & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & (-1)^2 \langle x \rangle_2 & \cdot & \dots & \cdot \\ 0 & 0 & 0 & (-1)^3 \langle x \rangle_3 & \dots & \cdot \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & (-1)^N \langle x \rangle_N \end{pmatrix}$$

Now, we give explicit expressions for $a_i(N + 1, x)$. By (2.14) and (2.15), we get

$$\begin{aligned} a_1(N + 1, x) &= -x \sum_{k_1=0}^N (x + 2)^{k_1} a_0(N - k_1, x) \\ &= -x \sum_{k_1=0}^{N-1} (x + 2)^{k_1} x^{N-k_1} \\ a_2(N + 1, x) &= -(x + 1) \sum_{k_2=0}^{N-1} (x + 2 \cdot 2)^{k_2} a_1(N - k_2, x) \\ &= (-1)^2 x(x + 1) \sum_{k_2=0}^{N-1} (x + 2 \cdot 2)^{k_2} (x + 2)^{k_1} x^{N-k_2-k_1-1}, \end{aligned}$$

and

$$\begin{aligned} a_3(N + 1) &= -(x + 2) \sum_{k_3=0}^{N-2} (x + 4)^{k_3} a_2(N - k_3, x) \\ &= (-1)^3 x(x + 1)(x + 2) \\ &\quad \times \sum_{k_3=0}^{N-2} \sum_{k_2=0}^{N-k_3-2} \sum_{k_1=0}^{N-k_3-k_2-2} (x + 2 \cdot 3)^{k_3} (x + 2 \cdot 2)^{k_2} (x + 2)^{k_1} x^{N-k_3-k_2-k_1-2}. \end{aligned}$$

Continuing this process, we have

$$\begin{aligned} a_i(N + 1) &= (-1)^i \langle x \rangle_i \sum_{k_i=0}^{N-i+1} \sum_{k_{i-1}=0}^{N-k_i-i+1} \cdots \sum_{k_1=0}^{N-k_i-\cdots-k_2-i+1} \prod_{l=1}^i (x + 2l)^{k_l} x^{N-k_i-k_{i-1}-\cdots-k_2-k_1-i+1}. \end{aligned} \tag{2.16}$$

Therefore, by (2.16), we obtain the following theorem.

Theorem 1. For $N = 0, 1, 2, \dots$, the linear functional equations

$$F^{(N)} = \left(\sum_{i=0}^N a_i(N, x) \left(\frac{2e^t}{e^{2t} + 1} \right)^i e^{it} \right) F$$

have a solution

$$F = F(t, x) = \left(\frac{2e^t}{e^{2t} + 1} \right)^x,$$

where

$$\begin{aligned} a_0(N, x) &= x^N, \\ a_N(N, x) &= (-1)^N \langle x \rangle_N, \\ a_i(N) &= (-1)^i \langle x \rangle_i \sum_{k_i=0}^{N-i} \sum_{k_{i-1}=0}^{N-k_i-i} \cdots \sum_{k_1=0}^{N-k_i-\cdots-k_2-i} \prod_{l=1}^i (x + 2l)^{k_l} x^{N-k_i-k_{i-1}-\cdots-k_2-k_1-i}, \\ &(1 \leq i \leq N - 1). \end{aligned}$$

From (1.1), we note that

$$F^{(N)} = \left(\frac{d}{dt} \right)^N F(t, x) = \sum_{k=0}^{\infty} \mathcal{E}_{k+N}(x) \frac{t^k}{k!}. \tag{2.17}$$

From Theorem 1 and (2.17), we can derive the following equation:

$$\begin{aligned}
 \sum_{k=0}^{\infty} \mathcal{E}_{k+N}(x) \frac{t^k}{k!} &= F^{(N)} \\
 &= \sum_{i=0}^N a_i(N, x) e^{it} \left(\frac{2e^t}{e^{2t} + 1} \right)^{x+i} \\
 &= \sum_{i=0}^N a_i(N, x) \left(\sum_{k=0}^{\infty} i^k \frac{t^k}{k!} \right) \left(\sum_{k=0}^{\infty} \mathcal{E}_k(x+i) \frac{t^k}{k!} \right) \\
 &= \sum_{k=0}^{\infty} \left(\sum_{i=0}^N \sum_{l=0}^k \binom{k}{l} a_i(N, x) i^{k-l} \mathcal{E}_l(x+i) \right) \frac{t^k}{k!}.
 \end{aligned} \tag{2.18}$$

By comparing the coefficients on both sides of (2.18), we obtain the following theorem.

Theorem 2. For $k = 0, 1, \dots$, and $N = 0, 1, 2, \dots$, we have

$$\mathcal{E}_{k+N}(x) = \sum_{i=0}^N \sum_{l=0}^k \binom{k}{l} a_i(N, x) i^{k-l} \mathcal{E}_l(x+i), \tag{1.19}$$

$$a_0(N, x) = x^N,$$

$$a_N(N, x) = (-1)^N \langle x \rangle_N,$$

$$a_i(N) = (-1)^i \langle x \rangle_i \sum_{k_i=0}^{N-i} \sum_{k_{i-1}=0}^{N-k_i-i} \cdots \sum_{k_1=0}^{N-k_i-\cdots-k_2-i} \prod_{l=1}^i (x+2l)^{k_l} x^{N-k_i-k_{i-1}-\cdots-k_2-k_1-i},$$

$$(1 \leq i \leq N-1).$$

Let us take $k = 0$ in (2.19). Then, we have the following corollary.

Corollary 3. For $N = 0, 1, 2, \dots$, we have

$$\mathcal{E}_N(x) = \sum_{i=0}^N a_i(N, x).$$

The first few of them are

$$\mathcal{E}_0(x) = 1,$$

$$\mathcal{E}_1(x) = 0, \quad \mathcal{E}_2^{(x)} = -x,$$

$$\mathcal{E}_3(x) = 0, \quad \mathcal{E}_4(x) = 2x + 3x^2,$$

$$\mathcal{E}_5(x) = 0, \quad \mathcal{E}_6(x) = -16x - 30x^2 - 15x^3,$$

$$\mathcal{E}_7(x) = 0, \quad \mathcal{E}_8(x) = 272x + 588x^2 + 420x^3 + 105x^4,$$

$$\mathcal{E}_9(x) = 0, \quad \mathcal{E}_{10}(x) = -7936x - 18960x^2 - 16380x^3 - 6300x^4 - 945x^5.$$

For $N = 0, 1, 2, \dots$, the linear functional equations

$$F^{(N)} = \left(\sum_{i=0}^N a_i(N, x) \left(\frac{2e^t}{e^{2t} + 1} \right)^i e^{it} \right) F$$

have a solution

$$F = F(t, x) = \left(\frac{2e^t}{e^{2t} + 1} \right)^x.$$

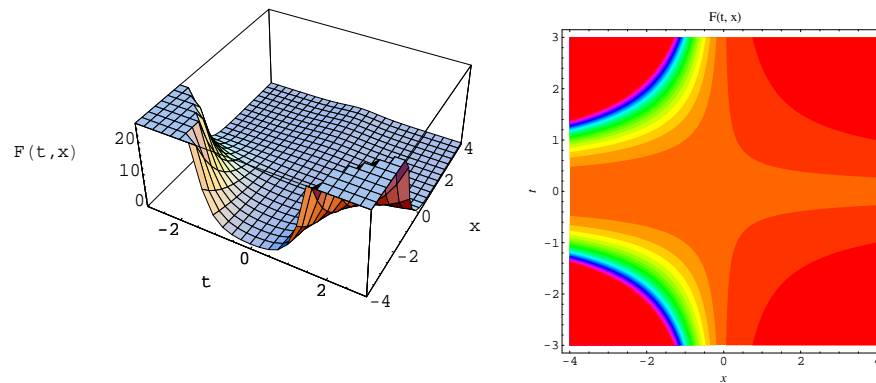


Figure 1: The surface for the solution $F(t, x)$

Here is a plot of the surface for this solution. In Figure 1(left), we plot of the surface for this solution. In Figure 1(right), we shows a higher-resolution density plot of the solution.

The author has no doubt that investigations along this line will lead to a new approach employing numerical method in the research field of the generalized Euler polynomials of the second kind to appear in mathematics and physics(see [5, 6, 7]).

REFERENCES

1. A. Bayad, T. Kim, *Higher recurrences for Apostol-Bernoulli-Euler numbers*, Russ. J. Math. Phys. **19(1)**, (2012), 1-10.
2. A. Erdelyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, *Higher Transcendental Functions*, Vol 3. New York: Krieger, 1981.
3. D.S. Kim, T. Kim, *Some identities of Bell polynomials*, Sci. China Math. **58(10)**(2015), 2095-2104.
4. T. Kim, D.S. Kim, *Identities involving degenerate Euler numbers and polynomials arising from non-linear differential equations*, J. Nonlinear Sci. Appl., **9**(2016), 2086-2098.
5. Y.H. Kim, H.Y. Jung, C.S. Ryoo, *On the generalized Euler polynomials of the second kind*, J. Appl. Math. & Informatics, **31**(2013), 623 - 630
6. C. S. Ryoo, Y.H. Kim, H.Y. Jung, *Distribution of the zeros of the generalized Euler polynomials of second kind*, Int. Journal of Math. Analysis, **7(43)**(2013), 2129 - 2136.
7. C.S. Ryoo, *Calculating zeros of the second kind Euler polynomials*, Journal of Computational Analysis and Applications, **12** (2010), 828-833.
8. P. T. Young, *Degenerate Bernoulli polynomials, generalized factorial sums, and their applications*, Journal of Number Theorey., **128**(2008), 738-758.

Chlodowsky Variant of Bernstein-Schurer Operators Based on (p,q) -Integers

Eser Gemikonakli · Tuba Vedi-Dilek

Abstract In this paper, we introduce the Chlodowsky variant of Bernstein-Schurer operators based on (p,q) -integers. By obtaining first few moments of these operators, we prove well-known Korovkin-type approximation theorems in different function spaces. We also compute the error of approximation by using modulus of continuity and Lipschitz-type functionals. Moreover, we study the generalization of the Chlodowsky variant of Bernstein-Schurer operators based on (p,q) -integers and investigate their approximations. Finally, numerical results are presented in detail.

Keywords (p,q) -integers, q -Bernstein operators, q -Bernstein-Schurer operators.

1 Introduction

The classical Bernstein-Chlodowsky operators were defined by Chlodowsky [4] as

$$C_n(f; x) = \sum_{r=0}^n f\left(\frac{r}{n}b_n\right) \binom{n}{r} \left(\frac{x}{b_n}\right)^r \left(1 - \frac{x}{b_n}\right)^{n-r},$$

where the function f is defined on $[0, \infty)$ and (b_n) is a positive increasing sequence with $b_n \rightarrow \infty$ and $\frac{b_n}{n} \rightarrow 0$ as $n \rightarrow \infty$.

In 2008, Karsh and Gupta [9] defined q -analogue of Chlodowsky operators as follows:

$$C_n(f; q; x) = \sum_{k=0}^n f\left(\frac{[k]_q}{[n]_q}b_n\right) \begin{bmatrix} n \\ k \end{bmatrix}_q \left(\frac{x}{b_n}\right)^k \prod_{s=0}^{n-k-1} \left(1 - q^s \frac{x}{b_n}\right), \quad 0 \leq x \leq b_n$$

where (b_n) has the same properties of Bernstein-Chlodowsky operators.

Eser Gemikonakli
 Institute of Applied Sciences, University of Kyrenia, Girne, Mersin 10, Turkey
 E-mail: eser.gemikonakli@kyrenia.edu.tr

Tuba Vedi-Dilek
 Institute of Applied Sciences, University of Kyrenia, Girne, Mersin 10, Turkey
 E-mail: tuba.vedi@kyrenia.edu.tr

In 1987, Lupaş [11] defined the q -based Bernstein operators and obtained the Korovkin-type approximation theorem. Over the past several years, there has been a considerable amount of research on the q -based operators (see [2], [3], [7], [12], [13],[16],[17], [20], [21], [22], [24]). To date, the focus of published work has been largely on (p,q) based operators.

In 2015, (p,q) -analogue of Bernstein operators were introduced by Mursaleen et al [14] as

$$B_{n,p,q}(f;x) = \frac{1}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}} x^k \prod_{s=0}^{n-k-1} (p^s - q^s x) f\left(\frac{[k]_{p,q}}{p^{k-n} [n]_{p,q}}\right), \quad x \in [0, 1]. \tag{1.1}$$

For $0 < q \leq p < 1$, the (p,q) -numbers are given as [6]

$$[k]_{p,q} = \frac{p^k - q^k}{p - q}.$$

For each $k \in \mathbb{N}_0$ the (p,q) -factorial is represented by

$$[k]_{p,q}! = \begin{cases} [k][k-1] \dots [1], & k = 1, 2, 3, \dots, \\ 1, & k = 0 \end{cases}$$

and (p,q) -binomial coefficients are defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[n-k]_{p,q}! [k]_{p,q}!}$$

where $n \geq k \geq 0$. Note that, as it is introduced in [18], the operators are reduced to the q -Bernstein operators for $p = 1$ in Eq.(1.1).

Recently, the (p,q) -analogue of Bernstein-Schurer operators have been introduced by Sidharth and Agrawal [1] as

$$\tilde{B}_{n,s}(f;p,q;x) = \sum_{k=0}^{n+s} p^{\frac{k(k-1)}{2} - \frac{(n+s)(n+s-1)}{2}} \begin{bmatrix} n+s \\ k \end{bmatrix}_{p,q} x^k (1-x)_{p,q}^{n+s-k} f\left(p^{n-k} \frac{[k]_{p,q}}{[n]_{p,q}}\right)$$

where $f \in C[0, 1+s]$, $s \in \mathbb{N}_0$ and $n \in \mathbb{N}$.

Corollary 1 *If we use the properties of (p,q) integers, we have*

$$[n]_{p,q} = p^{n-1} [n]_{q/p}$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = p^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{q/p}.$$

This paper is structured in the following way;

The next section introduces the Chlodowsky variant of Bernstein-Schurer operators based on (p,q) -integers and investigate the moments of the operator. Section 3 discusses several Korovkin-type theorems in different function spaces. In section 4, we obtain the order of convergence of the Chlodowsky variant of Bernstein-Schurer operators based on (p,q) -integers by means of Lipschitz class functions and the first modulus of continuity. Section 5 provides the generalization of the Chlodowsky variant of Bernstein-Schurer operators based on (p,q) -integers and investigate their approximations. Finally, in section 6, numerical results to illustrate the contribution of the Chlodowsky Variant of Bernstein-Schurer Operators based on (p,q) -integers are presented.

2 Construction of the Operators

We construct the Chlodowsky variant of Bernstein-Schurer operators based on (p, q) -integers as

$$\begin{aligned} \bar{C}_{n,s}(f; p, q; x) & \quad (2.1) \\ & := \sum_{k=0}^{n+s} f\left(p^{n-k} \frac{[k]_{p,q}}{[n]_{p,q}} b_n\right) p^{\frac{k(k-1)}{2} - \frac{(n+s-1)(n+s)}{2}} \begin{bmatrix} n+s \\ k \end{bmatrix}_{p,q} \left(\frac{x}{b_n}\right)^k \prod_{j=0}^{n+s-k-1} \left(p^j - q^j \frac{x}{b_n}\right), \end{aligned}$$

where $n, s \in \mathbb{N}, 0 \leq x \leq b_n$ and $0 < q < p \leq 1$. Note that, in case $p=1$ in Eq.(2.1), Chlodowsky variant of Bernstein-Schurer reduces to the Chlodowsky variant of q -Bernstein-Schurer operators.

First of all, we obtained the following lemma and used it throughout the paper.

Lemma 1 *Let $\bar{C}_{n,s}(f; p, q; x)$ be given in Eq.(2.1). Then we get,*

$$(i) \bar{C}_{n,s}(1; p, q; x) = 1,$$

$$(ii) \bar{C}_{n,s}(t; p, q; x) = \frac{[n+s]_{p,q}}{p^s [n]_{p,q}} x,$$

$$(iii) \bar{C}_{n,s}(t^2; p, q; x) = \frac{p^{1-2s} q [n+s-1]_{p,q} [n+s]_{p,q}}{[n]_{p,q}^2} x^2 + \frac{p^{n-s-1} [n+s]_{p,q}}{[n]_{p,q}^2} b_n x,$$

$$(iv) \bar{C}_{n,s}((t-x); p, q; x) = \left(\frac{[n+s]_{p,q}}{p^s [n]_{p,q}} - 1\right) x,$$

$$(v) \bar{C}_{n,s}((t-x)^2; p, q; x) = \left(\frac{p^{1-2s} q [n+s-1]_{p,q} [n+s]_{p,q}}{[n]_{p,q}^2} - 2 \frac{[n+s]_{p,q}}{[n]_{p,q}} + 1\right) x^2 + \frac{p^{n-s-1} [n+s]_{p,q}}{[n]_{p,q}^2} b_n x.$$

Proof Applying the Corollary 1, we have

$$\begin{aligned} \bar{C}_{n,s}(1; p, q; x) &= \sum_{k=0}^{n+s} p^{\frac{k(k-1)}{2} - \frac{(n+s-1)(n+s)}{2}} \begin{bmatrix} n+s \\ k \end{bmatrix}_{p,q} \left(\frac{x}{b_n}\right)^k (1-x)_{p,q}^{n+s-k} \\ &= \sum_{k=0}^{n+s} p^{\frac{k(k-1)}{2} - \frac{(n+s-1)(n+s)}{2}} p^{k(n+s-k)} \begin{bmatrix} n+s \\ k \end{bmatrix}_{q/p} \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)_{q/p}^{n+s-k} \\ &= 1 \end{aligned}$$

when $0 < \frac{q}{p} \leq 1$.

Using the linearity of the operators and Corollary 1 with some basic calculations, we can obtain the assertions (ii), (iii). Then, from (i) and (ii), we have

$$\begin{aligned} \bar{C}_{n,s}((t-x); p, q; x) &= \bar{C}_{n,s}(t; p, q; x) - x \bar{C}_{n,s}(1; p, q; x) \\ &= \left(\frac{[n+s]_{p,q}}{p^s [n]_{p,q}} - 1\right) x. \end{aligned}$$

It is known that

$$\bar{C}_{n,s}((t-x)^2; p, q; x) = \bar{C}_{n,s}(x^2; p, q; x) - 2x \bar{C}_{n,s}(x; p, q; x) + x^2 \bar{C}_{n,s}(1; p, q; x).$$

Hence, the result proposed is acquired.

Lemma 2 For the second central moment we obtain the following estimate

$$\begin{aligned} & \sup_{0 \leq x \leq b_n} \bar{C}_{n,s}((t-x)^2; p, q; x) \\ & \leq \left(\frac{p^{1-2s} q [n+s-1]_{p,q} [n+s]_{p,q} + p^{n-s-1} [n+s]_{p,q}}{[n]_{p,q}^2} - 2 \frac{[n+s]_{p,q}}{[n]_{p,q}} + 1 \right) b_n^2. \end{aligned}$$

3 Korovkin Type Approximation Theorem

In this section, we give Korovkin-type approximation theorem for the Chlodowsky variant of Bernstein-Schurer operators based on (p, q) -integers. Let us denote C_ρ as the space of all continuous functions f , which satisfies the following condition.

$$|f(x)| \leq M_f \rho(x), \quad -\infty < x < \infty.$$

Therefore, C_ρ is a linear normed space with the norm

$$\|f\|_\rho = \sup_{-\infty < x < \infty} \frac{|f(x)|}{\rho(x)}.$$

Theorem 1 (See [10]) There exists a sequence of positive linear operators Q_n , acting from C_ρ to C_ρ , satisfying the conditions.

$$\lim_{n \rightarrow \infty} \|Q_n(1; x) - 1\|_\rho = 0 \tag{3.1}$$

$$\lim_{n \rightarrow \infty} \|Q_n(\phi; x) - \phi\|_\rho = 0 \tag{3.2}$$

$$\lim_{n \rightarrow \infty} \|Q_n(\phi^2; x) - \phi^2\|_\rho = 0 \tag{3.3}$$

where $\phi(x)$ is continuous and increasing function on $(-\infty, \infty)$ such that $\lim_{x \rightarrow \pm\infty} \phi(x) = \pm\infty$ and $\rho(x) = 1 + \phi^2$ and there exists a function $f^* \in C_\rho$ for which $\overline{\lim}_{n \rightarrow \infty} \|Q_n f^* - f^*\|_\rho > 0$.

The following theorem has been given in [10] and can be used in the investigation of approximation properties of $\bar{C}_{n,s}(f; q; x)$ in weighted spaces.

Theorem 2 (See [10]) The conditions Eqs.(3.1),(3.2) and (3.3) imply $\lim_{n \rightarrow \infty} \|Q_n f - f\|_\rho = 0$ for any function f belonging to the subset C_ρ^0 of C_ρ for which

$$\lim_{|x| \rightarrow \infty} \frac{f(x)}{\rho(x)}$$

exists finitely.

Particularly, choosing $\rho(x) = 1 + x^2$ and performing Theorem 2 to the operators

$$Q_n(f; p, q; x) = \begin{cases} \bar{C}_{n,s}(f; p, q; x) & \text{if } 0 \leq x \leq b_n \\ f(x) & \text{if } x \notin [0, b_n] \end{cases},$$

we can state the following theorem.

Theorem 3 For all $f \in C^0_{1+x^2}$ we have

$$\lim_{n \rightarrow \infty} \sup_{0 \leq x \leq b_n} \frac{|Q_n(f; q_n; x) - f(x)|}{1+x^2} = 0$$

provided that $q := (q_n)$, $p := (p_n)$ with $0 < q_n < p_n \leq 1$, $\lim_{n \rightarrow \infty} q_n = 1$, $\lim_{n \rightarrow \infty} p_n = 1$ and $\lim_{n \rightarrow \infty} \frac{b_n}{[n]} = 0$ as $n \rightarrow \infty$.

Proof In view of Theorem 2, by using Lemma 1 (i), (ii) and (iii) we get the following inequalities, respectively,

$$\begin{aligned} \sup_{0 \leq x \leq b_n} \frac{|Q_n(t; p_n, q_n; x) - x|}{1+x^2} &\leq \sup_{0 \leq x \leq b_n} \frac{\left(\frac{[n+s]_{p,q}}{p^s [n]_{p,q}} - 1\right)x}{(1+x^2)} \\ &\leq \left| \frac{[n+s]_{p,q}}{p^s [n]_{p,q}} - 1 \right| \rightarrow 0 \end{aligned}$$

and

$$\begin{aligned} \sup_{0 \leq x \leq b_n} \frac{|Q_n(t^2; p_n, q_n; x) - x^2|}{1+x^2} &\leq \sup_{0 \leq x \leq b_n} \frac{\left| \left(\frac{[n+s]_{p,q}}{[n]_{p,q}} - 1\right)^2 x^2 + \frac{p^{n-s-1} [n+s]_{p,q}}{[n]_{p,q}^2} b_n x \right|}{(1+x^2)} \\ &\leq \left| \left(\frac{[n+s]_{p,q}}{[n]_{p,q}} - 1\right)^2 + \frac{p^{n-s-1} [n+s]_{p,q}}{[n]_{p,q}^2} b_n \right| \rightarrow 0 \end{aligned}$$

is satisfied since $\lim_{n \rightarrow \infty} q_n = 1$, $\lim_{n \rightarrow \infty} p_n = 1$ and $\frac{b_n}{[n]} \rightarrow 0$ as $n \rightarrow \infty$.

Lemma 3 Let B be a positive real number independent of n and f be a continuous function which vanishes on $[B, \infty]$. Assume that $q := (q_n)$, $p := (p_n)$ with $0 < q_n < p_n \leq 1$, $\lim_{n \rightarrow \infty} p_n^n =$

$N < \infty$, $\lim_{n \rightarrow \infty} q_n^n = K < \infty$ and $\lim_{n \rightarrow \infty} \frac{b_n^2}{[n]} = 0$. Then we have

$$\lim_{n \rightarrow \infty} \sup_{0 \leq x \leq b_n} |\bar{C}_{n,s}(f; p_n, q_n; x) - f(x)| = 0.$$

Proof From the hypothesis on f , one can write $|f(x)| \leq M$ ($M > 0$). For arbitrary small $\varepsilon > 0$, we have

$$\left| f\left(p^{n-k} \frac{[k]_{p,q}}{[n]_{p,q}} b_n\right) - f(x) \right| < \varepsilon + \frac{2M}{\delta^2} \left(p^{n-k} \frac{[k]_{p,q}}{[n]_{p,q}} b_n - x\right)^2,$$

where $x \in [0, b_n]$ and $\delta = \delta(\varepsilon)$ are independent of n . With the help of the following equality

$$\begin{aligned} \sum_{k=0}^{n+s} \left(p^{n-k} \frac{[k]_{p,q}}{[n]_{p,q}} b_n - x\right)^2 p^{\frac{k(k-1)}{2} - \frac{(n+s-1)(n+s)}{2}} \left[\begin{matrix} n+s \\ k \end{matrix} \right]_{p,q} \left(\frac{x}{b_n}\right)^k \prod_{j=0}^{n+s-k-1} \left(p^j - q^j \frac{x}{b_n}\right) \\ = \bar{C}_{n,s}(f; p, q; x), \end{aligned}$$

we get by Theorem 3 and Lemma 1 that

$$\begin{aligned} & \sup_{0 \leq x \leq b_n} |\bar{C}_{n,s}(f; p, q; x) - f(x)| \\ & \leq \varepsilon + \frac{2M}{\delta^2} \left[\left(\frac{p^{1-2s} q [n+s-1]_{p,q} [n+s]_{p,q}}{[n]_{p,q}^2} - 2 \frac{[n+s]_{p,q}}{[n]_{p,q}} + 1 \right) b_n^2 + \frac{p^{n-s-1} [n+s]_{p,q}}{[n]_{p,q}^2} b_n^2 \right]. \end{aligned}$$

Since $\frac{b_n^2}{[n]} \rightarrow 0$ as $n \rightarrow \infty$, we have the desired result.

4 Order of Convergence

In this section, we compute the rate of convergence of the operators in terms of the elements of Lipschitz classes and the modulus of continuity of the function. Now, we give the rate of convergence of the operators $\bar{C}_{n,s}$ in terms of the Lipschitz class $Lip_M(\gamma)$, for $0 < \gamma \leq 1$. Let $C_B[0, \infty)$ denotes the space of bounded continuous functions on $[0, \infty)$. A function $f \in C_B[0, \infty)$ belongs to $Lip_M(\gamma)$ if

$$|f(t) - f(x)| \leq M |t - x|^\gamma \quad (t, x \in [0, \infty))$$

is satisfied.

Theorem 4 Let $f \in Lip_M(\gamma)$

$$|\bar{C}_{n,s}(f; p, q; x) - f(x)| \leq M (\lambda_n(x))^{\gamma/2}$$

where $\lambda_{n,q}(x) = \left(\frac{p^{1-2s} q [n+s-1]_{p,q} [n+s]_{p,q}}{[n]_{p,q}^2} - 2 \frac{[n+s]_{p,q}}{[n]_{p,q}} + 1 \right) x^2 + \frac{p^{n-s-1} [n+s]_{p,q}}{[n]_{p,q}^2} b_n x$.

Proof Considering the monotonicity and the linearity of the operators, and taking into account that $f \in Lip_M(\gamma)$

$$\begin{aligned} & |\bar{C}_{n,s}(f; p, q; x) - f(x)| \\ & = \left| \sum_{k=0}^{n+s} (f(p^{n-k} \frac{[k]_{p,q}}{[n]_{p,q}} b_n) - f(x)) p^{\frac{k(k-1)}{2} - \frac{(n+s-1)(n+s)}{2}} \left[\begin{matrix} n+s \\ k \end{matrix} \right]_{p,q} \left(\frac{x}{b_n} \right)^{k+n+s-k-1} \prod_{j=0}^{n+s-k-1} \left(p^j - q^j \frac{x}{b_n} \right) \right| \\ & \leq \sum_{k=0}^{n+s} \left| f(p^{n-k} \frac{[k]_{p,q}}{[n]_{p,q}} b_n) - f(x) \right| p^{\frac{k(k-1)}{2} - \frac{(n+s-1)(n+s)}{2}} \left[\begin{matrix} n+s \\ k \end{matrix} \right]_{p,q} \left(\frac{x}{b_n} \right)^{k+n+s-k-1} \prod_{j=0}^{n+s-k-1} \left(p^j - q^j \frac{x}{b_n} \right) \\ & \leq M \sum_{k=0}^{n+s} |p^{n-k} \frac{[k]_{p,q}}{[n]_{p,q}} b_n - x|^\gamma p^{\frac{k(k-1)}{2} - \frac{(n+s-1)(n+s)}{2}} \left[\begin{matrix} n+s \\ k \end{matrix} \right]_{p,q} \left(\frac{x}{b_n} \right)^{k+n+s-k-1} \prod_{j=0}^{n+s-k-1} \left(p^j - q^j \frac{x}{b_n} \right). \end{aligned}$$

Using Hölder's inequality with $p = \frac{2}{\gamma}$ and $q = \frac{2}{2-\gamma}$, we get by the statement (Lemma 2)

$$\begin{aligned} & |\bar{C}_{n,s}(f; p, q; x) - f(x)| \\ & \leq M \sum_{k=0}^{n+s} \left\{ \left[(p^{n-k} \frac{[k]_{p,q}}{[n]_{p,q}} b_n - x)^2 p^{\frac{k(k-1)}{2} - \frac{(n+s-1)(n+s)}{2}} \left[\begin{matrix} n+s \\ k \end{matrix} \right]_{p,q} \left(\frac{x}{b_n} \right)^{k+n+s-k-1} \prod_{j=0}^{n+s-k-1} \left(p^j - q^j \frac{x}{b_n} \right) \right]^{\frac{\gamma}{2}} \right. \\ & \left. \times \left[p^{\frac{k(k-1)}{2} - \frac{(n+s-1)(n+s)}{2}} \left[\begin{matrix} n+s \\ k \end{matrix} \right]_{p,q} \left(\frac{x}{b_n} \right)^{k+n+s-k-1} \prod_{j=0}^{n+s-k-1} \left(p^j - q^j \frac{x}{b_n} \right) \right]^{\frac{2-\gamma}{2}} \right\} \end{aligned}$$

$$\begin{aligned} &\leq M \left[\left\{ \sum_{k=0}^{n+s} \left[(p^{n-k} \frac{[k]_{p,q}}{[n]_{p,q}} b_n - x)^2 p^{\frac{k(k-1)}{2} - \frac{(n+s-1)(n+s)}{2}} \left[\begin{matrix} n+s \\ k \end{matrix} \right]_{p,q} \left(\frac{x}{b_n} \right)^{k_{n+s-k-1}} \prod_{j=0}^{k_{n+s-k-1}} \left(p^j - q^j \frac{x}{b_n} \right) \right] \right\}^{\frac{\gamma}{2}} \right. \\ &\times \left. \left\{ \sum_{k=0}^{n+s} \left[p^{\frac{k(k-1)}{2} - \frac{(n+s-1)(n+s)}{2}} \left[\begin{matrix} n+s \\ k \end{matrix} \right]_{p,q} \left(\frac{x}{b_n} \right)^{k_{n+s-k-1}} \prod_{j=0}^{k_{n+s-k-1}} \left(p^j - q^j \frac{x}{b_n} \right) \right] \right\}^{\frac{2-\gamma}{\gamma}} \right] \\ &= M[\bar{C}_{n,s}((t-x)^2; p, q; x)]^{\frac{\gamma}{2}} \\ &\leq M(\lambda_{n,q}(x))^{\frac{\gamma}{2}}. \end{aligned}$$

Now we give the rate of convergence of the operators by means of the modulus of continuity which is denoted by $\omega(f; \delta)$. Let $f \in C_B[0, \infty)$ and $x \geq 0$. Then the definition of the modulus of continuity of f is given by

$$\omega(f; \delta) = \max_{\substack{|t-x| \leq \delta \\ t, x \in [0, \infty)}} |f(t) - f(x)|. \tag{4.1}$$

It is following that for any $\delta > 0$ the following inequality

$$|f(x) - f(y)| \leq \omega(f; \delta) \left(\frac{|x-y|}{\delta} + 1 \right) \tag{4.2}$$

is satisfied ([5]).

Theorem 5 *If $f \in C_B[0, \infty)$, we have*

$$|\bar{C}_{n,s}(f; p, q; x) - f(x)| \leq 2\omega\left(f; \sqrt{\lambda_{n,p,q}(x)}\right)$$

where $\omega(f; \cdot)$ is modulus of continuity of f which is defined in Eq.(4.1) and $\lambda_{n,q}(x)$ be the same as in Theorem 4.

