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Global existence and blow-up of solutions to strongly damped wave equations with nonlinear degenerate damping and source terms *

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Abstract This paper deals with the initial-boundary value problem of a class of strongly damped wave equations with nonlinear degenerate damping and source terms. By potential well theory, the global existence of weak solutions is proved if the initial data enter into the stable set. By Nakao inequality, the asymptotic behavior is obtained. Moreover, by establishing a new second order differential inequality, we prove a finite-time blow-up result under arbitrary positive initial energy.

Keywords strong damped wave equations; global existence; asymptotic behavior; blow-up

AMS Classification (2010): 35L35,35B40,35B44.

1 Introduction

In this paper, we are concerned with the following initial boundary value problem:

$$u_{tt} - \Delta u_{tt} - \Delta u_t - \Delta u + (|u|^k u)_t = |u|^q u, \ (x,t) \in \Omega \times (0,T), \tag{1.1}$$

$$u(x,t) = 0, \ x \in \partial\Omega, t \in (0,T), \tag{1.2}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \ x \in \Omega, \tag{1.3}$$

where Ω is a bounded domain in R^n with sufficiently smooth boundary $\partial\Omega$.

Evolution equation (1.1) is a simple prototype of the more general equation

$$\frac{\partial^2}{\partial t^2}(\triangle u - u) + \frac{\partial}{\partial t}(\triangle u - g(x,u)) + \triangle u + f(x,u) = 0, \tag{1.4}$$

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which describes ion acoustic waves in a plasma taking account of strong nonlinear dissipation and nonlinear sources [1, 2], where $f(x,u)$ and $g(x,u)$ describe the distribution of the sources of bound charges and the 'sinks' of free charges, respectively. Korpusov [2] proved that for any initial data in $H_0^1(\Omega)$ the problem (1.4), (1.2), (1.3) has a local strong generalized solution and obtained sufficient conditions for the blow-up of a solution in finite time provided that

$$\int_\Omega [|\nabla u_0|^2 + |\nabla u_1|^2 + |u_1|^2]dx - 2\int_\Omega \int_0^{u_0(x)} f(x,s)dsdx < 0. \tag{1.5}$$

Korpusov [3] gave sufficient conditions for finite-time blow-up of solutions of the following abstract Cauchy problem for a formally hyperbolic equation with double non-linearity

$$A\frac{d^2u}{dt^2} + \frac{du}{dt}(A_0 u + \sum_{j=1}^n A_j(u)) + H_f'(u) = F_f'(u).$$

As far as we know, there is little information about the equation (1.4).

It is worth noting here that if the damping terms $(|u|^k u)_t$ is absent, the problem (1.1)-(1.3) is studied extensively in the literature. For example, Shang [4, 5] investigated the existence, uniqueness, asymptotic behavior, and the blow up phenomenon of the solutions under some specific assumptions on f for the fourth-order wave equation

$$u_{tt} - \Delta u - \Delta u_t - \Delta u_{tt} = f(u). \tag{1.6}$$

For the initial boundary value problem of equation (1.6), Zhang and Hu [6] proved the existence and the stability of global weak solution. Xie and Zhong [7] established the existence of global attractors. Xu et al. [8] investigated the asymptotic behavior of solutions by using the multiplier method. For more results on the long-time behaviors of global strong solutions of the initial boundary value problem of equation (1.6), the reader is referred to [9, 10, 11, 12, 13, 14, 15, 16].

We mention also that, recently, in the absence of dispersive term and strong damped term (i.e. the terms Δu_{tt} and $-\Delta u_t$ are absence), equation (1.1) can be take the more general form

$$u_{tt} - \Delta u + |u|^k \partial j(u_t) = |u|^q u. \tag{1.7}$$

Under suitable conditions on j and the parameters in equation (1.7), Barbuet al. [17] established the existence and uniqueness of global weak solutions. They also obtained a nonexistence result of global solutions with negative initial energy on condition that q is greater than the critical value. In [18], the same authors established the blow-up result for the generalized solution with additional regularity for equation (1.7) under more restrictions on j provided that q is greater than the critical value and the initial energy is negative. A special case of equation (1.7) is the following polynomial-damped wave equation

$$u_{tt} - \Delta u + |u|^k |u_t|^{m-1} u_t = |u|^q u, \tag{1.8}$$

which has been studied extensively in the literature. We refer the reader to [19, 20, 21, 22, 23, 24, 25, 26, 27] and the references therein. But much less work is known for the initial boundary problem for a wave equation with degenerate damping term $(|u|^k u)_t$, dispersive term Δu_{tt}, strong damping term Δu_t and nonlinear source term $|u|^q u$.

In this paper, we prove the global existence of weak solutions for problem (1.1)-(1.3) by potential well theory [28] if the initial data enter into the stable set. By Nakao's inequality [29], the asymptotic behavior is obtained, and this method is different from that in [22, 26]. Moreover, we will extend the results in [2, 3] that the sufficient conditions for finite-time blow-up of solutions with negative initial energy to that with positive initial energy. We will establish a different differential inequality and prove a finite-time blow-up result under arbitrary positive initial energy. This different differential inequality and blow-up result are, to the best of our knowledge, new in the literature. The concavity method of Levine [30] is one of the most powerful methods for proving finite time blow-up of the solutions to nonlinear wave equations. The main idea of Levine is to replace the investigations of the equation with the study of ordinary differential inequality

$$\phi\phi'' - \alpha(\phi')^2 + \beta\phi^2 \geq 0, \alpha > 1, \beta \geq 0.$$

Later on, the generalization of this inequality

$$\phi\phi'' - \alpha(\phi')^2 + \beta\phi^2 + \gamma\phi\phi' \geq 0, \alpha > 1, \beta \geq 0, \gamma \geq 0,$$

was obtained in [31]. On the other hand, as pointed in [2], Levine's method as presented in [30, 31] cannot be used here due to the term $-\Delta u_t + (|u|^k u)_t$, so a different differential inequality was used to prove Theorem 3 in [3] for negative initial energy case (i.e. in the case of (1.5))

$$\phi\phi'' - \alpha(\phi')^2 + \beta\phi^2 + \gamma\phi^{2+q_1} \geq 0, \alpha > 1, \beta \geq 0, \gamma \geq 0, q_1 \geq 0.$$

But the above inequality cannot be used to the positive initial energy case. In this paper, we will establish a new differential inequality and prove a finite-time blow-up result under arbitrary positive initial energy.

This article is organized as follows. In Section 2, we are concerned with some notations and state our main results. Following the potential well theory introduced by [28], we get global existence in Section 3. Section 3 gives also an asymptotic stability results of the problem (1.1)-(1.3). In section 4, it is shown that the weak solution of the problem (1.1) -(1.3) blow-up in the case of positive initial energy $E(0) > 0$ and $q > k$.

2 Preliminaries

In this section we present some notations and state our main results. We use the standard Lebesgue space $L^p(\Omega)(1 \leq p \leq \infty)$ and Sobolev space $H_0^1(\Omega)$. We denote by $||u||_p$ the $L^p(\Omega)$

norm and by $||\nabla\cdot||$ the norm in $H_0^1(\Omega)$. Moreover, for later use we denote by $(\cdot,\cdot)$ the duality paining between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$. In this paper, we will always assume that

$$0<q<\infty,\quad if\ n=1,\ 2;\qquad 0<q<\frac{4}{n-2},\quad if\ n\geq 3,\tag{2.1}$$

and then the Sobolev imbedding $H_0^1(\Omega)\hookrightarrow L^{q+2}(\Omega)$ holds. Furthermore, we denote C_* is the embedding constant, that is

$$||\nabla u||\leq C_*||u||_{q+2}.$$

The constants C_i $(i=1,2,\cdots)$ used throughout this paper are positive generic constants, which may be different in various occurrences.

Now, we give the definition of a weak solution to problem (1.1)-(1.3).

A weak solution to the initial boundary value problem (1.1)-(1.3) over $\Omega\times[0,T)$ is a function $u\in L^\infty(0,T;H_0^1(\Omega))$ with $u_t,u_{tt}\in L^\infty(0,T;H_0^1(\Omega))$ such that

$$(u_{tt},\varphi)+(\nabla u_{tt},\nabla\varphi)+(\nabla u,\nabla\varphi)+(\nabla u_t,\nabla\varphi)+((|u|^ku)_t,\varphi)=\int_\Omega u|u|^q\varphi dx,$$

for all test functions $\varphi\in H_0^1(\Omega)$ and for almost all $t\in[0,T)$ and

$$u(x,0)=u_0\in H_0^1(\Omega),u_t(x,0)=u_1\in H_0^1(\Omega).$$

We introduce the following functionals:

$$I(t)=I(u)=||\nabla u||^2-||u||_{q+2}^{q+2},\tag{2.2}$$

$$J(t)=J(u)=\frac{1}{2}||\nabla u||^2-\frac{1}{q+2}||u||_{q+2}^{q+2},\tag{2.3}$$

$$E(t)=E(u)=\frac{1}{2}||u_t||^2+\frac{1}{2}||\nabla u_t||^2+J(u),\tag{2.4}$$

and the level

$$d=\inf_{u\in H_0^1(\Omega)\backslash\{0\}}\max_{\lambda\geq 0}J(\lambda u).$$

Moreover the value d is shown to be the Mountain Pass level associated to the underlying Dirichlet problem $-\Delta u=|u|^qu$ in Ω, $u=0$ on $\partial\Omega$ [32].

Remark 2.1 By multiplying Equation (1.1) by u_t, integrating over Ω, and using integration by parts, we get

$$E'(t)=-||\nabla u_t||^2-(k+1)\int_\Omega |u|^k|u_t|^2dx\leq 0,\qquad for\ t\geq 0.\tag{2.5}$$

Therefore,

$$E(t)\leq E(0),\qquad for\ t\geq 0.\tag{2.6}$$

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We can now define the stable set [32, 33, 34]

$$W=\{u\in H_0^1|I(u)>0,J(u)<d\}$$

and one can easily see that the stable set can also be defined by

$$W=\{(\lambda,E)\in[0,+\infty)\times R:0<g(\lambda)\leq E<d,0<\lambda<\lambda_0\},$$

where $g(\lambda)=\frac{1}{2}\lambda^2-C_*^{q+2}\frac{\lambda^{q+2}}{q+2}$, $\lambda_0=C_*^{-\frac{q+2}{q}}$ is the absolute maximum point of g, and finally $d=g(\lambda_0)=(\frac{1}{2}-\frac{1}{q+2})\lambda_0^2>0$.

In order to get the energy decay of the solution, we introduce the following set

$$W_1=\{u\in H_0^1|I(u)>0,J(u)<E_1\},$$

where $\lambda_1=((q+2)C_*^{q+2})^{-\frac{1}{q}},E_1=(\frac{1}{2}-\frac{1}{q+2})\lambda_1^2$. Obviously, $d>E_1$ and $W_1\subset W$.

Our main results read as follows. The first result is concerned with the global existence of weak solutions to the problem (1.1)-(1.3). Namely, we have the following theorem.

Theorem 2.1 Let $k>1,\ if\ n=1,\ 2;\ k<\frac{2n}{n-2},\ if\ n\geq 3;\ u_0,u_1\in H_0^1(\Omega)$, assuming that $k>q,E(0)<d$ and $u_0\in W$, then the problem (1.1)-(1.3) admits a global weak solution u and $u(\cdot)\in W$ for $t\geq 0$.

The second result is about the asymptotic stability results of the weak solutions.

Theorem 2.2 Under the assumptions of Theorem 2.1, $k>q$ and $u_0\in W_1$, and $||\nabla u_0||<\lambda_1,E(0)<E_1$, then there exist positive constant α such that the energy $E(t)$ satisfies the energy estimates

$$E(t)\leq E(0)\exp\{-\alpha[t-1]^+\}\qquad for\ large\ t,$$

where $[t-1]^+=\max\{t-1,0\}$.

Remark 2.2 Let us mention that the special polynomial form of the dissipation and source terms in (1.1)-(1.3) is not essential. The results can be extend to the case of more general nonlinearities under suitable assumptions.

Our final result provides a finite time blowup property of the weak solutions to problem (1.1)-(1.3).

Theorem 2.3 Assume that $k<q$. If the initial data are such that

$$E(0)>0, \tag{2.7}$$

$$(\nabla u_0,\nabla u_1)+(u_0,u_1)>0, \tag{2.8}$$

and

$$(\phi'(0))^2-\frac{\beta_0}{\alpha_0-1}\phi^2(0)+\frac{2\gamma_0}{(\alpha_0-1)\delta_1'}\phi^{2\alpha_0+\delta_1'(1-\alpha_0)}(0)$$
$$+\frac{2\delta_0}{(\alpha_0-1)\delta_2'}\phi^{2\alpha_0+\delta_2'(1-\alpha_0)}(0)=B_0>0, \tag{2.9}$$

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where $\alpha_0 = 2 + \frac{k}{2}$, $\beta_0 = \frac{2}{q-k}$, $\gamma_0 = \frac{2(k+1)^2 C_0^{2(k+1)}}{q-k}$, $\delta_0 = (q+2)E(0)$, $\delta_1' = \frac{2\alpha_0-2-k}{\alpha_0-1}$, $\delta_2' = \frac{2\alpha_0-1}{\alpha_0-1}$ and C_0 is embedding constant from H_0^1 to $L^{2(k+1)}$, then there exists

$$T_\infty \le \phi^{1-\alpha_0}(0)A^{-1} \quad such\ that \quad \lim_{t\to T_\infty^-} \phi(t) = \infty,$$

where $A^2 = (\alpha_0-1)^2\phi^{-2\alpha_0}(0)B_0$ and $\phi(t) = \frac{1}{2}||\nabla u||^2 + \frac{1}{2}||u||^2$.

Finally, we state the local existence result of problem (1.1)-(1.3).

Theorem2.4 Let $u_0, u_1 \in H_0^1(\Omega)$, then problem (1.1)-(1.3) has a unique weak solution on $[0, T_0)$ for some $T_0 > 0$, and we have either $T_0 = +\infty$ or $T_0 < +\infty$ and

$$\lim_{t\to T_0+} sup[||u_t||^2_{H_0^1(\Omega)} + ||u||^2_{H_0^1(\Omega)}] = +\infty.$$

This lemma can be established by combining the arguments of Theorem 8.1 (or Theorem 6.8) and Example 9.5 in [3], Theorem 2 in [2] and [35], so we omit it.

3 Global existence and asymptotic stability of the solutions

In this section we study the existence and asymptotic stability of global solutions for problem (1.1)-(1.3). We start by the following lemma.

Lemma 3.1 Suppose that u is the solution of problem (1.1)-(1.3), and $u_0, u_1 \in H_0^1$, if $u_0 \in W$ and $E(0) < d$, then $u(t)$ remains inside the set $\in W$ for any $t \ge 0$.

The proof is similar to that of Lemma 2.2 in [33], so we omit it.

Proof of Theorem 2.1. By Lemma 3.1, we have $u(t) \in W$ for all $t \in [0, T_0)$, then $I(u) > 0, J(u) < d$ for all $t \in [0, T_0)$. Therefore,

$$(\frac{1}{2} - \frac{1}{q+2})||u_m||_{q+2}^{q+2} = \frac{1}{2}||\nabla u||^2 - \frac{1}{q+2}||u||_{q+2}^{q+2} - \frac{1}{2}I(u) \le J(u) < d, \tag{3.1}$$

then

$$||u||_{q+2}^{q+2} < d. \tag{3.2}$$

By the energy equation (2.6), definition of $J(u)$ and (3.1), we arrive

$$\frac{1}{2}||u_t||^2 + \frac{1}{2}||\nabla u_t||^2 + \frac{1}{2}||\nabla u||^2 \le E(0) + \frac{1}{q+2}||u||_{q+2}^{q+2} \le Cd, \quad for\ 0 \le t < T_0, \tag{3.3}$$

It follows from (3.3) and from a standard continuous argument that the local solution u furnished by Theorem 2.4 can be extended to the whole interval $[0, +\infty)$, that is, u is a global solution. Finally from Lemma 3.1 we get $u \in W$ for $t \in [0, \infty)$.

In order to get the energy decay of the solution, we prepare the following lemma.

Lemma 3.2[29] Let $\varphi(x)$ be a nonnegative and non-increasing function defined on $[0, \infty)$, satisfying

$$\varphi^{1+r}(t) \le k_0(\varphi(t) - \varphi(t+1)), \qquad t \in [0, T],$$

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for $k_0 > 1$, and $r \geq 0$. Then we have, for each $t \in [0, T]$,

$$\varphi(t) \leq \varphi(0)\exp(-\alpha[t-1]^+), \qquad if\ r = 0,$$
$$\varphi(t) \leq (\varphi(0)^{-r} + k_0 r[t-1]^+)^{-\frac{1}{r}}, \qquad if\ r > 0,$$

where $[t-1]^+ = \max\{t-1, 0\}$, and $\alpha = \ln(\frac{k_0}{k_0-1})$.

Adapting the idea of Vitillaro[34], we have the following lemma.

Lemma 3.3 Suppose that u is the solution of problem (1.1)-(1.3), and $u_0, u_1 \in H_0^1$, if $u_0 \in W_1$ and $||\nabla u_0|| < \lambda_1, E(0) < E_1$, then $u(t)$ remains inside the set W_1 and $||\nabla u|| < \lambda_1, E(t) < E_1$ for any $t \geq 0$.

Lemma 3.4 Under the condition of Theorem 2.2. and $q > 0$, then, for $t \geq 0$,

$$||\nabla u||^2 \geq 2||u||_{q+2}^{q+2}. \tag{3.4}$$

$$E(t) \geq \frac{q+1}{2(q+2)}||\nabla u||^2 \geq \frac{q+1}{q+2}||u||_{q+2}^{q+2}, \tag{3.5}$$

Proof By the definition $E(t)$ and embedding theorem, we have

$$\begin{aligned} E(t) &\geq \tfrac{1}{2}||\nabla u||^2 - \tfrac{1}{q+2}||u||_{q+2}^{q+2} \geq \tfrac{1}{2}||\nabla u||^2 - ||u||_{q+2}^{q+2} \\ &\geq \tfrac{1}{2}||\nabla u||^2 - C_*^{q+2}||\nabla u||^{q+2} = g_1(||\nabla u||). \end{aligned} \tag{3.6}$$

where $g_1(\lambda) = \frac{1}{2}\lambda^2 - C_*^{q+2}\lambda^{q+2}$. Note that $g_1(\lambda)$ has the maximum at $\lambda_1 = ((q+2)C_*^{q+2})^{-\frac{1}{q}}$ and the maximum value $g_1(\lambda_1) = E_1$. We see that $g_1(\lambda)$ is increasing in $(0, \lambda_1)$, decreasing in $(\lambda_1, +\infty)$ and $g_1(\lambda) \to -\infty$ as $\lambda \to \infty$. Since $||\nabla u_0||^2 < \lambda_1, E(0) < E_1$,then $||\nabla u||^2 < \lambda_1$, for any $t \geq 0$, so $g_1(||\nabla u||) \geq 0$. By (3.6), we have

$$\begin{aligned} ||\nabla u||^2 - ||u||_{q+2}^{q+2} &= \tfrac{1}{2}||\nabla u||^2 + (\tfrac{1}{2}||\nabla u||^2 - ||u||_{q+2}^{q+2}) \\ &\geq \tfrac{1}{2}||\nabla u||^2 + g_1(||\nabla u||), \end{aligned}$$

then (3.4) holds since $g_1(||\nabla u||) > 0$. Furthermore, we have

$$E(t) \geq \frac{1}{2}||\nabla u||^2 - \frac{1}{q+2}||u||_{q+2}^{q+2} \geq \frac{q+1}{2(q+2)}||\nabla u||^2.$$

So (3.5) hold.

Proof of Theorem 2.2 From (2.5), we know that and $E(t)$ is nonincreasing. Setting $F(t) = \sqrt{E(t) - E(t+1)}$, then we have

$$F^2(t) = \int_t^{t+1}[||\nabla u_s||^2 + (k+1)\int_\Omega |u|^k|u_s|^2]ds \geq \int_t^{t+1}||\nabla u_s||^2 ds, \tag{3.7}$$

$$F^2(t) \geq (k+1)\int_t^{t+1}\int_\Omega |u|^k|u_s|^2 ds. \tag{3.8}$$

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Applying the mean value theorem in (3.7), there exists $t_1 \in [t, t+\frac{1}{4}]$ and $t_2 \in [t+\frac{3}{4}, t+1]$ such that

$$||\nabla u_t(t_i)||^2 \leq 2F^2(t), \qquad i = 1, 2. \tag{3.9}$$

Multiplying u in (1.1), intergrating over $[t_1, t_2] \times \Omega$, we have

$$\begin{aligned}
&\int_{t_1}^{t_2} [||\nabla u||^2 - ||u||_{q+2}^{q+2}] dt \\
&= -\int_\Omega u_t u|_{t=t_1}^{t=t_2} dx - \int_\Omega \nabla u_t \nabla u|_{t=t_1}^{t=t_2} dx + \int_{t_1}^{t_2} (||u_t||^2 + ||\nabla u_t||^2) dt \\
&- \int_{t_1}^{t_2} \int_\Omega \nabla u \nabla u_t dx dt - (k+1) \int_{t_1}^{t_2} \int_\Omega |u|^k u u_t dx dt \\
&= \sum_{i=1}^{5} M_i.
\end{aligned} \tag{3.10}$$

Now we estimate the terms of the right-hand side of (3.10). By Hölder inequality, Poincaré inequality, (3.9), Lemma 3.4, the fact that the $E(t)$ is non-increasing, and Young inequality with $\varepsilon > 0$, we have

$$\begin{aligned}
|M_1| &= |-\int_\Omega u_t u|_{t=t_1}^{t=t_2} dx| \leq \sum_{i=1}^{2} ||u_t(t_i)|| ||u(t_i)|| \\
&\leq \sum_{i=1}^{2} C_2 ||u_t(t_i)|| ||\nabla u(t_i)|| \leq C_3 E^{\frac{1}{2}}(t) F(t) \\
&\leq C_1(\varepsilon) F^2(t) + \varepsilon E(t), \qquad (3.11) \\
|M_2| &= |-\int_\Omega \nabla u_t \nabla u|_{t=t_1}^{t=t_2} dx| \\
&\leq \sum_{i=1}^{2} ||\nabla u_t(t_i)|| ||\nabla u(t_i)|| \leq C_4 E^{\frac{1}{2}}(t) F(t) \\
&\leq C_2(\varepsilon) F^2(t) + \varepsilon E(t). \qquad (3.12)
\end{aligned}$$

By Poincaré inequality and (3.7), we have

$$|M_3| = |\int_{t_1}^{t_2} (||u_t||^2 + ||\nabla u_t||^2) dt| \leq C_5 \int_t^{t+1} ||\nabla u_s||^2 ds \leq C_5 F^2(t). \tag{3.13}$$

From Lemma 3.4 and the fact that the $E(t)$ is non-increasing, we arrive

$$\begin{aligned}
|M_4| = |\textstyle\int_{t_1}^{t_2} \nabla u \nabla u_s ds| &\leq \textstyle\int_t^{t+1} [C_3(\varepsilon) ||\nabla u_s||^2 + \varepsilon \frac{2(q+2)}{q+1} ||\nabla u||^2] ds \\
&\leq C_3(\varepsilon) F^2(t) + \varepsilon E(t).
\end{aligned} \tag{3.14}$$

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According to Hölder inequality, embedding theorem, the assumption $k > q$ and the Lemma3.2, the fact that the $E(t)$ is non-increasing and Young inequality, we have

$$\begin{aligned}|M_5| &\leq (k+1)\int_{t_1}^{t_2}\int_\Omega |u|^{k+1}|u_t|dxdt\\ &= (k+1)\int_{t_1}^{t_2}\int_\Omega |u|^{\frac{k}{2}}|u_t||u|^{\frac{k+2}{2}}dxdt\\ &\leq (k+1)(\int_{t_1}^{t_2}\int_\Omega |u|^k|u_t|^2dxdt)^{\frac{1}{2}}(\int_{t_1}^{t_2}\int_\Omega |u|^{k+2}dxdt)^{\frac{1}{2}}\\ &\leq C_7F(t)(\int_{t_1}^{t_2}||u||_{k+2}^{k+2}dt)^{\frac{1}{2}}\\ &\leq C_9F(t)(\int_{t_1}^{t_2}||\nabla u||^k||\nabla u||^2dt)^{\frac{1}{2}}\\ &\leq C_{10}F(t)(\int_{t_1}^{t_2}E(t)dt)^{\frac{1}{2}}\\ &\leq C_4(\varepsilon)F^2(t)+\varepsilon E(t).\end{aligned} \tag{3.15}$$

Substituting (3.11)-(3.15) into (3.10), we get the estimate

$$\int_{t_1}^{t_2}[||\nabla u||^2 - ||u||_{q+2}^{q+2}]dt \leq C_5(\varepsilon)F^2(t) + 4\varepsilon E(t). \tag{3.16}$$

On the other hand, it follows from the definition of $E(t)$ and (3.5) in Lemma 3.4 that

$$\begin{aligned}E(t) &= \frac{1}{2}(||\nabla u||^2 - ||u||_{q+2}^{q+2}) + \frac{q}{2(q+2)}||u||_{q+2}^{q+2} + \frac{1}{2}||u_t||^2 + \frac{1}{2}||\nabla u_t||^2\\ &\leq \frac{1}{2}(||\nabla u||^2 - ||u||_{q+2}^{q+2}) + \frac{1}{2}||u_t||^2 + \frac{1}{2}||\nabla u_t||^2 + \frac{q}{2(q+1)}E(t).\end{aligned} \tag{3.17}$$

Then we have

$$\frac{q+2}{2(q+1)}E(t) \leq \frac{1}{2}(||\nabla u||^2 - ||u||_{q+2}^{q+2}) + \frac{1}{2}||u_t||^2 + \frac{1}{2}||\nabla u_t||^2. \tag{3.18}$$

Therefore, by (3.18), (3.16) and (3.13), we arrive that

$$\begin{aligned}\int_{t_1}^{t_2} E(s)ds &\leq \tfrac{2(q+1)}{q+2}\int_{t_1}^{t_2}(||\nabla u||^2 - ||u||_{q+2}^{q+2})ds + \tfrac{q+2}{2(q+1)}\int_{t_1}^{t_2}(||u_t||^2 + ||\nabla u_t||^2)ds\\ &\leq C_6(\varepsilon)F^2(t) + 4\varepsilon\tfrac{2(q+1)}{q+2}E(t).\end{aligned} \tag{3.19}$$

Since $E(t)$ is non-increasing, we can choose $t_3 \in [t_1, t_2]$ such that

$$E(t_3) \leq C\int_{t_1}^{t_2}E(s)ds. \tag{3.20}$$

Then, using (2.5), $t_3 < t+1$, it follows from (3.20) and the fact that $E(t)$ is non-increasing that

$$\begin{aligned}E(t) &= E(t+1) + \int_t^{t+1}||\nabla u_s||^2ds + (k+1)\int_t^{t+1}\int_\Omega |u|^k|u_s|^2ds\\ &\leq E(t_3) + \int_t^{t+1}||\nabla u_s||^2ds + (k+1)\int_t^{t+1}\int_\Omega |u|^k|u_s|^2ds\\ &\leq C\int_{t_1}^{t_2}E(s)ds + \int_t^{t+1}||\nabla u_s||^2ds + (k+1)\int_t^{t+1}\int_\Omega |u|^k|u_s|^2ds.\end{aligned} \tag{3.21}$$

Combining (3.21) with (3.19), (3.7) and (3.8), we have

$$E(t) \le C_7(\varepsilon)F^2(t) + 4\varepsilon\frac{2(q+1)}{q+2}E(t). \tag{3.22}$$

Choosing ε sufficiently small, (3.22) leads to

$$E(t) \le C_8F^2(t),$$

where $C_8 = 2(1 + 2C_6 + \frac{C_7^2}{2})$.

Since $E(t)$ is nonincreasing, using Lemma 3.2, we conclude that for each $t \in [0, \infty)$,

$$E(t) \le E(0)\exp(-\alpha[t-1]^+)$$

where $[t-1]^+ = \max\{t-1, 0\}$, and $\alpha = \ln(\frac{C_8}{C_8-1})$. Then the exponential decay of the energy is obtained. The proof of Theorem 2.2 is completed.

4 Blowup of the solutions

In this section our aim is to establish sufficient condition for blow-up of solutions to problem (1.1)-(1.3). We assume that $k < q$ and u be a weak solution to the problem (1.1)-(1.3) on the interval $[0, T]$. We note that the Levine energy method [30] is one of basic methods for studying the blow-up phenomenon. The role of the differential inequality [30]

$$\phi\phi'' - \alpha(\phi')^2 + \beta\phi^2 \ge 0, \alpha > 1, \beta \ge 0,$$

in the standard Levine method is known. The generalization of this inequality

$$\phi\phi'' - \alpha(\phi')^2 + \beta\phi^2 + \gamma\phi\phi' \ge 0, \alpha > 1, \beta \ge 0, \gamma \ge 0,$$

was obtained in [31]. On the other hand, a somewhat different differential inequality was used to prove Theorem 3 in [3]

$$\phi\phi'' - \alpha(\phi')^2 + \beta\phi^2 + \gamma\phi^{2+q_1} \ge 0, \alpha > 1, \beta \ge 0, \gamma \ge 0, q_1 \ge 0.$$

Now, we consider our main differential inequality

$$\phi\phi'' - \alpha(\phi')^2 + \beta\phi^2 + \gamma\phi^{2+l} + \delta\phi \ge 0, \tag{4.1}$$

where $\phi(t) \in C^2([0, T])$ and

$$\alpha > 1, \beta \ge 0, \gamma \ge 0, \delta \ge 0, l \ge 0. \tag{4.2}$$

Lemma 4.1 Suppose $\phi(t) \in C^2([0, T])$. Let conditions (4.1) and (4.2) be satisfied and moreover the following conditions hold:

$$2\alpha > 2 + l, \tag{4.3}$$

$$\phi(t) \ge 0, \phi'(0) > 0, \phi(0) > 0, \tag{4.4}$$

$$(\phi'(0))^2 - \tfrac{\beta}{\alpha-1}\phi^2(0) + \tfrac{2\gamma}{(\alpha-1)\delta_1}\phi^{2\alpha+\delta_1(1-\alpha)}(0) + \tfrac{2\delta}{(\alpha-1)\delta_2}\phi^{2\alpha+\delta_2(1-\alpha)}(0) = B > 0, \tag{4.5}$$

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where $\delta_1 = \frac{2\alpha-2-l}{\alpha-1}$ and $\delta_2 = \frac{2\alpha-1}{\alpha-1}$, then any solution to the differential inequality (4.1) satisfi es the condition

$$T_\infty \le \phi^{1-\alpha}(0)A^{-1}, \lim_{t\to T_\infty^-} \phi(t) = \infty$$

where $A^2 = (\alpha-1)^2\phi^{-2\alpha}(0)B$.

Proof Condition (4.4) imply the existence of a time $t_1 > 0$ for which the inequality $\phi'(t) > 0$ for $t \in [0, t_1)$ holds. Hence, $\phi(t) > \phi(0) > 0$ for $t \in [0, t_1)$. Dividing both sides of (4.1) by $\phi^{1+\alpha}$ we obtain

$$\phi^{-\alpha}\phi'' - \alpha\phi^{-(1+\alpha)}(\phi')^2 + \beta\phi^{1-\alpha} + \gamma\phi^{1+l-\alpha} + \delta\phi^{-\alpha} \ge 0. \tag{4.6}$$

Noting $\frac{d^2}{dt^2}\phi^{1-\alpha} = (1-\alpha)\phi^{-\alpha}\phi'' - \alpha(1-\alpha)\phi^{-(1+\alpha)}(\phi')^2$, from (4.6) it is easy to derive the inequality

$$\frac{1}{1-\alpha}\frac{d^2}{dt^2}\phi^{1-\alpha} + \beta\phi^{1-\alpha} + \gamma\phi^{1+l-\alpha} + \delta\phi^{-\alpha} \ge 0. \tag{4.7}$$

We introduce the new function

$$\psi = \phi^{1-\alpha}, \tag{4.8}$$

then we obtain

$$\frac{1}{1-\alpha}\psi'' + \beta\psi + \gamma\psi^{\frac{1+l-\alpha}{1-\alpha}} + \delta\psi^{\frac{-\alpha}{1-\alpha}} \ge 0. \tag{4.9}$$

Note now that by (4.8), we have

$$\psi'(t) = (1-\alpha)\phi'\phi^{-\alpha}, \tag{4.10}$$

so in view of $\phi'(t) > 0$ and $\alpha > 1$, it follows that

$$\psi'(t) \le 0. \tag{4.11}$$

Now multiplying (4.9) by $\psi'(t)$ we obtain

$$\frac{1}{1-\alpha}\psi'(t)\psi'' + \beta\psi'(t)\psi + \gamma\psi'(t)\psi^{\frac{\alpha-1-l}{\alpha-1}} + \delta\psi'(t)\psi^{\frac{\alpha}{\alpha-1}} \le 0,$$

since $\alpha > 1$, which gives us

$$\psi'(t)\psi'' \ge \beta(\alpha-1)\psi'(t)\psi + \gamma(\alpha-1)\psi'(t)\psi^{\frac{\alpha-1-l}{\alpha-1}} + \delta(\alpha-1)\psi'(t)\psi^{\frac{\alpha}{\alpha-1}}.$$

Hence

$$\frac{1}{2}\frac{d}{dt}(\psi'(t))^2 \ge \frac{\beta(\alpha-1)}{2}\frac{d}{dt}\psi^2 + \frac{\gamma(\alpha-1)}{\delta_1}\frac{d}{dt}\psi^{\delta_1} + \frac{\delta(\alpha-1)}{\delta_2}\frac{d}{dt}\psi^{\delta_2}, \tag{4.12}$$

where $\delta_1 = 1 + \frac{\alpha-1-l}{\alpha-1} = \frac{2\alpha-2-l}{\alpha-1}$, $\delta_2 = 1 + \frac{\alpha}{\alpha-1} = \frac{2\alpha-1}{\alpha-1}$. Since $2\alpha > 2 + l$ and $\alpha > 1$, it follows that $\delta_1 > 0$ and $\delta_2 > 0$.

Integrating (4.12), we obtain that

$$\frac{1}{2}(\psi'(t))^2 \ge \frac{1}{2}(\psi'(0))^2 + \frac{\beta(\alpha-1)}{2}(\psi^2 - \psi^2(0)) + \frac{\gamma(\alpha-1)}{\delta_1}(\psi^{\delta_1} - \psi^{\delta_1}(0)) + \frac{\delta(\alpha-1)}{\delta_2}(\psi^{\delta_2} - \psi^{\delta_2}(0)),$$

then, we obtain the following equivalent inequality

$$(\psi'(t))^2 \geq A^2 + \beta(\alpha-1)\psi^2 + \tfrac{2\gamma(\alpha-1)}{\delta_1}\psi^{\delta_1} + \tfrac{2\delta(\alpha-1)}{\delta_2}\psi^{\delta_2} \geq A^2, \tag{4.13}$$

where

$$A^2 = (\psi'(0))^2 - \beta(\alpha-1)\psi^2(0) - \tfrac{2\gamma(\alpha-1)}{\delta_1}\psi^{\delta_1}(0) - \tfrac{2\delta(\alpha-1)}{\delta_2}\psi^{\delta_2}(0).$$

From (4.8), (4.10) and the initial condition (4.5) we get

$$\begin{aligned} A^2 = &(1-\alpha)^2\phi^{-2\alpha}(0)(\phi'(0))^2 - \beta(\alpha-1)\phi^{2(1-\alpha)}(0) \\ &-\tfrac{2\gamma(\alpha-1)}{\delta_1}\phi^{\delta_1(1-\alpha)}(0) - \tfrac{2\delta(\alpha-1)}{\delta_2}\phi^{\delta_2(1-\alpha)}(0) \\ &= (\alpha-1)^2\phi^{-2\alpha}(0)B > 0. \end{aligned}$$

A further analysis of inequality (4.13) yields

$$|\psi'| \geq A > 0, \forall t \in [0, t_0), \tag{4.14}$$

where t_0 is the life time of the solution. Since $\psi' < 0$ by (4.11), hence,

$$\psi' \leq -A < 0. \tag{4.15}$$

Integrating the inequality (4.15) we obtain

$$\psi(t) \leq \psi(0) - At,$$

in view of (4.8) and $\alpha > 1$, therefore

$$\phi^{1-\alpha}(t) \leq \phi^{1-\alpha}(0) - At.$$

As a result, we obtain the lower estimate

$$\phi(t) \geq (\phi^{1-\alpha}(0) - At)^{\frac{-1}{\alpha-1}},$$

which implies that $T \neq +\infty$, since otherwise there exists $T_\infty < \phi^{1-\alpha}(0)A^{-1}$ such that $\lim\limits_{t\to T_\infty^-}\phi(t) = \infty$. Then the proof is completed.

Proof of Theorem 2.3 We introduce the following notations

$$\phi(t) = \frac{1}{2}||\nabla u||^2 + \frac{1}{2}||u||^2, G(t) = ||\nabla u_t||^2 + ||u_t||^2.$$

If we multiply the equation (1.1) by u, we obtain the following equality:

$$\phi'' - G + (\nabla u_t, \nabla u) + ((|u|^k u)_t, u) + ||\nabla u||^2 = ||u||_{q+2}^{q+2}. \tag{4.16}$$

The third and fourth terms in (4.16) can be estimated as follows:

$$|(\nabla u_t, \nabla u)| \leq \tfrac{\epsilon}{2}||\nabla u_t||^2 + \tfrac{1}{2\epsilon}||\nabla u||^2 \leq \tfrac{\epsilon}{2}G + \tfrac{1}{\epsilon}\phi \tag{4.17}$$

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and

$$|((|u|^k u)_t, u)| \leq (k+1)||u_t||||u||_{2(k+1)}^{k+1}$$
$$\leq \frac{\epsilon}{2}||u_t||^2 + \frac{(k+1)^2 C_0^{2(k+1)}}{2\epsilon}||\nabla u||^{2(k+1)} \leq \frac{\epsilon}{2}G + \frac{(k+1)^2 C_0^{2(k+1)}}{\epsilon}\phi^{k+1} \tag{4.18}$$

for any $\epsilon > 0$.

Hence, by the estimate (4.17) and (4.18), equality (4.16) yields

$$\phi'' - G + \epsilon G + \frac{1}{\epsilon}\phi + \frac{(k+1)^2 C_0^{2(k+1)}}{\epsilon}\phi^{k+1} + ||\nabla u||^2 \geq ||u||_{q+2}^{q+2}. \tag{4.19}$$

Integrating (2.5) with respect to t gives

$$E(t) = \frac{1}{2}||u_t||^2 + \frac{1}{2}||\nabla u_t||^2 + \frac{1}{2}||\nabla u||^2 - \frac{1}{q+2}||u||_{q+2}^{q+2} \leq E(0).$$

Then we get

$$||u||_{q+2}^{q+2} \geq \frac{q+2}{2}G + \frac{q+2}{2}||\nabla u||^2 - (q+2)E(0). \tag{4.20}$$

Then, (4.20) and (4.19) yield

$$\phi'' - \frac{q+4}{2}G + \epsilon G + \frac{1}{\epsilon}\phi + \frac{(k+1)^2 C_0^{2(k+1)}}{\epsilon}\phi^{k+1} - \frac{q}{2}||\nabla u||^2 + (q+2)E(0) \geq 0. \tag{4.21}$$

Now, we choose $\epsilon = \epsilon_0 = \frac{q-k}{2} > 0$ so that

$$\alpha_0 = \frac{q+4}{2} - \frac{\epsilon_0}{2} = 2 + \frac{k}{2} > 1, 2\alpha_0 = 4 + k > k+2, \tag{4.22}$$

where we have used the fact $q > k$. Then (4.21) becomes

$$\phi'' - \alpha_0 G + \frac{1}{\epsilon_0}\phi + \frac{(k+1)^2 C_0^{2(k+1)}}{\epsilon_0}\phi^{k+1} + (q+2)E(0) \geq \frac{q}{2}||\nabla u||^2 \geq 0. \tag{4.23}$$

From the Cauchy-Schwarz inequality, we have

$$(\phi')^2 \leq \phi G. \tag{4.24}$$

Then using (4.24) and (4.23) we arrive at the following second order differential inequality

$$\phi\phi'' - \alpha_0(\phi')^2 + \frac{1}{\epsilon_0}\phi^2 + \frac{(k+1)^2 C_0^{2(k+1)}}{\epsilon_0}\phi^{k+2} + (q+2)E(0)\phi \geq 0. \tag{4.25}$$

Comparing this differential inequality with inequality (4.1), we find that

$$\alpha = \alpha_0, \beta = \frac{1}{\epsilon_0}, \gamma = \frac{(k+1)^2 C_0^{2(k+1)}}{\epsilon_0}, \delta = (q+2)E(0), l = k. \tag{4.26}$$

By (4.26) and (4.22), we know that (4.2) and (4.3) are satisfied. By (2.8) and (2.9), we know that (4.4) and (4.5) are satisfied. Then by Lemma 4.1, we see that $\phi(t)$ blows up in finite time. This theorem is proved.

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COMPACT AND MATRIX OPERATORS ON THE SPACE $|C,-1|_k$

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Abstract. According to Hardy [5], Cesàro summability is usually considered for the range $\alpha \geq -1$. In a more recent paper [14], the space $|C_\alpha|_k$ is studied for $\alpha > -1$. In this paper we define $|C_{-1}|_k$ using the Cesàro mean $(C,-1)$ of Thorpe [26] , compute its α-, β- and γ- duals, give some algebraic and topological properties, and characterize related matrix operators, and also obtain some identities or estimates for the their operator norms and the Hausdorff measure of noncompactness. Further, by applying the Hausdorff measure of noncompactness, we establish the necessary and sufficient conditions for such operators to be compact. So some results in [14] is also extended to the range $\alpha \geq -1$.

1. Introduction

Let ω be the set of all complex sequences, $c, \ell_\infty \subset w$ be the set of convergent and bounded sequences. For c_s, b_s and ℓ_k $(k \geq 1,\ \ell_1 = \ell)$, we write the sets of all convergent, bounded, k-absolutely convergent series, respectively. Let $A = (a_{nj})$ be an arbitrary infinite matrix of complex numbers. By $A(x) = (A_n(x))$, we denote the A-transform of the sequence x, i.e., $A_n(x) = \sum_{j=0}^{\infty} a_{nj}x_j$, provided that the series converges for $n \geq 0$. We say that A defines a matrix transformation from U into V, and it denote by $A \in (U,V)$ or $A : U \to V$ if sequence $Ax = (A_n(x)) \in V$ for every sequence $x \in U$, where U and V are subspace of w and also the sets $U^\alpha = \{\varepsilon \in w : (\varepsilon_v x_v) \in \ell \text{ for all } x \in U\}$, $U^\beta = \{\varepsilon \in w : (\varepsilon_v x_v) \in c_s \text{ for all } x \in U\}$, $U^\gamma = \{\varepsilon \in w : (\varepsilon_v x_v) \in b_s \text{ for all } x \in U\}$ and

$$U_A = \{x \in w : A(x) \in U\} \tag{1.1}$$

are said to be the α-, β-, γ- duals of U and the domain of the matrix A in U, respectively. Further, U is said to be an BK-space if it is a complete normed space with continuous coordinates $p_n : U \to \mathbb{C}$ defined by $p_n(x) = x_n$ for $n \geq 0$. The sequence (e^n) is called a Schauder base (or briefly base) for a normed sequence space U if for each $x \in U$ there exist unique sequence of

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coefficients (x_v) such that $\|x-\sum_{v=0}^{m}x_v e^v\| \to 0$ $(m\to\infty)$, and in this case we write $x=\sum_{v=0}^{\infty}x_v e_v$. For example, the sequence $\left(e^{(n)}\right)$ is a base of l_k with respect to the norm $\|x\|_{\ell_k}=\left(\sum_{v=0}^{\infty}|x_v|^k\right)^{1/k}$, $k\geq 1$, where $e^{(n)}$ is the sequence whose only non-zero term is 1 in the nth place for each n.Throughout k^* denotes the conjugate of $k>1$, i.e., $1/k+1/k^*=1$, and $1/k^*=0$ for $k=1$.

Let Σa_n be an infinite series with partial sum s_n. Let (σ_n^α) be the nth Cesàro mean (C,α) of order $\alpha>-1$ of the sequence (s_n), $e.i.$, $\sigma_n^\alpha=\frac{1}{A_n^\alpha}\sum_{v=0}^{n}A_{n-v}^{\alpha-1}s_v$.The summability $|C,\alpha|_k$ was defined by Flett [4] as follows. The series Σa_n is said to be summable $|C,\alpha|_k$ with index $k\geq 1$ if

$$\sum_{n=1}^{\infty}n^{k-1}\left|\sigma_n^\alpha-\sigma_{n-1}^\alpha\right|^k<\infty.$$

More recently the series space $|C_\alpha|_k$ has been studied by the second author for $\alpha>-1$, [14]. The Cesàro summability (C,α) is studied usually for range $\alpha\geq -1$ (see [5]). Since the above definition does not work for $\alpha=-1$, so it was separately defined by Thorpe [26] as follows. If the series to sequence transformation

$$T_n=\sum_{\nu=0}^{n-1}a_\nu+(n+1)\,a_n \tag{1.2}$$

tends to s as n tends to infinity, then the series Σa_n is summable by Cesàro summability $(C,-1)$ [26]. Now we define the space $|C_{-1}|_k$, $k\geq 1$, as the set of all series summable by the method $|C,-1|_k$. Then, it can be written that $|C_{-1}|_k=\left\{a=(a_v):\sum_{n=1}^{\infty}n^{k-1}|T_n-T_{n-1}|^k<\infty\right\}$, where (T_n) is defined by (1.2), or

$$|C_{-1}|_k=\left\{a=(a_\nu):\sum_{n=1}^{\infty}n^{k-1}\left|(n+1)\,a_n-(n-1)\,a_{n-1}\right|^k<\infty\right\}.$$

The problems of absolute summability factors and comparision of these methods goes to old rather and uptill now were widely examined by many authors, (see, [1-3], [6], [10-12], [16-22] ,[24-25])) et al. There are a close relation between these problems and some special matrix transformations such as an identity matrix I and a matrix $W=(w_{nv})$ defined by $w_{vv}=\varepsilon_v$ and $w_{nv}=0$ for $v\neq n$.

In this paper we derive a series space $|C_{-1}|_k$ using the Cesàro summability $(C,-1)$ of Thorpe [26] , compute its α-, β- and γ- duals, give some algebraic and topological properties, and characterize certain matrix operators defined on that space, and also obtain some identities or estimates for the their operator norms and the Hausdorff measure of noncompactness. Moreover, by applying the Hausdorff measure of noncompactness, we establish the necessary and sufficient

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conditions for such operators to be compact. So we also complete some open problems in the paper of Sarıgöl [14].

The following lemmas play important roles to prove our theorems.

Lemma 1.1. Let $1<k<\infty$. Then, $A\in(\ell_k,\ell)$ if and only if

$$\|A\|_{(\ell_k,\ell)}=\sup_N\left\{\sum_{v=0}^{\infty}\left|\sum_{n\in N}a_{nv}\right|^{k^*}\right\}^{1/k^*}<\infty,$$

where N is any finite set of positive numbers [23].

The following lemma is more useful in many cases, which gives equivalent norm.

Lemma 1.2. Let $1<k<\infty$. Then, $A\in(\ell_k,\ell)$ if and only if

$$\|A\|'_{(\ell_k,\ell)}=\left\{\sum_{v=0}^{\infty}\left(\sum_{n=0}^{\infty}|a_{nv}|\right)^{k^*}\right\}^{1/k^*}<\infty,$$

and there exists $1\leq\xi\leq 4$ such that $\|A\|'_{(\ell_k,\ell)}=\xi\|A\|_{(\ell_k,\ell)}$ [15].

The second part of this lemma is easily seen by following the lines in [15] that $\|A\|_{(\ell_k,\ell)}\leq\|A\|'_{(\ell_k,\ell)}\leq 4\|A\|_{(\ell_k,\ell)}$.

Lemma 1.3. Let $1\leq k<\infty$. Then, $A\in(\ell,\ell_k)$ if and only if

$$\|A\|_{(\ell,\ell_k)}=\sup_v\left\{\sum_{n=0}^{\infty}|a_{nv}|^k\right\}^{1/k}<\infty,\ [7].$$

Lemma 1.4.

a-) $A\in(\ell,c)\Leftrightarrow(i)$ $\lim_n a_{nv}$ exists, $v\geq 0$, (ii) $\sup_{n,v}|a_{nv}|<\infty$.

b-) $A\in(\ell,\ell_\infty)\Leftrightarrow(ii)$ holds.

c-) Let $1<k<\infty$.Then, $A\in(\ell_k,\ell_\infty)\Leftrightarrow(iii)$ $\sup_n\sum_{v=0}^{\infty}|a_{nv}|^{k^*}<\infty$;

d-) $A\in(\ell_k,c)\Leftrightarrow(i)$ and (iii) hold [23].

2. The Hausdorff measure of noncompactness

If S and H are subsets of a metric space (X,d) and $\varepsilon>0$ then S is called an ε-net of H , if, for every $h\in H$, there exists an $s\in S$ such that $d(h,s)<\varepsilon$; if S is finite, then the ε-net S of H is called a finite ε-*net* of H. Let X and Y be Banach spaces. A linear operator $L:X\to Y$ is called compact its domain is all of X and, for every bounded sequence (x_n) in X, the sequence $(L(x_n))$ has a convergent subsequence in Y. We denote the class of such operators by $\mathcal{C}(X,Y)$. If Q is a bounded subset of the metric space X, then the Hausdorff measure of noncompactness of Q is defined by $\chi(Q)=\{\varepsilon>0:Q$ has a finite ε-net in $X\}$,and χ is called the Hausdorff measure of noncompactness.

The following result is an important tool to compute the Hausdorff measure of noncompactness of a bounded subset of the BK space $\ell_k,k\geq 1$.

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Lemma 2.1. Let Q be a bounded subset of the normed space X where $X = \ell_k$, for $1 \leq k < \infty$ or $X = c_0$.If $P_n : X \to X$ is the operator defined by $P_r(x) = (x_0, x_1, ..., x_r, 0, ...)$ for all $x \in X$, then $\chi(Q) = \lim_{r\to\infty} \left(\sup_{x\in Q} \|(I - P_r)(x)\|\right)$,where I is the identity operator on X, [13].

If X and Y be Banach spaces and χ_1 and χ_2 be Hausdorff measures on X and Y, then, the linear operator $L : X \to Y$ is said to be (χ_1, χ_2)-bounded if $L(Q)$ is bounded subset of Y for every bounded subset Q of X and there exists a positive constant M such that $\chi_2(L(Q)) \leq M\ \chi_1(Q)$ for every bounded Q of X. If an operator L is (χ_1, χ_2)-bounded then the number $\|L\|_{(\chi_1,\chi_2)} = \inf\{M > 0 : \chi_2(L(Q)) \leq M\chi_1(Q) \text{ for all bounded } Q \subset X\}$ is called the (χ_1, χ_2)-measure of noncompactness of L. In particular, we write $\|L\|_{(\chi,\chi)} = \|L\|_\chi$ for $\chi_1 = \chi_2 = \chi$.

Lemma 2.2. Let X and Y be Banach spaces, $L \in B(X, Y)$ and $S_X = \{x \in X : \|x\| \leq 1\}$ denote the unit sphere in X. Then, $\|L\|_\chi = \chi(L(S_X))$, and

$$L \in \mathcal{C}(X, Y) \text{ if and only if } \|L\|_\chi = 0, [8].$$

Lemma 2.3. Let X be normed sequence space and χ_T and χ denote the Hausdorff measures of noncompactness on $\mathcal{M}_{x_T}$ and $\mathcal{M}_X$, the collections of all bounded sets in X_T and X, respectively. Then, $\chi_T(Q) = \chi(T(Q)$ for all $Q \in \mathcal{M}_{x_T}$, where $T = (t_{nv})$ is a triangular infinite matrix, [9].

3. Continuous and compact matrix operators on the space $|C_{-1}|_k$

In this section by giving some topological properties, α-, β-, γ- duals, base of $|C_{-1}|_k$,we characterize matrix operators on that space, determine their norms, and also establish the necessary and sufficient conditions for such operators to be compact by applying the Hausdorff measure of noncompactness, which also extends some results of Sarıgöl [14] to $\alpha \geq -1$. First of all, we define the matrix $T^{(k)} = (t^{(k)}_{nv})$ by $t^{(k)}_{00} = 1$,

$$t^{(k)}_{nv} = \begin{cases} -n^{1/k^*}(n-1), & v = n-1 \\ n^{1/k^*}(n+1), & v = n \\ 0, & \text{otherwise}, \end{cases} \tag{3.1}$$

for $1 \leq k < \infty$. Then it is clear that $|C_{-1}|_k = (\ell_k)_{T^{(k)}}$ according to (1.1). Further, since every triangular matrix has a unique inverse which is also triangle [27, p. 9]. there exists the inverse of $T^{(k)}$ and denote this inverse by $S^{(k)}$. Now it follows from (1.2) that $a_0 = y_0$ and $a_n = [n(n+1)]^{-1} \sum_{v=1}^{n} v^{1/k} y_\nu$, $n \geq 1$, and so $s^{(k)}_{00} = 1$,

$$s^{(k)}_{nv} = \begin{cases} \dfrac{\nu^{1/k}}{n(n+1)}, & 1 \leq v \leq n \\ 0, & v > n. \end{cases} \tag{3.2}$$

Finally, we define the following notations:

$$D_1 = \left\{ \varepsilon = (\varepsilon_v) \in w : \sum_{v=r}^{\infty} s_{vr}^{(1)} \varepsilon_v \text{ converges, } r \geq 1 \right\},$$

$$D_2 = \left\{ \varepsilon = (\varepsilon_v) \in w : \sup_{m,r} \left| \sum_{v=r}^{m} s_{vr}^{(1)} \varepsilon_v \right| < \infty \right\},$$

$$D_3 = \left\{ \varepsilon = (\varepsilon_v) \in w : \sup_{m} \sum_{r=1}^{m} \left| \sum_{v=r}^{m} s_{vr}^{(k)} \varepsilon_v \right|^{k^*} < \infty \right\},$$

$$D_4 = \left\{ \varepsilon = (\varepsilon_v) \in w : \sup_{\nu} \sum_{n=\nu}^{\infty} \left| s_{nv}^{(1)} \varepsilon_n \right| < \infty \right\},$$

$$D_5 = \left\{ \varepsilon = (\varepsilon_v) \in w : \sum_{\nu=1}^{\infty} \left\{ \sum_{n=\nu}^{\infty} \left| s_{nv}^{(k)} \varepsilon_n \right| \right\}^{k^*} < \infty \right\}.$$

Theorem 3.1. Let $1 \leq k < \infty$. Then,

a-) The space $|C_{-1}|_k$ is BK-space with respect to the norm $\|x\|_{|C_{-1}|_k} = \left\|T^{(k)}(x)\right\|_{l_k}$ and isomorphic to the space ℓ_k, i.e., $|C_{-1}|_k \cong \ell_k$, where $T^{(k)}$ is defined by (3.1)

b-) $|C_{-1}|_k^{\beta} = D_1 \cap D_3$ for $1 < k < \infty$ and $|C_{-1}|^{\beta} = D_1 \cap D_2$ for $k = 1$.

c-) $|C_{-1}|_k^{\gamma} = D_3$ for $1 < k < \infty$ and $|C_{-1}|_k^{\gamma} = D_2$ for $k = 1$.

d-) $|C_{-1}|_k^{\alpha} = D_5$ for $1 < k < \infty$ and $|C_{-1}|_k^{\alpha} = D_4$ for $k = 1$.

e-) The sequence $b^{(v)} = (b_n^{(v)})$ is the Schauder base of the space $|C_{-1}|_k$ for $v, n \geq 0$, where $b_n^{(v)} = s_{nv}^{(k)}$.

Proof a-) Since l_k is BK-spaces with its usual norm, $|C_{-1}|_k = (\ell_k)_{T^{(k)}}$ and $T^{(k)}$ is a triangle matrix, it follows from Theorem 4.3.2 of Wilansky [27, p. 61] that $|C_{-1}|_k$ is BK-spaces for $1 \leq k < \infty$. To prove the second part, define $T^{(k)} : |C_{-1}|_k \to l_k$ by

$$T_n^{(k)}(x) = n^{1/k^*} \left[(n+1) x_n - (n-1) x_{n-1} \right], \ n \geq 1.$$

Then, it is clear that $T^{(k)}$ is linear operator and surjective, since, if $y \in \ell_k$, then $x_n = [n(n+1)]^{-1} \sum_{v=1}^{n} v^{1/k} y_\nu$, where $y = T^{(k)}(x)$, and also one to one. Further, it preserves the norm, since $\left\|T^{(k)}(x)\right\|_{l_k} = \|x\|_{|C_{-1}|_k}$,which completes the proof**.**

b-) Let $1 < k < \infty$. Now, $\varepsilon \in |C_{-1}|_k^{\beta}$ if and only if $\Sigma \varepsilon_n x_n$ is convergent for every $x \in |C_{-1}|_k$. Let $y = T^{(k)}(x)$. Then, $y \in \ell_k$ if and only if $x \in |C_{-1}|_k$, where $x_n = (n(n+1))^{-1} \sum_{v=1}^{n} v^{1/k} y_\nu$ for $n \geq 1$, $x_0 = y_0$, and also it can be written that

$$\sum_{v=0}^{m} \varepsilon_\nu x_\nu = \varepsilon_0 y_0 + \sum_{r=1}^{m} \left(\sum_{v=r}^{m} s_{vr}^{(k)} \varepsilon_v \right) y_r = \sum_{r=0}^{m} \mu_{mr} y_r$$

where, $\mu_{m0} = \varepsilon_0$,

$$\mu_{mr} = \begin{cases} \sum_{v=r}^{m} s_{vr}^{(k)} \varepsilon_v, 1 \le r \le m \\ 0, \quad r > m. \end{cases}$$

So it follows from Lemma 1.4. that $\varepsilon \in |C_{-1}|_k^{\beta}$ iff $\mu \in (\ell_k, c)$, or equivalently, $\varepsilon \in D_1 \cap D_3$, which completes the proof.

Since (c) and (d) can be proved easily as in (b), so we omit the detail.

e-) Since $|C_{-1}|_k = (\ell_k)_{T^{(k)}}$ and the sequence $\left(e^{(v)}\right)$ is a base of l_k, where $e^{(v)} = \left(e_n^{(v)}\right)$, it is clear that the sequence $\left(b^{(v)}\right)$ is the base for$|C_{-1}|_k$. In fact, if $x \in |C_{-1}|_k$, then there exists $y \in l_k$ such that $y = T^{(k)}(x)$, and so it follows

$$\left\| x - \sum_{v=0}^{m} x_v b^{(v)} \right\|_{|C_{-1}|_k} = \left\| y - \sum_{v=0}^{m} y_v e^{(v)} \right\|_{l_k} \to 0 \text{ as } m \to \infty,$$

where $x_v = \left(T^{(k)}\right)_v^{-1}(y)$, $v \ge 0$. Further, it has the unique representation by $x = \sum_{v=0}^{\infty} x_v b^{(v)}$, which is immediately seen from the triange inequality of norm.

Also, we state the following result which is immediate by Theorem 3.1.

Theorem 3.2. Let $1 \le k < \infty$ and $A = (a_{nv})$ is an infinite matrix with complex terms for all $n, v \ge 0$. Define the matrices $\overline{L} = \left(\overline{\ell}_{nr}\right)$, $D = (d_{nv})$ and $D^{(r)} = \left(\bar{d}_{n\nu}^{(r)}\right)$ by

$$\overline{\ell}_{nr} = r \sum_{v=r}^{\infty} \frac{a_{nv}}{\nu(\nu+1)},$$

$$d_{nv} = n^{1/k^*} \left[(n+1)\,\overline{\ell}_{n\nu} - (n-1)\,\overline{\ell}_{n-1,\nu}\right],\ n \ge 1, v \ge 1. \tag{3.3}$$

$$\bar{d}_{n\nu}^{(r)} = \begin{cases} 0,\ 1 \le n \le r \\ d_{nv}, \quad n > r \end{cases}$$

Then each matrix in the class $A \in (|C_{-1}|, |C_{-1}|_k)$ defines a bounded linear operator $L_A : |C_{-1}| \to |C_{-1}|_k$ such that $L_A(x) = A(x)$ for all $x \in |C_{-1}|$, and $A \in (|C_{-1}|, |C_{-1}|_k)$ if and only if

$$\overline{L} \text{ is well defined } n, r \ge 1, \tag{3.4}$$

$$\sup_{m,r} \left| r \sum_{v=r}^{m} \frac{a_{nv}}{\nu(\nu+1)} \right| < \infty \text{ for each } n, \tag{3.5}$$

$$\sup_{v} \sum_{n=1}^{\infty} |d_{nv}|^k < \infty. \tag{3.6}$$

Morever, if $A \in (|C_{-1}|, |C_{-1}|_k)$, then

$$\|L_A\|_{\left(|C_{-1}|,|C_{-1}|_k\right)} = \|D\|_{(\ell,\ell_k)} \text{ and } \quad \|L_A\|_{\chi} = \lim_{r\to\infty} \left\|D^{(r)}\right\|_{(l,l_k)} \tag{3.7}$$

Proof. The first part is immediate by Theorem 4.2.8 of Wilansky [27, p.57], since $|C_{-1}|$ and $|C_{-1}|_k$ are a BK-spaces by Theorem 3.1. For second part, $A \in$

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$(|C_{-1}|, |C_{-1}|_k)$ iff $(a_{nv})_{v=0}^{\infty} \in (|C_{-1}|)^{\beta}$ for each n, and $A(x) \in |C_{-1}|_k$ for every $x \in |C_{-1}|$. Now, by Theorem 3.1 (b), $(a_{nv})_{v=0}^{\infty} \in (|C_{-1}|)^{\beta}$ iff $(a_{n\nu})_{\nu=0}^{\infty} \in D_1 \cap D_2$, or, equivalently, (3.4) and (3.5) hold for each n. Now, to obtain (3.6), consider operator $T^{(1)} : |C_{-1}| \to \ell_1$ by using (3.1) for $k = 1$ by

$$T_n^{(1)}(x) = (n+1)\, x_n - (n-1)\, x_{n-1} \tag{3.8}$$

As in the proof of Theorem 3.1, $T^{(1)}$ is bijection and the matrix corresponding to this operator is triangle. Further, let $x \in |C_{-1}|$ be given, then $y \in \ell$, where $y = T^{(1)}(x)$, for $n \geq 0$, $x_{-1} = 0$, and also, $x_n = (n(n+1))^{-1} \sum_{v=1}^{n} v y_\nu$ for $n \geq 1$ and $x_0 = y_0$. Now, we can write

$$\sum_{v=1}^{m} a_{nv} x_v = \sum_{r=1}^{m} \left(r \sum_{v=r}^{m} \frac{a_{n\nu}}{\nu(\nu+1)} \right) y_r = \sum_{r=1}^{m} \bar{\ell}_{mr}^{(n)} y_r,$$

where

$$\bar{\ell}_{mr}^{(n)} = \begin{cases} r \sum_{v=r}^{m} \dfrac{a_{n\nu}}{\nu(\nu+1)}, 1 \leq \nu \leq m \\ \qquad 0, \qquad \nu > m. \end{cases}$$

Moreover, if any matrix $H = (h_{nv}) \in (\ell, c)$, then, the series $H_n(x) = \Sigma_v h_{nv} x_v$ converges informly in n, since, by Lemma 1.4, the remaining term tends to zero informly in n, that is,

$$\left| \sum_{v=m}^{\infty} h_{nv} x_v \right| \leq \sup_{n,v} |h_{nv}| \sum_{v=m}^{\infty} |x_v| \to 0 \text{ as } m \to \infty,$$

and so

$$\lim_n H_n(x) = \sum_{v=0}^{\infty} \lim_n h_{nv} x_v \tag{3.9}$$

Hence, since (3.4) and (3.5) hold, $\overline{L}^{(n)} = \left(\bar{\ell}_{mr}^{(n)} \right) \in (\ell, c)$, then by (3.9), we get

$$A_n(x) = \sum_{r=0}^{\infty} \left(\lim_m \bar{\ell}_{mr}^{(n)} \right) y_r = \sum_{r=0}^{\infty} \bar{\ell}_{nr} y_r = \overline{L}_n(y),$$

This shows that $A(x) \in |C_{-1}|_k$ for every $x \in |C_{-1}|$ iff $\overline{L}(y) \in |C_{-1}|_k$ for every $y \in \ell$, or, equivalently, $D(y) \in \ell_k$, since $|C_{-1}|_k = (\ell_k)_{T^{(k)}}$, where $D = T^{(k)} \overline{L}$. So, it is clear that $A \in (|C_{-1}|, |C_{-1}|_k)$ iff (3.4), (3.5) hold, and $D \in (\ell, \ell_k)$. A few calculations show that

$$d_{nv} = \sum_{r=1}^{n} t_{nr}^{(k)} \bar{\ell}_{r\nu} = n^{1/k^*} \left[(n+1)\, \bar{\ell}_{n\nu} - (n-1)\, \bar{\ell}_{n-1,\nu} \right], \quad v, n \geq 1,$$

Therefore, it can be obtained from Lemma 1.3 that $D \in (\ell, \ell_k)$ iff (3.6) is satisfied, which completes the proof of the second part.

Now, to compute norm of A, $A \in (|C_{-1}|, |C_{-1}|_k)$ iff $D \in (\ell, \ell_k)$, where D is defined by (3.3). On the other hand, consider the isomorphisms $T^{(1)} : |C_{-1}| \to \ell$ and $T^{(k)} : |C_{-1}|_k \to \ell_k$ defined in (3.8) and Theorem 3.1 (a), respectively. Then we get $A = \left(T^{(k)}\right)^{-1} oDoT^{(1)}$. So, by Theorem 3.1, $x \in |C_{-1}|$ iff $y = T^{(1)}(x) \in \ell$ and $\|A(x)\|_{|C_{-1}|_k} = \|D(y)\|_{\ell_k}$ for all $x \in |C_{-1}|$ and $y \in \ell$, which gives $\|A\|_{\left(|C_{-1}|, |C_{-1}|_k\right)} = \|D\|_{(l,l_k)}$.

Finally, let $S = \{x \in |C_{-1}| : \|x\| \le 1\}$. Then, by Lemma 2.1-Lemma 2.3, and from the definition of $D^{(r)}$ we get

$$\begin{aligned} \|L_A\|_\chi &= \chi(L_A S) = \chi(DTS) = \lim_{r\to\infty} \sup_{y \in TS} \|(I - P_r) D(y)\|_{l_k} \\ &= \lim_{r\to\infty} \left\|D^{(r)}\right\|_{(l,l_k)} = \lim_{r\to\infty} \sup_v \left\{ \sum_{n=r+1}^{\infty} |d_{nv}|^k \right\}^{1/k} \end{aligned}$$

where $P_r : l_k \to l_k$ is defined by $P_r(y) = (y_0, y_1, ..., y_r, 0, ...)$ and the matrix $D^{(r)} = \left(\bar{d}_{n\nu}\right)$ is defined as: $\bar{d}_{n\nu} = d_{n\nu}$ for $n > r$, 0 otherwise. Thus, the proof is completed together with Lemma 1.3.

It is directly characterized from Theorem 3.2 the compact operators in the class $(|C_{-1}|, |C_{-1}|_k)$.

Corollary 3.3. Under hypotheses of Theorem 3.2, $L_A \in (|C_{-1}|, |C_{-1}|_k)$ is compact if and only if $\|L_A\|_\chi = \lim_{r\to\infty} \left\|D^{(r)}\right\|_{(l,l_k)} = 0$.

Now, If a matrix A is chosen as a special matrix W and I, then $A \in (|C_{-1}|, |C_{-1}|_k)$ means the form of summability factors that $\Sigma \varepsilon_v x_v$ is summable $|C_{-1}|_k$ when Σx_v is summable $|C_{-1}|$, and $I \in (|C_{-1}|, |C_{-1}|_k)$ means the comparisons of these methods, i.e., $|C_{-1}| \subset |C_{-1}|_k$. If we put $A = I$, then, since

$$\bar{\ell}_{nr} = \begin{cases} \dfrac{r}{n(n+1)}, & 1 \le r \le n \\ 0, & n < r \end{cases}$$

and

$$d_{nv} = \begin{cases} n^{1/k^*} \left[(n+1)\,\bar{\ell}_{n\nu} - (n-1)\,\bar{\ell}_{n-1,\nu}\right], & 1 \le v \le n \\ 0, & v > n, \end{cases}$$

it is also clear that condition $(3.4), (3.5)$ and (3.6) are satisfied. So one can easily derive from Theorem 3.2 the following result.

Corollary 3.4. Let $1 \le k < \infty$. Then, $|C_{-1}| \subset |C_{-1}|_k$.

Theorem 3.5. Let $1 < k < \infty$ and $A = (a_{nv})$ is an infinite matrix with complex terms for all $n, v \ge 0$. Let be define the matrices $\overline{L} = \left(\bar{\ell}_{n\nu}\right)$ as in Theorem 3.2 and $F = (f_{n\nu})$ and $F^{(r)} = \left(\tilde{f}^{(n)}_{n\nu}\right)$ by

$$f_{n\nu} = \nu^{-1/k^*} \left[(n+1)\,\bar{\ell}_{n\nu} - (n-1)\,\bar{\ell}_{n-1,\nu}\right], \ \nu \ge 1, n \ge 1, \tag{3.10}$$

and

$$\tilde{f}_{n\nu}^{(r)} = \begin{cases} 0, \ 1 \le n \le r \\ f_{n\nu}, \quad n > r \end{cases}$$

Then each matrix in the class $A \in (|C_{-1}|_k, |C_{-1}|)$ defines a bounded linear operator $L_A : |C_{-1}|_k \to |C_{-1}|$ such that $L_A(x) = A(x)$ for all $x \in |C_{-1}|_k$, and $A \in (|C_{-1}|_k, |C_{-1}|)$ if and only if (3.4)

$$\sup_m \sum_{r=1}^{m} \left| r^{1/k} \sum_{v=r}^{m} \frac{a_{n\nu}}{\nu(\nu+1)} \right|^{k^*} < \infty, \tag{3.11}$$

$$\sum_{\nu=1}^{\infty} \left(\sum_{n=1}^{\infty} |f_{n\nu}| \right)^{k*} < \infty. \tag{3.12}$$

Morever, if $A \in (|C_{-1}|_k, |C_{-1}|)$, then, there exists $1 \le \xi \le 4$ such that

$$\|L_A\|_{(|C_{-1}|_k, |C_{-1}|)} = \frac{1}{\xi} \|F\|'_{(\ell_k, \ell)} \text{ and } \|L_A\|_\chi = \frac{1}{\xi} \lim_{r\to\infty} \left\| F^{(r)} \right\|'_{(l_k, l)}. \tag{3.13}$$

Proof. The first part is immediate by Theorem 4.2.8 of Wilansky [27, p.57], since $|C_{-1}|$ and $|C_{-1}|_k$ are a BK-spaces by Theorem 3.1. For the second part, let $A \in (|C_{-1}|_k, |C_{-1}|)$. Then, $(a_{n\nu})_{\nu=0}^{\infty} \in (|C_{-1}|_k)^\beta$ and $A(x) \in |C_{-1}|$ for every $x \in |C_{-1}|_k$. Now by Theorem 3.1 (b), $(a_{n\nu})_{\nu=0}^{\infty} \in (|C_{-1}|_k)^\beta$ iff $(a_{n\nu})_{\nu=0}^{\infty} \in D_1 \cap D_3$ for each n. This also means that $(a_{n\nu})_{\nu=0}^{\infty} \in (|C_{-1}|_k)^\beta$ iff conditions (3.4) and (3.11) hold. Also to obtain (3.12), we consider the operators $T^{(k)} : |C_{-1}|_k \to \ell_k$ same as Theorem 3.1 (a). Then, since the space $|C_{-1}|_k$ is izomorf to ℓ_k, it can be written that $x \in |C_{-1}|_k$ iff $y \in \ell_k$, where $T^{(k)}(x) = y$, $i.e., y_0 = x_0$ and $y_n = T_n^{(k)}(x) = n^{1/k^*}[(n+1)x_n - (n-1)x_{n-1}]$ for $n \ge 1$, $x_{-1} = 0$. So by (3.2), as in the proof of Theorem 3.2 we get

$$\sum_{v=1}^{m} a_{nv} x_v = \sum_{r=1}^{m} r^{-1/k^*} \ell_{mr}^{(n)} y_r = \sum_{r=1}^{m} \overline{f}_{mr}^{(n)} y_r$$

where, $\overline{f}_{mr}^{(n)} = r^{-1/k^*} \ell_{mr}^{(n)}$ for $1 \le r \le m$, and $\overline{f}_{mr}^{(n)} = 0$ for $r > m$, and $\overline{L}^{(n)} = \left(\overline{\ell}_{mr}^{(n)} \right)$ is defined as in Theorem 3.2. Moreover, if any matrix $H = (h_{nv}) \in (\ell_k, c)$, then by Lemma1.4 and using Hölder's inequality, we get

$$\left| \sum_{v=m}^{\infty} h_{nv} x_v \right| \le \sup_n \left(\sum_{v=0}^{\infty} |h_{nv}|^{k^*} \right)^{1/k^*} \left(\sum_{v=m}^{\infty} |x_v|^k \right)^{1/k}$$

Also, since $x \in \ell_k$, we obtain that the series $H_n(x) = \Sigma_v h_{nv} x_v$ converges informly in n, which implies

$$\lim_n H_n(x) = \sum_{v=0}^{\infty} \lim_n h_{nv} x_v. \tag{3.14}$$

Therefore, since (3.4) and (3.11) hold, $F^{(n)}=\left(\overline{f}_{mr}^{(n)}\right)\in(\ell_k,c)$, then by (3.14), we get

$$A_n(x)=\sum_{r=1}^{\infty}\left(\lim_m \overline{f}_{mr}^{(n)}\right)y_r=\sum_{r=1}^{\infty}r^{-1/k^*}\overline{\ell}_{nr}y_r=\sum_{r=1}^{\infty}\overline{f}_{nr}y_r=\overline{F}_n(y),\ n\geq 1$$

where, $\overline{f}_{nr}=\lim_m \overline{f}_{mr}^{(n)}$. This shows that $A(x)\in|C_{-1}|$ for every $x\in|C_{-1}|_k$ iff $\overline{F}(y)\in|C_{-1}|$ for every $y\in\ell_k$ or, equivalently $\left(T^{(1)}\overline{F}\right)(y)\in\ell$, since $|C_{-1}|=(\ell)_{T^{(1)}}$, so we obtain that $F\in(\ell_k,\ell)$, where $F=T^{(1)}\overline{F}$. Hence, it is clear that $A\in\left(|C_{-1}|_k,|C_{-1}|\right)$ iff (3.4), (3.11) hold, and $F\in(\ell_k,\ell)$. With a few calculations, it can be easily seen that F is the same as (3.10), and so it follows from applying Lemma 1.2 to the matrix F that $F\in(\ell_k,\ell)$ iff (3.12) is satisfied, and this proves the second part.

Also, considering that $T^{(k)}:|C_{-1}|_k\to\ell_k$ and $T^{(1)}:|C_{-1}|\to\ell$ are norm isomorphism, as in Theorem 3.2, it follows that $A=\left(T^{(1)}\right)^{-1}oFoT^{(k)}$ and so, by Lemma 1.2,

$$\|L_A\|_{\left(|C_{-1}|_k,|C_{-1}|\right)}=\|F\|_{(\ell_k,\ell)}=\frac{1}{\xi}\|F\|'_{(\ell_k,\ell)}$$

Finally, $S=\left\{x\in|C_{-1}|_k:\|x\|\leq 1\right\}$. Then, by considering Lemma 2.1-Lemma 2.3, and Lemma 1.2, there exists $1\leq\xi\leq 4$ such that

$$\|L_A\|_{\chi}=\lim_{r\to\infty}\sup_{y\in T^{(k)}S}\|(I-P_r)F(y)\|_l=\lim_{r\to\infty}\left\|F^{(r)}\right\|_{(l_k,l)}=\frac{1}{\xi}\lim_{r\to\infty}\left\|F^{(r)}\right\|'_{(l_k,l)}$$

where $P_r:l\to l$ is defined by $P_r(y)=(y_0,y_1,...,y_r,0,...)$ and the matrix $F^{(r)}=\left(\tilde{f}_{n\nu}^{(n)}\right)$ is defined as: $\tilde{f}_{n\nu}^{(n)}=0$ for $1\leq n\leq r$, and $f_{n\nu}$ for $n>r$, which proves the theorem together with Lemma 1.2.