Proof By triangular inequality, we get

$$\begin{aligned} &|\bar{C}_{n,s}(f; p, q; x) - f(x)| \\ &= \left| \sum_{k=0}^{n+s} f(p^{n-k} \frac{[k]_{p,q}}{[n]_{p,q}} b_n) p^{\frac{k(k-1)}{2} - \frac{(n+s-1)(n+s)}{2}} \left[\begin{matrix} n+s \\ k \end{matrix} \right]_{p,q} \left(\frac{x}{b_n} \right)^{k_{n+s-k-1}} \prod_{j=0}^{k_{n+s-k-1}} \left(p^j - q^j \frac{x}{b_n} \right) - f(x) \right| \\ &\leq \sum_{k=0}^{n+s} \left| (f(p^{n-k} \frac{[k]_{p,q}}{[n]_{p,q}} b_n) - f(x)) \right| p^{\frac{k(k-1)}{2} - \frac{(n+s-1)(n+s)}{2}} \left[\begin{matrix} n+s \\ k \end{matrix} \right]_{p,q} \left(\frac{x}{b_n} \right)^{k_{n+s-k-1}} \prod_{j=0}^{k_{n+s-k-1}} \left(p^j - q^j \frac{x}{b_n} \right). \end{aligned}$$

Now, using Eq.(4.2) and Hölder inequality, we can write

$$\begin{aligned} &|\bar{C}_{n,s}(f; p, q; x) - f(x)| \\ &\leq \sum_{k=0}^{n+s} \left(\frac{\left| p^{n-k} \frac{[k]_{p,q}}{[n]_{p,q}} b_n - x \right|}{\lambda} + 1 \right) \omega(f; \lambda) p^{\frac{k(k-1)}{2} - \frac{(n+s-1)(n+s)}{2}} \left[\begin{matrix} n+s \\ k \end{matrix} \right]_{p,q} \left(\frac{x}{b_n} \right)^{k_{n+s-k-1}} \prod_{j=0}^{k_{n+s-k-1}} \left(p^j - q^j \frac{x}{b_n} \right) \\ &= \omega(f; \lambda) \sum_{k=0}^{n+s} p^{\frac{k(k-1)}{2} - \frac{(n+s-1)(n+s)}{2}} \left[\begin{matrix} n+s \\ k \end{matrix} \right]_{p,q} \left(\frac{x}{b_n} \right)^{k_{n+s-k-1}} \prod_{j=0}^{k_{n+s-k-1}} \left(p^j - q^j \frac{x}{b_n} \right) \\ &+ \frac{\omega(f; \lambda)}{\lambda} \sum_{k=0}^{n+s} \left| p^{n-k} \frac{[k]_{p,q}}{[n]_{p,q}} b_n - x \right| p^{\frac{k(k-1)}{2} - \frac{(n+s-1)(n+s)}{2}} \left[\begin{matrix} n+s \\ k \end{matrix} \right]_{p,q} \left(\frac{x}{b_n} \right)^{k_{n+s-k-1}} \prod_{j=0}^{k_{n+s-k-1}} \left(p^j - q^j \frac{x}{b_n} \right) \end{aligned}$$

$$\begin{aligned}
 &= \omega(f; \lambda) \\
 &+ \frac{\omega(f; \lambda)}{\lambda} \left\{ \sum_{k=0}^{n+s} \left(p^{n-k} \frac{[k]_{p,q}}{[n]_{p,q}} b_n - x \right)^2 p^{\frac{k(k-1)}{2} - \frac{(n+s-1)(n+s)}{2}} \left[\begin{matrix} n+s \\ k \end{matrix} \right]_{p,q} \left(\frac{x}{b_n} \right)^{k_{n+s-k-1}} \prod_{j=0}^{n+s-k-1} \left(p^j - q^j \frac{x}{b_n} \right) \right\}^{1/2} \\
 &= \omega(f; \lambda) + \frac{\omega(f; \lambda)}{\lambda} \left\{ \bar{C}_{n,s} \left((t-x)^2 \right); p, q; x \right\}^{1/2}.
 \end{aligned}$$

Now, choosing $\lambda_{n,q}(x)$ same as in Theorem 4, then we have

$$\left| \bar{C}_{n,s}(f; p, q; x) - f(x) \right| \leq 2\omega \left(f; \sqrt{\lambda_{n,p,q}(x)} \right).$$

5 Generalization of the Chlodowsky Variant of q -Bernstein-Schurer-Stancu Operators

In this section, we introduce generalization of Chlodowsky variant of Bernstein-Schurer operators based on (p, q) -integers and this provides us to obtain approximate continuous functions on more general weighted spaces. For $x \geq 0$, consider any continuous function $\omega(x) \geq 1$ and define

$$G_f(t) = f(t) \frac{1+t^2}{\omega(t)}.$$

Let us consider the generalization of the $\bar{C}_{n,s}(f; p, q; x)$ as follows

$$\begin{aligned}
 &L_n(f; p, q; x) \\
 &= \frac{\omega(x)}{1+x^2} \sum_{k=0}^{n+s} G_f \left(p^{n-k} \frac{[k]_{p,q}}{[n]_{p,q}} b_n \right) p^{\frac{k(k-1)}{2} - \frac{(n+s-1)(n+s)}{2}} \left[\begin{matrix} n+s \\ k \end{matrix} \right]_{p,q} \left(\frac{x}{b_n} \right)^{k_{n+s-k-1}} \prod_{j=0}^{n+s-k-1} \left(p^j - q^j \frac{x}{b_n} \right),
 \end{aligned}$$

where $0 \leq x \leq b_n$ and (b_n) has the same properties of Chlodowsky variant of q -Bernstein-Schurer-Stancu operators.

Theorem 6 For the continuous functions satisfying

$$\lim_{x \rightarrow \infty} \frac{f(x)}{\omega(x)} = K_f < \infty,$$

we have

$$\lim_{n \rightarrow \infty} \sup_{0 \leq x \leq b_n} \frac{|L_n(f; p, q; x) - f(x)|}{\omega(x)} = 0.$$

Proof Clearly,

$$\begin{aligned}
 &L_n(f; p, q; x) - f(x) \\
 &= \frac{\omega(x)}{1+x^2} \left(\sum_{k=0}^{n+p} G_f \left(p^{n-k} \frac{[k]_{p,q}}{[n]_{p,q}} b_n \right) p^{\frac{k(k-1)}{2} - \frac{(n+s-1)(n+s)}{2}} \left[\begin{matrix} n+s \\ k \end{matrix} \right]_{p,q} \left(\frac{x}{b_n} \right)^k \right. \\
 &\quad \times \left. \prod_{j=0}^{n+s-k-1} \left(p^j - q^j \frac{x}{b_n} \right) - G_f(x) \right)
 \end{aligned}$$

thus

$$\sup_{0 \leq x \leq b_n} \frac{|L_n(f; p, q; x) - f(x)|}{\omega(x)} = \sup_{0 \leq x \leq b_n} \frac{|\bar{C}_{n,s}(G_f; p, q; x) - G_f(x)|}{1+x^2}.$$

By using $|f(x)| \leq M_f \omega(x)$ and continuity of the function f , we get that $|G_f(x)| \leq M_f(1+x^2)$ for $x \geq 0$ and $G_f(x)$ is continuous function on $[0, \infty)$. Thus, from the Theorem 1 we get the result.

Finally note that, the operators $L_n(f; p, q; x)$ reduces to $\bar{C}_{n,s}(G_f; p, q; x)$ by taking $\omega(x) = 1+x^2$.

6 Numerical Results and Discussions

In order to show the effectiveness and accuracy of $\bar{C}_{n,s}(f; p, q; x)$ to $f(x)$ with different values of parameters, numerical results are presented in this section. Sensitivity analysis is carried out to minimise the error of approximation of $\bar{C}_{n,s}(f; p, q; x)$ to the function $f(x) = 1 - \cos(4e^x)$ for minimum n and s values by taken into account different p and q values.

In Figure 1, $\bar{C}_{n,s}(f; p, q; x)$ results are given as a function of x for different n values. To minimise the error of the approximation of $\bar{C}_{n,s}(f; p, q; x)$ to $f(x)$, two different sequences $b_n = 1 + \log(\frac{n}{n+12})$ and $b_n = \frac{n^2+4}{n^2+18n}$ are considered respectively. It is evident that the $\bar{C}_{n,s}(f; p, q; x)$ converges to $f(x)$ for both sequences as the value of q and p approaches towards 1, while $s = 2$. However, using $b_n = 1 + \log(\frac{n}{n+12})$ for $\bar{C}_{n,s}(f; p, q; x)$ rather than $b_n = \frac{n^2+4}{n^2+18n}$ results better approximation results. Therefore, the effect of increasing n further than 20 is less evident for $x < 0.5$ for the convergence of $\bar{C}_{n,s}(f; p, q; x)$ to the function $f(x)$. On the other hand, it is required to increase the value of n further than 50 for $x > 0.5$ in order to have more accurate results for each sequences. Comparative results are given in Table 1 and Table 2, for the errors of the approximation of $\bar{C}_{n,s}(f; p, q; x)$ to $f(x)$, considering each sequences for different n values.

x	$f(x)$	$ f(x) - \bar{C}_{20,2}(f; p, q; x) $	$ f(x) - \bar{C}_{30,2}(f; p, q; x) $	$ f(x) - \bar{C}_{50,2}(f; p, q; x) $	$ f(x) - \bar{C}_{80,2}(f; p, q; x) $
0.1	1.2876	0.0460	0.0310	0.0190	0.0125
0.2	0.8276	0.0720	0.0432	0.0207	0.0090
0.3	0.3657	0.0543	0.0200	0.0049	0.0172
0.4	0.0494	0.0180	0.0411	0.0544	0.0595
0.5	0.0482	0.1369	0.1224	0.1050	0.0933
0.6	0.4642	0.2913	0.1998	0.1280	0.0898
0.7	1.1997	0.4942	0.2723	0.1210	0.0489
0.8	1.8665	0.7628	0.3627	0.1242	0.0224
0.9	1.9157	0.9275	0.3925	0.1406	0.0521

Table 1 Errors of approximation $\bar{C}_{n,s}(f; p, q; x)$ to $f(x)$ ($s = 2, b_n = \frac{n^2+4}{n^2+18n}, p = 1, q = 0.98$)

Figure 2 demonstrates the convergence of $\bar{C}_{n,s}(f; p, q; x)$ to $f(x)$ but this time considering different p and q values, when $n = 50$ for $b_n = 1 + \log(\frac{n}{n+12})$. In Figures 2(a), and 2(b), as q values are increased, the errors of the approximation of $\bar{C}_{n,s}(f; p, q; x)$ to $f(x)$ is

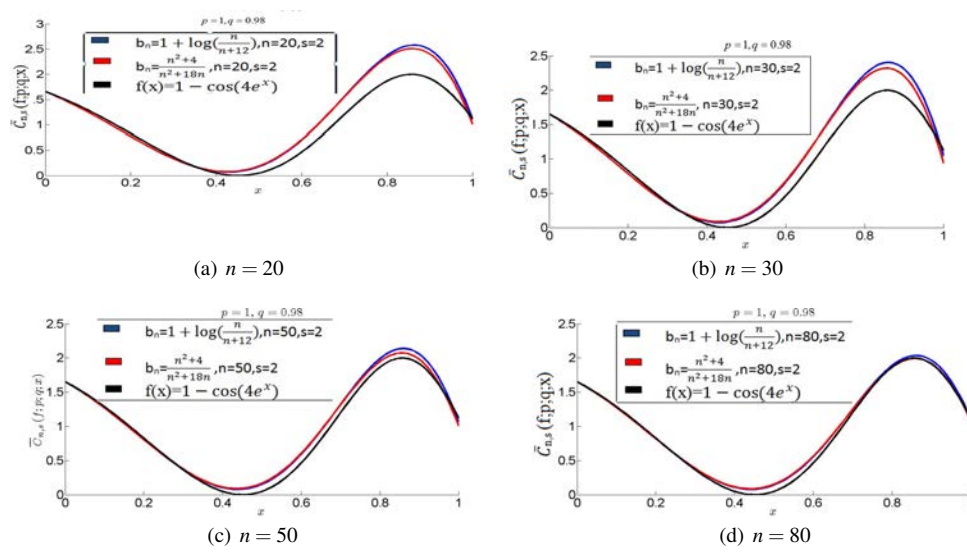


Fig. 1 Convergence of $\tilde{C}_{n,s}(f; p, q; x)$ for different n values

x	$f(x)$	$ f(x) - \tilde{C}_{20,2}(f; p, q; x) $	$ f(x) - \tilde{C}_{30,2}(f; p, q; x) $	$ f(x) - \tilde{C}_{50,2}(f; p, q; x) $	$ f(x) - \tilde{C}_{80,2}(f; p, q; x) $
0.1	1.2876	0.0459	0.0317	0.0196	0.0130
0.2	0.8276	0.0721	0.0427	0.0204	0.0089
0.3	0.3657	0.0546	0.0156	0.0088	0.0198
0.4	0.0494	0.0173	0.0520	0.0644	0.0663
0.5	0.0482	0.1358	0.1390	0.1203	0.1040
0.6	0.4642	0.2903	0.2187	0.1412	0.0993
0.7	1.1997	0.4947	0.2650	0.1157	0.0461
0.8	1.8665	0.7665	0.3110	0.0825	0.0045
0.9	1.9157	0.9361	0.2897	0.0640	0.0038

Table 2 Errors of approximation $\tilde{C}_{n,s}(f; p, q; x)$ to $f(x)$ ($s = 2, b_n = 1 + \log(\frac{n}{n+12}), p = 1, q = 0.98$)

minimised for $x < 0.5$ and $x > 0.8$ for any given p values. On the other hand, the results show that decreasing q values has an effect on the convergence of $\tilde{C}_{n,s}(f; p, q; x)$ which provide better approximate results for $x > 0.6$ and $x < 0.8$ (See Table 2).

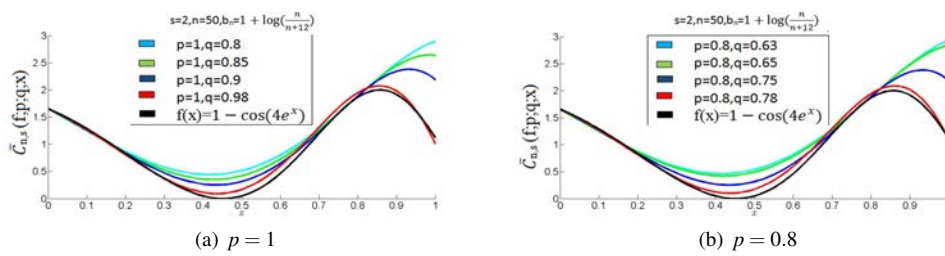


Fig. 2 Convergence of $\tilde{C}_{n,s}(f; p, q; x)$ for different p and q values

References

1. Sidharth, M. and Agrawal, P., N.: Bernstein-Schurer operators based on (p, q) -integers for functions of one and two variables, arXiv: 1510.00405v2.
2. Anastassiou GA, Gal, SG: Approximation by complex Bernstein-Schurer and Kantorovich-Schurer polynomials in compact disks, *Comput. Math Appl.*, **58** (4), 734-743 (2009).
3. Büyükyazıcı, İ, Sharma, H: Approximation properties of two-dimensional q -Bernstein-Chlodowsky-Durrmeyer operators, *Numer. Funct. Anal. Optim.*, 33 (2), 1351-1371 (2012).
4. Chlodowsky, I: Sur le developement des fonctions definiées dans un interval infini en series de polynomes de M. S. Bernstein, *Compositio Math.*, **4**, 380-393 (1937).
5. DeVore, RA, Lorentz, GG: *Constructive Approximation*, Springer-Verlag, Berlin (1993).
6. Hounkonnou, MN, Desire, J, Kyemba, B: $\mathcal{R}(p, q)$ -calculus: differentiation and integration, *SUT J. Math.* **49**, 145-167 (2013).
7. İbikli, E: Approximation by Bernstein-Chlodowsky polynomials, *Hacettepe Journal of Mathematics and Statics*, **32**, 1-5 (2003).
8. Kac, V, Cheung, P: *Quantum Calculus*, Springer, 2002.
9. Karlı, H, Gupta, V: Some approximation properties of q -Chlodowsky operators, *Applied Mathematics and Computation*, **195**, 220-229 (2008).
10. Gadjiev, A,D: The convergence problem for a sequence of positive linear operators on unbounded sets and theorems analogues to that of P.P. Korovkin, *Dokl. Akad. Nauk SSSR* **218** (5), English Translation in *Soviet Math. Dokl.* **15** (5) (1974).
11. Lupaş, AA: q -analogue of the Bernstein operators, university of Cluj-Napoca, Seminar on numerical and statistical calculus, **9**, 85-92 (1987).
12. Mahmudov, NI, Sabancıgil, P: q -parametric Bleimann Butzer and Hahn operators, *J. of Ineq. and Appl.*, 816367 (2008).
13. Muraru, CV: Note on q -Bernstein-Schurer operators, *Babeş-Bolyai Math.*, **56**, 489-495 (2011).
14. Mursaleem, M, Ansari, J., Khurshed and Khan A.: On (p, q) -analogue of Bernstein operators, *Applied Mathematics and Computation*, **266**, 874-882 (2015).
15. Mursaleem, M, Nasiruzzaman, MD and Nurgali, A: Some approximation results on Bernstein-Schurer operators defined by (p, q) -integers, *J. of Ineq. and Appl.*, 2015, 2015:249 doi:10.1186/s13660-015-0767-4.
16. Özarlan, MA: q -Szász Schurer operators, *Miscolc Mathematical Notes*, 12, 225-235 (2011).
17. Özarlan, MA, Vedi, T: q - Bernstein-Schurer-Kantorovich Operators, *J. of Ineq. and Appl.*, 2013, 2013:444 doi:10.1186/1029-242X-2013-444.
18. Phillips, GM: On Generalized Bernstein polynomials, *Numerical analysis*, World Sci. Publ., River Edge, **98**, 263-269 (1996).
19. Phillips, GM: *Interpolation and Approximation by Polynomials*, Newyork, (2003).
20. Ren, MY, Zeng, XM: Approximation by complex q -Bernstein-Schurer operators in compact disks, *Georgian Math. J.*, **20** (2), 377-395 2013.
21. Schurer, F: *Linear Positive Operators in Approximation Theory*, Math. Inst., Techn. Univ. Delf Report, 1962.
22. Sucu, S, İbikli, E, Szász-Schurer operators on a domain in complex plane, *Mathematical Sciences*, 7:40 8 pp (2013).
23. Quing-Bo, C, Guorong, Z: On (p, q) -analogue of Kantorovich type Bernstein-Stancu-Schurer operators, *Appl. Math. and Comp.*, **276**, 12-20 (2016)
24. Vedi, T, Özarlan, MA: Some Properties of q -Bernstein-Schurer operators, *J. Applied Functional Analysis*, **8** (1), 45-53 (2013).

Dynamical behavior of a general HIV-1 infection model with HAART and cellular reservoirs

Ahmed M. Elaiw, Abdullah M. Althiabi, Mohammed A. Alghamdi and Nicola Bellomo
 Department of Mathematics, Faculty of Science, King Abdulaziz University,
 P.O. Box 80203, Jeddah 21589, Saudi Arabia.
 Email: a_m_elaiw@yahoo.com (A. Elaiw)

Abstract

This paper study the dynamical behavior of a general HIV-1 infection model under the effect of Highly active antiretroviral therapies (HAART). The model includes three types of infected cells (i) long-lived productively infected cells which live for long time and creat small amount of HIV-1 particles, (ii) latently infected cells which do not creat HIV-1 until they have been activated (iii) short-lived productively infected cells which live for long time and creat large amount of HIV-1. The model incorporates humoral immune response and general nonlinear forms for the incidence rate of infection, the generation and removal rates of all compartments. The nonnegativity and boundedness of the solutions of the model as well as global stability of the steady states are studied. The global stability are established using Lyapunov method. Using MATLAB we conduct some numerical simulations to confirm our results.

Keywords: HIV-1 infection; HAART; global stability; humoral immune response; latency; viral reservoirs

1 Introduction

Human immunodeficiency virus type 1 (HIV-1) infects the CD4⁺ T cells which play the central role in the immune system of the human body. HIV-1 causes gradual depletion in the concentration of the uninfected CD4⁺ T cells which decreases the efficiency of the immune system against other infections. During the last decades, substantial efforts have been paid to propose treatment strategies for HIV-1 [1], [2]. Highly active antiretroviral therapies (HAART) which combines two classes of antiviral drugs, reverse transcriptase inhibitor (RTI) and protease inhibitor (PI), can rapidly decrease the concentration of the HIV-1 and increase the concentration of the healthy CD4⁺ T cells in the plasma. However, HAART can not eradicate the HIV-1 completely due to the presence of viral reservoirs such as latently infected cells. Mathematical modeling and analysis of the dynamics of HIV-1 are helpful in understanding the virus dynamics and improving diagnosis and treatment strategies [3]-[23]. Modeling the HIV-1 dynamics with latent infection has been studied by several researchers [24]-[29]. The HIV-1 dynamics model with latently infected cells consists of four compartments: uninfected CD4⁺ T cells, latently infected cells, actively infected cells and free HIV-1 particles [24].

$$\dot{x} = \rho - dx - (1 - \varepsilon_r)\beta xv, \tag{1}$$

$$\dot{w} = h(1 - \varepsilon_r)\beta xv - (a_1 + \delta_1)w, \tag{2}$$

$$\dot{y} = (1 - h)(1 - \varepsilon_r)\beta xv + a_1w - \delta_2y, \tag{3}$$

$$\dot{v} = \bar{N}\delta_2y - \delta_4v, \tag{4}$$

where x , w , y , v represent the concentrations of the uninfected CD4⁺ T cells, latently infected cells, actively infected cells and free HIV-1 particles, respectively. $\rho > 0$ is the replenished rate of uninfected CD4⁺ T cells from

body's sources such as bone marrow and thymus. The parameters d , δ_1 , δ_2 and δ_4 are the death rate constants of the uninfected $CD4^+$ T cells, latently infected cells, actively infected cells and HIV-1 particles, respectively. The uninfected $CD4^+$ T cells become infected by viruses with infectivity β . The efficacy of the RTI drugs is given by ε_r , where $\varepsilon_r \in [0, 1]$. The latently infected cells are activated at rate a_1w . The parameter \bar{N} is the average number of HIV-1 particles generated in the lifetime of the actively infected. A fraction $h \in (0, 1)$ of infection events result in latent infection. The global stability analysis of model (1)-(4) has been studied by Wang et al. in [26].

As reported in [24], [2] and [30] there are two types of productively infected cells, the first is short-lived productively infected cells which live for short time and produce high numbers of HIV-1 particles, and the second is the long-lived productively infected cells which live for long time and produce small numbers of HIV-1 particles. Long-lived productively infected cells can be seen as another reservoirs which a major obstacle to eliminate the HIV-1 completely by HAART. Model (1)-(4) has been modified by including: (i) mitotic proliferation of the uninfected $CD4^+$ T cells, (ii) three types of infected cells, latently infected cells (w), short-lived productively infected cells (y), and long-lived productively infected cells (u) [30].

$$\dot{x} = \rho - dx + px \left(1 - \frac{x}{x_{\max}}\right) - (1 - \varepsilon_r)(\bar{\beta}_1 + \bar{\beta}_2 + \bar{\beta}_3)xv, \tag{5}$$

$$\dot{w} = (1 - \varepsilon_r)\bar{\beta}_1xv - (a_1 + \delta_1)w, \tag{6}$$

$$\dot{y} = (1 - \varepsilon_r)\bar{\beta}_2xv + a_1w - \delta_2y, \tag{7}$$

$$\dot{u} = (1 - \varepsilon_r)\bar{\beta}_3xv - \delta_3u, \tag{8}$$

$$\dot{v} = (1 - \varepsilon_p)\bar{N}\delta_2y + (1 - \varepsilon_p)\bar{M}\delta_3u - \delta_4v. \tag{9}$$

Uninfected $CD4^+$ T cells can be produced by proliferation of existing healthy cells in the body. The parameter $p > 0$ is the maximum proliferation rate of uninfected cells. The parameter $x_{\max} > 0$ is the maximum level of uninfected cell concentration in the body. If the concentration arrives at x_{\max} , it should decrease. The parameters δ_2 and δ_3 are the death rate constants of the short-lived productively infected cells and long-lived productively infected cells, respectively. The uninfected $CD4^+$ T cells become infected by viruses with infectivity $\bar{\beta}_1 + \bar{\beta}_2 + \bar{\beta}_3$. The efficacy of the PI drugs is given by ε_p , where $\varepsilon_p \in [0, 1]$. The parameter \bar{M} is the average number of HIV-1 particles generated in the lifetime of the long-lived productively infected cells.

In model (5)-(9) we note the following (i) the infection rate is given by bilinear incidence which may not describe the virus dynamics accurately, (ii) the death rate of all compartments, the production rate of viruses and the latent-to-active transmission rate are given by linear functions, however, these rates are generally not known, (iii) the effect of immune response has been neglecting. The aim of this paper is to propose an HIV-1 infection model which improves model (5)-(9) by taking into account the humoral immune response and by assuming that the intrinsic growth rate of uninfected $CD4^+$ T cells as well as the death rate of HIV-1 and infected cells are given by general nonlinear functions. We study the qualitative behavior of the proposed model. The existence and global stability of all the steady states of the model is established. Lyapunov functionals and LaSalle's invariance principle are used to prove the global stability of the model.

2 Mathematical HIV-1 dynamics model

Based on the above discussion we formulate a general nonlinear HIV dynamics model with humoral immunity. The model can be considered as a generalization of several existing HIV-1 models.

$$\dot{x} = \pi(x) - (1 - \varepsilon_r)(\bar{\beta}_1 + \bar{\beta}_2 + \bar{\beta}_3)\xi(x, v), \tag{10}$$

$$\dot{w} = (1 - \varepsilon_r)\bar{\beta}_1\xi(x, v) - (a_1 + \delta_1)g_1(w), \tag{11}$$

$$\dot{y} = (1 - \varepsilon_r)\bar{\beta}_2\xi(x, v) + a_1g_1(w) - \delta_2g_2(y), \tag{12}$$

$$\dot{u} = (1 - \varepsilon_r)\bar{\beta}_3\xi(x, v) - \delta_3g_3(u), \tag{13}$$

$$\dot{v} = (1 - \varepsilon_p)\bar{N}\delta_2g_2(y) + (1 - \varepsilon_p)\bar{M}\delta_3g_3(u) - \delta_4g_4(v) - qg_4(v)g_5(z), \tag{14}$$

$$\dot{z} = rg_4(v)g_5(z) - \delta_5g_5(z), \tag{15}$$

where z represents the concentration of the B cells. Function $\pi(x)$ represents the intrinsic growth rate of uninfected CD4⁺ T cells accounting for both production and natural mortality. The viruses are neutralized at rate $qg_5(z)g_4(v)$ and die at rate $\delta_4g_4(v)$, where q and δ_4 are positive constants. The B cells are activated at rate $rg_5(z)g_4(v)$ and die at rate $\delta_5g_5(z)$. All the parameters are positive. Let us define $\beta_i = (1 - \varepsilon_r)\bar{\beta}_i$, $i = 1, 2, 3$, $N = (1 - \varepsilon_p)\bar{N}$ and $M = (1 - \varepsilon_p)\bar{M}$. Functions π , ξ , g_i , $i = 1, \dots, 5$ are continuously differentiable, moreover, they satisfy some hypotheses:

(H1). (i) there exists x_0 such that $\pi(x_0) = 0$, $\pi(x) > 0$ for $x \in [0, x_0)$,

(ii) $\pi'(x) < 0$ for $x \in (0, \infty)$,

(iii) there are $b > 0$ and $\bar{b} > 0$ such that $\pi(x) \leq b - \bar{b}x$ for $x \in [0, \infty)$.

(H2). (i) $\xi(x, v) > 0$ and $\xi(0, v) = \xi(x, 0) = 0$ for $x, v \in (0, \infty)$,

(ii) $\frac{\partial \xi(x, v)}{\partial x} > 0$, $\frac{\partial \xi(x, v)}{\partial v} > 0$ and $\frac{\partial \xi(x, 0)}{\partial v} > 0$ for all $x, v \in (0, \infty)$,

(iii) $\left(\frac{\partial \xi(x, 0)}{\partial v}\right)' > 0$ for $x \in (0, \infty)$.

(H3). (i) $g_j(\eta) > 0$ for $\eta \in (0, \infty)$, $g_j(0) = 0$, $j = 1, \dots, 5$

(ii) $g'_j(\eta) > 0$ for $\eta \in (0, \infty)$, $j = 1, 2, 3, 5$, $g'_4(\eta) > 0$, for $\eta \in [0, \infty)$,

(iii) there are $\alpha_j > 0$, $j = 1, \dots, 5$ such that $g_j(\eta) \geq \alpha_j\eta$ for $\eta \in [0, \infty)$.

(H4). $\frac{\partial}{\partial v} \left(\frac{\xi(x, v)}{g_4(v)}\right) \leq 0$ for $v \in (0, \infty)$.

3 Basic properties

In this section we study the basic properties of model (10)-(15). The non-negativity and boundedness of the solutions of the model will be established in the next theorem:

Theorem 1. Let Hypotheses (H1)-(H3) be hold true, then there exist a set

$$\Delta = \{(x, w, y, u, v, z) \in \mathbb{R}_{\geq 0}^6 : 0 \leq x, w, y, u \leq \kappa_1, 0 \leq v \leq \kappa_2, 0 \leq z \leq \kappa_3\}$$

which is positively invariant with respect to system (10)-(15), where κ_1 , κ_2 and κ_3 are positive numbers.

Proof. First, we show that $\mathbb{R}_{\geq 0}^6$ is positively invariant for system (10)-(15) as:

$$\dot{x}|_{x=0} = \pi(0) > 0,$$

$$\dot{w}|_{w=0} = \beta_1\xi(x, v) \geq 0 \quad \text{for } x, v \in [0, \infty),$$

$$\dot{y}|_{y=0} = \beta_2\xi(x, v) + a_1g_1(w) \geq 0 \quad \text{for } x, w, v \in [0, \infty),$$

$$\dot{u}|_{u=0} = \beta_3\xi(x, v) \geq 0 \quad \text{for } x, v \in [0, \infty),$$

$$\dot{v}|_{v=0} = N\delta_2g_2(y) + M\delta_3g_3(u) \geq 0 \quad \text{for } y, u \in [0, \infty),$$

$$\dot{z}|_{z=0} = 0.$$

Hence, all the state variables of system (10)-(15) are non-negative.

Let $T_1(t) = x(t) + w(t) + y(t) + u(t)$, then

$$\begin{aligned} \dot{T}_1 &= \pi(x) - \delta_1 g_1(w) - \delta_2 g_2(y) - \delta_3 g_3(u) \\ &\leq b - \bar{b}x - \delta_1 \alpha_1 w - \delta_2 \alpha_2 y - \delta_3 \alpha_3 u. \\ &\leq b - \sigma_1 (x + w + y + u) = b - \sigma_1 T_1, \end{aligned}$$

where $\sigma_1 = \min\{\bar{b}, \delta_1 \alpha_1, \delta_2 \alpha_2, \delta_3 \alpha_3\}$. Hence, $T_1(t) \leq \kappa_1$, if $T_1(0) \leq \kappa_1$, where $\kappa_1 = \frac{b}{\sigma_1}$. The non-negativity of $x(t)$, $w(t)$, $y(t)$ and $u(t)$ implies $0 \leq x(t), w(t), y(t), u(t) \leq \kappa_1$, if $0 \leq x(0) + w(0) + y(0) + u(0) \leq \kappa_1$. Moreover, we let $T_2(t) = v(t) + \frac{q}{r}z(t)$. Then

$$\begin{aligned} \dot{T}_2 &= N\delta_2 g_2(y) + M\delta_3 g_3(u) - \delta_4 g_4(v) - \frac{q\delta_5}{r}g_5(z) \\ &\leq N\delta_2 g_2(\kappa_1) + M\delta_3 g_3(\kappa_1) - \delta_4 \alpha_4 v - \frac{q\delta_5}{r}\alpha_5 z \\ &\leq N\delta_2 g_2(\kappa_1) + M\delta_3 g_3(\kappa_1) - \sigma_2 T_2, \end{aligned}$$

where $\sigma_2 = \min\{\delta_4 \alpha_4, \delta_5 \alpha_5\}$. Hence, $T_2(t) \leq \kappa_2$ if $T_2(0) \leq \kappa_2$, where $\kappa_2 = \frac{N\delta_2 g_2(\kappa_1) + M\delta_3 g_3(\kappa_1)}{\sigma_2}$. The non-negativity of $v(t)$ and $z(t)$ implies $0 \leq v(t) \leq \kappa_2$ and $0 \leq z(t) \leq \kappa_3$ if $0 \leq v(0) + \frac{q}{r}z(0) \leq \kappa_2$, where $\kappa_3 = \frac{r\kappa_2}{q}$.

Theorem 2. Suppose that Hypotheses (H1)-(H4) are valid, then there exist two bifurcation parameters \mathcal{R}_0 and \mathcal{R}_1 with $\mathcal{R}_0 > \mathcal{R}_1 > 0$ such that

- (i) if $\mathcal{R}_0 \leq 1$, then the system has only one positive steady state $S_0 \in \Delta$,
- (ii) if $\mathcal{R}_1 \leq 1 < \mathcal{R}_0$, then the system has only two positive steady states $S_0 \in \Delta$ and $S_1 \in \Delta$,
- (iii) if $\mathcal{R}_1 > 1$, then the system has three positive steady states $S_0 \in \Delta$, $S_1 \in \Delta$ and $S_2 \in \overset{\circ}{\Delta}$.