The compact operators in this class are obtained from Theorem 3.5 as follows.

Corollary 3.6. Under hypotheses of Theorem 3.5,

$$L_A\in\left(|C_{-1}|_k,|C_{-1}|\right)\text{ is compact if and only if }\lim_{r\to\infty}\left\|F^{(r)}\right\|'_{(l_k,l)}=0.$$

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On value distribution of meromorphic solutions of a certain second order difference equations *

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Abstract. In this paper, we consider difference equation $f(z+1)+f(z-1)=\frac{A(z)f(z)+C}{1-f^2(z)}$, where C is a non-zero constant, $A(z)=\frac{m(z)}{n(z)}$, $m(z)$ and $n(z)$ are irreducible polynomials, we obtain the forms of rational solutions.

To the difference equation $f(z+1)+f(z-1)=\frac{A(z)f(z)+C(z)}{1-f^2(z)}$, where $A(z)$, $C(z)$ are non-constant small functions with respect to $f(z)$, the Borel exceptional value, the exponents of convergence of zeros, poles and fixed points of finite order transcendental meromorphic solution $f(z)$, and the exponents of convergence of poles of differences $\Delta f(z)=f(z+1)-f(z)$, $\Delta^2 f(z)=\Delta f(z+1)-\Delta f(z)$ and divided differences $\frac{\Delta f(z)}{f(z)}$, $\frac{\Delta^2 f(z)}{f(z)}$ are estimated.

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1 Introduction and Results

Halburd and Korhonen [6, 7] used value distribution theory and a reasoning related to the singularity confinement to single out the difference Painlevé I and II equations from difference equation

$$f(z+1)+f(z-1)=R(z,f), \tag{1.1}$$

where R is rational in both of its arguments. They obtained that if (1.1) has an admissible meromorphic solutions of finite order, then either f satisfies a difference Riccati equation, or (1.1) may be transformed into some classical difference equations, which include difference Painlevé I equations

$$f(z+1)+f(z-1)=\frac{az+b}{f(z)}+c, \tag{1.2}$$

$$f(z+1)+f(z-1)=\frac{az+b}{f(z)}+\frac{c}{f^2(z)}, \tag{1.3}$$

$$f(z+1)+f(z)+f(z-1)=\frac{az+b}{f(z)}+c, \tag{1.4}$$

and difference Painlevé II equation

$$f(z+1)+f(z-1)=\frac{(az+b)f(z)+c}{1-f^2(z)}, \tag{1.5}$$

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where a, b and c are constants.

Recently, as the difference analogues of Nevanlinna's theory are investigated [2, 5], many results on the complex difference equations are rapidly obtained, such as [1, 3, 10−12]. However, there are a few papers concerning with the existence of rational solution of difference Painlevé equations. In this paper, we will consider the forms of rational solutions, and investigate the value distribution of meromorphic solution of finite order of a certain type of difference equation which originates from the difference Painlevé II equation.

We assume that the reader is familiar with the basic Nevanlinna's value distribution theory of meromorphic functions (see[4, 9]). In addition, we use the notation $\sigma(f)$ to denote the order of growth of the meromorphic function $f(z)$, $\lambda(f)$ and $\lambda(\frac{1}{f})$ to denote, respectively, the exponent of convergence of zeros and poles of $f(z)$. We also use the notation $\tau(f)$ to denote the exponent of convergence of fixed points of $f(z)$ which is defined as

$$\tau(f) = \overline{\lim_{r\to\infty}} \frac{\log N\left(r, \frac{1}{f(z)-z}\right)}{\log r}.$$

We denote by $S(r, f)$ any quantify satisfying $S(r, f) = o(T(r, f))$, as $r \to \infty$, possibly outside a set with finite measure. For every $n \in \mathbb{N}^*$, the forward difference $\Delta^n f(z)$ are defined in the standard way [14] by $\Delta f(z) = f(z+1) - f(z)$, $\Delta^{n+1} f(z) = \Delta^n f(z+1) - \Delta^n f(z)$.

Chen and Shon [3] considered the difference Painlevé II equation (1.5) and proved the following result.

Theorem A. (See [3]) *Let a, b, c be constants, $ac \neq 0$. Suppose that a rational function*

$$f(z) = \frac{P(z)}{Q(z)} = \frac{pz^m + p_{m-1}z^{m-1} + \cdots + p_0}{qz^n + q_{n-1}z^{n-1} + \cdots + q_0}$$

is a solution of (1.5)*, where $P(z)$ and $Q(z)$ are relatively prime polynomials, $p, p_{m-1}, \ldots, p_0$ and $q, q_{n-1}, \ldots, q_0$ are constants. Then*

$$n = m + 1 \quad and \quad p = -\frac{c}{a}q.$$

In equation (1.5), if we replace $az + b$ with $A(z) = \frac{m(z)}{n(z)}$, where $m(z)$ and $n(z)$ are mutually prime polynomials, we still consider the rational solutions of equation (1.5), what will happen? Here, we obtain the following result.

Theorem 1.1. *Let C be non-zero constant, and $A(z) = \frac{m(z)}{n(z)}$ be a rational function, where $m(z)$ and $n(z)$ are mutually prime polynomials with* $\deg m(z) = m$, $\deg n(z) = n$*. If difference equation*

$$f(z+1) + f(z-1) = \frac{A(z)f(z) + C}{1 - f^2(z)} \tag{1.6}$$

has a rational solution $f(z) = \frac{P(z)}{Q(z)}$, where $P(z)$ and $Q(z)$ are relatively prime polynomials with $\deg P(z) = p$, $\deg Q(z) = q$*, then*

(i) Suppose that $m > n$, then $p - q = \frac{m-n}{2}$ and $m - n$ must be even or $q - p = m - n \geq 1$;

(ii) Suppose that $m = n$, then $p - q = 0$ and

$$\lim_{z\to\infty} \frac{m(z)}{n(z)} = A \in \mathbb{C} \setminus \{0\}, \quad \lim_{z\to\infty} \frac{P(z)}{Q(z)} = B \in \mathbb{C} \setminus \{0\},$$

$C = B[2(1 - B^2) - A]$ or $C = \pm A$;

(iii) Suppose that $m < n$, then $p - q = 0$ and

$$\lim_{z\to\infty} \frac{P(z)}{Q(z)} = B \in \mathbb{C} \setminus \{0, \pm 1\}, \quad C = 2B(1 - B^2).$$

The following examples show that the difference equation (1.6) has rational solutions satisfying Theorem 1.1 (i), (ii) and (iii).

Example 1.1. *The difference equation*

$$f(z+1)+f(z-1)=\frac{-\frac{2z^6+2z^4+Cz^3+2z^2-Cz}{z^4-1}f(z)+C}{1-f^2(z)}$$

has a rational solution $f(z)=z+\frac{1}{z}$*, where* $C\neq 0,\ m=6,\ n=4,\ p=2,\ q=1$ *and* $p-q=\frac{m-n}{2}=1$.

Example 1.2. *The difference equation*

$$f(z+1)+f(z-1)=\frac{(-Cz+2)f(z)+C}{1-f^2(z)}$$

has a rational solution $f(z)=\frac{1}{z}$*, where* $C\neq 0,\ m=1,\ n=0,\ p=0,\ q=1$ *and* $q-p=m-n=1$.

Example 1.3. *The difference equation*

$$f(z+1)+f(z-1)=\frac{\frac{-2(z^3+5z+10)}{z^3+2z^2-z-2}f(z)+2}{1-f^2(z)}$$

has a rational solution $f(z)=\frac{z-1}{z+1}$*, where* $m=n=3,\ p=q=1$*,*

$$\lim_{z\to\infty}\frac{-2(z^3+5z+10)}{z^3+2z^2-z-2}=-2,\quad \lim_{z\to\infty}\frac{z-1}{z+1}=1,$$

and $C=2=B[2(1-B^2)-A]=1\cdot[2\cdot(1-1)-(-2)]$.

Example 1.4. *The difference equation*

$$f(z+1)+f(z-1)=\frac{\frac{-44z^3+36z^2+28z+12}{z^4-2z^3-z^2+2z}f(z)-12}{1-f^2(z)}$$

has a rational solution $f(z)=\frac{2(z+1)}{z-1}$*, where* $m=3,\ n=4,\ p=q=1$ *and*

$$\lim_{z\to\infty}\frac{2(z+1)}{z-1}=2,\quad C=-12=2B(1-B^2)=2\cdot 2\cdot(1-2^2).$$

In [3], Chen and Shon also investigated some properties of meromorphic solutions of finite order of difference Painlevé II equation (1.5) and obtained the following result.

Theorem B. (See [3]) *Let* a,b,c *be constants with* $ac\neq 0$*. If* $f(z)$ *is a finite-order transcendental meromorphic solution of the difference Painlevé II equation* (1.5)*, then:*
(i) $f(z)$ *has at most one non-zero finite Borel exceptional value;*
(ii) $\lambda\left(\frac{1}{f}\right)=\lambda(f)=\sigma(f)$*;*
(iii) $f(z)$ *has infinitely many fixed points and satisfies* $\tau(f)=\sigma(f)$*.*

In this paper, we investigate the properties of a transcendental meromorphic solution of the difference equation

$$f(z+1)+f(z-1)=\frac{A(z)f(z)+C(z)}{1-f^2(z)}, \tag{1.7}$$

where $A(z)$, $C(z)$ are nonconstant small functions with respect to $f(z)$. And we obtain the following result.

Theorem 1.2. *Suppose that the difference equation* (1.7) *admits a transcendental meromorphic solution of finite order, then*
(i) $\lambda\left(\frac{1}{f}\right)=\lambda(f)=\sigma(f)$*;*
(ii) $\lambda\left(\frac{1}{\Delta f(z)}\right)=\lambda\left(\frac{1}{\frac{\Delta f(z)}{f(z)}}\right)=\sigma(f),\quad \lambda\left(\frac{1}{\Delta^2 f(z)}\right)=\lambda\left(\frac{1}{\frac{\Delta^2 f(z)}{f(z)}}\right)=\sigma(f)$*;*
(iii) If $2z(z^2-1)+zA(z)+C(z)\not\equiv 0$*, then* $\tau(f)=\sigma(f)$*;*
(iv) In particular, if $A(z)\pm C(z)\equiv 0$*, then* $f(z)$ *has at most one non-zero finite Borel exceptional value.*

2 Lemmas for the Proof of Theorems

Lemma 2.1. (See [10, Theorem 2.4],[5]) *Let $f(z)$ be a transcendental meromorphic solution of finite order σ of the difference equation*

$$P(z,f)=0,$$

where $P(z,f)$ is a difference polynomial in $f(z)$ and its shifts. If $P(z,a)\not\equiv 0$ for a slowly moving target meromorphic function a, that is, $T(r,a)=S(r,f)$, then

$$m\left(r,\frac{1}{f-a}\right)=O(r^{\sigma-1+\varepsilon})+S(r,f),$$

outside of a possible exceptional set of finite logarithmic measure.

Lemma 2.2. (See [10, *Theorem* 2.3]) *Let $f(z)$ be a transcendental meromorphic solution of finite order σ of a difference equation of the form*

$$U(z,f)P(z,f)=Q(z,f),$$

where $U(z,f),P(z,f)$ and $Q(z,f)$ are difference polynomials such that the total degree $\deg_f U(z,f)=n$ in $f(z)$ and its shifts, and $\deg_f Q(z,f)\le n$. Moreover, we assume that $U(z,f)$ contains just one term of maximal total degree in $f(z)$ and its shifts. Then for each $\varepsilon>0$,

$$m(r,P(z,f))=O(r^{\sigma-1+\varepsilon})+S(r,f),$$

possibly outside of an exceptional set of finite logarithmic measure.

Lemma 2.3. (See [13]) *Let $f(z)$ be a meromorphic function. Then for all irreducible rational functions in $f(z)$,*

$$R(z,f(z))=\frac{\sum_{i=0}^{m}a_i(z)f^i(z)}{\sum_{j=0}^{n}b_j(z)f^j(z)},$$

with meromorphic coefficients $a_i(z),b_j(z)(a_m(z)b_n(z)\not\equiv 0)$ being small with respect to $f(z)$, the characteristic function of $R(z,f(z))$ satisfies

$$T(r,R(z,f(z)))=\max\{m,n\}T(r,f)+S(r,f).$$

Lemma 2.4. (See [2, *Corollary* 2.5]) *Let $f(z)$ be a meromorphic function of finite order σ and let η be a non-zero complex number. Then for each $\varepsilon>0$, we have*

$$m\left(r,\frac{f(z+\eta)}{f(z)}\right)+m\left(r,\frac{f(z)}{f(z+\eta)}\right)=O(r^{\sigma-1+\varepsilon}).$$

Lemma 2.5. (See [2, *Theorem* 2.1]) *Let $f(z)$ be a meromorphic function with order $\sigma=\sigma(f),\sigma<+\infty$, and let η be a fixed non-zero complex number, then for each $\varepsilon>0$, we have*

$$T(r,f(z+\eta))=T(r,f(z))+O(r^{\sigma-1+\varepsilon})+O(\log r).$$

Lemma 2.6. (See [2, *Theorem* 2.2]) *Let $f(z)$ be a meromorphic function with exponent of convergence of poles $\lambda\left(\frac{1}{f}\right)=\lambda<\infty,\eta\ne 0$ be fixed, then for each $\varepsilon>0$,*

$$N(r,f(z+\eta))=N(r,f(z))+O(r^{\lambda-1+\varepsilon})+O(\log r).$$

Lemma 2.7. (See [8, *Remark* 1]) *Let $f(z)$ be a transcendental meromorphic function. If $P(z,f)$ and $Q(z,f)$ are mutually prime polynomials in $f(z)$, there exist polynomials of $f(z)$, $U(z,f)$ and $V(z,f)$ such that*

$$U(z,f)P(z,f)+V(z,f)Q(z,f)=s(z),$$

where $s(z)$ and coefficients of $U(z,f)$ and $V(z,f)$ are small functions with respect to $f(z)$.

Lemma 2.8. *Let $f(z)$ be a transcendental meromorphic function and $A(z) \pm C(z) \not\equiv 0$, then $A(z)f(z)+C(z)$ and $f(z)(1-f^2(z))$ are mutually prime polynomials in $f(z)$, where $A(z)$, $C(z)$ are nonzero small functions with respect to $f(z)$.*

Proof. Since $A(z)$, $C(z)$ are non-zero small functions with respect to $f(z)$ and $A(z)\pm C(z) \not\equiv 0$, then $C(z)(C^2(z) - A^2(z)) \not\equiv 0$, $T(r, C(z)(C^2(z) - A^2(z))) = S(r, f)$. There exist polynomials of $f(z)$, $U(z, f) = A^2(z)f^2(z) - A(z)C(z)f(z) + C^2(z) - A^2(z)$ and $V(z, f) = A^3(z)$ such that

$$U(z, f)(A(z)f(z) + C(z)) + V(z, f)f(z)(1 - f^2(z)) = C(z)(C^2(z) - A^2(z)).$$

By Lemma 2.7, we see that $A(z)f(z) + C(z)$ and $f(z)(1 - f^2(z))$ are mutually prime.

Lemma 2.9. *Let $f(z)$ be a transcendental meromorphic function and $A(z) \pm C(z) \not\equiv 0$, then $2f^3(z) + (A(z) - 2)f(z) + C(z)$ and $1 - f^2(z)$ are mutually prime polynomials in $f(z)$, where $A(z)$, $C(z)$ are non-zero small functions with respect to $f(z)$.*

Proof. Since $A(z)$, $C(z)$ are non-zero small functions with respect to $f(z)$ and $A(z)\pm C(z) \not\equiv 0$, then $A^2(z) - C^2(z) \not\equiv 0$, $T(r, A^2(z) - C^2(z)) = S(r, f)$. There exist polynomials of $f(z)$, $U(z, f) = A(z)f(z) - C(z)$ and $V(z, f) = 2A(z)f^2(z) - 2C(z)f(z) + A^2(z)$ such that

$$U(z, f)(2f^3(z) + (A(z) - 2)f(z) + C(z)) + V(z, f)(1 - f^2(z)) = A^2(z) - C^2(z).$$

By Lemma 2.7, we see that $2f^3(z) + (A(z) - 2)f(z) + C(z)$ and $1 - f^2(z)$ are mutually prime.

Lemma 2.10. (See [15, *Theorem* 1.51]) *Suppose that $f_1(z), f_2(z), \dots, f_n(z) (n \geq 2)$ are meromorphic functions and $g_1(z), g_2(z), \dots, g_n(z)$ are entire functions satisfying the following conditions,*
(1) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv 0$;
(2) $g_j(z) - g_k(z)$ are not constants for $1 \leq j < k \leq n$;
(3) For $1 \leq j \leq n, 1 \leq h < k \leq n$, $T(r, f_j(z)) = o(T(r, e^{g_h(z)-g_k(z)}))(r \to \infty, r \notin E)$, where $E \subset [1, \infty)$ is finite linear measure or finite logarithmic measure.
Then $f_j(z) \equiv 0$ $(j = 1, \dots, n)$.

Lemma 2.11. *Suppose that $f(z)$ is a transcendental meromorphic solution of finite order of the difference equation*

$$f(z + 1) + f(z - 1) = \frac{\delta A(z)}{1 - \delta f(z)}, \tag{2.1}$$

where $\delta = \pm 1$, $A(z)$ is a non-constant small function with respect to $f(z)$, then
(i) $\lambda\left(\frac{1}{f}\right) = \lambda(f) = \sigma(f)$;
(ii) $f(z)$ has at most one non-zero finite Borel exceptional value.

Proof. (i) Since $f(z)$ is a finite order transcendental meromorphic solution of the equation (2.1), then we have

$$P(z, f) := \delta f(z)(f(z + 1) + f(z - 1)) - (f(z + 1) + f(z - 1)) + \delta A(z) \equiv 0. \tag{2.2}$$

By (2.2), we obtain

$$P(z, 0) = \delta A(z) \not\equiv 0. \tag{2.3}$$

It follows from (2.3) and Lemma 2.1 that

$$m\left(r, \frac{1}{f}\right) = S(r, f),$$

outside of a finite exceptional set of logarithmic measure. Then

$$N\left(r, \frac{1}{f}\right) = T(r, f) + S(r, f),$$

that is, $\lambda(f)=\sigma(f)$.

Next, we will prove $\lambda\left(\frac{1}{f}\right)=\sigma(f)$. It follows from (2.1) that

$$\delta f(z)(f(z+1)+f(z-1))=f(z+1)+f(z-1)-\delta A(z). \tag{2.4}$$

Set $\sigma(f)=\sigma<\infty$, by Lemma 2.2, we have

$$m(r,f(z+1)+f(z-1))=O(r^{\sigma-1+\varepsilon})+S(r,f), \tag{2.5}$$

possibly outside of an exceptional set of finite logarithmic measure. By (2.1) and Lemma 2.3, we obtain

$$T(r,f(z+1)+f(z-1))=T\left(r,\frac{\delta A(z)}{1-\delta f(z)}\right)=T(r,f)+S(r,f). \tag{2.6}$$

Since $\lambda\left(\frac{1}{f}\right)\le\sigma(f)<\infty$, it follows from (2.5), (2.6) and Lemma 2.6 that

$$\begin{aligned} T(r,f)+O(r^{\sigma-1+\varepsilon})+S(r,f)&=N(r,f(z+1)+f(z-1))\\ &\le 2N(r,f)+O(r^{\lambda(\frac{1}{f})-1+\varepsilon})+O(\log r), \end{aligned} \tag{2.7}$$

then $\lambda\left(\frac{1}{f}\right)=\sigma(f)$.

(ii) By (i), we see that $0,\ \infty$ are not the Borel exceptional values of $f(z)$. Suppose that $f(z)$ has two distinct finite Borel exceptional values $a(\neq 0)$ and $b(\neq 0,a)$.

Set

$$g(z)=\frac{f(z)-a}{f(z)-b}. \tag{2.8}$$

Then

$$\sigma(g)=\sigma(f),\ \lambda(g)=\lambda(f-a)<\sigma(g),\ \lambda\left(\frac{1}{g}\right)=\lambda(f-b)<\sigma(g).$$

That is, 0 and ∞ are the Borel exceptional values of $g(z)$. By [15, Theorem 2.11], we see that $g(z)$ is of regular growth, then $g(z)$ can be rewritten as

$$g(z)=P(z)e^{dz^{\sigma}}, \tag{2.9}$$

where $d(\neq 0)$ is a constant, $\sigma(g)=\sigma(\ge 1)$ is a positive integer, $P(z)$ is a meromorphic function with $\sigma(P)<\sigma(g)=\sigma$. By (2.8) and (2.9), we have

$$f(z)=b+\frac{b-a}{g(z)-1}=b+\frac{b-a}{P(z)e^{dz^{\sigma}}-1} \tag{2.10}$$

and

$$\begin{aligned} f(z+1)&=b+\frac{b-a}{P(z+1)e^{d(z+1)^{\sigma}}-1}=b+\frac{b-a}{P(z+1)P_{+1}(z)e^{dz^{\sigma}}-1},\\ f(z-1)&=b+\frac{b-a}{P(z-1)e^{d(z-1)^{\sigma}}-1}=b+\frac{b-a}{P(z-1)P_{-1}(z)e^{dz^{\sigma}}-1}, \end{aligned} \tag{2.11}$$

where

$$P_{+1}(z)=\exp\left\{d\sum_{i=1}^{\sigma}\binom{\sigma}{i}z^{\sigma-i}\right\},\quad P_{-1}(z)=\exp\left\{d\sum_{i=1}^{\sigma}(-1)^{i}\binom{\sigma}{i}z^{\sigma-i}\right\}.$$

Substituting (2.10) and (2.11) into (2.1) yields

$$C_1(z)e^{3dz^{\sigma}}+C_2(z)e^{2dz^{\sigma}}+C_3(z)e^{dz^{\sigma}}+C_4(z)=0, \tag{2.12}$$

where

$$\begin{cases} C_1(z) & = [\delta A(z) - 2b(1-\delta b)]P(z)P(z+1)P_{+1}(z)P(z-1)P_{-1}(z); \\ C_2(z) & = [(a+b)(1-\delta b) - \delta A(z)]P(z)[P(z+1)P_{+1}(z) + P(z-1)P_{-1}(z)] \\ & \quad +[2b(1-\delta a) - \delta A(z)]P(z+1)P_{+1}(z)P(z-1)P_{-1}(z); \\ C_3(z) & = [\delta A(z) - (a+b)(1-\delta a)][P(z+1)P_{+1}(z) + P(z-1)P_{-1}(z)] \\ & \quad +[\delta A(z) - 2a(1-\delta b)]P(z); \\ C_4(z) & = 2a(1-\delta a) - \delta A(z). \end{cases} \tag{2.13}$$

It follows from (2.13) and Lemma 2.10 that

$$C_1(z) \equiv C_2(z) \equiv C_3(z) \equiv C_4(z) \equiv 0.$$

Since $P(z)P(z+1)P_{+1}(z)P(z-1)P_{-1}(z) \not\equiv 0$ and $C_1(z) \equiv C_4(z) \equiv 0$, we obtain

$$\delta A(z) \equiv 2b(1-\delta b) \quad \textit{and} \quad \delta A(z) \equiv 2a(1-\delta a). \tag{2.14}$$

Note that $A(z)$ is a non-constant function, which shows that (2.14) is a contradiction.

3 Proof of Theorems

Proof of Theorem 1.1

Suppose that $f(z) = \frac{P(z)}{Q(z)}$ is a rational solution of (1.6). Then (1.6) can be rewritten as

$$\left[\frac{P(z+1)}{Q(z+1)} + \frac{P(z-1)}{Q(z-1)}\right]\left[1 - \frac{P^2(z)}{Q^2(z)}\right] = \frac{m(z)}{n(z)} \cdot \frac{P(z)}{Q(z)} + C \tag{3.1}$$

or

$$\begin{aligned} & n(z)[Q^2(z) - P^2(z)][P(z+1)Q(z-1) + P(z-1)Q(z+1)] \\ & = m(z)P(z)Q(z)Q(z+1)Q(z-1) + Cn(z)Q^2(z)Q(z+1)Q(z-1). \end{aligned} \tag{3.2}$$

(i) Suppose that $m > n$. If $p > q$, (3.2) yields

$$\begin{cases} \deg(n(z)[Q^2(z) - P^2(z)][P(z+1)Q(z-1) + P(z-1)Q(z+1)]) = n + 3p + q, \\ \deg(m(z)P(z)Q(z)Q(z+1)Q(z-1)) = m + 3q + p, \\ \deg(Cn(z)Q^2(z)Q(z+1)Q(z-1)) = n + 4q. \end{cases} \tag{3.3}$$

By $n + 3p + q > n + 4q$ and $m + 3q + p > n + 4q$, then we must have $n + 3p + q = m + 3q + p$, that is $p - q = \frac{m-n}{2}$, $m - n$ must be even.

If $p = q$, then $\frac{P(z)}{Q(z)} \to B$, $\frac{P(z+1)}{Q(z+1)} \to B$, $\frac{P(z-1)}{Q(z-1)} \to B$ as $z \to \infty$, where B is a nonzero constant, while $\frac{m(z)}{n(z)} \to \infty$ as $z \to \infty$. If $B = 1$ or $B = -1$, then

$$\left[\frac{P(z+1)}{Q(z+1)} + \frac{P(z-1)}{Q(z-1)}\right]\left[1 - \frac{P^2(z)}{Q^2(z)}\right] \to 0, \quad \frac{m(z)}{n(z)} \cdot \frac{P(z)}{Q(z)} + C \to \infty \textit{ as } z \to \infty.$$

If $B \neq \pm 1$, then

$$\left[\frac{P(z+1)}{Q(z+1)} + \frac{P(z-1)}{Q(z-1)}\right]\left[1 - \frac{P^2(z)}{Q^2(z)}\right] \to 2B(1-B^2), \quad \frac{m(z)}{n(z)} \cdot \frac{P(z)}{Q(z)} + C \to \infty \textit{ as } z \to \infty.$$

These show that (3.1) is a contradiction.

If $p < q$, (3.2) yields

$$\begin{cases} \deg(n(z)[Q^2(z) - P^2(z)][P(z+1)Q(z-1) + P(z-1)Q(z+1)]) = n + 3q + p, \\ \deg(m(z)P(z)Q(z)Q(z+1)Q(z-1)) = m + 3q + p, \\ \deg(Cn(z)Q^2(z)Q(z+1)Q(z-1)) = n + 4q. \end{cases} \tag{3.4}$$

By $m+3q+p > n+3q+p$ and $n+4q > n+3q+p$, then we must have $m+3q+p = n+4q$, that is $q-p = m-n \geq 1$.

(ii) Suppose that $m = n$. If $p > q$, by (3.3), we see that $n+3p+q > m+3q+p > n+4q$. This shows that there is only one term in (3.2) which has the highest degree, a contradiction.

If $p = q$, then $\frac{P(z)}{Q(z)} \to B$, $\frac{P(z+1)}{Q(z+1)} \to B$, $\frac{P(z-1)}{Q(z-1)} \to B$ as $z \to \infty$, where B is a nonzero constant, while $\frac{m(z)}{n(z)} \to A$ as $z \to \infty$, where A is a nonzero constant. If $B = 1$ or $B = -1$, then

$$\left[\frac{P(z+1)}{Q(z+1)} + \frac{P(z-1)}{Q(z-1)}\right]\left[1 - \frac{P^2(z)}{Q^2(z)}\right] \to 0, \quad \frac{m(z)}{n(z)} \cdot \frac{P(z)}{Q(z)} + C \to AB + C \ as\ z \to \infty.$$

So we have $C \pm A = 0$. If $B \neq \pm 1$, then

$$\left[\frac{P(z+1)}{Q(z+1)} + \frac{P(z-1)}{Q(z-1)}\right]\left[1 - \frac{P^2(z)}{Q^2(z)}\right] \to 2B(1-B^2), \quad \frac{m(z)}{n(z)} \cdot \frac{P(z)}{Q(z)} + C \to AB + C \ as\ z \to \infty.$$

So we have $C = B[2(1-B^2) - A]$.

If $p < q$, by (3.4), we see that $n+4q > n+3q+p = m+3q+p$. This also shows that there is only one term in (3.2) which has the highest degree, a contradiction.

(iii) Suppose that $m < n$. If $p > q$, by (3.3), we see that $n+3p+q > m+3q+p$, $n+3p+q > n+4q$, that is, there is only one term in (3.2) which has the highest degree, a contradiction.

If $p = q$, then $\frac{P(z)}{Q(z)} \to B$, $\frac{P(z+1)}{Q(z+1)} \to B$, $\frac{P(z-1)}{Q(z-1)} \to B$ as $z \to \infty$, where B is a nonzero constant, while $\frac{m(z)}{n(z)} \to 0$ as $z \to \infty$. If $B = 1$ or $B = -1$, then

$$\left[\frac{P(z+1)}{Q(z+1)} + \frac{P(z-1)}{Q(z-1)}\right]\left[1 - \frac{P^2(z)}{Q^2(z)}\right] \to 0, \quad \frac{m(z)}{n(z)} \cdot \frac{P(z)}{Q(z)} + C \to C \ as\ z \to \infty.$$

So we have $C = 0$, a contradiction. If $B \neq \pm 1$, then

$$\left[\frac{P(z+1)}{Q(z+1)} + \frac{P(z-1)}{Q(z-1)}\right]\left[1 - \frac{P^2(z)}{Q^2(z)}\right] \to 2B(1-B^2), \quad \frac{m(z)}{n(z)} \cdot \frac{P(z)}{Q(z)} + C \to C \ as\ z \to \infty.$$

So we have $C = 2B(1-B^2)$.

If $p < q$, by (3.4), we see that $n+4q > n+3q+p > m+3q+p$. That is, there is only one term in (3.2) which has the highest degree, a contradiction.

This completes the proof of Theorem 1.1.

Proof of Theorem 1.2

In what follows, we consider two cases: $A(z) \pm C(z) \not\equiv 0$ and $A(z) \pm C(z) \equiv 0$.

Case 1, $A(z) \pm C(z) \not\equiv 0$.

(i) Using the same method as in the proof of Lemma 2.11 (i), we can obtain that $\lambda\left(\frac{1}{f}\right) = \lambda(f) = \sigma(f)$.

(ii) First, we will prove $\lambda\left(\frac{1}{\frac{\Delta f(z)}{f(z)}}\right) = \sigma(f)$. By equation (1.7), Lemmas 2.3, 2.5 and 2.8, we have

$$\begin{aligned}
3T(r, f(z)) &= T\left(r, \frac{A(z)f(z) + C(z)}{f(z)(1-f^2(z))}\right) + S(r,f) \\
&= T\left(r, \frac{f(z+1) + f(z-1)}{f(z)}\right) + S(r,f) \\
&\leq T\left(r, \frac{f(z+1)}{f(z)}\right) + T\left(r, \frac{f(z)}{f(z-1)}\right) + S(r,f) \\
&= 2T\left(r, \frac{f(z+1)}{f(z)}\right) + S\left(r, \frac{f(z+1)}{f(z)}\right) + S(r,f) \\
&\leq 2T\left(r, \frac{f(z+1)}{f(z)}\right) + S(r,f) \\
&= 2T\left(r, \frac{\Delta f(z)}{f(z)}\right) + S(r,f),
\end{aligned}$$

that is,

$$\frac{3}{2}T(r,f(z)) \leq T\left(r,\frac{\Delta f(z)}{f(z)}\right)+S(r,w). \tag{3.5}$$

It follows from (3.5) and Lemma 2.4 that

$$\begin{aligned} N\left(r,\frac{\Delta f(z)}{f(z)}\right) &= T\left(r,\frac{\Delta f(z)}{f(z)}\right)-m\left(r,\frac{\Delta f(z)}{f(z)}\right) \\ &\geq \frac{3}{2}T(r,f(z))+S(r,f). \end{aligned}$$

Thus, $\lambda\left(\frac{1}{\frac{\Delta f(z)}{f(z)}}\right)=\sigma(f)$.

Next, we will prove $\lambda\left(\frac{1}{\Delta f(z)}\right)=\sigma(f)$. By equation (1.7),

$$\begin{aligned} \Delta f(z)-\Delta f(z-1) &= f(z+1)+f(z-1)-2f(z) \\ &= \frac{A(z)f(z)+C(z)}{1-f^2(z)}-2f(z) \\ &= \frac{2f^3(z)+(A(z)-2)f(z)+C(z)}{1-f^2(z)}. \end{aligned} \tag{3.6}$$

From (3.6), Lemmas 2.3, 2.5 and 2.9, we have

$$\begin{aligned} 3T(r,f(z)) &= T\left(r,\frac{2f^3(z)+(A(z)-2)f(z)+C(z)}{1-f^2(z)}\right)+S(r,f) \\ &= T(r,\Delta f(z)-\Delta f(z-1))+S(r,f) \\ &\leq 2T(r,\Delta f(z))+S(r,\Delta f(z))+S(r,f) \\ &\leq 2T(r,\Delta f(z))+S(r,f), \end{aligned}$$

that is,

$$\frac{3}{2}T(r,f(z)) \leq T(r,\Delta f(z))+S(r,f). \tag{3.7}$$

It follows from (3.7) and Lemma 2.4 that

$$\begin{aligned} N(r,\Delta f(z)) &= T(r,\Delta f(z))-m(r,\Delta f(z)) \\ &\geq T(r,\Delta f(z)-\left(m\left(r,\frac{\Delta f(z)}{f(z)}\right)+m(r,f(z))\right) \\ &\geq \frac{3}{2}T(r,f(z))-T(r,f(z))+S(r,f) \\ &= \frac{1}{2}T(r,f(z))+S(r,f). \end{aligned}$$

Hence, $\lambda\left(\frac{1}{\Delta f(z)}\right)=\sigma(f)$.

Furthermore, we will prove $\lambda\left(\frac{1}{\Delta^2 f(z)}\right)=\sigma(f)$. By (3.6), we have

$$\Delta^2 f(z-1)=\Delta f(z)-\Delta f(z-1)=\frac{2f^3(z)+(A(z)-2)f(z)+C(z)}{1-f^2(z)}. \tag{3.8}$$

From (3.8), Lemmas 2.3, 2.5 and 2.9, we have

$$\begin{aligned} 3T(r,f(z)) &= T\left(r,\frac{2f^3(z)+(A(z)-2)f(z)+C(z)}{1-f^2(z)}\right)+S(r,f) \\ &= T(r,\Delta^2 f(z-1))+S(r,f) \\ &= T(r,\Delta^2 f(z))+S(r,f). \end{aligned} \tag{3.9}$$

It follows from (3.9) and Lemma 2.4 that

$$\begin{aligned} N(r,\Delta^2 f(z)) &= T(r,\Delta^2 f(z)) - m(r,\Delta^2 f(z)) \\ &\geq 3T(r,f(z)) - \left(m\left(r,\frac{\Delta f(z)}{f(z)}\right) + m(r,f(z))\right) \\ &\geq 3T(r,f(z)) - T(r,f(z)) + S(r,f) \\ &= 2T(r,f(z)) + S(r,f). \end{aligned}$$

Thus, $\lambda\left(\frac{1}{\Delta^2 f(z)}\right) = \sigma(f)$.

Finally, we will prove $\lambda\left(\frac{1}{\frac{\Delta^2 f(z)}{f(z)}}\right) = \sigma(f)$. It follows from (3.9) and Lemma 2.4 that

$$\begin{aligned} N\left(r,\frac{\Delta^2 f}{f}\right) &= T\left(r,\frac{\Delta^2 f}{f}\right) - m\left(r,\frac{\Delta^2 f}{f}\right) \\ &\geq T(r,\Delta^2 f(z)) - T(r,f(z)) + S(r,f) \\ &= 3T(r,f(z)) - T(r,f(z)) + S(r,f) \\ &= 2T(r,f(z)) + S(r,f). \end{aligned}$$

Then, $\lambda\left(\frac{1}{\frac{\Delta^2 f(z)}{f(z)}}\right) = \sigma(f)$.

(iii) Suppose that $f(z)$ is a finite order transcendental meromorphic solution of equation (1.7). Set $g(z) = f(z) - z$. Then $g(z)$ is a transcendental meromorphic function with $\sigma(g) = \sigma(f) < \infty$ and $\tau(f) = \lambda(g)$. Substituting $f(z) = g(z) + z$ into (1.7) yields

$$P(z,g) := [g(z+1) + g(z-1) + 2z][(g(z)+z)^2 - 1] + A(z)g(z) + zA(z) + C(z) \equiv 0. \quad (3.10)$$

Since $P(z,0) = 2z(z^2-1) + zA(z) + C(z) \not\equiv 0$, it follows from (3.10) and Lemma 2.1 that

$$N\left(r,\frac{1}{g}\right) = T(r,g) + S(r,g) = T(r,f) + S(r,f).$$

Then, $\tau(f) = \lambda(g) = \sigma(g) = \sigma(f)$.

Case 2, $A(z) \pm C(z) \equiv 0$. Rewriting equation (1.7) as

$$f(z+1) + f(z-1) = \frac{\delta A(z)}{1 - \delta f(z)}, \quad (3.11)$$

where $\delta = \pm 1$.

By Lemma 2.11, we see that (i) and (iv) hold. Using the same method as in the proof of Case 1 (ii) and (iii), we can also obtain (ii) and (iii). If $2z(1-\delta z) - \delta A(z) \not\equiv 0$, then $\tau(f) = \sigma(f)$.

This completes the proof of Theorem 1.2.

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Catalan Numbers, k-Gamma and k-Beta Functions, and Parametric Integrals

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Abstract

In the paper, the authors establish some new explicit formulas and integral representations of the Catalan numbers and a class of parametric integrals in terms of the k-gamma and k-beta functions.

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1 Introduction and main results

The Catalan numbers C_n for $n \ge 0$ form a sequence of natural numbers that occur in various counting problems in combinatorial mathematics. The nth Catalan number can be expressed in terms of the central binomial coefficients $\binom{2n}{n}$ by

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n!(n+1)!} = \frac{4^n\Gamma\bigl(n+\frac{1}{2}\bigr)}{\sqrt{\pi}\,\Gamma(n+2)}, \tag{1}$$

where $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}\,t$ for $x > 0$ is the classical Euler gamma function, or say, the Euler integral of the second kind.

The Catalan numbers C_n were described in the 18th century by Leonhard Euler and are named after the Belgian mathematician Eugéne Charles Catalan. In 1988, it came to light that the Catalan numbers C_n had been used in China by the Mongolian mathematician Ming Antu by 1730, see [11, 12, 14, 16, 17, 18, 19, 20, 35]. In recent years, the Catalan numbers C_n has been analytically generalized and studied in [15, 21, 23, 25, 26, 27, 28, 30, 29, 31, 33, 36] and closely-related references therein. For more information on the Catalan numbers C_n, please refer to the monographs [2, 3, 8, 9, 10, 32, 34] and closely-related references therein.

It is common knowledge [1, p. 4] that the beta function, or say, the Euler integral of the first kind $B(x,y)$ can be defined by $B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}\,t$ and satisfies $B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ for $x,y>0$. The rising factorial, denoted by $(x)_n$, is defined [13, p. 13] by $(x)_n=x(x+1)\cdots(x+n-1)=\frac{\Gamma(x+n)}{\Gamma(x)}$ which is frequently called the Pochhammer symbol in mathematics.

Diaz and Pariguan [5, p. 180] introduced the Pochhammer k-symbol as

$$(x)_{n,k}=x(x+k)(x+2k)\cdots[x+(n-1)k].$$

It is clear that $(x)_{n,1}=(x)_n$.

Diaz et al. [4, 5, 6] introduced the k-gamma and k-beta functions and proved a number of their properties. They also studied the k-zeta function and the k-hypergeometric functions based on the Pochhammer k-symbols. The k-gamma function is defined in [5, p. 180] by

$$\Gamma_k(x)=\lim_{n\to\infty}\frac{n!k^n(kn)^{x/k-1}}{(x)_{n,\alpha}},\quad k>0.$$

It was showed [5, p. 180] that the Mellin transform of the exponential function $e^{-t^k/k}$ is the k-gamma function, that is,

$$\Gamma_k(x)=\int_0^\infty t^{x-1}e^{-t^k/k}\,\mathrm{d}\,t.$$

It is easy to see that

$$\Gamma(x)=\Gamma_1(x),\quad \Gamma_k(x)=k^{x/k-1}\Gamma\left(\frac{x}{k}\right),\quad \Gamma_k(x+k)=x\Gamma_k(x). \tag{2}$$

This gives rise to the k-beta function

$$B_k(x,y)=\frac{1}{k}\int_0^1 t^{x/k-1}(1-t)^{y/k-1}\,\mathrm{d}\,t \tag{3}$$

which satisfies

$$B_k(x,y)=\frac{1}{k}B\left(\frac{x}{k},\frac{y}{k}\right)\quad\text{and}\quad B_k(x,y)=\frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}. \tag{4}$$

The aim of this paper is to establish some new explicit formulas and integral representations of the Catalan numbers C_n and parametric integrals

$$I_{\alpha,\beta;k}(a)=\int_0^a x^{(\alpha+1)k-1}\left(a^{2k}-x^{2k}\right)^\beta\,\mathrm{d}\,x$$

for $a,k>0$ and some $\alpha,\beta>-1$ in terms of the k-gamma function $\Gamma_k(x)$ and the k-beta function $B_k(x,y)$.

Our main results in this paper can be stated as the following theorems.

Theorem 1.1. *For $k>0$ and $n\in\mathbb{N}$, we have*

$$C_n=k^{3/2}\frac{4^n\Gamma_k\left(\frac{2n+1}{2}k\right)}{\sqrt{\pi}\,\Gamma_k((n+2)k)}=\frac{k2^{1+2n(1-k)}}{\pi(n+1)}\int_0^2\frac{x^{(2n+1)k-1}}{\sqrt{2^{2k}-x^{2k}}}\,\mathrm{d}\,x. \tag{5}$$

Theorem 1.2. *For $a,k>0$ and $n\ge 0$, we have*

$$I_{n,1/2;k}(a)=\frac{k^{1/2}\sqrt{\pi}}{4}a^{(n+2)k}\frac{\Gamma_k\left(\frac{n+1}{2}k\right)}{\Gamma_k\left(\frac{n+4}{2}k\right)}=\frac{k^{1/2}}{2}a^{(n+2)k}B_k\left(\frac{n+1}{2}k,\frac{3}{2}k\right) \tag{6}$$

and

$$I_{n,-1/2;k}(a)=\sqrt{\frac{\pi}{k}}\,\frac{a^{nk}}{2}\frac{\Gamma_k\left(\frac{n+1}{2}k\right)}{\Gamma_k\left(\frac{n+2}{2}k\right)}=\frac{a^{nk}}{2}B\left(\frac{n+1}{2}k,\frac{1}{2}k\right). \tag{7}$$

For $a,k>0$ and $\alpha,\beta>-1$, we have

$$I_{\alpha,\beta;k}(a)=\frac{a^{\alpha+2\beta k+1}}{2}B_k\left(\frac{\alpha+1}{2},(\beta+1)k\right). \tag{8}$$

2 Proofs of main results

We are now in a position to prove our main results stated in Theorems 1.1 and 1.2.

From the second relation in (2), we have

$$\Gamma_k\left(\frac{2n+1}{2}k\right)=k^{(2n+1)k/2k-1}\Gamma\left(\frac{(2n+1)k}{2k}\right)=k^{(2n-1)/2}\Gamma\left(n+\frac{1}{2}\right)=k^{(2n-1)/2}\frac{(2n)!\sqrt{\pi}}{4^n n!}.$$

Accordingly, it follows that

$$\frac{\Gamma_k\left(\frac{2n+1}{2}k\right)}{\Gamma_k((n+2)k)}=\frac{k^{(2n-1)/2}\frac{(2n)!\sqrt{\pi}}{4^n n!}}{k^{n+1}(n+1)!}=k^{-3/2}\frac{(2n)!\sqrt{\pi}}{4^n n!(n+1)!}=k^{-3/2}\frac{\sqrt{\pi}\,C_n}{4^n}.$$

The explicit formulas in (5) thus follow.

By changing variables $x=at^{1/2k}$ for $t\in[0,1]$, we have

$$\begin{aligned}I_{n,1/2;k}(a)&=\int_0^a\left(at^{1/2k}\right)^{(n+1)k-1}\left[a^{2k}-\left(at^{1/2k}\right)^{2k}\right]^{1/2}\frac{a}{2k}t^{1/2k-1}\,\mathrm{d}\,t\\&=\frac{a^{(n+2)k}}{2k}\int_0^1 t^{(n+1)k/2k-1}(1-t)^{1/2+1-1}\,\mathrm{d}\,t\\&=\frac{a^{(n+2)k}}{2k}\int_0^1 t^{(n+1)k/2k-1}(1-t)^{3k/2k-1}\,\mathrm{d}\,t=\frac{a^{(n+2)k}}{2}B_k\left(\frac{n+1}{2}k,\frac{3}{2}k\right).\end{aligned}$$

Utilizing the second relation in (4) gives

$$I_{n,1/2;k}(a)=\frac{a^{(n+2)k}}{2}\frac{\Gamma_k\left(\frac{n+1}{2}k\right)\Gamma_k\left(\frac{3}{2}k\right)}{\Gamma_k\left(\frac{n+4}{2}k\right)}.$$

Further using

$$\Gamma_k\left(\frac{3}{2}k\right)=k^{1/2}\Gamma\left(\frac{3}{2}\right)=\frac{\sqrt{k\pi}}{2}\quad\text{and}\quad\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}\,,$$

we obtain the formulas in (6).

By the Leibniz rule for differentiation

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\int_{g(t)}^{h(t)} F(x,t)\,\mathrm{d}\,x = h'(t)F(h(t),t) - g'(t)F(g(t),t) + \int_{g(t)}^{h(t)} \frac{\partial}{\partial t}F(x,t)\,\mathrm{d}\,x$$

in [7, p. 615], differentiating with respect to a on both sides of (6) gives

$$I'_{n,1/2;k}(a) = ka^{2k-1}\int_0^a \frac{x^{(n+1)k-1}}{\left(a^{2k}-x^{2k}\right)^{1/2}}\,\mathrm{d}\,x = \frac{\sqrt{k\pi}}{4}(n+2)ka^{(n+2)k-1}\frac{\Gamma_k\left(\frac{n+1}{2}k\right)}{\Gamma_k\left(\frac{(n+4)k}{2}\right)}.$$

Therefore, it follows that

$$I_{n,-1/2;k}(a) = \frac{\sqrt{k\pi}}{4}(n+2)a^{nk}\frac{\Gamma_k\left(\frac{n+1}{2}k\right)}{\Gamma_k\left(\frac{n+4}{2}k\right)}. \tag{9}$$

The formulas in (7) are thus acquired.

Letting $a=2$ and replacing n by $2n$ in (9) derive

$$\frac{k2^{1+2n(1-k)}}{\pi(n+1)}\int_0^2 \frac{x^{(2n+1)k-1}}{\sqrt{2^{2k}-x^{2k}}}\,\mathrm{d}\,x = k^{3/2}\frac{4^n\Gamma_k\left(\frac{2n+1}{2}k\right)}{\sqrt{\pi}\,\Gamma_k\left((n+2)k\right)} = C_n.$$

The integral representation in (5) is thus proved.

By changing variables $x = a\sin^{1/k}\theta$ for $\theta\in\left[0,\frac{\pi}{2}\right]$, we have

$$\begin{aligned} I_{n,k}(a) &= \int_0^{\pi/2}\left(a\sin^{1/k}\theta\right)^{(\alpha+1)k-1}\left(a^{2k}-a^{2k}\sin^2\theta\right)^{\beta}\frac{a}{k}\sin^{1/k-1}\theta\cos\theta\,\mathrm{d}\,\theta \\ &= \frac{a^{(\alpha+2\beta+1)k}}{k}\int_0^{\pi/2}\sin^{\alpha}\theta\cos^{2\beta+1}\theta\,\mathrm{d}\,\theta \\ &= \frac{a^{(\alpha+2\beta+1)k}}{k}\int_0^{\pi/2}\sin^{2k(\alpha+1)/2k-1}\theta\cos^{2k(\beta+1)/k-1}\theta\,\mathrm{d}\,\theta = \frac{a^{(\alpha+2\beta+1)k}}{2}B_k\left(\frac{\alpha+1}{2}k,(\beta+1)k\right), \end{aligned}$$

where we used in the last step the formula

$$\int_0^{\pi/2}\sin^{2x/k-1}\theta\cos^{2y/k-1}\theta\,\mathrm{d}\,\theta = \frac{k}{2}B_k(x,y), \quad \Re(x),\Re(y)>0,$$

which can be derived from using the change of the variable $t=\sin^2\theta$ in (3) by

$$\begin{aligned} B_k(x,y) &= \frac{1}{k}\int_0^{\pi/2}\left(\sin^2\theta\right)^{x/k-1}\left(1-\sin^2\theta\right)^{y/k-1}2\sin\theta\cos\theta\,\mathrm{d}\,\theta \\ &= \frac{2}{k}\int_0^{\pi/2}\sin^{2x/k-1}\theta\cos^{2y/k-1}\theta\,\mathrm{d}\,\theta. \end{aligned}$$

The proof of the formula (8) is thus complete.

3 Remarks

Finally we give some remarks about connections between our main results and some know conclusions.

Remark 3.1. When $k = 1$, the expression (5) becomes the last expression in (1).

Letting $k = 1$ in (6), we can deduce [22, Theorem 2.1].

Letting $k = 1$ in (7), we can obtain [22, Theorem 3.1].

Letting $k = 1$ in (8), we can recover [22, Theorem 5.1].

Letting $k = 1, \alpha = n$, and $\beta = \frac{1}{2}$ in (8), we recover [22, Remark 6.1].

Letting $k = 1, \alpha = 2n$, and $\beta = \frac{1}{2}$ in (8), we can obtain [22, Remark 6.2].

Let $k = 1, \alpha = n$, and $\beta = -\frac{1}{2}$ in (6), we can derive [22, Remark 6.3].

The Wallis ratio $W_n = \frac{(2n)!}{4^n (n!)^2}$ can also be expressed by

$$W_n = \sqrt{\frac{k}{\pi}} \frac{\Gamma_k\left(\frac{2n+1}{2}k\right)}{\Gamma_k((n+1)k)}$$

for $n \in \mathbb{N}$ and $k > 0$, which is a generalization of [22, Remark 6.5].

Remark 3.2. This paper is a slightly revised version of the preprint [24].

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Generalized von Neumann-Jordan and James Constants for Quasi-Banach Spaces

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Abstract

The generalized von Neumann-Jordan constant $C_{NJ}^{(p)}(\mathcal{B})$ and the James constant $J(\mathcal{B})$ for a quasi-Banach space were introduced in [7]. In this note, it is shown that $C_{NJ}^{(p)}(\mathcal{B}) \leq 2$ for any quasi-Banach space $\mathcal{B}$ and $C_{NJ}^{(p)}(\mathcal{B}) < 2$ if and only if $\mathcal{B}$ is uniformly non-square. Along with relationship between $J(\mathcal{B})$ and $C_{NJ}^{(p)}(\mathcal{B})$, the criterion for the uniformly smooth quasi-Banach space is also established.

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1 Introduction

Among various geometric constants of a Banach space $\mathcal{B}$, the von Neumann-Jordan constant $C_{NJ}(\mathcal{B})$ for a Banach space $\mathcal{B}$ introduced by Clarkson [2] as the smallest constant C, for which the estimates

$$\frac{1}{C} \leq \frac{\|x_1 + x_2\|^2 + \|x_1 - x_2\|^2}{2(\|x_1\|^2 + \|x_2\|^2)} \leq C$$

hold for any $x_1, x_2 \in \mathcal{B}$ with $(x_1, x_2) \neq (0, 0)$. Equivalently

$$C_{NJ}(\mathcal{B}) = \sup\left\{\frac{\|x_1 + x_2\|^2 + \|x_1 - x_2\|^2}{2(\|x_1\|^2 + \|x_2\|^2)} : x_1, x_2 \in \mathcal{B} \text{ with } (x_1, x_2) \neq (0, 0)\right\}.$$

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This idea was further enhanced by many authors in [2, 4, 5, 8, 9].

The James constant $J(\mathcal{B})$ of a Banach space $\mathcal{B}$ is defined by

$$J(\mathcal{B}) = \sup\left\{\min(\|x_1 + x_2\|, \|x_1 - x_2\|) : x_1, x_2 \in S_{\mathcal{B}}\right\},$$

where $S_{\mathcal{B}}$ is unit sphere.

In [3], the authors introduced the generalized von Neumann-Jordan constant $C_{NJ}^{(p)}(\mathcal{B})$ which is defined as

$$C_{NJ}^{(p)}(\mathcal{B}) = \sup\left\{\frac{\|x_1 + x_2\|^p + \|x_1 - x_2\|^p}{2^{p-1}(\|x_1\|^p + \|x_2\|^p)} : x_1, x_2 \in \mathcal{B} \text{ with } (x_1, x_2) \neq (0, 0)\right\}$$

and obtained the relationship between $C_{NJ}^{(p)}(\mathcal{B})$ and $J(\mathcal{B})$.

This has an analog in the quasi-Banach space, that was considered in [7]. In this note, It is shown that $C_{NJ}^{(p)}(\mathcal{B}) \leq 2$ for any quasi-Banach space $\mathcal{B}$ and $C_{NJ}^{(p)}(\mathcal{B}) < 2$ if and only if $\mathcal{B}$ is uniformly non-square. A relationship between $J(\mathcal{B})$ and $C_{NJ}^{(p)}(\mathcal{B})$ is established. We also give the criterion for the uniformly smooth quasi-Banach space.

2 Preliminaries

Recall [1], that a quasi-norm on $\|\cdot\|$ on vector space $\mathcal{B}$ over a field K ($\mathbb{R}$ or $\mathbb{C}$) is a mapping $\mathcal{B} \to [0, \infty)$ with properties

- $\|x\| = 0 \iff x = 0.$
- $\|\alpha x\| = |\alpha|\|x\|$ for all $\alpha \in K$ and $x \in \mathcal{B}$.
- There exists a constant $C \geq 1$ such that $\forall x_1, x_2 \in \mathcal{B}$ we have

$$\|x_1 + x_2\| \leq C(\|x_1\| + \|x_2\|).$$

Definition 2.1. The generalized von Neumann-Jordan constant $C_{NJ}^{(p)}(\mathcal{B})$ for a quasi-Banach space is defined by

$$C_{NJ}^{(p)}(\mathcal{B}) = \sup\left\{\frac{\|x_1 + x_2\|^p + \|x_1 - x_2\|^p}{2^{p-1}C^p(\|x_1\|^p + \|x_2\|^p)} : x_1, x_2 \in \mathcal{B} \text{ with } (x_1, x_2) \neq (0, 0)\right\},$$

where $1 \leq p < \infty$.

The parametrized formula for the constant $C_{NJ}^{(p)}(\mathcal{B})$ is given as

$$C_{NJ}^{(p)}(\mathcal{B}) = \sup\left\{\frac{\|x_1 + tx_2\|^p + \|x_1 - tx_2\|^p}{C^p 2^{p-1}(1 + t^p)} : x_1, x_2 \in S_{\mathcal{B}},\ 0 \leq t \leq 1\right\},$$

By taking $t = 1$ and $x_1 = x_2$, we obtain the estimate

$$C_{NJ}^{(p)}(\mathcal{B}) \geq \frac{\|2x_1\|^p}{C2^p(1+1)} \geq \frac{2^p}{C2^{p-1}(1+1)} = \frac{1}{C}.$$

Definition 2.2. In a quasi-Banach space $\mathcal{B}$ the James constant is defined as

$$J(\mathcal{B}) = \sup\left\{\frac{1}{C}\min(\|x_1 + x_2\|, \|x_1 - x_2\|) : x_1, x_2 \in S_{\mathcal{B}}\right\}.$$

Definition 2.3. A quasi-Banach space $\mathcal{B}$ is said to be uniformly non-square if there exists a positive number $\delta < 2$ such that for any $x_1, x_2 \in S_{\mathcal{B}}$, we have

$$\min\left(\left\|\frac{x_1 + x_2}{C}\right\|, \left\|\frac{x_1 + x_2}{C}\right\|\right) \leq \delta.$$

Remark 2.4. As in classical case, the quasi-Banach space $\mathcal{B}$ is uniformly non-square if and only if $J(\mathcal{B}) < 2$

Definition 2.5. The modulus of uniform smoothness of a quasi-Banach space $\mathcal{B}$ is defined as

$$\rho_{\mathcal{B}}(t) = \sup\left\{\frac{\|x_1 + tx_2\| + \|x_1 - tx_2\|}{2C} - \frac{1}{C} : x_1, x_2 \in S_{\mathcal{B}},\ t \geq 0\right\}.$$

Definition 2.6. A quasi-Banach space $\mathcal{B}$ is said to be uniformly smooth if $(\rho_{\mathcal{B}})'_+(0) = \lim_{t\to 0^+} \frac{\rho_{\mathcal{B}}(t)}{t} = 0$.

Definition 2.7. For any quasi-Banach space $\mathcal{B}$ and a real number $p \in [0, \infty)$, $J_{\mathcal{B},p}(t)$ is defined by

$$J_{\mathcal{B},p}(t) = \sup\left\{\left(\frac{\|x_1 + tx_2\|^p + \|x_1 - tx_2\|^p}{2C^p}\right)^{\frac{1}{p}} : x_1, x_2 \in S_{\mathcal{B}}\right\}.$$

3 Main results

Theorem 3.1. *Let $\mathcal{B}$ be a non-trivial quasi-Banach space and $p \in [1, \infty)$. Then*

$$J_{\mathcal{B},p}(t) \geq \rho_{\mathcal{B}}(t) + \frac{1}{C}.$$

Proof. By using convexity of the function $f(u) = u^p$ on $(0, \infty)$, one can easily obtained

$$\left(\frac{\|x_1 + tx_2\| + \|x_1 - tx_2\|}{2}\right)^p \leq \frac{\|x_1 + tx_2\|^p + \|x_1 - tx_2\|^p}{2},$$

therefore

$$\left(\frac{\|x_1 + tx_2\| + \|x_1 - tx_2\|}{2C}\right)^p \leq \frac{\|x_1 + tx_2\|^p + \|x_1 - tx_2\|^p}{2C^p},$$

which implies that

$$\frac{1}{C} + \rho_{\mathcal{B}}(t) \leq J_{\mathcal{B},p}(t).$$

For $p = 1$, we have $J_{\mathcal{B},1}(t) = \frac{1}{C} + \rho_{\mathcal{B}}(t)$ and for $p = 2$, we have $2C^2 J^2_{\mathcal{B},2}(t) = E(t, \mathcal{B})$, where

$$E(t, \mathcal{B}) = \sup\left\{(\|x_1 + tx_2\|^2 + \|x_1 - tx_2\|^2) : x_1, x_2 \in S_{\mathcal{B}}\right\}.$$

This completes the proof. ∎

Theorem 3.2. *For any quasi-Banach space $\mathcal{B}$, we have*
(a) $J_{\mathcal{B},p}(t)$ *is a non-decreasing function.*
(b) $J_{\mathcal{B},p}(t)$ *is convex.*
(c) $J_{\mathcal{B},p}(t)$ *is continuous function.*
(d) $\frac{J_{\mathcal{B},p}(t)-1}{t}$ *is a non-decreasing function.*

Proof. We only prove (a) and remaining are analogous to it. Let $g(t) = \|x_1 + tx_2\|^p + \|x_1 - tx_2\|^p$ be a convex and even function. Let $0 < t_1 \leq t_2$ and $x_1, x_2 \in S_{\mathcal{B}}$. Then we have

$$\begin{aligned}\|x_1 + t_1x_2\|^p + \|x_1 - t_1x_2\|^p &= g(t_1) = g\left(\frac{t_2+t_1}{2t_2}t_2 + \frac{t_2-t_1}{2t_2}(-t_2)\right)\\ &\leq \frac{t_2+t_1}{2t_2}g(t_2) + \frac{t_2-t_1}{2t_2}g(t_2)\\ &= g(t_2) = \|x_1 + t_2x_2\|^p + \|x_1 - t_2x_2\|^p\\ &\leq 2C^pJ^p_{\mathcal{B},p}(t_2),\end{aligned}$$

which implies that

$$\frac{\|x_1 + t_1x_2\|^p + \|x_1 - t_1x_2\|^p}{2C^p} \leq J^p_{\mathcal{B},p}(t_2).$$

Hence $J_{\mathcal{B},p}(t_1) \leq J_{\mathcal{B},p}(t_2)$. □

Theorem 3.3. *For any quasi-Banach space $\mathcal{B}$ and $1 \leq p < \infty$, the generalized von Neumann-Jordan constant $C^{(p)}_{NJ}(\mathcal{B})$ satisfy the inequality $C^{(p)}_{NJ}(\mathcal{B}) \leq 2$.*

Proof. As we have already defined

$$C^{(p)}_{NJ}(\mathcal{B}) = \sup\left\{\frac{\|x_1 + tx_2\|^p + \|x_1 - tx_2\|^p}{C^p2^{p-1}(1+t^p)} : x_1, x_2 \in S_{\mathcal{B}} \text{ with } (x_1, x_2) \neq (0,0)\right\},$$

where $0 \leq t \leq 1$.

By using definition of a quasi-Banach space, we have the following inequality

$$\begin{aligned}\|x_1 + tx_2\|^p + \|x_1 - tx_2\|^p &\leq C^p(\|x_1\| + t\|x_2\|)^p + C^p(\|x_1\| + t\|x_2\|)^p\\ &= 2C^p(\|x_1\| + t\|x_2\|)^p\\ &= 2C^p(1+t)^p,\end{aligned}$$

therefore

$$\frac{\|x_1 + tx_2\|^p + \|x_1 - tx_2\|^p}{C^p2^{p-1}(1+t^p)} \leq \frac{2(1+t)^p}{2^{p-1}(1+t^p)}. \tag{3.1}$$

The function $f(u) = u^p$ is convex, which leads

$$(1+t)^p = \left(2 \cdot \frac{1+t}{2}\right)^p \leq 2^p\left(\frac{1+t^p}{2}\right) = 2^{p-1}(1+t^p).$$

Using above inequality (3.1) become

$$\frac{\|x_1 + tx_2\|^p + \|x_1 - tx_2\|^p}{C^p2^{p-1}(1+t^p)} \leq \frac{1}{2^{p-2}}2^{p-1} = 2.$$

Hence

$$C_{NJ}^{(p)}(\mathcal{B}) = \sup\left\{\frac{\|x_1+tx_2\|^p+\|x_1-tx_2\|^p}{C^p2^{p-1}(1+t^p)} : x_1, x_2 \in S_{\mathcal{B}},\, 0 \le t \le 1\right\}$$
$$\le 2.$$

This completes the proof. □

Next theorem presents the relationship between $C_{NJ}^{(p)}(\mathcal{B})$ and $J(\mathcal{B})$.

Theorem 3.4. *Let $\mathcal{B}$ be a non-trivial qausi-Banach space and $p \in (1,\infty)$. Then the following inequality hold:*

$$J(\mathcal{B}) \le 2^{\frac{p-1}{p}} \sqrt[p]{C_{NJ}^{(p)}(\mathcal{B})}.$$

Proof. For any $x_1, x_2 \in S_{\mathcal{B}}$, we have

$$\begin{aligned}
2(\min\{\|x_1+x_2\|, \|x_1-x_2\|\})^p &\le 2\left(\frac{\|x_1+x_2\|+\|x_1-x_2\|}{2}\right)^p \\
&\le 2\left(\frac{\|x_1+x_2\|^p+\|x_1-x_2\|^p}{2}\right) \\
&= \frac{\|x_1+x_2\|^p+\|x_1-x_2\|^p}{C^p2^{p-1}(\|x_1\|^p+\|x_2\|^p)}.C^p2^{p-1}(\|x_1\|^p+\|x_2\|^p) \\
&\le C^pC_{NJ}^{(p)}(\mathcal{B})2^{p-1}(\|x_1\|^p+\|x_2\|^p) \\
&= 2\cdot 2^{p-1}C^pC_{NJ}^{(p)}(\mathcal{B}),
\end{aligned}$$

$$(\{\min\{\|x_1+x_2\|, \|x_1-x_2\|\})^p \le 2^{p-1}C^pC_{NJ}^{(p)}(\mathcal{B}),$$

$$\frac{1}{C}\{\min\{\|x_1+x_2\|, \|x_1-x_2\|)\} \le 2^{\frac{p-1}{p}} \sqrt[p]{C_{NJ}^{(p)}(\mathcal{B})}.$$

Taking supremum both side,

$$\sup\left(\frac{1}{C}\min\{\|x_1+x_2\|, \|x_1-x_2\|\}\right) \le 2^{\frac{p-1}{p}} \sqrt[p]{C_{NJ}^{(p)}(\mathcal{B})}.$$

Therefore

$$J(\mathcal{B}) \le 2^{\frac{p-1}{p}} \sqrt[p]{C_{NJ}^{(p)}(\mathcal{B})}.$$

This completes the proof. □

Theorem 3.5. *For $p \in (1,\infty)$, a quasi-Banach space $\mathcal{B}$ is uniformly non-square if and only if there exists $\delta \in (0,1)$ such that for any $x_1, x_2 \in \mathcal{B}$, we have*

$$\left\|\frac{x_1+x_2}{2C}\right\|^p + \left\|\frac{x_1-x_2}{2C}\right\|^p \le (2-\delta)\frac{\|x_1\|^p+\|x_2\|^p}{2}. \tag{3.2}$$

Proof. Let $\mathcal{B}$ be an uniformly non-square quasi-Banach space and on contrary assume that (3.2) is not hold. Therefore for every positive integer n, there exists x_n and y_n in $\mathcal{B}$ such that

$$\left\|\frac{x_n+y_n}{2C}\right\|^p+\left\|\frac{x_n-y_n}{2C}\right\|^p>\left(2-\frac{1}{n}\right)\frac{\|x_n\|^p+\|y_n\|^p}{2}.$$

Let $x_n \in S_{\mathcal{B}}$ and $y_n \in B_{\mathcal{B}}=\{x\in\mathcal{B}:\|x\|\le 1\}$ for all n. With out loss of generality we assume that $\{\|y_n\|\}$ converges to some γ, where $0\le\gamma\le 1$ (by Bolzano-Wiestrass) we have

$$\begin{aligned}\left(2-\frac{1}{n}\right)\frac{1+\|y_n\|^p}{2} &< \left\|\frac{x_n+y_n}{2C}\right\|^p+\left\|\frac{x_n-y_n}{2C}\right\|^p\\ &\le 2\left(\frac{C(\|x_n\|^p+\|y_n\|^p)}{2C}\right)^p\\ &= 2\left(\frac{1+\|y_n\|}{2}\right)^p\\ &\le 2\left(\frac{1+\|y_n\|^p}{2}\right),\end{aligned}$$

letting $n\to\infty$, we obtain

$$\frac{(1+\gamma)^p}{1+\gamma^p}=2^{p-1}\quad\Longrightarrow\quad\gamma=1,$$

therefore

$$\left\|\frac{x_n+y_n}{2C}\right\|^p+\left\|\frac{x_n-y_n}{2C}\right\|^p\longrightarrow 2,$$

which implies that

$$\left\|\frac{x_n+y_n}{2C}\right\|^p\longrightarrow 1\quad\text{and}\quad\left\|\frac{x_n-y_n}{2C}\right\|^p\longrightarrow 1.$$

This contradiction to the fact that $\mathcal{B}$ is uniformly non-square.

Conversely, suppose that

$$\left\|\frac{x_1+x_2}{2C}\right\|^p+\left\|\frac{x_1-x_2}{2C}\right\|^p\le(2-\delta)\frac{\|x_1\|^p+\|x_2\|^p}{2}.$$

In particularly, we have

$$\left\|\frac{x_1+x_2}{2C}\right\|^p+\left\|\frac{x_1-x_2}{2C}\right\|^p\le(2-\delta),$$

which implies that

$$\min\left\{\left\|\frac{x_1+x_2}{2C}\right\|^p,\left\|\frac{x_1-x_2}{2C}\right\|^p\right\}\le\left(1-\frac{\delta}{2}\right)^{\frac{1}{p}}.$$

Hence $\mathcal{B}$ is uniformly non-square. This completes the proof. □

Theorem 3.6. *For $p\in(1,\infty)$, a quasi-Banach space $\mathcal{B}$ is uniformly non-square if and only if $C_{NJ}^{(p)}(\mathcal{B})<2$.*

Proof. From the Theorem 3.5, $\mathcal{B}$ is uniformly non-square if and only if there exists $0 < \delta < 1$ such that

$$\left\|\frac{x_1+x_2}{2C}\right\|^p + \left\|\frac{x_1-x_2}{2C}\right\|^p \leq (2-\delta)\frac{\|x_1\|^p+\|x_2\|^p}{2}.$$

Therefore

$$\frac{\|x_1+x_2\|^p+\|x_1-x_2\|^p}{C^p2^{p-1}(\|x_1\|^p+\|x_2\|^p)} \leq 2, \quad \forall (x_1,x_2)\neq(0,0).$$

Hence $C_{NJ}^{(p)}(\mathcal{B}) < 2$. □

Theorem 3.7. *For any quasi-Banach space $\mathcal{B}$ and $p \in (1,\infty)$, the inequalities $C_{NJ}^{(p)}(\mathcal{B}) < 2$ and $J(\mathcal{B}) < 2$ are equivalent.*

Proof. From the Remark 2.4, $J(\mathcal{B}) < 2$ if and only if $\mathcal{B}$ is uniformly non-square. Therefore by using Theorem 3.6, we have $C_{NJ}^{(p)}(\mathcal{B}) < 2$.

Suppose that $C_{NJ}^{(p)}(\mathcal{B}) < 2$. Then by using the Theorem 3.4, we have

$$J(\mathcal{B}) < 2^{\frac{p-1}{p}}2^{\frac{1}{p}} = 2.$$

This completes the proof. □

Theorem 3.8. *Let $\mathcal{B}$ be a quasi-Banach space, $p \in [1,\infty)$ and $t > 0$. Then the following conditions are equivalent:*

(*a*) $J_{\mathcal{B},p}(t) < 1+t$.

(*b*) $J(t,x_1) < 1+t$.

Proof. (*a*) ⇒ (*b*) Suppose on contrary that $J(t,x_1) \geq 1+t$. Then it is enough to take $J(t,x_1) = 1+t$.
Since

$$J(t,\mathcal{B}) = \sup\left\{\frac{1}{C}\min(\|x_1+x_2\|,\|x_1-x_2\|) : x_1,x_2\in S_{\mathcal{B}}\right\},$$

by using the definition of supremum, for any $\epsilon > 0$, there exist $x_1, x_2 \in S_{\mathcal{B}}$ such that

$$\begin{aligned}\frac{\|x_1+tx_2\|+\|x_1-tx_2\|}{2} &\geq \min\{\|x_1+tx_2\|,\|x_1-tx_2\|\} \\ &\geq c(1+t-\epsilon).\end{aligned} \tag{3.3}$$

Applying convexity of the function $f(u)=u^p$, we get

$$\left(\left(\frac{\|x_1+tx_2\|+\|x_1-tx_2\|}{2}\right)^p\right)^{\frac{1}{p}} \leq \left(\frac{\|x_1+tx_2\|^p+\|x_1-tx_2\|^p}{2}\right)^{\frac{1}{p}},$$

therefore from (3.3)

$$\begin{aligned}\left(\frac{\|x_1+tx_2\|^p+\|x_1-tx_2\|^p}{2}\right)^{\frac{1}{p}} &\geq \min\{\|x_1+tx_2\|,\|x_1-tx_2\|\} \\ &\geq c(1+t-\epsilon).\end{aligned}$$

Since ϵ is any arbitrary so

$$J_{\mathcal{B},p}(t) \geq 1+t,$$

which leads a contradiction.

$(b) \Rightarrow (a)$ Suppose on contrary that $J_{\mathcal{B},p}(t) \geq 1+t$. Then it is enough to take $J_{\mathcal{B},p}(t) = 1+t$. Again using the definition of supremum, for any $\epsilon > 0$ there exists $x_1, x_2 \in S_{\mathcal{B}}$ such that

$$\|x_1 + tx_2\|^p + \|x_1 - tx_2\|^p \geq 2C^p(1+t-\epsilon)^p,$$

also using

$$\|x_1 + tx_2\|^p + \|x_1 - tx_2\|^p \leq 2C^p(1+t)^p,$$

$$2C^p(1+t)^p \geq \|x_1 + tx_2\|^p + \|x_1 - tx_2\|^p \geq 2C^p(1+t-\epsilon)^p$$

since ϵ is arbitrary, so

$$\|x_1 + tx_2\|^p + \|x_1 - tx_2\|^p = 2C^p(1+t)^p,$$

which implies that

$$\|x_1 + tx_2\| = \|x_1 - tx_2\| = C(1+t).$$

So using the definition of $J(t, x_1)$, we get $J(t, x_1) \geq 1+t$, which lead to a contradiction. □

Corollary 3.9. *Let $\mathcal{B}$ be a quasi-Banach space, $p \in [1, \infty)$ and $t > 0$. Then the following conditions are equivalent:*

(a) $\mathcal{B}$ is uniformly non-square.
(b) $J_{\mathcal{B},p}(t) < 1+t$.
(c) $J(t, x_1) < 1+t$.

Theorem 3.10. *A quasi-Banach apace $\mathcal{B}$ is uniformly smooth if*

$$\lim_{t \to 0} \left(\frac{J_{\mathcal{B},p}(t) - \frac{1}{C}}{t} \right) = 0.$$

Proof. Suppose that

$$\lim_{t \to 0} \left(\frac{J_{\mathcal{B},p}(t) - \frac{1}{C}}{t} \right) = 0.$$

From Theorem 3.1 we know that

$$J_{\mathcal{B},p}(t) \geq \rho_{\mathcal{B}}(t) + \frac{1}{C},$$

which implies that

$$J_{\mathcal{B},p}(t) - \frac{1}{C} \geq \rho_{\mathcal{B}}(t),$$

dividing both side by t and applying the $\lim_{t \to 0}$

$$\lim_{t \to 0} \frac{\rho_{\mathcal{B}}(t)}{t} \leq \lim_{t \to 0} \left(\frac{J_{\mathcal{B},p}(t) - \frac{1}{C}}{t} \right) = 0.$$

So by definition $\mathcal{B}$ is uniformly smooth. □

4 Quasi-Banach fixed point theorem (Contraction theorem)

Definition 4.1. ([6]) Let $\mathcal{B} = (\mathcal{B}, \tilde{d})$ be a quasi-metric space. A mapping $T : \mathcal{B} \to T$ is called a contraction on $\mathcal{B}$ if there exists a positive real number $\alpha < 1$ such that for all $x_1, x_2 \in \mathcal{B}$

$$\tilde{d}(Tx_1, Tx_2) \leq \alpha \tilde{d}(x_1, x_2).$$

Theorem 4.2. *Consider a quasi-metric space $\mathcal{B} = (\mathcal{B}, \tilde{d})$, where $\mathcal{B} \neq \emptyset$. Suppose $\mathcal{B}$ is complete and $T : \mathcal{B} \to T$ be a contraction on $\mathcal{B}$ and suppose that $C^n \beta^m \to 0$, where $\beta = \frac{\alpha}{C}$. Then T has exactly one fixed point.*

Proof. Let any $x_0 \in \mathcal{B}$ and define the iterative sequence by

$$x_0, \quad x_1 = Tx_0, \quad x_2 = T^2 x_0, \quad x_3 = T^3 x_0, \quad \cdots, \quad x_n = T^n x_0, \quad \cdots.$$

First, we show that this iterative sequence (x_n) is cauchy. For this we take

$$\begin{aligned} \tilde{d}(x_{m+1}, x_m) &= \tilde{d}(Tx_m, Tx_{m-1}) \\ &\leq \alpha \tilde{d}(x_m, x_{m-1}) \\ &\leq \alpha \tilde{d}(Tx_{m-1}, Tx_{m-2}) \\ &\leq \alpha^2 \tilde{d}(x_{m-1}, x_{m-2}) \\ &\quad \cdots\cdots \\ &\leq \alpha^m \tilde{d}(x_1, x_0). \end{aligned}$$

Suppose that $n > m$ and using the definition of quasi-metric

$$\begin{aligned} \tilde{d}(x_m, x_n) &\leq C\big(\tilde{d}(x_m, x_{n-1}) + \tilde{d}(x_{n-1}, x_n)\big) \\ &\leq C^2\big(\tilde{d}(x_m, x_{n-2}) + \tilde{d}(x_{n-2}, x_{n-1})\big) + C\tilde{d}(x_{n-1}, x_n) \\ &\leq C^3\big(\tilde{d}(x_m, x_{n-3}) + \tilde{d}(x_{n-3}, x_{n-2})\big) + C^2\tilde{d}(x_{n-2}, x_{n-1}) + C\tilde{d}(x_{n-1}, x_n) \\ &\quad \cdots\cdots \\ &\leq C^{n-m}\tilde{d}(x_m, x_{m+1}) + \cdots + C\tilde{d}(x_{n-1}, x_n) \\ &\leq C^{n-m}\big(\tilde{d}(x_m, x_{m+1}) + \cdots + \tilde{d}(x_{n-1}, x_n)\big) \\ &\leq C^{n-m}\big[\alpha^m + \alpha^{m+1} + \cdots + \alpha^{n-1}\big]\tilde{d}(x_1, x_0) \\ &\leq C^{n-m}\frac{\alpha^m}{1-\alpha}\tilde{d}(x_1, x_0) \\ &= C^n\left(\frac{\alpha}{C}\right)^m \frac{1}{1-\alpha}\tilde{d}(x_1, x_0). \end{aligned}$$

Since

$$\tilde{d}(x_m, x_n) \leq C^n\left(\frac{\alpha}{C}\right)^m \frac{1}{1-\alpha}\tilde{d}(x_1, x_0)$$

from our supposition $\tilde{d}(x_m, x_n) \to 0$ as $m \to \infty$. Since $\mathcal{B}$ is complete so $x_m \to x \in \mathcal{B}$.

Next we show that this limit x is the fixed point of T. For this using the definition of quasi-metric we have

$$\begin{aligned} \tilde{d}(x, Tx) &\leq C(\tilde{d}(x, x_m) + \tilde{d}(x_m, Tx)) \\ &\leq C(\tilde{d}(x, x_m) + \alpha\tilde{d}(x_{m-1}, x)) \end{aligned}$$

because (x_m) converges to x, therefore we have $\tilde{d}(x, Tx) = 0$. This implies $Tx = x$. Now we prove that x is the unique fixed point of T. For this let $Tx = x$ and $Ty = y$, therefore we have

$$\tilde{d}(x, y) = \tilde{d}(Tx, Ty) \leq \alpha \tilde{d}(x, y),$$

which implies $\tilde{d}(x, y) = 0$ because $\alpha < 1$. Hence $x = y$. This completes the proof. □

5 Conclusion

Generalized von Neumann-Jordan and James constants studied by many researcher for Banach space for example in [2, 4, 5, 8, 9] and the references therein. In this paper, we introduce the generalized von Neumann-Jordan constant and the James constant for a quasi-Banach space. Relationships between James constant and generalized von Neumann-Jordan constant are also presented.

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Shared-values of meromorphic functions on annuli *

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Abstract

In this paper, we study the shared values and uniqueness of meromorphic functions on annulus, and obtain one theorem about meromorphic functions on annulus sharing some distinct values, and this result is an improvement of some theorems given by Cao, Yi [4, 5], Kondratyuk and Laine[9].
Key words: Meromorphic function, Nevanlinna theory, the annulus.
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1 Introduction and main resut

In 1929, R.Nevanlinna(see [14]) first investigated the uniqueness of meromorphic functions in the whole complex plane and obtained the well-known theorem—5 IM theorem of two meromorphic functions sharing five distinct values.

After his theorem, there are vast references on the uniqueness of meromorphic functions sharing values and sets in the whole complex plane(see [2, 16, 18]). In recent, the uniqueness problem of meromorphic functions with shared values in some angular domain attracted many investigations (see [3, 11, 19, 20]). Thus, we always assumed that the reader is familiar with the notations of the Nevanlinna theory such as $T(r,f), m(r,f), N(r,f)$ and so on (see [6, 16, 17]).

We use $\mathbb{C}$ to denote the open complex plane, $\overline{\mathbb{C}}$ to denote the extended complex plane, and $\mathbb{X}$ to denote the subset of $\mathbb{C}$. Let S be a set of distinct elements in $\widehat{\mathbb{C}}$ and $\mathbb{X} \subseteq \mathbb{C}$.

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Define

$$E_{\mathbb{X}}(S,f)=\bigcup_{a\in S}\{z\in\mathbb{X}|f_a(z)=0,\quad counting\quad multiplicities\},$$

$$\overline{E}_{\mathbb{X}}(S,f)=\bigcup_{a\in S}\{z\in\mathbb{X}|f_a(z)=0,\quad ignoring\quad multiplicities\},$$

where $f_a(z)=f(z)-a$ if $a\in\mathbb{C}$ and $f_\infty(z)=1/f(z)$.

For $a\in\overline{\mathbb{C}}$, we say that two meromorphic functions f and g share the value a CM (IM) in $\mathbb{X}$ (or $\mathbb{C}$), if $f(z)-a$ and $g(z)-a$ have the same zeros with the same multiplicities(ignoring multiplicities) in $\mathbb{X}$ (or $\mathbb{C}$). In addition, we also use $f=a\rightleftharpoons g=a$ in $\mathbb{X}$ (or $\mathbb{C}$) to express that f and g share the value a CM in $\mathbb{X}$ (or $\mathbb{C}$), $f=a\Longleftrightarrow g=a$ in $\mathbb{X}$ (or $\mathbb{C}$) to express that f and g share the value a IM in $\mathbb{X}$ (or $\mathbb{C}$), and $f=a\Longrightarrow g=a$ in $\mathbb{X}$ (or $\mathbb{C}$) to express that $f=a$ implies $g=a$ in $\mathbb{X}$ (or $\mathbb{C}$).

As we know, the whole complex plane $\mathbb{C}$ and angular domain all can be regarded as simply-connected regions, many results about the uniqueness of shared values and sets in the complex plane and angular domain can also be regarded as the uniqueness of meromorphic functions in simply-connected regions. *Thus, it arises naturally an interesting subject on the uniqueness for the meromorphic functions in the multiply connected region?*

The main purpose of this paper is to study the uniqueness of meromorphic functions in doubly connected domains of complex plane $\mathbb{C}$. From the Doubly Connected Mapping Theorem [1], we can get that each doubly connected domain is conformally equivalent to the annulus $\{z: r<|z|<R\}$, $0\le r<R\le+\infty$. For two cases: $r=0$, $R=+\infty$ simultaneously and $0<r<R<+\infty$, the latter case the homothety $z\mapsto\frac{z}{\sqrt{rR}}$ reduces the given domain to the annulus $\{z:\frac{1}{R_0}<|z|<R_0\}$, where $R_0=\sqrt{\frac{R}{r}}$. Thus, every annulus is invariant with respect to the inversion $z\mapsto\frac{1}{z}$ in two cases. *The basic notions of the Nevanlinna theory on annuli will be showed in the next section.*

In recent, there have some results on the Nevannlina Theory of meromorphic functions on the annulus (see [7, 8, 10, 12, 13, 15]). In 2005, Khrystiyanyn and Kondratyuk [7, 8] proposed the Nevanlinna theory for meromorphic functions on annuli (see also [9]). Lund and Ye [12] in 2009 studied functions meromorphic on the annuli with the form $\{z: R_1<|z|<R_2\}$, where $R_1\ge 0$ and $R_2\le\infty$. *However, there are few results about the uniqueness of meromorphic functions on the annulus.* In 2009 and 2011, Cao [4, 5] investigated the uniqueness of meromorphic functions on annuli sharing some values and some sets, and obtained an analog of Nevanlinna's famous five-value theorem as follows:

Theorem 1.1 ([5, Thereom 3.2] or [4, Corollary 3.3]). Let f_1 and f_2 be two transcendental or admissible meromorphic functions on the annulus $\mathbb{A}=\{z:\frac{1}{R_0}<|z|<R_0\}$, where $1<R_0\le+\infty$. Let a_j $(j=1,2,3,4,5)$ be five distinct complex numbers in $\overline{\mathbb{C}}$. If f_1,f_2 share a_j IM for $j=1,2,3,4,5$, then $f_1(z)\equiv f_2(z)$.

Remark 1.1 *For the case $R_0=+\infty$, the assertion was proved by Kondratyuk and Laine [9].*

In this paper, we will focus on the uniqueness problem of meromorphic functions in the field of complex analysis and obtain the main result below which improve Theorem 1.1.

Theorem 1.2 *Let f and g be two transcendental or admissible meromorphic functions on the annulus $\mathbb{A}=\{z:\frac{1}{R_0}<|z|<R_0\}$, where $1<R_0\le+\infty$, $a_j\in\mathbb{C}(j=1,2,3,4)$ be four distinct values. We assume that f and g share four distinct values $a_j(j=1,2,3,4)$ IM on $\mathbb{A}$ and $\overline{E}_{\mathbb{A}}(S,f)\subset\overline{E}_{\mathbb{A}}(S,g)$, where $S=\{b_1,\ldots,b_m\},m\ge1$ and $b_1,\ldots,b_m\in\widehat{\mathbb{C}}\setminus\{a_1,a_2,a_3,a_4\}$. Then f and g share all values CM on $\mathbb{A}$, thus it follows that either $f\equiv g$ or f is a Möbius transformation of g. Furthermore, if the number of the values in S is odd, then $f\equiv g$.*

Remark 1.2 *The special case $m=1$ of Theorem 1.2 immediately yields Theorem 1.1. In fact, when $m=1$, set $S=\{a_5\}$. If f,g share a_5 IM on $\mathbb{A}$, which implies $\overline{E}_{\mathbb{A}}(S,f)\subset\overline{E}_{\mathbb{A}}(S,g)$, then by Theorem 1.2, we can get $f\equiv g$.*

2 Basic notions in the Nevanlinna theory on annuli

For a meromorphic function f on whole plane $\mathbb{C}$, the classical notations of Nevanlinna theory are denoted as follows

$$N(R,f)=\int_0^R\frac{n(t,f)-n(0,f)}{t}dt+n(0,f)\log R,$$
$$m(R,f)=\frac{1}{2\pi}\int_0^{2\pi}\log^+|f(Re^{i\theta})|d\theta,\quad T(R,f)=N(R,f)+m(R,f),$$

where $\log^+x=\max\{\log x,0\}$, and $n(t,f)$ is the counting function of poles of the function f in $\{z:|z|\le t\}$.

Let f be a meromorphic function on the annulus $\mathbb{A}=\{z:\frac{1}{R_0}<|z|<R_0\}$, where $1<R<R_0\le+\infty$, the notations of the Nevanlinna theory on annuli will be introduced as follows, let

$$N_1(R,f)=\int_{\frac{1}{R}}^1\frac{n_1(t,f)}{t}dt,\quad N_2(R,f)=\int_1^R\frac{n_2(t,f)}{t}dt,$$
$$m_0(R,f)=m(R,f)+m(\frac{1}{R},f),\quad N_0(R,f)=N_1(R,f)+N_2(R,f),$$

where $n_1(t,f)$ and $n_2(t,f)$ are the counting functions of poles of the function f in $\{z:t<|z|\le1\}$ and $\{z:1<|z|\le t\}$, respectively. Similarly, for $a\in\overline{\mathbb{C}}$, we have

$$\begin{aligned}\overline{N}_0(r,\frac{1}{f-a})&=\overline{N}_1(R,\frac{1}{f-a})+\overline{N}_2(R,\frac{1}{f-a})\\&=\int_{\frac{1}{R}}^1\frac{\overline{n}_1(t,\frac{1}{f-a})}{t}dt+\int_1^R\frac{\overline{n}_2(t,\frac{1}{f-a})}{t}dt\end{aligned}$$

in which each zero of the function $f-a$ is counted only once. In addition, we use $\overline{n}_1^{k)}(t,\frac{1}{f-a})$ (or $\overline{n}_1^{(k}(t,\frac{1}{f-a})$) to denote the counting function of poles of the function $\frac{1}{f-a}$ with multiplicities $\le k$ (or $>k$) in $\{z:t<|z|\le1\}$, each point counted only once. Similarly, we have the notations $\overline{N}_1^{k)}(t,f)$, $\overline{N}_1^{(k}(t,f)$, $\overline{N}_2^{k)}(t,f)$, $\overline{N}_2^{(k}(t,f)$, $\overline{N}_0^{k)}(t,f)$, $\overline{N}_0^{(k}(t,f)$.

The Nevanlinna characteristic of f on the annulus $\mathbb{A}$ is defined by

$$T_0(R,f) = m_0(R,f) - 2m(1,f) + N_0(R,f).$$

For a nonconstant meromorphic function f on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R < R_0 \leq +\infty$, the following properties will be used in this paper (see [7])

$$(i)\quad T_0(R,f) = T_0\left(R, \frac{1}{f}\right),$$

$$(ii)\quad \max\{T_0(R, f_1 \cdot f_2), T_0(R, \frac{f_1}{f_2}), T_0(R, f_1 + f_2)\} \leq T_0(R, f_1) + T_0(R, f_2) + O(1),$$

$$(iii)\quad T_0(R, \frac{1}{f-a}) = T_0(R,f) + O(1),\ \ for\ \ every\ \ fixed\ \ a \in \mathbb{C}.$$

In 2005, the lemma on the logarithmic derivative on the the annulus $\mathbb{A}$ was obtained by Khrystiyanyn and Kondratyuk [8].

Theorem 2.1 ([8]) (Lemma on the logarithmic derivative) Let f be a nonconstant meromorphic function on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $R_0 \leq +\infty$, and let $\lambda > 0$. Then

(i) in the case $R_0 = +\infty$,

$$m_0\left(R, \frac{f'}{f}\right) = O(\log(RT_0(R,f)))$$

for $R \in (1, +\infty)$ except for the set $\triangle_R$ such that $\int_{\triangle_R} R^{\lambda-1} dR < +\infty$;

(ii) if $R_0 < +\infty$, then

$$m_0\left(R, \frac{f'}{f}\right) = O(\log(\frac{T_0(R,f)}{R_0 - R}))$$

for $R \in (1, R_0)$ except for the set $\triangle_R^{'}$ such that $\int_{\triangle_R^{'}} \frac{dR}{(R_0-R)^{\lambda-1}} < +\infty$.

In 2005, the second fundamental theorem on the the annulus $\mathbb{A}$ was first obtained by Khrystiyanyn and Kondratyuk [8]. Later, the other forms of the second fundamental theorem on annuli were given by Cao, Yi and Xu [5].

Theorem 2.2 ([5, Theorem 2.3]) (The second fundamental theorem) Let f be a nonconstant meromorphic function on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \leq +\infty$. Let $a_1, a_2, \ldots, a_q$ be q distinct complex numbers in the extended complex plane $\overline{\mathbb{C}}$. Let $k_1, k_2, \ldots, k_q$ be q positive integers, and let $\lambda \geq 0$. Then

$$(i)\quad (q-2)T_0(R,f) < \sum_{j=1}^{q} N_0(R, \frac{1}{f-a_j}) - N_0^{(1)}(R,f) + S(R,f),$$

$$(ii)\quad (q-2)T_0(R,f) < \sum_{j=1}^{q} \overline{N}_0(R, \frac{1}{f-a_j}) + S(R,f),$$

where

$$N_0^{(1)}(R,f)=N_0(R,\frac{1}{f'})+2N_0(R,f)-N_0(R,f^{'}),$$

and (i) in the case $R_0=+\infty$,

$$S(R,f)=O(\log(RT_0(R,f)))$$

for $R\in(1,+\infty)$ except for the set $\triangle_R$ such that $\int_{\triangle_R}R^{\lambda-1}dR<+\infty$;
(ii) if $R_0<+\infty$, then

$$S(R,f)=O(\log(\frac{T_0(R,f)}{R_0-R}))$$

for $R\in(1,R_0)$ except for the set $\triangle_R^{'}$ such that $\int_{\triangle_R^{'}}\frac{dR}{(R_0-R)^{\lambda-1}}<+\infty$.

Definition 2.1 *Let $f(z)$ be a non-constant meromorphic function on the annulus $\mathbb{A}=\{z:\frac{1}{R_0}<|z|<R_0\}$, where $1<R_0\le+\infty$. The function f is called a transcendental or admissible meromorphic function on the annulus $\mathbb{A}$ provided that*

$$\limsup_{R\to\infty}\frac{T_0(R,f)}{\log R}=\infty,\quad 1<R<R_0=+\infty$$

or

$$\limsup_{R\to R_0}\frac{T_0(R,f)}{-\log(R_0-R)}=\infty,\quad 1<R<R_0<+\infty,$$

respectively.

Then for a transcendental or admissible meromorphic function on the annulus $\mathbb{A}$, $S(R,f)=o(T_0(R,f))$ holds for all $1<R<R_0$ except for the set $\triangle_R$ or the set $\triangle_R^{'}$ mentioned in Theorem 2.1, respectively.

3 Some lemmas

To prove the above theorems, we need some Lemmas as follows.

From Theorem 2.1 and the definition of $m_0(R,f)$, f is transcendental or admissible function on $\mathbb{A}$, we can get Lemma 3.1 by using the same argument as in Lemma 4.3 in [16]

Lemma 3.1 *Suppose that f is a transcendental or admissible meromorphic function on the annulus $\mathbb{A}=\{z:\frac{1}{R_0}<|z|<R_0\}$, where $1<R_0\le+\infty$. Let $P(f)=a_0f^p+a_1f^{p-1}+\cdots+a_p(a_0\neq0)$ be a polynomial of f with degree p, where the coefficients $a_j(j=0,1,\ldots,p)$ are constants, and let $b_j(j=1,2,\ldots,q)$ be $q(q\ge p+1)$ distinct finite complex numbers. Then*

$$m_0\left(R,\frac{P(f)\cdot f'}{(f-b_1)(f-b_2)\cdots(f-b_q)}\right)=S(R,f).$$

Lemma 3.2 *Let f, g be two distinct transcendental or admissible meromorphic functions on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \le +\infty$. Suppose that f and g share four distinct values a_1, a_2, a_3, a_4 IM on the annulus $\mathbb{A}$. Then*

(i) $T_0(R, f) = T_0(R, g) + S(R, f)$ and $T_0(R, g) = T_0(R, f) + S(R, g)$;

(ii) $\sum_{j=1}^{4} \overline{N}_0\left(R, \frac{1}{f-a_j}\right) = 2T_0(R, f) + S(R, f)$;

(iii) $\overline{N}_0(R, \frac{1}{f-b}) = T_0(R, f) + S(R, f)$, $\overline{N}_0(r, \frac{1}{g-b}) = T_0(R, g) + S(R, g)$, where $b \neq a_j (j = 1, 2, 3, 4)$.

Proof: From Theorem 2.2(ii), we have

$$\begin{aligned} 2T_0(R, f) &\le \sum_{j=1}^{4} \overline{N}_0(R, \frac{1}{f-a_j}) + S(R, f) \le N_0(R, \frac{1}{f-g}) + S(R, f) \\ &\le T_0(R, f) + T_0(R, g) + S(R, f), \end{aligned}$$

then we can get $T_0(R, f) = T_0(R, g) + S(R, f)$ and $\sum_{j=1}^{4} \overline{N}_0\left(R, \frac{1}{f-a_j}\right) = 2T_0(R, f) + S(R, f)$. Similarly, we can get $T_0(R, g) = T_0(R, f) + S(R, g)$. Thus, we prove (i) and (ii).

Again by Theorem 2.2 and (ii), we get

$$\begin{aligned} 3T_0(R, f) &\le \sum_{j=1}^{4} \overline{N}_0(R, \frac{1}{f-a_j}) + \overline{N}_0(R, \frac{1}{f-b}) + S(R, f) \\ &= 2T_0(R, f) + \overline{N}_0(R, \frac{1}{f-b}) + S(R, f), \end{aligned}$$

that is,

$$\overline{N}_0(R, \frac{1}{f-b}) = T_0(R, f) + S(R, f).$$

Similarly, we can get

$$\overline{N}_0(R, \frac{1}{g-b}) = T_0(R, g) + S(R, g).$$

Thus, we obtain (iii).

Therefore, we complete the proof of this lemma. □

Lemma 3.3 *Let f, g be two distinct transcendental or admissible meromorphic functions on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \le +\infty$. Suppose that f and g share four distinct values a_1, a_2, a_3, a_4 CM on the annulus $\mathbb{A}$. Then f is a Möbius transformation of g, two of the shared values, say a_1 and a_2, are Picard values on $\mathbb{A}$, and the cross ratio $(a_1, a_2, a_3, a_4) = -1$.*

Proof: Since f, g share a_1, a_2, a_3, a_4 CM on $\mathbb{A}$, from Lemma 3.2(i) and f, g are transcendental or admissible, we have $S(R, f) = S(R, g)$. We assume that there exist three of $\overline{N}_0(R, \frac{1}{f-a_j})(j = 1, 2, 3, 4)$, say $\overline{N}_0(R, \frac{1}{f-a_j})(j = 1, 2, 3)$, such that $\overline{N}_0(R, \frac{1}{f-a_j}) = S(R, f)$, then from Theorem 2.2(ii) we have

$$T_0(R, f) \le \sum_{j=1}^{3} \overline{N}_0(R, \frac{1}{f-a_j}) + S(R, f) = S(R, f).$$

Thus, we can get a contradiction with the condition of the lemma. Therefore, there are at least two of $\overline{N}_0(R, \frac{1}{f-a_j})(j = 1, 2, 3, 4)$, say $\overline{N}_0(R, \frac{1}{f-a_j})(j = 1, 2)$, such that

$$\overline{N}_0(R, \frac{1}{f - a_1}) = S(R, f), \quad \overline{N}_0(R, \frac{1}{f - a_2}) = S(R, f). \tag{1}$$

Set

$$L(z) = \frac{z - a_3}{z - a_4}\frac{a_2 - a_4}{a_2 - a_3}.$$

Then $L(a_3) = 0, L(a_4) = \infty, L(a_2) = 1$ and

$$L(a_1) = \frac{z - a_3}{z - a_4}\frac{a_2 - a_4}{a_2 - a_3} = (a_1, a_2, a_3, a_4),$$

which is the cross ratio of a_1, a_2, a_3, a_4. Let $\Phi(z) = L(f(z)), \Psi(z) = L(g(z))$. From $f(z) \not\equiv g(z)$, we get $\Phi(z) \not\equiv \Psi(z)$. From the assumptions of this lemma, we have $L(a_j)(j = 1, 2, 3, 4)$ are shared CM by $\Phi(z)$ and $\Psi(z)$ on $\mathbb{A}$. Thus, we can get that $\Phi(z)$ and $\Psi(z)$ share $0, 1, \infty, b$ CM on $\mathbb{A}$, where $b = L(a_1)$. From Lemma 3.2(i) and (1), we have $S(R, \Phi) = S(R, \Psi)$ and

$$\overline{N}_0(R, \frac{1}{\Phi}) \neq S(R, f), \qquad \overline{N}_0(R, \Phi) \neq S(R, f). \tag{2}$$

Set

$$H_1 = \frac{\Phi'}{\Phi(\Phi - 1)(\Phi - b)} - \frac{\Psi'}{\Psi(\Psi - 1)(\Psi - b)}. \tag{3}$$

Suppose that $H_1(z) \not\equiv 0$, from Lemma 3.1, we have $m_0(R, H_1) = S(R, \Phi)$. If $z_0 \in \mathbb{A}$ is a point such that $\Phi(z_0) = \Psi(z_0) = L(a_j)$ for some $j = 1, 2, 3, 4$, then from (3) we can get that H_1 has no pole on $\mathbb{A}$. Thus, from we have $T_0(R, H_1) = m_0(R, H_1) + N_0(R, H_1) - 2m(1, H_1) = S(R, \Phi)$. If $z_1 \in \mathbb{A}$ is a pole of Φ with multiplicity p, then it must be a pole of Ψ with multiplicity p. Hence from (3) we have that z_1 is a zero of H with multiplicities as least $3p - (p + 1) = 2p - 1$. Thus, we get

$$\overline{N}_0(R, \frac{1}{\Phi}) \leq N_0(R, \frac{1}{H_1}) \leq T_0(R, H_1) + 2m(1, H_1) + O(1) = S(R, f).$$

Therefore, we can get a contradiction with (2). Thus, we can get that $H_1(z) \equiv 0$.

Set

$$H_2 = \frac{\Phi\Phi'}{(\Phi - 1)(\Phi - b)} - \frac{\Psi\Psi'}{(\Psi - 1)(\Psi - b)}. \tag{4}$$

By using the same argument as in the above, we can get that $H_2(z) \equiv 0$. From $H_1(z) \equiv H_2(z) \equiv 0$ we have $\Phi^2(z) \equiv \Psi^2(z)$. Since $\Phi(z) \not\equiv \Psi(z)$, we have $\Phi(z) = -\Psi(z)$. Thus, both 1 and -1 are Picard values of Φ and Ψ on $\mathbb{A}$. From Lemma 3.2(iii), we get that $b = -1$. Hence we have $L(a_1) = (a_1, a_2, a_3, a_4) = -1$. Therefore we get that a_1 and a_2 are Picard values of f and g on $\mathbb{A}$ and $L(f(z)) = L(g(z))$. Thus, we get that f is a Möbius transformation of g.

This completes the proof of this lemma. □

4 The proof of Theorem 1.2

Suppose that $f \not\equiv g$ and none of the $a_j (j = 1, 2, 3, 4)$ is ∞. From Lemma 3.2, we have $S(R, f) = S(R, g)$. Set $S(R) := S(R, f) = S(R, g)$. Let φ be the function expressed as follows

$$\varphi = \frac{f'g'(f-g)^2}{(f-a_1)(f-a_2)(f-a_3)(f-a_4)(g-a_1)(g-a_2)(g-a_3)(g-a_4)}. \tag{5}$$

Then we can get $\varphi \not\equiv 0$. We will show that $T_0(R, \varphi) = S(R)$ as follows.

Suppose $z_0 \in \mathbb{A}$ and $f(z_0) = a_1$ (or a_2, a_3, a_4) with multiplicity p and $g(z_0) = a_1$ (or a_2, a_3, a_4) with multiplicity q. From (5), we can get

$$\varphi(z) = O\left((z-z_0)^{2\min(p,q)-2}\right).$$

Hence φ is an analytic function on $\mathbb{A}$. Then, we have

$$\begin{aligned}
T_0(R,\varphi) &= m_0(R,\varphi)\\
&\leq m_0\left(R, \frac{f'}{(f-a_2)(f-a_3)(f-a_4)}\right) + m_0\left(R, \frac{f'}{(f-a_1)(f-a_2)(f-a_3)}\right)\\
&\quad + m_0\left(R, \frac{f'}{(f-a_1)(f-a_2)(f-a_3)(f-a_4)}\right)\\
&\quad + m_0\left(R, \frac{f'P_1(f)}{(f-a_1)(f-a_2)(f-a_3)(f-a_4)}\right)\\
&\quad + m_0\left(R, \frac{g'}{(g-a_2)(g-a_3)(g-a_4)}\right) + m_0\left(R, \frac{g'}{(g-a_1)(g-a_2)(g-a_3)}\right)\\
&\quad + m_0\left(R, \frac{g'}{(g-a_1)(g-a_2)(g-a_3)(g-a_4)}\right)\\
&\quad + m_0\left(R, \frac{g'P_2(g)}{(g-a_1)(g-a_2)(g-a_3)(g-a_4)}\right) + O(1)\\
&= S(R,f) + S(R,g) = S(R),
\end{aligned}$$

where $P_1(f)$ is a polynomial of degree no more than 2 in f and $P_2(g)$ is a polynomial of degree no more than 2 in g. By Lemma 3.2 (iii), we have

$$m_0\left(R, \frac{1}{f-b_j}\right) = S(R,f), \quad m_0\left(R, \frac{1}{g-b_j}\right) = S(R,g), \tag{6}$$

for any $b_j \in S (j = 1, 2, \ldots, m)$.

Set

$$\varphi_1 := \frac{(g-b_1)\cdots(g-b_m)}{(f-b_1)\cdots(f-b_m)} \cdot \left(\frac{g'(f-g)}{(g-a_1)\cdots(g-a_4)}\right)^m$$

and

$$\varphi_2 := \frac{(f-b_1)\cdots(f-b_m)}{(g-b_1)\cdots(g-b_m)} \cdot \left(\frac{f'(f-g)}{(f-a_1)\cdots(f-a_4)}\right)^m.$$

By Lemma 3.1 and (6), we can get that

$$m_0\left(R, \frac{1}{f-b_j}\cdot\frac{g'(f-g)(g-b_j)}{(g-a_1)\cdots(g-a_4)}\right)=S(R)$$

and

$$m_0\left(R, \frac{1}{g-b_j}\cdot\frac{f'(f-g)(f-b_j)}{(f-a_1)\cdots(f-a_4)}\right)=S(R).$$

From the definitions of φ_1 and φ_2, we get $m_0(R,\varphi_j)=S(R), j=1,2$. By Lemma 3.2(iii), we see that "almost all" of poles and b_j-points of f and g on the annulus $\mathbb{A}$ are simple. Since f,g share the four distinct values $a_j, j=1,2,3,4$ on the annulus $\mathbb{A}$ and $\overline{E}_{\mathbb{A}}(S,f)\subset\overline{E}_{\mathbb{A}}(S,g)$, we can easily get that $N_0(R,\varphi_1)=S(R)$. Therefore, we have

$$T_0(R,\varphi_1)=S(R). \tag{7}$$

Since $\varphi_1\varphi_2\equiv\varphi^m$ and $T_0(R,\varphi)=S(R)$, we can have

$$T_0(R,\varphi_2)=S(R). \tag{8}$$

Let $\Gamma^{pq}_{\mathbb{A}}(a_j)$ be the set of those a_j-points of f and g on the annulus $\mathbb{A}$ such that the multiplicities of f and g at these points are p and q, respectively. For any $z_0\in\Gamma^{pq}_{\mathbb{A}}(a_1)$, by simple computation, we have

$$\varphi_1(z_0)=\left(q\cdot\frac{f'(z_0)-g'(z_0)}{(a_1-a_2)(a_1-a_3)(a_1-a_4)}\right)^m$$

and

$$\varphi_2(z_0)=\left(p\cdot\frac{f'(z_0)-g'(z_0)}{(a_1-a_2)(a_1-a_3)(a_1-a_4)}\right)^m.$$

Hence

$$\frac{1}{q^m}\varphi_1(z_0)-\frac{1}{p^m}\varphi_2(z_0)=0. \tag{9}$$

Similarly, we can see that (9) holds for any $z_0\in\Gamma^{pq}_{\mathbb{A}}(a_j), j=2,3,4$.

Now we discuss two cases as follows.

Case 1. Suppose that $\varphi^{pq}:=\frac{1}{q^m}\varphi_1-\frac{1}{p^m}\varphi_2\not\equiv 0$, for all positive integers p,q.

Next, for $j=1,2,3,4$, we denote by

$$N^{pq}_0(R,\frac{1}{f-a_j})=N^{pq}_1(R,\frac{1}{f-a_j})+N^{pq}_2(R,\frac{1}{f-a_j}),$$

$$N^{pq}_1(R,\frac{1}{f-a_j})=\int_{\frac{1}{R}}^{1}\frac{n^{pq}_1(t,\frac{1}{f-a_j})}{t}dt,\quad N^{pq}_2(R,\frac{1}{f-a_j})=\int_{1}^{R}\frac{n^{pq}_2(t,\frac{1}{f-a_j})}{t}dt,$$

where $n^{pq}_1(t,\frac{1}{f-a_j})$($n^{pq}_2(t,\frac{1}{f-a_j})$) is the counting function of zeros of the function $f-a_j$ in $\{z:t<|z|\le 1\}$ (resp. $\{z:1<|z|\le t\}$) with respect to the set $\Gamma^{pq}_{\mathbb{A}}(a_j)$, similarly, we have $\overline{N}_0^{pq}(R,\frac{1}{f-a_j}),\overline{N}_1^{pq}(R,\frac{1}{f-a_j})$ and $\overline{N}_2^{pq}(R,\frac{1}{f-a_j})$. Thus, we have

$$\begin{aligned} N_0\left(R,\frac{1}{f-a_j}\right) &= N_1\left(R,\frac{1}{f-a_j}\right)+N_2\left(R,\frac{1}{f-a_j}\right)\\ &= \sum_{p,q=1}^{\infty}\left(N_1^{pq}\left(R,\frac{1}{f-a_j}\right)+N_2^{pq}\left(R,\frac{1}{f-a_j}\right)\right)\\ &= \sum_{p,q=1}^{\infty}N_0^{pq}\left(R,\frac{1}{f-a_j}\right)\end{aligned}$$

and

$$\overline{N}_0\left(R,\frac{1}{f-a_j}\right)=\sum_{p,q=1}^{\infty}\overline{N}_0^{pq}\left(R,\frac{1}{f-a_j}\right).$$

Since f,g are transcendental meromorphic on the annulus $\mathbb{A}$, and from the above two equations, (6),(7) and (8), we can see that $T_0(R,\varphi^{pq})=S(R,f)+S(R,g)$. And by (9) each zero of $f-a_j$ is a zero of φ^{pq}, so with the help of $\varphi^{pq}\not\equiv 0$, we can get

$$\begin{aligned}\overline{N}_0^{pq}\left(R,\frac{1}{f-a_j}\right) &\le \overline{N}_0^{pq}\left(R,\frac{1}{\varphi^{pq}}\right)\le T_0\left(R,\frac{1}{\varphi^{pq}}\right)+O(1)\\ &\le T_0(R,\varphi^{pq})+O(1)=S(R,f)+S(R,g):=S(R),\end{aligned}$$

for some p,q. By Lemma 3.2 (ii), we have $T_0(R,f)+S(R,f)=T_0(R,g)+S(R,g)$. Thus, from the definition of $S(R)$, we can get $T_0(R,f)=T_0(R,g)+S(R)$. Therefore, for a positive integer $k(>4)$, we have

$$\begin{aligned}\overline{N}_0\left(R,\frac{1}{f-a_j}\right) &= \sum_{\max(p,q)\ge k}\overline{N}_0^{pq}\left(R,\frac{1}{f-a_j}\right)+S(R,f)\\ &\le \frac{1}{k}\left(\sum_{\max(p,q)\ge k}N_0^{pq}\left(R,\frac{1}{f-a_j}\right)+\sum_{\max(p,q)\ge k}N_0^{pq}\left(R,\frac{1}{g-a_j}\right)\right)\\ &\quad +S(R,f)\\ &\le \frac{1}{k}\left(N_0\left(R,\frac{1}{f-a_j}\right)+N_0\left(R,\frac{1}{g-a_j}\right)\right)+S(R,f)\\ &\le \frac{2}{k}T(R,f)+S(R),\quad j=1,2,3,4.\end{aligned}$$

By the above inequality and Lemma 3.2(ii), we can get

$$T_0(R,f)\le\frac{4}{k}T_0(R,f)+S(R). \tag{10}$$

Since $k(>4)$ is a positive integer, that is, $\frac{4}{k}<1$, from f is transcendental or admissible on $\mathbb{A}$, thus, we can get a contradiction.

Case 2. Suppose that $\varphi^{pq} := \frac{1}{q^m}\varphi_1 - \frac{1}{p^m}\varphi_2 \equiv 0$, for some positive integers p, q. Thus, we have

$$\left(\frac{p}{q}\right)^m \cdot \frac{(g-b_1)^2\cdots(g-b_m)^2}{(f-b_1)^2\cdots(f-b_m)^2} \equiv \left(\frac{f'(g-a_1)\cdots(g-a_4)}{g'(f-a_1)\cdots(f-a_4)}\right)^m. \tag{11}$$

We will consider the two following subcases:

Subcase 2.1. $p \neq q$. Without loss of generality, we may assume that $p < q$. For some two positive integers p_1 and q_1, if $z_1 \in \Gamma_{\mathbb{A}}^{p_1 q_1}(a_j)$ for some $j \in \{1,2,3,4\}$, then (11) implies that $\frac{p}{q} = \frac{p_1}{q_1}$. Hence $q_1 > p_1 \geq 1$, and $q_1 \geq 2$ which means that any a_j-points $(j = 1,2,3,4)$ of g on $\mathbb{A}$ are multiple. By Lemma 3.2 and f, g are transcendental or admissible on $\mathbb{A}$, we can get

$$\begin{aligned} 2T_0(R,g) &= \sum_{j=1}^{4} \overline{N}_0\left(R, \frac{1}{g-a_j}\right) + S(R,g) \\ &\leq \frac{1}{2}\sum_{j=1}^{4} N_0\left(R, \frac{1}{g-a_j}\right) + S(R,g) \\ &\leq 2T_0(R,g) + S(R,g). \end{aligned}$$

Thus, we can get the following equalities easily

$$T_0(R,g) = N_0\left(R, \frac{1}{g-a_j}\right) + S(R), \quad j = 1,2,3,4 \tag{12}$$

and

$$N_0\left(R, \frac{1}{g-a_j}\right) = 2\overline{N}_0\left(R, \frac{1}{g-a_j}\right) + S(R), \quad j = 1,2,3,4. \tag{13}$$

From (12) and (13), we can see that "almost all" of a_j-points of g have multiplicity 2, and "almost all" of a_j-points of f are simple on the annulus $\mathbb{A}$. Without loss of generality, we may assume that f and g attain the values a_3 and a_4 on the annulus $\mathbb{A}$. Set

$$\phi_1 := \frac{2f'(f-a_4)}{(f-a_1)(f-a_2)(f-a_3)} - \frac{g'(g-a_4)}{(g-a_1)(g-a_2)(g-a_3)}$$

and

$$\phi_2 := \frac{2f'(f-a_3)}{(f-a_1)(f-a_2)(f-a_4)} - \frac{g'(g-a_3)}{(g-a_1)(g-a_2)(g-a_4)}.$$

Since $\phi_i (i = 1,2)$ is analytic at the poles of f and of g and also at those common a_j-points of f and g which have multiplicity 1 with respect to f and multiplicity 2 with respect to g, by Lemma 3.1, we have $T_0(R,\phi_i) = S(R,f), i = 1,2$. If $\phi_i \not\equiv 0$, then $N_0\left(r, \frac{1}{f-a_4}\right) \leq N_0\left(R, \frac{1}{\phi_1}\right) = S(R,f)$, which contradicts to equation (13). Then $\phi_1 \equiv 0$. Similarly, we have $\phi_2 \equiv 0$. Therefore, from the definitions of ϕ_1 and ϕ_2, we have

$$\left(\frac{f-a_4}{f-a_3}\right)^2 \equiv \left(\frac{g-a_4}{g-a_3}\right)^2. \tag{14}$$

Since $f \not\equiv g$, and from (14), we have

$$\frac{f-a_4}{f-a_3} \equiv -\frac{g-a_4}{g-a_3},$$

which implies that f and g share a_3, a_4 CM on the annulus $\mathbb{A}$. Since f and g assume the value a_3 there exist positive integers p_1, q_1 such that $\Gamma_{\mathbb{A}}^{p_1 q_1}(a_3) \neq \emptyset$. From the considerations above we get $q_1 > p_1$, contradicting the fact that f and g share a_3 CM.

Subcase 2.2. $p = q$.

In this subcase, (10) becomes

$$\frac{(g-b_1)^2 \cdots (g-b_m)^2}{(f-b_1)^2 \cdots (f-b_m)^2} \equiv \left(\frac{f'(g-a_1) \cdots (g-a_4)}{g'(f-a_1) \cdots (f-a_4)} \right)^m .$$

which implies that f and g share the four values $a_j (j = 1, 2, 3, 4)$ CM on the annulus $\mathbb{A}$. From the conditions of this lemma and applying Lemma 3.3, g is a Möbius transformation of f on $\mathbb{A}$. Furthermore, two of the four values, say a_1, a_2 are Picard exceptional values of f and g on the annulus $\mathbb{A}$. Set

$$\Delta_1 := \frac{f'(f-a_4)}{(f-a_1)(f-a_2)(f-a_3)} - \frac{g'(g-a_4)}{(g-a_1)(g-a_2)(g-a_3)}$$

and

$$\Delta_2 := \frac{f'(f-a_3)}{(f-a_1)(f-a_2)(f-a_4)} - \frac{g'(g-a_3)}{(g-a_1)(g-a_2)(g-a_4)}.$$

Using the same argument as in **Subcase** 2.1 for Δ_1, Δ_2, we can get

$$\frac{f-a_3}{f-a_4} \equiv -\frac{g-a_3}{g-a_4}.$$

We take the Möbius transformations T, M and L satisfying

$$T(\omega) := \frac{w-a_3}{w-a_4}, \qquad M(w) := -w \qquad \textit{and} \qquad L := T^{-1} \circ M \circ T.$$

Then we have

$$T \circ f = -T \circ g, \qquad \textit{hence} \qquad g = L \circ f.$$

Thus, we can see that a_3 and a_4 are the fixed points of L. Therefore, there exist no fixed points of L in the set S. If some $b \in S$ is given. Then from $b \neq a_1, a_2$, there exists a $z_0 \in \mathbb{C}$ such that $b = f(z_0)$, and from $\overline{E}_{\mathbb{A}}(S, f) \subseteq \overline{E}_{\mathbb{A}}(S, g)$ we obtain

$$L(b) = L(f(z_0)) = g(z_0) \in S.$$

So S is invariant under L. Furthermore, we have $L \circ L = I$ where I denotes the identical transformation. Thus, we can get that S must contain an even number of values.

Thus, we complete the proof of Theorem 1.2.

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On the Henstock-Pettis Integral for Fuzzy Number Valued Functions

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Abstract. In this paper, we show that the fuzzy Henstock-Pettis integration of $n-$dimensional fuzzy-number-valued function could be translated into the sum of the fuzzy Henstock-Pettis integration of a $n-$dimensional fuzzy-number-valued function and the Henstock-Pettis integral of a vector-valued function which valued in the kernel sets. In addition, we give the Komlós-type convergence theorems for such integrals.

Keywords. Fuzzy number, fuzzy Henstock-Pettis integral, representation theorems, convergence theorems.

AMS (MOS) subject classification: Primary 26E50; Secondary 28B20.

1 Introduction

It is well-known that the Henstock integral includes the Riemann, improper Riemann, Lebesgue and Newton integrals [9, 12]. It is also equal to the Denjoy and Perron integrals [14]. In the theory of integrals, there are some integrals based on the Banach space-valued functions such as Pettis and Bochner integrals [3, 14, 21]. In particular, Ziat [28, 29] and Amri and Hess [1] presented a characterization of Pettis integral having as their values convex weakly compact subsets of a Banach space. Bochner and Pettis integrals are all defined by using the Lebesgue integrability of the support functions. The integrals of fuzzy-number-valued functions, as a natural generalization of set-valued functions, have been discussed by Puri and Ralescu [19], Kaleva [10], and other authors [7, 24, 25, 27]. Recently, Wu and Gong [6, 8] discussed the fuzzy Henstock integrals of fuzzy-number-valued functions which extended Kaleva [10] integration. However, for a fuzzy valued function in the $n-$dimensional fuzzy number space E^n, the integral and its characteristic theorems have not defined or discussed. In [2], the authors shown that a fuzzy-number valued function is fuzzy Henstock integrable if and only if it can be represented by a sum of a fuzzy McShane integrable fuzzy-number valued function and a fuzzy Henstock integrable fuzzy number valued function generated by a Henstock integrable function. In 2014, K. Musiał [18] established the following decomposition theorem for fuzzy mappings with values in a Banach space: a fuzzy mapping is fuzzy Henstock integrable if and only if it can be represented as a sum of a fuzzy McShane integrable fuzzy mapping and of a fuzzy Henstock integrable fuzzy mapping generated by a Henstock integrable function. As a continuation of our previous work [16, 17, 22], in this paper, we continue to develop the theory of Henstock-Pettis integrals in fuzzy number spaces. By means of replacing the Lebesgue integrability of support functions with their Henstock integrability, we give the definitions of Henstock-Pettis integral and Aumann-Henstock-Pettis integral for compact convex set-valued functions. In addition, the relationships among Henstock-Pettis integral, Aumann-Henstock integral and Pettis integral are investigated. Furthermore, we present the Henstock-Pettis integral, fuzzy Henstock-Pettis integral and Aumann-Henstock-Pettis integral of $n-$dimensional fuzzy-number-valued functions, and the relationships among them are studied. At the same time, the representation theorems and the calculations of fuzzy Henstock-Pettis integral are given. It shows that the fuzzy Henstock-Pettis integration of a $n-$dimensional fuzzy-number-valued function equals the sum of the Henstock-Pettis integration of an $n-$dimensional fuzzy-number-valued function and the Henstock-Pettis integration of a vector-valued function which valued in the kernel sets.

The rest of the paper is organized as follows. In section 2, the definitions of Pettis integral, Henstock-Pettis integral and Aumann-Henstock integral for compact convex set-valued functions are given. In section 3, we disscus the characterization of Henstock-Pettis integral, fuzzy Henstock-Pettis integral and Aumann-Henstock integral for fuzzy-number-valued functions, and we give the representation theorems of fuzzy Henstock-Pettis integral. In section 4, we present a Komlós-type convergence theorem to the fuzzy Henstock-Pettis integral and give an existence theorem for a kind of fuzzy integral inclusion. And in section 5, we present some concluding remarks.

2 Preliminaries

Let T be the closed interval on the real line R, i.e., $T = [a, b]$ $(a, b \in R)$. $|T|$ denotes the length of T.

Throughout this paper, we use $P_k(R^n)$ to denote the family of all nonempty compact convex subsets of R^n. For $A, B \in P_k(R^n), k \in R$, the addition and scalar multiplication are defined by the equations as follows respectively:

$$A + B = \{x + y \mid x \in A, y \in B\}, \qquad aA = \{ax \mid x \in A\}.$$

In addition, for $A, B \in P_k(R^n)$, the Hausdorff metric between them defined by:

$$d(A, B) = \max\{\sup_{a\in A}\inf_{b\in B}\|a - b\|, \sup_{b\in B}\inf_{a\in A}\|b - a\|\}.$$

A compact convex set-valued function $F : T \to P_k(R^n)$ is said to be measurable if $\{t \in T | F(t) \cap O \neq \phi\}$ is a measurable set for any open subset $O \subset R^n$. F is said to be scalarly measurable if the map $\sigma(x, F(\cdot))$ is measurable for every $x \in S^{n-1}$. Certainly, a compact convex set-valued function $F : T \to P_k(R^n)$ is measurable if it is scalarly measurable.

A function $f : T \to R^n$ is called a selection of F if $f(t) \in F(t)$ for any $t \in T$. A selection f is said to be measurable if the function f is strongly measurable, i.e., f is a limit of an almost everywhere convergent sequence of measurable simple functions.

A compact convex set-valued function $F : T \to P_k(R^n)$ is said to be graph measurable if the set $\{(t, x) \in T \times R^n | x \in F(t)\}$ is a member of the product $\sigma-$algebra generated by $\mathcal{L}$ and the Borel subsets of R^n in the norm topology. Here $\mathcal{L}$ denotes the family of all Lebesgue measurable subsets of T.

Definition 2.1 ([27])**.** For $A \in P_k(R^n)$, $x \in S^{n-1}$, the support function of A is defined by

$$\sigma(x, A) = \sup_{y\in A}\langle y, x\rangle,$$

where S^{n-1} denotes the unit sphere of R^n, $\langle\cdot, \cdot\rangle$ is the inner product in R^n.

Next, we shall give the definitions of Pettis integral, Henstock-Pettis integral and Aumann-Henstock-Pettis integral for compact convex set-valued functions.

Definition 2.2. A set-valued function $F : T \to P_k(R^n)$ is said to be Henstock integrable to $I \in P_k(R^n)$ if for every $\varepsilon > 0$ there is a function $\delta(x) > 0$ such that for any $\delta-$fine division $\Pi = \{\xi_i, [x_{i-1}, x_i]\}$ of T, we have

$$d(I, \sum_i F(\xi_i)(x_i - x_{i-1})) < \varepsilon,$$

and write $(H)\int_T F(x)dx = I$.

Lemma 2.3 ([27])**.** *If $A \in P_k(R^n), x \in S^{n-1}$, then $A = \{y \in R^n | \langle y, x\rangle \leq \sigma(x, A), x \in S^{n-1}\}$.*

Lemma 2.4 ([26])**.** *If $A_r \in P_k(R^n), \{A_{r_m}\} \subset P_k(R^n)$, where r_m is converging nondecreasingly to r and $A_{r_m} \supset A_{r_{m+1}} \supset A_r (m = 1, 2, \cdots)$ for any $x \in S^{n-1}$, then $A_r = \bigcap_{m=1}^{\infty} A_{r_m}$ if $\sigma(x, A_{r_m})$ converge to $\sigma(x, A_r)$.*

Theorem 2.5. *A set-valued function $F : T \to P_k(R^n)$ is Henstock integrable on T iff the real-valued function $\sigma\big(x, F(t)\big)$ is Henstock integrable uniformly on T for any $x \in S^{n-1}$, and*

$$\sigma\big(x, (H)\int_T F(t)\mathrm{d}t\big) = (H)\int_T \sigma\big(x, F(t)\big)\mathrm{d}t. \tag{2.1}$$

Definition 2.6. Let $F : T \to P_k(R^n)$ be a measurable set-valued function. F is said to be Pettis (Henstock-Pettis) integrable on T if there is a nonempty set $A \in P_k(R^n)$ such that for any $x \in S^{n-1}$ we have

$$\sigma(x, A) = (L)\int_T \sigma(x, F(t))\mathrm{d}t$$

$$(\sigma(x, A) = (H)\int_T \sigma(x, F(t))\mathrm{d}t),$$

and write $A = (P)\int_T F(t)\mathrm{d}t \ \ (A = (wH)\int_T F(t)\mathrm{d}t)$.

In particular, if the set-valued function above is defined by $F : T \to R^n$, then the set A will become a vector in R^n, and for any $x \in S^{n-1}$ we have

$$< x, A >= (L)\int_T < x, F(t) > \mathrm{d}t$$

$$(< x, A >= (H)\int_T < x, F(t) > \mathrm{d}t).$$

In this case, F is also said to be Pettis (Henstock-Pettis) integrable on T.

Theorem 2.7. *If $F : T \to P_k(R^n)$ is a measurable set-valued function, then the family of measurable selections of F is not empty.*

Proof. Since R^n is a separable space, we can prove the theorem easily. □

Now, we use $s_H(F)$ to denote the family of Henstock-Pettis integrable selections and $s_P(F)$ to denote the family of Pettis integrable selections of F.

Definition 2.8. The Aumann-Henstock integral of a measurable set-valued function $F : T \to P_k(R^n)$ defined by

$$(AH)\int_T F(t)\mathrm{d}t = \{(HP)\int_T f(t)\mathrm{d}t | f \in s_H(F)\}.$$

Definition 2.9. Pick a set-valued function $F : T \to P_k(R^n)$ and let $I \subset T$. The function $f : A \to P_k(R^n)$ is the weak derivative of F on T if the Banach valued function $(\sigma(x, F))'$ is differentiable almost everywhere on I and $(\sigma(x, F))' = \sigma(x, f))$ almost everywhere on T.

Example. Let $F : T \to P_k(R^n)$ be weakly differentiable. Then its weak derivative F' is Henstock-Pettis integrable and

$$(HP)\int_a^s F'(t)\mathrm{d}t = F(s) - F(a), s \in T.$$

Indeed, F has the weak derivative at a point t means that there is a point $F'(t) \in R^n$ such that for any $x \in S^{n-1}$, we have

$$\lim_{\triangle t \to 0} \frac{< x, F(t + \triangle t) > - < x, F(t) >}{\triangle t} =< x, F'(t) > .$$

Since $< x, F >$ is differentiable, so we have

$$< x, F(s) > - < x, F(a) >= (H)\int_a^s < x, F >' \mathrm{d}t, s \in T.$$

On the other hand, $< x, F >' = < x, F' >$, it implies that

$$< x, F(s) > - < x, F(a) > = (H) \int_a^s < x, F'(t) > \mathrm{d}t.$$

That is

$$< x, F(s) - F(a) > = (H) \int_a^s < x, F'(t) > \mathrm{d}t.$$

Hence

$$F(s) - F(a) = (HP) \int_a^s F'(t) \mathrm{d}t. \qquad \square$$

Theorem 2.10. *If all measurable selections of $F : T \to P_k(R^n)$ are Henstock-Pettis integrable and $\sigma(x, F(t))$ is Henstock integrable, then $(AH) \int_T F(t)\mathrm{d}t$ is a compact convex set.*

Proof. Since $\sigma(x, F(t))$ is Henstock integrable, so $\sigma(x, F(t))$ is measurable. Now fix a measurable selection f of F and let $G(t) = F(t) - f(t)$. Since f is Henstock-Pettis integrable, so G is Aumann-Henstock integrable.

Let $I_T = (AH) \int_T G(t)\mathrm{d}t$ and D be a countable dense subset of S^{n-1}. We prove the convexity of I_T first. It can be proved that

$$(AH) \int_T G(t)\mathrm{d}t = \{(wH) \int_T g(t)\mathrm{d}t | g \in s_H(F - f)\}$$

is a convex set. In fact, for any $A, B \in I_T$, there exist $g_1(t), g_2(t) \in s_H(F - f)$ such that

$$A = (HP) \int_T g_1(t)\mathrm{d}t,\ B = (HP) \int_T g_2(t)\mathrm{d}t. \tag{2.2}$$

In addition, for any $\lambda \in [0, 1]$,

$$\begin{aligned} \lambda A + (1 - \lambda)B &= \lambda(HP) \int_T g_1(t)\mathrm{d}t + (1 - \lambda) \int_T g_2(t)\mathrm{d}t \\ &= (HP) \int_T (\lambda g_1(t) + (1 - \lambda)g_2(t))\mathrm{d}t. \end{aligned}$$

That is, $\lambda A + (1 - \lambda)B \in I_T$.

In order to prove the compactness of I_T, we take a sequence of points $x_n \in I_T$, and then there exists $g_n \in s_H(G)$ with $x_n = (HP) \int_T g_n(t)\mathrm{d}t$. For each $n \in N, t \in T, x \in S^{n-1}$, we have the inequalities

$$-\sigma(-x, G(t)) \leq < x, g_n(t) > \leq \sigma(x, G(t)). \tag{2.3}$$

Since $f \in s_H(F)$ and the null function is included in $G(t)$, $\sigma(x, G(t))$ is nonnegative Henstock integrable. It implies that the support function $\sigma(x, G(t))$ is Lebesgue integrable. Thus, each $< x, g_n >$ is Lebesgue integrable and

$$(L) \int_T | < x, g_n(t) > |\mathrm{d}t \leq (L) \int_T \sigma(x, G(t))\mathrm{d}t + (L) \int_T \sigma(-x, G(t))\mathrm{d}t.$$

Furthermore, due to the countability of D and L_1-boundeness of each $< x, g_n >$ we can find there exist $h_n \in conv\{g_n, g_{n+1}, \cdots\}$, such that for each $x \in D$ the sequence $< x, h_n >$ is almost everywhere convergent to a measurable function h_x.

As for each t and n we have $h_n(t) \in G(t)$ and $G(t)$ is compact, there is a cluster point $h(t) \in G(t)$. It follows that there is a set N of Lebesgue measure zero such that for any $x \in D$ and $t \notin N$ we have

$$< x, h(t) > = \lim_{n\to\infty} < x, h_n(t) > = h_x(t).$$

Taking into formula Eq. (2.3) and the Lebesgue dominated convergence theorem, we have for any $x \in S^{n-1}$,

$$\begin{aligned}\lim_{n\to\infty} < x, (HP)\int_T h_n(t)\mathrm{d}t > &= \lim_{n\to\infty}(L)\int_T < x, h_n(t) > \mathrm{d}t \\ &= (L)\int_T < x, h(t) > \mathrm{d}t \\ &= < x, (HP)\int_T h(t)\mathrm{d}t > . \end{aligned} \tag{2.4}$$

We write $y_n = (HP)\int_T h_n(t)\mathrm{d}t$, then $y_n \in I_T, y_n \in conv\{x_n, x_{n+1}, \cdots\}$ and the sequence $\{y_n\}$ convergent to $y_0 = (HP)\int_T h(t)\mathrm{d}t$. Thus, given an arbitrary sequence $\{x_n\}$, $x_n \in I_T$, there is a convex combination of points $y_n \in conv\{x_n, x_{n+1}, \cdots\}$ and $y_0 \in I_T$ such that y_n converge to y_0. Consequently, the set I_T is compact, i.e., there exists $y_0 = (HP)\int_T h(t)\mathrm{d}t \in I_T$ such that $\lim_{n\to\infty} y_n = y_0$. □

3 The Henstock-Pettis integral for fuzzy number valued functions

In this section, we give the definition of Henstock-Pettis integral of fuzzy-number-valued functions and its representation theorems.

Definition 3.1 ([4, 23])**.** Let $E^n = \{u | u : R^n \to [0,1]\}$. For any $u \in E^n$, u is said to be a $n-$dimensional fuzzy number if the following conditions are satisfied:
(1) u is a normal fuzzy set, i.e., there exists an $x_0 \in R^n$, such that $u(x_0) = 1$;
(2) u is a convex fuzzy set, i.e., $u\big(tx + (1-t)y\big) \geq \min\{u(x), u(y)\}$ for any $x, y \in R^n, t \in [0,1]$;
(3) u is upper semi-continuous;
(4) supp$u = \overline{\{x \in R^n \mid u(x) > 0\}}$ is compact, here $\overline{A}$ denotes the closure of A.

For $r \in (0,1]$, denote $[u]^r = \{x \in R^n \mid u(x) \geq r\}$ and we call it the $r-$level set of u, and $[u]^0 = \overline{\bigcup_{r\in(0,1]} [u]^r}$.

E^n denotes the $n-$dimensional fuzzy number space. If $u \in E^n$, then $[u]^r$ is a nonempty compact convex subset of R^n for each $r \in [0,1]$.

Theorem 3.2 ([4, 23])**.** *Define $D : E^n \times E^n \to [0,\infty)$ by the equation*

$$D(u,v) = \sup_{r\in[0,1]} d([u]^r, [v]^r), u, v \in E^n,$$

then
(1) (E^n, D) is a complete metric space;
(2) $D(\lambda u, \lambda v) = |\lambda| D(u,v), \lambda \in R$;
(3) $D(u+w, v+w) = D(u,v)$;
(4) $D(u+v, w+e) \leq D(u,w) + D(v,e)$;
(5) $D(u+v, \widetilde{0}) = D(u,\widetilde{0}) + D(v,\widetilde{0})$;
(6) $D(u+v, w) \leq D(u,w) + D(v,\widetilde{0})$.
where $u, v, w, e, \widetilde{0} \in E^n, \widetilde{0} = \chi_{(\{0\})}$.

The metric space (E^n, D) has a linear structure, it can be imbedded isomorphically as a convex cone with vertex θ into the Banach space of functions $u^* : I \times S^{n-1} \longrightarrow R$, where S^{n-1} is the unit sphere in R^n, with an imbedding function $u^* = j(u)$ defined by

$$u^*(r,x) = \sup_{\alpha\in[u]^\alpha} < \alpha, x >$$

for all $< r, x >\in I \times S^{n-1}$.

Theorem 3.3 ([23]). *There exists a real Banach space X such that E^n can be imbedding as a convex cone C with vertex θ into X. Furthermore the following conditions hold true:*
(1) the imbedding j is isometric,
(2) addition in X induces addition in E^n,
(3) multiplication by nonnegative real number in X induces the corresponding operation in E^n,
(4) $C - C$ is dense in X,
(5) C is closed.

A fuzzy-number-valued function $\tilde{F} : [a, b] \to E^n$ is said to satisfy the condition (H) on $[a, b]$, if for any $x_1 < x_2 \in [a, b]$ there exists $u \in E^n$ such that $f(x_2) = f(x_1) + u$. We call u is the H-difference of $\tilde{F}(x_2)$ and $\tilde{F}(x_1)$, denoted $\tilde{F}(x_2) -_H \tilde{F}(x_1)$ ([10]).

For brevity, we always assume that the condition (H) is satisfied when dealing with the operation of subtraction of fuzzy numbers throughout this paper.

Definition 3.4 ([24, 25]). A fuzzy-number-valued function $\tilde{F} : T \to E^n$ is said to be fuzzy Henstock integrable on T if there exists a fuzzy number $\tilde{A} \in E^n$ such that for every $\varepsilon > 0$ there is a function $\delta(x) > 0$ such that for any $\delta-$fine division $\Pi = \{\xi_i, [x_{i-1}, x_i]\}$ of T, we have

$$D(\tilde{A}, \sum_i \tilde{F}(\xi_i)(x_i - x_{i-1})) < \varepsilon.$$

We write $(FH) \int_T \tilde{F}(x)dx = \tilde{A}$.

Definition 3.5 ([22]). A fuzzy-number-valued function $\tilde{F} : T \to E^n$ is said to be Pettis (Henstock-Pettis) integrable on T if $[F(t)]^r$ is Pettis (Henstock-Pettis) integrable on T for every $r \in [0, 1]$, and there exists a fuzzy number $\tilde{A} \in E^n$ such that for any $x \in S^{n-1}$ we have

$$\sigma(x, [A]^r) = (L) \int_T \sigma(x, [F(t)]^r)\mathrm{d}t$$

$$(\sigma(x, [A]^r) = (H) \int_T \sigma(x, [F(t)]^r)\mathrm{d}t).$$

We write $\tilde{A} = (FP) \int_T \tilde{F}(t)\mathrm{d}t$ ($\tilde{A} = (FHP) \int_T \tilde{F}(t)\mathrm{d}t$).

Remark 3.6. In particular, if $\tilde{F}$ is degenerated into $F : T \to R^n$ and $\tilde{A}$ is degenerated into $A \in R^n$, then

$$\sigma(x, [A]^r) =< x, A > .$$

Remark 3.7. When $n = 1$, if the fuzzy-number-valued function $\tilde{F} : T \to E^1$ is Kaleva integrable on T (refer to [25]), then $\tilde{F}$ is also Pettis integrable.

Remark 3.8. When $n = 1$, if the fuzzy-number-valued function $\tilde{F} : T \to E^1$ is fuzzy Henstock integrable on T (refer to the Definition 3.2 of [24]), then $\tilde{F}$ is also fuzzy Henstock-Pettis integrable.

A fuzzy-number-valued function $\tilde{F} : T \to E^n$ is said to be measurable on T iff the compact convex set-valued function $F^r : T \to P_k(R^n)$ is measurable for any $r \in [0, 1]$.

Definition 3.9. Let $\tilde{F} : T \to E^n$ be a measurable fuzzy-number-valued function, $\tilde{F}$ is said to be fuzzy Aumann-Henstock-Petiss integrable on T if

$$(FAHP) \int_T [F(t)]^r \mathrm{d}t = \{(HP) \int_T f(t)\mathrm{d}t | f \in s_{HP}[F(t)]^r\}$$

determines a unique fuzzy number $\tilde{A} \in E^n$, where $s_H[F(t)]^r$ denotes the family of all fuzzy Henstock-Petiss integrable selections of $[F(t)]^r$. We write $(FAH) \int_T \tilde{F}(t)\mathrm{d}t = \tilde{A}$.

Theorem 3.10. *Let $\tilde{F} : T \to E^n$ be a fuzzy Aumann-Henstock-Pettis integrable function on T. If the set-valued function $[F(t)]^r$ is measurable and the measurable selections of $[F(t)]^r$ are Henstock-Pettis integrable for any $r \in [0,1]$, then for every $x \in S^{n-1}$ we have*

$$\sigma\big(x, (AHP)\int_T [F(t)]^r \mathrm{d}t\big) = (H)\int_T \sigma(x, [F(t)]^r)\mathrm{d}t.$$

Proof. Since the measurable selections of $[F(t)]^r$ are Henstock-Pettis integrable for any $r \in [0,1]$, we have

$$\sigma\big(x, (AHP)\int_T [F(t)]^r \mathrm{d}t\big) = (H)\int_T \sigma(x, [F(t)]^r)\mathrm{d}t$$

for any $x \in S^{n-1}, r \in [0,1]$. □

Furthermore, by Theorem 2.10, we can easily obtain the following Theorem 3.11.

Theorem 3.11. *Let $\tilde{F} : T \to E^n$ be a fuzzy-number-valued function. If all measurable selections of $[F(t)]^r$ are Henstock-Pettis integrable and $\sigma(x, [F(t)]^r)$ is Henstock integrable for any $r \in [0,1]$, then $(AHP)\int_T [F(t)]^r \mathrm{d}t$ is a compact convex set.*

Theorem 3.12. *Let $\tilde{F} : T \to E^n$ be a fuzzy-number-valued function on T. If $\tilde{F}$ is fuzzy Henstock-Pettis integrable on T, then each measurable selection of $[F(t)]^r$ is Henstock-Pettis integrable for any $r \in [0,1]$ and $t \in T$.*

Proof. Since $[F(t)]^r$ is Henstock-Pettis integrable on T, for any $r \in [0,1]$ and $t \in T$, by Lemma 3 of [5], the conclusion holds. □

Theorem 3.13. *If $\tilde{A}, \tilde{B} \in E^n$, then $\tilde{A} \subset \tilde{B}$ if and only if $\sigma(x, [A]^r) \le \sigma(x, [B]^r)$ for any $r \in [0,1]$ and $x \in S^{n-1}$.*

Proof. Necessity: If $\tilde{A} \subset \tilde{B}$, then for any $r \in [0,1]$ and $x \in S^{n-1}$ we have

$$\begin{aligned}\sigma(x, [A]^r) &= \sup\{< x, a > | a \in [A]^r\} \\ &\le \sup\{< x, b > | b \in [B]^r\} \\ &= \sigma(x, [B]^r). \end{aligned} \tag{3.1}$$

Sufficiency: If $\sigma(x, [A]^r) \le \sigma(x, [B]^r)$ for any $r \in [0,1]$ and $x \in S^{n-1}$, then for every $a \in A$, by the Lemma 2.3 we have $< x, a > \le \sigma(x, A) \le \sigma(x, B)$, thus $a \in [B]^r$, that is $[A]^r \subset [B]^r$. Hence, $\tilde{A} \subset \tilde{B}$. □

Theorem 3.14. *Let $\tilde{F} : T \to E^n$ be a fuzzy-number-valued function on T. If the integration $(FH)\int_T \tilde{F}(t)\mathrm{d}t$ exists, then the following statements are equivalent:*

(1) $\tilde{F}$ is fuzzy Henstock-Pettis integrable on T;

(2) For every Henstock-Pettis integrable function $f \in s_{HP}([F(t)]^1)$, there exists a fuzzy-number-valued function $\tilde{G} : T \to E^n$ such that $\tilde{F}(t) = \tilde{G}(t) + \tilde{f}(t)$ and $\tilde{G}$ is fuzzy Pettis integrable on T;

(3) For every $f, h \in s_H([F(t)]^1)$, $h - f$ is Pettis integrable;

(4) $\tilde{F}(x)$ is fuzzy Aumann-Henstock-Pettis integrable on T and for any $x \in S^{n-1}$, we have

$$\sigma(x, (AHP)\int_T [F(t)]^r \mathrm{d}t) = (H)\int_T \sigma(x, [F(t)]^r)\mathrm{d}t \ (r \in [0,1]).$$

Proof. (1) ⇒ (2): For every $f(t) \in s_H([F(t)]^1)$, since $[F(t)]^1$ is Henstock-Pettis integrable on T, by Theorem 3.10, wa can infer that $f(t)$ is Henstock-Pettis integrable on T. Define $[G(t)]^r = [F(t)]^r - f(t)$, $\quad [G]^r : T \to P_k(R^n)$, then for any $x \in S^{n-1}, t \in T$ we have $\sigma(x, [G(t)]^r) \ge 0$, and

$$\sigma(x, [F(t)]^r) = \sigma(x, [G(t)]^r) + < x, f(t) > .$$

By Theorem 3.12, $[G(t)]^r$ is Pettis integrable. In addition, we can prove that $\{[G(t)]^r, r \in [0,1]\}$ determines a fuzzy number. In fact, $\{[G(t)]^r\}$ satisfies the following conditions:

(i) $[G(t)]^r$ is a nonempty compact convex set;

(ii) if $0 \leq r_1 \leq r_2 \leq 1$, then

$$\begin{aligned} \sigma(x,[G(t)]^{r_1}) &= \sigma(x,[F(t)]^{r_1}) - <x, f(t)> \\ &\geq \sigma(x,[F(t)]^{r_2}) - <x, f(t)> \\ &= \sigma(x,[G(t)]^{r_2}) \end{aligned} \tag{3.2}$$

That is $[G(t)]^{r_1} \supseteq [G(t)]^{r_2}$;

(iii) for any $\{r_m\}$ converging increasingly to $r \in (0,1]$, since for any $x \in S^{n-1}$ we have

$$\sigma(x,[F(t)]^{r_m}) \downarrow \sigma(x,[F(t)]^r),$$

so

$$\begin{aligned} \sigma(x,[G(t)]^{r_m}) &= \sigma(x,[F(t)]^{r_m}) - <x, f(t)> \\ &\downarrow \sigma(x,[F(t)]^{r}) - <x, f(t)> \\ &= \sigma(x,[G(t)]^{r}). \end{aligned} \tag{3.3}$$

(2) $\Rightarrow$ (3): Let $f \in s_{HP}([F(t)]^1)$, $[G(t)]^r = [F(t)]^r - f(t)$. If $h \in s_H([F(t)]^1)$, then $g = h - f$ is a measurable selection of $[G]^r$, and

$$-\sigma(-x,[G(t)]^r) \leq <x, g(t)> \leq \sigma(x,[G(t)]^r).$$

For every $E \in \mathcal{L}$, we denotes $w_E = (P)\int_E [G(t)]^r \mathrm{d}t \in P_k(R^n)$, then

$$\begin{aligned} -\sigma(-x,w_E) &= -(L)\int_E \sigma(-x,[G(t)]^r)\mathrm{d}t \\ &\leq (L)\int_E <x, g(t)> \mathrm{d}t \\ &\leq (L)\int_E \sigma(x,[G(t)]^r)\mathrm{d}t = \sigma(x,w_E). \end{aligned} \tag{3.4}$$

On the other hand, w_E is compact and its support function $\sigma(x, w_E)$ is Lipschitz continuous uniformly with respect to x, therefore $x \to (L)\int_E <x, g(t)> \mathrm{d}t$ is continuous uniformly, it follows that $g = h - f$ is Pettis integrable.

(3) $\Rightarrow$ (2): For $f \in s_{HP}([F(t)]^1)$, define $[G(t)]^r = [F(t)]^r - f(t)$, then by assumption, each measurable selection g of $[G(t)]^r$ is Pettis integrable, and by Theorem 3.12, $[G(t)]^r$ is also Pettis integrable on T. Furthermore, we can prove $\{[G(t)]^r, r \in [0,1]\}$ determines a fuzzy number similar to (1) $\Rightarrow$ (2). It shows that $\tilde{G}$ is Pettis integrable on T.

(2) $\Rightarrow$ (4) For $f \in s_H([F(t)]^1)$, the set-valued function $[G(t)]^r = [F(t)]^r - f(t)$ is Pettis integrable on T. By the Theorem 3.12, $[G(t)]^r$ is Aumann-Henstock-Pettis integrable on T, and

$$(P)\int_T [G(t)]^r \mathrm{d}t = \{(P)\int_T g(t)\mathrm{d}t | g \in s_P([G(t)]^r)\}.$$

Note that $(P)\int_T [G(t)]^r \mathrm{d}t$ is a compact convex set, then

$$(AHP)\int_T [F(t)]^r \mathrm{d}t = (P)\int_T [G(t)]^r \mathrm{d}t + (HP)\int_T f(t)\mathrm{d}t$$

is also a compact convex set.

We can prove that $\{(AHP)\int_T [F(t)]^r \mathrm{d}t \mid r \in [0,1]\}$ determines a unique fuzzy number. In fact, $\{(AHP)\int_T [F(t)]^r \mathrm{d}t \mid r \in [0,1]\}$ satisfies the following conditions:

(i) $(AHP)\int_T [F(t)]^r \mathrm{d}t$ is a compact convex set;

(ii) if $0 \le r_1 \le r_2 \le 1$, then

$$\begin{aligned}\sigma(x,(AHP)\int_T [F(t)]^{r_1}\mathrm{d}t) &= \sigma(x,(P)\int_T [G(t)]^{r_1}\mathrm{d}t + (HP)\int_T f(t)\mathrm{d}t)\\ &= \sigma(x,(P)\int_T [G(t)]^{r_1}\mathrm{d}t) + \sigma(x,(HP)\int_T f(t)\mathrm{d}t)\\ &\ge \sigma(x,(P)\int_T [G(t)]^{r_2}\mathrm{d}t) + \sigma(x,(HP)\int_T f(t)\mathrm{d}t)\\ &= \sigma(x,(AHP)\int_T [F(t)]^{r_2}\mathrm{d}t). \end{aligned} \tag{3.5}$$

(iii) for any $\{r_m\}$ converging increasingly to $r \in (0,1]$, since for every $x \in S^{n-1}$ we have

$$\sigma(x,[G(t)]^{r_m}) \downarrow \sigma(x,[G(t)]^r).$$

Consequently,

$$\begin{aligned} &\sigma(x,(AHP)\int_T [F(t)]^{r_m}\mathrm{d}t)\\ =\ &\sigma(x,(P)\int_T [G(t)]^{r_m}\mathrm{d}t) + \sigma(x,(HP)\int_T f(t)\mathrm{d}t)\\ \downarrow\ &\sigma(x,(P)\int_T [G(t)]^{r}\mathrm{d}t) + \sigma(x,(HP)\int_T f(t)\mathrm{d}t)\\ =\ &\sigma(x,(AHP)\int_T [F(t)]^{r}\mathrm{d}t). \end{aligned}$$

Thus, $\{(AHP)\int_T [F(t)]^r \mathrm{d}t \mid r \in [0,1]\}$ determines a unique fuzzy number. That is, $\tilde{F}(x)$ is fuzzy Aumann-Henstock-Pettis integrable on T, and

$$\sigma(x,(AHP)\int_T [F(t)]^r \mathrm{d}t) = (H)\int_T \sigma(x,[F(t)]^r)\mathrm{d}t.$$

(4) $\Rightarrow$ (1): Since $\tilde{F}(t)$ is fuzzy Aumann-Henstock-Pettis integrable on T, so $[F(t)]^r$ is Aumann-Henstock-Pettis integrable on T for any $r \in [0,1]$. By Theorem 3.12, $[F(t)]^r$ is Henstock-Pettis integrable on T. Similar to the proof of (2) $\Rightarrow$ (4), we can prove that $\{[F(t)]^r, r \in [0,1]\}$ determines a unique fuzzy number $(FAHP)\int_T \tilde{F}(t)\mathrm{d}t \in E^n$, i.e., $\tilde{F}(t)$ is fuzzy Henstock-Pettis integrable on T.

Corollary 3.15. *If $\tilde{F}: T \to E^n$, $\tilde{G}: T \to E^n$, $f \in s_{HP}([F(t)]^1$, then the fuzzy Henstock-Pettis integration of $\tilde{F}$ could be translated into the Henstock-Pettis integration of $\tilde{G}$, and*

$$(FHP)\int_T \tilde{F}(t)\mathrm{d}t = (FP)\int_T \tilde{G}(t)\mathrm{d}t + (HP)\int_T f(t)\mathrm{d}t.$$

Theorem 3.16. *Let $\tilde{F}: T \to E^n$ be a measurable fuzzy-number-valued function, $\sigma(x,[F(t)]^r)$ Henstock integrable on T. If $\tilde{F}$ is fuzzy Henstock-Pettis integrable on T, then $[G(t)]^r = [F(t)]^r - f(t)$ is Pettis integrable on T for any measurable selection f of $[F(t)]^1$, and*

$$(H)\int_T \sigma(x,[F(t)]^r)\mathrm{d}t = (L)\int_T \sigma(x,[G(t)]^r)\mathrm{d}t + (H)\int_T < x, f(t) > \mathrm{d}t.$$

for every $t \in T, x \in S^{n-1}$.