Proof. Let $S(x, w, y, u, v, z)$ be any steady state of (10)-(15) satisfying the following equations:

$$0 = \pi(x) - (\beta_1 + \beta_2 + \beta_3)\xi(x, v), \tag{16}$$

$$0 = \beta_1 \xi(x, v) - (a_1 + \delta_1)g_1(w), \tag{17}$$

$$0 = \beta_2 \xi(x, v) + a_1 g_1(w) - \delta_2 g_2(y), \tag{18}$$

$$0 = \beta_3 \xi(x, v) - \delta_3 g_3(u), \tag{19}$$

$$0 = N\delta_2 g_2(y) + M\delta_3 g_3(u) - \delta_4 g_4(v) - qg_4(v)g_5(z), \tag{20}$$

$$0 = rg_4(v)g_5(z) - \delta_5 g_5(z). \tag{21}$$

From Eq. (21) we have two possible solutions, $g_5(z) = 0$ and $g_4(v) = \delta_5/r$. Let us consider the case $g_5(z) = 0$, then from Hypothesis (H3) we get $z = 0$. Hypothesis (H3) implies that g_i^{-1} , $i = 1, \dots, 5$ exist, strictly increasing and $g_i^{-1}(0) = 0$. Let us define

$$\begin{aligned} \theta(x) &= g_1^{-1} \left(\frac{\beta_1}{\beta(a_1 + \delta_1)} \pi(x) \right), & \psi(x) &= g_2^{-1} \left(\frac{a_1 \beta_1 + (a_1 + \delta_1) \beta_2}{\delta_2 \beta(a_1 + \delta_1)} \pi(x) \right), \\ \mu(x) &= g_3^{-1} \left(\frac{\beta_3}{\delta_3 \beta} \pi(x) \right), & \ell(x) &= g_4^{-1} \left(\frac{\gamma}{\beta} \pi(x) \right), \end{aligned} \tag{22}$$

where $\beta = \beta_1 + \beta_2 + \beta_3$ and $\gamma = \frac{N(a_1 \beta_1 + (a_1 + \delta_1) \beta_2) + M\beta_3(a_1 + \delta_1)}{\delta_4(a_1 + \delta_1)}$. It follows from Eqs. (16)-(21) that:

$$w = \theta(x), \quad y = \psi(x), \quad u = \mu(x), \quad v = \ell(x). \tag{23}$$

Obviously, $\theta(x), \psi(x), \mu(x), \ell(x) > 0$ for $x \in [0, x_0)$ and $\theta(x_0) = \psi(x_0) = \mu(x_0) = \ell(x_0) = 0$. From Eqs. (16), (22) and (23) we obtain

$$\gamma \xi(x, \ell(x)) - g_4(\ell(x)) = 0. \tag{24}$$

Eq. (24) admits a solution $x = x_0$ which gives the infection-free steady state $S_0(x_0, 0, 0, 0, 0, 0)$. Let

$$\Psi_1(x) = \gamma\xi(x, \ell(x)) - g_4(\ell(x)) = 0.$$

It is clear from Hypotheses (H1) and (H2) that, $\Psi_1(0) = -g_4(\ell(0)) < 0$ and $\Psi_1(x_0) = 0$. Moreover,

$$\Psi'_1(x_0) = \gamma \left[\frac{\partial\xi(x_0, 0)}{\partial x} + \ell'(x_0) \frac{\partial\xi(x_0, 0)}{\partial v} \right] - g'_4(0)\ell'(x_0).$$

We note from Hypothesis (H2) that $\frac{\partial\xi(x_0, 0)}{\partial x} = 0$. Then,

$$\Psi'_1(x_0) = \ell'(x_0)g'_4(0) \left(\frac{\gamma}{g'_4(0)} \frac{\partial\xi(x_0, 0)}{\partial v} - 1 \right).$$

From Eq. (22), we get

$$\Psi'_1(x_0) = \frac{\gamma}{\beta} \pi'(x_0) \left(\frac{\gamma}{g'_4(0)} \frac{\partial\xi(x_0, 0)}{\partial v} - 1 \right).$$

Therefore, from Hypothesis (H1), we have $\pi'(x_0) < 0$. Therefore, if $\frac{\gamma}{g'_4(0)} \frac{\partial\xi(x_0, 0)}{\partial v} > 1$, then $\Psi'_1(x_0) < 0$ and there exists $x_1 \in (0, x_0)$ such that $\Psi_1(x_1) = 0$. Hypotheses (H1)-(H3) imply that

$$w_1 = \theta(x_1) > 0, \quad y_1 = \psi(x_1) > 0, \quad u_1 = \mu(x_1) > 0, \quad v_1 = \ell(x_1) > 0. \tag{25}$$

It means that, a humoral-inactivated infection steady state $S_1(x_1, w_1, y_1, u_1, v_1, 0)$ exists when $\frac{\gamma}{g'_4(0)} \frac{\partial\xi(x_0, 0)}{\partial v} > 1$.

Let us define

$$\mathcal{R}_0 = \frac{\gamma}{g'_4(0)} \frac{\partial\xi(x_0, 0)}{\partial v},$$

The other solution of Eq. (21) is $g_4(v_2) = \frac{\delta_5}{r}$ which yields $v_2 = g_4^{-1} \left(\frac{\delta_5}{r} \right) > 0$. Substituting $v = v_2$ in Eq. (16) and letting $\Psi_2(x) = \pi(x) - \beta\xi(x, v_2) = 0$. According to Hypotheses (H1) and (H2), Ψ_2 is strictly decreasing, $\Psi_2(0) = \pi(0) > 0$ and $\Psi_2(x_0) = -\beta\xi(x_0, v_2) < 0$. Thus, there exists a unique $x_2 \in (0, x_0)$ such that $\Psi_2(x_2) = 0$. It follows from Eqs. (20) and (23) that,

$$w_2 = \theta(x_2) > 0, \quad y_2 = \psi(x_2) > 0, \quad u_2 = \mu(x_2) > 0, \quad v_2 = g_4^{-1} \left(\frac{\delta_5}{r} \right) > 0,$$

$$z_2 = g_5^{-1} \left(\frac{\delta_4}{q} \left(\gamma \frac{\xi(x_2, v_2)}{g_4(v_2)} - 1 \right) \right).$$

Thus, $z_2 > 0$ when $\gamma \frac{\xi(x_2, v_2)}{g_4(v_2)} > 1$. Now we define the parameter \mathcal{R}_1 as:

$$\mathcal{R}_1 = \gamma \frac{\xi(x_2, v_2)}{g_4(v_2)}.$$

If $\mathcal{R}_1 > 1$, then $z_2 = g_5^{-1} \left(\frac{\delta_4}{q} (\mathcal{R}_1 - 1) \right) > 0$ and exists a humoral-activated infection steady state $S_2(x_2, w_2, y_2, u_2, v_2, z_2)$.

Now we show that $S_0 \in \Delta$, $S_1 \in \Delta$ and $S_2 \in \overset{\circ}{\Delta}$. Clearly, $S_0 \in \Delta$. Now we show that $S_1 \in \Delta$. We have $x_1 \in (0, x_0)$, then from Hypothesis (H1) we obtain

$$0 = \pi(x_0) < \pi(x_1) \leq b - \bar{b}x_1.$$

It follows that

$$0 < x_1 < \frac{b}{\bar{b}} \leq \frac{b}{\sigma_1} = \kappa_1.$$

From Hypothese (H3) and Eqs. (22)-(23), we get the following:

$$\begin{aligned} \alpha_1 w_1 \leq g_1(w_1) &= \frac{\beta_1}{\beta(a_1 + \delta_1)} \pi(x_1) < \frac{\beta_1}{\beta(a_1 + \delta_1)} \pi(0) < \frac{\beta_1 b}{\beta(a_1 + \delta_1)} \\ \Rightarrow 0 < w_1 &< \frac{\beta_1 b}{\beta \alpha_1 (a_1 + \delta_1)} < \frac{\beta_1 b}{\beta \alpha_1 \delta_1} < \frac{b}{\alpha_1 \delta_1} \leq \kappa_1, \\ \alpha_2 y_1 \leq g_2(y_1) &= \frac{a_1 \beta_1 + (a_1 + \delta_1) \beta_2}{\delta_2 \beta (a_1 + \delta_1)} \pi(x_1) < \frac{a_1 \beta_1 + (a_1 + \delta_1) \beta_2}{\delta_2 \beta (a_1 + \delta_1)} \pi(0) \\ &\leq \frac{(a_1 \beta_1 + (a_1 + \delta_1) \beta_2) b}{\delta_2 \beta (a_1 + \delta_1)} < \frac{(a_1 + \delta_1) (\beta_1 + \beta_2) b}{\delta_2 \beta (a_1 + \delta_1)} < \frac{b}{\delta_2} \\ \Rightarrow 0 < y_1 &< \frac{b}{\alpha_2 \delta_2} \leq \kappa_1, \\ \alpha_3 u_1 \leq g_3(u_1) &= \frac{\beta_3}{\delta_3 \beta} \pi(x_1) < \frac{\beta_3}{\delta_3 \beta} \pi(0) < \frac{\beta_3 b}{\delta_3 \beta} < \frac{b}{\delta_3} \\ \Rightarrow 0 < u_1 &< \frac{b}{\alpha_3 \delta_3} \leq \kappa_1, \end{aligned}$$

Eq. (20) implies that

$$\begin{aligned} \delta_4 \alpha_4 v_1 \leq \delta_4 g_4(v_1) &= N \delta_2 g_2(y_1) + M \delta_3 g_3(u_1) < N \delta_2 g_2(\kappa_1) + M \delta_3 g_3(\kappa_1) \\ \Rightarrow 0 < v_1 &< \frac{N \delta_2 g_2(\kappa_1) + M \delta_3 g_3(\kappa_1)}{\delta_4 \alpha_4} \leq \kappa_2. \end{aligned}$$

We have $z_1 = 0$ then, $S_1 \in \Delta$.

It is clear that $0 < x_2, w_2, y_2, u_2 < \kappa_1$. Next we show that $0 < v_2 < \kappa_2$ and $0 < z_2 < \kappa_3$ when $\mathcal{R}_1 > 1$. From the steady state conditions of S_2 we have,

$$\delta_4 g_4(v_2) + q g_4(v_2) g_5(z_2) = N \delta_2 g_2(y_2) + M \delta_3 g_3(u_2).$$

Then if $\mathcal{R}_1 > 1$ we get

$$\begin{aligned} \delta_4 g_4(v_2) &< N \delta_2 g_2(y_2) + M \delta_3 g_3(u_2) \\ \Rightarrow \delta_4 \alpha_4 v_2 &< N \delta_2 g_2(\kappa_1) + M \delta_3 g_3(\kappa_1) \\ \Rightarrow 0 < v_2 &< \frac{N \delta_2 g_2(\kappa_1) + M \delta_3 g_3(\kappa_1)}{\delta_4 \alpha_4} \leq \kappa_2, \end{aligned}$$

and

$$\begin{aligned} q g_4(v_2) g_5(z_2) &\leq N \delta_2 g_2(y_2) + M \delta_3 g_3(u_2) \\ \Rightarrow \frac{q \delta_5}{r} \alpha_5 z_2 &\leq N \delta_2 g_2(\kappa_1) + M \delta_3 g_3(\kappa_1) \\ \Rightarrow 0 < z_2 &< \frac{r [N \delta_2 g_2(\kappa_1) + M \delta_3 g_3(\kappa_1)]}{q \delta_5 \alpha_5} \leq \kappa_3. \end{aligned}$$

Then, $S_2 \in \overset{\circ}{\Delta}$. Clearly from Hypotheses (H2) and (H4), we have

$$\mathcal{R}_1 = \gamma \frac{\xi(x_2, v_2)}{g_4(v_2)} \leq \gamma \lim_{v \rightarrow 0^+} \frac{\xi(x_2, v)}{g_4(v)} = \frac{\gamma}{g_4'(0)} \frac{\partial \xi(x_2, 0)}{\partial v} < \frac{\gamma}{g_4'(0)} \frac{\partial \xi(x_0, 0)}{\partial v} = \mathcal{R}_0. \square$$

4 Global properties

In this section, we established the global stability of the three steady states by of system (10)-(15) by constructing suitable Lyapunov functions.

Theorem 3. Suppose that Hypotheses (H1)-(H4) are valid and $\mathcal{R}_0 \leq 1$, then S_0 is globally asymptotically stable in Δ .

Proof. Define

$$W_0 = x - x_0 - \int_{x_0}^x \lim_{v \rightarrow 0^+} \frac{\xi(x_0, v)}{\xi(\eta, v)} d\eta + k_1 w + k_2 y + k_3 u + k_4 v + k_5 z, \tag{26}$$

where

$$\begin{aligned} \beta_1 k_1 + \beta_2 k_2 + \beta_3 k_3 &= \beta, & (a_1 + \delta_1) k_1 &= a_1 k_2, \\ k_2 &= N k_4, & k_3 &= M k_4, & q k_4 &= r k_5. \end{aligned} \tag{27}$$

The solution of Eqs. (27) is given by

$$k_1 = \frac{a_1 N \beta}{\gamma \delta_4 (a_1 + \delta_1)}, \quad k_2 = \frac{N \beta}{\gamma \delta_4}, \quad k_3 = \frac{M \beta}{\gamma \delta_4}, \quad k_4 = \frac{\beta}{\gamma \delta_4}, \quad k_5 = \frac{q \beta}{r \gamma \delta_4}. \tag{28}$$

We evaluate $\frac{dW_0}{dt}$ along the solutions of (10)-(15) as:

$$\begin{aligned} \frac{dW_0}{dt} &= \left(1 - \lim_{v \rightarrow 0^+} \frac{\xi(x_0, v)}{\xi(x, v)} \right) (\pi(x) - \beta \xi(x, v)) \\ &\quad + k_1 (\beta_1 \xi(x, v) - (a_1 + \delta_1) g_1(w)) + k_2 (\beta_2 \xi(x, v) + a_1 g_1(w) - \delta_2 g_2(y)) \\ &\quad + k_3 (\beta_3 \xi(x, v) - \delta_3 g_3(u)) + k_4 (N \delta_2 g_2(y) + M \delta_3 g_3(u) - \delta_4 g_4(v) - q g_4(v) g_5(z)) \\ &\quad + k_5 (r g_4(v) g_5(z) - \delta_5 g_5(z)). \end{aligned} \tag{29}$$

Collecting terms of Eq. (29) and using $\pi(x_0) = 0$, we obtain

$$\begin{aligned} \frac{dW_0}{dt} &= (\pi(x) - \pi(x_0)) \left(1 - \lim_{v \rightarrow 0^+} \frac{\xi(x_0, v)}{\xi(x, v)} \right) + \left(\frac{\beta \xi(x, v)}{g_4(v)} \lim_{v \rightarrow 0^+} \frac{\xi(x_0, v)}{\xi(x, v)} - k_4 \delta_4 \right) g_4(v) - k_5 \delta_5 g_5(z) \\ &\leq (\pi(x) - \pi(x_0)) \left(1 - \lim_{v \rightarrow 0^+} \frac{\xi(x_0, v)}{\xi(x, v)} \right) + \left(\lim_{v \rightarrow 0^+} \frac{\beta \xi(x, v)}{g_4(v)} \lim_{v \rightarrow 0^+} \frac{\xi(x_0, v)}{\xi(x, v)} - k_4 \delta_4 \right) g_4(v) - k_5 \delta_5 g_5(z) \\ &= (\pi(x) - \pi(x_0)) \left(1 - \frac{\partial \xi(x_0, 0) / \partial v}{\partial \xi(x, 0) / \partial v} \right) + k_4 \delta_4 \left(\frac{\beta}{k_4 \delta_4 g_4'(0)} \frac{\partial \xi(x_0, 0)}{\partial v} - 1 \right) g_4(v) - k_5 \delta_5 g_5(z) \\ &= (\pi(x) - \pi(x_0)) \left(1 - \frac{\partial \xi(x_0, 0) / \partial v}{\partial \xi(x, 0) / \partial v} \right) + k_4 \delta_4 (\mathcal{R}_0 - 1) g_4(v) - k_5 \delta_5 g_5(z). \end{aligned} \tag{30}$$

From Hypotheses (H1) and (H2), we have

$$(\pi(x) - \pi(x_0)) \left(1 - \frac{\partial \xi(x_0, 0, 0) / \partial v}{\partial \xi(x, 0, 0) / \partial v} \right) \leq 0.$$

Therefore, if $\mathcal{R}_0 \leq 1$, then $\frac{dW_0}{dt} \leq 0$ for $x, v, z \in (0, \infty)$. Moreover, $\frac{dW_0}{dt} = 0$ if and only if $x(t) = x_0, v(t) = 0$ and $z(t) = 0$ for all t . It is easy to show that, the largest invariant set $\Gamma_0 \subseteq \Gamma = \{(x, w, y, u, v, z) : \frac{dW_0}{dt} = 0\}$ is the singleton $\{S_0\}$ [31]. LaSalle's invariance principle provides that S_0 is globally asymptotically stable. \square

Lemma 1. Let $\mathcal{R}_0 > 1$ and Hypotheses (H1)-(H4) are satisfied, then

$$\text{sgn}(\mathcal{R}_1 - 1) = \text{sgn}(v_1 - v_2) = \text{sgn}(x_2 - x_1).$$

Proof. Using Hypotheses (H1) and (H2), that for $x_1, x_2, v_1, v_2 > 0$, we get

$$(x_1 - x_2) (\pi(x_2) - \pi(x_1)) > 0, \tag{31}$$

$$(x_2 - x_1) (\xi(x_2, v_2) - \xi(x_1, v_2)) > 0, \tag{32}$$

$$(v_2 - v_1) (\xi(x_1, v_2) - \xi(x_1, v_1)) > 0, \tag{33}$$

and from Hypothese (H4), we obtain

$$(v_1 - v_2) \left(\frac{\xi(x_1, v_2)}{g_2(v_2)} - \frac{\xi(x_1, v_1)}{g_2(v_1)} \right) > 0. \tag{34}$$

First, we show that $sgn(v_1 - v_2) = sgn(x_2 - x_1)$. Suppose that $sgn(v_2 - v_1) = sgn(x_2 - x_1)$. Using the steady states conditions of S_1 and S_2 we obtain

$$\pi(x_2) - \pi(x_1) = \beta [\xi(x_2, v_2) - \xi(x_1, v_1)] = \beta [(\xi(x_2, v_2) - \xi(x_1, v_2)) + (\xi(x_1, v_2) - \xi(x_1, v_1))].$$

Therefore, from inequalities (31)-(33) we obtain $sgn(x_2 - x_1) = sgn(x_1 - x_2)$, which is a contradiction, hence, $sgn(v_1 - v_2) = sgn(x_2 - x_1)$. Using Eq. (25) and the definition of \mathcal{R}_1 we get

$$\mathcal{R}_1 - 1 = \gamma \left(\frac{\xi(x_2, v_2)}{g_4(v_2)} - \frac{\xi(x_1, v_1)}{g_4(v_1)} \right) = \gamma \left[\frac{1}{g_4(v_2)} (\xi(x_2, v_2) - \xi(x_1, v_2)) + \frac{\xi(x_1, v_2)}{g_4(v_2)} - \frac{\xi(x_1, v_1)}{g_4(v_1)} \right].$$

Thus, from Eqs. (32) and (34) we obtain $sgn(\mathcal{R}_1 - 1) = sgn(v_1 - v_2)$.

Theorem 4. Suppose that Hypotheses (H1)-(H4) are satisfied and $\mathcal{R}_1 \leq 1 < \mathcal{R}_0$, then S_1 is globally asymptotically stable in Δ .

Proof. We introduce Lyapunov function

$$\begin{aligned} W_1 = & x - x_1 - \int_{x_1}^x \frac{\xi(x_1, v_1)}{\xi(\eta, v_1)} d\eta + k_1 \left(w - w_1 - \int_{w_1}^w \frac{g_1(w_1)}{g_1(\eta)} d\eta \right) + k_2 \left(y - y_1 - \int_{y_1}^y \frac{g_2(y_1)}{g_2(\eta)} d\eta \right) \\ & + k_3 \left(u - u_1 - \int_{u_1}^u \frac{g_3(u_1)}{g_3(\eta)} d\eta \right) + k_4 \left(v - v_1 - \int_{v_1}^v \frac{g_4(v_1)}{g_4(\eta)} d\eta \right) + k_5 z, \end{aligned}$$

and evaluate $\frac{dW_1}{dt}$ along the trajectories of (10)-(15):

$$\begin{aligned} \frac{dW_1}{dt} = & \left(1 - \frac{\xi(x_1, v_1)}{\xi(x, v_1)} \right) (\pi(x) - \beta\xi(x, v)) + k_1 \left(1 - \frac{g_1(w_1)}{g_1(w)} \right) (\beta_1\xi(x, v) - (a_1 + \delta_1)g_1(w)) \\ & + k_2 \left(1 - \frac{g_2(y_1)}{g_2(y)} \right) (\beta_2\xi(x, v) + a_1g_1(w) - \delta_2g_2(y)) + k_3 \left(1 - \frac{g_3(u_1)}{g_3(u)} \right) (\beta_3\xi(x, v) - \delta_3g_3(u)) \\ & + k_4 \left(1 - \frac{g_4(v_1)}{g_4(v)} \right) (N\delta_2g_2(y) + M\delta_3g_3(u) - \delta_4g_4(v) - qg_4(v)g_5(z)) \\ & + k_5 (rg_4(v)g_5(z) - \delta_5g_5(z)). \end{aligned} \tag{35}$$

Collecting terms of Eq. (35) and applying $\pi(x_1) = \beta\xi(x_1, v_1)$ we get

$$\begin{aligned} \frac{dW_1}{dt} = & (\pi(x) - \pi(x_1)) \left(1 - \frac{\xi(x_1, v_1)}{\xi(x, v_1)} \right) + \beta\xi(x_1, v_1) \left(1 - \frac{\xi(x_1, v_1)}{\xi(x, v_1)} \right) \\ & + \beta\xi(x, v) \frac{\xi(x_1, v_1)}{\xi(x, v_1)} - k_1\beta_1\xi(x, v) \frac{g_1(w_1)}{g_1(w)} + k_1(a_1 + \delta_1)g_1(w_1) - k_2\beta_2\xi(x, v) \frac{g_2(y_1)}{g_2(y)} \\ & - k_2a_1 \frac{g_2(y_1)g_1(w)}{g_2(y)} + k_2\delta_2g_2(y_1) - k_3\beta_3\xi(x, v) \frac{g_3(u_1)}{g_3(u)} + k_3\delta_3g_3(u_1) - k_4N\delta_2g_2(y) \frac{g_4(v_1)}{g_4(v)} \\ & - k_4M\delta_3g_3(u) \frac{g_4(v_1)}{g_4(v)} - k_4\delta_4g_4(v) + k_4\delta_4g_4(v_1) + k_4qg_4(v_1)g_5(z) - k_5\delta_5g_5(z). \end{aligned} \tag{36}$$

Utilizing conditions of the steady state S_1 , we obtain

$$\begin{aligned} (a_1 + \delta_1)g_1(w_1) &= \beta_1\xi(x_1, v_1), & k_2\delta_2g_2(y_1) &= (k_1\beta_1 + k_2\beta_2)\xi(x_1, v_1), \\ \delta_3g_3(u_1) &= \beta_3\xi(x_1, v_1), & k_4\delta_4g_4(v_1) &= \beta\xi(x_1, v_1), \end{aligned}$$

$$\begin{aligned} \frac{dW_1}{dt} &= (\pi(x) - \pi(x_1)) \left(1 - \frac{\xi(x_1, v_1)}{\xi(x, v_1)} \right) + \beta \xi(x_1, v_1) \left(1 - \frac{\xi(x_1, v_1)}{\xi(x, v_1)} \right) \\ &+ \beta \xi(x_1, v_1) \left(\frac{\xi(x, v)}{\xi(x, v_1)} - \frac{g_4(v)}{g_4(v_1)} \right) - k_1 \beta_1 \xi(x_1, v_1) \frac{\xi(x, v) g_1(w_1)}{\xi(x_1, v_1) g_1(w)} + k_1 \beta_1 \xi(x_1, v_1) \\ &- k_2 \beta_2 \xi(x_1, v_1) \frac{\xi(x, v) g_2(y_1)}{\xi(x_1, v_1) g_2(y)} - k_1 \beta_1 \xi(x_1, v_1) \frac{g_2(y_1) g_1(w)}{g_2(y) g_1(w_1)} + (k_1 \beta_1 + k_2 \beta_2) \xi(x_1, v_1) \\ &- k_3 \beta_3 \xi(x_1, v_1) \frac{\xi(x, v) g_3(u_1)}{\xi(x_1, v_1) g_3(u)} + k_3 \beta_3 \xi(x_1, v_1) - (k_1 \beta_1 + k_2 \beta_2) \xi(x_1, v_1) \frac{g_2(y) g_4(v_1)}{g_2(y_1) g_4(v)} \\ &- k_3 \beta_3 \xi(x_1, v_1) \frac{g_3(u) g_4(v_1)}{g_3(u_1) g_4(v)} + \beta \xi(x_1, v_1) + k_5 r \left(g_4(v_1) - \frac{\delta_5}{r} \right) g_5(z). \end{aligned} \tag{37}$$

Equation (37) can be simplified as:

$$\begin{aligned} \frac{dW_1}{dt} &= (\pi(x) - \pi(x_1)) \left(1 - \frac{\xi(x_1, v_1)}{\xi(x, v_1)} \right) + \frac{\beta \xi(x_1, v_1) g_4(v)}{\xi(x, v) \xi(x, v_1)} \left(\frac{\xi(x, v)}{g_4(v)} - \frac{\xi(x, v_1)}{g_4(v_1)} \right) (\xi(x, v) - \xi(x, v_1)) \\ &+ k_1 \beta_1 \xi(x_1, v_1) \left[5 - \frac{\xi(x_1, v_1)}{\xi(x, v_1)} - \frac{\xi(x, v) g_1(w_1)}{\xi(x_1, v_1) g_1(w)} - \frac{g_2(y_1) g_1(w)}{g_2(y) g_1(w_1)} - \frac{g_2(y) g_4(v_1)}{g_2(y_1) g_4(v)} - \frac{g_4(v) \xi(x, v_1)}{g_4(v_1) \xi(x, v)} \right] \\ &+ k_2 \beta_2 \xi(x_1, v_1) \left[4 - \frac{\xi(x_1, v_1)}{\xi(x, v_1)} - \frac{\xi(x, v) g_2(y_1)}{\xi(x_1, v_1) g_2(y)} - \frac{g_2(y) g_4(v_1)}{g_2(y_1) g_4(v)} - \frac{g_4(v) \xi(x, v_1)}{g_4(v_1) \xi(x, v)} \right] \\ &+ k_3 \beta_3 \xi(x_1, v_1) \left[4 - \frac{\xi(x_1, v_1)}{\xi(x, v_1)} - \frac{\xi(x, v) g_3(u_1)}{\xi(x_1, v_1) g_3(u)} - \frac{g_3(u) g_4(v_1)}{g_3(u_1) g_4(v)} - \frac{g_4(v) \xi(x, v_1)}{g_4(v_1) \xi(x, v)} \right] \\ &+ k_5 r (g_4(v_1) - g_4(v_2)) g_5(z). \end{aligned} \tag{38}$$

Hypotheses (H1), (H2), (H4), Lemma 1 and the condition $\mathcal{R}_1 \leq 1$ imply that

$$\begin{aligned} (\pi(x) - \pi(x_1)) \left(1 - \frac{\xi(x_1, v_1)}{\xi(x, v_1)} \right) &\leq 0, \\ \left(\frac{\xi(x, v)}{g_4(v)} - \frac{\xi(x, v_1)}{g_4(v_1)} \right) (\xi(x, v) - \xi(x, v_1)) &\leq 0, \\ g_4(v_1) - g_4(v_2) &\leq 0. \end{aligned}$$

It is known that the arithmetical mean is greater than or equal to the geometrical mean. It follows that for all $x, y, v, z > 0$ we have $\frac{dW_1}{dt} \leq 0$. Clearly, the largest invariant set $\Gamma_0 \subseteq \Gamma = \{(x, w, y, u, v, z) : \frac{dW_1}{dt} = 0\}$ is the singleton $\{S_1\}$. By LaSalle's invariance principle, S_1 is globally asymptotically stable. \square

Theorem 5. Let $\mathcal{R}_1 > 1$ and Hypotheses (H1)-(H4) are satisfied, then S_2 is globally asymptotically stable in $\overset{\circ}{\Delta}$.

Proof. Define a Lyapunov functional

$$\begin{aligned} W_2 &= x - x_2 - \int_{x_2}^x \frac{\xi(x_2, v_2)}{\xi(\eta, v_2)} d\eta + k_1 \left(w - w_2 - \int_{w_2}^w \frac{g_1(w_2)}{g_1(\eta)} d\eta \right) + k_2 \left(y - y_2 - \int_{y_2}^y \frac{g_2(y_2)}{g_2(\eta)} d\eta \right) \\ &+ k_3 \left(u - u_2 - \int_{u_2}^u \frac{g_3(u_2)}{g_3(\eta)} d\eta \right) + k_4 \left(v - v_2 - \int_{v_2}^v \frac{g_4(v_2)}{g_4(\eta)} d\eta \right) + k_5 \left(z - z_2 - \int_{z_2}^z \frac{g_5(z_2)}{g_5(\eta)} d\eta \right). \end{aligned}$$

$$\begin{aligned} \frac{dW_2}{dt} = & \left(1 - \frac{\xi(x_2, v_2)}{\xi(x, v_2)}\right) (\pi(x) - \beta\xi(x, v)) + k_1 \left(1 - \frac{g_1(w_2)}{g_1(w)}\right) (\beta_1\xi(x, v) - (a_1 + \delta_1)g_1(w)) \\ & + k_2 \left(1 - \frac{g_2(y_2)}{g_2(y)}\right) (\beta_2\xi(x, v) + a_1g_1(w) - \delta_2g_2(y)) + k_3 \left(1 - \frac{g_3(u_2)}{g_3(u)}\right) (\beta_3\xi(x, v) - \delta_3g_3(u)) \\ & + k_4 \left(1 - \frac{g_4(v_2)}{g_4(v)}\right) (N\delta_2g_2(y) + M\delta_3g_3(u) - \delta_4g_4(v) - qg_4(v)g_5(z)) \\ & + k_5 \left(1 - \frac{g_5(z_2)}{g_5(z)}\right) (rg_4(v)g_5(z) - \delta_5g_5(z)). \end{aligned} \tag{39}$$

Collecting terms of Eq. (39) and applying $\pi(x_2) = \beta\xi(x_2, v_2)$ we get

$$\begin{aligned} \frac{dW_1}{dt} = & (\pi(x) - \pi(x_2)) \left(1 - \frac{\xi(x_2, v_2)}{\xi(x, v_2)}\right) + \beta\xi(x_2, v_2) \left(1 - \frac{\xi(x_2, v_2)}{\xi(x, v_2)}\right) \\ & + \beta\xi(x, v) \frac{\xi(x_2, v_2)}{\xi(x, v_2)} - k_1\beta_1\xi(x, v) \frac{g_1(w_2)}{g_1(w)} + k_1(a_1 + \delta_1)g_1(w_2) \\ & - k_2\beta_2\xi(x, v) \frac{g_2(y_2)}{g_2(y)} - k_2a_1 \frac{g_2(y_2)g_1(w)}{g_2(y)} + k_2\delta_2g_2(y_2) - k_3\beta_3\xi(x, v) \frac{g_3(u_2)}{g_3(u)} \\ & + k_3\delta_3g_3(u_2) - k_4N\delta_2g_2(y) \frac{g_4(v_2)}{g_4(v)} - k_4M\delta_3g_3(u) \frac{g_4(v_2)}{g_4(v)} - k_4\delta_4g_4(v) + k_4\delta_4g_4(v_2) \\ & + k_4qg_4(v_2)g_5(z) - k_5\delta_5g_5(z) - k_5rg_5(z_2)g_4(v) + k_5\delta_5g_5(z_2) \end{aligned}$$

Using the following steady state conditions for S_1 :

$$\begin{aligned} (a_1 + \delta_1)g_1(w_2) &= \beta_1\xi(x_2, v_2), & k_2\delta_2g_2(y_2) &= (k_1\beta_1 + k_2\beta_2)\xi(x_2, v_2), \\ \delta_3g_3(u_2) &= \beta_3\xi(x_2, v_2), & k_4\delta_4g_4(v_2) &= \beta\xi(x_2, v_2) - k_4qg_5(z_2)g_4(v_2), \end{aligned}$$

we obtain

$$\begin{aligned} \frac{dW_2}{dt} = & (\pi(x) - \pi(x_2)) \left(1 - \frac{\xi(x_2, v_2)}{\xi(x, v_2)}\right) + \beta\xi(x_2, v_2) \left(1 - \frac{\xi(x_2, v_2)}{\xi(x, v_2)}\right) \\ & + \beta\xi(x_2, v_2) \left(\frac{\xi(x, v)}{\xi(x, v_2)} - \frac{g_4(v)}{g_4(v_2)}\right) - k_1\beta_1\xi(x_2, v_2) \frac{\xi(x, v)g_1(w_2)}{\xi(x_2, v_2)g_1(w)} + k_1\beta_1\xi(x_2, v_2) \\ & - k_2\beta_2\xi(x_2, v_2) \frac{\xi(x, v)g_2(y_2)}{\xi(x_2, v_2)g_2(y)} - k_1\beta_1\xi(x_2, v_2) \frac{g_2(y_2)g_1(w)}{g_2(y)g_1(w_2)} + (k_1\beta_1 + k_2\beta_2)\xi(x_2, v_2) \\ & - k_3\beta_3\xi(x_2, v_2) \frac{\xi(x, v)g_3(u_2)}{\xi(x_2, v_2)g_3(u)} + k_3\beta_3\xi(x_2, v_2) \\ & - (k_1\beta_1 + k_2\beta_2)\xi(x_2, v_2) \frac{g_2(y)g_4(v_2)}{g_2(y_2)g_4(v)} - k_3\beta_3\xi(x_2, v_2) \frac{g_3(u)g_4(v_2)}{g_3(u_2)g_4(v)} + \beta\xi(x_2, v_2). \end{aligned} \tag{40}$$

Equation (40) can be simplified as:

$$\begin{aligned} \frac{dW_2}{dt} = & (\pi(x) - \pi(x_2)) \left(1 - \frac{\xi(x_2, v_2)}{\xi(x, v_2)}\right) + \frac{\beta\xi(x_2, v_2)g_4(v)}{\xi(x, v)\xi(x, v_2)} \left(\frac{\xi(x, v)}{g_4(v)} - \frac{\xi(x, v_2)}{g_4(v_2)}\right) (\xi(x, v) - \xi(x, v_2)) \\ & + k_1\beta_1\xi(x_2, v_2) \left[5 - \frac{\xi(x_2, v_2)}{\xi(x, v_2)} - \frac{\xi(x, v)g_1(w_2)}{\xi(x_2, v_2)g_1(w)} - \frac{g_2(y_2)g_1(w)}{g_2(y)g_1(w_2)} - \frac{g_2(y)g_4(v_2)}{g_2(y_2)g_4(v)} - \frac{g_4(v)\xi(x, v_2)}{g_4(v_2)\xi(x, v)}\right] \\ & + k_2\beta_2\xi(x_2, v_2) \left[4 - \frac{\xi(x_2, v_2)}{\xi(x, v_2)} - \frac{\xi(x, v)g_2(y_2)}{\xi(x_2, v_2)g_2(y)} - \frac{g_2(y)g_4(v_2)}{g_2(y_2)g_4(v)} - \frac{g_4(v)\xi(x, v_2)}{g_4(v_2)\xi(x, v)}\right] \\ & + k_3\beta_3\xi(x_2, v_2) \left[4 - \frac{\xi(x_2, v_2)}{\xi(x, v_2)} - \frac{\xi(x, v)g_3(u_2)}{\xi(x_2, v_2)g_3(u)} - \frac{g_3(u)g_4(v_2)}{g_3(u_2)g_4(v)} - \frac{g_4(v)\xi(x, v_2)}{g_4(v_2)\xi(x, v)}\right]. \end{aligned} \tag{41}$$