Proof. Suppose f is a measurable selection of $[F(t)]^1$, $[G(t)]^r = [F(t)]^r - f(t)$. Since the support function $\sigma(x, [G(t)]^r)$ is a Henstock integrable set-valued function and $< x, f(t) >$ is Henstock integrable, we see that

$$[F(t)]^r = [G(t)]^r + f(t).$$

On the other hand, since $[G(t)]^r$ has at least one Bochner integrable selection (the null function), for every $E \in \mathcal{L}$ there is $w_E \in R^n$ with

$$\sigma(x, w_E) = (L) \int_E \sigma(x, [G(t)]^r) \mathrm{d}t$$

for any $x \in S^{n-1}$. Hence, we have for every $x \in S^{n-1}$

$$\sigma(x, w_T) + (H) \int_T < x, f(t) > \mathrm{d}t = (H) \int_T \sigma(x, [F(t)]^r) \mathrm{d}t \neq \pm\infty.$$

It follows that $\sigma(x, w_T) \neq \pm\infty$ for all $x \in S^{n-1}$. By Banach-Steinhaus Theorem, $w_T \in P_k(R^n)$. And we get that every w_E is bounded, thus $[G(t)]^r$ is Pettis integrable on T. □

4 The Komlós-type convergence theorem for the Fuzzy Henstock-Pettis integrals and a fuzzy integral inclusion

The Komlós's classical theorem (see[11]) yields that from any L^1-bounded sequence of real functions one can extract a subsequence such that the arithmetic averages of all its subsequence converge point almost everywhere. In [20], the author extended these results by providing a Komlós-type theorem for set-valued functions under Henstock-Pettis integrability assumptions. In this section, we extend the Komlós theorem to the case of the fuzzy-number-valued Henstock-Pettis integrals. As an application, an existence theorem for a fuzzy integral inclusion involving the fuzzy Henstock-Pettis integral is obtained.

Definition 4.1. A sequence $(\tilde{F}_n)_n$ of fuzzy-number-valued functions is said to be Komlós convergent ($K-$convergent for short) to a fuzzy-number-valued function $\tilde{F}$ if for every subsequence $(\tilde{F}_{k_n})_n$ there exists a $\mu-$null set $N \subset T$, such that for all $t \in T \backslash N$,

$$\sigma(x, [F(t)]^r) = \lim_n \sigma(x, \frac{1}{n} \sum_{i=1}^{n} [F_{k_i}(t)]^r).$$

Theorem 4.2. *Let $\tilde{F}_n : T \to E^n$ be a sequence of $(FHP)-$integrable functions. Suppose*

(i) *there exists a real Henstock integrable function $\bar{f}$, such that*

$$\bar{f}(t) \leq \sigma(x, [F_n(t)]^r), \quad \forall t \in T, \forall n \in \mathbb{N};$$

and

$$\sup_{n \in \mathbb{N}} (H) \int_T \sigma(x, [F_n(t)]^r) \mathrm{d}t < +\infty;$$

(ii) *there exists a function $h : T \times R \to [0, +\infty)$ such that, for every $t \in T$, $h(t, \cdot)$ is convex and compact, and a countable measurable partition $(B_m)_m$ of T satisfying:*

(a) $\sup_n (H) \int_{B_m} |\sigma(x, [F_n(t)]^r)| \mathrm{d}t < +\infty$;

(b) $\sup_n (H) \int_{B_m} h(t, [F_n(t)]^r) \mathrm{d}t$.

Then there exist a $(FHP)-$integrable function $\tilde{F}$ and a subsequence of $(\tilde{F}_n)_n$ which $K-$converges to $\tilde{F}$. Moreover, $\int_{B_m} h(t, [F(t)]^r) \mathrm{d}t$ exist for each $m \in \mathbb{N}$.

Proof. Consider the convex $Y \subset E^n$, let the function $g(t,C) = \sigma(x,C)$ be continuous on Y. By (ii) and defn 5.1, the fuzzy sequence $(\tilde{F}_n)_n$ $K-$converges to $\tilde{F}$ which is $(FHP)-$integrable and $\int_{B_m} h(t,[F(t)]^r)\mathrm{d}t$ exist for each $m \in \mathbb{N}$.

By (i), the function $-\bar{f}(t) + \sigma(x, \frac{1}{n}\sum_{i=1}^{n}[F_{k_i}(t)]^r)$ is Henstock integrable for every $n \in \mathbb{N}$. We are now able to apply Fatou's Lemma to the sequence $(-\bar{f}(t) + \sigma(x, \frac{1}{n}\sum_{i=1}^{n}[F_{k_i}(t)]^r))_n$, and have

$$\begin{aligned} & \int_T (-\bar{f}(t) + \sigma(x,[F(t)]^r)\mathrm{d}t \\ \leq\ & \liminf_n (H)\int_T (-\bar{f}(t) + \sigma(x, \frac{1}{n}\sum_{i=1}^{n}[F_{k_i}(t)]^r))\mathrm{d}t \\ =\ & (H)\int_T (-\bar{f}(t)\mathrm{d}t + \liminf_n (H)\int_T \sigma(x, \frac{1}{n}\sum_{i=1}^{n}[F_{k_i}(t)]^r)\mathrm{d}t \\ \leq\ & (H)\int_T (-\bar{f}(t)\mathrm{d}t + \sup_{n\in\mathbb{N}}\int_T \sigma(x,[F_n(t)]^r)\mathrm{d}t \leq +\infty. \end{aligned} \tag{4.1}$$

Consequently, $-\bar{f}(t)+\sigma(x,[F(t)]^r$ is (H)-integrable and, since $\bar{f}(t)$ is (H)-integrable, the H-integrability of $\sigma(x,[F(t)]^r$ follows. Every measurable selection $\tilde{f}$ of $\tilde{F}$ is $HP-$integrable, so we have

$$-\sigma(-x,[F(t)]^r) \leq \langle x, f(t)\rangle \leq \sigma(x, F(t)]^r), \quad a.e. t \in T.$$

For every $[a,b] \subset T$, there exist A, such that $\langle x, A\rangle = (H)\int_a^b \langle x, f(t)\rangle$. Thus every measurable selection of $\tilde{F}$ is Hestock-Pettis integrable.

Finally, by implication (4) $\Rightarrow$ (1) in Theorem 3.14, we have

$$\lim_{n\to\infty}(FHP)\int_T \tilde{F}_n(t)\mathrm{d}t = (FHP)\int_T \tilde{F}(t)\mathrm{d}t.$$

□

Corollary 4.3. *Let $(\tilde{F}_n)_n$ be a sequence of $(FHP)-$integrable functions satisfying hypothesis (i) in Theorem 4.2 and for every $n \in \mathbb{N}$ there exist $\tilde{F}'_n(t)$, such that $\tilde{F}_n(t) \subset \tilde{F}'_n$ a.e. Then there exist a $FHP-$integrable function $\tilde{F}$ and a subsequence of $(\tilde{F}_n)_n$ which $K-$converges to $\tilde{F}$.*

Proof. Let $B_m = \{t \in T | m-1 \leq D(\tilde{F}'_n(t), \tilde{0}) < m, \quad \forall m \in \mathbb{N}\}$ satisfy hypothesis (ii) in Theorem 4.2. Then, for every $m \in \mathbb{N}$, we have

$$\begin{aligned} \sup_{n\in\mathbb{N}}(H)\int_{B_m} |\sigma(x,[F_n(t)]^r)|\mathrm{d}t &\leq (H)\int_{B_m} |\sigma(x,[F'_n(t)]^r)|\mathrm{d}t \\ &\leq (H)\int_{B_m} D(\tilde{F}'_n(t), \tilde{0}) < +\infty. \end{aligned}$$

By Theorem 4.2, the conclusion holds. □

Theorem 4.4. *Let $(\tilde{F}_n)_n$ be a sequence of $(FHP)-$integrable functions satisfying hypothesis (i) in Theorem 4.2 and*

(ii′) there exists a measurable countable partition $(B_m)_m$ of T such that, for each $m \in \mathbb{N}$,

$$\sup_{n\in\mathbb{N}}(H)\int_{B_m} D(\tilde{F}_n(t), \tilde{0})\mathrm{d}t < +\infty.$$

Then there exist a $(FHP)-$integrable function $\tilde{F}$ and a subsequence of $(\tilde{F}_n)_n$ which $K-$converges to $\tilde{F}$ Moreover, $(H)\int_{B_m} D(\tilde{F}(t), \tilde{0})\mathrm{d}t < +\infty$ for every $m \in \mathbb{N}$.

In the sequel, using above Komlós-type convergence theorem, we give an existence theorem of a fuzzy integral inclusion as the following:

$$x(t) \in \xi + (FHP)\int_0^t \tilde{F}(s, x(s))\mathrm{d}s, \quad t \in T$$

Theorem 4.5. *Let U an open subset of fuzzy number space (E^n, D), and FHP-integrable function $\tilde{F} : T \times U \to E^n$ and $\tilde{\Gamma} : T \to E^n$ satisfy*

(1) $\tilde{F}(t,x) \subset \tilde{\Gamma}(t), \quad \forall t \in T, \forall x \in U$;

(2) $\tilde{F}(t,x)$ is upper semi-continuous for $t \in T$;

(3) $\sigma(x, [F(\cdot,x)]^r)$ is measurable for every $x \in U$.

Then, for every fixed $\xi \in U$, there exist $t_0 \in T$ such that $\xi + (FHP)\int_0^{t_0} \tilde{\Gamma}(s)\mathrm{d}s \subset U$ and

$$x(t) \in \xi + (FHP)\int_0^t \tilde{F}(s, x(s))\mathrm{d}s$$

has a solution in $C([0,t_0], E^n)$.

Proof. By Theorem 3.14, for all $\tilde{f} \in s_{HP}([F(t)]^1)$, there exists $\tilde{G} : T \to E^n$ such that $\tilde{F}(t) = \tilde{G}(t) + \tilde{f}(t)$, and $\tilde{G}$ is fuzzy Pettis integrable on T, then $\tilde{f}$ is measurable.

Fixing $\xi \in U$, we consider the open subset U_1 and U_2 of E^n such that $\xi \in U_1$ and $U_1 + U_2 \subset U$. Since $(FHP)\int_0^{(\cdot)} \tilde{f}(t)\mathrm{d}t$ is continuous, there exist $t_1 \in T$ such that $(FHP)\int_0^t \tilde{f}(t)\mathrm{d}t \in U_2$ for every $t \in [0,t_1]$. We define a fuzzy-number-valued function $\tilde{F'} : [0,t_1] \times U_1 \longrightarrow E^n$ as the following:

$$\tilde{F'}(t,x) = (-1)\cdot \tilde{f} + \tilde{F}(t, x + (FHP)\int_0^t \tilde{f}(\tau)\mathrm{d}\tau),$$

which satisfies the following conditions:

(1) $\tilde{F'}(t,x) \subset \tilde{G}(t), \quad \forall t \in T, \forall x \in U$;

(2) for evry $t \in T$, $\tilde{F'}(t,x)$ is upper semi-continuous;

(3) $\sigma(x, [F'(\cdot,x)]^r)$ is measurable for every $x \in U$.

Then we obtain that there exist $t_0 \in [0,t_1]$ such that $\xi + (FP)\int_0^{t_0} \tilde{G}(s)\mathrm{d}s \in U_1$, the integral inclusion

$$y(t) \in \xi + (FHP)\int_0^t \tilde{F'}(s, y(s))\mathrm{d}s \tag{4.2}$$

has a solution in $C([0,t_0], E^n)$ and the set of solution is compact in $C([0,t_0], E^n)$.

Therefore, we have

$$\xi + (FHP)\int_0^{t_0} \tilde{\Gamma}(s)\mathrm{d}s = \xi + (FHP)\int_0^{t_0} \tilde{f}(s)\mathrm{d}s + (FP)\int_0^{t_0} \tilde{G}(s)\mathrm{d}s \subset U$$

and we find $y(t) \in C([0,t_0], E^n)$ such that

$$y(t) \in \xi + (FP)\int_0^t (-1)\cdot \tilde{f}(s) + \tilde{F}(s, y(s) + (FHP)\int_0^s \tilde{f}(\tau)\mathrm{d}\tau)\mathrm{d}s,$$

That is

$$y(t) + (FHP)\int_0^t \tilde{f}(s)\mathrm{d}s \in \xi + (FHP)\int_0^t \tilde{F}(s, y(s) + (FHP)\int_0^s \tilde{f}(\tau)\mathrm{d}\tau)\mathrm{d}s.$$

Thus $x(\cdot) = y(\cdot) + (FHP)\int_0^{(\cdot)} \tilde{f}(\tau)\mathrm{d}\tau$ is a solution of the integral inclusion. □

5 Conclusions

In this paper, we study the Henstock-Pettis integral of compact convex set-valued functions and fuzzy-number-valued function and the $K-$convergence theorem of fuzzy Henstock-Pettis integrals. We emphasize that the outcomes of the second part in our paper are different from the results in L. Di Piazza's paper [5]. In the future research, we shall deals with a new derivative and Hestock-Pettis-Δ-integral for fuzzy-number-valued functions on time scales. Also, we shall study and investigate fuzzy differential equations and fuzzy integral equations with Δ_H-derivative and $FHP-\Delta-$integral on time scales.

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Some Inequalities for Riemann Diamond Integrals on Time Scales

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Abstract

In this paper, we investigate the Diamond integral on time scales. By using Darboux approach, we define the Riemann Diamond integral on time scales and prove the corresponding theorems. Our results extend and improve the corresponding results on inequality of [8].

Keywords: Diamond integral, generalized Hölder's inequality, generalized Jensen's inequality, time scales

2010 MSC: 26D15, 26E70

1. Introduction

The theory of time scales was born in 1988 with the Ph.D. thesis of Stefan Hilger, done under the supervision of Bernd Aulbach [9]. The aim of this theory was to unify various definitions and results from the theories of discrete and continuous dynamical systems, and to extend such theories to more general classes of dynamical systems. It has been extensively studied on various aspects by several authors [1,4,5,6,7,10,14,16].

Two versions of the calculus on time scales, the delta and nabla calculus, are now standard in the theory of time scales [5,6]. In 2006, the Diamond-alpha

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integral on time scales was introduced by Sheng, Fadag, Henderson, and Davis [16], as a linear combination of the delta and nabla integrals. The Diamond-alpha integral reduces to the standard delta integral for $\alpha = 1$ and to the standard nabla integral for $\alpha = 0$. We refer the reader to [2,3,11,12,13,15,16] for a complete account of the recent Diamond-alpha integral on time scales. In 2015, the Diamond integral on time scales, as a refined version of the diamond-alpha integral, was introduced by Artur M. C. Brito da Cruz et al., [8]. In this paper we define and study the Riemann Diamond integral on time scales. Basic properties of the theory are proved.

The paper is organized as follows. Section 2 contains basic concepts of time scales theory. In Section 3, definition of the Riemann diamond integral will be introduced. We will investigate basic properties of the Riemann diamond integral. In Section 4, we will establish generalized Hölder's inequality, Cauchy-Schwarz's inequality, Minkowski's inequality and Jensen's inequality on time scales.

2. Preliminaries

Let $\mathbb{T}$ be a time scale, i.e. a nonempty closed subset of $\mathbb{R}$. For $a, b \in \mathbb{T}$ we define the closed interval $[a, b]_{\mathbb{T}}$ by $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\}$. The open and half-open intervals are defined in an similar way. For $t \in \mathbb{T}$ we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ where $\inf \emptyset = \sup \mathbb{T}$, while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$ where $\sup \emptyset = \inf \mathbb{T}$.

If $\sigma(t) > t$, we say that t is right-scattered, while if $\rho(t) < t$, we say that t is left-scattered. If $\sigma(t) = t$, we say that t is right-dense, while if $\rho(t) = t$, we say that t is left-dense. A point $t \in \mathbb{T}$ is dense if it is right and left dense; isolated if it is right and left scattered. The forward graininess function $\mu : \mathbb{T} \to [0, \infty)$ and the backward graininess function $\eta : \mathbb{T} \to [0, \infty)$ are defined by $\mu(t) = \sigma(t) - t$, $\eta(t) = t - \rho(t)$ for all $t \in \mathbb{T}$ respectively. If $\sup \mathbb{T}$ is finite and left-scattered, then we define $\mathbb{T}^k := \mathbb{T} \backslash \sup \mathbb{T}$, otherwise $\mathbb{T}^k := \mathbb{T}$; if $\inf \mathbb{T}$ is finite and right-scattered, then $\mathbb{T}_k := \mathbb{T}\backslash \inf \mathbb{T}$, otherwise $\mathbb{T}_k := \mathbb{T}$. We set $\mathbb{T}^k_k := \mathbb{T}^k \bigcap \mathbb{T}_k$.

A function $f : \mathbb{T} \to \mathbb{R}$ is called *regulated* provided its right-sided limits exist (finite) at all right-dense point of $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense point of $\mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* provided it is continuous at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$.

Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Then we define $f^{\Delta}(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there exists a neighborhood U of t such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \leq \epsilon|\sigma(t) - s|$$

for all $s \in U$. We call $f^{\Delta}(t)$ the delta derivative of f at t and we say that f is delta differentiable on $\mathbb{T}^k$ provided $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^k$.

let $t \in \mathbb{T}_k$. We define $f^{\nabla}(t)$ to be the number with the property that given any $\varepsilon > 0$, there exists a neighborhood U of t such that

$$|f(\rho(t)) - f(s) - f^{\nabla}(t)(\rho(t) - s)| \leq \epsilon|\rho(t) - s|$$

for all $s \in U$. We call $f^{\nabla}(t)$ the nabla derivative of f at t and we say that f is nabla differentiable on $\mathbb{T}_k$ provided $f^{\nabla}(t)$ exists for all $t \in \mathbb{T}_k$.

Let $t, s \in \mathbb{T}$ and define $\mu_{t,s} := \sigma(t) - s$ and $\eta_{t,s} := \rho(t) - s$. We define $f^{\diamondsuit_\alpha}(t)$ to be the number with the property that given any $\varepsilon > 0$, there exists a neighborhood U of t such that

$$|\alpha(f(\sigma(t)) - f(s))\eta_{t,s} + (1-\alpha)(f(\rho(t)) - f(s))\mu_{t,s} - f^{\diamondsuit_\alpha}(t)\mu_{t,s}\eta_{t,s}| \leq \epsilon|\mu_{t,s}\eta_{t,s}|$$

for all $s \in U$. We call $f^{\diamondsuit_\alpha}(t)$ the diamond-α derivative of f at t and we say that f is diamond-α differentiable on $\mathbb{T}^k_k$ provided $f^{\diamondsuit_\alpha}(t)$ exists for all $t \in \mathbb{T}^k_k$.

The real function

$$\gamma(t) := \lim_{s \to t} \frac{\sigma(t) - s}{\sigma(t) + 2t - 2s - \rho(t)}.$$

3. The Riemann Diamond integral

A partition of $[a,b]_{\mathbb{T}}$ is any finite ordered subset

$$P = \{t_0, t_1, \ldots, t_n\} \subset [a,b]_{\mathbb{T}}, \quad where \;\; a = t_0 < t_1 < \ldots < t_n = b.$$

Each partition $P = \{t_0, t_1, \ldots, t_n\}$ of $[a,b]_{\mathbb{T}}$ decomposes it into subintervals $[t_{i-1}, t_i)_{\mathbb{T}},\ i = 1, 2, \ldots, n$, such that for $i \neq j$ one has $[t_{i-1}, t_i)_{\mathbb{T}} \cap [t_{j-1}, t_j)_{\mathbb{T}} = \emptyset$.

By $\mathcal{P}([a,b]_{\mathbb{T}})$ we denote the set of all partitions of $[a,b]_{\mathbb{T}}$. Let $P_n, P_m \in \mathcal{P}([a,b]_{\mathbb{T}})$. If $P_n \subset P_m$ we call P_n a refinement of P_m. If P_n, P_m are independently chosen, then the partition $P_n \cup P_m$ is a common refinement of P_n and P_m.

Let $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be a real-valued bounded function on $[a,b]_{\mathbb{T}}$. We denote

$$\overline{M} = \sup\{\gamma(t)f(t) : t \in [a,b)_{\mathbb{T}}\}, \quad \overline{m} = \inf\{\gamma(t)f(t) : t \in [a,b)_{\mathbb{T}}\},$$

$$\underline{M} = \sup\{(1-\gamma(t))f(t) : t \in (a,b]_{\mathbb{T}}\}, \quad \underline{m} = \inf\{(1-\gamma(t))f(t) : t \in (a,b]_{\mathbb{T}}\},$$

and for $1 \leq i \leq n$,

$$\overline{M_i} = \sup\{\gamma(t)f(t) : t \in [t_{i-1}, t_i)_{\mathbb{T}}\}, \quad \overline{m_i} = \inf\{\gamma(t)f(t) : t \in [t_{i-1}, t_i)_{\mathbb{T}}\},$$

$$\underline{M_i} = \sup\{(1-\gamma(t))f(t) : t \in (t_{i-1}, t_i]_{\mathbb{T}}\}, \quad \underline{m_i} = \inf\{(1-\gamma(t))f(t) : t \in (t_{i-1}, t_i]_{\mathbb{T}}\},$$

Let $\gamma(t) \in [0,1]$. The upper Darboux $\diamondsuit$-sum of f with respect to the partition P, denoted by $U(f,P)$, is defined by

$$U(f,P) = \sum_{i=1}^{n} (\overline{M_i} + \underline{M_i})(t_i - t_{i-1}),$$

while the lower Darboux $\diamondsuit$-sum of f with respect to the partition P, denoted by $L(f,P)$, is defined by

$$L(f,P) = \sum_{i=1}^{n} (\overline{m_i} + \underline{m_i})(t_i - t_{i-1}).$$

Note that

$$U(f,P) \leq \sum_{i=1}^{n} (\overline{M} + \underline{M})(t_i - t_{i-1}) = (\overline{M} + \underline{M})(b-a)$$

and

$$L(f,P) \geq \sum_{i=1}^{n} (\overline{m} + \underline{m})(t_i - t_{i-1}) = (\overline{m} + \underline{m})(b-a).$$

Thus, we have $\quad (\overline{m} + \underline{m})(b-a) \leq L(f,P) \leq U(f,P) \leq (\overline{M} + \underline{M})(b-a)$.

Definition 3.1 Let $I = [a,b]_{\mathbb{T}}$, where $a, b \in \mathbb{T}$. The upper Darboux $\diamondsuit-$integral of f from a to b is defined by

$$\overline{\int_a^b} f(t)\diamondsuit t = \inf_{P \in \mathcal{P}([a,b]_{\mathbb{T}})} U(f,P);$$

The lower Darboux $\diamondsuit-$integral of f from a to b is defined by

$$\underline{\int_a^b} f(t)\diamondsuit t = \sup_{P \in \mathcal{P}([a,b]_{\mathbb{T}})} L(f,P).$$

If $\overline{\int_a^b} f(t)\diamondsuit t = \underline{\int_a^b} f(t)\diamondsuit t$, then we say that f is Riemann $\diamondsuit-$integrable on $[a,b]_{\mathbb{T}}$, and the common value of the integrals, denoted by $\int_a^b f(t)\diamondsuit t$, is called the Riemann $\diamondsuit-$ integral.

Definition 3.2 Let $I = [a,b]_{\mathbb{T}}$, where $a, b \in \mathbb{T}$. The upper Darboux $\Delta-$integral of f from a to b is defined by

$$\overline{\int_a^b} f(t)\Delta t = \inf_{P \in \mathcal{P}([a,b]_{\mathbb{T}})} U(f,P)$$

where $U(f,P)$ denote the upper Darboux sum of f with respect to the partition P and

$$U(f,P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1}), M_i = \sup\{f(t) : t \in [t_{i-1}, t_i)_{\mathbb{T}}\}.$$

The lower Darboux $\Delta-$integral of f from a to b is defined by

$$\underline{\int_a^b} f(t)\nabla t = \sup_{P \in \mathcal{P}([a,b]_{\mathbb{T}})} L(f,P).$$

where $L(f,P)$ denote the lower Darboux sum of f with respect to the partition P and

$$L(f,P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1}), m_i = \inf\{f(t) : t \in [t_{i-1}, t_i)_{\mathbb{T}}\}.$$

If $\overline{\int_a^b} f(t)\Delta t = \underline{\int_a^b} f(t)\Delta t$, then we say that f is $\Delta-$integrable on $[a,b]_{\mathbb{T}}$, and the common value of the integrals, denoted by $\int_a^b f(t)\Delta t$, is called the Riemann $\Delta-$ integral. Similarly, we can give the definition of the Riemann $\nabla-$ integral.

We can easily get the following two theorems.

Theorem 3.1 If $\gamma f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is Riemann $\Delta-$integrable and $(1-\gamma)f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ Riemann $\nabla-$integrable on the interval $[a,b]_{\mathbb{T}}$, then $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is Riemann $\diamondsuit-$integral on $[a,b]_{\mathbb{T}}$ and

$$\int_a^b f(t)\diamondsuit t = \int_a^b \gamma(t)f(t)\Delta t + \int_a^b (1-\gamma(t))f(t)\nabla t.$$

Theorem 3.2 Let $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is Riemann $\diamondsuit-$integrable on the interval $[a,b]_{\mathbb{T}}$.

(1) If $\gamma(t) \equiv 1$, then f is Riemann $\Delta-$integrable on $[a,b]_{\mathbb{T}}$.

(2) If $\gamma(t) \equiv 0$, then f is Riemann $\nabla-$integrable on $[a,b]_{\mathbb{T}}$.

(3) If $0 < \gamma(t) < 1$, then f is Riemann $\Delta-$integrable and Riemann $\nabla-$integrable on $[a,b]_{\mathbb{T}}$.

The proofs of the following two Theorem are standard and similar to [6, Theorem 5.5 and Theorem 5.6].

Theorem 3.3 Let $L(f,P) = U(f,P)$ for some $P \in \mathcal{P}([a,b]_{\mathbb{T}})$, then the function f is Riemann $\diamondsuit-$integrable on the interval $[a,b]_{\mathbb{T}}$ and

$$\int_a^b f(t)\diamondsuit t = L(f,P) = U(f,P).$$

Theorem 3.4 (Cauchy criterion) Let $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be a bounded function on the interval $[a,b]_{\mathbb{T}}$. Then the function f is Riemann $\diamondsuit-$integrable on the interval $[a,b]_{\mathbb{T}}$ if and only if for every $\epsilon > 0$ there exists a partition $P \in \mathcal{P}([a,b]_{\mathbb{T}})$ such that $U(f,P) - L(f,P) < \epsilon$.

The following Lemma can be found in [7].

Lemma 3.5 Let $I = [a,b]_{\mathbb{T}}$ be a closed (bounded) interval in $\mathbb{T}$. For every $\delta > 0$ there is a partition $P_\delta = \{t_0, t_1, \dots, t_n\} \in \mathcal{P}([a,b]_{\mathbb{T}})$ such that for each i one has:

$$t_i - t_{i-1} \le \delta \ \ or \ \ t_i - t_{i-1} > \delta \wedge \rho(t_i) = t_{i-1}.$$

6

The next theorem gives another Cauchy criterion for integrability.

Theorem 3.6 A bounded function f on $[a,b]_{\mathbb{T}}$ is Riemann $\diamondsuit-$integrable if and only if for each $\epsilon > 0$ there exists $\delta > 0$ such that $P_\delta \in \mathcal{P}([a,b]_{\mathbb{T}})$ implies

$$U(f,P_\delta) - L(f,P_\delta) < \epsilon.$$

Proof. If for each $\epsilon > 0$ there exists $\delta > 0$ such that $P_\delta \in \mathcal{P}([a,b]_{\mathbb{T}})$ implies

$$U(f,P_\delta) - L(f,P_\delta) < \epsilon,$$

then we have that f is integrable on $[a,b]_{\mathbb{T}}$ by Theorem 3.4.

Conversely, suppose that f is Riemann $\diamondsuit-$integrable on $[a,b]_{\mathbb{T}}$. If $\gamma(t) \equiv 1$ or $\gamma(t) \equiv 0$, then f is Riemann $\Delta-$integrable or $\nabla-$integrable on $[a,b]_{\mathbb{T}}$. Therefore condition holds from [6, Theorem 5.9]. Now, let $0 < \gamma(t) < 1$, f is Riemann $\diamondsuit-$ integrable, then γf is Riemann $\Delta-$integrable and $(1-\gamma)f$ is Riemann $\nabla-$integrable. For each $\epsilon > 0$ there exists $\delta' > 0$ and $\delta'' > 0$ such that $P_{\delta'} \in \mathcal{P}([a,b]_{\mathbb{T}})$, $P_{\delta''} \in \mathcal{P}([a,b]_{\mathbb{T}})$ we have

$$\overline{U}(\gamma f, P_{\delta'}) - \overline{L}(\gamma f, P_{\delta'}) < \frac{\epsilon}{2}, \quad \underline{U}((1-\gamma)f, P_{\delta''}) - \underline{L}((1-\gamma)f, P_{\delta''}) < \frac{\epsilon}{2}.$$

If $P_\delta \in \mathcal{P}([a,b]_{\mathbb{T}})$ where $\delta = \min\{\delta', \delta''\}$, then we have

$$U(f,P_\delta) - L(f,P_\delta) = \overline{U}(\gamma f, P_\delta) - \overline{L}(\gamma f, P_\delta) + \underline{U}((1-\gamma)f, P_\delta) - \underline{L}((1-\gamma)f, P_\delta) < \epsilon.$$

The Riemann $\diamondsuit-$integral has the following properties. Here we will not dwell with the proofs.

Theorem 3.7 Let functions $f,\ g : \mathbb{T} \to \mathbb{R}$ be Riemann $\diamondsuit-$integrable on the interval $[a,b]_{\mathbb{T}}$, $a < b < c$ and α, β be arbitrary real numbers. Then,

(1) $\alpha f \pm\ \beta g$ is Riemann $\diamondsuit-$integrable on the interval $[a,b]_{\mathbb{T}}$ and

$$\int_a^b (\alpha f(t) \pm\ \beta g(t))\diamondsuit t = \alpha \int_a^b f(t)\diamondsuit t \pm \beta \int_a^b g(t)\diamondsuit t.$$

(2) $\int_a^c f(t)\diamondsuit t + \int_c^b f(t)\diamondsuit t = \int_a^b f(t)\diamondsuit t$.

(3) if $f \le g$ for $t \in [a,b]_{\mathbb{T}}$, then $\int_a^b f(t)\diamondsuit t \le \int_a^b g(t)\diamondsuit t$.

(4) $|f|$ is Riemann $\diamondsuit-$integrable on the interval $[a,b]_{\mathbb{T}}$ and $|\int_a^b f(t)\diamondsuit t| \le \int_a^b |f(t)|\diamondsuit t$.

(5) fg is Riemann $\diamondsuit-$integrable on the interval $[a,b]_{\mathbb{T}}$.

The following theorem may be proved in much the same way as [6, Theorem 5.18, 5.19, 5.20, 5.21.].

Theorem 3.8 Let $I=[a,b]_{\mathbb{T}}$, where $a,b\in\mathbb{T}$.

(¡) Every monotone function f is Riemann $\diamondsuit-$integrable on $[a,b]_{\mathbb{T}}$.

(¡¡) Every continuous function f is Riemann $\diamondsuit-$integrable on $[a,b]_{\mathbb{T}}$.

(¡¡¡) Every bounded function f with only finitely many discontinuity points is Riemann $\diamondsuit-$integrable on $[a,b]_{\mathbb{T}}$.

(¡¡¡¡) Every regulated function f is Riemann $\diamondsuit-$integrable on $[a,b]_{\mathbb{T}}$.

Theorem 3.9 Let $f:\mathbb{T}\to\mathbb{R}$ and $t\in\mathbb{T}$. Then, f is Riemann $\diamondsuit-$integrable on $[t,\sigma(t)]_{\mathbb{T}}$ and

$$\int_t^{\sigma(t)} f(s)\diamondsuit s=\mu(t)(f(t)+f^{\sigma}(t)).$$

Proof If $t=\sigma(t)$, then the equality is obvious. If $t<\sigma(t)$, then $\mathcal{P}([t,\sigma(t)]_{\mathbb{T}})$ contains only one element given by

$$t=s_0<s_1=\sigma(t).$$

Since $[s_0,s_1)_{\mathbb{T}}=\{t\}$ and $(s_0,s_1]_{\mathbb{T}}=\{\sigma(t)\}$, we have

$$U(f,P)=L(f,P)=\gamma(t)f(t)(\sigma(t)-t)+(1-\gamma(\sigma(t)))f^{\sigma}(t)(\sigma(t)-t)=\mu(t)(f(t)+f^{\sigma}(t)).$$

By Theorem 3.3, f is Riemann $\diamondsuit-$integrable on $[t,\sigma(t)]_{\mathbb{T}}$ and

$$\int_t^{\sigma(t)} f(s)\diamondsuit s=\mu(t)(f(t)+f^{\sigma}(t)).$$

Theorem 3.10 Let $f:\mathbb{T}\to\mathbb{R}$ and $t\in\mathbb{T}$. Then, f is Riemann $\diamondsuit-$integrable on $[\rho(t),t]_{\mathbb{T}}$ and

$$\int_{\rho(t)}^{t} f(s)\diamondsuit s=\eta(t)(f^{\rho}(t)+f(t)).$$

Proof If $t=\rho(t)$, then the equality is obvious. If $t>\rho(t)$, then $[\rho(t),t]_{\mathbb{T}}$ contains only one element given by

$$\rho(t)=s_0<s_1=t.$$

Since $[s_0, s_1)_{\mathbb{T}} = \{\rho(t)\}$ and $(s_0, s_1]_{\mathbb{T}} = \{t\}$, we have

$$U(f,P) = L(f,P) = \gamma(\rho(t))f^{\rho}(t)(t-\rho(t))+(1-\gamma(t))f(t)(t-\rho(t)) = \eta(t)(f^{\rho}(t)+f(t)).$$

By Theorem 3.3, f is Riemann $\Diamond-$integrable on $[\rho(t), t]_{\mathbb{T}}$ and

$$\int_{\rho(t)}^{t} f(s)\Diamond s = \eta(t)(f^{\rho}(t) + f(t)).$$

By the definition of the Riemann $\Diamond-$integral, we have the following Corollary:

Corollary Let $a, b \in \mathbb{T}$ and $a < b$. Then we have the following:

(1) If $\mathbb{T} = \mathbb{R}$, then a bounded function f is Riemann $\Diamond-$integrable on the interval $[a, b]_{\mathbb{T}}$ if and only if f is Riemann integrable on $[a, b]_{\mathbb{T}}$ in the classical sense, and in this case

$$\int_a^b f(t)\Diamond t = \int_a^b f(t)dt.$$

(2) If $\mathbb{T} = \mathbb{Z}$, then each function $f : \mathbb{Z} \to \mathbb{R}$ is Riemann $\Diamond-$integrable on the interval $[a, b]_{\mathbb{T}}$. Moreover

$$\int_a^b f(t)\Diamond t = \sum_{t=a+1}^{b-1} f(t) + f(a) + f(b).$$

(3) If $\mathbb{T} = h\mathbb{Z}$, then each function $f : h\mathbb{Z} \to \mathbb{R}$ is Riemann $\Diamond-$integrable on the interval $[a, b]_{\mathbb{T}}$. Moreover

$$\int_a^b f(t)\Diamond t = \sum_{k=\frac{a}{h}+1}^{\frac{b}{h}-1} f(kh) + f(a)h + f(b)h.$$

Example Let $f : \{1, 2, 3\} \to \mathbb{R}$ be defined by $f(t) = t$. Then,

$$\int_1^3 f(t)\Diamond t = \gamma(1)f(1)+(1-\gamma(2))f(2)+\gamma(2)f(2)+(1-\gamma(3))f(3) = 1+2+3 = 6.$$

4. Generalized Inequalities

In this section, we will establish generalized Hölder's inequality, Cauchy-Schwarz's inequality, Minkowski's inequality and Jensen's inequality on time scales.

Theorem 4.1 (Generalized Hölder's inequality) Let $f, g, h \in C_{ld}([a,b]_{\mathbb{T}}, R)$ and $\frac{1}{p}+\frac{1}{q}=1$ with $p>1$; then

$$\int_a^b |h(t)||f(t)g(t)|\Diamond t \leq \Big(\int_a^b |h(t)||f(t)|^p \Diamond t\Big)^{\frac{1}{p}} \Big(\int_a^b |h(t)||g(t)|^q \Diamond t\Big)^{\frac{1}{q}}.$$

Proof For nonnegative real numbers α, β and p, q such that $\frac{1}{p}+\frac{1}{q}=1$ with $p>1$, we have the well-known Young's inequality $\alpha^{\frac{1}{p}}\beta^{\frac{1}{q}} \leq \frac{\alpha}{p}+\frac{\beta}{q}$.

Without loss of generality, we suppose that

$$\Big(\int_a^b |h(t)||f(t)|^p \Diamond t\Big)\Big(\int_a^b |h(t)||g(t)|^q \Diamond t\Big) \neq 0.$$

Let

$$\alpha(t) = \frac{|h(t)||f(t)|^p}{\int_a^b |h(t)||f(t)|^p \Diamond t}, \quad \beta(t) = \frac{|h(t)||g(t)|^q}{\int_a^b |h(t)||g(t)|^q \Diamond t}.$$

Consequently we have that

$$\begin{aligned} & \int_a^b \frac{|h(t)|^{\frac{1}{p}}|f(t)|}{\Big(\int_a^b |h(t)||f(t)|^p \Diamond t\Big)^{\frac{1}{p}}} \frac{|h(t)|^{\frac{1}{q}}|g(t)|}{\Big(\int_a^b |h(t)||g(t)|^q \Diamond t\Big)^{\frac{1}{q}}} \Diamond t \\ = & \int_a^b \alpha^{\frac{1}{p}}(t)\beta^{\frac{1}{q}}(t)\Diamond t \\ \leq & \int_a^b \Big(\frac{\alpha(t)}{p}+\frac{\beta(t)}{q}\Big)\Diamond t \\ = & \int_a^b \Big(\frac{1}{p}\frac{|h(t)||f(t)|^p}{\int_a^b |h(t)||f(t)|^p \Diamond t}+\frac{1}{q}\frac{|h(t)||g(t)|^q}{\int_a^b |h(t)||g(t)|^q \Diamond t}\Big)\Diamond t \\ = & \frac{1}{p}+\frac{1}{q}=1, \end{aligned}$$

which completes the proof. For the particular case $p=q=2$ in Theorem 4.1, we obtain the Cauchy-Schwarz's inequality.

Theorem 4.2 (Generalized Cauchy-Schwarz's Inequality) Let f, g, h be $\Diamond-$integrable on the interval $[a,b]_{\mathbb{T}}$, then

$$\int_a^b |h(t)||f(t)g(t)|\Diamond t \leq \sqrt{\Big(\int_a^b |h(t)||f(t)|^2 \Diamond t\Big)\Big(\int_a^b |h(t)||g(t)|^2 \Diamond t\Big)}.$$

Theorem 4.3 (Generalized Minkowski's inequality) Let f, g, h be $\Diamond-$integrable on the interval $[a,b]_{\mathbb{T}}$ and $p>1$, then

$$\Big(\int_a^b |h(t)||f(t)+g(t)|^p \Diamond t\Big)^{\frac{1}{p}} \leq \Big(\int_a^b |h(t)||f(t)|^p \Diamond t\Big)^{\frac{1}{p}} + \Big(\int_a^b |h(t)||g(t)|^p \Diamond t\Big)^{\frac{1}{p}}.$$

Proof It follows from Theorem 4.1 that

$$
\begin{aligned}
& \int_a^b |h(t)||f(t)+g(t)|^p \diamondsuit t \\
= & \int_a^b |h(t)||f(t)+g(t)|^{p-1}(|f(t)+g(t)|)\diamondsuit t \\
\leq & \int_a^b |h(t)||f(t)+g(t)|^{p-1}(|f(t)|+|g(t)|)\diamondsuit t \\
= & \int_a^b |h(t)||f(t)+g(t)|^{p-1}|f(t)|\diamondsuit t + \int_a^b |h(t)||f(t)+g(t)|^{p-1}|g(t)|\diamondsuit t \\
\leq & \Big\{\int_a^b |h(t)|\Big(|f(t)+g(t)|^{p-1}\Big)^q \diamondsuit t\Big\}^{\frac{1}{q}}\Big(\int_a^b |h(t)||f(t)|^p \diamondsuit t\Big)^{\frac{1}{p}} \\
& +\Big\{\int_a^b |h(t)|\Big(|f(t)+g(t)|^{p-1}\Big)^q \diamondsuit t\Big\}^{\frac{1}{q}}\Big(\int_a^b |h(t)||g(t)|^p \diamondsuit t\Big)^{\frac{1}{p}} \\
= & \Big\{\int_a^b |h(t)||f(t)+g(t)|^p \diamondsuit t\Big\}^{\frac{1}{q}}\Big\{\Big(\int_a^b |h(t)||f(t)|^p \diamondsuit t\Big)^{\frac{1}{p}} + \Big(\int_a^b |h(t)||g(t)|^p \diamondsuit t\Big)^{\frac{1}{p}}.
\end{aligned}
$$

Dividing both sides by

$$\Big\{\int_a^b |h(t)||f(t)+g(t)|^p \diamondsuit t\Big\}^{\frac{1}{q}},$$

we arrive to Minkowskis inequality:

$$\Big(\int_a^b |h(t)||f(t)+g(t)|^p \diamondsuit t\Big)^{\frac{1}{p}} \leq \Big(\int_a^b |h(t)||f(t)|^p \diamondsuit t\Big)^{\frac{1}{p}} + \Big(\int_a^b |h(t)||g(t)|^p \diamondsuit t\Big)^{\frac{1}{p}}.$$

Theorem 4.4 (Jensen's inequality) Let $a, b \in \mathbb{T}$ and $c, d \in \mathbb{R}$. If $g : [a,b]_{\mathbb{T}} \to (c,d)$ is rd-continuous and $f : (c,d) \to \mathbb{R}$ is continuous and convex, then

$$f\Big(\frac{\int_a^b g(t)\diamondsuit t}{b-a}\Big) \leq \frac{\int_a^b f(g(t))\diamondsuit t}{b-a}.$$

Proof Let $x_0 \in (c,d)$. Then for each $x \in (c,d)$, there exists β such that

$$f(x) - f(x_0) \geq \beta(x - x_0).$$

Let $x_0 = \frac{\int_a^b g(t)\diamondsuit t}{b-a}$, Thus,

$$\begin{aligned} & \int_a^b f(g(t))\diamondsuit t - (b-a)f\Big(\frac{\int_a^b g(t)\diamondsuit t}{b-a}\Big) \\ = & \int_a^b f(g(t))\diamondsuit t - (b-a)f(x_0) \\ = & \int_a^b (f(g(t)) - f(x_0))\diamondsuit t \\ \geq & \ \beta\int_a^b (g(t) - x_0)\diamondsuit t \\ = & \ \beta\int_a^b g(t)\diamondsuit t - (b-a)x_0 = 0, \end{aligned}$$

which completes our proof.

Similarly, we have the following Generalized Jensen's inequality.

Theorem 4.5 (Generalized Jensen's inequality) Let $a, b \in \mathbb{T}$ and $c, d \in \mathbb{R}$. If $g : [a, b]_{\mathbb{T}} \to (c, d)$, $h : [a, b]_{\mathbb{T}} \to \mathbb{R}$ is rd-continuous with $\int_a^b |h(t)|\diamondsuit t > 0$ and $f : (c, d) \to \mathbb{R}$ is continuous and convex, then

$$f\Big(\frac{\int_a^b |h(t)|g(t)\diamondsuit t}{\int_a^b |h(t)|\diamondsuit t}\Big) \leq \frac{\int_a^b |h(t)|f(g(t))\diamondsuit t}{\int_a^b |h(t)|\diamondsuit t}.$$

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CONTINUITY AND CONTINUOUS HOMOGENEOUS SELECTIONS OF SET-VALUED METRIC GENERALIZED INVERSE IN BANACH SPACES

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Abstract. In this paper, upper semicontinuity and continuity for the set-valued metric generalized inverses T^{∂} in Banach spaces are investigated by metric projection operator. Moreover, criteria for the set-valued metric generalized inverses to have continuous homogeneous selections are given. Finally, the relation of continuity and continuous selection of the set-valued metric generalized inverse are given.

1. Introduction and preliminaries

Let $(X, \|\cdot\|)$ be a real Banach space. Let $S(X)$ and $B(X)$ denote the unit sphere and the unit ball of X, respectively. By X^* we denote the dual space of X. Let T denote a linear bounded operator from subspace of X into Banach space Y. Let $D(T)$, $R(T)$ and $N(T)$ denote the domain, range and null space of T, respectively. Let L be a subspace of X. The set-valued mapping $P_L : X \to L$

$$P_L(x) = \left\{ z \in L : \|x - z\| = \operatorname{dist}(x, L) := \inf_{y \in L} \|x - y\| \right\}$$

is said to be the metric projection operator from X onto L. A subspace L is said to be proximinal if $P_L(x) \neq \emptyset$ for all $x \in X$. Continuity of metric projection operator is an important content in geometry of Banach spaces. Moreover, metric projection operator plays an important role in the optimization, computational mathematics, theory of equation and control theory.

The concept of generalized inverses has been extensively studied in the last decades, which has its genetic in the context of the so-called "ill-posed" linear problems. If $N(T) \neq \{0\}$ or $R(T) \neq Y$, the operator equation $Tx = y$ is generally ill-posed, i.e., there exists $y_0 \in Y$ such that $\|Tx - y_0\| \neq 0$ for any $x \in D(T)$. In order to solve the best approximation problems for ill-posed linear operator equations in Banach spaces, it is necessary to study the set-valued metric generalized inverses of linear operators between Banach spaces. In 1974, Nashed and Votruba [9] introduced the concept of the set-valued metric generalized inverse of a linear operator between Banach spaces and they raised the following research

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suggestion "The problem of obtaining selections with nice properties for the metric generalized inverse merits study". Moreover, it is well known that set-valued metric generalized inverses is a set-valued mapping. Hence upper semicontinuity and continuity of set-valued metric generalized inverses merit study. In 2015, Shang and Cui [8] gave a criteria for upper semicontinuity of the set-valued metric generalized inverses in approximative compact spaces. In this paper, upper semicontinuity and continuity for the set-valued metric generalized inverses T^{∂} in Banach spaces are investigated by metric projection operator. Moreover, criteria for the set-valued metric generalized inverses to have continuous homogeneous selection are given. Finally, the relation of continuity and continuous selections of the set-valued metric generalized inverse are given. First let us recall some definitions that will be used in the further part of the paper.

Definition 1.1. (see [7]) A subspace $L \subset X$ is said to be k-Chebyshev subspace if L is proximinal and for any $x \in X$, we have $\dim(\operatorname{span}\{x - P_L(x)\}) \leq k$.

I. Singer defined the k-strictly convex spaces in [15]. He proved that if X is reflexive and k-strictly convex, then every closed subspace of X is k-Chebyshev subspace. Moreover, it is easy to see that if L is a Chebyshev subspace, then L is k-Chebyshev.

Definition 1.2. (see [4]) Set-valued mapping $F : X \to Y$ is said to be upper semicontinuous at x_0, if for each norm open set W with $F(x_0) \subset W$, there exists a norm neighborhood U of x_0 such that $F(x) \subset W$ for all x in U. F is called lower semicontinuous at x_0, if for any $y \in F(x_0)$ and any $\{x_n\}_{n=1}^{\infty}$ in X with $x_n \to x_0$, there exists $y_n \in F(x_n)$ such that $y_n \to y$ as $n \to \infty$. F is called continuous at x_0, if F is upper semicontinuous and is lower semicontinuous at x_0.

Definition 1.3. (see [14]) A closed subspace N of X is said to be a topologically complemented subspace of X, if there exists a closed subspace M of X such that $M \oplus N = X$.

Definition 1.4. A subspace $L \subset X$ is said to be maximal subspace of X if there exists $x^* \in S(X^*)$ such that $L = \{x \in X : x^*(x) = 0\}$.

Definition 1.5. (see [6]) A Banach space X is said to be nearly convex, if every closed convex set of $S(X)$ is compact.

Definition 1.6. (see [9]) A point $x_0 \in D(T)$ is said to be the best approximative solution to the operator equation $Tx = y$, if

$$\|Tx_0 - y\| = \inf\{\|Tx - y\| : x \in D(T)\}$$

and

$$\|x_0\| = \min\left\{\|v\| : v \in D(T),\ \|Tv - y\| = \inf_{x \in D(T)} \|Tx - y\|\right\}.$$

Definition 1.7. (see [9]) Let X, Y be Banach spaces, T be a linear bounded operator from subspace of X to Y and $D(T)$ be the domain of T. The set-valued mapping $T^{\partial} : D(T^{\partial}) \to X$ defined by

$$T^{\partial}(y) = \{x_0 \in D(T) : x_0 \text{ is a best approximative solution to } T(x) = y\}$$

for any $y \in D(T^\partial)$, is said to be the (set-valued) metric generalized inverse of T, where

$$D(T^\partial) = \{y \in Y : T(x) = y \text{ has a best approximative solution in } X\}.$$

2. Continuity of the set-valued metric generalized inverse in Banach spaces

Theorem 2.1. *Let X be a nearly convex space, Y be a Banach space, T be a linear bounded operator from subspace of X into Y, $D(T)$ be a closed subspace of X and $R(T)$ be a 2-Chebyshev maximal subspace of Y. Then*

(1) T^∂ is upper semicontinuous on Y if and only if $P_{N(T)}$ is upper semicontinuous on $D(T)$;

(2) T^∂ is continuous if and only if TT^∂ is lower semicontinuous and $P_{N(T)}$ is continuous on $D(T)$.

Proof. (1) "$\Rightarrow$" We first will prove that the metric projector operator $P_{R(T)}$ is continuous and $P_{R(T)}(y)$ is a line segment. In fact, by Lemma 1 of [8], we know that if closed subspace H is a 2-Chebyshev subspace of X, then $P_H(x)$ is a line segment for any $x \in X$. Hence $P_{R(T)}(y)$ is a line segment.

Since $R(T)$ is an 2-Chebyshev maximal subspace of Y, there exists $f \in S(X^*)$ such that $R(T) = \{y \in Y : f(y) = 0\}$. Let $y \in Y$. Pick $z \in R(T)$ and $h \in S(Y)$. Then there exists $\alpha \in R$ such that $y-z = \alpha h$. It is easy to see that $\alpha = f(y)/f(h)$. Then $y - z = (f(y)/f(h))h$. Hence $\|y - z\| = |f(y)| / |f(h)| \geq |f(y)|$. Then it is easy to see that $z \in P_{R(T)}(y)$ if and only if $h \in A_f$. Hence $P_{R(T)}(y) = y - f(y)A_f$, where $A_f = \{y \in S(Y) : f(y) = 1\}$.

Suppose that metric projection operator $P_{R(T)}$ is not upper semicontinuous at y_0. Then there exist a sequence $\{y_n\}_{n=1}^{\infty} \subset Y$ and an open set $W \supset P_{R(T)}(y_0)$ such that $P_{R(T)}(y_n) \not\subset W$ and $y_n \to y_0$ as $n \to \infty$. Hence there exists $z_n \in P_{R(T)}(y_n)$ such that $z_n \notin W$. Then $z_n = y_n - f(y_n)h_n$, where $h_n \in A_f$. Since $P_{R(T)}(y)$ is a line segment and $P_{R(T)}(y) = y - f(y)A_f$, we obtain that A_f is a line segment. Hence there exists a subsequence $\{h_{n_k}\}_{k=1}^{\infty}$ of $\{h_n\}_{n=1}^{\infty}$ such that $h_{n_k} \to h_0 \in A_f$ as $k \to \infty$. Let $z_0 = y_0 - f(y_0)h_0$. Then $z_0 \in P_{R(T)}(y_0)$ and

$$\lim_{k\to\infty} z_{n_k} = \lim_{k\to\infty} (y_{n_k} - f(y_{n_k})h_{n_k}) = y_0 - f(y_0)h_0 = z_0,$$

a contradiction. This implies that $P_{R(T)}$ is upper semicontinuous.

Let $y_n \to y_0$ as $n \to \infty$. Pick $z_0 \in P_{R(T)}(y_0)$. Then there exists $h_0 \in A_f$ such that $z_0 = y_0 - f(y_0)h_0$. Hence $z_n = y_n - f(y_n)h_0 \in P_{R(T)}(x_n)$ and

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} (y_n - f(y_n)h_0) = y_0 - f(y_0)h_0 = z_0$$

This implies that $P_{R(T)}$ is lower semicontinuous at y_0. Hence we obtain that $P_{R(T)}$ is continuous.

Pick $y_0 \in Y$. Suppose that T^∂ is not upper semicontinuous at y_0. Then there exist a sequence $\{y_n\}_{n=1}^{\infty} \subset Y$, $y_n \to y_0 \in Y$ and norm open set W with $T^\partial(y_0) \subset W$ such that $T^\partial(y_n) \not\subset W$ for all $n \in N$. Hence there exists $x_n \in T^\partial(y_n) \subset X$ such that $x_n \notin W$. Since T is a bounded linear operator, we obtain that $N(T)$ is a closed subspace of $D(T)$. Let

$$\overline{T}: D(T)/N(T) \to R(T), \qquad \overline{T}[x] = Tx,$$

where $[x] \in D(T)/N(T)$ and $x \in D(T)$. Then it is easy to see that $R(\overline{T}) = R(T)$. Moreover, $\overline{R(T)} = R(T)$. In fact, suppose that $\overline{R(T)} \neq R(T)$. Then there exists $y' \in \overline{R(T)}$ such that $y' \notin R(T)$. It is easy to see that $\{y \in R(T) : \|y' - y\| = \mathrm{dist}(y', R(T))\} = \emptyset$. This implies that $R(T)$ is not a 2-Chebyshev subspace of Y, a contradiction. Since $\overline{R(T)} = R(T)$, we obtain that $R(T)$ is a Banach space. Moreover, it is easy to see that $\overline{T}$ is a bounded linear operator and $N(\overline{T}) = \{0\}$. This implies that the bounded linear operator $\overline{T}$ is both injective and surjective. Therefore, by the inverse operator theorem, we obtain that the operator $\overline{T}^{-1}$ is a bounded linear operator.

Let $P_{R(T)}(y_0) = [y(1,0), y(2,0)]$. Since $\overline{T}^{-1}$ is a bounded linear operator and $[y(1,0), y(2,0)]$ is a compact set, we obtain that the infimum

$$\inf\left\{\left\|\overline{T}^{-1}(z)\right\| : z \in [y(1,0), y(2,0)]\right\}$$

is attainable on $[y(1,0), y(2,0)]$. Let

$$A(0) = \left\{y \in [y(1,0), y(2,0)] : \left\|\overline{T}^{-1}(y)\right\| = \inf\{\left\|\overline{T}^{-1}(z)\right\| : z \in [y(1,0), y(2,0)]\}\right\}.$$

It is easy to see that $A(0)$ is a closed set. Moreover, if $z_1 \in A(0)$ and $z_2 \in A(0)$, then

$$\begin{aligned}\left\|\overline{T}^{-1}(\lambda z_1 + (1-\lambda) z_2)\right\| &\leq \lambda \left\|\overline{T}^{-1}(z_1)\right\| + (1-\lambda)\left\|\overline{T}^{-1}(z_2)\right\| \\ &= \inf\left\{\left\|\overline{T}^{-1}(z)\right\| : z \in [y(1,0), y(2,0)]\right\},\end{aligned}$$

where $\lambda \in [0,1]$. This implies that the set $A(0)$ is a closed convex set. Hence there exist $z(1,0) \in [y(1,0), y(2,0)]$ and $z(2,0) \in [y(1,0), y(2,0)]$ such that $A(0) = [z(1,0), z(2,0)]$. Let $P_{R(T)}(y_n) = [y(1,n), y(2,n)]$. Then there exist $z(1,n) \in [y(1,n), y(2,n)]$ and $z(2,n) \in [y(1,n), y(2,n)]$ such that

$$\begin{aligned}& A(n) \\ = & \left\{y \in [y(1,n), y(2,n)] : \left\|\overline{T}^{-1}(y)\right\| = \inf\left\{\left\|\overline{T}^{-1}(z)\right\| : z \in [y(1,n), y(2,n)]\right\}\right\} \\ = & [z(1,n), z(2,n)].\end{aligned}$$

Since $P_{R(T)}$ is continuous, we may assume without loss of generality that

$$\lim_{n\to\infty} y(1,n) = z_1 \in [y(1,0), y(2,0)] \text{ and } \lim_{n\to\infty} y(2,n) = z_2 \in [y(1,0), y(2,0)]. \tag{2.1}$$

We claim that $[z_1, z_2] \in [z(1,0), z(2,0)]$. Otherwise, we may assume without loss of generality that $z_1 \notin [z(1,0), z(2,0)]$. Hence there exists $r > 0$ such that

$$\left\|\overline{T}^{-1}(z_1)\right\| > \inf\left\{\left\|\overline{T}^{-1}(z)\right\| : z \in [y(1,0), y(2,0)]\right\} + 4r.$$

Since $\overline{T}^{-1}$ is a bounded linear operator and $y(1,n) \to z_1$ as $n \to \infty$, we may assume without loss of generality that

$$\left\|\overline{T}^{-1}(z(1,n))\right\| > \inf\left\{\left\|\overline{T}^{-1}(z)\right\| : z \in [y(1,0), y(2,0)]\right\} + 2r \tag{2.2}$$

for every $n \in N$. Since the metric projection operator $P_{R(T)}$ is continuous, there exists $z(n) \in [y(1,n), y(2,n)]$ such that $z(n) \to z(1,0)$ as $n \to \infty$. Since $\overline{T}^{-1}$ is a bounded linear operator and $z(n) \to z(1,0)$ as $n \to \infty$, we have

$$\lim_{n\to\infty}\left\|\overline{T}^{-1}(z(n))\right\| = \left\|\overline{T}^{-1}(z(1,0))\right\| = \inf\left\{\left\|\overline{T}^{-1}(z)\right\| : z \in [y(1,0), y(2,0)]\right\}.$$

Therefore, by formula (2.2), we may assume without loss of generality that

$$\left\|\overline{T}^{-1}(z(1,n))\right\| > \left\|\overline{T}^{-1}(z(n))\right\| + r$$

for all $n \in N$, a contradiction.

Pick $x(1,0) \in \overline{T}^{-1}(z_1)$, $x(2,0) \in \overline{T}^{-1}(z_2)$, $x(1,n) \in \overline{T}^{-1}(z(1,n))$ and $x(2,n) \in \overline{T}^{-1}(z(2,n))$. Then we have $[x(1,n)] = \overline{T}^{-1}(z(1,n))$ and $[x(2,n)] = \overline{T}^{-1}(z(2,n))$. Since $\overline{T}^{-1}$ is a bounded linear operator, by formula (2.1), we have

$$\begin{aligned}\|[x(1,n) - x(1,0)]\| &= \|[x(1,n)] - [x(1,0)]\| \le \left\|\overline{T}^{-1}(z(1,n)) - \overline{T}^{-1}z_1\right\| \\ &\le \left\|\overline{T}^{-1}\right\| \|z(1,n) - z_1\| \to 0 \ \text{ as } \ n \to \infty.\end{aligned}$$

Hence we may assume without loss of generality that $x(1,n) \to x(1,0)$ as $n \to \infty$. Similarly, we may assume without loss of generality that $x(2,n) \to x(2,0)$ as $n \to \infty$. Moreover, by the definition of set-valued metric generalized inverse, there exists a sequence $\{\lambda_n\}_{n=1}^{\infty} \subset [0,1]$ such that

$$x_n = \lambda_n x(1,n) + (1-\lambda_n)x(2,n) - \pi_{N(T)}(\lambda_n x(1,n) + (1-\lambda_n)x(2,n)),$$

where

$$\pi_{N(T)}(\lambda_n x(1,n) + (1-\lambda_n)x(2,n)) \in P_{N(T)}(\lambda_n x(1,n) + (1-\lambda_n)x(2,n)).$$

We may assume without loss of generality that $\lambda_n \to \lambda$ as $n \to \infty$. Then

$$\lim_{n\to\infty} \lambda_n x(1,n) + (1-\lambda_n)x(2,n) = \lambda z_1 + (1-\lambda)z_2 \in [z_1, z_2]. \tag{2.3}$$

Since $P_{N(T)}$ is upper semicontinuous, by formula (2.3), we obtain that for any $\varepsilon > 0$, there exists $n_0 \in N$ such that

$$\pi_{N(T)}(\lambda_n x(1,n) + (1-\lambda_n)x(2,n)) \in \bigcup_{x \in P_{N(T)}(\lambda z_1 + (1-\lambda)z_2)} B(x,\varepsilon)$$

whenever $n > n_0$. This implies that

$$\operatorname{dist}\left(\{\lambda_n x(1,n) + (1-\lambda_n)x(2,n)\}_{n=1}^{\infty}, P_{N(T)}(\lambda z_1 + (1-\lambda)z_2)\right) = 0.$$

Hence, for any $k > 0$, there exists $h_{n_k} \in P_{N(T)}(\lambda z_1 + (1-\lambda)z_2)$ such that

$$\left\|\pi_{N(T)}(\lambda_{n_k} x(1,n_k) + (1-\lambda_{n_k})x(2,n_k)) - h_{n_k}\right\| < \frac{1}{k}. \tag{2.4}$$

Moreover, there exists $r > 0$ such that

$$\lambda z_1 + (1-\lambda)z_2 - P_{N(T)}(\lambda z_1 + (1-\lambda)z_2) \subset rS(X).$$

Since X is a nearly convex space, we obtain that $\lambda z_1+(1-\lambda)z_2-P_{N(T)}(\lambda z_1+(1-\lambda)z_2)$ is compact. Then the set $P_{N(T)}(\lambda z_1+(1-\lambda)z_2)$ is compact. Hence we may assume without loss of generality that $h_{n_k}\to h\in P_{N(T)}(\lambda z_1+(1-\lambda)z_2)$ as $n\to\infty$. Therefore, by formula (2.4), we have

$$\lim_{k\to\infty}\pi_{N(T)}\left(\lambda_{n_k}x(1,n_k)+(1-\lambda_{n_k})x(2,n_k)\right)=h\in P_{N(T)}(\lambda z_1+(1-\lambda)z_2).$$

This implies that

$$\begin{aligned} x_k &= \lambda_{n_k}x(1,n_k)+(1-\lambda_{n_k})x(2,n_k)-\pi_{N(T)}\left(\lambda_{n_k}x(1,n_k)+(1-\lambda_{n_k})x(2,n_k)\right)\\ &\to \lambda z_1+(1-\lambda)z_2-h\\ &\in \lambda z_1+(1-\lambda)z_2-P_{N(T)}(\lambda z_1+(1-\lambda)z_2).\end{aligned}$$

Moreover, by the definition of set-valued metric generalized inverse, we obtain that $\lambda z_1+(1-\lambda)z_2-P_{N(T)}(\lambda z_1+(1-\lambda)z_2)\subset T^{\partial}(y)\subset W$. Since W is a norm open set, we have $x_k\in W$ for k large enough, a contradiction.

"$\Leftarrow$" Suppose that $P_{N(T)}$ is not upper semicontinuous on $D(T)$. Then there exist $\{x_n\}_{n=1}^{\infty}\subset D(T)$, $x_0\in D(T)$ and a norm open set W such that $x_n\to x_0$, $P_{N(T)}(x_0)\subset W$ and $P_{N(T)}(x_n)\not\subset W$. Hence there exists $\pi_{N(T)}(x_n)\in P_{N(T)}(x_n)$ such that $\pi_{N(T)}(x_n)\notin W$. We claim that there exists $\delta>0$ such that

$$\bigcup_{z\in P_{N(T)}(x_0)}B(z,2\delta)\subset W.$$

Otherwise, there exists $z_n\in P_{N(T)}(x_0)$ such that $B(z_n,1/n)\not\subset W$. Since $P_{N(T)}(x_0)$ is compact, we may assume that $z_n\to z_0\in P_{N(T)}(x_0)$ as $n\to\infty$. Hence there exists $\eta>0$ such that $B(z_0,4\eta)\subset W$. Moreover, there exists $n_0\in N$ such that $1/n_0<\eta$ and $\|z_{n_0}-z_0\|\le\eta$. Hence, for any $z\in B(z_{n_0},1/n_0)$, we have

$$\|z-z_0\|\le\|z-z_{n_0}\|+\|z_{n_0}-z_0\|\le\frac{1}{n_0}+\eta<\eta+\eta<4\eta.$$

This implies that $z\in W$. Then $B(z_{n_0},1/n_0)\subset W$, a contradiction. Let $y_n=Tx_n$ and $y_0=Tx_0$. Then

$$T^{\partial}(y_n)=x_n-P_{N(T)}(x_n),\quad T^{\partial}(y_0)=x_0-P_{N(T)}(x_0)\ \text{ and }\ \lim_{n\to\infty}y_n=y_0.$$

Since $P_{N(T)}(x_0)\subset W$, we obtain that $T^{\partial}(y_0)=x_0-P_{N(T)}(x_0)\subset x_0-W$. We claim that

$$x_n-\pi_{N(T)}(x_n)\notin x_0-\bigcup_{z\in P_{N(T)}(x_0)}B(z,\delta)$$

whenever $\|x_n-x_0\|<\delta$. In fact, suppose that $x_n-\pi_{N(T)}(x_n)\in x_0-\bigcup_{z\in P_{N(T)}(x_0)}B(z,\delta)$ whenever $\|x_n-x_0\|<\delta$. Then

$$\begin{aligned}\pi_{N(T)}(x_n) &= x_n-\big(x_n-\pi_{N(T)}(x_n)\big)\\ &\in x_n-\left(x_0-\bigcup_{z\in P_{N(T)}(x_0)}B(z,\delta)\right)\\ &= \bigcup_{z\in P_{N(T)}(x_0)}B(z,\delta)+(x_n-x_0)\\ &\subset \bigcup_{z\in P_{N(T)}(x_0)}B(z,2\delta)\subset W,\end{aligned}$$

a contradiction. Since $x_n-\pi_{N(T)}(x_n)\notin x_0-\underset{z\in P_{N(T)}(x_0)}{\cup} B(z,\delta)$ whenever $\|x_n-x_0\| < \delta$, we obtain that T^∂ is not upper semicontinuous at y_0, a contradiction.

(2) "$\Rightarrow$" Let $y_0\in Y$ and $y_n\to y_0$ as $n\to\infty$. Then, by the previous proof, there exist $z(1,n)\in P_{R(T)}(y_n)$ and $z(2,n)\in P_{R(T)}(y_n)$, $z(1,0)\in P_{R(T)}(y_0)$ and $z(2,0)\in P_{R(T)}(y_0)$ such that

$$[z(1,n),z(2,n)]=\left\{z:\left\|\overline{T}^{-1}(z)\right\|=\inf\{\left\|\overline{T}^{-1}(y)\right\|:y\in P_{R(T)}(y_n)\}\right\}$$

and

$$[z(1,0),z(2,0)]=\left\{z:\left\|\overline{T}^{-1}(z)\right\|=\inf\{\left\|\overline{T}^{-1}(y)\right\|:y\in P_{R(T)}(y_0)\}\right\}.$$

Moreover, by the previous proof, we may assume without loss of generality that

$$\lim_{n\to\infty} z(1,n)=z_1\in[z(1,0),z(2,0)] \text{ and } \lim_{n\to\infty} z(2,n)=z_2\in[z(1,0),z(2,0)]. \tag{2.5}$$

From the previous proof, there exist $[x(1,n),x(2,n)]\subset X$ and $[x(1,0),x(2,0)]\subset X$ such that

$$T\{x:x\in[x(1,n),x(2,n)]\}=[z(1,n),z(2,n)]$$

and

$$T\{x:x\in[x(1,0),x(2,0)]\}=[z(1,0),z(2,0)].$$

Moreover, by the definition of set-valued metric generalized inverse, we obtain that

$$TT^\partial(y_0)=[z(1,0),z(2,0)] \text{ and } TT^\partial(y_n)=[z(1,n),z(2,n)].$$

Since the set-valued mapping TT^∂ is lower semicontinuous, by formula (2.5), we have

$$\lim_{n\to\infty} z(1,n)=z_1=z(1,0) \text{ and } \lim_{n\to\infty} z(2,n)=z_2=z(2,0).$$

Therefore, by the previous proof, we obtain that

$$\lim_{n\to\infty} x(1,n)=x(1,0) \text{ and } \lim_{n\to\infty} x(2,n)=x(2,0), \tag{2.6}$$

Moreover, by the definition of set-valued metric generalized inverse, we obtain that for any $x\in T^\partial(y_0)$, there exist $\lambda\in[0,1]$ and $h\in P_{N(T)}(\lambda x(1,0)+(1-\lambda)x(2,0))$ such that $x=\lambda x(1,0)+(1-\lambda)x(2,0)-h$. Since $P_{N(T)}$ is continuous, there exists $h_n\in P_{N(T)}(\lambda x(1,n)+(1-\lambda)x(2,n))$ such that $h_n\to h$ as $n\to\infty$. Therefore, by formula (2.6), we obtain that

$$\lim_{n\to\infty}\lambda x(1,n)+(1-\lambda)x(2,n)=\lambda x(1,0)+(1-\lambda)x(2,0).$$

This implies that

$$\lim_{n\to\infty}(\lambda x(1,n)+(1-\lambda)x(2,n)-h_n)=\lambda x(1,0)+(1-\lambda)x(2,0)-h=x. \tag{2.7}$$

Noticing that $\lambda x(1,n)+(1-\lambda)x(2,n)-h_n\in T^\partial(y_n)$ and formula (2.7), we obtain that T^∂ is lower semicontinuous at y_0. Therefore, by (1), we obtain that T^∂ is upper semicontinuous at y_0. Hence T^∂ is continuous at y_0.

"$\Leftarrow$" Let $y_0\in Y$ and $y_n\to y_0$ as $n\to\infty$. Then, by the previous proof, there exist $x_0\in X$ and $\{x_n\}_{n=1}^\infty\subset X$ such that $P_{R(T)}(y_0)=Tx_0$, $P_{R(T)}(y_n)=Tx_n$ and

$x_n \to x_0$ as $n \to \infty$. Then $T^\partial(y_0) = x_0 - P_{N(T)}(x_0)$ and $T^\partial(y_n) = x_n - P_{N(T)}(x_n)$. Since T^∂ is continuous, we obtain that for any $x_0 - z \in x_0 - P_{N(T)}(x_0)$, there exists $x_n - z_n \in x_n - P_{N(T)}(x_n)$ such that $x_n - z_n \to x_0 - z$ as $n \to \infty$, where $z_n \in P_{N(T)}(x_n)$. Hence, for any $z \in P_{N(T)}(x_0)$, there exists $z_n \in P_{N(T)}(x_n)$ such that $z_n \to z_0$ as $n \to \infty$. This implies that $P_{N(T)}$ is lower semicontinuous at y_0. Therefore, by (1), we obtain that $P_{N(T)}$ is continuous at y_0.

We next will prove that TT^∂ is lower semicontinuous. Let $y_0 \in Y$ and $y_n \to y_0$ as $n \to \infty$. Pick $z \in TT^\partial(y_0)$. Then there exists $x_0 \in D(T)$ such that $Tx_0 = y_0$ and $x_0 \in T^\partial(y_0)$. Since T^∂ is continuous, we obtain that T^∂ is lower semicontinuous. Hence there exists $x_n \in T^\partial(y_n)$ such that $x_n \to x_0$ as $n \to \infty$. This implies that $Tx_n \to Tx_0 = y_0$ as $n \to \infty$. Since $x_n \in T^\partial(y_n)$, we obtain that $Tx_n \in TT^\partial(y_n)$. Hence TT^∂ is lower semicontinuous at y_0, which completes the proof. □

3. Continuous selections of the set-valued metric generalized inverse in Banach spaces

Theorem 3.1. *Let X, Y be Banach spaces, T be a linear bounded operator from subspace of X into Y, $D(T)$ be a closed subspace of X, $N(T)$ be a topologically complemented subspace of $D(T)$ and $R(T)$ be a proximinal subspace of Y. Then the following statements are equivalent:*

(1) T^∂ has a continuous homogeneous selection on Y;

(2) $P_{N(T)}$ has a continuous homogeneous selection on $D(T)$ and the set-valued mapping TT^∂ has a continuous homogeneous selection on Y.

Proof. (2) ⇒ (1). Since $N(T)$ is a topologically complemented subspace of $D(T)$, there exists a closed subspace $M(T)$ of $D(T)$ such that $M(T) \oplus N(T) = D(T)$. Moreover, by $M(T) \oplus N(T) = D(T)$, we obtain that $T^{-1}(y) \cap M(T)$ is singleton for any $y \in R(T)$. Define the mapping $G : R(T) \to M(T)$ such that

$$G(y) = T^{-1}(y) \cap M(T), \quad y \in R(T).$$

Since T be a linear bounded operator from $D(T)$ into $R(T)$, by $M(T) \oplus N(T) = D(T)$, we obtain that G is a linear bounded operator from $R(T)$ into $M(T)$. Let TT^σ be a continuous homogeneous selection of TT^∂ and $\pi_{N(T)}$ be a continuous homogeneous selection of $P_{N(T)}$. Therefore, by the definition of set-valued metric generalized inverse, we obtain that the mapping

$$G \circ TT^\sigma - \pi_{N(T)} \circ G \circ TT^\sigma : \quad Y \to D(T)$$

is a continuous homogeneous selection of T^∂.

(1) ⇒ (2). Since T^∂ has a continuous homogeneous selection on Y, by the definition of set-valued metric generalized inverse, we obtain that $N(T)$ is a proximinal subspace of $D(T)$. Let T^σ be a continuous homogeneous selection of T^∂. Pick $x \in D(T)$. Let

$$\pi_{N(T)}(x) = x - T^\sigma T(x).$$

We next will prove that $\pi_{N(T)}$ is a continuous homogeneous selection of $P_{N(T)}$. In fact, since

$$T\left(\pi_{N(T)}(x)\right) = T\left(x - T^\sigma T(x)\right) = Tx - TT^\sigma T(x) = Tx - Tx = 0, \tag{3.1}$$

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we obtain that $\pi_{N(T)}(x) \in N(T)$. Moreover, by the definition of set-valued metric generalized inverse, we obtain that $\|T^\sigma T(x)\| = \operatorname{dist}(x, N(T))$. Therefore, by

$$x - \pi_{N(T)}(x) = x - x + T^\sigma T(x) = T^\sigma T(x)$$

and formula (3.1), we obtain that $\pi_{N(T)}(x) \in P_{N(T)}(x)$. Moreover, since T^σ is a continuous homogeneous selection of T^∂ and $\pi_{N(T)}(x) = x - T^\sigma T(x)$, we obtain that $\pi_{N(T)}$ is a homogeneous selection. This implies that $P_{N(T)}$ has a continuous homogeneous selection on $D(T)$.

Since T^σ is a continuous homogeneous selection of T^∂, we obtain that TT^σ is a continuous homogeneous selection of TT^∂, which completes the proof. □

Definition 3.2. (see [5]) A nonempty subset C of X is said to be approximatively compact if for any $\{y_n\}_{n=1}^\infty \subset C$ and any $x \in X$ satisfying $\|x - y_n\| \to \inf_{y\in C} \|x - y\|$ as $n \to \infty$, there exists a subsequence of $\{y_n\}_{n=1}^\infty$ converging to an element in C. X is called approximatively compact if every nonempty closed convex subset of X is approximatively compact.

Definition 3.3. (see [5]) A Banach space X is be said to be strictly convex if for any $x, y \in S(X)$ and $\|x + y\| = 2$ we have $x = y$.

Theorem 3.4. *Let X be approximatively compact and strictly convex, Y be a Banach spaces, T be a linear bounded operator from subspace of X into Y, $D(T)$ be a closed subspace of X, $N(T)$ be a topologically complemented subspace of $D(T)$ and $R(T)$ be a proximinal subspace of Y. Then the following statements are equivalent:*

(1) T^∂ has a continuous homogeneous selection on Y;

(2) TT^∂ has a continuous homogeneous selection on Y.

Proof. Since X is approximatively compact, we obtain that $P_{N(T)}$ is upper semicontinuous. Since X is a strictly convex space, we obtain that $P_{N(T)}$ is single value mapping. This implies that $P_{N(T)}$ is continuous. Therefore, by Theorem 3.1, it is easy to see that Theorem 3.4 is true, which completes the proof. □

Theorem 3.5. *Let X be approximatively compact and strictly convex, Y be a Banach space, T be a linear bounded operator from subspace of X into Y, $D(T)$ be a closed subspace of X and $R(T)$ be a proximinal subspace of Y. Then the following statements are equivalent:*

(1) T^∂ has a continuous selection on Y;

(2) TT^∂ has a continuous selection on Y.