According to Hypotheses (H1), (H2) and (H4) and the relation between the geometrical and arithmetical means we get $\frac{dW_2}{dt} \leq 0$. Clearly, the largest invariant set $\Gamma_0 \subseteq \Gamma = \{(x, w, y, u, v, z) : \frac{dW_2}{dt} = 0\}$ is the singleton $\{S_2\}$. By LaSalle's invariance principle S_2 is globally asymptotically stable. \square

5 Numerical simulations

We now perform some computer simulations on the following application:

$$\dot{x} = \rho - dx + px \left(1 - \frac{x}{x_{\max}} \right) - \frac{(1 - \varepsilon_r)\bar{\beta}xv}{1 + \eta_1x + \eta_2v}, \tag{42}$$

$$\dot{w} = \frac{(1 - \varepsilon_r)\bar{\beta}_1xv}{1 + \eta_1x + \eta_2v} - (a_1 + \delta_1)w, \tag{43}$$

$$\dot{y} = \frac{(1 - \varepsilon_r)\bar{\beta}_2xv}{1 + \eta_1x + \eta_2v} + a_1w - \delta_2y, \tag{44}$$

$$\dot{u} = \frac{(1 - \varepsilon_r)\bar{\beta}_3xv}{1 + \eta_1x + \eta_2v} - \delta_3u, \tag{45}$$

$$\dot{v} = (1 - \varepsilon_p)\bar{N}\delta_2y + (1 - \varepsilon_p)\bar{M}\delta_3u - \delta_4v - qvz, \tag{46}$$

$$\dot{z} = rvz - \delta_5z. \tag{47}$$

We assume that $p < d$. In this application, we consider the following specific forms of the general functions:

$$\pi(x) = \rho - dx + px \left(1 - \frac{x}{x_{\max}} \right), \quad \xi(x, v) = \frac{xv}{1 + \eta_1x + \eta_2v}, \quad g_i(\theta) = \theta, \quad i = 1, \dots, 5.$$

First we verify Hypotheses (H1)-(H4) for the chosen forms, then we solve the system using MATLAB. Clearly, $\pi(0) = \rho > 0$ and $\pi(x_0) = 0$, where

$$x_0 = \frac{x_{\max}}{2p} \left(p - d + \sqrt{(p - d)^2 + \frac{4\rho p}{x_{\max}}} \right).$$

We have

$$\pi'(x) = -d + p - \frac{2px}{x_{\max}} < 0. \tag{48}$$

Clearly, $\pi(x) > 0$, for $x \in [0, x_0)$ and

$$\pi(x) = \rho - (d - p)x - p\frac{x^2}{x_{\max}} \leq \rho - (d - p)x$$

Then Hypothesis (H1) is satisfied. We also have $\xi(x, v) > 0$, $\xi(0, v) = \xi(x, 0) = 0$ for $x, v \in (0, \infty)$, and

$$\frac{\partial \xi(x, v)}{\partial x} = \frac{v(1 + \delta v)}{(1 + \eta_1x + \eta_2v)^2}, \quad \frac{\partial \xi(x, v)}{\partial v} = \frac{x(1 + \eta_1x)}{(1 + \eta_1x + \eta_2v)^2}, \quad \frac{\partial \xi(x, 0)}{\partial v} = \frac{x}{1 + \eta_1x}.$$

Then, $\frac{\partial \xi(x, v)}{\partial x} > 0$, $\frac{\partial \xi(x, v)}{\partial v} > 0$ and $\frac{\partial \xi(x, 0)}{\partial v} > 0$ for $x, v \in (0, \infty)$. Therefore, Hypothesis (H1) is satisfied. In addition

$$\xi(x, v) = \frac{xv}{1 + \eta_1x + \eta_2v} \leq \frac{xv}{1 + \eta_1x} = v\frac{\partial \xi(x, 0)}{\partial v},$$

$$\left(\frac{\partial \xi(x, 0)}{\partial v} \right)' = \frac{1}{(1 + \eta_1x)^2} > 0 \text{ for all } x > 0.$$

It follows that, (H2) is satisfied. Clearly Hypothesis (H3) holds true. Moreover,

$$\frac{\partial}{\partial v} \left(\frac{\xi(x, v)}{g_4(v)} \right) = \frac{-\eta_2x}{(1 + \eta_1x + \eta_2v)} < 0.$$

Therefore, Hypothesis (H4) hold true and Theorems 3-5 are applicable. The parameters \mathcal{R}_0 and \mathcal{R}_1 for this application are given by:

$$\mathcal{R}_0 = \frac{(1 - \varepsilon_r)(1 - \varepsilon_p) \{ \bar{N}(a_1\bar{\beta}_1 + (a_1 + \delta_1)\bar{\beta}_2) + \bar{M}\bar{\beta}_3(a_1 + \delta_1) \}}{\delta_4(a_1 + \delta_1)} \frac{x_0}{1 + \eta_1x_0},$$

$$\mathcal{R}_1 = \frac{(1 - \varepsilon_r)(1 - \varepsilon_p) \{ \bar{N}(a_1\bar{\beta}_1 + (a_1 + \delta_1)\bar{\beta}_2) + \bar{M}\bar{\beta}_3(a_1 + \delta_1) \}}{\delta_4(a_1 + \delta_1)} \frac{x_2}{1 + \eta_1x_2 + \eta_2v_2}.$$

Now we are ready to perform some numerical simulations for system (42)-(47). The data of system (42)-(47) are provided in Table 1.

• **Effect of the drug efficacy on the stability of the steady states**

Now we verify our theoretical results given in Theorems 3-5 by numerical simulation. To discuss our global results we choose three different initial conditions:

IC1: $(x(0), w(0), y(0), u(0), v(0), z(0)) = (900, 10, 12, 60, 40, 1.6)$.

IC2: $(x(0), w(0), y(0), u(0), v(0), z(0)) = (700, 7, 8, 30, 25, 1.0)$.

IC3: $(x(0), w(0), y(0), u(0), v(0), z(0)) = (500, 4, 5, 10, 15, 0.6)$.

Let us address three scenarios for three different groups of the parameters $\varepsilon_r, \varepsilon_p$ and r .

Scenario (I): In this case we choose $\varepsilon_r = 0.6, \varepsilon_p = 0.6$ and $r = 0.001$ which gives $\mathcal{R}_0 = 0.4941 < 1$ and $\mathcal{R}_1 = 0.4430 < 1$. Therefore, based on Theorems 2 and 3, the system has unique steady state, that is S_0 and it is globally asymptotically stable. As we can see from Figures 1-6 that the concentration of the uninfected CD4⁺ T cells is increased and approached its normal value before infection that is $x_0 = 1083.9$, while concentrations of the other compartments converge to zero for all the three initial conditions. This case corresponds to the uninfected state where the HIV-1 is removed from the plasma.

Scenario (II): By taking $\varepsilon_r = 0.2, \varepsilon_p = 0.5$ and $r = 0.001$. With such choice we get, $\mathcal{R}_1 = 0.9351 < 1 < \mathcal{R}_0 = 1.2352$. Consequently, based on Theorems 2 and 4, the humoral-inactivated infection steady state S_1 is positive and is globally asymptotically stable. Figures 1-6 confirm that the numerical results support the theoretical results presented in Theorem 4. It can be observed that, the variables of the model eventually converge to $S_1 = (309.165, 13.2492, 15.4574, 94.0263, 72.0754, 0.0)$ for all the three initial conditions. This case corresponds to a chronic HIV-1 infection in the absense of immune response.

Scenario (III): $\varepsilon_r = 0.2, \varepsilon_p = 0.5$ and $r = 0.003$. Then, we calculate $\mathcal{R}_0 = 1.2352 > 1$ and $\mathcal{R}_1 = 1.19604 > 1$. According to Lemma 1 and Theorem 3, the humoral-activated infection steady state S_2 is positive and is globally asymptotically stable. We can see from Figures 1-6 that, there is a consistency between the numerical results and theoretical results of Theorem 5. The states of the system converge to $S_2 = \{820.603, 5.8629, 6.8401, 41.6079, 26.6667, 1.1762\}$ for all the three initial conditions, in the same time frame. In this case the humoral immune response is activated and can control the disease.

• **Effect of the HAART on the basic reproduction number:**

Let us define the overall HAART effect as $\varepsilon_e = \varepsilon_r + \varepsilon_p - \varepsilon_r \varepsilon_p$ [9]. If $\varepsilon_e = 0$, then the HAART has no effect, if $\varepsilon_e = 1$, the HIV-1 growth is completely halted. Consequently, the parameter \mathcal{R}_0 is given by

$$\mathcal{R}_0(\varepsilon_e) = \frac{(1 - \varepsilon_e) \{ \overline{N}(a_1 \overline{\beta}_1 + (a_1 + \delta_1) \overline{\beta}_2) + \overline{M} \overline{\beta}_3 (a_1 + \delta_1) \}}{\delta_4 (a_1 + \delta_1)} \frac{x_0}{1 + \eta_1 x_0}.$$

We note that, the value of $\mathcal{R}_0(\varepsilon_e)$ does not depend on the values of the parameters q, r and δ_5 . This means that, humoral immune response can play a significant role in reducing the infection progress but do not play a role in clearing the HIV-1 from the body. Since the goal is to clear the HIV-1 from the body, then we have to determine the drug efficacies that make $\mathcal{R}_0(\varepsilon_e) \leq 1$ for system (42)-(47). Now, we calculate the critical overall treatment effect ε_e^{crit} (i.e, the minimum overall treatment effect required to stabilize the system around the infection-free steady state). Let $\mathcal{R}_0(\varepsilon_e) \leq 1$, then

$$\varepsilon_e^{crit} \leq \varepsilon_e < 1, \quad \varepsilon_e^{crit} = \max \left\{ 0, \frac{\mathcal{R}_0(0) - 1}{\mathcal{R}_0(0)} \right\},$$

Figure 15 shows the effect of the HAART on the basic reproduction number $\mathcal{R}_0(\varepsilon)$. We note that, if $\varepsilon_e^{crit} \leq \varepsilon_e < 1$, then $\mathcal{R}_0(\varepsilon_e) \leq 1$ and S_0 is globally asymptotically stable. Moreover, if $0 \leq \varepsilon_e < \varepsilon_e^{crit}$, then S_0 is unstable.

Table 1: The values of the parameters of example (42)-(47).

Parameter	Value	Parameter	Value	Parameter	Value	Parameter	Value
ρ	10	δ_1	0.02	$\bar{\beta}_2$	0.0625	μ	0.08
d	0.01	δ_2	0.36	$\bar{\beta}_3$	0.0625	\bar{N}	62
p	0.008	δ_3	0.031	a_1	0.2	\bar{M}	30
x_{\max}	1200	δ_4	3.0	η_1	1	$\varepsilon_r, \varepsilon_p$	Varied
q	0.5	$\bar{\beta}_1$	0.0625	η_2	1	r	Varied

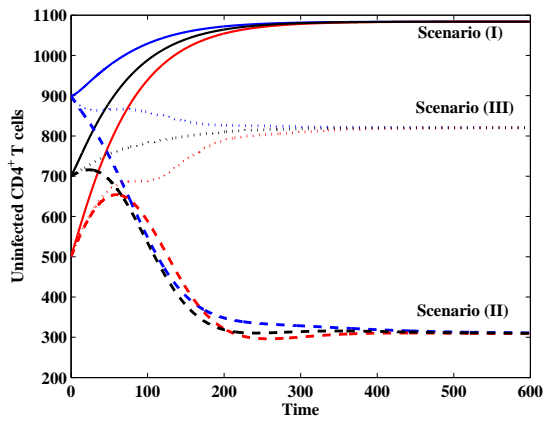


Figure 1: The concentration of uninfected CD4⁺ T cells for system (42)-(47).

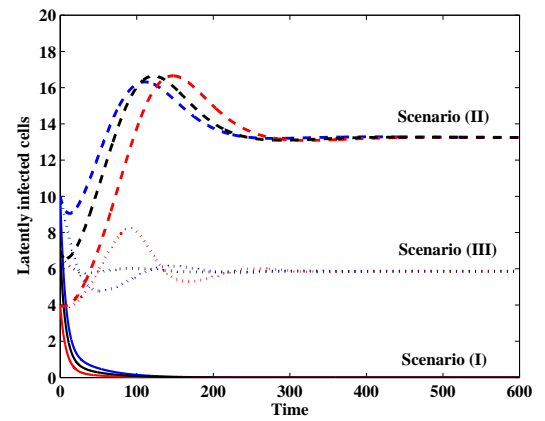


Figure 2: The concentration of latently infected cells for system (42)-(47).

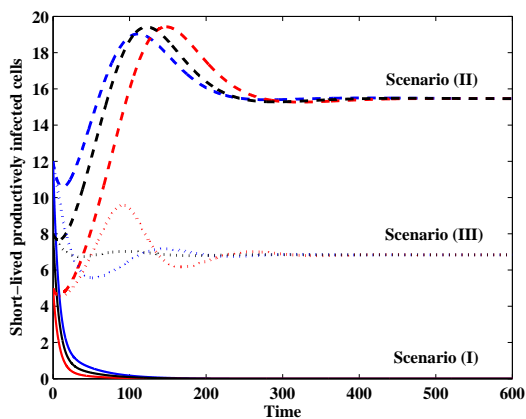


Figure 3: The concentration of short-lived productively infected cells for system (42)-(47).

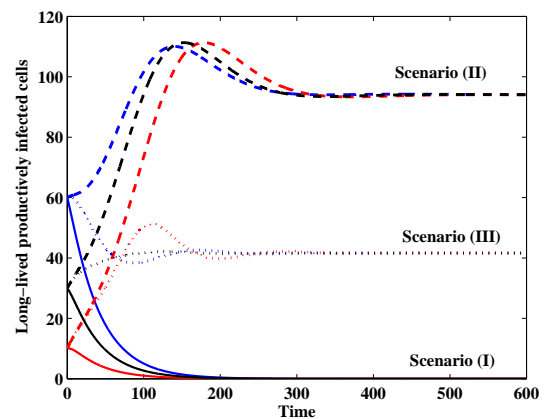


Figure 4: The concentration of long-lived productively infected cells for system (42)-(47).

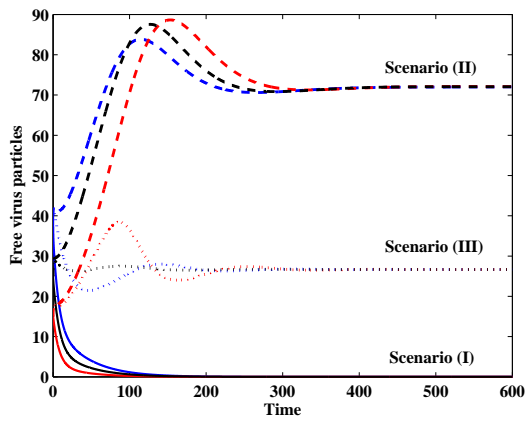


Figure 5: The concentration of free virus particles for system (42)-(47).

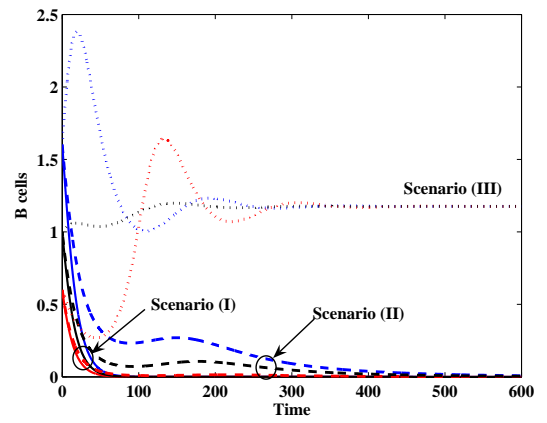


Figure 6: The concentration of B cells for system (42)-(47).

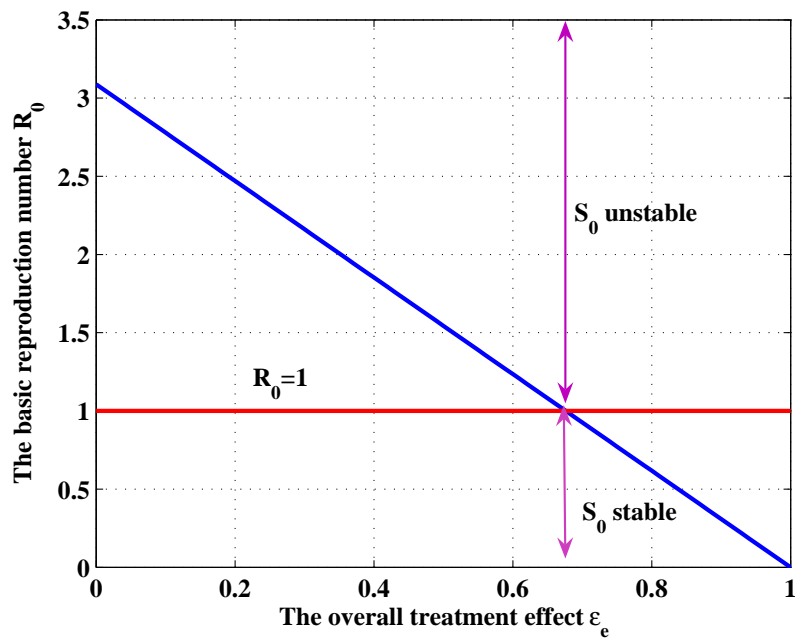


Figure 7: The basic reproduction number as a function of the overall treatment effect ϵ_e of system (42)-(47).

6 Acknowledgment

This research is supported by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, under Grant No. (29-130-36-HiCi). The authors, therefore, acknowledge with thanks DSR technical and financial support.

References

- [1] D. Wodarz and M. A. Nowak, Mathematical models of HIV pathogenesis and treatment, *BioEssays*, **24**, (2002), 1178-1187.
- [2] L. Rong and A. S. Perelson, Modeling HIV persistence, the latent reservoir, and viral blips, *Journal of Theoretical Biology*, **260** (2009), 308-331.
- [3] M. A. Nowak and C. R. M. Bangham, *Population dynamics of immune responses to persistent viruses*, *Science*, **272** (1996) 74-79.
- [4] M. A. Nowak, and R. M. May, *Virus dynamics: Mathematical Principles of Immunology and Virology*, Oxford Uni., Oxford, 2000.
- [5] X. Li, and S. Fu, *Global stability of a virus dynamics model with intracellular delay and CTL immune response*. *Mathematical Methods in the Applied Sciences* 2015; **38**:420-430.
- [6] A. S. Perelson, and P. W. Nelson, *Mathematical analysis of HIV-1 dynamics in vivo*, *SIAM Rev.*, 41 (1999), 3-44.
- [7] P. K. Roy, A. N. Chatterjee, D. Greenhalgh, and Q. J.A. Khan, *Long term dynamics in a mathematical model of HIV-1 infection with delay in different variants of the basic drug therapy model*, *Nonlinear Anal. Real World Appl.*, **14** (2013), 1621-1633.
- [8] Y. Yang, Y. Xu, *Global stability of a diffusive and delayed virus dynamics model with Beddington–DeAngelis incidence function and CTL immune response*, *Computers and Mathematics with Applications*, 71(4) (2016), 922-930.
- [9] Huang, Y., S. L. Rosenkranz, H.Wu, *Modeling HIV dynamic and antiviral response with consideration of time-varying drug exposures, adherence and phenotypic sensitivity*, *Mathematical Biosciences*, 184(2), pp. 165-186, 2003.
- [10] C. Lv, L. Huang, and Z. Yuan, *Global stability for an HIV-1 infection model with Beddington-DeAngelis incidence rate and CTL immune response*, *Communications in Nonlinear Science and Numerical Simulation*, **19** (2014), 121-127.
- [11] H. Shu, L. Wang and J. Watmough, *Global stability of a nonlinear viral infection model with infinitely distributed intracellular delays and CTL immune responses*, *SIAM Journal of Applied Mathematics*, **73**(3) (2013), 1280-1302.
- [12] K. Hattaf, N. Yousfi and A. Tridane, *Stability analysis of a virus dynamics model with general incidence rate and two delays*, *Applied Mathematics and Computation*, **221**, (2013), 514-521.
- [13] K. Hattaf, N. Yousfi and A. Tridane, *Mathematical analysis of a virus dynamics model with general incidence rate and cure rate*, *Nonlinear Anal. Real World Appl.*, **13**(4) (2012), 1866-1872.
- [14] X. Tian, R. Xu, *Global stability and Hopf bifurcation of an HIV-1 infection model with saturation incidence and delayed CTL immune response*, *Applied Mathematics and Computation*, **237**, (2014), 146-154.

- [15] G. Huang, Y. Takeuchi and W. Ma, *Lyapunov functionals for delay differential equations model of viral infections*, SIAM J. Appl. Math., **70**(7) (2010), 2693-2708.
- [16] A.M. Elaiw and S.A. Azoz, *Global properties of a class of HIV infection models with Beddington-DeAngelis functional response*, Mathematical Methods in the Applied Sciences, **36** (2013), 383-394.
- [17] A.M. Elaiw, *Global properties of a class of HIV models*, Nonlinear Analysis: Real World Applications, **11** (2010), 2253-2263.
- [18] D. Huang, X. Zhang, Y. Guo, and H. Wang, *Analysis of an HIV infection model with treatments and delayed immune response*, Applied Mathematical Modelling, **40**(4) (2016), 3081-3089.
- [19] C. Monica and M. Pitchaimani, *Analysis of stability and Hopf bifurcation for HIV-1 dynamics with PI and three intracellular delays*, Nonlinear Analysis: Real World Applications, **27** (2016), 55-69.
- [20] M. Y. Li and L. Wang, *Backward bifurcation in a mathematical model for HIV infection in vivo with anti-retroviral treatment*, Nonlinear Analysis: Real World Applications, **17** (2014), 147-160.
- [21] S. Liu, and L. Wang, *Global stability of an HIV-1 model with distributed intracellular delays and a combination therapy*, Mathematical Biosciences and Engineering, **7**(3) (2010), 675-685.
- [22] Elaiw AM, Almualllem NA. Global dynamics of delay-distributed HIV infection models with differential drug efficacy in cocirculating target cells. Mathematical Methods in the Applied Sciences 2016; **39**:4-31.
- [23] X. Wang, A. M. Elaiw, X. Song, *Global properties of a delayed HIV infection model with CTL immune response*, Applied Mathematics and Computation, **218** (2012), 9405-9414.
- [24] D.S. Callaway, and A.S. Perelson, *HIV-1 infection and low steady state viral loads*, Bull. Math. Biol., **64** (2002), 29-64.
- [25] B. Buonomo, and C. Vargas-De-Le, *Global stability for an HIV-1 infection model including an eclipse stage of infected cells*, Journal of Mathematical Analysis and Applications, **385** (2012), 709-720.
- [26] H. Wang, R. Xu, Z. Wang, H. Chen, *Global dynamics of a class of HIV-1 infection models with latently infected cells*, Nonlinear Analysis: Modelling and Control, **20** (1) (2012), 21-37.
- [27] A. Korobeinikov, *Global properties of basic virus dynamics models*, Bull. Math. Biol. **66**, (2004), 879-883.
- [28] S. Pankavich, *The effects of latent infection on the dynamics of HIV*, Differential Equations and Dynamical Systems (2015), doi: 10.1007/s12591-014-0234-6.
- [29] A. M. Elaiw and N. H. AlShamrani, *Global stability of humoral immunity virus dynamics models with nonlinear infection rate and removal*, Nonlinear Analysis: Real World Applications, **26**, (2015), 161-190.
- [30] W. S. Hlavacek, N. I. Stilianakis, A. S. Perelson, *Influence of follicular dendritic cells on HIV dynamics*, Philos.Trans. R. Soc. London B Biol. Sci. **355** (2000) 1051-1058.
- [31] J. K. Hale, and S. M. V. Lunel, *Introduction to functional differential equations*, Springer-Verlag, New York, (1993).

Ideal theory of pre-logics based on the theory of falling shadows

Young Bae Jun¹ and Sun Shin Ahn^{2,*}

¹ Department of Mathematics Education, Gyeongsang National University, Jinju, 52828, Korea

²Department of Mathematics Education, Dongguk University, Seoul 04620, Korea

Abstract. Based on the theory of a falling shadow which was first formulated by Wang [8], a theoretical approach of the ideal structure in pre-logics is established. The notions of a falling subalgebra, a falling and a positive implicative falling ideal of a pre-logic are introduced. Some fundamental properties are investigated. Relations among a falling subalgebra, a falling ideal and a positive implicative falling ideal are stated. Characterizations of falling deals and positive implicative falling ideals are discussed.

1. Introduction

In the study of a unified treatment of uncertainty modelled by means of combining probability and fuzzy set theory, Goodman [3] pointed out the equivalence of a fuzzy set and a class of random sets. Wang and Sanchez [7] introduced the theory of falling shadows which directly relates probability concepts to the membership function of fuzzy sets. Falling shadow representation theory shows us a method of selection relied on the joint degree distributions. It is a reasonable and convenient approach for the theoretical development and the practical applications of fuzzy sets and fuzzy logics. The mathematical structure of the theory of falling shadows is formulated in [8]. Y. B. Jun and C. H. Park [5] discussed the notion of a falling fuzzy subalgebra/ideal of a *BCK/BCI*-algebra. Y. B. Jun and M. S. Kang [4] established a theoretical approach for defining a fuzzy positive implicative ideal in a *BCK*-algebra based on the theory of falling shadows. I. Chajda and R. Halas [2] introduced the concept of a pre-logic which is an algebra weaker than a Hilbert algebra (an algebraic counterpart of intuitionistic logic) but strong enough to have deductive systems. Y. B. Jun and S. S. Ahn [1] defined the notion of pseudo-valuations (valuation) on pre-logics and induced a pseudo-metric by using a pseudo-valuation on pre-logics.

In this paper, we introduce the notions of a falling subalgebra, a falling ideal and a positive implicative falling ideal of a pre-logic. We investigate some fundamental properties. Also we give relations among a falling subalgebra, a falling ideal and a positive implicative falling ideal. We establish characterizations of falling ideals and positive implicative falling ideals.

2. Preliminaries

⁰**2010 Mathematics Subject Classification:** 06F35; 03G25; 08A72.

⁰**Keywords:** falling shadow; falling subalgebra; falling ideal; positive implicative falling ideal.

* The corresponding author. Tel: +82 2 2260 3410, Fax: +82 2 2266 3409

⁰**E-mail:** skywine@gmail.com (Y. B. Jun); sunshine@dongguk.edu (S. S. Ahn)

Young Bae Jun and Sun Shin Ahn

We recall some definitions and results (see [1, 2, 6]).

Definition 2.1. ([2]) By a *pre-logic*, we mean a triple $(X; *, 1)$ where X is a non-empty set, $*$ is a binary operation on X and $1 \in X$ is a constant such that the following identities hold:

- (P1) $(\forall x \in X) (x * x = 1)$,
- (P2) $(\forall x \in X) (1 * x = x)$,
- (P3) $(\forall x \in X) (x * (y * z) = (x * y) * (x * z))$,
- (P4) $(\forall x, y, z \in X) (x * (y * z) = y * (x * z))$.

In what follows, let X denote a pre-logic unless otherwise specified.

Lemma 2.2. ([2]) *Let X be a pre-logic. Then the following hold:*

- (a) $(\forall x \in X) (x * 1 = 1)$;
- (b) $(\forall x, y \in X) (x * (y * x) = 1)$;
- (c) *an order relation \leq on X defined by*

$$(\forall x, y \in X) (x \leq y \text{ if and only if } x * y = 1)$$

is a quasiorder on X (i.e., a reflexive and transitive order relation on X);

- (d) $1 \leq x$ for all $x \in X$ implies $x = 1$.

Remark 2.3. ([2]) The quasiorder \leq of Lemma 2.2(c) is called the *induced quasiorder* of a pre-logic X .

Lemma 2.4. ([2]) *Let \leq be the induced quasiorder of a pre-logic X and let $x, y, z \in X$. If $x \leq y$, then $z * x \leq z * y$ and $y * z \leq x * z$.*

Definition 2.5. ([2]) Let $X = (X; *, 1)$ be a pre-logic. A non-empty subset D of X is called a *deductive system* of X if the following conditions hold:

- (d1) $1 \in D$,
- (d2) if $x \in D$ and $x * y \in D$, then $y \in D$.

Definition 2.6. ([2]) Let X be a pre-logic. A non-empty subset I of X is called an *ideal* of X if the following conditions are satisfied:

- (I1) $x \in X$ and $y \in I$ imply $x * y \in I$;
- (I2) $x \in X$ and $y_1, y_2 \in I$ imply $(y_2 * (y_1 * x)) * x \in I$.

Lemma 2.7. ([2]) *Let X be a pre-logic and \leq its induced quasiorder. Then the following hold:*

- (a) $(\forall x, y \in X) (x * ((x * y) * y) = 1)$,
- (b) $(\forall x, y, z \in X) ((y * z) * ((x * y) * (x * z)) = 1)$,
- (c) *if D is a deductive system of X , $a \in D$, and $a \leq b$, then $b \in D$.*

Ideal theory of pre-logics based on the theory of falling shadows

Theorem 2.8. ([1]) *A non-empty subset I of a pre-logic X is an ideal of X if and only if it satisfies the following two conditions:*

- (I1') $(1 \in I)$;
- (I2') $(\forall x, z \in X)(\forall y \in I) (x * (y * z) \in I \Rightarrow x * z \in I)$.

Definition 2.9. ([6]) *A non-empty subset I of a pre-logic X is a positive implicative ideal of X if it satisfies (I1') and*

- (I3) $(\forall y, z \in X)(\forall x \in I) (x * ((y * z) * y) \in I \Rightarrow y \in I)$.

Theorem 2.10. ([6]) *Every positive implicative ideal of a pre-logic X is an ideal of X .*

We now display the basic theory on falling shadows. We refer the reader to the papers [3, 4, 5, 7, 8] for further information regarding the theory of falling shadows.

Given a universe of discourse U , let $\mathcal{P}(U)$ denote the power set of U . For each $u \in U$, let

$$\dot{u} := \{E \mid u \in E \text{ and } E \subseteq U\}, \tag{2.1}$$

and for each $E \in \mathcal{P}(U)$, let

$$\dot{E} := \{\dot{u} \mid u \in E\}. \tag{2.2}$$

An ordered pair $(\mathcal{P}(U), \mathcal{B})$ is said to be a hyper-measurable structure on U if \mathcal{B} is a σ -field in $\mathcal{P}(U)$ and $\dot{U} \subseteq \mathcal{B}$. Given a probability space (Ω, \mathcal{A}, P) and a hyper-measurable structure $(\mathcal{P}(U), \mathcal{B})$ on U , a random set on U is defined to be a mapping $\xi : \Omega \rightarrow \mathcal{P}(U)$ which is \mathcal{A} - \mathcal{B} measurable, that is,

$$(\forall C \in \mathcal{B}) (\xi^{-1}(C) = \{\omega \mid \omega \in \Omega \text{ and } \xi(\omega) \in C\} \in \mathcal{A}). \tag{2.3}$$

Suppose that ξ is a random set on U . Let $\tilde{H}(u) := P(\omega \mid u \in \xi(\omega))$ for each $u \in U$. Then \tilde{H} is a kind of fuzzy set in U . We call \tilde{H} a falling shadow of the random set ξ , and ξ is called a cloud of \tilde{H} .

For example, $(\Omega, \mathcal{A}, P) = ([0, 1], \mathcal{A}, m)$, where \mathcal{A} is a Borel field on $[0, 1]$ and m is the usual Lebesgue measure. Let \tilde{H} be a fuzzy set in U and $\tilde{H}_t := \{u \in U \mid \tilde{H}(u) \geq t\}$ be a t -cut of \tilde{H} . Then $\xi : [0, 1] \rightarrow \mathcal{P}(U)$, $t \mapsto \tilde{H}_t$ is a random set and ξ is a cloud of \tilde{H} . We shall call ξ defined above as the cut-cloud of \tilde{H} (see [3]).

3. Falling subalgebras and falling ideals

Definition 3.1. Let (Ω, \mathcal{A}, P) be a probability space, and let $\xi : \Omega \rightarrow \mathcal{P}(X)$ be a random set, where X is a pre-logic. If $\xi(\omega)$ is a subalgebra (resp. ideal) of X for any $\omega \in \Omega$ with $\xi(\omega) \neq \emptyset$, then the falling shadow \tilde{H} of the random set ξ , i.e., $\tilde{H}(x) = P(\omega \mid x \in \xi(\omega))$ is called a falling subalgebra (resp. falling ideal) of X .

In what follows, let \tilde{H} denote a falling shadow of the random set $\xi : \Omega \rightarrow \mathcal{P}(X)$.

Young Bae Jun and Sun Shin Ahn

Example 3.2. (1) Let $X := \{1, a, b, c, d\}$ be a set with the following Cayley table:

*	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

Then $(X; *, 1)$ is a pre-logic (see [6]). Let $(\Omega, \mathcal{A}, P) = ([0, 1], \mathcal{A}, m)$ and define a random set $\xi : \Omega \rightarrow \mathcal{P}(X)$ as follows:

$$\xi(\omega) := \begin{cases} \{1, a, b\} & \text{if } \omega \in [0, 0.6) \\ \emptyset & \text{if } \omega \in [0.6, 0.7), \\ X & \text{if } \omega \in [0.7, 1]. \end{cases}$$

Then the falling shadow \tilde{H} of ξ is both a falling subalgebra of X and a falling ideal of X .