Proof. (2)⇒(1). Since X is approximatively compact, we obtain that $P_{N(T)}$ is upper semicontinuous. Since X is a strictly convex space, we obtain that $P_{N(T)}$ is a single value mapping. This implies that $P_{N(T)}$ is a continuous and single value mapping. Let TT^σ be a continuous selection of TT^∂ and $f^{-1}TT^\sigma$ be a selection of $T^{-1}TT^\sigma$. This implies that the mapping

$$f^{-1}TT^\sigma - P_{N(T)}f^{-1}TT^\sigma$$

is a selection of T^∂. We next will prove that if $y_n \to y$ as $n \to \infty$, then

$$\lim_{n\to\infty}[f^{-1}TT^\sigma(y_n) - P_{N(T)}f^{-1}TT^\sigma(y_n)] = f^{-1}TT^\sigma(y) - P_{N(T)}f^{-1}TT^\sigma(y). \quad (3.2)$$

In fact, by the proof of Theorem 2.1, there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset D(T)$ such that

$$x_n \in f^{-1}TT^{\sigma}(y_n) - P_{N(T)}f^{-1}TT^{\sigma}(y_n) + N(T) \text{ and } \lim_{n\to\infty} x_n = f^{-1}TT^{\sigma}(y).$$

Since the mapping $P_{N(T)}$ and $f^{-1}TT^{\sigma}$ are single value mappings, we obtain that

$$Tx_n = T\left(f^{-1}TT^{\sigma}(y_n) - P_{N(T)}f^{-1}TT^{\sigma}(y_n)\right).$$

Hence there exists a sequence $\{z_n\}_{n=1}^{\infty} \subset N(T)$ such that

$$x_n - P_{N(T)}(x_n) - z_n = f^{-1}TT^{\sigma}(y_n) - P_{N(T)}f^{-1}TT^{\sigma}(y_n).$$

Moreover, by the definition of set-valued metric generalized inverse, we have

$$\left\|x_n - P_{N(T)}(x_n) - z_n\right\| = \left\|f^{-1}TT^{\sigma}(y_n) - P_{N(T)}f^{-1}TT^{\sigma}(y_n)\right\| = \operatorname{dist}(x_n, N(T)).$$

This implies that $P_{N(T)}(x_n)+z_n \subset P_{N(T)}(x_n)$. Since the mapping $P_{N(T)}$ is a single value mapping, we have $z_n = 0$. Hence

$$x_n - P_{N(T)}(x_n) = f^{-1}TT^{\sigma}(y_n) - P_{N(T)}f^{-1}TT^{\sigma}(y_n).$$

Since $P_{N(T)}$ is a continuous single value mapping, by $x_n \to f^{-1}TT^{\sigma}(y)$ as $n \to \infty$, we obtain that

$$\lim_{n\to\infty}[x_n - P_{N(T)}(x_n)] = f^{-1}TT^{\sigma}(y) - P_{N(T)}f^{-1}TT^{\sigma}(y).$$

Noticing that $x_n - P_{N(T)}(x_n) = f^{-1}TT^{\sigma}(y_n) - P_{N(T)}f^{-1}TT^{\sigma}(y_n)$, we obtain that (3.2) is true. Hence $f^{-1}TT^{\sigma} - P_{N(T)}f^{-1}TT^{\sigma}$ is a continuous selection of T^{∂}.

(1)⇒(2). Let T^{σ} be a continuous selection of T^{∂}. Then TT^{σ} is a continuous selection of TT^{∂}, which completes the proof. □

Theorem 3.6. *Let X be approximatively compact and strictly convex, Y be a Banach spaces, T be a linear bounded operator from subspace of X into Y, $D(T)$ be a closed subspace of X, $N(T)$ be a topologically complemented subspace of $D(T)$ and $R(T)$ be a 2-Chebyshev maximal subspace of Y. Then the following statements are equivalent:*

(1) For any $x \in T^{\partial}(y)$, there exists a selection T^{σ} of T^{∂} such that $T^{\sigma}(y) = x$ and T^{σ} is continuous at y;

(2) TT^{∂} is lower semicontinuous at y.

Proof. (1)⇒(2). From the proof of Theorem 2.1, there exist $z(1) \in P_{R(T)}(y)$ and $z(2) \in P_{R(T)}(y)$ such that

$$TT^{\partial}(y) = [z(1,y), z(2,y)] \subset P_{R(T)}(y).$$

Since $N(T)$ is a topologically complemented subspace of $D(T)$, there exists a closed subspace $M(T)$ of $D(T)$ such that $M(T) \oplus N(T) = D(T)$. Pick $x(1,y) \in \overline{T}^{-1}(z(1,y))$ and $x(2,y) \in \overline{T}^{-1}(z(2,y))$. Then, by $M(T) \oplus N(T) = D(T)$, there exist $G(y(1)), G(y(2)) \in M(T)$ and $h(y(1)), h(y(2) \in N(T)$ such that

$$x(1,y) = G(y(1)) + h(y(1)) \text{ and } x(2,y) = G(y(2)) + h(y(2)).$$

Therefore, by $x(1,y) \in \overline{T}^{-1}(z(1,y))$ and $x(2,y) \in \overline{T}^{-1}(z(2,y))$, we obtain that

$$T(G(y(1))) = z(1,y) \text{ and } T(G(y(2))) = z(2,y).$$

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Hence, for any $z \in TT^{\partial}(y)$, there exists $\lambda \in [0,1]$ such that

$$z = T\left(\lambda G(y(1)) + (1-\lambda)G(y(2))\right).$$

Pick $x \in T^{\sigma}(y)$. Then there exists a selection T^{σ} of T^{∂} such that $T^{\sigma}(y) = x$ and T^{σ} is continuous at y. This implies that

$$Tx = T\left(\lambda G(y(1)) + (1-\lambda)G(y(2))\right) = z.$$

Therefore, by the definition of set-valued metric generalized inverse, we have

$$T^{\sigma}(y) = \lambda G(y(1)) + (1-\lambda)G(y(2)) - P_{N(T)}\left(\lambda G(y(1)) + (1-\lambda)G(y(2))\right).$$

We next will prove that set-valued mapping TT^{∂} is lower semicontinuous at y. Let $y_n \to y$ as $n \to \infty$. Then, by $M(T) \oplus N(T) = D(T)$ and the previous proof, there exist $G(y_n(1)) \in M(T)$ and $G(y_n(2)) \in M(T)$ such that

$$T(G(y_n(1))) = z(1, y_n), \quad T(G(y_n(2))) = z(2, y_n)$$

and

$$TT^{\partial}(y_n) = [z(1, y_n), z(2, y_n)] \subset P_{R(T)}(y_n).$$

Hence there exists a sequence $\{\lambda_n\}_{n=1}^{\infty} \subset [0,1]$ such that

$$T^{\sigma}(y_n) = \lambda_n G(y_n(1)) + (1-\lambda_n)G(y_n(2)) - P_{N(T)}\left(\lambda_n G(y_n(1)) + (1-\lambda_n)G(y_n(2))\right).$$

Since

$$\lambda_n G(y_n(1)) + (1-\lambda_n)G(y_n(2)) \in M(T), \quad \lambda G(y(1)) + (1-\lambda)G(y(2)) \in M(T),$$
$$P_{N(T)}\left(\lambda_n G(y_n(1)) + (1-\lambda_n)G(y_n(2))\right) \in N(T)$$

and

$$P_{N(T)}\left(\lambda G(y(1)) + (1-\lambda)G(y(2))\right) \in N(T),$$

by $M(T) \oplus N(T) = D(T)$ and $T^{\sigma}(y_n) \to T^{\sigma}(y)$ as $n \to \infty$, we obtain that

$$\lim_{n\to\infty} \lambda_n G(y_n(1)) + (1-\lambda_n)G(y_n(2)) = \lambda G(y(1)) + (1-\lambda)G(y(2)).$$

This implies that

$$\lim_{n\to\infty} T\left(\lambda_n G(y_n(1)) + (1-\lambda_n)G(y_n(2))\right) = T\left(\lambda G(y(1)) + (1-\lambda)G(y(2))\right) = z.$$

where $T\left(\lambda_n G(y_n(1)) + (1-\lambda_n)G(y_n(2))\right) \in TT^{\sigma}(y_n)$. Hence the set-valued mapping TT^{∂} is lower semicontinuous at y.

(2)⇒(1). Let $x \in T^{\partial}(y)$. Then, by the previous proof, there exists $\lambda \in [0,1]$ such that

$$x = \lambda G(y(1)) + (1-\lambda)G(y(2)) - P_{N(T)}\left(\lambda G(y(1)) + (1-\lambda)G(y(2))\right).$$

Define the set-valued mapping $Y \to TT^{\partial}(Y)$ such that

$$F(z) = c, \quad \text{where} \quad \|c - Tx\| = \inf_{h \in TT^{\partial}(z)} \|h - Tx\| \quad \text{and} \quad c \in TT^{\partial}(z).$$

It is easy to see that $F(y) = Tx$. Let f be a selection of F. Moreover, let $y_n \to y$ and $f(y_n) = c_n$. Since TT^{∂} is lower semicontinuous at y, we obtain that $c_n \to Tx$ as $n \to \infty$. This implies that f is continuous at y. Since $N(T)$ is a topologically complemented subspace of $D(T)$, there exists a closed subspace $M(T)$ of $D(T)$

such that $M(T) \oplus N(T) = D(T)$. Define the mapping $G : R(T) \to M(T)$ such that

$$G(y) = T^{-1}(y) \cap M(T), \quad y \in R(T).$$

Then, by the previous proof, we obtain that G is a linear bounded operator from $R(T)$ into $M(T)$. Since $M(T) \oplus N(T) = D(T)$, we have $x = x_1 - x_2$, where $x_1 \in M(T)$ and $x_2 \in N(T)$. Since X is a strictly convex space, we obtain that $P_{N(T)}$ is a single value mapping. Since $x \in T^{\partial}(y)$, by the definition of set-valued metric generalized inverse, we have

$$\|x\| = \|x_1 - x_2\| = \inf\left\{\|z\| : z \in T^{-1}x = T^{-1}x_1\right\} = \inf\left\{\|x_1 - h\| : h \in N(T)\right\}.$$

Therefore, by $x_1 \in M(T)$ and $x_2 \in N(T)$, we obtain that $x_2 = P_{N(T)}(x_1)$. Since f is a selection of F and $F(y) = Tx$, we have $G \circ f(y) = G(Tx) = x_1$ Define the mapping

$$T^{\sigma} = G \circ f - P_{N(T)} \circ G \circ f.$$

Then T^{σ} is a selection of T^{∂}. Moreover, by $G \circ f(y) = G(Tx) = x_1$, we have

$$\begin{aligned} T^{\sigma}(y) &= G \circ f(y) - P_{N(T)} \circ G \circ f(y) \\ &= G(Tx) - P_{N(T)}(G(Tx)) \\ &= x_1 - P_{N(T)}(x_1) = x. \end{aligned}$$

Since X is approximatively compact, we obtain that $P_{N(T)}$ is upper semicontinuous. Since $P_{N(T)}$ is a single value mapping, we obtain that $P_{N(T)}$ is continuous. Since f and G is continuous at y, we obtain that $T^{\sigma} = G \circ f - P_{N(T)} \circ G \circ f$ is continuous at y, which completes the proof. □

4. Relation of continuity and continuous selections of the set-valued metric generalized inverse in Banach spaces

Theorem 4.1. *Let X be approximatively compact and strictly convex, Y be a Banach space, T be a linear bounded operator from subspace of X into Y, $D(T)$ be a closed subspace of X, $N(T)$ be a topologically complemented subspace of $D(T)$ and $R(T)$ be a 2-Chebyshev maximal subspace of Y. Then the following statements are equivalent:*

(1) For any $x \in \cup_{y\in Y} T^{\partial}(y)$, there exists a selection T^{σ} of T^{∂} such that $T^{\sigma}(y) = x$ and T^{σ} is continuous at y;

(2) TT^{∂} is lower semicontinuous;

(3) T^{∂} is continuous.

Proof. Since X is approximatively compact, we obtain that $P_{N(T)}$ is upper semicontinuous. Since X is a strictly convex space, we obtain that $P_{N(T)}$ is single value mapping. This implies that $P_{N(T)}$ is continuous. Therefore, by Theorem 2.1, we obtain that (2)⇔ (3) is true. Moreover, by Theorem 3.5, we obtain that (1)⇔ (2) is true, which completes the proof. □

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Locally and globally small Riemann sums and Henstock-Stieltjes integral of fuzzy-number-valued functions

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Abstract: In this paper, we introduce and study locally and globally small Riemann sums with respect to α on $[a,b]$ for fuzzy-number-valued functions and obtain some of it's characterizations. Also, we shall prove two main theorems: (i) If a fuzzy-number-valued functions $\tilde{f}(x)$ is Henstock-Stieltjes integrable on $[a,b]$ then it has $(LSRS)$ and the converse is always true. (ii) If a fuzzy-number-valued functions $\tilde{f}(x)$ is Henstock-Stieltjes integrable on $[a,b]$ then it has $(GSRS)$ and the converse is always true. Finally, by Egorov's Theorem, we obtain the dominated convergence theorem for globally small Riemann sums $(GSRS)$ with respect to α on $[a,b]$ for fuzzy-number-valued functions.

Keywords : Fuzzy numbers; fuzzy integrals; Henstock-Stieltjes integral; locally small Riemann sums $(LSRS)$ with respect to α on $[a,b]$; globally small Riemann sums $(GSRS)$ with respect to α on $[a,b]$.

1 Introduction

Since the concept of fuzzy sets was firstly introduced by Zadeh in 1965 [21], it has been studied extensively from many different aspects of the theory and applications, such as fuzzy topology, fuzzy analysis, fuzzy decision making and fuzzy logic, information science and so on. It's well known that the concept of the Stieltjes integral for fuzzy-number-valued functions was originally introduced by Nanda [12] in 1989. Nonetheless, as Wu et al. [17] pointed out that the existence of supremum and infimum for a finite set of fuzzy numbers wasn't easy at first thought. That is, Nanda's concept of fuzzy Riemann-Stieltjes (FRS) integral in [12] was incorrect. In 1998, Wu [18] introduced the notion of (FRS) integral by means of the representation theorem of fuzzy-number-valued functions, whose membership function could be obtained by solving a nonlinear programming problem, but it's difficult to calculate and extend to the higher-dimensional space. In 2006, Ren et al. proposed the notion of two types of (FRS) integral for fuzzy-number-valued functions [13, 14] and showed that a continuous fuzzy-number-valued function was (FRS) integrable with respect to a real-valued increasing function. Gong et al. [2] defined and discussed the (HS) integral for fuzzy-number-valued functions and proved two convergence theorems for sequences of the (FHS) integrable functions in 2012. The locally and globally small Riemann sums have been introduced by many authors from different points of views. In 1986, Schurle characterized the Lebesgue integral in $(LSRS)$ (locally small Riemann sums) property [15]. The $(LSRS)$ property has been used to characterized the Perron (P) integral on $[a,b]$ [16]. By considering the equivalency between the (P) integral and the Henstock-Kurzweil (HK) integral, the $(LSRS)$ property has been used to characterized the (HK) integral on $[a,b]$ [10].

The $(LSRS)$ property brought a research to have global characterization on the Riemann sums of an (HK) integrable function on $[a,b]$. This research has been done by considering the following fact: Every (HK) integrable function on $[a,b]$ is measurable, however, there is no guarantee the boundedness of the function. A measurable function f is (HK) integrable on $[a,b]$ depends on it behaves on the set of x in which $|f(x)|$ is large, i.e. $|f(x)| \geq N$

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for some N. This fact has been characterized in $(GSRS)$ (globally small Riemann sums) property [10]. The $(GSRS)$ property involves one characteristic of the primitive of an (HK) integrable function. That is the primitive of the (HK) integral on $[a, b]$ is ACG^* (generalized strongly absolutely continuous) on $[a, b]$. This is not a simple concept. In 2015, Indrati [8] introduced a countably Lipschitz condition of a function which is simpler than the ACG^*, and proved that the (HK) integrable function or it's primitive could be characterized in countably Lipschitz condition. Also, by considering the characterization of the (HK) integral in the $(GSRS)$ property, it showed that the relationship between $(GSRS)$ property and countably Lipschitz condition of an (HK) integrable function on $[a, b]$.

In 2018, Hamid et al. [6] investigated locally and globally small Riemann sums for fuzzy-number-valued functions and proved two main theorems: (1) A fuzzy-number-valued functions $\tilde{f}(x)$ is Henstock integrable on $[a, b]$ if and only if $\tilde{f}(x)$ has $(LSRS)$. (2) A fuzzy-number-valued functions $\tilde{f}(x)$ is Henstock integrable on $[a, b]$ if and only if $\tilde{f}(x)$ has $(GSRS)$.

In this paper, we introduce and study the locally and globally small Riemann sums with respect to α on $[a, b]$ for fuzzy-number-valued functions. We show that a fuzzy-number-valued functions is Henstock-Stieltjes integrable with respect to α on $[a, b]$ iff it has $(LSRS)$ with respect to α on $[a, b]$. Also it is shown that a fuzzy-number-valued functions is Henstock-Stieltjes integrable on $[a, b]$ iff it has $(GSRS)$ with respect to α on $[a, b]$. Finally, by Egorov's Theorem, we get the dominated convergence theorem for globally small Riemann sums $(GSRS)$ with respect to α on $[a, b]$ for fuzzy-number-valued functions.

The rest of this paper is organized as follows, in Section 2 we shall review the relevant concepts and properties of fuzzy sets and the definition of Henstock-Stieltjes integrals for fuzzy-number-valued functions. Section 3 is devoted to discussing the locally small Riemann sums $(LSRS)$ with respect to α on $[a, b]$ for fuzzy-number-valued functions. In Section 4 we shall investigate the globally small Riemann sums $(GSRS)$ with respect to α on $[a, b]$ for fuzzy-number-valued functions.

2 Preliminaries

Definition 2.1 [7, 10] Let $\delta : [a, b] \to \mathbb{R}^+$ be a positive real-valued function. $P = \{[x_{i-1}, x_i]; \xi_i\}$ is said to be a δ-fine division, if the following conditions are satisfied:

(1) $a = x_0 < x_1 < x_2 < ... < x_n = b$;

(2) $\xi_i \in [x_{i-1}, x_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))(i = 1, 2, \cdots, n)$.

For brevity, we write $P = \{[u, v]; \xi\}$, where $[u, v]$ denotes a typical interval in P and ξ is the associated point of $[u, v]$.

Definition 2.2 [4] Let $\alpha : [a, b] \to \mathbb{R}$ be an increasing function. A real function $f : [a, b] \to R$ is Henstock-Stieltjes (HS) integrable to $A \in R$ with respect to α on $[a, b]$ if for every $\varepsilon > 0$, there is a function $\delta(x) > 0$, such that for any δ-fine division $P = \{[u_i, v_i]; \xi_i\}_{i=1}^n$ we have

$$\Big| \sum_{i=1}^{n} f(\xi_i)[\alpha(v_i) - \alpha(u_i)] - A\Big| < \varepsilon. \tag{2.1}$$

We write $(HS)\int_a^b f(x)\mathrm{d}\alpha = A$, and $f \in HS_\alpha[a, b]$.

For the results about fuzzy number space E^1. we recall that $E^1 = \{u : R \to [0, 1] : u$ satisfies (1)-(4) below$\}$:

(1) u is normal, i.e., there exists a $x_0 \in R$ such that $u(x_0) = 1$;

(2) u is a convex fuzzy set, i.e., $u(rx + (1 - r)y) \geqslant \min(u(x), u(y)),\ x, y \in R,\ r \in [0, 1]$;

(3) u is upper semi-continuous;

(4) $cl\{x \in R : u(x) > 0\}$ is compact, where clA denotes the closure of A.

For $0 < r \leqslant 1$, denote $[u]^r = \{x : u(x) \geqslant r\}$. Then from (1)-(4), it follows that the $r-$level set $[u]^r$ is a close interval for all $r \in [0, 1]$ (refer to [1, 3, 5, 9, 11, 19, 20]). We write $u^r = [u]^r = [u^r_-, u^r_+]$ or $[u_-(r), u_+(r)]$.

For $u, v \in E^1$, the addition and scalar multiplication are defined by the equations:

$$[u + v]^r = [u]^r + [v]^r,\ \text{i.e.,}\ u^r_- + v^r_- = [u + v]^r_- \text{ and } u^r_+ + v^r_+ = [u + v]^r_+;$$

$$[ku]^r = k[u]^r, \text{ i.e., } [ku]^r_- = \min\{ku^r_-, ku^r_+\} \text{ and } [ku]^r_+ = \max\{ku^r_-, ku^r_+\},$$

respectively.

Define $D(u,v) = \sup_{r\in[0,1]} d([u]^r,[v]^r) = \sup_{r\in[0,1]} \max\{|u^r_- - v^r_-|, |u^r_+ - v^r_+|\}$, where d is Hausdorff metric. Furthermore, we write

$$\|\tilde{u}\|_{E^1} = D(\tilde{u},\tilde{0}) = \sup_{\lambda\in[0,1]} \max\{|u^-_\lambda|, |u^+_\lambda|\}.$$

Notice that $\|\cdot\|_{E^1} = D(\cdot,\tilde{0})$ doesn't stands for the norm of E^1.

For $u,v\in E^1$, $u \leqslant v$ means $u^r_- \leqslant v^r_-, u^r_+ \leqslant v^r_+$ (see [1, 3, 5, 9, 11, 19, 20]).

Using the results of [1, 3, 5, 9, 11, 19, 20], we recall that:

(1) (E^1, D) is a complete metric space,

(2) $D(u+w, v+w) = D(u,v)$,

(3) $D(u+v, w+e) \leqslant D(u,w) + D(v,e)$,

(4) $D(ku,kv) = |k|D(u,v), k\in R$,

(5) $D(u+v,\tilde{0}) \leqslant D(u,\tilde{0}) + D(v,\tilde{0})$,

(6) $D(u+v,w) \leqslant D(u,w) + D(v,\tilde{0})$, where $\tilde{0} = \chi_{\{0\}}$ and $u,v,w,e\in E^1$.

Definition 2.3 [2] Let $\alpha : [a,b] \to \mathbb{R}$ be an increasing function. A fuzzy-number-valued function $\tilde{f}(x)$ is said to be fuzzy Henstock-Stieltjes (FHS) integrable with respect to α on $[a,b]$ if there exists a fuzzy number $\widetilde{H}\in E^1$ such that for every $\varepsilon > 0$, there is a function $\delta(x) > 0$ such that for any δ-fine division $P = \{[u_i,v_i];\xi_i\}_{i=1}^n$, we have

$$D\Big(\sum_{i=1}^{n} \tilde{f}(\xi_i)[\alpha(v_i) - \alpha(u_i)], \widetilde{H}\Big) < \varepsilon. \tag{2.2}$$

We write $(FHS)\int_a^b \tilde{f}(x)\mathrm{d}\alpha = \widetilde{H}$, and $\tilde{f}\in FHS_\alpha[a,b]$.

Definition 2.4 [6] A fuzzy-number-valued function $\tilde{f} : [a,b] \to E^1$ is said to be have locally small Riemann sums or ($LSRS$) if for every $\varepsilon > 0$ there is a $\delta(\xi) > 0$ such that for every $t\in[a,b]$, we have

$$\big\|\sum \tilde{f}(\xi)(v-u)\big\|_{E^1} < \varepsilon, \tag{2.3}$$

whenever $P = \{[u,v];\xi\}$ is a δ-fine division of an interval $[r,s] \subset (t-\delta(t), t+\delta(t))$, $t\in[r,s]$ and Σ sums over P.

3 Locally small Riemann sums and Henstock-Stieltjes integral of fuzzy-number-valued functions

In this section, we shall define locally small Riemann sums with respect to α on $[a,b]$ for fuzzy-number-valued functions. Furthermore, we prove that a fuzzy-number-valued functions is Henstock-Stieltjes integrable with respect to α on $[a,b]$ if and only if it has ($LSRS$) with respect to α on $[a,b]$. We begin with the following definition.

Definition 3.1 Let $\alpha : [a,b] \to \mathbb{R}$ be an increasing function. A fuzzy-number-valued function $\tilde{f} : [a,b] \to E^1$ is said to be have locally small Riemann sums (or $LSRS$) with respect to α on $[a,b]$, if for every $\varepsilon > 0$ there is a $\delta(\xi) > 0$ such that for every $t\in[a,b]$, we have

$$\Big\|\sum \tilde{f}(\xi)[\alpha(v) - \alpha(u)]\Big\|_{E^1} < \varepsilon, \tag{3.1}$$

whenever $P = \{[u,v];\xi\}$ is a δ-fine division of an interval $[r,s] \subset (t-\delta(t), t+\delta(t))$, $t\in[r,s]$ and Σ sums over P.

If there exists a $z\in E^1$ such that $x = y+z$, then we call z the $H-$ difference of x and y, denoted by $x-y$. According to the additivity of FHS, we have the following Lemma.

Lemma 3.1 [2] Let $\tilde{f}\in FHS_\alpha[a,b]$ and $\tilde{F}$ be the primitive of $\tilde{f}(x)$ then $\tilde{F}$ satisfies the $H-$ difference.

Lemma 3.2 (Henstock Lemma). Let $\alpha : [a,b] \to \mathbb{R}$ be an increasing function. If a fuzzy-number-valued function $\tilde{f} : [a,b] \to E^1$ is Henstock-Stieltjes integrable with respect to α on $[a,b]$ with primitive $\tilde{F}$, i.e., for every $\varepsilon > 0$ there is a positive function $\delta(\xi) > 0$ such that for any δ-fine division $P = \{[u,v];\xi\}$ of $[a,b]$, we have

$$D\Big(\sum \tilde{f}(\xi)[\alpha(v) - \alpha(u)], \sum \tilde{F}(u,v)\Big) < \varepsilon. \tag{3.2}$$

Then for any sum of parts $\sum_1$ from $\sum$, we have

$$D\Big(\sum_1 \tilde{f}(\xi)[\alpha(v) - \alpha(u)], \sum_1 \tilde{F}(u,v)\Big) < \varepsilon. \tag{3.3}$$

The proof is similar to the Theorem 3.7 [10].

Theorem 3.1 Let $\alpha : [a,b] \to \mathbb{R}$ be an increasing function. If $\tilde{f}(x)$ is Henstock-Stieltjes integrable with respect to α on $[a,b]$ then $\tilde{f}(x)$ it has $LSRS$ with respect to α on $[a,b]$.

Proof Let $\tilde{F}$ be the primitive of $\tilde{f}(x)$. Given $\varepsilon > 0$ there is a $\delta(\xi) > 0$ such that for any δ-fine division $P = \{[u,v];\xi\}$ of $[a,b]$, we have

$$D\Big(\sum \tilde{f}(\xi)[\alpha(v) - \alpha(u)], \sum \tilde{F}(u,v)\Big) < \varepsilon. \tag{3.4}$$

Where $\tilde{F}(u,v) = \tilde{F}(v) - \tilde{F}(u)$. By the continuity of $\tilde{F}$ at ξ,

$$D\big(\tilde{F}(u), \tilde{F}(v)\big) < \varepsilon \quad \text{whenever} \quad [u,v] \subset (\xi - \delta(\xi), \xi + \delta(\xi)).$$

Therefore for $t \in [a,b]$ and any δ-fine division $P = \{[u,v];\xi\}$ of $[r,s] \subset (t - \delta(t), t + \delta(t))$, we have

$$\begin{aligned} \Big\| \sum \tilde{f}(\xi)[\alpha(v) - \alpha(u)] \Big\|_{E^1} &\leq D\Big(\sum \tilde{f}(\xi)[\alpha(v) - \alpha(u)], \sum \tilde{F}(u,v)\Big) + D\Big(\tilde{F}(r), \tilde{F}(s)\Big) \\ &< 2\varepsilon. \end{aligned}$$

That is $\tilde{f}(x)$ has $LSRS$ with respect to α on $[a,b]$.
This completes the proof. □

Lemma 3.3 [2] (Cauchy criterion). Let $\alpha : [a,b] \to \mathbb{R}$ be an increasing function. A fuzzy-number-valued function $\tilde{f} : [a,b] \to E^1$ is Henstock-Stieltjes integrable with respect to α on $[a,b]$ if and only if for every $\varepsilon > 0$ there is a positive function $\delta(\xi) > 0$ such that whenever $P_1 = \{[u_1,v_1];\xi_1\}$, $P_2 = \{[u_2,v_2];\xi_2\}$ are two δ-fine divisions, we have

$$D\Big(\sum_{(P_1)} \tilde{f}(\xi_1)[\alpha(v_1) - \alpha(u_1)], \sum_{(P_2)} \tilde{f}(\xi_2)[\alpha(v_2) - \alpha(u_2)]\Big) < \varepsilon. \tag{3.5}$$

Theorem 3.2 Let $\alpha : [a,b] \to \mathbb{R}$ be an increasing function. If a fuzzy-number-valued function $\tilde{f} : [a,b] \to E^1$ has $LSRS$ with respect to α on $[a,b]$ then $\tilde{f}(x)$ is Henstock-Stieltjes integrable with respect to α on any closed sub-interval $C \subset (a,b)$. (Where $C = [r,s]$).

Proof A fuzzy-number-valued function $\tilde{f} : [a,b] \to E^1$ has $LSRS$ with respect to α on $[a,b]$ means that for every $\varepsilon > 0$ there is a $\delta(\xi) > 0$ such that for every $t \in [a,b]$, we have

$$\Big\| \sum \tilde{f}(\xi)[\alpha(v) - \alpha(u)] \Big\|_{E^1} < \varepsilon, \tag{3.6}$$

whenever $P = \{[u,v];\xi\}$ is a δ-fine division of an interval $C \subset (t - \delta(t), t + \delta(t))$, $t \in C$ and Σ sums over P.

(i) If there $t \in [a,b]$ with $C \subset (t - \delta(t), t + \delta(t))$ we have the following discussion:

(1) If $t \in C$ then for every $\varepsilon > 0$ there is a two δ-fine divisions $P_1 = \{[u_1,v_1];\xi_1\}$, $P_2 = \{[u_2,v_2];\xi_2\}$ on C, such that

$$D\Big(\sum_{(P_1)} \tilde{f}(\xi_1)[\alpha(v_1) - \alpha(u_1)], \sum_{(P_2)} \tilde{f}(\xi_2)[\alpha(v_2) - \alpha(u_2)]\Big) < \varepsilon. \tag{3.7}$$

According to the Cauchy criterion, then $\tilde{f}(x)$ is Henstock-Stieltjes integrable on C.

(2) If $t \notin C$ then there is a closed interval $E \subset (t-\delta(t), t+\delta(t))$, with the result that $t \in E$ and $C \subset E$ (where $E = [g,h]$). As a result, for every $\varepsilon > 0$ there is a two δ-fine divisions $P_1 = \{[u_1, v_1]; \xi_1\}$, $P_2 = \{[u_2, v_2]; \xi_2\}$ on E, such that

$$D\bigg(\sum_{(P_1)} \tilde{f}(\xi_1)[\alpha(v_1) - \alpha(u_1)], \sum_{(P_2)} \tilde{f}(\xi_2)[\alpha(v_2) - \alpha(u_2)]\bigg) < \varepsilon. \tag{3.8}$$

According to the Cauchy criterion, then $\tilde{f}(x)$ is Henstock-Stieltjes integrable on E. Because $C \subset E$ and $\tilde{f}(x)$ is Henstock-Stieltjes integrable on E then $\tilde{f}(x)$ is Henstock-Stieltjes integrable on C.

(ii) If $C \not\subseteq (t-\delta(t), t+\delta(t))$ then there is a positive function δ on $[a,b]$ which resulted in the presence that $P = \{(C_i, t_i) : i = 1, 2, \cdots, k\}$ is a δ-fine division of the interval C. It follows that $\tilde{f}(x)$ is Henstock-Stieltjes integrable on C_i for $i = 1, 2, \cdots, k$.

Then $\tilde{f}(x)$ is Henstock-Stieltjes integrable on C.

This completes the proof. □

Corollary 3.1 Let $\alpha : [a,b] \to \mathbb{R}$ be an increasing function. If a fuzzy-number-valued function $\tilde{f} : [a,b] \to E^1$ has $LSRS$ with respect to α on $[a,b]$ then $\tilde{f}(x)$ is Henstock-Stieltjes integrable with respect to α on C for any simple set $C \subset (a,b)$.

Notice that a simple set C means that there exists finite closed sub-interval C_i which belongs to (a,b) such that $C = \bigcup_{i=1}^{k} C_i$.

Theorem 3.3 Let $\alpha : [a,b] \to \mathbb{R}$ be an increasing function. If a fuzzy-number-valued function $\tilde{f} : [a,b] \to E^1$ has $LSRS$ with respect to α on $[a,b]$ then $\tilde{f}(x)$ is Henstock-Stieltjes integrable with respect to α on $[a,b]$.

Proof A fuzzy-number-valued function $\tilde{f} : [a,b] \to E^1$ has $LSRS$ with respect to α on $[a,b]$, then for every $\varepsilon > 0$ there is $\delta^*(\xi) > 0$ such that for every $t \in [a,b]$, we have

$$\bigg\| \sum \tilde{f}(\xi)[\alpha(v) - \alpha(u)] \bigg\|_{E^1} < \varepsilon, \tag{3.9}$$

whenever $P = \{[u,v]; \xi\}$ is a δ^*-fine division of an interval $C \subset (t-\delta(t), t+\delta(t))$, $t \in C$ and Σ sums over P. According to the Corollary 3.1, $\tilde{f}(x)$ is Henstock-Stieltjes integrable on C for any simple set $C \subset (a,b)$.

Rows set $\{E_i\}$, $E_i \bigcap E_j = \phi$, $\forall i \neq j$ with property $(a,b) = \bigcup E_i$, E_i is a closed interval. Thus for a bove $\varepsilon > 0$, there is a positive numbers n_0 with property

$$\mu\{[a,b] - \bigcup_{i \leq n_0} E_i\} < \varepsilon, \tag{3.10}$$

where μ is Lebesgue measure.

For any i, there is a positive function δ_i such that for any δ_i-fine division on E_i, we have

$$D\bigg(\sum \tilde{f}(\xi)[\alpha(v) - \alpha(u)], (HS)\int_{E_i} \tilde{f}(x)\mathrm{d}\alpha\bigg) < \varepsilon. \tag{3.11}$$

Define a positive function δ by the formula:

$$\delta(\xi) = \begin{cases} \min\{\delta^*(\xi), \frac{1}{2}d(\xi, \partial[a,b])\} & \text{if } \xi \in \bigcup\limits_{i > n_0} E_i, \\ \min\{\delta^*(\xi), \delta_i(\xi)\}, & \text{if } \xi \in \bigcup\limits_{i \leq n_0} E_i. \end{cases}$$

For each $\mathcal{C} = \{C\} = \{C_1, C_2, \cdots, C_k\}$ with $C_j = E_i \bigcap Q$ (where $Q = [u,v]$), for one $i \leq n_0$ and one Q with $\{[u,v]; \xi\}$ is a δ-fine division and $\xi \in (a,b)$, we have

(i) If $C_j = E_i$ for $i \leq n_0$. Because $\tilde{f}(x)$ is Henstock-Stieltjes integrable on E_i and $\tilde{f}(x)$ is Henstock-Stieltjes integrable on C_j consequently $\tilde{f}(x)$ is Henstock-Stieltjes integrable on $\bigcup_{j=1}^{k} C_j$. Selected a positive function δ_* with $\delta_*(\xi) = \min\{\delta_j(\xi) : j = 1, 2, \cdots, k\}$, then for each δ_*-fine division $P = \{[u,v]; \xi\}$ on $\bigcup_{j=1}^{k} C_j$, we have

$$D\bigg((HS)\int_{\bigcup_{j=1}^{k} C_j} \tilde{f}(x)\mathrm{d}\alpha, \sum \tilde{f}(\xi)[\alpha(v) - \alpha(u)]\bigg) < \varepsilon. \tag{3.12}$$

Thus obtained:

$$\begin{aligned}\Big\|\mathcal{C}\sum(HS)\int_C \tilde f(x)\mathrm{d}\alpha\Big\|_{E^1} &\le D\Big((HS)\int_{\bigcup_{j=1}^k C_j}\tilde f(x)\mathrm{d}\alpha, \sum \tilde f(\xi)[\alpha(v)-\alpha(u)]\Big)+\sum_{j=1}^k\Big\|\sum \tilde f(\xi)[\alpha(v)-\alpha(u)]\Big\|_{E^1}\\ &< \varepsilon+k\varepsilon.\end{aligned}$$

According to the properties of Cauchy, $\tilde f(x)$ is Henstock-Stieltjes integrable on $[a,b]$.

(ii) If $C_j=E_i\bigcap Q$, for $i\le n_0$ and one δ-fine Q with $\{[u,v];\xi\}$ and $\xi\in(a,b)$ then $C_j\subset(\xi-\delta(\xi),\xi+\delta(\xi))$. According to the Theorem 3.2, then $\tilde f(x)$ is Henstock-Stieltjes integrable on C_j. As the result $\tilde f(x)$ is Henstock-Stieltjes integrable on $\bigcup_{j=1}^k C_j$. Selected a positive function δ_1 with property $\delta_1(\xi)\le\delta(\xi)$ then for each δ_*-fine division $P=\{[u,v];\xi\}$ on $\bigcup_{j=1}^k C_j$, we have

$$D\Big((HS)\int_{\bigcup_{j=1}^k C_j}\tilde f(x)\mathrm{d}\alpha, \sum \tilde f(\xi)[\alpha(v)-\alpha(u)]\Big)<\varepsilon. \tag{3.13}$$

Thus obtained:

$$\begin{aligned}\Big\|\mathcal{C}\sum(HS)\int_C \tilde f(x)\mathrm{d}\alpha\Big\|_{E^1} &\le D\Big((HS)\int_{\bigcup_{j=1}^k C_j}\tilde f(x)\mathrm{d}\alpha, \sum \tilde f(\xi)[\alpha(v)-\alpha(u)]\Big)+\sum_{j=1}^k\Big\|\sum \tilde f(\xi)[\alpha(v)-\alpha(u)]\Big\|_{E^1}\\ &< \varepsilon+k\varepsilon.\end{aligned}$$

According to the properties of Cauchy, $\tilde f(x)$ is Henstock-Stieltjes integrable on $[a,b]$.
This completes the proof. □

Corollary 3.2 Let $\alpha:[a,b]\to\mathbb{R}$ be an increasing function. A fuzzy-number-valued function $\tilde f:[a,b]\to E^1$ has $LSRS$ with respect to α on $[a,b]$ if and only if $\tilde f(x)$ is Henstock-Stieltjes integrable with respect to α on $[a,b]$.

4 Globally small Riemann sums and Henstock-Stieltjes integral of fuzzy-number-valued functions

In this section, we shall define globally small Riemann sums with respect to α on $[a,b]$ for fuzzy-number-valued functions. Furthermore, we prove that a fuzzy-number-valued functions is Henstock-Stieltjes integrable with respect to α on $[a,b]$ if and only if it has $(GSRS)$ with respect to α on $[a,b]$. We begin with the following definition.

Definition 4.1 Let $\alpha:[a,b]\to\mathbb{R}$ be an increasing function. A fuzzy-number-valued function $\tilde f:[a,b]\to E^1$ is said to be have globally small Riemann sums or $(GSRS)$ with respect to α on $[a,b]$ if for every $\varepsilon>0$ there exists a positive integer N such that for every $n\geqslant N$ there is a $\delta_n(\xi)>0$ and for every δ_n-fine division $P=\{[u,v];\xi\}$ of $[a,b]$, we have

$$\Big\|\sum_{\|\tilde f(\xi)\|_{E^1}>n}\tilde f(\xi)[\alpha(v)-\alpha(u)]\Big\|_{E^1}<\varepsilon, \tag{4.1}$$

where the $\sum$ is taken over P and for which $\|\tilde f(\xi)\|_{E^1}>n$.

Theorem 4.1 Let $\alpha:[a,b]\to\mathbb{R}$ be an increasing function and let $\tilde f(x)$ be Henstock-Stieltjes integrable to $\tilde F(a,b)$ with respect to α on $FHS_\alpha[a,b]$ and $\tilde F_n(a,b)$ the integral of $\tilde f_n(x)$ on $FHS_\alpha[a,b]$, where $\tilde f_n(x)=\tilde f(x)$ when $\|\tilde f(x)\|_{E^1}\leqslant n$ and $\tilde 0$ otherwise. If $\tilde F_n(a,b)\to\tilde F(a,b)$ as $n\to\infty$ then $\tilde f(x)$ has $GSRS$ with respect to α on $[a,b]$.

Proof Given $\varepsilon>0$ there is a $\delta_n(\xi)>0$ such that for every δ_n-fine division $P=\{[u,v];\xi\}$ of $[a,b]$, we have

$$D\big(\sum \tilde f_n(\xi)[\alpha(v)-\alpha(u)],\tilde F_n(a,b)\big)<\varepsilon, \tag{4.2}$$

where $(FHS)\int_a^b \tilde{f}_n(x)\mathrm{d}\alpha = \tilde{F}_n(a,b)$.

$$D\big(\sum \tilde{f}(\xi)[\alpha(v)-\alpha(u)], \tilde{F}(a,b)\big) < \varepsilon, \tag{4.3}$$

where $(FHS)\int_a^b \tilde{f}(x)\mathrm{d}\alpha = \tilde{F}(a,b)$.

Choose N so that whenever $n \geqslant N$

$$D\big(\tilde{F}_n(a,b), \tilde{F}(a,b)\big) < \varepsilon. \tag{4.4}$$

Therefore for $n \geqslant N$ and δ_n-fine division $P = \{[u,v];\xi\}$ of $[a,b]$, we have

$$\begin{aligned}
&\Big\| \sum_{\|\tilde{f}(\xi)\|_{E^1} > n} \tilde{f}(\xi)[\alpha(v)-\alpha(u)] \Big\|_{E^1} = D\Big(\sum \tilde{f}_n(\xi)[\alpha(v)-\alpha(u)], \sum \tilde{f}(\xi)[\alpha(v)-\alpha(u)]\Big) \\
\leqslant\ & D\Big(\sum \tilde{f}_n(\xi)[\alpha(v)-\alpha(u)], \tilde{F}_n(a,b)\Big) + D\Big(\tilde{F}_n(a,b), \tilde{F}(a,b)\Big) + D\Big(\tilde{F}(a,b), \sum \tilde{f}(\xi)[\alpha(v)-\alpha(u)]\Big) \\
<\ & 3\varepsilon.
\end{aligned}$$

Hence $\tilde{f}(x)$ has $GSRS$ with respect to α on $[a,b]$.
This completes the proof. □

Theorem 4.2 Let $\alpha : [a,b] \to \mathbb{R}$ be an increasing function. A fuzzy-number-valued function $\tilde{f}(x)$ has $GSRS$ with respect to α on $[a,b]$ if and only if $\tilde{f}(x)$ is Henstock-Stieltjes integrable with respect to α on $[a,b]$ and $\tilde{F}_n(a,b) \to \tilde{F}(a,b)$ as $n \to \infty$ where $\tilde{F}_n(a,b)$ and $\tilde{F}(a,b)$ are defined as in Theorem 4.1.

Proof Theorem 4.1 proves the sufficiency. We shall prove only the necessity. Suppose $\tilde{f}(x)$ has $GSRS$ with respect to α on $[a,b]$. Note that $\tilde{f}_n(x)$, as defined in Theorem 4.1, is fuzzy Henstock-Stieltjes integrable on $[a,b]$ for all n. Then for $n,m \geqslant N$ and a suitably chosen δ-fine division $P = \{[u,v];\xi\}$, we have

$$\begin{aligned}
&D\Big(\tilde{F}_n(a,b), \tilde{F}_m(a,b)\Big) \\
\leqslant\ & D\Big(\tilde{F}_n(a,b), \sum_{\|\tilde{f}(\xi)\|_{E^1} \leqslant n} \tilde{f}(\xi)[\alpha(v)-\alpha(u)]\Big) + D\Big(\sum_{\|\tilde{f}(\xi)\|_{E^1} \leqslant m} \tilde{f}(\xi)[\alpha(v)-\alpha(u)], \tilde{F}_m(a,b)\Big) \\
+\ & \Big\| \sum_{\|\tilde{f}(\xi)\|_{E^1} > n} \tilde{f}(\xi)[\alpha(v)-\alpha(u)] \Big\|_{E^1} + \Big\| \sum_{\|\tilde{f}(\xi)\|_{E^1} > m} \tilde{f}(\xi)[\alpha(v)-\alpha(u)] \Big\|_{E^1} \\
<\ & 4\varepsilon.
\end{aligned}$$

That is, $\tilde{F}_n(a,b)$ converge to a fuzzy number, say $\tilde{F}(a,b)$, as $n \to \infty$. Again, for suitably chosen N and $\delta(\xi)$ and for every δ-fine division $P = \{[u,v];\xi\}$, we have

$$\begin{aligned}
&D\Big(\sum \tilde{f}(\xi)[\alpha(v)-\alpha(u)], \tilde{F}(a,b)\Big) \leqslant D\Big(\tilde{F}(a,b), \tilde{F}_N(a,b)\Big) \\
+\ & D\Big(\tilde{F}_N(a,b), \sum_{\|\tilde{f}(\xi)\|_{E^1} \leqslant N} \tilde{f}(\xi)[\alpha(v)-\alpha(u)]\Big) + \Big\| \sum_{\|\tilde{f}(\xi)\|_{E^1} > N} \tilde{f}(\xi)[\alpha(v)-\alpha(u)] \Big\|_{E^1} \\
<\ & 3\varepsilon.
\end{aligned}$$

That is, $\tilde{f}(x)$ is fuzzy Henstock-Stieljes integrable on $[a,b]$.
This completes the proof. □

Theorem 4.3 Let $\alpha : [a,b] \to \mathbb{R}$ be an increasing function and let $\tilde{f}_n(x) \in FHS_\alpha[a,b],\ n = 1,2,3\cdots$ and satisfy:

(1) $\lim\limits_{n\to\infty} \tilde{f}_n(x) = \tilde{f}(x)$ almost everywhere in $[a,b]$;

(2) there exists a Lebesgue-Stieljes integrable (Henstock-Stieljes integrable) function $h(x)$ on $[a,b]$ such that

$$D\big(\tilde{f}_n(x), \tilde{f}_m(x)\big) < h(x). \tag{4.5}$$

Then, $\tilde{f}_n(x)$ has $GSRS$ with respect to α on $[a,b]$ uniformly for any n. Naturally, $\tilde{f}$ is (FHS) integrable with respect to α on $[a,b]$. Furthermore,

$$\lim_{n\to\infty}(FHS)\int_a^b \tilde{f}_n(x)\mathrm{d}\alpha = (FHS)\int_a^b \tilde{f}(x)\mathrm{d}\alpha. \tag{4.6}$$

Proof Let $\varepsilon > 0$. Since $H(x) = (LS)\int_a^x h(t)\mathrm{d}\alpha$ is absolutely continuous with respect to stieljes measurable on $[a,b]$, there exists a positive number $\eta > 0$ such that $\sum |H(b_i) - H(a_i)| < \varepsilon$ whenever $\{[a_i,b_i]\}$ is a finite collection of non-overlapping intervals in $[a,b]$ that satisfy $\sum \big(\alpha(b_i) - \alpha(a_i)\big) < \eta$. Since $\lim_{n\to\infty} \tilde{f}_n(x) = \tilde{f}(x)$ almost everywhere in $[a,b]$, and

$$\begin{aligned} D(\tilde{f}_n,\tilde{f}) &= \sup_{r\in[0,1]} \max\{|(f_n(x))_-^r - (f(x))_-^r|, |(f_n(x))_+^r - (f(x))_+^r|\} \\ &= \sup_{r_k\in[0,1]} \max\{|(f_n(x))_-^{r_k} - (f(x))_-^{r_k}|, |(f_n(x))_+^{r_k} - (f(x))_+^{r_k}|\} \end{aligned}$$

is a sequence of Lebesgue-Stieljes measurable functions, where $r_k \in [0,1]$ is the set of rational numbers, by Egorov's Theorem, there exists an open set G with $LS(G) < \eta$ such that $\lim_{n\to\infty} \tilde{f}_n(x) = \tilde{f}(x)$ uniformly for $x \in [a,b]\setminus G$. Then, there is an natural number N, such that for any $n,m > N$, and for any $x \in [a,b]\setminus G$, we have $D(\tilde{f}_n(x),\tilde{f}_m(x)) < \varepsilon$. Since $h(x)$ is Henstock-Stieltjes integrable on $[a,b]$, there is a $\delta_h(\xi) > 0$ such that for any δ_h-fine division $P = \{[u,v];\xi\}$ of $[a,b]$, we have

$$\left|\sum h(\xi)[\alpha(v) - \alpha(u)] - (LS)\int_a^b h(t)\mathrm{d}\alpha\right| < \varepsilon. \tag{4.7}$$

Define

$$\delta(\xi) = \begin{cases} \delta_h(\xi), & \text{if } \xi \in [a,b]\setminus G, \\ \delta(\xi),\ \text{satisfying } (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i)) \subset G, & \text{if } \xi \in [a,b]. \end{cases}$$

Then, it follows that for a δ-fine division $P_0 = \{[x_{i-1},x_i];\xi_i\}$ of $[a,b]$,

$$\begin{aligned} & D\Big(\sum \tilde{f}_n(\xi_i)[\alpha(x_i) - \alpha(x_{i-1})], \sum \tilde{f}_m(\xi_i)[\alpha(x_i) - \alpha(x_{i-1})]\Big) \\ \leq\ & D\Big(\sum_{\xi_i\in[a,b]\setminus G} \tilde{f}_n(\xi_i)[\alpha(x_i) - \alpha(x_{i-1})], \sum_{\xi_i\in[a,b]\setminus G} \tilde{f}_m(\xi_i)[\alpha(x_i) - \alpha(x_{i-1})]\Big) \\ +\ & D\Big(\sum_{\xi_i\in G} \tilde{f}_n(\xi_i)[\alpha(x_i) - \alpha(x_{i-1})], \sum_{\xi_i\in G} \tilde{f}_m(\xi_i)[\alpha(x_i) - \alpha(x_{i-1})]\Big) \\ \leq\ & \sum_{\xi_i\in[a,b]\setminus G} D\big(\tilde{f}_n(\xi_i),\tilde{f}_m(\xi_i)\big)[\alpha(x_i) - \alpha(x_{i-1})] + \sum_{\xi_i\in G} D\big(\tilde{f}_n(\xi_i),\tilde{f}_m(\xi_i)\big)[\alpha(x_i) - \alpha(x_{i-1})] \\ <\ & \varepsilon(b-a) + \Big|\sum_{\xi_i\in G} h(\xi_i)[\alpha(x_i) - \alpha(x_{i-1})] - \int_G h(t)\mathrm{d}\alpha\Big| + \Big|\int_G h(t)\mathrm{d}\alpha\Big| \\ <\ & \varepsilon(b-a) + 3\varepsilon. \end{aligned}$$

Hence, there is an natural number N such that for any $n,m > N$, we have

$$\begin{aligned} & D\Big(\tilde{F}_n[a,b],\tilde{F}_m[a,b]\Big) \\ \leq\ & D\Big(\tilde{F}_n[a,b], \sum \tilde{f}_n(\xi_i)[\alpha(x_i) - \alpha(x_{i-1})]\Big) + D\Big(\tilde{F}_m[a,b], \sum \tilde{f}_m(\xi_i)[\alpha(x_i) - \alpha(x_{i-1})]\Big) \\ +\ & D\Big(\sum \tilde{f}_n(\xi_i)[\alpha(x_i) - \alpha(x_{i-1})], \sum \tilde{f}_m(\xi_i)[\alpha(x_i) - \alpha(x_{i-1})]\Big) \\ <\ & 3\varepsilon. \end{aligned}$$

Thus, $\tilde{F}_n[a,b]$ is a Cauchy sequence, and there is an natural number N_1 such that for any $n > N_1$, we have $D(\tilde{F}_n[a,b],\tilde{A}) < \varepsilon$. According to the (FHS) integrability of $\tilde{f}_{N_1}(x)$, there is a $\delta_{N_1}(\xi) > 0$ such that for any δ_{N_1}-fine division $P = \{[u,v];\xi\}$ of $[a,b]$, for any $n > N_{N_1}$, we have

$$\begin{aligned} & D\Big(\sum \tilde{f}_n(\xi)[\alpha(v) - \alpha(u)], \tilde{F}_n[a,b]\Big) \\ \leq\ & D\Big(\tilde{F}_n[a,b],\tilde{F}_{N_1}[a,b]\Big) + D\Big(\sum \tilde{f}_{N_1}(\xi)[\alpha(v) - \alpha(u)], \tilde{F}_{N_1}[a,b]\Big) \\ +\ & D\Big(\sum \tilde{f}_n(\xi)[\alpha(v) - \alpha(u)], \sum \tilde{f}_{N_1}(\xi)[\alpha(v) - \alpha(u)]\Big) \\ <\ & 3\varepsilon. \end{aligned}$$

This completes the proof. □

5 conclusions

In this paper, we have investigated locally and globally small Riemann sums with respect to α on $[a, b]$ for fuzzy number-valued functions. Also we have stated and proved two main theorems: (1) If a fuzzy number-valued functions $\tilde{f}(x)$ is Henstock-Stieltjes integrable with respect to α on $[a, b]$ then $\tilde{f}(x)$ has $(LSRS)$ with respect to α on $[a, b]$ and the converse is always true. (2) If a fuzzy number-valued functions $\tilde{f}(x)$ is Henstock-Stieltjes integrable with respect to α on $[a, b]$ then $\tilde{f}(x)$ has $(GSRS)$ with respect to α on $[a, b]$ and the converse is always true. Finally, by Egorov's Theorem, we got the dominated convergence theorem for globally small Riemann sums $(GSRS)$ with respect to α on $[a, b]$ for fuzzy-number-valued functions.

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A note on special fuzzy differential subordinations using multiplier transformation and Ruschewehy derivative

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Abstract

In this paper we establish some fuzzy differential subordinations regarding the differential operator $RI^{\alpha}_{m,\lambda,l}: \mathcal{A}_n \to \mathcal{A}_n$, $RI^{\alpha}_{m,\lambda,l}f(z) = (1-\alpha)R^m f(z) + \alpha I(m,\lambda,l) f(z)$, where R^m is the Ruscheweyh derivative,$I(m,\lambda,l)$ is the multiplier transformation and $\mathcal{A}_n = \{f \in \mathcal{H}(U),\ f(z) = z + a_{n+1}z^{n+1} + \dots,\ z \in U\}$ is the class of normalized analytic functions. We introduce a fuzzy class $RI^{\delta}_{\mathcal{F}}(\alpha,m,\lambda,l)$ and by using the fuzzy differential subordinations we derive some properties of this class. Also, several fuzzy differential subordinations are established regarding the studied differential operator.

Keywords: fuzzy differential subordination, convex function, fuzzy best dominant, differential operator.
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1 Introduction

The differential subordination method was introduced and developed by S.S. Miller and P.T. Mocanu. G.I. Oros and Gh.Oros in [6], [7] combine the notions from the complex functions domain with the fuzzy sets theory.

In this paper we obtain fuzzy differential subordinations regarding the differential operator studied in [4] using the methods from [2], [3].

Consider $U = \{z \in \mathbb{C} : |z| < 1\}$ the unit disc of the complex plane, $\mathcal{H}(U)$ the space of holomorphic functions in U, $\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1}z^{n+1} + \dots,\ z \in U\}$ and $\mathcal{H}[a,n] = \{f \in \mathcal{H}(U) : f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \dots,\ z \in U\}$ for $a \in \mathbb{C}$ and $n \in \mathbb{N}$. The class of normalized convex functions in U is denoted by $\mathcal{K} = \left\{ f \in \mathcal{A}_n : \mathrm{Re}\, \frac{zf''(z)}{f'(z)} + 1 > 0,\ z \in U \right\}$.

We need the following.

Definition 1.1 *([6]) Let $D \subset \mathbb{C}$, $z_0 \in D$ be a fixed point and let the functions $f, g \in \mathcal{H}(D)$. The function f is said to be fuzzy subordinate to g and write $f \prec_{\mathcal{F}} g$ or $f(z) \prec_{\mathcal{F}} g(z)$, if are satisfied the conditions $f(z_0) = g(z_0)$ and $F_{f(D)}f(z) \le F_{g(D)}g(z)$, $z \in D$.*

Definition 1.2 *([7, Definition 2.2]) Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and h univalent in U, with $\psi(a,0;0) = h(0) = a$. If p is analytic in U, with $p(0) = a$ and satisfies the fuzzy differential subordination*

$$F_{\psi(\mathbb{C}^3\times U)}\psi(p(z), zp'(z), z^2p''(z); z) \le F_{h(U)}h(z), \quad z \in U, \tag{1.1}$$

then p is called a fuzzy solution of the fuzzy differential subordination. The univalent function q is called a fuzzy dominant of the fuzzy solutions of the fuzzy differential subordination, if $F_{p(U)}p(z) \le F_{q(U)}q(z)$, $z \in U$, for all p satisfying (1.1). A fuzzy dominant $\widetilde{q}$ that satisfies $F_{\widetilde{q}(U)}\tilde{q}(z) \le F_{q(U)}q(z)$, $z \in U$, for all fuzzy dominants q of (1.1) is said to be the fuzzy best dominant of (1.1).

Lemma 1.1 *([5, Corollary 2.6g.2, p. 66]) Let $h \in \mathcal{A}_n$ and $L[f](z) = G(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t)\, t^{\frac{1}{n}-1} dt$, $z \in U$. If $Re\left(\frac{zh''(z)}{h'(z)} + 1\right) > -\frac{1}{2}$, $z \in U$, then $L(f) = G \in \mathcal{K}$.*

Lemma 1.2 *([8]) Let h be a convex function with $h(0) = a$, and let $\gamma \in \mathbb{C}^*$ be a complex number with $Re\, \gamma \ge 0$. If $p \in \mathcal{H}[a,n]$ with $p(0) = a$, $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$, $\psi(p(z), zp'(z); z) = p(z) + \frac{1}{\gamma}zp'(z)$ an analytic function in U and*

$$F_{\psi(\mathbb{C}^2\times U)}\left(p(z) + \frac{1}{\gamma}zp'(z)\right) \le F_{h(U)}h(z), \quad z \in U, \tag{1.2}$$

then $F_{p(U)}p(z) \le F_{g(U)}g(z) \le F_{h(U)}h(z)$, $z \in U$, where $g(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t)t^{\gamma/n-1}dt$, $z \in U$. The function q is convex and is the fuzzy best dominant.

Lemma 1.3 *([8]) Let g be a convex function in U and let $h(z) = g(z) + n\alpha z g'(z)$, $z \in U$, where $\alpha > 0$ and n is a positive integer. If $p(z) = g(0) + p_n z^n + p_{n+1} z^{n+1} + \dots$, $z \in U$, is holomorphic in U and $F_{p(U)}\left(p(z) + \alpha z p'(z)\right) \leq F_{h(U)} h(z)$, $z \in U$, then $F_{p(U)} p(z) \leq F_{g(U)} g(z)$, $z \in U$, and this result is sharp.*

We need the following diferential operators.

Definition 1.3 *(Ruscheweyh [9]) For $f \in \mathcal{A}_n$, $m, n \in \mathbb{N}$, the operator R^m is defined by $R^m : \mathcal{A}_n \to \mathcal{A}_n$,*

$$\begin{aligned} R^0 f(z) &= f(z), \ R^1 f(z) = z f'(z), \ \dots \\ (m+1) R^{m+1} f(z) &= z \left(R^m f(z)\right)' + m R^m f(z), \quad z \in U. \end{aligned}$$

Remark 1.1 *If $f \in \mathcal{A}_n$, $f(z) = z + \sum_{j=n+1}^{\infty} a_j z^j$, then $R^m f(z) = z + \sum_{j=n+1}^{\infty} \frac{(n+j-1)!}{n!(j-1)!} a_j z^j$, $z \in U$.*

Definition 1.4 *([1]) For $f \in \mathcal{A}_n$, $m, n \in \mathbb{N}$, $\lambda, l \geq 0$, the operator $I(m, \lambda, l) f(z)$ is defined by the following infinite series $I(m, \lambda, l) f(z) = z + \sum_{j=n+1}^{\infty} \left(\frac{\lambda(j-1)+l+1}{l+1}\right)^m a_j z^j$.*

Remark 1.2 *We have $I(0, \lambda, l) f(z) = f(z)$, $(l+1) I(m+1, \lambda, l) f(z) = (l+1-\lambda) I(m, \lambda, l) f(z) + \lambda z \left(I(m, \lambda, l) f(z)\right)'$, $z \in U$.*

Definition 1.5 *([4]) Let $\alpha, \lambda, l \geq 0$, $m, n \in \mathbb{N}$. Denote by $RI^{\alpha}_{m,\lambda,l}$ the operator given by $RI^{\alpha}_{m,\lambda,l} : \mathcal{A}_n \to \mathcal{A}_n$, $RI^{\alpha}_{m,\lambda,l} f(z) = (1-\alpha) R^m f(z) + \alpha I(m, \lambda, l) f(z)$, $z \in U$.*

Remark 1.3 *If $f \in \mathcal{A}_n$, $f(z) = z + \sum_{j=n+1}^{\infty} a_j z^j$, then*
$RI^{\alpha}_{m,\lambda,l} f(z) = z + \sum_{j=n+1}^{\infty} \left\{\alpha \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m + (1-\alpha) \frac{(n+j-1)!}{n!(j-1)!}\right\} a_j z^j$, $z \in U$.

2 Main results

Using the operator $RI^{\alpha}_{m,\lambda,l}$ we define the class $RI^{\delta}_{\mathcal{F}}(\alpha, m, \lambda, l)$ and we study fuzzy subordinations.

Definition 2.1 *The class $RI^{\delta}_{\mathcal{F}}(\alpha, m, \lambda, l)$ contains all the functions $f \in \mathcal{A}_n$ which satisfy the inequality*

$$F_{\left(RI^{\alpha}_{m,\lambda,l} f\right)'(U)} \left(RI^{\alpha}_{m,\lambda,l} f(z)\right)' > \delta, \quad z \in U, \tag{2.1}$$

where $\delta \in (0, 1]$, $\alpha, \lambda, l \geq 0$ and $m, n \in \mathbb{N}$.

Theorem 2.1 *$RI^{\delta}_{\mathcal{F}}(\alpha, m, \lambda, l)$ is a convex set.*

Proof. Let $f_1, f_2 \in RI^{\delta}_{\mathcal{F}}(\alpha, m, \lambda, l)$, $f_k(z) = z + \sum_{j=n+1}^{\infty} a_{jk} z^j$, $k = 1, 2$, $z \in U$. We show that the function $h(z) = \eta_1 f_1(z) + \eta_2 f_2(z)$ is in the class $RI^{\delta}_{\mathcal{F}}(\alpha, m, \lambda, l)$, where η_1 and η_2 are nonnegative such that $\eta_1 + \eta_2 = 1$.

Differentiating, we obtain $h'(z) = (\mu_1 f_1 + \mu_2 f_2)'(z) = \mu_1 f_1'(z) + \mu_2 f_2'(z)$, $z \in U$, and $\left(RI^{\alpha}_{m,\lambda,l} h(z)\right)' = \left(RI^{\alpha}_{m,\lambda,l} (\mu_1 f_1 + \mu_2 f_2)(z)\right)' = \mu_1 \left(RI^{\alpha}_{m,\lambda,l} f_1(z)\right)' + \mu_2 \left(RI^{\alpha}_{m,\lambda,l} f_2(z)\right)'$, so we have also
$F_{\left(RI^{\alpha}_{m,\lambda,l} h\right)'(U)} \left(RI^{\alpha}_{m,\lambda,l} h(z)\right)' = F_{\left(RI^{\alpha}_{m,\lambda,l} (\mu_1 f_1 + \mu_2 f_2)\right)'(U)} \left(RI^{\alpha}_{m,\lambda,l} (\mu_1 f_1 + \mu_2 f_2)(z)\right)' =$
$F_{\left(RI^{\alpha}_{m,\lambda,l} (\mu_1 f_1 + \mu_2 f_2)\right)'(U)} \left(\mu_1 \left(RI^{\alpha}_{m,\lambda,l} f_1(z)\right)' + \mu_2 \left(RI^{\alpha}_{m,\lambda,l} f_2(z)\right)'\right) =$
$\frac{F_{\left(\mu_1 RI^{\alpha}_{m,\lambda,l} f_1\right)'(U)} \left(\mu_1 \left(RI^{\alpha}_{m,\lambda,l} f_1(z)\right)'\right) + F_{\left(\mu_2 RI^{\alpha}_{m,\lambda,l} f_2\right)'(U)} \left(\mu_2 \left(RI^{\alpha}_{m,\lambda,l} f_2(z)\right)'\right)}{2} =$
$\frac{F_{\left(RI^{\alpha}_{m,\lambda,l} f_1\right)'(U)} \left(RI^{\alpha}_{m,\lambda,l} f_1(z)\right)' + F_{\left(RI^{\alpha}_{m,\lambda,l} f_2\right)'(U)} \left(RI^{\alpha}_{m,\lambda,l} f_2(z)\right)'}{2}$.

Since $f_1, f_2 \in RI^{\delta}_{\mathcal{F}}(\alpha, m, \lambda, l)$, we have $\delta < F_{\left(RI^{\alpha}_{m,\lambda,l} f_1\right)'(U)} \left(RI^{\alpha}_{m,\lambda,l} f_1(z)\right)' \leq 1$ and
$\delta < F_{\left(RI^{\alpha}_{m,\lambda,l} f_2\right)'(U)} \left(RI^{\alpha}_{m,\lambda,l} f_2(z)\right)' \leq 1$, $z \in U$.

Therefore $\delta < \frac{F_{\left(RI^{\alpha}_{m,\lambda,l} f_1\right)'(U)} \left(RI^{\alpha}_{m,\lambda,l} f_1(z)\right)' + F_{\left(RI^{\alpha}_{m,\lambda,l} f_2\right)'(U)} \left(RI^{\alpha}_{m,\lambda,l} f_2(z)\right)'}{2} \leq 1$ and we obtain that
$\delta < F_{\left(RI^{\alpha}_{m,\lambda,l} h\right)'(U)} \left(RI^{\alpha}_{m,\lambda,l} h(z)\right)' \leq 1$, which means that $h \in RI^{\delta}_{\mathcal{F}}(\alpha, m, \lambda, l)$ and $RI^{\delta}_{\mathcal{F}}(\alpha, m, \lambda, l)$ is a convex set. ■

Theorem 2.2 *Consider g a convex function in U and $h(z) = g(z) + \frac{1}{c+2} z g'(z)$, with $z \in U$, $c > 0$. When $f \in RI_{\mathcal{F}}^{\delta}(\alpha, m, \lambda, l)$ and $G(z) = \frac{c+2}{z^{c+1}} \int_0^z t^c f(t)\, dt$, $z \in U$, then*

$$F_{\left(RI_{m,\lambda,l}^{\alpha} f\right)'(U)} \left(RI_{m,\lambda,l}^{\alpha} f(z)\right)' \leq F_{h(U)} h(z), \quad z \in U, \tag{2.2}$$

implies $F_{\left(RI_{m,\lambda,l}^{\alpha} G\right)'(U)} \left(RI_{m,\lambda,l}^{\alpha} G(z)\right)' \leq F_{g(U)} g(z)$, $z \in U$, and this result is sharp.

Proof. Differentiating with respect to z the following relation $z^{c+1} G(z) = (c+2) \int_0^z t^c f(t)\, dt$. We obtain $(c+1) G(z) + z G'(z) = (c+2) f(z)$ and $(c+1) RI_{m,\lambda,l}^{\alpha} G(z) + z \left(RI_{m,\lambda,l}^{\alpha} G(z)\right)' = (c+2) RI_{m,\lambda,l}^{\alpha} f(z)$, $z \in U$. Differentiating again we get $\left(RI_{m,\lambda,l}^{\alpha} G(z)\right)' + \frac{1}{c+2} z \left(RI_{m,\lambda,l}^{\alpha} G(z)\right)'' = \left(RI_{m,\lambda,l}^{\alpha} f(z)\right)'$, $z \in U$, and the fuzzy differential subordination (2.2) becomes

$$F_{RI_{m,\lambda,l}^{\alpha} G(U)} \left(\left(RI_{m,\lambda,l}^{\alpha} G(z)\right)' + \frac{1}{c+2} z \left(RI_{m,\lambda,l}^{\alpha} G(z)\right)'' \right) \leq F_{g(U)} \left(g(z) + \frac{1}{c+2} z g'(z) \right). \tag{2.3}$$

Consider

$$p(z) = \left(RI_{m,\lambda,l}^{\alpha} G(z)\right)', \quad z \in U, \tag{2.4}$$

evidently $p \in \mathcal{H}[1, n]$ and from relation (2.3) we get $F_{p(U)} \left(p(z) + \frac{1}{c+2} z p'(z) \right) \leq F_{g(U)} \left(g(z) + \frac{1}{c+2} z g'(z) \right)$, $z \in U$.

By using Lemma 1.3 we obtain $F_{\left(RI_{m,\lambda,l}^{\alpha} G\right)'(U)} \left(RI_{m,\lambda,l}^{\alpha} G(z)\right)' \leq F_{g(U)} g(z)$, $z \in U$, and g is the fuzzy best dominant. ■

Theorem 2.3 *Let $h(z) = \frac{1 + (2\beta - 1) z}{1 + z}$, $\beta \in [0, 1)$ and $c > 0$. If $\lambda, l \geq 0$, $m, n \in \mathbb{N}$ and $I_c(f)(z) = \frac{c+2}{z^{c+1}} \int_0^z t^c f(t)\, dt$, $z \in U$, then*

$$I_c \left[RI_{\mathcal{F}}^{\beta}(\alpha, m, \lambda, l) \right] \subset RI_{\mathcal{F}}^{\beta^*}(\alpha, m, \lambda, l), \tag{2.5}$$

where $\beta^* = 2\beta - 1 + \frac{(c+2)(2-2\beta)}{n} \int_0^1 \frac{t^{\frac{c+2}{n} - 1}}{t+1} dt$.

Proof. A similar proof with Theorem 2.2 for the convex function h get us $F_{p(U)} \left(p(z) + \frac{1}{c+2} z p'(z) \right) \leq f_{h(U)} h(z)$, with $p(z)$ defined in (2.4).

Applying Lemma 1.2 we obtain $F_{\left(RI_{m,\lambda,l}^{\alpha} G\right)'(U)} \left(RI_{m,\lambda,l}^{\alpha} G(z)\right)' \leq F_{g(U)} g(z) \leq F_{h(U)} h(z)$, where $g(z) = \frac{c+2}{n z^{\frac{c+2}{n}}} \int_0^z t^{\frac{c+2}{n} - 1} \frac{1 + (2\beta - 1) t}{1 + t} dt = 2\beta - 1 + \frac{(c+2)(2-2\beta)}{n z^{\frac{c+2}{n}}} \int_0^z \frac{t^{\frac{c+2}{n} - 1}}{t+1} dt$. Since g is convex and $g(U)$ is symmetric with respect to the real axis, we have

$$F_{I(m,\lambda,l) G(U)} \left(I(m, \lambda, l) G(z)\right)' \geq \min_{|z|=1} F_{g(U)} g(z) = F_{g(U)} g(1) \tag{2.6}$$

and $\beta^* = g(1) = 2\beta - 1 + \frac{(c+2)(2-2\beta)}{n} \int_0^1 \frac{t^{\frac{c+2}{n} - 1}}{t+1} dt$. The inclusion (2.5) follows from (2.6). ■

Theorem 2.4 *Consider g a convex function with $g(0) = 1$ and $h(z) = g(z) + z g'(z)$, $z \in U$. For $\lambda, l \geq 0$, $m, n \in \mathbb{N}$, $f \in \mathcal{A}_n$ and the fuzzy differential subordination*

$$F_{\left(RI_{m,\lambda,l}^{\alpha} f\right)'(U)} \left(RI_{m,\lambda,l}^{\alpha} f(z)\right)' \leq F_{h(U)} h(z), \quad z \in U, \tag{2.7}$$

holds, we obtain $F_{RI_{m,\lambda,l}^{\alpha} f(U)} \frac{RI_{m,\lambda,l}^{\alpha} f(z)}{z} \leq F_{g(U)} g(z)$, $z \in U$, and this result is sharp.

Proof. For $p(z) = \frac{RI_{m,\lambda,l}^{\alpha} f(z)}{z}$, $z \in U$, we have $p \in \mathcal{H}[1, n]$ and differentiating the relation $RI_{m,\lambda,l}^{\alpha} f(z) = z p(z)$, $z \in U$, we obtain $\left(RI_{m,\lambda,l}^{\alpha} f(z)\right)' = p(z) + z p'(z)$, $z \in U$.

The fuzzy differential subordination (2.7) becomes $F_{p(U)} \left(p(z) + z p'(z)\right) \leq F_{h(U)} h(z) = F_{g(U)} \left(g(z) + z g'(z)\right)$, $z \in U$, and applying Lemma 1.3, we deduce $F_{\left(RI_{m,\lambda,l}^{\alpha} f\right)'(U)} \frac{RI_{m,\lambda,l}^{\alpha} f(z)}{z} \leq F_{g(U)} g(z)$, $z \in U$. ■

Theorem 2.5 *Let h be a holomorphic function with* Re $\left(1 + \frac{z h''(z)}{h'(z)}\right) > -\frac{1}{2}$, *$z \in U$, and $h(0) = 1$. When the fuzzy differential subordination*

$$F_{\left(RI_{m,\lambda,l}^{\alpha} f\right)'(U)} \left(RI_{m,\lambda,l}^{\alpha} f(z)\right)' \leq F_{h(U)} h(z), \quad z \in U, \tag{2.8}$$

holds, where $\lambda, l \geq 0$, $m, n \in \mathbb{N}$, $f \in \mathcal{A}_n$, then $F_{RI_{m,\lambda,l}^{\alpha} f(U)} \frac{RI_{m,\lambda,l}^{\alpha} f(z)}{z} \leq F_{q(U)} q(z)$, $z \in U$. The function $q(z) = \frac{1}{n z^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n} - 1} dt$ is convex and it is the fuzzy best dominant.

Proof. Since Re $\left(1+\frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}$, $z \in U$, from Lemma 1.1, we deduce that $q(z) = \frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t)t^{\frac{1}{n}-1}dt$ is a convex function and verifies the differential equation associated to the fuzzy differential subordination (2.8) $q(z) + zq'(z) = h(z)$, therefore it is the fuzzy best dominant.

Keeping $p(z) = \frac{RI^{\alpha}_{m,\lambda,l}f(z)}{z}$, $p \in \mathcal{H}[1,n]$ and differentiating it, we obtain $\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)' = p(z) + zp'(z)$, $z \in U$ and (2.8) becomes $F_{p(U)}(p(z) + zp'(z)) \le F_{h(U)}h(z)$, $z \in U$.

By using Lemma 1.3, we deduce $F_{RI^{\alpha}_{m,\lambda,l}f(U)}\frac{RI^{\alpha}_{m,\lambda,l}f(z)}{z} \le F_{q(U)}q(z)$, $z \in U$. ■

Example 2.1 *Let* $h(z) = \frac{1-z}{1+z}$ *with* $h(0) = 1$, $h'(z) = \frac{-2}{(1+z)^2}$ *and* $h''(z) = \frac{4}{(1+z)^3}$.

Since $Re\left(\frac{zh''(z)}{h'(z)}+1\right) = Re\left(\frac{1-z}{1+z}\right) = Re\left(\frac{1-\rho\cos\theta - i\rho\sin\theta}{1+\rho\cos\theta+i\rho\sin\theta}\right) = \frac{1-\rho^2}{1+2\rho\cos\theta+\rho^2} > 0 > -\frac{1}{2}$, *the function* h *is convex in* U.

Let $f(z) = z - z^2$, $z \in U$. *For* $n = 1$, $m = 1$, $l = 1$, $\lambda = 2$, $\alpha = \frac{2}{3}$, *we obtain* $RI^{\frac{2}{3}}_{1,2,1}f(z) = \frac{1}{3}R^1 f(z) + \frac{2}{3}I(1,2,1)f(z) = \frac{1}{3}zf'(z) + \frac{2}{3}zf'(z) = z - 2z^2$. *Then* $\left(RI^{\frac{2}{3}}_{1,2,1}f(z)\right)' = 1 - 4z$ *and* $\frac{RI^{\frac{2}{3}}_{1,2,1}f(z)}{z} = 1 - 2z$. *Function* $q(z) = \frac{1}{z}\int_0^z \frac{1-t}{1+t}dt = -1 + \frac{2\ln(1+z)}{z}$.

From Theorem 2.5 we have $F_U(1-4z) \le F_U\left(\frac{1-z}{1+z}\right)$, $z \in U$, *imply* $F_U(1-2z) \le F_U\left(-1+\frac{2\ln(1+z)}{z}\right)$, $z \in U$.

Theorem 2.6 *Let* g *be a convex function with* $g(0) = 1$ *and* $h(z) = g(z) + zg'(z)$, $z \in U$. *When the fuzzy differential subordination*

$$F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(\frac{zRI^{\alpha}_{m+1,\lambda,l}f(z)}{RI^{\alpha}_{m,\lambda,l}f(z)}\right)' \le F_{h(U)}h(z), \quad z \in U \tag{2.9}$$

holds, for $\lambda, l \ge 0$, $m, n \in \mathbb{N}$, $f \in \mathcal{A}_n$, *then* $F_{RI^{\alpha}_{m,\lambda,l}f(U)}\frac{RI^{\alpha}_{m+1,\lambda,l}f(z)}{RI^{\alpha}_{m,\lambda,l}f(z)} \le F_{g(U)}g(z)$, $z \in U$, *and this result is sharp.*

Proof. Diferentiating $p(z) = \frac{RI^{\alpha}_{m+1,\lambda,l}f(z)}{RI^{\alpha}_{m,\lambda,l}f(z)}$ we get $p'(z) = \frac{\left(RI^{\alpha}_{m+1,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m,\lambda,l}f(z)} - p(z)\cdot\frac{\left(RI^{\alpha}_{m+1,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m,\lambda,l}f(z)}$ and $p(z) + z\cdot p'(z) = \left(\frac{zRI^{\alpha}_{m+1,\lambda,l}f(z)}{RI^{\alpha}_{m,\lambda,l}f(z)}\right)'$.

In this conditions, relation (2.9) becomes $F_{p(U)}(p(z) + zp'(z)) \le F_{h(U)}h(z) = F_{g(U)}(g(z) + zg'(z))$, $z \in U$, and aplying Lemma 1.3, we obtain $F_{RI^{\alpha}_{m,\lambda,l}f(U)}\frac{RI^{\alpha}_{m+1,\lambda,l}f(z)}{RI^{\alpha}_{m,\lambda,l}f(z)} \le F_{g(U)}g(z)$, $z \in U$. ■

Theorem 2.7 *For a convex function* g, *with* $g(0) = 1$, *consider* $h(z) = g(z) + \frac{z}{\delta}g'(z)$, $z \in U$. *The fuzzy differential subordination*

$$F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(\left(\frac{RI^{\alpha}_{m,\lambda,l}f(z)}{z}\right)^{\delta-1}\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'\right) \le F_{h(U)}h(z), \quad z \in U, \tag{2.10}$$

is satisfied, when $\alpha, \lambda, l, \delta \ge 0$, $m, n \in \mathbb{N}$, $f \in \mathcal{A}_n$, *and implies the following fuzzy sharp differential subordination* $F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(\frac{RI^{\alpha}_{m,\lambda,l}f(z)}{z}\right)^{\delta} \le F_{g(U)}g(z)$, $z \in U$.

Proof. Differentiating $p(z) = \left(\frac{RI^{\alpha}_{m,\lambda,l}f(z)}{z}\right)^{\delta}$, $z \in U$, we get $\left(\frac{RI^{\alpha}_{m,\lambda,l}f(z)}{z}\right)^{\delta-1}\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)' = p(z) + \frac{1}{\delta}zp'(z)$, $z \in U$. Evidently $p \in \mathcal{H}[1,n]$, and (2.10) becomes $F_{p(U)}\left(p(z) + \frac{1}{\delta}zp'(z)\right) \le F_{h(U)}h(z) = F_{g(U)}\left(g(z) + \frac{z}{\delta}g'(z)\right)$, $z \in U$.

By using Lemma 1.3, we have $F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(\frac{RI^{\alpha}_{m,\lambda,l}f(z)}{z}\right)^{\delta} \le F_{g(U)}g(z)$, $z \in U$. ■

Theorem 2.8 *Consider an holomorphic function* h *which satisfies the inequality* Re $\left(1+\frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}$, $z \in U$, *and* $h(0) = 1$. *When the fuzzy differential subordination*

$$F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(\left(\frac{RI^{\alpha}_{m,\lambda,l}f(z)}{z}\right)^{\delta-1}\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'\right) \le F_{h(U)}h(z), \quad z \in U, \tag{2.11}$$

holds, with $\alpha, \lambda, l, \delta \ge 0$, $m, n \in \mathbb{N}$, $f \in \mathcal{A}_n$, *then* $F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(\frac{RI^{\alpha}_{m,\lambda,l}f(z)}{z}\right)^{\delta} \le F_{q(U)}q(z)$, $z \in U$, *and* $q(z) = \frac{\delta}{nz^{\frac{\delta}{n}}}\int_0^z h(t)t^{\frac{\delta}{n}-1}dt$.

Proof. Since Re $\left(1+\frac{zh''(z)}{h'(z)}\right)>-\frac{1}{2}$, $z\in U$, from Lemma 1.1, we deduce that $q(z)=\frac{\delta}{nz^{\frac{\delta}{n}}}\int_0^z h(t)t^{\frac{\delta}{n}-1}dt$ is a convex function and verifies the differential equation asscociated to the fuzzy differential subordination (2.11) $q(z)+\frac{1}{\delta}zq'(z)=h(z)$, therefore it is the fuzzy best dominant.

Consider $p(z)=\left(\frac{RI^{\alpha}_{m,\lambda,l}f(z)}{z}\right)^{\delta}$, $z\in U$, then $p\in\mathcal{H}[1,n]$. Differentiating, we obtain $\left(\frac{RI^{\alpha}_{m,\lambda,l}f(z)}{z}\right)^{\delta-1}\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'=p(z)+\frac{1}{\delta}zp'(z)$, $z\in U$, and the fuzzy differential subordination (2.11) becomes $F_{p(U)}\left(p(z)+\frac{1}{\delta}zp'(z)\right)\le F_{h(U)}h(z)$, $z\in U$. Aplying Lemma 1.2, we get $F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(\frac{RI^{\alpha}_{m,\lambda,l}f(z)}{z}\right)^{\delta}\le F_{q(U)}q(z)$ and $q(z)=\frac{\delta}{nz^{\frac{\delta}{n}}}\int_0^z h(t)t^{\frac{\delta}{n}-1}dt$, $z\in U$, is the best dominant. ■

Theorem 2.9 *For a convex function g, with $g(0)=1$, define the function $h(z)=g(z)+\frac{z}{\delta}g'(z)$, $z\in U$. If $\alpha,\lambda,l,\delta\ge 0$, $m,n\in\mathbb{N}$, $f\in\mathcal{A}_n$ and the fuzzy differential subordination*

$$F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(\frac{z^2}{\delta}\frac{RI^{\alpha}_{m,\lambda,l}f(z)}{\left(RI^{\alpha}_{m+1,\lambda,l}f(z)\right)^2}\left[\frac{\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m,\lambda,l}f(z)}-2\frac{\left(RI^{\alpha}_{m+1,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m+1,\lambda,l}f(z)}\right]\right.$$
$$\left.+z\frac{\delta+1}{\delta}\frac{RI^{\alpha}_{m,\lambda,l}f(z)}{\left(RI^{\alpha}_{m+1,\lambda,l}f(z)\right)^2}\right)\le F_{h(U)}h(z),\ z\in U, \tag{2.12}$$

holds, then $F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(z\frac{RI^{\alpha}_{m,\lambda,l}f(z)}{\left(RI^{\alpha}_{m+1,\lambda,l}f(z)\right)^2}\right)\le F_{g(U)}g(z)$, $z\in U$, and this result is sharp.

Proof. Differentiating $p(z)=z\frac{RI^{\alpha}_{m,\lambda,l}f(z)}{\left(RI^{\alpha}_{m+1,\lambda,l}f(z)\right)^2}$, we get
$p(z)+\frac{z}{\delta}p'(z)=z\frac{\delta+1}{\delta}\frac{RI^{\alpha}_{m,\lambda,l}f(z)}{\left(RI^{\alpha}_{m+1,\lambda,l}f(z)\right)^2}+\frac{z^2}{\delta}\frac{RI^{\alpha}_{m,\lambda,l}f(z)}{\left(RI^{\alpha}_{m+1,\lambda,l}f(z)\right)^2}\left[\frac{\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m,\lambda,l}f(z)}-2\frac{\left(RI^{\alpha}_{m+1,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m+1,\lambda,l}f(z)}\right]$.

The fuzzy differential subordination (2.12) becomes
$F_{p(U)}\left(p(z)+\frac{z}{\delta}p'(z)\right)\le F_{h(U)}h(z)F_{g(U)}\left(g(z)+\frac{z}{\delta}g'(z)\right)$, $z\in U$, and by using Lemma 1.3, we derive $F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(z\frac{RI^{\alpha}_{m,\lambda,l}f(z)}{\left(RI^{\alpha}_{m+1,\lambda,l}f(z)\right)^2}\right)\le F_{g(U)}g(z)$, $z\in U$. ■

Theorem 2.10 *Let h be a holomorphic function with properties* Re $\left(1+\frac{zh''(z)}{h'(z)}\right)>-\frac{1}{2}$, *$z\in U$, and $h(0)=1$. The fuzzy differential subordination*

$$F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(\frac{z^2}{\delta}\frac{RI^{\alpha}_{m,\lambda,l}f(z)}{\left(RI^{\alpha}_{m+1,\lambda,l}f(z)\right)^2}\left[\frac{\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m,\lambda,l}f(z)}-2\frac{\left(RI^{\alpha}_{m+1,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m+1,\lambda,l}f(z)}\right]\right.$$
$$\left.+z\frac{\delta+1}{\delta}\frac{RI^{\alpha}_{m,\lambda,l}f(z)}{\left(RI^{\alpha}_{m+1,\lambda,l}f(z)\right)^2}\right)\le F_{h(U)}h(z),\quad z\in U, \tag{2.13}$$

implies the following fuzzy differential subordination $F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(z\frac{RI^{\alpha}_{m,\lambda,l}f(z)}{\left(RI^{\alpha}_{m+1,\lambda,l}f(z)\right)^2}\right)\le F_{q(U)}q(z)$, $z\in U$, where $\alpha,\lambda,l,\delta\ge 0$, $m,n\in\mathbb{N}$, $f\in\mathcal{A}_n$, and $q(z)=\frac{\delta}{nz^{\frac{\delta}{n}}}\int_0^z h(t)t^{\frac{\delta}{n}-1}dt$.

Proof. Since Re $\left(1+\frac{zh''(z)}{h'(z)}\right)>-\frac{1}{2}$, $z\in U$, from Lemma 1.1, we deduce that $q(z)=\frac{\delta}{nz^{\frac{\delta}{n}}}\int_0^z h(t)t^{\frac{\delta}{n}-1}dt$ is a convex function and verifies the differential equation associated to the fuzzy differential subordination (2.13) $q(z)+\frac{1}{\delta}zq'(z)=h(z)$, therefore it is the fuzzy best dominant.

Considering $p(z)=z\frac{RI^{\alpha}_{m,\lambda,l}f(z)}{\left(RI^{\alpha}_{m+1,\lambda,l}f(z)\right)^2}$, $z\in U$, $p\in\mathcal{H}[1,n]$. Since
$p(z)+\frac{z}{\delta}p'(z)=z\frac{\delta+1}{\delta}\frac{RI^{\alpha}_{m,\lambda,l}f(z)}{\left(RI^{\alpha}_{m+1,\lambda,l}f(z)\right)^2}+\frac{z^2}{\delta}\frac{RI^{\alpha}_{m,\lambda,l}f(z)}{\left(RI^{\alpha}_{m+1,\lambda,l}f(z)\right)^2}\left[\frac{\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m,\lambda,l}f(z)}-2\frac{\left(RI^{\alpha}_{m+1,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m+1,\lambda,l}f(z)}\right]$, $z\in U$, from (2.13) we have $F_{p(U)}\left(p(z)+\frac{z}{\delta}p'(z)\right)\le F_{h(U)}h(z)$, $z\in U$, and from Lemma 1.2, we obtain $F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(z\frac{RI^{\alpha}_{m,\lambda,l}f(z)}{\left(RI^{\alpha}_{m+1,\lambda,l}f(z)\right)^2}\right)\le F_{q(U)}q(z)$, $z\in U$, and the best dominant is $q(z)=\frac{\delta}{nz^{\frac{\delta}{n}}}\int_0^z h(t)t^{\frac{\delta}{n}-1}dt$. ■

Theorem 2.11 *For a convex function g with $g(0)=0$ define $h(z)=g(z)+\frac{z}{\delta}g'(z)$, $z\in U$. If $\alpha,\lambda,l,\delta\geq 0$, $m,n\in\mathbb{N}$, $f\in\mathcal{A}_n$ and the fuzzy differential subordination*

$$F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(z^2\frac{\delta+2}{\delta}\frac{\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m,\lambda,l}f(z)}+\right.$$

$$\left.\frac{z^3}{\delta}\left[\frac{\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)''}{RI^{\alpha}_{m,\lambda,l}f(z)}-\left(\frac{\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m,\lambda,l}f(z)}\right)^2\right]\right)\leq F_{h(U)}h(z),\ z\in U, \tag{2.14}$$

holds, then the following result is sharp $F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(z^2\frac{\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m,\lambda,l}f(z)}\right)\leq F_{g(U)}g(z)$, $z\in U$.

Proof. Considering $p(z)=z^2\frac{\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m,\lambda,l}f(z)}$, we deduce that $p\in\mathcal{H}[0,1]$ and differentiating it, we obtain $p(z)+\frac{z}{\delta}p'(z)=z^2\frac{\delta+2}{\delta}\frac{\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m,\lambda,l}f(z)}+\frac{z^3}{\delta}\left[\frac{\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)''}{RI^{\alpha}_{m,\lambda,l}f(z)}-\left(\frac{\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m,\lambda,l}f(z)}\right)^2\right]$, $z\in U$.

The fuzzy differential subordination becomes $F_{p(U)}\left(p(z)+\frac{1}{\delta}zp'(z)\right)\leq F_{h(U)}h(z)=F_{g(U)}\left(g(z)+\frac{z}{\delta}g'(z)\right)$ and by using Lemma 1.3, we deduce $F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(z^2\frac{\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m,\lambda,l}f(z)}\right)\leq F_{g(U)}g(z)$, $z\in U$, and this result is sharp. ■

Theorem 2.12 *Consider h a holomorphic function with* Re $\left(1+\frac{zh''(z)}{h'(z)}\right)>-\frac{1}{2}$, *$z\in U$, and $h(0)=0$. If the fuzzy differential subordination*

$$F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(z^2\frac{\delta+2}{\delta}\frac{\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m,\lambda,l}f(z)}+\right.$$

$$\left.\frac{z^3}{\delta}\left[\frac{\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)''}{RI^{\alpha}_{m,\lambda,l}f(z)}-\left(\frac{\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m,\lambda,l}f(z)}\right)^2\right]\right)\leq F_{h(U)}h(z),\ z\in U, \tag{2.15}$$

holds, for $\alpha,\lambda,l,\delta\geq 0$, $m,n\in\mathbb{N}$, $f\in\mathcal{A}_n$, then $F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(z^2\frac{\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m,\lambda,l}f(z)}\right)\leq F_{q(U)}q(z)$, $z\in U$, where $q(z)=\frac{\delta}{nz^{\frac{\delta}{n}}}\int_0^z h(t)t^{\frac{\delta}{n}-1}dt$.

Proof. Since Re $\left(1+\frac{zh''(z)}{h'(z)}\right)>-\frac{1}{2}$, $z\in U$, from Lemma 1.1, we deduce that $q(z)=\frac{\delta}{nz^{\frac{\delta}{n}}}\int_0^z h(t)t^{\frac{\delta}{n}-1}dt$ is a convex function and verifies the differential equation asscociated to the fuzzy differential subordination (2.15) $q(z)+\frac{1}{\delta}zq'(z)=h(z)$, therefore it is the fuzzy best dominant.

Considering $p(z)=z^2\frac{\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m,\lambda,l}f(z)}$, $z\in U$, $p\in\mathcal{H}[0,n]$. Differentiating it, we obtain $p(z)+\frac{z}{\delta}p'(z)=z^2\frac{\delta+2}{\delta}\frac{\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m,\lambda,l}f(z)}+\frac{z^3}{\delta}\left[\frac{\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)''}{RI^{\alpha}_{m,\lambda,l}f(z)}-\left(\frac{\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m,\lambda,l}f(z)}\right)^2\right]$, $z\in U$, and (2.15) becomes $F_{p(U)}\left(p(z)+\frac{1}{\delta}zp'(z)\right)\leq F_{h(U)}h(z)$, $z\in U$.

Applying Lemma 1.2, we get $F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(z^2\frac{\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m,\lambda,l}f(z)}\right)\leq F_{q(U)}q(z)$, $z\in U$, and the best dominant is $q(z)=\frac{\delta}{nz^{\frac{\delta}{n}}}\int_0^z h(t)t^{\frac{\delta}{n}-1}dt$. ■

Theorem 2.13 *Let $h(z)=g(z)+zg'(z)$, $z\in U$, where g is a convex function such that $g(0)=1$. When the fuzzy differential subordination*

$$F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(1-\frac{RI^{\alpha}_{m,\lambda,l}f(z)\cdot\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)''}{\left[\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'\right]^2}\right)\leq F_{h(U)}h(z),\quad z\in U \tag{2.16}$$

holds, for $\alpha,\lambda,l\geq 0$, $m,n\in\mathbb{N}$, $f\in\mathcal{A}_n$, then the followinf result is sharp $F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(\frac{RI^{\alpha}_{m,\lambda,l}f(z)}{z\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}\right)\leq F_{g(U)}g(z)$, $z\in U$.

Proof. Let $p(z)=\frac{RI^{\alpha}_{m,\lambda,l}f(z)}{z\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}$. We deduce that $p\in\mathcal{H}[1,n]$ and differentiating it, we get $1-\frac{RI^{\alpha}_{m,\lambda,l}f(z)\cdot\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)''}{\left[\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'\right]^2}=p(z)+zp'(z)$, $z\in U$.

The fuzzy differential subordination becomes $F_{p(U)}\left(p(z)+zp'(z)\right)\leq F_{h(U)}h(z)=F_{g(U)}\left(g(z)+zg'(z)\right)$ and we apply Lemma 1.3 to deduce the following sharp result $_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(\frac{RI^{\alpha}_{m,\lambda,l}f(z)}{z\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}\right)\leq F_{g(U)}g(z)$, $z\in U$. ■

Theorem 2.14 *Let h be a holomorphic function with $h(0)=1$ and* Re $\left(1+\frac{zh''(z)}{h'(z)}\right)>-\frac{1}{2}$, $z\in U$. *The fuzzy differential subordination*

$$F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(1-\frac{RI^{\alpha}_{m,\lambda,l}f(z)\cdot\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)''}{\left[\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'\right]^2}\right)\leq F_{h(U)}h(z),\quad z\in U, \tag{2.17}$$

induce $F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(\frac{RI^{\alpha}_{m,\lambda,l}f(z)}{z\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}\right)\leq F_{q(U)}q(z)$, $z\in U$, *where* $q(z)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t)t^{\frac{1}{n}-1}dt$, *for* $\alpha,\lambda,l\geq 0$, $m,n\in\mathbb{N}$, $f\in\mathcal{A}_n$.

Proof. Since Re $\left(1+\frac{zh''(z)}{h'(z)}\right)>-\frac{1}{2}$, $z\in U$, from Lemma 1.1, we deduce that $\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t)t^{\frac{1}{n}-1}dt$ is a convex function and verifies the differential equation asscociated to the fuzzy differential subordination (2.17) $q(z)+zq'(z)=h(z)$, therefore it is the fuzzy best dominant.

Consider $p(z)=\frac{RI^{\alpha}_{m,\lambda,l}f(z)}{z\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}$, $z\in U$, $p\in\mathcal{H}[1,n]$. Differentiating it, we obtain $1-\frac{RI^{\alpha}_{m,\lambda,l}f(z)\cdot\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)''}{\left[\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'\right]^2}=p(z)+zp'(z)$, $z\in U$, and the fuzzy differential subordination (2.17) is written $F_{p(U)}\left(p(z)+zp'(z)\right)\leq F_{h(U)}h(z)$, $z\in U$.

By using Lemma 1.2, we get $F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(\frac{RI^{\alpha}_{m,\lambda,l}f(z)}{z\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}\right)\leq F_{q(U)}q(z)$, $z\in U$, and the best dominant is $q(z)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t)t^{\frac{1}{n}-1}dt$. ■

Example 2.2 *Let $h(z)=\frac{1-z}{1+z}$ a convex function in U with $h(0)=1$ and Re* $\left(\frac{zh''(z)}{h'(z)}+1\right)>-\frac{1}{2}$.

Let $f(z)=z-z^2$, $z\in U$. For $n=1$, $m=1$, $l=1,\lambda=2$, $\alpha=\frac{2}{3}$, we obtain $RI^{\frac{2}{3}}_{1,2,1}f(z)=z-2z^2$, $z\in U$. Then $\left(RI^{\frac{2}{3}}_{1,2,1}f(z)\right)'=1-4z$ *and* $\left(RI^{\frac{2}{3}}_{1,2,1}f(z)\right)''=-4$, $\frac{RI^{\frac{2}{3}}_{1,2,1}f(z)}{z\left(RI^{\frac{2}{3}}_{1,2,1}f(z)\right)'}=\frac{z-2z^2}{z(1-4z)}=\frac{1-2z}{1-4z}$,

$1-\frac{RI^{\frac{2}{3}}_{1,2,1}f(z)\cdot\left(RI^{\frac{2}{3}}_{1,2,1}f(z)\right)''}{\left[\left(RI^{\frac{2}{3}}_{1,2,1}f(z)\right)'\right]^2}=1-\frac{\left(z-2z^2\right)\cdot(-4)}{(1-4z)^2}=\frac{8z^2-4z+1}{(1-4z)^2}$. *We have* $q(z)=\frac{1}{z}\int_0^z\frac{1-t}{1+t}dt=-1+\frac{2\ln(1+z)}{z}$.

Using Theorem 2.14 we obtain $F_U\left(\frac{8z^2-4z+1}{(1-4z)^2}\right)\leq F_U\left(\frac{1-z}{1+z}\right)$, $z\in U$, *induce* $F_U\left(\frac{1-2z}{1-4z}\right)\leq F_U\left(-1+\frac{2\ln(1+z)}{z}\right)$, $z\in U$.

Theorem 2.15 *Consider $h(z)=g(z)+zg'(z)$, $z\in U$, where g is a convex function with $g(0)=1$. When the fuzzy differential subordination*

$$F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(\left[\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'\right]^2+RI^{\alpha}_{m,\lambda,l}f(z)\cdot\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)''\right)\leq F_{h(U)}h(z),\quad z\in U \tag{2.18}$$

holds, for $\alpha,\lambda,l\geq 0$, $m,n\in\mathbb{N}$, $f\in\mathcal{A}_n$, then the following result is sharp $F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(\frac{RI^{\alpha}_{m,\lambda,l}f(z)\cdot\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{z}\right)\leq F_{g(U)}g(z)$, $z\in U$.

Proof. Let $p(z)=\frac{RI^{\alpha}_{m,\lambda,l}f(z)\cdot\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{z}$. We deduce that $p\in\mathcal{H}[1,n]$.

Differentiating it, we obtain $\left[\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'\right]^2+RI^{\alpha}_{m,\lambda,l}f(z)\cdot\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)''=p(z)+zp'(z)$, $z\in U$, and the fuzzy differential subordination becomes $F_{p(U)}\left(p(z)+zp'(z)\right)\leq F_{h(U)}h(z)=F_{g(U)}\left(g(z)+zg'(z)\right)$.

By using Lemma 1.3, we obtain the following result sharp $F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(\frac{RI^{\alpha}_{m,\lambda,l}f(z)\cdot\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{z}\right)\leq F_{g(U)}g(z)$, $z\in U$. ■

Theorem 2.16 *For a holomorphic function h which satisfies the inequality* $\operatorname{Re}\left(1+\frac{zh''(z)}{h'(z)}\right)>-\frac{1}{2}$, $z\in U$, *and* $h(0)=1$, *the fuzzy differential subordination*

$$F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(\left[\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'\right]^2+RI^{\alpha}_{m,\lambda,l}f(z)\cdot\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)''\right)\leq F_{h(U)}h(z),\ z\in U, \tag{2.19}$$

induce $F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(\frac{RI^{\alpha}_{m,\lambda,l}f(z)\cdot\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{z}\right)\leq F_{q(U)}q(z)$, $z\in U$, *where* $q(z)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t)t^{\frac{1}{n}-1}dt$, *and* $\alpha,\lambda,l\geq 0$, $m,n\in\mathbb{N}$, $f\in\mathcal{A}_n$

Proof. Since $\operatorname{Re}\left(1+\frac{zh''(z)}{h'(z)}\right)>-\frac{1}{2}$, $z\in U$, from Lemma 1.1, we deduce that $q(z)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t)t^{\frac{1}{n}-1}dt$ is a convex function and verifies the differential equation asscociated to the fuzzy differential subordination (2.19) $q(z)+zq'(z)=h(z)$, therefore it is the fuzzy best dominant.

Let $p(z)=\frac{RI^{\alpha}_{m,\lambda,l}f(z)\cdot\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{z}$, $z\in U$, evidently $p\in\mathcal{H}[1,n]$.

We have $p(z)+zp'(z)=\left[\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'\right]^2+RI^{\alpha}_{m,\lambda,l}f(z)\cdot\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)''$, $z\in U$, and (2.19) means $F_{p(U)}(p(z)+zp'(z))\leq F_{h(U)}h(z)$, $z\in U$.

Applying Lemma 1.2, we get $F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(\frac{RI^{\alpha}_{m,\lambda,l}f(z)\cdot\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{z}\right)\leq F_{q(U)}q(z)$, $z\in U$, and the best dominant is $q(z)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t)t^{\frac{1}{n}-1}dt$. ■

Example 2.3 *Let* $h(z)=\frac{1-z}{1+z}$ *a convex function in U with* $h(0)=1$ *and* $Re\left(\frac{zh''(z)}{h'(z)}+1\right)>-\frac{1}{2}$.

Consider $f(z)=z-z^2$, $z\in U$. *For* $n=1$, $m=1$, $l=1$, $\lambda=2$, $\alpha=\frac{2}{3}$, *we obtain* $RI^{\frac{2}{3}}_{1,2,1}f(z)=z-2z^2$, $z\in U$.

Then $\left(RI^{\frac{2}{3}}_{1,2,1}f(z)\right)'=1-4z$, $\left(RI^{\frac{2}{3}}_{1,2,1}f(z)\right)''=-4$, $\frac{RI^{\frac{2}{3}}_{1,2,1}f(z)\cdot\left(RI^{\frac{2}{3}}_{1,2,1}f(z)\right)'}{z}=\frac{(z-2z^2)(1-4z)}{z}=8z^2-6z+1$, $\left[\left(RI^{\frac{2}{3}}_{1,2,1}f(z)\right)'\right]^2+RI^{\frac{2}{3}}_{1,2,1}f(z)\cdot\left(RI^{\frac{2}{3}}_{1,2,1}f(z)\right)''=(1-4z)^2+(z-2z^2)\cdot(-4)=24z^2-12z+1$.

We have $q(z)=\frac{1}{z}\int_0^z\frac{1-t}{1+t}dt=-1+\frac{2\ln(1+z)}{z}$.

From Theorem 2.16 we obtain $F_U\left(24z^2-12z+1\right)\leq F_U\left(\frac{1-z}{1+z}\right)$, $z\in U$, *induce* $F_U\left(8z^2-6z+1\right)\leq F_U\left(-1+\frac{2\ln(1+z)}{z}\right)$, $z\in U$.

Theorem 2.17 *Consider* $h(z)=g(z)+\frac{z}{1-\delta}g'(z)$, $z\in U$, *where g is a convex function with* $g(0)=1$. *If the fuzzy differential subordination*

$$F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(\left(\frac{z}{RI^{\alpha}_{m,\lambda,l}f(z)}\right)^{\delta}\frac{RI^{\alpha}_{m+1,\lambda,l}f(z)}{1-\delta}\left(\frac{\left(RI^{\alpha}_{m+1,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m+1,\lambda,l}f(z)}-\delta\frac{\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m,\lambda,l}f(z)}\right)\right)\leq F_{h(U)}h(z), \tag{2.20}$$

$z\in U$, *holds, for* $\alpha,\lambda,l\geq 0$, $\delta\in(0,1)$, $m,n\in\mathbb{N}$, $f\in\mathcal{A}_n$, *then the following result is sharp* $F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(\frac{RI^{\alpha}_{m+1,\lambda,l}f(z)}{z}\cdot\left(\frac{z}{RI^{\alpha}_{m,\lambda,l}f(z)}\right)^{\delta}\right)\leq F_{g(U)}g(z)$, $z\in U$.

Proof. Let $p(z)=\frac{RI^{\alpha}_{m+1,\lambda,l}f(z)}{z}\cdot\left(\frac{z}{RI^{\alpha}_{m,\lambda,l}f(z)}\right)^{\delta}$. We deduce that $p\in\mathcal{H}[1,n]$ and differentiating the function p, we obtain $\left(\frac{z}{RI^{\alpha}_{m,\lambda,l}f(z)}\right)^{\delta}\frac{RI^{\alpha}_{m+1,\lambda,l}f(z)}{1-\delta}\left(\frac{\left(RI^{\alpha}_{m+1,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m+1,\lambda,l}f(z)}-\delta\frac{\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m,\lambda,l}f(z)}\right)=p(z)+\frac{1}{1-\delta}zp'(z)$, $z\in U$.

The fuzzy differential subordination means $F_{p(U)}\left(p(z)+\frac{1}{1-\delta}zp'(z)\right)\leq F_{h(U)}h(z)=F_{g(U)}\left(g(z)+\frac{z}{1-\delta}g'(z)\right)$ and by using Lemma 1.3, we obtain the following sharp result $F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(\frac{RI^{\alpha}_{m+1,\lambda,l}f(z)}{z}\cdot\left(\frac{z}{RI^{\alpha}_{m,\lambda,l}f(z)}\right)^{\delta}\right)\leq F_{g(U)}g(z)$, $z\in U$. ■

Theorem 2.18 *Let h be an holomorphic function with* $h(0)=1$ *and* $\operatorname{Re}\left(1+\frac{zh''(z)}{h'(z)}\right)>-\frac{1}{2}$, $z\in U$. *The fuzzy differential subordination*

$$F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(\left(\frac{z}{RI^{\alpha}_{m,\lambda,l}f(z)}\right)^{\delta}\frac{RI^{\alpha}_{m+1,\lambda,l}f(z)}{1-\delta}\left(\frac{\left(RI^{\alpha}_{m+1,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m+1,\lambda,l}f(z)}-\delta\frac{\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m,\lambda,l}f(z)}\right)\right)\leq F_{h(U)}h(z), \tag{2.21}$$

$z \in U$, *induce* $F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(\frac{RI^{\alpha}_{m+1,\lambda,l}f(z)}{z}\cdot\left(\frac{z}{RI^{\alpha}_{m,\lambda,l}f(z)}\right)^{\delta}\right)\leq F_{q(U)}q(z)$, $z\in U$, *where* $q(z)=\frac{1-\delta}{nz^{\frac{1-\delta}{n}}}\int_0^z h(t)t^{\frac{1-\delta}{n}-1}dt$, *for* $\alpha,\lambda,l\geq 0$, $\delta\in(0,1)$, $m,n\in\mathbb{N}$, $f\in\mathcal{A}_n$.

Proof. Since Re $\left(1+\frac{zh''(z)}{h'(z)}\right)>-\frac{1}{2}$, $z\in U$, from Lemma 1.1, we deduce that $q(z)=\frac{1-\delta}{nz^{\frac{1-\delta}{n}}}\int_0^z h(t)t^{\frac{1-\delta}{n}-1}dtt$ is a convex function and verifies the differential equation asscociated to the fuzzy differential subordination (2.21) $q(z)+\frac{1}{1-\delta}zq'(z)=h(z)$, therefore it is the fuzzy best dominant.

Consider $p(z)=\frac{RI^{\alpha}_{m+1,\lambda,l}f(z)}{z}\cdot\left(\frac{z}{RI^{\alpha}_{m,\lambda,l}f(z)}\right)^{\delta}$, $z\in U$, $p\in\mathcal{H}[1,n]$.

Differentiating it, we get $\left(\frac{z}{RI^{\alpha}_{m,\lambda,l}f(z)}\right)^{\delta}\frac{RI^{\alpha}_{m+1,\lambda,l}f(z)}{1-\delta}\left(\frac{\left(RI^{\alpha}_{m+1,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m+1,\lambda,l}f(z)}-\delta\frac{\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m,\lambda,l}f(z)}\right)=p(z)+\frac{1}{1-\delta}zp'(z)$, $z\in U$, and (2.21) becomes $F_{p(U)}\left(p(z)+\frac{1}{1-\delta}zp'(z)\right)\leq F_{h(U)}h(z)$, $z\in U$.

From Lemma 1.2, we get $F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(\frac{RI^{\alpha}_{m+1,\lambda,l}f(z)}{z}\cdot\left(\frac{z}{RI^{\alpha}_{m,\lambda,l}f(z)}\right)^{\delta}\right)\leq F_{q(U)}q(z)$, $z\in U$, and the best dominant is $q(z)=\frac{1-\delta}{nz^{\frac{1-\delta}{n}}}\int_0^z h(t)t^{\frac{1-\delta}{n}-1}dt$. ■

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About some differential sandwich theorems involving a multiplier transformation and Ruscheweyh derivative

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Abstract

In this work we study a new operator $IR_{\lambda,l}^{m;n}$ defined as the Hadamard product of the multiplier transformation $I(m,\lambda,l)$ and Ruscheweyh derivative R^n, given by $IR_{\lambda,l}^{m;n}:\mathcal{A}\to\mathcal{A}$, $IR_{\lambda,l}^{m;n}f(z)=(I(m,\lambda,l)*R^n)f(z)$ and $\mathcal{A}_n=\{f\in\mathcal{H}(U):f(z)=z+a_{n+1}z^{n+1}+...,\ z\in U\}$ is the class of normalized analytic functions with $\mathcal{A}_1=\mathcal{A}$. The purpose of this paper is to derive certain subordination and superordination results involving the operator $IR_{\lambda.l}^{m;n}$ and we establish differential sandwich-type theorems.

Keywords: analytic functions, differential operator, differential subordination, differential superordination.
2010 Mathematical Subject Classification: 30C45.

1 Introduction

Let $\mathcal{H}(U)$ be the class of analytic function in the open unit disc of the complex plane $U=\{z\in\mathbb{C}:|z|<1\}$. Let $\mathcal{H}(a,n)$ be the subclass of $\mathcal{H}(U)$ consisting of functions of the form $f(z)=a+a_nz^n+a_{n+1}z^{n+1}+\ldots$.

Let $\mathcal{A}_n=\{f\in\mathcal{H}(U):f(z)=z+a_{n+1}z^{n+1}+\ldots,\ z\in U\}$ and $\mathcal{A}=\mathcal{A}_1$.

Let the functions f and g be analytic in U. We say that the function f is subordinate to g, written $f\prec g$, if there exists a Schwarz function w, analytic in U, with $w(0)=0$ and $|w(z)|<1$, for all $z\in U$, such that $f(z)=g(w(z))$, for all $z\in U$. In particular, if the function g is univalent in U, the above subordination is equivalent to $f(0)=g(0)$ and $f(U)\subset g(U)$.

Let $\psi:\mathbb{C}^3\times U\to\mathbb{C}$ and h be an univalent function in U. If p is analytic in U and satisfies the second order differential subordination

$$\psi(p(z),zp'(z),z^2p''(z);z)\prec h(z),\quad \text{for } z\in U, \tag{1.1}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant, if $p\prec q$ for all p satisfying (1.1). A dominant $\widetilde{q}$ that satisfies $\widetilde{q}\prec q$ for all dominants q of (1.1) is said to be the best dominant of (1.1). The best dominant is unique up to a rotation of U.

Let $\psi:\mathbb{C}^2\times U\to\mathbb{C}$ and h analytic in U. If p and $\psi\left(p(z),zp'(z),z^2p''(z);z\right)$ are univalent and if p satisfies the second order differential superordination

$$h(z)\prec\psi(p(z),zp'(z),z^2p''(z);z),\qquad z\in U, \tag{1.2}$$

then p is a solution of the differential superordination (1.2) (if f is subordinate to F, then F is called to be superordinate to f). An analytic function q is called a subordinant if $q\prec p$ for all p satisfying (1.2). An univalent subordinant $\widetilde{q}$ that satisfies $q\prec\widetilde{q}$ for all subordinants q of (1.2) is said to be the best subordinant.

Miller and Mocanu [6] obtained conditions h, q and ψ for which the following implication holds $h(z)\prec\psi(p(z),zp'(z),z^2p''(z);z)\Rightarrow q(z)\prec p(z)$.

For two functions $f(z)=z+\sum_{j=2}^{\infty}a_jz^j$ and $g(z)=z+\sum_{j=2}^{\infty}b_jz^j$ analytic in the open unit disc U, the Hadamard product (or convolution) of $f(z)$ and $g(z)$, written as $(f*g)(z)$ is defined by $f(z)*g(z)=(f*g)(z)=z+\sum_{j=2}^{\infty}a_jb_jz^j$.

Definition 1.1 *[5] For $f\in\mathcal{A}$, $m\in\mathbb{N}\cup\{0\}$, $\lambda,l\geq0$, the multiplier transformation $I(m,\lambda,l)f(z)$ is defined by the following infinite series $I(m,\lambda,l)f(z):=z+\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{1+l}\right)^m a_jz^j$.*

Remark 1.1 *We have $(l+1)I(m+1,\lambda,l)f(z)=(l+1-\lambda)I(m,\lambda,l)f(z)+\lambda z(I(m,\lambda,l)f(z))'$, $z\in U$.*

Remark 1.2 *For $l = 0$, $\lambda \geq 0$, the operator $D_\lambda^m = I(m, \lambda, 0)$ was introduced and studied by Al-Oboudi , which reduced to the Sălăgean differential operator $S^m = I(m, 1, 0)$ for $\lambda = 1$.*

Definition 1.2 *(Ruscheweyh [8]) For $f \in \mathcal{A}$ and $n \in \mathbb{N}$, the Ruscheweyh derivative R^n is defined by $R^n : \mathcal{A} \to \mathcal{A}$,*

$$\begin{aligned} R^0 f(z) &= f(z), \quad R^1 f(z) = z f'(z), \quad ... \\ (n+1) R^{n+1} f(z) &= z (R^n f(z))' + n R^n f(z), \quad z \in U. \end{aligned}$$

Remark 1.3 *If $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $R^n f(z) = z + \sum_{j=2}^{\infty} \frac{(n+j-1)!}{n!(j-1)!} a_j z^j$ for $z \in U$.*

Definition 1.3 *([2]) Let $\lambda, l \geq 0$ and $n, m \in \mathbb{N}$. Denote by $IR_{\lambda,l}^{m,n} : \mathcal{A} \to \mathcal{A}$ the operator given by the Hadamard product of the multiplier transformation $I(m, \lambda, l)$ and the Ruscheweyh derivative R^n, $IR_{\lambda,l}^{m,n} f(z) = (I(m, \lambda, l) * R^n) f(z)$, for any $z \in U$ and each nonnegative integers m, n.*

Remark 1.4 *If $f \in \mathcal{A}$ and $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $IR_{\lambda,l}^{m,n} f(z) = z + \sum_{j=2}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m \frac{(n+j-1)!}{n!(j-1)!} a_j^2 z^j$, $z \in U$.*

Using simple computation one obtains the next result.

Proposition 1.1 *[1]For $m, n \in \mathbb{N}$ and $\lambda, l \geq 0$ we have*

$$(n+1) IR_{\lambda,l}^{m,n+1} f(z) - n IR_{\lambda,l}^{m,n} f(z) = z \left(IR_{\lambda,l}^{m,n} f(z)\right)'. \tag{1.3}$$

The purpose of this paper is to derive the several subordination and superordination results involving a differential operator. Furthermore, we studied the results of Selvaraj and Karthikeyan [10], Shanmugam, Ramachandran, Darus and Sivasubramanian [11] and Srivastava and Lashin [12].

In order to prove our subordination and superordination results, we make use of the following known results.

Definition 1.4 *[7] Denote by Q the set of all functions f that are analytic and injective on $\overline{U} \backslash E(f)$, where $E(f) = \{\zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty\}$, and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \backslash E(f)$.*

Lemma 1.1 *[7] Let the function q be univalent in the unit disc U and θ and ϕ be analytic in a domain D containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = z q'(z) \phi(q(z))$ and $h(z) = \theta(q(z)) + Q(z)$. Suppose that Q is starlike univalent in U and $Re\left(\frac{z h'(z)}{Q(z)}\right) > 0$ for $z \in U$.*

If p is analytic with $p(0) = q(0)$, $p(U) \subseteq D$ and $\theta(p(z)) + z p'(z) \phi(p(z)) \prec \theta(q(z)) + z q'(z) \phi(q(z))$, then $p(z) \prec q(z)$ and q is the best dominant.

Lemma 1.2 *[4] Let the function q be convex univalent in the open unit disc U and ν and ϕ be analytic in a domain D containing $q(U)$. Suppose that $Re\left(\frac{\nu'(q(z))}{\phi(q(z))}\right) > 0$ for $z \in U$ and $\psi(z) = z q'(z) \phi(q(z))$ is starlike univalent in U.*

If $p(z) \in \mathcal{H}[q(0), 1] \cap Q$, with $p(U) \subseteq D$ and $\nu(p(z)) + z p'(z) \phi(p(z))$ is univalent in U and $\nu(q(z)) + z q'(z) \phi(q(z)) \prec \nu(p(z)) + z p'(z) \phi(p(z))$, then $q(z) \prec p(z)$ and q is the best subordinant.

2 Main results

We begin with the following

Theorem 2.1 *Let $\frac{z\left(IR_{\lambda,l}^{m,n} f(z)\right)'}{IR_{\lambda,l}^{m,n} f(z)} \in \mathcal{H}(U)$ and let the function $q(z)$ be analytic and univalent in U such that $q(z) \neq 0$, for all $z \in U$. Suppose that $\frac{z q'(z)}{q(z)}$ is starlike univalent in U. Let*

$$Re\left(\frac{\xi}{\beta} q(z) + \frac{2\mu}{\beta} q^2(z) + 1 + z\frac{q''(z)}{q(z)} - z\frac{q'(z)}{q(z)}\right) > 0, \tag{2.1}$$

for $\alpha, \xi, \beta, \mu \in \mathbb{C}$, $\beta \neq 0$, $z \in U$ and

$$\psi_{\lambda,l}^{m,n}(\alpha, \xi, \mu, \beta; z) := \left(\alpha - \xi n + \mu n^2\right) + (n+1)(\xi - 2n\mu - \beta) \frac{IR_{\lambda,l}^{m,n+1} f(z)}{IR_{\lambda,l}^{m,n} f(z)} \tag{2.2}$$

$$+\mu (n+1)^2 \left(\frac{IR_{\lambda,l}^{m,n+1} f(z)}{IR_{\lambda,l}^{m,n} f(z)} \right)^2 + \beta (n+1) \frac{(n+2) IR_{\lambda,l}^{m,n+2} f(z) - (n+1) IR_{\lambda,l}^{m,n+1} f(z)}{(n+1) IR_{\lambda,l}^{m,n+1} f(z) - n IR_{\lambda,l}^{m,n} f(z)}.$$

If q satisfies the following subordination

$$\psi_{\lambda,l}^{m,n} (\alpha, \beta, \mu; z) \prec \alpha + \xi q(z) + \mu (q(z))^2 + \beta \frac{z q'(z)}{q(z)}, \tag{2.3}$$

for $\alpha, \xi, \beta, \mu \in \mathbb{C}$, $\beta \neq 0$, then

$$\frac{z \left(IR_{\lambda,l}^{m,n} f(z) \right)'}{IR_{\lambda,l}^{m,n} f(z)} \prec q(z), \tag{2.4}$$

and q is the best dominant.

Proof. Let the function p be defined by $p(z) := \frac{z\left(IR_{\lambda,l}^{m,n} f(z)\right)'}{IR_{\lambda,l}^{m,n} f(z)}$, $z \in U$, $z \neq 0$, $f \in \mathcal{A}$. We have $p'(z) = (n+1) \frac{\left(IR_{\lambda,l}^{m,n+1} f(z)\right)'}{IR_{\lambda,l}^{m,n} f(z)} - (n+1) \frac{IR_{\lambda,l}^{m,n+1} f(z)}{IR_{\lambda,l}^{m,n} f(z)} \cdot \frac{\left(IR_{\lambda,l}^{m,n} f(z)\right)'}{IR_{\lambda,l}^{m,n} f(z)}$.

By using the identity (1.3), we obtain

$$\frac{z p'(z)}{p(z)} = (n+1) \frac{(n+2) IR_{\lambda,l}^{m,n+2} f(z) - (n+1) IR_{\lambda,l}^{m,n+1} f(z)}{(n+1) IR_{\lambda,l}^{m,n+1} f(z) - n IR_{\lambda,l}^{m,n} f(z)} - (n+1) \frac{IR_{\lambda,l}^{m,n+1} f(z)}{IR_{\lambda,l}^{m,n} f(z)}. \tag{2.5}$$

By setting $\theta(w) := \alpha + \xi w + \mu w^2$ and $\phi(w) := \frac{\beta}{w}$, it can be easily verified that θ is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C} \backslash \{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C} \backslash \{0\}$.

Also, by letting $Q(z) = z q'(z) \phi(q(z)) = \beta \frac{z q'(z)}{q(z)}$ and $h(z) = \theta(q(z)) + Q(z) = \alpha + \xi q(z) + \mu (q(z))^2 + \beta \frac{z q'(z)}{q(z)}$, we find that $Q(z)$ is starlike univalent in U.

We have $h'(z) = \xi q'(z) + 2\mu q(z) q'(z) + \beta \frac{q'(z)}{q(z)} + \beta z \frac{q''(z)}{q(z)} - \beta z \left(\frac{q'(z)}{q(z)} \right)^2$ and $\frac{z h'(z)}{Q(z)} = \frac{\xi}{\beta} q(z) + \frac{2\mu}{\beta} q^2(z) + 1 + z \frac{q''(z)}{q(z)} - z \frac{q'(z)}{q(z)}$.

We deduce that $Re \left(\frac{z h'(z)}{Q(z)} \right) = Re \left(\frac{\xi}{\beta} q(z) + \frac{2\mu}{\beta} q^2(z) + 1 + z \frac{q''(z)}{q(z)} - z \frac{q'(z)}{q(z)} \right) > 0$.

By using (2.5), we obtain

$\alpha + \xi p(z) + \mu (p(z))^2 + \beta \frac{z p'(z)}{p(z)} = \left(\alpha - \xi n + \mu n^2 \right) + (n+1)(\xi - 2n\mu - \beta) \frac{IR_{\lambda,l}^{m,n+1} f(z)}{IR_{\lambda,l}^{m,n} f(z)} + \mu (n+1)^2 \left(\frac{IR_{\lambda,l}^{m,n+1} f(z)}{IR_{\lambda,l}^{m,n} f(z)} \right)^2 + \beta (n+1) \frac{(n+2) IR_{\lambda,l}^{m,n+2} f(z) - (n+1) IR_{\lambda,l}^{m,n+1} f(z)}{(n+1) IR_{\lambda,l}^{m,n+1} f(z) - n IR_{\lambda,l}^{m,n} f(z)}$.

By using (2.3), we have $\alpha + \xi p(z) + \mu (p(z))^2 + \beta \frac{z p'(z)}{p(z)} \prec \alpha + \xi q(z) + \mu (q(z))^2 + \beta \frac{z q'(z)}{q(z)}$.

By an application of Lemma 1.1, we have $p(z) \prec q(z)$, $z \in U$, i.e. $\frac{z\left(IR_{\lambda,l}^{m,n} f(z)\right)'}{IR_{\lambda,l}^{m,n} f(z)} \prec q(z)$, $z \in U$ and q is the best dominant. ■

Corollary 2.2 *Let $m, n \in \mathbb{N}$, $\lambda, l \geq 0$. Assume that (2.1) holds. If $f \in \mathcal{A}$ and $\psi_{\lambda,l}^{m,n} (\alpha, \beta, \mu; z) \prec \alpha + \xi \frac{1+Az}{1+Bz} + \mu \left(\frac{1+Az}{1+Bz} \right)^2 + \frac{\beta (A-B) z}{(1+Az)(1+Bz)}$, for $\alpha, \beta, \mu, \xi \in \mathbb{C}$, $\beta \neq 0$, $-1 \leq B < A \leq 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.2), then $\frac{z\left(IR_{\lambda,l}^{m,n} f(z)\right)'}{IR_{\lambda,l}^{m,n} f(z)} \prec \frac{1+Az}{1+Bz}$, and $\frac{1+Az}{1+Bz}$ is the best dominant.*

Proof. For $q(z) = \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$ in Theorem 2.1 we get the corollary. ■

Corollary 2.3 *Let $m, n \in \mathbb{N}$, $\lambda, l \geq 0$. Assume that (2.1) holds. If $f \in \mathcal{A}$ and $\psi_{\lambda,l}^{m,n} (\alpha, \beta, \mu; z) \prec \alpha + \xi \left(\frac{1+z}{1-z} \right)^\gamma + \mu \left(\frac{1+z}{1-z} \right)^{2\gamma} + \frac{2\beta\gamma z}{1-z^2}$, for $\alpha, \beta, \mu, \xi \in \mathbb{C}$, $0 < \gamma \leq 1$, $\beta \neq 0$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.2), then $\frac{z\left(IR_{\lambda,l}^{m,n} f(z)\right)'}{IR_{\lambda,l}^{m,n} f(z)} \prec \left(\frac{1+z}{1-z} \right)^\gamma$, and $\left(\frac{1+z}{1-z} \right)^\gamma$ is the best dominant.*

Proof. Corollary follows by using Theorem 2.1 for $q(z) = \left(\frac{1+z}{1-z} \right)^\gamma$, $0 < \gamma \leq 1$. ■

Theorem 2.4 *Let q be analytic and univalent in U such that $q(z) \neq 0$ and $\frac{z q'(z)}{q(z)}$ be starlike univalent in U. Assume that*

$$Re \left(\frac{\xi}{\beta} q(z) q'(z) + \frac{2\mu}{\beta} q^2(z) q'(z) \right) > 0, \ \textit{for } \xi, \beta, \mu \in \mathbb{C}, \ \beta \neq 0. \tag{2.6}$$

If $f \in \mathcal{A}$, $\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)} \in \mathcal{H}[q(0),1] \cap Q$ *and* $\psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z)$ *is univalent in* U*, where* $\psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z)$ *is as defined in (2.2), then*

$$\alpha+\xi q(z)+\mu(q(z))^2+\frac{\beta z q'(z)}{q(z)} \prec \psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z) \tag{2.7}$$

implies

$$q(z) \prec \frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}, \quad z \in U, \tag{2.8}$$

and q *is the best subordinant.*

Proof. Let the function p be defined by $p(z) := \frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}$, $z \in U$, $z \neq 0$, $f \in \mathcal{A}$.

By setting $\nu(w) := \alpha+\xi w+\mu w^2$ and $\phi(w) := \frac{\beta}{w}$ it can be easily verified that ν is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C}\backslash\{0\}$.

Since $\frac{\nu'(q(z))}{\phi(q(z))} = \frac{q'(z)q(z)[\xi+2\mu q(z)]}{\beta}$, it follows that $Re\left(\frac{\nu'(q(z))}{\phi(q(z))}\right) = Re\left(\frac{\xi}{\beta}q(z)q'(z)+\frac{2\mu}{\beta}q^2(z)q'(z)\right) > 0$, for $\alpha,\beta,\mu \in \mathbb{C}$, $\mu \neq 0$.

By using (2.5) and (2.7) we obtain $\alpha+\xi q(z)+\mu(q(z))^2+\frac{\beta z q'(z)}{q(z)} \prec \alpha+\xi p(z)+\mu(p(z))^2+\frac{\beta z p'(z)}{p(z)}$.

Using Lemma 1.2, we have $q(z) \prec p(z) = \frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}$, $z \in U$, and q is the best subordinant. ■

Corollary 2.5 *Let* $m,n \in \mathbb{N}$, $\lambda,l \geq 0$*. Assume that (2.6) holds. If* $f \in \mathcal{A}$, $\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)} \in \mathcal{H}[q(0),1] \cap Q$ *and* $\alpha+\xi\frac{1+Az}{1+Bz}+\mu\left(\frac{1+Az}{1+Bz}\right)^2+\frac{\beta(A-B)z}{(1+Az)(1+Bz)} \prec \psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z)$*, for* $\alpha,\beta,\xi,\mu \in \mathbb{C}$, $\beta \neq 0$, $-1 \leq B < A \leq 1$*, where* $\psi_{\lambda,l}^{m,n}$ *is defined in (2.2), then* $\frac{1+Az}{1+Bz} \prec \frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}$*, and* $\frac{1+Az}{1+Bz}$ *is the best subordinant.*

Proof. For $q(z) = \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$ in Theorem 2.4 we get the corollary. ■

Corollary 2.6 *Let* $m,n \in \mathbb{N}$, $\lambda,l \geq 0$*. Assume that (2.6) holds. If* $f \in \mathcal{A}$, $\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)} \in \mathcal{H}[q(0),1] \cap Q$ *and* $\alpha+\xi\left(\frac{1+z}{1-z}\right)^\gamma+\mu\left(\frac{1+z}{1-z}\right)^{2\gamma}+\frac{2\beta\gamma z}{1-z^2} \prec \psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z)$*, for* $\alpha,\beta,\mu,\xi \in \mathbb{C}$, $\beta \neq 0$, $0 < \gamma \leq 1$*, where* $\psi_{\lambda,l}^{m,n}$ *is defined in (2.2), then* $\left(\frac{1+z}{1-z}\right)^\gamma \prec \frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}$*, and* $\left(\frac{1+z}{1-z}\right)^\gamma$ *is the best subordinant.*

Proof. For $q(z) = \left(\frac{1+z}{1-z}\right)^\gamma$, $0 < \gamma \leq 1$ in Theorem 2.4 we get the corollary. ■

Combining Theorem 2.1 and Theorem 2.4, we state the following sandwich theorem.

Theorem 2.7 *Let* q_1 *and* q_2 *be analytic and univalent in* U *such that* $q_1(z) \neq 0$ *and* $q_2(z) \neq 0$*, for all* $z \in U$*, with* $\frac{zq_1'(z)}{q_1(z)}$ *and* $\frac{zq_2'(z)}{q_2(z)}$ *being starlike univalent. Suppose that* q_1 *satisfies (2.1) and* q_2 *satisfies (2.6). If* $f \in \mathcal{A}$, $\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)} \in \mathcal{H}[q(0),1] \cap Q$ *and* $\psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z)$ *is as defined in (2.2) univalent in* U*, then*

$$\alpha+\xi q_1(z)+\mu(q_1(z))^2+\frac{\beta z q_1'(z)}{q_1(z)} \prec \psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z) \prec \alpha+\xi q_2(z)+\mu(q_2(z))^2+\frac{\beta z q_2'(z)}{q_2(z)},$$

for $\alpha,\beta,\mu,\xi \in \mathbb{C}$, $\beta \neq 0$*, implies* $q_1(z) \prec \frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)} \prec q_2(z)$*, and* q_1 *and* q_2 *are respectively the best subordinant and the best dominant.*

For $q_1(z) = \frac{1+A_1z}{1+B_1z}$, $q_2(z) = \frac{1+A_2z}{1+B_2z}$, where $-1 \leq B_2 < B_1 < A_1 < A_2 \leq 1$, we have the following corollary.

Corollary 2.8 *Let* $m,n \in \mathbb{N}$, $\lambda,l \geq 0$*. Assume that (2.1) and (2.6) hold. If* $f \in \mathcal{A}$, $\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)} \in \mathcal{H}[q(0),1] \cap Q$ *and* $\alpha+\xi\frac{1+A_1z}{1+B_1z}+\mu\left(\frac{1+A_1z}{1+B_1z}\right)^2+\frac{\beta(A_1-B_1)z}{(1+A_1z)(1+B_1z)} \prec \psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z) \prec \alpha+\xi\frac{1+A_2z}{1+B_2z}+\mu\left(\frac{1+A_2z}{1+B_2z}\right)^2+\frac{\beta(A_2-B_2)z}{(1+A_2z)(1+B_2z)}$*, for* $\alpha,\beta,\mu,\xi \in \mathbb{C}$, $\beta \neq 0$, $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$*, where* $\psi_{\lambda,l}^{m,n}$ *is defined in (2.2), then* $\frac{1+A_1z}{1+B_1z} \prec \frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)} \prec \frac{1+A_2z}{1+B_2z}$*, hence* $\frac{1+A_1z}{1+B_1z}$ *and* $\frac{1+A_2z}{1+B_2z}$ *are the best subordinant and the best dominant, respectively.*

For $q_1(z) = \left(\frac{1+z}{1-z}\right)^{\gamma_1}$, $q_2(z) = \left(\frac{1+z}{1-z}\right)^{\gamma_2}$, where $0 < \gamma_1 < \gamma_2 \leq 1$, we have the following corollary.

Corollary 2.9 *Let $m,n\in\mathbb{N}$, $\lambda,l\geq 0$. Assume that (2.1) and (2.6) hold. If $f\in\mathcal{A}$, $\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}\in\mathcal{H}\left[q\left(0\right),1\right]\cap Q$ and $\alpha+\xi\left(\frac{1+z}{1-z}\right)^{\gamma_1}+\mu\left(\frac{1+z}{1-z}\right)^{2\gamma_1}+\frac{2\beta\gamma_1 z}{1-z^2}\prec\psi_{\lambda,l}^{m,n}\left(\alpha,\beta,\mu;z\right)\prec\alpha+\xi\left(\frac{1+z}{1-z}\right)^{\gamma_2}+\mu\left(\frac{1+z}{1-z}\right)^{2\gamma_2}+\frac{2\beta\gamma_2 z}{1-z^2}$, for $\alpha,\beta,\mu,\xi\in\mathbb{C}$, $\beta\neq 0$, $0<\gamma_1<\gamma_2\leq 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.2), then $\left(\frac{1+z}{1-z}\right)^{\gamma_1}\prec\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}\prec\left(\frac{1+z}{1-z}\right)^{\gamma_2}$, hence $\left(\frac{1+z}{1-z}\right)^{\gamma_1}$ and $\left(\frac{1+z}{1-z}\right)^{\gamma_2}$ are the best subordinant and the best dominant, respectively.*

We have also

Theorem 2.10 *Let $\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}\in\mathcal{H}\left(U\right)$, $f\in\mathcal{A}$, $z\in U$, $m,n\in\mathbb{N}$, $\lambda,l\geq 0$ and let the function $q\left(z\right)$ be convex and univalent in U such that $q\left(0\right)=1$, $z\in U$. Assume that*

$$Re\left(\frac{\alpha+\beta}{\beta}+z\frac{q''\left(z\right)}{q'\left(z\right)}\right)>0, \tag{2.9}$$

for $\alpha,\beta\in\mathbb{C}$, $\beta\neq 0$, $z\in U$, and

$$\psi_{\lambda,l}^{m,n}\left(\alpha,\beta;z\right):=\beta\left(n+1\right)\left(n+2\right)\frac{IR_{\lambda,l}^{m,n+2}f\left(z\right)}{IR_{\lambda,l}^{m,n}f\left(z\right)}- \tag{2.10}$$

$$\beta\left(n+1\right)^2\left(\frac{IR_{\lambda,l}^{m,n+1}f\left(z\right)}{IR_{\lambda,l}^{m,n}f\left(z\right)}\right)^2+\left(\alpha-\beta\right)\left(n+1\right)\frac{IR_{\lambda,l}^{m,n+1}f\left(z\right)}{IR_{\lambda,l}^{m,n}f\left(z\right)}-\alpha n.$$

If q satisfies the following subordination

$$\psi_{\lambda,l}^{m,n}\left(\alpha,\beta;z\right)\prec\alpha q\left(z\right)+\beta zq'\left(z\right), \tag{2.11}$$

for $\alpha,\beta\in\mathbb{C}$, $\beta\neq 0$, $z\in U$, then

$$\frac{z\left(IR_{\lambda,l}^{m,n}f\left(z\right)\right)'}{IR_{\lambda,l}^{m,n}f\left(z\right)}\prec q\left(z\right),\quad z\in U, \tag{2.12}$$

and q is the best dominant.

Proof. Let the function p be defined by $p\left(z\right):=\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}$, $z\in U$, $z\neq 0$, $f\in\mathcal{A}$. The function p is analytic in U and $p\left(0\right)=1$

We have $p'\left(z\right)=\left(n+1\right)\frac{\left(IR_{\lambda,l}^{m,n+1}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}-\left(n+1\right)\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\cdot\frac{\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}$.

By using the identity (1.3), we obtain

$$zp'\left(z\right)=\left(n+1\right)\left(n+2\right)\frac{IR_{\lambda,l}^{m,n+2}f\left(z\right)}{IR_{\lambda,l}^{m,n}f\left(z\right)}-\left(n+1\right)^2\left(\frac{IR_{\lambda,l}^{m,n+1}f\left(z\right)}{IR_{\lambda,l}^{m,n}f\left(z\right)}\right)^2-\left(n+1\right)\frac{IR_{\lambda,l}^{m,n+1}f\left(z\right)}{IR_{\lambda,l}^{m,n}f\left(z\right)}. \tag{2.13}$$

By setting $\theta\left(w\right):=\alpha w$ and $\phi\left(w\right):=\beta$, it can be easily verified that θ is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi\left(w\right)\neq 0$, $w\in\mathbb{C}\backslash\{0\}$.

Also, by letting $Q\left(z\right)=zq'\left(z\right)\phi\left(q\left(z\right)\right)=\beta zq'\left(z\right)$, we find that $Q\left(z\right)$ is starlike univalent in U.

Let $h\left(z\right)=\theta\left(q\left(z\right)\right)+Q\left(z\right)=\alpha q\left(z\right)+\beta zq'\left(z\right)$.

We have $Re\left(\frac{zh'(z)}{Q(z)}\right)=Re\left(\frac{\alpha+\beta}{\beta}+z\frac{q''(z)}{q'(z)}\right)>0$.

By using (2.13), we obtain $\alpha p\left(z\right)+\beta zp'\left(z\right)=\beta\left(n+1\right)\left(n+2\right)\frac{IR_{\lambda,l}^{m,n+2}f(z)}{IR_{\lambda,l}^{m,n}f(z)}-$
$\beta\left(n+1\right)^2\left(\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^2+\left(\alpha-\beta\right)\left(n+1\right)\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}-\alpha n$.

By using (2.11), we have $\alpha p\left(z\right)+\beta zp'\left(z\right)\prec\alpha q\left(z\right)+\beta zq'\left(z\right)$.

From Lemma 1.1, we have $p\left(z\right)\prec q\left(z\right)$, $z\in U$, i.e. $\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}\prec q\left(z\right)$, $z\in U$, and q is the best dominant. ■

Corollary 2.11 *Let $q\left(z\right)=\frac{1+Az}{1+Bz}$, $z\in U$, $-1\leq B<A\leq 1$, $m,n\in\mathbb{N}$, $\lambda,l\geq 0$. Assume that (2.9) holds. If $f\in\mathcal{A}$ and $\psi_{\lambda,l}^{m,n}\left(\alpha,\beta;z\right)\prec\alpha\frac{1+Az}{1+Bz}+\frac{\beta(A-B)z}{(1+Bz)^2}$, for $\alpha,\beta\in\mathbb{C}$, $\beta\neq 0$, $-1\leq B<A\leq 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.10), then $\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}\prec\frac{1+Az}{1+Bz}$, and $\frac{1+Az}{1+Bz}$ is the best dominant.*

Proof. For $q(z) = \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$, in Theorem 2.10 we get the corollary. ∎

Corollary 2.12 *Let* $q(z) = \left(\frac{1+z}{1-z}\right)^{\gamma}$, $m, n \in \mathbb{N}$, $\lambda, l \geq 0$. *Assume that (2.9) holds. If* $f \in \mathcal{A}$ *and* $\psi_{\lambda,l}^{m,n}(\alpha, \beta; z) \prec \alpha\left(\frac{1+z}{1-z}\right)^{\gamma} + \frac{2\beta\gamma z}{1-z^2}\left(\frac{1+z}{1-z}\right)^{\gamma}$, *for* $\alpha, \beta \in \mathbb{C}$, $0 < \gamma \leq 1$, $\beta \neq 0$, *where* $\psi_{\lambda,l}^{m,n}$ *is defined in (2.10), then* $\frac{z\left(IR_{\lambda,l}^{m,n} f(z)\right)'}{IR_{\lambda,l}^{m,n} f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\gamma}$, *and* $\left(\frac{1+z}{1-z}\right)^{\gamma}$ *is the best dominant.*

Proof. Corollary follows by using Theorem 2.10 for $q(z) = \left(\frac{1+z}{1-z}\right)^{\gamma}$, $0 < \gamma \leq 1$. ∎

Theorem 2.13 *Let* q *be convex and univalent in* U *such that* $q(0) = 1$. *Assume that*

$$Re\left(\frac{\alpha}{\beta} q'(z)\right) > 0, \text{ for } \alpha, \beta \in \mathbb{C},\ \beta \neq 0. \tag{2.14}$$

If $f \in \mathcal{A}$, $\frac{z\left(IR_{\lambda,l}^{m,n} f(z)\right)'}{IR_{\lambda,l}^{m,n} f(z)} \in \mathcal{H}[q(0), 1] \cap Q$ *and* $\psi_{\lambda,l}^{m,n}(\alpha, \beta; z)$ *is univalent in* U, *where* $\psi_{\lambda,l}^{m,n}(\alpha, \beta; z)$ *is as defined in (2.10), then*

$$\alpha q(z) + \beta z q'(z) \prec \psi_{\lambda,l}^{m,n}(\alpha, \beta; z) \tag{2.15}$$

implies

$$q(z) \prec \frac{z\left(IR_{\lambda,l}^{m,n} f(z)\right)'}{IR_{\lambda,l}^{m,n} f(z)}, \quad \delta \in \mathbb{C},\ \delta \neq 0,\ z \in U, \tag{2.16}$$

and q *is the best subordinant.*

Proof. Let the function p be defined by $p(z) := \frac{z\left(IR_{\lambda,l}^{m,n} f(z)\right)'}{IR_{\lambda,l}^{m,n} f(z)}$, $z \in U$, $z \neq 0$, $f \in \mathcal{A}$. The function p is analytic in U and $p(0) = 1$.

By setting $\nu(w) := \alpha w$ and $\phi(w) := \beta$ it can be easily verified that ν is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C}\backslash\{0\}$.

Since $\frac{\nu'(q(z))}{\phi(q(z))} = \frac{\alpha}{\beta} q'(z)$, it follows that $Re\left(\frac{\nu'(q(z))}{\phi(q(z))}\right) = Re\left(\frac{\alpha}{\beta} q'(z)\right) > 0$, for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$.

Now, by using (2.15) we obtain $\alpha q(z) + \beta z q'(z) \prec \alpha p(z) + \beta z p'(z)$, $z \in U$. From Lemma 1.2, we have $q(z) \prec p(z) = \frac{z\left(IR_{\lambda,l}^{m,n} f(z)\right)'}{IR_{\lambda,l}^{m,n} f(z)}$, $z \in U$, and q is the best subordinant. ∎

Corollary 2.14 *Let* $q(z) = \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$, $z \in U$, $m, n \in \mathbb{N}$, $\lambda, l \geq 0$. *Assume that (2.14) holds. If* $f \in \mathcal{A}$, $\frac{z\left(IR_{\lambda,l}^{m,n} f(z)\right)'}{IR_{\lambda,l}^{m,n} f(z)} \in \mathcal{H}[q(0), 1] \cap Q$, *and* $\alpha\frac{1+Az}{1+Bz} + \frac{\beta(A-B)z}{(1+Bz)^2} \prec \psi_{\lambda,l}^{m,n}(\alpha, \beta; z)$, *for* $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$, $-1 \leq B < A \leq 1$, *where* $\psi_{\lambda,l}^{m,n}$ *is defined in (2.10), then* $\frac{1+Az}{1+Bz} \prec \frac{z\left(IR_{\lambda,l}^{m,n} f(z)\right)'}{IR_{\lambda,l}^{m,n} f(z)}$, *and* $\frac{1+Az}{1+Bz}$ *is the best subordinant.*

Proof. For $q(z) = \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$, in Theorem 2.13 we get the corollary. ∎

Corollary 2.15 *Let* $q(z) = \left(\frac{1+z}{1-z}\right)^{\gamma}$, $m, n \in \mathbb{N}$, $\lambda, l \geq 0$. *Assume that (2.14) holds. If* $f \in \mathcal{A}$, $\frac{z\left(IR_{\lambda,l}^{m,n} f(z)\right)'}{IR_{\lambda,l}^{m,n} f(z)} \in \mathcal{H}[q(0), 1] \cap Q$ *and* $\alpha\left(\frac{1+z}{1-z}\right)^{\gamma} + \frac{2\beta\gamma z}{1-z^2}\left(\frac{1+z}{1-z}\right)^{\gamma} \prec \psi_{\lambda,l}^{m,n}(\alpha, \beta; z)$, *for* $\alpha, \beta \in \mathbb{C}$, $0 < \gamma \leq 1$, $\beta \neq 0$, *where* $\psi_{\lambda,l}^{m,n}$ *is defined in (2.10), then* $\left(\frac{1+z}{1-z}\right)^{\gamma} \prec \frac{z\left(IR_{\lambda,l}^{m,n} f(z)\right)'}{IR_{\lambda,l}^{m,n} f(z)}$, *and* $\left(\frac{1+z}{1-z}\right)^{\gamma}$ *is the best subordinant.*

Proof. Corollary follows by using Theorem 2.13 for $q(z) = \left(\frac{1+z}{1-z}\right)^{\gamma}$, $0 < \gamma \leq 1$. ∎

Combining Theorem 2.10 and Theorem 2.13, we state the following sandwich theorem.

Theorem 2.16 *Let* q_1 *and* q_2 *be convex and univalent in* U *such that* $q_1(z) \neq 0$ *and* $q_2(z) \neq 0$, *for all* $z \in U$. *Suppose that* q_1 *satisfies (2.9) and* q_2 *satisfies (2.14). If* $f \in \mathcal{A}$, $\frac{z\left(IR_{\lambda,l}^{m,n} f(z)\right)'}{IR_{\lambda,l}^{m,n} f(z)} \in \mathcal{H}[q(0), 1] \cap Q$, *and* $\psi_{\lambda,l}^{m,n}(\alpha, \beta; z)$ *is as defined in (2.10) univalent in* U, *then* $\alpha q_1(z) + \beta z q_1'(z) \prec \psi_{\lambda,l}^{m,n}(\alpha, \beta; z) \prec \alpha q_2(z) + \beta z q_2'(z)$, *for* $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$, *implies* $q_1(z) \prec \frac{z\left(IR_{\lambda,l}^{m,n} f(z)\right)'}{IR_{\lambda,l}^{m,n} f(z)} \prec q_2(z)$, $z \in U$, *and* q_1 *and* q_2 *are respectively the best subordinant and the best dominant.*

For $q_1(z) = \frac{1+A_1 z}{1+B_1 z}$, $q_2(z) = \frac{1+A_2 z}{1+B_2 z}$, where $-1 \leq B_2 < B_1 < A_1 < A_2 \leq 1$, we have the following corollary.

Corollary 2.17 *Let $m, n \in \mathbb{N}$, $\lambda, l \geq 0$. Assume that (2.9) and (2.14) hold for $q_1(z) = \frac{1+A_1 z}{1+B_1 z}$ and $q_2(z) = \frac{1+A_2 z}{1+B_2 z}$, respectively. If $f \in \mathcal{A}$, $\frac{z\left(IR_{\lambda,l}^{m,n} f(z)\right)'}{IR_{\lambda,l}^{m,n} f(z)} \in \mathcal{H}[q(0), 1] \cap Q$ and $\alpha \frac{1+A_1 z}{1+B_1 z} + \frac{\beta(A_1-B_1)z}{(1+B_1 z)^2} \prec \psi_{\lambda,l}^{m,n}(\alpha, \beta; z) \prec \alpha \frac{1+A_2 z}{1+B_2 z} + \frac{\beta(A_2-B_2)z}{(1+B_2 z)^2}$, $z \in U$, for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$, $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.2), then $\frac{1+A_1 z}{1+B_1 z} \prec \frac{z\left(IR_{\lambda,l}^{m,n} f(z)\right)'}{IR_{\lambda,l}^{m,n} f(z)} \prec \frac{1+A_2 z}{1+B_2 z}$, $z \in U$, hence $\frac{1+A_1 z}{1+B_1 z}$ and $\frac{1+A_2 z}{1+B_2 z}$ are the best subordinant and the best dominant, respectively.*

For $q_1(z) = \left(\frac{1+z}{1-z}\right)^{\gamma_1}$, $q_2(z) = \left(\frac{1+z}{1-z}\right)^{\gamma_2}$, where $0 < \gamma_1 < \gamma_2 \leq 1$, we have the following corollary.

Corollary 2.18 *Let $m, n \in \mathbb{N}$, $\lambda, l \geq 0$. Assume that (2.9) and (2.14) hold for $q_1(z) = \left(\frac{1+z}{1-z}\right)^{\gamma_1}$ and $q_2(z) = \left(\frac{1+z}{1-z}\right)^{\gamma_2}$, respectively. If $f \in \mathcal{A}$, $\frac{z\left(IR_{\lambda,l}^{m,n} f(z)\right)'}{IR_{\lambda,l}^{m,n} f(z)} \in \mathcal{H}[q(0), 1] \cap Q$ and $\alpha \left(\frac{1+z}{1-z}\right)^{\gamma_1} + \frac{2\beta\gamma_1 z}{1-z^2}\left(\frac{1+z}{1-z}\right)^{\gamma_1} \prec \psi_{\lambda,l}^{m,n}(\alpha, \beta; z)$ $\prec \alpha \left(\frac{1+z}{1-z}\right)^{\gamma_2} + \frac{2\beta\gamma_2 z}{1-z^2}\left(\frac{1+z}{1-z}\right)^{\gamma_2}$, $z \in U$, for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$, $0 < \gamma_1 < \gamma_2 \leq 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.2), then $\left(\frac{1+z}{1-z}\right)^{\gamma_1} \prec \frac{z\left(IR_{\lambda,l}^{m,n} f(z)\right)'}{IR_{\lambda,l}^{m,n} f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\gamma_2}$, $z \in U$, hence $\left(\frac{1+z}{1-z}\right)^{\gamma_1}$ and $\left(\frac{1+z}{1-z}\right)^{\gamma_2}$ are the best subordinant and the best dominant, respectively.*

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Global stability of a quadratic anti-competitive system of rational difference equations in the plane with Allee effects.

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Abstract. We investigate global dynamics of the following systems of difference equations

$$\begin{cases} x_{n+1} &= \frac{y_n^2}{a+x_n^2} \\ y_{n+1} &= \frac{x_n^2}{b+y_n^2} \end{cases}, \quad n=0,1,2,\ldots$$

where the parameters a, b are positive numbers and initial conditions x_0 and y_0 are arbitrary nonnegative numbers. We find all possible dynamical scenario for this system. We show that this system has substantially different behavior than the corresponding linear fractional system.

Keywords. Competitive map, global stable manifold, monotonicity, period-two solution.

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1 Introduction and Preliminaries

We investigate global dynamics of the following systems of difference equations

$$\begin{cases} x_{n+1} &= \frac{y_n^2}{a+x_n^2} \\ y_{n+1} &= \frac{x_n^2}{b+y_n^2} \end{cases}, \quad n=0,1,\ldots \tag{1}$$

where the parameters a, b are positive numbers and initial conditions x_0 and y_0 are arbitrary nonnegative numbers. System (1) is related to an anti-competitive system considered in [21]

$$x_{n+1} = \frac{\gamma_1 y_n}{A_1 + x_n}, \quad y_{n+1} = \frac{\beta_2 x_n}{A_2 + y_n}, \quad n = 0, 1, ..., \tag{2}$$

where the parameters A_1, γ_1, A_2 and β_2 are positive numbers and the initial conditions (x_0, y_0) are arbitrary nonnegative numbers. In the classification of all linear fractional systems in [3], System (2) was mentioned as system $(16, 16)$.

The main result on the global behavior of System (2) is summarized in the following theorem, see [21].