Define a random set $\eta : \Omega \rightarrow \mathcal{P}(X)$ as follows:

$$\eta(\omega) := \begin{cases} \emptyset & \text{if } \omega \in [0, 0.3), \\ \{1, b, c\} & \text{if } \omega \in [0.3, 0.8), \\ X & \text{if } \omega \in [0.8, 1]. \end{cases}$$

Then $\eta(\omega)$ is a subalgebra of X for all $\omega \in \Omega$ with $\eta(\omega) \neq \emptyset$, but not an ideal of X , since $(b * (a * a)) * a = (b * 1) * a = 1 * a = a \notin \{1, b, c\}$. Hence the falling shadow \tilde{H} of ξ is a falling subalgebra of X , but not a falling ideal of X .

For a probability space (Ω, \mathcal{A}, P) and any element x of a *BCC*-algebra X , let

$$\Omega(x; \xi) := \{\omega \in \Omega \mid x \in \xi(\omega)\}. \tag{3.1}$$

Then $\Omega(x; \xi) \in \mathcal{A}$.

Lemma 3.3. *If \tilde{H} is a falling subalgebra of a pre-logic X , then $(\forall x \in X) (\Omega(x; \xi) \subseteq \Omega(1; \xi))$.*

Proof. If $\Omega(x; \xi) = \emptyset$, then it is clear. Assume that $\Omega(x; \xi) \neq \emptyset$ and let $\omega \in \Omega$ be such that $\omega \in \Omega(x; \xi)$. Then $x \in \xi(\omega)$, and so $1 = x * x \in \xi(\omega)$ since $\xi(\omega)$ is a subalgebra of X . Hence $\omega \in \Omega(1; \xi)$, and therefore $\Omega(x; \xi) \subseteq \Omega(1; \xi)$ for all $x \in X$. □

Proposition 3.4. *Every falling ideal of a pre-logic X is a falling subalgebra of X .*

Proof. Let \tilde{H} be a falling ideal of X . Then $\xi(\omega)$ is an ideal of X for any $\omega \in \Omega$ with $\xi(\omega) \neq \emptyset$. Let $x, y \in \xi(\omega)$. Using (I1), we have $x * y \in \xi(\omega)$. Hence $\xi(\omega)$ is a subalgebra of X . Thus \tilde{H} is a falling subalgebra of X . □

Ideal theory of pre-logics based on the theory of falling shadows

The converse of Proposition 3.4 is not true in general (see Example 3.2). We provide a characterization of a falling ideal.

Theorem 3.5. *Let X be a pre-logic. Then \tilde{H} is a falling ideal of X if and only if the following conditions are valid:*

- (i) $(\forall x, y \in X) (\Omega(x * (y * z); \xi) \cap \Omega(y; \xi) \subseteq \Omega(x * z; \xi))$,
- (ii) $(\forall x \in X) (\Omega(x; \xi) \subseteq \Omega(1; \xi))$.

Proof. Assume that \tilde{H} is a falling ideal of X . For any $x, y, z \in X$, if $\omega \in \Omega(x * (y * z); \xi) \cap \Omega(y; \xi)$, then $x * (y * z) \in \xi(\omega)$ and $y \in \xi(\omega)$. It follows from (I2') that $x * z \in \xi(\omega)$ since $\xi(\omega)$ is an ideal of X . Hence $\omega \in \Omega(x * z; \xi)$. Therefore $\Omega(x * (y * z); \xi) \cap \Omega(y; \xi) \subseteq \Omega(x * z; \xi)$, for any $x, y, z \in X$. Thus (i) is valid. The second condition (ii) follows from Lemma 3.3 and Proposition 3.4.

Conversely, suppose that two conditions (i) and (ii) are valid. Let $x, y, z \in X$ and $\omega \in \Omega$ be such that $x * (y * z) \in \xi(\omega)$ and $y \in \xi(\omega)$. Then $\omega \in \Omega(x * (y * z); \xi)$ and $\omega \in \Omega(y; \xi)$. It follows from (i) that $\omega \in \Omega(x * z; \xi)$. Hence $x * z \in \xi(\omega)$. Now, assume that $x \in \xi(\omega)$ for every $x \in X$ and for all $\omega \in \Omega$. Then $\omega \in \Omega(x; \xi) \subseteq \Omega(1; \xi)$ and so $1 \in \xi(\omega)$ for all $\omega \in \Omega$. Therefore $\xi(\omega)$ is an ideal of X for all $\omega \in \Omega$ with $\xi(\omega) \neq \emptyset$. Hence \tilde{H} is a falling ideal of X . \square

Theorem 3.6. *Let X be a pre-logic. Then \tilde{H} is a falling ideal of X if and only if the following conditions are valid:*

- (i) $(\forall x, y \in X) (\Omega(y; \xi) \subseteq \Omega(x * y; \xi))$,
- (ii) $(\forall x, y, z \in X) (\Omega(x; \xi) \cap \Omega(y; \xi) \subseteq \Omega((x * (y * z)) * z; \xi))$.

Proof. Assume \tilde{H} satisfies two conditions (i) and (ii). Let $x, y \in X$ and $\omega \in \Omega$ such that $y \in \xi(\omega)$. Then $\omega \in \Omega(y; \xi)$. Using (i), we have $\omega \in \Omega(x * y; \xi)$. Hence $x * y \in \xi(\omega)$. Now, let $x, y, z \in X$ and $\omega \in \Omega$ such that $x, y \in \xi(\omega)$. Then $\omega \in \Omega(x; \xi)$ and $\omega \in \Omega(y; \xi)$ and so $\omega \in \Omega(x; \xi) \cap \Omega(y; \xi)$. It follows from (ii) that $\omega \in \Omega((x * (y * z)) * z; \xi)$. Hence $(x * (y * z)) * z \in \xi(\omega)$ and so $\xi(\omega)$ is an ideal of X . Therefore \tilde{H} is a falling ideal of X .

Conversely, suppose that \tilde{H} is a falling ideal of X . Let $x, y \in X$ and $\omega \in \Omega$ be such that $\omega \in \Omega(y; \xi)$. Then $y \in \xi(\omega)$. Since $\xi(\omega)$ is an ideal of X , we have $x * y \in \xi(\omega)$. Hence $\omega \in \Omega(x * y; \xi)$. Therefore (i) is valid. For any $x, y, z \in X$, if $\omega \in \Omega(x; \xi) \cap \Omega(y; \xi)$, then $x \in \xi(\omega)$ and $y \in \xi(\omega)$. Since $\xi(\omega)$ is an ideal of X , we get $(x * (y * z)) * z \in \xi(\omega)$. Therefore $\omega \in \Omega((x * (y * z)) * z; \xi)$. Thus (ii) is true. \square

Proposition 3.7. *Every falling ideal of a pre-logic satisfies the following assertions:*

- (i) $(\forall x \in X) (\Omega(x; \xi) \subseteq \Omega(1; \xi))$,
- (ii) $(\forall x, y \in X) (\Omega(x; \xi) \subseteq \Omega((x * y) * y; \xi))$,
- (iii) $(\forall x, y \in X) (x \leq y \Rightarrow \Omega(x; \xi) \subseteq \Omega(y; \xi))$.

Proof. (i) Using (P1) and Theorem 3.6(i), we have $\Omega(x; \xi) \subseteq \Omega(1; \xi)$.

Young Bae Jun and Sun Shin Ahn

(ii) Taking $x := x, y := 1$, and $z := y$ in Theorem 3.6(ii) and using (P2) and (i), we get $\Omega(x; \xi) = \Omega(x; \xi) \cap \Omega(1; \xi) \subseteq \Omega((x * (1 * y)) * y; \xi) = \Omega((x * y) * y; \xi)$.

(iii) Let $x, y \in X$ be such that $x \leq y$. Then $x * y = 1$. Using (P2), we have $\Omega(x; \xi) \subseteq \Omega((x * y) * y; \xi) = \Omega(1 * y; \xi) = \Omega(y; \xi)$. \square

Lemma 3.8. *Every falling ideal of \tilde{H} of a pre-logic X satisfies the following property:*

$$(\forall x, y \in X)(\Omega(x * y; \xi) \cap \Omega(x; \xi) \subseteq \Omega(y; \xi)). \tag{3.2}$$

Proof. Using (P1), (P2), and Theorem 3.6(ii), we have $\Omega(x * y; \xi) \cap \Omega(x; \xi) \subseteq \Omega(((x * y) * (x * y)) * y; \xi) = \Omega(1 * y; \xi) = \Omega(y; \xi)$ for all $x, y \in X$. \square

Corollary 3.9. *Let X be a pre-logic. Then \tilde{H} is a falling ideal of X if and only if it satisfies the condition (3.2) and*

$$(i) (\forall x \in X) (\Omega(x; \xi) \subseteq \Omega(1; \xi)).$$

Proof. Assume that \tilde{H} is a falling ideal of X . Using Proposition 3.7, (i) holds. By Lemma 3.8, the condition (3.2) holds.

Conversely, suppose that \tilde{H} satisfies two conditions (3.2) and (i). Using (3.2), we have $\Omega(y * (x * z); \xi) \cap \Omega(y; \xi) \subseteq \Omega(x * z; \xi)$. Using (P4), we have $\Omega(x * (y * z); \xi) \cap \Omega(y; \xi) \subseteq \Omega(x * z; \xi)$. By Theorem 3.5, \tilde{H} is a falling ideal of X . \square

Lemma 3.10. *For any falling ideal \tilde{H} of a pre-logic X , the following are equivalent:*

- (i) $(\forall x, y \in X) (\Omega(x * y; \xi) \cap \Omega(x; \xi) \subseteq \Omega(y; \xi))$.
- (ii) $(\forall x, y, z \in X) (\Omega(x * (y * z); \xi) \cap \Omega(x * y; \xi) \subseteq \Omega(x * z; \xi))$.

Proof. Assume that \tilde{H} satisfies (i). For any $x, y, z \in X$, using (P3), we have $\Omega(x * (y * z); \xi) \cap \Omega(x * y; \xi) = \Omega((x * y) * (x * z); \xi) \cap \Omega(x * y; \xi) \subseteq \Omega(x * z; \xi)$. Thus (ii) is valid.

Conversely, suppose that \tilde{H} satisfies (ii). Putting $x := 1$ in (ii) and using (P2), we have $\Omega(y * z; \xi) \cap \Omega(y; \xi) = \Omega(1 * (y * z); \xi) \cap \Omega(1 * y; \xi) \subseteq \Omega(1 * z; \xi) = \Omega(z; \xi)$. Thus (i) is true. \square

Proposition 3.11. *Let X be a pre-logic. Then \tilde{H} is a falling ideal of X if and only if the following conditions are valid:*

- (i) $(\forall x \in X) (\Omega(x; \xi) \subseteq \Omega(1; \xi))$.
- (ii) $(\forall x, y, z \in X) (\Omega(x * (y * z); \xi) \cap \Omega(x * y; \xi) \subseteq \Omega(x * z; \xi))$.

Proof. It follows from Corollary 3.9 and Lemma 3.10. \square

Corollary 3.12. *Every falling ideal \tilde{H} of a pre-logic X satisfies the following property:*

$$(\forall x, y \in X)(\Omega(x * (x * y); \xi) \subseteq \Omega(x * y; \xi)).$$

Ideal theory of pre-logics based on the theory of falling shadows

Proof. Putting $x := x, z := y$ and $y := x$ in Proposition 3.11(ii), we have $\Omega(x * (x * y); \xi) = \Omega(x * (x * y); \xi) \cap \Omega(1; \xi) = \Omega(x * (x * y); \xi) \cap \Omega(x * x; \xi) \subseteq \Omega(x * y; \xi)$, for all $x, y \in X$. \square

4. Positive implicative falling ideals

Definition 4.1. Let (Ω, \mathcal{A}, P) be a probability space, and let $\xi : \Omega \rightarrow \mathcal{P}(X)$ be a random set, where X is a pre-logic. If $\xi(\omega)$ is a positive implicative ideal of X for any $\omega \in \Omega$ with $\xi(\omega) \neq \emptyset$, then the falling shadow \tilde{H} of the random set ξ , i.e., $\tilde{H}(x) = P(\omega \mid x \in \xi(\omega))$ is called a *positive implicative falling ideal* of X .

Example 4.2. Let $X = \{1, a, b, c, d\}$ be a pre-logic as in Example 3.2.

(1) Consider a random set ξ as in Example 3.2. Then the falling shadow \tilde{H} of ξ is a positive implicative falling ideal of X , since $\{1, a, b\}$ is a positive implicative ideal of X .

(2) Define a random set $\eta : \Omega \rightarrow \mathcal{P}(X)$ as follows:

$$\eta(\omega) := \begin{cases} \emptyset & \text{if } \omega \in [0, 0.3), \\ \{1, b\} & \text{if } \omega \in [0.3, 0.7), \\ X & \text{if } \omega \in [0.7, 1]. \end{cases}$$

Note that $J := \{1, b\}$ is an ideal of X but not a positive implicative ideal of X since $b * ((a * d) * a) = b * (d * a) = b * 1 = 1 \in J$ and $b \in J$ but $a \notin J$. Hence \tilde{H} is a falling ideal of X , but not a positive implicative falling ideal of X .

Proposition 4.3. Every positive implicative falling ideal of a pre-logic X is a falling ideal of X .

Proof. Straightforward by Definition 4.1 and Theorem 2.10. \square

The converse of Proposition 4.3 is not true in general (see Example 4.2(2)).

Theorem 4.4. Let X be a pre-logic. Then \tilde{H} is a positive implicative falling ideal of X if and only if \tilde{H} satisfies the following two conditions:

- (i) $(\forall x \in X)(\Omega(x; \xi) \subseteq \Omega(1; \xi))$,
- (ii) $(\forall x, y, z \in X)(\Omega(x * ((y * z) * y); \xi) \cap \Omega(x; \xi) \subseteq \Omega(y; \xi))$.

Proof. Assume that \tilde{H} satisfies two conditions (i) and (ii). Let $x \in \xi(\omega)$ for every $x \in X$ and for all $\omega \in \Omega$. Then $\omega \in \Omega(x; \xi) \subseteq \Omega(1; \xi)$ and so $1 \in \xi(\omega)$. Let $x, y, z \in X$ be such that $x \in \xi(\omega)$ and $x * ((y * z) * y) \in \xi(\omega)$. Then $\omega \in \Omega(x; \xi)$ and $\omega \in \Omega(x * ((y * z) * y); \xi)$. Using (ii), we have $\omega \in \Omega(y; \xi)$. Hence $y \in \xi(\omega)$ and so $\xi(\omega)$ is a positive implicative ideal of X . Therefore \tilde{H} is a positive implicative falling ideal of X .

Conversely, suppose that \tilde{H} is a positive implicative falling ideal of X . The first condition (i) follows from Proposition 3.7(i) and Proposition 4.3. For any $x, y, z \in X$, if $\omega \in \Omega(x * ((y * z) * y); \xi) \cap \Omega(x; \xi)$, then $x * ((y * z) * y) \in \xi(\omega)$ and $x \in \xi(\omega)$. Since $\xi(\omega)$ is a positive implicative ideal

Young Bae Jun and Sun Shin Ahn

of X , we have $\omega \in \Omega(y; \xi)$. Therefore $\Omega(x * ((y * z) * y); \xi) \cap \Omega(x; \xi) \subseteq \Omega(y; \xi)$ for any $x, y, z \in X$. Thus (ii) holds. \square

Theorem 4.5. *Let \tilde{H} be a falling ideal of a pre-logic X . Then the following are equivalent:*

- (i) \tilde{H} is a positive implicative falling ideal of X .
- (ii) $(\forall x, y \in X)(\Omega((x * y) * x; \xi) \subseteq \Omega(x; \xi))$.

Proof. Assume that \tilde{H} is a positive implicative ideal of X . Putting $x := 1, y := x$, and $z := y$ in Theorem 4.4(ii), we have $\Omega(1 * ((x * y) * x); \xi) \cap \Omega(1; \xi) = \Omega((x * y) * x; \xi) \cap \Omega(1; \xi) = \Omega((x * y) * x; \xi) \subseteq \Omega(x; \xi)$. Hence (ii) holds.

Conversely, suppose that a falling ideal \tilde{H} satisfies (ii). By Lemma 3.8, for any $x, y, z \in X$, we have $\Omega(x * ((y * z) * y); \xi) \cap \Omega(x; \xi) \subseteq \Omega((y * z) * y; \xi) \subseteq \Omega(y; \xi)$. By Theorem 4.4, \tilde{H} is a positive implicative falling ideal of X . Thus (i) is true. \square

Corollary 4.6. *Any positive implicative falling ideal of a pre-logic X satisfies the following property:*

$$(\forall x, y \in X)(\Omega((x * y) * y; \xi) \subseteq \Omega((y * x) * x; \xi)).$$

Proof. Since $x \leq (y * x) * x$ for all $x, y \in X$, it follows from Lemma 2.4 that $((y * x) * x) * y \leq x * y$. Then $(x * y) * y \leq (y * x) * ((x * y) * x) = (x * y) * ((y * x) * x) \leq (((y * x) * x) * y) * ((y * x) * x)$. By Proposition 3.7(iii) and Proposition 4.5, we have $\Omega((x * y) * y; \xi) \subseteq \Omega(((y * x) * x) * y * ((y * x) * x); \xi) \subseteq \Omega((y * x) * x; \xi)$, for any $x, y \in X$. This completes the proof. \square

REFERENCES

- [1] S. S. Ahn and Y. B. Jun, Pseudo-valuations on pre-logics, *Acta Math. Hungar.*, **134**(2012), 499-510.
- [2] I. Chajada and R. Halas, Algebraic properties of pre-logics, *Math. Slovaca*, **52**(2002), 157-751.
- [3] I. R. Goodman, Fuzzy sets as equivalence classes of random sets, in: R. Yager(Ed.), *Recent Ddevelopments in Fuzzy Sets and Possibility Theory*, Pergamon, New York, 1982.
- [4] Y. B. Jun and M. S. Kang, Fuzzy positive implicative ideals of BCK-algebras based on the theory of falling shadows, *Comput. Math. Appl.*, **61** (2011), 62–67.
- [5] Y. B. Jun and C. H. Park, Falling shadows applied to subalgebras and ideals of *BCK/BCI*-algebras, *Chaos Solutions Fractals*, (submitted for publication).
- [6] Y. H. Kim and S. S. Ahn, Ideal theory of pre-logics based on \mathcal{N} -structures, *Honam Math. J.*, **33** (2011), 535-546.
- [7] P. Z. Wang and E. Sanchez, Treating a fuzzy subset as a projectable random set, in: M. M. Gupta, E. Sanchez, Eds., “Fuzzy Information and Decision” (Pergamon, New York, 1982), 212–219.
- [8] P. Z. Wang, *Fuzzy Sets and Falling Shadows of Random Sets*, Beijing Normal Univ. Press, People’s Republic of China, 1985 [In Chinese].

QUADRATIC ρ -FUNCTIONAL EQUATIONS IN NON-ARCHIMEDEAN BANACH SPACES

CHOONKIL PARK, GANG LU, YINHUA CUI, AND MING FANG*

ABSTRACT. In this paper, we solve the quadratic ρ -functional equations

$$f(x+y) + f(x-y) - 2f(x) - 2f(y) = \rho \left(4f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - 2f(y) \right), \quad (0.1)$$

where ρ is a fixed non-Archimedean number with $|\rho| < |2|$, and

$$4f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - 2f(y) = \rho(f(x+y) + f(x-y) - 2f(x) - 2f(y)), \quad (0.2)$$

where ρ is a fixed non-Archimedean number with $|\rho| < |2|$.

Furthermore, we prove the Hyers-Ulam stability of the quadratic ρ -functional equations (0.1) and (0.2) in non-Archimedean Banach spaces.

1. INTRODUCTION AND PRELIMINARIES

A *valuation* is a function $|\cdot|$ from a field K into $[0, \infty)$ such that 0 is the unique element having the 0 valuation, $|rs| = |r| \cdot |s|$ and the triangle inequality holds, i.e.,

$$|r+s| \leq |r| + |s|, \quad \forall r, s \in K.$$

A field K is called a *valued field* if K carries a valuation. The usual absolute values of \mathbb{R} and \mathbb{C} are examples of valuations.

Let us consider a valuation which satisfies a stronger condition than the triangle inequality. If the triangle inequality is replaced by

$$|r+s| \leq \max\{|r|, |s|\}, \quad \forall r, s \in K,$$

then the function $|\cdot|$ is called a *non-Archimedean valuation*, and the field is called a *non-Archimedean field*. Clearly $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$. A trivial example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except for 0 into 1 and $|0| = 0$.

Throughout this paper, we assume that the base field is a non-Archimedean field, hence call it simply a field.

Definition 1.1. ([8]) Let X be a vector space over a field K with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\| : X \rightarrow [0, \infty)$ is said to be a *non-Archimedean norm* if it satisfies the following conditions:

2010 *Mathematics Subject Classification*. Primary 46S10, 39B62, 39B52, 47S10, 12J25.

Key words and phrases. Hyers-Ulam stability; non-Archimedean normed space; quadratic ρ -functional equation.

*Corresponding author.

C. PARK, G. LU, Y. CUI, AND M. FANG

- (i) $\|x\| = 0$ if and only if $x = 0$;
- (ii) $\|rx\| = |r|\|x\|$ ($r \in K, x \in X$);
- (iii) the strong triangle inequality

$$\|x + y\| \leq \max\{\|x\|, \|y\|\}, \quad \forall x, y \in X$$

holds. Then $(X, \|\cdot\|)$ is called a *non-Archimedean normed space*.

Definition 1.2. (i) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X . Then the sequence $\{x_n\}$ is called *Cauchy* if for a given $\varepsilon > 0$ there is a positive integer N such that

$$\|x_n - x_m\| \leq \varepsilon$$

for all $n, m \geq N$.

(ii) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X . Then the sequence $\{x_n\}$ is called *convergent* if for a given $\varepsilon > 0$ there are a positive integer N and an $x \in X$ such that

$$\|x_n - x\| \leq \varepsilon$$

for all $n \geq N$. Then we call $x \in X$ a limit of the sequence $\{x_n\}$, and denote by $\lim_{n \rightarrow \infty} x_n = x$.

(iii) If every Cauchy sequence in X converges, then the non-Archimedean normed space X is called a *non-Archimedean Banach space*.

The stability problem of functional equations originated from a question of Ulam [18] concerning the stability of group homomorphisms. The functional equation $f(x + y) = f(x) + f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [7] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [11] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [6] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

The functional equation $f(x + y) + f(x - y) = 2f(x) + 2f(y)$ is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The stability of quadratic functional equation was proved by Skof [17] for mappings $f : E_1 \rightarrow E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [5] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. The functional equation $4f\left(\frac{x+y}{2}\right) + (x - y) = f(x) + f(y)$ is called a *Jensen type quadratic equation*. The stability problems of various functional equations have been extensively investigated by a number of authors (see [1, 3, 4, 9, 10, 12, 13, 14, 15, 16, 19, 20]).

In Section 2, we solve the quadratic ρ -functional equation (0.1) and prove the Hyers-Ulam stability of the quadratic ρ -functional equation (0.1) in non-Archimedean Banach spaces.

In Section 3, we solve the quadratic ρ -functional equation (0.2) and prove the Hyers-Ulam stability of the quadratic ρ -functional equation (0.2) in non-Archimedean Banach spaces.

QUADRATIC ρ -FUNCTIONAL EQUATIONS

Throughout this paper, assume that X is a non-Archimedean normed space and that Y is a non-Archimedean Banach space. Let $|2| \neq 1$.

2. QUADRATIC ρ -FUNCTIONAL EQUATION (0.1) IN NON-ARCHIMEDEAN NORMED SPACES

Throughout this section, assume that ρ is a fixed non-Archimedean number with $|\rho| < |2|$.

In this section, we solve the quadratic ρ -functional equation (0.1) in non-Archimedean normed spaces.

Lemma 2.1. *If a mapping $f : G \rightarrow Y$ satisfies $f(0) = 0$ and*

$$\begin{aligned} f(x + y) + f(x - y) - 2f(x) - 2f(y) \\ = \rho \left(4f\left(\frac{x + y}{2}\right) + f(x - y) - 2f(x) - 2f(y) \right) \end{aligned} \tag{2.1}$$

for all $x, y \in G$, then $f : G \rightarrow Y$ is quadratic.

Proof. Assume that $f : G \rightarrow Y$ satisfies (2.1).

Letting $y = x$ in (2.1), we get $f(2x) - 4f(x) = 0$ for all $x \in G$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{4}f(x) \tag{2.2}$$

for all $x \in G$.

It follows from (2.1) and (2.2) that

$$\begin{aligned} f(x + y) + f(x - y) - 2f(x) - 2f(y) \\ = \rho \left(4f\left(\frac{x + y}{2}\right) + f(x - y) - 2f(x) - 2f(y) \right) \\ = \rho(f(x + y) + f(x - y) - 2f(x) - 2f(y)) \end{aligned}$$

and so

$$f(x + y) + f(x - y) = 2f(x) + 2f(y)$$

for all $x, y \in G$. □

Now, we prove the Hyers-Ulam stability of the quadratic ρ -functional equation (2.1) in non-Archimedean Banach spaces.

Theorem 2.2. *Let $r < 2$ and θ be nonnegative real numbers and let $f : X \rightarrow Y$ be a mapping satisfying*

$$\begin{aligned} \left\| f(x + y) + f(x - y) - 2f(x) - 2f(y) - \rho \left(4f\left(\frac{x + y}{2}\right) + f(x - y) - 2f(x) - 2f(y) \right) \right\| \\ \leq \theta(\|x\|^r + \|y\|^r) \end{aligned} \tag{2.3}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \rightarrow Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{2}{|2|^r} \theta \|x\|^r \tag{2.4}$$

for all $x \in X$.

C. PARK, G. LU, Y. CUI, AND M. FANG

Proof. Letting $x = y = 0$ in (2.3), we get $-2f(0) = \rho f(0)$. Since $|\rho| < |2|$, $f(0) = 0$.

Letting $y = x$ in (2.3), we get

$$\|f(2x) - 4f(x)\| \leq 2\theta\|x\|^r \tag{2.5}$$

for all $x \in X$. So $\|f(x) - 4f\left(\frac{x}{2}\right)\| \leq \frac{2}{|2|^r}\theta\|x\|^r$ for all $x \in X$. Hence

$$\begin{aligned} & \left\| 4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right) \right\| \tag{2.6} \\ & \leq \max \left\{ \left\| 4^l f\left(\frac{x}{2^l}\right) - 4^{l+1} f\left(\frac{x}{2^{l+1}}\right) \right\|, \dots, \left\| 4^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 4^m f\left(\frac{x}{2^m}\right) \right\| \right\} \\ & = \max \left\{ |4|^l \left\| f\left(\frac{x}{2^l}\right) - 4f\left(\frac{x}{2^{l+1}}\right) \right\|, \dots, |4|^{m-1} \left\| f\left(\frac{x}{2^{m-1}}\right) - 4f\left(\frac{x}{2^m}\right) \right\| \right\} \\ & \leq \max \left\{ \frac{|4|^l}{|2|^{rl}}, \dots, \frac{|4|^{m-1}}{|2|^{r(m-1)}} \right\} \frac{2}{|2|^r} \theta \|x\|^r = \frac{2\theta}{|2|^{(r-2)l}} \frac{1}{|2|^r} \|x\|^r \end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.6) that the sequence $\{4^n f\left(\frac{x}{2^n}\right)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{4^n f\left(\frac{x}{2^n}\right)\}$ converges. So one can define the mapping $Q : X \rightarrow Y$ by

$$Q(x) := \lim_{n \rightarrow \infty} 4^n f\left(\frac{x}{2^n}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \rightarrow \infty$ in (2.6), we get (2.4).

It follows from (2.3) that

$$\begin{aligned} & \left\| Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y) - \rho \left(4Q\left(\frac{x+y}{2}\right) + Q(x-y) - 2Q(x) - 2Q(y) \right) \right\| \\ & = \lim_{n \rightarrow \infty} |4|^n \left\| f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right) \right. \\ & \quad \left. - \rho \left(4f\left(\frac{x+y}{2^{n+1}}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right) \right) \right\| \leq \lim_{n \rightarrow \infty} \frac{|4|^n \theta}{|2|^{nr}} (\|x\|^r + \|y\|^r) = 0 \end{aligned}$$

for all $x, y \in X$. So

$$Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y) = \rho \left(4Q\left(\frac{x+y}{2}\right) + Q(x-y) - 2Q(x) - 2Q(y) \right)$$

for all $x, y \in X$. By Lemma 2.1, the mapping $h : X \rightarrow Y$ is quadratic.

Now, let $T : X \rightarrow Y$ be another quadratic mapping satisfying (2.4). Then we have

$$\begin{aligned} \|Q(x) - T(x)\| & = \left\| 4^q Q\left(\frac{x}{2^q}\right) - 4^q T\left(\frac{x}{2^q}\right) \right\| \\ & \leq \max \left\{ \left\| 4^q Q\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right) \right\|, \left\| 4^q T\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right) \right\| \right\} \leq \frac{2}{|2|^{(r-2)q+r}} \theta \|x\|^r, \end{aligned}$$

which tends to zero as $q \rightarrow \infty$ for all $x \in X$. So we can conclude that $Q(x) = T(x)$ for all $x \in X$. This proves the uniqueness of Q . Thus the mapping $Q : X \rightarrow Y$ is a unique quadratic mapping satisfying (2.4). \square

QUADRATIC ρ -FUNCTIONAL EQUATIONS

Theorem 2.3. *Let $r > 2$ and θ be positive real numbers, and let $f : X \rightarrow Y$ be a mapping satisfying (2.3). Then there exists a unique quadratic mapping $Q : X \rightarrow Y$ such that*

$$\|f(x) - Q(x)\| \leq \frac{2\theta}{|4|} \|x\|^r$$

for all $x \in X$.

Proof. It follows from (2.5) that

$$\left\| f(x) - \frac{1}{4}f(2x) \right\| \leq \frac{2\theta}{|4|} \|x\|^r$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

3. QUADRATIC ρ -FUNCTIONAL EQUATION (0.2)

Throughout this section, assume that ρ is a fixed non-Archimedean number with $|\rho| < |2|$.

In this section, we solve the quadratic ρ -functional equation (0.2) in non-Archimedean normed spaces.

Lemma 3.1. *If a mapping $f : G \rightarrow Y$ satisfies $f(0) = 0$ and*

$$\begin{aligned} 4f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - 2f(y) & \quad (3.1) \\ & = \rho(f(x+y) + f(x-y) - 2f(x) - 2f(y)) \end{aligned}$$

for all $x, y \in G$, then $f : G \rightarrow Y$ is quadratic.

Proof. Assume that $f : G \rightarrow Y$ satisfies (3.1).

Letting $y = 0$ in (3.1), we get

$$4f\left(\frac{x}{2}\right) = f(x) \quad (3.2)$$

for all $x \in G$.

It follows from (3.1) and (3.2) that

$$\begin{aligned} & f(x+y) + f(x-y) - 2f(x) - 2f(y) \\ & = 4f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - 2f(y) \\ & = \rho(f(x+y) + f(x-y) - 2f(x) - 2f(y)) \end{aligned}$$

and so

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in G$. □

Now, we prove the Hyers-Ulam stability of the quadratic ρ -functional equation (3.1) in non-Archimedean Banach spaces.

C. PARK, G. LU, Y. CUI, AND M. FANG

Theorem 3.2. *Let $r < 2$ and θ be nonnegative real numbers, and let $f : X \rightarrow Y$ be a mapping satisfying*

$$\left\| 4f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - 2f(y) - \rho(f(x+y) + f(x-y) - 2f(x) - 2f(y)) \right\| \leq \theta(\|x\|^r + \|y\|^r) \tag{3.3}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \rightarrow Y$ such that

$$\|f(x) - Q(x)\| \leq \theta\|x\|^r \tag{3.4}$$

for all $x \in X$.

Proof. Letting $x = y = 0$ in (3.3), we get $f(0) = 2\rho f(0)$. Since $|\rho| < |2|$, $f(0) = 0$.

Letting $y = 0$ in (3.3), we get

$$\left\| 4f\left(\frac{x}{2}\right) - f(x) \right\| \leq \theta\|x\|^r \tag{3.5}$$

for all $x \in X$. So

$$\begin{aligned} & \left\| 4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right) \right\| \tag{3.6} \\ & \leq \max \left\{ \left\| 4^l f\left(\frac{x}{2^l}\right) - 4^{l+1} f\left(\frac{x}{2^{l+1}}\right) \right\|, \dots, \left\| 4^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 4^m f\left(\frac{x}{2^m}\right) \right\| \right\} \\ & = \max \left\{ |4|^l \left\| f\left(\frac{x}{2^l}\right) - 4f\left(\frac{x}{2^{l+1}}\right) \right\|, \dots, |4|^{m-1} \left\| f\left(\frac{x}{2^{m-1}}\right) - 4f\left(\frac{x}{2^m}\right) \right\| \right\} \\ & \leq \max \left\{ \frac{|4|^l}{|2|^{rl}}, \dots, \frac{|4|^{m-1}}{|2|^{r(m-1)}} \right\} \theta\|x\|^r = \frac{\theta}{|2|^{(r-2)l}} \|x\|^r \end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.6) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $Q : X \rightarrow Y$ by

$$Q(x) := \lim_{n \rightarrow \infty} 4^n f\left(\frac{x}{2^n}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \rightarrow \infty$ in (3.6), we get (3.4).

The rest of the proof is similar to the proof of Theorem 2.2. □

Theorem 3.3. *Let $r > 2$ and θ be positive real numbers, and let $f : X \rightarrow Y$ be an even mapping satisfying (3.3). Then there exists a unique quadratic mapping $Q : X \rightarrow Y$ such that*

$$\|f(x) - Q(x)\| \leq \frac{|2|^r \theta}{|4|} \|x\|^r \tag{3.7}$$

for all $x \in X$.

Proof. It follows from (3.5) that

$$\left\| f(x) - \frac{1}{4}f(2x) \right\| \leq \frac{|2|^r \theta}{|4|} \|x\|^r$$

QUADRATIC ρ -FUNCTIONAL EQUATIONS

for all $x \in X$. Hence

$$\begin{aligned} & \left\| \frac{1}{4^l} f(2^l x) - \frac{1}{4^m} f(2^m x) \right\| \tag{3.8} \\ & \leq \max \left\{ \left\| \frac{1}{4^l} f(2^l x) - \frac{1}{4^{l+1}} f(2^{l+1} x) \right\|, \dots, \left\| \frac{1}{4^{m-1}} f(2^{m-1} x) - \frac{1}{4^m} f(2^m x) \right\| \right\} \\ & = \max \left\{ \frac{1}{|4|^l} \left\| f(2^l x) - \frac{1}{4} f(2^{l+1} x) \right\|, \dots, \frac{1}{|4|^{m-1}} \left\| f(2^{m-1} x) - \frac{1}{4} f(2^m x) \right\| \right\} \\ & \leq \max \left\{ \frac{|2|^{rl}}{|4|^{l+1}}, \dots, \frac{|2|^{r(m-1)}}{|4|^{(m-1)+1}} \right\} |2|^r \theta \|x\|^r = \frac{|2|^{r\theta}}{|2|^{(2-r)l+2}} \|x\|^r \end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.8) that the sequence $\{\frac{1}{4^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n} f(2^n x)\}$ converges. So one can define the mapping $Q : X \rightarrow Y$ by

$$Q(x) := \lim_{n \rightarrow \infty} \frac{1}{4^n} f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \rightarrow \infty$ in (3.8), we get (3.7).

The rest of the proof is similar to the proofs of Theorems 2.2 and 3.2. □

REFERENCES

- [1] M. Adam, *On the stability of some quadratic functional equation*, J. Nonlinear Sci. Appl. **4** (2011), 50–59.
- [2] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64–66.
- [3] L. Cădariu, L. Găvruta and P. Găvruta, *On the stability of an affine functional equation*, J. Nonlinear Sci. Appl. **6** (2013), 60–67.
- [4] A. Chahbi and N. Bounader, *On the generalized stability of d’Alembert functional equation*, J. Nonlinear Sci. Appl. **6** (2013), 198–204.
- [5] P. W. Cholewa, *Remarks on the stability of functional equations*, Aequationes Math. **27** (1984), 76–86.
- [6] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–43.
- [7] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. U.S.A. **27** (1941), 222–224.
- [8] M.S. Moslehian and Gh. Sadeghi, *A Mazur-Ulam theorem in non-Archimedean normed spaces*, Nonlinear Anal.-TMA **69** (2008), 3405–3408.
- [9] C. Park, K. Ghasemi, S. G. Ghaleh and S. Jang, *Approximate n -Jordan $*$ -homomorphisms in C^* -algebras*, J. Comput. Anal. Appl. **15** (2013), 365–368.
- [10] C. Park, A. Najati and S. Jang, *Fixed points and fuzzy stability of an additive-quadratic functional equation*, J. Comput. Anal. Appl. **15** (2013), 452–462.
- [11] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.
- [12] S. Schin, D. Ki, J. Chang and M. Kim, *Random stability of quadratic functional equations: a fixed point approach*, J. Nonlinear Sci. Appl. **4** (2011), 37–49.
- [13] S. Shagholi, M. Bavand Savadkouhi and M. Eshaghi Gordji, *Nearly ternary cubic homomorphism in ternary Fréchet algebras*, J. Comput. Anal. Appl. **13** (2011), 1106–1114.

C. PARK, G. LU, Y. CUI, AND M. FANG

- [14] S. Shagholi, M. Eshaghi Gordji and M. Bavand Savadkouhi, *Stability of ternary quadratic derivation on ternary Banach algebras*, J. Comput. Anal. Appl. **13** (2011), 1097–1105.
- [15] D. Shin, C. Park and Sh. Farhadabadi, *On the superstability of ternary Jordan C^* -homomorphisms*, J. Comput. Anal. Appl. **16** (2014), 964–973.
- [16] D. Shin, C. Park and Sh. Farhadabadi, *Stability and superstability of J^* -homomorphisms and J^* -derivations for a generalized Cauchy-Jensen equation*, J. Comput. Anal. Appl. **17** (2014), 125–134.
- [17] F. Skof, *Propriet locali e approssimazione di operatori*, Rend. Sem. Mat. Fis. Milano **53** (1983), 113–129.
- [18] S. M. Ulam, *A Collection of the Mathematical Problems*, Interscience Publ. New York, 1960.
- [19] C. Zaharia, *On the probabilistic stability of the monomial functional equation*, J. Nonlinear Sci. Appl. **6** (2013), 51–59.
- [20] S. Zolfaghari, *Approximation of mixed type functional equations in p -Banach spaces*, J. Nonlinear Sci. Appl. **3** (2010), 110–122.

CHOONKIL PARK

DEPARTMENT OF MATHEMATICS, RESEARCH INSTITUTE FOR NATURAL SCIENCES, HANYANG UNIVERSITY, SEOUL 133-791, KOREA

E-mail address: baak@hanyang.ac.kr

GANG LU

DEPARTMENT OF MATHEMATICS, SCHOOL OF SCIENCE, SHENYANG UNIVERSITY OF TECHNOLOGY, SHENYANG 110870, P.R. CHINA; DEPARTMENT OF MATHEMATICS, ZHEJIANG UNIVERSITY, HANGZHOU 310027, P.R. CHINA

E-mail address: lvgang1234@hanmail.net

YINHUA CUI

DEPARTMENT OF MATHEMATICS, YANBIAN UNIVERSITY, YANJI 133001, P.R. CHINA

E-mail address: cuiyh@ybu.edu.cn

MING FANG

DEPARTMENT OF MATHEMATICS, YANBIAN UNIVERSITY, YANJI 133001, P.R. CHINA

E-mail address: fangming@ybu.edu.cn

The Incomplete Global GMERR Algorithm to Solve $AX = B^*$

Yu-Hui Zheng^{a†}, Jian-Lei Li^b, Dong-Xu Cheng^a, Ling-Ling Lv^b

^a College of Science, Zhongyuan University of Technology,
Zhengzhou, Henan, 450007, PR China.

^b College of Mathematics and Information Science, North China University of
Water Resources and Electric Power, Zhengzhou, Henan, 450011, PR China.

Abstract

To reduce the computational cost and storage requirement of global generalized minimal error (GLGMERR) method, in this paper, we propose a truncated version of GLGMERR method, which is termed as incomplete global generalized minimal error method. The proposed approach uses only a few rather than all of the prior computed matrices in recurrences to generate the next matrix. Moreover a quasi-minimum error solution is obtained as well. Finally, we present the numerical results by comparing with the traditional global GMERR method in CPU time and storage requirements to show the effectiveness and advantages of our method.

Key words: matrix equation; incomplete global generalized minimal error.
AMSC(2000): 65F10, 49M15, 65H10, 15A24

1 Introduction

Consider the following problem:

$$Ax^{(i)} = b^{(i)}, i = 1, 2, \dots, s, \quad (1.1)$$

*This research was supported by National Natural Science Foundation of China(11501200,11601152,11501525) and National Natural Science Tianyuan Foundation of China(11626238), the Natural Science Foundation of Henan Province(16A110018,17A110037,15A110050), the Key Scientific Research Project of colleges and universities in Henan Province(No.15A110045), Growth Funds for Scientific Research team of NCWU(320009-00200), Youth Science and Technology Innovation talents of NCWU(70491), Doctoral Research Project of NCWU(201119).

[†]Corresponding author. E-mail:zhuoer2008@sohu.com.

 Yu-Hui Zheng et al: The incomplete global GMERR algorithm to Solve $AX = B$

where A is a $n \times n$ unsymmetric matrix, $x^{(i)}$, $b^{(i)}$ are all $n \times 1$ real vectors, $s \leq n$. In our daily life, sometimes we have to solve this problem. Therefore, it is of importance that researchers are interested in the study of the numerical solutions, algorithms design and software development for solving problem (1.1).

Krylov subspace method, as one of the effective method for solving $Ax = b$, can be used to solve s linear systems one by one. However, when the order of A is large, it is not enough to use this method to solve this problem. Therefore, we have to find the other new method to solve it. It should be noticed that when all $b^{(i)}$ do impact on the whole system, the problem (1.1) can be rewritten as

$$AX = B, \quad (1.2)$$

where $X = [x^{(1)}, x^{(2)}, \dots, x^{(s)}]^T$, $B = [b^{(1)}, b^{(2)}, \dots, b^{(s)}]^T$.

In the past decades, some related works have been achieved to solve the problem (1.2). In 1999, Jbilou [1] et al proposed the global Arnoldi method and moreover, they proposed global FOM and GMRES methods based on global Arnoldi method, which extended the Krylov subspace method. Among all Krylov subspace methods, GMERR method is one of the most effective methods, because it can minimize the error norm of this method on Krylov subspace. The literature [2] presented the global GMRES method for solving unsymmetric linear systems, which maps the initial residual matrix to the Krylov subspace. In some sense, global GMERR and global GMERS [1] methods have similar structures.

The global generalized minimal error(GMERR) algorithm is an effective Krylov subspace method to solve the linear equations with multiple right-hand sides. As the global GMERR method and the GMERR method have the long recurrence, which result in the dramatic increase of the calculation and storage along with the increase of the step numbers. At present, there are many truncation strategies. For example, Young [3] presented the truncated forms of the orthogonal direction method and the orthogonal residual method. In [4, 5], a truncated forms of FOM method has been given. The truncated forms of IGMRES method or QGMRES method are presented in [6–10]. In this paper, we use the truncation strategy to improve the global GMERR algorithm, and propose a incomplete global GMERR algorithm, which use a few of the previously generated matrix to construct the new basis matrix, and we also give the quasi global minimum error solution on the Krylov subspace.

The remainder of this paper is organized as follows. In Section 2, we present the incomplete global GMERR algorithm. Section 3 and 4 give some numerical experiments to test the effectiveness of the incomplete global GMERR algorithm and conclusions, respectively

2 Incomplete global GMERR algorithm

The incomplete global GMERR algorithm, which is based on the incomplete orthogonality of the Krylov subspace matrix, is to seek the quasi global minimum error solution. The basis matrices $\{V_i\}(i = 1, 2, \dots, m)$ of the Krylov subspace $K_m(A^T, R_0)$ can be obtained through the incomplete orthogonal process. $A^T V_i(i = 1, 2, \dots, m)$ is carried out in the orthogonal process with the first $q(q < m)$ matrices $V_{i_0}, \dots, V_i(i_0 = \max\{1, i - q + 1\})$, The incomplete global GMERR algorithm is to seek the approximate solution $X_m = X_0 + Z_m, Z_m \in A^T K_m(A^T, R_0)$. Moreover $R_m = B - AX_m \perp K_m(A^T, R_0)$, i.e.,

$$R_0 - AZ_m \perp K_m(A^T, R_0). \tag{2.1}$$

Note that $U_m = [V_1, V_2, \dots, V_m]$. let $Z_m = A^T U_m * y_m$, then we can have

$$X_m = X_0 + A^T U_m * y_m, \quad R_m = R_0 - AA^T U_m * y_m.$$

Since $R_0 - AZ_m$ is orthogonal to $K_m(A^T, R_0)$ from equation (2.1). Therefore, for $i = 1, 2, \dots, m$, we can obtain

$$\langle V_i, R_0 \rangle = \langle V_i, AA^T U_m * y_m \rangle. \tag{2.2}$$

Let $V_1 = R_0 / \|R_0\|_F$, then for the formula (2.2), when $i = 1$, it means $tr(V_1^T R_0) = tr(V_1^T AA^T U_m * y_m)$, i.e.,

$$\|R_0\|_F = (tr(V_1^T AA^T V_1), tr(V_1^T AA^T V_2), \dots, tr(V_1^T AA^T V_m)) y_m;$$

when $i = q + 2, \dots, m$, $(tr(V_i^T AA^T V_1), tr(V_i^T AA^T V_2), \dots, tr(V_i^T AA^T V_m)) y_m = 0$;

when $i = 2, \dots, q + 1$, $tr(V_i^T R_0) = (tr(V_i^T AA^T V_1), tr(V_i^T AA^T V_2), \dots, tr(V_i^T AA^T V_m)) y_m$.

Therefore, through the above discussion, we can obtain y_m by solving the following linear system:

$$\begin{pmatrix} tr(V_1^T AA^T V_1) & tr(V_1^T AA^T V_2) & \cdots & tr(V_1^T AA^T V_m) \\ tr(V_2^T AA^T V_1) & tr(V_2^T AA^T V_2) & \cdots & tr(V_2^T AA^T V_m) \\ \vdots & \vdots & \ddots & \vdots \\ tr(V_m^T AA^T V_1) & tr(V_m^T AA^T V_2) & \cdots & tr(V_m^T AA^T V_m) \end{pmatrix} y_m = \begin{pmatrix} \|R_0\|_F \\ 0 \\ \vdots \\ 0 \\ tr(V_{q+2}^T R_0) \\ \vdots \\ tr(V_m^T R_0) \end{pmatrix} \tag{2.3}$$

To sum up, we obtain the following restarting incomplete GMERR Algorithm.

Algorithm 1 (The restarting in complete GMERR Algorithm)

Step 1. Choose the restarting step number m , let $2 \leq q \leq m$, set the precision tol and initial estimation $n \times s$ moment X_0 . Then calculate $R_0 = B - AX_0$, Let $V_1 = R_0 / \|R_0\|_F$;

Yu-Hui Zheng et al: The incomplete global GMERR algorithm to Solve $AX = B$

Now, let us state some results which are indispensable for our subsequent discussions.

Lemma 1 For inner product $\langle X, Y \rangle$, we have $\langle X, Y \rangle \leq \|X\|_F \|Y\|_F$, where $X, Y \in \mathbb{Z}_{n \times s}$, $\mathbb{Z}_{n \times s}$ represents the $n \times s$ matrix space over \mathbb{R}

Proof. Obviously, $\langle X, Y \rangle = \text{tr}(X^T Y) = \sum_{i=1}^n \sum_{j=1}^s x_{ij} y_{ij}$, By Cauchy-Schwarz inequality, we have

$$\begin{aligned} \sum_{i=1}^n \sum_{j=1}^s x_{ij} y_{ij} &\leq \sum_{i=1}^n \left[\left(\sum_{j=1}^s x_{ij}^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{j=1}^s y_{ij}^2 \right)^{\frac{1}{2}} \right] \\ &\leq \left(\sum_{i=1}^n \sum_{j=1}^s x_{ij}^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{i=1}^n \sum_{j=1}^s y_{ij}^2 \right)^{\frac{1}{2}} \\ &= \|X\|_F \|Y\|_F . \end{aligned}$$

Hence, the proof of the theorem is completed. \square

Lemma 2 If $j + 2 \leq i \leq m + 1$, then $\text{tr}(V_i^T A^T V_j) = \sum_{k=j-q+1}^{\min\{i-q-1, j+1\}} h_{k,j} \text{tr}(V_i^T V_k)$.

Proof. By analyzing the algorithm and components of H_m , we have $A^T V_j = \sum_{k=1}^{m+1} h_{k,j} V_k$, where $j = 1, 2, \dots, m$. Multiplying V_i^T left to the two sides of the above

formula, we have $V_i^T A^T V_j = \sum_{k=1}^{\min\{i-q-1, j+1\}} h_{k,j} V_i^T V_k$.

Taking the trace, we can have

$$\begin{aligned} \text{tr}(V_i^T A^T V_j) &= \sum_{k=1}^{i-q-1} h_{k,j} \text{tr}(V_i^T V_k) + h_{i,j} + \sum_{k=i+q+1}^{m+1} h_{k,j} \text{tr}(V_i^T V_k) \\ &= \sum_{k=1}^{i-q-1} h_{k,j} \text{tr}(V_i^T V_k) + h_{i,j} + \sum_{k=i+q+1}^{j+1} h_{k,j} \text{tr}(V_i^T V_k) \quad (k > 1, h_{k,j} = 0) \\ &= \sum_{k=1}^{\min\{i-q-1, j+1\}} \text{tr}(h_{k,j} V_i^T V_k) \\ &= \sum_{k=j-q+1}^{\min\{i-q-1, j+1\}} h_{k,j} \text{tr}(V_i^T V_k) \end{aligned}$$

If $k \leq 0$, let $\text{tr}(V_i^T V_k)$, we have $h_{k,j} = 0$. \square

Theorem 2 Suppose $q \geq 2, i \leq m + 1$ and $i - j \geq q + 1$, if the incomplete global

Yu-Hui Zheng et al: The incomplete global GMERR algorithm to Solve $AX = B$

GMERR algorithm doesn't interrupt before the step, we have

$$|tr(V_j^T V_i)| \leq c_i \|A^T - A\|_F / h_{i,i-1}, \tag{2.4}$$

where $c_{i+1} = \max_{1 \leq j \leq i-q} \gamma_{i+1,j}$,

$$\gamma_{i+1,j} = 1 + c_i \left(\sum_{k=j-q+1}^{\min\{i-q-1,j+1\}} |h_{k,j}| \right) / h_{i,i-1} + \left(\sum_{k=j+q+1}^i c_k |h_{k,i}| \right) / h_{k,k-1}$$

and if $k \leq 0$, then $\gamma_{i+1,j} = 1$.

Proof. Let $U_{m+1} = [V_1, V_2, \dots, V_{m+1}]$, $B_{m+1} = (tr(V_i^T V_j))_{(m+1) \times (m+1)}$. From Algorithm 1 and Lemma 2, when $i + 1 \leq m + 1$ and $i + 1 - j \geq q + 1$, we obtain that

$$h_{i+1,i} V_{i+1} = A^T V_i - \sum_{k=i_0}^i h_{k,i} V_k, \tag{2.5}$$

Left-multiplying V_j^T to both sides of equation (2.5) and taking trace, we can get

$$\begin{aligned} tr(V_j^T V_{i+1} h_{i+1,i}) &= tr \left(V_j^T \left(A^T V_i - \sum_{k=i_0}^i h_{k,i} V_k \right) \right) \\ &= tr(V_j^T (A^T - A) V_i) + \sum_{k=j-q+1}^{\min\{i-q-1,j+1\}} h_{k,j} tr(V_k^T V_i) - \sum_{k=i_0}^i h_{k,i} tr(V_j^T V_k), \end{aligned} \tag{2.6}$$

where $i_0 = \max\{1, i - q + 1\}$. In the following part, the inductive method is used to prove our theorem.

When $i + 1 = q + 2 \leq m + 1$, $j = 1$, from equation (2.2) and Lemma 2, we can obtain

$$h_{q+2,q+1} tr(V_1^T V_{q+2}) = tr(V_1^T (A^T - A) V_{q+1}).$$

Assume that the incomplete global GMERR algorithm doesn't interrupt, and then we can have

$$\begin{aligned} |tr(V_1^T V_{q+2})| &= |tr(V_1^T (A^T - A) V_{q+1})| / h_{q+2,q+1} \\ &= |\langle V_1^T, (A^T - A) V_{q+1} \rangle| / h_{q+2,q+1} \\ &\leq \|V_1\|_F \|A^T - A\|_F \|V_{q+1}\|_F / h_{q+2,q+1} \\ &= \|A^T - A\|_F / h_{q+2,q+1}, \end{aligned}$$

which shows that it satisfies formula (2.5), where $c_{q+2} = 1$.

Yu-Hui Zheng et al: The incomplete global GMERR algorithm to Solve $AX = B$

Assume the first i columns in the upper right corner of matrix B_{m+1} satisfy equation (2.5). In the following we process the $(i+1)$ th column, where $1 \leq j \leq j-q$, i.e. $i-j \geq q$. For formula (2.6), we can separately get the following inequations

$$\begin{aligned} \left| \sum_{k=j-q+1}^{\min\{i-q-1, j+1\}} h_{k,j} \text{tr}(V_k^T V_i) \right| &\leq \sum_{k=j-q+1}^{\min\{i-q-1, j+1\}} |h_{k,j} \text{tr}(V_k^T V_i)| \\ &\leq c_i \sum_{k=j-q+1}^{\min\{i-q-1, j+1\}} |h_{k,j}| \|A^T - A\|_F / h_{i,i-1}, \end{aligned}$$

and

$$\begin{aligned} \left| \sum_{k=i_0}^i h_{k,i} \text{tr}(V_j^T V_k) \right| &\leq \sum_{k=i_0}^i |h_{k,i} \text{tr}(V_j^T V_k)| = \sum_{k=j+q+1}^i |h_{k,i} \text{tr}(V_j^T V_k)| \\ &\leq \sum_{k=j+q+1}^i c_k |h_{k,i}| \|A^T - A\|_F / h_{k,k-1}. \end{aligned}$$

Then, we can conclude that

$$\begin{aligned} |\text{tr}(V_j^T V_{i+1})| &\leq \frac{\left(\|V_j\|_F \|A^T - A\|_F \|V_i\|_F + \left| \sum_{k=j-q+1}^{\min\{i-q-1, j+1\}} h_{k,j} \text{tr}(V_k^T V_i) \right| + \left| \sum_{k=i_0}^i h_{k,i} \text{tr}(V_j^T V_k) \right| \right)}{h_{i+1,i}} \\ &\leq \frac{\left[1 + \left(c_i \sum_{k=j-q+1}^{\min\{i-q-1, j+1\}} |h_{k,j}| \right) / h_{i,i-1} + \left(\sum_{k=j+q+1}^i c_k |h_{k,i}| / h_{k,k-1} \right) \right] \|A^T - A\|_F}{h_{i+1,i}} \\ &= \gamma_{i+1,j} \|A^T - A\|_F / h_{i+1,i}, \end{aligned}$$

where $c_{i+1} = \max_{1 \leq j \leq i-q} \gamma_{i+1,j}$. The proof the theorem is completed. \square

Theorem 3 Any singular value $\sigma(B_m)$ of $B_m = (\text{tr}(V_i^T V_j))_{m \times m}$ satisfies

$$\max \{0, 1 - (m - q - 1)c \|A^T - A\|_F\} \leq \sigma(B_m) \leq 1 + (m - q - 1)c \|A^T - A\|_F$$

where c is a function generated by H_m .

Proof. By Gerschgorin Circular disc Theorem and Theorem 1, for any singular value $\sigma(B_m)$ of B_m , there must exist i , satisfying

$$|\sigma(B_m) - 1| \leq \sum_{j \neq i} |\text{tr}(V_i^T V_j)| = \sum_{|i-j| \geq q+1} |\text{tr}(V_i^T V_j)|.$$

Yu-Hui Zheng et al: The incomplete global GMERR algorithm to Solve $AX = B$

Therefore, we can get that

$$1 - \sum_{|i-j| \geq q+1} |tr(V_i^T V_j)| \leq \sigma(B_m) \leq 1 + \sum_{|i-j| \geq q+1} |tr(V_i^T V_j)|.$$

By analyzing the algorithm and the structure of B_m , we have

$$\sum_{|i-j| \geq q+1} |tr(V_i^T V_j)| = \sum_{j=1}^{i-q-1} |tr(V_i^T V_j)| + \sum_{j=i+q+1}^m |tr(V_i^T V_j)|$$

If $i \leq q + 1$, $\sum_{j=1}^{i-q-1} |tr(V_i^T V_j)| = 0$, then

$$\begin{aligned} \sum_{|i-j| \geq q+1} |tr(V_i^T V_j)| &\leq (m - i - q) \max_{i+q+1 \leq j \leq m} |tr(V_i^T V_j)| \\ &\leq (m - q - 1) \max_{q+2 \leq j \leq m} |tr(V_i^T V_j)| \\ &\leq (m - q - 1)c \|A^T - A\|_F, \end{aligned}$$

where $c = \max_{q+2 \leq j \leq m} c_j/h_{j,j-1}$.

If $i \geq m - q$, $\sum_{j=i+q+1}^m |tr(V_i^T V_j)| = 0$, then

$$\begin{aligned} \sum_{|i-j| \geq q+1} |tr(V_i^T V_j)| &\leq (i - q - 1) \max_{1 \leq j \leq i-q-1} |tr(V_i^T V_j)| \\ &\leq (m - q - 1) \max_{1 \leq j \leq m-q-1} |tr(V_j^T V_i)| \\ &\leq (m - q - 1)c \|A^T - A\|_F, \end{aligned}$$

where $c = \max_{m-q \leq i \leq m} c_i/h_{i,i-1}$.

If $q + 2 \leq i \leq m - q - 1$, based on Lemma 2, we have

$$\begin{aligned} \sum_{|i-j| \geq q+1} |tr(V_i^T V_j)| &\leq (i - q - 1) \max_{1 \leq j \leq i-q-1} |tr(V_i^T V_j)| + (m - i - q) \max_{i+q+1 \leq j \leq m} |tr(V_i^T V_j)| \\ &\leq (i - q - 1) \|A^T - A\|_F \max_{q+2 \leq i \leq m-q-1} c_i/h_{i,i-1} \\ &\quad + (m - i - q) \|A^T - A\|_F \max_{i+q+1 \leq j \leq m} c_j/h_{j,j-1} \\ &\leq (m - 2q - 1)c \|A^T - A\|_F \leq (m - q - 1)c \|A^T - A\|_F, \end{aligned}$$

where $c = \max \left\{ \max_{q+2 \leq i \leq m-q-1} c_i/h_{i,i-1}, \max_{i+q+1 \leq j \leq m} c_j/h_{j,j-1} \right\}$. Based the above disscusion, the proof the theorem is completed. \square

Yu-Hui Zheng et al: The incomplete global GMERR algorithm to Solve $AX = B$

Theorem 3 presents that the orthogonal degree of basis matrices is determined by the symmetry degree of the coefficient matrix. By the above theorems, we can obtain the conclusion on the algorithm convergence. If the algorithm is interrupted, which means $h_{1+i,i} = 0(1 \leq i \leq m)$ at i th step, the invariant subspace of A^T could be generated, moreover the approximate solution X_i generated by incomplete global GMERR algorithm is the exact solution of $AX = B$. Meanwhile, the error $R_i = 0$.

During the incomplete orthogonal process, the generated basis matrix may lose the orthogonality to some extent. From theorem 2 and theorem 3, we can find that the incomplete global GMERR algorithm can not control the orthogonal degree of the generated basis matrix when the coefficient matrix is far away from the symmetric property. And so, the algorithm may not converge.

3 Numerical experiments

In this section, we give numerical experiments to test the effectiveness of the incomplete global GMERR algorithm. Moreover we compare it with the global GMERR algorithm and find that our proposed method is more effective than the traditional method when they are set in the same accuracy.

Example Consider the two-dimension Convection-Diffusion Equation which is defined on the domain $\Omega = [0, 1] \times [0, 1]$

$$\begin{cases} -\Delta u(x, y) + \alpha \frac{\partial}{\partial x} u(x, y) = f(x, y), \\ u(x, y) = 0. \end{cases}$$

In this paper, we use the central difference method with grid length $h = 1/(l + 1)$ to discrete the above equation, and then we obtain the $n = l^2$ order non-symmetric matrix $A(\alpha)$

$$A(\alpha) = \begin{pmatrix} B(\alpha) & -I & & \\ -I & \ddots & \ddots & \\ & \ddots & \ddots & -I \\ & & -I & B(\alpha) \end{pmatrix},$$

where $B(\alpha) = \begin{pmatrix} 4 & a & & \\ b & \ddots & \ddots & \\ & \ddots & \ddots & a \\ & & b & 4 \end{pmatrix}$ is l order matrix, $a = -1 + \frac{\alpha}{2(l+1)}$, $b = -1 - \frac{\alpha}{2(l+1)}$,

I is l order identity matrix, parameter α control the $A(\alpha)$ and deviation of symmetry.

Yu-Hui Zheng et al: The incomplete global GMERR algorithm to Solve $AX = B$

Table 1: The incomplete global GMERR(m) algorithm with $n = 2500, \alpha = 0.25, \|A(0.25)^T - A(0.25)\|_F = 0.3431$

m	q	CPU	$ratio$	IT	$\ R\ _F$
40	2	43.8590	0.7221	28	9.2951e-7
	5	45.7030	0.7725	28	9.2951e-7
	10	50.7810	0.8361	28	9.2951e-7
	20	55.6400	0.9161	28	9.2951e-7
	30	57.4530	0.9459	28	9.2951e-7
	40	60.7350	1	28	9.2951e-7
50	2	34.4850	0.6876	15	7.3342e-7
	5	37.0470	0.7386	15	7.3342e-7
	10	39.9530	0.7966	15	7.3342e-7
	20	43.9370	0.8760	15	7.3342e-7
	30	47.7030	0.9511	15	7.3342e-7
	40	49.4220	0.9854	15	7.3342e-7
	50	50.1560	1	15	7.3342e-7

Table 2: The incomplete global GMERR(m) algorithm with $n = 2500, \alpha = 2.5, \|A(2.5)^T - A(2.5)\|_F = 3.4314$

m	q	CPU	$ratio$	IT	$\ R\ _F$
40	2	44.6410	0.7431	24	9.1938e-7
	5	47.2960	0.7872	24	9.1938e-7
	10	48.1100	0.8008	24	9.1938e-7
	20	58.7340	0.9778	24	9.1938e-7
	30	59.6250	0.9812	24	9.1938e-7
	40	60.0780	1	24	9.1938e-7
50	2	34.4690	0.7204	13	6.5037e-7
	5	33.4690	0.6995	13	6.5037e-7
	10	38.4060	0.8027	13	6.5037e-7
	20	39.2660	0.8207	13	6.5037e-7
	30	42.0780	0.8795	13	6.5037e-7
	40	44.0630	0.9209	13	6.5037e-7
	50	47.8440	1	13	6.5037e-7

Yu-Hui Zheng et al: The incomplete global GMERR algorithm to Solve $AX = B$

Table 3: The incomplete global GMERR(m) algorithm with $n = 2500, \alpha = 25, \| A(25)^T - A(25) \|_F = 34.3137$

m	q	CPU	$ratio$	IT	$\ R \ _F$
40	2	393.7810	0.7567	208	8.2432e-7
	5	381.4370	0.7330	208	9.6001e-7
	10	420.1400	0.8074	208	8.4919e-7
	20	461.5940	0.8871	208	9.3441e-7
	30	489.7580	0.9412	208	9.6380e-7
	40	520.3590	1	209	8.2174e-7
50	2	431.6090	0.8871	192	7.7878e-7
	5	324.9540	0.6679	137	8.8358e-7
	10	352.1880	0.7238	137	8.4919e-7
	20	392.1410	0.8059	137	9.1390e-7
	30	427.0160	0.8776	137	9.0901e-7
	40	442.0160	0.9084	136	9.7113e-7
	50	486.5620	1	136	9.1955e-7

Table 4: The incomplete global GMERR(m) algorithm with $n = 2500, \alpha = 2500, \| A(\alpha)^T - A(\alpha) \|_F = 3.4314e + 3$

m	q	CPU	$ratio$	IT	$\ R \ _F$
40	2	153.0620	0.7777	90	9.6087e-7
	5	145.2810	0.7382	92	9.8770e-7
	10	162.4370	0.8253	92	9.8773e-7
	20	180.1560	0.9154	92	9.8773e-7
	30	191.8900	0.9750	92	9.8773e-7
	40	196.8130	1	92	9.8773e-7
50	2	196.8280	0.7695	71	9.5397e-7
	5	183.8900	0.7189	71	9.3412e-7
	10	205.8750	0.8049	71	9.5319e-7
	20	228.0320	0.8915	71	9.5319e-7
	30	256.4370	1.0026	71	9.5319e-7
	40	244.1870	0.9547	71	9.5319e-7
	50	255.7810	1	71	9.5319e-7

The purpose of this section is to demonstrate that, with the same accuracy, the incomplete global GMERR algorithm is more effective than the global GMERR algorithm when we solve the large linear equations with multiple right-hand sides. Without loss of generality, we set $s = 2$, which means the two right-hand sides. Assuming $B = rand(n, s)$, $X_0 = 0$, $l = 50$, $n = 2500$, $tol = 10^{-6}$, we test the incomplete global GMERR algorithm for numerical analysis for $m = 40, 50$. Moreover, we find that the incomplete global GMERR algorithm degenerate to global GMERR algorithm when $q = m$. Table 1-4 show us the numerical results, where CPU is denoted as the algorithm running time (in seconds), IT represents the iterate times, Ratio means the running time ratio of the incomplete global GMERR algorithm to the GMERR algorithm with the same accuracy requirements.

For $\alpha = 0.25$, $A(\alpha)$ is approximately symmetric. Thus, the loss of the orthogonality of the basis matrices is not serious. The CPU time of incomplete global GMERR algorithm is shorter than the global GMERR algorithm, which shows effective of our proposed method.

For $\alpha = 2.5$, we can see the incomplete global GMERR algorithm is more effective than the global GMERR algorithm from table 2 and table 3. For $\alpha = 2500$, although $A(\alpha)$ is far away from the symmetric property and the loss of the orthogonality of the basis matrices is serious, we can find that the incomplete global GMERR algorithm is still effective than the global GMERR algorithm from table 4.

The experimental results show that, with the same accuracy, the incomplete global GMERR algorithm is more effective than the global GMERR algorithm. With the same computational cost, operation time and storage of our method is less than these of traditional method.

4 Conclusion

The incomplete global GMERR algorithm can overcome the long recurrence of the global GMERR algorithm by truncation strategy, which can save the computation and storage requirements effectively. In this paper, we present the incomplete global GMERR algorithm theoretically. Finally, the experimental results show effectiveness of the incomplete global GMERR algorithm by comparing with the traditional global GMERR algorithm.

References

- [1] Jbilou. K, Messaoudi. A, Sadok. H. Global FOM and GMRES algorithms for matrix equations. *Appl Numer Math.*, 1999, 31, pp. 49–63.