Theorem 1 *(a) If $\beta_2\,\gamma_1 - A_1\,A_2 < 0$, then $E_0(0,0)$ is a unique equilibrium and it is globally asymptotically stable.*

(b) If $\beta_2\,\gamma_1 - A_1\,A_2 > 0$, then there exist two equilibrium points, namely a repeller E_0 and an interior saddle E_+. There exists a set $\mathcal{C} \subset \mathcal{R} = [0,\infty) \times [0,\infty)$ which is invariant subset of the basin of attraction of E_+. The set $\mathcal{C}$ is a graph of a strictly increasing continuous function of the first variable on an interval, and $E_0 \in \mathcal{C}$ and separates $\mathcal{R}$ into two connected and invariant components, namely

$$\begin{aligned} \mathcal{W}_- &:= \{x \in \mathcal{R}\backslash\mathcal{C} : \exists y \in \mathcal{C} \text{ with } x \preceq_{se} y\}, \\ \mathcal{W}_+ &:= \{x \in \mathcal{R}\backslash\mathcal{C} : \exists y \in \mathcal{C} \text{ with } y \preceq_{se} x\}. \end{aligned}$$

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which satisfy:

i) If $(x_0, y_0) \in \mathcal{W}_+$*, then*

$$\lim_{n\to\infty}(x_{2n}, y_{2n}) = (\infty, 0) \quad and \quad \lim_{n\to\infty}(x_{2n+1}, y_{2n+1}) = (0, \infty).$$

ii) If $(x_0, y_0) \in \mathcal{W}_-$*,*

$$\lim_{n\to\infty}(x_{2n}, y_{2n}) = (0, \infty) \quad and \quad \lim_{n\to\infty}(x_{2n+1}, y_{2n+1}) = (\infty, 0).$$

(c) If $\beta_2\,\gamma_1 - A_1\,A_2 = 0$*, then,*

i. $E_0(0,0)$ *is the unique equilibrium, and every point of the positive semiaxes is a period-two point. Orbits of period-two solutions consist of the points* $(x, 0)$ *and* $(0, \frac{\beta_2}{A_2}\,x)$*, for some* $x > 0$*.*

ii. *All minimal period-two solutions and the equilibrium are stable but not asymptotically stable.*

iii. *There exists a family of strictly increasing curves* $\mathcal{C}_0$*,* $\mathcal{C}_x$ *and* $\mathcal{C}^x$ *for* $x > 0$*, that emanate from* E_0*,* $E_x := (x, 0)$ *and* $E^x := (0, \frac{\beta_2}{A_2}\,x)$ *respectively, such that the curves are pairwise disjoint, the union of all the curves equals* $\mathbb{R}^2_+$*, and solutions with initial point in* $\mathcal{C}_0$ *converge to* E_0*, solutions with initial point in* $\mathcal{C}_x$ *have even-indexed terms converging to* E_x *and odd-indexed terms converging to* E^x*, and, solutions with initial point in* $\mathcal{C}^x$ *have even-indexed terms converging to* E^x *and odd-indexed terms converging to* E_x*.*

As we will show in this paper System (1) has very different behavior than System (2), showing that introduction of quadratic terms can significantly change behavior of the system. As we will show in Section 4 there are three dynamic scenarios for System (1), each different than one of the three scenarios for System (2). For example System (1) always possesses the unique period-two solution which substantially effects the global behavior. Second System (1) exhibits the Allee's effect which is nonexistent in System (2). Third major difference between two systems lies in the techniques of the proof used in two results. While the results about the global stable and unstable manifolds in [18, 19, 20] were sufficient for the proofs of global dynamics of System (2), these results are not effective in the case of System (1) as the eigenvectors which correspond to the period-two solution of System (1) are parallel to the coordinate axes. Thus we used new techniques based on the properties of the basins of attraction of the period-two solution or the points at infinity $(0, \infty)$ and $(\infty, 0)$. Furthermore, we used the real algebraic geometry to prove some basic facts about the local stabilty of the equilibrium points and the period-two solutions. Our results show that the introduction of quadratic terms in the linear fractional systems of difference equations change substantially their behavior, see [2, 10] for similar results. In particular, introduction of quadratic terms creates the Allee's effect and introduces the periodic solutions.

The rest of this section contains some known results about competitive systems. Section 2 gives some basic facts about the global behavior of System (1). Section 3 presents local stability analysis of the equilibrium solutions and the period-two solution. Finally, Section 4 gives complete global dynamics of System (1).

A first order system of difference equations

$$\begin{cases} x_{n+1} = f(x_n, y_n) \\ y_{n+1} = g(x_n, y_n) \end{cases}, \quad n = 0, 1, ..., (x_0, y_0) \in \mathcal{R}, \tag{3}$$

where $\mathcal{R} \subset \mathbb{R}^2, (f, g) : \mathcal{R} \to \mathcal{R}, f, g$ are continuous functions is *competitive* if $f(x, y)$ is non-decreasing in x and non-increasing in y, and $g(x, y)$ is non-increasing in x and non-decreasing in y.

System (3) where the functions f and g have monotonic character opposite of the monotonic character in competitive system will be called *anti-competitive.* In other words (3) is anti-competitive if $f(x, y)$ is non-increasing in x and non-decreasing in y, and $g(x, y)$ is non-decreasing in x and non-increasing in y.

Consider a partial ordering $\preceq$ on $\mathbb{R}^2$. Two points $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are said to be *related* if $\mathbf{v} \preceq \mathbf{w}$ or $\mathbf{w} \preceq \mathbf{v}$. Also, a strict inequality between points may be defined as $\mathbf{v} \prec \mathbf{w}$ if $\mathbf{v} \preceq \mathbf{w}$ and $\mathbf{v} \neq \mathbf{w}$. A stronger inequality may be defined as $\mathbf{v} = (v_1, v_2) \ll \mathbf{w} = (w_1, w_2)$ if $\mathbf{v} \preceq \mathbf{w}$ with $v_1 \neq w_1$ and $v_2 \neq w_2$. For $\mathbf{u}$, $\mathbf{v}$ in $\mathbb{R}^2$, the *order interval* $[\![\mathbf{u}, \mathbf{v}]\!]$ is the set of all $\mathbf{x} \in \mathbb{R}^2$ such that $\mathbf{u} \preceq \mathbf{x} \preceq \mathbf{v}$. The interior of a set A is deoned as $intA$.

A map T on a nonempty set $\mathcal{S} \subset \mathbb{R}^2$ is a continuous function $T : \mathcal{S} \to \mathcal{S}$. The map T is *monotone* if $\mathbf{v} \preceq \mathbf{w}$ implies $T(\mathbf{v}) \preceq T(\mathbf{w})$ for all $\mathbf{v}, \mathbf{w} \in \mathcal{S}$, and it is *strongly monotone* on $\mathcal{S}$ if $\mathbf{v} \prec \mathbf{w}$ implies that $T(\mathbf{v}) \ll T(\mathbf{w})$ for all $\mathbf{v}, \mathbf{w} \in \mathcal{S}$. The map is *strictly monotone* on $\mathcal{S}$ if $\mathbf{v} \prec \mathbf{w}$ implies that $T(\mathbf{v}) \prec T(\mathbf{w})$ for all $\mathbf{v}, \mathbf{w} \in \mathcal{S}$. Clearly, being related is invariant under iteration of a strongly monotone map.

Throughout this paper we shall use the *North-East ordering* (NE) for which the positive cone is the first quadrant, i.e. this partial ordering is defined by $(x_1, y_1) \preceq_{ne} (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \leq y_2$ and the South-East (SE) ordering defined as $(x_1, y_1) \preceq_{se} (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \geq y_2$.

A map T on a nonempty set $\mathcal{S} \subset \mathbb{R}^2$ which is monotone with respect to the North-East ordering is called *cooperative* and a map monotone with respect to the South-East ordering is called *competitive.* A map T on a nonempty set $\mathcal{S} \subset \mathbb{R}^2$ which second iterate T^2 is monotone with respect to the North-East ordering is called *anti-cooperative* and a map which second iterate T^2 is monotone with respect to the South-East ordering is called *anti-competitive.* A map T that corresponds to System (3) is defined as $T = (f, g)$. An equilibrium $\mathbf{x}$ of anti-competitive system (3) is said to be nonhyperbolic of stable (resp. unstable) type if one of the eigenvalues of the Jacobian matrix evaluated at $\mathbf{x}$ is by absolute value 1 and the second one is by absolute value less (resp. bigger) than 1.

Next, we give three results for competitive maps in the plane. There is an extensive literature on competitive systems in the plane, see [1, 2, 4, 5, 6, 8, 9, 18, 19, 20, 23] for different examples of planar competitive systems and their applications. The following definition is from [24].

Definition 1 *Let $\mathcal{S}$ be a nonempty subset of $\mathbb{R}^2$. A competitive map $T : \mathcal{S} \to \mathcal{S}$ is said to satisfy condition $(O+)$ if for every x, y in $\mathcal{S}$, $T(x) \preceq_{ne} T(y)$ implies $x \preceq_{ne} y$, and T is said to satisfy condition $(O-)$ if for every x, y in $\mathcal{S}$, $T(x) \preceq_{ne} T(y)$ implies $y \preceq_{ne} x$.*

The following theorem was proved by de Mottoni-Schiaffino [7] for the Poincaré map of a periodic competitive Lotka-Volterra system of differential equations. Smith generalized the proof to competitive and cooperative maps [23].

Theorem 2 *Let $\mathcal{S}$ be a nonempty subset of $\mathbb{R}^2$. If T is a competitive map for which $(O+)$ holds then for all $x \in \mathcal{S}$, $\{T^n(x)\}$ is eventually componentwise monotone. If the orbit of x has compact closure, then it converges to a fixed point of T. If instead $(O-)$ holds, then for all $x \in \mathcal{S}$, $\{T^{2n}\}$ is eventually componentwise monotone. If the orbit of x has compact closure in $\mathcal{S}$, then its omega limit set is either a period-two orbit or a fixed point.*

The following result is from [24], with the domain of the map specialized to be the cartesian product of intervals of real numbers. It gives a sufficient condition for conditions $(O+)$ and $(O-)$.

Theorem 3 *Let $\mathcal{R} \subset \mathbb{R}^2$ be the cartesian product of two intervals in $\mathbb{R}$. Let $T : \mathcal{R} \to \mathcal{R}$ be a C^1 competitive map. If T is injective and $\det J_T(x) > 0$ for all $x \in \mathcal{R}$ then T satisfies $(O+)$. If T is injective and $\det J_T(x) < 0$ for all $x \in \mathcal{R}$ then T satisfies $(O-)$.*

The following result is a direct consequence of the Trichotomy Theorem of Dancer and Hess, see [11, 19], and is helpful for determining the basins of attraction of the equilibrium points.

Corollary 1 *If the nonnegative cone of $\preceq$ is a generalized quadrant in $\mathbb{R}^n$, and if T has no fixed points in $[\![u_1, u_2]\!]$ other than u_1 and u_2, then the interior of $[\![u_1, u_2]\!]$ is either a subset of the basin of attraction of u_1 or a subset of the basin of attraction of u_2.*

2 Some Basic Facts

Let

$$T_1(x, y) = \frac{y^2}{a + x^2}, \quad T_2(x, y) = \frac{x^2}{b + y^2}.$$

The map $T(x, y)$ associated to system (1) is given by

$$T(x, y) = (T_1(x, y), T_2(x, y)) = \left(\frac{y^2}{a + x^2}, \frac{x^2}{b + y^2}\right), \quad (x, y) \in [0, \infty) \times [0, \infty) \tag{4}$$

and the Jacobian matrix of the map T at the point (x, y) is given by:

$$J_T(x, y) = \begin{pmatrix} -\frac{2xy^2}{(x^2+a)^2} & \frac{2y}{x^2+a} \\ \frac{2x}{y^2+b} & -\frac{2x^2y}{(y^2+b)^2} \end{pmatrix}. \tag{5}$$

Determinant of the Jacobian matrix (5) is given by

$$\det J_T(x, y) = -\frac{4xy(bx^2 + a(y^2 + b))}{(x^2 + a)^2(y^2 + b)^2}, \tag{6}$$

and the trace of the Jacobian matrix (5) is given by

$$tr J_T(x, y) = -\left(\frac{2yx^2}{(y^2 + b)^2} + \frac{2y^2x}{(x^2 + a)^2}\right). \tag{7}$$

The map T^2 is given by $T^2(x, y) = (F(x, y), G(x, y))$, where

$$F(x, y) = \frac{x^4}{(b + y^2)^2 \left(\frac{y^4}{(a+x^2)^2} + a\right)}, \quad G(x, y) = \frac{y^4}{(a + x^2)^2 \left(\frac{x^4}{(b+y^2)^2} + b\right)}. \tag{8}$$

The Jacobian matrix of T^2 is given as

$$J_{T^2}(x,y) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{9}$$

where

$$A = \frac{4x^3\left(\frac{(2x^2+a)y^4}{(x^2+a)^3}+a\right)}{(y^2+b)^2\left(\frac{y^4}{(x^2+a)^2}+a\right)^2}, \quad B = -\frac{4x^4\left(x^2+a\right)^2 y\left(2y^4+by^2+a\left(x^2+a\right)^2\right)}{(y^2+b)^3\left(y^4+a\left(x^2+a\right)^2\right)^2}, \tag{10}$$

$$C = -\frac{4xy^4\left(y^2+b\right)^2\left(\left(2x^2+a\right)x^2+b\left(y^2+b\right)^2\right)}{(x^2+a)^3\left(x^4+b\left(y^2+b\right)^2\right)^2}, \quad D = \frac{4y^3\left(\frac{(2y^2+b)x^4}{(y^2+b)^3}+b\right)}{(x^2+a)^2\left(\frac{x^4}{(y^2+b)^2}+b\right)^2}. \tag{11}$$

Determinant of the Jacobian matrix (9) is

$$\det J_{T^2}(x,y) = \frac{\mathcal{A}}{(a+x^2)(b+y^2)\left(a\left(a+x^2\right)^2+y^4\right)^2\left(b\left(b+y^2\right)^2+x^4\right)^2}, \tag{12}$$

where

$$\mathcal{A} = 16x^3y^3\Big(a\left(a+x^2\right)^2\left(b^3+x^4\right) + by^4\left(a\left(a+x^2\right)^2+b^2\right)+2ab^2y^2\left(a+x^2\right)^2+2b^2y^6+by^8\Big)\left(a\left(b+y^2\right)+bx^2\right).$$

The following lemma summarizes some basic facts about System (1).

Lemma 1 *Let $(x_n, y_n) := T^n(x_0, y_0)$ be any solution of System (1). Then*

(i) Assume that $x_0 = 0$ and $y_0 > 0$. Then the following holds:

- *(i.1) $x_{2n} = 0$ and $y_{2n-1} = 0$ for all $n \in \mathbb{N}$.*
- *(i.2) If $0 < y_0 < \sqrt[3]{a^2b}$, then $0 < x_1 < \sqrt[3]{ab^2}$ and $y_{2n+2} \le y_{2n}$ for all $n \in \mathbb{N}$.*
- *(i.3) If $y_0 > \sqrt[3]{a^2b}$, then $x_1 > \sqrt[3]{ab^2}$ and $y_{2n} \le y_{2n+2}$ for all $n \in \mathbb{N}$.*

(ii) Assume that $x_0 > 0$ and $y_0 = 0$. Then the following holds

- *(ii.1) $x_{2n-1} = 0$ and $y_{2n} = 0$ for all $n \in \mathbb{N}$.*
- *(ii.2) If $0 < x_0 < \sqrt[3]{ab^2}$, then $y_1 < \sqrt[3]{a^2b}$ and $x_{2n+2} \le x_{2n}$ for all $n \in \mathbb{N}$.*
- *(ii.3) If $x_0 > \sqrt[3]{ab^2}$, then $y_1 > \sqrt[3]{a^2b}$ and $x_{2n} \le x_{2n+2}$ for all $n \in \mathbb{N}$.*

(iii) For all $n > 0$ we have $x_n y_n < 1$.

Proof. We prove statement $(i.1)$. Take $x_0 = 0$ and $y_0 > 0$. The statement (i) follows from

$$(0, y_0) - T^2(0, y_0) = (0, y_0) - \left(0, \frac{y_0^4}{a^2b}\right) = \left(0, \frac{a^2by_0 - y_0^4}{a^2b}\right),$$

$$x_1 - \sqrt[3]{ab^2} = \frac{y_0^2}{a} - \sqrt[3]{ab^2} = \frac{y_0^2 - \sqrt[3]{a^4b^2}}{a}$$

and monotonicity of T^2. Similarly, we prove $(ii.1)$. Proofs of $(i.2), (ii.2), (i.3)$ and $(ii.3)$ are immediate.

Take $x_0, y_0 \in [0, \infty) \times [0, \infty)$. Let $(x_n, y_n) := T^n(x_0, y_0)$. The proof of the statement (iii) follows from the fact

$$x_{n+1}y_{n+1} = \frac{y_n^2}{a+x_n^2}\frac{x_n^2}{b+y_n^2} = \frac{x_n^2}{a+x_n^2}\frac{y_n^2}{b+y_n^2} < 1.$$

□

Lemma 2 *The map T is injective.*

Proof. We have to prove that

$$T(x_1, y_1) = T(x_2, y_2) \Rightarrow x_1 = x_2 \quad \text{and} \quad y_1 = y_2.$$

Let $x_1, y_1, x_2, y_2 > 0$. Then

$$T(x_1, y_1) - T(x_2, y_2) = \left(\frac{x_2^2 y_1^2 + a y_1^2 - x_1^2 y_2^2 - a y_2^2}{(x_1^2 + a)(x_2^2 + a)}, \frac{y_2^2 x_1^2 + b x_1^2 - b x_2^2 - x_2^2 y_1^2}{(y_1^2 + b)(y_2^2 + b)} \right).$$

From $T(x_1, y_1) - T(x_2, y_2) = 0$ we have

$$x_2^2 y_1^2 + a y_1^2 - x_1^2 y_2^2 - a y_2^2 = 0 \tag{13}$$

and

$$y_2^2 x_1^2 + b x_1^2 - b x_2^2 - x_2^2 y_1^2 = 0. \tag{14}$$

Equation (13) implies that

$$x_1^2 = \frac{x_2^2 y_1^2 + a y_1^2 - a y_2^2}{y_2^2}. \tag{15}$$

Substituting x_1^2 from (15) into equation (14) we have

$$\frac{(y_1 - y_2)(y_1 + y_2)\left(b x_2^2 + a y_2^2 + ab\right)}{y_2^2} = 0,$$

from which it follows that $y_1 = y_2$. Substituting in (15) we have $x_1 = x_2$. This proves Lemma. □

Theorem 4 *Every bounded solution of System (1) converges to a period-two solution.*

Proof. In view of Lemma 2 the map T associated to System (1) is injective which implies that the map T^2 is also injective. Relation (12) implies that determinant of the Jacobian matrix (9) is positive for all $x \in (0, \infty) \times (0, \infty)$. By using Theorem 3 we have that the condition $(O+)$ is satisfied for the map T^2 (T^2 is competitive). Theorem 2 implies that odd and even subsequences $\{x_{2n}\}_{n=0}^{\infty}, \{x_{2n+1}\}_{n=-1}^{\infty}, \{y_{2n}\}_{n=0}^{\infty}, \{y_{2n+1}\}_{n=-1}^{\infty}$ of any solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ are eventually monotonic, from which the proof follows. □

3 The local stability of the equilibrium solutions and the period-two solution

The equilibrium points $(\bar{x}, \bar{y})$ of System (1) satisfy equations

$$\frac{\bar{y}^2}{a + \bar{x}^2} = \bar{x}, \quad \frac{\bar{x}^2}{b + \bar{y}^2} = \bar{y}. \tag{16}$$

By eliminating $\bar{x}$ from (16) we get

$$\bar{y}^9 + 3b\bar{y}^7 + 2a\bar{y}^6 + 3b^2\bar{y}^5 + (4ab - 1)\bar{y}^4 + \left(b^3 + a^2\right)\bar{y}^3 + 2ab^2\bar{y}^2 + a^2 b\bar{y} = 0. \tag{17}$$

Similarly, we can eliminate variable $\bar{y}$ from system (16) to obtain

$$\bar{x}^9 + 3a\bar{x}^7 + 2b\bar{x}^6 + 3a^2\bar{x}^5 + (4ab - 1)\bar{x}^4 + \left(a^3 + b^2\right)\bar{x}^3 + 2a^2 b\bar{x}^2 + ab^2\bar{x} = 0. \tag{18}$$

In view of Descartes' Rule of Signs we obtain that Eq. (18) has zero equilibrium always and either zero, one or two positive equilibrium points if $4ab - 1 < 0$. By using (16) all its real roots are positive numbers. These equilibrium points will be denoted $E_0(0, 0)$, $E(\bar{x}, \bar{y})$, $E_{SW}(\bar{x}, \bar{y})$ and $E_{NE}(\bar{x}, \bar{y})$.

Lemma 3 *Let*

$$\begin{aligned}\Delta_1 = {} & 186624b^2a^{15} + 55296b^5a^{13} + 2657664b^4a^{12} + 4096b^8a^{11} + 1619712b^3a^{11} \\ & + 754688b^7a^{10} - 12500b^2a^{10} + 10767632b^6a^9 + 55296b^{10}a^8 - 11550400b^5a^8 + 2657664b^9a^7 \\ & + 1980000b^4a^7 + 1619712b^8a^6 - 84375b^3a^6 + 186624b^{12}a^5 - 12500b^7a^5 \end{aligned} \tag{19}$$

and

$$\Delta_2 = 4ab - 1.$$

Then the following holds:

a) If $\Delta_2 \geq 0$, then equation (18) has one real root and four pairs of conjugate imaginary roots. Consequently, System (1) has one equilibrium point $E_0(0,0)$;

b) If $\Delta_2 < 0$, and $\Delta_1 < 0$, then equation (18) has three distinct real roots and three pairs of conjugate imaginary roots. Consequently, System (1) has three equilibrium points $E_0(0,0)$; $E_{SW}(\bar{x},\bar{y})$ and $E_{NE}(\bar{x},\bar{y})$;

c) If $\Delta_2 < 0$, and $\Delta_1 > 0$, then equation (18) has four pairs of conjugate imaginary roots and one real root. Consequently, System (1) has one equilibrium point $E_0(0,0)$;

d) If $\Delta_2 < 0$ and $\Delta_1 = 0$ then equation (18) has three pairs of conjugate imaginary roots and one real root of multiplicity two and one root of multiplicity one. Consequently, System (1) has two equilibrium points $E_0(0,0)$ and $E(\bar{x},\bar{y})$.

Proof. Let Δ be discriminant of

$$\tilde{f}(x) = x^9 + 3ax^7 + 2bx^6 + 3a^2x^5 + (4ab-1)x^4 + \left(a^3+b^2\right)x^3 + 2a^2bx^2 + ab^2x.$$

Then $\Delta = a^2b\Delta_1$. The rest of the proof follows from the fact that equation (18) has at most three real roots and Theorem 5.1 from [13]. □

Period-two solution $\{(\Phi,\Psi), T(\Phi,\Psi)\}$ satisfies the system

$$F(\Phi,\Psi) = \Phi, \ G(\Phi,\Psi) = \Psi,$$

which is equivalent to

$$\Phi\left(\Phi^3 - \left(b+\Psi^2\right)^2\left(\frac{\Psi^4}{\left(a+\Phi^2\right)^2} + a\right)\right) = 0, \quad \Psi\left(\Psi^3 - \left(a+\Phi^2\right)^2\left(\frac{\Phi^4}{\left(b+\Psi^2\right)^2} + b\right)\right) = 0. \tag{20}$$

For $\Phi = 0$ we have $\Psi = 0$ or $\Psi = \sqrt[3]{a^2b}$, and for $\Psi = 0$ we have $\Phi = 0$ or $\Phi = \sqrt[3]{ab^2}$. Hence, we have two minimal period-two points $P_1\left(0, \sqrt[3]{a^2b}\right)$ and $P_2\left(\sqrt[3]{ab^2}, 0\right)$.

Lemma 4 *The period-two solution $\{P_1, P_2\}$ is a saddle point with corresponding eigenvectors which are coordinate axes.*

Proof. The proof follows from the fact that $J_{T^2}(P_1) = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}$. □

Lemma 5 *Let $\mathcal{C}_F := \{(x,y) : F(x,y) = x\}$ and $\mathcal{C}_G := \{(x,y) : G(x,y) = y\}$ be the period-two curves, that is the curves which intersection is a period-two solution. Then for all $y > \sqrt[3]{a^2b}$ there exists exactly one $x_G(y) > 0$ such that $G(x_G(y), y) = y$ and for all $x > \sqrt[3]{ab^2}$ there exists exactly one $y_F(x) > 0$ such that $F(x, y_F(x)) = x$. Furthermore, $x_G(y)$ and $y_F(x)$ are continuous functions and $x'_G(y) > 0$, $y'_F(x) > 0$.*

Proof. Since $F(x,y) = x$ and $G(x,y) = y$ if and only if

$$-xy^4\left(a^3+2a^2x^2+ax^4+b^2\right) + x\left(a+x^2\right)^2\left(x^3-ab^2\right) - 2abxy^2\left(a+x^2\right)^2 - 2bxy^6 - xy^8 = 0,$$
$$-x^4\left(a^2+b\left(b+y^2\right)^2\right) - \left(b+y^2\right)^2\left(a^2b-y^3\right) - 2abx^2\left(b+y^2\right)^2 - 2ax^6 - x^8 = 0,$$

respectively, in view of Descartes' Rule of Signs we have that for all $y > \sqrt[3]{a^2b}$ there exists exactly one $x_G(y) > 0$ such that $G(x_G(y), y) = y$ and for all $x > \sqrt[3]{ab^2}$ there exists exactly one $y_F(x) > 0$ such that $F(x, y_F(x)) = x$. Taking derivatives of $F(x,y) = x$ with respect to x we get

$$y'_F(x) = \frac{1 - F'_x(x,y)}{F'_y(x,y)}.$$

From $F(x,y) = x$ we have that $\left(b+y^2\right)^2 = \frac{x^3}{\frac{y^4}{\left(a+x^2\right)^2}+a}$, which implies

$$F'_x(x,y) = \frac{4y^4\left(a+2x^2\right) + 4a\left(a+x^2\right)^3}{y^4\left(a+x^2\right) + a\left(a+x^2\right)^3} > 1.$$

Since $F'_y(x,y) < 0$ we get $x'_F(y) > 0$. Taking derivatives of $G(x,y) = y$ with respect to y we get

$$x'_G(y) = \frac{1 - G'_y(x,y)}{G'_x(x,y)}.$$

From $G(x,y)=y$ we have that $\left(b+y^2\right)^2=\frac{x^3}{\frac{y^4}{\left(a+x^2\right)^2}+a}$, which implies

$$G'_y(x,y)=\frac{4x^4y^2}{(b+y^2)\left(b\left(b+y^2\right)^2+x^4\right)}+4>4.$$

Since $G'_x(x,y)<0$ we get $x'_G(y)>0$. □

Theorem 5 *If T has a period-two solution $\{(\Phi,\Psi),T(\Phi,\Psi)\}$, then it is unstable. If μ_1 and μ_2, $(0<\mu_1<\mu_2)$ are the eigenvalues of $J_{T^2}(\Phi,\Psi)$ then $\mu_1>0$ and $\mu_2>1$. All period-two-solutions are ordered with respect to the North-East ordering.*

Proof. Since

$$F'_x(\Phi,\Psi)=\frac{4\Psi^4\left(a+2\Phi^2\right)+4a\left(a+\Phi^2\right)^3}{\Psi^4\left(a+\Phi^2\right)+a\left(a+\Phi^2\right)^3}>1$$

and

$$G'_y(\Phi,\Psi)=\frac{4\Phi^4\Psi^2}{(b+\Psi^2)\left(b\left(b+\Psi^2\right)^2+\Phi^4\right)}+4>4,$$

we obtain

$$trJ_{T^2}(\Phi,\Psi)=\mu_1+\mu_2>5.$$

The rest of the proof follows from the fact that $detJ_{T^2}(\Phi,\Psi)=\mu_1\mu_2>0$ and Lemma 5. □

Theorem 6 *If map T has a minimal period-two point $\{(\Phi_1,\Psi_1),T(\Phi_1,\Psi_1)\}$, which is non-hyperbolic, then $Dis(p)=0$, where $Dis(p)$ is the discriminant of polynomial*

$$p(x):=p_{16}x^{16}+p_{15}x^{15}+\cdots+p_1x+p_0,$$

where the coefficients $p_i,\ i=0,...,16$ are given in appendix A. If $\{(\Phi_1,\Psi_1),T(\Phi_1,\Psi_1)\}$ and $\{(\Phi_2,\Psi_2),T(\Phi_2,\Psi_2)\}$ are two period-two points such that T has no other period-two points in $[[(\Phi_1,\Psi_1),(\Phi_2,\Psi_2)]]=\{(x,y):(\Phi_1,\Psi_1)\preceq_{ne}(x,y)\preceq_{ne}(\Phi_2,\Psi_2)\}$, $Dis(\tilde{f})\neq 0$ and $Dis(p)\neq 0$, then one of them is a saddle point and the other one is repeller.

Proof. Period-two solution curves $\mathcal{C}_F=\{(x,y)\in\mathcal{R}:\tilde{F}(x,y)=0\}$ and $\mathcal{C}_G=\{(x,y)\in\mathcal{R}:\tilde{G}(x,y)=0\}$, where

$$\begin{aligned}\tilde{F}(x,y)=&-a^3b^2x-2a^3bxy^2-a^3xy^4-2a^2b^2x^3-4a^2bx^3y^2+a^2x^4\\&-2a^2x^3y^4-ab^2x^5-2abx^5y^2+2ax^6-ax^5y^4-b^2xy^4-2bxy^6+x^8-xy^8,\\\tilde{G}(x,y)=&-a^2b^3y-2a^2b^2y^3-a^2by^5-a^2x^4y-2ab^3x^2y-4ab^2x^2y^3\\&-2abx^2y^5-2ax^6y-b^3x^4y-2b^2x^4y^3+b^2y^4-bx^4y^5+2by^6-x^8-y+y^8,\end{aligned}$$

are algebraic curves. By using software *Mathematica* one can see that the resultant of the polynomials $\tilde{F}(x,y)$ and $\tilde{G}(x,y)$ in variable y is given by

$$\begin{aligned}R(\tilde{F},\tilde{G})=&x^{20}\left(a+x^2\right)^{16}\left(ab^2-x^3\right)\\&\left(a^3x^2+2a^2bx+3a^2x^4+ab^2+4abx^3+3ax^6+b^2x^2+2bx^5+x^8-x^3\right)p(y)\\=&x^{19}(a+x^2)^{16}(ab^2-x^3)\tilde{f}(x)p(x).\end{aligned}$$

The rest of the proof is the same as the proof of Theorem 15 in [10] so we skip it. □

It is easy to see that the following holds:

Lemma 6 *The equilibrium point E_0 is locally asymptotically stable.*

Let $\mathcal{C}_1:=\{(x,y):T_1(x,y)=x\}$ and $\mathcal{C}_2:=\{(x,y):T_2(x,y)=y\}$ be the equilibrium curves, that is the curves which intersection is an equilibrium solution. Then for all $x\geq 0$ there exist exactly one $y_1(x)>0$ such that $T_1(x,y_1(x))=x$ and exactly one $y_2(x)>0$ such that $T_2(x,y_2(x))=y$. Furthermore, it can be seen that $y_1(x)$ and $y_2(x)$ are continuous increasing functions.

Theorem 7 *Assume that $\Delta_1 < 0$ and $\Delta_2 < 0$. Then there exist two positive equilibrium solutions E_{SW} and E_{NE} and the following holds true:*

(i) The equilibrium solution E_{SW} is repeller.

(ii) The equilibrium solution E_{NE} is a saddle point.

Proof. The existence of the equilibrium solution follows from Lemma 3.

(i) Since the map T is anti-competitive then by results in [15], the eigenvalues λ_1 and λ_2 of the Jacobian matrix associated to the map T at $E_{SW}(\bar{x},\bar{y}) \in int(\mathbb{R}^2_+)$ are real and distinct, and the following holds $|\lambda_2| < -\lambda_1$. By using (6), we see that $det_{J_T}(\bar{x},\bar{y}) = \lambda_1\lambda_2 < 0$, which implies $\lambda_2 > 0$. Let

$$J_T(\bar{x},\bar{y}) = \begin{pmatrix} e & f \\ g & h \end{pmatrix}. \tag{21}$$

Taking derivatives of $T_1(x,y) = x$ and $T_2(x,y) = y$ with respect to x in the neighborhood of $\bar{x}$, we have

$$y_1'(\bar{x}) - y_2'(\bar{x}) = \frac{1-e}{f} - \frac{g}{1-h} = \frac{p(1)}{f(1-h)} = \frac{(\lambda_1 - 1)(\lambda_2 - 1)}{f(1-h)}, \tag{22}$$

where

$$p(\lambda) = \lambda^2 - (e+f)\lambda + (eh - fg)$$

is the characteristic equation of (21). One can see that $y_1'(\bar{x}) - y_2'(\bar{x}) < 0$. Since $\lambda_1 < 0$, and $f > 0$, $h < 0$, we obtain $\lambda_2 > 1$. In view of $|\lambda_2| < -\lambda_1$ we have $\lambda_1 < -1$ from which the proof follows.

(ii) Now, we consider the equilibrium point $E_{NE}(\bar{x},\bar{y})$. Same as in the previous case we have that $y_1'(\bar{x}) - y_2'(\bar{x}) > 0$ which implies that $0 < \lambda_2 < 1$. By Theorem 5 we have $\lambda_1^2 + \lambda_2^2 > 5$ from which it follows $\lambda_1^2 > 4$, i.e. $\lambda_1 < -2$. This completes the proof.

□

Theorem 8 *Assume that $\Delta_2 < 0$ and $\Delta_1 = 0$. Then there exist one positive equilibrium point $E(\bar{x},\bar{y})$ which is a non-hyperbolic equilibrium point of unstable type. If λ_1 and λ_2 are the eigenvalues of the Jacobian matrix associated to the map T at $E(\bar{x},\bar{y}) \in int(\mathbb{R}^2_+)$ then $\lambda_1 < -1$ and $\lambda_2 = 1$.*

Proof. In view of Lemmas 6 and 7 from [1], the curves C_F and C_G intersect tangentially at $E(\bar{x},\bar{y})$ (i.e. $y_1'(\bar{x}) - y_2'(\bar{x}) = 0$) if and only if $\bar{x}$ is zero of $\tilde{f}(x)$ of multiplicity greater then one. By Lemma 3, $\bar{x}$ is a root of $\tilde{f}(x)$ of multiplicity two. In view of

$$y_1'(\bar{x}) - y_2'(\bar{x}) = \frac{(\lambda_1 - 1)(\lambda_2 - 1)}{f(1-h)} = 0, \tag{23}$$

we obtain $\lambda_2 = 1$. Since $|\lambda_2| < -\lambda_1$ we have $\lambda_1 < -1$. □

4 The global behavior

Let $\mathcal{R} = [0,\infty)^2$, $\mathcal{C}_F = \{(x,y) \in \mathcal{R} : F(x,y) = x\}$, $\mathcal{C}_G = \{(x,y) \in \mathcal{R} : G(x,y) = y\}$ and

$$\begin{aligned}
\mathcal{R}_{T^2}(-,-) &= \{\ (x,y) \in \mathcal{R} : F(x,y) < x,\ G(x,y) < y\ \},\\
\mathcal{R}_{T^2}(+,-) &= \{\ (x,y) \in \mathcal{R} : F(x,y) > x,\ G(x,y) < y\ \},\\
\mathcal{R}_{T^2}(+,+) &= \{\ (x,y) \in \mathcal{R} : F(x,y) > x,\ G(x,y) > y\ \},\\
\mathcal{R}_{T^2}(-,+) &= \{\ (x,y) \in \mathcal{R} : F(x,y) < x,,\ G(x,y) > y\ \}.
\end{aligned}$$

By Lemma 5, $\mathcal{C}_F$ na $\mathcal{C}_G$ are the graphs of continuous strictly increasing functions y_F and y_G, i.e. $\mathcal{C}_F = \{\ (x, y_F(x)) : x \geq \sqrt[3]{ab^2}\ \}$ and $\mathcal{C}_G = \{\ (x, y_G(x)) : x \geq 0\ \}$.

In view of Lemma 4 [15] we have that $T(\mathcal{R}_{T^2}(+,-)) \subseteq \mathcal{R}_{T^2}(-,+)$ and $T(\mathcal{R}_{T^2}(-,+)) \subseteq \mathcal{R}_{T^2}(+,-)$ and $T^2(\mathcal{R}_{T^2}(-,+)) \subseteq \mathcal{R}_{T^2}(-,+)$ and $T^2(\mathcal{R}_{T^2}(+,-)) \subseteq \mathcal{R}_{T^2}(+,-)$. Since T^2 is competitive map, by using (iii) of Lemma 1, we obtain $T^2(x_0,y_0) \to (0,\infty)$ if $(x_0,y_0) \in \mathcal{R}_{T^2}(-,+)$ and $T^2(\tilde{x}_0,\tilde{y}_0) \to (\infty,0)$ if $(\tilde{x}_0,\tilde{y}_0) \in \mathcal{R}_{T^2}(+,-)$.

Lemma 7 *$int[[P_1,P_2]] \subset B(E_0)$.*

Proof. If $P \in int[[P_1, P_2]]$ then there exist x_0 and y_0 such that

$$P_1 \prec_{se} (0, y_0) \prec_{se} P \prec_{se} (x_0, 0) \prec_{se} P_2.$$

Since T^2 is competitive map we have

$$P_1 \prec_{se} T^{2n}(0, y_0) \prec_{se} T^{2n}(P) \prec_{se} T^{2n}(x_0, 0) \prec_{se} P_2.$$

From Lemma 1 we obtain $T^{2n}(0, y_0) \to E_0$ and $T^{2n}(x_0, 0) \to E_0$ as $n \to \infty$ from which the proof follows. □

4.1 The case $\Delta_2 \geq 0$ or ($\Delta_2 < 0$ and $\Delta_1 > 0$)

In this case, by Lemma 3, there exists one equilibrium point, E_0 which is locally asymptotically stable and the minimal period two solution $\{P_1, P_2\}$ which is a saddle point. In this case we have that $y_F(x) < y_G(x)$ for $x \geq 0$ and

$$\begin{aligned} \mathcal{R}_{T^2}(-,-) &= \{\ (x,y) \in \mathcal{R} : y_F(x) < y < y_G(x)\ \}, \\ \mathcal{R}_{T^2}(+,-) &= \{\ (x,y) \in \mathcal{R} : y_F(x) > y\ \} \subseteq \mathcal{B}(\infty, 0), \\ \mathcal{R}_{T^2}(-,+) &= \{\ (x,y) \in \mathcal{R} : y > y_G(x)\ \} \subseteq \mathcal{B}(0, \infty). \end{aligned}$$

Let $\mathcal{B}(0,\infty)$ denote the basin of attraction of $(0,\infty)$ and $\mathcal{B}(\infty,0)$ denote the basin of attraction of $(\infty,0)$ with respect to the map T^2.

In view of Theorem 4 and iii) of Lemma 1 it is clear that $\{T^n(x_0,y_0)\}$ is either asymptotic to $\{(0,\infty),(\infty,0)\}$ or converges to a period-two solution, for all $(x_0,y_0) \in \mathcal{R} = [0,\infty)^2$. Let $\mathcal{S}_1$ denote the boundary of $\mathcal{B}(0,\infty)$ and let $\mathcal{S}_2$ denote the boundary of $\mathcal{B}(\infty,0)$. It is easy to see that $P_1 \in \mathcal{S}_1$, $P_2 \in \mathcal{S}_2$, and $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathcal{R}_{T^2}(-,-)$. Similarly as in [10] it follows that $T^2(\mathcal{S}_1) \subseteq \mathcal{S}_1$, $T^2(\mathcal{S}_2) \subseteq \mathcal{S}_2$ and $T(\mathcal{S}_1) = \mathcal{S}_2$, $T(\mathcal{S}_2) = \mathcal{S}_1$. Further, $\mathcal{S}_1$ and $\mathcal{S}_2$ are the graphs of continuous strictly increasing functions. Since $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathcal{R}_{T^2}(-,-)$, we have, by the uniqueness of the global stable manifold of the map T^2, that $\mathcal{W}^s(P_1) = \mathcal{S}_1$ and $\mathcal{W}^s(P_2) = \mathcal{S}_2$.

Theorem 9 *Assume that $\Delta_2 \geq 0$. Then System (1) possesses one equilibrium point $E_0(0,0)$ and one minimal period-two solution $\{P_1,P_2\}$. Equilibrium E_0 is locally asymptotically stable and $\{P_1,P_2\}$ is a saddle point. Global stable manifold $\mathcal{W}^s(\{P_1,P_2\})$, which is a union of two continuous increasing curves $\mathcal{S}_1$ and $\mathcal{S}_2$, divides the first quadrant such that the following holds:*

i) Every initial point $(x_0,y_0) \in \mathcal{R}$ such that $(\tilde{x}_0,\tilde{y}_0) \ll_{se} (x_0,y_0) \ll_{se} (\bar{x}_0,\bar{y}_0)$ for some $(\tilde{x}_0,\tilde{y}_0) \in \mathcal{S}_1$ and $(\bar{x}_0,\bar{y}_0) \in \mathcal{S}_2$ is attracted to E_0.

ii) If $(x_0,y_0) \in \mathcal{R}$ such that $(x_0,y_0) \ll_{se} (\tilde{x}_0,\tilde{y}_0)$ for some $(\tilde{x}_0,\tilde{y}_0) \in \mathcal{S}_1$ then the subsequence of even-indexed terms $\{(x_{2n},y_{2n})\}$ is asymptotic to $(0,\infty)$, and the subsequence of odd-indexed terms $\{(x_{2n+1},y_{2n+1})\}$ is asymptotic to $(\infty,0)$.

iii) If $(x_0,y_0) \in \mathcal{R}$ such that $(\bar{x}_0,\bar{y}_0) \ll_{se} (x_0,y_0)$ for some $(\bar{x}_0,\bar{y}_0) \in \mathcal{S}_2$ then the subsequence of even-indexed terms $\{(x_{2n},y_{2n})\}$ is asymptotic to $(\infty,0)$, and the subsequence of odd-indexed terms $\{(x_{2n+1},y_{2n+1})\}$ is asymptotic to $(0,\infty)$.

See Figure 1 (a) for visual illustration.

Proof. Since S_1 is invariant under T^2 and subset of $\mathcal{R}_{T^2}(-,-)$ we have that if $(x_0,y_0) \in S_1$ then $T^{2n+2}(x_0,y_0) \preceq_{ne} T^{2n}(x_0,y_0)$. This implies that subsequences $\{x_{2n}\}$ and $\{y_{2n}\}$ are decreasing and since they are bounded sequences, they are convergent. It must be that $T^{2n}(x_0,y_0) \to P_1$ as $n \to \infty$. By the uniqueness of the global stable manifold of T^2 we obtain $\mathcal{W}^s(P_1) = S_1$. Similarly we get $\mathcal{W}^s(P_2) = S_2$ from which it follows that $\mathcal{W}^s(\{P_1,P_2\}) = S_1 \cup S_2$. Take $(x_0,y_0) \in \mathcal{R}$, $(\tilde{x}_0,\tilde{y}_0) \in \mathcal{S}_1$ and $(\bar{x}_0,\bar{y}_0) \in \mathcal{S}_2$ such that $(\tilde{x}_0,\tilde{y}_0) \ll_{se} (x_0,y_0) \ll_{se} (\bar{x}_0,\bar{y}_0)$. By monotonicity of T^2 we have $T^{2n}(\tilde{x}_0,\tilde{y}_0) \ll_{se} T^{2n}(x_0,y_0) \ll_{se} T^{2n}(\bar{x}_0,\bar{y}_0)$. Since $T^{2n}(\tilde{x}_0,\tilde{y}_0) \to P_1$ and $T^{2n}(\bar{x}_0,\bar{y}_0) \to P_2$ as $n \to \infty$ and by the uniqueness of the global stable manifold we obtain that $T^{2n}(x_0,y_0)$ eventually enters $int[[P_1,P_2]]_{se}$. So it is enough to prove that $[[P_1,P_2]]_{se} \subseteq \mathcal{B}(E_0)$. Indeed, for $(x_0,y_0) \in int[[P_1,P_2]]_{se}$ there exist $\sqrt[3]{ab^2} > \bar{\bar{x}}_0 > 0$ and $\sqrt[3]{a^2b} > \bar{\bar{y}}_0 > 0$ such that $(0,\bar{\bar{y}}_0) \preceq_{se} (x_0,y_0) \preceq_{se} (\bar{\bar{x}}_0,0)$. By Lemma 1 we have $T^{2n}(0,\bar{\bar{y}}_0) \to E_0$ and $T^{2n}(\bar{\bar{x}}_0,0) \to E_0$ as $n \to \infty$. By monotonicity of T^2 we get $T^{2n}(x_0,y_0) \to E_0$ from which the proof follows. By construction of the sets S_1 and S_2 the statements ii) and iii) are valid. □

4.2 The case $\Delta_2 < 0$ and $\Delta_1 < 0$

In this case, by Lemma 3, there exist three equilibrium points E_0, E_{SW} and E_{NE}. By Lemma 6 E_0 is locally asymptotically stable and by Theorem 7 E_{SW} is repeller and E_{NE} is a saddle point. If $Q_2(E_{SW}) = \{(x,y) : 0 \le x \le \bar{x}_{SW} \text{ and } y \ge \bar{x}_{SW}\}$ and $Q_4(E_{SW}) = \{(x,y) : x \ge \bar{x}_{SW} \text{ and } 0 \le y \le \bar{x}_{SW}\}$, then one can see that $Q_2(E_{SW}) \subseteq \mathcal{R}_{T^2}(-,+)$ and $Q_4(E_{SW}) \subseteq \mathcal{R}_{T^2}(+,-)$.

Let $\mathcal{B}(0,\infty)$ denote the basin of attraction of $(0,\infty)$ and $\mathcal{B}(\infty,0)$ denote the basin of attraction of $(\infty,0)$ with respect to the map T^2.

In view of Theorem 4 and iii) of Lemma 1 it is easy to see that $\{T^n(x_0,y_0)\}$ is either asymptotic to $(0,\infty)$ or $(\infty,0)$ or converges to a period-two solution, for all $(x_0,y_0) \in \mathcal{R} = [0,\infty)^2$. Let $\mathcal{S}_1$ denote the boundary of $\mathcal{B}(0,\infty)$ considered as a subset of $Q_3(E_{SW})$ and $\mathcal{S}_2$ denote the boundary of $\mathcal{B}(\infty,0)$ considered as a subset of $Q_3(E_{SW})$. It follows that $P_1, E_{SW} \in \mathcal{S}_1$ and $P_2, E_{SW} \in \mathcal{S}_2$. It is easy to see $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathcal{R}_{T^2}(-,-)$, from which, similarly as in [10], it follows that $T^2(\mathcal{S}_1) \subseteq \mathcal{S}_1$, $T^2(\mathcal{S}_2) \subseteq \mathcal{S}_2$ and $T(\mathcal{S}_1) = \mathcal{S}_2$, $T(\mathcal{S}_2) = \mathcal{S}_1$. Further, $\mathcal{S}_1$ and $\mathcal{S}_2$ are the graphs of continuous strictly increasing functions. Since $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathcal{R}_{T^2}(-,-)$, we have, by the uniqueness of the global stable manifold of T^2, that $\mathcal{W}^s(P_1) = \mathcal{S}_1$ and $\mathcal{W}^s(P_2) = \mathcal{S}_2$.

Lemma 8 $\mathcal{B}(E_0) = \{P \in [0,\infty)^2 : \widetilde{P} \prec_{se} P \prec_{se} \overline{P}$ *for* $\widetilde{P} \in \mathcal{S}_1$ *and* $\overline{P} \in \mathcal{S}_2\}$.

Proof. Assume that $\widetilde{P} \prec_{se} P \prec_{se} \overline{P}$ for $\widetilde{P} \in \mathcal{S}_1$ and $\overline{P} \in \mathcal{S}_2$. By monotonicity of T we get $T^{2n}(\widetilde{P}) \prec_{se} T^{2n}(P) \prec_{se} T^{2n}(\overline{P})$. Since $T^{2n}(\widetilde{P}) \to P_1$ and $T^{2n}(\overline{P}) \to P_2$ as $n \to \infty$. By the uniqueness of the global stable manifold we have that T^{2n} eventually enters $int[[P_1, P_2]]$. The rest of the proof follows from Lemma 7. □

Lemma 9 *Assume that $Dis(P) \neq 0$. Then System (1) does not have minimal period-two solution.*

Proof. For contradiction, assume that P is a minimal period-two solution of System (1). It is clear from previous discussions that $P \in (Q_1(E_{SW}) \cap Q_3(E_{NE})) \cup Q_1(E_{NE})$. Furthermore, assume that $P \in Q_1(E_{SW}) \cap Q_3(E_{NE})$ and T has no other minimal period-two solutions in $[[E_{SW}, P]]_{\preceq_{ne}}$. Since E_{SW} is a repeller by Lemma 6 we obtain that P is a saddle point. The map T^2 satisfy all conditions of Theorem 5 [20], which yields the existence of the global stable manifolds $\mathcal{W}^s(\{P, T(P)\})$, which is the union of two curves $\mathcal{W}^s(P)$ and $\mathcal{W}^s(T(P))$. Since $\mathcal{W}^s(T(P)) = T(\mathcal{W}^s(P))$ we have that these two curves have a common endpoint E_{SW} and there exists minimal period-two solution $\{\tilde{P}, T(\tilde{P})\}$ such that $P \preceq_{ne} \tilde{P} \preceq_{ne} E_{NE}$ and the curve $\mathcal{W}^s(P)$ has the second endpoint at $\tilde{P}$ while the curve $\mathcal{W}^s(T(P))$ has the second endpoint at $T(\tilde{P})$. Furthermore, the minimal period-two solution $\{\tilde{P}, T(\tilde{P})\}$ is a repeller. Since all positive period-two solutions are ordered with respect to the North-East ordering it must be $\mathcal{W}^s(T(P)) \preceq_{ne} \mathcal{W}^s(P)$, i.e. $\mathcal{W}^s(T(P)) \subset Q_3(E_{SW})$ which is in contradiction to $\mathcal{W}^s(T(P)) \subset Q_1(E_{SW})$. Similarly, we have contradiction if $P \in Q_1(E_{NE})$. Hence, T has no minimal period-two solutions. □

Theorem 10 *Assume that $\Delta_2 < 0$ and $\Delta_1 < 0$. Then System (1) has three equilibrium solutions $E_0 \prec_{ne} E_{SW} \prec_{ne} E_{NE}$, where E_0 is locally asymptotically stable, E_{SW} is a repeller and E_{NE} is a saddle point and the minimal period-two solution $\{P_1, P_2\}$ which is a saddle point. In this case there exist three invariant continuous curves $\mathcal{W}^s(E_{NE})$, $\mathcal{W}^s(P_1)$, $\mathcal{W}^s(P_2)$, which have end point at E_{SW} and they are graphs of increasing functions. Every solution $\{(x_n, y_n)\}$ which starts below $\mathcal{W}^s(E_{NE}) \cup \mathcal{W}^s(P_1)$ in South-East ordering is asymptotic to $(0,\infty)$ and every solution $\{(x_n, y_n)\}$ which starts above $\mathcal{W}^s(E_{NE}) \cup \mathcal{W}^s(P_2)$ in South-East ordering is asymptotic to $(\infty, 0)$. Every solution $\{(x_n, y_n)\}$ which starts below $\mathcal{W}^s(P_2)$ and above $\mathcal{W}^s(P_1)$ in South-East ordering converges to E_0. The first quadrant of the initial conditions $Q_1 = \{(x_0, y_0) : x_0 \ge 0, y_0 \ge 0\}$ is the union of six disjoint basins of attraction, i.e.*

$$Q_1 = \mathcal{B}(0,\infty) \cup \mathcal{B}(\infty,0) \cup \mathcal{B}(E_0) \cup \mathcal{B}(\{P_1,P_2\}) \cup \mathcal{B}(E_{NE}) \cup \mathcal{B}(E_{SW}),$$

where

$$\begin{aligned}
\mathcal{B}(E_{SW}) = \{E_{SW}\}, \quad \mathcal{B}(E_{NE}) &= \mathcal{W}^s(E_{NE}), \quad \mathcal{B}(\{P_1,P_2\}) = \mathcal{W}^s(P_1) \cup \mathcal{W}^s(P_2),\\
\mathcal{B}(0,\infty) &= \{(x,y) | (x,y) \preceq_{se} (\tilde{x}_0, \tilde{y}_0) \quad \textit{for some} \quad (\tilde{x}_0, \tilde{y}_0) \in \mathcal{W}^s(E_{NE}) \cup \mathcal{W}^s(P_1)\},\\
\mathcal{B}(\infty,0) &= \{(x,y) | (\tilde{x}_1, \tilde{y}_1) \preceq_{se} (x,y) \quad \textit{for some} \quad (\tilde{x}_1, \tilde{y}_1) \in \mathcal{W}^s(E_{NE}) \cup \mathcal{W}^s(P_2)\},\\
\mathcal{B}(E_0) &= \{(x,y) | (\tilde{x}_1, \tilde{y}_1) \preceq_{se} (x,y) \preceq_{se} (\tilde{x}_2, \tilde{y}_2) \quad \textit{for some} \quad (\tilde{x}_1, \tilde{y}_1) \in \mathcal{W}^s(P_1), (\tilde{x}_2, \tilde{y}_2) \in \mathcal{W}^s(P_2)\}.
\end{aligned}$$

Proof. The proof which follows from previous discussions and Theorem 5 [20] will be ommited. See Figure 1 (c) for visual illustration. □

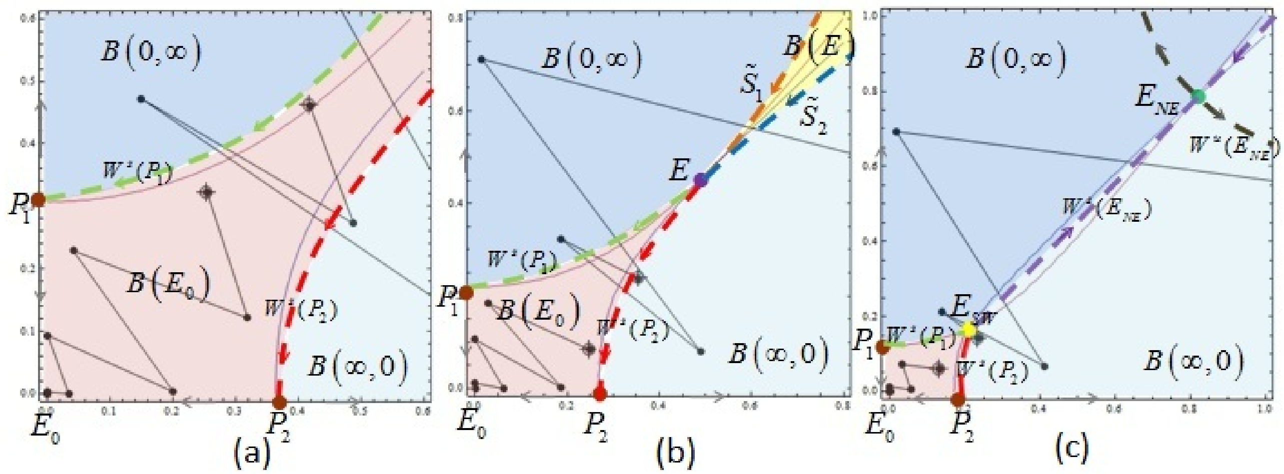

Figure 1: Visual illustration of (a) Theorem 9 (b) Theorem 11 and (c) Theorem 10.

4.3 The case $\Delta_2 < 0$ and $\Delta_1 = 0$

Let S_1 and S_2 be defined as in the previous case where $E_{SW} = E_{NE} = E$. In this case, by Lemma 3, there exist two equilibrium points E_0 and E. By Lemma 6 we have that E_0 is locally asymptotically stable and E is a non-hyperbolic equilibrium point of unstable type. Let $\widetilde{\mathcal{S}}_1$ denote the boundary of $\mathcal{B}(0,\infty)$ considered as a subset of $Q_1(E)$ and $\widetilde{\mathcal{S}}_2$ denote the boundary of $\mathcal{B}(\infty,0)$ considered as a subset of $Q_1(E)$. It is clear that $E \in \widetilde{\mathcal{S}}_1 \cup \widetilde{\mathcal{S}}_2$. Furthermore, one can see $\widetilde{\mathcal{S}}_1, \widetilde{\mathcal{S}}_2 \subseteq \mathcal{R}_{T^2}(-,-)$. Similarly as in [10], we have that $T^2(\bar{S}_1) \subseteq \widetilde{S}_1$, $T^2(\widetilde{S}_2) \subseteq \widetilde{S}_2$ and $T(\widetilde{S}_1) = \widetilde{S}_2$, $T(\widetilde{S}_2) = \widetilde{S}_1$. Further, $\widetilde{S}_1$ and $\widetilde{S}_2$ are the graphs of continuous strictly increasing functions.

Theorem 11 *Assume that $\Delta_2 < 0$ and $\Delta_1 = 0$. Then System (1) possesses two equilibrium solutions E_0 and E and one minimal period-two solution $\{P_1, P_2\}$. Equilibrium E_0 is locally asymptotically stable, E is non-hyperbolic of unstable type and $\{P_1, P_2\}$ is a saddle point. Global stable manifold $\mathcal{W}^s(\{P_1, P_2\})$ is the union of two continuous increasing curves $\mathcal{S}_1$ and $\mathcal{S}_2$ and the following holds*

i) Every initial point $(x_0, y_0) \in \mathcal{R}$ such that $(\tilde{x}_0, \tilde{y}_0) \ll_{se} (x_0, y_0) \ll_{se} (\bar{x}_0, \bar{y}_0)$ for some $(\tilde{x}_0, \tilde{y}_0) \in \mathcal{S}_1$ and $(\bar{x}_0, \bar{y}_0) \in \mathcal{S}_2$ is attracted to E_0.

ii) Every initial point $(x_0, y_0) \in \mathcal{R}$ such that $(\tilde{x}_0, \tilde{y}_0) \ll_{se} (x_0, y_0) \ll_{se} (\bar{x}_0, \bar{y}_0)$ for some $(\tilde{x}_0, \tilde{y}_0) \in \widetilde{\mathcal{S}}_1$ and $(\bar{x}_0, \bar{y}_0) \in \widetilde{\mathcal{S}}_2$ is attracted to E.

iii) If $(x_0, y_0) \in \mathcal{R}$ such that $(x_0, y_0) \ll_{se} (\tilde{x}_0, \tilde{y}_0)$ for some $(\tilde{x}_0, \tilde{y}_0) \in \mathcal{S}_1 \cup \widetilde{\mathcal{S}}_1$ then the subsequence of even-indexed terms $\{(x_{2n}, y_{2n})\}$ is asymptotic to $(0,\infty)$, and the subsequence of odd-indexed terms $\{(x_{2n+1}, y_{2n+1})\}$ is asymptotic to $(\infty, 0)$.

iv) If $(x_0, y_0) \in \mathcal{R}$ such that $(\bar{x}_0, \bar{y}_0) \ll_{se} (x_0, y_0)$ for some $(\bar{x}_0, \bar{y}_0) \in \mathcal{S}_2 \cup \widetilde{\mathcal{S}}_2$ then the subsequence of even-indexed terms $\{(x_{2n}, y_{2n})\}$ is asymptotic to $(\infty, 0)$, and the subsequence of odd-indexed terms $\{(x_{2n+1}, y_{2n+1})\}$ is asymptotic to $(0,\infty)$.

See Figure 1 (b) for visual illustration.

Proof. The proof of the statement i) is the same as the proof of the statement i) of Theorem 9. Since $\widetilde{S}_1$ is invariant under T^2 and subset of $\mathcal{R}_{T^2}(-,-)$ we have that if $(x_0, y_0) \in \widetilde{S}_1$ then $T^{2n+2}(x_0, y_0) \preceq_{ne} T^{2n}(x_0, y_0)$. This implies that subsequences $\{x_{2n}\}$ and $\{y_{2n}\}$ are decreasing and since they are bounded sequences, they are convergent. It must be that $T^{2n}(x_0, y_0) \to E$ as $n \to \infty$. Since T is continuous map and E is an equilibrium point we obtain $T^{2n+1}(x_0, y_0) \to E$ as $n \to \infty$ and $\widetilde{S}_2 = T(\widetilde{S}_1)$. Similarly we obtain that if $(x_0, y_0) \in \widetilde{S}_2$ then $T^{2n}(x_0, y_0) \in \widetilde{S}_2$, $T^{2n}(x_0, y_0) \to E$ as $n \to \infty$. Further, $T^{2n+1}(x_0, y_0) \in \widetilde{S}_1$, $T^{2n+1}(x_0, y_0) \to E$ as $n \to \infty$. Take $(x_0, y_0) \in \mathcal{R}$ and $(\tilde{x}_0, \tilde{y}_0) \in \widetilde{\mathcal{S}}_1$ and $(\bar{x}_0, \bar{y}_0) \in \widetilde{\mathcal{S}}_2$ such that $(\tilde{x}_0, \tilde{y}_0) \ll_{se} (x_0, y_0) \ll_{se} (\bar{x}_0, \bar{y}_0)$. By monotonicity of T^2 we have $T^{2n}(\tilde{x}_0, \tilde{y}_0) \ll_{se} T^{2n}(x_0, y_0) \ll_{se} T^{2n}(\bar{x}_0, \bar{y}_0)$. Since $T^{2n}(\tilde{x}_0, \tilde{y}_0) \to E$ and $T^{2n}(\bar{x}_0, \bar{y}_0) \to E$ as $n \to \infty$ we obtain that $T^{2n}(x_0, y_0) \to E$, which implies the statement ii). The statements iii) and iv) follow by construction of the sets $\widetilde{S}_1$ and $\widetilde{S}_2$. □

Remark 1 The major results of this paper, Theorems 9 - 11, are actually the general results for general anti-competitive system (3). In fact, any anti-competitive system (3) with same configuration and local stability of the equilibrium and period-two solutions will have the same global dynamics. So System (1) is actually an example of a global dynamics described in Theorems 9 - 11.

Remark 2 In [22] we consider system

$$\begin{cases} x_{n+1} &= \frac{x_n^2}{a+y_n^2} \\ y_{n+1} &= \frac{y_n^2}{b+x_n^2} \end{cases}, \quad n = 0, 1, \ldots \tag{24}$$

where the parameters a, b are positive numbers and initial conditions x_0 and y_0 are arbitrary nonnegative numbers, and obtain global dynamics similar to one described in Theorems 9 - 11, with the major difference that P_1 and P_2 are saddle point equilibrium solutions. Since the eigenvectors of the linearized system at P_1 and P_2 are parallel to the coordinate axes one can not apply at this time the results from [19, 20] to prove the existence of stable manifolds at these two points. However, existence of stable manifolds at these two points can be proved as in Theorems 9 - 11, where these two manifolds are obtained as Julia sets of the points $(\infty, 0)$ and $(0, \infty)$. Thus all results in [22] are correct with this adjustment of the proof.

A Values of coefficients p_i for $i = 0, ..., 16$.

$$\begin{aligned}
p_{16} =& a^4b^4 - 4a^3b^3 + 6a^2b^2 - 4ab + 1 \\
p_{15} =& 4a^3b^5 - 12a^2b^4 + 12ab^3 - 4b^2 \\
p_{14} =& 6a^2b^6 - 12ab^5 + 7a^5b^4 + 6b^4 - 20a^4b^3 + 18a^3b^2 - 4a^2b - a \\
p_{13} =& 4ab^7 - 4b^6 + 26a^4b^5 - 52a^3b^4 + 24a^2b^3 + 4ab^2 - 2b \\
p_{12} =& b^8 + 38a^3b^6 - 40a^2b^5 + 21a^6b^4 - 10ab^4 - 40a^5b^3 + 12b^3 + 10a^4b^2 + 8a^3b + a^7 + a^2 \\
p_{11} =& 28a^2b^7 - 4ab^6 + 72a^5b^5 - 16b^5 - 87a^4b^4 - 30a^3b^3 + 26a^2b^2 + 4a^6b + 10ab + 7a^5 + 1 \\
p_{10} =& 5a^8 + 35b^4a^7 - 40b^3a^6 - 12b^2a^5 + 101b^6a^4 + 32ba^4 - 44b^5a^3 + 13a^3 - 100b^4a^2 \\
& + 11b^8a + 16b^3a + 4b^7 + 11b^2 \\
p_9 =& 2b^9 + 76a^3b^7 + 10a^2b^6 + 110a^6b^5 - 58ab^5 - 68a^5b^4 - 6b^4 - 102a^4b^3 \\
& + 34a^3b^2 + 18a^7b + 28a^2b + 21a^6 + 3a \\
p_8 =& 10a^9 + 35b^4a^8 - 20b^3a^7 + 2b^2a^6 + 145b^6a^5 + 58ba^5 - 12b^5a^4 + 15a^4 - 163b^4a^3 + 33b^8a^2 \\
& - 20b^3a^2 + 8b^7a + 13b^2a + 8b^6 + 4b \\
p_7 =& 8ab^9 + b^8 + 104a^4b^7 + 16a^3b^6 + 100a^7b^5 - 66a^2b^5 - 22a^6b^4 - 33ab^4 - 66a^5b^3 + 29a^4b^2 \\
& + 32a^8b + 22a^3b + 22a^7 + 3a^2 \\
p_6 =& 10a^{10} + 21b^4a^9 - 4b^3a^8 + 35b^2a^7 + 120b^6a^6 + 48ba^6 + 8b^5a^5 + 6a^5 - 90b^4a^4 + 43b^8a^3 \\
& - 42b^3a^3 + 6b^7a^2 - b^2a^2 + 16b^6a + 4ba + b^{10} + 2b^5 + 1 \\
p_5 =& 28ba^9 + 54b^5a^8 + 9a^8 + 10b^3a^6 + 76b^7a^5 + 16b^2a^5 + 6b^6a^4 + 10ba^4 - 22b^5a^3 \\
& + a^3 + 10b^9a^2 - 37b^4a^2 + 4b^8a - 10b^3a + b^2 \\
p_4 =& 5a^{11} + 7b^4a^{10} + 33b^2a^8 + 56b^6a^7 + 14ba^7 + 4b^5a^6 + a^6 - 5b^4a^5 + 26b^8a^4 - 18b^3a^4 \\
& + 6b^7a^3 - b^2a^3 + 14b^6a^2 + 2ba^2 + b^{10}a - 2b^5a + 2b^9 + b^4 \\
p_3 =& 12ba^{10} + 16b^5a^9 + a^9 + b^4a^8 + 20b^3a^7 + 28b^7a^6 + 3b^2a^6 + 4ba^5 + 2b^5a^4 \\
& + 4b^9a^3 - 9b^4a^3 + 8b^8a^2 - 6b^3a^2 - 4b^7a - 2b^2a + b^6 \\
p_2 =& a^{12} + b^4a^{11} + 11b^2a^9 + 13b^6a^8 + 6b^4a^6 + 6b^8a^5 + 2b^3a^5 \\
& + 6b^7a^4 + 3b^2a^4 + 3b^6a^3 + 2b^5a^2 + 2b^9a - b^4a + b^8 \\
p_1 =& 2ba^{11} + 2b^5a^{10} + 4b^3a^8 + 4b^7a^7 + 4b^4a^4 + 4b^8a^3 - 2b^3a^3 - 2b^7a^2 \\
p_0 =& ab^8 + 2a^5b^7 + a^9b^6 + a^2b^4 + 2a^6b^3 + a^{10}b^2
\end{aligned}$$

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Solutions to Periodic Sylvester Matrix Equations Based on Matrices Splitting *

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Abstract

New iterative algorithms are introduced to solve periodic Sylvester matrix equations in this paper. The iterative algorithms are based on the principle of matrix splitting and gradient iteration method. Detailed iterative steps for solving equations are presented and their convergence property are strictly verified. A numerical test is employed to prove the correctness and effectiveness of the iterative algorithms.

Keywords: Periodic Sylvester matrix equations; iterative algorithm; matrix splitting.

1 Introduction

Analysis and design of time-varying systems are more charllenging than that of time-invariant dynamic systems since their coefficients are changing according to time. Take stability and stabilization for example, the stability concepts and criterion for (linear) time-varying systems are very difficult to characterize as they generally have no direct relationship with their coefficients (see [18] and [19] for detailed introductions). The periodic linear system as a special case of linear time-varying systems is thus important since it helps to understand that methods built for time-invariant systems can be generalized to time-varying setting. On the other, periodic linear systems also have important applications in engineering since they can be frequently used to describe cyclic temporal variation (seasonal or interannual) and to account for the operation of multiple processes. For example, Caswell analyzed in [1] the periodic models that must trace the effects of parameter changes and they applied the method to periodic system for periodic environments, and Verstraete introduced in [13] a picture to analyse the density matrix renormalization group (DMRG) numerical method from a quantum information prespective, which leads to a variational formulation of DMRG that allows for dramatic improvements in the case of problems with periodic boundary conditions. Therefore, in recent years, periodic linear systems have attracted significant attention in the literature.

Periodic Sylvester matrix equations play a major role in the analysis and design of discrete-time periodic linear systems. A general form of the periodic Sylvester matrix equation is as follows

$$A_tX_t + X_{t+1}B_t = C_t, \tag{1}$$

and

$$A_tX_{t+1} + X_tB_t = C_t, \tag{2}$$

where the coefficient matrices $A_t,\ B_t,\ C_t \in \mathbb{R}^{n\times n},\ t = 0,\ 1,\ \cdots$, are given matrices and $X_t \in \mathbb{R}^{n\times n}$ are unknown matrices. These matrices are periodic with period T, i.e., $A_{t+T} = A_t$, $B_{t+T} = B_t$, $C_{t+T} = C_t$ and $X_{t+T} = X_t$. In [8], Korotyaev showns that it is related with the periodic matrix-valued Jacobi operators. We have shown recently that the aboveperiodic Sylvester matrix equation are helpful in the design of periodic

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Luenberger observers [10] and output regulator [11]. For more applications of this class of periodic Sylvester matrix equations, see [10], [11] and the refereces therein.

The (generalized periodic) Sylvester matrix equations were first put forward by Sylvester and applied in mathematical control theory. As the development of science and technology, they become more and more important. Now many scholars and experts have analyzed the existence and uniqueness of solutions to the Sylvester matrix equation. In [12], Sreedhar proposed an elegant and simple method for computing the periodic solution of Sylvester matrix equations. In [3], Chen used the matrix sign function to solve periodic Sylvester equations. In [16], Zhang offered a finite iterative algorithm for solving the complex generalized coupled Sylvester matrix equations. In [2], the author pays attention to solving the Lyapunov matrix equations and Sylvester matrix equations in control theory by numerical methods. [4] constructs an iterative algorithm to solve the generalized coupled Sylvester matrix equations over reflexive matrices. In [9], a comprehensive theory of the matrix linear equation $AX + XB = C$ is presented. In [6], Gu applied Jacobi iteration of solving linear equations to solve Sylvester matrix equations. In [5], M Dehghan propose two iterative algorithms for finding the Hermitian reflexive and skew-Hermitian solutions of the Sylvester matrix equation $AX + XB = C$.

Furthermore, it should be pointed that gradient iterative algorithm is attracting more and more researchers. Many experts apply it to solve the Sylvester matrix equations and a lot of cases show that it is a better way to solve matrix equations. In [17], zhang present a gradient iterative algorithm for solving coupled matrix equations based on the hierarchical identification principle. In [14], Li study solutions of general matrix equations by using the iterative method and present gradient iterative algorithms by applying the hierarchical identification principle. In [7], Hoskins discussed an iterative method for solving the matrix equation $XA + AY = F$ and compared it with existing techniques. In [15], an iterative algorithm is construct to solve the general coupled matrix equations over reflexive matrix solution. Of course, researchers have given numerical examples to demonstrate the correctness of the proposed algorithm.

However, to the best of our knlowedge, the iterative algorithms for the periodic Sylvester matrix equations have not been fully researched in the literature. Therefore, in this paper, we dedicate to give iterative algorithms for solving equations (1) and (2). The iterative algorithms are based on the principle of matrix splitting and gradient iteration method. Detailed iterative steps for solving equations are presented and their convergence property are strictly verified. A numerical test is employed to prove the correctness and effectiveness of the iterative algorithms.

The rest of this paper is arranged in the following ways. In section 2, new iterative algorithms are proposed to solve the Sylvester matrix equations and the convergences are validated. In section 3, a numerical example is provided to verify the correctness of the iterative algorithm. And in section 4, we draw some conclusions.

2 Main results

2.1 Iterative algorithm for equation (1)

Firstly, for $t = 0, 1, \cdots, T-1$, define $W_t^{'}$ and $W_t^{''}$ as

$$W_t^{'} = C_t - X_{t+1}B_t \tag{3}$$

$$W_t^{''} = C_t - A_tX_t \tag{4}$$

Rewrite matrices A_t and B_t as

$$A_t = \alpha_t I_{n\times n} + T_t^{'}, \tag{5}$$

$$B_t = \beta_t I_{n\times n} + T_t^{''}, \tag{6}$$

where α_t and β_t are arbitrary constant numbers, $I_{n\times n}$ is the unit matrix and $T_t^{'}$, $T_t^{''}$ are the remaining matrices of A_t, B_t.

Based on the above division, it is easy to obtain that

$$W_t^{'} = (\alpha_t I_{n\times n} + T_t^{'})X_t \tag{7}$$

$$W_t^{''} = X_{t+1}(\beta_t I_{n\times n} + T_t^{''}) \tag{8}$$

Construct the following iteration:

$$X_t^{'}(k) = X_t^{'}(k-1) + \theta_t\alpha_t\left(C_t - X_{t+1}^{'}(k-1)B_t - A_tX_t^{'}(k-1)\right) \tag{9}$$

$$X_t^{''}(k) = X_t^{''}(k-1) + \theta_t\beta_t\left(C_t - X_{t+1}^{''}(k-1)B_t - A_tX_t^{''}(k-1)\right) \tag{10}$$

Where k is iterative step and bigger than 1.

Further, let

$$\begin{aligned} X_t(k) &= \frac{X_t^{'}(k) + X_t^{''}(k)}{2} \\ &= X_t(k-1) + \frac{1}{2}\theta_t(\alpha_t+\beta_t)\left(C_t - X_{t+1}(k-1)B_t - A_tX_t(k-1)\right) \end{aligned} \tag{11}$$

In addition, denote

$$R_t(k) = \|X_t(k) - X_t(k-1)\| \tag{12}$$

Algorithm 1 *(An iterative algorithm for equation (1))*

1. *Set error upper limit ε, freely select initial matrices $X_t^{'}(0)$ and $X_t^{''}(0)$, calculate*

$$X_t(0) = \frac{X_t^{'}(0) + X_t^{''}(0)}{2}$$

2. *Choose parameters of α_t, β_t and θ_t for $t = 0,\ 1,\ \cdots,\ T-1$, calculate*

$$\lambda_t = \sum_{t=0}^{T-1}\|(\mathrm{I}_t - \frac{1}{2}\theta_t(\alpha_t+\beta_t)A_t)\| + \sum_{t=0}^{T-1}\|\frac{1}{2}\theta_t(\alpha_t+\beta_t)\|\sum_{t=0}^{T-1}\|B_t\|) \tag{13}$$

$$k := 0;$$

3. *If $\lambda_t < 1$ for $t = 0,\ 1,\ \cdots,\ T-1$, go to next step; else, return to step 2.*
4. *Set k=$k+1$, according to (9),(10),(11), compute $X_t(k)$; Further more, compute $R_t(k)$ by (12).*
5. *If $R_t(k) \le \varepsilon$, stop; else, go to step 4.*

The convergence of the iterative algorithm will be proved by the following theorem.