- [2] Y. H. Zheng, D. X. Cheng, X. H. Qian. Global GMERR algorithm for linear systems with multiple right-hand sides. *Journal of Northeast Normal University(Natural Science Edition)*, 2012, 44(3), pp. 41–45.
- [3] D. M. Young, K. C. Jea. Generalized conjugate-gradient acceleration of nonsymmetrizable iterative methods. *Linear Algebra Appl.*, 1980, 34, pp. 159–194.
- [4] Y. Saad. Krylov subspace methods for solving large unsymmetric linear systems. *Math Comput.*, 1981, 37, pp. 105–126.
- [5] Z. Jia. On IOM(q): the incomplete orthogonalization method for large unsymmetric linear system. *Numer Linear Alg Appl.*, 1996, 3(6), pp. 491–512.
- [6] P. N. Brown, A. C. Hindmarsh. Reduced storage methods in stiff ODE systems. *Appl Math Comput.*, 1989, 31, pp. 40–91.
- [7] P. N. Brown, Y. Saad. Hybrid Krylov subspace methods for nonlinear systems of equations. *SIAM J Sci Stat Comput.*, 1990, 11, pp. 450–481.
- [8] P. N. Brown. A theoretical comparison of the Arnoldi and GMRES algorithms. *SIAM J Sci Stat Comput.*, 1991, 12, pp. 58–78.
- [9] Y. Saad K. Wu. DQGMRES: a direct quasi-minimal residual algorithms based on incomplete orthogonalization. *Numer Linear Alg Appl.*, 1996, 3, pp. 329–344.
- [10] Z. Jia. On IGMRES: an incomplete generalized minimal residual method for large unsymmetric linear systems. *Science in China(Series A)*, 1998, 41(12), pp. 1278–1288.

DOUBLE DIFFERENCE SPACES OF ALMOST NULL AND ALMOST CONVERGENT SEQUENCES FOR ORLICZ FUNCTION

KULDIP RAJ AND RENU ANAND

ABSTRACT. The objective of this paper is to introduce and study some double difference spaces of almost null and almost convergent sequences defined by a Musielak-Orlicz function. We prove that these spaces are Banach, Barreled and Bornological spaces. An attempt is also made to prove that these spaces are BDK spaces and prove some interrelationship between these spaces.

1. INTRODUCTION AND PRELIMINARIES

The initial work on double sequences is found in Bromwich [4]. Later on, it was studied by Hardy [10], Móricz [15], Móricz and Rhoades [16], Tripathy ([27],[28]), Başarır and Sonalcan [2] and many others. Hardy [10] introduced the notion of regular convergence for double sequences. Quite recently, Zeltser [30] in her Ph.D thesis has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [17] have recently introduced the statistical convergence and Cauchy convergence for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Next, Mursaleen [21] and Mursaleen and Edely [18] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the M -core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{kl})$ into one whose core is a subset of the M -core of x . The set of all complex valued double sequences is a vector space with coordinatewise addition and scalar multiplication which is denoted by Ω .

By the convergence of a double sequence we mean the *convergence* in the Pringsheim sense i.e. a double sequence $x = (x_{kl})$ has Pringsheim limit L (denoted by $P - \lim x = L$) provided that given $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $|x_{kl} - L| < \epsilon$ whenever $k, l > n_0$. We shall write more briefly as P -convergent. The space of all convergent double sequences in Pringsheim's sense is denoted by C_p .

A double sequence $x = (x_{kl})$ of complex numbers is said to be *bounded* if $\|x\|_\infty = \sup_{k,l \in \mathbb{N}} |x_{kl}| < \infty$, where $\mathbb{N} = \{0, 1, 2, \dots\}$. The space of all bounded double sequences is denoted by \mathcal{M}_u , which is a Banach space with the norm $\|\cdot\|_\infty$.

It is well known that there are such sequences in the space C_p but not in the space \mathcal{M}_u . Indeed, if we define the sequence $x = (x_{kl})$ by

$$x_{kl} = \begin{cases} k, & k \in \mathbb{N} \\ l, & l \in \mathbb{N} \\ 0, & k, l \in \mathbb{N} \setminus \{0\}, \end{cases}$$

2010 *Mathematics Subject Classification.* 46A45, 40C05.

Key words and phrases. Orlicz function, Musielak-Orlicz function, sequence space, double sequence, P-convergent, almost convergence.

for all $k, l \in \mathbb{N}$, then, it is trivial that $x \in Cp \setminus \mathcal{M}_u$, since $P - \lim_{k,l \rightarrow \infty} x_{kl} = 0$ but $\|x\|_\infty = \infty$.

Therefore, we can consider the space C_{bp} of the double sequences that are both convergent in Pringsheim's sense and bounded which we write $C_{bp} = C_p \cap \mathcal{M}_u$.

A sequence in the space C_p is said to be *regularly convergent* if it is a single convergent sequence with respect to each index and denote the space of all such sequences by C_r . Also by C_{bp0} and C_{r0} , we denote the spaces of all double sequences converging to 0 contained in the sequence spaces C_{bp} and C_r , respectively. Moricz [15] proved that C_{bp}, C_{bp0}, C_r and C_{r0} are Banach spaces with the norm $\|\cdot\|_\infty$.

The concept of almost convergence for single sequences was introduced by Lorentz [13] and for double sequences by Móríciz and Rhoades [16]. A double sequence $x = (x_{kl})$ of complex numbers is said to be almost convergent to a generalized limit α if

$$P - \lim_{q,r \rightarrow \infty} \sup_{s,t > 0} \left| \frac{1}{(q+1)(r+1)} \sum_{k=s}^{s+q} \sum_{l=t}^{t+r} (x_{kl} - \alpha) \right| = 0. \quad (\text{see [29]})$$

Here, α is called the f_2 - limit of x . The space of all almost convergent double sequences is denoted by C_f . A P - convergent double sequence need not to be almost convergent. However, every bounded convergent double sequence is almost convergent and every almost convergent double sequence is also bounded.

Definition 1.1. [6] A bounded double sequence $x = (x_{kl})$ of real numbers is said to σ -convergent to a limit L if

$$P - \lim_{q,r} \tau_{qrst}(x) = L \text{ uniformly in } s, t \in \mathbb{N},$$

where

$$\tau_{qrst}(x) = \frac{1}{(q+1)(r+1)} \sum_{k=0}^q \sum_{l=0}^r x_{\sigma^k(s), \sigma^l(t)}.$$

In this case, we write $\sigma_2 - \lim x = L$. The set of all bounded σ - convergent double sequences is denoted by V_σ^2 . Clearly, $C_{bp} \subset V_\sigma^2$.

Definition 1.2. [26] A topological vector space λ over \mathbb{R} or \mathbb{C} is called locally convex if it is a Hausdorff space such that every neighbourhood of any $x \in \lambda$ contains a convex neighbourhood of x .

Definition 1.3. [30] A locally convex double sequence space λ is called a DK - space if all of the seminorms $r_{kl} : \lambda \rightarrow \mathbb{R}, x = (x_{kl}) \rightarrow |x_{kl}|$ for all $k, l \in \mathbb{N}$ are continuous. A DK - space with a Frechet topology is called an FDK - space. A normed FDK - space is called a BDK - space.

Definition 1.4. [26] Let λ be a vector space over the field \mathbb{C} and let A, B be subsets of λ . Then A absorbs B if there exists $\alpha_0 \in \mathbb{C}$ such that $B \subset \alpha A$ whenever $|\alpha| > |\alpha_0|$. A subset C of λ is circled if $\alpha C \subset C$ whenever $|\alpha| \leq 1$.

Definition 1.5. [26] A locally convex space λ is bornological if every circled, convex subset $A \subset \lambda$ that absorbs every bounded set in λ is a neighbourhood of 0 in λ .

Definition 1.6. [5] Let λ be a locally convex space. Then a subset is called barrel if it is absolutely convex, absorbing and closed in λ . Moreover, λ is called a barreled space if each barrel is a neighbourhood of zero.

Lemma 1.7. [26] *Every Banach space and every Fréchet space is a barreled space.*

Lemma 1.8. [26] *Every Fréchet space and hence every Banach space is a bornological.*

Lemma 1.9. [5] *Let (X, p) be a seminormed space and q be a seminorm on X . Then the following are equivalent:*

- (a) q is continuous.
- (b) q is continuous at zero.
- (c) There exists $M > 0$ such that $q(x) \leq Mp(x)$ for all $x \in X$.

Altay and Başar [1] introduced the space BS of bounded series as follows:

$$BS = \left\{ x = (x_{kl}) \in \Omega : \|x\|_{BS} = \sup_{m,n \in \mathbb{N}} \left| \sum_{k,l=0}^{m,n} x_{kl} \right| < \infty \right\}.$$

The space is also a Banach space with the norm $\|\cdot\|_{BS}$.

One can refer to Mursaleen and Mohiuddine [19] for relevant terminology and required details on the spaces of double sequences and related topics.

The notion of difference sequence spaces was introduced by Kızmaz [11], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [7] by introducing the spaces $l_\infty(\Delta^m)$, $c(\Delta^m)$ and $c_0(\Delta^m)$. Later the concept have been studied by Bektaş et al. [3] and Et et al. [8]. Another type of generalization of the difference sequence spaces is due to Tripathy and Esi [27] who studied the spaces $l_\infty(\Delta_v)$, $c(\Delta_v)$ and $c_0(\Delta_v)$ where m, v are non-negative integers. Now, for $Z = c, c_0$ and l_∞ , we have sequence spaces

$$Z(\Delta^m) = \{x = (x_k) \in \Omega : (\Delta^m x_k) \in Z\},$$

where $\Delta^m x = (\Delta^m x_k) = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1})$ and $\Delta^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta^m x_k = \sum_{v=0}^m (-1)^v \binom{m}{v} x_{k+v}.$$

Taking $m = 1$, we get the spaces studied by Et and Çolak [7].

An Orlicz function M is a function, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \rightarrow \infty$ as $x \rightarrow \infty$.

Lindenstrauss and Tzafriri [12] used the idea of Orlicz function to define the following sequence space. Let w be the space of all real or complex sequences $x = (x_k)$, then

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}$$

which is called as an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}.$$

It is shown in [12] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (p \geq 1)$. The Δ_2 -condition is equivalent to $M(Lx) \leq kLM(x)$ for all values of $x \geq 0$ and for $L > 1$. For more details about sequence spaces (see [9], [20], [23], [24], [25]) and references therein .

A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is called a Musielak-Orlicz function (see [14],[22]). A sequence $\mathcal{N} = (N_k)$ is defined by

$$N_k(v) = \sup\{|v|u - (M_k) : u \geq 0\}, \quad k = 1, 2, \dots$$

is called the complementary function of a Musielak-Orlicz function \mathcal{M} . For a given Musielak-Orlicz function \mathcal{M} , the Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows:

$$t_{\mathcal{M}} = \left\{ x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for some } c > 0 \right\},$$

$$h_{\mathcal{M}} = \left\{ x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for all } c > 0 \right\},$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k), x = (x_k) \in t_{\mathcal{M}}.$$

We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$\|x\| = \inf \left\{ k > 0 : I_{\mathcal{M}}\left(\frac{x}{k}\right) \leq 1 \right\}$$

or equipped with the Orlicz norm

$$\|x\|^0 = \inf \left\{ \frac{1}{k} (1 + I_{\mathcal{M}}(kx)) : k > 0 \right\}.$$

Let $\mathcal{M} = (M_{kl})$ be Musielak-Orlicz function, $p = (p_{kl})$ be a bounded sequence of positive real numbers and $u = (u_{kl})$ be a double sequence of strictly positive real numbers. In the present paper we define the following classes of sequences:

$$C_f(\mathcal{M}, u, \Delta^m, p) =$$

$$\left\{ x = (x_{kl}) \in \Omega : P - \lim_{q,r \rightarrow \infty} \sup_{s,t > 0} \left| \frac{1}{(q+1)(r+1)} \sum_{k=s}^{s+q} \sum_{l=t}^{t+r} M_{kl} \left(\frac{u_{kl} \Delta^m x_{kl} - \alpha}{\varrho} \right) \right|^{p_{kl}} = 0, \right.$$

for some $\varrho > 0$ } and

$$C_{f_0}(\mathcal{M}, u, \Delta^m, p) =$$

$$\left\{ x = (x_{kl}) \in \Omega : P - \lim_{q,r \rightarrow \infty} \sup_{s,t > 0} \left| \frac{1}{(q+1)(r+1)} \sum_{k=0}^q \sum_{l=0}^r M_{kl} \left(\frac{u_{kl} \Delta^m x_{k+s, l+t}}{\varrho} \right) \right|^{p_{kl}} = \right.$$

0, for some $\varrho > 0$ }.

Remark 1.10. Let us consider a few special cases of the above sequence spaces:

(i) If we take $\mathcal{M}(x) = x$, then the above classes of sequences reduces to following sequence spaces:

$$C_f(u, \Delta^m, p) =$$

$$\left\{ x = (x_{kl}) \in \Omega : P - \lim_{q,r \rightarrow \infty} \sup_{s,t > 0} \left| \frac{1}{(q+1)(r+1)} \sum_{k=s}^{s+q} \sum_{l=t}^{t+r} \left(\frac{u_{kl} \Delta^m x_{kl} - \alpha}{\varrho} \right) \right|^{p_{kl}} = 0, \right.$$

for some $\varrho > 0$ } and

$$C_{f_0}(u, \Delta^m, p) =$$

$$\left\{ x = (x_{kl}) \in \Omega : P - \lim_{q,r \rightarrow \infty} \sup_{s,t > 0} \left| \frac{1}{(q+1)(r+1)} \sum_{k=0}^q \sum_{l=0}^r \left(\frac{u_{kl} \Delta^m x_{k+s, l+t}}{\varrho} \right) \right|^{p_{kl}} = 0, \right.$$

for some $\varrho > 0$ }.

(ii) If we take $p = (p_{kl}) = 1$, then the above sequence space becomes

$$C_f(\mathcal{M}, u, \Delta^m) = \left\{ x = (x_{kl}) \in \Omega : P - \lim_{q,r \rightarrow \infty} \sup_{s,t > 0} \left| \frac{1}{(q+1)(r+1)} \sum_{k=s}^{s+q} \sum_{l=t}^{t+r} M_{kl} \left(\frac{u_{kl} \Delta^m x_{kl} - \alpha}{\varrho} \right) \right| = 0, \right.$$

for some $\varrho > 0$ } and

$$C_{f_0}(\mathcal{M}, u, \Delta^m) = \left\{ x = (x_{kl}) \in \Omega : P - \lim_{q,r \rightarrow \infty} \sup_{s,t > 0} \left| \frac{1}{(q+1)(r+1)} \sum_{k=0}^q \sum_{l=0}^r M_{kl} \left(\frac{u_{kl} \Delta^m x_{k+s,l+t}}{\varrho} \right) \right| = 0, \right.$$

for some $\varrho > 0$ }.

(iii) If we take $\mathcal{M}(x) = x$, $p = (p_{kl}) = 1$, $u = (u_{kl}) = 1$, $m = 0$ and $\varrho = 1$, then we have

$$C_f = \left\{ x = (x_{kl}) \in \Omega : P - \lim_{q,r \rightarrow \infty} \sup_{s,t > 0} \left| \frac{1}{(q+1)(r+1)} \sum_{k=s}^{s+q} \sum_{l=t}^{t+r} (x_{kl} - \alpha) \right| = 0 \text{ and} \right.$$

$$C_{f_0} = \left\{ x = (x_{kl}) \in \Omega : P - \lim_{q,r \rightarrow \infty} \sup_{s,t > 0} \left| \frac{1}{(q+1)(r+1)} \sum_{k=0}^q \sum_{l=0}^r (x_{k+s,l+t}) \right| = 0 \right\}.$$

which were introduced and studied by Yeşilkayağıil and Başar [29].

The main purpose of the present paper is to show that the sequence spaces $C_f(\mathcal{M}, u, \Delta^m, p)$ and $C_{f_0}(\mathcal{M}, u, \Delta^m, p)$ are BDK-spaces, Barreled and Bornological. Furthermore, we also studied some inclusion relations between these spaces.

2. MAIN RESULTS

Theorem 2.1. *Let $\mathcal{M} = (M_{kl})$ be Musielak-Orlicz function, $p = (p_{kl})$ be a bounded sequence of positive real numbers and $u = (u_{kl})$ be a double sequence of strictly positive real numbers. Then the sequence spaces $C_f(\mathcal{M}, u, \Delta^m, p)$ and $C_{f_0}(\mathcal{M}, u, \Delta^m, p)$ are Banach spaces with the supremum norm.*

Proof. We are going to prove this for the space $C_f(\mathcal{M}, u, \Delta^m, p)$ and the other can be proved in the similar way. Define norm $\|\cdot\|$ on $C_f(\mathcal{M}, u, \Delta^m, p)$ as:

$$\|x\|_{C_f}^{\mathcal{M}, u, \Delta^m, p} = \sup_{q,r,s,t \in \mathbb{N}} \left| \frac{1}{(q+1)(r+1)} \sum_{k=0}^q \sum_{l=0}^r M_{kl} \left(\frac{u_{kl} \Delta^m x_{k+s,l+t}}{\varrho} \right) \right|^{p_{kl}}.$$

Clearly, $C_f(\mathcal{M}, u, \Delta^m, p)$ is a normed linear space by the above defined norm. Now, we have to prove that $C_f(\mathcal{M}, u, \Delta^m, p)$ is complete. For this, let $(\Delta^m x^{(b)})$ be a Cauchy sequence in $C_f(\mathcal{M}, u, \Delta^m, p)$. Then $(\Delta^m x_{kl}^{(b)})$ is a Cauchy sequence in \mathbb{C} , for each k, l . Therefore, $\Delta^m x_{kl}^{(b)} \rightarrow \Delta^m x_{kl}$ (say). Put $\Delta^m x = (\Delta^m x_{kl})$. For given ϵ , there exists an integer $N(\epsilon) = N$ (say) such that for each $b, n > N$

$$\|\Delta^m x^{(b)} - \Delta^m x^{(n)}\| < \frac{\epsilon}{2}.$$

Hence,

$$\sup_{q,r,s,t} |\tau_{qrst}(\Delta^m x^{(b)} - \Delta^m x^{(n)})| < \frac{\epsilon}{2}.$$

Then, for each q, r, s, t and $b, n > N$, we have

$$(2.1) \quad |\tau_{qrst}(\Delta^m x^{(b)} - \Delta^m x^{(n)})| < \frac{\epsilon}{2}.$$

Now, for fixed b , the above inequality holds. Since for fixed b , $\Delta^m x^{(b)} \in C_f(\mathcal{M}, u, \Delta^m, p)$, we get

$$\lim_{q,r \rightarrow \infty} \tau_{qrst}(\Delta^m x^{(b)}) = L,$$

uniformly in s, t .

For given $\epsilon > 0$, there exists positive integers q_0, r_0 such that

$$(2.2) \quad |\tau_{qrst}(\Delta^m x^{(b)}) - L| < \frac{\epsilon}{2},$$

for $q \geq q_0, r \geq r_0$ and for all s, t . Here q_0, r_0 are independent of s, t but depend upon ϵ . Now, by using (2.1) and (2.2), we obtain

$$\begin{aligned} |\tau_{qrst}(\Delta^m x) - L| &= |\tau_{qrst}(\Delta^m x) - \tau_{qrst}(\Delta^m x^{(b)}) + \tau_{qrst}(\Delta^m x^{(b)}) - L| \\ &\leq |\tau_{qrst}(\Delta^m x) - \tau_{qrst}(\Delta^m x^{(b)})| + |\tau_{qrst}(\Delta^m x^{(b)}) - L| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

for $q \geq q_0, r \geq r_0$ and for all s, t .

Hence, $\Delta^m x = (\Delta^m x_{kl}) \in C_f(\mathcal{M}, u, \Delta^m, p)$ and so $C_f(\mathcal{M}, u, \Delta^m, p)$ is complete. This completes the proof. \square

Corollary 2.2. *Let $\mathcal{M} = (M_{kl})$ be Musielak-Orlicz function, $p = (p_{kl})$ be a bounded sequence of positive real numbers and $u = (u_{kl})$ be a double sequence of strictly positive real numbers. Then the sequence spaces $C_f(\mathcal{M}, u, \Delta^m, p)$ and $C_{f_0}(\mathcal{M}, u, \Delta^m, p)$ are barreled spaces.*

Proof. Since $C_f(\mathcal{M}, u, \Delta^m, p)$ and $C_{f_0}(\mathcal{M}, u, \Delta^m, p)$ are Banach spaces with the supremum norm and every Banach space is a barreled space, it follows that $C_f(\mathcal{M}, u, \Delta^m, p)$ and $C_{f_0}(\mathcal{M}, u, \Delta^m, p)$ are also barreled spaces. \square

Corollary 2.3. *Let $\mathcal{M} = (M_{kl})$ be Musielak-Orlicz function, $p = (p_{kl})$ be a bounded sequence of positive real numbers and $u = (u_{kl})$ be a double sequence of strictly positive real numbers. Then the sequence spaces $C_f(\mathcal{M}, u, \Delta^m, p)$ and $C_{f_0}(\mathcal{M}, u, \Delta^m, p)$ are bornological spaces.*

Proof. By the same argument given in the above corollary, the above spaces are bornological spaces. \square

Theorem 2.4. *Let $\mathcal{M} = (M_{kl})$ be Musielak-Orlicz function, $p = (p_{kl})$ be a bounded sequence of positive real numbers and $u = (u_{kl})$ be a double sequence of strictly positive real numbers. Then the sequence spaces $C_f(\mathcal{M}, u, \Delta^m, p)$ and $C_{f_0}(\mathcal{M}, u, \Delta^m, p)$ are BDK-spaces with the norm defined as:*

$$\|x\|_{C_f}^{\mathcal{M}, u, \Delta^m, p} = \sup_{q,r,s,t \in \mathbb{N}} \left| \frac{1}{(q+1)(r+1)} \sum_{k=0}^q \sum_{l=0}^r M_{kl} \left(\frac{u_{kl} \Delta^m x_{k+s, l+t}}{\varrho} \right) \right|^{p_{kl}}.$$

Proof. We will prove it for the space $C_f(\mathcal{M}, u, \Delta^m, p)$ and rest can be proved in a similar way. Since every normed space is a seminormed space, it follows that $C_f(\mathcal{M}, u, \Delta^m, p)$ is also a seminormed space with respect to the given norm. Now for $x = (x_{kl})$ in $C_f(\mathcal{M}, u, \Delta^m, p)$, we define some new seminorm $r_{kl} : C_f(\mathcal{M}, u, \Delta^m, p) \rightarrow \mathbb{R}$ as $r_{kl}(x_{kl}) = |x_{kl}|$ for all $k, l \in \mathbb{N}$. We have to show that the each one is continuous. By Lemma (1.9), there exists $T > 0$ for all $x \in C_f(\mathcal{M}, u, \Delta^m, p)$ such that

$$r_{kl}(x) = |x_{kl}| \leq T \|x\|_{C_f}^{\mathcal{M}, u, \Delta^m, p} \quad \forall k, l \in \mathbb{N}.$$

So, the seminorm is continuous. Therefore, $C_f(\mathcal{M}, u, \Delta^m, p)$ is a DK -space and so is Banach space, it follows that $C_f(\mathcal{M}, u, \Delta^m, p)$ has Frechet topology. Thus, it is BDK -space with the above given norm. Hence, the proof is complete. \square

Theorem 2.5. *Let $\mathcal{M} = (M_{kl})$ be Musielak-Orlicz function, $p = (p_{kl})$ be a bounded sequence of positive real numbers and $u = (u_{kl})$ be a double sequence of strictly positive real numbers. Suppose that $\beta = \lim_{w \rightarrow \infty} \frac{M_{kl}(w)}{w} < \infty$. Then, $C_{f_0}(u, \Delta^m, p) = C_{f_0}(\mathcal{M}, u, \Delta^m, p)$.*

Proof. In order to prove that $C_{f_0}(u, \Delta^m, p) = C_{f_0}(\mathcal{M}, u, \Delta^m, p)$, it is sufficient to show that $C_{f_0}(\mathcal{M}, u, \Delta^m, p) \subset C_{f_0}(u, \Delta^m, p)$. Now, let $\beta > 0$. By definition of β , we have $w \leq \frac{1}{\beta} M_{kl}(w), \forall w \geq 0$. Let $x = (x_{kl}) \in C_{f_0}(\mathcal{M}, u, \Delta^m, p)$. Thus, we have

$$\begin{aligned} \lim_{q,r \rightarrow \infty} \sup_{s,t > 0} \left| \frac{1}{(q+1)(r+1)} \sum_{k=0}^q \sum_{l=0}^r \left(\frac{u_{kl} \Delta^m x_{k+s,l+t}}{\varrho} \right) \right|^{p_{kl}} \\ \leq \frac{1}{\beta} \lim_{q,r \rightarrow \infty} \sup_{s,t > 0} \left| \frac{1}{(q+1)(r+1)} \sum_{k=0}^q \sum_{l=0}^r M_{kl} \left(\frac{u_{kl} \Delta^m x_{k+s,l+t}}{\varrho} \right) \right|^{p_{kl}} \end{aligned}$$

which implies that $x = (x_{kl}) \in C_{f_0}(u, \Delta^m, p)$. \square

Theorem 2.6. *Let $\mathcal{M} = (M_{kl})$ be Musielak-Orlicz function and $u = (u_{kl})$ be a double sequence of strictly positive real numbers. If $p = (p_{kl})$ and $v = (v_{kl})$ are bounded sequences of positive real numbers with $0 \leq p_{kl} \leq v_{kl} < \infty \forall k, l$; then*

$$C_{f_0}(\mathcal{M}, u, \Delta^m, p) \subset C_{f_0}(\mathcal{M}, u, \Delta^m, v).$$

Proof. Let $x = (x_{kl}) \in C_{f_0}(\mathcal{M}, u, \Delta^m, p)$. Then

$$\sup_{s,t > 0} \left| \frac{1}{(q+1)(r+1)} \sum_{k=0}^q \sum_{l=0}^r M_{kl} \left(\frac{u_{kl} \Delta^m x_{k+s,l+t}}{\varrho} \right) \right|^{p_{kl}} \rightarrow 0 \text{ as } q, r \rightarrow \infty.$$

This implies that

$\left| M_{kl} \left(\frac{u_{kl} \Delta^m x_{k+s,l+t}}{\varrho} \right) \right|^{p_{kl}} \leq 1$, for sufficiently large values of k and l . Since (M_{kl}) is increasing and $p_{kl} \leq v_{kl}$, we have

$$\begin{aligned} \sup_{s,t > 0} \left| \frac{1}{(q+1)(r+1)} \sum_{k=0}^q \sum_{l=0}^r M_{kl} \left(\frac{u_{kl} \Delta^m x_{k+s,l+t}}{\varrho} \right) \right|^{v_{kl}} \\ \leq \sup_{s,t > 0} \left| \frac{1}{(q+1)(r+1)} \sum_{k=0}^q \sum_{l=0}^r M_{kl} \left(\frac{u_{kl} \Delta^m x_{k+s,l+t}}{\varrho} \right) \right|^{p_{kl}}. \end{aligned}$$

Thus, $x = (x_{kl}) \in C_{f_0}(\mathcal{M}, u, \Delta^m, v)$. This completes the proof. \square

Theorem 2.7. *Let $\mathcal{M} = (M_{kl})$ be Musielak-Orlicz function, $p = (p_{kl})$ be a bounded sequence of positive real numbers and $u = (u_{kl})$ be a double sequence of strictly positive real numbers. Then the following inclusions hold:*

- (i) *If $0 < \inf p_{kl} < p_{kl} \leq 1$ then $C_{f_0}(\mathcal{M}, u, \Delta^m, p) \subset C_{f_0}(\mathcal{M}, u, \Delta^m)$;*
- (ii) *If $1 \leq p_{kl} \leq \sup p_{kl} < \infty$ then $C_{f_0}(\mathcal{M}, u, \Delta^m) \subset C_{f_0}(\mathcal{M}, u, \Delta^m, p)$.*

Proof. (i) Let $x = (x_{kl}) \in C_{f_0}(\mathcal{M}, u, \Delta^m, p)$. Then, since $0 < \inf p_{kl} < p_{kl} \leq 1$, we obtain the following:

$$\begin{aligned} \lim_{q,r \rightarrow \infty} \sup_{s,t > 0} \left| \frac{1}{(q+1)(r+1)} \sum_{k=0}^q \sum_{l=0}^r M_{kl} \left(\frac{u_{kl} \Delta^m x_{k+s,l+t}}{\varrho} \right) \right| \\ \leq \lim_{q,r \rightarrow \infty} \sup_{s,t > 0} \left| \frac{1}{(q+1)(r+1)} \sum_{k=0}^q \sum_{l=0}^r M_{kl} \left(\frac{u_{kl} \Delta^m x_{k+s,l+t}}{\varrho} \right) \right|^{p_{kl}}. \end{aligned}$$

Thus, $x = (x_{kl}) \in C_{f_0}(\mathcal{M}, u, \Delta^m)$.

(ii) Let $p = (p_{kl}) \geq 1$ for each k and l and $\sup p_{kl} < \infty$. Let $x = (x_{kl}) \in C_{f_0}(\mathcal{M}, u, \Delta^m)$. Then for each $0 < \epsilon < 1$, there exists a positive integer N such that

$$\lim_{q,r \rightarrow \infty} \sup_{s,t > 0} \left| \frac{1}{(q+1)(r+1)} \sum_{k=0}^q \sum_{l=0}^r M_{kl} \left(\frac{u_{kl} \Delta^m x_{k+s,l+t}}{\varrho} \right) \right| \leq \epsilon < 1 \quad \forall \quad q, r \geq N.$$

This implies that

$$\begin{aligned} \lim_{q,r \rightarrow \infty} \sup_{s,t > 0} \left| \frac{1}{(q+1)(r+1)} \sum_{k=0}^q \sum_{l=0}^r M_{kl} \left(\frac{u_{kl} \Delta^m x_{k+s,l+t}}{\varrho} \right) \right|^{p_{kl}} \\ \leq \lim_{q,r \rightarrow \infty} \sup_{s,t > 0} \left| \frac{1}{(q+1)(r+1)} \sum_{k=0}^q \sum_{l=0}^r M_{kl} \left(\frac{u_{kl} \Delta^m x_{k+s,l+t}}{\varrho} \right) \right|. \end{aligned}$$

Therefore, $x = (x_{kl}) \in C_{f_0}(\mathcal{M}, u, \Delta^m, p)$. This concludes the proof. \square

Theorem 2.8. Let $\mathcal{M} = (M_{kl})$ and $\mathcal{M}' = (M'_{kl})$ be two Musielak-Orlicz functions, $p = (p_{kl})$ be a bounded sequence of positive real numbers and $u = (u_{kl})$ be a double sequence of strictly positive real numbers. Then,

$$C_{f_0}(\mathcal{M}, u, \Delta^m, p) \cap C_{f_0}(\mathcal{M}', u, \Delta^m, p) \subset C_{f_0}(\mathcal{M} + \mathcal{M}', u, \Delta^m, p).$$

Proof. Let $x = (x_{kl}) \in C_{f_0}(\mathcal{M}, u, \Delta^m, p) \cap C_{f_0}(\mathcal{M}', u, \Delta^m, p)$. Then

$$\sup_{s,t > 0} \left| \frac{1}{(q+1)(r+1)} \sum_{k=0}^q \sum_{l=0}^r M_{kl} \left(\frac{u_{kl} \Delta^m x_{k+s,l+t}}{\varrho} \right) \right|^{p_{kl}} \rightarrow 0 \quad \text{as } q, r \rightarrow \infty$$

and

$$\sup_{s,t > 0} \left| \frac{1}{(q+1)(r+1)} \sum_{k=0}^q \sum_{l=0}^r M'_{kl} \left(\frac{u_{kl} \Delta^m x_{k+s,l+t}}{\varrho} \right) \right|^{p_{kl}} \rightarrow 0 \quad \text{as } q, r \rightarrow \infty.$$

Then, we have

$$\begin{aligned} \sup_{s,t > 0} \left| \frac{1}{(q+1)(r+1)} \sum_{k=0}^q \sum_{l=0}^r (M_{kl} + M'_{kl}) \left(\frac{u_{kl} \Delta^m x_{k+s,l+t}}{\varrho} \right) \right|^{p_{kl}} \\ \leq K \left[\sup_{s,t > 0} \left| \frac{1}{(q+1)(r+1)} \sum_{k=0}^q \sum_{l=0}^r M_{kl} \left(\frac{u_{kl} \Delta^m x_{k+s,l+t}}{\varrho} \right) \right|^{p_{kl}} \right] \\ + K \left[\sup_{s,t > 0} \left| \frac{1}{(q+1)(r+1)} \sum_{k=0}^q \sum_{l=0}^r M'_{kl} \left(\frac{u_{kl} \Delta^m x_{k+s,l+t}}{\varrho} \right) \right|^{p_{kl}} \right] \rightarrow 0 \quad \text{as } q, r \rightarrow \infty. \end{aligned}$$

$$\text{Thus, } \sup_{s,t > 0} \left| \frac{1}{(q+1)(r+1)} \sum_{k=0}^q \sum_{l=0}^r (M_{kl} + M'_{kl}) \left(\frac{u_{kl} \Delta^m x_{k+s,l+t}}{\varrho} \right) \right|^{p_{kl}} \rightarrow 0 \quad \text{as } q, r \rightarrow \infty.$$

Therefore, $x = (x_{kl}) \in C_{f_0}(\mathcal{M} + \mathcal{M}', u, \Delta^m, p)$. This completes the proof. \square

Theorem 2.9. Let $\mathcal{M} = (M_{kl})$ and $\mathcal{M}' = (M'_{kl})$ be two Musielak-Orlicz functions, $p = (p_{kl})$ be a bounded sequence of positive real numbers and $u = (u_{kl})$ be a double sequence of strictly positive real numbers. Then,

$$C_{f_0}(\mathcal{M}', u, \Delta^m, p) \subset C_{f_0}(\mathcal{M} \circ \mathcal{M}', u, \Delta^m, p).$$

Proof. Let $x = (x_{kl}) \in C_{f_0}(\mathcal{M}', u, \Delta^m, p)$. Then, we have

$$\lim_{q,r \rightarrow \infty} \sup_{s,t > 0} \left| \frac{1}{(q+1)(r+1)} \sum_{k=0}^q \sum_{l=0}^r M'_{kl} \left(\frac{u_{kl} \Delta^m x_{k+s,l+t}}{\varrho} \right) \right|^{p_{kl}} = 0.$$

Let $\epsilon > 0$ and choose $\delta > 0$ with $0 < \delta < 1$ such that $M_{kl}(n) < \epsilon$ for $0 \leq n \leq \delta$.

Write $y_{kl} = \left[M'_{kl} \left(\frac{u_{kl} \Delta^m x_{k+s, l+t}}{\rho} \right) \right]$ and consider

$$\begin{aligned} & \sup_{s, t > 0} \left| \frac{1}{(q+1)(r+1)} \sum_{k, l=0, 0}^{q, r} M_{kl}(y_{kl}) \right|^{p_{kl}} \\ &= \sup_{s, t > 0} \left| \frac{1}{(q+1)(r+1)} \sum_1 M_{kl}(y_{kl}) \right|^{p_{kl}} + \sup_{s, t > 0} \left| \frac{1}{(q+1)(r+1)} \sum_2 M_{kl}(y_{kl}) \right|^{p_{kl}} \end{aligned}$$

where the first summation is over $y_{kl} \leq \delta$ and second summation is over $y_{kl} > \delta$. Since M_{kl} is continuous, we have

$$(2.3) \quad \sup_{s, t > 0} \left| \frac{1}{(q+1)(r+1)} \sum_1 M_{kl}(y_{kl}) \right|^{p_{kl}} < \epsilon^H$$

where $H = \sup p_{kl}$ and for $y_{kl} > \delta$, we use the fact that

$$y_{kl} < \frac{y_{kl}}{\delta} < 1 + \frac{y_{kl}}{\delta}$$

By the definition, we have for $y_{kl} > \delta$, $M_{kl}(y_{kl}) < 2M_{kl}(1) \frac{y_{kl}}{\delta}$. Hence,

$$(2.4) \quad \begin{aligned} & \sup_{s, t > 0} \left| \frac{1}{(q+1)(r+1)} \sum_2 M_{kl}(y_{kl}) \right|^{p_{kl}} \\ & \leq \max \left(1, (2F_k(1)\delta^{-1})^H \right) \sup_{s, t > 0} \left| \frac{1}{(q+1)(r+1)} \sum_{k, l=0, 0}^{q, r} y_{kl} \right|^{p_{kl}} \end{aligned}$$

Therefore, from equations (2.3) and (2.4), we have

$$C_{f_0}(\mathcal{M}', u, \Delta^m, p) \subset C_{f_0}(\mathcal{M} \circ \mathcal{M}', u, \Delta^m, p).$$

This completes the proof. □

Theorem 2.10. *Let BS be the space of bounded series of double sequences and $C_{f_0}(\mathcal{M}, u, \Delta^m, p)$ be the space of all almost null double sequences. Then the inclusion relation $BS \subset C_{f_0}(\mathcal{M}, u, \Delta^m, p)$ holds.*

Proof. Let $x = (x_{kl}) \in BS$. Then $T = \sup_{s, t \in \mathbb{N}} \left| \sum_{k, l=0, 0}^{s, t} x_{kl} \right| < \infty$.

Therefore, for all $q, r, s, t \in \mathbb{N}$, we have

$$\begin{aligned} & \left| \sum_{k=s}^{s+q} \sum_{l=t}^{t+r} x_{kl} \right| = \left| \sum_{k=0}^{s+q} \sum_{l=0}^{t+r} x_{kl} - \sum_{k=0}^{s-1} \sum_{l=0}^{t+r} x_{kl} - \sum_{k=s}^{s+q} \sum_{l=0}^{t-1} x_{kl} \right| \\ & \leq \left| \sum_{k=0}^{s+q} \sum_{l=0}^{t+r} x_{kl} \right| + \left| \sum_{k=s}^{s+q} \sum_{l=0}^{t+r} x_{kl} \right| + \left| \sum_{k=s}^{s+q} \sum_{l=0}^{t-1} x_{kl} \right| \\ & \leq 2T + \left| \sum_{k=s}^{s+q} \sum_{l=0}^{t-1} x_{kl} \right| \\ & = 2T + \left| \sum_{k=0}^{s+q} \sum_{l=0}^{t-1} x_{kl} - \sum_{k=0}^s \sum_{l=0}^{t-1} x_{kl} \right| \\ & \leq 2T + \left| \sum_{k=0}^{s+q} \sum_{l=0}^{t-1} x_{kl} \right| + \left| \sum_{k=0}^s \sum_{l=0}^{t-1} x_{kl} \right| \leq 4T, \end{aligned}$$

which implies that

$$\left| \frac{1}{(q+1)(r+1)} \sum_{k=s}^{s+q} \sum_{l=t}^{t+r} M_{kl} \left(\frac{u_{kl} \Delta^m x_{kl}}{\rho} \right) \right|^{p_{kl}} \leq \left| \frac{4T}{(q+1)(r+1)} \sum_{k=s}^{s+q} \sum_{l=t}^{t+r} M_{kl} \left(\frac{u_{kl} \Delta^m x_{kl}}{\rho} \right) \right|^{p_{kl}}$$

Further, if we take supremum over $s, t \in \mathbb{N}$ in the above relation and also apply the P-limit as $q, r \rightarrow \infty$, then, we have

$$P - \lim_{q, r \rightarrow \infty} \sup_{s, t > 0} \left| \frac{1}{(q+1)(r+1)} \sum_{k=s}^{s+q} \sum_{l=t}^{t+r} M_{kl} \left(\frac{u_{kl} \Delta^m x_{kl}}{\rho} \right) \right|^{p_{kl}} = 0,$$

therefore, $x \in C_{f_0}(\mathcal{M}, u, \Delta^m, p)$. Hence, the result holds. □

REFERENCES

- [1] A. Altay and F. Başar, *Some new spaces of double sequences*, J Math Anal Appl, **309** (2005), 70-90.
- [2] M. Başarır and O. Sonalcan, *On some double sequence spaces*, J. Indian Acad. Math., **21** (1999), 193-200.
- [3] Ç. A. Bektaş, M. Et and R. Çolak, *Generalized difference sequence spaces and their dual spaces*, J. Math. Anal. Appl., **292** (2004), 423-432.
- [4] T. J. Bromwich, *An introduction to the theory of infinite series*, Macmillan and co. Ltd., New York (1965).
- [5] J. Boos, *Classical and Modern Methods in Summability*, NewYork, NY, USA: Oxford University Press Inc., 2000.
- [6] C. Çakan, B. Altay and M. Mursaleen, *The σ -convergence and σ -core of double sequences*, Appl. Math. Letters, **19** (2006), 1122-1128.
- [7] M. Et and R. Çolak, *On generalized difference sequence spaces*, Soochow J. Math., **21** (1995), 377-386.
- [8] M. Et and A. Esi, *On Köthe-Toeplitz duals of generalized difference sequence spaces*, Bull. Malays. Math. Sci. Soc., **23** (2000), 25-32.
- [9] A. Esi, B. C. Tripathy and B. Sarma, *On some new type generalized difference sequence spaces*, Math. Slovaca., **57** (2007), 475-482.
- [10] G. H. Hardy, *On the convergence of certain multiple series*, Proc. Camb. Phil. Soc., **19** (1917), 86-95.
- [11] H. Kizmaz, *On certain sequence spaces*, Canad. Math. Bull., **24** (1981), 169-176.
- [12] J. Lindenstrauss and L. Tzafriri, *On Orlicz sequence spaces*, Israel J. Math., **10** (1971), 379-390.
- [13] G.G. Lorentz, *A contribution to the theory of divergent sequences*, Acta Math., **80** (1948), 167-190.
- [14] L. Maligranda, *Orlicz spaces and interpolation*, Seminars in Mathematics **5**, Polish Academy of Science, 1989.
- [15] F. Móricz, *Statistical convergence of multiple sequences*, Arch. Math. (Basel), **81** (2003), 82-89.
- [16] F. Móricz and B. E. Rhoades, *Almost convergence of double sequences and strong regularity of summability matrices*, Math. Proc. Camb. Phil. Soc., **104** (1988), 283-294.
- [17] M. Mursaleen and O. H. H. Edely, *Statistical convergence of double sequences*, J. Math. Anal. Appl., **288**(1), (2003), 223-231.
- [18] M. Mursaleen and O. H. H. Edely, *Almost convergence and a core theorem for double sequences*, J. Math. Anal. Appl., **293**, (2004), 532-540.
- [19] M. Mursaleen and S.A. Mohiuddine, *Invariant mean and some core theorems for double sequences*, Taiwanese J Math, **14** (2010), 21-33.
- [20] M. Mursaleen and S.A. Mohiuddine, *Convergence Methods For Double Sequences and Applications*, New Delhi, India: Springer, 2014.
- [21] M. Mursaleen, *Almost strongly regular matrices and a core theorem for double sequences*, J Math Anal Appl., **293** (2004), 523-531.
- [22] J. Musielak, *Orlicz spaces and modular spaces*, Lecture notes in Mathematics, **1034** (1983).
- [23] K. Raj, A. Azimhan and K. Ashirbayev, *Some generalized difference sequence spaces of ideal convergence and Orlicz functions*, J. Comput. Anal. Appl., **22** (2017), 52-63.
- [24] K. Raj and S. K. Sharma, *Applications of double lacunary sequences to n-norm*, Acta Univ. Sapientiae Mathematica, **7** (2015), 67-88.
- [25] K. Raj, S. Jamwal and S. K. Sharma, *New classes of generalized sequence spaces defined by an Orlicz function*, J. Comput. Anal. Appl., **15** (2013), 730-737.
- [26] H. H. Schaefer, *Topological vector spaces*, Graduate Texts in Mathematics, Vol. 3, 5th printing, 1986.

DOUBLE DIFFERENCE SPACES OF ALMOST NULL AND ALMOST CONVERGENT SEQUENCES11

- [27] B. C. Tripathy and A. Esi, *A new type of difference sequence spaces*, Int. J. Sci. Tech., **1** (2006), 11-14.
- [28] B. C. Tripathy, A. Esi and B. Tripathy, *On a new type of generalized difference Cesàro sequence spaces*, Soochow J. Math., **31** (2005), 333-340.
- [29] M. Yeşilkayagil and Feyzi Başar, *Some topological properties of the spaces of almost null and almost convergent double sequences* Turk. J. Math. **40** (2016), 624-630.
- [30] M. Zeltser, *Investigation of double sequence spaces by soft and hard analytical methods*, Diss. Math. Univ. Tartu., **25**, Tartu University Press, Univ. of Tartu, Faculty of Mathematics and Computer Science, Tartu (2001).

SCHOOL OF MATHEMATICS SHRI MATA VAISHNO DEVI UNIVERSITY, KATRA-182320, J & K (INDIA)

E-mail address: kuldipraj68@gmail.com

E-mail address: renuanand71@gmail.com

ADDITIVE ρ -FUNCTIONAL INEQUALITIES

HARIN LEE, JAE YOUNG CHA, MIN WOO CHO, MYUNGJUN KWON AND CHOONKIL PARK*

ABSTRACT. In this paper, we solve the additive ρ -functional inequalities

$$\|f(2x - y) + f(y - x) - f(x)\| \leq \|\rho(f(x + y) - f(x) - f(y))\|, \tag{0.1}$$

where ρ is a fixed complex number with $|\rho| < 1$, and

$$\|f(x + y) - f(x) - f(y)\| \leq \|\rho(f(2x - y) + f(y - x) - f(x))\|, \tag{0.2}$$

where ρ is a fixed complex number with $|\rho| < \frac{1}{2}$.

Furthermore, we prove the Hyers-Ulam stability of the additive ρ -functional inequalities (0.1) and (0.2) in complex Banach spaces.

1. INTRODUCTION AND PRELIMINARIES

The stability problem of functional equations originated from a question of Ulam [17] concerning the stability of group homomorphisms.

The functional equation $f(x + y) = f(x) + f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [6] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [10] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

The stability of quadratic functional equation was proved by Skof [16] for mappings $f : E_1 \rightarrow E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [3] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. See [2, 4, 7, 8, 9, 11, 12, 13, 14, 15, 18] for more information on the stability problems of functional equations.

In Section 2, we solve the additive ρ -functional inequality (0.1) and prove the Hyers-Ulam stability of the additive ρ -functional inequality (0.1) in complex Banach spaces.

In Section 3, we solve the additive ρ -functional inequality (0.2) and prove the Hyers-Ulam stability of the additive ρ -functional inequality (0.2) in complex Banach spaces.

2010 *Mathematics Subject Classification*. Primary 39B62, 39B72, 39B52.

Key words and phrases. Hyers-Ulam stability; additive ρ -functional inequality.

*Corresponding author.

H. LEE, J. CHA, M. W. CHO, M. KWON AND C. PARK

Throughout this paper, let G be a 2-divisible abelian group. Assume that X is a real or complex normed space with norm $\|\cdot\|$ and that Y is a complex Banach space with norm $\|\cdot\|$.

2. ADDITIVE ρ -FUNCTIONAL INEQUALITY (0.1)

Throughout this section, assume that ρ is a fixed complex number with $|\rho| < 1$.

In this section, we solve and investigate the additive ρ -functional inequality (0.1) in complex Banach spaces.

Lemma 2.1. *If a mapping $f : G \rightarrow Y$ satisfies*

$$\|f(2x - y) + f(y - x) - f(x)\| \leq \|\rho(f(x + y) - f(x) - f(y))\| \tag{2.1}$$

for all $x, y \in G$, then $f : G \rightarrow Y$ is additive.

Proof. Assume that $f : G \rightarrow Y$ satisfies (2.1).

Letting $x = 0$ and $y = 0$ in (2.1), we get $\|f(0)\| \leq \|\rho(f(0))\|$ and so $f(0) = 0$ with $|\rho| < 1$.

Letting $x = 0$ in (2.1), we get $\|f(-y) + f(y)\| \leq 0$ and so f is an *odd mapping*.

Letting $x = z$ and $y = z - w$ in (2.1), we get

$$\|f(z + w) - f(z) - f(w)\| \leq \|\rho(f(2z - w) + f(w - z) - f(z))\| \tag{2.2}$$

for all $z, w \in G$.

It follows from (2.1) and (2.2) that

$$\begin{aligned} \|f(2x - y) + f(y - x) - f(x)\| &\leq \|\rho(f(x + y) - f(x) - f(y))\| \\ &\leq |\rho|^2 \|f(2x - y) + f(y - x) - f(x)\| \end{aligned}$$

and so $f(2x - y) + f(y - x) = f(x)$ for all $x, y \in G$. It is easy to show that f is additive. □

We prove the Hyers-Ulam stability of the additive ρ -functional inequality (2.1) in complex Banach spaces.

Theorem 2.2. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \rightarrow Y$ be a mapping such that*

$$\begin{aligned} \|f(2x - y) + f(y - x) - f(x)\| & \tag{2.3} \\ & \leq \|\rho(f(x + y) - f(x) - f(y))\| + \theta(\|x\|^r + \|y\|^r) \end{aligned}$$

for all $x, y \in X$. Then there exists a unique additive mapping $h : X \rightarrow Y$ such that

$$\|f(x) - h(x)\| \leq \frac{2\theta}{2^r - 2} \|x\|^r \tag{2.4}$$

for all $x \in X$.

ADDITIVE ρ -FUNCTIONAL INEQUALITIES

Proof. Letting $x = y = 0$, in (2.3), we get $\|f(0)\| \leq 0$. So $f(0) = 0$.

Letting $y = 0$ in (2.3), we get

$$\|f(2x) + f(-x) - f(x)\| \leq \theta \|x\|^r \tag{2.5}$$

for all $x \in X$.

Letting $x = 0$ in (2.3), we get

$$\|f(y) + f(-y)\| \leq \theta \|y\|^r \tag{2.6}$$

for all $y \in X$.

From (2.5) and (2.6), we get

$$\begin{aligned} \|f(2x) - 2f(x)\| &\leq \|f(2x) + f(-x) - f(x)\| + \|f(x) + f(-x)\| \\ &\leq 2\theta \|x\|^r \end{aligned} \tag{2.7}$$

for all $x \in X$. So,

$$\left\| f(x) - 2f\left(\frac{x}{2}\right) \right\| \leq \frac{2}{2^r} \theta \|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned} \left\| 2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right) \right\| &\leq \sum_{j=l}^{m-1} \left\| 2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right) \right\| \\ &\leq \frac{2}{2^r} \sum_{j=l}^{m-1} \frac{2^j}{2^{rj}} \theta \|x\|^r \end{aligned} \tag{2.8}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.8) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $h : X \rightarrow Y$ by

$$h(x) := \lim_{n \rightarrow \infty} 2^n f\left(\frac{x}{2^n}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \rightarrow \infty$ in (2.8), we get (2.4).

It follows from (2.3) that

$$\begin{aligned} &\|h(2x - y) + h(y - x) - h(x)\| \\ &= \lim_{n \rightarrow \infty} 2^n \left\| f\left(\frac{2x - y}{2^n}\right) + f\left(\frac{y - x}{2^n}\right) - f\left(\frac{x}{2^n}\right) \right\| \\ &\leq \lim_{n \rightarrow \infty} 2^n |\rho| \left\| f\left(\frac{x + y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) \right\| + \lim_{n \rightarrow \infty} \frac{2^n \theta}{2^{nr}} (\|x\|^r + \|y\|^r) \\ &= |\rho| \|h(x + y) - h(x) - h(y)\| \end{aligned}$$

for all $x, y \in X$. So

$$\|h(2x - y) + h(y - x) - h(x)\| \leq |\rho| \|h(x + y) - h(x) - h(y)\|$$

for all $x, y \in X$. By Lemma 2.1, the mapping $h : X \rightarrow Y$ is additive.

H. LEE, J. CHA, M. W. CHO, M. KWON AND C. PARK

Now, let $T : X \rightarrow Y$ be another additive mapping satisfying (2.4). Then we have

$$\begin{aligned} \|h(x) - T(x)\| &= 2^n \left\| h\left(\frac{x}{2^n}\right) - T\left(\frac{x}{2^n}\right) \right\| \\ &\leq 2^n \left(\left\| h\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right) \right\| + \left\| T\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right) \right\| \right) \\ &\leq \frac{4 \cdot 2^n}{(2^r - 2)2^{nr}} \theta \|x\|^r, \end{aligned}$$

which tends to zero as $n \rightarrow \infty$ for all $x \in X$. So we can conclude that $h(x) = T(x)$ for all $x \in X$. This proves the uniqueness of h . Thus the mapping $h : X \rightarrow Y$ is a unique additive mapping satisfying (2.4). \square

Theorem 2.3. *Let $r < 1$ and θ be positive real numbers, and let $f : X \rightarrow Y$ be a mapping satisfying $f(0) = 0$ and (2.3). Then there exists a unique additive mapping $h : X \rightarrow Y$ such that*

$$\|f(x) - h(x)\| \leq \frac{2\theta}{2 - 2^r} \|x\|^r \tag{2.9}$$

for all $x \in X$.

Proof. It follows from (2.7) that

$$\left\| f(x) - \frac{1}{2}f(2x) \right\| \leq \theta \|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned} \left\| \frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x) \right\| &\leq \sum_{j=l}^{m-1} \left\| \frac{1}{2^j}f(2^j x) - \frac{1}{2^{j+1}}f(2^{j+1} x) \right\| \\ &\leq \sum_{j=l}^{m-1} \frac{2^{rj}}{2^j} \theta \|x\|^r \end{aligned} \tag{2.10}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.10) that the sequence $\{\frac{1}{2^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n}f(2^n x)\}$ converges. So one can define the mapping $h : X \rightarrow Y$ by

$$h(x) := \lim_{n \rightarrow \infty} \frac{1}{2^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \rightarrow \infty$ in (2.10), we get (2.9).

The rest of the proof is similar to the proof of Theorem 2.2. \square

Remark 2.4. If ρ is a real number such that $-1 < \rho < 1$ and Y is a real Banach space, then all the assertions in this section remain valid.

ADDITIVE ρ -FUNCTIONAL INEQUALITIES

3. ADDITIVE ρ -FUNCTIONAL INEQUALITY (0.2)

Throughout this section, assume that ρ is a fixed complex number with $|\rho| < \frac{1}{2}$.

In this section, we solve and investigate the additive ρ -functional inequality (0.2) in complex Banach spaces.

Lemma 3.1. *If a mapping $f : G \rightarrow Y$ satisfies*

$$\|f(x + y) - f(x) - f(y)\| \leq \|\rho(f(2x - y) + f(y - x) - f(x))\| \tag{3.1}$$

for all $x, y \in G$, then $f : G \rightarrow Y$ is additive.

Proof. Assume that $f : G \rightarrow Y$ satisfies (3.1).

Letting $x = y = 0$ in (3.1), we get $\|f(0)\| \leq 0$. So $f(0) = 0$.

Letting $y = x$ in (3.1), we get $\|f(2x) - 2f(x)\| \leq 0$ and so

$$2f(x) = f(2x) \tag{3.2}$$

for all $x \in G$.

Letting $y = 2x$ in (3.1), we get $\|f(3x) - f(x) - f(2x)\| \leq 0$ and from (3.2),

$$3f(x) = f(3x) \tag{3.3}$$

for all $x \in G$.

Letting $y = -x$ in (3.1), we get $\|f(x) + f(-x)\| \leq \|\rho(f(3x) + f(-2x) - f(x))\|$. From (3.2) and (3.3), $f(3x) + f(-2x) - f(x) = 2f(x) + 2f(-x)$, so $\|f(x) + f(-x)\| \leq 0$, and we get

$$f(x) + f(-x) = 0 \tag{3.4}$$

for all $x \in G$. So f is an *odd mapping*.

Letting $x = z, y = z - w$ in (3.1), we get

$$\|f(2z - w) - f(z) - f(z - w)\| \leq \|\rho(f(z + w) + f(-w) - f(z))\|$$

and from (3.4),

$$\|f(2z - w) + f(w - z) - f(z)\| \leq \|\rho(f(z + w) - f(z) - f(w))\| \tag{3.5}$$

for all $z, w \in G$.

It follows from (3.1) and (3.5) that

$$\begin{aligned} \|f(x + y) - f(x) - f(y)\| &\leq \|\rho(f(2x - y) + f(y - x) - f(x))\| \\ &\leq |\rho|^2 \|f(x + y) - f(x) - f(y)\| \end{aligned}$$

and so $f(x + y) = f(x) + f(y)$ for all $x, y \in G$. So f is *additive*. □

We prove the Hyers-Ulam stability of the additive ρ -functional inequality (3.1) in complex Banach spaces.

Theorem 3.2. *Let $r > 1$ and θ be positive real numbers, and let $f : X \rightarrow Y$ be a mapping such that*

$$\begin{aligned} \|f(x + y) - f(x) - f(y)\| & \tag{3.6} \\ & \leq \|\rho(f(2x - y) + f(y - x) - f(x))\| + \theta(\|x\|^r + \|y\|^r) \end{aligned}$$

H. LEE, J. CHA, M. W. CHO, M. KWON AND C. PARK

for all $x, y \in X$. Then there exists a unique additive mapping $h : X \rightarrow Y$ such that

$$\|f(x) - h(x)\| \leq \frac{2\theta}{2^r - 2} \|x\|^r \tag{3.7}$$

for all $x \in X$.

Proof. Letting $x = y = 0$ in (3.4), we get $\|f(0)\| \leq 0$. So $f(0) = 0$.

Letting $y = x$ in (3.4), we get

$$\|f(2x) - 2f(x)\| \leq 2\theta \|x\|^r \tag{3.8}$$

for all $x \in X$. So

$$\begin{aligned} \left\| 2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right) \right\| &\leq \sum_{j=l}^{m-1} \left\| 2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right) \right\| \\ &\leq \frac{2}{2^r} \sum_{j=l}^{m-1} \frac{2^j}{2^{rj}} \theta \|x\|^r \end{aligned} \tag{3.9}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.9) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $h : X \rightarrow Y$ by

$$h(x) := \lim_{n \rightarrow \infty} 2^n f\left(\frac{x}{2^n}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \rightarrow \infty$ in (3.9), we get (3.7).

It follows from (3.6) that

$$\begin{aligned} &\|h(x + y) - h(x) - h(y)\| \\ &= \lim_{n \rightarrow \infty} 2^n \left\| f\left(\frac{x + y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) \right\| \\ &\leq \lim_{n \rightarrow \infty} 2^n \left\| \rho\left(f\left(\frac{2x - y}{2^n}\right) + f\left(\frac{y - x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right) \right\| + \lim_{n \rightarrow \infty} \frac{2^n \theta}{2^{nr}} (\|x\|^r + \|y\|^r) \\ &= \|\rho(h(2x - y) + h(y - x) - h(x))\| \end{aligned}$$

for all $x, y \in X$. So

$$\|h(x + y) - h(x) - h(y)\| \leq \|\rho(h(2x - y) + h(y - x) - h(x))\|$$

for all $x, y \in X$. By Lemma 3.1, the mapping $h : X \rightarrow Y$ is additive.

Now, let $T : X \rightarrow Y$ be another additive mapping satisfying (3.7). Then we have

$$\begin{aligned} \|h(x) - T(x)\| &= 2^n \left\| h\left(\frac{x}{2^n}\right) - T\left(\frac{x}{2^n}\right) \right\| \\ &\leq 2^n \left(\left\| h\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right) \right\| + \left\| T\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right) \right\| \right) \\ &\leq \frac{2 \cdot 2^n \cdot 2}{(2^r - 2) 2^{nr}} \theta \|x\|^r, \end{aligned}$$

ADDITIVE ρ -FUNCTIONAL INEQUALITIES

which tends to zero as $n \rightarrow \infty$ for all $x \in X$. So we can conclude that $h(x) = T(x)$ for all $x \in X$. This proves the uniqueness of h . Thus the mapping $h : X \rightarrow Y$ is a unique additive mapping satisfying (3.7). \square

Theorem 3.3. *Let $r < 1$ and θ be positive real numbers, and let $f : X \rightarrow Y$ be a mapping satisfying (3.4). Then there exists a unique additive mapping $h : X \rightarrow Y$ such that*

$$\|f(x) - h(x)\| \leq \frac{2\theta}{2 - 2^r} \|x\|^r \tag{3.10}$$

for all $x \in X$.

Proof. It follows from (3.8) that

$$\left\| f(x) - \frac{1}{2} f(2x) \right\| \leq \theta \|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned} \left\| \frac{1}{2^l} f(2^l x) - \frac{1}{2^m} f(2^m x) \right\| &\leq \sum_{j=l}^{m-1} \left\| \frac{1}{2^j} f(2^j x) - \frac{1}{2^{j+1}} f(2^{j+1} x) \right\| \\ &\leq \sum_{j=l}^{m-1} \frac{2^{rj}}{2^j} \|x\|^r \end{aligned} \tag{3.11}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.11) that the sequence $\{\frac{1}{2^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n} f(2^n x)\}$ converges. So one can define the mapping $h : X \rightarrow Y$ by

$$h(x) := \lim_{n \rightarrow \infty} \frac{1}{2^n} f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \rightarrow \infty$ in (3.11), we get (3.10).

The rest of the proof is similar to the proof of Theorem 3.2. \square

Remark 3.4. If ρ is a real number such that $-\frac{1}{2} < \rho < \frac{1}{2}$ and Y is a real Banach space, then all the assertions in this section remain valid.

REFERENCES

- [1] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64–66.
- [2] A. Chahbi and N. Bounader, *On the generalized stability of d'Alembert functional equation*, J. Nonlinear Sci. Appl. **6** (2013), 198–204.
- [3] P. W. Cholewa, *Remarks on the stability of functional equations*, Aequationes Math. **27** (1984), 76–86.
- [4] G. Z. Eskandani and P. Găvruta, *Hyers-Ulam-Rassias stability of peroxidized Cauchy functional equation in 2-Banach spaces*, J. Nonlinear Sci. Appl. **5** (2012), 459–465.
- [5] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.
- [6] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. U.S.A. **27** (1941), 222–224.

H. LEE, J. CHA, M. W. CHO, M. KWON AND C. PARK

- [7] C. Park, *Orthogonal stability of a cubic-quartic functional equation*, J. Nonlinear Sci. Appl. **5** (2012), 28–36.
- [8] C. Park, K. Ghasemi, S. G. Ghaleh and S. Jang, *Approximate n -Jordan $*$ -homomorphisms in C^* -algebras*, J. Comput. Anal. Appl. **15** (2013), 365–368.
- [9] C. Park, A. Najati and S. Jang, *Fixed points and fuzzy stability of an additive-quadratic functional equation*, J. Comput. Anal. Appl. **15** (2013), 452–462.
- [10] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.
- [11] K. Ravi, E. Thandapani and B. V. Senthil Kumar, *Solution and stability of a reciprocal type functional equation in several variables*, J. Nonlinear Sci. Appl. **7** (2014), 18–27.
- [12] S. Shagholi, M. Bavand Savadkouhi and M. Eshaghi Gordji, *Nearly ternary cubic homomorphism in ternary Fréchet algebras*, J. Comput. Anal. Appl. **13** (2011), 1106–1114.
- [13] S. Shagholi, M. Eshaghi Gordji and M. Bavand Savadkouhi, *Stability of ternary quadratic derivation on ternary Banach algebras*, J. Comput. Anal. Appl. **13** (2011), 1097–1105.
- [14] D. Shin, C. Park and Sh. Farhadabadi, *On the superstability of ternary Jordan C^* -homomorphisms*, J. Comput. Anal. Appl. **16** (2014), 964–973.
- [15] D. Shin, C. Park and Sh. Farhadabadi, *Stability and superstability of J^* -homomorphisms and J^* -derivations for a generalized Cauchy-Jensen equation*, J. Comput. Anal. Appl. **17** (2014), 125–134.
- [16] F. Skof, *Propriet locali e approssimazione di operatori*, Rend. Sem. Mat. Fis. Milano **53** (1983), 113–129.
- [17] S. M. Ulam, *A Collection of the Mathematical Problems*, Interscience Publ. New York, 1960.
- [18] C. Zaharia, *On the probabilistic stability of the monomial functional equation*, J. Nonlinear Sci. Appl. **6** (2013), 51–59.

HARIN LEE, JAE YOUNG CHA, MIN WOO CHO, MYUNGJUN KWON
 MATHEMATICS BRANCH, SEOUL SCIENCE HIGH SCHOOL, SEOUL 03066, REPUBLIC OF KOREA
E-mail address: harinboy@naver.com; ckwodud99@naver.com; whalsdn9903@naver.com

CHOONKIL PARK
 RESEARCH INSTITUTE FOR NATURAL SCIENCES, HANYANG UNIVERSITY, SEOUL 04763, REPUBLIC OF KOREA
E-mail address: baak@hanyang.ac.kr

TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 24, NO. 4, 2018

Zeros of functions in weighted Dirichlet spaces and Carleson type measure, Ruishen Qian, Songxiao Li, and Yanhua Zhang,.....	609
Smarandache fuzzy BCI-algebras, Sun Shin Ahn and Young Joo Seo,.....	619
On Fibonacci derivative equations, Hee Sik Kim, J. Neggers, and Keum Sook So,.....	628
A note on symmetric identities for twisted Daehee polynomials, Jongkyum Kwon and Jin-Woo Park,.....	636
Dynamics and Behavior of $x_{n+1} = ax_n + bx_{n-1} + \frac{\alpha + cx_{n-2}}{\beta + dx_{n-2}}$, M. M. El-Dessoky and Aatef Hobiny,.....	644
Algebraic and Order Properties of Tracy-Singh Products for Operator Matrices, Arnon Ploymukda, Pattawut Chansangiam, and Wicharn Lewkeeratiyutkul,.....	656
Analytic Properties of Tracy-Singh Products for Operator Matrices, Arnon Ploymukda, Pattawut Chansangiam, and Wicharn Lewkeeratiyutkul,.....	665
On the radial distribution of Julia set of solutions of $f'' + Af' + Bf = 0$, Jianren Long,.....	675
Common Fixed Point Results for the Family of Multivalued Mappings Satisfying Contractions on a Sequence in Hausdorff Fuzzy Metric Space, Abdullah Shoaib, Akbar Azam, and Aqeel Shahzad,.....	692
Additive-quadratic ρ -functional inequalities in non-Archimedean Banach spaces, Choonkil Park, Jung Rye Lee, and Dong Yun Shin,.....	700
Differential equations associated with the generalized Euler polynomials of the second kind, C. S. Ryou,.....	711
Chlodowsky Variant of Bernstein-Schurer Operators Based on (p,q)-Integers, Eser Gemikonakli and Tuba Vedi-Dilek,.....	717
Dynamical behavior of a general HIV-1 infection model with HAART and cellular reservoirs, Ahmed M. Elaiw, Abdullah M. Althiabi, Mohammed A. Alghamdi and Nicola Bellomo,....	728

**TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL
ANALYSIS AND APPLICATIONS, VOL. 24, NO. 4, 2018**

(continued)

Ideal theory of pre-logics based on the theory of falling shadows, Young Bae Jun and Sun Shin Ahn,.....	744
Quadratic ρ -functional equations in non-Archimedean Banach spaces, Choonkil Park, Gang Lu, Yinhua Cui, and Ming Fang,.....	752
The Incomplete Global GMERR Algorithm to Solve $AX = B^*$, Yu-Hui Zheng, Jian-Lei Li, Dong-Xu Cheng, and Ling-Ling Lv,.....	760
Double difference spaces of almost null and almost convergent sequences for Orlicz function, Kuldip Raj and Renu Anand,.....	773
Additive ρ -functional inequalities, Harin Lee, Jae Young Cha, Min Woo Cho, Myungjun Kwon and Choonkil Park,.....	784