Theorem 1 *If equation (1) has solutions X_t^* and λ_t shown in (13) is less than 1, the iterative sequence of $X_t(k)$ generated by Algorithm 1 converges to the true solution X_t^*, which means, for any initial $X_t(0)$, there is*

$$\lim_{k\to\infty} X_t(k) = X_t^*$$

Proof. Define error matrix $\bar{X}_t(k)$=$X_t(k) - X_t^*$, where X_t^* act as the real matrix, $X_t(k)$ is the iterative solution to k by the algorithm, then

$$\bar{X}_t^{'}(k) = X_t{'}(k) - X_t^* \tag{14}$$

$$\bar{X}_t^{''}(k) = X_t{''}(k) - X_t^* \tag{15}$$

We can easily get

$$\bar{X}_t^{'}(k) = \bar{X}_t^{'}(k-1) + \frac{1}{2}\theta_t(\alpha_t+\beta_t)(-\bar{X}_{t+1}^{'}(k-1)B_t - A_t\bar{X}_t^{'}(k-1)) \tag{16}$$

$$\bar{X}_t^{''}(k) = \bar{X}_t^{''}(k-1) + \frac{1}{2}\theta_t(\alpha_t+\beta_t)(-\bar{X}_{t+1}^{''}(k-1)B_t - A_t\bar{X}_t^{''}(k-1)) \tag{17}$$

Then, we get

$$\begin{aligned}
\bar{X}_t(k) &= \frac{\bar{X}_t^{'}(k) + \bar{X}_t^{''}(k)}{2} \\
&= \bar{X}_t(k-1) + \frac{1}{2}\theta_t(\alpha_t+\beta_t)(-\bar{X}_{t+1}(k-1)B_t - A_t\bar{X}_t(k-1)) \\
&= \bar{X}_t(k-1) - \frac{1}{2}\theta_t\alpha_t\bar{X}_{t+1}(k-1)B_t - \frac{1}{2}\theta_t\beta_t\bar{X}_{t+1}(k-1)B_t \\
&- \frac{1}{2}\theta_t\alpha_t A_t\bar{X}_t(k-1)) - \frac{1}{2}\theta_t\beta_t A_t\bar{X}_t(k-1)) \\
&= (I_t - \frac{1}{2}\theta_t\alpha_t A_t - \frac{1}{2}\theta_t\beta_t A_t)\bar{X}_t(k-1) - \frac{1}{2}\theta_t\alpha_t\bar{X}_{t+1}(k-1)B_t \\
&- \frac{1}{2}\theta_t\beta_t\bar{X}_{t+1}(k-1)B_t \\
&= (I_t - \frac{1}{2}\theta_t(\alpha_t+\beta_t)A_t)\bar{X}_t(k-1) - \frac{1}{2}\theta_t(\alpha_t+\beta_t)\bar{X}_{t+1}(k-1)B_t
\end{aligned}$$

Let

$$\begin{aligned}
\|\bar{X}_t(k)\| &= \|(I_t - \frac{1}{2}\theta_t(\alpha_t+\beta_t)A_t)\bar{X}_t(k-1) - \frac{1}{2}\theta_t(\alpha_t+\beta_t)\bar{X}_{t+1}(k-1)B_t\| \\
&\le \|(I_t - \frac{1}{2}\theta_t(\alpha_t+\beta_t)A_t)\bar{X}_t(k-1)\| + \|\frac{1}{2}\theta_t(\alpha_t+\beta_t)\bar{X}_{t+1}(k-1)B_t\| \\
&\le \|(I_t - \frac{1}{2}\theta_t(\alpha_t+\beta_t)A_t)\|\|\bar{X}_t(k-1)\| + \|\frac{1}{2}\theta_t(\alpha_t+\beta_t)\|\|\bar{X}_{t+1}(k-1)\|\|B_t\|
\end{aligned}$$

So we can obtain

$$\begin{aligned}
\sum_{t=0}^{T-1}\|\bar{X}_t(k)\| &\le \sum_{t=0}^{T-1}\|(I_t - \frac{1}{2}\theta_t(\alpha_t+\beta_t)A_t)\|\|\bar{X}_t(k-1)\| + \|\frac{1}{2}\theta_t(\alpha_t \\
&+ \beta_t)\|\|\bar{X}_{t+1}(k-1)\|\|B_t\| \\
&\le \sum_{t=0}^{T-1}\|(I_t - \frac{1}{2}\theta_t(\alpha_t+\beta_t)A_t)\|\sum_{t=0}^{T-1}\|\bar{X}_t(k-1)\| + \sum_{t=0}^{T-1}\|\frac{1}{2}\theta_t(\alpha_t \\
&+ \beta_t)\|\sum_{t=0}^{T-1}\|\bar{X}_{t+1}(k-1)\|\sum_{t=0}^{T-1}\|B_t\| \\
&\le (\sum_{t=0}^{T-1}\|(I_t - \frac{1}{2}\theta_t(\alpha_t+\beta_t)A_t)\| + \sum_{t=0}^{T-1}\|\frac{1}{2}\theta_t(\alpha_t \\
&+ \beta_t)\|\sum_{t=0}^{T-1}\|B_t\|)\sum_{t=0}^{T-1}\|\bar{X}_t(k-1)\|
\end{aligned}$$

According to assumption $\lambda_t < 1$, where λ_t are shown in (14), we can obtain that

$$\sum_{t=0}^{T-1}\|\bar{X}_t(k)\| \le \lambda_t\sum_{t=0}^{T-1}\|\bar{X}_t(k-1)\| \le \lambda_t^2\sum_{t=0}^{T-1}\|\bar{X}_t(k-2)\| \le \ldots \le \lambda_t^k\sum_{t=0}^{T-1}\|\bar{X}_t(0)\| \tag{18}$$

By controlling parameters of α_t, β_t, θ_t to make $\lambda_t < 1$.

When k is towards infinity,

$$\lim_{k\to\infty}\bar{X}_t(k) = 0$$

So

$$\lim_{k\to\infty} X_t(k) = X_t^*$$

■

2.2 Iterative algorithm for equation (2)

On periodic Sylvester matrix equation (2),we can also build an convergent algorithm which is similar to Algorithm 1.

Firstly construct the following iteration:

$$X_t^{'}(k) = X_t^{'}(k-1) + \theta_t\alpha_t \left(C_t - X_t^{'}(k-1)B_t - A_tX_{t+1}^{'}(k-1)\right) \tag{19}$$

$$X_t^{''}(k) = X_t^{''}(k-1) + \theta_t\beta_t \left(C_t - X_t^{''}(k-1)B_t - A_tX_{t+1}^{''}(k-1)\right) \tag{20}$$

Let

$$\begin{aligned} X_t(k) &= \frac{X_t^{'}(k) + X_t^{''}(k)}{2} \\ &= X_t(k-1) + \frac{1}{2}\theta_t(\alpha_t+\beta_t)\left(C_t - X_t(k-1)B_t - A_tX_{t+1}(k-1)\right) \end{aligned} \tag{21}$$

Algorithm 2 *(An iterative algorithm for equation (2))*

1. *Set error upper limit ε, arbitrary select initial matrices $X_t^{'}(0)$ and $X_t^{''}(0)$, calculate $X_t(0)$ as*

$$X_t(0) = \frac{X_t^{'}(0) + X_t^{''}(0)}{2}$$

2. *Choose parameters of α_t, β_t and θ_t for $t = 0, 1, \cdots, T-1$, calculate λ_t according to (13), and set*

$$k := 0;$$

3. *If $\lambda_t < 1$ for $t = 0, 1, \cdots, T-1$, go to next step; else, return to step 2.*
4. *Set $k{=}k+1$, according to (19),(20),(21), compute $X_t(k)$.*
5. *Compute $R_t(k)$ according to (12). If $R_t(k) \le \varepsilon$, stop; else, go to step 4.*

We can make use of the following theorem to prove the convergence of the iterative algorithm.

Theorem 2 *If equation (2) has solutions X_t^* and λ_t shown in (13) is less than 1, the iterative sequence of $X_t(k)$ generated by Algorithm 2 converges to the true solutions X_t^*, which means, for any initial $X_t(0)$, there is*

$$\lim_{k\to\infty} X_t(k) = X_t^*$$

Proof. According to Algorithm 2, we can acquire the following results

$$\bar{X}_t^{'}(k) = \bar{X}_t^{'}(k-1) + \frac{1}{2}\theta_t(\alpha_t+\beta_t)(-\bar{X}_t^{'}(k-1)B_t - A_t\bar{X}_{t+1}^{'}(k-1)) \tag{22}$$

$$\bar{X}_t^{''}(k) = \bar{X}_t^{''}(k-1) + \frac{1}{2}\theta_t(\alpha_t+\beta_t)(-\bar{X}_t^{''}(k-1)B_t - A_t\bar{X}_{t+1}^{''}(k-1)) \tag{23}$$

$$\bar{X}_t(k) = \bar{X}_t(k-1) + \frac{1}{2}\theta_t(\alpha_t+\beta_t)(-\bar{X}_t(k-1)B_t - A_t\bar{X}_{t+1}(k-1)) \tag{24}$$

Let

$$\begin{aligned}\|\bar{X}_t(k)\| &= \|(I_t - \frac{1}{2}\theta_t(\alpha_t+\beta_t)A_t)\bar{X}_{t+1}(k-1) - \frac{1}{2}\theta_t(\alpha_t+\beta_t)\bar{X}_t(k-1)B_t\| \\ &\le \|(I_t - \frac{1}{2}\theta_t(\alpha_t+\beta_t)A_t)\bar{X}_{t+1}(k-1)\| + \|\frac{1}{2}\theta_t(\alpha_t+\beta_t)\bar{X}_t(k-1)B_t\| \\ &\le \|(I_t - \frac{1}{2}\theta_t(\alpha_t+\beta_t)A_t)\|\|\bar{X}_{t+1}(k-1)\| + \|\frac{1}{2}\theta_t(\alpha_t+\beta_t)\|\|\bar{X}_t(k-1)\|\|B_t\|\end{aligned}$$

Further, let

$$\begin{aligned}\sum_{t=0}^{T-1}\|\bar{X}_t(k)\| \le& \sum_{t=0}^{T-1}\|(I_t - \frac{1}{2}\theta_t(\alpha_t+\beta_t)A_t)\|\|\bar{X}_{t+1}(k-1)\| + \|\frac{1}{2}\theta_t(\alpha_t \\ &+ \beta_t)\|\|\bar{X}_t(k-1)\|\|B_t\| \\ &\le \sum_{t=0}^{T-1}\|(I_t - \frac{1}{2}\theta_t(\alpha_t+\beta_t)A_t)\|\sum_{t=0}^{T-1}\|\bar{X}_{t+1}(k-1)\| + \sum_{t=0}^{T-1}\|\frac{1}{2}\theta_t(\alpha_t \\ &+ \beta_t)\|\sum_{t=0}^{T-1}\|\bar{X}_t(k-1)\|\sum_{t=0}^{T-1}\|B_t\| \\ &\le (\sum_{t=0}^{T-1}\|(I_t - \frac{1}{2}\theta_t(\alpha_t+\beta_t)A_t)\| + \sum_{t=0}^{T-1}\|\frac{1}{2}\theta_t(\alpha_t \\ &+ \beta_t)\|\sum_{t=0}^{T-1}\|B_t\|)\sum_{t=0}^{T-1}\|\bar{X}_t(k-1)\|\end{aligned}$$

According to assumption $\lambda_t < 1$, where λ_t are shown in (13), we can obtain that

$$\sum_{t=0}^{T-1}\|\bar{X}_t(k)\| \le \lambda_t\sum_{t=0}^{T-1}\|\bar{X}_t(k-1)\| \le \lambda_t^2\sum_{t=0}^{T-1}\|\bar{X}_t(k-2)\| \le \ldots \le \lambda_t^k\sum_{t=0}^{T-1}\|\bar{X}_t(0)\| \qquad (25)$$

When k is towards infinity and $\lambda < 1$, we can obtain

$$\lim_{k\to\infty}\bar{X}_t(k) = 0$$

So

$$\lim_{k\to\infty}X_t(k) = X_t^*$$

■

3 A numerical example

In this section, we will give an example to illustrate the correctness and effectiveness of the iterative algorithm.

Example 1 *In this example, we consider the following periodic Sylvester matrix equation with $T = 3$:*

$$A_tX_t + X_{t+1}B_t = C_t$$

For given matrices

$$A_0 = \begin{bmatrix} 2.1 & 0.8 \\ -1.0 & 1.3 \end{bmatrix}, A_1 = \begin{bmatrix} 3.2 & 1.3 \\ 0.9 & 3.1 \end{bmatrix}, A_2 = \begin{bmatrix} 5.2 & 2.8 \\ -3.1 & 5.3 \end{bmatrix}$$
$$B_0 = \begin{bmatrix} 0.5 & -0.2 \\ 0.3 & 1.0 \end{bmatrix}, B_1 = \begin{bmatrix} 1.1 & -0.4 \\ 0.3 & 1.0 \end{bmatrix}, B_2 = \begin{bmatrix} 2.1 & -1.6 \\ 0.7 & 2.5 \end{bmatrix}$$
$$C_0 = \begin{bmatrix} 12.2 & 10.6 \\ 0.6 & 7.4 \end{bmatrix}, C_1 = \begin{bmatrix} 25.6 & 21.4 \\ 1.2 & 15.1 \end{bmatrix}, C_2 = \begin{bmatrix} 37.4 & 30.2 \\ 1.6 & 24.4 \end{bmatrix}$$

Set the corresponding parameters as follows:

$$\theta_0 = 0.22, \alpha_0 = 1, \beta_0 = 1$$

$$\theta_1 = 0.44, \alpha_1 = 1, \beta_1 = 0$$

$$\theta_2 = 0.44, \alpha_2 = 0, \beta_2 = 1$$

$$\varepsilon = 0.0000001$$

By applying the iterative algorithm given in Algorithm 1 with $X_0(0) = X_1(0) = X_2(0) = 10^{-6}\mathbf{1}(2)$*, we can compute the sequences* $X_0(k)$ *,* $X_1(k)$ *and* $X_2(k)$ *and finally obtain the convergent solution as*

$$X_0^* = \begin{bmatrix} 2.2792084 & 2.1443643 \\ -0.0053164232 & 2.8578095 \end{bmatrix}$$
$$X_1^* = \begin{bmatrix} 3.8959794 & 3.0173837 \\ 0.91457065 & 4.3687527 \end{bmatrix}$$
$$X_2^* = \begin{bmatrix} 3.7974423 & 2.1832934 \\ 1.8076325 & 3.3274102 \end{bmatrix}$$

In order to demonstrate the convergent effectiveness,we define the relative iteration error as

$$\delta(k) = \sqrt{\frac{\sum_{t=0}^{T-1} \|X_t(k) - X_t^*\|^2}{\sum_{t=0}^{T-1} \|X_t^*\|^2}}.$$

The varying trajectory of relative iteration error with the time is shown in 1. It is cleared that $\delta(k)$ *decreases quickly and converges to zero as* k *increases.*

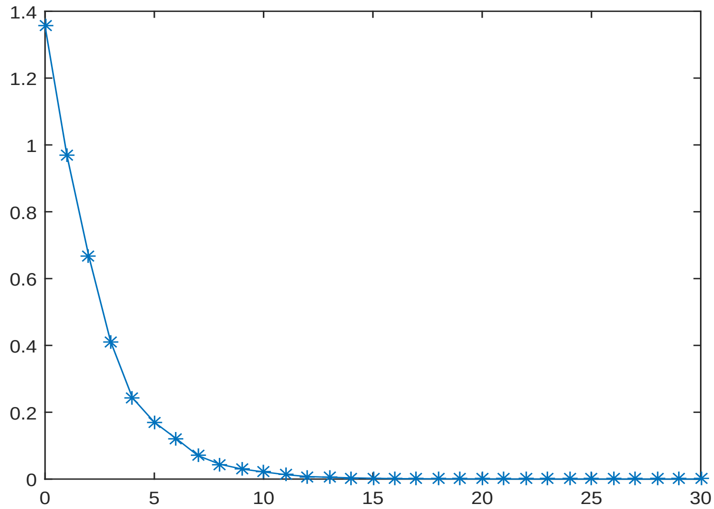

Figure 1: The changed trend of relative error

4 Conclusions

In this paper, we introduce a new iterative algorithm to solve a kind of periodic Sylvester matrix equation. The iterative algorithm is proven to converge the exact solutions in finite iteration steps without round-off errors. Finally, we give a numerical example to check the convergence and performance of the iterative algorithm.

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FOURIER SERIES OF SUMS OF PRODUCTS OF EULER AND GENOCCHI FUNCTIONS AND THEIR APPLICATIONS

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Abstract. We study three types of sums of products of Euler and Genocchi functions and derive Fourier series expansions for them. Further, we will be able to express each of those functions in terms of Bernoulli functions.

1. Introduction

The *Genocchi polynomials* $G_m(x)$ are given by the generating function

$$\frac{2t}{e^t+1}e^{xt}=\sum_{m=0}^{\infty}G_m(x)\frac{t^m}{m!},\ \text{(see [1-5, 7, 11])}.$$

The first few Genocchi polynomials are as follows:

$$\begin{aligned}&G_0(x)=0,\ G_1(x)=1,\ G_2(x)=2x-1,\\&G_3(x)=3x^2-3x,\ G_4(x)=4x^3-6x^2+1,\\&G_5(x)=5x^4-10x^3+5x,\ G_6(x)=6x^5-15x^4+15x^2-3,\\&G_7(x)=7x^6-21x^5+35x^3-21x.\end{aligned}$$

The *Euler polynomials* $E_m(x)$ are defined by the generating function

$$\frac{2}{e^t+1}e^{xt}=\sum_{m=0}^{\infty}E_m(x)\frac{t^m}{m!},\ \text{(see [2-4, 6-8, 10])}.$$

When $x=0$, $E_m(0)=E_m$ are called the *Euler numbers*.

From the relation $G_m(x)=mE_{m-1}(x)$ $(m\geq 1)$, we have

$$\deg G_m(x)=m-1\ (m\geq 1),\ G_m=mE_{m-1}\ (m\geq 1),$$
$$G_0=0,\ G_1=1,\ G_{2m+1}=0\ (m\geq 1),\ \text{and}\ G_{2m}\neq 0\ (m\geq 1).$$

Moreover,

$$\frac{d}{dx}G_m(x)=mG_{m-1}(x),\ (m\geq 1),$$
$$G_m(x+1)+G_m(x)=2mx^{m-1},\ (m\geq 0).$$

From these, we have

$$G_m(1)+G_m(0)=2\delta_{m,1},\ (m\geq 0),$$

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$$\begin{aligned}\int_0^1 G_m(x)dx =& \frac{1}{m+1}\left(G_{m+1}(1) - G_{m+1}(0)\right)\\ =& \frac{2}{m+1}\left(-G_{m+1}(0) + \delta_{m,0}\right)\\ =& \begin{cases} 0, & \text{for } m \text{ even},\\ -\dfrac{2}{m+1}G_{m+1}, & \text{for } m \text{ odd}.\end{cases}\end{aligned}$$

For any real number x, let

$$< x >= x - \lfloor x \rfloor \in [0,1)$$

denote the fractional part of x.

Let $B_m(x)$ denote the Bernoulli polynomials given by $\frac{t}{e^t-1}e^{tx} = \sum_{m=0}^{\infty} B_m(x)\frac{t^m}{m!}$. Then we recall the following about Bernoulli functions $B_m(< x >)$:

(a) for $m \geq 2$,

$$B_m(< x >) = -m! \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} \frac{e^{2\pi i n x}}{(2\pi i n)^m},$$

(b) for $m = 1$,

$$-\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} \frac{e^{2\pi i n x}}{2\pi i n} = \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z},\\ 0, & \text{for } x \in \mathbb{Z}.\end{cases}$$

Fourier series expansion of higher-order Bernoulli functions were treated in the recent paper [9]. Here we will study three types of sums of products of Euler and Genocchi functions and derive Fourier series expansions for them. Further, we will be able to express each of those functions in terms of Bernoulli functions.

2. Sums of products of Euler and Genocchi functions of the first type

Let

$$\alpha_m(x) = \sum_{k=0}^{m-1} E_k(x)G_{m-k}(x),\ (m \geq 2).$$

Note that deg $\alpha_m(x) = m - 1$. Then we will consider the function

$$\alpha_m(< x >) = \sum_{k=0}^{m-1} E_k(< x >)G_{m-k}(< x >),\ (m \geq 2)$$

defined on $(-\infty,\ \infty)$, which is periodic with period 1.

The Fourier series of $\alpha_m(< x >)$ is

$$\sum_{n=-\infty}^{\infty} A_n^{(m)} e^{2\pi i n x},$$

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where

$$\begin{aligned} A_n^{(m)} &= \int_0^1 \alpha_m(< x >)e^{-2\pi inx}dx \\ &= \int_0^1 \alpha_m(x)e^{-2\pi inx}dx. \end{aligned}$$

To proceed further, we need the following

$$\begin{aligned} \alpha'(x) &= \sum_{k=0}^{m-1} \left(kE_{k-1}(x)G_{m-k}(x) + (m-k)E_k(x)G_{m-k-1}(x)\right) \\ &= \sum_{k=1}^{m-1} kE_{k-1}(x)G_{m-k}(x) + \sum_{k=0}^{m-2}(m-k)E_k(x)G_{m-k-1}(x) \\ &= \sum_{k=0}^{m-2}(k+1)E_k(x)G_{m-1-k}(x) + \sum_{k=0}^{m-2}(m-k)E_k(x)G_{m-1-k}(x) \\ &=(m+1)\sum_{k=0}^{m-2} E_k(x)G_{m-1-k}(x) \\ &=(m+1)\alpha_{m-1}(x). \end{aligned}$$

So, $\alpha'_m(x) = (m+1)\alpha_{m-1}(x)$. From this,

$$\begin{aligned} \left(\frac{\alpha_{m+1}(x)}{m+2}\right)' &=\alpha_m(x), \\ \int_0^1 \alpha_m(x)dx &=\frac{1}{m+2}\left(\alpha_{m+1}(1) - \alpha_{m+1}(0)\right), \end{aligned}$$

and

$$\begin{aligned} \alpha_m(1) - \alpha_m(0) &= \sum_{k=0}^{m-1} \left(E_k(1)G_{m-k}(1) - E_kG_{m-k}\right) \\ &= \sum_{k=0}^{m-1} \left(\left(-E_k + 2\delta_{0,k}\right)\left(-G_{m-k} + 2\delta_{m-1,k}\right) - E_kG_{m-k}\right) \\ &= \sum_{k=0}^{m-1} \left(-2E_k\delta_{m-1,k} - 2\delta_{0,k}G_{m-k} + 4\delta_{k,0}\delta_{m-1,k}\right) \\ &= -2E_{m-1} - 2G_m + 4\delta_{m-1,0} \\ &= -2\left(E_{m-1} + G_m\right) \\ &= -2(m+1)E_{m-1}. \end{aligned}$$

Recall that

$$E_{2n} = 0 \ (n \geq 1), E_{2n-1} \neq 0 \ (n \geq 1), \text{ and } E_0 = 1.$$

So

$$\alpha_m(0) = \alpha(1) \ \Leftrightarrow m = 2n+1 \ (n \geq 1).$$

Also,

$$\int_0^1 \alpha_m(x)dx = \frac{1}{m+2}\left(-2(m+2)E_m\right) \\ = -2E_m.$$

Now, we are going to determine the Fourier coefficients $A_n^{(m)}$.

Case 1: $n \neq 0$.

$$\begin{aligned} A_n^{(m)} &= \int_0^1 \alpha_m(x)e^{-2\pi inx}dx \\ &= -\frac{1}{2\pi in}\left[\alpha_m(x)e^{-2\pi inx}\right]_0^1 + \frac{1}{2\pi in}\int_0^1 \alpha_m'(x)e^{-2\pi inx}dx \\ &= -\frac{1}{2\pi in}(\alpha_m(1)-\alpha_m(0)) + \frac{m+1}{2\pi in}\int_0^1 \alpha_{m-1}(x)e^{-2\pi inx}dx \\ &= \frac{m+1}{2\pi in}A_n^{(m-1)} + \frac{2(m+1)}{2\pi in}E_{m-1} \\ &= \frac{m+1}{2\pi in}\left(\frac{m}{2\pi in}A_n^{(m-2)} + \frac{2m}{2\pi in}E_{m-2}\right) + \frac{2(m+1)}{2\pi in}E_{m-1} \\ &= \frac{(m+1)_2}{(2\pi in)^2}A_n^{(m-2)} + \sum_{k=1}^{2}\frac{2(m+1)_k}{(2\pi in)^k}E_{m-k} \\ &= \cdots \\ &= \frac{(m+1)_{m-1}}{(2\pi in)^{m-1}}A_n^{(1)} + \sum_{k=1}^{m-1}\frac{2(m+1)_k}{(2\pi in)^k}E_{m-k} \\ &= \sum_{k=1}^{m-1}\frac{2(m+1)_k}{(2\pi in)^k}E_{m-k}, \end{aligned}$$

where $A_n^{(1)} = \int_0^1 e^{-2\pi inx}dx = 0$.

Case 2: $n = 0$.

$$A_0^{(m)} = \int_0^1 \alpha_m(x)dx = -2E_m.$$

$\alpha_m(< x >)$, $(m \geq 2)$ is piecewise C^∞. Moreover, $\alpha_m(< x >)$ is continuous for all odd integers ≥ 3 and is discontinuous with jump discontinuities at integers for all even integers ≥ 2.

Assume the first that m is an odd integer ≥ 3 . Then $\alpha_m(0) = \alpha_m(1)$. $\alpha_m(< x >)$ is piecewise C^∞, and continuous. So the Fourier series of $\alpha_m(< x >)$ converges uniformly to $\alpha_m(<x>)$, and

$$\begin{aligned}\alpha_m(<x>) =& -2E_m + \sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\left(\sum_{k=1}^{m-1}\frac{2(m+1)_k}{(2\pi in)^k}E_{m-k}\right)e^{2\pi inx}\\ =& -2E_m - 2\sum_{k=1}^{m-1}\binom{m+1}{k}E_{m-k}\left(-k!\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^k}\right)\\ =& -2E_m - 2\sum_{k=2}^{m-1}\binom{m+1}{k}E_{m-k}B_k(<x>) - 2(m+1)E_{m-1}\\ &\times\begin{cases}B_1(<x>), & \text{for } x\notin\mathbb{Z},\\ 0, & \text{for } x\in\mathbb{Z}.\end{cases}\end{aligned}$$

Now, we can state our first theorem.

Theorem 2.1. *Let m be an odd integer ≥ 3. Then we have the following.*

(i) $\sum_{k=0}^{m-1}E_k(<x>)G_{m-k}(<x>)$ *has the Fourier series expansion*

$$\begin{aligned}&\sum_{k=0}^{m-1}E_k(<x>)G_{m-k}(<x>)\\ =& -2E_m + \sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\left(\sum_{k=1}^{m-1}\frac{2(m+1)_k}{(2\pi in)^k}E_{m-k}\right)e^{2\pi inx},\end{aligned}$$

for all $x\in(-\infty,\infty)$. Here the convergence is uniform.

(ii)

$$\begin{aligned}&\sum_{k=0}^{m-1}E_k(<x>)G_{m-k}(<x>)\\ =& -2E_m - 2\sum_{k=2}^{m-1}\binom{m+1}{k}E_{m-k}B_k(<x>).\end{aligned}$$

Here $B_k(<x>)$ is the Bernoulli function.

Assume next that m is an even integer ≥ 2. Then $\alpha_m(0)\neq\alpha_m(1)$. Hence $\alpha_m(<x>)$ is piecewise C^∞, and discontinuous with jump discontinuities at integers. The Fourier series of $\alpha_m(<x>)$ converges pointwise to $\alpha_m(<x>)$, for $x\notin\mathbb{Z}$, and converges to

$$\begin{aligned}&\frac{1}{2}\left(\alpha_m(0)+\alpha_m(1)\right)\\ =&\alpha_m(0)-(m+1)E_{m-1}\\ =&\sum_{k=0}^{m-1}E_kG_{m-k}-(m+1)E_{m-1},\end{aligned}$$

for $x\in\mathbb{Z}$. Hence we have the following theorem.

Theorem 2.2. *Let m be an even integer ≥ 2. Then we have the following.*

(i)

$$-2E_m+\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(\sum_{k=1}^{m-1}\frac{2(m+1)_k}{(2\pi in)^k}E_{m-k}\right)e^{2\pi inx}$$
$$=\begin{cases}\sum_{k=0}^{m-1}E_k(<x>)G_{m-k}(<x>), & \text{for } x\notin\mathbb{Z},\\ \sum_{k=0}^{m-1}E_kG_{m-k}-(m+1)E_{m-1}, & \text{for } x\in\mathbb{Z}.\end{cases}$$

Here the convergence is pointwise.

(ii)

$$-2E_m-2\sum_{k=1}^{m-1}\binom{m+1}{k}E_{m-k}B_k(<x>)$$
$$=\sum_{k=0}^{m-1}E_k(<x>)G_{m-k}(<x>),\ \text{for } x\notin\mathbb{Z};$$

$$-2E_m-2\sum_{k=2}^{m-1}\binom{m+1}{k}E_{m-k}B_k(<x>)$$
$$=\sum_{k=0}^{m-1}E_kG_{m-k}-(m+1)E_{m-1},\ \text{for } x\in\mathbb{Z}.$$

Here $B_k(<x>)$ is the Bernoulli function.

3. Sums of products of Euler and Genocchi functions of the second type

Let

$$\beta_m(x)=\sum_{k=0}^{m-1}\frac{1}{k!(m-k)!}E_k(x)G_{m-k}(x),\ (m\geq 2).$$

Then we will consider the function

$$\beta_m(<x>)=\sum_{k=0}^{m-1}\frac{1}{k!(m-k)!}E_k(<x>)G_{m-k}(<x>),\ (m\geq 2)$$

defined on $(-\infty,\infty)$, which is periodic with period 1.

The Fourier series of $\beta_m(<x>)$ is

$$\sum_{n=-\infty}^{\infty}B_n^{(m)}e^{2\pi inx},$$

where

$$B_n^{(m)}=\int_0^1\beta_m(<x>)e^{-2\pi inx}dx$$
$$=\int_0^1\beta_m(x)e^{-2\pi inx}dx.$$

To proceed further, we need to observe the following.

$$\begin{aligned}\beta'_m(x) =& \sum_{k=0}^{m-1}\left\{\frac{k}{k!(m-k)!}E_{k-1}(x)G_{m-k}(x)+\frac{m-k}{k!(m-k)!}E_k(x)G_{m-k-1}(x)\right\}\\ =& \sum_{k=1}^{m-1}\frac{1}{(k-1)!(m-k)!}E_{k-1}(x)G_{m-k}(x)+\sum_{k=0}^{m-2}\frac{1}{k!(m-k-1)!}E_k(x)G_{m-k-1}(x)\\ =& \sum_{k=0}^{m-2}\frac{1}{k!(m-1-k)!}E_k(x)G_{m-1-k}(x)+\sum_{k=0}^{m-2}\frac{1}{k!(m-1-k)!}E_k(x)G_{m-1-k}(x)\\ =& 2\sum_{k=0}^{m-2}\frac{1}{k!(m-1-k)!}E_k(x)G_{m-1-k}(x)\\ =& 2\beta_{m-1}(x).\end{aligned}$$

So $\beta'_m(x)=2\beta_{m-1}(x)$, and from this we obtain

$$\left(\frac{\beta_{m+1}(x)}{2}\right)'=\beta_m(x).$$

Thus

$$\int_0^1\beta_m(x)dx=\frac{1}{2}\left(\beta_{m+1}(1)-\beta_{m+1}(0)\right),$$

and

$$\begin{aligned}&\beta_m(1)-\beta_m(0)\\ =& \sum_{k=0}^{m-1}\frac{1}{k!(m-k)!}\left(E_k(1)G_{m-k}(1)-E_kG_{m-k}\right)\\ =& \sum_{k=0}^{m-1}\frac{1}{k!(m-k)!}\left((-E_k+2\delta_{k,0})(-G_{m-k}+2\delta_{m-1,k})-E_kG_{m-k}\right)\\ =& \sum_{k=0}^{m-1}\frac{1}{k!(m-k)!}\left(-2E_k\delta_{m-1,k}-2\delta_{k,0}G_{m-k}+4\delta_{k,0}\delta_{m-1,k}\right)\\ =& -\frac{2E_{m-1}}{(m-1)!}-\frac{2G_m}{m!}+\frac{4\delta_{m-1,0}}{m!}\\ =& -\frac{2E_{m-1}}{(m-1)!}-\frac{2mE_{m-1}}{m!}\\ =& -\frac{4}{(m-1)!}E_{m-1}.\end{aligned}$$

So,

$$\begin{aligned}\beta_m(0)=\beta_m(1)&\Longleftrightarrow E_{m-1}=0\\ &\Longleftrightarrow m=2n+1\ (n\geq 1).\end{aligned}$$

Also,

$$\begin{aligned}\int_0^1\beta_m(x)dx=&\frac{1}{2}\left(\beta_{m+1}(1)-\beta_{m+1}(0)\right)\\ =&-\frac{2}{m!}E_m.\end{aligned}$$

Now, we are ready to determine the Fourier coefficients $B_n^{(m)}$.

Case 1 : $n \neq 0$.

$$\begin{aligned}
B_n^{(m)} &= \int_0^1 \beta_m(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n}\left[\beta_m(x) e^{-2\pi i n x}\right]_0^1 + \frac{1}{2\pi i n}\int_0^1 \beta_m'(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n}\left(\beta_m(1) - \beta_m(0)\right) + \frac{2}{2\pi i n}\int_0^1 \beta_{m-1}(x) e^{-2\pi i n x} dx \\
&= \frac{2}{2\pi i n} B_n^{(m-1)} + \frac{4}{2\pi i n (m-1)!} E_{m-1} \\
&= \frac{2}{2\pi i n}\left(\frac{2}{2\pi i n} B_n^{(m-2)} + \frac{4}{2\pi i n (m-2)!} E_{m-2}\right) + \frac{4}{2\pi i n (m-1)!} E_{m-1} \\
&= \left(\frac{2}{2\pi i n}\right)^2 B_n^{(m-2)} + \sum_{k=1}^{2} \frac{2^{k+1}}{(2\pi i n)^k (m-k)!} E_{m-k} \\
&= \cdots \\
&= \left(\frac{2}{2\pi i n}\right)^{m-1} B_n^{(1)} + \sum_{k=1}^{m-1} \frac{2^{k+1}}{(2\pi i n)^k (m-k)!} E_{m-k} \\
&= \sum_{k=1}^{m-1} \frac{2^{k+1}}{(2\pi i n)^k (m-k)!} E_{m-k},
\end{aligned}$$

where $B_n^{(1)} = \int_0^1 e^{-2\pi i n x} dx = 0$.

Case 2 : $n = 0$.

$$B_0^{(m)} = \int_0^1 \beta_m(x) dx = -\frac{2}{m!} E_m.$$

$\beta_m(< x >)$, $(m \geq 2)$ is piecewise C^∞. Moreover, $\beta_m(< x >)$ is continuous for all odd integers ≥ 3 and discontinuous with jump discontinuities at integers for all even integers ≥ 2.

Assume first that m is an odd integer ≥ 3. Then $\beta_m(0) = \beta_m(1)$. $\beta_m(< x >)$ is piecewise C^∞, and continuous. So the Fourier series of $\beta_m(< x >)$ converges uniformly to $\beta_m(< x >)$, and

$$\begin{aligned}
\beta_m(< x >) &= -\frac{2}{m!} E_m + 2\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(\sum_{k=1}^{m-1} \frac{2^k}{(2\pi i n)^k (m-k)!} E_{m-k}\right) e^{2\pi i n x} \\
&= -\frac{2}{m!} E_m - \frac{2}{m!}\sum_{k=1}^{m-1} 2^k \binom{m}{k} E_{m-k} \left(-k! \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{e^{2\pi i n x}}{(2\pi i n)^k}\right) \\
&= -\frac{2}{m!} E_m - \frac{2}{m!}\sum_{k=2}^{m-1} 2^k \binom{m}{k} E_{m-k} B_k(< x >) \\
&\quad - \frac{4}{(m-1)!} E_{m-1} \times \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases}
\end{aligned}$$

We can now state our first theorem.

Theorem 3.1. *Let m be an odd integer ≥ 3. Then we have the following.*

(i)

$$\sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} E_k(< x >) G_{m-k}(< x >)$$

has the Fourier series expansion

$$\begin{aligned}
&\sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} E_k(< x >) G_{m-k}(< x >) \\
&= -\frac{2}{m!} E_m + 2 \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left(\sum_{k=1}^{m-1} \frac{2^k}{(2\pi i n)^k (m-k)!} E_{m-k} \right) e^{2\pi i n x}
\end{aligned}$$

for all $x \in (-\infty, \infty)$. Here the convergence is uniform.

(ii)

$$\begin{aligned}
&\sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} E_k(< x >) G_{m-k}(< x >) \\
&= -\frac{2}{m!} E_m - \frac{2}{m!} \sum_{k=2}^{m-1} 2^k \binom{m}{k} E_{m-k} B_k(< x >)
\end{aligned}$$

Here $B_k(< x >)$ is the Bernoulli function.

Assume next that m is an even integer ≥ 2. Then $\beta_m(0) \neq \beta_m(1)$. Hence $\beta_m(< x >)$ is piecewise C^∞, and discontinuous with jump discontinuities at integers. The Fourier series of $\beta_m(< x >)$ converges pointwise to $\beta_m(< x >)$, for $x \notin \mathbb{Z}$, and converges to

$$\begin{aligned}
&\frac{1}{2} (\beta_m(0) + \beta_m(1)) = \beta_m(0) - \frac{2}{(m-1)!} E_{m-1} \\
&= \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} E_k G_{m-k} - \frac{2}{(m-1)!} E_{m-1},
\end{aligned}$$

for $x \in \mathbb{Z}$.

Now, we can state our second theorem.

Theorem 3.2. *Let m be an even integer ≥ 2. Then we have the following.*

(i)

$$\begin{aligned}
&-\frac{2}{m!} E_m + 2 \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left(\sum_{k=1}^{m-1} \frac{2^k}{(2\pi i n)^k (m-k)!} E_{m-k} \right) e^{2\pi i n x} \\
&= \begin{cases} \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} E_k(x) G_{m-k}(x), & \text{for } x \notin \mathbb{Z}, \\ \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} E_k G_{m-k} - \frac{2}{(m-1)!} E_{m-1}, & \text{for } x \in \mathbb{Z}. \end{cases}
\end{aligned}$$

Here the convergence is pointwise.

(ii)

$$
\begin{aligned}
&-\frac{2}{m!}E_m-\frac{2}{m!}\sum_{k=1}^{m-1}2^k\binom{m}{k}E_{m-k}B_k(<x>)\\
&=\sum_{k=0}^{m-1}\frac{1}{k!(m-k)!}E_k(<x>)G_{m-k}(<x>),\ for\ x\notin\mathbb{Z};
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{2}{m!}E_m-\frac{2}{m!}\sum_{k=2}^{m-1}2^k\binom{m}{k}E_{m-k}B_k(<x>)\\
&=\sum_{k=0}^{m-1}\frac{1}{k!(m-k)!}E_kG_{m-k}-\frac{2}{(m-1)!}E_{m-1},\ for\ x\in\mathbb{Z}.
\end{aligned}
$$

Here $B_k(<x>)$ is the Bernoulli function.

4. Sums of products of Euler and Genocchi functions of the third type

Let $\gamma_m(x)=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}E_k(x)G_{m-k}(x)$, $(m\geq 3)$. Then we will consider the function

$$
\gamma_m(<x>)=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}E_k(<x>)G_{m-k}(<x>)
$$

defined on $(-\infty,\infty)$, which is periodic with period 1.

The Fourier series of $\gamma_m(<x>)$ is

$$
\sum_{n=-\infty}^{\infty}C_n^{(m)}e^{2\pi inx},
$$

where

$$
\begin{aligned}
C_n^{(m)}&=\int_0^1\gamma_m(<x>)e^{-2\pi inx}dx\\
&=\int_0^1\gamma_m(x)e^{-2\pi inx}dx.
\end{aligned}
$$

To proceed further, we need to note the following.

$$\gamma_m'(x) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)} \{kE_{k-1}(x)G_{m-k}(x) + (m-k)E_k(x)G_{m-k-1}(x)\}$$

$$= \sum_{k=1}^{m-1} \frac{1}{m-k} E_{k-1}(x)G_{m-k}(x) + \sum_{k=1}^{m-2} \frac{1}{k} E_k(x)G_{m-k-1}(x)$$

$$= \sum_{k=0}^{m-2} \frac{1}{m-1-k} E_k(x)G_{m-1-k}(x) + \sum_{k=1}^{m-2} \frac{1}{k} E_k(x)G_{m-1-k}(x)$$

$$= \frac{1}{m-1} G_{m-1}(x) + \sum_{k=1}^{m-2} \frac{1}{m-1-k} E_k(x)G_{m-1-k}(x) + \sum_{k=1}^{m-2} \frac{1}{k} E_k(x)G_{m-1-k}(x)$$

$$= \frac{1}{m-1} G_{m-1}(x) + (m-1) \sum_{k=1}^{m-2} \frac{1}{k(m-1-k)} E_k(x)G_{m-1-k}(x)$$

$$= \frac{1}{m-1} G_{m-1}(x) + (m-1)\gamma_{m-1}(x).$$

So $\gamma_m'(x) = \frac{1}{m-1} G_{m-1}(x) + (m-1)\gamma_{m-1}(x)$, and from this, we have

$$\left(\frac{1}{m} \left(\gamma_{m+1}(x) - \frac{1}{m(m+1)} G_{m+1}(x) \right) \right)' = \gamma_m(x).$$

Since $G_{m+1}(1) + G_{m+1}(0) = 2\delta_{m,0}$,

$$\int_0^1 \gamma_m(x)dx = \frac{1}{m} \left[\gamma_{m+1}(x) - \frac{1}{m(m+1)} G_{m+1}(x) \right]_0^1$$

$$= \frac{1}{m} \left(\gamma_{m+1}(1) - \gamma_{m+1}(0) - \frac{1}{m(m+1)} \left(G_{m+1}(1) - G_{m+1}(0) \right) \right)$$

$$= \frac{1}{m} \left(\gamma_{m+1}(1) - \gamma_{m+1}(0) + \frac{2}{m(m+1)} G_{m+1} \right),$$

and

$$\gamma_m(1) - \gamma_m(0) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)} \left(E_k(1)G_{m-k}(1) - E_kG_{m-k} \right)$$

$$= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} \left((-E_k + 2\delta_{k,0}) (-G_{m-k} + 2\delta_{m-1,k}) - E_kG_{m-k} \right)$$

$$= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} \left(-2E_k\delta_{m-1,k} \right)$$

$$= -\frac{2E_{m-1}}{m-1}.$$

So

$$\gamma_m(0) = \gamma_m(1) \iff E_{m-1} = 0$$
$$\iff m = 2n+1, \ (n \geq 1).$$

Also,

$$\int_0^1 \gamma_m(x)dx = \frac{1}{m}\left(-\frac{2E_m}{m} + \frac{2}{m(m+1)}G_{m+1}\right) = \frac{1}{m}\left(-\frac{2E_m}{m} + \frac{2E_m}{m}\right) = 0.$$

Now, we are going to determine the Fourier coefficients $C_n^{(m)}$.

Case 1 : $n \neq 0$.

$$\begin{aligned}
C_n^{(m)} &= \int_0^1 \gamma_m(x)e^{-2\pi inx}dx \\
&= -\frac{1}{2\pi in}\left[\gamma_m(x)e^{-2\pi inx}\right]_0^1 + \frac{1}{2\pi in}\int_0^1 \gamma_m'(x)e^{-2\pi inx}dx \\
&= -\frac{1}{2\pi in}(\gamma_m(1)-\gamma_m(0)) + \frac{1}{2\pi in}\int_0^1 \left\{\frac{1}{m-1}G_{m-1}(x) + (m-1)\gamma_{m-1}(x)\right\} e^{-2\pi inx}dx \\
&= -\frac{1}{2\pi in}(\gamma_m(1)-\gamma_m(0)) + \frac{1}{2\pi in(m-1)}\int_0^1 G_{m-1}(x)e^{-2\pi inx}dx \\
&\quad + \frac{m-1}{2\pi in}\int_0^1 \gamma_{m-1}(x)e^{-2\pi inx}dx \\
&= \frac{2E_{m-1}}{2\pi in(m-1)} + \frac{2}{2\pi in(m-1)}\Phi_m + \frac{m-1}{2\pi in}C_n^{(m-1)},
\end{aligned}$$

where $\Phi_m = \sum_{k=1}^{m-2} \frac{(m-1)_{k-1}G_{m-k}}{(2\pi in)^k}$, and, for $l \geq 2$,

$$\int_0^1 G_l(x)e^{-2\pi inx}dx = \begin{cases} 2\sum_{k=1}^{l-1}\frac{(l)_{k-1}G_{l-k+1}}{(2\pi in)^k}, & \text{for } n \neq 0, \\ -\frac{2G_{l+1}}{l+1}, & \text{for } n = 0. \end{cases}$$

Continuing our argument, we have

$$\begin{aligned}
C_n^{(m)} &= \frac{m-1}{2\pi in}C_n^{(m-1)} + \frac{2E_{m-1}}{2\pi in(m-1)} + \frac{2}{2\pi in(m-1)}\Phi_m \\
&= \frac{m-1}{2\pi in}\left(\frac{m-2}{2\pi in}C_n^{(m-2)} + \frac{2E_{m-2}}{2\pi in(m-2)} + \frac{2}{2\pi in(m-2)}\Phi_{m-1}\right) \\
&\quad + \frac{2E_{m-1}}{2\pi in(m-1)} + \frac{2}{2\pi in(m-1)}\Phi_m \\
&= \frac{(m-1)_2}{(2\pi in)^2}C_n^{(m-2)} + \sum_{j=1}^{2}\frac{2(m-1)_{j-1}}{(2\pi in)^j(m-j)}E_{m-j} + \sum_{j=1}^{2}\frac{2(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Phi_{m-j+1} \\
&= \cdots \\
&= \frac{(m-1)!}{(2\pi in)^{m-2}}C_n^{(2)} + \sum_{j=1}^{m-2}\frac{2(m-1)_{j-1}}{(2\pi in)^j(m-j)}E_{m-j} + \sum_{j=1}^{m-2}\frac{2(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Phi_{m-j+1} \\
&= -\frac{(m-1)!}{(2\pi in)^{m-1}} + \sum_{j=1}^{m-2}\frac{2(m-1)_{j-1}}{(2\pi in)^j(m-j)}E_{m-j} + \sum_{j=1}^{m-2}\frac{2(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Phi_{m-j+1},
\end{aligned}$$

where $C_n^{(2)} = \int_0^1 (x - \frac{1}{2}) e^{-2\pi inx} dx = -\frac{1}{2\pi in}$.
Here we note that

$$\begin{aligned}
&\sum_{j=1}^{m-2} \frac{2(m-1)_{j-1}}{(2\pi in)^j (m-j)} \Phi_{m-j+1} \\
=&\sum_{j=1}^{m-2} \frac{2(m-1)_{j-1}}{(2\pi in)^j (m-j)} \sum_{k=1}^{m-j-1} \frac{(m-j)_{k-1} G_{m-j-k+1}}{(2\pi in)^k} \\
=&\sum_{j=1}^{m-2} \sum_{k=1}^{m-j-1} \frac{2(m-1)_{j+k-2}}{(2\pi in)^{j+k}(m-j)} G_{m-j-k+1} \\
=&\frac{2}{m} \sum_{j=1}^{m-2} \sum_{s=j+1}^{m-1} \frac{(m)_{s-1}}{(2\pi in)^s (m-j)} G_{m-s+1} \\
=&\frac{2}{m} \sum_{s=2}^{m-1} \frac{(m)_{s-1}}{(2\pi in)^s} G_{m-s+1} \sum_{j=1}^{s-1} \frac{1}{m-j} \\
=&\frac{2}{m} \sum_{s=1}^{m-1} \frac{(m)_s}{(2\pi in)^s} \frac{G_{m-s+1}}{m-s+1} (H_{m-1} - H_{m-s}).
\end{aligned}$$

Thus

$$\begin{aligned}
C_n^{(m)} =& -\frac{(m-1)!}{(2\pi in)^{m-1}} + \frac{2}{m} \sum_{s=1}^{m-2} \frac{(m)_s}{(2\pi in)^s (m-s)} E_{m-s} \\
&+ \frac{2}{m} \sum_{s=1}^{m-1} \frac{(m)_s}{(2\pi in)^s} \frac{G_{m-s+1}}{m-s+1} (H_{m-1} - H_{m-s}) \\
=& -\frac{(m-1)!}{(2\pi in)^{m-1}} + \frac{2}{m} \sum_{s=1}^{m-2} \frac{(m)_s}{(2\pi in)^s (m-s)} E_{m-s} \\
&+ \frac{2}{m} \sum_{s=1}^{m-2} \frac{(m)_s}{(2\pi in)^s} \frac{G_{m-s+1}}{m-s+1} (H_{m-1} - H_{m-s}) - \frac{(m-1)!}{(2\pi in)^{m-1}} (H_{m-1} - 1) \\
=& -\frac{(m-1)!}{(2\pi in)^{m-1}} H_{m-1} + \frac{2}{m} \sum_{s=1}^{m-2} \frac{(m)_s}{(2\pi in)^s} \left\{ \frac{G_{m-s+1}}{m-s+1} (H_{m-1} - H_{m-s}) + \frac{E_{m-s}}{m-s} \right\}.
\end{aligned}$$

Case 2 : $n = 0$.

$$C_0^{(m)} = \int_0^1 \gamma_m(x) dx = 0.$$

$\gamma_m(< x >)$, $(m \geq 2)$ is piecewise C^∞. Moreover, $\gamma_m(< x >)$ is continuous for all odd integers ≥ 3, and discontinuous with jump discontinuities at integers for all even integers ≥ 2.

Assume first that m is an odd integer ≥ 3. Then $\gamma_m(0) = \gamma_m(1)$. $\gamma_m(< x >)$ is piecewise C^∞, and continuous. So the Fourier series of $\gamma_m(< x >)$ converges uniformly to $\gamma_m(< x >)$, and

$$\begin{aligned}\gamma_m(< x >) =& \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\Bigg\{-\frac{(m-1)!}{(2\pi i n)^{m-1}}H_{m-1}\\ &+\frac{2}{m}\sum_{s=1}^{m-2}\frac{(m)_s}{(2\pi i n)^s}\left(\frac{G_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s})+\frac{E_{m-s}}{m-s}\right)\Bigg\}e^{2\pi i n x}\\ =&H_{m-1}\left(-(m-1)!\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{e^{2\pi i n x}}{(2\pi i n)^{m-1}}\right)\\ &-\frac{2}{m}\sum_{s=1}^{m-2}\binom{m}{s}\left(\frac{G_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s})+\frac{E_{m-s}}{m-s}\right)\left(-s!\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{e^{2\pi i n x}}{(2\pi i n)^s}\right)\\ =&H_{m-1}B_{m-1}(< x >)-\frac{2}{m}\sum_{s=2}^{m-2}\binom{m}{s}\left(\frac{G_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s})+\frac{E_{m-s}}{m-s}\right)\\ &\times B_s(< x >)-\frac{2E_{m-1}}{m-1}\times\begin{cases}B_1(< x >), & \text{for } x\notin\mathbb{Z},\\ 0, & \text{for } x\in\mathbb{Z}.\end{cases}\end{aligned}$$

Now, we can state our first theorem.

Theorem 4.1. *Let m be an odd integer ≥ 3. Then we have the following.*

(i)

$$\sum_{k=1}^{m-1}\frac{1}{k(m-k)}E_k(< x >)G_{m-k}(< x >)$$

has the Fourier series expansion

$$\begin{aligned}&\sum_{k=1}^{m-1}\frac{1}{k(m-k)}E_k(< x >)G_{m-k}(< x >)\\ =&\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\Bigg\{-\frac{(m-1)!}{(2\pi i n)^{m-1}}H_{m-1}\\ &+\frac{2}{m}\sum_{s=1}^{m-2}\frac{(m)_s}{(2\pi i n)^s}\left(\frac{G_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s})+\frac{E_{m-s}}{m-s}\right)\Bigg\}e^{2\pi i n x},\end{aligned}$$

for all $x\in(-\infty,\infty)$. Here the convergence is uniform.

(ii)

$$\begin{aligned}&\sum_{k=1}^{m-1}\frac{1}{k(m-k)}E_k(< x >)G_{m-k}(< x >)\\ =&H_{m-1}B_{m-1}(< x >)-\frac{2}{m}\sum_{s=2}^{m-2}\binom{m}{s}\left(\frac{G_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s})+\frac{E_{m-s}}{m-s}\right)B_s(< x >).\end{aligned}$$

Here $B_s(< x >)$ is the Bernoulli function.

Assume next that m is an even integer ≥ 4. Then $\gamma_m(0) \neq \gamma_m(1)$. Hence $\gamma_m(< x >)$ is piecewise C^∞ and discontinuous with jump discontinuities at integers. The Fourier series of $\gamma_m(< x >)$ converges pointwise to $\gamma_m(< x >)$, for $x \notin \mathbb{Z}$, and converges to

$$\begin{aligned}&\frac{1}{2}(\gamma_m(0)+\gamma_m(1)) = \gamma_m(0) - \frac{E_{m-1}}{m-1}\\ =&\sum_{k=1}^{m-2}\frac{1}{k(m-k)}E_k G_{m-k},\end{aligned}$$

for $x \in \mathbb{Z}$. Now, we can state our second theorem.

Theorem 4.2. *Let m be an even integer ≥ 4. Then we have the following.*

(i)

$$\begin{aligned}&\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\Bigg\{-\frac{(m-1)!}{(2\pi i n)^{m-1}}H_{m-1}\\ &+\frac{2}{m}\sum_{s=1}^{m-2}\frac{(m)_s}{(2\pi i n)^s}\left(\frac{G_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s})+\frac{E_{m-s}}{m-s}\right)\Bigg\}e^{2\pi i n x}\\ =&\begin{cases}\sum_{k=1}^{m-1}\frac{1}{k(m-k)}E_k(< x >)G_{m-k}(< x >), & \text{for } x \notin \mathbb{Z},\\ \sum_{k=1}^{m-2}\frac{1}{k(m-k)}E_k G_{m-k}, & \text{for } x \in \mathbb{Z}.\end{cases}\end{aligned}$$

Here the convergence is pointwise.

(ii)

$$\begin{aligned}&-\frac{2}{m}\sum_{s=1}^{m-1}\binom{m}{s}\left(\frac{G_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s})+\frac{E_{m-s}}{m-s}\right)B_s(< x >)\\ =&\sum_{k=1}^{m-1}\frac{1}{k(m-k)}E_k(< x >)G_{m-k}(< x >),\ \text{for } x \notin \mathbb{Z};\\ &-\frac{2}{m}\sum_{s=2}^{m-1}\binom{m}{s}\left(\frac{G_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s})+\frac{E_{m-s}}{m-s}\right)B_s(< x >)\\ =&\sum_{k=1}^{m-2}\frac{1}{k(m-k)}E_k G_{m-k},\ \text{for } x \in \mathbb{Z}.\end{aligned}$$

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MAJORIZATION PROPERTIES FOR CERTAIN FAMILIES OF ANALYTIC FUNCTIONS IN THE UNIT DISK

ADEL A. ATTIYA, M. F. YASSEN, AND MAHER I. ABDELHAFIZ

ABSTRACT. The main object of this paper is to introduce the majorization properties for certain families of analytic functions associated with generalized Srivastava-Attiya operator in the unit disk. Also, some applications of our results are discussed which give a number of new results.

1. INTRODUCTION

Let $A(p)$ denote the class of functions $f(z)$ of the from

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p}, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Also, let $A = A(1)$.

Definition 1.1. *Let f and F be analytic functions in $\mathbb{U}$, f is said to be majorized by F in $\mathbb{U}$ (see [15], [18]), written $f \ll F$, $z \in \mathbb{U}$, if there exists a function φ, analytic in $\mathbb{U}$ such that*

$$|\varphi(z)| \leq 1 \quad \text{and} \quad f(z) = \varphi(z)F(z) \quad (z \in \mathbb{U}). \tag{1.2}$$

Noting that the concept of majorization is closely related to the concept of quasi-subordination between analytic functions (see [18]).

Definition 1.2. *Let f and F be analytic functions. The function f is said to be subordinate to F, written $f \prec F$, if there exists a function w analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$, and such that $f(z) = g(w(z))$, in particular, if F is univalent, then $f \prec F$ if and only if $f(0) = F(0)$ and $f(\mathbb{U}) \subset F(\mathbb{U})$.*

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A general Hurwitz-Lerch Zeta function $\Phi(z, s, b)$ defined by (*cf., e.g.,* [20, P. 121 et seq.])

$$\Phi(z, s, b) = \sum_{k=0}^{\infty} \frac{z^k}{(k+b)^s}, \tag{1.3}$$

$(b \in \mathbb{C} \setminus \mathbb{Z}_0^-,\ \mathbb{Z}_0^- = \mathbb{Z}^- \cup \{0\} = \{0, -1, -2, \dots\},\ s \in \mathbb{C}$ when $z \in \mathbb{U},\ \operatorname{Re}(s) > 1$ when $|z| = 1)$.

Many authors studied and invistagated various properties of $\Phi(z, s, b)$, see *e.g.* [2], [6], [5], [7], [8], [14], [10], [11], [19], [21], [22] and [17].

Now, let us define, the operator $J_{s,b}(f)$ which has been introduced by Srivastava and Attiya [19]

$$J_{s,b}(f)(z) = G_{s,b}(z) * f(z) \tag{1.4}$$

$$\left(z \in \mathbb{U};\ f \in A;\ b \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ s \in \mathbb{C}\right)$$

where

$$G_{s,b}(z) = (1+b)^s \left[\Phi(z, s, b) - b^{-s}\right] \tag{1.5}$$

and $*$ denotes the Hadamard product .

Moreover, Attiya and Hakami [2] defined the function $G_{s,b,t}$ by

$$G_{s,b,t} = 1 + (t+b)^s\ z\Phi(z, s, 1+t+b) \tag{1.6}$$

$$\left(z \in \mathbb{U};\ b \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ s \in \mathbb{C};\ t \in \mathbb{R}\right),$$

we denote by

$$\mathcal{J}_{s,b}^t(f):\ A(p) \longrightarrow A(p),$$

Attiya and Hakami [2] defined the operator $\mathcal{J}_{s,b}^t(f)(z)$ by:

$$\mathcal{J}_{s,b}^t(f)(z) = z^p G_{s,b,t} * f(z) \tag{1.7}$$

$$\left(z \in \mathbb{U};\ f \in A(p); b \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ s \in \mathbb{C};\ t \in \mathbb{R}\right),$$

where $*$ denotes the Hadamard product
Noting that

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$$\mathcal{J}_{s,b}^{t}(f)(z) = z^p + \sum_{k=1}^{\infty} \left(\frac{t+b}{k+t+b} \right)^s a_{k+p}\ z^{k+p}\ (z \in \mathbb{U}) \tag{1.8}$$

we note that

$$\mathcal{J}_{s,b}^{1}(f) = J_{s,b}^{\ p}(f)\ , \tag{1.9}$$

where $J_{s,b}^{\ p}(f)$ introduced by Liu [13] .

The operator $\mathcal{J}_{s,b}^{t}(f)$ generalizes many operators *e.g.*, Srivastava and Attiya operator [19], Liu operator [13], Alexander operator [1], Libera operator [12], Bernardi operator [4] and Jung-Kim-Srivastava integral operator [9].

Now, we begin by the following lemma due to Attiya and Hakami [2].

Lemma 1.1. *Let $f(z) \in A(p)$, then*

$$z \left(\mathcal{J}_{s+1,b}^{t}\ f(z) \right)' = (t+b)\mathcal{J}_{s,b}^{t} f(z) - (t+b-p)\mathcal{J}_{s+1,b}^{t} f(z), \tag{1.10}$$
$$\left(z \in \mathbb{U};\ b \in \mathbb{C} \backslash \mathbb{Z}_0^-;\ s \in \mathbb{C};\ t \in \mathbb{R} \right)$$

Definition 1.3. *A function $f(z) \in A(p)$ is said to be in the class $S_{s,b,p}^{n,t}(A,B,\zeta)$ if it satisfies*

$$1 + \frac{1}{\zeta} \left(\frac{z \left(\left(\mathcal{J}_{s+1,b}^{t} \right)^{(n+1)} (f)(z) \right)}{\left(\mathcal{J}_{s+1,b}^{t} \right)^{(n)} (f)(z)} - p + n \right) \prec \frac{1+Az}{1+Bz}, \tag{1.11}$$

where $n \in \mathbb{N}_0 = \{0, 1, ...\}, -1 \leq B < A \leq 1, \zeta \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}, s \in \mathbb{C}, t \in \mathbb{R}$ and $b \in \mathbb{C} \setminus \mathbb{Z}_0^-$.

We note that $S_{s-1,b,1}^{0,1}(A, B, 1-\alpha) = H_{s,b,\alpha}(A, B)$ the class which introduced by Kutbi and Attiya [10], $S_{-1,b,1}^{0,1}(1, -1, 1-\alpha)$ the well known class of starlike function of order α. Also, using special cases of n, b, p, t, A, B, ζ we have many various classes associated with Alexander operator, Libera operator, Bernardi and Jung-Kim-Srivastava operator.

Also, we define the following classes:

(1) $S_{s,b,p}^{n,t}(-1, 1, \zeta) = S_{s,b,p}^{n,t}(\zeta)$.
(2) $S_{s,b,p}^{n,t}(-1, 1, 1) = S_{s,b,p}^{n,t}$.
(3) $S_{0,0,p}^{n,t}(A, B, \zeta) = \mathcal{A}_p^{n,t}(A, B, \zeta)$.
(4) $S_{0,1,p}^{n,t}(A, B, \zeta) = \mathcal{L}_p^{n,t}(A, B, \zeta)$.
(5) $S_{0,\gamma,p}^{n,t}(A, B, \zeta) = \mathcal{L}_{\gamma,p}^{n,t}(A, B, \zeta)$ (γ real ; $\gamma > -1$)

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(6) $S^{n,t}_{\sigma,1,p}(A,B,\zeta) = \mathcal{I}^{n,t}_{\sigma,p}(A,B,\zeta)$ (σ real ; $\sigma > 0$)

2. Main Results

To introduce our results, we need the following lemma which can be proved by using Lemma 1.1 and induction .

Lemma 2.1. *Let $f(z) \in A(p)$, then*

(2.1)

$$z\left(\mathcal{J}^t_{s+1,b}\right)^{(n+1)}(f)(z) = (t+b)\left(\mathcal{J}^t_{s,b}\right)^{(n)}(f)(z) - (t+n+b-p)\left(\mathcal{J}^t_{s+1,b}\right)^{(n)}(f)(z)$$

$$\left(n \in \mathbb{N}_0,\ z \in \mathbb{U};\ b \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ s \in \mathbb{C};\ t \in \mathbb{R}\right)$$

We begin by proving the following main result.

Theorem 2.1. *Let the function $g(z) \in S^n_{s,b}(A,B,\zeta)$, if*

$$\mathcal{J}^{(n)}_{s+1,b}(f)(z) \ll \mathcal{J}^{(n)}_{s+1,b}(g)(z), \quad (z \in \mathbb{U}), \tag{2.2}$$

then

$$\left|\left(\mathcal{J}^t_{s+1,b}\right)^{(n)}(f)(z)\right| \leq \left|\left(\mathcal{J}^t_{s+1,b}\right)^{(n)}(g)(z)\right| \ (|z| \leq r_0), \tag{2.3}$$

where $f(z) \in A(p)$ and $r_0 = r_0(\zeta, b, A, B)$ is the positive root and the smallest of the equation

(2.4)

$$r^3|\zeta(A-B)+(t+b)B| - [|t+b|+2|B|]r^2 - [|\zeta(A-B)+(t+b)B|+2]r + |t+b| = 0,$$

$$(-1 \leq B < A \leq 1, \zeta \in \mathbb{C}^*, s \in \mathbb{C}, b \in \mathbb{C}\setminus\mathbb{Z}_0^-, t \in \mathbb{R}),$$

Proof. Since $g(z) \in S^{n,t}_{s,b,p}(A,B,\zeta)$, then (1.11) gives

$$1 + \frac{1}{\zeta}\left(\frac{z\left(\left(\mathcal{J}^t_{s+1,b}\right)^{(n+1)}(g)(z)\right)}{\left(\mathcal{J}^t_{s+1,b}\right)^{(n)}(g)(z)} - p + n\right) = \frac{1 + A\,\omega(z)}{1 + B\,\omega(z)}, \tag{2.5}$$

where $\omega(z)$ is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \text{ and } |\omega(z)| < 1 \quad (z \in \mathbb{U}).$$

From (2.5), we get

(2.6)

$$\frac{z\left(\left(\mathcal{J}^t_{s+1,b}\right)^{(n+1)}(g)(z)\right)}{\left(\mathcal{J}^t_{s+1,b}\right)^{(n)}(g)(z)} = \frac{(p-n) + [(p-n)B + \zeta(A-B)]\omega(z)}{1 + B\,\omega(z)}.$$

by using Lemma 2.1 and (2.6), we get

(2.7)

$$\left|\left(\mathcal{J}^t_{s+1,b}\right)^{(n)}(g)(z)\right| \leq \frac{(t+b)[1+|B||z|]}{(t+b) - |\zeta(A-B)+(t+b)B||z|}\left|\left(\mathcal{J}^t_{s,b}\right)^{(n)}(g)(z)\right|.$$

Since $\left(\mathcal{J}_{s+1,b}^{t}\right)^{(n)}(f)(z))$ is majorized by $\left(\mathcal{J}_{s+1,b}^{t}\right)^{(n)}(g)(z)$, in $\mathbb{U}$, therefore, the equation (2.2), gives

$$\left(\mathcal{J}_{s+1,b}^{t}\right)^{(n)}(f)(z)) = \varphi(z)\left(\mathcal{J}_{s+1,b}^{t}\right)^{(n)}(g)(z), \tag{2.8}$$

where $|\varphi(z)| \le 1$. Differentiating (2.8) with respect to z , we have the following relation:

(2.9)

$$z(\left(\mathcal{J}_{s+1,b}^{t}\right)^{(n+1)}(f)(z)) = z\varphi'(z)\left(\mathcal{J}_{s+1,b}^{t}\right)^{(n)}(g)(z)+z\varphi(z)\left(\mathcal{J}_{s+1,b}^{t}\right)^{(n+1)}(g)(z).$$

Using (2.1) in the above equation, we get

(2.10)

$$\left(\mathcal{J}_{s,b}^{t}\right)^{(n)}(f)(z)) = \frac{z\varphi'(z)}{(t+b)}\left(\mathcal{J}_{s+1,b}^{t}\right)^{(n)}(g)(z) + \varphi(z)\left(\mathcal{J}_{s,b}^{t}\right)^{(n)}(g)(z).$$

since $\varphi \in \mathcal{P}$ satisfies the inequality (See, e.g., Nehari [16])

$$|\varphi'(z)| \le \frac{1-|\varphi(z)|^2}{1-|z|^2}, \quad (z \in \mathbb{U}), \tag{2.11}$$

and making use of (2.7) and (2.11) in(2.10), it yields

(2.12)

$$|\left(\mathcal{J}_{s,b}^{t}\right)^{(n)}(f)(z)| \le \left(|\varphi(z)| + \frac{1-|\varphi(z)|^2}{1-|z|^2}\frac{[1+|B||z|]|z|}{|t+b| - |\zeta(A-B)+(t+b)B||z|}\right)|\left(\mathcal{J}_{s,b}^{t}\right)^{(n)}(g)(z)|.$$

Setting

$$|z| = r \quad \textit{and} \quad |\varphi(z)| = \eta \quad (0 \le \eta \le 1)$$

this gives us the following inequality

(2.13)

$$|\left(\mathcal{J}_{s,b}^{t}\right)^{(n)}(f)(z)| \le \frac{\Phi(\eta)}{(1-r^2)\left[|t+b| - |\zeta(A-B)+(t+b)B|r\right]}|\left(\mathcal{J}_{s,b}^{t}\right)^{(n)}(g)(z)|,$$

where

(2.14)

$$\Phi(\eta) = -r(1+|B|r)\eta^2 + (1-r^2)\left[|t+b| - |\zeta(A-B)+(t+b)B|r\right]\eta + r(1+|B|r),$$

the function $\Phi(\eta)$ takes the maximum value at $\eta = 1$, with $r_0 = r_0(A,B,s,b,t)$ where r_0 is the smallest positive root of (2.4). Moreover, if $0 \le \eta \le r_0(A,B,s,b,t)$ then the function $\Psi(\eta)$ defined by

$$\Psi(\eta) = -\sigma(1+|B|\sigma)\eta^2 + (1-\sigma^2)\left[|t+b| - |\zeta(A-B)+(t+b)B|\sigma\right]\eta + \sigma(1+|B|\sigma),$$

is an increasing function on the interval $[0,1]$, therefore

$$\Psi(\eta) \le \Psi(1) = (1-\sigma^2)[|t+b| - |\zeta(A-B)+(t+b)B|\sigma], \tag{2.15}$$

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$$(0 \le \eta \le 1; 0 \le \sigma \le r_0(A, B, s, b)).$$

Hence putting $\eta = 1$, in (2.14), we conclude that (2.3) of Theorem 2.1 holds true for

$$|z| \le r_0 = r_0(A, B, s, b),$$

where r_0 is the smallest positive root of equation (2.4). This completes the proof of Theorem 2.1. □

Remark 2.1. Putting $t = 1$ in Theorem 2.1, we have the result due to Attiya and Yassen [3].

Putting $A = 1$ and $B = -1$ in Theorem 2.1, we have the following result.

Corollary 2.1. *Let the function* $g(z) \in \mathcal{S}^{n,t}_{s,b,p}(\zeta)$, *if*

$$\left(\mathcal{J}^t_{s+1,b}\right)^{(n)}(f)(z) \ll \left(\mathcal{J}^t_{s+1,b}\right)^{(n)}(g)(z), \quad (z \in \mathbb{U}), \tag{2.16}$$

then

$$\left|\left(\mathcal{J}^t_{s+1,b}\right)^{(n)}(f)(z)\right| \le \left|\left(\mathcal{J}^t_{s+1,b}\right)^{(n)}(g)(z)\right| \; (|z| \le r_0), \tag{2.17}$$

where $f(z) \in A(p)$ *and* r_0 *given by*

$$r_0 = \begin{cases} \frac{m - \sqrt{m^2 - 4|b+t||2\zeta - b - t|}}{2|2\zeta - b - t|}, & \zeta \ne \frac{b+t}{2} \\ \frac{\sqrt{1 + |b+t|(2 + |b+t|)} - 1}{2 + |b+t|}, & \zeta = \frac{b+t}{2} \end{cases}, \tag{2.18}$$

$m = 2 + |b+t| + |2\zeta - b - t|$, $\zeta \in \mathbb{C}^*, s \in \mathbb{C}$ *and* $b \in \mathbb{C} \setminus \mathbb{Z}^-_0$).

Letting $A = 1$, $B = -1$ and $\zeta = 1$ in Theorem 2.1, we get the following property.

Corollary 2.2. *Let the function* $g(z) \in \mathcal{S}^{n,t}_{s,b,p}$, *if*

$$\left(\mathcal{J}^t_{s+1,b}\right)^{(n)}(f)(z) \ll \left(\mathcal{J}^t_{s+1,b}\right)^{(n)}(g)(z), \quad (z \in \mathbb{U}), \tag{2.19}$$

then

$$\left|\left(\mathcal{J}^t_{s+1,b}\right)^{(n)}(f)(z)\right| \le \left|\left(\mathcal{J}^t_{s+1,b}\right)^{(n)}(g)(z)\right| \; (|z| \le r_0), \tag{2.20}$$

where $f(z) \in A(p)$ *and* r_0 *given by*

$$r_0 = \begin{cases} \frac{m - \sqrt{m^2 - 4|b+t||2 - b - t|}}{2|2 - b - t|}, & b \ne 2 - t \\ \frac{1}{2}, & b = 2 - t \end{cases}, \tag{2.21}$$

$m = 2 + |b+t| + |2 - b - t|$, $s \in \mathbb{C}$ *and* $b \in \mathbb{C} \setminus \mathbb{Z}^-_0$).

Letting $s = b = 0$, in Theorem 2.1, we have the following corollary.

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Corollary 2.3. *Let the function* $g(z) \in \mathcal{A}_p^{n,t}(A, B, \zeta)$*, if*

$$\left(\mathcal{A}_p^t\right)^{(n)}(f)(z) \ll \left(\mathcal{A}_p^t\right)^{(n)}(g)(z), \quad (z \in \mathbb{U}), \tag{2.22}$$

then

$$|\left(\mathcal{A}_p^t\right)^{(n)}(f)(z)| \leq |\left(\mathcal{A}_p^t\right)^{(n)}(g)(z)| \ (|z| \leq r_0), \tag{2.23}$$

where $f(z) \in A(p)$ *and* $r_0 = r_0(\zeta, A, B)$ *is the smallest positive root of the equation*

$$r^3|\zeta(A-B)+Bt|-[|t|+2|B|]r^2-[|\zeta(A-B)+Bt|+2]r+|t| = 0, \tag{2.24}$$
$$(-1 \leq B < A \leq 1, \zeta \in \mathbb{C}^*),$$

If we put $s = 0$ and $b = 1$,in Theorem 2.1, then we have the following result.

Corollary 2.4. *Let* $g(z) \in \mathcal{L}_p^{n,t}(A, B, \zeta)$*, if*

$$\left(\mathcal{L}_p^t\right)^{(n)}(f)(z) \ll \left(\mathcal{L}_p^t\right)^{(n)}(g)(z), \quad (z \in \mathbb{U}), \tag{2.25}$$

then

$$|\left(\mathcal{L}_p^t\right)^{(n)}(f)(z)| \leq |\left(\mathcal{L}_p^t\right)^{(n)}(g)(z)| \ (|z| \leq r_0), \tag{2.26}$$

where $f(z) \in A(p)$ *and* $r_0 = r_0(\zeta, A, B)$ *is the smallest positive root of the equation*

(2.27)

$$r^3|\zeta(A-B)+(t+1)B|-[|1+t|+2|B|]r^2-[|\zeta(A-B)+(1+t)B|+2]r+|1+t| = 0,$$
$$(-1 \leq B < A \leq 1, \zeta \in \mathbb{C}^*).$$

Putting $s = 0$ and $b = \gamma > -t$ in Theorem 2.1, we get the following corollary.

Corollary 2.5. *Let the function* $g(z) \in \mathcal{L}_\gamma^n(A, B, \zeta)$*, if*

$$\left(\mathcal{L}_{\gamma,p}^t\right)^{(n)}(f)(z) \ll \left(\mathcal{L}_{\gamma,p}^t\right)^{(n)}(g)(z), \quad (z \in \mathbb{U}, \ \gamma > -1), \tag{2.28}$$

then

$$|\left(\mathcal{L}_{\gamma,p}^t\right)^{(n)}(f)(z)| \leq |\left(\mathcal{L}_{\gamma,p}^t\right)^{(n)}(g)(z)| \ (|z| \leq r_0), \tag{2.29}$$

where $f(z) \in A(p)$ *and* $r_0 = r_0(\zeta, b, A, B)$ *is the smallest positive root of the equation*

(2.30)

$$r^3|\zeta(A-B)+(t+\gamma)B|-[t+\gamma+2|B|]r^2-[|\zeta(A-B)+(t+\gamma)B|+2]r+(t+\gamma) = 0,$$
$$(-1 \leq B < A \leq 1, \ \gamma > -1, \ \zeta \in \mathbb{C}^*, s \in \mathbb{C}),$$

Putting $s = \sigma$ (σ; real, $\sigma > 0$) and $b = 1$ in Theorem 2.1, we get the following corollary.

Corollary 2.6. *Let the function $g(z) \in \mathcal{I}_{\sigma}^{n}(A, B, \zeta)$, if*

$$\left(\mathcal{I}_{\sigma,p}^{t}\right)^{(n)}(f)(z) \ll \left(\mathcal{I}_{\sigma,p}^{t}\right)^{(n)}(g)(z), \quad (z \in \mathbb{U}; \sigma > 0), \tag{2.31}$$

then

$$|\left(\mathcal{I}_{\sigma,p}^{t}\right)^{(n)}(f)(z)| \leq |\left(\mathcal{I}_{\sigma,p}^{t}\right)^{(n)}(g)(z)| \ (|z| \leq r_0), \tag{2.32}$$

where $f(z) \in A(p)$ and $r_0 = r_0(\zeta, A, B)$ is the smallest positive root of the equation

$$r^3|\zeta(A-B)+(t+1)B| - [|1+t|+2|B|]r^2 - [|\zeta(A-B)+(1+t)B|+2]r + |1+t| = 0,$$
$$(-1 \leq B < A \leq 1, \zeta \in \mathbb{C}^*, s \in \mathbb{C}).$$

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